2003
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time- 5 minutes
- Working Time – 2 hours
- Write using a blue or black pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a fresh sheet of paper.

Total marks (**84**)

- Attempt Questions 1-7
- All questions are of equal value

WESTERN REGION

Question 1 12 Marks Start a fresh sheet of paper. **Marks**

a) Find the horizontal asymptote for $y = \dfrac{3x^2 + 4x + 5}{x^2}$ **2**

b) Solve the inequality $\dfrac{x}{2 - x} \leq 4$ **3**

c) Find $\sum_{n=2}^{20} 3n - 4$ **2**

d) Use the substitution $u = x^2 - 3$ to evaluate **3**

$$\int_2^6 \frac{x}{\sqrt{x^2 - 3}}\,dx$$

e) A parabola is defined by the parametric equations

$$x = 3t$$
$$y = 6t^2$$

i) What point is defined when $t = 5$? **1**

ii) What is the Cartesian equation of the parabola? **1**

Question 3   12 Marks   Start a fresh sheet of paper.   **Marks**

a) Find $\int_0^{\pi/16} \cos^2 4x\,dx$ **3**

b) i) How many arrangements can be made from the letters of the word EXCESSIVE ? **1**

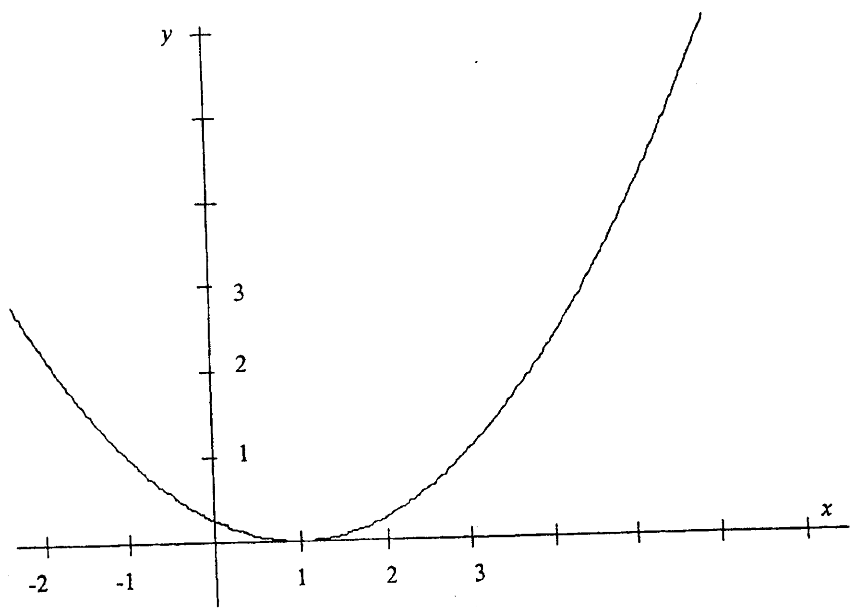
ii) Find the probability that such an arrangement has the consonants and vowels in alternating positions. **2**

c) Calculate the solutions to $4\cos\theta + 3\sin\theta = 2$ in the range $0 \le \theta \le 2\pi$ **4**
Express your answers to the nearest hundredth of a radian.

d) The graph below shows a function $y = f(x)$.

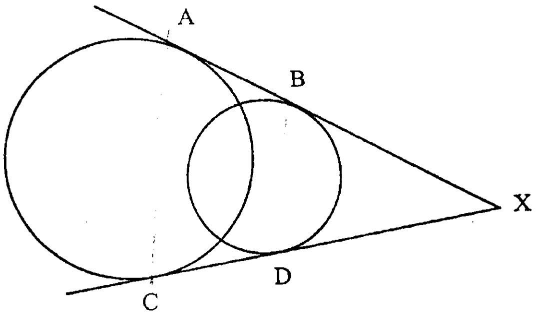
i) Specify a portion of the domain for which $f(x)$ has an inverse **1**

ii) Copy the graph of the curve onto your answer sheet and neatly draw $y = f^{-1}(x)$ for the domain you specified in i) **1**



Question 4 12 Marks Start a fresh sheet of paper. **Marks**

a) **3**



In the diagram AB is common tangent to the two circles.

