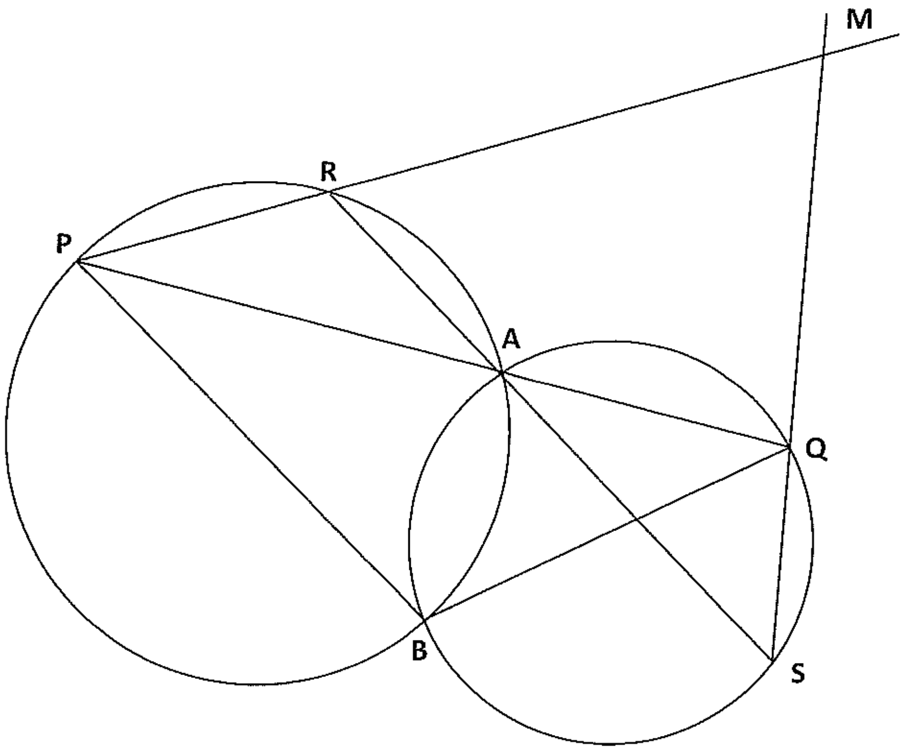
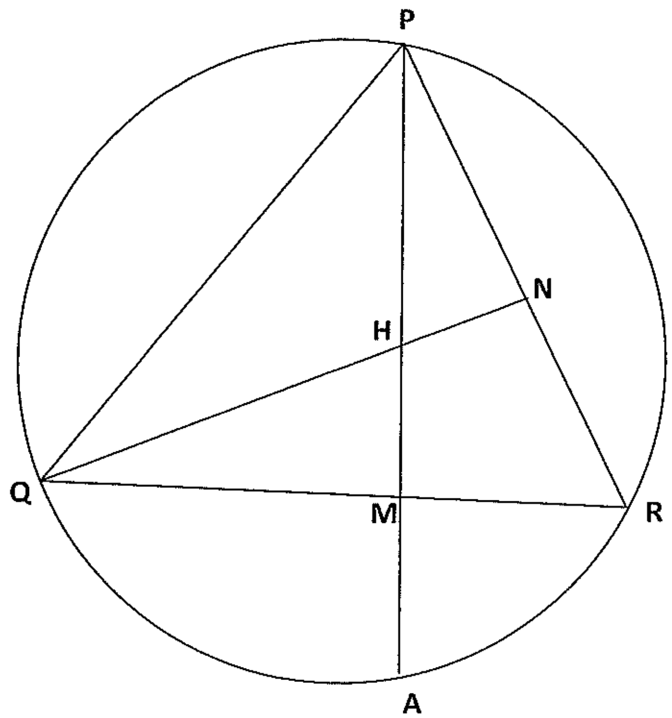
Harder Circle Geometry

Extension 2



Two circles ARPB and AQSB intersect at A and B. PAQ and RAS are straight lines. PR and SQ are produced to meet at M.

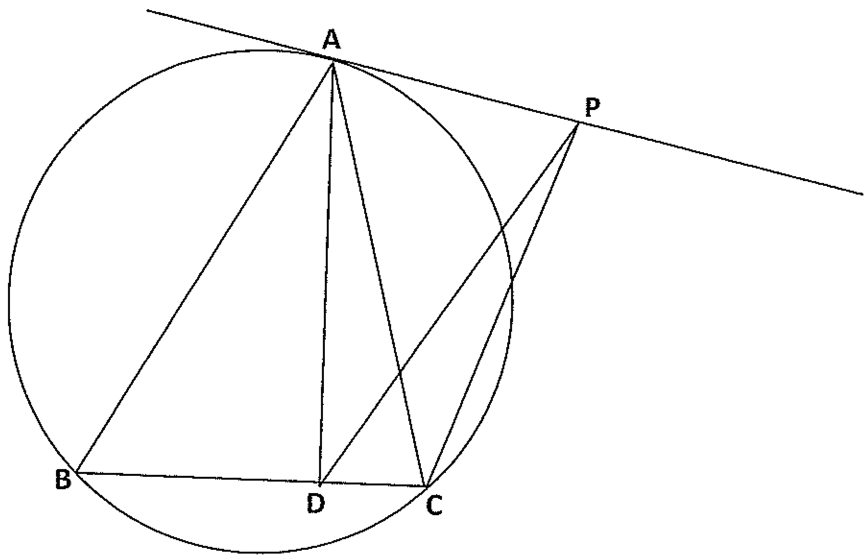
Prove that MPBQ is a cyclic quadrilateral.



