2015 Bored of Studies Trial Examinations

Mathematics Extension 2

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General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11 – 16.

Total Marks – 100

Section I Pages 1 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 7 – 22

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.

Total marks – 10
Attempt Questions 1 – 10
All questions are of equal value

Shade your answers in the appropriate box in the Multiple Choice answer sheet provided.

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1 The equation $x^2 - 2ix + k = 0$ has a root of $-1+2i$? What is the value of *k*?

(A) $-1+2i$.

(B) $-1-2i$.

(C) 1.

(D) 5.

2 Let *e* be the eccentricity of a conic section with both foci on the *x* axis.

Which of the following is NOT always true?

(A) For the hyperbola, as $e \to \infty$, the asymptotes approach the *y* axis.

(B) If two ellipses have equal eccentricity, then they have the same equation.

(C) If two hyperbolae have equal eccentricity, then they share the same asymptotes.

(D) For the ellipse, as $e \to 0$, the directrices move further away from the origin whilst the foci approach the origin.

3 The hyperbolae $\mathcal{H}_1$ and $\mathcal{H}_2$ have equations $x^2 - y^2 = a^2$ and $xy = a^2$ respectively, where a is a positive constant.

Which of the following is a point of intersection of one of the directrices of $\mathcal{H}_2$ and one of the asymptotes of $\mathcal{H}_1$?

(A) $\left(\dfrac{a}{2}, \dfrac{a}{2}\right)$.

(B) $\left(\dfrac{a}{\sqrt{2}}, \dfrac{a}{\sqrt{2}}\right)$.

(C) $(2a, 2a)$.

(D) $\left(a\sqrt{2}, a\sqrt{2}\right)$.

4 Sophia is given four options to transform any given curve.

(I) Translate 1 unit to the left, then double the x coordinates.

(II) Double the x coordinates, then translate 1 unit to the left.

(III) Reflect in the line $x = -1/2$, then multiply the x coordinates by -2.

(IV) Multiply the x coordinates by -2, then reflect in the line $x = -1/2$.

She is then given a sketch of the curve $y = f(2x-1)$.

Which of the above options should she select to obtain the graph of $y = f(x)$?

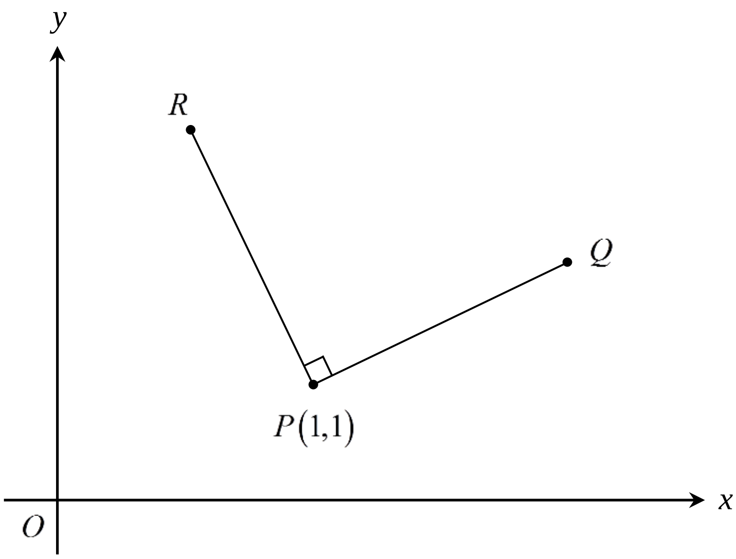
(A) Options (I) or (III).

(B) Options (I) or (IV).

(C) Options (II) or (III).

(D) Options (II) or (IV).

5 Let P be a point on the Argand diagram with coordinates $(1,1)$. Let Q be another point such that the complex number represented by vector PQ is given by $a+bi$.



The vector PQ is rotated about the point P anti-clockwise by an angle of $\frac{\pi}{2}$ to form the vector PR.

What are the coordinates of R ?

(A) $(-b,a)$.

(B) $(1-b,1+a)$.

(C) $(b,-a)$.

(D) $(1+b,1-a)$.

6 Let α be a non-zero complex number and $P(x)$ be a monic polynomial with real coefficients such that $P(\alpha) = P'(\alpha) = 0$.

