UNIT 3

Q1. 1. $xy = 9 \quad c^2 = 9 \quad \therefore c = 3 \qquad x = 3t,\ y = \frac{3}{t}$

Q1. 2. $xy = 16 \quad c^2 = 16 \quad c = 4 \qquad x = 4t \quad y = \frac{4}{t}$

Q1. 3. $xy = \frac{25}{4} \quad c = \frac{5}{2} \qquad x = \frac{5t}{2} \quad y = \frac{5}{2t}$

Q1. 4. $xy = \frac{1}{9} \quad c = \frac{1}{3} \qquad x = \frac{t}{3} \quad y = \frac{1}{3t}$

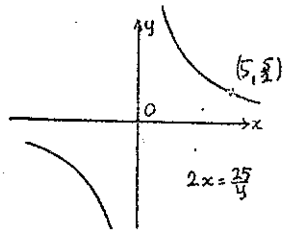
Q1. 5. $xy = 2 \quad c = \sqrt{2} \qquad x = \sqrt{2}t \quad y = \frac{\sqrt{2}}{t}$

Q1. 6. $-xy = +4 \quad c = 2 \qquad x = 2t \quad y = -\frac{2}{t}$

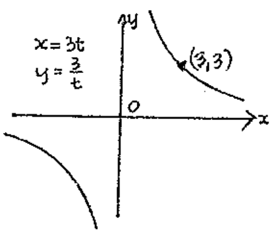
Q1. 7. $x = 5t \ y = \frac{5}{t} \iff xy = 25$

Q1. 10. $(t, -\frac{1}{t}) \Rightarrow xy = -1$

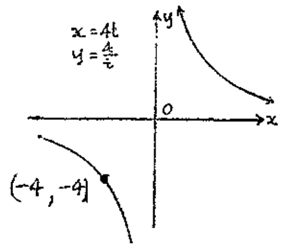
Q1. 11.



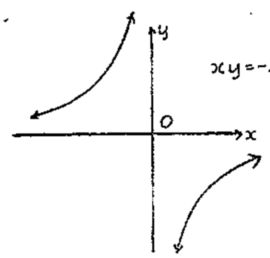
Q1. 12.



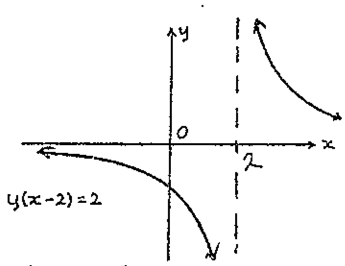
Q1. 13.



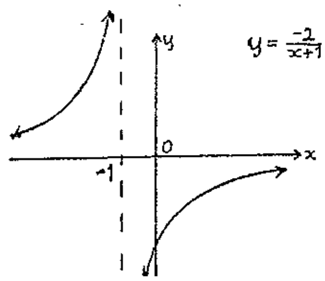
Q1. 14.



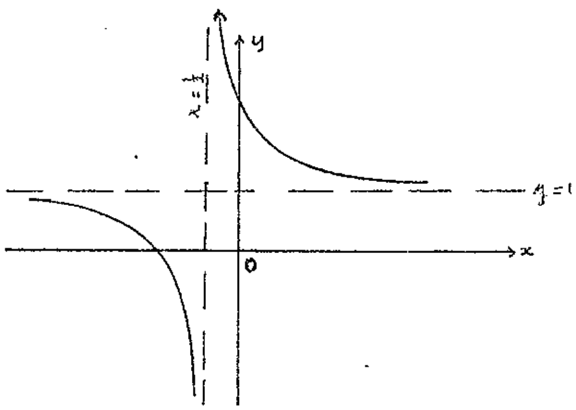
Q1. 15



Q1. 16



Q1. 17. $(y-1)(2x+1) = 16$



UNIT 3

Q1. 18. $xy = 18$ $\quad c^2 = 18 \quad \therefore 18 = \frac{a^2}{2}$ i.e. $a = 6$, $\underline{e = \sqrt{2}}$

(i) length of transv. axis = 12 units

$ae = 6\sqrt{2}$ $\qquad 2x^2 = (6\sqrt{2})^2$

$x = 6$

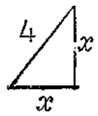
(ii) focus; $S(6,6)$ $\quad S^1(-6,6)$

Q1. 19. $xy = 4 \quad c^2 = \frac{a^2}{2} = 4 \quad \therefore a = 2\sqrt{2}$

(i) transv. axis is $4\sqrt{2}$ units

(ii) $ae = 2\sqrt{2}\sqrt{2}$

$= 4$ foci $S(2\sqrt{2}, 2\sqrt{2})$ $S^1(-2\sqrt{2}, -2\sqrt{2})$



$x^2 = 8$

$x = 2\sqrt{2}$

focus (ae·cos 45°, ae cos 45°)

$= (a,a)$

Q1. 20. $x = 8t$

$y = \frac{8}{t}$ $\quad xy = 64 \quad c^2 = \frac{a^2}{2} = 64 \quad a = 8\sqrt{2}$

(i) transverse axis; $8\sqrt{2} \times 2 = 16\sqrt{2}$ units.

