Hills Grammar 2004 Ext 2 trial

Question One — Marks

(a) Find:

(i) $\int x\sqrt{x^2-5}\,dx$ — 2

(ii) $\int (1-x^2)^3\,dx$ — 2

(b) Evaluate:

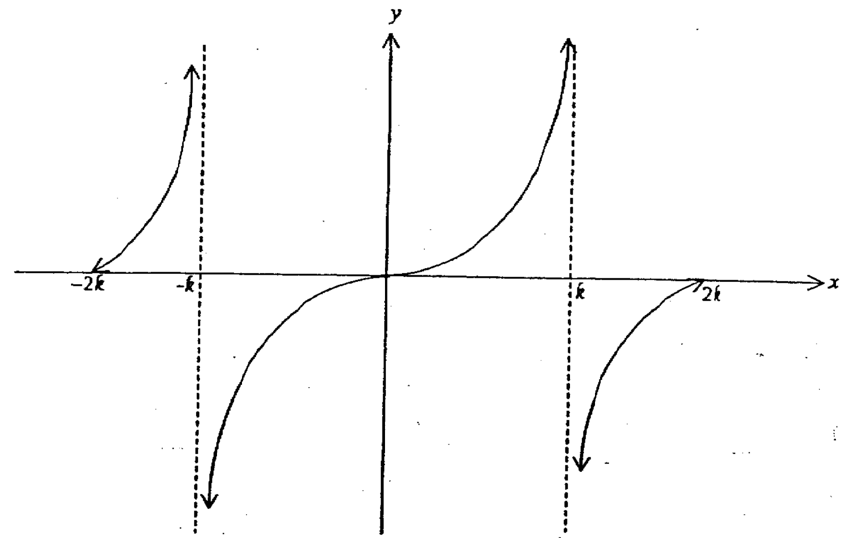
(i) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\,d\theta$ — 3

(ii) $\int_0^1 \frac{x^2-5x-2}{(2-x)(4+x^2)}\,dx$ — 4

(iii) $\int_1^2 x^2.e^x\,dx$ — 4

Question Two (Start a new booklet)

(a) The graph of $f(x)$ is shown below:



Draw neat sketches of the following: — 8

(i) $y = f(x-k)$

(ii) $y = [f(x)]^2$

(iii) $y = \frac{1}{f(x)}$

(iv) $y = f'(x)$

(b) Consider the curve $y = 4x^2(2-x^2)$.

(i) Sketch the curve, clearly indicating the important features. — 4

(ii) Hence sketch the curve $y^2 = 4x^2(2-x^2)$ — 1

(iii) Sketch the curve $y = \log_e 4x^2(2-x^2)$ — 2

Question Three **(Start a new booklet)**

(a) Express $w = 1 + i$ and $z = \sqrt{3} - i$ in the form $r(\cos\theta + i\sin\theta)$.

Hence find the modulus and argument of

(i) wz

(ii) $w^{-1}z$ **4**

(b) An equilateral triangle has its vertices on the circle $|z| = 2$. One vertex is the point representing $\sqrt{3} + i$. Find the other two vertices and make a neat sketch. **4**

(c) Solve for z:

$$\frac{z - 2i}{1 + iz} = \frac{4}{3}$$

expressing your answer in modulus-argument form. **4**

(d) Shade the region of the Argand diagram consisting of those points z for which

(i) $R(z) \le 2$ and $I(z) > -1$ **1**

(ii) $|z - 1 - i| \le 1, \quad 0 \le \arg z \le \frac{\pi}{4}$ **2**

Question Four **(Start a new booklet)**

(a) Given that $P(x) = (x^4 - 1)(x^2 - 2)$, factorise $P(x)$ completely over:

(i) The real numbers R

(ii) The complex numbers C. **3**

(b) (i) If $x = \alpha$ is a double root of the polynomial equation $Q(x) = 0$, show that $x = \alpha$ is a root of the equation $Q'(x) = 0$. **2**

(ii) If the polynomial $P(x) = x^4 + x^2 + 6x + 4$ has a rational zero of multiplicity 2, find all the zeros of $P(x)$ over the complex field. **4**

(c) Consider the polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$.

