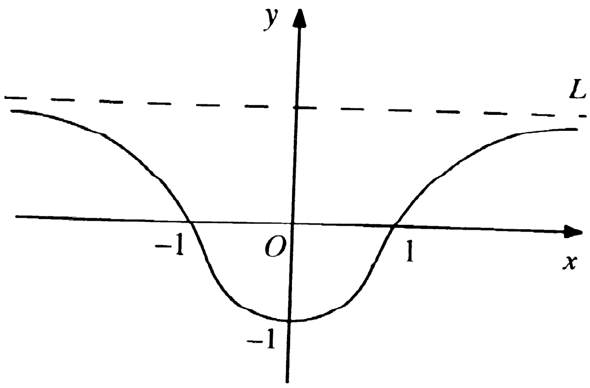
CSSA Trial HSC Examination

Mathematics Extension 2

2002

1. (a)



The diagram shows the graph of $y = f(x)$ where $f(x) = \frac{x^2-1}{x^2+1}$.
(i) Find the equation of the asymptote L.
(ii) On separate diagrams sketch the following graphs, showing any intercepts on the coordinate axes and the equations of any asymptotes:

$$y = \{f(x)\}^2,\ y = \sqrt{f(x)},\ y = \frac{1}{f(x)},\ \text{and } y = e^{f(x)}.$$

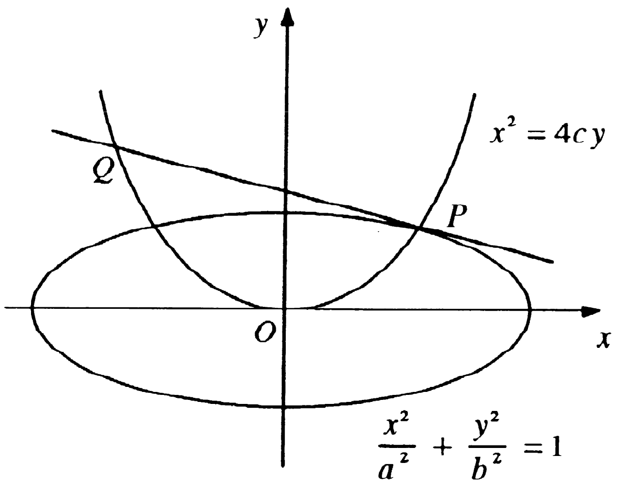
(iii) The function $f(x)$ with its domain restricted to $x \geq 0$ has an inverse $f^{-1}(x)$. Find $f^{-1}(x)$ as a function of x.
(iv) If $g(x) = e^{f(x)},\ x \geq 0$, write the inverse function g^{-1} in terms of f^{-1} and hence find $g^{-1}(x)$ as a function of x.
(b) Consider the curve defined by the parametric equations $\begin{cases} x &= t + t - 1 \\ y &= te^{2t} \end{cases}$.
(i) Show that $\frac{dy}{dx} = e^{2t}$
(ii) Hence show that the tangent to the curve at the point on the curve where $t = -1$ passes through the origin.

2. (a) (i) Find $\int \frac{\cos^2 x}{1-\sin x}\ dx$.
(ii) Find $\int \frac{1}{x(x^2+1)}\ dx$.
(b) (i) Use the substitution $u = e^x$ to find $\int \frac{e^x}{\sqrt{e^{2x}+1}}\ dx$.
(ii) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{1-\cos x}\ dx$.

(c) **(i)** If $I_n = \int_0^t \frac{1}{(1+x^2)^n}\,dx,\ \ n = 1, 2, 3, \ldots$, show that $2nI_{n+1} = (2n-1)I_n + \frac{t}{(1+t^2)^n}$ for $n = 1, 2, 3, \ldots$.
(ii) Hence find the value of I_3 in terms of t.

3. **(a)** Let $z = \sqrt{3} + i$
(i) Express z in modulus/argument form.
(ii) Show that $z^7 + 64z = 0$.
(b) Find the complex number $z = a+ib$, where a and b are real, such that $\Im(z) + \overline{z} = \frac{1}{1-i}$.
(c) The complex number z satisfies the condition $|z-8| = 2\Re(z-2)$
(i) Sketch the locus defined by this equation on an Argand diagram, showing any important features of the curve. State the type of curve and write down its equation.
(ii) Write down the value of $|z+8| - |z-8|$.
(iii) Find the possible values of $\arg z$.
(d) P, Q represent complex numbers α, β respectively in an Argand diagram, where O is the origin and O, P, Q are not collinear. In $\triangle OPQ$, the median from O to the midpoint M of PQ meets the median from Q to the midpoint N of OP in the point R, where R represents the complex number z.
(i) Show this information on a sketch.
(ii) Explain why there are positive real numbers k, l so that $kz = \frac{1}{2}(\alpha+\beta)$ and $l(z-\beta) = \frac{1}{2}\alpha - \beta$.
(iii) Show that $z = \frac{1}{3}(\alpha+\beta)$
(iv) Deduce that R is the point of concurrence of the three medians of $\triangle OPQ$

