CHAPTER **ONE**

# Complex Numbers

Chapter Overview: One of the significant properties of the real numbers is that any of the four arithmetic operations of addition, subtraction, multiplication and division can be applied to any pair of real numbers, with the exception that division by zero is undefined. As a result, every linear equation

$$ax + b = 0 \qquad \text{where} \qquad a \neq 0$$

can be solved.

The situation is not so satisfactory when quadratic equations are considered. There are some quadratic equations that can be solved, but others, like

$$x^2 + 2x + 3 = 0\,,$$

have no real solution. This apparent inconsistency, that some quadratics have a solution whilst others do not, can be resolved by the introduction of a new type of number, the complex number.

But there is more to complex numbers than just solving quadratic equations. In this chapter the reader is shown an application to geometry and later in the course, complex numbers will be used in the study of polynomials. These new numbers have many applications beyond this course, such as in evaluating certain integrals and in solving problems in electrical engineering. Complex numbers also provide links between seemingly unrelated quantities and areas of mathematics. Here is a stunning example. The four most significant real numbers encountered so far are 0, 1, $e$ and $\pi$. Although the proof is beyond the scope of this course, these four are connected in a remarkably simple equation involving the special complex number $i$, namely

$$e^{i\pi} + 1 = 0\,.$$

## 1A The Arithmetic of Complex Numbers

**Introducing A New Type of Number:** We begin by examining the roots of various quadratic equations. For convenience in presenting the new work, we will solely use the method of completing the square.

Suppose that initially we restrict our attention to those quadratic equations with rational solutions such as the equation $x^2 - 4x - 12 = 0$. Completing the square:

$$(x-2)^2 = 16$$

so $\quad x - 2 = 4$ or $-4$

which leads to the two roots

$$\alpha = 6 \text{ and } \beta = -2\,.$$

Note that $\alpha + \beta = 4$ and $\alpha\beta = -12$.

Repeating this process for a number of quadratics with rational solutions, it soon becomes evident that if $ax^2 + bx + c = 0$ has solutions $\alpha$ and $\beta$ then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}\,.$$

Further investigation reveals that there are some quadratic equations which do not have rational solutions, such as $x^2 - 4x - 1 = 0$. Completing the square:

$$(x-2)^2 = 5\,.$$

Herein lies a problem since there is no rational number whose square is $5$. We seek to overcome this problem by introducing a new type of number, in this case the irrational number $\sqrt{5}$ which has the property that $\left(\sqrt{5}\right)^2 = 5$. Assuming that the normal rules of algebra apply to this new number, we further note that $\left(-\sqrt{5}\right)^2 = \left(\sqrt{5}\right)^2 = 5$, so that 5 has two square roots, namely $\sqrt{5}$ and $-\sqrt{5}$. We hope that the introduction of this new type of number makes sense of our calculations and proceed with the solution. Thus

$$x - 2 = \sqrt{5} \text{ or } -\sqrt{5}$$

which leads to the two roots

$$\alpha = 2 + \sqrt{5} \text{ and } \beta = 2 - \sqrt{5}\,.$$

Note that $\alpha + \beta = 4$ and $\alpha\beta = -1$.

Repeating this process for a number of quadratics with irrational solutions, it soon becomes evident that if $ax^2 + bx + c = 0$ has irrational roots $\alpha$ and $\beta$ then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}\,.$$

Since this is consistent with the quadratic equations with rational solutions, it seems that the introduction of surds into our number system is valid. Indeed we have used surds since Year 9 and are now quite comfortable manipulating them.

Yet further investigation reveals that there are some quadratic equations which have neither rational nor irrational solutions, such as $x^2 - 4x + 5 = 0$. Completing the square yields:

$$(x-2)^2 = -1\,.$$

Again there is a problem since there is no known number whose square is $-1$. Just as before, we seek to overcome this problem by introducing a new type of number. In this case we introduce the so called imaginary number $i$ which has the property that $i^2 = -1$. Assuming that the normal rules of algebra apply to this new number, we further note that $(-i)^2 = i^2 = -1$, so that $-1$ has two square roots, namely $i$ and $-i$. We hope that the introduction of this new type of number makes sense of our calculations and proceed with the solution. Thus

$$x - 2 = i \text{ or } -i$$

which leads to the two roots

$$\alpha = 2 + i \text{ and } \beta = 2 - i\,.$$

Note that $\alpha + \beta = 4$ and

$$\begin{aligned}\alpha\beta &= (2 - i)(2 + i)\\ &= 2^2 - i^2 \quad \text{(difference of two squares)}\\ &= 4 + 1\\ &= 5\,.\end{aligned}$$

Repeating this process for a number of quadratics with solutions which involve the imaginary number $i$, it soon becomes evident that if $ax^2 + bx + c = 0$ has solutions $\alpha$ and $\beta$ then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}\,.$$

Since this is consistent with all previously encountered quadratic equations, it seems reasonable to include the imaginary number $i$ in our number system.

**A New Number in Arithmetic:** We will introduce the imaginary number $i$ into our system of numbers, which has the special property that $i^2 = -1$. We will treat the number $i$ as if it were an algebraic pronumeral when it is combined with real numbers using the four arithmetic operations of addition, subtraction, multiplication and division.

**1** A NEW NUMBER: The new number $i$ has the special property that

$$i^2 = -1\,.$$

It may be used like a pronumeral with real numbers in addition, subtraction, multiplication and division.

It is instructive to write out the first four positive powers of $i$. They are:

$$i^1 = i \qquad i^2 = -1 \text{ (by definition)} \qquad \begin{aligned}i^3 &= i^2 \times i\\ &= -1 \times i\\ &= -i\end{aligned} \qquad \begin{aligned}i^4 &= i^3 \times i\\ &= -i \times i\\ &= 1\end{aligned}$$

Writing out the next four powers of $i$, we see that this sequence repeats.

$$\begin{aligned}i^5 &= i^4 \times i\\ &= 1 \times i\\ &= i\end{aligned} \qquad \begin{aligned}i^6 &= i^4 \times i^2\\ &= 1 \times (-1)\\ &= -1\end{aligned} \qquad \begin{aligned}i^7 &= i^4 \times i^3\\ &= 1 \times (-i)\\ &= -i\end{aligned} \qquad \begin{aligned}i^8 &= \left(i^4\right)^2\\ &= 1\end{aligned}$$

It should be clear from these calculations that the sequence continues to cycle. In general we only need to look at the remainder after the index has been divided by 4 in order to determine the result.

**2** POWERS OF THE IMAGINARY NUMBER: A power of $i$ may take only one of four possible values. If $k$ is an integer, then these values are:

$$i^{4k} = 1\,,\quad i^{4k+1} = i\,,\quad i^{4k+2} = -1\,,\quad i^{4k+3} = -i\,.$$

**WORKED EXERCISE:** Simplify: (a) $i^{23}$ (b) $i^7 + i^9$

**SOLUTION:**

(a) Since $23 = 4 \times 5 + 3$

$$i^{23} = -i$$

(b)

$$\begin{aligned}i^7 + i^9 &= -i + i\\ &= 0\end{aligned}$$

**Complex Numbers:** Since we have included $i$ in our number system and since it is to be treated as a pronumeral, our number system must now include the real numbers plus new quantities like

$$2i\,,\qquad -7i\,,\qquad 5+4i\qquad\text{and}\qquad \sqrt{6}-3i\,.$$

The set which includes all such quantities as well as the real numbers is given the symbol $\mathbf{C}$. Each quantity in $\mathbf{C}$ is called a *complex number*. Thus 5, $2i$ and $\sqrt{6}-3i$ are all examples of complex numbers. In the first case, 5 is also a real number, and the real numbers form a special subset of the complex numbers. The number $2i$ is an example of another special subset of the complex numbers. This set consists of all the real multiples of $i$, which are called *imaginary numbers*. Thus $-7i$ is another example of an imaginary number.

3 TWO NEW TYPES OF NUMBERS: Let $a$ and $b$ be real numbers.
COMPLEX NUMBERS: Numbers of the form $a+ib$ are called *complex numbers*. The set of all complex numbers is given the symbol $\mathbf{C}$.
IMAGINARY NUMBERS: Numbers of the form $ib$, that is the complex numbers for which $a=0$, are called *imaginary numbers*.

Again noting that $i$ is treated as a pronumeral, the addition, subtraction and multiplication of complex numbers presents no problem.

$$(2-3i)+(5+7i)=7+4i\,,\qquad (7+2i)-(5-3i)=2+5i\,,$$
$$3(-5+7i)=-15+21i\,,\qquad \sqrt{3}(2+i\sqrt{3})=2\sqrt{3}+3i\,.$$

In some cases of multiplication we will also need to use binomial expansion and the property that $i^2=-1$.

$$\begin{aligned}(2-3i)(5+7i)&=10-i-21i^2\\&=10-i+21\\&=31-i\end{aligned}\qquad\begin{aligned}(4+3i)^2&=16+24i+9i^2\\&=16+24i-9\\&=7+24i\end{aligned}$$

$$\begin{aligned}(3-2i)^2&=9-12i+4i^2\\&=9-12i-4\\&=5-12i\end{aligned}\qquad\begin{aligned}(2+5i)(2-5i)&=4-25i^2\\&=4+25\\&=29\end{aligned}$$

The last three examples above demonstrate the expansions of $(x+iy)^2$, $(x-iy)^2$ and $(x+iy)(x-iy)$ for real values of $x$ and $y$. Note that in the final example, the result is the sum of two squares and is a real number. This will always be the case, regardless of the values of $x$ and $y$.

4 THE SUM OF TWO SQUARES: Let $x$ and $y$ be real numbers, then

$$(x+iy)(x-iy)=x^2+y^2$$

which is always a real number.

**Complex Conjugates:** The last result is significant and will be used frequently. Clearly the pair of numbers $x+iy$ and $x-iy$ are special, and consequently they are given a special description. We say that the numbers $x+iy$ and $x-iy$ are *complex conjugates*. Thus the complex conjugate of $3+2i$ is $3-2i$. Similarly the conjugate of $7-5i$ is $7+5i$.

In order to indicate that the conjugate is required, we write the complex number with a bar above it. Thus:

$$\overline{2+i} = 2-i \qquad \overline{-3i} = 3i$$

$$\overline{-1+4i} = -1-4i \qquad \overline{-3-5i} = -3+5i$$

5 COMPLEX CONJUGATES: Let $x$ and $y$ be real numbers, then the two complex numbers $x+iy$ and $x-iy$ are called complex conjugates.
A: The conjugate of $x+iy$ is $\overline{x+iy} = x-iy$.
B: The conjugate of $x-iy$ is $\overline{x-iy} = x+iy$.

**Division:** Just like real numbers, division by zero is undefined. Dividing a complex number by any other real number presents no problem. As with rational numbers, fractions should be simplified wherever possible by cancelling out common factors.

$$\frac{6+8i}{2} = 3+4i \qquad \frac{-2-6i}{3} = -\tfrac{2}{3} - 2i$$

$$\frac{\sqrt{2}-2i}{\sqrt{2}} = 1 - i\sqrt{2} \qquad \frac{-12+21i}{15} = \frac{-4+7i}{5} \text{ or } -\tfrac{4}{5} + \tfrac{7}{5}\,i$$

There is a potential problem if one complex number is divided by another, such as in $\frac{2+i}{3-i}$. As it stands, it is not clear that this sort of quantity is even allowed in our new number system, since it is not in the standard form, $x+iy$.

The problem is resolved by taking a similar approach to that used to deal with surds in the denominator. The process here is called *realising the denominator*. Thus if the divisor is an imaginary number then simply multiply the fraction by $i/i$, as in the following two examples.

$$\begin{aligned}\frac{1}{4i} &= \frac{1}{4i} \times \frac{i}{i} \\ &= \frac{i}{4i^2} \\ &= -\tfrac{1}{4}\,i\end{aligned} \qquad \begin{aligned}\frac{1+2i}{3i} &= \frac{1+2i}{3i} \times \frac{i}{i} \\ &= \frac{i+2i^2}{3i^2} \\ &= \frac{2-i}{3}\end{aligned}$$

If on the other hand the denominator is a complex number then the method is to multiply top and bottom by its conjugate, as demonstrated here.

$$\begin{aligned}\frac{5}{2+i} &= \frac{5}{2+i} \times \frac{2-i}{2-i} \\ &= \frac{5(2-i)}{4+1} \\ &= 2-i\end{aligned} \qquad \begin{aligned}\frac{5+2i}{3-4i} &= \frac{5+2i}{3-4i} \times \frac{3+4i}{3+4i} \\ &= \frac{15+26i-8}{9+16} \\ &= \frac{7+26i}{25}\end{aligned}$$

6 REALISING THE DENOMINATOR: There are two cases.
A: If the denominator is an imaginary number, multiply top and bottom by $i$.
B: If the denominator is complex, multiply top and bottom by its conjugate.

We should now be satisfied that the complex numbers form a valid number system since we have seen on the previous pages that the four basic arithmetic operations of addition, subtraction, multiplication and division all behave in a sensible way.

**A Convention for Pronumerals:** It is often necessary in developing the theory of complex numbers to perform algebraic manipulations with unknown complex numbers. In order to help distinguish between real and complex variables, the convention that we shall use in this text is that the pronumerals $x$, $y$, $a$ and $b$ will represent real numbers and the pronumerals $z$ and $w$ will represent complex numbers. Thus we will often write $z = x + iy$, where it is understood that $x$ and $y$ are real whilst $z$ is complex.

**Real and Imaginary Parts:** Given the complex number $z = x + iy$, the real part of $z$ is the real number $x$, and the imaginary part of $z$ is the real number $y$. It is convenient to define two new functions of the complex variable $z$ for these two quantities. Thus

$$\mathrm{Re}(z) = x \qquad \text{and} \qquad \mathrm{Im}(z) = y$$

from which it is clear that

$$z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)\,.$$

**WORKED EXERCISE:** Determine $\mathrm{Re}(z^2 - iz)$, where $z = 3 - i$.

**SOLUTION:** Expanding the quadratic in $z$ first,

$$\begin{aligned} z^2 - iz &= (3-i)^2 - i(3-i) \\ &= 8 - 6i - 3i - 1 \\ &= 7 - 9i\,, \end{aligned}$$

so $\qquad \mathrm{Re}(z^2 - iz) = 7\,.$

If two complex numbers $z$ and $w$ are equal, by analogy with surds, we expect that $\mathrm{Re}(z) = \mathrm{Re}(w)$ and $\mathrm{Im}(z) = \mathrm{Im}(w)$. This is in fact the case.

**7** THEOREM — EQUALITY OF COMPLEX NUMBERS: If two complex numbers $z$ and $w$ are equal then $\quad \mathrm{Re}(z) = \mathrm{Re}(w) \quad$ and $\quad \mathrm{Im}(z) = \mathrm{Im}(w)\,.$

PROOF: Let $z = x + iy$ and $w = a + ib$, and suppose that $z = w$ then

$$x + iy = a + ib\,.$$

Rearranging $\qquad i(y - b) = a - x \qquad (**)$

and if $y - b \neq 0$ then $\qquad i = \dfrac{a - x}{y - b}$, which is a real number.

This contradicts $i$ being an imaginary number. Thus $y - b = 0$ and hence $y = b$, whence by equation $(**)$ $x = a$, and the proof is complete.

The careful reader will have noticed that the definitions of $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ given above are not in terms of the variable $z$. Both of these functions can be expressed in terms of $z$ by first writing down $z$ and its conjugate.

$$\begin{aligned} z &= x + iy \\ \overline{z} &= x - iy \end{aligned}$$

Thus we have a pair of simultaneous equations which can be solved for $x$ and $y$ to obtain:

$$\mathrm{Re}(z) = \tfrac{1}{2}(z + \overline{z}) \qquad \text{and} \qquad \mathrm{Im}(z) = \tfrac{1}{2i}(z - \overline{z})\,.$$

**The Arithmetic of Conjugates:** Since taking the complex conjugate of $z$ simply changes the sign of the imaginary part, when it is applied twice in succession the end result leaves $z$ unchanged. Thus

$$\overline{(\overline{z})} = \overline{\left(\overline{x+iy}\right)} = \overline{x-iy} = x+iy = z\,.$$

Another important property of taking conjugates is that it commutes with the four basic arithmetic operations. For example, with addition,

$$\begin{aligned} \overline{(3+i)+(2-4i)} &= \overline{5-3i} \\ &= 5+3i\,, \\ \text{and} \qquad \overline{3+i}+\overline{2-4i} &= 3-i+2+4i \\ &= 5+3i\,. \\ \text{Thus} \qquad \overline{(3+i)+(2-4i)} &= \overline{3+i}+\overline{2-4i}\,. \end{aligned}$$

Notice that it does not matter whether the addition is done first or second, the result is the same. Here is an example with multiplication.

$$\begin{aligned} \overline{(3+i)(2-4i)} &= \overline{10-10i} \\ &= 10+10i\,, \\ \text{and} \qquad \overline{3+i}\times\overline{2-4i} &= (3-i)(2+4i) \\ &= 10+10i\,. \\ \text{Thus} \qquad \overline{(3+i)(2-4i)} &= \overline{3+i}\times\overline{2-4i}\,. \end{aligned}$$

Again notice that it does not matter whether the multiplication is done first or second, the result is the same. This is always the case for addition, subtraction, multiplication and division.

**8** THE ARITHMETIC OF CONJUGATES: The taking of complex conjugates is commutative with addition, subtraction, multiplication and division.

(a) $\overline{w+z} = \overline{w}+\overline{z}$ (c) $\overline{wz} = \overline{w}\times\overline{z}$

(b) $\overline{w-z} = \overline{w}-\overline{z}$ (d) $\overline{w\div z} = \overline{w}\div\overline{z}$

The proof of these results is left as a question in the exercise. There are two special cases of these results. To get the conjugate of a negative, put $w=0$ into (b).

$$\begin{aligned} \overline{(-z)} &= \overline{0-z} \\ &= \overline{0}-\overline{z} \\ \text{thus} \qquad \overline{(-z)} &= -\overline{z}\,. \end{aligned}$$

For the conjugate of a reciprocal, put $w=1$ in (d) to get

$$\begin{aligned} \overline{z^{-1}} &= \overline{1\div z} \\ &= \overline{1}\div\overline{z} \\ &= 1\div\overline{z} \\ \text{thus} \qquad \overline{z^{-1}} &= (\overline{z})^{-1}\,. \end{aligned}$$

**Integer Powers:** The careful reader will have noted that several of the examples used above involve powers of a complex number despite the fact that the meaning of $z^n$ has not yet been properly defined. If the index $n$ is a positive integer then the meaning of $z^n$ is analogous to the real number definition. Thus

$$z^n = \underbrace{z\times z\times\ldots\times z}_{n\text{ factors}}$$

or, the recursive definition,

$$z^1 = z\,,$$
$$z^n = z \times z^{n-1} \qquad \text{for} \qquad n > 1\,.$$

Just like the real numbers, if $z = 0$ then $z^0$ is undefined. For all other complex numbers, $z^0 = 1$. Again continuing the analogy with the real numbers, a negative integer power yields a reciprocal. Thus if $n$ is a positive integer then

$$z^{-n} = \frac{1}{z^n}\,, \qquad z \neq 0\,.$$

As with other division by complex numbers, the denominator is usually realised by multiplying by the conjugate. The case when $n = 1$ occurs frequently and should be learnt.

$$z^{-1} = \frac{1}{z} = \frac{\overline{z}}{z\overline{z}}$$

Indices which are not integers will not be considered in this text.

## Exercise **1A**

**1.** Use the rule given in Box 2 to simplify:
(a) $i^2$ (c) $i^7$ (e) $i^{29}$ (g) $i^3 + i^4 + i^5$
(b) $i^4$ (d) $i^{13}$ (f) $i^{2010}$ (h) $i^7 + i^{16} + i^{21} + i^{22}$

**2.** Evaluate:
(a) $\overline{2i}$ (b) $\overline{3+i}$ (c) $\overline{1-i}$ (d) $\overline{5-3i}$ (e) $\overline{-3+2i}$

**3.** Express in the form $a + ib$, where $a$ and $b$ are real:
(a) $(7 + 3i) + (5 - 5i)$ (c) $(4 - 2i) - (3 - 7i)$
(b) $(-8 + 6i) + (2 - 4i)$ (d) $(3 - 5i) - (-4 + 6i)$

**4.** Express in the form $x + iy$, where $x$ and $y$ are real:
(a) $(4 + 5i)i$ (d) $(-7 + 5i)(8 - 6i)$ (g) $(2 + i)^3$
(b) $(1 + 2i)(3 - i)$ (e) $(5 + i)^2$ (h) $(1 - i)^4$
(c) $(3 + 2i)(4 - i)$ (f) $(2 - 3i)^2$ (i) $(3 - i)^4$

**5.** Use the rule for the sums of two squares given in Box 4 to simplify:
(a) $(1 + 2i)(1 - 2i)$ (c) $(5 + 2i)(5 - 2i)$
(b) $(4 + i)(4 - i)$ (d) $(-4 - 7i)(-4 + 7i)$

**6.** Express in the form $x + iy$, where $x$ and $y$ are real:
(a) $\dfrac{1}{i}$ (c) $\dfrac{5-i}{1-i}$ (e) $\dfrac{-11+13i}{5+2i}$
(b) $\dfrac{2+i}{i}$ (d) $\dfrac{6-7i}{4+i}$ (f) $\dfrac{(1+i)^2}{3-i}$

**7.** Let $z = 1 + 2i$ and $w = 3 - i$. Find, in the form $x + iy$:
(a) $\overline{(iz)}$ (b) $w + \overline{z}$ (c) $2z + iw$ (d) $\text{Im}(5i - z)$ (e) $z^2$

**8.** Let $z = 8 + i$ and $w = 2 - 3i$. Find, in the form $x + iy$:
(a) $\overline{z} - w$ (b) $\text{Im}(3iz + 2w)$ (c) $zw$ (d) $65 \div z$ (e) $z \div w$

**9.** Let $z = 2 - i$ and $w = -5 - 12i$. Find, in the form $x + iy$:
(a) $-zw$ (b) $(1 + i)\overline{z} - w$ (c) $\dfrac{10}{\overline{z}}$ (d) $\dfrac{w}{2 - 3i}$ (e) $\text{Re}\Big((1 + 4i)z\Big)$

**DEVELOPMENT**

**10.** By equating real and imaginary parts, find the real values of $x$ and $y$ given that:
(a) $(x + yi)(2 - 3i) = -13i$
(b) $(1 + 4i)(x + yi) = 6 + 7i$
(c) $(1 + i)x + (2 - 3i)y = 10$
(d) $x(1 + 2i) + y(2 - i) = 4 + 5i$
(e) $\dfrac{x}{2 + i} + \dfrac{y}{2 + 3i} = 4 + i$

**11.** Express in the form $x + iy$, where $x$ and $y$ are real:
(a) $\dfrac{1}{1 + i} + \dfrac{2}{1 + 2i}$
(b) $\dfrac{1 + i\sqrt{3}}{2} + \dfrac{2}{1 + i\sqrt{3}}$
(c) $\dfrac{3 + 2i}{2 - 5i} + \dfrac{3 - 2i}{2 + 5i}$
(d) $\dfrac{-8 + 5i}{-2 - 4i} - \dfrac{3 + 8i}{1 + 2i}$

**12.** Given that $z = x + iy$ and $w = a + ib$, where $a$, $b$, $x$ and $y$ are real, prove that:
(a) $\overline{z + w} = \overline{z} + \overline{w}$
(b) $\overline{z - w} = \overline{z} - \overline{w}$
(c) $\overline{zw} = \overline{z}\,\overline{w}$
(d) $\overline{z^2} = (\overline{z})^2$
(e) $\overline{\left(\dfrac{1}{z}\right)} = \dfrac{1}{\overline{z}}$, $z \neq 0$
(f) $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$, $w \neq 0$

**13.** Let $z = a + ib$, where $a$ and $b$ are real and non-zero. Prove that:
(a) $z + \overline{z}$ is real,
(b) $z - \overline{z}$ is imaginary,
(c) $z^2 + (\overline{z})^2$ is real,
(d) $z\overline{z}$ is real and positive.

**14.** Let $z = a + ib$, where $a$ and $b$ are real. If $\dfrac{z}{z - i}$ is real, show that $z$ is imaginary.

**15.** Prove that if $z^2 = (\overline{z})^2$ then $z$ is either real or imaginary but not complex.

**16.** If $z = x + iy$, where $x$ and $y$ are real, express in the form $a + ib$, where $a$ and $b$ are written in terms of $x$ and $y$:
(a) $z^{-1}$ (b) $z^{-2}$ (c) $\dfrac{z - 1}{z + 1}$

**EXTENSION**

**17.** If both $z + w$ and $zw$ are real, prove that either $z = \overline{w}$ or $\text{Im}(z) = \text{Im}(w) = 0$.

**18.** Given that $z = 2(\cos\theta + i\sin\theta)$, show that $\text{Re}\left(\dfrac{1}{1 - z}\right) = \dfrac{1 - 2\cos\theta}{5 - 4\cos\theta}$.

**19.** Show that $\dfrac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$.

**20.** If $z = \cos\theta + i\sin\theta$, show that $\dfrac{2}{1 + z} = 1 - it$, where $t = \tan\frac{\theta}{2}$.

# 1B Quadratic Equations

Now that the arithmetic of complex numbers has been satisfactorily developed, it is appropriate to return to the original problem of solving quadratic equations. To reflect the fact that the solutions may be complex, the variable $z$ will be used.

**Quadratic Equations with Real Coefficients:** The simplest quadratic equations are the perfect square

$$z^2 = 0$$

for which $z = 0\,,$

and the difference of two squares

$$z^2 - \lambda^2 = 0$$

for which $z = -\lambda$ or $\lambda\,.$

It is now also possible to solve equations involving the sum of two squares, using the result of Box 4 in Section 1A.

Given $z^2 + \lambda^2 = 0$

$$(z + i\lambda)(z - i\lambda) = 0 \qquad \text{(the sum of two squares)}$$

so $z = -i\lambda$ or $i\lambda\,.$

Thus there are three possible cases for a simple quadratic equation: a perfect square, the difference of two squares, or the sum of two squares.

**WORKED EXERCISE:** Find the two imaginary solutions of $z^2 + 10 = 0$.

**SOLUTION:** Factoring the sum of two squares

$$(z + i\sqrt{10})(z - i\sqrt{10}) = 0$$

so $z = -i\sqrt{10}$ or $i\sqrt{10}$

For more general quadratic equations, it is simply a matter of completing the square in $z$ to obtain one of the same three situations: a perfect square, the difference of two squares, or the sum of two squares.

**WORKED EXERCISE:** Find the complex solutions of $z^2 + 6z + 25 = 0$.

**SOLUTION:** Completing the square:

$$(z + 3)^2 + 16 = 0$$

so $(z + 3 + 4i)(z + 3 - 4i) = 0$ (sum of two squares)

thus $z = -3 - 4i$ or $-3 + 4i\,.$

Notice that the sum of two squares situation always yields two roots which are complex conjugates.

9 QUADRATIC EQUATIONS WITH REAL COEFFICIENTS: Complete the square in $z$ to obtain one of the following situations:

A. A PERFECT SQUARE: There is only one real root.
B. THE DIFFERENCE OF TWO SQUARES: There are two real roots.
C. THE SUM OF TWO SQUARES: There are two conjugate complex roots.

There are several of ways of proving the assertion that complex solutions to quadratic equations with real coefficients must occur as conjugate pairs. The approach presented here will later be extended to encompass all polynomials with real coefficients.

PROOF: Let $Q(z) = az^2 + bz + c$, where $a$, $b$ and $c$ are real numbers. Suppose that the equation $Q(z) = 0$ has at least one complex solution $z = w$, then

$$aw^2 + bw + c = 0\,.$$

Take the conjugate of both sides of this equation to get

$$\overline{aw^2 + bw + c} = \overline{0}\,.$$

Now the conjugate of a real number is the same real number. Further, as noted in Box 9, taking a conjugate is commutative with addition and multiplication. Thus the last equation becomes

$$a(\overline{w})^2 + b(\overline{w}) + c = 0$$

that is $$Q(\overline{w}) = 0\,.$$

Hence if $z = w$ is one complex root of $Q(z) = 0$ then it follows that $z = \overline{w}$ is the other root of the equation, and the proof is complete.

**WORKED EXERCISE:** Find a quadratic equation with real coefficients given that one of the roots is $w = 5 - i$.

**SOLUTION:** Complex roots are in conjugate pairs, so the other root is $\overline{w} = 5 + i$.

$$\big(z - (5 - i)\big)\big(z - (5 + i)\big) = 0$$

or $$\big((z - 5) + i\big)\big((z - 5) - i\big) = 0$$

thus $$(z - 5)^2 + 1 = 0\,.$$

Finally $$z^2 - 10z + 26 = 0\,.$$

In general, the quadratic equation with real coefficients which has a complex root $z = \alpha$ is

$$z^2 - 2\,\mathrm{Re}(\alpha)\,z + \alpha\,\overline{\alpha} = 0\,,$$

as can be observed in the three Worked Exercises above. The proof is quite straight forward, and is one of the questions in the exercise.

**10** REAL QUADRATIC EQUATIONS WITH COMPLEX ROOTS: The quadratic equation with real coefficients which has a complex root $z = \alpha$ is

$$z^2 - 2\,\mathrm{Re}(\alpha)\,z + \alpha\,\overline{\alpha} = 0\,.$$

**The Quadratic Method:** Many readers will know the quadratic formula as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,.$$

There is a problem with this formula when complex numbers are involved. When applied to real numbers, the symbol $\sqrt{\phantom{x}}$ means the positive square root, but it is unclear what "positive" means when applied to complex numbers. It might be tempting to say that $i$ is positive and $-i$ is negative, but then what is to be said about numbers like $(-1+i)$ or $(1-i)$? In short, it does not make sense to speak of positive and negative complex numbers, and so the positive square root has no meaning. Thus it is not appropriate to blindly use the quadratic formula to solve an equation with complex roots.

Recall that the quadratic formula arose from applying the method of completing the square. Let us review this process.

Given $az^2 + bz + c = 0\,,$

$$z^2 + \tfrac{b}{a}z = -\tfrac{c}{a}$$

so $\left(z + \frac{b}{2a}\right)^2 = \frac{\Delta}{(2a)^2}\,,$ where $\Delta = b^2 - 4ac\,.$

Now suppose there exists a number $\lambda$, possibly complex, such that $\Delta = \lambda^2$.

Then $\left(z + \frac{b}{2a}\right)^2 - \left(\frac{\lambda}{2a}\right)^2 = 0$

whence $\left(z + \frac{b+\lambda}{2a}\right)\left(z + \frac{b-\lambda}{2a}\right) = 0$ (the difference of two squares)

and so $z = \frac{-b-\lambda}{2a}$ or $\frac{-b+\lambda}{2a}\,.$

Thus if we can find a number $\lambda$, possibly complex, where $\lambda^2 = \Delta$, then we can write down the solution to the quadratic equation using the last line above. If the quadratic formula is to be applied then this is the method that should always be used.

**11** THE QUADRATIC METHOD: Use the following steps to solve $az^2 + bz + c = 0\,.$
1. First find $\Delta = b^2 - 4ac\,.$
2. Next find a number $\lambda$, possibly complex, such that $\lambda^2 = \Delta\,.$
3. Finally, the roots are $z = \dfrac{-b-\lambda}{2a}$ or $\dfrac{-b+\lambda}{2a}\,.$

**WORKED EXERCISE:** Solve $z^2 + 2z + 6 = 0\,.$

**SOLUTION:**
$$\begin{aligned}\Delta &= 2^2 - 4 \times 1 \times 6\\ &= -20\\ &= \left(2i\sqrt{5}\right)^2\,,\end{aligned}$$

hence
$$\begin{aligned}z &= \frac{-2-2i\sqrt{5}}{2} \text{ or } \frac{-2+2i\sqrt{5}}{2}\\ &= -1-i\sqrt{5} \text{ or } -1+i\sqrt{5}\,.\end{aligned}$$

**Complex Square Roots:** Before extending the above work to the case of a quadratic equation with complex coefficients, it is necessary to develop methods for finding the square roots of complex numbers.

The first thing to notice is that, just like real numbers, every complex number has two square roots. The proof is quite straight forward. Suppose that the complex number $z$ is a square root of another complex number $w$ then

$$z^2 = w\,.$$

Further
$$\begin{aligned}(-z)^2 &= z^2\\ &= w\,.\end{aligned}$$

Hence $w$ has a second square root which is the opposite of the first, namely $(-z)$. Thus for example $-2i$ has two opposite square roots, $(1-i)$ and $(-1+i)$. This is not really very surprising since all real numbers (other than zero) have two opposite square roots. For example, 9 has square roots 3 and $-3$, whilst $-5$ has square roots $i\sqrt{5}$ and $-i\sqrt{5}$. The proof that there are no more than two square roots is left as an exercise.

**Complex Square Roots and Pythagoras:** At this point in the course, the method is to equate the real and imaginary parts of $z^2 = w$ in order to obtain a pair of simultaneous equations.

Given $(x + iy)^2 = a + ib\,,$ where $x$, $y$, $a$ and $b$ are real,

$$x^2 - y^2 + 2ixy = a + ib\,.$$

Equating real and imaginary parts yields

$$x^2 - y^2 = a$$

and $$xy = \tfrac{1}{2}\,b\,.$$

In simple cases this pair of equations should be solved by inspection, as in the following example.

**WORKED EXERCISE:** Find the square roots of $7 + 24i$.

**SOLUTION:** Let $(x + iy)^2 = 7 + 24i\,,$ where $x$ and $y$ are real,

then $(x^2 - y^2) + 2ixy = 7 + 24i\,.$

Equating real and imatinary parts yields the simultaneous equations

$$x^2 - y^2 = 7$$

and $$xy = 12\,.$$

These equations can be solved by inspecting the factors of 12. Thus $x = 4$ and $y = 3$, or $x = -4$ and $y = -3$. Hence the square roots of $7 + 24i$ are the opposites

$$4 + 3i \quad \text{and} \quad -4 - 3i\,.$$

Some readers will have noticed in the above example that 7 and 24 are the first two numbers of the Pythagorean triad 7, 24, 25. This is no coincidence. It is often the case that if $b$ is even and the numbers $|a|$, $|b|$ and $\sqrt{a^2 + b^2}$ form a Pythagorean triad then the resulting equations for $x$ and $y$ can simply be solved by inspecting the factors of $\frac{1}{2}\,b$.

**12**

> COMPLEX SQUARE ROOTS AND PYTHAGORAS: Given $(x + iy)^2 = a + ib$, equate the real and imaginary parts to get the simultaneous equations
>
> $$x^2 - y^2 = a$$
> $$xy = \tfrac{1}{2}\,b\,.$$
>
> If $b$ is even and the numbers $|a|$, $|b|$ and $\sqrt{a^2 + b^2}$ form a Pythagorean triad then the these equations can often be solved by inspecting the factors of $\frac{1}{2}\,b$.

**Quadratic Equations with Complex Coefficients:** We are now ready to solve simple quadratic equations with complex coefficients. All that is needed is to combine the above method for finding the roots of a complex number with either the method of completing the square or the quadratic formula method.

**WORKED EXERCISE:** Solve $z^2 - (2+6i)z + (-5+2i) = 0$ by completing the square.

**SOLUTION:** Rearranging

$$z^2 - 2(1 + 3i)z = 5 - 2i$$

so
$$\begin{aligned}(z - (1 + 3i))^2 &= (1 + 3i)^2 + 5 - 2i\\ &= -8 + 6i + 5 - 2i\,,\end{aligned}$$

thus $(z-(1+3i))^2 = -3+4i\,.$
Let $(x+iy)^2 = -3+4i$
then $x^2-y^2 = -3$
and $xy = 2$
so by inspection one solution is $x=1$ and $y=2\,.$
Hence $(z-(1+3i))^2 = (1+2i)^2$
and thus $z = (1+3i)+(1+2i)$ or $(1+3i)-(1+2i)$
that is $z = 2+5i$ or $i\,.$

**Extension — Harder Complex Square Roots:** Often the simultaneous equations given in Box 12 cannot be solved by inspection. Fortunately there is an identity that can be used to help easily solve these equations. Recall that if

$$(x+iy)^2 = a+ib$$

then $x^2-y^2 = a$ (1)

and $2xy = b\,.$ (2)

Squaring these and adding:

$$\begin{aligned} a^2+b^2 &= (x^2-y^2)^2+(2xy)^2 \\ &= (x^2)^2+2x^2y^2+(y^2)^2 \\ &= (x^2+y^2)^2\,. \end{aligned}$$

Hence $x^2+y^2 = \sqrt{a^2+b^2}$ (3)

Equations (1) and (3) now form a very simple pair of simultaneous equations to solve. Equation (2) is used to determine whether $x$ and $y$ have the same sign, when $b>0$, or opposite sign, when $b<0$.

**13**

SQUARE ROOTS OF A COMPLEX NUMBER: Given $(x+iy)^2 = a+ib$ then $x$ and $y$ are solutions of the pair of simultaneous equations

$$x^2-y^2 = a$$

$$x^2+y^2 = \sqrt{a^2+b^2}$$

with the same sign if $b$ is positive, and opposite sign if $b$ is negative.

**WORKED EXERCISE:** Determine the two square roots of $-4+2i\,.$

**SOLUTION:** Let $(x+iy)^2 = -4+2i\,.$ Since $\mathrm{Im}(-4+2i)>0$, $x$ and $y$ have the same sign. Further, $a^2+b^2=20$, so we solve

$x^2-y^2 = -4$ (1)

and $x^2+y^2 = 2\sqrt{5}$ (2)

Adding (1) and (2) yields

$$2x^2 = -4+2\sqrt{5}$$

so $x = -\sqrt{-2+\sqrt{5}}$ or $\sqrt{-2+\sqrt{5}}\,.$

Subtracting (1) from (2) yields

$$2y^2 = 4+2\sqrt{5}$$

so $\qquad y = -\sqrt{2+\sqrt{5}}$ or $\sqrt{2+\sqrt{5}}\,.$

Hence $\qquad x + iy = -\sqrt{-2+\sqrt{5}} - i\sqrt{2+\sqrt{5}}$ or $\sqrt{-2+\sqrt{5}} + i\sqrt{2+\sqrt{5}}\,.$

In fact, the result in Box 13 can be used to develop a formula for the square roots of any complex number, which is derived in one of the Exercise questions. However that formula is not part of the course and should not be memorised.

**Extension — Harder Quadratic Equations:** Any quadratic equation can now be solved, including those with complex discriminants. Box 13 is used to find the square roots of discriminants that cannot be found by inspection.

**WORKED EXERCISE:** [A HARD EXAMPLE] Solve $z^2 + (4-2i)z + 1 = 0$ by using the quadratic formula method.

**SOLUTION:**

$$\begin{aligned}\Delta &= (4-2i)^2 - 4\\ &= 12 - 16i - 4\\ &= 8 - 16i\,.\end{aligned}$$

Let $\qquad (x+iy)^2 = 8 - 16i.$

Now $\mathrm{Im}(8-16i) < 0$ so $x$ and $y$ have opposite sign, with

$$x^2 - y^2 = 8 \qquad (1)$$

and $\qquad x^2 + y^2 = \sqrt{8^2 + 16^2}$

or $\qquad x^2 + y^2 = 8\sqrt{5} \qquad (2)$

Adding and subtracting equtions (1) and (2) yields

$$\begin{aligned}2x^2 &= 8 + 8\sqrt{5}\\ x^2 &= 4(1+\sqrt{5})\end{aligned} \qquad \text{and} \qquad \begin{aligned}2y^2 &= -8 + 8\sqrt{5}\\ y^2 &= 4(-1+\sqrt{5})\,.\end{aligned}$$

Thus $\qquad \Delta = \left(2\sqrt{1+\sqrt{5}} - 2i\,\sqrt{-1+\sqrt{5}}\right)^2$

and so $\qquad z = \frac{1}{2}\left(-4 + 2i + 2\sqrt{1+\sqrt{5}} - 2i\,\sqrt{-1+\sqrt{5}}\right)$

$\qquad\qquad$ or $\frac{1}{2}\left(-4 + 2i - 2\sqrt{1+\sqrt{5}} + 2i\,\sqrt{-1+\sqrt{5}}\right)$

that is $\qquad z = \left(\left(-2 + \sqrt{1+\sqrt{5}}\right) + i\left(1 - \sqrt{-1+\sqrt{5}}\right)\right)$

$\qquad\qquad$ or $\left(\left(-2 - \sqrt{1+\sqrt{5}}\right) + i\left(1 + \sqrt{-1+\sqrt{5}}\right)\right)$

## Exercise **1B**

**1.** Solve for $z$:

(a) $z^2 + 9 = 0$ (c) $z^2 + 2z + 5 = 0$ (e) $16z^2 - 16z + 5 = 0$

(b) $(z-2)^2 + 16 = 0$ (d) $z^2 - 6z + 10 = 0$ (f) $4z^2 + 12z + 25 = 0$

**2.** Write as a product of two complex linear factors:

(a) $z^2 + 36$ (c) $z^2 - 2z + 10$ (e) $z^2 - 6z + 14$

(b) $z^2 + 8$ (d) $z^2 + 4z + 5$ (f) $z^2 + z + 1$

**3.** Form a quadratic equation with real coefficients given that one root is:
(a) $i\sqrt{2}$ (b) $1-i$ (c) $-1+2i$ (d) $2-i\sqrt{3}$

**4.** In each case, find the two square roots of the given number by the inspection method.
(a) $2i$ (b) $3+4i$ (c) $-8-6i$ (d) $35+12i$ (e) $-5+12i$ (f) $24-10i$ (g) $-15-8i$ (h) $9-40i$

DEVELOPMENT

**5.** (a) Find the two square roots of $-3-4i$.
(b) Hence solve $z^2-3z+(3+i)=0$.

**6.** (a) Find the two square roots of $-8+6i$.
(b) Hence solve $z^2-(7-i)z+(14-5i)=0$.

**7.** Solve for $z$:
(a) $z^2-z+(1+i)=0$
(b) $z^2+3z+(4+6i)=0$
(c) $z^2-6z+(9-2i)=0$
(d) $(1+i)z^2+z-5=0$
(e) $z^2+(2+i)z-13(1-i)=0$
(f) $iz^2-2(1+i)z+10=0$

**8.** (a) Find the value of $w$ if $i$ is a root of the equation $z^2+wz+(1+i)=0$.
(b) Find the real numbers $a$ and $b$ given that $3-2i$ is a root of the equation $z^2+az+b=0$.
(c) Given that $1-2i$ is a root of the equation $z^2-(3+i)z+k=0$, find $k$ and the other root of the equation.

**9.** Find the two complex numbers $z$ satisfying $z\,\overline{z}=5$ and $\dfrac{z}{\overline{z}}=\dfrac{1}{5}(3+4i)$.

**10.** (a) Solve $z^2-2z\cos\theta+1=0$ for $z$ by completing the square.
(b) Rearranging the equation in part (a) gives $\cos\theta=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$. Confirm this result for each of the solutions to part (a) by substitution.

**11.** By first factoring the sum or difference of two cubes, solve for $z$:
(a) $z^3=-1$ (b) $z^3+i=0$

**12.** Consider the quadratic equation $az^2+bz+c=0$, where $a$, $b$ and $c$ are real and $b^2-4ac<0$. Suppose that $\omega$ is one of the complex roots of the equation.
(a) Explain why $a\omega^2+b\omega+c=0$.
(b) By taking the conjugate of both sides of the result in (a), and using the properties of conjugates, show that $a\left(\overline{\omega}\right)^2+b\,\overline{\omega}+c=0$.
(c) What have you just proved about the two complex roots of the equation?

**13.** Suppose that $z=\alpha$ is a complex solution to a quadratic equation with real coefficients.
(a) Which other number is also a solution of this quadratic equation?
(b) Hence prove that the quadratic equation is $z^2-2\,\mathrm{Re}(\alpha)z+\alpha\,\overline{\alpha}=0$.

**14.** Let $(x+iy)^2=a+ib$, then we have $x^2-y^2=a$ and $2xy=b$.
(a) For the moment, assume that both $a$ and $b$ are positive.
(i) Sketch the graphs of these two equations on the same number plane.
(ii) What feature of your sketch indicates that there are two square roots of $a+ib$?
(b) Investigate how the sketch changes when either $a$ or $b$ or both are negative or zero.

**EXTENSION**

**15.** Use the results of Box 13 to find the two square roots of:
(a) $-i$ (b) $-6+8i$ (c) $2+2i\sqrt{3}$ (d) $10-24i$ (e) $2-4i$

**16.** Find the discriminant and its square roots, and hence solve:
(a) $z^2+(4+2i)z+(1+2i)=0$
(b) $z^2-2(1+i)z+(2+6i)=0$
(c) $z^2+2(1-i\sqrt{3})z+2+2i\sqrt{3}=0$
(d) $z^2+(1-i)z+(i-1)=0$

**17.** Let $\alpha$ and $\beta$ be the two complex roots of $z^3=1$. Show that:
(a) $\beta=\overline{\alpha}$, (b) $\alpha^2=\beta$ and $\beta^2=\alpha$, (c) $1+\alpha+\alpha^2=0$,
(d) the sum of the first $n$ terms of the series $1+\alpha+\alpha^2+\alpha^3+\ldots$ is either 0, 1 or $-\alpha^2$, depending on the remainder when $n$ is divided by 3.

**18.** Let $a$, $b$ and $c$ be real with $b^2-4ac<0$, and suppose that the quadratic equation $az^2+bz+c=0$ has complex solutions $\alpha=x+iy$ and $\beta=u+iv$.
(a) By considering the sum and product of the roots, show that

$$\operatorname{Im}(\alpha+\beta)=0 \quad \text{and} \quad \operatorname{Im}(\alpha\beta)=0\,.$$

(b) Hence show that $\alpha=\overline{\beta}$.

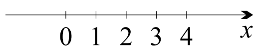
**19.** Let $(x+iy)^2=a+ib$, where $b\neq 0$. Use the result of Box 13 to prove the formula:

$$x+iy=\pm\left(\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}+a\right)}+i\,\frac{b}{|b|}\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)}\right).$$

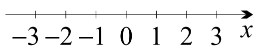
Explain the significance of the term $b/|b|$ in this formula.

# 1C The Argand Diagram

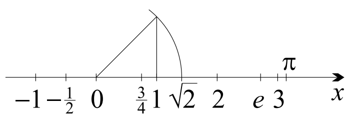
Mathematics requires a knowledge of numbers, and in our course of study at school our understanding of numbers has been enhanced by being able to plot them on a number line, to visualise their properties and relationships. Initially there were the natural numbers, shown at discrete intervals on the number line.




When negative numbers were included to create the integers, the number line was extended to the left of the origin to show these new numbers.



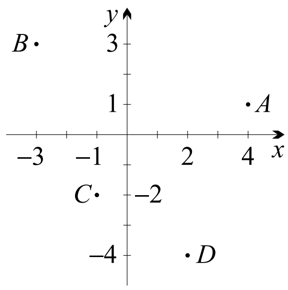

Next came the rationals, the fractions which exist in the spaces between integers. Eventually we became aware of strange numbers called irrationals which fit in the "gaps" that are somehow left between rationals. Some irrational numbers like $\sqrt{2}$ can be constructed geometrically, but others like $e$ and $\pi$ can only be approximated to so many decimal places. The construction for $\sqrt{2}$ is shown here along with the positions of $-\frac{1}{2}$, $\frac{3}{4}$, $e$ and $\pi$.



The number line is now full, the reals have filled it up, and there is no space left for any new objects like complex numbers. Further, since complex numbers come in two parts, real and imaginary, there is no satisfactory way of representing them on a number line. A two dimensional representation is needed.

**The Complex Number Plane:** Keeping to things that are familiar, the number plane would seem to be a convenient way to represent complex numbers. More formally, for each complex number $z = x + iy$ there corresponds a point $Z(x, y)$ in the Cartesian plane. Equally, given any point $W(a, b)$ in the real number plane, the associated complex number is $w = a + ib$.

Thus in the diagram on the right, the complex numbers $4 + i$ and $-3 + 3i$ are represented by the points $A$ and $B$ respectively. The points $C$ and $D$ represent the complex numbers $-1 - 2i$ and $2 - 4i$. Several different names are used to describe a coordinate plane that is used to represent complex numbers. One name is the *Argand diagram*, after the French mathematician Jean-Robert Argand, born in Geneva in 1768. The terms *complex number plane* or $z$-plane are also used.




**14** THE ARGAND DIAGRAM: The complex number $z = x + iy$ is associated with the point $Z(x, y)$ in the real number plane. A complex number may be represented by a point, and a point may be represented by a complex number.

As a convenient abbreviation, the point $Z(x, y)$ will sometimes be simply referred to as the point $z$ in the Argand diagram. It is important to remember that the complex number plane is just a real number plane which is conveniently used to display complex numbers. By the nature of this representation, if two complex numbers are equal then they represent the same point. The converse is also true.

**The Real and Imaginary Axes:** If $\mathrm{Im}(z) = 0$, that is $z = x + 0i$, then $z$ is a real number and the corresponding point $Z(x, 0)$ in the Argand diagram lies on the horizontal axis. Thus the horizontal axis is called the *real axis*.

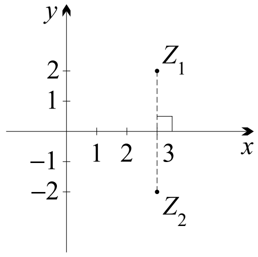
Likewise, if $\mathrm{Re}(z) = 0$, that is $z = 0 + iy$, then $z$ is an imaginary number and the corresponding point $Z(0, y)$ in the Argand diagram lies on the vertical axis. Thus the vertical axis is called the *imaginary axis*.

**Some Simple Geometry:** Now that the complex plane has been introduced, it is immediately possible to observe the geometry of some simple complex number operations. In particular we will consider the geometry of conjugates, opposites, and multiplication by $i$.

Let $\quad z = x + iy \quad$ then the conjugate is

$\qquad \overline{z} = x - iy\,,$

that is, $y$ has been replaced by $-y$. This was encountered in the work on graphs and is known to be a reflection in the real axis. This is clearly evident in the example of $z_1 = 3 + 2i$ and $z_2 = 3 - 2i = \overline{z_1}$ shown on the right.

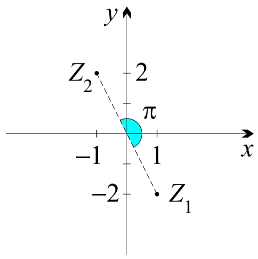



**15** THE GEOMETRY OF CONJUGATES: The points representing $z$ and $\overline{z}$ in the Argand diagram are reflections of each other in the real axis.

Let $\quad z = x + iy \quad$ then the opposite is

$$-z = -x - iy\,.$$

In this case, $x$ and $y$ have been replaced by $-x$ and $-y$ respectively. Thus the result is obtained by reflecting the point in both axes in succession. Alternatively, it is a rotation by $\pi$ about the origin. The diagram on the right with $z_1 = 1 - 2i$ and $z_2 = -1 + 2i = -z_1$ demonstrates this rotation.



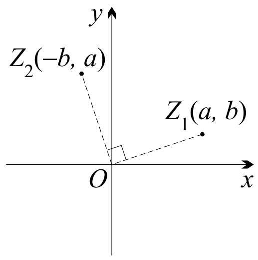

**16** | THE GEOMETRY OF OPPOSITES: The points representing $z$ and $-z$ in the Argand diagram are rotations of each other by $\pi$ about the origin.

Let $\quad z_1 = a + ib \quad$ then $z_2 = i\,z_1$ is given by

$$z_2 = -b + ia\,.$$

Consider the corresponding points $Z_1$ and $Z_2$ shown in the Argand diagram on the right, where neither $a$ nor $b$ is zero. The product of the gradients of $OZ_1$ and $OZ_2$ is

$$\frac{b}{a} \times \frac{a}{-b} = -1\,.$$




Hence $OZ_2$ is perpendicular to $OZ_1$ and the conclusion is that multiplication by $i$ is equivalent to an anticlockwise rotation by $\frac{\pi}{2}$ about the origin. The situation is the same whenever $z_1$ is real or imaginary, but not zero, and the proof is left as an exercise.

**17** | THE GEOMETRY OF MULTIPLICATION BY $i$: The point representing $iz$ is the result of rotating the point representing $z$ by $\frac{\pi}{2}$ anticlockwise about the origin.

Note that multiplication by $i$ twice in succession yields a rotation of $2 \times \frac{\pi}{2} = \pi$. This is consistent with the geometry of opposites, since $i(iz) = i^2 z = -z$.

**WORKED EXERCISE:** Let $z = x + iy$. Determine $i\,\overline{z}$ and hence give a geometric interpretation of the result.

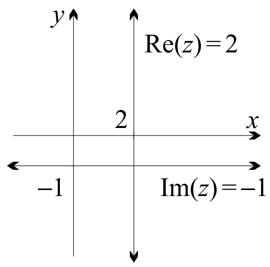
**SOLUTION:**
$$\begin{aligned} i\,\overline{z} &= i(x - iy) \\ &= y + ix\,. \end{aligned}$$

This is just $z$ with $x$ and $y$ swapped. Thus it is a reflection in the line $y = x$.

**Simple Locus Problems:** So far we have concentrated our attention on individual points in the complex plane. Often an equation in $z$ will correspond to a whole collection of points, that is a locus. In the simple cases dealt with here, the equation of that locus can be found by putting $z = x + iy$.

**WORKED EXERCISE:** Graph the following loci:
(a) $\mathrm{Re}(z) = 2$, (b) $\mathrm{Im}(z) = -1$.

**SOLUTION:** The two loci are:
(a) the vertical line $x = 2$, and
(b) the horizontal line $y = -1$,
as shown in the diagram on the right.
Note that these two lines intersect at $z = 2 - i$.

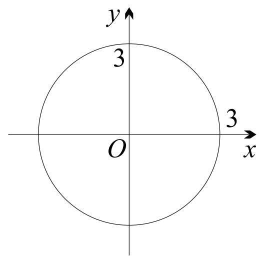


**18**

VERTICAL AND HORIZONTAL LINES: In the Argand diagram:
- the equation $\text{Re}(z) = a$ is the vertical line $x = a$
- the equation $\text{Im}(z) = b$ is the horizontal line $y = b$
- these two lines intersect at $z = a + ib$.

**WORKED EXERCISE:** Let the point $P$ in the complex plane represent the number $z = x + iy$. Given that $z\,\overline{z} = 9$, find the locus of $P$ and sketch it.




**SOLUTION:** The given equation becomes

$$(x + iy)(x - iy) = 9$$

so $\quad x^2 + y^2 = 3^2$

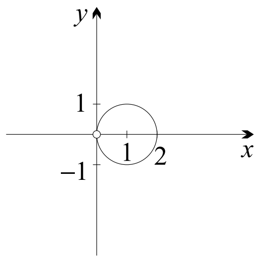
that is a circle with centre the origin and radius 3.

In some examples it is best to manipulate the given equation in $z$ first, and then substitute $x+iy$. It is also important to note any restrictions on $z$ before starting. Both of these points feature in the following example.

**WORKED EXERCISE:** Find and describe the locus of $z$ in the Argand diagram given

$$\frac{1}{z} + \frac{1}{\overline{z}} = 1\,.$$

**SOLUTION:** Note that in the given equation $z \neq 0$, since the LHS is undefined there. Multiply both sides by the lowest common denominator to get




$$\overline{z} + z = z\,\overline{z}$$

so $\quad 2x = x^2 + y^2$

or $\quad 0 = x^2 - 2x + y^2$

thus $\quad 1 = (x-1)^2 + y^2$

that is, the circle with radius 1 and centre $(1, 0)$, excluding the origin.

## Exercise **1C**

**1.** Write down the coordinates of the point in the complex plane that represents:

(a) $2$ (b) $i$ (c) $-3 + 5i$ (d) $\overline{2 + 2i}$ (e) $-5(1 + i)$ (f) $(2 + i)i$

**2.** Write down the complex number that is represented by the point:

(a) $(-3, 0)$ (b) $(0, 3)$ (c) $(7, -5)$ (d) $(a, b)$

**3.** Let $z = 1 + 3i$, and let $A$, $B$, $C$ and $D$ be the points representing $z$, $iz$, $i^2z$ and $i^3z$ respectively.

(a) Plot the points $A$, $B$, $C$ and $D$ in the complex plane.

(b) What type of special quadrilateral is $ABCD$?

(c) What appears to be the geometric effect of multiplying a complex number by $i$?

**4.** Let $z = 3 + i$ and $w = 1 + 2i$. Plot the points representing each group of complex numbers on separate Argand diagrams.

(a) $z$, $iz$, $-z$, $-iz$
(b) $w$, $iw$, $-w$, $-iw$
(c) $z$, $\overline{z}$, $w$, $\overline{w}$
(d) $z$, $w$, $z + w$
(e) $z$, $w$, $z - w$
(f) $z$, $w$, $w - z$

**5.** Graph the following loci:

(a) $\operatorname{Re}(z) = -3$
(b) $\operatorname{Im}(z) = 2$
(c) $\operatorname{Im}(z) < 1$
(d) $\operatorname{Re}(z) \geq -2$
(e) $\operatorname{Re}(z) = \operatorname{Im}(z)$
(f) $2\operatorname{Re}(z) = \operatorname{Im}(z)$
(g) $\operatorname{Re}(z) \leq 2\operatorname{Im}(z)$
(h) $\operatorname{Re}(z) > -\operatorname{Im}(z)$

**DEVELOPMENT**

**6.** Let the point $P$ represent the complex number $z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$, and let the points $Q$, $R$, $S$ and $T$, represent $\overline{z}$, $-z$, $iz$ and $\frac{1}{z}$ respectively. Plot all these points on an Argand diagram.

**7.** Show that the point representing $-\overline{z}$ is a reflection of the point representing $z$ in the $y$-axis.

**8.** Consider the points represented by the complex numbers $z$, $\overline{z}$, $-z$ and $-\overline{z}$.
Show that these points form a rectangle by using:

(a) coordinate geometry to show that the diagonals are equal and bisect each other,
(b) the geometry of conjugates and opposites.

**9.** In the text it was proven that when $z$ is complex, $iz$ is a rotation by $\frac{\pi}{2}$ about the origin. Prove the same result when $z$ is: (a) real, (b) imaginary.

**10.** The numbers $z = a + ib$ and $w = iz$ are plotted in the complex plane at $A$ and $B$ respectively.

(a) By considering the gradients, show that $OA \perp OB$.
(b) Use the distance formula to show that $OA = OB$.
(c) What type of triangle is $\triangle OAB$?

**11.** The point $P$ in the complex plane represents the number $z$. Find and describe the locus of $P$ given that

$$\frac{1}{z} - \frac{1}{\overline{z}} = i\,.$$

**12.** The complex number $z$ is represented by the point $C$ in the Argand diagram. Find and describe the locus of $C$ if

$$\operatorname{Re}\left(\frac{z-6}{z}\right) = 0\,.$$

**13.** Show that $(z - 2)\,\overline{(z - 2)} = 9$ represents a circle in the Argand diagram.

**14.** Find and describe the locus of points in the Argand diagram which correspond to

$$z\,\overline{z} = \Big(\operatorname{Re}(z - 1 + 3i)\Big)^2\,.$$

**EXTENSION**

**15.** Let the point $H$ represent $z$ in the complex plane. Draw the loci of $H$ if:

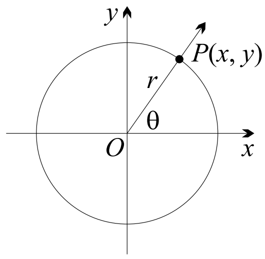
(a) $\operatorname{Im}\left(z^2\right) = 2c^2$
(b) $\operatorname{Re}\left(z^2\right) = c^2$

**16.** Show that the point representing $-i\overline{z}$ is a reflection of the point representing $z$ in $y = -x$.

**17.** Show that $\frac{1}{z}$ is a reflection and enlargement of $z$.

# 1D Modulus-Argument Form

**The Modulus and Argument of a Complex Number:** Recall that in the study of trigonometry it was found that the location of a point $P$ could be expressed either in terms of its horizontal and vertical positions, $x$ and $y$, or in terms of its distance $OP = r$ from the origin and the angle $\theta$ that the ray $OP$ makes with the positive $x$-axis. The situation is shown in the number plane on the right.




In the complex number plane the distance $r$ is called the *modulus* of $z$, and owing to its geometric definition as a distance it is written as $|z|$. On squaring:

$$\begin{aligned}|z|^2 &= r^2\\ &= x^2 + y^2\\ &= (x+iy)(x-iy) \quad \text{(sum of two squares)}\end{aligned}$$

hence $|z|^2 = z\overline{z}\,.$

The angle $\theta$ is called the *argument* of $z$, and is written $\theta = \arg(z)$. Just as with trigonometry, $\theta$ can take infinitely many values for the same point $P$, but the convention in this course is to choose the value for which $-\pi < \arg(z) \le \pi$. Note the strict inequality on the left hand side, and the use of radian measure.

19 MODULUS AND ARGUMENT: Let $P$ represent the complex number $z = x + iy$ in the Argand diagram, with origin $O$.
- The *modulus* of $z$ is the distance $|z| = r = OP$. Note that $|z|^2 = z\overline{z}$.
- The *argument* of $z$ is the angle $\arg(z) = \theta$ that the ray $OP$ makes with the positive real axis. By convention, we choose the value of $\theta$ for which $-\pi < \theta \le \pi\,.$

From the trigonometric definitions it is clear that

$$x = r\cos\theta \qquad (1)$$

and $$y = r\sin\theta\,, \qquad (2)$$

from which it follows that

$$z = r\cos\theta + ir\sin\theta\,.$$

Notice that the modulus $r$ is a common factor in this last expression and it is more commonly written as

$$z = r(\cos\theta + i\sin\theta)$$

or $$z = r\operatorname{cis}\theta \qquad \text{for short.}$$

In order to contrast the two ways of writing a complex number, $z = x + iy$ is called *real-imaginary* or *Cartesian* form whilst $z = r(\cos\theta + i\sin\theta)$ is called *modulus-argument* form, or *mod-arg* form for short. Equations (1) and (2) above serve to link the two forms.

**WORKED EXERCISE:** Express each complex number in real-imaginary form.

(a) $z = 4\operatorname{cis}\pi$ (b) $z = 2\operatorname{cis}\frac{\pi}{6}$ (c) $z = \operatorname{cis}\frac{2\pi}{3}$

**SOLUTION:**

(a) $z = 4\cos\pi + 4i\sin\pi$
$= -4$

(b) $z = 2\cos\frac{\pi}{6} + 2i\sin\frac{\pi}{6}$
$= \sqrt{3} + i$

(c) $z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$
$= -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$

**WORKED EXERCISE:** Express each complex number in mod-arg form. In part (c) give $\arg(z)$ correct to two decimal places.

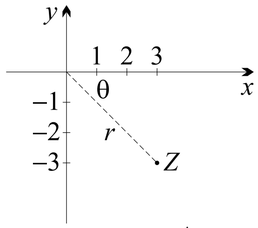
(a) $z = 5i$ (b) $z = 3 - 3i$ (c) $z = -4 - 3i$

**SOLUTION:** In each case let $z = r \operatorname{cis} \theta$ with $Z$ the point in the Argand diagram.

(a) In this case $Z$ is on the positive imaginary axis so

$$r = 5 \qquad \text{and} \qquad \theta = \tfrac{\pi}{2},$$

hence $z = 5 \operatorname{cis} \tfrac{\pi}{2}$.



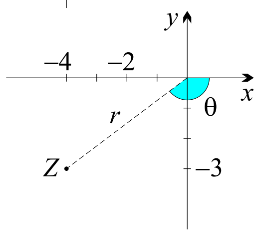

(b) $r^2 = 3^2 + 3^2$

so $r = 3\sqrt{2}$.

Now $\cos\theta = \dfrac{1}{\sqrt{2}}$ and $Z$ is in the fourth quadrant

so $\theta = -\tfrac{\pi}{4}$,

hence $z = 3\sqrt{2} \operatorname{cis}\left(-\tfrac{\pi}{4}\right)$.




(c) $r = 5$ (Pythagorian triad)

Now $\cos\theta = -\tfrac{4}{5}$ and $Z$ is in the third quadrant

so $\theta = -\pi + \cos^{-1}\tfrac{4}{5} \doteqdot -2{\cdot}50$ radians,

hence $z \doteqdot 5 \operatorname{cis}(-2{\cdot}50)$.

**20** FORMS OF A COMPLEX NUMBER:

- $x + iy$ is called the *real-imaginary* form or *Cartesian* form of $z$.
- $r(\cos\theta + i\sin\theta) = r \operatorname{cis}\theta$ is called the *modulus-argument* form of $z$.
- The equations relating the two forms are:

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

**Some Simple Algebra:** As we shall see in the remainder of this chapter, the use of mod-arg form is a powerful tool, both in simplifying much algebra and in providing geometric interpretations. The first thing to notice is that $|0| = 0$ but that $\arg(0)$ is undefined. This is because $0 = 0 \operatorname{cis}\theta$ for all values of $\theta$.

The second thing to notice is that $|\operatorname{cis}\theta| = 1$. The geometry of the situation makes the result obvious since if $z = \cos\theta + i\sin\theta$ then the point $Z(\cos\theta, \sin\theta)$ lies on the unit circle. Hence $|z| = OZ = 1$. Here is an algebraic derivation of the same result.

$$\begin{aligned} |\cos\theta + i\sin\theta|^2 &= \cos^2\theta + \sin^2\theta \\ &= 1, \end{aligned}$$

hence $|\operatorname{cis}\theta| = 1$.

This identity has immediate applications in quadratic equations.

**WORKED EXERCISE:** Find a quadratic equation with real coefficients given that one root is $z = \operatorname{cis}\theta$.

**SOLUTION:** The other root must be $\overline{\operatorname{cis}\theta}$. Thus the quadratic equation is

$$(z - \operatorname{cis}\theta)(z - \overline{\operatorname{cis}\theta}) = 0$$

or $z^2 - (\operatorname{cis}\theta + \overline{\operatorname{cis}\theta})z + \operatorname{cis}\theta \times \overline{\operatorname{cis}\theta} = 0$

that is $z^2 - 2\operatorname{Re}(\operatorname{cis}\theta)z + |\operatorname{cis}\theta|^2 = 0$

thus $z^2 - 2z\cos\theta + 1 = 0$.

**The Product of Two Complex Numbers:** The modulus-argument form of the product of two numbers is a particularly important result. Let $w = a \operatorname{cis}\theta$ and $z = b \operatorname{cis}\phi$, with $a \neq 0$ and $b \neq 0$, then

$$\begin{aligned} wz &= a(\cos\theta + i\sin\theta) \times b(\cos\phi + i\sin\phi) \\ &= ab\Big((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)\Big) \\ &= ab\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big)\,, \end{aligned}$$

that is $\quad a \operatorname{cis}\theta \times b \operatorname{cis}\phi = ab \operatorname{cis}(\theta+\phi)\,.$

Thus $\quad |wz| = ab \quad$ and $\quad \arg(wz) = \theta + \phi\,.$

This yields the following two significant results:

$$|wz| = |w|\,|z|$$

and $$\arg(wz) = \arg(w) + \arg(z)\,.$$

**WORKED EXERCISE:** Let $w = \sqrt{3} + i$ and $z = 1 + i$.

(a) Evaluate $wz$ in real-imaginary form.

(b) Express $w$ and $z$ in mod-arg form and hence evaluate $wz$ in mod-arg form.

(c) Hence find the exact value of $\cos\frac{5\pi}{12}$.

**SOLUTION:**

(a) $$\begin{aligned} wz &= (\sqrt{3}+i)(1+i) \\ &= (\sqrt{3}-1) + i(\sqrt{3}+1)\,. \end{aligned}$$

(b) Now $$\begin{aligned} w &= 2\operatorname{cis}\tfrac{\pi}{6} \\ z &= \sqrt{2}\operatorname{cis}\tfrac{\pi}{4}\,, \end{aligned}$$

hence $$\begin{aligned} wz &= 2\sqrt{2}\operatorname{cis}(\tfrac{\pi}{6} + \tfrac{\pi}{4}) \\ &= 2\sqrt{2}\operatorname{cis}\tfrac{5\pi}{12}\,. \end{aligned}$$

(c) Equating the real parts of parts (a) and (b) yields

$$2\sqrt{2}\cos\tfrac{5\pi}{12} = \sqrt{3} - 1$$

hence $$\cos\tfrac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}\,.$$

21 THE PRODUCT OF TWO COMPLEX NUMBERS: Let $w$ and $z$ be two complex numbers.

- The modulus of the product is the product of the moduli, that is:

$$|wz| = |w|\,|z|\,.$$

- The argument of the product is the sum of the arguments, that is:

$$\arg(zw) = \arg(w) + \arg(z) \qquad \text{(provided } w \neq 0 \text{ and } z \neq 0\,.)$$

**Some Simple Geometry Again:** It is instructive to re-examine the geometry of conjugates, opposites and multiplication by $i$ using mod-arg form. Beginning with the conjugate, recall that the geometrical interpretation is a reflection in the real axis. Thus the modulus should be unchanged, and the argument should be opposite.

Let $z = r\operatorname{cis}\theta$ then $$\begin{aligned} \overline{z} &= r\cos\theta - ir\sin\theta \\ &= r\cos(-\theta) + ir\sin(-\theta) \end{aligned}$$

$$= r\,\mathrm{cis}(-\theta)$$

hence $\quad |\overline{z}| = |z|$

and $\quad \arg(\overline{z}) = -\arg(z)$

that is, the modulus is unchanged and the angle is opposite, as expected.

The cases of opposites and multiplication by $i$ are more simply dealt with. Recall that these operations represented rotations in the complex plane by $\pi$ and $\frac{\pi}{2}$ respectively. Thus, again, the modulus should be the same, and the argument should be increased appropriately. Looking at opposites first:

$$|-z| = |(-1)\times z| = |-1|\times|z| = |z|\,,$$

and $\quad \arg(-z) = \arg(-1\times z) = \arg(-1) + \arg(z) = \pi + \arg(z)\,.$

That is, the moduli of opposites are equal and the arguments differ by $\pi$.

Similarly $\quad |iz| = |i|\,|z| = |z|\,,$

and $\quad \arg(iz) = \arg(i) + \arg(z) = \frac{\pi}{2} + \arg(z)\,.$

That is, the moduli of $z$ and $iz$ are equal and the arguments differ by $\frac{\pi}{2}$. In both cases the results are exactly as expected.

**The Geometry of Multiplication and Division:** Aside from the special cases above, the general geometry of multiplication and division is evident in the results of Box 21. The product of the moduli indicates an enlargement with centre the origin, and the sum of the arguments represents an anticlockwise rotation about the origin.

Consider these two transformations individually and let $w = r\,\mathrm{cis}\,\theta$. When $\theta = 0$ the product $wz$ reduces to $wz = rz$, which is an enlargement without any rotation. Thus both $z$ and $rz$ lie on the same ray.

When $|w| = r = 1$ the product $wz$ becomes $wz = z\,\mathrm{cis}\,\theta$. Using Box 21:

$$\begin{aligned} |wz| &= |w||z| \\ &= |z| \end{aligned}$$

and

$$\begin{aligned} \arg(wz) &= \arg w + \arg z \\ &= \theta + \arg z\,. \end{aligned}$$

This is simply a rotation without any enlargement. Thus $z$ and $z\,\mathrm{cis}\,\theta$ both lie on a circle of radius $|z|$. The following example serves to demonstrate the situation.

**WORKED EXERCISE:** Let $w = \frac{1}{5}(3+4i)$ and $z = 1+i$. (a) Show that $|w| = 1$.
(b) Evaluate $wz$ and hence confirm that $|wz| = |z|$.
(c) Plot $z$ and $wz$ on the Argand diagram.
(d) What is the angle subtended by these two points at the origin, correct to two decimal places.

**SOLUTION:**

(a)
$$\begin{aligned} |w|^2 &= (\tfrac{3}{5})^2 + (\tfrac{4}{5})^2 \\ &= 1 \end{aligned}$$

hence $\quad |w| = 1\,.$

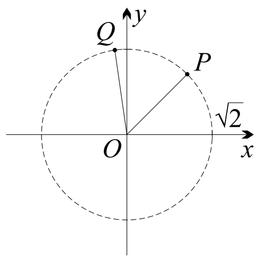
(b)
$$wz = \tfrac{1}{5}(-1+7i)$$

so
$$|wz|^2 = \frac{1+49}{25}$$

$$= 2$$

hence $|wz| = \sqrt{2}$

$$= |z|\,.$$




(c) The number $z$ is shown at $P$ and $wz$ is at $Q$.

(d) Let $\angle POQ = \theta = \arg(w)$, then

$$\cos\theta = \tfrac{3}{5} \qquad \text{and} \qquad \sin\theta = \tfrac{4}{5}$$

whence $\theta$ is acute and

$$\theta = \cos^{-1}\tfrac{3}{5}$$

$$\doteqdot 0{\cdot}93 \text{ radians}$$

**22** THE GEOMETRY OF MULTIPLICATION: Let $w = r\operatorname{cis}\theta$ then the complex number $wz$ is the result of a rotation of $z$ by $\theta$ about the origin and an enlargement of $z$ by factor $r$ with centre the origin.

The corresponding explanation for division is obtained by first writing

$$\frac{z}{w} = \frac{z\overline{w}}{|w|^2} = z \times \frac{1}{r}\operatorname{cis}(-\theta)\,.$$

Thus dividing by $w$ yields an enlargement by factor $\frac{1}{r}$ and a rotation about the origin of $-\theta$. In the special case where $z = 1$ we get the reciprocal of $w$ with

$$\frac{1}{w} = \frac{1}{r}\operatorname{cis}(-\theta)\,,$$

whence $|w^{-1}| = |w|^{-1}$ and $\arg(w^{-1}) = -\arg(w)\,.$

## Exercise **1D**

**1.** Express each complex number in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$.

(a) $2i$ (b) $-4$ (c) $1 + i$ (d) $\sqrt{3} - i$ (e) $-1 + \sqrt{3}\,i$ (f) $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

**2.** Repeat the previous question for each of these complex numbers, writing $\theta$ in radians correct to two decimal places.

(a) $3 + 4i$ (b) $12 - 5i$ (c) $-2 + i$ (d) $-1 - 3i$

**3.** Express in the form $a + ib$, where $a$ and $b$ are real:

(a) $3\operatorname{cis}0$ (b) $5\operatorname{cis}\left(-\frac{\pi}{2}\right)$ (c) $4\operatorname{cis}\frac{\pi}{4}$ (d) $6\operatorname{cis}\left(-\frac{\pi}{6}\right)$ (e) $2\operatorname{cis}\frac{3\pi}{4}$ (f) $2\operatorname{cis}\left(-\frac{2\pi}{3}\right)$

**4.** Given that $z = 1 - i$, express in mod-arg form:

(a) $z$ (b) $\overline{z}$ (c) $-z$ (d) $iz$ (e) $z^2$ (f) $(\overline{z})^{-1}$

**5.** Simplify each expression, leaving your answer in mod-arg form:

(a) $5\operatorname{cis}\frac{\pi}{12} \times 2\operatorname{cis}\frac{\pi}{4}$ (b) $3\operatorname{cis}\theta \times 3\operatorname{cis}2\theta$ (c) $6\operatorname{cis}\frac{\pi}{2} \div 3\operatorname{cis}\frac{\pi}{6}$ (d) $\dfrac{3\operatorname{cis}5\alpha}{2\operatorname{cis}4\alpha}$ (e) $\left(4\operatorname{cis}\frac{\pi}{5}\right)^2$ (f) $\left(2\operatorname{cis}\frac{2\pi}{7}\right)^3$

**DEVELOPMENT**

**6.** In the complex plane, mark a point $K$ to represent a complex number $z = r\operatorname{cis}\theta$ where $1 < r < 2$ and $\frac{\pi}{4} < \theta < \frac{\pi}{2}$. Hence indicate clearly the points $M$, $N$, $P$, $Q$, and $R$ representing $\overline{z}$, $-z$, $2z$, $iz$ and $\frac{1}{z}$ respectively.

**7.** Let $z$ be a non-zero complex number such that $0 < \arg z < \frac{\pi}{2}$. Indicate points $A$, $B$, $C$ and $D$ in the complex plane representing the complex numbers $z$, $-iz$, $(2\operatorname{cis}\frac{\pi}{3})z$ and $\left(\frac{1}{2}\operatorname{cis}(-\frac{\pi}{4})\right)z$.

**8.** Replace $z$ with $z \div w$ in Box 21 to prove that for $z \neq 0$ and $w \neq 0$:

(a) $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ (b) $\arg\left(\dfrac{z}{w}\right) = \arg z - \arg w$

**9.** Given that $z_1 = \sqrt{3} + i$ and $z_2 = 2\sqrt{2} + 2\sqrt{2}\,i$,

(a) write $z_1$ and $z_2$ in mod-arg form, (b) hence write $z_1z_2$ and $\dfrac{z_2}{z_1}$ in mod-arg form.

**10.** Repeat the previous question for $z_1 = -\sqrt{3} + i$ and $z_2 = -1 - i$.

**11.** (a) Express $\dfrac{1 + i\sqrt{3}}{1 + i}$ in real-imaginary form.

(b) Write $1 + i$ and $1 + i\sqrt{3}$ in mod-arg form and hence express $\dfrac{1 + i\sqrt{3}}{1 + i}$ in mod-arg form.

(c) Hence find $\cos\frac{\pi}{12}$ in surd form.

**12.** Let $z = (\sqrt{3} + 1) + (\sqrt{3} - 1)i$.

(a) By writing $\frac{\pi}{12}$ as $\frac{\pi}{3} - \frac{\pi}{4}$, show that $\tan\frac{\pi}{12} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$.

(b) Hence write $z$ in mod-arg form.

**13.** Let $z_1 = 1 + 5i$ and $z_2 = 3 + 2i$, and let $z = \dfrac{z_1}{z_2}$.

(a) Find $|z|$ without finding $z$.

(b) Find $\tan(\tan^{-1}5 - \tan^{-1}\frac{2}{3})$, and hence find $\arg z$ without finding $z$.

(c) Hence write $z$ in the form $x + iy$, where $x$ and $y$ are real.

**14.** Show that for any non-zero complex number $z = r\operatorname{cis}\theta$:

(a) $z\overline{z} = |z|^2$, (b) $\arg(z^2) = 2\arg(z)$, (c) if $|z| = 1$ then $\overline{z} = z^{-1}$.

**15.** Let $z$ be any non-zero complex number. By considering $\arg(|z|^2)$, use the result in part (a) of the previous question to prove that $\arg\overline{z} = -\arg z$.

**16.** The complex number $z$ satisfies the equation $|z - 1| = 1$. Square both sides and hence show that $|z|^2 = 2\operatorname{Re}(z)$.

**17.** If $z$ is a complex number and $|2z - 1| = |z - 2|$, prove that $|z| = 1$.

**18.** Let $z = \cos\theta + i\sin\theta$. Determine $z^2$ in two different ways and hence show that:

(a) $\cos 2\theta = \cos^2\theta - \sin^2\theta$ (b) $\sin 2\theta = 2\sin\theta\cos\theta$

**19.** Let $z = 1 + \cos\theta + i\sin\theta$.

(a) Show that $|z| = 2\cos\frac{\theta}{2}$ and $\arg z = \frac{\theta}{2}$. (b) Hence show that $z^{-1} = \frac{1}{2} - \frac{1}{2}i\tan\frac{\theta}{2}$.

**EXTENSION**

**20.** Let $z = \operatorname{cis}\theta$ and $w = \operatorname{cis}\phi$, that is $|z| = |w| = 1$. Evaluate $z + w$ in mod-arg form and hence show that $\arg(z + w) = \frac{1}{2}(\arg z + \arg w)$. [HINT: Use sums to products.]

**21.** [CIRCLE GEOMETRY] The three complex numbers $z_0$, $z_1$ and $z_2$ are related to each other by the equations $z_2 = z_0 + i\lambda z_0$ and $z_2 = z_1 - i\lambda z_1$, where $\lambda$ is real.

(a) Show that $|z_2 - z_0| = |z_2 - z_1|$. (b) Show that $|z_0| = |z_1|$.

(c) Use circle geometry to describe the situation in the Argand diagram.

**22.** (a) Prove that $\text{Re}(z) \le |z|$. Under what circumstances are they equal?

(b) Prove that $|z+w| \le |z| + |w|$. Begin by writing $|z+w|^2 = (z+w)\overline{(z+w)}$.

**23.** (a) Let $z_1 = r_1 \operatorname{cis}\theta_1$ and $z_2 = r_2 \operatorname{cis}\theta_2$ be any two complex numbers. Prove that:

(i) $|z_1 z_2| = |z_1||z_2|$ (ii) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

(b) Let $z_1, z_2, z_3, \ldots, z_n$ be complex numbers. Prove by induction that for integers $n \ge 2$:

(i) $|z_1 z_2 z_3 \ldots z_n| = \prod_{i=1}^{n} |z_i|$ (ii) $\arg(z_1 z_2 z_3 \ldots z_n) = \sum_{i=1}^{n} \arg(z_i)$

# 1E Vectors and the Complex Plane

The goemetry of multiplication and division became evident with the introduction of the modulus-argument form in the previous section. Since the arguments are added or subtracted, it is clear that a rotation is involved. Since the moduli are multiplied or divided, it is clear that an enlargement is involved.

So far, the observed geometry of addition and subtraction has been limited. A better understanding of these two operations is desirable and can be achieved by yet another representation of complex numbers, this time as vectors.

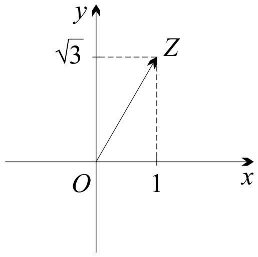
**Vectors:** In the simple definition used in this course, a vector has two characteristics, a magnitude and a direction. Thus the instruction on a pirate treasure map "walk 40 paces east" is an example of a displacement vector. The magnitude is "40 paces" and the direction is "east". A train travelling from Sydney to Perth across the Nullabor at 120 km/h is an example of a velocity vector. The magnitude is 120 km/h and the direction is west.

On the number plane, a vector is represented by an arrow, or more formally a directed line segment. The length of the arrow indicates the magnitude of the vector and the direction of the arrow is the direction of the vector. In particular, in the Argand diagram we will use an arrow joining two points to represent the vector from one complex number to another. When naming a vector, the two letter name of the line segment is used with an arrow above it to indicate the direction, as in the following two examples.

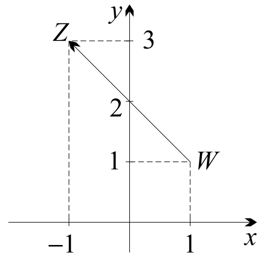
**WORKED EXERCISE:** In the Argand diagram, draw the vectors which represent:
(a) $\overrightarrow{OZ}$ where $z = 1 + i\sqrt{3}$, (b) $\overrightarrow{WZ}$ where $w = 1 + i$ and $z = -1 + 3i$.

**SOLUTION:**

(a)




(b)

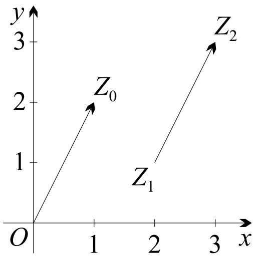


It should be clear from part (a) of the above exercise that the magnitude of the vector $\overrightarrow{OZ}$ is $|z| = 2$ and the direction is $\arg(z) = \frac{\pi}{3}$. By analogy with shifting in the number plane, the magnitude of the vector $\overrightarrow{WZ}$ in part (b) is $|z - w| = 2\sqrt{2}$ and the direction is $\arg(z - w) = \frac{3\pi}{4}$ radians.

**23** VECTORS: The vector $\overrightarrow{WZ}$ represents the complex number $(z - w)$ in the Argand diagram. It has magnitude $|z - w|$ and direction $\arg(z - w)$.

If a vector is translated in the number plane, its length and direction do not change. In the diagram on the right $Z_0$, $Z_1$ and $Z_2$ represent the complex numbers $1+2i$, $2+i$ and $3+3i$ respectively. If the vector $\overrightarrow{OZ_0}$ is shifted so that its tail is at $Z_1$ then its head is at $Z_2$. Since the vectors $\overrightarrow{OZ_1}$ and $\overrightarrow{Z_1Z_2}$ have the same magnitude and direction, we say that they are equal and write

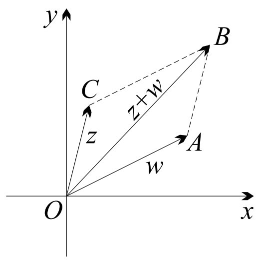



$$\overrightarrow{OZ_0} = \overrightarrow{Z_1Z_2}\ .$$

**24** EQUAL VECTORS: Two vectors are said to be equal if they have the same magnitude and direction.

It may be tempting to say that two vectors are equal if they have the same magnitude and are parallel, but this is wrong. In the above example, the vectors $\overrightarrow{OZ_0}$ and $\overrightarrow{Z_2Z_1}$ are parallel but not equal since they have the opposite direction.

**Addition and Subtraction:** Consider the three points $A$, $B$ and $C$ which represent the complex numbers $w$, $w + z$ and $z$. The direction of $\overrightarrow{AB}$ is




$$\arg\big((w + z) - w\big) = \arg(z)\,,$$

thus $AB||OC$. Likewise, the direction of $\overrightarrow{CB}$ is

$$\arg\big((w + z) - z\big) = \arg(w)\,,$$

thus $CB||OA$. Hence $OABC$ is a parallelogram, from which it follows that the opposite sides are equal in length. That is the corresponding vectors have the same magnitude. We now have two pairs of vectors with the same magnitude and direction, so

$$\overrightarrow{OC} = \overrightarrow{AB}$$

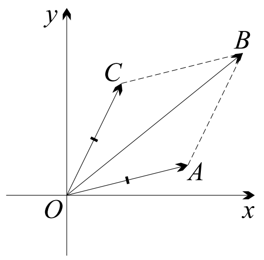
and $$\overrightarrow{OA} = \overrightarrow{CB}\ .$$

Thus we observe that in order to add two complex numbers geometrically, we simply construct the parallelogram $OABC$ from the vectors $\overrightarrow{OA}$ and $\overrightarrow{OC}$, with the sum being the diagonal $\overrightarrow{OB}$. This result is most useful in solving certain algebraic problems geometrically.

**WORKED EXERCISE:** Given two non-zero complex numbers $w$ and $z$ with equal moduli, show that $\arg(w + z) = \frac{1}{2}\big(\arg(w) + \arg(z)\big)$.

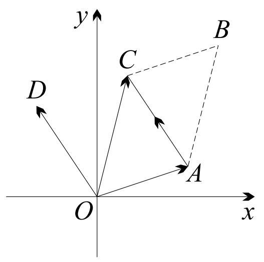
**SOLUTION:** Consider the points $OABC$ in the $z$-plane representing the complex numbers 0, $w$, $w+z$ and $z$ respectively. Now $OABC$ is a parallelogram. Further $OA = OC$, since $|w| = |z|$. Thus in fact $OABC$ is a rhombus. Since the diagonal $OB$ of the rhombus bisects the angle at the vertex $O$, it follows that

$$\arg(w+z) = \tfrac{1}{2}\big(\arg(w) + \arg(z)\big).$$




25 THE GEOMETRY OF ADDITION: Let the points $O$, $A$ and $C$ represent the complex numbers 0, $w$ and $z$. Construct the parallelogram $OABC$. The diagonal vector $\overrightarrow{OB}$ represents the complex number $z + w$.

Given that one diagonal of the parallelogram represents the sum of two complex numbers, it is logical to ask what the other diagonal represents. The vector $\overrightarrow{AC}$ is from $w$ to $z$, thus it has magnitude $|z - w|$ and its direction is $\arg(z - w)$. That is, it represents the complex number $z-w$. The position $D$ of this point in the Argand diagram is determined by translating the vector $\overrightarrow{AC}$ so that its tail is at the origin, as shown on the right.




**WORKED EXERCISE:** The points $OABC$ represent the complex numbers 0, $w$, $w + z$ and $z$. Given that $z - w = i(z + w)$, explain why $OABC$ is a square.

**SOLUTION:** Firstly $OABC$ is a parallelogram, where $\overrightarrow{OB}$ represents $z + w$ and $\overrightarrow{AC}$ represents $z - w$. Since $z - w = i(z + w)$ it follows that

$$\begin{aligned}\arg(z-w) &= \arg(i(z+w))\\ &= \arg(i) + \arg(z+w)\\ &= \tfrac{\pi}{2} + \arg(z+w),\end{aligned}$$

and

$$\begin{aligned}|z-w| &= |i(z+w)|\\ &= |i| \times |z+w|\\ &= |z+w|.\end{aligned}$$

Thus the diagonals $OB$ and $AC$ are at right angles to each other and have the same length. Hence $OABC$ is both a rhombus and a rectangle, that is, a square.

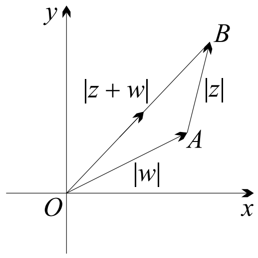
26 THE GEOMETRY OF SUBTRACTION: Let $OABC$ represent the complex numbers 0, $w$, $w + z$ and $z$. Then the diagonal $\overrightarrow{AC}$ represents the complex number $z - w$.

**The Triangle Inequality:** An important identity encountered with the absolute value of real numbers is the triangle inequality

$$\big||x| - |y|\big| \le |x + y| \le |x| + |y|.$$

Given that the absolute value of a real number is analogous to the modulus of a complex number, it is not surprising that the same result holds for the modulus of complex numbers, that is

$$\Big||z| - |w|\Big| \le |z + w| \le |z| + |w|\,.$$




This result can be explained in terms of the geometry of the addition of complex numbers. Consider only the points $O$, $A$ and $B$ as defined previously and shown in the diagram on the right. Recall that the three vectors $\overrightarrow{OA}$, $\overrightarrow{AB}$ and $\overrightarrow{OB}$ represent the complex numbers $w$, $z$ and $z + w$ respectively. Hence the three moduli $|w|$, $|z|$ and $|z + w|$ are the lengths of the sides of $\triangle OAB$.

It is a well known result of Euclidean geometry that the length of one side of a triangle must be less than or equal to the sum of the other two, thus

$$|z + w| \le |z| + |w|\,,$$

with equality when $O$, $A$ and $B$ are collinear. Similarly the length of one side is greater than or equal to the difference of the other two, thus

$$\Big||z| - |w|\Big| \le |z + w|\,,$$

with equality again when the points are collinear. Combining these two yields

$$\Big||z| - |w|\Big| \le |z + w| \le |z| + |w|\,,$$

and replacing $w$ with $-w$ throughout gives

$$\Big||z| - |w|\Big| \le |z - w| \le |z| + |w|\,.$$

These inequalities are called the *triangle inequalities*, after their geometric origins.

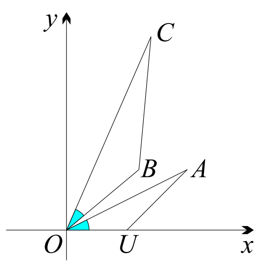
**27** THE TRIANGLE INEQUALITIES: For all complex numbers $z$ and $w$,

- $\Big||z| - |w|\Big| \le |z + w| \le |z| + |w|$
- $\Big||z| - |w|\Big| \le |z - w| \le |z| + |w|$

**Multiplication and Division:** The geometry of these two operations has already been satisfactorily explained as a rotation and enlargement. This interpretation is further demonstrated by the following example.

The diagram below shows the points $O$, $U$, $A$, $B$ and $C$ which correspond to the complex numbers 0, 1, $w$, $z$ and $wz$. In $\triangle UOA$ and $\triangle BOC$,

$$\begin{aligned}\angle BOC &= \arg(wz) - \arg(z)\\ &= \arg(w) + \arg(z) - \arg(z)\\ &= \arg(w)\\ &= \angle UOA\,,\end{aligned}$$




and
$$\begin{aligned}\frac{OC}{OB} &= \frac{|wz|}{|z|}\\ &= \frac{|w||z|}{|z|}\\ &= |w|\\ &= \frac{OA}{OU}\,.\end{aligned}$$

Hence $\triangle BOC ||| \triangle UOA$ (SAS)

Note that the similarity ratio is $OB : OU = |z| : 1$.

This provides us with a novel way of constructing the point $C$ for any given complex numbers $w$ and $z$. First construct $\triangle UOA$, then use the base $OB$ to construct the similar triangle $\triangle BOC$ by applying the similarity ratio $|z| : 1$.

Other than being an application of similar triangles, this construction method is not particularly enlightening, and it is rarely used. The geometry of the situation should always be remembered as a rotation of $w$ by $\arg(z)$ and an enlargement by factor $|z| : 1$.

**Two Special Cases:** A vector approach is very helpful in analysing the geometry in two special cases of division. Let $z_1$, $z_2$, $z_3$ and $z_4$ be four complex numbers corresponding to the points $A$, $B$, $C$ and $D$, and let

$$\frac{z_2 - z_1}{z_3 - z_4} = \lambda \,.$$

Suppose that $\lambda$ is real, then

$$z_2 - z_1 = \lambda(z_3 - z_4)\,,$$

that is, one vector is a multiple of the other. Hence both vectors have the same direction if $\lambda > 0$ (but differ in length) and opposite direction if $\lambda < 0$. In either case the lines $AB$ and $CD$ are parallel.

In the case where $\lambda$ is imaginary, the two vectors are perpendicular, since multiplication by $i$ is equivalent to a rotation by $\frac{\pi}{2}$. Hence $AB \perp CD$. If $\lambda < 0$ then the rotation is in the clockwise direction.

**WORKED EXERCISE:** Let $z_1$ and $z_2$ be any two complex numbers representing the points $A$ and $B$ in the complex plane. Consider the complex number $z$ given by the equation $\dfrac{z - z_1}{z_2 - z_1} = t$ where $t$ is real. Let the point $C$ represent $z$.

(a) Show that $A$, $B$ and $C$ are collinear.

(b) Hence show that $C$ divides $AB$ in the ratio $t : 1 - t$.

**SOLUTION:**

(a) First note that $\overrightarrow{AB}$ represents $z_2 - z_1$ and that $\overrightarrow{AC}$ represents $z - z_1$. Since $\dfrac{z - z_1}{z_2 - z_1}$ is real it follows that $AB$ and $AC$ are parallel. Further since $A$ is common to both lines, it follows that $A$, $B$ and $C$ are collinear.

(b) If $t < 0$ then $\overrightarrow{AC}$ has the opposite direction to $\overrightarrow{AB}$ and the order of the points is $CAB$. If $t = 0$ then $A$ and $C$ coincide. If $0 < t < 1$ then both $\overrightarrow{AC}$ and $\overrightarrow{AB}$ have the same direction and $\overrightarrow{AB}$ has the greater magnitude. Hence the order of the points is $ACB$. If $t = 1$ then $C$ and $B$ coincide. If $t > 1$ then the vectors again have the same direction but $\overrightarrow{AC}$ has the greater magnitude, hence the order is $ABC$. In all these cases the ratio of the magnitudes is

$$\begin{aligned} AC : AB &= |z - z_1| : |z_2 - z_1| \\ &= |t| : 1\,. \end{aligned}$$

Hence in all cases the point $C$ divides $AB$ in the ratio $t : 1 - t$.

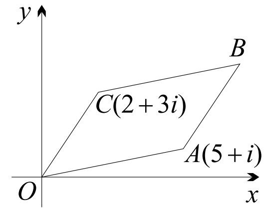
# Exercise **1E**

**1.** In the diagram on the right, $OABC$ is a parallelogram. The points $A$ and $C$ represent $5 + i$ and $2 + 3i$ respectively. Find the complex numbers represented by:

(a) the vector $OB$,

(b) the vector $AC$,

(c) the vector $CA$.



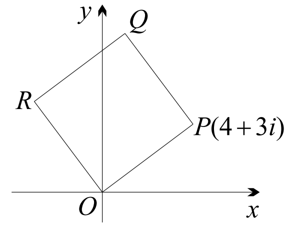

**2.** In the diagram on the right, $OPQR$ is a square. The point $P$ represents $4 + 3i$. Find the complex numbers represented by:

(a) the point $R$,

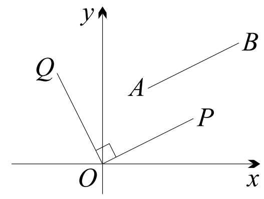
(b) the point $Q$,

(c) the vector $QR$,

(d) the vector $PR$.



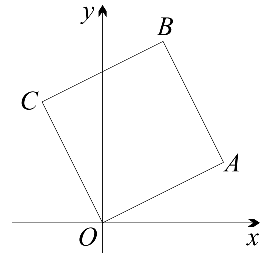

**3.** In the diagram on the right, intervals $AB$, $OP$ and $OQ$ are equal in length, $OP$ is parallel to $AB$ and $\angle POQ = \frac{\pi}{2}$. If $A$ and $B$ represent the complex numbers $3+5i$ and $9+8i$ respectively, find the complex number which is represented by $Q$.




**4.** In the diagram on the right, $OABC$ is a square. The point $A$ represents the complex number $2 + i$.

(a) Find the numbers represented by $B$ and $C$.

(b) If the square is rotated $45^\circ$ anticlockwise about $O$ to give $OA'B'C'$, find the number represented by $B'$.

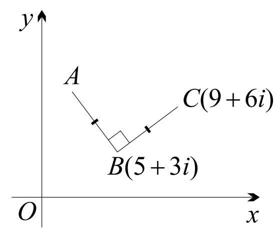



**5.** In the diagram on the right, $AB = BC$ and $\angle ABC = 90^\circ$. The points $B$ and $C$ represent $5+3i$ and $9+6i$ respectively. Find the complex numbers represented by:

(a) the vector $BC$,

(b) the vector $BA$,

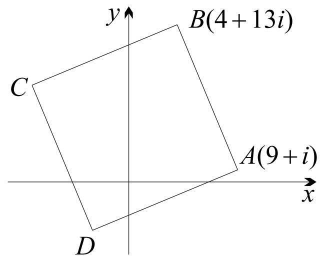
(c) the point $A$.




**6.** The diagram on the right shows a square $ABCD$ in the complex plane. The vertices $A$ and $B$ represent the complex numbers $9 + i$ and $4 + 13i$ respectively. Find the complex numbers that correspond to:

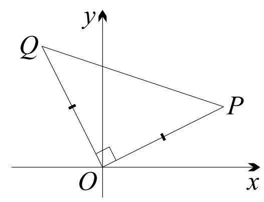
(a) the vector $AB$,

(b) the vertex $D$.

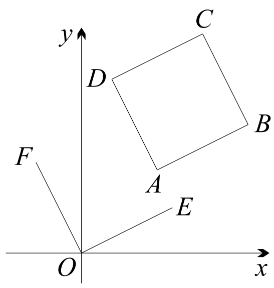



**DEVELOPMENT**

**7.** In the diagram on the right, the points $P$ and $Q$ correspond to the complex numbers $z$ and $w$ respectively. The triangle $OPQ$ is isosceles and the angle $POQ$ is a right angle. Prove that $z^2 + w^2 = 0$.



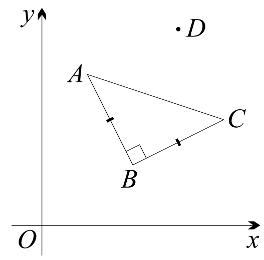
**8.** In the Argand diagram on the right, $ABCD$ is a square, and $OE$ and $OF$ are parallel and equal in length to $AB$ and $AD$ respectively. The vertices $A$ and $B$ correspond to the complex numbers $w_1$ and $w_2$ respectively. What complex numbers correspond to the points $E$, $F$, $C$ and $D$?




**9.** In the diagram on the right, the vertices of a triangle $ABC$ are represented by the complex numbers $z_1$, $z_2$ and $z_3$ respectively. The triangle is isosceles, and right-angled at $B$.

(a) Explain why $(z_1 - z_2)^2 + (z_3 - z_2)^2 = 0$.

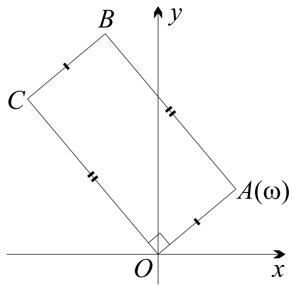
(b) Suppose that $D$ is the point such that $ABCD$ is a square. Find, in terms of $z_1$, $z_2$ and $z_3$, the complex number that the point $D$ represents.




**10.** In the Argand diagram on the right, $OABC$ is a rectangle, with $OC = 2OA$. The vertex $A$ corresponds to the complex number $\omega$.

(a) What complex number corresponds to the vertex $C$?

(b) What complex number corresponds to the point of intersection $D$ of the diagonals $OB$ and $AC$?




**11.** The vertices of an equilateral triangle are equidistant from the origin. One of its vertices is at $1 + \sqrt{3}i$. Find the complex numbers represented by the other two vertices.
[HINT: What is the angle subtended by the vertices at the origin?]

**12.** Given $z = 3 + 4i$, find the two possible values of $w$ so that the points representing 0, $z$ and $w$ form a right-angled isosceles triangle with right-angle at the point representing:

(a) 0 (b) $z$ (c) $w$

**13.** If $z_1 = 4 - i$ and $z_2 = 2i$, find in each case the two possible values of $z_3$ so that the points representing $z_1$, $z_2$ and $z_3$ form an isosceles right-angled triangle witg right-angle at:

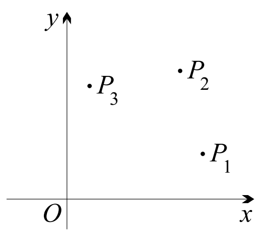
(a) $z_1$ (b) $z_2$ (c) $z_3$

**14.** Given that $z_1 = 1 + i$, $z_2 = 2 + 6i$ and $z_3 = -1 + 7i$, find the three possible values of $z_4$ so that the points representing $z_1$, $z_2$, $z_3$ and $z_4$ form a parallelogram.

**15.** A triangle in the Argand diagram has vertices at the points representing the complex numbers $z_1$, $z_2$ and $z_3$. If $\dfrac{z_2 - z_1}{z_3 - z_1} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$, show that the triangle is equilateral.

**16.** In an Argand diagram, $O$ is the origin, and the points $P$ and $Q$ represent the complex numbers $z_1$ and $z_2$ respectively. The triangle $OPQ$ is equilateral.
Prove that ${z_1}^2 + {z_2}^2 = z_1 z_2$.

**17.** In the diagram on the right, the points $P_1$, $P_2$ and $P_3$ represent the complex numbers $z_1$, $z_2$ and $z_3$ respectively. If $\dfrac{z_2}{z_1} = \dfrac{z_3}{z_2}$, show that $OP_2$ bisects $\angle P_1OP_3$.



**18.** If $z_1$ and $z_2$ are complex numbers such that $|z_1| = |z_2|$, show that:

$$\arg(z_1 z_2) = \arg\left((z_1 + z_2)^2\right) .$$

**19.** Let $z_1 = 2i$ and $z_2 = 1 + \sqrt{3}\,i$.

(a) Express $z_1$ and $z_2$ in mod-arg form.

(b) Plot in the complex plane the points $P$, $Q$, $R$ and $S$ representing $z_1$, $z_2$, $z_1 + z_2$ and $z_1 - z_2$ respectively.

(c) Find the exact values of: (i) $\arg(z_1 + z_2)$ (ii) $\arg(z_1 - z_2)$

**20.** Suppose that the complex number $z$ has modulus one, and that $0 < \arg z < \frac{\pi}{2}$.
Prove that $2\arg(z + 1) = \arg z$.

**21.** The vertices of the quadrilateral $ABCD$ in the complex plane represent the complex numbers $z_1$, $z_2$, $z_3$ and $z_4$ respectively.

(a) If $z_1 - z_2 = z_4 - z_3$, show that the quadrilateral $ABCD$ is a parallelogram.

(b) If $z_1 - z_2 = z_4 - z_3$ and $z_1 - z_3 = i(z_4 - z_2)$, show that $ABCD$ is a square.

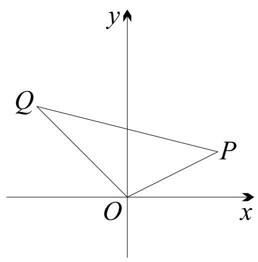
**22.** (a) Prove that for any complex number $z$, $|z|^2 = z\overline{z}$.

(b) Hence prove that for any complex numbers $z_1$ and $z_2$:

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$$

(c) Explain this result geometrically.

**23.** In the diagram on the right, the points $P$ and $Q$ represent the complex numbers $z$ and $w$ respectively.




(a) Explain why $|z - w| \le |z| + |w|$.

(b) Indicate on the diagram the point $R$ representing $z + w$.

(c) What type of quadrilateral is $OPRQ$?

(d) If $|z - w| = |z + w|$, what can be said about the complex number $\dfrac{w}{z}$?

**24.** (a) Prove that the points $z_1$, $z_2$ and $z_3$ are collinear if $\dfrac{z_3 - z_1}{z_2 - z_1}$ is real.

(b) Hence show that the points representing $5 + 8i$, $13 + 20i$ and $19 + 29i$ are collinear.

**EXTENSION**

**25.** The complex numbers $\omega_1$ and $\omega_2$ have modulus 1, and arguments $\alpha_1$ and $\alpha_2$ respectively, where $0 < \alpha_1 < \alpha_2 < \frac{\pi}{2}$.
Show that $\arg(\omega_1 - \omega_2) = \frac{1}{2}(\alpha_1 + \alpha_2 - \pi)$.

**26.** [CIRCLE GEOMETRY] It is known that $\arg\left(\dfrac{z_4 - z_1}{z_2 - z_1}\right) + \arg\left(\dfrac{z_2 - z_3}{z_4 - z_3}\right) = \pi$. Explain why the points representing these complex numbers are concyclic.

**27.** [CIRCLE GEOMETRY] The points representing the complex numbers 0, $z_1$, $z_2$ and $z_3$ are concyclic. Prove that the points representing $\dfrac{1}{z_1}$, $\dfrac{1}{z_2}$ and $\dfrac{1}{z_3}$ are collinear.

[Hint: Show that $\dfrac{z_2^{-1} - z_1^{-1}}{z_3^{-1} - z_1^{-1}}$ is real.]

# 1F Locus Problems

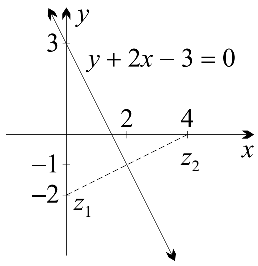
In many situations a set of equations or conditions on a variable complex number $z$ yields a locus of points in the Argand diagram which is a familiar geometric object, such as a line or a circle. The main aim of this section is to provide a geometric description for a locus specified algebraically. Therefore the examples in this text have been grouped by the various geometries.

There are two basic approaches to identifying a locus, algebraic or geometric. The advantage of the algebraic approach is that most readers will already be proficient at manipulating equations in $x$ and $y$. Unfortunately the geometry of the situation may be obscured by the algebra. The advantage of the geometric approach is that it will often provide a very elegant solution to the problem, but may also require a keen insight. Both methods should be practised, with the aim to become proficient at the geometric approach.

**Straight Lines:** Some simple straight lines have already been encountered in 1C, such as the vertical line $\text{Re}(z) = a$. Here are some other examples and their geometric interpretations.

In coordinate geometry, given the coordinates of two points $A$ and $B$, the task of finding the equation of the perpendicular bisector of $AB$ is a lengthy one. The equivalent complex equation is remarkably simple.

**WORKED EXERCISE:** Let $z_1 = 4$ and $z_2 = -2i$, and let the variable point $z$ satisfy the equation $|z - z_1| = |z - z_2|$.
(a) Put $z = x + iy$ and hence show that $z$ lies on the straight line $y + 2x - 3 = 0$.
(b) Describe this line geometrically in terms of $z_1$ and $z_2$.




**SOLUTION:** (a) Substitute the values of $z_1$ and $z_2$, then square to get

$$|z - 4|^2 = |z + 2i|^2$$

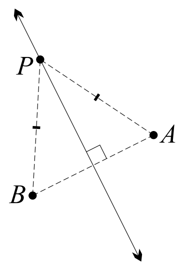
or $\quad (x-4)^2 + y^2 = x^2 + (y+2)^2$

whence $\quad (x-4)^2 - x^2 = (y+2)^2 - y^2$

thus $\quad -4(2x-4) = 2(2y+2)$

so $\quad 4 - 2x = y + 1$

hence $\quad y + 2x - 3 = 0\,.$




(b) Let $z$, $z_1$ and $z_2$ be the points $P$, $A$ and $B$ in the Argand diagram. Since the modulus is a distance, the given equation yields

$$PA = PB$$

Thus the locus of $P$ is the set of all points equidistant from $A$ and $B$, that is, the perpendicular bisector of $AB$.

**28** THE PERPENDICULAR BISECTOR OF AN INTERVAL: Let $z_1$ and $z_2$ be the fixed points $A$ and $B$ in the Argand diagram, and let $z$ be a variable point $P$. If

$$|z - z_1| = |z - z_2|$$

then the locus of $P$ is the perpendicular bisector of $AB$.

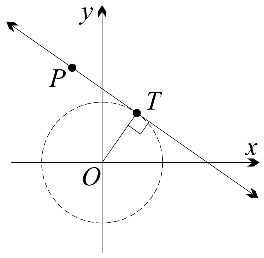
**WORKED EXERCISE:** Let $z_0 = a + ib$ be the fixed point $T$ and let $z = x + iy$ be a variable point $P$ in the complex plane. It is known that $z - z_0 = ikz_0$, where $k$ is a real number.

(a) It is also known that as $k$ varies, the locus of $P$ is a straight line. Find the equation of that straight line in terms of $x$ and $y$.

(b) What is the geometry of the situation?

**SOLUTION:** (a) The given equation expands to

$$x + iy - (a + ib) = ik(a + ib)\,.$$




Equating real and imaginary parts yields

$$x - a = -kb$$

and

$$y - b = ka\,.$$

Eliminating $k$ from this pair of equations, we get

$$b(y - b) = -a(x - a)$$

or

$$ax + by = a^2 + b^2$$

(b) Some readers will recognise this equation as the tangent to a circle. This geometry is confirmed by examining the given equation more closely.

Since multiplication by $i$ represents a rotation of $\frac{\pi}{2}$, it follows that for $k \neq 0$ the vector $\overrightarrow{TP}$ is perpendicular to $\overrightarrow{OT}$. That is, $P$ lies on a line perpendicular to $OT$. Further, when $k = 0$, $z = z_0$, so this line passes through $T$. That is, $PT$ is the tangent to the circle with radius $OT$, as shown above.

**Rays:** The horizontal and vertical lines in 1C and the first example above demonstrate some of the geometry of the recently introduced functions $\text{Re}(z)$, $\text{Im}(z)$ and $|z|$. The new function $\arg(z)$ describes a ray in the $z$-plane.

**WORKED EXERCISE:** The complex number $z$ satisfies the equation $\arg(z) = \frac{\pi}{3}$.

(a) Let $|z| = r$. Write $z$ in modulus-argument form.

(b) Plot $z$ when $r = 1, 2, 3, 4$, and observe that $z$ lies on a ray.

(c) Explain why the origin is not part of this locus.

(d) Use shifting to sketch $\arg(z - 2 - i) = \frac{\pi}{3}$.

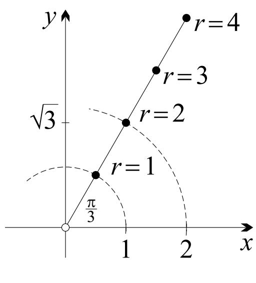
**SOLUTION:**

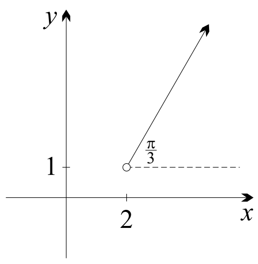
(a) $z = r(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$.

(b) See the first graph on the right.

(c) $\arg(0)$ is undefined so the origin is not included.

(d) $\arg(z - 2 - i) = \arg(z - (2 + i))$ so the ray has been shifted to the point $2 + i$, as shown on the right.




**29**

> RAYS IN THE ARGAND DIAGRAM:
> - The equation $\arg(z) = \theta$ represents the ray which makes an angle $\theta$ with the positive real axis, omitting the origin.
> - The locus $\arg(z - z_0) = \theta$ is the result of shifting the above ray from the origin to the point $z_0$.

**Conics Sections:** The conics, that is the circle, parabola, ellipse and hyperbola, may each be written as equations of complex variables. The geometric definitions of each conic in terms of distance or the ratio of two distances is often the key. Here are three examples.

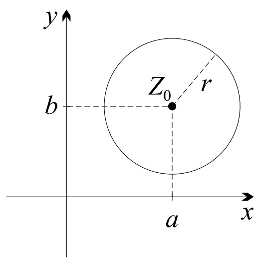
**WORKED EXERCISE:** Consider the locus specified by the equation $|z-z_0| = r$, for some fixed complex number $z_0 = a+ib$ and positive real number $r$.

(a) Explain why this represents a circle. State the centre and radius.

(b) Confirm your answer to part a) by putting $z = x+iy$ and finding the cartesian equation.

(c) Expand $|z-1|^2$ in terms of $z$ and $\overline{z}$. Hence determine the locus specified by $|z|^2 = z + \overline{z}$.

**SOLUTION:** (a) The equation tells us that the distance between $z$ and $z_0$ is fixed. This is the geometric definition of a circle. The centre is $z_0$ and the radius is $r$.




(b) Begin by squaring both sides:

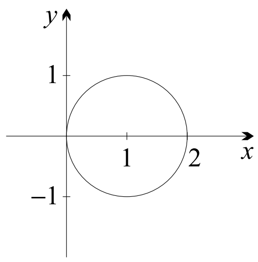
$$|z-z_0|^2 = r^2$$

so $\quad |(x-a)+i(y-b)|^2 = r^2$

thus $\quad (x-a)^2 + (y-b)^2 = r^2\,.$

(c) Noting that $|w|^2 = w\,\overline{w}$, we may write

$$\begin{aligned} |z-1|^2 &= (z-1)\,\overline{(z-1)} \\ &= (z-1)(\overline{z}-1) \\ &= |z|^2 - (z+\overline{z}) + 1\,. \end{aligned}$$




Since we are told that $|z|^2 = z + \overline{z}$, it follows that

$$|z-1|^2 = 1\,,$$

that is, the circle with centre $z = 1$ and radius $1$.

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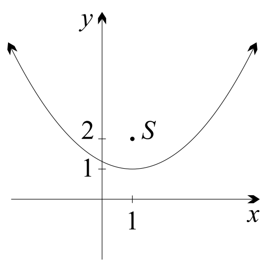
> CIRCLES IN THE ARGAND DIAGRAM: Let $z_0$ be the fixed point $C$ in the Argand diagram, and let $z$ a variable point $P$. If
>
> $$|z-z_0| = r$$
>
> then the locus of $P$ is the circle with centre $C$ and radius $r$.

**WORKED EXERCISE:** Let $S$ be the fixed point $1+2i$ and $z$ be the variable point $P$ in the Argand diagram.

(a) Describe the locus of $P$ given that $|z-(1+2i)| = \mathrm{Im}(z)$.

(b) Confirm your answer algebraically by putting $z = x+iy$.

**SOLUTION:** (a) Clearly $\mathrm{Im}(z) \geq 0$ so is the distance to the real axis. Further, $|z-(1+2i)|$ is the distance $PS$. Thus $P$ is equidistant from a point and a line. That is, the locus of $P$ is a parabola. The focus is $S$ and the directrix is the real axis, hence the vertex is $1+i$ and the focal length is 1.




(b) Squaring both sides of the given equation

$$(x-1)^2 + (y-2)^2 = y^2$$

so

$$\begin{aligned} (x-1)^2 &= y^2 - (y-2)^2 \\ &= 4y - 4 \end{aligned}$$

thus $$(x-1)^2 = 4(y-1)\,.$$
This is the equation of a parabola with vertex $1+i$ and focal length 1, as before.

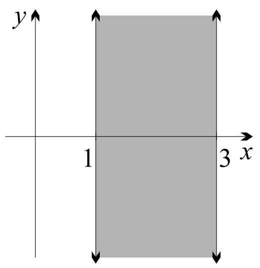
**WORKED EXERCISE:** Given that $z^2 - \overline{z^2} = 4i$, put $z = x+iy$ and hence determine the corresponding locus in the complex plane.

**SOLUTION:** The left hand side of the given equation is $2i\,\mathrm{Im}(z^2)$ so
$$2i\,\mathrm{Im}(z^2) = 4i$$
or $$\mathrm{Im}(z^2) = 2$$
thus $$2xy = 2\,.$$
Hence the locus is the rectangular hyperbola
$$xy = 1\,.$$

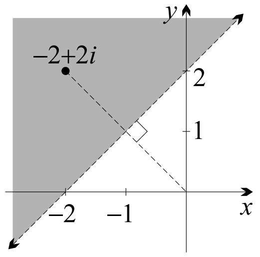
**Regions:** In many instances a curve divides the plane into two or more regions. In simple cases a region is defined by the corresponding inequation. Two or more inequations will result in the union or intersection of the regions. Some common examples follow.

**WORKED EXERCISE:** Sketch the following loci:
(a) $1 \le \mathrm{Re}(z) \le 3$
(b) $|z| > |z+2-2i|$
(c) $|z-2+i| < 1$
(d) $0 \le \arg(z) \le \frac{\pi}{4}$

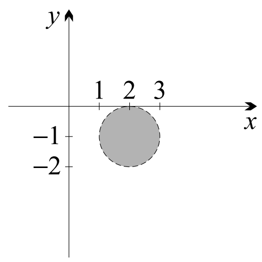
**SOLUTION:** The first three can be easily explained geometrically.

(a)


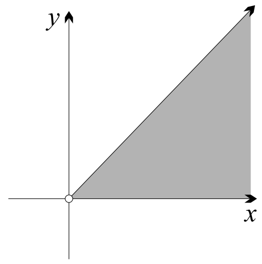

This is $1 \le x \le 3$, the vertical strip between $x = 1$ and $x = 3$.

(b)



The perpendicular bisector of the segment from 0 to $-2+2i$ is the boundary, and is not included. The region includes the point $-2+2i$, since the RHS of the inequality is zero there.

(c)



The boundary curve is the circle with radius 1 and centre $2-i$, which is not included. The region includes the centre of the circle since the LHS of the inequality is zero there.

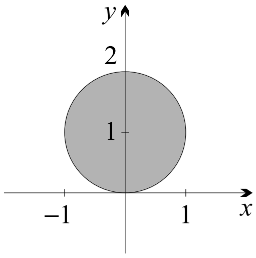
(d)



Put $z = r\,\mathrm{cis}\,\theta$ to get $0 \le \theta \le \frac{\pi}{4}$, which defines a wedge excluding the origin, since $\arg(0)$ is undefined.

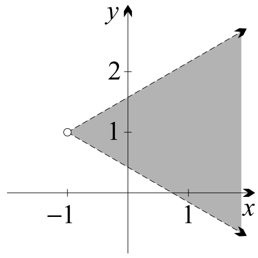
**WORKED EXERCISE:**
(a) Sketch the regions (i) $|z-i| \le 1$ and (ii) $-\frac{\pi}{6} < \arg(z+1-i) < \frac{\pi}{6}$.
(b) Hence sketch (i) the union and (ii) the intersection of these regions.

