UNIT 1

Q.1. 1. $x^2/2 + y^2 = 1$ Here $a = \sqrt{2}$ and $b = 1$

(i) Since $b^2 = a^2(1-e^2)$ the eccentricity

$$e = \frac{\sqrt{a^2-b^2}}{a}, \quad \text{So } e = \frac{\sqrt{2-1}}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

(ii) foci are; $(\pm ae, 0) = (\pm 1, 0)$

(iii) directrixes; $x = \pm \frac{a}{e}$ i.e. $x = \pm \frac{\sqrt{2}}{1/\sqrt{2}}$ so $x = \pm 2$

Q.1. 2. $x^2/_6 + y^2/_4 = 1$ Here $a = \sqrt{6}$, $b = 2$

(i) $e = \frac{\sqrt{a^2 - b^2}}{a}$ So $e = \frac{\sqrt{6-4}}{\sqrt{6}} = \frac{1}{\sqrt{3}}$

(ii) foci are; $(\pm ae, 0) = (\pm \frac{\sqrt{6}}{\sqrt{3}}, 0) = (\pm \sqrt{2}, 0)$

(iii) directrixes; $x = \pm \frac{a}{e}$ i.e. $x = \pm \frac{\sqrt{6}}{1/\sqrt{3}}$ so $x = \pm 3\sqrt{2}$

Q.1. 3. $2x^2 + y^2 = 8$ i.e. $x^2/_4 + y^2/_8 = 1$

Here $a = \sqrt{8} = 2\sqrt{2}$ and $b = 2$. (Note: This ellipse has its major axis on the y axis!)

(i) $e = \frac{\sqrt{8-4}}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$

(ii) foci are; $(0, \pm ae) = (0, \pm 2\sqrt{2}/_{\sqrt{2}}) = (0, \pm 2)$

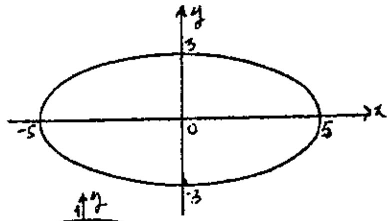
(iii) directrixes; $y = \pm \frac{a}{e}$ i.e. $y = \pm 2\sqrt{2} \times \sqrt{2}$ so $y = \pm 4$

Q.1.4. $\frac{x^2}{4} + \frac{y^2}{16/_9} = 1.$ Here $a = 2$ $b = {}^4/_3$

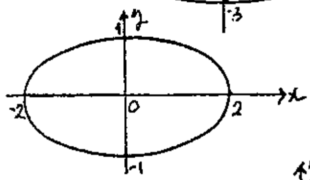
Q.1.5. $\frac{x^2}{25} + \frac{y^2}{25/_{16}} = 1.$ Here $a = 5$ and $b = {}^5/_4$

UNIT 1

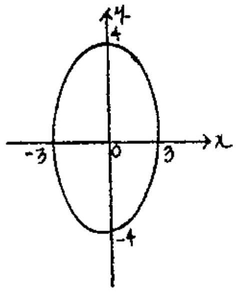
Q.2. 1. $(5 \cos \theta,\ 3 \sin \theta)$, $a = 5,\ b = 3$



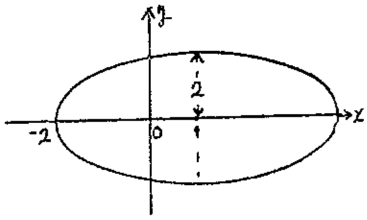
Q.2. 2. $(2 \cos \theta,\ \sin \theta)$, $a = 2,\ b = 1$



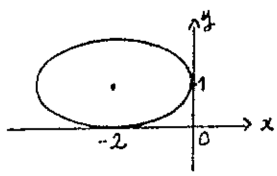
Q.2. 3. $(3 \sin \theta,\ 4 \cos \theta)$, $\left.\begin{aligned} \frac{x}{3} &= \sin \theta \\ \frac{y}{4} &= \cos \theta \end{aligned}\right\}$ $\therefore\ \frac{x^2}{9} + \frac{y^2}{16} = 1$



Q.2. 4. $(1 + 3 \cos \theta,\ 2 \sin \theta)$, $a = 3,\ b = 2$
centre $(1,0)$



Q.2. 5. $(2 \cos \theta - 2,\ \sin \theta + 1)$, $a = 2,\ b = 1$
centre $(-2,1)$



Q.2. 6. $\left.\begin{aligned} x &= 6 \cos \theta \\ y &= 2 \sin \theta \end{aligned}\right\} \begin{aligned} a &= 6 \\ b &= 2 \end{aligned}$

(i) $\frac{b^2}{a^2} = 1 - e^2$

$$\therefore\ e^2 = \frac{a^2 - b^2}{a^2} \text{ so } e = \frac{\sqrt{a^2 - b^2}}{a}$$

$$= \frac{\sqrt{36 - 4}}{6} = \frac{4\sqrt{2}}{6}$$

eccentricity $= \underline{\frac{2\sqrt{2}}{3}}$

(ii) focus $(ae,0)$, $(-ae,0)$

$$ae = \pm\sqrt{a^2 - b^2} = \pm\ 4\sqrt{2}$$

foci $(4\sqrt{2},0)$ and $(-4\sqrt{2},0)$

UNIT 1

(iii) directrix $x = \pm \frac{a}{e}$

$x = \pm \frac{6}{2\sqrt{2}} \cdot \frac{3}{1} = \pm \frac{9}{\sqrt{2}}$

$x = \pm \frac{9}{\sqrt{2}}$

Q.2.8. $\left.\begin{array}{l} x = \sqrt{2}\cos\theta \\ y = \sin\theta \end{array}\right\} \begin{array}{l} a = \sqrt{2} \\ b = 1 \end{array}$

(i) $e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{2-1}}{\sqrt{2}}$

eccentricity $= \frac{1}{\sqrt{2}}$

(ii) foci; $ae = \pm\sqrt{2} \times \frac{1}{\sqrt{2}} = 1$

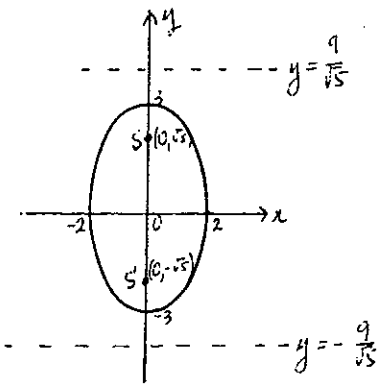
foci are $(-1,0)$ and $(1,0)$

(iii) $\frac{a}{e} = \frac{\pm\sqrt{2}}{\frac{1}{\sqrt{2}}}$

directrix are $x = \pm(\sqrt{2})^2$
$= \pm 2$

Q.2.9. $\left.\begin{array}{l} x = 2\sin\theta \\ y = 3\cos\theta \end{array}\right\} \begin{array}{l} a = 3 \\ b = 2 \end{array}$

$\begin{array}{c} b\sin\theta \\ a\cos\theta \end{array}$



(i) $e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{9-4}}{3} = \frac{\sqrt{5}}{3}$

eccentricity is $\sqrt{5}/3$

(ii) $ae = \pm 3 \times \frac{\sqrt{5}}{3}$

foci $(0, \pm\sqrt{5})$

(iii) $\frac{a}{e} = \frac{3}{\sqrt{5}} \cdot \frac{3}{1}$

directrix $y = \pm \frac{9}{\sqrt{5}}$

Q.2.10. $x = 2 + 3\cos\theta \Rightarrow \quad \cos\theta = \frac{x-2}{3}$

$y = 2\sin\theta - 3 \Rightarrow \quad \sin\theta = \frac{y+3}{2}$

$\frac{(x-2)^2}{9} + \frac{(y+3)^2}{4} = 1$

The equations represent an ellipse with centre $(2,-3)$

Area $= \pi ab$

$= \pi \times 3 \times 2$

$= 6\pi$ sq. units.

