UNIT 2

Q1. 1. $\frac{x^2}{4} - y^2 = 1$ $\qquad \left.\begin{matrix} a = 2 \\ b = 1 \end{matrix}\right\}$ $b^2 = a^2(e^2 - 1)$

$\frac{b^2}{a^2} = e^2 - 1 \quad \therefore e^2 = 1 + \frac{b^2}{a^2}$

$e = \sqrt{\frac{a^2 + b^2}{a}}$

(i) eccentricity $e = \sqrt{\frac{4+1}{2}} = \sqrt{\frac{5}{2}}$

(ii) foci ae $= \frac{2}{1} \times \frac{\sqrt{5}}{2}$ $\quad S(\sqrt{5}, 0)\ S^1(-\sqrt{5}, 0)$

(iii) directrixes $x = \pm \frac{a}{e} \quad x = \pm \frac{4}{\sqrt{5}}$

(iv) asymptotes $y = \pm \frac{b}{a} \quad y = \pm \frac{1}{2}x$

Q1. 2. $x^2 - y^2 = 4 \quad a = b = 2 \quad e = \frac{\sqrt{a^2 + a^2}}{a} = \frac{a\sqrt{2}}{a} = \sqrt{2}$

(i) eccentricity $e = \sqrt{2}$

(ii) foci, ae $= 2\sqrt{2} \quad (\pm 2\sqrt{2}, 0)$

(iii) directrixes $x = \pm \frac{a}{e} \quad \therefore x = \mp \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \mp\sqrt{2}$

(iv) asymptotes $y = \pm x$

Q1. 3. $\frac{x^2}{12} - \frac{y^2}{4} = 1 \quad \begin{matrix} a = 2\sqrt{3} \\ b = 2 \end{matrix} \quad e = \frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{12 + 4}}{\sqrt{12}} = \frac{4}{2\sqrt{3}} = \frac{2}{\sqrt{3}}$

(i) eccentricity $e = \frac{2}{\sqrt{3}}$

(ii) foci, ae $= \frac{2\sqrt{3}}{3} \quad (\mp 4, 0)$

(iii) directrices $x = \pm\ 3$

(iv) asymptotes $y = \pm \frac{1}{\sqrt{3}} x$

Q1. 4. $\frac{x^2}{144} - \frac{y^2}{25} = 1 \qquad \begin{matrix} a = 12 \\ b = 5 \end{matrix}$

(i) $e = \frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{169}}{12} = \frac{13}{12}$

(ii) $ae = \frac{12}{1} \cdot \frac{13}{12} \quad (\mp 13, 0)$

(iii) $x = \pm \frac{a}{e} \quad x = \pm \frac{144}{13}$

(iv) $y = \pm \frac{5}{12} x$

UNIT 2

Q1. 5. $x^2 - 4y^2 = 36 \iff \frac{x^2}{36} - \frac{y^2}{9} = 1$ $\quad \begin{cases} a = 6 \\ b = 3 \end{cases}$

(i) $e = \frac{\sqrt{a^2 + b^2}}{a} = \frac{3\sqrt{5}}{6} = \frac{\sqrt{5}}{2}$

(ii) $ae = \cancel{6}^{3} \cdot \frac{\sqrt{5}}{\cancel{2}_1} = 3\sqrt{5}$ $\quad (\mp 3\sqrt{5}, 0)$

(iii) $x = \pm \frac{a}{e}$ $\quad x = \pm \frac{12}{\sqrt{5}}$

(iv) $y = \pm \frac{1}{2}x$

Q1. 6. $4x^2 - 4y^2 = 9$ $\quad a = \frac{3}{2} = b$

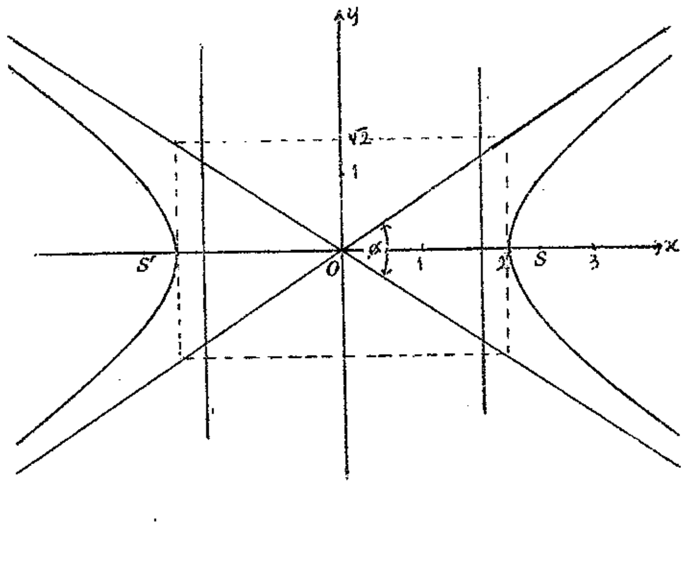
(i) $e = \sqrt{2}$

(ii) $ae = \frac{3\sqrt{2}}{2}$ $\quad$ foci $(\mp \frac{3\sqrt{2}}{2}, 0)$

(iii) $x = \pm \frac{3}{2\sqrt{2}}$

(iv) $y = \pm x$

Q1. 7. $\frac{x^2}{4} - \frac{y^2}{2} = 1$ $\quad \begin{cases} a = 2 \\ b = \sqrt{2} \end{cases}$



ϕ = angle between asymptotes $= 70^\circ 32'$

$e = \frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{6}}{2}$; $\quad ae = \frac{\sqrt{6}}{2} \cdot \frac{2}{1} = \sqrt{6}$; $\quad \frac{a}{e} = \frac{4}{\sqrt{6}} = 1.6$

UNIT 2

Q1. 8. Asymptotes of $\frac{(x-3)^2}{8} - \frac{(y+1)^2}{8} = 1 \quad a = b = 2\sqrt{2}$

The hyperbola is rectangular. $\therefore m = 1$

Has center at $(3,-1)$

$\therefore$ Equ. is $y + 1 = \pm 1(x-3)$

$y = -x + 2$ and

$y = x - 4$ are the asymptotes.