The altitudes* PM and QN of an acute-angled triangle PQR meet at H. PM produced cuts the circle PQR at A.

Show that HM = MA.

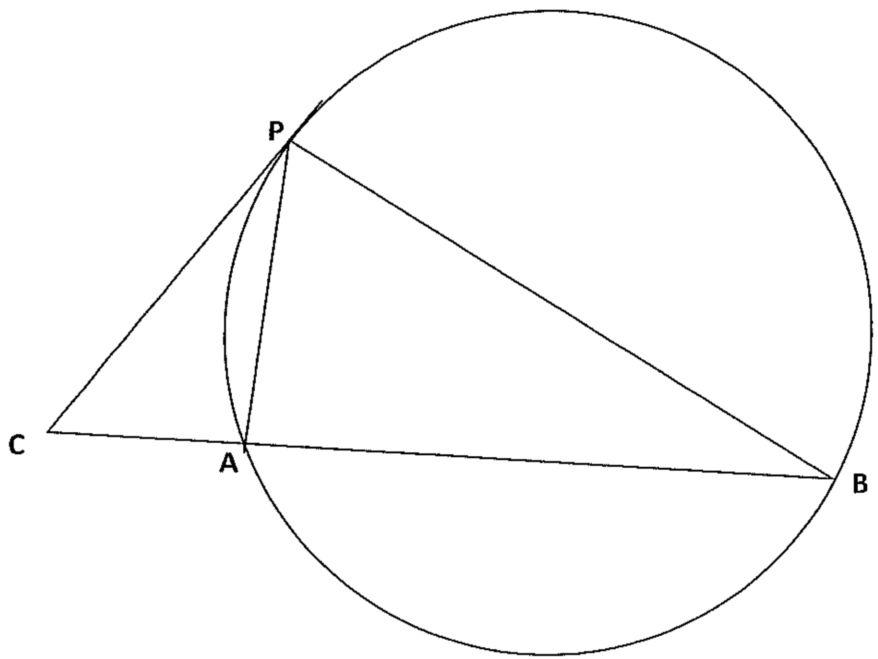
* altitudes are at right-angled to the base of a triangle.



AD is an altitude of the triangle ABC which is inscribed in a circle. DP is drawn parallel to BA and meets the tangent at A at P.

i. Explain why <PAC =<ABC.

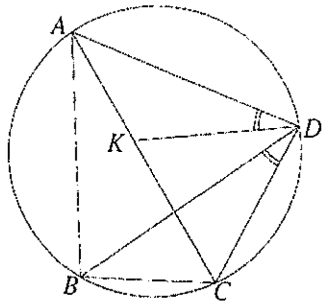
Ii. Show that <CPA is a right-angle.



CP is the tangent to the circle at P and CAB is any secant.

Prove that $CP^2 = CA \times CB$

In the diagram $ABCD$ is a cyclic quadrilateral. The point K is on AC such that $\angle ADK = \angle CDB$, and hence ΔADK is similar to ΔBDC.

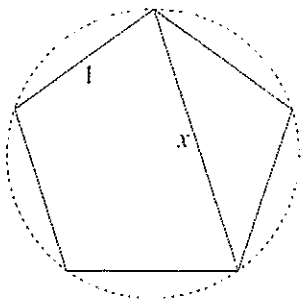


Copy or trace the diagram into your writing booklet.

(i) Show that $\triangle ADB$ is similar to $\triangle KDC$. **2**

(ii) Using the fact that $AC = AK + KC$, show that $BD \times AC = AD \times BC + AB \times DC$. **2**

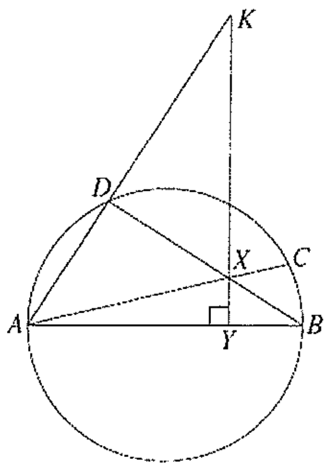
(iii) A regular pentagon of side length 1 is inscribed in a circle, as shown in the diagram. **2**



Let x be the length of a chord in the pentagon.

Use the result in part (ii) to show that $x = \dfrac{1+\sqrt{5}}{2}$.

n the diagram AB is the diameter of the circle. The chords AC and BD intersect
it X. The point Y lies on AB such that XY is perpendicular to AB. The point K
s the intersection of AD produced and YX produced.

