STHS EXT1 MATHS T.H.S.C. 2005

QUESTION 1

Marks

a) Find $\frac{d}{dx}\left(\frac{1}{4+x^2}\right)$ **2**

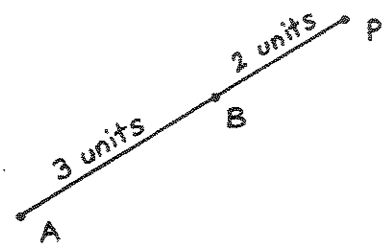
b) Find $\int \frac{1}{4+x^2}\,dx$ **2**

c) Solve $\frac{1-x}{1+x} \le 1$ **3**

d) The polynomial equation $3x^3 - 2x^2 + 3x - 4 = 0$ has roots α, β and δ. **3**

Find the value of $\frac{1}{\alpha\beta} + \frac{1}{\alpha\delta} + \frac{1}{\beta\delta}$

e) **2**



The point P divides the interval AB in the ratio 1:K. Find K

QUESTION 2

Marks

a) Show that $\sin x - \cos 2x = 2\sin^2 x + \sin x - 1$ **5**

Hence or otherwise solve

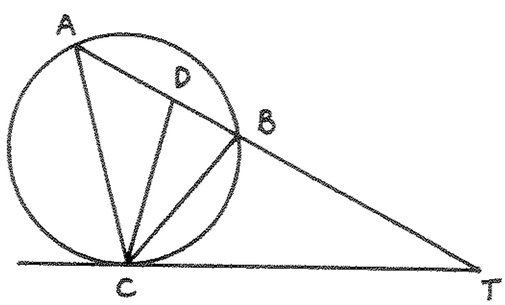
$\sin x - \cos 2x = 0$ for $0 \le x \le 2\pi$

b) ABC is a triangle inscribed in a circle. The tangent at C meets AB at T. **3**

The bisector of $\angle$ACB cuts AB at D

i) Copy the diagram

ii) Prove TC = TD



Question 2 (cont.)

c) Consider the sequence 4

$\log_{10}(x-2), \quad \log_{10}(x-2)^2, \quad \log_{10}(x-2)^3$

i) Is this sequence arithmetic or geometric? Justify your answer.

ii) Show that the sum to n terms is given by

$\frac{n}{2}\log_{10}(x-2)^{n+1}$

QUESTION 3 **Marks**

a) i) Find x such that $\sin^{-1} x = \cos^{-1} x$ 4

ii) On the same number plane, sketch the graphs of

$y = \sin^{-1} x$ *and* $y = \cos^{-1} x$.

Label important points clearly.

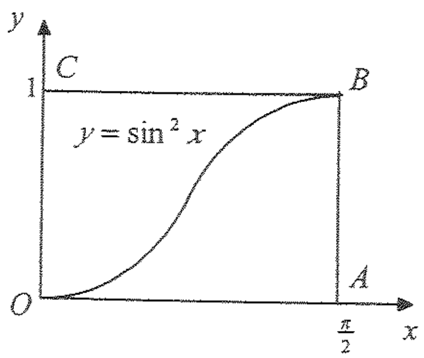
iii) On the same diagram as ii, sketch $y = \sin^{-1} x + \cos^{-1} x$

b) i) By considering the graph of $y = e^x$ show that the equation 4

$e^x + x + 1 = 0$ has only one real root and that this root is negative.

ii) Taking $x = -1.5$ as a first approximation to this root, use one application of Newton's method to find a better approximation.

c) 4



The rectangle $OABC$ has vertices $O(0,0)$, $A(\frac{\pi}{2},0)$, $B(\frac{\pi}{2},1)$ and $C(0,1)$.

The curve $y = \sin^2 x$ is shown passing through the points O and B. Show that this curve divides the rectangle $OABC$ into two regions of equal area.