Likewise CD is also a common tangent.

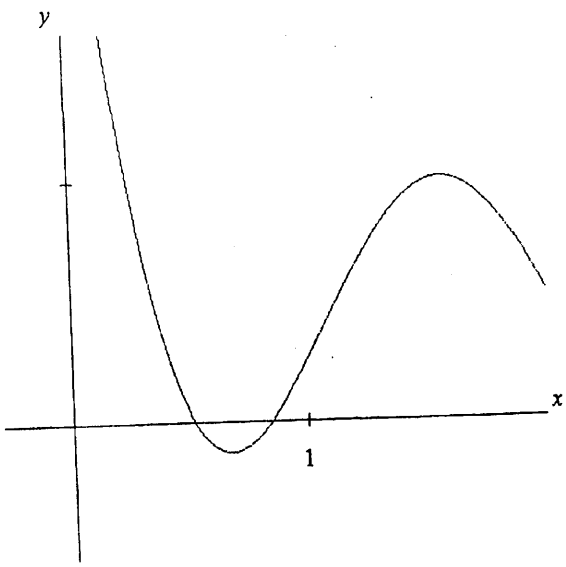
The two tangents meet externally at X.

Explain why AC || BD.

b) Given that $\cos 3\theta = \cos(\theta + 2\theta)$, use the double angle formulae to express $\cos 3\theta$ in terms of $\cos\theta$. **2**

c) **5**

The graph represents a part of the curve $y = 8\sin^2 x - 10\sin x + 3$. Calculate the two roots shown in the diagram and evaluate the minimum value shown in the graph.



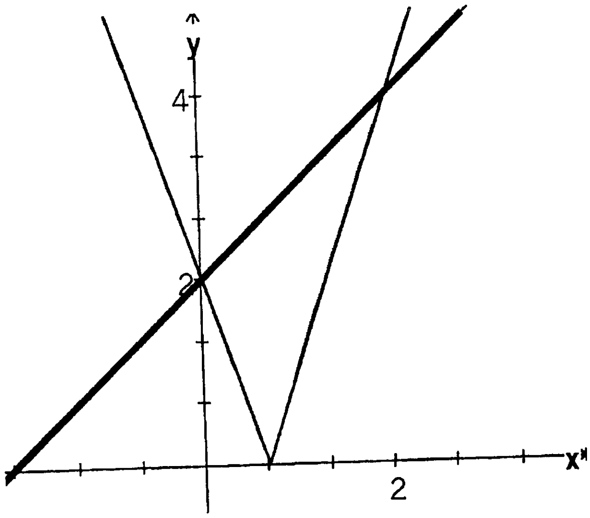
Question 4 is continued on page 6

Marks

Question 4 **continued**

d) i) Specify the equation graphed by the thinner of the two lines. **1**

ii) What values of x are defined by $x+2 \geq |3x-2|$? **1**



Question 5 12 Marks Start a fresh sheet of paper. **Marks**

a) i) Use the method of proof by induction to show that 3

$1+7+19+\ldots\ldots+(3n^2-3n+1)=n^3$

ii) Show that the rule $T_n = S_n - S_{n-1}$ holds true in part (i). 1

b) i) Use the Chain Rule to show that $\frac{dv}{dt}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ 1

ii) The acceleration due to gravity is inversely proportional to the square of the distance x from the centre of the earth. 1

This can be written as $a=\frac{-k}{x^2}$.

