Induction Work sheet

All questions assume we are dealing with real numbers only.

Proof by mathematical induction usually consists of the following four steps
Let the general statement be $S(n)$

Step 1: Basis	Prove that the general statement is true for the smallest value of n which is usually $n=1$ i.e. prove that $S(1)$ is true
Step 2: Assumption	Assume that the general statement is true for $n=k$ i.e. assume that $S(k)$ is true
Step 3: Inductive	Show that the general statement is true for $n=k+1$ .e. prove that $S(k+1)$ is true using the fact that $S(k)$ is true
Step 4 : Conclusion	The general statement is then true for all positive integers, n

Most of the work in induction (and hence the marks) comes from Step 3
It is always a good idea when doing step 3 to write down what it is that you want to prove.

Syllabus Content :
8.2 Induction
The student is able to:
• carry out proofs by mathematical induction in which $S(1)$, $S(2)\ldots S(k)$ are assumed to be true in order to prove $S(k+1)$ is true
• use mathematical induction to prove results in topics which include geometry, inequalities, sequences and series, calculus and algebra.

Questions 1 to 4 are the syllabus questions

1: Prove that the angle sum of an n sided figure is equal to $2n-4$ right angles.

step 1: prove S(3) true
$n=3$
$2n-4 = 2(3)-4 = 2$
$2\times 90 = 180^\circ$
true

step 2: assume true for $n=k$
$\therefore S(k) = 2k-4$

Step 3: $n=k+1$
RTP the angle of $k+1$ sided figure is
$2(k+1)-2$ right angles
To get $k+1$ sided figure from a
k sided shape [illegible] one side
[illegible] sides to add a triangle.

$\therefore$ angle sum of $k+1$ sided
= angle sum of k
sided figure + angle
sum of Δ
$= 2k-4+2$
$= 2k-2$

2:

A sequence $\{u_n\}$ is such that

$u_{n+3} = 6u_{n+2} - 5u_{n+1}$, and $u_1 = 2, u_2 = 6$.

Prove that $u_n = 5^{n-1} + 1$.

$n = 1$

$u_1 = 5^0 + 1$

$= 2$

true for $n = 1$

$n = 2$

$u_2 = 5^1 + 1$

$= 6$

$n = 3$

$u_3 = 5^2 + 1$

$= 26$

Assume

$n = k$ [illegible]

$u_k = 5^{k-1} + 1$

$u_{k+1} = 5^k + 1$

$u_{k+2} = 5^{k+1} + 1$

Prove $u_{k+3} = 5^{k+2} + 1$

$u_{k+3} = 6u_{k+2} - 5u_{k+1}$

$= 6(5^{k+1} + 1) - 5(5^k + 1)$

$= 30(5^k) + 6 - 5(5^k) - 5$

$= 25(5^k) + 1$

$= 5^{k+2} + 1$

[illegible]

Question 8(ii) *(1985)*

(a) Show that for $k \geq 0$,

$$2k + 3 > 2\sqrt{\{(k+1)(k+2)\}}.$$

(b) Hence prove that for $n \geq 1$,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > 2[\sqrt{(n+1)} - 1]$$

(c) Is the statement that, for all positive integers N,

$$\sum_{k=1}^{N} \frac{1}{\sqrt{k}} < 10^{10},$$

true? Give reasons for your answer.

a) $2k+3 > 2\sqrt{(k+1)(k+2)}$

LHS

$(2k+3)^2 - 4(k+1)(k+2) > 0$

$1 > 0$

true

$\therefore 2k+3 > 2\sqrt{(k+1)(k+2)}$

b) $n=1$

$1 > 2(\sqrt{2} - 1)$

true

$n=k$

$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} > 2(\sqrt{k+1} - 1)$

$n=k+1$

$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \ldots \frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2} - 1)$

$2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2} - 1)$

$\frac{2\sqrt{k+1}(\sqrt{k+1} - 1) + 1}{\sqrt{k+1}} > 2(\sqrt{k+2} - 1)$

LHS $= \frac{2k+3 - 2\sqrt{k+1}}{\sqrt{k+1}}$

$> \frac{2\sqrt{(k+1)(k+2)} - 2\sqrt{k+1}}{\sqrt{k+1}}$

$= 2(\sqrt{k+2} - 1)$

$=$ RHS.

c) $10^{10} = 2(\sqrt{n+1} - 1)$

$\frac{10^{10}}{2} + 1 = \sqrt{n+1}$

$\left(\frac{10^{10}}{2} + 1\right)^2 = n+1$

$n = \left(\frac{10^{10}}{2} + 1\right)^2 - 1$

when $n = \left(\frac{10^{10}}{2} + 1\right)^2 - 1$

$\sum_{k=1}^{N} \frac{1}{\sqrt{k}} > 10^{10}$

$\therefore$ false.

