Independent Trial HSC 2009 Mathematics Extension 2 Marking Guidelines

Question 1

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• rearranges integrand into appropriate sum of terms	1
• finds primitive	1

Answer

$$\int \frac{(x+1)^2}{x}\,dx = \int\left(x+2+\frac{1}{x}\right)dx = \tfrac{1}{2}x^2+2x+\ln x+c$$

b. Outcomes assessed : H8, PE3, E8

Marking Guidelines

Criteria	Marks
i • writes and solves a pair of simultaneous equations for A and C	1
• writes and solves a pair of simultaneous equations for B and D	1
ii • finds and evaluates definite integral for one term	1
• finds and evaluates definite integral for second term	1

Answer

i.

$$\frac{x^3+2x^2+4x+2}{(x^2+1)(x^2+4)} \equiv \frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+4}$$

$$x^3+2x^2+4x+2 \equiv (Ax+B)(x^2+4)+(Cx+D)(x^2+1)$$

equating coefficients of x^3 : $1=A+C$ (1) $\quad (2)-(1)\Rightarrow 3=3A$

equating coefficients of x : $4=4A+C$ (2) $\quad \therefore A=1,\ C=0$

equating coefficients of x^2: $2=B+D$ (3) $\quad (4)-(3)\Rightarrow 3B=0$

putting $x=0$: $2=4B+D$ (4) $\quad \therefore B=0,\ D=2$

ii.

$$\int_0^2 \frac{x^3+2x^2+4x+2}{(x^2+1)(x^2+4)}\,dx = \int_0^2\left(\frac{x}{(x^2+1)}+\frac{2}{(x^2+4)}\right)dx$$

$$=\left[\tfrac{1}{2}\ln(x^2+1)+\tan^{-1}\tfrac{x}{2}\right]_0^2$$

$$=\tfrac{1}{2}\ln 5+\tan^{-1}1$$

$$=\tfrac{1}{2}\ln 5+\tfrac{\pi}{4}$$

c. Outcomes assessed : H5, HE6

Marking Guidelines

Criteria	Marks
• expresses dx in terms of dt and substitutes expressions for $\cos x$ and $\sin x$ in terms of t	1
• simplifies integrand as a function of t and finds primitive in terms of t	1
• writes primitive as a function of x	1

Answer

$$t=\tan\tfrac{x}{2}$$
$$dt=\tfrac{1}{2}\sec^2\tfrac{x}{2}\,dx$$
$$dx=\frac{2}{1+t^2}\,dt$$

$$5+4\cos x+3\sin x$$
$$=\frac{5(1+t^2)+4(1-t^2)+6t}{1+t^2}$$
$$=\frac{t^2+6t+9}{1+t^2}$$
$$=\frac{(t+3)^2}{1+t^2}$$

$$\int\frac{1}{5+4\cos x+3\sin x}\,dx$$
$$=\int\frac{1+t^2}{(t+3)^2}\cdot\frac{2}{1+t^2}\,dt$$
$$=-\frac{2}{t+3}+c$$
$$=-\frac{2}{3+\tan\frac{x}{2}}+c$$

d. Outcomes assessed : H8, HE6

Marking Guidelines

Criteria	Marks
• expresses du in terms of dx and converts x limits to u limits	1
• simplifies integrand as a function of u	1
• uses table of integrals to write primitive then evaluates by substitution	1

Answer

$$u=\sin x \qquad x=0\Rightarrow u=0$$
$$du=\cos x\,dx \qquad x=\tfrac{\pi}{2}\Rightarrow u=1$$

$$\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx=\int_0^1\frac{1}{\sqrt{1+u^2}}\,du$$
$$=\left[\ln\left(u+\sqrt{u^2+1}\right)\right]_0^1$$
$$=\ln(1+\sqrt{2})$$

e. Outcomes assessed : H8, HE6

Marking Guidelines

Criteria	Marks
• expresses du in terms of dx and converts x limits to u limits	1
• simplifies and rearranges integrand as function of u	1
• evaluates definite integral	1

Answer

$$u=-x$$
$$du=-dx$$
$$x=-1\Rightarrow u=1$$
$$x=1\Rightarrow u=-1$$

$$I=\int_{-1}^{1}\frac{1}{e^x+1}\,dx$$
$$=\int_{1}^{-1}\frac{1}{e^{-u}+1}\,.-du$$
$$=\int_{-1}^{1}\frac{1}{e^{-u}+1}\,du$$
$$=\int_{-1}^{1}\frac{e^u}{1+e^u}\,du$$

$$\therefore 2I=\int_{-1}^{1}\frac{1}{e^x+1}\,dx+\int_{-1}^{1}\frac{e^x}{1+e^x}\,dx$$
$$=\int_{-1}^{1}\frac{1+e^x}{1+e^x}\,dx$$
$$=\int_{-1}^{1}1\,dx$$
$$\therefore 2I=2$$
$$I=1$$

Question 2

a. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • finds modulus	1
ii • simplifies sum	1
iii • realises denominator to simplify quotient	1

Answer

$z_1 = 3i\,,\quad z_2 = 1+i$

i.
$$|z_1 - z_2| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

ii.
$$z_1 + \bar{z}_2 = 3i + 1 - i = 1 + 2i$$

iii.
$$\frac{z_1}{z_2} = \frac{3i\,(1-i)}{(1+i)(1-i)} = \tfrac{3}{2} + \tfrac{3}{2}\,i$$

b. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • writes z in modulus-argument form	1
• writes square of z in modulus-argument form	1
• writes reciprocal of z in modulus-argument form	1
ii • uses a diagram or subtracts arguments to show points collinear	1

