Term 2 – 2006 – Ext. I

Question 1

(a) $a = \sin t + e^{3t}$

$v = -\cos t + \frac{1}{3}e^{3t} + c$

$t = 0, v = 0 \quad \therefore 0 = -1 + \frac{1}{3} + c$

$c = \frac{2}{3}$

$\therefore v = -\cos t + \frac{1}{3}e^{3t} + \frac{2}{3}$

$x = -\sin t + \frac{1}{9}e^{3t} + \frac{2}{3}t + K$

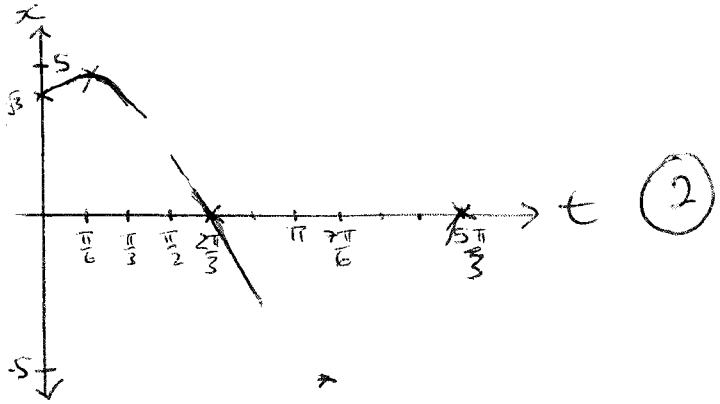
$x = \frac{10}{9}, t = 0$

$\therefore \frac{10}{9} = 0 + \frac{1}{9} + 0 + K$

$\therefore K = 1$

$\therefore x = -\sin t + \frac{1}{9}e^{3t} + \frac{2}{3}t + 1$ (3)

(b) (i) $x = 5\sin(t + \frac{\pi}{3})$



(2)

(ii) First changes direction at $t = \frac{\pi}{6}$ seconds when $x = 5$m (2)

(iii) $\text{dist} = (5 - \frac{5}{2}\sqrt{3}) + 5 + 5$

$= 10.67$m (2)

Question 2

(a) (i) $\frac{d^2x}{dt^2} = \frac{1}{2}x^3 + 2x$

$\frac{1}{2}v^2 = \int(\frac{1}{2}x^3 + 2x)dx$

$\frac{1}{2}v^2 = \frac{1}{8}x^4 + x^2 + c$

$v^2 = \frac{1}{4}x^4 + 2x^2 + 2c$

at $t = 0, x = 0, v = 2$

$\therefore 4 = 0 + 0 + 2c$

$\therefore c = 2$

$\therefore v^2 = \frac{1}{4}x^4 + 2x^2 + 4$

$v^2 = \frac{1}{4}(x^4 + 8x^2 + 16)$

$v^2 = \frac{1}{4}(x^2 + 4)^2$

$v = \pm\frac{1}{2}(x^2 + 4)$ (4)

but initially $v = 2$, so it's moving to the right and $a > 0$ for all $x > 0$ $\therefore$ it continues to move to the right with increasing speed

$\therefore v = \frac{1}{2}(x^2 + 4)$ only.

(ii) $\frac{dx}{dt} = \frac{1}{2}(x^2 + 4)$

$\frac{dt}{dx} = \frac{2}{x^2 + 4}$

$t = \tan^{-1}(\frac{x}{2}) + c_1$ (2)

$t = 0, x = 0 \quad \therefore c_1 = 0$

$t = \tan^{-1}(\frac{x}{2})$

$\tan t = \frac{x}{2}$

$\therefore x = 2\tan t$ (2)

(iii) at $t = \pi/4$ seconds

$x = 2\tan \pi/4 = 2$

$v = \frac{8}{2} = 4$

$a = \frac{8}{2} + 4 = 8$ (3)

$\therefore$ At $t = \pi/4$ seconds, the particle is 2m to the right of the origin travelling with a velocity of 4m/s and acceleration of $8 m/s^2$

Question 3

(a) (i) $v^2 = 108 - 36x - 9x^2$

$\frac{1}{2}v^2 = 54 - 18x - \frac{9}{2}x^2$

$\frac{d}{dx}(\frac{1}{2}v^2) = -18 - 9x$

$\therefore \ddot{x} = -18 - 9x$

$= -9(x+2)$ (2)

(ii) amp when $v = 0$

ie $0 = 108 - 36x - 9x^2$

$0 = x^2 + 4x - 12$

$0 = (x+6)(x-2)$

$\therefore x = -6$ or $x = 2$

(3) $\therefore$ amplitude is 4 units

c. of m. is at $x = -2$.

