

### 3/4 UNIT MATHEMATICS FORM VI

**Time allowed:** 2 hours (plus 5 minutes reading)

**Exam date:** 16th August, 1999

**Instructions:**

- All questions may be attempted.
- All questions are of equal value.
- Part marks are shown in boxes in the left margin.
- All necessary working must be shown.
- Marks may not be awarded for careless or badly arranged work.
- Approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

**Collection:**

- Each question will be collected separately.
- Start each question in a new answer booklet.
- If you use a second booklet for a question, place it inside the first. Don't staple.
- Write your candidate number on each answer booklet.

QUESTION ONE (Start a new answer booklet)

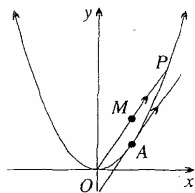
Marks

- 2 (a) Find and simplify the term in  $x^5$  in the expansion of  $(2 - x)^7$ .
- 2 (b) Differentiate  $e^{2x} \sin x$ .
- 2 (c) Find the gradient of the tangent to  $y = \sin^{-1} \frac{x}{2}$  at the point where  $x = 1$ .
- 2 (d) Solve  $x^2 - x - 6 > 0$ .
- 2 (e) Find, correct to the nearest minute, the acute angle between the lines  $x - y + 3 = 0$  and  $2x + y + 1 = 0$ .
- 2 (f) Find:
- (i)  $\int \frac{1 + e^x}{e^x} dx$ ,
- (ii)  $\int \frac{e^x}{1 + e^x} dx$ .

**QUESTION TWO** (Start a new answer booklet)

Marks

- 2** (a) Find the general solution of  $\cos x = -\frac{1}{2}$ .
- 2** (b) What are the coordinates of the focus of the parabola  $(x + 3)^2 = 8(y - 1)$ ?
- 4** (c)



The point  $P(2ap, ap^2)$  and the origin  $O$  lie on the parabola  $x^2 = 4ay$ .  $M$  is the mid-point of the chord  $OP$ .

- (i) Find the gradient of  $OP$ .
- (ii) Show that the tangent at a point  $T(2at, at^2)$  on the parabola has gradient  $t$ .
- (iii) Hence find the point  $A$  on the parabola where the tangent is parallel with the chord  $OP$ , and show that  $A$  is equidistant from  $M$  and the  $x$ -axis.
- 4** (d) (i) Show  $\alpha^3 + \beta^3 = (\alpha + \beta)((\alpha + \beta)^2 - 3\alpha\beta)$ .
- (ii) Given  $\alpha$  and  $\beta$  are roots of the quadratic equation  $x^2 + 3x - 2 = 0$ , find the value of  $\alpha^3 + \beta^3$  without finding the values of the roots.

**QUESTION THREE** (Start a new answer booklet)

Marks

- 4** (a) (i) Prove that  $\sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta$ .
- (ii) Hence determine  $\int_0^\pi \sin^2 \theta \, d\theta$ .
- 4** (b) Use the substitution  $u = 1 - x$  to help evaluate  $\int_0^1 (1 + 3x)(1 - x)^7 \, dx$ .
- 4** (c) (i) Write down a value of  $\theta$  for which  $\frac{1}{1 + \sin \theta}$  is undefined.
- (ii) Show that  $\frac{1}{1 + \sin \theta} = \sec^2 \theta - \sec \theta \tan \theta$ .
- (iii) Hence find  $\int \frac{1}{1 + \sin \theta} \, d\theta$ . [HINT: You may want to consult the list of standard integrals.]

QUESTION FOUR (Start a new answer booklet)

Marks

- 3 (a) (i) Use sigma notation to express  $(1+x)^{2n}$  as a sum of powers of  $x$ .

(ii) Hence show that  $\sum_{r=0}^{2n} {}^{2n}C_r \left(-\frac{1}{2}\right)^r = \left(\frac{1}{2}\right)^{2n}$ .

