2015 Bored of Studies Trial Examinations

Mathematics Extension 1

Solutions

Section I

1. B	3. D	5. C	7. A	9. A
2. B	4. A	6. B	8. D	10. B

Working/Justification

Question 1

It can be deduced that the lengths of PA and PB are km and kn respectively for some positive constant k. Similarly it can be deduced that the lengths PQ and QB have the lengths rm and rn for some positive constant r. Hence

$$\frac{PQ}{PA} = \frac{r}{k}$$

But $PB = PQ + QB$ so $kn = r(m+n)$ or equivalently $\dfrac{r}{k} = \dfrac{n}{m+n}$. Hence

$$\frac{PQ}{PA} = \frac{n}{m+n}$$

So the answer is (B)

Question 2

Rewriting the differential equation in the more standard form

$$\frac{dN}{dt} = -k(N-P)$$

This has a general solution $N = P + Ae^{-kt}$

Since $t > 0$ then if $k < 0$ then when $t \to \infty$, $N \to \infty$ since $e^{-kt} \to \infty$

If $k > 0$ then when $t \to \infty$, $N \to P$ since $e^{-kt} \to 0$

Hence the answer is (B)

Question 3

(A) is not always true as the only triangle where an angle is bisected by the median is an isosceles triangle, which is not necessarily the case for ΔABC.

(B) is not always true as it suggests that ΔABC is an isosceles right angled triangle which is not necessarily the case.

(C) is not always true because a possible counterexample is where ΔABC is an isosceles right angled triangle, which suggests that ΔAMB would also be an isosceles right angled triangle since $\angle BAM = \angle ABM = 45°$ with hypotenuse AB which cannot equal BM.

(D) is always true because the points A, B and C are concyclic as a circle can be drawn around those points with AC as the diameter, thus satisfying the theorem that an angle in a semi-circle is a right angle. Since M represents the centre of this circle then $BM = AM$.

Question 4

(B) and (C) are not always true because it is possible that $P(x) = kQ(x)$ for some non-zero constant k since we only know that $P(x)$ and $Q(x)$ share the same set of roots. This means that $P(x)$ and $Q(x)$ are not necessarily identical and therefore do not necessarily have the same remainder when divided by any polynomial when considering the remainder theorem.

(D) is not always true because if $P(x) = kQ(x)$ as described above then $P(x)Q(x) = k[Q(x)]^2$. It is possible that $P(x)Q(x) < 0$ if $k < 0$.

(A) is always correct because if α_i is a common root of $P(x)$ and $Q(x)$ for every $i = 1, 2, ..., n$ then since $P(\alpha_i) = Q(\alpha_i) = 0$ this means that $P(\alpha_i) + Q(\alpha_i) = 0$ hence by the factor theorem the polynomial $P(x) + Q(x)$ has root of α_i as well for all $i = 1, 2, ..., n$.

Question 5

Using the fact that $\cos 2x = 2\cos^2 x - 1$

$$\begin{aligned}\int_0^\pi (4\cos^4 x - \cos^2 2x)\,dx &= \int_0^\pi (2\cos^2 x - \cos 2x)(2\cos^2 x + \cos 2x)\,dx \\ &= \int_0^\pi (1 + 2\cos 2x)\,dx \\ &= [x + \sin 2x]_0^\pi \\ &= \pi \qquad \text{hence the answer is } (C)\end{aligned}$$

Question 6

Deriving vertical equations for general projectile motion launched from the origin

$$\ddot{y} = -g$$

$$\dot{y} = -gt + V\sin\theta$$

$$y = -\frac{gt^2}{2} + Vt\sin\theta$$

To solve for the time of flight set $y = 0$ and $t > 0$

$$t\left(V\sin\theta - \frac{gt}{2}\right) = 0$$

$$t = \frac{2V\sin\theta}{g}$$

Note that particle A is the special case when $\theta = \frac{\pi}{2}$ so the time of flights t_A and t_B for particles A and B respectively are

$$t_A = \frac{2V}{g}$$

$$t_B = \frac{2V\sin\theta}{g}$$

$$t_A > t_B \quad \text{since } 0 < \theta < \frac{\pi}{2} \Rightarrow 0 < \sin\theta < 1$$

Hence particle A takes longer to land on the surface than particle B so the answer is (B).

Question 7

From the angle between two lines

$$\tan 45^\circ = \left|\frac{2m - m}{1 + (m)(2m)}\right|$$

$$|1 + 2m^2| = |m|$$

$$2m^2 - m + 1 = 0 \quad \text{or} \quad 2m^2 + m + 1 = 0$$

Both equations have a discriminant of -7 which means there are no solutions, so correct answer is (A).

Question 8

If there are an odd number of tosses there must be either more heads than tails or more tails than heads. Given the symmetrical nature of the scenario they must be equally likely (the binomial probabilities can be explicitly calculated). Hence (A) and (B) cannot be true.