UNIT 1

Q.2.11. $x = 1 + 4\cos\theta$, $y = 1 + \sin\theta$

$SS^1 = 2ae$

$= 2 \times 4 \times \frac{\sqrt{15}}{4}$

$= 2\sqrt{15}$

$a = 4$

$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{4^2 - 1}}{4}$

The distance between the foci is $2\sqrt{15}$ units

Q.2.14. $4x^2 + 9y^2 = 9$ and $\frac{x^2}{9/4} + \frac{y^2}{1} = 1$

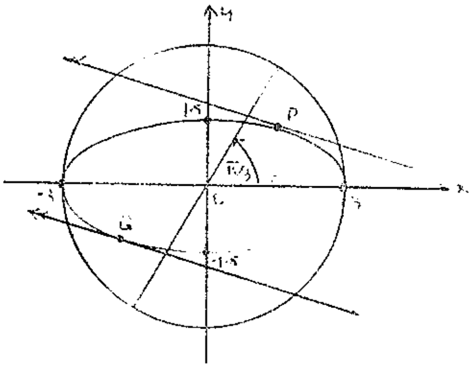
$\therefore a = \frac{3}{2}$, $b = 1$

$x = \frac{3}{2}\cos\theta$, $y = \sin\theta$

Q.2.16. $\frac{(x+2)^2}{16} + \frac{(y-1)^2}{9} = 1 \iff x = -2 + 4\cos\theta$, $y = 1 + 3\sin\theta$

$a = 4$, $b = 3$

Q.2.17.



$x^2 + 4y^2 = 9 \iff \frac{x^2}{9} + \frac{y^2}{9/4} = 1$

$x = 3\cos\theta$, $y = \frac{3}{2}\sin\theta$ $\therefore$ The coordinates of the point

Q are $(3\cos\frac{4\pi}{3}, \frac{3}{2}\sin\frac{4\pi}{3})$

$Q(-\frac{3}{2}, -\frac{3\sqrt{3}}{4})$ $P(\frac{3}{2}, \frac{3\sqrt{3}}{4})$

$\therefore$ eccentric angle of Q is $\frac{4\pi}{3}$ or $-\frac{2\pi}{3}$

$m \equiv 2x + 8y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2x}{8y} = -\frac{x}{4y}$

m at $P(\frac{3}{2}, \frac{3\sqrt{3}}{4})$ is $-\frac{3/2}{4 \times 3\sqrt{3}/4} = -\frac{\cancel{3}^1}{2} \cdot \frac{\cancel{4}^1}{\cancel{4}_1 \times \cancel{3}_1\sqrt{3}} = -\frac{1}{2\sqrt{3}}$

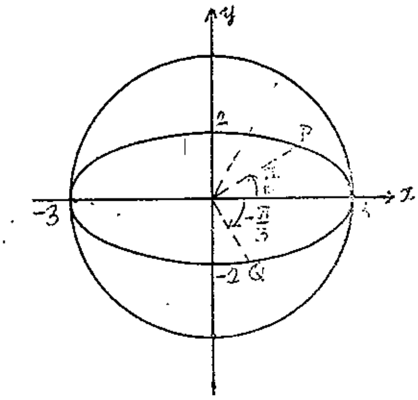
m at $Q(-\frac{3}{2}, -\frac{3\sqrt{3}}{4})$ is also $-\frac{1}{2\sqrt{3}}$

The tangents at the extremities of a diameter are //
This is true for the other conics also (except parabola).

UNIT 1

Q.2.18. $4x^2 + 9y^2 = 36 \iff \frac{x^2}{9} + \frac{y^2}{4} = 1$

$a = 3$
$b = 2$



$x = 3 \cos \theta$
$y = 2 \sin \theta$

(i) The point at $\pi/6$

$x = 3 \cos \pi/6 = \frac{3\sqrt{3}}{2}$

$y = 2 \sin \pi/6 = 3 \times \frac{1}{2}$

$P(\frac{3\sqrt{3}}{2}, \frac{3}{2})$

(ii) The point at $-\frac{\pi}{3}$

$x = 3 \times \cos(-\frac{\pi}{3}) = \frac{3}{2}$

$y = 2 \times \sin(-\frac{\pi}{3} = -\sqrt{3}$

$Q(\frac{3}{2}, -\sqrt{3})$

Q.2.19. $\frac{\pi ab}{\pi a^2} = \frac{5}{9} \quad \therefore \quad \frac{a}{b} = \frac{9}{5}$

$e^2 = 1 - \frac{b^2}{a^2} \iff e^2 = 1 - \frac{25}{81}$

$e = 0.83$ or $\qquad e = \sqrt{\frac{81 - 25}{81}} = \frac{\sqrt{56}}{9} = \frac{2\sqrt{14}}{9}$

Q.2.20. $2ae = 8, \quad e = \frac{3}{4} \qquad \therefore \frac{3}{2}a = 8$

$a = 8 \times \frac{2}{3}$

$b^2 = a^2(1 - e^2)$

$= \frac{16^2}{3^2}(1 - \frac{9}{16})$

$= \frac{\overset{16}{\cancel{256}}}{9} \cdot \frac{7}{\underset{1}{\cancel{16}}}$

$= \frac{112}{9} \qquad \therefore b = \frac{4\sqrt{7}}{3}$

Area of ellipse $= \pi ab$

$= \pi \times \frac{16}{3} \times \frac{4\sqrt{7}}{3}$

$= \frac{64\pi\sqrt{7}}{9}$ sq. units.

