SOLUTIONS TO FORM VI EXTENSION 1

TRIAL HSC 2008

TOTAL = 12

(a) $\dfrac{n!}{(n-1)!} = n$ ✓

(b) $\dfrac{-2x}{\sqrt{1-x^4}}$ ✓

(c) $\displaystyle\int \frac{1}{40+x^2}\,dx = \frac{1}{2\sqrt{10}} \tan^{-1}\frac{x}{2\sqrt{10}} + c$ ✓

(d) $\ln e^{\frac{1}{2}} = \frac{1}{2} \ln e$

$= \frac{1}{2}$ ✓

No penalty for omission of c.

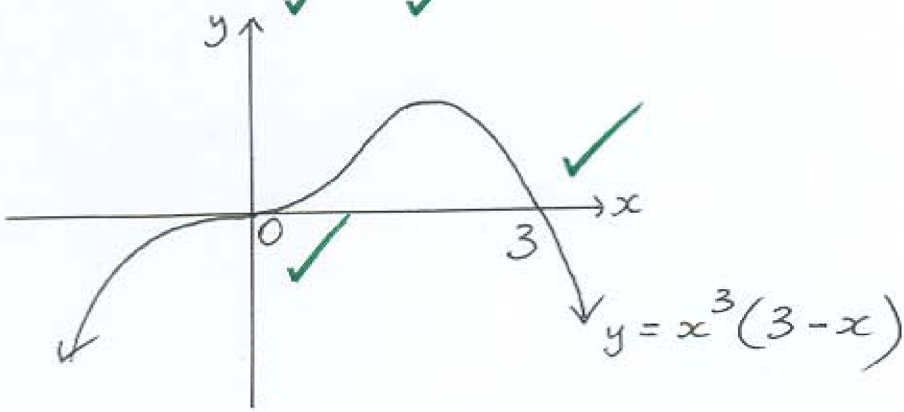
(e) $\displaystyle\int 2x e^{x^2}\,dx = e^{x^2} + c$ ✓

(f) $\cos 2\theta = \dfrac{1-t^2}{1+t^2}$ ✓ (where $t = \tan\theta$)

(g) $P = \left(\dfrac{28-24}{11}, \dfrac{70+8}{11}\right)$

$= \left(\dfrac{4}{11}, 7\dfrac{1}{11}\right)$ ✓ ✓

(h)



(i) Substitute $x = 1$ into the identity: ✓

$$\sum_{r=0}^{n} {}^nC_r \cdot (1)^r = (1+1)^n$$

i.e. ${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$ ✓

(2)

(a) $\displaystyle\int \frac{x}{(x+2)^2}\,dx = \int \frac{u-2}{u^2}\,du$

$\displaystyle = \int \left(\frac{1}{u} - 2u^{-2}\right)du$

$\displaystyle = \ln u + \frac{2}{u} + c$

$\displaystyle = \ln(x+2) + \frac{2}{x+2} + c$

Let $x = u-2$

$\therefore \dfrac{dx}{du} = 1$

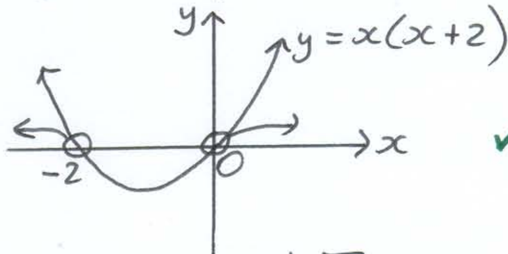
$\therefore dx = du$

(b) $\dfrac{x}{x+2} > 0 \quad (x \neq -2)$

Multiply both sides by $(x+2)^2$:

$x(x+2) > 0$

$x < -2$ or $x > 0$



(c) Let $\alpha = \tan^{-1} 2$ and $\beta = \tan^{-1}\sqrt{2}$.