Question 8(ii) *(1981)*

Using induction, show that for each positive integer n there are unique positive integers p_n and q_n such that

$$(1+\sqrt{2})^n = p_n + q_n\sqrt{2}.$$

Show also that $p_n{}^2 - 2q_n{}^2 = (-1)^n$.

$n=1$

$1+\sqrt{2} = p_1 + q_1\sqrt{2}$

$= \text{LHS}$ when $p_1 = 1$, $q_1 = 1$

$n=k$

$(1+\sqrt{2})^k = p_k + q_k\sqrt{2}$

$n=k+1$

$(1+\sqrt{2})^{k+1} = p_{k+1} + q_{k+1}\sqrt{2}$

$(1+\sqrt{2})(1+\sqrt{2})^k = p_{k+1}$ [illegible]

$(1+\sqrt{2})(p_k + q_k\sqrt{2}) =$ [illegible] $q_{k+1}\sqrt{2}$

$\text{LHS} = p_k + q_k\sqrt{2} + p_k\sqrt{2} + 2q_k$

$= p_k + 2q_k + \sqrt{2}(p_k + q_k)$

$= p_{k+1} + q_{k+1}\sqrt{2}$

where $p_{k+1} = p_k + 2q_k$

[illegible]

[illegible] q_k

are [illegible]

[illegible] $p_k + q_k$

[illegible]

[illegible] q_k

[illegible]

(3 Unit HSC 1972, Question 9)

(a) Write down an expression for $\cos(a+b)$ and hence prove that $\cos(2q) = 1 - 2\sin^2 q$.

(b) Prove the identity

$$\frac{\cos y - \cos(y+2q)}{2\sin q} = \sin(y+q).$$

(c) Use mathematical induction and the result of part (b) to prove the identity:

$$\sin q + \sin 3q + \sin 5q + \ldots + \sin(2n-1)q$$
$$= \frac{1-\cos 2nq}{2\sin q}.$$

6: For $n=1,2,3,\ldots\ldots.,$ let $s_n = 1+\sum_{r=1}^{1}\frac{1}{r!}$

(i) Prove by mathematical induction that $e - s_n = e\int_0^1 \frac{x^n}{n!}e^{-x}\,dx$

(ii) from (i), deduce that $0 < e - s_n < \frac{3}{(n+1)!}$ for $n=1,2,3,\ldots\ldots.,$

Remember that $e<3$ and $e^{-x}\leq 1$ for $x \geq 0$

7:

The numbers p, q and s are fixed and positive. Also $p > 1$, $q > 1$ and $p = \dfrac{q}{q-1}$.

(i) What positive value of t minimises the expression

$$f(t) = \frac{s^p}{p} + \frac{t^q}{q} - st\ ?$$

(ii) Show that for all $t > 0$,

$$\frac{s^p}{p} + \frac{t^q}{q} \geq st.$$

(iii) Prove by induction that

$$\left(x_1 x_2 \cdots x_n\right)^{\frac{1}{n}} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

for all $x_1, \ldots, x_n > 0$.

(iv) Deduce that, for all $y_1, y_2, \ldots, y_n > 0$,

$$\frac{y_1}{y_2} + \frac{y_2}{y_3} + \cdots + \frac{y_{n-1}}{y_n} + \frac{y_n}{y_1} \geq n.$$

(3 Unit HSC 1984, Question 7)

It is given that $A > 0$, $B > 0$ and n is a positive integer.

(a) Divide $A^{n+1} - A^nB + B^{n+1} - B^nA$

by $A-B$, and deduce that

$A^{n+1} + B^{n+1} > A^nB + B^nA$.

