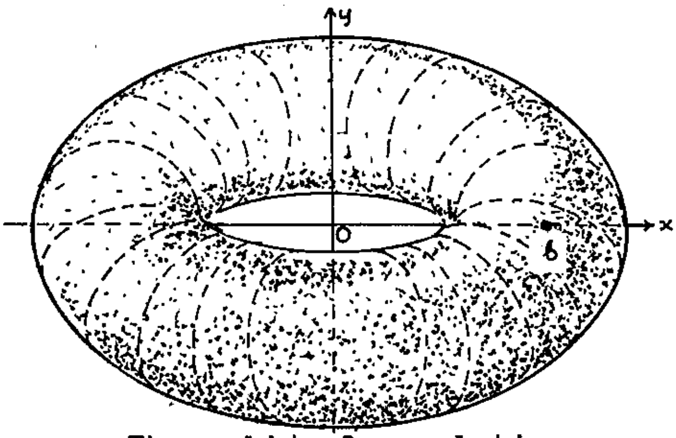


The solid of revolution

Step (2) The cross-sectional area of this washer like slice - at a distance of y units above the x axis is

$$A = \pi(r_2^2 - r_1^2)$$

$$A(x) = \pi(x_2^2 - x_1^2)$$

Note: Since the thickness of a slice is Δy we need to express A as a function in y.

Step (3) The volume of a slice is

$$\Delta V = A(x).\Delta y$$

$$= \pi(x_2^2 - x_1^2).\Delta y$$

Step (4) The volume of the solid of revolution is

$$V = \lim_{\Delta y \to 0} \sum_{y=-a}^{a} \pi(x_2^2 - x_1^2).\Delta y$$

$$V = \int_{-a}^{a} \pi(x_2^2 - x_1^2)dy$$

Step (5) We can evaluate this integral as soon as we find x in terms of y.

Consider the expanded form of the equation $(x - b)^2 + y^2 = a^2$ i.e. $x^2 - 2bx + b^2 + y^2 = a^2$. Since y is a constant when we deal with an individual slice, we obtain the quadratic equation

$$x^2 - 2bx + (b^2 + y^2 - a^2) = 0 \qquad (1)$$

whose roots are x_1 and x_2.

The sum of these roots can be expressed as

$$x_1 + x_2 = 2b \qquad (2)$$

and the product of the roots

$$x_1 \cdot x_2 = y^2 + b^2 - a^2 \qquad (3)$$

Step (5) cont.

Also $x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1)$ (4)

Since we already have an expression for the sum of the roots in terms of the coefficients of the quadratic equation (1) we need to find another one for the difference of the roots $x_2 - x_1$ in terms of the coefficients of (1). Consider

$$x_2 - x_1 = \sqrt{(x_2 - x_1)^2}$$
$$= \sqrt{[(x_2 + x_1)^2 - 4x_2x_1]} \qquad (5)$$
$$= \sqrt{[4b^2 - 4(y^2 + b^2 - a^2)]} \text{ by putting } (2)\&(3) \rightarrow (5)$$
$$= 2\sqrt{(a^2 - y^2)} \qquad (6)$$

Finally, by putting (2)&(6)$\rightarrow$(4) we obtain

$$x_2^2 - x_1^2 = 4b\sqrt{(a^2 - y^2)} \qquad (7)$$

So the volume of the torus is

$$V = 4b\pi \int_{-a}^{a} \sqrt{(a^2 - y^2)}dy$$

(This integral represents the area of a semi circle radius=a)

$$= 4b\pi . \tfrac{1}{2}\pi a^2$$
$$V = 2b\pi^2 a^2 \text{ cubic unit.}$$

ALTERNATIVELY, result (7) can be obtained by using the general quadratic formula for $x^2 - 2bx + b^2 + y^2 - a^2 = 0$. So

$$x = \frac{2b \pm \sqrt{[4b^2 - 4(b^2 + y^2 - a^2)]}}{2} = b \pm \sqrt{(a^2 - y^2)}. \text{ So}$$

$x_2 = b + \sqrt{(a^2 - y^2)}$ and $x_1 = b - \sqrt{(a^2 - y^2)}$. So

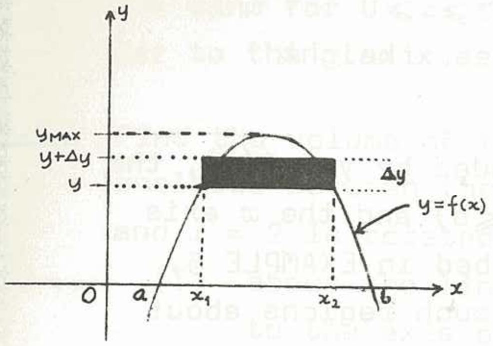
$$x_2 + x_1 = 2b$$
$$x_2 - x_1 = 2\sqrt{(a^2 - y^2)}$$
$$x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1)$$
$$= 2\sqrt{(a^2 - y^2)}.2b$$
$$= 4b\sqrt{(a^2 - y^2)} \text{ as above.}$$

The volume integral can be evaluated by using trigonometric substitution. Consider that $y = a\sin\theta$, then $dy = a\cos\theta.d\theta$ and when $y = a$, $\theta = \frac{\pi}{2}$ and when $y = -a$ then $\theta = -\frac{\pi}{2}$. So

$$V = 4b\pi \int_{-a}^{a} \sqrt{(a^2 - y^2)}dy = 4b\pi \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} a^2\cos^2\theta.d\theta = 2b\pi^2 a^2$$

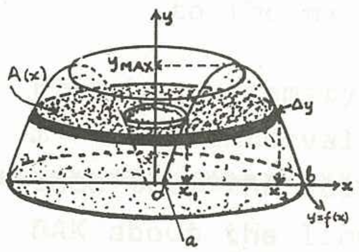
EXAMPLE 5.

Let f be a continuous function over [a,b] such that f(a) = f(b) = 0 where $f(x) \geqslant 0$ for each x in [a,b] and $0 < a < b$. Show that if the region between $y = f(x)$ and the x axis is rotated about the y axis then the volume of the solid of revolution - by slicing perpendicular to the y axis is given by



$$V = \int_0^{y_{max.}} \pi(x_2^2 - x_1^2).dy$$

Solution: This integral represents the sum of the volumes of the "washer" like horizontal slices of thickness Δy.



$A(x) = \pi(x_2^2 - x_1^2)$ is the area of the region formed, which is y unit distant from the x axis.

Let the volume of a washer be ΔV, then

$\Delta V = A(x).\Delta y = \pi(x_2^2 - x_1^2).\Delta y$

If V is the volume of the solid of revolution generated then

$$V \doteqdot \Sigma\pi(x_2^2 - x_1^2).\Delta y$$

$$V = \lim_{\Delta y \to 0} \Sigma\pi(x_2^2 - x_1^2).\Delta y \quad \text{which can be written as}$$

$$V = \int_0^{y_{max.}} \pi.(x_2^2 - x_1^2).dy$$

(The next step is to express x as a function of y then evaluate the above integral.)

NOTES: (a) In all the five examples so far the "slicing" has been done perpendicular to the axis of rotation hence all plane sections obtained this way are - in general - either CIRCLES or ANNULI.

(b) The volume of a "slice" which is a "circular disc" or a thin cylinder (or a slice of a loaf of bread for example) is given by

$V = A(x).\Delta x$ if the rotation is about the x axis

$V = A(y).\Delta y$ if the rotation is about the y axis

V = base area x height in both cases

NOTES cont.

The volume of the solid (or of a loaf of bread for example) is the sum of the volumes of the individual slices which is given by

$$V = \int_a^b A(x).dx \doteqdot \sum \text{base area} \times \text{height}$$

(c) All regions in the first quadrant bounded by $y = f(x)$, the vertical lines $x = a$, $x = b$ (i.e. $0 \leqslant a \leqslant b$) and the x axis in general satisfy the condition described in EXAMPLE 5. Hence the volume generated by rotating such regions about the y axis is given by

$$V = \int_0^{y_{max.}} \pi(x_2^2 - x_1^2).dy$$

EXERCISES

UNIT ONE

Q.1. Find the volume of the solid generated by rotating about the x axis the regions described below. Use slices perpendicular to the x axis. Follow the 5 steps shown in the examples. (Note that drawing diagrams is part of step 1.)

(1) $y = x^2$, $y = x$

(2) $y = x^3$, $x = 2$ and the x axis.

(3) $y = x^{\frac{2}{3}}$, $x = 8$, $x = -8$ and the x axis.

(4) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ the x and y axes.

(5) $y = 4 - x^2$, $x = 2$ and $y = 4$

Q.2. Find the volume of the solid of revolution generated by rotating about the y axis the regions described below. Use slices perpendicular to the y axis. Follow the 5 steps shown in the examples.