**SOLUTION:**

(a) (i)

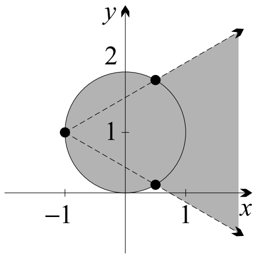



(ii)

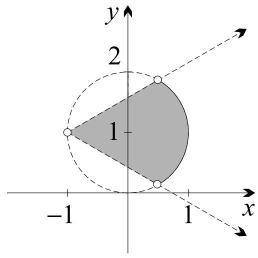



(b) The boundaries intersect at $-1+i$ and, from trigonometry, they interesect again at $\frac{1}{2}+i\,(1+\frac{\sqrt{3}}{2})$ and $\frac{1}{2}+i\,(1-\frac{\sqrt{3}}{2})$. Here are the graphs.

(i)




(ii)




**Loci and Circle Geometry:** Many of the circle geometry theorems encountered in the Mathematics Extension 1 course may be expressed in terms of the locus of a complex number. One significant result is included here, with other examples to be found in the exercise.

**WORKED EXERCISE:** Let $z_1 = 3+i$ and $z_2 = 1-i$.

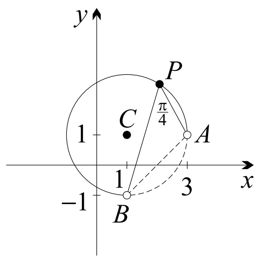
(a) Describe and sketch the locus of $z$, where $\arg\left(\dfrac{z-z_1}{z-z_2}\right) = \frac{\pi}{4}$.

(b) What happens to the locus if $z_1$ and $z_2$ are swapped?

**SOLUTION:**

(a) Let $z_1$, $z_2$ and $z$ represent the points $A$, $B$ and $P$ respectively. The given equation indicates that the angle between the vectors $AP$ and $BP$ is fixed, that is the angle at $P$ subtended by $AB$ is $\frac{\pi}{4}$. Using the converse of the angles in the same segment theorem, it follows that $P$ must lie on the arc of a circle with chord $AB$. Since

$$\angle APB = \tfrac{\pi}{4} < \tfrac{\pi}{2}$$



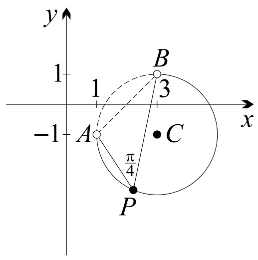

it is a major arc. Since $\arg(z-z_1) > \arg(z-z_2)$ the arc is taken anticlockwise from $A$ to $B$. Lastly, since $\arg(0)$ is undefined, the endpoints of the arc are not included. It simply remains to find the centre and radius of this circle. Let $C$ be the centre of the circle then

$\angle ACB = \frac{\pi}{2}$ (Angles at the centre and circumference)

whence $\angle CAB = \frac{\pi}{4}$ (base angles of isosceles triangle.)

Since $\arg(z_1 - z_2) = \frac{\pi}{4}$

it follows that $AC$ is horizontal and $BC$ is vertical. Thus $C = 1+i$ is the centre of the circle and $AC = 2$ is its radius.




(b) Since the points $A$ and $B$ have been swapped the horizontal line $AC$ and the vertical line $BC$ intersect at a different point, namely the new centre $C = 3-i$. Effectively the locus has been rotated by $\pi$ about the mid-point of $AB$.

**31** THE ARC OF A CIRCLE: Let points $A$ and $B$ represent the complex numbers $z_1$ and $z_2$. Let the variable point $P$ represent $z$. The equation

$$\arg\left(\frac{z-z_1}{z-z_2}\right) = \alpha$$

implies that the angle at $P$ subtended by $AB$ is $\alpha$. Thus the locus of $P$ is the arc $AB$ of a circle, taken anticlockwise. The end points of the arc are excluded.

## Exercise **1F**

**1.** Sketch these straight lines by using the result of Box 28.

(a) $|z+3| = |z-5|$ (b) $|z-i| = |z+1|$ (c) $|z+2-2i| = |z|$ (d) $|z-i| = |z-4+i|$

**2.** Graph the rays specified in the following equations. Box 29 may be of help.

(a) $\arg(z-4) = \frac{3\pi}{4}$ (b) $\arg(z+1) = \frac{\pi}{4}$ (c) $\arg(z-1-i\sqrt{3}) = \frac{\pi}{3}$

**3.** Use Box 30 to sketch these circles.

(a) $|z+1-i| = 1$ (b) $|z-3-2i| = 2$ (c) $|z-1+i| = \sqrt{2}$

**4.** In each case determine the locus of the corresponding boundary equations and hence sketch the indicated region.

(a) $|z-8i| \ge |z-4|$
(b) $|z-2+i| \le |z-4+i|$
(c) $|z+1-i| \ge |z-3+i|$
(d) $0 \le \arg(z) \le \frac{3\pi}{4}$
(e) $-\frac{\pi}{3} < \arg(z) < \frac{\pi}{6}$
(f) $-\frac{\pi}{4} \le \arg(z+2+i) < \frac{\pi}{4}$
(g) $|z| > 2$
(h) $|z+2i| \le 1$
(i) $1 < |z-2+i| \le 2$

**DEVELOPMENT**

**5.** In each case sketch (i) the intersection and (ii) the union of the given pair of regions.

(a) $|z-2+i| \le 2$, $\text{Im}(z) \ge 0$
(b) $0 \le \text{Re}(z) \le 2$, $|z-1+i| \le 2$
(c) $|z-\overline{z}| < 2$, $|z-1| \ge 1$
(d) $\text{Re}(z) \le 4$, $|z-4+5i| \le 3$
(e) $|z-1-i| \le 2$, $0 \le \arg(z-1-i) \le \frac{\pi}{4}$
(f) $|z| \le 1$, $0 \le \arg(z+1) \le \frac{\pi}{4}$
(g) $|z+1-2i| \le 3$, $-\frac{\pi}{3} \le \arg z \le \frac{\pi}{4}$
(h) $|z-3-i| \le 5$, $|z+1| \le |z-1|$

**6.** Put $z = x+iy$ to help sketch these hyperbolae.

(a) $z^2 - (\overline{z})^2 = 16i$ (b) $z^2 - (\overline{z})^2 = 12i$

**7.** In each case the given equation is that of a parabola. (i) Show this algebraically by putting $z = x+iy$, and hence draw each parabola. (ii) Use the fact that $|z|$ is a distance to help determine the focus and directrix.

(a) $|z-3i| = \text{Im}(z)$ (b) $|z+2| = -\text{Re}(z)$ (c) $|z| = \text{Re}(z+2)$ (d) $|z-i| = \text{Im}(z+i)$

**8.** By putting $z = x + iy$ or otherwise, determine the locus specified by:

(a) $\mathrm{Im}(z) = |z|$ (b) $\mathrm{Re}\left(1 - \dfrac{4}{z}\right) = 0$ (c) $\mathrm{Re}\left(1 - \dfrac{1}{z}\right) = 0$

**9.** Determine the arcs specified by the following equations. Sketch each one, showing the centre and radius of the associated circle.

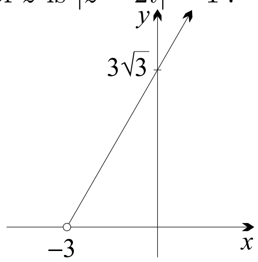
(a) $\arg\left(\dfrac{z-2}{z}\right) = \dfrac{\pi}{2}$ (c) $\arg\left(\dfrac{z-i}{z+i}\right) = \dfrac{\pi}{4}$ (e) $\arg\left(\dfrac{z-2i}{z+2i}\right) = \dfrac{\pi}{6}$

(b) $\arg\left(\dfrac{z-1+i}{z-1-i}\right) = \dfrac{\pi}{2}$ (d) $\arg\left(\dfrac{z+1}{z-3}\right) = \dfrac{\pi}{3}$ (f) $\arg\left(\dfrac{z}{z+4}\right) = \dfrac{3\pi}{4}$

**10.** A complex number $z$ satisfies $\arg z = \frac{\pi}{3}$.

(a) Use a diagram to show that $|z-2i| \geq 1$. (b) For which value of $z$ is $|z - 2i| = 1$?

**11.** The diagram on the right show the locus of a variable point $P$ which represents the complex number $z$.




(a) Write down an equation for this locus.

(b) Find the modulus and argument of $z$ at the point where $|z|$ takes its minimum value.

(c) Hence find $z$ in Cartesian form when $|z|$ takes its least value.

**12.** (a) A complex number $z$ satisfies $|z-1| = 2$. Draw a diagram and hence find the greatest and least possible values of $|z|$.

(b) If $z$ is a complex number such that $\mathrm{Re}(z) \leq 2$ and $|z - 3| = 2$, show with the aid of a diagram that $1 \leq |z| \leq \sqrt{7}$.

**13.** (a) The variable complex number $z$ satisfies $|z - 2 - i| = 1$. Use a diagram to find the maximum and minimum values of: (i) $|z|$, (ii) $|z - 3i|$.

(b) A complex number $z$ satisfies $|z| = 3$. Use a sketch to find the greatest and least values of $|z + 5 - i|$.

(c) The variable complex number $z$ satisfies $|z - z_0| = r$. Use a similar approach to parts (a) and (b) to find the maximum and minimum values of: (i) $|z|$, (ii) $|z-z_1|$.

(d) Confirm your answers to the previous parts by using the triangle inequality.

**14.** (a) A complex number $z$ satisfies $|z - 2| = 1$.

(i) Sketch the locus of $z$. (ii) Show that $-\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{6}$.

(b) The complex number $z$ is such that $|z| = 1$. Use your answers to part (a) to explain why $-\frac{\pi}{6} \leq \arg(z + 2) \leq \frac{\pi}{6}$.

**15.** The variable complex number $w$ satisfies $|w| = 10$ and $0 \leq \arg w \leq \frac{\pi}{2}$. The variable complex number $z$ is given by $z = 3 + 4i + w$.

(a) Sketch the locus of $z$.

(b) Use your diagram to determine the maximum value of $|z|$.

(c) What is the value of $z$ for which this maximum occurs?

**16.** (a) Show that the circle equation $|z - z_0| = r$ is equivalent to

$$z\,\overline{z} - (z\,\overline{z_0} + \overline{z}\,z_0) + z_0\,\overline{z_0} - r^2 = 0\,.$$

[HINT: Square both sides of $|z - z_0| = r$ and use the result $|w|^2 = w\,\overline{w}$.]

(b) Use the result of part (a) to help identify these circles.

(i) $z\,\overline{z}+2(z+\overline{z}) = 0$ (ii) $z\,\overline{z}-(1+i)\overline{z}-(1-i)z+1 = 0$ (iii) $\dfrac{1}{z} + \dfrac{1}{\overline{z}} = 1$

**17.** Find the locus of $z$ if the value of $\dfrac{z-1}{z-i}$ is: (a) real, (b) imaginary.

**18.** Sketch the locus of $z$ given that: (a) $\arg(z+i) = \arg(z-1)$, (b) $\arg(z+i) = \arg(z-1)+\pi$.

**19.** Suppose that $|z| = 1$ and that $\arg z = 2\theta$, where $0 < \theta < \frac{\pi}{2}$. Show that:

(a) $|z^2 - z| = |z - 1|$ (b) $\arg(z^2 - z) = \frac{\pi}{2} + 3\theta$

**EXTENSION**

**20.** It is known that the locus specified by $\arg\left(\dfrac{z-z_1}{z-z_2}\right) = \alpha$ is an arc taken anticlockwise from $z_1$ to $z_2$. What else can be said about the locus when:

(a) $\alpha = 0$ (b) $0 < \alpha < \frac{\pi}{2}$ (c) $\alpha = \frac{\pi}{2}$ (d) $\frac{\pi}{2} < \alpha < \pi$ (e) $\alpha = \pi$

**21.** Put $z = x + iy$ to help sketch the hyperbola $z^2 + (\overline{z})^2 = 2$.

**22.** [CIRCLE GEOMETRY] Let $z_1$ and $z_2$ be two fixed points in the Argand diagram, and for simplicity suppose that $0 < \arg(z_2 - z_1) < \pi$. The variable point $z$ satisfies the equation

$$\arg\left(\frac{z-z_2}{z-z_1}\right) = \arg(z_2 - z_1)\,.$$

(a) Use a theorem in circle geometry to help determine the locus of $z$.

(b) Investigate the situation if the restriction on $\arg(z_2 - z_1)$ is removed.

**23.** [CONICS] Let $a$ and $e$ be two positive real numbers, with $0 < e < 1$. Describe the locus of the points $z$ in the complex plane which satisfy $|z - ae| + |z + ae| = 2a$.

**24.** [VERY DIFFICULT] Suppose that $k|z - z_1| = \ell|z - z_2|$, where $k \neq \ell$ and both are positive real numbers.

(a) Show that the locus of $z$ in the Argand diagram is a circle with centre $\dfrac{k^2 z_1 - \ell^2 z_2}{k^2 - \ell^2}$ and radius $\dfrac{kl|z_2 - z_1|}{|k^2 - \ell^2|}$,

(i) by letting $z = x + iy$, (ii) by geometric means.

(b) What happens in the limit as $k$ approaches $\ell$?

# Chapter One

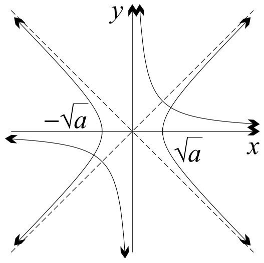
## Exercise 1A (Page 8)

**1(a)** $-1$ **(b)** $1$ **(c)** $-i$ **(d)** $i$
**(e)** $i$ **(f)** $-1$ **(g)** $1$ **(h)** $0$
**2(a)** $-2i$ **(b)** $3-i$ **(c)** $1+i$ **(d)** $5+3i$ **(e)** $-3-2i$
**3(a)** $12-2i$ **(b)** $-6+2i$ **(c)** $1+5i$ **(d)** $7-11i$
**4(a)** $-5+4i$ **(b)** $5+5i$ **(c)** $14+5i$ **(d)** $-26+82i$
**(e)** $24+10i$ **(f)** $-5-12i$ **(g)** $2+11i$ **(h)** $-4$
**(i)** $28-96i$
**5(a)** $5$ **(b)** $17$ **(c)** $29$ **(d)** $65$
**6(a)** $-i$ **(b)** $1-2i$ **(c)** $3+2i$ **(d)** $1-2i$ **(e)** $-1+3i$
**(f)** $-\frac{1}{5}+\frac{3}{5}i$
**7(a)** $-2-i$ **(b)** $4-3i$ **(c)** $3+7i$ **(d)** $3$ **(e)** $-3+4i$
**8(a)** $6+2i$ **(b)** $18$ **(c)** $19-22i$ **(d)** $8-i$ **(e)** $1+2i$
**9(a)** $22+19i$ **(b)** $6+15i$ **(c)** $4-2i$ **(d)** $2-3i$
**(e)** $6$
**10(a)** $x=3$ and $y=-2$ **(b)** $x=2$ and $y=-1$
**(c)** $x=6$ and $y=2$ **(d)** $x=\frac{14}{5}$ and $y=\frac{3}{5}$
**(e)** $x=\frac{35}{2}$ and $y=-\frac{39}{2}$
**11(a)** $\frac{9}{10}-\frac{13}{10}i$ **(b)** $1$ **(c)** $-\frac{8}{29}$ **(d)** $-4-\frac{5}{2}i$
**16(a)** $\frac{x-iy}{x^2+y^2}$ **(b)** $\frac{x^2-y^2-2ixy}{(x^2+y^2)^2}$ **(c)** $\frac{x^2+y^2-1+2iy}{(x+1)^2+y^2}$

## Exercise 1B (Page 15)

**1(a)** $z=\pm 3i$ **(b)** $z=2\pm 4i$ **(c)** $z=-1\pm 2i$
**(d)** $z=3\pm i$ **(e)** $z=\frac{1}{2}\pm\frac{1}{4}i$ **(f)** $z=-\frac{3}{2}\pm 2i$
**2(a)** $(z-6i)(z+6i)$ **(b)** $(z-2\sqrt{2}\,i)(z+2\sqrt{2}\,i)$
**(c)** $(z-1-3i)(z-1+3i)$ **(d)** $(z+2-i)(z+2+i)$
**(e)** $(z-3+\sqrt{5}\,i)(z-3-\sqrt{5}\,i)$ **(f)** $(z+\frac{1}{2}-\frac{\sqrt{3}}{2}i)(z+\frac{1}{2}+\frac{\sqrt{3}}{2}i)$
**3(a)** $z^2+2=0$ **(b)** $z^2-2z+2=0$ **(c)** $z^2+2z+5=0$ **(d)** $z^2-4z+7=0$
**4(a)** $\pm(1+i)$ **(b)** $\pm(2+i)$ **(c)** $\pm(-1+3i)$ **(d)** $\pm(6+i)$ **(e)** $\pm(2+3i)$ **(f)** $\pm(5-i)$ **(g)** $\pm(1-4i)$
**(h)** $\pm(5-4i)$
**5(a)** $\pm(1-2i)$ **(b)** $z=2-i$ or $1+i$
**6(a)** $\pm(1+3i)$ **(b)** $z=4+i$ or $3-2i$
**7(a)** $z=1-i$ or $i$ **(b)** $z=-3+2i$ or $-2i$ **(c)** $z=4+i$ or $2-i$ **(d)** $z=-2+i$ or $\frac{1}{2}(3-i)$ **(e)** $z=-5+i$ or $3-2i$ **(f)** $z=3+i$ or $-1-3i$
**8(a)** $w=-1$ **(b)** $a=-6$ and $b=13$ **(c)** $k=8-i$ and the other root is $2+3i$.
**9** $z=\pm(2+i)$
**10(a)** $\cos\theta+i\sin\theta$ or $\cos\theta-i\sin\theta$
**11(a)** $z=-1$ or $\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$ **(b)** $z=i$ or $\pm\frac{\sqrt{3}}{2}-\frac{1}{2}i$
**12(a)** $x=\omega$ satisfies the equation. **(c)** They are complex conjugates.

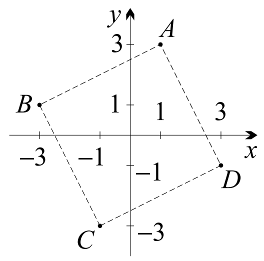
**13(a)** $\overline{\alpha}$
**14(a)(i)**




**15(a)** $\pm\frac{1}{\sqrt{2}}(1-i)$ **(b)** $\pm\sqrt{2}(1+2i)$ **(c)** $\pm(\sqrt{3}+i)$
**(d)** $\pm\sqrt{2}(3-2i)$
**(e)** $\pm\left(\sqrt{\sqrt{5}+1}-i\sqrt{\sqrt{5}-1}\right)$
**16(a)** $-2-i\pm\left(\sqrt{\sqrt{2}+1}+i\sqrt{\sqrt{2}-1}\right)$
**(b)** $1+i\pm\left(\sqrt{\sqrt{5}-1}-i\sqrt{\sqrt{5}+1}\right)$
**(c)** $-1+i\sqrt{3}\pm\left(\sqrt{2}-i\sqrt{6}\right)$
**(d)** $\frac{1}{2}\left(-1+i\pm\left(\sqrt{\sqrt{13}+2}-i\sqrt{\sqrt{13}-2}\right)\right)$

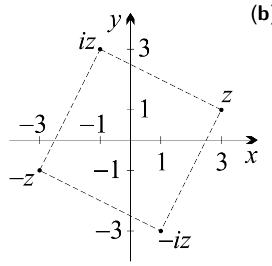
## Exercise 1C (Page 20)

**1(a)** $(2,0)$ **(b)** $(0,1)$ **(c)** $(-3,5)$ **(d)** $(2,-2)$
**(e)** $(-5,-5)$ **(f)** $(-1,2)$
**2(a)** $-3+0i=-3$ **(b)** $0+3i=3i$ **(c)** $7-5i$
**(d)** $a+bi$
**3(a)**

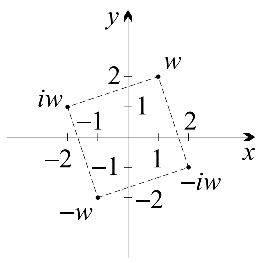



**(b)** A square. **(c)** An anticlockwise rotation of $90°$ about the origin.

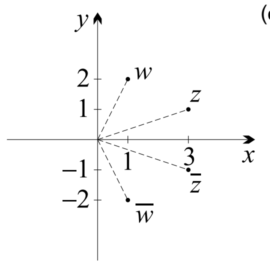
**4(a)**



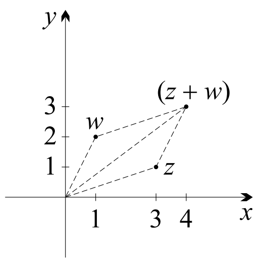

**(b)**



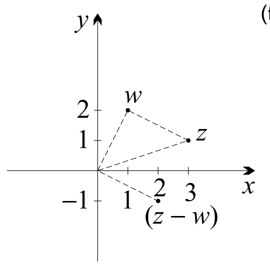
**(c)**



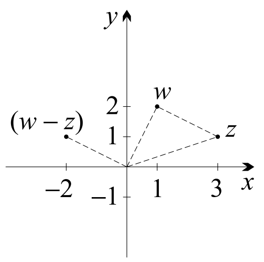

**(d)**



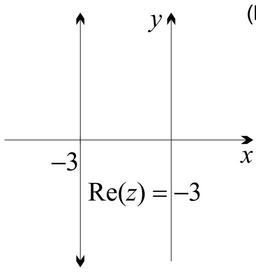

**(e)**



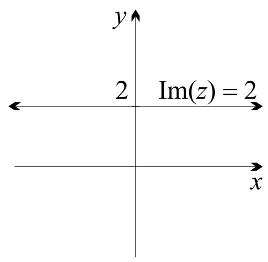

**(f)**



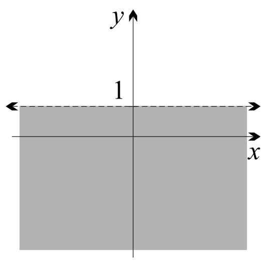

**5(a)**



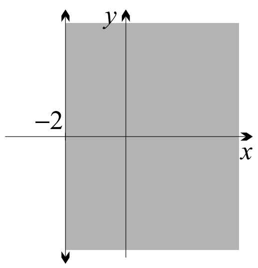

**(b)**



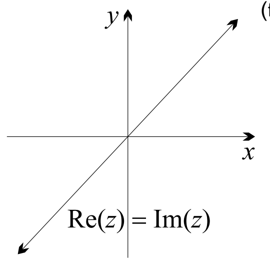

**(c)**



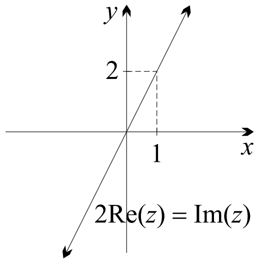

**(d)**



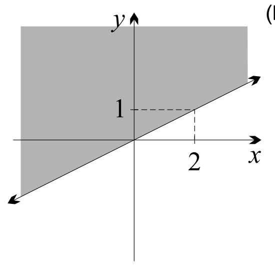

**(e)**



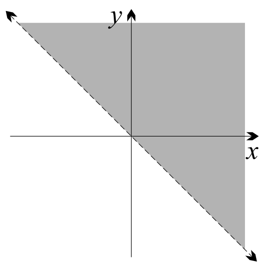

**(f)**



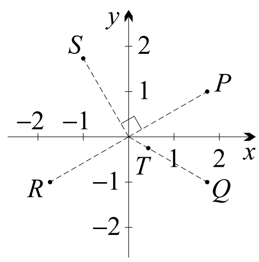

**(g)**




**(h)**




**6**



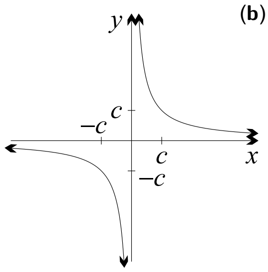

**10(c)** right-isosceles

**11** It is the circle centre $(0, -1)$ with radius 1, omitting the origin.

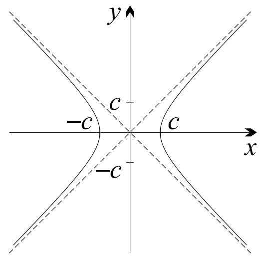
**12** It is the circle centre $(3, 0)$ with radius, omitting the origin.

**14** It is a parabola with focus the origin and directrix $x = 1$.

**15(a)**



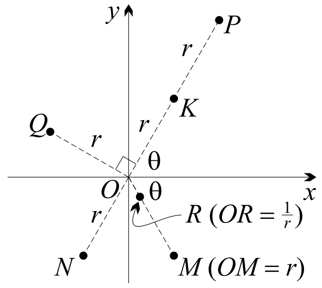

**(b)**



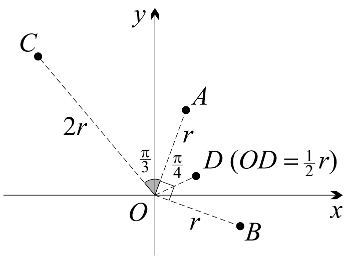

## Exercise **1D** (Page 26)

**1(a)** $2\,\mathrm{cis}\,\frac{\pi}{2}$ **(b)** $4\,\mathrm{cis}\,\pi$ **(c)** $\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}$ **(d)** $2\,\mathrm{cis}\left(-\frac{\pi}{6}\right)$
**(e)** $2\,\mathrm{cis}\,\frac{2\pi}{3}$ **(f)** $\mathrm{cis}\left(-\frac{3\pi}{4}\right)$
**2(a)** $5\,\mathrm{cis}(0{\cdot}93)$ **(b)** $13\,\mathrm{cis}(-0{\cdot}39)$
**(c)** $\sqrt{5}\,\mathrm{cis}(2{\cdot}68)$ **(d)** $\sqrt{10}\,\mathrm{cis}(-1{\cdot}89)$
**3(a)** $3$ **(b)** $-5i$ **(c)** $2\sqrt{2} + 2\sqrt{2}\,i$ **(d)** $3\sqrt{3} - 3i$
**(e)** $-\sqrt{2} + \sqrt{2}\,i$ **(f)** $-1 - \sqrt{3}\,i$
**4(a)** $\sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$ **(b)** $\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}$ **(c)** $\sqrt{2}\,\mathrm{cis}\,\frac{3\pi}{4}$
**(d)** $\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}$ **(e)** $2\,\mathrm{cis}\left(-\frac{\pi}{2}\right)$ **(f)** $\frac{1}{\sqrt{2}}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$
**5(a)** $10\,\mathrm{cis}\,\frac{\pi}{3}$ **(b)** $9\,\mathrm{cis}\,3\theta$ **(c)** $2\,\mathrm{cis}\,\frac{\pi}{3}$ **(d)** $\frac{3}{2}\,\mathrm{cis}\,\alpha$
**(e)** $16\,\mathrm{cis}\,\frac{2\pi}{5}$ **(f)** $8\,\mathrm{cis}\,\frac{6\pi}{7}$

**6**



**7**




**9(a)** $z_1 = 2\operatorname{cis}\frac{\pi}{6}$ and $z_2 = 4\operatorname{cis}\frac{\pi}{4}$ **(b)** $z_1z_2 = 8\operatorname{cis}\frac{5\pi}{12}$ and $\frac{z_2}{z_1} = 2\operatorname{cis}\frac{\pi}{12}$
**10** $z_1 = 2\operatorname{cis}\frac{5\pi}{6}$, $z_2 = \sqrt{2}\operatorname{cis}(-\frac{3\pi}{4})$,
$z_1z_2 = 2\sqrt{2}\operatorname{cis}\frac{\pi}{12}$ and $\frac{z_2}{z_1} = \frac{\sqrt{2}}{2}\operatorname{cis}\frac{5\pi}{12}$
**11(a)** $\frac{1}{2}\Big((\sqrt{3}+1) + i(\sqrt{3}-1)\Big)$ **(b)** $\sqrt{2}\operatorname{cis}\frac{\pi}{12}$
**(c)** $\frac{1}{2\sqrt{2}}(\sqrt{3}+1)$
**12(b)** $2\sqrt{2}\operatorname{cis}\frac{\pi}{12}$
**13(a)** $\sqrt{2}$ **(b)** $\frac{\pi}{4}$ **(c)** $1+i$
**20** $z + w = 2\cos\left(\frac{\theta-\phi}{2}\right)\operatorname{cis}\left(\frac{\theta+\phi}{2}\right)$
**21(c)** The tangents at $z_0$ and $z_1$ to the circle with centre the origin meet at $z_2$.
**22(a)** When $\operatorname{Im}(z) = 0$.

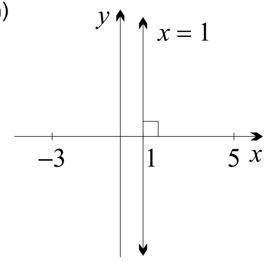
## Exercise **1E** (Page 32)

**1(a)** $7+4i$ **(b)** $-3+2i$ **(c)** $3-2i$
**2(a)** $-3+4i$ **(b)** $1+7i$ **(c)** $-4-3i$ **(d)** $-7+i$
**3** $-3+6i$
**4(a)** $B$ represents $1+3i$, $C$ represents $-1+2i$
**(b)** $-\sqrt{2}+2\sqrt{2}\,i$
**5(a)** $4+3i$ **(b)** $-3+4i$ **(c)** $2+7i$
**6(a)** $-5+12i$ **(b)** $-3-4i$
**8** $E$ represents $w_2 - w_1$, $F$ represents $i(w_2 - w_1)$, $C$ represents $w_2 + i(w_2 - w_1)$ and $D$ represents $w_1 + i(w_2 - w_1)$.
**9(a)** Vectors $BA$ and $BC$ represent $z_1 - z_2$ and $z_3 - z_2$ respectively, and $BA$ is the anticlockwise rotation of $BC$ through $90°$ about $B$. So $z_1 - z_2 = i(z_3 - z_2)$. Squaring both sides gives the result.
**(b)** $z_1 - z_2 + z_3$
**10(a)** $2\omega i$ **(b)** $\frac{1}{2}\omega(1+2i)$
**11** $-2$ and $1-\sqrt{3}\,i$
**12(a)** $w = -4+3i$ or $4-3i$ **(b)** $w = -1+7i$ or $7+i$ **(c)** $w = \frac{1}{2}(7+i)$ or $\frac{1}{2}(-1+7i)$
**13(a)** $1-5i$, $7+3i$ **(b)** $3+6i$, $-3-2i$ **(c)** $\frac{7}{2}+\frac{5}{2}i$, $\frac{1}{2}-\frac{3}{2}i$
**14** $-2+2i$, $12i$, $4$
**19(a)** $z_1 = 2\operatorname{cis}\frac{\pi}{2}$, $z_2 = 2\operatorname{cis}\frac{\pi}{3}$ **(c)(i)** $\frac{5\pi}{12}$ **(ii)** $\frac{11\pi}{12}$

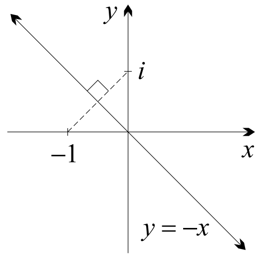
**22(c)** The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.
**23(c)** parallelogram **(d)** $\arg\frac{w}{z} = \frac{\pi}{2}$, so $\frac{w}{z}$ is purely imaginary.
**26** Use the converse of the opposite angles of a cyclic quadrilateral.

## Exercise **1F** (Page 41)

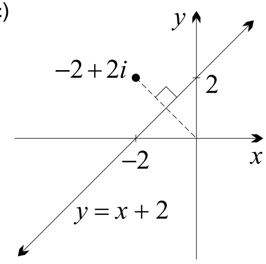
**1(a)**



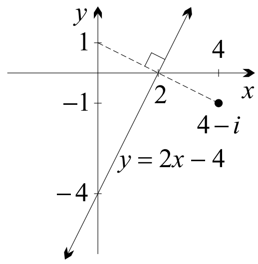

**(b)**



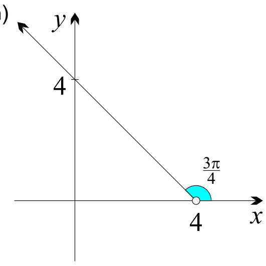

**(c)**



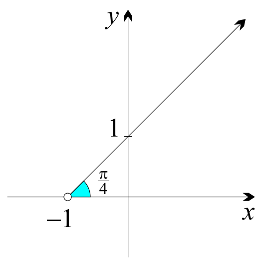

**(d)**



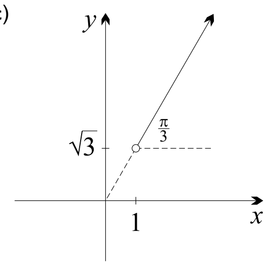

**2(a)**



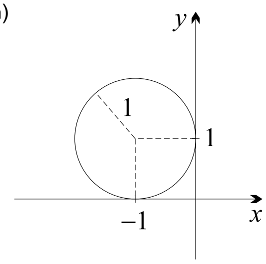

**(b)**



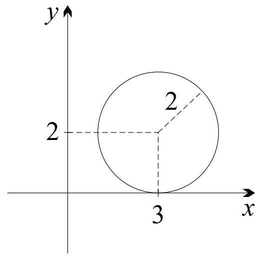

**(c)**




**3(a)**

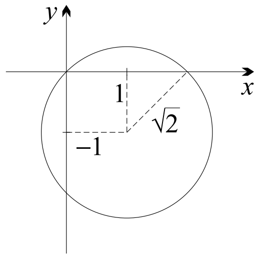



**(b)**

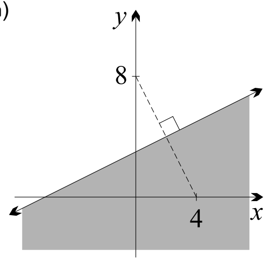


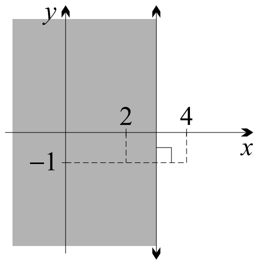
**(c)**



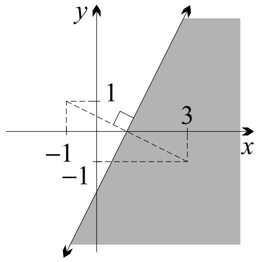

**4(a)**



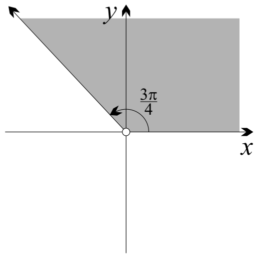

**(b)**



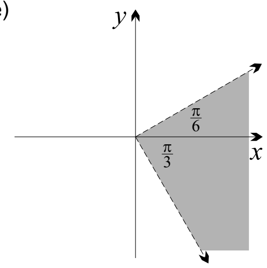

**(c)**




**(d)**




**(e)**

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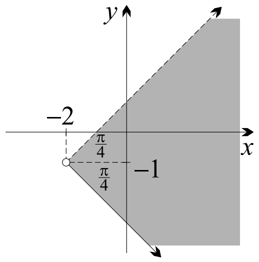
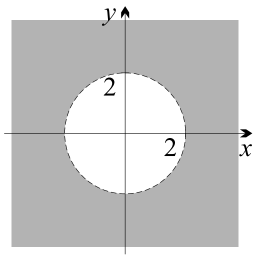
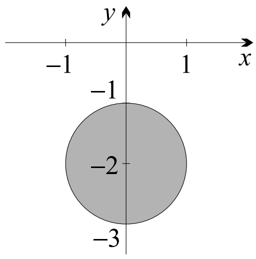


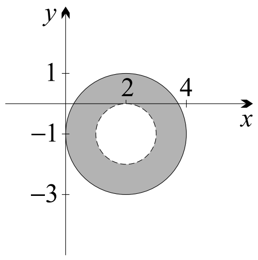

**(g)**



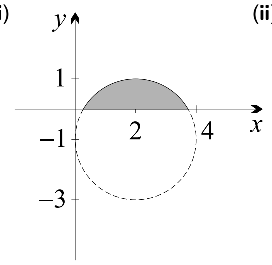

**(h)**



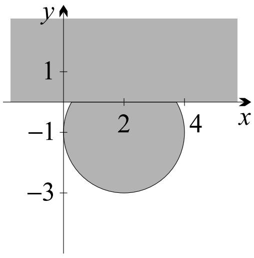

**(i)**



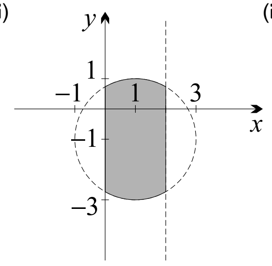

**5(a)(i)**



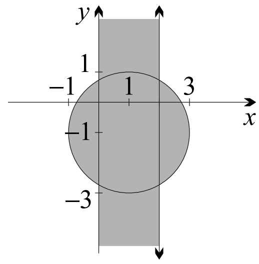

**(ii)**



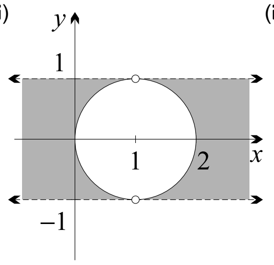

**(b)(i)**




**(ii)**




**(c)(i)**

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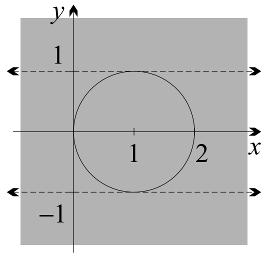
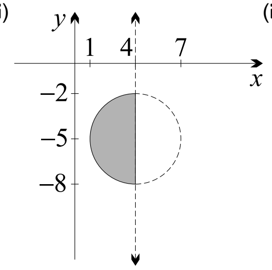
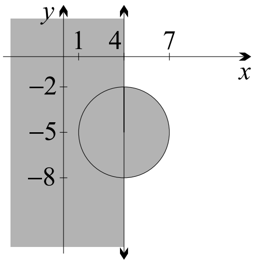


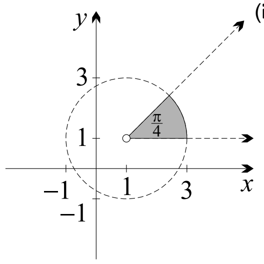

**(d)(i)**




**(ii)**




**(e)(i)**

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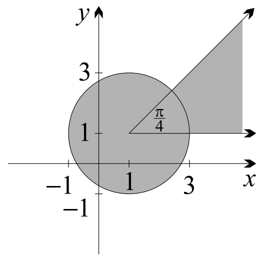
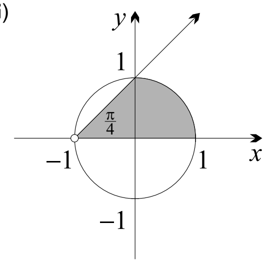
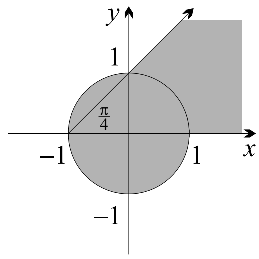


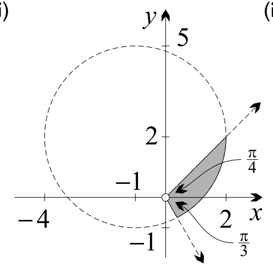

**(f)(i)**



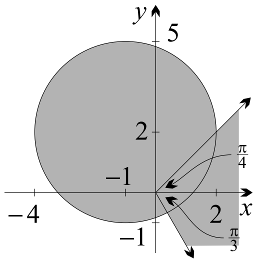

**(ii)**



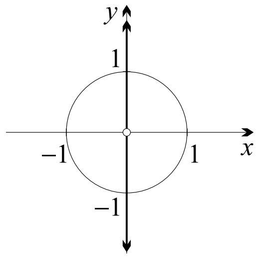
**(g)(i)**



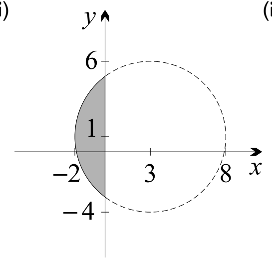

**(ii)**



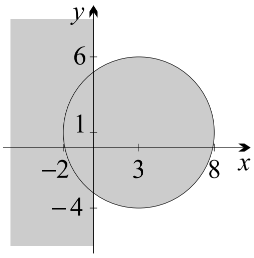

**(c)**



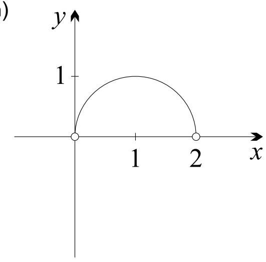

**(h)(i)**



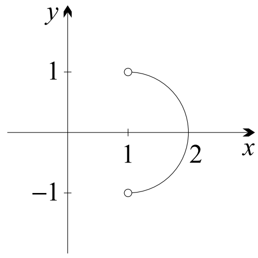

**(ii)**



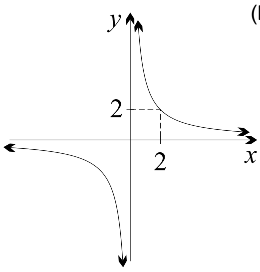

**9(a)**




**(b)**




**6(a)**

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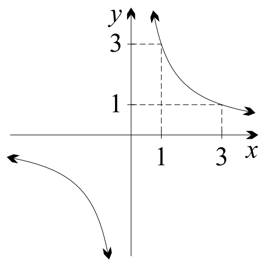
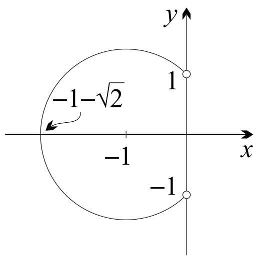
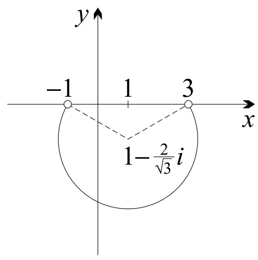


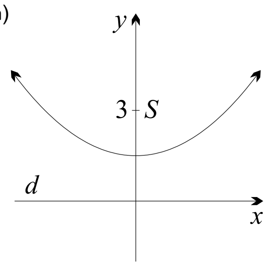

**(c)**



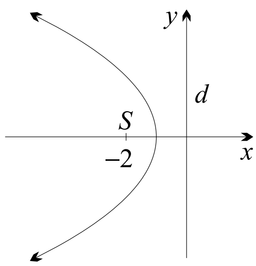

**(d)**



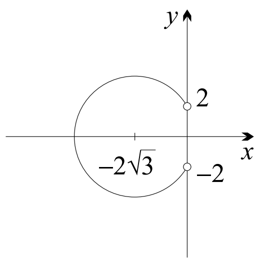

**7(a)**



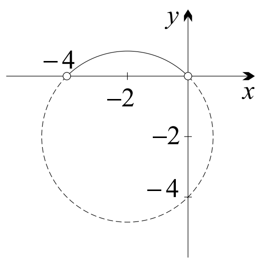

**(b)**



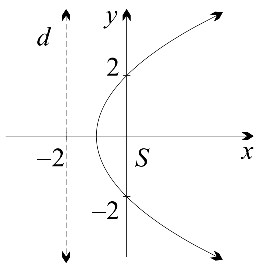

**(e)**



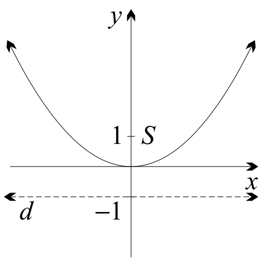

**(f)**




**(c)**




**(d)**

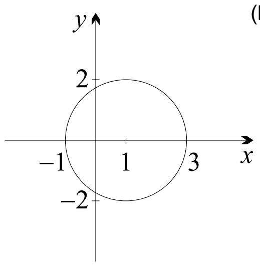



**10(b)** $\sqrt{3}\,\mathrm{cis}\,\frac{\pi}{3} = \frac{\sqrt{3}}{2}(1 + i\sqrt{3}\,)$

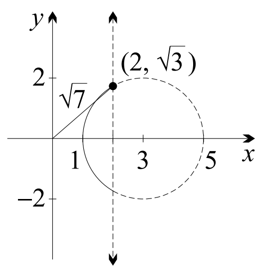
**11(a)** $\arg(z+3) = \frac{\pi}{3}$ **(b)** $|z| = \frac{3\sqrt{3}}{2}$, $\arg z = \frac{5\pi}{6}$

**(c)** $-\frac{9}{4} + \frac{3\sqrt{3}}{4}i$

**12(a)**

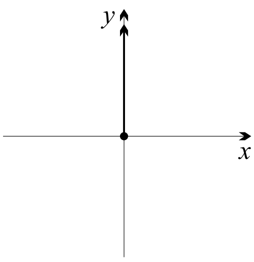



**(b)**

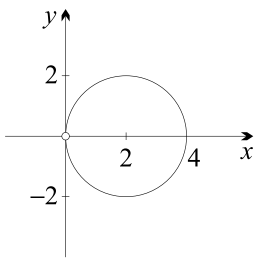



3 and 1

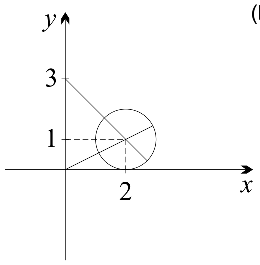
**8(a)**



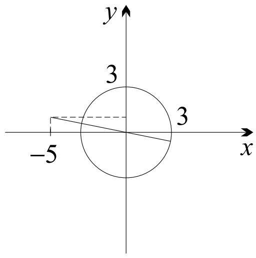

**(b)**



**13(a)**




**(b)**



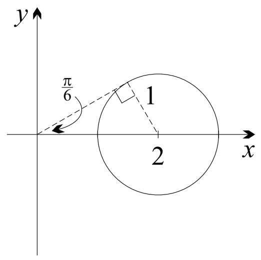

**(i)** $\sqrt{5}+1$ and $\sqrt{5}-1$ $\quad\sqrt{26}+3$ and $\sqrt{26}-3$

**(ii)** $2\sqrt{2}+1$ and $2\sqrt{2}-1$

**(c)(i)** $\Big||z_0|-r\Big| \le |z| \le |z_0|+r$

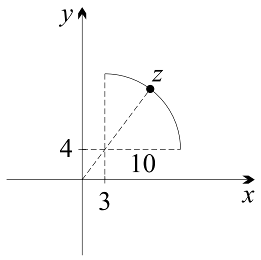
**(ii)** $\Big||z_0-z_1|-r\Big| \le |z-z_1| \le |z_0-z_1|+r$

**14(a)(i)**




**(b)** This is simply part (a) shifted left by 2.

**15(a)**




**(b)** 15 **(c)** $9+12i$

**16(b)(i)** $|z+2|=2$, centre $-2$, radius 2

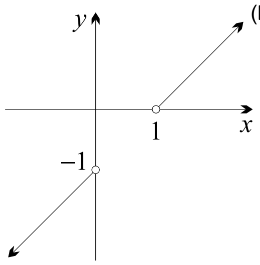
**(ii)** $|z-(1+i)|=1$, centre $1+i$, radius 1

**(iii)** $|z-1|=1$, centre 1, radius 1

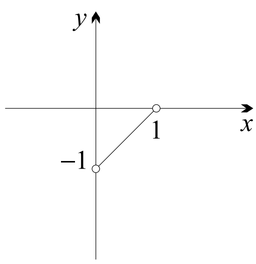
**17(a)** The line through 1 and $i$, omitting $i$.

**(b)** The circle with diameter joining 1 and $i$, omitting these two points.

**18(a)**

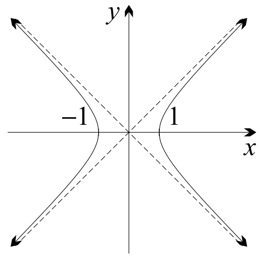



**(b)**




**20(a)** straight line external to $z_1$ and $z_2$ **(b)** major arc **(c)** semi-circle **(d)** minor arc **(e)** straight line between $z_1$ and $z_2$

**21**




**22(a)** Angle in the alternate segment theorem: it is the arc taken anticlockwise from $z_2$ to $z_1$ of the circle tangent to $y=\text{Im}(z_1)$ and through $z_2$.

**23** The ellipse with eccentricity $e$, semi-major axis $a$ and semi-minor axis $b$, where $b^2=a^2(1-e^2)$.

**24(b)** The locus is the perpendicular bisector of the line joining $z_1$ and $z_2$.

CHAPTER **SEVEN**

# De Moivre's Theorem

In the chapter on polynomials it was discovered that there is a close link with complex numbers. In this chapter that link is investigated further by considering specific equations of the type

$$z^n - 1 = 0\,,$$

which have as their solutions the complex roots of 1.

The key to solving these equations is De Moivre's theorem, which is presented in Section A. One consequence of this theorem is that trigonometric identities can be quickly and easily developed, and some common identities are investigated in Section B. The chapter concludes in Section C with the main goal of finding the complex roots of unity, and deducing relationships between those roots.

## 7A De Moivre's Theorem

Recall that when complex numbers are mulitplied the arguments are added, viz:

$$\arg(wz) = \arg(w) + \arg(z)\,.$$

Now put $w = z = \operatorname{cis}\theta$. That is, both are equal and have modulus 1. Then:

$$\begin{aligned} z^2 &= z \times z \\ &= \operatorname{cis}(\theta + \theta) \qquad \text{(adding arguments)} \\ &= \operatorname{cis} 2\theta\,. \end{aligned}$$

Next put $w = z^2$, so that

$$\begin{aligned} z^3 &= z^2 \times z \\ &= \operatorname{cis}(2\theta + \theta) \qquad \text{(again by adding arguments)} \\ &= \operatorname{cis} 3\theta\,. \end{aligned}$$

These initial calculations suggest the simple relationship

$$z^n = \operatorname{cis} n\theta\,,$$

whenever $|z| = 1$, at least for positive integers $n$. In fact the result is true for all integers, which is now proven.

**de Moivre's Theorem:** Let $z = \cos\theta + i\sin\theta$. It can be proven that

$$z^n = \cos n\theta + i \sin n\theta$$

for all integers $n$. The proof is in two parts, beginning with a proof by induction for $n \ge 0$. Conjugates are then used to extend the proof to negative integers.

PROOF: As always with proof by induction, first prove the result true for the starting value.

A. When $n = 0$

$$\text{LHS} = z^0 \qquad\qquad \text{RHS} = \cos 0 + i \sin 0$$
$$= 1\,, \qquad\qquad = 1 + 0i$$
$$= \text{LHS}\,.$$

Hence the statement is true for $n = 0$.

B. Suppose that the result is true for some integer $k \geq 0$, that is

$$z^k = \cos k\theta + i \sin k\theta\,. \qquad (**)$$

Now prove the statement for $n = k + 1$. That is, prove that

$$z^{k+1} = \cos\big((k+1)\theta\big) + i \sin\big((k+1)\theta\big)\,.$$

$$\begin{aligned} \text{LHS} &= z^k \times z \\ &= (\cos k\theta + i \sin k\theta) \times (\cos\theta + i \sin\theta) && \text{(by the hypothesis $(**)$)} \\ &= \cos\big((k+1)\theta\big) + i \sin\big((k+1)\theta\big) && \text{(by the sum of arguments)} \\ &= \text{RHS}\,. \end{aligned}$$

Hence the result is true for $n = k + 1$.

C. It follows from parts A and B by mathematical induction that the statement is true for all integers $n \geq 0$.

D. Next, recall that if $|w| = 1$ then $w^{-1} = \overline{w}$.

Now consider the value of $z^{-n}$ when $n$ is a positive integer.

$$\begin{aligned} z^{-n} &= (z^n)^{-1} \\ &= (\cos n\theta + i \sin n\theta)^{-1} && \text{(by part C.)} \\ &= \overline{(\cos n\theta + i \sin n\theta)} && \text{(since $|\operatorname{cis} n\theta| = 1$)} \\ &= \cos(-n\theta) + i \sin(-n\theta)\,, \end{aligned}$$

and the proof is complete.

1 DE MOIVRE'S THEOREM: Let $z = \cos\theta + i \sin\theta$ be a complex number with modulus 1. Then for all integers $n$,

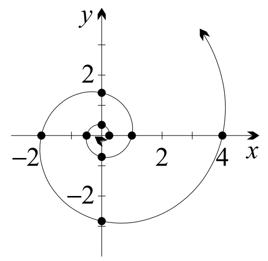
$$z^n = \cos n\theta + i \sin n\theta\,.$$

One immediate consequence of the above theorem is that if $z = r \operatorname{cis}\theta$ then $z^n = r^n \operatorname{cis} n\theta$. Thus if $r > 1$ and $\theta > 0$ then as $n$ increases so too does the modulus and argument of $z^n$. That is, the points representing $z^n$ lie on an anticlockwise spiral.

**WORKED EXERCISE:** Let $z = i\sqrt{2}$. Plot the points corresponding to $z^n$ for values of $n$ in the domain $-4 \leq n \leq 4$, and draw the spiral that these points lie on.

**SOLUTION:** Here is the table for $z^n$.

| $n$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|---|---|---|---|---|
| $z^n$ | $\frac{1}{4}$ | $i\frac{1}{2\sqrt{2}}$ | $-\frac{1}{2}$ | $-i\frac{1}{\sqrt{2}}$ | $1$ | $i\sqrt{2}$ | $-2$ | $-i2\sqrt{2}$ | $4$ |




Notice that in the graph the spiral does not cut the axes at right angles.

A more practical application is to quickly simplify integer powers of complex numbers, as in the following example.

**WORKED EXERCISE:** (a) Write $z = -\sqrt{3} + i$ in modulus-argument form.
(b) Hence express $z^7$ in factored real-imaginary form.

**SOLUTION:** (a) It should be clear that $z = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$.
(b) Using de Moivre's theorem,

$$\begin{aligned} z^7 &= 2^7(\cos\tfrac{5\pi}{6} + i\sin\tfrac{5\pi}{6})^7 \\ &= 128(\cos\tfrac{35\pi}{6} + i\sin\tfrac{35\pi}{6}) \\ &= 128(\cos\tfrac{-\pi}{6} + i\sin\tfrac{-\pi}{6}) \\ &= 64(\sqrt{3} - i)\,. \end{aligned}$$

**WORKED EXERCISE:** For which values of $k$ is $(1+i)^k$ imaginary?

**SOLUTION:** Now $(1+i) = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$
so $(1+i)^k = \sqrt{2^k}(\cos\frac{k\pi}{4} + i\sin\frac{k\pi}{4})$ (by de Moivre)
which is imaginary when $\frac{k\pi}{4}$ is an odd multiple of $\frac{\pi}{2}$.
Thus $\frac{k\pi}{4} = \frac{(2n+1)\pi}{2}$ where $n$ is an integer,
that is $k = 4n + 2\,,$
hence $k = \ldots, -6, -2, 2, 6, 10, \ldots$

## Exercise **7A**

**1.** Write each expression in the form $\operatorname{cis} n\theta$:
(a) $(\cos\theta + i\sin\theta)^5$
(b) $(\cos\theta + i\sin\theta)^{-3}$
(c) $(\cos 2\theta + i\sin 2\theta)^4$
(d) $\cos\theta - i\sin\theta$
(e) $(\cos\theta - i\sin\theta)^{-7}$
(f) $(\cos 3\theta - i\sin 3\theta)^2$

**2.** Simplify as fully as possible:
(a) $\dfrac{(\cos\theta + i\sin\theta)^6(\cos\theta + i\sin\theta)^{-3}}{(\cos\theta - i\sin\theta)^4}$
(b) $\dfrac{(\cos 3\theta + i\sin 3\theta)^5(\cos 2\theta - i\sin 2\theta)^{-4}}{(\cos 4\theta - i\sin 4\theta)^{-7}}$

**3.** Write each expression in the form $a + ib$, where $a$ and $b$ are real:
(a) $(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})^4$
(b) $(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})^3$
(c) $(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})^5$
(d) $(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})^{-2}$
(e) $(\cos\frac{3\pi}{8} - i\sin\frac{3\pi}{8})^{-6}$
(f) $(\cos\frac{5\pi}{12} - i\sin\frac{5\pi}{12})^4$

**4.** (a) Write $1 + i$ in the form $r(\cos\theta + i\sin\theta)$.
(b) Hence, or otherwise, find $(1+i)^{17}$ in the form $a + ib$, where $a$ and $b$ are integers.

**5.** Let $z = 1 + i\sqrt{3}$.
(a) Express $z$ in mod-arg form.
(b) Express $z^{11}$ in the form $a + ib$, where $a$ and $b$ are real.

**6.** Let $z = -\sqrt{3} + i$.
(a) Find the values of $|z|$ and $\arg z$.
(b) Hence, or otherwise, show that $z^7 + 64z = 0$.

**7.** (a) Express $\sqrt{3} - i$ in mod-arg form.
(b) Express $(\sqrt{3} - i)^7$ in mod-arg form.
(c) Hence express $(\sqrt{3} - i)^7$ in the form $x + iy$, where $x$ and $y$ are real.

**8.** (a) Express $-1 - i\sqrt{3}$ in mod-arg form.
(b) Express $(-1 - i\sqrt{3})^5$ in mod-arg form.
(c) Hence express $(-1 - i\sqrt{3})^5$ in the form $x + iy$, where $x$ and $y$ are real.

**9.** (a) Express $z = \sqrt{2} - i\sqrt{2}$ in mod-arg form.
(b) Hence write $z^{22}$ in the form $a + ib$, where $a$ and $b$ are real.

**DEVELOPMENT**

**10.** Show that:
(a) $(1+i)^{10}$ is purely imaginary
(b) $(1 - i\sqrt{3})^9$ is real
(c) $-1 + i$ is a fourth root of $-4$
(d) $-\sqrt{3} - i$ is a sixth root of $-64$

**11.** If $k$ is a multiple of 4, prove that $(-1+i)^k$ is real.

**12.** (a) Find the minimum value of the positive integer $m$ for which $(\sqrt{3}+i)^m$ is:
(i) real, (ii) purely imaginary.
(b) Evaluate $(\sqrt{3}+i)^m$ for each of the above values of $m$.

**13.** (a) Prove that $(1+i)^n + (1-i)^n$ is real for all positive integer values of $n$.
(b) Determine the values of $n$ for which $(1+i)^n + (1-i)^n = 0$.

**14.** Use de Moivre's theorem to prove that:

$$(-\sqrt{3}+i)^n - (-\sqrt{3}-i)^n = 2^{n+1} \sin \tfrac{5\pi n}{6}\, i$$

**15.** (a) Show that if $n$ is divisible by 3 then

$$(1+\sqrt{3}\,i)^{2n} + (1-\sqrt{3}\,i)^{2n} = 2^{2n+1}.$$

(b) Simplify the expression if $n$ is not divisible by 3.

**16.** Show that $\left(\dfrac{1+\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i \sin 2\theta}\right)^n = \operatorname{cis} 2n\theta$.

**17.** Prove that $(1+\cos\alpha + i\sin\alpha)^k + (1+\cos\alpha - i\sin\alpha)^k = 2^{k+1}\cos\tfrac{1}{2}k\alpha \cos^k \tfrac{1}{2}\alpha$.

**EXTENSION**

**18.** Let $z = \operatorname{cis}\frac{\pi}{n}$, where $n$ is a positive integer.
Show that:
(a) $1 + z + z^2 + \cdots + z^{2n-1} = 0$
(b) $1 + z + z^2 + \cdots + z^{n-1} = 1 + i\cot\frac{\pi}{2n}$

# 7B Trigonometric Identities

De Moivre's theorem is particularly useful when combined with the binomial theorem to obtain various trigonometric identities.

**WORKED EXERCISE:**
(a) Express $\cos 3\theta$ in terms of powers of $\cos\theta$.
(b) Hence show that $x = \cos\frac{\pi}{9}$ is a solution of $8x^3 - 6x - 1 = 0$.
(c) Find the value of $\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9}$.

**SOLUTION:**

(a) Let $z = \cos\theta + i\sin\theta$, then

$$z^3 = (\cos\theta + i\sin\theta)^3$$

so by de Moivre's theorem

$$\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta\,.$$

Take the real part to get

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta\\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)\\ &= 4\cos^3\theta - 3\cos\theta\,.\end{aligned}$$

(b) Now put $\theta = \frac{\pi}{9}$ so that

$$\cos\tfrac{\pi}{3} = 4\cos^3\tfrac{\pi}{9} - 3\cos\tfrac{\pi}{9}$$

or $\quad \frac{1}{2} = 4x^3 - 3x \qquad$ where $x = \cos\frac{\pi}{9}$,

thus $\quad 8x^3 - 6x - 1 = 0\,.$

(c) Since $\cos 3\theta = \frac{1}{2}$ for $\theta = \frac{\pi}{9}, \frac{5\pi}{9}$, and $\frac{7\pi}{9}$, it follows that the given expression is the product of the roots of the equation in part (b). Hence

$$\cos\tfrac{\pi}{9}\cos\tfrac{5\pi}{9}\cos\tfrac{7\pi}{9} = \tfrac{1}{8}\,.$$

**WORKED EXERCISE:**

(a) Let $z = \cos\theta + i\sin\theta$. Show that $z^n - z^{-n} = 2i\sin n\theta$.

(b) Expand $(z - z^{-1})^5$.

(c) Use parts (a) and (b) to show that $16\sin^4\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta\,.$

(d) Hence find $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^5\theta\,d\theta\,.$

**SOLUTION:**

(a)
$$\begin{aligned}z^n - z^{-n} &= z^n - \overline{z^n} \qquad \text{(since } |z| = 1)\\ &= 2i\,\mathrm{Im}(z^n)\\ &= 2i\sin n\theta\,.\end{aligned}$$

(b) $(z - z^{-1})^5 = z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}\,.$

(c) Rearranging part (b),

$$(z - z^{-1})^5 = (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$$

so $\quad (2i\sin\theta)^5 = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta \qquad$ by part (a)

thus $\quad 16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta\,.$

(d) Dividing by 16 and integrating yields

$$\begin{aligned}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^5\theta\,d\theta &= \frac{1}{16}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 5\theta - 5\sin 3\theta + 10\sin\theta\,d\theta\\ &= \frac{1}{16}\left[-\frac{\cos 5\theta}{5} + \frac{5\cos 3\theta}{3} - 10\cos\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\\ &= 0 - \frac{1}{16}\left(\frac{1}{5\sqrt{2}} - \frac{5}{3\sqrt{2}} - \frac{10}{\sqrt{2}}\right)\\ &= \frac{43\sqrt{2}}{120}\,.\end{aligned}$$

# Exercise 7B

**1.** (a) Use the identity $\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$ to show that:

(i) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(ii) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(b) Show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

**2.** Use similar methods to the previous question to show that:

(a) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

(b) $\sin 4\theta = 4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$

(c) $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

**3.** Let $z = \cos\theta + i\sin\theta$.

(a) Use de Moivre's theorem to show that $z^n + z^{-n} = 2\cos n\theta$.

(b) Show that $(z + z^{-1})^4 = (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$.

(c) Hence show that $\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$.

**4.** Repeat the methods of the previous question to show that:

$$\sin^4\theta = \tfrac{1}{8}\cos 4\theta - \tfrac{1}{2}\cos 2\theta + \tfrac{3}{8}$$

(Start by showing that $z^n - z^{-n} = 2i\sin n\theta$.)

**5.** (a) Use the methods of questions 1 and 2 to show that:

$$\cos 6\alpha = 32\cos^6\alpha - 48\cos^4\alpha + 18\cos^2\alpha - 1$$

(b) Hence show that the polynomial equation $32x^6 - 48x^4 + 18x^2 - 1 = 0$ has roots of the form $x = \cos\frac{n\pi}{12}$, where $n = 1, 3, 5, 7, 9, 11$.

(c) Use the product of these six roots to deduce that $\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = \frac{1}{4}$.

**6.** (a) Use the methods of question 3 to show that:

$$\cos^5\theta = \tfrac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

(b) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos^5\theta\,d\theta$.

**DEVELOPMENT**

**7.** (a) Use de Moivre's theorem to show that:

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

(b) Hence show that the equation $16x^5 - 20x^3 + 5x - 1 = 0$ has roots $x = 1$, $\sin\frac{\pi}{10}$, $\sin\frac{9\pi}{10}$, $\sin\frac{13\pi}{10}$, $\sin\frac{17\pi}{10}$.

(c) By equating coefficients, or otherwise, find the values of $b$ and $c$ for which $16x^4 + 16x^3 - 4x^2 - 4x + 1 = (4x^2 + bx + c)^2$, and hence explain why the equation $16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$ has two double roots.

(d) Use part (b) to show that the equation $16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$ has roots $x = \sin\frac{\pi}{10}$, $\sin\frac{9\pi}{10}$, $\sin\frac{13\pi}{10}$, $\sin\frac{17\pi}{10}$. Does this contradict part (c) which asserts that the equation has two double roots?

(e) Hence find exact values for $\sin\frac{\pi}{10}$ and $\sin\frac{3\pi}{10}$.

**8.** (a) Show that $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

(b) Hence solve the equation $16\sin^5\theta = \sin 5\theta$ for $0 \le \theta < 2\pi$.

**9.** (a) Use de Moivre's theorem to show that $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$.

(b) Hence show that the equation $x^4 - 10x^2 + 5 = 0$ has roots $x = \pm\tan\frac{\pi}{5}$, $\pm\tan\frac{2\pi}{5}$.

(c) Deduce that $\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}$ and that $\tan^2\frac{\pi}{5} + \tan^2\frac{2\pi}{5} = 10$.

**10.** Let $z = \cos\theta + i\sin\theta$.

(a) Show that $2\cos n\theta = z^n + \dfrac{1}{z^n}$ and that $2i\sin n\theta = z^n - \dfrac{1}{z^n}$.

(b) Hence show that:

$$128\cos^3\theta\sin^4\theta = \left(z^7 + \frac{1}{z^7}\right) - \left(z^5 + \frac{1}{z^5}\right) - 3\left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)$$

(c) Deduce that $\cos^3\theta\sin^4\theta = \frac{1}{64}(\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta)$.

**11.** Consider the polynomial equation $5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$, which has four complex roots with modulus one.
Let $z = \operatorname{cis}\theta$.

(a) Show that $5\cos 2\theta - 11\cos\theta + 8 = 0$.

(b) Hence determine the four roots of the equation in the form $a + ib$, where $a$ and $b$ are real.

**12.** (a) Use de Moivre's theorem to express $\dfrac{\sin 8\theta}{\sin\theta\cos\theta}$ as a polynomial in $s$, where $s = \sin\theta$.

(b) Hence solve the equation $x^6 - 6x^4 + 10x^2 - 4 = 0$, leaving the roots in trigonometric form.

**EXTENSION**

**13.** Let $n$ be a positive integer.

(a) Use de Moivre's theorem to show that:

$$\sin(2n+1)\theta = {}^{2n+1}\mathrm{C}_1\cos^{2n}\theta\sin\theta - {}^{2n+1}\mathrm{C}_3\cos^{2n-2}\theta\sin^3\theta + \cdots + (-1)^n\sin^{2n+1}\theta$$

(b) Hence show that the polynomial $P(x) = {}^{2n+1}\mathrm{C}_1\,x^n - {}^{2n+1}\mathrm{C}_3\,x^{n-1} + \cdots + (-1)^n$ has roots of the form $\cot^2\left(\dfrac{k\pi}{2n+1}\right)$ where $k = 1, 2, 3, \ldots, n$.

(c) Deduce that $\cot^2\left(\dfrac{\pi}{2n+1}\right) + \cot^2\left(\dfrac{2\pi}{2n+1}\right) + \cdots + \cot^2\left(\dfrac{n\pi}{2n+1}\right) = \dfrac{n(2n-1)}{3}$.

(d) Use the fact that $\cot\theta < \frac{1}{\theta}$ for $0 < \theta < \frac{\pi}{2}$ to show that:

$$\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right)\frac{(2n+1)^2}{2n(2n-1)} > \frac{\pi^2}{6}$$

# 7C Roots of Unity

Recall from a previous worked exercise that the points in the Argand diagrm which represent $z^n$, where $n$ is an integer, lie on a spiral whenever $|z| \neq 1$. When $|z| = 1$, it should be clear that the points lie on the unit circle. Further, if $z = \cos\theta + i\sin\theta$ then the angle at the origin subtended by successive points is

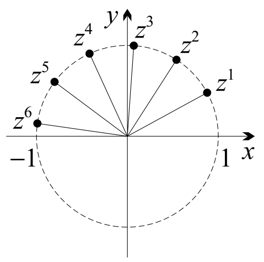
$$\begin{aligned} \arg(z^n) - \arg(z^{n-1}) &= \arg\left(\frac{z^n}{z^{n-1}}\right) \\ &= \arg(z) \\ &= \theta\,. \end{aligned}$$

That is, the angle is constant. Thus successive points are regularly spaced about the unit circle.

For example, the sketch on the right shows the points $z^n$ for $n = 1, 2, 3, 4, 5, 6$, where $z = \cos\frac{1}{2} + i\sin\frac{1}{2}$. Note that

$$\arg(z) = \tfrac{1}{2} \doteqdot 28°39'\,,$$

which is the angle subtended at the origin by any pair of successive points. It should be clear that $2\pi \div \frac{1}{2} = 4\pi$ is irrational, and hence none of the points coincide, even for larger values of $n$. In that sense, this is not a very interesting example.




Significant configurations of points arise when equations of the form $z^n = w$ are solved, where $|w| = 1$. There are always $n$ solutions and the points are equally spaced about the unit circle. Further, if $z = 1$ is a solution and if $\alpha$ is another solution, then $\alpha^k$ will always coincide with one of the points, regardless of the integer value of $k$.

**WORKED EXERCISE:**

(a) Solve $z^6 = 1$.

(b) Plot the solutions on the unit circle in the complex plane.
  (i) What is the angle subtended at the origin by successive roots?
  (ii) What regular polygon has these points as vertices?

(c) Let $\alpha = \text{cis}(-\frac{\pi}{3})$. Show that the list 1, $\alpha$, $\alpha^2$, $\alpha^3$, $\alpha^4$ and $\alpha^5$ includes all six roots of $z^6 = 1$.

(d) Let $\beta = \text{cis}\,\frac{2\pi}{3}$. Which roots of $z^6 = 1$ can be written in the form $\beta^k$, where $k$ is an integer?

**SOLUTION:**

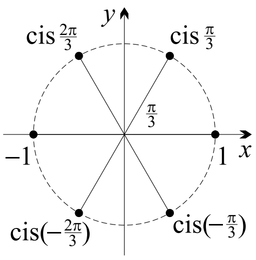
(a) Let $z = \text{cis}\,\theta$ and note that $1 = \text{cis}\,2n\pi$, where $n$ is an integer. Thus

$$\text{cis}\,6\theta = \text{cis}\,2n\pi \qquad \text{(by de Moivre)}$$
so $$6\theta = 2n\pi$$
hence $$\theta = \tfrac{n\pi}{3}\,.$$

Apply the restriction $-\pi < \theta \leq \pi$ to obtain all the distinct solutions. Thus

$$-\pi < \tfrac{n\pi}{3} \leq \pi$$
so $$-3 < n \leq 3\,.$$

Hence the six roots of $z^6 = 1$ are
$\operatorname{cis}(-\frac{2\pi}{3})$, $\operatorname{cis}(-\frac{\pi}{3})$, 1, $\operatorname{cis}\frac{\pi}{3}$, $\operatorname{cis}\frac{2\pi}{3}$ and $-1$.




(b) The graph on the right shows these six roots.
 (i) Clearly the angle at the centre is $\frac{\pi}{3}$.
 (ii) These are the vertices of a regular hexagon.

(c) Using de Moivre's theorem, the given list is:
$$1\,,\operatorname{cis}(-\tfrac{\pi}{3})\,,\operatorname{cis}(-\tfrac{2\pi}{3})\,,\operatorname{cis}(-\tfrac{3\pi}{3}) = -1\,,$$
$$\operatorname{cis}(-\tfrac{4\pi}{3}) = \operatorname{cis}\tfrac{2\pi}{3} \text{ and } \operatorname{cis}(-\tfrac{5\pi}{3}) = \operatorname{cis}\tfrac{\pi}{3}\,.$$
This is the same list as given in the answer to part (a), but simply in a different order.

(d) Now $\beta^k = \operatorname{cis}\frac{2k\pi}{3}$ by de Moivre's theorem. Hence $\arg(\beta^k)$ is a multiple of $\frac{2\pi}{3}$. Thus the possible values that $\beta^k$ may take are:
$$\operatorname{cis}(-\tfrac{2\pi}{3}), 1 \text{ and } \operatorname{cis}\tfrac{2\pi}{3}\,.$$
That is, only these three roots can be written as a power of $\beta$.

**WORKED EXERCISE:** Consider the equation $z^5 + 1 = 0$.

(a) Find the roots of this equation and show them on the Argand diagram.

(b) Factorise $z^5 + 1$:
 (i) as a product of linear factors,
 (ii) as a product of linear and quadratic factors with real coefficients.

(c) Evauate $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$.

(d) Let $\alpha$ be a complex root of $z^5 + 1 = 0$, that is $\alpha \neq -1$.
 (i) Show that $1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4 = 0$.
 (ii) Find a quadratic equation with roots $(\alpha^4 - \alpha)$ and $(\alpha^2 - \alpha^3)$.

(e) Put $\alpha = \operatorname{cis}\frac{\pi}{5}$ in part (d), and hence evaluate $\cos\frac{\pi}{5}$.

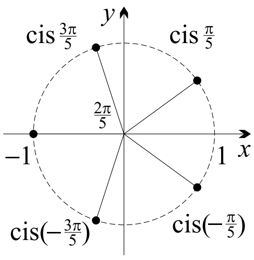
**SOLUTION:**

(a) Let $z = \operatorname{cis}\theta$ and note that $-1 = \operatorname{cis}(2n+1)\pi$, where $n$ is an integer. Thus
$$\operatorname{cis}5\theta = \operatorname{cis}(2n+1)\pi \qquad \text{(by de Moivre)}$$
so $\quad 5\theta = (2n+1)\pi$

hence $\quad \theta = \frac{(2n+1)\pi}{5}$.

Apply the restriction $-\pi < \theta \leq \pi$ to obtain all the distinct solutions. Thus
$$-\pi < \tfrac{(2n+1)\pi}{5} \leq \pi$$
so $\quad -5 < (2n+1) \leq 5$

or $\quad -3 < n \leq 2$.




Hence the five roots are:
$$\operatorname{cis}(-\tfrac{3\pi}{5}), \operatorname{cis}(-\tfrac{\pi}{5}), \operatorname{cis}\tfrac{\pi}{5}, \operatorname{cis}\tfrac{3\pi}{5} \text{ and } -1\,,$$
or in conjugate pairs,
$$\operatorname{cis}\tfrac{\pi}{5}, \overline{\operatorname{cis}\tfrac{\pi}{5}}, \operatorname{cis}\tfrac{3\pi}{5}, \overline{\operatorname{cis}\tfrac{3\pi}{5}} \text{ and } -1\,.$$

(b) Using the roots of the given equation,

$$\begin{aligned} z^5+1 &= (z+1)(z-\operatorname{cis}\tfrac{\pi}{5})(z-\overline{\operatorname{cis}\tfrac{\pi}{5}})(z-\operatorname{cis}\tfrac{3\pi}{5})(z-\overline{\operatorname{cis}\tfrac{3\pi}{5}}) \\ &= (z+1)(z^2-2z\cos\tfrac{\pi}{5}+1)(z^2-2z\cos\tfrac{3\pi}{5}+1)\,. \end{aligned}$$

(c) By the sum of the roots

$$\operatorname{cis}\tfrac{\pi}{5}+\overline{\operatorname{cis}\tfrac{\pi}{5}}+\operatorname{cis}\tfrac{3\pi}{5}+\overline{\operatorname{cis}\tfrac{3\pi}{5}}-1=0$$

whence $2\cos\tfrac{\pi}{5}+2\cos\tfrac{3\pi}{5}=1\,,$

that is $\cos\tfrac{\pi}{5}+\cos\tfrac{3\pi}{5}=\tfrac{1}{2}\,.$

(d) (i) Since $\alpha$ is a complex root,

$$\alpha^5+1=0$$

so $(\alpha+1)(1-\alpha+\alpha^2-\alpha^3+\alpha^4)=0$ (from GP theory)

thus $1-\alpha+\alpha^2-\alpha^3+\alpha^4=0$ (since $\alpha\neq -1$)

(ii) The sum of the roots is $-\alpha+\alpha^2-\alpha^3+\alpha^4=-1$ from part (i). The product of the roots is

$$\begin{aligned} (\alpha^4-\alpha)(\alpha^2-\alpha^3) &= \alpha^6-\alpha^7-\alpha^3+\alpha^4 \\ &= -\alpha+\alpha^2-\alpha^3+\alpha^4 \qquad \text{(since } \alpha^5=-1\text{)} \\ &= -1\,. \end{aligned}$$

Hence the required quadratic is $z^2+z-1=0\,.$

(e) With $\alpha=\operatorname{cis}\tfrac{\pi}{5}$ the roots of the equation in part (d) are

$$\begin{aligned} \alpha^4-\alpha &= -\alpha^{-1}-\alpha \qquad \text{(since } \alpha^5=-1\text{)} \\ &= -(\overline{\alpha}+\alpha) \qquad \text{(since } |\alpha|=1\text{)} \\ &= -2\cos\tfrac{\pi}{5}\,, \end{aligned}$$

and

$$\begin{aligned} \alpha^2-\alpha^3 &= \alpha^2+\alpha^{-2} \qquad \text{(since } \alpha^5=-1\text{)} \\ &= \alpha^2+\overline{\alpha^2} \qquad \text{(since } |\alpha|=1\text{)} \\ &= 2\cos\tfrac{2\pi}{5}\,. \end{aligned}$$

Also, by direct calculation

$$z=\tfrac{-1-\sqrt5}{2} \text{ or } \tfrac{-1+\sqrt5}{2}\,,$$

hence $\cos\tfrac{\pi}{5}=\tfrac{1+\sqrt5}{4}$ and $\cos\tfrac{2\pi}{5}=\tfrac{-1+\sqrt5}{4}\,.$

## Exercise **7C**

**1.** (a) Find the three cube roots of unity, expressing the complex roots in both $r\operatorname{cis}\theta$ and $x+iy$ form.

(b) Show that the points in the complex plane representing these three roots form an equilateral triangle.

(c) If $\omega$ is one of the complex roots, show that the other complex root is $\omega^2$.

(d) Write down the values of:

(i) $\omega^3$ (ii) $1+\omega+\omega^2$

(e) Show that:

(i) $(1+\omega^2)^3=-1$

(ii) $(1-\omega-\omega^2)(1-\omega+\omega^2)(1+\omega-\omega^2)=8$

(iii) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)=9$

**2.** (a) Solve the equation $z^6 = 1$, expressing the complex roots in the form $a + ib$, where $a$ and $b$ are real.
(b) Plot these roots on an Argand diagram, and show that they form a regular hexagon.
(c) If $\alpha$ is the complex root with smallest positive principal argument, show that the other three complex roots are $\alpha^2$, $\alpha^{-1}$ and $\alpha^{-2}$.
(d) Show that $z^6 - 1 = (z^2 - 1)(z^4 + z^2 + 1)$.
(e) Hence write $z^4 + z^2 + 1$ as a product of quadratic factors with real coefficients.

**3.** (a) Find, in the form $a + ib$, the four fourth roots of $-1$.
(b) Hence write $z^4 + 1$ as a product of two quadratic factors with real coefficients.

**4.** (a) Find, in the form $a + ib$, the six roots of the equation $z^6 + 1 = 0$.
(b) Hence show that $z^6 + 1 = (z^2 + 1)(z^2 - \sqrt{3}\,z + 1)(z^2 + \sqrt{3}\,z + 1)$.
(c) Divide both sides of this identity by $z^3$, and then let $z = \operatorname{cis}\theta$ to show that:

$$\cos 3\theta = 4\cos\theta(\cos\theta - \cos\tfrac{\pi}{6})(\cos\theta - \cos\tfrac{5\pi}{6})$$

**5.** (a) Find, in mod-arg form, the five fifth roots of $i$.
(b) Find, in mod-arg form, the four fourth roots of $-i$.
(c) Find, in the form $a + ib$, the four fourth roots of $-8 - 8\sqrt{3}\,i$.
(d) Find, in mod-arg form, the five fifth roots of $16\sqrt{2} - 16\sqrt{2}\,i$.

**DEVELOPMENT**

**6.** (a) Find the five fifth roots of $-1$, writing the complex roots in mod-arg form.
(b) If $\alpha$ is the complex root with least positive principal argument, show that $\alpha^3$, $\alpha^7$ and $\alpha^9$ are the other three complex roots.
(c) Show that $\alpha^7 = -\alpha^2$ and that $\alpha^9 = -\alpha^4$.
(d) Use the sum of the roots to show that $\alpha + \alpha^3 = 1 + \alpha^2 + \alpha^4$.

**7.** (a) Find the seven seventh roots of unity.
(b) By considering the sum of the real parts of these seven roots, show that:

$$\cos\tfrac{2\pi}{7} + \cos\tfrac{4\pi}{7} + \cos\tfrac{6\pi}{7} = -\tfrac{1}{2}$$

(c) Write $z^7 - 1$ as a product of one linear and three quadratic factors, all with real coefficients.
(d) If $\alpha$ is the complex seventh root of unity with the least positive principal argument, show that $\alpha^2$, $\alpha^3$, $\alpha^4$, $\alpha^5$ and $\alpha^6$ are the other five complex roots.
(e) By considering the relationships between the roots and the coefficients, show that the cubic equation $x^3 + x^2 - 2x - 1 = 0$ has roots $\alpha + \alpha^6$, $\alpha^2 + \alpha^5$ and $\alpha^3 + \alpha^4$.

**8.** (a) (i) Find the five fifth roots of unity, writing the complex roots in mod-arg form.
(ii) Show that the points in the complex plane representing these roots form a regular pentagon.
(iii) By considering the sum of these five roots, show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.
(b) (i) Show that $z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1)$.
(ii) Hence show that $z^4 + z^3 + z^2 + z + 1 = (z^2 - 2\cos\frac{2\pi}{5}\,z + 1)(z^2 - 2\cos\frac{4\pi}{5}\,z + 1)$.
(iii) By equating the coefficients of $z$ in this identity, show that $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$.
(c) (i) Use the substitution $x = u + \dfrac{1}{u}$ to show that the equation $x^2 + x - 1 = 0$ has roots $2\cos\frac{2\pi}{5}$ and $2\cos\frac{4\pi}{5}$.
(ii) Deduce that $\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}$.

**9.** (a) Find the ninth roots of unity.

(b) Hence show that:

$$z^6 + z^3 + 1 = (z^2 - 2\cos\tfrac{2\pi}{9}\,z + 1)(z^2 - 2\cos\tfrac{4\pi}{9}\,z + 1)(z^2 - 2\cos\tfrac{8\pi}{9}\,z + 1)$$

(c) Deduce that:

$$2\cos 3\theta + 1 = 8(\cos\theta - \cos\tfrac{2\pi}{9})(\cos\theta - \cos\tfrac{4\pi}{9})(\cos\theta - \cos\tfrac{8\pi}{9})$$

**10.** Let $\omega = \operatorname{cis}\frac{2\pi}{9}$.

(a) Show that $\omega^k$, where $k$ is an integer, is a solution of the equation $z^9 = 1$.

(b) Show that $\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = -1$.

(c) Hence show that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$.

(d) Deduce that $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$.

**11.** Let $\rho = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$. The complex number $\alpha = \rho + \rho^2 + \rho^4$ is a root of the quadratic equation $x^2 + ax + b = 0$, where $a$ and $b$ are real.

(a) Prove that $1 + \rho + \rho^2 + \ldots + \rho^6 = 0$.

(b) The second root of the quadratic equation is $\beta$. Express $\beta$ in terms of positive powers of $\rho$. Justify your answer.

(c) Find the values of the coefficients $a$ and $b$.

(d) Deduce that $-\sin\frac{\pi}{7} + \sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} = \frac{\sqrt{7}}{2}$.

**EXTENSION**

**12.** (a) Show that the equation $(z+1)^8 - z^8 = 0$ has roots $z = -\frac{1}{2}, -\frac{1}{2}\left(1 \pm i\cot\frac{k\pi}{8}\right)$, where $k = 1, 2, 3$.

(b) Hence show that:

$$(z+1)^8 - z^8 = \tfrac{1}{8}(2z+1)(2z^2+2z+1)(4z^2+4z+\operatorname{cosec}^2\tfrac{\pi}{8})(4z^2+4z+\operatorname{cosec}^2\tfrac{3\pi}{8})$$

(c) By making a suitable substitution into this identity, deduce that:

$$\cos^{16}\theta - \sin^{16}\theta = \tfrac{1}{16}\cos 2\theta(\cos^2 2\theta + 1)(\cos^2 2\theta + \cot^2\tfrac{\pi}{8})(\cos^2 2\theta + \cot^2\tfrac{3\pi}{8})$$

**13.** Suppose that $\omega^3 = 1$, and $\omega \neq 1$.
Let $k$ be a positive integer.

(a) What are the two possible values of $1 + \omega^k + \omega^{2k}$?

(b) Use the binomial theorem to expand $(1+\omega)^n$ and $(1+\omega^2)^n$, where $n$ is a positive integer.

(c) Let $\ell$ be the largest integer for which $3\ell \le n$.
Show that:

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots + \binom{n}{3\ell} = \tfrac{1}{3}\left(2^n + (1+\omega)^n + (1+\omega^2)^n\right)$$

(d) If $n$ is a multiple of 6, show that:

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots + \binom{n}{n} = \tfrac{1}{3}\left(2^n + 2\right)$$

**14.** Consider the equation $(z+1)^{2n} + (z-1)^{2n} = 0$, where $n$ is a positive integer.

(a) Show that every root of the equation is purely imaginary.

(b) Let the roots be represented by the points $P_1, P_2, \ldots, P_{2n}$ in the Argand diagram, and let $O$ be the origin.
Show that:

$$OP_1{}^2 + OP_2{}^2 + \cdots + OP_{2n}{}^2 = 2n(2n-1)$$

# Chapter Seven

## Exercise 7A (Page 45)

**1(a)** $\operatorname{cis} 5\theta$ **(b)** $\operatorname{cis}(-3\theta)$ **(c)** $\operatorname{cis} 8\theta$ **(d)** $\operatorname{cis}(-\theta)$
**(e)** $\operatorname{cis} 7\theta$ **(f)** $\operatorname{cis}(-6\theta)$
**2(a)** $\operatorname{cis} 7\theta$ **(b)** $\operatorname{cis}(-5\theta)$
**3(a)** $-1$ **(b)** $-i$ **(c)** $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$ **(d)** $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$
**(e)** $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ **(f)** $\frac{1}{2} + \frac{\sqrt{3}}{2}i$
**4(a)** $\sqrt{2} \operatorname{cis} \frac{\pi}{4}$ **(b)** $256 + 256i$
**5(a)** $2 \operatorname{cis} \frac{\pi}{3}$ **(b)** $1024 - 1024\sqrt{3}\, i$
**6(a)** $2,\ \frac{5\pi}{6}$
**7(a)** $2 \operatorname{cis}\left(-\frac{\pi}{6}\right)$ **(b)** $128 \operatorname{cis} \frac{5\pi}{6}$ **(c)** $-64\sqrt{3} + 64i$
**8(a)** $2 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$ **(b)** $32 \operatorname{cis} \frac{2\pi}{3}$ **(c)** $-16 + 16\sqrt{3}\, i$
**9(a)** $2 \operatorname{cis}\left(-\frac{\pi}{4}\right)$ **(b)** $2^{22} i$
**12(a)(i)** $6$ **(ii)** $3$ **(b)** $-64,\ 8i$
**13(b)** $n = 2,\ 6,\ 10,\ \ldots$
**15(b)** $-2^{2n}$

## Exercise 7B (Page 47)

**6(b)** $\frac{8}{15}$
**7(c)** $b = 2,\ c = -1$
**(d)** No, since $\sin \frac{\pi}{10} = \sin \frac{9\pi}{10}$ and $\sin \frac{13\pi}{10} = \sin \frac{17\pi}{10}$
**(e)** $\sin \frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$, $\sin \frac{3\pi}{10} = \frac{\sqrt{5}+1}{4}$
**8(b)** $\theta = 0,\ \frac{\pi}{6},\ \frac{5\pi}{6},\ \pi,\ \frac{7\pi}{6},\ \frac{11\pi}{6}$
**11(b)** $z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ or $\frac{3}{5} \pm \frac{4}{5}i$
**12(a)** $8(1 - 10s^2 + 24s^4 - 16s^6)$
**(b)** $x = 2\sin \frac{n\pi}{8}$ for $n = 1,\ 2,\ 3,\ 5,\ 6,\ 7$

## Exercise 7C (Page 52)

**1(a)** $1,\ \operatorname{cis} \frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\, i,\ \operatorname{cis} \frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}\, i$
**(d)(i)** $1$ **(ii)** $0$
**2(a)** $z = \pm 1,\ \frac{1}{2} + \frac{\sqrt{3}}{2}\, i,\ \frac{1}{2} - \frac{\sqrt{3}}{2}\, i,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\, i,$
$-\frac{1}{2} - \frac{\sqrt{3}}{2}\, i$ **(e)** $(z^2 - z + 1)(z^2 + z + 1)$
**3(a)** $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\, i,\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\, i,\ -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\, i,\ -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\, i$
**(b)** $(z^2 - \sqrt{2}\, z + 1)(z^2 + \sqrt{2}\, z + 1)$
**4(a)** $i,\ -i,\ \frac{\sqrt{3}}{2} + \frac{1}{2}\, i,\ \frac{\sqrt{3}}{2} - \frac{1}{2}\, i,\ -\frac{\sqrt{3}}{2} + \frac{1}{2}\, i,\ -\frac{\sqrt{3}}{2} - \frac{1}{2}\, i$
**5(a)** $\operatorname{cis}\left(-\frac{7\pi}{10}\right),\ \operatorname{cis}\left(-\frac{3\pi}{10}\right),\ \operatorname{cis} \frac{\pi}{10},\ \operatorname{cis} \frac{\pi}{2} = i,\ \operatorname{cis} \frac{9\pi}{10}$
**(b)** $\operatorname{cis}\left(-\frac{5\pi}{8}\right),\ \operatorname{cis}\left(-\frac{\pi}{8}\right),\ \operatorname{cis} \frac{3\pi}{8},\ \operatorname{cis} \frac{7\pi}{8}$
**(c)** $1 + \sqrt{3}\, i,\ -1 - \sqrt{3}\, i,\ \sqrt{3} - i,\ -\sqrt{3} + i$
**(d)** $2 \operatorname{cis}\left(-\frac{17\pi}{20}\right),\ 2 \operatorname{cis}\left(-\frac{9\pi}{20}\right),\ 2 \operatorname{cis}\left(-\frac{\pi}{20}\right),\ 2 \operatorname{cis} \frac{7\pi}{20},$
$2 \operatorname{cis} \frac{3\pi}{4}$
**6(a)** $-1,\ \operatorname{cis} \frac{\pi}{5},\ \operatorname{cis}\left(-\frac{\pi}{5}\right),\ \operatorname{cis} \frac{3\pi}{5},\ \operatorname{cis}\left(-\frac{3\pi}{5}\right)$
**7(a)** $1,\ \operatorname{cis}\left(\pm\frac{2\pi}{7}\right),\ \operatorname{cis}\left(\pm\frac{4\pi}{7}\right),\ \operatorname{cis}\left(\pm\frac{6\pi}{7}\right)$
**(c)** $(z - 1) \times (z^2 - 2\cos \frac{2\pi}{7}\, z + 1) \times$
$(z^2 - 2\cos \frac{4\pi}{7}\, z + 1) \times (z^2 - 2\cos \frac{6\pi}{7}\, z + 1)$
**8(a)(i)** $1,\ \operatorname{cis} \frac{2\pi}{5},\ \operatorname{cis}\left(-\frac{2\pi}{5}\right),\ \operatorname{cis} \frac{4\pi}{5},\ \operatorname{cis}\left(-\frac{4\pi}{5}\right)$
**9(a)** $\operatorname{cis} \frac{2k\pi}{9}$ for $k = -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4$
**13(a)** $3$, when $k$ is a multiple of $3$, $0$ otherwise.
**(b)** $(1 + \omega)^n = \sum_{r=0}^{n} \binom{n}{r} \omega^r$ and
$(1 + \omega^2)^n = \sum_{r=0}^{n} \binom{n}{r} \omega^{2r}$
**14(a)** The roots are $-i \cot \frac{(2k-1)\pi}{4n}$
for $k = 1,\ 2,\ 3,\ \ldots,\ 2n$.

CHAPTER **FIVE**

# Polynomials

This chapter is an extension of the work already done on polynomials in the Mathematics Extension 1 course. That work is assumed knowledge though some parts of the theory are repeated here for the sake of convenience. The focus of the chapter is on polynomials with real coefficients and the relationships with the zeroes, particularly when they are either complex, or real and repeated.

The crux of the chapter is in Section 5B where the Fundamental Theorem of Algebra is presented along with some of its consequences. The theorem is left unproven as any proof is beyond the scope of the course.

The chapter concludes with harder questions on the relationships between the zeroes and coefficients of a polynomial, and simple examples of how the zeroes may be transformed by use of suitable substitutions.

## 5A Zeroes and Remainders

**Polynomials with Integer Coefficients:** If a polynomial with integer coefficients has an integer zero $x = k$, then $k$ is a factor of the constant term. This is a significant aid in factorising a polynomial.

**Worked Exercise:** It is known that the polynomial $P(x) = x^3 - x^2 - 8x - 6$ has only one integer zero. Find it and hence factorise $P(x)$ completely.

**Solution:** Since the zero is a factor of 6, the possible values are: $\pm1, \pm2, \pm3$. Testing these one by one:

$$P(1) = -14\,, \qquad P(-1) = 0\,,$$

and there is no need to continue further. By the factor theorem, $(x + 1)$ is a factor of $P(x)$. Performing the long division:

$$\begin{array}{r|l}
 & \phantom{x^3 -}\ x^2 - 2x - 6 \\ \hline
(x+1) & x^3 - \phantom{2}x^2 - 8x - 6 \\
 & \underline{x^3 + \phantom{2}x^2} \\
 & \phantom{x^3} - 2x^2 - 8x - 6 \\
 & \phantom{x^3} \underline{- 2x^2 - 2x} \\
 & \phantom{x^3 - 2x^2} - 6x - 6 \\
 & \phantom{x^3 - 2x^2} \underline{- 6x - 6} \\
 & \phantom{x^3 - 2x^2 - 6x - } 0
\end{array}$$

Thus
$$\begin{aligned} P(x) &= (x+1)(x^2-2x-6) \\ &= (x+1)\Big((x-1)^2-7\Big) \qquad \text{(completing the square)} \\ &= (x+1)(x-1-\sqrt{7})(x-1+\sqrt{7}) \qquad \text{(difference of two squares.)} \end{aligned}$$

1 INTEGER COEFFICIENTS AND ZEROES: If the polynomial

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

with integer coefficients $a_0$, $a_1$, $a_2, \ldots, a_n$, has an integer zero $x = k$, then $k$ is a factor of the constant term $a_0$.

PROOF: Since $P(k) = 0$, it follows that

$$\begin{aligned} a_0 + a_1k + a_2k^2 + \ldots + a_nk^n &= 0 \\ \text{so} \qquad a_1k + a_2k^2 + \ldots + a_nk^n &= -a_0 \\ \text{thus} \qquad k \times (a_1 + a_2k + \ldots + a_nk^{n-1}) &= -a_0\,. \end{aligned}$$

Since all the terms in the brackets are integers, it follows that the result is also an integer. Thus the left hand side is the product of two integers. Hence, as asserted, $k$ is a factor of $a_0$.

**Polynomials and Complex Numbers:** Consider the general polynomial

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n\,.$$

Each term in this expression involves an integer power and multiplication by a constant. The terms are then simply added. Since integer powers, multiplication and addition are all natural operations with complex numbers, it follows that the polynomial can be evaluated when $x$ is a complex number. For example if $P(x) = x^2 - 2x + 4$ then at $x = i$ its value is

$$\begin{aligned} P(i) &= i^2 - 2i + 4 \\ &= 3 - 2i\,. \end{aligned}$$

In some examples the polynomial will be written as a function of $z$ in order to emphasise the fact that complex numbers may be substituted. Thus the above example may be written as $P(z) = z^2 - 2z + 4$.

It will be necessary in this course to occasionally consider polynomials where the coefficients are also complex numbers. For example, $P(z) = 2z^2 + (1+i)z + 3i$, for which

$$\begin{aligned} P(i) &= 2i^2 + (1+i)i + 3i \\ &= -3 + 4i\,. \end{aligned}$$

**Remainders and Factors:** Here is a quick summary of certain important results from the Mathematics Extension 1 course. In the usual notation, let $P(x)$ and $D(x)$ be any pair of polynomials, where $D(x) \neq 0$. There is a unique pair of polynomials $Q(x)$ and $R(x)$, such that

$$P(x) = D(x) \times Q(x) + R(x)\,,$$

and where either

$$\deg(D) > \deg(R) \qquad \text{or} \qquad R(x) = 0\,.$$

This is known as the division algorithm. As a consequence, if $D(x) = (x - \alpha)$ then $R(x)$ must be a constant, either zero or non-zero. Let this constant be $r$. Re-writing the division algorithm:

$$P(x) = (x - \alpha) \times Q(x) + r\,,$$

whence $\quad P(\alpha) = r\,,$

which is known as the remainder theorem.

If $R(x) = 0$ then from the division algorithm

$$P(x) = D(x) \times Q(x)\,,$$

so that $P(x)$ is a product of the factors $D(x)$ and $Q(x)$. In particular, $x - \alpha$ is a factor of $P(x)$ if and only if $P(\alpha) = 0$. This is known as the factor theorem.

The division algorithm, the remainder theorem and the factor theorem are valid for complex numbers as well as real numbers. Though these claims will not be proven here, the results may be freely applied to solve problems.

**WORKED EXERCISE:** Let $P(x) = x^3 - 2x^2 - x + k$, where $k$ is real.

(a) Show that $P(i) = (2 + k) - 2i$.

(b) When $P(x)$ is divided by $x^2 + 1$ the remainder is $4 - 2x$. Find the value of $k$.

**SOLUTION:** (a)
$$\begin{aligned} P(i) &= i^3 - 2i^2 - i + k \\ &= -i + 2 - i + k \\ &= (2 + k) - 2i\,. \end{aligned}$$

(b) By the division algorithm,

$$P(x) = (x^2 + 1) \times Q(x) + 4 - 2x\,.$$

Thus $\quad P(i) = 4 - 2i$

whence $\quad (2 + k) - 2i = 4 - 2i\,.$

Equating the real parts gives $k = 2$.

**Real Coefficients and Remainders:** Suppose that the polynomial $P(z)$ has real coefficients. If the remainder when $P(z)$ is divided by $(z - \alpha)$ is $\beta$ then the remainder when $P(z)$ is divided by $(z - \overline{\alpha})$ is $\overline{\beta}$. Using the remainder theorem, this is equivalent to the statement that if $P(\alpha) = \beta$ then $P(\overline{\alpha}) = \overline{\beta}$.

**WORKED EXERCISE:**

(a) Use the remainder theorem to find the remainder when $P(z) = z^3 - 2z^2 + 3z - 1$ is divided by $(z - i)$.

(b) Hence find the remainder when $P(z)$ is divided by $(z + i)$.

**SOLUTION:**

(a) The remainder is:
$$\begin{aligned} P(i) &= i^3 - 2i^2 + 3i - 1 \\ &= 1 + 2i\,. \end{aligned}$$

(b) It is:
$$\begin{aligned} P(-i) &= P(\overline{i}\,) \\ &= \overline{1 + 2i} \\ &= 1 - 2i\,. \end{aligned}$$

**2** | REAL COEFFICIENTS AND REMAINDERS: If the polynomial $P(z)$ has real coefficients and if $P(\alpha) = \beta$ then $P(\overline{\alpha}) = \overline{\beta}$.

The proof is not too difficult and is dealt with in a question of the exercise.

**Real Coefficients and Complex Zeroes:** Suppose that the polynomial $P(z)$ has real coefficients. If $P(z)$ has a complex zero $z = \alpha$ then it is guaranteed to have a second complex zero $z = \overline{\alpha}$. Further, by the factor theorem, there exists another polynomial $Q(z)$ such that:

$$\begin{aligned} P(z) &= (z-\alpha)(z-\overline{\alpha}) \times Q(z) \\ &= (z^2 - (\alpha+\overline{\alpha})z + \alpha\overline{\alpha}) \times Q(z) \\ &= (z^2 - 2\,\mathrm{Re}(\alpha)z + |\alpha|^2) \times Q(z)\,. \end{aligned}$$

Thus $P(z)$ has a quadratic factor with real coefficients: $(z^2 - 2\,\mathrm{Re}(\alpha)z + |\alpha|^2)$.

**WORKED EXERCISE:** Consider the polynomial $P(z) = 2z^3 - 3z^2 + 18z + 10$.

(a) Given that $1 - 3i$ is a zero of $P(z)$, explain why $1 + 3i$ is another zero.

(b) Find the third zero of the polynomial.

(c) Hence write $P(z)$ as a product of:

(i) linear factors,

(ii) a linear factor and a quadratic factor, both with real coefficients.

**SOLUTION:** (a) Since $P(z)$ has real coefficients, $\overline{(1-3i)} = 1 + 3i$ is also a zero.

(b) Let the third zero be $a$, then by the sum of the roots:

$$\begin{aligned} a + (1-3i) + (1+3i) &= \tfrac{3}{2} \\ \text{so} \qquad a + 2 &= \tfrac{3}{2} \\ \text{and} \qquad a &= -\tfrac{1}{2}\,. \end{aligned}$$

(c) (i) By the factor theorem:

$$\begin{aligned} P(z) &= 2(z + \tfrac{1}{2})(z - 1 + 3i)(z - 1 - 3i) \\ &= (2z+1)(z-1+3i)(z-1-3i)\,. \end{aligned}$$

(ii) $P(z) = (2z+1)(z^2 - 2z + 10)$.

3 REAL COEFFICIENTS AND ZEROES: If the polynomial $P(z)$ has real coefficients and a complex zero $z = \alpha$ then it is guaranteed to have a second complex zero $z = \overline{\alpha}$.

Consequently $P(z)$ has $(z^2 - 2\,\mathrm{Re}(\alpha)z + |\alpha|^2)$ as a factor, which is a quadratic with real coefficients.

PROOF: Suppose that the complex number $z = \alpha$ is a zero of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n\,,$$

where the coefficients $a_0, a_1, \ldots, a_n$ are all real. That is $P(\alpha) = 0$. Then

$$\begin{aligned} P(\overline{\alpha}) &= a_0 + a_1\overline{\alpha} + a_2\overline{\alpha}^{\,2} + \ldots + a_n\overline{\alpha}^{\,n} \\ &= a_0 + a_1\overline{\alpha} + a_2\overline{\alpha^2} + \ldots + a_n\overline{\alpha^n} \quad (\text{since } \overline{z}^{\,n} = \overline{z^n}) \\ &= \overline{a_0} + \overline{a_1\alpha} + \overline{a_2\alpha^2} + \ldots + \overline{a_n\alpha^n} \quad (\text{since } c\,\overline{z} = \overline{cz} \text{ for real } c) \\ &= \overline{a_0 + a_1\alpha + a_2\alpha^2 + \ldots + a_n\alpha^n} \quad (\text{since } \overline{w} + \overline{z} = \overline{w+z}) \\ &= \overline{P(\alpha)} \\ &= \overline{0} \\ &= 0\,. \end{aligned}$$

Hence $z = \overline{\alpha}$ is also a zero of the polynomial $P(z)$. Further, as shown above:

$$P(z) = (z^2 - 2\,\mathrm{Re}(\alpha)z + |\alpha|^2) \times Q(z)\,.$$

# Exercise **5A**

**1.** It is known that in each case the given polynomial $P(x)$ has only one integer zero. Find it and hence factorise $P(x)$ completely.

(a) $P(x) = x^3 - 6x + 4$ (b) $P(x) = x^3 + 3x^2 - 2x - 2$ (c) $P(x) = x^3 - 3x^2 - 2x + 4$

**2.** It is known that $1 + i$ is a zero of the polynomial $P(x) = x^3 - 8x^2 + 14x - 12$.

(a) Why is $1 - i$ also a zero of $P(x)$?

(b) Use the sum of the zeroes to find the third zero of $P(x)$.

**3.** It is known that $1 - 2i$ is a zero of the polynomial $P(x) = x^3 + x + 10$.

(a) Write down another complex zero of $P(x)$, and give a reason for your answer.

(b) Hence show that $x^2 - 2x + 5$ is a factor of $P(x)$.

(c) Find the third zero, and hence write $P(x)$ as a product of factors with real coefficients.

**4.** It is known that $-3i$ is a zero of the polynomial $P(z) = 2z^3 + 3z^2 + 18z + 27$.

(a) Write down another complex zero of $P(z)$. Justify your answer.

(b) Hence write down a quadratic factor of $P(z)$ with real coefficients.

(c) Write $P(x)$ as a product of factors with real coefficients.

**5.** Let $P(z) = 2z^3 - 13z^2 + 26z - 10$.

(a) Show that $P(3 + i) = 0$.

(b) State the value of $P(3 - i)$, and give a reason for your answer.

(c) Hence write $P(z)$ as a product of:

(i) linear factors,

(ii) a linear factor and a quadratic factor, both with real coefficients.

**DEVELOPMENT**

**6.** Consider the polynomial $Q(x) = x^4 - 6x^3 + 8x^2 - 24x + 16$.

(a) It is known that $Q(2i) = 0$. Why does it follow immediately that $Q(-2i) = 0$?

(b) By using the sum and the product of the zeroes of $Q(x)$, or otherwise, find the other two zeroes of $Q(x)$.

(c) Hence write $Q(x)$ as a product of:

(i) four linear factors,

(ii) three factors with real coefficients,

(iii) two factors with integer coefficients.

**7.** (a) Solve the equation $x^4 - 3x^3 + 6x^2 + 2x - 60 = 0$ given that $x = 1 + 3i$ is a root.

(b) Solve the equation $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$ given that $x = 1 - i$ is a root.

**8.** Two of the zeroes of $P(z) = z^4 - 12z^3 + 59z^2 - 138z + 130$ are $a + ib$ and $a + 2ib$, where $a$ and $b$ are real and $b > 0$.

(a) Find the value of $a$ by considering the sum of the zeroes.

(b) Use the product of the zeroes to show that $4b^4 + 45b^2 - 49 = 0$, and hence find $b$.

(c) Hence express $P(z)$ as the product of quadratic factors with real coefficients.

**9.** Suppose that $P(x) = x^3 + kx^2 + 6$, where $k$ is real.

(a) Show that $P(2i) = (6 - 4k) - 8i$.

(b) When $P(x)$ is divided by $x^2 + 4$ the remainder is $-4x - 6$. Find the value of $k$.

**10.** Let $P(x) = x^3 - x^2 + mx + n$, where both $m$ and $n$ are integers.
(a) Show that $P(-i) = (1 + n) + i(1 - m)$.
(b) When $P(x)$ is divided by $x^2 + 1$ the remainder is $6x - 3$. Find the values of $m$ and $n$.

**11.** Suppose that $P(x) = x^3 + x^2 + 6x - 3$.
(a) Use the remainder theorem to find the remainder when $P(x)$ is divided by $x + 2i$.
(b) Hence find the remainder when $P(x)$ is divided by: (i) $x - 2i$, (ii) $x^2 + 4$.

**12.** Let $P(z) = z^8 - \frac{5}{2}z^4 + 1$. Suppose that $w$ is a root of $P(z) = 0$.
(a) Show that $iw$ and $\frac{1}{w}$ are also roots of $P(z) = 0$.
(b) Find one of the roots of $P(z) = 0$ in exact form.
(c) Hence find all the roots of $P(z) = 0$.

**13.** Suppose that $P(x) = x^4 + Ax^2 + B$, where $A$ and $B$ are positive real numbers.
(a) Explain why $P(x)$ has no real zeroes.
(b) Given that $ic$ and $id$, where $c$ and $d$ are real, are zeroes of $P(x)$, write down the other two zeroes of $P(x)$, and give a reason.
(c) Prove that $c^4 + d^4 = A^2 - 2B$.

**14.** The polynomial $P(x) = x^3 + cx + d$, where $c$ and $d$ are real and non-zero, has a negative real zero $k$, and two complex zeroes. The graph of $y = P(x)$ has two turning points.
(a) What can be said about the two complex zeroes of $P(x)$, and why?
(b) By considering $P'(x)$, show that $c < 0$.
(c) Sketch the graph of $y = P(x)$.
(d) If $a \pm ib$, where $a$ and $b$ are real, are the complex zeroes of $P(x)$, deduce that $a > 0$.
(e) Prove that $d = 8a^3 + 2ac$.

**15.** Consider the polynomial function $f(x) = x^3 - 3x + k$, where $k$ is an integer greater than 2.
(a) Show that $f(x)$ has exactly one real zero $r$, and explain why $r < -1$.
(b) Give a reason why the two complex zeroes of $f(x)$ form a conjugate pair.
(c) If the complex zeroes are $a + ib$ and $a - ib$, use the result for the sum of the roots two at a time to show that $b^2 = 3(a^2 - 1)$.
(d) Find the three zeroes of $f(x)$ given that $k = 2702$, and that $a$ and $b$ are integers.

**EXTENSION**

**16.** In the text it was proven that if $P(z)$ is a polynomial with real coefficients and if $P(\alpha) = 0$ then $P(\overline{\alpha}) = 0$. Use a similar approach to prove that if $P(\alpha) = \beta$ then $P(\overline{\alpha}) = \overline{\beta}$.

**17.** Let $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be a polynomial with integer coefficients. Suppose that $P(x)$ has a rational zero $x = \frac{p}{q}$ where $p$ and $q$ have highest common factor 1. Show that $p$ is a factor of $a_0$ and that $q$ is a factor of $a_n$.

**18.** (a) Let $u$ and $v$ be two numbers of the form $u = a + b\sqrt{c}$, where $a$, $b$ and $c$ are rational numbers but where $\sqrt{c}$ is irrational. Let the notation $u^*$ indicate the value of $u$ when the sign of $b$ is reversed. That is, $u^* = a - b\sqrt{c}$.
(i) Show that $u^* + v^* = (u + v)^*$.
(ii) Show that $\lambda u^* = (\lambda u)^*$ whenever $\lambda$ is a rational number.
(iii) Prove by induction that $(u^n)^* = (u^*)^n$ for positive integers $n$.
(b) Suppose that $u = a + b\sqrt{c}$ is a zero of a certain polynomial with rational coefficients. Use the results of part (a) to show that $u^* = a - b\sqrt{c}$ is also a zero of this polynomial.

# 5B Multiple Zeroes

Multiple zeroes of a polynomial were encountered in the Mathematics Extension 1 course. Recall that for the polynomial $P(x) = (x+2)^3x^2(x-2)$ the value $x = -2$ is called a *triple zero*, the value $x = 0$ is called a *double zero*, and the value $x = 2$ is called a *simple zero*. The general situation is summarised here.

**4**

MULTIPLE ZEROES: Suppose that the polynomial $P(x)$ may be factored as

$$P(x) = (x-\alpha)^m Q(x)\,, \text{ where } Q(\alpha) \neq 0\,.$$

Then $x = \alpha$ is called a *zero of multiplicity* $m$.

A zero of multiplicity 1 is called a *simple zero*, and a zero of multiplicity greater than 1 is called a *multiple zero*.

In the special case of a polynomial $P(x)$ with real coefficients which has a real zero $x = \alpha$ with multiplicity $m > 1$ it can be shown that the derivative $P'(x)$ has the same real zero $x = \alpha$ but with multiplicity $(m-1)$.

**WORKED EXERCISE:** The polynomial $P(x) = x^3 - 3x^2 + 4$ has a double zero.

(a) Find the double zero and hence factor $P(x)$.

(b) Sketch the graph of $y = P(x)$.

**SOLUTION:**

(a) Since $P(x)$ has a double zero, it is a solution of $P'(x) = 0$, that is:

$$3x^2 - 6x = 0$$

so $\quad 3x(x-2) = 0\,.$

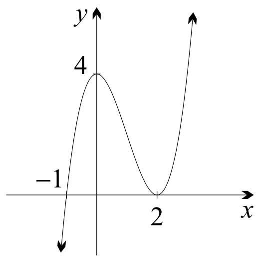
Thus $\quad x = 0$ or $2$.

Now $\quad P(0) = 4$ and $P(2) = 0\,.$

Thus $x = 2$ is the double zero.

By the product of roots, the third zero is $x = -1$.

Hence $P(x) = (x-2)^2(x+1)\,.$




(b) The graph is shown on the right.

**5**

MULTIPLE ZEROES AND THE DERIVATIVE: Suppose that the polynomial $P(x)$ with real coefficients has a real zero $x = \alpha$ with multiplicity $m > 1$.

Then $x = \alpha$ is a zero of $P'(x)$ with multiplicity $(m-1)$.

PROOF: Let $P(x) = (x-\alpha)^m Q_0(x)$, where $Q_0(\alpha) \neq 0$ and where $m > 1$. Then

$$\begin{aligned} P'(x) &= m(x-\alpha)^{m-1}Q_0(x) + (x-\alpha)^m Q_0'(x) \qquad \text{(by the product rule)} \\ &= (x-\alpha)^{m-1}\Big(mQ_0(x) - (x-\alpha)Q_0'(x)\Big)\,. \end{aligned}$$

Let $Q_1(x)$ be the term in brackets involving $Q_0$ and $Q_0'$. Then:

$$P'(x) = (x-\alpha)^{m-1}Q_1(x)\,,$$

where $Q_1(x) = mQ_0(x) - (x-\alpha)Q_0'(x)\,.$

Now $\quad Q_1(\alpha) = mQ_0(\alpha)$

$\qquad\qquad \neq 0 \qquad$ (since $Q_0(\alpha) \neq 0$ and $m > 1\,$.)

Hence $x = \alpha$ is a zero of $P'(x)$ with multiplicity exactly equal to $(m-1)\,.$

In fact this result is also true for polynomials with complex zeroes but the proof is beyond the scope of this course.

**Multiple Zeroes and Higher Derivatives:** Suppose that the polynomial $P(x)$ with real coefficients has a triple zero $x = \alpha$ which is real. Applying the above theorem repeatedly gives:

$x = \alpha$ is a double zero of $P'(x)$, and
$x = \alpha$ is a simple zero of $P''(x)$.

**WORKED EXERCISE:** It is known that the polynomial $P(x) = x^4 - 6x^2 - 8x - 3$ has a triple zero. (a) Find the triple zero. (b) Hence factorise $P(x)$.

**SOLUTION:** (a) Differentiating:

$$\begin{aligned} P'(x) &= 4x^3 - 12x - 8 \\ \text{and} \quad P''(x) &= 12x^2 - 12 \\ &= 12(x-1)(x+1)\,. \end{aligned}$$

Thus the possible values of the triple zero are $x = 1$ or $x = -1$.
Since $P'(-1) = 0$ and $P(-1) = 0$ it follows that $x = -1$ is the triple zero.

(b) Let $x = \alpha$ be the remaining zero, then by the sum of the zeroes,

$$\begin{aligned} 3 \times (-1) + \alpha &= 0 \\ \text{thus} \qquad \alpha &= 3\,. \\ \text{Hence} \qquad P(x) &= (x+1)^3(x-3)\,. \end{aligned}$$

This example of a triple zero can be extended to the general case of a polynomial $P(x)$ with real coefficients which has a real zero $x = \alpha$ of multiplicity $m$. The value $x = \alpha$ is also a zero of each of the derivatives $P^{(j)}(x)$, for $j = 1, \ldots, (m-1)$.

**6** MULTIPLE ZEROES AND HIGHER DERIVATIVES: Suppose that the polynomial $P(x)$ with real coefficients has a real zero $x = \alpha$ with multiplicity $m > 1$.

Then $x = \alpha$ is a zero of each of the derivatives $P^{(j)}(x)$, for $j = 1, \ldots, (m-1)$.

This result can be proved relatively easily by induction and is left as an exercise. It is also true for polynomials with complex zeroes but again the proof is beyond the scope of this course.

**The Fundamental Theorem of Algebra:** All the work encountered so far in this chapter deals with finding the zeroes of various polynomials. Up to this point it has been possible to sidestep an important question: does every polynomial have a zero? For there is no point in searching for one if none exists.

In order to emphasise this point, consider the polynomial $P(x) = x^2 + 1$. Clearly this function has no real zero, and there is no point in searching for one. Yet the polynomial does indeed have two zeroes, both of which happen to be complex numbers: namely $i$ and $-i$. Could it be that there is another polynomial which has neither real nor complex zeroes?

The answer to this question is: every polynomial with degree $\geq 1$ has at least one zero, though that zero may be complex. This is such an important and basic fact in the study of mathematics that it is given a title — *The Fundamental Theroem of Algebra.*

**7** THE FUNDAMENTAL THEOREM OF ALGEBRA: Every polynomial with degree $\geq 1$ has at least one zero, though that zero may be complex.

Several eminent mathematicians worked on this theorem including Leibniz, Euler and Argand. But credit is usually given to Gauss for the first proof, which he presented in his doctoral thesis in 1799. This, or any other proof of the theorem, is beyond the scope of this course.

Although the wording given in the box above is imprecise, it is usually sufficent for the problems encountered at this level. Those who have read more widely will know that this theorem may be formally stated in a number of different ways, including: *every polynomial with complex coefficients and degree $\geq 1$ has at least one complex zero.*

**The Degree and the Number of Zeroes:** Although the Fundamental Theorem of Algebra cannot be proven here, it is possible to prove two significant consequences of the theorem. The first is that every polynomial of degree $n \geq 1$ with complex coefficents has precisely $n$ zeroes, as counted by their multiplicities.

This is also true for polynomials with real coefficients. To demonstrate the result, recall that the cubic $P(x) = x^3 - 3x^2 + 4$ encountered in the first worked exercise has three zeroes: the simple zero $x = -1$ and the double zero $x = 2$.

**8** THE DEGREE AND THE NUMBER OF ZEROES: Every polynomial of degree $n \geq 1$ with complex coefficents has precisely $n$ zeroes, as counted by their multiplicities.

PROOF: This proof uses induction.

A. Consider the general polynomial of degree one with complex coefficients:

$$P_1(x) = a_0 + a_1 x\,, \qquad \text{where } a_1 \neq 0\,.$$

Clearly this polynomial has one zero $x = \alpha_1$, where

$$\alpha_1 = -\frac{a_0 \overline{a_1}}{|a_1|^2}\,.$$

Thus the result is true for $n = 1$.

B. Suppose that the result is true for some integer $k \geq 1$. That is, suppose that every polynomial of degree $k$ with complex coefficients

$$P_k(x) = a_0 + a_1 x + \ldots + a_k x^k\,, \qquad \text{where } a_k \neq 0\,,$$

has $k$ zeroes, $x = \alpha_1, \ldots, \alpha_k$, as counted by their multiplicities. $(**)$

The statement is now proven true for $n = k+1$. That is, it is proven that every polynomial of degree $k+1$ with complex coefficents

$$P_{k+1}(x) = a_0 + a_1 x + \ldots + a_{k+1} x^{k+1}\,, \qquad \text{where } a_{k+1} \neq 0\,,$$

has $k+1$ zeroes.