Which of the following statements is always true?

(A) The complex number α must be a real number.

(B) The polynomial $P(x)$ has a stationary point at $(\alpha, 0)$.

(C) If α is non-real, then the polynomial $P(x)$ must have even degree.

(D) If α is non-real, then the smallest possible degree of $P(x)$ is 4.

7 A solid of revolution is formed by rotating the region bounded by $y = 2x - x^2$, the line $x = 1$ and the x axis about the line $x = 1$.

By using the method of cylindrical shells, which of the following integrals is equal to the volume of the solid?

(A) $\pi \int_0^1 (1-x)(2x - x^2)\, dx.$

(B) $2\pi \int_0^1 x(2x - x^2)\, dx.$

(C) $\pi \int_0^1 x\, dx.$

(D) $2\pi \int_0^1 x\, dx.$

8 A particle moves in uniform circular motion about the origin with angular velocity ω.

Let x be the horizontal displacement of the particle.

Which of the following is the correct expression for the horizontal component of the particle's acceleration?

(A) $\ddot{x} = -\omega x$.

(B) $\ddot{x} = \omega x$.

(C) $\ddot{x} = -\omega^2 x$.

(D) $\ddot{x} = \omega^2 x$.

9 Let a, b and c be the lengths of the sides of $\triangle ABC$, where the longest side has length a.

If $\angle BAC$ is obtuse, which of the following inequalities is always true?

(A) $a^2 > 2bc$.

(B) $a^2 > b + c$.

(C) $a^2 < 2bc$.

(D) $a^2 < b + c$.

10 Let n and k be positive integers.

Which of the following is equivalent to $\int_0^{\pi} \sin^n(x)\cos(2kx)\,dx$?

(A) $\int_0^{\frac{\pi}{2}} \sin^n\left(\frac{x}{2}\right)\cos(kx)\,dx$.

(B) $\int_{\frac{\pi}{2}}^{\pi} \sin^n\left(\frac{x}{2}\right)\cos(kx)\,dx$.

(C) $\int_0^{\pi} \sin^n\left(\frac{x}{2}\right)\cos(kx)\,dx$.

(D) $\int_0^{2\pi} \sin^n\left(\frac{x}{2}\right)\cos(kx)\,dx$.

Total marks – 90
Attempt Questions 11 – 16
All questions are of equal value

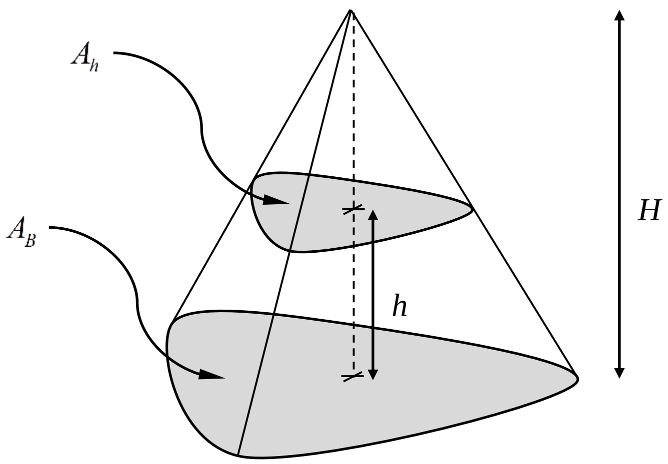
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int \frac{\sqrt{\sin\theta}}{\cos\theta\sin\theta}\,d\theta.$ **3**

(b) The diagram below shows a right pyramid with height H and an irregularly shaped base with area A_B. **3**

Cross sections taken parallel to the ground at height h have area A_h.



Show that the volume of the pyramid is $\frac{1}{3}A_B H$.

