2015 Bored of Studies Trial Examinations

Mathematics Extension 1

Written by Carrotsticks & Trebla

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11 – 14.

Total Marks – 70

Section I Pages 1 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 7 – 17

60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section.

Total marks – 10
Attempt Questions 1 – 10
All questions are of equal value

Shade your answers in the appropriate box in the Multiple Choice answer sheet provided.

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1 Suppose that *P* divides the interval *AB* internally in the ratio $m:n$.

A point *Q* lies on the interval *PB* such that it divides it internally in the ratio $m:n$.

What is the ratio of *PQ* to *PA*?

(A) $\dfrac{m}{m+n}$.

(B) $\dfrac{n}{m+n}$.

(C) $\dfrac{m+n}{m}$.

(D) $\dfrac{m+n}{n}$.

2 Consider a quantity *N* which behaves over time according to differential equation at time *t*

$$\frac{dN}{dt}=k\left(P-N\right)$$

where *k* and *P* are constants. Which of the following statements is correct?

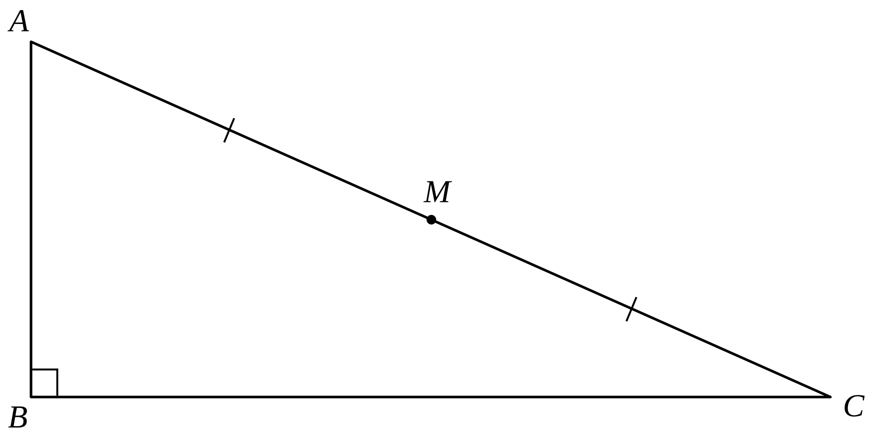
(A) If $k>0$, then $N\to 0$ as $t\to\infty$.

(B) If $k>0$, then $N\to P$ as $t\to\infty$.

(C) If $k<0$, then $N\to 0$ as $t\to\infty$.

(D) If $k<0$, then $N\to P$ as $t\to\infty$.

3 The diagram below shows a right angled $\triangle ABC$, where M is the midpoint of AC.



Which of the following statements is always true?

(A) $\angle ABM = \angle CBM$.

(B) $\angle BAM = \angle BCM$.

(C) $BM = AB$.

(D) $BM = AM$.

4 Consider two polynomials $P(x)$ and $Q(x)$ which both have degree n and have the same set of roots.

Which of the following statements is always true?

(A) $P(x)+Q(x)$ shares the same set of roots with either $P(x)$ or $Q(x)$.

(B) $P(x)$ and $Q(x)$ are identical polynomials.

(C) $P(x)$ and $Q(x)$ have the same remainder when divided by any other polynomial.

(D) $P(x)Q(x) \geq 0$ for all real x.

5 What is the value of $\int_0^{\pi} 4\cos^4 x - \cos^2 2x\,dx$?

(A) $\frac{\pi}{4}$.

(B) $\frac{\pi}{2}$.

(C) π .

(D) 2π .

6 A particle *A* is projected vertically upwards at an initial speed of *V* on a flat surface. Another particle *B* is projected from the same position at angle of θ for some $0 < \theta < \frac{\pi}{2}$ to the horizontal at an initial speed of *V* at the same time. Assume there is no air resistance.

Which of the following statements is correct?

(A) Particle *A* will land on the surface before particle *B*.