Q1. 9. $x^2 - y^2 + 2x + 4y = 0$

$x^2 + 2x + 1 - (y^2 - 4y + 4) = 1 - 4$

$(x+1)^2 - (y-2)^2 = -3$

$\frac{(y-2)^2}{3} - \frac{(x+1)^2}{3} = 1$

Hence the hyperbola is rectangular $(a = b)$

$\therefore e = \sqrt{\frac{a^2 + b^2}{a}} = \sqrt{2}$ and the asymptotes have equs. $y = \pm x$

i.e. are $\perp$.

Q1. 10. $S(5,0)$, $e = \frac{5}{4}$

$\therefore$ equ. $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Working

$ae = 5$

$a = \frac{5}{e} = \frac{5 \times 4}{5} = 4$

$b^2 = a^2(e^2 - 1)$

$= 16\left(\frac{25-16}{16}\right)$

$b^2 = 9$

$a^2 = 16$

Q2. 1. $x^2 - 2y^2 = 2 \iff \frac{x^2}{2} - y^2 = 1 \quad \begin{cases} a = \sqrt{2} \\ b = 1 \end{cases}$

Tangent at $(2,1)$

$\frac{2x}{2} - y = 1 \Rightarrow x - y = 1$

Focus is $S(ae,0)$ $\because (2,1)$ is on the positive branch

$e = \sqrt{\frac{a^2 + b^2}{a}} = \sqrt{\frac{2+1}{\sqrt{2}}} = \frac{\sqrt{3}}{\sqrt{2}}$

$\therefore ae = \sqrt{3}$

$\therefore S(\sqrt{3},0)$ and $S^1(-\sqrt{3},0)$

distance of S from $x - y - 1 = 0 \qquad d = \left|\frac{\sqrt{3} - 1}{\sqrt{2}}\right|$

distance of S^1 from $x - y - 1 = 0 \qquad d^1 = \left|\frac{-\sqrt{3} - 1}{\sqrt{2}}\right|$

$dd^1 = \frac{(\sqrt{3}+1)(\sqrt{3}-1)}{2}$

$dd^1 = 1$ as required.

ween
s

UNIT 2

Q2. 2. $4x^2 - 9y^2 = 36 \cap 2y = x + 1$

$4(2y-1)^2 - 9y^2 = 36$

$16y^2 - 16y + 4 - 9y^2 - 36 = 0$

$7y^2 - 16y - 32 = 0$

$$y = \frac{8 + 12\sqrt{2}}{7} \text{ or } \frac{8 - 12\sqrt{2}}{7}$$

$$x = 2y - 1$$

$$= \frac{16 + 24\sqrt{2}}{7} - 1$$

$$= \frac{9 + 24\sqrt{2}}{7} \text{ or } \frac{9 - 24\sqrt{2}}{7}$$

$$\text{Midpt. } M = \left[\left(\frac{9+24\sqrt{2}}{7} + \frac{9 - 24\sqrt{2}}{7}\right) \div 2, \left(\frac{8+12\sqrt{2}}{7}\right) + \frac{8 - 12\sqrt{2}}{7}\right) \div 2\right]$$

$$= \left(\frac{18}{14}, \frac{16}{14}\right)$$

$$= \left(\frac{9}{7}, \frac{8}{7}\right)$$

A more "elegant" method.

Let M be (x,y) then $x_1 + x_2 = 2x$

$y_1 + y_2 = 2y$, then y_1 and y_2

are the roots of $7y^2 - 16y - 32 = 0$

$$y_1 + y_2 = -\frac{b}{a} = \frac{16}{7}$$

$$\frac{y_1 + y_2}{2} = y = \frac{8}{7} \therefore x = 2 \times \frac{8}{7} - 1$$

$$= \frac{9}{7}$$

So midpt. is $M\left(\frac{9}{7}, \frac{8}{7}\right)$

Q2. 3. $x^2 - 2y^2 = 1 \iff x^2 - \dfrac{-y^2}{(1/\sqrt{2})^2} = 1$ $\qquad a = 1$, $b = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

Equ. of tangent: $y = mx \pm \sqrt{a^2m^2 - b^2}$

$\parallel$ to $y = \frac{3}{4}x$.

$$y = \frac{3}{4}x \pm \sqrt{\frac{3^2}{4^2} - \frac{1}{2}} \qquad \sqrt{\frac{9}{16} - \frac{8}{16}}$$

$$y = \frac{3}{4}x \pm \frac{1}{4}$$

$$4y = 3x \pm 1.$$

UNIT 2

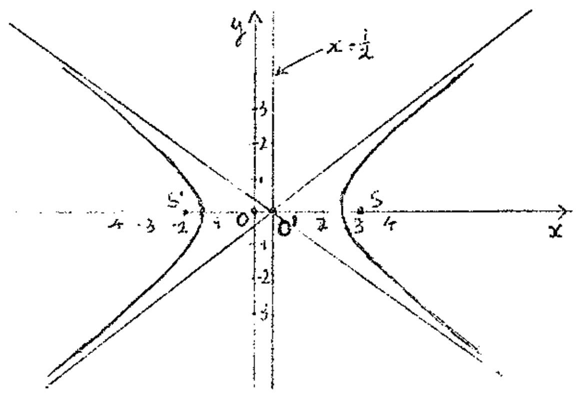
Q2. 4. $S(3,0)$ $S^1(-2,0)$ $\therefore$ equ. is $\dfrac{(x-\frac{1}{2})^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$a = 2.$ Centre $(\frac{1}{2},0)$

Translate centre to $(\frac{1}{2},0)$ $\therefore$ equ. is $\dfrac{x^2}{4} + \dfrac{y^2}{b^2} = 1$

Then $S(2\frac{1}{2},0)$, $S^1(-2\frac{1}{2},0)$

$$ae = \frac{5}{2} \quad \therefore e = \frac{5}{4} \quad \text{then } b = a\sqrt{e^2 - 1}$$
$$= 2\sqrt{\frac{25-16}{16}}$$
$$= \frac{3}{2}$$



