Q1

a $\int \cos^2(2x)\,dx$

$\cos 4x = 2\cos^2(2x) - 1$

$\therefore \cos^2(2x) = \frac{1}{2} + \frac{\cos 4x}{2}$

$\int \cos^2(2x)\,dx = \int \frac{1}{2} + \frac{1}{2}\cos 4x\,dx$

$= \frac{1}{2}x + \frac{1}{8}\sin 4x + c$ 2

---

b $\int \frac{e^x}{1+e^{2x}}\,dx$

$u = e^x$

$\frac{du}{dx} = e^x$

$\therefore du = e^x . dx$

$\int \frac{e^x\,dx}{1+e^{2x}} = \int \frac{du}{1+u^2}$

$= \tan^{-1}u + c$ 3

$= \tan^{-1}(e^x) + c$

---

c $\lim_{x \to 0} \frac{\sin \frac{x}{2}}{4x}$

$= \frac{1}{8} . \lim_{x \to 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}}$

$= \frac{1}{8}$ 2

---

d $x = \frac{k_1 x_2 + k_2 x_1}{k_1 + k_2}$

$-3 = \frac{k \times 0 + 1 \times 6}{k+1}$

$-3k - 3 = 6$

$-3k = 9$

$\therefore k = -3$

$y = \frac{k_1 y_2 + k_2 y_1}{k_1 + k_2}$

$8 = \frac{k \times 4 + 1 \times (-4)}{k+1}$

$8k + 8 = 4k - 4$

$4k = -12$

$\therefore k = -3$ 3

---

e Prove

$\frac{2\tan A}{1+\tan^2 A} = \sin 2A$

$L.H.S = \frac{2\tan A}{1+\tan^2 A}$

$= \frac{2\tan A}{\sec^2 A}$

$= \frac{2\sin A}{\cos A} . \cos^2 A$

$= 2\sin A\cos A$

$= \sin 2A$

$= R.H.S.$ 2

Q2

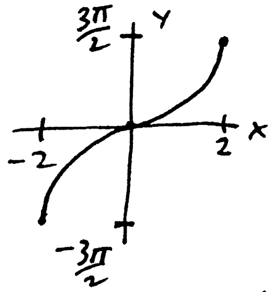
(a) $f(x) = 3\sin^{-1}\left(\frac{x}{2}\right)$

i $f(2) = 3\sin^{-1}\left(\frac{2}{2}\right)$

$= 3 \times \pi/2$

$= \frac{3\pi}{2}$ (1)

ii



$-1 \le \frac{x}{2} \le 1$

$\therefore -2 \le x \le 2$

$-\frac{\pi}{2} \le \sin^{-1}\left(\frac{x}{2}\right) \le \frac{\pi}{2}$ (1)

$\therefore -\frac{3\pi}{2} \le 3\sin^{-1}\left(\frac{x}{2}\right) \le \frac{3\pi}{2}$

iii Dom $-2 \le x \le 2$ (1)

Range $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$ (1)

(b) $x^3 + 6x^2 - x - 30 = 0$

Roots $= \alpha, \beta, \gamma$

$\alpha = \beta + \gamma$ (given)

Sum of roots

$= \alpha + \beta + \gamma = -6$

$\alpha + \alpha = -6$

$\therefore \underline{\alpha = -3}$

Product in pairs

$\alpha\beta + \alpha\gamma + \beta\gamma = -1$

$-3\beta - 3\gamma + \beta\gamma = -1$

$-3(\beta+\gamma) + \beta\gamma = -1$

$-3(\alpha) + \beta\gamma = -1$

$9 + \beta(-3-\beta) = -1$

$9 - 3\beta - \beta^2 = -1$

$0 = \beta^2 + 3\beta - 10$

$(\beta+5)(\beta-2) = 0$

$\therefore \beta = -5$ or 2

$\alpha = \beta + \gamma$

$-3 = -5 + \gamma$

$\therefore \gamma = 2$

or $-3 = 2 + \gamma$

$\gamma = -5$ (3)