(B) Particle *B* will land on the surface before particle *A*.

(C) Both particles will land on the surface at the same time.

(D) There is not enough information to conclude which particle lands on the surface first.

7 Suppose that there are two lines on the number plane such that one line has twice the gradient of the other line. Let m be the gradient of one of the lines.

How many values of m are there if the angle between the two lines is equal to 45° ?

(A) 0.

(B) 1.

(C) 2.

(D) 4.

8 A coin is flipped n times and has equal chance of landing heads or tails.

Which of the following is true?

(A) The probability of having more heads than tails, after an odd number of tosses is greater than 50%.

(B) The probability of having more heads than tails, after an odd number of tosses is less than 50%.

(C) The probability of having more heads than tails, after an even number of tosses is greater than 50%.

(D) The probability of having more heads than tails, after an even number of tosses is less than 50%.

9 Let k be a positive constant.

Which of the following has the same derivative as $\dfrac{\sin(3kx)}{\sin(kx)}$, with respect to x ?

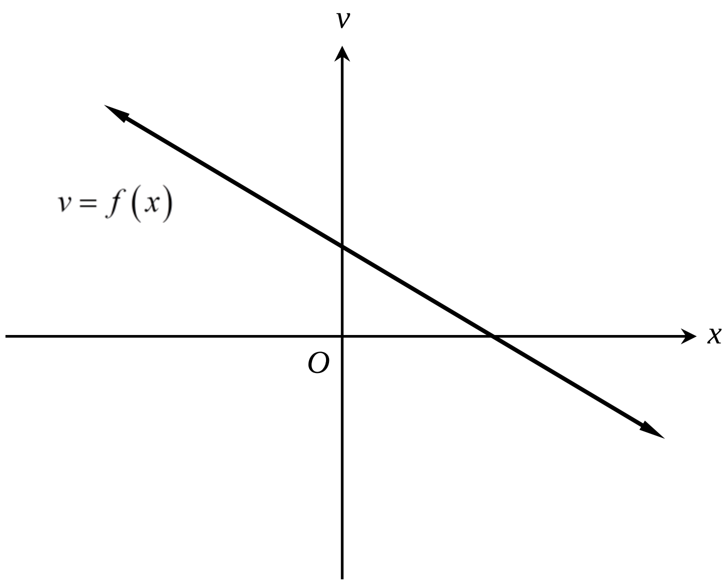
(A) $\dfrac{\cos 3kx}{\cos kx}$.

(B) $\dfrac{\cos 3kx}{\sin kx}$.

(C) $\dfrac{\tan 3kx}{\tan kx}$.

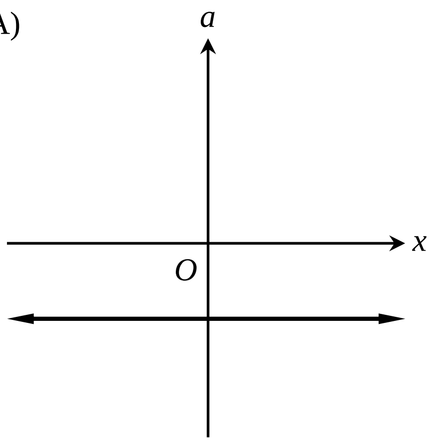
(D) $\dfrac{\tan 3kx}{\sin kx}$.

10 The following is a sketch of a particle's velocity as a function of displacement.

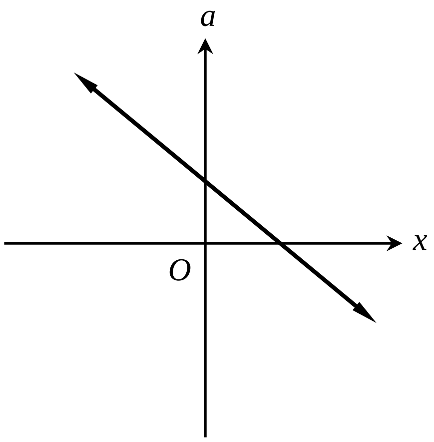


Which of the following graphs best represents the particle's acceleration as function of displacement?

