2015 Bored of Studies Trial Examinations

Mathematics Extension 2

SOLUTIONS

Multiple Choice

1. A
2. B
3. B
4. D
5. B
6. D
7. C
8. C
9. A
10. C

Brief Explanations

Question 1 The value of k can be found either by substituting it directly into the quadratic, as it is a root, or by considering the sum and product of roots.

Question 2 (B) is not always true because the eccentricity determines the 'shape' of the ellipse. So if one ellipse is a scaled version of the other, they will have the same eccentricity but different equations. For example, $\frac{x^2}{9}+\frac{x^2}{4}=1$ and $\frac{x^2}{36}+\frac{x^2}{16}=1$ both have eccentricity $\frac{5}{9}$. This can also be observed by considering $e^2=1-\frac{b^2}{a^2}$, where we can still generate the same value of e using differing values of b and a, as the eccentricity relies on the ratio of the two.

Question 3 The asymptotes of $\mathcal{H}_1$ are $y=\pm x$ and the directrices of $\mathcal{H}_2$ are $x+y=\pm a\sqrt{2}$. Observe that all of the options are in the first quadrant, so we solve $y=x$ simultaneously with $x+y=a\sqrt{2}$ to get (B).

Question 4 Option (I) yields $f\left(2\left(x+1\right)-1\right)=f\left(2x+1\right)$ and then $f\left(2\left(\frac{x}{2}\right)+1\right)=f\left(x+1\right)$. Option (II) yields $f\left(2\left(\frac{x}{2}\right)-1\right)=f\left(x-1\right)$ and then $f\left(\left(x+1\right)-1\right)=f\left(x\right)$, so this is one possible option. Option (III) yields $f\left(2\left(-x-1\right)-1\right)=f\left(-2x-3\right)$ and then $f\left(-2\left(-\frac{x}{2}\right)-3\right)=f\left(x-3\right)$. Option (IV) yields $f\left(2\left(-\frac{x}{2}\right)-1\right)=f\left(-x-1\right)$ and then $f\left(-\left(-x-1\right)-1\right)=f\left(x\right)$, so this is also a possible option. Hence options (II) or (IV) work.

Question 5 We have the vector equation $\overrightarrow{PR} = i\overrightarrow{PQ}$. Let *r* be the complex number represented by *R*, so $r-(1+i)=i(a+bi)$ noting that $a+bi$ represents the vector *PQ*. Hence $r=i(a+bi)+(1+i)=(1-b)+i(1+a)$, which is (B).

Question 6 A counter example for (A) is $P(x)=x^4+2x^2+1$, which has $P(i)=P'(i)=0$. This also serves as a counter example for (B) since α is not real. The polynomial is not necessarily even because we can have any real linear factor multiplied to our counter example above, thus making it have odd degree. (D) is always true because if α is non-real, then both $(x-\alpha)^2$ and $(x-\alpha)^2$ must be factors for the conjugate root theorem to hold. So the polynomial must be in the form $P(x)=(x-\alpha)^2(x-\bar{\alpha})^2Q(x)$, which has degree $n \geq 4$.

Question 7 The method of cylindrical shells yields $V = 2\pi\int_0^1 (1-x)\, y\, dx$. Using the substitution $y=2x-x^2$, we have $dy = 2(1-x)\,dx$ and so $V=\pi\int_0^1 y\, dy$, which is the same value as option (C) as *y* is just a dummy variable.

Question 8 As *x* moves along a circle about the origin, it's *x* coordinate can be parameterised as $x = r\cos\theta$. Differentiate both sides with respect to *t* twice, noting that $\dot{\theta}=\omega$, we obtain (C). Alternatively, one could observe that in a one dimensional context, the circular motion is just simple harmonic motion so the answer is either (A) or (C). But only (C) will yield an appropriate period for the circular motion/simple harmonic motion.

Question 9 From the cosine rule, we have $\cos\angle BAC = \dfrac{b^2+c^2-a^2}{2bc} < 0$ since $\angle BAC$ is obtuse. So $a^2 > b^2+c^2 = (b-c)^2+2bc \geq 2bc$. So the answer is (A).

Question 10 Using the substitution $u=2x$, the given integral *I* is equivalent to $\dfrac{1}{2}\int_0^{2\pi}\sin^n\left(\dfrac{x}{2}\right)\cos(kx)\,dx$, which eliminates (D). Observe that this integral is symmetric about $x=\pi$, so $\int_0^{2\pi}\sin^n\left(\dfrac{x}{2}\right)\cos(kx)\,dx = 2\times\int_0^{\pi}\sin^n\left(\dfrac{x}{2}\right)\cos(kx)\,dx$ so for the original integral, we have $I=\dfrac{1}{2}\int_0^{2\pi}\sin^n\left(\dfrac{x}{2}\right)\cos(kx)\,dx = \int_0^{\pi}\sin^n\left(\dfrac{x}{2}\right)\cos(kx)\,dx$, which is (C).

Written Response

Question 11 (a)

Let $u^2 = \sin\theta \Rightarrow 2u\,du = \cos\theta\,d\theta$.

$$\int \frac{\sqrt{\sin\theta}}{\sin\theta\cos\theta}\,d\theta = \int \frac{\cos\theta\sqrt{\sin\theta}}{\sin\theta\cos^2\theta}\,d\theta$$

$$= \int \frac{\cos\theta\sqrt{\sin\theta}}{\sin\theta\left(1-\sin^2\theta\right)}\,d\theta$$

$$= 2\int \frac{u^2}{u^2\left(1-u^4\right)}\,du$$

$$= 2\int \frac{du}{1-u^4}$$

$$= \int \frac{1}{1-u^2} + \frac{1}{1+u^2}\,du$$

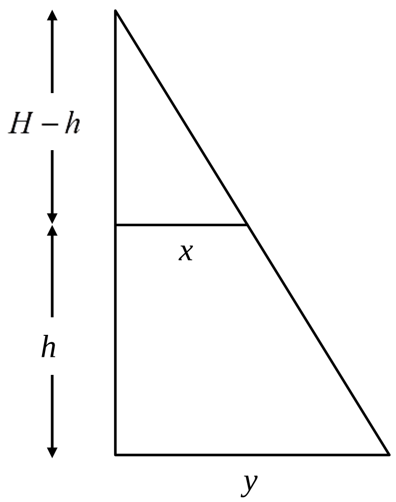
$$= \int \frac{1/2}{1+u} + \frac{1/2}{1-u} + \frac{1}{1+u^2}\,du$$

$$= \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right) + \tan^{-1}(u) + C$$

$$= \frac{1}{2}\ln\left(\frac{1+\sqrt{\sin\theta}}{1-\sqrt{\sin\theta}}\right) + \tan^{-1}\left(\sqrt{\sin\theta}\right) + C$$

Question 11 (b)

Consider an arbitrary ‘dimension’ *x* from the slice and the corresponding dimension *y* from the base.



By similar triangles, we have $\dfrac{H-h}{h}=\dfrac{x}{y}$.

But recall that the square of the ratio of corresponding sides is equal to the ratio of the areas, in similar shapes. Hence $\left(\dfrac{x}{y}\right)^2=\dfrac{A_h}{A_B}$ and so $\left(\dfrac{H-h}{H}\right)^2=\dfrac{A_h}{A_B}$ or equivalently

$$A_h=A_B\left(\frac{H-h}{H}\right)^2.$$

The volume of an arbitrary slice at height *h* is $\delta V=A_h\,\delta h$.

$$\begin{aligned}
V&=\int_0^H A_B\left(\frac{H-h}{H}\right)^2 dh\\
&=\frac{A_B}{H^2}\int_0^H (H-h)^2\,dh\\
&=-\frac{A_B}{3H^2}(H-h)^3\Big|_0^H\\
&=\frac{1}{3}A_BH
\end{aligned}$$

Question 11 (c)

First obtain a polynomial with roots $A=\alpha+\beta$, $B=\beta+\gamma$ and $C=\alpha+\gamma$. We will then find the value of $\frac{1}{A^2}+\frac{1}{B^2}+\frac{1}{C^2}$.

