2007

Higher School Certificate

Preliminary Examination

Mathematics

Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- All necessary working should be shown in every question
- Board approved calculators may be used
- A table of standard integrals is provided
- Write your student number and/or name at the top of every page

Total marks - 72

Attempt All Questions 1 – 6

All Questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME.......................................

Student name / number ....

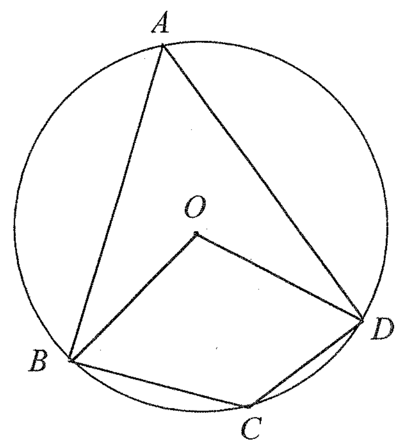
Marks

Question 1 **Begin a new booklet**

(a)(i) Show that $\dfrac{1}{p^2+pq}+\dfrac{1}{q^2+pq}=\dfrac{1}{pq}$. **1**

(ii) Hence express $\dfrac{1}{5}$ in the form $\dfrac{1}{a}+\dfrac{1}{b}$ for some positive integers a and b. **1**

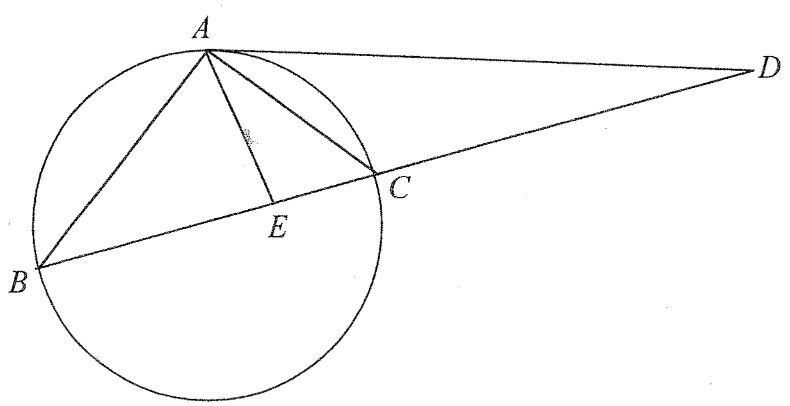
(b)



$ABCD$ is a quadrilateral inscribed in a circle with centre O. $\angle DAB = 36°$.
Find, giving reasons

(i) the size of $\angle DOB$. **1**
(ii) the size of $\angle BCD$. **1**

(c)



BC is a diameter of a circle. The tangent to the circle at A meets BC produced at D.
E is the point on BC such that AC bisects $\angle DAE$.

(i) Give a reason why $\angle DAC = \angle ABC$. **1**
(ii) Hence show that AE is perpendicular to BC. **3**

Student name / number ....

Marks

(d) The equation $x^3 + px^2 + qx + pq = 0$, where $p \neq 0$ and $q \neq 0$, has three real roots α, β and γ.

(i) By considering the relationships between the roots and the coefficients of the equation, show that $(\alpha+\beta+\gamma)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)=1$. **2**

(ii) Show that $-p$ is a root of the equation. Hence show that $q<0$. **2**

Question 2 **Begin a new booklet**

(a)(i) Show that $x^2+bx+c = \left(x+\frac{b}{2}\right)^2+\frac{4c-b^2}{4}$. **1**

(ii) Hence find the coordinates of the vertex of the parabola $y=x^2+bx+c$. **1**

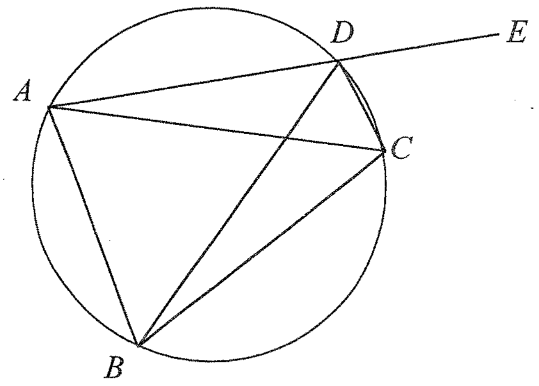
(b) The point $P(2,5)$ lies on the graph of the odd polynomial function $y=P(x)$. Find, with reasons,