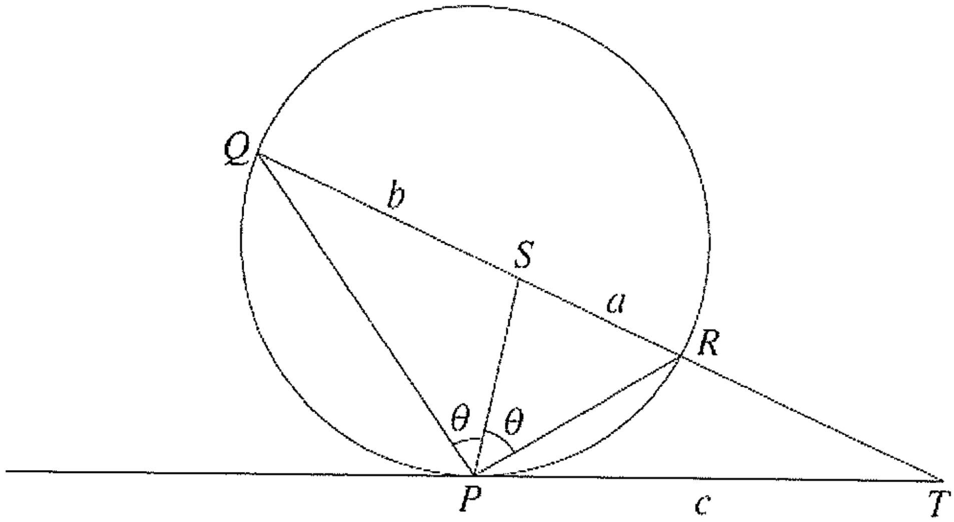


'opy or trace the diagram into your writing booklet.

(i) Show that $\angle AKY = \angle ABD$. 2

(ii) Show that $CKDX$ is a cyclic quadrilateral. 2

(iii) Show that B, C and K are collinear. 2

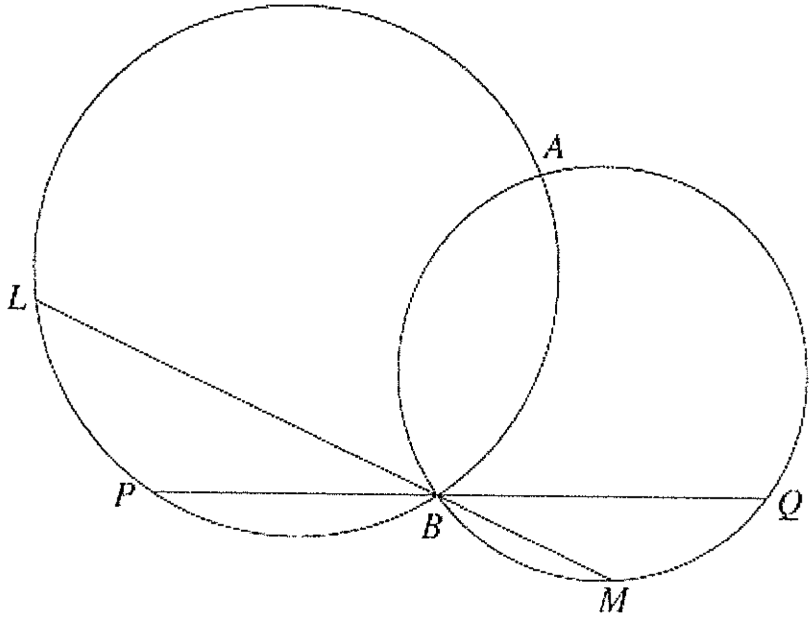


In the diagram, the points P, Q and R lie on a circle. The tangent at P and the secant QR intersect at T. The bisector of $\angle QPR$ meets QR at S so that $\angle QPS = \angle RPS = \theta$. The intervals RS, SQ and PT have lengths a, b and c respectively.

(i) Show that $\angle TSP = \angle TPS$. **2**

(ii) Hence show that $\dfrac{1}{a} = \dfrac{1}{b} + \dfrac{1}{c}$. **2**

(a)



Two circles intersect at A and B.

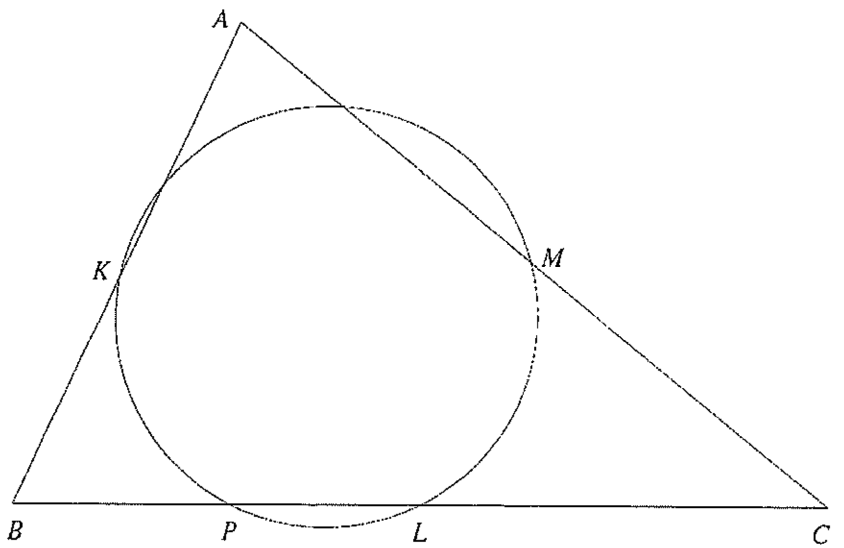
The lines LM and PQ pass through B, with L and P on one circle and M and Q on the other circle, as shown in the diagram.