Let $u=\alpha+\beta$.

From the sum of roots, we have $\alpha+\beta+\gamma=p$ and so $u=p-\gamma \Rightarrow \gamma=p-u$.

Since γ is a root, $P(\gamma)=0$.

$$\begin{aligned} P(p-u) &= (p-u)^3 - p(p-u)^2 + p \\ &= -u^3 + 2pu^2 - p^2u + p \end{aligned}$$

Hence, the polynomial $R(x)=x^3-2px^2+p^2x-p$ has roots $A=\alpha+\beta$, $B=\beta+\gamma$ and $C=\alpha+\gamma$.

$$\begin{aligned} \frac{1}{A^2}+\frac{1}{B^2}+\frac{1}{C^2} &= \frac{A^2B^2+A^2C^2+B^2C^2}{A^2B^2C^2} \\ &= \frac{(AB+AC+BC)^2-2ABC(A+B+C)}{(ABC)^2} \\ &= \frac{p^4-4p^2}{p^2} \\ &= p^2-4 \\ &= 0 \qquad \ldots(\text{from the question}) \end{aligned}$$

Hence, the values of p such that $\frac{1}{A^2}+\frac{1}{B^2}+\frac{1}{C^2}=0$ are $p=\pm 2$.

Question 11 (d) (i)

As vectors *AP* and AA' are collinear, we can say that $\overrightarrow{AP} = k\overrightarrow{AA'}$. If a' is the complex number representing A', this becomes

$$p - a = k\left(a' - a\right).$$

But as A' is the midpoint of *BC*, we have $a' = \dfrac{b+c}{2}$.

$$\begin{aligned} p - a &= k\left(\frac{b+c}{2} - a\right) \\ p &= (1-k)a + \frac{k}{2}(b+c) \end{aligned}$$

Question 11 (d) (ii)

Similarly to (i), using B' instead, we have $p = (1-k)b + \dfrac{k}{2}(a+c)$, so equating *p* yields

$$\begin{aligned} (1-k)a + \frac{k}{2}(b+c) &= (1-k)b + \frac{k}{2}(a+c) \\ (1-k)(a-b) &= \frac{k}{2}(a+c-b-c) \\ (1-k)(a-b) &= \frac{k}{2}(a-b) \\ k &= \frac{2}{3} \end{aligned}$$

Substituting this back into $p = (1-k)a + \dfrac{k}{2}(b+c)$ yields the required result.

Question 11 (d) (iii)

Let *Q* be the point on the interval CC', represented by the complex number *q*, such that $\overrightarrow{CQ} = r\overrightarrow{CC'}$ for some $0 < r < 1$.

Then by a similar argument to part (i), we have $q = (1-r)c + \dfrac{r}{2}(a+b)$

Substituting in $r = \dfrac{2}{3}$ yields $q = \dfrac{a+b+c}{3} = p$, and so *P* lies on CC'.

Question 12 (a) (i)

First split the log to obtain $x^3 - y^3 = 3(\ln x + \ln y)$.

Differentiate both sides with respect to *x*.

$$3x^2 - 3y^2\frac{dy}{dx} = 3\left(\frac{1}{x} + \frac{1}{y}\cdot\frac{dy}{dx}\right)$$

Rearrange to make $\frac{dy}{dx}$ the subject.

$$\frac{dy}{dx} = \frac{x^3 y - y}{xy^3 + x}$$

Substituting $(1,1)$ yields $\frac{dy}{dx} = 0$ and hence it is a critical point.

Question 12 (a) (ii)

Substitute in $(-y_0, -x_0)$.

$$\begin{aligned}\text{LHS} &= -y_0^3 + x_0^3 \\ &= x_0^3 - y_0^3 \\ &= 3\ln(x_0 y_0) \quad \ldots\big(\text{as } (x_0, y_0) \text{ lies on the original curve}\big) \\ &= 3\ln\big((-x_0)(-y_0)\big) \\ &= \text{RHS}\end{aligned}$$

Geometrically, this means for every point (x_0, y_0) on the curve, there exists a corresponding point $(-y_0, -x_0)$ that also lies on the curve, and the effect of that is that the curve is symmetric in $y = -x$.

Question 12 (a) (iii)

Rearrange the equation as

$$x^3 - 3\ln(x) = y^3 + 3\ln(y)$$

$$x^3\left[1-3\left(\frac{\ln(x)}{x^3}\right)\right] = y^3 + 3\ln(y)$$

As $x \to \infty$, $\dfrac{\ln(x)}{x^3} \to 0$ and so $\text{LHS} \to \infty$. So it must be the case that $y \to \infty$.

Question 12 (a) (iv)

First, recall from (iii) that $x^3 - 3\ln(x) = y^3 + 3\ln(y)$ or equivalently $\dfrac{x^3 - 3\ln(x)}{y^3 + 3\ln(y)} = 1$.

So $\left(\dfrac{y^3}{y^3 + 3\ln y}\right)\left(\dfrac{x^3 - 3\ln x}{x^3}\right) = \dfrac{y^3}{x^3}$. By dividing the numerator and denominator of the left and right factors by y^3 and x^3 respectively, we have

$$\left[\frac{1}{1+3\left(\frac{\ln y}{y^3}\right)}\right]\left[1-3\left(\frac{\ln x}{x^3}\right)\right] = \frac{y^3}{x^3}$$

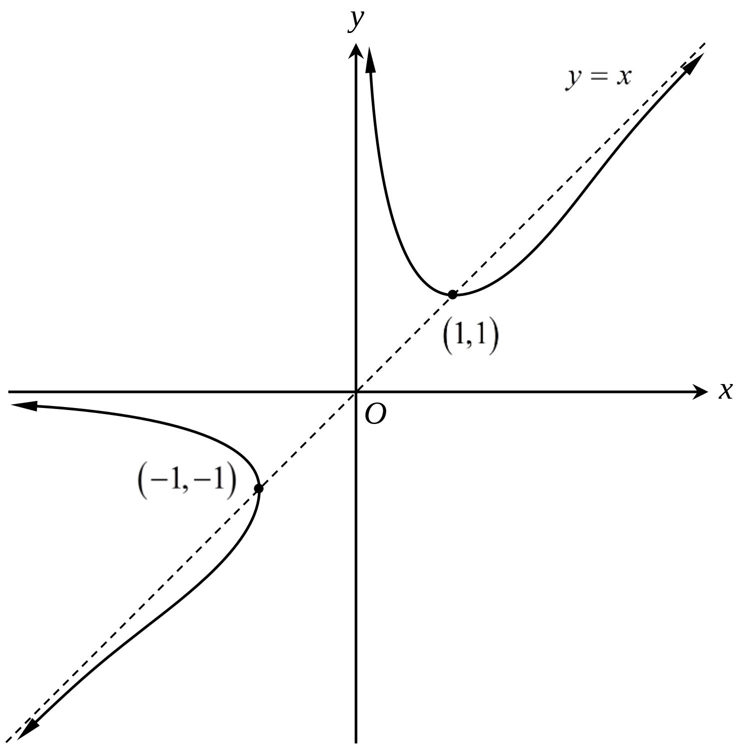
As $x \to \infty$, we know from part (iii) that $y \to \infty$. Using the given limit in part (iii), the left factor approaches 1 and so the right factor also approaches 1.

Hence $\left(\dfrac{y}{x}\right)^3 \to 1$ and therefore $y \to x$, meaning that $y = x$ is an asymptote.

Question 12 (a) (v)

We know that the curve has the following features:

- ✓ Critical point at $(1,1)$.
- ✓ Symmetric about $y=-x$
  - So $(-1,-1)$ is also a critical point.
- ✓ Has an oblique asymptote $y=x$.