$\therefore$ Roots are $-3, -5, 2$

(c) $x = 2\sin x$

$\therefore x - 2\sin x = 0$

Let $f(x) = x - 2\sin x$

$f'(x) = 1 - 2\cos x$

Let $x_1 = 1.7$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1.7 - \frac{[1.7 - 2\times\sin 1.7]}{[1 - 2\times\cos 1.7]}$

$= 1.9$ (3)

(d) $\frac{2}{x-1} < 1$

Let $x - 1 = 0$

$\therefore x = 1$ (1st c.v)

Let $\frac{2}{x-1} = 1$

$2 = x - 1$

$3 = x$ (2nd c.v)

$\overset{1}{\vert}\quad\overset{3}{\vert}$

Test $x = 0$, $\frac{2}{-1} < 1$ True

Test $x = 2$, $\frac{2}{1} \not< 1$

Test $x = 4$, $\frac{2}{3} < 1$ True

$\therefore$ Ans $x < 1$, $x > 3$ (2)

Q3

a i $y = x$, $y = x^3$

$x^3 = x$

$x^3 - x = 0$

$x(x-1)(x+1) = 0$

$\therefore x = 0, \quad x = 1, \quad x = -1$

$\therefore$ In 1st Quad Intersection pt $= (1,1)$ 1

ii $y = x^3$, $\frac{dy}{dx} = 3x^2$

$x = 1, \quad dy/dx = 3 = m_1$

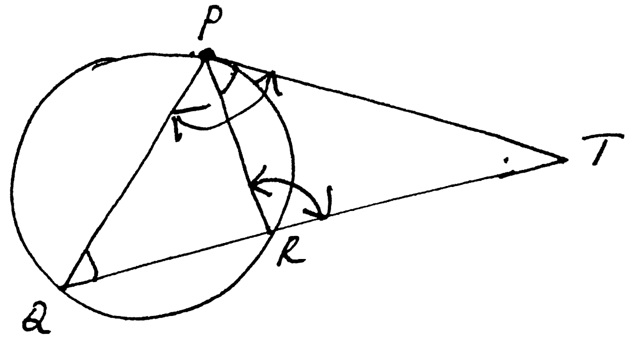
$y = x, \quad dy/dx = 1 = m_2$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \frac{3-1}{1+3\times1} = \frac{2}{4} = \frac{1}{2}$

$\theta = 27^\circ$ 2

b i



1

ii Aim. Prove ΔPRT ||| ΔQPT

Proof. In ΔPRT and ΔQPT

$\angle T$ is common

$\angle TPR = \angle Q$ (angle in alt seg)

$\therefore \angle PRT = \angle QPT$ ($\angle$ sum of Δ)

$\therefore \Delta PRT$ ||| ΔQPT (equiangular) 2

iii $\frac{PT}{QT} = \frac{RT}{PT}$ (eq ratios sim Δs)

$\therefore PT^2 = QT \times RT$ 2

c i $\frac{dT}{dt} = k(T-A)$

Proposed Solution is

$T = A + Ce^{kt}$

LHS $= \frac{dT}{dt} = 0 + Cke^{kt}$

R.H.S $= k(T-A)$

$= k(Ce^{kt})$

$\therefore$ L.H.S = R.H.S

1

ii $T = A + Ce^{kt}$

$T = 5 + Ce^{kt}$

$t = 0, \quad T = 20$

$20 = 5 + Ce^{k\times0}$

$\therefore C = 15$

$T = 5 + 15e^{kt}$

$17 = 5 + 15e^{0.5k}$

$\log_e\left(\frac{12}{15}\right) = 0.5k$

$k = -0.446287$

Finally put $T = 10$

$10 = 5 + 15e^{kt}$

$\log_e\left(\frac{5}{15}\right) = kt$

$\therefore t = 2.46 \doteqdot 2.5$ hours. 3

Q4

a i $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x-3$

$\frac{1}{2}v^2 = \int 2x-3 \, dx$

$= x^2-3x+c$

$t=0, \; x=4, \; v=0$

$0 = 16-12+c$

$c = -4$

$\therefore \frac{1}{2}v^2 = x^2-3x-4$ (2)

$\therefore v^2 = 2(x^2-3x-4)$

ii at origin, $x=0$

$\therefore v^2 = -8$ No soln.