(ii) foci; $S(8\sqrt{2}, 8\sqrt{2})$

$S^1(-8\sqrt{2}, -8\sqrt{2})$

Q1. 22. $xy = 8 \quad \therefore a^2 = 16 \quad a = 4$ $\qquad c = 2\sqrt{2}$

$a^1 = 4 \cos 45^\circ$

$= \frac{4}{\sqrt{2}}$

$\therefore V(4/\sqrt{2}, 4/\sqrt{2}) \qquad V^1(-4/\sqrt{2}, -4/\sqrt{2})$

Equ. of tangent;

$x + t^2y = 2ct$

$c = 2\sqrt{2}$

$x = 4/\sqrt{2}$

$y = 4/\sqrt{2}$

$x = ct$

$4/\sqrt{2} = 2\sqrt{2}t$

$t = \frac{4}{\sqrt{2}} \cdot \frac{1}{2\sqrt{2}} = 1$ $\qquad$ where $t = \pm 1$

tangents; $x + y = \pm 4\sqrt{2}$

UNIT 3

Q1. 23. Tang. and normal at $(4t, \frac{4}{t})$ on $xy = 16$ $\therefore c = 4$.

tangent; $x + t^2y = 8t$

normal; gradient $= t^2$

$$y - \frac{c}{t} = t^2(x - ct)$$

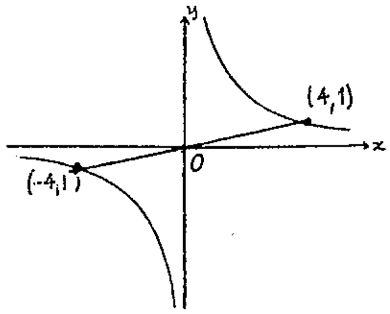
$$(y - \frac{4}{t}) = t^2(x - 4t)$$

$$t^3(x - 4t) = ty - 4$$

$$ty - t^3x = 4 - 4t^4$$

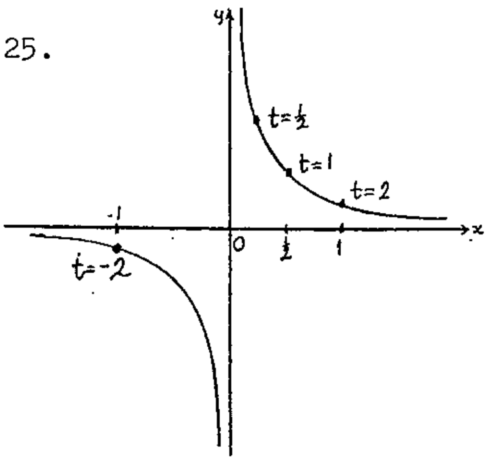
$$ty - t^3x = 4(1 - t^4)$$

Q1. 24. Length of diameter. $xy = 4$, through $(4,1)$



$d = 2\sqrt{17}$

Q1. 25.



$$\left.\begin{aligned} x &= \frac{t}{2} \\ y &= \frac{1}{2t} \end{aligned}\right\}$$

Q1. 26. $xy = 3$; tangents // $y + 3x = 0$

Equ. of tang; $x + t^2y = 2ct$. $m = -\frac{1}{t^2}$

$$\therefore -\frac{1}{t^2} = -3$$

hence $t = \pm\frac{1}{\sqrt{3}}$ $c = \sqrt{3}$

$\therefore$ equation of tangent // $y + 3x = 0$ is

$3x + y = \pm 6$ $d = 2\left|\frac{6}{\sqrt{10}}\right| = \frac{12}{\sqrt{10}}$

UNIT 3

Q1. 27. Pt. of contact of 2 tangents from $(-5,1)$ to $xy = 4$

Method 1: (non parametric)

Let $y = mx + c$ be the equation of the tangent.

$x(mx + c) = 4 \iff mx^2 + cx - 4 = 0$

if tangent, then $b^2 - 4ac = 0$ i.e. $c^2 + 16m = 0$ i.e. $m = -\frac{c^2}{16}$

Now we have $y = -\frac{c^2}{16}x + c$ as tangent. But it is through

$(-5,1)$ $\therefore$ $1 = \frac{5}{16}c^2 + c \iff (5c - 4)(c + 4)$ $\therefore c = -4$ or $\frac{4}{5}$

and the tangents are $y = -x - 4$ and $y = \frac{x + 20}{25}$.

$xy = 4 \cap y = -x - 4 \implies (x+2)(x+2) = 0$ $\therefore x = -2$ and $y = -2$

$xy = 4 \cap y = \frac{-x + 20}{25} \implies x^2 + 20x + 100 = 0$ $\therefore x = +10$ and $y = \frac{2}{5}$

So the pt. of contacts are $(10,\frac{2}{5})$ and $-2,-2)$.

Method 2:

Let $x + t^2y = 2ct$ be the tangent, which is a quadratic in t. ($\therefore$ it has 2 roots.)

Since $yt^2 - 2ct + x = 0$ is through $(-5,1)$ and $c = 2$ we have

$t^2 - 4t - 5 = 0 \iff (t-5)(t+1) = 0$

$\therefore$ $t = 5$ or $t = -1$.

$$\text{if } t = 5;\ \left.\begin{array}{l} x = ct \iff x = 10 \\ y = \frac{c}{t} \iff y = \frac{2}{5} \end{array}\right\} (10, 2/5)$$

$$\text{if } t = -1;\ \left.\begin{array}{l} x = ct \iff x = -2 \\ y = \frac{c}{t} \iff y = 2 \end{array}\right\} (-2,-2)$$

$\therefore$ the points of contact are $(10,\frac{2}{5})$, $(-2,-2)$.

UNIT 3

Q1. 28. Normal at $P(8,2)$ cuts $(4t, \frac{4}{t})$ at Q.

$x = 8 = 4t \therefore t = 2$

Gradient of normal; $t^2 = 4$

Equation of normal; $y - 2 = 4(x - 8)$

$$4x - y = 30$$

Normal $\cap\ xy = 16$ when $x(4x - 30) = 16$

$$2x^2 - 15x - 8 = 0$$

$$(2x+1)(x-8) = 0$$

$\therefore x = -\frac{1}{2}$ is the other abscissa. $y = -32$.