$\therefore \tan\alpha = 2$, where $0 < \alpha < \frac{\pi}{2}$,

and $\tan\beta = \sqrt{2}$, where $0 < \beta < \frac{\pi}{2}$.

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

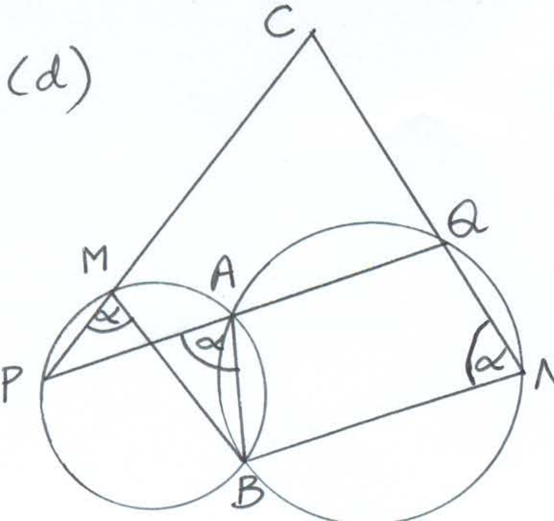
$$= \frac{2-\sqrt{2}}{1+2\sqrt{2}} \cdot \frac{1-2\sqrt{2}}{1-2\sqrt{2}}$$

$$= \frac{2 - 4\sqrt{2} - \sqrt{2} + 4}{1-8}$$

$$= \frac{6-5\sqrt{2}}{-7}$$

$$= \frac{5\sqrt{2}-6}{7}$$

(d)



(i) Exterior angle of cyclic quad ABNQ is equal to the interior opposite angle.

(ii) $\angle PMB = \alpha$ (angles at circumference standing on same arc)

$\therefore \angle PMB = \angle BNQ = \alpha$

$\therefore$ quad CMBN is cyclic (converse of reason in (i))

(3)(a) Let V mm^3 be the volume of the ice-cube, and x mm its edge length.

We are given $\frac{dx}{dt} = -2$ mm/min.

We want $\frac{dV}{dt}$ when $x = 15$.

$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$ ✓, where $V = x^3$.

$\therefore \frac{dV}{dt} = 3x^2 \cdot (-2)$ ✓

$= -6x^2$

So when $x = 15$, $\frac{dV}{dt} = -6(15)^2$ ✓

$= -1350$.

So when the edge is 15mm, the volume is decreasing at 1350 mm^3/min. ✓ ✓

(b) Let the roots be α, $-\frac{2}{\alpha}$ and β.

(i) The product of the roots is $\frac{-d}{a} = -1$.

$\therefore \alpha \cdot -\frac{2}{\alpha} \cdot \beta = -1$

$\therefore \beta = \frac{1}{2}$

So one of the roots is $\frac{1}{2}$. ✓

(ii) The sum of the roots is $\frac{-b}{a} = \frac{17}{6}$.

$\therefore \alpha - \frac{2}{\alpha} + \frac{1}{2} = \frac{17}{6}$ ✓

$\alpha - \frac{2}{\alpha} = \frac{7}{3}$

$3\alpha^2 - 7\alpha - 6 = 0$ ✓

$(3\alpha + 2)(\alpha - 3) = 0$

$\alpha = -\frac{2}{3}$ or 3 ✓

So the other two roots are $-\frac{2}{3}$ and 3.

(c) $\int_0^{\frac{\pi}{2}} (\cos x - \cos^2 x)\,dx = \int_0^{\frac{\pi}{2}} \left(\cos x - \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right)\right)dx$ ✓

$= \left[\sin x - \frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$ ✓

$= \sin\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{4}\sin\pi - (0 - 0 - 0)$ ✓

$= 1 - \frac{\pi}{4}$ ✓

(4)(a) when $n=1$, $LHS = 1 \times 2^2$
$= 4$
$RHS = \frac{1}{12} \times 1 \times 2 \times 3 \times 8$
$= 4$

So the result is true for $n=1$.

