Year 12 Term 2 Assessment 2007 - EXTENSION II - SOLUTIONS

Question 1

Marks

(a) Sketch the hyperbola $xy = 16$ clearly showing the coordinates of the foci and the equations of the directrices. 4

$xy = \frac{1}{2}a^2$

focii (a, a) and $(-a, -a)$

directrices $x + y = \pm a$

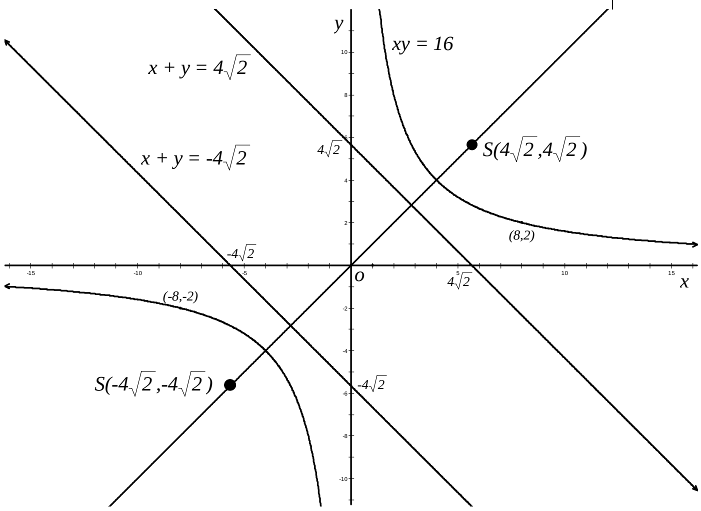
$xy = 16 = \frac{1}{2}(32)$

$xy = \frac{1}{2}\left(4\sqrt{2}\right)^2$

$\therefore a = 4\sqrt{2}$

focii $S\left(4\sqrt{2}, 4\sqrt{2}\right)$ and $S'\left(-4\sqrt{2}, -4\sqrt{2}\right)$

directrices $x + y = \pm 4\sqrt{2}$



(b) The region bounded by the curve $y = (4 - x)\sqrt{x}$ and the $x-$axis is rotated one revolution about the $y-$axis. By considering cylindrical shells with their generators parallel to the $y-$axis, find the volume of the solid formed. 4

$$V = 2\pi\int_0^4 x(4-x)\sqrt{x}\,dx$$

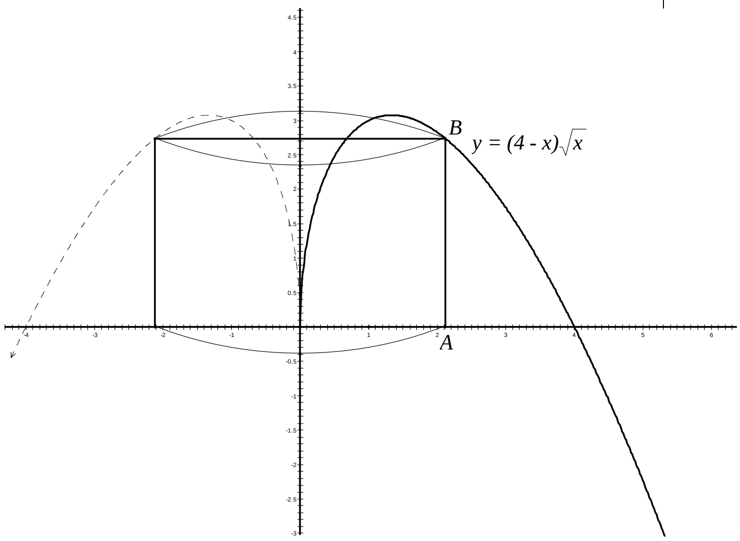
$$= 2\pi\int_0^4 \left(4x\sqrt{x} - x^2\sqrt{x}\right)dx$$

$$= 2\pi\left[\frac{8}{5}x^{2.5} - \frac{2}{7}x^{3.5}\right]_0^4$$

$$= 2\pi\left\{\left(\frac{8}{5}(4)^{2.5} - \frac{2}{7}(4)^{3.5}\right) - 0\right\}$$

$$= \frac{1024\pi}{35}$$

Volume $= \frac{1024\pi}{35}u^3$



(c) An object is moving on a horizontal plane and at position x metres from the origin it has acceleration ($\ddot{x}$ m/s^2) given by $\ddot{x} = 0.01\left(x + \frac{64}{x^3}\right)$. Initially the object is 4 metres to the left of the origin and moving to the left at a speed of $\frac{\sqrt{3}}{5}$ m/s .

