HSC 2003 - 3 Unit Solutions

Question 1

(a) $x = \dfrac{mx_2 + nx_1}{m+n} = \dfrac{1\times 5 + 3\times -3}{4} = -1$

$y = \dfrac{my_2 + ny_1}{m+n} = \dfrac{1\times 6 + 3\times 4}{4} = 4\frac{1}{2}$

$\therefore$ P has coordinates $(-1, 4\frac{1}{2})$

(b) $\dfrac{3}{x-2} \leq 1, \quad x \neq 2$

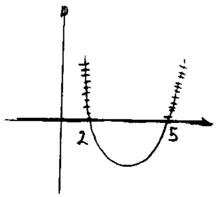
$3(x-2) \leq (x-2)^2 = x^2 - 4x + 4$

$3x - 6 \leq x^2 - 4x + 4$

$0 \leq x^2 - 7x + 10$

$0 \leq (x-5)(x-2)$

$\therefore \quad x < 2 \quad \text{or} \quad x \geq 5$



(c) $\lim\limits_{x\to 0} \dfrac{3x}{\sin 2x} = \dfrac{3}{2} \lim\limits_{x\to 0} \dfrac{2x}{\sin 2x} = \dfrac{3}{2}$

(d) $x = \dfrac{t}{2} \rightarrow t = 2x \quad \therefore \; y = 3(2x)^2 = 12x^2$

(e) $\dfrac{1}{2}\displaystyle\int_1^5 \dfrac{du}{u^3}$

$u = x^2 + 1$

$du = 2x\,dx$

$\dfrac{du}{2} = x\,dx$

At $x = 0, u = 1$

$x = 2, u = 5$

$= -\dfrac{1}{4}\left[\dfrac{1}{u^2}\right]_1^5$

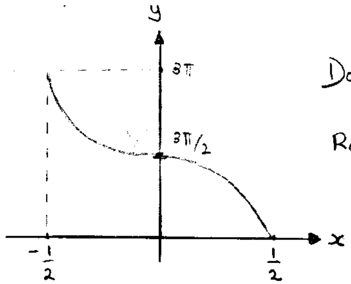
$= -\dfrac{1}{4}\left[\dfrac{1}{25} - 1\right]$

$= -\dfrac{1}{4}\left[-\dfrac{24}{25}\right]$

$= \dfrac{6}{25}$

Question 2

(a)



Domain: $-\frac{1}{2} \leq x \leq \frac{1}{2}$

Range: $0 \leq y \leq 4\pi$

(b) $\dfrac{d}{dx}\left(x\tan^{-1}x\right) = \dfrac{x}{1+x^2} + \tan^{-1}x$

(c) $\displaystyle\int_0^1 \dfrac{1}{\sqrt{2-x^2}}\,dx = \left[\sin^{-1}\dfrac{x}{\sqrt{2}}\right]_0^1 = \dfrac{\pi}{4}$

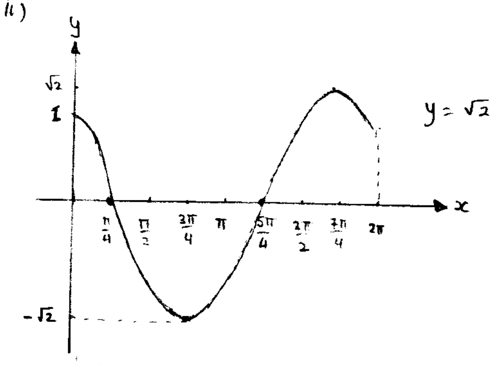
(d) ${}^5C_2\,(2)^3 = 80$

(e) (i) $R = \sqrt{a^2+b^2} = \sqrt{2}$

$\alpha = \tan^{-1}\dfrac{b}{a} = \tan^{-1}1 = \dfrac{\pi}{4}$

$\therefore \quad \cos x - \sin x = \sqrt{2}\cos\left(x + \dfrac{\pi}{4}\right)$

(ii)



$y = \sqrt{2}\cos\left(x + \dfrac{\pi}{4}\right)$

Question 3

(a) $\dfrac{9!}{3!\,2!} = 30\,240$

(b) (i) $x = 4\sin\left(2t + \dfrac{\pi}{3}\right)$

$\dot{x} = 8\cos\left(2t + \dfrac{\pi}{3}\right)$

$\ddot{x} = -16\sin\left(2t + \dfrac{\pi}{3}\right) = -4x$

Since $\ddot{x}$ is of the form $-n^2x$

$\therefore$ motion is Simple Harmonic.

