Mathematics Extension 1

Name: Squar3root

Contents:

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Polynomials

General:

- Polynomials have only *real*, *positive, integer* powers
- A degree 'n' polynomial can generally be written as $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ where
  - n is the degree
  - a_n is the leading coefficient,
  - $a_n x^n$ is the leading term and
  - a_0 is the constant term
- Coefficients can be anything (3, e, π, $\frac{1}{2}$, $\sqrt{7}$) but only have real, positive integer powers, ie 5 or 3 and *NOT* $\frac{1}{2}$ or $\sqrt{2}$
- The domain of *any* polynomial is *all real* x and the curve is always continuous
- A monic polynomial is one where the coefficient of the term with the highest degree (AKA, the leading coefficient) is 1. ie, $x^4 + 2x^3 + \pi x^2 + e$
- A "zero" of a polynomial means that it satisfies it and hence is a root of the polynomial. eg,

  $P(x) = 3x^4 + 20x + 8$

  Now let $x = -2$

  $P(-2) = 3(-2)^4 + 20(-2) + 8$

  $= 0$

  Hence, we can say that -2 is a root of $P(x)$ or $(x + 2)$ is a factor of $P(x)$

Examples of Polynomials

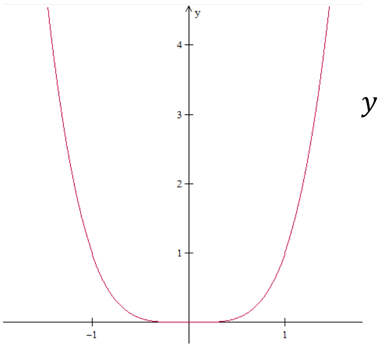
- $x^3 + 3x^2 + 6x + 9$
- $\frac{2}{7}x^3 + \pi x^2 - ex + \sqrt{34}$

Examples of ***NOT*** Polynomials

- $4x^2 + x^e + 3x^{-1}$
- $\sqrt{x} + \ln(x) + \frac{1}{x^3} + \sin x$

Graphing:

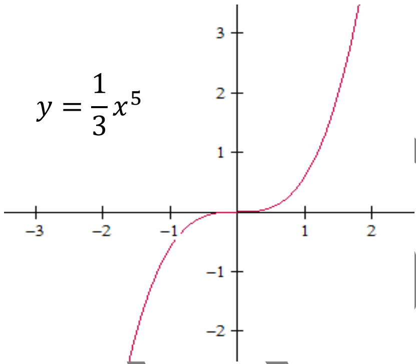
- If a polynomial is of *even degree*, the ends of the "head" and the "tail" of the graph will point in the *same direction*



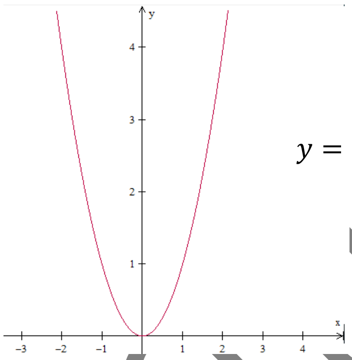
$y = x^4$

- Furthermore, if the polynomial is of *odd degree*; the "head" and the "tail" of the graph will point in *opposite directions*

$y = \frac{1}{3}x^5$

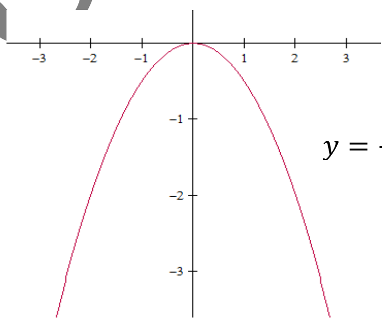


- Now, if the *leading coefficient* of the graph is *positive*, on an *even degree* polynomial, the "head" and "tail" of the graph will be in the *1st and 2nd quadrants*



$y = 4x^2$

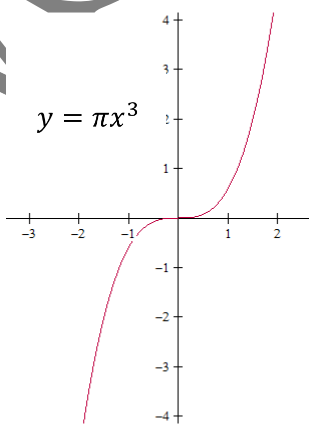
- But if the *leading coefficient* of the graph is *negative*, on an *even degree* polynomial, the "head" and "tail" of the graph will be in the *3st and 4nd quadrants*



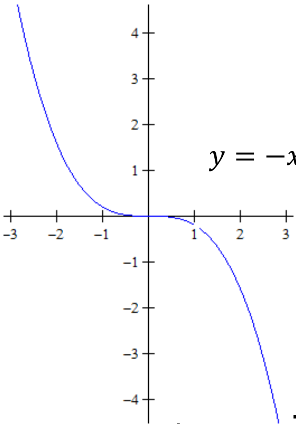
$y = -ex^2$

- Moreover, if the *leading coefficient* of the graph is *positive*, on an *odd degree* polynomial, the "head" and "tail" of the graph will be in the *1st and 3nd quadrants*

$y = \pi x^3$



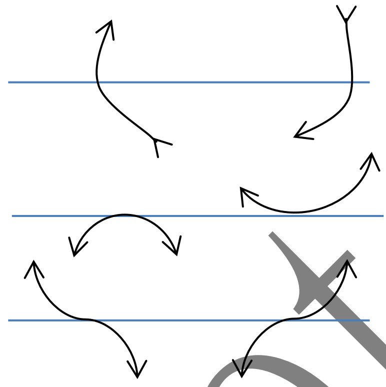
- Now, if the if the *leading coefficient* of the graph is *negative*, on an *odd degree* polynomial, the "head" and "tail" of the graph will be in the *2st and 4nd quadrants*



$y = -x^3$

Types of roots:

Degree 1 factor	Cuts
Degree 2 factor	Turns on
Degree 3 factor	Horizontal Point of inflection



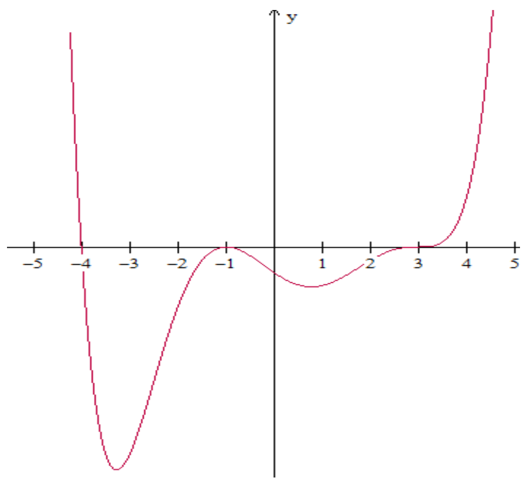
Example, graph: $P(x) = (x+1)^2(x+4)(x-3)^3$

We can say that the "zero's" of $P(x)$ are: -4, -1 and 3 (think of it like this: what makes the polynomial "zero?" well just put the negative of the number in the bracket) (this can be checked with a simple substitution into the function)

- What would the degree be? If we were to expand this out we would get: $y = x^6 - 3x^5 - 18x^4 + 58x^3 + 45x^2 - 135x - 108$ and clearly we can see that it is x^6 ($\therefore$ even degree $\Rightarrow$ ends point in the same direction) and that it is also positive so the "head" and "tail" of the graph are in the 1st and 2nd quadrants
- To "quickly expand it out and get the degree" multiply the powers together, ie 2x1x3=6 it is a degree 6 polynomial (ends in the same direction) and positive (1st and 2nd quadrants) as above
- Now we have to classify these roots. On inspection we can see that:

$x = -4$	Cut
$x = -1$	Turn
$x = 3$	Inflection

- Now we can graph it:



Polynomial division:

- Polynomials can be written in the form of:

$$\begin{array}{l} \quad\quad\; Q(x) \\ A(x)\sqrt{P(x)} + R(x) \end{array}$$

Where:

- $P(x)$ is the dividend
- $A(x)$ is the divisor
- $Q(x)$ is the quotient
- $R(x)$ is the remainder

It's basically long division

This is best explained through an example: Divide (x^3+6x^2-2) by $(x+3)$

Step 1: Set it up like this,

$$(x+3)\sqrt{x^3+6x^2-2}$$

Step 2: Now, look at the first term of $P(x)$ in this case it is x^3. So ask yourself, what term will I need to get x^3 when I multiply x and something together. Essentially, $x^3 = x.x^2$

Step 3: So write this where the quotient is

$$\begin{array}{l} \quad\quad\quad\;\; x^2 \\ (x+3)\sqrt{x^3+6x^2-2} \end{array}$$

Step 4: Multiply the quotient out by the divisor

$$\begin{array}{l} \quad\quad\quad\;\; x^2 \\ (x+3)\sqrt{x^3+6x^2-2} \\ \quad\quad\quad\;\; x^3+3x^2 \end{array}$$

Step 5: Now subtract this part from the dividend, so we get

$$\left.\begin{array}{l} \quad\quad\quad\;\; x^2 \\ (x+3)\sqrt{x^3+6x^2-2} \\ \quad\quad\quad\;\; \underline{x^3+3x^2\quad\;\;} \\ \quad\quad\quad\;\; 0+3x^2-2 \end{array}\right\} \text{Carry the '2'}$$

Step 6: Now the new dividend is $3x^2-2$ and we keep doing this until the degree of $R(x)$ is strickly less than the degree of $A(x)$. Sometimes the use of "ghost terms" are benificial

Step 7: Eventually we get $x^3+6x^2-2 \equiv (x+3)(x^2+3x-9)+25$

Renainder theorm:

- Remainder theorm states that "*if a polynomial is divided by* $(x-a)$*, then the remainder will be* $P(a)$"
- Example; find the remainder when $2x^3-4x^2+x+7$ is divided by $(x-2)$

$$P(x) = 2x^3 - 4x^2 + x + 7$$
$$P(2) = 2(2)^3 - 4(2)^2 + (2) + 7$$
$$= 9$$

- Typical question: $P(x) = x^3 + kx + 1$ has remaider 5 when divided by $(x-4)$ Find k

$$P(4) = (4)^3 + k(4) + 1 = 5$$

$$-4k = -60$$

$$k = 15$$

Factor theorm:

- Factor theorm states that "*if a polynomial is divided by* $(x-a)$*, and the remainder is zero, then* $(x-a)$ *is a root of the polynomial*"
- Factors are usually multiples of the constant term

Roots:

- A degree 2 polynomial can be written as $x^2-(\alpha+\beta)x+\alpha\beta$ so,

$$\alpha+\beta = -\frac{b}{a} \qquad\qquad \alpha\beta = \frac{c}{a}$$

- Similarily for higher degrees, ax^3+bx^2+cx+d

$\alpha+\beta+\gamma = -\frac{b}{a}$ can also be written as:$\sum\alpha = -\frac{b}{a}$

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$ can also be written as: $\sum\alpha\beta = \frac{c}{a}$

$\alpha\beta\gamma = -\frac{d}{a}$ can also be written as: $\prod\alpha\beta\gamma = -\frac{d}{a}$

- Note the alternating signs
- The product of roots is always $\frac{constant}{leading\ term\ coefficent}$ and work from there
- For suming any number of roots; eg 3 at a time on a degree 4 look at the coefficent of degree $(4-3)=1$ so look at the coefficent of x
- Don't make a mistake by looking at the wrong coefficent, sometimes it is zero
- Algebraic manipulation is usually required

Inequalities

Inequalities with variables on the numerator and denominator:

General steps to solve a problem:

1. Move the entire inequality to one side (so that the RHS is zero)
2. Decompose into one expression (using algebra)
3. Multiply both sides of the inequality by the denominator of the fraction squared
4. Draw graph (even if the questions doesn't say to, do it anyway)
5. Exclude restrictions

Let's apply this to a problem: Solve $\frac{5x-3}{x+1} \geq 2$

Step 1:

$$\frac{5x-3}{x+1} - 2 \geq 0$$

Step 2:

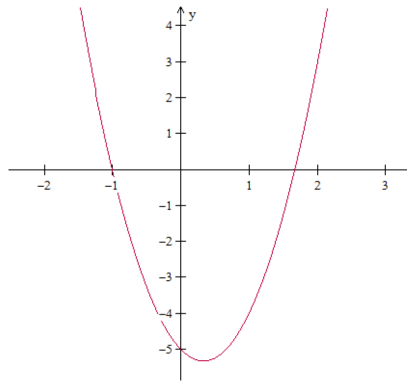
$$\frac{3x-5}{x+1} \geq 0$$

Step 3:

$$\frac{3x-5}{x+1} \times (x+1)^2 \geq 0 \times (x+1)^2$$

$$(3x-5)(x+1) \geq 0$$

Step 4:



On inspection we can see that there is a cut at $x = -1$ and $x = \frac{5}{3}$

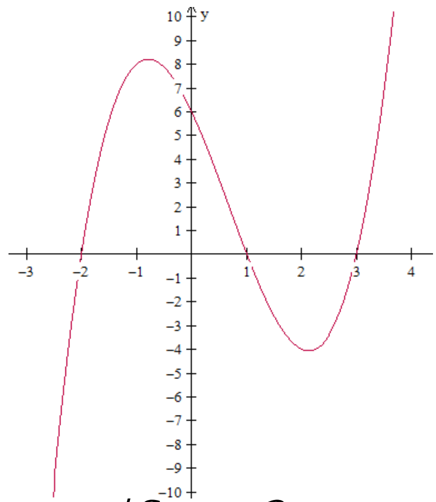
Now; when are the y values greater than zero? When:

$$x \geq \frac{5}{3} \text{ OR } x \leq -1$$

Step 5: But in this case; $x \neq -1$ so solutions are $x \geq \frac{5}{3}$ OR $x < -1$

Polynomial inequalities:

Solve: $(x-1)(x+2)(x-3) > 0$



Just like any other inequality from now on, we draw a graph and solve from there

From the graph we can see that it satisfies when

$$2 < x < 1 \; OR \; x > 3$$

Trigonometry

3D Trigonometry:

- There really is no way to teach this and nothing new to learn here. This area just requires sophisticated use of trigonometry. Some of the problems require the following formulae.

