CATHOLIC SECONDARY SCHOOLS ASSOCIATION

2009 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS EXTENSION 1

Question 1 (12 marks)
(a) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• applies the Remainder Theorem or equivalent progress towards solution	1
• finds correct remainder	1

Sample Answer:

$$P(x) = x^3 - 3x^2 + 3x - 5$$

By the Remainder Theorem $P(2) =$ remainder

$$\therefore \text{remainder} = 8 - 12 + 6 - 5$$
$$= -3$$

OR

Correct division of polynomial.

(b) (2 marks)
Outcomes assessed: HE6, HE7
Targeted Performance Bands: E2-E3

Criteria	Marks
• correct trigonometric substitution in integral	1
• finds a correct primitive (+*C* not necessary)	1

Sample Answer:

$$\int \sin^2 6x\, dx = \frac{1}{2}\int (1 - \cos 12x)\, dx$$
$$= \frac{1}{2}\left(x - \frac{1}{12}\sin 12x\right) + C$$
$$= \frac{x}{2} - \frac{\sin 12x}{24} + C$$

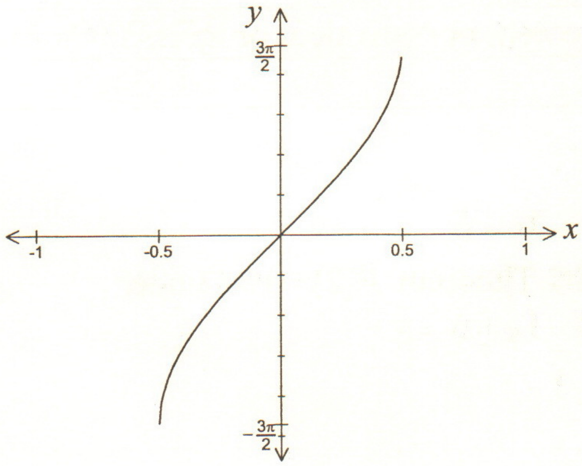
(c) (3 marks)
Outcomes assessed: HE4
Targeted Performance Bands: E2-E3

Criteria	Marks
• draws correctly shaped graph	1
• identifies correct domain	1
• identifies correct range	1

Sample Answer:

$$y = 3\sin^{-1}(2x)$$

domain: $\frac{-1}{2} \le x \le \frac{1}{2}$ range: $\frac{-3\pi}{2} \le y \le \frac{3\pi}{2}$



(d) (i) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• uses correct trigonometric identity	1
• substitutes correctly and determines correct equation	1

Sample Answer:

$x = \cos t$

$y = 3 + \sin t \quad \Rightarrow \quad \sin t = y - 3$

substitute into $\cos^2 t + \sin^2 t = 1$

$x^2 + (y-3)^2 = 1$

(d) (ii) (1 mark)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• correctly describes locus	1

Sample Answer:

Geometrically the locus is a circle with centre $(0, 3)$ and radius 1.

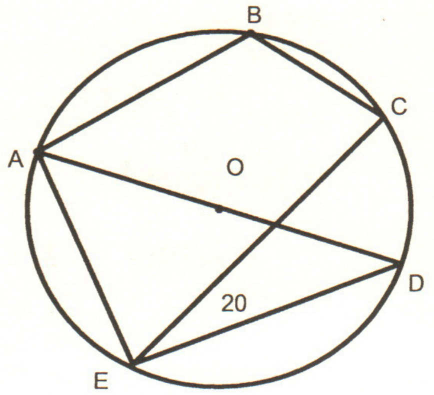
(e) (2 marks)

Outcomes assessed: PE2, PE3

Targeted Performance Bands: E2-E3

Criteria	Marks
• finds $\angle AED$, giving correct reason	1
• finds $\angle ABC$, giving correct reason	1

Sample Answer:



$\angle AED = 90^0$ (angle in a semicircle, AD is a diameter)