(ii) $x = a\sin(nt + \alpha) \;\therefore\;$ amplitude $= 4$

(iii) Maximum speed occurs at centre of motion

i.e. when $x = 0$

i.e. $4\sin\left(2t + \dfrac{\pi}{3}\right) = 0$

i.e. $2t + \dfrac{\pi}{3} = \pi \rightarrow t = \dfrac{\pi}{3}$

(c)(i) There are 4 possible outcomes:

$(1,4)$, $(4,1)$, $(2,3)$, $(3,2)$

Total possible outcomes $= 36$

$\therefore$ P(sum = 5) : $\dfrac{4}{36} = \dfrac{1}{9}$

(ii) P(at least twice) $= 1 - P(\text{none}) - P(\text{one})$

$= 1 - \left(\dfrac{8}{9}\right)^7 - {}^7C_1\left(\dfrac{1}{9}\right)\left(\dfrac{8}{9}\right)^6$

$= 0.178$ to 3 d.p.

(d) Step 1: Test for $n = 1$.

$LHS = \dfrac{1}{3}, \quad RHS = \dfrac{1}{2\times 1 + 1} = \dfrac{1}{3} = LHS$

Step 2: Assume result true for $n = k$

i.e. $\dfrac{1}{1\times 3} + \dfrac{1}{3\times 5} + \dots + \dfrac{1}{(2k-1)(2k+1)} = \dfrac{k}{2k+1}$

Step 3: Prove true for $n = k+1$

$S_k + T_{k+1} = \dfrac{k}{2k+1} + \dfrac{1}{(2k+1)(2k+3)}$

$= \dfrac{2k^2 + 3k + 1}{(2k+1)(2k+3)} = \dfrac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \dfrac{k+1}{2k+3}$

Step 4: If result is true for $n = k$ it is true for $n = k+1$. It has been shown true for $n = 1$, hence it is true for $n = 1+1 = 2$ and so on for all +ve integers n.

Question 4

(a) $^{14}C_6 = 3003$

(b) $f(x) = \sin x - \frac{2x}{3}$, $f(1.5) = -0.002505013396$

$f'(x) = \cos x - \frac{2}{3}$, $f'(1.5) = -0.595929465$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.496$ to 3 d.p

(c) Let roots be $\alpha, \frac{1}{\alpha}$ and β

$\alpha\left(\frac{1}{\alpha}\right)\beta = \beta = -\frac{6}{2} = -3$

$\alpha + \frac{1}{\alpha} + \beta = -\frac{1}{2}$ i.e. $\alpha + \frac{1}{\alpha} = \frac{5}{2}$

$\alpha\left(\frac{1}{\alpha}\right) + \alpha\beta + \beta\left(\frac{1}{\alpha}\right) = -\frac{k}{2}$

$1 - 3\left(\alpha + \frac{1}{\alpha}\right) = -\frac{k}{2}$

$\therefore k = 6\left(\alpha + \frac{1}{\alpha}\right) - 2 = 6\left(\frac{5}{2}\right) - 2 = 13$

(d) (i) $\angle CPB = \angle CQB = 90^\circ$

$\therefore$ CPQB is a cyclic quadrilateral

(If an interval subtends = ∠'s on same side of it then the end points of the interval and the 2 points are concyclic).

(ii) $\angle APT = \angle AQT = 90^\circ$ (∠'s in straight line)

$\therefore$ PAQT is a cyclic quadrilateral

(opposite ∠'s supplementary)

(iii) $\angle TPQ = \angle TAQ = \alpha$ (∠'s in same segment on chord TQ)

$\angle QCB = \angle TPQ = \alpha$ (∠'s in same segment on chord BQ)

$\therefore \angle TAQ = \angle QCB$ (from above).

(iv) $\angle ATQ = 90^\circ - \alpha$ (∠ sum of Δ AQT)

$\angle CTR = 90^\circ - \alpha$ (vert. opp. ∠'s)

$\angle TCR = \alpha$ (from iii, noting $\angle QCB = \angle TCR$)

$\therefore \angle TRC = 90^\circ$ (∠ sum of Δ TCR)

Hence $AR \perp CB$

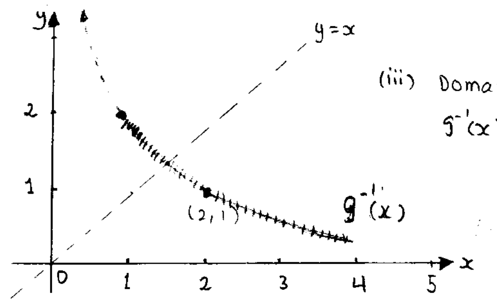
Question 5

(a) $\int \cos^2 3x \, dx = \frac{1}{2}\int (1 + \cos 6x)\, dx$

$= \frac{1}{2}\left[x + \frac{1}{6}\sin 6x\right] + C$

(b) (i) Doesn't pass horizontal line test i.e. horizontal line cuts it in more than one point.