UNIT 1

Q.2.21. $b^2x^2 + a^2y^2 = a^2b^2 \iff \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$x = a\cos\theta$ is the x coord. of the point on the auxilliary circle $x^2 + y^2 = a^2$

$\therefore\ y = a\sin\theta$

$\therefore\ P(a\cos\theta,\ a\sin\theta)$

Q.3.1. $x^2 + 4y^2 = 9 \iff \frac{x^2}{9} + \frac{y^2}{9/4} = 1$

$\therefore\ a = 3$

$b = \frac{3}{2}$

The equation of the tangent in terms of its gradient is
$(y = mx \pm\sqrt{a^2m^2 + b^2})$

$y = mx \pm\sqrt{9m^2 + \frac{9}{4}}$

If $\parallel$ to line $2x + 3y = 0$ then $m = -\frac{2}{3}$

$\therefore\ y = -\frac{2}{3}x \pm\sqrt{\frac{\cancel{9}^1}{1}\cdot\frac{4}{\cancel{9}_1} + \frac{9}{4}} \iff \sqrt{\frac{16+9}{4}}$

$y = -\frac{2}{3}x + \frac{5}{2}$ or $y = -\frac{2}{3}x - \frac{5}{2}$

$4x + 6y = 15$ or $4x + 6y = -15$

or using $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$, i.e. $\frac{x}{3}\cos\theta + \frac{2y\sin\theta}{3} = 1$

$m = -\frac{\cos\theta}{\cancel{3}_1}\cdot\frac{\cancel{3}^1}{2\sin\theta} = \frac{1}{2}\cot\theta = -\frac{2}{3}$

$\cot\theta = \frac{4}{3}\ \therefore\ \tan\theta = \frac{3}{4}$

and $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$ (when θ is in Quad 3. $\sin\theta$, $\cos\theta$ are neg.)

$\frac{x \times \frac{4}{5}}{3} + \frac{y \times \frac{3}{5}}{3/2} = \pm 1 \iff 4x + 6y = \pm 15$

UNIT 1

Q.3.2. $16x^2 + y^2 = 169$

$$\frac{x^2}{169/16} + \frac{y^2}{169} = 1 \qquad \left.\begin{array}{l} a = 13 \\ b = \frac{13}{4} \end{array}\right\} \text{ at } (3,5)$$

Equ. of tangent; $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$ however this ellipse is the in the form of $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$. Therefore the tangent will take the form; $\dfrac{xx_1}{b^2} + \dfrac{yy_1}{a^2} = 1$

$$\frac{3x}{169/16} + \frac{5y}{169} = 1 \iff 48x + 5y = 169$$

Equ. of the normal; $\dfrac{xa^2}{x_1} - \dfrac{yb^2}{y_1} = a^2 - b^2$

so we have $\dfrac{xb^2}{x_1} - \dfrac{ya^2}{y_1} = a^2 - b^2$

$$\frac{169}{16} \times \frac{x}{3} - \frac{169y}{5} = -169 + \frac{169}{16}$$

$$5 \times 169x - 169y \times 48 = -169 \times 16 \times 15 + 169 \times 15$$

$$5x - 48y = -\frac{\cancel{169}^{1} \times 15(16-1)}{\cancel{169}_{1}}$$

$$5x - 48Y + 225 = 0$$

Equ. of diameter three $(3,5)$ $y = \frac{5}{3}x$

$$5x - 3y = 0.$$

Q.3.3. $2x - 3y + 1 = 0 \cap x^2 + 3y^2 = 1$

$$x = \frac{3y-1}{2} \longrightarrow x^2 + 3y^2 = 1$$

$$\frac{(3y-1)^2}{4} + 3y^2 = 1$$

$$9y^2 - 6y + 1 + 12y^2 = 4$$

$$21y^2 - 6y - 3 = 0$$

$$7y^2 - 2y - 1 = 0$$

$$y = \frac{2 \pm \sqrt{4+28}}{14} = \frac{\cancel{2}^1 \pm \cancel{4}^2\sqrt{2}}{\cancel{14}_7} = \frac{1 \pm 2\sqrt{2}}{7}$$

$$x = \frac{\frac{3 \pm 6\sqrt{2}}{7} - 1}{2} = \frac{3 \pm 6\sqrt{2} - 7}{7 \times 2}$$

$$x = \frac{-4 \pm 6\sqrt{6}}{14} = \frac{-2 \pm 3\sqrt{6}}{7}$$

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UNIT 1

Q.3.3 (cont'd)

The points are $\left(\frac{-2+3\sqrt{6}}{7}, \frac{1+2\sqrt{2}}{7}\right)$ and

$\left(\frac{-2-3\sqrt{6}}{7}, \frac{1-2\sqrt{2}}{7}\right)$

Midpoint of the chord $\left(-\frac{2}{7}, \frac{1}{7}\right)$

$$\left.\begin{array}{l} x = -\frac{2}{7} \\ y = \frac{1}{7} \end{array}\right\} \longrightarrow x + 2y = 0 \quad \text{True}$$

$\therefore$ diameter bisects the chord $2x - 3y + 1 = 0$.

Q.3.4. Putting $(2\cos\theta, 3\sin\theta) \longrightarrow 3x\cos\theta + 2y\sin\theta = 6$ is insufficient as proof!! (may be a secant!)

Rather; Equ. through $(2\cos\theta, 3\sin\theta)$..(2) to the ellipse $\underset{\uparrow b^2}{\frac{x^2}{4}} + \underset{\uparrow a^2}{\frac{y^2}{9}} = 1$ is;

$$\frac{x\cos\theta}{2} + \frac{y\sin\theta}{3} = 1 \quad .....(1)$$

$a = 9$

$b = 2$

$$(2)\longrightarrow(1) \quad \frac{2\cos^2\theta}{2} + \frac{3\sin^2\theta}{3} = 1$$

LHS = RHS.

$\therefore$ $(2\cos\theta, 3\sin\theta)$ is on the tangent, and is an ellipse.

Hence $\frac{x\cos\theta}{2} + \frac{y\sin\theta}{3} = 1 \iff 3x\cos\theta + 2y\sin\theta = 6$

is the tangent to $\frac{x^2}{4} + \frac{y^2}{9} = 1$ at $(2\cos\theta, 3\sin\theta)$.

0

(diameter) $y = 2x \cap \frac{x^2}{4} + \frac{y^2}{9} = 1 \implies \frac{x^2}{4} + \frac{4x^2}{9} = 1$

$$25x^2 = 36$$

$$x = \pm\frac{6}{5} \text{ and } y = \pm\frac{12}{5}$$

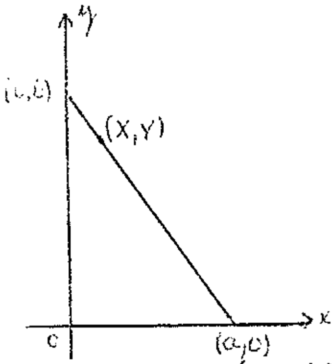
Tangents at $\left(\frac{6}{5}, \frac{12}{5}\right)$ and at $\left(-\frac{6}{5}, -\frac{12}{5}\right)$

$$\frac{xx_1}{4} + \frac{yy_1}{9} = 1 \iff \frac{\cancel{6}^{3}x}{\cancel{20}_{10}} + \frac{\cancel{12}^{4}y}{5\times\cancel{9}_{3}} = \pm 1$$

$$45x + 40y = 150$$

$$9x + 8y = \pm 30$$

Q.3.5.