If there are an even number of tosses there can either be more heads than tails, more tails than heads or an equal number of heads and tails. Given the symmetrical nature of the scenario having more heads than tails is equally likely as having more tails than heads. However, since it possible to have an equal number of heads and tails with a non-zero probability (which can only be achieved in an even number of tosses) then the probability of having more heads than tails is less than 50%. Hence the answer is (D)

Question 9

Note that if $f'(x) = g'(x)$ then $f(x) = g(x) + c$. So without actually computing the derivatives explicitly, one can check if the difference between the two functions is a constant and therefore they will have the same derivative.

Set k to be a fixed value (say 1) and apply a simple substitution of two different values of x (e.g. $x = \frac{\pi}{3}$ and $x = \frac{\pi}{6}$) for $\dfrac{\sin(3kx)}{\sin(kx)} - f(x)$ where $f(x)$ is one of the options. If the two values are not equal then clearly the choice of $f(x)$ does not make $\dfrac{\sin(3kx)}{\sin(kx)} - f(x)$ a constant.

The only choice of $f(x)$ where the two substitutions yield equal values should be (A) and in fact

$$\begin{aligned}\frac{\sin(3kx)}{\sin(kx)} - \frac{\cos(3kx)}{\cos(kx)} &= \frac{\sin(3kx)\cos(kx) - \cos(3kx)\sin(kx)}{\sin(kx)\cos(kx)} \\ &= \frac{\sin(3kx - kx)}{\frac{1}{2}\sin(2kx)} \\ &= 2\end{aligned}$$

Question 10

From the sketch, the velocity function has the equation $v = mx + b$ where $b > 0$ and $m < 0$. Note that for acceleration a

$$\begin{aligned}a &= v\frac{dv}{dx} \\ &= m^2x + bm\end{aligned}$$

This is a linear function with gradient m^2 and y-intercept bm. Note that $m^2 > 0$ and $bm < 0$ as $b > 0$ and $m < 0$ so the graph has a positive gradient and a negative y-intercept. Hence the answer is (B).

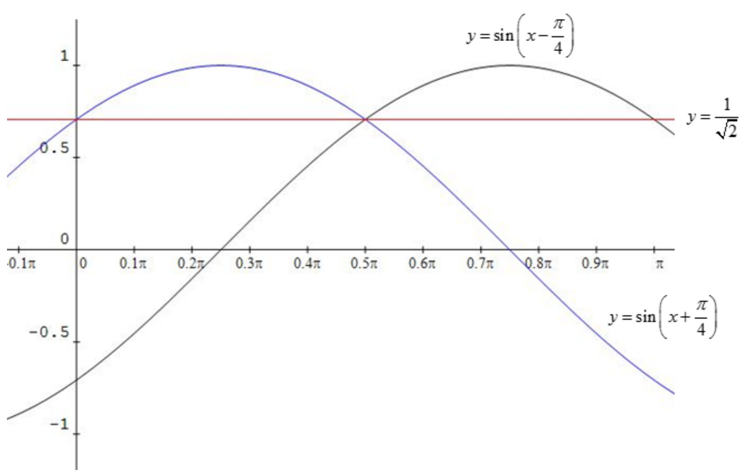
Section II

Question 11

(a) For $\dfrac{\pi}{4} \le x < \dfrac{\pi}{2}$, $\cos x < 1$ and $\sin x < 1$. So the inequality can be rearranged as follows

$$\begin{aligned}
\frac{\sin x}{\cos x - 1} &> \frac{\cos x}{\sin x - 1} \\
\frac{\sin x}{1 - \cos x} &< \frac{\cos x}{1 - \sin x} \\
\sin x - \sin^2 x - \cos x + \cos^2 x &< 0 \\
(\sin x - \cos x) - (\sin x - \cos x)(\sin x + \cos x) &< 0 \\
(\sin x - \cos x)(1 - \sin x - \cos x) &< 0 \\
(\sin x - \cos x)(\sin x + \cos x - 1) &> 0 \\
\sqrt{2}\left(\sin x \cos \frac{\pi}{4} - \cos x \sin \frac{\pi}{4}\right)\sqrt{2}\left(\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4} - \frac{1}{\sqrt{2}}\right) &> 0 \\
\sin\left(x - \frac{\pi}{4}\right)\left(\sin\left(x + \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}}\right) &> 0
\end{aligned}$$

To solve this consider a sketch of $y = \sin\left(x - \dfrac{\pi}{4}\right)$ and $y = \sin\left(x + \dfrac{\pi}{4}\right)$ on the same set of axes.



From the sketch it can be seen that in the domain $\dfrac{\pi}{4} \le x < \dfrac{\pi}{2}$ that $\sin\left(x - \dfrac{\pi}{4}\right) \ge 0$ and $\sin\left(x + \dfrac{\pi}{4}\right) > \dfrac{1}{\sqrt{2}}$.
Hence the solution is $\dfrac{\pi}{4} < x < \dfrac{\pi}{2}$.