(i) If the velocity of the object is v m/s, show that $v^2 = 0.01\left(\frac{x^4 - 64}{x^2}\right)$. 2

$$\ddot{x} = 0.01\left(x + \frac{64}{x^3}\right)$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 0.01\left(x + \frac{64}{x^3}\right)$$

$$\frac{1}{2}v^2 = 0.01\left(\frac{1}{2}x^2 - \frac{64}{2x^2}\right) + c$$

$$v^2 = 0.01\left(x^2 - \frac{64}{x^2}\right) + 2c$$

$$x = -4,\ v = -\frac{\sqrt{3}}{5}$$

$$\frac{3}{25} = 0.01\left(16 - \frac{64}{16}\right) + 2c$$

$$\frac{3}{25} = \frac{1}{100}(12) + 2c$$

$$c = 0$$

$$v^2 = 0.01\left(x^2 - \frac{64}{x^2}\right)$$

$$v^2 = 0.01\left(\frac{x^4 - 64}{x^2}\right)$$

(ii) Explain why the velocity of the object at a position x metres from the origin is given by $v = \frac{\sqrt{x^4 - 64}}{10x}$. 1

Object will be stationary when $v = 0$

$\therefore\ x^4 - 64 = 0$

$x = \pm 2\sqrt{2}$

Now, since object starts at $x = -4$ and moves left it will never come to rest (at rest at $x = -2\sqrt{2}$), therefore velocity is always negative so when we take the square root we need to choose an expression that is always negative and since $x < 0$ we choose the positive square roots of $x^4 - 64$ and x^2

Or: For $x < 0$, x and $\frac{1}{x^3}$ are both <0 therefore the acceleration is always <0 for $x < 0$ and since the initial direction of motion is to the left from position $x = -4$ then the object will always move towards the left, therefore the velocity will always be <0. Therefore choose the positive square roots of $x^4 - 64$ and x^2

(iii) Find, correct to the nearest second, the time to reach a position 50 metres to the left of the origin. 4

$$v = \frac{\sqrt{x^4 - 64}}{10x}$$

$$\frac{dx}{dt} = \frac{\sqrt{x^4 - 64}}{10x}$$

$$\frac{dt}{dx} = \frac{10x}{\sqrt{x^4 - 64}}$$

$$\int_0^T dt = \int_4^{50} \frac{10x}{\sqrt{x^4 - 64}}\, dx$$

let $u = x^2$

$x = 4, u = 16$ and $x = 50, u = 2500$

$du = 2xdx$

$$[t]_0^T = \int_{16}^{2500} \frac{5du}{\sqrt{u^2 - 64}}$$

$$\begin{aligned} T &= 5\left[\ln\left(u + \sqrt{u^2 - 64}\right)\right]_{16}^{2500} \\ &= 5\left\{\ln\left(2500 + \sqrt{2500^2 - 64}\right) - \ln\left(16 + \sqrt{16^2 - 64}\right)\right\} \\ &\approx 25.604 \end{aligned}$$

time $= 26$ sec (to nearest second)

Question 2 (START A NEW PAGE) (15 MARKS)

Marks

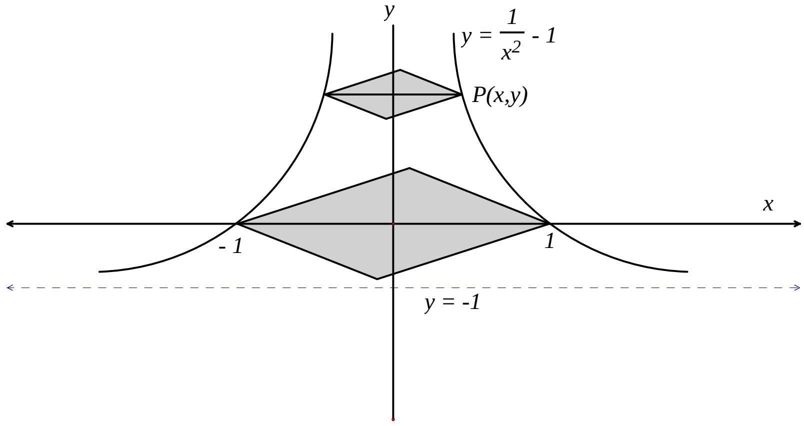
(a) (i) A solid is formed so that every cross-section perpendicular to the $y-$axis is a square with the opposite ends of one of its diagonals on each branch of the curve $y = \frac{1}{x^2} - 1$. Show that a cross-section of the solid, y units above the $x-$axis has area $\frac{2}{y+1}$ units^2. 3