Question 12 (b) (i)

We can factorise

$$\begin{aligned} P(z) &= z^n - 1 \\ &= (z-1)\left(1+z+z^2+\dots+z^{n-1}\right) \end{aligned}$$

This polynomial has roots $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$.

But $\alpha_n = \cos(2\pi) + i\sin(2\pi) = 1$, so the roots of $S(z) = 1+z+z^2+\dots+z^{n-1} = 0$ must be $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-1}$.

So the linear decomposition of $S(z)$ is

$$S(z) = 1+z+z^2+z^3+\dots+z^{n-1}$$

$$= (z-\alpha_1)(z-\alpha_2)(z-\alpha_3)\dots(z-\alpha_{n-1})$$

and evaluating $S(1)$ yields

$$\begin{aligned} (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\dots(1-\alpha_{n-1}) &= \underbrace{1+1+1+\dots+1}_{n \text{ times}} \\ &= n \end{aligned}$$

Question 12 (b) (ii)

Consider the general factor $1-\alpha_k = 1-\cos\left(\frac{2k\pi}{n}\right) - i\sin\left(\frac{2k\pi}{n}\right)$ for $k = 1, 2, 3, \ldots, n-1$.

$$\begin{aligned}
1-\cos\left(\frac{2k\pi}{n}\right) - i\sin\left(\frac{2k\pi}{n}\right) &= 1-\left(1-2\sin^2\left(\frac{k\pi}{n}\right)\right) - 2i\sin\left(\frac{k\pi}{n}\right)\cos\left(\frac{k\pi}{n}\right) \\
&= 2\sin^2\left(\frac{k\pi}{n}\right) - 2i\sin\left(\frac{k\pi}{n}\right)\cos\left(\frac{k\pi}{n}\right) \\
&= 2\sin\left(\frac{k\pi}{n}\right)\left[\sin\left(\frac{k\pi}{n}\right) - i\cos\left(\frac{k\pi}{n}\right)\right] \\
&= 2\sin\left(\frac{k\pi}{n}\right)\left[\cos\left(\frac{\pi}{2}-\frac{k\pi}{n}\right) - i\sin\left(\frac{\pi}{2}-\frac{k\pi}{n}\right)\right] \\
1-\alpha_k &= 2\sin\left(\frac{k\pi}{n}\right)\left[\cos\left(\frac{\pi}{2n}(2k-n)\right) + i\sin\left(\frac{\pi}{2n}(2k-n)\right)\right]
\end{aligned}$$

Compute the modulus of both sides to get $|1-\alpha_k| = \left|2\sin\left(\frac{k\pi}{n}\right)\right|$. Since $k = 1, 2, 3, \ldots, n-1$, the expression $\sin\left(\frac{k\pi}{n}\right)$ is always positive and so $|1-\alpha_k| = 2\sin\left(\frac{k\pi}{n}\right)$.

Multiplying these expressions for $k = 1, 2, 3, \ldots, n-1$, we get

$$\begin{aligned}
2^{n-1}\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\ldots\sin\left(\frac{n-1}{n}\pi\right) &= |1-\alpha_1||1-\alpha_2||1-\alpha_3|\ldots|1-\alpha_{n-1}| \\
2^{n-1}\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\ldots\sin\left(\frac{n-1}{n}\pi\right) &= |(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\ldots(1-\alpha_{n-1})| \\
2^{n-1}\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\ldots\sin\left(\frac{n-1}{n}\pi\right) &= |n| \qquad \ldots(\text{from part (i)}) \\
2^{n-1}\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\ldots\sin\left(\frac{n-1}{n}\pi\right) &= n \\
\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\ldots\sin\left(\frac{n-1}{n}\pi\right) &= \frac{n}{2^{n-1}}
\end{aligned}$$

Question 12 (c) (i)

Resolving forces vertically and horizontally, we have

$$F\sin\theta + N\cos\theta = mg \qquad ...(1)$$
$$F\cos\theta - N\sin\theta = \frac{mv^2}{r} \qquad ...(2)$$

By considering $(1)\times\cos\theta - (2)\times\sin\theta$, we have

$$N = mg\cos\theta - \frac{mv^2}{r}\sin\theta .$$

From this expression, we can observe that there must exist some velocity $v = v_0$ where the car leaves the surface of the cone. To find this velocity let $N = 0$, which yields $v_0^2 = rg\cot\theta$.

Question 12 (c) (ii)

By considering $(1)\times\sin\theta + (2)\times\cos\theta$, we have $F = mg\sin\theta + \frac{mv^2}{r}\cos\theta$

To find the amount of lateral thrust F_0 that the car experiences at the maximum velocity v_0, we can substitute the result from part (i) back into F.

$$\begin{aligned}F_0 &= mg\sin\theta + \frac{mv_0^2}{r}\cos\theta \\ &= mg\sin\theta + \frac{mgr\cot\theta}{r}\cos\theta \\ &= mg\sin\theta + mg\frac{\cos^2\theta}{\sin\theta} \\ &= mg\left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta}\right) \\ &= \frac{mg}{\sin\theta}\end{aligned}$$

Alternatively

From our resolution of vertical forces, we had $F\sin\theta + N\cos\theta = mg$.

By letting $N = 0$, we obtain the result immediately.

Question 13 (a) (i)

First we'll find an expression for the terminal velocity *w*. Particle *B* has force equation

$$\begin{aligned} F_{net} &= ma \\ &= mg - mkv^2 \\ a &= g - kv^2 \end{aligned}$$

For terminal velocity, we have $a = 0,\ v = w$ and so $w^2 = \dfrac{g}{k}$.

The force equation for Particle *A* is $F_{net} = ma = -\left(mg + kv^2\right)$, so we have the acceleration-velocity equation $a = -\left(g + kv^2\right)$. Using the fact that $w^2 = \dfrac{g}{k}$, this is equivalent to

$$\begin{aligned} v\frac{dv}{dx} &= -\left(g + kv^2\right) \\ &= -k\left(w^2 + v^2\right) \end{aligned}$$

So by separating variables and integrating both sides with respect to appropriate limits, we have

$$\begin{aligned} -k\int_0^{x_A} dx &= \int_u^{v_A} \frac{v}{w^2 + v^2}\,dv \\ &= \frac{1}{2}\ln\left(w^2 + v^2\right)\Big|_u^{v_A} \\ -kx_A &= \frac{1}{2}\ln\left(\frac{w^2 + v_A^2}{w^2 + u^2}\right) \\ x_A &= -\frac{1}{2k}\ln\left(\frac{w^2 + v_A^2}{w^2 + u^2}\right) \\ &= \frac{1}{2k}\ln\left(\frac{w^2 + u^2}{w^2 + v_A^2}\right) \end{aligned}$$

Question 13 (a) (ii)

From part (i), we can find an expression for H by setting $v_A = 0,\ x = H$.

$$H = \frac{1}{2k}\ln\left(\frac{w^2+u^2}{w^2}\right)$$

From the diagram, we can observe that when the particles collide, $x_A + x_B = H$ since x_A is the distance from the ground and x_B is the distance from maximum height *H*. Also, noting that they collide when $v_A = w$, we have

$$\begin{aligned}
\frac{1}{2k}\ln\left(\frac{w^2+u^2}{w^2+w^2}\right)+\frac{1}{2k}\ln\left(\frac{w^2}{w^2-v_B^2}\right) &= \frac{1}{2k}\ln\left(\frac{w^2+u^2}{w^2}\right)\\
\ln\left(\frac{w^2+u^2}{w^2+w^2}\times\frac{w^2}{w^2-v_B^2}\right) &= \ln\left(\frac{w^2+u^2}{w^2}\right)\\
\frac{w^2+u^2}{2w^2}\times\frac{w^2}{w^2-v_B^2} &= \frac{w^2+u^2}{w^2}\\
\frac{1}{2}\times\frac{w^2}{w^2-v_B^2} &= 1\\
w^2 &= 2w^2-2v_B^2\\
w^2 &= 2v_B^2\\
v_B &= \frac{w}{\sqrt{2}} \qquad \ldots(\text{as } v_B > 0)
\end{aligned}$$

Question 13 (b) (i)

We first use the sine rule in triangles *APR* and *BPR* to respectively obtain

$$\frac{\sin \angle ARP}{AP} = \frac{\sin \angle APR}{AR} \qquad \text{...}(1)$$

$$\frac{\sin \angle BRP}{BP} = \frac{\sin \angle BPR}{BR} \qquad \text{...}(2)$$

Since, $\angle ARP$ and $\angle BRP$ are supplementary, $\sin \angle ARP = \sin(\pi - \angle BRP) = \sin \angle BRP$.