$sin\,\theta = \frac{opposite}{hypotenuse}$ $cos\,\theta = \frac{adjacent}{hypotenuse}$ $tan\,\theta = \frac{opposite}{adjacent}$

$\frac{a}{sinA} = \frac{b}{sinB}$ $c^2 = a^2 + b^2 - 2abcosC$ $c^2 = a^2 + b^2$

Compound angles:

- These need to be memorised

Easy way to remember expansions:

$\sin(\alpha \pm \beta) = sin\alpha cos\beta \pm cos\alpha sin\beta$

$\sin(first)\cos(second) \pm \cos(first)\sin(second)$

$\cos(\alpha \pm \beta) = cos\alpha cos\beta \mp sin\alpha sin\beta$

$\cos(first)\cos(second) \mp \sin(first)\sin(second)$

$\tan(\alpha \pm \beta) = \frac{tan\alpha \pm tan\beta}{1 \mp tan\alpha tan\beta}$

$\tan(first) \pm \tan(second)$ $over$ 1
$\mp \tan(first)\tan(second)$

Double Angle:

$sin2\theta = 2sin\theta cos\theta$

$cos2\theta = cos^2\theta - sin^2\theta$

$tan2\theta = \frac{2tan\theta}{1 - \tan^2\theta}$

$= 2cos^2\theta - 1$

$= 1 - 2sin^2\theta$

- Derive for triple roots
- You can do anything to the angle, ie, $sin\theta = 2sin\frac{\theta}{2}cos\frac{\theta}{2}$

General solution:

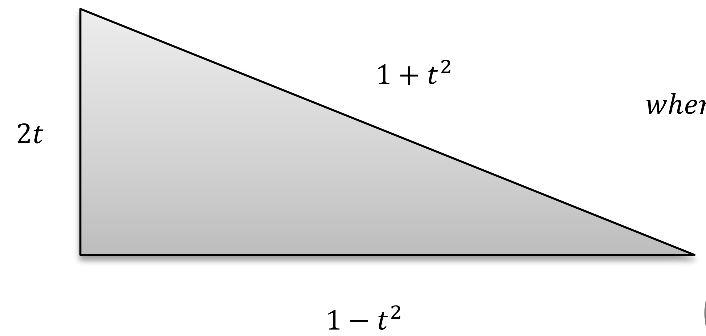
If $sin\theta = c$ then $\theta = k\pi + (-1)^k \sin^{-1} c\ \ where\ k \in R$

If $cos\theta = c$ then $\theta = 2k\pi \pm \cos^{-1} c\ \ where\ k \in R$

If $tan\theta = c$ then $\theta = k\pi + \tan^{-1} c\ \ where\ k \in R$

T-formulae:

- This is used to simplify trigonometric expressions and integration
- Try to minimise the use of this and use identities to simplify
- Memorise the triangle and then derive ratios (Do ***NOT*** attempt to memorise ratios)



$$where\ t = tan\frac{\theta}{2}$$

Trigonometric equations:

- Get the equation in terms of one variable and solve. This is best explained through an example. Solve: $2cos^2\theta + 5sin\theta + 1 = 0\ for\ 0 \leq \theta \leq 2\pi$

$$2cos^2\theta + 5sin\theta + 1 = 0$$

$$2(1 - sin^2\theta) + 5sin\theta + 1 = 0$$

$$2sin^2\theta - 5sin\theta - 3 = 0$$

$$let\ u = sin\theta$$

$$\therefore 2u^2 - 5u - 3 = 0$$

$$(2u+1)(u-3) = 0$$

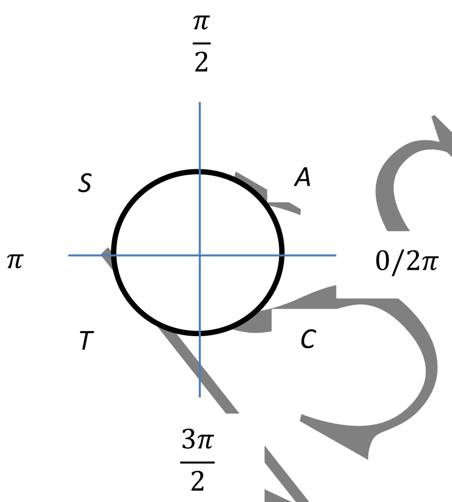
$$u = -\tfrac{1}{2}\ OR\ u = 3$$

$$\therefore \sin\theta = -\tfrac{1}{2}\ OR\ sin\theta = 3$$

$$sin\theta = 3\ is\ no\ solutions\ as -1 \leq sin\theta \leq 1$$

Quick Δ check

$5^2 - 4 \times 2 \times -3 = 49$



From above we can see that sin is negative in the 3rd and 4th quadrants and $Acute\ \theta = \frac{\pi}{6}$, so

$$\theta = \pi + \tfrac{\pi}{6}\ \text{OR}\ \theta = 2\pi - \tfrac{\pi}{6}$$

$$\theta = \frac{7\pi}{6}\ OR\ \frac{11\pi}{6}$$

Auxiliary angle technique:

- This method is handy when solving equations in the form $acos\theta + bsin\theta = c$
- We need to decompose the equation into the form $Rsin(\theta \pm \alpha)\ OR\ Rcos(\theta \pm \alpha)$ where $R = \sqrt{a^2 + b^2}\ and\ 0 < \alpha < \frac{\pi}{2}$
- Example, Solve: $\sqrt{3}sin\theta - cos\theta = 1$

Firstly $R = \sqrt{\sqrt{3}^2 + 1^2} = 2$

Now we need to factorise this out of the expression to get:

$$2(\frac{\sqrt{3}}{2}sin\theta - \frac{1}{2}cos\theta) = 1$$

We need to find an acute angle that gives the exact result. In this case it would be $\frac{\pi}{6}$ so

$$2(cos\ \frac{\pi}{6}sin\theta - sin\ \frac{\pi}{6}cos\theta) = 1$$

This is a compound angle expansion, so we can do the reverse

$$2\left[\sin\left(\theta - \frac{\pi}{6}\right)\right] = 1$$

And we can solve this from here

$$\theta = \frac{\pi}{3} OR\ \theta = \pi$$

Trigonometric Limits:

$$\lim_{\theta \to 0} \frac{sin\theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{tan\theta}{\theta} = 1$$

$$\lim_{\theta \to 0} cos\theta = 1$$

Multiply by a constant to get the bottom to equal the top

Example:

$$\lim_{\theta \to 0} \frac{sin3\theta}{2\theta}$$

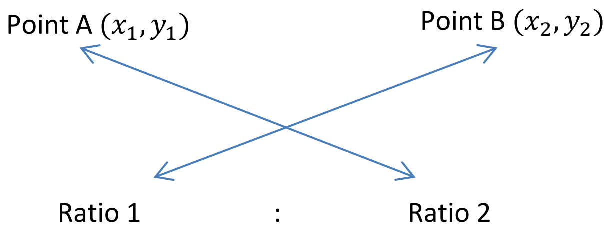
$$\lim_{\theta \to 0} \frac{sin3\theta}{3\theta} \times \frac{3}{2}$$

$$\frac{3}{2} \times 1 = \frac{3}{2}$$

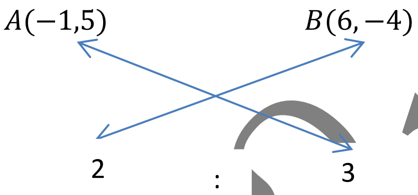
Division ratio:

This is best explained through an example: find the coordinates which divide the points $A(-1,5)$ and $B(6,-4)$ in the ratio 2:3, **i)** internally and **ii)** externally

i) Firstly, with any problem, set up a diagram like this:



So in the above case, we have

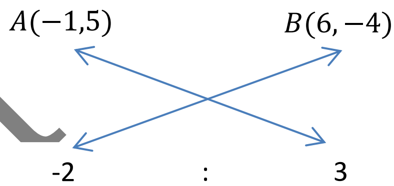


Now, what we need to do is multiply the ratio with the opposite x value for each coordinate and divide it by the addition of the 2 ratios. Do the same with the y values. So:

$$x = \frac{(3 \times -1) + (2 \times 6)}{2+3} \qquad y = \frac{(3 \times 5) + (2 \times -4)}{2+3}$$

Hence the coordinates are $P(\frac{9}{5}, \frac{7}{5})$

ii) As for externally, we do the exact same thing, *<u>except we put a negative in front of one of the ratios</u>*



$$x = \frac{(3 \times -1) + (2 \times 6)}{3 + -2} \qquad y = \frac{(3 \times 5) + (2 \times -4)}{3 + -2}$$

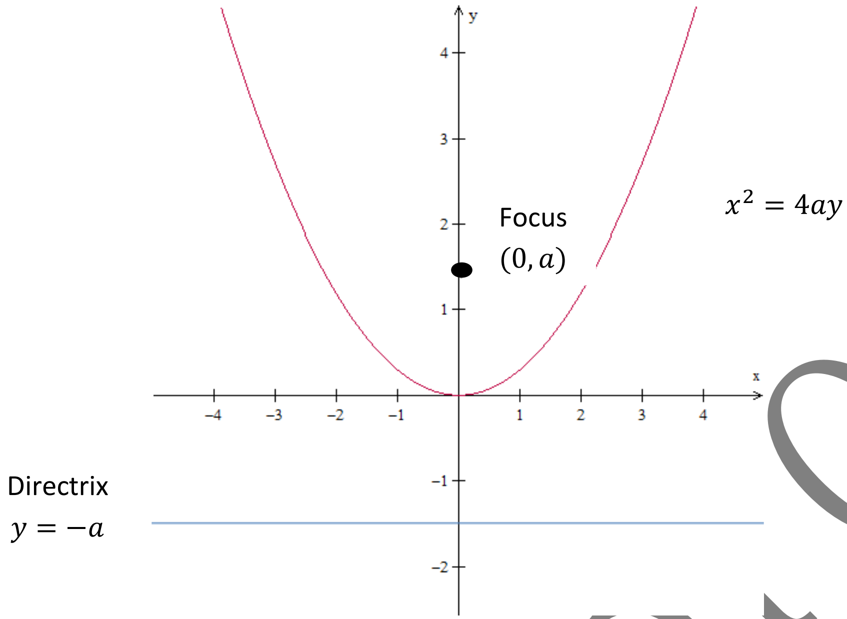
Hence the coordinates are $P(-9,23)$

Angles between 2 lines:

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

- Sometimes the use of calculus is required to find the gradient at a point

Parametrics

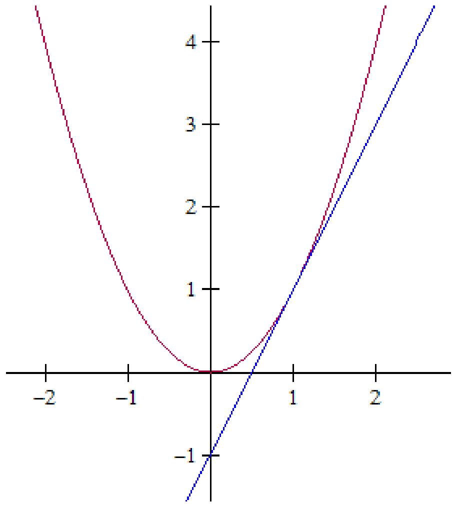


- The parametric coordinates for the parabola $x^2 = 4ay$ is $x = 2at$ and $y = at^2$
- The arbitrary point $P(2ap, ap^2)$ is used

Few tips:

- Do ***NOT*** memorise any equations, 99.99% of the time, the exam will ask you to *either* derive the equation or it will be given to you
- Hopefully you can see and appreciate that you actually haven't learnt anything entirely "new" here. Instead, you just built on your algebra and coordinate geometry skills in this section
- Squar3root is the best
- You won't explicitly be asked to only derive an equation, questions usually have a conceptual component such as being able to show that the locus is restricted by a particular boundary or being able to describe something

Equation of a *tangent* on a parabola at P



$$y = \frac{x^2}{4a}$$

$$\frac{dy}{dx} = \frac{x}{2a}$$

At $x = 2ap$

$$\frac{dy}{dx} = \frac{2ap}{2a} = p$$

Now equation of a line:

$$y - y_1 = m(x - x_1)$$

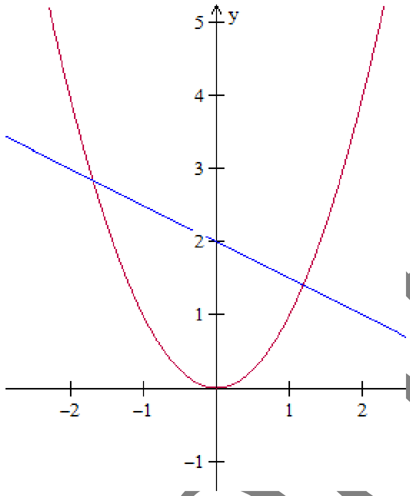
Substituting points/gradient:

$$y - ap^2 = p(x - 2ap)$$

Expanding and cleaning up;

$$y = px - ap^2$$

Equation of a *normal* on a parabola at P



$$y = \frac{x^2}{4a}$$

$$\frac{dy}{dx} = \frac{x}{2a}$$

At $x = 2ap$

$$\frac{dy}{dx} = \frac{2ap}{2a} = p$$

For the gradient of the normal, we must take the reciprocal of the gradient if the tangent, hence:

$$\frac{dy}{dx} = -\frac{1}{p}$$

Now equation of a line:

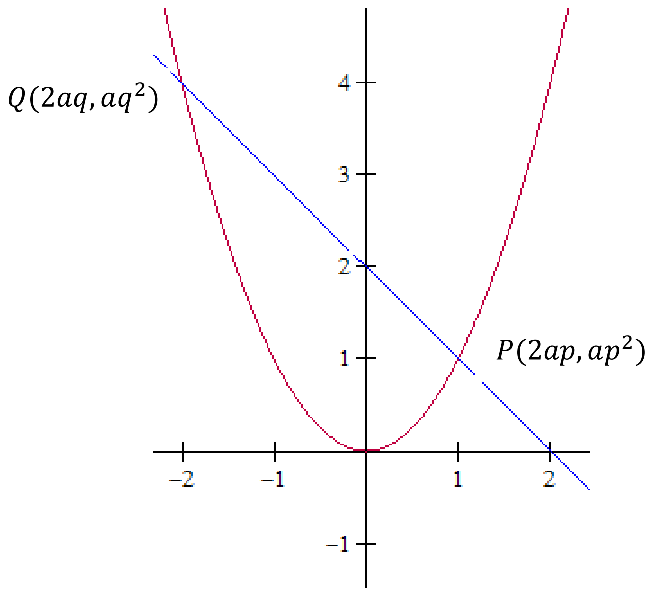
$$y - y_1 = m(x - x_1)$$

Substituting points/gradient:

$$y - ap^2 = -\frac{1}{p}(x - 2ap)$$

$$x + py = 2ap + ap^3$$

Equation of a chord



The gradient of the graph is obtained with:

$$\frac{\Delta y}{\Delta x} = \frac{ap^2 - aq^2}{2ap - 2aq}$$

$$= \frac{a(p+q)(p-q)}{2a(p-q)}$$

$$= \frac{1}{2}(p+q)$$

Now the equation of a line

$$y - y_1 = m(x - x_1)$$

Substituting points/gradient:

$$y - ap^2 = \frac{1}{2}(p+q)(x - 2ap)$$

Note: we could have used the coordinates of "Q"

$$y = \frac{1}{2}(p+q)x - apq\#$$

A further application is the focal chord. The focal chord passes through the focus (duh) which means that the equation of the chord must satisfy the coordinates of the focus. So,

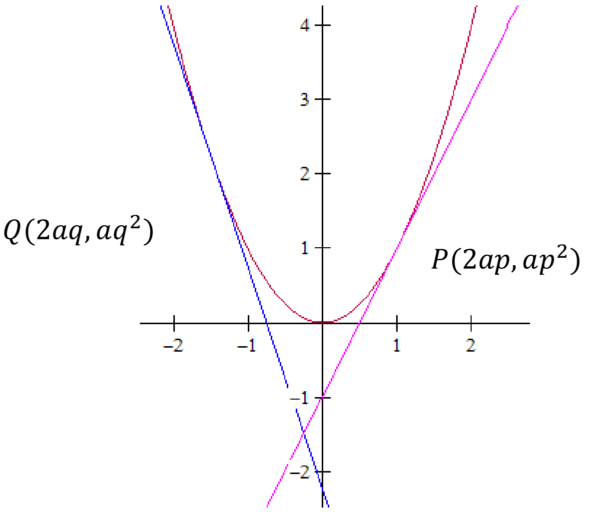
$$a = \frac{1}{2}(p+q) \times 0 - apq\#$$

$$pq = -1$$

So the condition for a chord to be a focal chord is if $pq = -1$

I've skipped most of the algebra to save paper :P but take some time now to derive the equation of a tangent, normal and chord in the space below

Intersection of 2 tangents



This is simply solving the equations of 2 tangents simultaneously

The tangent at P: $y = px - ap^2$

The tangent at Q: $y = qx - aq^2$

$$\therefore px - ap^2 = qx - aq^2$$

$$x(q-p) = a(p+q)(p-q)$$

$$x = a(p+q)$$

Now the y ordinate:

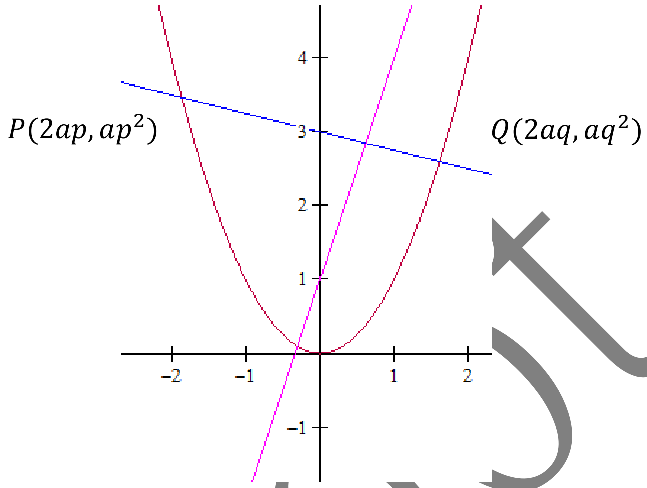
$$f([a(p+q)]) = apq$$

Hence 2 tangents intersect at

$$(a(p+q), apq)$$

Furthermore, if PQ is a focal chord, the 2 tangents will intersect on the directrix of the parabola at 90° to each other.

Intersection of 2 normals'



Again, this is just simply solving the equation of 2 normals simultaneously

The normal at P: $x + py = 2ap + ap^3$

The normal at Q: $x + qy = 2aq + aq^3$

$$\therefore 2ap + ap^3 - py = 2aq + aq^3 - qy$$

$$y(p-q) = a(p^3 - q^3) + 2a(p-q)$$

$$= a(p-q)(p^2 + pq + q^2) + 2a(p+q)$$

$$y = a(p^2 + pq + q^2 + 2)$$

Subbing in for y

$$x + p[a(p^2 + pq + q^2 + 2)] = 2ap + ap^3$$

$$x = -apq(p+q)$$

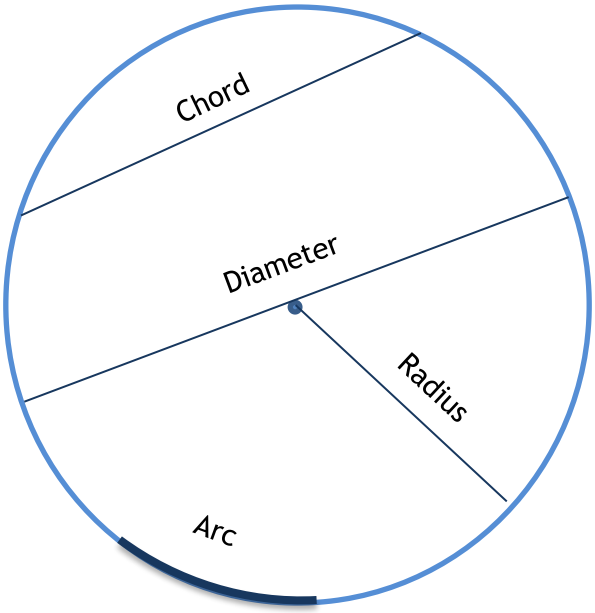
Hence 2 normals intersect at

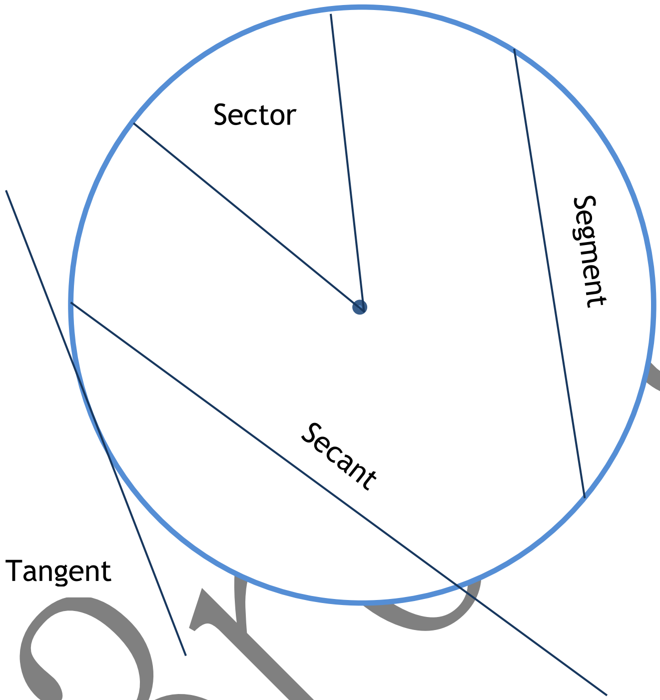
$$(-apq(p+q), a[p^2 + pq + q^2 + 2])$$

Quick Recap

Equation of a Tangent	$y = px - ap^2$
Equation of a Normal	$x + py = 2ap + ap^3$
Equation of a Chord	$y = \frac{1}{2}(p+q)x - apq$
Intersection of 2 Tangents	$(a(p+q), apq)$
Intersection of 2 Normals	$(-apq(p+q), a[p^2 + pq + q^2 + 2])$

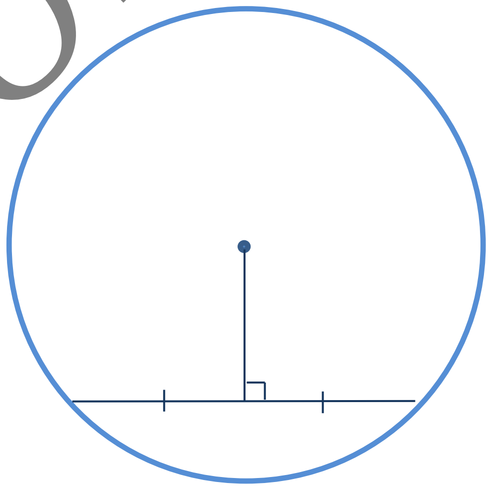
Circle Geometry

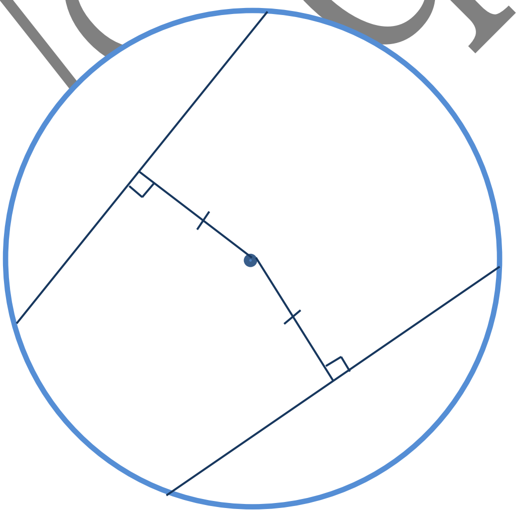




Theorems:

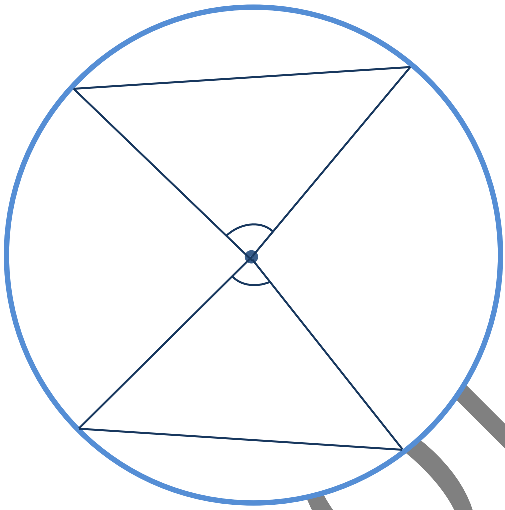
- The perpendicular line from the centre of a circle, bisects the chord
- The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord
- A perpendicular line through the midpoint of a chord passes through the centre
- Key words: Centre, chord and perpendicular. If 2 of the words come up, then the 3rd is also true

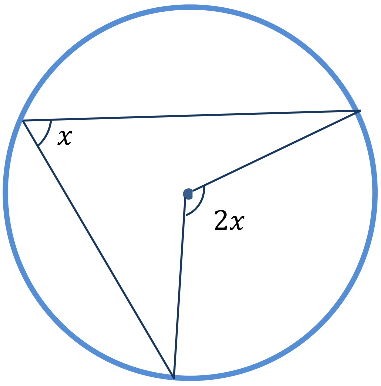


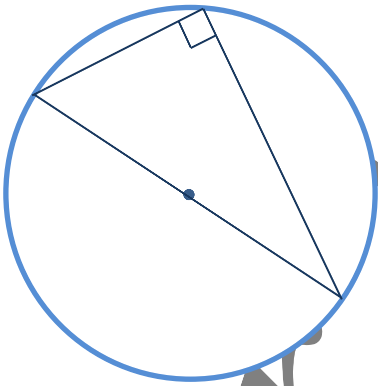


- Equal chords are equidistant from the centre OR chords in a circle which are equidistant from the centre are equal