$$\text{Area of square} = \frac{1}{2}(\text{diagonal})^2$$
$$= \frac{1}{2}(2x)^2$$
$$= 2x^2$$

but $y = \frac{1}{x^2} - 1$

$$x^2 = \frac{1}{y+1}$$

$$\text{Area} = \frac{2}{y+1}$$



(ii) Find the volume of the solid in (i) if the $x-$axis lies along a diameter of its base and the solid is 3 units high. 3

$$V = \int_0^3 \frac{2}{y+1}\, dy$$
$$= 2\left[\ln(y+1)\right]_0^3$$
$$= 2\ln 4 - 2\ln 1$$
$$= 2\ln 4$$

$$\text{volume} = 2\ln 4\, u^3$$

(b) (i) Prove that the equation of the tangent to the hyperbola $xy = 36$ at the point $T\left(6t, \frac{6}{t}\right)$ is $x + t^2 y - 12t = 0$. 2

$y = \frac{36}{x}$

$\frac{dy}{dx} = -\frac{36}{x^2}$

when $x = 6t, \frac{dy}{dx} = -\frac{36}{36t^2}$

$= -\frac{1}{t^2}$

tangent is $y - \frac{6}{t} = -\frac{1}{t^2}(x - 6t)$

$t^2 y - 6t = -x + 6t$

$x + t^2 y - 12t = 0$

(ii) If the tangents to the hyperbola $xy = 36$ at the points $P\left(6p, \frac{6}{p}\right)$ and $Q\left(6q, \frac{6}{q}\right)$ meet at the point $R(x_o, y_o)$, prove that $pq = \frac{x_o}{y_o}$ and $p + q = \frac{12}{y_o}$. 3

at $P: x + p^2 y - 12p = 0$............(1)

at $Q: x + q^2 y - 12q = 0$.............(2)

$(1) - (2) \Rightarrow (p^2 - q^2)y = 12(p - q)$

$y = \frac{12}{p+q}$

sub. into $(1) \Rightarrow x = 12p - p^2\left(\frac{12}{p+q}\right)$

$x = \frac{12p(p+q) - 12p^2}{p+q}$

$x = \frac{12pq}{p+q}$

$x_o = \frac{12pq}{p+q} \quad y_o = \frac{12}{p+q}$

$\frac{x_o}{y_o} = \left(\frac{12pq}{p+q}\right) \div \left(\frac{12}{p+q}\right)$

$\therefore \frac{x_o}{y_o} = pq$ and $p + q = \frac{12}{y_o}$

(iii) Find an expression for the length of the chord PQ in terms of p and q. 1

$$PQ = \sqrt{(6p-6q)^2 + \left(\frac{6}{p} - \frac{6}{q}\right)^2}$$

(iv) If the length of the chord PQ is always 12 units, prove that the locus of point R lies on the curve with equation $(x^2+y^2)(36-xy) = x^2y^2$. 3

$$\sqrt{(6p-6q)^2 + \left(\frac{6}{p} - \frac{6}{q}\right)^2} = 12$$

$$(6p-6q)^2 + \left(\frac{6}{p} - \frac{6}{q}\right)^2 = 144$$

$$(p-q)^2 + \left(\frac{1}{p} - \frac{1}{q}\right)^2 = 4$$

$$(p-q)^2 + \left(\frac{q-p}{pq}\right)^2 = 4$$

$$\left[(p+q)^2 - 4pq\right] + \left[\frac{(p+q)^2 - 4pq}{p^2q^2}\right] = 4$$

$$\left[\left(\frac{12}{y}\right)^2 - 4\left(\frac{x}{y}\right)\right] + \left[\frac{\left(\frac{12}{y}\right)^2 - 4\left(\frac{x}{y}\right)}{\left(\frac{x}{y}\right)^2}\right] = 4$$