(1) $y = 4 - x^2$, $x = 2$, $y = 4$

(2) $y = \sqrt{x}$, $y = 2$ and the y axis

(3) $y = \sqrt{x}$, $x = 4$ and the x axis

(4) $y^2 = \frac{1}{2}x^3$, and $x = 2$

Q.3. Find the volume of the solid of revolution obtained by rotating about the x axis the region bounded by the curve $y = \sin x$ for $0 \leqslant x \leqslant \pi$ and the x axis. Use slices perpendicular to the x axis.

Q.4. Find the volume of the solid of revolution when the area enclosed between the curve $x^2 = 4 - y$ and the lines $y = 4$ and $x = 2$ is rotated

(i) about the line $y = 4$ using slices perpendicular to the axis of revolution,

(ii) about the line $x = 2$ using slices perpendicular to the axis of revolution.

Q.5. From an extremity A of the latus rectum AB of the parabola $x^2 = 4ay$ interval AK is drawn perpendicular to the x axis. Show that the volume formed by the rotation of the region OAK about the line AK is $\frac{2}{3}\pi a^3$ cubic unit.

Q.6. The parabola $y^2 = 4ax$ is rotated around its latus rectum. Using slices perpendicular to the axis of rotation, find the volume of the solid generated.

Q.7. Find the volume of the solid - called the prolate spheroid - by rotating the region enclosed by the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ about the x axis. Use slices perpendicular to the x axis.

Q.8. Find the volume of the solid - called the oblate spheroid - by rotating the region bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$ about the y axis. Use slices perpendicular to the y axis.

Q.9. Sketch the region in the Cartesian plane satisfying the conditions: $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant x^2 - \frac{1}{4}x^4$.
Rotate this region about the y axis, using slices perpendicular to the y axis and calculate the volume of the solid of revolution generated.
(Hint: (a) Use EXAMPLE 4 for reference, (b) In step 4 you need to use calculus to find the limits of the definite integral)

Q.10. Find the volume of the solid of revolution in EXAMPLE 6. by using slices perpendicular to the y axis.

Q.11. The region between the curve $x = 4y^2 + 2$ and the vertical line $x = 6$ is rotated about the y axis. Find the volume of the solid

Q.12. Find the volume generated when the area between the curve $y = \frac{1}{2}\sqrt{(x - 2)}$, the line $y = 1$ and the coordinate axes is rotated about the y axis. Use slices perpendicular to the axis of rotation.

Q.13. A hole of radius r is bored through a sphere of radius R. Find the remaining volume if

(i) R = 8 cm, r = 2 cm.

(ii) The radius of the sphere is 12 cm and the diameter of the hole is 6 cm.

(iii) $R/_r = 2/_1$

(iv) $R/_r = 4/_1$

Q.14. Find the volume of the solid of revolution - called the TORUS - obtained when the circle $x^2 + (y - b)^2 = a^2$ $(b > a)$ is rotated about the x axis. Use slices perpendicular to the x axis.

Q.15. Let n be a positive integer and a be a positive real number. Consider the graph of the function $y = x^n$ which divides the rectangle formed by the lines $x = a$, $y = a^n$, $x = 0$ and $y = 0$ into two regions bounded by two straight lines and a section of the curve itself.

Show that the two solids obtained by rotating these regions about the x axis is 1 : 2n.

Find the volume of the solid obtained by rotating the region formed by the curve $y = x^3$, $y = 8$ and the y axis. Compare this to the answer of Q.1.(2). and use this to verify the result above.

Q.16. Use the slicing technique to show that the volume of a spherical cap of height h and radius r (where r > h) is

$$V = \tfrac{1}{3}\pi h^2(3r - h)$$

Water is pumped into a hemispherical container of diameter 1 metre at a rate of 0.1 cm^3/sec. Show that the rate at which the water level in the container is rising when the water is

(a) 10 cm deep is $\frac{1}{9000\pi}$ cm/sec

(b) 40 cm deep is $\frac{1}{24000\pi}$ cm/sec.

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UNIT TWO

VOLUME OF SOLIDS OF REVOLUTION BY CYLINDRICAL SHELLS.

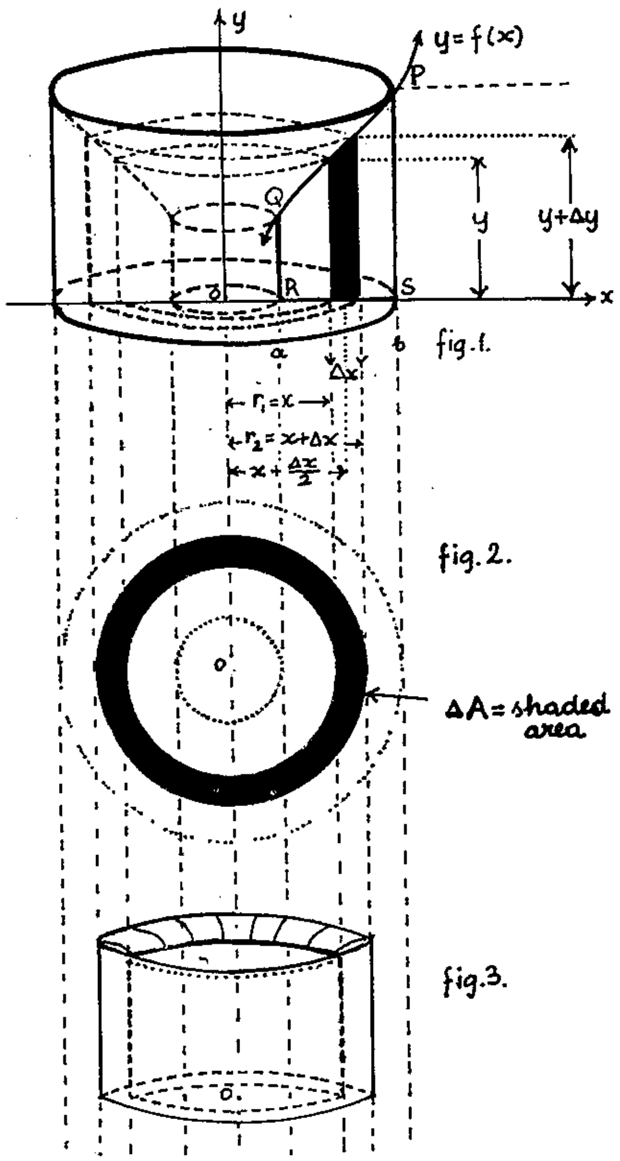


fig.1. fig.2. fig.3.

Consider the region PQRS in the first quadrant bounded by the graph of a continuous function f, the x axis and the lines x=a and x=b. Assume that $0 \leqslant a \leqslant b$ and $f(x) \geqslant 0$ for each $x \in [a,b]$

Rotate PQRS around the y-axis to obtain a solid of revolution (see fig.1) Take a strip of area i.e. a typical rectangle (shaded in fig.1) and revolve it about the y-axis. The rectangle sweeps out a hollow cylindrical shell with a thin wall of thickness Δx (see fig.3). The base of the shell is an annulus (see fig.2) with inner and outer radii of $r_1 = x$ and $r_2 = x + \Delta x$ respectively, and volume ΔV.

Let the base area of the shell be denoted by ΔA. Then

$$\begin{aligned}\Delta A &= \pi(r_2^2 - r_1^2)\\ &= \pi(r_2 + r_1)(r_2 - r_1)\\ &= 2\pi\left(\frac{r_2 + r_1}{2}\right)(r_2 - r_1) \qquad (1)\end{aligned}$$

However $r_2 - r_1 = \Delta x$ (2)

Let the average radius $\dfrac{r_2 + r_1}{2} = r$ (3)

Put (2) and (3) $\longrightarrow$ (1) so $\Delta A = 2\pi r.\Delta x$ which is the shaded area in fig.2.

Noting that $\dfrac{r_2 + r_1}{2} = \dfrac{2x + \Delta x}{2} = x + \dfrac{\Delta x}{2}$, so $\Delta A = 2\pi(x + \dfrac{\Delta x}{2})\Delta x.$

The altitude of the cylindrical shells varies between y and $y + \Delta y$. The volume of a cylindrical shell lies between that of a shell of base ΔA, altitude y and that of a shell of base ΔA, altitude $y + \Delta y$, i.e. $y.\Delta A < \Delta V < (y + \Delta y).\Delta A$

The total volume V of the solid of revolution from $x = a$ to $x = b$ is the sum of the volumes of all the cylindrical shells (typical slices or elements) ΔV as $\Delta x \to 0$.