Assume that the result is true for $n=k$, where k is a positive integer.

i.e. assume that $1 \times 2^2 + 2 \times 3^2 + \ldots + k(k+1)^2 = \frac{1}{12}k(k+1)(k+2)(3k+5)$.

Prove that the result is true for $n=k+1$.

i.e. prove that

$1 \times 2^2 + 2 \times 3^2 + \ldots + k(k+1)^2 + (k+1)(k+2)^2 = \frac{1}{12}(k+1)(k+2)(k+3)(3k+8)$

$LHS = 1 \times 2^2 + 2 \times 3^2 + \ldots + k(k+1)^2 + (k+1)(k+2)^2$

$= \frac{1}{12}k(k+1)(k+2)(3k+5) + (k+1)(k+2)^2$

(using the assumption)

$= \frac{1}{12}(k+1)(k+2)\left(k(3k+5) + 12(k+2)\right)$

$= \frac{1}{12}(k+1)(k+2)(3k^2+17k+24)$

$= \frac{1}{12}(k+1)(k+2)(k+3)(3k+8)$

$= RHS$

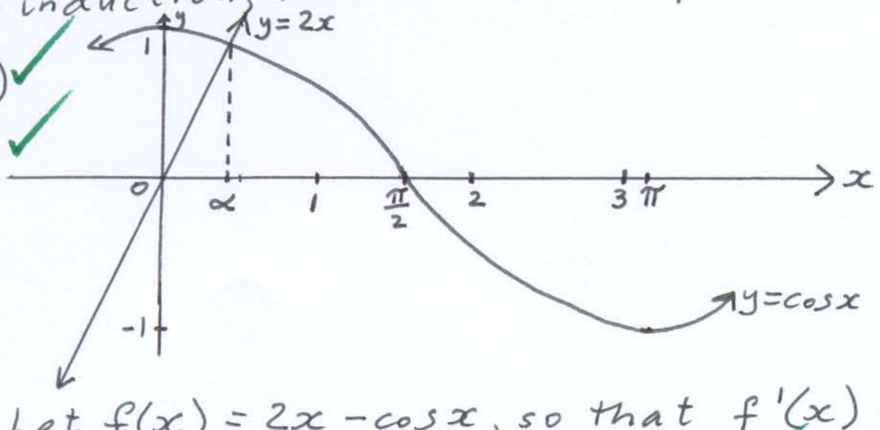
So the result is true for $n=k+1$ if it is true for $n=k$.

But the result is true for $n=1$.

So, by induction, it is true for all positive integer values of n.

(b)(i)

(ii)



(iii) Let $f(x) = 2x - \cos x$, so that $f'(x) = 2 + \sin x$.

$x_2 = 0.5 - \frac{1 - \cos 0.5}{2 + \sin 0.5}$

$= 0.4506\ldots$

$\doteq 0.45$

(4)(c) RHS of identity $= (1+x)^{100}$

$$= \sum_{r=0}^{100} \binom{100}{r} x^r.$$

The coefficient of x^4 is $\binom{100}{4}$.

LHS of identity

$$= \left(\binom{4}{0} + \binom{4}{1}x + \binom{4}{2}x^2 + \binom{4}{3}x^3 + \binom{4}{4}x^4\right)$$

$$\cdot \left(\binom{96}{0} + \binom{96}{1}x + \binom{96}{2}x^2 + \binom{96}{3}x^3 + \binom{96}{4}x^4 + \dots + \binom{96}{96}x^{96}\right)$$

The coefficient of x^4 is

$$\binom{4}{0}\binom{96}{4} + \binom{4}{1}\binom{96}{3} + \binom{4}{2}\binom{96}{2} + \binom{4}{3}\binom{96}{1} + \binom{4}{4}\binom{96}{0}$$

$$= \binom{96}{4} + \binom{4}{1}\binom{96}{3} + \binom{4}{2}\binom{96}{2} + \binom{4}{3}\binom{96}{1} + 1,$$

since $\binom{4}{0} = \binom{4}{4} = \binom{96}{0} = 1$.