QUESTION 4 **Marks**

a) Prove by Mathematical Induction that 5

$$1 \times 3 + 2 \times 3^2 + \ldots\ldots.+ n \times 3^n = \frac{(2n-1)\ 3^{n+1} + 3}{4}$$

where n is an interger, $n \geq 1$

b) If $\tan^{-1} y = 2 \tan^{-1} x$ show that 3

$$y = \frac{2x}{1-x^2}$$

c) Using the substitution $u = e^{2x}$ find a and b such that 4

$$\int_0^{\ln 2} \frac{e^{2x}}{1+e^{4x}}\, dx = \frac{1}{2} \tan^{-1} a - b$$

QUESTION 5 **Marks**

a) The rate at which a body cools in air is proportional to the difference between its temperature T and the constant temperature 20°C (in this case) of the surrounding air. This can be expressed by the differential equation 6

$$\frac{dT}{dt} = -k\ (T - 20)$$

The original temperature of a heated metal bar was 100°C. The bar cools to 70°C in 10 minutes.

i) Show that $T = 20 + Ae^{-kt}$ is a solution to the differential equation.

ii) Show A = 80

iii) Find the exact value of k

iv) Find the time taken for the temperature of the bar to reach 60°C.
(Give your answer to the nearest minute).

b) M (2am, am ^{2}) and N (2an, an ^{2}) are points on the parabola $x^2 = 4ay$ 6

i) Find the equation of the chord MN

ii) Find the co-ordinates of the midpoint of the chord MN

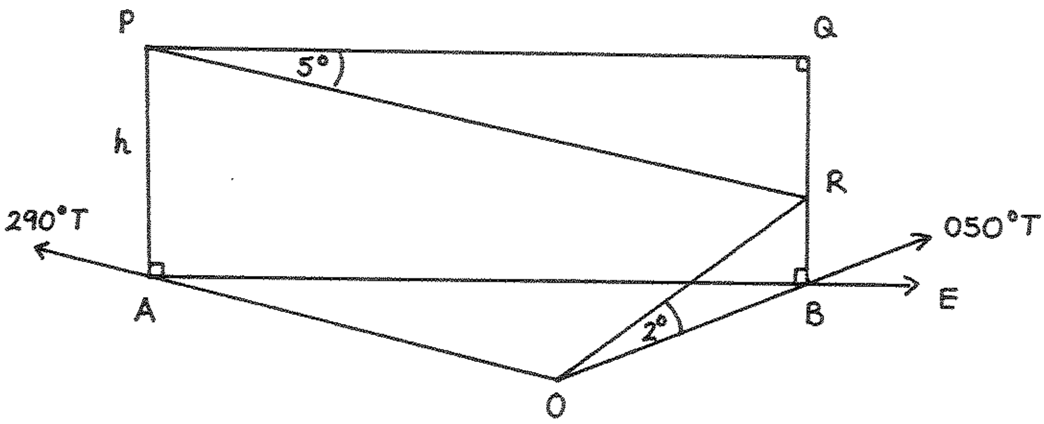
iii) If the chords all pass through the point (0,2), show that the locus of the midpoint of MN is

$$x^2 = 2a\,(y-2)$$

QUESTION 6 **Marks**

a) $P(x)$ is an odd polynomial of degree 3. It has $x+4$ as a factor, and when it is divided by $x-3$ the reminder is 21. Find $P(x)$. 3

b) 5



In the diagram above an aircraft is flying along the path PR. It has a constant speed of 300 km/h and is descending at a steady angle of 5°. It flies directly over beacons at A and B where B is due East of A. An observer at O first sights the aircraft over A at a bearing of 290°T. The observer sights the aircraft again 10 minutes later over B at a bearing of 050°T and with an angle of elevation of 2°. O is on the same horizontal plane as A and B.

i) Show that the aircraft has travelled 50km in the 10 minutes between observations.

ii) Show that $\angle AOB = 120°$.

iii) Prove that the observer at O is 19 670 metres, to the nearest 10 metres, from the beacon at B.

(iv) Find the altitude h of the aircraft, to the nearest 10 metres, when it was originally sighted over A.

