Mathematics Trial HSC 2010 Solutions.

Question 1

(a) $1 - \frac{p-q}{p+q}$

$= \frac{p+q-p+q}{p+q}$

$= \frac{2q}{p+q}$. # (2)

(b) $\frac{4x-5}{x} = 2$

$4x - 5 = 2x$

$4x - 2x = 5$

$2x = 5$

$\therefore x = \frac{5}{2}$ or $2\frac{1}{2}$ # (2)

(c) $|x-1| = 5$

$x - 1 = 5$ or $-(x-1) = 5$

$x = 6$ or $-x+1 = 5$

$\therefore x = 6$ or $x = -4$ # (2)

(d) $y = x^3 - 4x$

$\therefore \frac{dy}{dx} = 3x^2 - 4$

when $x = 1$, $\frac{dy}{dx} = 3 \times 1^2 - 4$

$\therefore \frac{dy}{dx} = -1$ # (2)

(e) $2\sin\theta = 1$

$\therefore \sin\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{6}$. # (2)

(f) $\ln x = 3$

$\therefore x = e^3$

$\therefore x = 20.086$ # (2)

Question 2

(a) (i) $\frac{d}{dx}(x\tan x)$

$= x \times \frac{d}{dx}(\tan x) + \tan x \times \frac{d}{dx}(x)$

$= x\sec^2 x + \tan x$ # (2)

(ii) $\frac{d}{dx}\left[(e^x+1)^3\right]$

$= 3(e^x+1)^2 \times \frac{d}{dx}(e^x+1)$

$= 3e^x(e^x+1)^2$. # (2)

(b) (i) $\int 4\,dx = 4x + c$ # (1)

(ii) $\int \frac{2}{(x-5)^2}\,dx$

$= 2\int (x-5)^{-2}\,dx$

$= -2(x-5)^{-1} + c$

$= \frac{-2}{x-5} + c$. # (2)

(iii) $\int_0^3 \sqrt{5x+1}\,dx$

$= \left[\frac{(5x+1)^{3/2}}{5 \times 3/2}\right]_0^3$

$= \left[\frac{2}{15}(5x+1)^{3/2}\right]_0^3$

$= \frac{2}{15} \times 16^{3/2} - \frac{2}{15} \times 1^{3/2}$

$= \frac{2}{15} \times 64 - \frac{2}{15}$

$= 8\frac{2}{5}$ # (3)

(c) $\sum_{k=2}^{5} \frac{(-1)^k}{k+1}$

$= \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \frac{(-1)^4}{4+1} + \frac{(-1)^5}{5+1}$

$= \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$

$= \frac{7}{60}.$ # (2)

Question 3

(a) $n = 20,\ a = 1,\ d = 7.$

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$\therefore S_{20} = \frac{20}{2}\left[2\times 1 + (20-1)\times 7\right]$

$= 10(2 + 133)$

$= 1350.$ # (2)

(b) (i) $A(-4,-2);\ O(0,0)$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{0-(-2)}{0-(-4)}$

$= \frac{1}{2}.$

$y - y_1 = m(x - x_1)$

$y - 0 = \frac{1}{2}(x - 0)$

$y = \frac{1}{2}x$ # (2)

(ii) $m_{AB} = \frac{1}{2}$

$\therefore \frac{1}{2} \times m_{BC} = -1$

$\therefore m_{BC} = -2$

equation of BC is

$y - 6 = -2(x-2)$

$y - 6 = -2x + 4$

$\therefore y = -2x + 10$ # (2)

(iii) $y = \frac{1}{2}x$

$\therefore x = 2y.$

Sub. $x = 2y$ into $y = -2x + 10$

$\therefore y = -2(2y) + 10$

$y = -4y + 10$

$5y = 10$

$\therefore y = 2$

$\therefore x = 2\times 2 = 4$

$\therefore B(4,2)$ # (2)

(iv) $AC = \sqrt{(2+4)^2 + (6+2)^2}$

$= \sqrt{36 + 64}$

$= \sqrt{100}$

$\therefore AC = 10$ units # (1)

(v) $M\left(\frac{2-4}{2}, \frac{6-2}{2}\right)$

$= M(-1, 2)$ # (1)

(vi) $MB = 4 + 1 = 5$ units

$MA = MC = 5$ units $= MB$

$\therefore$ a circle, centre M, passes through A, B and C. # (1)

(vii) $(x+1)^2 + (y-2)^2 = 25.$ # (1)