(ii)



(iii) Domain of $g^{-1}(x)$: $x \geq 1$

(iv) $y = x^2 - 4x + 5$, interchanging x with y:

$x = y^2 - 4y + 5 \Rightarrow y^2 - 4y + (5 - x) = 0$

$y = \frac{4 \pm \sqrt{16 - 4(5-x)}}{2} = \frac{4 \pm 2\sqrt{x-1}}{2} = 2 \pm \sqrt{x-1}$

But since $y \leq 2$ $\therefore g^{-1}(x) = 2 - \sqrt{x-1}$.

(c) (i) $T = A + Be^{kt} \rightarrow \frac{dT}{dt} = kBe^{kt} = k(T - A)$

(ii) $T = A + Be^{kt} \rightarrow T = 20 + Be^{kt}$

At $t = 6$, $T = 80^\circ C$

At $t = 8$, $T = 50^\circ C$

$80 = 20 + Be^{k6} \rightarrow 60 = Be^{6k}$ ..... (1)

$50 = 20 + Be^{8k} \rightarrow 30 = Be^{8k}$ ..... (2)

$\frac{(2)}{(1)}$: $\frac{1}{2} = e^{2k} \rightarrow k = \frac{\ln(\frac{1}{2})}{2} = -\frac{\ln 2}{2}$

(iii) Substituting $k = -\frac{\ln 2}{2}$ into (1) gives:

$B = 480^\circ C$

At $t = 0$: $T = A + B = 20^\circ C + 480^\circ C = 500^\circ C$

Question 6

(a) (i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 8x(x^2 + 4) = 8x^3 + 32x$

Integrating w.r.t. x gives:

$\therefore \frac{1}{2}v^2 = 2x^4 + 16x^2 + C$

At $x = 0$, $v = 8$ $\therefore C = 32$

$\therefore v^2 = 4x^4 + 32x^2 + 64 = 4(x^2 + 4)^2$

Hence speed $= |v| = 2(x^2 + 4)$ m/s

(ii) $\frac{dx}{dt} = 2(x^2 + 4)$

$\frac{dt}{dx} = \frac{1}{2(x^2 + 4)}$

Integrating w.r.t. x gives:

$t = \frac{1}{2}\int \frac{1}{x^2 + 4}\, dx = \frac{1}{4}\tan^{-1}\frac{x}{2} + C$

At $t = 0$, $x = 0$ $\therefore C = 0$

i.e. $t = \frac{1}{4}\tan^{-1}\frac{x}{2}$

At $x = 2$, $t = \frac{1}{4}\tan^{-1} 1 = \frac{1}{4} \times \frac{\pi}{4} = \frac{\pi}{16}$ sec

(b) (i) $\tan\alpha = \frac{1}{p}$ i.e. $\alpha = \tan^{-1}\left(\frac{1}{p}\right)$ [Note: $\angle BFC = \beta$]

$\tan\beta = \frac{1}{q}$ i.e. $\beta = \tan^{-1}\left(\frac{1}{q}\right)$

(ii) $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \frac{\frac{1}{p} + \frac{1}{q}}{1 - \frac{1}{pq}} = \frac{p+q}{pq} \times \frac{pq}{pq - 1} = \frac{p+q}{pq-1}$

noting $\angle EAG = 45^\circ$ (diagonals of square bisect ∠'s)

$\therefore \alpha + \beta = 135^\circ$ i.e. $\tan 135^\circ = -1$

$\therefore -1 = \frac{p+q}{pq-1} \rightarrow p + q = 1 - pq$

(iii) Area of EBFD $= 1 - \frac{p}{2} - \frac{q}{2}$

from (ii) $p + q = 1 - pq$

$q(1+p) = 1 - p$

$q = \left(\frac{1-p}{1+p}\right)$

$\therefore$ Area of EBFD $= 1 - \frac{p}{2} + \frac{p-1}{2(1+p)}$

(iv) $A = 1 - \frac{p}{2} + \frac{p-1}{2(1+p)}$

$$\frac{dA}{dp} = -\frac{1}{2} + \frac{2(1+p).1 - (p-1).2}{4(1+p)^2}$$

$$= -\frac{1}{2} + \frac{4}{4(1+p)^2} = \frac{1}{(1+p)^2} - \frac{1}{2}$$

Let $\frac{dA}{dp} = 0$ to find maxima/minima.