Answer

i. $z = 1 + i\sqrt{3}$

$$z = 2\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right) = 2(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3})$$

$$z^2 = 4(\cos\tfrac{2\pi}{3} + i\sin\tfrac{2\pi}{3})$$

$$\frac{1}{z} = \tfrac{1}{2}\left(\cos(-\tfrac{\pi}{3}) + i\sin(-\tfrac{\pi}{3})\right)$$

ii. $\angle AOB = \arg z^2 - \arg\frac{1}{z} = \pi \qquad \therefore A, O, B$ are collinear

c. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • explains why sides OA, OB are equal	1
• finds size of angle at O and deduces triangle equilateral	1
ii • recognizes $\overrightarrow{AB}$ as rotation of $\overrightarrow{OB}$ and expresses $z_2 - z_1$ as multiple of z_2	1
• expresses $z_2 - z_1$ in modulus-argument form	1

Answer

$z_1 = 2(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5})\,,\quad z_2 = 2(\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15})$

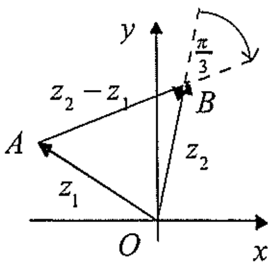
i. $|z_1| = |z_2| \quad \therefore OA = OB$

$\angle AOB = \arg z_1 - \arg z_2 = \frac{\pi}{3}$

$\therefore$ all $\angle$'s of ΔAOB are $\frac{\pi}{3}$

($\angle$ sum is π and $\angle$'s opp. equal sides are equal)

$\therefore \Delta AOB$ is equilateral.



$\overrightarrow{AB}$ represents $z_2 - z_1$.

$\overrightarrow{AB}$ is the rotation of $\overrightarrow{OB}$ clockwise by $\frac{\pi}{3}$.

$$\therefore z_2 - z_1 = z_2(\cos\tfrac{-\pi}{3} + i\sin\tfrac{-\pi}{3}) = 2(\cos\tfrac{2\pi}{15} + i\sin\tfrac{2\pi}{15})$$

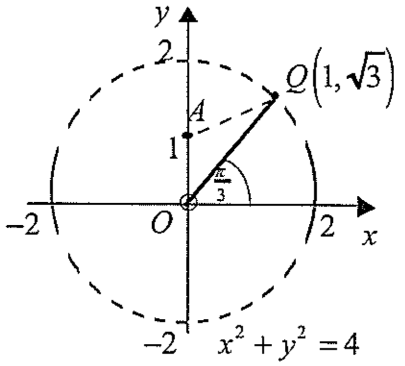
d. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • shows a ray from O making angle $\frac{\pi}{3}$ with positive x-axis, excluding O	1
• restricts ray to an interval inside the circle with centre O and radius 2.	1
ii • finds the lower bound for the required argument as a strict inequality	1
• finds the upper bound for the required argument	1

Answer

i. P represents z such that $\arg z = \frac{\pi}{3}$ and $|z| \le 2$. Locus of P is the interval OQ on the graph below, with O excluded.



ii. Point A represents the complex number i.

Using trigonometry, Q has coordinates $\left(2\cos\frac{\pi}{3}, 2\sin\frac{\pi}{3}\right)$, giving $Q\left(1,\sqrt{3}\right)$ as shown.

Gradient of AQ is $\sqrt{3}-1$.

$\therefore\ -\frac{\pi}{2} < \text{Arg}(z-i) \le \tan^{-1}\left(\sqrt{3}-1\right)$

Question 3

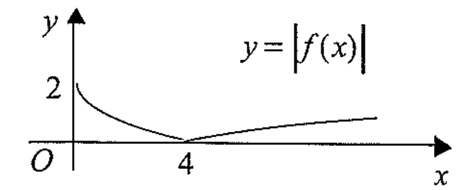
a. Outcomes assessed : E6

Marking Guidelines

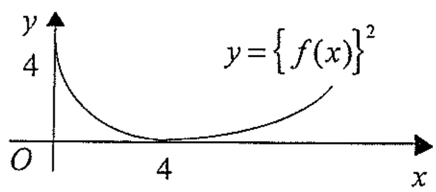
Criteria	Marks
i • sketches curve with correct shape and intercepts	1
ii • sketches curve with correct shape and intercepts	1
iii • shows correct shape and y-intercept for branch to left of vertical asymptote at $x = 4$	1
• shows correct shape and position of branch to right of vertical asymptote	1
iv • shows correct shape, position and behaviour near vertical asymptote at $x = 4$	1
• shows x-intercept	1

Answer

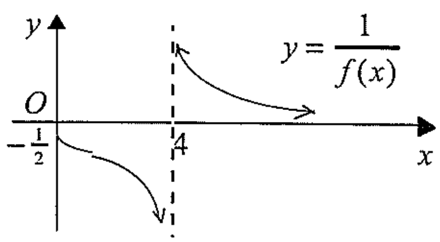
i.