The coefficients of x^4 on both sides of the identity are equal, so

$$\binom{96}{4} + \binom{4}{1}\binom{96}{3} + \binom{4}{2}\binom{96}{2} + \binom{4}{3}\binom{96}{1} = \binom{100}{4} - 1.$$

(5) (a) General term $= {}^{5n}C_r . (ax^3)^{5n-r} . (bx^{-2})^r$

$= {}^{5n}C_r . a^{5n-r} . b^r . x^{15n-3r} . x^{-2r}$

$= {}^{5n}C_r . a^{5n-r} . b^r . x^{15n-5r}$

We require $15n - 5r = 0$,

i.e. $r = 3n$.

So the constant term is

${}^{5n}C_{3n} . a^{2n} . b^{3n}$.

(b) (i) $\frac{dH}{dt} = -kAe^{-kt}$

$= -k(H - S)$

(ii) When $t = 0$, $H = 80$.

$\therefore 80 = A + 20$

$\therefore A = 60$

When $t = 5$, $H = 70$.

$\therefore 70 = 60e^{-5k} + 20$

$\frac{5}{6} = e^{-5k}$

$k = -\frac{1}{5} \ln \frac{5}{6}$

$\therefore H = 60e^{\frac{1}{5} t \ln \frac{5}{6}} + 20$

$= 20 + 60e^{\ln (\frac{5}{6})^{\frac{t}{5}}}$

$= 20 + 60\left(\frac{5}{6}\right)^{\frac{t}{5}}$, as required.

(iii) When $t = 60$,

$H = 20 + 60\left(\frac{5}{6}\right)^{12}$

$= 26.729...$

So after one hour, the temperature of the body is $26.7\,^\circ C$, correct to one decimal place

(c) (i) $P(-b) = b^2(b+c) + b^2(c-b) + c^2(-b+b) - 2b^2c$

$= b^3 + b^2c + b^2c - b^3 - 2b^2c$

$= 0$

$\therefore$ $a+b$ is a factor of $P(a)$

(ii) $P(a)$ is symmetric in a, b and c, so $b+c$ and $c+a$ are also factors of $P(a)$.

So $P(a) = (a+b)(b+c)(c+a)$.

Other methods, such as long division, are acceptable.

(6) (a) $\ddot{x} = -4\left(x + \frac{1}{x^3}\right)$

$$\therefore \frac{1}{2}v^2 = -4\int (x + x^{-3})\,dx$$

$$= -4\left(\frac{x^2}{2} + \frac{x^{-2}}{-2}\right) + c$$

$$= -4\left(\frac{x^2}{2} - \frac{1}{2x^2}\right) + c$$

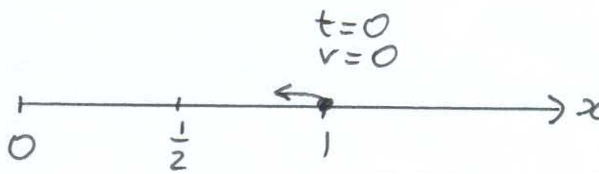
When $t = 0$, $x = 1$ and $v = 0$.

$$\therefore 0 = -4\left(\frac{1}{2} - \frac{1}{2}\right) + c$$

$$\therefore c = 0$$

$$\therefore v^2 = -8\left(\frac{x^2}{2} - \frac{1}{2x^2}\right)$$

$$= -4x^2 + \frac{4}{x^2}$$



When $t = \frac{1}{2}$,

$$v^2 = -4 \cdot \frac{1}{4} + \frac{4}{\frac{1}{4}}$$

$$= 15$$

$\therefore v = -\sqrt{15}$, because the particle is travelling in the negative direction.

