S.C.H.S 3U Trial Solns 1999

1) a) $3\cot x = 4$

$\cot x = \frac{4}{3}$

$\frac{6\sin x - 4\cos x}{\operatorname{cosec} x + \sec x}$

$= \left[6\left(\frac{3}{5}\right) - 4\left(\frac{4}{5}\right)\right] \div \left[\frac{5}{3} + \frac{5}{4}\right]$

$= \frac{24}{175}$

b) $\frac{d}{dx} e^{x^2} = 2x e^{x^2}$

$\therefore \int 2x e^{x^2}\,dx = e^{x^2} + c,$

$\therefore \int_0^2 x e^{x^2}\,dx = \frac{1}{2}\left[e^{x^2}\right]_0^2$

$= \frac{1}{2}(e^4 - 1)$

c) $y = x^3 \sin^{-1} 4x$

put $u = x^3$, $\frac{du}{dx} = 3x^2$

$v = \sin^{-1} 4x$, $\frac{dv}{dx} = \frac{4}{\sqrt{1-16x^2}}$

$\frac{dy}{dx} = 3x^2 \sin^{-1} 4x + \frac{4x^3}{\sqrt{1-16x^2}}$

or $\frac{x^3}{\sqrt{\frac{1}{16}-x^2}} + 3x^2 \sin^{-1} 4x$

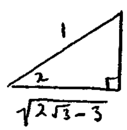
d) $\log_a\left(\frac{b}{c}\right) + \log_a ac$

$= (\log_a b - \log_a c) + (\log_a a + \log_a c)$

$= (0.3 - 0.4) + (1 + 0.4)$

$= 1.3$

e)



$\sin x = \sqrt{3} - 1$

$= \frac{\sqrt{3}-1}{1}$

$\cos 2x = \cos^2 x - \sin^2 x$

$= 2\sqrt{3} - 3 - (\sqrt{3}-1)^2$

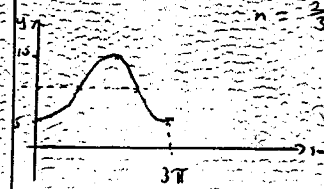
$= 2\sqrt{3} - 3 - (4 - 2\sqrt{3})$

$= 2\sqrt{3} - 3 - 4 + 2\sqrt{3}$

$= 4\sqrt{3} - 7$

f) $a = 5$, $\frac{2\pi}{n} = 3\pi$

$n = \frac{2}{3}$



$y = 10 - 5\cos\frac{2x}{3}$

or $y = 10 + 5\cos\left(\frac{2x}{3} - \pi\right)$

or $y = 5\sin\left(\frac{2x}{3} - \frac{\pi}{2}\right) + 10$

Q2) a) all real except $x^2 + 5x + 6 = 0$

ie all real except $x = -2, x = -3$

b) $P(x) = x^3 + 5x^2 + 8x + 2$

$\alpha + \beta + \gamma = -\frac{b}{a}$

$= -5$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$= 8$

$\alpha\beta\gamma = \frac{-d}{a}$

$= -2$

i) $(\alpha+1) + (\beta+1) + (\gamma+1)$

$= \alpha + \beta + \gamma + 3$

$= -5 + 3$

$= -2$

ii) $(\alpha+1)(\beta+1)(\gamma+1)$

$= (\alpha+1)(\beta\gamma + \beta + \gamma + 1)$

$= (\alpha\beta\gamma) + (\alpha\beta + \alpha\gamma + \gamma\beta) + (\alpha + \beta + \gamma) + 1$