i.e. $\frac{1}{2} = \frac{1}{(1+p)^2} \rightarrow (1+p)^2 = 2$

$1 + p = \sqrt{2}$ (as $p > 0$)

$p = \sqrt{2} - 1$

$\frac{d^2A}{dp^2} = \frac{-2}{(1+p)^3} < 0$ for $p = \sqrt{2} - 1$

$\therefore$ $p = \sqrt{2} - 1$ gives area of maximum value

Thus maximum area $= 1 - \frac{\sqrt{2}-1}{2} + \frac{\sqrt{2}-2}{2\sqrt{2}}$

$$= \frac{2\sqrt{2} - 2 + \sqrt{2} + \sqrt{2} - 2}{2\sqrt{2}}$$

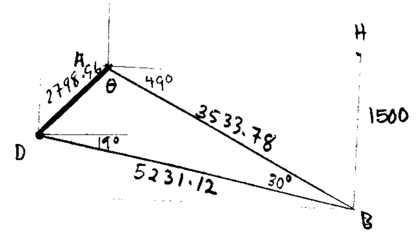
$$= \frac{4\sqrt{2} - 4}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{8 - 4\sqrt{2}}{4}$$

$$= 2 - \sqrt{2} \text{ units}^2$$

Question 7

(a)



$AD^2 = AB^2 + BD^2 - 2(BD)(AB)\cos 30°$

$AB = \frac{1500}{\tan 23°} = 3533.778549$

$BD = \frac{1500}{\tan 16°} = 5231.121666$

$\therefore$ $AD = 2798.96$ m

$\frac{BD}{\sin\theta} = \frac{AD}{\sin 30°} \Rightarrow \theta = 110°51'$ (note $\angle > 90°$)

$\therefore$ Bearing of D from A $= 250°T$ to the nearest degree.

(b) $y = vt\sin\alpha - \frac{1}{2}gt^2$, $x = vt\cos\alpha$

(i) $\dot{y} = v\sin\alpha - gt$, max. height occurs when $\dot{y} = 0$

i.e. $t = \frac{v\sin\alpha}{g}$, substituting into y gives:

$$y = \frac{v^2\sin^2\alpha}{g} - \frac{1}{2}g\left(\frac{v^2\sin^2\alpha}{g^2}\right) = \frac{v^2\sin^2\alpha}{2g}$$

(ii) Particle returns to initial height when $y = 0$

i.e. $vt\sin\alpha - \frac{1}{2}gt^2 = 0$

$t(v\sin\alpha - \frac{1}{2}gt) = 0$

$t = 0$, $v\sin\alpha = \frac{1}{2}gt$

$t = \frac{2v\sin\alpha}{g}$

substituting into x, gives:

$$x = v\left(\frac{2v\sin\alpha}{g}\right)\cos\alpha = \frac{v^2\, 2\sin\alpha\cos\alpha}{g} = \frac{v^2\sin 2\alpha}{g}$$

(iii) maximum separation occurs when $\alpha = 45°$ provided that max. height is not exceeded

$$(H - S) \geq \frac{v^2\sin^2 45°}{2g} \rightarrow v^2 \leq 4g(H-S)$$

in that case $d = \frac{v^2}{g}\sin 90° = \frac{v^2}{g}$

If on the other hand $v^2 \geq 4g(H-S)$, then maximum separation will occur when ball reaches maximum height of $(H-S)$ for some angle α.

Let $h = (H-S)$ i.e. $\sin\alpha = \frac{\sqrt{2gh}}{v}$

$$d = \frac{v^2}{g}.2\sin\alpha\cos\alpha$$

$$= \frac{v^2}{g}.2.\frac{\sqrt{2gh}}{v}.\sqrt{1 - \frac{2gh}{v^2}}$$

$$= \frac{v^{\cancel{2}}}{g}.\frac{2\sqrt{2gh}}{\cancel{v}}.\frac{\sqrt{v^2 - 2gh}}{\cancel{v}}$$

$$= \frac{2}{g}.\sqrt{2g}\sqrt{h}.\sqrt{2g}\sqrt{\frac{v^2}{2g} - h}$$

$$= 4\sqrt{h}\sqrt{\frac{v^2}{2g} - h}$$

$$= 4\sqrt{h\left(\frac{v^2}{2g}\right) - h^2}$$

$$= 4\sqrt{(H-S)\left(\frac{v^2}{2g}\right) - (H-S)^2}$$

$\therefore$ In summary

$d = 4 \times \sqrt{(H-S)\left(\frac{v^2}{2g}\right) - (H-S)^2}$ if $v^2 \geq 4g(H-S)$ and

$d = \frac{v^2}{2g}$ if $v^2 \leq 4g(H-S)$