Also, it is given that $\angle APR = \angle BPR$, so we have $\sin \angle APR = \sin \angle BPR$.

Calculating $(2) \div (1)$:

$$\frac{\sin \angle BRP}{BP} \times \frac{AP}{\sin \angle ARP} = \frac{\sin \angle BPR}{BR} \times \frac{AR}{\sin \angle APR}$$

$$\frac{AP}{BP} = \frac{AR}{BR}$$

Question 13 (b) (ii)

From the given ratio, and from part (i), we have

$$\frac{AS}{BS} = \frac{AR}{BR} = k,$$

which is a fixed ratio.

As *A* and *B* are fixed points, and *R* and *S* are concurrent with *AB*, it must be the case that *R* and *S* are also fixed points regardless of the position of *P*.

Question 13 (b) (iii)

Since $\angle APT = \pi$ and $\angle RPS = \frac{1}{2}\angle APT$, we have $\angle RPS = \frac{\pi}{2}$.

But the points *R* and *S* are fixed, with *P* varying. Hence, as *P* varies, it must move along the circumference of a circle with diameter *RS* (angle in a semi-circle).

Question 13 (c) (i)

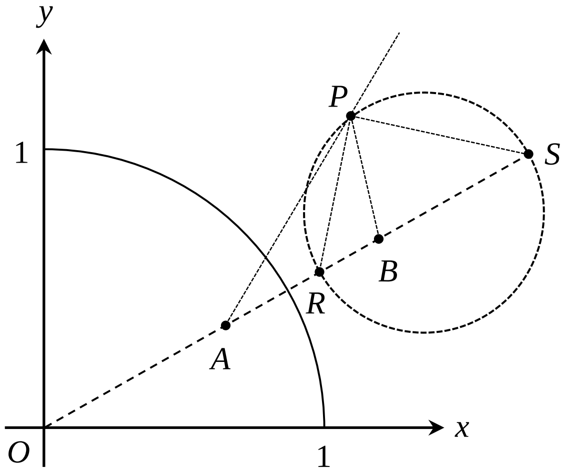
By first noting that $1-\bar{a}z=\bar{a}\left(\frac{1}{\bar{a}}-z\right)$, and using the fact $|a|=|\bar{a}|$, we can deduce that the given locus condition is equivalent to

$$\frac{|z-a|}{\left|z-\frac{1}{\bar{a}}\right|}=r|a|.$$

As $r|a|>1$, we can observe that this is similar to ratio $\frac{PA}{PB}=k$ from part (b), where $P\Rightarrow z$, $A\Rightarrow a$ and $B\Rightarrow\frac{1}{\bar{a}}$.

Hence, using the result of (b) (iii), we can deduce that the locus of z is a circle.

The diagram below shows this result, where A represents the complex number a and B represents the complex number $\frac{1}{\bar{a}}$. Note that the complex number $\frac{1}{\bar{a}}$ has the same argument as the complex number a, but with a reciprocal modulus. As a is inside the unit circle, the complex number $\frac{1}{\bar{a}}$ must be outside the unit circle.



Question 13 (c) (ii)

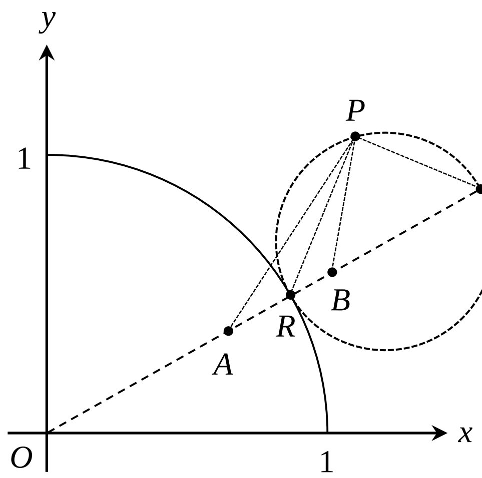


Diagram not to scale

From the ratio in part (b) (i), and from the locus condition, we have

$$\frac{PA}{PB} = \frac{RA}{RB} = r|a|,$$

where R lies on the unit circle in order for the locus of z to touch the unit circle.

So in the context of moduli, this translates to

$$\frac{RA}{RB} = \frac{1-|a|}{\left|\frac{1}{\overline{a}}\right|-1} = r|a|$$

So working with this, we have

$$\frac{1-|a|}{\frac{1}{|a|}-1} = r|a|$$

$$1-|a| = r|a|\left(\frac{1}{|a|}-1\right)$$

$$1-|a| = r\left(1-|a|\right)$$

$$r = 1$$

Question 14 (a) (i)

We have $CB = CD$ (given), and $OB = OD$ (radii), and OC is common.

Hence $\triangle COB \equiv \triangle COD$ (SSS).

Question 14 (a) (ii)

Since triangles are congruent, we have $\angle CBO = \angle CDO$. Adding all the angles in $\triangle COB$ and $\triangle COD$, we have

$$\left(\angle CBO + \angle BOC + \angle OCB\right) + \left(\angle CDO + \angle DOC + \angle OCD\right) = 2\pi$$

$$2\angle CBO + \angle BCD_{\text{reflex}} + \angle BOD = 2\pi$$

as $BODC$ is a quadrilateral.

But

$$\begin{aligned} \angle BCD &= 2 \times \angle BAD \quad &&\ldots(\text{given}) \\ &= \angle BOD &&\ldots(\text{angle at the centre} = 2 \times \text{angle at the circumference}) \end{aligned}$$

So then we have

$$\begin{aligned} 2\angle CBO + \left(2\pi - \angle BCD\right) + \angle BOD &= 2\pi \\ 2\angle CBO + \angle BOD - \angle BCD &= 0 \\ 2\angle CBO &= 0 \quad \ldots(\text{since } \angle BOD = \angle BCD) \\ \angle CBO &= 0 \end{aligned}$$

And this forces O and C to coincide.

Question 14 (b) (i)

The x intercept of the tangent is $x = \dfrac{a}{\cos\theta}$ and so

$$\begin{aligned} TS &= ae - \frac{a}{\cos\theta} \\ &= \frac{a}{\cos\theta}\left(e\cos\theta - 1\right) \end{aligned}$$

Similarly, $TS' = \dfrac{a}{\cos\theta}\left(e\cos\theta + 1\right)$ so $\dfrac{TS}{TS'} = \dfrac{e\cos\theta - 1}{e\cos\theta + 1}$.

Recall that $PS = ePM$ and $PS' = ePM'$.

$$\begin{aligned} \frac{PS}{PS'} &= \frac{PM}{PM'} \\ &= \frac{a\cos\theta - \frac{a}{e}}{a\cos\theta + \frac{a}{e}} \\ &= \frac{e\cos\theta - 1}{e\cos\theta + 1} \end{aligned}$$

And so we have the required result.

Question 14 (b) (ii)

First, observe that

$\angle STQ = \angle S'TQ'$ (vertically opposite angles)

$\angle TQS = \angle TQ'S' = \dfrac{\pi}{2}$ (given).