(i) the remainder when $P(x)$ is divided by $(x-2)$. **1**

(ii) the remainder when $P(x)$ is divided by $(x+2)$. **1**

Marks

(c)

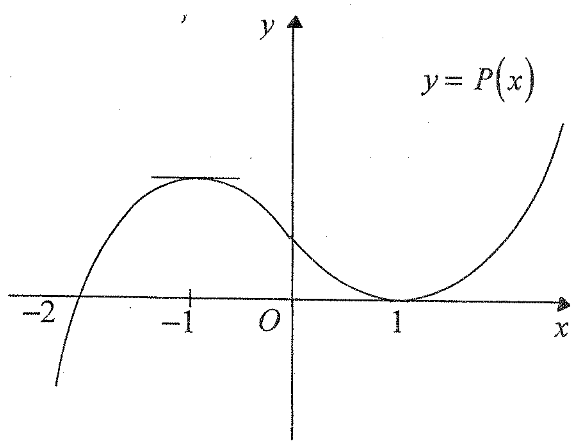


$ABCD$ is a quadrilateral inscribed in a circle. $CA = CB$. AD is produced to E.

(i) Give a reason why $\angle BDC = \angle BAC$. **1**

(ii) Hence show that DC bisects $\angle EDB$. **3**

(d)



The graph of the monic cubic polynomial $P(x)$ cuts the x-axis at $x = -2$, touches the x-axis at $x = 1$ and has a maximum turning point at $x = -1$.

(i) Show that $P(x) = x^3 - 3x + 2$. **2**

(ii) Find the set of values of k such that the equation $P(x) = k$ has three distinct real roots. **2**

Student name / number ....

Marks

Question 3 **Begin a new booklet**

(a) Solve the inequality $\dfrac{6}{x^2} \leq \dfrac{x-5}{x}$. **3**

(b) $A(-3,2)$ and $B(5,6)$ are two vertices of an acute angled triangle ABC. The side BC has equation $x+2y-17=0$. Find the size of the angle between the sides AB and BC correct to the nearest degree. **3**

(c) Solve the equation $\sin 2x + \cos x = 0$ for $0° \leq x \leq 360°$. **3**

(d)(i) Express $\tan\left(45° + \dfrac{x}{2}\right)$ in terms of t where $t = \tan\dfrac{x}{2}$. **1**

(ii) Hence show that $\dfrac{1+\sin x}{\cos x} = \tan\left(45° + \dfrac{x}{2}\right)$. **2**

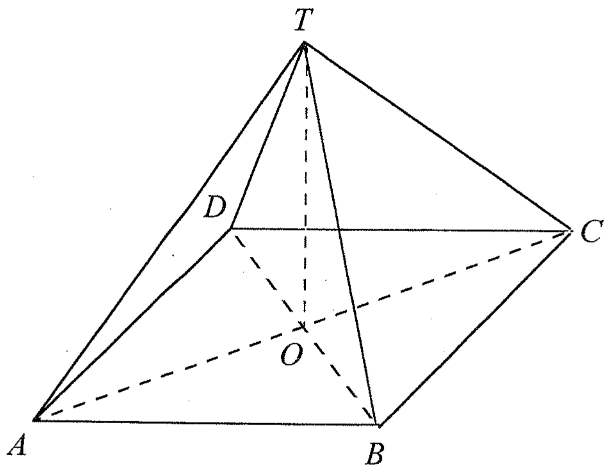
Student name / number ....

Marks

Question 4 **Begin a new booklet**

(a) $A(24, \log_{10} 24)$ and $B(3, \log_{10} 3)$ are two points. Find in simplest exact form the coordinates of the point P which divides the interval AB internally in the ratio $2 : 1$. **3**

(b)



The standard model Egyptian pyramid has a square base $ABCD$ whose diagonals intersect at O. The top of the pyramid lies directly above O. Its height OT is x units and the perimeter of its base $ABCD$ is $2\pi x$ units. Find, correct to the nearest minute, the angle of elevation of T from A. **3**

(c) Express $\tan 45°$ in terms of $\tan 22\frac{1}{2}°$ and hence find the value of $\tan 22\frac{1}{2}°$ in simplest exact form. **3**

(d)(i) Show that $\sqrt{2}\cos(x - 45°) = \cos x + \sin x$. **1**

(ii) Hence solve the equation $\cos x + \sin x = \dfrac{1}{\sqrt{2}}$ for $0° \le x \le 360°$. **2**

Student name / number ....