Copy or trace this diagram into your writing booklet.

Show that $\angle LAM = \angle PAQ$.

2

HSC 2007



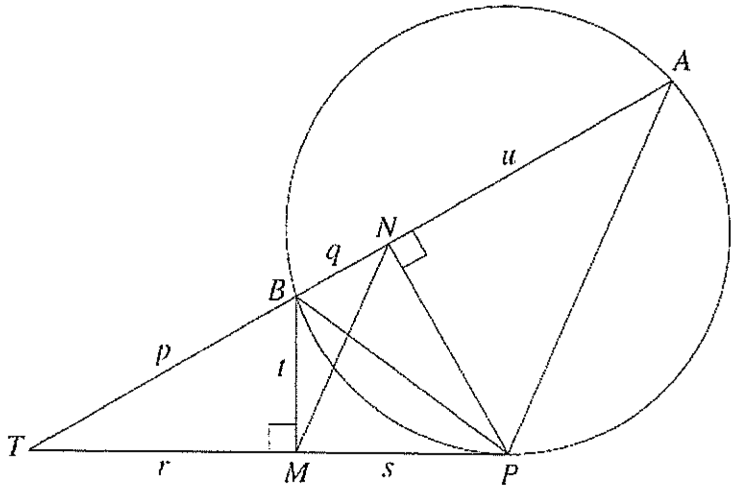
In the acute-angled triangle ABC, K is the midpoint of AB, L is the midpoint of BC and M is the midpoint of CA. The circle through K, L and M also cuts BC at P as shown in the diagram.

Copy or trace the diagram into your writing booklet.

(i) Prove that $KMLB$ is a parallelogram. **1**

(ii) Prove that $\angle KPB = \angle KML$. **1**

(iii) Prove that $AP \perp BC$. **2**



The points A, B and P lie on a circle.

The chord AB produced and the tangent at P intersect at the point T, as shown in the diagram. The point N is the foot of the perpendicular to AB through P, and the point M is the foot of the perpendicular to PT through B.

Copy or trace this diagram into your writing booklet.

(i) Explain why $BNPM$ is a cyclic quadrilateral. **1**

(ii) Prove that MN is parallel to PA. **3**

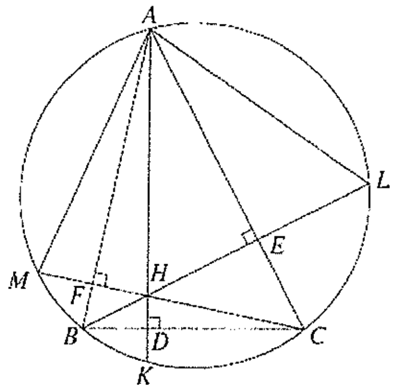
Let $TB = p$, $BN = q$, $TM = r$, $MP = s$, $MB = t$ and $NA = u$.

(iii) Show that $\frac{s}{u} < \frac{r}{p}$. **2**

(iv) Deduce that $s < u$. **1**

(b) The vertices of an acute-angled triangle ABC lie on a circle. The perpendiculars from A, B and C meet BC, AC and AB at D, E and F respectively. These perpendiculars meet at H.

The perpendiculars AD, BE and CF are produced to meet the circle at K, L and M respectively.

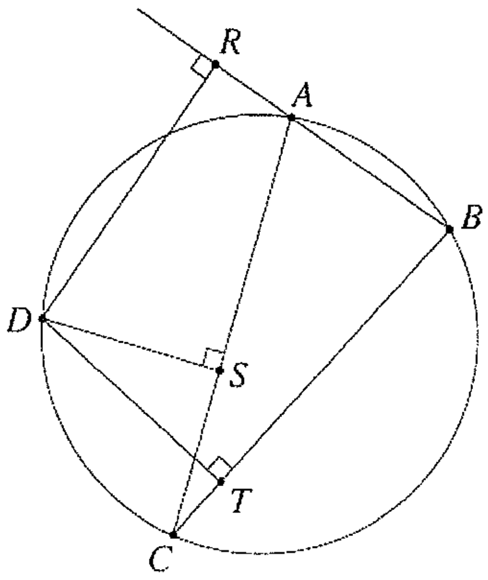


(i) Prove that $\angle AHE = \angle DCE$. 2

(ii) Deduce that $AH = AL$. 1

(iii) State a similar result for triangle AMH. 1

(iv) Show that the length of the arc BKC is half the length of the arc MKL. 2



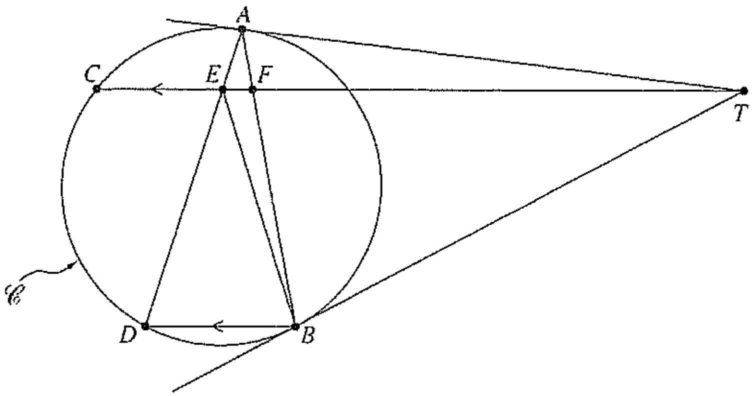
In the diagram, A, B, C and D are concyclic, and the points R, S, T are the feet of the perpendiculars from D to BA produced, AC and BC respectively.