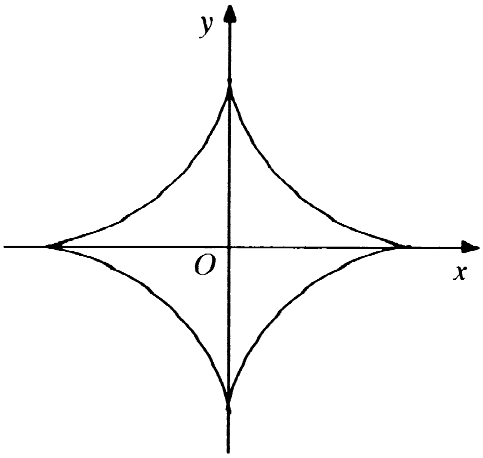
4.



The parabola $x^2 = 4cy$ intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(2cp, cp^2)$ in the first quadrant, where $a > b > 0$ and $c > 0$. The tangent to the ellipse at P meets the parabola again at $Q(2cq, cq^2)$.
(i) Show that the tangent to the ellipse at P has equation $\frac{2cpx}{a^2} + \frac{cp^2y}{b^2} = 1$

(ii) If this tangent meets the parabola at $(2ct, ct^2)$ show that $\frac{p^2t^2}{b^2} + \frac{4pt}{a^2} - \frac{1}{c^2} = 0$ and deduce that, considered as a quadratic in t, this equation has roots p and q.
(iii) If PQ subtends a right angle at the origin, show that $pq = -4$ and deduce that $\frac{1}{b^2} = \frac{1}{a^2} + \frac{1}{(4c)^2}$.
(iv) Hence show that if PQ subtends a right angle at the origin, then $p = 2e$, where e is the eccentricity of the ellipse.
(v) Find a set of positive rational numbers a, b, c with $a > b$ so that PQ subtends a right angle at the origin, and sketch the corresponding parabola and ellipse showing the equations of the curves, the equation of the tangent at P, and the coordinates of P and Q.

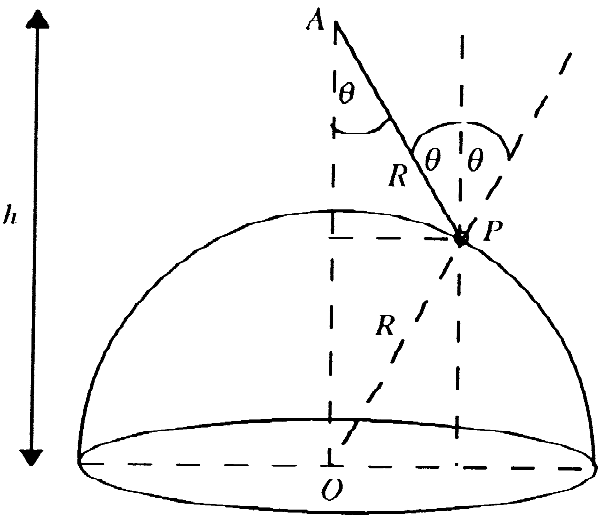
5. (a)



The diagram shows the graph of the relation $|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} = L^{\frac{1}{2}}$ for some $L > 0$.
(i) Show that the area of the region enclosed by the curves is $\frac{2}{3}L^2$.
(ii) A stone building has height H metres. Its base is the region enclosed by the curve $|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} = L^{\frac{1}{2}}$, and the cross section taken parallel to the base at height h metres us a similar region enclosed by the curve $|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} = l^{\frac{1}{2}}$ where $l = L\left(1 - \frac{h}{H}\right)$. Find the volume of the building.
(b) (i) Use De Moivre's Theorem to show that $(\cot\theta + i)^n + (\cot\theta - i)^n = \frac{2\cos n\theta}{\sin^n\theta}$
(ii) Show that the equation $(x+i)^5 + (x-i)^5 = 0$ has roots $0 \pm \cot\frac{\pi}{10}, \pm\cot\frac{3\pi}{10}$.
(iii) Hence show that the equation $x^4 - 10x^2 + 5 = 0$ has roots $\pm\cot\frac{\pi}{10}, \pm\cot\frac{3\pi}{10}$.
(iv) Hence show that $\cot\frac{\pi}{10} = \sqrt{5 + 2\sqrt{5}}$.

6.

