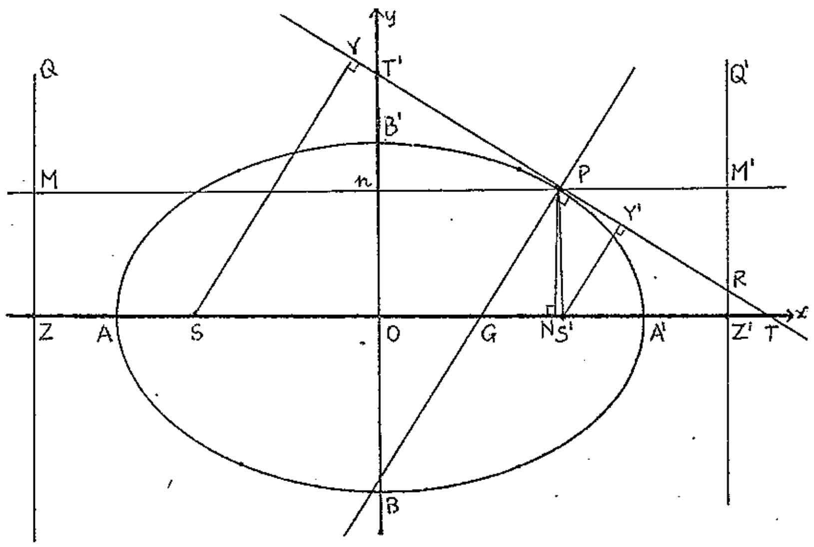
UNIT 4



(Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. S, S^1 are the foci, ZQ and Z^1Q^1 the directrixes, PG, PT, PN are the normal, tangent and the ordinate at P.)

Q1. 1.

$ON = a \cos\theta$

$OT = {}^{a}/\cos\theta$

$ON \cdot OT = a\cos\theta \cdot \frac{a}{\cos\theta}$

$\therefore ON \cdot OT = a^2$

Tangent; $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

$y = 0 \therefore x = \frac{a}{\cos\theta}$

Q1. 2.

$On = b\sin\theta$

$OT^1 = \frac{b}{\sin\theta}$

$\therefore On \cdot OT^1 = b\sin\theta \cdot \frac{b}{\sin\theta}$

$= b^2$

$x = 0 \therefore \frac{y\sin\theta}{b} = 1$

Q1. 3.

Equ. of normal $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$
$= a^2e^2$

$SG = SO + OG$

at G $y = 0 \therefore \frac{ax}{\cos\theta} = a^2 - b^2 \quad \therefore x = \frac{\cos\theta(a^2 - b^2)}{a}$

$\therefore SG = ae + ae^2\cos\theta$
$= e(a + ae\cos\theta) \implies ae(1 + e\cos\theta)$

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Q1. 3. (cont'd)

$$SP^2 = (-ae - a\cos\theta)^2 + (-b\sin\theta)^2$$

$$= a^2e^2 + 2a^2e\cos\theta + a^2\cos^2\theta + (b^2\sin^2\theta) \qquad b^2 = a^2 - a^2e^2$$

$$= a^2e^2 + 2a^2e\cos\theta + a^2\cos\theta + (a^2 - a^2e^2)\sin^2\theta$$

$$= a^2(e^2 + 2e\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} - e^2\sin^2\theta)$$

$$= a^2(1 + 2e\cos\theta + e^2(1 - \sin^2\theta))$$

$$= a^2(1 + 2e\cos\theta + e^2\cos^2\theta)$$

$$= (a(1 + e\cos\theta)^2) \qquad \therefore\ SP = a(1 + e\cos\theta)$$

$$eSP = ae(1 + e\cos\theta) =$$

$$= SG$$

Q1. 4. $S^1G = eS^1P$ (to prove)

$$S^1G = OG - OS^1$$

$$= ae^2\cos\theta - ae$$

$$= ae(e\cos\theta - 1)$$

$$eS^1P^2 = e((a\cos\theta - ae)^2 + (b\sin\theta)^2) \qquad \text{from } 1.(3)$$

$$= e(a(\cos\theta - 1))^2$$

$$eS^1P = ae(\cos\theta - 1)$$

$$= S^1G$$

Q1. 5. $SG = e^2PM$ (to prove)

$$SG = ae + ae^2\cos\theta$$

$$PM = \frac{a}{e} + a\cos\theta$$

$$e^2PM = ae + ae^2\cos\theta$$

$$= SG$$

Q1. 6. Prove: $\angle GPS^1 = \angle PM^1S^1$

Note: $1 - e^2 = \frac{b^2}{a^2}$

$$\text{Gradient of } S^1M = \frac{b\sin\theta}{\frac{a}{e} - ae} = \frac{eb\sin\theta}{a(1 - e^2)} = \frac{ae\sin\theta}{b}$$

$$= \tan\angle PM^1S^1$$

$$\text{Gradient of } GP = \frac{b\sin\theta}{a\cos\theta - ae^2\cos\theta}$$

$$= \frac{a\sin\theta}{b\cos\theta}$$

$$\text{Gradient of } S^1P = \frac{b\sin\theta}{a\cos\theta - ae}$$

$$\tan\angle GPS^1 = \frac{\frac{a\sin\theta}{b\cos\theta} + \frac{b\sin\theta}{ae - a\cos\theta}}{1 + \frac{ab\sin^2\theta}{-abe\cos\theta + ab\cos^2\theta}}$$

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Q1. 6. (cont'd)

$$= \frac{a^2e\sin\theta + a^2\sin\theta\cos\theta + b^2\sin\theta\cos\theta}{-abe\cos\theta = ab\cos^2\theta + ab\sin^2\theta}$$

$$= \frac{a^2e\sin\theta + \sin\theta\cos\theta\,(b^2 - a^2)}{-abe\cos\theta + ab(\cos^2\theta + \sin^2\theta)}$$

$$= \frac{a^2e\sin\theta - a^2e^2\sin\theta\cos\theta}{ab - abe\cos\theta}$$

$$= \frac{a^2e\sin\theta(1 - e\cos\theta)}{ab(1 - e\cos\theta)}$$

$$= \frac{ae\sin\theta}{b}$$

$$= \tan\angle PM^1S^1$$

$$\therefore \angle GPS^1 = \angle PM^1S^1$$

Q1. 7. $\angle PS^1R = 90^\circ$ (to prove)

Gradient of $PS^1 = \dfrac{b\sin\theta}{a\cos\theta - ae} = m_1$

Gradient of $RS^1 = \dfrac{k - 0}{\frac{a}{e} - ae}$

$R(\frac{a}{e}, k)$

to find k;

$\dfrac{\cos\theta}{e} + \dfrac{y\sin\theta}{b} = 1$

$\therefore y = k = \dfrac{b(e-\cos\theta)}{e\sin\theta}$

$$= \frac{\frac{b(e-\cos\theta)}{c\sin\theta}}{\frac{a}{e} - ae}$$

$$= \frac{be(e - \cos\theta)}{ae\sin\theta(1 - e^2)}$$

$$= \frac{\overset{1}{\cancel{b}}\overset{1}{\cancel{e}}(e - \cos\theta)}{\underset{1}{\cancel{a}}\underset{1}{\cancel{e}}\sin\theta\ (b)^{\cancel{2}}}(a^{\cancel{2}})^1$$

$$= \frac{a(e - \cos\theta)}{b\sin\theta} = m_2$$

$$m_1m_2 = \frac{b\sin\theta}{a(\cos - e)} \cdot \frac{a(e - \cos\theta)}{b\sin\theta}$$

$$= -1$$

$$\therefore \angle PS^1R = 90^\circ$$

Q1. 8. $SY \cdot SY^1 = b^2$ (to prove)

Tangent $YY^1 \equiv \dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$

$S(-ae, 0)$, $S^1(ae, 0)$

$$d_1 = \left|\frac{-aeb\cos\theta - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right|$$

$$d_2 = \left|\frac{aeb\cos\theta - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right|$$

$$\left|\frac{a^2e^2b^2\cos^2\theta - a^2b^2}{\quad}\right|$$

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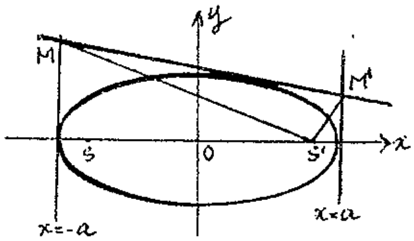
$$= \left|\frac{a^2b^2(e^2\cos^2\theta - 1)}{(b^2 - a^2)\cos^2\theta + a^2}\right|$$

$$= \left|\frac{a^2b^2(e^2\cos^2\theta - 1)}{-a^2e^2\cos^2\theta + a^2}\right|$$

$$= \left|\frac{a^2b^2(e^2\cos^2\theta - 1)}{a^2(1 - e^2\cos^2\theta - 1)}\right|$$

$SY \cdot SY^1 = b^2$

Q1. 9.