- Equal angles at the centre stand on equal chords and conversely
- Equal arcs on circles subtend equal angels at the centre and conversely

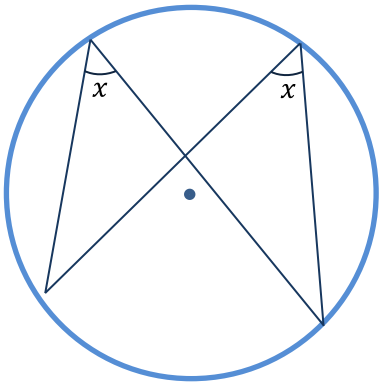


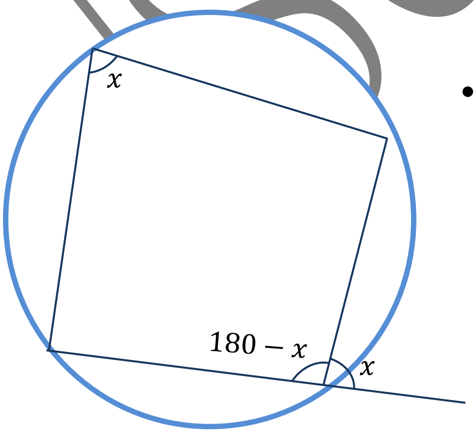




- The angle at the centre is twice the angle at the circumference subtended by the same arc
- Furthermore, angle in a semi-circle is a right angle

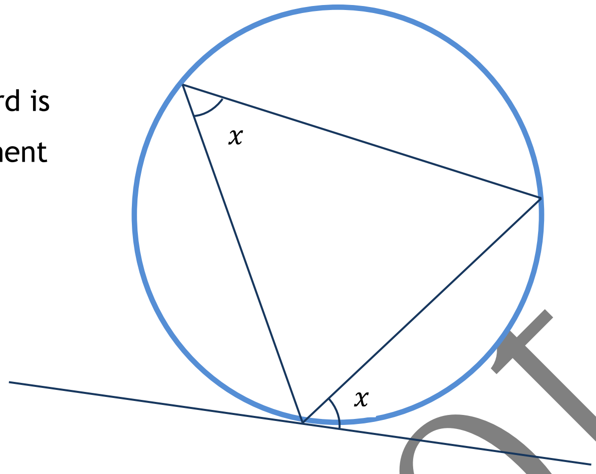
- Angles standing in the same arc are equal and conversely
- Equal angles that subtend on the same are equal and conversely
- Can show that 4 points are concyclic from this

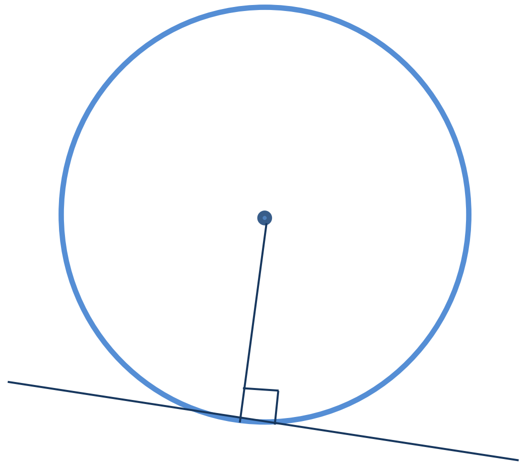




- Opposite angles of a cyclic quadrilateral are supplementary (add to 180°)
- Interior angles equals the exterior opposite angle

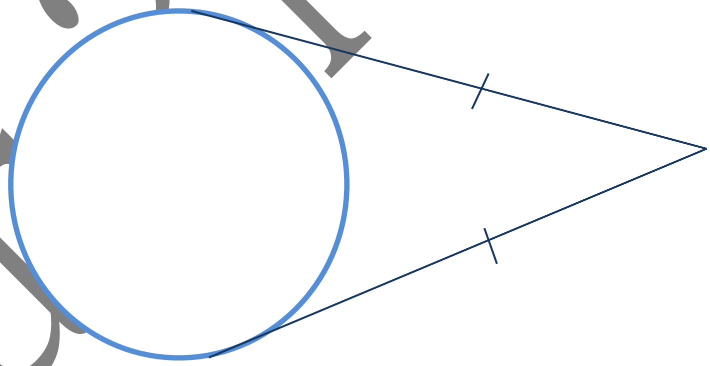
- The angle between a tangent and a chord is equal to the angle in the alternate segment

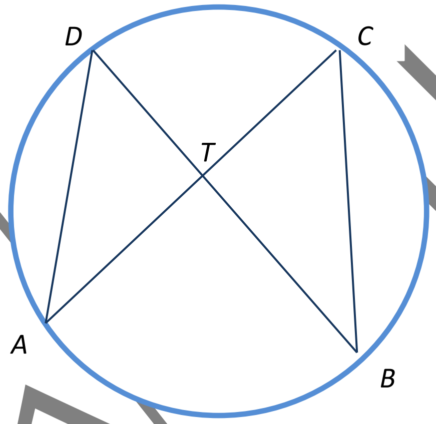




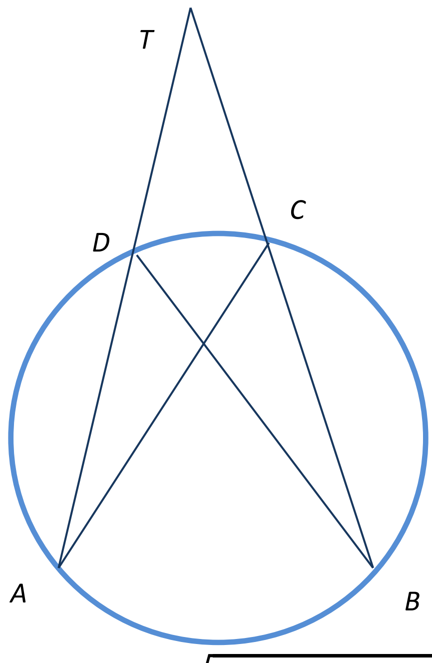
- The angle between the radii and a tangent is 90°

- Tangents to a circle from an external point are equal

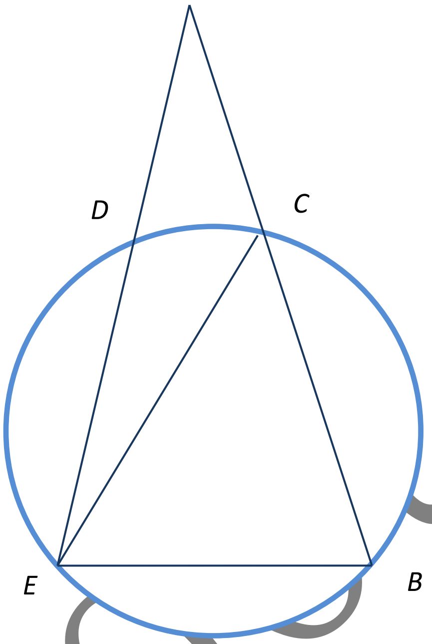




- The product of the intercepts of 2 intersecting chords are equal
- $AT \times CT = BT \times DT$



- Similarly, even when 'T' lies outside the circle
- $TA \times TD = TB \times TC$



- From the above example, as $D \to A$ we get the point E
- This makes it $TE^2 = TB \times TC$

Some general points

- Usually students don't have a problem with memorising the theorems but rather not being able to apply them to a specific question simply because they couldn't "see it." This reiterates the notion that problems and questions should be practised on a regular basis to ensure that you are still able to solve these problems
- Draw big clear diagrams in pen (you're not paying for the paper :P)
- Once you have drawn the diagram, mark all the given information in pen and the inferred information in pencil. This will allow you to have a nice clean diagram and keep your work tidy if you make a mistake
- When writing something down (eg $\angle ABC = \angle DEF$), always provide a reason (eg, equal angles on the same arc) to show the marker you're not just guessing

Integration

$$\frac{d}{dx}[sinx] = cosx \qquad \frac{d}{dx}[tanx] = \sec^2 x \qquad \frac{d}{dx}[secx] = secxtanx$$

$$\frac{d}{dx}[cosx] = sinx \qquad \frac{d}{dx}[cotx] = -cosec^2 x \qquad \frac{d}{dx}[cosecx] = -cosecxcotx$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = cosec^2\theta$$

$$\sin^2 x = \frac{1}{2}(1 - cos2x) \qquad cos^2 x = \frac{1}{2}(1 + cos2x)$$

Simple Integrands:

$$\int \cos^2\frac{x}{3}dx = \int \frac{1}{2}(1 + cos\frac{2x}{3}) = \frac{x}{2} + \frac{3}{4}sin\frac{2x}{3} + c$$

$$\int \tan^2 2x\,dx = \int (\sec^2 2x - 1)\,dx = \frac{1}{2}tan2x - x + c$$

Using Substitution:

1. Use/Pick the substitution
2. Differentiate or find du
3. Change limits (if necessary)
4. Use an appropriate method for the integration
5. Sub in limits *OR* replace with the initial substitution

Example: utilising the substitution $u = 2x^2 + 1$, or otherwise, Evaluate:

$$\int_0^1 x(2x^2 + 1)^{\frac{5}{4}}$$

1. Let $u = 2x^2 + 1$
2. $du = 4x\,dx \to \frac{du}{4} = xdx$
3. At $x = 0; u = 1$ | At $x = 1;\ u = 3$
4. $\therefore \int_1^3 u^{\frac{5}{4}}\frac{du}{4} = \frac{1}{9}[u^{\frac{9}{4}}]_1^3$
5. $\frac{1}{9}\left[(3)^{\frac{9}{4}} - (1)^{\frac{9}{4}}\right] = 2.85$

Induction

Some General Points:

- There are 5 types of induction
    1. Summations
        - Used when adding up an infinite number of terms
    2. Divisibility
        - Proving that something is divisible by a positive integer
    3. Inequalities
        - Proving that something is greater than (or equal to) something else
    4. Other Types
        - Require an inquisitive method and have a different operating function in Step 4
    5. Harder Types
        - Hard applications of any of the above
- $T(n)$ is the left side and $S(n)$ is the right side
- Marks are usually not awarded for earlier steps; rather they are given in the actual algebraic proof (Step 4/5)
- If you are stuck somewhere in the proof, ask yourself, "Have I used the assumption somewhere?"
- $1^2 + 2^2 + 3^2 + \cdots + n^2$ can be written as $\sum_{r=1}^{n} r^2$ This is called sigma notation
- The general steps involved and what they mean
    1. Define $S(n)$
        - Rewrite the question, nicely
    2. Let $n = 1$ (or first non-restriction)
        - Seeing if what you have to prove works (it always 100% will)
    3. Assume true for $n = k$ [AKA. The Assumption]
        - Replace the n's with k's
    4. Show true for $n = k + 1$
        - Though algebra, to show that it works for all integers
    5. Statement
        - A quick sentence summarising what you've found from the induction (not really necessary, just shows the marker you can use your time well)

Summations:

- General steps:
    1. Define $S(n)$
    2. Let $n = 1$ (or first non-restriction)
    3. Assume true for $n = k$ [AKA. The Assumption]
    4. Show true for $n = k + 1$
    5. Show $S(k+1) = S(k) + T(k+1)$
    6. Statement:

Example: Prove by mathematical induction that:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

for all $n \geq 1$

How to write it in an exam:

1. *Define* $S(n)$*:*
    - To prove; $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for $n \geq 1$
2. *Let* $n = 1$*:*
    - $LHS = 1^2 = 1$ $\quad RHS = \frac{1}{6}(1)(2)(3) = 1$ $\quad \therefore LHS = RHS$
3. *Assume true for* $n = k$:
    - $S(k) = \frac{1}{6}(k)(k+1)(2k+1)$
4. *Show true for* $n = k + 1$*:*
    - $S(k+1) = \frac{1}{6}[k+1]([k+1]+1)(2[k+1]+1)$

      $= \frac{1}{6}(k+1)(2k+3)(k+2)$
5. *Show* $S(k+1) = S(k) + T(k+1)$:
    - Now; $S(k+1) = S(k) + T(k+1)$

      $$\begin{aligned} S(k+1) &= \frac{1}{6}(k)(k+1)(2k+1) + (k+1)^2 \\ &= \frac{1}{6}(k+1)[(k)(2k+1) + 6(k+1)] \\ &= \frac{1}{6}(k+1)[2k^2 + 7k + 6] \\ &= \frac{1}{6}(k+1)(2k+3)(k+2) \\ &= S(k+1) \end{aligned}$$
6. *Statement*:
    - Hence, by the principle of mathematical induction $S(n)$ is true for all $n \geq 1$

Divisibility:

- General Steps:
    1. Define S(n)
    2. Let $n = 1$
    3. Assume true for $n = k$ [AKA. The Assumption]
    4. Show true for $n = k + 1$
    5. Statement

Example: Prove by mathematical induction that: $7^n - 1$ is divisible by 6 for all $n \geq 1$

How to write it in an exam:

1. *Define* $S(n)$:
    - To prove: $7^n - 1$ is divisible by 6 for all $n \geq 1$
2. *Let* $n = 1$
    - $S(1) = 7^1 - 1 = 6$ which is divisible by 6
3. *Assume true for* $n = k$
    - $S(k) = 7^k - 1 = 6M$ *where M is an integer*

      $7^k = 6M + 1$ [AKA. The Assumption]
4. *Show true for* $n = k + 1$
    - $S(k + 1) = 7^{k+1} - 1$

      $= 7^k \times 7^1 - 1$

      $= (6M + 1) \times 7 - 1$ [Using assumption]

      $= 42M + 6$

      $= 6(7M + 1)$ [Factorise the divisible constant]

      $= 6Q$ Where Q is an integer
5. *Statement:*
    - Hence, by the principle of mathematical induction, $7^n - 1$ is divisible by 6 for all $n \geq 1$

FUN FACT: $(x + 1)^n - 1$ is always divisible by x. Maybe you can prove this with induction?