Q2. 5. $\dfrac{x^2}{4} - \dfrac{y^2}{2} = 1$ Equation of tangent; at any pt. (x_1,y_1)

$$\frac{xx_1}{4} - \frac{yy_1}{2} = 1$$

Equation of asymptotes; $a = 2$, $b = \sqrt{2}$

$$y = \pm\frac{1}{\sqrt{2}}x \iff y = \pm\frac{\sqrt{2}x}{2}$$

$$\frac{xx_1}{4} - \frac{yy_1}{2} = 1 \cap y = \frac{\sqrt{2}x}{2} \qquad \Bigg| \qquad \frac{xx_1}{4} - \frac{yy_1}{2} = 1 \cap y = -\frac{\sqrt{2}x}{2}$$

$$\frac{xx_1}{4} - \frac{\sqrt{2}xy_1}{4} = 1 \qquad \Bigg| \qquad \frac{xx_1}{4} + \frac{\sqrt{2}xy_1}{4} = 1$$

$$x(x_1 - \sqrt{2}y_1) = 4 \qquad \Bigg| \qquad x = \frac{4}{x_1 + \sqrt{2}y_1}$$

$$x = \frac{4}{x_1 - \sqrt{2}y_1} \qquad \Bigg| \qquad y = -\frac{2\sqrt{2}}{x_1 + \sqrt{2}y_1}$$

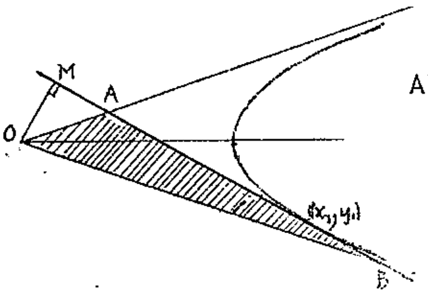
$$y = \frac{2\sqrt{2}}{x_1 - \sqrt{2}y_1}$$

$OM = \left|\dfrac{-4}{\sqrt{x_1^2 + 4y_1^2}}\right|$ dist. of $(0,0)$ from $xx_1 - 2yy_1 - 4 = 0$

$$AB^2 = \frac{(4(\cancel{x_1} + \sqrt{2}y_1 - \cancel{x_1} + \sqrt{2}y_1))^2}{x_1^2 - 2y_1^2} + \frac{(2\sqrt{2}(x_1 + \cancel{\sqrt{2}y_1} + x_1 - \cancel{\sqrt{2}y_1}))^2}{x_1^2 - 2y_1^2}$$

(cont'd)

UNIT 2



$$AB = \frac{\sqrt{128y_1^2 + 32x_1^2}}{x_1^2 - 2y_1^2}$$

$$= \frac{4\sqrt{2}}{x_1^2 - 2y_1^2}\sqrt{4y_1^2 + x_1^2}$$

Note that $x^2 - 2y^2 = 4$

$$AB = \sqrt{2}\ \sqrt{4y_1^2 + x_1^2}$$

$$\text{Area of } \triangle OAB = \frac{1}{2} \cdot OM \cdot AB$$

$$= \frac{1}{2} \cdot \frac{4}{\sqrt{4y_1^2 + x_1^2}} \cdot \frac{\sqrt{2}\sqrt{4_1^2 + x_1^2}}{1}$$

$$= 2\sqrt{2} \text{ sq units which is constant.}$$

(i.e., independent of x and y.)

Q2. 6. $x^2 - 9y^2 = 9 \iff \frac{x^2}{9} - y^2 = 1$ $\quad \begin{cases} a = 3 \\ b = 1 \end{cases}$

Gradient of asymptotes $m_1 = \frac{1}{3}$ $\quad m_2 = -\frac{1}{3}$

Equation of tangents $\perp$ to these asymptotes have $m_1 = -3$, $m_2 = 3$.

General equ. of tang. $y = mx \pm \sqrt{a^2m^2 - b^2}$

$$y = -3x \pm \sqrt{9.9-1} \iff y = -3x \pm 4\sqrt{5} \quad \text{——(1)}$$

$$\text{or} \quad y = 3x \pm \sqrt{9.9-1} \iff y = 3x \pm 4\sqrt{5} \quad \text{——(2)}$$

(i) $y = -3x + 4\sqrt{5} \cap x^2 - 9y^2 = 9 \iff x^2 - 9(9x^2 - 24\sqrt{5}x + 80) - 9 = 0$

$80x^2 - 216\sqrt{5}x + 729 = 0$

$(4\sqrt{5}x - 27)^2 = 0 \quad \therefore\ x = \frac{27}{4\sqrt{5}}$

$y = -\frac{1}{4\sqrt{5}}$ $\quad \therefore A\left(\frac{27}{4\sqrt{5}}, \frac{-1}{4\sqrt{5}}\right)$

(ii) $y = -3x - 4\sqrt{5} \cap x^2 - 9y^2 = 9 \implies x^2 - 9(9x^2 + 24\sqrt{5}x + 80) - 9 = 0 \implies 80x^2 + 216\sqrt{5}x + 729 = 0$

$\implies (4\sqrt{5}x + 27)^2 = 0 \quad \therefore\ x = -\frac{27}{4\sqrt{5}},\ y = \frac{1}{4\sqrt{5}} \therefore B\left(\frac{-27}{4\sqrt{5}}, \frac{1}{4\sqrt{5}}\right)$

(iii) $y = 3x + 4\sqrt{5} \cap x^2 - 9y^2 = 9 \implies x^2 - 9(9x^2 + 24\sqrt{5}x + 80) - 9 = 0 \implies 80x^2 + 216\sqrt{5}x + 729 = 0$

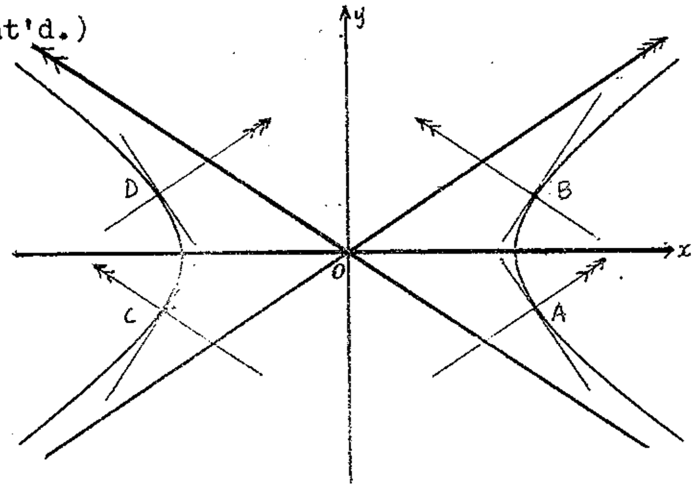
$\implies (4\sqrt{5}x + 27)^2 = 0 \quad \therefore\ x = \frac{-27}{4\sqrt{5}},\ y = \frac{-1}{4\sqrt{5}} \quad C\left(\frac{-27}{4\sqrt{5}}, \frac{-1}{4\sqrt{5}}\right)$