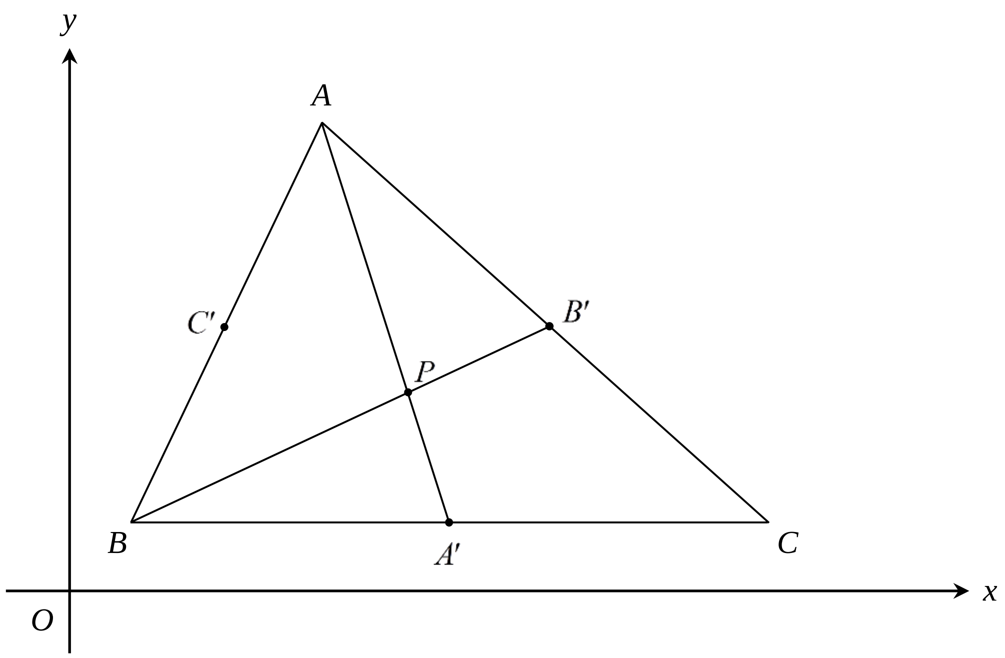
(c) The polynomial $P(x) = x^3 - px^2 + p$ has roots α, β, γ. **4**

Find the values of p such that

$$\frac{1}{(\alpha+\beta)^2} + \frac{1}{(\beta+\gamma)^2} + \frac{1}{(\alpha+\gamma)^2} = 0.$$

Question 11 continues on page 8

(d) Points A, B and C represent the complex numbers a, b and c respectively on the Argand diagram, as shown below.



Let A', B' and C' be the midpoints of BC, AC and AB respectively.

Let the intersection of AA' and BB' be P, representing the complex number p.

You may assume, without proof, that $\dfrac{AP}{AA'} = \dfrac{BP}{BB'} = k$, where $0 < k < 1$.

(i) Show that any point P on the interval AA' can be expressed as **1**

$$p = (1-k)a + \frac{k}{2}(b+c).$$

(ii) Use (i) to show that $p = \dfrac{a+b+c}{3}$. **3**

(iii) Deduce that P also lies on CC'. **1**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the curve $\mathcal{C}$ implicitly defined by the equation

$$x^3 - y^3 = 3\ln(xy).$$

(i) Show that $(1,1)$ is a critical point. **2**

(ii) Let (x_0, y_0) be any point on $\mathcal{C}$. **1**

Show that $(-y_0, -x_0)$ also lies on $\mathcal{C}$.

(iii) By using the fact that **1**

$$\lim_{x\to\infty}\left(\frac{\ln x}{x^3}\right) = 0,$$

or otherwise, show that if $x \to \infty$, then $y \to \infty$.

(iv) By considering the expression **2**

$$\left(\frac{y^3}{y^3 + 3\ln y}\right)\left(\frac{x^3 - 3\ln x}{x^3}\right),$$

or otherwise, show that $y = x$ is an asymptote.

(v) Hence, sketch the graph of the curve. **2**

Question 12 continues on page 10

(b) Consider the equation $z^n = 1$ with roots $\alpha_k = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$ for $k = 1, 2, 3, \ldots, n$.