Now for any particular polynomial $P_{k+1}(x)$, that polynomial has at least one zero by the Fundamental Theorem of Algebra. Let this zero be $x = \alpha_{k+1}$. Then, by the factor theorem, it follows that

$$P_{k+1}(x) = (x - \alpha_{k+1}) Q_k(x)$$

for some polynomial $Q_k(x)$ of degree $k$. But by the induction hypothesis above $(**)$, $Q_k(x)$ has $k$ zeroes, all of which are thus inherited by $P_{k+1}(x)$.

Hence $P_{k+1}(x)$ has $k+1$ zeroes, $x = \alpha_1, \ldots, \alpha_k, \alpha_{k+1}$, as counted by their multiplicities. Clearly this follows for each and every polynomial $P_{k+1}(x)$.

C. It follows from parts A and B by mathematical induction that the statement is true for all integers $n \geq 1$.

**Real Linear and Quadratic Factors:** The second significant consequence of the Fundamental Theorem of Algebra is that every polynomial of degree $n \geq 1$ with real coefficients can be written as a product of factors which are either linear or *irreducible* quadratics, each with real coefficients. In this context the word *irreducible* is used to indicate that the quadratic has no real zero.

In order to demonstrate the result, notice that the polynomial $P(x) = x^3 - 1$ can be written as the product

$$P(x) = (x-1)(x^2+x+1)\,.$$

The quadratic factor $(x^2+x+1)$ is irreducible since it has no real zero.

**9** REAL LINEAR AND QUADRATIC FACTORS: Every polynomial of degree $n \geq 1$ which has real coefficents can be written as a product of factors which are either linear or irreducible quadratics, each with real coefficients.

PROOF: Let $P_n(x) = a_0 + a_1x + \ldots + a_nx^n$ be a polynomial with degree $n \geq 1$ which has real coefficients. By the previous result, this polynomial has $n$ zeroes. Let these zeroes be $x = \alpha_1, \ldots, \alpha_n$.

If some of the zeroes are complex numbers then they occur as conjugate pairs, since the coefficients of $P_n(x)$ are real. Let the number of conjugate pairs be $j$, where $1 < 2j \leq n$. Now re-order and re-label the zeroes with the conjugate pairs listed first. Thus the first conjugate pair is $x = \alpha_1, \overline{\alpha_1}$, and the last conjugate pair is $x = \alpha_j, \overline{\alpha_j}$.

If there are any other zeroes then they are real. The first of these is $x = \alpha_{2j+1}$ and the last is $x = \alpha_n$. So by the factor theorem, and using product notation:

$$\begin{aligned} P_n(x) &= a_n \times \prod_{k=1}^{j} \Big((x-\alpha_k)(x-\overline{\alpha_k})\Big) \times \prod_{\ell=2j+1}^{n} (x-\alpha_\ell) \\ &= a_n \times \prod_{k=1}^{j} \Big(x^2 - 2\,\mathrm{Re}(\alpha_k)x + |\alpha_k|^2\Big) \times \prod_{\ell=2j+1}^{n} (x-\alpha_\ell)\,, \end{aligned}$$

which is a product of linear and irreducible quadratic factors with real coefficients. Put more simply, multiply all the complex factors together in conjugate pairs to get irreducible quadratic factors with real coefficients, and any remaining factors are both linear and real.

## Exercise **5B**

**1.** Consider the polynomial $P(x) = x^3 - 4x^2 - 3x + 18$.

(a) (i) Show that $P(3)$ and $P'(3)$ are both zero.

(ii) What can be deduced from the results in part (i)?

(b) Use part (a) and the sum of zeroes to find all the zeroes of $P(x)$.

(c) Hence factorise $P(x)$.

**2.** Consider the polynomial $P(x) = x^4 + 8x^3 + 18x^2 + 16x + 5$.
(a) (i) Show that $P(-1)$, $P'(-1)$ and $P''(-1)$ are all zero.
(ii) What can be deduced from the results in part (i)?
(b) Use part (a) and the product of zeroes to find all the zeroes of $P(x)$.
(c) Hence factorise $P(x)$.

**3.** The polynomial $P(x) = x^3 - 27x + 54$ has a double zero.
(a) Find the zeroes of $P'(x)$.
(b) Determine which of the zeroes of $P'(x)$ is the double zero of $P(x)$.
(c) Find the remaining simple zero of $P(x)$.

**4.** The polynomial $P(x) = x^4 + 5x^3 - 75x^2 - 625x - 1250$ has a triple zero.
(a) Find the zeroes of $P''(x)$.
(b) Determine which of the zeroes of $P''(x)$ is the triple zero of $P(x)$.
(c) Find the remaining simple zero of $P(x)$.

**5.** The polynomial $P(x) = 2x^3 + 5x^2 - 4x - 12$ has a double zero.
(a) Find the double zero.
(b) Find the remaining simple zero, and hence factorise $P(x)$.

**6.** The polynomial $P(x) = 8x^4 - 28x^3 + 30x^2 - 13x + 2$ has a triple zero.
(a) Find the triple zero.
(b) Find the remaining simple zero, and hence factorise $P(x)$.

**DEVELOPMENT**

**7.** Consider the polynomial equation $x^4 - 5x^3 + 4x^2 + 3x + 9 = 0$.
(a) Show that $x = 3$ is a double root of the equation.
(b) Hence solve the equation.

**8.** The polynomial $P(x) = x^3 - 3x^2 - 9x + k$ has a double zero.
(a) Find the two possible values of $k$.
(b) For each of the possible values of $k$, factorise $P(x)$.

**9.** The coefficients of the polynomial $P(x) = ax^3 + bx + c$ are real and $P(x)$ has a multiple zero at $x = 1$. When $P(x)$ is divided by $x + 1$ the remainder is 4. Find the values of $a$, $b$ and $c$.

**10.** The polynomial $P(x) = x^4 + 7x^3 + 9x^2 - 27x + c$ has a triple zero.
(a) Determine the value of the triple zero.
(b) Hence find the value of $c$. (c) Factorise $P(x)$.

**11.** (a) Find the values of $b$ and $c$ if $x = 1$ is a double root of the equation
$$x^4 + bx^3 + cx^2 - 5x + 1 = 0\,.$$
(b) Find the other roots of the equation.

**12.** Consider the constant polynomial $P(x) = 1$. Clearly $P(x)$ has no zero, which may appear to contradict the Fundamental Theorem of Algebra. Explain why it does not.

**13.** It is known that $(x - 1)^2$ is a factor of the polynomial $P(x) = ax^{n+1} + bx^n + 1$. Show that $a = n$ and $b = -(1 + n)$.

**14.** The equation $Ax^3 + Bx^2 + D = 0$ has a double root. If $D \neq 0$, prove that $27A^2D + 4B^3 = 0$.

**15.** Prove that $P(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!}$, where $n \geq 2$, has no multiple zeroes.

**16.** (a) Prove that the polynomial equation $x^4 + mx^2 + n = 0$, where $m \neq 0$, cannot have a root of multiplicity greater than 2.

(b) Let $x = \alpha$ be a double root of the equation in part (a).

(i) Prove that $x = -\alpha$ is also a double root. (ii) For what values of $m$ is $\alpha$ real?

(iii) Prove that $n = \frac{1}{4}m^2$. Hence write down the roots of the equation in terms of $m$.

**EXTENSION**

**17.** The polynomial $P(x)$ with real coefficients has a real zero $x = \alpha$ of multiplicity $m > 1$. Use induction on the value of $j$ to prove that $x = \alpha$ is a zero of each of the derivatives $P^{(j)}(x)$, for $j = 0, 1, \ldots, (m-1)$.

**18.** Use the Fundamental Theorem of Algebra to carefully explain why every polynomial of odd degree with real coefficients has at least one real zero.

**19.** The polynomial $P(x) = x^3 + 3px^2 + 3qx + r$ has a double zero.

(a) Prove that the double zero is $\alpha = \dfrac{pq - r}{2(q - p^2)}$.

(b) Hence, or otherwise, prove that $4(p^2 - q)(q^2 - pr) = (pq - r)^2$.

**20.** The polynomial $P(z)$ has real coefficients and a double complex zero $z = \alpha$.

(a) Prove that $z = \overline{\alpha}$ is also a double zero.

(b) Explain why $\big(z - 2\,\mathrm{Re}(\alpha) + |\alpha|^2\big)^2$ is a factor of $P(z)$.

(c) Hence prove that $P'(\alpha) = 0$.

(d) Can you generalise this result to complex zeros with higher multiplicity?

# 5C The Zeroes and The Coefficients

The relationships between the zeroes and the coefficients of quadratics, cubics and quartics with real coefficients were encountered in the Mathematics Extension 1 course. It is not difficult to prove that those relationships are also valid for polynomials with complex coefficients or complex zeroes, or both. As a matter of convenience, those relationships are repeated here. In each case, let the zeroes be $\alpha_1, \ldots, \alpha_n$ and the coefficients be $a_0, \ldots, a_n$, where $n$ is the degree.

**10** ZEROES AND COEFFICIENTS OF A QUADRATIC:

$$\alpha_1 + \alpha_2 = -\frac{a_1}{a_2} \qquad \text{and} \qquad \alpha_1\alpha_2 = \frac{a_0}{a_2}.$$

**11** ZEROES AND COEFFICIENTS OF A CUBIC:

$$\alpha_1 + \alpha_2 + \alpha_3 = -\frac{a_2}{a_3}, \quad \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = \frac{a_1}{a_3} \quad \text{and} \quad \alpha_1\alpha_2\alpha_3 = -\frac{a_0}{a_3}$$

**12**

ZEROES AND COEFFICIENTS OF A QUARTIC:

$$\begin{aligned}
\alpha_1+\alpha_2+\alpha_3+\alpha_4 &= -\frac{a_3}{a_4}\\
\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_4 &= +\frac{a_2}{a_4}\\
\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4 &= -\frac{a_1}{a_4}\\
\alpha_1\alpha_2\alpha_3\alpha_4 &= +\frac{a_0}{a_4}
\end{aligned}$$

In practice, often only the first and last formulae of each box are required.

**The General Case:** Looking carefully at these results, it is evident that in each box the left hand side of successive equations is the sum of the zeroes taken one at a time, then two at a time, then three at a time and so on. Thus it is possible to generalise the formulae as follows.

**13**

ZEROES AND COEFFICIENTS OF A POLYNOMIAL:

$$\text{sum of roots taken } j \text{ at a time} = (-1)^j \times \frac{a_{n-j}}{a_n} \qquad \text{for } 1 \le j \le n\,.$$

**WORKED EXERCISE:** The polynomial equation $3x^3+7x^2+11x+51=0$ has roots $\alpha$, $\beta$ and $\gamma$.

(a) Evaluate $\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2$.

(b) (i) Find $\alpha^2+\beta^2+\gamma^2$.

(ii) Use part (i) to determine how many of the roots are real.

(c) Determine the value of $3(\alpha^3+\beta^3+\gamma^3)$.

**SOLUTION:** (a) Using the sum and product of roots:

$$\alpha+\beta+\gamma = -\tfrac{7}{3}$$

and

$$\begin{aligned}\alpha\beta\gamma &= -\tfrac{51}{3}\\ &= -17\,.\end{aligned}$$

Thus

$$\begin{aligned}\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2 &= \alpha\beta\gamma(\alpha+\beta+\gamma)\\ &= (-\tfrac{7}{3})\times(-17)\\ &= \tfrac{119}{3}\,.\end{aligned}$$

(b) (i)

$$\begin{aligned}\alpha^2+\beta^2+\gamma^2 &= (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)\\ &= (-\tfrac{7}{3})^2-2(\tfrac{11}{3})\\ &= -\tfrac{17}{9}\,.\end{aligned}$$

(ii) Since the sum of the squares of the roots is negative, at least one of them is complex.

Since the coefficients of the polynomial equation are real, complex roots come in conjugate pairs, and hence exactly two of them are complex.

Thus there is precisely one real root of the polynomial equation.

(c) Re-arranging the given equation

$$3x^3 = -7x^2-11x-51$$

which is true for each of the three roots. Thus:

$$3\alpha^3 = -7\alpha^2 - 11\alpha - 51\,,$$
$$3\beta^3 = -7\beta^2 - 11\beta - 51\,,$$
$$3\gamma^3 = -7\gamma^2 - 11\gamma - 51\,.$$

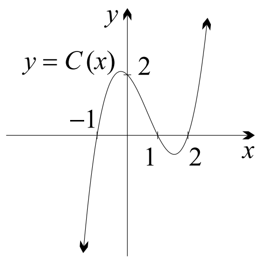
Adding these three yields:

$$\begin{aligned}3(\alpha^3+\beta^3+\gamma^3) &= -7(\alpha^2+\beta^2+\gamma^2) - 11(\alpha+\beta+\gamma) - 3\times 51\\ &= -7\times(-\tfrac{17}{9}) - 11\times(-\tfrac{7}{3}) - 3\times 51\\ &= -114\tfrac{1}{9}\,.\end{aligned}$$

**Transforming Roots:** Let the polynomial $P(x)$ have real zeroes and real coefficients. If the graph of $y = P(x)$ is transformed by a horizontal shift or by a horizontal enlargement then the zeroes will be similarly transformed. To demonstrate this characteristic, consider the cubic polynomial

$$\begin{aligned}C(x) &= x^3 - 2x^2 - x + 2\\ &= (x+1)(x-1)(x-2)\end{aligned}$$

which has zeroes at $x = -1, 1$ and $2$.



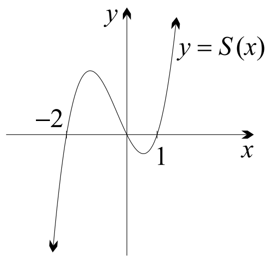

Shifting $y = C(x)$ by 1 unit to the left yields:

$$\begin{aligned}S(x) &= C(x+1)\\ &= (x+1)^3 - 2(x+1)^2 - (x+1) + 2\\ &= x^3 + x^2 - 2x\,.\end{aligned}$$

The new polynomial may be factored as

$$S(x) = (x+2)x(x-1)\,,$$

which clearly has zeroes at $x = -2, 0$ and $1$. This verifies that the zeroes have been shifted to the left by the same amount. The graph on the right also confirms the fact.

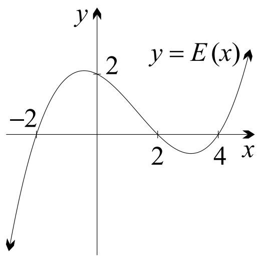



Enlarging $y = C(x)$ horizontally by a factor of 2 yields:

$$\begin{aligned}E(x) &= C(\tfrac{1}{2}x)\\ &= (\tfrac{1}{2}x+1)(\tfrac{1}{2}x-1)(\tfrac{1}{2}x-2)\end{aligned}$$

or $\quad E(x) = \frac{1}{8}(x+2)(x-2)(x-4)\,.$

The zeroes of $E(x)$ are $x = -2, 2$ and $4$. This verifies that the zeroes of $C(x)$ have been enlarged by the same factor. The graph on the right also confirms this fact.




Although it will not be proven here, the same procedures may be applied to transform complex zeroes of a polynomial.

**14**

TRANSFORMED ROOTS:

SHIFTING: To shift the zeroes by an amount $k$, replace $x$ with $(x-k)$.

STRETCHING: To enlarge the zeroes by a factor of $a$, replace $x$ with $\dfrac{x}{a}$.

**WORKED EXERCISE:** The polynomial $P(x) = x^3 - 5x + 3$ has zeroes $\alpha$, $\beta$ and $\gamma$. Find a cubic polynomial with integer coefficients which has zeroes $3\alpha$, $3\beta$ and $3\gamma$.

**SOLUTION:** A polynomial with the required zeroes is:

$P(\frac{1}{3}x) = \frac{1}{27}x^3 - \frac{5}{3}x + 3\,.$

Thus a polynomial with integer coefficients is:

$I(x) = 27P(\frac{1}{3}x),$

so $I(x) = x^3 - 45x + 81\,.$

**Reciprocal of the Roots:** In the worked exercise above, the result of the substitution is a new polynomial. Some substitutions, however, lead to a new function instead of a polynomial. Nevertheless, the corresponding equation can be re-arranged into a polynomial equation. The precise effect on the roots of the equation will depend on the particular substitution. One special transformation is the reciprocal of the roots.

For the sake of simplicity, begin by considering the monic quadratic

$$P(x) = (x - \alpha)(x - \beta)\,.$$

It should be clear that the roots of the corresponding quadratic equation $P(x) = 0$ are $x = \alpha$ and $\beta$. What happens to these roots if $x$ is replaced with $\frac{1}{x}$?

When written out in full, the equation $P(\frac{1}{x}) = 0$ becomes:

$$(\tfrac{1}{x} - \alpha)(\tfrac{1}{x} - \beta) = 0\,.$$

Although this is not a polynomial equation, it is equivalent to one since it may be multiplied by $x^2$ to get:

$$(1 - \alpha x)(1 - \beta x) = 0\,.$$

Thus the roots of the equation are $x = \frac{1}{\alpha}$ and $\frac{1}{\beta}$. Hence the substitution yields a new polynomial equation with roots which are the reciprocal of the original.

The same result is obtained for all polynomials, not just quadratics. There is of course a problem if one of the roots is zero, since the reciprocal of zero is undefined. This issue is dealt with in one of the exercise questions.

**15** RECIPROCAL OF THE ROOTS: To get a new polynomial equation with roots which are the reciprocal of the original, replace $x$ with $\frac{1}{x}$ and rearrange.

**WORKED EXERCISE:** The equation $x^3 - x^2 - 7x + 15 = 0$ has roots $\alpha$, $\beta$ and $\gamma$.

(a) Find a polynomial equation with integer coefficients which has roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$.

(b) Hence write down the value of $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$.

**SOLUTION:** (a) An equation with reciprocal roots is:

$\frac{1}{x^3} - \frac{1}{x^2} - \frac{7}{x} + 15 = 0\,.$

Multiply this by $x^3$ to get:

$1 - x - 7x^2 + 15x^3 = 0\,.$

(b) The required expression is the sum of the reciprocal roots taken two at a time. Thus, from the polynomial equation in part (a), its value is:

$\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} = \frac{-1}{15}\,.$

**Square of the Roots:** The other special substitution encountered in this is course is when $x$ is replaced by $\sqrt{x}$. Again, for the sake of simplicity, begin by considering the quadratic

$$P(x) = (x - \alpha)(x - \beta)\,.$$

Writing out $P(\sqrt{x}) = 0$ in full:

$$(\sqrt{x} - \alpha)(\sqrt{x} - \beta) = 0\,.$$

Expand this to get:

$$x - (\alpha + \beta)\sqrt{x} + \alpha\beta = 0$$

or

$$x + \alpha\beta = (\alpha + \beta)\sqrt{x}\,.$$

Squaring yields:

$$x^2 + 2\alpha\beta x + \alpha^2\beta^2 = (\alpha^2 + 2\alpha\beta + \beta^2)x$$

so $\quad x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0$

or $\quad (x - \alpha^2)(x - \beta^2) = 0\,.$

Thus the roots of the equation are $x = \alpha^2$ and $\beta^2$. Hence the substitution yields a new polynomial equation with roots which are the square of the original.

The same result is obtained for all polynomials, not just quadratics.

**16** SQUARE OF THE ROOTS: To get a new polynomial equation with roots which are the square of the original, replace $x$ with $\sqrt{x}$ and rearrange.

**WORKED EXERCISE:** The equation $3x^3 + 7x^2 + 11x + 51 = 0$ has roots $\alpha$, $\beta$ and $\gamma$.

(a) Find a polynomial equation with integer coefficients that has roots $\alpha^2$, $\beta^2$ and $\gamma^2$.

(b) Hence evaluate $\alpha^2 + \beta^2 + \gamma^2$.

**SOLUTION:**

(a) Substitute $\sqrt{x}$ for $x$ to get:

$$\sqrt{x}(3x + 11) + 7x + 51 = 0$$

or

$$\sqrt{x}(3x + 11) = -(7x + 51)\,.$$

Squaring both sides yields:

$$x(3x + 11)^2 = (7x + 51)^2$$

so $\quad 9x^3 + 66x^2 + 121x = 49x^2 + 714x + 2601$

or $\quad 9x^3 + 17x^2 - 593x - 2601 = 0\,.$

(b) From the coefficients of $x^2$ and $x^3$ it follows that $\alpha^2 + \beta^2 + \gamma^2 = -\frac{17}{9}$.

**Extension — Other Transformations:** Some readers will have already discerned an apparent relationship between the substitution and the effect on the roots. They are inverse functions, and this is born out in the above worked exercises. Thus the substitution $\frac{1}{3}x$ trebled the roots, whilst the substitution $\sqrt{x}$ squared the roots. The relationship holds for all substitutions and can be stated as follows.

Let the polynomial equation $P(x) = 0$ have a root $x = \alpha$. Given a function $g(x)$ which has an inverse function $g^{-1}(x)$, the corresponding root of the equation $P(g(x)) = 0$ is $g^{-1}(\alpha)$, provided that $g^{-1}(\alpha)$ exists.

**17** TRANSFORMED ROOTS: Let the polynomial equation $P(x) = 0$ have a root $x = \alpha$. The corresponding root of the equation $P(g(x)) = 0$ is $g^{-1}(\alpha)$, provided that $g^{-1}(\alpha)$ exists.

PROOF: Since $P(x) = 0$ has a root $x = \alpha$ it follows from the Fundamental Theorem of Algebra that

$$P(x) = (x - \alpha) \times Q(x)\,.$$

The equation $P(g(x)) = 0$ thus becomes

$$(g(x) - \alpha) \times Q(g(x)) = 0\,.$$

One solution of this equation is:

$$g(x) = \alpha$$

whence $x = g^{-1}(\alpha)$ (if it exists.)

**WORKED EXERCISE:** The equation $2x^2 - 3x - 2 = 0$ has roots $x = -\frac{1}{2}$ and 2. Find the roots of $2\cos^2 x - 3\cos x - 2 = 0$ for $0 \le x \le \pi$.

**SOLUTION:** Clearly, $x$ has been replaced with $\cos x$. Since there is no real value for $\cos^{-1} 2$, there is only one real solution, viz:

$$\begin{aligned} x &= \cos^{-1}(-\tfrac{1}{2}) \\ &= \tfrac{2\pi}{3} \end{aligned}$$

## Exercise **5C**

**1.** Consider the polynomial $P(x) = x^2 + 9$, and let $Q(x) = P(x - 2)$.
(a) Write down the zeroes of $P(x)$.
(b) Hence write down the zeroes of $Q(x)$. (c) Find $Q(x)$.
(d) By solving the equation $Q(x) = 0$, confirm your answer to part (b).

**2.** Consider the polynomial $P(x) = x^2 + 8x + 20$, and let $Q(x) = P(2x)$.
(a) Find the zeroes of $P(x)$.
(b) Hence write down the zeroes of $Q(x)$. (c) Find $Q(x)$.
(d) By solving the equation $Q(x) = 0$, confirm your answers to part (b).

**3.** The polynomial $P(x) = x^3 - 5x + 3$ has zeroes $\alpha$, $\beta$ and $\gamma$. By replacing $x$ with $\frac{1}{2}x$, find a polynomial with integer coefficients which has zeroes $2\alpha$, $2\beta$ and $2\gamma$.

**4.** Let $\alpha$, $\beta$ and $\gamma$ be the roots of the equation $x^3 - 5x^2 + 5 = 0$. By replacing $x$ with $(x + 1)$, find a polynomial equation with integer coefficients which has roots $(\alpha - 1)$, $(\beta - 1)$ and $(\gamma - 1)$.

**5.** Consider the polynomial equation $x^3 - 4x^2 + 6x - 8 = 0$ with roots $\alpha$, $\beta$ and $\gamma$. Find the value of:
(a) $\alpha + \beta + \gamma$ (c) $\alpha^3 + \beta^3 + \gamma^3$ (e) $\alpha^5 + \beta^5 + \gamma^5$
(b) $\alpha^2 + \beta^2 + \gamma^2$ (d) $\alpha^4 + \beta^4 + \gamma^4$

**DEVELOPMENT**

**6.** Suppose that the roots of the polynomial equation $x^3 - 3x + 1 = 0$ are $\alpha$, $\beta$ and $\gamma$.
(a) Replace $x$ with $\dfrac{1}{x}$ to find a polynomial equation with roots $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$.
(b) Replace $x$ with $\sqrt{x}$ to find a polynomial equation with roots $\alpha^2$, $\beta^2$ and $\gamma^2$.

**7.** Suppose that $\alpha$, $\beta$, $\gamma$ and $\delta$ are the roots of the polynomial equation

$$x^4 + 2x^3 - x^2 + 4x - 3 = 0\,.$$

(a) (i) Find a polynomial equation with roots $\dfrac{\alpha}{3}$, $\dfrac{\beta}{3}$, $\dfrac{\gamma}{3}$ and $\dfrac{\delta}{3}$.

(ii) Hence find a polynomial equation with the reciprocal roots $\dfrac{3}{\alpha}$, $\dfrac{3}{\beta}$, $\dfrac{3}{\gamma}$ and $\dfrac{3}{\delta}$.

(b) (i) Find a polynomial equation with roots $(\alpha - 3)$, $(\beta - 3)$, $(\gamma - 3)$ and $(\delta - 3)$.

(ii) Hence find a polynomial equation with the opposite roots $(3-\alpha)$, $(3-\beta)$, $(3-\gamma)$ and $(3-\delta)$.

**8.** Let the roots of the polynomial equation $x^3 + mx + n = 0$ be $\alpha$, $\beta$ and $\gamma$.

(a) Find a cubic polynomial equation, with coefficients in terms of $m$ and $n$, which has roots $\alpha^2$, $\beta^2$ and $\gamma^2$,

(b) Use part (a) and a suitable substitution to find a cubic polynomial equation which has roots $\dfrac{1}{\alpha^2}$, $\dfrac{1}{\beta^2}$ and $\dfrac{1}{\gamma^2}$.

**9.** (a) Expand $(\sqrt{3}+1)^2$.

(b) The polynomial equation $x^4 + 4x^3 - 2x^2 - 12x - 3 = 0$ has roots $\alpha$, $\beta$, $\gamma$ and $\delta$. Find the polynomial equation with roots $(\alpha+1)$, $(\beta+1)$, $(\gamma+1)$ and $(\delta+1)$.

(c) Hence, or otherwise, solve the equation $x^4 + 4x^3 - 2x^2 - 12x - 3 = 0$.

**10.** (a) Find the quartic equation whose roots exceed by 3 the roots of the equation $x^4 + 12x^3 + 49x^2 + 78x + 42 = 0$.

(b) Hence or otherwise solve the equation given in part (a).

**11.** The polynomial equation $2x^3 + 8x^2 + 3x - 6 = 0$ has roots $\alpha$, $\beta$ and $\gamma$.

(a) Evaluate: (i) $\alpha+\beta+\gamma$ (ii) $\alpha\beta\gamma$

(b) Hence find a polynomial equation with roots:

(i) $2\alpha+\beta+\gamma$, $\alpha+2\beta+\gamma$ and $\alpha+\beta+2\gamma$ (ii) $\alpha^2\beta\gamma$, $\alpha\beta^2\gamma$ and $\alpha\beta\gamma^2$

**12.** The polynomial equation $x^4 - ax^3 + bx^2 - abx + 1 = 0$ has roots $\alpha$, $\beta$, $\gamma$ and $\delta$.

(a) Explain why $\alpha+\beta+\gamma = a - \delta$.

(b) Hence show that $(\alpha+\beta+\gamma)(\alpha+\beta+\delta)(\alpha+\gamma+\delta)(\beta+\gamma+\delta) = 1$.

**13.** The numbers $\alpha$, $\beta$ and $\gamma$ satisfy the three equations

$$\begin{aligned} \alpha+\beta+\gamma &= 5 \\ \alpha^2+\beta^2+\gamma^2 &= 9 \\ \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} &= \tfrac{4}{3} \end{aligned}$$

(a) Find the value of: (i) $\alpha\beta+\alpha\gamma+\beta\gamma$ (ii) $\alpha\beta\gamma$

(b) Explain why $\alpha$, $\beta$ and $\gamma$ are the roots of $x^3 - 5x^2 + 8x - 6 = 0$.

(c) Find the values of $\alpha$, $\beta$ and $\gamma$.

**14.** The equation $x^4 + px^3 + qx^2 + rx + s = 0$ has roots $\alpha$, $\beta$, $\gamma$ and $\delta$.

(a) Find the values of the following in terms of $p$, $q$, $r$ and $s$.

(i) $\alpha+\beta+\gamma+\delta$ (ii) $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta$

(b) Show that $\alpha^2+\beta^2+\gamma^2+\delta^2 = p^2 - 2q$.

(c) (i) Let $P(x) = x^4 - 3x^3 + 5x^2 + 7x - 8$. Use part (b) to show that $P(x) = 0$ cannot have four real roots.

(ii) Evaluate $P(0)$ and $P(1)$, and hence explain why $P(x) = 0$ has exactly two real roots.

**15.** Suppose that the polynomial $P(z) = z^3 + fz^2 + gz + h$ has zeroes $\alpha$, $-\alpha$ and $\beta$. The numbers $f$, $g$ and $h$ are real.

(a) Prove that $fg = h$.

(b) It is known that $P(z)$ does not have three real zeroes. Explain why two of them are purely imaginary. [A complex number $z$ is purely imaginary if it has the form $z = iy$ where $y$ is real.]

**16.** Let $\alpha$, $\beta$ and $\gamma$ be the roots of the cubic equation $x^3 - px - q = 0$, and define $S_n$ by

$$S_n = \alpha^n + \beta^n + \gamma^n \qquad \text{for } n = 1, 2, 3, \ldots$$

(a) Explain why $S_1 = 0$, and show that $S_2 = 2p$ and $S_3 = 3q$.

(b) Prove that for $n > 3$, $S_n = pS_{n-2} + qS_{n-3}$.

(c) Deduce that $\dfrac{\alpha^5 + \beta^5 + \gamma^5}{5} = \left(\dfrac{\alpha^2 + \beta^2 + \gamma^2}{2}\right)\left(\dfrac{\alpha^3 + \beta^3 + \gamma^3}{3}\right)$.

**17.** Consider the polynomial $P(x) = x^3 + qx^2 + qx + 1$, where $q$ is real. It is known that $-1$ is a zero of $P(x)$.

(a) Show that if $\alpha$ is a zero of $P(x)$ then so too is $\dfrac{1}{\alpha}$.

(b) Let $\alpha$ be a complex number where $\text{Im}(\alpha) \neq 0$ and $P(\alpha) = 0$.

(i) Show that $|\alpha| = 1$. (ii) Show that $\text{Re}(\alpha) = \dfrac{1-q}{2}$.

**EXTENSION**

**18.** The equation $3x^3 - 5x^2 - 2x = 0$ has roots $x = 2, 0$ and $-\frac{1}{3}$. Investigate what happens to these roots when $x$ is replaced by $\frac{1}{x}$. Can you generalise your conclusions?

**19.** The equation $x^3 + px^2 + qx + r = 0$ has roots $a$, $b$ and $c$.

(a) Find a cubic polynomial equation with roots $\dfrac{a+b}{c}$, $\dfrac{b+c}{a}$ and $\dfrac{c+a}{b}$.

(b) Hence or otherwise show that $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a} + \dfrac{b}{a} + \dfrac{c}{b} + \dfrac{a}{c} = \dfrac{pq}{r} - 3$.

**20.** The monic degree $n$ polynomial $P(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ has zeroes 1, 2, 3, ..., $n$.

(a) Find expressions for: (i) $a_{n-1}$ (ii) $a_0$

(b) (i) Prove by induction that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integer values of $n$.

(ii) Hence prove that $a_{n-2} = \frac{1}{24}n(n-1)(n+1)(3n+2)$.

**21.** Let $f(x) = x^3 + cx + d$ have three distinct zeroes $t_1$, $t_2$ and $t_3$.

(a) By considering the graph of $y = f(x)$, explain why $f'(t_1)f'(t_2)f'(t_3) < 0$.

(b) Show that a cubic with zeroes ${t_1}^2$, ${t_2}^2$ and ${t_3}^2$ is $g(x) = x^3 + 2cx^2 + c^2x - d^2$.

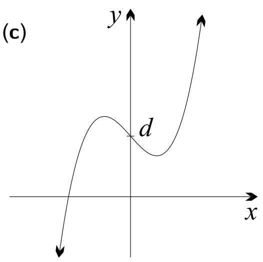
(c) Show that $f'(t_1)f'(t_2)f'(t_3) = -27 \times g(-\frac{c}{3})$.

(d) Hence show that $4c^3 + 27d^2 < 0$.

# Chapter Five

## Exercise 5A (Page 4)

**1(a)** $(x-2)(x+1-\sqrt{3})(x+1+\sqrt{3})$
**(b)** $(x-1)(x+2-\sqrt{2})(x+2+\sqrt{2})$
**(c)** $(x-1)(x-1-\sqrt{5})(x-1+\sqrt{5})$
**2(a)** The coefficients of $P(x)$ are real, so complex zeroes occur in conjugate pairs. **(b)** 6
**3(a)** $1+2i$; the coefficients of $P(x)$ are real, so complex zeroes occur in conjugate pairs.
**(c)** $P(x)=(x+2)(x^2-2x+5)$
**4(a)** $3i$; the coefficients of $P(z)$ are real, so complex zeroes occur in conjugate pairs. **(b)** $z^2+9$
**(c)** $P(z)=(2z+3)(z^2+9)$
**5(b)** 0; the coefficients of $P(z)$ are real, so complex zeroes occur in conjugate pairs.
**(c)(i)** $P(z)=(2z-1)(z-3-i)(z-3+i)$
**(ii)** $P(z)=(2z-1)(z^2-6z+10)$
**6(a)** The coefficients of $Q(x)$ are real, so complex zeroes occur in conjugate pairs. **(b)** $3+\sqrt{5}$, $3-\sqrt{5}$
**(c)(i)** $(x-2i)(x+2i)(x-3-\sqrt{5})(x-3+\sqrt{5})$
**(ii)** $(x^2+4)(x-3-\sqrt{5})(x-3+\sqrt{5})$
**(iii)** $(x^2+4)(x^2-6x+4)$
**7(a)** $x=1\pm 3i$, 3 or $-2$ **(b)** $x=1\pm i$ or $2\pm i$
**8(a)** $a=3$ **(b)** $b=1$
**(c)** $(x^2-6x+10)(x^2-6x+13)$
**9(b)** $k=3$
**10(b)** $m=7$, $n=-4$
**11(a)** $-7-4i$ **(b)(i)** $-7+4i$ **(ii)** $2x-7$
**12(b)** $P(z)=\frac{1}{2}(z^4-2)(2z^4-1)$ so one root is $z=\sqrt[4]{2}$. **(c)** $\sqrt[4]{2}$, $\frac{1}{\sqrt[4]{2}}$, $-\sqrt[4]{2}$, $-\frac{1}{\sqrt[4]{2}}$, and $i\sqrt[4]{2}$, $\frac{1}{\sqrt[4]{2}}i$, $-i\sqrt[4]{2}$, $-\frac{1}{\sqrt[4]{2}}i$
**13(a)** $P(x)$ has minimum value $B$, when $x=0$. Since $B>0$, it follows that $P(x)>0$ for all real values of $x$. **(b)** $-ic$, $-id$; the coefficients of $P(x)$ are real, so complex zeroes occur in conjugate pairs.
**14(a)** They form a conjugate pair, since $P(x)$ has real coefficients. **(c)**




**15(a)** The minimum stationary point is at $x=1$. $f(1)=k-2>0$. Hence the graph of $f(x)$ has only one $x$-intercept which lies to the left of the maximum stationary point at $x=-1$.
**(b)** $f(x)$ has real coefficients **(d)** $-14$, $7\pm 12i$

## Exercise 5B (Page 10)

**1(a)(ii)** 3 is a double zero of $P(x)$ **(b)** 3, 3, $-2$
**(c)** $P(x)=(x-3)^2(x+2)$
**2(a)(ii)** $-1$ is a triple zero of $P(x)$
**(b)** $-1$, $-1$, $-1$, $-5$ **(c)** $P(x)=(x+1)^3(x+5)$
**3(a)** $-3$ and 3 **(b)** 3 **(c)** $-6$
**4(a)** $\frac{5}{2}$ and $-5$ **(b)** $-5$ **(c)** 10
**5(a)** $-2$ **(b)** $\frac{3}{2}$, $P(x)=(x+2)^2(2x-3)$
**6(a)** $\frac{1}{2}$ **(b)** 2, $P(x)=(2x-1)^3(x-2)$
**7(b)** $x=3$, $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$
**8(a)** $k=27$ or $-5$
**(b)** When $k=27$, $P(x)=(x-3)^2(x+3)$ and when $k=-5$, $P(x)=(x+1)^2(x-5)$.
**9** $a=1$, $b=-3$, $c=2$
**10(a)** $-3$ **(b)** $c=-54$ **(c)** $P(x)=(x+3)^3(x-2)$
**11(a)** $b=-5$ and $c=8$
**(b)** $x=\frac{1}{2}(3-\sqrt{5})$ or $\frac{1}{2}(3+\sqrt{5})$
**12** The Fundamental Theroem of Algebra only applies to polynomials of degree $\geq 1$.
**15** HINT: consider $P(x)-P'(x)$
**16(b)(ii)** $m<0$ **(iii)** $x=-\sqrt{-\frac{m}{2}}$ or $\sqrt{-\frac{m}{2}}$
**19(a)** HINT: $x^2=-(2px+q)$
**(b)** HINT: $P'(\alpha)=0$.
**20(b)** $(z-\alpha)^2(z-\overline{\alpha})^2$ is a factor. **(c)** HINT: Begin by writing: $P(z)=\big(z-2\operatorname{Re}(\alpha)+|\alpha|^2\big)^2\times Q(z)$

## Exercise 5C (Page 17)

**1(a)** $\pm 3i$ **(b)** $2\pm 3i$ **(c)** $Q(x)=x^2-4x+13$
**2(a)** $-4\pm 2i$ **(b)** $-2\pm i$ **(c)** $Q(x)=4x^2+16x+20$
**3** $8P(\frac{x}{2})=x^3-20x+24$
**4** $x^3-2x^2-7x+1=0$
**5(a)** 4 **(b)** 4 **(c)** 16 **(d)** 72 **(e)** 224
**6(a)** $x^3-3x^2+1=0$ **(b)** $x^3-6x^2+9x-1=0$
**7(a)(i)** $27x^4+18x^3-3x^2+4x-1=0$
**(ii)** $x^4-4x^3+3x^2-18x-27=0$
**(b)(i)** $x^4+14x^3+71x^2+160x+135=0$
**(ii)** $x^4-14x^3+71x^2-160x+135=0$
**8(a)** $x^3+2mx^2+m^2x-n^2=0$
**(b)** $n^2x^3-m^2x^2-2mx-1=0$
**9(a)** $4+2\sqrt{3}$ **(b)** $x^4-8x^2+4=0$
**(c)** $x=\sqrt{3}$, $-\sqrt{3}$, $-2+\sqrt{3}$ or $-2-\sqrt{3}$
**10(a)** $x^4-5x^2+6=0$ **(b)** $x=-3\pm\sqrt{2}$ or $-3\pm\sqrt{3}$
**11(a)(i)** $-4$ **(ii)** 3 **(b)(i)** $2x^3+32x^2+163x+262=0$

**(ii)** $2x^3 + 24x^2 + 27x - 162 = 0$

**12(a)** Use the sum of roots.

**13(a)(i)** $8$ **(ii)** $6$ **(c)** $3,\, 1+i,\, 1-i$

**14(a)(i)** $-p$ **(ii)** $-r$ **(c)(ii)** $P(0) = -8,\, P(1) = 2$

**19(a)** Replace $x$ with $-\frac{p}{x+1}$ to get

$rx^3 + (3r - pq)x^2 + (p^3 - 2pq + 3r)x + (r - pq) = 0$

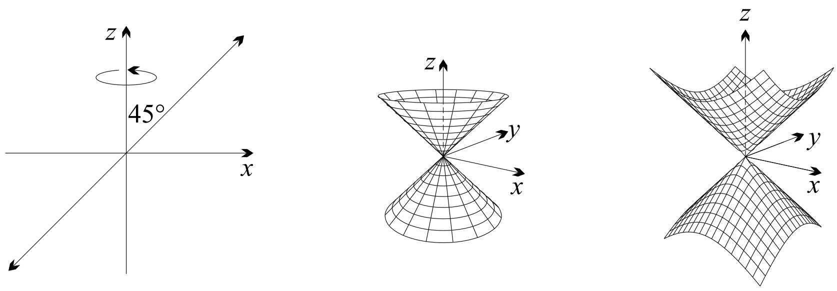
**20(a)(i)** $-\frac{1}{2}n(n+1)$ **(ii)** $(-1)^n\, n!$

CHAPTER **THREE**

# Conics

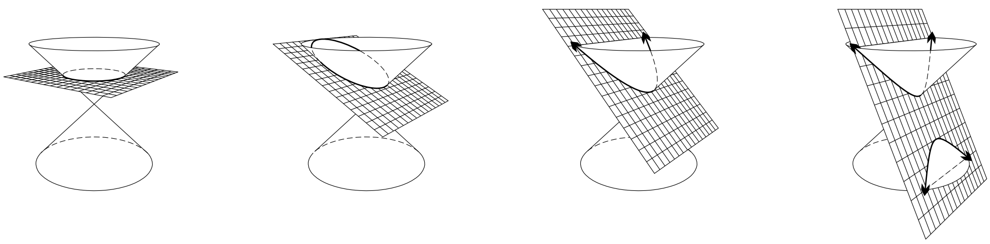
## 3A The Geometry of Conics

**Conic Sections:** Conics, or more precisely conic sections, are so called because each curve is formed by the intersection of a plane with a cone. The orientation of the plane relative to the axis of symmetry of the cone determines whether the conic section is a circle, ellipse, parabola or hyperbola.




As is found in the chapter on volumes of revolution, a cone is formed when a line through the origin, which is neither vertical nor horizontal, is rotated about the vertical axis, as shown in the first two figures above. Notice therefore that by this construction a cone has two parts, one part above the apex and the other below. For simplicity, the line $z = x$ was chosen to generate the cone, so that the semi-vertical angle (between the line and the axis of symmetry) is $45^\circ$. The third diagram is the same cone as in the second diagram, but shows the lines on the cone which lie above or below a rectangular grid in the $xy$-plane.

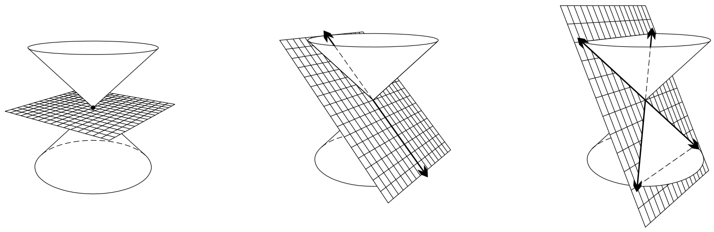
The next four diagrams show four planes intersecting with the cone at different angles to the axis of symmetry, to demonstrate the various conic sections.



In the first case, the plane is perpendicular to the axis. The resulting conic is a circle. In the second diagram, the angle between the axis and the plane has been reduced. The resulting conic is an ellipse, which will be studied in detail in this chapter. When the angle between the plane and the axis is equal to the semi-vertical angle of the cone the result is a parabola, as shown in the third diagram. When the angle is reduced still further the result is a hyperbola, as seen in the final diagram. Note that in this case, the plane intersects both parts of the cone which explains why hyperbolae have two branches.

The rectangular hyperbola $y = \dfrac{1}{x}$ met previously is a special type of hyperbola. It arises when the plane is parallel with the axis of the cone. That is, when the plane is vertical in the last diagram. Both rectangular hyperbolae and general hyperbolae will be studied in detail in this chapter.

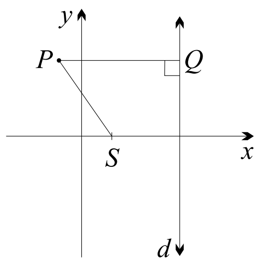
**Degenerate Cases:** The careful reader will have noted that in each case presented so far, the plane of intersection does not pass through the apex of the cone. If it does pass through this point then three degenerate configurations arise which are shown in the diagrams below.



In the first instance, a plane through the apex perpendicular to the axis intersects the cone at but one point, the apex. The same is true if the angle between the plane and the axis is greater than the semi-vertical angle. When those two angles are equal, the plane is tangent to the curved surface of the cone as shown in the second diagram, and the intersection is a single straight line. When the angle between plane and axis is decreased further, the intersection is a pair of straight lines with equal but opposite gradients, as shown in the third figure.

**A Geometric Definition:** Whilst the above descriptions are instructive with regard to the origins of conic sections, they are algebraically inconvenient as they rely on three dimensions. Though the equations of the conic sections can be obtained from the three dimensional model, the derivation is complex and is not part of this course. The geometric definitions used in this course for the conic sections are two dimensional, and therefore algebraically simpler.

We have already seen that the geometric definition of a parabola is the locus of points equidistant from a given point $S$ called the focus, and a fixed line $d$ called the directrix. This definition can be easily modified to obtain the equations of the ellipse and the hyperbola.




In this course, for the sake of convenience, let $S$ be on the $x$-axis and let the line $d$ be vertical, as shown in the diagram on the right. Let $P$ be a variable point in the plane, and let $PQ$ be the perpendicular distance from $P$ to the directrix.

We now define a new quantity $e$, called the *eccentricity*, which is the ratio of the distances from $P$ to each of $S$ and $Q$. That is:

$$e = \frac{PS}{PQ}\,.$$

(Note that $e$ is a variable here, not to be confused with Euler's number $e \doteqdot 2{\cdot}7183$, which is a constant.) It should be clear that when $e = 1$, the locus of $P$ is the familiar parabola. The other conic sections correspond to other values of $e$.

**1** ECCENTRICITY: Let $P$ be a point in the locus of a conic with focus $S$ and directrix $d$, and let $PQ$ be the distance from $P$ to the directrix. The eccentricity $e$ of the conic is defined to be the ratio:

$$e = \frac{PS}{PQ}\,.$$

Since $e$ is a ratio of distances, it can never be negative. Thus the domain of $e$ is $e \ge 0$. The four conic sections correspond to four cases of the value of $e$, namely:

| | |
|---|---|
| $e = 0$ | circle |
| $0 < e < 1$ | ellipse |
| $e = 1$ | parabola |
| $e > 1$ | hyperbola |

## Exercise **3A**

In the Appendix to this chapter are three special grids to help you draw an example each of the parabola, ellipse and hyperbola.

On each grid is marked a directrix $d$, which is horizontal. Moving away from $d$, each horizontal line is another unit. Thus the third line above $d$ is 3 units from $d$.

Also marked on each grid is a focus $S$ at the centre of concentric circles. Moving away from $S$, each circle is another unit. Thus the third circle out is 3 units from $S$.

The lone vertical line on the grid is an axis of symmetry of the conic, as will be seen after the graphs are completed. This line through $S$ is called the major axis.

**1.** THE PARABOLA: Use the first special grid to answer these questions.
(a) Mark both points where the line 3 units above $d$ intersects the circle with radius 3.
(b) Why must these two points lie on a parabola?
(c) Mark both points where the line 4 units above $d$ intersects the circle with radius 4.
(d) Continue to mark all the points on the grid which are equidistant from both $d$ and $S$.
(e) It should be evident when you look at the points that you have marked, that they all lie on a parabola. Join the points with a smooth curve to complete the parabola.

**2.** THE ELLIPSE: Use the second special grid to answer these questions. Notice that the directrix is towards the bottom of the grid. The eccentricity is $e = \frac{1}{2}$.
(a) Mark both points where the line 10 above $d$ intersects the circle with radius 5.
(b) Why must these two points lie on an ellipse with $e = \frac{1}{2}$?
(c) Mark both points where the line 12 units above $d$ intersects the circle with radius 6.
(d) Continue to mark all points $P$ on the grid for which $\frac{PS}{PQ} = \frac{1}{2}$.
(e) It should be evident when you look at the points that you have marked, that they all lie on an ellipse. Join the points with a smooth curve to complete the ellipse.

**3.** THE HYPERBOLA: Use the third special grid to answer these questions. In this case the eccentricity is $e = 2$.

(a) Mark both points where the line 1 unit above $d$ intersects the circle with radius 2.

(b) Why must these two points lie on a hyperbola with $e = 2$?

(c) Mark both points where the line 2 units above $d$ intersects the circle with radius 4.

(d) Continue to mark all points $P$ above $d$ on the grid for which $\frac{PS}{PQ} = 2$.

(e) It should be evident when you look at the points that you have marked, that they all lie on a hyperbola. Join the points with a smooth curve to complete the hyperbola.

(f) Locate the points on the grid which lie on the other branch of the hyperbola and join them with a smooth curve.
[HINT: On the axis of symmetry you might use the external ratio division formula.]

DEVELOPMENT

**4.** Using the parabola grid, investigate what happens to the shape of the parabola if a different line above $S$ is chosen as the directrix.

**5.** Look carefully at your graph of the ellipse with eccentricity $e = \frac{1}{2}$.

(a) It should be clear that the ellipse is symmetric about one of the horizontal lines in the grid. Mark that line. This line is called the minor axis.

(b) Locate $S'$, the image of the focus when reflected in this line, and locate $d'$, the image of the directrix.

(c) Choose any point $P$ on the ellipse and measure both $PS$ and $PS'$. Next measure the length of the ellipse along the axis of symmetry and compare it with $PS + PS'$. What do you notice?

(d) Now measure $PQ'$, the distance from $P$ to $d'$. What do you notice about the ratio $\frac{PS'}{PQ'}$? Is the same true of other points on the ellipse?

**6.** Using the ellipse grid, investigate what happens to the shape of the ellipse when a smaller value is used for the eccentricity. Try using $e = \frac{1}{3}$.

**7.** Using the hyperbola grid, investigate what happens to the shape of the hyperbola when a smaller value is used for the eccentricity. Try using $e = \frac{3}{2}$.

**8.** Each ellipse and hyperbola cuts the axis of symmetry twice. The *centre* of the conic is the mid-point of these. Locate the centre for each ellipse and for each hyperbola you drew.

EXTENSION

**9.** The asymptotes of a hyperbola pass through its centre. Try to add asymptotes for each hyperbola you drew.

# 3B Circles

Although a circle is a conic with eccentricity $e = 0$, it is more convenient to continue to use the traditional definition of a circle. Thus in this section a circle is the locus of points in the plane which are a fixed distance from a given point. The fixed distance is called the radius, and the given point is called the centre. The equation of the circle with centre the origin and radius $r$ is:

$$x^2 + y^2 = r^2 .$$

**Parametric Form:** Recall that the trigonometric functions were defined in terms of the coordinates of a point $P(x, y)$ on this circle, where the angle $\theta$ was measured anti-clockwise about the origin from the $x$-axis to the ray $OP$. Thus:

$$\cos\theta = \frac{x}{r}$$
$$\sin\theta = \frac{y}{r}\,.$$

Hence the angle $\theta$ is the natural parameter of a circle with radius $r$, and the corresponding parametric equations are:

$$x = r\cos\theta$$
$$y = r\sin\theta\,.$$

**2** PARAMETRIC EQUATIONS: The parametric equations of a circle with radius $r$ and centre the origin are:

$$x = r\cos\theta$$
$$y = r\sin\theta\,.$$

The parametric equations for circles with different centres may be found by simple translations.

**WORKED EXERCISE:** Write down the parametric equations for the circle

$$(x-1)^2 + (y+2)^2 = r^2\,.$$

**SOLUTION:** The given circle has centre at $(1, -2)$. The corresponding parametric equations are:

$$x - 1 = r\cos\theta$$
$$y + 2 = r\sin\theta\,,$$

or more conveniently:

$$x = 1 + r\cos\theta$$
$$y = -2 + r\sin\theta\,.$$

**Tangents and Normals:** The equation of the normal to the circle $x^2 + y^2 = r^2$ at any point $P$ is simply the equation of the radius $OP$, and is trivial to determine. It is therefore left as a question in the exercises.

The equation of a tangent can also be found without the aid of calculus. From geometry we know that the tangent at $P$ is perpendicular to the radius $OP$, so its gradient can easily be found. If $P$ is the point $(r\cos\theta, r\sin\theta)$ then the gradient of $OP$ is $\tan\theta$, and the gradient of the tangent is $-\cot\theta$. Hence the equation of the tangent is:

$$y - r\sin\theta = -\cot\theta(x - r\cos\theta)$$
$$y\sin\theta - r\sin^2\theta = -x\cos\theta + r\cos^2\theta$$

so $\quad x\cos\theta + y\sin\theta = r(\cos^2\theta + \sin^2\theta)$

or $\quad x \times r\cos\theta + y \times r\sin\theta = r^2\,.$

By analogy, the cartesian form of the equation of the tangent at $P(x_1, y_1)$ is

$$x_1x + y_1y = r^2$$

and the proof is left as a question in the exercises.

| | |
|---|---|
| **3** | TANGENT IN PARAMETRIC FORM: $x\,r\cos\theta + y\,r\sin\theta = r^2$ <br> TANGENT IN CARTESIAN FORM: $x_1x + y_1y = r^2$ |

**WORKED EXERCISE:** Find the cartesian equation of the tangent to the circle with centre $C(h,k)$ at the point $P(x_1,y_1)$ by:

(a) translating the second equation in the Box 3 above,

(b) letting $T(x,y)$ be a point on the tangent and noting that $PT$ is perpendicular to $PC$.

**SOLUTION:**

(a) The circle has been shifted horizontally by $h$ and vertically by $k$. Thus the equation is

$$(x_1-h)(x-h)+(y_1-k)(y-k)=r^2$$

where $r^2=(x_1-h)^2+(y_1-k)^2$.

(b) Since $PT$ is perpendicular to $PC$, the product of their gradients is $-1$. Thus,

$$\frac{y-y_1}{x-x_1}\times\frac{y_1-k}{x_1-h}=-1$$

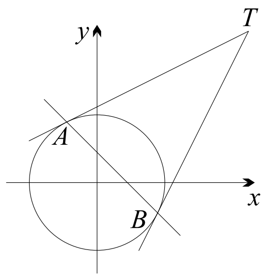
so $$(y-y_1)(y_1-k)=-(x-x_1)(x_1-h)$$

or $$(x-x_1)(x_1-h)+(y-y_1)(y_1-k)=0\,.$$

The proof that the answers to both parts are equivalent is left as an exercise.

**The Chord of Contact:** Suppose that the tangents at $A(x_1,y_1)$ and $B(x_2,y_2)$ meet at $T(x_0,y_0)$. From above, the equation of the tangent at $A$ is

$$x_1x+y_1y=r^2\,.$$




Since $T(x_0,y_0)$ lies on this line, it follows that

$$x_1x_0+y_1y_0=r^2\,.$$

From this equation, $A(x_1,y_1)$ must lie on the line

$$x_0x+y_0y=r^2\,.$$

Similarly, the equation of the tangent at $B(x_2,y_2)$ is

$$x_2x+y_2y=r^2\,.$$

Since $T(x_0,y_0)$ lies on this line, it follows that

$$x_2x_0+y_2y_0=r^2\,.$$

From this equation, $B(x_2,y_2)$ must also lie on the line

$$x_0x+y_0y=r^2\,.$$

Hence both $A$ and $B$ lie on the line $x_0x+y_0y=r^2$, which must therefore be the equation of the chord of contact.

| | |
|---|---|
| **4** | THE CHORD OF CONTACT: The equation of the chord of contact for the tangents from the external point $T(x_0,y_0)$ to the circle $x^2+y^2=r^2$ is <br> $x_0x+y_0y=r^2\,.$ |

**The Equation of a Circle on a Given Diameter:** It is sometimes necessary to determine the equation of a circle where only the coordinates of the endpoints of the diameter $AB$ are known. One method is to calculate the length of $AB$ and to find the coordinates of $C$, the midpoint of $AB$. It should be clear that the required circle has centre $C$ and radius $\frac{1}{2}|AB|$.

**WORKED EXERCISE:** Find the equation of the circle with diameter $AB$, where $A = (1, 1)$ and $B = (5, 3)$.

**SOLUTION:** Clearly the mid-point of $AB$ is $C = (3, 2)$, and this is the centre.

$$\begin{aligned} \text{Now} \qquad |AB|^2 &= (5-1)^2 + (3-2)^2 \\ &= 20 \\ \text{so} \qquad r &= \tfrac{1}{2}|AB| \\ &= \sqrt{5}\,. \end{aligned}$$

Hence the equation of the circle is $(x-3)^2 + (y-2)^2 = 5$.

When knowledge of the centre and radius is not required, there is an alternate method. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the endpoints of a diameter of a circle, and let $P(x, y)$ be another point on the circle. Clearly the angle at $P$ is a right angle, being the angle in a semi-circle. Thus the product of the gradients of $AP$ and $PB$ must equal $-1$. Hence:

$$\frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1$$

so $\quad (y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$

or $\quad (x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0\,.$

5 THE EQUATION OF A CIRCLE WITH DIAMETER $AB$: Given the fixed points $A(x_1, y_1)$ and $B(x_2, y_2)$, the equation of the circle with diameter $AB$ is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0\,.$$

NOTE: The careful reader will have realised that the gradients of $AP$ and $PB$ are not defined when $x = x_1$ or $x = x_2$. When $AB$ is vertical or horizontal there are two such points on the circle, namely when $P = A$ and when $P = B$. When $AB$ is neither horizontal nor vertical there are four such points, which correspond to the vertices of a rectangle with $AB$ as one diagonal and with its sides parallel with the axes. Nevertheless, the formula given for the equation of the circle is valid for all points on the circle, and the proof of this is left as an exercise.

**WORKED EXERCISE:** (a) Write down the equation of the circle with diameter $AB$, where $A = (1, 1)$ and $B = (5, 3)$.
(b) Hence show that it passes through $P(4, 4)$.

**SOLUTION:**

(a) The equation of the circle is $(x-1)(x-5) + (y-1)(y-3) = 0$.

(b) At $P$,
$$\begin{aligned} \text{LHS} &= 3 \times (-1) + 3 \times 1 \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

hence $P$ is on the circle.

## Exercise **3B**

**1.** (a) Show that the equation of the normal to $x^2 + y^2 = r^2$ at the point $P(x_1, y_1)$ is $x_1 y = y_1 x$.
(b) What is the equation of the normal to $(x-a)^2 + (y-b)^2 = r^2$ at the point $P(x_1, y_1)$?

**2.** Determine the chord of contact for each given point and circle.
(a) $x^2 + y^2 = 4$, $(3, -1)$
(b) $x^2 + y^2 = 12$, $(2, 4)$
(c) $x^2 + y^2 = 9$, $(-1, -4)$
(d) $x^2 + y^2 = 5$, $(-2, 2)$

**3.** Find the centre $C$ and radius $r$ of each circle.
(a) $x^2 + y^2 + 2x - 4y = 0$
(b) $x^2 + y^2 - 6x + 1 = 0$
(c) $x^2 + y^2 + 6x + 4y - 3 = 0$
(d) $x^2 + y^2 - x + 3y + 2 = 0$

**4.** In each case find the equation of the circle with diamter $AB$ by first finding the centre $C$ and radius $\frac{1}{2}|AB|$.
(a) $A = (0, 0)$, $B = (-2, 2)$
(b) $A = (5, 0)$, $B = (-1, -2)$
(c) $A = (1, 7)$, $B = (5, 1)$
(d) $A = (-5, -7)$, $B = (-1, 1)$

**5.** Repeat the previous Question using the formula
$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0\,,$$
and check that the answers are equivalent.

**DEVELOPMENT**

**6.** Use implicit differentiation to help derive the equation of the tangent to a circle, with radius $r$ and centre the origin, in: (a) parametric form, (b) cartesian form.

**7.** Use the results of the previous question to find the equation of the tangent to each circle at the indicated point.
(a) $x^2 + y^2 = 25$, $(3, 4)$
(b) $x^2 + y^2 = 2$, $\theta = \frac{\pi}{4}$
(c) $x^2 + y^2 = 4$, $\theta = \frac{2\pi}{3}$
(d) $x^2 + y^2 = 50$, $(-1, -7)$

**8.** Find the equations of the two circles through $A(-1, 1)$ and $B(3, 1)$ with radius $\sqrt{10}$.

**9.** Find the equation of the circle through $A(-1, 1)$ and $B(3, 1)$ given that its centre is on the line $y = x + 4$.

**10.** (a) (i) The line $y = mx + b$ is tangent to the circle $x^2 + y^2 = r^2$. Show that $b^2 = r^2(m^2 + 1)$.
(ii) Hence show that $y = 2x + 5$ is tangent to $x^2 + y^2 = 5$
(b) (i) Generalise this result for the circle with centre $(h, k)$.
(ii) Hence show that $x + 2y - 12 = 0$ is tangent to $(x-3)^2 + (y-2)^2 = 5$.

**11.** Let $A = (-r, 0)$ and $B = (r, 0)$ be the end-points of a diameter of the circle $x^2 + y^2 = r^2$. Let $P$ be any other point on the circle. Prove that the angle in a semicircle is $\frac{\pi}{2}$ using:
(a) cartesian form $P = (x_1, y_1)$
(b) parametric form $P = (r\cos\theta, r\sin\theta)$

**12.** The line $y = mx$ intersects the circle $(x-b)^2 + y^2 = r^2$, where $b > r$, at the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.
(a) Show that $OP = x_1\sqrt{m^2+1}\,$.
(b) Use the product of the roots of a suitable quadratic equation to determine $x_1 x_2$.
(c) Hence show that $PO \times OQ$ does not depend on $m$.
(d) What circle geometry theorem have you confirmed in this question?
(e) Investigate the situation when $0 < b < r$.

**EXTENSION**

**13.** Let $A(r\cos\phi, r\sin\phi)$ and $B(r\cos\phi, -r\sin\phi)$ be two fixed points on the circle $x^2+y^2=r^2$. Let $P$ be the variable point $(r\cos\theta, r\sin\theta)$, where $\theta \neq \phi$ and $\theta \neq -\phi$.

(a) Find the gradient of $PA$.

(b) Let $\alpha$ be the acute angle between the lines $PA$ and $PB$. By considering the gradients of $PA$ and $PB$, show that $\tan\alpha = |\tan\phi|$.

(c) What circle geometry theorem have you confirmed in this question?

**14.** Let $A(x_1, y_1)$ be the point of contact of a tangent from the external point $P(x_0, y_0)$ to the circle $x^2+y^2=r^2$.

(a) Use the distance formula and the equation of the chord of contact to prove that $PA$ does not depend on the coordinates of $A$.

(b) What circle geometry theorem have you confirmed in this question?

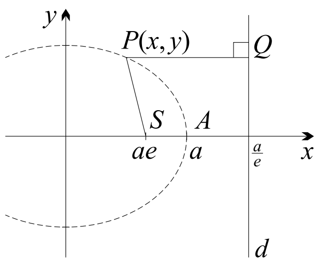
**15.** It is known that the parabola with parametric equations $x = 2at$ and $y = at^2$ intersects the circle $(x-h)^2+(y-k)^2=r^2$ at four points with parameters $t_1$, $t_2$, $t_3$ and $t_4$. By considering an appropriate polynomial equation, show that $t_1+t_2+t_3+t_4=0$.

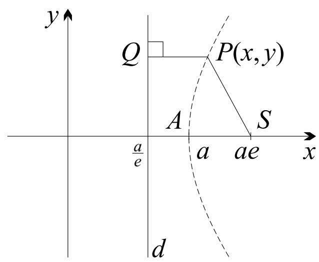
**16.** The circles $x^2+y^2+2x+4y+1=0$ and $x^2+y^2-2x+6y+5=0$ intersect at two points $A$ and $B$. Find the equation of the circle through $A$, $B$ and the point $C=(4,1)$, without finding the points $A$ and $B$.

# 3C The Equations of the Ellipse and Hyperbola

The derivations of the equations of the ellipse and hyperbola differ only at the last step, and so it is appropriate to derive both together up to that point. A careful choice of the focus and directrix is the key.

The positive numbers $\frac{a}{e}$, $a$ and $ae$ form a geometric progression with ratio $e$. It can be shown that the consecutive differences are in the same ratio, and the proof is left as an exercise. So let the focus be $S=(ae,0)$ and the directrix be $x=\frac{a}{e}$. Then it is guaranteed that the point $A=(a,0)$ is in the locus since the distances between these points is in the ratio $e$. The two configurations below show the situation on the left for $0<e<1$ and on the right for $e>1$.




Now let $P=(x,y)$ so that $Q=(\frac{a}{e},y)$, then from the definition of the eccentricity:

$$PS^2 = e^2PQ^2$$

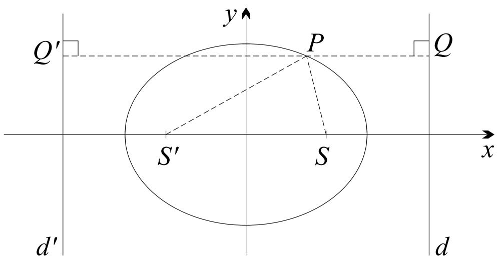
so $$(x-ae)^2+y^2 = e^2(x-\tfrac{a}{e})^2$$

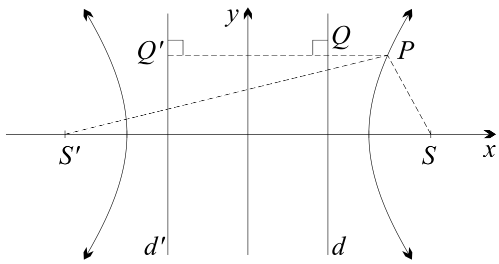
or $$x^2-2aex+(ae)^2+y^2 = e^2x^2-2aex+a^2\,.$$

Thus $$x^2(1-e^2)+y^2 = a^2(1-e^2)$$

hence $$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)} = 1\,.$$

Notice that the final equation above is unchanged if we replace $a$ with $-a$. Thus it is evident that every ellipse and every hyperbola has two foci at $S(ae, 0)$ and $S'(-ae, 0)$ with two corresponding directrices $d$ at $x = \frac{a}{e}$ and $d'$ at $x = -\frac{a}{e}$.




The two situations are shown above. Also note that corresponding with the focus $S(ae, 0)$ is the point $Q(\frac{a}{e}, y)$, and corresponding with the focus $S'(-ae, 0)$ is the point $Q'(-\frac{a}{e}, y)$, so that both

$$\frac{PS}{PQ} = e \quad \text{and} \quad \frac{PS'}{PQ'} = e\,.$$

It is at this point that the treatment of hyperbolae and ellipses differs, and the remainder of this section will be devoted to the equation of an ellipse.

**The Ellipse:** The value of $(1 - e^2)$ is positive for ellipses since $0 < e < 1$, so let

$$b^2 = a^2(1 - e^2)\,,$$

where $a > b > 0$. The equation of the ellipse is now

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

Clearly this intersects the axes at $A(a, 0)$, $A'(-a, 0)$, $B(0, b)$ and $B'(0, -b)$. These are called the *vertices* of the ellipse, being the points furthest away from, and closest to, the origin. The origin is called the *centre* of the ellipse. $A'A$ is called the *major axis*, being the longer of the two, and has length $2a$. Also note that the major axis is perpendicular to both directrices and passes through both foci. $B'B$ is called the *minor axis* and has length $2b$. It should be clear that both the major axis and the minor axis are also axes of symmetry.

**WORKED EXERCISE:** Find the values of $a$, $b$, $e$, the coordinates of the foci and the equations of the directrices for the ellipse

$$\frac{x^2}{4} + y^2 = 1\,.$$

Sketch the ellipse showing these features.

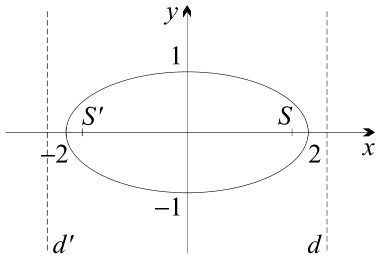
**SOLUTION:** Clearly $a = 2$ and $b = 1$ so that

$1 = 4(1 - e^2)$

or $\quad \frac{1}{4} = 1 - e^2$

thus $\quad e^2 = \frac{3}{4}$

so $\quad e = \frac{\sqrt{3}}{2}$.




Thus the foci are $S'(-\sqrt{3}, 0)$ and $S(\sqrt{3}, 0)$.
The directrices are $d' : x = -\frac{4}{\sqrt{3}}$ and $d : x = \frac{4}{\sqrt{3}}$.

Note that $\sqrt{3} \doteqdot 1{\cdot}73$ and $\frac{4}{\sqrt{3}} \doteqdot 2{\cdot}31$, hence the relative locations of the foci and directrices in the above graph.

**6** THE EQUATION OF AN ELLIPSE: The equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $b^2 = a^2(1 - e^2)$, and $0 < e < 1$ is the eccentricity.

**Parametric Equations:** The equation of an ellipse can be thought of as the result of stretching the unit circle by factor $a$ horizontally and factor $b$ vertically. Applying the same stretches to the parametric equations of a circle yields the parametric equations for an ellipse:

$$x = a\cos\theta \qquad \text{and} \qquad y = b\sin\theta\,.$$

To confirm this, substitute these equations into the left hand side to get:

$$\begin{aligned}\text{LHS} &= \frac{a^2\cos^2\theta}{a^2} + \frac{b^2\sin^2\theta}{b^2}\\ &= \cos^2\theta + \sin^2\theta\\ &= 1\end{aligned}$$

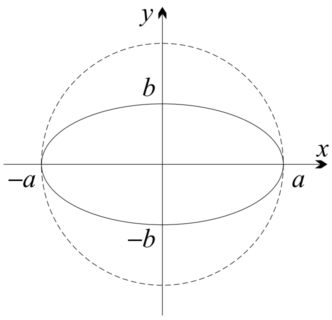
as required.

**7** THE PARAMETRIC EQUATIONS: The parametric equations of an ellipse are:

$$x = a\cos\theta \qquad \text{and} \qquad y = b\sin\theta\,.$$

**WORKED EXERCISE:** Find the coordinates of the point with parameter $\theta = -\frac{\pi}{3}$ on the ellipse with equation $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1\,.$

**SOLUTION:** Since $a = 2$ and $b = \sqrt{3}$, the point is
$(2\cos(-\frac{\pi}{3}), \sqrt{3}\sin(-\frac{\pi}{3})) = (1, -\frac{3}{2})\,.$

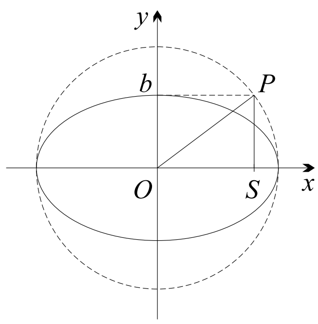



**The Auxiliary Circle:** The circumcircle of an ellipse has its centre at the origin and radius $a$. That is:

$$x^2 + y^2 = a^2\,.$$

This circle has several important features and is called the *auxiliary circle*.

The first feature is that the auxiliary circle can be used to find the minor axis. Let $P$ be the point on the auxiliary circle in the first quadrant for which $x = ae$. That is, the point $P$ is vertically aligned with the focus. The configuration is shown in the diagram on the right. The $y$-coordinate of $P$ is given by:

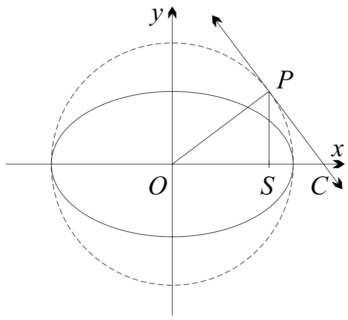



$$\begin{aligned}y^2 &= a^2 - a^2e^2\\ &= a^2(1 - e^2)\,.\end{aligned}$$

That is $y = b$, the semi-minor axis length.

A second feature is revealed by investigating the tangent at $P$. Let this tangent intersect the $x$-axis at $C$, as shown in the diagram below on the right. Recall from Section 3B that the equation of the tangent to the auxiliary circle at $(x_1, y_1)$ is given by

$$x_1 x + y_1 y = a^2 .$$




Hence at $P(ae, b)$ this yields:

$$aex + by = a^2 .$$

Thus at the $x$-intercept $C$:

$$aex = a^2$$

so $$x = \frac{a}{e} .$$

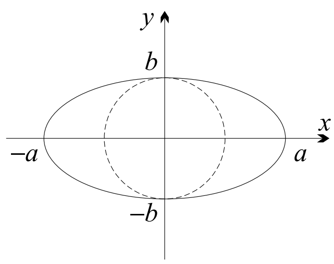
That is, the tangent to the auxiliary circle at $P$, the $x$-axis and the directrix are concurrent. This result can also be deduced using the similar triangles $\triangle OSP$ and $\triangle OPC$, and is left as an exercise.

8 THE AUXILIARY CIRCLE: The auxiliary circle is the circumcircle of the ellipse. It has centre the origin and radius equal to the semi-major axis. Its equation is

$$x^2 + y^2 = a^2 .$$

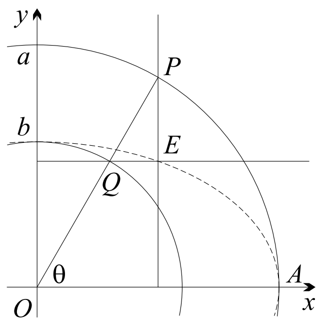
**The Auxiliary Circle and Parameters:** The incircle of an ellipse has its centre at the origin and radius $b$. That is:

$$x^2 + y^2 = b^2 .$$




The incircle and the auxiliary circle can be used to quickly plot points specified parametrically.

Construct the auxiliary circle and incircle on a set of axes. Let $A(a, 0)$ be the $x$-intercept of the auxiliary circle. Construct $P$ on the auxiliary circle so that $\angle AOP = \theta$, the required parameter. Let $OP$ intersect the incircle at $Q$. Construct the vertical line through $P$ and the horizontal line through $Q$, and let these lines intersect at $E$. Then point $E$ is on the ellipse and has coordinates $(a\cos\theta, b\sin\theta)$. The situation is shown on the right and the proof is left as an exercise. Notice from this that $\theta$ is the angle in the auxiliary circle.




**The Latus Rectum:** This is a chord through $S$ or $S'$ which is perpendicular to the major axis. Let the end points be at $A$ and $B$. The $x$-coordinate is $x = ae$.

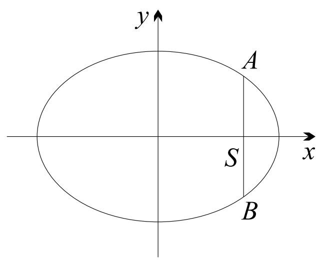
Hence $$e^2 + \frac{y^2}{b^2} = 1$$

so $$y^2 = b^2(1 - e^2)$$

or $$y^2 = a^2(1 - e^2)^2 \quad (\text{since } b^2 = a^2(1 - e^2))$$

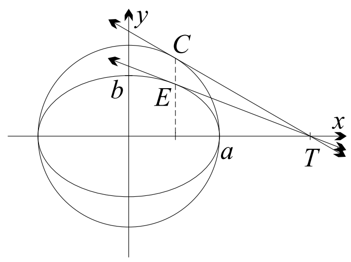
thus $$y = \pm a(1 - e^2) .$$

Hence $$AB = 2a(1 - e^2) = \frac{2b^2}{a} .$$



**Tangents:** Let $C(a\cos\theta, a\sin\theta)$ be a point on the auxiliary circle with corresponding point $E(a\cos\theta, b\sin\theta)$ on the ellipse. The equation of the tangent at $E$ can be obtained from the tangent at $C$ by a vertical stretch of factor $\frac{b}{a}$. Note that because only vertical stretching is involved, both tangents have the same $x$-intercept at $T$.




Now from Section 3B, the equation of $CT$ is

$$x\,a\cos\theta + y\,a\sin\theta = a^2\,.$$

Thus the equation of $ET$ is, after stretching,

$$x\,a\cos\theta + \frac{a}{b}\,y\,a\sin\theta = a^2$$

or $$\frac{x\,a\cos\theta}{a^2} + \frac{y\sin\theta}{b} = 1$$

so $$\frac{x\,a\cos\theta}{a^2} + \frac{y\,b\sin\theta}{b^2} = 1\,.$$

Notice that effectively, in the equation of the ellipse, an $x$ and a $y$ have been replaced with the coordinates of $E$.

If instead $E$ has the Cartesian coordinates $(x_1, y_1)$ then the Cartesian form of the equation of the tangent is:

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1\,.$$

| | | |
|---|---|---|
| **9** | TANGENT IN PARAMETRIC FORM: | $\dfrac{x\,a\cos\theta}{a^2} + \dfrac{y\,b\sin\theta}{b^2} = 1$ |
| | TANGENT IN CARTESIAN FORM: | $\dfrac{x_1x}{a^2} + \dfrac{y_1y}{b^2} = 1$ |

Both these equations can also be derived via calculus. In the Cartesian case, begin with the equation of the ellipse and use implicit differentiation, as follows.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

so $$\frac{2x}{a^2} + \frac{2y}{b^2}\,\frac{dy}{dx} = 0\,.$$

Thus $$\frac{dy}{dx} = -\frac{x}{y} \times \frac{b^2}{a^2}$$

and at $E$ $$\frac{dy}{dx} = -\frac{x_1}{y_1} \times \frac{b^2}{a^2}\,.$$

Hence the equation of the tangent is

$$y - y_1 = -\frac{x_1}{y_1} \times \frac{b^2}{a^2} \times (x - x_1)$$

so $$\frac{y_1y - {y_1}^2}{b^2} = \frac{{x_1}^2 - x_1x}{a^2}$$

or $$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = \frac{{x_1}^2}{a^2} + \frac{{y_1}^2}{b^2}\,.$$

Thus $$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1 \qquad \text{(since } E \text{ is on the ellipse)}$$

exactly as before.

**WORKED EXERCISE:** Derive the equation of the tangent at $(1, -\frac{3}{2})$ on the ellipse

$$\frac{x^2}{4} + \frac{y^2}{3} = 1\,.$$

**SOLUTION:** Differentiate implicitly to get:

$$\frac{2x}{4} + \frac{2y}{3} \times \frac{dy}{dx} = 0$$

so $$\frac{dy}{dx} = -\frac{3x}{4y}\,.$$

At the given point

$$\frac{dy}{dx} = \frac{1}{2}\,.$$

The equation of the tangent is

$$y + \tfrac{3}{2} = \tfrac{1}{2}(x-1)$$

or $$2y + 3 = x - 1$$

so $$x - 2y = 4\,.$$

Note that if the formula is used, the same answer is obtained.

**Normals:** Using the gradient of the tangent found above, the Cartesian equation of the normal is given by:

$$y - y_1 = \frac{y_1}{x_1} \times \frac{a^2}{b^2} \times (x - x_1)$$

so $$\frac{b^2 y}{y_1} - b^2 = \frac{a^2 x}{x_1} - a^2$$

thus $$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

or $$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 e^2\,.$$

The parametric form is easily obtained by replacing $(x_1, y_1)$ with $(a\cos\theta, b\sin\theta)$.

**10**

| | |
|---|---|
| NORMAL IN PARAMETRIC FORM: | $\dfrac{a^2 x}{a\cos\theta} - \dfrac{b^2 y}{b\sin\theta} = a^2 - b^2$ |
| NORMAL IN CARTESIAN FORM: | $\dfrac{a^2 x}{x_1} - \dfrac{b^2 y}{y_1} = a^2 - b^2$ |

**WORKED EXERCISE:**

(a) Use the formula to find the equation of the normal at $P(1, -\frac{3}{2})$ on the ellipse

$$\frac{x^2}{4} + \frac{y^2}{3} = 1\,.$$

(b) The normal in part (a) has $x$-intercept at $A$ and $y$-intercept at $B$. Show that $PA : PB = 1 - e^2 : 1$.

**SOLUTION:**

(a) From above: $$\frac{4x}{1} - \frac{3y}{(-\frac{3}{2})} = 4 - 3$$

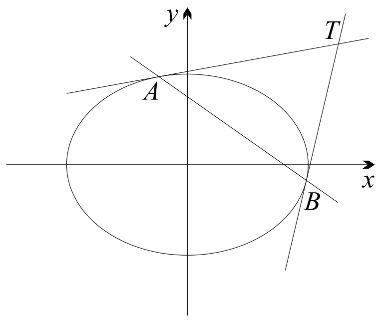
so $$4x + 2y = 1\,.$$

(b) From (a) $A = (\frac{1}{4}, 0)$
so $PA^2 = \frac{9}{16} + \frac{9}{4}$
$= \frac{45}{16}$,
and $B = (0, \frac{1}{2})$
so $PB^2 = 1 + 4$
$= 5$.
Hence $PA : PB = \frac{3}{4}\sqrt{5} : \sqrt{5}$
$= \frac{3}{4} : 1$
$= 1 - e^2 : 1$ (since $\frac{b^2}{a^2} = 1 - e^2$.)

**Chord of Contact:** The derivation of the chord of contact follows a similar argument to that for the circle. Suppose that the tangents at $A(x_1, y_1)$ and $B(x_2, y_2)$ meet at $T(x_0, y_0)$. From above, the equation of the tangent at $A$ is

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1\,.$$




Since $T(x_0, y_0)$ lies on this line, it follows that

$$\frac{x_1 x_0}{a^2} + \frac{y_1 y_0}{b^2} = 1\,.$$

From this equation, $A(x_1, y_1)$ must lie on the line

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1\,.$$

Similarly, the equation of the tangent at $B(x_2, y_2)$ is

$$\frac{x_2 x}{a^2} + \frac{y_2 y}{b^2} = 1\,.$$

Since $T(x_0, y_0)$ lies on this line, it follows that

$$\frac{x_2 x_0}{a^2} + \frac{y_2 y_0}{b^2} = 1\,.$$

From this equation, $B(x_2, y_2)$ must also lie on the line

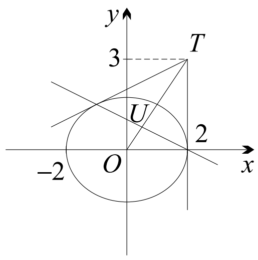
$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1\,.$$

Hence both $A$ and $B$ lie on the line $\dfrac{x_0 x}{a^2} + \dfrac{y_0 y}{b^2} = 1$, which must therefore be the equation of the chord of contact.

**11**

> THE CHORD OF CONTACT: The equation of the chord of contact for the tangents from the external point $T(x_0, y_0)$ to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is
>
> $$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1\,.$$

**WORKED EXERCISE:** The chord of contact for the point $T(2, 3)$ and the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ meets $OT$ at $U$. Find the coordinates of $U$.




**SOLUTION:** Since $U$ is on $OT$, let $U = (2\lambda, 3\lambda)$.
The equation of the chord of contact is

$$\tfrac{1}{2}x + y = 1\,.$$

Since $U$ is on this line

$$\lambda + 3\lambda = 1$$

thus $\lambda = \frac{1}{4}$

and $U = (\frac{1}{2}, \frac{3}{4})$.

**The Auxiliary Circle Again:** Notice that

$$\begin{aligned}\lim_{e\to 0^+} b^2 &= \lim_{e\to 0^+} a^2(1-e^2)\\ &= a^2(1-0)\\ &= a^2\,.\end{aligned}$$

Thus in the limit as $e \to 0$ the equation of the ellipse becomes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

or $x^2 + y^2 = a^2$.

That is, the limit of the ellipse as $e \to 0$ is the auxiliary circle. It should also be clear that the two foci at $(ae, 0)$ and $(-ae, 0)$ have coalesced at the origin. Strictly speaking the equations of the directrices are undefined, but the lines can be thought of as having moved infinitely away from the origin along the $x$-axis. It is in this limiting sense that $e = 0$ is assigned to be the eccentricity of a circle.

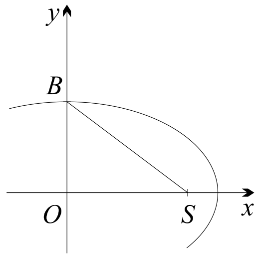
## Exercise **3C**

**1.** The ellipse $\mathcal{E}$ has equation $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$.
(a) Find its eccentricity. (b) Find the coordinates of its foci.
(c) Find the equations of its directrices.
(d) Sketch $\mathcal{E}$, showing the foci and directrices.
(e) Show that the parametric equations $x = 5\cos\theta$, $y = 4\sin\theta$ represent $\mathcal{E}$.
(f) Find the coordinates of the point on $\mathcal{E}$ corresponding to $\theta = \frac{\pi}{3}$.

**2.** (a) Show that the ellipse $4x^2 + 9y^2 = 16$ has foci $(\frac{2\sqrt{5}}{3}, 0)$ and $(-\frac{2\sqrt{5}}{3}, 0)$, and directrices $x = \frac{6\sqrt{5}}{5}$ and $x = -\frac{6\sqrt{5}}{5}$.
(b) Write down a pair of parametric equations representing the ellipse.

**3.** Consider the ellipse $\mathcal{E}$ with equation $5x^2 + 9y^2 = 45$.
(a) Determine the foci and directrices of $\mathcal{E}$.
(b) Sketch $\mathcal{E}$, showing its foci, directrices and auxiliary circle.
(c) Show that the parametric equations $x = 3\cos\alpha$, $y = \sqrt{5}\sin\alpha$ represent $\mathcal{E}$.
(d) Show on your diagram how to construct the point where $\alpha = \frac{\pi}{3}$.

**4.** Consider the ellipse with parametric equations $x = 5\cos\theta$, $y = 3\sin\theta$.
(a) Show that $(-\frac{5\sqrt{2}}{2}, \frac{3\sqrt{2}}{2})$ is the point on the ellipse corresponding to $\theta = \frac{3\pi}{4}$.
(b) Find the Cartesian equation of the ellipse.
(c) Find the eccentricity of the ellipse.
(d) Show that each latus rectum is of length $3\frac{3}{5}$ units.




**5.** (a) Let $O(0,0)$ be the centre, $S(ae,0)$ a focus and let $B(0,b)$ be a vertex of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Use Pythagoras' Theorem in $\triangle OSB$ to find $SB$ and hence show that $SB+S'B=2a$.

(b) (i) An ellipse has centre $O(0,0)$ and focus at $S(3,0)$. The point $B(0,1)$ is a vertex on the minor axis. Find the equation of the ellipse.

(ii) An ellipse has centre $(0,0)$ and foci at $(4,0)$ and $(-4,0)$. Its minor axis has length 6 units. Find its equation.

**6.** An ellipse has eccentricity $\frac{2}{3}$ and centre at the origin. Its major axis lies along the $x$-axis and has length 12 units. Find the equation of the ellipse.

**DEVELOPMENT**

**7.** A variable point $P$ in the number plane moves in such a way that the sum of its distances from $(3,0)$ and $(-3,0)$ is always 10 units. Thus, the locus of $P$ is an ellipse.

(a) Draw a diagram showing the two foci and the point $P$ at the vertex $(a,0)$. Use the sum of the distances to show that $a=5$.

(b) Hence show that the equation of the ellipse is $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$.

**8.** The orbit of the earth about the sun is an ellipse with the sun at one of the foci. It is known that the maximum and minimum distances of the earth from the sun are in the ratio $30:29$. Use a diagram to show that the eccentricity of the earth's orbit is $\frac{1}{59}$.

**9.** The equation of the chord of contact in the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ from the point $(x_0,y_0)$ is

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1\,.$$

Use this formula to answer the following questions.

(a) Find the chord of contact in the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$ from the point $(4,-3)$.

(b) Find the chord of contact in the ellipse $x^2+4y^2=4$ from the point $(-5,-2)$.

(c) Show that the chord of contact in the ellipse with parametric equations $x=5\cos\theta$ and $y=2\sin\theta$, from the point $(3,-3)$, has equation $12x-75y=100$.

**10.** In each case, begin by using implicit differentiation.

(a) Show that the tangent to $x^2+4y^2=100$ at $(6,4)$ has equation $3x+8y=50$.

(b) Show that the tangent to $\dfrac{x^2}{9}+\dfrac{y^2}{5}=1$ at $(2,\frac{5}{3})$ has equation $2x+3y=9$.

(c) Show that the normal to $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$ at $(3,\frac{12}{5})$ has equation $100x-45y=192$.

**11.** (a) Show that the tangent to the ellipse given by $x=2\cos\theta$ and $y=\sin\theta$ at the point where $\theta=\frac{\pi}{4}$ has equation $x+2y=2\sqrt{2}$. Begin by writing $\dfrac{dy}{dx}=\dfrac{dy}{d\theta}\div\dfrac{dx}{d\theta}$.

(b) Likewise show that the tangent to the ellipse given by $x=4\cos\theta$, $y=3\sin\theta$ at the point where $\theta=\frac{2\pi}{3}$ has equation $3x-4\sqrt{3}\,y+24=0$.

(c) Show that the normal to the ellipse specified by $x=3\cos\theta$ and $y=2\sin\theta$ at the point where $\theta=\frac{\pi}{6}$ has equation $2\sqrt{3}\,x-4y=5$.

**12.** Suppose that $\mathcal{E}$ is the ellipse $\dfrac{x^2}{100}+\dfrac{y^2}{25}=1$, and let $P$ be the point $(6,4)$ on $\mathcal{E}$.

(a) Show that the normal at $P$ has equation $8x-3y=36$.

(b) Let the normal meet the major axis at $G$, and let $H$ be the foot of the perpendicular drawn from the origin to the tangent at $P$. Show that $PG\times OH=25$.

**13.** (a) Show that the line $x+2y+5=0$ is a tangent to the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$, and show that the point of contact is $(-1\frac{4}{5},-1\frac{3}{5})$.

(b) Show that the line $2x-2y+3=0$ is a tangent to the ellipse $2x^2+4y^2=3$, and show that the point of contact is $(-1,\frac{1}{2})$.

**14.** (a) Show that the two tangents to the ellipse $\dfrac{x^2}{8}+\dfrac{y^2}{4}=1$ with gradient 2 have equations $y=2x-6$ and $y=2x+6$.

(b) Show that the two tangents to the ellipse $x^2+16y^2=25$ which are perpendicular to the line $6x+2y+3=0$ have equations $4x-12y=-25$ and $4x-12y=25$.

**15.** The ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ has foci at $S(5\sqrt{3},0)$ and $S'(-5\sqrt{3},0)$, and passes through $P(8,3)$.

(a) (i) Show that $a^2-b^2=75$ and that $\dfrac{64}{a^2}+\dfrac{9}{b^2}=1$.

(ii) Hence show that the ellipse has equation $\dfrac{x^2}{100}+\dfrac{y^2}{25}=1$.

(b) Alternatively, begin by showing that $SP=2(5-2\sqrt{3})$. Find a similar value for $S'P$, and hence determine the equation of the ellipse.

**16.** The line $y=mx+b$ is a tangent to the ellipse $\dfrac{x^2}{9}+y^2=1$.

(a) Show that $b^2=9m^2+1$.

(b) Hence show that the tangents to the ellipse from the point $(2,1)$ have equations

$$y=1 \qquad \text{and} \qquad 4x+5y=13\,.$$

**17.** The relationship $b^2=a^2(1-e^2)$ can be re-written as

$$(\tfrac{b}{a})^2=1-e^2$$

where the quantity $\frac{b}{a}$ is called the aspect ratio. Describe what happens to the aspect ratio and the shape of the ellipse as (a) $e\to 0^+$, (b) $e\to 1^-$.

**18.** The complex number $z$ satisfies the equation $\left|z+1-i\sqrt{5}\right|+\left|z+5-i\sqrt{5}\right|=6$, which represents an ellipse in the Argand diagram.

(a) Write down the complex number that corresponds to the centre of the ellipse.

(b) Sketch the ellipse and mark the locations of the foci. Clearly show the lengths of the major and minor axes.

(c) Write down the range of values that $\arg(z)$ may take.

**EXTENSION**

**19.** (a) Determine the values of $\lambda$ for which $\dfrac{x^2}{4-\lambda}+\dfrac{y^2}{2-\lambda}=1$ represents an ellipse.

(b) Describe how the shape of the ellipse changes as $\lambda$ increases from 1 to 2.

(c) What happens to the ellipse in the limit as $\lambda\to 2$?

**20.** Let $S = (ae, 0)$ and $S' = (-ae, 0)$. The variable point $P(x, y)$ moves in such a way that $SP + S'P = 2a$. Use the distance formula, and judicious squaring, to show that the locus of $P$ is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

**21.** The equation of an ellipse is undefined in the limit as $e \to 1^-$. By considering the choice of focus, vertex and directrix used at the start of this Section, explain why this happens.

**22.** (a) In a certain ellipse the distance between the focus $S(ae, 0)$ and the vertex $A(a, 0)$ is 1 unit. Show that $b^2 = \dfrac{1+e}{1-e}$.

(b) Suppose that the ellipse in part (a) is shifted so that the vertex is at the origin.

(i) Show that the ellipse has equation $x^2(1-e^2) + 2x(1+e) + y^2 = 0$.

(ii) Describe what happens in the limit as $e \to 1^-$.

(c) Investigate the situation when $SA = f$ units.

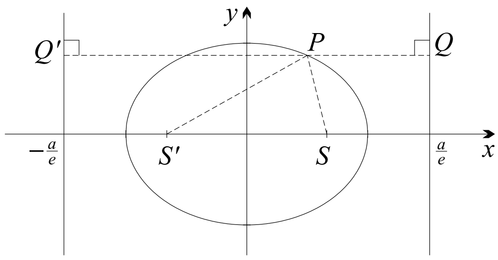
# 3D Geometrical Properties of the Ellipse

Both geometric and algebraic methods may be required in these problems. As a general rule, if geometry can be applied then it will provide a more efficient solution than an algebraic approach.

**WORKED EXERCISE:** Let $P$ be any point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, with foci at $S$ and $S'$. Use the geometric definition of the ellipse to show that $PS + PS' = 2a$.

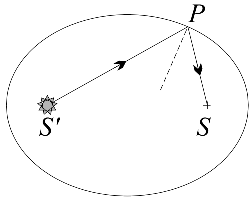
**SOLUTION:** The situation is shown on the right. The points $Q$ and $Q'$ are the feet of the perpendiculars from $P$ to the directrices. Now by definition

$$\frac{PS}{PQ} = \frac{PS'}{PQ'} = e\,.$$




Thus
$$\begin{aligned} PS + PS' &= e \times (PQ + PQ') \\ &= e \times Q'Q \\ &= e \times \tfrac{2a}{e} \\ &= 2a\,. \end{aligned}$$

**The Reflection Property of an Ellipse:** When a light source is placed at one focus of an ellipse, rays reflected from the ellipse pass through the other focus. Measuring reflection angles as in physics, the normal at any point on the ellipse bisects the angle at that point subtended by the foci.



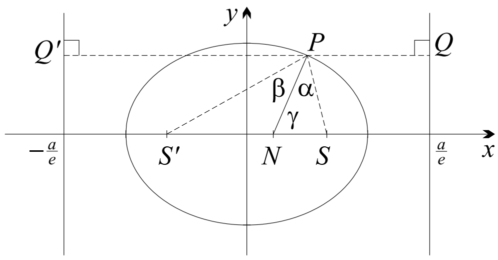

**12** THE REFLECTION PROPERTY OF AN ELLIPSE: The normal at any point on an ellipse bisects the angle at that point subtended by the foci.

PROOF: Let $P(a\cos\theta, b\sin\theta)$ be a point on the ellipse, not on the $x$-axis, and let the normal at $P$ intersect the $x$-axis at $N$. Label the foci $S$ and $S'$, and let $\angle NPS = \alpha$, $\angle NPS' = \beta$ and $\angle SNP = \gamma$. The situation is shown below.

Now from Section 3C, the equation of the normal is

$$\frac{a^2x}{a\cos\theta} - \frac{b^2y}{b\sin\theta} = a^2e^2\,,$$




so the $x$-coordinate of $N$ is

$$x = e^2a\cos\theta\,.$$

Thus $NS = ea(1 - e\cos\theta)$

and $NS' = ea(1 + e\cos\theta)\,.$

Further
$$\begin{aligned} PS &= ePQ \\ &= e\left(\frac{a}{e} - a\cos\theta\right) \\ &= a(1 - e\cos\theta)\,. \end{aligned}$$

Similarly $PS' = a(1 + e\cos\theta)\,.$

Now apply the sine rule in $\triangle NPS$ and $\triangle NPS'$.

$$\begin{aligned} \sin\alpha &= \frac{NS}{PS}\sin\gamma \\ &= e\sin\gamma\,, \end{aligned}$$

and
$$\begin{aligned} \sin\beta &= \frac{NS'}{PS'}\sin(\pi - \gamma) \\ &= e\sin\gamma\,. \end{aligned}$$

Hence $\sin\alpha = \sin\beta\,.$

Now since $\angle SPS'$ is not a straight angle, it follows that

$$\alpha = \beta\,.$$

Thus the normal bisects the angle at $P$ subtended by the foci, as required. The proof for the case when $P$ is on the $x$-axis is left as an exercise.

Finally, note that the reflection property is equivalent to showing that the tangent at $P$ is equally and oppositely inclined to $PS$ and $PS'$.

**13** ALTERNATIVE REFLECTION PROPERTY OF AN ELLIPSE: The tangent at any point on an ellipse is equally and oppositely inclined to the focal chords through that point.

The proof of this result is left as an exercise.

## Exercise **3D**

**1.** As an exercise in algebra, use the distance formula to show that in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ the distance from $P(a\cos\theta, b\sin\theta)$ to the focus $S(ae, 0)$ is $PS = a(1 - e\cos\theta)$. Likewise find $S'P$ and hence show that $S'P + PS = 2a$.
[NOTE: The geometric method shown in the worked exercise is far simpler.]

**2.** Let $P(x_1, y_1)$ be a point on $\mathcal{E}$, the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, with $x_1 \neq 0$ and $y_1 \neq 0$.

(a) Show that the tangent to $\mathcal{E}$ at $P$ has equation $\dfrac{x_1 x}{a^2} + \dfrac{y_1 y}{b^2} = 1$.

(b) The tangent meets the $x$-axis at $T$. Show that $T$ is the point $(\frac{a^2}{x_1}, 0)$.

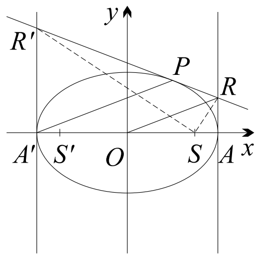
(c) Show that the normal to $\mathcal{E}$ at $P$ has equation $\dfrac{a^2 x}{x_1} - \dfrac{b^2 y}{y_1} = a^2 - b^2$.

(d) The normal meets the $x$-axis at $N$. Show that $N$ is the point $(e^2 x_1, 0)$.

(e) Let $S$ be a focus and $O$ the origin. Show that $OT \times ON = OS^2$.

(f) Let $F$ be the foot of the perpendicular from $P$ to the $x$-axis. Show that $OT \times NF = b^2$.

**3.** Let $\mathcal{E}$ be the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$, and let $P(5\cos\theta, 3\sin\theta)$ be a point on $\mathcal{E}$ with $\sin\theta \neq 0$. Also let $A$ and $A'$ be the vertices $(5, 0)$ and $(-5, 0)$ respectively, and let $O$ be the origin.




(a) Show that the tangent at $P$ is $3x\cos\theta + 5y\sin\theta = 15$.

(b) The tangents at $A$ and $P$ meet at $R$.
Show that $R$ has coordinates $\left(5, \frac{3(1-\cos\theta)}{\sin\theta}\right)$.

(c) Show that $OR$ is parallel to $A'P$.

(d) The tangents at $A'$ and $P$ meet at $R'$. Let $S$ be the focus $(4, 0)$.
Use the gradients of $RS$ and $R'S$ to show that $\angle RSR' = 90°$.

**4.** $\mathcal{E}$ is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and $P(a\cos\theta, b\sin\theta)$ is a point on $\mathcal{E}$ in the first quadrant.

(a) Show that the normal at $P$ has equation $ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$.

(b) The normal at $P$ cuts the $x$-axis at $A$ and the $y$-axis at $B$. Find $A$ and $B$.

(c) Hence show that $\dfrac{PA}{PB} = 1 - e^2$.

**5.** The foci of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ are $S(ae, 0)$ and $S'(-ae, 0)$. The latus rectum through $S$ meets the ellipse in the first quadrant at $P$.

(a) Show that $P$ has coordinates $\big(ae, a(1 - e^2)\big)$.

(b) Show that the tangent at $P$ has equation $ex + y = a$.

(c) The tangent at $P$ meets the $y$-axis at $Q$. Show that $Q$ is on the auxiliary circle, and that the line $QS'$ is parallel to the normal at $P$.

**6.** $\mathcal{E}$ is the ellipse $x = a\cos\theta$, $y = b\sin\theta$. Let $P$ and $Q$ be the points on $\mathcal{E}$ where $\theta = \alpha + \beta$ and $\theta = \alpha - \beta$ respectively. Let $R$ be the point on $\mathcal{E}$ where $\theta = \alpha$.

(a) Show that the chord $PQ$ has gradient $-\frac{b\cos\alpha}{a\sin\alpha}$.

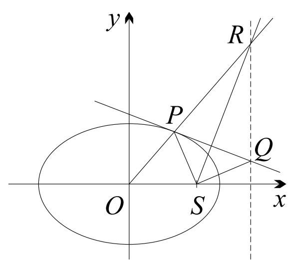
(b) Hence show that the chord $PQ$ is parallel to the tangent to $\mathcal{E}$ at $R$.

(c) Let $O$ be the origin and $M$ be the mid-point of $PQ$. Show that $M$ lies on $OR$.

**7.** (a) The $x$-coordinates of the directrix, vertex and focus of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ form the GP $\frac{a}{e}$, $a$, $ae$. Show that if $b$ is the next term in this sequence then $e^2 = \frac{1}{2}(\sqrt{5} - 1)$. Find the value of the aspect ratio $\frac{a}{b}$.

(b) Suppose that the foci of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ lie on the incircle $x^2 + y^2 = b^2$. What is the eccentricity?

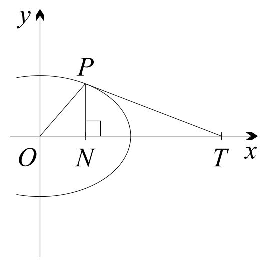
**8.** $\mathcal{E}$ is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and $P$ is the point $(x_1, y_1)$ on $\mathcal{E}$.




(a) If $y_1 \neq 0$ then the tangent to $\mathcal{E}$ at $P$ meets the directrix $x = \frac{a}{e}$ at $Q$. Let $S$ be the focus $(ae, 0)$. Determine the $y$-coordinate of $Q$ and hence show that $SP \perp SQ$.

(b) $O$ is the centre of $\mathcal{E}$. If $x_1 \neq 0$ then the line $OP$ meets the same directrix at $R$. Show that $SR \perp PQ$.

**9.** $\mathcal{E}$ is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and $P$ is the point $(x_1, y_1)$ on $\mathcal{E}$. If $x_1 \neq 0$ then the tangent to $\mathcal{E}$ at $P$ meets the major axis at $T$. Let $N$ be the foot of the perpendicular from $P$ to the major axis.

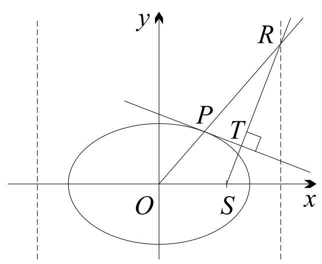



(a) Find the equation of the tangent at $P$ and hence show that $ON \times OT = a^2$.

(b) The segment $NP$ produced meets the auxiliary circle $\mathcal{C}$ at $Q$. Thus $QT$ is tangent to $\mathcal{C}$. Use similar triangles in the auxiliary circle to again show that $ON \times OT = a^2$.

**DEVELOPMENT**

**10.** $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, with focus $S(ae, 0)$. The point $T$ is the foot of the perpendicular from $S$ to the tangent at $P$.




(a) Show that $ST$ has equation $ax\sin\theta - by\cos\theta = a^2 e\sin\theta$.

(b) The line $ST$ meets the directrix $x = \frac{a}{e}$ at $R$. Find the coordinates of $R$.

(c) Let $O$ be the centre of the ellipse. Show that $O$, $P$ and $R$ are collinear.

**11.** The ellipse $\mathcal{E}$ with equation $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ has foci $S(4, 0)$ and $S'(-4, 0)$.

(a) Show that the tangent at $P(x_1, y_1)$ on $\mathcal{E}$ has equation $9x_1x + 25y_1y = 225$.

(b) Suppose that $Q(x_2, y_2)$ is another point on $\mathcal{E}$, where the chord $PQ$ passes through $S$. Show that $4(y_2 - y_1) = x_1y_2 - x_2y_1$.

(c) Show that the tangents at $P$ and $Q$ intersect on the directrix corresponding to $S$, except when $y_1 = 0$.

(d) Show that the normal at $P$ has equation $25y_1x - 9x_1y = 16x_1y_1$, and decide under what circumstances, if any, it passes through $S$ or $S'$.

**12.** $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. A line is drawn through the centre of the ellipse parallel to the tangent at $P$. This line intersects the ellipse at $R$ and $R'$.

(a) Show that the tangent at $P$ has equation $(b\cos\theta)x + (a\sin\theta)y - ab = 0$.

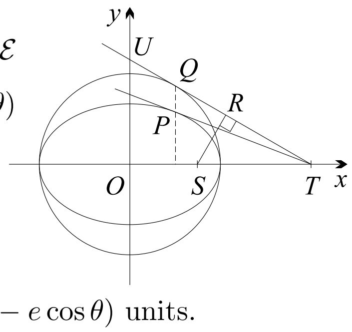
(b) Use the perpendicular distance formula to find the distance from the centre of the ellipse to the tangent at $P$.

(c) Put $R = (a\cos\phi, b\sin\phi)$ and show that $\cos(\theta - \phi) = 0$. Hence find the coordinates of $R$ and $R'$ in terms of $\theta$.

(d) Use the previous two parts to show that the area of $\triangle RPR'$ is independent of where $P$ lies on the ellipse. That is, show that the area does NOT depend on $\theta$.

(e) Now prove the same result geometrically by considering an appropriate triangle in the auxiliary circle and using stretching.




**13.** The point $P(a\cos\theta, b\sin\theta)$, with $\cos\theta \neq 0$, is on the ellipse $\mathcal{E}$ with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and $Q$ is the point $(a\cos\theta, a\sin\theta)$ on the auxiliary circle $\mathcal{C}$. $S$ is the focus $(ae, 0)$.

(a) Show that the tangent at $P$ is $bx\cos\theta + ay\sin\theta = ab$.

(b) Show that the tangent at $Q$ is $x\cos\theta + y\sin\theta = a$.

(c) These tangents meet at $T$. Show that $T$ is on the $x$-axis.

(d) Use the definition of eccentricity to show that $SP = a(1 - e\cos\theta)$ units.

(e) Let $R$ be the foot of the perpendicular from $S$ to the tangent to $\mathcal{C}$ at $Q$. Use the perpendicular distance formula to show that $SR = SP$.

(f) Now prove the same result with the aid of some geometry. Let the tangent at $Q$ meet the $x$-axis at $T$ and the $y$-axis at $U$. Begin by showing that $\triangle UOT \,|||\, \triangle SRT$ and that $\angle TUO = \theta$.

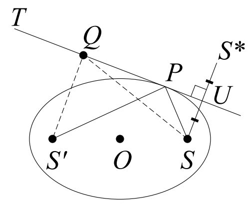
**14.** $\mathcal{E}$ is the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$, and $P(4\cos\theta, 3\sin\theta)$ is a point on $\mathcal{E}$, with $\cos\theta \neq 0$. $B$ and $B'$ are the endpoints of the minor axis.

(a) Show that the tangent to $\mathcal{E}$ at $P$ has equation $3x\cos\theta + 4y\sin\theta = 12$.

(b) The tangent at $P$ meets the tangents at $B$ and $B'$ at $C$ and $C'$ respectively.
Show that $BC \times B'C' = 16$.

(c) The circle with diameter $CC'$ meets the $x$-axis at $D$ and $D'$.
Show that $OD \times OD' = 7$.

(d) Let $N$ be a point on the minor axis of $\mathcal{E}$. Prove that $\angle CNC'$ cannot be a right-angle.

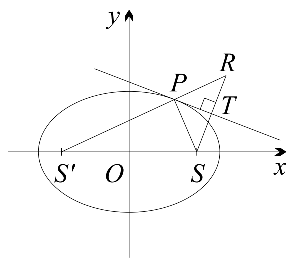



**15.** In the diagram, $P$ is a point on an ellipse and $TP$ is the tangent at that point. The foci of the ellipse are at $S$ and $S'$. When $S$ is reflected in $PT$ the result is $S^*$, and $S^*S$ intersects $PT$ at $U$. Let $Q$ be an arbitrary point on $PT$.

(a) Explain why $S'Q + QS \geq S'P + PS$. When are they equal?

(b) The shortest path between $S'$ and $S^*$ is a straight line. Justify why this line must pass through $P$.

(c) Hence show that the normal at $P$ bisects the angle subtended by the foci at $P$.



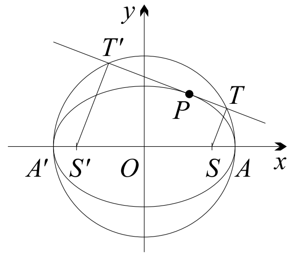

**16.** In the diagram, $P$ is a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with foci at $S$ and $S'$ on the $x$-axis. $T$ is the foot of the perpendicular from $S$ to the tangent at $P$. $ST$ and $S'P$ intersect at $R$.

(a) Use the alternative reflection property to prove that

$$ST = RT\,.$$

(b) Explain why $S'R = 2a$.

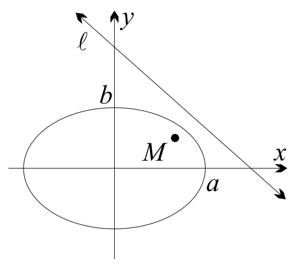
(c) Hence prove that $T$ lies on the auxiliary circle.




**17.** In the diagram, $P$ is a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with foci at $S(ae, 0)$ and $S'(-ae, 0)$. The corresponding vertices on the major axis are at $A(a, 0)$ and $A'(-a, 0)$. $T$ and $T'$ are the feet of the perpendiculars from $S$ and $S'$ respectively to the tangent at $P$. As proven in the previous question, the points $T$ and $T'$ lie on the auxiliary circle.

Use symmetry and circle geometry to prove that $ST \times S'T' = b^2$.

**EXTENSION**




**18.** (a) Show that $pq \le \dfrac{p^2+q^2}{2}$ for all real values of $p$ and $q$.

(b) The ellipse $\mathcal{E}$ has equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The point $M(x_0, y_0)$ lies inside $\mathcal{E}$, so that $\dfrac{{x_0}^2}{a^2}+\dfrac{{y_0}^2}{b^2}<1$.

The line $\ell$ has equation $\dfrac{x_0x}{a^2}+\dfrac{y_0y}{b^2}=1$.

(i) Use the result in part (a) to show that the line $\ell$ lies entirely outside $\mathcal{E}$. That is, show that if $P(x_1, y_1)$ is any point on $\ell$, then $\dfrac{{x_1}^2}{a^2}+\dfrac{{y_1}^2}{b^2}>1$.

(ii) The chord of contact to $\mathcal{E}$ from any point $Q(x_2, y_2)$ outside $\mathcal{E}$ has equation

$$\frac{x_2x}{a^2}+\frac{y_2y}{b^2}=1\,.$$

Show that if $M$ lies on this chord then $Q$ must be on $\ell$.

**19.** Consider the concentric ellipses $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and $\dfrac{x^2}{c^2}+\dfrac{y^2}{d^2}=1$, where $b<a<c$ and $b<d<c$. The tangent at $E(x_0, y_0)$ on the inner ellipse intersects the outer ellipse at the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

(a) By considering the roots of an appropriate quadratic, show that

$$x_1x_2=\frac{a^4c^2(b^4-d^2{y_0}^2)}{a^4d^2{y_0}^2+b^4c^2{x_0}^2}\,.$$

(b) Similarly show that

$$y_1y_2=\frac{b^4d^2(a^4-c^2{x_0}^2)}{a^4d^2{y_0}^2+b^4c^2{x_0}^2}\,.$$

(c) It is known that the tangents at $P$ and $Q$ are perpendicular for all points $E$ on the inner ellipse. Show that

$$c^4=a^2(c^2+d^2) \qquad \text{and} \qquad d^4=b^2(c^2+d^2)\,.$$

(d) Solve these equations simultaneously to show that

$$c^2=a(a+b) \qquad \text{and} \qquad d^2=b(a+b)\,.$$

(e) The tangents at $P$ and $Q$ meet at $T$. Show that $T$ lies on the circle with centre the origin and radius $(a+b)$.

(f) Show that if $a$, $b$, $c$ and $d$ are integers, with $a$ and $b$ having no common factors, then

$$a=p^2\,,\ b=q^2\,,\ c=rp \text{ and } d=rq$$

where $(p, q, r)$ is a Pythagorean triad.

# 3E Hyperbolas

Recall that the equation of a conic with eccentricity $e$ and $x$-intercept at $A(a, 0)$ is given by

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1\,.$$

Now if $e > 1$ then the second denominator is guaranteed to be negative, so write

$$b^2 = a^2(e^2 - 1)$$

and the equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,.$$

This is the equation of a hyperbola.

**14** THE EQUATION OF A HYPERBOLA: The equation of a hyperbola is

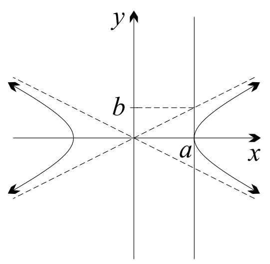
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b^2 = a^2(e^2 - 1)$, and $e > 1$ is the eccentricity.

Like the rectangular hyperbola $xy = 1$, all hyperbolae have two branches. Unlike the ellipse, there are only two vertices at $A(a, 0)$ and $A'(-a, 0)$. $A'A$ is again called the major axis and has length $2a$. It is also called the *transverse axis*, being the line that passes through the two closest points on opposite branches. The $y$-axis is called the minor axis or *conjugate axis*. Both the transverse and conjugate axes are of course axes of symmetry.

**Asymptotes:** As with all hyperbolae, there are two asymptotes. Rearranging the equation of the hyperbola yields

$$\frac{b^2}{a^2} - \frac{y^2}{x^2} = \frac{b^2}{x^2}$$




so in the limit as $x \to \infty$ the result is

$$\frac{b^2}{a^2} - \frac{y^2}{x^2} = 0\,.$$

Thus the equations of the two asymptotes are

$$\frac{y}{x} = \frac{b}{a} \quad \text{and} \quad \frac{y}{x} = -\,\frac{b}{a}\,.$$

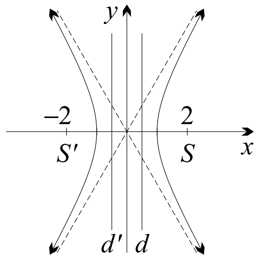
Notice therefore that the tangent to the hyperbola at $A(a, 0)$ meets the asymptote at $(a, b)$ in the first quadrant, as shown in the diagram above.

**15** THE ASYMPTOTES OF A HYPERBOLA: The equations of the asymptotes are

$$\frac{y}{x} = \frac{b}{a} \quad \text{and} \quad \frac{y}{x} = -\,\frac{b}{a}\,.$$

**WORKED EXERCISE:** Sketch the hyperbola $x^2 - \dfrac{y^2}{3} = 1$, showing the asymptotes, foci and directrices.




**SOLUTION:** Now $a = 1$ and $b = \sqrt{3}$ so the asymptotes are

$$y = x\sqrt{3} \quad \text{and} \quad y = -x\sqrt{3}\,.$$

From $a$ and $b$, the eccentricity $e$ is given by

$$3 = 1 \times (e^2 - 1)$$

so $\quad e = 2\,.$

Thus the directrices are $x = \frac{1}{2}$ and $x = -\frac{1}{2}$, and the foci are $S(2, 0)$ and $S'(-2, 0)$. These features are shown in the diagram above on the right.

**Parameters:** The parametric equations used in this course for the hyperbola come from one of the Pythagorean trigonometric identities. Recall that

$$\tan^2\theta + 1 = \sec^2\theta$$

or $\quad \sec^2\theta - \tan^2\theta = 1\,.$

This suggests the parametric equations

$$x = a\sec\theta \quad \text{and} \quad y = b\tan\theta$$

so that
$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \sec^2\theta - \tan^2\theta$$
$$= 1$$

as expected.

16 | THE PARAMETRIC EQUATIONS: The parametric equations of a hyperbola are:

$$x = a\sec\theta \quad \text{and} \quad y = b\tan\theta\,.$$

**WORKED EXERCISE:** Find the coordinates of the point where $\theta = \frac{3\pi}{4}$ on

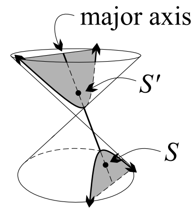
$$\frac{x^2}{4} - y^2 = 1\,.$$

**SOLUTION:** Since $a = 2$ and $b = 1$, the point is

$$(2 \times \sec\tfrac{3\pi}{4}, 1 \times \tan\tfrac{3\pi}{4}) = (-2\sqrt{2}, -1)\,.$$

Notice that this point is in the third quadrant despite the angle $\frac{3\pi}{4}$ being in the second quadrant. Clearly the relationship between the angle and the location of the point is more complicated than in the cases of the circle and ellipse. This relationship will be investigated later.

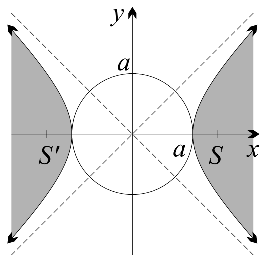
**The Auxiliary Circle:** The two branches of the hyperbola divide the Cartesian plane into three regions. The two regions where the the foci are located are said to be inside the hyperbola. This is because in the three dimensional model those two regions are inside the cone, as shown shaded in the diagram on the right. Since the foci are inside the cone, it makes sense to say that they are inside the conic section.



The circumcircle of a hyperbola is outside and tangent to it. Hence the circumcircle is

$$x^2 + y^2 = a^2\,.$$

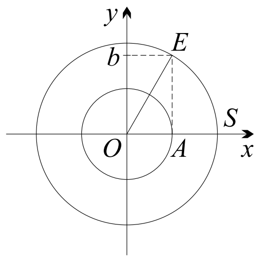
Once again the circumcircle is called the *auxiliary circle* and has several important features, some of which will be investigated here.




| | |
|---|---|
| **17** | THE AUXILIARY CIRCLE: The auxiliary circle is the circumcircle of the hyperbola. It has centre the origin and radius equal to the semi-major axis. Its equation is $x^2 + y^2 = a^2.$ |

The auxiliary circle and the circle with $S'S$ as diameter can be used to locate the asymptotes. Draw the tangent to the auxiliary circle at $A(a, 0)$ and let it intersect the focal circle at $E$ in the first quadrant. The $y$-coordinate of $E$ is given by

$$\begin{aligned} y^2 &= (ae)^2 - a^2 \\ &= a^2(e^2 - 1)\,. \end{aligned}$$




That is $y = b$, and hence $OE$ is an asymptote.