(iv) $y = 3x - 4\sqrt{5} \cap x^2 - 9x^2 = 9 \implies D\left(\frac{-27}{4\sqrt{5}}, \frac{1}{4\sqrt{5}}\right)$

(diagram on following page)

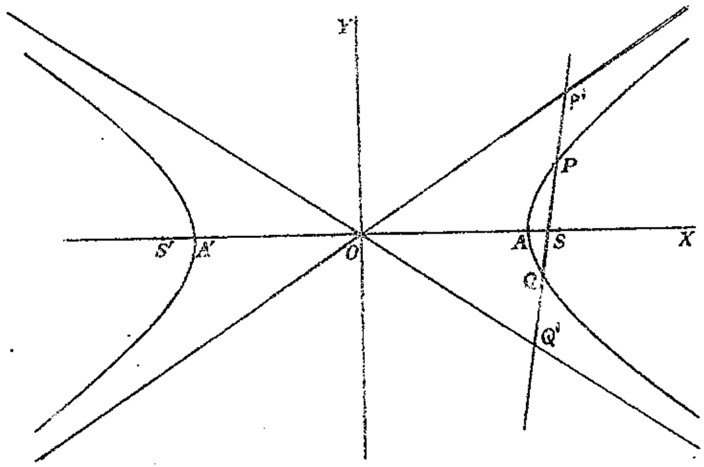
UNIT 2

Q2. 6. (cont'd.)



Q2. 7. $\left.\begin{array}{l} 4x^2 - 8y^2 = 64 \\ \frac{x^2}{16} - \frac{y^2}{8} = 1 \end{array}\right\}$ $a = 4,\ b = 2\sqrt{2}$

Equation of asymptotes $y = \pm \frac{\sqrt{2}}{2}x$



Let the equation of the straight line be $y = mx + c$ intersecting the curve $\frac{x^2}{16} - \frac{y^2}{8} = 1$ at P, Q and intersecting the asymptotes at P^1, Q^1.

If P and Q are the points (x_1, y_1), (x_2, y_2) then x_1 and x_2 are the roots of the equation;

$$4x^2 - 8(mx + c)^2 = 64$$

$$x^2 - 2(m^2x^2 + 2mxc + c^2) - 16 = 0$$

$$x^2(1-2m^2) - 4mcx - 2c^2 - 16 = 0$$

$\therefore$ Sum of roots,

$$x_1 + x_2 = \frac{4mc}{1 - 2m^2}$$

(continued on next page)

UNIT 2

Let R be the midpoint of PQ, where R is $(x_3,\ y_3)$

Then $x_3 = \dfrac{x_1 + x_2}{2} = \dfrac{2mc}{1 - 2m^2}$

(Solving for y it can be shown that $y^2(1-2m^2)-2cy + c^2 - 8m^2=0$

$\therefore \dfrac{y_1 + y_2}{2} = \dfrac{c}{1 - 2m^2}$)

If P^1 and Q^1 are the points (x_1', y_1') and (x_2', y_2') then x_1' is the root of

$$mx + c = \frac{\sqrt{2}}{2}x$$

$$\text{so} \quad x_1' = \frac{-2c}{2m - \sqrt{2}} \quad \text{and } y_1' = \frac{\sqrt{2}c}{\sqrt{2} - 2m}$$

x_2' is the root of $mx + c = -\dfrac{\sqrt{2}}{2}x$

$$\text{so} \quad x_2' = \frac{-2c}{2m + \sqrt{2}} \quad \text{and } y_2' = \frac{\sqrt{2}c}{\sqrt{2} + 2m}$$

Let S be the midpoint of P^1Q^1, where S is (x_4, y_4)

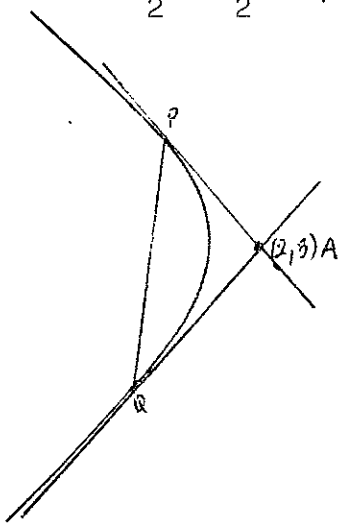
then $x_4 = \dfrac{x_1' + x_2'}{2} = \dfrac{\dfrac{-2c}{2m - \sqrt{2}} + \dfrac{-2c}{2m + \sqrt{2}}}{2} = \dfrac{-2mc}{2m^2-1} = \dfrac{2mc}{1-2m^2}$

$y_4 = \dfrac{y_1' + y_2'}{2} = \dfrac{c}{1 - 2m^2}$

Now we find that R and S are the same point, i.e. PQ and P^1Q^1 have the same midpoint.

$\therefore PP^1 = QQ^1$ as required.

Q2. 8. Find P and Q the points of contact of the tangents from A to $\dfrac{x^2}{2} - \dfrac{y^2}{2} = 1$ $\quad (a = b = \sqrt{2})$



Equ. of PQ $\Rightarrow \dfrac{2x}{2} - \dfrac{3y}{2} = 1$

$2x - 3y = 2 \Rightarrow x = 1 + \dfrac{3y}{2}$

Chord of contact PQ $\cap$ $\dfrac{x^2}{2} - \dfrac{y^2}{2} = 1$;

$(1 + \dfrac{3y}{2})^2 - y^2 - 2 = 0$

$1 + \dfrac{9y^2}{4} + 3y - y^2 - 2 = 0$

$9y^2 + 12y - 4y^2 - 4 = 0$

$5y^2 + 12y - 4 = 0$

$y = \dfrac{12 + \overset{4\sqrt{14}}{\sqrt{224}}}{10}$ or $\dfrac{-12 - \overset{4\sqrt{14}}{\sqrt{224}}}{10} \Rightarrow y = \dfrac{-6 + 2\sqrt{14}}{5}$ or $y = \dfrac{-6 - 2\sqrt{14}}{5}$