Question 4

(a) $\$200 + \$150 + \$\frac{3}{4}\times 150 + \dots$

a geometric series with

$a = \$200,\ r = \frac{3}{4}$

Since $|x|<1$ then | $\therefore \angle QPR = 60^\circ$ # (3)

$\therefore \triangle ABC \,|||\, \triangle ADB$ (equiangular) # (2)

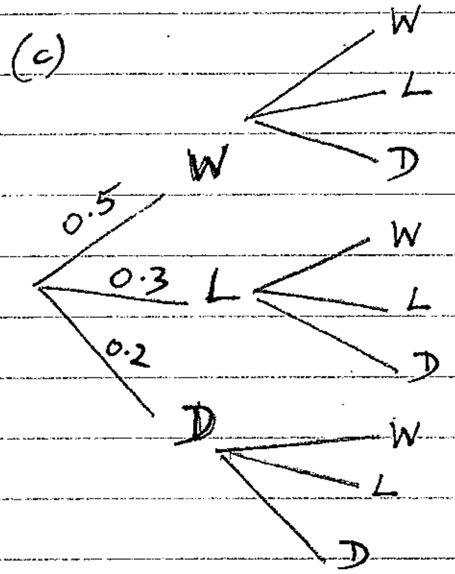
(ii) $\frac{AB}{AC} = \frac{AD}{AB}$ (matching sides of similar $\triangle$s are proportional)

$\therefore \frac{AB}{16} = \frac{4}{AB}$

$AB^2 = 64$

$\therefore AB = 8$ cm. # (2)

(c)



(i) $P(DD) = 0.2 \times 0.2$

$= 0.04$ # (1)

(ii) P(win at least 1 match)

$= 1 - P(LL + LD + DL + DD)$

$= 1 - [0.3^2 + 2\times0.3\times0.2 + 0.2^2]$

$= 1 - 0.25$

$= 0.75$ # (2)

(iii) P(not win either match)

$= P(LL + LD + DL + DD)$

$= 0.25$ # (2)

Question 6

(a) (i) $l = r\theta$.

$\therefore \frac{\pi}{4} = r \times \frac{\pi}{6}$

$\therefore r = \frac{\pi}{4} \times \frac{6}{\pi}$

$\therefore r = 1.5$ m. # (2)

(ii) $A = \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \times 1.5^2 \times \frac{\pi}{6}$

$= 0.59 \text{ m}^2$ # (1)

(iii) $AB^2 = 1.5^2 + 1.5^2 - 2\times1.5^2 \times \cos 30°$

$= 4.5 - 4.5 \cos 30°$

$= 0.602885083\ldots$

$\therefore AB = 0.78$ m # (2)

(b) $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

$= \frac{|5\times2 - 12\times-1 + 4|}{\sqrt{5^2+(-12)^2}}$

$= \frac{26}{13}$

$= 2$ units # (2)