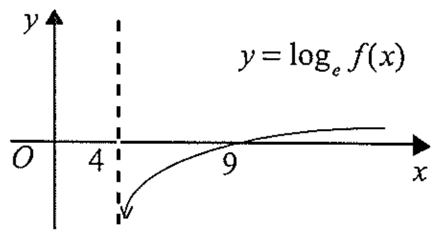
ii.



iii.



iv.



b. Outcomes assessed : E6

Marking Guidelines

Criteria	Marks
• uses differentiation to find the gradient of the tangent at P	1
• uses coordinates of O and P to find gradient of OP and hence writes equation for x_1	1
• solves this equation to find coordinates of P	1

Answer

$$y = \sqrt{x-a}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x-a}}$$

$\therefore$ tangent at P has gradient $\dfrac{1}{2\sqrt{x_1 - a}}$

$$\therefore \frac{1}{2\sqrt{x_1-a}} = \frac{y_1}{x_1}$$

$$x_1 = 2y_1\sqrt{x_1 - a}$$

$$x_1 = 2(x_1 - a)$$

$$x_1 = 2x_1 - 2a$$

Hence P has coordinates $\left(2a\,,\sqrt{a}\right)$

c. Outcomes assessed : H8, PE3

Marking Guidelines

Criteria	Marks
i • writes expression for approximate area using trapezoidal rule	1
• simplifies this expression	1
ii • differentiates given expression	1
• uses fact that integration is the inverse operation to evaluate required definite integral	1
iii • compares total area enclosed by trapezia with area under curve	1
• uses this to write and simplify inequality	1

Answer

i.

$$\int_1^n \ln x\, dx$$

$$\approx \tfrac{1}{2}\left\{\ln 1 + 2\left[\ln 2 + \ln 3 + \ldots + \ln(n-1)\right] + \ln n\right\}$$

$$= \ln 1 + \ln 2 + \ln 3 + \ldots + \ln n - \tfrac{1}{2}(\ln 1 + \ln n)$$

$$= \ln n! \, - \tfrac{1}{2}\ln n$$

ii.

$$\frac{d}{dx}\left(x\ln x - x\right) = 1\,.\ln x + x\,.\frac{1}{x} - 1$$

$$= \ln x$$

$$\left[x\ln x - x\right]_1^n = \int_1^n \ln x\, dx$$

$$\therefore \int_1^n \ln x\, dx = n\ln n - n + 1$$

iii. The total area of the trapezia fitted under the curve is less than the area under the curve.

$$\therefore \ln n! - \tfrac{1}{2}\ln n < n\ln n - n + 1$$

$$\ln n! < (n + \tfrac{1}{2})\ln n - n + 1$$

Question 4

a. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
• writes division transformation of $P(x)$ when divided by $(x-2)(x-3)$ with remainder $(ax+b)$	1
• uses remainder theorem to write simultaneous equations for a, b then finds required remainder	1

Answer

Using the division transformation, $P(x) \equiv (x-2)(x-3)Q(x) + ax + b$ for some polynomial $Q(x)$ and real constants a, b, where $(ax+b)$ is the remainder when $P(x)$ is divided by $(x-2)(x-3)$.

Then $\left.\begin{array}{l} P(2) = 9 \Rightarrow 2a + b = 9 \\ P(3) = 4 \Rightarrow 3a + b = 4 \end{array}\right\}$ $\quad \therefore \begin{array}{l} a = -5 \\ b = 19 \end{array}$ $\quad \therefore$ remainder is $(-5x+19)$.

b. Outcomes assessed : E3, E4

Marking Guidelines

Criteria	Marks
i • finds equation of tangent to hyperbola by differentiation	1
• finds gradient of tangent to circle in terms of θ	1
• completes equation of tangent to circle using appropriate trig. identity	1
ii • compares coefficients for two forms of equation of PQ to obtain results for $\tan\theta$, $\sec\theta$	1
• obtains quartic equation for t	1
• factors then rearranges to get required cubic equation	1
iii • graphs $y = x^3 - 4x$ showing intercepts and stationary point in 4th quadrant	1
• compares vertical translations to deduce existence of such a point on 3rd quad. branch	1
• uses turning point to deduce existence of such points on 1st quad. branch for stated c	1
iv • sketches hyperbola and circle, touching in first quadrant, with two common tangents	1
• gives coordinates of point where curves touch and equation of this common tangent	1
• gives coordinates of point of contact on hyperbola for second common tangent	1
• gives equation of this tangent and coordinates of its point of contact with circle	1

Answer

i.

$x = ct \qquad y = \frac{c}{t}$

$\frac{dx}{dt} = c \qquad \frac{dy}{dt} = -\frac{c}{t^2}$

$\therefore \frac{dy}{dx} = -\frac{c}{t^2} \div c = -\frac{1}{t^2}$

Tangent to hyperbola has equation

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$x + t^2 y = 2ct$

$x = 1 + \cos\theta \qquad y = \sin\theta$

$\frac{dx}{d\theta} = -\sin\theta \qquad \frac{dy}{d\theta} = \cos\theta$

$\therefore \frac{dy}{dx} = -\frac{\cos\theta}{\sin\theta}$

Tangent to circle has equation

$y - \sin\theta = -\frac{\cos\theta}{\sin\theta}\left(x - 1 - \cos\theta\right)$

$x\cos\theta + y\sin\theta = \cos^2\theta + \sin^2\theta + \cos\theta$

$x\cos\theta + y\sin\theta = 1 + \cos\theta$

ii. When these two tangents are in fact the same line, PQ is tangent to both curves. Comparing the equations $x + t^2 y = 2ct$ and $x + y\tan\theta = \sec\theta + 1$ (rearrangement of tangent to circle at Q), these equations give the same line when $t^2 = \tan\theta$ and $2ct - 1 = \sec\theta$.