$\therefore \angle AEC = 70^0$

$\angle ABC = 110^0$ (opposite angles of cyclic quadrilateral $ABCE$ are supplementary)

Question 2 (12 marks)

(a) (1 mark)

Outcomes assessed: PE2

Targeted Performance Bands: E2-E3

Criteria	Marks
• gives correct result	1

Sample Answer:

$$\lim_{x\to 0}\frac{\sin 3x}{x} = 3\lim_{x\to 0}\frac{\sin 3x}{3x}$$

$$= 3\times 1 \qquad \text{using } \lim_{x\to 0}\tfrac{\sin x}{x} = 1$$

$$= 3$$

(b) (3 marks)

Outcomes assessed: HE6

Targeted Performance Bands: E2-E3

Criteria	Mark
• rewrites the integral using the substitution	1
• finds the new limits	1
• evaluates the integral correctly (correct numerical equivalence)	1

Sample Answer:

$$\int_1^2 \frac{x}{3x-1}\,dx = \frac{1}{9}\int_1^2 \frac{3x}{3x-1}\times 3dx$$

$$= \frac{1}{9}\int_2^5 \frac{u+1}{u}\,du$$

$$= \frac{1}{9}\int_2^5 \left(1+\frac{1}{u}\right)du$$

$$= \frac{1}{9}\left[u+\ln u\right]_2^5$$

$$= \frac{1}{9}\left[5+\ln 5-(2+\ln 2)\right]$$

$$= \frac{1}{9}\left(3+\ln\tfrac{5}{2}\right)$$

$$= \tfrac{1}{3}+\tfrac{1}{9}\ln\tfrac{5}{2}$$

$u = 3x-1$ $\quad 3x = u+1$

$\frac{du}{dx} = 3$

Limits $\quad x = 2 \Rightarrow u = 5$

$\quad x = 1 \Rightarrow u = 2$

(c) (4 marks)

Outcomes assessed: HE7

Targeted Performance Bands: E2-E3

Criteria	Mark
• uses logarithmic laws	1
• establishes the quadratic equation	1
• solves the quadratic equation	1
• gives correct solution	1

Sample Answer

for valid solutions $x > 2$

$$\ln(2x+3)+\ln(x-2) = 2\ln(x+4)$$

$$\ln(2x+3)(x-2) = \ln(x+4)^2$$

$$2x^2 - x - 6 = x^2 + 8x + 16$$

$$x^2 - 9x - 22 = 0$$

$$(x+2)(x-11) = 0$$

$$\therefore x = -2 \text{ or } x = 11$$

but $x = -2$ is not valid $\therefore x = 11$ is the only solution

(d) (i) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Mark
• uses combinations correctly or significant progress towards answer	1
• gives correct answer	1

Sample Answer:

Girls can be selected in ${}^7C_3 = 35$ ways

Boys can be selected in ${}^6C_2 = 15$ ways

There are ${}^7C_3 \times {}^6C_2 = 525$ groups of 5.

(d) (ii) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• calculates the number of ways that the boys can stand together	1
• finds the correct probability	1

Sample Answer:

If the boys stand together then there are $2! = 2$ ways to arrange themselves.
In the line there are 3 girls and the group of boys to be arranged $\Rightarrow 4! = 24$ arrangements.
$\therefore 2! \times 4! = 48$ ways of the boys standing together in the line.

If no restrictions the 5 can be arranged in $5! = 120$ ways in a line.

$$\text{P(boys stand together)} = \frac{48}{120} = \frac{2}{5}.$$

Question 3 (12 marks)

(a) (3 marks)

Outcomes assessed: PE3

Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes correct quadratic or other correct significant step towards solution	1
• further significant step towards solution	1
• finds solution	1

