SOLUTIONS

QUESTION 1

(a) $u = \log x$ $\qquad$ $x = e, u = 1$

$du = \frac{1}{x}dx$ $\qquad$ $x = e^2, u = 2$

$\therefore \int_1^2 \frac{1}{u}du$

$= [\log u]_1^2$

$= \log 2 - \log 1$

$= \log 2$

(b) $$\frac{5}{(2-x)(x+2)} > 1$$

$$\frac{5}{(2-x)(x+2)} - 1 > 0$$

$$\frac{5-(4-x^2)}{(2-x)(x+2)} > 0$$

$$\frac{x^2+1}{(2-x)(x+2)} > 0$$

i.e. $(2-x)(x+2) > 0$

$-2 \qquad 2$

Test $x = 0$, true

$\therefore$ Solution is $-2 < x < 2$

(c) Line PQ has equation $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$$\frac{y+3}{x+3} = \frac{5+3}{1+3}$$

$$y+3 = 2(x+3)$$

$$y = 2x+3$$

A lies on PQ since, when $x = \frac{1}{2}$, $y = 2(\frac{1}{2}) + 3$

$= 4$

$x_A = \frac{mx_Q + nx_P}{m+n}$ $\qquad$ or $\qquad$ $y_A = \frac{my_Q + ny_P}{m+n}$

$\frac{1}{2} = \frac{m(1)+n(-3)}{m+n}$ $\qquad$ $4 = \frac{m(5)+n(-3)}{m+n}$

$m + n = 2m - 6n$ $\qquad$ $4m + 4n = 5m - 3n$

$m = 7n$ $\qquad$ $m = 7n$

$\frac{m}{n} = 7$ $\qquad$ $\frac{m}{n} = 7$

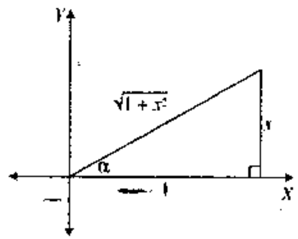
i.e. $m : n = 7 : 1$ $\qquad$ $m : n = 7 : 1$

i.e. A divides the line segment PQ in the ratio 7 : 1

(d) Let $\tan^{-1} x = \alpha$

$\therefore \tan\alpha = x$ $\qquad$ for $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

and α can be represented as a first quadrant angle.



Then $\cos\alpha = \frac{1}{\sqrt{1+x^2}}$

so that $\cos^{-1}\frac{1}{\sqrt{1+x^2}} = \alpha$

$\therefore \tan^{-1} x = \cos^{-1}\frac{1}{\sqrt{1+x^2}}$

(e) Remainder $= P(-4) = -64 + 16 + 2$

$= -46$

QUESTION 2

(a) $7! \times {}^4C_2$

(b) $(1-2x)^6 = \sum_{k=0}^{6} \binom{6}{k}(-2x)^k$

$\therefore (1-3x+2x^3)(1-2x)^6$

$= (1-3x+2x^3)[1 + \binom{6}{1}(-2x) + \binom{6}{2}(-2x)^2 + \binom{6}{3}(-2x)^3 + \binom{6}{4}(-2x)^4 + \binom{6}{5}(-2x)^5 + \binom{6}{6}(-2x)^6]$

The x^5 terms arise from

$1 \times \binom{6}{5}(-2x)^5 - 3x[\binom{6}{4}(-2x)^4] + 2x^3[\binom{6}{2}(-2x)^2]$

$= -192x^5 - 720x^5 + 120x^5$

$= -792x^5$

$\therefore$ Coefficient of x^5 term is -792

(c) $\cos 54° \cos\alpha + \sin 54° \sin\alpha = \sin 2\alpha$

$\cos(54° - \alpha) = \cos(90° - 2\alpha)$

$\therefore 54° - \alpha = \pm(90° - 2\alpha) + 360°n$

$54° - \alpha = 90° - 2\alpha + 360°n$	$54° - \alpha = -(90° - 2\alpha) + 360°n$
$\alpha = 36° + 360°n$	$54° - \alpha = -90° + 2\alpha + 360°n$
	$3\alpha = 144° - 360°n$
	$\alpha = 48° - 120°n$

