Exercise 6N Proof by Mathematical Induction

1. Use mathematical induction to prove that for all positive integers n:
   (a) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$
   (b) $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$
   (c) $1 + 3 + 5 + 7 + \cdots + (2n-1) = n^2$
   (d) $1 + 2 + 3 + 4 + \cdots + n = \frac{1}{2}n(n+1)$
   (e) $1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$
   (f) $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$
   (g) $\dfrac{1}{1 \times 2} + \dfrac{1}{2 \times 3} + \dfrac{1}{3 \times 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$
   (h) $\dfrac{1}{1 \times 4} + \dfrac{1}{4 \times 7} + \dfrac{1}{7 \times 10} + \cdots + \dfrac{1}{(3n-2)(3n+1)} = \dfrac{n}{3n+1}$
   (i) $2 \times 2^0 + 3 \times 2^1 + 4 \times 2^2 + \cdots + (n+1)2^{n-1} = n \times 2^n$
   (j) $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \cdots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$
   (k) $\dfrac{1}{1 \times 2 \times 3} + \dfrac{1}{2 \times 3 \times 4} + \dfrac{1}{3 \times 4 \times 5} + \cdots + \dfrac{1}{n(n+1)(n+2)} = \dfrac{n(n+3)}{4(n+1)(n+2)}$
   (l) $a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(r^n - 1)}{r-1}$, provided $r \neq 1$
   (m) $a + (a+d) + (a+2d) \cdots + (a + (n-1)d) = \frac{1}{2}n(2a + (n-1)d)$
2. Hence find the limiting sums of the series in parts (g), (h) and (k) of the previous question.

DEVELOPMENT

3. Use mathematical induction to prove these divisibility results for all positive integers n:
   (a) $5^n + 3$ is divisible by 4
   (b) $9^n - 3$ is a multiple of 6
   (c) $11^n - 1$ is divisible by 10
   (d) $5^n + 2 \times 11^n$ is a multiple of 3
   (e) $5^{2n} - 1$ is a multiple of 24
   (f) $x^n - 1$ is divisible by $x - 1$
4. Prove these divisibility results, advancing in part B of the proof from k to $k+2$:
   (a) For even n: (i) $n^3 + 2n$ is divisible by 12 (ii) $n^2 + 2n$ is a multiple of 8
   (b) For odd n: (i) $3^n + 7^n$ is divisible by 10 (ii) $7^n + 6^n$ is divisible by 13
5. Examine the divisors of $n^3 - n$ for low odd values of n, make a judgement about the largest integer divisor, and prove your result by induction.
6. Prove these inequalities by mathematical induction:
   (a) $n^2 > 10n + 7$, for $n \geq 11$
   (b) $2^n > 3n^2$, for $n \geq 8$
   (c) $3^n > n^2$, for $n \geq 2$ (and also for $n = 0$ and 1)
   (d) $(1+\alpha)^n \geq 1 + n\alpha$, for $n \geq 1$, where $\alpha > -1$
7. Examine 2^n and $2n^3$ for low values of n, make a judgement about which is eventually bigger, and prove your result by induction.
8. Prove: (a) $\displaystyle\sum_{r=1}^{n} \frac{1}{r^2} \leq 2 - \frac{1}{n}$, for $n \geq 1$ (b) $\dfrac{1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times \cdots \times 2n} \geq \dfrac{1}{2n}$, for $n \geq 1$
9. (a) Given that $T_n = 2T_{n-1} + 1$ and $T_1 = 5$, prove that $T_n = 6 \times 2^{n-1} - 1$.
   (b) Given that $T_n = \dfrac{3T_{n-1} - 1}{4T_{n-1} - 1}$ and $T_1 = 1$, prove that $T_n = \dfrac{n}{2n-1}$.
10. Prove by induction that the sum of the angles of a polygon with n sides is $n - 2$ straight angles. [HINT: Dissect the $(k+1)$-gon into a k-gon and a triangle.]
11. Prove by induction that n lines in the plane, no two being parallel and no three concurrent, divide the plane into $\frac{1}{2}n(n+1) + 1$ regions. [HINT: The $(k+1)$th line will cross k lines in k distinct points, and so will add $k+1$ regions.]
12. Prove by mathematical induction that every set with n members has 2^n subsets. [HINT: When a new member is added to a k-member set, then every subset of the resulting $(k+1)$-member set either contains or does not contain the new member.]
13. Defining $n! = 1 \times 2 \times 3 \times \cdots \times n$ (this is called 'n factorial'), prove that:
    (a) $\displaystyle\sum_{r=1}^{n} r \times r! = (n+1)! - 1$ (b) $\displaystyle\sum_{r=1}^{n} \frac{r-1}{r!} = 1 - \frac{1}{n!}$
14. Prove: (a) $\displaystyle\sum_{r=1}^{n} r^4 = \tfrac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$ (b) $\displaystyle\sum_{r=1}^{n} r^5 = \tfrac{1}{12}n^2(n+1)^2(2n^2+2n-1)$
15. (a) By rationalising the numerator, prove that $\sqrt{n+1} - \sqrt{n} > \dfrac{1}{2\sqrt{n+1}}$.
    (b) Hence prove by induction that $1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} < \sqrt{n}$, for $n \geq 7$.
16. (a) Show that $f(n) = n^2 - n + 17$ is prime for $n = 0, 1, 2, \ldots, 16$. Show, however, that $f(17)$ is not prime. Which step of proof by induction *does this counterexample show* is necessary?
    (b) Begin to show that $f(n) = n^2 + n + 41$ is prime for $n = 0, 1, 2, \ldots, 40$, but not for 41.

    NOTE: There is no formula for generating prime numbers — these quadratics are interesting because of the long unbroken sequences of primes they produce.

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2 $\dfrac{1}{1 \times 2} + \dfrac{1}{2 \times 3} + \dfrac{1}{3 \times 4} + \cdots = 1$,
$\dfrac{1}{1 \times 4} + \dfrac{1}{4 \times 7} + \dfrac{1}{7 \times 10} + \cdots = \frac{1}{3}$,
$\dfrac{1}{1 \times 2 \times 3} + \dfrac{1}{2 \times 3 \times 4} + \dfrac{1}{3 \times 4 \times 5} + \cdots = \frac{1}{4}$

5 $n^3 - n$ is divisible by 24, for odd cardinals n.

7 $2^n > 2n^3$, for $n \geq 12$.

17(d) $L_n = \left(\dfrac{1+\sqrt{5}}{2}\right)^n + \left(\dfrac{1-\sqrt{5}}{2}\right)^n$