Sample Answer:

$$\frac{x^2-4}{x+3} < x-4 \qquad \times (x+3)^2 \qquad x \neq -3$$

$$(x+3)(x^2-4) < (x-4)(x+3)^2$$

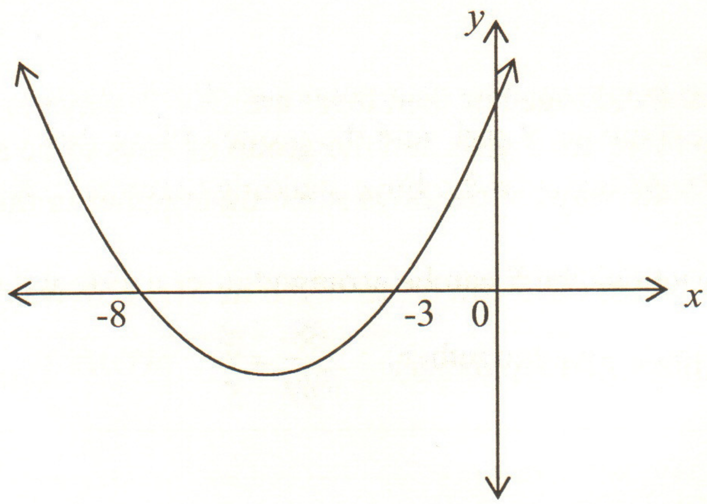
$$(x+3)(x^2-4) - (x-4)(x+3)^2 < 0$$

$$(x+3)(x^2-4-(x-4)(x+3)) < 0$$

$$(x+3)(x^2-4-(x^2-x-12)) < 0$$

$$(x+3)(x+8) < 0$$

$$-8 < x < -3$$



(b) (4 marks)

Outcomes assessed: HE2

Targeted Performance Bands: E2-E3

Criteria	Marks
• establishes the truth of $S(1)$	1
• establishes the result for $S(k)$	1
• substitutes result in $S(k+1)$	1
• deduces the required result	1

Sample Answer:

Let $S(n)$ be the statement $3^{3n} + 2^{n+2}$ is divisible by 5

Consider $S(1)$: $3^3 + 2^3 = 35$ which is divisible by 5.

Hence $S(1)$ is true

If $S(k)$ is true: $3^{3k} + 2^{k+2} = 5M$ where M is an integer *

RTP $S(k+1)$ is true i.e. prove $3^{3(k+1)} + 2^{(k+1)+2} = 5Q$ where Q is an integer

$LHS = 3^{3k+3} + 2^{k+3}$

$= 3^3 \times 3^{3k} + 2 \times 2^{k+2}$

$= 27(5M - 2^{k+2}) + 2 \times 2^{k+2}$ if $S(k)$ is true using *

$= 27 \times 5M - 27 \times 2^{k+2} + 2 \times 2^{k+2}$

$= 5 \times 27M - 25 \times 2^{k+2}$

$= 5(27M - 5 \times 2^{k+2})$

$= 5Q$ where Q is an integer since M and k are integers

Hence if $S(k)$ then $S(k+1)$ is true. Thus since $S(1)$ is true it follows by induction that $S(n)$ is true for positive integral n.

OR

$LHS = 3^{3k+3} + 2^{k+3}$

$= 3^3 \times 3^{3k} + 2 \times 2^{k+2}$

$= 25 \times 3^{3k} + 2 \times 3^{3k} + 2 \times 2^{k+2}$

$= 25 \times 3^{3k} + 2(3^{3k} + 2^{k+2})$

$= 25 \times 3^{3k} + 2 \times 5M$ if $S(k)$ is true using *

$= 5(5 \times 3^{3k} + 2M)$

$= 5Q$ where Q is an integer since M and k are integers

Conclusion as above

(c) (i) (3 marks)
Outcomes assessed: HE5
Targeted Performance Bands: E3-E4

Criteria	Marks
• progress towards correct differentiation	1
• finds a correct expression for acceleration	1
• shows correct relationship	1