$\therefore$ Particle does NOT pass through origin. (1)

iii $v = 10$

$100 = 2(x^2-3x-4)$

$0 = x^2-3x-54$

$(x-9)(x+6) = 0$

$x = 9$ or -6 (1)

Since partilcle starts at $x=4$ and can't reach $x=-6$ (other side of origin)

$\therefore$ $x = 9$ m when $v = 10$

---

b i $f: \; y = 2x-x^2$

Dom $x \geq 1$
Range $y \leq 1$] Restric

$f^{-1}: \; x = 2y-y^2$

$y^2-2y = -x$

$y^2-2y+1 = 1-x$

$(y-1)^2 = 1-x$

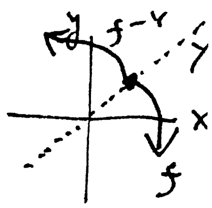
$y-1 = \pm\sqrt{1-x}$

$y = 1 \pm \sqrt{1-x}$

$\therefore$ Inv. f^{-1} is $y = 1+\sqrt{1-x}$

f^{-1} Dom $x \leq 1$
Range $y \geq 1$ (3)

ii Common point solve $y = x$ with $y = 2x-x^2$



$x = 2x-x^2$

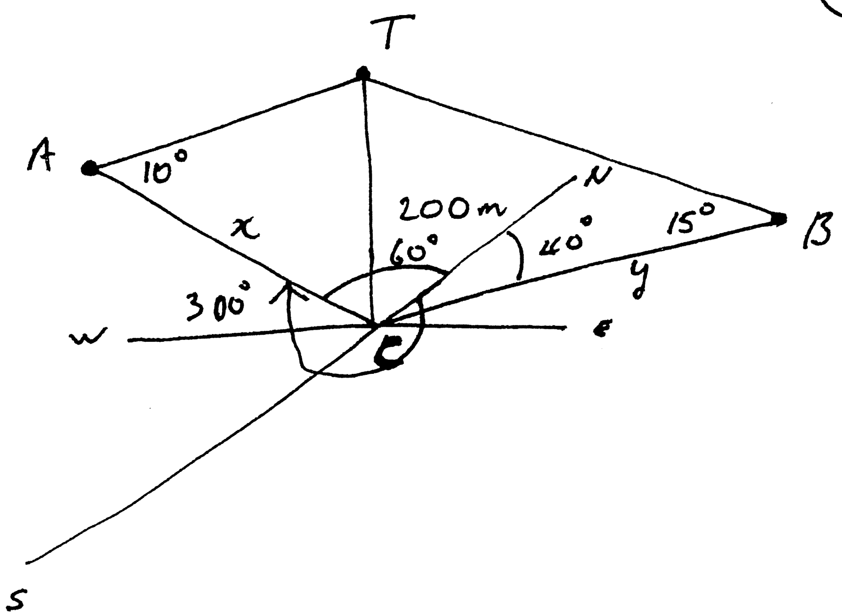
$x^2-x = 0$

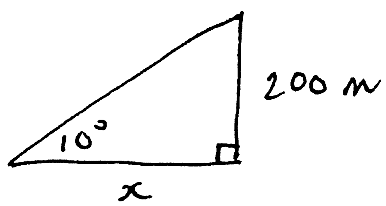
$x(x-1) = 0$

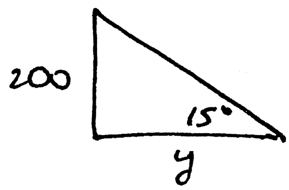
$\therefore (1,1)$ = Common pt. (1)

Q4 (c)