$$\left(\frac{144-4xy}{y^2}\right) + \left(\frac{144-4xy}{y^2}\right)\left(\frac{y^2}{x^2}\right) = 4$$

$$\left(\frac{36-xy}{y^2}\right) + \left(\frac{36-xy}{x^2}\right) = 1$$

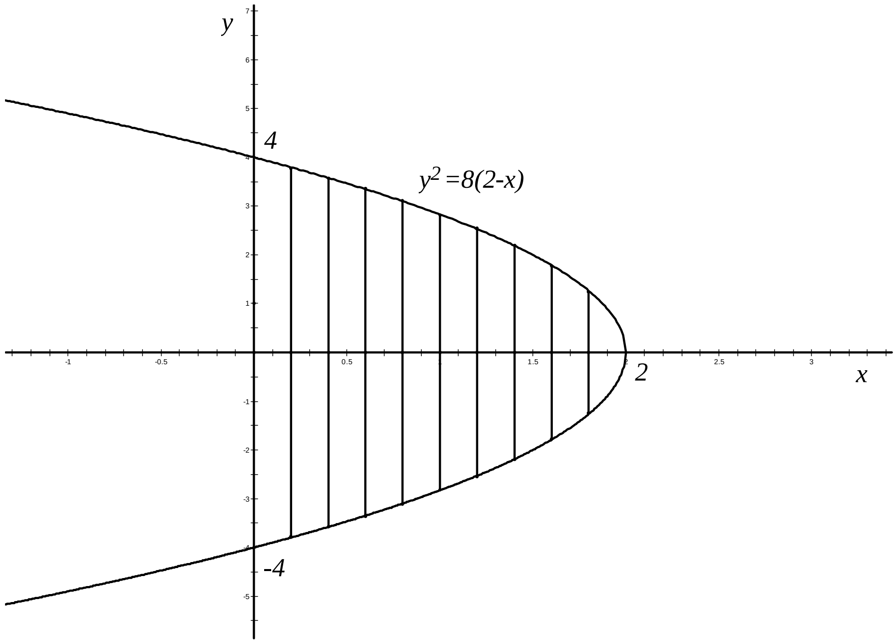
$$(36-xy)x^2 + (36-xy)y^2 = x^2y^2$$

$$(36-xy)(x^2+y^2) = x^2y^2$$

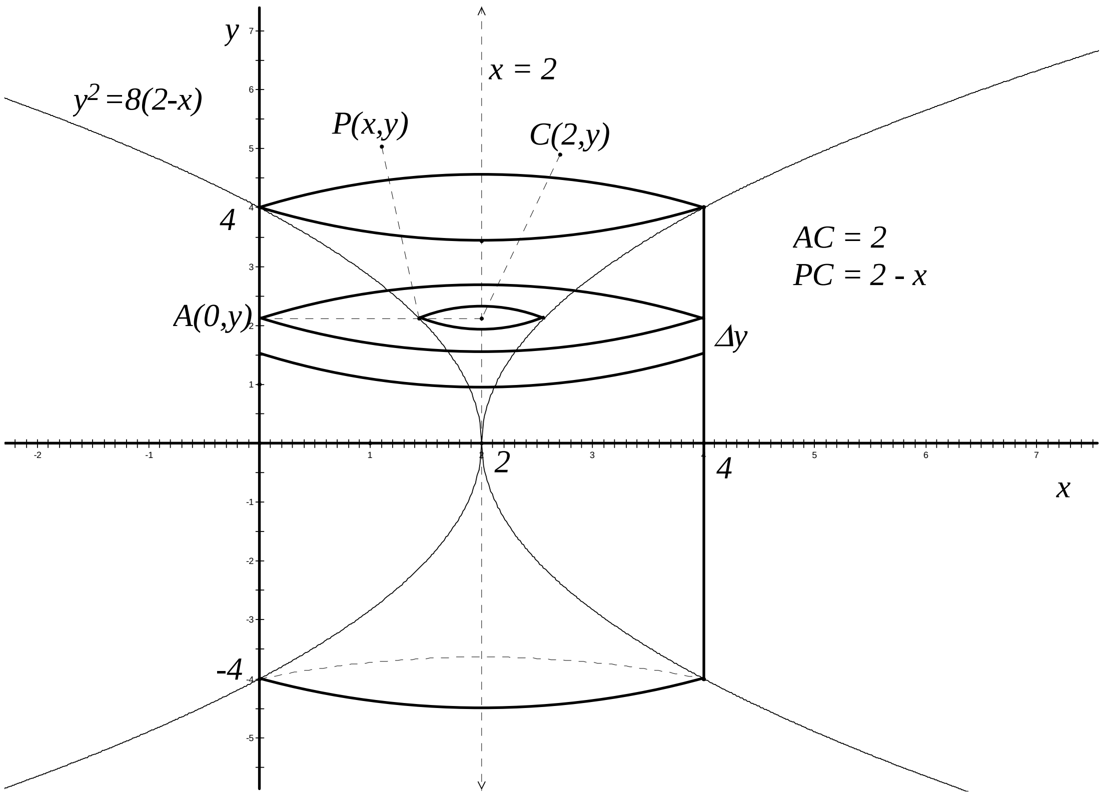
Question 3 (START A NEW PAGE) (15 MARKS)

