Balmain High School

4 unit mathematics

Trial HSC Examination 1986

1. **(i)** Evaluate **(a)** $\int_{-1}^{2} \frac{x^2}{\sqrt{x^3+2}}\,dx$ **(b)** $\int_0^1 xe^{-x}\,dx$ **(c)** $\int_0^{\pi} \sin^2(\frac{x}{4})\,dx$ **(d)** $\int_2^4 \frac{dx}{x^2-4x+8}$
(ii) Find all solutions in the domain $\theta : -2\pi \leq \theta \leq 2\pi$, $\cos\theta + \cos 2\theta + \cos 3\theta = 0$.

2. **(i)** Define the absolute value of x ($|x|$) for positive, negative and zero values of x. Sketch the following curves (not on graph paper).
(a) $y = |\sin x|$ for $x : -2\pi \leq x \leq 2\pi$
(b) $y = \sin|x|$ for $x : -2\pi \leq x \leq 2\pi$
(c) $|x| + |y| = 1$
(ii) Find the complete factorization of $P(z) = z^6 - 1$
(a) over the complex field $\mathbb{C}$
(b) over the real field $\mathbb{R}$

3. **(i)** Express $\frac{1}{(x-1)(x^2+1)}$ as a sum of partial fractions and hence find $\int \frac{dx}{(x-1)(x^2+1)}$
(ii) When the polynomial $P(x)$ is divided by $(x-2)$ and by $(x-3)$ the respective remainders are 4 and 9. Determine what the remainder must be when the polynomial is divided by $(x-2)(x-3)$.
(iii) The roots of the equation $x^3 + ax^2 + bx + c = 0$ are α, β, γ. Find the values of the following (in terms of a, b, c)
(a) $\alpha + \beta + \gamma$ **(b)** $\alpha^2 + \beta^2 + \gamma^2$ **(c)** $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$
(d) Write an equation which has $\alpha - 1, \beta - 1$, and $\gamma - 1$ as its roots.

4. **(i)** Prove $|z_1 + z_2| \leq |z_1| + |z_2|$.
(ii) If $z = 3 + 2i$ show on the Argand diagram **(a)** z **(b)** $\overline{z}$ **(c)** $z\overline{z}$ **(d)** iz
(iii) In the Argand diagram, P represents the complex number z and Q the complex number w given by $w = \frac{3z-1}{z-1}$. If P describes the circle of unit radius with centre at the origin find the locus by Q.

5. Show that the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) has equation $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$. The tangent to the ellipse at any point P meets the x-axis at T, the foot of the perpendicular from P to the x-axis in N, and the normal at P meets the x-axis at G. If O is the centre of the ellipse show that $OT.NG = b^2$.

6. **(i)** A rectangle is inscribed in a semi-circle of radius a. Find the maximum area of the rectangle.
(ii) Find the turning points of the curve $y = x^4 - 4x^3 + c$. Show that for $0 < c < 27$ the curve crosses the x-axis between $x = 0$ and $x = 3$. What is the condition that

the curve does not intersect the x-axis?

7. (a) Find the volume of the torus generated by revolving the circle $x^2 + y^2 = 16$ about the line $x = 6$ by using the 'slicing method'.
(b) Confirm your answer by using a different method or approach.

8. (i) Prove that if n is a positive integer and $x > 0$, then $x^n + \frac{1}{x^n} > x^{n-1} + \frac{1}{x^{n-1}}$ (provided $x \neq 1$).
(ii) Given a triangle whose sides are in the ratio $4 : 5 : 6$ prove (without use of calculators or tables) that one angle is twice another.
(iii) From the top of a hill of uniform slope the angle of depression of a point in the plane below is $30°$, and from a spot $3/4$ of the way down it is $15°$. Find the slope of the hill.