To find M^1 and M. $\xrightarrow{x\,=\,a}$ $\frac{x}{a}\cos\theta + \frac{y}{b}(\sin\theta) = 1$

$$y = \frac{b(1-\cos\theta)}{\sin\theta}$$

$M^1(a, \frac{b(1-\cos\theta)}{\sin\theta}$ and $M(-a. \frac{b(1+\cos\theta)}{\sin\theta}$

$$m_{S^1M^1} = \frac{b(1-\cos\theta)}{a\sin\theta\,(1-e)}$$

$$m_{S^1M} = \frac{-b(1+\cos\theta)}{(1+e)a\sin\theta}$$

$$m_{S^1M^1}\cdot m_{S^1M} = \frac{-b^2(1-\cos^2\theta)}{a^2\sin^2\theta(1-e^2)}$$

$$= -\frac{b^2}{a^2}\cdot\frac{1}{{}^2/{}_{b^2}}$$

$= -1$ (The result is identical when using S instead of S^1.)

Hence MM^1 subtends a right angle at either focus.

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Q1. 10. $y = mx - c \cap \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{x^2}{a^2} + \frac{m^2x^2 + c^2 - 2mcx}{b^2} = 1$$

$$bx^2 + a^2m^2x^2 + a^2c^2 - 2a^2mc - a^2b^2 = 0$$

$$x^2(b^2 + a^2m^2) - 2a^2mcx + a^2(c^2 - b^2) = 0$$

If $y = mx - c$ is a tangent then

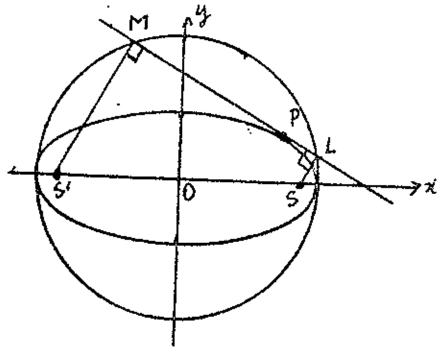
$$\Delta = b^2 - 4ac = 0 \quad \text{i.e.}$$

$$\cancel{a^2m^2c^2} - b^2c^2 + b^4 - \cancel{a^2m^2c^2} + a^2b^2m^2 = 0$$

$$b^2c^2 = a^2b^2m^2 + b^4$$

$$\therefore c = \pm\sqrt{a^2m^2 + b^2}$$

Q1. 11.



Aim: to prove that the foot of the perpendicular from the foci lie on the auxiliary circle $x^2 + y^2 = a^2$.

(This question lends itself to a rather straightforward geometric solution, which is not part of the syllabus.) It is complicated to find the point of $\cap$ of $x^2 + y^2 = a^2$ with $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$.

Instead; find the point of intersection of the tangent with the perpendiculars through S and S^1, then show that those points lie on the circle $x^2 + y^2 = a^2$.

Equation of tangent at P(a cos θ, b sin θ);

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \qquad \text{———} (1)$$

Gradient of a line $\perp$ to tangent $= \dfrac{a\sin\theta}{b\cos\theta}$

Equation of perpendicular through S;

$$y = \frac{a\sin\theta}{b\cos\theta}(x - ae)$$

$$y = \frac{ax\sin\theta - a^2e\sin\theta}{b\cos\theta} \qquad \text{———} (2)$$

$$(2) \longrightarrow (1) \quad \frac{x\cos\theta}{a} + \frac{(xa\sin\theta - a^2e\sin\theta)\sin\theta}{b^2\cos\theta} = 1$$

$$xb^2\cos^2\theta + xa^2\sin^2\theta - a^3e\sin^2\theta = ab^2\cos\theta$$

$$\therefore\ x = \frac{ab^2\cos\theta + a^3e\sin^2\theta}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$\text{From (2)} \quad x = \frac{by\cos\theta + a^2}{a\sin\theta} \qquad \text{———} (3)$$

$$(3) \rightarrow (1) \frac{\cos\theta}{a} \cdot \frac{by\cos\theta + a^2e\sin\theta}{a\sin\theta} + \frac{y\sin\theta}{b} = 1$$

$$b^2y\cos^2\theta + ya^2\sin^2\theta + a^2be\cos\theta\sin\theta - a^2b\sin\theta = 0$$

(cont'd)

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$$y = \frac{a^2b\sin\theta - a^2be\cos\theta\sin\theta}{b^2\cos^2\theta + a^2\sin^2\theta}$$

Now substitute $\dfrac{ab^2\cos\theta + a^3e\sin^2\theta}{b^2\cos^2\theta + a^2\sin^2 O}$, $\dfrac{a^2b\sin\theta - a^2be\cos\theta\sin\theta}{b^2\cos^2\theta + a^2\sin^2\theta}$

into $x^2 + y^2 = a^2$

LHS $= x^2 + y^2$

$$\text{LHS} = \frac{a^2b^4\cos^2\theta + a^6e^2\sin^4\theta + 2a^4b^2e\cos\theta\sin^2\theta}{(b^2\cos^2\theta + a^2\sin^2\theta)^2} +$$

$$\frac{a^4b^2\sin^2\theta + a^4b^2e^2\cos^2\theta\sin^2\theta - 2a^4b^2e\sin^2\theta\cos\theta}{(b^2\cos^2\theta + a^2\sin^2\theta)^2}$$

substitute $a^2e^2 = a^2 - b^2$

$$= \frac{a^2\left[(a^4 - a^2b^2)\sin^4\theta + b^4\cos^2\theta + (a^2b^2 - b^4)\cos^2\theta\sin^2\theta + a^2b^2\sin^2\theta\right]}{a^4\sin^4\theta + 2a^2b^2\cos^2\theta\sin^2\theta + b^4\cos^4\theta}$$

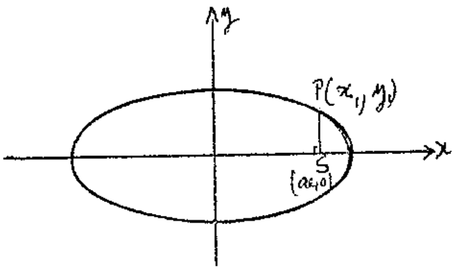
$$= \frac{a^2\left[a^4\sin^4\theta + (a^2b^2\sin^2\theta - a^2b^2\sin^4\theta) + a^2b^2\cos^2\theta\sin^2\theta + (b^4\cos^2\theta - b^4\cos^2\theta\sin^2\theta)\right]}{a^4\sin^4\theta + 2a^2b^2\cos^2\theta\sin^2\theta + b^4\cos^4\theta}$$

$$= \frac{a^2(a^4\sin^4\theta + a^2b^2\cos^2\sin^2\theta + a^2b^2\cos^2\sin^2\theta + b^4\cos^4\theta}{a^4\sin^4\theta + 2a^2b^2\cos^2\theta\sin^2\theta + b^4\cos^4\theta}$$

$= a^2$

$=$ RHS Hence M,L lie on $x^2 + y^2 = a^2$

Q1. 12.



(i) If PS is a semi latus rectum

$x_1 = ae$ and

$$\frac{a^2e^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

$$\therefore\ y_1^2 = \left(\frac{a^2 - a^2e^2}{a^2}\right)b^2$$

$$= \frac{a^2(1 - e^2)}{a^2}\,b^2$$

$$= \frac{b^4}{a^2}$$

$$y_1 = \frac{b^2}{a}$$

So the length of PS $= \dfrac{b^2}{a}$ units.

(continued on next page)

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Q1. 12. (cont'd)

OR (ii) If $P(a\cos\theta,\ b\sin\theta)$

$$PS^2 = (ae - a\cos\theta)^2 + b^2\sin^2\theta$$
$$= a^2e^2 + a^2\cos^2\theta + b^2\sin^2\theta$$
$$= a^2[e^2 + \cos^2\theta - 2e\cos\theta + (1-e^2)\sin^2\theta] \qquad (b^2 = a^2 - a^2e^2)$$
$$= a^2[e^2 + \cos^2\theta - 2e\cos\theta + (1-e^2)(1-\cos^2\theta)]$$
$$= a^2[\cancel{e^2} + \cancel{\cos^2\theta} - 2e\cos\theta + 1 - \cancel{\cos^2\theta} - \cancel{e^2} + e^2\cos^2\theta]$$
$$= a^2[1 - e\cos\theta]^2$$
$$= a^2(1-e^2)^2$$
$$= a^2\frac{b^4}{a^4}$$
$$\therefore PS = \frac{b^2}{a}$$

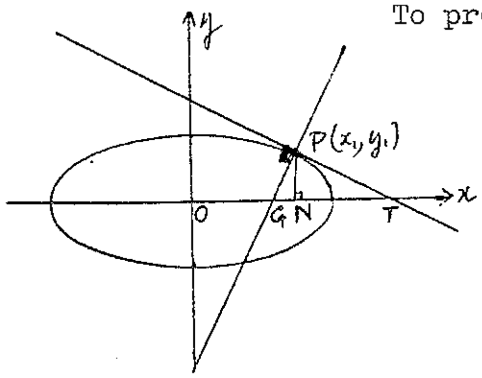
Note

but $ae = a\cos\theta$

$\therefore \ e = \cos\theta$

but $1 - e^2 = \frac{b^2}{a^2}$

Q1. 13.