Marks

(a) (i) Shade the region bounded by the parabola $y^2 = 8(2-x)$ and the y-axis. 2



(ii) The region in (i) is rotated one revolution about the line $x = 2$. Show that the area A units^2 of a cross-section by a plane parallel to the $x-$axis and distance y units from it, is given by $A = \frac{\pi}{64}\left(256 - y^4\right)$, hence find the volume of this solid. 5



Area of cross-section

$$A = \pi(2)^2 - \pi(2-x)^2$$
$$= 4\pi - \pi\left(4 - 4x + x^2\right)$$
$$= \pi\left(4 - 4 + 4x - x^2\right)$$
$$= \pi\left(4x - x^2\right)$$
$$= \pi x(4 - x)$$
$$= \pi\left(\frac{16-y^2}{8}\right)\left(4 - \frac{16-y^2}{8}\right) \qquad \text{since } x = \frac{16-y^2}{8}$$
$$= \pi\left(\frac{16-y^2}{8}\right)\left(\frac{16+y^2}{8}\right)$$
$$A = \frac{\pi}{64}\left(256 - y^4\right)$$

$$V = \frac{\pi}{64}\int_{-4}^{4} \left(256 - y^4\right) dy$$
$$= \frac{\pi}{32}\int_{0}^{4}\left(256 - y^4\right) dy$$
$$= \frac{\pi}{32}\left[256y - \frac{1}{5}y^5\right]_0^4$$
$$= \frac{\pi}{32}\left\{\left(256 \times 4 - \frac{4^5}{5}\right) - 0\right\}$$
$$= \frac{128\pi}{5}$$

volume $= \dfrac{128\pi}{5} u^3$

(b) A particle of mass 2kg is found to experience a resistive force, in newtons, of $\frac{1}{10}$ of its velocity in metres per second, when moving through air. The particle is projected vertically upwards with velocity 60 m/s from a point *O* on the ground. Assume the value of *g* is 10m/s^2

(i) Draw a diagram showing the forces acting on the particle, and show that the equation of motion of the particle is given by $\ddot{x} = -\frac{1}{20}(200 + v)$. 2

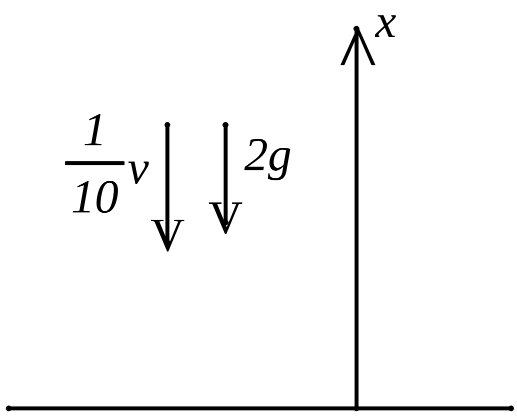
Resultant force $\uparrow = -2g - \frac{1}{10}v$

$$2\ddot{x} = -2g - \frac{1}{10}v$$

$$\ddot{x} = -g - \frac{1}{20}v$$

$$= -\frac{1}{20}(20g + v)$$

$$\ddot{x} = -\frac{1}{20}(200 + v)$$



(ii) Find the time for the particle to reach the highest point above the ground. 3

