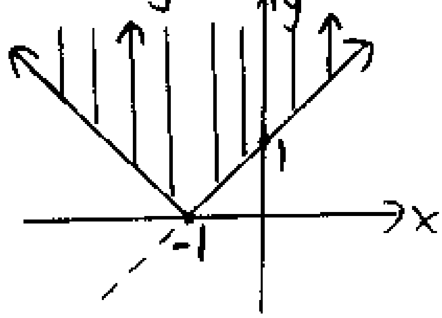
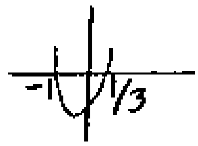
Question 1

(a) draw $y = |x+1|$

draw $y = x+1$



(b) $\frac{4}{x+1} < 3$



$\cdot (x+1)^2$: $4(x+1) < 3(x+1)^2$

$4x+4 < 3x^2+6x+3$.

concave up since 3 +ve $\leftarrow$ $3x^2+2x-1 > 0$

$(x+1)(3x-1) > 0$

final solution:

$x < -1$ or

$x > 1/3$.

Alternative approach:

consider separate cases.

when $(x+1) > 0$: $4 < 3(x+1)$

$x > -1$ $\quad$ $x > \frac{1}{3}$

when $(x+1) < 0$: $4 > 3(x+1)$

$x < -1$ $\quad$ $x < \frac{1}{3}$

but domain is $x < -1$

Solution:

$x > \frac{1}{3}$

or

$x < -1$

(c) $x_p = \frac{-2(3)+5(9)}{-2+5} = 13$

$y_p = \frac{-2(-1)+5(2)}{-2+5} = 4$

$P(13, 4)$.

note: you can also do

$x_p = \frac{2(3)-5(9)}{2-5}$

(d) $\int_0^1 \frac{dx}{\sqrt{2^2-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_0^1$

$= \sin^{-1}\frac{1}{2} - \sin^{-1}0$

$= \pi/6$.

Question 1

(e) $\int_3^4 x\sqrt{x-3}\,dx$

let $u = x-3$.
$du = dx$.
$x = u+3$.
$x=4,\ u=1$
$x=3,\ u=0$

$= \int_0^1 (u+3)\sqrt{u}\,du$

$= \int_0^1 \left(u^{3/2}+3u^{1/2}\right)du$

$= \left[\frac{2}{5}u^{5/2}+2u^{3/2}\right]_0^1$

$= \frac{2}{5}+2$ or 2.4

Question 2

(a) $\lim_{x\to 0} \frac{\sin\left(\frac{x}{5}\right)}{2x} = \lim_{x\to 0}\left(\frac{\sin\left(\frac{x}{5}\right)}{\frac{x}{5}} \cdot \frac{1}{10}\right)$

$= \frac{1}{10}\lim_{x\to 0}\frac{\sin\left(\frac{x}{5}\right)}{\frac{x}{5}}$ since $\frac{1}{10}$ is not dependent on x.

$= \frac{1}{10}\lim_{\frac{x}{5}\to 0}\frac{\sin\left(\frac{x}{5}\right)}{\frac{x}{5}}$ since as $x\to 0$, $\frac{x}{5}\to 0$.

$= \frac{1}{10}$

(b) $\frac{d}{dx}\cos^{-1}(3x^2)$

$= -\frac{1}{\sqrt{1+(3x^2)^2}} \cdot \frac{d}{dx}(3x^2)$

$= \frac{-6x}{\sqrt{1+9x^4}}$

(c) $AT^2 = BT.TC$ (secant and tangent theorem)

$12^2 = (7+x)x$.

$x^2+7x-144=0$

$(x-9)(x+16)=0$

but $x>0$ so $x=9$

(d) (i) set up an identity:

$$8\cos x + 6\sin x \equiv A\cos(x-\alpha),\ A>0,\ 0\le\alpha\le\pi/2.$$
$$\equiv A\cos x\cos\alpha + A\sin x\sin\alpha$$

(compound angle expansion)

equating coefficients of $\cos x$ and of $\sin x$:

$8 = A\cos\alpha$ — ①

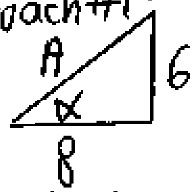
$6 = A\sin\alpha$ — ②

Using ① or ②:

$8 = 10\cos\alpha$

$\alpha = \cos^{-1} 4/5$

Approach #1:



$A = \sqrt{8^2+6^2}$

$= 10.$

Approach #2:

$①^2 + ②^2$: $8^2+6^2 = A^2(\cos^2\alpha+\sin^2\alpha)$

$100 = A^2(1).$

$A = 10.$

If $0\le\alpha\le\pi/2$ was not given:
we know A is in quadrant 1
(since both $\cos\alpha$ and $\sin\alpha$ are positive from ① and ②)

$$\therefore 8\cos x + 6\sin x \equiv 10\cos(x-\cos^{-1}4/5).$$

(ii) solve $10\cos(x-\cos^{-1}4/5) = 5$, $\cos^{-1}4/5 \doteqdot 0.6435$

Related angle $(x-\cos^{-1}4/5) = \pi/3 \doteqdot 1.0472$

for $0\le x\le 2\pi$, $x = \frac{\pi}{3}+\cos^{-1}\frac{4}{5}$ or $\frac{5\pi}{3}+\cos^{-1}\frac{4}{5}$

$x = 1.691$ or $x = 5.879$

note: if you don't like $\cos^{-1}4/5$, write its value to $(n+2)$ decimal places, $n=3$.

(e) (i) choose 4 from $(9+7)$ people.

${}^{16}C_4 = 1820$ ways.

(ii) number of favourable outcomes $= {}^9C_4$

(choose 4 from the 9 women)

$$P = \frac{{}^9C_4}{{}^{16}C_4} = 0.069$$

Question 3

(a) $\int \cos^2 4x \, dx$

$= \int \left(\frac{1}{2} + \frac{1}{2}\cos 8x\right) dx$ using $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

$= \frac{1}{2}x + \frac{1}{16}\sin 8x + C$

(b) (i) using the remainder theorem:

$P(-1) = -11$

$P(3) = 1$

now putting this to $P(x) = (x+1)(x-3)Q(x) + a(x+1) + b$:

$P(-1)$: $(-1+1)(-1-3)Q(-1) + a(-1+1) + b = -11$

$b = -11$

(ii) $P(3)$: $(3+1)(3-3)Q(3) + a(3+1) - 11 = 1$

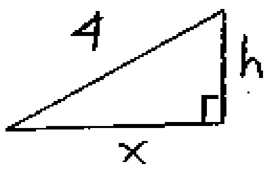
$a = 3$

from its form, $P(x) = (x+1)(x-3)Q(x) + R(x)$,

remainder $R(x)$ is $a(x+1) + b$

$= 3(x+1) - 11 = 3x - 8$

note: remainder can contain x.

(c) (i)


using Pythagoras' theorem:

$4^2 = h^2 + x^2$

$x^2 = 16 - h^2$

but $x > 0$ $\therefore$ $x = \sqrt{16 - h^2} = (16 - h^2)^{1/2}$

(ii) the easiest approach is to look at what the question asks, which is $\frac{dx}{dt}$. $\frac{dx}{dt} = \frac{dx}{da} \cdot \frac{da}{dt}$, a = something.

You are given $\frac{dh}{dt}$. Also from (i) you can find $\frac{dx}{dh}$.

So $a = h$. $\frac{dx}{dh} = \frac{1}{2}(16 - h^2)^{-1/2}(-2h) = \frac{-h}{\sqrt{16 - h^2}} = \frac{-1}{\sqrt{15}}$ when $h = 1$

$\frac{dx}{dt} = \frac{dx}{dh} \cdot \frac{dh}{dt} = \frac{-1}{\sqrt{15}} \cdot -0.3 = \frac{0.3}{15}\sqrt{15}$

$= \frac{\sqrt{15}}{50}$ metres/hour

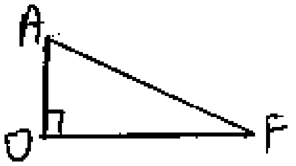
Question 3

(d) (i) all faces of the cube are identical squares.
So, diagonals AF, AC and FC are all equal.
$\therefore \Delta FAC$ is equilateral
$\angle FAC = 60°$.

(ii) $\Delta FAO \equiv \Delta FCO$

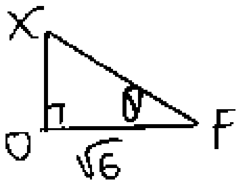
$\left(\begin{array}{l} FA = FC \text{, diagonals} \\ AO = CO \text{, O midpoint of AC} \\ FO \text{ is common} \end{array} \right)$

$\therefore \angle FOA = \angle FOC$, corresponding angles.
but they add up to $180°$.
$\angle FOA = 90°$, ΔFOA right-angled.