(i) Show that $\angle DSR = \angle DAR$. **2**

(ii) Show that $\angle DST = \pi - \angle DCT$. **2**

(iii) Deduce that the points R, S and T are collinear. **2**

(b)



In the diagram, $\mathscr{C}$ is a circle with exterior point T. From T, tangents are drawn to the points A and B on $\mathscr{C}$ and a line TC is drawn, meeting the circle at C. The point D is the point on $\mathscr{C}$ such that BD is parallel to TC. The line TC cuts the line AB at F and the line AD at E.

Copy or trace the diagram into your writing booklet.

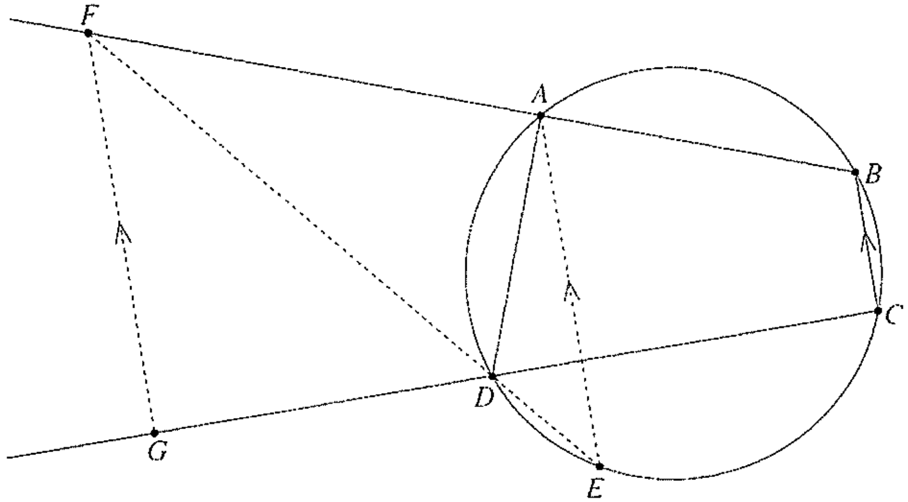
(i) Prove that ΔTFA is similar to ΔTAE. **3**

(ii) Deduce that $TE.TF = TB^2$. **2**

(iii) Show that ΔEBT is similar to ΔBFT. **2**

(iv) Prove that ΔDEB is isosceles. **1**

(b) In the diagram, $ABCD$ is a cyclic quadrilateral. The point E lies on the circle through the points A, B, C and D such that $AE \parallel BC$. The line ED meets the line BA at the point F. The point G lies on the line CD such that $FG \parallel BC$.



Copy or trace the diagram into your writing booklet.

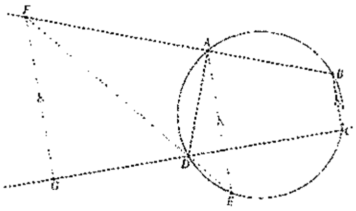
(i) Prove that $FADG$ is a cyclic quadrilateral. **2**

(ii) Explain why $\angle GFD = \angle AED$. **1**

(iii) Prove that GA is a tangent to the circle through the points A, B, C and D. **2**

2011

(b) (i)



Let $\angle BCD = x$

$\angle FAD = x$ (exterior $\angle$ of cyclic quad)

$\angle FGC = \pi - x$ (co-interior $\angle$'s, with $FG \parallel BC$)

$\angle FAD$ and $\angle FGD$ are supplementary, thus $FADG$ is a cyclic quadrilateral.

(ii) $\angle GFD = \angle AED$ (alternate $\angle$'s, with $FG \parallel BC$).

(iii) $\angle GFD = \angle GAD$ ($\angle$'s on circumference)

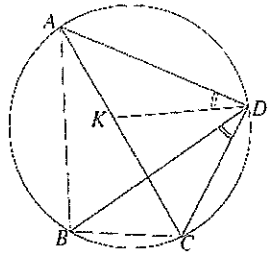
$\angle GFD = \angle AED$ (alternate $\angle$'s, FG||AE)

$\therefore \angle AED = \angle GAD$ (both equal $\angle GFD$)

Since $\angle AED = \angle GAD$, this satisfies the condition for the angle in the alternate segment, hence GA is a tangent to the circle $ABCD$.

