Mrs Gibson
Mr Keanan Brown
Mrs Lee
Mrs Choong
Mrs Leslie

PYMBLE LADIES' COLLEGE

YEAR 12

MATHEMATICS EXTENSION 1

HSC TRIAL EXAMINATION 2001

Time Allowed: 2 hours + 5 mins reading time **Test date: 16 August 2001**

Instructions:

- All questions should be attempted.
- Write your name and your teacher's name on each page
- Start each question on a new page.
- **DO NOT** staple the questions together.
- Only approved calculators may be used.
- A standard integral sheet is attached.
- Marks might be deducted for careless or untidy work.
- Hand this question paper in with your answers.
- ALL rough working paper must be attached to the back of the last question.
- Staple a coloured sheet of paper to the back of each question.
- There are seven (7) questions in this paper.
- All questions are of equal value.

MARKING GUIDELINES

- **Provide answers which are complete, accurate and comprehensive.**
- **Leave your answers in exact form unless otherwise stated.**
- **Include all necessary working. Correct answers will not necessarily gain full marks unless necessary working is shown. Relevant working <u>might</u> gain marks even if your answer is wrong.**
- **Take care with mathematical notation.**
- **Show relevant information clearly and unambiguously on sketches if required.**
- **Present well set out solutions using a logical set of steps in which justification is included where necessary.**

QUESTION 1 **Marks**

(a) Differentiate $\dfrac{1}{1+x^2}$ **1**

(b) The polynomial $P(x) = 2x^3 - x + a$ is divisible by $x + 2$. **1**

Find the value of a.

(c) A, B and P are the points (-1,8), (6,-6) and (4,-2) respectively. **2**

The point P divides the interval AB internally in the ratio $k{:}1$.

Find the value of k.

(d) Solve $\mathrm{x} - 1 = \sqrt{\mathrm{x} + 1}$ **3**

(e) Evaluate $\displaystyle\int_1^{\sqrt{2}} \frac{1}{\sqrt{4 - x^2}}\, dx$ **3**

(f) Solve $|\,3 - 3x\,| > x + 3$ **2**

QUESTION 2 **Start a new page** **Marks**

(a) Find the exact value of $cos\left(sin^{-1}(-\frac{1}{3})\right)$ **2**

(b) Given that $\log_b a = 2$ and $\log_c b = 3$, find the value of $\log_a$ c. **2**

(c) Find the value of $\int_0^3 \frac{t}{\sqrt{1+t}}\,dt$ **4**

using the substitution $t = u^2 - 1$ where $u > 0$

(d) A and B are acute angles such that $cos\,A = \frac{3}{5}$ and $sin\,B = \frac{1}{\sqrt{5}}$. **4**

Without finding the size of either angle, show that $A = 2B$, and

use this result to find the exact value of $sin\,3B$.

QUESTION 3 **Start a new page** **Marks**

(a) Write down the value of the **prime** number b such that

$$\sum_{n=1}^{3} \log_2 2n = a + \log_2 b$$ 1

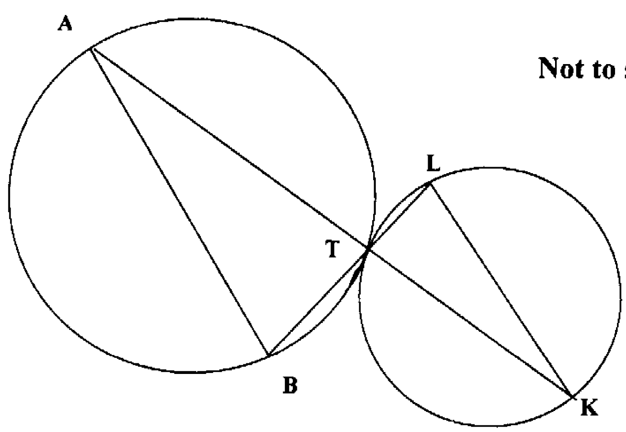
(b) The diagram shows two circles touching externally at T.

AB is any diameter of the first circle, and AT and BT are produced to meet the second circle again at K and L respectively.