$X = \frac{1\times a + 3\times 0}{1+3} \quad Y = \frac{1\times 0 + 3\times b}{1+3}$

$X = \frac{a}{b} \therefore a = 4X \quad Y = \frac{3b}{4} \therefore b = \frac{4Y}{3}$

By Pythagoras $a^2 + b^2 = \text{constant}$

$$a^2 + b^2 = 4^2$$

$$\therefore 16X^2 + \frac{16Y^2}{9} = 16$$

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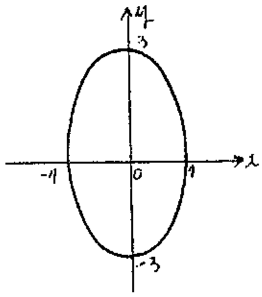
UNIT 1

Q.3.5 (cont'd)

The locus has equation $X^2 + \frac{Y^2}{9} = 1 \iff 9X^2 + Y^2 = 9$



OR

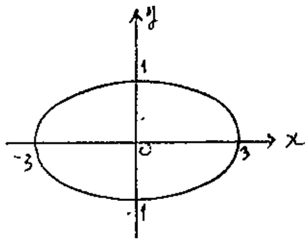
Instead of 3 : 1, divide the 4 unit interval in the ratio 1 : 3 then

$$X = \frac{3a}{4} \text{ and } Y = \frac{b}{4} \iff a = \frac{4X}{3}, \quad b = 4Y.$$

$$a^2 + b^2 = 16 \iff \frac{16X^2}{9} + 16Y^2 = 16$$

$$\frac{X^2}{9} + Y^2 = 1 \text{ or}$$

we have $X^2 + 9Y^2 = 9 \longrightarrow$



Q.3.6. $x^2 + 16y^2 = 25 \iff \frac{x^2}{25} + \frac{y^2}{\frac{25}{16}} = 1$

Semi major axis is $a = 5$

Semi minor axis is $b = \frac{5}{4}$

Tangent at (3,1) $\frac{3x}{25} + \frac{y}{\frac{25}{16}} = 1$

$$\therefore 3x + 16y = 25$$

Normal at (3,1) $16(x-3) - 3(y-1) = 0$

$$16x - 3y = 45$$

Q.3.7. $9x^2 + 16y^2 = 36$

$\frac{x^2}{4} + \frac{y^2}{9/4} = 1 \qquad a = 2, \; b = \frac{3}{2}$

$\perp$ to $x + y = 4 \quad \therefore m = 1$

Tangent is $y = m \pm \sqrt{a^2m^2 + b^2}$

$$y = x \pm \sqrt{4 + \frac{9}{4}}$$

$$y = x \pm \frac{5}{2} \iff 2x - 2y = \pm 5$$

(cont. on next p.)

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UNIT 1

Equ. of tangent $\perp$ to $x + y = 4$ is

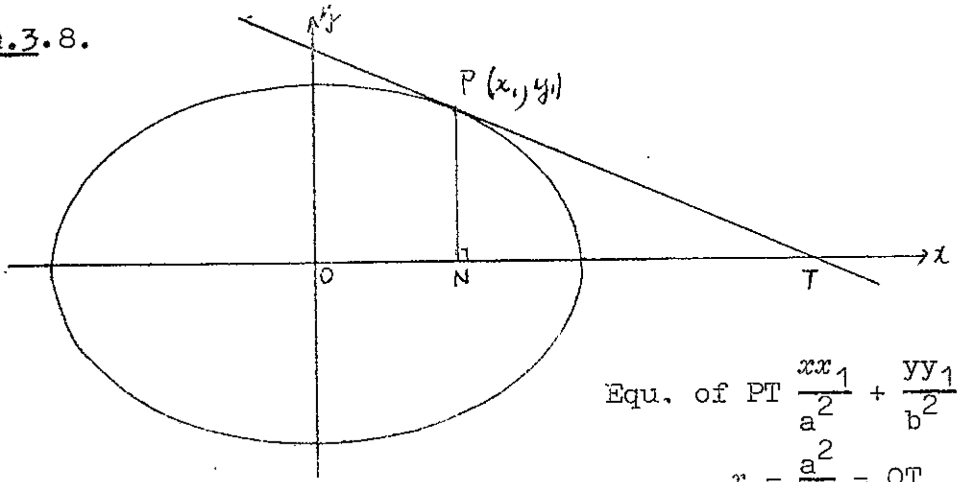
$y = x \pm 5/2 \rightarrow 9x^2 + 16y^2 = 36 \Leftrightarrow 9x^2 + 16x^2 \pm 80x + 100 - 36 = 0$

$25x^2 \pm 80x + 64 = 0 \Leftrightarrow (8 \pm 5x)(8 \pm 5x) = 0$

$$\therefore x = -\frac{8}{5} \quad y = \frac{9}{10}$$

$$\text{or} \quad x = \frac{8}{5} \quad y = -\frac{9}{10}$$

Q.3.8.



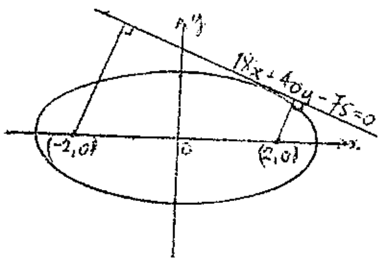
Equ. of PT $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$ at $y = 0$

$$x = \frac{a^2}{x_1} = OT$$

$$ON = x_1$$

$$\therefore ON.OT = x_1 . \frac{a^2}{x_1} = a^2$$

Q.3.9.



$$36x^2 + 100y = 225$$

$$\frac{x^2}{\frac{225}{36}} + \frac{y^2}{\frac{9}{4}} = 1 \qquad \therefore a = \frac{\cancel{15}^{5}}{\cancel{6}_{2}} ; \; b = \frac{3}{2}$$

$$\frac{x^2}{25/4} + \frac{y^2}{9/4} = 1$$

Equ. of tangent at $(\frac{3}{2}, \frac{6}{5})$ is

$$\frac{3x}{2 \times \frac{25}{4}} + \frac{6y}{5 \times \frac{9}{4}} = 1$$

$$\frac{\overset{6}{\cancel{12x}}}{\underset{25}{\cancel{50}}} + \frac{\overset{8}{\cancel{24y}}}{\underset{15}{\cancel{45}}} = 1 \Leftrightarrow 18x + 40y = 75 \text{ is the}$$

equation of the tangent.

Product of $\perp$ distances $= \left|\dfrac{18 \times 2 - 75}{\sqrt{18^2 + 40^2}}\right| \cdot \left|\dfrac{18 \times -2 - 75}{\sqrt{18^2 + 40^2}}\right|$

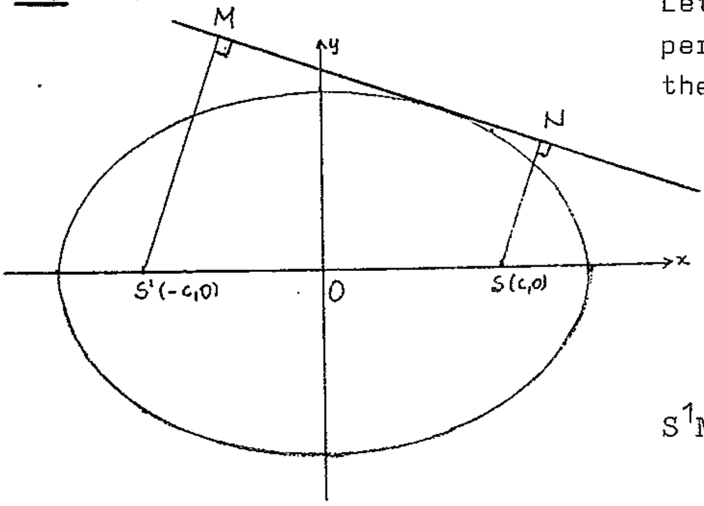
$$= \frac{39 \times 111}{\sqrt{1924^2}} = \frac{4329}{1924} = \frac{13 \times 333}{13 \times 148} = \frac{37 \times 9}{37 \times 4}$$

$$= \frac{9}{4}$$

$\therefore$ The product of the $\perp$ dist. from $(\pm 2, 0)$ to the tangent $= \dfrac{9}{4}$

UNIT 1

Q.3. 10.