Then

$1 + \tan^2\theta = \sec^2\theta \Rightarrow 1 + t^4 = (2ct-1)^2$

$t^4 - 4c^2t^2 + 4ct = 0$

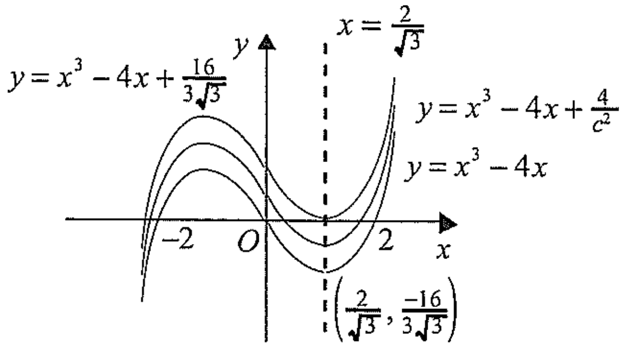
$t(t^3 - 4c^2t + 4c) = 0$

$t \neq 0 \Rightarrow t^3 - 4c^2t + 4c = 0$

$\therefore \quad \left(\frac{t}{c}\right)^3 - 4\left(\frac{t}{c}\right) + \frac{4}{c^2} = 0 \quad *$

iii.

$$y = x^3 - 4x$$
$$\frac{dy}{dx} = 3x^2 - 4$$
$$\frac{dy}{dx} = 0 \Rightarrow x = \pm\frac{2}{\sqrt{3}}$$
$$x = \frac{2}{\sqrt{3}} \Rightarrow y = -\frac{16}{3\sqrt{3}}$$

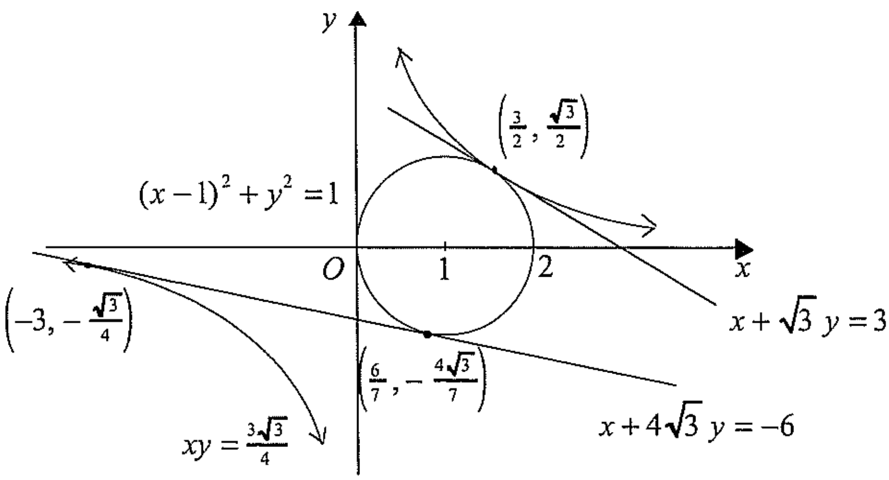


The graph of $y = x^3 - 4x + k, \quad k > 0$, is an upward vertical translation of $y = x^3 - 4x$. Hence for all $c > 0$, $y = x^3 - 4x + \frac{4}{c^2}$ has exactly one negative x-intercept corresponding to one negative t value satisfying *, thus giving exactly one point P on the third-quadrant branch of the hyperbola where the tangent to the hyperbola is also tangent to the circle.
For $\frac{4}{c^2} < \frac{16}{3\sqrt{3}}$, $y = x^3 - 4x + \frac{4}{c^2}$ also has two distinct positive x-intercepts. Hence for $c^2 > \frac{3\sqrt{3}}{4}$, there are two distinct positive t values satisfying *, giving two such points P on the first-quadrant branch of the hyperbola.

iv. When $c^2 = \frac{3\sqrt{3}}{4}$, $x^3 - 4x + \frac{4}{c^2} = 0$ becomes $x^3 - 4x + \frac{16}{3\sqrt{3}} = 0$. From the graph, this equation has a double root $\frac{2}{\sqrt{3}}$, and a third root $\frac{-4}{\sqrt{3}}$ (since the sum of the roots is zero).

$\frac{t}{c} = \frac{2}{\sqrt{3}} \quad \Rightarrow \quad ct = \frac{3}{2}, \quad \frac{c}{t} = \frac{\sqrt{3}}{2}$ $\quad \therefore$ curves touch at $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$ with common tangent $x + \sqrt{3}\, y = 3$.

$\frac{t}{c} = \frac{-4}{\sqrt{3}} \quad \Rightarrow \quad ct = -3, \quad \frac{c}{t} = -\frac{\sqrt{3}}{4}$ $\quad$ and $\quad \sec\theta = 2ct - 1 = -7, \quad \tan\theta = t^2 = 4\sqrt{3}$ $\quad \therefore$ points of contact of second common tangent $x + 4\sqrt{3}\, y = -6$ are $\left(-3, -\frac{\sqrt{3}}{4}\right)$ on the hyperbola and $\left(\frac{6}{7}, -\frac{4\sqrt{3}}{7}\right)$ on the circle.



Question 5

a. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • uses de Moivre's theorem to show result	1
ii • expands $(z^1+z^{-1})(z^2+z^{-2})(z^3+z^{-3})$	1
• rearranges, regroups and applies result from i. to obtain required trig. identity	1
iii • uses double angle formula and identity from ii. to simplify equation	1
• obtains general solution for $\cos\theta=0$, or obtains all solutions for $-\pi<\theta\le\pi$	1
• obtains remaining general solutions	1

Answer

i. $z=\cos\theta+i\sin\theta$. By de Moivre's theorem, $z^n=\cos n\theta+i\sin n\theta$ for $n=1,2,3,\ldots$.