The volume V is contained between the two limits L and U where

L = sum of the volume of all cyl. shells with altitude = y

U = sum of the volume of all cyl. shells with altitude = $y + \Delta y$

Briefly we can say that; $L < V < U$ so that $\lim\limits_{\Delta x \to 0} L \leqslant V \leqslant \lim\limits_{\Delta x \to 0} U$ (4)

$$\lim_{\Delta x \to 0} L = \lim_{\Delta x \to 0} \sum_{x=a}^{b} \Delta A.y = \lim_{\Delta x \to 0} \sum_{a}^{b} 2\pi(x + \tfrac{1}{2}\Delta x)\Delta x.y$$

$$= \lim_{\Delta x \to 0} \sum_{a}^{b} 2\pi\{xy.\Delta x + \tfrac{1}{2}y(\Delta x)^2\}$$

$$= \int_a^b 2\pi xy \, dx$$

(As $\Delta x \to 0$ terms of the second order such as $(\Delta x)^2$ will vanish)

$$\lim_{\Delta x \to 0} U = \lim_{\Delta x \to 0} \sum_{x=a}^{b} \Delta A(y + \Delta y) = \lim_{\Delta x \to 0} \sum_{a}^{b} 2\pi(x + \tfrac{1}{2}\Delta x).\Delta x.(y + \Delta y)$$

$$= \lim_{\Delta x \to 0} \sum_{a}^{b} 2\pi\{xy.\Delta x + x.\Delta x.\Delta y + \tfrac{1}{2}(\Delta x)^2.y + \tfrac{1}{2}(\Delta x)^2.\Delta y\}$$

$$= \int_a^b 2\pi xy \, dx$$

(As $\Delta x \to 0$ terms of the second and third order such as $\Delta x.\Delta y$, $(\Delta x)^2$, $(\Delta x)^2.\Delta y$ will vanish.)

Therefore (4) becomes $\int_a^b 2\pi xy dx \leqslant V \leqslant \int_a^b 2\pi xy dx$. Since the outer limits are equal $V = \int_a^b 2\pi xy.dx$

The volume V obtained by rotating the region PQRS about the y axis using CYLINDRICAL SHELLS is given by

$$V = \int_a^b 2\pi xy.dx = \int_a^b 2\pi x.f(x).dx$$

NOTES:

(a) Though the rotation of region PQRS is about the y - axis we still integrate with respect to x.(!!)

(b) $\boxed{\text{Volume} = 2\pi r h.\Delta x \quad h=y \quad 2\pi r = 2\pi x}$

Consider the thickness of this page to be Δx. By cutting out the "rectangle" on the left hand side you would hold in your hand the flattened out cylindrical shell of fig.3. of volume ΔV. The volume of this shell (or element) which is a rectangular prism is;

ΔV = Base area x height

$= \Delta A \times y$ (see fig.1 and fig.2)

$= 2\pi.\frac{1}{2}(r_2 + r_1)(r_2 - r_1).y$ (from (1))

$= 2\pi r.\Delta x.y = 2\pi xy.\Delta x$ (using $r = \lim_{\Delta x \to 0}(x + \frac{1}{2}\Delta x) = x$)

$= 2\pi$ x average radius x altitude x thickness (see (2),(3))

So the result $V = \int_a^b 2\pi x.f(x).dx$ can be considered as

$V = \int_a^b$ length x height x breadth of the "rectangle"

This may help you to visualize and understand better the result

$$V = \int_a^b 2\pi x.f(x).dx \qquad (5)$$

(c) Let $\int_a^b f(x).dx = A$ (region PQRS), then the result (5) can be written as $V = 2\pi x.\int_a^b f(x).dx = 2\pi x A$, where x = the radius of the circle described by the centre of gravity (centroid) of the figure whose area is A. Note that in (b) we called this radius the "average radius") So the volume of a solid of revolution V can be calculated as

****
$V = 2\pi x.A$ = circumference described by the centroid of the figure x area of the figure (Pappus c.200 A.D.)
****

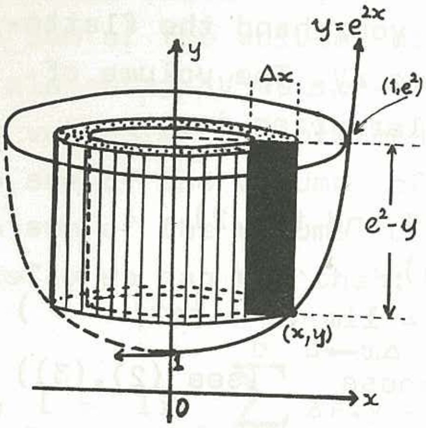
The centre of gravity of a figure which has line symmetry lies on its axis of symmetry. When such figures are rotated, the average radius i.e. the radius of the circle described by the centre of gravity can be considered to be the distance between 2 parallel lines; the axis of symmetry and the axis of revolution. (The above is true only if the axis of rotation doesn't intersect the area.) Applying these ideas to Example 4 we can conclude that $x = b$, $A(x) = \pi a^2$, so $V = 2\pi.b.\pi a^2 = 2\pi^2 a^2 b$.

(d)
****
The volume V obtained by rotating a figure of area A about the x axis using CYLINDRICAL SHELLS is given by

$V = \int^c 2\pi yx.dy = \int^c 2\pi y.f(y).dy = 2\pi y.A$

EXAMPLE 6: (rotating around the y axis and "slicing parallel to the y axis thus obtaining cylindrical shells.)

Find the volume of the solid of revolution when the area enclosed by the graph of the function $y = e^{2x}$, the y axis and the horizontal line $y = e^2$ is rotated about the y axis.



Solution:

(1) The thickness of a slice is Δx

(2) The solid generated by revolving this typical rectangle about the y axis is a thin cylindrical shell of area 2πrh. Verify that $r = x$, $h = e^2 - y$, hence the area $A = 2\pi x.(e^2 - y)$ and $A(x) = 2\pi x(e^2 - e^{2x})$

(3) The volume ΔV of a cylindrical shell is $\Delta V = 2\pi x(e^2 - e^{2x}).\Delta x$

(4) The volume V of the solid of revolution i.e. the sum of the volumes of all cylindrical shells is

$$V = \lim_{\Delta x \to 0} \sum_0^1 2\pi x(e^2 - e^{2x}).\Delta x \quad \text{which can be written as}$$

$$V = \int_0^1 2\pi x(e^2 - e^{2x}).dx$$

(5) $$V = \int_0^1 (2\pi e^2 x - 2\pi x e^{2x}).dx = [2\pi e^2 . \tfrac{1}{2}x^2]_0^1 - \pi\int_0^1 2xe^{2x}.dx$$

$$= \pi e^2 - \pi\int_0^1 x.\frac{d}{dx}(e^{2x}).dx$$

$$= \pi e^2 - \pi[x.e^{2x}]_0^1 + \pi\int_0^1 e^{2x}.dx$$

$$= \pi e^2 - \pi e^2 + \tfrac{1}{2}\pi[e^{2x}]_0^1$$

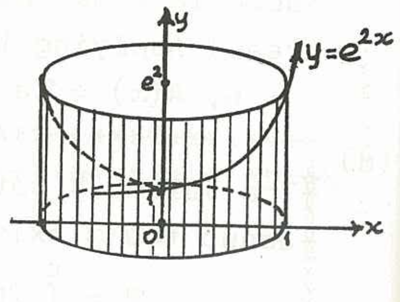
$$= \tfrac{1}{2}\pi(e^2 - 1)$$

This example demonstrates the advantage of using the slicing technique by cylindrical shells. By slicing perpendicular to the y axis you obtain:

$$V = \tfrac{1}{4}\int_1^{e^2} (\log y)^2.dy$$

which is not as simple to evaluate as the integral above.

Note: By using $V = \int_0^1 2\pi xy.dx$ indiscriminately and/or without a neat sketch you would obtain $V = \tfrac{1}{2}\pi(e^2 + 1)$ which is the volume generated by rotating the region bounded by $y = e^{2x}$, the x axis and the lines $x = 0$ and $x = 1$. See figure

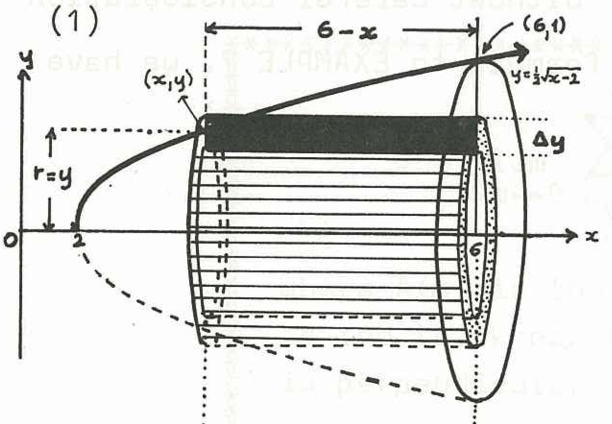


EXAMPLE 7. (Rotating about the x axis, slicing parallel to the x axis thus obtaining cylindrical shells.)