Sample Answer:

$$v = \frac{2}{1+3x}$$

$$\frac{1}{2}v^2 = \frac{1}{2}\frac{4}{(1+3x)^2}$$

$$= 2(1+3x)^{-2}$$

Now

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= 2\times -2(1+3x)^{-3}\times 3$$

$$= \frac{-12}{(1+3x)^3}$$

$$= -12\times\frac{8}{(1+3x)^3}\times\frac{1}{8}$$

$$= -\frac{12}{8}v^3$$

$$= -\frac{3}{2}v^3$$

$\therefore a$ varies directly as v^3

(c) (ii) (2 marks)
Outcomes assessed: HE7
Targeted Performance Bands: E2-E3

Criteria	Marks
• describes initial motion	1
• describes motion as $t \to \infty$	1

Sample Answer:

Initially $v = 2$ cms^{-1} $\therefore$ the particle moves in a positive direction from the origin.

As t increases, x increases and v decreases.
As $t \to \infty$, the particle continues in a positive direction with $v \to 0$.

Question 4 (12 marks)

(a) (2 marks)

Outcomes assessed: PE3, HE7

Targeted Performance Bands: E2-E3

Criteria	Marks
• progress towards solution	1
• finds correct approximation (correct numerical equivalence)	1

Sample Answer:

$$f(x) = e^x - x - 2$$

$$\therefore f'(x) = e^x - 1$$

Let $x_1 = 1.2$

$$f(x_1) = e^{1.2} - 1.2 - 2 = 0.1201169...$$

$$f'(x_1) = e^{1.2} - 1 = 2.3201169...$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.2 - \frac{0.1201169...}{2.3201169...}$$

$$= 1.14822...$$

$$= 1.15$$

(b) (i) (2 marks)

Outcomes assessed: PE6

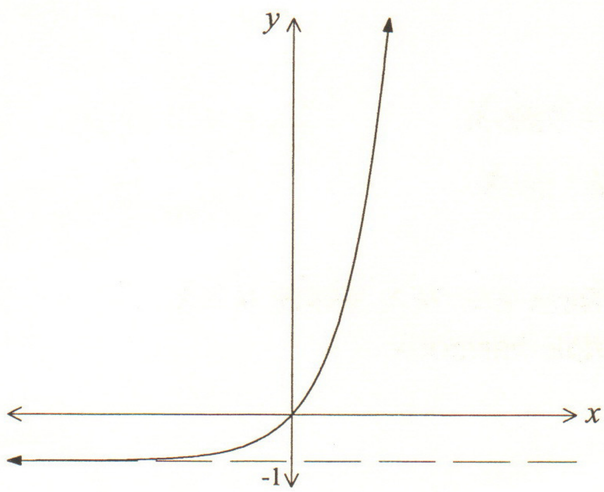
Targeted Performance Bands: E2-E3

Criteria	Marks
• draws correct graph	1
• states correct range	1

Sample Answer:

$$y = e^{3x} - 1$$

Range: $y > -1$



(b) (ii) (3 marks)
Outcomes assessed: HE4
Targeted Performance Bands: E2-E3

Criteria	Marks
• interchanges variables or progress towards solution	1
• changes subject of equation or further progress towards solution	1
• states inverse function with correct restriction	1

Sample Answer:

$y = e^{3x} - 1$

Swap x and y

$x = e^{3y} - 1$

$e^{3y} = x + 1$

$3y = \ln(x+1)$

$y = \frac{1}{3}\ln(x+1)$

$f^{-1}(x) = \frac{1}{3}\ln(x+1), \quad x > -1$

(c) (i) (2 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• differentiates correctly	1
• shows motion is simple harmonic	1

Sample Answer:

$x = \sqrt{3}\cos 3t - \sin 3t$

$v = \frac{dx}{dt}$

$= -3\sqrt{3}\sin 3t - 3\cos 3t$

$a = \frac{dv}{dt}$

$= -9\sqrt{3}\cos 3t + 9\sin 3t$

$= -9(\sqrt{3}\cos 3t - \sin 3t)$

$= -9x$

which is of the form $a = -n^2 x$ where $n = 3$

$\therefore$ motion is simple harmonic

(c) (ii) (3 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes result using auxiliary angle or other progress toward solution	1
• solves correctly for time	1
• finds correct velocity (correct numerical equivalence)	1