$AF = 2\sqrt{2}$ (diagonal of square)
$OA = \sqrt{2}$ (half diagonal)
$OF^2 = (2\sqrt{2})^2 - (\sqrt{2})^2$
$= 8 - 2$
$OF = \sqrt{6}$ metres

(iii) $\Delta FXO \equiv \Delta FCO$ (RHS congruency test)



$XO = 1$ (radius of circle, in a square of side length 2)

$\tan\theta = \frac{1}{\sqrt{6}}$. $\theta = 22.207...°$

$\angle XFY = 2\theta = 44°$.

Question 4.

(a) when $n=3$, $LHS = \left(1 - \frac{2}{3}\right) = \frac{1}{3}$
$RHS = \frac{2}{3(3-1)} = \frac{1}{3}$ } true for $n=3$.

suppose it's true for $n=k$. For $n=k+1$:

$LHS = \frac{2}{k(k-1)}\left(1 - \frac{2}{k+1}\right)$ since $n=k$ is true

$= \frac{2}{k(k-1)} - \frac{4}{k(k-1)(k+1)} = \frac{2(k+1)-4}{k(k-1)(k+1)} = \frac{2k-2}{k(k-1)(k+1)}$

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(a) (continued)

$$LHS = \frac{2(k-1)}{k(k-1)(k+1)} = \frac{2}{k(k+1)} = \frac{2}{(k+1)([k+1]-1)} = RHS.$$

$\therefore$ Whenever $n=k$ is true, $n=k+1$ is also true. But $n=3$ is true. So the statement is true for $n=3, 4, 5, 6, \ldots$

(b) (i) tangent at P: $y = px - ap^2$ — ①

tangent at Q: $y = qx - aq^2$ — ②

solve simultaneously:

$$px - ap^2 = qx - aq^2$$
$$x(p-q) = a(p^2-q^2)$$
$$= a(p-q)(p+q).$$
$$x = a(p+q).$$

put this to ①: $y = p(a(p+q)) - ap^2$

$= p^2 \cdot a - apq - ap^2$

$= apq.$

$\therefore R = (a(p+q), apq)$

(ii) $\angle POQ = 90° \Rightarrow PO \perp QO.$

$m_{PO} = \frac{ap^2}{2ap} = \frac{p}{2}$. $m_{QO} = \frac{q}{2}$.

$m_{PO} \cdot m_{QO} = -1 \Rightarrow pq = -4.$

$x_R = a(p+q) = a(p + \frac{-4}{p}) = a(p - \frac{4}{p})$ or $a(q - \frac{4}{q})$

$y_R = apq = a(-4) = -4a.$

$\therefore$ locus of R is the straight line $y = -4a$, with x-coordinate given by $x_R = a(p - 4p^{-1})$

R doesn't exist when $p = 0$, P is at $(0, 0)$.

(c) (i) P(she wins in any particular week) = $\frac{1}{10} = 0.1$

P(she never wins in 7 weeks) = 0.9^7

P(she wins at least once in 7 weeks) = $1 - 0.9^7 = 0.5217$

(c) (ii) you need binomial probability

$$P = {}^nC_k \, p^k q^{n-k}.$$

let X be the number of wins in 20 weeks.

$$P(X=2) = {}^{20}C_2 \, 0.1^2 \, 0.9^{18} = 0.2852$$

$$P(X=1) = {}^{20}C_1 \, 0.1^1 \, 0.9^{19} = 0.2702$$

$$P(X=2) > P(X=1).$$

(iii) let Y be the number of wins in n weeks.

$$P(Y=3) = {}^nC_3 \, 0.1^3 \, 0.9^{n-3}$$

$$P(Y=2) = {}^nC_2 \, 0.1^2 \, 0.9^{n-2}.$$

we want $P(Y=3) > P(Y=2)$, i.e. $\dfrac{P(Y=3)}{P(Y=2)} > 1$

$$\frac{0.1}{0.9} \cdot \frac{\cancel{n!}}{3\cancel{!}\cancel{(n-3)!}} \cdot \frac{\cancel{2!}(n-2)\cancel{!}}{\cancel{n!}} > 1$$

$$\frac{0.1}{0.9} \, \frac{(n-2)}{3} > 1$$

$$n-2 > 27$$

$$n > 29.$$

$\therefore$ Katie must participates in at least 30 weeks.