$Q(-\frac{1}{2}, -32)$

$$PQ^2 = (8+\tfrac{1}{2})^2 + (2+32)^2$$

$$= (\tfrac{17}{4})^2 + 1156$$

$$= \frac{4913}{4}$$

$$PQ = \frac{17\sqrt{17}}{2}$$

Q1. 29. $xy = 9 \quad \therefore \quad c = 3 \qquad$ Tangent through $(-9,3)$

Let the equation of the tangent be $y = mx + b$.

Condition; $mx^2 + bx - 9 = 0$ has $b^2 - 4ac = 0$

$$b^2 + 36m = 0$$

$$m = \frac{-b^2}{36}$$

$\therefore$ The tangent is $y = \frac{-b^2}{36}x + b$

Through $(-9,3)$ $\therefore$ we have $3 = \frac{b^2}{4} + b$

$$12 = b^2 + 4b$$

$b^2 + 4b - 12 = 0 \iff (b+6)(b-2) = 0$

$\therefore b = 2$ or -6

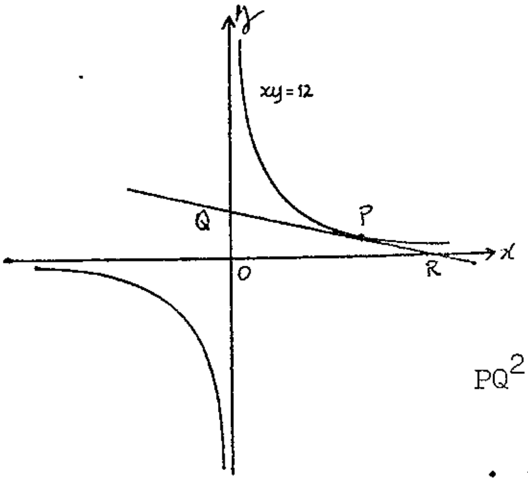
Hence the tangents are

$y = -\frac{1}{9}x + 2$ or $y = -x - 6$

$x + 9y = 18$ or $x + y + 6 = 0$

UNIT 3

Q2. 1. Tangent at P(6,2) of $xy = 12$



$x = ct$ $\qquad c = 2\sqrt{3}$

$6 = 2\sqrt{3}t$

$\therefore\ t = \sqrt{3}$

Tangent; $x + 3y = 2 \times 2\sqrt{3} \times \sqrt{3}$

$x + 3y = 12$

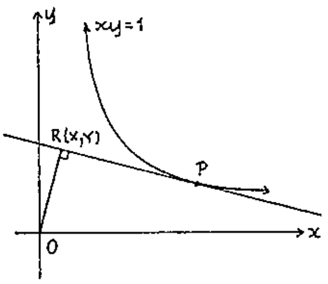
Q(12,0) $\quad$ R(0,4)

$PQ^2 = (6-12)^2 + (2-0)^2 \qquad PR^2 = (6-0)^2 + (2-4)^2$

$= 40 \qquad\qquad = 40$

$\therefore\ PQ = PR$

Q2. 2. $\qquad x = t,\ y = \frac{1}{t}$



Equation of tangent at P

$t^2y + x = 2ct \qquad c = 1$

$t^2y + x = 2t$ ....(A)

Equation of OR, where R(X,Y) is the foot of the $\perp$ through O to the tangent; $y = t^2x$ ....(B)

(B)$\rightarrow$(A) $\quad t^4x - 2t + x = 0 \implies X = \dfrac{2t}{t^4 + 1}$ and $Y = \dfrac{2t^3}{t^4 + 1}$

OR $\perp$ PR $\therefore\ \frac{Y}{X} = t^2$........................(1) Note $m_1m_2=-1$

$X = \dfrac{2t}{t^4 + 1} \qquad Xt^4 + X = 2t$ ....(2)

(1) $\rightarrow$ (2) $\quad t = \frac{1}{2}\left(\dfrac{X^2 + Y^2}{X}\right)$

$\therefore\ t^2 = \dfrac{1}{4X^2}(X^2 + Y^2)^2$ ....(3)

$Y = t^2X$ from (1)

(3)$\rightarrow$(1) $\quad Y = \dfrac{1}{4X^2}(X^2 + Y^2)^2 . X$

$\therefore\ 4XY = (X^2 + Y^2)^2$ is the locus of R.

(Showing that $t^3 = \dfrac{1}{8X^3}(X^2 + Y^2)^3$ ....(4) and subst.

(1) and (4) into $Y = \dfrac{2t^3}{t^4 + 1}$ would also yield the same)

UNIT 3

Q2. 3. $A(m,0)$ $B(0,\frac{4}{m})$ $\qquad xy = 1$ ....(2)

$$AB \Leftrightarrow \frac{x}{m} + \frac{y}{4/m} = 1$$

$$4x + m^2y = 4m$$

$$y = \frac{4}{m^2}(m - x) \quad \text{.........................(1)}$$

Then (1) $\longrightarrow$ (2) gives $x\left[\frac{4}{m^2}(m - x)\right] = 1$

$$4x^2 - 4mx + m^2 = 0 \quad \text{...........(3)}$$

If (1) is a tangent, then $\Delta = b^2 - 4ac = 0$

i.e. $\Delta = 16m^2 - 16m^2$

$= 0$ for all m.

$\therefore$ AB touches the hyperbola $xy = 1$ for all values of m.