Then $z^{-n}=\cos(-n\theta)+i\sin(-n\theta) \quad\Rightarrow\quad z^{-n}=\cos n\theta-i\sin n\theta$

Hence $z^n+z^{-n}=2\cos n\theta$

ii.
$$\begin{aligned}(z^1+z^{-1})(z^2+z^{-2})(z^3+z^{-3}) &= (z^3+z^{-3}+z^1+z^{-1})(z^3+z^{-3})\\ &= 2+z^2+z^{-2}+z^4+z^{-4}+z^6+z^{-6}\end{aligned}$$

$\therefore\ 2\cos\theta\,.\,2\cos2\theta\,.\,2\cos3\theta=2+2\cos2\theta+2\cos4\theta+2\cos6\theta$

$4\cos\theta\cos2\theta\cos3\theta=1+\cos2\theta+\cos4\theta+\cos6\theta$

iii.
$$\begin{aligned}\cos^2\theta+\cos^2 2\theta+\cos^2 3\theta&=1\\ 2\cos^2\theta+2\cos^2 2\theta+2\cos^2 3\theta-2&=0\\ (1+\cos2\theta)+(1+\cos4\theta)+(1+\cos6\theta)-2&=0\\ 1+\cos2\theta+\cos4\theta+\cos6\theta&=0\\ 4\cos\theta\cos2\theta\cos3\theta&=0\end{aligned}$$

$\cos\theta=0 \Rightarrow \theta=2m\pi\pm\frac{\pi}{2}=(4m\pm1)\frac{\pi}{2}$

$\therefore\ \theta=(4m\pm1)\frac{\pi}{2},\quad (4m\pm1)\frac{\pi}{4},\quad (4m\pm1)\frac{\pi}{6}$

for $m=0,\pm1,\pm2,\ldots$

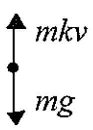
b. Outcomes assessed : E5

Marking Guidelines

Criteria	Marks
i • uses Newton's 2nd law to derive equation of motion	1
• uses $\ddot{x}\to0$ as $v\to U$ to express equation of motion in required form	1
ii • finds primitive function for t	1
• evaluates constant of integration to establish required expression for t in terms of v	1
• rearranges to find expression for v in terms of t	1
iii • integrates to find x in terms of t	1
• substitutes and rearranges to obtain required expression for x	1
iv • calculates percentage of terminal velocity gained in first second	1
• calculates distance travelled in first second.	1

Answer

i.



i. By Newton's 2nd Law,

$m\ddot{x}=mg-mkv$

$\ddot{x}=g-kv$

$\ddot{x}\to0$ as $v\to\frac{g}{k} \Rightarrow U=\frac{g}{k} \quad \therefore k=\frac{g}{U}$

$\therefore\ \ddot{x}=\frac{g}{U}(U-v)$

ii.

$$\frac{dv}{dt}=\frac{g}{U}(U-v)$$

$$-\frac{g}{U}\frac{dt}{dv}=-\frac{1}{U-v}$$

$$-\frac{g}{U}t=\ln A(U-v),\ \ A\ const.$$

$$\left.\begin{array}{l} t=0 \\ v=\frac{1}{2}U \end{array}\right] \Rightarrow A=\frac{2}{U}$$

$$\therefore\ -\frac{g}{U}t=\ln 2\left(1-\frac{v}{U}\right)$$

$$e^{-\frac{g}{U}t}=2\left(1-\frac{v}{U}\right)$$

$$\therefore \frac{v}{U}=1-\frac{1}{2}e^{-\frac{g}{U}t}$$

iii.

$$v=U-\tfrac{1}{2}U\,e^{-\frac{g}{U}t}$$

$$x=Ut+\frac{U^2}{2g}e^{-\frac{g}{U}t}+c,\quad c\ const$$

$$\left.\begin{array}{l} t=0 \\ x=0 \end{array}\right] \Rightarrow\ 0=0+\frac{U^2}{2g}+c \qquad \therefore c=-\frac{U^2}{2g}$$

$$x=Ut+\frac{U^2}{g}\left(\tfrac{1}{2}e^{-\frac{g}{U}t}-\tfrac{1}{2}\right)$$

$$\therefore\ x=Ut-\frac{U^2}{g}\left(\frac{v}{U}-\frac{1}{2}\right)$$

iv. $t=1 \Rightarrow \dfrac{v}{U}-\dfrac{1}{2}=\dfrac{1}{2}\left(1-e^{-0\cdot 1}\right)\approx 0\cdot 04758$

$$x=100-1000\times\left(\frac{v}{U}-\frac{1}{2}\right)\approx 52\cdot 4187$$

$\therefore$ particle has gained $4\cdot 8\%$ of its terminal velocity and travelled $52\cdot 4$ metres during the first second.

Question 6

a. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
• uses relationship between coefficients and sum of roots to find one root	1
• substitutes this root into the equation to find the value of k	1

Answer

Let $x^3+3x^2+7x+k=0$ have roots $\alpha-d,\ \alpha,\ \alpha+d$. Then considering the sum of the roots
$3\alpha=-3 \quad \therefore\ \alpha=-1$

Then $(-1)^3+3(-1)^2+7(-1)+k=0\ \Rightarrow k=5$

b. Outcomes assessed : E7

Marking Guidelines

Criteria	Marks
i • finds area of square cross section in terms of x	1
• expresses volume as limiting sum of slices and hence as integral	1
ii • expresses dx in terms of du	1
• converts x limits to u limits and simplifies new integrand after substitution	1
• evaluates resulting definite integral	1

Answer

i. Area of square cross section is $A=(2y)^2=4\left(1-|x|^{\frac{1}{2}}\right)^4$, since $y=\left(1-|x|^{\frac{1}{2}}\right)^2$, $-1\le x\le 1$.