Next let $D$ be the point in the first quadrant where the directrix $x = \frac{a}{e}$ intersects the auxiliary circle. Let $C$ be the $x$-intercept of the tangent there. At $D$:

$$\begin{aligned} y^2 &= a^2 - (\tfrac{a}{e})^2 \\ &= \frac{a^2}{e^2}(e^2 - 1) \\ &= \frac{b^2}{e^2}\,. \end{aligned}$$

Thus $y = \frac{b}{e}\,.$

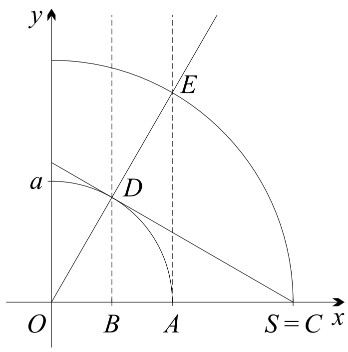
Notice therefore that $D(\frac{a}{e}, \frac{b}{e})$ lies on the asymptote $OE$.

Next, the equation of the tangent at $D$ is

$$\tfrac{a}{e}x + \tfrac{b}{e}y = a^2$$

and the $x$-intercept is

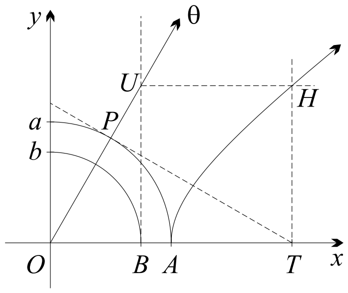
$$\begin{aligned} &\tfrac{a}{e}x = a^2 \\ \text{or}\quad &x = ae\,. \end{aligned}$$



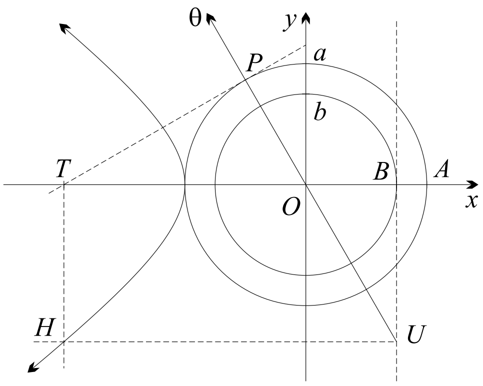

That is, the point $C$ coincides with the focus, as shown in the diagram above. This result can also be obtained by the congruent triangles $\triangle OAE$ and $\triangle ODC$, and is left as an exercise.

**The Auxiliary Circle and Parameters:** The circle $x^2 + y^2 = b^2$ will be called the complementary circle. The complementary circle and the auxiliary circle can be used to quickly plot points specified parametrically.

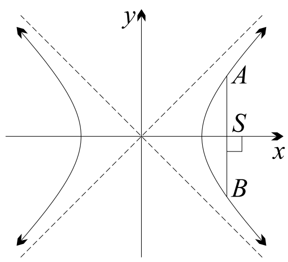
Construct the auxiliary circle and complementary circle on a set of axes. Let $A(a, 0)$ be the $x$-intercept of the auxiliary circle and $B(b, 0)$ be the $x$-intercept of the complementary circle. Construct $P$ on the auxiliary circle so that $\angle AOP = \theta$, the required parameter. Let the tangent at $P$ meet the $x$-axis at $T$ and let the tangent at $B$ meet $OP$ at $U$. Let the vertical line through $T$ and the horizontal line through $U$ intersect at $H$. Then the point $H$ is on the hyperbola and has coordinates $(a \sec\theta, b \tan\theta)$. The situation is shown on the right and the proof is left as an exercise.




The careful reader will have noticed that when $\theta$ is obtuse the ray $OP$ and tangent at $B$ do not intersect. In this case the ray is extended to meet the tangent in the opposite quadrant. Thus when $\theta$ lies in the second quadrant, the point $H$ on the hyperbola lies in the third quadrant, and vice versa. The situation is shown in the second diagram on the right.




**The Latus Rectum:** Like the ellipse, the latus rectum passes through $S$ or $S'$ and is perpendicular to the major axis. Let the end points of the latus rectum through $S$ be at $A$ and $B$. The $x$-coordinate of these points is $x = ae$.




Hence $\dfrac{(ae)^2}{a^2} - \dfrac{y^2}{b^2} = 1$

so $\dfrac{y^2}{b^2} = e^2 - 1$

or $y^2 = a^2(e^2 - 1)^2$

since $b^2 = a^2(e^2 - 1)$.

Thus $y = \pm a(e^2 - 1)$.

Hence
$$\begin{aligned} AB &= 2a(e^2 - 1) \\ &= \frac{2b^2}{a}. \end{aligned}$$

**Tangents:** Following the patterns of the equations of tangents to circles, parabolae and ellipses, the equation of the tangent to the hyperbola at the point $H(x_1, y_1)$ should be

$$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1\,,$$

and indeed it is. If instead $H = (a \sec\theta, b \tan\theta)$ then the parametric form is obtained, viz:

$$\frac{x\, a \sec\theta}{a^2} - \frac{y\, b \tan\theta}{b^2} = 1\,.$$

**18**

| | |
|---|---|
| TANGENT IN PARAMETRIC FORM: | $\dfrac{x\,a\sec\theta}{a^2} - \dfrac{y\,b\tan\theta}{b^2} = 1$ |
| TANGENT IN CARTESIAN FORM: | $\dfrac{x_1 x}{a^2} - \dfrac{y_1 y}{b^2} = 1$ |

Both these equations can be derived via calculus. In the parametric case, begin with the chain rule as follows.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{d\theta} \div \frac{dx}{d\theta}\\ &= b\sec^2\theta \div a\sec\theta\tan\theta\\ &= \frac{b\sec\theta}{a\tan\theta}.\end{aligned}$$

Hence the equation of the tangent is

$$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta} \times (x - a\sec\theta)$$

so $$\frac{y\tan\theta}{b} - \tan^2\theta = \frac{x\sec\theta}{a} - \sec^2\theta$$

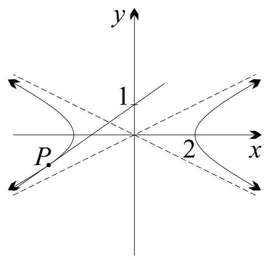
or $$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = \sec^2\theta - \tan^2\theta\,.$$

Thus $\dfrac{x\,a\sec\theta}{a^2} - \dfrac{y\,b\tan\theta}{b^2} = 1$ (by the Pythagorean trigonometric identity)

exactly as expected.

**WORKED EXERCISE:** Derive the equation of the tangent at $P(-2\sqrt{2}, -1)$ on the hyperbola $\frac{1}{4}x^2 - y^2 = 1\,$.

**SOLUTION:** Differentiate implicitly to get




$$\frac{x}{2} - 2y\,\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = \frac{x}{4y}\,.$$

So at $P$ $$\frac{dy}{dx} = \frac{1}{\sqrt{2}}\,.$$

Hence the equation of the tangent is

$$y + 1 = \tfrac{1}{\sqrt{2}}(x + 2\sqrt{2}\,)$$

so $$2y + 2 = x\sqrt{2} + 4$$

or $$2y - x\sqrt{2} = 2\,.$$

**Normals:** Using the gradient of the tangent, the Cartesian equation of the normal at the point $(x_1, y_1)$ is

$$y - y_1 = -\frac{a^2 y_1}{b^2 x_1} \times (x - x_1)$$

so $$\frac{b^2 y}{y_1} - b^2 = -\frac{a^2 x}{x_1} + a^2$$

thus $$\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2\,,$$

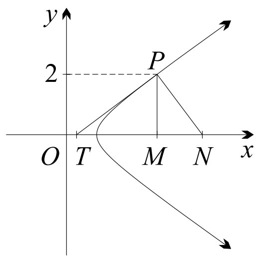
or $$\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 e^2\,.$$

The parametric form is easily obtained by replacing $(x_1, y_1)$ with $(a\sec\theta, b\tan\theta)$.

**19**

| | |
|---|---|
| NORMAL IN PARAMETRIC FORM: | $\dfrac{a^2x}{a\sec\theta} + \dfrac{b^2y}{b\tan\theta} = a^2 + b^2$ |
| NORMAL IN CARTESIAN FORM: | $\dfrac{a^2x}{x_1} + \dfrac{b^2y}{y_1} = a^2 + b^2$ |

**WORKED EXERCISE:** The normal at $P(3,2)$ on the hyperbola $x^2 - 2y^2 = 1$ meets the $x$-axis at $N$. The point $M$ is the foot of the perpendicular from $P$ to the $x$-axis. The tangent at $P$ meets the $x$-axis at $T$. Show that $OT \times MN = b^2$.




**SOLUTION:** Differentiating implicitly:

$$2x - 4y\frac{dy}{dx} = 0$$

so $$\frac{dy}{dx} = \frac{x}{2y}\,.$$

At $P$ $$\frac{dy}{dx} = \frac{3}{4}$$

thus the equation of the normal at $P$ is

$$y - 2 = -\tfrac{4}{3}(x-3)$$

or $$\frac{x}{3} + \frac{y}{4} = \frac{3}{2}\,.$$

Thus $$N = (4\tfrac{1}{2}, 0)\,.$$

The equation of the tangent at $P$ is

$$y - 2 = \tfrac{3}{4}(x-3)$$

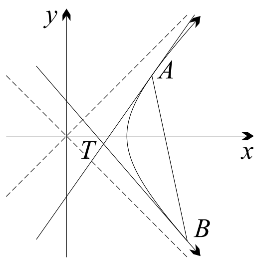
or $$3x - 4y = 1\,.$$

Thus $$T = (\tfrac{1}{3}, 0)\,.$$

Clearly $M = (3,0)$ and so $MN = \frac{3}{2}$, $OT = \frac{1}{3}$, and hence $MN \times OT = \frac{1}{2} = b^2$.

**Chord of Contact:** Proceeding as usual, suppose that the tangents at $A(x_1, y_1)$ and $B(x_2, y_2)$ meet at $T(x_0, y_0)$. From above, the equation of the tangent at $A$ is

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1\,.$$




Since $T(x_0, y_0)$ lies on this line, it follows that

$$\frac{x_1x_0}{a^2} - \frac{y_1y_0}{b^2} = 1\,.$$

From this equation, $A(x_1, y_1)$ must lie on the line

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1\,.$$

Similarly, the equation of the tangent at $B(x_2, y_2)$ is

$$\frac{x_2x}{a^2} - \frac{y_2y}{b^2} = 1\,.$$

Since $T(x_0, y_0)$ lies on this line, it follows that

$$\frac{x_2x_0}{a^2} - \frac{y_2y_0}{b^2} = 1\,.$$

From this equation, $B(x_2, y_2)$ must also lie on the line

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1\,.$$

Hence both $A$ and $B$ lie on the line $\dfrac{x_0 x}{a^2} - \dfrac{y_0 y}{b^2} = 1$, which must therefore be the equation of the chord of contact.

**20** THE CHORD OF CONTACT: The equation of the chord of contact for the tangents from the external point $T(x_0, y_0)$ to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1\,.$$

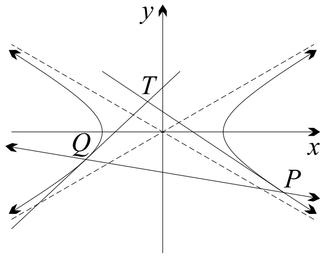
**WORKED EXERCISE:** Use the formula above to determine the equation of the chord of contact from $(-\frac{1}{2}, 1)$ in $x^2 - 3y^2 = 4$. By considering the gradients of the chord and the asymptotes, explain why the two points of contact must be on separate branches of the hyperbola.

**SOLUTION:** Applying the formula in Box 20, the chord of contact is:

$$\frac{(-\frac{1}{2})x}{4} - \frac{3(1)y}{4} = 1$$

so $\qquad x + 6y + 8 = 0\,.$

The gradient of this line is $-\frac{1}{6}$ and the gradients of the asymptotes are $\frac{1}{\sqrt{3}}$ and $-\frac{1}{\sqrt{3}}$. Since $-\frac{1}{\sqrt{3}} \le -\frac{1}{6} \le \frac{1}{\sqrt{3}}$, it follows that the chord of contact must intersect both asymptotes, and hence it also intersects both branches of the hyperbola. The graph on the right confirms this.




## Exercise 3E

**1.** The hyperbola $\mathcal{H}$ has equation $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$.

(a) Find its eccentricity, the coordinates of its foci, and the equations of its directrices.

(b) Write down the equations of its asymptotes.

(c) Sketch $\mathcal{H}$, showing these features.

(d) Show that the parametric equations $x = 2\sec\theta$, $y = \sqrt{5}\tan\theta$ represent $\mathcal{H}$.

(e) Find the coordinates of the point on $\mathcal{H}$ corresponding to $\theta = \frac{\pi}{4}$.

**2.** Consider the hyperbola $\mathcal{H}$ with equation $x^2 - y^2 = 4$.

(a) Find the foci and directrices of $\mathcal{H}$.

(b) Sketch $\mathcal{H}$, showing its foci, directrices and asymptotes.

(c) Show that the parametric equations $x = 2\sec\alpha$, $y = 2\tan\alpha$ represent $\mathcal{H}$.

(d) Show on your diagram the point where $\alpha = \frac{3\pi}{4}$.

**3.** The hyperbola $\mathcal{H}$ has equation $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$.

(a) Find its eccentricity, the coordinates of its foci and the equations of its directrices.
(b) Write down the equations of its asymptotes.
(c) Sketch $\mathcal{H}$, showing these features.
(d) Write down the standard pair of parametric equations representing $\mathcal{H}$.
(e) Find the coordinates of the point on $\mathcal{H}$ corresponding to $\theta = -\frac{\pi}{3}$.

**4.** Consider the hyperbola defined by the parametric equations $x = \sec\theta$, $y = \sqrt{3}\tan\theta$.

(a) Find the coordinates of the point on the hyperbola corresponding to $\theta = -\frac{5\pi}{6}$.
(b) Write down the Cartesian equation of the hyperbola.
(c) What is its eccentricity?
(d) Show that each latus rectum is of length 6 units.

**5.** (a) A certain hyperbola has its centre at the origin, a focus at $(4, 0)$, and the corresponding directrix is $x = 1$. Find its equation.

(b) Another hyperbola has eccentricity 3 and centre at the origin. Its transverse axis is on the $x$-axis. Its two vertices are 10 units apart. Find its equation.

DEVELOPMENT

**6.** A variable point $P$ in the number plane moves in such a way that the difference of its distances from $(5, 0)$ and $(-5, 0)$ is always 6 units. Thus, the locus of $P$ is a hyperbola.

(a) Draw a diagram showing the two foci and the point $P$ at the vertex $(a, 0)$. Use the difference of distances to show that $a = 3$.
(b) Hence show that the equation of the hyperbola is $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$.

**7.** The chord of contact for the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ from the point $(x_0, y_0)$ is

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1\,.$$

Use this formula to answer the following questions.

(a) Find the chord of contact for the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$ from the point $(1, 1)$.
(b) Show that the chord of contact for the hyperbola with parametric equations $x = 4\sec\theta$ and $y = \tan\theta$, from the point $(2, -3)$, has equation $x + 24y = 8$.

**8.** In each case, begin by using implicit differentiation.

(a) Show that the tangent and normal to the hyperbola $x^2 - 2y^2 = 2$ at the point $(-2, 1)$ have equations $x + y + 1 = 0$ and $x - y + 3 = 0$ respectively.
(b) Show that the tangent and normal to the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$ at the point $(4, -3\sqrt{3})$ have equations $\sqrt{3}\,x + y = \sqrt{3}$ and $x - \sqrt{3}\,y = 13$ respectively.

**9.** (a) Show that the tangent to the hyperbola given by $x = 2\sec\theta$ and $y = \tan\theta$ at the point where $\theta = \frac{\pi}{4}$ has equation $x - \sqrt{2}\,y = \sqrt{2}$. Begin by writing $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \div \dfrac{dx}{d\theta}$.

(b) Likewise show that the tangent to the hyperbola given by $x = \sec\theta$ and $y = \tan\theta$ at the point where $\theta = \frac{\pi}{3}$ has equation $2x - \sqrt{3}\,y = 1$.

**10.** (a) The point $P(9,-3)$ is on the hyperbola $\dfrac{x^2}{54} - \dfrac{y^2}{18} = 1$. The normal at $P$ meets the hyperbola again at $Q$. Show that the tangents at $P$ and $Q$ meet at the point $(\frac{9}{2}, \frac{3}{2})$.

(b) The line $y = 2x - 4$ intersects the hyperbola $\dfrac{x^2}{3} - \dfrac{y^2}{2} = 1$ at $P$ and $Q$. Show that the tangents at $P$ and $Q$ intersect at the point $(\frac{3}{2}, \frac{1}{2})$.

**11.** The tangent at the point $P(2,1)$ on the hyperbola $9x^2 - 4y^2 = 32$ meets the asymptotes at $A$ and $B$.

(a) Show that the interval $AB$ has length $\frac{2\sqrt{85}}{3}$ units.

(b) Show that $P$ is the mid-point of $AB$.

**12.** (a) Show that the line $x - 2y + 1 = 0$ is a tangent to the hyperbola $x^2 - 6y^2 = 3$, and show that the point of contact is $(-3,-1)$.

(b) Show that the line $4x - 3y = 5$ is a tangent to the hyperbola $2x^2 - 3y^2 = 5$, and show that the point of contact is $(2,1)$.

**13.** (a) Show that the two tangents to the hyperbola $\dfrac{x^2}{3} - \dfrac{y^2}{2} = 1$ with gradient $-1$ have equations $x + y = 1$ and $x + y = -1$.

(b) Show that the two tangents to the hyperbola $x^2 - 3y^2 = 6$ which are parallel to the line $2x - y = 7$ have equations $2x - y = \sqrt{22}$ and $2x - y = -\sqrt{22}$.

**14.** The line $y = mx + c$ is a tangent to the hyperbola $2x^2 - y^2 = 1$.

(a) Show that $m^2 = 2(c^2 + 1)$.

(b) Hence show that the tangents to the hyperbola from the point $(2,3)$ have equations

$$y = 2x - 1 \qquad \text{and} \qquad y = \tfrac{10}{7}x + \tfrac{1}{7}\,.$$

**15.** (a) By considering the asymptotes, investigate how the shape of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ changes as: (i) $e \to 1^+$, (ii) $e \to \infty$.

(b) Describe what happens to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ in the limit as $a \to 0^+$. You may assume that $e$ is fixed.

**16.** The complex number $z$ satisfies the equation $|z-2| - |z+2| = 2\sqrt{3}$. That is, if the point $P$ represents $z$ in the argand diagram then $P$ lies on a hyperbola.

(a) Sketch the hyperbola and asymptotes, and mark the locations of the foci.

(b) Carefully explain why $-\frac{\pi}{6} < \arg(z) < \frac{\pi}{6}$.

**EXTENSION**

**17.** (a) In a certain hyperbola the distance between the focus $S(ae,0)$ and the vertex $A(a,0)$ is 1 unit. Show that $b^2 = \dfrac{e+1}{e-1}$.

(b) Suppose that the hyperbola in part (a) is shifted so that the vertex is at the origin.

(i) Show that the hyperbola has equation $x^2(e^2-1) + 2x(e+1) - y^2 = 0$.

(ii) Describe what happens in the limit as $e \to 1^+$.

(c) Investigate the situation when $SA = f$ units.

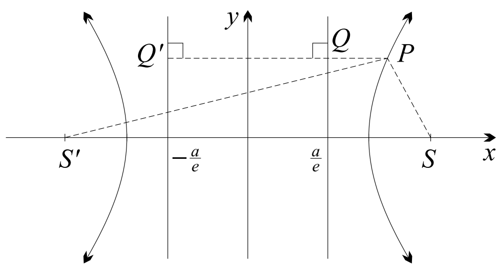
# 3F Geometrical Properties of the Hyperbola

Both geometric and algebraic methods may be required in these problems. As a general rule, if geometry can be applied then it will provide a more efficient solution than an algebraic approach.

**WORKED EXERCISE:** Let $P$ be any point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, with foci at $S$ and $S'$. Use the geometric definition of the hyperbola to show that

$$|PS - PS'| = 2a\,.$$

**SOLUTION:** The situation is shown on the right. The points $Q$ and $Q'$ are the feet of the perpendiculars from $P$ to the directrices. Now by definition

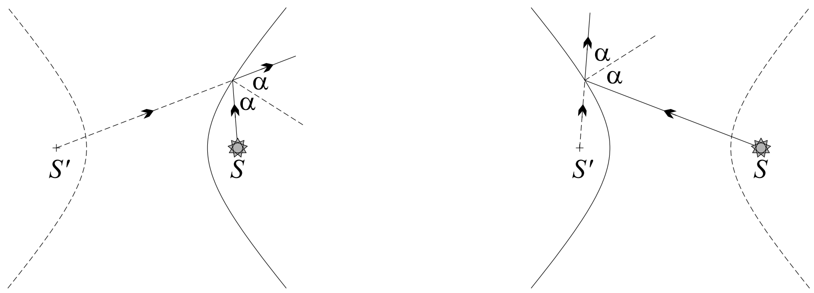



$$\frac{PS}{PQ} = \frac{PS'}{PQ'} = e\,.$$

Thus
$$\begin{aligned} |PS - PS'| &= e \times |PQ - PQ'| \\ &= e \times Q'Q \\ &= e \times \tfrac{2a}{e} \\ &= 2a\,. \end{aligned}$$

Note that in the diagram, since $P$ is shown on the right hand branch, both the quantities $(PS - PS')$ and $(PQ - PQ')$ are positive so the absolute value sign is superfluous. In particular, $(PQ - PQ')$ is simply the distance between the directirices. However, if $P$ were to be on the left branch then both quantities are negative and the absolute value sign is essential.

**The Reflection Property of a Hyperbola:** When a light source is placed at one focus of a hyperbola, the rays reflected from the hyperbola are seen to have originated at the other focus. Measuring reflection angles as in physics, the normal at any point on the hyperbola is equally and oppositely inclined to the focal chords through that point.



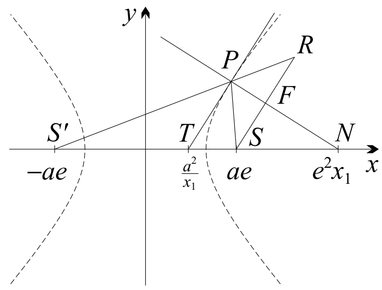

There are two configurations of the situation depending on which branch of the hyperbola is the reflective surface, and these are shown in the diagrams above.

**21** | THE REFLECTION PROPERTY OF A HYPERBOLA: The normal at any point on a hyperbola is equally and oppositely inclined to the focal chords through that point.

PROOF: It is convenient to use complex numbers and vectors in this proof. As is the usual convention, capital letters will be used to represent points in the Argand diagram and the corresponding complex numbers will be written in lower case.

Let $P(x_1, y_1)$ be a point on the hyperbola $\mathcal{H}$ with foci $S(ae, 0)$ and $S'(-ae, 0)$ and eccentricity $e$. The point $P$ is not on the real axis, and for simplicity $P$ is on the right hand branch of $\mathcal{H}$. Let the tangent at $P$ intersect the real axis at $T$, and let the normal intersect at $N$. Let $R$ be the reflection of $S$ in $PN$ so that $\angle SPN = \angle RPN$. That is, light from a source at $S$ will be seen at $R$. Finally let $F$ be the mid-point of $SR$. The aim is to prove that $S'$, $P$ and $R$ are collinear.

By the similar triangles $\triangle PTN$ and $\triangle FSN$,




$$\begin{aligned}\overrightarrow{SF} &= \overrightarrow{TP} \times \frac{SN}{TN}\\ &= (p-t) \times \frac{SN}{TN},\end{aligned}$$

so
$$\begin{aligned}r &= s + 2\,\overrightarrow{SF}\\ &= s + 2(p-t) \times \frac{SN}{TN}.\end{aligned}$$

Thus
$$\begin{aligned}\overrightarrow{PR} &= r - p\\ &= s + 2(p-t) \times \frac{SN}{TN} - p\\ &= p \times \left(2\,\frac{SN}{TN} - 1\right) - \left(2t\,\frac{SN}{TN} - s\right).\end{aligned}$$

Next consider the second bracketed term. Now

$$\begin{aligned}\frac{SN}{TN} &= \frac{e^2x_1 - ae}{e^2x_1 - \frac{a^2}{x_1}}\\ &= \frac{e\,x_1(e\,x_1 - a)}{e^2x_1^2 - a^2}\\ &= \frac{e\,x_1}{e\,x_1 + a} \qquad \text{(difference of two squares.)}\end{aligned}$$

Thus
$$\begin{aligned}2t\,\frac{SN}{TN} - s &= 2\left(t\,\frac{SN}{TN} - s\right) + s\\ &= 2\left(\frac{a^2}{x_1} \times \frac{e\,x_1}{e\,x_1 + a} - ae\right) + ae\\ &= \frac{2(a^2e - ae^2x_1 - a^2e)}{e\,x_1 + a} + ae\\ &= -ae\,\frac{2e\,x_1}{e\,x_1 + a} + ae\\ &= s' \times \left(2\,\frac{SN}{TN} - 1\right).\end{aligned}$$

Hence
$$\begin{aligned}\overrightarrow{PR} &= p \times \left(2\,\frac{SN}{TN} - 1\right) - s' \times \left(2\,\frac{SN}{TN} - 1\right)\\ &= (p - s') \times \left(2\,\frac{SN}{TN} - 1\right)\\ &= \overrightarrow{S'P} \times \left(2\,\frac{SN}{TN} - 1\right).\end{aligned}$$

Now since $\overrightarrow{PR}$ is a real multiple of $\overrightarrow{S'P}$ it follows that $S'$, $P$ and $R$ are collinear. The case where $P$ is on the real axis, and the case where $P$ is on the left branch of $\mathcal{H}$ are left as exercises.

Finally note that the reflection property is equivalent to showing that the tangent at $P$ bisects the angle subtended by the foci.

**22** THE ALTERNATIVE REFLECTION PROPERTY OF A HYPERBOLA: The tangent at any point on a hyperbola bisects the angle at that point subtended by the foci.

The proof of this alternative result is much simpler than the one above, and is the subject of a question in the exercise.

## Exercise 3F

**1.** (a) As an exercise in algebra, show that in the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ the distance from $P(a\sec\theta, b\tan\theta)$, where $\sec\theta > 0$, to the focus $S(ae, 0)$ is $PS = a(e\sec\theta - 1)$. Likewise find $S'P$ and hence show that $|SP - S'P| = 2a$.
[NOTE: The geometric method shown in the worked exercise is far simpler.]
(b) Show that $|SP - S'P| = 2a$ is also valid when $\sec\theta < 0$.

**2.** (a) Suppose that the directrix of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is tangent to the circle $x^2 + y^2 = b^2$. Show that $e^2 = \frac{1}{2}(1 + \sqrt{5})$, that is, the golden ratio.
(b) In a certain hyperbola the asymptotes are at right-angles. What is the eccentricity?
(c) Determine the eccentricity of the hyperbola where $a$, $b$ and $ae$ form a GP.

**3.** Let $\mathcal{H}$ be the hyperbola $9x^2 - 16y^2 = 144$, and let $P(x_1, y_1)$ be a point on $\mathcal{H}$ with $x_1 > 0$.
(a) Show that the foci of $\mathcal{H}$ are at $S(5, 0)$ and $S'(-5, 0)$.
(b) Use the eccentricity of $\mathcal{H}$ to show that $SP = \frac{5x_1 - 16}{4}$ and that $S'P = \frac{5x_1 + 16}{4}$.
(c) Show that the tangent to $\mathcal{H}$ at $P$ has equation $9x_1x - 16y_1y = 144$.
(d) The tangent meets the $x$-axis at $G$. Show that $G$ is the point $(\frac{16}{x_1}, 0)$.
(e) Hence prove that $\dfrac{SP}{S'P} = \dfrac{SG}{S'G}$.

**4.** Let $\mathcal{H}$ be the hyperbola $5x^2 - 4y^2 = 20$, and let $P(2\sec\theta, \sqrt{5}\tan\theta)$ be a point on $\mathcal{H}$.
(a) Show that the foci of $\mathcal{H}$ are at $S(3, 0)$ and $S'(-3, 0)$.
(b) Determine the equations of the directrices and asymptotes.
(c) Show that the tangent at $P$ has equation $\sqrt{5}\,x\sec\theta - 2y\tan\theta = 2\sqrt{5}$.
(d) The tangent at $P$ meets a directrix at $T$. Use gradients to show that $PT$ subtends a right-angle at the corresponding focus.

**5.** $\mathcal{H}$ is the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $P(a\sec\theta, b\tan\theta)$ is a point on $\mathcal{H}$ with $\tan\theta \neq 0$. Let $O$ be the centre of $\mathcal{H}$.
(a) Show that the normal at $P$ has equation $ax\sin\theta + by = (a^2 + b^2)\tan\theta$.
(b) The normal meets the $x$-axis at $N$. Show that $N$ has coordinates $\left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$.
(c) Let $F$ be the foot of the perpendicular from $P$ to the $x$-axis. Show that $ON = e^2 \times OF$.

**6.** The line $x = 1$ is a directrix and $(2, 0)$ is a focus of a hyperbola with eccentricity $\sqrt{2}$.
(a) Show that the hyperbola has equation $x^2 - y^2 = 2$.
(b) Show that the normal at the point $P(x_1, y_1)$ has equation $\dfrac{x}{x_1} + \dfrac{y}{y_1} = 2$, provided $x_1 \neq 0$ and $y_1 \neq 0$.
(c) The normal at $P$ meets the $x$-axis at $(n_1, 0)$ and the $y$-axis at $(0, n_2)$. Let $N$ be the point $(n_1, n_2)$. Show that as $P$ varies on the hyperbola $x^2 - y^2 = 2$, $N$ always lies on the hyperbola $x^2 - y^2 = 8$.

**DEVELOPMENT**

**7.** Let $\mathcal{H}$ be the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and let $S$ be the focus $(ae, 0)$. $P(a\sec\theta, b\tan\theta)$ is the point on $\mathcal{H}$ in the first quadrant such that $PS$ is parallel to the asymptote $y = -\frac{b}{a}x$.
(a) Show that $\sec\theta + \tan\theta = e$, and hence that $\tan\frac{\theta}{2} = \frac{e-1}{e+1}$.
(b) Show that the tangent at $P$ has equation $bx\sec\theta - ay\tan\theta = ab$.
(c) Deduce that this tangent meets $y = -\frac{b}{a}x$ on the directrix corresponding to $S$.

**8.** The tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at $P(a\sec\theta, b\tan\theta)$ meets the asymptotes of the hyperbola at $A$ and $B$.
(a) Show that the tangent at $P$ has equation $bx\sec\theta - ay\tan\theta = ab$.
(b) Show that the two points are $A = \big(a(\sec\theta + \tan\theta), b(\sec\theta + \tan\theta)\big)$ and $B = \big(a(\sec\theta - \tan\theta), -b(\sec\theta - \tan\theta)\big)$.
(c) Hence show that $P$ is the midpoint of $AB$.

**9.** Let $\mathcal{H}$ be the hyperbola $3x^2 - y^2 = 3$ with centre $O$.
(a) Show that the foci of $\mathcal{H}$ are $S(2, 0)$ and $S'(-2, 0)$, and that the directrices are $x = \frac{1}{2}$ and $x = -\frac{1}{2}$.
(b) Show that the line through $S$ perpendicular to the asymptote with positive gradient has equation $x + \sqrt{3}\,y = 2$.
(c) The line and the asymptote in part (ii) meet at the point $Q$. Show that $Q$ lies on the directrix corresponding to $S$.
(d) Show that $Q$ also lies on the auxiliary circle of $\mathcal{H}$.

**10.** Let $P(x_1, y_1)$ be a point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and let $d_1$ and $d_2$ be the distances from $P$ to the asymptotes. Show that $d_1 d_2 = \frac{a^2 b^2}{a^2 + b^2}$.

**11.** Let $P(a\sec\theta, b\tan\theta)$ be a point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with centre $O$. The tangent at $P$ meets the asymptotes at $Q$ and $R$, and meets the $x$-axis at $T$.
(a) Find the vertical distance between $Q$ and $R$.
(b) Find $OT$ and hence show that the area of $\triangle OQR$ is $ab$.

**12.** Let $P(a\sec\theta, b\tan\theta)$ be a point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with $\tan\theta \neq 0$. The tangent at $P$ meets a directrix at $Q$, and $S$ is the corresponding focus. $O$ is the origin.
(a) Prove that $SP \perp SQ$.
(b) Deduce that the tangents at the endpoints of a focal chord meet on a directrix.
(c) Prove that line through $S$ perpendicular to the tangent at $P$ meets the line $OP$ on the corresponding directrix.

**13.** $P(a\sec\theta, b\tan\theta)$ is a point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with $\tan\theta \neq 0$. The vertical line through $P$ intersects an asymptote at $Q$. The tangent at $P$ meets the same asymptote at $T$ and the normal at $P$ meets the $x$-axis at $N$. Prove that $Q$ lies on the circle with diameter $NT$.

**14.** $\mathcal{H}$ is the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and $P(a\sec\theta, b\tan\theta)$ is a point on $\mathcal{H}$ with $\tan\theta \neq 0$.

(a) (i) Show that the tangent at $P$ has equation $bx\sec\theta - ay\tan\theta = ab$.

(ii) Show that the normal at $P$ has equation $by\sec\theta + ax\tan\theta = a^2e^2\sec\theta\tan\theta$.

(b) The tangent and the normal to $\mathcal{H}$ at $P$ cut the $y$-axis at $T$ and $N$ respectively. Use gradients to prove that the circle with diameter $NT$ passes through the foci of $\mathcal{H}$.

(c) The tangent and normal at $P$ meet the $x$-axis at $A$ and $B$ respectively. Prove that the midpoint of the interval $AB$ is never at a focus of $\mathcal{H}$.

**15.** $\mathcal{H}$ is the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and $P(a\sec\theta, b\tan\theta)$ is a point on $\mathcal{H}$. The tangent to $\mathcal{H}$ at $P$ meets an asymptote of $\mathcal{H}$ at $A$. The feet of the perpendiculars from $A$ to the $x$ and $y$ axes are at $B$ and $C$ respectively. Prove that the line $BC$ passes through $P$.

**16.** The points $(r\cos\theta, r\sin\theta)$ and $\left(s\cos(\theta+\frac{\pi}{2}), s\sin(\theta+\frac{\pi}{2})\right)$ lie on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with centre $O$.

(a) Show that $\dfrac{1}{r^2} + \dfrac{1}{s^2} = \dfrac{1}{a^2} - \dfrac{1}{b^2}$, and hence that $e > \sqrt{2}$.

(b) $P$ and $Q$ are points on the hyperbola such that $OP \perp OQ$. Deduce that the value of the expression $\dfrac{1}{OP^2} + \dfrac{1}{OQ^2}$ does not depend on the positions of $P$ and $Q$.

**17.** $\mathcal{H}$ is the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with centre $O$, and $\mathcal{C}$ is the auxiliary circle $x^2 + y^2 = a^2$. $T$ is a point on $\mathcal{C}$ in the first quadrant, and the tangent there meets the $x$-axis at $M$. $MP$ is perpendicular to the $x$-axis and $P$ lies on $\mathcal{H}$ in the first quadrant. Let $\angle TOM = \theta$, where $0 < \theta < \frac{\pi}{2}$.

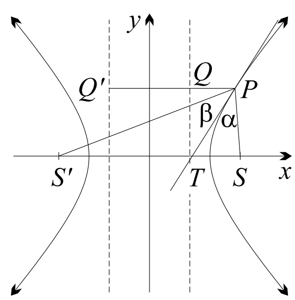
(a) Show that $P$ has coordinates $(a\sec\theta, b\tan\theta)$.

(b) The point $Q(a\sec\phi, b\tan\phi)$ is also on $\mathcal{H}$ with $\theta + \phi = \frac{\pi}{2}$ and $\theta \neq \frac{\pi}{4}$. Show that the chord $PQ$ has equation $ay = b(\cos\theta + \sin\theta)x - ab$.

(c) Every such chord $PQ$ passes through a fixed point. Write down its coordinates.

(d) Show that as either $\theta \to \frac{\pi}{2}^-$ or $\theta \to 0^+$, the chord $PQ$ approaches a line parallel to an asymptote of $\mathcal{H}$.

**18.** In the diagram, $P(x_0, y_0)$ is a point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with $y_0 \neq 0$. The foci are at $S$ and $S'$ on the $x$-axis. The horizontal line through $P$ meets the directrices at $Q$ and $Q'$. The tangent at $P$ intersects the $x$-axis at $T$. Let $\angle SPT = \alpha$ and $\angle S'TP = \beta$.

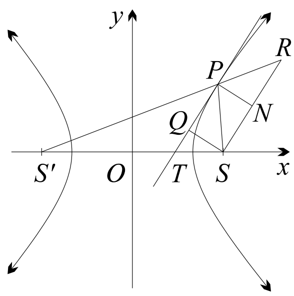



(a) Find the $x$-coordinate of $T$.

(b) Use the focus-directrix definition of a hyperbola to show that
$$\frac{SP}{S'P} = \frac{ST}{S'T}.$$

(c) Use the sine rule to show that $\alpha = \beta$, and hence prove the reflection property of a hyperbola, that the tangent at $P$ bisects the angle subtended by the foci.

**19.** In the diagram, $P$ is a point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with foci at $S$ and $S'$, and centre at $O$. The tangent at $P$ intersects the $x$-axis at $T$. The line through $S$ parallel with the tangent meets $S'P$ at $R$. The normal at $P$ meets $SR$ at $N$, and the line through $S$ parallel with the normal meets the tangent at $Q$.

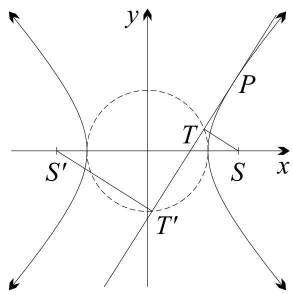



(a) Use the reflection property to prove that $ON||S'R$.

(b) Use the fact that quadrilateral $SNPQ$ is a rectangle to show that $Q$ is on $ON$.

(c) Hence show that $Q$ is on the auxiliary circle. That is, show that $OQ = a$.

**20.** In the diagram, $P$ is a point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with foci at $S$ and $S'$. The lines through $S$ and $S'$ perpendicular to the tangent at $P$ intersect that tangent at $T$ and $T'$ respectively. As shown in the previous question, both $T$ and $T'$ lie on the auxiliary circle. Use symmetry and circle geometry to show that




$$S'T' \times ST = b^2 .$$

**EXTENSION**

**21.** (a) Let $H(a\cos\theta, b\sin\theta)$ be a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ for some fixed value of $\theta$. Now suppose that both the eccentricity $e$ and the vertex $a$ vary in such a way that the focus $c = ae$ remains constant. Find and describe the locus of $H$ for $e < 1$.

(b) Let $E(a\sec\theta, b\tan\theta)$ be a point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ for some fixed value of $\theta$. Now suppose that both the eccentricity $e$ and the vertex $a$ vary in such a way that the focus $c = ae$ remains constant. Find and describe the locus of $E$ for $e > 1$.

**22.** (a) The hyperbolic trigonometric functions are defined to be

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2} .$$

(i) Show that $\cosh^2 x - \sinh^2 x = 1$ .

(ii) Show that $\dfrac{d}{dx}\cosh x = \sinh x$ .

(iii) Show that $\dfrac{d}{dx}\sinh x = \cosh x$ .

(b) Consider the parametric equations $x = f\cos\theta\cosh u$ and $y = f\sin\theta\sinh u$, where $f$ is a positive constant and $\theta$ and $u$ are the parameters.

(i) Show that if the value of $u$ is non-zero and constant then these are the parametric equations for an ellipse with foci at $(f, 0)$ and $(-f, 0)$.

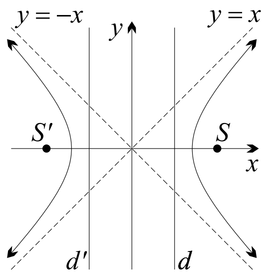
(ii) Show that if the value of $\theta$ is constant and not a multiple of $\frac{\pi}{2}$ then these are the equations for one branch of a hyperbola, also with foci at $(f, 0)$ and $(-f, 0)$.

(iii) Show that the tangents to the ellipse and hyperbola through the point $P$ where $u = 1$ and $\theta = \frac{\pi}{4}$ are perpendicular.

(iv) Show that the same is true for any other choice of $u$ and $\theta$.

# 3G Rectangular Hyperbolas

When the asymptotes of a hyperbola are at right angles the curve is called a rectangular hyperbola. From the gradients of the asymptotes, this will happen when




$$\tfrac{b}{a} \times \tfrac{-b}{a} = -1$$

so $\quad b^2 = a^2$

or $\quad b = a\,.$

Also $\quad a^2(e^2-1) = a^2$

so $\quad e^2 = 2$

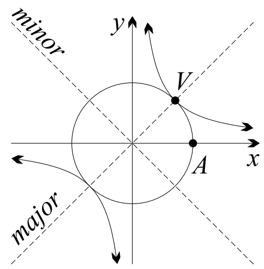
thus $\quad e = \sqrt{2}\,.$

Thus the foci are at $S(a\sqrt{2},0)$ and $S'(-a\sqrt{2},0)$ and the directrices have equations $d : x = \frac{1}{\sqrt{2}}a$ and $d' : x = -\frac{1}{\sqrt{2}}a$. The equation of the hyperbola is usually written without fractions as

$$x^2 - y^2 = a^2\,.$$

It is often more convenient to study these hyperbolae after an anti-clockwise rotation of 45° about the origin, so that the equation has the form

$$xy = c^2 \qquad \text{or} \qquad y = \frac{c^2}{x}\,.$$




After this rotation, the major axis is $y = x$ and the minor axis is $y = -x$. The auxiliary circle is unchanged and so has the same radius. The image of the vertex $A(a,0)$ will be on the line $y = x$. Let this point be $V(c,c)$. Since $V$ lies on the auxiliary circle it follows that

$$a^2 = c^2 + c^2$$

so $\quad a = c\sqrt{2}\,,$

and $\quad V = \left(\frac{1}{\sqrt{2}}\,a, \frac{1}{\sqrt{2}}\,a\right)\,.$

The coordinates of the foci are easily found by multiplying the coordinates of the vertex by the eccentricity, since $OS : OV = e : 1$. Thus the foci are:

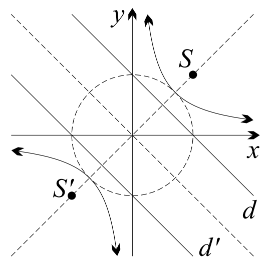
$$S(c\sqrt{2}, c\sqrt{2}\,) \quad \text{and} \quad S'(-c\sqrt{2}, -c\sqrt{2}\,)$$
$$\text{or} \quad S(a,a) \quad \text{and} \quad S'(-a,-a)\,.$$

One vertical directrix of $x^2 - y^2 = a^2$ passes through $(\frac{1}{\sqrt{2}}a, 0)$. Thus after rotation it has gradient $-1$ and passes through $(\frac{1}{\sqrt{2}}c, \frac{1}{\sqrt{2}}c)$. A line which has the required gradient is $x + y = k$. The value of $k$ is found by substituting the point to get $k = c\sqrt{2}$ and so

$$x + y = c\sqrt{2} \qquad \text{or} \qquad x + y = a\,.$$

likewise, the other directrix is

$$x + y = -c\sqrt{2} \qquad \text{or} \qquad x + y = -a\,.$$



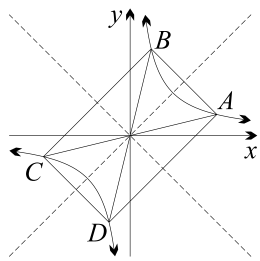

Thus the intercepts of the directrices coincide with the intercepts of the auxiliary circle, as shown in the diagram.

**Parametric Equations:** There are several ways of parameterising the hyperbola $xy = c^2$. The one used in this course is simply

$$x = ct \quad \text{and} \quad y = \frac{c}{t}\,.$$

Unlike the parabola, where the parameter is the gradient of the tangent, this parameter for the rectangular hyperbola does not have any practical significance.

**Parameters and Symmetry:** A hyperbola is symmetric in its major axis. Thus it follows that the points $A(cp, \frac{c}{p})$ and $B(\frac{c}{p}, cp)$ are symmetric in the line $y = x$. Also, a hyperbola is symmetric in its minor axis and so the points $A(cp, \frac{c}{p})$ and $D(-\frac{c}{p}, -cp)$ are symmetric in the line $y = -x$. Further, $xy = c^2$ is an odd function and hence the points $A$ and $C(-cp, -\frac{c}{p})$ are diametrically opposite. That is $AC$ passes through the origin.




Thus the quadrilateral $ABCD$, with parameters $p$, $\frac{1}{p}$, $-p$ and $-\frac{1}{p}$ respectively, is a rectangle with pairs of vertices on opposite branches of the hyperbola. Its diagonals are diameters of the hyperbola. The sitation is shown in the diagram.

**The Parametric Equation of a Chord:** The chords of ellipses and hyperbolas are in general quite complex, and hence have been omitted from the theory of Sections 3C and 3E. The equation of a chord of a rectangular hyperbola with parametric equations $x = ct$ and $y = \frac{c}{t}$ is more straightforward.

Let $P = (cp, \frac{c}{p})$ and $Q = (cq, \frac{c}{q})$ lie on the hyperbola. The gradient of $PQ$ is

$$\begin{aligned}\text{grad } PQ &= \frac{\frac{c}{q} - \frac{c}{p}}{cq - cp} \times \frac{pq}{pq} \\ &= \frac{c(p-q)}{-c(p-q)pq} \\ &= -\frac{1}{pq}\,.\end{aligned}$$

Using the point $P$, the equation of the chord is thus

$$y - \frac{c}{p} = -\frac{1}{pq}(x - cp)$$

or $\quad pqy - cq = -x + cp$

so $\quad x + pqy = c(p+q)\,.$

**23** THE PARAMETRIC EQUATION OF THE CHORD: The chord through $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ has equation

$$x + pqy = c(p+q)\,.$$

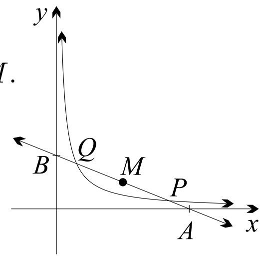
**WORKED EXERCISE:** The chord $PQ$ on the hyperbola $xy = c^2$ intersects the $x$-axis at $A$ and $y$-axis at $B$.

(a) Show that the mid-point of $PQ$ is the mid-point of $AB$.

(b) Hence show that $AP = BQ$

**SOLUTION:** Let $P = (cp, \frac{c}{p})$ and $Q = (cq, \frac{c}{q})$

(a) The mid-point of $PQ$ is $M = \left(\frac{c}{2}(p+q), \frac{c}{2}(\frac{1}{p}+\frac{1}{q})\right)$.
From above the equation of $PQ$ is $x + pqy = c(p+q)$.
Thus $A = (c(p+q), 0)$ and $B = (0, c(\frac{1}{p}+\frac{1}{q}))$.
Hence the mid-point of $AB$ is $\left(\frac{c}{2}(p+q), \frac{c}{2}(\frac{1}{p}+\frac{1}{q})\right) = M$.




(b) From part (a),

$$\begin{aligned} AP &= AM - PM \\ &= BM - QM \\ &= BQ\,. \end{aligned}$$

**The Latus Rectum:** Recall that the general formula for the length of the latus rectum of the hyperbola is $\frac{2b^2}{a}$, so in this case the length is

$$\begin{aligned} \frac{2b^2}{a} &= \frac{2a^2}{a} \qquad \text{(since } a = b\text{)} \\ &= 2a \\ &= 2c\sqrt{2} \qquad \text{(since } a = c\sqrt{2}.\text{)} \end{aligned}$$

**Tangents:** Differentiating parametrically:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \div \frac{dx}{dt} \\ &= -\frac{c}{t^2} \div c \\ &= -\frac{1}{t^2}\,. \end{aligned}$$

Hence the equation of the tangent is:

$$\begin{aligned} & y - \tfrac{c}{t} = -\tfrac{1}{t^2}(x - ct) \\ \text{so}\quad & t^2y - ct = -x + ct \\ \text{or}\quad & x + t^2y = 2ct\,. \end{aligned}$$

**24** TANGENT IN PARAMETRIC FORM: $x + t^2y = 2ct$

**Normals:** From the derivation of the tangent above

$$\frac{dy}{dx} = -\frac{1}{t^2}\,.$$

Hence the equation of the normal is:

$$\begin{aligned} & y - \tfrac{c}{t} = t^2(x - ct) \\ \text{so}\quad & ty - c = t^3x - ct^4 \\ \text{or}\quad & t^3x - ty = c(t^4 - 1)\,. \end{aligned}$$

**25** NORMAL IN PARAMETRIC FORM: $t^3x - ty = c(t^4 - 1)$

**WORKED EXERCISE:**

(a) The normal at $P(2c, \frac{c}{2})$ on $xy = c^2$ intersects the hyperbola again at $Q(ct, \frac{c}{t})$. Show that $t = -\frac{1}{8}$.

(b) The line $OP$ intersects the hyperbola again at $R$. Show that $\angle PRQ = \frac{\pi}{2}$.

**SOLUTION:**

(a) From above, the equation of the normal at $P$ is $8x - 2y = 15c$. The point $Q$ is on this line so

$$8ct - \tfrac{2c}{t} = 15c$$

or $\quad 8t^2 - 15t - 2 = 0\,.$

Now $(8t+1)(t-2) = 0$

so $\quad t = -\frac{1}{8}$ or $2\,.$

But $t = 2$ corresponds to the point $P$, so $Q$ is the point where $t = -\frac{1}{8}$.

(b) $PR$ is a diameter of the hyperbola, hence $R = (-2c, -\frac{c}{2})$. Now

$$\begin{aligned}\text{gradient } PR &= \frac{\frac{c}{2} + \frac{c}{2}}{2c + 2c}\\ &= \tfrac{1}{4}\,.\\ \text{gradient } RQ &= \frac{-8c + \frac{c}{2}}{-\frac{c}{8} + 2c}\\ &= -\tfrac{15c}{2} \div \tfrac{15c}{8}\\ &= -4\,.\end{aligned}$$

Since the product of the gradients is $-1$, it follows that $PR \perp RQ$.

**Chord of Contact:** By now, the argument to derive the equation of the chord of contact should be very familiar. As expected the result is

$$x_0x + y_0y = 2c^2 \qquad \text{or} \qquad x_0x + y_0y = a^2\,,$$

and the proof is left as an exercise.

**Extension:** The familiar equation $xy = c^2$ can be easily derived from the general form of the rectangular hyperbola, $x^2 - y^2 = a^2$, by considering the problem in the complex plane. Begin by letting $w = u + iv$ be a variable point in the Argand diagram. Next, suppose that the locus of $w$ is the general form of the rectangular hyperbola, so that $u^2 - v^2 = a^2$. Put $z = w \times \operatorname{cis}\frac{\pi}{4}$, so that $z$ is the result of rotating $w$ by $45^\circ$ about the origin.

Rearranging this last equation yields, $w = z \times \operatorname{cis}(-\frac{\pi}{4})$. Also let $z = x + iy$ to get

$$\begin{aligned}u + iv &= (x + iy) \times \tfrac{1}{\sqrt{2}}(1 - i)\\ &= \tfrac{1}{\sqrt{2}}(x + y) + \tfrac{1}{\sqrt{2}}(x - y)i\,.\end{aligned}$$

Equating real and imaginary parts yields

$$u = \tfrac{1}{\sqrt{2}}(x + y) \qquad \text{and} \qquad v = \tfrac{1}{\sqrt{2}}(x - y)\,.$$

Thus the hyperbola $u^2 - v^2 = a^2$ becomes

$$\tfrac{1}{2}(x + y)^2 - \tfrac{1}{2}(x - y)^2 = a^2$$

or $\quad (x + y)^2 - (x - y)^2 = 2a^2$

so $\quad 2x \times 2y = 2a^2 \qquad$ (difference of two squares)

thus $\quad xy = c^2$

where $a^2 = 2c^2$. That is, the equation $xy = c^2$ is just the result of rotating $x^2 - y^2 = a^2$ by 45° about the origin, exactly as claimed at the beginning of this section. There are other means of establishing this relationship, such as by geometrical argument in the Cartesian plane, which could be the subject of further investigation for interested readers.

## Exercise **3G**

**1.** (a) Sketch the hyperbola $x^2 - y^2 = 2$ showing the foci, vertices, directrices and asymptotes.

(b) Sketch each hyperbola showing the foci, vertices, directrices, auxiliary circle, and major and minor axes.

(i) $xy = 1$ (ii) $xy = 2$ (iii) $xy = -1$

**2.** The point $P(ct, \frac{c}{t})$, where $c$ is a positive constant and $t \neq 0$, is on the hyperbola $xy = c^2$, and $O$ is the centre of the hyperbola. $PM$ and $PN$ are the respective perpendiculars from $P$ to the horizontal and vertical asymptotes. The tangent at $P$ meets the horizontal and vertical asymptotes at $A$ and $B$ respectively.

(a) Show that $PM \times PN$ is constant (that is, independent of $t$).

(b) Show that the tangent at $P$ has equation $x + t^2 y = 2ct$.

(c) Show that $P$ is the midpoint of $AB$.

(d) Show that the area of $\triangle OAB$ is constant.

**3.** $\mathcal{R}$ is the rectangular hyperbola $xy = c^2$. Let $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ be two points on $\mathcal{R}$, and let $O$ be the origin.

(a) Show that the tangent at $P$ has equation $x + p^2 y = 2cp$, and hence write down the equation of the tangent at $Q$.

(b) The tangents at $P$ and $Q$ intersect at $T$. Show that $T$ has coordinates $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$.

(c) Show that the line $OT$ bisects the chord $PQ$.

**4.** $\mathcal{H}$ is the hyperbola $xy = c^2$ and $P(ct, \frac{c}{t})$ is a point on $\mathcal{H}$.

(a) Show that the normal at $P$ has equation $t^2 x - y = ct^3 - \frac{c}{t}$.

(b) This normal meets $\mathcal{H}$ again at $Q$. Show that the $x$-coordinates of $P$ and $Q$ satisfy

$$x^2 - c\left(t - \tfrac{1}{t^3}\right)x - \tfrac{c^2}{t^2} = 0\,.$$

Hence find the coordinates of $Q$.

(c) If $|t| \neq 1$ and $PR$ is a diameter of $\mathcal{H}$, show that $PR \perp QR$.

(d) What happens when $|t| = 1$?

**5.** Consider the rectangular hyperbola given by $x = ct$ and $y = \frac{c}{t}$. Let $P$, $Q$ and $R$ be the points where $t = p$, $t = q$ and $t = r$ respectively. Chord $PQ$ subtends a right-angle at $R$.

(a) Show that $pqr^2 = -1$, and hence explain why $P$ and $Q$ are on separate branches.

(b) Hence show that $PQ$ is parallel to the normal at $R$.

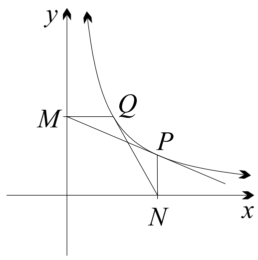
**6.** (a) Show that the rectangular hyperbola $xy = c^2$ can be parameterised by

$$x = c(\sec\theta + \tan\theta) \quad \text{and} \quad y = c(\sec\theta - \tan\theta)\,.$$

(b) Show that, using these parametric equations, the two vertices are at $\theta = 0$ and $\theta = \pi$.

(c) The point $P$ is at $\theta = \alpha$ and $Q$ is at $\theta = -\alpha$. What is the geometrical relationship between $P$ and $Q$?

**7.** $\mathcal{H}$ is the hyperbola $xy = c^2$, and $P(ct, \frac{c}{t})$ is a point on $\mathcal{H}$. $N$ is the foot of the perpendicular from $P$ to the $x$-axis, and $M$ is the point where the tangent at $P$ cuts the $y$-axis. The line through $M$ parallel to the $x$-axis meets $\mathcal{H}$ at $Q$.




(a) Show that the tangent at $P$ has equation $x + t^2y = 2ct$.

(b) Show that $Q$ is the point $\left(\frac{ct}{2}, \frac{2c}{t}\right)$.

(c) Show that the tanget at $Q$ passes through $N$.

**DEVELOPMENT**

**8.** Let $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ be two points on the same branch of the hyperbola $xy = c^2$. Let $M$ be the mid-point of $PQ$ and let $OM$ intersect the hyperbola at $N$.

(a) Show that the coordinates of $P$, $N$ and $Q$ form two sets of geometric progressions.

(b) Show that the tangent at $N$ is parallel with $PQ$.

**9.** $\mathcal{H}$ is the rectangular hyperbola $xy = c^2$. Let $P(ct, \frac{c}{t})$ be a point on $\mathcal{H}$, and let $PP'$ be the corresponding diameter. The tangent at $P$ meets the horizontal and vertical lines through $P'$ at $Q$ and $Q'$ respectively.

(a) Write down the coordinates of $P'$.

(b) Find the coordinates of $Q$ and $Q'$.

(c) Show that $P$ is the midpoint of $QQ'$.

(d) Show that as $P$ varies on $\mathcal{H}$, the locus of both $Q$ and $Q'$ is the hyperbola $xy = -3c^2$.

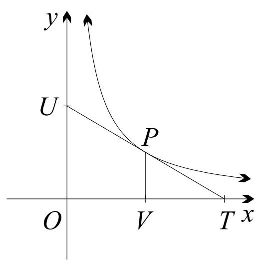
**10.** Let $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ be two points on the same branch of the hyperbola $xy = c^2$.

(a) Find the equation of the chord $PQ$.

(b) The chord $PQ$ intersect the $x$-axis at $A$ and the $y$-axis at $B$. Show that the mid-point of $PQ$ is the mid-point of $AB$.

(c) By taking an appropriate limit, prove that $P$ is the mid-point of the intercepts of the tangent at $P$.

**11.** Let $P(a, b)$ be in the first quadrant on the hyperbola $xy = k^2$. Let the tangent at $P$ intersect the $x$-axis at $T(c, 0)$ and the $y$-axis at $U(0, d)$. Let $V(a, 0)$ be the foot of the perpendicular from $P$ to the $x$-axis.




(a) Use the similar triangles $\triangle OTU$ and $\triangle VTP$ to show that

$$d\,a^2 - (dc)a + k^2c = 0\,.$$

(b) Since $UT$ is tangent at $P$, this quadratic equation in $a$ has only one solution. Prove that $P$ is the mid-point of $TU$.

**12.** $\mathcal{R}$ is the rectangular hyperbola $xy = c^2$. $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ are variable points on $\mathcal{R}$, and $N$ is the foot of the perpendicular from $P$ to the $y$-axis. The tangent at $Q$ passes through $N$.

(a) Show that the tangents at $P$ and $Q$ intersect at $T\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$.

(b) Show that $2p = q$, and hence show that the locus of $T$ has Cartesian equation $xy = \frac{8c^2}{9}$.

**13.** $\mathcal{H}$ is the hyperbola $xy = c^2$. $P(ct, \frac{c}{t})$ is a point on $\mathcal{H}$. The normal at $P$ meets the $x$-axis at $A$, while the tangent at $P$ meets the $y$-axis at $B$. $M$ is the midpoint of $AB$.

(a) Show that the normal at $P$ has equation $t^3x - ty = c(t^4 - 1)$.

(b) Find the coordinates of $M$.

(c) Hence show that the locus of $M$ as $P$ varies has Cartesian equation $2c^2xy + y^4 = c^4$.

**14.** $\mathcal{R}$ is the rectangular hyperbola $xy = c^2$, with centre $O$. Let $P(ct_1, \frac{c}{t_1})$ and $Q(ct_2, \frac{c}{t_2})$ be two variable points on $\mathcal{R}$, with $t_1 \neq -t_2$. The tangents to $\mathcal{R}$ at $P$ and $Q$ intersect at $T$.

(a) Show that $T$ has coordinates $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$.

(b) Given that the product $t_1t_2 = k^2$, where $k$ is a constant, prove that the locus of $T$ is a diameter of $\mathcal{R}$, omitting the origin.

**15.** The hyperbola $\mathcal{H}$ has equation $xy = 16$. $P(4p, \frac{4}{p})$, where $p > 0$, and $Q(4q, \frac{4}{q})$, where $q > 0$, are distinct arbitrary points on $\mathcal{H}$. The tangents to $\mathcal{H}$ at $P$ and $Q$ intersect at $T$.

(a) Show that the chord $PQ$ has equation $x + pqy = 4(p + q)$.

(b) Show that the tangent at $P$ has equation $x + p^2y = 8p$.

(c) Show that the tangents at $P$ and $Q$ intersect at the point $T\left(\frac{8pq}{p+q}, \frac{8}{p+q}\right)$.

(d) The chord $PQ$, when extended, passes through the point $(0, 8)$. Deduce that the locus of $T$ is the line $x = 4$, but only for $0 < y < 4$.

**16.** The line $y = mx + b$ is a tangent to the rectangular hyperbola $xy = c^2$.

(a) Prove that $b^2 = -4mc^2$.

(b) Tangents are drawn to the hyperbola from the point $(-8c, c)$. Show that the equations are $x + 4y + 4c = 0$ and $x + 16y - 8c = 0$.

**17.** (a) Use the results of symmetry in the parameters to show that it is not possible to place a square so that its diagonals are diameters of a rectangular hyperbola.

(b) Use the gradients of chords to show that it is not possible to place a square in any manner so that its vertices are on a rectangular hyperbola.

**18.** $\mathcal{H}$ is the rectangular hyperbola defined by the parametric equations $x = ct$, $y = \frac{c}{t}$. $P$, $Q$ and $R$ are the points where $t = t_1$, $t = t_2$ and $t = t_3$ on $\mathcal{H}$. Let $A$ be the point of intersection of the three altitudes of $\triangle PQR$. ($A$ is called the orthocentre of $\triangle PQR$.)

(a) Show that $A$ has coordinates $\left(-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$.

(b) Hence prove the theorem that if a rectangular hyperbola passes through the three vertices of a triangle, then it also passes through the orthocentre of the triangle.

(c) Further prove that if $P$, $Q$ and $R$ lie on the same branch of the hyperbola then the orthocentre is outside the triangle, and is otherwise inside the triangle.

**19.** $\ell$ is the line $ax + by = 1$, and $\mathcal{H}$ is the rectangular hyperbola $xy = c^2$. The line $\ell$ cuts $\mathcal{H}$ in two distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Let $M(x_0, y_0)$ be the midpoint of $P_1P_2$.

(a) Find a quadratic equation whose roots are $x_1$ and $x_2$.

(b) Show that the equation of $\ell$ may be written as $y_0x + x_0y = 2x_0y_0$.

(c) Suppose that $\mathcal{L}$ is a line with gradient $m_1 \neq 0$ through the centre of a $\mathcal{H}$. Deduce that $\mathcal{L}$ bisects any chord with gradient $m_2 = -m_1$.

**20.** $\mathcal{H}$ is the rectangular hyperbola $xy = c^2$. Let $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ be points on opposite branches of $\mathcal{H}$. The circle $\mathcal{C}$ has diameter $PQ$. Let $A$ and $B$ be the points where $\mathcal{C}$ cuts $\mathcal{H}$ again, and let $\alpha$ and $\beta$ be the respective $x$-coordinates of $A$ and $B$.

(a) Prove that $\alpha + \beta = 0$.

(b) Hence deduce that $AB$ is a diameter of $\mathcal{H}$.

**EXTENSION**

**21.**

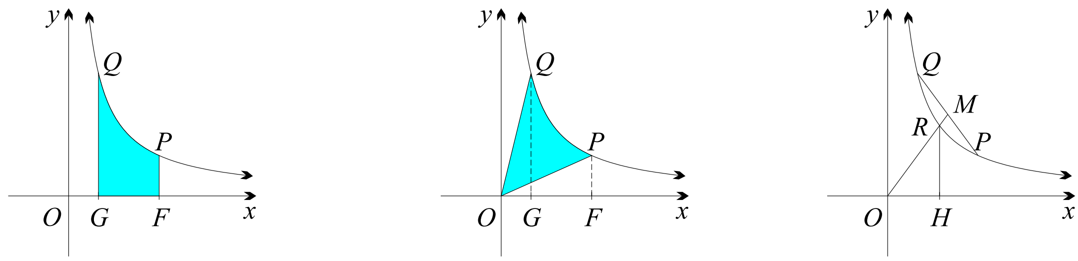



Figure 1 Figure 2 Figure 3

In Figure 1 above, the points $P(p, \frac{1}{p})$ and $Q(q, \frac{1}{q})$ are on the hyperbola $xy = 1$, with $p > q > 0$. The lines $PF$ and $QG$ are perpendiculars to the $x$-axis and $O$ is origin.

(a) Find the area of $\triangle OFP$ in Figure 2.

(b) Prove that the area of region $OPQ$ shaded in Figure 2 is equal to the area of region $PQGF$ shaded in Figure 1.

In Figure 3, $M$ is the mid-point of $PQ$ and $OM$ intersects hyperbola at $R(r, \frac{1}{r})$. Line $RH$ is perpendicular to the $x$-axis

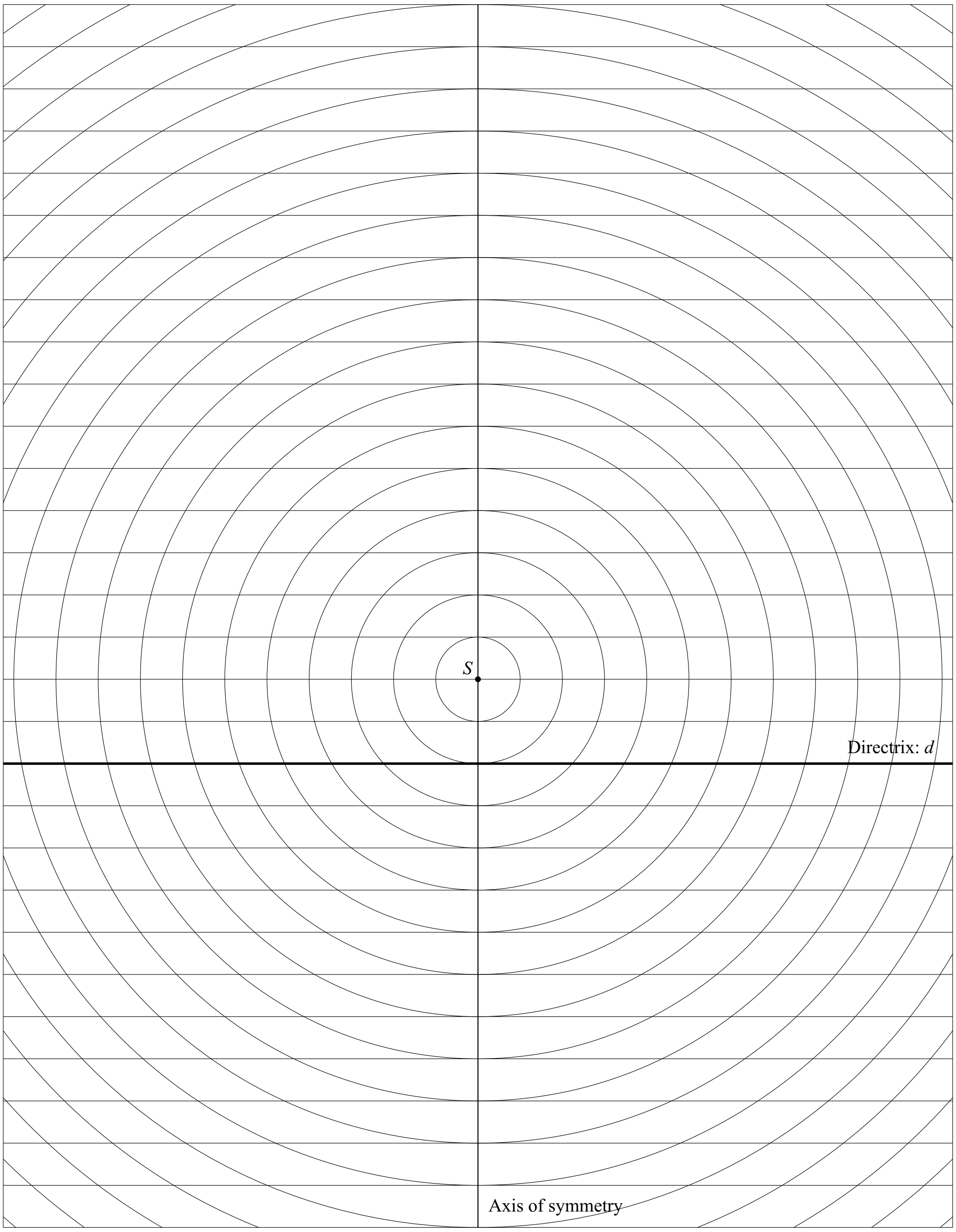
(c) Use similar triangles to show that $r^2 = pq$.

(d) Use integration to show that $RH$ divides $PQGF$ into equal areas.

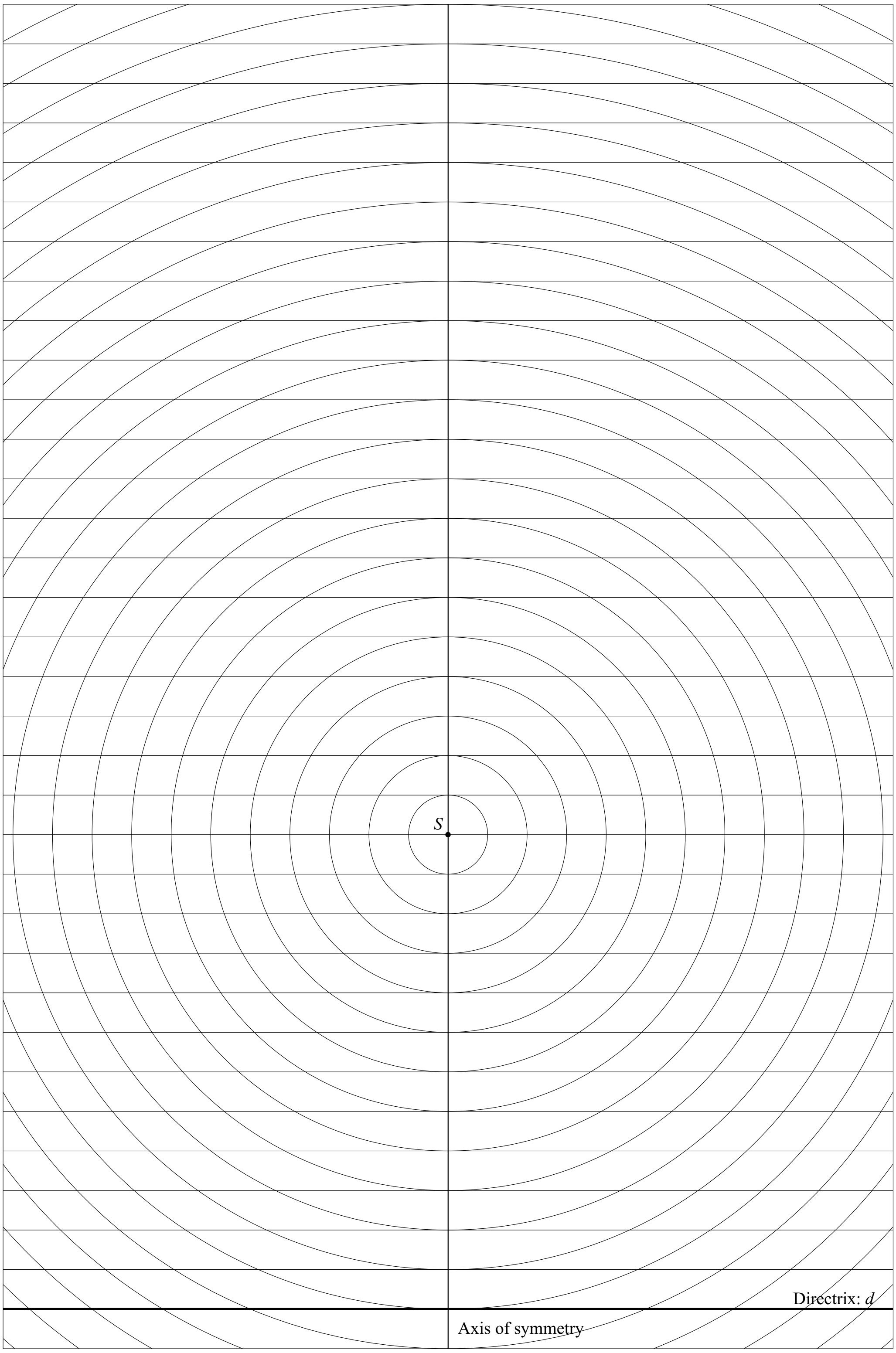
(e) Show that $OR$ divides region $OPQ$ into equal areas.

# Appendix — Conic Grids

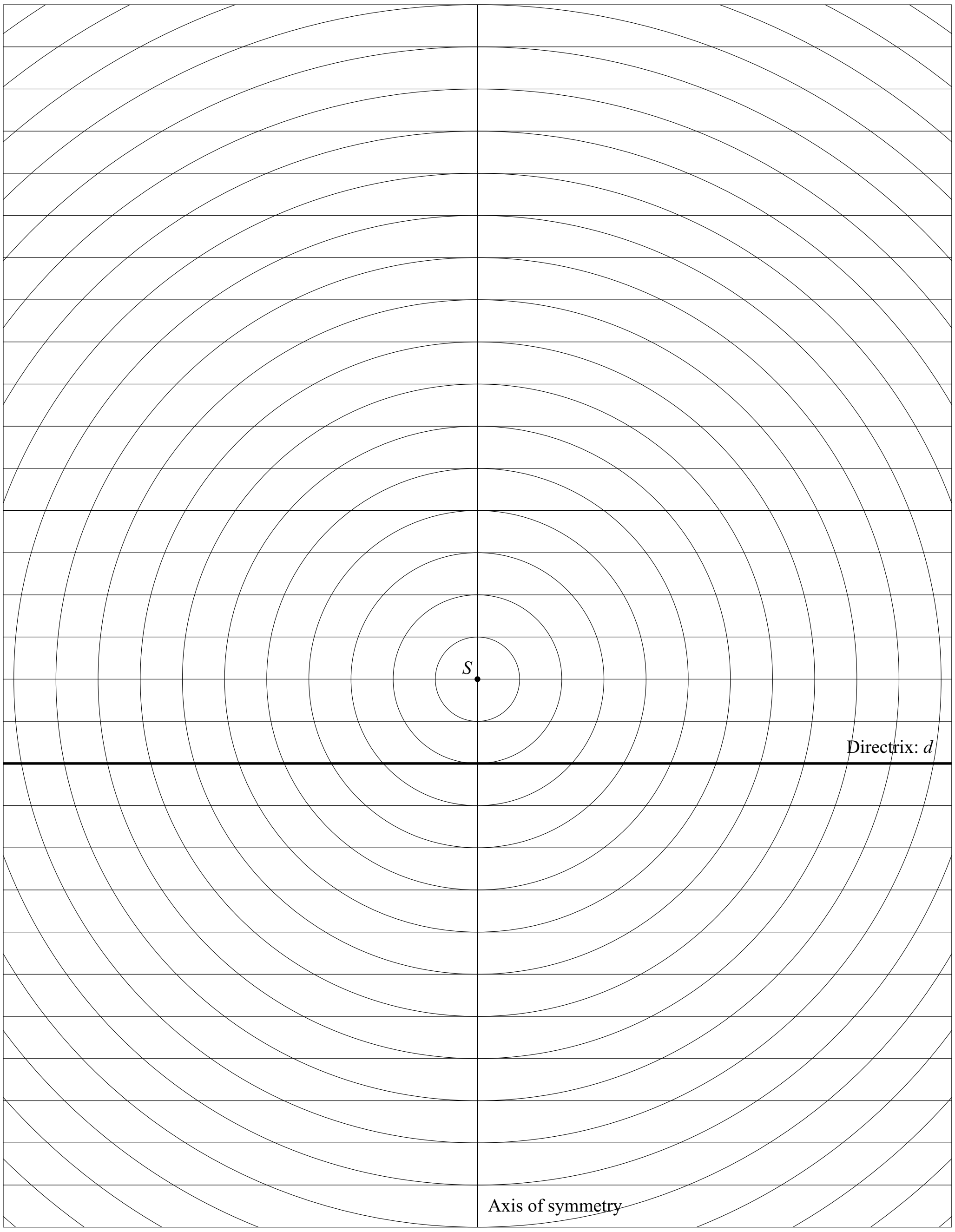
## Parabola:



## Ellipse:



## Hyperbola:



# Chapter Three

## Exercise **3A** (Page 3)

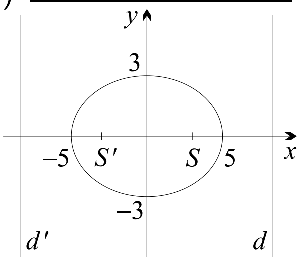
**1(b)** Both points are equidistant from $d$ and $S$.
**2(b)** In both cases $\frac{PS}{PQ} = \frac{1}{2}$.
**3(b)** In both cases $\frac{PS}{PQ} = 2$.
**5(c)** They are the same: $PS + PS' = 16$ units.
**(d)** $\frac{PS'}{PQ'} = \frac{PS}{PQ} = \frac{1}{2}$, always.

## Exercise **3B** (Page 7)

**1(b)** $x_1(y - b) = y_1(x - a)$
**2(a)** $3x - y = 4$ **(b)** $x + 2y = 6$ **(c)** $x + 4y = -9$
**(d)** $2y - 2x = 5$
**3(a)** $C = (-1, 2),\ r = \sqrt{5}$ **(b)** $C = (3, 0),\ r = 2\sqrt{2}$
**(c)** $C = (-3, -2),\ r = 4$ **(d)** $C = (\frac{1}{2}, -\frac{3}{2}),\ r = \frac{1}{\sqrt{2}}$
**4(a)** $(x + 1)^2 + (y - 1)^2 = 2$
**(b)** $(x - 2)^2 + (y + 1)^2 = 10$
**(c)** $(x - 3)^2 + (y - 4)^2 = 13$
**(d)** $(x + 3)^2 + (y + 3)^2 = 20$
**5(a)** $x(x + 2) + y(y - 2) = 0$
**(b)** $(x - 5)(x + 1) + y(y + 2) = 0$
**(c)** $(x - 1)(x - 5) + (y - 7)(y + 1) = 0$
**(d)** $(x + 5)(x + 1) + (y + 7)(y - 1) = 0$
**6(a)** $x \times r\cos\theta + y \times r\sin\theta = r^2$ **(b)** $x_1x + y_1y = r^2$.
**7(a)** $3x + 4y = 25$ **(b)** $x + y = 2$ **(c)** $y\sqrt{3} - x = 4$
**(d)** $x + 7y = -50$
**8** $x^2 + (y - 2)^2 = 10$ or $(x - 4)^2 + (y + 2)^2 = 10$
**9** $(x + 1)^2 + (y - 3)^2 = 20$
**10(b)(i)** $(b + mh - k)^2 = r^2(m^2 + 1)$
**12(b)** $\frac{b^2 - r^2}{m^2 + 1}$ **(c)** $b^2 - r^2$ **(d)** The product of intercepts of intersecting secants is constant.
**13(a)** $\frac{\sin\theta - \sin\phi}{\cos\theta - \cos\phi}$ **(c)** Angles in the same segment are equal.
**14(a)** $PA^2 = {x_0}^2 + {y_0}^2 - r^2$ **(b)** Tangents to a circle from an external point are equal.
**16** $(x - 5)^2 + (y + 5)^2 = 37$

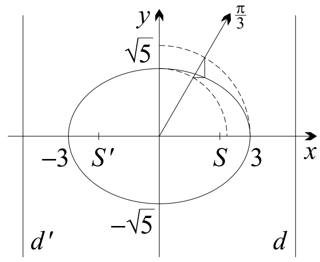
## Exercise **3C** (Page 16)

**1(a)** $\frac{3}{5}$
**(b)** $(3, 0)$ and $(-3, 0)$
**(c)** $x = \frac{25}{3}$ and $x = -\frac{25}{3}$
**(f)** $(\frac{5}{2}, 2\sqrt{3})$

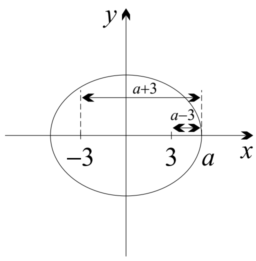



**2(b)** $x = 2\cos\theta,\ y = \frac{4}{3}\sin\theta$

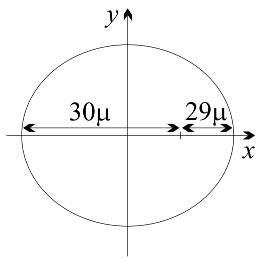
**3(a)** $(2, 0),\ (-2, 0)$,
$x = \frac{9}{2},\ x = -\frac{9}{2}$



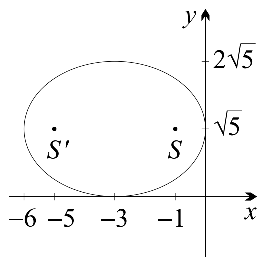

**4(b)** $\frac{x^2}{25} + \frac{y^2}{9} = 1$ **(c)** $\frac{4}{5}$
**5(a)** $SB = a$ **(b)(i)** $\frac{x^2}{10} + y^2 = 1$ **(ii)** $\frac{x^2}{25} + \frac{y^2}{9} = 1$
**6** $\frac{x^2}{36} + \frac{y^2}{20} = 1$
**7**




**8**




**9(a)** $3x - 4y = 12$ **(b)** $5x + 8y + 4 = 0$
**15(b)** $S'P = 2(5 + 2\sqrt{3})$
**17(a)** $\frac{b}{a} \to 1^-$ and the ellipse becomes more circular. **(b)** $\frac{b}{a} \to 0^+$ and the ellipse becomes long and slender.
**18(a)** $-3 + i\sqrt{5}$
**(c)** $\frac{\pi}{2} \le \arg(z) \le \pi$




**19(a)** $\lambda < 2$ **(b)** The length of the major axis increases from $2\sqrt{3}$ to $2\sqrt{2}$, while the length of the minor axis starts at 2 and approaches zero. **(c)** When $\lambda = 2$, $b = 0$, so the ellipse has collapsed onto the interval joining $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$.
**21** All three coalesce at $x = a$.
**22(b)(ii)** In the limit, $y^2 = -4x$ is obtained. This is a parabola with focal length 1.
**(c)** A parabola with focal length $f$ is obtained.

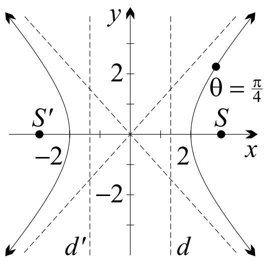
## Exercise **3D** (Page 20)

**4(b)** $A = \left(\frac{a^2 - b^2}{a}\cos\theta, 0\right),\ B = \left(0, \frac{b^2 - a^2}{b}\sin\theta\right)$
**7(a)** $\frac{a}{b} = \frac{1}{e^2} = \frac{1}{2}(\sqrt{5} + 1)$ — the golden ratio.
**(b)** $e = \frac{1}{\sqrt{2}}$
**8(a)** $\frac{b^2(ae - x_1)}{aey_1}$
**9(a)** $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$

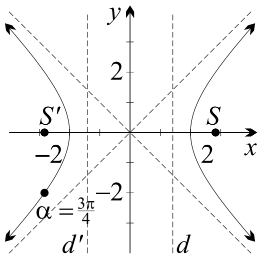
**10(b)** $R = \left(\frac{a}{e}, \frac{b\sin\theta}{e\cos\theta}\right)$
**11(d)** Only if $y_1 = 0$, since $|e^2 x_1| < ae$.
**12(b)** $\dfrac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$
**(c)** $R = (a\sin\theta, -b\cos\theta)$, $R' = (-a\sin\theta, b\cos\theta)$
**(d)** $|\triangle RPR'| = ab$
**15(a)** $Q$ is outside the ellipse except when $Q = P$.
**(b)** $\triangle SUP \equiv \triangle S^*UP$ so $S'P + PS^* = S'P + PS$. If the line passed through any other point $Q$, the distance would be greater than this.
**16(a)** Matching sides, $\triangle STP \equiv \triangle RTP$
**(b)** $S'P + PR = S'P + PS = 2a$, since $\triangle STP \equiv \triangle RTP$.
**(c)** $\triangle S'RS \ ||| \ \triangle OTS$ so $OT = a$.
**17** Let $TS$ intersect the auxiliary circle again at $T^*$. By symmetry in the circle, $ST^* = S'T'$. So $ST \times S'T' = ST \times ST^* = AS \times SA'$ (intersecting chords)

## Exercise **3E** (Page 31)

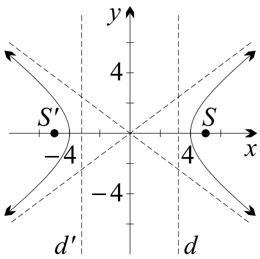
**1(a)** $e = \frac{3}{2}$, $S = (3, 0)$, $S' = (-3, 0)$, $d : x = \frac{4}{3}$, $d' : x = \frac{4}{3}$
**(b)** $y = \frac{\sqrt{5}}{2}x$, $y = -\frac{\sqrt{5}}{2}x$
**(e)** $(2\sqrt{2}, \sqrt{5})$ $\doteqdot (2{\cdot}83, 2{\cdot}24)$




**2(a)** $S = (2\sqrt{2}, 0)$, $S = (-2\sqrt{2}, 0)$, $d : x = \sqrt{2}$, $d : x = -\sqrt{2}$
**(b)** $y = x$, $y = -x$
**(d)** $(2\sqrt{2}, -2)$ $\doteqdot (2{\cdot}28, -2)$




**3(a)** $e = \frac{5}{4}$, $S = (5, 0)$, $S' = (-5, 0)$, $d : x = \frac{16}{5}$, $d' : x = -\frac{16}{5}$
**(b)** $y = \frac{3}{4}x$, $y = -\frac{3}{4}x$
**(d)** $x = 4\sec\theta$, and $y = 3\tan\theta$
**(e)** $(8, -3\sqrt{3})$




**4(a)** $(-\frac{2}{\sqrt{3}}, 1)$ **(b)** $x^2 - \frac{y^2}{3} = 1$ **(c)** 2
**5(a)** $\frac{x^2}{4} - \frac{y^2}{12} = 1$ **(b)** $\frac{x^2}{25} - \frac{y^2}{200} = 1$
**7(a)** $4x - 9y = 36$
**15(b)** It collapses onto the asymptotes: $\frac{y^2}{x^2} = e^2 - 1$.

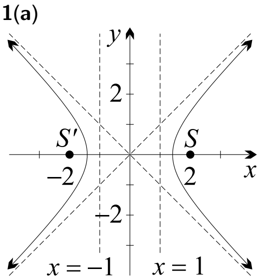
**16(b)** Since $z$ is closer to 2 than $-2$, the left branch is omitted. Thus $\arg(z)$ is in the fourth and first quadrants between the asymptotes.
**17(b)(ii)** In the limit, $y^2 = 4x$ is obtained. This is a parabola with focal length 1.
**(c)** A parabola with focal length $f$ is obtained.

## Exercise **3F** (Page 36)

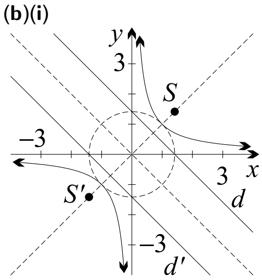
**2(b)** $e = \sqrt{2}$ **(c)** $e = \frac{1}{2}(1 + \sqrt{5})$, the golden ratio.
**4(b)** $x = \frac{4}{3}$, $x = -\frac{4}{3}$, $y = \frac{\sqrt{5}}{2}x$, $y = -\frac{\sqrt{5}}{2}x$
**11(a)** $2b|\sec\theta|$ **(b)** $OT = \frac{a}{|\sec\theta|}$
**17(c)** $(-b, 0)$
**18(a)** $\frac{a^2}{x_0}$
**21(a)** It is the hyperbola $\dfrac{x^2}{c^2\cos^2\theta} - \dfrac{y^2}{c^2\sin^2\theta} = 1$ with eccentricity $e = \sec\theta$ and the same foci as the ellipse.
**(b)** It is the ellipse $\dfrac{x^2}{c^2\sec^2\theta} + \dfrac{y^2}{c^2\tan^2\theta} = 1$ with eccentricity $e = \cos\theta$ and the same foci as the hyperbola.

## Exercise **3G** (Page 44)

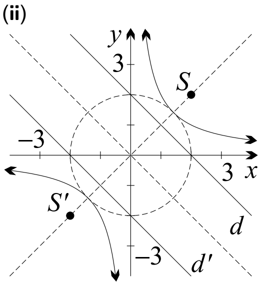
**1(a)**



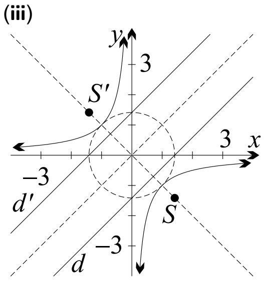

**(b)(i)**




**(ii)**




**(iii)**




**4(b)** $Q\left(-\frac{c}{t^3}, -ct^3\right)$ **(d)** $Q \equiv R$
**5(a)** $p$ and $q$ have opposite sign.
**6(c)** $Q$ is the result of reflecting $P$ in $y = x$.
**9(b)** $Q = \left(3ct, -\frac{c}{t}\right)$, $Q' = \left(-ct, \frac{3c}{t}\right)$
**10(a)** $x + pqy = c(p + q)$
**13(b)** $\left(\frac{c}{2t^3}(t^4 - 1), \frac{c}{t}\right)$

CHAPTER **EIGHT**

# Graphs

Chapter Overview: This chapter reviews and extends the work done in previous years on graphing functions and relations, including the application of the calculus. In particular the relationships between algebra and geometry are further revealed by examining how graphs are transformed when some common algebraic operations are applied, such as squaring or the taking of reciprocals. The graphs of the transformations of both known and unknown functions are obtained by identifying significant features of both the original function and the transformed function.

The previous work on standard functions and relations done in Years 10 and 11 is assumed knowledge, however Section 8A contains a brief review of this work, omitting the calculus. The following sections deal with superposition (that is, addition), modulation (multiplication), reciprocals, reflections, natural powers, square roots, and composite functions, and the chapter concludes with graphs of relations, where the calculus requires use of implicit differentiation. Each topic is investigated through the use of the relevant parts of the curve sketching menu.

Wherever possible in the text, an accurate graph of a function is shown so that the reader may know what it should look like. However, it must be remembered that the aim of a sketch is not an accurate plot, but rather to show the significant features and the correct general shape. Therefore, students should overlook minor discrepencies when marking their work.

No book could possibly cover the infinite variety of transformations that exist. Thus this text does not attempt to deal with every eventuality. Instead, familiar problems from previous study are combined with pertinant practical examples to intruduce the new work. Comments, observations and deductions about the transformations of functions are confined to general cases, and the functions themselves are assumed to be 'nice'. Thus, for example, if the analysis requires differentiation then it is assumed that the function in question is differentiable. Some exceptions and special cases are presented in the exercises.

## 8A Review

**The Standard Graphs:** The graphs of the common functions and relations studied in both Mathematics Extension 1 and Mathematics Extension 2 courses are assumed knowledge. Where the corresponding equations have more than one standard form, such as $ax+by+c=0$ and $y=mx+b$ for the equation of a straight line, then they too are assumed knowledge. A sample list is given below for convenience.

If any one of these should prove unfamiliar then it should be reviewed in detail before proceeding with the rest of this chapter.

| | |
|---|---|
| Linear | $ax + by + c = 0$ |
| Quadratic | $y = ax^2 + bx + c$ |
| Polynomial | $y = P_n(x)$ |
| Rectangular Hyperbola | $xy = c^2$ |
| Exponential | $y = e^{kx}$ |
| Logarithmic | $y = a \log x$ |
| Trigonometric | $y = \cos x$ |
| Inverse Trigonometric | $y = \sin^{-1} x$ |
| Absolute Value | $y = \lvert x \rvert$ |
| Circle | $x^2 + y^2 = r^2$ |
| Conic | $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ |

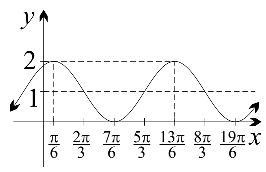
The simplest types of transformations have already been encountered. These are translations, stretches and reflections. They are briefly reviewed here.

**Translations:** A graph may be translated by shifting it horizontally, vertically, or in both directions. When $x$ is replaced by $(x - h)$ in the equation, the graph is shifted $h$ units to the right. When $y$ is replaced by $(y - k)$, the graph is shifted $k$ units up. Thus the parabolas $x^2 = 4ay$ and $(x - h)^2 = 4a(y - k)$ have the same focal length but the latter has been shifted so that its vertex is at $(h, k)$. Similarly $(x - a)^2 + (y - b)^2 = r^2$ is the result of shifting the circle $x^2 + y^2 = r^2$ so that its centre is at $(a, b)$.

**1** HORIZONTAL SHIFT: To shift $h$ units to the right, replace $x$ by $(x - h)$.
VERTICAL SHIFT: To shift $k$ units up, replace $y$ by $(y - k)$.

**WORKED EXERCISE:** Sketch $y = \sin\left(x + \frac{\pi}{3}\right) + 1$.

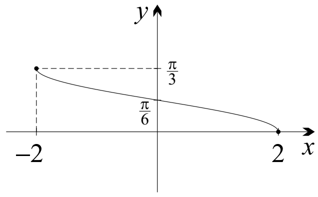
**SOLUTION:** Rearranging, $(y - 1) = \sin\left(x + \frac{\pi}{3}\right)$.
This is the result of translating $y = \sin x$ left by $\frac{\pi}{3}$ and up by 1, and is sketched on the right. Note the maximum at $x = \frac{\pi}{6}$ and $x$-intercept at $\frac{7\pi}{6}$. Also note that the height is 1 at $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{3}$. The wavelength is $2\pi$ so these features are repeated every $2\pi$.




**Stretches:** A graph may be stretched horizontally, vertically, or in both directions. When $x$ is replaced by $\frac{x}{a}$ in the equation, the graph is stretched horizontally by factor $a$. When $y$ is replaced by $\frac{y}{b}$, the graph is stretched vertically by factor $b$. Thus whilst the sine wave $y = \sin x$ has wavelength $2\pi$, the graph of $y = \sin \pi x$ has wavelength 2, since the wave has been stretched by factor $\frac{1}{\pi}$. Similarly, the circle $x^2 + y^2 = r^2$ stretched vertically by factor $\lambda$ becomes the ellipse $x^2 + \left(\frac{y}{\lambda}\right)^2 = r^2$.

**2** HORIZONTAL STRETCH: To stretch horizontally by factor $a$, replace $x$ by $\frac{x}{a}$.
VERTICAL STRETCH: To stretch vertically by factor $b$, replace $y$ by $\frac{y}{b}$.

**WORKED EXERCISE:** Sketch $y = \frac{1}{3}\cos^{-1}\frac{x}{2}$ .



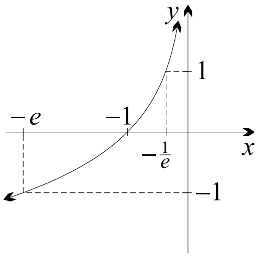

**SOLUTION:** Rearranging, $3y = \cos^{-1}\frac{x}{2}$ .

This is the result of stretching $y = \cos^{-1} x$ horizontally by factor 2 and vertically by factor $\frac{1}{3}$ . It is graphed on the right. Note the $x$-intercept at $x = 2$ and $y$-intercept at $y = \frac{\pi}{6}$ .

**Reflections in the Axes:** A graph may be reflected in the $y$-axis, in the $x$-axis or in both axes. When $x$ is replaced by $-x$ in the equation, the graph is reflected in the $y$-axis. When $y$ is replaced by $-y$ , the graph is reflected in the $x$-axis. Thus the graphs of the exponential functions $y = e^x$ and $y = e^{-x}$ are reflections of each other in the $y$-axis. Likewise, the graphs of $y = \tan x$ and $-y = \tan x$ are reflections of each other in the $x$-axis.

**3** HORIZONTAL REFLECTION: To reflect in the $y$-axis, replace $x$ by $-x$ .
VERTICAL REFLECTION: To reflect in the $x$-axis, replace $y$ by $-y$ .

**WORKED EXERCISE:** Sketch $y = \log\left(\dfrac{-1}{x}\right)$ .




**SOLUTION:** Rearranging, $-y = \log(-x)$ .
This is the result of reflecting the log graph in both axes, and is sketched on the right. Note the $x$-intercept at $-1$ , and that the vertical asymptote appears unchanged. Also note that this graph could have been obtained by rotating the log graph by $180°$ about the origin.

**Odd and Even Functions:** Two special cases of reflections in the axes are odd and even functions. In the case of an even function, the graph is unaltered by a reflection in the $y$-axis. That is $y = f(x)$ and $y = f(-x)$ appear the same, whence $f(-x) = f(x)$ . A simple example is $y = x^2$ . In the case of an odd function, the graph is unaltered after reflecting in each of the coordinate axes. That is $y = f(x)$ and $-y = f(-x)$ appear the same, whence $f(-x) = -f(x)$ . A simple example is $y = x^3$ .

**4** EVEN FUNCTIONS: $f(x)$ is called *even* if $f(-x) = f(x)$ , for all $x$ in its domain.
ODD FUNCTIONS: $f(x)$ is called *odd* if $f(-x) = -f(x)$ , for all $x$ in its domain.

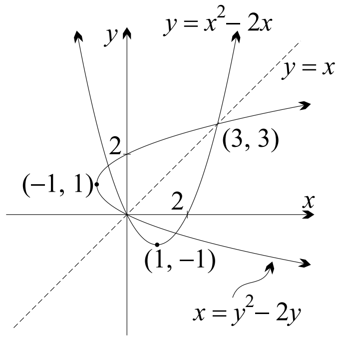
Finally notice that the geometry of odd functions can be interpreted in two ways. By the definition, they are unaltered by a reflection in each of the coordinate axes. Alternatively, they are unaltered by a rotation of $180°$ about the origin.

**Inverses:** The third type of reflection encountered in the Mathematics Extension 1 course is in the line $y = x$ , as used to find the inverse of a function or relation. Algebraically this is achieved by swapping $x$ and $y$ throughout the equation. Thus the parabola $x^2 = 4ay$ and its inverse $y^2 = 4ax$ are symmetric in the line $y = x$ . One method to sketch the inverse is to begin by plotting those points which correspond to the significant features of the original relation. Thus reversing the coordinates of a specific point yields the corresponding point on the inverse.

**5** INVERSES: The inverse relation is obtained by reflecting in the line $y = x$. Algebraically, swap $x$ and $y$. On the number plane, reverse each ordered pair.

**WORKED EXERCISE:** (a) Sketch $y = x^2 - 2x$.
(b) Hence sketch $x = y^2 - 2y$.

**SOLUTION:** Part (a) is a parabola, and part (b) is the result when reflected in the line $y = x$. The first one passes through three significant points, the intercepts at $(0, 0)$ and $(2, 0)$, and the vertex at $(1, -1)$. Thus the corresponding points on the inverse are $(0, 0)$, $(0, 2)$ and $(-1, 1)$. Both parabolas are drawn on the right.




**Reflection in the line $x = a$:** A myriad of new transformations can be created by combining translations, stretches and the reflections encountered so far. One combination has a significant application in integration and so is included in this course. When $y = f(x)$ is reflected in the $y$-axis, the result is $y = f(-x)$. If this is then shifted $2a$ units to the right, the result is

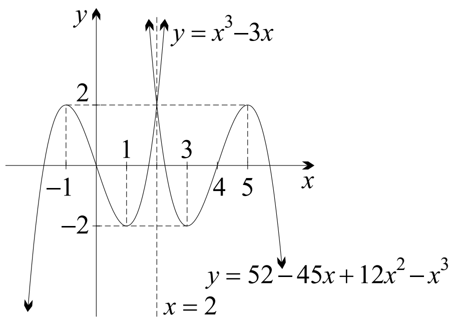
$$y = f\big(-(x - 2a)\big)$$
or $$y = f(2a - x).$$

This combination of reflection and shift is equivalent to a simple reflection in the line $x = a$, as the following example demonstrates.

The graphs of $y = x^3 - 3x$ and $y = (4-x)^3 - 3(4-x)$ are sketched below the table of values. Note that the latter equation expands to $y = 52 - 45x + 12x^2 - x^3$.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ |
|---|---|---|---|---|---|---|---|---|---|
| $x^3 - 3x$ | $-2$ | $2$ | $0$ | $-2$ | $2$ | $21$ | $52$ | $110$ | $198$ |
| $(4-x)^3 - 3(4-x)$ | $198$ | $110$ | $52$ | $21$ | $2$ | $-2$ | $0$ | $2$ | $-2$ |

It should be clear that the third line of the table of values is just the reverse of the second line. That is, there is symmetry about the middle value $x = 2$. The graph also makes it clear that $y = (4-x)^3 - 3(4-x)$ is obtained by reflecting $y = x^3 - 3x$ in the line $x = 2$. In particular, it shows that the local minima and local maxima are equally spaced either side of the reflection line.




**6** REFLECT IN A VERTICAL LINE: To reflect in the vertical line $x = a$ replace $x$ by $(2a - x)$.

Just as even functions are symmetric in the $y$-axis, there are functions which are symmetric in the vertical line $x = a$ and these functions satisfy the equation

$$f(x) = f(2a - x).$$

There are numerous such functions, but three significant examples have been met in this course, namely the parabola, and the sine and cosine waves. In each case it is easy to prove the symmetry algebraically.

**WORKED EXERCISE:** Prove that $f(x) = \sin x$ is symmetric in the line $x = \frac{\pi}{2}$.

**SOLUTION:** Replacing $x$ with $(\pi - x)$ yields

$$\begin{aligned} f(\pi - x) &= \sin(\pi - x) \\ &= \sin\pi\cos x - \cos\pi\sin x \\ &= 0 + \sin x \\ &= f(x)\,. \end{aligned}$$

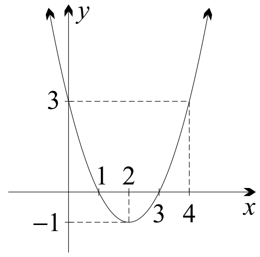
Thus $f(x) = \sin x$ is symmetric in the line $x = \frac{\pi}{2}$.

**7** SYMMETRY IN A VERTICAL LINE: A function which is symmetric in the vertical line $x = a$ satisfies the equation $f(x) = f(2a - x)$.

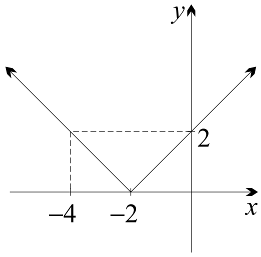
## Exercise **8A**

**1.** In each case the graph of $y = f(x)$ is given. Sketch the graphs of: (i) $y = f(x+1)$, (ii) $y = f(x) + 1$, (iii) $y = f(\frac{1}{2}x)$, (iv) $y = \frac{1}{2}f(x)$, (v) $y = f(-x)$, (vi) $y = -f(x)$, (vii) $y = f(2-x)$, (viii) $y = 2 - f(x)$.

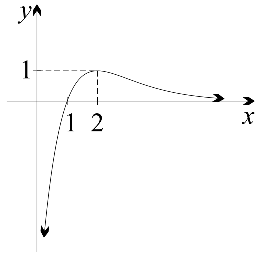
(a)




(b)




(c)




**2.** (a) (i) State the equation of the axis of the parabola with equation $y = 2x - x^2$.

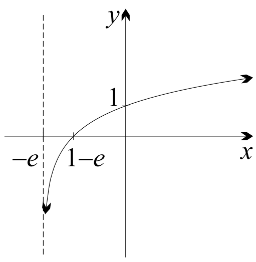
(ii) Prove the result algebraicly by replacing $x$ by $(2-x)$ and showing that the equation is unchanged.

(b) Similarly prove algebraically that each of the following parabolas is symmetric by using an appropriate substitution.

(i) $y = x^2 - 4x + 3$ (ii) $y = 1 - 3x - x^2$

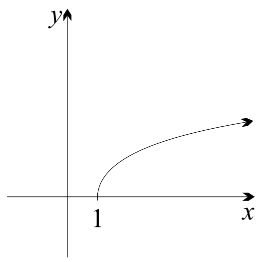
**3.** In each case, the graph of $y = f(x)$ is given. Sketch the graph of $y = f^{-1}(x)$ then determine $f^{-1}(x)$.

(a)



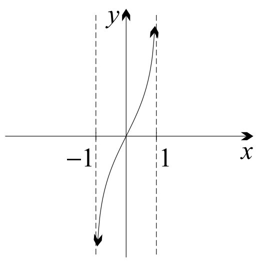

$f(x) = \log(e + x)$

(b)




$f(x) = \log(x + \sqrt{x^2 - 1})$

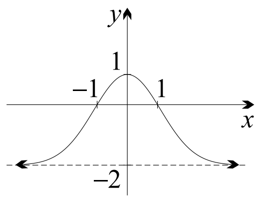
(c)




$f(x) = \log(1+x) - \log(1-x)$

DEVELOPMENT

**4.** For the given sketch of $f(x)$, sketch the graph of $y = g(x)$ where




(a) $g(x) = \begin{cases} f(x) & \text{for } x \geq 1 \\ f(2-x) & \text{for } x < 1 \end{cases}$

(b) $g(x) = \begin{cases} f(x) & \text{for } x \geq -1 \\ f(-2-x) & \text{for } x < -1 \end{cases}$

**5.** (a) Describe geometrically two ways of transforming the graph of the circle $x^2+(y-1)^2 = 4$ to get the circle $x^2 + (y+1)^2 = 4$.

(b) Describe geometrically three ways of transforming the graph of the wave $y = \sin(2x)$ to get the wave $y = \sin(2x + \pi)$.

**6.** Use a suitable substitution to prove that $Q(x) = ax^2 + bx + c$ is symmetric in the line $x = -\frac{b}{2a}$.

**7.** (a) (i) The graph of $y = \cos x$ is symmetric in the $y$-axis. What other vertical lines are lines of symmetry.

(ii) Prove your result with a suitable substitution.

(b) Do likewise for $y = \sin x$.

**8.** The function $f(x)$ has the property $f(x) = f(2a-x)$. Prove algebraicly that this function is symmetric in the line $x = a$ by showing that $f(a+t) = f(a-t)$.

EXTENSION

**9.** (a) If $f(x)$ is odd then prove that $f'(x)$ is even.

(b) Is the converse true?

(c) Investigate the situation when $f(x)$ is even.

**10.** Show that every function can be written as the sum of an odd and even function.
HINT: Begin by investigating the function $h(x) = f(x) + g(x)$, where $f(x)$ is even and $g(x)$ is odd.

# 8B Superposition

Superposition is simply the addition of two functions to create a new function. Thus if $f(x)$ and $g(x)$ are two functions then the result will be $h(x) = f(x)+g(x)$. A simple example might be the sum of the quadratic $f(x) = x^2$ and the linear function $g(x) = 2x$ to obtain $h(x) = x^2 + 2x$. Thus every quadratic with more than one term is an example of superposition.

When one of the functions is constant, for example $h(x) = f(x)+b$, the situation reduces to a vertical shift of $b$ units as reviewed in Section 8A, and so will not be considered here. Although this section does not deal explicitly with the difference of two functions, the theory applies equally to differences since the function

$$h(x) = f(x) - g(x)$$

can be written as a sum, as follows:

$$h(x) = f(x) + \Big(-g(x)\Big).$$

Note that in the remainder of this chapter, the function notation will often be dropped for brevity. Thus $h(x) = f(x) + g(x)$ may be written as $h = f + g$.

**Domain:** The domain of $h$ is the intersection of the individual domains of $f$ and $g$.

Thus if $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x-2}$

then the domain of $h = f + g$ is $x \neq 0, 2$.

**Intercepts:** In most cases, the $y$-intercept is trivially found, provided $x = 0$ is in the domain. Thus there is no further discussion of the $y$-intercept here, nor in the remainder of this chapter.

Since the $x$-intercepts of $y = h(x)$ are solutions of $h = 0$, it follows that

$$f + g = 0$$

or $$f = -g$$

at these points. That is, the $x$-intercepts occur wherever the ordinates of the constituent functions are opposites or both zero.

8 INTERCEPTS: The $x$-intercepts of $y = f(x) + g(x)$ occur wherever $f$ and $g$ are opposite or both zero.

**Symmetry:** In general, the sum of two even functions is even, the sum of two odd functions is odd, and a mixture is neither. These results are summarised in the addition table below, and the proofs are left to the exercise.

| + | odd | even |
|---|---|---|
| odd | odd | neither |
| even | neither | even |

Combinations of other functions may yield odd or even symmetry, and there may be other symmetries to investigate. Every function should be routinely checked.

**The Calculus:** At stationary points, $h' = 0$ so

$$f' + g' = 0$$

or $$f' = -g'$$

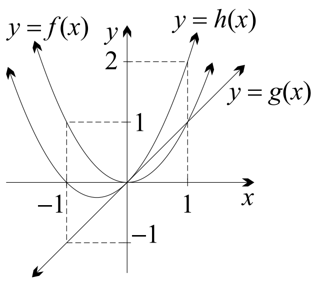
That is, the gradients of the constituent functions are opposite.

**Other:** If $y = f(x)$ and $y = g(x)$ intersect at $x = a$ then $f(a) = g(a)$ and

$$\begin{aligned} h(a) &= f(a) + g(a) \\ &= 2f(a) = 2g(a)\,. \end{aligned}$$

That is, the ordinate of $h$ is double that of $f$ or $g$. Such points should be plotted.

**WORKED EXERCISE:** Let $f(x) = x^2$ and $g(x) = x$, with $h(x) = f(x) + g(x)$. Graph $y = f(x)$ and $y = g(x)$ on the same set of axes and hence draw $y = h(x)$.




**SOLUTION:** At $x = 0$ both $f$ and $g$ are zero so $y = h(x)$ has an intercept there.

At $x = -1$ we find $f = 1$ and $g = -1$ are opposite, so $y = h(x)$ has another intercept at $x = -1$.

At $x = -\frac{1}{2}$ the gradients of $f$ and $g$ are opposite, so $y = h(x)$ has a stationary point there.

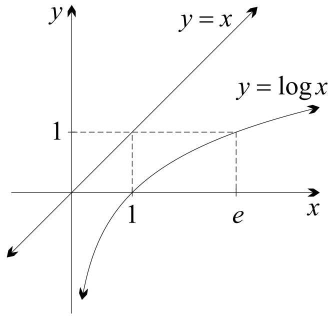
Finally $f$ and $g$ intersect at $(1, 1)$, hence $y = h(x)$ passes through $(1, 2)$, at double the height.

**9** | INTERSECTIONS: At the intersection points of $y = f(x)$ and $y = g(x)$ the height of $y = f(x) + g(x)$ is double.

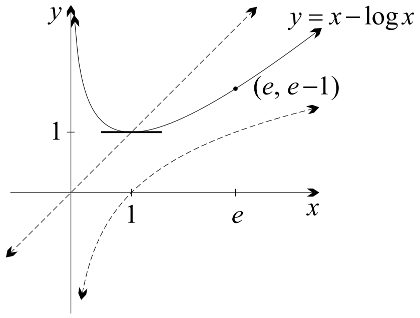
Finally, if $f(x)$ has a zero at $x = a$ then $h(a) = g(a)$, with similar results at the intercepts of $y = g(x)$. Thus it is usual to plot $y = h(x)$ at the $x$-intercepts of both $f$ and $g$.

**WORKED EXERCISE:** (a) Sketch $y = x$ and $y = \log x$ on the same graph.
(b) Hence Sketch $y = x - \log x$.

**SOLUTION:**
(a)




(b)




The graph on the left shows $y = x$ and $y = \log x$. The latter has an intercept at $x = 1$, so the graph of $y = x - \log x$ on the right has height $1$.

Also at $x = 1$ both curves on the left have gradient 1, so the difference has a stationary point there. The second derivative is $y'' = x^{-2}$ so the curve is everywhere concave up, and the stationary point is a global minimum.

Since $x > \log x$, it follows that $x - \log x > 0$ for all $x$ in the domain. The domain is $x > 0$, so $y = x - \log x$ must lie entirely in the first quadrant. Lastly, $\log e = 1$ so the point $(e, e - 1)$ is in the graph.

**10** | INTERCEPTS AGAIN: It is usual to plot $y = f(x) + g(x)$ at the $x$-intercepts of $f$ and $g$ if they are known or easy to find.

**WORKED EXERCISE:** [A HARD EXAMPLE]
From the sketches of

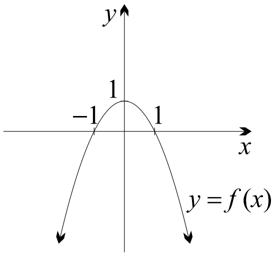
$$y = f(x)$$

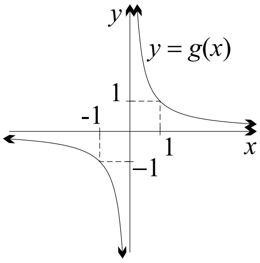
and $$y = g(x)$$

given on the right, sketch

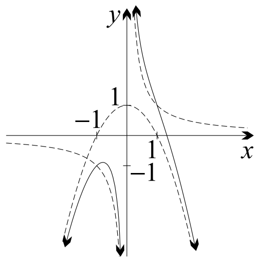
$$y = h(x)$$

where $h(x) = f(x) + g(x)$.




**SOLUTION:** Clearly the domain of $h$ is $x \neq 0$ and hence there is no $y$-intercept.




Since $f(x) = 0$ at $x = -1$ and 1, the graph of $y = h(x)$ crosses $y = g(x)$ at these points.

For $-1 < x < 1$, $f(x) > 0$ so the graph of $y = h(x)$ is above $y = g(x)$ in this region.

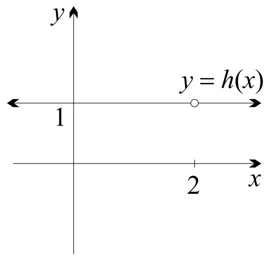
For large value of $x$, $g(x) \to 0^+$, so the graph of $y = h(x)$ approaches $y = f(x)$ from above.

For large negative values of $x$, $g(x) \to 0^-$, so the graph of $y = h(x)$ approaches $y = f(x)$ from below.

All these details are shown in the graph of $y = h(x)$ above. The graphs of $y = f(x)$ and $y = g(x)$ are shown dashed on the same set of axes for comparison.

**Discontinuities:** Care must be taken when analysing functions at discontinuities. Consider the following example.

$$\begin{aligned}
\text{Let} \quad & f(x) = 1 + \frac{1}{2-x} \\
\text{and} \quad & g(x) = \frac{1}{x-2}\,. \\
\text{Adding} \quad & h(x) = f(x) + g(x) \\
& \phantom{h(x)} = 1 + \frac{1}{2-x} + \frac{1}{x-2} \\
& \phantom{h(x)} = 1 - \frac{1}{x-2} + \frac{1}{x-2} \\
\text{hence} \quad & h(x) = 1\,.
\end{aligned}$$




Thus it appears that $h(x)$ is a continuous function. This is an incorrect conclusion however, and a common mistake to make. Since $h = f + g$ and since neither $f$ nor $g$ are defined at $x = 2$, it follows that the domain of $h$ is $x \neq 2$. Hence the graph of $y = h(x)$ has a hole at $x = 2$, as is shown above. This example demonstrates that the domain of the function should always be checked.

## Exercise **8B**

**1.** (a) (i) Draw the graphs of $y = |x+1|$ and $y = |x-2|$ on the same number plane.

(ii) Hence sketch $y = |x+1| - |x-2|$.

(b) Do likewise for the following.

(i) $y = |x-1| - |x+1|$ (ii) $y = |x-1| + |x-2|$

**2.** In each case, graph the functions $y = f(x)$, $y = -g(x)$ and $y = g(x)$. Hence graph $y = f(x) - g(x)$ and $y = f(x) + g(x)$.

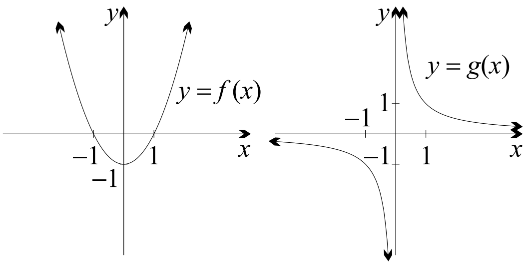
(a) $f(x) = x^2$, $g(x) = 2x$ (c) $f(x) = x$, $g(x) = \sin x$

(b) $f(x) = x$, $g(x) = e^{-x}$ (d) $f(x) = e^x$, $g(x) = e^{-x}$

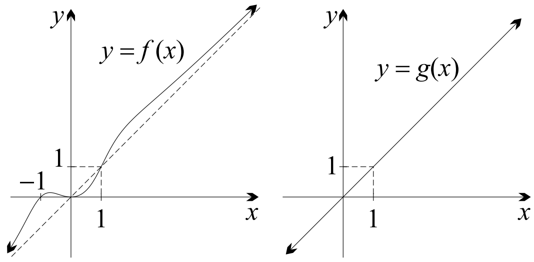
 

**3.** In each case, use the graphs of $y = f(x)$ and $y = g(x)$ to help sketch the required function.

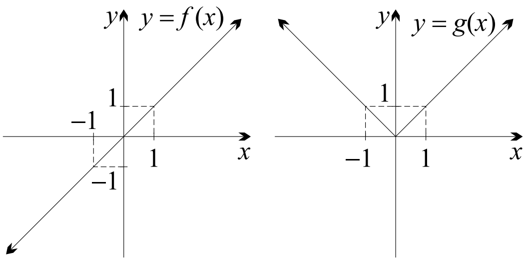
(a) $y = f(x) + g(x)$



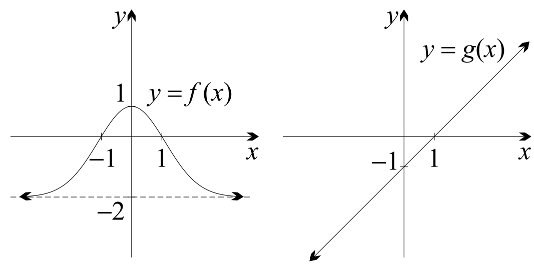

(b) $y = f(x) - g(x)$




(c) $y = f(x) + g(x)$




(d) $y = f(x) - g(x)$




**DEVELOPMENT**

**4.** The piecewise continuous function in Question 1(a) can be written as

$$y = \begin{cases} -3 & \text{for } x < -1 \\ 2x - 1 & \text{for } -1 \le x < 2 \\ 3 & \text{for } x \ge 2 \end{cases}$$

Find similar expressions for the functions in part (b).

**5.** (a) Sketch $y = \log(1 + x)$ and $y = \log(1 - x)$ on the same number plane.

(b) Hence sketch $y = \log(1 + x) - \log(1 - x)$ .

**6.** (a) Sketch $y = \cos^{-1} x$ and $y = \sin^{-1} x$ on the same axes, and observe the symmetry in the line $y = \frac{\pi}{4}$ .

(b) Hence sketch $y = \cos^{-1} x + \sin^{-1} x$ .

**7.** (a) Sketch $f(x) = \dfrac{1}{x}$ for $x > 0$ , and $g(x) = \log x$ on the same number plane.

(b) Notice that the behaviour of $y = f(x) + g(x)$ near $x = 0$ cannot be determined from the graph in part (a). Use $y'$ to determine the behaviour of $y$ as $x \to 0^+$ .