To prove: $OT \cdot NG = b^2$

At T we have $\frac{xx_1}{a^2} = 1$

$$x = \frac{a^2}{x_1}$$

at G we have $\frac{xa^2}{x_1} = a^2 - b^2$

$$x = \frac{(a^2-b^2)}{a^2}x_1$$

$$\therefore \quad OT = \frac{a^2}{x_1}$$

$$NG = ON - OG$$
$$= x_1 - \frac{x_1}{a^2}(a^2 - b^2)$$
$$= x_1 - \frac{x_1}{\cancel{a^2}_{1}}(\cancel{a^2}^{1}e^2)$$
$$= x_1(1 - e^2)$$
$$= x_1\frac{b^2}{a^2}$$
$$OT \cdot NG = \frac{a^2}{x_1} \cdot \frac{b^2x_1}{a^2}$$
$$= b^2$$

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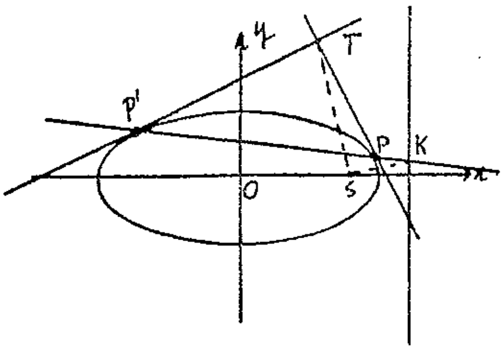
Q1. 14.



Prove: $T\hat{S}K = 90^o$

Note that PP^1 is a chord of contact with equation

$$\frac{xx_o}{a^2} + \frac{yy_o}{b^2} = 1 \quad [\text{and } T(x_o, y_o)]$$

PP^1 intersects directrix $x = \frac{a}{e}$ when $\frac{x_o}{ae} + \frac{yy_o}{b^2} = 1$

$$\therefore\ y = \frac{b^2(ae - x_o)}{y_o ae}$$

Hence $K\left(\frac{a}{e}, \frac{b^2(ae - x_o)}{y_o ae}\right)$

Then the gradient of the line joining $T(x_o, y_o)$ $S(ae, 0)$ is

$$m_1 = \frac{y_o}{x_o - ae}$$

The gradient of the line joining $S(ae, 0)$

$K\left(\frac{a}{e}, \frac{b^2(ae - x_o)}{y_o ae}\right)$ is

$$m_2 = \frac{\frac{b^2(ae - x_o)}{y_o ae} - 0}{\frac{a}{e} - ae}$$

$$m_2 = \frac{ae - x_o}{y_o}$$

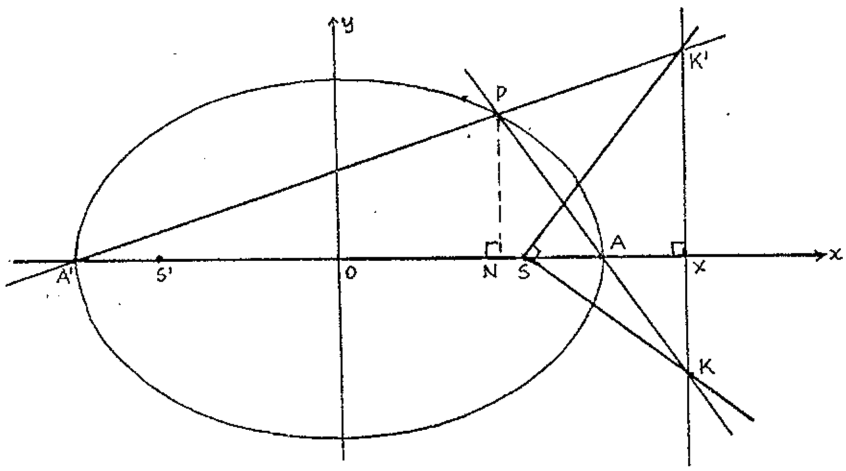
$$m_1 m_2 = \frac{y_o}{x_o - ae} \cdot \frac{ae - x_o}{y_o}$$

$$= -1$$

Hence $T\hat{S}K = 90^o$

(Notice; as $P^1 \longrightarrow P$; $T \longrightarrow P$, $TS \longrightarrow PS$, so this property applies for a single tangent at P also. See example page 42.)

Q1. 15.



(i) (a) Let P be $(a \cos \theta, b \sin \theta)$

Gradient of $PA^1 = \frac{b \sin \theta}{a(\cos \theta + 1)}$

Equation of $PA^1 \Rightarrow y = \frac{b \sin \theta}{a(1 + \cos \theta)} \cdot (x + a)$

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At K^1 $x = \frac{a}{e}$, hence we have $y = \frac{b\sin\theta(1+e)}{e(1+\cos\theta)}$ $\therefore K^1\left(\frac{a}{e}, \frac{b\sin\theta(1+e)}{e(1+\cos\theta)}\right)$

Gradient of PA $= \frac{b\sin\theta}{a(\cos\theta - 1)}$

Equation of PA; $y = \frac{b\sin\theta}{a(\cos\theta - 1)}(x - a)$

At K, $x = \frac{a}{e}$ hence we have $y = \frac{(1-e)b\sin\theta}{e(\cos\theta - 1)}$ $\therefore K\left(\frac{a}{e}, \frac{(1-e)b\sin\theta}{e(\cos\theta - 1)}\right)$

$$m_1 = \text{gradient of PK} = \frac{(1-e)\,b\sin\theta}{e(\cos\theta - 1)}\Big/\left(\frac{a}{e} - ae\right) = \frac{(1-e)\,b\sin\theta}{a(1-e^2)(\cos\theta - 1)}$$

$$m_2 = \text{gradient of } P^1K = \frac{(1+e)\,b\sin\theta}{e(\cos\theta + 1)}\Big/\left(\frac{a}{e} - ae\right) = \frac{(1+e)\,b\sin\theta}{a(1-e^2)(\cos\theta + 1)}$$

$$m_1m_2 = \frac{(1-e^2)b^2\sin^2\theta}{a^2(1-e^2)^2(\cos^2\theta - 1)} = \frac{b^2}{a^2}\cdot\frac{\sin^2\theta}{(-\sin^2\theta)}\cdot\frac{1}{1-e^2} = -1$$

Hence $\angle KSK^1 = 90^\circ$

(b) <u>if $P(x_1, y_1)$ instead of $(a\cos\theta, b\sin\theta)$</u>

Gradient of $A^1P = \frac{y_1}{x_1 + a}$

Equ. of A^1P $\quad y = \frac{y_1}{x_1 + a}(x + a)$

Cuts $x = \frac{a}{e}$ where $y = \frac{y_1 a(1+e)}{e(x_1 + a)}$

$\therefore K^1$ $\frac{a}{e}, \frac{ay_1(1+e)}{e(x_1 + a)}$

m_1 = gradient of SK^1

$$= \frac{\frac{y_1 a(1+e)}{e(x_1 + a)}}{\frac{a}{e} - ae}$$

$$= \frac{y_1 \cancel{a}\cancel{e}(1+e)}{\cancel{e}(x_1 + a)\cancel{a}(1-e^2)}$$

$$= \frac{y_1(1+e)}{(x_1 + a)(1-e^2)}$$

Gradient of AP $= \frac{y_1}{x_1 - a}$

Equ. of AP $\quad y = \frac{y_1}{x_1 - a}(x - a)$

Cuts $x = \frac{a}{e}$ when $y = \frac{ay_1(1-e)}{e(x_1 - a)}$

$\therefore K\left(\frac{a}{e}, \frac{ay_1(1-e)}{e(x_1 - a)}\right)$

m_2 = gradient of SK

$$= \frac{\frac{ay_1(1-e)}{e(x_1 - a)}}{\frac{a}{e} - ae}$$

$$= \frac{y_1 \cancel{a}\cancel{e}(1-e)}{\cancel{e}(x_1 - a)\cancel{a}(1-e^2)}$$

$$= \frac{y_1(1-e)}{(x_1 - a)(1-e^2)}$$

$$m_1 \times m_2 = \frac{y_1(1+e)}{(x_1 + a)(1-e^2)}\cdot\frac{y_1(1-e)}{(x_1 - a)(1-e^2)}$$

$$= \frac{y_1^2(1-e^2)}{(x_1^2 - a^2)(1-e^2)^2}$$

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Q1. 15. (i) (b) cont'd)

$$= \frac{y_1^2}{(x_1^2 - a^2)(1 - e^2)}$$

$$= \frac{b^2(a^2 - x_1^2)}{a^2(x_1^2 - a^2)} \cdot \frac{a^2}{b^2}$$

$$= -1 \quad \therefore \; K\widehat{S}K = 90^\circ$$

$\{$Note
$1 - e^2 = \dfrac{b^2}{a^2}$
$y_1^2 = \dfrac{b^2}{a^2}(a^2 - x_1^2)\}$

(ii) $K^1X \cdot KX = XS^2$ (to prove)