Copy the diagram onto your answer paper, then prove that

(i) KL is a diameter of the second circle 1

(ii) LK is parallel to AB 2



Not to scale

(c) Evaluate $\int_0^{\frac{\pi}{4}} (\cos x + \sec x)^2 \, dx$ 4

(d) The perimeter of an equilateral triangle of side a *cm* is increasing at a constant rate of 6 *cm/sec* as the triangle is being enlarged. 4

Find the rate at which the area of the triangle is increasing at the instant the perimeter is 24 *cm*. (The triangle remains equilateral.)

QUESTION 4 **Start a new page** **Marks**

(a) A certain population N is changing at a rate given by the equation

$$\frac{dN}{dt} = 0\cdot5\,(N - 100).$$

(i) Show that $N = 100 + Ae^{0\cdot5t}$ is a solution of this equation, and find the value of A given that the initial value of N is 500. **2**

(ii) Find the value of N when $t = 10$. **1**

(b) A function $f(x)$ has an inverse whose equation is $f^{-1}(x) = \frac{2x-2}{x-2}$. **3**

What is the equation of $f(x)$?

Explain the geometrical significance of your answer.

(c) (i) Sketch $f(x) = \sin x$ and its inverse $g(x) = \sin^{-1} x$ on the same axes for $0 \le x \le \frac{\pi}{2}$. **1**

(ii) Show that the tangent at $x = 1$ on $f(x)$ and the tangent at $y = 1$ on $g(x)$ are equally inclined to $y = x$. **4**

(iii) What is the angle between these two tangents? **1**

QUESTION 5 **Start a new page** **Marks**

(a) A particle travels in a straight line executing simple harmonic motion about O according to the equation $x = a\cos nt$.

(i) Show that the velocity v and displacement x of the particle at any time t are related by the equation $v^2 = n^2(a^2 - x^2)$. **2**

(ii) **Hence** show that the acceleration of the particle can be given as $\ddot{x} = -n^2 x$. **1**

(b) A particle executes simple harmonic motion about O according to the above equations. Initially it is at $x = 2$. As it passes through O its speed is 2 m/sec. How long does it take to get to O for the first time? **3**

(c) Draw a large and accurate sketch of the curve $y = \dfrac{x+4}{x(x+8)}$, showing all essential features such as intercepts on axes and asymptotes. **4**

Show that there are no stationary points. (You do not need to find the coordinates of any inflection points.)

(d) Find the area bound by the curve $y = \dfrac{x+4}{x(x+8)}$ and the x - axis between $x=1$ and $x=2$. **2**

You may use the substitution $u = x(x+8)$ to evaluate this area if you wish.

QUESTION 6 **Start a new page** **Marks**

(a) A curve has equation $f(x) = 3x - 4x^3$.

(i) Show that the equation of a tangent at the point on the curve where $x = a$ is $y = (3 - 12a^2)x + 8a^3$. **2**

(ii) How many tangents can be drawn to this curve from the point (1,0)? (You must show full working to substantiate your answer.) **3**

(b) P $\left(2ap, ap^2\right)$ and Q $\left(2aq, aq^2\right)$ are two points on the parabola $x^2 = 4ay$.

The tangent at P and a line through Q parallel to the y axis meet at point R.

The tangent at Q and a line through P parallel to the y axis meet at point S.

(i) Draw a neat diagram showing all information given above. **1**

(ii) Show that the equation of the tangent at P is $y = px - ap^2$. **2**

(iii) Show that PQRS is a parallelogram **2**

(iv) Show that the area of PQRS is $2a^2|p - q|^3$ square units. **2**

QUESTION 7 **Start a new page** **Marks**

(a) A particle moves in a straight line towards the centre O experiencing an acceleration that is inversely proportional to the cube of the distance from O, namely $a = -\dfrac{4}{x^3}$.

(i) If the particle starts from rest at $x = 2$, find an expression for the velocity of the particle in terms of x. **3**

Make sure you justify the sign of your expression.

(ii) Hence find an expression that relates elapsed time t and displacement x, and find the time the particle takes to reach $x = 1$ (for the first time, if it does so more than once). **3**

(b) (i) Prove by induction that for all integers $n \geq 1$ **3**

$$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(ii) Use this result to evaluate $2^2 + 4^2 + 6^2 + \ldots + 100^2$ **2**

(iii) Hence evaluate $1^2 + 3^2 + 5^2 + \ldots\ldots + 99^2$ **1**

END OF PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$