$$\frac{dv}{dt} = -\frac{1}{20}(200 + v)$$

$$\frac{dt}{dv} = \frac{-20}{200 + v}$$

$$\int_0^T dt = \int_{60}^0 \frac{-20}{200 + v}\,dv$$

$$T = -20\left[\ln(200 + v)\right]_{60}^{0}$$

$$= -20(\ln 200 - \ln 260)$$

$$= 20\ln 1.3$$

$$\text{time} = 20\ln(1.3)\,\text{seconds}$$

(iii) Find the maximum height of the particle above the ground. 3

$$\ddot{x} = -\frac{1}{20}(200 + v)$$

$$v\frac{dv}{dx} = \frac{200 + v}{-20}$$

$$\frac{dv}{dx} = \frac{200 + v}{-20v}$$

$$\frac{dx}{dv} = \frac{-20v}{200 + v}$$

$$= \frac{-20(200 + v - 200)}{200 + v}$$

$$= -20\left(1 - \frac{200}{200 + v}\right)$$

$$\int_0^H dx = -20\int_{60}^0 \left(1 - \frac{200}{200 + v}\right) dv$$

$$H = -20\left[v - 200\ln(200 + v)\right]_{60}^{0}$$

$$= -20\{(0 - 200\ln 200) - (60 - 200\ln 260)\}$$

$$= 20\{60 - 200\ln(1.3)\}$$

$$\text{height} = 400\{3 - 10\ln(1.3)\}\text{ metres}$$

Question 4 (START A NEW PAGE) (15 MARKS)

Marks

(a) When an aircraft touches down, two different retarding forces combine to bring it to rest. For a particular aircraft with mass M kg and speed v m/s there is a constant frictional force of $\frac{1}{5}M$ newtons and a force of $\frac{1}{120}Mv^2$ newtons due to the reverse thrust of the engines. The reverse thrust of the engines does not take effect until 20 seconds after touch down.
By considering the forces it can be shown that the equations of motion for $0 < t < 20$ is given by $\ddot{x} = -\frac{1}{5}$ for $0 < t < 20$.

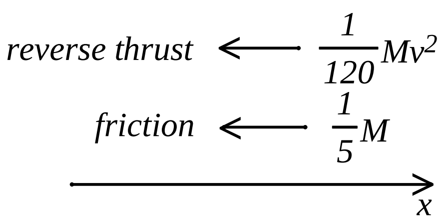
(i) Explain why the equation of motion for $t > 20$, and until the aircraft stops, is given by $\ddot{x} = -\frac{1}{120}\left(24 + v^2\right)$. 1

Resultant force $\rightarrow = -\frac{1}{5}M - \frac{1}{120}Mv^2$

$M\ddot{x} = -\frac{1}{5}M - \frac{1}{120}Mv^2$

$\ddot{x} = -\frac{1}{5} - \frac{1}{120}v^2$

$\ddot{x} = -\frac{1}{120}\left(24 + v^2\right)$



(ii) If the aircraft's speed at touch down is 50 m/s, show that $v = 46$ and $x = 960$ at the instant that the engines are reversed. 4