(a) using similar triangles ($\triangle KXS \sim \triangle SXK^1$)

$$\frac{KX}{XS} = \frac{XS}{K^1X} \quad \therefore \; K^1X \cdot KX = XS^2$$

OR

(b) $K^1X = \left|\dfrac{b \sin\theta(1 + e)}{e(1 + \cos\theta)}\right|, \quad KX = \left|\dfrac{(1 - e)\, b \sin\theta}{e(\cos\theta - 1)}\right|$

$$SX = \left|\frac{a}{e} - ae\right| = \left|\frac{a}{e}(1 - e^2)\right| = \frac{a}{e} \cdot \frac{b^2}{a^2} = \frac{b^2}{ae}$$

$$K^1X \cdot KX = \left|\frac{b^2 \sin^2\theta(1 - e^2)}{e^2(-\sin^2\theta)}\right|$$

$$= \frac{b^2(1 - e^2)}{e^2}$$

$$= \frac{b^4}{a^2e^2}$$

$$= SX^2$$

(iii) $PN : NA^1 = XK^1 : XA^1$ (to prove)

(a) Using similar triangles ($\triangle A^1PN \sim \triangle A^1K^1X$)

$$\frac{PN}{NA^1} = \frac{K^1X}{XA^1}$$

OR (b) $PN : NA^1 = XK^1 : XA^1 \iff PN \cdot XA^1 = XK^1 \cdot NA^1$

LHS $= PN \cdot XA^1$

$= b \sin\theta \, (a + \frac{a}{e})$

$= \frac{ab}{e}\sin\theta\,(e + 1)$

RHS $= XK^1 \cdot NA^1$

$= \dfrac{b\sin\theta(1+e)}{e(1 + \cos\theta)} \cdot (a + a\cos\theta)$

$= \dfrac{ab \sin\theta(1 + \cos\theta)(1 + e)}{e\,(1 + \cos\theta)}$

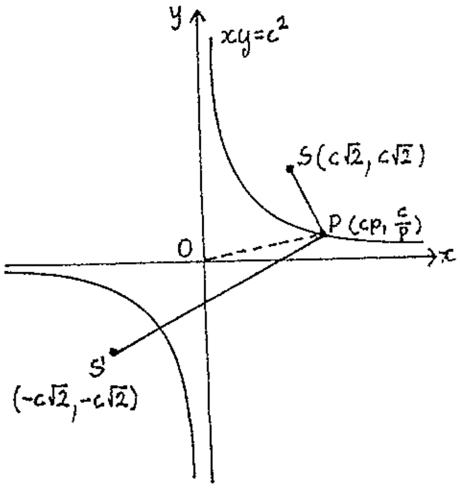
$= \dfrac{ab \sin\theta(1 + e)}{e}$

$=$ LHS

$$\therefore PN : NA^1 = XK^1 : XA^1$$

UNIT 4

Q.2. 1.



(a) Let P be the point $(cp, \frac{c}{p})$, deduce that S and S' (foci) are $(c\sqrt{2}, c\sqrt{2})$ and $(-c\sqrt{2}, -c\sqrt{2})$ respectively.

Verify that $p^2 + \frac{1}{p^2} = (p + \frac{1}{p})^2 - 2 \ldots (1)$

$$p^4 + \frac{1}{p^4} = (p + \frac{1}{p})^4 - 4(p + \frac{1}{p})^2 + 2 \ldots (2)$$

To prove: $SP.S'P = OP^2$ i.e. that $SP^2.S'P^2 = OP^4$.

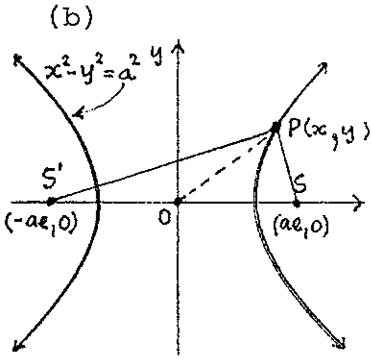
$$SP^2 = (c\sqrt{2} - cp)^2 + (c\sqrt{2} - {}^{c}/p)^2$$
$$= 4c^2 - 2\sqrt{2}c^2(p + {}^{1}/p) + c^2(p^2 + {}^{1}/{}_{p^2})$$

$$S'P^2 = (cp + c\sqrt{2})^2 + ({}^{c}/p + c\sqrt{2})^2 = 4c^2 + 2\sqrt{2}c^2(p + {}^{1}/p) + c^2(p^2 + {}^{1}/{}_{p^2})$$

$$LHS = SP^2.\ S'P^2$$
$$= c^2\left[(p^2 + \frac{1}{p^2}) - 2\sqrt{2}(p + {}^{1}/p) + 4\right] c^2\left[(p^2 + {}^{1}/{}_{p^2}) + 2\ 2(p + {}^{1}/p) + 4\right]$$
$$= c^4\left[(p + {}^{1}/p)^2 - 2\sqrt{2}(p + {}^{1}/p) + 2\right]\left[(p + {}^{1}/p)^2 + 2\sqrt{2}\ (p + {}^{1}/p) + 2\right]$$
$$= c^4\left[(p + {}^{1}/p) - \sqrt{2}\right]^2\left[(p + {}^{1}/p) + \sqrt{2}\right]^2$$
$$= c^4\left[(p + {}^{1}/p)^2 - 2\right]^2$$

$$RHS = OP^4 = (c^2p^2 + {}^{c^2}/{}_{p^2})^2 = c^4(p^4 + \frac{1}{p^4} + 2)$$
$$= c^4\left[(p + {}^{1}/p)^4 - 4(p + {}^{1}/p)^2 + 4\right] \quad \text{from (2)}$$
$$= c^4\left[(p + {}^{1}/p)^2 - 2\right]^2$$
$$= LHS$$

(b)



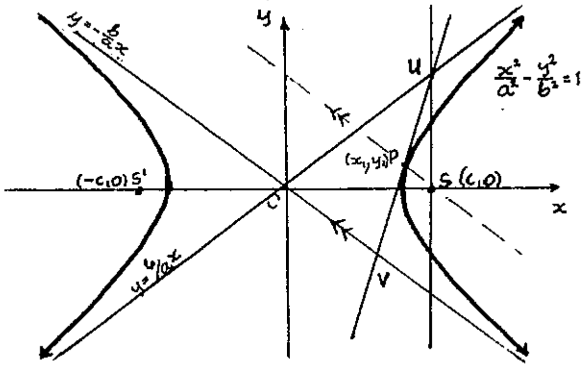
Let $P(x,y)$ be any point on $x^2 - y^2 = a^2$ with foci $S(ae,o)$, $S'(-ae,o)$.

$$SP^2.S'P^2 = \left[(x - ae)^2 + y^2\right]\left[(x + ae)^2 + y^2\right]$$
$$= \left[(x - ae)(x + ae)\right]^2 + y^2\left[(x+ae)^2 + (x-ae)^2\right] + y^4$$
$$= (x^2 - a^2e^2)^2 + 2y^2(x^2 + a^2e^2) + y^4$$

$$= \left[x^2 - 2(x^2-y^2)\right]^2 + 2y^2\left[x^2 + 2(x^2-y^2)\right] + y^4 \quad (\text{Using } a^2 = x^2 - y^2;\ e^2 = 2)$$
$$= x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2 \quad (\text{Simplifying and factorizing}).$$
$$SP^2.S'P^2 = x^2 + y^2$$
$$= OP^2 \quad \text{as required.}$$

UNIT 4

Q2. 2.



Let (c,0) be the coordinates of S, (x_1, y_1) the coordinates of P and U is the point of concurrence of the tangent at P, the asymptote and of the perpendicular through S.

The tangent at P $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ intersects the asymptote

$y = \frac{b}{a}x$ at U_1 so we have

$$\frac{xx_1}{a^2} - \frac{bxy_1}{b^2a} = 1$$

$$x = \frac{a^2b}{bx_1 - ay_1} \qquad \left(\text{and } y = \frac{ab^2}{bx_1 - ay_1}\right)$$

but at U $x = c$ $\quad \therefore \frac{a^2b}{bx_1 - ay_1} = c$ is the condition for concurrency.