(i) Show that **1**

$$(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\ldots(1-\alpha_{n-1}) = n.$$

(ii) Hence, show that **3**

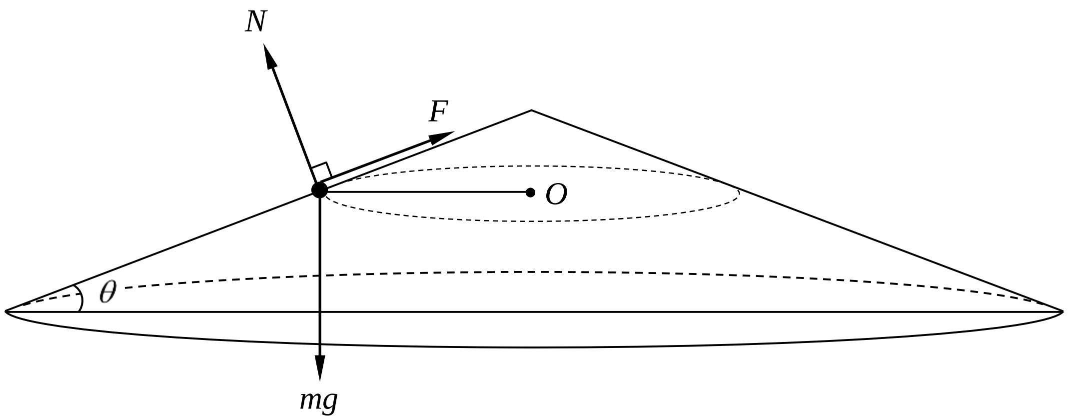
$$\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)\ldots\sin\left(\frac{n-1}{n}\pi\right) = \frac{n}{2^{n-1}}.$$

Question 12 continues on page 11

Question 12 (continued)

(c) The diagram below shows a car of mass m on the cross section of a bend in a highway, which is part of a circle of radius r and centre O. The road surface is banked at an angle θ to the horizontal.

The car experiences a normal force N, a lateral force F directed up the bend and a gravitational force g.



(i) Find the maximum possible velocity of the car before it leaves the surface of the cone. **2**

(ii) Hence, or otherwise, show that the maximum amount of lateral force the car will experience is **1**

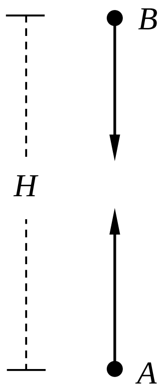
$$F = \frac{mg}{\sin\theta}.$$

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A particle A of mass m is projected vertically with initial velocity u and reaches maximum height H. It experiences gravitational force mg and a resistive force mkv_A^2, where v_A is the velocity of the particle and k is a constant.

At the same time, particle B of equal mass is dropped from height H above the ground. It experiences gravitational force mg and resistive force mkv_B^2.



Let w be the terminal velocity of particle B.

(i) Show that the displacement of particle A from the origin is **3**

$$x_A = \frac{1}{2k}\ln\left(\frac{w^2+u^2}{w^2+v_A^2}\right),$$

where v_A is the velocity of particle A.

(ii) The two particles collide when particle A reaches velocity w. **2**

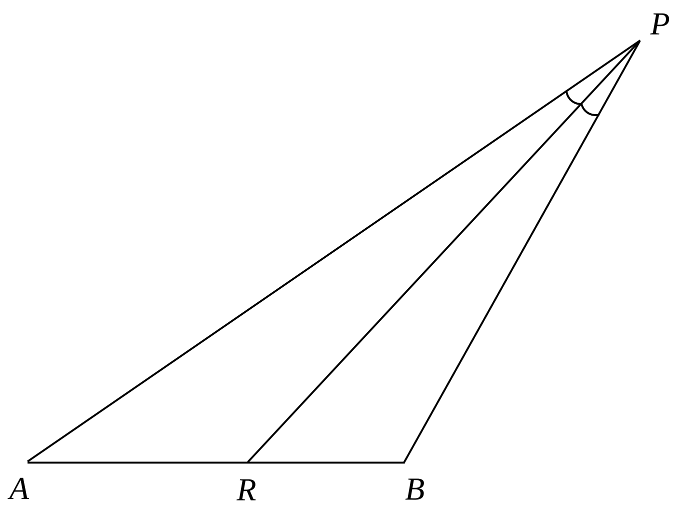
The displacement of particle B from the point of release is given by

$$x_B = \frac{1}{2k}\ln\left(\frac{w^2}{w^2-v_B{}^2}\right)$$ **(Do NOT prove this)**

Use this result to show that when the particles collide, particle B has velocity $v_B = \dfrac{w}{\sqrt{2}}$.