(c) Hence sketch $y = f(x) + g(x)$ .

**8.** This question demonstrates that the domain must be noted when a function is simplified.

(a) Show that $\log(x + \frac{3}{2}) + \log x = \log\left(x^2 + \frac{3}{2}x\right)$ for $x > 0$ , but not for $x < 0$ .

(b) Graph $y = \log(x + \frac{3}{2}) + \log x$ and $y = \log\left(x^2 + \frac{3}{2}x\right)$ on separate number planes, in each case using the natural domain. Observe that the two graphs differ.

**9.** The addition table for odd and even functions is given in the text. There are essentially three cases: both odd, both even, or one even and one odd.

(a) Find an example to demonstrate each result.

(b) Prove each result in general.

(c) In general, the sum of two odd functions is an odd function. What is the one exception to this rule?

**EXTENSION**

**10.** The Heaviside step function is also called the unit step function, and is defined as follows.

$$u(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

(a) Sketch $y = u(x)$ and $y = u(x-1)$ on separate number planes

(b) Hence sketch (i) $y = u(x) - u(x-1)$, and (ii) $y = u(x) + u(x-1)$.

**11.** Use the graphs of $y = x^4 - 1$ and $y = mx$ to determine the number of zeros of the function $f(x) = x^4 + mx - 1$.

# 8C Modulation

Modulation is the product of two functions to create a new function. Thus if $f(x)$ and $g(x)$ are two functions then the result will be $h(x) = f(x) \times g(x)$. A simple example might be the product of the exponential $f(x) = e^x$ and the linear function $g(x) = x$ to obtain $h(x) = xe^x$.

When one of the functions is constant, for example $h(x) = af(x)$, the situation reduces to a vertical stretch by factor $a$ as reviewed in Section 8A, and so will not be considered here. Although this section does not deal explicitly with rational functions, many of the points made here apply to quotients since the function

$$h(x) = \frac{f(x)}{g(x)}$$

can be written as a product, as follows:

$$h(x) = f(x) \times \frac{1}{g(x)}.$$

The Mathematics Extension 1 course includes the study of rational functions, consequently some examples have been included in the exercises. A suitable text book should be consulted for a full exposition of that topic.

**Domain:** As with superposition, the domain of $h$ is the intersection of the individual domains of $f$ and $g$.

**Intercepts:** The $x$-intercepts of $h$ will occur at the $x$-intercepts of $f$ and $g$, provided that those values of $x$ lie in the domain of $h$.

**11** INTERCEPTS: The $x$-intercepts of $h(x) = f(x) \times g(x)$ occur at the $x$-intercepts of $f$ and $g$, provided that those values of $x$ lie in the domain of $h$.

**Symmetry:** The symmetries that result from the products of odd and even functions are summarised in the following multiplication table, and the proofs are left to the exercise.

| $\times$ | odd | even |
|---|---|---|
| odd | even | odd |
| even | odd | even |

These symmetries are very important in the study of integration and should have already been encountered in the Mathematics Extension 1 course. For example:

$$\int_{-\frac{\pi}{5}}^{\frac{\pi}{5}} \cos 2x \sin 3x \, dx = 0$$

since the limits are symmetric and the integrand is odd.

Products of other functions may yield odd or even symmetry, and there may be other symmetries to investigate. For example, the functions $f(x) = e^x$ and $g(x) = e^{-x}$ have no symmetry, yet $h(x) = 1$ which is even. As always, every function should be routinely checked for symmetry.

**The Calculus:** Apart from a few special cases there is little to be said in this course about the calculus of the products of functions.

**Other:** Wherever $f = 1$ the product reduces to $h(x) = g(x)$ and wherever $f = -1$ it reduces to $h(x) = -g(x)$, with similar results wherever $|g| = 1$. Thus whenever possible, points where $|f| = 1$ or $|g| = 1$ should be plotted.

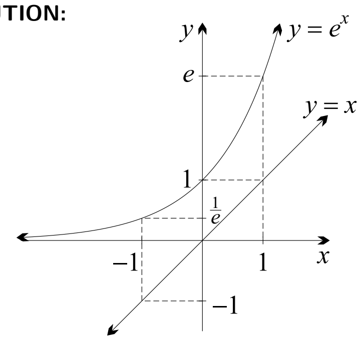
**12** SPECIAL POINTS: If the solutions of $|f(x)| = 1$ and $|g(x)| = 1$ are easy to find and in the domain, then plot the corresponding points for $h(x) = f(x) \times g(x)$.

**WORKED EXERCISE:** (a) Sketch $y = x$ and $y = e^x$. (b) Hence sketch $y = xe^x$.

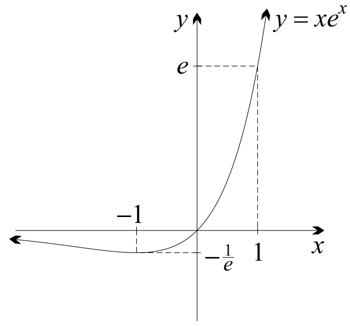
(c) Without the use of calculus, explain why $f(x) = xe^x$ must have a global minimum for a negative value of $x$.

**SOLUTION:**

(a)




(b)




Note that $|x| = 1$ at $x = 1$ or $-1$, and $|e^x| = 1$ at $x = 0$.

Note the corresponding points plotted at $x = -1, 0$ and 1.

(c) The functions $f(x)$ is continuous and has the three properties:

$$f(0) = 0,$$
$$f(x) < 0 \text{ for } x < 0,$$
and $$\lim_{x \to -\infty} f(x) \to 0^-.$$

Hence there exists a value $x = a$, $-\infty < a < 0$, for which $f(a)$ is a local minimum. Further, since $f(x) \geq 0$ wherever $x \geq 0$, it follows that $f(a)$ is a global minimum. [Calculus reveals that $a = -1$.]

## Exercise 8C

**1.** (a) Let $f(x) = x$ and $g(x) = \sin x$, and let $h(x) = f(x) \times g(x)$.

(i) Show that $h(x)$ is even.

(ii) Find the $x$-intercepts of $y = g(x)$ and hence plot $y = h(x)$ at those values.

(iii) Find the values of $x$ where $g(x) = 1$ and where $g(x) = -1$, and hence plot $y = h(x)$ at those values.

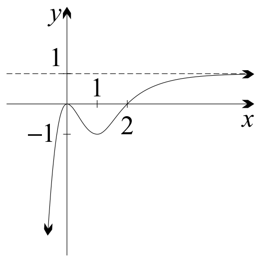
(iv) Complete the graph of $y = h(x)$.

(b) Similiarly sketch $y = h(x)$ when $f(x) = e^{-x}$ and $g(x) = \cos \pi x$.

**2.** In each case, graph the functions $y = f(x)$ and $y = g(x)$. Hence graph $y = f(x) \times g(x)$ without the use of calculus.

(a) $f(x) = x^2$, $g(x) = e^x$ (b) $f(x) = x^2 - 1$, $g(x) = e^x$

**3.** (a) The diagram on the right shows $y = f(x)$.
Let $h(x) = xf(x)$.




(i) Plot $y = h(x)$ at the values of $x$ where $f(x) = 0$.

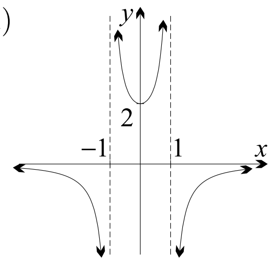
(ii) Locate any point where $|f(x)| = 1$. Hence plot $y = h(x)$ at those values.

(iii) Explain why $h(x) \to x$ as $x \to \infty$.

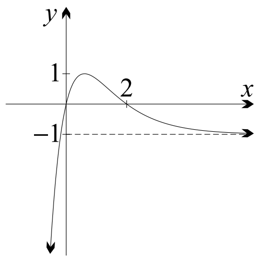
(iv) Hence complete the sketch of $y = h(x)$.

(b) In each case use a similar approach to part (a) to sketch $y = xf(x)$ for the given graph of $y = f(x)$.

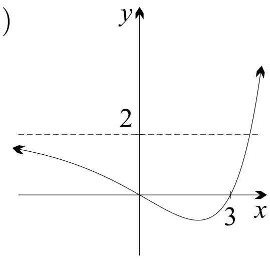
(i)




(ii)

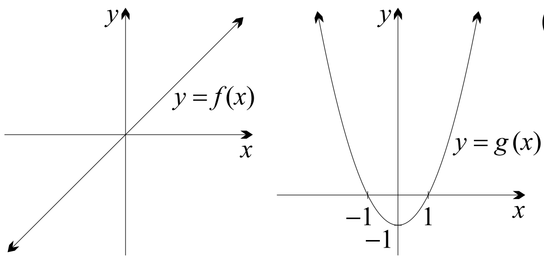



(iii)




**4.** Use the graphs of $y = f(x)$ and $y = g(x)$ to sketch $y = f(x) \times g(x)$. In part (e) you may assume that $\lim_{x\to\infty} f(x) \times g(x) = 0$.

(a)

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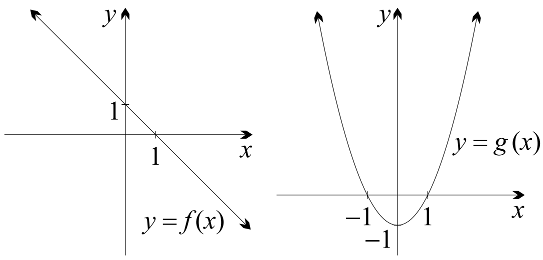
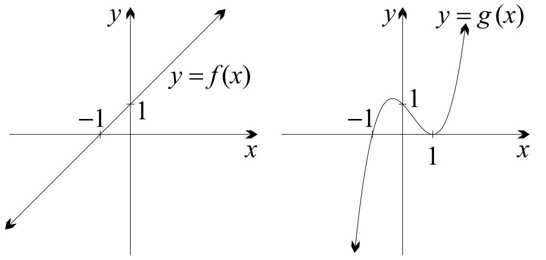
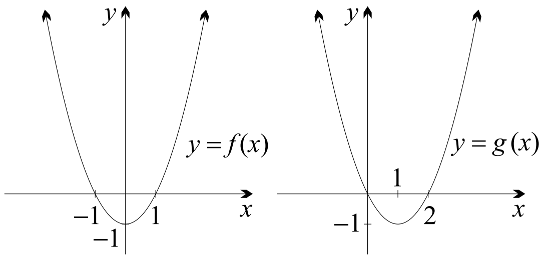


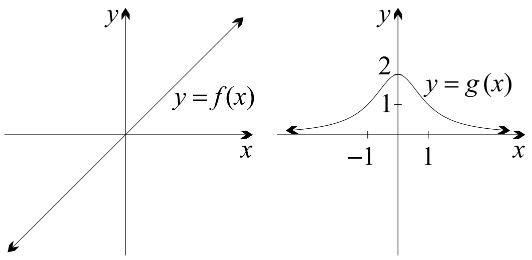

(c)



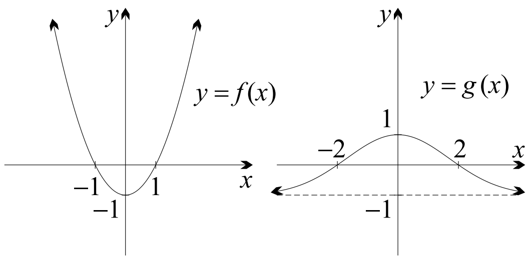

(d)



(e)




(f)




**DEVELOPMENT**

**5.** (a) Graph $y = x^{\frac{1}{3}}$, paying particular attention to the behaviour near the origin.

(b) Hence sketch $y = x^{\frac{1}{3}} e^x$.

**6.** Let $f(x) = \dfrac{x^2}{x^2 - 9}$.

(a) Show that $f(x) = 1 + \dfrac{3}{2}\left(\dfrac{1}{x-3} - \dfrac{1}{x+3}\right)$.

(b) Hence determine the vertical and horiztonal asymptotes of the graph of $y = f(x)$.

(c) Show that $x^2 - 9 < 0$ for $-3 < x < 3$. Hence show that $f(x)$ has a local maximum at $x = 0$. There is no need to resort to the calculus.

(d) Sketch $y = \dfrac{x^2}{x^2 - 9}$.

**7.** (a) Locate and classify the stationary points of $y = \dfrac{2x}{1 + x^2}$

(b) Use the second derivative to show that there is an inflexion point at the origin.

(c) Hence sketch $y = \dfrac{2x}{1 + x^2}$.

**8.** (a) Graph $y = \dfrac{x^2 - 1}{x^2 - 4}$.

(b) Hence solve $\dfrac{x^2 - 1}{x^2 - 4} > 1$.

**9.** Consider $y = \dfrac{(x^2 - 4)(x^2 - 1)}{x^4}$

(a) Write down the $x$-intercepts.

(b) State the equations of any vertical asymptotes.

(c) Show that $y = 1 - \dfrac{5}{x^2} + \dfrac{4}{x^4}$. Hence answer the following.

(i) State the equation of any horiztonal asymptotes.

(ii) Locate any stationary points.

(d) Hence sketch $y = \dfrac{(x^2 - 4)(x^2 - 1)}{x^4}$

(e) For what values of $b$ does the equation $(x^2 - 4)(x^2 - 1) = bx^4$ have four solutions?

(f) What would be a better way of solving this problem?

**10.** The multiplication table for odd and even functions is given in the text. Prove each entry in the multiplication table.

**11.** Let $f(x) = x^3$ and $g(x) = x^2$, with $h(x) = \dfrac{f(x)}{g(x)}$. Explain how the graphs of $y = x$ and $y = h(x)$ differ.

**EXTENSION**

**12.** The Heaviside step function was defined in Section 8B and is

$$u(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

Graph the following functions

(a) $u(x) \times (x^3 - x)$

(b) $u(x) \times e^{-x}$

(c) $u(x) \times \sin \pi x$

(d) $u(x+1) \times e^{-x}$

**13.** One particular application of modulation is used in sending radio signals. The 'AM' in AM Radio stands for amplitude modulation. A high frequency carrier wave has its amplitude modulated by a signal wave, which forms an envelope around the carrier wave.

(a) Use a computer to graph $y = \sin\left(\frac{x}{2}\right)\cos(5x)$.

(b) Which is the carrier wave?

# 8D Reciprocals

A significant number of problems involve graphing the reciprocal of a function. That is, the graph of $y = g(x)$, where $g(x) = \dfrac{1}{f(x)}$ and either $f(x)$ itself is known or the graph of $y = f(x)$ has been given. The classic example is of course when $f(x) = x$ and $g(x) = \dfrac{1}{x}$, that is, the rectangular hyperbola.

**Domain and Sign:** The domain of $g$ is the domain of $f$ intersecting with $f \neq 0$, since division by zero is undefined. The sign of $g$ is everywhere the same as the sign of $f$.

**Intercepts and Asymptotes:** The zeros of $f$ are the vertical asymptotes of $g$, but the converse is not true. If $y = f(x)$ has a vertical asymptote at $x = a$ then $\lim\limits_{x \to a} g(x) = 0$. That is, the graph approaches the $x$-axis but does not have an intercept at $x = a$.

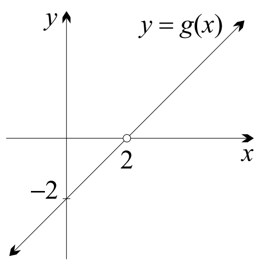
To demonstrate the point, consider the example

$$f(x) = \frac{1}{x-2} \quad \text{with } x \neq 2,$$

for which

$$g(x) = x - 2 \quad \text{with } x \neq 2.$$

The graph of $y = g(x)$ is shown on the right and has a hole at $x = 2$ instead of an $x$-intercept.



**13** INTERCEPTS AND ASYMPTOTES: The zeros of $y = f$ are the asymptotes of $y = \frac{1}{f}$, but if $f(x)$ has a vertical asymptote at $x = a$ then $\lim\limits_{x \to a} \frac{1}{f} = 0$.

If $y = 0$ is a horizontal asymptote of $y = f(x)$ then $|g| \to \infty$, and vice versa. Other horizontal asymptotes behave as expected. As an example let $f(x) = 2 + \frac{1}{x}$. Clearly $f \to 2$ as $x \to \infty$, so $g \to \frac{1}{2}$ as $x \to \infty$.

**Symmetry:** Odd and even symmetry is preserved. The proof is left as an exercise.

**The Calculus:** As before, let $g(x) = \dfrac{1}{f(x)}$. Differentiating the function $g(x)$ yields:

$$\begin{aligned} g' &= \frac{dg}{dx} \\ &= \frac{dg}{df} \times \frac{df}{dx} \quad \text{(by the chain rule)} \\ &= -\frac{1}{f^2} \times \frac{df}{dx} \\ \text{so} \quad g' &= -\frac{f'}{f^2}\,. \end{aligned}$$

Thus $y = g(x)$ will have stationary points with the same $x$-coordinates as the stationary points of $y = f(x)$, provided $f$ is not simultaneously zero. Further, the above result can be used to show that the nature of the stationary points is reversed. A question in the exercise deals with this.

**14** STATIONARY POINTS: The stationary points of $y = f(x)$ have the same $x$-coordinates as those of $y = g(x) = \dfrac{1}{f(x)}$ provided $f(x) \neq 0$ there.

The nature of the stationary points can also be determined from the second derivative, as follows.

$$g'' = \frac{2(f')^2 - ff''}{f^3} \quad \text{(by the quotient rule)}$$

so at stationary points where $f' = 0$ and $f \neq 0$ this becomes

$$g'' = -\frac{f''}{f^2}\,.$$

Thus at the stationary points of $f$ the sign of $g''$ is opposite that of $f''$. Hence the types of extrema of $f$ and $g$ are reversed, as asserted earlier. That is, $g$ has a maximum where $f$ has a minimum and vice versa.

**15** TYPES OF STATIONARY POINTS: If $g(x) = \dfrac{1}{f(x)}$ then the nature of the stationary points of $f$ and $g$ are reversed, provided the ordinates are non-zero. That is, $g$ has a maximum where $f$ has a minimum and vice versa.

**Other:** If both $y = f(x)$ and $y = g(x)$ are sketched on the same number plane then they intersect when

$$f = g\,,$$

so $$f = \frac{1}{f}$$

or $$f^2 = 1\,,$$

whence $f(x) = 1$ or $-1$.

Thus it is usual to plot these points if the $x$-coordinates are easy to find.

**WORKED EXERCISE:**

(a) Sketch the graph of $y = \dfrac{2(x^2-1)}{x^2+2}$ without the aid of the calculus.
Use the identity $\dfrac{2(x^2-1)}{x^2+2} = 2 - \dfrac{6}{x^2+2}$ to locate the minimum. Indicate on the sketch the intercepts with the axes, the horizontal asymptote and any points where $y = 1$.

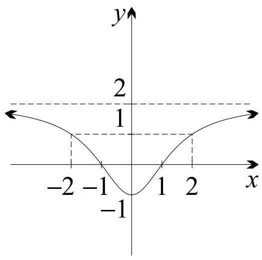
(b) Hence graph $y = \dfrac{x^2+2}{2(x^2-1)}$

**SOLUTION:** (a) Clearly there is no restriction on the domain and the function is even. There are intercepts with the axes at $(-1,0)$, $(1,0)$ and $(0,-1)$.

From the given identity, the minimum of $y$ occurs when the denominator is a minimum, that is at the $y$-intercept.

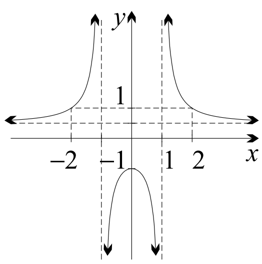
Now $$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{2(1-\frac{1}{x^2})}{1+\frac{2}{x^2}} = 2$$

so $y = 2$ is a horizontal asymptote.




Solving $$\frac{2(x^2-1)}{x^2+2} = 1$$
$$2x^2 - 2 = x^2 + 2$$
thus $$x^2 = 4$$
so $$x = 2 \text{ or } -2\,.$$

Hence $(-2,1)$ and $(2,1)$ are on the curve as shown.




(b) From part (a) it is clear that the domain is $x \neq -1, 1$ and the graph is symmetric in the $y$-axis. The $y$-intercept remains unchanged and there is no $x$-intercept. There are three asymptotes, namely $y = \frac{1}{2}$, $x = -1$ and $x = 1$. The behaviour either side of the vertical asymptotes is determined by the sign of $y$, and from part (a) this gives

$$\lim_{x\to 1^+} y \to \infty$$

and $$\lim_{x\to 1^-} y \to -\infty$$

Finally the curve passes through $(-2,1)$ and $(2,1)$ as shown.

**16** INTERSECTION POINTS: The graphs of $y = f$ and $y = \dfrac{1}{f}$ intersect wherever $|f| = 1$.



## Exercise **8D**

**1.** Let $f(x) = x - 1$.

(a) Graph $y = f(x)$ showing the intercepts with the axes and the points where $|f(x)| = 1$.

(b) Hence on the same number plane sketch $y = \dfrac{1}{f(x)}$.

**2.** Let $y = f(x)$ where $f(x) = \frac{1}{3}(x+1)(x-3)$.

(a) Show that $y = 1$ at $x = 1 - \sqrt{7}$ and $x = 1 + \sqrt{7}$. Plot these points.

(b) Complete the graph of $y = f(x)$ showing the vertex, the intercepts with the axes and the points where $f(x) = -1$.

(c) Hence on the same number plane sketch $y = \dfrac{1}{f(x)}$.

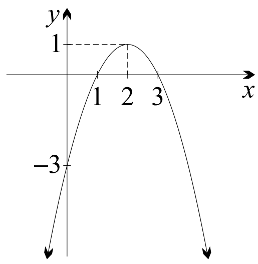
**3.** Sketch each polynomial without the aid of calculus and hence sketch its reciprocal.

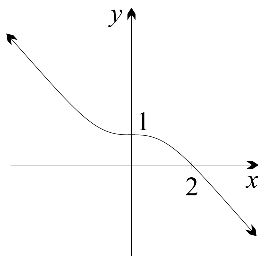
(a) $y = 1 - x^2$ (b) $y = x^3 - x$ (Note that $y = 1$ at $x \doteqdot 1{\cdot}3$.)

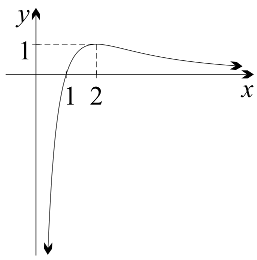
**4.** Graph $y = f(x)$ and $y = \dfrac{1}{f(x)}$ for each of the following functions. Take care to show all points where $|f(x)| = 1$.

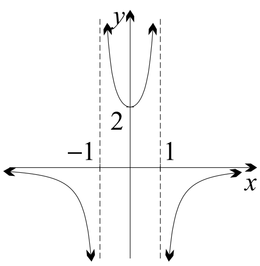
(a) $f(x) = e^x$ (b) $f(x) = \log x$ (c) $f(x) = \cos x$ (d) $f(x) = \tan x$

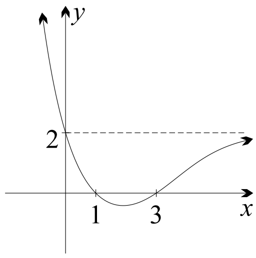
**5.** Sketch the reciprocal of each function shown.

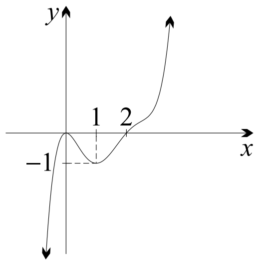
(a)



(b)



(c)



(d)



(e)



(f)



**DEVELOPMENT**

**6.** (a) Graph $y = \dfrac{2+x}{x}$ by first noting that $y = 1 + \dfrac{2}{x}$.

(b) Hence graph $y = \dfrac{x}{2+x}$.

**7.** (a) Graph $y = \dfrac{(x^2-1)}{x^2+1}$ by first noting that $y = 1 - \dfrac{2}{x^2+1}$.

(b) Hence graph $y = \dfrac{x^2+1}{x^2-1}$.

**8.** Consider the function $f(x) = \frac{1}{2}\left(3x - x^3\right)$.

(a) (i) Show that $f(x)$ is odd.

(ii) Find the $x$-intercepts.

(iii) Show that $f(1) = 1$. Hence solve $f(x) = 1$.

(iv) Use the theory of polynomials to explain why there is a stationary point at $(1, 1)$.

(v) Graph $y = f(x)$, showing all relevant features.

(b) Hence graph $y = \dfrac{1}{f(x)}$.

**9.** Follow similar steps to the previous Question to sketch $y = \dfrac{6}{x^3 - 7x}$.

**10.** If the graph of $y = f(x)$ has a vertical asymptote at $x = a$, then the graph of the reciprocal approaches an $x$-intercept at $x = a$. Further, the nature of the asymptote determines the behaviour of the reciprocal, as the following two examples demonstrates. In each case graph $y = g(x)$ where $g(x) = \dfrac{1}{f(x)}$, paying particular attention to the shape near $x = 2$.

(a) $f(x) = \dfrac{1}{(x-2)^2}$.

(b) $f(x) = \dfrac{1}{(x-2)^3}$.

**11.** Prove that odd and even symmetry is preserved for $y = \dfrac{1}{f(x)}$.

**12.** (a) Given that $g(x) = \dfrac{1}{f(x)}$, use the chain rule to differentiate $g(x)$ and thus explain why the sign of $g'$ is the opposite of the sign of $f'$.

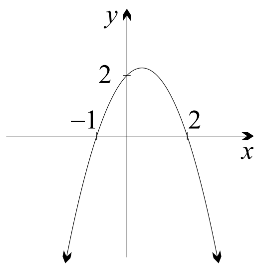
(b) Hence explain why the nature of stationary points is reversed for $y = g(x)$.

**EXTENSION**

**13.** Use differentiation from first principles to show that if $g = \dfrac{1}{f}$ then $g' = -\dfrac{f'}{f^2}$.

# 8E More Reflections

This section examines the seven cases of reflections in the axes that can occur when the absolute value function is applied. In some cases the result is a symmetric graph. As a means of comparison, the same example function, $f(x) = 2 + x - x^2$, is used throughout the theory and is graphed on the right. Other reflections and symmetries may be obtained by combining this section with previous work. Some examples are given in the exercise.

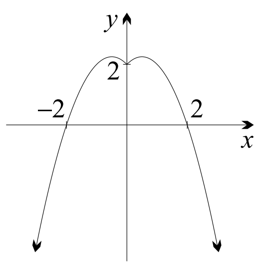



**Symmetry in the *y*-axis:** Let $g(x) = f(|x|)$ then by definition

$$\begin{aligned} g(x) &= f(|x|) \\ &= f(x) \quad \text{for } x \ge 0. \end{aligned}$$

Further $g$ is an even function since

$$\begin{aligned} g(-x) &= f(|-x|) \\ &= f(|x|) \\ &= g(x). \end{aligned}$$



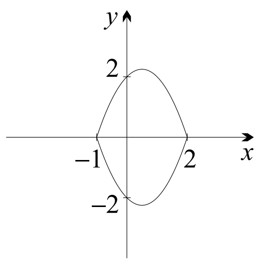
Thus the graph of $y = f(|x|)$ is that part of $y = f(x)$ for which $x \geq 0$, plus its reflection in the $y$-axis. The graph above shows the situation.

| | |
|---|---|
| **17** | SYMMETRY IN THE $Y$-AXIS: The graph of $y = f(|x|)$ is that part of $y = f(x)$ for which $x \geq 0$, plus its reflection in the $y$-axis. |

**Symmetry in the *x*-axis:** Consider the relation $|y| = f(x)$. Clearly $|y|$ cannot be negative, thus the domain of $f(x)$ is restricted so that $f(x) \geq 0$. Then, by the definition of the absolute value function

$$y = f(x) \quad \text{or} \quad -y = f(x)$$

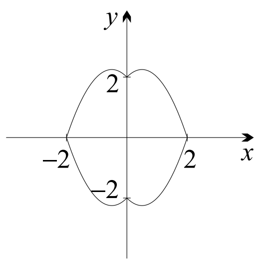
Thus the result is that part of $y = f(x)$ which lies above the $x$-axis, plus its reflection in the $x$-axis, as shown on the right for $f(x) = 2 + x - x^2$.




| | |
|---|---|
| **18** | SYMMETRY IN THE $X$-AXIS: The graph of $|y| = f(x)$ is that part of $y = f(x)$ for which $f(x) \geq 0$, plus its reflection in the $x$-axis. |

**Symmetry in Both Axes:** The relation $|y| = f(|x|)$ is identical to $y = f(x)$ whenever both $x$ and $y$ are positive, so begin by graphing that part of $y = f(x)$ which lies in the first quadrant.

As was seen above, the presence of $|x|$ means that the graph is symmetric in the $y$-axis, so add in the second quadrant the reflection of the first quadrant. Lastly, the presence of $|y|$ means that the graph is also symmetric in the $x$-axis, so finally in the third and fourth quadrants add the reflection of what has already been drawn. Thus the result is symmetric in both axes.

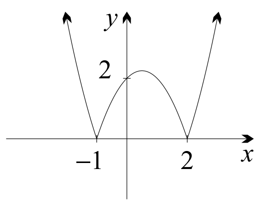



| | |
|---|---|
| **19** | SYMMETRY IN BOTH AXES: The graph of $|y| = f(|x|)$ is that part of $y = f(x)$ which lies in the first quadrant, plus its reflection in the axes. |

**Reflection in the *x*-axis:** The function $y = \big|f(x)\big|$ can be written as:

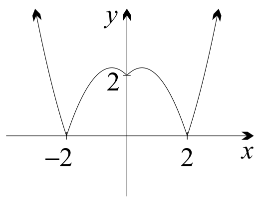
$$y = \begin{cases} f(x) & \text{for } f(x) \geq 0\,, \\ -f(x) & \text{for } f(x) < 0\,. \end{cases}$$

Thus the graph is unchanged wherever $f(x) \geq 0$ and is reflected in the $x$-axis wherever $f(x) < 0$, which is shown in the graph on the right.




| | |
|---|---|
| **20** | REFLECTION IN THE $X$-AXIS: The graph of $y = \big|f(x)\big|$ is the same as $y = f(x)$ at points where $f \geq 0$, and is the result of reflecting $y = f(x)$ in the $x$-axis wherever $f < 0$. |

**Reflection in the *x*-axis and Symmetry in the *y*-axis:** The function $y = \big|f(|x|)\big|$ is the same as $y = \big|f(x)\big|$ for $x \geq 0$. The graph of $y = \big|f(|x|)\big|$ is thus the portion of $y = \big|f(x)\big|$ to the right of the $y$-axis, plus its reflection in the $y$-axis, as shown on the right.



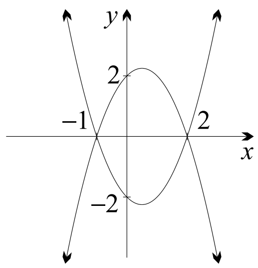

**21** REFLECTION IN THE $X$-AXIS AND SYMMETRY IN THE $Y$-AXIS: In order to graph $y = \big|f(|x|)\big|$ first graph $y = |f(x)|$ for $x \geq 0$, then add the reflection in the $y$-axis.

**Reflection Symmetry in the *x*-axis:** One definition of the absolute value function is $|x| = \sqrt{x^2}$. It is this definition which is most useful to analyse the relation $|y| = \big|f(x)\big|$. Thus begin by squaring both sides to get:

$$y^2 = f^2$$

so $\quad y = f(x)$ or $-f(x)$.

Hence the graph of $|y| = \big|f(x)\big|$ is the graph of $y = f(x)$ plus its reflection in the $x$-axis.

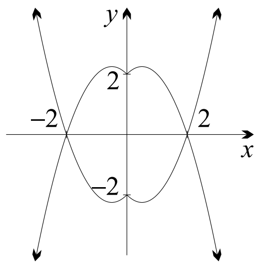



**22** REFLECTION SYMMMETRY IN THE $X$-AXIS: The graph of $|y| = \big|f(x)\big|$ is the graph of $y = f(x)$ plus its reflection in the $x$-axis.

**Symmetry in the *y*-axis and Reflection Symmetry in the *x*-axis:** There are several ways to interpret the relation $|y| = \big|f(|x|)\big|$. One approach is to use the previous case, since

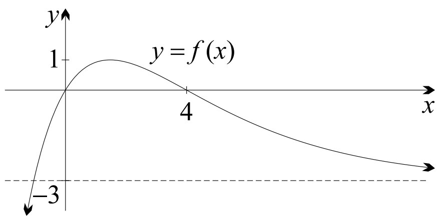
$$\big|f(|x|)\big| = \big|f(x)\big| \text{ for } x \geq 0.$$

Thus, begin by graphing $y = f(x)$ for $x \geq 0$. Then add its reflection in the $x$-axis. Next note that $|y| = \big|f(|x|)\big|$ is symmetric in the $y$-axis, so finally add in the second and third quadrants the reflection of what has already been drawn.




**23** SYMMETRY IN THE $Y$-AXIS AND REFLECTION SYMMETRY IN THE $X$-AXIS: In order to graph $|y| = \big|f(|x|)\big|$ first graph $y = f(x)$ for $x \geq 0$, then add the reflection in the $x$-axis. Finally add the reflection in the $y$-axis.

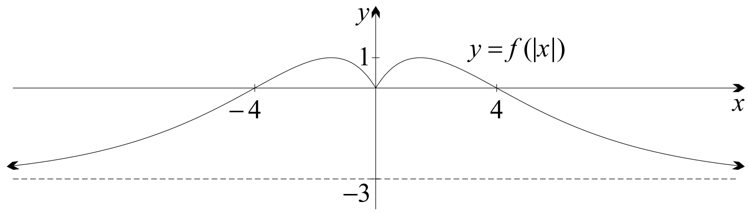
**WORKED EXERCISE:** The graph of $y = f(x)$ is shown below and has $x$-intercepts at the origin and $(4, 0)$. The maximum is $y = 1$ and $y = -3$ is an asymptote.



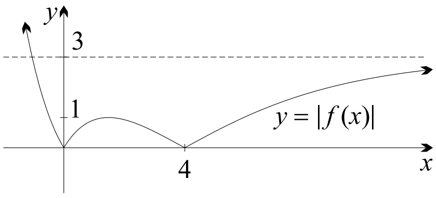

Sketch graphs of: (a) $y = f(|x|)$, (b) $y = |f(x)|$, (c) $|y| = \big|f(|x|)\big|$

**SOLUTION:**

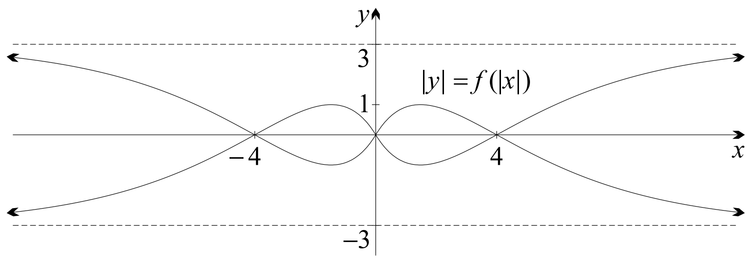
(a) The graph of $y = f(|x|)$ is the same as $y = f(x)$ for $x \geq 0$, plus its reflection in the $y$-axis, as shown below.




(b) The graph of $y = |f(x)|$ is the result of reflecting $y = f(x)$ in the $x$-axis wherever $f < 0$.



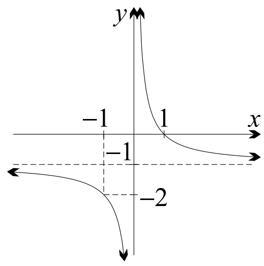

(c) The graph of $|y| = \Big|f(|x|)\Big|$ is the same as (b) for $x \geq 0$, plus its reflection both axes.



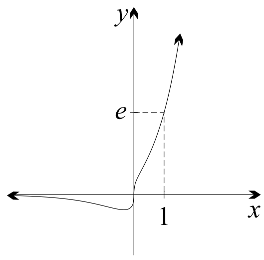

## Exercise **8E**

**1.** Use the given graph of $y = f(x)$ to sketch (i) $|y| = f(x)$, (ii) $y = |f(x)|$, (iii) $|y| = |f(x)|$.

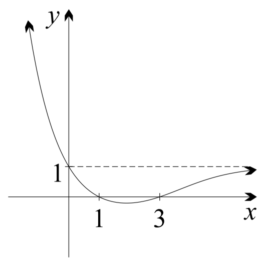
(a)



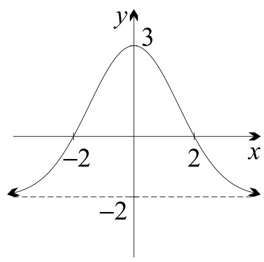

(b)




(c)



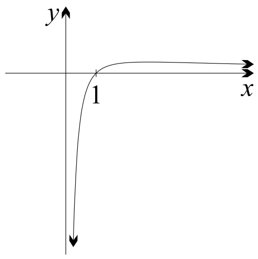

(d)



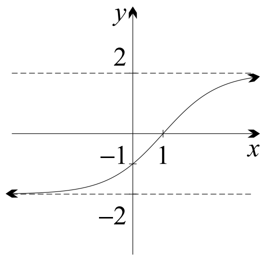

 

**2.** In each case use the given graph of $y = f(x)$ to sketch (iv) $y = f(|x|)$, (v) $|y| = f(|x|)$, (vi) $y = \big|f(|x|)\big|$ and (vii) $|y| = \big|f(|x|)\big|$.

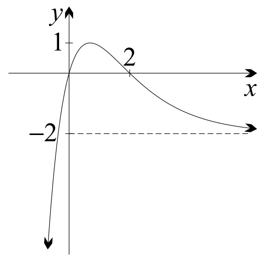
(a)




(b)




(c)




**DEVELOPMENT**

**3.** In each case sketch the graphs of (i) $y = f(x)$, (ii) $y = f(|x|)$, (iii) $|y| = f(x)$, (iv) $|y| = f(|x|)$, (v) $y = \big|f(x)\big|$, (vi) $y = \big|f(|x|)\big|$, (vii) $|y| = \big|f(x)\big|$, (viii) $|y| = \big|f(|x|)\big|$.

(a) $f(x) = 2x - x^2$ (b) $f(x) = x^3 - 3x$

**4.** Repeat Question 3 for the following functions:

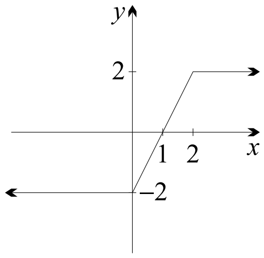
(a) $f(x) = \log x$ (b) $f(x) = 1 - \dfrac{1}{x}$ (c) $f(x) = \sin x$

**5.** In each case sketch the graphs of (i) $y = f(x) + \big|f(x)\big|$, and (ii) $y = f(x) - \big|f(x)\big|$.

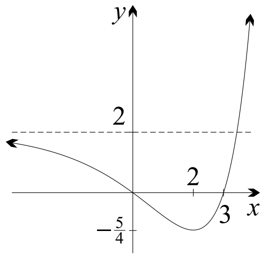
(a) $f(x) = x^2 - 1$ (b) $f(x) = 1 + \dfrac{1}{x}$ (c) $f(x) = e^x - 1$ (d) $f(x) = \cos x$

**6.** In each case use the given graph of $y = f(x)$ to sketch graphs of (i) $y = f(x) + \big|f(x)\big|$, and (ii) $y = f(x) - \big|f(x)\big|$.

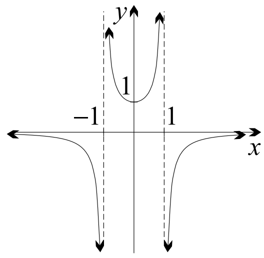
(a)




(b)




(c)




**7.** Sketch the following.

(a) $|x| + |y| = 1$ (b) $|x| - |y| = 1$ (c) $\big||x| - |y|\big| = 1$

**EXTENSION**

**8.** (a) If $f(x)$ is odd, prove that the graphs of $y = |f(x)|$ and $y = \big|f(|x|)\big|$ are identical.
(b) Which other pairs of graphs are identical whenever $f(x)$ is odd?
(c) Prove the result for each pair you find.
(d) Investigate the situation when $f(x)$ is even.
(e) Is it possible for a pair to be identical when the function is neither even nor odd?

# 8F Integer Powers

If the graph of $y = f(x)$ is known then it is possible to determine the shape of the graph of $y = \big(f(x)\big)^n$, where $n$ is an integer.

Despite the title of this section, only positive integer powers will be considered. Negative indices can be accounted for by splitting the operation into two steps, a positive integer power and a reciprocal. Thus in order to graph $y = f^{-3}$ first graph $y = f^3$ and then graph its reciprocal.

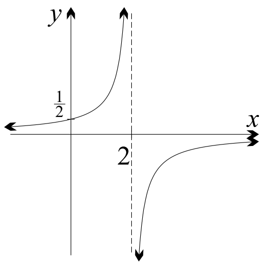
**Domain, Intercepts, Sign and Asymptotes:** If $n$ is a positive integer then the domain, $x$-intercepts and location of vertical asymptotes do not change. Clearly the $y$-intercept and ordinate of any horizontal asymptote will be affected. Note that $f^n \geq 0$ for all $x$ whenever $n$ is even, and when $n$ is odd $f^n$ has the same sign as $f$. Thus the nature of a vertical asymptote may change if $n$ is even. Consider the following example:

if $\quad f(x) = \dfrac{1}{2-x}$

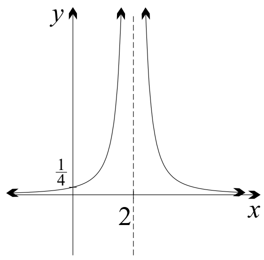
then $\quad f \to -\infty \quad$ as $\quad x \to 2^+$

but $\quad f^2 \to +\infty \quad$ as $\quad x \to 2^+$.

This change in nature is clearly evident in the graphs below.




$y = \dfrac{1}{2-x}$




$y = \dfrac{1}{(2-x)^2}$

**Symmetry:** The symmetries that result from the powers of odd and even functions are summarised in the following table, and the proofs are left to the exercise.

| $f^n$ | $n$ odd | $n$ even |
|---|---|---|
| $f$ odd | odd | even |
| $f$ even | even | even |

Other symmetries should be investigated as the need arises.

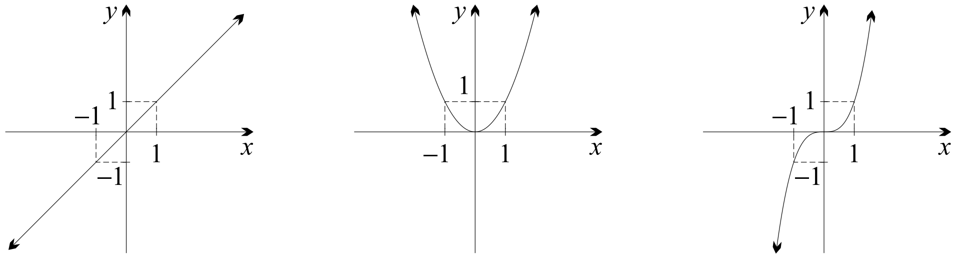
**The Calculus:** Let $g(x) = \big(f(x)\big)^n$, then

$$\frac{dg}{dx} = \frac{dg}{df} \times \frac{df}{dx} \quad \text{(by the chain rule)}$$
$$= nf^{n-1} \times f'.$$

Thus the stationary points of $f$ are also stationary points of $g$. Additionally there are stationary points located at the solutions of $f = 0$, that is at the $x$-intercepts of $y = f(x)$. The nature of a stationary point depends on $n$ and the nature of $f$ and $f'$. Various cases are presented in the exercise questions.

**24** STATIONARY POINTS:
The stationary points of $y = f(x)$ are stationary points of $y = \big(f(x)\big)^n$.
In addition, the zeros of $f(x)$ are also stationary points of $y = \big(f(x)\big)^n$.

This feature of additional stationary points at the $x$-intercepts of powers has been met before, as is demonstrated in the graphs below of $y = x$, $y = x^2$ and $y = x^3$.




**Other:** Three other observations can be made about the above graphs of $y = x$, $y = x^2$ and $y = x^3$, which apply to all graphs of positive integer powers.

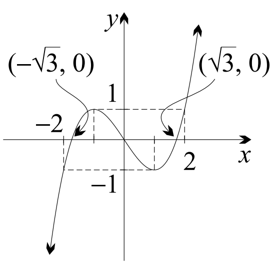
Firstly, if $|f| = 1$ then $|f^n| = 1$. Thus it is usual to plot points where $|f| = 1$ if the $x$-coordinates are easy to find. The other two features are: if $0 < |f| < 1$ then $|f^n| < |f|$, and if $|f| > 1$ then $|f^n| > |f|$. The proofs are in the exercise.

**25** RELATIVE HEIGHT: The ordinates of $y = f(x)$ and $y = \big(f(x)\big)^n$ satisfy the following.
1. If $|f| = 1$ then $|f^n| = 1$.
2. If $0 < |f| < 1$ then $|f^n| < |f|$.
3. If $|f| > 1$ then $|f^n| > |f|$.

**WORKED EXERCISE:** (a) Sketch the graph of $y = f(x)$, where $f(x) = \frac{1}{2}x(x^2 - 3)$, showing the intercepts with the axes and the stationary points.
(b) Hence sketch the graphs of (i) $y = f^2$ and (ii) $y = f^3$.

**SOLUTION:** (a) Clearly the intercepts are at the origin, $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$. It should also be clear that $f(x)$ is odd since it is a polynomial with odd powers.

| | |
|---|---|
| Now | $f'(x) = \frac{3}{2}(x^2 - 1)$ |
| and | $f''(x) = 3x$, |
| so | $f'(x) = 0 \quad \text{at} \quad x = 1$ |
| and | $f''(1) = 3$. |

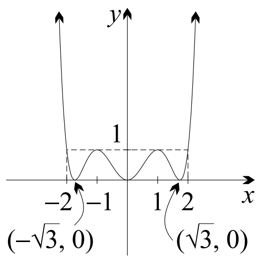



Hence there is a minimum stationary point at $(1, -1)$ and, by the odd symmetry, there is a maximum at $(-1, 1)$. Finding other points where $f(x) = 1$ leads to

$$x^3 - 3x - 2 = 0.$$

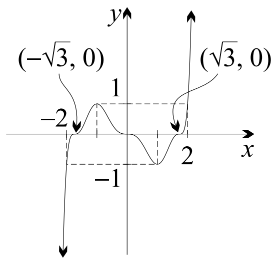
Since there is a stationary point at $(-1, 1)$, it is evident that $(x+1)^2$ is a factor of the left hand side. Thus by the product of the zeros we know that the remaining zero is $x = 2$, and $(2, 1)$ is on the curve. Again, symmetry yields the point $(-2, -1)$ on the curve, which is sketched above.

(b) (i) Clearly the resulting function is even and either positive or zero. Thus it is efficient to begin by sketching $y = f^2$ in the first quadrant and then add the reflection in the $y$-axis. There is a stationary point at $(1, 1)$, and additional stationary points at the $x$-intercepts, $(0, 0)$ and $(\sqrt{3}, 0)$. The graph also has height 1 at $(2, 1)$. Here it is.



(ii) Since $y = f$ is odd, its cube is also odd and has the same sign everywhere. Thus it is efficient to begin by sketching $y = f^3$ in the first and fourth quadrants, and then use symmetry to draw the remainder. There is a stationary point at $(1, -1)$, and additional stationary points at the $x$-intercepts, $(0, 0)$ and $(\sqrt{3}, 0)$. The graph has height 1 at $(2, 1)$ and is drawn on the right.




## Exercise 8F

**1.** Let $f(x) = x - 1$. Graph the following.

(a) $y = f(x)$ (b) $y = \big(f(x)\big)^2$ (c) $y = \big(f(x)\big)^3$ (d) $y = \big(f(x)\big)^4$

**2.** (a) Sketch $y = f(x)$, where $f(x) = \frac{1}{4}(4 - x^2)$, showing points where $|y| = 1$.

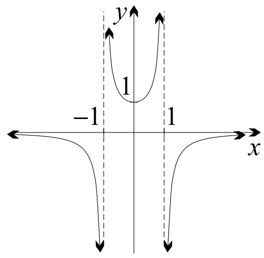
(b) Hence sketch $y = \big(f(x)\big)^2$.

**3.** Use the graph of $y = x^2 - 1$ to help sketch:

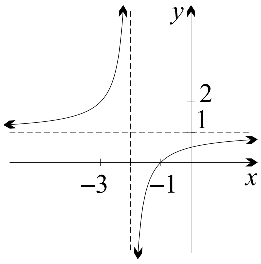
(a) $y = x^4 - 2x^2 + 1$ (b) $y = x^6 - 3x^4 + 3x^2 - 1$

**4.** In each case use the given graph of $y = f(x)$ to help sketch $y = f^2$.

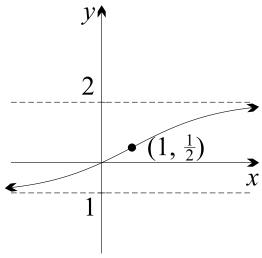
(a)



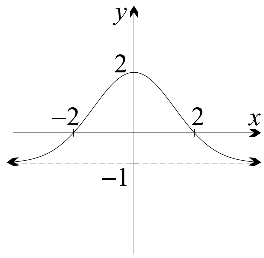

(b)



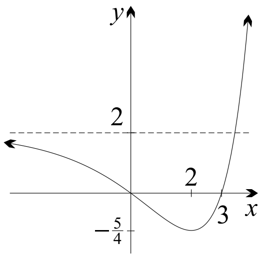

(c)



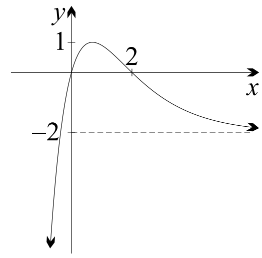

(d)




(e)




(f)




**DEVELOPMENT**

**5.** In each case draw $y = f^2$ for the given funtion. It may help to draw $y = f(x)$ first.

(a) $f(x) = \log x$ (b) $f(x) = \cos x$ (c) $f(x) = e^x - 1$ (d) $f(x) = \sqrt{x+2}$

**6.** The cubic $f(x) = \frac{1}{4}(x + 1)^2(2 - x)$ has zeros at $x = -1$ and $2$.

(a) Show that the graph of $y = f(x)$ has a maximum turning point at $(1, 1)$. Hence find the other point on the graph where $y = 1$. (HINT: Use symmetry, or use the sum and product of the roots of $f(x) = 1$. Do NOT attempt to find the point where $y = -1$.)

(b) Where is the local minimum?

(c) Graph $y = f(x)$, showing these features. (d) Hence sketch $y = \frac{1}{16}(x + 1)^4(x - 2)^2$.

**7.** The cubic $f(x) = \frac{1}{2}(x-2)^2(x+1)$ has zeros at $x = -1$ and $2$.

(a) Show that the graph of $y = f(x)$ passes through $(1, 1)$ and hence find the other points on the graph where $y = 1$. (Do NOT attempt to find the point where $y = -1$.)

(b) Show that the $y$-intercept is a local maximum. Where is the local minimum?

(c) Graph $y = f(x)$, showing these features.

(d) Hence sketch $y = f^2$.

(e) Hence determine the number of real roots of $\big(f(x)\big)^2 = 2$.

**8.** Prove the following two results, as asserted in Box 19.

(a) If $0 < |f| < 1$ then $|f^n| < |f|$.

(b) If $|f| > 1$ then $|f^n| > |f|$.

**9.** Prove the symmetry results in the table for powers of odd and even functions.

**EXTENSION**

**10.** Suppose that the function $f(x)$ is continuous and differentiable everywhere, and that $f(2) = 0$. If the graph of $y = \big(f(x)\big)^n$ is drawn, under what circumstances will the point $(2, 0)$ be:

(a) a local minimum, (b) a local maximum, (c) a horiztonal inflexion point?

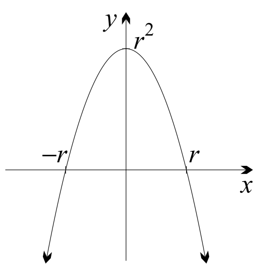
# 8G Square Roots

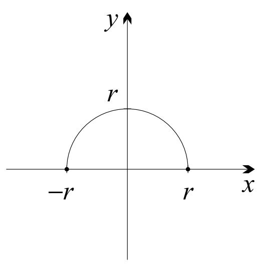
This section looks at how the graph of $y = \sqrt{f(x)}$ might be determined from the graph of $y = f(x)$. The relation $y^2 = f(x)$ is included here for completeness. Algebraicly, observe that if $y^2 = f(x)$ then

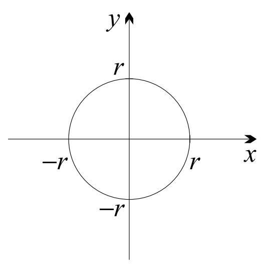
$$y = \sqrt{f(x)} \quad \text{or} \quad -\sqrt{f(x)}\,.$$

Thus the graph of $y^2 = f(x)$ is the union of the graph of $y = \sqrt{f(x)}$ with its reflection in the $x$-axis.

The case when $f(x)$ is a quadratic is significant. Let $f(x) = r^2 - x^2$, then the graph of $y = f(x)$ is a concave down parabola. The graph of $y = \sqrt{f(x)}$ is a semi-circle whilst $y^2 = f(x)$ is the entire circle, centre the origin and radius $r$.








Other fractional indices may be encountered in this course, and candidates should be able to analyse straight forward examples. Consequently, some questions on other fractional indices have been included in the exercise.

**Domain and Intercepts:** Since the square root of a negative is not real, it follows that the domain of $\sqrt{f}$ is the domain of $f$ with the additional restriction that $f \geq 0$. The same is true for $y^2 = f$, since $y^2$ also cannot be negative. This feature of the graphs is evident in the example $f(x) = r^2 - x^2$, above. The domain for both $y = \sqrt{r^2 - x^2}$ and $y^2 = r^2 - x^2$ is clearly $-r \leq x \leq r$.

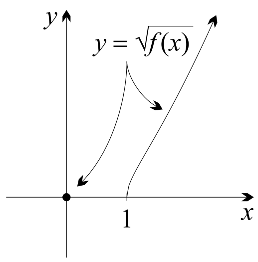
**26** DOMAIN: The domain of $y = \sqrt{f(x)}$ is the domain of $f(x)$ with the additional restriction that $f(x) \geq 0$.

The $x$-intercepts of $y = \sqrt{f}$ are the same as for $y = f(x)$. For some functions this leads to isolated points. Consider the function $f(x) = x^3 - x^2$. Solving $f(x) \geq 0$

$$x^2(x-1) \geq 0$$

so $\qquad x = 0 \text{ or } x \geq 1$.

Thus the domain of $y = \sqrt{f(x)}$ is the continous region $x \geq 1$ plus the isolated point $x = 0$. Care must be taken to clearly mark these isolated points on the graph, as shown on the right.




**Symmetry:** If $f(x)$ is an even function then so too is $y = \sqrt{f(x)}$. The proof is straight forward and is left as an exercise. If $f(x)$ is an odd function then the graph of $y = \sqrt{f(x)}$ is neither even nor odd. The proof is in the exercise.

The same observations can be made about $y^2 = f(x)$, but this graph has an additional symmetry, namely symmetry in the $x$-axis, as was stated earlier.

**The Calculus:** Let $g(x) = \sqrt{f(x)}$, then

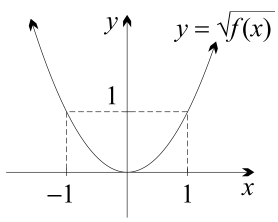
$$\frac{dg}{dx} = \frac{dg}{df} \times \frac{df}{dx} \quad \text{(by the chain rule)}$$
$$= \frac{f'}{2\sqrt{f}}.$$

Thus the stationary points of $f$ have the same $x$-coordinate as the stationary points of $g$, provided $f \neq 0$ simultaneously. If $f(x)$ has a stationary point at $x = a$ and $f(a) = 0$, the precise behaviour of $y = g(x)$ is unclear. In that case, further investigation is required, such as finding $\lim\limits_{x \to a} g'(x)$ or differentiation from first principles.

**27** STATIONARY POINTS:
(a) If $y = f(x)$ has a stationary point at $x = a$ and $f(a) \neq 0$ then $y = \sqrt{f(x)}$ also has a stationary point at $x = a$.
(b) If $y = f(x)$ has a stationary point at $x = a$ and $f(a) = 0$ then the behaviour of $y = \sqrt{f(x)}$ requires further investigation.

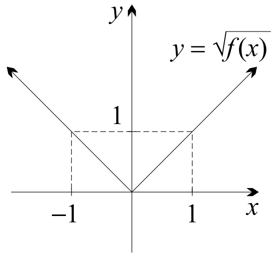
To demonstrate the problem, consider the following three short examples.

The function $f(x) = x^4$ clearly has a minimum turning point at $x = 0$ and $f(0) = 0$. The function $g(x) = \sqrt{f(x)}$ is identically $g(x) = x^2$. Thus in this case $g(x)$ also has a minimum turning point at $x = 0$.



The function $f(x) = x^2$ has a minimum turning point at $x = 0$, but the behaviour of $g(x)$ differs.

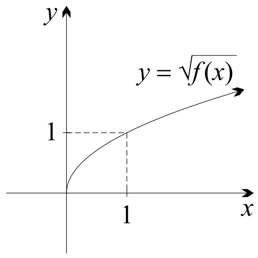
$$\begin{aligned} g(x) &= \sqrt{f(x)} \\ &= \sqrt{x^2} \\ &= |x| \quad \text{(by definition)} \end{aligned}$$




Thus $g(x)$ does indeed have a minimum, but its nature is different. It is no longer a stationary point, but instead is a critical point.

To complete the sequence, consider $f(x) = x$, despite the fact that it does not have a stationary point. In this case

$$\begin{aligned} y &= g(x) \\ &= \sqrt{x} \end{aligned}$$




or, rearranging,

$$y^2 = x \ \text{ for } y \ge 0\,.$$

Thus the graph is the upper half of the parabola $y^2 = x$. In this case the derivative function is undefined at $x = 0$. Nevertheless, from the geometry of the curve, the tangent there is known to be the vertical line $x = 0$.

Strictly speaking, the calculus of $y^2 = f(x)$ requires implicit differentiation, which is considered in detail in Section 8I. Fortunately all that is required in this case is a simple application of the chain rule. Thus

$$\frac{d}{dx}\big(y^2\big) = \frac{d}{dx} f(x)$$

yields $$\frac{d}{dy}\big(y^2\big)\frac{dy}{dx} = f'(x) \quad \text{(by the chain rule)}$$

so $$2y\,\frac{dy}{dx} = f'(x)$$

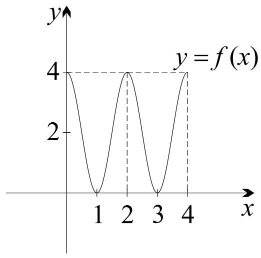
hence $$\frac{dy}{dx} = \frac{f'(x)}{2y}\,.$$

Observe that the derivative is now a function of both $x$ and $y$, and is undefined whenever $y = 0$. Thus, provided $y \ne 0$, it is clear from this that the $x$-coordinates of the stationary points of $y = f(x)$ and $y^2 = f(x)$ are the same.

**Other:** It should be clear that if $y = \sqrt{f(x)}$ then $y = 1$ when $f(x) = 1$. Thus it is usual to plot points where $f(x) = 1$ if the $x$-coordinates are easy to find. Lastly, it should be clear that if $0 < f < 1$ then $\sqrt{f} > f$, and if $f > 1$ then $\sqrt{f} < f$. These features are evident when the graph of $y = x$ is compared with the graph of $y = \sqrt{x}$ above.

**WORKED EXERCISE:** Consider the function $f(x) = 2\big(1 + \cos(\pi x)\big)$ for $0 \le x \le 4$.
(a) Sketch $y = f(x)$.
(b) (i) Use the double angle identities to rewrite $f(x)$ in terms of $\cos\left(\frac{\pi}{2}x\right)$.
(ii) Hence sketch $y = \sqrt{f(x)}$.
(iii) At which values of $x$ do the two graphs intersect?

**SOLUTION:**



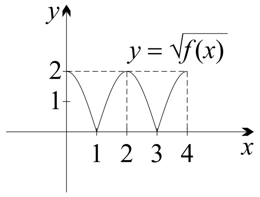

(a) This is a wave with amplitude 2, centre $y = 2$ and wavelength 2, as shown on the right.

(b) (i) Since $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ it follows that

$$\begin{aligned} f(x) &= 2\big(1 + \cos(\pi x)\big) \\ &= 4\cos^2\left(\tfrac{\pi}{2}x\right). \end{aligned}$$

(ii) Note that $\sqrt{x^2} = |x|$, hence

$$\begin{aligned} \sqrt{f(x)} &= \sqrt{4\cos^2\left(\tfrac{\pi}{2}x\right)} \\ &= 2\left|\cos\left(\tfrac{\pi}{2}x\right)\right|. \end{aligned}$$




The is a wave of amplitude 2, centre $y = 0$ and wavelength 4, with those portions below the $x$-axis reflected in the $x$-axis. Here it is sketched on the right. Note that in this case the stationary points at the $x$-intercepts of $y = f(x)$ become corners where the derivative is undefined.

(iii) The two graphs will cross when $y = 1$. The solutions of this are given by

$$\begin{aligned} &\left|\cos\left(\tfrac{\pi}{2}x\right)\right| = \tfrac{1}{2} \\ \text{so}\quad &\cos\left(\tfrac{\pi}{2}x\right) = \tfrac{1}{2} \text{ or } -\tfrac{1}{2} \\ \text{thus}\quad &\tfrac{\pi}{2}x = \tfrac{\pi}{3}, \tfrac{2\pi}{3}, \tfrac{4\pi}{3} \text{ or } \tfrac{5\pi}{3}. \\ \text{Hence}\quad &x = \tfrac{2}{3}, \tfrac{4}{3}, \tfrac{8}{3} \text{ or } \tfrac{10}{3}. \end{aligned}$$

The two graphs will also intersect when $y = 0$, which from the previous graphs occurs at $x = 1$ and $x = 3$.

## Exercise 8G

**1.** For each of the following functions, graph (i) $y = \sqrt{f(x)}$, and (ii) $y^2 = f(x)$.

(a) $f(x) = 9 - x^2$

(b) $f(x) = x + 1$

(c) $f(x) = (x - 2)^2$

(d) $f(x) = \dfrac{2}{x^2 + 1}$

**2.** (a) Let $f(x) = \cos x$, which is even. Graph $y = \sqrt{f(x)}$ and observe that it is also even.

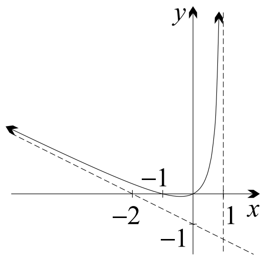
(b) Let $f(x) = \sin x$, which is odd.

(i) Graph $y = \sqrt{f(x)}$ and observe that it is neither even nor odd.

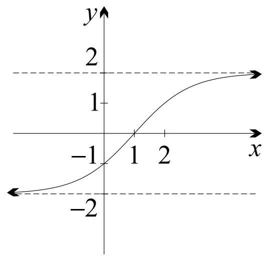
(ii) Is there any symmetry in this graph?

**3.** In each case use the graph of $y = f(x)$ to sketch (i) $y = \sqrt{f(x)}$, and (ii) $y^2 = f(x)$.

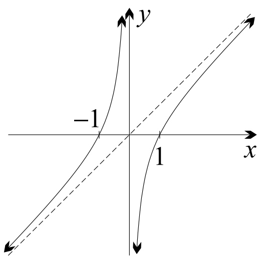
(a)




(b)



(c)

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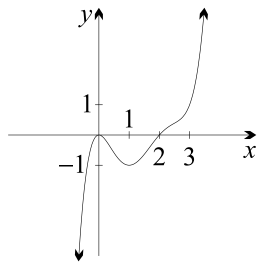



**DEVELOPMENT**

**4.** Consider the function $f(x) = 2\big(1 - \cos(\pi x)\big)$ for $0 \le x \le 4$.

(a) Sketch $y = f(x)$.

(b) (i) Use the double angle identities to rewrite $f(x)$ in terms of $\sin\left(\frac{\pi}{2}x\right)$. Hence sketch the graph of $y = \sqrt{f(x)}$.

(ii) At which values of $x$ do the two graphs intersect?

**5.** Sketch the following graphs for the function $f(x) = -x(1-x)^2$, taking care to clearly identify the isolated points.

(a) $y = \sqrt{f}$

(b) $y^2 = f$

**6.** (a) By first considering the graph of $y = x$, sketch $y = x^{\frac{1}{3}}$, taking care with the shape at the origin. [Do NOT use inverse functions as an aid.]

(b) The equation $y = x^{\frac{2}{3}}$ can be rewritten as $y = \left(x^{\frac{1}{3}}\right)^2$. Use this result and your answer to part (a) to sketch $y = x^{\frac{2}{3}}$.

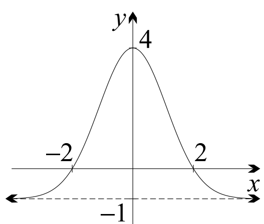
**7.** Sketch $y = (4 - x^2)^{\frac{1}{4}}$, clearly marking the intercepts with the axes.

**8.** Prove that if $f(x)$ is even then so too is $\sqrt{f(x)}$.

**9.** Suppose that $f(a) = 0$ and $f'(a) \ne 0$. Prove that the graph of $y = \sqrt{f(x)}$ has a vertical tangent at $x = a$.

**10.** Use the result in Question 9 to show that $y = \sqrt{\log x}$ has a vertical tangent at $x = 1$. Hence Sketch $y = \sqrt{\log x}$.

**11.** The graph of $y = f(x)$ is on the right. There are $x$-intercepts at $x = 2$ and $-2$, a $y$-intercept at $y = 4$, and $y = -1$ is a horizontal asymptote.




(a) Sketch $y = \sqrt{f}$.

(b) Hence sketch $y = \dfrac{1}{\sqrt{f}}$.

**12.** This question further demonstrates that the conics are intrinsicly linked. Consider the function $f(x) = r^2 - x^2$.

(a) It was noted in the text that the graph of $y^2 = f(x)$ is a circle with centre the origin and radius $r$. Show that the graph of $y^2 = -f(x)$ is a rectangular hyperbola with auxilliary circle radius $r$.

(b) The graph of $y = \sqrt{f(x)}$ is the upper semicircle. What is the graph of $y = \sqrt{-f(x)}$?

(c) Now consider the graphs of $y^2 = f(x)$ and $y^2 = -f(x)$ when $r = 0$.

(i) Draw the two graphs.

(ii) Explain goemetricly how these conics are obtained from the intersection of a plane with a cone which has a vertical axis and a semivertical angle of 45°.

EXTENSION

**13.** Under what circumstances will the graphs of $y = \sqrt{f(x)}$ and $y^2 = f(x)$ be identical?

**14.** Prove that if $f(x)$ is odd then the graph of $y = \sqrt{f(x)}$ is neither even nor odd.

**15.** The curve with equation $(x-1)(x^2+y^2) = x^2$ is one example of a type of curve called the Conchoid of De Sluze.

(a) Show that the equation may be rewritten as $y^2 = \dfrac{x^2(2-x)}{x-1}$.

(b) By noting that squares cannot be negative and denominators cannot be zero, find the domain of this curve.

(c) Graph $y = \dfrac{x^2(2-x)}{x-1}$ for this domain.

(d) Hence sketch $(x-1)(x^2+y^2) = x^2$.

# 8H Composition of Functions

A function which is built up from simpler functions by applying one followed by another is called a *composition of functions*. Thus the function $h(x) = \log(\sin e^x)$ is a composition of functions. It is the result of applying the exponential function followed by the sine function and finally the logarithmic function. The focus of this section is on compositions of just two functions, that is

$$h(x) = g\big(f(x)\big),$$

for some functions $f(x)$ and $g(x)$. For example, the composition of the square root function $g(x) = \sqrt{x}$ and the quadratic function $f(x) = 4 - x^2$ yields

$$\begin{aligned} h(x) &= g\big(f(x)\big) \\ &= g(4-x^2) \\ &= \sqrt{4-x^2}, \end{aligned}$$

which is a semi-circle with radius 2 and centre the origin when graphed.

**Domain:** Care must be taken when determining the domain of the composition of two functions. One approach is to first write down the domain of $f(x)$ and then remove any points were $g(f)$ is undefined.

**WORKED EXERCISE:** Find the domain of $\tan\sqrt{x}$.

**SOLUTION:** The domain of $f(x) = \sqrt{x}$ is $x \ge 0$ and the domain of $g(x) = \tan x$ is $x \ne \frac{(2n+1)\pi}{2}$. Thus the domain of $g\big(f(x)\big)$ is:

$$x \ge 0 \quad \text{and} \quad \sqrt{x} \ne \tfrac{(2n+1)\pi}{2}$$

$$\text{or} \quad x \ge 0 \quad \text{and} \quad x \ne \left(\tfrac{(2n+1)\pi}{2}\right)^2.$$

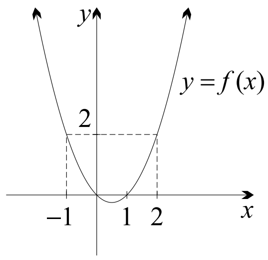
**Intercepts:** The $x$-intercepts of $h(x) = g\big(f(x)\big)$ can be found if the zeros of $g(x)$ are known and if $f(x)$ has an inverse function. Suppose that $g(a) = 0$, then $x$ is a zero of $h(x)$ provided

$$f(x) = a$$

$$\text{or} \quad x = f^{-1}(a).$$

Alternatively, if the graph of $y = f(x)$ is known then add the line $y = a$ and read off the $x$-coordinate at any point of intersection.

**WORKED EXERCISE:** (a) State the zeros of $g(x) = (x-2)^2(x+2)$.
(b) Graph $y = f(x)$ where $f(x) = x(x-1)$.
(c) Hence determine the zeros of $y = g\big(f(x)\big)$.




**SOLUTION:** (a) Clearly the zeros are at $x = 2$ or $-2$.
(b) The parabola is shown on the right.
(c) From the graph it is clear that $f(x) = -2$ has no solution and that $f(x) = 2$ has two solutions, namely $x = -1$ or $2$. These are the zeros of $y = g\big(f(x)\big)$.

**Symmetry:** The symmetries that result from compositions of odd and even functions are summarised in the following table, and the proofs are left to the exercise.

| $g\big(f(x)\big)$ | $g$ odd | $g$ even |
|---|---|---|
| $f$ odd | odd | even |
| $f$ even | even | even |

The careful reader will note that the entries in this table are the same as for integer powers of functions encountered earlier in this chapter, which is hardly surprising since that topic is just a particular example of composite functions.

**The Calculus:** Finally, differentiation of $h(x)$ yields

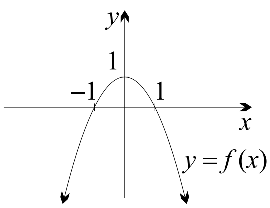
$$\begin{aligned} h'(x) &= \frac{d}{dx}\Big(g(f)\Big) \\ &= \frac{dg}{df} \times \frac{df}{dx} \quad \text{(by the chain rule)} \end{aligned}$$

So the stationary points of $f(x)$ are stationary points of $h(x)$ provided $\dfrac{dg}{df}$ is defined at those points. Extra stationary points may be introduced at the zeros of $\dfrac{dg}{df}$, again as was noted in the section on integer powers of functions. Nevertheless, in most examples encountered in this course the calculus is not required.

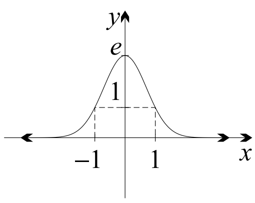
**WORKED EXERCISE:** (a) Sketch the graph of $y = f(x)$ where $f(x) = 1 - x^2$
(b) Hence sketch $y = e^{f(x)}$ without resorting to the calculus.

**SOLUTION:**

(a)




(b)




Clearly the $y$-intercept of $y = e^{f(x)}$ is $(0, e)$, and $y > 0$ for all real $x$. Since $f(x)$ is even, it follows that $e^f$ is also even. Hence it is only necessary to determine the features of the graph for $x \geq 0$ and then reflect these in the $y$-axis. If $f(x) = 0$ then $e^f = 1$, hence the point $(1, 1)$ lies on the graph. As $x \to \infty$, $f(x) \to -\infty$ and hence $e^f \to 0^+$. Thus the $x$-axis is an asymptote. The graph is shown above.

## Exercise **8H**

**1.** In each case, use a graph of $y = f(x)$ to help sketch the given composite function. The use of calculus is not required.

(a) $y = e^f$, where $f(x) = 2x - x^2$

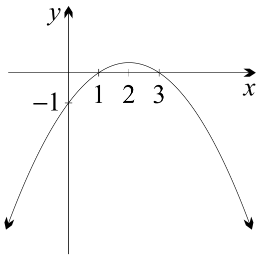
(b) $y = e^f$, where $f(x) = \cos \pi x$

(c) $y = e^f$, where $f(x) = \dfrac{1}{x}$

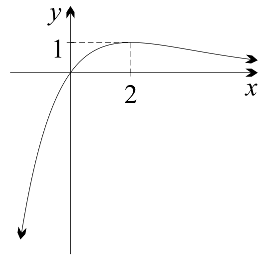
(d) $y = \log f$, where $f(x) = e - x^2$

**2.** In each case, use the given graph of $y = f(x)$ to help sketch $y = e^f$.

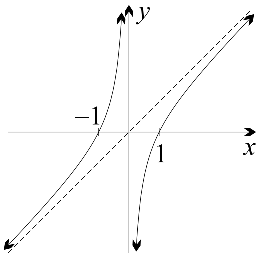
(a)



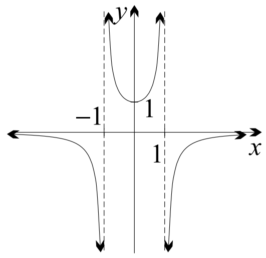

(b)




(c)

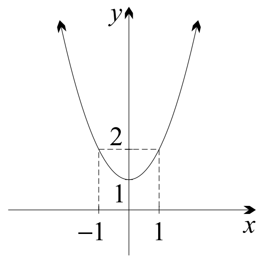



(d)

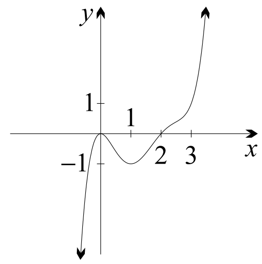



**3.** In each case, use the given graph of $y = f(x)$ to help sketch $y = \log f$.

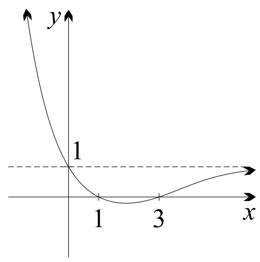
(a)



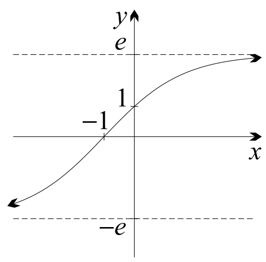

(b)




(c)




(d)




**DEVELOPMENT**

**4.** Carefully sketch the following composite functions. The use of calculus is not required.

(a) $y = \cos(2^x \times \frac{\pi}{2})$

(b) $y = \tan^{-1}\left(\dfrac{1}{x}\right)$

(c) $y = \log(\sin x)$

(d) $y = \sin\left(\dfrac{\pi}{x}\right)$

**5.** Sketch the following composite trigonometric functions.
(a) $y = \sin\left(\sin^{-1} x\right)$ (c) $y = \cos\left(\sin^{-1} x\right)$ (e) $y = \sin\left(\cos^{-1} x\right)$
(b) $y = \sin^{-1}(\sin x)$ (d) $y = \sin^{-1}(\cos x)$ (f) $y = \cos^{-1}(\sin x)$

**6.** (a) Carefully graph $y = \log\left(x^2 + \frac{3}{2}x\right)$.
(b) Also graph $y = \log(x + \frac{3}{2}) + \log x$, and hence show that the two functions are not equivalent, unless the domain in part (a) is restricted.

**7.** (a) Graph $y = e^{\tan x}$, clearly showing what happens at odd multiples of $\frac{\pi}{2}$.
(b) Sketch $y = e^{(1-x^2)^{-1}}$, clearly indicating the behaviour at $x = 1$ and $x = -1$.

**8.** Prove the results in the table for compositions of odd and even functions.

**9.** Sketch the following where $f(x) = \dfrac{2x}{1+x^2}$. It may be useful to sketch $y = f(x)$ first.
(a) $y = \log f$ (b) $y = e^f$ (c) $y = \tan^{-1} f$ (d) $y = \sin^{-1} f$

**EXTENSION**

**10.** (a) In Question 7(a), the graph appears to be horizontal in the limit as $x \to \left(\frac{\pi}{2}\right)^+$. Prove that this is the case.
(b) Prove a similar result for the graph in Question 7(b).

**11.** (a) Show that $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = 1$ and that $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x} = -1$.
(b) Hence graph $y = x + \sqrt{x^2+1}$.
(c) Show that $\log(x + \sqrt{x^2+1})$ is odd.
(d) Hence graph $\log(x + \sqrt{x^2+1})$.
(e) What is the inverse of this function?

# 8I Simple Implicit Equations

In many cases it is necessary or convenient to specify the equation of a curve implicitly, that is to say, $y$ is not given as a function of $x$. A familiar example is the equation of a circle, $x^2 + y^2 = r^2$. Like the equation of a circle, an implicit equation often represents the graph of a relation, and thus $y$ cannot be written explicitly as a function of $x$. There are exceptions however, such as the hyperbola $xy = c^2$, which can be written in function form as $y = \dfrac{c^2}{x}$.

**Implicit Differentiation:** When differentiating implicit expressions and equations, the usual rules of differentiation apply, with the added condition that $y$ is treated as an unknown function of $x$.

**WORKED EXERCISE:** Given that $y$ is a function of $x$, differentiate $x^3y$.

**SOLUTION:**
$$\begin{aligned}\frac{d}{dx}\left(x^3y\right) &= \frac{d}{dx}\left(x^3\right) \times y + x^3 \times \frac{d}{dx}(y) \quad \text{(by the product rule)}\\ &= 3x^2y + x^3\,\frac{dy}{dx}.\end{aligned}$$

Often the chain rule is required to simplify the derivatives of functions of $y$, as in the following worked exercise.

**WORKED EXERCISE:** Given that $y$ is a function of $x$, differentiate $x^2 + xy + y^2$.

**SOLUTION:**
$$\begin{aligned}&\frac{d}{dx}\left(x^2 + xy + y^2\right)\\ &= 2x + y + x \times \frac{dy}{dx} + \frac{d}{dy}\left(y^2\right)\frac{dy}{dx} \quad \text{(by the chain rule)}\\ &= 2x + y + xy' + 2yy'.\end{aligned}$$

**28** IMPLICIT DIFFERENTIATION: The usual rules of differentiation apply, with the added condition that $y$ is treated as an unknown function of $x$. The chain rule may be required to simplify the derivatives of functions of $y$.

**Tangents and Normals:** When the derivative is found implicitly the result will generally involve both $x$ and $y$. Thus if the coordinates of a point on the curve are known then the equations of the tangent and normal may be found.

**WORKED EXERCISE:** Consider the hyperbola with equation $x^2 - 7y^2 = 9$.

(a) Find the gradient of the tangent at a point $P(x, y)$ on the hyperbola.

(b) What is the equation of the tangent to this hyperbola at $A(4, 1)$?

(c) Where on the hyperbola is the gradient of the tangent undefined?

**SOLUTION:** (a) Differentiating the equation implicitly yields

$$2x - 14y\frac{dy}{dx} = 0$$

so
$$\frac{dy}{dx} = \frac{x}{7y}.$$

(b) At $A(4, 1)$, $y' = \frac{4}{7}$, hence the equation of the tangent is

$$y - 1 = \tfrac{4}{7}(x - 4)$$

or $\quad 4x - 7y - 9 = 0$.

(c) Clearly the derivative is undefined wherever $y = 0$, that is at the $x$-intercepts. This is expected since the tangents are vertical there.

**Curve Sketching:** Implicit differentiation can be used to help graph the curves of simple implicit equations. In most cases the derivative will be a fraction and, as with functions, it is usual to look for stationary points where the numerator is zero and the denominator is non-zero. The curve will have a vertical tangent at points where the numerator is non-zero and the denominator is zero. There may be other points where the tangent is horizontal or vertical, but these are not dealt with in this text.

**29** HORIZONTAL AND VERTICAL TANGENTS: If the derivative is a fraction then:

- the curve has horizontal tangents at points where the numerator is zero and the denominator is non-zero.
- the curve has vertical tangents at points where the numerator is non-zero and the denominator is zero.
- there may be other points on the curve with horizontal or vertical tangents.

**WORKED EXERCISE:** Consider the ellipse with equation $25x^2-32xy+16y^2=144$.

(a) (i) The given equation is a quadratic in $y$. Use the discriminant to determine the domain, and find the coordinates of the endpoints.

(ii) Similarly find the range and the coordinate of the endpoints.

(iii) Find the intercepts with the axes.

(b) Show that the relation is unchanged when both $x$ and $y$ are replaced by $-x$ and $-y$. What symmetry does this indicate?

(c) (i) Show that $\dfrac{dy}{dx}=\dfrac{25x-16y}{16(x-y)}$.

(ii) Hence locate the points on the ellipse with horizontal tangents.

(iii) Investigate the derivative at the end points of the domain.

(d) Sketch the curve.

**SOLUTION:**

(a) (i) Rearranging,

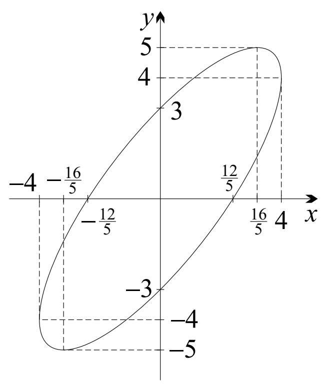
$$16y^2-32xy+(25x^2-144)=0$$

so $$\Delta_y = 32^2x^2-4\times 16\times(25x^2-144)$$
$$=576(16-x^2)\ge 0\,,$$

hence $-4\le x\le 4$.

Substitution yields the endpoints

$(-4,-4)$ and $(4,4)$.




(ii) Likewise,

$$\Delta_x = 32^2y^2-4\times 25\times(16y^2-144)$$
$$=576(25-y^2)\ge 0\,,$$

hence $-5\le y\le 5$.

Substitution yields the endpoints

$(-\frac{16}{5},-5)$ and $(\frac{16}{5},5)$.

(iii) At $x=0$

$$16y^2=144$$

so $y=3$ or $-3$.

At $y=0$

$$25x^2=144$$

so $x=\frac{12}{5}$ or $-\frac{12}{5}$.

(b) Applying the substitutions,

$$LHS = 25(-x)^2-32(-x)(-y)+16(-y)^2$$
$$=25x^2-32xy+16y^2$$

and hence the equation is unchanged.

Thus the graph is unchanged by a rotation of 180° about the origin.

(c) (i) Differentiating implicitly,

$$50x-32y-32xy'+32yy'=0$$

so $$32y'(x-y)=2(25x-16y)$$

or $$\frac{dy}{dx}=\frac{25x-16y}{16(x-y)}\,.$$

(ii) The derivative is zero when the numerator is zero (and the denominator is non-zero). This yeilds $y=\frac{25}{16}x$. Notice that the endpoints of the range lie on this line and so the tangents are horizontal at those two points. From the geometry of the ellipse, we know there are no other points on the curve with horizontal tangents.

(iii) The endpoints of the domain lie on $y = x$. Thus the denominator is zero and the derivative is undefined at these points. This is expected from the geometry of an ellipse. The tangents are vertical at the endpoints of the domain.

(d) The curve is sketched above.

**Deriving Explicit Functions:** Like the circle and other conic sections, some simple implicit equations can be solved for $y$ to obtain a set of explicit functions. These functions can then be investigated further to determine additional information about the graph. For example, in the last worked exercise we may write:

$$9x^2 + 16(x^2 - 2xy + y^2) = 144$$

so $$16(y-x)^2 = 9(16 - x^2)$$

thus $y = \frac{1}{4}\left(4x + 3\sqrt{16 - x^2}\right)$ or $y = \frac{1}{4}\left(4x - 3\sqrt{16 - x^2}\right)$.

The two functions correspond to the parts of the curve above and below $y = x$.

## Exercise **8I**

**1.** Differentiate the following, where $y$ is an unknown function of $x$.

(a) $y + x$ (d) $y^3 + 3xy$ (g) $y(2x + 3y)$
(b) $xy$ (e) $\log y$ (h) $(x + y)^3$
(c) $x^2 - y^2$ (f) $e^y$ (i) $(x^2 + y^2)^2$

**2.** Solve the following for $y$ to obtain a set of functions to describe the curve.

(a) $x^2 - y^2 = 9$ (c) $x^2 + y^2 - 2y = 0$
(b) $x^2 + y^2 = 4$ (d) $2x^2 + 2xy + y^2 = 1$

**3.** In each case, find an expression for the derivative in terms of $x$ and $y$. List any points on the curve where this derivative is undefined.

(a) $x^2 + y^2 = 36$ (c) $x^2 - 2xy + 2y^2 = 2$
(b) $x^2 - y^2 = 16$ (d) $x(x^2 + y^2) = 2y^2$

**4.** List any points on the curves in Question 4 where it is guaranteed that the tangent is horizontal, where the numerator of the derivative is zero and the denominator is non-zero.

**5.** In each case determine the gradient of the curve at the given point.

(a) $x^2 - y^2 = 9$, $(5, 4)$ (c) $x^4 = 2x^2 - 4y^2$, $(1, \frac{1}{2})$ (e) $x^2 - 2xy - y^2 = 2$, $(-3, -1)$
(b) $x^2 + 2y^2 = 9$, $(1, -2)$ (d) $x^3 + y^3 = 7$, $(-1, 2)$ (f) $x^4 - 5xy^2 + y^4 = 7$, $(2, 3)$

**6.** Find the equation of the tangent to each curve at the given point.

(a) $x^2 + 3y^2 = 12$, $(-3, 1)$ (d) $y^4 - x^4 = 2(xy - 1)$, $(-1, -1)$
(b) $2x^2 - y^2 = 1$, $(5, 7)$ (e) $2y(x^2 - y^2) = x^4 - 10$, $(2, 1)$
(c) $y^4 - 10y^2 - 6 = x^4 - x^2$, $(1, -2)$ (f) $(x^2 - 1)^2 - 4 = y^2(3 + 2y)$, $(-2, 1)$

**DEVELOPMENT**

**7.** Consider the curve with equation $x^2 + 2xy + y^5 = 4$.

(a) Find the equation of the tangent at $(-3, 1)$.
(b) Explain why the curve has no vertical tangents.
(c) Show that if the curve has a horizontal tangent at $(x, y)$ then $x^5 + x^2 + 4 = 0$.
(d) Hence determine how many points on the curve have a horizontal tangent.

**8.** The curve with equation $(x^2 + y^2 + x)^2 = x^2 + y^2$ is an example of a cardioid.

(a) Show that $\dfrac{dy}{dx} = \dfrac{x - (1 + 2x)(x + x^2 + y^2)}{2y(x + x^2 + y^2) - y}$.

(b) Hence find the equation of the tangent to the cardioid at $(0, 1)$.

**9.** Consider the ellipse with equation $x^2 + y^2 + xy = 3$.

(a) The given equation is a quadratic in $y$. Use the discriminant to determine the domain of the relation.

(b) Find the $x$-intercepts.

(c) (i) Show that the curve is symmetric in the line $y = x$.

(ii) Where does the curve cross this line?

(d) Use implicit differentiation to show that $\dfrac{dy}{dx} = -\dfrac{2x + y}{2y + x}$.

(e) Hence find the points on the curve where the tangent is horizontal.

(f) Use symmetry to locate the points where the tangent is vertical.

(g) Sketch the curve, showing all these features.

**10.** Consider the hyperbola with equation $4y(x\sqrt{3} - y) = 3$.

(a) The given equation is a quadratic in $y$. Use the discriminant to determine the domain of the relation.

(b) Are there any intercepts with the axes?

(c) Show that the relation has odd symmetry.

(d) Make $x$ the subject of the given equation, and hence determine the equations of the two asymptotes.

(e) Show that $\dfrac{dy}{dx} = \dfrac{y\sqrt{3}}{2y - x\sqrt{3}}$, and hence show that the derivative is undefined at the end-points of the domain. [In fact, the two tangents are vertical.]

(f) Are there any points on the curve where the tangent is horizontal?

(g) Sketch the curve, showing all these features.

(h) It may have been much simpler to sketch this graph by first sketching the inverse relation. Investigate this possibility.

**EXTENSION**

**11.** A simple example of a lemniscate is the curve with equation $x^4 = x^2 - y^2$.

(a) Determine the domain of the relation.

(b) Find the intercepts with the axes.

(c) Explain why the curve is symmetric in both axes.

(d) Use implicit differentiation to show that $\dfrac{dy}{dx} = \dfrac{x - 2x^3}{y}$.

(e) Hence show that in the first quadrant $\dfrac{dy}{dx} = \dfrac{1 - 2x^2}{\sqrt{1 - x^2}}$, $0 < x < 1$.

(f) Use this last result to show that in the first quadrant $\lim\limits_{x \to 0^+} y' = 1$, and $\lim\limits_{x \to 1^-} y' \to -\infty$.

(g) Find the point in the first quadrant where the tangent is horizontal.

(h) Sketch the curve, showing all these features.

**12.** The folium of Descartes has the equation $x^3 + y^3 = 3xy$.

(a) Determine any intercepts with the axes.

(b) (i) Show that the curve is symmetric in the line $y = x$.

(ii) Where does the curve cross this line?

(c) (i) Show that $\dfrac{dy}{dx} = \dfrac{x^2 - y}{x - y^2}$.

(ii) Hence find where the tangent is guaranteed to be horizontal on this curve.

(d) (i) Show that the curve can be parameterised by $x = \dfrac{3t}{1+t^3}$ and $y = \dfrac{3t^2}{1+t^3}$.

(ii) Hence show that $\dfrac{dy}{dx} = \dfrac{t(2-t^3)}{1-2t^3}$.

(iii) Using this version of the derivative, what is the gradient at the origin?

(e) (i) Applying the symmetry of part (b) to the results in parts (c) and (d), where on the curve is the tangent vertical?

(ii) What do you conclude happens at the origin?

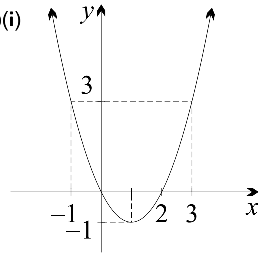
(f) (i) Using the parametric equations, show that $|x| \to \infty$ and $\frac{y}{x} \to -1$ as $t \to -1$.

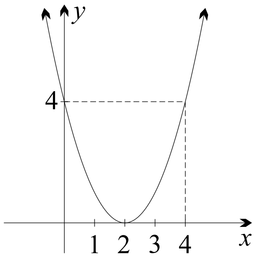
(ii) Show that $(x+y)(\frac{x}{y} - 1 + \frac{y}{x}) = 3$, and hence find the oblique asymptote.

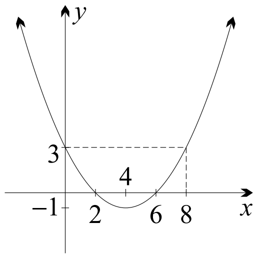
(g) Sketch the curve, showing all these features.

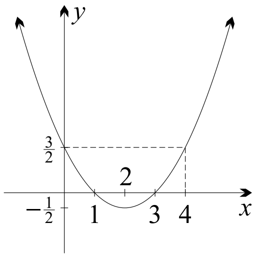
# Chapter Eight

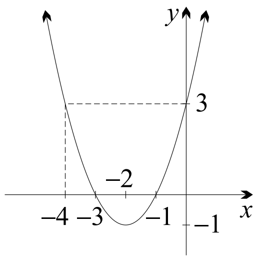
## Exercise **8A** (Page 60)

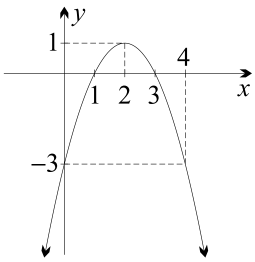
**1(a)(i)**


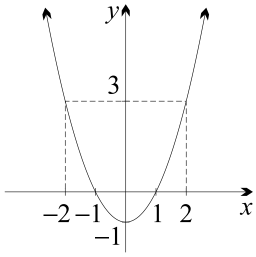

**(ii)**


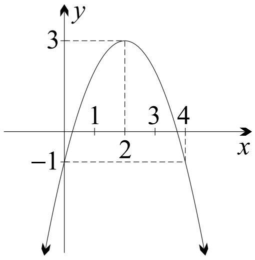

**(iii)**


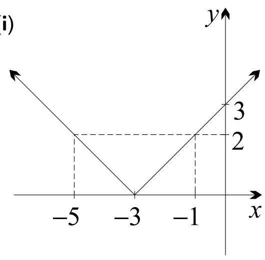

**(iv)**


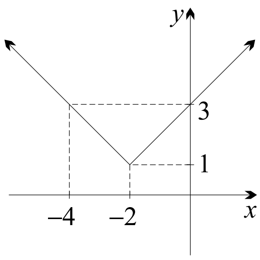

**(v)**


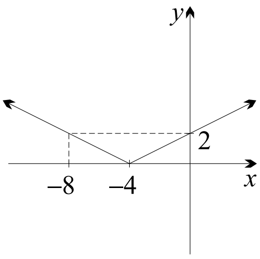

**(vi)**


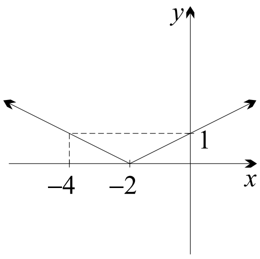

**(vii)**


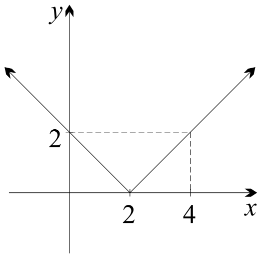

**(viii)**


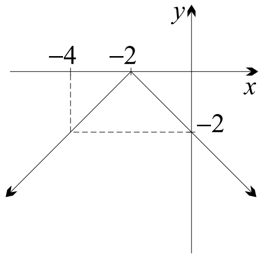

**(b)(i)**


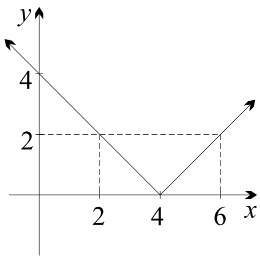

**(ii)**


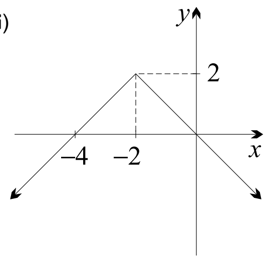

**(iii)**


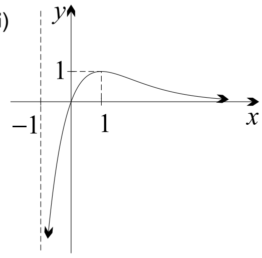

**(iv)**


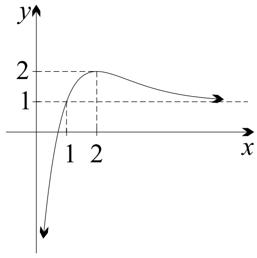

**(v)**


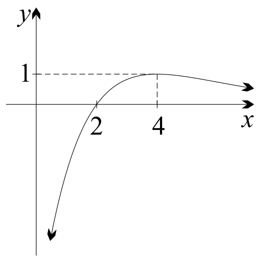

**(vi)**


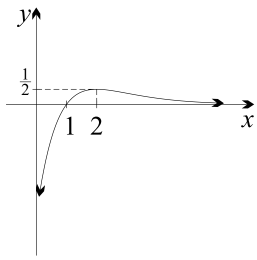

**(vii)**


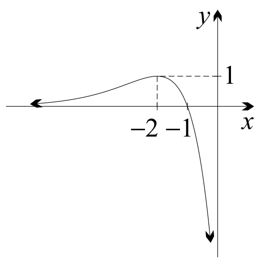

**(viii)**


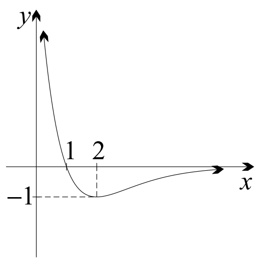

**(c)(i)**


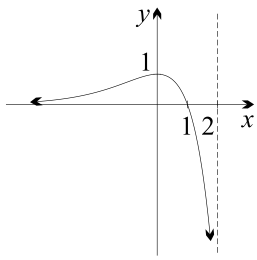

**(ii)**


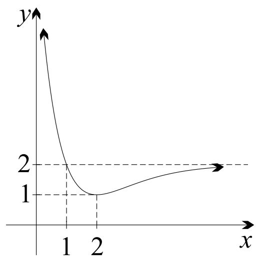

**(iii)**



**(iv)**


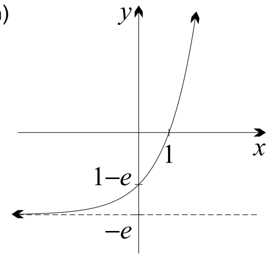

**(v)**



**(vi)**



**(vii)**


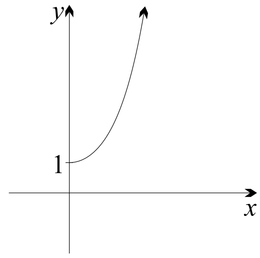

**(viii)**


**3(a)**



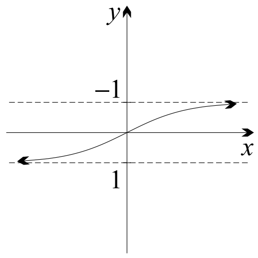

$f^{-1}(x) = e^x - e$

**(b)**



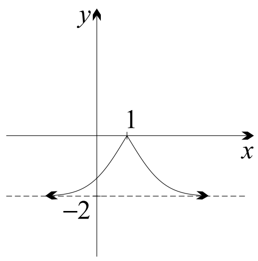

$f^{-1}(x) = \frac{1}{2}(e^x + e^{-x})$

**(c)**

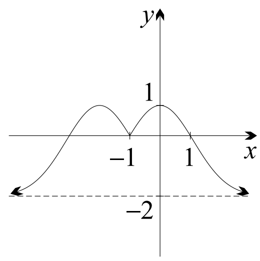



$$f^{-1}(x) = \frac{e^x - 1}{e^x + 1}$$

**4(a)**




**(b)**



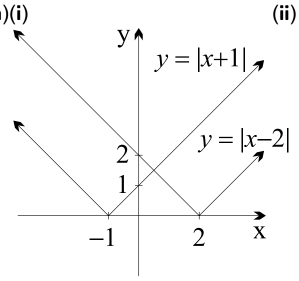

**5(a)** It could be a vertical shift of two down or a reflection in the $x$-axis. **(b)** It could be a shift left by $\frac{\pi}{2}$ or a reflection in the $x$-axis or a reflection in the $y$-axis.

**7(a)(i)** $x = n\pi$ for integer $n$.

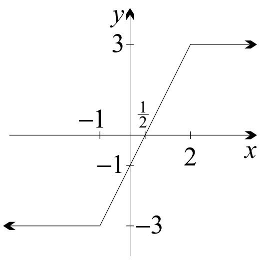
**9(b)** The converse is not true. For example, a primitive of $3x^2$ is $x^3 + 1$ which is neither even nor odd. **(c)** The converse is true in this case.

## Exercise **8B** (Page 64)

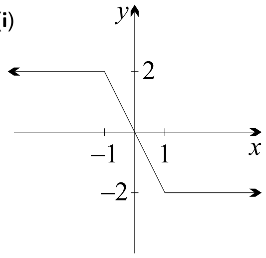
**1(a)(i)**



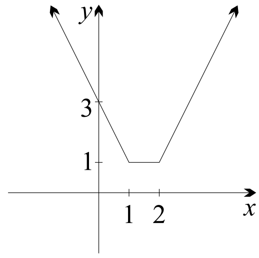

**(ii)**



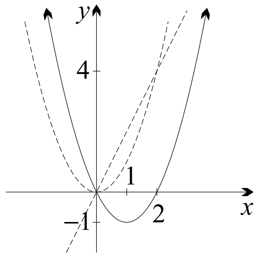

**(b)(i)**

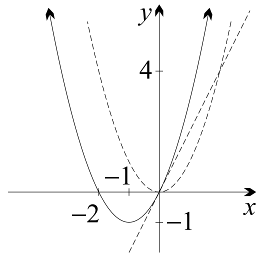



**(ii)**

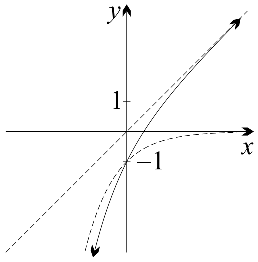


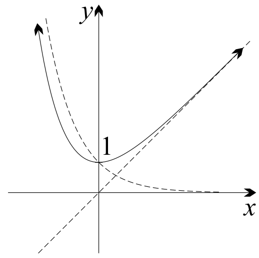

**2(a)**



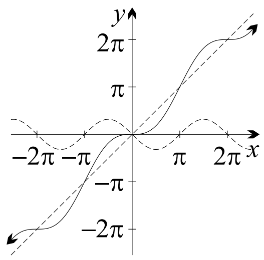



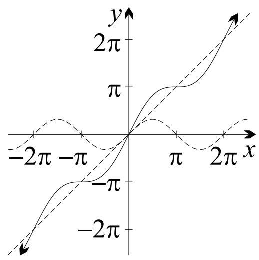

**(b)**



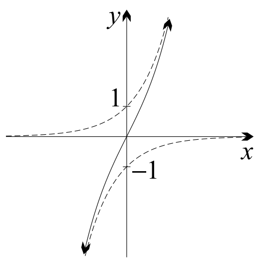



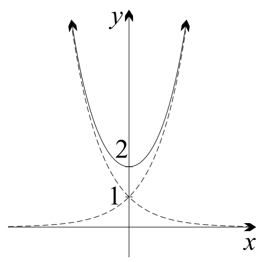

**(c)**



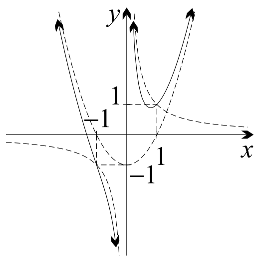




**(d)**

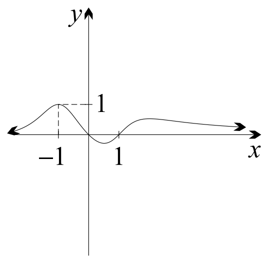


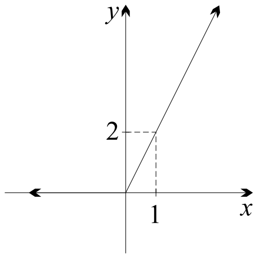

**3(a)**



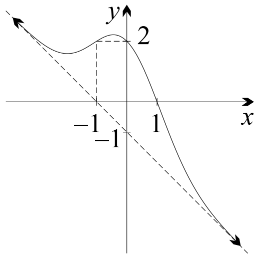

**(b)**



**(c)**



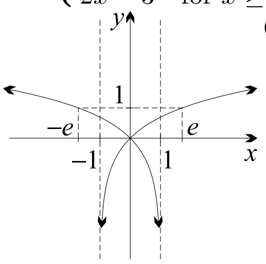

**(d)**



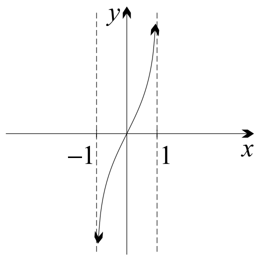

**4(b)(i)** $y = \begin{cases} 2 & \text{for } x < -1 \\ -2x & \text{for } -1 \le x < 1 \\ -2 & \text{for } x \ge 1 \end{cases}$

**(ii)** $y = \begin{cases} 3 - 2x & \text{for } x < 1 \\ 1 & \text{for } 1 \le x < 2 \\ 2x - 3 & \text{for } x \ge 2 \end{cases}$

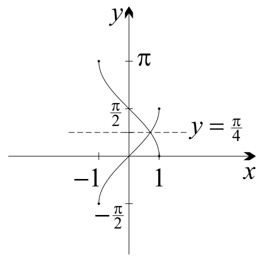
**5(a)**



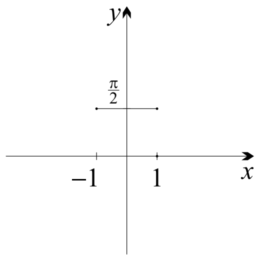

**(b)**



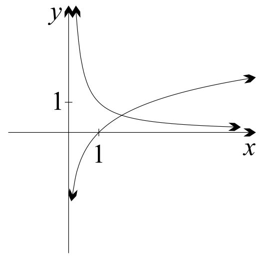

**6(a)**



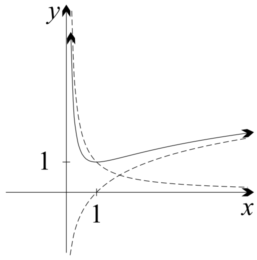

**(b)**



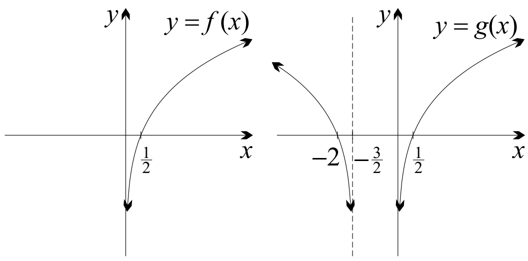

**7(a)**




**(c)**

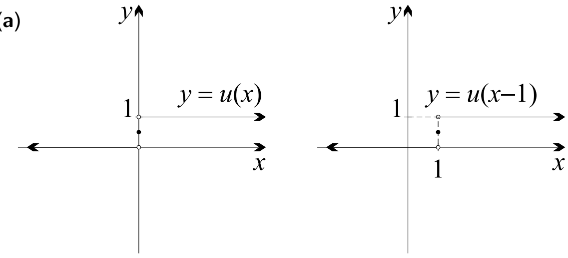



**8(b)**

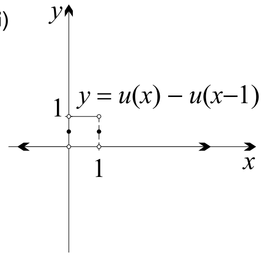



**9(a)** One possible selection is as follows.
Both odd: $f(x) = x$, $g(x) = \sin x$.
Both even: $f(x) = x^2$, $g(x) = \cos x$.
Odd and even: $f(x) = x$, $g(x) = \cos x$.
**(c)** When $g(x) = -f(x)$, $h(x) \equiv 0$ which is even.