(c) $2\log x = \log(5x+6)$

$\therefore \log x^2 = \log(5x+6)$

$\therefore x^2 = 5x+6$

$x^2 - 5x - 6 = 0$

$(x-6)(x+1) = 0$

$\therefore x = 6$ or $x = -1$

$\therefore \; x = 6$ since $x > 0$ # ②

(d) $V = \pi \int_{\pi/4}^{\pi/3} y^2 \, dx$

$= \pi \int_{\pi/4}^{\pi/3} \sec^2 x \, dx$

$= \pi \left[\tan x \right]_{\pi/4}^{\pi/3}$

$= \pi \left(\tan \frac{\pi}{3} - \tan \frac{\pi}{4} \right)$

$= \pi (\sqrt{3} - 1)$ units^3 # ③

Question 7

(a) $-x^2 + 13x - 36 = 0$

$\therefore \; x^2 - 13x + 36 = 0$

$(x-4)(x-9) = 0$

$\therefore \; x = 4$ or $x = 9$ # ②

(b) $y = -x^2 + 13x - 36$

at $x = 6$; $y = -6^2 + 13 \times 6 - 36$

$= 6$

$\frac{dy}{dx} = -2x + 13$

at $x = 6$, $\frac{dy}{dx} = -2 \times 6 + 13 = 1.$

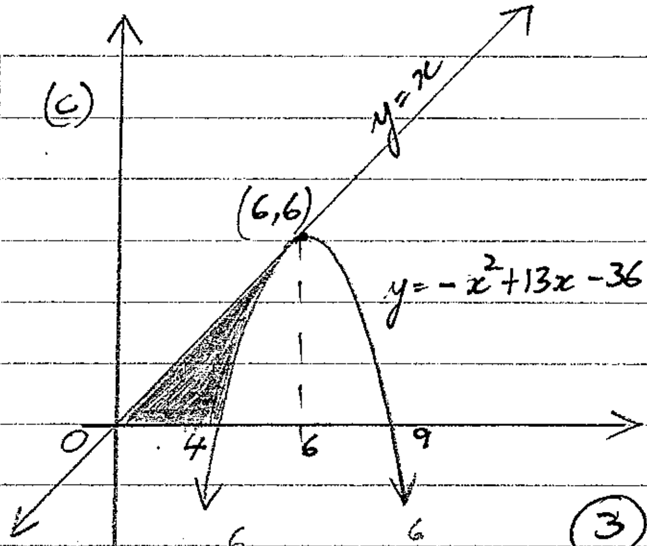
$\therefore$ equation of tangent is

$y - 6 = 1(x - 6)$

$y - 6 = x - 6$

$\therefore \; y = x$ # ③

(c)



③

or $\int_0^6 x \, dx - \int_4^6 (-x^2 + 13x - 36) \, dx$

(d) Area shaded

$= \frac{1}{2} \times 6 \times 6 - \int_4^6 (-x^2 + 13x - 36) \, dx$

$= 18 - \left[\frac{-x^3}{3} + \frac{13x^2}{2} - 36x \right]_4^6$

$= 18 - \left[\frac{-6^3}{3} + \frac{13 \times 6^2}{2} - 36 \times 6 \right]$

$- \left[\frac{-4^3}{3} + \frac{13 \times 4^2}{2} - 36 \times 4 \right]$

$= 18 - [-54] - [61\frac{2}{3}]$

$= 18 + 54 - 61\frac{2}{3}$

$= 10\frac{2}{3}$ units^2 # ④

Question 8

(a) (i) $f(x) = \frac{1}{3}x^3 + x^2 - 3x + 5$

$f'(x) = x^2 + 2x - 3.$

$f''(x) = 2x + 2$

$f'(x) = 0$ for stationary points

$\therefore \; x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$\therefore \; x = -3$ or $x = 1.$

When $x=-3$,

$f(x) = \frac{1}{3}(-3)^3 + (-3)^2 - 3\times -3 + 5$

$= -9+9+9+5$

$= 14.$

when $x=-3$,

$f''(x) = 2\times -3 + 2 = -4.$

$\therefore (-3, 14)$ is a maximum turning point. #

When $x=1$,

$f(x) = \frac{1}{3} + 1 - 3 + 5$

$= 3\frac{1}{3}.$

when $x=1$,

$f''(x) = 2\times 1 + 2 = 4.$

$\therefore (1, 3\frac{1}{3})$ is a minimum turning point. # ④

(ii) For points of inflection

$f''(x) = 0$

$\therefore 2x+2=0$

$2x=-2$

$\therefore x=-1$

when $x=-1$,

$f(x) = \frac{1}{3}(-1)^3 + (-1)^2 - 3(-1) + 5$

$= -\frac{1}{3} + 1 + 3 + 5$

$= 8\frac{2}{3}.$

$\therefore (-1, 8\frac{2}{3})$ is a possible point of inflection.

Test for concavity:

when $x=-6.9$, $f''(x) = 2\times -0.9 + 2$

$= 0.2$

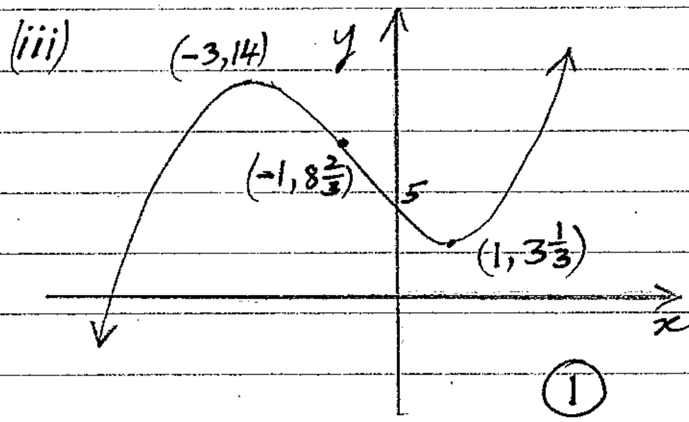
when $x=-1.1$, $f''(x) = 2\times -1.1 + 2$

$= -0.2$

Since concavity changes about $x=-1$,

$(-1, 8\frac{2}{3})$ is a point of inflection. # ②

(iii)