Question 13 continues on page 13

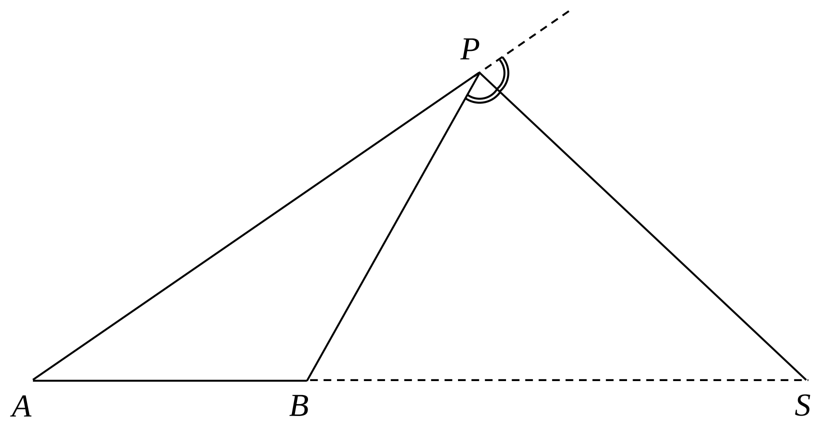
(b) In ΔPAB, the point R is chosen so that PR bisects $\angle APB$.



(i) Prove that $\dfrac{AP}{BP} = \dfrac{AR}{BR}$. **2**

It can similarly be shown that if PS bisects the exterior angle of $\angle APB$, then

$$\frac{AP}{BP} = \frac{AS}{BS}.$$ **(Do NOT prove this)**



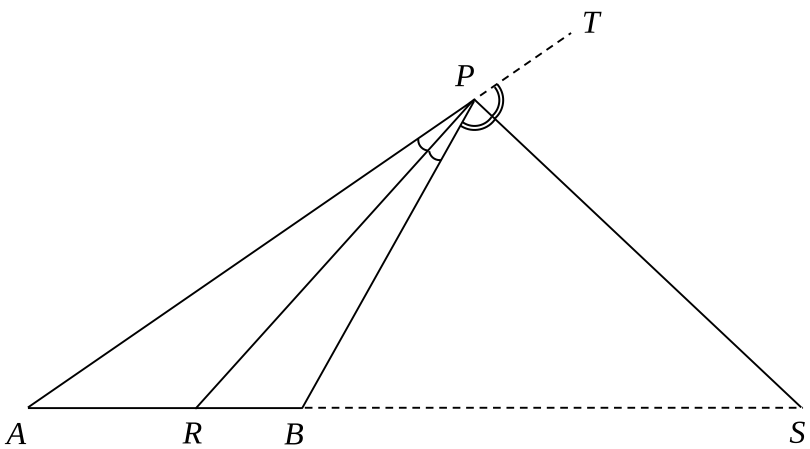
Question 13 continues on page 14

Question 13 (continued)

(ii) The diagram below shows fixed points A and B, and a point P so that **2**

$$\frac{PA}{PB} = k .$$

for some fixed $k > 1$.



The interval AP is produced to T and the points R and S are chosen such that PR and PS bisect $\angle APB$ and $\angle BPT$ respectively.

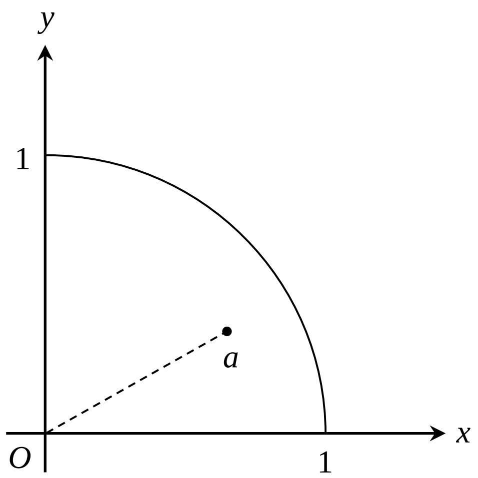
Use (i), or otherwise, to explain why as P varies, R and S remain fixed.

(iii) Deduce that the locus of P is a circle with diameter RS. **2**

Question 13 continues on page 15

Question 13 (continued)

(c) The diagram below shows a fixed complex number a strictly inside the unit circle on the Argand diagram.