Question 5 **Begin a new booklet** **Marks**

(a) Sketch the graph of the function $f(x) = 2^{-|x|}$. **2**

(b) Find $\lim_{h \to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h}$. **2**

(c) Find the number of ways in which the letters of the word SQUARE can be arranged in a straight line

(i) without restriction. **1**

(ii) so that consonants occupy the two end positions. **1**

(iii) so that exactly two vowels are next to each other. **2**

(d) A group of students comprises 3 Year 11 girls, 2 Year 11 boys, 2 Year 12 girls and 2 Year 12 boys. Find the number of ways in which 6 members of this group can be chosen

(i) without restriction. **1**

(ii) so as to include more boys than girls. **1**

(iii) so as to include an equal number of Year 11 girls and Year 12 girls. **2**

Student name / number ....

Marks

Question 6 **Begin a new booklet**

(a) $P(2ap\,,ap^2)$ and $Q(2aq\,,aq^2)$ are two points which move on the parabola $x^2=4ay$ so that the chord PQ subtends a right angle at the origin.

(i) Show that PQ has equation $(p+q)x-2y=2apq$. **2**

(ii) Show that $pq=-4$ and hence show that PQ always passes through a fixed point on the y-axis. **2**

(b)(i) Use differentiation to show that the normal to the parabola $x^2=4ay$ at the point $T(2at\,,at^2)$ has equation $x+ty=2at+at^3$. **2**

(ii) Hence show that for $t\neq 0$ this normal meets the parabola again at the point $R(2ar\,,ar^2)$ where $r=\dfrac{-(t^2+2)}{t}$. **2**

(c)(i) Show that $\cos(k-1)x-\cos(k+1)x=2\sin kx\sin x$. **1**

(ii) Hence show that for $\sin x\neq 0$ **3**

$$\sin x+\sin 2x+\sin 3x+\ \dots\ +\sin nx\ =\ \frac{1+\cos x-\cos nx-\cos(n+1)x}{2\sin x}.$$

Question 1

a. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
i • selects appropriate common denominator and simplifies	1
ii • substitutes appropriate values for p and q then simplifies denominators	1

Answer

i. $$\frac{1}{p^2+pq}+\frac{1}{q^2+pq}=\frac{1}{p(p+q)}+\frac{1}{q(p+q)}$$
$$=\frac{q+p}{pq(p+q)}$$
$$=\frac{1}{pq}$$

ii. $p=5,\ q=1 \Rightarrow \frac{1}{5}=\frac{1}{5^2+5}+\frac{1}{1^2+5}$

$$\therefore \frac{1}{5}=\frac{1}{30}+\frac{1}{6}$$

b. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
i • states size of angle quoting appropriate circle property	1
ii • states size of angle quoting appropriate circle property	1

Answer

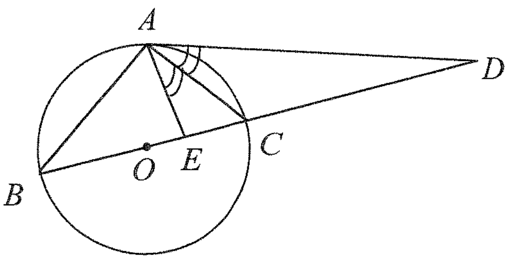
i. $\angle DOB = 72°$ (*$\angle$ subtended at the centre is twice $\angle$ subtended at the circumference by arc DB*)
ii. $\angle BCD = 144°$ (*opposite $\angle$'s of cyclic quadrilateral ABCD are supplementary*)

c. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
i • quotes alternate segment theorem	1
ii • deduces $\angle ABC = \angle EAC$	1
• explains why $\angle BAE + \angle ABE = 90°$	1
• deduces $AE \perp BC$ giving a reason	1

Answer



O is the centre of the circle.

i. The angle between a tangent to a circle and a chord drawn from the point of contact is equal to any angle subtended by that chord in the alternate segment.