(6)(b)(i) $y = \frac{x^2}{4a}$

$\therefore y' = \frac{x}{2a}$

When $x = 2ap$,

$y' = \frac{2ap}{2a}$

$= p$.

So the normal at P has gradient $-\frac{1}{p}$.

So the normal at P has equation

$$y - ap^2 = -\frac{1}{p}(x - 2ap)$$
$$py - ap^3 = -x + 2ap$$
$$x + py = 2ap + ap^3$$

(ii) When $x = -ap$ and $y = 3a + ap^2$,

$$\begin{aligned} LHS &= x + py \\ &= -ap + p(3a + ap^2) \\ &= 2ap + ap^3 \\ &= RHS \end{aligned}$$

So the normal at P passes through R.

(iii) The normal at Q has equation $x + qy = 2aq + aq^3$.

Substitute $x = -ap$ and $y = 3a + ap^2$:

$$-ap + 3aq + ap^2q = 2aq + aq^3$$
$$aq^3 - ap^2q - aq + ap = 0$$
$$aq(q^2 - p^2) - a(q - p) = 0$$

$\boxed{\div a}$ $\quad q(q-p)(q+p) - 1(q-p) = 0 \quad (a \neq 0)$

$$(q-p)(q^2 + pq - 1) = 0$$

$q \neq p$ since P and Q are distinct points,

so $q^2 + pq - 1 = 0$.

(iv) Consider the equation $q^2 + pq - 1 = 0$ as a quadratic equation in q.

$\therefore \Delta = p^2 + 4 > 0$ for all real values of p.

So the equation always has two real roots.

(6)(b)(v) Consider again the quadratic equation

$q^2 + pq - 1 = 0$. Let the roots be q_1 and q_2 $(q_1 \neq q_2)$

The product of the roots is -1.

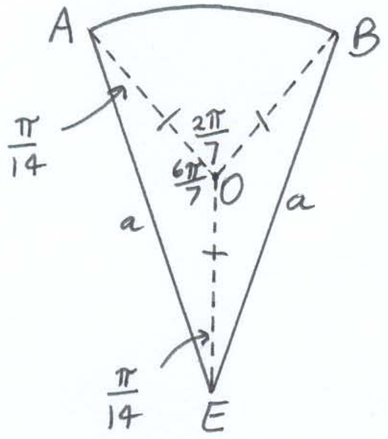
$\therefore \; q_1 q_2 = -1$

$$\therefore \; \frac{-1}{q_1} \cdot \frac{-1}{q_2} = -1$$

So the normals at the points $(2aq_1, aq_1^2)$ and $(2aq_2, aq_2^2)$ pass through R, and these normals (whose gradients are $\frac{-1}{q_1}$ and $\frac{-1}{q_2}$) are perpendicular.

From (ii), we also know that the normal at P passes through R.

(7)(a)



Let O be the centre of the coin.

$\therefore OA = OB = OE$

$\angle AOB = \frac{1}{7}$ of a revolution

$= \frac{2\pi}{7}$

$\therefore \angle AOE = \angle BOE = \frac{6\pi}{7}$ (angles at a point)

$\therefore \angle OAE = \angle OEA = \frac{\pi}{14}$ (angle sum of isosceles triangle)

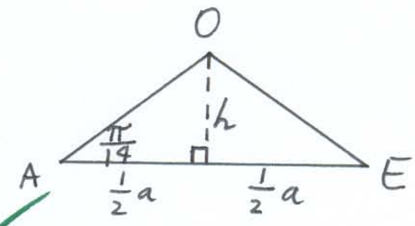
(i) $\angle AEB = \frac{\pi}{7}$ (with some justification) ✓

So area of sector AEB $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \cdot a^2 \cdot \frac{\pi}{7}$

$= \frac{1}{14} \pi a^2$ ✓

(ii) In ΔOAE,



$\frac{h}{\frac{1}{2}a} = \tan\frac{\pi}{14}$

$\therefore h = \frac{1}{2} a \tan\frac{\pi}{14}$

So ΔOAE has area $\frac{1}{4}a^2 \tan\frac{\pi}{14}$. ✓

So area of portion AOB = area of sector AEB $- 2 \times$ area of ΔOAE

$= \frac{1}{14}\pi a^2 - \frac{1}{2}a^2 \tan\frac{\pi}{14}$.