Let M and N be the foot of the perpendiculars from S' and S to the tangent MN.

$$S^1M = \left| \frac{-b^2cx_1 - a^2b^2}{\sqrt{(b^2x_1)^2 + (a^2y_1)^2}} \right|$$

$$SN = \left| \frac{b^2cx_1 - a^2b^2}{\sqrt{(b^2x_1)^2 + (a^2y_1)^2}} \right|$$

$$S'M \cdot SN = \frac{-b^2(x_1c + a^2)b^2(x_1c - a^2)}{(b^4x_1^2 + a^4y_1^2)}$$

$$= \frac{-b^4(x_1^2c^2 - a^4)}{b^4x_1^2 + a^4y_1^2}$$

$$= \frac{-b^4(x_1^2a^2 - x_1^2b^2 - a^4)}{b^4x_1^2 + \cancel{a^4}^{2}\,\frac{b^2}{\cancel{a^2}_{1}}(a^2 - x_1^2)} \qquad c^2 = a^2 - b^2, \quad y_1^2 = \frac{a^2b^2 - b^2x^2}{a^2} = \frac{b^2}{a^2}(a^2 - x_1^2)$$

$$= \frac{b^4(a^4 + x_1^2b^2 - x_1^2a^2)}{b^4x_1^2 + a^4b^2 - a^2b^2x_1^2}$$

$$= \frac{b^4\cancel{(a^4 + x_1^2b^2 - x_1^2a^2)}}{b^2\cancel{(a^4 + x_1^2b^2 - x_1^2a^2)}}$$

$$= b^2$$

The product of the ⊥ distances from (c,0) and (-c,0) (i.e. from the foci) to the tangent of the ellipse is b^2.

UNIT 1

Q.3.11. $9x^2 + 25y^2 = 225$

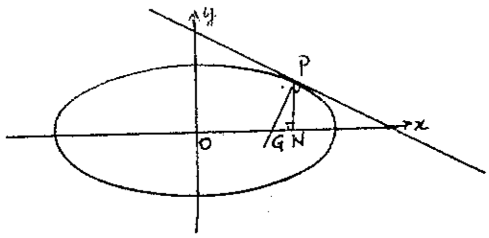
$\frac{x^2}{25} + \frac{y^2}{9} = 1 \qquad \therefore$ the semi major axis $a = 5$
the semi minor axis $b = 3$

The equ. of the normal at $(3, 12/5)$ is $\left(\frac{xa^2}{x_1} - \frac{yb^2}{y_1} = a^2 - b^2\right)$

$$\frac{25}{3}x - \frac{\overset{3}{\cancel{9}} \times 5}{\underset{4}{\cancel{12}}}y = 25 - 9$$

$100x - 45y = 192$ is the normal at $(3, \frac{12}{5})$

At G $y = 0 \quad \therefore \quad \frac{\overset{48}{\cancel{192}}}{\underset{25}{\cancel{100}}} = x$



$$\begin{aligned} GN &= ON - OG \\ &= 3 - 1\tfrac{23}{25} \\ &= 1\tfrac{2}{25} \end{aligned}$$

Q.3.12. $5x^2 + 9y^2 = 45 \iff \frac{x^2}{9} + \frac{y^2}{5} = 1$

Tangent at $(2, \frac{5}{3})$ is $\frac{2x}{9} + \frac{\cancel{5y}}{\underset{3}{\cancel{15}}} = 1$

$$2x + 3y = 9$$

m of $\perp = \frac{3}{2}$. Thru $(-2,0)$	Thru $(2,0)$ the equ. is
$y + 0 = \frac{3}{2}(x+2)$	$y = \frac{3x}{2} - 3$
$6y = 3x + 6$	
$y = \frac{3x}{2} + 3$	

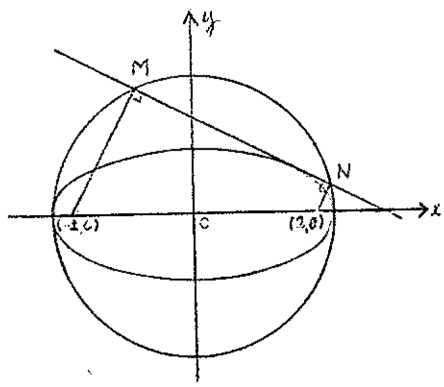
$3x - 2y = 6 \cap 2x + 3y = 9$

$$\begin{aligned} 9x - 6y &= 18 \\ 4x + 6y &= 18 \\ \hline 13x \quad &= 36 \end{aligned}$$

$x = \frac{36}{13} \qquad y = \frac{3}{\underset{1}{\cancel{2}}} \times \frac{\overset{18}{\cancel{36}}}{13} - 3$

$= \frac{54}{13} - \frac{39}{13}$

$= \frac{15}{13}$



$M(\frac{36}{13}, \frac{15}{13}) \longrightarrow x^2 + y^2 = 9 \iff \frac{36^2}{13^2} + \frac{15^2}{13^2} = \frac{1521}{169}$

$= 9$

LHS = RHS

(continued on next page)

Q.3. 12 (continued)

$$3x - 2y = -6 \cap 2x + 3y = 9$$

$$\begin{array}{rcr} 9x - 6y & = & -18 \\ 4x + 6y & = & 18 \\ \hline 13x & = & 0 \end{array}$$

$$x = 0 \qquad \therefore y = 3$$

$$N(0,3) \longrightarrow x^2 + y^2 = 9 \qquad 0 + 9 = 9$$

$$\text{LHS} = \text{RHS}.$$

The feet of the $\perp$ from the points $(-2,0)$ and $(2,0)$ (i.e., from the foci) lie on the circle $x^2 + y^2 = 9$ (i.e. on the auxilliary circle).