$= -2 + 8 - 5 + 1$

$= 2$

c) $M = M_0 e^{-kt}$

when $t = 24$, $M = \frac{M_0}{2}$

ie $\frac{M_0}{2} = M_0 e^{-24k}$

$0.5 = e^{-24k}$

$-24k = \log_e 0.5$

$k = \frac{\log_e 0.5}{-24}$ $[\doteqdot 0.029]$

If 15% remains

$M = 0.15 M_0$

ie $0.15 = e^{-kt}$

$\log_e(0.15) = \log_e e^{-kt}$

$-kt = \log(0.15)$

$t = \frac{\log(0.15)}{-k}$

$= 65 \text{ hrs } 41' \, 14''$

d) $y = e^{\tan^{-1} x}$

On the y axis $x = 0$

$y = e^{\tan^{-1}(0)}$

$= e^0$

$= 1$ ie $(0,1)$

$\frac{dy}{dx} = \frac{1}{1+x^2} \times e^{\tan^{-1} x}$

$= \frac{e^{\tan^{-1} x}}{1+x^2}$

when $x = 0$

$m = \frac{e^{\tan^{-1}(0)}}{1+0}$

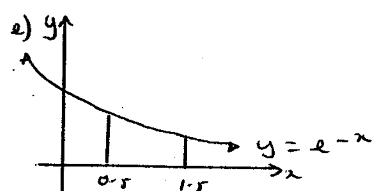
$= 1$

$y - y_1 = m(x - x_1)$

$y - 1 = 1(x - 0)$

$y = x + 1$

e)



$V = \pi \int_{0.5}^{1.5} (e^{-x})^2\,dx$

$= \pi \int^{1.5} e^{-2x}\,dx$

$\pi\left[-\frac{1}{2} e^{-2x}\right]_{0.5}^{1.5}$

$-\frac{\pi}{2}\left[e^{-3} - e^{-1}\right]$

$= \frac{\pi}{2}\left[\frac{1}{e} - \frac{1}{e^3}\right]$ units^3

$= 0.50$ (2d.p)

Q3) a) i) $x = 3 + Ae^{5t}$

$\frac{dx}{dt} = 5Ae^{5t}$ (2)

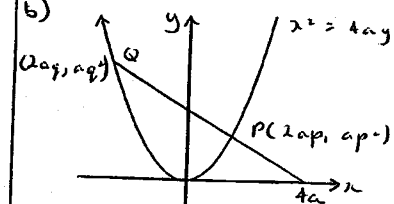
$= 5(x - 3)$ [since $Ae^{5t} = x - 3$]

ii) $x = 20$ when $t = 0$

$20 = 3 + A$

$A = 17$ (1)

b)



i) Gradient $PQ = \frac{ap^2 - aq^2}{2ap - 2aq}$

$= \frac{p+q}{2}$

Eqn PQ

$y - y_1 = m(x - x_1)$

$y - ap^2 = \frac{p+q}{2}(x - 2ap)$

$2y - 2ap^2 = (p+q)(x - 2ap)$

subst $x = 4a$, $y = 0$

$-2ap^2 = (p+q)(4a - 2ap)$

$-\cancel{2ap^2} = 4ap - \cancel{2ap^2} + 4aq - 2apq$

$2apq = 4ap + 4aq$

$pq = 2p + 2q$

$= 2(p+q)$ (2)

ii) Co-ords of M

$x = \frac{2ap + 2aq}{2}$, $y = \frac{ap^2 + aq^2}{2}$

$x = a(p+q)$ $\qquad y = \frac{a}{2}(p^2+q^2)$ (B)

$\therefore p+q = \frac{x}{a}$ (A)

Now $y = \frac{a}{2}[p^2+q^2]$

$= \frac{a}{2}[(p+q)^2 - 2pq]$

$= \frac{a}{2}[(\frac{x}{a})^2 - 2pq]$ from A

$= \frac{a}{2}[(\frac{x}{a})^2 - (2)(2)(p+q)]$ from part i)

$= \frac{a}{2}[(\frac{x}{a})^2 - 4(p+q)]$

$= \frac{a}{2}[\frac{x^2}{a^2} - 4\frac{x}{a}]$ from A

or $2ay = x^2 - 4ax$ (3)

or $(x-2a)^2 = 2a(y+2a)$

c) $y = -x$, $y = \frac{2}{\sqrt{3}}x$

$\therefore m_1 = -1$, $m_2 = \frac{2}{\sqrt{3}}$

$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

$= (-1 - \frac{2}{\sqrt{3}}) \div (1 - \frac{2}{\sqrt{3}})$

$\theta = 85° \, 54'$ (2)