Since $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ i.e., $bx_1^2 - ay_1^2 = a^2b^2$ the above equation

can be simplified to $\frac{a^2b}{bx_1 - ay_1} \cdot \frac{bx_1 + ay_1}{bx_1 + ay_1} = \frac{a^2b(bx_1 + ay_1)}{b^2x_1^2 - a^2y_1^2} = \frac{bx_1 + ay_1}{b}$

Hence we have $\frac{bx_1 + ay_1}{b} = c$ ....(1)

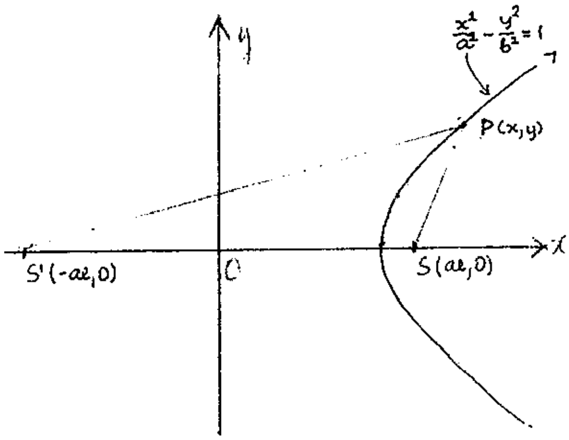
The gradient of SP is $\frac{y_1}{x_1 - c} = m$ ....(2)

$$(1) \longrightarrow (2) \quad m = \frac{y_1}{x_1 - \frac{(bx_1 + ay_1)}{b}} = \frac{by_1}{bx_1 - bx_1 - ay_1} \quad \therefore m = -\frac{b}{a} \text{ which}$$

is the same as the gradient of the other asymptote.

So PS is parallel to $y = -\frac{b}{a}x$.

Q2. 3.



Prove that as $P(x,y)$ moves so that $S^1P - SP =$ constant, then the equation of the locus is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Let S and S^1 be the points (ae,0) and (-ae,0) respectively, and the constant 2a.

UNIT 4

Q2. 3. (cont'd)

Hence $S^1P - SP = 2a$

$$\text{i.e., } \sqrt{(x+ae)^2+y^2} - \sqrt{(x-ae)^2+y^2} = 2a$$

$$\sqrt{(x+ae)^2+y^2} = 2a + \sqrt{(x-ae)^2+y^2}$$

$$x^2+2aex+a^2e^2+y^2 = 4a^2+(x-ae)^2+y^2+4a\sqrt{(x-ae)^2+y^2}$$

$$x^2+2aex+a^2e^2+y^2 = 4a^2+x^2-2aex+a^2e^2+y^2+4a\sqrt{(x-ae)^2+y^2}$$

$$4aex-4a^2 = 4a\sqrt{(x-ae)^2+y^2}$$

$$(ex-a)^2 = (x-ae)^2+y^2$$

$$e^2x^2-2aex+a^2 = x^2-2aex+a^2e^2+y^2$$

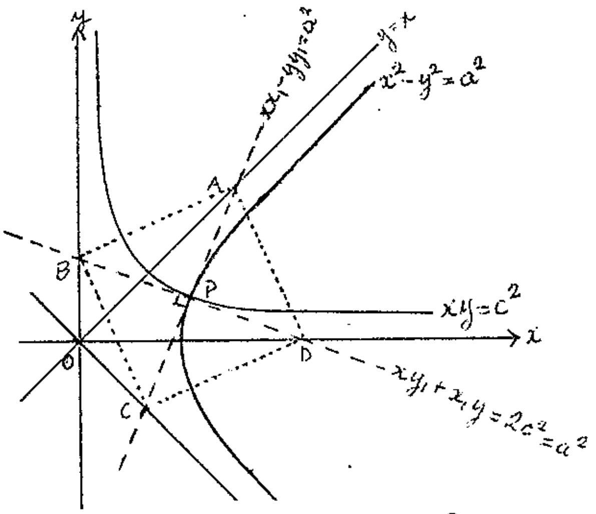
$$a^2(1-e^2) = x^2(1-e^2)+y^2 \qquad \div \text{ by } a^2(1-e^2)$$

$$1 = \frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}$$

Let $b^2 = (1-e^2)a^2$, but for the hyperbola $e > 1$ $\therefore 1-e^2$ is negative and since b^2 must be positive, we must let $b^2 = -a^2(1-e^2)$ or $b^2 = a^2(e^2-1)$, so the equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{as required.}$$

Q2. 4.



For the hyperbola $x^2-y^2 = a^2$ the equation of tangent at $P(x_1,y_1)$ is $xx_1-yy_1 = a^2$, and the asymptotes are $y = \pm x$.

The tangent at $P(x_1,y_1)$ meets the asymptotes when

$$xx_1 \pm xy_1 = a^2$$

$$\therefore x = \frac{a^2}{x_1-y_1} \text{ or } x = \frac{a^2}{x_1+y_1}$$

$$\text{then } y = \frac{a^2}{x_1-y_1} \text{ or } y = \frac{a^2}{x_1+y_1}$$

Hence A is $\left(\frac{a^2}{x_1-y_1}, \frac{a^2}{x_1-y_1}\right)$ and B is $\left(\frac{a^2}{x_1+y_1}, \frac{-a^2}{x_1+y_1}\right)$

$$AB^2 = \left(\frac{a^2}{x_1-y_1} - \frac{a^2}{x_1+y_1}\right)^2 + \left(\frac{a^2}{x_1-y_1} + \frac{a^2}{x_1+y_1}\right)^2$$

$$= \frac{4a^4(x_1^2+y_1^2)}{(x_1^2-y_1^2)} \quad \text{but } x_1^2-y_1^2 = a^2 \quad \therefore AB^2 = 4(x_1^2+y_1^2)$$

UNIT 4

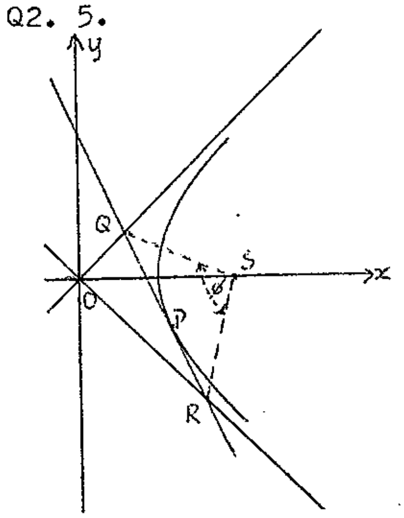
Q.2. 4. cont.

For the hyperbola $xy = c^2$ the equation of tangent at $P(x_1, y_1)$ is $xy_1 + yx_1 = 2c^2$ and it meets the asymptote $x = 0$ at $B(\frac{2c^2}{y_1}, 0)$ and the asymptote $y = 0$ at $D(0, \frac{2c^2}{x_1})$.

But $2c^2 = 2x_1y_1$ $\therefore$ $B(2x_1, 0)$ and $D(0, 2y_1)$ and $BD^2 = 4(x_1^2 + y_1^2)$.

Hence the diagonals of quadrilateral ABCD are equal. Gradient of AC is $\frac{x^1}{y_1}$ and of BD is $-\frac{y_1}{x_1}$ $\therefore$ AC $\perp$ BD and ABCD is a square.

Q2. 5.



Let $(c, 0)$ and $(a \sec\theta, b\tan\theta)$ be the coordinates of S and P respectively.

The equation of the tangent at P is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$ ....(1)
and the equations of the asymptotes are $y = \pm\frac{b}{a}x$ ....(2)
Then solving (1) and (2) we obtain $\frac{x\sec\theta}{a} \pm \frac{x\tan\theta}{a} = 1$

$$\therefore\ x = \frac{a}{\sec\theta - \tan\theta} \qquad \text{or } x = \frac{a}{\sec\theta + \tan\theta}$$

$$x = \frac{a\cos\theta}{1 - \sin\theta} \qquad \text{or } x = \frac{a\cos\theta}{1 + \sin\theta}$$

Since $y = \pm\frac{bx}{a}$

$$y = \frac{b\cos\theta}{1 - \sin\theta} \qquad \text{or } y = \frac{-b\cos\theta}{1 + \sin\theta}$$

Thus the coordinates of the points of intersection are;

$$Q\left(\frac{a\cos\theta}{1 - \sin\theta}, \frac{b\cos\theta}{1 - \sin\theta}\right) \quad R\left(\frac{a\cos\theta}{1 + \sin\theta}, \frac{b\cos\theta}{1 + \sin\theta}\right)$$

The gradient of QS is $\dfrac{\frac{b\cos\theta}{1 - \sin\theta}}{\frac{a\cos\theta}{1 - \sin\theta} - c} = \dfrac{b\cos\theta}{a\cos\theta - c + c\sin\theta} = m_1$

The gradient of RS is $\dfrac{-b\cos\theta}{a\cos\theta - c - c\sin\theta} = m_2$