①

(iv) Concave upwards when

$f''(x) > 0.$

$\therefore 2x+2>0$

$2x>-2$

$\therefore x>-1.$ # ①

(b) (i) $M = 175e^{-kt}$

$\therefore 87.5 = 175e^{-6k}$

$0.5 = e^{-6k}$

$\ln 0.5 = -6k \ln e$

$\therefore k = \frac{\ln 0.5}{-6}$ $(\ln e = 1)$

$\therefore k = 0.11552$ # ②

(ii) $\frac{dM}{dt} = -175ke^{-kt}$

$\therefore$ when $t=10$,

$\frac{dM}{dt} = -175\times\frac{\ln 0.5}{-6}e^{-10\frac{\ln 0.5}{-6}}$

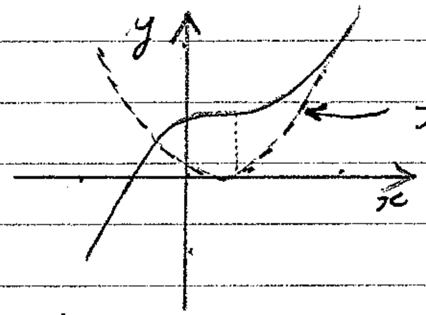
$= -6.367890692\ldots$

$= -6.4$ g/day

i.e. disintegrating at 6.4 g/day. # ②

Question 9

(a)



(2)

(b) $\frac{dy}{dx} = 6x - \frac{2}{2x-1}$

$\therefore y = 3x^2 - \ln(2x-1) + c$

Sub. (1,7)

$\therefore 7 = 3 - \ln 1 + c$

$\therefore c = 4$

$\therefore y = 3x^2 - \ln(2x-1) + 4$ #

(2)

(c)

(i) \$10000, $r = 1.01$

Amount owing after 1 month, A_1, is given by

$A_1 = \$10000 \times 1.01 - M$

Amount owing after 2 months, A_2, is given by

$A_2 = A_1(1.01) - M$

$= [\$10000(1.01) - M]\,1.01 - M$

$= \$10000(1.01)^2 - M(1 + 1.01)$

Amount owing after 3 months, A_3, is given by

$A_3 = A_2(1.01) - M$

$= [\$10000(1.01)^2 - M(1+1.01)]$

$\times 1.01 - M$

$= \$10000(1.01)^3 - M(1 + 1.01 + 1.01^2)$

$\therefore$

$A_n = \$10000(1.01)^n - M(1 + 1.01 + \dots + 1.01^{n-1})$

Now $1 + 1.01 + 1.01^2 + \dots + 1.01^{n-1}$ is a geometric series with $a = 1$, $r = 1.01$, n terms

Now $S = \frac{a(r^n - 1)}{r-1}$

$\therefore S = \frac{1(1.01^n - 1)}{1.01 - 1}$

$\therefore A_n = \$10000(1.01)^n - M\frac{(1.01^n - 1)}{0.01}$

\# (4)

(ii) $A_{60} = 0$.

$\therefore \$10000(1.01)^{60} = \frac{M(1.01^{60} - 1)}{0.01}$

$\therefore M = \frac{\$10000(1.01)^{60} \times 0.01}{1.01^{60} - 1}$

$= \$222.44$ # (2)

(iii) $M = \frac{\$10000(1.01)^{84} \times 0.01}{1.01^{84} - 1}$

$= \$176.53$

$\therefore$ repayments over 7 years

$= \$176.53 \times 84$

$= \$14828.52$

repayments over 5 years

$= \$222.44 \times 60$

$= \$13346.40$

$\therefore$ extra $= \$14828.52 - \$13346.40$

$= \$1482.12$ # (2)

Question 10

(a)

x	0	0.5	1	1.5	2
$f(x)$	2	$\sqrt{4.25}$	$\sqrt{5}$	2.5	$\sqrt{8}$
w	1	4	2	4	1

$$A \doteqdot (2-0) \times \left\{ \frac{2 + 4\sqrt{4.25} + 2\sqrt{5} + 10 + \sqrt{8}}{1+4+2+4+1} \right\}$$

$$= 4.5911129055\ldots$$

$$= 4.6 \ \#$$ (3)

(b) (i) $V = \dfrac{6}{\sqrt{2t+1}} = 6(2t+1)^{-1/2}$

$$\therefore x = \int 6(2t+1)^{-1/2}\,dt$$

$$= \frac{6(2t+1)^{1/2}}{\frac{1}{2} \times 2} + c$$

$$= 6\sqrt{2t+1} + c$$

when $t = 0$, $x = 0$

$\therefore 0 = 6 + c$

$\therefore c = -6$

$\therefore x = 6\sqrt{2t+1} - 6 \ \#$ (1)