$x = \dfrac{-8 + 6\sqrt{14}}{10}$ or $x = \dfrac{-8 - 6\sqrt{14}}{10}$

(Cont'd on next page)

UNIT 2

$P\left(\frac{-8+6\sqrt{14}}{10}, \frac{-6+2\sqrt{14}}{5}\right) \doteqdot (1.44,\ 0.296)$

$Q\left(\frac{-8-6\sqrt{14}}{10}, \frac{-6-2\sqrt{14}}{5}\right) \doteqdot (-3.04,\ -2.7)$

$A(2,3)$

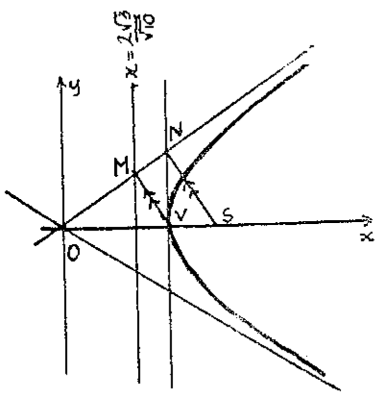
$m_1 = \frac{3-0296}{2-1.44} = 4.$

$m_2 = \frac{3+2.7}{2+3.04} = 1.131$

$$\tan\phi = \frac{4.829 - 1.131}{1 + 4879 \times 1.131} = \frac{3.696}{6.462} = 0.5723$$

$$\phi \doteqdot 29^\circ 47'$$

Q2. 9. $2x^2 - 8y^2 = 3 \Longrightarrow \frac{x^2}{3/2} - \frac{y^2}{3/8} = 1 \qquad a = \frac{\sqrt{3}}{\sqrt{2}},\ b = \frac{\sqrt{3}}{2\sqrt{2}}$



Equation of asymptote OM; $y = \frac{1}{2}x$

$$e = \frac{\sqrt{a^2+b^2}}{a} = \sqrt{2}\,\frac{\sqrt{3/2+3/8}}{\sqrt{3}} = \frac{\sqrt{3+3/4}}{\sqrt{3}} = \frac{\sqrt{5}}{2}$$

$S \Leftrightarrow (ae,0) \Longrightarrow \left(\frac{\sqrt{15}}{2\sqrt{2}}, 0\right) \qquad ae = \frac{\sqrt{5}}{2}\cdot\frac{\sqrt{3}}{\sqrt{2}}$

$\frac{a}{e} = \frac{\sqrt{3}}{\sqrt{2}}\cdot\frac{2}{\sqrt{5}} = \frac{2\sqrt{3}}{\sqrt{10}}$, directric; $x = \frac{2\sqrt{3}}{\sqrt{10}}$

$x = \frac{2\sqrt{3}}{\sqrt{10}} \qquad y = \frac{1}{2}x \quad \therefore\ y = \frac{\sqrt{3}}{\sqrt{10}}$

i.e. $M\left(\frac{2\sqrt{3}}{\sqrt{10}}, \frac{\sqrt{3}}{\sqrt{10}}\right)$

$V\left(\frac{\sqrt{3}}{\sqrt{2}}, 0\right)$

$N\left(\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}\right)$

$$\text{Gradient of MV} = \frac{\frac{\sqrt{3}}{\sqrt{10}} - 0}{\frac{2\sqrt{3}}{\sqrt{10}} - \frac{\sqrt{3}}{\sqrt{2}}} = \frac{\sqrt{3}}{2\sqrt{3}-\sqrt{15}}$$

$$\text{Gradient of NS} = \frac{\frac{\sqrt{3}}{2\sqrt{2}} - 0}{\frac{\sqrt{3}}{\sqrt{2}} - \frac{\sqrt{15}}{2\sqrt{2}}} = \frac{\sqrt{3}}{2\sqrt{3}-\sqrt{15}}$$

$\therefore$ MV $\parallel$ NS

UNIT 2

Q2. 10. $\frac{x^2}{16} - \frac{y^2}{9} = 1$ $\left.\begin{array}{l} a = 4 \\ b = 3 \end{array}\right\}$ Equation of asymptotes $y = \pm \frac{3}{4}x$

Equation of tangent at (x_1, y_1);

$$\frac{xx_1}{16} - \frac{yy_1}{9} = 1$$

$$9xx_1 - 16yy_1 = 144 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

A $\Rightarrow$ $y = \frac{3}{4}x \longrightarrow (1)$ $\quad 9xx_1 - \cancel{16}^{4}x\,\frac{3}{\cancel{4}_1}xy_1 = 144$

$$3xx_1 - 4xy_1 = 48$$

$$x(3x_1 - 4y_1) = 48 \quad \therefore\ x = \frac{48}{3x_1 - 4y_1}$$

$$y = \frac{3}{4}\cdot\frac{48}{3x_1 - 4y_1} \qquad\qquad y = \frac{36}{3x_1 - 4y_1}$$

i.e. $A\left(\frac{48}{3x_1 - 4y_1}, \frac{36}{3x_1 - 4y_1}\right)$

B $\Rightarrow$ $y = -\frac{3}{4}x \longrightarrow (1)$ $\quad 3xx_1 + 4xy_1 = 48 \ \therefore\ x = \frac{48}{3x_1 + 4y_1}$

$$y = -\frac{3}{4}\cdot\frac{48}{3x_1 + 4y_1} \ \therefore\ y = \frac{-36}{3x_1 + 4y_1}$$

i.e. $B\left(\frac{48}{3x_1 + 4y_1}, \frac{-36}{3x_1 + 4y_1}\right)$

Midpt. of AB;

$$\left(\frac{48}{3x_1 - 4y_1} + \frac{48}{3x_1 + 4y_1}\right) \div 2 = \frac{48(3x_1 + \cancel{4y_1} + 3x_1 - \cancel{4y_1})}{2(9x_1^2 - 16y_1^2)} = \frac{144x_1}{9x_1^2 - 16y_1^2}$$

$$= x_1 \qquad (\text{Since } 9x_1^2 - 16y_1^2 = 144)$$

$$\left(\frac{36}{3x_1 - 4y_1} - \frac{36}{3x_1 + 4y_1}\right) \div 2 = \frac{36(\cancel{3x_1} + 4y_1 - \cancel{3x_1} + 4y_1}{2(9x_1^2 - 16y_1^2)} = \frac{144y_1}{9x_1^2 - 16y_1^2}$$

$$= y_1$$

$\therefore$ $P(x_1, y_1)$ is the midpt. of AB.