(i) If $P(x)$ has roots $a + bi$ and $a - 2bi$ [where a and b are real], find the values of a and b. **3**

(ii) Hence find the zeros of $P(x)$ over the complex field and express $P(x)$ as the product of two quadratic factors. **3**

Question Five (Start a new booklet)

(a) A solid has its base in the shape of an ellipse with major axis 8 units and minor axis 6 units. If every section perpendicular to the major axis is an equilateral triangle, show that the volume of the solid formed is $48\sqrt{3}$ cubic units. **5**

(b) Using the method of cylindrical shells find the volume of the solid of revolution obtained by rotating about the y-axis, the region bounded by the curve $y = \sin x$ and the x-axis from $x = 0$ to $x = \frac{\pi}{2}$. **5**

(c) Find the volume obtained by rotating the area enclosed by the x-axis, the curve $y = \tan^{-1} x$ and the line $x = 1$ about the line $x = 1$. **5**

Question Six (Start a new booklet)

(a) Consider the curves $\frac{x^2}{16} + \frac{y^2}{7} = 1$ and $x^2 - \frac{y^2}{8} = 1$.

(i) Show that both curves have the same foci. **4**

(ii) Find the equation of the circle through the points of intersection of the two curves. **4**

(b) (i) Show that the tangent to the ellipse $\frac{x^2}{12} + \frac{y^2}{4} = 1$ at the point $P(3, 1)$ has equation $x + y = 4$. **3**

(ii) If this tangent cuts the directrix in the fourth quadrant at the point T, and S is the corresponding focus, show that SP and ST are at right angles to each other. **4**

Question Seven (Start a new booklet)

(a) Consider the function $f(x) = \dfrac{3x}{(x-1)(4-x)}$.

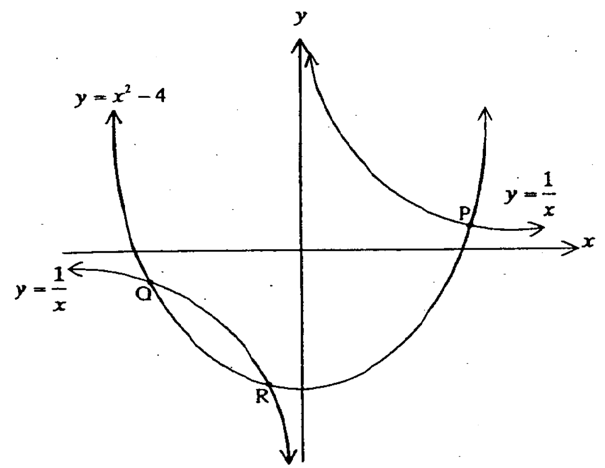
(i) Express $f(x)$ in partial fractions. **2**

(ii) Find the co-ordinates and the nature of any turning points of the graph $y = f(x)$. **4**

(iii) Sketch the graph of $y = f(x)$ showing clearly the co-ordinates of any turning points and the equation of asymptotes. **3**

(iv) Find the area of the region bounded by the curve $y = f(x)$ and the x-axis between the lines $x = 2$ and $x = 3$. **2**

(b)



The curves $y = x^2 - 4$ and $y = \dfrac{1}{x}$ intersect at the points P, Q and R whose x-coordinates are α, β and λ respectively.

(i) Show that α, β and λ are roots of the equation $x^3 - 4x - 1 = 0$. **1**

(ii) Find a polynomial equation which has roots α^2, β^2 and λ^2. **3**

Question Eight (Start a new booklet)

(a) Use the substitution $x = \dfrac{2}{3}\sin\theta$ to prove that $\int_0^{\frac{2}{3}} \sqrt{4-9x^2}\,dx = \dfrac{\pi}{3}$.

Hence, or otherwise, find the area enclosed by the ellipse $9x^2 + y^2 = 4$. **5**

(b) (i) If $z = \cos\theta + i\sin\theta$ use de Moivres Theorem to show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta.$$

(ii) By expanding $\left(z + \dfrac{1}{z}\right)^4$ show that $\cos^4\theta = \dfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. **5**

(c) Show that $\dfrac{d}{dx}\left[\ln(x + \sqrt{x^2+4})\right] = \dfrac{1}{\sqrt{x^2+4}}$.

Hence or otherwise, prove that $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 x\,dx}{\sqrt{\tan^2 x + 4}} = 2\ln\left(\frac{\sqrt{5}+1}{2}\right)$. **5**

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