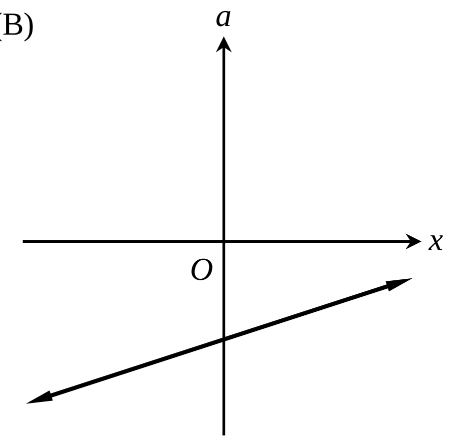
(A)



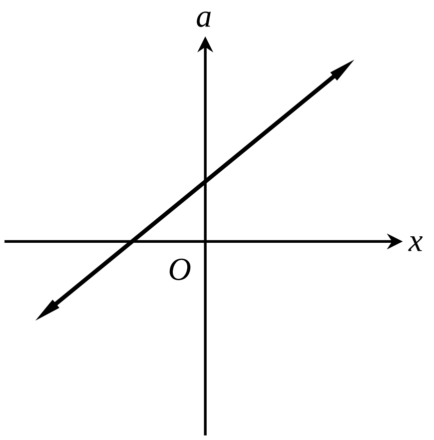
(C)



(B)



(D)



Total marks – 90
Attempt Questions 11 – 16
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find the set of all values of x satisfying **3**

$$\frac{\sin x}{\cos x - 1} > \frac{\cos x}{\sin x - 1}$$

in the domain $\frac{\pi}{4} \le x < \frac{\pi}{2}$.

(b) It can be shown that **2**

$$2016 - 1 = 2015 \text{ \textbf{(Do NOT prove this)}}$$

Use the above theorem, or otherwise, to find the least positive value of k such that

$$2015^{2015} + k$$

is divisible by 3.

(c) Let a, b, and c be constants where $a, c \neq 0$.

The polynomial $P(x) = ax^3 + bx + c$, has a real quadratic factor in the form

$$x^2 + kx + 1,$$

where k is some constant.

(i) Explain why the roots of $P(x)$ can be expressed as α, $\frac{1}{\alpha}$ and β. **2**

(ii) Deduce that **2**

$$a^2 - c^2 = ab.$$

Question 11 continues on page 8

Question 11 (continued)

(d) The cubic polynomial $P(x)$ has one of its roots being $x = \alpha$, so that **2**

$$P(x) = (x - \alpha)Q(x)$$

where $Q(x)$ is a quadratic polynomial.

Let $x = r$ be a root of $Q'(x)$.

Prove that for every cubic polynomial $P(x)$, the tangent drawn from $x = r$ intersects the x axis exactly at the root $x = \alpha$.

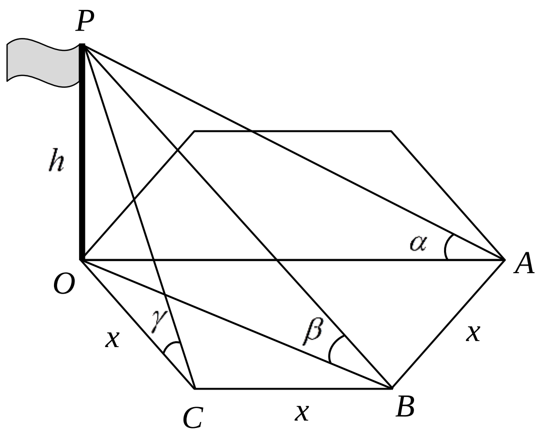
(e) Use the substitution $x = \dfrac{1-u}{1+u}$ to show that **4**

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2.$$

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows vertices A, B, C and D on a regular hexagon of side length x. The point O is the base of a vertical flagpole OP of height h. **3**



From A, B and C on the hexagon, the angle of elevations of the flag from the ground are α, β and γ respectively.