Find the volume of the PARABOLOID obtained when the region between the curve $y = \frac{1}{2}\sqrt{x - 2}$, the x axis and the line $x = 6$ is rotated about the x axis.

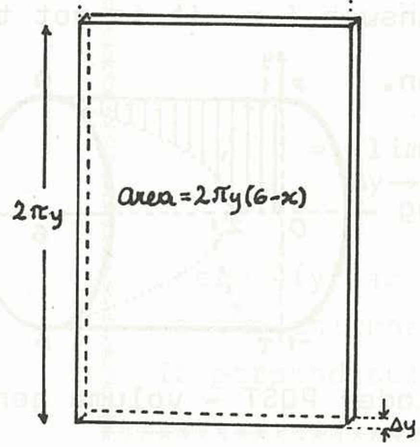
Solution:

(1)



The thickness of a slice i.e. the width of the typical rectangle is Δy.

(2)



The solid generated by revolving this strip about the x axis is a hollow cylindrical shell of inner radius y, inner circumference $2\pi y$ and height $6 - x$. The area of a flattened shell is $2\pi rh$ which can be written as $A = 2\pi y(6 - x)$. Since the thickness is Δy we need to express A as a function of y:

$$A(y) = 2\pi y\,[6 - (4y^2 + 2)]$$
$$= 8\pi y(1 - y^2)$$

(3) Hence the volume of a cylindrical shell is

$$\Delta V = 8\pi y(1 - y^2).\Delta y$$

(4) Consider the volume of the solid of revolution to be the integral which is suggested by the sum of the volumes of these hollow cylindrical shells;

$$V = \lim_{\Delta y \to 0} \sum_{0}^{1} 8\pi y(1 - y^2).\Delta y$$

$$V = 8\pi \int_0^1 (y - y^3).dy$$

(5)

$$= 8\pi\left[\tfrac{1}{2}y^2 - \tfrac{1}{4}y^4\right]_0^1$$

$$= 2\pi \text{ cubic units.}$$

<u>NOTE:</u> $V = \int_a^b 2\pi x . f(x) . dx$ can easily be extended to find the volume of the solid of revolution by the cylindrical shell method when the rotation is about the x axis.

Then
$$V = \int_c^d 2\pi y . f(y) . dy$$

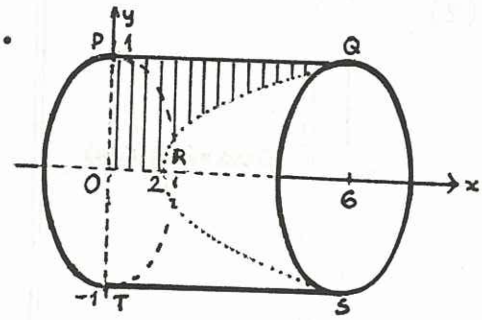
Using this result as a "formula" without careful consideratio can be misleading. Applying the formula to EXAMPLE 7. we have $c = 0$, $d = 1$, $f(y) = 4y^2 + 2$, so

$$V = \int_0^1 2\pi . y . (4y^2 + 2) . dy$$

$$= \left[2\pi y^4 + 2\pi y^2 \right]_0^1$$

$$= 4\pi \text{ cubic units.}$$

Obviously this is not the correct answer i.e. it is not the volume of the paraboloid in question.



By definition the above sum represents the solid obtained by rotating the region OPQR about the x axis. However not all the effort made is wasted since the volume of the cylinder PQST - volume generated by the rotation of region OPQR = volume of the required paraboloid i.e.

$$V = \pi . 1 . 6 - 4 . \pi = 2\pi \text{ cu. units.}$$

VOLUMES OF SOLIDS OTHER THAN SOLIDS OF REVOLUTION

The computation of the volume of a solid which is not a solid of revolution has been derived on page 5 - 6. The volume of such a solid is given by

$$V = \lim_{\Delta x \to 0} \sum_{x=a}^{b} A(x).\Delta x = \int_a^b A(x)dx$$

where $A(x)$ is the cross-sectional area, Δx is the thickness of a slice and the slicing is perpendicular to the x axis.

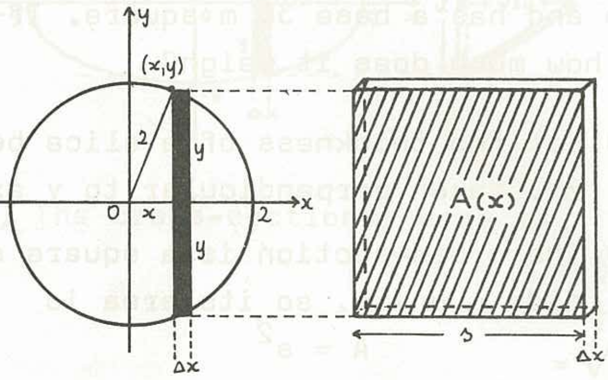
or

$$V = \lim_{\Delta y \to 0} \sum_{y=c}^{d} A(y).\Delta y = \int_c^d A(y)dy$$

where $A(y)$ is the cross-sectional area, Δy is the thickness of a slice and the slicing is perpendicular to the y axis.

EXAMPLE 8.

A solid is constructed with a circular base of radius 2 cm and such that every cross section perpendicular to a chosen diameter of the base is a square, one side of it lies in the base of the solid. Find the volume of this solid.



Step (1) The figure shows a top view of the solid. Take the diameter to be an interval of the x axis. The strip of width Δx represents the thickness of this "slice" at the point (x,y). The "slice" is taken perpendicular to the x axis and the point (x,y) lies on the circle $x^2 + y^2 = 4$.

NOTE: It is possible to find such a volume as the above without visualizing the actual solid. However a proper diagram is a definite advantage.

Step (2) Let the length of a side of the square be s units. The area of a cross-section is

$$A = s^2$$

Since the thickness of a slice is Δx we need to express A as a function in x.

$$A = (2y)^2 \qquad \text{(see diagram)}$$

However $x^2 + y^2 = 4$ therefore $y^2 = 4 - x^2$ and

$$A(x) = 4(4 - x^2)$$

Step (3) The volume of a slice is

$$\Delta V = (16 - 4x^2).\Delta x$$

Step (4) The volume of the solid V is the sum of the volumes of all slices, i.e.

$$V \doteqdot \sum_{x=-2}^{2} A(x).\Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=-2}^{2} (16 - 4x^2).\Delta x$$

$$V = \int_{-2}^{2} (16 - 4x^2)dx$$

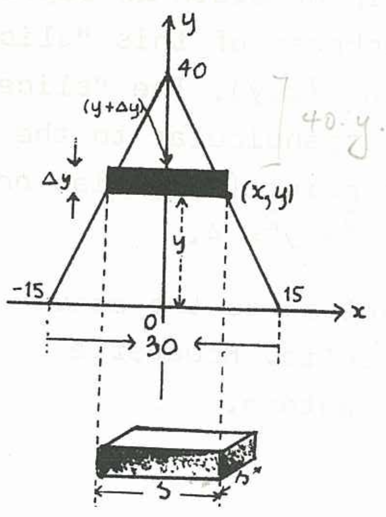
Step (5) The value of this integral is

$$V = 2\left[16x - {}^{4x^3}/_{3}\right]_0^2$$

$$= {}^{128}/_{3} \text{ cm}^3$$

EXAMPLE 9.

A regular pyramid is 40 metres high and has a base 30 m square. If it is made of rock weighing 0.5 t/m^3, how much does it weigh?



Step (1) Let the thickness of a slice be Δy, taken perpendicular to y axis.

Step (2) The cross-section is a square of side s units, so its area is

$$A = s^2$$

and is y unit distant from the x axis, at the point (x,y). Using similar triangles to find the value of s in terms of y we have

Step (2) cont.

so
$$s = 30 - \tfrac{3}{4}y$$
$$A(y) = (30 - \tfrac{3}{4}y)^2$$

Step (3) The volume of a slice is
$$\Delta V = A(y).\Delta y$$
$$= (30 - \tfrac{3}{4}y)^2.\Delta y$$

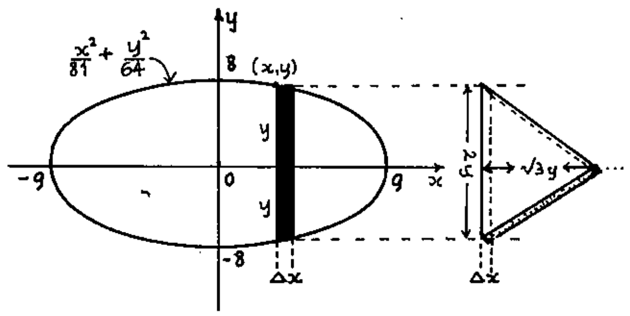
Step (4) The sum of the volumes of all slices represents the volume V of the pyramid
$$V = \lim_{\Delta y \to 0} \sum_{y=0}^{40} (30 - \tfrac{3}{4}y)^2.\Delta y$$
$$V = \int_0^{40} (30 - \tfrac{3}{4}y)^2 dy$$

Step (5) The value of this integral is
$$V = -\tfrac{4}{9}\,[(30 - \tfrac{3}{4}y)^3]_0^{40}$$
$$= 12000$$

If the volume is 12000 m^3 then its weight is 6000 tonnes.

