Question 1:

(a) The complex numbers $z_1 = \dfrac{a}{1+i}$ and $z_2 = \dfrac{b}{1+2i}$ where a and b are real, satisfy the condition $z_1 + z_2 = 1$. Find the value of a and b. **3**

(b) The complex number z has modulus r and argument θ where $0 \le \theta \le \pi$. Find in terms of r and θ the modulus and arguments of

(i) z^2 **1**

(ii) $\dfrac{1}{z}$ **1**

(iii) iz **1**

(c) (i) Sketch (without using calculus) the curve $y = \dfrac{x^2 + 2x - 3}{x-2}$ clearly showing its intercepts with the coordinate axes and the position of all its asymptotes. **5**

(ii) Find the area bounded by the curve $y = \dfrac{x^2 + 2x - 3}{x-2}$ and the x-axis. **4**

Question 2: (START A NEW PAGE)

(a) Evaluate:

(i) $\displaystyle\int_0^{\frac{\pi}{6}} \cos\theta \sin^3\theta \; d\theta$. **2**

(ii) $\displaystyle\int_0^3 \frac{\sqrt{x}}{1+x} \; dx$. $\left(\text{Let } u^2 = x\right)$. **3**

(iii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{5 + 3\cos\theta} \; d\theta$ **4**

(b) Given that $I_n = \displaystyle\int_0^{\frac{\pi}{2}} x^n \sin x \; dx$.

(i) Prove that $I_n + n(n-1)I_{n-2} = n\left(\dfrac{\pi}{2}\right)^{n-1}$. **4**

(ii) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} x^4 \sin x \; dx$. **2**

Question 3: (START A NEW PAGE)

(a) Sketch the ellipse $9x^2 + 25y^2 = 225$ clearly showing: **4**

(i) the coordinates of the intercepts with the x and y-axes,

(ii) the coordinates of the foci,

(iii) the equation of the directrices.

(b) Prove that the curves $x^2 - y^2 = c^2$ and $xy = c^2$ meet at right angles. **4**

(c) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the ellipse meets the x-axis at the points A and A'.

(i) Prove that the tangent at P has the equation $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$. **3**

(ii) The tangent at P meets the tangents from A and A' at points Q and Q' respectively. Find the coordinates of Q and Q'. **2**

(iii) Prove that the product $AQ \times A'Q'$ is independent of the position of P. **2**

Question 4: (START A NEW PAGE)

(a) Prove that $\dfrac{d}{dx}\left[\sqrt{bx - x^2} + \dfrac{b}{2}\cos^{-1}\left(\dfrac{2x - b}{b}\right)\right] = -\sqrt{\dfrac{x}{b - x}}$ for $x \geq 0$. **3**

(b) A particle of mass m is attracted towards the origin by a force of magnitude $\dfrac{\mu m}{x^2}$ for $x \neq 0$, where the distance from the origin is x and μ is a positive constant.

(i) If the particle starts from rest at a distance b to the right of the origin, show that its velocity v is given by $v^2 = 2\mu\left(\dfrac{b - x}{bx}\right)$. **3**

(ii) Find the time required for the particle to reach a point halfway towards the origin. **4**

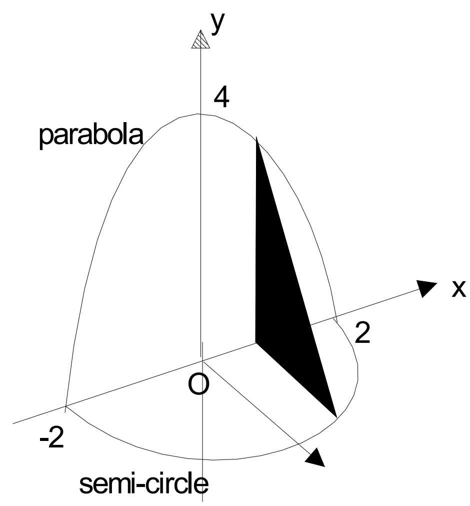
(c) Using the Principle of mathematical induction, prove that $(x+1)^n - nx - 1$ is divisible by x^2 for all integer $n \geq 2$. **5**

Question 5: (START A NEW PAGE)

(a) (i) Using the substitution $x = 2\sin\theta$, prove that $\int_0^2 \left(4-x^2\right)^{\frac{3}{2}} dx = 16\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta$ **2**

(ii) A solid (see diagram) sits on a semi-circular base of radius 2 units. Vertical cross-sections perpendicular to the diameter of the semi-circle are right-angled triangles with their heights being bounded by the parabola $y = 4 - x^2$. By slicing the solid perpendicular to the x-axis, show that the volume $\left(V \ unit^3\right)$ of the solid formed is given by **2**

$$V = \int_0^2 \left(4-x^2\right)^{\frac{3}{2}} \, dx$$



(iii) Find the volume of the solid. **5**

(b) A tourist is walking along a straight road. At one point he observes a vertical tower standing on a large flat plain. The tower is on a bearing $053°$ with an angle of elevation of $21°$. After walking 230 metres, the tower is on a bearing $342°$ with an angle of elevation of $26°$.

(i) Draw a neat diagram showing the above information. **1**

(ii) Find the height of the tower correct to the nearest metre. **5**

Question 6: (START A NEW PAGE)

(a) The tangent to the hyperbola $xy = c^2$ at the point $T\left(ct, \frac{c}{t}\right)$ meets the x and y axes at F and G respectively and the normal at T meets the line $y = x$ at H.