Prove that $\cot^2\alpha - \cot^2\beta = \cot^2\gamma$.

Question 12 continues on page 10

(b) A particle moves in simple harmonic motion about the origin with amplitude a and period T seconds. **4**

When the particle is at $x = a$, it is given a push with speed u towards the centre of motion. The particle then continues to move in simple harmonic motion with the same period and centre of motion, but with amplitude A, where $A > a$.

Show that it will first arrive at $x = -a$

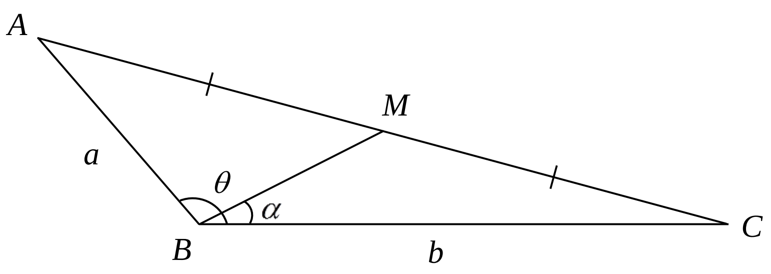
$$\frac{T}{\pi}\tan^{-1}\left(\frac{uT}{2\pi a}\right)$$

seconds faster than if it had not been pushed.

Question 12 continues on page 11

(c) The diagram below shows a triangle ABC with two sides of fixed length a and b, where $\angle ABC = \theta$ as shown in the diagram below.

Let M be the midpoint of AC, and let $\angle MBC = \alpha$.



(i) Show that **4**

$$\frac{d\theta}{d\alpha} = \frac{\sin\theta}{\sin\alpha\cos(\theta-\alpha)}.$$

(ii) The angle α decreases at a rate of $\frac{a}{b}$ radians per second. **1**

Show that the rate of change of θ, with respect to time, is

$$\frac{d\theta}{dt} = -\frac{\sin\theta}{\sin(\theta-\alpha)\cos(\theta-\alpha)}.$$

(d) Use mathematical induction to prove that **3**

$$n^{n+1} > (n+1)^n,$$

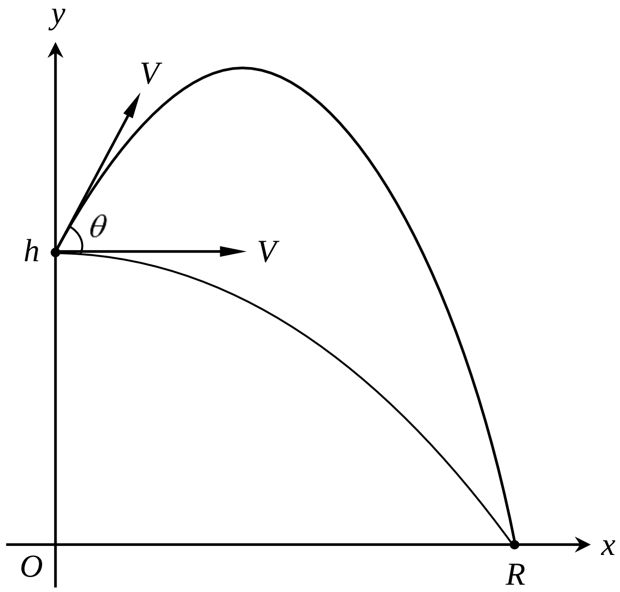
for all integers $n \geq 3$.

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A particle is fired horizontally with initial speed V from the top of a cliff of height h. It hits a target on the ground R metres away from the base of the cliff, after travelling for T seconds. **3**

From the same point of projection, the target can also be hit by firing the particle with the same initial speed but with an angle θ to the horizontal, where $0 \le \theta \le \dfrac{\pi}{2}$.