4





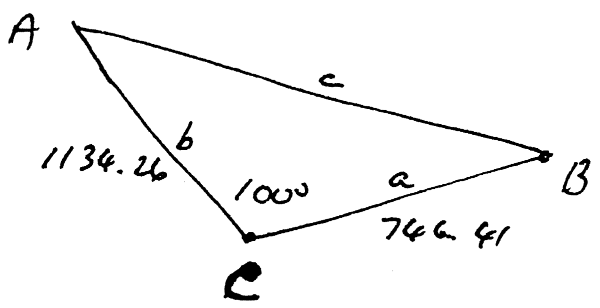


$$\tan 10^\circ = \frac{200}{x}$$

$$x = \frac{200}{\tan 10^\circ} = 1134.26$$

$$\tan 15^\circ = \frac{200}{y}$$

$$y = \frac{200}{\tan 15} = 746.41$$



using Cosine Rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 746.41^2 + 1134.26^2 - 2 \times 746.41 \times 1134.26 \times \cos 100^\circ$$

$$c = 1462.09$$

ans $AB = 1462$ m

Q5 a

i $\sqrt{3}\sin\theta - \cos\theta$

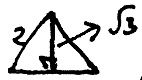
$= A\sin(\theta - \alpha)$

$= A\sin\theta\cos\alpha - A\cos\theta\sin\alpha$

$A\cos\alpha = \sqrt{3}$, $A\sin\alpha = 1$

$A^2 = 4 \quad \therefore A = 2$

$\frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \frac{1}{\sqrt{3}}$



$\therefore \alpha = 30^\circ = \pi/6$

$1\frac{1}{2}$

$\therefore A\sin(\theta-\alpha) = 2\sin(\theta - \pi/6)$

ii $2\sin(\theta - \pi/6) = 1$

$\sin(\theta - \pi/6) = 1/2$

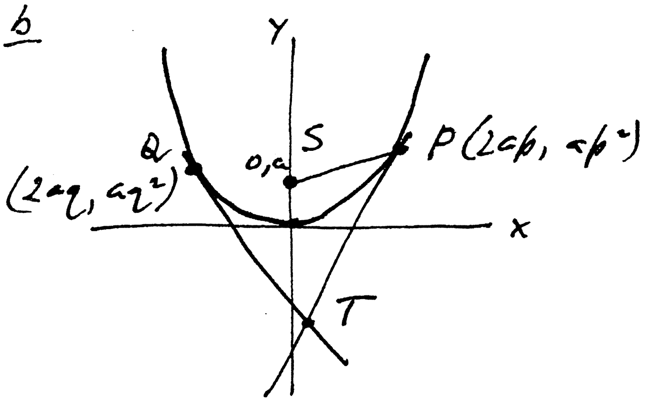
$\theta - \frac{\pi}{6} = \pi/6$ or $5\pi/6$

$1\frac{1}{2}$

$\therefore \theta = \pi/3$ or π

---

b



i $px - y - ap^2 = 0$ ①

$qx - y - aq^2 = 0$ ②

① - ②

$(p-q)x = a(p^2 - q^2)$

$\therefore x = a(p+q)$

$ap(p+q) - y - ap^2 = 0$

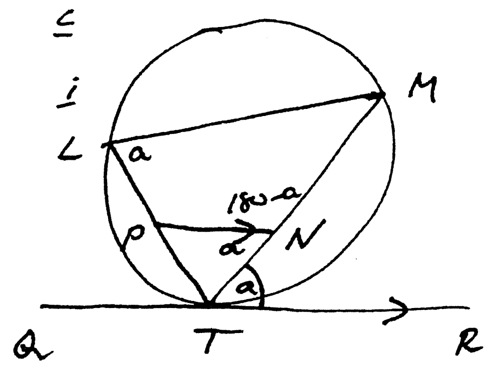
2

$\therefore y = apq$

$\therefore T = [\, a(p+q), apq \,]$

c

i



4

Aim. Prove LMNP is a cyclic quadrilateral.