Find k if $a = -g$ when $x = \mathrm{R}$.

iii) If the initial velocity of a rocket is u ms^{-1}, show that 2

$v^2=\frac{2R^2g}{x}+u^2-2gR$ where g is the acceleration due to gravity and R is the radius of the earth.

iv) Find the maximum distance that the rocket will travel from the centre of the earth. 1

(Answer in terms of g, R and u)

v) Taking $g = 9.8$, $R = 6400$ km find the value of u in ms^{-1} for which the rocket will escape the gravity of the earth. 1

c) Given that $f(x)=ax^3+bx^2+cx+d$ is a function with a double root at $x = -1$ and with a minimum value of -4 when $x = 1$, find the values of a, b, c and d. 2

Marks

Question 6 12 Marks Start a fresh sheet of paper.

a) A body is moving in a straight line and its position x is given by

$x = 2\sin^2 t$.

i) What are the extremities of its position? **1**

ii) Express the acceleration of the particle in terms of x. **2**

iii) Show the particle is undergoing SHM. **1**

iv) Find its maximum speed. **1**

b) The binomial theorem states that $(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k$ **2**

Show that ${}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \ldots\ldots . n\,{}^nC_n = n \times 2^{n-1}$

c) $\left(\frac{1}{2}+\frac{1}{2}\right)^7$ represents the outcomes in terms of gender of children **1**

for a family with 7 children.

Calculate the probability of 5 boys and 2 girls.

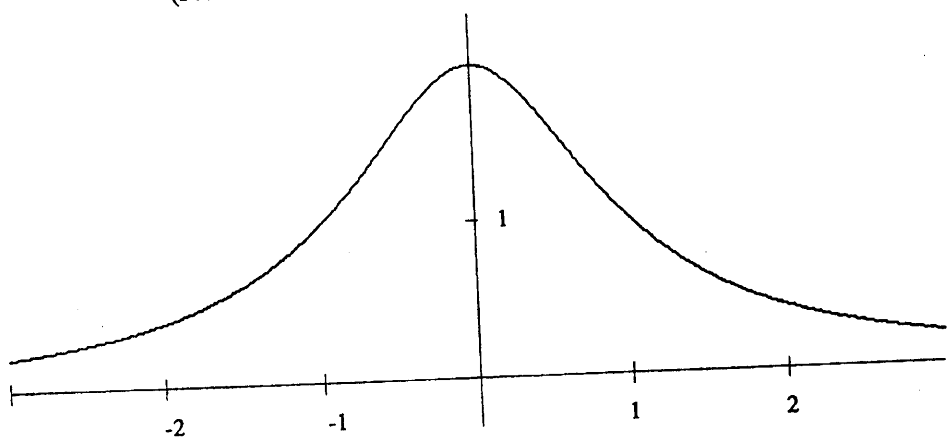
d) The graph below shows the derivative of $y = 2\tan^{-1} x$.

i) Where does $y = 2\tan^{-1} x$ have its greatest slope and what is this slope? **1**

ii) What x values correspond with $\frac{dy}{dx} = \frac{1}{3}$ **1**

iii) What is the total area bounded by this curve and the x axis? **2**

(Note: Domain of the function is $-\infty \le x \le \infty$)



Question 2 12 Marks Start a fresh sheet of paper. **Marks**

a) Consider the points A (-1, -1) B (2, 4) C (8, 14)

i) Find the ratio AB:BC **1**

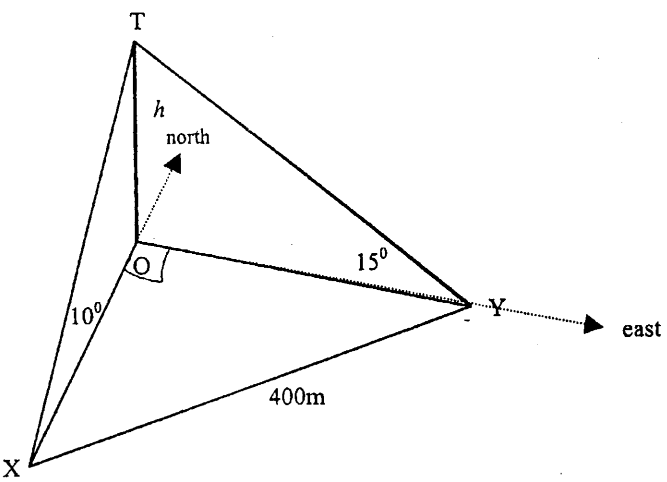
ii) Complete the statement "C divides AB externally in the ratio ..." **1**