d) $y = \log_e\left(\frac{3+x}{3-x}\right)$

$= \log_e(3+x) - \log_e(3-x)$

$\frac{dy}{dx} = \frac{1}{3+x} - \frac{-1}{3-x}$

$= \frac{1}{3+x} + \frac{1}{3-x}$

$= \frac{6}{(3-x)(3+x)}$

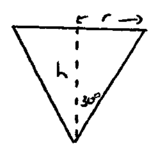
Q4 Let depth be h

then $\frac{dh}{dt} = 4 \text{ cms}^{-1}$

Find $\frac{dV}{dt}$ when $h = 9$

$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dh} = \frac{1}{3}\pi r^2$



$\frac{r}{h} = \tan 30°$

$r = \frac{h}{\sqrt{3}}$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h$

$= \frac{\pi h^3}{9}$

$\frac{dV}{dh} = \frac{\pi h^2}{3}$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

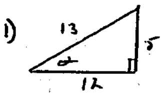
$= \frac{\pi h^2}{3} \times 4$

when $h = 9$

$\frac{dV}{dt} = \frac{\pi(81)(4)}{3}$

$= 108\pi \text{ cm}^3\text{s}^{-1}$

b) i)

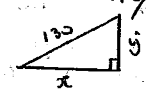


$\tan\alpha = \frac{5}{12}$

$\sin\alpha = \frac{5}{13}$

$\cos\alpha = \frac{12}{13}$

Initially



$t = 0$, $x = 0, y = 0$

$\dot{x} = 130\cos\alpha$

$= 130 \times \frac{12}{13}$

$= 120$

$\dot{y} = 130\sin\alpha$

$= 130 \times \frac{5}{13}$

$= 50$

Horizontal Motion

$\ddot{x} = 0$

$\dot{x} = c_1$

when $t = 0$ $\dot{x} = 120$

$\dot{x} = 120$

$x = 120t + c_2$

when $t = 0$, $x = 0$

$x = 120t$

Vertical Motion

$\ddot{y} = -10$

$\dot{y} = -10t + c_1$

when $t = 0$, $\dot{y} = 50$

$\dot{y} = -10t + 50$

$y = -5t^2 + 50t + c$

when $t = 0$, $y = 0$

$y = -5t^2 + 50t$

ii) at max range $y = 0$

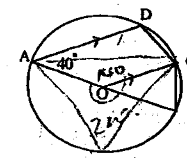
i.e. $-5t^2 + 50t = 0$

$-5t(t - 10)$, $t = 10$

$x_{max} = 120(10)$

$= 1200$ metres

c)



$\angle AOC + 40° = 180°$ (co-int ∠s

$\angle AOC = 140°$

Major $\angle AOC = 360° - 140°$

$= 220°$ (∠s at

$\angle ADC = 220 \div 2$ (∠ at cen

twice ∠

circ)

$= 110°$

$\angle OCO = 180° - 110°$ (co-int

$= 70°$ AD ∥ C

Q5

a) i) $R = P(3)$

$= 3^3 - (k+1)9 + 3k + 1$

$= 30 - 6k$

ii) If divisible $P(3) = 0$

$0 = 30 - 6k$

$k = 5$

iii) $P(x) = x^3 - 6x^2 + 5x + 12$

$$\begin{array}{r} x^2 - 3x - 4 \\ x - 3 \overline{)\, x^3 - 6x^2 + 5x + 12} \\ x^3 - 3x^2 \qquad\qquad \\ \hline -3x^2 + 5x \qquad \\ -3x^2 + 9x \qquad \\ \hline -4x + 12 \\ -4x + 12 \\ \hline 0 \end{array}$$

$\therefore P(x) = (x-3)(x^2 - 3x - 4)$

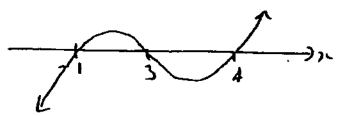
$= (x-3)(x-4)(x+1)$

zeros at $x = 3$, $x = 4$

$x = -1$

Q 5 1999 3U

iv)