Q2. 11. $b^2x^2 - a^2y^2 = a^2b^2$ if $x = a\sec\phi$

Then $b^2a^2\sec^2\phi - a^2y^2 = a^2b^2$

$a^2y^2 = a^2b^2(\sec^2\phi - 1)$

$y^2 = b^2\tan^2\phi$ (for all ϕ)

$\therefore = y = b\tan\phi$

$x = a\sec\phi$, $y = b\tan\phi$ is the parametric equation of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$.

UNIT 2

Q2. 11. (cont'd)

$$\frac{dy}{dx} = \frac{dy}{d\phi}\Big/\frac{dx}{d\phi}$$

$$= \frac{b\ \sec^2\phi}{a\ \tan\phi\ \sec\phi} \qquad\qquad \frac{d}{d\phi}(\cos\phi)^{-1} = -(\cos\phi)^{-2}(-\sin\phi)$$

$$= \frac{\sin\phi}{\cos^2\phi}$$

$$= \frac{b}{a}\,\frac{\sec\phi}{\tan\phi} = \frac{b}{a}\,\frac{\frac{1}{\cos\phi}}{\frac{\sin\phi}{\cos\phi}} = \frac{b}{a\ \sin\phi} \qquad\qquad = \tan\phi\ \sec\phi$$

tangent; $y - b\tan\phi = \frac{b}{a\sin\theta}(x - a\sec\phi)$

$$ay\sin\phi - ab\frac{\sin^2\phi}{\cos\phi} = bx - ab\frac{1}{\cos\phi}$$

$$bx - ay\sin\phi = \frac{ab}{\cos\phi} - \frac{ab\sin^2\phi}{\cos\phi}$$

$$= \frac{ab}{\cos\phi}(1 - \sin^2\phi)$$

$bx - ay\sin\phi = ab\cos\phi$ is the rqd. equ. of the tangent at $x = a\sec\phi$, $y = b\tan\phi$

Q2. 12. $x = \frac{at}{2} + \frac{a}{2t} \longrightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$y^2 = b^2\left(\frac{x^2}{a^2} - 1\right)$$

$$= b^2\left[\frac{1}{\cancel{a^2}}\cdot\frac{\cancel{a^2}}{4}\left(t + \frac{1}{t}\right)^2 - \frac{a^2}{a^2}\right]$$

$$= \frac{b^2}{4a^2}\left[a^2\left(t^2 + 2 + \frac{1}{t^2}\right) - 4a^2\right]$$

$$= \frac{b^2}{4}\left[\left(t + \frac{1}{t}\right)^2 - 4\right]$$

$$= \frac{b^2}{4}\left(t^2 + 2 + \frac{1}{t^2} - 4\right)$$

$$= \frac{b^2}{4}\left(t^2 - 2 + \frac{1}{t^2}\right)$$

$$= \frac{b^2}{4}\left(t - \frac{1}{t}\right)^2$$

$$\therefore\ y = \frac{b}{2}\left(t - \frac{1}{t}\right)$$

$y = \frac{b}{2}\overline{t - \frac{1}{t}}$ i.e. the pt. $\left(\frac{a}{2}\left[t + \frac{1}{t}\right], \frac{b}{2}\left[t - \frac{1}{t}\right]\right)$ lies on the hyperbola for varying values of t.

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UNIT 2

Q2. 12 (cont'd)

The equ. of the tangent;

$$\frac{dy}{dx} \Rightarrow \frac{\cancel{2}^{1}x}{a^2} - \frac{\cancel{2}^{1}y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{a^2}\,\frac{b^2}{y}$$

$$= \frac{b^{\cancel{2}1}}{a^{\cancel{2}1}} \cdot \frac{\cancel{a}^{1}}{\cancel{2}_{1}}\left(t + \frac{1}{t}\right)\frac{\cancel{2}^{1}}{\cancel{b}_{1}}\;\frac{1}{t - \frac{1}{t}}$$

$$= \frac{b}{a}\,\frac{t^2+1}{\cancel{t}_1} \cdot \frac{\cancel{t}^1}{t^2-1}$$

$$= \frac{b}{a}\left(\frac{t^2+1}{t^2-1}\right)$$

$$y - \frac{b}{2}\left(\frac{t^2-1}{t}\right) = \frac{b}{a}\,\frac{t^2+1}{t^2-1}\left(x - \frac{a}{2}\,\frac{t^2+1}{t}\right)$$

$$ay(t^2-1) - \frac{ab}{2}\frac{(t^2-1)^2}{t} = bx(t^2+1) - \frac{ab}{2}\frac{(t^2+1)^2}{t}$$

$$bx(t^2+1) - ay(t^2-1) = \frac{ab}{2}\frac{(t^2+1)^2}{t} - \frac{ab}{2}\frac{(t^2-1)^2}{t}$$

$$= \frac{ab}{2t}(\cancel{t^4} + 2t^2 + \cancel{1} - \cancel{t^4} + 2t^2 - \cancel{1})$$

$$= \frac{ab4t^2}{2t}$$

$$bx(t^2+1) - ay(t^2-1) = 2abt$$

Q3. 1. $x^2 - 4y^2 = 9 \Rightarrow \frac{x^2}{9} - \frac{y^2}{9/4} = 1$

$a = 3, \quad b = \frac{3}{2}.$

Tang. $\parallel$ to $2x + 3y = 0$ i.e. has $m = -\frac{2}{3}$.

$$y = -\frac{2}{3}x \pm \sqrt{\frac{\cancel{4}^1}{1} \cdot \frac{4}{\cancel{9}_1} - \frac{9}{4}}$$

$$= -\frac{2}{3}x \pm \sqrt{\frac{7}{4}}$$

$$3y = -2x \pm \frac{3\sqrt{7}}{2} \Rightarrow 6y + 4x = \pm 3\sqrt{7}$$

Q3. 2. $16x^2 - y^2 = 12 \quad P(1, -2) \quad \frac{x^2}{3/4} - \frac{y^2}{12} = 1$

Tangent; $\frac{4x}{3} + \frac{\cancel{2}^1 y}{\cancel{12}_6} = 1$

$$8x + y = 6$$

(continued on next page)

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UNIT 2

Q3. 2. (continued)