Sample Answer:

when $x = 1$, $\sqrt{3}\cos 3t - \sin 3t = 1$

Let $\sqrt{3}\cos 3t - \sin 3t = R\cos(3t + \alpha)$

$R\cos(3t + \alpha) = R\cos 3t\cos\alpha - R\sin 3t\sin\alpha$

$\therefore R\cos\alpha = \sqrt{3}$

$R\sin\alpha = 1$

i.e. $\tan\alpha = \dfrac{1}{\sqrt{3}} \quad \Rightarrow \quad \alpha = \dfrac{\pi}{6}$

$R^2 = 1 + 3 \quad \Rightarrow \quad R = 2$

$\sqrt{3}\cos 3t - \sin 3t = 2\cos\left(3t + \dfrac{\pi}{6}\right)$

i.e. solve $2\cos\left(3t + \dfrac{\pi}{6}\right) = 1$

$\cos\left(3t + \dfrac{\pi}{6}\right) = \dfrac{1}{2}$

$3t + \dfrac{\pi}{6} = \dfrac{\pi}{3}$ (first oscillation)

$t = \dfrac{\pi}{18}$ seconds

When $t = \dfrac{\pi}{18}$ $\quad v = -3\sqrt{3}\sin\dfrac{\pi}{6} - 3\cos\dfrac{\pi}{6}$

$= -3\sqrt{3} \times \dfrac{1}{2} - 3 \times \dfrac{\sqrt{3}}{2}$

$= -3\sqrt{3}$ cms^{-1}

Question 5 (12 marks)

(a) (i) (2 marks)

Outcomes assessed: PE3

Targeted Performance Bands: E3-E4

Criteria	Marks
• defines roots in arithmetic series	1
• uses sum of roots to show result	1

Sample Answer:

Let the roots be $\alpha - d$, α and $\alpha + d$

$x^3 - 6x^2 + 3x + k = 0$

sum of roots $= \dfrac{-b}{a} = 6$

Also sum of roots $= \alpha - d + \alpha + \alpha + d = 3\alpha$

$\therefore 3\alpha = 6$

$\alpha = 2$

i.e. one of the roots is 2

(a) (ii) (3 marks)

Outcomes assessed: PE3

Targeted Performance Bands: E2-E3

Criteria	Mark
• finds correct value for k	1
• progress toward solution	1
• finds correct roots	1

Sample Answer:

Since one root is 2 substitute into equation to find k.

$2^3 - 6 \times 2^2 + 3 \times 2 + k = 0$

$\therefore k = 10$

i.e. equation is $x^3 - 6x^2 + 3x + 10 = 0$

product of roots $= \dfrac{-d}{a} = -10$

product of roots $= \alpha(\alpha - d)(\alpha + d)$ from (i)

$= \alpha(\alpha^2 - d^2)$

$\therefore -10 = 2 \times (2^2 - d^2)$

$-5 = 4 - d^2$

$d^2 = 9$

$d = \pm 3$

$\therefore$ roots are $-1, 2, 5$

(b) (3 marks)

Outcomes assessed: PE2

Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes correct t-formula or other progress towards result	1
• significant progress toward the result	1
• completes the proof	1

Sample Answer:

Let $t = \tan\theta$, $\therefore \tan 2\theta = \dfrac{2t}{1-t^2}$

$$\begin{aligned}
\text{LHS} &= \frac{\tan 2\theta - \tan\theta}{\tan 2\theta + \cot\theta} \\
&= \left(\frac{2t}{1-t^2} - t\right) \div \left(\frac{2t}{1-t^2} + \frac{1}{t}\right) \\
&= \frac{2t - t + t^3}{1-t^2} \div \left(\frac{2t^2 + 1 - t^2}{t(1-t^2)}\right) \\
&= \frac{t(1+t^2)}{1-t^2} \times \frac{t(1-t^2)}{t^2+1} \\
&= t^2 \\
&= \tan^2\theta \\
&= \text{RHS}
\end{aligned}$$