Q.3. 13. $a^2 = \frac{1}{4} \quad b^2 = \frac{1}{9} \quad m = -\frac{1}{2}$

Tangents with $m = -\frac{1}{2} \Longrightarrow y = -\frac{1}{2}x \pm \sqrt{\frac{1}{4} \cdot \frac{1}{4} + \frac{1}{9}}$

$$y = -\frac{1}{2}x \pm \frac{5}{12}$$

$$6x + 12y = \pm 5$$

Q.3. 14. $x^2 + 4y^2 = 65 \Longleftrightarrow \frac{x^2}{65} + \frac{y^2}{\frac{65}{4}} = 1$

Normal at $(1,4)$ $\quad 65x - \frac{65y}{16} = 65 - \frac{65}{4}$

$$1040x - 65y = 1040 - 260$$

$$1040x - 65y = 780$$

$$208x - 13y = 156$$

$$16x - y = 12 \quad \text{——— (1)}$$

Normal at $(7,2)$ $\quad \frac{65x}{7} - \frac{65y}{8} = \frac{195}{4}$

$$520x - 455y = 2730$$

$$104x - 91y = 546$$

$$8x - 7y = 42 \quad \text{——— (2)}$$

$$\begin{array}{rcl} 16x - y & = & 12 \\ 16x - 14y & = & 84 \\ \hline 13y & = & -72 \end{array} \qquad y = -\frac{72}{13}$$

$$x = \left(42 + \frac{7 \times 72}{13}\right)\frac{1}{8} = \frac{\overset{21}{\cancel{44}}}{13} \cdot \frac{1}{\cancel{8}_4} = \frac{21}{52}$$

Normals $\cap$ at $\left(\frac{21}{52}, \frac{-72}{13}\right)$ equ. thru 0

$$\frac{y}{x} = \frac{\overset{24}{\cancel{72}}}{\cancel{13}_1} \cdot \frac{\overset{4}{\cancel{52}}}{\cancel{21}_7} \qquad 96x + 7y = 0$$

UNIT 1

Q.3. 15. $x = 2y - 3 \cap x^2 + 2y^2 = 9$ $\quad a^2 = 9$

$b^2 = \frac{9}{2}$

$4y^2 - 12y + 9 + 2y^2 - 9 = 0$

$6y^2 - 12y = 0$

$y(y-2) = 0$ $\quad x = 2 \times 0 - 3 = -3$

$\therefore y = 0$ or $y = 2$ $\quad x = 2 \times 2 - 3 = 1$

The line cuts the ellipse at (-3,0) and at (1,2)

(i) Tangent at (-3,0) $\frac{-3x}{9} = 1 \quad x = -3$

(ii) Tangent at (1,2) $\frac{x}{9} + \frac{22y}{9} = 1$

$x + 4y = 9$

The tangents $\cap$ when $-3 + 4y = 9$

$y = 3$

i.e. at (-3,3)

Q.3. 16. $9x^2 + 25y^2 = 169$

$\frac{x^2}{\frac{169}{9}} + \frac{y^2}{\frac{169}{25}} = 1 \qquad a = \frac{13}{3}, \quad b = \frac{13}{5}$

Tangent at (4,-1) $\frac{4x}{\frac{169}{9}} + \frac{-y}{\frac{169}{25}} = 1$

$36x + -25y = 169$

$36x - 25y = 169$

Circle $x^2 + y^2 + 28x - 23y = 152 \cap 9x^2 + 25y^2 = 169 = \{(4,-1)\}$

$\underline{(4,-1)} \rightarrow$ Circle LHS $= 16 + 1 + 112 + 23$

$= 152$

$\therefore \quad = \text{RHS}$

$\therefore$ circle cuts ellipse at (4,-1)

If this (4,-1) is the only $\cap$ pt. between the curves then the tangent must be a tangent to the circle also. $\therefore \quad \Delta = 0$

$x = \frac{169 + 25y}{36} \longrightarrow$ circle

$\frac{(169 + 25y)^2}{36} + y^2 + \cancel{28}^{7}\left(\frac{169 + 25y}{\cancel{36}_{9}}\right) - 23y - 152 = 0$

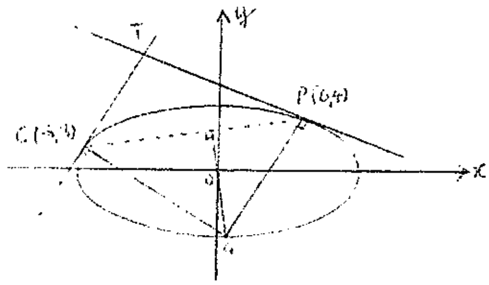
$1921y^2 + 3842y + 1921 = y^2 + 2y + 1 = 0$. So $\Delta = 0$ $\therefore$ tangent is common to both ellipse and circle. So circle <u>touches</u> ellipse at (4, 1).

UNIT 1

Q.3. 17. P(6,4)

Q(-8,3) are pts. on $x^2 + 4y^2 = 100$

$$\frac{x^2}{100} + \frac{y^2}{25} = 1$$



Tangent at P $\frac{\overset{3}{\cancel{6}x}}{\underset{50}{\cancel{100}}} + \frac{4y}{25} = 1 \iff 3x + 8y = 50$

Tangent at Q $\frac{\overset{2}{\cancel{-8}x}}{\underset{25}{\cancel{100}}} + \frac{3y}{25} = 1 \iff 2x - 3y = -25$

$$\begin{aligned} 6x + 16y &= 100 \\ 6x - 9y &= -75 \\ \hline 25y &= 175 \\ y &= 7 \\ x &= -2 \end{aligned}$$

$\therefore$ T (-2,7)

Normal at P $8(x-6) -3(y-4) = 0 \Rightarrow 8x - 3y = 36$ ——— (1)

Normal at Q $3(x+8) +2(y-3) = 0 \Rightarrow 3x + 2y = -18$ ——— (2)

$16x - 6y = 72$ ——— (1) x 2

$9x + 6y = -54$ ——— (2) x 3

$25x = 18$

$x = \frac{18}{25}$ $2y = -18 - \frac{54}{25}$

$y = \frac{-504}{100}$

$y = \frac{-252}{25}$

G $\left(\frac{18}{25}, \frac{-252}{25}\right)$

m_1 of diameter OG $= \frac{-252}{25} \cdot \frac{25}{18}$

$= -14$

m_2 of chord PQ $= \frac{4-3}{6+8}$

$= \frac{1}{14}$

$m_1 \times m_2 = -1$ $\therefore$ PQ is $\perp$ to OG

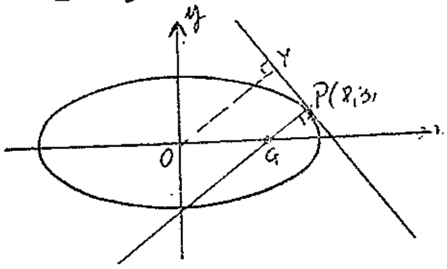
UNIT 1

Q.3.18. Normal to $x^2 + 4y^2 = 100$

$$\frac{x^2}{100} + \frac{y^2}{25} = 1 \quad \text{at } (8,3)P$$

$$\frac{\cancel{100}^{\,25}}{\cancel{8}_{\,2}} - \frac{25y}{3} = 100 - 25$$

$$\frac{x}{2} - \frac{y}{3} = 3 \Rightarrow 3x - 2y = 18$$



at G, y = 0 $\therefore$ $3x = 18$

$x = 6$

$\therefore$ Point G is (6,0)

$$PG = \sqrt{(8-6)^2 + (3-0)^2} = \sqrt{4+9} = \sqrt{13}$$

Equ. of OY. $m = \frac{3}{2}$

$$y = \frac{3}{2}x$$

Tangent at P $\dfrac{{}^{2}\cancel{8x}}{\cancel{100}_{\,25}} + \dfrac{3y}{25} = 1 \Longleftrightarrow 2x + 3y = 25$

