Exercise 6C

Find the equations of the (a) tangent and (b) normal to the following curves at the points given:

1. $\frac{x^2}{25} + \frac{y^2}{16} = 1$, $P(\frac{5}{2}, 2\sqrt{3})$

2. $x^2 + 4y^2 = 1$, at $x = \frac{1}{2}$

3. $x^2 - 2y^2 = 2$, at $x = 2$

4. $x = 4\cos\theta$, $y = 3\sin\theta$, $\theta = \frac{\pi}{4}$

5. $x = 5\sec\theta$, $y = 4\tan\theta$, $\theta = \frac{\pi}{4}$

6. Show that the equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

   (a) at $P(x_1, y_1)$ is $\frac{x - x_1}{b^2x_1} = \frac{y - y_1}{a^2y_1}$

   (b) at $P(a\cos\theta, b\sin\theta)$ is $ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$.

7. Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

   and at the point $(a\sec\theta, b\tan\theta)$ is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$.

8. Show that the equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

   (a) at $P(x_1, y_1)$ is $\frac{a^2}{x_1} \cdot x + \frac{b^2}{y_1} \cdot y = a^2 + b^2$

   (b) at $P(a\sec\theta, b\tan\theta)$ is $ax\cos\theta + by\cot\theta = a^2 + b^2$.

9. P is any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with foci S and S', prove that

   $PS + PS' = 2a$.

10. $P(x, y)$ is any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with foci S and S', prove that $|PS' - PS| = 2a$.

11. Find the equations of the tangent and normal to

(a) the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ at $x = 1$

(b) the hyperbola $x^2 - y^2 = 4$ at $x = 3$

12. Find the equations of two tangents to the ellipse $16x^2 + 25y^2 = 400$ which are parallel to the line $y = x + 2$.

13. The tangent to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at $P(4\sqrt{2}, 3)$ meets the asymptotes of the hyperbola at A and B. Show that P is the mid-point of AB. Find the length of AB in the exact form.

14. Find the equation of the tangent to the curve whose parametric equations are $x = 2\cos\theta$ and $y = 3\sin\theta$ at $\theta = \frac{\pi}{3}$. This tangent meets the x-axis in A and y-axis in B. Find the length AB.

15. Show that the equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P(a\cos\theta, b\sin\theta)$ is given by $ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$. The normal at P meets the x-axis at M and N is the foot of the perpendicular PN to the x-axis. Prove that $MN = \frac{b^2\cos\theta}{a}$.

16. The tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(a\sec t, b\tan t)$ meets the asymptotes in A and B. Prove that P is the mid-point of AB.

17. The chord through the focus $S(ae, o)$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, at right angles to the x-axis meets the ellipse at $P(a\cos\theta, b\sin\theta)$. The normal at P passes through the end-point B' of the minor axis, of the ellipse. Prove that:

(a) $\cos\theta = e$ and $\sin\theta = \sqrt{1 - e^2}$

(b) equation of the normal at P is $ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$

(c) $e^4 + e^2 - 1 = 0$. Hence, find e in the exact form.

18. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intersects the x-axis at A and A'. Find the co-ordinates of A and A'. Write down the equations of the tangents at A, A' and $P(x_1, y_1)$. Let the tangents at A and P intersect in Q and those at A' and P at Q'. Prove that the ~~area~~ product $AQ \cdot A'Q'$ is independent of the position of P.

19. Show that the condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 + b^2$. Prove that the pair of tangents from the point $P(4, 5)$ to this ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ are at right angles to one another.

20. Show that the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$ has equation $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

This tangent meets the x-axis at T, $PN \perp$ x-axis, and the normal at P, meets the x-axis at G. Show that $OT \times NG = b^2$, where O is the centre of the ellipse.

21. The line $y = mx + c$ is a tangent to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$.

Show that $c^2 = 25m^2 - 16$. The tangents from $P(x_1, y_1)$ to this hyperbola meet at right angles. Prove that the locus of P is the circle $x^2 + y^2 = 9$.

22. $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$ are two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, such that $\theta + \phi = 90°$.

Find the co-ordinates of the mid-point R of PQ and hence show that the locus of R is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{y}{b}$.

23. $P(x_1, y_1)$ is a point on the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$.

Prove that the equation of the tangent at P is $16xx_1 - 25yy_1 = 400$

(a) Find the co-ordinates of the point G at which this tangent cuts the x-axis.

(b) Hence prove that $\frac{SP}{S'P} = \frac{SG}{S'G}$ where S and S' are the foci of the hyperbola.

24. Show that the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$ is given by

$$a^2 y_1 x - b^2 x_1 y = (a^2 - b^2)\, x_1 y_1$$

(a) This normal meets the x-axis at G. Prove that $GS = e \, . \, PS$ and $GS' = e\, PS'$, where S and S' are the foci of the hyperbola.

25. Write down the equation of the normal at $P(5\cos\theta, 3\sin\theta)$ to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1.$

This normal cuts the x-axis and the y-axis at G and H respectively. Show that the locus of the mid-point of GH is another ellipse with the same eccentricity as the first.
Sketch both ellipses on the same co-ordinate axes.

