CSSA - NSW

4 Unit Mathematics

Trial HSC Examination 1989

1. **(i)** The function f is defined for $x > 0$ by $f(x) = x \ln x$. Show that the graph of $y = f(x)$ has a stationary point at $x = -\frac{1}{e}$ and determine the nature of this point.
(ii) The function g is defined for $x \geq \frac{1}{e}$ by $g(x) = x \ln x$. Sketch the graph of $y = g(x)$ showing clearly the coordinates of its end point and the coordinates of its points of intersection with the x axis and the line $y = x$.
(iii) On the same axes as the graph of $y = g(x)$ sketch the graph of the inverse function $y = g^{-1}(x)$ showing clearly the coordinates of its end point and the coordinates of its points of intersection with the y axis and the line $y = x$. (Do not try to find an expression for the inverse function g^{-1}.)
(iv) Evaluate $\int_{\frac{1}{e}}^{e} x(1 - \ln x)dx$. Shade a region on the graphs with area given by this definite integral.
(v) Hence find the area of the region bounded by the line $y = -x$ and the curves $y = g(x)$ and $y = g^{-1}(x)$.

2. **(a)** If $y = \tan^{-1} e^x$ show that $\frac{d^2y}{dx^2} = 2(\frac{dy}{dx})^2 \cot 2y$.
(b) Find $\int \frac{e^{2x}}{e^2+1}\, dx$.
(c) **(i)** By using partial fraction show that $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{x(\pi-2x)}\, dx = \frac{2}{\pi} \ln 2$.
(ii) By using the substitution $u = a+b-x$ show that $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$.
(iii) Hence evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x}{x(\pi-2x)}\, dx$.

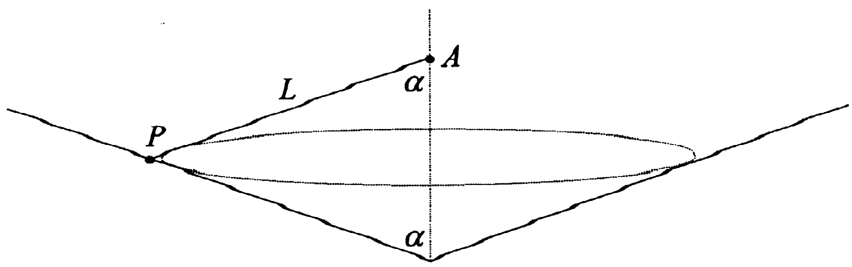
3. **(a)** **(i)** If $z_1 = 1 - i$ and $z_2 = -1 + i\sqrt{3}$ find $|z_1|$ and $|z_2|$ and write down $|z_1 z_2|$ in surd form. Find also $\arg z_1$ and $\arg z_2$ and write down $\arg z_1 z_2$ in terms of π.
(ii) Use the given forms of z_1 and z_2 to find $z_1 z_2$ in the form $a + ib$. Deduce that $\cos \frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.
(b) If z is any complex number such that $|z| = 1$ show using an Argand diagram or otherwise that:
(i) $1 \leq |z + 2| \leq 2$;
(ii) $-\frac{\pi}{6} \leq \arg(z + 2) \leq \frac{\pi}{6}$.
(c) **(i)** Let $z = x + iy$ be any non-zero complex number. Express $z + \frac{1}{z}$ in the form $a + ib$.
(ii) Given that $z + \frac{1}{z} = k$ where k is real, show that either $y = 0$ or $x^2 + y^2 = 1$. Show that if $y = 0$ then $|k| \geq 2$ and that if $x^2 + y^2 = 1$ then $|k| \leq 2$.

4. **(a)** **(i)** Show that the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ intersect at right angles.

(ii) Find the equation of the circle through the points of intersection of the two conics.
(b) (i) Show that the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (where $a > b > 0$) at the point $P(a\sec\theta, b\tan\theta)$ has equation $bx\sec\theta - ay\tan\theta = ab$.
(ii) If this tangent passes through a focus of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (where $a > b > 0$) show that it is parallel to one of the lines $y = x, y = -x$ and that its point of contact with the hyperbola lies on a directrix of the ellipse.