Question 6 (cont.)

c) If $f(x) = u(x) - \ln[u(x)+1]$ **4**

i) Show that $f'(x) = \dfrac{u(x)u'(x)}{1 + u(x)}$

ii) Hence or otherwise evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\cos x \sin x}{1 + \sin x}\, dx$$

QUESTION 7 **Marks**

a) The velocity v m/s of a particle at time t seconds is given in terms of position x m by **6**

$$v = \frac{4}{x} \ (where\ x > 0)$$

Initially $x = 8$.

i) Find the acceleration of the particle when $x = 1$

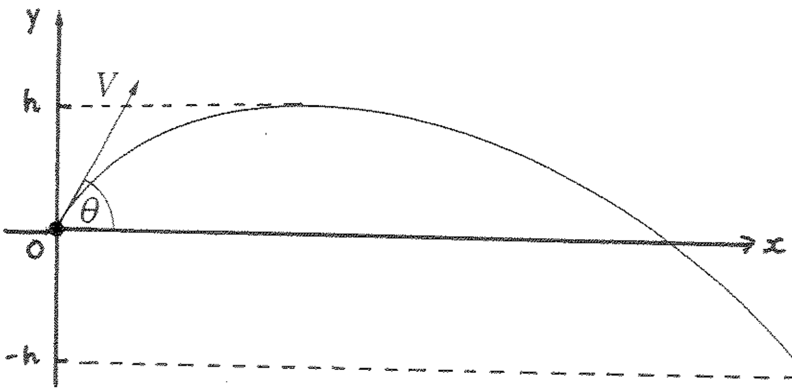
ii) Find an expression for x in terms of t.

iii) What is the position of the particle when $t = 2$?

iv) Describe the motion of the particle.

Question 7 (cont.)

b) 6



The diagram above shows the path of a projectile fired from the top O of a cliff. Its initial velocity is V m/s, its initial angle of elevation is θ and it rises to a maximum height h metres above O. It strikes a target T situated on a horizontal plane h metres below O.

The horizontal and vertical components of displacement in metres at time t seconds are given by $x = Vt\ \cos\theta$ *and* $y = Vt\sin\theta - \frac{1}{2}gt^2$ respectively.

i) Prove that $h = \dfrac{V^2\sin^2\theta}{2g}$.

ii) Prove that the time taken for the projectile to reach its target is:

$$\frac{V\sin\theta(1+\sqrt{2})}{g} \text{ seconds}$$

iii) Hence show that the distance from the target to the base of the cliff is:

$$\frac{V^2(1+\sqrt{2})\sin 2\theta}{2g} \text{ metres}$$

End of Exam

2005 EXTENSION 1 TRIAL HSC

Question 1

a) $\frac{d}{dx}\left(\frac{1}{4+x^2}\right)$

$= \frac{d}{dx}(4+x^2)^{-1}$

$= -1(4+x^2)^{-2} \cdot 2x$

$= \frac{-2x}{(4+x^2)^2}$

b) $\int \frac{1}{4+x^2}\,dx = \frac{1}{2}\tan^{-1}\frac{x}{2} + c$

c) $\frac{1-x}{1+x} \le 1$

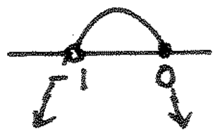
$(1+x)^2 \times \frac{1-x}{1+x} \le 1 \times (1+x)^2$

$(1+x)(1-x) \le (1+x)^2$

$(1+x)(1-x) - (1+x)^2 \le 0$

$(1+x)[(1-x)-(1+x)] \le 0$

$(1+x)(-2x) \le 0$



$\therefore\ x < -1,\ x \geq 0$

d) $3x^3 - 2x^2 + 3x - 4 = 0$

$\alpha + \beta + \delta = \frac{2}{3}$

$\alpha\beta\delta = \frac{4}{3}$

$\frac{1}{\alpha\beta} + \frac{1}{\alpha\delta} + \frac{1}{\beta\delta} = \frac{\delta+\beta+\alpha}{\alpha\beta\delta}$

$= \frac{2}{3} \div \frac{4}{3}$

$= \frac{1}{2}$

c) $5 : -2 = 1 : K$

$\therefore\ K = -\frac{2}{5}$

Question 2

a) LHS $= \sin x - \cos 2x$

$= \sin x - (1 - 2\sin^2 x)$

$= \sin x - 1 + 2\sin^2 x$

$= 2\sin^2 x + \sin x - 1$

$=$ RHS

$\sin x - \cos 2x = 0$

$2\sin^2 x + \sin x - 1 = 0$

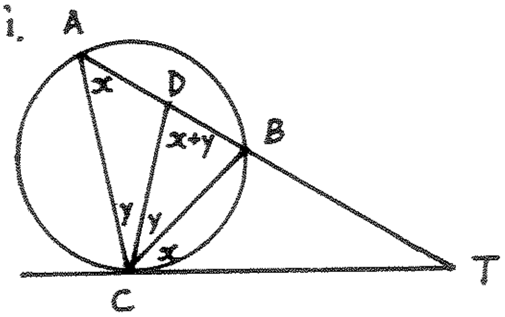
$(2\sin x - 1)(\sin x + 1) = 0$

$\sin x = \frac{1}{2}$ or $\sin x = -1$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ $\qquad x = \frac{3\pi}{2}$