(b) Using (a), show by mathematical induction that

$$\left(\frac{A+B}{2}\right)^n \le \frac{A^n + B^n}{2}$$

Question 3 (continued)

(c) Use mathematical induction to prove that $(2n)! \geq 2^n (n!)^2$ for all positive integers n. 3

8: HSC '10 q 6

(b) A sequence a_n is defined by **3**

$$a_n = 2a_{n-1} + a_{n-2},$$

for $n \geq 2$, with $a_0 = a_1 = 2$.

Use mathematical induction to prove that

$$a_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n \text{ for all } n \geq 0.$$

9: HSC '07 q6

(a) (i) Use the binomial theorem **1**

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \cdots + b^n$$

to show that, for $n \geq 2$,

$$2^n > \binom{n}{2}.$$

(ii) Hence show that, for $n \geq 2$, **2**

$$\frac{n+2}{2^{n-1}} < \frac{4n+8}{n(n-1)}.$$

(iii) Prove by induction that, for integers $n \geq 1$, **3**

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

(iv) Hence determine the limiting sum of the series **1**

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots.$$

9: HSC '08

(a) It is given that $2\cos A\sin B = \sin(A+B) - \sin(A-B)$. (Do NOT prove this.) **3**

Prove by induction that, for integers $n \geq 1$,

$$\cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}.$$

Question 7 (continued)

(c) The sequence $\{x_n\}$ is given by

$$x_1 = 1 \text{ and } x_{n+1} = \frac{4 + x_n}{1 + x_n} \text{ for } n \geq 1.$$

(i) Prove by induction that for $n \geq 1$ **4**

$$x_n = 2\left(\frac{1 + \alpha^n}{1 - \alpha^n}\right),$$

where $\alpha = -\frac{1}{3}$.

(ii) Hence find the limiting value of x_n as $n \to \infty$. **1**

11: HSC 05 q6

(a) For each integer $n \geq 0$, let $I_n(x) = \int_0^x t^n e^{-t}\, dt$.

(i) Prove by induction that **4**

$$I_n(x) = n!\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)\right].$$

(ii) Show that **1**

$$0 \leq \int_0^1 t^n e^{-t}\, dt \leq \frac{1}{n+1}.$$

(iii) Hence show that **1**

$$0 \leq 1 - e^{-1}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) \leq \frac{1}{(n+1)!}.$$

(iv) Hence find the limiting value of $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ as $n \to \infty$. **1**

12: HSC 04 q7

(a) (i) Let a be a positive real number. Show that $a+\frac{1}{a}\geq 2$. **2**

(ii) Let n be a positive integer and $a_1, a_2, \ldots, a_n$ be n positive real numbers. **4**

Prove by induction that $(a_1+a_2+\ldots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)\geq n^2$.

(iii) Hence show that $\operatorname{cosec}^2\theta+\sec^2\theta+\cot^2\theta\geq 9\cos^2\theta$. **1**

12: HSC 03 q6

(b) A sequence s_n is defined by $s_1=1$, $s_2=2$ and, for $n>2$,

$$s_n=s_{n-1}+(n-1)s_{n-2}.$$

(i) Find s_3 and s_4. **1**

(ii) Prove that $\sqrt{x}+x\geq\sqrt{x(x+1)}$ for all real numbers $x\geq 0$. **2**

(iii) Prove by induction that $s_n\geq\sqrt{n!}$ for all integers $n\geq 1$. **3**

13: HSC 01 q8

(b) (i) Explain why, for $\alpha>0$, **2**

$$\int_0^1 x^\alpha e^x\,dx<\frac{3}{\alpha+1}.$$

(You may assume $e<3$.)

(ii) Show, by induction, that for $n=0, 1, 2, \ldots$ there exist integers a_n and b_n such that **2**

$$\int_0^1 x^n e^x\,dx = a_n+b_n e .$$

(iii) Suppose that r is a positive rational, so that $r=\frac{p}{q}$ where p and q are positive integers. Show that, for all integers a and b, either **2**

$$|a+br|=0 \quad\text{or}\quad |a+br|\geq\frac{1}{q}.$$

(iv) Prove that e is irrational. **2**