You may assume the equations of motion

$$x = Vt\cos\theta$$

$$y = -\frac{1}{2}gt^2 + Vt\sin\theta + h$$

$$y = -\frac{gx^2}{2V^2}\sec^2\theta + x\tan\theta + h$$

Show that the horizontal distance of the target from the base of the cliff is

$$R = \frac{1}{2}gT^2\tan\theta.$$

Question 13 continues on page 13

Question 13 (continued)

(b) Define the function

$$f(x) = \frac{x}{\sin^{-1} x}.$$

(i) Show that $\lim_{x \to 0} \left(\frac{x}{\sin^{-1} x} \right) = 1$. **1**

(ii) State the domain of $f(x)$. **1**

(iii) Use the fact that $\sin x < x$ for all $x > 0$ to show that $f(x) < 1$ for all x in the domain. **2**

(iv) It can be shown that **3**

$$\theta < \tan \theta,$$

for all $0 < \theta < \frac{\pi}{2}$. **(Do NOT prove this)**

Use this, or otherwise, to show that $f(x)$ is decreasing for all $0 < x < 1$.

(v) Describe the behaviour of $f'(x)$ as $x \to 1$. **1**

(vi) Hence, find the range of $f(x)$. **2**

(vii) Sketch the graph of $y = f(x)$. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) A bag contains m black balls and n white balls. Balls are drawn randomly from the bag without replacement.

(i) Show that the probability of drawing k black balls consecutively before the first white ball is drawn is **2**

$$\frac{\binom{m+n-k-1}{m-k}}{\binom{m+n}{m}}.$$

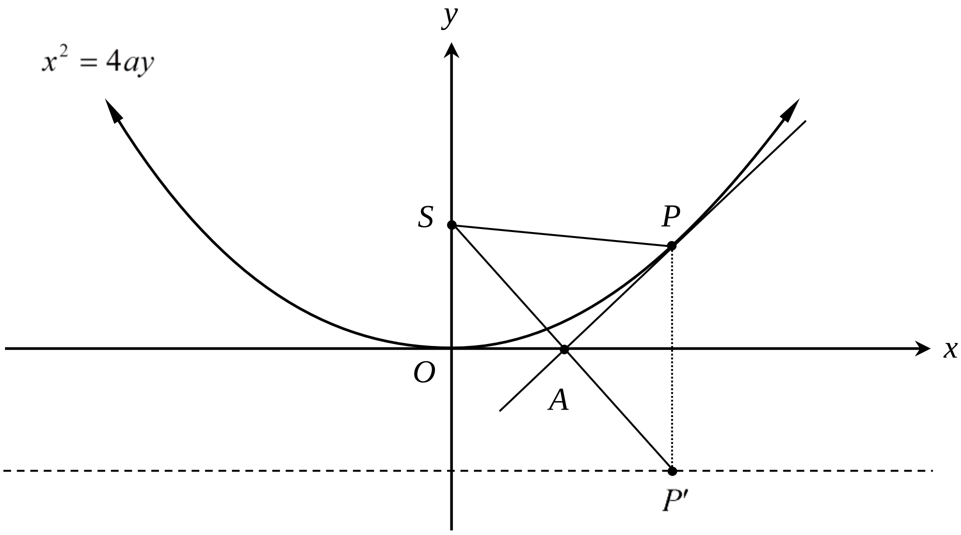
(ii) Hence, or otherwise, simplify the sum **3**

$$\frac{\binom{m}{0}}{\binom{m+n-1}{0}}+\frac{\binom{m}{1}}{\binom{m+n-1}{1}}+\frac{\binom{m}{2}}{\binom{m+n-1}{2}}+\ldots+\frac{\binom{m}{m}}{\binom{m+n-1}{m}}.$$

Question 14 continues on page 15

(b) The diagram below shows a tangent with x intercept A, drawn from a point $P\left(2ap, ap^2\right)$, on the parabola $x^2 = 4ay$, where $a > 0$.

Let P' be the foot of the perpendicular from P to the directrix.