$4x^2 - 4mx + m^2 = 0$ ....(3)

$(2x - m)^2 = 0$

$\therefore \frac{m}{2} = x, \; y = \frac{2}{m}$

Hence the point of contact is;

$$\left(\frac{m}{2}, \frac{2}{m}\right)$$

Q2. 4. Tangent at $(2t, \frac{1}{t})$ to $xy = 2$

$y + x\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{y}{x}$

m at $(2t, \frac{1}{t})$ is $= -\frac{1}{2t^2}$

Equation of tangent;

$$y - \frac{1}{t} = -\frac{1}{2t^2}(x - 2t)$$

$$x + 2t^2y - 4t = 0$$

Distance of $(2,2)$ from tangent $= \left|\frac{2 + 4t^2 - 4t}{\sqrt{1 + 4t^4}}\right| = d_1$

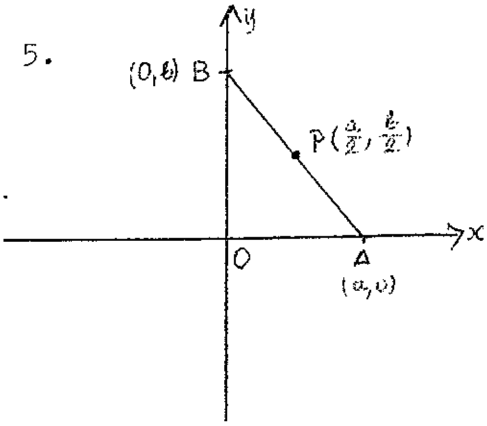
Distance of $(-2,2)$ from tangent $= \left|\frac{-2 - 4t^2 - 4t}{\sqrt{1 + 4t^4}}\right| = d_2$

$$d_1d_2 = \frac{4\left|(2t^2 - 2t + 1)(2t^2 + 2t + 1)\right|}{1 + 4t^4}$$

$= 4$ as required.

UNIT 3

Q2. 5.



A(a,0)
B(0,b)

Midpt. P is $(\frac{a}{2}, \frac{b}{2})$

$\triangle$ BOA $= 2c^2$

i.e. $\frac{ab}{2} = 2c^2$

$\therefore \quad ab = 4c^2$ ....(1)

Let $X = \frac{a}{2}$, $Y = \frac{b}{2}$ i.e. $a = 2X$

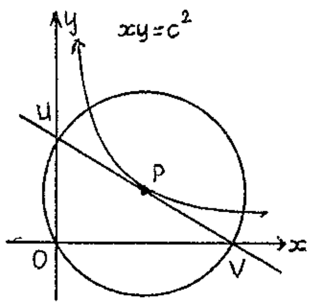
$b = 2Y$

$\therefore \quad ab = 4XY$ ....(2)

$(1) \longrightarrow (2) \quad 4c^2 = 4XY$

$\therefore$ $XY = c^2$ is the locus of P.

Q2. 6.



Equation of tangent at P.

$$x + t^2y = 2ct.$$

Tangent meets asymptotes at

$U(0, \frac{2c}{t})$ and at

$V(2ct, 0)$,

$O(0,0)$

$$PU^2 = (ct-0)^2 + (\frac{c}{t} - \frac{2c}{t})^2 \longleftrightarrow PU^2 = c^2t^2 + \frac{c^2}{t^2}$$

$$PV^2 = (ct - 2ct)^2 + (\frac{c}{t} - 0)^2 \longleftrightarrow PV^2 = c^2t^2 + \frac{c^2}{t^2}$$

$$PO^2 = c^2t^2 + \frac{c^2}{t^2}$$

So the points O, U and V are equidistant from P, i.e. they lie on a circle whose centre is P.

Q2. 7. Normal to $xy = 6$ at A(2,3)

$c = \sqrt{6} \qquad x = ct$

$\therefore \quad 2 = \sqrt{6}t \qquad t = \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}$

Normal; $\quad t^3x - ty = c(t^4 - 1)$

$$\frac{2\sqrt{6}}{9}x - \frac{\sqrt{6}}{3}y = \sqrt{6}(\frac{36}{81} - 1)$$

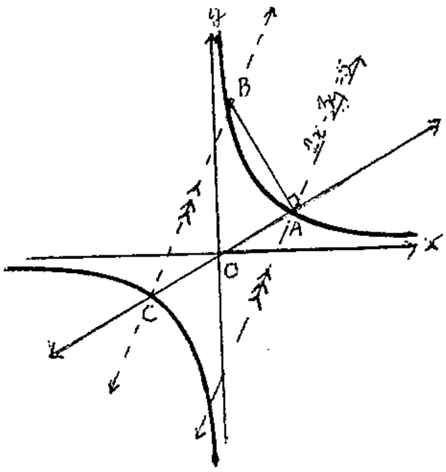
$$2x - 3y = -5$$

B(1,6) $\quad m_{AB} = \frac{6-3}{1-2} = -3 \quad \therefore m_{AC} = \frac{1}{3}$

(continued on next page)

UNIT 3

Q2. 7. (cont'd)



Equ. of AC $x - 3y + 7 = 0$

to find C;

$y(3y-7) = 6$

$(3y+2)(y-3) = 0$

$\therefore\ y = -\,^2/_3$ or $y = 3$

and $x = -9$ or $x = 2$

$\therefore\ C\ (-9,\ -\frac{2}{3})$

Q2. 8. $xy = 4$ Normal // to $4x - y = 2$

$\left.\begin{array}{l}\text{Gradient of normal} = t^2\\ \text{Gradient of line} \quad = 4\end{array}\right\}\ \therefore\ t^2 = 4$

$t = \pm 2$

Equation of normal;