Question 5

(a) (i) $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 2x$

$$\frac{1}{2}v^2 = \frac{x^4}{2} + x^2 + C_1$$

$$v^2 = x^4 + 2x^2 + C_2.$$

initially, $x=2$ and $v=5$.

$$25 = 16 + 8 + C_2 \Rightarrow C_2 = 1$$

$v^2 = (x^2+1)^2$ (completing the square)

$v = x^2+1$ since initially $v>0$.

(ii) $\dfrac{dx}{dt} = x^2+1 \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{1+x^2} \Rightarrow t = \tan^{-1}x + C_3.$

$t=0, x=2, C_3 = -\tan^{-1}2$

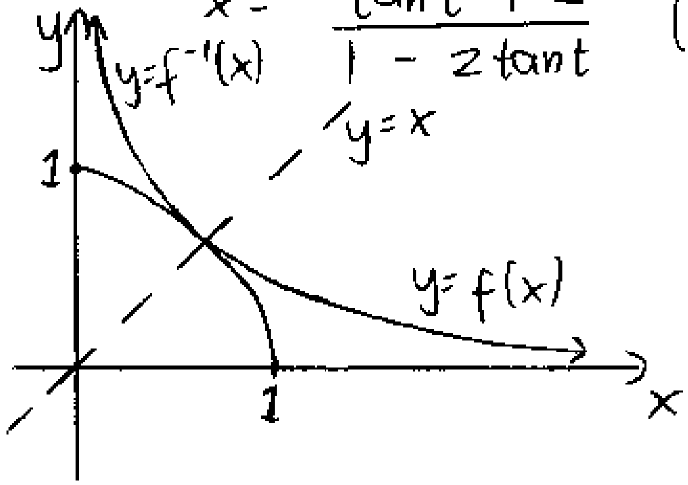
$$\tan^{-1}x = t + \tan^{-1}2 \Rightarrow x = \tan(t + \tan^{-1}2)$$

Question 5

(a)(ii) (continued). using compound angle expansion:

$$x = \frac{\tan t + z}{1 - z\tan t} \quad (\text{treat } \tan^{-1} z \text{ as an angle})$$

(b)(i)



reflect in $y=x$, swap x and y coordinates

(ii) range of $f(x)$: $0 < y \leq 1$

domain of $f^{-1}(x)$: $0 < x \leq 1$

(iii) $f^{-1}(x)$: $x = \frac{1}{1+y^2}$ (swap x and y)

$$x + xy^2 = 1$$
$$xy^2 = 1 - x$$
$$y^2 = \frac{1-x}{x} \Rightarrow y = \sqrt{\frac{1-x}{x}} \text{ since } y \geq 0$$

(iv) $f(x)$: $y = \frac{1}{1+x^2}$. $f^{-1}(x)$: $y = \sqrt{\frac{1-x}{x}}$.

$\Rightarrow$ Bad approach: solve simultaneously.

$$\frac{1}{1+x^2} = \sqrt{\frac{1-x}{x}}, \text{ giving } x_p.$$

squaring: $\frac{1}{1+2x^2+x^4} = \frac{1-x}{x}$

moving stuff: $x = (1-x)(1+2x^2+x^4)$

$$x = 1 + 2x^2 + x^4 - x - 2x^3 - x^5$$
$$x^5 - x^4 + 2x^3 - 2x^2 + 2x - 1 = 0.$$

Perform long division with (x^3+x-1) and show remainder is zero and the quotient has no zeroes.

$\Rightarrow$ Good approach:

They will intersect at line $y=x$.

$y=x$: $\frac{1}{1+x^2} = x$. (or you can use $\sqrt{\frac{1-x}{x}} = x$)

$$1 = x(1+x^2) \Rightarrow x^3 + x - 1 = 0.$$

Question 6.

(a) (i) $\angle ABD = \angle DCA$ (face same chord AD).

$= 180° - \theta$, θ is $\angle DBF$.

but $\angle DCA = \angle EBF$ (external angle of cyclic BECF)

$= \angle DBF$.

$180° - \theta = \theta$.

$180° = 2\theta \Rightarrow \theta = 90°$

(ii) this means $\angle DCA = 180° - \theta$

$= 90°$.