b) i) If $\log_3 12 = 2.26186$, find $\log_3 4$ **1**

ii) Find $\log_e e^{1.09}$ **1**

c) Find the quotient and the remainder when $x^3 + 4x^2 - 2x + 3$ is divided by $x^2 - 1$. **2**

d) A surveyor at X observes a tower due north.
The angle of elevation to the top of the tower is 10^0.
He then walks 400m to a position Y which is due east of the tower.
The angle of elevation from Y to the top of the tower is 15^0.



i) Write an expression for OY in terms of h. **1**

ii) Calculate h to the nearest metre. **3**

iii) Find the bearing of Y from X. **2**

Question 7 12 Marks Start a fresh sheet of paper. **Marks**

a) A projectile has an initial velocity V and an angle of projection θ.

i) Assuming $\frac{d^2y}{dt^2} = -10$, $\frac{d^2x}{dt^2} = 0$ and the initial point of projection is 10m above the origin, find expressions for x and y in terms of t. **3**

ii) If $V = 13\text{ms}^{-1}$ and $\theta = \tan^{-1}\frac{5}{12}$ find the range of the projectile. **2**

b) P $(2ap, ap^2)$ and Q $(2aq, aq^2)$ are extremities of a focal chord for the parabola $x^2 = 4ay$.

i) Form the equation of the chord PQ and deduce the constraint on p and q. **2**

ii) Find where the tangents at P and Q meet. **2**

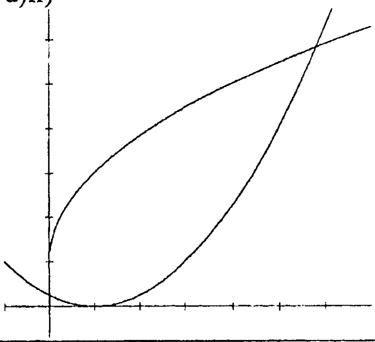
iii) Show that the chord PQ has length $a\left(p+\frac{1}{p}\right)^2$. **3**

End of Paper

Solutions Question 1 2003	Marks/Comments
1 a) $\lim_{x\to\pm\infty}\frac{3x^2+4x+5}{x^2}$ $=\lim_{x\to\infty}(3+\frac{4}{x}+\frac{5}{x^2})$ $=3+0+0$ $=3$	1 for realising x tends to infinity represents horizontal asymptote 1 for answer
1 b) $\frac{x}{2-x}\le 4, x\ne 2$ $x=8-4x$ $5x=8$ $x=1.6$ $x=2$, 1.6 are critical points test points x=0 ✓, x=5✓, x=1.75 × $x\le 1.6$ *or* $x>2$	1 for CPs by either method 1 for test 1 for statement. not 3rd mark if $x\ge 2$
1 c) $2+5+8+\ldots\ldots+56$ has 19 terms with common difference = 3 $\frac{n}{2}(a+l)=9.5\times 58=551$	1 for clear expression 1 for correct answer
1 d) $\int_2^6\frac{x}{\sqrt{x^2-3}}dx$ $\frac{du}{dx}=2x,\ x=2, u=1,\ x=6, u=33$ $I=\frac{1}{2}\int_1^{33}\frac{du}{\sqrt{u}}$ $=\frac{1}{2}\left[2u^{½}\right]_1^{33}$ $=\sqrt{33}-1$	1 not all reqd 1 for clear statement of integral 1 for completion
1 e) i) (15,150)	1
1 e) ii) $9t^2=x^2=1.5y$	1 /12