(b) Using the fact that $2015 = 2016 - 1$

$$2015^{2015} + k = (2016-1)^{2015} + k$$

$$= \sum_{k=0}^{2015} \binom{2015}{k} (2016)^k (-1)^{2015-k} + k$$

$$= k + \binom{2015}{0} (-1)^{2015} + \sum_{k=1}^{2015} \binom{2015}{k} (2016)^k (-1)^{2015-k}$$

$$= k - 1 + \sum_{k=1}^{2015} \binom{2015}{k} (2016)^k (-1)^{2015-k}$$

But note every term in $\displaystyle\sum_{k=1}^{2015} \binom{2015}{k} (2016)^k (-1)^{2015-k}$ contains a factor 2016.

Since 2016 is divisible by 3 ($2016 = 3 \times 672$) then every term is divisible by 3. This implies that the entire sum is divisible by 3 so we can express $2015^{2015} + k$ in the form below

$$2015^{2015} + k = k - 1 + 3M \quad \text{where } M \text{ is a positive integer}$$

To ensure $2015^{2015} + k$ is divisible by 3 the lowest value of k must be 1.

(c)

(i) The polynomial can be written as the following for some constants p and q

$$P(x) = (px + q)(x^2 + kx + 1)$$

Equating the coefficient of x^3 we get $p = a$ and equating the constant term we get $q = c$ hence

$$P(x) = (ax + c)(x^2 + kx + 1)$$

From this we can deduce that one of the roots must be $-\dfrac{c}{a}$ which is also the product of the roots. This means the other two roots must multiply to give 1 hence the roots are of the form α, $\frac{1}{\alpha}$ and β

(ii) Consider the sum of the roots noting that $\beta = -\dfrac{c}{a}$

$$\alpha + \frac{1}{\alpha} + \beta = 0$$

$$\alpha + \frac{1}{\alpha} = \frac{c}{a}$$

Now consider the sum of the roots in pairs

$$\alpha \times \frac{1}{\alpha} + \alpha \times \beta + \frac{1}{\alpha} \times \beta = \frac{b}{a}$$

$$1 - \frac{c}{a}\left(\alpha + \frac{1}{\alpha}\right) = \frac{b}{a}$$

$$1 - \frac{c^2}{a^2} = \frac{b}{a}$$

$$a^2 - c^2 = ab$$

(d) First note that

$$P'(x) = Q(x) + (x-\alpha)Q'(x) \quad \text{sub } x = r$$

$$P'(r) = Q(r) + (r-\alpha)Q'(r) \quad \text{but } Q'(r) = 0$$

$$= Q(r)$$

At $x = r$, the equation of the tangent is

$$y - P(r) = P'(r)(x - r)$$

This tangent intersects the x-axis when $y = 0$

$$x = r - \frac{P(r)}{P'(r)}$$

$$= r - \frac{(r-\alpha)Q(r)}{Q(r)}$$

$$= r - (r - \alpha)$$

$$= \alpha$$

(e) Let $x = \dfrac{1-u}{1+u}$

$$dx = \frac{-(1+u)-(1-u)}{(1+u)^2}\,du$$

$$= -\frac{2}{(1+u)^2}\,du$$

When $x = 1, u = 0$ When $x = 0, u = 1$

Substituting these in

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = -2\int_1^0 \frac{\ln\left(1+\frac{1-u}{1+u}\right)}{1+\frac{(1-u)^2}{(1+u)^2}} \times \frac{1}{(1+u)^2}\,du$$

$$= 2\int_0^1 \frac{\ln\left(\frac{1+u+1-u}{1+u}\right)}{(1+u)^2+(1-u)^2}\,du$$

$$= 2\int_0^1 \frac{\ln\left(\frac{2}{1+u}\right)}{1+2u+u^2+1-2u+u^2}\,du$$

$$= 2\int_0^1 \frac{\ln 2 - \ln(1+u)}{2(1+u^2)}\,du$$

$$= \int_0^1 \frac{\ln 2}{1+u^2}\,du - \int_0^1 \frac{\ln(1+u)}{1+u^2}\,du$$

$$= \int_0^1 \frac{\ln 2}{1+x^2}\,dx - \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$$

$$2\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \ln 2\left[\tan^{-1} x\right]_0^1$$

$$= \frac{\pi}{4}\ln 2$$

$$\therefore \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2$$

Question 12

(a) By trigonometry

$$\tan\alpha = \frac{h}{OA}$$

$$OA = h\cot\alpha$$

similarly $\quad OB = h\cot\beta$

and $\quad OC = h\cot\gamma$

But $AB = OC$ (sides of regular hexgaon) so $AB = h\cot\gamma$.