AD is diameter $\Rightarrow AD = 2r$.

Question 5

(b)(v) let $P(x) = x^3 + x - 1 = 0$. $P(0.5) = -0.375$

$P'(x) = 3x^2 + 1$. $P'(0.5) = 1.75$

$x_1 = x_0 - \frac{P(x_0)}{P'(x_0)} = 0.5 - \frac{-0.375}{1.75} = 0.7$

Question 6

(b)(i) this happens when $y = 0$.

$0 = vt\sin\theta - \frac{1}{2}gt^2$

$= t\,(v\sin\theta - \frac{1}{2}gt)$

either $t = 0$ or $\frac{1}{2}gt = v\sin\theta$

$t = \frac{2v\sin\theta}{g}$ —①

put ① to $x = vt\cos\theta$:

$x = \frac{v^2.(2\sin\theta\cos\theta)}{g} = \frac{v^2\sin 2\theta}{g}$ —②

(ii) put $\theta = 15°$ and $x = 40$ to ②:

$40 = \frac{v^2(0.5)}{g} \Rightarrow v^2 = 80g$.

(iii) from $x = vt\cos\theta$, $t = \frac{x}{v\cos\theta}$

$y = vt\sin\theta - \frac{1}{2}gt^2 = v\left(\frac{x}{v\cos\theta}\right)\sin\theta - \frac{1}{2}g\left(\frac{x}{v\cos\theta}\right)^2$

$= x\tan\theta - \frac{x^2\sec^2\theta\, g}{2v^2}$, but $v^2 = 80g$.

Question 6

(b)(iv) put $y = 20$ at $x = 40$;

$$20 = 40\tan\theta - \frac{(40)^2\sec^2\theta}{160}$$

$2 = 4\tan\theta - \sec^2\theta$. but $\sec^2\theta = \tan^2\theta + 1$.

$2 = 4\tan\theta - \tan^2\theta - 1$

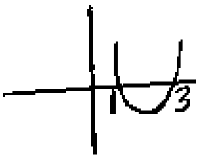
$\tan^2\theta - 4\tan\theta + 3 = 0$ gives the angle.

(v) it hits the front of the wall when $0 < y < 20$ when $x = 40$.

$\Rightarrow$ solve $y \le 20$:

by looking at the derivation process in (iv), it will be $\tan^2\theta - 4\tan\theta + 3 > 0$.

$(\tan\theta - 1)(\tan\theta - 3) > 0$.



$\tan\theta < 1$ or $\tan\theta > 3$

since $\tan x$ is increasing for $0 < x < \pi/2$; $\theta < 45^\circ$ or $\theta > \tan^{-1}3$

$\doteqdot 71^\circ 34'$

$\Rightarrow$ solve $y > 0$:

- long approach: $0 < 4\tan\theta - \tan^2\theta - 1$

$\tan^2\theta - 4\tan\theta + 1 < 0$.

if equality: $\tan\theta = \dfrac{4 \pm \sqrt{16-4}}{2}$

$= 2 \pm \sqrt{3}$

$\theta = 15^\circ$ or 75°

$y = \tan^2\theta - 4\tan\theta + 1$ is concave up so $15^\circ < \theta < 75^\circ$.

- shorter approach: use the formula in part (i) for range.

$$40 = \frac{80g\sin 2\theta}{g} \Rightarrow \theta = 15^\circ \text{ or } 75^\circ$$

then since max range is at $\theta = 45^\circ$, it's $15^\circ < \theta < 75^\circ$

note: notice how $(45 - 15) = (75 - 45)$.

this is a general result for range.

$\Rightarrow$ since $0 < y < 20$, answer is

$15^\circ < \theta < 45^\circ$ or $\tan^{-1}3 < \theta < 75^\circ$

Question 7

(a) (i) amplitude $= \frac{1}{2}(10-4) = 3.$

10 ↓ depth at high tide

4 ↘ depth at low tide.

t is defined after a high tide so it's $+\cos$.