5. (a) (i) Show that the complex number $z = 1 + \cos\theta + i\sin\theta$ has modulus $2\cos\frac{\theta}{2}$ and argument $\frac{\theta}{2}$. Hence find the modulus and the argument of the complex number $(1 + \cos\theta + i\sin\theta)^n$ where n is a positive integer.
(ii) Hence show that $1 + {}^4C_1\cos\theta + {}^4C_2\cos 2\theta + {}^4C_3\cos 3\theta + \cos 4\theta = 16\cos^4\frac{\theta}{2}\cos 2\theta$, and obtain a similar expression for ${}^4C_1\sin\theta + {}^4C_2\sin 2\theta + {}^4C_3\sin 3\theta + \sin 4\theta$.
(b) (i) On the same axes sketch the graphs of the functions $y = \frac{e^x + e^{-x}}{2}$ and $y = \frac{e^x - e^{-x}}{2}$, showing clearly the coordinates of any points of intersection with the x axis and the y axis.
(ii) The region between the two curves bounded by the y axis and the line $x = 1$ is rotated through one complete revolution about the y axis. Use cylindrical shells to show that the volume V of the solid of revolution so formed is given by $V = 2\pi\int_0^1 xe^{-x}dx$ and hence find this volume.

6.



A smooth conical shell with semi-vertical angle α, $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$, is fixed with its vertex down, and axis vertical. A particle P of mass m is attached to a fixed point A, vertically above the vertex of the cone, by a light inextensible string of length L which makes an angle α with the vertical (as shown in the diagram above). The particle P is observed to move in a horizontal circle on the inner surface of the conical shell, with constant angular velocity ω, and with the string taut.
(i) Draw a diagram showing all the forces on the particle P.
(ii) Show that if T is the tension in the string, and R the magnitude of the force the surface exerts on the particle, then
$T + R\tan\alpha = mg\sec\alpha$
$T + R\cot\alpha = mL\omega^2$
(iii) Find expressions for R and T.
(iv) Show that if ω exceeds a certain critical value, the particle loses contact with the surface. State this critical value of ω, and describe qualitatively what would be observed if ω were to exceed this value.

(v) Show that the string goes slack when the linear speed of the particle is $\sqrt{gr\cot\alpha}$, where r is the radius of the circle of motion.
(vi) Suppose that initially ω is such that the particle is moving in a circle in contact with the surface and with the string taut, but that the surface is now rough, producing a friction force which slows the linear speed v of the particle.
• Describe qualitatively what motion you would now observe as v decreases.
• What difference would it have made if α had been less than $\frac{\pi}{4}$?

7. **(a)** $P(x)$ is a polynomial of degree 4 with real coefficients.
(i) Show that if the complex number α is one zero of $P(x)$, then its complex conjugate $\overline{\alpha}$ is also a zero of $P(x)$.
(ii) The complex number α satisfies $\Im(\alpha) \neq 0, \Re(\alpha) = a$, and $|\alpha| = r$. Show that if α is a zero of $P(x)$, then $P(x)$ has a factor $x^2 - 2ax + r^2$ over $\mathbb{R}$, the field of real numbers.
(iii) α is a non-real double zero of $P(x) = x^4-8x^3+30x^2-56x+49$. Factor $P(x)$ into irreducible factors over $\mathbb{R}$, and find the four roots of $x^4 - 8x^3 + 30x^2 - 56x + 49 = 0$.
(b) a curve has parametric equations $\begin{cases} x &= \theta - \sin\theta \\ y &= 1 - \cos\theta \end{cases}$
(i) Show that $\frac{dy}{dx} = \cot\frac{\theta}{2}$ and hence show $\frac{d^2y}{dx^2} = -\frac{1}{y^2}, y \neq 0$.
(ii) Write down the coordinates of any stationary points on the curve and state the nature of each such point.
(iii) Sketch the curve, showing the stationary points, the intercepts on the coordinate axes, and the direction of the tangents at the points where the curve meets the x axis.

8. **(a)** The vertices of a quadrilateral $ABCD$ lie on a circle of radius r. The angles subtended at the centre of the circle by the sides of $ABCD$ taken in order are in arithmetic progression with first term α and common difference β.
(i) Show that $2\alpha + 3\beta = \pi$ and interpret this result geometrically.
(ii) Show that the area of the quadrilateral is $2r^2\cos\beta\cos\frac{\beta}{2}$. If required you may use without proof the results:
$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$
$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$
$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$
$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$.
(b) n co-planar lines are such that the number of intersection points is a minimum.
(i) How many intersection points are there?
(ii) If n such lines divide the plane into u_n regions, show that $u_n = u_{n-1} + n$. Hence deduce that $u_n = 1 + \frac{1}{2}n(n+1)$. How many of these regions have finite area?