$t^3x - ty = c(t^4 - 1)$ $c = 2$

$t^3x - t2y = 2(t^4 - 1)$

when $t = 2$ $8x - 2y = 30$

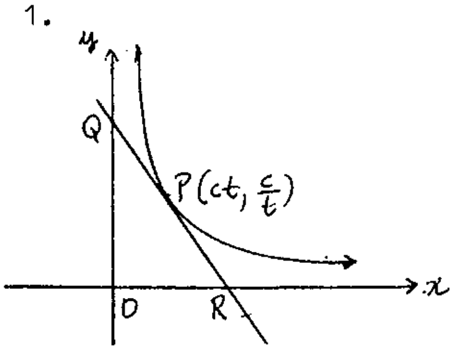
$4x - y = 15$

when $t = -2$ $-8x + 2y = 30$

$4x - y = -15$

$\therefore$ Equation of normals // to $4x - y = 2$ are $4x - y = \pm 15$.

Q3. 1.



Equation of tangent at P

$x + t^2y = 2ct$

$\therefore\ Q \Longrightarrow (0, \frac{2c}{t})$

$R \Longrightarrow (2ct, 0)$

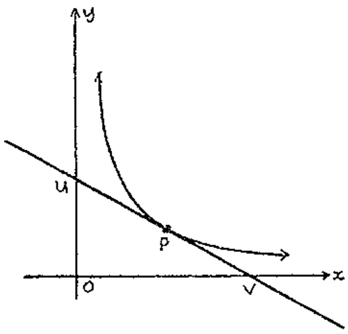
$QR^2 = 4c^2t^2 + \frac{4c^2}{t^2} = 4(c^2t^2 + \frac{c^2}{t^2})$

$OP^2 = c^2t^2 + \frac{c^2}{t^2}$

$\therefore\ 2OP = QR$

UNIT 3

Q3. 2.



Let P be the point $(ct, \frac{c}{t})$

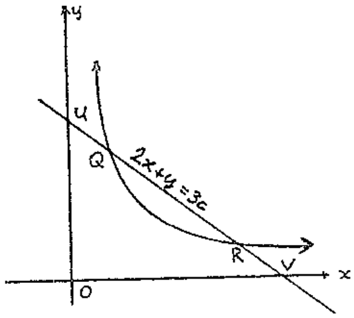
$\therefore$ $U(0, \frac{2c}{t})$

$V(2ct, 0)$

$\Delta\, UOV = \frac{1}{2} \cdot 2c\cancel{t}^{1} \cdot \frac{2c}{\cancel{t}_1}$

$= 2c^2$ which is constant.

Q3. 3.



$U(0, 3c)$

$V(\frac{3c}{2}, 0)$

Let $P(x, y)$ be the pt. of trisection of UV.

$x = \dfrac{2 \cdot \frac{3c}{2} + 1.0}{2 + 1}$ $\qquad$ $y = \dfrac{2.0 + 1.3c}{2 + 1}$

$x = c$ $\qquad$ $y = c$ $\qquad$ So $P(c, c)$

$\therefore x \Longrightarrow ct = c$ $\qquad$ or $\qquad$ $y \Longrightarrow \frac{c}{t} = c$

$t = 1$ $\qquad$ $t = 1$

$\therefore$ the point (c, c) is on $xy = c^2$

OR $x = \dfrac{1 \cdot \frac{3c}{2} + 2.0}{2 + 1}$ $\qquad$ $y = \dfrac{2.3c + 1.0}{2 + 1}$

$x = \frac{c}{2}$ $\qquad$ $y = 2c \longrightarrow (\frac{c}{2}, 2c)$

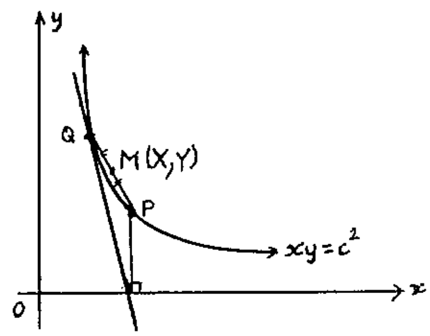
$\frac{c}{2} \cdot 2c = c^2$ $\qquad$ $\therefore$ the point $(\frac{c}{2}, 2c)$ is

also on $xy = c^2$.

$\therefore$ Both pts. of trisection of UV lies on $xy = c^2$.

UNIT 3

Q3. 4.



Let Q and P have coordinates $(cq,\frac{c}{q})$, $(cp,\frac{c}{p})$ respectively and let M(X,Y) be the midpoint of PQ.

So $X = \frac{c}{2}(\ p + q\ )$ ....(1)

$Y = \frac{c}{2}(\ \frac{p + q}{pq}\ )$ ....(2)

The tangent at Q is $x + q^2y = 2cq$ cuts the x axis when $y = 0$ i.e. when $x = 2cq$. This is equivalent to the abscissa of point P: $x = cp$.

So $2cq = cp$ i.e. $p = 2q$ ....(3)

(3)⟶ (1) $X = \frac{3cq}{2}$ i.e. $q = \frac{2X}{3c}$ ....(4)

(3)⟶ (2) $Y = \frac{3c}{4q}$ ....(5)

(5)⟶ (4) $XY = \frac{9c^2}{8}$. Hence the locus is a hyperbola having the same asymptotes as $xy = c^2$.

Q4. 1.(a) See p. 45 Example 3.

<u>Alternatively</u> using the same diagram as in example 3.

Let the equation of the family of parallel chords be in the form $y = mx$ + constant. Let $y = mx + d$ be the equation of chord PQ (a member of the family of parallel chords) with T(X,Y) being its midpoint.