Hence $\Delta TQS \,|||\, \Delta TQ'S'$ (equiangular).

Hence, by similar triangle ratios, we have

$$\begin{aligned} \frac{QS}{Q'S'} &= \frac{TS}{TS'} \\ &= \frac{PS}{PS'} \quad \ldots\text{(from part (i))} \end{aligned}$$

Question 14 (b) (iii)

In $\triangle PQS$ and $\triangle PQ'S'$, we have $\sin(\angle SPQ) = \dfrac{QS}{PS}$ and $\sin(\angle S'PQ') = \dfrac{Q'S'}{PS'}$ respectively.

If we divide these expressions, we have

$$\frac{\sin(\angle SPQ)}{\sin(\angle S'PQ')} = \frac{QS}{PS} \times \frac{PS'}{Q'S'}$$

$$= 1 \qquad \ldots(\text{from part (ii)})$$

So $\sin(\angle SPQ) = \sin(\angle S'PQ')$.

As the two angles must be acute (they both exist in right angled triangles), it must be the case that $\angle SPQ = \angle S'PQ'$ and hence the line ℓ bisects $\angle SPS'$.

Question 14 (c) (i)

Integrating both sides of the given identity, over the domain $0 \le x \le \dfrac{\pi}{2}$, we have

$$\int_0^{\frac{\pi}{2}} \frac{\sin(n+1/2)x}{\sin(x/2)}\,dx = \int_0^{\frac{\pi}{2}} 1 + 2(\cos x + \cos 2x + \cos 3x + \ldots + \cos nx)\,dx$$

$$= x + 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots + \frac{1}{n}\sin nx\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + 2\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots + \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\right) \quad \ldots(\text{as } \sin k\pi = 0,\ k \in \mathbb{Z})$$

Question 14 (c) (ii)

We can use integration by parts on $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \frac{\sin(n+1/2)x}{\sin(x/2)}\,dx$ by setting $u = \dfrac{1}{\sin\left(\frac{x}{2}\right)}$ and

$dv = \sin\left(n+\dfrac{1}{2}\right)x$. Noting that $\cos\left(n+\dfrac{1}{2}\right)\pi = 0$, we then have

$$\int_{\frac{\pi}{2}}^{\pi} \frac{\sin(n+1/2)x}{\sin(x/2)}\,dx = -\frac{2}{2n+1}\left.\frac{\cos\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}\right|_{\frac{\pi}{2}}^{\pi} - \frac{1}{2n+1}\int_{\frac{\pi}{2}}^{\pi} \frac{\cos(x/2)}{\sin^2(x/2)}\cos\left(n+\frac{1}{2}\right)x\,dx$$

$$= -\frac{2}{2n+1}\left[0-\sqrt{2}\cos\left(\frac{2n+1}{4}\pi\right)\right] - \frac{1}{2n+1}\int_{\frac{\pi}{2}}^{\pi} \frac{\cos(x/2)}{\sin^2(x/2)}\cos\left(n+\frac{1}{2}\right)x\,dx$$

$$= \frac{1}{2n+1}\left[2\sqrt{2}\cos\left(\frac{2n+1}{4}\pi\right) - \int_{\frac{\pi}{2}}^{\pi} \frac{\cos(x/2)}{\sin^2(x/2)}\cos\left(n+\frac{1}{2}\right)x\,dx\right]$$

Taking the absolute value of both sides and using the triangle inequality, we have

$$\left|\int_{\frac{\pi}{2}}^{\pi} \frac{\sin(n+1/2)x}{\sin(x/2)}\,dx\right| = \frac{1}{2n+1}\left|2\sqrt{2}\cos\left(\frac{2n+1}{4}\pi\right) - \int_{\frac{\pi}{2}}^{\pi} \frac{\cos(x/2)}{\sin^2(x/2)}\cos\left(n+\frac{1}{2}\right)x\,dx\right|$$

$$\leq \frac{1}{2n+1}\left[2\sqrt{2}\left|\cos\left(\frac{2n+1}{4}\pi\right)\right| + \left|\int_{\frac{\pi}{2}}^{\pi} \frac{\cos(x/2)}{\sin^2(x/2)}\cos\left(n+\frac{1}{2}\right)x\,dx\right|\right]$$

$$\leq \frac{1}{2n+1}\left[2\sqrt{2} + \left|\int_{\frac{\pi}{2}}^{\pi} \frac{\cos(x/2)}{\sin^2(x/2)}\,dx\right|\right]$$

$$= \frac{1}{2n+1}\left[2\sqrt{2} + \int_{\frac{\pi}{2}}^{\pi} \frac{\cos(x/2)}{\sin^2(x/2)}\,dx\right] \quad \ldots\left(\begin{array}{c}\text{since the integrand is}\\ \text{positive for all } \frac{\pi}{2} \leq x \leq \pi\end{array}\right)$$

Question 14 (c) (iii)

First consider

$$\int_0^{\pi} \frac{\sin\left(n+1/2\right)x}{\sin\left(x/2\right)}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin\left(n+1/2\right)x}{\sin\left(x/2\right)}\,dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin\left(n+1/2\right)x}{\sin\left(x/2\right)}\,dx$$

We can calculate what the left hand side becomes directly, using the given identity.

$$\int_0^{\pi} \frac{\sin\left(n+1/2\right)x}{\sin\left(x/2\right)}\,dx = \int_0^{\pi} 1+2\left(\cos x+\cos 2x+\cos 3x+\ldots+\cos nx\right)dx$$

$$= x+2\left(\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\ldots+\frac{1}{n}\sin nx\right)\Bigg|_0^{\pi}$$

$$= \pi$$

So we now have

$$\pi = \int_0^{\frac{\pi}{2}} \frac{\sin\left(n+1/2\right)x}{\sin\left(x/2\right)}\,dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin\left(n+1/2\right)x}{\sin\left(x/2\right)}\,dx$$

From part (ii) we can deduce that as $n\to\infty$, $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \frac{\sin\left(n+1/2\right)x}{\sin\left(x/2\right)}\,dx \to 0$ and so we have the required result.

Question 14 (c) (iv)

From part (i), we had

$$\int_0^{\frac{\pi}{2}} \frac{\sin\left(n+1/2\right)x}{\sin\left(x/2\right)}\,dx = \frac{\pi}{2}+2\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots+\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\right)$$

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots+\frac{1}{n}\sin\left(\frac{n\pi}{2}\right) = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin\left(n+1/2\right)x}{\sin\left(x/2\right)}\,dx - \frac{\pi}{4}$$

Taking the limit of both sides as $n\to\infty$, we have $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots = \frac{1}{2}(\pi)-\frac{\pi}{4} = \frac{\pi}{4}$

Question 15 (a) (i)

Regardless of whether it is a *U* or a *D*, the graph will progress to the right by 1 unit. As the sequence contains $a+b$ letters in total, the *x* coordinate will end at $x=a+b$.

As there are more *U*'s than *D*'s (given), the net vertical movement will be the number of *U*'s less by the number of *D*'s, which is $a-b$. Hence the *y* coordinate will end at $y=a-b$.

Question 15 (a) (ii)

The number of paths from the origin to *T* is the same as the number of ways of arranging a sequence of *a* copies of *U* and *b* copies of *D*, which is $\dfrac{(a+b)!}{a!b!}$.

Alternatively

Out of a sequence of $a+b$ letters, choose either *a* or *b* to become *U*'s, thus forcing the remainder of the letters to be *D*'s. So we have $\binom{a+b}{a}$ or $\binom{a+b}{b}$.

Question 15 (a) (iii)

Suppose the sequence from $(1,1)$ to *T* either touches or crosses the *x* axis. There must exist a 'first time' $(c,0)$ where it makes contact with the *x* axis. Reflect the path up until $(c,0)$ about the *x* axis. By doing so, we obtain a unique path from $(1,-1)$ to *T*.