$\therefore\ x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

b) i.



ii. let $\angle BCT = x$

$\angle BCD = \angle DCA = y$

(given CD bisects $\angle ACB$)

$\angle CAB = x$ (alternate segment theorem)

$\angle TCD = x + y$

$\angle TDC = x + y$ (exterior angle of Δ ACD)

$\therefore$ TC = TD (equal sides are opposite equal angles)

c) $\log_{10}(x-2)$, $2\log_{10}(x-2)$, $3\log_{10}(x-2)$

i. arithmetic sequence

$d = \log_{10}(x-2)$

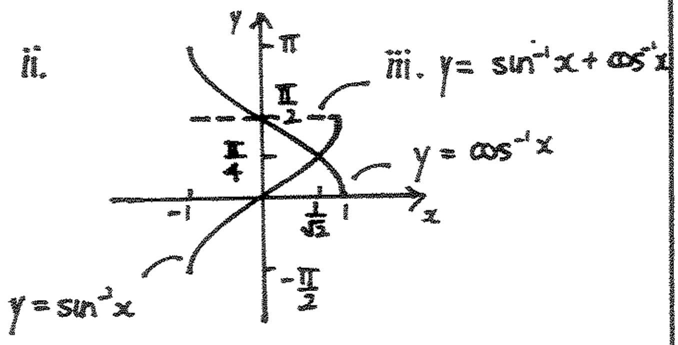
$T_2 - T_1 = T_3 - T_2$

ii.
$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$
$$= \frac{n}{2}\left[2\log_{10}(x-2) + (n-1)\log_{10}(x-2)\right]$$
$$= \frac{n}{2}\left[\log_{10}(x-2)^2 + \log_{10}(x-2)^{n-1}\right]$$
$$= \frac{n}{2}\left[\log_{10}(x-2)^{2+n-1}\right]$$
$$= \frac{n}{2}\log_{10}(x-2)^{n+1}$$

Question 3

a) i. $\underline{\underline{x = \frac{1}{\sqrt{2}}}}$

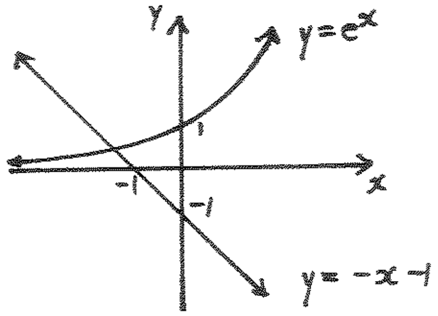
ii.



iii. $y = \sin^{-1}x + \cos^{-1}x$

$y = \pi$

b) i. Solution of $e^x + x + 1 = 0$ is the point of intersection of $y = e^x$ and $y = -x - 1$.



ie. one point of intersection with $x < -1$

∴ the equation has one real and negative root.

ii. see below

c)
$$A_{OABC} = \frac{\pi}{2} \times 1 = \frac{\pi}{2}$$
$$A_{OAB} = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 - \cos 2x \, dx$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$
$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right]$$
$$= \frac{1}{2} \times \frac{\pi}{2}$$
$$= \frac{\pi}{4}$$
$$A_{OBC} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

∴ $A_{OBC} = A_{OAB}$

∴ curve divides rectangle into two regions of equal area.

ii.
$$f(x) = e^x + x + 1$$
$$f'(x) = e^x + 1$$
$$x_2 = -1\cdot5 - \frac{f(-1\cdot5)}{f'(-1\cdot5)}$$