EXAMPLE 10.

The base of a particular solid is an ellipse whose major and minor axes are 18 and 16 cm respectively. Every cross-section perpendicular to the major axis is an equilateral triangle, one side of which lies in the base of the solid. Find the volume of the solid.



(1) The figure shows a top view of the solid. Take a strip of width Δx at the point (x,y) perpendicular to the x axis. The equilateral triangle "stands" on the shaded strip of length 2y units.

(2) The cross-sectional area is
$$A = \tfrac{1}{2}b.h$$
$$= \tfrac{1}{2}.2y.\sqrt{3}y$$
$$= \sqrt{3}y^2$$

Since the width of the strip is Δx we need to express A as a function in x. The eqation of the ellipse is $x^2/_{81} + y^2/_{64} = 1$, hence $y^2 = \frac{64}{81}(81 - x^2)$, so
$$A(x) = \frac{64\sqrt{3}}{81}(81 - x^2)$$

(3) The volume of a slice is

$$\Delta V = A(x).\Delta x$$

$$\Delta V = \frac{64\sqrt{3}}{81}(81 - x^2).\Delta x$$

(4) The sum of the volumes of all "slices" represents the volume of the solid

$$V = \lim_{\Delta x \to 0} \sum_{x=-9}^{9} \frac{64\sqrt{3}}{81}(81 - x^2).\Delta x$$

$$V = \int_{-9}^{9} \frac{64\sqrt{3}}{81}(81 - x^2).dx$$

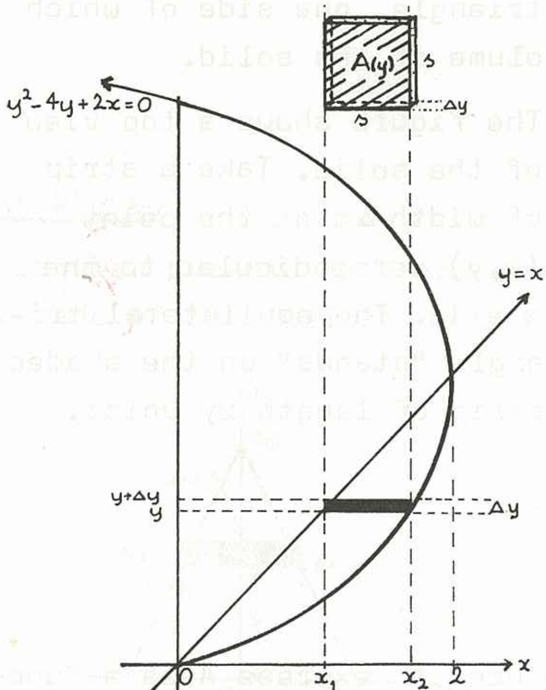
(5) The value of this integral is

$$V = 2.\frac{64\sqrt{3}}{81}[81x - \tfrac{1}{3}x^3]_0^9$$

$$= 768\sqrt{3}\ \text{cm}^3$$

EXAMPLE 11.

The solid S has its base the region bounded by the curves $y = x$ and $y^2 - 4y + 2x = 0$. Cross-sections parallel to the x axis are squares one side of which lies in the base of the solid. Find the volume of the solid.



(1) Consider a strip of width Δy and take points (x_1,y_1) and (x_2,y_2) y unit distant from the x axis on $y = x$ and $y^2 - 4y + 2x = 0$ respectively. The length of the strip is $x_2 - x_1$ units representing the length of a side of the square shaped cross-section. The width of the strip represents the thickness of a "slice" as the thin square slab "stands" on the shaded strip.

(2) If the legth of the side of the square is $x_2 - x_1$ then its area

$$A = (x_2 - x_1)^2$$

Since (x_2,y_2) lies on $y^2 - 4y + 2x = 0$, $x_2 = 2y - \frac{1}{2}y^2$ and since (x_1,y_1) lies on $y = x$, $x_1 = y_1$. The cross-sectional area now can be written as a function of y, i.e.

$$A(y) = [(2y - \tfrac{1}{2}y^2) - y]^2$$
$$= y^2 - y^3 + \tfrac{1}{4}y^4$$

(3) The volume of a slice is

$$\Delta V = A(y).\Delta y$$
$$= (y^2 - y^3 + \tfrac{1}{4}y^4).\Delta y$$

(4) The simultaneous solution of the equations of the two curves $y = x$ and $y^2 - 4y + 2x = 0$ gives (0,0) and (2,2). The y components are representing the lower and the upper limits of the sums respectively. So the sum of the volumes of all slices is the volume of the solid S, in the limit $\Delta y \to 0$

$$V = \lim_{\Delta y \to 0} \sum_{y=0}^{2} (y^2 - y^3 + \tfrac{1}{4}y^4).\Delta y$$

$$V = \int_0^2 (y^2 - y^3 + \tfrac{1}{4}y^4)\,dy$$

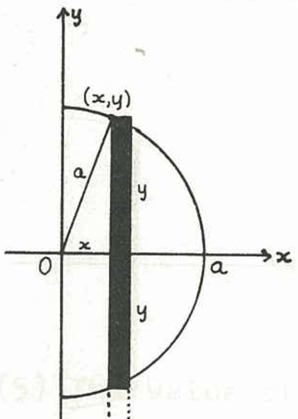
$$= \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{20}\Big]_0^2 = \frac{8}{3} - 4 + \frac{8}{5} = \frac{4}{15}$$

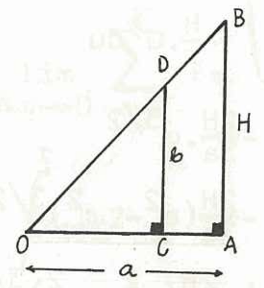
(5) The value of the integral is

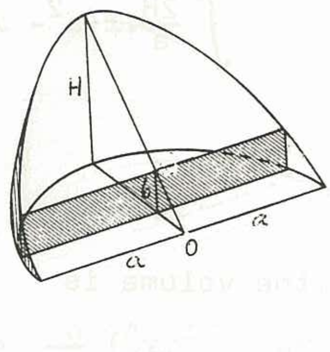
$$V = 4/15 \text{ cubic unit}$$

EXAMPLE 12.

A CYLINDRICAL SEGMENT (or a wedge shape) is a solid bounded by the base of a given cylinder, a plane passing through a diameter of the base and of the lateral surface of the cylinder.
Find the volume of a cylindrical segment with base radius a and altitude H units.







(1) Take a slice of the solid parallel to the diameter of the base of

(1) continued.
thickness Δx. Place this semicircular base in the x-y plane. The equation of this semicircle of radius a is $x^2 + y^2 = a^2$ and the strip of width Δx, taken at the point (x,y) on the circle, represents the thickness of the slice as the thin rectangular slice "stands" on this shaded strip.

(2) The cross-sectional region is a rectangle of length = 2y and breadth = b, so its area is

$$A = 2y.b$$

Since the thickness is Δx, we must express A as a function in x. Using $x^2 + y^2 = a^2$ we obtain $y = \sqrt{(a^2 - x^2)}$. By using similar triangles OAB and OCD we can express b in terms of x, i.e.

$$b/_H = x/_a$$

$$b = xH/_a$$

Hence $$A(x) = \frac{2H}{a}.x\sqrt{(a^2 - x^2)}$$

(3) The volume of a slice is

$$\Delta V = \frac{2H}{a}.x\sqrt{(a^2 - x^2)}.\ \Delta x$$

(4) The volume of the solid is the sum of the volumes of all slices of thickness Δx as $\Delta x \rightarrow 0$. This can be written as

$$V = \lim_{\Delta x \rightarrow 0} \sum_{x=0}^{a} \frac{2H}{a}.x\sqrt{(a^2 - x^2)}.\Delta x$$

$$V = \int_0^a \frac{2H}{a}.x\sqrt{(a^2 - x^2)}dx$$

(5) Let $u = a^2 - x^2$. Therefore $du = -2x.dx$ hence our integral can be written as

$$\int \frac{2H}{a}.x(a^2 - x^2)^{\frac{1}{2}}dx = \int -\frac{H}{a}.u^{\frac{1}{2}}du$$

$$= -\frac{2H}{3a}.u^{3/2}$$

$$= -\frac{2H}{3a}(a^2 - x^2)^{3/2}$$

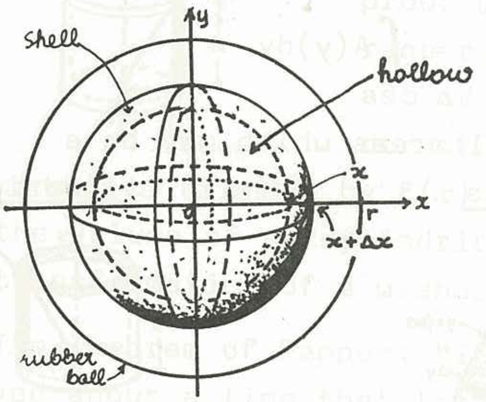
So the volume is $$V = [-\frac{2H}{3a}(a^2 - x^2)^{3/2}]_0^a$$