ii. $\angle DAC = \angle EAC$ (*given AC bisects $\angle DAE$*)
$\therefore \angle ABC = \angle EAC$ (*both equal to $\angle DAC$*)
But $\angle BAC = 90°$ (*angle in a semicircle is a right angle*)
$\therefore \angle BAE + \angle EAC = 90°$ (*by addition of adjacent angles*)
$\therefore \angle BAE + \angle ABE = 90°$ (*$\angle ABE$, $\angle ABC$ same angle*)
$\therefore \angle AEB = 90°$ (*$\angle$ sum of ΔABE is 180°*)
$\therefore AE \perp BC$

d. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
i • expresses the sum of the reciprocals of the roots in terms of $\alpha\beta\gamma$ and $\alpha\beta+\beta\gamma+\gamma\alpha$	1
• substitutes for $\alpha+\beta+\gamma$, $\alpha\beta\gamma$ and $\alpha\beta+\beta\gamma+\gamma\alpha$ then simplifies given product	1
ii • shows $-p$ is a root of the equation	1
• deduces remaining roots are opposites and their product is q, hence $q<0$	1

Answer

i. α, β, γ are real roots of $x^3+px^2+qx+pq=0, \quad p\neq 0, q\neq 0$

$$\begin{aligned}(\alpha+\beta+\gamma)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)&=(\alpha+\beta+\gamma)\left(\frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma}\right)\\ &=(-p)\left(\frac{q}{-pq}\right)\\ &=1\end{aligned}$$

ii. $(-p)^3+p(-p)^2+q(-p)+pq=-p^3+p^3-qp+pq=0$

Hence $-p$ is a root of the equation.

Let α be the root $-p$. Then $\alpha+\beta+\gamma=-p \Rightarrow \beta+\gamma=0 \quad \therefore \gamma=-\beta$

and $\alpha\beta\gamma=-pq \Rightarrow \beta\gamma=q \quad \therefore q=-\beta^2$

But β is real and $pq\neq 0 \Rightarrow \beta\neq 0$. $\therefore \beta^2>0$ and hence $q<0$.

Question 2

a. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
i • completes the square or expands RHS and simplifies	1
ii • writes the coordinates of the vertex	1

Answer

i. $$\begin{aligned}x^2+bx+c&=x^2+2\frac{b}{2}x+\left(\frac{b}{2}\right)^2+c-\left(\frac{b}{2}\right)^2\\ &=\left(x+\frac{b}{2}\right)^2+c-\frac{b^2}{4}\\ &=\left(x+\frac{b}{2}\right)^2+\frac{4c-b^2}{4}\end{aligned}$$

ii. This quadratic expression has a minimum value of $\frac{4c-b^2}{4}$ when $x=-\frac{b}{2}$.

Hence the parabola has vertex $\left(-\frac{b}{2}, \frac{4c-b^2}{4}\right)$

b. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
i • applies remainder theorem	1
ii • deduces $P(-2)=-5$ and applies remainder theorem	1

Answer

i. $P(2)=5$. Hence by remainder theorem, division of $P(x)$ by $(x-2)$ leaves a remainder of 5.

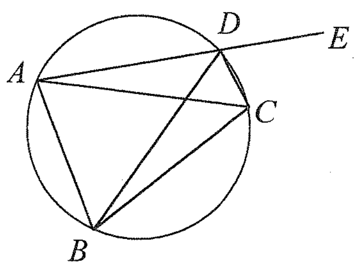
ii. $P(x)$ odd $\Rightarrow P(-2)=-P(2)=-5$. Hence, applying the remainder theorem, remainder on division by $(x+2)$ is -5.

c. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
i • quotes appropriate circle property	1
ii • deduces $\angle BAC = \angle ABC$, quoting appropriate property of an isosceles triangle	1
• deduces $\angle EDC = \angle ABC$, quoting appropriate property of cyclic quadrilateral	1
• deduces $\angle BDC = \angle EDC$ to prove required result	1

Answer



i. $\angle BDC = \angle BAC$ *since angles subtended at the circumference by the same arc BC are equal.*

ii. $\angle BAC = \angle ABC$ (*in* ΔABC, $\angle$'s *opp. equal sides CA, CB are equal*)

$\angle EDC = \angle ABC$ (*exterior* $\angle$ *of cyclic quad. ABCD is equal to opposite interior* $\angle$)

$\therefore \angle BAC = \angle EDC$ (*both equal to* $\angle ABC$)