Observe that by doing this, we obtain a one-to-one correspondence between the paths from $(1,1)$ to *T*, which touch or intersect the *x* axis, to the total number of paths from $(1,-1)$ to *T*, and visa versa. Hence the two sets of paths must be equal in size.

Question 15 (a) (iv)

The total number of paths from $(1,1)$ to T is $\binom{a+b-1}{a-1}=\dfrac{(a+b-1)!}{(a-1)!b!}$ ...(1).

The total number of paths from $(1,-1)$ to T is $\binom{a+b-1}{a}=\dfrac{(a+b-1)!}{a!(b-1)!}$ ...(2).

The number of paths from $(1,1)$ to T that do NOT touch or cross the x axis is the value in equation (1) less by the paths that DO touch or cross the x axis, which from part (iii) is given by equation (2).

Hence, we have

$$\frac{(a+b-1)!}{(a-1)!b!}-\frac{(a+b-1)!}{a!(b-1)!}=\frac{(a+b-1)!}{(a-1)!(b-1)!}\left[\frac{1}{b}-\frac{1}{a}\right]$$

$$=\frac{(a+b-1)!}{(a-1)!(b-1)!}\left[\frac{a-b}{ab}\right]$$

$$=\frac{(a+b-1)!}{a!b!}(a-b)$$

$$=\frac{a-b}{a+b}\frac{(a+b)!}{a!b!}$$

$$=\frac{a-b}{a+b}\binom{a+b}{a}$$

Question 15 (b)

Let an A card being drawn correspond to a U from part (a), and a B card being drawn correspond to a D. Observe that for Candidate A to always have a higher tally than Candidate B, the first card drawn is necessarily an A card. In other words, the sequence must begin with a U.

Candidate A always having a higher tally than Candidate B corresponds to a path from $(1,1)$ to $(p+q, p-q)$ that does not touch or cross the x axis, which is $\frac{p-q}{p+q}\binom{p+q}{p}$, from part (a) (iv).

The total number of possible vote sequences (sample space) corresponds to the total number of paths from the origin to $(p+q, p-q)$, which is $\binom{p+q}{p}$.

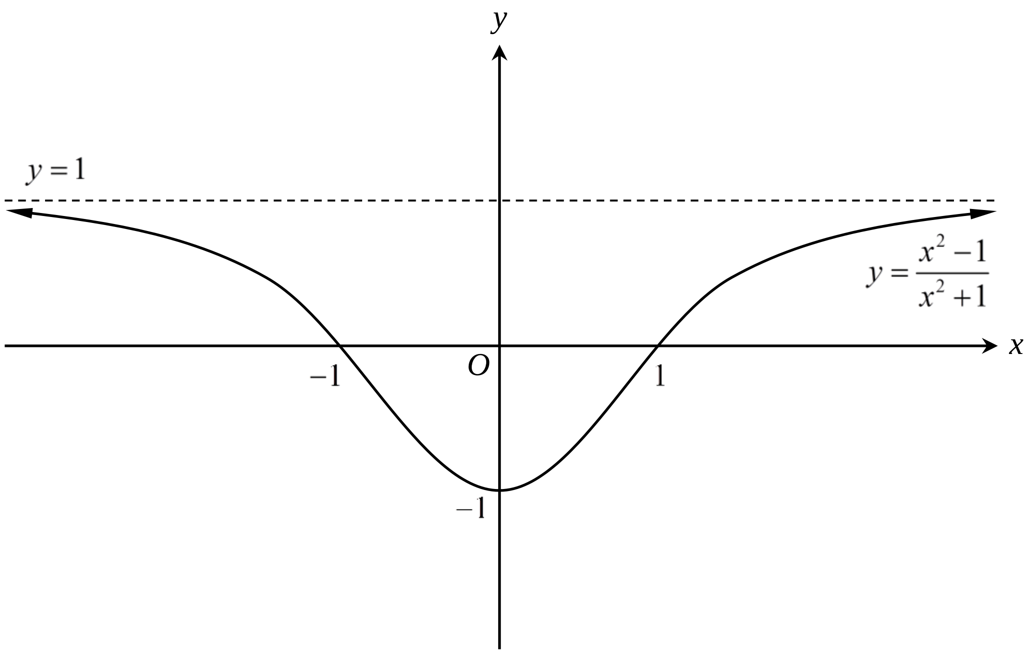
Hence, the required probability is $\dfrac{\frac{p-q}{p+q}\binom{p+q}{p}}{\binom{p+q}{p}} = \dfrac{p-q}{p+q}$.

Question 15 (c)

By observing the equation, we can immediately acquire the following features.

- ✓ y intercept at $(0,-1)$.
- ✓ x intercepts at $(\pm 1, 0)$.
- ✓ Horizontal asymptote $y=1$.
- ✓ The asymptotes are approached from below, as the numerator is always less than the denominator.
- ✓ The function is even.

And this is sufficient to deduce the shape of the curve.



Question 15 (d) (i)

By equating the coefficient of x, we have $ak = k$, so $a = 1$.

By equating the constant term, we have $ab + a = 2$ and hence $b = 1$.

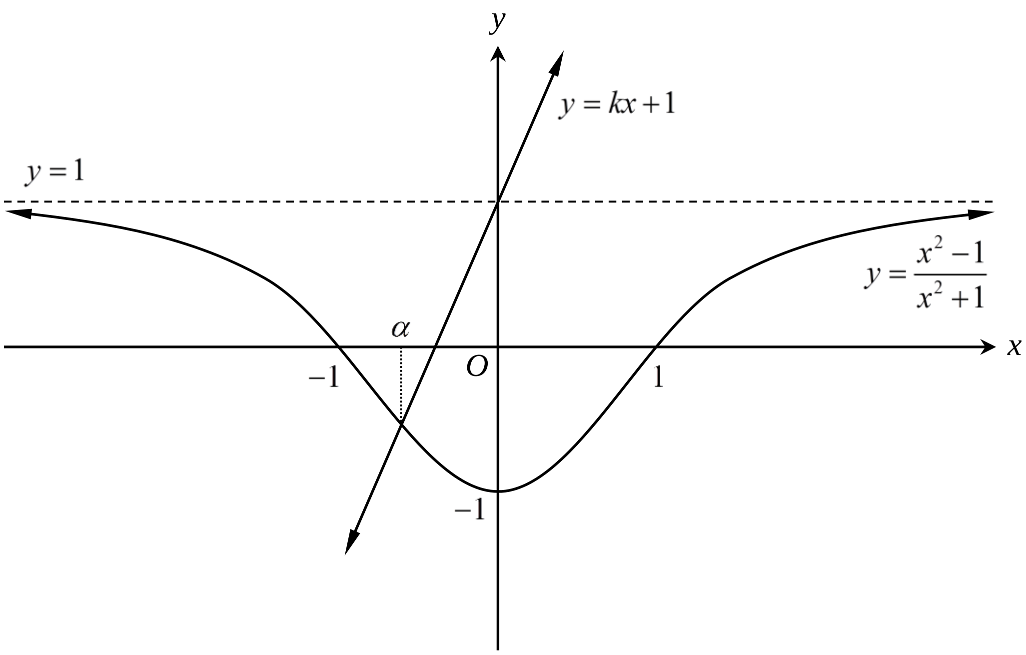
So $P(x) = (x^2+1)(kx+1) - (x^2-1)$.

Question 15 (d) (ii)

Let $P(x)=0$.

$$\left(x^2+1\right)\left(kx+1\right)=\left(x^2-1\right)$$
$$kx+1=\frac{x^2-1}{x^2+1}$$

So if $P(x)$ has exactly one real root α, it will correspond to the intersection point of the line $y=kx+1$ with the curve $y=\frac{x^2-1}{x^2+1}$. From the diagram below, we can observe that the line intersects the curve exactly once for any non-zero value of *k*. Hence, there is only one real root.