Inequalities:

- General Steps:
    1. Define $S(n)$
    2. Let $n = 1$
    3. Assume true for $n = k$ [AKA. The Assumption]
    4. Show true for $n = k + 1$
    5. Statement

Example: Prove by mathematical induction that: $3^n > 2n + 1$ for all $n > 1$

How to write it in an exam:

1. *Define* $S(n)$
    - To prove: $3^n > 2n + 1$
2. *Let* $n = 2$ [Note restriction]
    - $LHS = 3^2 = 9$ $RHS = 2(2) + 1 = 5$ $\therefore LHS > RHS$
3. *Assume true for* $n = k$
    - $3^k > 2k + 1$
4. *Show true for* $n = k + 1$
    - $3^{(k+1)} > 2(k + 1) + 1$

      $> 2k + 3$

      Now $LHS = 3^{(k+1)}$

      $= 3^k \times 3^1$

      $> 3 \times (2k + 1)$ [Using assumption]

      $= 6k + 3$

      $> 2k + 1$ [As $6k > 2k$ and $3 > 1$]

      $> RHS$
5. *Statement*
    - Hence, by the principle of mathematical induction, $3^n > 2n + 1$ for all $n > 1$

Other Types:

- General Steps:
  1. Define $S(n)$
  2. Let $n = 1$
  3. Assume true for $n = k$ [AKA. The Assumption]
  4. Show true for $n = k + 1$
  5. Statement

Example: Prove by mathematical induction that:

$$\left(\frac{1 \times 5}{3^2}\right)\left(\frac{3 \times 7}{5^2}\right)\dots\left(\frac{(2n-1)(2n+3)}{(2n+1)^2}\right) = \frac{2n+3}{3(2n+1)}$$

for all $n \geq 1$

1. *Define* $S(n)$
   - To prove: $\left(\frac{1\times5}{3^2}\right)\left(\frac{3\times7}{5^2}\right)\dots\left(\frac{(2n-1)(2n+3)}{(2n+1)^2}\right) = \frac{2n+3}{3(2n+1)}$
2. *Let* $n = 1$
   - $LHS = \left(\frac{1\times5}{3^2}\right) = \frac{5}{9}$ $\quad RHS = \frac{2(1)+3}{3(2(1)+1)} = \frac{5}{9}$ $\quad \therefore LHS = RHS$
3. *Assume true for* $n = k$
   - $\left(\frac{1\times5}{3^2}\right)\dots\left(\frac{(2k-1)(2k+3)}{(2k+1)^2}\right) = \frac{2k+3}{3(2k+1)}$
4. *Show true for* $n = k + 1$
   - $S(k+1) = \frac{2(k+1)+3}{3(2(k+1)+1)} = \frac{2k+5}{6k+9}$

     Now, $S(k+1) = S(k) \times T(k+1)$ [Note how this is different to the others']

$$= \frac{2k+3}{3(2k+1)} \times \left(\frac{(2(k+1)-1)(2(k+1)+3)}{(2(k+1)+1)^2}\right)$$

$$= \frac{2k+3}{3(2k+1)} \times \frac{(2k+1)(2k+5)}{(2k+3)^2}$$

$$= \frac{2k+5}{6k+9}$$

$$= S(k+1)$$

5. *Statement*
   - Hence, by the principle of mathematical induction,

     $\left(\frac{1\times5}{3^2}\right)\left(\frac{3\times7}{5^2}\right)\dots\left(\frac{(2n-1)(2n+3)}{(2n+1)^2}\right) = \frac{2n+3}{3(2n+1)}$ for all $n > 1$

Harder Types:

- General Steps:
  1. Define $S(n)$
  2. Let $n = 1$
  3. Assume true for $n = k$ [AKA. The Assumption]
  4. Show true for $n = k + 1$
  5. Statement

Example: Prove by mathematical induction that: $\sin(n\pi + x) = (-1)^n sinx$ for all $n \geq 1$

1. *Define* $S(n)$
   - To prove: $sin(n\pi + x) = (-1)^n sinx$ for all $n \geq 1$
2. *Let* $n = 1$
   - $LHS = \sin(\pi + x) = -sinx \qquad RHS = -sinx \qquad \therefore LHS = RHS$
3. *Assume true for* $n = k$
   - $sin(k\pi + x) = (-1)^k sinx$
4. *Show true for* $n = k + 1$
   - $S(k + 1) = sin([k + 1]\pi + x) = (-1)^{(k+1)} sinx$

     Now, $LHS = sin([k\pi + x] + \pi)$

     $= sin(k\pi + x)\, cos\pi + cos(k\pi + x)\, sin\pi$

     $= -\, sin(k\pi + x) + 0$

     $= (-1)(-1)^k sin\, x$ [Using Assumption]

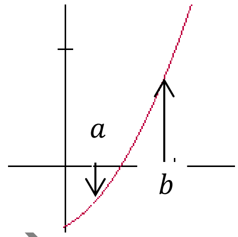
     $= (-1)^{k+1} sinx$

     $= RHS$
5. *Statement*
   - Hence, by the principle of mathematical induction, $\sin(n\pi + x) = (-1)^n sinx$ for all $n \geq 1$

This is pretty easy once you see the steps involved (which is the hard bit)

Approximating Roots

Bisection Method (AKA. Halving the Interval)



We can clearly see that this graph has a root between 'a' and 'b' so can utilise this property to find roots closer to the real value

Example: Let $f(x) = e^x - x - 2$

a) Show that there is a root between $x = 1$ and $x = 1.5$ on $f(x)$
b) Use one application of the bisection method to find a smaller interval for the root

a) To show a root has a root in a specified domain; we must show that there is a change in polarity of the y value around the possible root, so:

$$f(1) = e^1 - 1 - 2 = -0.281718\ldots(-ve)$$

$$\text{and } f(1.5) = e^{1.5} - 1.5 - 2 = 0.981689\ (+ve)$$

Since there is a change in sign, and the curve is continuous, a root must lie between $x = 1$ and $x = 1.5$

b) The first thing we need to do is average the x values. We get: $\frac{1+1.5}{2} = 1.25$

Now we need to substitute this value into $f(x)$

$$f(1.25) = e^{1.25} - 1.25 - 2 = 0.24034\ldots$$

Since $f(1.25) > 0$ we reject $x = 1.5$ as $x = 1.25$ is closer to the solution. Hence; the root lies in the interval $1 < x < 1.25$

- If you are not sure whether you are "doing it right" test the value you got and put it onto the function. It *must* be closer to 0

Newton's Method

$$x = x_o - \frac{f(x_o)}{f'(x_o)}$$

Example: let $f(x) = e^x - x - 2$. Beginning with $x = 1$, use one application of Newton's method to find a better approximation, correct to 4 significant figures

In this case x_o is 1, $f(x) = e^x - x - 2$ and $f'(x) = e^x - 1$

Now, $f(1) = e^1 - 1 - 2 = -0.281718\ldots$ (Calculator memory A)

And $f'(x) = e^1 - 1 = 1.71828\ldots$ (Calculator memory B) and substituting into the formula:

$$x = 1 - \frac{-0.281718}{1.71828}$$

$$= 1.16395$$

$$= 1.164 \text{ (Correct to 4 sig. fig.)}$$

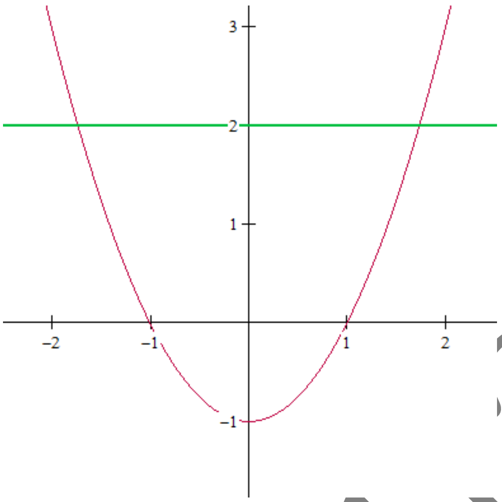
- Don't use this technique if the x value is close to a stationary point. This will result in an invalid answer

Inverse Functions

General

- An inverse function is created by interchanging the x and y values
- The *domain* of $f(x)$ become the *range* of $f^{-1}(x)$
- The *range* of $f(x)$ become the *domain* of $f^{-1}(x)$
- The graphs $f(x)$ and $f^{-1}(x)$ are reflections about the line $y = x$

- Every function has inverse *BUT* that inverse may not necessarily be a function as well. To show that an function has an inverse function, it must:
  - Pass the horizontal line test (only cuts $f(x)$ once in the horizontal direction)
  - Be monotone



- Consider the graph $y = x^2$, the horizontal line cuts it twice. This means that the graph does not have an inverse function.
- We would need to restrict the domain of $f(x)$ if we want to make this an inverse function
- So in this case, $f(x)$ is an inverse function for $x \geq 0$ *OR* $x \leq 0$ because it is only increasing or decreasing in the domains respectively

- To *prove* a function has an inverse; show that it is monotone by proving that it is an only increasing function *OR* an only decreasing function (by taking the derivative)
- Alternatively, you can graph the function and draw a horizontal line and show it only cuts it once. Don't draw some random graph and say "hence, this is true"
- To *disprove* a function has an inverse; show that a horizontal line cuts it twice OR that the function is not monotone (polytone) (basically do the opposite of above dot points)
- To find the inverse of a function, interchange the x's and y's and solve for y
- To find where a function meets it inverse (if any) solve the function simultaneously with x so, $f(x) = x$

Let us consider $f(x) = e^{2x}$

a) Show that an inverse function exists for $f(x)$ [*1* mark]
b) State the domain and range of $f(x)$ and $f^{-1}(x)$ [*1* mark]
c) Find the inverse function [2 mark]
d) Find the point(s), if any, were $f(x)$ and $f^{-1}(x)$ intersect [2 mark]
e) Hence, sketch $f(x)$ and $f^{-1}(x)$ on the same axes [2 mark]

a) $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

Now $2e^{2x} > 0$

Hence $f(x)$ is an only increasing function and hence an inverse function exists

b) The domain of $f(x)$ is $all\ real\ x$

The range of $f(x)$ is $y > 0$

So this means that the domain of $f^{-1}(x)$ is $y > 0$

And the range of $f^{-1}(x)$ is $all\ real\ x$

Essentially we are just interchanging the domain and range with the functions

c) Let $y = e^{2x}$

Now $x = e^{2y}$ [interchanging the x's and y's]

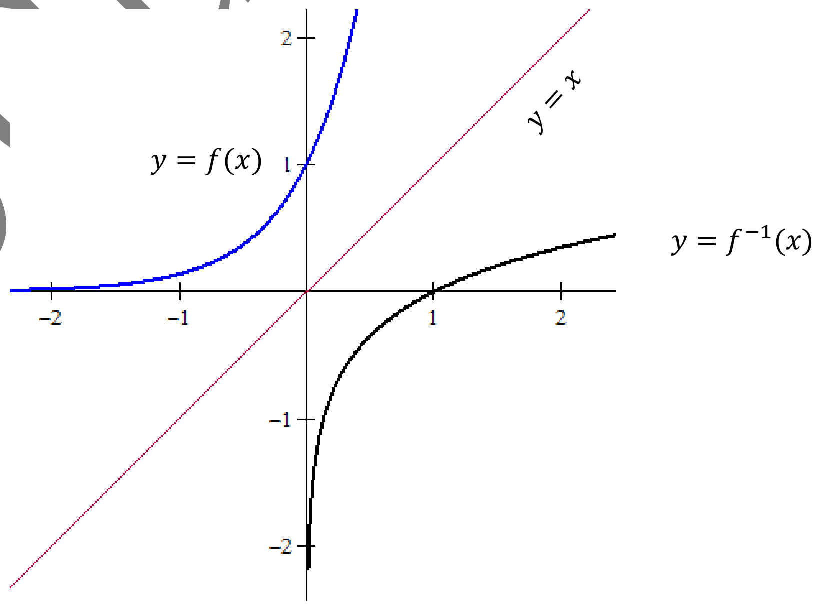
Solving for y we get: $y = \frac{1}{2}lnx$

Hence $f^{-1}(x) = \frac{1}{2}lnx$

d) To solve simatenousely, $e^{2x} = x$

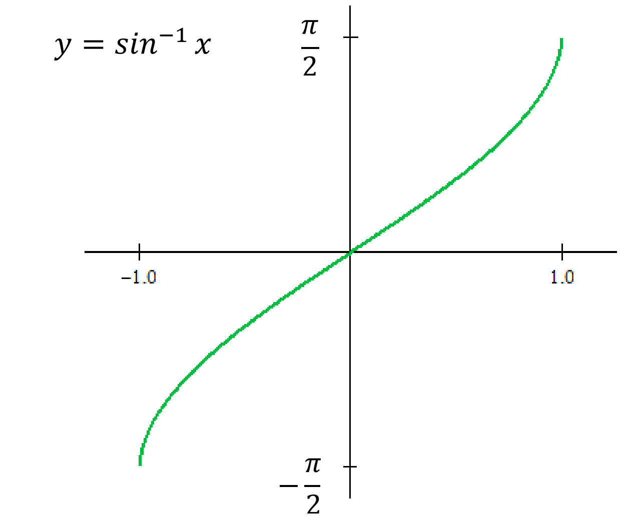
This equation cannot be solved so the function does not meet its inverse

e)