So area of coin is $7 \times$ area of AOB

$= 7\left(\frac{1}{14}\pi a^2 - \frac{1}{2}a^2 \tan\frac{\pi}{14}\right)$

$= \frac{1}{2}a^2\left(\pi - 7\tan\frac{\pi}{14}\right)$. ✓

(7)(b)(i) P has coordinates $(d\cos\beta, d\sin\beta)$.

(ii) This point lies on the parabola, so

$d\cos\beta = Vt\cos\alpha$ ① and $d\sin\beta = Vt\sin\alpha - \frac{1}{2}gt^2$ ②

From ①, $t = \dfrac{d\cos\beta}{V\cos\alpha}$.

Substitute into ②:

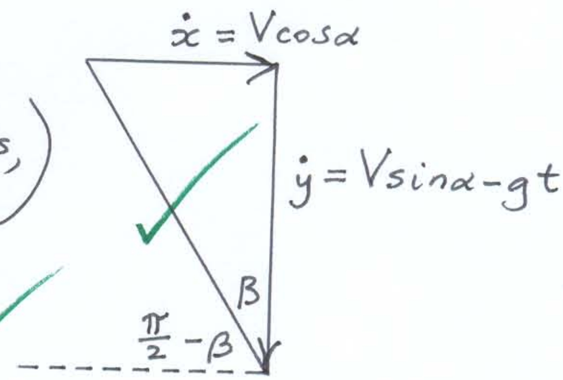
$$d\sin\beta = V\sin\alpha \cdot \frac{d\cos\beta}{V\cos\alpha} - \frac{g}{2}\cdot\frac{d^2\cos^2\beta}{V^2\cos^2\alpha}$$

Dividing by d ($d \neq 0$, since $d = 0$ corresponds to the particle being at the origin),

$$\sin\beta = \tan\alpha\cos\beta - d\cdot\frac{g\cos^2\beta}{2V^2\cos^2\alpha}$$

$$\therefore d = \frac{2V^2\cos^2\alpha}{g\cos^2\beta}\left(\tan\alpha\cos\beta - \sin\beta\right)$$

(iii) $\cot\beta = \dfrac{-\dot{y}}{\dot{x}}$ ($\dot{y}$ is negative because the particle is moving downwards, $\dot{x}$ is positive.)



$$= \frac{gt - V\sin\alpha}{V\cos\alpha}$$

$$= \frac{g}{V\cos\alpha}\cdot\frac{d\cos\beta}{V\cos\alpha} - \frac{V\sin\alpha}{V\cos\alpha}$$

$$= \frac{gd\cos\beta}{V^2\cos^2\alpha} - \tan\alpha$$

(iv) From (iii),

$$\tan\alpha = \frac{gd\cos\beta}{V^2\cos^2\alpha} - \cot\beta$$

Using (ii),

$$\tan\alpha = \frac{g\cos\beta}{V^2\cos^2\alpha}\cdot\frac{2V^2\cos^2\alpha}{g\cos^2\beta}\left(\tan\alpha\cos\beta - \sin\beta\right) - \cot\beta$$

$$= \frac{2}{\cos\beta}\left(\tan\alpha\cos\beta - \sin\beta\right) - \cot\beta$$

$$= 2\tan\alpha - 2\tan\beta - \cot\beta$$

$$\therefore \tan\alpha = \cot\beta + 2\tan\beta$$