$y = mx + d \cap xy = c^2$

$x(mx+d) = c^2$

$\therefore mx^2 + dx - c^2 = 0.$ If x_1 and x_2 are the roots of this equation then $x_1 + x_2 = -\frac{d}{m}$

hence $X = \frac{x_1 + x_2}{2} = -\frac{d}{2m}$

$Y = m\ (\frac{-d}{2m}) + d$

ie. $Y = \frac{d}{2}$

Now $\frac{X}{Y} = \frac{d/2}{-d/2m}$ ⟷ $Y = -mX$ which is the equation of the locus of T.

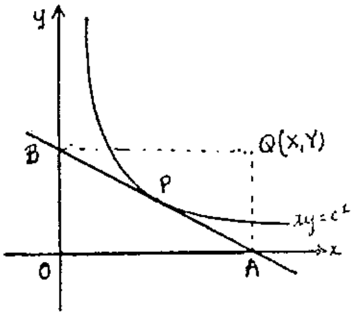
This equation represents a straight line through O
∴ it is the <u>diameter</u> conjugate to the chord $y = mx + d$

Conclusion: as above.

UNIT 3

Q4. 1.(b) See page 46 Example 4.

Q4. 2.(a)



$x = ct \qquad \frac{dy}{dt} = -\frac{c}{t^2}$

$y = \frac{c}{t} \qquad \frac{dx}{dt} = c$

$\therefore \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = -\frac{1}{t^2}$

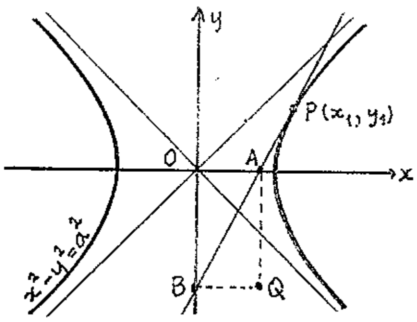
Tangent is $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

As $x = 0$ $y = \frac{2c}{t}$ at $y = 0$ $x = 2ct$

$\therefore$ Equ. of locus $Y = \frac{2c}{X/2c}$

$XY = 4c^2$ is the equation of Q.

Q4. 2.(b) The equation of the tangent on the hyperbola



$x^2 - y^2 = a^2$ .... (1)

at $P(x_1, y_1)$ is $xx_1 - yy_1 = a^2$ .... (2)

Put $y = 0 \longrightarrow$ (2) $\therefore A(a^2/x_1, 0)$

Put $x = 0 \longrightarrow$ (2) $\therefore B(0, -a^2/y_1)$

Let the coordinates of Q be (X,Y)

So $X = a^2/x_1$ i.e. $x_1 = a^2/X$ .... (3)

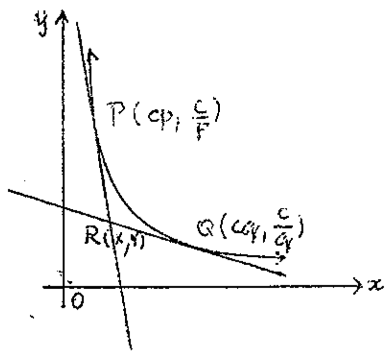
and $Y = a^2/y_1$ ie. $y_1 = -a^2/Y$ .... (4)

Since (x_1, y_1) lies on (1) $x_1^2 - y_1^2 = a^2$ .... (5)

Put (3) and (4) $\longrightarrow$ (5) $a^4/X^2 - a^4/Y^2 = a^2$

Simplify to get $Y^2 - X^2 = \frac{X^2 Y^2}{a^2}$ as required.

Q4. 3.(a)



Let R (X,Y) be any point on the locus.
By solving the equations of the tangents
at P; $x + p^2y = 2cp$ (1)
at Q; $x + q^2y = 2cq$ (2) simultaneously
we obtain;

$Y = \frac{2c}{p + q}$ (3)

Substitute (3) into (1) then

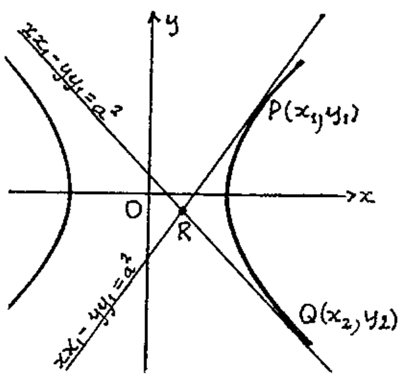
$X = 2cp\left(1 - \frac{p}{p + q}\right)$

$X = \frac{2cpq}{p + q} = pqY$ (using (3))

So $X = kY$ is the equation of the locus, which is a straight line through the origin. (i.e. a diameter)

UNIT 3

Q4. 3.(b) Derive that the equations of the tangent at



$P(x_1,y_1)$ $\quad xx_1 - yy_1 = a^2$ .... (1)

as $Q(x_2,y_2)$ is $xx_2 - yy_2 = a^2$.......... (2)

(1)x y_2 $\quad xx_1y_2 - yy_1y_2 = a^2y_2$.......... (3)

(2)x $(-y_1)$ $\quad xx_2y_1 - yy_2y_1 = a^2y_1$........ (4)

$$x(x_1y_2 - x_2y_1) = a^2(y_2 - y_1)$$

So $x = \dfrac{a^2(y_2-y_1)}{x_1y_2-x_2y_1}$ ...... (5)

Similarly (1) x x_2 and (2) x $(-x_1)$ gives $y = \dfrac{a^2(x_2 - x_1)}{x_1y_2 - x_2y_1}$....... (6)

If the gradient of PQ is constant then $\dfrac{y_2 - y_1}{x_2 - x_1} = k$ .... (7)