Trig

Graphs of Inverse trig functions



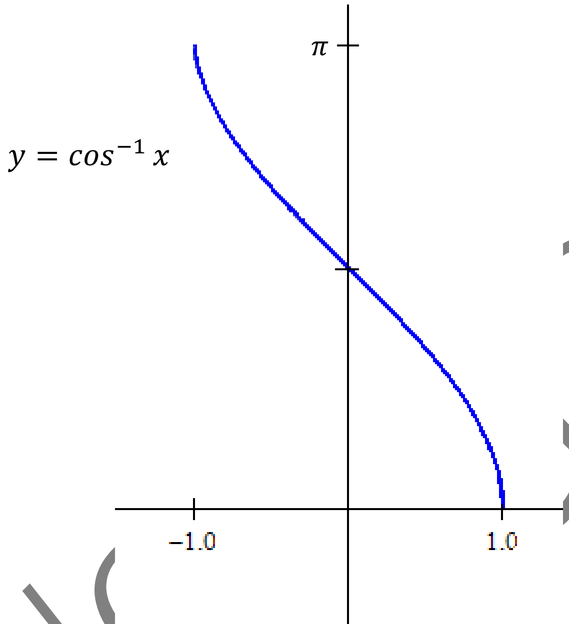
$$y = sin^{-1} x$$

Domain: $-1 \leq x \leq 1$

Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\sin^{-1}(-x) = -\sin^{-1}(x)$ ∴ sine is an odd function

$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$ ∴ sine is an only increasing function



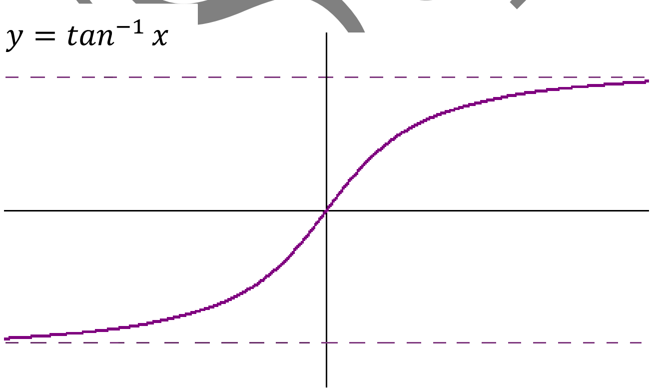
$$y = cos^{-1} x$$

Domain: $-1 \leq x \leq 1$

Range: $0 \leq y \leq \pi$

$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$ ∴ cosine is neither an odd or even function

$\frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}}$ ∴ cosine is an only decreasing function



$$y = tan^{-1} x$$

Domain: $all\ real\ x$

Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ (asymptotes $\pm\frac{\pi}{2}$)

$\tan^{-1}(-x) = -\tan^{-1}(x)$ ∴ tan is an odd function

$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$ ∴ tan is an only increasing function

Graphing inverse trig functions

- The coefficient of x affects the *domain*. To find this, set the stuff with the x squeezed between the usual domain of the function and solve
- The coefficient of the function affects it in the y *range*
- External constants (such as $+\pi$, -7) affect the function in the y range

Example: Graph $y = 3\cos^{-1}\left(\frac{x}{2}\right) - \pi$

Always set it out like this:

Domain: $-1 \leq \frac{x}{2} \leq 1$

$$-2 \leq x \leq 2$$

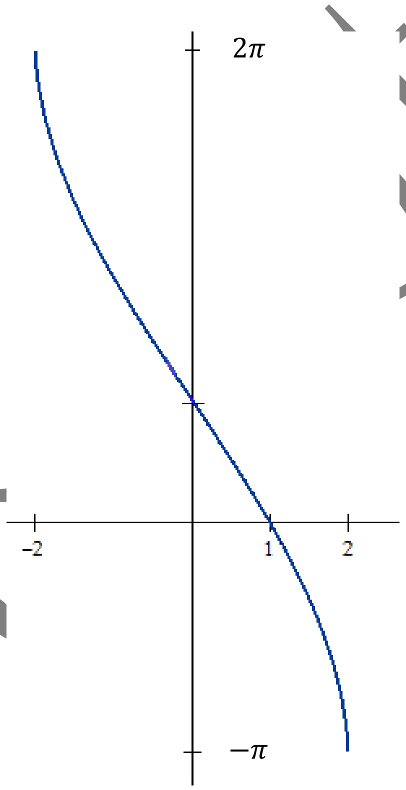
Range: $0 \leq \cos^{-1}(x) \leq \pi$

$$0 \leq \cos^{-1}\left(\frac{x}{2}\right) \leq \pi$$

$$0 \leq 3\cos^{-1}\left(\frac{x}{2}\right) \leq 3\pi$$

$$0 - \pi \leq 3\cos^{-1}\left(\frac{x}{2}\right) - \pi \leq 3\pi - \pi$$

$$-\pi \leq y \leq 2\pi$$



Example: Graph $2\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{\pi}{4}$

Domain: $-1 \leq \frac{x-2}{3} \leq 1$

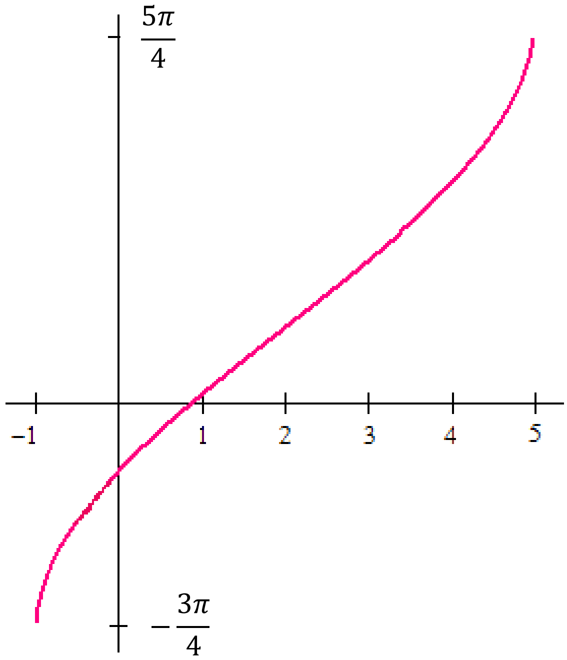
$$-1 \leq x \leq 5$$

Range: $-\frac{\pi}{2} \leq \sin^{-1}\left(\frac{x-2}{3}\right) \leq \frac{\pi}{2}$

$$-\pi \leq 2\sin^{-1}\left(\frac{x-2}{3}\right) \leq \pi$$

$$-\pi + \frac{\pi}{4} \leq 2\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{\pi}{4} \leq \pi + \frac{\pi}{4}$$

$$-\frac{3\pi}{4} \leq y \leq \frac{5\pi}{4}$$



Applications of Calculus to the Physical world

Related Rates

- In this topic there really is nothing new to learn here. It is just further applications of what you already know
- You must be able to use the chain rule effectively by manipulating it in order to solve a problem
- Watch out for small things like unit changes, letters, etc
- Be sure to know all the formulae of basic figures, eg area of a circle, volume of a cone, etc
- Always add units to your answers, so the marker knows what you're talking about

Example: A copper circular plate is heated such that it expands uniformly. When the radius is 30cm, the radius is increasing at 5mm per second. Find the rate of change of the

a) Area
b) Circumference

a) Firstly, write down all the given *and* implied information

Circular plate $\Longrightarrow A = \pi r^2 \rightarrow \frac{dA}{dr} = 2\pi r$

$\Longrightarrow C = 2\pi r \rightarrow \frac{dC}{dr} = 2\pi$

$\Longrightarrow \frac{dr}{dt} = 5mm/sec = 0.5cm/sec$

[Note the unit change. For it to work, you need to use consistent units]

Use the units as a guide to figure out what we want. The question says we want the rate of change of area; so this is $\frac{dA}{dt}$. So far we have $\frac{dr}{dt}$ and we need to set up a chain rule that involves this expression. We get:

$\frac{dA}{dt} = \frac{dr}{dt} \times \frac{dA}{dr}$ Think of the 'dr' as cancelling

Now we have $\frac{dA}{dt} = 0.5 \times 2\pi r = \pi r$

When $r = 30cm$; $\frac{dA}{dt} = 30\pi\ cm^2/second$

$\therefore$ The rate of change of area when $r = 30$ is $30\pi\ cm^2/second$ [Note, the area is increasing because it is positive. If it had a negative in front, this would mean it is decreasing]

b) The rate of change of circumference means $\frac{dC}{dt}$. Now set up another chain rule where we would get this expression:

$$\frac{dC}{dt} = \frac{dr}{dt} \times \frac{dC}{dr}$$

Substituting out values,

$$\frac{dC}{dt} = 0.5 \times 2\pi = \pi$$

Since there is no 'r' variable in $\frac{dC}{dt}$ we do not need to sub in $r = 30$.

The circumference of the plate is expanding uniformly at $\pi\ cm/sec$

Try this question:

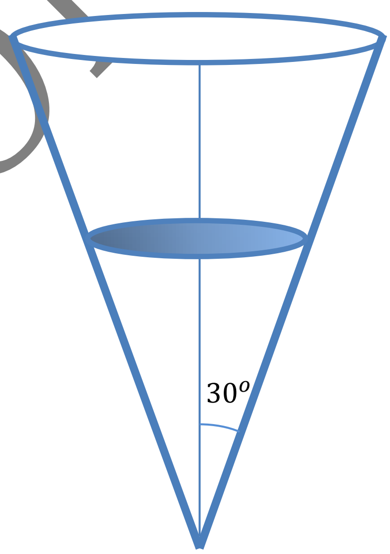
Water is pouring into a conical reservoir at a rate of $2m^3/minute$ whose semi-vertical angle is 30°

Show that:

a) $r = \frac{h}{\sqrt{3}}$

b) $V = \frac{1}{9}\pi h^3$

c) $V = \frac{3}{\sqrt{3}}\pi r^3$



When the water is 3 meters deep; find the rate at which:

d) The water is rising

e) The area of the surface of the water is increasing

f) The area of the curved surface of the reservoir is increasing

Acceleration in terms of x

- *Where* means "where is the particle"
- *When* means "what time does this occur"
- *Initially* means "$t = 0$"
- *Origin* means "$x = 0$"
- *Rest* means "$v = 0$"

$$\frac{d^2x}{dt^2} = \frac{dv}{dt}$$
$$= \frac{dv}{dx} \times \frac{dx}{dt}$$
$$= v\frac{dv}{dx}$$
$$= \frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx}$$
$$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

- There is a choice on which one of these to use. Most of the time $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ is used and $v\frac{dv}{dx}$ is used more in 4U mechanics
- But, sometimes it is not necessary to use any of the above; instead it just requires 'basic' algebra and calculus

$$\boxed{\frac{d^2x}{dt^2} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)}$$

Consider $\frac{d^2x}{dt^2} = 18x(x^2+1)$. Given that when $t = 0$, $x = 0$ and $v = 3$, Find

a) v in terms of x
b) x in terms of t

a) It would be best to use $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ in this case. So,

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 18x(x^2+1)$$

Integrating both sides with respect to dx

$$\int \frac{d}{dx}\left(\frac{1}{2}v^2\right) dx = \int 18x^3 + 18x \, dx$$

$$v^2 = 9x^4 + 18x^2 + c$$

When $x = 0$; $v = 3$

$$(3)^2 = 9(0)^4 + 18(0)^2 + c \Longrightarrow c = 9$$

$$v^2 = 9(x^2+1)^2$$ We have found v in terms of x

b) In this case it would be best to approach this algebraically. So,

$$v^2 = 9(x^2+1)^2$$

$$v = 3(x^2+1)$$ [Taking the positive since $v > 0$ when $x = 0$]

Since we want x in terms of t, it would be best to make use of the $v = \frac{dx}{dt}$ notation. So,

$$\frac{dx}{dt} = 3(x^2+1)$$

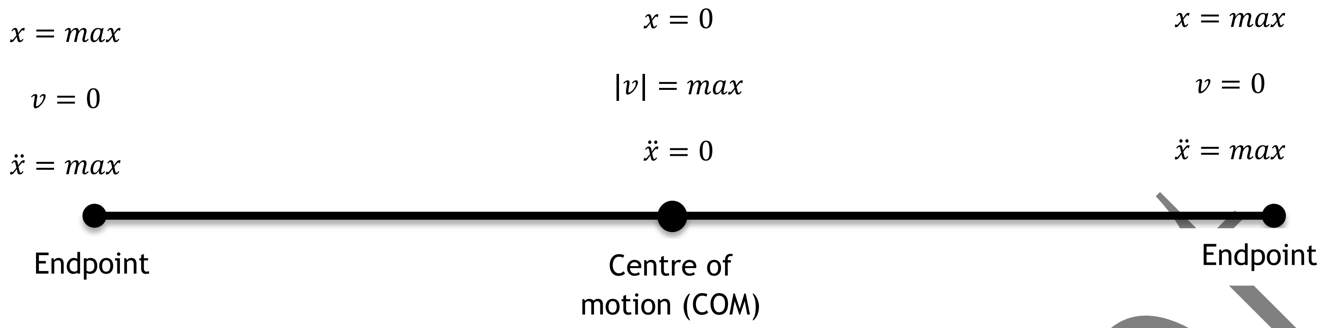
$$\int \frac{dt}{dx} dx = \frac{1}{3}\int \frac{dx}{x^2+1}$$

$$t = \frac{1}{3}\tan^{-1} x + d$$

When $t = 0$; $x = 0$

$$(0) = \frac{1}{3}\tan^{-1} 0 + d \Longrightarrow d = 0$$
$$3t = \tan^{-1} x \Longrightarrow x = \tan(3t)$$