2010

(a) (i)



$AC = AD = BD = x$

$BD \times AC = AD \times BC + AB \times DC$

$x \times x = x \times 1 + 1 \times 1$

$x^2 = x + 1$

$x^2 - x - 1 = 0$

$x = \frac{1 \pm \sqrt{5}}{2}$

$x = \frac{1+\sqrt{5}}{2}$ since $x > 0$

In $\Delta's$ ADB, KDC

$\angle ADB = \angle KDC$

(given angle + common angle)

$\angle ABD = \angle KCD$ (standing on arc AD)

$\therefore \Delta ABD \quad \Delta KCD$ (equi-angular)

$\frac{AB}{KC} = \frac{AD}{KD} = \frac{BD}{CD}$ (sides in proportion)

(ii) Since $\Delta ADK \quad \Delta BDC$

$\frac{AD}{BD} = \frac{AK}{BC} = \frac{DK}{DC}$

$\therefore AD \times BC = BD \times AK$

from (i) $AB \times CD = BD \times KC$

adding we get:

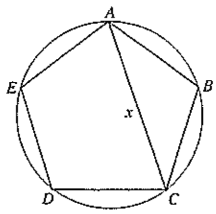
$AB \times CD + AD \times BC$

$= BD \times KC + BD \times AK$

$AB \times CD + AD \times BC = BD \times (KC + AK)$

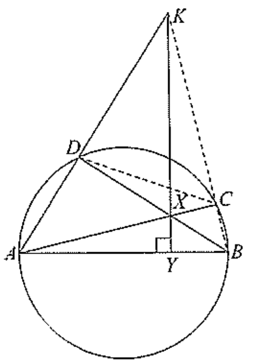
$\therefore BD \times AC = AD \times BC + AB \times DC$

(iii)



$AB = BC = CD = 1$

2009



(i) In ΔAKY and ΔABD,

$\angle ADB = 90°$ (angle in a semi-circle)

$\angle KYA = 90°$ (given)

$\therefore \angle ADB = \angle KYA$

$\angle KAY = \angle DAB$ (common angle)

$\therefore \angle AKY = \angle ABD$ (angle sum of Δs).

(ii) From (i), $\angle AKY = \angle ABD$

$\angle ABD = \angle DCA$ (angles in the same segment on chord AD)

$\therefore \angle DKX = \angle DCX$

$\therefore \angle DKX$ and $\angle DCX$ are on chord DX

$\therefore CKDX$ is a cyclic quadrilateral.

(iii) $\angle KDX = 90°$ (angles on a straight line are supplementary)

$\therefore \angle KDX + \angle KCX = 180°$

(opposite angles of cyclic quadrilateral $CKDX$ are supplementary)

$\therefore \angle KCX = 90°$

Now, $\angle ACB = 90°$ (angle in a semi-circle)

$\therefore \angle KCB =$ straight angle

$\therefore B, C, K$ are collinear.

2008

(b) (i) $\angle TSP = \angle SQP + \angle SPQ$

(exterior $\angle$ of a triangle equals sum of interior opposite $\angle$'s).

$= \angle SQP + \angle RPS$ (both θ)

$= \angle TPR + \angle RPS$

$= \angle TPS$

($\angle$ between a chord and tangent is equal to the $\angle$ in the alternate segment).

(ii) The square of the tangent is equal to the product of the secants, so

$$TR \times TQ = TP^2$$

From (i), ΔTSP is isosceles, so

$$TR = c - a$$

$$TQ = c + b$$

$$\therefore\ (c-a)(c+b) = c^2$$

$$c^2 - ac + bc - ab = c^2$$

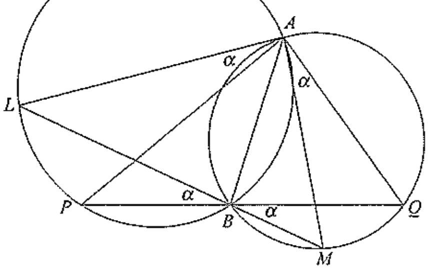
$$bc = ac + ab$$

$$\frac{bc}{abc} = \frac{ac}{abc} + \frac{ab}{abc}$$

$$\therefore \frac{1}{a} = \frac{1}{b} + \frac{1}{c}.$$

2007

(a)



Let $\angle LAP = \alpha$

$\therefore \quad \angle LBP = \angle LAP$ ($\angle$s in the same segment on LP)

$= \alpha$

$\therefore \; \angle QBM = \angle LBP$ (Vertically opposite $\angle$s)

$= \alpha$

$\therefore \; \angle QAM = \angle QBM$ ($\angle$s in the same segment on QM)

$= \alpha$

$\therefore \quad \angle LAP = \angle QAM$

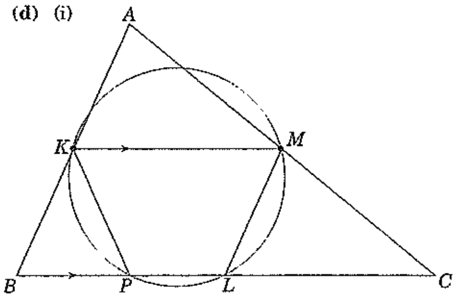
Now $\angle LAM = \angle LAP + \angle PAM$ (from above)

$= \angle QAM + \angle PAM$

$\therefore \; \angle LAM = \angle PAQ.$

2006

(d) (i)



In triangles ABC and AKM,

$$\frac{AB}{AC} = \frac{2AK}{2AM} = \frac{AK}{AM}$$

and the included angle $\angle A$ is common.