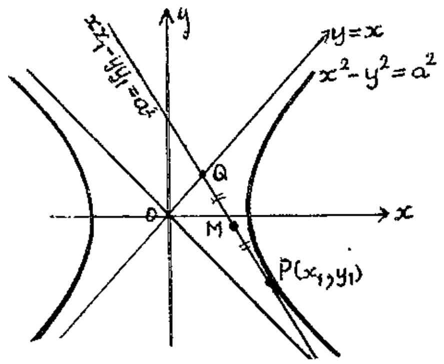
and if R has components of (X,Y) then

$$\frac{X}{Y} = \frac{a^2(y_2 - y_1)}{x_1y_2 - x_2y_1} \cdot \frac{x_1y_2 - x_2y_1}{a^2(y_2 - y_1)}$$

$\dfrac{X}{Y} = \dfrac{y_2 - y_1}{x_2 - x_1}$ .... (8)

(7) $\longrightarrow$ (8) $\quad \dfrac{X}{Y} = k$ i.e. $X = kY$ which represents a straight line through the origin.

Q4. 4.



Deduce that the equation of the tangent at $P(x_1,y_1)$ is $xx_1 - yy_1 = a^2$.........(1)

$y = x$ ....(2) (2) $\longrightarrow$ (1)

$x(x_1 - y_1) = a^2$ i.e. $x = \dfrac{a^2}{x_1 - y_1}$

but $a^2 = x_1^2 - y_1^{\,2}$ so $x = x_1 + y_1 = y$

Hence Q has coordinates $(x_1 + y_1,\ x_1 + y_1)$

Let M(X,Y) be the midpoint of PQ.

So $X = \dfrac{2x_1 + y_1}{2}$ ....(3) and $Y = \dfrac{2y_1 + x_1}{2}$ ....(4)

<u>Aim</u> is to express x_1 and y_1 in terms of X and Y so we can substitute x_1 and x_2 into $x^2 - y^2 = a^2$ .... (5) in order to find the required locus. (This can be done since $P(x_1,\ y_1)$ is a point on (5).)

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from (3) $y_1 = 2X - 2x_1$ ....(6) from (4) $x_1 = 2Y - 2y_1$ .....(7)

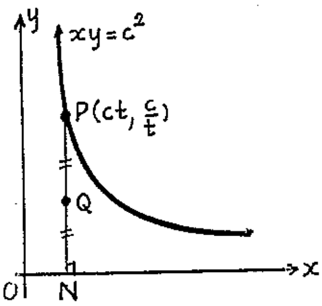
$(7) \longrightarrow (3)\quad X = \dfrac{4Y - 4y_1 + y_1}{2} = \dfrac{4Y - 3y_1}{2}$ i.e. $y_1 = \dfrac{4Y - 2X}{3}$ ....(8)

$(6) \longrightarrow (4)\quad Y = \dfrac{4X - 4x_1 + x_1}{2} = \dfrac{4X - 3x_1}{2}$ i.e. $x_1 = \dfrac{4X - 2Y}{3}$ ....(9)

Put (8) and (9) into (5) $\dfrac{(4X - 2Y)^2}{9} - \dfrac{(4Y - 2X)^2}{9} = a^2$

Which simplifies to $X^2 - Y^2 = \dfrac{3a^2}{4}$ as required.

Q4. 5. (a) Let P be the point $(ct, c/_t)$ on the hyperbola $xy = c^2$.



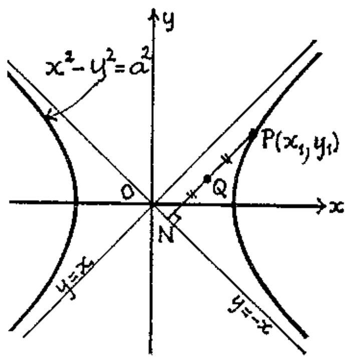
N(ct, 0) is the foot of the perpendicular from P, Q(X,Y) is the midpoint of interval PN.

$X\ ct/_2$ i.e. $t = 2X/_c$ ....(1)

$Y = c/_t$ ....(2)

Put (1) (2). So $Y = c^2/_{2X}$ i.e. $XY = \dfrac{c^2}{2}$ is the equation of the locus of Q, which is a rectangular hyperbola and has the same asymptotes as $xy = c^2$.

(b) Let $P(x_1, y_1)$ be any point on $x^2 - y^2 = a^2$ .... (1)



N is the foot of the perpendicular to the asymptote $y = -x$ .... (2)
so the gradient of PN is 1.
Eqn. of PN is $y - y_1 = x - x_1$ .... (3)

$(2) \longrightarrow (3)\ -x - y = x - x_1$ i.e. $x = \dfrac{x_1 - y_1}{2}$ .. (4)

$(4) \longrightarrow (3)$ gives $y = \dfrac{y_1 - x_1}{2}$, so $N \equiv \left(\dfrac{x_1 - y_1}{2}, \dfrac{y_1 - x_1}{2}\right)$

The of Q is obtained by using the midpoint formula, so $X = \dfrac{2x_1 - y_1}{2}$ i.e. $x_1 = X + y_{1}/_2$ .... (5)

and $Y = \dfrac{2y_1 - x_1}{2}$ i.e. $y_1 = Y + {}^{x_1}/_2$ .... (6)

$(6) \rightarrow (5)\quad x_1 = X + \dfrac{Y}{2} + \dfrac{x_1}{4}$ i.e. $x_1 = \dfrac{4X + 2Y}{3}$ .... (7)

$(5) \rightarrow (6)\quad y_1 = Y + \dfrac{X}{2} + \dfrac{y_1}{4}$ i.e. $y_1 = \dfrac{4Y + 2X}{3}$ .... (8)

These last two steps were necessary to express x_1 and y_1 in terms of X, Y so x_1 and y_1 can be eliminated by putting (7) and (8) into (1) since P $(x_1 y_1)$ lies on (1).