$\therefore \angle BDC = \angle EDC$ (*both equal to* $\angle BAC$)

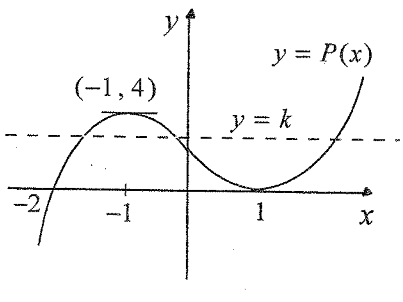
Hence DC bisects $\angle EDB$.

d. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
i • writes $P(x)$ in factored form	1
• expands and simplifies this expression	1
ii • finds y coordinate of maximum turning point	1
• deduces $0<k<4$ by considering nature of intersections of line $y=k$ with the curve	1

Answer



i. $P(x)=(x-1)^2(x+2)$

$=(x^2-2x+1)(x+2)$

$=x^3-2x^2+x+2x^2-4x+2$

$=x^3-3x+2$

ii. $P(x)=k$ has 3 distinct real roots if line $y=k$ cuts the curve in 3 distinct points. Since $P(-1)=4$, this occurs for $0<k<4$.

Question 3

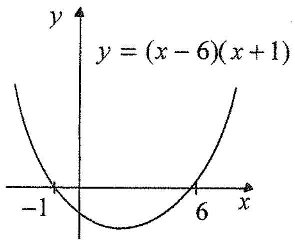
a. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• writes equivalent quadratic inequality	1
• applies an appropriate method of solution, obtaining at least one inequality for x	1
• writes two inequalities for x, indicating how they are to be combined	1

Answer

$$\frac{6}{x^2} \le \frac{x-5}{x}$$
$$6 \le x(x-5), \quad x \ne 0$$
$$0 \le x^2 - 5x - 6$$
$$0 \le (x-6)(x+1)$$



$\therefore$ by inspection of the graph,

$$x \le -1 \quad \text{or} \quad x \ge 6$$

b. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
• finds the gradients of AB and BC	1
• writes a numerical expression for $\tan \angle ABC$	1
• finds the size of the angle to the nearest degree	1

Answer

$A(-3, 2)$ and $B(5, 6)$

Gradient of side AB is $\frac{4}{8} = \frac{1}{2}$.

BC: $x + 2y - 17 = 0$ has gradient $-\frac{1}{2}$.

Since the triangle is acute angled,

$$\tan \angle ABC = \left| \frac{\frac{1}{2} - (-\frac{1}{2})}{1 + \frac{1}{2}(-\frac{1}{2})} \right|$$

$$\therefore \tan \angle ABC = \frac{1}{(\frac{3}{4})} = \frac{4}{3}$$

$\therefore \angle ABC \approx 53°$ (to the nearest degree)

c. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
• writes an equivalent equation in terms of $\sin x$ and $\cos x$	1
• shows one possibility is $\cos x = 0$ giving corresponding solutions for x	1
• writes remaining solutions for x derived from $\sin x = -\frac{1}{2}$	1

Answer

$$\sin 2x + \cos x = 0, \quad 0° \le x \le 360°$$
$$2\sin x \cos x + \cos x = 0$$
$$\cos x\,(2\sin x + 1) = 0$$

$$\therefore \cos x = 0 \ \text{or} \ \sin x = -\tfrac{1}{2}, \quad 0° \le x \le 360°$$
$$\therefore x = 90°,\ 270°,\ 210°,\ 330°$$

d. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
i • uses compound angle formula to obtain required result	1
ii • substitutes for $\sin x$ and $\cos x$ in terms of t	1
• rearranges and simplifies to show required result	1

Answer

i. $$\tan\left(45° + \frac{x}{2}\right) = \frac{\tan 45° + \tan\frac{x}{2}}{1 - \tan 45° \tan\frac{x}{2}}$$
$$= \frac{1+t}{1-t}$$

ii. $$\frac{1+\sin x}{\cos x} = \left(1 + \frac{2t}{1+t^2}\right) \div \frac{1-t^2}{1+t^2}$$
$$= \frac{1+t^2+2t}{1+t^2} \times \frac{1+t^2}{1-t^2}$$
$$= \frac{(1+t)^2}{(1+t)(1-t)}$$
$$= \frac{(1+t)}{(1-t)}$$
$$\therefore \frac{1+\sin x}{\cos x} = \tan\left(45° + \frac{x}{2}\right)$$