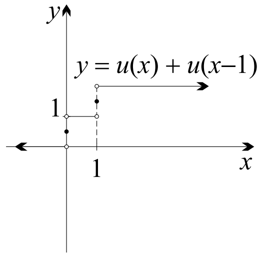
**10(a)**




**(b)(i)**



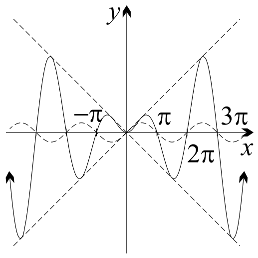

**(ii)**



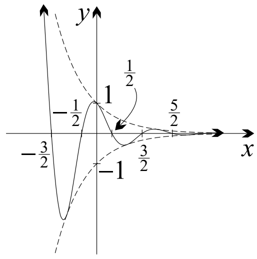

**11** 2

## Exercise **8C** (Page 68)

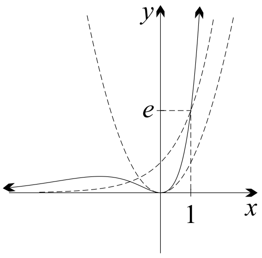
**1(a)**



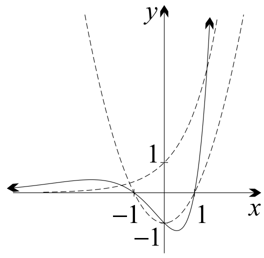

**(b)**



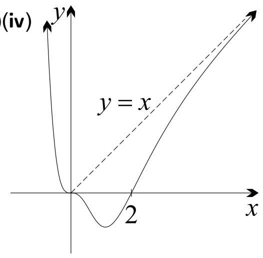

**2(a)**



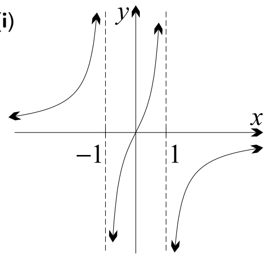

**(b)**



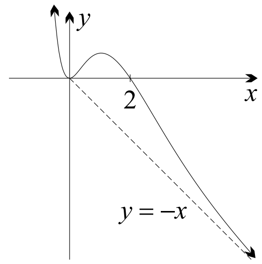

**3(a)(iv)**



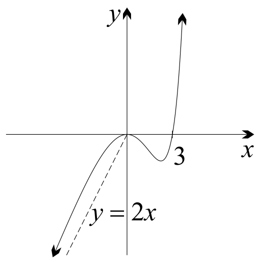

**(b)(i)**



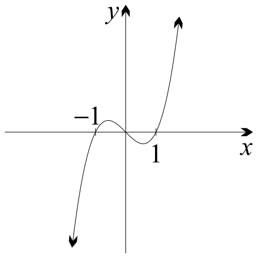

**(ii)**



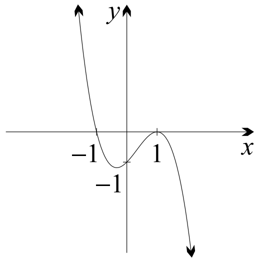

**(iii)**



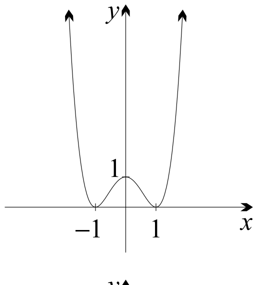
**4(a)**



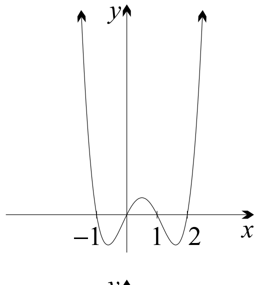

**(b)**



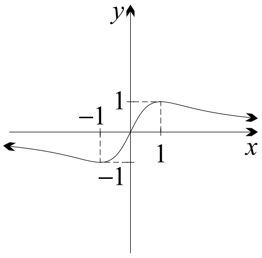

**(c)**



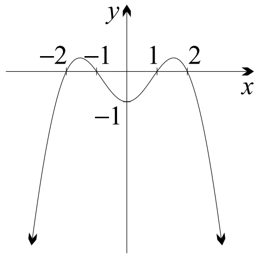

**(d)**



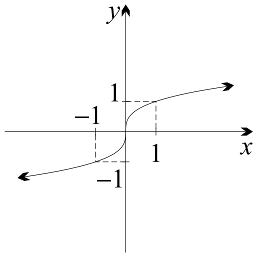

**(e)**



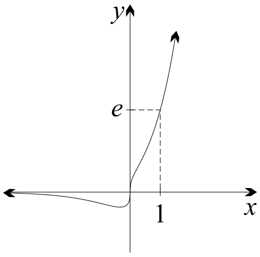

**(f)**



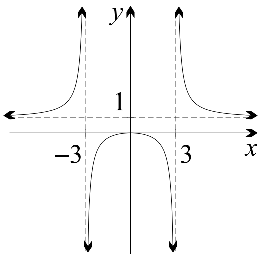

**5(a)**



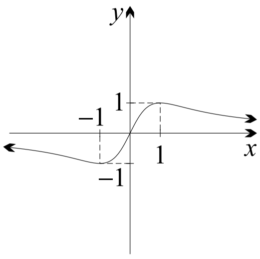

**(b)**



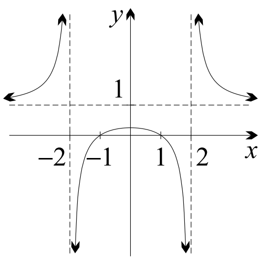

**6**




**7**




**8(a)**




**(b)** $x < -2$ or $x > 2$

**9(a)** $x = -2, -1, 1, 2$

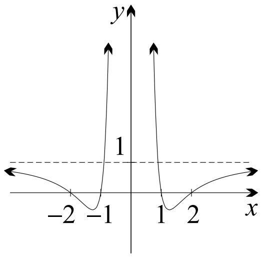
**(b)** $x = 0$

**(c)(i)** $y = 1$

**(ii)** $\left(\frac{\pm 4}{\sqrt{10}}, -\frac{9}{16}\right)$

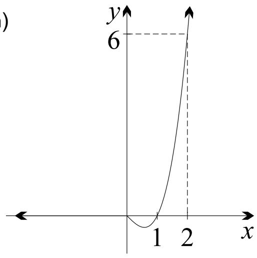
**(d)** $-\frac{9}{16} < b < 1$

**(d)**

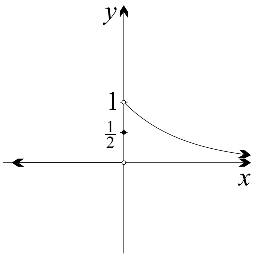



**11** $y = h(x)$ has a hole at the origin.

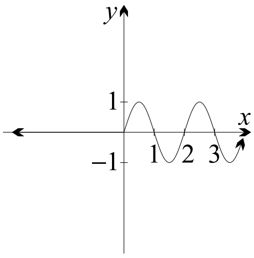
**12(a)**



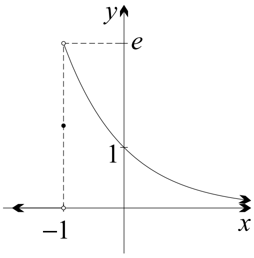

**(b)**



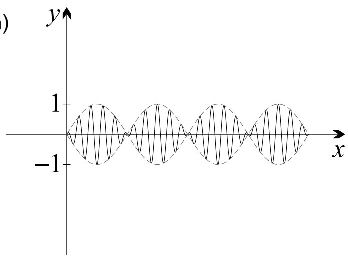

**(c)**




**(d)**



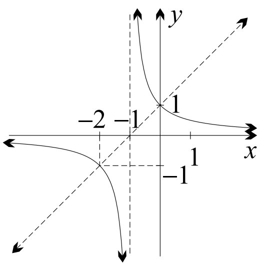

**13(a)**



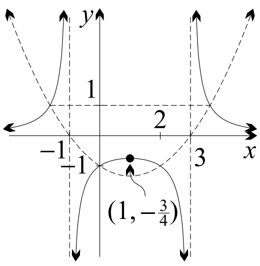

**(b)** $y = \cos 5x$

## Exercise **8D** (Page 72)

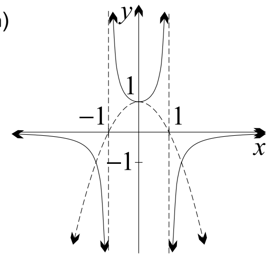
**1**



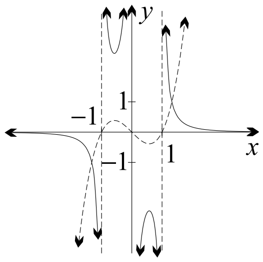

**2**



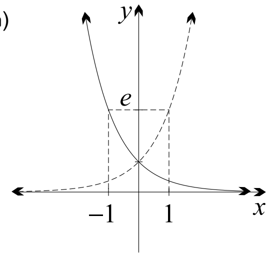
**3(a)**



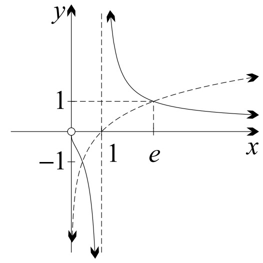

**(b)**



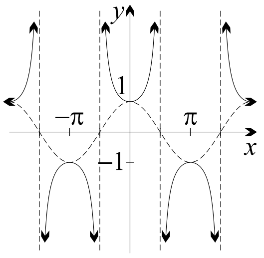

**4(a)**



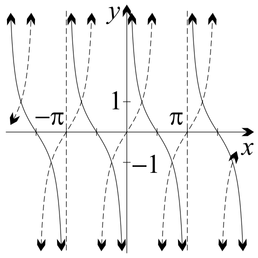

**(b)**



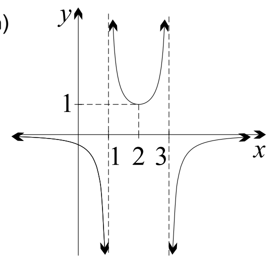

**(c)**



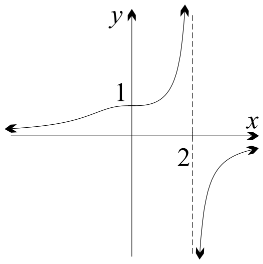

**(d)**



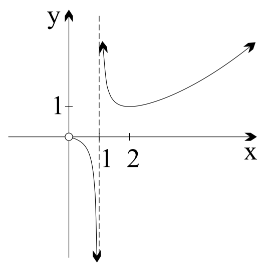

**5(a)**



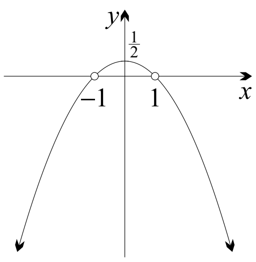

**(b)**



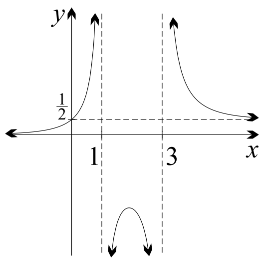

**(c)**



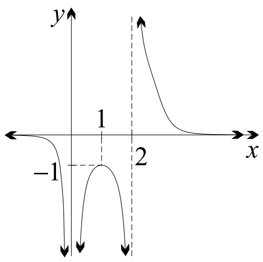

**(d)**



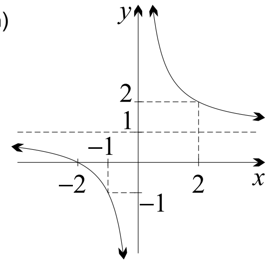

**(e)**



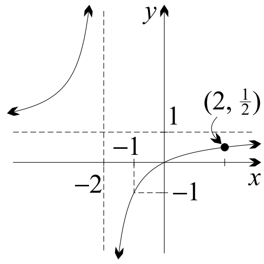

**(f)**



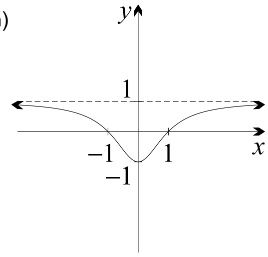

**6(a)**



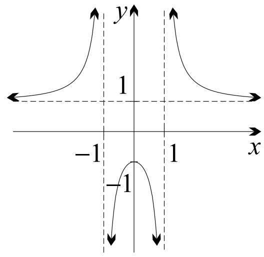

**(b)**




**7(a)**



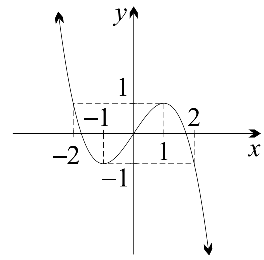

**(b)**



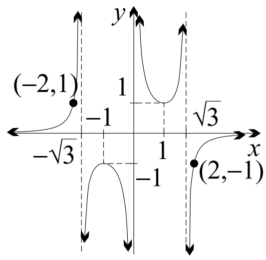

**8(a)(ii)** $x=-\sqrt{3}, 0, \sqrt{3}$ **(iii)** $x=1, 1, -2$

**(iv)** $f(x)=1$ has a double root at $x=1$.

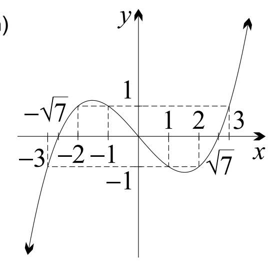
**(v)**



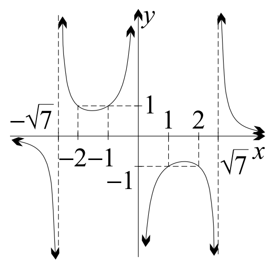

**(b)**



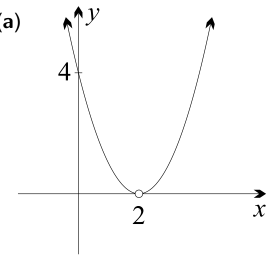

**9(a)**



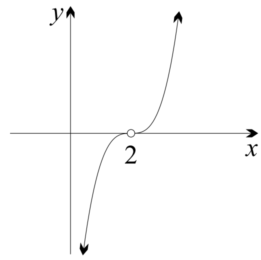

**(b)**




**10(a)**



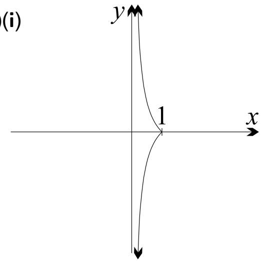

**(b)**



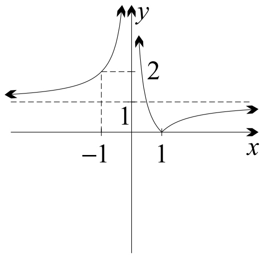

 

## Exercise **8E** (Page 77)

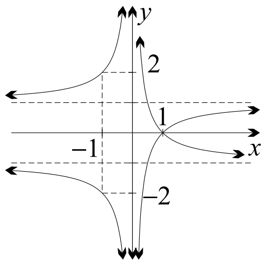
**1(a)(i)**



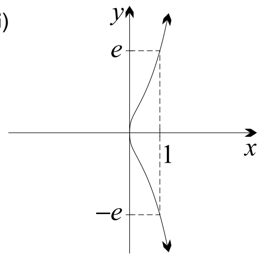

**(ii)**



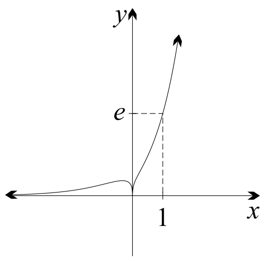

**(iii)**



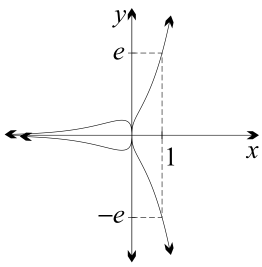

**(b)(i)**



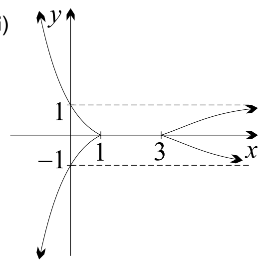

**(ii)**



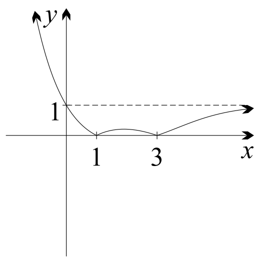

**(iii)**



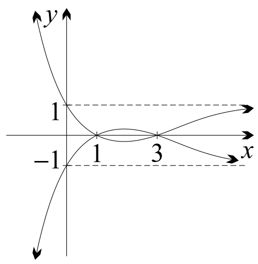

**(c)(i)**



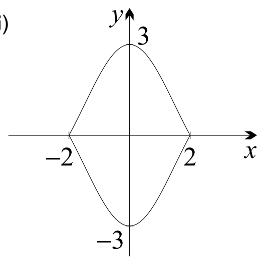

**(ii)**



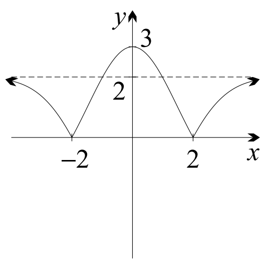

**(iii)**



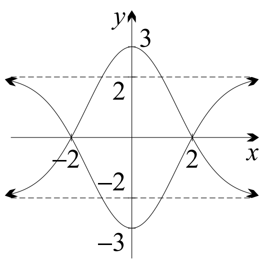

**(d)(i)**



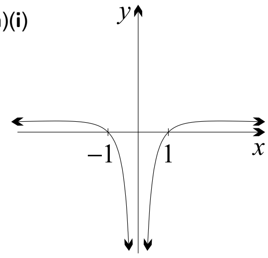

**(ii)**



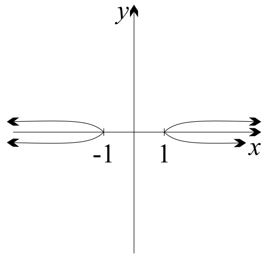

**(iii)**



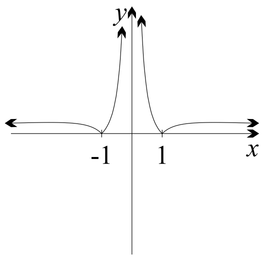

**2(a)(i)**



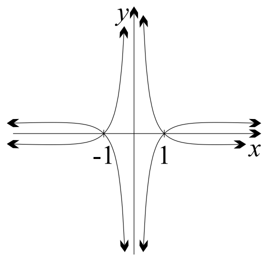

**(ii)**



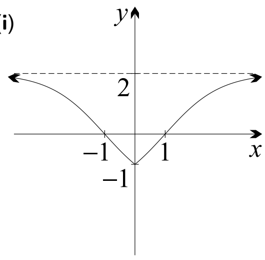

**(iii)**



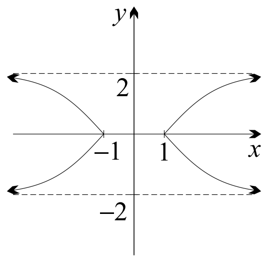

**(iv)**



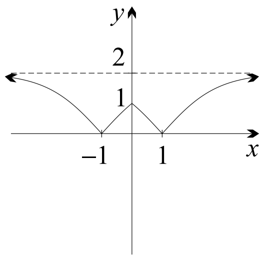

**(b)(i)**



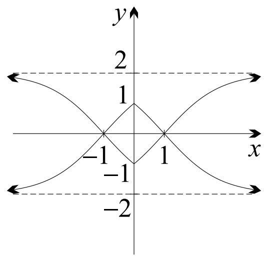

**(ii)**



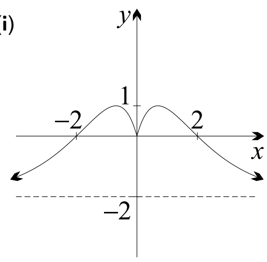

**(iii)**



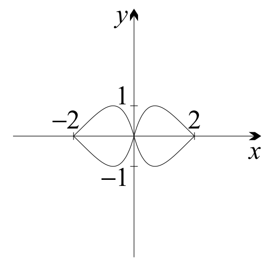

**(iv)**



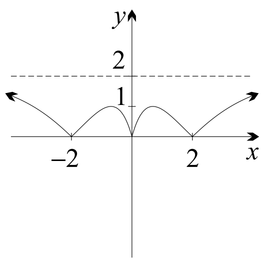

**(c)(i)**



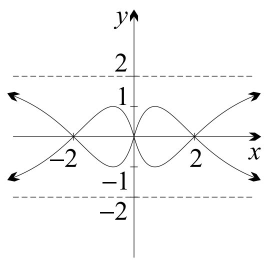

**(ii)**

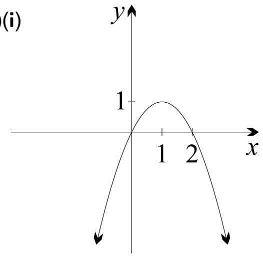


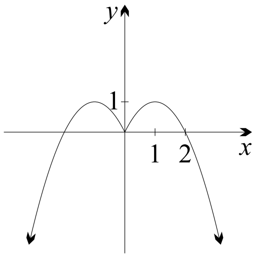

**(iii)**

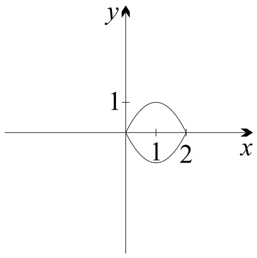


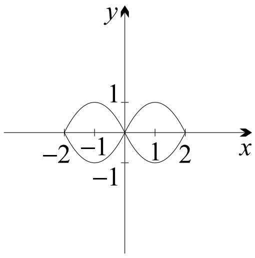

**(iv)**

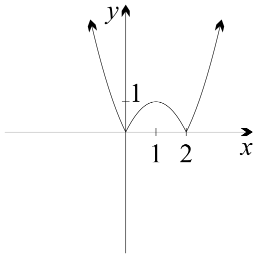


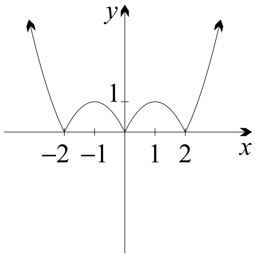
**3(a)(i)**


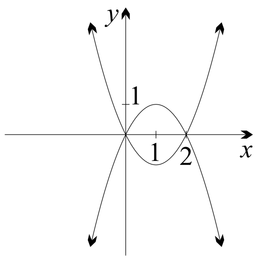

**(ii)**


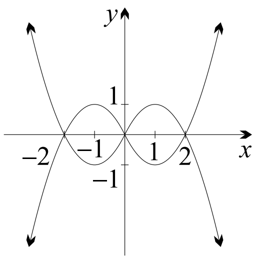

**(iii)**


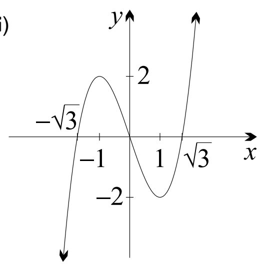

**(iv)**


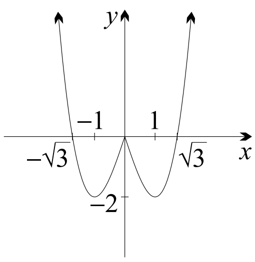

**(v)**


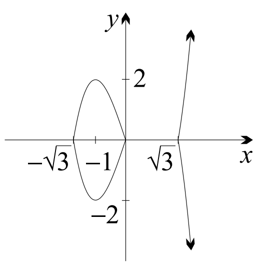

**(vi)**


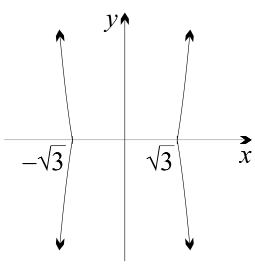

**(vii)**


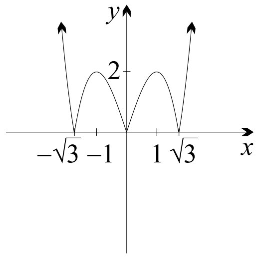

**(viii)**


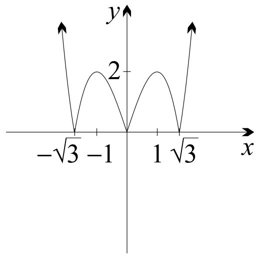

**(b)(i)**


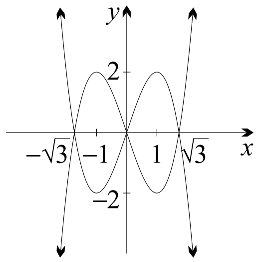

**(ii)**


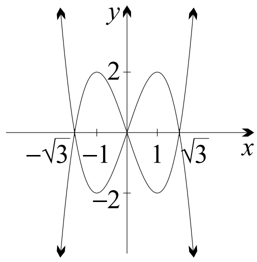

**(iii)**


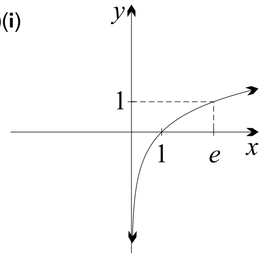

**(iv)**


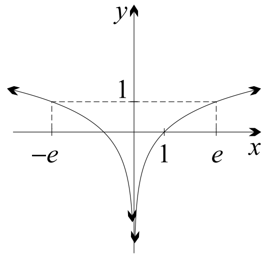

**(v)**


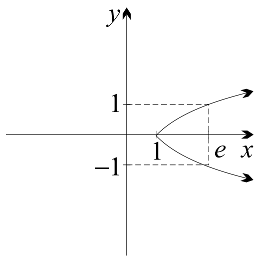

**(vi)**


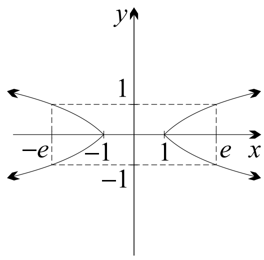

**(vii)**


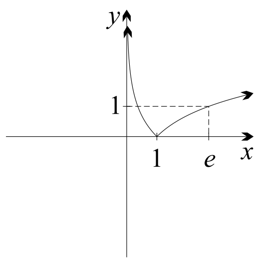

**(viii)**


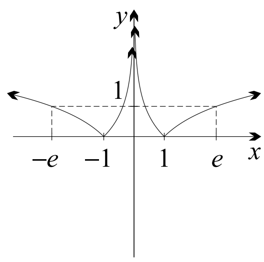

**4(a)(i)**


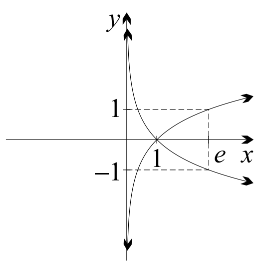

**(ii)**


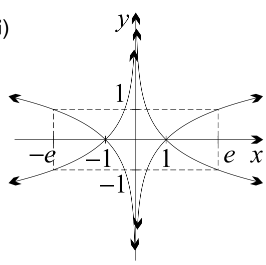

**(iii)**


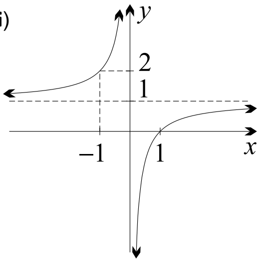

**(iv)**


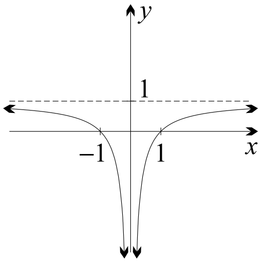

**(v)**


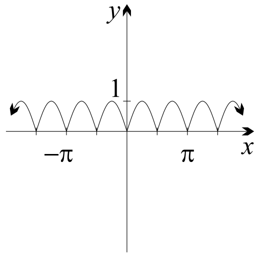

**(vi)**


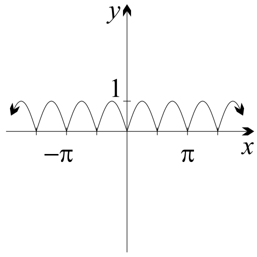

**(vii)**


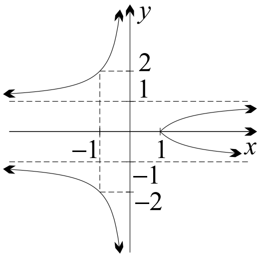

**(viii)**


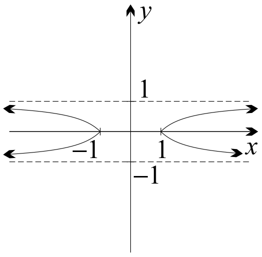
**(b)(i)**


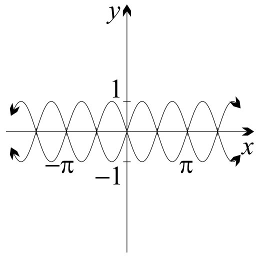

**(ii)**


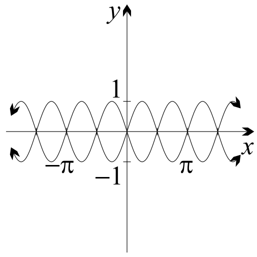

**(v)**


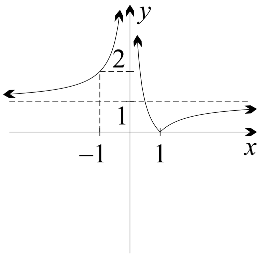

**(vi)**


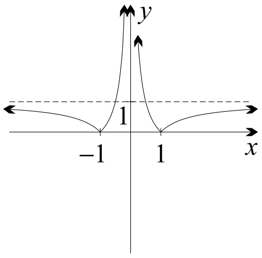

**(iii)**


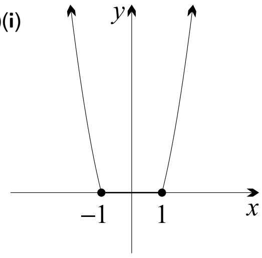

**(iv)**


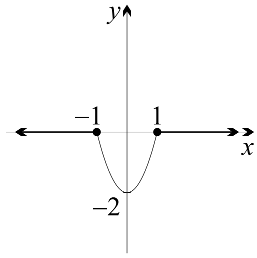

**(vii)**


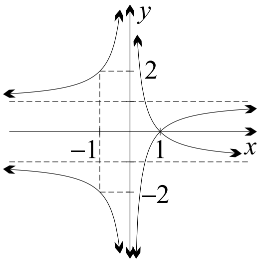

**(viii)**


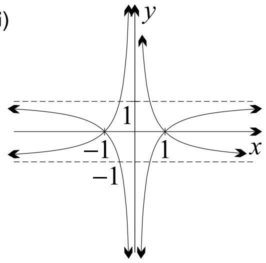

**(v)**


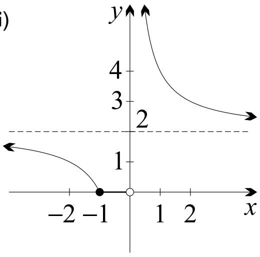

**(vi)**


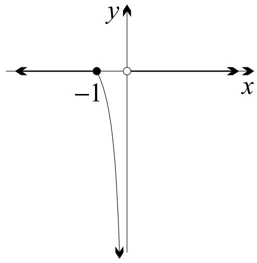

**5(a)(i)**


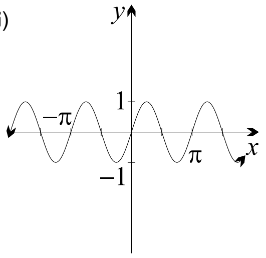

**(ii)**


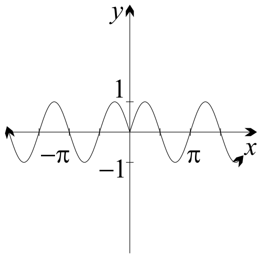

**(vii)**


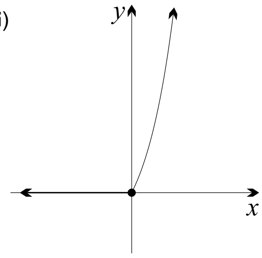

**(viii)**


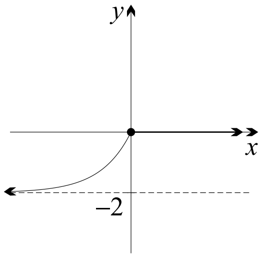

**(b)(i)**


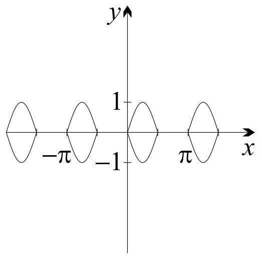

**(ii)**


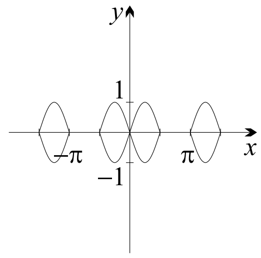

**(c)(i)**


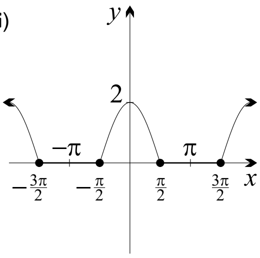

**(ii)**


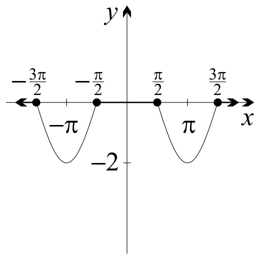

**(c)(i)**


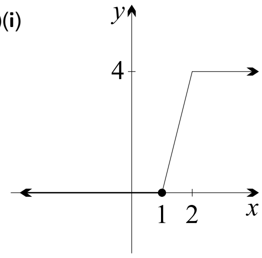

**(ii)**


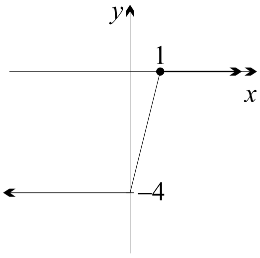

**(iii)**


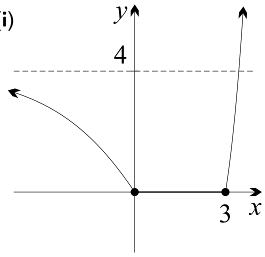

**(iv)**


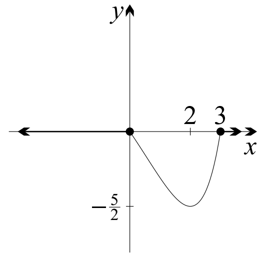

**(d)(i)**


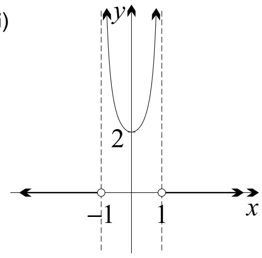

**(ii)**


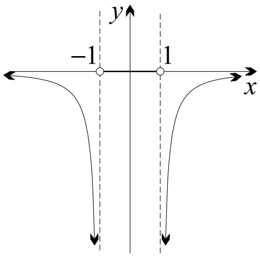
**6(a)(i)**


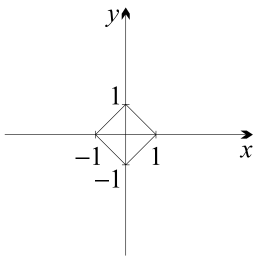

**(ii)**


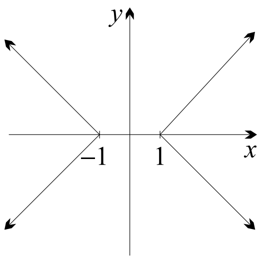

**(b)(i)**


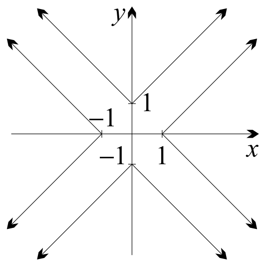

**(ii)**



**(c)(i)**



**(ii)**



**7(a)**


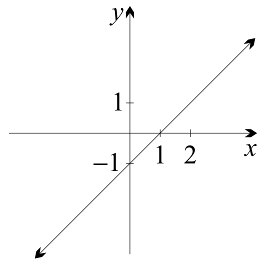

**(b)**


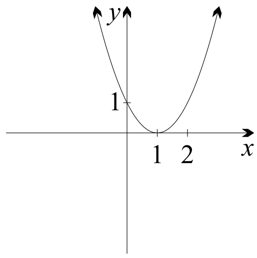

**(c)**


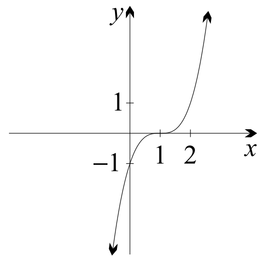

**8(b)** $|y| = |f(x)|$ and $|y| = \Big|f(|x|)\Big|$

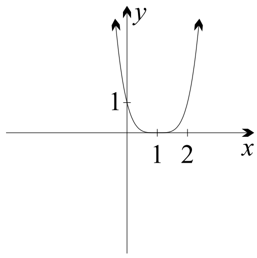
**(e)** Yes: $|y| = f(|x|)$ and $|y| = \Big|f(|x|)\Big|$ are the same if $f(|x|) \geq 0$, for example $f(x) = e^x - 1$.

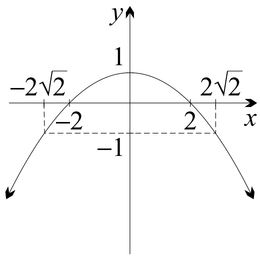
## Exercise **8F** (Page 81)

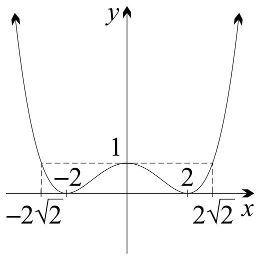
**1(a)**


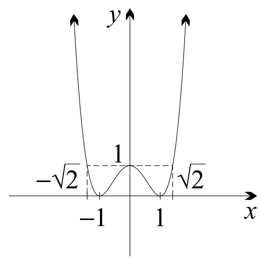

**(b)**


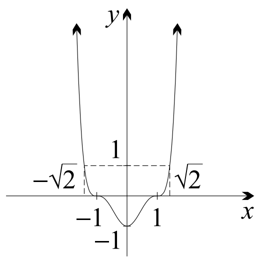

**(c)**


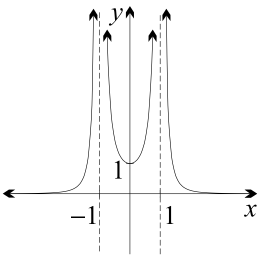

**(d)**


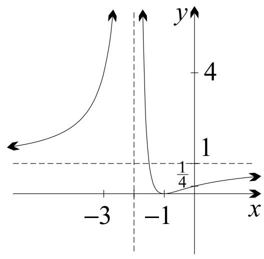

**2(a)**


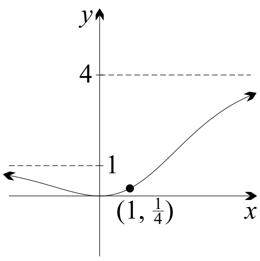

**(b)**


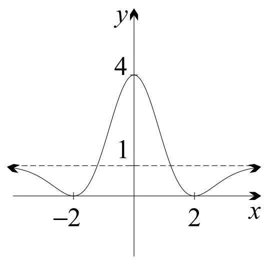

**3(a)**



**(b)**


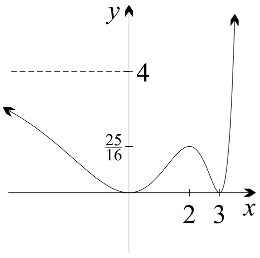

**4(a)**


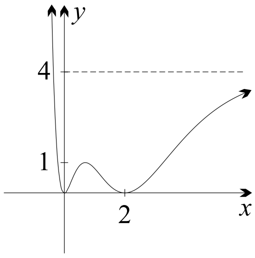

**(b)**


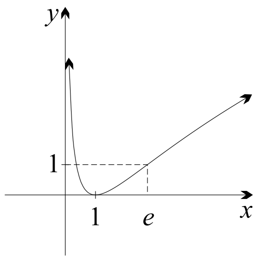

**(c)**


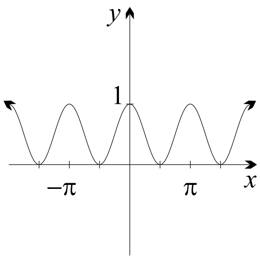

**(d)**


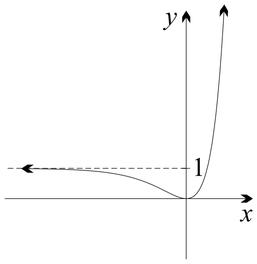

 

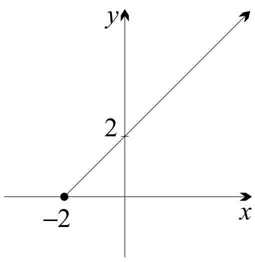
**(e)**



**(f)**


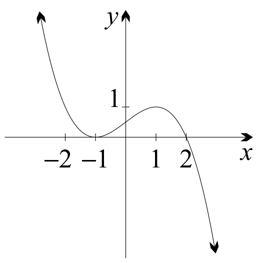

**5(a)**


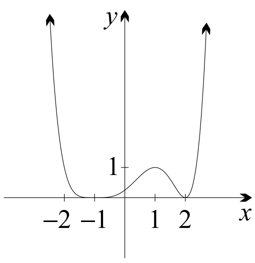

**(b)**



**(c)**


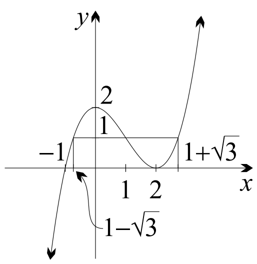

**(d)**


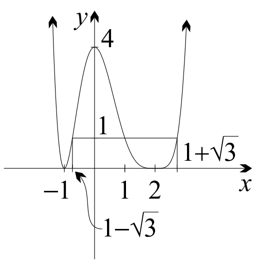

**6(a)** $(-2, 1)$ **(b)** $(-1, 0)$

**(c)**



**(d)**



**7(a)** $(1-\sqrt{3}, 1)$ and $(1+\sqrt{3}, 1)$ **(b)** $(2, 0)$

**(c)**



**(d)**


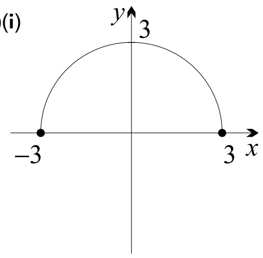

**(e)** $4$

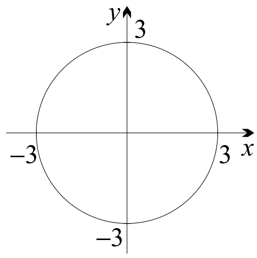
**10(a)** Either $(2, 0)$ is a minimum of $f(x)$, or $n$ is even and $f(x)$ changes sign across $x = 2$.

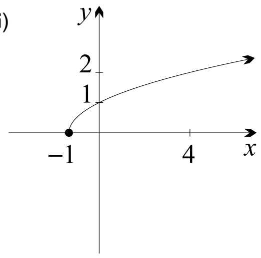
**(b)** $n$ is odd and $f(x)$ has a maximum at $(2, 0)$.

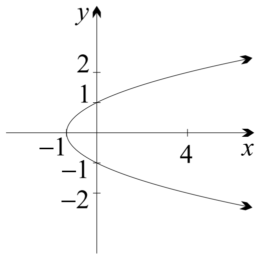
**(c)** $n$ is odd and $f(x)$ changes sign across $x = 2$.

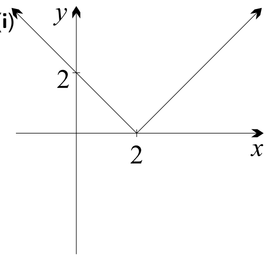
## Exercise **8G** (Page 85)

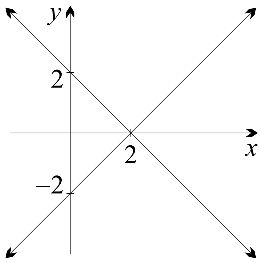
**1(a)(i)**


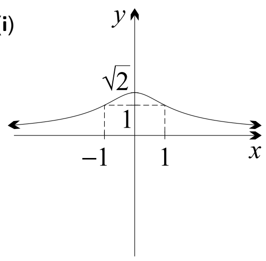

**(ii)**


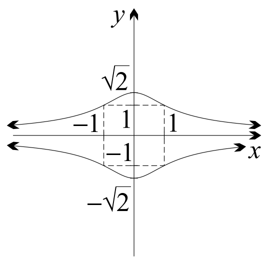

**(b)(i)**


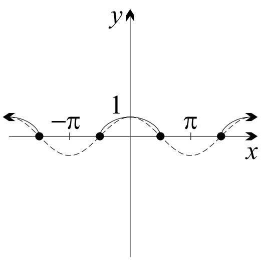

**(ii)**


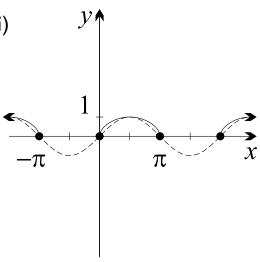

**(c)(i)**



**(ii)**


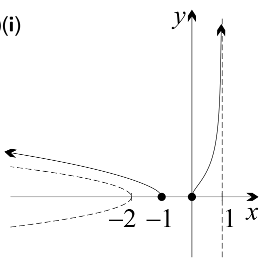

**(d)(i)**


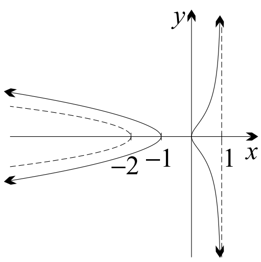

**(ii)**



**2(a)**


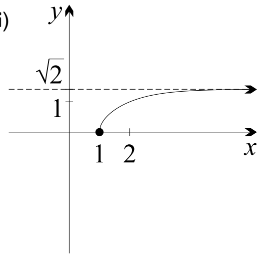

**(b)(i)**



**(ii)** There is symmetry in $x = \frac{\pi}{2}$

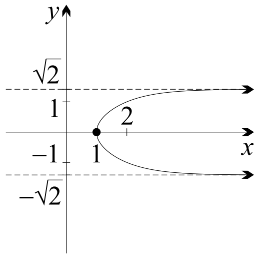
**3(a)(i)**



**(ii)**


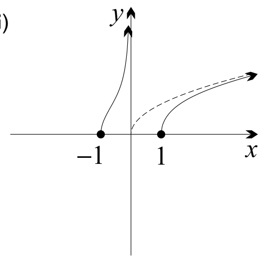
**(b)(i)**



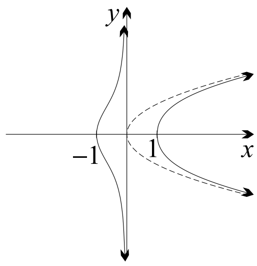

**(ii)**



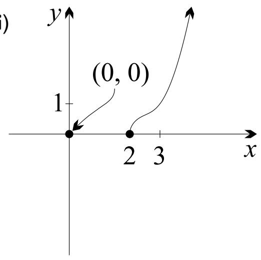

**(c)(i)**



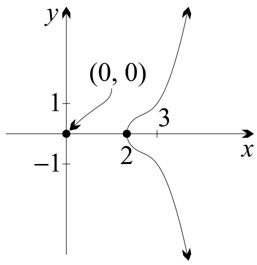

**(ii)**



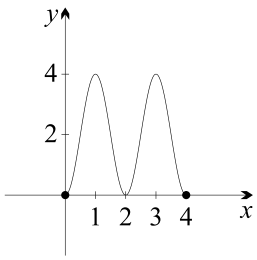

**(d)(i)**



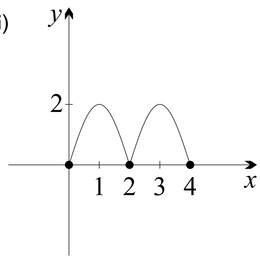

**(ii)**




**4(a)**



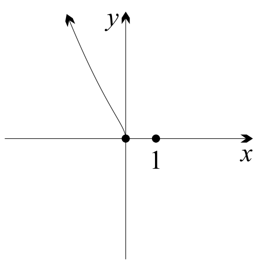

**(b)(i)**



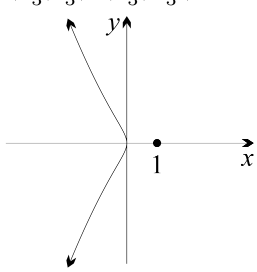

$f(x) = 4\sin^2\left(\frac{\pi}{2}x\right)$

**(ii)** $0,\ \frac{1}{3},\ \frac{5}{3},\ 2,\ \frac{7}{3},\ \frac{11}{3},\ 4$

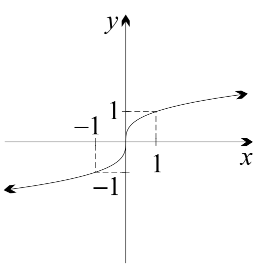
**5(a)**



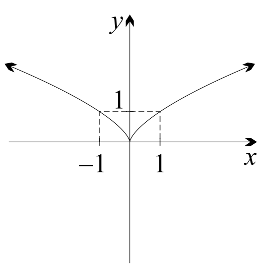

**(b)**



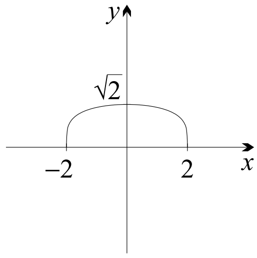

**6(a)**



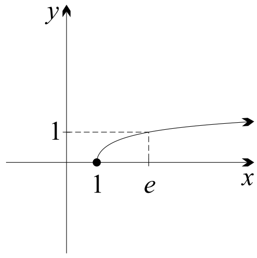

**(b)**



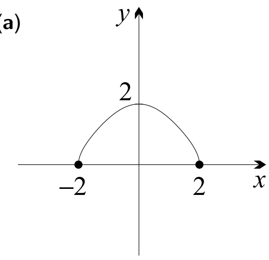

**7**



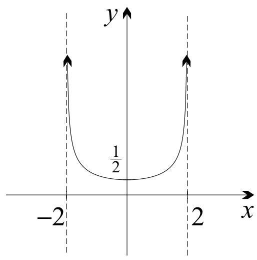

**10**




**11(a)**

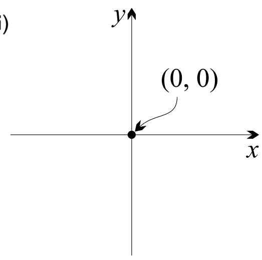


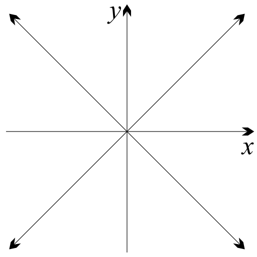

**(b)**




**12(b)** The upper half of the hyperbola in part (a).

**(c)(i)**



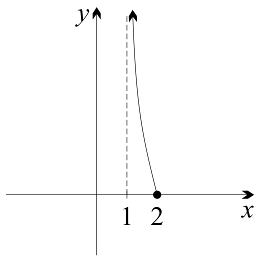




**(ii)** A horizontal plane through the apex yields a solitary point at the origin. A vertical plane through the apex yields a pair of perpendicular lines through the origin.

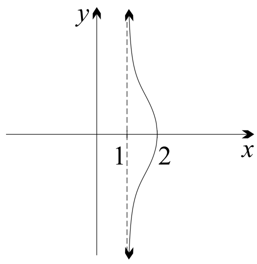
**13** When $f(x) \le 0$ for all $x$ in the natural domain.

**15(b)** $1 < x \le 2$

**(c)**

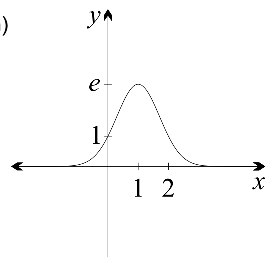



**(d)**

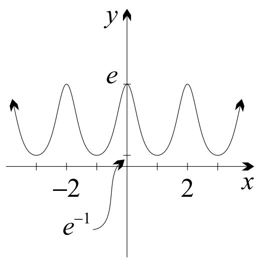



## Exercise **8H** (Page 88)

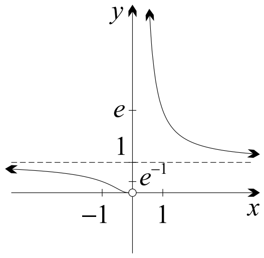
**1(a)**



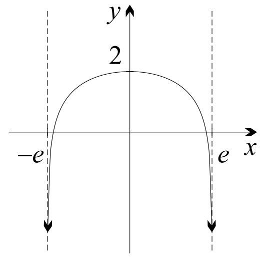

**(b)**



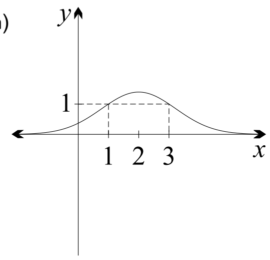
**(c)**



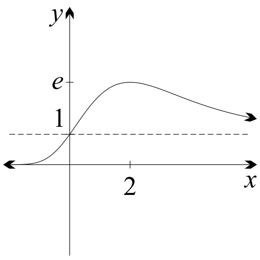

**(d)**



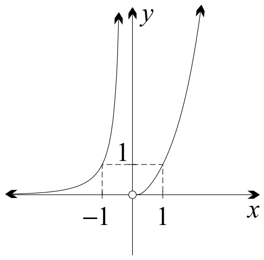

**2(a)**



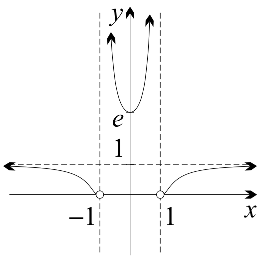

**(b)**



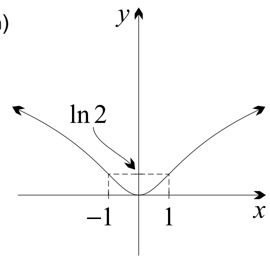

**(c)**



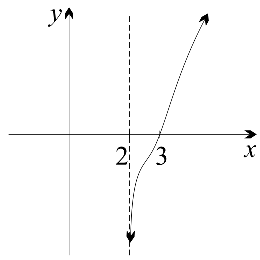

**(d)**



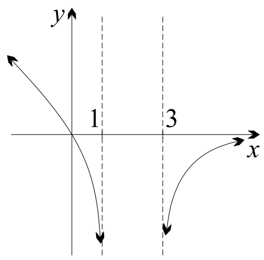

**3(a)**



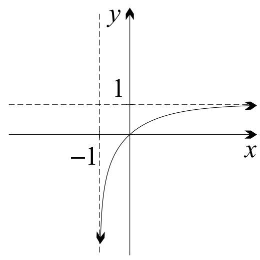

**(b)**



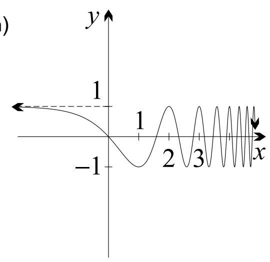

**(c)**



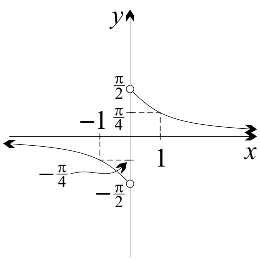

**(d)**



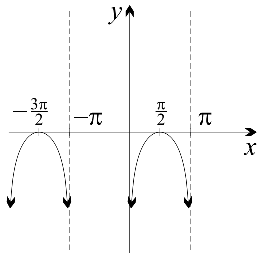

**4(a)**



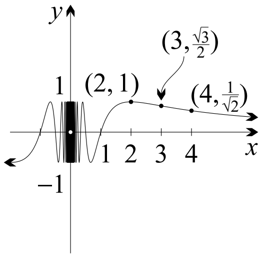

**(b)**



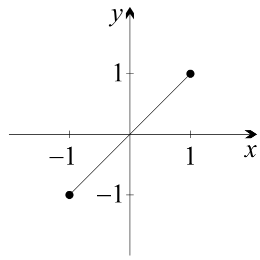

**(c)**



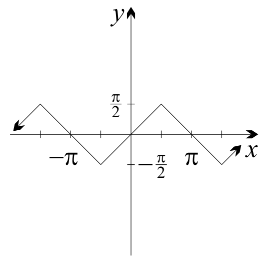

**(d)**



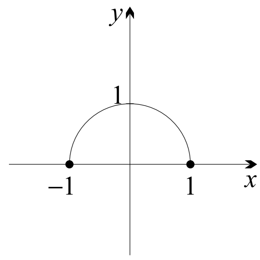

**5(a)**



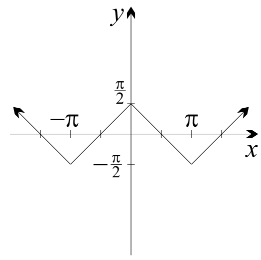

**(b)**



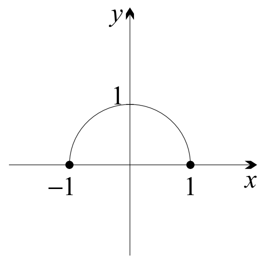

**(c)**



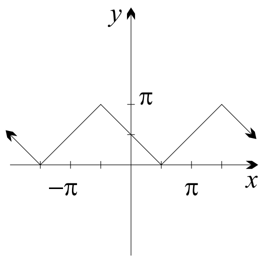

**(d)**



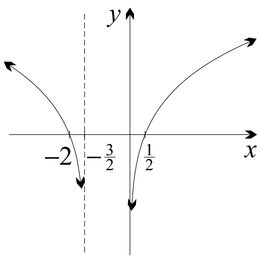

**(e)**



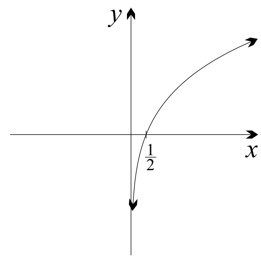

**(f)**



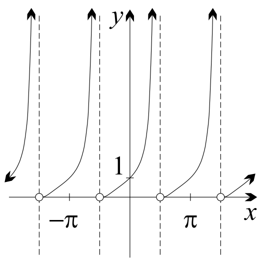

**6(a)**



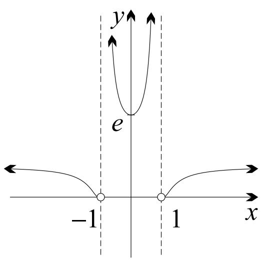

**(b)**



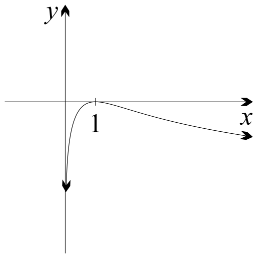

**7(a)**



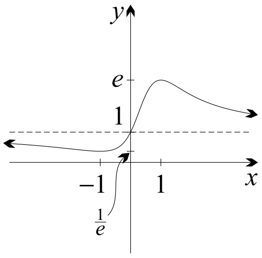

**(b)**



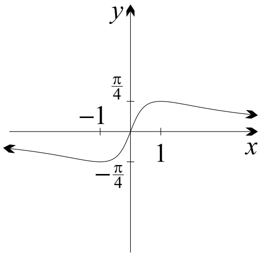
**9(a)**



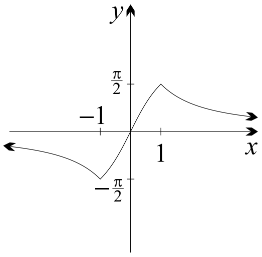
**(b)**



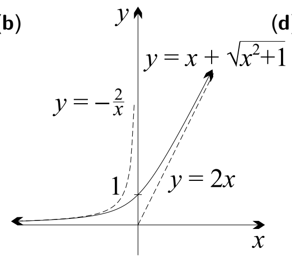

**(c)**



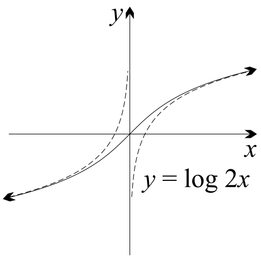

**(d)**




**11(b)**




**(d)**



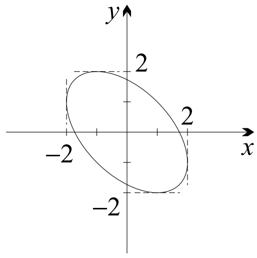

**(e)** $\sinh x = \dfrac{e^x - e^{-x}}{2}$

## Exercise **8I** (Page 93)

**1(a)** $y' + 1$ **(b)** $y + xy'$ **(c)** $2x - 2yy'$
**(d)** $3y^2y' + 3y + 3xy'$ **(e)** $y^{-1}y'$ **(f)** $e^y y'$
**(g)** $y'(2x + 3y) + y(2 + 3y')$ **(h)** $3(x + y)^2(1 + y')$
**(i)** $4(x^2 + y^2)(x + yy')$
**2(a)** $y = \sqrt{x^2 - 9}$ or $y = -\sqrt{x^2 - 9}$
**(b)** $y = \sqrt{4 - x^2}$ or $y = -\sqrt{4 - x^2}$
**(c)** $y = 1 + \sqrt{1 - x^2}$ or $y = 1 - \sqrt{1 - x^2}$
**(d)** $y = -x + \sqrt{1 - x^2}$ or $y = -x - \sqrt{1 - x^2}$
**3(a)** $y' = -\frac{x}{y}$, $(-6, 0)$, $(6, 0)$ **(b)** $y' = \frac{x}{y}$, $(-4, 0)$, $(4, 0)$ **(c)** $y' = \frac{x-y}{x-2y}$, $(-2, -1)$, $(2, 1)$ **(d)** $y' = \frac{3x^2+y^2}{2y(2-x)}$, $(0, 0)$
**4(a)** $(0, -6)$, $(0, 6)$ **(b)** none **(c)** $(-\sqrt{2}, -\sqrt{2})$, $(\sqrt{2}, \sqrt{2})$ **(d)** none
**5(a)** $\frac{5}{4}$ **(b)** $\frac{1}{4}$ **(c)** $0$ **(d)** $-\frac{1}{4}$ **(e)** $\frac{1}{2}$ **(f)** $\frac{13}{48}$
**6(a)** $y = x + 4$ **(b)** $10x - 7y = 1$ **(c)** $x - 2y - 5 = 0$
**(d)** $y = 3x + 2$ **(e)** $y = 12x - 23$ **(f)** $y = 2x - 3$
**7(a)** $4x - 7y + 19 = 0$ **(b)** The denominator of $y'$ is never zero. **(d)** $1$
**8(b)** $y = 1 - x$

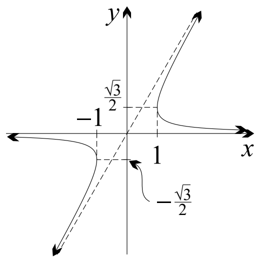
**9(a)** $-2 \le x \le 2$
**(b)** $(-\sqrt{3}, 0)$, $(\sqrt{3}, 0)$
**(c)(ii)** $(-1, -1)$, $(1, 1)$
**(e)** $(-1, 2)$, $(1, -2)$
**(f)** $(-2, 1)$, $(2, -1)$

**(g)**



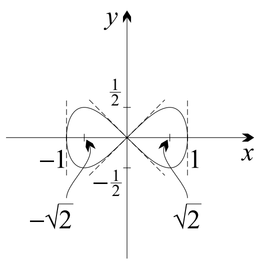

**10(a)** $x \le -1$ or $x \ge 1$
**(b)** no
**(d)** $y = 0$, $y = x\sqrt{3}$
**(f)** no

**(g)**



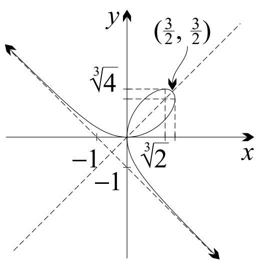

**11(a)** $-1 \le x \le 1$
**(b)** $(-1, 0)$, $(0, 0)$, $(1, 0)$
**(c)** The relations is even in both $x$ and $y$.
**(g)** $(\frac{1}{\sqrt{2}}, \frac{1}{2})$

**(h)**




**12(a)** $(0, 0)$ **(b)(ii)** $(\frac{3}{2}, \frac{3}{2})$ **(c)(ii)** $(\sqrt[3]{2}, \sqrt[3]{4})$ **(d)(iii)** $0$
**(e)(i)** $(0, 0)$, $(\sqrt[3]{4}, \sqrt[3]{2})$
**(ii)** The curve crosses itself. **(f)(ii)** $x + y + 1 = 0$

**(g)**



CHAPTER **TWO**

# Integration

Chapter Overview: The art of integration is a skill that all mathematicians must possess, as integrals arise in all areas of mathematics. For example, in the seemingly unrelated topic of prime numbers the integral $\displaystyle\int \frac{dx}{\log x}$ appears.

As integration is an art form, it requires plenty of practice to become proficient. Thus students are encouraged to attempt as many of the exercise questions as possible in the time they have available.

The work in this chapter builds on the content of the Mathematics Extension 1 course. A methodical approach is needed to study the material. In particular, it is important to be able to recognise the different forms of integrals, and to quickly determine which method is appropriate to apply.

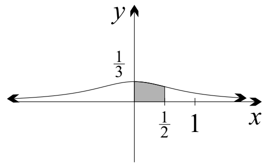
The first four sections are relatively straightforward, being based on algebraic manipulation. In Section 2E the new method of integration by parts is introduced, which is based on the product rule for differentiation. Section 2F covers various types of harder Trignometric integrals. Section 2G introduces the concept of integrals that can be referenced by an index, and the corresponding reduction formulae. The chapter concludes with Section 2I which deals with theorems about integrals that can be used to simplify certain problems.

## 2A Algebraic Manipulation

**Standard integrals:** Students will know that each examination is accompanied by a table of Standard Integrals, located at the end of each paper. A copy of a table is included in the appendix to this chapter. Most of the results in that table will have already been encountered in the Mathematics Extension 1 course. The ability to make simple manipulations to the integrals in this table is expected.

**Worked Exercise:** Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{3+4x^2}$.

**Solution:** Take out a factor of $\frac{1}{4}$ to get:




$$\int_0^{\frac{1}{2}} \frac{dx}{3+4x^2} = \frac{1}{4}\int_0^{\frac{1}{2}} \frac{dx}{(\frac{\sqrt{3}}{2})^2 + x^2}$$

$$= \frac{1}{4} \times \frac{2}{\sqrt{3}} \left[\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)\right]_0^{\frac{1}{2}} \quad \text{(table of Standard Integrals)}$$

$$= \frac{1}{2\sqrt{3}} \tan^{-1} \tfrac{1}{\sqrt{3}}$$
$$= \frac{\pi}{12\sqrt{3}} .$$

**Algebraic Manipulation:** Many of the integrals encountered contain fractions which require some sort of rearrangement before proceeding. In the first worked exercise the numerator is almost identical to the denominator.

**WORKED EXERCISE:** Determine $\displaystyle\int \frac{x^2-1}{x^2+1}\,dx$.

**SOLUTION:** Noting that $x^2 - 1 = (x^2 + 1) - 2$ we write:

$$\begin{aligned}\int \frac{x^2-1}{x^2+1}\,dx &= \int \frac{x^2+1}{x^2+1} - \frac{2}{x^2+1}\,dx\\ &= \int 1 - \frac{2}{x^2+1}\,dx\\ &= x - 2\tan^{-1} x + C .\end{aligned}$$

In harder problems long division is required, though in some cases the numerator is almost a multiple of the denominator, as in the next worked exercise.

**WORKED EXERCISE:** Find $\displaystyle\int \frac{4x^3-2x^2+1}{2x-1}\,dx$.

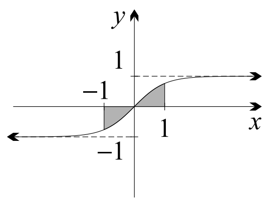
**SOLUTION:**
$$\begin{aligned}\int \frac{4x^3-2x^2+1}{2x-1}\,dx &= \int \frac{2x^2(2x-1)+1}{2x-1}\,dx\\ &= \int 2x^2 + \frac{1}{2x-1}\,dx\\ &= \tfrac{2}{3}x^3 + \tfrac{1}{2}\log(2x-1) + C .\end{aligned}$$

**A Hard Example:** The final worked exercise demonstrates a fraction which first requires multiplication or division by a common factor. The result is a numerator which is the derivative of the denominator.

**WORKED EXERCISE:** Evaluate $\displaystyle\int_{-1}^{1} \frac{e^{2x}-1}{e^{2x}+1}\,dx$.

**SOLUTION:** Divide numerator and denominator by $e^x$ to get:

$$\begin{aligned}\int_{-1}^{1} \frac{e^{2x}-1}{e^{2x}+1}\,dx &= \int_{-1}^{1} \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\,dx\\ &= \Big[\log(e^x + e^{-x})\Big]_{-1}^{1}\\ &= \log(e + e^{-1}) - \log(e^{-1} + e)\\ &= 0 .\end{aligned}$$




**Two New Integrals:** The final two integrals in the standard table will be new to most readers. Here, the result for the last integral is proven using a similar approach to the previous worked exercise.

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \int \frac{(x+\sqrt{x^2+a^2})}{\sqrt{x^2+a^2}\,(x+\sqrt{x^2+a^2})}\,dx$$

$$= \int \frac{(\frac{x}{\sqrt{x^2+a^2}} + 1)}{(x + \sqrt{x^2 + a^2})}\, dx$$
$$= \int \frac{(1 + \frac{x}{\sqrt{x^2+a^2}})}{(x + \sqrt{x^2 + a^2})}\, dx\,.$$

Looking carfeully at the last line, notice that the numerator is the derivative of the denominator and hence we have the desired result:

$$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \log(x + \sqrt{x^2 + a^2}) + C\,.$$

The other new integral in the list may be done in a similar way and is one of the questions in the Exercise.

## Exercise **2A**

**1.** Use a table of Standard Integrals to determine the following. A copy of a table of Standard Integrals may be found in the appendix to this chapter.

(a) $\displaystyle\int \cos 2x\, dx$ (c) $\displaystyle\int \frac{1}{25 + x^2}\, dx$ (e) $\displaystyle\int \frac{1}{\sqrt{x^2 + 3}}\, dx$

(b) $\displaystyle\int \sec^2 \tfrac{x}{3}\, dx$ (d) $\displaystyle\int \frac{1}{\sqrt{4 - x^2}}\, dx$ (f) $\displaystyle\int \frac{1}{\sqrt{x^2 - 5}}\, dx$

**2.** Evaluate the following with the aid of a table of Standard Integrals. A copy of a table of Standard Integrals may be found in the appendix to this chapter.

(a) $\displaystyle\int_0^4 e^{\frac{x}{2}}\, dx$ (c) $\displaystyle\int_{-4}^4 \frac{1}{16 + x^2}\, dx$ (e) $\displaystyle\int_{\sqrt{5}}^3 \frac{1}{\sqrt{x^2 - 4}}\, dx$

(b) $\displaystyle\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x\, dx$ (d) $\displaystyle\int_0^1 \frac{1}{\sqrt{2 - x^2}}\, dx$ (f) $\displaystyle\int_{-4}^4 \frac{1}{\sqrt{x^2 + 9}}\, dx$

**3.** Determine these logarithmic integrals.

(a) $\displaystyle\int \frac{x}{1 - x^2}\, dx$ (b) $\displaystyle\int \frac{1 + \sec^2 x}{x + \tan x}\, dx$ (c) $\displaystyle\int \frac{\cos 3x}{1 + \sin 3x}\, dx$

**4.** Evaluate:

(a) $\displaystyle\int_0^1 \frac{x^2}{1 + x^3}\, dx$ (b) $\displaystyle\int_0^1 \frac{e^{2x}}{e^{2x} + 1}\, dx$ (c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos x}\, dx$

**DEVELOPMENT**

**5.** Evaluate the following with the aid of a table of Standard Integrals.

(a) $\displaystyle\int_0^1 \frac{dx}{1 + 3x^2}$ (b) $\displaystyle\int_0^{\frac{1}{3}} \frac{dx}{\sqrt{4 - 9x^2}}$ (c) $\displaystyle\int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{dx}{\sqrt{4x^2 + 9}}$ (d) $\displaystyle\int_1^{\frac{7}{\sqrt{5}}} \frac{dx}{\sqrt{5x^2 - 4}}$

**6.** Determine the following by rewriting the numerator in terms of the denominator.

(a) $\displaystyle\int \frac{x}{x - 1}\, dx$ (b) $\displaystyle\int \frac{x - 1}{x + 1}\, dx$ (c) $\displaystyle\int \frac{x + 1}{x - 1}\, dx$

**7.** Evaluate the following.

(a) $\displaystyle\int_0^1 \frac{x - 1}{x + 1}\, dx$ (b) $\displaystyle\int_0^2 \frac{x}{2x + 1}\, dx$ (c) $\displaystyle\int_0^1 \frac{3 - x^2}{1 + x^2}\, dx$

**8.** Evaluate the following. In each case, begin by rewriting the given fraction as two fractions by separating the terms in the numerator.

(a) $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \frac{1 - x}{\sqrt{1 - x^2}}\, dx$ (b) $\displaystyle\int_0^1 \frac{2x + 1}{1 + x^2}\, dx$ (c) $\displaystyle\int_0^1 \frac{1 - x}{1 + x^2}\, dx$ (d) $\displaystyle\int_0^2 \frac{1 + x}{4 + x^2}\, dx$

**9.** Use a similar approach to that shown in the text to prove that
$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left(x+\sqrt{x^2-a^2}\right)+C\,.$$

**10.** (a) Given that $x^3 = (x^3+1)-1$, determine $\displaystyle\int \frac{x^3}{x+1}\,dx$.

(b) Given that $x^3 = (x^3+x)-x$, determine $\displaystyle\int \frac{x^3}{x^2+1}\,dx$.

(c) Use similar approaches to those shown in parts (a) and (b) to determine the following.

(i) $\displaystyle\int \frac{x^3}{x-1}\,dx$ (iii) $\displaystyle\int \frac{1}{1+e^x}\,dx$ (v) $\displaystyle\int \frac{x}{\sqrt{1-x}}\,dx$

(ii) $\displaystyle\int \frac{x^4}{x^2+1}\,dx$ (iv) $\displaystyle\int \frac{x}{\sqrt{2+x}}\,dx$ (vi) $\displaystyle\int \frac{x^3}{x^2+4}\,dx$

**11.** Evaluate these by first muliplying or dividing by an appropriate factor.

(a) $\displaystyle\int_1^2 \frac{e^{2x}+1}{e^{2x}-1}\,dx$ (b) $\displaystyle\int_0^1 \frac{e^x}{e^x+e^{-x}}\,dx$ (c) $\displaystyle\int_1^{\sqrt{3}} \frac{2+\frac{1}{x}}{x+\frac{1}{x}}\,dx$

**12.** By using long division or otherwise, determine:

(a) $\displaystyle\int \frac{x^2+x+1}{x+1}\,dx$ (b) $\displaystyle\int \frac{x^3-2x^2+3}{x-2}\,dx$ (c) $\displaystyle\int \frac{(x+1)^2}{1+x^2}\,dx$

**EXTENSION**

**13.** Divide numerator and denominator by an appropriate factor to help determine
$$\int \frac{1}{x+\sqrt{x}}\,dx\,.$$

# 2B Substitution

Many of the techniques used in integration are derived from differentiation. This is not so surprising since the two processes are essentially mutually inverse. One particularly useful technique is substitution which is the integration equivalent of the chain rule for differentiation, and is sometimes called the reverse chain rule.

**The Chain Rule:** Suppose that $F$ is a function of $u$, which is in turn a function of $x$. Further suppose that $F(u)$ is a primitive of $f(u)$. Differentiating $F$ with respect to $x$ and treating the derivative like a fraction we get:
$$\frac{d}{dx}F(u) = \frac{dF}{du}\times\frac{du}{dx}$$
so
$$\frac{d}{dx}F(u) = f(u)\times u'\,.$$

Integrating both sides of this result
$$\int\left(\frac{d}{dx}F(u)\right)dx = \int f(u)\times u'\,dx$$
or
$$F(u)+C = \int f(u)\times u'\,dx\,.$$

It is this last result which proves most useful for integration. Thus if an integrand can be expressed as a product, where one factor is a chain of functions $f(u)$ and the other factor is $u'$ then we can immediately write down the primitive.

**Substitution:** In the simplest examples, the primitive can be determined mentally. For example, a standard integral in the exponential function topic is

$$\int 2x\, e^{x^2}\, dx = e^{x^2} + C\,.$$

In harder examples a formal procedure should be followed.

**WORKED EXERCISE:** Determine $\displaystyle\int \frac{x^2}{\sqrt{x^3+1}}\, dx$ by using a suitable substitution.

**SOLUTION:** Let $I = \displaystyle\int \frac{x^2}{\sqrt{x^3+1}}\, dx$ and put $u = x^3 + 1$, then

$$\frac{du}{dx} = 3x^2$$

or $\quad \tfrac{1}{3}du = x^2\, dx \qquad$ (treating the derivative like a fraction.)

Thus $\quad I = \displaystyle\int \frac{1}{3\sqrt{u}}\, du$

$\qquad = \tfrac{2}{3}\sqrt{u} + C\,.$

Hence $\quad I = \tfrac{2}{3}\sqrt{x^3+1} + C\,.$

Notice that the final step of the solution is a back substitution to get the integral $I$ in terms of $x$. It is important to remember to do this.

It is equally important to follow this formal procedure when definite integrals are involved, paying particular attention to the limits of integration.

**WORKED EXERCISE:** Use a suitable substitution to find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)^3}\, dx\,.$

**SOLUTION:** Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)^3}\, dx$ and put $u = 1 + \cos x$ to get

$$\frac{du}{dx} = -\sin x$$

so $\quad -du = \sin x\, dx\,.$

When $\quad x = 0\,, \qquad u = 2\,,$

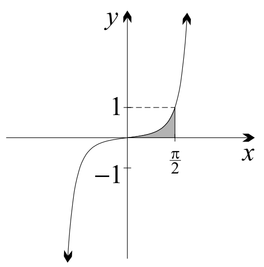
and when $\quad x = \tfrac{\pi}{2}\,, \qquad u = 1\,,$

thus $\quad I = \displaystyle\int_2^1 \frac{-1}{u^3}\, du$

$\qquad = \left[\dfrac{1}{2u^2}\right]_2^1$

$\qquad = \tfrac{1}{2} - \tfrac{1}{8}$

$\qquad = \tfrac{3}{8}\,.$




The step where the limits are expressed in terms of the substitute variable is important. Had this step not been done then the wrong answer is obtained since

$$\int_0^{\frac{\pi}{2}} \frac{-1}{u^3}\, du = \left[\frac{1}{2u^2}\right]_0^{\frac{\pi}{2}}$$

which is undefined at the lower limit. Again notice that the derivative is treated like a fraction in the third line of the solution.

In simple examples like those above, candidates are expected to determine the appropriate substitution for themselves. In harder problems the substitution will be given in the question. Implicit differentiation may also be convenient.

**WORKED EXERCISE:** Use the substitution $u = \sqrt{x}$ to determine $\displaystyle\int \frac{1}{x+\sqrt{x}}\,dx$.

**SOLUTION:** Let $I = \displaystyle\int \frac{dx}{x+\sqrt{x}}$ and note that $u^2 = x$, so:

$$2u\,\frac{du}{dx} = 1$$

or $\quad 2u\;du = dx\,.$

Hence
$$\begin{aligned} I &= \int \frac{2u\;du}{u^2+u} \\ &= \int \frac{2\,du}{u+1} \\ &= 2\log(u+1) + C \\ &= 2\log(\sqrt{x}+1) + C\,. \end{aligned}$$

**Take Care with Substitutions:** There are many integrals which require a careful choice of substitution so as to avoid subsequent difficulties. For example, the correct choice of substitution in the previous worked exercise is $u = \sqrt{x}$.

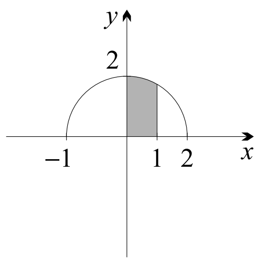
On first inspection, it would seem to make no difference to make the alternate substitution $u^2 = x$, however observe what happens in the denominator.

$$x + \sqrt{x} = u^2 + \sqrt{u^2} = u^2 + |u|\,.$$

Thus in this case a new complication has been introduced, namely the absolute value function. In general, the best choice of substitution is of the form $u = f(x)$.

**WORKED EXERCISE:** Evaluate $\displaystyle\int_0^1 \sqrt{4-x^2}\,dx$ by applying a suitable substitution.

**SOLUTION:** Let $I = \displaystyle\int_0^1 \sqrt{4-x^2}\;dx$ and put $\theta = \sin^{-1}(\frac{x}{2})$ so that $\cos\theta \ge 0$.




Rearranging $\quad x = 2\sin\theta$

so $\quad dx = 2\cos\theta\;d\theta\,.$

When $\quad x = 0\,,\quad \theta = 0\,,$

and when $\quad x = 1\,,\quad \theta = \sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}\,.$

Thus
$$\begin{aligned} I &= \int_0^{\frac{\pi}{6}} 2\cos\theta\,\sqrt{4-4\sin^2\theta}\;d\theta \\ &= \int_0^{\frac{\pi}{6}} 4\cos\theta\,\sqrt{\cos^2\theta}\;d\theta && \text{(by the Pythagorean identity)} \\ &= \int_0^{\frac{\pi}{6}} 4\cos^2\theta\;d\theta && \text{(since } \cos\theta \ge 0) \\ &= \int_0^{\frac{\pi}{6}} 2(1+\cos 2\theta)\,d\theta && \text{(by the double-angle formula)} \\ &= \Big[2\theta + \sin 2\theta\Big]_0^{\frac{\pi}{6}} \\ &= \tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2}\,. \end{aligned}$$

On first inspection, the alternate substitution $x = 2\sin\theta$ would seem to make no difference. However in this case the limits of integration are indeterminate. For example when $x = 1$, there are multiple solutions, namely $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$, and the problem then is to find the correct choice of limits.

**Two Guidelines for Substitutions:** The infinite variety of integrals that may be encountered make it impractical to give a specific recipe for making the correct substitution. However the following two guidelines may help, and can be observed in practice in the previous worked exercises.

- Try to replace the part of the integral which causes difficulty, such as the innermost function in a chain of functions. In particular, if the integral involves square-roots of sums or differences of squares then a trigonometric substitution is likely to work.
- It is better to use a substitution which is a function $u = f(x)$ rather than a relation $x = g(u)$. Substituting a relation such as $x = u^2$ can lead to problems later in the calculations, as demonstrated above.

## Exercise **2B**

**1.** (a) Use the result $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log\big(f(x)\big) + C$ to help determine these indefinite integrals.

(i) $\displaystyle\int \frac{x}{1-x^2}\,dx$ (ii) $\displaystyle\int \frac{\cos x}{1+\sin x}\,dx$ (iii) $\displaystyle\int \frac{1}{x\log x}\,dx$

(b) Do likewise for these definite integrals.

(i) $\displaystyle\int_0^1 \frac{e^{2x}}{e^{2x}+1}\,dx$ (ii) $\displaystyle\int_0^1 \frac{x^2}{1+x^3}\,dx$ (iii) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x}\,dx$

**2.** (a) Use the result $\displaystyle\int f'(x)e^{f(x)}\,dx = e^{f(x)} + C$ to help determine these indefinite integrals.

(i) $\displaystyle\int 6x^2 e^{x^3}\,dx$ (ii) $\displaystyle\int \sec^2 x\, e^{\tan x}\,dx$ (iii) $\displaystyle\int \tfrac{1}{x^2} e^{\frac{1}{x}}\,dx$

(b) Do likewise for these definite integrals.

(i) $\displaystyle\int_0^1 x\,e^{1-x^2}\,dx$ (ii) $\displaystyle\int_0^{\frac{\pi}{2}} \cos x\, e^{\sin x}\,dx$ (iii) $\displaystyle\int_1^4 \tfrac{1}{\sqrt{x}} e^{\sqrt{x}}\,dx$

**3.** Try to find these integrals mentally, otherwise use a suitable substitution.

(a) $\displaystyle\int 2x(x^2+1)^4\,dx$ (c) $\displaystyle\int \frac{6x^2}{(1+x^3)^2}\,dx$ (e) $\displaystyle\int \frac{x}{\sqrt{x^2-2}}\,dx$

(b) $\displaystyle\int 3x^2(1+x^3)^6\,dx$ (d) $\displaystyle\int \frac{4x}{(3-x^2)^5}\,dx$ (f) $\displaystyle\int \frac{x^3}{\sqrt{1+x^4}}\,dx$

**DEVELOPMENT**

**4.** Use a suitable substitution where necessary to find:

(a) $\displaystyle\int \frac{\cos x}{\sin^3 x}\,dx$ (c) $\displaystyle\int \frac{(\log x)^2}{x}\,dx$ (e) $\displaystyle\int \frac{x}{1+x^4}\,dx$

(b) $\displaystyle\int \frac{\sec^2 x}{(1+\tan x)^2}\,dx$ (d) $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ (f) $\displaystyle\int \frac{x^2}{\sqrt{1-x^6}}\,dx$

**5.** Use a suitable substitution where necessary to evaluate:

(a) $\displaystyle\int_0^1 x^3(1+3x^4)^2\,dx$ (c) $\displaystyle\int_3^4 \frac{x+1}{\sqrt{x^2+2x+3}}\,dx$ (e) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\sec^2 x\,dx$

(b) $\displaystyle\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx$ (d) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^4 x\cos x\,dx$ (f) $\displaystyle\int_1^{e^2} \frac{\log x}{x}\,dx$

**6.** (a) Use a suitable substitution to help evaluate $\displaystyle\int_0^1 x(x-1)^5\,dx$.

(b) How could this integral have been evaluated using just algebraic manipultaion?

**7.** Use the given substitution to find:

(a) $\displaystyle\int x\sqrt{x+1}\,dx$ [put $u=\sqrt{x+1}$] (c) $\displaystyle\int \frac{1}{1+x^{\frac{1}{4}}}\,dx$ [put $u=x^{\frac{1}{4}}$]

(b) $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$ [put $u=1+\sqrt{x}$] (d) $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$ [put $u=\sqrt{e^{2x}-1}$]

**8.** In each case, use the given substitution to evaluate the integral.

(a) $\displaystyle\int_0^1 \frac{2-x}{(2+x)^3}\,dx$ [put $u=2+x$] (c) $\displaystyle\int_0^4 \frac{1}{5+\sqrt{x}}\,dx$ [put $u=\sqrt{x}$]

(b) $\displaystyle\int_0^4 x\sqrt{4-x}\,dx$ [put $u=\sqrt{4-x}$] (d) $\displaystyle\int_4^{12} \frac{1}{(4+x)\sqrt{x}}\,dx$ [put $u=\sqrt{x}$]

**9.** In each case, use the given substitution to determine the primitive.

(a) $\displaystyle\int \frac{1}{(1+x)\sqrt{x}}\,dx$ [put $u=\sqrt{x}$] (b) $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$ [put $u=\sqrt{x+1}$]

**10.** In each case use the given trigonometric substitution to evaluate the integral. You may assume that $0 \le \theta < \frac{\pi}{2}$.

(a) $\displaystyle\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$ [put $x=\tan\theta$] (c) $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$ [put $x=5\cos\theta$]

(b) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ [put $x=2\sin\theta$] (d) $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$ [put $x=\tan\theta$]

**11.** (a) Use a suitable substitution to help evaluate $\displaystyle\int_0^{\sqrt{2}} \frac{x^3}{\sqrt{x^2+1}}\,dx$.

(b) How could this integral have been evaluated using just algebraic manipulation?

**12.** (a) Use a suitable substitution to show that $\displaystyle\int_1^2 \sqrt{4-x^2}\,dx = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}$.

(b) Redo this problem by geometric means.

**EXTENSION**

**13.** (a) Use a trigonometric substitution to show that $\displaystyle\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx = \tfrac{\pi}{12} - \tfrac{\sqrt{3}}{8}$.

(b) How could this integral have been evaluated using algebra then geometry?

**14.** Consider the indefinite integral $I = \displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$. Clearly the domain of the integrand is disjoint, being $x>1$ or $x<-1$. Thus it seems appropriate to use a different substitution in each part of the domain.

(a) Find $I$ for $x>1$ by using the substitution $u=\sqrt{x^2-1}$.

(b) Find $I$ for $x<-1$ by using the substitution $u=-\sqrt{x^2-1}$.

**15.** (a) Use a suitable substitution to determine $\displaystyle\int_{2+\epsilon}^{4} \frac{dx}{x^2\sqrt{x^2-4}}$, where $\epsilon > 0$.

(b) Take the limit of this result as $\epsilon \to 0^+$ and hence find $\displaystyle\int_{2}^{4} \frac{dx}{x^2\sqrt{x^2-4}}$.

# 2C Partial Fractions

In arithmetic, when given the sum of two fractions, the normal procedure is to combine them into a single fraction using the lowest common denominator. Thus

$$\frac{1}{3}+\frac{1}{2}=\frac{5}{6}.$$

Unfortunately when the fractions are functions and integration is involved, this is exactly the wrong thing to do. Whilst it is true that

$$\frac{3}{x+2}+\frac{2}{x-1}=\frac{5x+1}{x^2+x-2},$$

when considering the corresponding integrals,

$$\int \frac{3}{x+2}+\frac{2}{x-1}\,dx=\int \frac{5x+1}{x^2+x-2}\,dx,$$

it should be clear that the left hand side is far simpler to determine than the right hand integral. So:

$$\begin{aligned}\int \frac{5x+1}{x^2+x-2}\,dx &= \int \frac{3}{x+2}+\frac{2}{x-1}\,dx\\ &= 3\log(x+2)+2\log(x-1)+C.\end{aligned}$$

This example is typical of integrals of rational functions. It is easiest to first split the fraction into its simpler components. In mathematical terminology, the fraction is *decomposed into partial fractions*.

**A Theorem About Partial Fractions:** Consider the rational function

$$\frac{P(x)}{A(x)\times B(x)},$$

where $P$, $A$ and $B$ are polynomials, with no common factors between any pair, and where $\deg P < \deg A + \deg B$. It can be shown that it is always possible to write

$$\frac{P(x)}{A(x)\times B(x)}=\frac{R_A(x)}{A(x)}+\frac{R_B(x)}{B(x)},$$

where the remainders $R_A$ and $R_B$ are polynomials with $\deg R_A < \deg A$ and $\deg R_B < \deg B$. The proof is beyond the scope of this course.

**Linear Factors:** In the simplest examples, $A(x)$ and $B(x)$ are linear. Since the degrees of $R_A$ and $R_B$ are less, they must be constants, yet to be found.

**WORKED EXERCISE:** (a) Decompose $\dfrac{x+1}{(x-1)(x+3)}$ into its partial fractions.

(b) Hence evaluate $\displaystyle\int_{2}^{6} \frac{x+1}{(x-1)(x+3)}\,dx$.

**SOLUTION:** (a) Let $\dfrac{x+1}{(x-1)(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+3}$, where $A$ and $B$ are unknown constants. Multiply this equation by $(x-1)(x+3)$ to get:

$$x+1 = A(x+3) + B(x-1)$$

or

$$x+1 = (A+B)x + (3A-B)\,.$$

Equating coefficients of like powers of $x$ yields the simultaneous equations

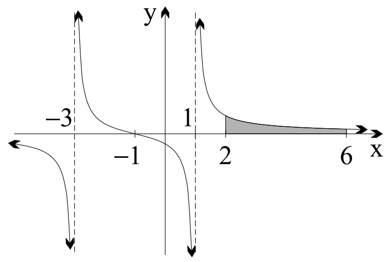
$$A+B = 1$$
$$3A-B = 1\,.$$

These can be solved mentally to get $A = \frac{1}{2}$ and $B = \frac{1}{2}$. Thus

$$\frac{x+1}{(x-1)(x+3)} = \frac{(\frac{1}{2})}{x-1} + \frac{(\frac{1}{2})}{x+3}\,.$$

(b) Hence $\displaystyle\int_2^6 \frac{x+1}{(x-1)(x+3)}\,dx$

$$= \tfrac{1}{2}\int_2^6 \frac{1}{x-1} + \frac{1}{x+3}\,dx$$
$$= \tfrac{1}{2}\Big[\log(x-1) + \log(x+3)\Big]_2^6$$
$$= \tfrac{1}{2}\Big((\log 9 + \log 5) - (\log 5 + \log 1)\Big)$$
$$= \log 3\,.$$




This method of equating coefficients of like powers of $x$ is usually only convenient in straight forward examples like this one.

**Finding the Constants by Substitution:** A more generalised method of finding the unknown constants in partial fractions uses substitution. In many cases it is also a quicker method.

**WORKED EXERCISE:** Decompose $\dfrac{3x-5}{(x-3)(x+1)}$ into partial fractions.

**SOLUTION:** Let $\dfrac{3x-5}{(x-3)(x+1)} = \dfrac{A}{x-3} + \dfrac{B}{x+1}$, where $A$ and $B$ are unkown constants. Multiply this equation by $(x-3)(x+1)$ to get:

$$3x-5 = A(x+1) + B(x-3)\,.$$

When $x=3$, $\quad 4 = 4A$

so $\quad A = 1\,.$

When $x=-1$, $\quad -8 = -4B$

so $\quad B = 2\,.$

Thus $\quad \dfrac{3x-5}{(x-3)(x+1)} = \dfrac{1}{x-3} + \dfrac{2}{x+1}\,.$

The careful reader will have noticed a point of contention with the solution. The fraction is undefined when $x=3$ and when $x=-1$, yet these values were used in the substitution steps. How can this be valid? The answer is that some of the detail of the solution has been omitted. Here is a more complete explanation.

Since $\quad \dfrac{3x-5}{(x-3)(x+1)} = \dfrac{A}{x-3} + \dfrac{B}{x+1}$ where $x \neq -1, 3\,,$

it follows that $\quad 3x-5 = A(x+1) + B(x-3)$ where $x \neq -1, 3\,.$

Now this last equation is true whenever $x \neq -1, 3$. That is, it is a linear equation which is true for at least two other values of $x$. Hence, by the work done in Year 11 on identities, it is true for all $x$, including $x = -1$ and $x = 3$. Thus these values can be substituted to determine $A$ and $B$. It is not necessary to give this complete explanation as part of a solution, but students should be aware of it.

**Numerators with Higher Degree:** In slightly harder problems, the degree of the numerator is greater than or equal to the degree of the denominator. In such cases, the fraction should be expressed as a sum of a polynomial and the partial fractions. Long division may be used at this step, but it is often easier to use a polynomial with unkown coefficients, as in the following worked exercise.

**WORKED EXERCISE:** Determine $\displaystyle\int \frac{x^3+x-3}{x^2-3x+2}\,dx$.

**SOLUTION:** First note that $\displaystyle\frac{x^3+x-3}{x^2-3x+2} = \frac{x^3+x-3}{(x-2)(x-1)}$,

so let $\displaystyle\frac{x^3+x-3}{(x-2)(x-1)} = Ax + B + \frac{C}{x-2} + \frac{D}{x-1}$,

thus $x^3+x-3 = (Ax+B)(x-2)(x-1) + C(x-1) + D(x-2)$.

Equating the coefficients of $x^3$, $A = 1$.

At $x = 1$ $\quad -1 = -D$ so $D = 1$.

At $x = 2$ $\quad 7 = C$.

At $x = 0$ $\quad -3 = 2B - 7 - 2$

so $\quad B = 3$.

Finally
$$\int \frac{x^3+x-3}{x^2-3x+2}\,dx = \int x + 3 + \frac{7}{x-2} + \frac{1}{x-1}\,dx$$
$$= \tfrac{1}{2}x^2 + 3x + 7\log(x-2) + \log(x-1) + C\,.$$

**A Special Case:** There is an even quicker method to find the unknown constants of the partial fractions, provided that the original denominator is a product of distinct linear factors, and provided that the degree of the numerator is less than the degree of the denominator. The trick is to multiply by just one linear factor at a time.

**WORKED EXERCISE:** Express $\displaystyle\frac{7-5x}{(x+1)(x-2)(x-3)}$ as a sum of partial fractions.

**SOLUTION:** Let
$$\frac{7-5x}{(x+1)(x-2)(x-3)} = \frac{C_1}{x+1} + \frac{C_2}{x-2} + \frac{C_3}{x-3}. \qquad (*)$$

$(*) \times (x+1)$ gives
$$\frac{7-5x}{(x-2)(x-3)} = C_1 + \frac{C_2(x+1)}{x-2} + \frac{C_3(x+1)}{x-3}$$

so at $x = -1$
$$C_1 = \frac{12}{(-3)(-4)} = 1\,.$$

$(*) \times (x-2)$ gives
$$\frac{7-5x}{(x+1)(x-3)} = \frac{C_1(x-2)}{x+1} + C_2 + \frac{C_3(x-2)}{x-3}$$

so at $x = 2$
$$C_2 = \frac{-3}{3 \times (-1)} = 1\,.$$

Finally
$$\frac{7-5x}{(x+1)(x-2)} = \frac{C_1(x-3)}{x+1} + \frac{C_2(x-3)}{x-2} + C_3$$

so at $x = 3$ $$C_3 = \frac{-8}{4 \times 1} = -2\,.$$

Hence $$\frac{7-5x}{(x+1)(x-2)(x-3)} = \frac{1}{x+1} + \frac{1}{x-2} - \frac{2}{x-3}\,.$$

This method of finding the constants is sometimes called the *cover up rule*. Look carefully at how the three constants are determined. For each constant, the matching linear factor is effectively omitted, or "covered up". Thus for $C_1$, $(x+1)$ is left out of the original fraction. For $C_2$, $(x-2)$ is excluded, and for $C_3$, $(x-3)$ is omitted from the original fraction. In each case, the resulting rational function is then evaluated at the corresponding value of $x$. With practice, most students should be able to determine the constants mentally using this method.

**Extension — Proof of the Cover up Rule:** Here is a proof for the general case.

PROOF: Consider the rational function $\dfrac{P(x)}{Q(x)}$ where $\deg P < \deg Q$, and where $Q(x)$ is a product of distinct linear factors, that is
$$\begin{aligned} Q(x) &= C \times (x-a_1) \times (x-a_2) \times \ldots \times (x-a_n) \\ &= C \prod_{i=1}^{n} (x-a_i) \qquad \text{(note the use of product notation, } \textstyle\prod\text{.)} \end{aligned}$$

Let $$\frac{P(x)}{Q(x)} = \frac{C_1}{x-a_1} + \frac{C_2}{x-a_2} + \ldots + \frac{C_k}{x-a_k} + \ldots + \frac{C_n}{x-a_n}$$

Multiply this last equation by $(x-a_k)$ to get
$$\frac{P(x)(x-a_k)}{Q(x)} = \frac{C_1(x-a_k)}{x-a_1} + \frac{C_2(x-a_k)}{x-a_2} + \ldots + C_k + \ldots + \frac{C_n(x-a_k)}{x-a_n}\,.$$
Now take the limit as $x \to a_k$. All terms except $C_k$ on the right hand side are zero and so:
$$\begin{aligned} C_k &= \lim_{x \to a_k} \frac{P(x)(x-a_k)}{Q(x)} \\ &= \lim_{x \to a_k} \frac{P(x)}{C \displaystyle\prod_{\substack{i=1 \\ i \neq k}}^{n} (x-a_i)} \qquad \text{(that is, cancel the } k\text{th linear factor)} \end{aligned}$$
hence $$C_k = \frac{P(a_k)}{C \displaystyle\prod_{\substack{i=1 \\ i \neq k}}^{n} (a_k-a_i)}\,.$$
The mathematical notation may seem difficult, but the result is exactly as before. To get the $k$th coefficient $C_k$, omit the $k$th linear factor from the denominator and evaluate the rest of the fraction at $x = a_k$.

**Quadratic Factors:** In certain instances, the denominator of the rational function being considered will have a quadratic factor with no real zero. For example, in
$$\frac{3x+10}{(x-2)(x^2+4)}$$
the quadratic factor $(x^2+4)$ has no real zero. Thus the denominator of the rational function cannot be expressed as a product of real linear factors.

Nevertheless, the method for finding the partial fraction decomposition remains essentially the same. And since the only requirement is that the degree of the numerator is less than the degree of the denominator, it follows that for any quadratic factor the numerator can be a linear polynomial.

**WORKED EXERCISE:** (a) Rewrite $\dfrac{3x+10}{(x-2)(x^2+4)}$ in its partial fractions.

(b) Hence determine $\displaystyle\int \frac{3x+10}{(x-2)(x^2+4)}\,dx$.

**SOLUTION:** (a) Let $\dfrac{3x+10}{(x-2)(x^2+4)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+4}$, where $A$, $B$ and $C$ are unknown constants. Then

$$3x+10 = A(x^2+4) + (Bx+C)(x-2)$$

At $x=2$ $\quad 16 = 8A$ so $A = 2$.

Equating coefficients of $x^2$ yields

$$0 = 2 + B \quad \text{so} \quad B = -2.$$

At $x=0$ $\quad 10 = 8 - 2C$

so $\quad C = -1$.

Thus $\quad \dfrac{4x+10}{(x-2)(x^2+4)} = \dfrac{2}{x-2} - \dfrac{2x+1}{x^2+4}$.

(b) Hence
$$\int \frac{4x+10}{(x-2)(x^2+4)}\,dx = \int \frac{2}{x-2} - \frac{2x}{x^2+4} - \frac{1}{x^2+4}\,dx$$
$$= 2\log(x-2) - \log(x^2+4) - \tfrac{1}{2}\tan^{-1}(\tfrac{x}{2}) + C.$$

**Repeated Factors:** In a polynomial, a factor which has degree greater than one is called a repeated factor. For example in the denominator of the fraction

$$\frac{8-x}{(x-2)^2(x+1)},$$

the factor $(x-2)^2$ is a repeated factor since its index is two. When a partial fraction question involves repeated factors, normally the initial decomposition is given in the question and it is simply a matter of finding the values of the unknown constants.

**WORKED EXERCISE:** (a) Find the real numbers $A$, $B$ and $C$ such that

$$\frac{8-x}{(x-2)^2(x+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1}.$$

(b) Hence evaluate $\displaystyle\int_0^1 \frac{8-x}{(x-2)^2(x+1)}\,dx$.

**SOLUTION:**

(a) Now $\quad 8 - x = A(x-2)(x+1) + B(x+1) + C(x-2)^2$.

At $x=-1$ $\quad 9 = 9C$ so $C = 1$.

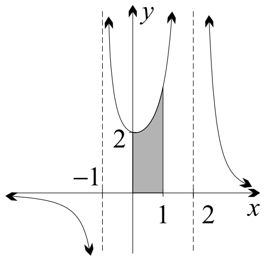
At $x=2$ $\quad 6 = 3B$ so $B = 2$.

At $x=3$ $\quad 5 = 4A + 8 + 1$

so $\quad A = -1$.

(b) Hence $\displaystyle\int_0^1 \frac{8-x}{(x-2)^2(x+1)}\,dx$

$$= \int_0^1 \frac{1}{x+1} + \frac{2}{(x-2)^2} - \frac{1}{x-2}\,dx$$
$$= \left[\log(x+1) - \frac{2}{x-2} - \log|x-2|\right]_0^1$$
$$= (\log 2 + 2 - \log 1) - (\log 1 + 1 - \log 2)$$
$$= 1 + 2\log 2\,.$$




## Exercise **2C**

**1.** Decompose the following fractions into partial fractions.

(a) $\dfrac{2}{(x-1)(x+1)}$ (c) $\dfrac{4x}{x^2-9}$ (e) $\dfrac{x-1}{x^2+x-6}$

(b) $\dfrac{1}{(x-4)(x-1)}$ (d) $\dfrac{x}{x^2-3x+2}$ (f) $\dfrac{3x+1}{(x-1)(x^2+3)}$

**2.** Find:

(a) $\displaystyle\int \frac{2}{(x-4)(x-2)}\,dx$ (c) $\displaystyle\int \frac{3x-2}{(x-1)(x-2)}\,dx$ (e) $\displaystyle\int \frac{4x+5}{(2x+3)(x+1)}\,dx$

(b) $\displaystyle\int \frac{4}{x^2+4x+3}\,dx$ (d) $\displaystyle\int \frac{2x+10}{x^2+2x-3}\,dx$ (f) $\displaystyle\int \frac{10x}{2x^2-x-3}\,dx$

**3.** Evaluate:

(a) $\displaystyle\int_4^6 \frac{1}{x^2-4}\,dx$ (c) $\displaystyle\int_2^5 \frac{11}{2x^2+5x-12}\,dx$

(b) $\displaystyle\int_2^4 \frac{3}{x^2+x-2}\,dx$ (d) $\displaystyle\int_{-1}^0 \frac{1}{3x^2-4x+1}\,dx$

**4.** Determine:

(a) $\displaystyle\int \frac{x^2-2x+5}{(x-2)(x^2+1)}\,dx$ (b) $\displaystyle\int \frac{6-x}{(2x+1)(x^2+3)}\,dx$ (c) $\displaystyle\int \frac{x^2+x+3}{x^3+x}\,dx$

**5.** Find the value of:

(a) $\displaystyle\int_0^{\frac{1}{2}} \frac{1+2x-4x^2}{(x+1)(4x^2+1)}\,dx$ (b) $\displaystyle\int_{-1}^1 \frac{7-x}{(x+3)(x^2+1)}\,dx$ (c) $\displaystyle\int_1^{\sqrt{2}} \frac{x^2-4}{x^3+2x}\,dx$

**DEVELOPMENT**

**6.** Find:

(a) $\displaystyle\int \frac{2x+3}{(x-1)(x-2)(2x-3)}\,dx$ (b) $\displaystyle\int \frac{4x+12}{x^3-6x^2+8x}\,dx$

**7.** Evalute:

(a) $\displaystyle\int_2^7 \frac{3x+5}{(x-1)(x+2)(x+1)}\,dx$ (b) $\displaystyle\int_1^2 \frac{13x+6}{x^3-x^2-6x}\,dx$

**8.** (a) (i) Let $\dfrac{2x^2+1}{(x-1)(x+2)} = A + \dfrac{B}{x-1} + \dfrac{C}{x+2}$. Find the values of $A$, $B$ and $C$.

(ii) Hence find $\displaystyle\int \frac{2x^2+1}{(x-1)(x+2)}\,dx$

(b) Use a similar technique to part (a) in order to find:

(i) $\displaystyle\int \frac{x^2-2x+3}{(x+1)(x-2)}\,dx$ (ii) $\displaystyle\int \frac{3x^2-66}{(x+4)(x-5)}\,dx$

**9.** (a) (i) Find the values of $A$, $B$, $C$ and $D$ such that

$$\frac{x^3 - 3x^2 - 4}{(x+1)(x-3)} = Ax + B + \frac{C}{x+1} + \frac{D}{x-3}\,.$$

(ii) Hence evaluate $\displaystyle\int_0^1 \frac{x^3 - 3x^2 - 4}{(x+1)(x-3)}\,dx\,.$

(b) Use a similar method to evaluate $\displaystyle\int_2^4 \frac{x^3 + 4x^2 + x - 3}{(x+2)(x-1)}\,dx\,.$

**10.** (a) (i) Find the values of $A$ and $B$ such that

$$\frac{3x^2 - 10}{x^2 - 4x + 4} = 3 + \frac{A}{x-2} + \frac{B}{(x-2)^2}\,.$$

(ii) Hence find $\displaystyle\int \frac{3x^2 - 10}{x^2 - 4x + 4}\,dx.$

(b) (i) Find the integers $A$, $B$, $C$ and $D$ such that

$$\frac{3x+7}{(x-1)^2(x-2)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}\,.$$

(ii) Hence find $\displaystyle\int \frac{3x+7}{(x-1)^2(x-2)^2}\,dx\,.$

**11.** Show that:

(a) $\displaystyle\int_4^6 \frac{x^2 - 8}{x^3 + 4x}\,dx = \tfrac{3}{2}\log 2 - 2\log\tfrac{3}{2}.$

(b) $\displaystyle\int_0^2 \frac{1+4x}{(4-x)(x^2+1)}\,dx = \tfrac{1}{2}\log 20.$

**12.** (a) Let $\dfrac{x^2 - 1}{x^4 + x^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx + D}{x^2 + 1}$. Find $A$, $B$, $C$ and $D$.

(b) Hence show that $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{x^2 - 1}{x^4 + x^2}\,dx = \tfrac{1}{3}(\pi - 2\sqrt{3}).$

**13.** Use appropriate methods to find:

(a) $\displaystyle\int \frac{x^2+1}{x^2-1}\,dx$

(b) $\displaystyle\int \frac{x^2+1}{x^2-x}\,dx$

(c) $\displaystyle\int \frac{x^3+1}{x^3+x}\,dx$

(d) $\displaystyle\int \frac{x^2}{x^2-5x+6}\,dx$

(e) $\displaystyle\int \frac{x^3+5}{x^2+x}\,dx$

(f) $\displaystyle\int \frac{x^4}{x^2-3x+2}\,dx$

**EXTENSION**

**14.** Use a similar approach to Question 10 for repeated factors to show that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{5x - x^2}{(x+1)^2(x-1)}\,dx = 4 - 3\log 3\,.$$

**15.** (a) In the notation of the text, if $Q(x)$ is a product of distinct linear factors then:

$$C_k = \lim_{x \to a_k} \frac{P(x)(x - a_k)}{Q(x)}\,.$$

Use this result to prove that

$$C_k = \frac{P(a_k)}{Q'(a_k)}\,.$$

[HINT: What is the value of $Q(a_k)$?]

(b) Use this formula to redo Questions 6(b) and 7(b).

# 2D Quadratics in the Denominator

Many practical applications yield integrals with a quadratic in the denominator. In the simplest cases it is a matter of applying the four standard integral results:

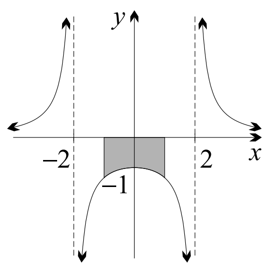
$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} \qquad \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right)$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} \qquad \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Another common integral is $\displaystyle\int \frac{dx}{x^2-a^2}$. Although a formula exists for this, it is not part of the course. It is expected that candidates determine the primitive by use of partial fractions whenever this type of integral is encountered.

**WORKED EXERCISE:** Evaluate $\displaystyle\int_{-1}^{1} \frac{4}{x^2-4}\,dx$.

**SOLUTION:** Let $\dfrac{4}{x^2-4} = \dfrac{A}{x-2} + \dfrac{B}{x+2}$, then




$$4 = A(x+2) + B(x-2)$$

At $x = 2$ $\quad 4 = 4A\,,\quad$ so $\quad A = 1\,.$

At $x = -2$ $\quad 4 = -4B\,,\quad$ so $\quad B = -1\,.$

Hence
$$\begin{aligned}\int_{-1}^{1} \frac{4}{x^2-4}\,dx &= \int_{-1}^{1} \frac{1}{x-2} - \frac{1}{x+2}\,dx \\ &= \Big[\log|x-2| - \log(x+2)\Big]_{-1}^{1} \\ &= (\log 1 - \log 3) - (\log 3 - \log 1) \\ &= -2\log 3\,.\end{aligned}$$

**Quadratics with Linear Terms:** Frequently the quadratic will have a linear term, such as in $3+2x-x^2$. In these instances the method is to complete the square to obtain either the sum of two squares or the difference of two squares.

**WORKED EXERCISE:** Find $\displaystyle\int \frac{1}{\sqrt{3+2x-x^2}}\,dx$.

**SOLUTION:** Completing the square in the denominator:

$$\begin{aligned}\int \frac{1}{\sqrt{3+2x-x^2}}\,dx &= \int \frac{1}{\sqrt{4-(x-1)^2}}\,dx \\ &= \int \frac{1}{\sqrt{4-u^2}}\,du \qquad \text{where } u = x-1 \\ &= \sin^{-1}\tfrac{u}{2} + C \\ &= \sin^{-1}\tfrac{x-1}{2} + C\,.\end{aligned}$$

Notice that the solution uses a substitution. This step may be omitted by using a result from the Mathematics Extension 1 course. Recall that if $F(x)$ is a primitive of $f(x)$ then

$$\int f(ax+b)\,dx = \tfrac{1}{a}F(ax+b) + C\,.$$

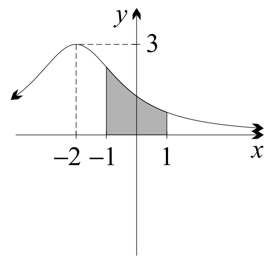
In this particular instance, $f(x) = \dfrac{1}{\sqrt{4-x^2}}$, the primitive is $F(x) = \sin^{-1}\frac{x}{2}$ with $a = 1$ and $b = 1$. Thus it is permissible to write

$$\int \frac{1}{\sqrt{4-(x-1)^2}}\,dx = \sin^{-1}\tfrac{x-1}{2} + C\,,$$

without showing any working. Here is a similar example.

**WORKED EXERCISE:** Find the value of $\displaystyle\int_{-1}^{1} \frac{9}{7+4x+x^2}\,dx$.

**SOLUTION:** Completing the square in the denominator:




$$\begin{aligned}\int_{-1}^{1} \frac{9}{7+4x+x^2}\,dx &= \int_{-1}^{1} \frac{9}{3+(4+4x+x^2)}\,dx\\ &= \int_{-1}^{1} \frac{9}{3+(2+x)^2}\,dx\\ &= \tfrac{9}{\sqrt3}\left[\tan^{-1}\tfrac{x+2}{\sqrt3}\right]_{-1}^{1}\\ &= 3\sqrt3\left(\tfrac{\pi}{3}-\tfrac{\pi}{6}\right)\\ &= \tfrac{\pi\sqrt3}{2}\,.\end{aligned}$$

1 **QUADRATICS WITH LINEAR TERMS:** Complete the square, then use the result

$$\int f(ax+b)\,dx = \tfrac{1}{a}F(ax+b) + C\,,$$

where $F(x)$ is the primitive of $f(x)$.

**Linear Numerators:** So far in all the worked exercises the numerator has been a constant. When the numerator is linear it is best to carefully split it into two parts. The first term should be a multiple of the derivative of the quadratic in the denominator. The second term will then be a constant.

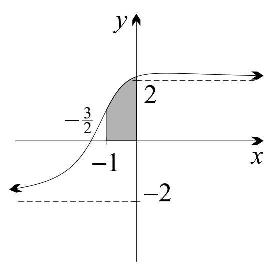
**WORKED EXERCISE:** Determine $\displaystyle\int \frac{4x+3}{x^2+9}\,dx$.

**SOLUTION:**

$$\begin{aligned}\int \frac{4x+3}{x^2+9}\,dx &= 2\int \frac{2x}{x^2+9}\,dx + \int \frac{3}{x^2+9}\,dx\\ &= 2\log(x^2+9) + \tan^{-1}\tfrac{x}{3} + C\,.\end{aligned}$$

In harder examples the quadratic will also contain a linear term.

**WORKED EXERCISE:** Evaluate $\displaystyle\int_{-1}^{0} \frac{2x+3}{\sqrt{x^2+2x+2}}\,dx$.




**SOLUTION:**

$$\begin{aligned}&\int_{-1}^{0} \frac{2x+3}{\sqrt{x^2+2x+2}}\,dx\\ &= \int_{-1}^{0} \frac{2x+2}{\sqrt{x^2+2x+2}}\,dx + \int_{-1}^{0} \frac{1}{\sqrt{(x+1)^2+1}}\,dx\end{aligned}$$

$$= \Big[2\sqrt{x^2+2x+2}\,\Big]_{-1}^{0} + \Big[\log\big((x+1)+\sqrt{(x+1)^2+1}\,\big)\Big]_{-1}^{0}$$
$$= 2\sqrt{2} - 2 + \log(1+\sqrt{2}) - \log 1$$
$$= 2(\sqrt{2}-1) + \log(1+\sqrt{2})\,.$$

**2** LINEAR NUMERATORS: When the numerator is linear it is best to split it into a multiple of the derivative of the quadratic in the denominator plus a constant.

**Rationalising the Numerator:** In much previous work it has been convenient to rationalise the denominator when a surd appears. In contrast, when calculus is involved it is often convenient to rationalise the numerator instead.

**WORKED EXERCISE:** Find $\displaystyle\int \sqrt{\frac{x+1}{x+7}}\,dx\,.$

**SOLUTION:** Rationalising the numerator

$$\begin{aligned}\int \sqrt{\frac{x+1}{x+7}}\,dx &= \int \frac{x+1}{\sqrt{x^2+8x+7}}\,dx\\ &= \int \frac{x+4}{\sqrt{x^2+8x+7}}\,dx - \int \frac{3}{\sqrt{x^2+8x+7}}\,dx\\ &= \int \frac{x+4}{\sqrt{x^2+8x+7}}\,dx - \int \frac{3}{\sqrt{(x+4)^2-3^2}}\,dx\\ &= \sqrt{x^2+8x+7} - 3\log\Big((x+4)+\sqrt{(x+4)^2-3^2}\Big) + C\,.\end{aligned}$$

Notice that in the first line of working, by rationalising, the numerator has become linear. This is typical of the questions done in this section.

Care is needed when applying this technique to definite integrals. For example whilst $\displaystyle\int_{-1}^{1} \sqrt{\frac{x+1}{x+7}}\,dx$ is well defined, the resulting integral $\displaystyle\int_{-1}^{1} \frac{x+1}{\sqrt{x^2+8x+7}}\,dx$ is not, since the denominator is zero at the lower limit. Definite integrals of this type are dealt with in the last section of this chapter.

**3** RATIONALISING THE NUMERATOR: When calculus is involved it is often convenient to rationalise the numerator.

## Exercise **2D**

**1.** Find:

(a) $\displaystyle\int \frac{1}{9+x^2}\,dx$ (c) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$ (e) $\displaystyle\int \frac{1}{x^2-9}\,dx$

(b) $\displaystyle\int \frac{1}{\sqrt{9+x^2}}\,dx$ (d) $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx$ (f) $\displaystyle\int \frac{1}{9-x^2}\,dx$

**2.** Determine:

(a) $\displaystyle\int \frac{1}{x^2+4x+5}\,dx$ (c) $\displaystyle\int \frac{1}{\sqrt{x^2-6x+13}}\,dx$ (e) $\displaystyle\int \frac{1}{\sqrt{9+8x-x^2}}\,dx$

(b) $\displaystyle\int \frac{1}{x^2-4x+20}\,dx$ (d) $\displaystyle\int \frac{1}{\sqrt{x^2+8x+12}}\,dx$ (f) $\displaystyle\int \frac{1}{\sqrt{4x^2+8x+6}}\,dx$

**3.** Evaluate:

(a) $\displaystyle\int_1^3 \frac{1}{x^2-2x+5}\,dx$ (c) $\displaystyle\int_{-1}^0 \frac{1}{\sqrt{3-2x-x^2}}\,dx$ (e) $\displaystyle\int_{-1}^3 \frac{1}{\sqrt{x^2+2x+10}}\,dx$

(b) $\displaystyle\int_1^5 \frac{4}{x^2-6x+13}\,dx$ (d) $\displaystyle\int_0^1 \frac{3}{\sqrt{3+4x-4x^2}}\,dx$ (f) $\displaystyle\int_{\frac{1}{2}}^1 \frac{2}{\sqrt{x^2-x+1}}\,dx$

**DEVELOPMENT**

**4.** Find:

(a) $\displaystyle\int \frac{2x+1}{x^2+2x+2}\,dx$ (c) $\displaystyle\int \frac{x}{\sqrt{x^2+2x+10}}\,dx$ (e) $\displaystyle\int \frac{x}{\sqrt{6x-x^2}}\,dx$

(b) $\displaystyle\int \frac{x}{x^2+2x+10}\,dx$ (d) $\displaystyle\int \frac{x+3}{\sqrt{x^2-2x-4}}\,dx$ (f) $\displaystyle\int \frac{x+3}{\sqrt{4-2x-x^2}}\,dx$

**5.** Find the value of:

(a) $\displaystyle\int_0^2 \frac{x+1}{x^2+4}\,dx$ (c) $\displaystyle\int_1^2 \frac{2x-3}{x^2-2x+2}\,dx$ (e) $\displaystyle\int_{-1}^3 \frac{1-2x}{\sqrt{x^2+2x+3}}\,dx$

(b) $\displaystyle\int_1^2 \frac{x+1}{x^2-4x+5}\,dx$ (d) $\displaystyle\int_{-1}^0 \frac{x}{\sqrt{3-2x-x^2}}\,dx$ (f) $\displaystyle\int_0^1 \frac{x+3}{\sqrt{x^2+4x+1}}\,dx$

**6.** Determine each primitive.

(a) $\displaystyle\int \sqrt{\frac{x-1}{x+1}}\,dx$ (b) $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx$ (c) $\displaystyle\int \sqrt{\frac{3-x}{2+x}}\,dx$

**7.** Evaluate:

(a) $\displaystyle\int_{-1}^0 \sqrt{\frac{1-x}{x+3}}\,dx$ (b) $\displaystyle\int_{-1}^0 \sqrt{\frac{x+2}{1-x}}\,dx$ (c) $\displaystyle\int_0^1 \sqrt{\frac{x+1}{x+3}}\,dx$

**EXTENSION**

**8.** (a) Why is it not valid to evalute $\displaystyle\int_0^2 \sqrt{\frac{x}{4-x}}\,dx$ using the techniques of this section?

(b) Nevertheless, show that its value is $\displaystyle\lim_{\epsilon\to 0^+}\int_\epsilon^2 \sqrt{\frac{x}{4-x}}\,dx = \pi - 2\,.$

**9.** (a) Show that $x^3+3x^2+5x+1 = (x+1)(x^2+2x+2)+(x-1)\,.$

(b) Hence or otherwise show that

$$\int_{-1}^0 \frac{x^3+3x^2+5x+1}{\sqrt{x^2+2x+2}}\,dx = \tfrac{1}{3}(5\sqrt{2}-4) - 2\log(1+\sqrt{2}\,)\,.$$

# 2E Integration by Parts

Whilst there are well known and relatively simple formulae for the derivatives of products and quotients of functions, there are no such general formulae for the integrals of products and quotients. Nevertheless, as was found in the previous two sections, certain quotients can be integrated relatively easily. In this section, a method of integration is developed that can be applied to certain types of products. It begins with the product rule for differentiation.

Now $$\frac{d}{dx}(uv) = u'v + u\,v'\,.$$

Swapping sides and integrating yields
$$\int u'v\,dx + \int u\,v'\,dx = uv\,,$$
hence $$\int u\,v'\,dx = uv - \int u'v\,dx\,.$$

This last equation provides a way to rearrange an integral of one product into an integral of a different product. The formula is applied with the aim that the new integral is in some way simpler. The process is called *integration by parts*.

**WORKED EXERCISE:** Use integration by parts to find $\displaystyle\int xe^x\,dx$.

**SOLUTION:**

Let $$\begin{aligned} I &= \int xe^x\,dx \\ &= \int u\,v'\,dx\,, \end{aligned}$$
where $u = x$ and $v' = e^x$
so $u' = 1$ and $v = e^x$.

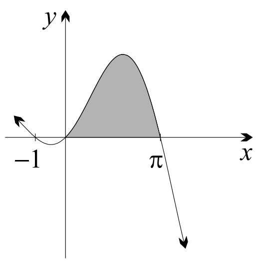
Hence $$\begin{aligned} I &= uv - \int u'v\,dx \\ &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C \end{aligned}$$
or $I = e^x(x-1) + C$.

Notice the lack of any constant of integration until the process is finished.

4 INTEGRATION BY PARTS: The integral of the product $u\,v'$ can be rearranged using the integration by parts formula:
$$\int u\,v'\,dx = uv - \int u'v\,dx\,.$$

**Reducing Polynomials:** When one of the factors of the integrand is a polynomial, it is common to let $u$ be that polynomial. In that way the new integral, which depends on $u'$, will contain a polynomial of lesser degree. That is, the aim is to reduce the degree of the polynomial.

**WORKED EXERCISE:** Evaluate $\displaystyle\int_0^{\pi} (x+1)\sin x\,dx$.




**SOLUTION:**

Let $$\begin{aligned} I &= \int_0^{\pi} (x+1)\sin x\,dx \\ &= \int_0^{\pi} u\,v'\,dx\,, \end{aligned}$$

where $u = (x+1)$ and $v' = \sin x$
so $u' = 1$ and $v = -\cos x$.

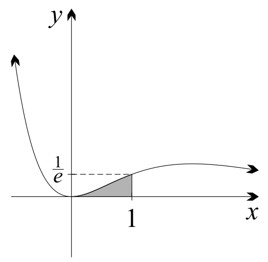
$$\text{Thus } I = \Big[uv\Big]_0^{\pi} - \int_0^{\pi} u'v\,dx$$
$$= \Big[-(x+1)\cos x\Big]_0^{\pi} + \int_0^{\pi} \cos x\,dx$$
$$= (\pi+1) + 1 + \Big[\sin x\Big]_0^{\pi},$$

hence $I = \pi + 2$.

**Repeated Applications:** It may be necessary to apply integration by parts more than once in order to complete the process of integration. In simpler examples it may be possible to do some of the steps mentally.

**WORKED EXERCISE:** Evaluate $\displaystyle\int_0^1 x^2 e^{-x}\,dx$.

**SOLUTION:**




Let $\displaystyle I = \int_0^1 x^2 e^{-x}\,dx$

and put $u = x^2$ and $v' = e^{-x}$
so $u' = 2x$ and $v = -e^{-x}$.

$$\text{Then } I = \Big[-x^2 e^{-x}\Big]_0^1 + \int_0^1 2xe^{-x}\,dx \qquad \text{(by parts.)}$$

Now put $u = 2x$ and $v' = e^{-x}$
so $u' = 2$ and $v = -e^{-x}$.

$$\text{Thus } I = -e^{-1} + \left(\Big[-2xe^{-x}\Big]_0^1 + \int_0^1 2e^{-x}\,dx\right) \qquad \text{(by parts again)}$$
$$= -e^{-1} - 2e^{-1} - \Big[2e^{-x}\Big]_0^1$$
$$= 2 - 5e^{-1}.$$

**Exceptions with Polynomials:** Although it is common to reduce the degree of a polynomial using integration by parts, there are many exceptions. In this course these exceptions typically involve the logarithm function.

**WORKED EXERCISE:** Determine $\displaystyle\int x\log x\,dx$.

**SOLUTION:**

Let $\displaystyle I = \int x\log x\,dx$

and put $u = \log x$ and $v' = x$
so $u' = \frac{1}{x}$ and $v = \frac{1}{2}x^2$.

$$\text{Thus } I = \tfrac{1}{2}x^2\log x - \int \tfrac{1}{2}x^2 \times \tfrac{1}{x}\,dx \qquad \text{(by parts)}$$
$$= \tfrac{1}{2}x^2\log x - \int \tfrac{1}{2}x\,dx$$
$$= \tfrac{1}{2}x^2\log x - \tfrac{1}{4}x^2 + C$$

or $I = \frac{1}{4}x^2(2\log x - 1) + C$.

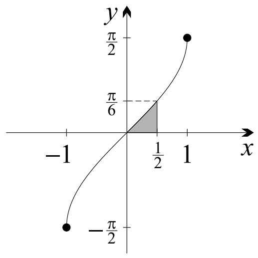
**Integrands where $v' = 1$:** The prime number 5 has only two distinct factors, namely $1 \times 5$. In the same way we may treat a function like a prime and write:

$$\sin^{-1} x = 1 \times \sin^{-1} x\,.$$

This somewhat artificial form of factoring is applied to facilitate integration by parts. It is then usual to put $u$ equal to the function and $v' = 1$.

**WORKED EXERCISE:** Find the value of $\displaystyle\int_0^{\frac{1}{2}} \sin^{-1} x\,dx$.

**SOLUTION:**




Let $\displaystyle I = \int_0^{\frac{1}{2}} \sin^{-1} x\,dx$

$\displaystyle \quad = \int_0^{\frac{1}{2}} 1 \times \sin^{-1} x\,dx\,.$

Put $u = \sin^{-1} x$ and $v' = 1$

so $u' = \dfrac{1}{\sqrt{1-x^2}}$ and $v = x$.

$$\text{Thus } I = \Big[x\sin^{-1} x\Big]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx \qquad \text{(by parts)}$$

$$= \Big[x\sin^{-1} x + \sqrt{1-x^2}\,\Big]_0^{\frac{1}{2}}$$

$$= \left(\tfrac{1}{2} \times \tfrac{\pi}{6} + \sqrt{\tfrac{3}{4}}\right) - (0+1)$$

$$= \tfrac{\pi}{12} + \tfrac{\sqrt{3}}{2} - 1\,.$$

**A Recurrence of the Integral:** Integration by parts may lead to a recurrence of the original integral. It is then simply a matter of collecting like terms.

**WORKED EXERCISE:** Find a primitive of $e^x \sin x$.

**SOLUTION:**

Let $\displaystyle I = \int e^x \sin x\,dx$

and put $u = \sin x$ and $v' = e^x$

so $u' = \cos x$ and $v = e^x$.

Then $\displaystyle I = e^x \sin x - \int e^x \cos x\,dx$ (by parts)

Now put $u = \cos x$ and $v' = e^x$

so $u' = -\sin x$ and $v = e^x$.

$$\text{Thus } I = e^x \sin x - \left(e^x \cos x + \int e^x \sin x\,dx\right) \qquad \text{(by parts again)}$$

$$= e^x(\sin x - \cos x) - I$$

or $2I = e^x(\sin x - \cos x)$

hence $I = \frac{1}{2}e^x(\sin x - \cos x) + C$ is the general primitive.

In this example it was important to apply the method consistently. Notice that $u$ was always the trigonometric function and $v'$ was always the exponential function. As an exercise to highlight the significance of these choices, repeat the worked exercise but put $u = e^x$ and $v' = \cos x$ at the second integration by parts.

# Exercise **2E**

**1.** Find:

(a) $\int xe^x \, dx$
(b) $\int xe^{-x} \, dx$
(c) $\int (x+1)e^{3x} \, dx$
(d) $\int x \cos x \, dx$
(e) $\int (x-1)\sin 2x \, dx$
(f) $\int (2x-3)\sec^2 x \, dx$

**2.** Evaluate:

(a) $\int_0^{\pi} x \sin x \, dx$
(b) $\int_0^{\frac{\pi}{2}} x \cos x \, dx$
(c) $\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$
(d) $\int_0^{1} xe^{2x} \, dx$
(e) $\int_0^{1} (1-x)e^{-x} \, dx$
(f) $\int_{-2}^{0} (x+2)e^{x} \, dx$

**3.** In these questions put $v' = 1$.

(a) $\int \log x \, dx$
(b) $\int \log(x^2) \, dx$
(c) $\int \cos^{-1} x \, dx$

**4.** Find the value of:

(a) $\int_0^1 \tan^{-1} x \, dx$
(b) $\int_1^e \log x \, dx$
(c) $\int_1^e \log \sqrt{x} \, dx$

**5.** In each case use integration by parts to increase the power of $x$.

(a) $\int x \log x \, dx$
(b) $\int x^2 \log x \, dx$
(c) $\int \frac{\log x}{x^2} \, dx$

**DEVELOPMENT**

**6.** Use repeated applications of integration by parts in order to find:

(a) $\int x^2 e^x \, dx$
(b) $\int x^2 \cos x \, dx$
(c) $\int (\log x)^2 \, dx$

**7.** These integrals are more naturally done by substitution. Nevertheless they can also be done by parts. Use integration by parts here and then compare your answers with similar questions in Exercise 2B.