(iii) Hence evaluate  $\sum_{r=0}^{2n-1} {}^{2n}C_r \left(-\frac{1}{2}\right)^r$ .

- 4 (b) (i) Expand  $\left(x - \frac{1}{x}\right)^2$ .

(ii) Show that  $\left(x^2 + \frac{1}{x^2}\right)^{14} = \sum_{r=0}^{14} {}^{14}C_r x^{28-4r}$ .

(iii) Hence show that the coefficient of  $x^6$  in the expansion of  $\left(x - \frac{1}{x}\right)^2 \left(x^2 + \frac{1}{x^2}\right)^{14}$  is equal to  ${}^{15}C_6$ .

- 5 (c) (i) An amount  $P$  is borrowed from a bank at an interest rate of  $R$  per month compounded monthly. At the end of each month, an instalment  $M$  is paid back to the bank. Let  $A_n$  be the amount owed at the end of the  $n^{\text{th}}$  month, after the instalment is paid. Show that:

$$A_n = P(1+R)^n - \frac{M((1+R)^n - 1)}{R}.$$

- (ii) A couple want to borrow \$20 000 from the bank, for a new car. After all charges are taken into account, the effective interest rate for the personal loan is 1.2% per month, compounded monthly, with the loan to be repaid over 5 years. The couple can only afford to make repayments of \$450 per month. Will the bank give them the loan? Justify your answer.

**QUESTION FIVE** (Start a new answer booklet)

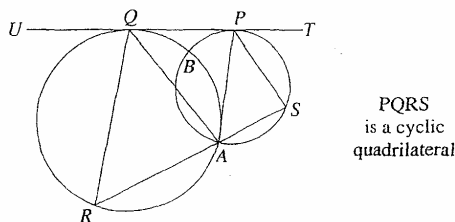
Marks

- 5** (a) An object moves so that its displacement  $x$  metres at time  $t$  seconds is given by:

$$x = \cos 3t + 2 \sin 3t .$$

- Show that the motion is simple harmonic by showing that it satisfies the differential equation  $\ddot{x} = -n^2 x$ , for some  $n > 0$ .
- Express  $x$  in the form  $r \sin(3t + \alpha)$ , where  $r > 0$  and  $0 \leq \alpha < \frac{\pi}{2}$ .
- Hence find at what time, to the nearest second, the object first reaches  $x = 2$ .

- 5** (b)

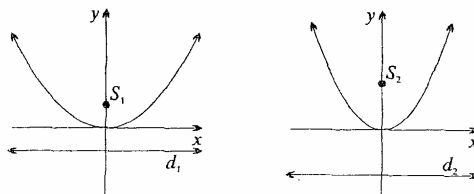


In the diagram above, two circles intersect at  $A$  and  $B$ . The common tangent  $TU$  touches the circles at  $P$  and  $Q$  respectively. A line through  $A$  cuts the left-hand circle at  $R$  and the right-hand circle at  $S$ , and it is found that  $PQRS$  is a cyclic quadrilateral. Copy the diagram into your answer booklet.

- Give a reason why  $\angle UQR = \angle PSA$ .
- Use the angle in the alternate segment theorem to prove that  $PS \parallel AQ$ .
- Thus show that  $\triangle PAS \parallel \triangle QRA$ .

- 2** (c) (i) The *latus rectum* of a parabola is the focal chord parallel with the directrix. The parabola  $x^2 = 4ay$  has focal length  $a$ . Write down the length of its latus rectum.

- (ii)

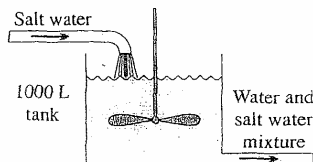


A pupil drew the graphs of two different parabolas, shown above, using the same scale on both graphs and using the given foci and directrices. The left hand graph is correct. Explain why the right hand graph must be incorrect.