$\therefore \; \Delta ABC \,|||\, \Delta AKM$ (corresponding sides in proportion and included $\angle$ equal).

$\angle AKM = \angle ABC$ (corresponding $\angle$s in similar Δs)

$\therefore \quad KM \,||\, BL$ (corresponding $\angle$s are equal).

The scale ratio $\dfrac{AK}{AB} = \dfrac{1}{2}$ (K is the midpoint of AB).

$\therefore \quad KM = \frac{1}{2}BC.$

But $\quad BL = \frac{1}{2}BC$ (L is the midpoint of BC).

$\therefore \quad KM = BL.$

$\therefore$ $KMLB$ is a parallelogram (one pair of opposite sides are equal and parallel).

(ii) $KMLP$ is a cyclic quadrilateral.

$\angle KPB = \angle KML$ (exterior $\angle$ of a cyclic quadrilateral is equal to the opposite interior $\angle$).

(iii) $\angle KPB = \angle KML$, from (ii)

$\angle KML = \angle KBP$, from (i) (opposite $\angle$s in a parallelogram are equal).

Hence $\angle KPB = \angle KBP$.

$\therefore$ ΔKBP is isosceles with $KB = KP$.

$KB = KA$ (K is the midpoint of AB).

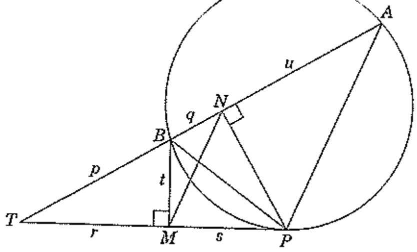
Therefore $KB = KA = KP$, so that K is the centre of a circle with diameter AB and passing through P.

$\therefore \; \angle APB = 90°$ ($\angle$ in a semi-circle is a right angle).

$\therefore \quad AP \perp BC.$

2005

Question 7



(i) $\angle BMP + \angle BNP = 90° + 90°$

$= 180°.$

The two angles are opposite angles in the quadrilateral $BNPM$, and since they are supplementary, this quadrilateral is cyclic.

(ii) $\angle BAP = \angle BPM$ (alternate segment theorem)

$= \angle BNP.$ ($\angle$s at circumference subtended by the same arc in circle $BNPM$)

Hence $NM \,||\, AP.$ ($\angle$s BNM and BAP are both corresponding and equal)

(iii) Consider the parallel lines NM, AP and a third line through T parallel to these. Intercepts between these parallel lines are in proportion.

Hence $\dfrac{MP}{NA} = \dfrac{TM}{TN}$

$\dfrac{s}{u} = \dfrac{r}{p+q}$

$< \dfrac{r}{p}$, since $r, p, q > 0.$

Hence $\dfrac{s}{u} < \dfrac{r}{p}.$

(iv) In ΔTBM, p is the length of the hypotenuse and r is the length of a shorter side.

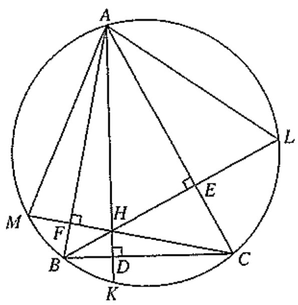
Hence $r < p$

$\dfrac{r}{p} < 1$

$\dfrac{s}{u} < 1$, using (iii)

$s < u.$

2004



(i) $\angle AHE + \angle AEH + \angle HAE$ ($\angle$ sum, ΔAEH

$= \angle DCE + \angle ADC + \angle CAD$ and ΔADC)

Now $\angle HAE = \angle CAD$ (common $\angle$)

and $\angle AEH = \angle ACD$ (both right $\angle$s)

$\therefore \quad \angle AHE = \angle DCE.$

This can also be done many other ways, such as by showing $\Delta AEH \,|||\, \Delta ADC$, or by showing $\Delta HBD \,|||\, \Delta CBE$, or by showing that quadrilateral $EHDC$ is cyclic.

(ii) $\angle ALB = \angle ACB$ ($\angle$s in same segment standing on same arc AB)

that is, $\angle ALE = \angle ECD$ (same $\angle$s)

but $\angle AHE = \angle DCE$, from (i)

$\therefore \quad \angle AHE = \angle ALE$

$\therefore \quad AH = AL$ (sides opposite equal $\angle$s are equal)

Again, other methods are possible, such as showing $\Delta AEH \equiv \Delta AEL$.

(iii) Similarly, $AH = AM$ (diagram is symmetric)

(iv) $\angle AHE = \angle ALE$, from (ii)

$\therefore \; \angle HAE = \angle LAE$ ($\angle$ sum of Δ)

$\therefore$ arc KC = arc CL (equal $\angle$s stand on equal arcs)

Similarly from (iii), arc KB = arc BM.