Solutions Question 2 2003	Marks/Comments
a) i) 1:2	1
a) ii) 3:2	1
b) i) $\log_3 4 = \log_3 \frac{12}{3}$ $= \log_3 12 - \log_3 3$ $= 2.26186 - 1$ $= 1.26186$	1
b) ii) 1.09	1
c) $x^2 - 1\overline{)x^3 + 4x^2 - 2x + 3}$ $Q(x) = x + 4, \quad R(x) = 7 - x$	1 for setting up the division 1
d) i) $\frac{h}{OY} = \tan 15^0$ $OY = \frac{h}{\tan 15^0}$ or $h\cot 15^0$	1
d) ii) Likewise $OX = h\cot 10^0$ Now right angle at O in Δ OXY so $400^2 = h^2(\cot^2 15^0 + \cot^2 10^0)$ $h = \frac{400}{\sqrt{\cot^2 15^0 + \cot^2 10^0}}$ $h = \frac{400}{\sqrt{46.09164071}}$ $= 59m$	1 1 1
d) iii) $\tan \angle OXY$ $= \frac{h\cot 15^0}{h\cot 10^0}$ $= \frac{\tan 10^0}{\tan 15^0}$ $= .658...$ $\angle OXY = 33^0 21'$	1 1 /12

Solutions Question 3 2003	Marks/Comments
a) Now	1
$\cos 2x = 2\cos^2 x - 1$	
$\cos^2 x = \dfrac{\cos 2x + 1}{2}$	
$\cos^2 4x = \dfrac{\cos 8x + 1}{2}$	
$I = \dfrac{1}{2}\displaystyle\int_0^{\pi/16} \cos 8x + 1 dx$	1
$= \dfrac{1}{2}\left[\dfrac{1}{8}\sin 8x + x\right]_0^{\pi/16}$	
$= \dfrac{1}{2}\left(\dfrac{1}{8}\sin\dfrac{\pi}{2} + \dfrac{\pi}{16} - 0\right)$	
$= \dfrac{2+\pi}{32}$	1
b) i) 9 letters, E appears 3 times and S appears twice $\dfrac{9!}{3!2!} = 30240$	1 simplification not reqd
b) ii) The requirement is C V C V C V C V C	1
Vowels can be ordered in $\dfrac{4!}{3!} = 4$ ways	
Consonants can be ordered in $\dfrac{5!}{2!} = 60$ ways	
Probability $= \dfrac{240}{30240} = \dfrac{1}{126}$	1 simplification not reqd
c) $4\cos\theta + 3\sin\theta = 2$	
then $\dfrac{4}{5}\cos\theta + \dfrac{3}{5}\sin\theta = \dfrac{2}{5}$	1
noting $\sin(\alpha+\theta) = \sin\alpha\cos\theta + \cos\alpha\sin\theta$	
we have $\sin\alpha = \dfrac{4}{5}$ *and* $\cos\alpha = \dfrac{3}{5}$	1
$\alpha = 0.9273^c$ ~~3.3~~ 2.73	1
$\therefore 0.9273 + \theta = 0.41151^c$ *or* $\cancel{1.982313^c}$	1 ignore failure to conform to accuracy. (Answer in degrees acceptable)
and so $\theta = \cancel{1.06^c}, \cancel{5.77^c}$ [illegible] wrong 5.77, 1.8 ~~1~~.85	
d) i) $x \le 1$ *or* $x \ge 1$	1