Copy the diagram into your writing booklet.

(i) Use part (b), or otherwise, to explain why the locus of z satisfying **2**

$$\left|\frac{z-a}{1-\bar{a}z}\right| = r\,,$$

where $r|a| > 1$, is a circle.

You do not need to find the centre and radius of the circle.

(ii) Find the value of r such that the locus of z touches the unit circle. **2**

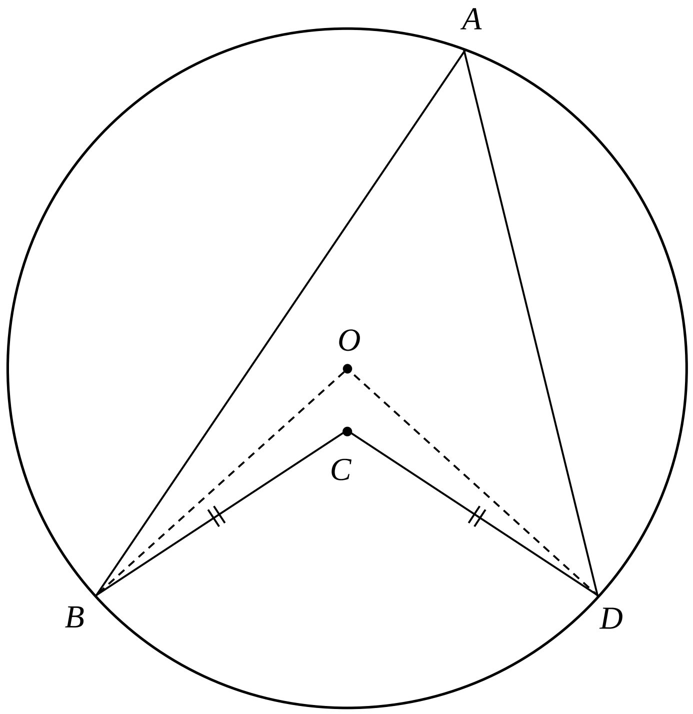
End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows a quadrilateral $ABCD$, where $BC = CD$ and $\angle BCD = 2 \times \angle BAD$.

A circle is drawn to pass through points A, B and D.

Let the centre of this circle be O and assume that the point C is distinct from O.



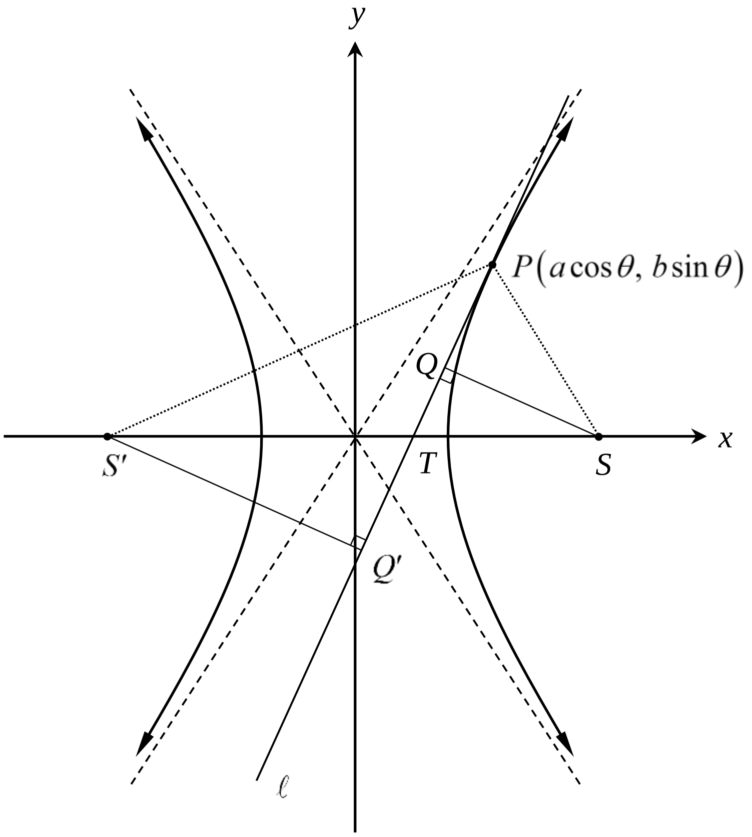
Copy the diagram into your writing booklet.