(a) $\int_0^1 x(x-1)^5 \, dx$
(b) $\int_0^1 x\sqrt{x+1} \, dx$
(c) $\int_0^4 x\sqrt{4-x} \, dx$

**8.** Determine: (a) $\int e^x \cos x \, dx$ (b) $\int e^{-x} \sin x \, dx$

**9.** Evaluate: (a) $\int_0^{\frac{\pi}{2}} e^{2x} \cos x \, dx$ (b) $\int_0^{\frac{\pi}{4}} e^{x} \sin 2x \, dx$

**10.** Use integration by parts to evaluate:

(a) $\int_0^{\frac{\sqrt{3}}{2}} \sin^{-1} x \, dx$
(b) $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \cos^{-1} x \, dx$
(c) $\int_0^1 4x \tan^{-1} x \, dx$

**11.** Show that:

(a) $\int_0^{\pi} x^2 \cos 2x \, dx = \frac{\pi}{2}$
(b) $\int_0^{\pi} x^2 \sin \frac{1}{2}x \, dx = 8\pi - 16$
(c) $\int_1^e \sin(\log x) \, dx = \frac{1}{2}e(\sin 1 - \cos 1) + \frac{1}{2}$
(d) $\int_1^e \cos(\log x) \, dx = \frac{1}{2}e(\sin 1 + \cos 1) - \frac{1}{2}$

**12.** Determine formulae for the following:

(a) $\int \sqrt{a^2 - x^2} \, dx$
(b) $\int \log(x + \sqrt{x^2 + a^2}\,) \, dx$
(c) $\int \log(x + \sqrt{x^2 - a^2}\,) \, dx$

**13.** (a) Determine $\displaystyle\int x\log x\,dx$. (b) Hence find $\displaystyle\int x(\log x)^2\,dx$.

**14.** Use trigonometric identities and then integration by parts to show that:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} x\sin x\cos x\,dx = \tfrac{\pi}{8}$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} x\sin^2 x\,dx = \tfrac{1}{16}(\pi^2+4)$

(c) $\displaystyle\int_0^{\frac{\pi}{4}} x\tan^2 x\,dx = \tfrac{\pi}{4} - \tfrac{\pi^2}{32} - \tfrac{1}{2}\log 2$

(d) $\displaystyle\int_0^{\pi} x^2(\cos^2 x - \sin^2 x)\,dx = \tfrac{\pi}{2}$

**EXTENSION**

**15.** Determine:

(a) $\displaystyle\int x\sin x\cos 3x\,dx$ (b) $\displaystyle\int x\cos 2x\cos x\,dx$ (c) $\displaystyle\int e^x\sin 2x\cos x\,dx$

**16.** Determine: (a) $\displaystyle\int_0^{\frac{1}{2}} x\sin^{-1} x\,dx$ (b) $\displaystyle\int_0^{1} x^2\tan^{-1} x\,dx$

**17.** Let $s$ be a positive constant. Show that $\displaystyle\lim_{N\to\infty}\int_0^N te^{-st}\,dt = \frac{1}{s^2}$.

# 2F Trigonometric Integrals

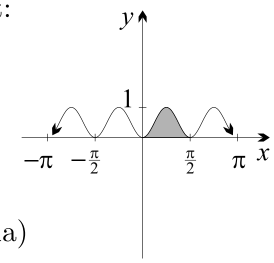
**Powers of Cosine and Sine:** There are two methods for the integral

$$\int \cos^m x\sin^n x\,dx$$

depending on whether the constants $m$ and $n$ are odd or even. If both are even then it is best to use the double angle identities.

**WORKED EXERCISE:** Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} 4\cos^2 x\sin^2 x\,dx$

**SOLUTION:** Apply the double angle formula for sine to get:




$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} 4\cos^2 x\sin^2 x\,dx\\
&= \int_0^{\frac{\pi}{2}} \sin^2 2x\,dx\\
&= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} 1-\cos 4x\,dx \qquad \text{(cosine double angle formula)}\\
&= \tfrac{1}{2}\Big[x - \tfrac{1}{4}\sin 4x\Big]_0^{\frac{\pi}{2}}\\
&= \tfrac{\pi}{4}\,.
\end{aligned}$$

In the second method one or both of $m$ and $n$ is odd. Work with cosine if $m$ is odd, otherwise work with sine. The odd index of the chosen trigonometric function can be reduced to 1 via the Pythagorean identity, $\cos^2 x + \sin^2 x = 1$. It is then a matter of making a substitution for the other trigonometric function. The result is a polynomial integral.

**WORKED EXERCISE:** Determine $\int \cos^3 x \sin^2 x \, dx$.

**SOLUTION:**

Let $I = \int \cos^3 x \sin^2 x \, dx$

$= \int \cos x (1 - \sin^2 x) \sin^2 x \, dx$ (by Pythagoras.)

Put $u = \sin x$,

so that $du = \cos x \, dx$,

then $I = \int (1 - u^2) u^2 \, du$

$= \int u^2 - u^4 \, dx$

$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$

$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$.

**5** POWERS OF COSINE AND SINE: Given an integral of the form $\int \cos^m x \sin^n x \, dx$:

- if $m$ and $n$ are both even then use the double angle formulae,
- if either $m$ or $n$ is odd then use the Pythagorean identity and a substitution.

**Powers of Secant and Tangent:** There are three general methods for the integral

$$\int \sec^m x \tan^n x \, dx,$$

again depending on whether the constants $m$ and $n$ are odd or even. There are also two special cases which should be dealt with first.

When $m = 0$ and $n = 1$ the situation is trivial, viz:

$$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\ &= -\log(\cos x) + C.\end{aligned}$$

A very clever trick is required for the other special case when $m = 1$ and $n = 0$.

$$\begin{aligned}\int \sec x \, dx &= \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} \, dx \\ &= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \, dx \\ &= \log(\sec x + \tan x) + C.\end{aligned}$$

Notice that in both special cases the result is a logarithmic function since the numerator of the integrand can be written as the derivative of the denominator.

**6** THE INTEGRALS OF THE TANGENT AND SECANT FUNCTIONS:

$$\int \tan x \, dx = -\log(\cos x) + C$$

$$\int \sec x \, dx = \log(\sec x + \tan x) + C$$

Now for the general cases. If $m$ and $n$ are both even then separate out a factor of $\sec^2 x$ and substitute $u = \tan x$. The Pythagorean identity $1 + \tan^2 x = \sec^2 x$ may be required, particularly when $m = 0$.

**WORKED EXERCISE:** Find $\displaystyle\int \tan^4 x\,dx$.

**SOLUTION:**

$$\begin{aligned}
\int \tan^4 x\,dx &= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx && \text{(by Pythagoras)}\\
&= \int \tan^2 x \sec^2 x\,dx - \int \sec^2 x - 1\,dx && \text{(by Pythagoras again)}\\
&= \int u^2\,du - \int \sec^2 x\,dx + \int 1\,dx && \text{where } u = \tan x\\
&= \tfrac{1}{3}u^3 - \tan x + x + C\\
&= \tfrac{1}{3}\tan^3 x - \tan x + x + C\,.
\end{aligned}$$

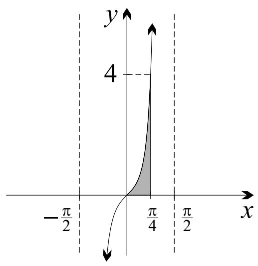
**WORKED EXERCISE:** Show that $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x \tan^2 x\,dx = \tfrac{8}{15}$.

**SOLUTION:** Let $\displaystyle I = \int_0^{\frac{\pi}{4}} \sec^4 x \tan^2 x\,dx$

so $\displaystyle I = \int_0^{\frac{\pi}{4}} \sec^2 x\,(\tan^2 x + 1)\tan^2 x\,dx$ (by Pythagoras.)

Put $u = \tan x$,

then
$$\begin{aligned}
I &= \int_0^1 (u^2+1)u^2\,du\\
&= \int_0^1 u^4 + u^2\,du\\
&= \Big[\tfrac{1}{5}u^5 + \tfrac{1}{3}u^3\Big]_0^1\\
&= \tfrac{8}{15}\,.
\end{aligned}$$




If $n$ is odd then factor out the term $\sec x \tan x$ and substitute $u = \sec x$. The Pythagorean identity may be required.

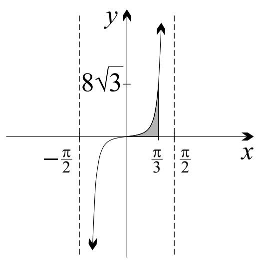
**WORKED EXERCISE:** Determine the value of $\displaystyle\int_0^{\frac{\pi}{3}} \sec^3 x \tan x\,dx$.

**SOLUTION:** Let $\displaystyle I = \int_0^{\frac{\pi}{3}} \sec^3 x \tan x\,dx$,

so $\displaystyle I = \int_0^{\frac{\pi}{3}} \sec^2 x \times \sec x \tan x\,dx$.

Put $u = \sec x$,

then
$$\begin{aligned}
I &= \int_1^2 u^2\,du\\
&= \Big[\tfrac{1}{3}u^3\Big]_1^2\\
&= \tfrac{7}{3}\,.
\end{aligned}$$



Whenever $m$ is odd and $n$ is even it is best to integrate by parts. Once again the Pythagorean identity may be required.

**WORKED EXERCISE:** Find $\displaystyle\int \sec^3\, dx$.

**SOLUTION:**

$$\begin{aligned}
\text{Let}\quad I &= \int \sec^3\, dx \\
&= \int \sec^2 x \times \sec x\, dx\,.
\end{aligned}$$

Put $u = \sec x$ and $v' = \sec^2 x$
so $u' = \sec x \tan x$ and $v = \tan x$.

$$\begin{aligned}
\text{Thus}\quad I &= \sec x \tan x - \int \sec x \tan^2 x\, dx \qquad \text{(by parts)} \\
&= \sec x \tan x - \int \sec x(\sec^2 x - 1)\, dx \qquad \text{(by Pythagoras)} \\
&= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx\,.
\end{aligned}$$

$$\begin{aligned}
\text{So}\quad I &= \sec x \tan x - I + \log(\sec x + \tan x) \qquad \text{(from the special case)} \\
\text{or}\quad 2I &= \sec x \tan x + \log(\sec x + \tan x)\,, \\
\text{hence}\quad I &= \tfrac{1}{2}\big(\sec x \tan x + \log(\sec x + \tan x)\big) + C\,.
\end{aligned}$$

**7**

> POWERS OF SECANT AND TANGENT: Given an integral of the form $\displaystyle\int \sec^m x \tan^n x\, dx$:
>
> - if $m$ and $n$ are both even then factor out $\sec^2 x$ and substitute $u = \tan x$
> - if $n$ is odd then factor out the term $\sec x \tan x$ and substitute $u = \sec x$
> - if $m$ is odd and $n$ is even then use integration by parts

**Products to Sums:** There are three standard formulae for converting products of trigonometric functions to sums. These will be familiar to some readers and are easily proved by expanding each right hand side.

**8**

> PRODUCTS TO SUMS:
>
> $$\begin{aligned}
> \sin A \cos B &= \tfrac{1}{2}\big(\sin(A-B) + \sin(A+B)\big) \\
> \cos A \cos B &= \tfrac{1}{2}\big(\cos(A-B) + \cos(A+B)\big) \\
> \sin A \sin B &= \tfrac{1}{2}\big(\cos(A-B) - \cos(A+B)\big)
> \end{aligned}$$

These formulae can be applied to simplify an integral, as in the following example.

**WORKED EXERCISE:** Find $\displaystyle\int \cos 3x \cos 2x\, dx$.

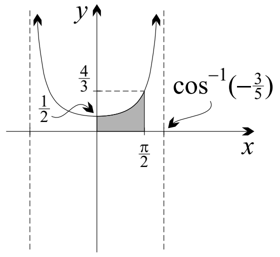
**SOLUTION:**

$$\begin{aligned}
\int \cos 3x \cos 2x\, dx &= \tfrac{1}{2}\int \cos x + \cos 5x\, dx \qquad \text{(products to sums)} \\
&= \tfrac{1}{2}\sin x + \tfrac{1}{10}\sin 5x + C\,.
\end{aligned}$$

**The $t$-substitution:** The $t$-substitution, namely $t = \tan\frac{x}{2}$, should be well known to readers, being part of the Mathematics Extension 1 course. Here it is applied to some harder integral problems.

**WORKED EXERCISE:** Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{4}{3+5\cos x}\,dx = \log 3\,.$

**SOLUTION:** Let $\displaystyle I = \int_0^{\frac{\pi}{2}} \frac{4}{3+5\cos x}\,dx\,.$




Put $\quad t = \tan\frac{x}{2}$

so that $\quad dx = \dfrac{2\,dt}{1+t^2}$

and $\quad \cos x = \dfrac{1-t^2}{1+t^2}\,,$

$$\text{then}\qquad \begin{aligned} I &= \int_0^1 \frac{4}{3+5\,\frac{1-t^2}{1+t^2}} \times \frac{2\,dt}{1+t^2} \\ &= \int_0^1 \frac{8}{8-2t^2}\,dt \\ &= \int_0^1 \frac{4}{4-t^2}\,dt\,. \end{aligned}$$

Let $\quad \dfrac{4}{(2-t)(2+t)} = \dfrac{A}{2-t} + \dfrac{B}{2+t} \qquad$ (partial fractions)

then $\qquad A = 1 \quad$ and $\quad B = 1 \qquad$ (by the cover up method.)

$$\begin{aligned} \text{Thus }\ I &= \int_0^1 \frac{1}{2-t} + \frac{1}{2+t}\,dt \\ &= \Big[\log(2+t) - \log(2-t)\Big]_0^1 \\ &= \log 3\,. \end{aligned}$$

As a final note, take care with this method if the limits of integration include odd multiples of $\pi$ since $\tan\frac{x}{2}$ is undefined there. Definite integrals of this type are dealt with in the last section of this chapter.

## Exercise **2F**

**1.** Find:

(a) $\displaystyle\int \cos x\,dx$ (b) $\displaystyle\int \sin x\,dx$ (c) $\displaystyle\int \tan x\,dx$ (d) $\displaystyle\int \cot x\,dx$

**2.** Each of the following can be found with a substitution; either $u = \sin x$ or $u = \cos x$. You may also need to apply the Pythagorean identity $\cos^2 x + \sin^2 x = 1$.

(a) $\displaystyle\int \cos x \sin^2 x\,dx$ (c) $\displaystyle\int \sin^3 x\,dx$ (e) $\displaystyle\int \cos^5 x\,dx$

(b) $\displaystyle\int \cos^2 x \sin x\,dx$ (d) $\displaystyle\int \cos^3 x\,dx$ (f) $\displaystyle\int \sin^3 x \cos^3 x\,dx$

**3.** Use the double angle formulae to evaluate:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$ (b) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 x\,dx$ (c) $\displaystyle\int_0^{\pi} \sin^2 x \cos^2 x\,dx$

**4.** Use the substitution $u = \tan x$ to find the following. You may also need to apply the Pythagorean identity $1 + \tan^2 x = \sec^2 x$.

(a) $\displaystyle\int \sec^2 x\,dx$ (b) $\displaystyle\int \tan^2 x\,dx$ (c) $\displaystyle\int \sec^4 x\,dx$ (d) $\displaystyle\int \tan^4 x\,dx$

**5.** Use the substitution $u = \sec x$ to help evaluate the following. You may also need to apply the Pythagorean identity $1 + \tan^2 x = \sec^2 x$.

(a) $\displaystyle\int_0^{\frac{\pi}{4}} \sec x \tan x\,dx$ (c) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^3 x\,dx$ (e) $\displaystyle\int_0^{\frac{\pi}{4}} \sec x \tan^3 x\,dx$

(b) $\displaystyle\int_0^{\frac{\pi}{6}} \sec^3 x \tan x\,dx$ (d) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \tan x\,dx$ (f) $\displaystyle\int_0^{\frac{\pi}{3}} \sec^3 x \tan^3 x\,dx$

**DEVELOPMENT**

**6.** Evaluate:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^3 x \sin x\,dx$ (c) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 x \cos x\,dx$ (e) $\displaystyle\int_0^{\pi} \sin^3 x \cos^2 x\,dx$

(b) $\displaystyle\int_0^{\frac{\pi}{6}} \cos^3 x\,dx$ (d) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^5 x\,dx$ (f) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 x \cos^3 x\,dx$

**7.** Determine:

(a) $\displaystyle\int \cos^4 x\,dx$ (b) $\displaystyle\int \sin^4 x\,dx$ (c) $\displaystyle\int \sin^4 x \cos^4 x\,dx$

**8.** Show that:

(a) $\displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x \tan^2 x\,dx = \sqrt{3}$ (c) $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x \tan x\,dx = \tfrac{3}{4}$

(b) $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \tan^3 x\,dx = 2\tfrac{2}{9}$ (d) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^5 x\,dx = \tfrac{1}{4}(2\log 2 - 1)$

**9.** Use the $t$-substitution to help evaluate:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{4+5\cos x}\,dx$ (c) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5+3\sin x}\,dx$

**10.** In each case use a suitable trigonometric substitution to evaluate the integral.

(a) $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$ (b) $\displaystyle\int_0^1 x^3\sqrt{1+x^2}\,dx$ (c) $\displaystyle\int_0^1 x^2\sqrt{1-x^2}\,dx$

**11.** Let $I = \displaystyle\int \sin x \cos x\,dx$.

(a) Find $I$ using a suitable substitution. (b) Find $I$ by the double angle formulae.

(c) Show that the answers to parts (a) and (b) are equivalent.

**12.** (a) Use the $t$-substitution to show that $\displaystyle\int \sec x\,dx = \log\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right) + C$

(b) Show that this answer and the result in Box 6 are equivalent.

**13.** Use integration by parts to find the following. You may also need to apply the Pythagorean identity $1 + \tan^2 x = \sec^2 x$.

(a) $\displaystyle\int \sec x \tan^2 x\,dx$. (b) $\displaystyle\int \sec^3 x\,dx$ (c) $\displaystyle\int 8\sec^5 x\,dx$

**14.** Evaluate:

(a) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 x \sec^2 x\, dx$ (c) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^3 x + \tan x\, dx$

(b) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 x \sec^4 x\, dx$ (d) $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos x - \cos^3 x\, dx$

**15.** Find these integrals by first converting the products to sums:

(a) $\displaystyle\int \sin 3x \cos x\, dx$ (b) $\displaystyle\int \cos 3x \sin x\, dx$ (c) $\displaystyle\int \cos 6x \cos 2x\, dx$

**16.** Evaluate these integrals by first converting the products to sums.

(a) $\displaystyle\int_0^{\frac{\pi}{4}} \sin 3x \sin x\, dx$ (b) $\displaystyle\int_0^{\frac{\pi}{4}} \cos 4x \cos 2x\, dx$ (c) $\displaystyle\int_0^{\frac{\pi}{3}} \sin 4x \cos 2x\, dx$

**17.** Use the substitution $t = \tan \frac{x}{2}$ to determine:

(a) $\displaystyle\int \frac{1}{1+\cos x}\, dx$ (b) $\displaystyle\int \frac{1}{1+\sin x - \cos x}\, dx$ (c) $\displaystyle\int \frac{1}{3\sin x + 4\cos x}\, dx$

**EXTENSION**

**18.** Find $\displaystyle\int x \sec x \tan x\, dx$.

**19.** Repeat Question 16 for the integral $\displaystyle\int \sin^3 x \cos^3 x\, dx$.

**20.** In the chapter on complex numbers it was shown that $(\operatorname{cis}\theta)^3 = \operatorname{cis} 3\theta$. Use this result to help determine $\displaystyle\int \cos^3 \theta\, d\theta$.

**21.** Show that $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x \sec^3 x\, dx = \tfrac{1}{8}\big(3\sqrt{2} - \log(\sqrt{2}+1)\big)$.

# 2G Reduction Formulae

The reader should already be familiar with sequences and series, such as the sequence of odd numbers,

$$1,\ 3,\ 5,\ 7,\ \ldots \qquad \text{or} \qquad u_n = 2n-1\,,$$

or the powers of 2,

$$1,\ 2,\ 4,\ 8,\ \ldots \qquad \text{or} \qquad u_n = 2^{n-1}\,.$$

In this section the sequences are sequences of integrals, such as

$$\int_0^{\frac{\pi}{2}} \sin x\, dx\,,\ \int_0^{\frac{\pi}{2}} \sin^2 x\, dx\,,\ \int_0^{\frac{\pi}{2}} \sin^3 x\, dx\,,\ \ldots \qquad \text{or} \qquad I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx\,.$$

Of particular interest are the equations which relate the terms of the sequence.

Continuing with the above example, if

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx\,,$$

it can be shown that

$$I_n = \tfrac{n-1}{n} \times I_{n-2}\,.$$

Such equations are called *reduction formulae*, because they enable the index to be reduced, in this case from $n$ to $n-2$. In practical terms, this means that if one of the integrals in the sequence is known then other terms can be simply calculated from it without the need for further integration. Returning to the above example, since

$$I_1 = \int_0^{\frac{\pi}{2}} \sin x\,dx = 1\,,$$

it follows that $I_3 = \frac{2}{3}I_1 = \frac{2}{3}\,,$

and $I_5 = \frac{4}{5}I_3 = \frac{8}{15}\,.$

This is obviously a significant saving of effort since it was not necessary to find the primitives of $\sin^3 x$ and $\sin^5 x$ in order to evaluate $I_3$ and $I_5$. It should now be clear that reduction formulae are of particular importance.

Also note that by convention the index of the sequence is applied before the integral is evaluated. Thus

$$\begin{aligned} I_0 &= \int_0^{\frac{\pi}{2}} 1\,dx \\ &= \tfrac{\pi}{2}\,. \end{aligned}$$

**Identities:** In a few cases the reduction formula can be generated by use of an identity.

**WORKED EXERCISE:** Let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x\,dx\,.$

(a) Show that $I_n = \dfrac{1}{n-1} - I_{n-2}$ for $n \ge 2$. (b) Evaluate $I_1$ and hence find $I_5$.

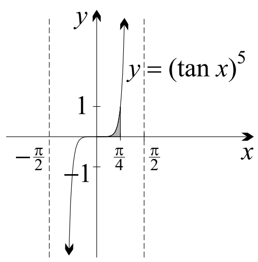
**SOLUTION:**

(a)
$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,(\sec^2 x - 1)\,dx \qquad \text{(by Pythagoras)} \\ &= \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,\sec^2 x\,dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx \\ &= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2} \\ &= \frac{1}{n-1} - I_{n-2}\,. \end{aligned}$$

(b)
$$\begin{aligned} I_1 &= \int_0^{\frac{\pi}{4}} \tan x\,dx \\ &= \Big[-\log(\cos x)\Big]_0^{\frac{\pi}{4}} \\ &= \tfrac{1}{2}\log 2\,. \end{aligned}$$

Thus
$$\begin{aligned} I_3 &= \tfrac{1}{2} - I_1 \\ &= \tfrac{1}{2} - \tfrac{1}{2}\log 2\,, \end{aligned}$$

and
$$\begin{aligned} I_5 &= \tfrac{1}{4} - I_3 \\ &= \tfrac{1}{2}\log 2 - \tfrac{1}{4}\,. \end{aligned}$$




**By Parts:** Many examples of reduction formulae use integration by parts.

**WORKED EXERCISE:**

(a) Let $I_n = \displaystyle\int_1^e (\log x)^n \, dx$ and show that $I_n = e - nI_{n-1}$ for $n \geq 1$.

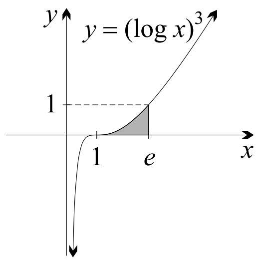
(b) Find $I_0$ and hence show that $I_3 = 6 - 2e$.

**SOLUTION:**

(a)
$$\begin{aligned} I_n &= \int_1^e 1 \times (\log x)^n \, dx \\ &= \Big[ x(\log x)^n \Big]_1^e - \int_1^e x \times \tfrac{n}{x} (\log x)^{n-1} \, dx \qquad \text{(by parts)} \\ &= (e - 0) - n \int_1^e (\log x)^{n-1} \, dx \\ &= e - nI_{n-1} \,. \end{aligned}$$

(b)
$$\begin{aligned} I_0 &= \int_1^e 1 \, dx \\ &= e - 1 \,. \end{aligned}$$

Thus
$$\begin{aligned} I_1 &= e - I_0 \\ &= 1 \,, \\ I_2 &= e - 2I_1 \\ &= e - 2 \,, \end{aligned}$$
and
$$\begin{aligned} I_3 &= e - 3I_2 \\ &= e - 3(e - 2) \\ &= 6 - 2e \,. \end{aligned}$$




**By Parts with an Identity:** In harder examples, integration by parts is used along with an identity.

**WORKED EXERCISE:** Let $I_n = \displaystyle\int_0^1 x^2(1 - x^2)^n \, dx$.

(a) Use the identity $x^2 \equiv 1 - (1 - x^2)$ to show that $I_n = \frac{2n}{2n+3} I_{n-1}$ for $n \ge 1$.

(b) Evaluate $I_0$ and hence find $I_3$.

**SOLUTION:**

(a) Apply integration by parts to get:
$$\begin{aligned} I_n &= \Big[ \tfrac{1}{3} x^3 (1 - x^2)^n \Big]_0^1 - \int_0^1 \tfrac{1}{3} x^3 \times (-2nx)(1 - x^2)^{n-1} \, dx \\ &= 0 + \tfrac{2n}{3} \int_0^1 x^2 \times x^2 (1 - x^2)^{n-1} \, dx \\ &= \tfrac{2n}{3} \int_0^1 x^2 (1 - x^2)^{n-1} - x^2 (1 - x^2)^n \, dx \qquad \text{(by part the identity)} \end{aligned}$$

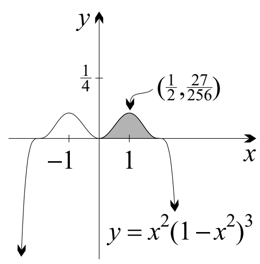
so $\quad I_n = \frac{2n}{3} I_{n-1} - \frac{2n}{3} I_n \,.$

thus $\quad \frac{2n+3}{3} I_n = \frac{2n}{3} I_{n-1}$

or $\quad I_n = \frac{2n}{2n+3} I_{n-1} \,.$

(b)
$$I_0 = \int_0^1 x^2 \, dx = \tfrac{1}{3} \,.$$

Thus
$$\begin{aligned} I_1 &= \tfrac{2}{5} I_0 = \tfrac{2}{15} \,, \\ I_2 &= \tfrac{4}{7} I_1 = \tfrac{8}{105} \,, \end{aligned}$$
and
$$I_3 = \tfrac{6}{9} I_2 = \tfrac{16}{315} \,.$$




# Exercise **2G**

**1.** (a) Given that $I_n = \int \tan^n x\,dx$, prove that $I_n = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$ for $n \ge 2$.
(b) Hence show that $I_6 = \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C$

**2.** (a) If $I_n = \int x^n e^x\,dx$, show that $I_n = x^n e^x - nI_{n-1}$ for $n \ge 1$.
(b) Hence show that $\int x^3 e^x\,dx = (x^3 - 3x^2 + 6x - 6)e^x + C$.

**3.** (a) If $I_n = \int_1^e x(\log x)^n\,dx$, show that $I_n = \frac{1}{2}e^2 - \frac{1}{2}nI_{n-1}$ for $n \ge 1$.
(b) Find $I_0$ and hence show that $I_4 = \frac{1}{4}(e^2 - 3)$.

**4.** Let $u_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$.
(a) Use integration by parts and the Pythagorean identity to prove that $u_n = \frac{n-1}{n}u_{n-2}$ for $n \ge 2$.
(b) Hence evaluate $u_5$.

**DEVELOPMENT**

**5.** Let $T_n = \int_0^{\frac{\pi}{4}} \sec^n x\,dx$.
(a) Show that $T_n = \dfrac{(\sqrt{2})^{n-2}}{n-1} + \dfrac{n-2}{n-1}T_{n-2}$ for $n \ge 2$. (b) Deduce that $T_6 = \frac{28}{15}$.

**6.** Let $C_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$, where $n \ge 0$.
(a) Prove that $C_n = (\frac{\pi}{2})^n - n(n-1)C_{n-2}$, for $n \ge 2$. (b) Hence evaluate $C_6$.

**7.** (a) If $I_n = \int_0^1 (1 - x^2)^n\,dx$, show that $I_n = \dfrac{2n}{2n+1}\,I_{n-1}$ for $n \ge 1$.
(b) Evaluate $I_0$ and hence find $I_4$.

**8.** (a) If $u_n = \int_0^1 x(1 - x^3)^n\,dx$, show that $u_n = \dfrac{3n}{3n+2}\,u_{n-1}$ for $n \ge 1$.
(b) Show that $u_0 = \frac{1}{2}$ and hence evaluate $u_4$.

**9.** Suppose that $J_n = \int \dfrac{x^n}{\sqrt{1-x^2}}\,dx$.
(a) Show that $J_n = \frac{1}{n}\left((n-1)J_{n-2} - x^{n-1}\sqrt{1-x^2}\right)$ for $n \ge 2$.
[HINT: Do by parts with $u = x^{n-1}$ and $v' = \dfrac{x}{\sqrt{1-x^2}}$.]
(b) Hence determine $\int \dfrac{x^2}{\sqrt{1-x^2}}\,dx$.

**10.** Let $u_n = \int_0^{\frac{\pi}{2}} \sin^n x \cos^2 x\,dx$.
(a) Show that $u_n = \left(\frac{n-1}{n+2}\right) u_{n-2}$, for $n \ge 2$.
[HINT: Do by parts with $u = \sin^{n-1} x$ and $v' = \sin x \cos^2 x$.]
(b) Hence show that $u_4 = \frac{\pi}{32}$.

**11.** Let $T_n = \displaystyle\int_0^1 x^n\sqrt{1-x}\,dx$.

(a) Deduce the reduction formula $T_n = \dfrac{2n}{2n+3}T_{n-1}$ for $n \ge 1$. (b) Show that $T_3 = \frac{32}{315}$.

(c) Use the reduction formula to help prove by induction that $T_n = \dfrac{n!(n+1)!}{(2n+3)!}\,4^{n+1}$.

**12.** Consider the integral $I_n = \displaystyle\int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx$.

(a) Show that $I_0 = 2\sqrt{2} - 2$.

(b) Show that $I_{n-1} + I_n = \displaystyle\int_0^1 x^{n-1}\sqrt{1+x}\,dx$ for $n \ge 1$.

(c) Use integration by parts to show that $I_n = \dfrac{2\sqrt{2} - 2nI_{n-1}}{2n+1}$ for $n \ge 1$.

(d) Hence evaluate $I_2$.

**13.** (a) Show that $(1+t^2)^{n-1} + t^2(1+t^2)^{n-1} = (1+t^2)^n$.

(b) Put $P_n = \displaystyle\int_0^x \left(1+t^2\right)^n dt$. Use integration by parts and the result in part (a) to show that $P_n = \frac{1}{2n+1}\Big(\left(1+x^2\right)^n x + 2nP_{n-1}\Big)$ for $n \ge 1$.

(c) Hence determine $P_4$:

(i) by the reduction formula, (ii) by using the binomial theorem.

(d) Hence write $1 + \frac{4}{3}x^2 + \frac{6}{5}x^4 + \frac{4}{7}x^6 + \frac{1}{9}x^8$ in powers of $(1+x^2)$.

**EXTENSION**

**14.** Let $I_n = \displaystyle\int_0^1 (1-x^2)^n\,dx$ and $J_n = \displaystyle\int_0^1 x^2(1-x^2)^n\,dx$.

(a) Apply integration by parts to $I_n$ to show that $I_n = 2n\,J_{n-1}$ for $n \ge 1$.

(b) Hence show that $I_n = \dfrac{2n}{2n+1}I_{n-1}$ for $n \ge 1$.

(c) Show that $J_n = I_n - I_{n+1}$, and hence deduce that $J_n = \dfrac{1}{2n+3}I_n$.

(d) Hence write down a reduction formula for $J_n$ in terms of $J_{n-1}$.

**15.** For $n = 0, 1, 2, \ldots$ let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n\theta\,d\theta$.

(a) Show that $I_1 = \frac{1}{2}\ln 2$.

(b) Show that, for $n \ge 2$, $I_n + I_{n-2} = \dfrac{1}{n-1}$.

(c) For $n \ge 2$, explain why $I_n < I_{n-2}$, and deduce that $\dfrac{1}{2(n+1)} < I_n < \dfrac{1}{2(n-1)}$.

(d) Use the reduction formula in part (b) to find $I_5$, and hence deduce that $\frac{2}{3} < \ln 2 < \frac{3}{4}$.

# 2H Miscellaneous Integrals

As was stated in the chapter overview, integration is an art form and requires much practice. In particular, it is important to be able to recognise the different forms of integrals, and to quickly determine which method is appropriate to apply. To that end, this section has been included. The exercise contains a mixture of all integral types encountered so far. Some questions can be done by more than one method. It is up to the reader to determine which method is most efficient.

## Exercise 2H

**1.** Evaluate:

(a) $\displaystyle\int_{-1}^{1} \frac{x^2}{(5+x^3)^2}\,dx$ (c) $\displaystyle\int_{2}^{3} \frac{2x+2}{(x+3)(x-1)}\,dx$ (e) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{3\cos x}{\sin^4 x}\,dx$

(b) $\displaystyle\int_{0}^{\pi} x\sin x\,dx$ (d) $\displaystyle\int_{0}^{2} \frac{x-1}{x+1}\,dx$ (f) $\displaystyle\int_{0}^{1} \frac{1}{\sqrt{4x^2+1}}\,dx$

**2.** Find:

(a) $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$ (d) $\displaystyle\int \frac{1}{2x^2+3x+1}\,dx$ (g) $\displaystyle\int \frac{1}{x^2+6x+25}\,dx$

(b) $\displaystyle\int \frac{1+x}{1+x^2}\,dx$ (e) $\displaystyle\int x^3\log x\,dx$ (h) $\displaystyle\int 3x\cos 3x\,dx$

(c) $\displaystyle\int \sin x\cos^4 x\,dx$ (f) $\displaystyle\int \sin^3 2x\,dx$ (i) $\displaystyle\int \frac{x}{\sqrt{4+x}}\,dx$

**3.** Show that:

(a) $\displaystyle\int_{0}^{1} x^2e^{-x}\,dx = 2-\tfrac{5}{e}$

(b) $\displaystyle\int_{0}^{\frac{\pi}{2}} \sin^3 x\cos^5 x\,dx = \tfrac{1}{24}$

(c) $\displaystyle\int_{0}^{1} \frac{x}{(x+1)(x^2+1)}\,dx = \tfrac{1}{8}(\pi-2\log 2)$

(d) $\displaystyle\int_{0}^{\frac{1}{2}} (1-x^2)^{-\frac{3}{2}}\,dx = \tfrac{1}{\sqrt{3}}$

(e) $\displaystyle\int_{0}^{1} \frac{1-x^2}{1+x^2}\,dx = \tfrac{\pi}{2}-1$

(f) $\displaystyle\int_{2}^{4} \frac{x}{\sqrt{6x-8-x^2}}\,dx = 3\pi$

(g) $\displaystyle\int_{0}^{1} \frac{\sqrt{x}}{1+x}\,dx = \tfrac{1}{2}(4-\pi)$

(h) $\displaystyle\int_{0}^{\frac{\pi}{3}} \sec x\,dx = \log(2+\sqrt{3})$

(i) $\displaystyle\int_{0}^{\frac{\pi}{4}} \sin 2x\cos 3x\,dx = \tfrac{1}{10}(3\sqrt{2}-4)$

(j) $\displaystyle\int_{0}^{\pi} e^{-x}\cos x\,dx = \tfrac{1}{2}(1+e^{-\pi})$

**DEVELOPMENT**

**4.** (a) Find the rational numbers $A$, $B$ and $C$ such that

$$\frac{x-1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}\,.$$

(b) Hence show that $\displaystyle\int_{0}^{1} \frac{x^3+x}{x^3+1}\,dx = 1-\tfrac{2}{3}\log 2$.

**5.** Use integration by parts to show that $\displaystyle\int x^3e^{-x^2}\,dx = -\tfrac{1}{2}e^{-x^2}(1+x^2)+C$.

**6.** Use integration by parts to evaluate $I = \displaystyle\int_{0}^{\frac{\pi}{4}} \sec^3 x\,dx$.

**7.** In each case let $t = \tan \frac{x}{2}$ in order to show that:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{3 + 5\cos x}\,dx = \tfrac{1}{4}\log 3$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\cos x - 2\sin x + 3}\,dx = \tfrac{\pi}{4}$

**8.** (a) Find the values of $A$, $B$, $C$, and $D$ such that

$$\frac{4t}{(1+t)^2(1+t^2)} = \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{Ct + D}{1+t^2}\,.$$

(b) Hence use the $t$-substitution to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \sin x}\,dx\,.$

**9.** Use the substitution $u = \sqrt[6]{x}$ to show that $\displaystyle\int_1^{64} \frac{1}{\sqrt{x} + \sqrt[3]{x}}\,dx = 11 - 6\log\tfrac{3}{2}\,.$

**10.** Find $\displaystyle\int \sqrt{a^2 - x^2}\,dx$ using:

(a) the substitution $\theta = \sin^{-1}\frac{x}{a}$, (b) integration by parts.

**11.** (a) Show that $\displaystyle\int_0^1 \frac{5 - 5x^2}{(1+2x)(1+x^2)}\,dx = \tfrac{1}{2}(\pi + \log\tfrac{27}{16})\,.$

(b) Hence find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \cos x + 2\sin x}\,dx$ using the substitution $t = \tan\frac{x}{2}$.

**12.** (a) Find integers $P$ and $Q$ such that

$$8\sin x + \cos x - 2 = P(3\sin x + 2\cos x - 1) + Q(3\cos x - 2\sin x)\,.$$

(b) Hence find $\displaystyle\int \frac{8\sin x + \cos x - 2}{3\sin x + 2\cos x - 1}\,dx\,.$

**13.** (a) If $T_n = \displaystyle\int_0^{\pi} \sin^n x\,dx$, show that $T_n = \frac{n-1}{n}T_{n-2}$. (b) Hence show that $T_5T_6 = \frac{\pi}{3}$.

**14.** (a) Let $I_n = \displaystyle\int_1^{e} (\log x)^n\,dx$ and show that $I_n = e - nI_{n-1}$. (b) Hence evaluate $I_3$.

**EXTENSION**

**15.** Let $I_n = \displaystyle\int_0^1 \frac{x^{n-1}}{(x+1)^n}\,dx$, for $n = 1, 2, 3, \ldots$

(a) Show that $I_1 = \ln 2$.

(b) Use integration by parts to show that $I_{n+1} = I_n - \dfrac{1}{n\,2^n}$.

(c) The maximum value of $\dfrac{x}{x+1}$, for $0 \le x \le 1$, is $\frac{1}{2}$.
Use this fact to show that $I_{n+1} < \frac{1}{2}I_n$.

(d) Deduce that $I_n < \dfrac{1}{n\,2^{n-1}}$.

(e) Use the reduction formula in part (b) and the inequality in part (d) to show that

$\frac{2}{3} < \ln 2 < \frac{17}{24}$.

**16.** Given that $\displaystyle\int_0^{\pi} \frac{1}{5 + 3\cos x}\,dx = \tfrac{\pi}{4}$, show that $\displaystyle\int_0^{\pi} \frac{\cos x + 2\sin x}{5 + 3\cos x}\,dx = \tfrac{1}{12}(16\log 2 - \pi)\,.$

**17.** (a) Use the substitution $u = t - t^{-1}$ to show that $\displaystyle\int \frac{1+t^2}{1+t^4}\,dt = \tfrac{1}{\sqrt{2}}\tan^{-1}\tfrac{\sqrt{2}(t^2-1)}{2t} + C\,.$

(b) Alternatively, use the result $(1+t^4) = (1+t^2)^2 - (\sqrt{2}\,t)^2$ and partial fractions to show that $\displaystyle\int \frac{1+t^2}{1+t^4}\,dt = \tfrac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\,t+1) + \tfrac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\,t-1) + C\,.$

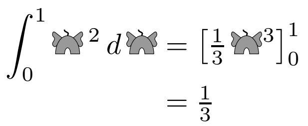
**18.** Consider the two new functions $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$.
Show that $\displaystyle\int_0^{\log 2} \frac{1}{5\cosh x - 3\sinh x}\,dx = \tfrac{1}{2}\tan^{-1}\tfrac{1}{3}\,.$

# 2I Further Integration

**Dummy Variables:** In the case of definite integrals it does not matter what variable is used in the integrand, provided that the variable chosen is used consistently during the calculations. Thus for example, the three integrals below all have the same value, despite using different variables in the calculations.

$$\int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3} \qquad \int_0^1 t^2\,dt = \left[\tfrac{1}{3}t^3\right]_0^1 = \tfrac{1}{3} \qquad \int_0^1 \theta^2\,d\theta = \left[\tfrac{1}{3}\theta^3\right]_0^1 = \tfrac{1}{3}$$

To be particularly absurd we could choose anything we like for the variable, such as a picture of an elephant from behind, . Thus:



which still gives the same value. In such cases as these, we say that the variable used is a *dummy variable*, since it is only seen in intermediate calculations and does not appear in the final answer.

In itself, the notion of dummy variables is not a particularly exciting result. However it is a feature that can be used to help prove some useful theorems about definite integrals. These theorems can then be used to help evaluate more complicated integrals.

**Odd and Even Symmetry:** If $f(x)$ exhibits odd or even symmetry then we may use the following to quickly simplify an integral:

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd,} \\ 2\displaystyle\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even.} \end{cases}$$

Clearly in the case where $f(x)$ is odd the result is immediate. In the case where the integrand is even let $F(x)$ be a primitive of $f(x)$. If the constant of integration is omitted then $F(0) = 0$. Hence

$$\begin{aligned} \int_{-a}^{a} f(x)\,dx &= 2\int_0^a f(x)\,dx \\ &= 2\big(F(a) - F(0)\big) \\ &= 2F(a)\,. \end{aligned}$$

Thus it is only necessary to evaluate the primitive at the upper limit.

**WORKED EXERCISE:** Evaluate $\displaystyle\int_{-1}^{1} \frac{x^2}{1+x^2}\, dx$.

**SOLUTION:** Clearly the integrand is even.

Now $$\int \frac{x^2}{1+x^2}\, dx = \int 1 - \frac{1}{1+x^2}\, dx$$
$$= x - \tan^{-1} x \qquad \text{(omitting the constant.)}$$

Hence $$\int_{-1}^{1} \frac{x^2}{1+x^2}\, dx = 2(1 - \tan^{-1} 1)$$
$$= 2 - \tfrac{\pi}{2}\,.$$

**WORKED EXERCISE:** Evaluate $\displaystyle\int_{-1}^{1} f(x)\, dx$, where $f(x) = \dfrac{x^2}{\sqrt{2+x}} - \dfrac{x^2}{\sqrt{2-x}}$.

**SOLUTION:** $$f(-x) = \frac{(-x)^2}{\sqrt{2+(-x)}} - \frac{(-x)^2}{\sqrt{2-(-x)}}$$
$$= \frac{x^2}{\sqrt{2-x}} - \frac{x^2}{\sqrt{2+x}}$$
$$= -f(x)\,.$$

Hence $f(x)$ is odd and thus $\displaystyle\int_{-1}^{1} f(x)\, dx = 0$.

9 ODD AND EVEN SYMMETRY: Let $F(x)$ be a primitive of $f(x)$ without constant, then:

$$\int_{-a}^{a} f(x)\, dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd,} \\ 2F(a) & \text{if } f(x) \text{ is even.} \end{cases}$$

Here is a proof of the case where $f(x)$ is even. The odd case is left as an exercise.

PROOF: Let $f(x)$ be even with primitive $F(x)$, then

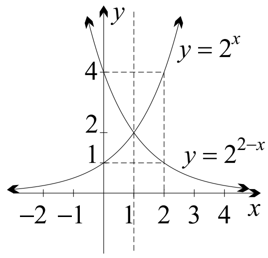
$$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_{0}^{a} f(x)\, dx$$
$$= -\int_{a}^{0} f(-t)\, dt + \int_{0}^{a} f(x)\, dx \quad \text{where } t = -x\,.$$

Thus $$\int_{-a}^{a} f(x)\, dx = \int_{0}^{a} f(-t)\, dt + \int_{0}^{a} f(x)\, dx \qquad \text{(reversing the limits)}$$
$$= \int_{0}^{a} f(t)\, dt + \int_{0}^{a} f(x)\, dx \qquad \text{(since } f \text{ is even)}$$
$$= 2\int_{0}^{a} f(x)\, dx \qquad \text{(since } x \text{ and } t \text{ are dummy variables)}$$
$$= 2F(a)\,.$$

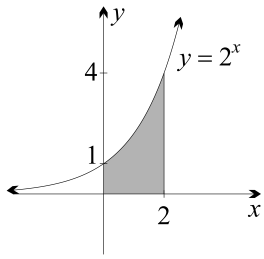
**Reflection in the Line $x = a$:** Integrals of the form $\displaystyle\int_{0}^{2a} f(x)\, dx$ can often be simplified by a reflection in the vertical line $x = a$. This is achieved by replacing $x$ with $(2a - x)$. Such reflections are dealt with in more detail in the chapter on Graphs. The following example demonstrates the situation.

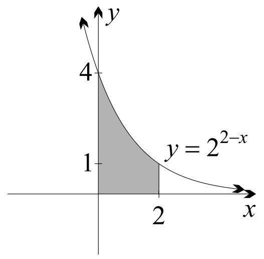
The graphs of $y = 2^x$ and $y = 2^{2-x}$ are to the right of the table of values.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|---|---|---|
| $2^x$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $1$ | $2$ | $4$ | $8$ | $16$ |
| $2^{2-x}$ | $16$ | $8$ | $4$ | $2$ | $1$ | $\frac{1}{2}$ | $\frac{1}{4}$ |




It should be clear that the third line of the table of values is just the reverse of the second line. That is, there is symmetry about the middle value $x = 1$. The graphs also make it clear that $y = 2^{2-x}$ is obtained by reflecting $y = 2^x$ in the line $x = 1$.




The second pair of graphs should further make it clear that since a reflection is involved, the areas under the exponential curves between $x = 0$ and $x = 2$ are the same. That is:

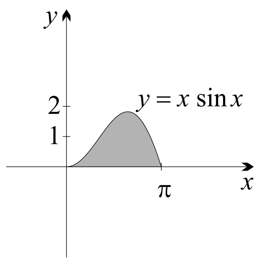
$$\int_0^2 2^x\,dx = \int_0^2 2^{2-x}\,dx\,.$$

Notice that in the integrand $x$ has been replaced with $2a-x = 2-x$, since $a = 1$.

**WORKED EXERCISE:** Determine $\displaystyle\int_0^\pi x\sin x\,dx$ by a suitable reflection.

**SOLUTION:** Let $I = \displaystyle\int_0^\pi x\sin x\,dx$. Reflect in the line $x = \frac{\pi}{2}$.

Thus replace $x$ with $(\pi - x)$ to get:




$$\begin{aligned}
I &= \int_0^\pi (\pi - x)\sin(\pi - x)\,dx \\
&= \int_0^\pi (\pi - x)\sin x\,dx \quad \text{(expanding } \sin(\pi - x)) \\
&= \int_0^\pi \pi\sin x\,dx - \int_0^\pi x\sin x\,dx \\
&= \int_0^\pi \pi\sin x\,dx - I\,.
\end{aligned}$$

Hence $2I = \displaystyle\int_0^\pi \pi\sin x\,dx$

$$\begin{aligned}
\text{thus}\quad I &= \tfrac{\pi}{2}\int_0^\pi \sin x\,dx \\
&= \tfrac{\pi}{2}\times 2 \\
&= \pi\,.
\end{aligned}$$

The integral can also be done using integration by parts. The method of reflection in $x = \frac{\pi}{2}$ provides a geometric alternative which in some ways is simpler.

**10** REFLECTION IN A VERTICAL LINE: The integral of a function $f(x)$ between $x = 0$ and $x = 2a$ is unchanged by a reflection in the line $x = a$, thus:

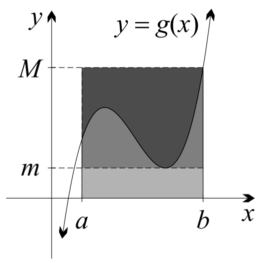
$$\int_0^{2a} f(x)\,dx = \int_0^{2a} f(2a - x)\,dx\,.$$

The proof is straight forward, and again makes use of dummy variables.

PROOF: Put $x = 2a - t$ so that $dx = (-1)\,dt$.
When $x = 0$, $t = 2a$, and when $x = 2a$, $t = 0$.

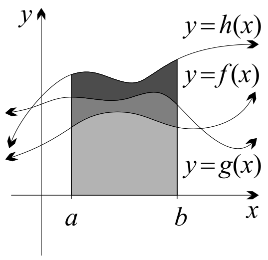
$$\begin{aligned}\text{Thus}\quad \int_0^{2a} f(x)\,dx &= \int_{2a}^{0} f(2a-t)\times(-1)\,dt\\ &= \int_0^{2a} f(2a-t)\,dt \quad \text{(reversing the limits)}\\ &= \int_0^{2a} f(2a-x)\,dx \quad \text{(since } x \text{ and } t \text{ are dummy variables)}\end{aligned}$$

**Bounding:** There are times when it is not necessary to know the exact value of an integral, just that it lies within certain bounds. For example, in the interval $a \le x \le b$ in the graph on the right the function $y = g(x)$ lies between its minimum value $y = m$ and its maximum value $y = M$. It should be clear then that the area under $y = g(x)$ in this interval is bigger than the lower rectangle and less than the upper rectangle, hence:




$$m(b-a) \le \int_a^b g(x)\,dx \le M(b-a)\,.$$

Again, by comparing areas, it should be clear in general that if $f(x) \le g(x) \le h(x)$ whenever $a \le x \le b$ then




$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx \le \int_a^b h(x)\,dx\,.$$

**WORKED EXERCISE:** (a) Prove that $\dfrac{1}{x+1} \le \dfrac{1}{x+\cos^2 x} \le \dfrac{1}{x}$ for $x > 0$.

(b) Hence show that $\log\frac{3}{2} \le \displaystyle\int_1^2 \frac{1}{x+\cos^2 x}\,dx \le \log 2$.

**SOLUTION:** (a) Now $0 \le \cos^2 x \le 1$, so:

$$x \le x + \cos^2 x \le x + 1 \qquad \text{for all } x\,,$$

$$\text{hence}\qquad \frac{1}{x+1} \le \frac{1}{x+\cos^2 x} \le \frac{1}{x} \qquad \text{for } x > 0\,.$$

(b) Integrating all three parts:

$$\int_1^2 \frac{1}{x+1}\,dx \le \int_1^2 \frac{1}{x+\cos^2 x}\,dx \le \int_1^2 \frac{1}{x}\,dx$$

$$\text{so}\qquad \Big[\log(x+1)\Big]_1^2 \le \int_1^2 \frac{1}{x+\cos^2 x}\,dx \le \Big[\log x\Big]_1^2$$

$$\text{hence}\qquad \log\tfrac{3}{2} \le \int_1^2 \frac{1}{x+\cos^2 x}\,dx \le \log 2\,.$$

**Improper Integrals and Limits:** A definite integral is called an *improper integral* if the integrand is undefined at some point in the interval or if the interval of integration is unbounded. Thus

$$\int_1^2 \frac{1}{x-1}\,dx$$

is an improper integral since $\dfrac{1}{x-1}$ is undefined at $x = 1$. The integral

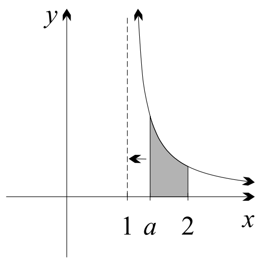
$$\int_0^\infty e^{-x}\,dx$$

is also an improper integral, in this case since the interval, $0 \le x < \infty$, is unbounded on the right hand side.

The value of an improper integral, if it exists, is found by taking the limit of a related integral.

**WORKED EXERCISE:** Find the value of $\displaystyle\int_1^2 \frac{1}{x-1}\,dx$, if it exists.

**SOLUTION:**



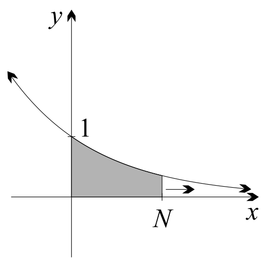

$$\begin{aligned} \text{Let}\quad I(a) &= \int_a^2 \frac{1}{x-1}\,dx \\ &= \Big[\log(x-1)\Big]_a^2 \\ &= \log 1 - \log(a-1) \\ &= -\log(a-1)\,. \end{aligned}$$

$$\begin{aligned} \text{Thus}\quad \int_1^2 \frac{1}{x-1}\,dx &= \lim_{a\to 1^+} I(a) \qquad \text{(if the limit exists)} \\ &= \lim_{a\to 1^+} -\log(a-1) \end{aligned}$$

which is undefined. Hence $\displaystyle\int_1^2 \frac{1}{x-1}\,dx$ is undefined.

**WORKED EXERCISE:** Determine $\displaystyle\int_0^\infty e^{-x}\,dx$.

**SOLUTION:**




$$\begin{aligned} \text{Let}\quad I(N) &= \int_0^N e^{-x}\,dx \\ &= \Big[e^{-x}\Big]_0^N \\ &= 1 - e^{-N}\,. \end{aligned}$$

$$\begin{aligned} \text{Thus}\quad \int_0^\infty e^{-x}\,dx &= \lim_{N\to\infty} I(N) \qquad \text{(if the limit exists)} \\ &= \lim_{N\to\infty} 1 - e^{-N} \\ &= 1\,. \end{aligned}$$

# Exercise **2I**

**1.** (a) Prove that $f(x) = \sqrt{2+x} - \sqrt{2-x}$ is odd and hence evaluate $\displaystyle\int_{-2}^{2} \sqrt{2+x} - \sqrt{2-x}\,dx$.

(b) Prove that $g(x) = e^x - e^{-x}$ is odd and hence evaluate $\displaystyle\int_{-1}^{1} e^x - e^{-x}\,dx$.

**2.** Use the formula $\displaystyle\int_0^{2a} f(x)\,dx = \int_0^{2a} f(2a-x)\,dx$ to help evaluate:

(a) $\displaystyle\int_0^1 x(1-x)^{10}\,dx$ (b) $\displaystyle\int_0^1 x^2\sqrt{1-x}\,dx$ (c) $\displaystyle\int_0^{\pi} x\sin^2 x\,dx$

**3.** (a) Use a graph to show that $\dfrac{1}{2x} \le \dfrac{\sin x}{x} \le \dfrac{1}{x}$ for $\frac{\pi}{6} \le x \le \frac{\pi}{2}$.

(b) Hence show that $\frac{1}{2}\log 3 < \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx < \log 3$.

**4.** (a) Use a graph to show that $\dfrac{1}{x} \le \dfrac{\tan x}{x} \le \dfrac{\sqrt{3}}{x}$ for $\frac{\pi}{4} \le x \le \frac{\pi}{3}$.

(b) Hence show that $\log\frac{4}{3} < \displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan x}{x}\,dx < \sqrt{3}\log\frac{4}{3}$.

**5.** (a) Use a graph to show that $\tan x \le \sqrt{\tan x} \le 1$ for $0 \le x \le \frac{\pi}{4}$.

(b) Hence show that $\frac{1}{2}\log 2 < \displaystyle\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\,dx < \frac{\pi}{4}$.

**6.** (a) Explain why $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}}$ is an improper integral.

(b) Find $I(a) = \displaystyle\int_0^a \frac{dx}{\sqrt{1-x}}$, where $a < 1$.

(c) Determine $\displaystyle\lim_{a\to 1^-} I(a)$ and hence state the value of $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}}$.

**7.** (a) Explain why $\displaystyle\int_0^{\infty} \frac{dx}{4+x^2}$ is an improper integral.

(b) Find $I(N) = \displaystyle\int_0^N \frac{dx}{4+x^2}$.

(c) Determine $\displaystyle\lim_{N\to\infty} I(N)$ and hence state the value of $\displaystyle\int_0^{\infty} \frac{dx}{4+x^2}$.

**DEVELOPMENT**

**8.** (a) Use the substitution $u = -x$ to prove that $\displaystyle\int_{-a}^0 f(x)\,dx = \int_0^a f(-x)\,dx$.

(b) Hence prove that $\displaystyle\int_{-a}^a f(x)\,dx = \int_0^a \big(f(x) + f(-x)\big)\,dx$.

(c) Use the theorem in part (b) to show that:

(i) $\displaystyle\int_{-a}^a f(x)\,dx = 0$ if $f(x)$ is odd

(ii) $\displaystyle\int_{-1}^1 \frac{1}{1+e^{-x}}\,dx = 1$

(iii) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}\,dx = 2$

(iv) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x\sin^2 x}{1+e^x}\,dx = \frac{\pi}{4}$

**9.** Use the result of Box 10 with a suitable choice of $a$ to evaluate:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\, dx$ (b) $\displaystyle\int_0^{\pi} \frac{\cos x}{3 + \sin^2 x}\, dx$ (c) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \sin 2x}\, dx$

**10.** (a) Show that $\displaystyle\int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\, dx = \tfrac{\pi}{2}$.

(b) Use the substitution $x = \pi - u$ to find $\displaystyle\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$.

**11.** Evaluate the following improper integrals by applying an appropriate limit.

(a) $\displaystyle\int_0^2 \frac{dx}{\sqrt{2 - x}}$ (c) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1 - x^2}}$ (e) $\displaystyle\int_0^1 \sqrt{\frac{1 + x}{1 - x}}\, dx$

(b) $\displaystyle\int_0^4 \frac{dx}{\sqrt{x}}$ (d) $\displaystyle\int_0^1 \frac{dx}{\sqrt{2x - x^2}}$ (f) $\displaystyle\int_0^e (\log x)^2 dx$

**12.** Evaluate the following improper integrals by applying an appropriate limit.

(a) $\displaystyle\int_1^{\infty} \frac{dx}{x^2}$ (c) $\displaystyle\int_1^{\infty} \frac{dx}{x^2 - 4x + 5}$ (e) $\displaystyle\int_{-\infty}^0 e^x\, dx$

(b) $\displaystyle\int_0^{\infty} \frac{dx}{1 + x^2}$ (d) $\displaystyle\int_0^{\infty} x e^{-x^2}\, dx$ (f) $\displaystyle\int_0^{\infty} \frac{2\, dx}{e^x + e^{-x}}$

**13.** (a) Given that $0 < t < 1$, show that $\tfrac{1}{2} < \dfrac{1}{1 + t} < 1$.

(b) Hence, for $0 < x < 1$, show that $\tfrac{1}{2}x < \log(1 + x) < x$.

**14.** (a) Prove that $y = \tfrac{1}{2}(x + 1)$ is the tangent to $y = \sqrt{x}$ at $x = 1$.

(b) Hence explain why $\tfrac{1}{2}(x + 1) \geq \sqrt{x}$ for $x \geq 0$.

(c) Hence prove that $x + \sqrt{x} + 1 \leq \tfrac{3}{2}(x + 1)$.

(d) Hence show that $\displaystyle\int_1^2 \frac{1}{x + \sqrt{x} + 1}\, dx \geq \tfrac{2}{3} \log \tfrac{3}{2}$.

**15.** (a) Given that $n > 2$ and $0 < x < 1$, show that $0 < x^n < x^2$.

(b) Hence, for $n > 2$ and $0 < x < 1$, show that $1 < \dfrac{1}{\sqrt{1 - x^n}} < \dfrac{1}{\sqrt{1 - x^2}}$.

(c) Deduce that $\tfrac{1}{2} < \displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1 - x^n}}\, dx < \tfrac{\pi}{6}$.

**16.** (a) Given that $\sin x > \tfrac{2x}{\pi}$ for $0 < x < \tfrac{\pi}{2}$, explain why $\displaystyle\int_0^{\frac{\pi}{2}} e^{-\sin x}\, dx < \int_0^{\frac{\pi}{2}} e^{-\frac{2x}{\pi}}\, dx$.

(b) Use the substitution $u = \pi - x$ to show that $\displaystyle\int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}\, dx = \int_0^{\frac{\pi}{2}} e^{-\sin x}\, dx$.

(c) Deduce that $\displaystyle\int_0^{\pi} e^{-\sin x}\, dx < \tfrac{\pi}{e}(e - 1)$.

**17.** (a) Show that $\displaystyle\int_0^1 x^2(1 - x)^2\, dx = \tfrac{1}{30}$ and that $\displaystyle\int_0^1 \frac{x^2(1 - x)^2}{x + 2}\, dx = 36 \ln \tfrac{3}{2} - \tfrac{175}{12}$.

(b) Explain why $\tfrac{1}{3}x^2(1 - x)^2 < \dfrac{x^2(1 - x)^2}{x + 2} < \tfrac{1}{2}x^2(1 - x)^2$, for $0 < x < 1$.

(c) Hence show that $\tfrac{2627}{6480} < \ln \tfrac{3}{2} < \tfrac{2628}{6480}$.

**18.** (a) Show that $\displaystyle\int_0^1 x^4(1-x)^4\,dx = \tfrac{1}{630}$ and that $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \tfrac{22}{7} - \pi$.

(b) Explain why $\tfrac{1}{2}x^4(1-x)^4 < \dfrac{x^4(1-x)^4}{1+x^2} < x^4(1-x)^4$, for $0 < x < 1$.

(c) Hence show that $\tfrac{22}{7} - \tfrac{1}{630} < \pi < \tfrac{22}{7} - \tfrac{1}{1260}$.

**19.** Explain why $0 \le \left|\displaystyle\int_a^b f(x)\,dx\right| \le \displaystyle\int_a^b |f(x)|\,dx$. A diagram may help.

**20.** (a) Let $I_n = \displaystyle\int x(\log x)^n\,dx$. Show that $I_n = \tfrac{1}{2}x^2(\log x)^n - \tfrac{1}{2}nI_{n-1}$.

(b) Given that $\displaystyle\lim_{x\to 0} x^n \log x = 0$ for $n > 0$, deduce a similar reduction formula for the improper integral $u_n = \displaystyle\int_0^1 x(\log x)^n\,dx$.

(c) Hence evaluate $u_4$.

**21.** (a) Use a suitable substitution to show that $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

(b) A function $g(x)$ has the property that $g(x) + g(a-x) = g(a)$. Use part (a) to prove that $\displaystyle\int_0^a g(x)\,dx = \tfrac{a}{2}g(a)$.

**EXTENSION**

**22.** Let $I_n(x) = \displaystyle\int_0^x t^n e^{-t}\,dt$, where $n$ is a positive integer.

(a) Prove by induction that $I_n(x) = n!\left(1 - e^{-x}\displaystyle\sum_{j=0}^{n} \frac{x^j}{j!}\right)$, where $0! = 1$.

(b) Show that $0 \le \displaystyle\int_0^1 t^n e^{-t}\,dt \le \frac{1}{n+1}$.

(c) Hence show that $0 \le 1 - e^{-1}\displaystyle\sum_{j=0}^{n} \frac{1}{j!} \le \frac{1}{(n+1)!}$.

(d) Hence find $\displaystyle\lim_{n\to\infty}\left(\sum_{j=0}^{n} \frac{1}{j!}\right)$.

**23.** (a) Given that $e < 3$, show that $\displaystyle\int_0^1 x^n e^x\,dx < \frac{3}{n+1}$.

(b) Show by induction that for $n = 0, 1, 2, \ldots$ there exist integers $a_n$ and $b_n$ such that

$$\int_0^1 x^n e^x\,dx = a_n + b_n e\,.$$

(c) Let $r$ be a positive rational number so that $r = \dfrac{p}{q}$, where $p$ and $q$ are positive integers. Show that for all integers $a$ and $b$, either $|a + br| = 0$ or $|a + br| \ge \dfrac{1}{q}$.

(d) Prove that $e$ is irrational.

**24.** Show that $\displaystyle\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \tfrac{1}{8}\pi\log 2$.

# Appendix — Table of Standard Integrals

Here is a table of standard integrals. A similar table is supplied on the last page of each examination paper.

**STANDARD INTEGRALS**

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

# Chapter Two

## Exercise **2A** (Page 53)

**1(a)** $\frac{1}{2}\sin 2x + C$ **(b)** $3\tan\frac{x}{3} + C$
**(c)** $\frac{1}{5}\tan^{-1}(\frac{x}{5}) + C$ **(d)** $\sin^{-1}(\frac{x}{2}) + C$
**(e)** $\log\left(x + \sqrt{x^2+3}\right) + C$
**(f)** $\log\left(x + \sqrt{x^2-5}\right) + C$
**2(a)** $2(e^2-1)$ **(b)** $\frac{1}{2}$ **(c)** $\frac{\pi}{8}$ **(d)** $\frac{\pi}{4}$
**(e)** $\log\left(\frac{3+\sqrt{5}}{1+\sqrt{5}}\right) = \log\left(\frac{1+\sqrt{5}}{2}\right)$ **(f)** $2\log 3$
**3(a)** $-\frac{1}{2}\log(1-x^2) + C$ **(b)** $\log(x + \tan x) + C$
**(c)** $\frac{1}{3}\log(1 + \sin 3x) + C$
**4(a)** $\frac{1}{3}\log 2$ **(b)** $\frac{1}{2}\log(\frac{e^2+1}{2})$ **(c)** $\log 2$
**5(a)** $\frac{\pi}{3\sqrt{3}}$ **(b)** $\frac{\pi}{18}$ **(c)** $\frac{1}{2}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = \log(\sqrt{2}+1)$
**(d)** $\frac{1}{\sqrt{5}}\log\left(\frac{15+7\sqrt{5}}{5+\sqrt{5}}\right) = \frac{1}{\sqrt{5}}\log(2+\sqrt{5})$
**6(a)** $x + \log(x-1) + C$ **(b)** $x - 2\log(x+1) + C$
**(c)** $x + 2\log(x-1) + C$
**7(a)** $1 - \log 4$ **(b)** $1 - \frac{1}{4}\log 5$ **(c)** $\pi - 1$
**8(a)** $\frac{\pi}{3} - \frac{1}{2}$ **(b)** $\frac{\pi}{4} + \log 2$ **(c)** $\frac{1}{4}(\pi - \log 4)$
**(d)** $\frac{\pi}{8} + \frac{1}{2}\log 2$
**10(a)** $\frac{x^3}{3} - \frac{x^2}{2} + x - \log(x+1) + C$
**(b)** $\frac{1}{2}\left(x^2 - \log(x^2+1)\right) + C$
**(c)(i)** $\frac{x^3}{3} + \frac{x^2}{2} + x + \log(x-1) + C$
**(ii)** $\frac{x^3}{3} - x + \tan^{-1}x + C$ **(iii)** $x - \log(1+e^x) + C$
**(iv)** $\frac{1}{3}(2x-8)\sqrt{2+x}+C$ **(v)** $-\frac{2}{3}(2+x)\sqrt{1-x}+C$
**(vi)** $\frac{1}{2}x^2 - 2\log(x^2+4) + C$
**11(a)** $\log(e+e^{-1})$ **(b)** $\frac{1}{2}\log\left(\frac{e^2+1}{2}\right)$ **(c)** $\frac{\pi}{12} + \log 2$
**12(a)** $\frac{1}{2}x^2 + \log(x+1) + C$ **(b)** $\frac{1}{3}x^3 + 3\log(x-2) + C$
**(c)** $x + \log(1+x^2) + C$
**13** $2\log(1+\sqrt{x}) + C$

## Exercise **2B** (Page 57)

**1(a)(i)** $-\frac{1}{2}\log(1-x^2) + C$ **(ii)** $\log(1+\sin x) + C$
**(iii)** $\log(\log x) + C$ **(b)(i)** $\frac{1}{2}\left(\log(e^2+1) - \log 2\right)$
**(ii)** $\frac{1}{3}\log 2$ **(iii)** $\frac{1}{2}\log 3$
**2(a)(i)** $2e^{x^3} + C$ **(ii)** $e^{\tan x} + C$ **(iii)** $-e^{\frac{1}{x}} + C$
**(b)(i)** $\frac{1}{2}(e-1)$ **(ii)** $e - 1$ **(iii)** $2e(e-1)$
**3(a)** $\frac{1}{5}(x^2+1)^5 + C$ **(b)** $\frac{1}{7}(1+x^3)^7 + C$
**(c)** $-\frac{2}{1+x^3} + C$ **(d)** $\frac{1}{2(x^2-3)^4} + C$
**(e)** $\sqrt{x^2-2} + C$ **(f)** $\frac{1}{2}\sqrt{1+x^4} + C$
**4(a)** $\frac{-1}{2\sin^2 x} + C$ **(b)** $\frac{-1}{1+\tan x} + C$ **(c)** $\frac{1}{3}(\log x)^3 + C$
**(d)** $2\sin\sqrt{x} + C$ **(e)** $\frac{1}{2}\tan^{-1}x^2 + C$
**(f)** $\sin^{-1}x^3 + C$
**5(a)** $\frac{7}{4}$ **(b)** $2 - \sqrt{3}$ **(c)** $3(\sqrt{3} - \sqrt{2})$
**(d)** $\frac{1}{5}$ **(e)** $\frac{1}{3}$ **(f)** $2$
**6(a)** $-\frac{1}{42}$ **(b)** Begin by writing $x = (x-1) + 1$.
**7(a)** $\frac{2}{15}(3x-2)(1+x)\sqrt{1+x} + C$
**(b)** $2\left(\sqrt{x} - \log(1+\sqrt{x})\right) + C$
**(c)** $4\left(x^{\frac{1}{4}} - \frac{1}{2}\sqrt{x} + \frac{1}{3}x^{\frac{3}{4}} - \log(1+x^{\frac{1}{4}})\right) + C$
**(d)** $\tan^{-1}\sqrt{e^{2x}-1} + C$
**8(a)** $\frac{1}{9}$ **(b)** $\frac{128}{15}$ **(c)** $4 + 10\log\frac{5}{7}$ **(d)** $\frac{\pi}{12}$
**9(a)** $2\tan^{-1}\left(\sqrt{x}\right) + C$ **(b)** $\frac{2}{3}(x-2)\sqrt{x+1} + C$
**10(a)** $\frac{x}{\sqrt{1+x^2}} + C$ **(b)** $2\sin^{-1}\frac{x}{2} - \frac{1}{2}x\sqrt{4-x^2} + C$
**(c)** $-\frac{\sqrt{25-x^2}}{25x} + C$ **(d)** $-\frac{1}{x}\sqrt{1+x^2} + C$
**11(a)** $\frac{2}{3}$ **(b)** Begin by writing $x^3 = x(x^2+1) - x$.
**12(b)** The region is half a segment.
**13(b)** Begin by writing $x^2 = 1 - (1-x^2)$.
**14(a)** $\tan^{-1}\sqrt{x^2-1} + C_1$ **(b)** $\tan^{-1}\sqrt{x^2-1} + C_2$
**15(a)** $\frac{\sqrt{3}}{8} - \frac{\sqrt{\epsilon(4+\epsilon)}}{4(2+\epsilon)}$ **(b)** $\frac{\sqrt{3}}{8}$

## Exercise **2C** (Page 64)

**1(a)** $\frac{1}{x-1} - \frac{1}{x+1}$ **(b)** $\frac{1}{3(x-4)} - \frac{1}{3(x-1)}$ **(c)** $\frac{2}{x-3} + \frac{2}{x+3}$
**(d)** $\frac{2}{x-2} - \frac{1}{x-1}$ **(e)** $\frac{1}{5(x-2)} + \frac{4}{5(x+3)}$ **(f)** $\frac{1}{x-1} + \frac{2-x}{x^2+3}$
**2(a)** $\ln(x-4) - \ln(x-2) + C$
**(b)** $2\ln(x+1) - 2\ln(x+3) + C$
**(c)** $4\log(x-2) - \log(x-1) + C$
**(d)** $3\log(x-1) - \log(x+3) + C$
**(e)** $\log(x+1) + \log(2x+3) + C$
**(f)** $2\log(x+1) + 3\log(2x-3) + C$
**3(a)** $\frac{1}{4}\log\frac{3}{2}$ **(b)** $\log 2$ **(c)** $\log\frac{14}{3}$ **(d)** $\frac{1}{2}\log 2$
**4(a)** $\log(x-2) - 2\tan^{-1}x + C$
**(b)** $\log(2x+1) - \frac{1}{2}\log(x^2+3) + C$
**(c)** $\tan^{-1}x + 3\log x - \log(x^2+1) + C$
**5(a)** $\frac{\pi}{4} - \log\frac{3}{2}$ **(b)** $\pi + \log 2$ **(c)** $\log 4 - \frac{3}{2}\log 3$
**6(a)** $5\log(x-1) + 7\log(x-2) - 12\log(2x-3) + C$
**(b)** $\frac{3}{2}\log(x) - 5\log(x-2) + \frac{7}{2}\log(x-4) + C$
**7(a)** $\frac{5}{3}\log 3 - \log 2$ **(b)** $2\log 3 - 8\log 2$
**8(a)(i)** $A = 2,\ B = 1,\ C = -3$
**(ii)** $2x + \log(x-1) - 3\log(x+2) + C$
**(b)(i)** $x + \log(x-2) - 2\log(x+1) + C$
**(ii)** $3x + 2\log(x+4) + \log(x-5) + C$
**9(a)(i)** $A = 1,\ B = -1,\ C = 2,\ D = -1$
**(ii)** $\log 3 + \log 2 - \frac{1}{2}$ **(b)** $12 + \log 2$
**10(a)(i)** $A = 12,\ B = 2$
**(ii)** $3x + 12\log(x-2) - \frac{2}{x-2} + C$
**(b)(i)** $A = 23,\ B = 10,\ C = -23,\ D = 13$
**(ii)** $23\log\left(\frac{x-1}{x-2}\right) - \frac{10}{x-1} - \frac{13}{x-2} + C$
**12(a)** $A = 0,\ B = 1,\ C = 0,\ D = 2$
**13(a)** $x + \log(x-1) - \log(x+1) + C$
**(b)** $x + 2\log(x-1) - \log x + C$
**(c)** $x - \tan^{-1}x + \log x - \frac{1}{2}\log(x^2+1) + C$
**(d)** $x + 9\log(x-3) - 4\log(x-2) + C$

**(e)** $\frac{1}{2}x^2 - x + 5\log(x) - 4\log(x+1) + C$
**(f)** $\frac{1}{3}x^3 + \frac{3}{2}x^2 + 7x + 16\log(x-2) - \log(x-1) + C$

## Exercise **2D** (Page 68)

**1(a)** $\frac{1}{3}\tan^{-1}\frac{x}{3} + C$ **(b)** $\log(x + \sqrt{9+x^2}) + C$
**(c)** $\sin^{-1}\frac{x}{3} + C$ **(d)** $\log(x + \sqrt{x^2-9}) + C$
**(e)** $\frac{1}{6}\Big(\log(x-3) - \log(x+3)\Big) + C$
**(f)** $\frac{1}{6}\Big(\log(3+x) - \log(3-x)\Big) + C$
**2(a)** $\tan^{-1}(x+2) + C$ **(b)** $\frac{1}{4}\tan^{-1}\left(\frac{x-2}{4}\right) + C$
**(c)** $\log(x - 3 + \sqrt{x^2-6x+13}\,) + C$
**(d)** $\log(x + 4 + \sqrt{x^2+8x+12}\,) + C$
**(e)** $\sin^{-1}\frac{x-4}{5} + C$
**(f)** $\frac{1}{2}\log\left(x + 1 + \sqrt{x^2+2x+\frac{3}{2}}\right) + C$
**3(a)** $\frac{\pi}{8}$ **(b)** $\pi$ **(c)** $\frac{\pi}{6}$ **(d)** $\frac{\pi}{2}$ **(e)** $\log 3$ **(f)** $\log 3$
**4(a)** $\log(x^2+2x+2) - \tan^{-1}(x+1) + C$
**(b)** $\frac{1}{2}\log(x^2+2x+10) - \frac{1}{3}\tan^{-1}\frac{x+1}{3} + C$
**(c)** $\sqrt{(x+1)^2+9} - \log\left(x + 1 + \sqrt{(x+1)^2+9}\right)$
**(d)** $\sqrt{x^2-2x-4} + 4\log\left(x - 1 + \sqrt{x^2-2x-4}\right)$
**(e)** $-\sqrt{6x-x^2} + 3\sin^{-1}\frac{x-3}{3} + C$
**(f)** $-\sqrt{4-2x-x^2} + 2\sin^{-1}\frac{x+1}{\sqrt{5}} + C$
**5(a)** $\frac{1}{2}\log 2 + \frac{\pi}{8}$ **(b)** $\frac{1}{4}(3\pi - \log 4)$ **(c)** $\log 2 - \frac{\pi}{4}$
**(d)** $2 - \sqrt{3} - \frac{\pi}{6}$ **(e)** $3\log(3+2\sqrt{2}\,) - 4\sqrt{2}$
**(f)** $\log\left(1 + \sqrt{\frac{2}{3}}\right) + \sqrt{6} - 1$
**6(a)** $\sqrt{x^2-1} - \log\left(x + \sqrt{x^2-1}\right) + C$
**(b)** $\sin^{-1}x - \sqrt{1-x^2} + C$
**(c)** $\sqrt{6+x-x^2} + \frac{5}{2}\sin^{-1}\frac{2x-1}{5} + C$
**7(a)** $\frac{\pi}{3} + \sqrt{3} - 2$ **(b)** $3\sin^{-1}\frac{1}{3}$
**(c)** $2\sqrt{2} - \sqrt{3} + \log\left(\frac{2+\sqrt{3}}{3+2\sqrt{2}}\right)$
**8(a)** $\frac{x}{\sqrt{4x-x^2}}$ is undefined at $x = 0$.

## Exercise **2E** (Page 72)

**1(a)** $e^x(x-1) + C$ **(b)** $-e^{-x}(x+1) + C$
**(c)** $\frac{1}{9}e^{3x}(3x+2) + C$ **(d)** $x\sin x + \cos x + C$
**(e)** $-\frac{1}{2}(x-1)\cos 2x + \frac{1}{4}\sin 2x + C$
**(f)** $(2x-3)\tan x + 2\log(\cos x) + C$
**2(a)** $\pi$ **(b)** $\frac{\pi}{2} - 1$ **(c)** $\frac{\pi}{4} - \frac{1}{2}\log 2$ **(d)** $\frac{1}{4}(e^2+1)$
**(e)** $e^{-1}$ **(f)** $1 + e^{-2}$
**3(a)** $x(\log x - 1) + C$ **(b)** $2x(\log x - 1) + C$
**(c)** $x\cos^{-1}x - \sqrt{1-x^2} + C$
**4(a)** $\frac{\pi}{4} - \frac{1}{2}\log 2$ **(b)** $1$ **(c)** $\frac{1}{2}$
**5(a)** $\frac{1}{4}x^2(2\log x - 1) + C$ **(b)** $\frac{1}{9}x^3(3\log x - 1) + C$
**(c)** $-\frac{1}{x}(\log x + 1) + C$
**6(a)** $(2 - 2x + x^2)e^x + C$
**(b)** $x^2\sin x + 2x\cos x - 2\sin x + C$
**(c)** $x(\log x)^2 - 2x\log x + 2x + C$
**7(a)** $-\frac{1}{42}$ **(b)** $\frac{4}{15}(1+\sqrt{2}\,)$ **(c)** $\frac{128}{15}$
**8(a)** $\frac{1}{2}e^x(\cos x + \sin x) + C$
**(b)** $-\frac{1}{2}e^{-x}(\cos x + \sin x) + C$
**9(a)** $\frac{1}{5}(e^\pi - 2)$ **(b)** $\frac{1}{5}(e^{\frac{\pi}{4}} + 2)$
**10(a)** $\frac{1}{2\sqrt{3}}(\pi - \sqrt{3})$ **(b)** $\frac{\sqrt{3}\pi}{2}$ **(c)** $\pi - 2$
**12(a)** $\frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}(\frac{x}{a})\right) + C$
**(b)** $x\log(x + \sqrt{x^2+a^2}\,) - \sqrt{x^2+a^2} + C$
**(c)** $x\log(x + \sqrt{x^2-a^2}\,) - \sqrt{x^2-a^2} + C$
**13(a)** $\frac{1}{4}x^2(2\log x - 1) + C$
**(b)** $\frac{1}{4}x^2\left(2(\log x)^2 - 2\log x + 1\right) + C$
**15(a)** $\frac{1}{32}(\sin 4x - 4x\cos 4x + 8x\cos 2x - 4\sin 2x) + C$
**(b)** $\frac{1}{18}(3x\sin 3x + \cos 3x + 9x\sin x + 9\cos x) + C$
**(c)** $\frac{1}{4}e^x(\sin 3x - 3\cos 3x + 5\sin x - 5\cos x) + C$
**16(a)** $\frac{1}{48}(3\sqrt{3} - \pi)$ **(b)** $\frac{1}{12}(\pi + 2\log 2 - 2)$

## Exercise **2F** (Page 78)

**1(a)** $\sin x + C$ **(b)** $-\cos x + C$ **(c)** $-\log(\cos x) + C$
**(d)** $\log(\sin x) + C$
**2(a)** $\frac{1}{3}\sin^3 x + C$ **(b)** $-\frac{1}{3}\cos^3 x + C$
**(c)** $\frac{1}{3}\cos^3 x - \cos x + C$ **(d)** $\sin x - \frac{1}{3}\sin^3 x + C$
**(e)** $\frac{1}{5}\sin^5 x - \frac{2}{3}\sin^3 x + \sin x + C$
**(f)** $\frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$
**3(a)** $\frac{\pi}{4}$ **(b)** $\frac{\pi}{12}$ **(c)** $\frac{\pi}{8}$
**4(a)** $\tan x + C$ **(b)** $\tan x - x + C$
**(c)** $\frac{1}{3}\tan^3 x + \tan x + C$ **(d)** $\frac{1}{3}\tan^3 x - \tan x + x + C$
**5(a)** $\sqrt{2} - 1$ **(b)** $\frac{1}{27}(8\sqrt{3} - 9)$ **(c)** $\frac{1}{2}(1 - \log 2)$
**(d)** $\frac{4}{3}$ **(e)** $\frac{1}{3}(2 - \sqrt{2}\,)$ **(f)** $\frac{58}{15}$
**6(a)** $\frac{1}{4}$ **(b)** $\frac{11}{24}$ **(c)** $\frac{9}{64}$ **(d)** $\frac{53}{480}$ **(e)** $\frac{4}{15}$ **(f)** $\frac{7}{60\sqrt{2}}$
**7(a)** $\frac{1}{32}(\sin 4x + 8\sin 2x + 12x) + C$
**(b)** $\frac{1}{32}(\sin 4x - 8\sin 2x + 12x) + C$
**(c)** $\frac{1}{1024}(24x - 8\sin 4x + \sin 8x) + C$
**9(a)** $1$ **(b)** $\frac{1}{3}\log 2$ **(c)** $\frac{\pi}{4}$
**10(a)** $\frac{\pi}{4}$ **(b)** $\frac{2}{15}(1 + sqrt2)$ **(c)** $\frac{\pi}{16}$
**11(a)** $\frac{1}{2}\sin^2 x + C_1$ **(b)** $-\frac{1}{4}\cos 2x + C_2$
**13(a)** $\frac{1}{2}\Big(\sec x\tan x - \log(\sec x + \tan x)\Big) + C$
**(b)** $\frac{1}{2}\Big(\sec x\tan x + \log(\sec x + \tan x)\Big) + C$
**(c)** $\sec x\tan x(2\sec^2 x + 3) + 3\log(\sec x + \tan x) + C$
**14(a)** $\frac{1}{2}$ **(b)** $\frac{4}{3}$ **(c)** $\frac{1}{2}$ **(d)** $\frac{\sqrt{3}}{4}$
**15(a)** $-\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + C$
**(b)** $-\frac{1}{8}\cos 4x + \frac{1}{4}\cos 2x + C$
**(c)** $\frac{1}{16}\sin 8x + \frac{1}{8}\sin 4x + C$
**16(a)** $\frac{1}{4}$ **(b)** $\frac{1}{6}$ **(c)** $\frac{3}{8}$
**17(a)** $\tan\frac{x}{2} + C$ **(b)** $\log\left(\frac{\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right) + C$
**(c)** $\frac{1}{5}\log\left(\frac{1+2\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right) + C$

**18** $x\sec x - \log(\sec x + \tan x) + C$
**20** $\frac{1}{3}\sin 3\theta + \sin^3\theta + C$

## Exercise **2G** (Page 83)

**3(b)** $\frac{1}{2}(e^2 - 1)$
**4(b)** $\frac{8}{15}$
**6(b)** $(\frac{\pi}{2})^6 - 30(\frac{\pi}{2})^4 + 360(\frac{\pi}{2})^2 - 720$
**7(b)** $I_0 = 1,\ I_4 = \frac{128}{315}$
**8(b)** $u_4 = \frac{243}{1540}$
**9(b)** $J_2 = -\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + C$
**12(d)** $\frac{1}{15}(14\sqrt{2} - 16)$
**13(d)** $\frac{1}{9}\Big((1+x^2)^4 + \frac{8}{7}(1+x^2)^3 + \frac{48}{35}(1+x^2)^2$
$+\frac{192}{105}(1+x^2) + \frac{384}{105}\Big)$
**14(d)** $J_n = \dfrac{2n}{2n+3}\,J_{n-1}$
**15(d)** $I_5 = \frac{1}{4}\,(2\ln 2 - 1)$

## Exercise **2H** (Page 86)

**1(a)** $\frac{1}{36}$ **(b)** $\pi$ **(c)** $\log\frac{12}{5}$ **(d)** $2-2\log 3$ **(e)** $2\sqrt{2}-1$
**(f)** $\frac{1}{2}\log(2+\sqrt{5}\,)$
**2(a)** $\sqrt{1+x^2} + C$ **(b)** $\tan^{-1}x + \frac{1}{2}\ln(1+x^2) + C$
**(c)** $-\frac{1}{5}\cos^5 x + C$ **(d)** $\log\left(\frac{2x+1}{x+1}\right) + C$
**(e)** $\frac{1}{4}x^4\log x - \frac{1}{16}x^4 + C$ **(f)** $\frac{1}{6}\cos^3 2x - \frac{1}{2}\cos 2x + C$
**(g)** $\frac{1}{4}\tan^{-1}\frac{x+3}{4} + C$ **(h)** $x\sin 3x + \frac{1}{3}\cos 3x + C$
**(i)** $\frac{2}{3}(x-8)\sqrt{4+x} + C$
**4(a)** $A = -\frac{2}{3}\,,\ B = \frac{2}{3}\,,\ C = -\frac{1}{3}$
**6** $\frac{1}{\sqrt{2}} + \frac{1}{2}\log(1+\sqrt{2})$
**8(a)** $A = 0,\ B = -2,\ C = 0,\ D = 2$ **(b)** $\frac{\pi}{2} - 1$
**10** $\frac{1}{2}a^2\sin^{-1}\frac{x}{a} + \frac{1}{2}x\sqrt{a^2-x^2} + C$
**11(b)** $\frac{1}{10}(\pi + \log\frac{27}{16})$
**12(a)** $P = 2,\ Q = -1$
**(b)** $2x - \log(3\sin x + 2\cos x - 1) + C$
**14(b)** $6 - 2e$

## Exercise **2I** (Page 93)

**1(a)** $0$ **(b)** $0$
**2(a)** $\frac{1}{132}$ **(b)** $\frac{16}{105}$ **(c)** $\frac{\pi^2}{4}$
**6(a)** The integrand is undefined at $x = 1$.
**(b)** $2(1-\sqrt{1-a})$ **(c)** $2$
**7(a)** The interval is unbounded.
**(b)** $\frac{1}{2}\tan^{-1}\left(\frac{N}{2}\right)$ **(c)** $\frac{\pi}{4}$
**9(a)** $\frac{\pi}{4}$ **(b)** $0$ **(c)** $1 - \frac{\pi}{4}$
**10(b)** $\frac{\pi^2}{4}$
**11(a)** $2\sqrt{2}$ **(b)** $4$ **(c)** $\frac{\pi}{2}$ **(d)** $\frac{\pi}{2}$ **(e)** $1 + \frac{\pi}{2}$ **(f)** $e$
**12(a)** $1$ **(b)** $\frac{\pi}{2}$ **(c)** $\frac{3\pi}{4}$ **(d)** $\frac{1}{2}$ **(e)** $1$ **(f)** $\frac{\pi}{2}$
**20(b)** $u_n = -\frac{n}{2}u_{n-1}$ **(c)** $\frac{3}{4}$
**22(d)** $e$

CHAPTER **SIX**

# Volumes

Chapter Overview: The work in this chapter demonstrates an application of integration to the real world, namely to find the volumes of certain solids. The solids encountered include those with rotational symmetry and those with known cross-sections. The volumes of solids with rotational symmetry is an extension of the work done in Mathematics Extension 1. The volumes of solids with known cross-sections will be new to most students.

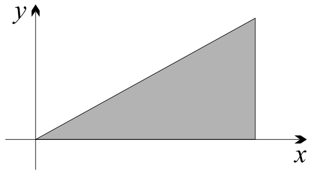
In the solution of a given problem, a full exposition would include the derivation of an expression for the tiny bits, called volume elements, that go to make up the solid. A limiting sum of these volume elements then gives the volume of the solid as an integral. Some of the Worked Exercises and some of the questions in the Exercises follow this approach, and more capable students are encouraged to attempt these harder questions. However the trend in recent HSC examinations is to obtain a formula for an area which is then integrated to yield the volume of the solid. The majority of the text and Exercise questions follows this format.

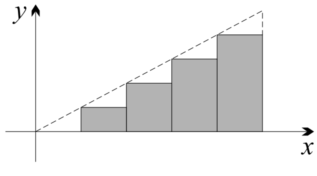
Each solid encountered in this topic has what is called a generating shape. For example, a cone is traced out when a right-angled triangle is rotated about its altitude. Thus it is said that the triangle generates the cone. Sketches of the generating shape and the solid are included in the Worked Exercises and students are encouraged to draw similar sketches when tackling questions.

There are three sections in the chapter. Simple volumes of revolution are dealt with in 6A, including volumes of revolution about lines other than the coordinate axes. In 6B the method of cylindrical shells is introduced, which is used to simplify certain harder types of volumes of revolution. The final section deals with solids with known cross-sections, where similarity is an important component.

## 6A Volumes of Revolution

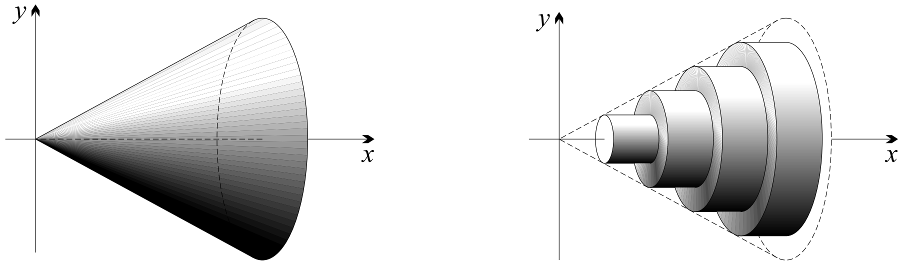
This section begins by reviewing the work done in Mathematics Extension 1 on volumes of revolution. In the Integration topic it was observed that an area can be approximated by a series of rectangles, as in the diagrams below.



As the number of rectangles is increased, the approximation is improved. The actual area is obtained by letting the number of rectangles tend to infinity, though this was only proven in a few simple cases. The result is an integral.

In a similar way, a volume of revolution can be approximated by a series of short cylinders, or disks, as shown in the diagrams below. The volume of each disk is easily calculated using $V = \pi r^2 h$.




It should be clear that as the number of disks is increased, the approximation is improved. The actual volume is obtained by letting the number of disks tend to infinity, though this was never proven. The result is again an integral.

**Volumes of Revolution About the Axes:** If the axis of of rotation is the $x$-axis, the radius of each disk is the $y$-coordinate and the height is $dx$. Thus if the solid exists between $x = a$ and $x = b$ then the volume integral is

$$V = \int_a^b \pi y^2 \, dx \,.$$

When the axis of rotation is the $y$-axis, the radius of each disk is the $x$-coordinate and the height is $dy$. Thus if the solid exists between $y = c$ and $y = d$ then the volume integral is

$$V = \int_c^d \pi x^2 \, dy \,.$$

Observe that in each case the radius is squared so that it does not matter if either $x$ or $y$ is negative. Both formulae should be familiar from the Mathematics Extension 1 course.

**A Generalised Approach:** Notice that in both cases the area of a circle $A = \pi r^2$ is integrated across the height of the solid. Thus it is possible to replace both formulae with the simpler result

$$V = \int_a^b \pi r^2 \, dh \,.$$

Further simplifying the notation,

$$V = \int_a^b A \, dh \,.$$

Reading this last line more carefully, if the area $A$ of a typical cross-section is integrated over the height of the solid, the result is the volume $V$ of the solid. Although it will not be proven, this formula is valid for any shaped solid and will be used in most problems for the remainder of this chapter.

1 A GENERAL FORMULA FOR VOLUME: If $A(h)$ is the area of a cross-section of a solid at height $h$ then the volume $V$ is given by

$$V = \int_a^b A\,dh\,,$$

where $h = a$ is the lowest point of the solid and $h = b$ is the highest point.

**Volumes of Revolution About Other Axes:** The previous results for volumes of revolution are now extended to problems where the axis of revolution is some other horizontal or vertical line. The approach is to carefully determine the radius of each circular cross-section, or slice, and hence find the area of that circle. Then, using the result of Box 1, integrate over the height of the solid to obtain its volume.

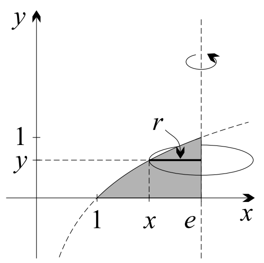
**WORKED EXERCISE:** The region bounded by $y = \log x$, the line $x = e$ and the $x$-axis is rotated about the line about $x = e$ to generate a solid.

(a) Show that the area of a slice at height $y$ is given by

$$A = \pi(e - e^y)^2\,.$$

(b) Hence find the volume of the solid, correct to one decimal place.

**SOLUTION:** The situation is shown on the right.



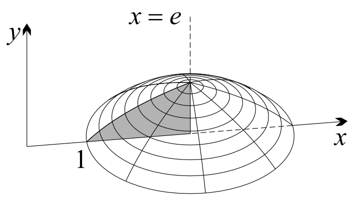

(a) First re-arrange $y = \log x$ to get $x = e^y$.
Consider the circular slice at height $y$. The radius is

$$\begin{aligned} r &= e - x \\ &= e - e^y \end{aligned}$$

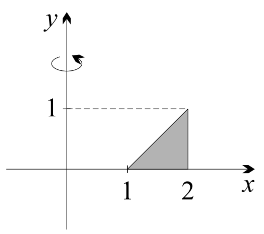
hence $A = \pi(e - e^y)^2\,.$

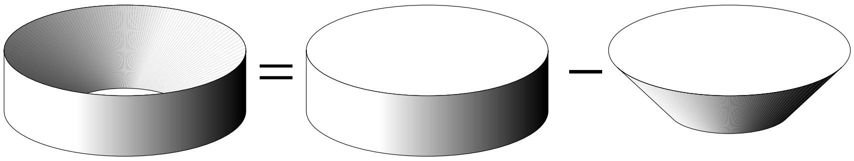
(b) Integrating part (a) from $y = 0$ to $y = 1$:

$$\begin{aligned} V &= \pi\int_0^1 (e - e^y)^2\;dy \\ &= \pi\int_0^1 e^2 - 2e^{y+1} + e^{2y}\;dy \\ &= \pi\left[e^2y - 2e^{y+1} + \tfrac{1}{2}e^{2y}\right]_0^1 \\ &= \pi\Big((e^2 - 2e^2 + \tfrac{1}{2}e^2) - (0 - 2e + \tfrac{1}{2})\Big) \\ &= \pi\Big(2e - \tfrac{1}{2}(e^2 + 1)\Big) \\ &\doteqdot 3{\cdot}9 \text{ cubic units} \end{aligned}$$




**Volumes by Subtraction:** Some volume problems are best solved by taking the difference between the volumes of two simpler solids. This is covered in the Mathematics Extension 1 course. As an example, consider the solid generated when the triangle on the right is rotated about the $y$-axis. The volume is easily found by subtracting the solid formed by rotating the region to the left of the hypotenuse from the cylinder with radius 2 and height 1. The situation is shown in the diagram on the next page .





Thus the volume required is the difference between the bigger outer volume and the smaller inner volume. That is:

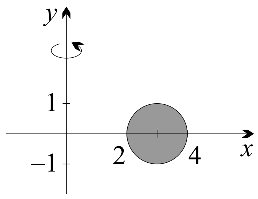
$$\begin{aligned} V &= V_{\text{outer}} - V_{\text{inner}} \\ &= \int_a^b \pi r_{\text{outer}}^2 \, dh - \int_a^b \pi r_{\text{inner}}^2 \, dh\,. \end{aligned}$$

In most problems it is easy to evaluate each integral and subtract. However in some instances it is algebraically advantageous to combine these two terms into a single integral, thus:

$$V = \int_a^b \pi (r_{\text{outer}}^2 - r_{\text{inner}}^2) \, dh\,.$$

The term $A = \pi(r_{\text{outer}}^2 - r_{\text{inner}}^2)$ represents the area of the region between two concentric circles, properly called an *annulus*, though sometimes called a washer. An annulus will result whenever the generating region does not contact the axis of revolution all the way from the lowest point to the highest point of the solid.

**WORKED EXERCISE:** A toroid (ring) is formed by rotating the circle $(x-3)^2 + y^2 = 1$ about the $y$-axis.



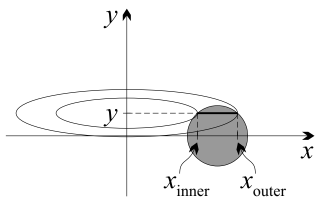

(a) Sketch a typical cross-section at height $y$ and describe its shape.

(b) Show that the area of this cross-section is

$$A = 12\pi\sqrt{1-y^2}$$

(c) Hence find the volume of the toroid.

**SOLUTION:** (a) The sketch is on the right.
The cross-section is an annulus.

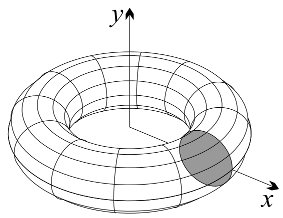



(b) First solve the equation of the circle for $x$.

$$\begin{aligned} (x-3)^2 &= 1-y^2 \\ \text{so} \quad x-3 &= \sqrt{1-y^2} \text{ or } -\sqrt{1-y^2} \\ \text{thus} \quad x_{\text{outer}} &= 3+\sqrt{1-y^2} \\ \text{and} \quad x_{\text{inner}} &= 3-\sqrt{1-y^2}\,. \\ \text{Hence} \quad A &= \pi(x_{\text{outer}}^2 - x_{\text{inner}}^2) \\ &= \pi(x_{\text{outer}} + x_{\text{inner}})(x_{\text{outer}} - x_{\text{inner}}) \quad \text{(difference of two squares)} \\ &= \pi \times 6 \times 2\sqrt{1-y^2} \\ &= 12\pi\sqrt{1-y^2}\,. \end{aligned}$$

(c) Integrating $A$ from $y = -1$ to $y = 1$

$$\begin{aligned} V &= \int_{-1}^{1} 12\pi\sqrt{1-y^2}\ dy \\ &= 12\pi \int_{-1}^{1} \sqrt{1-y^2}\ dy \\ &= 12\pi \times \tfrac{1}{2}\pi \quad \text{(area of a semi-circle, radius 1)} \\ &= 6\pi^2 . \end{aligned}$$




**2** VOLUMES BY SUBTRACTION: Whenever the cross-section of a volume of revolution is an annulus, use the formula

$$V = \int_a^b \pi(r_{\text{outer}}^2 - r_{\text{inner}}^2)\ dh .$$

where $h = a$ is the lowest point of the solid and $h = b$ is the highest point.

**Shifts and Reflections:** In some problems a judicious choice of a shift or reflection can simplify the situation. The aim is to move the axis of rotation to coincide with one of the coordinate axes.

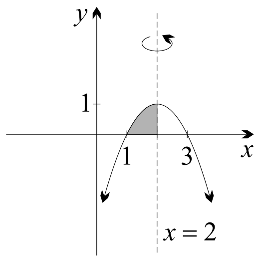
For example if a region is rotated about the line $x = 3$ then shifting the problem left by 3 units will make the $y$-axis the axis of rotation. In the following problem, reflection in the line $x = a$ is used, as encountered in the Mathematics Extension 2 topics Integration and Graphs. Recall that this means replacing $x$ with $(2a - x)$.

**WORKED EXERCISE:** The region under the parabola $y = (x-1)(3-x)$ to the left of $x = 2$ and above the $x$-axis is rotated about the line $x = 2$ to generate a solid.

(a) Draw the situation.
(b) What reflection will make the $y$-axis become the axis of rotation?
(c) What is the equation of the reflected parabola?
(d) Draw the new configuration of the problem.
(e) Hence find the volume of the solid.

**SOLUTION:**

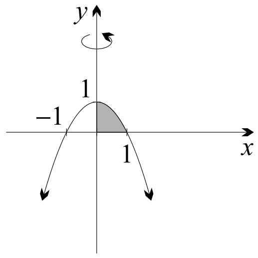
(a) The situation is shown on the right.




(b) Reflect in $x = 1$, so replace $x$ with $(2 - x)$.

(c) $$\begin{aligned} y &= \big((2-x)-1\big)\big(3-(2-x)\big) \\ &= (1-x)(1+x) \\ &= 1 - x^2 . \end{aligned}$$

(d) The new configuration is shown on the right.




(e) This is now a simple volume of revolution.

$$\begin{aligned} V &= \pi \int_0^1 x^2\ dy \\ &= \pi \int_0^1 1 - y\ dy \\ &= \pi \left[y - \tfrac{1}{2}y^2\right]_0^1 \\ &= \tfrac{\pi}{2} . \end{aligned}$$

**3** | SHIFTS AND REFLECTIONS: If a region is rotated about the line $x = 2a$, do one of the following to make the $y$-axis become the axis of rotation.
- Shift left by replacing $x$ with $(x + 2a)$.
- Reflect in the line $x = a$ by replacing $x$ with $(2a - x)$.

Similar vertical shifts and reflections can be applied when the axis of rotation is the horizontal line $y = 2a$, to make the $x$-axis become the axis of rotation.

## Exercise **6A**

**1.** (a) Draw the region bounded by $y = x^3 + 1$ and the coordinate axes.
(b) Find the volume of the solid formed when this region is rotated about:
(i) the $x$-axis, (ii) the $y$-axis.

**2.** The region bounded by $y = x^2$, $y = 2 - x$ and the $x$-axis is rotated about the $x$-axis to generate a solid. (a) Draw the region. (b) Find the volume of the solid.

**3.** (a) Sketch the region $\mathcal{R}$ bounded by the curve $y = x^2$, the line $x = 2$ and the $x$-axis.
(b) By slicing perpendicular to the axis of rotation, find the volume of the solid formed when the region $\mathcal{R}$ is rotated through $360^\circ$ about: (i) the $x$-axis, (ii) the $y$-axis.

**4.** (a) Graph the region bounded by the curve $y = \sqrt{x}$, the $x$-axis and the line $x = 4$.
(b) Us the result of Box 2 to find the volume of the solid generated when this region is rotated about the $y$-axis.

**5.** The region $\mathcal{R}$ is bounded by the curve $y = x^3$, and the lines $x = 0$ and $y = 1$. Use the result of Box 2 to find the volume of the solid formed when $\mathcal{R}$ is rotated about $y = 0$.

**6.** In each case find the volume of the solid formed when the region with the given boundaries is rotated about the $x$-axis:
(a) the parabola $y^2 = 4x$, the $y$-axis and the line $y = 2$,
(b) the parabola $y = 3 + x^2$ and the line $y = 4$,
(c) the parabola $y = 3x - x^2$ and the line $y = 2$.

**DEVELOPMENT**

**7.** The region bounded by $y = x - 1$, $y = 3 - x$ and the $x$-axis is rotated about the $y$-axis. Find the volume of the resulting solid.

**8.** A solid is formed by rotating the region bounded by $y = x^2 + 1$, and $y = 3 - x^2$ about the $x$-axis. What is its volume?

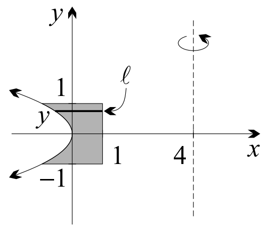
**9.** By taking slices perpendicular to the axis of rotation, find the volume of the solid generated when the region bounded by the curve $y = \sqrt{x}$, the $x$-axis and the line $x = 4$ is rotated about the line $x = 4$.

**10.** The region $\mathcal{R}$ is bounded by the curve $y = x^2$, the line $x = 2$ and the $x$-axis. By slicing perpendicular to the axis of rotation, find the volume of the solid formed when the region $\mathcal{R}$ is rotated through $360^\circ$ about:
(a) the line $x = 2$ (b) the line $y = 4$ (c) the line $x = 3$

**11.** The region $\mathcal{R}$ is bounded by the curve $y = x^3$, and the lines $x = 0$ and $y = 1$. By slicing perpendicular to the axis of rotation, find the volume of the solid formed when $\mathcal{R}$ is rotated about: (a) the line $y = 1$, (b) the line $y = 2$.

**12.** The region $\mathcal{A}$ is bounded by the parabola $y = 4 - x^2$, and the lines $x = 2$ and $y = 4$. Find, by taking slices perpendicular to the axis of rotation, the volume of the solid generated when $\mathcal{A}$ is rotated about: (a) the line $x = 2$, (b) the line $x = 3$.

**13.** The shaded region is bounded by the lines $x = 1$, $y = 1$ and $y = -1$ and by the curve $x + y^2 = 0$. The region is rotated through 360° about the line $x = 4$ to form a solid. When the region is rotated, the line segment $\ell$ at height $y$ sweeps out an annulus.



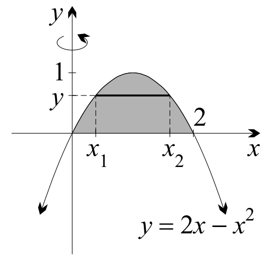

(a) Show that the area of the annulus is $\pi(y^4 + 8y^2 + 7)$.

(b) Hence find the volume of the solid.

**14.** The region $\mathcal{P}$ is bounded by the parabola $y^2 = 4ax$ and its latus rectum $x = a$. Find the volume of the solid generated when $\mathcal{P}$ is rotated about each of the following vertical lines:

(a) the latus rectum, (b) $x = 2a$, (c) the directrix.

**15.** The diagram on the right shows the region $\mathcal{R}$ bounded by the parabola $y = 2x - x^2$ and the $x$-axis.




(a) When the interval $AB$ at height $y$ is rotated about the $y$-axis, an annulus is generated. Show that its area is given by

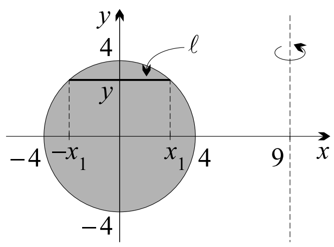
$$4\pi\sqrt{1 - y}\,.$$

(b) Hence find the volume of the solid formed when $\mathcal{R}$ is rotated about the $y$-axis.

**16.** (a) Sketch the curve $y = 2x^2 - x^4$, clearly showing the $x$-intercepts and the coordinates of the stationary points.

(b) Let $\mathcal{A}$ be the region in the first quadrant bounded by the curve $y = 2x^2 - x^4$ and the $x$-axis. Using the methods of the previous question, find the volume of the solid formed by rotating $\mathcal{A}$ about the $y$-axis.

**17.** The circle $x^2 + y^2 = 16$ is rotated about the line $x = 9$ to form a ring. When the circle is rotated, the line segment $\ell$ at height $y$ sweeps out an annulus. The endpoints of $\ell$ have $x$-values $x_1$ and $-x_1$.




(a) Show that the area of the annulus is $36\pi\sqrt{16 - y^2}$.

(b) Hence find the volume of the ring.

**18.** The circle $(x - c)^2 + y^2 = a^2$, where $c > a$, is rotated about the $y$-axis to form a torus. By slicing perpendicular to the $y$-axis, show that the torus has volume $2\pi^2 a^2 c$ cubic units.

**19.** Find the volume of the solid formed when the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is rotated about each of the following horizontal lines:

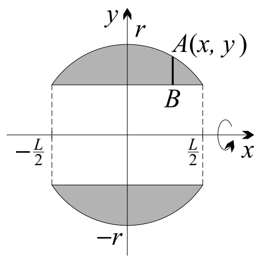
(a) $y = 0$, (b) $y = b$, (c) $y = c$. where $c > b$

**20.** (a) In $\triangle ABC$, $AB = 3\,\text{cm}$, $BC = 4\,\text{cm}$ and $AC = 5\,\text{cm}$. The point $D$ is on $AC$ such that $BD \perp AC$. Use similar triangles to find $AD$ and $CD$.

(b) A spherical cap of height $h$ is cut off a sphere of radius $r$ by a horizontal plane. Show that the cap has volume $\frac{1}{3}\pi h^2(3r - h)$.

(c) The centres of two intersecting spheres of radii 3 cm and 4 cm are 5 cm apart. Use the results in (a) and (b) to find the volume common to the two spheres.

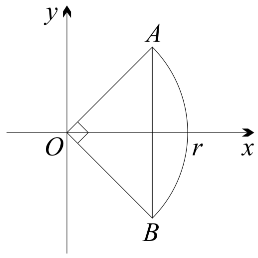
**21.** The diagram shows the cross-section of a sphere through which a cylindrical hole of length $L$ has been drilled.




(a) Show that the annulus formed by rotating the interval $AB$ about the $x$-axis has area $\frac{\pi}{4}(L^2 - 4x^2)$ square units, and thus is independent of $r$.

(b) Hence show that the volume of the sphere remaining is the same as the volume of a sphere of diameter $L$.

**22.** In the diagram, $AB$ is an arc of a quadrant of a circle centred at $O$ and of radius $r$. The chord $AB$ is parallel to the $y$-axis.

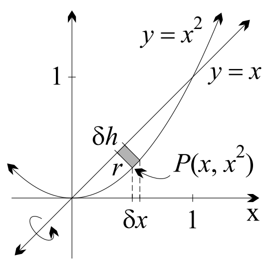



(a) The quadrant $AOB$ is rotated about the $y$-axis. Show that the solid formed has volume $\frac{2\sqrt{2}}{3}\pi r^3$.

(b) The region bounded by arc $AB$ and the chord $AB$ is rotated about the chord $AB$. Show that the solid formed has volume $\frac{10-3\pi}{6\sqrt{2}}\pi r^3$ cubic units.

**EXTENSION**

**23.** [A First Principles Approach] The region bounded by $y = x^2$ and $y = x$ is rotated about the line $y = x$ to generate a solid. In this question you will find the volume of this solid by first approximating a small portion with a disk.




Let $P(x, x^2)$ be a typical point on the parabola. From $x$ to $x + \delta x$ we will approximate the area between the curve and the line with a rectangle. One corner of this rectangle is at $P$ and a side is on the line $y = x$. Let the dimensions of this rectangle be $r \times \delta h$ as shown in the diagram.

(a) Show that $\delta h = \sqrt{2}\,\delta x$ and find $r$ as a function of $x$.

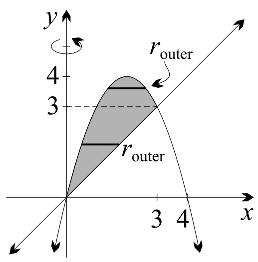
(b) When the rectangle is rotated about the line $y = x$ it generates a cylindrical prism with volume $\delta V$. Find $\delta V$.

(c) Divide $\delta V$ by $\delta x$. What is the limit of this ratio as $\delta x \to 0$?

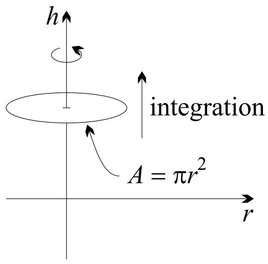
(d) Hence show that the volume of the solid is $\frac{\pi}{30\sqrt{2}}$.

# 6B The Method of Cylindrical Shells

Inevitably when tackling volume problems there will be certain questions which result in awkward integrals. As an example, consider the solid generated when the region between $y = 4x - x^2$ and $y = x$ is rotated about the $y$-axis, as shown on the right. The volume can be found by the methods of Section 6A since each cross-section is an annulus. However the situation is complicated by the fact that the formula for the outer radius changes. Below $y = 3$ the radius is $r_{\text{outer}} = y$ whilst above $y = 3$ it is $r_{\text{outer}} = 2 + \sqrt{4 - y}$. Hence the integral must be split into two parts. (Try this as an exercise.)

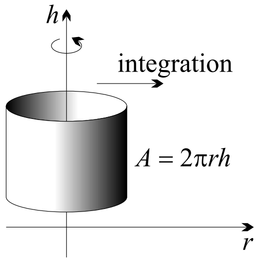


In this case, and in similar instances, these difficulties can be overcome by using the method of cylindrical shells. By way of contrast, first examine more closely how volumes of revolution were derived in Section 6A. At any height a slice perpendicular to the axis of revolution is a circle or annulus. The area of the slice is found. The result is then integrated in a direction perpendicular to the slice, that is along the $h$-axis. The situation for the circle is shown on the right. Thus the volume is:




$$V = \int_a^b \pi r^2 \, dh\,.$$

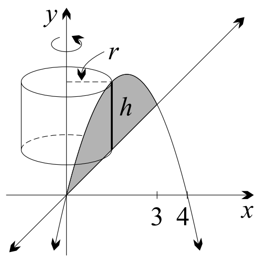
In the method of cylindrical shells a slice is taken parallel with the axis of revolution. When rotated about the axis the slice generates a cylindrical shell. The surface area of the cylinder is $A = 2\pi rh$. This result is now integrated in a direction perpendicular to the surface, that is along the $r$-axis. Again, the situation is shown on the right. Thus, by the method of cylindrical shells:




$$V = \int_a^b 2\pi rh \; dr$$

**WORKED EXERCISE:** The region between $y = 4x - x^2$ and $y = x$ is rotated about the $y$-axis to form a solid. Find its volume by the method of cylindrical shells.

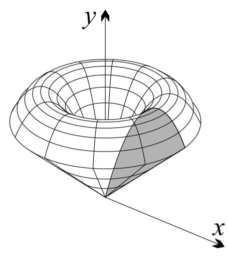
**SOLUTION:** Since the axis of revolution is the $y$-axis, the radius of a cylindrical shell is the $x$-coordinate, and the height is the difference between the $y$-coordinates on the two curves. Thus:




$$r = x$$

and
$$\begin{aligned} h &= (4x - x^2) - x \\ &= 3x - x^2\,. \end{aligned}$$

Hence
$$\begin{aligned} V &= \int_0^3 2\pi x(3x - x^2)\, dx \\ &= 2\pi \int_0^3 3x^2 - x^3 \, dx \\ &= 2\pi \Big[ x^3 - \tfrac{1}{4}x^4 \Big]_0^3 \\ &= \tfrac{27\pi}{2}\,. \end{aligned}$$



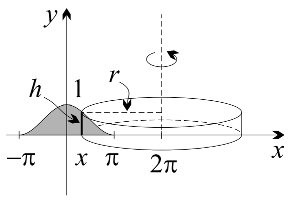

4 THE METHOD OF CYLINDRICAL SHELLS: In a volume of revolution, a slice taken parallel with the axis generates a cylindrical shell. The surface-area of the cylinder is $A = 2\pi rh$ and the total volume of the solid is:

$$V = \int_a^b 2\pi rh \; dr\,.$$

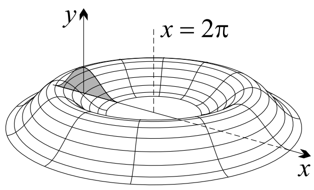
where $r = a$ is the innermost radius and $r = b$ is the outermost radius.

**Rotation about another axis:** Care needs to be taken that the expressions for $r$ and $h$ are positive. Otherwise the method is the same as before. That is, find the surface-area of a cylindrical shell then integrate to obtain the volume.

**WORKED EXERCISE:** The region between $y = 1 + \cos x$ and the $x$-axis is rotated about the line $x = 2\pi$ to form a solid. Find the volume of this solid by the method of cylindrical shells.




**SOLUTION:** The situation is shown on the right. The radius of the cylinder is $r = 2\pi - x$, so that $dr = -dx$. As $r$ increases, $x$ changes from $\pi$ to $-\pi$. Thus:

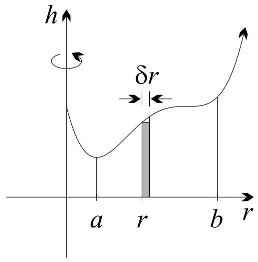


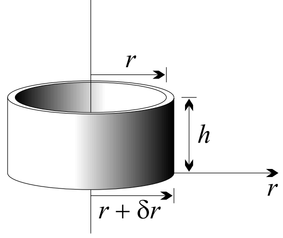

$$
\begin{aligned}
V &= -\int_{\pi}^{-\pi} 2\pi(2\pi - x)(1+\cos x)dx \\
&= 2\pi\int_{-\pi}^{\pi} 2\pi(1+\cos x) - x(1+\cos x)\,dx \\
&= 2\pi\int_{-\pi}^{\pi} 2\pi(1+\cos x)\,dx \qquad \text{(by odd symmetry)} \\
&= 4\pi^2\Big[x + \sin x\Big]_{-\pi}^{\pi} \\
&= 8\pi^3 .
\end{aligned}
$$

Note that this problem could also be solved by first reflecting in the line $x = \pi$, that is, replacing $x$ with $2\pi - x$. Try this as an exercise and compare the results.

**A First Principles Approach:** In harder problems you may be required to find the volume of a solid using a first principles approach. Here it is used to derive the formula in Box 4 for cylindrical shells. It should be noted that although this example is a more formal approach to the method of cylindrical shells, it is not a proper proof.

Suppose that the region below the curve $h = f(r)$ and between $r = a$ and $r = b$ is rotated about the vertical axis to generate a solid. For simplicity, it is assumed that $f(r)$, $a$ and $b$ are all positive. Now suppose that the region is approximated by a series of thin rectangular strips of width $\delta r$. One such strip is shown in the diagram on the right. The volume element generated when this thin strip is rotated about the axis is a pipe or cylindrical shell, as in the second diagram on the right. Let $\delta V$ be the volume of this shell then:




$$
\begin{aligned}
\delta V &= \pi(r+\delta r)^2 h - \pi r^2 h \\
&= \pi h\Big((r+\delta r)^2 - r^2\Big) \\
&= \pi h(2r + \delta r)\delta r \qquad \text{(difference of two squares.)}
\end{aligned}
$$

The volume of the original solid will be approximately equal to the sum of all such shells between $x = a$ and $x = b$, that is:

$$V \doteqdot \sum_{r=a}^{r=b} \pi h(2r + \delta r)\delta r$$

Although the result will not be proven, it should be intuitively clear that these two volumes will be equal in the limit as $\delta r \to 0$. Thus:

$$V = \lim_{\delta r \to 0} \left( \sum_{r=a}^{r=b} \pi h(2r + \delta r)\delta r \right)$$

From the previous work on integration, such a limiting sum yields an integral, and hence:

$$V = \int_a^b \pi h(2r + 0)\, dr\,,$$

or more simply

$$V = \int_a^b 2\pi r h\, dr\,,$$

exactly as before.