In ΔOCB, note that $\angle OCB = \dfrac{2\pi}{3}$ as an interior angle of a regular hexagon.

$$OC = BC = x \quad \text{(sides of regular hexagon)}$$

$$\therefore \angle CBO = \angle COB = \frac{\pi}{6} \qquad \text{(equal angles opposite equal sides)}$$

Since $\angle CBA = \dfrac{2\pi}{3}$ as it also an interior angle of the hexagon then

$$\begin{aligned}\angle OBA &= \angle CBA - \angle CBO \\ &= \frac{\pi}{2}\end{aligned}$$

This means that ΔOBA is a right angled triangle so

$$OA^2 = OB^2 + AB^2 \qquad \text{(Pythagoras' theorem)}$$

$$h^2\cot^2\alpha = h^2\cot^2\beta + h^2\cot^2\gamma$$

$$\therefore \cot^2\alpha - \cot^2\beta = \cot^2\gamma$$

(b) The displacement equation about the origin t seconds after being pushed is

$$x = A\cos(nt+\alpha) \qquad \text{for constants } n \text{ and } -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$$

$$\dot{x} = -An\sin(nt+\alpha)$$

When $t = 0$, $\dot{x} = -u$ and $x = a$. From the velocity equation

$$-u = -An\sin\alpha$$

$$\sin\alpha = \frac{u}{An}$$

From the displacement equation

$$\begin{aligned} a &= A\cos\alpha \\ \cos\alpha &= \frac{a}{A} \end{aligned}$$

Required to find the lowest value of t where $x = -a$

$$\begin{aligned} -a &= A\cos(nt+\alpha) \\ -\frac{a}{A} &= \cos(nt+\alpha) \\ -\cos\alpha &= \cos(nt+\alpha) \\ \cos(\pi-\alpha) &= \cos(nt+\alpha) \end{aligned}$$

The lowest value of t occurs when

$$\pi - \alpha = nt + \alpha$$

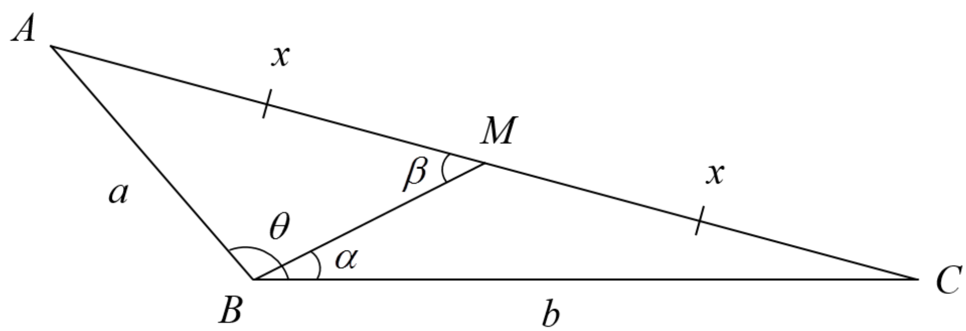
$$\begin{aligned} t &= \frac{\pi - 2\alpha}{n} \qquad \text{but } T = \frac{2\pi}{n} \\ &= \frac{T(\pi - 2\alpha)}{2\pi} \end{aligned}$$

If the particle was not pushed, it would take $\frac{T}{2}$ seconds to move from $x = a$ to $x = -a$ so the difference is

$$\begin{aligned} \frac{T}{2} - \frac{T(\pi-2\alpha)}{2\pi} &= \frac{\alpha T}{\pi} \\ &= \frac{T}{\pi}\tan^{-1}\left(\frac{u}{an}\right) \qquad \text{since } \sin\alpha = \frac{u}{An} \quad \text{and } \cos\alpha = \frac{a}{A} \quad \text{then } \tan\alpha = \frac{u}{an} \\ &= \frac{T}{\pi}\tan^{-1}\left(\frac{uT}{2\pi a}\right) \qquad \text{since } T = \frac{2\pi}{n} \end{aligned}$$

(c)

(i) Let $\angle AMB = \beta$ and hence $\angle BMC = \pi - \beta$ by angles on a line. Also, let $AM = MC = x$.



From the sine rule

$$\frac{a}{\sin\beta} = \frac{x}{\sin(\theta-\alpha)}$$

$$\frac{b}{\sin(\pi-\beta)} = \frac{x}{\sin\alpha}$$

Divide the two results and noting that $\sin\beta = \sin(\pi-\beta)$

$$\frac{a\sin(\pi-\beta)}{b\sin\beta} = \frac{\sin\alpha}{\sin(\theta-\alpha)}$$

$$\sin(\theta-\alpha) = \frac{b}{a}\sin\alpha \quad (*)$$

$$\theta = \alpha + \sin^{-1}\left(\frac{b}{a}\sin\alpha\right) \qquad \text{for } 0 < \theta-\alpha \leq \frac{\pi}{2}$$

$$\text{OR} \quad \theta = \alpha + \pi - \sin^{-1}\left(\frac{b}{a}\sin\alpha\right) \qquad \text{for } \frac{\pi}{2} < \theta-\alpha < \pi$$