(d) $\dfrac{d}{dx}\left[\dfrac{\tan^2 x}{x}\right]$

$= \dfrac{2x\tan x\sec^2 x - \tan^2 x}{x^2}$

(e) $f(x) = 2x^2 + x$

$f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$

$= \lim_{h\to 0} \dfrac{2(a+h)^2 + a + h - (2a^2 + a)}{h}$

$= \lim_{h\to 0} \dfrac{2a^2 + 4ah + 2h^2 + a + h - 2a^2 - a}{h}$

$= \lim_{h\to 0} \dfrac{4ah + 2h^2 + h}{h}$

$= \lim_{h\to 0} 4a + 2h + 1$

$= 4a + 1$

QUESTION 3

(a) $y = x^2 - 4x - 1$

$y + 1 = x^2 - 4x$

$x^2 - 4x + 4 = y + 5$

$(x-2)^2 = y + 5$

$(x-2)^2 = 4(\frac{1}{4})(y+5)$

$\therefore$ Vertex is $(2, -5)$

Focal length $= \dfrac{1}{4}$

$\therefore$ Focus is $(2, -4\frac{3}{4})$

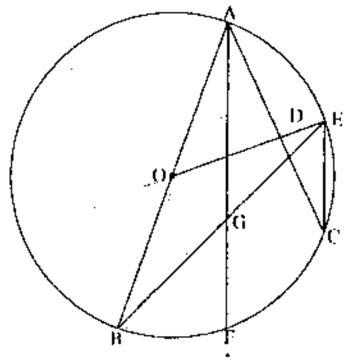
Directrix has equation $y = -5\frac{1}{4}$

(b) With one digit: 6

With two digits: ${}^6P_2 = 30$

With three digits: $\boxed{3}\,\boxed{5}\,\boxed{4} = 60$

Total = 96



(c) (i) Let $\angle BAF = x$

$\therefore \angle FAC = x$ (AF bisects $\angle BAC$)

$\therefore \angle AOD = 2x$ ($DA = DO$)

$\therefore \angle ABF = x$

(angle at centre = 2 × angle at circumference)

$\therefore \angle BAF = \angle ABE = x$

$\therefore GA = GB$

(ii) $\angle AGE = 2x$ (Exterior $\angle$ of $\triangle GAB$)

$\therefore \angle AGE = \angle AOD = 2x$

$\therefore AOGE$ is a cyclic quadrilateral (angles subtended by AE proved equal)

(iii) $\angle BEC = \angle BAC$ (angles subtended by BC)

$= 2x$

$\therefore \angle BEC = \angle AGE = 2x$

$\therefore EC \parallel FA$ (alternate $\angle$s proved equal)

QUESTION 4

(a) $\displaystyle\sum_{r=1}^{n} \frac{r^2}{(2r-1)(2r+1)} = \frac{1^2}{1\times3} + \frac{2^2}{3\times5} + \ldots + \frac{n^2}{(2n-1)(2n+1)}$

If $n = 1$, LHS $= \dfrac{1^2}{1\times3} = \dfrac{1}{3}$

RHS $= \dfrac{1(2)}{2(3)} = \dfrac{1}{3}$

$\therefore$ The statement is true for $n = 1$

Assume that the statement is true for $n = k$, a positive integer.

i.e. $\dfrac{1^2}{1\times3} + \dfrac{2^2}{3\times5} + \ldots + \dfrac{k^2}{(2k-1)(2k+1)} = \dfrac{k(k+1)}{2(2k+1)}$