OR

$$\begin{aligned}
\text{LHS} &= \frac{\tan 2\theta - \tan\theta}{\tan 2\theta + \cot\theta} \\
&= \left(\frac{2\tan\theta}{1-\tan^2\theta} - \tan\theta\right) \div \left(\frac{2\tan\theta}{1-\tan^2\theta} + \frac{1}{\tan\theta}\right) \\
&= \left(\frac{2\tan\theta - \tan\theta + \tan^3\theta}{1-\tan^2\theta}\right) \times \left(\frac{\tan\theta(1-\tan^2\theta)}{2\tan^2\theta + 1 - \tan^2\theta}\right) \\
&= \tan\theta(1+\tan^2\theta) \times \frac{\tan\theta}{\tan^2\theta + 1} \\
&= \tan^2\theta \\
&= \text{RHS}
\end{aligned}$$

(c) (i) (2 marks)
Outcomes assessed: PE4
Targeted Performance Bands: E3-E4

Criteria	Mark
• uses correct formula for division of interval or progress using other correct method	1
• finds correct coordinates from working	1

Sample Answer:

$P(2ap,\ ap^2)$, $S(0,\ a)$ and $PQ:QS = -4:3$

Let Q have coordinates $(x_q,\ y_q)$

$$x_q = \frac{3\times 2ap - 4\times 0}{-4+3} \qquad y_q = \frac{3\times ap^2 - 4\times a}{-4+3}$$

$$= \frac{6ap}{-1} \qquad = \frac{3ap^2 - 4a}{-1}$$

$$= -6ap \qquad = a\left(4-3p^2\right)$$

$\therefore$ Q has coordinates $\left(-6ap,\ a(4-3p^2)\right)$

(c) (ii) (2 marks)
Outcomes assessed: PE4
Targeted Performance Bands: E3-E4

Criteria	Marks
• makes progress to finding the locus	1
• shows locus is a parabola	1

Sample Answer:

From (i) $x = -6ap$

$$\therefore p = \frac{-x}{6a} \text{ and } p^2 = \frac{x^2}{36a^2}$$

$$\therefore y = a\left(4-3p^2\right)$$

$$= a\left(4 - \frac{3x^2}{36a^2}\right)$$

$$= 4a - \frac{x^2}{12a}$$

$$\frac{x^2}{12a} = 4a - y$$

$$x^2 = 48a^2 - 12ay$$

$$= -12a(y-4a)$$

which is the form of a parabola [with vertex $(0,\ 4a)$]

Question 6 (12 marks)

(a) (3 marks)

Outcomes assessed: PE2

Targeted Performance Bands: E2-E3

Criteria	Marks
• simplifies some indices	1
• further progress with simplifying indices	1
• gives correct expression	1

Sample Answer:

$$\begin{aligned}\frac{2^{4n}\times 3^{2n}}{8^n\times 6^n}+3^n &= \frac{2^{4n}\times 3^{2n}}{2^{3n}\times 2^n\times 3^n}+3^n\\ &= \frac{2^{4n}\times 3^n}{2^{4n}}+3^n\\ &= 3^n+3^n\\ &= 2\times 3^n\end{aligned}$$

(b) (i) (2 marks)

Outcomes assessed: PE5, HE7

Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes correct derivative	1
• shows the result	1

Sample Answer:

$$\begin{aligned}V &= \pi r^2 h\\ &= \pi r^3 k \qquad \text{since } h = kr\end{aligned}$$

$$\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}$$

$$\frac{dV}{dt}=3\pi r^2 k\times\frac{dr}{dt}$$

$$\frac{dV}{dt}=0.2 \text{ when } r=4$$

$$\therefore 0.2 = 3\pi\times 4^2 k\times\frac{dr}{dt}$$

$$\begin{aligned}\frac{dr}{dt} &= \frac{0.2}{48\pi k}\\ &= \frac{1}{240\pi k}\end{aligned}$$