Question 4

a. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
• writes x coordinate of P in simplest form	1
• writes numerical expression for exact y coordinate of P	1
• expresses y coordinate in simplest exact form	1

Answer

$A(24\,,\ \log_{10} 24)$ $\qquad$ $B(3\,,\ \log_{10} 3)$

2 : 1

$$P\left(\frac{6+24}{2+1}\ ,\ \frac{2\log_{10} 3 + \log_{10} 24}{2+1}\right)$$

But $$2\log_{10} 3 + \log_{10} 24 = \log_{10}(3^2 \times 24)$$
$$= \log_{10}(3^3 \times 2^3)$$
$$= 3\log_{10} 6$$

$\therefore P$ has coordinates $(10\,,\ \log_{10} 6)$

b. Outcomes assessed : P4, PE1

Marking Guidelines

Criteria	Marks
• finds AO in terms of x	1
• finds $\tan\angle TAO$	1
• finds the required angle of elevation correct to the nearest minute	1

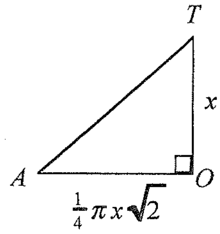
Answer

Using Pythagoras, the diagonal d of a square of side s is given by $d^2 = 2s^2 \Rightarrow d = s\sqrt{2}$.
Also the diagonals of a square bisect each other.

$\therefore AO = \frac{1}{2}AC = \frac{1}{2}\sqrt{2}\ AB$.

But $AB = \frac{1}{4}(2\pi x)$.

Hence $AO = \frac{1}{4}\pi x\sqrt{2}$.



$\therefore \tan\angle TAO = \dfrac{x}{\frac{1}{4}\pi x\sqrt{2}} = \dfrac{4}{\pi\sqrt{2}}$

Hence angle of elevation $\angle TAO$ is $42°0'$ (to the nearest minute).

c. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
• expresses $\tan 45°$ in terms of $\tan 22\frac{1}{2}°$	1
• writes a quadratic equation with one root $\tan 22\frac{1}{2}°$	1
• solves this equation to find this root in simplest exact form	1

Answer

$$\tan 45° = \frac{2\tan 22\frac{1}{2}°}{1-\tan^2 22\frac{1}{2}°}$$

Let $\tan 22\frac{1}{2}° = t$.

Then $\dfrac{2t}{1-t^2} = 1$ and $t > 0$.

$$2t = 1 - t^2$$
$$t^2 + 2t - 1 = 0$$
$$\therefore\ t = \frac{-2 \pm \sqrt{8}}{2}$$

$$\therefore t > 0 \Rightarrow t = \frac{-2+2\sqrt{2}}{2} = -1+\sqrt{2}$$

Hence $\tan 22\frac{1}{2}° = \sqrt{2} - 1$

d. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
i • uses compound angle formula to obtain result	1
ii • finds one solution for x from value of $\cos(x-45°)$	1
• finds second solution for x	1

Answer

i. $\cos(x-45°) = \cos x\cos 45° + \sin x\sin 45°$

$= \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x$

$\therefore \sqrt{2}\cos(x-45°) = \cos x + \sin x$

ii. $\cos x + \sin x = \dfrac{1}{\sqrt{2}}$, $0° \le x \le 360°$

$\cos(x-45°) = \frac{1}{2}$, $-45° \le x - 45° \le 315°$

$x - 45° = 60°,\ 300°$

$x = 105°,\ 345°$

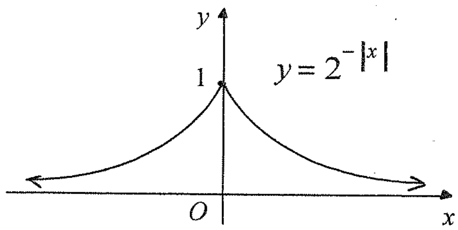
Question 5

a. Outcomes assessed : P5

Marking Guidelines

Criteria	Marks
• sketches curve of correct shape for $x \geq 0$ with y intercept 1 and x-axis as asymptote	1
• sketches correct shape for $x \leq 0$, with symmetry in the y-axis	1