UNIT 4

Q2. 5. (cont'd)

$$\tan \phi = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{\dfrac{b\cos\theta}{a\cos\theta - c + c\sin\theta} + \dfrac{b\cos\theta}{a\cos\theta - c - c\sin\theta}}{1 - \dfrac{b\cos\theta}{a\cos\theta - c + c\sin\theta}\cdot\dfrac{b\cos\theta}{a\cos\theta - c - c\sin\theta}}$$

$$= \frac{2b\cos\theta(a\cos\theta - c)}{(a\cos\theta - c + c\sin\theta)(a\cos\theta - c - c\sin\theta) - b^2\cos^2\theta}$$

$$= \frac{2b\cos\theta(a\cos\theta - c)}{a^2\cos\theta - 2ac\cos\theta + c^2 - c^2\sin^2\theta - b^2\cos^2\theta}$$

$$= \frac{2b\cos\theta(a\cos\theta - c)}{a^2\cos^2\theta - b^2\cos^2\theta + c^2\cos^2\theta - 2ac\cos\theta + c^2 - c^2}$$

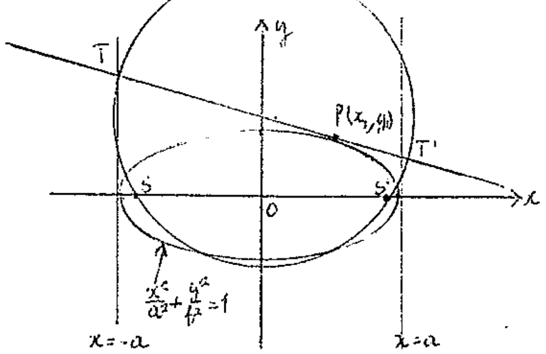
$$= \frac{2b\cos\theta(a\cos\theta - c)}{(a^2 + c^2 - b^2)\cos^2\theta - 2ac\cos\theta}$$ (since $ae = c$, $a^2+c^2-b^2 = 2a^2$)

$$= \frac{2b\cos\theta(a\cos\theta - c)}{2a^2\cos^2\theta - 2ac\cos\theta}$$

$$= \frac{2b\cos\theta(a\cos\theta - c)}{2a\cos\theta(a\cos\theta - c)}$$

$\therefore \tan\phi = \frac{b}{a}$. Hence QR subtends a constant angle; $\tan^{-1}(\frac{b}{a})$ at S.

Q3. 1.



Let the coordinates of S and P be $(ae, 0)$ and (x_1, y_1) respectively.

The tangent at P $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

cuts $x = \pm a$ at $y = \frac{b^2}{y_1}(1 \pm \frac{x_1}{a})$

$\therefore$ T$\left[-a, \frac{b^2}{y_1}(1 + \frac{x_1}{a})\right]$ and

T$^1\left[a, \frac{b^2}{y_1}(1 - \frac{x_1}{a})\right]$

UNIT 4

The equation of a circle on a given diameter $A(x_3,y_3)$, $B(x_4,y_4)$ is $(x-x_3)(x-x_4) + (y-y_3)(y-y_4) = 0$ (see page 21).

The equation of the circle with diameter TT^1 is

$$(x-a)(x+a) + \left[y - \frac{b^2}{y_1}\left(1 - \frac{x_1}{a}\right)\right]\left[y - \frac{b^2}{y_1}\left(1 + \frac{x_1}{a}\right)\right] = 0$$

The circle intersects the x axis when $y = 0$ i.e. when

$$x^2 - a^2 + \left[\frac{b^2}{y_1}\left(1 - \frac{x_1}{a}\right)\right]\left[\frac{b^2}{y_1}\left(1 + \frac{x_1}{a}\right)\right] = 0$$

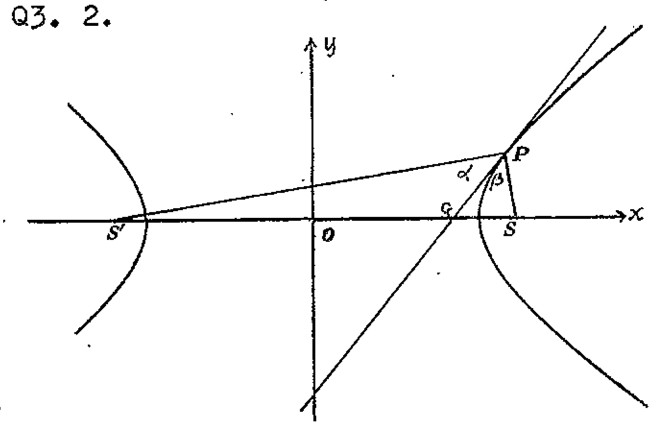
$$x^2 - a^2 + \frac{b^4}{y_1^2}\left(1 - \frac{x_1^2}{a^2}\right) = 0 \qquad \left(\text{but } \frac{y_1^2}{b^2} = 1 - \frac{x_1^2}{a^2}\right)$$

$$x^2 - a^2 + \frac{b^4}{y_1^2} \cdot \frac{y_1^2}{b^2} = 0 \qquad (b^2 = a^2(1-e^2) = a^2 - a^2e^2)$$

$$x^2 = a^2 - b^2$$

$\therefore \quad x^2 = a^2e^2$ so $x = \pm\, ae$. i.e. the circle passes through S and S^1.

Q3. 2.



Let P be the point (x_1,y_1) $S\ (ae,0)$ and $S^1(-ae,0)$.

The tangent at P

$\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1$ meets the transverse axis at G.

$\therefore \dfrac{xx_1}{a^2} = 1$ i.e. $x = \dfrac{a^2}{x_1}$

hence $G\left(\dfrac{a^2}{x_1}, 0\right)$

$$GS = ae - \frac{a^2}{x_1} \text{ and } GS^1 = ae + \frac{a^2}{x_1}$$

$$PS^2 = (x_1 - ae)^2 + y_1^2 = x_1^2 - 2aex_1 + a^2e^2 + y_1^2$$

$$= x_1^2 - 2aex_1 + a^2e^2 + \frac{b^2}{a^2}(x_1^2 - a^2) \text{ --------(NOTE: } y_1^2 = \frac{b^2}{2}(x_1^2 - a^2) \text{ and } b^2 = a\ (e^2-1))$$

$$= x_1^2 - 2aex_1 + a^2e^2 + (e^2-1)(x_1^2 - a^2)$$

$$= x_1^2e^2 - 2aex_1 + a^2$$

$$PS = \sqrt{(x_1e - a)^2} = x_1e - a \qquad \text{similarly } PS^1 = x_1e + a$$

UNIT 4

$$\frac{PS}{SG} = \frac{x_1 e - a}{ae - \frac{a^2}{x_1}} = \frac{x_1^2 e - x_1 a}{aex_1 - a^2} = \frac{x_1(x_1 e - a)}{a(x_1 e - a)} = \frac{x_1}{a}$$

$$\frac{PS^1}{GS^1} = \frac{x_1 e + a}{ae + \frac{a^2}{x_1}} = \frac{x_1^2 e + x_1 a}{aex_1 + a^2} = \frac{x_1(x_1 e + a)}{a(x_1 e + a)} = \frac{x_1}{a}$$

$$\therefore \frac{PS}{SG} = \frac{PS^1}{GS^1}$$

Assume that PG makes $S^1\widehat{PG} = \alpha$ and $\widehat{SPG} = \beta$ i.e. divides $\widehat{SPS}$ unequally.

Using the sine rule in $\triangle$'s S^1PG and SPG we have

$$\frac{GS^1}{\sin\alpha} = \frac{S^1P}{\sin S^1\widehat{GP}} \quad \text{and} \quad \frac{SG}{\sin\beta} \quad \frac{SP}{\sin \widehat{SGP}}$$

$$\text{i.e.,} \quad \frac{GS^1}{S^1P} = \frac{\sin\alpha}{\sin(180^\circ - \widehat{SGP})} \quad \text{and} \quad \frac{SG}{SP} = \frac{\sin\beta}{\sin \widehat{SGP}}$$

$$\text{but} \quad \frac{GS^1}{S^1P} = \frac{SG}{SP} \quad \therefore \frac{\sin\alpha}{\sin(180^\circ - \widehat{SGP})} = \frac{\sin\beta}{\sin \widehat{SGP}}$$

Since $\sin(180^\circ - \widehat{SGP}) = \sin \widehat{SGP}$

$$\therefore \sin\alpha = \sin\beta$$

and $\alpha = \beta$

i.e. the tangent PG bisects the angle S^1PS.

OR

<u>Theorem</u>: The bisector of one angle of a triangle divides the opposite side in the ratio of the sides about that angle.