For the case $0 < \theta - \alpha \leq \frac{\pi}{2}$

$$\frac{d\theta}{d\alpha} = 1 + \frac{1}{\sqrt{1-\frac{b^2}{a^2}\sin^2\alpha}} \times \frac{b}{a}\cos\alpha$$

$$= 1 + \frac{1}{\sqrt{1-\frac{\sin^2(\theta-\alpha)}{\sin^2\alpha}\times\sin^2\alpha}} \times \frac{\sin(\theta-\alpha)\cos\alpha}{\sin\alpha} \qquad \text{using } (*)$$

$$= 1 + \frac{\sin(\theta - \alpha)\cos\alpha}{\sin\alpha\sqrt{\cos^2(\theta - \alpha)}} \qquad \text{but } 0 < \theta - \alpha \leq \frac{\pi}{2} \qquad \text{so} \qquad \sqrt{\cos^2(\theta - \alpha)} = \cos(\theta - \alpha)$$

$$= \frac{\cos(\theta - \alpha)\sin\alpha + \sin(\theta - \alpha)\cos\alpha}{\cos(\theta - \alpha)\sin\alpha}$$

$$= \frac{\sin\theta}{\cos(\theta - \alpha)\sin\alpha}$$

For the case $\frac{\pi}{2} < \theta - \alpha < \pi$

$$\frac{d\theta}{d\alpha} = 1 - \frac{1}{\sqrt{1 - \frac{b^2}{a^2}\sin^2\alpha}} \times \frac{b}{a}\cos\alpha \qquad \text{which is similar to case above}$$

$$= 1 - \frac{\sin(\theta - \alpha)\cos\alpha}{\sin\alpha\sqrt{\cos^2(\theta - \alpha)}} \qquad \text{but } \frac{\pi}{2} < \theta - \alpha < \pi \qquad \text{so} \qquad \sqrt{\cos^2(\theta - \alpha)} = -\cos(\theta - \alpha)$$

$$= 1 + \frac{\sin(\theta - \alpha)\cos\alpha}{\sin\alpha\cos(\theta - \alpha)}$$

$$= \frac{\sin\theta}{\cos(\theta - \alpha)\sin\alpha}$$

Hence for all cases

$$\frac{d\theta}{d\alpha} = \frac{\sin\theta}{\sin\alpha\cos(\theta - \alpha)}$$

Alternatively, implict differentiation can be applied on $(*)$ to obtain the result.

(ii) Since α is decreasing at $\frac{a}{b}$ radians per second then $\frac{d\alpha}{dt} = -\frac{a}{b}$.

Using the chain rule

$$\frac{d\theta}{dt} = \frac{d\theta}{d\alpha} \times \frac{d\alpha}{dt}$$

$$= -\frac{\sin\theta}{\frac{b}{a}\sin\alpha\cos(\theta - \alpha)} \qquad \text{but from } (*) \quad \frac{b}{a}\sin\alpha = \sin(\theta - \alpha)$$

$$= -\frac{\sin\theta}{\sin(\theta - \alpha)\cos(\theta - \alpha)}$$

(d) When $n = 3$

$$\begin{aligned} LHS &= 3^4 \\ &= 81 \\ RHS &= 4^3 \\ &= 64 \end{aligned}$$

$LHS > RHS$ so the statement is true for $n = 3$

Assume the statement is true for $n = k$

$$k^{k+1} > (k+1)^k$$

Required to prove that the statement is true for $n = k + 1$

$$(k+1)^{k+2} > (k+2)^{k+1}$$

Since all terms are positive, first consider the expression

$$\begin{aligned} \frac{(k+2)^{k+1}}{(k+1)^{k+2}} &= \frac{(k+1+1)^{k+1}}{(k+1)(k+1)^{k+1}} \\ &= \frac{1}{k+1}\left(1 + \frac{1}{k+1}\right)^{k+1} \\ &< \frac{1}{k+1}\left(1 + \frac{1}{k}\right)^{k+1} \qquad \text{since } \frac{1}{k+1} < \frac{1}{k} \\ &= \frac{1}{k+1}\left(\frac{k+1}{k}\right)^{k+1} \\ &= \frac{1}{k+1} \times \frac{(k+1)^k}{k^{k+1}} \times (k+1) \qquad \text{but from the assumption } \frac{(k+1)^k}{k^{k+1}} < 1 \\ &< \frac{1}{k+1} \times 1 \times (k+1) \\ &= 1 \qquad \text{which is equivalent to } (k+1)^{k+2} > (k+2)^{k+1} \end{aligned}$$

Since the statement is true $n = 3$ then by induction it is true for all integers $n \geq 3$.

Question 13

(a) For the particle fired horizontally, the given equations of motion are as follows by setting $\theta = 0$

$$x = Vt$$

$$y = -\frac{1}{2}gt^2 + h$$

$$y = -\frac{gx^2}{2V^2} + h$$

When $t = T$ the particle hits the point $(R, 0)$ so substitute this into the equations of motion above

$$R = VT$$

$$\frac{1}{2}gT^2 = h \quad (*)$$

$$\frac{gR^2}{2V^2} = h \quad (**)$$

Now consider the Cartesian equation of the particle projected at angle $0 < \theta < \frac{\pi}{2}$ and substitute $(R, 0)$

$$-\frac{gR^2}{2V^2}\sec^2\theta + R\tan\theta + h = 0 \qquad \text{subsitute results } (*) \text{ and } (**)$$

$$-\frac{1}{2}gT^2\sec^2\theta + R\tan\theta + \frac{1}{2}gT^2 = 0$$

$$R\tan\theta - \frac{1}{2}gT^2(\sec^2\theta - 1) = 0$$

$$R\tan\theta - \frac{1}{2}gT^2\tan^2\theta = 0$$

$$\tan\theta\left(R - \frac{1}{2}gT^2\tan\theta\right) = 0$$

$$\therefore R = \frac{1}{2}gT^2\tan\theta \qquad \text{noting that } \tan\theta \neq 0 \text{ for } 0 < \theta < \frac{\pi}{2}.$$