(a)

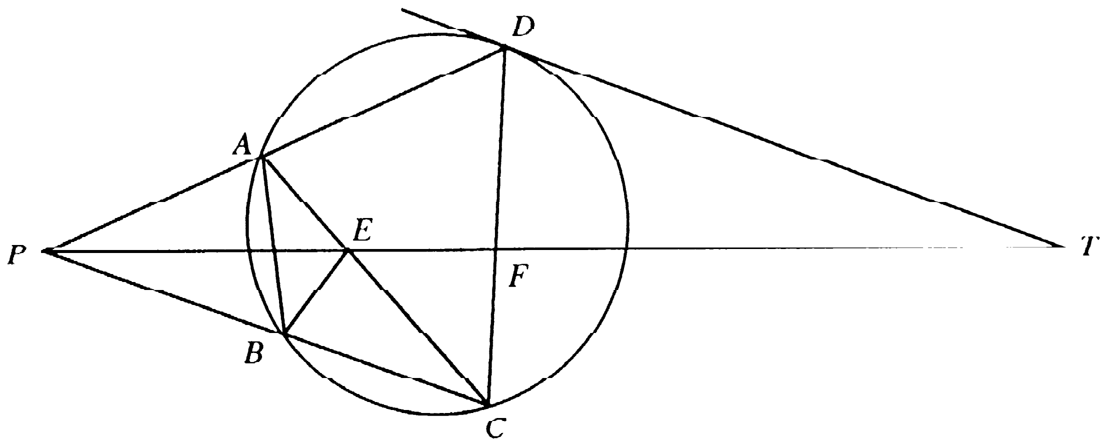


A particle P of mass m kg is connected to a fixed point A by a light, inextensible string. Point A is at a height h metres above the centre O of a smooth, hemispherical shell of radius R metres. P travels in a horizontal circle around the surface of the hemisphere with constant angular velocity ω radians per second. The length of the string is R metres, and the string makes an angle θ with OA.
(i) Draw a diagram showing the forces on P, and explain why the magnitude T of the tension in the string and the magnitude N of the normal reaction force between P and the surface of the sphere (measured in Newtons) satisfy the simulataneous equations

$$\begin{cases} T + N &= \frac{mg}{\cos\theta} \\ T - N &= mR\omega^2 \end{cases}$$

(ii) If ω_0 is the maximum angular velocity for which P stays in contact with the surface, and $\omega = \lambda\omega_0$ for some $0 < \lambda < 1$, show that $\frac{N}{T} = \frac{1-\lambda^2}{1+\lambda^2}$.
(iii) Describe qualitatively what would happen to the motion of the particle P if ω were to increase to ω_0 and then exceed ω_0.
(b) An object of mass m kg is dropped from rest from the top of a cliff 30 metres high. The resistance to its motion has magnitude $\frac{1}{20}mv^2$ when the velocity of the object is v m.s^{-1}. The object has fallen x metres after t seconds.
(i) Draw a diagram showing the forces on the object, and explain why $\ddot{x} = g - \frac{1}{20}v^2$.
(ii) Find v as a function of x.
(iii) What percentage of its terminal velocity V m.s^{-1} will the object attain just before it hits the ground?

7 (a)



$ABCD$ is a cyclic quadrilateral. DA produced and CB produced meet at P. T is a point on the tangent at D to the circle through A, B, C and D. PT cuts CA and CD at E and F respectively. $TF = TD$.

(i) Copy the diagram.

(ii) Show that $AEFD$ is a cyclic quadrilateral.

(iii) Show that $PBEA$ ia a cyclic quadrilateral.

(b) (i) Use the method of Mathematical Induction to show that
$a^2+(a+d)^2+(a+2d)^2+\cdots+\{a+(n-1)d\}^2 = \frac{1}{6}n\{6a^2+6ad(n-1)+d^2(n-1)(2n-1)\}$
for all positive integers $n \geq 1$.

(ii) Hence show that

$$1^2+3^2+5^2+\cdots+l^2 = \tfrac{1}{6}l(l+1)(l+2) \text{ if } l \text{ is odd, and}$$
$$2^2+4^2+6^2+\cdots+l^2 = \tfrac{1}{6}l(l+1)(l+2) \text{ if } l \text{ is even.}$$

8. (a) If $P(x) = x^m(b^n - c^n) + b^m(c^n - x^n) + c^m(x^n - b^n)$ where m and n are positive integers, show that $x^2 - (b+c)x + bc$ is a factor of $P(x)$.

(b) (i) If $f(x) = x - \ln(1 + x + \frac{1}{2}x^2)$ show that $f(x)$ is an increasing function of x for $x < 0$.

(ii) Hence show that $e^x < 1 + x + \frac{1}{2}x^2$ for $x < 0$.

(c) A goat grazes a rectangular paddock 10 metres by 20 metres. It is tethered to the fence at one corner of the paddock by an inextensible rope of length x metres, where $10 < x < 20$.

(i) Show that the goat can graze an area of $A\,\mathrm{m}^2$ where
$A = \frac{1}{2}x^2\sin^{-1}\left(\frac{10}{x}\right) + 5\sqrt{x^2-100}$.

(ii) If the goat can graze an area equal to half the area of the paddock, find the length of the rope using one application of Newton's method with an initial value of $x = 10$. Give your answer correct to one decimal place.