$\{Y\} = 2x + 3y = 25 \cap y = \frac{3}{2}x \Rightarrow 2x + \frac{9}{2}x = 25$

$$4x + 9x = 0$$

$$13x = 50$$

$$x = \frac{50}{13} \qquad y = \frac{\cancel{150}^{\,75}}{\cancel{2}_{\,1}\times 13}$$

$$Y\left(\frac{50}{13}, \frac{75}{13}\right)$$

$$OY = \sqrt{\left(\frac{50}{13}\right)^2 + \left(\frac{75}{13}\right)^2}$$

$$= \sqrt{\frac{2500 + 5625}{169}}$$

$$= \frac{25}{13}$$

$$PG \cdot OY = \sqrt{13} \times \frac{25}{\sqrt{13}} = 25$$

$b^2 = 25$

$$\therefore PG \cdot OY = b^2$$

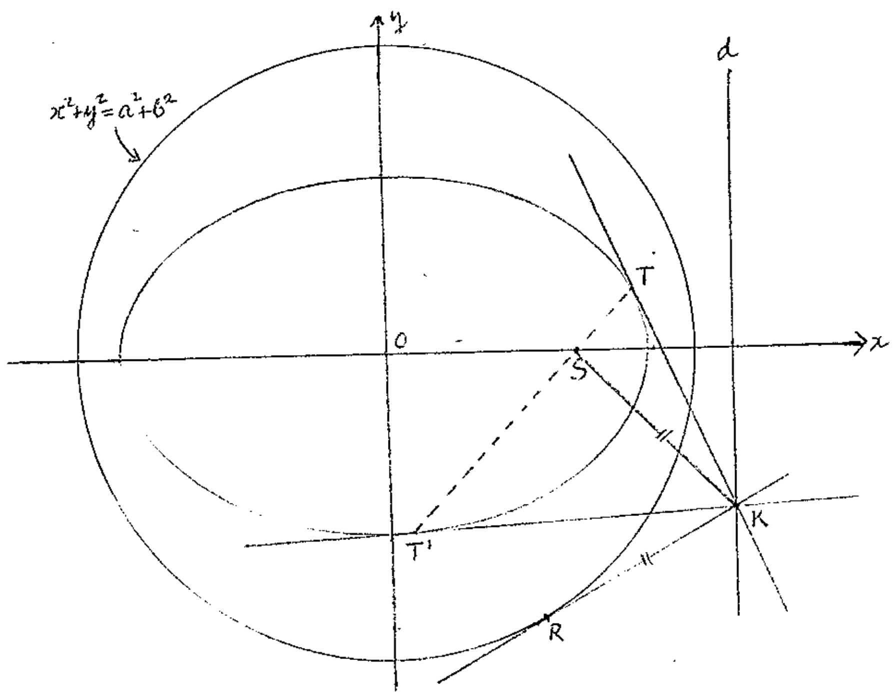
PG . OY is equal to the square on the minor semi axis.

(See diagram on next page)

8

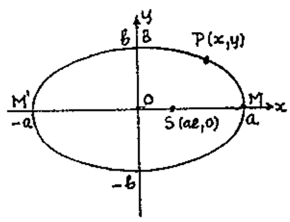
Q.4

UNIT 1



13

Q.4. 1. Let S be the earth, SM and SM^1 be the least and greatest distance of the earth from the moon respectively.



$$\text{Average (mean) dist.} = \frac{SM + SM^1}{2}$$

$$384\ 000 = \frac{(a-ae)+(a+ae)}{2}$$

$$a = 384\ 000$$

So $SM = a - ae = 384\ 000 - 384\ 000 \times 0.$

$SM \doteqdot 363\ 000$ km

and $SM^1 = a + ae = 384\ 000 + 384\ 000 \times 0.055$
$\doteqdot 405\ 000$ km as required

Note: Diagram is not drawn to scale and it may seem that other points of the ellipse are closer to S than M is.
(i.e. SM > SP where P is any point on it)
This can easily be disproved;

$SM = a - ae$ $\quad SP = a - ex_1$ (see the proof of this in Q.4. 5.) Now for all x_1; $0 \leqslant x_1 \leqslant a \therefore ex_1 \leqslant ea$

$\therefore$ SM < SP i.e. SM is the least distance of the earth from the moon. Also note that as $e \to 0$ ellipse becomes a circle so for e = 0.055 we have "almost" a circle.

UNIT 1

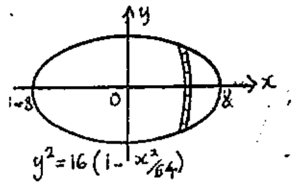
Q.4. 2. $4x^2 + 25y^2 = 100 \iff x^2/_{25} + y^2/_4 = 1$

Here $a = 5$ so external circle is $x^2 + y^2 = 25$

$b = 2$ so internally touching circle is $x^2 + y^2 = 4$.

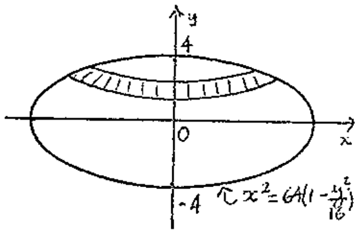
Q.4. 3. (i) $x^2/_{64} + y^2/_{16} = 1$ so $y^2 = 16(1 - x^2/_{64})$

$$V = 16\pi \int_{-8}^{8} (1 - x^2/_{64})dx = 16\pi\left[x - x^3/_{192}\right]_{-8}^{+8}$$



$= 512\pi/_3$ cu.units

(ii)



$x^2/_{64} + y^2/_{16} = 1$ so $x^2 = 64(1 - y^2/_{16})$

$$V = 64\pi \int_{-4}^{4} (1 - y^2/_{16})dy = 64\pi.\left[y - y^3/_{48}\right]_{-4}^{4}$$

$= 1024\pi/_3$ cu. units.

Q4. 4. Method (i) Differentiate $x^2/_{25} + y^2/_9 = 1$ -----(1) implicitly

so $2x/_{25} + 2y/_9\, dy/_{dx} = 0$ i.e. $dy/_{dx} = -9x/_{25y}$

If parallel to $y = 2x$ then $-9x/_{25y} = 2$

and $x = 50y/_9$ -----(2) (condition for tangent to be parallel to diameter).

(2) $\longrightarrow$ (1) $\dfrac{2500y^2}{81 \times 25} + \dfrac{y^2}{9} = 1$ so $y = \pm 9/\sqrt{109}$ ------(3)

(3) $\longrightarrow$ (2) so $x = \dfrac{50}{9} \times \pm \dfrac{9}{\sqrt{109}} = \pm \dfrac{50}{\sqrt{109}}$

So tangent touches ellipse at $\left(\pm \dfrac{50}{\sqrt{109}}\ \pm \dfrac{9}{\sqrt{109}}\right)$

Deduce that the equation of the tangent to the ellipse is

$\dfrac{xx_1}{25} + \dfrac{yy_1}{9} = 1$

i.e. $\pm \dfrac{2x}{\sqrt{109}} \pm \dfrac{y}{\sqrt{109}} = 1$

which is $y = 2x \pm \sqrt{109}$ as rqd.