(b) (ii) (3 marks)

Outcomes assessed: PE5, HE7

Targeted Performance Bands: E3-E4

Criteria	Marks
• finds expression for $\frac{dr}{dt}$ using surface area or progress toward result	1
• equates expressions using (i) or significant progress toward result	1
• finds correct value of k	1

Sample Answer:

$$S = 2\pi rh + 2\pi r^2$$

$$= 2\pi r^2 k + 2\pi r^2 \qquad \text{since } h = kr$$

$$= 2\pi r^2(k+1)$$

$$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$$

$$\frac{dS}{dt} = 4\pi r(k+1) \times \frac{dr}{dt}$$

$$\frac{dS}{dt} = 0.1 \text{ when } r = 4$$

$$\therefore 0.1 = 4\pi \times 4(k+1) \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{0.1}{16\pi(k+1)}$$

$$= \frac{1}{160\pi(k+1)}$$

$$\therefore \frac{1}{160\pi(k+1)} = \frac{1}{240\pi k} \qquad \text{from (i)}$$

$$240k = 160k + 160$$

$$80k = 160$$

$$k = 2$$

(c) (i) (2 marks)
Outcomes assessed: HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• differentiate LHS correctly	1
• differentiate RHS correctly	1

Sample Answer:

$$(1+x)^{2n} = \sum_{k=0}^{2n} {}^{2n}C_k\, x^k = {}^{2n}C_0 + {}^{2n}C_1\, x + {}^{2n}C_2\, x^2 + \ldots + {}^{2n}C_k\, x^k + \ldots + {}^{2n}C_{2n}\, x^{2n}$$

Differentiate both sides with respect to x.

$$\text{LHS} = 2n(1+x)^{2n-1}$$

$$RHS = {}^{2n}C_1 + {}^{2n}C_2\, 2x + \ldots + {}^{2n}C_k\, kx^{k-1} + \ldots + {}^{2n}C_{2n}\, 2nx^{2n-1}$$

$$= \sum_{k=1}^{2n} {}^{2n}C_k\, kx^{k-1}$$

$$\left[\therefore 2n(1+x)^{2n-1} = \sum_{k=1}^{2n} k\, {}^{2n}C_k\, x^{k-1}\right]$$

(c) (ii) (2 marks)
Outcomes assessed: HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• correct substitution into equation	1
• gives correct conclusion	1

Sample Answer:

Let $x = 1$ in the expansion of $2n(1+x)^{2n-1} = \sum_{k=1}^{2n} k\, {}^{2n}C_k\, x^{k-1}$.

$$\text{LHS} = 2n \times 2^{2n-1}$$

$$= n \times 2^{2n}$$

$$= n \times 4^n$$

$$\text{RHS} = \sum_{k=1}^{2n} k\, {}^{2n}C_k$$

$$\therefore \sum_{k=1}^{2n} k\, {}^{2n}C_k = n \times 4^n$$

Question 7 (12 marks)
(a) (i) (2 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• establishes correct binomial probability	1
• gives correct answer (correct numerical equivalence)	1

Sample Answer:

Let probability of correct guess, $p = 0.3$ and incorrect guess, $q = 0.7$

Binomial probability; $(0.7 + 0.3)^{50}$

$P(25 \text{ correct}) = {}^{50}C_{25}(0.7)^{25}(0.3)^{25}$

$[= 0.0014]$

(a) (ii) (3 marks)
Outcomes assessed: H5
Targeted Performance Bands: E3-E4

Criteria	Marks
• applies greatest coefficient method or some progress towards solution	1
• further progress towards solution (e.g. solution of inequality)	1
• gives correct answer	1

Sample Answer:

Most likely number correct $\Rightarrow$ find the greatest term in $(0.7 + 0.3)^{50}$