Centre is $\frac{10+4}{2} = 7.$

$T = 12.5$ (hours)

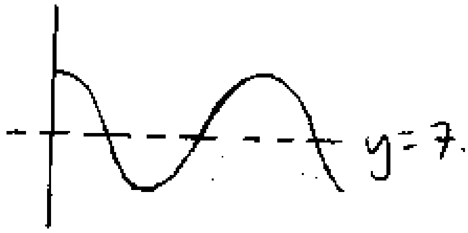
$n = \frac{2\pi}{T} = \frac{4\pi}{25}$

$\therefore y = 7 + 3\cos\left(\frac{4\pi}{25}t\right)$

(ii) $y \leq 8.5.$

$7 + 3\cos\left(\frac{4\pi}{25}t\right) \leq 8.5$

$\cos\left(\frac{4\pi}{25}t\right) \leq \frac{1}{2}$



If it's equality: $\frac{4\pi}{25}t = \frac{\pi}{3}, \frac{5\pi}{3}$, etc.

earliest time: $t = \frac{\pi}{3} \cdot \frac{25}{4\pi} = \frac{25}{12}$ hours

$t =$ 2 hours 5 minutes

at $t = 0$, it's 2 am. So it's 4:05 am.

(iii) It's best to define t as the number of hours after a high tide. So for this part, after 1 am.

$3\cos\left(\frac{4\pi}{25}t\right) \geq -1$

note: for easier visualisation, pretend that the depth at the entrance = the depth at the wharf. we want $y \geq 7$.

if equality:

$\cos\left(\frac{4\pi}{25}t\right) = -\frac{1}{3}$

$\frac{4\pi}{25}t = 1.9106, 4.37255$

we want it out by 7am so $t \leq 5$

$t = 1.906 \cdot \frac{25}{4\pi} = 3.8010$ hours, from 1am.

$= 2.8010$ from 2 am.

But it takes 20 minutes $= \frac{1}{3}$ hours to travel

time $= (2.8010 - \frac{1}{3})$ hours after 2 am

$=$ 4:28 am.

Question 7

(b)(i) the thing in [...] is a GP. Reading from right to left, $a=1$, $r=(1+x)$, number of terms $=n$.

$$\text{sum} = \frac{a(r^n-1)}{r-1} = \frac{1\left((1+x)^n-1\right)}{(1+x)-1}$$

$$LHS = x \cdot \frac{(1+x)^n-1}{x} = (1+x)^n-1 = RHS.$$

(ii) find the coefficients of x^k terms in (i).
for the LHS (of (i)), this will come from the x^{k-1} terms in the [...].

using (i) : $\binom{n-1}{k-1} + \binom{n-2}{k-1} + \dots + \binom{k-1}{k-1} = \binom{n}{k}$

(iii) This can be done using the definition

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

$$LHS = n \cdot \frac{(n-1)!}{k!(n-1-k)!}$$

$$RHS = (k+1) \cdot \frac{n!}{(k+1)!(n-k-1)!} = \frac{n(n-1)!}{k!(n-1-k)!} = LHS.$$

long approach

(iv) $\frac{d}{dx} LHS = \left[(1+x)^{n-1} + (1+x)^{n-2} + \dots + (1+x)^1 + (1+x)^0\right] + x\left[(n-1)(1+x)^{n-2} + (n-2)(1+x)^{n-3} + \dots + 1(1+x)^0 + 0\right]$ using product rule

$$\frac{d}{dx} RHS = n(1+x)^{n-1}$$

from the second [...] in $\frac{d}{dx} LHS$, you can guess that we need the coefficents of x^k. But we have this thing in the first [...]. Replace it with $\frac{(1+x)^n-1}{x}$ (from (i))

$\therefore$ coeff of x^k : $\left[\binom{n}{k+1}\right] + \left[(n-1)\binom{n-2}{k-1} + (n-2)\binom{n-3}{k-1} + \dots + k\binom{k-1}{k-1}\right] = n\binom{n-1}{k}$

$$(n-1)\binom{n-2}{k-1} + (n-2)\binom{n-3}{k-1} + \dots + k\binom{k-1}{k-1} = n\binom{n-1}{k} - \binom{n}{k+1}$$

$= (k+1)\binom{n}{k+1} - \binom{n}{k+1}$ from (iii) $= k\binom{n}{k+1}$

Shorter approach:

RHS in (iv) : ${}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_k x^k + \dots + {}^nC_n x^n$

$\frac{1}{x}(RHS)$: ${}^nC_1 x^0 + \dots + {}^nC_k x^{k-1} + {}^nC_{k+1} x^k + \dots + {}^nC_n x^{n-1}$

equate the coefficients of x^{k-1} in $\frac{d}{dx}\frac{LHS}{x} = \frac{d}{dx}\frac{RHS}{x}$