Hence $V=\lim\limits_{\delta x\to 0}\sum\limits_{x=-1}^{1}4\left(1-|x|^{\frac{1}{2}}\right)^4\delta x=8\lim\limits_{\delta x\to 0}\sum\limits_{x=0}^{1}\left(1-x^{\frac{1}{2}}\right)^4\delta x=8\int_0^1\left(1-\sqrt{x}\right)^4 dx$ (using symmetry) ii.

$$u = 1 - \sqrt{x}$$
$$x = (1-u)^2$$
$$dx = -2(1-u)du$$

$$x = 0 \Rightarrow u = 1$$
$$x = 1 \Rightarrow u = 0$$

$$\begin{aligned}\therefore V &= 8\int_1^0 u^4 . -2(1-u)\, du \\ &= 16\int_0^1 (u^4 - u^5)du \\ &= 16\left[\tfrac{1}{5}u^5 - \tfrac{1}{6}u^6\right]_0^1 \\ &= \tfrac{8}{15}\end{aligned}$$

c. Outcomes assessed : E5

Marking Guidelines

Criteria	Marks
i • finds gradient of tangent by differentiation	1
• finds gradient of normal at P and hence deduces required expression for $\tan\theta$	1
ii • draws diagram showing forces on P	1
iii • resolves vertically and horizontally to find simultaneous equations	1
• finds r in terms of g and ω	1
• finds N in terms of m, g and ω	1
iv • considers expressions for r, N and the height of the bowl to find limits for ω	1
	1

Answer

i. $y^2 - x^2 = 1$, $P\left(r,\ \sqrt{1+r^2}\right)$.

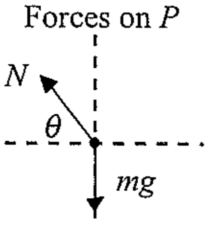
$$2y\frac{dy}{dx} - 2x = 0$$
$$\frac{dy}{dx} = \frac{x}{y}$$

$\therefore$ normal at P has gradient

$$-\frac{\sqrt{1+r^2}}{r}$$

$$\therefore \tan\theta = \frac{\sqrt{1+r^2}}{r}$$

ii. Forces on P



iii. Resolving vertically and horizontally, by Newton's 2nd law

$$N\sin\theta = mg \qquad (1)$$
$$N\cos\theta = mr\omega^2 \qquad (2)$$

$$\frac{(1)}{(2)} \Rightarrow \tan\theta = \frac{g}{r\omega^2}$$
$$\frac{\sqrt{1+r^2}}{r} = \frac{g}{r\omega^2}$$
$$\therefore r = \sqrt{\frac{g^2}{\omega^4} - 1}$$
$$r = \frac{\sqrt{g^2 - \omega^4}}{\omega^2}$$

Then from (2)

$$\begin{aligned}N^2 &= m^2r^2\omega^4\sec^2\theta \\ &= m^2r^2\omega^4(1+\tan^2\theta) \\ &= m^2\omega^4(r^2 + r^2\tan^2\theta) \\ &= m^2\omega^4\left(\frac{g^2}{\omega^4} - 1 + \frac{g^2}{\omega^4}\right) \\ &= m^2(2g^2 - \omega^4)\end{aligned}$$
$$\therefore N = m\sqrt{2g^2 - \omega^4}$$

iv. Considering expressions for r and N, $\omega^4 \le g^2 \quad \therefore \omega \le \sqrt{g}$

Also $y \le 5 \Rightarrow \sqrt{1+r^2} \le 5 \Rightarrow \dfrac{g}{\omega^2} \le 5 \qquad \therefore \sqrt{\dfrac{g}{5}} \le \omega \le \sqrt{g}$

Question 7
a. Outcomes assessed : E8, E9

Marking Guidelines

Criteria	Marks
i • executes integration by parts	1
• evaluates numerical term and simplifies new definite integral to obtain reduction formula	1
ii • finds value of I_0	1
• uses reduction formula to evaluate required integral	1
iii • obtains reduction formula for $\frac{1}{r!}I_r$, $r=1,2,\dots,n$	1
• obtains required result by summation and simplification	1
iv • shows integrand lies between 0 and 1 for $1\le x\le e$ and deduces required inequality	1
v • shows 0 is limiting value of $\frac{1}{n!}I_n$ as $n\to\infty$ then deduces required result	1

Answer

i. $I_n=\int_1^e(1-\ln x)^n\,dx$, $n=0,1,2,\dots$

For $n\ge1$,

$$I_n=\left[x(1-\ln x)^n\right]_1^e-\int_1^e x\,.\,n(1-\ln x)^{n-1}(-\tfrac{1}{x})dx$$

$$=-1+n\int_1^e(1-\ln x)^{n-1}\,dx$$

$$\therefore I_n=-1+nI_{n-1}\,,\quad n=1,2,3,\dots$$

ii. $I_0=\int_1^e 1\,dx=e-1$

$$\begin{aligned}I_3&=-1+3I_2\\&=-1+3(-1+2I_1)\\&=-4+6(-1+1I_0)\\&=-10+6(e-1)\\&=-16+6e\end{aligned}$$

iii.
$$\frac{I_r}{r!}=\frac{-1}{r!}+\frac{rI_{r-1}}{r!}\,,\quad r=1,2,\dots,n$$

$$\frac{I_r}{r!}=\frac{-1}{r!}+\frac{I_{r-1}}{(r-1)!}$$

$$\sum_{r=1}^n\frac{I_r}{r!}=-\sum_{r=1}^n\frac{1}{r!}+\sum_{r=1}^n\frac{I_{r-1}}{(r-1)!}$$