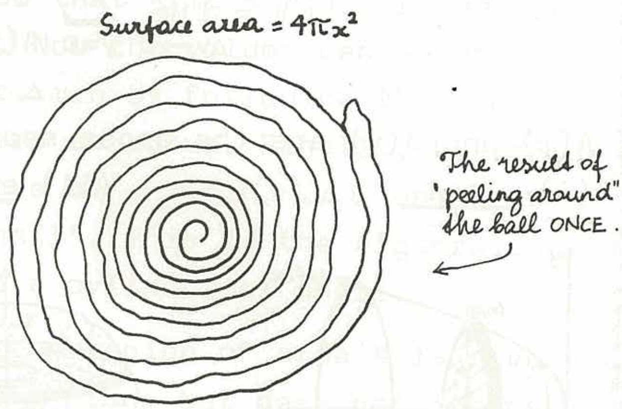
$$V = \tfrac{2}{3}Ha^2 \quad \text{cubic units.}$$

EXAMPLE 13.

Volumes of solids can be calculated by taking sections other than flat slices. This method is applied when using "cylindrical shells". (Note that it is possible to flatten a cylindrical shell) This notion can be further developed and used to compute the volume of a sphere by using "hollow spherical shells". (Note that a spherical shell can not be completely flattened without altering its surface area)

Consider a solid rubber ball (a "super-ball") of radius r. "Slice" this perfectly spherical ball into gradually decreasing hollow spheres by peeling the ball "around" at an even thickness, then by carefully putting this thin "skin" together to form a hollow spherical shell. Repeat this procedure until there is no more rubber left.





(1) Consider one of these shells of inner radius x and outer radius $x + \Delta x$. The thickness of the spherical shell is Δx.

(2) The "cross-sectional" area is

$$A = 4\pi r^2$$
$$A(x) = 4\pi x^2$$

(3) The volume of this spherical shell is

$$\Delta V(x) = A(x).\Delta x = 4\pi x^2.\Delta x$$

(4) The sum of the volumes of all spherical shells represents the volume of the spherical ball, which is written as

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{r} 4\pi x^2.\Delta x$$

$$V = \int_0^r 4\pi x^2\, dx$$

(5) The value of $V = [\frac{4}{3}\pi x^3]_0^r$ $\left(\text{Since } \frac{d}{dx}(\frac{4}{3}\pi x^3) = 4\pi x^2\right)$

$$= \frac{4}{3}\pi r^3 \text{ cubic units. (As expected)}$$

NOTE: The above can be used to explain the existence of the lovely formula

$$\boxed{\frac{d}{dr}(\frac{4}{3}\pi r^3) = 4\pi r^2}$$

S U M M A R Y .

The volume V of a solid can be considered as the sum of the volumes of all slices. When the slicing is perpendicular to the x axis, then

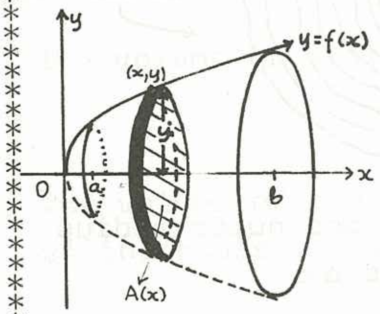
$$V = \lim_{\Delta x \to 0} \sum_{x=a}^{b} A(x).\Delta x = \int_a^b A(x)dx .$$

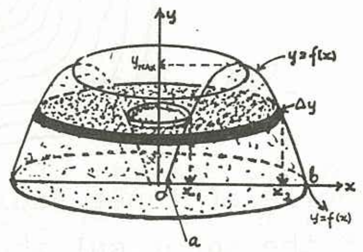
When the slicing is perpendicular to the y axis, then

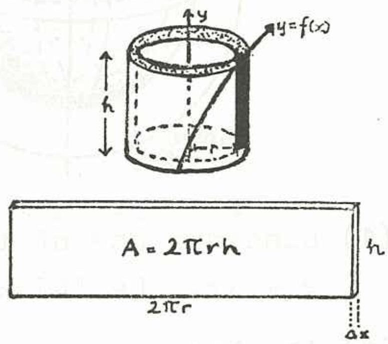
$$V = \lim_{\Delta y \to 0} \sum_{y=c}^{d} A(y).\Delta y = \int_c^d A(y)dy .$$

$A(x)$ and $A(y)$ are the cross-sectional areas which may be a

(i) circle (ii) annulus (iii) cylindrical shel







$A(x) = \pi[f(x)]^2$

$V = \int_a^b \pi[f(x)]^2 dx$

$A(y) = \pi(x_2^2 - x_1^2)$

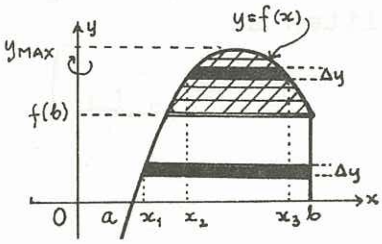
$V = \int_c^d \pi(x_2^2 - x_1^2)dy$

Here $c=0$ and $d=y_{MAX}$.

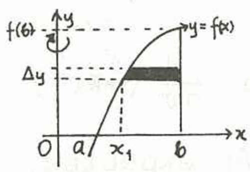
$A(x) = 2\pi x.f(x)$

$V = \int_a^b 2\pi x.f(x)dx$

NOTE: In the case of the cross-sectional area being (ii) annulus we assumed that f is continuous over [a,b], $f(x) \geqslant 0$ for x in [a,b] and that $f(a) = f(b) = 0$. One can further generalise by considering the case when $f(b) \neq 0$ and



(a) $y_{MAX.} > f(b)$. Then we must divide the volume into two by rotating the regio above (shaded) and below the line $y = f(b)$ separately. The volume of th solid is given by $V = \int_0^{f(b)} \pi(b^2 - x_1^2)dy + \int_{f(b)}^{y_{MAX.}} \pi(x_3^2 - x_2^2)dy.$



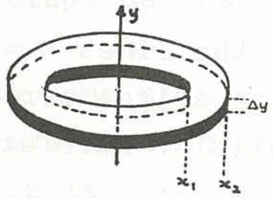
(b) $y_{MAX.} = f(b)$ then the volume is

$$V = \int_0^{f(b)} \pi(b^2 - x_1^2)dy.$$

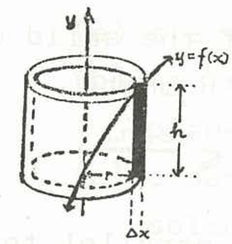
NOTE: One of the advantages of using cylindrical shells is that we can integrate with respect to x even though the rotation of th region is about the y axis.

SUMMARY CONTINUED.

NOTE: As a matter of interest one may investigate the relationship between the volumes of a "washer" and of a "cylindrical shell". Consider the volume of a washer, which can be written as



$$\Delta V = \text{base area} \times \text{height}$$
$$= \pi(x_2^2 - x_1^2).\Delta y$$
$$= 2\pi\tfrac{1}{2}(x_2 + x_1)(x_2 - x_1).\Delta y$$



Let us flatten the washer by hammering it perpendicular to its axis of rotation, then $(x_2 - x_1) \rightarrow \Delta x$ and $\Delta y \rightarrow h$. From previous proof we know that $\frac{1}{2}(x_2 + x_1) = r$ the average radius. Now the volume can be written as $\Delta V = 2\pi r.\Delta x.h$ by following the same order of factors as above. Finally, by replacing r by x and h by $f(x)$ we have $\Delta V = 2\pi x.f(x).\Delta x$ which is the volume of a "cylindrical shell". This is the algebraic transformation of a washer into a cylindrical shell.

NOTE: The Theorem of Pappus: "If a plane region of area A is revolved about a line that lies in its plane but does not intersect the region, then the volume generated is equal to the product of the area A and the distance travelled by its centroid"(See a "proof" of this at Misc.Q.1.W.Sol.) That is $V = 2\pi rA$. This formula is easy to use when the region being rotated has an axis of symmetry parallel to the axis of rotation. Since the centroid lies on the axis of symmetry of a figure, the radius of the circle on which the centroid moves is r, the distance between the two parallel axes! It is very important to observe that <u>A is not the cross-sectional area of a slice!!!</u>

NOTE: Rather than rely on formulae only, such as

$$V = \pi\int_a^b (x_2^2 - x_1^2)dy \qquad \text{or}$$

$$V = \int_a^b 2\pi x f(x)dx$$

follow the 5 steps introduced in the examples.