Normal; $\frac{xa^2}{x_1} + \frac{yb^2}{y_1} = a^2 + b^2$

$$\frac{3x}{4} - \frac{\overset{6}{\cancel{12}}y}{\cancel{12}} = 12\frac{3}{\cancel{4}}$$

$$3x - 24y = 51$$

$$x - 8y = 17$$

Diameter $y = -2x$

Q3. 3. $9x^2 - y^2 = 32 \iff \frac{x^2}{32/9} - \frac{y^2}{32} = 1$

$$a = \frac{4\sqrt{2}}{3}, \quad b = 4\sqrt{2}$$

P(2,-2) Q(-3,7)

Normal at P. $\frac{\overset{16}{\cancel{32}}x}{9\times\underset{1}{\cancel{2}}} - \frac{\overset{16}{\cancel{32}}y}{\underset{1}{\cancel{2}}} = 32 + \frac{32}{9}$

$$\frac{\overset{1}{\cancel{16}}x}{9} - \overset{1}{\cancel{16}}y = \frac{\overset{20}{\cancel{320}}}{9}$$

$$x - 9y = 20 \qquad (1)$$

Normal at Q - $\frac{32x}{9\times3} + \frac{32y}{7} = \frac{320}{9}$

$$-7x + 27y = 210$$

$$7x - 27y = -210 \qquad (2)$$

(1)x 7 $\quad 7x - 63y = 140$

(2) $\quad -7x + 27y = +210$

$$-36y = 350$$

$$y = -\frac{175}{18} \qquad x = -\frac{135}{2}$$

Normals intersect at T. $T(-\frac{135}{2}, -\frac{175}{18})$

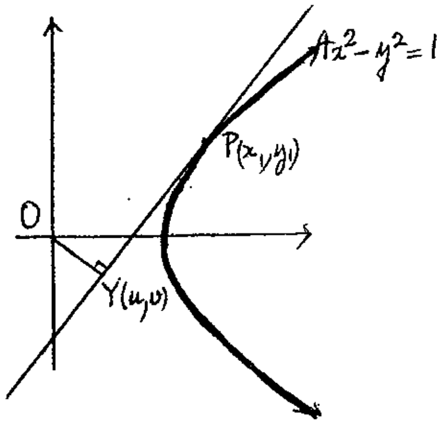
Equation of the chord of contact from T is

$$-\frac{135x}{2\times\frac{32}{9}} + \frac{175y}{18\times32} = 1$$

$$10935x - 175y + 576 = 0$$

UNIT 2

Q3. 4.



E quation of tangent at $P(x_1, y_1)$ is $4xx_1 - yy_1 = 1$.

Gradient of perpendicular through O is $-\dfrac{y_1}{4x_1}$

Equation of OY; $y = \dfrac{-y_1}{4x_1}x$

$OY \cap PY = \{Y\}$

$y = \dfrac{-y_1}{4x_1}x \cap 4xx_1 - yy_1 = 1$

$$x\left(4x_1 + \frac{y_1^2}{4x_1}\right) = 1 \Longleftrightarrow u = \frac{4x_1}{16x_1^2 + y_1^2} \quad \ldots\ldots(1) \text{ and } v = \frac{-y_1}{16x_1^2 + y_1^2} \quad \ldots\ldots(2)$$

i.e., $Y\left(\dfrac{4x_1}{16x_1^2 + y_1^2}, \dfrac{-y}{16x_1^2 + y_1^2}\right)$

Aim: 1. Find x_1 and y_1 in terms of u and v.

2. Express $4x^2 - y^2 = 1$ in terms of u,v in order to find equ. of locus.

$OY \perp PY \therefore \dfrac{v}{u} \times \dfrac{4x_1}{y_1} = -1$ i.e. $\dfrac{u}{v} = \dfrac{-4x_1}{y_1}$ $\ldots\ldots\ldots(3)$

From (1) $16x_1^2u + uy_1^2 = 4x_1$ From (2) $16x_1^2v + vy_1^2 = -y_1$

Subst $(3) \longrightarrow (1)$ $4x_1\left(u + \dfrac{v^2}{u}\right) = 1$. Subst $(3) \rightarrow (2)$ $y_1^2\left(\dfrac{u^2}{v} + v\right) = -y_1$

$$\therefore x_1 = \frac{u}{4(u^2 + v^2)} \quad \ldots\ldots(4) \qquad \therefore y_1 = \frac{-v}{u^2 + v^2} \quad \ldots\ldots(5)$$

Subst. (4) and (5) into $4x^2 - y^2 = 1$ since (x_1, y_1) lies on it.

$$\frac{u^2}{4(u^2 + v^2)^2} - \frac{v^2}{(u^2 + v^2)^2} = 1$$

$4(u^2 + v^2)^2 = u^2 - 4v^2$ which is the required equation of the locus of Y.

Q3. 5. $9x^2 - 4y^2 = 2 \Leftrightarrow \dfrac{x^2}{\frac{20}{9}} - \dfrac{y^2}{5} = 1$ $\quad \begin{cases} a = \sqrt{\dfrac{20}{3}} \\ b = \sqrt{5} \end{cases}$

Tangent at $P(-2, 2)$

$$-\frac{18x}{20} - \frac{8y}{20} = 1$$

$$9x + 4y = -10$$

Equation of asymptotes $y = \pm \dfrac{3\sqrt{5}}{\sqrt{20}}x \Longleftrightarrow y = \pm \dfrac{3}{2}x$

(continued on next page)

Q3. 5. (cont'd)

$L \Rightarrow \quad 9x + 4y = -10 \cap y = \frac{3}{2}x$

$$9x + 6x = -10$$

$$15x = -10$$

$$x = -\frac{2}{3} \qquad y = \frac{3}{2} \cdot \frac{-2}{3} = -1$$

$$\therefore L\left(-\frac{2}{3}, -1\right)$$

$M \Rightarrow 9x + 4y = -10 \cap y = -\frac{3}{2}x$

$$9x - 6x = -10$$

$$3x = -10$$

$$x = -\frac{10}{3} \qquad y = -\frac{10}{\cancel{3}_1} \cdot \frac{\cancel{3}^1}{2}$$

$$y = 5$$

$$M\left(-\frac{10}{3}, 5\right)$$

Midpoint of LM $\left(\dfrac{-\frac{2}{3} - \frac{10}{3}}{2}, \dfrac{-1 + 5}{2}\right) \Leftrightarrow (-2,2)P$

i.e., Midpt. of LM is P.