$$\sum_{r=1}^n\frac{I_r}{r!}=-\sum_{r=1}^n\frac{1}{r!}+\sum_{r=0}^{n-1}\frac{I_r}{r!}$$

$$\therefore\frac{I_n}{n!}=-\sum_{r=1}^n\frac{1}{r!}+\frac{I_0}{0!}$$

$$=-\sum_{r=1}^n\frac{1}{r!}+\frac{-1}{0!}+\frac{e}{0!}$$

$$=e-\sum_{r=0}^n\frac{1}{r!}$$

iv. $1\le x\le e\Rightarrow 0\le\ln x\le1\Rightarrow0\le(1-\ln x)^n\le1$

$$\therefore 0\le\int_1^e(1-\ln x)^n\,dx\le\int_1^e 1\,dx$$

$$0\le I_n\le e-1$$

v. $$0\le\frac{I_n}{n!}\le\frac{e-1}{n!}$$

But $\lim\limits_{n\to\infty}\frac{e-1}{n!}=0$ $\therefore\ \lim\limits_{n\to\infty}\frac{I_n}{n!}=0$

Then $\lim\limits_{n\to\infty}\left(e-\sum_{r=0}^n\frac{1}{r!}\right)=\lim\limits_{n\to\infty}\frac{I_n}{n!}=0$

$$\therefore\quad e-\lim_{n\to\infty}\sum_{r=0}^n\frac{1}{r!}=0$$

$$\therefore\quad \lim_{n\to\infty}\sum_{r=0}^n\frac{1}{r!}=e$$

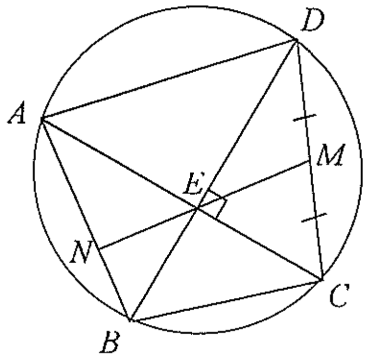
b. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
i • copies diagram, shows given information, realises a circle can be drawn through C, D, E	1
• explains why CD is a diameter of circle CDE.	1
• deduces that M is the centre of circle CDE and hence ME, MC are radii.	1
ii • uses i. to deduce $\angle ECM = \angle CEM$	1
• uses equality of vertically opposite angles to deduce $\angle NEA = \angle ECD$	1
• uses equality of angles subtended by same arc at circumference to deduce $\angle NAE = \angle EDC$	1
• uses angle sum property of a triangle to complete proof	1

Answer

i.



A unique circle can be drawn through any set of 3 non-collinear points.
Consider the circle that passes through C, D and E.
Since $\angle CED = 90°$, CD is a diameter of this circle.
Also, given M is the midpoint of DC, M is the centre of circle CDE.
$\therefore ME = MC$ (*radii of a circle are equal*)

ii. $\angle ECM = \angle CEM$ (*$\angle$'s opp. equal sides are equal in ΔMEC*)
$\angle NEA = \angle CEM$ (*vert. opp. $\angle$'s are equal*)
$\therefore \angle NEA = \angle ECM = \angle ECD$ (*D, M, C collinear*)
Also $\angle BAC = \angle BDC$ (*$\angle$'s subtended at the circumference by the same arc BC are equal*)
$\therefore \angle NAE = \angle EDC$ (*$\angle NAE$, $\angle BAC$ same angle; $\angle EDC$, $\angle BDC$ same angle*)
$\therefore \angle ANE = \angle DEC = 90°$ (*third $\angle$'s of Δ's NAE, EDC also equal since $\angle$ sum of each is* 180°)
$\therefore MN \perp AB$

Question 8

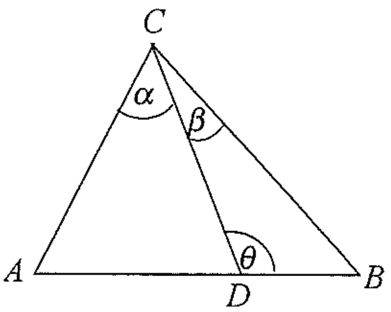
a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • uses sine rule in ΔCAD to relate sides CD, AD and opposite angles	1
• uses sine rule in ΔCDB to relate sides CD, DB and opposite angles	1
• combines resulting equalities to obtain ratio $AD : DB$ in terms of sine ratios of angles	1
• uses internal division information and trig. identity for sine to deduce required result	1
ii • expands sine of a sum and of a difference	1
• divides by product of cosine ratios to rearrange result in terms of tangent ratios	1
• rearranges to get required result	1

Answer

i.



In ΔCAD, by ext. $\angle$ theorem, $\angle CAD = \theta - \alpha$

$$\therefore \frac{CD}{\sin(\theta-\alpha)} = \frac{AD}{\sin\alpha} \quad (1)$$

In ΔCDB, by $\angle$ sum $180°$, $\angle CBD = 180° - (\theta+\beta)$

$$\therefore \frac{CD}{\sin\{180° - (\theta+\beta)\}} = \frac{DB}{\sin\beta} \quad (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)} = \frac{AD\sin\beta}{DB\sin\alpha}$$

$$\therefore \frac{\sin(\theta+\beta)\sin\alpha}{\sin(\theta-\alpha)\sin\beta} = \frac{AD}{DB} = \frac{m}{n}$$

ii. $n\sin\alpha(\sin\theta\cos\beta + \cos\theta\sin\beta) = m\sin\beta(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$