**A Derivative Approach:** Here is an alternative to the first principles approach. The initial steps are identical and so the volume of a cylindrical shell is:

$$\delta V = \pi h(2r + \delta r)\delta r$$

or $$\frac{\delta V}{\delta r} = 2\pi(r + \delta r)h\,.$$

Now take the limit as $\delta r \to 0$ to get

$$\frac{dV}{dr} = 2\pi r h\,.$$

Finally, integrate with respect to $r$ to once again get

$$V = \int_a^b 2\pi r h\, dr\,.$$

## Exercise **6B**

**1.** The region bounded by the curve $y = x^2$, the line $x = 2$ and the $x$-axis is rotated about the $y$-axis to form a solid.

(a) Sketch the region.

(b) Use the method of cylindrical shells to find the volume of the solid.

**2.** The region $\mathcal{A}$ lies in the first quadrant and is bounded by the curve $y = x^2$, the line $y = 4$ and the $y$-axis.

(a) Graph this region.

(b) Use the method of cylindrical shells to find the volume of the solid formed when region $\mathcal{A}$ is rotated through $360^\circ$ about the $x$-axis.

**3.** Find, using cylindrical shells, the volume of the solid generated when the region with the given boundaries is rotated about the $y$-axis:

(a) $y = 4x - x^2$ and the $x$-axis, (b) $y^2 = 4x$ and $x = 9$, (c) $y = x^2 - x^3$ and the $x$-axis.

**4.** The region $\mathcal{C}$ is bounded by the curve $y = 1 - x^3$, and the coordinate axes. By using cylindrical shells, find the volume of the solid formed when $\mathcal{C}$ is rotated about:

(a) the $y$-axis, (b) the $x$-axis (let $1 - y = u^3$).

**DEVELOPMENT**

**5.** The region $\mathcal{R}$ is bounded by the curve $y = x^2$, the line $x = 2$ and the $x$-axis. By using the method of cylindrical shells, find the volume of the solid formed when the region $\mathcal{R}$ is rotated through $360^\circ$ about: (a) the line $x = 2$, (b) the line $x = 3$.

**6.** The region $\mathcal{A}$ lies in the first quadrant and is bounded by the curve $y = x^2$, the line $y = 4$ and the $y$-axis. Use the method of cylindrical shells to find the volume of the solid formed when the region $\mathcal{A}$ is rotated through $360^\circ$ about: (a) the line $y = 4$, (b) the line $y = 6$.

**7.** The region $\mathcal{A}$ is bounded by the parabola $y = 4 - x^2$, and the lines $x = 2$ and $y = 4$. Find, by the method of cylindrical shells, the volume of the solid generated when $\mathcal{A}$ is rotated about: (a) the line $x = 2$, (b) the line $x = 3$.

**8.** The region bounded by the curves $y = x^3 + 8$ and $y = x^2 + 1$, and the lines $x = 0$ and $x = 2$, is rotated about the $y$-axis. Use cylindrical shells to show that the solid formed has volume $\frac{164\pi}{5}$ cubic units.

**9.** (a) Sketch the parabola $y = x^2$ and the line $y = 4x - 3$, showing their points of intersection.
(b) The region in part (a) bounded by the parabola and the line is rotated about the $y$-axis to generate a solid. Find the volume of this solid using cylindrical shells.

**10.** (a) Sketch the region enclosed by $y = x + 1$ and $y = (x - 1)^2$.
(b) The region is rotated about the $y$-axis. Find the volume of the solid formed.

**11.** The region bounded by the curve $y = e^x$ and the lines $y = 0$, $x = 0$ and $x = 1$ is rotated about the line $x = 0$. Use cylindrical shells to find the volume of the solid formed.

**12.** The region $\mathcal{R}$ is bounded by the parabolas $y = 3 - x^2$ and $y = x + x^2$ and the line $x = -1$, and lies to the right of the line $x = -1$. Use the method of cylindrical shells to find the volume of the solid generated when $\mathcal{R}$ is rotated about the line $x = -1$.

**13.** The region $\mathcal{R}$ is bounded by the parabola $y^2 = 9x$, the line $x = 1$ and the $x$-axis. A solid is formed by rotating $\mathcal{R}$ about the line $x = 2$. Find the volume of the solid by using:
(a) slices perpendicular to the axis of rotation, (b) cylindrical shells.

**14.** A solid is formed by rotating the region bounded by the parabola $y^2 = 16(1 - x)$ and the $y$-axis about the line $x = 2$.
(a) By slicing perpendicular to the axis of rotation, find the volume of the solid.
(b) (i) Use cylindrical shells to show that the volume is $V = 16\pi \displaystyle\int_0^1 (2 - x)\sqrt{1 - x}\, dx$.
(ii) Apply the substitution $u = 1 - x$ to evaluate this integral.

**15.** Find, using cylindrical shells, the volume of the solid formed when the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is rotated about each of the following horizontal lines:
(a) $y = 0$ (b) $y = b$ (c) $y = c$, where $c > b$

**16.** The region $\mathcal{P}$ is bounded by the parabola $y^2 = 4ax$ and its latus rectum $x = a$. Use the method of cylindrical shells to find the volume of the solid generated when $\mathcal{P}$ is rotated about each of the following vertical lines:
(a) the latus rectum, (b) $x = 2a$, (c) the directrix.

**17.** The circle $(x - c)^2 + y^2 = a^2$, where $c > a$, is rotated about the $y$-axis to form a torus. Use the method of cylindrical shells to show that the torus has volume $2\pi^2 a^2 c$ cubic units.

**18.** A hole of diameter $R$ is drilled through the centre of a solid sphere of diameter $2R$. Show that the remaining solid has volume $\frac{\sqrt{3}}{2}\pi R^3$ by using:

(a) the method of cylindrical shells, (b) slices perpendicular to the axis of rotation.

**19.** Consider the solid formed when the region under $y = f(x)$ between $x = a$ and $x = b$ is rotated about the line $x = c$. For simplicity assume that $f(x)$, $a$, $b$ and $c$ are positive, with $a < b < c$. In a typical cylindrical shell the radius is $r = c - x$ and the height is $h = f(x)$. As in the second Worked Exercise, it seems that the volume is:

$$V = \int 2\pi rh\, dr = \int_a^b 2\pi(c - x)f(x)\, dx\,.$$

The careful reader will have observed a problem with this, since if $r = c-x$ then $dr = -dx$. Nevertheless the above result is correct. Prove it.

**EXTENSION**

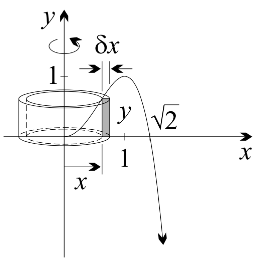
**20.** Let $y = f(x)$ be an even function. The region between this curve and the $x$-axis and between $x = -a$ and $x = a$ is rotated about the line $x = c$, where $0 < a < c$. Consider one cylindrical shell at $x = t$ and a second cylindrical shell at $x = -t$, where $0 \leq t \leq a$.

(a) Show that the total surface-area of the two cylinders is $A_T = 4\pi c f(t)$.

(b) Hence show that the volume of this solid is $V = 4\pi c \displaystyle\int_0^a f(t)\, dt$.

(c) Repeat the second Worked Exercise using this formula.

**21.** [A First Principals Approach] The region in the first quadrant below $y = 2x^2 - x^4$ is rotated about the $y$-axis to form a solid. Now suppose that the region is approximated by a series of thin rectangles of width $\delta x$. One such strip is shown in the diagram.



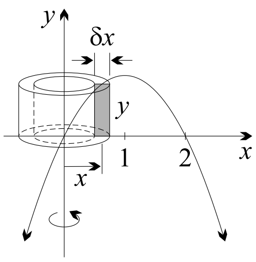

(a) Show that the volume of the cylindrical shell generated when this rectangle is rotated about the $y$-axis is:

$$\delta V = \pi(2x^2 - x^4)(2x + \delta x)\, \delta x\,.$$

(b) Write down a limiting sum for the volume of the solid.

(c) Rewrite the limiting sum as an integral and hence find the volume of the solid.

**22.** [A First Principals Approach] The region in the first quadrant below $y = 2x - x^2$ is rotated about the $y$-axis to form a solid. Consider the thin rectangle of width $\delta x$ which generates a shell with volume $\delta V$. Notice that in this case $x$ is the distance from the $y$-axis to the *midpoint* of the base of the rectangle.




(a) What are the inner and outer radii of the cylindrical shell?

(b) Show that $\delta V = 2\pi x(2x - x^2)\, \delta x$.

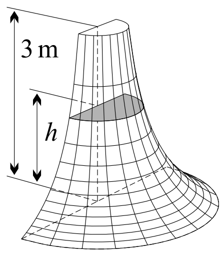
(c) Write down a limiting sum for the volume of the solid.

(d) Rewrite the limiting sum as an integral and hence find the volume of the solid.

# 6C Solids with Known Cross-Sections

Whenever an expression is known or can be determined for the area $A(h)$ of a cross-section of a solid at height $h$ then it is simply a matter of using the result in Box 1 to determine the volume.

**WORKED EXERCISE:** An artist creates a sculpture which is 3 metres high and has semi-circular cross-sections. The radius of the semi-circle at height $h$ above its base is known to be $r = \frac{2}{h+1}$ . Find the volume of the sculpture.




**SOLUTION:** The area of the semi-circle at height $h$ is

$$A(h) = \frac{1}{2} \times \pi \left(\frac{2}{h+1}\right)^2$$
$$= \frac{2\pi}{(h+1)^2}$$

hence
$$V = \int_0^3 \frac{2\pi}{(h+1)^2}\, dh$$
$$= \left[\frac{-2\pi}{h+1}\right]_0^3$$
$$= \frac{3\pi}{2} \text{ cubic metres.}$$

Note that this problem could have been done as a volume of revolution by rotating $y = \frac{2}{x+1}$ about the $x$-axis and then halving the answer. Try this as an exercise.

**Simple Geometry:** In many instances a knowledge of simple geometry will help to determine the expression for the area of the cross-section. The next Worked Exercise is an example of such a problem and the solution demonstrates two tricks that should be learnt to help make sketches look three dimensional.

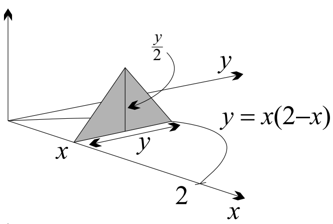
**WORKED EXERCISE:** A certain solid has a base which is the region between the parabola $y = x(2 - x)$ and the $x$-axis. Each cross-section perpendicular to the base and parallel with the $y$-axis is an isosceles right angled triangle with the hypotenuse lying along the base of the solid.

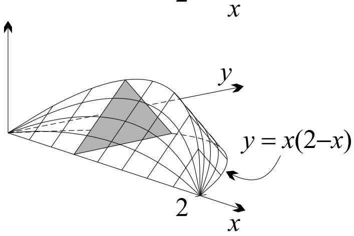
(a) Draw a diagram showing this information.

(b) Show that the area of the cross-section at $x$ is $A = \frac{1}{4}x^2(2 - x)^2$ .

(c) Hence find the volume of the solid.

**SOLUTION:** (a) The diagram is on the right. Notice that in order to help portray the perspective of the situation, the $x$- and $y$-axes have been skewed. The extra vertical axis indicates height above the $xy$-plane. These are two useful tricks to learn to help make your pictures look three dimensional. The solid is shown in the second diagram.




(b) In an isosceles right angled triangle, the distance from the hypotenuse to the oposite vertex is half the length of the hypotenuse. In this case the length of the hypotenuse is the $y$-coordinate hence:

$$A = \tfrac{1}{2} \times y \times \tfrac{1}{2}y$$
$$= \tfrac{1}{4}y^2$$
$$= \tfrac{1}{4}x^2(2 - x)^2 .$$

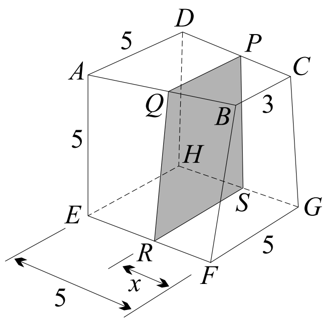
(c) Hence the volume is:

$$V = \int_0^2 \tfrac{1}{4}x^2(2 - x)^2\, dx$$

$$= \tfrac{1}{4}\int_0^2 4x^2 - 4x^3 + x^4\ dx$$
$$= \frac{1}{4}\left[\frac{4x^3}{3} - x^4 + \frac{x^5}{5}\right]_0^2$$
$$= \frac{4}{15}\,.$$

**Similarity:** In some harder problems similarity is used to find $A(h)$. It is often helpful to draw a separate diagram of the relevant part of the solid before applying the similarity argument.

**WORKED EXERCISE:** Two adjacent corners of a cube with edge length 5 cm are sliced off. The resulting solid is shown on the right. The top $ABCD$ and front $GCBF$ are congruent trapezia, with base 5 cm, top 3 cm and height 5 cm. Opposite these, $EFGH$ and $HDAE$ are square. The vertical slice $PQRS$ is taken $x$ cm from $GCBF$. You may assume that this cross-section is a trapezium.



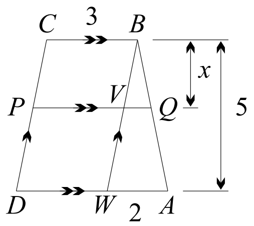

(a) Use similarity in trapezium $ABCD$ to show that

$$PQ = \tfrac{2}{5}x + 3\,.$$

(b) Find the area of $PQRS$ in terms of $x$.

(c) Hence find the volume of the solid.

**SOLUTION:**




(a) The diagram on the right has been provided for those unfamiliar with the geometry of the intercepts on transversals. Since $PQ||DA$, it should be clear that $\triangle BVQ\ |||\ \triangle BWA$. Hence:

$$\frac{PQ-3}{2} = \frac{x}{5} \quad \text{(ratio of matching bases and altitudes)}$$

so $\qquad PQ = \tfrac{2}{5}x + 3\,.$

(b)
$$\text{Area } PQRS = \tfrac{1}{2}(PQ+RS)\times 5$$
$$= \tfrac{1}{2}(\tfrac{2}{5}x + 8)\times 5$$
$$= (x+20)\ \text{cm}^2.$$

(c)
$$V = \int_0^5 x + 20\,dx$$
$$= \left[\tfrac{1}{2}x^2 + 20x\right]_0^5$$
$$= 112\tfrac{1}{2}\ \text{cm}^3.$$

## Exercise **6C**

**1.** A solid is 6 metres high. A horizontal cross-section at height $h$ metres has area $(30 + h - h^2)$ square metres. Find the volume of the solid.

**2.** A monument has height 12 metres. A horizontal cross-section $y$ metres from the top of the monument is an equilateral triangle of side length $\frac{y}{6}$ metres.

(a) Find the area of the cross-section. (b) Show that the monument has volume $4\sqrt{3}\,\text{m}^3$.

**3.** A solid is 20 cm high. A cross-section parallel to the base at height $h$ cm is a square of side $\left(10 - e^{\frac{1}{10}h}\right)$ cm. Show that the solid has volume $990\,\text{cm}^3$, correct to the nearest $\text{cm}^3$.

**4.** The horizontal base of a solid is the circle $x^2 + y^2 = 36$. A typical vertical cross-section of the solid perpendicular to the $x$-axis is a square with one side in the base.

(a) Draw a diagram showing this situation.

(b) Find the area of the cross-section at position $x$.

(c) Hence determine the volume of the solid.

**5.** The base of a circus "big-top" is the ellipse $4x^2+9y^2 = 360$, with a scale of 1 unit : 1 metre. The roof is a dome where each vertical cross-section of the "big-top" perpendicular to the $x$-axis is a semicircle with its diameter in the base.

(a) Draw a diagram showing this situation.

(b) Show that a typical such cross-section has area $\frac{2}{9}\pi(90 - x^2)$.

(c) Hence show that the "big-top" has a capacity approximately equal to $795\,\text{m}^3$.

**6.** The horizontal base of a solid is the circle $x^2 + y^2 = 36$. A typical vertical cross-section of the solid perpendicular to the $x$-axis is a right-angled isosceles triangle with its hypotenuse in the base.

(a) Draw a diagram showing this situation.

(b) Determine the area of the cross-section at position $x$.

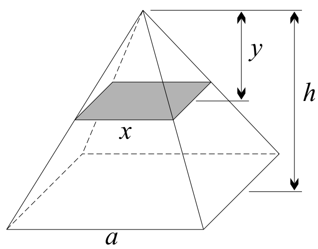
(c) Hence find the volume of the solid.

**DEVELOPMENT**

**7.** The horizontal base of a solid is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Vertical cross-sections of the solid parallel to the $y$-axis are equilateral triangles with one side in the base. Show that the solid has volume $\frac{4ab^2}{\sqrt{3}}$ cubic units.

**8.** The horizontal base of a solid is the region in the first quadrant bounded by the curves $y = x^2$ and $y = x^{\frac{1}{2}}$. Each vertical cross-section of the solid parallel to the $x$-axis is a right angled isosceles triangles with its hypotenuse lying in the base. Find the volume of the solid.

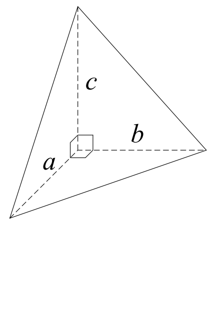
**9.** The diagram on the right shows a square pyramid of height $h$ units and base base length $a$ units. A typical square cross-section parallel to the base is shown. It is $y$ units from the top of the pyramid, and it has side length $x$ units.




(a) Show that $x = \frac{ay}{h}$.

(b) Hence prove that the pyramid has volume $\frac{1}{3}a^2h$ units$^3$.

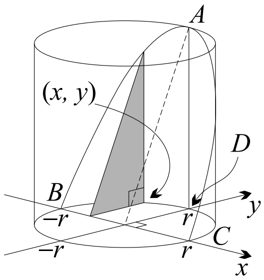
**10.** The diagram on the right shows a triangular pyramid. By slicing parallel to the base, prove that the pyramid has volume $\frac{1}{6}abc$.




**11.** (a) Show that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has area $\pi ab$ square units.

(b) Hence, by slicing parallel to the base, find the volume of a cone with elliptical base $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and perpendicular height $h$ units.

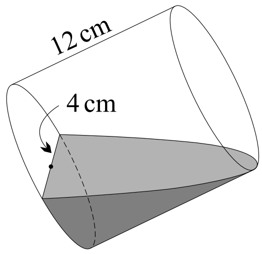
**12.** In the diagram on the right is a cylindrical wedge $ABCD$. The height of the cylinder is equal to the diameter of its base. Let the radius of the base be $r$ units.



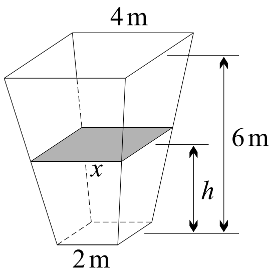

(a) Show that the typical triangular cross-section shaded has area $(r^2 - x^2)$ square units.

(b) Hence show that the wedge has volume $\frac{4}{3}r^3$ cubic units.

**13.** The diagram on the right shows a cylindrical drinking glass of interior radius 4 cm and perpendicular height 12 cm. The glass is filled with water which is then drunk slowly until half of the bottom of the glass is exposed. Use the methods of the previous question to find the volume of water remaining.




**14.** The diagram to the right shows a large tank of depth 6 metres. Its base and its top are squares with sides of 2 metres and 4 metres respectively. A typical square cross-section of side length $x$ metres is shown $h$ metres above the base.




(a) Show that $x = \frac{h}{3} + 2$.

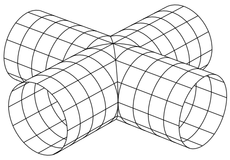
(b) Hence find the capacity of the tank.

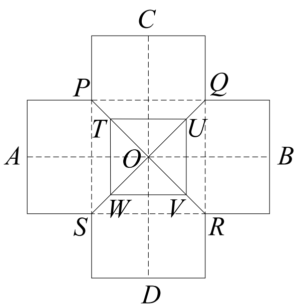
**15.** A rubbish skip on a building site has a rectangular base 6 metres by 3 metres, and its perpendicular height is 2 metres. Its sides are trapezia that slope outwards from bottom to top. The open top is a rectangle 7 metres by 4 metres.

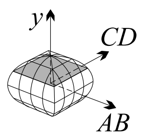
(a) Show that a rectangular cross-section $h$ metres above the base has area $\left(\frac{h}{2} + 6\right)\left(\frac{h}{2} + 3\right)$ square metres.

(b) Hence find the capacity of the skip.

**16.**





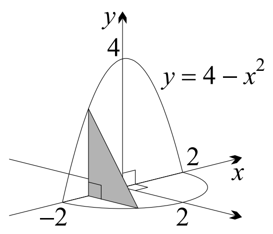




The first diagram shows two identical intersecting cylinders of radius $r$. The second diagram is the view from above. Their axes $AB$ and $CD$ intersect at $90^\circ$ at the point $O$. The third diagram shows the solid which is common to both cylinders, bounded at its widest point by the horizontal square $PQRS$.

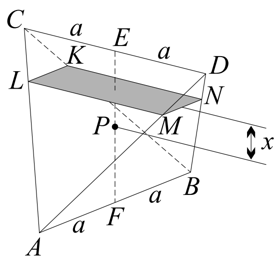
(a) The typical square cross-section $TUVW$ shown is parallel to the square $PQRS$ and $y$ units above it. Find an expression for the area of this typical cross-section of the solid in terms of $y$.

(b) Hence find the volume that is common to the two intersecting cylinders.

**17.** The solid shown on the right has a semicircular base of radius 2 cm. Vertical cross-sections of the solid perpendicular to the diameter of the semicircle are right-angled triangles, the heights of which are bounded by the parabola $y = 4 - x^2$. Show that the solid has volume $3\pi\,\text{cm}^3$.



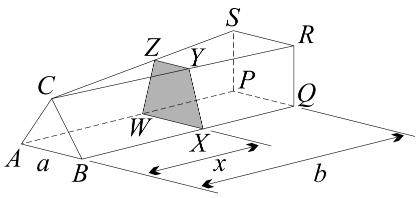

**18.** The diagram on the right shows tetrahedron $ABCD$. The lines $AB$ and $CD$ have length $2a$ and lie in horizontal planes at a distance $2a$ apart. The midpoint $E$ of $CD$ is vertically above the midpoint $F$ of $AB$, and $AB$ runs from South to North, whilst $CD$ runs from West to East. Rectangle $KLMN$ is the horizontal cross-section of the tetrahedron $ABCD$ at distance $x$ from the midpoint $P$ of $EF$ (so $PE = PF = a$).




(a) By considering the triangle $ABE$, deduce that $KL = a - x$, and find the area of the rectangle $KLMN$.

(b) Find the volume of the tetrahedron $ABCD$.

**19.** The diagram shows a sandstone solid with rectangular base $ABQP$ of length $b$ metres and width $a$ metres. The end $PQRS$ is a square, and the other end $ABC$ is an equilateral triangle. Both ends are perpendicular to the base.




Consider cross-section of the solid $WXYZ$ which is parallel with the ends. Let $BX = x$ metres.

(a) Find the height of the equilateral triangle $ABC$.

(b) Given that the triangles $CRS$ and $CYZ$ are similar, find $YZ$ in terms of $a$, $b$ and $x$.

(c) Let the perpendicular height of the trapezium $WXYZ$ be $h$ metres. Show that

$$h = \frac{a}{2}\left(\sqrt{3} + \left(2 - \sqrt{3}\right)\frac{x}{b}\right).$$

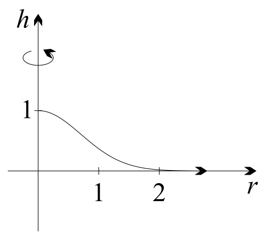
(d) Hence show that the cross-sectional area of $WXYZ$ is given by

$$\frac{a^2}{4b^2}\left((2 - \sqrt{3})x + b\sqrt{3}\right)(b + x).$$

(e) Find the volume of the solid.

**EXTENSION**

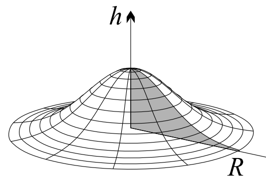
**20.** The graph of $h = e^{-r^2}$ is shown on the right. The region below the curve which lies in the first quadrant is rotated about the vertical axis to generate a solid which extends horizontally to infinity in all directions. A portion of this solid is shown in the second diagram.




(a) Consider the part of the solid which lies above the circle with radius $r = R$.

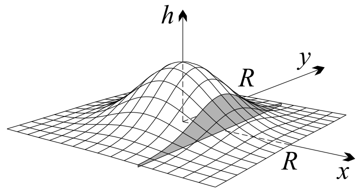
(i) Show that the volume of this is given by

$$\int_0^R 2\pi r e^{-r^2}\,dr\,.$$




(ii) Evaluate this integral.

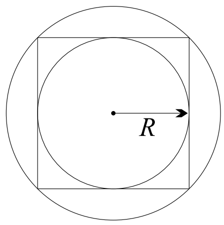
(b) Now in the Cartesian plane, $r^2 = x^2 + y^2$ so we may divide the base of our solid into a rectangular grid and write $h = e^{-(x^2+y^2)}$. Consider the part of the solid which lies above the square $-R \le x \le R$ and $-R \le y \le R$, as shown on the right.




(i) Let $I = \displaystyle\int_0^R e^{-t^2}\,dt$. Do not attempt to evaluate this integral. Show that the area of the slice parallel with the $y$-axis at any given value of $x$ is equal to $2e^{-x^2}I$.

(ii) Hence show that the volume of this portion of the solid is $4I^2$.

(c) Looking from above, the base of the solid in part (a) is a circle with radius $R$, and the base of part (b) is a square with side $2R$. Now consider the portion of our solid above a circular base which passes through the corners of the square in (b).




(i) What is the radius of the base?

(ii) Find the volume of this portion of the solid.

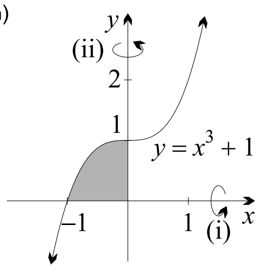
(d) Write down an inequality involving the volumes found in parts (a), (b) and (c).

(e) Hence evaluate $\displaystyle\int_0^\infty e^{-t^2}\,dt$ by taking the limit as $R \to \infty$.

# Chapter Six

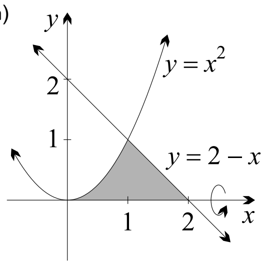
## Exercise **6A** (Page 27)

**1(a)**



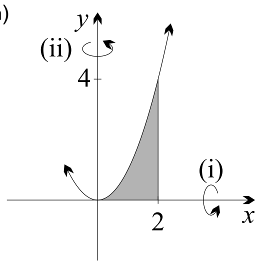

**(b)(i)** $\frac{9\pi}{14}$ **(ii)** $\frac{3\pi}{5}$

**2(a)**



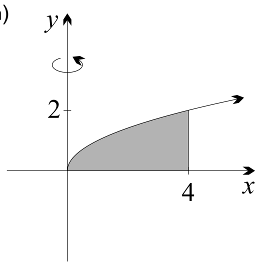

**(b)** $\frac{8\pi}{15}$

**3(a)**




**(b)(i)** $\frac{32\pi}{5}$ **(ii)** $8\pi$

**4(a)**




**(b)** $\frac{128\pi}{5}$

**5** $\frac{6\pi}{7}$

**6(a)** $2\pi$ **(b)** $\frac{48\pi}{5}$ **(c)** $\frac{7\pi}{10}$

**7** $4\pi$

**8** $\frac{32\pi}{3}$

**9** $\frac{256\pi}{15}$

**10(a)** $\frac{8\pi}{3}$ **(b)** $\frac{224\pi}{15}$ **(c)** $8\pi$

**11(a)** $\frac{9\pi}{14}$ **(b)** $\frac{15\pi}{7}$

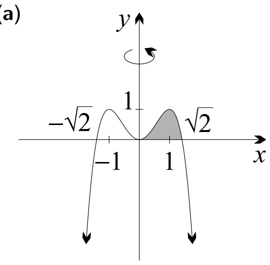
**12(a)** $\frac{8\pi}{3}$ **(b)** $8\pi$

**13(b)** $\frac{296\pi}{15}$

**14(a)** $\frac{32\pi a^3}{15}$ **(b)** $\frac{112\pi a^3}{15}$ **(c)** $\frac{128\pi a^3}{15}$

**15(b)** $\frac{8\pi}{3}$

**16(a)**




**(b)** $\frac{4\pi}{3}$

**17(b)** $288\pi^2$

**19(a)** $\frac{4}{3}\pi ab^2$ **(b)** $2\pi^2 ab^2$ **(c)** $2\pi^2 abc$

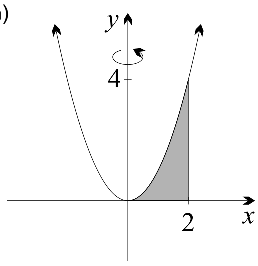
**20(a)** $AD = \frac{9}{5}$, $CD = \frac{16}{5}$ **(c)** $\frac{92\pi}{15}\text{cm}^3$

**23(a)** $\frac{1}{\sqrt{2}}(x - x^2)$ **(b)** $\delta V = \frac{\pi}{\sqrt{2}}(x - x^2)^2\,\delta x$

**(c)** $V' = \frac{\pi}{\sqrt{2}}(x - x^2)^2$

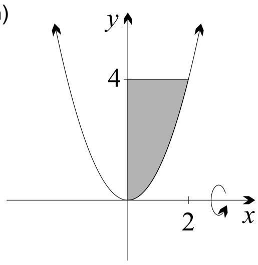
## Exercise **6B** (Page 32)

**1(a)**




**(b)** $8\pi$

**2(a)**




**(b)** $\frac{128\pi}{5}$

**3(a)** $\frac{128\pi}{3}$ **(b)** $\frac{3888\pi}{5}$ **(c)** $\frac{\pi}{10}$

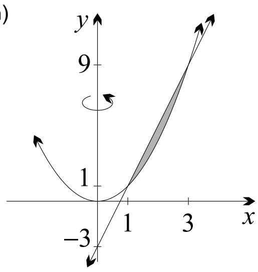
**4(a)** $\frac{3\pi}{5}$ **(b)** $\frac{9\pi}{14}$

**5(a)** $\frac{8\pi}{3}$ **(b)** $8\pi$

**6(a)** $\frac{256\pi}{15}$ **(b)** $\frac{192\pi}{5}$

**7(a)** $\frac{8\pi}{3}$ **(b)** $8\pi$

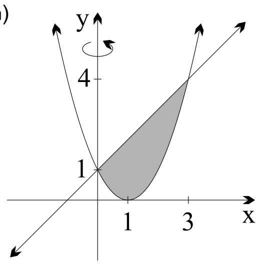
**9(a)**




**(b)** $\frac{16\pi}{3}$

**10(a)**

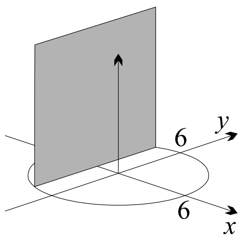



**(b)** $\frac{27\pi}{2}$

**11** $2\pi$
**12** $8\pi$
**13** $\frac{28\pi}{5}$
**14** $\frac{256\pi}{15}$
**15(a)** $\frac{4}{3}\pi ab^2$ **(b)** $2\pi^2 ab^2$ **(c)** $2\pi^2 abc$
**16(a)** $\frac{32\pi a^3}{15}$ **(b)** $\frac{112\pi a^3}{15}$ **(c)** $\frac{128\pi a^3}{15}$
**21(b)** $V = \lim\limits_{\delta x \to 0} \sum\limits_{x=0}^{x=\sqrt{2}} \pi(2x^2 - x^4)(2x + \delta x)\,\delta x$
**(c)** $\frac{4\pi}{3}$
**22(a)** $(x - \frac{1}{2}\delta x),\ (x + \frac{1}{2}\delta x)$
**(c)** $V = \lim\limits_{\delta x \to 0} \sum\limits_{x=0}^{x=2} 2\pi x(2x - x^2)\,\delta x$ **(d)** $\frac{8\pi}{3}$

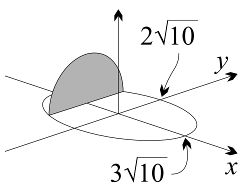
## Exercise **6C** (Page 36)

**1** $126\,\mathrm{m}^3$
**2(a)** $\frac{y^2\sqrt{3}}{144}\,\mathrm{m}^2$
**4(a)**

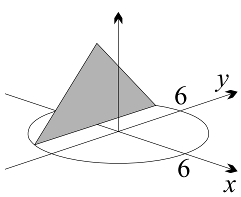



**(b)** $4(36 - x^2)$ **(c)** $1152$

**5(a)**




**6(a)**




**(b)** $36 - x^2$ **(c)** $288$

**8** $\frac{9}{280}$
**11(b)** $\frac{1}{3}\pi abh$
**13** $128\,\mathrm{ml}$
**14(b)** $56\,\mathrm{m}^3$

**15(b)** $45\frac{2}{3}\,\mathrm{m}^3$
**16(a)** $4(r^2 - y^2)$ **(b)** $\frac{16}{3}r^3$
**18(a)** $a^2 - x^2$ **(b)** $\frac{4}{3}a^3$
**19(a)** $\frac{\sqrt{3}}{2}a$ **(b)** $\frac{ax}{b}$ **(e)** $\frac{1}{12}a^2 b(5 + 2\sqrt{3})\,\mathrm{m}^3$
**20(a)(ii)** $\pi(1 - e^{-R^2})$ **(c)(i)** $R\sqrt{2}$ **(ii)** $\pi(1 - e^{-2R^2})$
**(d)** $\pi(1 - e^{-R^2}) \le 4I^2 \le \pi(1 - e^{-2R^2})$ **(e)** $\frac{\sqrt{\pi}}{2}$

CHAPTER **FOUR**

# Resisted Motion

## 4A Horizontal Resisted Motion

### Exercise 4A

**1.** A certain drag-racing car of mass $M$ kg is capable of a top speed of 288 km/h. After it reaches this top speed, two different retarding forces combine to bring it to rest. First there is a constant breaking force of magnitude $\frac{2}{3}M$ Newtons. Secondly there is a resistive force of magnitude $\dfrac{Mv^2}{180}$ Newtons, where $v$ m/s is the speed of the car, acting against a parachute released from the rear-end of the vehicle. Let $x$ metres be the distance of the car from the point at which the two retarding forces are activated.

(a) Show that $x = 90\ln\left(\dfrac{120+80^2}{120+v^2}\right)$.

(b) Hence calculate, to the nearest metre, the distance that the drag-racing car travels as it is brought from its top speed to rest.

**2.** A monorail of mass 10 000 kg is pulling out of a station $S$. Its motor provides a propelling force of magnitude 10 000 Newtons, and as it moves it experiences a resistive force of magnitude $100v^2$ Newtons, where $v$ metres per second is its velocity.

(a) Show that the maximum speed the monorail can attain is 36 km/h.

(b) Show that $x = 50\ln\left(\dfrac{100}{100-v^2}\right)$, where $x$ metres is the distance the monorail has travelled from $S$.

(c) What percentage (to the nearest per cent) of its maximum speed has the monorail reached when it has travelled 50 metres?

**3.** A particle of unit mass moves in a straight line against a resistance numerically equal to $v + v^3$, where $v$ is its velocity. Initially the particle is at the origin and is travelling with velocity $Q$, where $Q > 0$.

(a) Show that $v$ is related to the displacement $x$ by the formula $x = \tan^{-1}\left(\dfrac{Q-v}{1+Qv}\right)$.

(b) Show that the time $t$ which has elapsed when the particle is travelling with velocity $v$ is given by $t = \frac{1}{2}\log_e \dfrac{Q^2(1+v^2)}{v^2(1+Q^2)}$.

(c) Show that $v^2 = \dfrac{Q^2}{(1+Q^2)e^{2t} - Q^2}$.

(d) What are the limiting values of $v$ and $x$ as $t \to \infty$?

**4.** When a jet aircraft touches down two different retarding forces combine to bring it to rest. If the aircraft has mass $M$ kg and speed $v$ m/s there is a constant frictional force of $\frac{1}{4}M$ Newtons and a force of $\frac{1}{108}Mv^2$ Newtons due to the reverse thrust of the engines. The reverse thrust does not take effect until 20 seconds after touchdown.

Let $x$ be the distance in metres of the jet from its point of touchdown and let $t$ be the time in seconds after touchdown.

(a) If the jet's speed at touchdown is 60 m/s, show that $v = 55$ and $x = 1150$ at the instant the reverse thrust of the engines takes effect.

(b) Show that when $t > 20$, $x = 1150 + 54\left(\ln(27+55^2) - \ln(27+v^2)\right)$.

(c) How far from the point of touchdown, correct to the nearest metre, does the jet come to rest?

**5.** A particle of mass $m$ kg experiences a resistance of $kv^2$ Newtons when moving along the $x$-axis, where $k$ is a positive constant and $v$ is the speed of the particle in metres per second. The maximum speed attainable by the particle is $u$ metres per second under a variable propelling force of $\dfrac{P}{v}$ Newtons, where $P$ is a positive constant.

(a) Show that $k = \dfrac{P}{u^3}$.

(b) Show that $\dfrac{dv}{dx} = \dfrac{P}{m}\left(\dfrac{1}{v^2} - \dfrac{v}{u^3}\right)$.

(c) Prove that the distance travelled as the speed changes from $\dfrac{u}{3}$ m/s to $\dfrac{2u}{3}$ m/s is $\dfrac{mu^3}{3P}\ln\frac{26}{19}$ metres.

(d) When the brakes are applied, the propelling force is no longer in operation. If the maximum force exerted by the brakes is $B$ Newtons, prove that the minimum distance travelled in coming to rest from a speed of $u$ m/s is $\dfrac{mu^3}{2P}\ln\left(1+\dfrac{P}{Bu}\right)$ metres.

# 4B Vertical Resisted Motion

## Exercise **4B**

**1.** An object of mass 5 kg is projected vertically upwards with velocity 40 m/s and experiences a resistive force in Newtons of magnitude $0{\cdot}2v^2$, where $v$ is the velocity of the object at time $t$ seconds. Assume that $g = 10\,\text{m/s}^2$.

(a) Show that $\ddot{x} = \dfrac{-250 - v^2}{25}$.

(b) Find, correct to the nearest tenth of a second, the time that the object takes to reach its maximum height.

(c) Find the maximum height reached, correct to the nearest metre.

**2.** An object of mass $0{\cdot}5$ kg is projected upwards with velocity 40 m/s and experiences a resistive force in Newtons of magnitude $0{\cdot}2v$, where $v$ is the velocity of the object at time $t$ seconds. Assume that $g = 10\,\text{m/s}^2$.

(a) Show that $\ddot{x} = \dfrac{-50 - 2v}{5}$.

(b) Show that the object takes $\frac{5}{2}\ln\frac{13}{5}$ seconds to reach its maximum height.

(c) Show that the maximum height reached, in metres, is $100 + \frac{125}{2}\ln\frac{5}{13}$.

**3.** An object of mass 100 kg is found to experience a resistive force, in Newtons, of one-tenth the square of its velocity in metres per second when it moves through the air. Suppose that the object falls from rest under gravity, and take $g = 9{\cdot}8\,\text{m/s}^2$.

(a) Show that its terminal velocity is about 99 m/s.

(b) If the object reaches 80% of its terminal velocity before striking the ground, show that the point from which it was dropped was about 511 metres above the ground.

**4.** An object of mass 1 kg is projected vertically upwards from the ground at 20 m/s. The body is under the effect of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{40}v^2$, where $v$ is the velocity at time $t$. (Take $g = 10\,\text{m/s}^2$, and take upwards as the positive direction.)

(a) (i) Show that the greatest height reached by the object is $20\ln 2$ metres.

(ii) Show that the time taken to reach this greatest height is $\frac{\pi}{2}$ seconds.

(b) Having reached its greatest height the particle falls back to its starting point. The particle is still under the effect of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{40}v^2$.

(i) Write down the equation of motion of the object as it falls, this time taking downwards as the positive direction.

(ii) Find the speed of the object when it returns to its starting point.

**5.** A certain object, when projected vertically downwards with initial velocity $V$, experiences air resistance of magnitude $mkv$, where $k$ is a positive constant. Take downwards as the positive direction.

(a) Show that $t = \frac{1}{k}\log_e\left(\frac{g-kV}{g-kv}\right)$.

(b) Hence show that $v = \frac{g}{k}(1-e^{-kt}) + Ve^{-kt}$, and explain from this equation why the terminal velocity is $\frac{g}{k}$.

(c) Integrate again to show that $x = \frac{gt}{k} + \frac{kV-g}{k^2}\left(1-e^{-kt}\right)$.

(d) Suppose that the terminal velocity of this object is $20\,\text{m/s}$, and that $g = 10\,\text{m/s}^2$. One of these objects is thrown vertically downwards from a lookout at the top of a cliff at precisely the terminal velocity, and, at the same instant, another of these objects is dropped. Show that the distance between the two falling objects after $t$ seconds is, in metres, $40(1-e^{-\frac{1}{2}t})$, and hence state the limiting distance between the two falling objects.

**6.** A particle of mass 10 kg is found to experience a resistive force, in Newtons, of one-tenth of the square of its velocity in metres per second, when it moves through the air. The particle is projected vertically upwards from a point $O$ with a velocity of $u$ metres per second, and the point $A$, vertically above $O$, is the highest point reached by the particle before it starts to fall to the ground again. Assuming that $g = 10\,\text{m/s}^2$,

(a) show that the particle takes $\sqrt{10}\tan^{-1}\frac{u}{10\sqrt{10}}$ seconds to reach $A$ from $O$,

(b) show that the height $OA$ is $50\log_e\frac{1000+u^2}{1000}$ metres,

(c) show that the particle's velocity $w$ metres per second when it reaches $O$ again is given by $w^2 = \dfrac{1000u^2}{1000+u^2}$.

**7.** (a) A particle of mass $m$ falls from rest, from a point $O$, in a medium whose resistance is $mkv$, where $k$ is a positive constant and $v$ is the velocity at time $t$.

(i) Prove that the terminal velocity $V$ is $V = \dfrac{g}{k}$.

(ii) Prove that the speed at time $t$ is given by $\frac{g}{k}\left(1-e^{-kt}\right)$.

(b) An identical particle is projected upwards from $O$ with initial velocity $U$ in the same medium. Suppose that this second particle is released simultaneously with the first.

(i) Prove that the second particle reaches its maximum height at $t = \frac{1}{k}\ln\frac{g+kU}{g}$.

(ii) Prove that the speed of the first particle when the second particle is at its maximum height is $\dfrac{UV}{U+V}$.

**8.** A particle $P_1$ of mass $m$ kg is dropped from point $A$ and falls towards point $B$, which is directly underneath $A$. At the instant when $P_1$ is dropped, a second particle $P_2$, also of mass $m$ kg, is projected upwards from $B$ towards $A$ with an initial velocity equal to twice the terminal velocity of $P_1$. Each particle experiences a resistance of magnitude $mkv$ as it moves, where $v\,\text{ms}^{-1}$ is the velocity and $k$ is a constant.

(a) Show that the terminal velocity of $P_1$ is $\dfrac{g}{k}$, where $g$ is acceleration due to gravity.

(b) For particle $P_2$, show that $t = \dfrac{1}{k}\ln\left(\dfrac{3g}{g+kv}\right)$, where $v\,\text{ms}^{-1}$ is the velocity after $t$ seconds.

(c) Suppose that the particles collide at the instant when $P_1$ has reached 30% of its terminal velocity. Show that the velocity of $P_2$ when they collide is $\dfrac{11g}{10k}\,\text{ms}^{-1}$.

**9.** An object of mass 1 kg is dropped from a lookout on top of a high cliff.
Take the acceleration due to gravity to be $10\,\text{m/s}^2$.

(a) At first, air resistance causes a deceleration of magnitude $\dfrac{v}{10}$, where $v$ m/s is the speed of the object $t$ seconds after it is dropped.

(i) Taking downwards as positive, explain why its equation of motion is

$$\ddot{x} = 10 - \tfrac{v}{10}\,,$$

where $x$ is the distance that the object has fallen in the first $t$ seconds.

(ii) Show that $\dfrac{dv}{dx} = \dfrac{100-v}{10v}$, and hence show that the speed $V$ of the object when it is 40 metres below the lookout satisfies the equation

$$V + 100\log_e\left(1 - \tfrac{V}{100}\right) + 4 = 0.$$

(b) After the object has fallen 40 metres and reached this speed V, a very small parachute opens, and air resistance now causes a deceleration to its motion of magnitude $\dfrac{v^2}{10}$.

(i) Taking downwards as positive, write an expression for the new acceleration $\ddot{x}$ of the object, where $x$ now is the distance that the object has fallen in the first $t$ seconds after the parachute opens.

(ii) Show that $v^2 = 100 - (100 - V^2)e^{-\frac{1}{5}x}$, and hence find the terminal velocity of the object.

(iii) Show that $t$ seconds after the parachute opens,

$$t = \tfrac{1}{2}\log_e\frac{(v+10)(V-10)}{(v-10)(V+10)}\,.$$

(iv) Given that the solution to the equation in part (ii) of part (a) is $V \doteqdot 25{\cdot}7\,\text{m/s}$, how long after the parachute opens does the object reach 105% of its terminal velocity?

**10.** A particle of mass 2 kg experiences a resistive force, in Newtons, of 10% of the square of its velocity $v$ metres per second when it moves through the air. The particle is projected vertically upwards from a point $A$ with velocity $u$ metres per second. The highest point reached is $B$, directly above $A$. Assume that $g = 10\,\text{ms}^{-2}$, and take upwards as the positive direction.

(a) Show that the acceleration of the particle as it rises is given by

$$\ddot{x} = -\frac{v^2 + 200}{20}.$$

(b) Show that the distance $x$ metres of the particle from $A$ as it rises is given by

$$x = 10 \log_e \left( \frac{200 + u^2}{200 + v^2} \right).$$

(c) Show that the time $t$ seconds that the particle takes to reach a velocity of $v$ metres per second is given by

$$t = \sqrt{2} \left( \tan^{-1} \frac{u}{10\sqrt{2}} - \tan^{-1} \frac{v}{10\sqrt{2}} \right).$$

(d) Now suppose that we take two of the 2 kg particles described above.
One of the particles is projected upwards from $A$ with initial velocity $10\sqrt{2}\,\text{ms}^{-1}$, then, $\frac{3\sqrt{2}}{5}$ seconds later, the other particle is projected upwards from $A$ with initial velocity $30\sqrt{2}\,\text{ms}^{-1}$. Will the second particle catch up to the first particle before the first particle reaches its maximum height? You must explain your reasoning and show your working.

**11.** (a) Consider the function

$$f(x) = x - \frac{g^2}{x} - 2g \ln \left( \frac{x}{g} \right), \quad \text{for } x \geq g.$$

(i) Evaluate $f(g)$.

(ii) Show that $f'(x) = \left(1 - \frac{g}{x}\right)^2$.

(iii) Explain why $f(x) > 0$ for $x > g$.

(b) A body is moving vertically through a resisting medium, with resistance proportional to its speed. The body is initially fired upwards from the origin with speed $v_0$. Let $y$ metres be the height of the object above the origin at time $t$ seconds, and let $g$ be the constant acceleration due to gravity. Thus

$$\frac{d^2y}{dt^2} = -(g + kv) \quad \text{where} \quad k > 0.$$

(i) Find $v$ as a function of $t$, and hence show that

$$k^2 y = (g + kv_0)(1 - e^{-kt}) - gkt.$$

(ii) Find $T$, the time taken to reach the maximum height.

(iii) Show that when $t = 2T$,

$$k^2 y = (g + kv_0) - \frac{g^2}{g + kv_0} - 2g \ln \left( \frac{g + kv_0}{g} \right).$$

(iv) Use this result and part (a) to show that the downwards journey takes longer.

# Chapter Four

## Exercise 4A (Page 115)

**1(b)** 360 metres

**2(c)** 80%

**3(d)** 0 and $\tan^{-1} Q$

**4(c)** 1405 metres

## Exercise 4B (Page 117)

**1(b)** $1{\cdot}9$ seconds **(c)** 25 metres

**4(b)(ii)** $10\sqrt{2}\,\text{m/s}$

**5(d)** 40 metres

**10(d)** Yes.

**11(b)(ii)** $T = \frac{1}{k}\ln\left(\dfrac{g + kv_0}{g}\right)$

CHAPTER **NINE**

# Circular Motion

## 9A Introduction

### Exercise 9A

1. A wheel is rotating at 129 revolutions per minute about its centre. Show that the angular velocity of a point on the wheel about the centre is approximately $13{\cdot}5$ radians per second.

2. A wheel attached to an electric motor is rotating at 300 revolutions per minute. The radius of the wheel is 2 cm. Show that the speed of a notch on the edge of the wheel is about $0{\cdot}63$ m/s.

3. An object of mass 12 kg rests on a smooth horizontal surface and is attached by a string $1{\cdot}2$ m long to a fixed point $O$ on the surface. If the object moves in a horizontal circle at $3{\cdot}6$ m/s, show that the tension in the string is $129{\cdot}6$ Newtons.

4. A towel of mass 2 kg is spinning in a drier of diameter 1 metre at 50 rev/s. Show that the force exerted by the towel on the bearings of the drier is about 98 696 Newtons.

5. A go-kart is being driven around a flat circular track of radius 35 m at 54 km/h.
   (a) Show that the angular velocity of the go-kart is $\frac{3}{7}$ rad/s.
   (b) If the combined mass of the go-kart and driver is 120 kg, show that the centripetal force is approximately 771 Newtons.

6. A boy ties a rock to one end of a piece of string, then swings it in a horizontal circle of radius 2 metres. If the mass of the rock is 200 grams and the tension in the string is $2{\cdot}5$ Newtons, show that:
   (a) the speed of the stone is 5 m/s,
   (b) the angular velocity of the stone is $2{\cdot}5$ rad/s.

7. A string of length 50 cm can just sustain a mass of 20 kg without breaking. A mass of 4 kg is attached to one end of the string, while the other end is fixed to a point on a smooth horizontal table. The mass is then revolved at a uniform speed on the table. Take $g = 9{\cdot}8\ \text{m/s}^2$.
   (a) Show that the maximum tension the string can sustain is 196 N.
   (b) Show that the maximum number of complete revolutions per minute the mass can make without the string breaking is 94.

8. A nylon cord is 60 cm long and will break if a mass exceeding 40 kg is hung from it. A mass of 2 kg is attached to one end of the cord and the other end is attached to a fixed point. The mass then undergoes circular motion in a horizontal plane. Show that the greatest speed the mass can achieve without the cord breaking is approximately $10{\cdot}84$ m/s.

# 9B The Conical Pendulum

## Exercise 9B

**1.** A particle is attached to one end of a string, and the other end of the string is attached to a fixed point. The particle moves in a horizontal circle of radius 60 cm with a constant angular velocity of $\frac{4\pi}{3}$ rad/s. Show that the string is inclined at approximately $47^\circ$ to the vertical. (Assume that $g = 9{\cdot}8\text{m/s}^2$.)

**2.** A small object of mass 4 kg is attached by a string of length 35 cm to a fixed point and moves in a horizontal circle with uniform angular velocity 1 rev/s. Show that the tension in the string is approximately 55 N.

**3.** A 0·5 kg mass is attached to one end of a cord 1·5 m long. The cord and the mass rotate as a conical pendulum at 60 rev/min. Assume that $g = 9.8\text{m/s}^2$.
(a) Show that the tension in the cord is about 29·6 N.
(b) Show that the cord makes an angle of about $80^\circ 28'$ with the vertical.
(c) Show that the radius of the circle that the mass moves in is about 1·48 m.

**4.** A small object is attached by a string 1225 mm long to a fixed point, and moves with uniform speed in a horizontal circle. The tension in the string is twice the weight of the object. Show that the angular velocity is 4 rad/s.

**5.** The mass of the bob in a conical pendulum is 4 kg and the length of the string is 60 cm. The greatest tension that the string can sustain is 600 N. Show that the maximum number of revolutions per second of the bob without the string breaking is approximately 2·52.

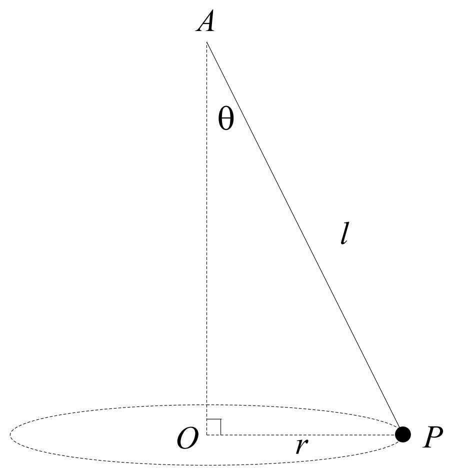
**6.** The number of revolutions per minute of a conical pendulum is increased from 75 to 80.
(a) Show that $h = \dfrac{g}{\omega^2}$, where $h$ is the perpendicular height of the cone.
(b) Hence show that the rise in the level of the bob is about 1·92 cm.

**7.** A particle is attached to one end of a string of length 80 cm, and the other end of the string is attached to a fixed point. The particle moves at a constant speed in a horizontal circle so that the string is inclined at $30^\circ$ to the vertical. Show that the particle is rotating at about 36 revolutions per minute.

**8.** A small mass is suspended by a light rod from a pivot $P$. The mass moves with constant speed in a horizontal circle. The rod has length 1 metre and makes an angle of $30^\circ$ with the vertical. Assume that $g = 9{\cdot}8\text{m/s}^2$.
(a) Show that the mass takes about 1·87 seconds to complete one revolution.
(b) Show that the speed of the mass is about 1·68 m/s.
(c) If the speed of the mass is doubled, show that the rod will make an angle of $54^\circ 44'$ with the vertical.

**9.** A particle of mass 4 kg is attached to a string 2 m long. The particle and string revolve as a conical pendulum. The constant speed of the particle is $\sqrt{g}$ m/s, where $g\,\text{m/s}^2$ is the acceleration due to gravity. Let $\theta$ be the angle of inclination of the string to the vertical, and let $r$ m be the radius of the horizontal circle in which the particle is revolving.
(a) Show that $\tan\theta = \frac{1}{r}$.
(b) Hence show that $\theta = \cos^{-1}\frac{\sqrt{17}-1}{4}$.
(c) Show that the tension in the string is about 50·2 Newtons. (Take $g = 9{\cdot}8\,\text{m/s}^2$.)

**10.**




In the diagram above, $A$ and $P$ are the endpoints of a light string of length $\ell$ metres. $A$ is a fixed point, while an object of mass $m$ kg is attached to the string at $P$. The object moves in a horizontal circle of radius $r$ metres about the point $O$. Let the constant angular velocity of the object be $w$ rad/s, and the acceleration due to gravity be $g$ m/s$^2$. Let $\theta$ be the angle between the string and the vertical, and $T$ Newtons the tension in the string.

(a) Draw a diagram showing the forces acting on the object.

(b) Deduce that $\cos\theta = \dfrac{g}{\ell w^2}$.

(c) Suppose the angular velocity of the object is increased to $w_1$ rad/s, so that the angle $\theta$ is doubled. Show that:

$$w_1 = \sqrt{\frac{g\ell w^4}{2g^2 - \ell^2 w^4}}$$

**11.** A particle $P$ of mass $m$ is attached to one end of a light inelastic string of length $\ell$, while the other end is fixed at $O$. The particle moves with velocity $v$ in a horizontal circle of radius $r$ so that the string describes a cone whose vertical axis passes through the centre $C$ of the circle. Let $T$ be the tension in the string as the particle moves, let $OC = h$ and let $\theta$ be the angle between the string and the vertical.

(a) Draw a diagram showing the forces acting on $P$.

(b) Show that $h = \dfrac{mg\ell}{T}$ and that $r^2 = \dfrac{mv^2\ell}{T}$.

(c) Hence show that $T = \dfrac{m}{2\ell}\left(v^2 + \sqrt{v^4 + 4g^2\ell^2}\right)$.

# 9C Banked Tracks

## Exercise 9C

**1.** A car travels round a circular bend in a road of radius 45 metres at a speed of 48 km/h. There is no sideways frictional force between the road surface and the tyres. Show that the circular bend is banked at $21°57'$ to the horizontal. (Assume $g = 9{\cdot}8\,\text{m/s}^2$.)

**2.** A railway line has been constructed around a circular bend of radius 400 m. The distance between the rails is 1·5 m and the outside rail is 8 cm above the inside rail. Show that the optimum speed of a train on this bend (that is, the speed at which the wheels exert no sideways force on the rails) is about 52 km/h. (Assume $g = 9{\cdot}8\,\text{m/s}^2$.)

**3.** A car of mass 1·2 tonnes is rounding a circular bend of radius 150 m. The bend is banked at $10°$ to the horizontal. Assume that $g = 9{\cdot}8\,\text{m/s}^2$.

(a) Show that the car must travel at about 58 km/h so that there is no tendency to skid sideways.

(b) Suppose that the car travels into the bend at 72 km/h. Show that the sideways frictional force exerted by the tyres on the road is approximately 1109 Newtons.

**4.** A railway track around a circular curve of radius 200 m is designed for an optimum speed of 50 km/h. Assume that $g = 9{\cdot}8\,\text{m/s}^2$.

(a) If the gauge of the track is 1·52 m, show that the difference in height between the outer and inner rails is about 15 cm.

(b) Show that the sideways thrust on the rails is about 110 577 Newtons if a train of mass 120 tonnes travels around the curve at 70 km/h.

**5.** The sleepers of a railway line at a point on a circular bend of radius 100 metres are sloped such that a train travelling at 48 km/h exerts no lateral force on the rails. Show that a locomotive of mass 100 tonnes at rest on this bend would exert a lateral force of about $1{\cdot}75 \times 10^5$ Newtons on the rails.

**6.** A car travels at $v$ m/s around a curved track of radius $R$ metres.

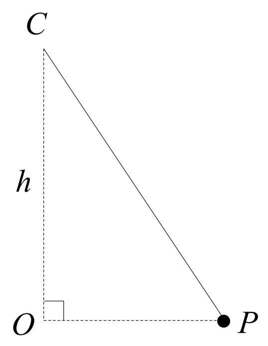
(a) Show that the inclination $\theta$ of the track to the horizontal satisfies $\tan\theta = \dfrac{v^2}{Rg}$ if there is no tendency for the car to slip sideways.

(b) A second car of mass $M$ kg is travelling around the same curved track at $V$ m/s. Prove that the sideways frictional force exerted by the surface of the track on the tyres of this car is $\dfrac{Mg(V^2 - v^2)}{\sqrt{v^4 + R^2g^2}}$ Newtons.

# 9D Miscellaneous Problems

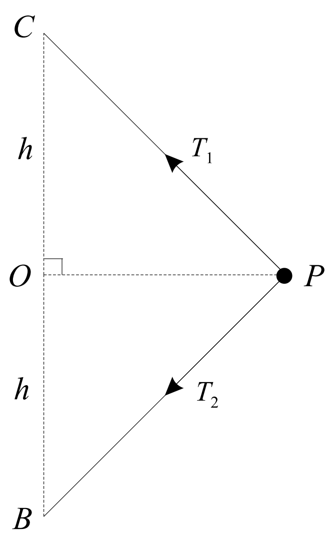
## Exercise **9D**

**1.** (a)




The diagram above shows a particle $P$ attached by an inelastic string to a fixed point $C$. The particle moves in uniform circular motion about a fixed point $O$ that is at a distance $h$ below $C$. Show that the angular velocity of $P$ about $O$ is $\sqrt{\frac{g}{h}}$, where $g$ is acceleration due to gravity.

(b)

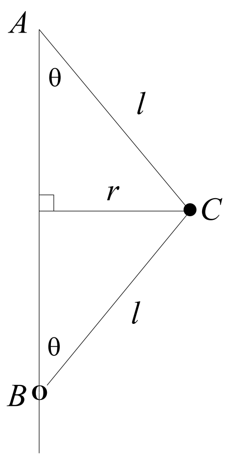



Suppose now, as shown in the diagram above, that $P$ is attached by a second string, identical to the first, to another fixed point $B$ which is at a distance $2h$ below $C$.

(i) Write down two equations of motion by resolving forces vertically and horizontally at $P$.

(ii) If the angular velocity of $P$ about $O$ is $3\sqrt{\frac{g}{h}}$, show that the ratio $T_1 : T_2$ of the tensions in the two strings is $5 : 4$.

**2.**



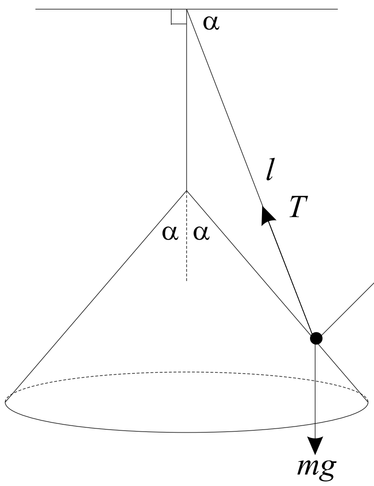

A mass $m$ at a point $C$ is freely jointed to two identical light rods $CA$ and $CB$ of length $\ell$, as shown in the diagram above. The point $A$ is fixed, and at $B$ there is a collar, also of mass $m$, which is free to slide along the smooth vertical bar $AB$. The mass at $C$ rotates in a horizontal circle with uniform angular velocity $\omega$. While this happens, the inclination of the rods to the vertical is $\theta$. Let $T_1$ be the tension in the rod $CA$, and $T_2$ the tension in the rod $CB$.

(a) By resolving forces vertically at $B$, show that $T_2 \cos\theta = mg$.

(b) By resolving forces vertically and horizontally at $C$, show that $T_1 \cos\theta = mg + T_2 \cos\theta$ and that $(T_1 + T_2)\sin\theta = mrw^2$.

(c) Deduce that $\cos\theta = \dfrac{3g}{\ell\omega^2}$.

**3.**



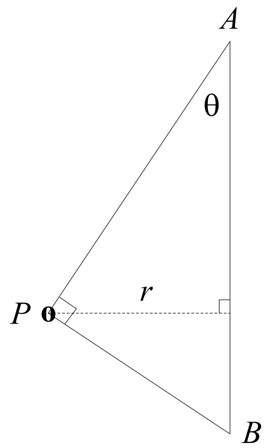

A particle of mass $m$ is suspended by a string of length $\ell$ from a point directly above the vertex of a smooth cone, which has a vertical axis. The particle remains in contact with the cone and rotates as a conical pendulum with angular velocity $\omega$. The angle of the cone at its vertex is $2\alpha$, where $\alpha > \frac{\pi}{4}$, and the string makes an angle of $\alpha$ with the horizontal. The forces acting on the particle are the tension in the string $T$, the normal reaction of the cone $N$ and the gravitational force $mg$.

(a) Show, with the aid of a diagram, that the vertical component of $N$ is $N \sin\alpha$.

(b) Show that $T + N = \dfrac{mg}{\sin\alpha}$, and find an expression for $T - N$ in terms of $m$, $\ell$ and $\omega$.

(c) The angular velocity is increased until $N = 0$, that is, when the particle is about to lose contact with the cone. Find an expression for this value of $\omega$ in terms of $\alpha$, $\ell$ and $g$.

**4.**




The ends of a light string are fixed to two points $A$ and $B$, as shown in the diagram above. The string passes through a small ring of mass $m$. The ring is fastened to the string at $P$. When the string is taut, $\angle APB = 90^\circ$, $\angle BAP = \theta$ and the distance of $P$ from $AB$ is $r$. Suppose that the ring revolves in a horizontal circle with constant angular velocity $\omega$ and that while this happens the string is taut.

(a) Draw a diagram showing the forces acting on the ring.

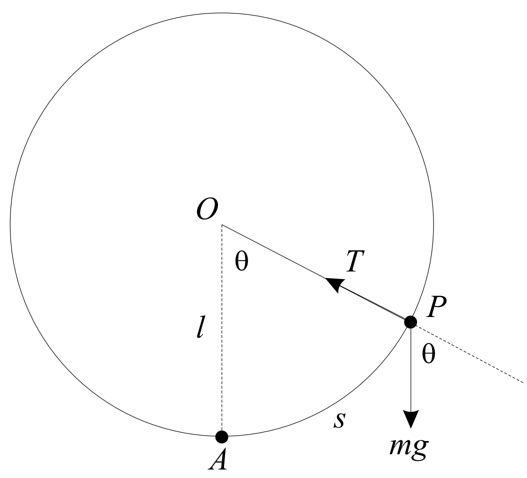
(b) Show that the tensions $T_1$ and $T_2$ in the parts $AP$ and $PB$ respectively of the string are $T_1 = m(r\omega^2 \sin\theta + g\cos\theta)$ and $T_2 = m(r\omega^2 \cos\theta - g\sin\theta)$.

(c) Given that $AB = 13$ units and $AP = 12$ units, show that $144\omega^2 > 13g$.

(d) Suppose that the ring is now free to move on the string instead of being fastened to the string. Show that the condition for the ring to remain at the point $P$ on the string is $420\omega^2 = 221g$.

**5.**




A string of length $\ell$ is initially vertical and has a mass $P$ of $m$ kg attached to it. The mass $P$ is given an initial velocity $V$ and begins to move along the arc of a circle in an anticlockwise direction. $O$ is the centre of the circle and $A$ is the initial position of $P$. Let $s$ denote the arc length $AP$, $v = \dfrac{ds}{dt}$ and $\theta = \angle AOP$ and let the tension in the string be $T$.

(a) Starting with $s = \ell\theta$, show that the tangential acceleration of $P$ is given by

$$\frac{d^2s}{dt^2} = \frac{1}{\ell}\frac{d}{d\theta}(\frac{1}{2}v^2).$$

(b) Show that $\dfrac{1}{\ell}\dfrac{d}{d\theta}(\dfrac{1}{2}v^2) = -g\sin\theta$.

(c) Deduce that $V^2 = v^2 + 2\ell g(1 - \cos\theta)$.

(d) Explain why $T - mg\cos\theta = \dfrac{1}{\ell}mv^2$.

(e) Suppose that $V^2 = 3g\ell$. Find the value of $\theta$ at which $T = 0$.

(f) Consider the situation in part (v). Briefly describe the path of $P$ after the tension $T$ becomes zero.

# Chapter Nine

Exercise **9A** (Page 1) ____________________

Exercise **9B** (Page 2) ____________________

Exercise **9C** (Page 4) ____________________

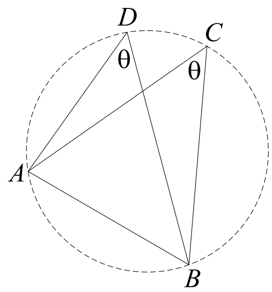
Exercise **9D** (Page 5) ____________________

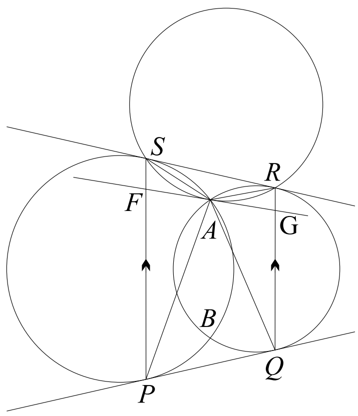
CHAPTER **TEN**

# Further Extension 1

## 10A Geometry




In the Mathematics Extension 2 course, candidates are expected to have have a thorough knowledge of the circle geometry theorems, and to be able to apply the results to complex diagrams. It is also expected that the converse of theorems are understood and able to be applied. For example, if an interval $AB$ subtends equal angles at $C$ and $D$ then quadrilateral $ABCD$ is cyclic by the converse of the angles in the same segment theorem.




**WORKED EXERCISE:** In the diagram on the right, two circles of differing radius intersect at $A$ and $B$. The lines $PQ$ and $RS$ are the common tangents with $PS||QR$. A third circle passes through the points $S$, $A$ and $R$. The tangent to this circle at $A$ meets the parallel lines at $F$ and $G$.

In the tiny triangle $RAG$,
let $\angle RAG = \alpha$, $\angle AGR = \beta$ and $\angle GRA = \gamma$.

(a) State why $\angle AFP = \beta$.

(b) Show that $\angle SPA = \alpha$.

(c) Hence prove that $FG$ is also tangent to the circle which passes through the points $A$, $P$ and $Q$.

**SOLUTION:**

(a) $\angle AFP = \beta$ (alternate angles, $PS||QR$.)

(b) $\angle RSA = \angle RAG$ (angle in the alternate segment of circle $SAR$)
$= \alpha\,.$
$\angle SPA = \angle RSA$ (angle in the alternate segment of circle $PBAS$)
$= \alpha\,.$

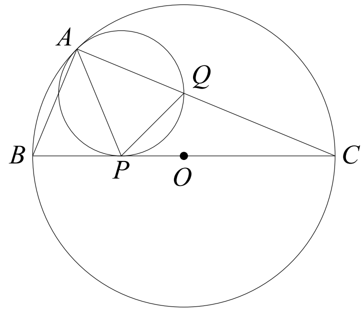
(c) $\angle FAP = \gamma$ (angle sum of $\triangle FAP$)
$\angle PQA = \angle QRA$ (angle in the alternate segment of circle $RABQ$)
$= \gamma\,.$
Thus $\angle FAP = \angle AQP$

Hence $FG$ is tangent to the circle through $APQ$ by the converse of the angles in the alternate segment theorem.

## Exercise **10A**

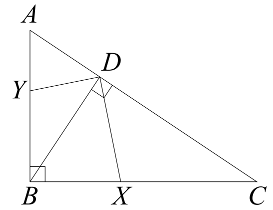
**1.** Two circles touch internally at $A$ where there is a common tangent. $BC$ is a diameter of the larger circle, touching the smaller circle at $P$. $AC$ cuts the smaller circle at $Q$.

Prove that $\angle APQ + \angle ACP = \frac{\pi}{2}$.




**2.** $\triangle ABC$ is right-angled at $B$. The point $D$ lies on $AC$ so that $BD \perp AC$. The bisector of $\angle CDB$ meets $CB$ at $X$ and the bisector of $\angle ADB$ meets $AB$ at $Y$.

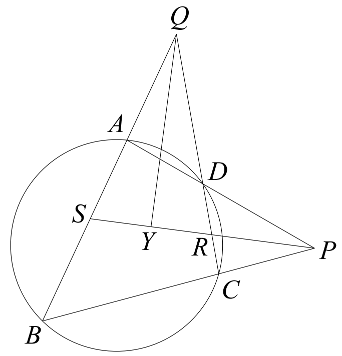
Prove that $BX = BY$.




**3.** In the diagram quadrilateral $ABCD$ is cyclic. $AD$ and $BC$ produced meet at $P$, whilst $BA$ and $CD$ produced meet at $Q$. The bisector of $\angle APB$ meets $CD$ at $R$ and $AB$ at $S$. The bisector of $\angle CQB$ meets $RS$ at $Y$.

(a) Prove that $\angle QRS = \angle QSR$.

(b) Prove that $QY \perp SR$.

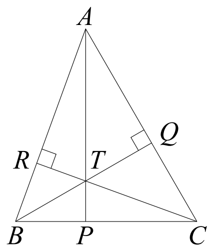



**4.** In $\triangle ABC$ the altitudes $CR$ and $BQ$ intersect at $T$. $AT$ produced meets $CB$ at $P$.

(a) Explain why $BCQR$ and $AQTR$ are cyclic.

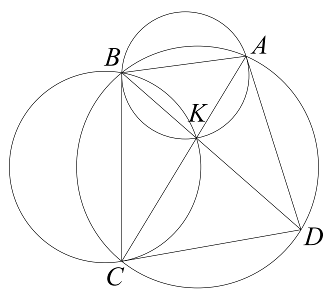
(b) Prove that $\angle TAQ = \angle QBC$.

(c) Prove that $AP \perp BC$.




**5.** The quadrilateral $ABCD$ shown is cyclic. Its diagonals intersect at $K$. Circles $ABK$ and $BCK$ are drawn. $CD$ is a tangent to circle $BCK$.

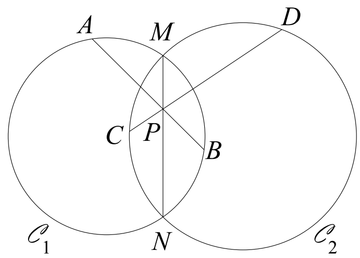
Prove that $AD$ is tangent to circle $ABK$.



**6.** Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ share a common chord $MN$. Let $P$ be a point on this chord. Chord $AB$ of $\mathcal{C}_1$ and chord $CD$ of $\mathcal{C}_2$ are drawn through $P$.

Prove that quadrilateral $ACBD$ is cyclic.



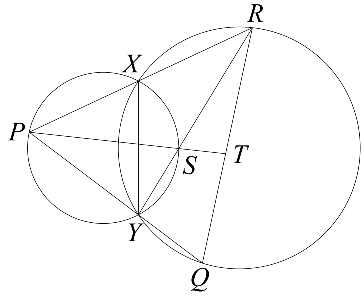

**DEVELOPMENT**

**7.** The circles $XPYS$ and $XYQR$ intersect at $X$ and $Y$. $PYQ$, $PXR$, $RSY$, $PST$ and $QTR$ are straight lines. Let $\angle QRP = \alpha$ and $\angle RPT = \beta$.

(a) Show that $\angle PTQ = \alpha + \beta$.

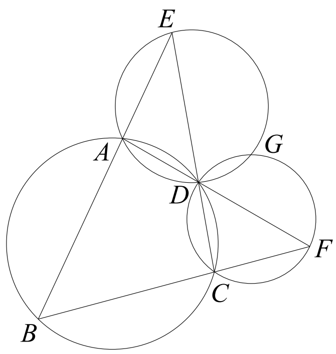
(b) Give a reason why $\angle XYS = \beta$.

(c) Show that $TSYQ$ is cyclic.




**8.** In the diagram $ABCD$ is a cyclic quadrilateral. $BA$ and $CD$ produced intersect at $E$, while $BC$ and $AD$ produced intersect at $F$. The circles $EAD$ and $FCD$ intersect at $D$ and $G$.

Prove that the points $E$, $G$ and $F$ are collinear.

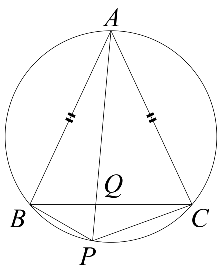



**9.** Isosceles triangle $ABC$ and its circumcircle are shown. The point $Q$ lies on the base $BC$, and $AQ$ produced meets the circumcircle at $P$.

(a) Prove that the triangles $BQP$ and $AQC$ are similar.

(b) Prove that $BP \times CQ = PQ \times AC$.

(c) Prove that $\dfrac{1}{BP} + \dfrac{1}{CP} = \dfrac{1}{PQ} \times \dfrac{BC}{AC}$.



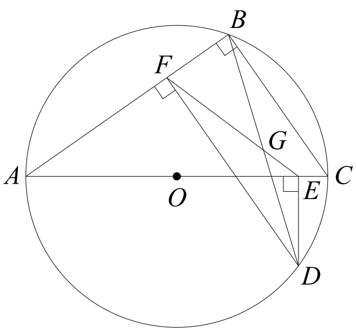

**10.** Triangle $ABC$ is right-angled at $B$. Its circumcircle has centre $O$. Point $D$ is chosen on the circumcircle. Points $E$ and $F$ are the feet of the perpendiculars from $D$ to $AC$ and $AB$ respectively. $DB$ and $EF$ intersect at $G$.

(a) Explain why $ADEF$ is a cyclic quadrilateral.

(b) Let $\angle DAE = \theta$. Prove that $\triangle FGB$ is isosceles.

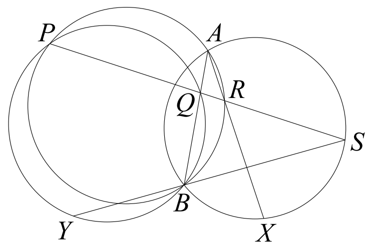
(c) Prove that $ODEG$ is a cyclic quadrilateral.

(d) Deduce that $OG \perp BD$.



**11.** Three circles $ABP$, $ABS$ and $PQB$ are shown. It is known that $PQRS$ and $YBS$ are straight lines.

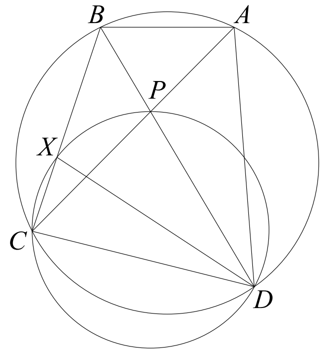
Prove that $SX$ and $QY$ are parallel.




**12.** The diagram shows a cyclic quadrilateral $ABCD$.
The diagonals intersect at $P$.
The circle $DPC$ cuts $BC$ at $X$.

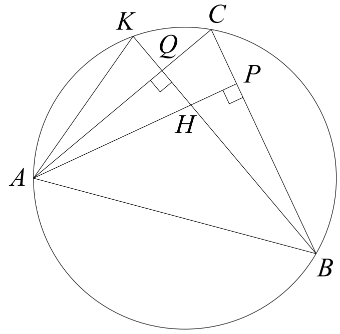
(a) Prove that $BD$ bisects $\angle ADX$.

(b) Prove that $\angle DXC = \angle PBC + \angle PCB$.



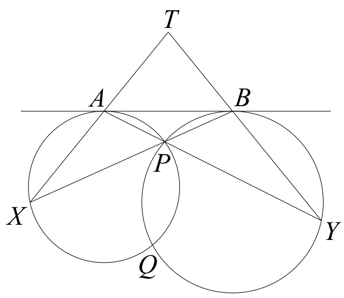

**13.** The diagram shows triangle $ABC$.
The altitudes $AP$ and $BQ$ intersect at $H$.
$BQ$ produced meets the circumcircle of $\triangle ABC$ at $K$.

Prove that $HQ = QK$.




**14.** In the diagram $AB$ is a common tangent of two circles that intersect at $P$ and $Q$.
$APY$ and $BPX$ are straight lines.
$XA$ produced and $YB$ produced meet at $T$.

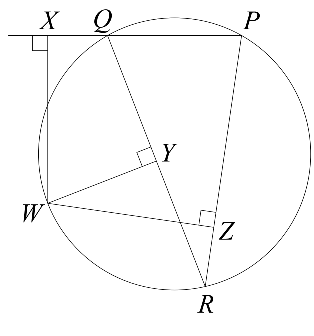
Prove that $AT = BT$.




**15.** The diagram shows $\triangle PQR$ inscribed in a circle. A point $W$ is chosen on arc $QR$. $WX$ is drawn perpendicular to $PQ$ produced, $WY$ is drawn perpendicular to $QR$ and $WZ$ is drawn perpendicular to $PR$.

(a) Explain why $WXQY$ and $WYZR$ are cyclic.

(b) Prove that the points $X$, $Y$ and $Z$ are collinear.



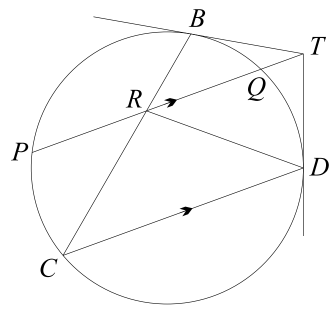
**16.** In the diagram $PQ$ and $CD$ are parallel chords of a circle. The tangent at $D$ intersects $PQ$ at $T$. The point $B$ is the point of contact of the other tangent from $T$. Chords $BC$ and $PQ$ intersect at $R$.

(a) Prove that quadrilateral $BTDR$ is cyclic.

(b) Prove that $\angle BRT = \angle DRT$.

(c) Prove that $\triangle RCD$ is isosceles.

(d) Deduce that $BC$ bisects $PQ$.



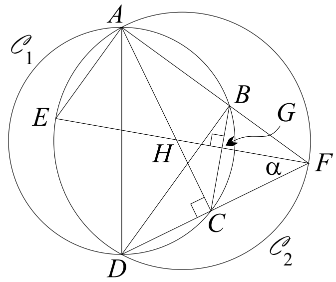

**17.** In the diagram $ABCD$ is a cyclic quadrilateral in circle $\mathcal{C}_1$ with $AC$ perpendicular to $DC$. Secants $AB$ and $DC$ meet at $F$. $\mathcal{C}_2$ is the circle through $A$, $D$ and $F$. The line from $F$ perpendicular to $BC$ cuts $BC$ at $G$, $AC$ at $H$ and meets $\mathcal{C}_2$ at $E$. Let $\angle DFE = \alpha$.

(a) Prove that $\angle HCG = \alpha$.

(b) Prove that $AB \perp DB$.

(c) prove that $AE \parallel BD$.

(d) Prove that the points $E$, $A$, $B$ and $G$ are concyclic.

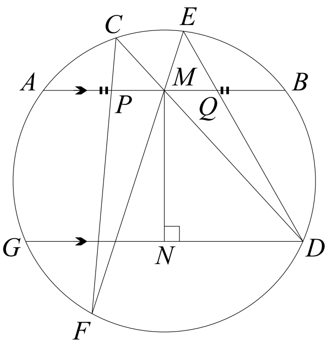



**18.** The chords $AB$, $CD$ and $EF$ of a circle are concurrent at $M$, where $M$ is the midpoint of $AB$. The lines $CF$ and $ED$ intersect $AB$ at $P$ and $Q$ respectively, while $GD$ is parallel to $AB$. The foot of the perpendicular from $M$ to $GD$ is at $N$.

(a) Prove that $\triangle MGD$ is isosceles.

(b) Prove that quadrilateral $PMFG$ is cyclic.

(c) Prove that $PM = MQ$.




**19.** The circles $PABC$ and $PMNO$ intersect at $P$. $APM$, $BPN$ and $CPO$ are straight lines. $BA$ and $MN$ produced meet at $X$, $CA$ and $MO$ produced meet at $Y$, while $CB$ and $NO$ produced meet at $Z$. Let $\angle YAX = \alpha$.

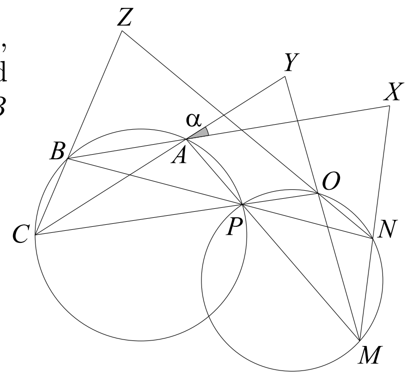
(a) Prove that $\angle BPC = \alpha$.

(b) Prove that $\angle OMN = \alpha$.

(c) Prove that $XYAM$ is a cyclic quadrilateral.

(d) Prove that $\angle XYM = \angle BCP$.

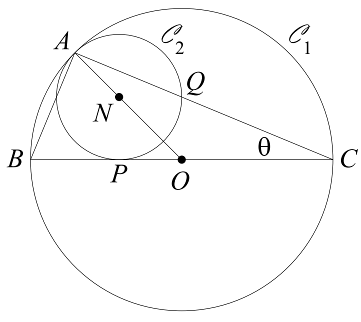
(e) Prove that the points $X$, $Y$ and $Z$ are collinear.




**EXTENSION**

**20.** Circle $\mathcal{C}_1$ has centre $O$ and radius $OA = a$. Circle $\mathcal{C}_2$ has centre $N$ on $OA$ and radius $NA = r$. It should be clear that $0 < r < \frac{1}{2}a$. The diameter $BC$ of circle $\mathcal{C}_1$ is tangent to $\mathcal{C}_2$ at $P$. Chord $AC$ intersects $\mathcal{C}_2$ at $Q$. Let $AB$ have length $2c$ and $AC$ have length $2b$. Let $\angle ACB = \theta$.

Show that $r = \dfrac{2abc}{(b+c)^2}$.



# 10B Algebraic Inequalities

There is an amazing number of types of algebraic inequalities, but most can be assigned to one of four broad categories.

**Everything on One Side:** In a few instances the inequality can easily be proved by moving all terms to one side and considering the sign of the result.

**WORKED EXERCISE:** Let $a < b$ be real numbers. Prove that the average of the squares of $a$ and $b$, is greater than the square of the average.

**SOLUTION:** The corresponding inequality to prove is

$$\frac{a^2+b^2}{2} > \left(\frac{a+b}{2}\right)^2 .$$

Now

$$\begin{aligned} \text{LHS} - \text{RHS} &= \frac{a^2+b^2}{2} - \frac{a^2+2ab+b^2}{4} \\ &= \frac{a^2-2ab+b^2}{4} \\ &= \frac{(a-b)^2}{4} \\ &> 0 \qquad \text{(since squares cannot be negative.)} \end{aligned}$$

Hence $\dfrac{a^2+b^2}{2} - \left(\dfrac{a+b}{2}\right)^2 > 0$

and so

$$\frac{a^2+b^2}{2} > \left(\frac{a+b}{2}\right)^2 .$$

**Squares Cannot be Negative:** The crucial step in the previous worked exercise was that the square of a real number cannot be negative. This important result can be used to solve numerous other inequalities.

Recall from the study of sequences and series that if $a$ and $b$ are positive real numbers then the sequence

$$a, x, b$$

will be arithmetic if

$$x = \frac{a+b}{2} .$$

The value of $x$ is called the *arithmetic mean* of $a$ and $b$. Likewise, the sequence

$$a, y, b$$

will be geometric if

$$y = \sqrt{ab} .$$

The value of $y$ is called the *geometric mean* of $a$ and $b$.

One very important result is that the arithmetic mean is at least as large as the geometric mean. This is sometimes called the *AM/GM result*. It can be proved by noting that squares cannot be negative.

**WORKED EXERCISE:** Prove the AM/GM result.

**SOLUTION:** Let $a$ and $b$ be two positive real numbers.
Since squares cannot be negative,

$$\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0\,.$$

Expanding $\quad a - 2\sqrt{ab} + b \geq 0$

so $\quad a + b \geq 2\sqrt{ab}$

or $\quad \dfrac{a+b}{2} \geq \sqrt{ab}\,.$

**Combinations of Inequalities:** Having established an inequality, it can be restated using other variables and the results combined to form a new inequality.

**WORKED EXERCISE:**

(a) It is known that if $1 \leq k \leq n$ then $n \leq k(n-k+1)$.

Explain why $\sqrt{n} \leq \sqrt{k(n-k+1)} \leq \dfrac{n+1}{2}\,.$

(b) Hence prove that, for all positive integers $n$,

$$\sqrt{n^n} \leq n! \leq \left(\frac{n+1}{2}\right)^n\,.$$

**SOLUTION:**

(a) The left hand inequality follows directly from the given result.
By the AM/GM result,

$$\begin{aligned}\sqrt{k(n-k+1)} &\leq \frac{k+(n-k+1)}{2}\\ &\leq \frac{n+1}{2}\,.\end{aligned}$$

(b) By part (a) it follows that

$$\sqrt{n} \leq \sqrt{1 \times n} \leq \frac{n+1}{2} \qquad (\text{k} = 1)$$

$$\sqrt{n} \leq \sqrt{2(n-1)} \leq \frac{n+1}{2} \qquad (\text{k} = 2)$$

$$\sqrt{n} \leq \sqrt{3(n-2)} \leq \frac{n+1}{2} \qquad (\text{k} = 3)$$

$$\vdots$$

$$\sqrt{n} \leq \sqrt{n \times 1} \leq \frac{n+1}{2} \qquad (\text{k} = \text{n})$$

Now multiply all these results together to get

$$\left(\sqrt{n}\right)^n \leq \sqrt{\Big(1 \times 2 \times \ldots \times n\Big) \times \Big(n \times (n-1) \times \ldots \times 1\Big)} \leq \left(\frac{n+1}{2}\right)^n$$

hence $\quad \sqrt{n^n} \leq \sqrt{n! \times n!} \leq \left(\dfrac{n+1}{2}\right)^n$

or $\quad \sqrt{n^n} \leq n! \leq \left(\dfrac{n+1}{2}\right)^n$

This example happened to use multiplication of inequalities. Other examples may require addition, subtraction or division. Multiplication and division should be avoided unless it is guaranteed that the quantities involved do not change sign. Otherwise it is not known whether the direction of the inequality is affected.

**Begin with a Known Result:** Many problems begin with a known result such as $x \le |x|$ from which another result is to be obtained, $|x+y| \le |x| + |y|$. In other problems certain restrictions are placed on the variables, such as $a+b+c=1$, from which it follows that $ab+ac+bc < \frac{1}{3}$. The usual approach is to begin manipulating the expression and at crucial steps apply the given restrictions.

In some instances the solution also involves changing the value of a fraction by altering the numerator or denominator. For example, decreasing the numerator or increasing the denominator will reduce the value of the fraction. This is clearly evident in the following numerical example.

$$\frac{3}{5} > \frac{2}{5} > \frac{2}{7}$$

The strategy is used twice in the next worked exercise, in conjunction with a restriction on the variables.

**WORKED EXERCISE:** Let $a$, $b$ and $c$ be three positive real numbers. It is known that $a+b \ge c$. Show that

$$\frac{a}{1+a} + \frac{b}{1+b} - \frac{c}{1+c} \ge 0\,.$$

**SOLUTION:** Firstly re-arrange the left hand side:

$$\begin{aligned}\frac{a}{1+a} + \frac{b}{1+b} - \frac{c}{1+c} &= 1 - \frac{1}{1+a} + 1 - \frac{1}{1+b} - 1 + \frac{1}{1+c}\\ &= 1 + \frac{1}{1+c} - \left(\frac{1}{1+a} + \frac{1}{1+b}\right)\end{aligned}$$

Use $a+b \ge c$ to increase the first denominator on the RHS so that

$$\begin{aligned}\frac{a}{1+a} + \frac{b}{1+b} - \frac{c}{1+c} &\ge 1 + \frac{1}{1+a+b} - \left(\frac{1}{1+a} + \frac{1}{1+b}\right)\\ &\ge \frac{2+a+b}{1+a+b} - \frac{2+a+b}{1+a+b+ab}\end{aligned}$$

Now decrease the second denominator on the RHS to get

$$\frac{2+a+b}{1+a+b} - \frac{2+a+b}{1+a+b+ab} \ge \frac{2+a+b}{1+a+b} - \frac{2+a+b}{1+a+b}$$

hence $$\frac{a}{1+a} + \frac{b}{1+b} - \frac{c}{1+c} \ge 0\,.$$

Careful thought is required at the last step. By decreasing the denominator, the fraction has increased. However, since subtraction is involved, the overall value of the expression has been reduced, in this case to zero.

## Exercise **10B**

**1.** Suppose that $a$ and $b$ are real numbers.
Prove that $\dfrac{a^2+b^2}{2} \ge \left(\dfrac{a+b}{2}\right)^2$.

**2.** If $a \geq b$, prove that $a^3 - b^3 \geq a^2b - ab^2$.

**3.** Let $a$ and $b$ be positive real numbers. Prove that $\dfrac{a^2}{b^2} + \dfrac{b^2}{a^2} + 6 \geq 4\left(\dfrac{a}{b} + \dfrac{b}{a}\right)$.

**4.** (a) Given that $x$ and $y$ are non-negative, prove that $\dfrac{x+y}{2} \geq \sqrt{xy}$.

(b) Hence prove that $(x+y)(x+z)(y+z) \geq 8xyz$.

**5.** Suppose that $p$, $q$ and $r$ are real and distinct.

(a) Prove that $p^2 + q^2 > 2pq$.

(b) Hence prove that $p^2 + q^2 + r^2 > pq + qr + rp$.

(c) Given that $p + q + r = 1$, prove that $pq + qr + rp < \frac{1}{3}$.

**DEVELOPMENT**

**6.** Suppose that $a$, $b$ and $c$ are real numbers.

(a) Prove that $a^4 + b^4 + c^4 \geq a^2b^2 + a^2c^2 + b^2c^2$.

(b) Hence show that $a^2b^2 + a^2c^2 + b^2c^2 \geq a^2bc + b^2ac + c^2ab$.

(c) Deduce that if $a + b + c = d$, then $a^4 + b^4 + c^4 \geq abcd$.

**7.** Suppose that $a$, $b$ and $c$ are positive.

(a) Prove that $a^2 + b^2 \geq 2ab$.

(b) Hence prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

(c) Given that $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$, prove that $a^3 + b^3 + c^3 \geq 3abc$.

(d) If $x$, $y$ and $z$ are positive, show that $x + y + z \geq 3\,(xyz)^{\frac{1}{3}}$.

(e) Suppose that $(1+x)(1+y)(1+z) = 8$. Prove that $xyz \leq 1$.

**8.** (a) Show that $a^2 + b^2 \geq 2ab$ for all real numbers $a$ and $b$.

(b) Hence deduce that for all positive real numbers $a$, $b$ and $c$:

(i) $(a+b+c)^2 \geq 3(ab+bc+ca)$

(ii) $ab(a+b) + bc(b+c) + ca(c+a) \geq 6abc$

(c) Suppose that $a$, $b$ and $c$ are the side lengths of a triangle.

(i) Explain why $(b-c)^2 \leq a^2$.

(ii) Deduce that $(a+b+c)^2 \leq 4(ab+bc+ca)$.

**9.** Suppose that $a$, $b$ and $c$ are positive.

(a) Prove that $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$.

(b) Hence show that $(a+b+c)\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right) \geq 9$.

(c) (i) Prove that $a^3 + b^3 \geq \left(\dfrac{a}{c} + \dfrac{b}{c}\right)abc$, and write down similar inequalities for $b^3 + c^3$ and $c^3 + a^3$.

(ii) Hence prove that $a^3 + b^3 + c^3 \geq 3abc$.

(iii) Deduce that $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a} \geq 3$.

**10.** (a) Suppose that $a$, $b$, $c$ and $d$ are positive.
Use the fact that $\dfrac{a^2+b^2}{2} \geq ab$ to show that $\dfrac{a^2+b^2+c^2+d^2}{4} \geq \sqrt{abcd}$.

(b) Hence show that for positive number $w$, $x$, $y$ and $z$, $\dfrac{w+x+y+z}{4} = \sqrt[4]{wxyz}$.

**11.** Suppose that $x$ and $y$ are positive. Prove that:

(a) $\dfrac{1}{x} + \dfrac{1}{y} \geq \dfrac{4}{x+y}$ (b) $\dfrac{1}{x^2} + \dfrac{1}{y^2} \geq \dfrac{8}{(x+y)^2}$

**12.** It is known that $a \leq |a|$ for any real number $a$.
Let $x$ and $y$ be any two real numbers. Use the above result to prove that

$$|x+y| \leq |x| + |y|\,.$$

Begin by squaring both sides.

**13.** (a) Let $z = x + iy$ be a complex number. Prove algebraically that $\mathrm{Re}(z) \leq |z|$.

(b) Let $z$ and $w$ be two complex numbers. Prove that $|z+w| \leq |z| + |w|$.
Begin by writing $|z+w|^2 = (z+w)\overline{(z+w)}$.

(c) Under what circumstances is $|z+w| = |z| + |w|$?

**14.** (a) Prove by induction that $2^n > n$, for all positive integers $n$.

(b) Hence show that $1 < \sqrt[n]{n} < 2$, if $n$ is a positive integer greater than 1.

(c) Suppose that $a$ and $n$ are positive integers. It is known that if $\sqrt[n]{a}$ is a rational number, then it is an integer. Explain why $\sqrt[n]{n}$, where $n$ is a positive integer greater than 1, is never a rational number.

**15.** (a) (i) Prove by induction that $(1+c)^n > 1 + cn$, for all integers $n \geq 2$, where $c$ is a nonzero constant greater than $-1$.

(ii) Hence show that $(1 - \frac{1}{2n})^n > \frac{1}{2}$, for all integers $n \geq 2$.

(b) (i) Solve the inequation $x^2 > 2x + 1$.

(ii) Hence prove by induction that $2^n > n^2$, for all integers $n \geq 5$.

(c) Suppose that $a > 0$, $b > 0$, and $n$ is a positive integer.

(i) Divide the expression $\left(a^{n+1} - a^n b + b^{n+1} - b^n a\right)$ by $(a-b)$, and hence show that

$$a^{n+1} + b^{n+1} \geq a^n b + b^n a\,.$$

(ii) Hence prove by induction that $\left(\dfrac{a+b}{2}\right)^n \leq \dfrac{a^n+b^n}{2}$.

**16.** (a) Given that $\sin x > \dfrac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$, show that:

(i) $e^{-\sin x} < e^{-\frac{2x}{\pi}}$ for $0 < x < \frac{\pi}{2}$, (ii) $\displaystyle\int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx < \int_0^{\frac{\pi}{2}} e^{-\frac{2x}{\pi}}\,dx$.

(b) Use the substitution $u = \pi - x$ to show that

$$\int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx = \int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}\,dx\,.$$

(c) Hence show that $\displaystyle\int_0^{\pi} e^{-\sin x}\,dx < \frac{\pi}{e}(e-1)$.

**17.** For $n = 0, 1, 2, \ldots$ let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n \theta \, d\theta$.

(a) Show that $I_1 = \frac{1}{2}\ln 2$.

(b) Show that, for $n \geq 2$, $I_n + I_{n-2} = \dfrac{1}{n-1}$.

(c) For $n \geq 2$, explain why $I_n < I_{n-2}$, and deduce that

$$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}\,.$$

(d) Use the reduction formula in part (b) to find $I_5$, and hence deduce that

$$\tfrac{2}{3} < \ln 2 < \tfrac{3}{4}\,.$$

**18.** Let $I_n = \displaystyle\int_0^1 \frac{x^{n-1}}{(x+1)^n}\, dx$, for $n = 1, 2, 3, \ldots$

(a) Show that $I_1 = \ln 2$.

(b) Use integration by parts to show that $I_{n+1} = I_n - \dfrac{1}{n\,2^n}$.

(c) The maximum value of $\dfrac{x}{x+1}$, for $0 \leq x \leq 1$, is $\frac{1}{2}$.
Use this fact to show that $I_{n+1} < \frac{1}{2}I_n$.

(d) Deduce that $I_n < \dfrac{1}{n\,2^{n-1}}$.

(e) Use the reduction formula in part (b) and the inequality in part (d) to show that $\frac{2}{3} < \ln 2 < \frac{17}{24}$.

**19.** (a) Write down the sum of the geometric series $1 - x^2 + x^4 - \ldots + x^{4n}$.

(b) Hence show that $\dfrac{1}{1+x^2} \leq 1 - x^2 + x^4 - \ldots + x^{4n} \leq \dfrac{1}{1+x^2} + x^{4n+2}$.

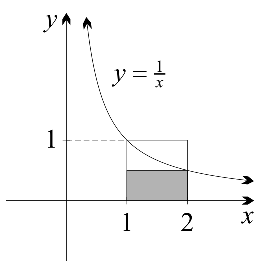
(c) Let $0 \leq y \leq 1$. Show that $\tan^{-1} \leq y - \frac{1}{3}y^3 + \frac{1}{5}y^5 - \ldots + \frac{1}{4n+1}y^{4n+1}$.

(d) Hence prove that $0 < \left(1 - \frac{1}{3} + \frac{1}{5} - \ldots + \frac{1}{1001}\right) - \frac{\pi}{4} < 10^{-3}$.

# 10C Inequalities in Geometry and Calculus

Many inequalities arise through the study of calculus and geometry. As a very simple example, using an upper and a lower rectangle to approximate the area between the hyperbola $y = \frac{1}{x}$ and the $x$-axis for $1 < x < 2$ gives

$$\tfrac{1}{2} < \log 2 < 1\,.$$




Often knowledge of other topics is also required. The following worked exercise makes use of calculus and a geometric series to find an approximation for $\log \frac{3}{2}$ which is accurate to two decimal places.

**WORKED EXERCISE:** Consider the geometric series

$$S_{2n} = 1 - h + h^2 - \ldots + h^{2n},$$

where $0 < h < 1$.

(a) Show that $S_{2n-1} < \dfrac{1}{1+h} < S_{2n}$.

(b) Integrate the previous result between $h = 0$ and $h = x$, where $0 < x < 1$, and hence write down a polynomial inequality for $\log(1+x)$.

(c) Use $n = 3$ to estimate the value of $\log \frac{3}{2}$.

**SOLUTION:**

(a) Now

$$\begin{aligned}
S_{2n-1} &= 1 - h + h^2 - h^3 + \ldots - h^{2n-1} \\
&= \frac{1-(-h)^{2n}}{1-(-h)} \qquad \text{(by GP theory)} \\
&= \frac{1-h^{2n}}{1+h} \\
&< \frac{1}{1+h} \qquad \text{(increase numerator)}
\end{aligned}$$

and

$$\begin{aligned}
S_{2n} &= 1 - h + h^2 - \ldots + h^{2n} \\
&= \frac{1-(-h)^{2n+1}}{1-(-h)} \qquad \text{(by GP theory)} \\
&= \frac{1+h^{2n+1}}{1+h} \\
&< \frac{1}{1+h} \qquad \text{(decrease numerator)}
\end{aligned}$$

hence $S_{2n-1} < \dfrac{1}{1+h} < S_{2n}$ .

(b)

$$\int_0^x S_{2n-1}\,dh < \int_0^x \frac{1}{1+h}\,dh < \int_0^x S_{2n}\,dh$$

or

$$\left[h - \frac{h^2}{2} + \ldots - \frac{h^{2n}}{2n}\right]_0^x < \Big[\log(1+h)\Big]_0^x < \left[h - \frac{h^2}{2} + \ldots + \frac{h^{2n+1}}{2n+1}\right]_0^x$$

so

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots - \frac{x^{2n}}{2n} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots - \frac{x^{2n+1}}{2n+1}$$

(c) Put $n = 3$ and $x = \frac{1}{2}$ to get

$$\tfrac{1}{2} - \tfrac{1}{8} + \tfrac{1}{24} - \tfrac{1}{64} + \tfrac{1}{160} - \tfrac{1}{384} < \log \tfrac{3}{2} < \tfrac{1}{2} - \tfrac{1}{8} + \tfrac{1}{24} - \tfrac{1}{64} + \tfrac{1}{160} - \tfrac{1}{384} + \tfrac{1}{896}$$

so

$$\tfrac{259}{640} < \log \tfrac{2}{3} < \tfrac{909}{2240}$$

or

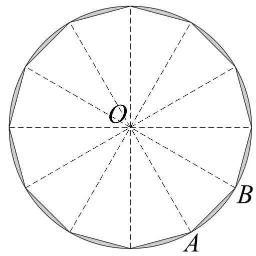
$$0{\cdot}4047 < \log \tfrac{3}{2} < 0{\cdot}4058$$

These calculations suggest that $\log \frac{3}{2}$ is about $0{\cdot}405$ correct to three decimal places. The actual value is $0{\cdot}40547$ correct to five decimal places.

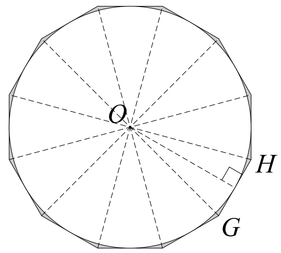
## Exercise **10C**

**1.** (a) A regular dodecagon is drawn inside a circle of radius 1 cm and centre $O$ so that its vertices lie on the circumference, as shown in the first diagram. Determine the area of $\triangle OAB$, and hence find the exact area of the inscribed dodecagon.




(b) (i) Use the formula for $\tan 2\theta$ to show that $\tan 15^\circ = 2 - \sqrt{3}$.

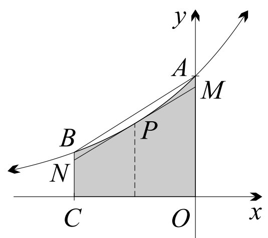
(ii) Another regular dodecagon is drawn with centre $O$, so that each side is tangent to the circle, as shown in the second diagram. Find the area of $\triangle OGH$ and hence find the exact area of the circumscribed dodecagon.




(c) By considering the results in parts (a) and (b), show that

$$3 < \pi < 12(2 - \sqrt{3}) \doteqdot 3{\cdot}24\,.$$

**2.** The diagram shows the points $A(0,1)$ and $B(-1, e^{-1})$ on the curve $y = e^x$, and the point $C(-1, 0)$ on the $x$-axis. The tangent at $P(-\frac{1}{2}, e^{-\frac{1}{2}})$ intersects $OA$ at $M$ and $BC$ at $N$.




(a) Determine the exact area of the region bounded by the curve, $BC$, $CO$ and $OA$.

(b) Find the area of trapezium:

(i) $OABC$,

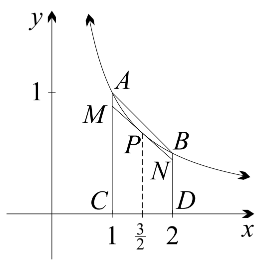
(ii) $OMNC$. [HINT: The equation of $MN$ is not needed.]

(c) Hence show that $\frac{1}{2}(3 + \sqrt{5}) < e < 3$.

**3.** (a) Use Simpson's rule with three function values to approximate the area under $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{6}$.

(b) Hence show that $\pi \doteqdot \frac{18}{13}(4 - \sqrt{3})$, which is accurate to two decimal places.

**4.** The points $A$, $P$ and $B$ on the curve $y = \dfrac{1}{x}$ have $x$-coordinates $1$, $1\frac{1}{2}$ and $2$ respectively. The points $C$ and $D$ are the feet of the perpendiculars drawn from $A$ and $B$ to the $x$-axis. The tangent to the curve at $P$ cuts $AC$ and $BD$ at $M$ and $N$ respectively.




(a) Find the areas of trapezia $ABDC$ and $MNDC$.

(b) Hence show that $\frac{2}{3} < \ln 2 < \frac{3}{4}$.

**5.** Suppose that $m \le f(x) \le M$ in the interval $a \le x \le b$. Use a diagram to help prove that

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)\,.$$

**DEVELOPMENT**

**6.** The diagram shows upper rectangles constructed on the graph of $y = \dfrac{1}{x}$.

(a) By considering appropriate areas, show that

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \ge \log(n+1)\,.$$

(b) What do you conclude about the infinite series

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots ?$$

**7.** (a) Show, using calculus, that the graph of $y = \ln x$ is concave down throughout its domain.
(b) Sketch the graph of $y = \ln x$, and mark two points $A(a, \ln a)$ and $B(b, \ln b)$ on the curve, where $0 < a < b$.
(c) Find the coordinates of the point $P$ that divides the interval $AB$ in the ratio $2:1$.
(d) Using parts (b) and (c), deduce that $\frac{1}{3}\ln a + \frac{2}{3}\ln b < \ln(\frac{1}{3}a + \frac{2}{3}b)$.

**8.** Let $f(x) = x^n\, e^{-x}$, where $n > 1$.
(a) Show that $f'(x) = x^{n-1}e^{-x}(n-x)$.
(b) Show that $(n, n^n e^{-n})$ is a maximum turning point of the graph of $f(x)$, and hence sketch the graph for $x \ge 0$. (Don't attempt to find points of inflexion.)
(c) Explain why $x^n e^{-x} < n^n e^{-n}$ for $x > n$. Begin by considering the graph of $f(x)$ for $x > n$,
(d) Deduce from part (c) that $(1 + \frac{1}{n})^n < e$.

**9.** The function $f(x)$ is defined by $f(x) = x - \log_e(1 + x^2)$.
(a) Show that $f'(x)$ is never negative.
(b) Explain why the graph of $y = f(x)$ lies completely above the $x$-axis for $x > 0$.
(c) Hence prove that $e^x > 1 + x^2$, for all positive values of $x$.

**10.** Consider the function $y = e^x\left(1 - \dfrac{x}{10}\right)^{10}$.
(a) Find the two turning points of the graph of the function.
(b) Discuss the behaviour of the function as $x \to \infty$ and as $x \to -\infty$.
(c) Sketch the graph of the function.
(d) From your graph, deduce that $e^x \le \left(1 - \dfrac{x}{10}\right)^{-10}$, for $x < 10$.
(e) Hence show that $\left(\dfrac{11}{10}\right)^{10} \le e \le \left(\dfrac{10}{9}\right)^{10}$.

**11.** (a) Let $a$, $b$ and $c$ be the lengths of the sides of a triangle and let $\angle A$ be opposite side $a$. Use the cosine rule to help prove that $|b - c| \le a \le b + c$.
That is, prove that one side of a triangle is longer than the difference between the other two sides and shorter than the sum of the other two sides.
(b) Hence prove for any two complex numbers $z$ and $w$ that
$$\Big||z| - |w|\Big| \le |z \pm w| \le |z| + |w|\,.$$
(c) Under what circumstances is $\Big||z| - |w|\Big| = |z + w|$?

**12.** In this question you may assume that simple exponential curves are concave up.
(a) Show by direct calculation that: (i) $6^6 < 3 \times 5^6$, (ii) $5 \times 6^6 < 2 \times 7^6$.
(b) The points $A\left(-\frac{1}{6}, 3^{-\frac{1}{6}}\right)$ and $B(0, 1)$ lie on the exponential curve $y = 3^x$. The points $B$ and $C\left(\frac{1}{6}, \left(\frac{5}{2}\right)^{\frac{1}{6}}\right)$ lie on the exponential curve $y = \left(\frac{5}{2}\right)^x$.
(i) Use part (a) to show that the gradient of chord $AB$ is greater than 1 and the gradient of chord $BC$ is less than 1.
(ii) Hence show that $\frac{5}{2} < e < 3$.

**13.** Let $|t| < 1$ and let $N$ be a positive integer.

(a) Show that $1 + t^2 + t^4 + \ldots + t^{2N} < \dfrac{1}{1-t^2}$.

(b) Show that the difference between the two is $\dfrac{t^{2N+2}}{1-t^2}$.

(c) Integrate the result in part (a) between 0 and $x$, where $|x| < 1$. Hence show that:

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2N+1}}{2N+1} < \tfrac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

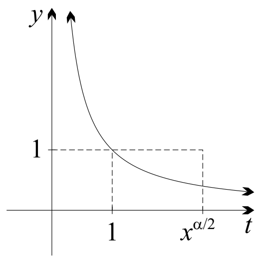
(d) Explain why $\displaystyle\int_0^x \frac{t^{2N+2}}{1-t^2}\,dt \le \int_0^x \frac{x^{2N+2}}{1-t^2}\,dt$.

(e) Use parts (b) to (d) to show that

$$\lim_{N\to\infty}\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2N+1}}{2N+1}\right) = \tfrac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

(f) Hence find $\log 2$ correct to three decimal places.

**14.** The diagram shows the graph of $y = \dfrac{1}{t}$, for $t > 0$.




Let $x > 1$ and $\alpha > 0$.

(a) By considering upper and lower rectangles, show that

$$0 < \tfrac{1}{2}\alpha\log x < x^{\alpha/2}.$$

(b) Hence show that $\displaystyle\lim_{x\to\infty}\left(\frac{\log x}{x^\alpha}\right) = 0$, for all $\alpha > 0$.

**15.** (a) Let $n > 1$ and $k$ be positive integers. Use lower rectangles to prove that

$$1 - \tfrac{1}{n} \le \int_{n^k}^{n^{k+1}} \frac{1}{x}\,dx.$$

(b) Hence prove that $\displaystyle\int_1^{n^k} \frac{1}{x}\,dx \to \infty$ as $k \to \infty$ regardless of the choice of $n$.

**16.** Consider the integral $\displaystyle\int_n^{n+x} \frac{1}{t}\,dt$.

(a) Use upper and lower rectangles to show that $\dfrac{x}{1+\frac{x}{n}} < n\log\left(1+\frac{x}{n}\right) < x$.

(b) Hence show that $\displaystyle\lim_{n\to\infty}\left(1+\tfrac{x}{n}\right)^n = e^x$ for any given value of $x$.

(c) Use trial and error to determine how big $n$ needs to be so that $\left(1+\frac{x}{n}\right)^n \doteqdot e^x$ correct to three decimal places when $x = 0{\cdot}1$.

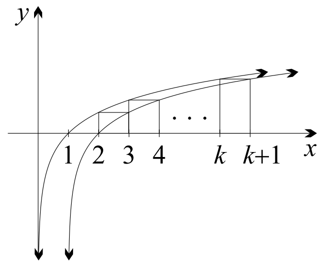
 

**EXTENSION**

**17.** The diagram shows the curves

$$y = \log x \quad \text{and} \quad y = \log(x-1)\,,$$

and $k-1$ rectangles constructed between $x = 2$ and $x = k+1$, where $k \geq 2$.



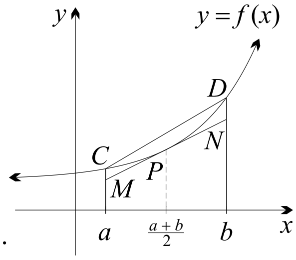

(a) Show that:

(i) $\displaystyle\int_2^{k+1} \log(x-1)\,dx = k\log k - k + 1$

(ii) $\displaystyle\int_2^{k+1} \log x\,dx = (k+1)\log(k+1) - \log 4 - k + 1$

(b) Deduce that $k^k < k!\,e^{k-1} < \frac{1}{4}(k+1)^{k+1}$, for all $k \geq 2$.

**18.** The diagram on the right shows the curve $y = f(x)$ in the interval $a \leq x \leq b$ where $f''(x) > 0$. The corresponding chord is $CD$ and $MN$ is tangent to $y = f(x)$ at $P$ where $x = \frac{a+b}{2}$ .




(a) Use areas to briefly explain why

$$(b-a)\,f\left(\frac{a+b}{2}\right) < \int_a^b f(x)\,dx < (b-a)\,\frac{f(a)+f(b)}{2}\,.$$

(b) Hence show that, for $n = 2, 3, 4, \ldots$,

$$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2}\left(\frac{1}{(n-1)^2} + \frac{1}{n^2}\right)\,.$$

(c) Deduce that

$$4\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)\,.$$

(d) Show that

$$\frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right) < \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\,.$$

(e) Hence show that $\displaystyle\frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4}$.

**19.** (a) Show that $\displaystyle\int_1^n \ln x\,dx = n\ln n - n + 1$.

(b) Use the trapezoidal rule on the intervals with endpoints $1, 2, 3, \ldots, n$ to show that

$$\int_1^n \ln x\,dx \doteqdot \tfrac{1}{2}\ln n + \ln(n-1)!$$

(c) Hence show that $n! < n^{n+\frac{1}{2}}\,e^{1-n}$. NOTE: This is a preparatory lemma in the proof of Stirling's formula $n! \doteqdot \sqrt{2\pi}\,n^{n+\frac{1}{2}}\,e^{-n}$, which gives an approximation for $n!$ whose percentage error converges to 0 for large integers $n$.

**20.** (a) Prove that $\log_e x \le x - 1$ for $x > 0$.

(b) Suppose that $p_1, p_2, p_3, \ldots, p_n$ are positive real numbers whose sum is 1.
Prove that $\displaystyle\sum_{r=1}^{n} \log_e(np_r) \le 0$.

(c) Let $x_1, x_2, x_3, \ldots, x_n$ be positive real numbers.
Prove that $\dfrac{x_1 + x_2 + x_3 + \cdots + x_n}{n} \ge (x_1 x_2 x_3 \ldots x_n)^{\frac{1}{n}}$.

# Chapter Ten

## Exercise 10A (Page 2)

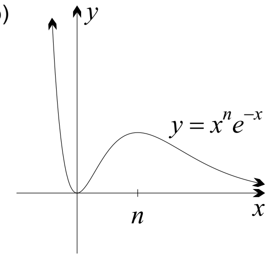
**15(a)** Converse of angles in a semi-circle at $X$, $Y$ and $Z$.
**20** Let $D$ be the foot of the perpendicular from $A$ to $BC$, then use trigonometry in $\triangle OAD$.

## Exercise 10B (Page 8)

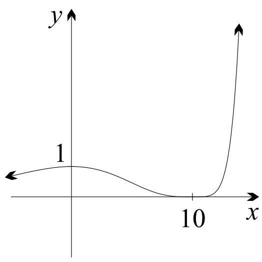
**1** Begin with LHS − RHS.
**2** Begin with LHS − RHS.
**4(a)** Begin with $(a-b)^2 \geq 0$ and put $a = \sqrt{x}$ and $b = \sqrt{y}$. **(b)** Use part (a) three times.
**5(a)** Begin with $(p-q)^2 \geq 0$.
**(b)** Use part (a) three times and add.
**(c)** Begin with $(p+q+r)^2$ and use part (b).
**6(a)** Use Question 4(b) with $p = a^2$ and so on.
**(b)** Use Question 4(b) with $p = ab$ and so on.
**(c)** Use parts (a) and (b).
**7(a)** See Question 4(a).
**(b)** See Question 4(b).
**(c)** Use part (b).
**(d)** Use part (c) and put $a^3 = x$ and so on.
**(e)** Expand and then used part (d). A more sophisticated approach is to use part (a) and in the first bracket put $a^2 = 1$ and $b^2 = x$, and so on.
**8(a)** See Question 4(a).
**(b)(i)** Expand the LHS and use part (a).
**(c)(i)** The triangle inequality: the length of any side is between the sum of the other two and the difference of the other two.
**(ii)** Begin with LHS − RHS and use part (i).
**9(a)** Use Question 7(a) and divide by $ab$.
**(b)** Expand the LHS and use part (a).
**(c)(i)** Begin with Question 7(a) and multiply by $(a+b)$.
**(ii)** Add part (i) and use part (a).
**(iii)** Begin with part (ii) and replace $a^3$ with $\frac{a}{b}$.
**10(a)** Replace $a^2$ with $a^2+b^2$ and so on.
**(b)** Use part (a) and put $a^2 = w$ and so on.
**11(a)** Begin with Question 7(a) and put $a^2 = \frac{1}{x}$ and so on.
**(b)** Begin with Question 7(a) and put $a^2 = \frac{1}{x^2}$ and so on.
**13(c)** When $z = kw$, with $k > 0$, or when either $z = 0$ or $w = 0$.
**15(b)(i)** $x > 1+\sqrt{2}$ or $x < 1-\sqrt{2}$
**17(d)** $I_5 = \frac{1}{4}(2\ln 2 - 1)$
**19(a)** $\dfrac{1+x^{4n+2}}{1+x^2}$

## Exercise 10C (Page 13)

**1(a)** $|\triangle OAB| = \frac{1}{4}$ sq. units, 3 sq. units
**(b)(ii)** $|\triangle OGH| = (2-\sqrt{3})$ sq. units, $12(2-\sqrt{3})$ sq. units
**2(a)** $(1-e^{-1})$ sq. units **(b)(i)** $\frac{1}{2}(1+e^{-1})$ sq. units
**(ii)** $e^{-\frac{1}{2}}$ sq. units
**3(a)** $\frac{\pi}{36}(4+\sqrt{3})$
**4(a)** $\frac{3}{4}$ and $\frac{2}{3}$ square units.
**6(b)** It diverges to infinity.
**7(c)** $\left(\frac{a+2b}{3}, \frac{\ln a + 2\ln b}{3}\right)$
**8(b)**




**10(a)** $(0,1)$ is a maximum turning point, $(10,0)$ is a minimum turning point.
**(b)** $y \to \infty$ as $x \to \infty$, and $y \to 0$ as $x \to -\infty$.
**(c)**




**11(c)** When $z = kw$, with $k < 0$, or when either $z = 0$ or $w = 0$.
**12(a)(i)** $6^6 = 46\,656$, $3 \times 5^6 = 46\,875$
**(ii)** $5 \times 6^6 = 233\,280$, $2 \times 7^6 = 235\,298$
**13(f)** 0·693
**16(c)** $n = 9$
**18(b)** In part (a), put $f(x) = x^{-2}$, $a = (n-1)$ and $b = n$.