Q3. 6. $(1) \longrightarrow 4x^2 - y^2 = 35 \Longrightarrow \dfrac{x^2}{\frac{35}{4}} - \dfrac{y^2}{35} = 1 \quad \begin{cases} a = \sqrt{\frac{35}{2}} \\ b = \sqrt{35} \end{cases}$

Equation of the asymptotes $y = \pm 2x$

$P(3\frac{1}{2}, 7)$ lies on the asymptote $y = 2x$

Equation of tangent to hyperbola at (x_1, y_1)

$4xx_1 - yy_1 = 35$; through $(3\frac{1}{2}, 7)$ is $14x_1 - 7y_1 = 35$

i.e. $y_1 = 2x_1 - 5$ ....(2)

To find pt. of intersection of (1) and (2)

$$4x_1^2 - (4x_1^2 - 20x_1 + 25) - 35 = 0$$

$\therefore \left.\begin{matrix} x_1 = 3 \\ y_1 = 1 \end{matrix}\right\}$ i.e. the "line" through $(3\frac{1}{2}, 7)$ is touching the hyperbola at Q(3,1).

The equation of PQ is $y - 1 = \dfrac{7 - 1}{\frac{7}{2} - 3}(x - 3)$

$$12x - y - 35 = 0$$

UNIT 2

Q3. 7. $9x^2 - y^2 = 5$ $\qquad\qquad$ $\left.\begin{array}{l} a = \sqrt{5}/3 \\ b = \sqrt{5} \end{array}\right\}$

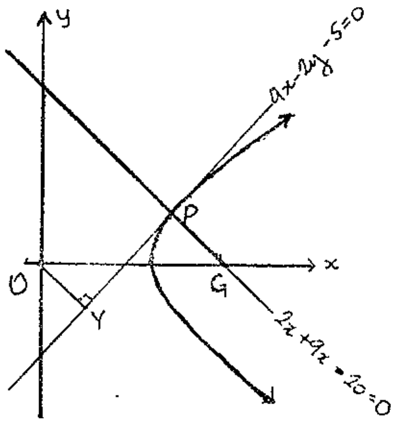
$$\frac{x^2}{5/9} - \frac{y^2}{5} = 1$$

Normal $\dfrac{xa^2}{x_1} + \dfrac{yb^2}{y_1} = a^2 + bb^2$ at P(1,2)

$$\frac{\cancel{5}^{1}x}{9} + \frac{\cancel{5}^{1}y}{2} = \frac{\cancel{50}^{10}}{9}$$

$2x + 9y = 20$ ....(1)

(1) meets x axis at G(10,0)



Equation of tangent at P

$9x - 2y = 5$ ....(2)

Equation of OY $y = -\frac{2}{9}x$ ...(3)

(3)→(2) $\qquad 9x + \frac{4}{9}x = 5$

$x = \frac{9}{17}$ $y = -\frac{2}{17}$ $\quad\therefore$ Y$\left(\frac{9}{17}, -\frac{2}{17}\right)$

$$OY \cdot PG = \left|\frac{5}{\sqrt{9^2+2^2}}\right| \cdot \left|\frac{10\times 9 - 5}{\sqrt{9^2+2^2}}\right|$$

$$= \left|\frac{-5 \times 85}{85}\right|$$

$$= 5$$

Q3. 8. See Answers

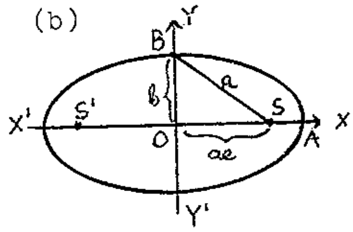
Q3. 9 See Answers

Q3. 10 See next page please.

UNIT 2

Q.3.10. (a) $p > -4$ for ellipse and $-9 < p < -4$ for hyperbola.

(b)



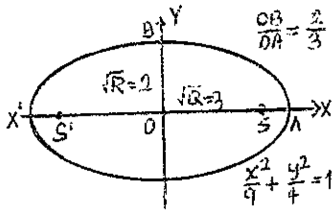
$b^2 = a^2(1 - e^2)$ for the ellipse,

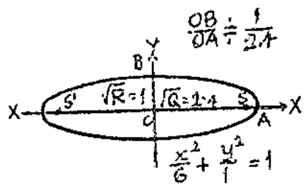
so $a^2 = b^2 + a^2e^2$ ....(1)

(c) From (1) we have $ae = \sqrt{(a^2 - b^2)}$ which is the focal distance OS.

Let $Q = 9 + p$ and $R = 4 + p$ for C. Hence the focal distance $OS = \sqrt{(Q - R)} = \sqrt{(9 + p - 4 - p)} = \sqrt{(5)}$ which is independent of p.

(d)





C becomes an extremely narrow ellipse (both axes become smaller, but the minor axis more rapidly.)

2/17) (e) As $p \to -4$ the semiminor axis $\sqrt{R} \to 0$ and $\sqrt{(Q - R)} \to Q$ i.e. the semimajor axis $\sqrt{Q} \to$ OS the focal distance. So the limiting length of the minor axis is 0 and of the major axis is the interval SS'.

(f) Shape is a straight line (actually 2 coincident lines). Position is the line through SS'. Equation of $C \equiv (4 + p)x^2 + (9 + p)y^2 - (4 + p)(9 + p) = 0$ becomes $y^2 = 0$. (Note that when (in general) b the semiminor axis is zero, eccentricity $e = \frac{\sqrt{(a^2 + b^2)}}{a} = 1$. In this case the conic is a parabola. In the previous section "Degenerate Cases of Conics" $y^2 = 0$ is the equation representing the degenerate parabola.)

(g) Shape is a "line like" extremely narrow hyperbola. Branches represented by the rays SX and S'X'.

(h) The semiminor axis $\sqrt{R}$ is imaginary since $R < 0$ when $p < -4$.

(i) The hyperbola flattens and since $\sqrt{Q}$ the semitransverse axis becomes less and less and the branches will approach the centre.

(j) Shape is a straight line (actually 2 coincident lines) perpendicular to the transverse axis through the centre. Equation of $C \equiv (4 + p)x^2 + (9 + p)y^2 - (4 + p)(9 + p) = 0$ becomes $x^2 = 0$.