(iii) $n = 3$, $a = 4$

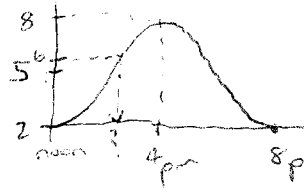
(1) max. speed $= 12^2 = 144 m/s$

(iv) $T = \frac{2\pi}{n}$

$= \frac{2\pi}{3}$ (1)

(b) $T = \frac{2\pi}{n} = 8$

$n = \pi/4$



$x = b + a\cos(nt)$

$x = 5 - 3\cos(\frac{\pi}{4}t)$

$6 = 5 - 3\cos(\frac{\pi}{4}t)$

$-\frac{1}{3} = \cos(\frac{\pi}{4}t)$

$t = 2.43\ 269\ 3792$

$t = 2 hrs\ 26 mins$

$\therefore$ Earliest time is 2:26pm.

Question 4

(a) $T = B + Ae^{kt}$

$15 = 180 + Ae^0$

$\therefore A = -165$

$\therefore T = 180 - 165e^{kt}$

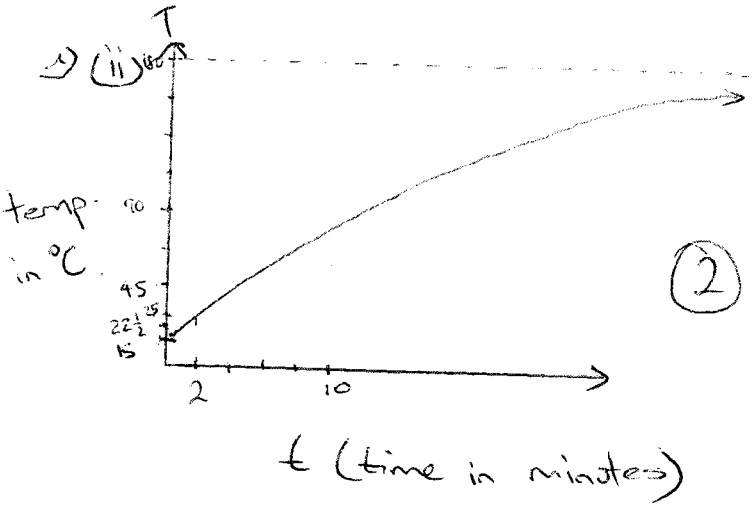
$25 = 180 - 165e^{2k}$

$\frac{-155}{-165} = e^{2k}$

$\ln(\frac{155}{165}) = 2k$ (2)

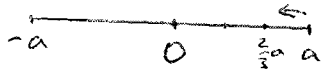
$\therefore k = \frac{1}{2}\ln(\frac{31}{33})$

$\therefore T = 180 - 165e^{\frac{1}{2}\ln(\frac{31}{33})t}$



(2)

(b)



period $= \pi$ $\therefore \pi = \frac{2\pi}{n}$

$n = 2$

$x = a\cos(2t)$

$\frac{2}{3}a = a\cos(2t)$

$\frac{2}{3} = \cos 2t$

$t = \frac{1}{2}\cos^{-1}(\frac{2}{3})$ (2)

$t = 0.420534335$

$t = 0.4$ seconds

(c) P(Mrs W not next to Mr W)

seat Mr W first – 1

seat Mrs W 2nd – 3

fill other men – 4!

fill other women – 4!

total $= 3 \times 4! \times 4!$

$= 1728$ (3)

$P = \frac{1728}{4!5!} = \frac{1728}{2880} = \frac{3}{5}$

Question 5

(a) $T = B + Ae^{-kt}$

$100^\circ = 27^\circ + A$

$\therefore A = 73^\circ$

$T = 27 + 73e^{-kt}$ (1)

$t = 1\frac{1}{2}$, $T = 85^\circ C$

$\therefore 85 = 27 + 73e^{-1\frac{1}{2}k}$

$\ln(\frac{58}{73}) = -\frac{3}{2}k$

$k = -\frac{2}{3}\ln(\frac{58}{73})$ (1)

$50 = 27 + 73e^{\frac{2}{3}\ln(\frac{58}{73})t}$

$\frac{23}{73} = e^{\frac{2}{3}\ln(\frac{58}{73})t}$

$\ln(\frac{23}{73}) = \frac{2}{3}\ln(\frac{58}{73})t$ (1)

$t = \frac{3\ln(\frac{23}{73})}{2\ln(\frac{58}{73})}$

$t = 7.53$ (1)