$P(x) > 0$ for $-1 < x < 3$ and $x > 4$

b) $y = \ln x + \sin x$

$a_2 = a_1 - \dfrac{f(a_1)}{f'(a_1)}$

$\dot{y} = \frac{1}{x} + \cos x$

$a_2 = 0.5 - \dfrac{(\ln 0.5 + \sin 0.5)}{\left(\frac{1}{0.5} + \cos 0.5\right)}$

$= 0.574$ (to 3 dp)

Question 6.

i) Step 1. Verify for $n=1$
ie $7^1 + 2 = 9$ which is divisible by 3

Step 2. a) Assume true for $n=k$
ie $7^k + 2 = 3P$ (P integer)

b) Prove true for $n = k+1$

$7^{k+1} + 2 = 7^k . 7 + 2$

$= 7(3P-2) + 2$ (from assp)

$= 21P - 14 + 2$

$= 3(7P-4)$

since P is an integer, $(7P-4)$ is an integer and $7^{k+1}+2$ is divisible by 3 if the assumption is true.

ie true for $n=k+1$ if true for $n=k$.

Missing soln's

Soln's

Step 3 Since statement is true for $n=1$, it is true for $n=2$. Since true for $n=2$, then true for $n=3$, and so on for all positive integers. (3)

(5)

b) $\cos 2x = \sin x$

$1 - 2\sin^2 x = \sin x$

$\therefore 2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$\sin x = \frac{1}{2}$ $\quad \sin x = -1$

$\therefore x = n\pi + (-1)^n \sin^{-1}\frac{1}{2}$ or $n\pi + (-1)^n \sin^{-1}(-1)$

ie $x = n\pi + (-1)^n\left(\frac{\pi}{6}\right)$ or $n\pi + (-1)^n\left(-\frac{\pi}{2}\right)$ (3)

c) $y > 0$ for all x
(ie does not cut x axis)

$\therefore A = \displaystyle\int_{-2}^{2} \frac{dx}{\sqrt{25-x^2}}$

$= 2\displaystyle\int_0^2 \frac{dx}{\sqrt{25-x^2}}$ since fn. is even

$= 2\left[\sin^{-1}\frac{x}{5}\right]_0^2 = 2\left(\sin^{-1}\frac{2}{5} - \right.$

Area $\doteqdot 0.82\ u^2$ (2)

d) $y = \log(\sec x + \tan x)$

let $u = \sec x + \tan x$

$= (\cos x)^{-1} + \tan x$

$\frac{du}{dx} = -(\cos x)^{-2} . -\sin x + \sec$

$= \dfrac{\sin x}{\cos^2 x} + \sec^2 x$

$= \tan x . \sec x + \sec^2 x$

(see bottom of next page)

7 (a)(i) $V^2 = 64 - 4x^2 + 24x$.

Q7) For SHM $\ddot{x} = -n^2 x$ or $\ddot{X} = -n^2 X$

Now $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = a = \ddot{x}$

$\therefore \frac{1}{2}v^2 = 32 - 2x^2 + 12x$.

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x + 12$

ie $\ddot{x} = -4(x-3)$ — is of for $\ddot{x} = -n^2 x$

$\therefore$ motion is SHM.

(ii) Centre of motion $X = 0$ ie. $x - 3 = 0$, $x = 3$.

or when $v = 0$
$4x^2 - 24x - 64 = 0$
$x^2 - 6x - 16 = 0$
$(x-8)(x+2) = 0$
$\therefore x = -2$ to $x = 8$
Centre then x

(iii) Period $T = \frac{2\pi}{n}$ where $n = 2$

$\therefore T = \frac{2\pi}{2} = \pi$ Period $= \pi$ secs

Amplitude: is from centre to end
ie from 3 to 8
$\therefore$ amplitude $= 5$ m.

or complete $v^2 = n^2(a^2 - x^2)$

$v^2 = n^2(a^2 - x^2)$
$v^2 = 4(16 + 6x - x^2$
$v = 4(25 - (9 - 6x$
$v = 4(25 - (x-3)^2$
$\therefore a = 5$

(iv)

$x = -5\sin 2t + 3$

$x = 5\sin(-2t) + 3$

$x = 5\cos\left(2t - \frac{3\pi}{2}\right) + 3$

Q6 (cont'd)

$\frac{dy}{dx} = \dfrac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$

$= \sec x$.