So, when $n = k+1$

$$\text{LHS} = \frac{1^2}{1\times3} + \frac{2^2}{3\times5} + \ldots + \frac{k^2}{(2k-1)(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)}$$

$$= \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)} \quad \text{by assumption}$$

$$= \frac{k(k+1)(2k+3) + 2(k+1)^2}{2(2k+1)(2k+3)}$$

$$= \frac{(k+1)(2k^2+3k+2k+2)}{2(2k+1)(2k+3)}$$

$$= \frac{(k+1)(2k^2+5k+2)}{2(2k+1)(2k+3)}$$

$$= \frac{(k+1)(2k+1)(k+2)}{2(2k+1)(2k+3)}$$

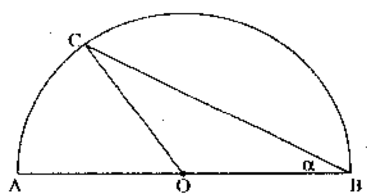
$$= \frac{(k+1)(k+2)}{2(2k+3)} = \text{RHS}$$

$\therefore$ If the statement is true for $n = k$, then it is true for $n = k + 1$.

But it is true for $n = 1$, and so true for $n = 2$, and hence by induction it is true for all positive integers.

(b) (i) Let O be the centre of the semi-circle and join OC.

$\angle OCB = \alpha$. ($OC = OB$)



and $\angle COB = \pi - 2\alpha$

$\therefore$ Area of segment cut off by CB

$= \frac{1}{2}(1)^2 [\pi - 2\alpha - \sin(\pi - 2\alpha)]$

$= \frac{1}{2}(\pi - 2\alpha - \sin 2\alpha)$

(ii) Area of segment $= \frac{1}{2}$(area of semi-circle)

$\frac{1}{2}(\pi - 2\alpha - \sin 2\alpha) = \frac{1}{2}(\frac{1}{2}\pi)$

$\pi - 2\alpha - \sin 2\alpha = \frac{\pi}{2}$

$2\pi - 4\alpha - 2\sin 2\alpha = \pi$

$\therefore 2\sin 2\alpha + 4\alpha = \pi$

(iii) Let $f(\alpha) = 2\sin 2\alpha + 4\alpha - \pi$

$f(0.4) = -0.106 < 0$

$f(0.5) = +0.541 > 0$

Change in sign proves that a root lies between $\alpha = 0.4$ and $\alpha = 0.5$

(iv) Taking $\alpha = 0.45, f(0.45) = 0.225 > 0$

But $f(0.4) < 0$

$\therefore$ Root lies closer to 0.4 than 0.5

QUESTION 5

(a) (i) $T = T_O + Ae^{-kt}$

$\therefore Ae^{-kt} = T - T_O$

Now $T = T_O + Ae^{-kt}$

$\frac{dT}{dt} = -kAe^{-kt}$

$-k(T - T_O)$

(ii) When $t = 0$, $T = 100$

When $t = 3$, $T = 70$

$T = T_O + Ae^{-kt}$

$100 = 25 + Ae^{o}$

$\therefore A = 75$

Now $T = 25 + 75e^{-kt}$

$70 = 25 + 75e^{-3k}$

$e^{-3k} = \frac{45}{75}$

$-3k = ln(0.6)$

$k = \frac{ln 0.6}{-3}$

$= 0.170$

(iii) $T = 25 + 75e^{-0.170t}$

$T = 50$

$50 = 25 + 75e^{-0.170t}$

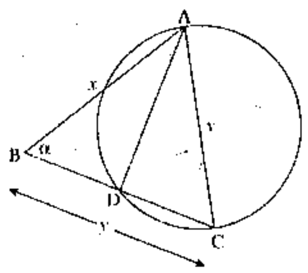
$e^{-0.170t} = \frac{25}{75}$

$-0.170t = ln(\frac{1}{3})$

$t = \frac{ln(\frac{1}{3})}{-0.170}$

$t = 6.45$ min

(b) (i) Applying cosine rule to triangle ABC:



$y^2 = x^2 + y^2 - 2xy\cos\alpha$

$2xy\cos\alpha = x^2$

$\cos\alpha = \dfrac{x^2}{2xy}$

$= \dfrac{x}{2y}$

(ii) $\angle BAC = \alpha$ ($\triangle ABC$ isosceles)

$\therefore \angle ACB = 180° - 2\alpha$ (angles of $\triangle ABC$)