(b)

(i) Let $y = \sin^{-1} x$ then $x = \sin y$. When $x \to 0$, $y \to 0$

$$\begin{aligned}\lim_{x\to 0} \frac{x}{\sin^{-1} x} &= \lim_{y\to 0} \frac{\sin y}{y} \\ &= 1\end{aligned}$$

(ii) The restriction on x is within the $\sin^{-1} x$ and that the denominator cannot be zero.

Hence the domain is $-1 \le x < 0$ and $0 < x \le 1$.

(iii) Since $\sin x < x$ for $x > 0$ then $x < \sin^{-1} x$ (noting that $\sin^{-1} x$ is an increasing function in its domain)

When $x > 0$ then $\sin^{-1} x > 0$ with $\dfrac{x}{\sin^{-1} x} < 1$. However, note that

$$f(-x) = \frac{-x}{\sin^{-1}(-x)} \qquad \text{but } \sin^{-1}(-x) = -\sin^{-1} x$$

$$= \frac{x}{\sin^{-1} x}$$

$$= f(x) \qquad \text{hence } f(x) \text{ is an even function}$$

So by symmetry this $f(x) < 1$ holds true for the entire domain $-1 \le x < 0$ and $0 < x \le 1$.

(iv) Using the quotient rule

$$f'(x) = \frac{\sin^{-1} x - \dfrac{x}{\sqrt{1-x^2}}}{\left(\sin^{-1} x\right)^2}$$

Given that $\theta < \tan\theta$ for $0 < \theta < \dfrac{\pi}{2}$ then

$$\theta - \frac{\sin\theta}{\cos\theta} < 0$$

Let $\theta = \sin^{-1} x$ for $0 < \theta < \frac{\pi}{2}$ or equivalently for $0 < x < 1$ which means that $x = \sin\theta$ and $\cos\theta = \sqrt{1-x^2}$ for $0 < \theta < \frac{\pi}{2}$. This means that for $0 < x < 1$

$$\sin^{-1} x - \frac{x}{\sqrt{1-x^2}} < 0$$

Since $\left(\sin^{-1} x\right)^2 > 0$ then $f'(x) < 0$ for $0 < x < 1$

(v) Rewriting the first derivative as

$$f'(x) = \frac{\sqrt{1-x^2}\sin^{-1}x - x}{\sqrt{1-x^2}\left(\sin^{-1}x\right)^2}$$

When $x \to 1$ then the denominator of $f'(x)$ approaches zero and the numerator approaches -1 hence $f'(x) \to \infty$.

(vi) Since $f(x)$ is an even function then the range for the domain $x > 0$ is also the range for $x < 0$

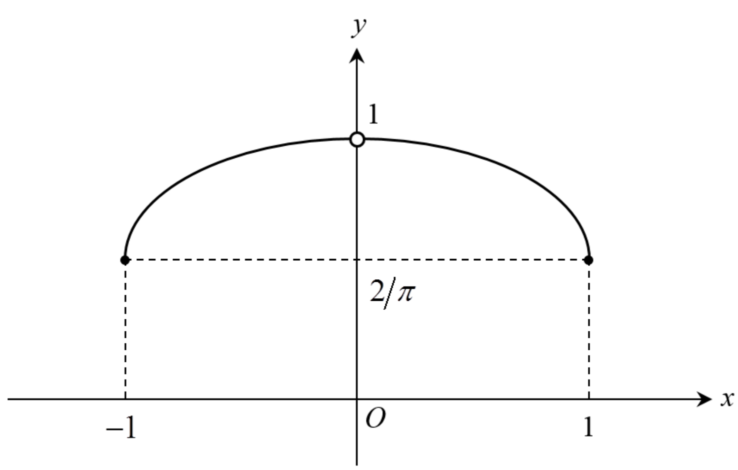
It is sufficient to just find the range for $x > 0$ to find the range for the entire domain in this case.

From part (i), $\lim_{x\to 0} f(x) = 1$ and from part (ii) $f(x) < 1$ for all x in the domain so the upper bound of the range is 1.

Since $f(x)$ is decreasing for $0 < x < 1$ then the minimum value occurs when $x = 1$ which is where a boundary of the domain. The minimum value is given by $f(1) = \dfrac{1}{\sin^{-1}1}$.

Hence the range is $\dfrac{2}{\pi} \leq f(x) < 1$

(vii)



Question 14

(a) (i)

The probability of getting a black ball on the 1st draw is $\dfrac{m}{m+n}$

The probability of getting a black ball afterwards on the 2nd draw is $\dfrac{m-1}{m+n-1}$

...