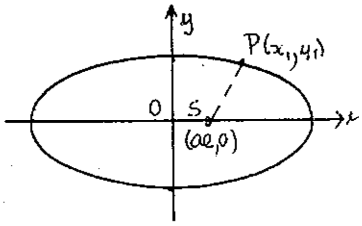
Method (ii) using $y = mx \pm \sqrt{a^2m^2 + b^2}$

$m = 2$, $a^2 = 25$, $b^2 = 9$ we have $y = 2x \pm \sqrt{25 \times 4 + 9}$

which simplifies to $y = 2x \pm \sqrt{109}$ as rqd.

UNIT 1

Q.4. 5.



Let $P(x_1, y_1)$ be any point on $x^2/a^2 + y^2/b^2 = 1$

$$
\begin{aligned}
(*PS)^2 &= (ae - x_1)^2 + (0 - y_1)^2 \\
&= a^2e^2 - 2aex_1 + x_1^2 + y_1^2 \\
&= a^2e^2 - 2aex_1 + x_1^2 + \frac{b^2}{a^2}(a^2 - x^2)
\end{aligned}
$$

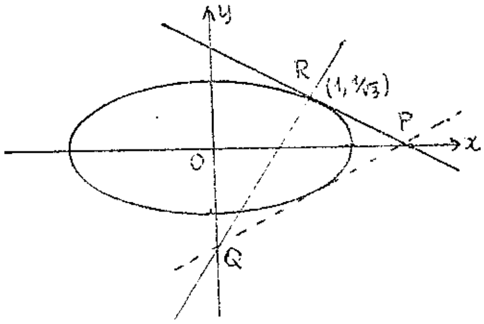
Aim: eliminate b, since the required result does not contain b.

$$= a^2e^2 - 2aex_1 + x_1^2 + (1-e^2)(a^2 - x_1^2) \qquad \text{Since } \frac{b^2}{a^2} = 1 - e^2$$

$$= a^2 - 2aex_1 + e^2x_1^2$$

$$= (a - ex_1)^2$$ so $PS = a - ex_1$ as required.

Q4. 6.



On differentiating $x^2+3y^2 = 2$ implicitly, $2x + 6y\, dy/_{dx} = 0$

$dy/_{dx} = -x/_{3y}$,

so at $x = 1$ the gradient of the tangent $m_1 = -\sqrt{3}/_3$. Gradient of PR is $-\sqrt{3}/_3 = \dfrac{1/\sqrt{3}}{1 - x}$, so

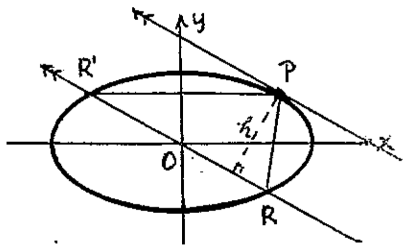
$x = 2$ and $P = (2,0)$

Similarly at $x = 1$ the gradient of the normal $m_2 = 3/\sqrt{3}$. Gradient of RQ is $3/\sqrt{3} = \dfrac{1/\sqrt{3} - y}{1}$, so $y = -2/\sqrt{3}$ and $Q = (0, -2/\sqrt{3})$.

Deduce that the equation of the tangent to the ellipse is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1.$ i.e. $\dfrac{xx_1}{2} + \dfrac{yy_1}{2/3} = 1.$ Show that both $P(2,0)$ and $Q(0, -2/\sqrt{3})$ satisfies this equation.

UNIT 1

Q.4. 7. The area of the triangle PRR' = $\frac{1}{2}$.h.RR'. By differentiating



$$x^2/_9 + y^2/_4 = 1 \qquad (1)$$

with respect to x we obtain $dy/_{dx} = -4x/_{9y}$

Let P be the point ($3\cos\phi$, $2\sin\phi$). Hence the gradient of the tangent at P $= \frac{-2\cos\phi}{3\sin\phi}$ = the gradient of RR'. So the equation of RR' is $(2\cos\phi)x + (3\sin\phi)y = 0$. (2)

from (2) $\qquad x = \frac{3y\sin\phi}{-2\cos\phi}$ (3)

Put (3)$\rightarrow$(1) to obtain $y^2\sin^2\phi + y^2\cos^2\phi = 4\cos^2\phi$

So $y^2 = 4\cos^2\phi$ i.e. $y = \pm 2\cos\phi$ (4)

Put (4)$\rightarrow$(3) $\quad x = \mp 3\sin\phi$. Hence the points R and R' are

$(3\sin\phi, -2\cos\phi)$ and $(-3\sin\phi, 2\cos\phi)$.

$RR' = \sqrt{[(6\sin\phi)^2 + (4\cos\phi)^2]} = 2\sqrt{(9\sin^2\phi + 4\cos^2\phi)}$ (5)

$h = \left|\frac{Ax_1 + By_1 + C}{\sqrt{(A^2 + B^2)}}\right|$ where $A = 2\cos\phi$, $B = 3\sin\phi$, $C = 0$ from

(2) and $x_1 = 3\cos\phi$, $y_1 = 2\sin\phi$. So $h = \left|\frac{6\cos^2\phi + 6\sin^2\phi}{\sqrt{(4\cos^2\phi + 9\sin^2\phi)}}\right|$

i.e. $h = \frac{6}{\sqrt{(4\cos^2\phi + 9\sin^2\phi)}}$ (6)

So the area of the triangle PRR' = $\frac{1}{2}$.h.RR' (7)

(5)&(6)$\rightarrow$(7) gives;

$$\text{triangle PRR'} = \tfrac{1}{2}.\frac{6}{\sqrt{(4\cos^2\phi+9\sin^2\phi)}}.2\sqrt{(9\sin^2\phi+4\cos^2\phi)}$$

$$= 6 \text{ sq. units}$$

ALTERNATIVELY: One may choose the point P to be (x_1 , y_1).

Then the equation of RR' is $4xx_1 + 9yy_1 = 0$. $R(\frac{3y_1}{2}, -\frac{2x_1}{3})$

and $R'(\frac{-3y_1}{2}, \frac{2x_1}{3})$. Then $RR' = \frac{1}{3}.\sqrt{(81y_1^2 + 16x_1^2)}$ and

$$h = \left|\frac{4x_1^2 + 9y_1^2 + 0}{\sqrt{(16x_1^2 + 81y_1^2)}}\right| = \frac{36}{\sqrt{(16x_1^2 + 81y_1^2)}}.$$

Since (x_1 , y_1) is on (1) i.e. on $4x^2 + 9y^2 = 36$ the area of

triangle PRR' $= \frac{1}{2}.\frac{1}{3}\sqrt{(81y_1^2 + 16x_1^2)}.\frac{36}{\sqrt{(81y_1^2+16x_1^2)}}$

$$= 6 \text{ sq. units}$$