$\angle ADC = 90°$ (angle in a semi-circle)

In $\triangle ADC$: $\cos(180 - 2\alpha) = \dfrac{DC}{y}$

$-\cos 2\alpha = \dfrac{DC}{y}$

$\therefore DC = -y\cos 2\alpha$

$= -y(2\cos^2\alpha - 1)$

$= -y(\dfrac{2x^2}{4y^2} - 1)$

i.e. $DC = y - \dfrac{x^2}{2y}$

QUESTION 6

(a) 11.00 a.m. → 5.20 p.m. = $6\frac{1}{3}$ h

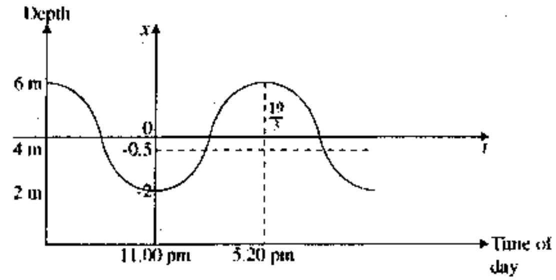
$\therefore$ Period $T = 12\frac{2}{3} = \frac{38}{3}$ h

$\therefore n = \dfrac{2\pi}{T} = \dfrac{2\pi}{\frac{38}{3}} = \dfrac{3\pi}{19}$

Mean tide = 4 m and amplitude = 2 m

Let x = the number of metres by which the water depth differs from 4 m at time t after 11.00 a.m.

So $x = -2\cos\dfrac{3\pi t}{19}$



The yacht may enter safely when $x \geq -0.5$

Consider $x = -0.5$

$-2\cos\dfrac{3\pi t}{19} = -0.5$

$\cos\dfrac{3\pi t}{19} = 0.25$

$\dfrac{3\pi t}{19} = 1.318$ or $\dfrac{3\pi t}{19} = 2\pi - 1.318$

$\therefore t = 2.66$ or $t = 10.00$ h

$= 2$ h 40 min

$\therefore$ The yacht may safely cross the lagoon between 1.40 p.m. and 9.00 p.m.

(b) (i) Number of ways of arranging n different objects in a circle is $(n-1)!$

$\therefore$ With no restrictions, number of arrangements $= (9-1)!$

$= 8!$

$= 40\,320$

(ii) Suppose that host and hostess do sit next to each other.

Then they may be arranged in 2! ways while the guests may be arranged in 7! ways.

$\therefore$ Number of ways $= 2! \times 7!$

$= 10\,080$

$\therefore$ Number of ways if host and hostess are separated

$= 40\,320 - 10\,080$

$= 30\,240$

(iii) Probability $= \dfrac{{}^{20}C_{13}}{{}^{32}C_{13}}$

$= 2.23 \times 10^{-4}$

QUESTION 7

(a) $v = \sqrt{8x - x^2}$

$\therefore v^2 = 8x - x^2$

$\frac{1}{2}v^2 = 4x - \frac{x^2}{2}$

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4 - x$

$\therefore$ When $x = 3$, $a = 1$

(b) (i) Substituting $t = \dfrac{x}{V\cos\alpha}$ into $y = Vt\sin\alpha - \frac{1}{2}gt^2$

gives $y = x\tan\alpha - \dfrac{gx^2}{2V^2\cos^2\alpha}$

i.e. $y = x\tan\alpha - \dfrac{gx^2\sec^2\alpha}{2V^2}$

(ii) $y = Vt\sin\alpha - \frac{1}{2}gt^2$

$\dot{y} = V\sin\alpha - gt$

The ball reaches its maximum height when $\dot{y} = 0$.

i.e. when $t = \dfrac{V\sin\alpha}{g}$

Substitution into $y = Vt\sin\alpha - \frac{1}{2}gt^2$ yields

$h = \dfrac{V^2\sin^2\alpha}{g} - \dfrac{1}{2}\dfrac{V^2\sin^2\alpha}{g}$

i.e. $h = \dfrac{V^2\sin^2\alpha}{2g}$