You may assume, without proof, that the equation of the tangent from P is

$$y = px - ap^2 .$$

(i) Show that A is the midpoint of SP'. **2**

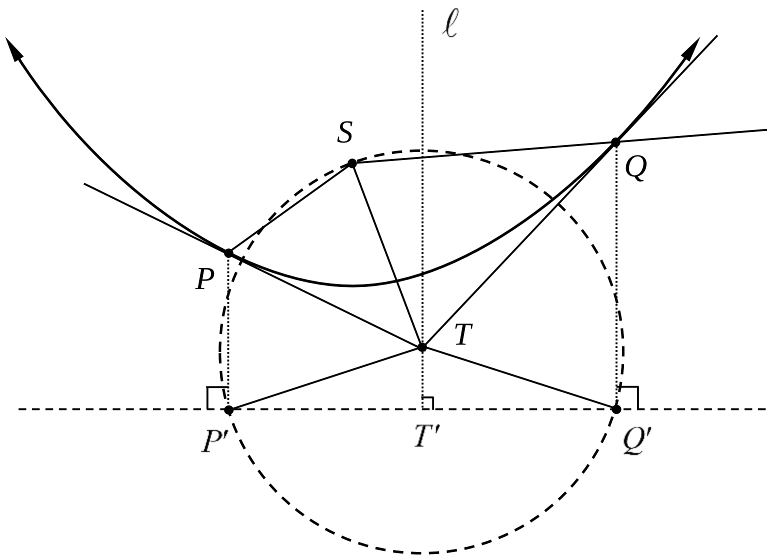
(ii) Deduce that $\triangle APS \equiv \triangle APP'$. **1**

Question 14 continues on page 16

(c) The diagram below two points P and Q on a parabola with focus S. Tangents are drawn from P and Q to intersect at T.

Let P' and Q' be the feet of the perpendiculars from P and Q to the directrix respectively.

A vertical line ℓ is drawn through T and intersects the directrix at T'.



It can be shown that the point T is the centre of the circle passing through $P'SQ'$

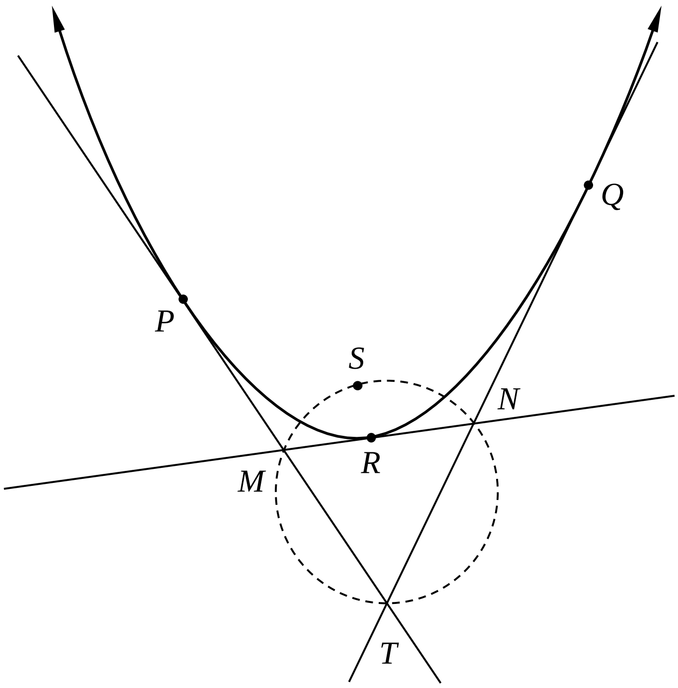
(Do NOT prove this)

(i) Use part (b) to prove that $\triangle PST \equiv \triangle PP'T$. **2**

(ii) Hence, show that $\angle PTS = \angle SQT$. **3**

Question 14 continues on page 17

(iii) A third tangent is drawn from a point R on the parabola to intersect the tangents drawn from P and Q at M and N respectively, as shown in the diagram below. **2**



Prove that M, N, T and S are concyclic.

End of Exam

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$