(ii) When $x = 0$, $t = 0$

when $x = 24$,

$24 = 6\sqrt{2t+1} - 6$

$\therefore 30 = 6\sqrt{2t+1}$

$\therefore 5 = \sqrt{2t+1}$

$\therefore 25 = 2t + 1$

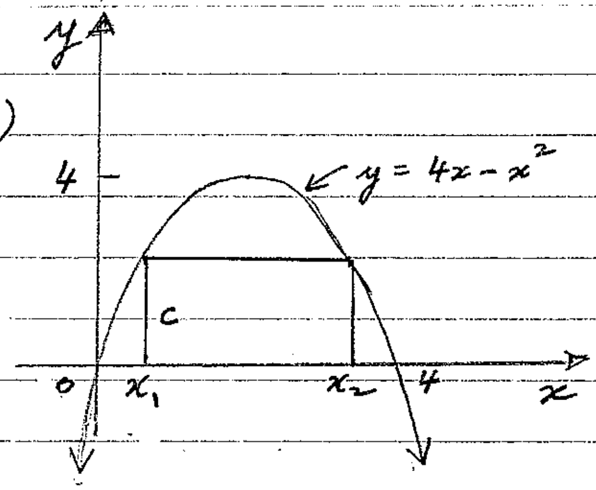
$2t = 24$

$\therefore t = 12$

$\therefore$ average velocity $= \dfrac{24-0}{12-0}$

$= 2 \text{ m/s} \ \#$ (2)

(c)



(i) If $y = c$, then $c = 4x - x^2$

$\therefore x^2 - 4x + c = 0$

$$\therefore x = \frac{4 \pm \sqrt{16 - 4c}}{2}$$

$$= \frac{4 \pm 2\sqrt{4-c}}{2}$$

$$= 2 \pm \sqrt{4-c}.$$

$\therefore x = 2 + \sqrt{4-c}$ or $2 - \sqrt{4-c}$

i.e. $x_1 = 2 - \sqrt{4-c}$ and

$x_2 = 2 + \sqrt{4-c}$

$\therefore$ Length of rectangle

$= (2 + \sqrt{4-c}) - (2 - \sqrt{4-c})$ cm

$= 2\sqrt{4-c}$ cm

$\therefore$ Area of rectangle

$= 2c\sqrt{4-c}$ cm² # (3)

(ii) Let A represent the area (in cm²).

$$\therefore A = 2c\sqrt{4-c}$$

$$\therefore \frac{dA}{dc} = 2c \times \tfrac{1}{2}(4-c)^{-\frac{1}{2}} \times -1 + (4-c)^{\frac{1}{2}} \times 2$$

$$= \frac{-c}{\sqrt{4-c}} + 2\sqrt{4-c}$$

For min./max A, $\frac{dA}{dc} = 0$

$\therefore \quad 2\sqrt{4-c} - \frac{c}{\sqrt{4-c}} = 0$

$\therefore \quad 2\sqrt{4-c} = \frac{c}{\sqrt{4-c}}$

Cross-multiplying,

$2(4-c) = c$

$8 - 2c = c$

$8 = 3c$

$\therefore \quad c = \frac{8}{3} = 2\frac{2}{3}.$

c	2	$2\frac{2}{3}$	3
$\frac{dA}{dc}$	$\sqrt{2}$	0	-1

Check: when $c = 2$

$\frac{dA}{dc} = 2\sqrt{4-2} - \frac{2}{\sqrt{4-2}}$

$= 2\sqrt{2} - \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$= 2\sqrt{2} - \sqrt{2}$

$= \sqrt{2} \qquad > 0.$

when $c = 3$,

$\frac{dA}{dc} = 2\sqrt{4-3} - \frac{3}{\sqrt{4-3}}$

$= 2 - 3$

$= -1 \qquad < 0.$

$\therefore$ maximum A occurs

when $c = 2\frac{2}{3}$ cm

$\therefore \quad A = 2 \times \frac{8}{3}\sqrt{4 - \frac{8}{3}}$

$= \frac{16}{3}\sqrt{\frac{4}{3}}$

$= \frac{16}{3} \times \frac{2}{\sqrt{3}}$

$= \frac{32}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{32\sqrt{3}}{9}$ cm^2 # 3