∴ $x_2 = \underline{-1\cdot27}$

Question 4

a) $1\times 3 + 2\times 3^2 + \ldots + n \times 3^n$

$$= \frac{(2n-1)\,3^{n+1} + 3}{4}$$

Step 1 : let $n=1$

$LHS = 1\times 3 = 3$ $\quad RHS = \frac{(2-1)3^2+3}{4} = 3$

$\therefore$ true for $n=1$

Step 2: assume true for $n=k$

ie $S_k = \frac{(2k-1)\,3^{k+1}+3}{4}$

Step 3 : hence show true for $n=k+1$

ie show

$$S_{k+1} = \frac{[2(k+1)-1]\,3^{k+1+1}+3}{4} = \frac{(2k+1)\,3^{k+2}+3}{4}$$

$S_{k+1} = S_k + T_{k+1}$

$$S_{k+1} = \frac{(2k-1)3^{k+1}+3}{4} + (k+1)3^{k+1}$$

$$= \frac{(2k-1)3^{k+1}+3+4(k+1)3^{k+1}}{4}$$

$$= \frac{3^{k+1}(2k-1+4k+4)+3}{4}$$

$$= \frac{3^{k+1}(6k+3)+3}{4}$$

$$= \frac{3^{k+1}\times 3^1(2k+1)+3}{4}$$

$$= \frac{3^{k+2}(2k+1)+3}{4}$$

Step 4 : Since true for $n=1$ then from step 3 true for $n=1+1=2$ and so on for all $n \geq 1$.

b) $\tan^{-1} y = 2\tan^{-1} x$

let $\alpha = \tan^{-1} y \quad \beta = \tan^{-1} x$

$\tan\alpha = y \quad \tan\beta = x$

so $\alpha = 2\beta$

$\tan\alpha = \tan 2\beta$

$$\tan\alpha = \frac{2\tan\beta}{1-\tan^2\beta}$$

$$\therefore\ y = \frac{2x}{1-x^2}$$

c) $u = e^{2x}$ $\quad$ when $x=\ln 2$, $u = e^{2\ln 2} = 4$

$\frac{du}{dx} = 2e^{2x}$

$\frac{du}{2} = e^{2x}\,dx$ $\quad$ when $x=0$, $u=e^0 = 1$

$$\int_0^{\ln 2} \frac{e^{2x}}{1+e^{4x}}\,dx$$

$$= \frac{1}{2}\int_1^4 \frac{1}{1+u^2}\,du$$

$$= \frac{1}{2}\left[\tan^{-1} u\right]_1^4$$

$$= \frac{1}{2}\left[\tan^{-1} 4 - \tan^{-1} 1\right]$$

$$= \frac{1}{2}\tan^{-1} 4 - \frac{\pi}{8}$$

$\therefore$ $a = 4$ & $b = \frac{\pi}{8}$

Question 5

a) i. $T = 20 + Ae^{-kt}$

$\frac{dT}{dt} = -kAe^{-kt}$

$= -k(T-20)$ since $Ae^{-kt} = T-20$

ii. when $t=0$, $T = 100$

$100 = 20 + Ae^{0}$

$A = 80$

iii. $T = 20 + 80e^{-kt}$

when $t = 10$, $T = 70$

$70 = 20 + 80e^{-10k}$

$50 = 80e^{-10k}$

$\frac{5}{8} = e^{-10k}$

$\ln\frac{5}{8} = -10k$

$\therefore \quad k = \frac{1}{10}\ln\frac{5}{8}$

iv. when $T = 60$, $t = ?$

$60 = 20 + 80e^{-kt}$

$40 = 80e^{-kt}$

$\frac{1}{2} = e^{-kt}$

$t = \ln\frac{1}{2} \div -k$

$\therefore \quad t = 15$ min

b) $M(2am, am^2)$

$N(2an, an^2)$

i. $m_{MN} = \frac{am^2 - an^2}{2am - 2an}$

$= \frac{a(m+n)(m-n)}{2a(m-n)}$

$= \frac{1}{2}(m+n)$

Equation of chord MN:

$y - am^2 = \frac{1}{2}(m+n)(x - 2am)$

$2y - 2am^2 = (m+n)x - 2am^2 - 2amn$

$2y = (m+n)x - 2amn$

$\therefore \quad y = \frac{1}{2}(m+n)x - amn$

ii. $\text{midpoint}_{MN} = \left(\frac{2am+2an}{2}, \frac{am^2+an^2}{2}\right)$

$= \left[a(m+n), \frac{a(m^2+n^2)}{2}\right]$

iii. $(0,2)$ satisfies eqtn i.