QUESTION SIX (Start a new answer booklet)

Marks

8 (a)



In the diagram above, a tank initially contains 1000 L of pure water. Salt water begins pouring into the tank from a pipe and a stirring blade ensures it is completely mixed with the pure water. A second pipe draws the water and salt water mixture off at the same rate, so that there is always a total of 1000 L in the tank.

- (i) If the salt water entering the tank contains 2 grams of salt per litre and is flowing in at the constant rate of  $w$  L/min, how much salt is entering the tank per minute?
- (ii) If  $Q$  grams is the amount of salt in the tank at time  $t$ , how much salt is in 1 L at time  $t$ ?
- (iii) Hence write down the amount of salt leaving the tank per minute.
- (iv) Use the previous parts to show that  $\frac{dQ}{dt} = -\frac{w}{1000}(Q - 2000)$ .
- (v) Show that  $Q = 2000 + Ae^{-\frac{wt}{1000}}$  is a solution of this differential equation.
- (vi) Determine the value of  $A$ .
- (vii) What happens to  $Q$  as  $t \rightarrow \infty$ ?
- (viii) If there is 1 kg of salt in the tank after  $5\frac{3}{4}$  hours, find  $w$ .

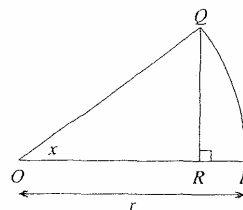
- 4 (b) A pupil investigated a differentiable function  $f(x)$  and found the following information:  $f(x)$  has its only zero at  $x = -1$ ,  $f(0) = 2$ ,  $\lim_{x \rightarrow \infty} f(x) = 0$ .

- (i) Draw a graph of the possible shape of  $f(x)$ .
- (ii) Use your graph to demonstrate that  $f(x)$  must have an inflexion point to the right of  $x = -1$ .

**QUESTION SEVEN** (Start a new answer booklet)

Marks

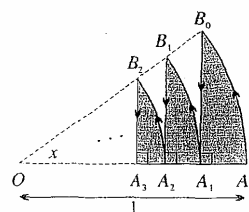
- 7** (a) (i) In the diagram on the right,  $PQ$  is the arc of a circle with radius  $r$  subtending an acute angle  $x$  at the centre  $O$ .  $R$  is the foot of the perpendicular from  $Q$  to the radius  $OP$ . Find lengths of the arc  $PQ$  and the interval  $QR$  in terms of  $x$  and  $r$ .



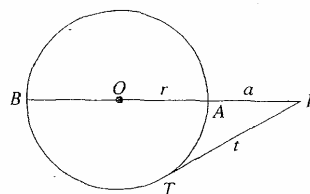
- (ii) An ant travels from  $A_0$  to  $O$  along the saw-tooth path as shown in the diagram on the right. Show that the total distance  $y$  travelled by the ant is:

$$y = \frac{x + \sin x}{1 - \cos x}$$

- (iii) Given  $0 < x \leq \frac{\pi}{2}$ , use the derivative of  $y$  to find the value of  $x$  that gives the shortest such distance.



- 5** (b) (i) In the diagram,  $P$  is a point outside a circle with centre  $O$  and radius  $r$ . The secant  $PO$  cuts the circle at  $A$  and  $B$  respectively, and  $PA = a$ .  $PT$  is tangent to the circle at  $T$  and  $PT = t$ .



(α) Give a reason why  $t^2 = a(a + 2r)$ .

- (β) Solve this equation for  $a$  and hence show the geometric mean of  $PA$  and  $PB$  is less than the arithmetic mean.

NOTE: The geometric mean of  $a$  and  $b$  is  $\sqrt{ab}$  and arithmetic mean is  $\frac{a+b}{2}$ .

- (ii) The diagram on the right shows the interval  $PAB$ . A circle is drawn to pass through  $A$  and  $B$ . A tangent is drawn from  $P$  to touch the circle at  $T$ . Find and describe the locus of  $T$  for all such circles and tangents.