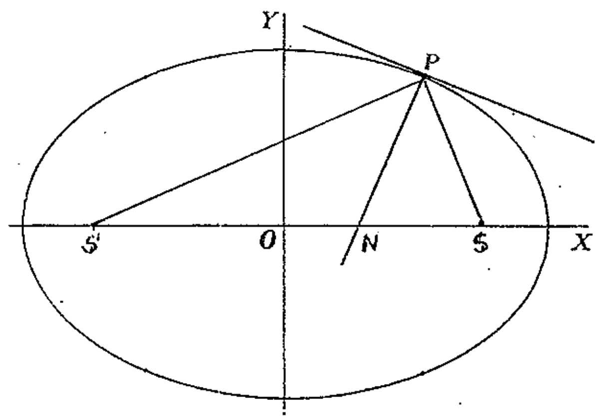
Hence it is sufficient to prove that

$$\frac{GS^1}{S^1P} = \frac{SG}{SP}$$

Conclusion: PG bisects the angle S^1PS.

UNIT 4

Q3. 3.



PN is the normal at P.

The normal
$$\frac{xa}{\cos\theta} - \frac{yb}{\sin\theta} = a^2-b^2 = a^2e^2$$
meets the x axis at N.

$\therefore$ we have $x = \dfrac{a^2e^2\cos\theta}{a}$

So $N(e^2a\cos\theta, 0)$. If S and S^1 are the points $(ae,0)$, $(-ae,0)$ respectively, P is $(a\cos\theta,\ b\sin\theta)$ as given.

Then the gradient of PN $= \dfrac{b\sin\theta}{a\cos\theta - ae^2\cos\theta} = \dfrac{b\sin\theta}{a\cos\theta(1-e^2)} = \dfrac{a^2b\sin\theta}{a\cos\theta\cdot b^2} = \dfrac{a\sin\theta}{b\cos\theta}$

The gradient of PS $= \dfrac{b\sin\theta}{a\cos\theta - ae}$ and

the gradient of PS^1 $= \dfrac{b\sin\theta}{a\cos\theta + ae}$

$$\tan N\hat{P}S^1 = \left|\frac{\dfrac{b\sin\theta}{a\cos\theta+ae} - \dfrac{a\sin\theta}{b\cos\theta}}{1 + \dfrac{b\sin\theta}{a\cos\theta+ae}\cdot\dfrac{a\sin\theta}{b\cos\theta}}\right| = \left|\frac{b^2\sin\theta\cos\theta - a^2\sin\theta\cos\theta - a^2e\sin\theta}{ab\cos^2\theta + ab\sin^2\theta + abe\cos\theta}\right|$$

$$= \left|\frac{(b^2-a^2)\sin\theta\cos\theta - a^2e\sin\theta}{ab(\cos^2\theta+\sin^2\theta) + abe\cos\theta}\right| = \left|\frac{-a^2e^2\sin\theta\cos\theta - a^2e\sin\theta}{ab(1+e\cos\theta)}\right|$$

$$= \left|\frac{-a^2e\sin\theta(e\cos\theta+1)}{ab(1+e\cos\theta)}\right|$$

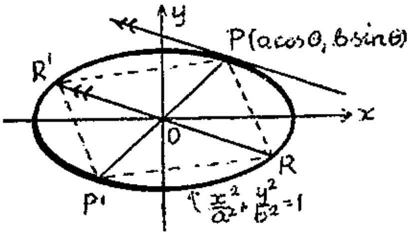
$\tan N\hat{P}S^1 = \dfrac{ae\sin\theta}{b}$. Similarly it can be shown that

$$\tan N\hat{P}S = \frac{\dfrac{b\sin\theta}{a\cos\theta - ae} - \dfrac{a\sin\theta}{b\cos\theta}}{1 + \dfrac{b\sin\theta}{a\cos\theta - ae}\cdot\dfrac{a\sin\theta}{b\cos\theta}} = \frac{a^2e\sin\theta(1-e\cos\theta)}{ab(1-e\cos\theta)} = \frac{ae\sin\theta}{b}$$
$$= \tan N\hat{P}S^1$$

i.e. the normal at P bisects the angle SPS.

UNIT 4 - MISCELLANEOUS QUESTIONS

Q1.



(a) Let $E \equiv x^2/a^2 + y^2/b^2 = 1$ ....(1)

Equation of tangent at

$P(a\cos\theta, b\sin\theta)$ is:

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ ....(2)

Equation of RR' parallel to (2)

is:

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 0$ i.e. $xb\cos\theta + ya\sin\theta = 0$ ....(3)

$E \cap RR'$: Subst. (3) into (1) $x^2/a^2 + \left[x^2b^2/b^2a^2\right]\cot^2\theta = 1$

$x^2(1 + \cot^2\theta) = a^2$

so $x = \pm a\sin\theta$ ....(4)

Substitute (4) into (3) yields: $y = \mp b\cos\theta$ ....

So $R = (a\sin\theta, -b\cos\theta)$ and $R'(-a\sin\theta, b\cos\theta)$

Let h = perpendicular distance of P from RR'

So $h = \frac{ab\cos^2\theta + ab\sin^2\theta}{\sqrt{(b^2\cos^2\theta + a^2\sin^2\theta)}}$ i.e. $h = \frac{ab}{\sqrt{(b^2\cos^2\theta + a^2\sin^2\theta)}}$

$RR' = \sqrt{((2a\sin\theta)^2 + (2b\cos\theta)^2)}$ i.e. $RR' = 2\sqrt{(b^2\cos^2\theta + a^2\sin^2\theta)}$

Area of triangle RPR' $= \frac{1}{2}.h.RR'$

$= ab$ which is independent of the

position of P.

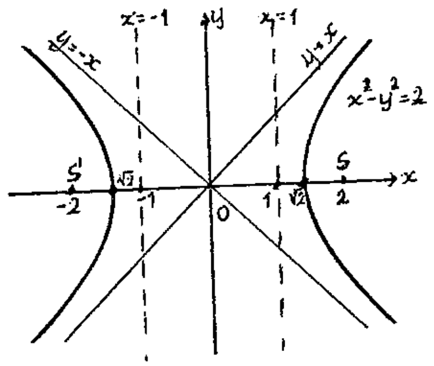
(b) Using symmetry $P'(-a\cos\theta, -b\sin\theta)$. Since O is the midpoint of RR' and of PP' PRP'R' is a parallelogram of area = 2 × triangle PRP'. So area of PRP'R' = 2ab.

Q2. (a) The equation of the directrix of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is $y = \pm\frac{b}{a}x$ which is the same as $y = \pm x$ hence $\frac{b}{a} = 1$ i.e. $a = b$ so the hyperbola is RECTANGULAR. $b^2 = a^2(e^2 - 1)$ becomes $1 = e^2 - 1$. So $e = \sqrt{2}$. The equation of the directrix is $x = \frac{a}{e}$ which is given to be $x = 1$

$\therefore$ $a = e$ i.e. $a = \sqrt{2}$ and the required equation is

$x^2/2 - y^2/2 = 1$ or $x^2 - y^2 = 2$...............(1)

(b)



(c) Differentiate $x^2 - y^2 = a^2$ ....(2)

w.r.t.x. $2x - 2y\ {}^{dy}\!/_{dx} = 0$

So ${}^{dy}\!/_{dx} = {}^{x}\!/_{y}$. Hence the gradient of the normal at

$P(x_1,y_1)$ is $-{}^{y_1}\!/_{x_1}$.

Equation of normal: $y - y_1 = -{}^{y_1}\!/_{x_1}\ (x - x_1)$ i.e.

$y_1x + x_1y - 2x_1y_1 = 0$ ....(3)

(d) Put $y = 0$ into (3) so $X = 2x_1$ i.e. $x_1 = \frac{X}{2}$ ....(4)

Put $x = 0$ into (3) so $Y = 2y_1$ i.e. $y_1 = \frac{Y}{2}$........(5)

But $T(x_1,y_1)$ is a point on (2) so it must satisfy (2).

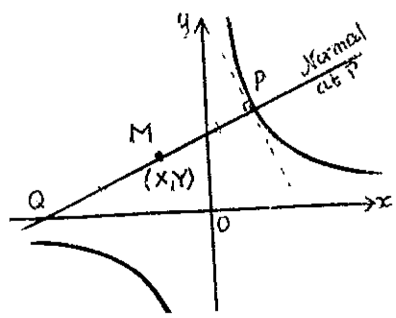
Hence $(\frac{X}{2})^2 - (\frac{Y}{2})^2 = a^2$ i.e. $X^2 - Y^2 = 4a^2$ as required.

(e) For H $a = \sqrt{2}$. So the equation of the locus of T is

$X^2 - Y^2 = 8$.

Q3. Let $P \equiv (ct, {}^{c}\!/_{t})$. If $y = {c^2}\!/_{x}$, ${}^{dy}\!/_{dx} = {}^{-c^2}\!/_{x^2}$ at $x = ct$

${}^{dy}\!/_{dx} = {}^{-1}\!/_{t^2}$



Hence the gradient of normal at P is t^2. Equation of normal

$y - {}^{c}\!/_{t} = t^2(x - ct)$.......(1)

Put $y = 0$ into (1)

so $x = c(t - {}^{1}\!/_{t^3})$

Hence $Q \equiv (c(t - {}^{1}\!/_{t^3}), 0)$.