Simple Harmonic Motion



- A particle in simple harmonic motion looks like: $x = a\cos(nt + \alpha)$ or $x = a\sin(nt + \alpha)$. This is assuming that the COM is the origin (x=0) but there may be a constant which shifts the COM
- To prove a particle is undergoing SHM, show that $\ddot{x} = -n^2 x$, ie

$$x = a\sin(nt + \alpha)$$

$$\dot{x} = an\cos(nt + \alpha)$$

$$\ddot{x} = -n^2 \times a\sin(nt + \alpha)$$

$$\ddot{x} = -n^2 x$$

- To find which points the particle is oscillating between, solve $v^2 \geq 0$

Some formulae to memorise:

If the COM is the origin:

$$v^2 = n^2(a^2 - x^2)$$

$$x = a\cos(nt + \alpha) \text{ Or } x = a\sin(nt + \alpha)$$

$$\ddot{x} = -n^2 x$$

If the COM is *not* the origin

$$v^2 = n^2(a^2 - (x - b)^2)$$

$$\ddot{x} = -n^2(x - b)$$

Miscellaneous formulae

$$T = \frac{2\pi}{n} \qquad v_{max} = an \qquad \frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{vdv}{dx}$$

Example:

A particle moves in a straight line and its displacement is given by $x = \cos^2 3t$

a) When is the particle first at $x = \frac{3}{4}$

b) In what direction is the particle travelling in when it is first at $x = \frac{3}{4}$

c) Find an expression for the acceleration of the particle in terms of x

d) Hence show that the particle is SHM

e) What is the period of the particle?

a)

$$\cos^2 3t = \frac{3}{4}$$

$$cos3t = \pm\frac{\sqrt{3}}{2}$$

$$3t = \cos^{-1}\frac{\sqrt{3}}{2}$$

$$3t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \ldots$$

$$t = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}, \ldots$$

The particle is first at $x = \frac{3}{4}$ when

$t = \frac{\pi}{18}$ *seconds*

b)

$$x = \cos^2 3t$$

$$\dot{x} = 2cos3t \times -3sin3t$$

$$\dot{x} = -3sin6t$$

When $x = \frac{3}{4}$, $t = \frac{\pi}{18}$

$$\frac{dx}{dt} = -3sin6\left(\frac{\pi}{18}\right)$$

$$= -0.866025\ldots$$

Since the velocity is negative, the particle is *moving to the left*

c)

$$\dot{x} = -3sin6t$$

$$\ddot{x} = -18cos6t$$

d)

$$\ddot{x} = -18cos6t$$

$$\ddot{x} = -18(2\cos^2 3t - 1)$$

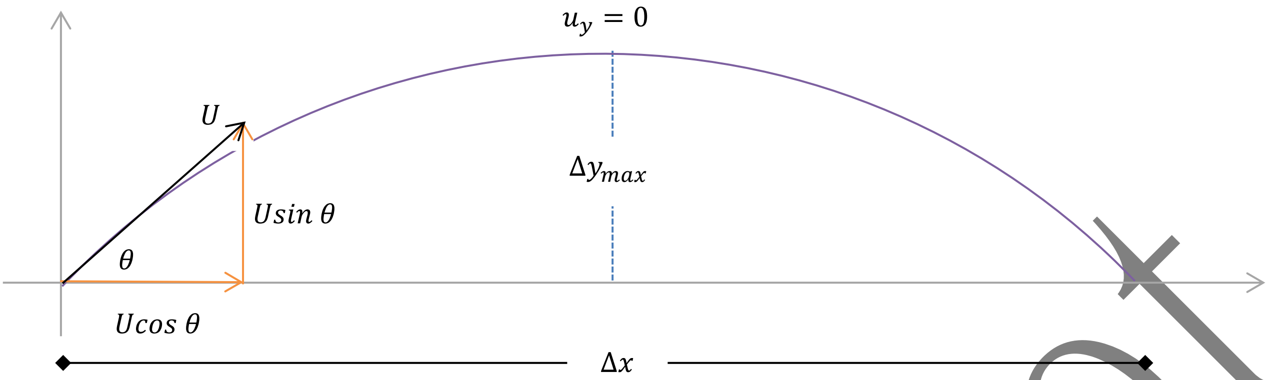
$$\ddot{x} = -18(2x - 1)$$

$$\ddot{x} = -36\left(x - \frac{1}{2}\right)$$

Since $\ddot{x}$ is in the form $-n^2(x - b)$ the particle is SHM

e) $T = \frac{2\pi}{6} = \frac{\pi}{3}$

Projectile Motion



Deriving the equations of motion

Consider *Acceleration*:

There is no acceleration that acts in the horizontal direction

$$\ddot{x} = 0$$

There is only gravity acting in the vertical direction (air resistance is ignored)

$$\ddot{y} = -g$$

Consider *Velocity*: Integrating acceleration with respect to time

$$\dot{x} = c_1$$

$$\dot{y} = -gt + c_2$$

Now using the initial velocity diagram

$$\dot{x} = Ucos\theta \Longrightarrow c_1 = Ucos\theta$$

$$\dot{x} = Ucos\theta$$

$$\dot{y} = Usin\theta$$

$$\therefore Usin\theta = -gt + c_2$$

When $t = 0$

$$Usin\theta = -g(0) + c_2$$

$$\therefore c_2 = Usin\theta$$

$$\therefore \dot{y} = -gt + Usin\theta$$

Consider *Displacement*: Integrating velocity with respect to time

$$x = (Ucos\theta)t + c_3$$

Initially there is no horizontal displacement, so $c_3 = 0$

$$x = Utcos\theta$$

$$\therefore y = -\frac{1}{2}gt^2 + (Usin\theta)t + c_4$$

Initially there is no vertical displacement, so $c_4 = 0$

$$\therefore y = Utsin\theta - \frac{1}{2}gt^2$$

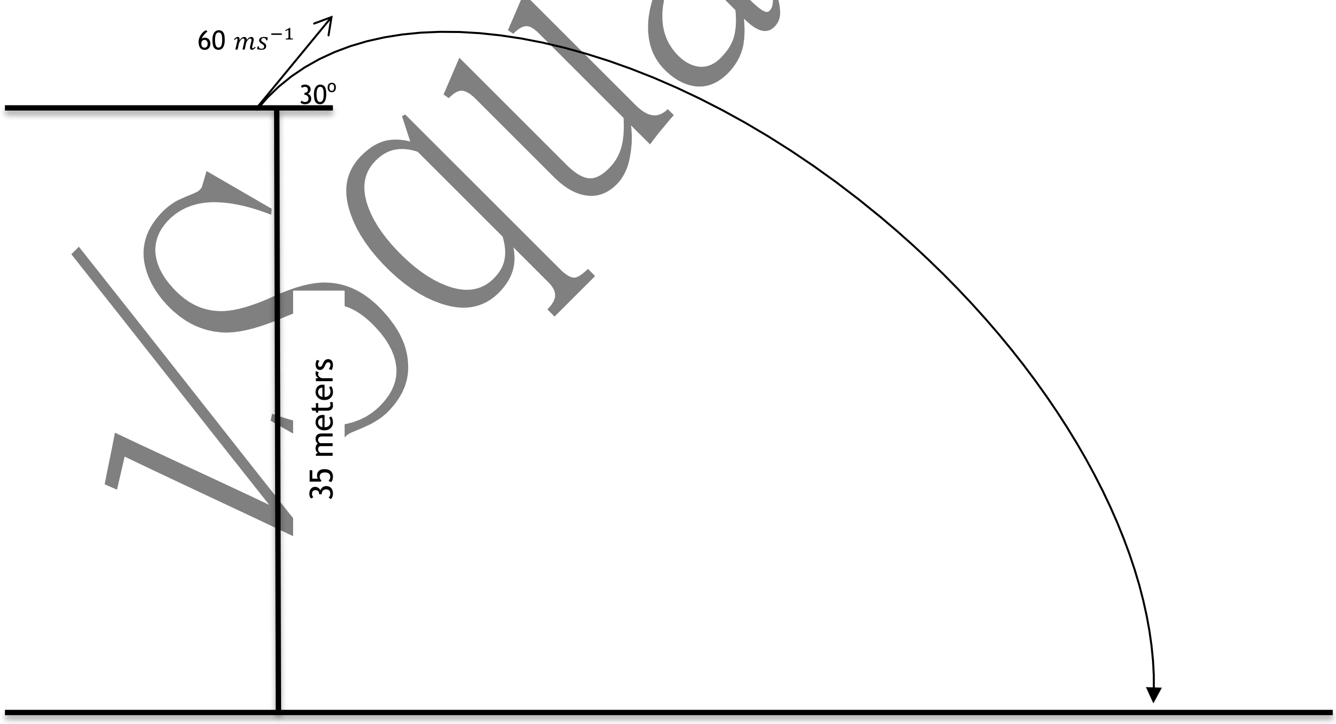
- This derivation is unique for all questions. You should be able to derive it from scratch for each one starting with acceleration, then velocity and finally displacement
- You will either be asked to prove the equations of motion which would be 2-3 marks or you will be given the equations of motion and asked to do something with it
- Do not assume the constants are zero (or something else). Sometimes the projectile is being launched from a building, etc and so initially it does have a vertical displacement

Example:

A particle is projected 35 meters from the top of a building at 30° with a velocity of $60ms^{-1}$ (assume $g = 10ms^{-2}$).

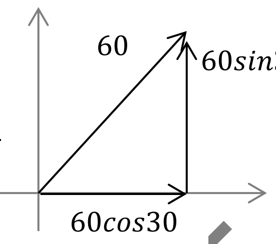
a) Derive the equations of motion
b) Find the equation of flight
c) Find the maximum height
d) Find the time of flight
e) Find the range
f) Find the velocity at which the particle strikes the ground at

With any problem; it helps to draw a diagram



a) In the horizontal direction:

Initial velocity diagram (IVD)



$$\ddot{x} = 0$$

Integrating acceleration with respect to time

$$\dot{x} = c_1$$

From the IVD, we can see that $c_1 = 60cos30 = 30\sqrt{3} \rightarrow \dot{x} = 30\sqrt{3}$

Integrating velocity with respect to time

$$x = \left(30\sqrt{3}\right)t + c_2$$

Initially there is no horizontal displacement so $c_2 = 0$

$$x = \left(30\sqrt{3}\right)t$$

In the vertical direction

$$\ddot{y} = -10$$

Integrating acceleration with respect to time

$$\dot{y} = -10t + c_3$$

From the IVD, we can see that $c_3 = 60sin30 = 30 \rightarrow \dot{y} = -10t + 30$

Integrating velocity with respect to time

$$y = -5t^2 + 30t + c_4$$

Initially there is a displacement of 35 meters in the vertical direction, so: $c_4 = 35$

$y = -5t^2 + 30t + 35$

b) Equation of flight

The formulae for the equations of flight are:

$$x = 30t\sqrt{3} \qquad\qquad y = -5t^2 + 30t + 35$$

Making t the subject of x and substituting it into y

$$t = \frac{x}{30\sqrt{3}} \longrightarrow y = -5\left(\frac{x}{30\sqrt{3}}\right)^2 + 30\left(\frac{x}{30\sqrt{3}}\right) + 35$$

$$y = \frac{x^2}{540} + \frac{x}{\sqrt{3}} + 35$$

In general the equation of flight is:

$$y = xtan\theta - \frac{gx^2}{2U^2}(1 + tan^2\theta)$$

c) The maximum height occurs when $\dot{y} = 0$

$$\therefore -10t + 30 = 0$$

$$t = 3$$

It takes 3 seconds to reach the maximum height

Now the vertical displacement:

$$y = -5(3)^2 + 30(3) + 35 = 80\, meters$$

$\therefore$ The maximum height reached is 80 meters

d) The projectile hits the ground when $y = 0$

$$y = -5t^2 + 30t + 35 = 0$$

Using the quadratic formula:

$$t = \frac{-30 \pm \sqrt{30^2 - 4 \times -5 \times 35}}{2 \times -5}$$

$$t = 7, -1$$

$$t = 7\ (t > 0)$$

$\therefore$ The projectile is in the air for 7 seconds

e) The projectile spends a total of 7 seconds in the air so we can immediately sub this into the formula for the horizontal displacement (range)

$$x = 30\sqrt{3} \times 7$$

$$x = 210\sqrt{3}\, meters$$

f) The particle's instantaneous velocity can be given by:

$$v^2 = \dot{x}^2 + \dot{y}^2$$

$$\dot{x}\ is\ constant\ throughout\ the\ motion$$

The velocity at when it hits the ground is $\dot{y} = -10(7) + 30 = -40ms^{-1}$

$$v^2 = \left(30\sqrt{3}\right)^2 + (-40)^2 = 4300$$

$$v = 65.6\, ms^{-1}$$

Growth and Decay

- This is just a continuation/extension of growth and decay from the 2 unit course.