$\therefore t = 7$ mins 32 seconds

(b) $T = B + Ae^{-kt}$

$2B = B + Ae^{0}$

$\therefore A = B$

$T = B + Be^{-kt}$

$t = 10$, $T = 6B$ $\therefore 6B = B + Be^{10k}$

$5 = e^{10k}$

$k = -\frac{\ln 5}{10}$

$T = 4B$, $t = ??$ $\therefore 4B = B + Be^{t\frac{\ln 5}{10}}$

$3 = e^{t\frac{\ln 5}{10}}$

$\ln 3 = \frac{\ln 5}{10}t$

$t = \frac{10\ln 3}{\ln 5}$

$t = 6$ mins 5

(3)

5(c) ${}^{n}P_{2} = 132$

$\frac{n!}{(n-2)!} = 132$

$n(n-1) = 132$

$n^2 - n - 132 = 0$

$(n-12)(n+11) = 0$ (2)

$\therefore n = 12$ or -11

but $n > 0$ $\therefore n = 12$ only

Question 6

a) $\ddot{x} = 0$ $\quad \ddot{y} = -g$

$\dot{x} = v\cos\theta$ $\quad \dot{y} = -gt + v\sin\theta$

$x = vt\cos\theta$ $\quad y = -\frac{g}{2}t^2 + vt\sin\theta$

max. height when $\dot{y} = 0$

ie $0 = -gt + v\sin\theta$

$t = \frac{v\sin\theta}{g}$

sub. into y

$\therefore y = -\frac{g}{2}v^2\frac{\sin^2\theta}{g^2} + \frac{v^2\sin^2\theta}{g}$

$= \frac{-v^2\sin^2\theta + 2v^2\sin^2\theta}{2g}$

$= \frac{v^2\sin^2\theta}{2g}$

now max height $= \frac{1}{3}$ of range

range is when $y = 0$

ie $0 = -\frac{g}{2}t^2 + vt\sin\theta$

$0 = t(-\frac{g}{2}t + v\sin\theta)$

$t = 0$ or $t = \frac{-2v\sin\theta}{g}$

$\therefore$ range $= \frac{2\cos\theta\sin\theta v^2}{g}$

$\therefore$ c $\frac{v^2\sin^2\theta}{2g} = \frac{2}{3}\frac{\cos\theta\sin\theta v^2}{g}$

$3\sin^2\theta = 4\sin\theta\cos\theta$

$0 = 4\sin\theta\cos\theta - 3\sin^2\theta$

(4) $0 = \sin\theta(4\cos\theta - 3\sin\theta)$

$\sin\theta = 0$ or $4\cos\theta = 3\sin\theta$

$\theta = 0°, 180°, 360°$ $\quad \tan\theta = 4/3$

$\theta = 53°8$

$\therefore$ angle of projection is $53°8$

(b) (i) $\ddot{x} = 0$ $\quad \ddot{y} = -g$

$\dot{x} = v\cos\theta$ $\quad \dot{y} = v\sin\theta - gt$

$x = vt\cos\theta$ $\quad y = vt\sin\theta - \frac{gt^2}{2}$

when $t = 2$, $x = 20$, $y = 10$

$20 = 2v\cos\theta$

$10 = v\cos\theta$ ... ①

$10 = 2v\sin\theta - 5(4)$

$15 = v\sin\theta$ ... ② (3)

② ÷ ① $\tan\theta = 1.5$

$\theta = 56°19'$

(ii) ①^2 + ②^2 $\quad 100 = v^2\cos^2\theta$

$225 = v^2\sin^2\theta$

$\therefore v^2 = 325$ (2)

$v = 18.0$ m/s

Question 7

(a) $\text{Prob(winning)} = \dfrac{{}^{6}C_{6}}{{}^{45}C_{6}}$ (2)

$= \dfrac{1}{8145060}$

(b) $\text{Prob.} = \dfrac{2\int_{2}^{3} \frac{1}{x}\,dx}{2\int_{1}^{2} \frac{1}{x}\,dx}$

$= \dfrac{2(\ln 3 - \ln 2)}{2(\ln 4 - \ln 1)}$

$= \dfrac{2\ln(3/2)}{2\ln 4}$

$= \dfrac{0.81093}{2.772589}$ (3)

$= 0.29248125$

$= 29.2\%$

(c)(i) arrangements $= \dfrac{11!}{2!2!2!}$ (2)

$= 4989600$

(ii) $\text{Prob.} = \dfrac{x}{4989600}$

$x \Rightarrow$ ⊗○○○⊗○○○○○

6 ways of placing "group"

$\frac{9!}{2!2!}$ ways of arranging rest.

$x = 6 \times \dfrac{9!}{2!2!} = 544320$ (1)

$\therefore \text{Prob.} = \dfrac{544320}{4989600}$

$= \dfrac{6}{55}$ (1)