d)ii)	1
	/12

Solutions Question 4 2003	Marks/Comments
a) BX =DX (tangents drawn from external point)	1
$\therefore \angle$DBX=$\angle$BDX	
likewise AX = CX and $\angle$CAX = $\angle$ACX	1
(Both these pairs of equal angles are equal since $\angle$X is common in both triangles)	
$\therefore$ AC ‖ BD (corresponding angles equal)	1
b)	
$\cos(\theta+2\theta)$	
$=\cos\theta\cos 2\theta-\sin\theta\sin 2\theta$	
$=\cos\theta(2\cos^2\theta-1)-\sin\theta(2\sin\theta\cos\theta)$	1
$=2\cos^3\theta-\cos\theta-2(1-\cos^2\theta)\cos\theta$	
$=4\cos^3\theta-3\cos\theta$	1
c) $\sin x=\dfrac{10\pm\sqrt{100-96}}{16}=0.5 \ \ or \ \ 0.75$	1
Then $x=.524^c$ or $.848^c$	1
Minimum when first deriv. = 0	
$16\sin x\cos x-10\cos x=0$	
$16\sin x=10$	1
since $\cos x\neq 0$ $\quad$ ($x=\frac{\pi}{2}$ but $\frac{\pi}{2}>1$, $\therefore$ not a solution)	
sin$x$=0.625 , $x=.675\ldots$	1
$y=8\left(\dfrac{5}{8}\right)^2-10\times\dfrac{5}{8}+3=-0.125$	1
d) i) $y=\lvert 3x-2\rvert$	1
d) ii) $0\leq x\leq 2$	1 /12

Solutions Question 5 2003	Marks/Comments
a) i) If $n=1$, $1=1^3$ true when $n=1$ Assume when $n=k$ ie. $1+7+19+.....+(3k^2-3k+1)=k^3$ Reqd to prove $1+7+...+(3k^2-3k+1)+(3(k+1)^2-3(k+1)+1)=(k+1)^3$ LHS = $k^3+3(k+1)^2-3(k+1)+1$ $=k^3+3k^2+6k+3-3k-3+1=k^3+3k^2+3k+1$ $=(k+1)^3$ The proposal holds when $n=1$. If assumed for a number it will hold for the next number, so it holds for n = 2 etc. Hence by induction the proposal holds for all $n\in J,\ n\geq 1$	1 1 1
a) ii) $n^3-(n-1)^3$ $=n^3-n^3+3n^2-3n+1$ $=3n^2-3n+1$	1
b) i) $\frac{dv}{dt}=\frac{dv}{dx}\cdot\frac{dx}{dt}$ $=v\cdot\frac{dv}{dx}$ $=\frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx}$ $=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$	1
b) ii) $-g=\frac{-k}{R^2}$ $\therefore k=gR^2$	1
b) iii) $a=\frac{-gR^2}{x^2}$ $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-gR^2x^{-2}$ $\frac{1}{2}v^2=\int -gR^2x^{-2}dx$ $v^2=2gR^2x^{-1}+c$ but when $x=R$, $v=u$, $\therefore c=u^2-2gR$ $\therefore\ v^2=\frac{2R^2g}{x}+u^2-2gR$ as reqd	1 1

b) iv) $v = 0$ for max distance $0 = \frac{2R^2 g}{x} + u^2 - 2Rg$ $2Rg - u^2 = \frac{2R^2 g}{x}$ $x = \frac{2R^2 g}{2Rg - u^2}$	1
b) v) $as \quad x \to \infty, \quad u^2 = 2gR$ $= 2 \times 9.8 \times 6400000$ $u = 11200 ms^{-1}$	1
c) $f(x) = ax^3 + bx^2 + cx + d \qquad f^{\prime}(x) = 3ax^2 + 2bx + c$ $f(-1) = -a + b - c + d \ ...(1) \quad f^{\prime}(-1) = 3a - 2b + c ...(2)$ (double root at x = -1) min value at (1,-4) $\therefore \ f(1) = a + b + c + d = -4...(3) \quad f^{\prime}(1) = 3a + 2b + c = 0...(4)$ (min turning point) Solving $(2)+(4)$, $(2)-(4)$, $(1)+(3)$, $(3)-(1)$ and subbing we get $a = 1$, $b = 0$, $c = -3$, $d = -2$	1 1 /12