The probability of getting a black ball afterwards on the kth draw is $\dfrac{m-k+1}{m+n-k+1}$

The probability of getting a white ball afterwards on the $(k+1)$th draw is $\dfrac{n}{m+n-k}$

Hence the probability of drawing k balls consecutively is given by

$$P = \frac{m}{m+n} \times \frac{m-1}{m+n-1} \times \frac{m-2}{m+n-2} \times ... \times \frac{m-k+1}{m+n-k+1} \times \frac{n}{m+n-k}$$

But note that

$$m \times (m-1) \times (m-2) \times ... \times (m-k+1)$$

$$= m \times (m-1) \times (m-2) \times ... \times (m-k+1) \times \frac{(m-k) \times (m-k-1) \times (m-k-2) \times ... \times 2 \times 1}{(m-k) \times (m-k-1) \times (m-k-2) \times ... \times 2 \times 1}$$

$$= \frac{m!}{(m-k)!}$$

Similarly

$$(m+n) \times (m+n-1) \times ... \times (m+n-k+1) \times (m+n-k) = \frac{(m+n)!}{(m+n-k-1)!}$$

$$\text{and} \quad n = \frac{n!}{(n-1)!}$$

Hence

$$P = \frac{m!}{(m-k)!} \times \frac{(m+n-k-1)!}{(m+n)!} \times \frac{n!}{(n-1)!}$$

$$= \frac{(m+n-k-1)!}{(m-k)!(n-1)!} \times \frac{m!n!}{(m+n)!}$$

$$= \frac{\binom{m+n-k-1}{m-k}}{\binom{m+n}{m}}$$

Alternative approach

Consider a sequence of k black balls and 1 white ball following it. The number of ways of arranging the remaining $(m+n-k-1)$ balls is $\dfrac{(m+n-k-1)!}{(m-k)!(n-1)!}$ ways or equivalently $\binom{m+n-k-1}{m-k}$ ways.

There are $\dfrac{(m+n)!}{m!n!}$ ways or equivalently $\binom{m+n}{m}$ ways to arrange m balls and n black balls in the draw. So the probability of obtaining k consecutive black balls is $\dfrac{\binom{m+n-k-1}{m-k}}{\binom{m+n}{m}}$.

(ii)

$$\frac{\binom{m}{0}}{\binom{m+n-1}{0}}+\frac{\binom{m}{1}}{\binom{m+n-1}{1}}+\frac{\binom{m}{2}}{\binom{m+n-1}{2}}+\ldots+\frac{\binom{m}{m}}{\binom{m+n-1}{m}}=\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{m+n-1}{k}}$$

But note that

$$\begin{aligned}
\frac{\binom{m}{k}}{\binom{m+n-1}{k}} &= \frac{m!}{k!(m-k)!}\times\frac{k!(m+n-k-1)!}{(m+n-1)!} \\
&= \frac{(m+n-k-1)!m!}{(m+n-1)!(m-k)!}\times\frac{n!}{(n-1)!}\times\frac{1}{m+n}\times\frac{m+n}{n} \\
&= \frac{(m+n-k-1)!}{(m-k)!(n-1)!}\times\frac{m!n!}{(m+n)!}\times\frac{m+n}{n} \\
&= \frac{\binom{m+n-k-1}{m-k}}{\binom{m+n}{m}}\times\frac{m+n}{n}
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{m+n-1}{k}} &= \sum_{k=0}^{m}\frac{\binom{m+n-k-1}{m-k}}{\binom{m+n}{m}}\times\frac{m+n}{n} \\
&= \frac{m+n}{n}\sum_{k=0}^{m}\frac{\binom{m+n-k-1}{m-k}}{\binom{m+n}{m}} \\
&= \frac{m+n}{n}
\end{aligned}$$

Since $\displaystyle\sum_{k=0}^{m}\frac{\binom{m+n-k-1}{m-k}}{\binom{m+n}{m}}$ is the total probability of drawing any number of black balls using the result in (i) which must be 1.

(b)

(i) The equation of the tangent to P is given as $y = px - ap^2$

To find A substitute $y = 0$ which gives $x = ap$ so A has coordinates $(ap, 0)$

Noting that the coordinates of S and P' are $(0, a)$ and $(2ap, -a)$, the midpoint of SP' is $\left(\frac{0+2ap}{2}, \frac{a-a}{2}\right)$ which is $(ap, 0)$ and coincides with A.