(i) Show that the tangent at T is $x + t^2 y = 2ct$. **3**

(ii) Show that the normal at T is $t^3 x - ty = c(t^4 - 1)$. **2**

(iii) Prove that $FH \perp HG$. **6**

(b) The area bounded by the curve $y = \frac{\ln x}{\sqrt{x}}$ and the x-axis for $1 \le x \le e$ is rotated through one revolution about the y-axis. Using the method of cylindrical shells, find the volume of the solid formed. **4**

Question 7: (START A NEW PAGE)

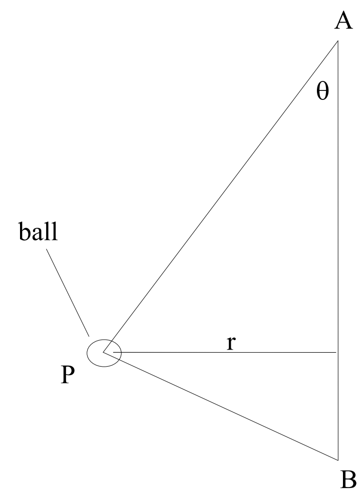
(a) In a state swimming championships, 12 swimmers (including the Jones twins) are chosen to represent their club and are divided into three teams of four swimmers to form 3 relay teams. Find the number of ways this can be done:

(i) with no restrictions. **2**

(ii) if the Jones twins (Angela and Bethany) are not to be in the same relay team. **3**

(b) The ends of a light string are fixed at 2 points A and B with B directly below A, as shown in the diagram. The string passes through a small ball of mass m which is then fastened to the string at point P. The angle PAB is θ and the distance from P to AB is r.

Suppose now that the ball revolves in a horizontal circle about the vertical through AB with constant angular velocity ω and while this happens both sections (AP and BP) of the string are taut and the angle APB is a right angle.



(i) Draw a diagram showing the forces acting on the ball. **2**

(ii) Show that the tensions T_1 and T_2 in the sections of the string AP and BP respectively are $T_1 = m\left(r\omega^2 \sin\theta + g\cos\theta\right)$ and $T_2 = m\left(r\omega^2 \cos\theta - g\sin\theta\right)$. **4**

(iii) Given that AB = 100cm and AP = 80cm, show that $\omega^2 > \dfrac{25g}{16}$. **2**

(iv) Suppose that the ball is free to slide on the string. Show that the condition for the ball to remain at point P on the string is $\omega^2 = \dfrac{175g}{12}$. **2**

Question 8: (START A NEW PAGE)

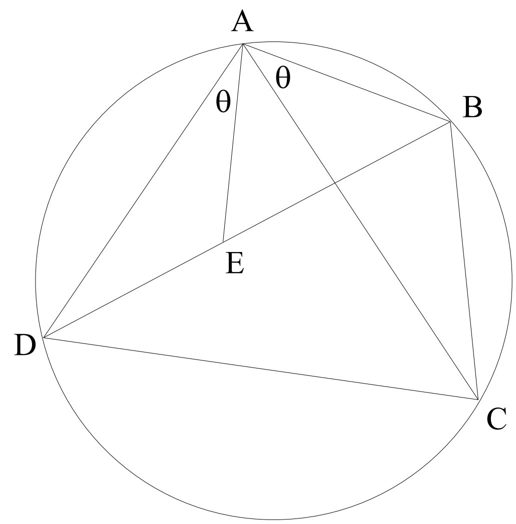
(a) (i) If $t = \tan x$ prove that $\tan 4x = \dfrac{4t\left(1-t^2\right)}{t^4 - 6t^2 + 1}$. **2**

(ii) If $\tan x \tan 4x = 1$ deduce that $5t^4 - 10t^2 + 1 = 0$. **1**

(iii) Prove that $x = 18°$ and $x = 54°$ satisfy the equation $\tan x \tan 4x = 1$. **2**

(iv) Deduce that $\tan 54° = \sqrt{\dfrac{5+2\sqrt{5}}{5}}$. **3**

(b) ABCD is a cyclic quadrilateral and E is on BD such that $\angle DAE = \angle BAC$.



(i) Copy the diagram onto your answer sheet and prove that ΔABE and ΔADC are similar. **2**

(ii) Prove that $AB \times CD = AC \times BE$. **1**

(iii) Hence by proving that another pair of triangles are similar, deduce that $AB \times CD + AD \times BC = AC \times BD$. **4**

THE END

(b) Given that x, y and z are distinct positive numbers, prove that

(i) $x + y > 2\sqrt{xy}$. **1**

(ii) $(x + y)(y + z)(z + x) > 8xyz$. **2**

(iii) $\dfrac{x + y}{z} + \dfrac{y + z}{x} + \dfrac{z + x}{y} > 6$. **3**

(a) (i) Prove that $2\sin\left(\dfrac{A + B}{2}\right)\cos\left(\dfrac{A - B}{2}\right) = \sin A + \sin B$. **3**

(ii) Write down a result similar to (i) for $\cos A + \cos B$. **1**

(iii) Prove that $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\left(\dfrac{A + B}{2}\right)$. **2**

(iv) If A, B and C are the angles of a triangle, prove that $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \cot\left(\dfrac{C}{2}\right)$. **2**