Question 15 (d) (iii)

From the diagram above, *k* approaching infinity corresponds to the line $y=kx+1$ becoming steeper, which will result in $\alpha\to 0^-$.

Question 15 (d) (iv)

Let the roots of $P(x)$ be α, β and $\overline{\beta}$.

$$\begin{aligned}\sum_i \alpha_i &= \alpha + \beta + \overline{\beta} \\ &= \alpha + 2\operatorname{Re}(\beta) \\ &= 0 \qquad \ldots(\text{by observing the coefficients})\end{aligned}$$

$$\operatorname{Re}(\beta) = -\frac{\alpha}{2}$$

Since $\alpha \to 0^-$, from part (ii), we can deduce that as $k \to \infty$, $\operatorname{Re}(\beta) \to 0^+$.

$$\begin{aligned}\sum_{i<j} \alpha_i\alpha_j &= \alpha\beta + \alpha\overline{\beta} + \beta\overline{\beta} \\ &= 2\alpha\operatorname{Re}(\beta) + |\beta|^2 \\ &= 1 \qquad \ldots(\text{by observing the coefficients})\end{aligned}$$

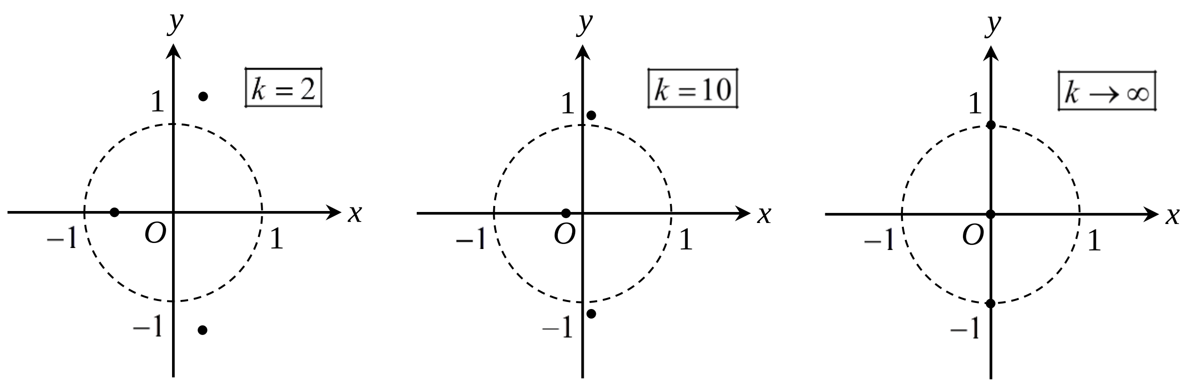
$$|\beta|^2 = 1 - 2\alpha\operatorname{Re}(\beta)$$

Since $\operatorname{Re}(\beta) \to 0^+$ and $\alpha \to 0^-$, we can deduce that $|\beta|^2 \to 1^+$.

So the non-real roots approach the unit circle from the outside, and the real component approaches zero. This means that the argument of the roots approach $\pm\frac{\pi}{2}$ as $k \to \infty$.

And so the non-real roots approach $\pm i$. This can be verified by expressing the polynomial as $x\left(x^2+1\right) = -\frac{2}{k}$ and taking the limit as $k \to \infty$.

The diagrams below show the behaviour of the roots for increasing values of *k*.



Question 16 (a) (i)

Using integration by parts on $\int_a^b f(x)\,dx,$ we have

$$\begin{aligned}\int_a^b f(x)\,dx &= x\,f(x)\Big|_a^b - \int_a^b x\,f'(x)\,dx \\ &= b\cdot f(b) - a\cdot f(a) - \int_a^b x\,f'(x)\,dx\end{aligned}$$

Substituting this into *E*, we have

$$\begin{aligned}E &= \frac{b-a}{2}\left[f(a)+f(b)\right] - b\cdot f(b) + a\cdot f(a) + \int_a^b x\,f'(x)\,dx \\ &= \int_a^b x\,f'(x)\,dx - \left(\frac{a+b}{2}\right)\left[f(b)-f(a)\right]\end{aligned}$$

Question 16 (a) (ii)

Using integration by parts on $I = \frac{1}{2}\int_a^b (b-x)(x-a)\, f''(x)\,dx$, we have

$$I = \frac{1}{2}\Big[(b-x)(x-a)\, f'(x)\Big]_a^b - \frac{1}{2}\int_a^b \big((a+b)-2x\big)\, f'(x)\,dx$$

$$= -\frac{1}{2}\int_a^b \big((a+b)-2x\big)\, f'(x)\,dx$$

$$= \int_a^b \left(x - \left(\frac{a+b}{2}\right)\right) f'(x)\,dx$$

$$= \int_a^b x\, f'(x)\,dx \;\; - \left(\frac{a+b}{2}\right)\int_a^b f'(x)\,dx$$

$$= \int_a^b x\, f'(x)\,dx \;\; - \left(\frac{a+b}{2}\right)\big[f(b) - f(a)\big]$$

$$= E$$

Question 16 (b) (iii)

Since $f(x)$ is concave up in the domain $a \le x \le b$, we can assume that $f''(x) > 0$.

But observe also that $(b-x)(x-a) \ge 0$ for all $a \le x \le b$.

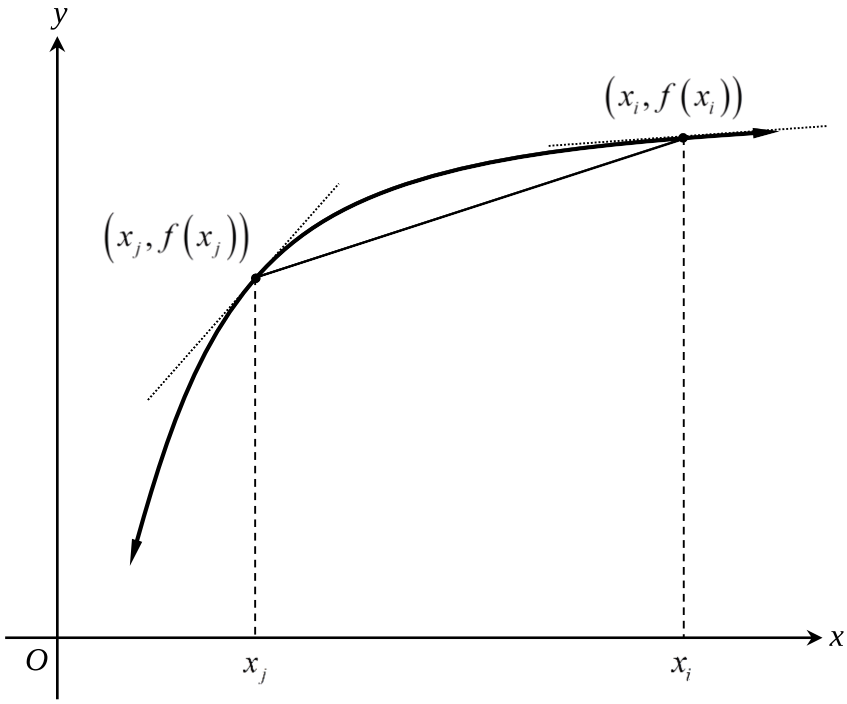
Hence $\int_a^b (b-x)(x-a)\, f''(x)\,dx > 0$ and so $E > 0$.

What this proves is that if a function is concave up in the domain $a \le x \le b$, applications of the trapezoidal rule within the same domain will always over-approximate the value of $\int_a^b f(x)\,dx$.

Question 16 (b) (i)

The given condition implies that the function is concave down, where equality holds when the function is linear. The diagram below shows an example of such a curve.

Observe that from the diagram, the gradient of the secant lies between the gradient of the tangents at x_j and x_i. And so we have the required inequality.