Answer



b. Outcomes assessed : P8

Marking Guidelines

Criteria	Marks
• simplifies algebraic expression in x and h	1
• takes limit	1

Answer

$$\frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \frac{x-(x+h)}{hx(x+h)} = \frac{-h}{hx(x+h)} = \frac{-1}{x(x+h)}$$

$$\therefore \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0}\frac{-1}{x(x+h)} = \frac{-1}{x^2}$$

c. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
i • writes the number of arrangements	1
ii • writes the number of arrangements	1
iii • applies a systematic counting procedure with some success	1
• calculates the number of arrangements.	1

Answer

i. $6! = 720$

ii. ${}^3P_2 \times 4! = 144$

iii. Select the two vowels V_1, V_2 to be side by side in order ${}^3P_2$ ways.

Then $\boxed{V_1, V_2}$, V_3, S, Q, R can be arranged in $5!$ ways.

In $2\times 4!$ of these arrangements, V_3 is next to $\boxed{V_1, V_2}$.

Hence required number of arrangements is ${}^3P_2\left(5!-2\times 4!\right) = 432$

d. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
i • writes down number of ways	1
ii • writes down number of ways	1
iii • applies a systematic counting procedure with some success	1
• calculates the number of ways	1

Answer

i. 6 to be chosen from a group of 9, hence ${}^{9}C_{6} = 84$ ways

ii. Group has 5 girls and 4 boys. Hence chosen 6 comprises 4 boys and 2 girls. $\therefore\ {}^{4}C_{4} \times {}^{5}C_{2} = 10$ ways.

iii. Group has 3 Yr 11 girls, 2 Yr 12 girls and 4 boys. The possible choices are listed below:

Yr 11 girls	*Yr 12 girls*	*boys*	*number of ways*
1	1	4	${}^{3}C_{1} \times {}^{2}C_{1} \times {}^{4}C_{4} = 6$
2	2	2	${}^{3}C_{2} \times {}^{2}C_{2} \times {}^{4}C_{2} = 18$

Hence $6 + 18 = 24$ ways.

Question 6

a. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
i • finds gradient of PQ	1
• shows equation of PQ has required form	1
ii • shows $pq = -4$	1
• shows y intercept of PQ is independent of p and q	1

Answer

i. $P(2ap, ap^2)$, $Q(2aq, aq^2)$

$$\text{gradient } PQ = \frac{a(p^2 - q^2)}{2a(p-q)} = \frac{(p-q)(p+q)}{2(p-q)} = \tfrac{1}{2}(p+q)$$

PQ has equation

$$y - ap^2 = \tfrac{1}{2}(p+q)(x - 2ap)$$
$$2y - 2ap^2 = (p+q)x - 2ap(p+q)$$
$$2y = (p+q)x - 2apq$$
$$2apq = (p+q)x - 2y$$

ii. gradient $OP = \dfrac{ap^2}{2ap} = \dfrac{p}{2}$

Similarly OQ has gradient $\dfrac{q}{2}$

$$\therefore OP \perp OQ \Rightarrow \frac{p}{2} \times \frac{q}{2} = -1$$
$$\therefore pq = -4$$

Hence PQ has equation $(p+q)x - 2y = -8a$
and y intercept $4a$, which is independent of p and q.

$\therefore PQ$ passes through the fixed point $(0, 4a)$.

b. Outcomes assessed : PE4

Marking Guidelines

Criteria	Marks
i • uses differentiation to find the gradient of the normal	1
• shows the equation of the normal has the required form	1
ii • substitutes coordinates of R into equation of normal	1
• rearranges to obtain r in terms of t	1

Answer

i. $x = 2at \qquad y = at^2$

$\dfrac{dx}{dt} = 2a \qquad \dfrac{dy}{dt} = 2at$

$\therefore \dfrac{dy}{dx} = \dfrac{2at}{2a} = t$

Normal at T has gradient $-\dfrac{1}{t}$ and equation

$$y - at^2 = -\frac{1}{t}\left(x - 2at\right)$$
$$t\,y - at^3 = -x + 2at$$
$$x + t\,y = 2at + at^3$$

ii. $R\left(2ar\,,\,ar^2\right)$ lies on this normal if

$$2ar + t\,ar^2 = 2at + at^3$$
$$2(r-t) = t\,(t^2 - r^2)$$
$$2(r-t) = -t\,(r-t)(r+t)$$