$\therefore$ arc MKL

= arc MB + arc BK + arc KC + arc CL

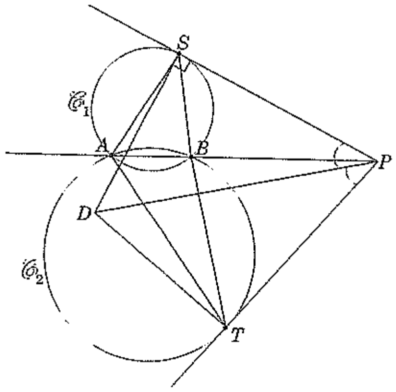
= 2 arc BK + 2 arc KC

= 2(arc BK + arc KC)

= 2 arc BKC.

$\therefore$ arc $BKC = \frac{1}{2}$ arc MKL.

2003



(i) In Δs ASP and SBP,

$\angle PSB = \angle PAS$ ($\angle$ in alt. segment)

$\angle SPB = \angle APS$ (common)

$\Delta ASP \parallel\!\!| \Delta SBP$ (equal $\angle$s).

(ii) $\dfrac{PS}{PA} = \dfrac{PB}{PS}$ (sides proportional in similar Δs)

$\therefore \quad PS^2 = AP \cdot BP.$

Likewise, $\Delta ATP \parallel\!\!| \Delta TBP$,

so $PT^2 = AP \cdot BP$

$\therefore \quad PT = PS.$

(iii) In Δs SPD and TPD,

DP is common

$TP = SP$, from (ii),

$\angle SPD = \angle TPD$ (DP bisects SPT)

$\therefore \Delta SPD \equiv \Delta TPD$ (SAS)

$\therefore \angle DTP = \angle DSP$ (corresp. $\angle$s)

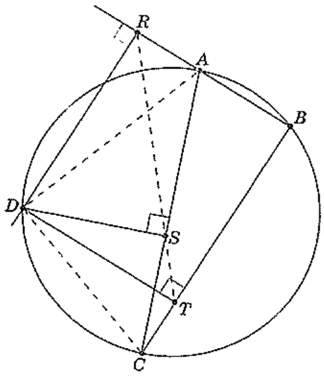
$= 90°$ ($SD \perp SP$).

But PT is a tangent to $\mathscr{C}_2$.

$\therefore$ DT passes through the centre of $\mathscr{C}_2$ (converse of tangent $\perp$ radius theorem).

2002

(b)



(i) In $DSAR$,

$\angle DSA + \angle ARD = 180°$ (since both right $\angle$s).

$\therefore$ Points D, S, A, R are concyclic (opp. $\angle$s are supplementary).

$\therefore \angle DSA = \angle DAR$ ($\angle$s at circumference of circle $DSAR$ subtended by same arc DR).

(ii) In $DSTC$,

$\angle DSC = \angle DTC$ (since both right $\angle$s).

$\therefore$ Points D, S, T, C are concyclic (converse of $\angle$s at circumference subtended by the same arc are equal).

$\therefore \angle DST = \pi - \angle DCT$ (opp. $\angle$s of cyclic quadrilateral $DSTC$ are supplementary).

(iii) $\angle DSR = \angle DAR$, (from (i)),

but $\angle DAR = \angle DCT$ (ext. $\angle$ of cyclic quad. $DABC$ equals int. opp. $\angle$).

$\therefore \quad \angle DSR = \angle DCT$

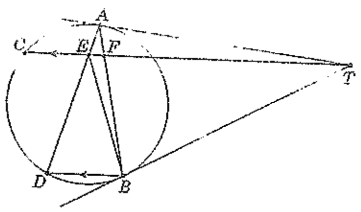
$\angle DST = \pi - \angle DCT$, (from (ii)),

$\therefore \angle DSR + \angle DST = \pi$

$\therefore$ R, S, T are collinear (adjacent $\angle$s are supplementary).

2001

(b)



(i) $\angle AET = \angle ADB$ (Corresponding, $CT \parallel DB$)

$\angle ADB = \angle FAT$ ($\angle$ between chord and tangent equals $\angle$ in alternate segment).

$\therefore \angle AET = \angle FAT$ (Both $= \angle ADB$).

Also,

$\angle ATF = \angle ETA$ (Common).

So $\Delta TFA \parallel\!\!| \Delta TAE$ (Same angles).

(ii) $\dfrac{TA}{TE} = \dfrac{TF}{TA}$ [Scale ratios of similar Δs in (i)].

$\therefore \quad TE \cdot TF = TA^2.$

But $TA = TB$ (Tangents from an external point are equal).

$\therefore \quad TE \cdot TF = TB^2.$

(iii) $\angle BTE = \angle FTB$ (Common)

$\dfrac{TE}{TB} = \dfrac{TB}{TF}$, from (ii).

$\therefore \quad \Delta EBT \parallel\!\!| \Delta BFT$ (2 pairs of corresponding sides are in proportion and their included $\angle$s are equal).

(iv) $\angle EDB = \angle FBT$ ($\angle$ between chord and tangent equals $\angle$ in alternate segment).

$\angle FBT = \angle BET$ (Corresponding, $\Delta EBT \parallel\!\!| \Delta BFT$)

$\angle BET = \angle EBD$ (Alternate, $CT \parallel DB$)

$\therefore \angle EDB = \angle EBD$

$\therefore \Delta DEB$ is isosceles (Base $\angle$s equal).