Proof. Let $\angle NTR = a$

$\angle NTR = \angle PNT = a$ (alt $\angle$'s PN // QR

also $\angle NTR = \angle TLM = a$ (angle in alt seg

$\angle PNM = 180 - a$ (adj supp $\angle$'s)

$\therefore$ LMNP is cyclic quad

since $\angle L + \angle PNM = 180^\circ$ (opp $\angle$'s supp

---

b ii $SP^2 = (2ap - 0)^2 + (ap^2 - a)^2$

$= 4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2$

$= a^2p^4 + 2a^2p^2 + a^2$

$= a^2(p^2+1)^2$

$\therefore SP = ap^2 + a$

1

(iii) (over)

Q5 Condition of locus is

b iii $SP + SQ = 4a$

$ap^2 + a + aq^2 + a = 4a$

$a(p^2 + q^2) = 2a$

$\therefore p^2 + q^2 = 2$

$x = a(p+q)$ $\qquad$ $y = apq$

$(p+q)^2 = p^2 + 2pq + q^2$

$\frac{x^2}{a^2} = 2 + \frac{2y}{a}$

$x^2 = 2a^2 + 2ay$

$\therefore x^2 = 2a(y+a)$ is locus of T

2

Q6
(a) Prove $n^3 + 2n$ is divisible by 3 for all positive integers n.

Step 1 Prove true for $n = 1$

$1^3 + 2 \times 1 = 3$ which is divisible by 3 $\therefore$ True for $n = 1$

Step 2 assume true for $n = k$ (= integer)

$k^3 + 2k = 3m$ (m = integer)

Step 3 Prove true for $n = k+1$

$(k+1)^3 + 2(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 2k + 2$

$= (k^3 + 2k) + (3k^2 + 3k + 3)$

$= 3m + 3(k^2 + k + 1)$

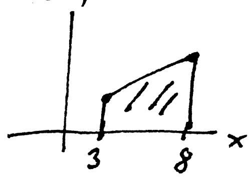
which is divisible by 3 since $k^2 + k + 1$ = integer.

$\therefore$ True for $n = k+1$

Step 4 Since true for $n = 1$ and having assumed true for $n = k$ and subsequently proven true for $n = k+1$, then result is true by Math. Induction for all positive integers n.

4

(b) i



$$\text{Area} = \int_3^8 \frac{x-1}{\sqrt{x+1}}\, dx$$

$u^2 = x + 1$

$u^2 - 1 = x$

$\frac{dx}{du} = 2u$

$\therefore dx = 2u\, du$

Change Limits

$x = 3, \quad u = 2$

$x = 8, \quad u = 3$

$$\text{Area} = \int_2^3 \frac{u^2 - 2}{u} \cdot 2u\, du$$

$$= 2\int_2^3 u^2 - 2\, du$$

$$= 2\left[\frac{u^3}{3} - 2u\right]_2^3$$

$$= 2\left[\left(\frac{27}{3} - 6\right) - \left(\frac{8}{3} - 4\right)\right]$$

$$= 8\tfrac{2}{3} \text{ units}^2$$

2

b ii

$$\text{Vol} = \pi \int_3^8 y^2\, dx$$

$$= \pi \int_3^8 \frac{(x-1)^2}{x+1}\, dx$$

$u^2 = x + 1$

$x - 1 = u^2 - 2$

$(x-1)^2 = (u^2 - 2)^2$

$= u^4 - 4u^2 + 4$

$$\text{Vol} = \pi \int_2^3 \frac{(u^4 - 4u^2 + 4)\, 2u\, du}{u^2}$$

$$= 2\pi \int_2^3 u^3 - 4u + \frac{4}{u}\, du$$

$$= 2\pi \left[\frac{u^4}{4} - 2u^2 + 4\log_e u\right]_2^3$$

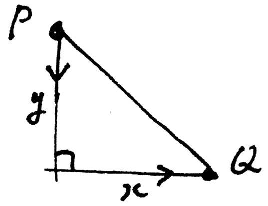
$$= 2\pi\left[\frac{81}{4} - 18 + 4\log_e 3 - 4 + 8 - 4\log_e 2\right]$$

$= 49.46$

$\doteqdot 49.5$ units^3

2

Q6 c



$$x^2 + y^2 = 100$$

$$y = \sqrt{100 - x^2}$$

$$= (100 - x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(-2x)(100 - x^2)^{-\frac{1}{2}}$$

$$= \frac{-x}{\sqrt{100 - x^2}}$$

$\frac{dx}{dt} = +60$

since moving Left to Right.