$2 = 0 - amn$

$mn = -\frac{2}{a}$

from ii.

$x = a(m+n)$ $\qquad y = \frac{a(m^2+n^2)}{2}$

$\frac{x}{a} = m+n$ $\qquad \frac{2y}{a} = m^2+n^2$

$(m+n)^2 = m^2 + 2mn + n^2$

$\left(\frac{x}{a}\right)^2 = \frac{2y}{a} + 2 \times -\frac{2}{a}$

$\frac{x^2}{a^2} = \frac{2y}{a} - \frac{4}{a}$

$x^2 = 2ay - 4a$

$\therefore \quad x^2 = 2a(y-2)$

Question 6

a) $P(x) = ax(x+4)(x-4)$

$P(3) = 21$

$3a(3+4)(3-4) = 21$

$a = -1$

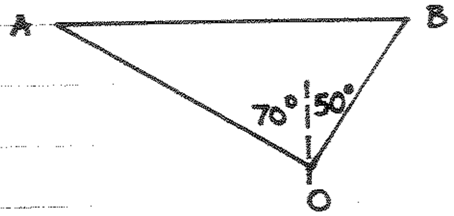
$\therefore P(x) = -x(x+4)(x-4)$

b) i. 300 km in 1 h

$= \dfrac{300}{6}$ km in 10 min

$= 50$ km

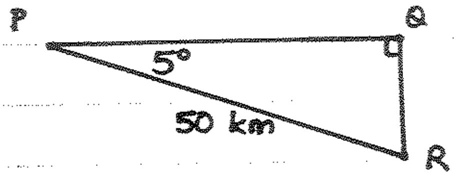
ii.



$\angle AOB = 70^\circ + 50^\circ$

$= 120^\circ$

iii.

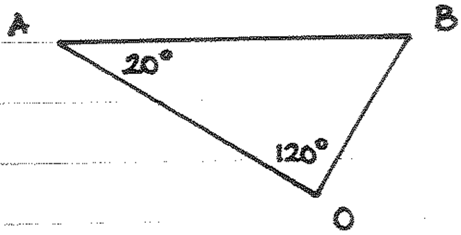


$\cos 5^\circ = \dfrac{PQ}{50}$

$PQ = 50 \cos 5^\circ$

but $PQ = AB$

so $AB = 50 \cos 5^\circ$



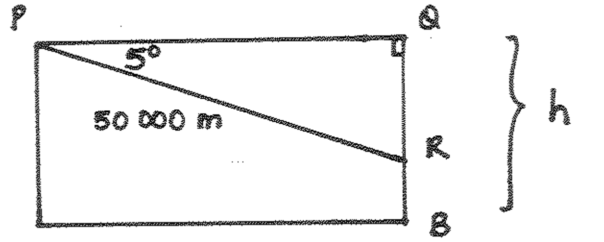
$\dfrac{OB}{\sin 20^\circ} = \dfrac{50 \cos 5^\circ}{\sin 120^\circ}$

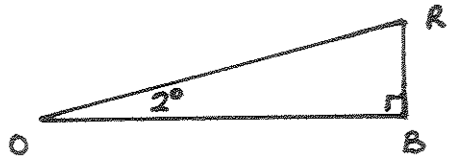
$OB = \dfrac{50 \cos 5^\circ \sin 20^\circ}{\sin 120^\circ}$