DRAWING DIAGRAMS IS A MUST !!!!!! A good diagram solves $\frac{1}{2}$ of the problem!!!!!

EXERCISES
UNIT TWO

Q.1. Find the volume of the solid in Example 2. by using slices parallel to the y axis.

Q.2. The region formed by the curve $y = 4 - x^2$, the lines $x = 2$ and $y = 4$ is revolved around the y axis. Use slices parallel to the y axis to find the volume of the solid of revolution.

Q.3. Using cylindrical shells find the volume of the solid obtained by rotating about the y axis the region R where

$$R = \{(x,y) : 0 \leqslant x \leqslant \pi, \quad 0 \leqslant y \leqslant \frac{\sin x}{x}\}.$$

Q.4. Repeat Example 4. but this time use slices parallel to the y axis. (Hint: substitute $x = b + a\cos\theta$ when integrating.)

Q.5. Suppose that $f(a) = f(b) = 0$ and the region

$$R = \{(x,y) : 0 \leqslant y \leqslant f(x), \ 0 \leqslant a \leqslant x \leqslant b\}$$

is rotated about the y axis. By considering a thin cylindrical slice (shell) of height $f(x)$ with an annulus base of radii x and $x + \Delta x$. Prove that the volume of the solid obtained is

$$V = \int_a^b 2\pi xy\,dx$$

Q.6. Repeat Ex. Unit One Q.9. by using a different method of slicing.

Q.7. The region between the curve $y = \frac{1}{2}\sqrt{(x - 2)}$, the line $x = 6$ and the x axis is revolved about the y axis. By using slices parallel to the y axis find the volume of the solid generated.

Q.8. The base of a certain solid is a circular disc with a radius of 3 cm. Cross-sections perpendicular to a diameter of the disc is

(a) an isosceles triangle of height k with the base being in the plane of the circle.

(b) an equilateral triangle with one side in the base of the solid,

(c) an isosceles right triangle with

(i) the unequal side

(ii) one of the equal side

in the base of the solid.

(d) a square, with one side in the base of the solid,

(e) a semicircle with a diameter in the base of the solid

Find the volumes of these solids. Draw diagrams and follow the 5 steps demonstrated in Example 8.

Q.9. The base of a certain solid is an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Every cross-section perpendicular to the major axis is

(a) a square with one side in the base of the solid,

(b) an isosceles right triangle with the base in the plane of the ellipse,

(c) a semicircle with its diameter in the base of the solid.

Find the volumes of these solids. Follow the steps shown in Example 10.

Q.10. Find the volume of such a solid which is constructed with a square base of side s. Cross-sections perpendicular to one of its diagonal are squares also, with one of its side in the base of the solid.

Q.11. Determine the volume of a wedge cut off a circular cylinder by a plane passing through the diameter of the base and inclined to the base at an angle of θ. If the radius of the base is R, show by using slices perpendicular to the line of intersection of the planes, that its volume is $\frac{2}{3}R^3 \tan\theta$.

Q.12. Calculate the volume of a solid ellipsoid such that three mutually perpendicular cross-sections are bounded by ellipses whose axes are 2a, 2b and 2c. You may assume that the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is πab square units.

Q.13. A solid has its base the region bounded by the curves $y = x$ and $x = 2y - \frac{1}{2}y^2$. Cross-sections parallel to the x axis are

(a) semicircles with diameter in the base

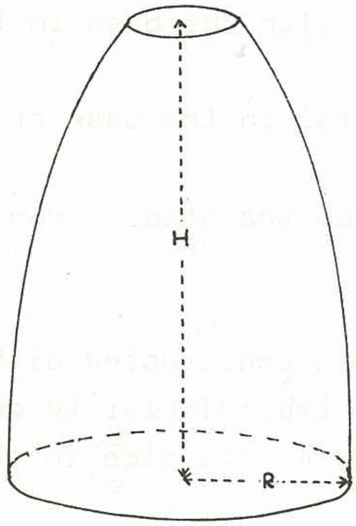
(b) equilateral triangles with a side in the base.

Find the volumes of these solids.

Q.14. The base of a certain solid is the region between the curves $y = x$ and $y = x^2$. Each plane section of the solid perpendicular to the x axis is a semicircle with its diameter in the base of the solid. Show that the volume of this solid is $\pi/240$

Q.15. The base of a solid is a parabolic segment of the parabola $y = x^2$, cut off by the chord $y = 4$. Each plane sections of the solid perpendicular to the axis of the parabola is a rectangle whose base is the chord and its height is $\frac{1}{2}(4 - y)$ units. Find the volume of this solid.

Q.16. A wooden pillar is carved out of a log of timber, having the shape of a truncated "cone". Its slant height is not a straight line but a curve. Every cross-section by a plane perpendicular to the axis of the solid is a circle of radius r. At height h



$$\frac{r(h)}{R} = \frac{1}{(1 + h^2)^{\frac{1}{4}}}$$

where R is the radius of the base and H is the height of the truncated cone.

Find the volume of the pillar in terms of R and H. Calculate the volume if R = 1 metre and H = 10 cm. (Answer in m^3 correct to 3 significant figures.)

Q.17. Show that the volume common to two right circular cylinders intersecting at right angles and both having the same radius of a units is

$$V = \frac{16a^3}{3} \qquad \text{(Archimedes)}$$

Q.18. Find the volume of the solid of revolution Ex.Unit One Q.4.(i) and (ii) by using slices parallel to the line $y = 4$ and parallel to $x = 2$ respectively.

Q.19. The side projection of "Katy's Fish" is the area enclosed by the curve $y^2 = (x - 1)(x - 3)^2$ and the line $x = 4$. All cross sections perpendicular to this dorsal - ventral plane are ellipses whose major axes are perpendicular to the spine of the fish. The minor axes are half of the length of the corresponding major axes. Find the volume of "Kathy's Fish". (By Katherine Merrick Ascham School 1982.)

Q.20. Make up shapes which can not be generated by rotation and find their volumes. Similar to Q.19. above.

Q.21. Find the volume of the solid of revolution formed when the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is rotated through one

Q.21. (cont.)

complete revolution about the line $y = b$.

Q.22. The Theorem of Pappus.

"If a plane region of area A is revolved about a line that lies in its plane but does not intersect the region, then the volume generated is

$$V = 2\pi r A$$

i.e. the product of the area A and the distance travelled by its centroid"

The proof of this theorem is rather complicated, but the end result provides you with a useful formula to check some of your answers. (See notes page 31 also.) In Example 4. the circle $(x - b)^2 + y^2 = a^2$ (where $a<b$) of area πa^2 was rotated about the y axis to form the Torus (doughnut shape). Its centroid (centre in this case) travels a distance of $2\pi b$ units, so its volume $V = 2\pi b.\pi a^2 = 2\pi^2 a^2 b$

Use Pappus' Theorem to verify your answers to Ex.Unit One Q.14., Ex. Unit Two Q.21.

Q.23. Prove that the area of the ellipse $E: x = a\cos\theta,\ y = b\sin\theta$ is πab square units. Hence or otherwise show that the volume of an elliptic cone of height h, standing on the ellipse E as its base is

$$V = \tfrac{\pi}{3}abh$$

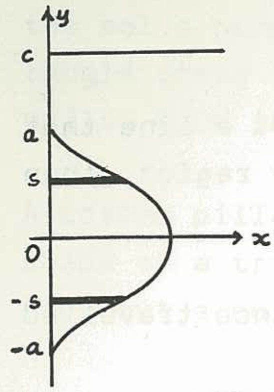
cubic units.

Q.24. Explain the existence of the lovely formula

$$\frac{d}{dr}(\pi r^2) = 2\pi r$$

in the "spirit" of EXAMPLE 13. (Hint: "Divide" the circle into thin annuli of width Δx. Consider the area of the circle to be the sum of the areas of all annuli.)

Question 1



(i) The diagram shows the area A between the smooth curve $x = f(y)$ and the y axis. $f(y) \geqq 0$ for $-a \leqq y \leqq a$ and $f(-a) = f(a) = 0$. The area A is rotated about the line $y = c$ (where $c > a$) to generate a solid whose volume is V. This volume is to be found by slicing A into horizontal strips, rotating these to obtain cylindrical shells, and adding the shells. Two of these strips (shaded) of width δs whose "centre lines" are distance s from the x axis are shown.

(a) Show that the indicated strips generate shells of approximate volume $2\pi.f(s)(c-s)\delta s$ and $2\pi.f(-s)(c+s)\delta s$.