For $t \le 20$

$\ddot{x} = -\frac{1}{5}$

$\frac{dv}{dt} = -\frac{1}{5}$

$\int_{50}^{v} dv = -\frac{1}{5}\int_{0}^{t} dt$

$[v]_{50}^{v} = -\frac{1}{5}[t]_{0}^{t}$

$v - 50 = -\frac{1}{5}t$

$v = 50 - 0.2t$

when $t = 20, v = 50 - 0.2(20)$

$= 46$

$$\frac{dx}{dt} = 50 - 0.2t$$

$$\int_0^x dx = \int_0^{20} (50 - 0.2t)\, dt$$

$$x = \left[50t - 0.1t^2\right]_0^{20}$$

$$= (1000 - 0.1 \times 400) - 0$$

$$= 960$$

Velocity = 46 m/s and distance = 960 m

(iii) Show that when $t > 20$, $x = 960 + 60\{\ln(2140) - \ln(24 + v^2)\}$. 3

$$\ddot{x} = -\frac{1}{120}(24 + v^2)$$

$$v\frac{dv}{dx} = -\frac{1}{120}(24 + v^2)$$

$$\frac{dv}{dx} = -\frac{(24 + v^2)}{120v}$$

$$\frac{dx}{dv} = \frac{-120v}{24 + v^2}$$

$$\int_{960}^x dx = -120\int_{46}^0 \frac{v}{24 + v^2}\, dv$$

$$[x]_{960}^x = -60\left[\ln(24 + v^2)\right]_{46}^v$$

$$x - 960 = -60\{\ln(24 + v^2) - \ln(24 + 46^2)\}$$

$$x = 960 - 60\{\ln(24 + v^2) - \ln 2140\}$$

$$x = 960 + 60\ln 2140 - 60\ln(24 + v^2)$$

(iv) Calculate how far from the touch down point the jet comes to rest. Give your answer correct to the nearest 10 metres. 1

when $v = 0$

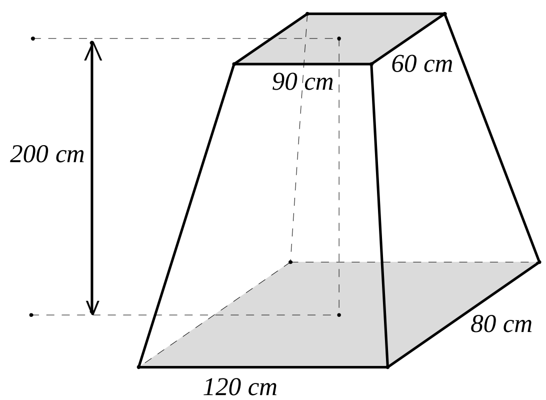
$$x = 960 + 60\ln 2140 - 60\ln(24)$$

$$\approx 1229.4$$

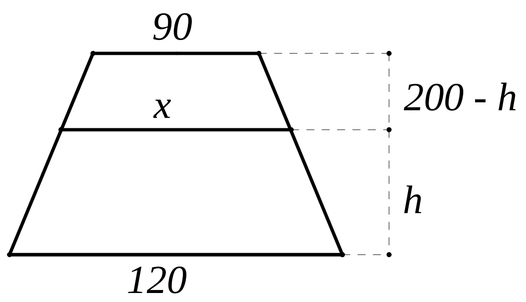
distance $= 1230$ (to nearest metre)

Marks

(b) The base on which a statue stands is made in the form of a pyramid with two rectangular faces (see diagram). The bottom of the base is 120cm by 80cm and the top is 90cm by 60cm. The distance between the two rectangular faces is 200 cm and the centre of the top face is directly above the centre of the bottom face.



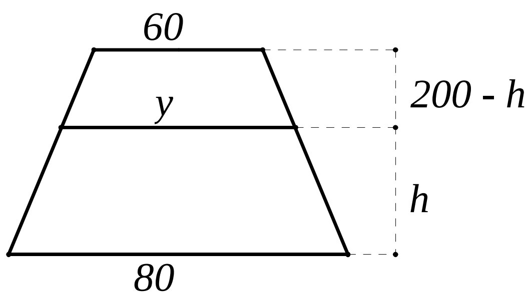
(i) Show that the area of a cross-section h cm above the base equals $\dfrac{3(800-h)^2}{200}$ cm^2. 3



Comparing areas

$$\frac{1}{2}(h)(120+90)=\frac{1}{2}(h)(120+x)+\frac{1}{2}(200-h)(x+90)$$

$$x=\frac{3(800-h)}{20}$$



$$\frac{1}{2}(h)(80+60)=\frac{1}{2}(h)(80+y)+\frac{1}{2}(200-h)(y+60)$$

$$y=\frac{(800-h)}{10}$$

$$\text{Area of cross-section} = xy\ cm^2$$
$$=\frac{3(800-h)}{20}\cdot\frac{(800-h)}{10}\ cm^2$$
$$=\frac{3(800-h)^2}{200}\ cm^2$$

(ii) Hence find the volume of the base of the statue. 3

$$V=\int_0^{200}\frac{3}{200}(800-h)^2\,dh$$
$$=-\frac{1}{200}\left[(800-h)^3\right]_0^{200}$$
$$=-\frac{1}{200}\left\{600^3-800^3\right\}$$
$$=1.48\times10^6$$
$$\text{Volume}=1.48\times10^6\ cm^3$$

THIS IS THE END OF THE EXAMINATION PAPER