(ii)

$AS = AP'$ since A is the midpoint of $P'S$

AP is common

$PS = PP'$ by definition of the locus of the parabola

$\therefore \Delta APS \equiv \Delta APP' \quad (SSS)$

(c)

(i) From part (b), it can be deduced that

$\angle APS = \angle APP'$ (corresponding angles of congruent triangles)

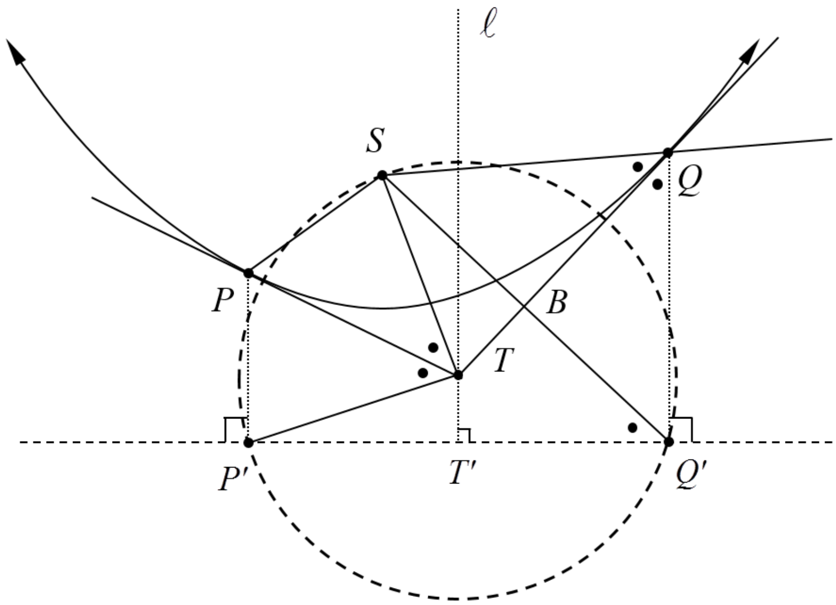
Translating this to the diagram in part (c), it can be deduced that $\angle TPP' = \angle TPS$. Also

PT is common

$PS = PP'$ by definition of the locus of the parabola

$\therefore \Delta PST \equiv \Delta PP'T \quad (SAS)$

(ii)



$$\angle PTP' = \angle PTS \quad \text{(corresponding angles of congruent triangles)}$$

Let $\angle PTS = x$ so $\angle P'TS = 2x$

$$\begin{aligned}\angle SQ'P' &= \frac{1}{2}\angle P'TS \quad \text{(angle at centre is twice angle at centre standing on the same arc)}\\ &= x\end{aligned}$$

From part (i) it be can similarly argued that $\Delta SQT \equiv \Delta Q'QT$ for point Q. Hence

$$\angle SQT = \angle Q'QT \quad \text{(corresponding angles of congruent triangles)} \quad (*)$$

Let B the point of intersection between SQ' and TQ. Since $QS = QQ'$ by definition of the locus of the parabola then $\Delta SQQ'$ is isosceles. Hence

$$BQ \perp SQ' \quad \text{(perpendicular bisector of an isosceles triangle)}$$

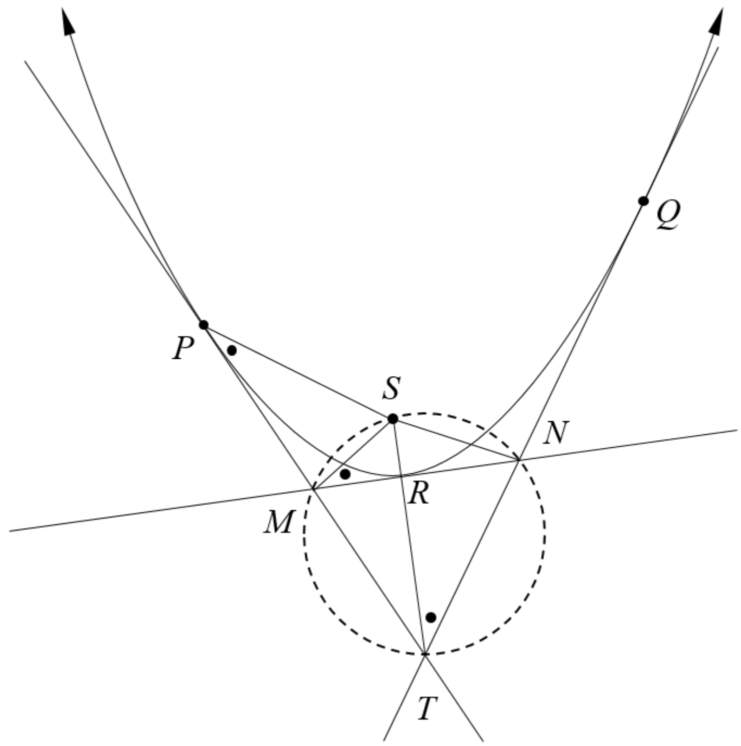
In $\Delta BQQ'$ which is a right angled triangle

$$\angle BQ'Q = \frac{\pi}{2} - x \quad \text{(Given } QQ' \perp T'Q'\text{)}$$

$$\angle BQQ' = x \quad \text{(angle sum of triangle)}$$

But from $(*)$ then $\angle SQT = x$ hence $\angle PTS = \angle SQT$.

(iii)



Consider the intersection of the two tangents from P and R. Similarly applying the result in part (ii)

$$\angle SMR = \angle MPS$$

Now consider the intersection of the two tangents from P and Q. Similarly applying the result in part (ii)

$$\angle MPS = \angle STN$$

This means that $\angle SMR = \angle STN$. However, the points M and T subtend from the same interval SN so S, M, N and T are concyclic.