$\therefore \displaystyle\int_0^{\pi/4} \sec x\, dx = \left[\ln(\sec x + \tan x)\right]$

$= \ln\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \ln(\sec 0 + \tan$

$= \ln(\sqrt{2} + 1) - \ln(1 + 0)$

$= \ln(\sqrt{2} + 1)$

(4)

(6)

i) $\sqrt{2}\cos\theta + \sin\theta$ 1999

7 3U Soln's $\cos\theta = \dfrac{1-t^2}{1+t^2}$ where $t = \tan\frac{\theta}{2}$

$\sin\theta = \dfrac{2t}{1+t^2}$

$\therefore \Rightarrow \sqrt{2}.\left(\dfrac{1-t^2}{1+t^2}\right) + \dfrac{2t}{1+t^2}$

$\Rightarrow \dfrac{\sqrt{2}(1-t^2)+2t}{1+t^2}$

(ii) Now $\sqrt{2}\cos\theta + \sin\theta = 1$. $\quad 0^\circ < \theta < 360^\circ$

$\therefore \dfrac{\sqrt{2}(1-t^2)+2t}{1+t^2} = 1$

$\sqrt{2}(1-t^2) + 2t = 1 + t^2$

$\sqrt{2} - \sqrt{2}t^2 + 2t = 1 + t^2$

$t^2(1+\sqrt{2}) - 2t + (1-\sqrt{2}) = 0$.

$\therefore t = \dfrac{2 \pm \sqrt{4 - 4(1+\sqrt{2})(1-\sqrt{2})}}{2(1+\sqrt{2})}$

$t = \dfrac{2 \pm \sqrt{4-4(1-2)}}{2(1+\sqrt{2})}$

$\therefore t = \dfrac{2(1+\sqrt{2})}{2(1+\sqrt{2})}$ or $t = \dfrac{1-\sqrt{2}}{1+\sqrt{2}} \times \dfrac{1-\sqrt{2}}{1-\sqrt{2}}$

$\underline{\underline{t = 1}}$ $\qquad t = \dfrac{2\sqrt{2}-3}{1}$

when $t = 1$

then $\tan\frac{\theta}{2} = 1$

i.e. $\frac{\theta}{2} = \frac{\pi}{4}$

* $\theta = \frac{\pi}{2}$

when $t = \dfrac{2\sqrt{2}-3}{1}$

$\tan\frac{\theta}{2} \doteqdot -0.1715$

$\theta = -19^\circ 28'$

But

$0 < \theta < 360^\circ$.

$\therefore \theta = 360 - 19^\circ 28'$

$\left\{ \begin{array}{l} \underline{\theta = 340^\circ 32'} \\ \theta = \frac{\pi}{2} \end{array} \right.$

(c) $\displaystyle\int \frac{x.dx}{(25+x^2)^{\frac{3}{2}}}$

$x = 5\tan\theta$.

$dx = 5\sec^2\theta.d\theta$.

$\therefore x.dx = 25\tan\theta.\sec^2\theta.d\theta$.

$25 + x^2 = 25 + 25\tan^2\theta$

$= 25\sec^2\theta$.

$(25+x^2)^{\frac{3}{2}} = 125\sec^3\theta$.

$I = \displaystyle\int \frac{25\tan\theta.\sec^2\theta.d\theta}{125\sec^3\theta}$.

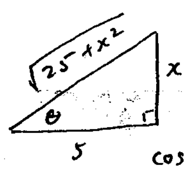
$I = \frac{1}{5}\displaystyle\int \frac{\tan\theta}{\sec\theta}.d\theta$

$I = \frac{1}{5}\displaystyle\int \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}.d\theta$.

$I = \frac{1}{5}\displaystyle\int \sin\theta.d\theta$

$I = -\frac{1}{5}\cos\theta + C$

But



$\cos\theta = \dfrac{5}{\sqrt{25+x^2}}$

$\therefore I = \dfrac{-1}{\sqrt{25+x^2}} + C$