This should be pretty straight forward, so I'll go through an example:

Square turns on his air conditioner to 20°C and makes his morning coffee and rests it on the kitchen table. He hears the doorbell ring and goes to answer it, forgetting his coffee. After 6 minutes the coffee cooled to 80^oC from 100°C. After another 2 minutes he takes the temperature again and it was 50°C. The coffee cooled in accordance with newton's law of cooling which states: $\frac{dT}{dt} = k(T - A)$

a) Show that $T = A + Be^{kt}$ satisfies the above differential equation
b) Find values for A, B, and k
c) Find the initial temperature
d) It is given that when the coffee was 60°C, the time was 7:22 am. What time did Square make his coffee?

a) $T = A + Be^{kt}$ which implies that $T - A = Be^{kt}$

Now $\frac{dT}{dt} = k \times Be^{kt}$

$= k(T - A)$ As required

b) We know that A is the difference in the temperatures so, $A = 100 - 80 = 20^oC$

When $t = 6\ min; T = 80^oC$ so subbing this in:

$$80 = 20 + Be^{k(6)}$$
$$Be^{6k} = 60$$

When $t = 8\ min; T = 50^oC$ so subbing this in:

$$50 = 20 + Be^{k(8)}$$
$$Be^{8k} = 30$$

Solving simultaneously;

$$\frac{Be^{8k}}{Be^{6k}} = \frac{30}{60}$$
$$e^{2k} = \frac{1}{2}$$
$$k = -\frac{1}{2}ln2$$

Plugging this value back into either equation:

$$Be^{8\left(-\frac{1}{2}ln2\right)} = 30$$

$$Be^{ln\frac{1}{16}} = 30$$

$$B = 480$$

$e^{\ln(x)} = x$

$$A = 20; B = 480 \; and \; k = -\frac{1}{2}ln2$$

$$\therefore T = 20 + 480e^{\left(-\frac{1}{2}ln2\right)t}$$

c) The initial temperature is when $t = 0$

$$\therefore T = 20 + 480e^{\left(-\frac{1}{2}ln2\right)\times 0}$$

$$\therefore T = 20 + 480e^{0} = 500$$

That's one hot cup of coffee

d) How long does it take the coffee to cool to 60°C?

$$\therefore 60 = 20 + 480e^{\left(-\frac{1}{2}ln2\right)t}$$

$$\frac{40}{480} = e^{\left(-\frac{1}{2}ln2\right)t}$$

$$\ln\left(\frac{1}{12}\right) = \left(-\frac{1}{2}ln2\right)t$$

$$t = \frac{\ln\left(\frac{1}{12}\right)}{\left(-\frac{1}{2}ln2\right)}$$

$$t = 7min$$

It takes 7 minutes to cool to 60°C from 7:22 am. So we can subtract this time from the time taken for it to cool, so: 7:22 am – 7 min = 7:15 am

Square has his morning coffee at 7:15am

Binomial Theorem

Permutations and Combinations

General tips:

- Order is important with permutations but not important with combinations
- Make it clear if the question is a permutation or combination. Some key words are distinct, committee, etc
- These types of questions require "critical thinking" rather than just applying a formula

Permutations:

Example: How many ways can Adam, Blake, Charlie, David and Earl be arranged in a line?

Well there are 5 people so ask ourselves, how many people can sit in the first seat?

Well there can be 5 people in that seat

5				

Now, how many people can sit in the next seat? Since 1 is already chosen; there are 4 remaining people, so:

5	4			

Continuing this pattern we can fill up the table as:

5	4	3	2	1

So the total number of ways is $5 \times 4 \times 3 \times 2 \times 1 = 120$. This can be written as $5! = 120$

What of there were only 3 seats?

We would continue with the same method as above

5	4	3

So the total number of ways is $5 \times 4 \times 3 = 60$. This can be written as ${}^{5}P_{3}$

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

How many ways can the word ENGINEER be arranged?

There are 8 letters and 'E' repeats 3 times and 'N' repeats 2 times, so:

$$\frac{8!}{3!\,2!} = 10080\, ways$$

- This method can also be used in other circumstances such as "if there are 3 blue cars, 4 red cars and 5 white cars, how many ways can they be arranged?" $\frac{12!}{3!4!5!}$

Example: Consider 6 ordinary people, of which 2 are Adam and Blake. How many ways can they be arranged if there:

a) Are no restrictions
b) Adam and Blake sit at the ends
c) Adam and Blake sit next to each other
d) Adam and Blake *do not* sit next to each other

a) Pretty straight forward, 6 people, so: $6! = 720\, ways$
b) Firstly, draw a table:

A	4	3	2	1	B

We can see that if Adam and Blake take the ends, there are $4!$ Ways the others can sit. Furthermore, Adam and Blake can switch places so they can be arranged in $2!$ ways. So the final answer is $4! \times 2! = 48\, ways$

c) In this case we should treat Adam and Blake as one element ("one person")

AB	4	3	2	1

Now, there are $5!$ ways to arrange these elements. But, Adam and Blake can be switched around so the total number of ways is $5! \times 2! = 240\, ways$

d) The number of ways that Adam and Blake can sit together is 240 ways, and the total number of ways everyone can sit is in 720 ways. So if Adam and Blake don't sit together, the total number of ways is $720 - 240 = 480$ ways

Combinations:

Example: Consider 5 people, Adam, Blake, Charlie, David and Earl. How many groups can be made if there are;

a) 5 people in each group
b) 3 people in each group?

a) Well, there are only 5 people so we can only make 1 group
b) We need to use a new notation called 'C' (universally it's not called this, but I can find a proper name for it). It is given by ${}^{n}C_r = \frac{{}^{n}P_r}{r!} = \frac{n!}{r!(n-r)!}$ Where 'n' is the sample space and 'r' is the restriction. So if we apply it to the above case, we get: ${}^{5}C_3 = 10\ ways$

Example: 6 students, 2 of which are Abigail and Beatrice are to be selected from 9 students. How many groups are possible if

a) No restriction applies
b) Beatrice is excluded
c) Abigail and Beatrice are always in the group
d) Beatrice is included and Abigail is excluded

a) This is straight forward; ${}^{9}C_6 = 84\ ways$
b) Since Beatrice is not included there are 8 more students to choose from, so: ${}^{8}C_6 = 28\ ways$
c) Since Abigail and Beatrice are always in the group, this means that there are $(6-2) = 4$ more students to choose from $(9-2) = 7$ students, so: ${}^{7}C_4 = 35\ ways$
d) Is Beatrice is already included this means that there are 8 students to now choose from, and Abigail is excluded so there now 7 students to choose from, but Beatrice has already been chosen, so: ${}^{7}C_5 = 21\ ways$

Arrangements in a ring: $(n-1)!$

General

Pascals Triangle:

$n = 0$	${}^{0}C_0$ (1)				
$n = 1$	${}^{1}C_0$ (1)	${}^{1}C_1$ (1)			
$n = 2$	${}^{2}C_0$ (1)	${}^{2}C_1$ (2)	${}^{2}C_2$ (1)		
$n = 3$	${}^{3}C_0$ (1)	${}^{3}C_1$ (3)	${}^{3}C_2$ (3)	${}^{3}C_3$ (1)	
$n = 4$	${}^{4}C_0$ (1)	${}^{4}C_1$ (4)	${}^{4}C_2$ (6)	${}^{4}C_3$ (4)	${}^{4}C_4$ (1)
$n = \cdots$	……				

- The number of terms in each row is $(n+1)$
- The sum of each row is 2^n

Using Pascal's triangle, or otherwise, expand $(1+x)^4$

$$(1+x)^4 = {}^{4}C_0(1)^4(x)^0 + {}^{4}C_1(1)^3(x)^1 + {}^{4}C_2(1)^2(x)^2 + {}^{4}C_3(1)^1(x)^3 + {}^{4}C_4(1)^0(x)^4$$

$$= 1 + 4x + 6x^2 + 4x^3 + x^4$$

Generally, the expansion of

$$(x+y)^n = {}^{n}C_0(x)^0(y)^n + {}^{n}C_1(x)^1(y)^{n-1} + {}^{n}C_2(x)^2(y)^{n-2} + \cdots + {}^{n}C_{n-1}(x)^{n-1}(y) + {}^{n}C_n(x)^n(y)^0$$

This can be written as:

$$(x+y)^n = \sum_{r=0}^{n} {}^{n}C_r\, x^r y^{n-r}$$

Some relationships you should memorize:

- ${}^{n}C_r = {}^{n}C_{n-r}$
- ${}^{n}C_r + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$

General expansion and terms (including independent)

Consider the operation: $\left(3x^2 - \frac{1}{x}\right)^{18}$

a) Find the coefficient of x^6
b) Find the term independent of x

a) Firstly we need to "generate" a general expansion. This can be done using the theory:

$$\sum_{r=0}^{n} {}^{n}C_r \, x^r y^{n-r}$$

So applying this to our specific operation, we get:

$$^{18}C_r (3x^2)^r \left(-\frac{1}{x}\right)^{18-r}$$

$$^{18}C_r 3^r x^{2r} (-1)^{18-r} (x)^{-(18-r)}$$

$$^{18}C_r 3^r (-1)^{18-r} x^{3r-18}$$

Now we need to equate the degree of the x with the coefficient associated with the degree we want which is 6, so:

$$3r - 18 = 6$$

$$r = 8$$

Now we can plug this value for r into the generator and we get:

$$^{18}C_8 3^8 (-1)^{18-8} x^6$$

So the coefficient is $^{18}C_8 3^8 = 287096238$

Hopefully you can see that this process is important, you don't want to be there expanding a degree 18 factor

b) This is a more commonly asked question. Independent of x simply means the constant term. So to do this, we need to set the degree of x equal to zero, so:

$$3r - 18 = 0$$

$$r = 3$$

So the term independent of x is:

$$^{18}C_3 3^3 (-1)^{18-3} x^0 = -22032$$

Probability

Generally the formula is given by:

$$P(X = r) = {}^{n}C_{r} s^{r} f^{n-r}$$

Where $s = success$ and $f = failure$

Example: Consider rolling a fair 6 sided die. What is the probability that:

a) I roll *exactly* three 5's in four rolls
b) I roll *at least* four 5's in six rolls

a) The probability of rolling a 5: $P(5) = \frac{1}{6}$

The probability of *not* rolling a 5: $P(not\ a\ 5) = \frac{5}{6}$

Now; to roll exactly three 5's in 4 rolls: $P(5) = {}^{4}C_{3}\left(\frac{1}{6}\right)^{3}\left(\frac{5}{6}\right)^{4-3} = \frac{5}{324}$

b) At least means we can roll four 5's, five 5's or six 5's, so:

$$P(four\ 5's) = {}^{6}C_{4}\left(\frac{1}{6}\right)^{4}\left(\frac{5}{6}\right)^{6-4} = \frac{125}{15552}$$

$$P(five\ 5's) = {}^{6}C_{5}\left(\frac{1}{6}\right)^{3}\left(\frac{5}{6}\right)^{6-5} = \frac{5}{7776}$$

$$P(six\ 5's) = {}^{6}C_{6}\left(\frac{1}{6}\right)^{6}\left(\frac{5}{6}\right)^{6-6} = \frac{1}{46656}$$

$$P(four\ 5's) + P(five\ 5's) + P(six\ 5's) = \frac{203}{23328}$$

General tips;

- Watch for language, (eg, 'exactly' and 'at least')

Proofs

- These pop up in the later part of the paper, usually around question 6-7 (now Q13-14)
- They require a lot of insight and experimenting (eg HSC 2003 Q7)
- ${}^{n}C_r$ is the same thing as $\binom{n}{r}$ without the line in the middle

Example: by writing $(1+x)^{2n}$ as $(1+x)^n(1+x)^n$ and considering the coefficient of x^n on both sides, deduce:

$$\sum_{r=0}^{n} ({}^{n}C_r)^2 = {}^{2n}C_n$$

The coefficient of x^n in $(1+x)^{2n}$ is ${}^{2n}C_n$

Expanding $(1+x)^n(1+x)^n$ we get:

$$[{}^{n}C_0 + {}^{n}C_1 x + \cdots + {}^{n}C_{n-1}x^{n-1} + {}^{n}C_n x^n][{}^{n}C_0 + {}^{n}C_1 x + \cdots + {}^{n}C_{n-1}x^{n-1} + {}^{n}C_n x^n]$$

Now; terms in x^n are given by multiplying the following terms together:

$${}^{n}C_0 \times {}^{n}C_n x^n + {}^{n}C_1 x \times {}^{n}C_{n-1}x^{n-1} + \cdots + {}^{n}C_{n-1}x^{n-1} \times {}^{n}C_1 x + {}^{n}C_0 \times {}^{n}C_n x^n$$

Using the relationship ${}^{n}C_r = {}^{n}C_{n-r}$ and only considering the coefficients, we get:

$${}^{n}C_0 \times {}^{n}C_0 + {}^{n}C_1 \times {}^{n}C_1 + \cdots + {}^{n}C_{n-1} \times {}^{n}C_{n-1} + {}^{n}C_n \times {}^{n}C_n$$

$$({}^{n}C_0)^2 + ({}^{n}C_1)^2 + \cdots + ({}^{n}C_{n-1})^2 + ({}^{n}C_n)^2$$

$$= \sum_{r=0}^{n} \left({}^{n}C_r\right)^2$$

Now since the coefficients of x^n is equal in both sides we can equate them, so:

$$\sum_{r=0}^{n} ({}^{n}C_r)^2 = {}^{2n}C_n$$

As required

- As you can probably see, there was a lot of “moves” that one would not necessarily know how to do without guidance. The only way to really be able to be successful with these questions is with practise