Find k such that $\dfrac{T_{k+1}}{T_k} \geq 1$

$$\frac{T_{k+1}}{T_k} = \frac{50 - k + 1}{k} \times \frac{0.3}{0.7}$$

i.e. $\dfrac{153 - 3k}{7k} \geq 1$

$153 - 3k \geq 7k$

$10k \leq 153$

$k \leq 15.3$

$\therefore k = 15$

Most likely number correct is 15. $\left[T_{16} = {}^{50}C_{15}(0.3)^{15}(0.7)^{35} = 0.122\right]$

(b)(i) (2 marks)

Outcomes assessed: HE3

Targeted Performance Bands: E2-E3

Criteria	Marks
• differentiates and equates to zero	1
• shows correct result	1

Sample Answer:

Particle reaches maximum height when $y' = 0$

$$y = Vt\sin\theta - \frac{1}{2}gt^2 \qquad \Rightarrow \qquad y' = V\sin\theta - gt$$

when $y' = 0$, $gt = V\sin\theta$ i.e. $t = \dfrac{V\sin\theta}{g}$

(b) (ii) (3 marks)

Outcomes assessed: HE3

Targeted Performance Bands: E3-E4

Criteria	Marks
• some progress toward solution	1
• further progress toward solution	1
• substitutes and simplifies to obtain desired result	1

Sample Answer:

At maximum height $t = \dfrac{V\sin\theta}{g}$, $x = c$ and $y = h$

$$h = \frac{V^2\sin^2\theta}{g} - \frac{1}{2}g\frac{V^2\sin^2\theta}{g^2} \qquad \text{and} \quad c = \frac{V^2\cos\theta\sin\theta}{g}$$

$$h = \frac{V^2\sin^2\theta}{2g} \qquad\qquad c^2 = \frac{V^4\cos^2\theta\sin^2\theta}{g^2}$$

$$\therefore \sin^2\theta = \frac{2gh}{V^2} \quad (1) \qquad\qquad = \frac{V^4\sin^2\theta(1-\sin^2\theta)}{g^2}$$

$$= \frac{V^4\dfrac{2gh}{V^2}\left(1-\dfrac{2gh}{V^2}\right)}{g^2} \quad \text{substituting for } \sin^2\theta \text{ from (1)}$$

$$= \frac{2h(V^2-2gh)}{g}$$

$$\therefore V^2 = 2gh + \frac{c^2g}{2h}$$

$$= \frac{4gh^2 + c^2g}{2h}$$

$$= \frac{g}{2h}\left(4h^2 + c^2\right)$$

(b) (iii) (2 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• significant progress towards solutions	1
• finds a correct expression for θ	1

Sample Answer:

$$c = \frac{V^2\cos\theta\sin\theta}{g} \qquad h = \frac{V^2\sin^2\theta}{2g}$$

$$\frac{h}{c} = \frac{V^2\sin^2\theta}{2g} \times \frac{g}{V^2\cos\theta\sin\theta}$$

$$\frac{h}{c} = \frac{\sin\theta}{2\cos\theta}$$

$$\frac{2h}{c} = \tan\theta$$

$$\therefore \theta = \tan^{-1}\left(\frac{2h}{c}\right)$$

OR

$$V^2 = \frac{g}{2h}\left(4h^2 + c^2\right) \qquad h = \frac{V^2\sin^2\theta}{2g} \quad \text{i.e. } \sin^2\theta = \frac{2gh}{V^2}$$

$$\sin^2\theta = \frac{2gh}{\frac{g}{2h}\left(4h^2 + c^2\right)}$$

$$\sin^2\theta = \frac{4h^2}{\left(4h^2 + c^2\right)}$$

$$\sin\theta = \frac{2h}{\sqrt{4h^2 + c^2}} \qquad (\theta \text{ acute})$$

$$\theta = \sin^{-1}\left(\frac{2h}{\sqrt{4h^2 + c^2}}\right)$$