Question 16 (b) (ii)

Let x_α be the value on the number line that divides the interval $x_2 \le x \le x_1$ internally in the ratio $\alpha : 1-\alpha$.

Hence we have $x_\alpha = \dfrac{\alpha x_1 + (1-\alpha)x_2}{\alpha + (1-\alpha)}$. As x_α internally divides the interval, it must satisfy

$x_2 < x_\alpha < x_1$ and so $x_2 < \alpha x_1 + (1-\alpha)x_2 < x_1$.

Question 16 (b) (iii)

From the right hand inequality in part (i), we have

$$f\left(x_i\right) \le f\left(x_j\right) + \left(x_i - x_j\right) f'\left(x_j\right)$$

Let $x_i = x_1$ and $x_j = \alpha x_1 + \left(1-\alpha\right) x_2$.

$$f\left(x_1\right) \le f\left(\alpha x_1 + \left(1-\alpha\right) x_2\right) + \left(x_1 - \alpha x_1 - \left(1-\alpha\right) x_2\right) f'\left(\alpha x_1 + \left(1-\alpha\right) x_2\right)$$

$$f\left(x_1\right) \le f\left(\alpha x_1 + \left(1-\alpha\right) x_2\right) + \left(1-\alpha\right)\left(x_1 - x_2\right) f'\left(\alpha x_1 + \left(1-\alpha\right) x_2\right) \quad ...(1)$$

From the left hand inequality from part (i), we have

$$f\left(x_j\right) \le f\left(x_i\right) - \left(x_i - x_j\right) f'\left(x_i\right)$$

Let $x_i = \alpha x_1 + \left(1-\alpha\right) x_2$ and $x_j = x_2$.

$$f\left(x_2\right) \le f\left(\alpha x_1 + \left(1-\alpha\right) x_2\right) - \left(\alpha x_1 + \left(1-\alpha\right) x_2 - x_2\right) f'\left(\alpha x_1 + \left(1-\alpha\right) x_2\right)$$

$$f\left(x_2\right) \le f\left(\alpha x_1 + \left(1-\alpha\right) x_2\right) - \alpha\left(x_1 - x_2\right) f'\left(\alpha x_1 + \left(1-\alpha\right) x_2\right) \quad ...(2)$$

Taking $\left(1\right)\times\alpha + \left(2\right)\times\left(1-\alpha\right)$:

$$\alpha f\left(x_1\right) + \left(1-\alpha\right) f\left(x_2\right) \le f\left(\alpha x_1 + \left(1-\alpha\right) x_2\right)$$

which is the required result.

Question 16 (b) (iv)

<u>Base Case:</u> $n = 2$

We have $a_1 + a_2 = 1$, so $a_2 = 1 - a_1$.

From part (ii), we have

$$\begin{aligned} f\left(a_1 x_1 + a_2 x_2\right) &= f\left(a_1 x_1 + \left(1 - a_1\right) x_2\right) \\ &\geq a_1 f\left(x_1\right) + \left(1 - a_1\right) f\left(x_2\right) \\ &= a_1 f\left(x_1\right) + a_2 f\left(x_2\right) \end{aligned}$$

as required.

<u>Inductive Hypothesis:</u> $n = k$

$$f\left(a_1 x_1 + a_2 x_2 + ... + a_k x_k\right) \geq a_1 f\left(x_1\right) + a_2 f\left(x_2\right) + ... + a_k f\left(x_k\right),$$

where $a_1,\ a_2,\ a_3,\ \dots,\ a_k$ is *any* set of positive numbers such that $a_1 + a_2 + ... + a_k = 1$

<u>Inductive Step:</u> $n = k \Rightarrow n = k + 1$

Required to prove

If $a_1,\ a_2,\ a_3,\ \dots,\ a_k,\ a_{k+1}$ is *any* set of positive numbers such that $a_1 + a_2 + ... + a_k + a_{k+1} = 1$,

$$f\left(a_1 x_1 + a_2 x_2 + ... + a_k x_k + a_{k+1} x_{k+1}\right) \geq a_1 f\left(x_1\right) + a_2 f\left(x_2\right) + ... + a_k f\left(x_k\right) + a_{k+1} f\left(x_{k+1}\right).$$

$$\text{LHS} = f\left(a_1x_1 + a_2x_2 + \ldots + a_kx_k + a_{k+1}x_{k+1}\right)$$

$$= f\left(\left(a_1x_1 + a_2x_2 + \ldots + a_kx_k\right) + \left(a_{k+1}x_{k+1}\right)\right)$$

$$= f\left(\left(a_1 + a_2 + \ldots + a_k\right)\left(\frac{a_1x_1 + a_2x_2 + \ldots + a_kx_k}{a_1 + a_2 + \ldots + a_k}\right) + \left(1 - \left(a_1 + a_2 + \ldots + a_k\right)\right)\left(x_{k+1}\right)\right)$$

$$\geq \left(a_1 + a_2 + \ldots + a_k\right) f\left(\frac{a_1x_1 + a_2x_2 + \ldots + a_kx_k}{1 - a_{k+1}}\right) + \left(1 - \left(a_1 + a_2 + \ldots + a_k\right)\right) f\left(x_{k+1}\right) \quad \ldots(\text{part (ii)})$$

$$= \left(1 - a_{k+1}\right) f\left(\frac{a_1}{1 - a_{k+1}}x_1 + \frac{a_2}{1 - a_{k+1}}x_2 + \frac{a_3}{1 - a_{k+1}}x_3 + \ldots + \frac{a_k}{1 - a_{k+1}}x_k\right) + a_{k+1} f\left(x_{k+1}\right)$$

$$= \left(1 - a_{k+1}\right) f\left(A_1x_1 + A_2x_2 + A_3x_3 + \ldots + A_kx_k\right) + a_{k+1} f\left(x_{k+1}\right) \quad \ldots\left(\text{where } A_k = \frac{a_k}{1 - a_{k+1}}\right)$$

However, observe that $A_1 + A_2 + A_3 + \ldots + A_k = \dfrac{a_1 + a_2 + a_3 + \ldots + a_k}{1 - a_{k+1}} = 1$.

Hence, we can apply the inductive hypothesis. So continuing on, we then have

$$\text{LHS} \geq \left(1 - a_{k+1}\right) f\left(A_1x_1 + A_2x_2 + A_3x_3 + \ldots + A_kx_k\right) + a_{k+1} f\left(x_{k+1}\right)$$

$$\geq \left(1 - a_{k+1}\right)\left[A_1 f\left(x_1\right) + A_2 f\left(x_2\right) + \ldots + A_k f\left(x_k\right)\right] + a_{k+1} f\left(x_{k+1}\right) \quad \ldots(\text{inductive hypothesis})$$

$$= \left(1 - a_{k+1}\right) A_1 f\left(x_1\right) + \left(1 - a_{k+1}\right) A_2 f\left(x_2\right) + \ldots + \left(1 - a_{k+1}\right) A_k f\left(x_k\right) + a_{k+1} f\left(x_{k+1}\right)$$

$$= a_1 f\left(x_1\right) + a_2 f\left(x_2\right) + \ldots + a_k f\left(x_k\right) + a_{k+1} f\left(x_{k+1}\right)$$

= RHS, as required.

Question 16 (b) (v)

The function $f(x) = \ln(x)$ is a suitable candidate because it is concave down.

Also, set $a_1 = a_2 = a_3 = \ldots = a_n = \dfrac{1}{n}$.

From the result in part (iii), and using log laws, we have

$$\ln\left(\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}\right) \geq \frac{1}{n}\left(\ln(x_1) + \ln(x_2) + \ldots + \ln(x_n)\right)$$

$$\ln\left(\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}\right) \geq \ln\left[\left(x_1 x_2 x_3 \ldots x_n\right)^{\frac{1}{n}}\right]$$

$$\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 x_3 \ldots x_n}$$