Solutions Question 6 2003	Marks/Comments
a) i) $-1 \le \sin t \le 1$ $0 \le \sin^2 t \le 1$ $0 \le 2\sin t \le 2$ $\therefore$ extremities are between $x = 0$ and $x = 2$	1
a) ii) $\frac{dx}{dt} = 2 \times 2\sin t \cos t$ $= 4\sin t \cos t$ $\frac{d^2x}{dt^2} = vu' + uv'$ $= 4(\cos^2 t - \sin^2 t)$ $= 4\left(1 - 2\sin^2 t\right)$ $= 4(1 - x)$	1 for clear intention to differentiate wrt t 1 for completion
a) iii) Particle has SHM since its acceleration has form $-n^2X$	1
a) iv) Maximum speed when $x = 1$, $t = \sin^{-1}(1/\sqrt{2})$ Then $\frac{dx}{dt} = 2 \times 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = 2\ \text{ms}^{-1}$	1
b) $(1+x)^n = {}^nC_o + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$ differentiating both sides wrt x $n(1+x)^{n-1} = {}^nC_1 + 2\,{}^nC_2 x + 3\,{}^nC_3 x^2 + \ldots + n\,{}^nC_n x^{n-1}$ letting $x = 1$ RHS $= n \times 2^{n-1}$	1 for clear expression of bin. th. and differentiating or letting x equal something 1
c) ${}^7C_5 \left(\frac{1}{2}\right)^5 \times \left(\frac{1}{2}\right)^2 = 0.1640625$	1
d) i) By inspection $m_{\max} = 2$ when $x = 0$	1
d) ii) The curve represents $\frac{dy}{dx} = \frac{2}{1+x^2}$ which equals $\frac{1}{3}$ when $x = \pm\sqrt{5}$	1
d)iii) $\int_{-\infty}^{\infty} \frac{2}{1+x^2}\,dx = 2\int_0^{\infty} \frac{2}{1+x^2}\,dx$ $= 4\int_0^{\infty} \frac{1}{1+x^2}\,dx$ $= 4\left[\tan^{-1} x\right]_0^{\infty}$ $= 4 \times \frac{\pi}{2}$ $= 2\pi$	1 1

Solutions Question 7 2003	Marks/Comments
a) i) $\frac{dy}{dt} = -10t + c$ but when t = 0 $y' = V\sin\theta$ so $y' = V\sin\theta - 10t$ Also $x' = V\cos\theta$ $x = Vt\cos\theta$ $y = \int V\sin\theta - 10t\,dt = Vt\sin\theta - 5t^2 + c$ and from the initial conditions $c = 10$	1 clear intention to integrate both wrt t 1 1 for correct constants
a) ii) By Pythagoras $\sin\theta = 5/13$ and $\cos\theta = 12/13$ We require x when $y = 0$ $y = 13t\frac{5}{13} - 5t^2 + 10$ which =0 when $t = \frac{-5 \pm \sqrt{25 + 4\times5\times10}}{-10}$ $= \frac{-20}{-10}$ *or* $\frac{10}{-10}$ When $t = 2$ $x = 13\times2\times\frac{12}{13} = 24$ m	1 1
b) i) PQ has eqn $\frac{aq^2 - ap^2}{2aq - 2ap} = \frac{y - ap^2}{x - 2ap}$ $= \frac{q+p}{2}$ which becomes $2y - 2ap^2 = (p+q)x - 2apq - 2ap^2$ when $x = 0$, $y = a$ $2a = -2apq$ $\therefore pq = -1$	1 1
b) ii) Tangent at P $y = px - ap^2$ Tangent at Q $y = qx - aq^2$ $q\times$ Tangent at P $qy = pqx - ap^2q$ $p\times$ Tangent at Q $py = pqx - aq^2p$ whence $(q-p)y = apq(q-p)$ $y = -a$ subbing $-a = px - ap^2$ $pqa + ap^2 = px$ $x = a(p+q)$	1 1
b)iii) $PQ = \sqrt{(2ap - 2aq)^2 + (ap^2 - aq^2)^2}$	1