Let $M(X,Y)$ be the midpoint of interval PQ.

$$X = \frac{ct + ct - {}^{c}\!/_{t^3}}{2} \quad \text{and} \quad Y = \frac{{}^{c}\!/_{t}}{2}$$

$$X = \frac{c(2t^4 - 1)}{2t^3}$$ ....(2) and $t = {}^{c}\!/_{2Y}$ ....(3)

Subst. (3) into (2) in order to eliminate the parameter t.

UNIT 4 - MISCELLANEOUS QUESTIONS

$$\text{So } X = \frac{c\left(\frac{2c^4}{16Y^4} - 1\right)}{2c^3/8Y^3} \qquad \text{i.e. } X = \frac{c(c^4 - 8Y^4)}{8Y^4} \times \frac{4Y^3}{c^3}$$

$2c^2YX = c^4 - 8Y^4$ i.e. $8y^4 = c^2(c^2 - 2xy)$ is the locus of M the midpoint of PQ.

Q4. Using (1) from Q3. (above) the equation of the normal to $xy = c^2$ is $ty - c = t^3(x - ct)$.

If the normal passes through an arbitrary point (x_o, y_o) then (1) (from Q3) becomes

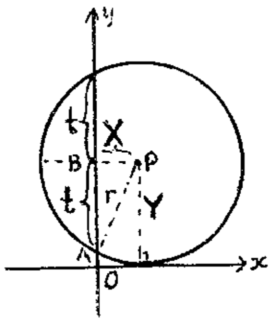
$$ty_o - c = t^3x_o - ct^4$$

$$\text{i.e. } ct^4 - x_ot^3 + y_ot - c = 0 \text{ ....(2)}$$

This equation is of the fourth degree (quartic) in t hence four normals can be drawn through an arbitrary point.

If t_1, t_2, t_3, t_4 are the 4 roots of (2) then it is true that $\Sigma t_1t_2 = 0$ and $t_1t_2t_3t_4 = -c/c = -1$.

Q5.



Let P(X,Y) be any point on the locus. We need to obtain expressions for X and Y. By eliminating the parameter from these we obtain the red. eqn.

$Y = r$ (by inspection)

$X^2 + t^2 = r^2$ (in triangle ABP).

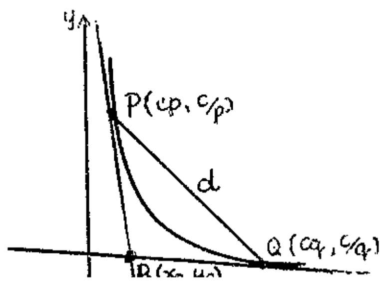
Eliminate the parameter r.

So $X^2 + t^2 = Y^2$ or $Y^2 - X^2 = t^2$ is the required locus.

Q6. (a) $y = c^2/x$ So $dy/dx = -c^2/x^2$. $m = f'(cp) = -1/p^2$ So the eqn. of tangent at P is $y = c/p = 1/p^2(x - cp)$ i.e.

$p^2y - cp = -x + cp$ or $x + q^2y = 2cp$ ....(1)

(b) Tangent at Q is $x + q^2y = 2cq$ ....(2)



$(p^2 - q^2)y = 2c(p - q)$ (1) - (2)

$(p + q)y = 2c$ $(p \neq q)$

$y = 2c/(p+q)$ ....(3)

Put (3) into (1) $x + \dfrac{p^2 2c}{p + q} = 2cp$

So $x = 2cpq/(p+q)$ ....(4)

UNIT 4 - MISCELLANEOUS QUESTIONS

(3) $y_o = {}^{2c}/_{(p+q)}$ i.e. $p + q = {}^{2c}/_{y_o}$ ....(5)

from (4) $x_o = {}^{2cpq}/_{(p+q)}$ ....(6)

Subst. (5) into (6) $x_o = 2cpq \ \dfrac{y_o}{2c}$

i.e. $\dfrac{x_o}{y_o} = pq$ ....(7)

(c) $d^2 = (cp - cq)^2 + ({}^{c}/_{p} - {}^{c}/_{q})^2$

$= c^2(p-q)^2 + c^2(\frac{q-p}{pq})^2$

$d^2 = c^2(p-q)^2 \ \{1 + \frac{1}{p^2q^2}\}$ ....(8)

(d) Note: $(p-q)^2 = (p+q)^2 - 4pq$ ....(9)

Put (5), (7) into (9). $(p-q)^2 = ({}^{2c}/_{y_o})^2 - {}^{4x_o}/_{y_o}$ ....(10)

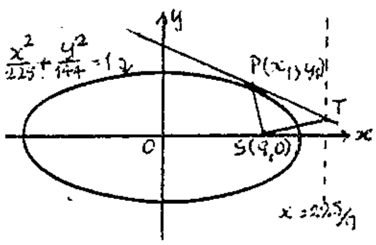
Put (10), (7) into (8) So $d^2 = c^2\left[{}^{4c^2}/_{y_o^2} - {}^{4x_o}/_{y_o}\right]\left[1 + {}^{y_o^2}/_{x_o^2}\right]$

$= c^2\left[\dfrac{4c^2 - 4x_oy_o}{y_o^2}\right]\left[\dfrac{x_o^2 + y_o^2}{x_o^2}\right]$

$x^2y^2d^2 = 4c^2(x^2 + y^2)(c^2 - x\ y\ )$

which is the required eqn. of the locus of R.

Q7. (a) ${}^{x^2}/_{225} + {}^{y^2}/_{144} = 1$. So $a = 15$, $b = 12$ $e = \dfrac{\sqrt{(a^2 - b^2)}}{a} = {}^{9}/_{15}$



Equation of tangent at $P(x_1,y_1)$ is:

${}^{x_1x}/_{225} + {}^{y_1y}/_{144} = 1$ ....(1)

Equation of directrix: $x = \frac{a}{e}$

i.e. $x = {}^{225}/_{9}$ ....(2)

Put (2) into (1) to find T the point where the tangent meets the directrix.

So $\dfrac{225}{9} \cdot \dfrac{x_1}{225} + \dfrac{yy_1}{144} = 1$ which yields $y = {}^{16(9 - x_1)}/_{y_1}$

i.e. T is the point $\{{}^{225}/_{9},\ {}^{16(9 - x_1)}/_{y_1}\}$

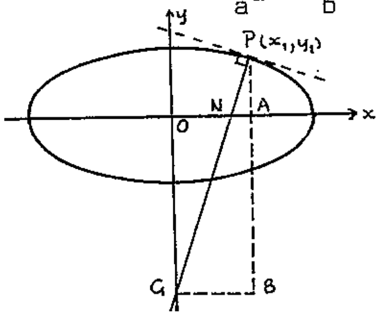
$ae = 15 \times {}^{9}/_{15} = 9$ So the focus S is the point (9,0)

$m_{PS} = {}^{y_1}/_{(x_1-9)}$ $m_{ST} = {}^{16(9-x_1)}/_{y_1} \times {}^{9}/_{144} = {}^{(9-x_1)}/_{y_1}$

Since $m_{PS} \cdot m_{ST} = -1$ PT subtends a right angle at the focus S.

MISCELLANEOUS QUESTIONS

Q.7. (b) (i) Differentiate $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ implicitly with respect to x to obtain $\frac{2x}{a^2} + \frac{2y}{b^2}.\frac{dy}{dx} = 0$ i.e. show that $\frac{dy}{dx} = \frac{-b^2x}{a^2y}$.



Therefore the gradient of the normal at $P(x_1,y_1)$ is $m = \frac{a^2y_1}{b^2x_1}$ and the equation of the normal at $P(x_1,y_1)$ is $y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1)$ i.e.

$a^2y_1x - b^2x_1y = x_1y_1(a^2 - b^2)$ (1)

However $a^2 - b^2 = a^2e^2$. (2)

(2)→(1) gives $a^2y_1x - b2x_1y = x_1y_1a^2e^2$ (3)

When $x = 0$ in (3) $y = \frac{-a^2e^2y_1}{b^2}$. So $N(0 , -\frac{a^2e^2y_1}{b^2})$

When $y = 0$ in (3) $x = x_1e^2$. So $G(x_1e^2 , 0)$

Using similar triangles PNA and PGB and noting that the coordinates of A and B are $(x_1 , 0)$ and $(0 , \frac{-a^2e^2y_1)}{b^2}$ respectively, we obtain:

$$\frac{PN}{NG} = \frac{PA}{AB} = \frac{y_1}{y_1e^2a^2/_{b^2}} = \frac{b^2}{a^2} . \frac{1}{e^2} = \frac{1 - e^2}{e^2}$$ as required

(ii)

$$\frac{PN}{PG} = \frac{PA}{PB} = \frac{y_1}{y_1 + y_1a^2e^2/_{b^2}} = \frac{y_1b^2}{y_1(b^2 + a^2e^2)} = \frac{b^2}{a^2}$$ as required