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= \frac{-x}{\sqrt{100 - x^2}} \times +60$$

Put $x = +8$ (since Left of O)

$$\frac{dy}{dt} = \frac{8 \times -60}{\sqrt{100 - 64}}$$

$$= \frac{8 \times -60}{\sqrt{36}}$$

$$= -80 \text{ km/h}$$

$\therefore$ Car P is travelling at 80 km/h when Car Q is 8 km from the intersection.

4

Q7 (a) $x = 2\cos(t + \frac{\pi}{4})$

i $\dot{x} = -2\sin(t + \pi/4)$

$\ddot{x} = -2\cos(t + \frac{\pi}{4})$

$\therefore \ddot{x} = -x$

Thus acceleration is proportional to the displacement (x)

$\ddot{x} = -n^2 x$

$\therefore$ Motion is S.H.M. (1)

ii let $t = 0$

$x = 2\cos(\frac{\pi}{4})$

$= 2 \times 1/\sqrt{2}$

$= \sqrt{2}$ m = initial position. (1)

iii $n^2 = 1 \quad \therefore n = 1$ (1)

$T = \frac{2\pi}{n} = \frac{2\pi}{1} = 2\pi$ secs.

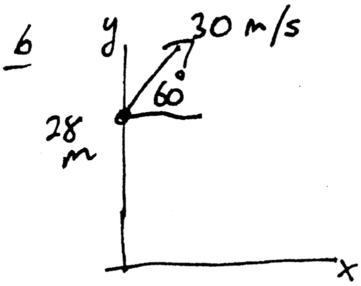
iv Max displacement $= a$

$= 2$ metres (1)

v $t = \frac{\pi}{2}$, $x = 2\cos(\frac{3\pi}{4}) = 2 \times -\frac{1}{\sqrt{2}} = -\sqrt{2}$ m (1)

---

b



Data $t = 0, x = 0, \dot{x} = 30 \times \frac{1}{2} = 15$

$t = 0, y = 28, \dot{y} = 30 \times \frac{\sqrt{3}}{2} = 15\sqrt{3} \qquad g = 10$

Horiz. Motion

$\dot{x} = 15$

$x = 15t$

Vertical Motion

$\ddot{y} = -g$

$\dot{y} = -gt + c$

$15\sqrt{3} = 0 + c$

$\dot{y} = -gt + 15\sqrt{3}$

$y = -g\frac{t^2}{2} + 15\sqrt{3}.t + 28$

i Put $y = 0$

$0 = -5t^2 + 15\sqrt{3}t + 28$

$5t^2 - 15\sqrt{3}t - 28 = 0$

$t = \frac{15\sqrt{3} \pm \sqrt{675 + 560}}{10}$

$= 6.1$

$= 6$ secs. (2)

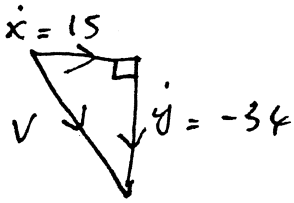
ii $x = 15 \times 6$

$= 90$ m (1)

(iii) $\dot{y} = -10 \times 6 + 15\sqrt{3}$

$= -34$

$\dot{x} = 15$

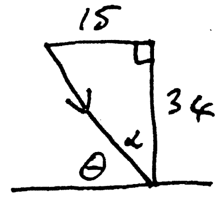


$V^2 = 15^2 + (-34)^2$

$= 1381$

$V = 37 \text{ ms}^{-1}$ (2)

iv



$\tan\alpha = \frac{15}{34}$

$\alpha = 23°48'$

$\therefore \theta = 90° - 23°48'$

$= 66°$ (2)