Dividing both sides by $\cos\alpha\cos\beta\cos\theta$ gives

$n\tan\alpha(\tan\theta + \tan\beta) = m\tan\beta(\tan\theta - \tan\alpha)$

$(n+m)\tan\alpha\tan\beta = \tan\theta(m\tan\beta - n\tan\alpha)$

$$\tan\theta = \frac{(n+m)\tan\alpha\tan\beta}{(m\tan\beta - n\tan\alpha)}$$

b. Outcomes assessed : H5, PE3, E9

Marking Guidelines

	Criteria	**Marks**
i	• realises that the definite integral cannot be negative for any real λ	1
	• expands the integrand to write a quadratic expression in λ	1
	• finds the discriminant of this quadratic expression	1
	• uses $\Delta \le 0$ to deduce the required result	1
ii	• substitutes $a=1$ and $g(x)=1$	1
	• evaluates integral involving $\{g(x)\}^2$ to obtain required result	1
iii	• squares both sides of result from ii.	1
	• applies result from ii. a second time with $f(x)$ replaced by $\{f(x)\}^2$	1

Answer

i. Since $a > 0$, $\int_0^a \{\lambda f(x) + g(x)\}^2\, dx \ge 0$ for all real λ.

$$\int_0^a \{\lambda f(x)+g(x)\}^2\, dx = \lambda^2 \int_0^a \{f(x)\}^2\, dx + 2\lambda\int_0^a f(x)g(x)\, dx + \int_0^a \{g(x)\}^2\, dx$$

Considered as a quadratic in λ, this expression has discriminant $\Delta \le 0$.

$$\therefore 4\left\{\int_0^a f(x)g(x)\, dx\right\}^2 - 4\int_0^a \{f(x)\}^2\, dx \,.\, \int_0^a \{g(x)\}^2\, dx \le 0$$

$$\therefore \left\{\int_0^a f(x)g(x)\, dx\right\}^2 \le \int_0^a \{f(x)\}^2\, dx \,.\, \int_0^a \{g(x)\}^2\, dx$$

ii. Let $a=1$ and $g(x)=1$. Then $\int_0^1 1^2 dx = 1 \Rightarrow \left\{\int_0^1 f(x)\, dx\right\}^2 \le \int_0^1 \{f(x)\}^2\, dx$

iii. $\left\{\int_0^1 f(x)\, dx\right\}^4 \le \left\{\int_0^1 \{f(x)\}^2\, dx\right\}^2 \le \int_0^1 \left\{\{f(x)\}^2\right\}^2 dx$ $\quad \therefore \left\{\int_0^1 f(x)\, dx\right\}^4 \le \int_0^1 \{f(x)\}^4\, dx$

Independent Trial Examination Mathematics Extension 2 2009 Mapping Grid

Question	Marks	Content	Syllabus Outcomes	Targeted Performance Bands
1 a	2	Integration	H5	E2-E3
b i	2	Polynomials	PE3	E2-E3
ii	2	Integration	H8, E8	E2-E3
c	3	Integration	H5, HE6	E2-E3
d	3	Integration	H8, HE6	E2-E3
e	3	Integration	H8, HE6	E2-E3
2 a i	1	Complex numbers	E3	E2-E3
ii	1	Complex numbers	E3	E2-E3
iii	1	Complex numbers	E3	E2-E3
b i	3	Complex numbers	E3	E2-E3
ii	1	Complex numbers	E3	E2-E3
c i	2	Complex numbers	E3	E2-E3
ii	2	Complex numbers	E3	E2-E3
d i	2	Complex numbers	E3	E2-E3
ii	2	Complex numbers	E3	E2-E3
3 a i	1	Graphs	E6	E2-E3
ii	1	Graphs	E6	E2-E3
iii	2	Graphs	E6	E2-E3
iv	2	Graphs	E6	E2-E3
b	3	Graphs	E6	E2-E3
c i	2	Integration	H8	E2-E3
ii	2	Integration	H8	E2-E3
iii	2	Inequalities	PE3	E2-E3
4 a	2	Polynomials	E4	E2-E3
b i	3	Conics	E4	E2-E3
ii	3	Conics	E4	E2-E3
iii	3	Conics	E3	E2-E3
iv	4	Conics	E3	E2-E3
5 a i	1	Complex numbers	E3	E3-E4
ii	2	Complex numbers	E3	E3-E4
iii	3	Complex numbers	E3	E3-E4
b i	2	Mechanics	E5	E3-E4
ii	3	Mechanics	E5	E3-E4
iii	2	Mechanics	E5	E3-E4
iv	2	Mechanics	E5	E3-E4
6 a	2	Polynomials	E4	E2-E3
b i	2	Volumes	E7	E3-E4
ii	3	Volumes	E7	E2-E3
c i	2	Mechanics	E5	E3-E4
ii	1	Mechanics	E5	E3-E4
iii	3	Mechanics	E5	E3-E4
iv	2	Mechanics	E5	E3-E4

7 a i	2	Integration	E8	E3-E4
ii	2	Integration	E8	E3-E4
iii	2	Integration	E8, E9	E3-E4
iv	1	Integration	E8	E3-E4
v	1	Integration	E9	E3-E4
b i	3	Circle geometry	PE2, PE3	E2-E3
ii	4	Circle geometry	PE2, PE3	E2-E3
8 a i	4	Trigonometry	H5	E2-E3
ii	3	Trigonometry	H5	E2-E3
b i	4	Inequalities	PE3	E3-E4
ii	2	Integration	H5	E3-E4
iii	2	Inequalities	E9	E3-E4