(b) Given that the graph of f is symmetrical about the x axis show that

$$V = 2\pi c A$$

(ii) Assuming the results of part (i), solve the following problems:

(a) a torus (doughnut shape) is formed by rotating a circular disc of radius r about an axis in its own plane at a distance c ($c>r$) from the centre of the disc. Find the volume of the torus.

(b) Find the volume of the solid generated when the region between the curve $x = 2 + 4y^2$ and the line $x = 6$ is rotated about the line $y = -1$.

(c) The shape of a certain cake can be represented by rotating the area between the curve $y = \sin x, -\pi \leqq x \leqq 0$, and the x axis about the line $x = \pi/4$.

Question 2

(i) A is the area of the Region R_1 bounded by the upper branch of the hyperbola $y^2/b^2 - x^2/a^2 = 1$ the x axis and the lines $x = \pm a$.

(a) Show that $A = \frac{Lb}{2}[\sqrt{2} + \ln(1 + \sqrt{2})]$ square unit where L

(b) S_1 is the solid whose base is the ellipse $x^2/a^2 + y^2/b^2 = 1$. Cross sections perpendicular to the base and to the minor axis, are plane figures similar to region R_1 where the line of intersection of the planes is the base length of R_1. Find the volume of S_1.

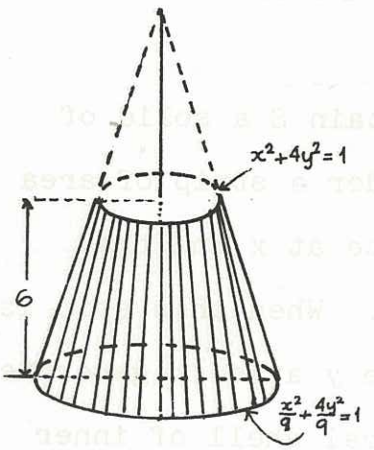
(ii) (a) R_2 is the region obtained by the reflection of region R_1 in the x axis. Find the volume of the biconcave lens which can be generated by rotating R_2 about the y axis. Use the cylindrical shell method.

(b) S_2 is the solid whose base is the region R_2. Cross sections perpendicular to the base and to the x axis are the semi ellipses of the form $x^2/a^2 + y^2/b^2 = 1$. The line of intersection of the planes i.e. the distance between the branches is the major axis of the semi elliptical cross section. Show that the area of the semi ellipse with semi major axis a, and semi minor axis b is $\frac{1}{2}\pi ab$. Hence calculate the volume of S_2.

Question 3

(a) Prove that the area of the conic $x = a\cos\theta$, $y = b\sin\theta$ is πab sq. units.

(b) A right elliptical cone has its top cut off by a plane parallel to its elliptical base. The remaining solid has an ellipse at its base of equation $x^2/9 + 4y^2/9 = 1$ and another ellipse at its top of equation $x^2 + 4y^2 = 1$. The height of the solid is 6 units. By finding the area of a cross sectional slice of the solid, calculate the volume of the solid.



Question 4

The plans for a special wave guide antenna are shown on the

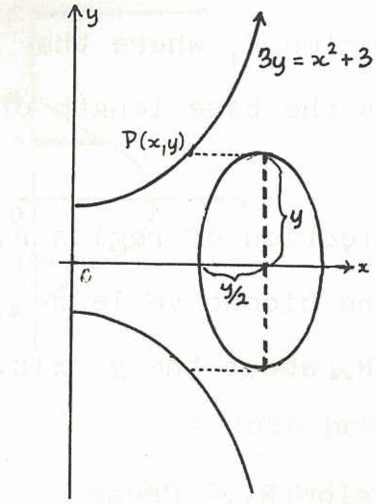


diagramme. Each cross section is a plane perpendicular to the central axis of the wave guide (i.e. to the x axis) is an ellipse whose major axis is twice the length of its minor axis. The upper boundary of the widest longitudinal cross section is the parabola $3y = x^2 + 3$. The entire antenna is 2 m. long.

(a) Show that the area of the ellipse is πab sq. units.

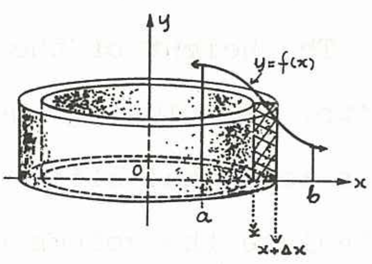
(b) Show that the area A(x) of a cross section cut from the antenna by a plane perpendicular to the x axis at a point (x,y) is $\pi(x^4/18 + x^2/3 + \frac{1}{2})$ sq. units.

(c) Hence find the volume of the region enclosecd by the wave guide.

Question 5

(a) Consider a region R in the first quadrant, bounded by the graph of a continuous function f, the x axis and the lines $x = a$ and $x = b$, where $0 \leqq a \leqq b$ and $f(x) \geqq 0$ for all x in [a,b].

Rotate region R about the y axis to obtain S a solid of revolution. Consider a strip of area between the ordinate at x and the ordinate at $x + \Delta x$. When this stip is revolved around the y axis it generates a hollow, cylindrical shell of inner radius x, outer radius $x + \Delta x$ and volume ΔV. By considering the volume of all



cylindrical shells prove that the volume of S is given by

$$V = \int^{b} 2\pi x f(x) dx .$$

(b) The volume generated by revolving about the y axis the region bounded by the coordinate axis, the line $y = h$ and the line $y = h(x-r_2)/(r_1-r_2)$ i.e. the volume of the truncated cone of height h and radii r_1 and r_2 where $r_1 \neq r_2$. Use the method in part (a) to show the volume of the solid generated is $\frac{1}{3}\pi h(r_1^2 + r_1 r_2 + r_2^2)$

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ANSWERS

MISC. EXAM. TYPE QUESTIONS

Q.1. (ii) (a) $2\pi^2 r^2 c$ (b) $32\pi/3$ (c) $3\pi^2$

Q.2. (i) (b) $\frac{\pi L b^2}{4}[\sqrt{2} + \ln(1 + \sqrt{2})]$ (ii) (a) $\frac{\pi}{3}L^2 b(2\sqrt{2} - 1)$

(ii) (b) $\pi ab^2[\sqrt{2} + \ln(1 + \sqrt{2})]$ cubic units

Q.3. (b) $2\pi ab$

Q.4. (x) $101\pi/45$

UNIT ONE EXERCISES

Q.1. (1) $\frac{2\pi}{15}$ (2) $\frac{128\pi}{7}$ (3) $\frac{768\pi}{7}$ (4) $\frac{\pi a^3}{15}$ (5) $\frac{224\pi}{15}$

Q.2. (1) 8π (2) $\frac{32\pi}{5}$ (3) $\frac{128\pi}{5}$ (4) $\frac{64\pi}{7}$ (5) $\frac{816\pi}{5}$

Q.3. $\frac{\pi^2}{2}$ Q.4. (i) $\frac{32\pi}{5}$ (ii) $\frac{8\pi}{3}$ Q.6. $\frac{32\pi a^3}{15}$

Q.7. $\frac{4\pi ab^2}{3}$ Q.8. $\frac{4\pi ba^2}{3}$ Q.9. $\frac{8\pi}{3}$ Q.11. $\frac{704\pi}{15}$

Q.12. $\frac{188\pi}{15}$ Q.13. (i) $160\pi\sqrt{15}$ cm^3 (ii) $540\pi\sqrt{15}$ cm^3

Q.13. (iii) $\frac{\sqrt{3}\pi R^3}{2}$ (iv) $\frac{5\pi\sqrt{15}R^3}{16}$ Q.14. $2\pi^2 a^2 b$

Q.15. $a = 2$, $n = 3$, $V = \frac{768\pi}{7}$, $V_1 : V_2 = 1 : 6 = 1 : 2n$.

UNIT TWO EXERCISES

Q.2. 8π Q.3. 4π Q.7. $\frac{352\pi}{15}$ Q.8. (a) $\frac{9\pi k}{2}$ cm^3

Q.8. (b) $36\sqrt{3}$ cm^3 (c) (i) 36 cm^3 (ii) 72 cm^3 (d) 144 cm^3

Q.8. (e) 18π cm^3 Q.9. (a) $\frac{16ab^2}{3}$ (b) $\frac{4ab^2}{3}$ (c) $\frac{2\pi ab^2}{3}$

Q.10. $\frac{4\sqrt{2}s^3}{3}$ Q.12. $\frac{4\pi abc}{3}$ Q.13. (a) $\frac{\pi}{30}$ (b) $\frac{\sqrt{3}}{15}$

Q.15. $\frac{128}{15}$ Q.16. $V = \pi R^2 \log[H + \sqrt{(1 + H^2)}]$, 9.42 m^3

Q.19. $\frac{9\pi}{8}$ Q.21. $2\pi^2 b^2 a$