R and T are distinct points on the parabola.
$\therefore r \neq t$. Hence, cancelling $(r-t)$,

$$r + t = -\frac{2}{t}$$
$$r = -\left(t + \frac{2}{t}\right)$$
$$r = \frac{-(t^2+2)}{t}$$

c. Outcomes assessed : P4, PE6

Marking Guidelines

Criteria	Marks
i • uses compound angle formula on each term on LHS then simplifies	1
ii • repeatedly uses result for $k = 1, 2, 3, \ldots, n$	1
• adds and simplifies	1
• divides by $2\sin x$ to obtain required result	1

Answer

i. $\cos(k-1)x = \cos(kx - x) = \cos kx \cos x + \sin kx \sin x$

$\cos(k+1)x = \cos(kx + x) = \cos kx \cos x - \sin kx \sin x$

$\therefore \cos(k-1)x \ - \ \cos(k+1)x = 2\sin kx \sin x$

ii.
$$2\sin x \sin x = 1 \ - \ \cos 2x$$
$$2\sin 2x \sin x = \cos x - \cos 3x$$
$$2\sin 3x \sin x = \cos 2x - \cos 4x$$
$$2\sin 4x \sin x = \cos 3x - \cos 5x$$
$$\ldots$$
$$2\sin(n-1)x \sin x = \cos(n-2)x - \cos nx$$
$$2\sin nx \sin x = \cos(n-1)x - \cos(n+1)x$$

$\therefore 2\sin x\,\sin x + 2\sin 2x\,\sin x + 2\sin 3x\,\sin x + \ldots + 2\sin nx\,\sin x = 1 + \cos x - \cos nx - \cos(n+1)x$

$$\therefore \sin x + \sin 2x + \sin 3x + \ldots + \sin nx\ x = \frac{1 + \cos x - \cos nx - \cos(n+1)x}{2\sin x}$$

Independent Preliminary Examination 2007 Mathematics Extension 1 Mapping Grid

Question	Marks	Content	Syllabus Outcomes	Targeted Performance Bands
1 a i	1	Basic arithmetic and algebra	P4	E2-E3
ii	1	Basic arithmetic and algebra	P4	E2-E3
b i	1	Circle geometry	PE3	E2-E3
ii	1	Circle geometry	PE3	E2-E3
c i	1	Circle geometry	PE3	E2-E3
ii	3	Circle geometry	PE2, PE3	E2-E3
d i	2	Polynomials	PE3	E2-E3
ii	2	Polynomials	PE2, PE3	E2-E3
2 a i	1	Quadratic polynomial and the parabola	P4	E2-E3
ii	1	Quadratic polynomial and the parabola	P4	E2-E3
b i	1	Polynomials	PE3	E2-E3
ii	1	Polynomials	PE3	E2-E3
c i	1	Circle geometry	PE3	E2-E3
ii	3	Circle geometry	PE2, PE3	E2-E3
d i	2	Polynomials	PE2, PE3	E2-E3
ii	2	Polynomials	PE2, PE3	E2-E3
3 a	3	Inequalities	PE3	E2-E3
b	3	Angle between two lines	P4	E2-E3
c	3	Further trigonometry	P4	E2-E3
d i	1	Further trigonometry	P4	E2-E3
ii	2	Further trigonometry	P4	E2-E3
4 a	3	Division of an interval	P4	E2-E3
b	3	3D trigonometry	P4, PE1	E3-E4
c	3	Further trigonometry	P4	E2-E3
d i	1	Further trigonometry	P4	E2-E3
ii	2	Further trigonometry	P4	E2-E3
5 a	2	Real functions	P5	E2-E3
b	2	Differentiation	P8	E2-E3
c i	1	Permutations and combinations	PE3	E2-E3
ii	1	Permutations and combinations	PE3	E2-E3
iii	2	Permutations and combinations	PE2, PE3	E3-E4
d i	1	Permutations and combinations	PE3	E2-E3
ii	1	Permutations and combinations	PE3	E2-E3
iii	2	Permutations and combinations	PE2, PE3	E3-E4
6 a i	2	Parametric representation	PE3	E3-E4
ii	2	Parametric representation	PE3	E3-E4
b i	2	Parametric representation	PE4	E3-E4
ii	2	Parametric representation	PE4	E3-E4
c i	1	Further trigonometry	P4	E2-E3
ii	3	Further trigonometry	PE6	E3-E4