26. Show that the gradient of the line joining the points $P(ct_1, \frac{c}{t_1})$ and $Q(ct_2, \frac{c}{t_2})$ on the hyperbola $xy = c^2$ is $\frac{-1}{t_1 t_2}$. The points P, Q, R lie on this hyperbola. The line through P perpendicular to QR meets the line through Q perpendicular to PR at M. Prove that M lies on the hyperbola $xy = c^2$.

27. Show that the line $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$. Hence obtain the quadratic equation satisfied by m, where m is the gradient of the tangent from the external point $P(x_1\ y_1)$.

Find the locus of P if the two tangents from P are at right angles.

28. Find the equation of the normal at $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$ PN $\perp$ x-axis, and this normal meets the x-axis at G.

Show that $NG : ON = b^2 : a^2$, where $O(o, o)$.

29. Show that the equations of the tangent and the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(a\sec\theta, b\tan\theta)$ are respectively:

(a) $bx \sec\theta - ay \tan\theta = ab$, and

(b) $by \sec\theta + ax \tan\theta = (a^2 + b^2) \sec\theta \tan\theta$.

The tangent and the normal cut the y-axis at M and N respectively. Show that the circle on MN as diameter passes through the foci of the hyperbola.

30. (a) Show that $ab = 2c^2$ if the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches the hyperbola $xy = c^2$.

(b) $P(x_1, y_1)$ moves on the line $y = mx$ and $Q(x_2, y_2)$ on the line $y = -mx$. Find the co-ordinates of R, the mid-point of PQ, in terms of x_1 , x_2 and m. Show that the locus of R is a certain ellipse, if $PQ = 2K$, where K is a constant.

31. Show that for all values of θ, the point $P(4\cos\theta, 3\sin\theta)$ lies on the ellipse and find the equation of this ellipse.

(a) Find the equations of the tangents at the points P and $Q(-4\sin\theta, 3\cos\theta)$

(b) Find the point of the intersection, T, of these tangents and show that as θ varies, the locus of T is the ellipse $9x^2 + 16y^2 = 288$.

32. The ordinate at $P(a\sec\theta, b\tan\theta)$ meets the asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at Q. The normal at P meets the x-axis at G. Prove that GQ is perpendicular to the asymptote.

(Hint: $ax\tan\theta + by\sec\theta = (a^2 + b^2)\sec\theta\tan\theta$ is the equation of the normal.)

33. Prove that the equation of the tangent to the hyperbola $x^2 - y^2 = c^2$ at $P(x_1, y_1)$ is $xx_1 - yy_1 = c^2$. This tangent meets the lines $y = x$ and $y = -x$ at Q and R respectively. Prove that area of ΔOQR is constant.

34. Show that $P(a\cos\theta, a\sin\theta)$ lies on the circle $x^2 + y^2 = a^2$. If $P(\theta)$ and $Q(\phi)$ are two points on the circle $x^2 + y^2 = a^2$, prove that the locus of the mid-point of PQ is the line $y = \sqrt{3}x$ given that $\theta + \phi = \frac{2\pi}{3}$ for all positions of P and Q.

35. Simplify $\cos(\theta + \frac{\pi}{2})$ and $\sin(\theta + \frac{\pi}{2})$. Show that if $P(r_1\cos\theta, r_1\sin\theta)$ and $Q[r_2\cos(\theta + \frac{\pi}{2}), r_2\sin(\theta + \frac{\pi}{2})]$ lie on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{1}{a^2} - \frac{1}{b^2}$, where O is the centre. Deduce that if $OP \perp OQ$, then $r_1^{-2} + r_2^{-2}$ is independent of the positions of P and Q.

36. The normal at $P(a\sec\theta, b\tan\theta)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the x-axis at G, and $PN \perp$ the x-axis. Prove that $OG : ON = e^2$, where O is $(0, 0)$.

37. $P(x_1, y_1)$ is any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the equation of the tangent at P. A line drawn from the centre $O(0, 0)$ parallel to the tangent at P, meets the ellipse at Q.
Prove that area of ΔOPQ is independent of the position of P. Find the area of ΔOPQ. (Hint: Use the parametric form $x_1 = a\cos\theta$, $y_1 = b\sin\theta$).

38. Find the area of largest rectangle that can be inscribed in the ellipse $9x^2 + 25y^2 = 225$.

39. A conic is a rectangular hyperbola with eccentricity $\sqrt{2}$, focus $(2, 0)$ and directrix $x = 1$. Find the equation of this hyperbola. Sketch the hyperbola with its asymptotes.
   (a) Find the equation of the normal to this hyperbola at a point $P(x_1, y_1) = (a\sec\phi, a\tan\phi)$.
   (b) This normal meets the x-axis at $Q(X, 0)$ and the y-axis at $R(o, Y)$. Show that the locus of a point $M(X, Y)$ is given by $x^2 - y^2 = 8$, as P varies.

40. The tangent at $P(a\cos\theta, b\sin\theta)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts the axes
(0, b cosec θ)
at M and N. Show that M and N are $(a\cancel{co}\sec\theta, 0)$ and $(0, b\,{}^{co}\sec\theta)$ respectively. Find the minimum value of the area of ΔOMN and the corresponding co-ordinates of P.