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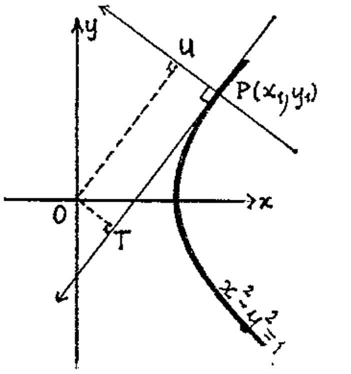
So $\frac{(4X + 2Y)^2}{9} - \frac{(4Y + 2X)^2}{9} = a^2$

$16X^2 + 16\,XY + 4Y^2 - 16Y^2 - 16XY - 4X^2 = 9$ which simplifies to $X^2 - Y^2 = \frac{3a^2}{4}$

Q4. 6. See example 5 on page 46. Replace a^2 by 1 in the solution.

Q4. 7. The locus of T: Let $P(x_1, y_1)$ be any point on



$x^2 - y^2 = 1$ .... (1)

Deduce that the equation of the tangent at P is
$xx_1 - yy_1 = 1$ ....(2)

The gradient of PT is ${}^{x_1}/_{y_1}$ so the gradient of OT is $-{}^{y_1}/_{x_1}$. So the equation of OT is

$y = \frac{y_1 x}{x_1}$ .... (3)

Since T(X,Y) is on (3) $Y = \frac{-y_1 X}{x_1}$ .... (4)

In order to express X and Y in terms of x_1 and y_1 solve (1) and (3) simultaneously:

(3) $\longrightarrow$ (2) $xx_1 + \frac{xy_1^2}{x_1} = 1$ i.e. $x(x_1{}^2 + y_1^2) = x_1$

So $X = \frac{x_1}{x_1^2 + y_1^2}$ .... (5)

From (3) $x_1 = \frac{-y_1 x}{Y}$ (2) gives $Y = \frac{-y_1}{x_1^2 + y_1^2}$ .... (6)

In order to eliminate x_1 and y_1 we express them in terms of X and Y.

From (4) $x_1 = \frac{-y_1 X}{Y}$ (7) $y_1 = \frac{x_1\,Y}{X}$ .... (8)

(8) $\longrightarrow$ (6) $X = \frac{x_1}{x_1^2 + x_1^2\, Y^2/X^2}$ i.e. $X = \frac{X^2 x_1}{x_1^2\,(X^2 + Y^2)}$ hence

$x_1 = \frac{X}{X^2 + Y^2}$ .... (9)

Similarly by (7) $\longrightarrow$ (5) we get $y_1 = \frac{-Y}{X^2 + Y^2}$ ....(10)

Since (x_1, y_1) is on (1) $x_1{}^2 - y_1^2 = 1$ ....(11)

Q.4. 7. continued.

Put (9) and (10) into (11) $\dfrac{X^2}{(X^2+Y^2)} - \dfrac{Y^2}{(X^2+Y^2)} = 1$

i.e. $X^2 - Y^2 = (X^2 + Y^2)^2$ as required.

The locus of U. Deduce that the equation of the normal at P is $xy_1 + yx_1 = 2x_1y_1$ ....(12)

The gradient of OU = grad. of PT = x_1/y_1 . So the equation of OU is $y = \dfrac{x_1 x}{y_1}$ ....(13)

Since U(X,Y) is on (13) $Y = Xx_1/y_1$ ....(14)

Express X and Y in terms of x_1 and y_1. So OU $\cap$ PU:

(13) $\longrightarrow$ (12) $xy_1 + x^2x/_{y_1} = 2x_1y_1$ i.e. $x(y_1^2 + x_1^2) = 2x_1y_1^2$ so

$$X = \frac{2x_1y_1^2}{x_1^2 + y_1^2} \qquad \text{..(15)}$$

From (13) $x = {}^{yy_1}/_{x_1} \longrightarrow$ (12) ${}^{yy_1^2}/_{x_1} + yx_1 = 2xy_1$ which

$$\text{yields } Y = \frac{2x_1^2\, y_1}{x_1^2 + y_1^2} \qquad \text{..(16)}$$

From (14) $y_1 = Xx_1/_Y \longrightarrow$ (16) so $Y = \dfrac{2x_1^3\; X/_Y}{x_1^2 + X^2x_1^2/_{Y^2}} = \dfrac{2x_1^3\; XY}{x_1^2(X^2 + Y^2)}$

i.e. $x_1 = \dfrac{X^2 + Y^2}{2X}$ ....(17)

From (14) $x_1 = Yy_1/_X \longrightarrow$ (15) $X = \dfrac{2y_1^3\; Y/_X}{y_1^2 + y_1^2\; Y^2/_{X^2}} = \dfrac{2y_1^3\; XY}{y_1^2(X^2 + Y^2)}$

i.e. $y_1 = \dfrac{X^2 + Y^2}{2X}$ ....(18)

Since (x_1, y_1) is on (1) $x_1^2 - y_1^2 = 1$ ....(19)

Put (17), (18) $\longrightarrow$ (19) $\dfrac{(X^2 + Y^2)^2}{4X^2} - \dfrac{(X^2 + Y^2)}{4Y^2} = 1$

i.e. $(X^2 - Y^2)\,(X^2 + Y^2)^2 = 4\,X^2Y^2$ as required.