(i) Show that $\triangle COB \equiv \triangle COD$. **1**

(ii) Hence, prove that the point C cannot be distinct from O. **2**

Question 14 continues on page 17

(b) The diagram below shows a point $P(a\cos\theta, b\sin\theta)$ on the positive branch of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with foci S and S'. From P, a tangent ℓ with equation $\dfrac{x\cos\theta}{a} - \dfrac{y\sin\theta}{b} = 1$ is drawn to intersect the x axis at T. Let Q and Q' be points on ℓ such that QS and $Q'S'$ are perpendicular to ℓ.



(i) Show that $\dfrac{TS}{TS'} = \dfrac{PS}{PS'}$. **3**

(ii) Show that $\dfrac{QS}{Q'S'} = \dfrac{PS}{PS'}$. **2**

(iii) Deduce that the line ℓ bisects $\angle SPS'$. **1**

Question 14 continues on page 18

(c) It can be shown that for positive integer values of n,

$$1+2(\cos x+\cos 2x+\cos 3x+...+\cos nx)=\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} \qquad \textbf{(Do NOT prove this)}$$

(i) Show that **1**

$$\int_0^{\frac{\pi}{2}} \frac{\sin(n+1/2)x}{\sin(x/2)}\,dx=\frac{\pi}{2}+2\left(1-\frac{1}{3}+\frac{1}{5}-...+\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\right).$$

(ii) Show that **2**

$$\left|\int_{\frac{\pi}{2}}^{\pi} \frac{\sin(n+1/2)x}{\sin(x/2)}\,dx\right|<\frac{1}{2n+1}\left[2\sqrt{2}+\int_{\frac{\pi}{2}}^{\pi}\frac{\cos(x/2)}{\sin^2(x/2)}\,dx\right].$$

(iii) Hence, show that as $n\to\infty$, **2**

$$\int_0^{\frac{\pi}{2}} \frac{\sin(n+1/2)x}{\sin(x/2)}\,dx\to\pi .$$

(iv) Deduce that **1**

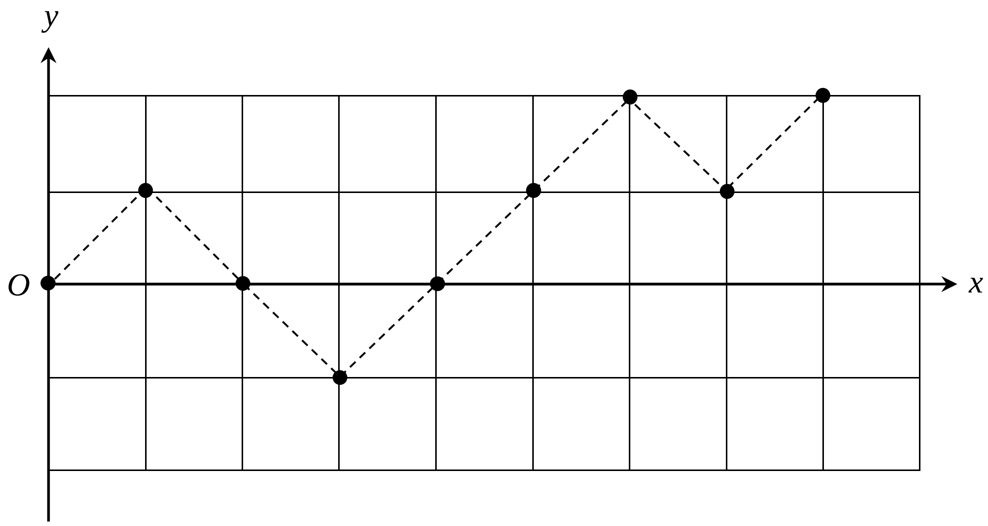
$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...$$

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) A sequence of letters is made using a copies of the letter U and b copies of the letter D in some order, where $a > b$. The sequence can be expressed graphically by matching the letters U and D to a series of movements. From the current position $P(x, y)$, the letter U moves P to $(x+1, y+1)$ whereas the letter D moves P to $(x+1, y-1)$.