$=\sqrt{4a^2(p-q)^2+a^2(p^2-q^2)^2}$ $=a\sqrt{4(p-q)^2+(p-q)(p+q)^2}$ $=a(p-q)\sqrt{4+(p+q)^2}\ ,\ p>q$ $=a(p-q)\sqrt{-4pq+(p+q)^2}\ ,\quad (pq=-1)$ $=a(p-q)\sqrt{(p-q)^2}$ $=a(p-q)^2$ but $q=\dfrac{-1}{p}$ $=a\left(p+\dfrac{1}{p}\right)^2$ as reqd	1 1 or 3 marks for other method

Mathematics Extension 1 2003 Trial HSC Examination Mapping Grid

Question	Marks	Content	Syllabus Outcomes	Targetted Performance Bands
1 a)	2	Real functions of a real variable and their geometrical representation	P5	E2 - E3
1 b)	3	Basic Arithmetic and Algebra	P4	E2 - E4
1 c)	2	Series and Applications	H5, H9	E2 - E3
1 d)	3	Integration	HE6	E2 - E3
1 e) i)	1	The Quadratic Polynomial and the Parabola	PE4	E2 - E3
1 e) ii)	1	The Quadratic Polynomial and the Parabola	PE4	E2 - E3
2 a) i)	1	Linear Function and lines	P4	E2 - E3
2 a) ii)	1	Linear Function and lines	P4	E2 - E3
2 b) i)	1	Logarithmic and Exponential Functions	H3	E2 - E3
2 b) ii)	1	Logarithmic and Exponential Functions	H3	E2 - E3
2 c)	2	Polynomials	PE3	E2 - E3
2 d) i)	1	Trigonometric Ratios	P4	E2 - E3
2 d) ii)	3	Trigonometric Ratios Basic Arithmetic and Algebra	HE3	E3 - E4
2 d) iii)	2	Trigonometric Ratios	H5	E2 - E3
3 a)	3	Integration, The Trigonometric Functions	H8	E2 - E3
3 b) i)	1	Permutations, Combinations and Further Probability	PE3	E2 - E3
3 b) ii)	2	Permutations, Combinations and Further Probability	PE3	E3 - E4
3 c)	4	Trigonometric Ratios, The Trigonometric Functions	H5, PE2	E2 - E3
3 d) i)	1	Inverse function, and Inverse Trigonometric Functions	HE4	E2 - E4
3 d) ii)	1	Inverse function, and Inverse Trigonometric Functions	H5, HE4	E2 - E3
4 a)	3	Circle Geometry	PE2,PE3,H5	E2 - E4
4 b)	2	Trigonometric Ratios, Basic Arithmetic and Algebra	P4	E3 - E4
4 c)	5	The Trigonometric Functions, Geometric Applications of Differentiation	H2, H5, H6	E3 - E4
4 d) i)	1	Basic Arithmetic and Algebra	P5	E2 - E4
4 d) ii)	1	Basic Arithmetic and Algebra	P5	E2 - E4
5 a) i)	3	Series and Applications	HE2	E2 - E3
5 a) ii)	1	Series and Applications	H5, HE2	E2 - E3
5 b) i)	1	Applications of Calculus to the Physical World	HE3, HE5	E2 - E3
5 b) ii)	1	Applications of Calculus to the Physical World	P3	E3 - E4
5 b) iii)	2	Applications of Calculus to the Physical World	HE3, HE7	E3 - E4
5 b) iv)	1	Applications of Calculus to the Physical World	HE3	E3 - E4
5 b) v)	1	Applications of Calculus to the Physical World	HE3	E2 - E4
5 c)	2	Polynomials	HE7, PE3	E2 - E3
6 a) i)	1	Applications of Calculus to the Physical World, Trigonometric Functions	H5	E2 - E4
6 a) ii)	2	Applications of Calculus to the Physical World	HE3	E2 - E3
6 a) iii)	1	Applications of Calculus to the Physical World	H5	E3 - E4
6 a) iv)	1	Applications of Calculus to the Physical World	HE3	E3 - E4