$= 19.6714..$ km

$= 19\,671.4..$ m

$\therefore$ $OB = 19\,670$ m





$\sin 5^\circ = \dfrac{QR}{50\,000}$

$QR = 50\,000 \sin 5^\circ$

$\tan 2^\circ = \dfrac{RB}{19670}$

$RB = 19\,670 \tan 2^\circ$

$h = QR + RB$

$= 5044.67..$

$\therefore$ $\underline{\underline{h = 5040}}$ m

c) $f(x) = u(x) - \ln\left[u(x) + 1\right]$

i. $f'(x) = u'(x) - \dfrac{u'(x)}{u(x)+1}$

$= \dfrac{u'(x)u(x) + u'(x) - u'(x)}{u(x)+1}$

$= \dfrac{u'(x)\,u(x)}{u(x)+1}$

ii. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x \sin x}{1 + \sin x}\, dx$

ie $u(x) = \sin x$

$$= \left[\sin x - \ln[\sin x + 1]\right]_0^{\frac{\pi}{2}}$$

$$= \left[\sin\frac{\pi}{2} - \ln\left[\sin\frac{\pi}{2} + 1\right]\right] - \left[\sin 0 - \ln[\sin 0 + 1]\right]$$

$$= \underline{\underline{1 - \ln 2}}$$

Question 7

a) i.

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$
$$= \frac{d}{dx}\left(\frac{1}{2} \times \frac{16}{x^2}\right)$$
$$= \frac{d}{dx}\left(8x^{-2}\right)$$
$$= -16x^{-3}$$

when $x = 1$

$$\underline{\underline{\ddot{x} = -16}}\ \text{m/s}^2$$

ii.

$$\frac{dx}{dt} = \frac{4}{x}$$
$$\frac{dt}{dx} = \frac{x}{4}$$
$$t = \frac{x^2}{8} + c$$

when $t = 0,\ x = 8$

$$0 = \frac{8^2}{8} + c$$
$$c = -8$$
$$t = \frac{x^2}{8} - 8$$

$$8t = x^2 - 64$$
$$x^2 = 8t + 64$$
$$x = \pm\sqrt{8t + 64}$$

$\therefore\ \underline{\underline{x = \sqrt{8t+64}}}$ (given $x > 0$)

iii. when $t = 2$

$$x = \sqrt{8.2 + 64}$$
$$= \sqrt{80}$$
$$= \underline{\underline{4\sqrt{5}}}$$

iv. The particle is moving to the right from $x = 8$ and is slowing down (since $v > 0$ and $a < 0$). The particle continues to slow but does not come to rest (since $v \neq 0$).

b) i.

$$x = Vt\cos\theta$$
$$y = Vt\sin\theta - \frac{1}{2}gt^2$$

$$\dot{y} = V\sin\theta - gt$$

At max pt : $\dot{y} = 0$

$$0 = V\sin\theta - gt$$
$$t = \frac{V\sin\theta}{g}$$

$$y = V \times \frac{V\sin\theta}{g} \times \sin\theta - \frac{1}{2}g \times \left(\frac{V\sin\theta}{g}\right)^2$$
$$= \frac{V^2\sin^2\theta}{g} - \frac{V^2\sin^2\theta}{2g}$$
$$= \frac{2V^2\sin^2\theta - V^2\sin^2\theta}{2g}$$

$$h = \frac{V^2\sin^2\theta}{2g}$$

ii. At T : $y = \dfrac{-V^2 \sin^2\theta}{2g}$

$$\frac{-V^2 \sin^2\theta}{2g} = Vt \sin\theta - \frac{1}{2} g t^2$$

$$-V^2 \sin^2\theta = 2g\,Vt \sin\theta - g^2 t^2$$

$$g^2t^2 - 2g\,Vt \sin\theta - V^2 \sin^2\theta = 0$$

$$t = \frac{2gV\sin\theta \pm \sqrt{4g^2V^2\sin^2\theta + 4g^2V^2\sin^2\theta}}{2g^2}$$

$$= \frac{2gV\sin\theta \pm \sqrt{8g^2V^2\sin^2\theta}}{2g^2}$$

$$= \frac{2gV\sin\theta \pm 2\sqrt{2}\,gV\sin\theta}{2g^2}$$

$$= \frac{V\sin\theta \pm \sqrt{2}\,V\sin\theta}{g}$$

$$\therefore t = \frac{V\sin\theta\,(1+\sqrt{2})}{g} \qquad (t > 0 \text{ only})$$

iii. $x = Vt\cos\theta$

$$= V \times \frac{V\sin\theta\,(1+\sqrt{2})}{g} \times \cos\theta$$

$$= \frac{V^2(1+\sqrt{2})\sin\theta\cos\theta}{g}$$

$$\therefore x = \frac{V^2(1+\sqrt{2})\sin 2\theta}{2g}$$