The diagram below shows the corresponding graph of the sequence *UDDUUUDU*.



(i) Explain why if the sequence starts from the origin, it will then terminate at the point $T(a+b,\ a-b)$. **1**

(ii) Write down the number of possible paths from the origin to T. **1**

(iii) Explain why the number of paths from $(1,1)$ to T that touches or crosses the x axis is equal to the number of paths from $(1,-1)$ to T. **1**

(iv) Hence, show that the number of paths from $(1,1)$ to T that do NOT touch or cross the x axis is **2**

$$\frac{a-b}{a+b}\binom{a+b}{a}.$$

Question 15 continues on page 20

(b) Two candidates A and B are polled against each other in a vote. Each vote is written on identical cards and placed in a bag. The tally is done by pulling a card randomly from the bag, recording it, then discarding the card. **2**

Candidate A has p votes and Candidate B has q votes, where $p > q$.

Using part (a), or otherwise, find the probability that at all times during the vote count, Candidate A has a higher tally than Candidate B.

(c) Sketch the graph of $y = \dfrac{x^2 - 1}{x^2 + 1}$, labelling any asymptotes, stationary points and intercepts. **2**

(d) Let $P(x) = kx^3 + kx + 2$ be a cubic polynomial with real and non-zero coefficients.

(i) Express the polynomial $P(x)$ in the form **1**

$$P(x) = \left(x^2 + a\right)\left(kx + b\right) - \left(x^2 - a\right),$$

for appropriate values of a and b.

(ii) By considering the sketch from part (c), explain why $P(x)$ has exactly one real root. **1**

(iii) Describe the behaviour of the real root of $P(x)$ as $k \to \infty$. **1**

(iv) Hence describe the behaviour of the modulus and argument of the non-real roots of $P(x)$ as $k \to \infty$. **3**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Let $f(x)$ be a non-linear continuous function in the interval $a \le x \le b$, where $a \ne b$.

Define

$$E = \frac{b-a}{2}\left[f(a)+f(b)\right] - \int_a^b f(x)\,dx.$$

(i) Show that **2**

$$E = \int_a^b x\,f'(x)\,dx - \frac{a+b}{2}\left[f(b)-f(a)\right].$$

(ii) Show that **2**

$$E = \frac{1}{2}\int_a^b (b-x)(x-a)\,f''(x)\,dx.$$

(iii) Suppose $f(x)$ is concave up for $a \le x \le b$. **2**

Prove that $E > 0$ and explain the significance of this result.

Question 16 continues on page 22

Question 16 (continued)

(b) A continuous function $f(x)$ has the property that for any $x_i > x_j$,

$$f'(x_i) \le f'(x_j).$$

(i) Explain, with the use of a graph, why for any $x_i > x_j$, **1**

$$f'(x_i) \le \frac{f(x_i) - f(x_j)}{x_i - x_j} \le f'(x_j).$$

(ii) Let α be a constant such that $0 < \alpha < 1$ and suppose that $x_1 > x_2$. **1**

Show that $x_2 < \alpha x_1 + (1-\alpha)x_2 < x_1$.

(iii) Let α be a constant such that $0 < \alpha < 1$. **2**

Use part (i), or otherwise, to show that for any x_1 and x_2,

$$f(\alpha x_1 + (1-\alpha)x_2) \ge \alpha f(x_1) + (1-\alpha) f(x_2).$$

(iv) Let $a_1, a_2, a_3, \dots, a_n$ be *any* set of positive numbers such that **4**

$$a_1 + a_2 + \dots + a_n = 1.$$

Use mathematical induction to prove that for $n \ge 2$

$$f(a_1x_1 + a_2x_2 + \dots + a_nx_n) \ge a_1 f(x_1) + a_2 f(x_2) + \dots + a_n f(x_n).$$

(v) By choosing an appropriate function for $f(x)$, show that **1**

$$\frac{x_1 + x_2 + \dots + x_n}{n} \ge \sqrt[n]{x_1x_2 \dots x_n},$$

for positive values of $x_1, x_2, \dots, x_n$.

End of Exam

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\, dx = \ln x, \quad x > 0$$

$$\int e^{ax}\, dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$