JRAHS Ext^n 1 TRIAL 2005 ANSWERS

(1) a) By Factor theorem $P(-2)=0$

$\therefore (-2)^3 - 2(-2)^2 + a(-2) + 4 = 0$

$-12 - 2a = 0$

$\underline{\underline{a = -6}}$ $\boxed{1}$

b) i) $\dfrac{d}{dx}\ln(\cos 2x) = \dfrac{-2\sin 2x}{\cos 2x}$ $\boxed{1}$

$(= -2\tan 2x)$

ii) $\displaystyle\int_0^{\pi/6} \tan 2x = \left[-\tfrac{1}{2}\ln(\cos 2x)\right]_0^{\pi/6}$

$= -\tfrac{1}{2}\ln\left(\cos\tfrac{\pi}{3}\right) + \tfrac{1}{2}\ln 1$

$= -\tfrac{1}{2}\ln\left(\tfrac{1}{2}\right)$ $\boxed{2}$

$= \underline{\underline{\ln\sqrt{2}}}$ or $\underline{\underline{\tfrac{1}{2}\ln 2}}$

c) i) $\displaystyle\int \frac{e^{3x}\,dx}{2+e^{3x}} = \tfrac{1}{3}\ln(2+e^{3x}) + c$

where c is an undetermined constant. $\boxed{1}$

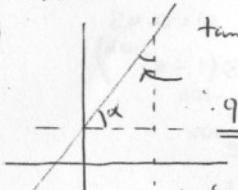
ii) $\displaystyle\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{9/4 - x^2}}$

$= \tfrac{1}{2}\sin^{-1}\left(\tfrac{2x}{3}\right) + c$ $\boxed{2}$

$\left[\text{or } \tfrac{1}{2}\cos^{-1}\left(\tfrac{2x}{3}\right) + c\right]$

where c is an undetermined constant

d)



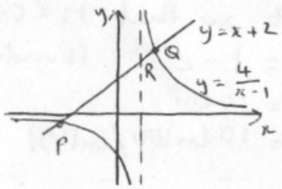
$\tan\alpha = \sqrt{3}$

$\alpha = 60^\circ\ (\pi/3)$

$\underline{90 - \alpha = 30^\circ}\ (\pi/6)$ $\boxed{2}$

1 for correct notion or a correct formula.

e)



From the diagram, $x+2 < \dfrac{4}{x-1}$ for values of x less than that at P and for values between R (non-inclusive) and Q.

P, Q found by solving $(x+2)(x-1) = 4$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

P is $(-3,-1)$ Q is $(2,4)$ $\boxed{3}$

Soln. is $\underline{\underline{\{x : x < -3\} \cup \{x : 1 < x < 2\}}}$

(2) a) Let $u = x^{1/2}$

$\dfrac{du}{dx} = \dfrac{1}{2x^{1/2}} = \dfrac{1}{2u}$

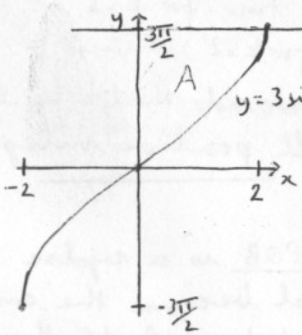
$\displaystyle\int_0^{\pi^2/16} \frac{\sin\sqrt{x}\,dx}{\sqrt{x}} = \int_0^{\pi/4} 2\sin u\,du$

$= \left[-2\cos u\right]_0^{\pi/4}$

$= \left[-\dfrac{2}{\sqrt{2}} + 2\right]$

$= \underline{\underline{2-\sqrt{2}}}$ $\boxed{3}$

b) i)



$\boxed{2}$

ii) Area $A = \displaystyle\int_0^{3\pi/2} x\,dy$

$= \displaystyle\int_0^{3\pi/2} 2\sin\frac{y}{3}\,dy$

$= \left[-6\cos\dfrac{y}{3}\right]_0^{3\pi/2}$

$= \underline{\underline{6}}$ sq units. $\boxed{2}$

2 c) $\alpha+\beta+\gamma = 2/3$

$(\alpha\beta+\beta\gamma+\gamma\alpha = 1)$

$\alpha\beta\gamma = 4/3$

$\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\gamma\alpha} = \dfrac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$

$= \dfrac{2/3}{4/3} = \underline{\underline{\dfrac{1}{2}}}$ $\boxed{2}$

d) i) $3! = \underline{\underline{6}}$ $\boxed{1}$

ii) Initially treat vowels as 1 unit $\to 3!$

But vowels can be arranged in $3!$ ways.

$\therefore$ Total $= 3!\,3! = \underline{\underline{36}}$ $\boxed{1}$

iii) Both answers will be divided by 2. $\boxed{1}$

(3) a) $\cos 2x - \cos x = 0$

$2\cos^2 x - \cos x - 1 = 0$

$(2\cos x + 1)(\cos x - 1) = 0$

$\underline{\cos x = -\tfrac{1}{2} \text{ or } \cos x = 1}$

But $\cos\pi/3 = -\tfrac{1}{2}$ and $\cos 0 = 1$

$\therefore x = \underline{\underline{2n\pi \pm \tfrac{2\pi}{3} \text{ or } 2m\pi}}$ $\boxed{3}$

when $n = \ldots -2, -1, 0, 1, 2, \ldots$

$m = \ldots -2, -1, 0, 1, 2, \ldots$

b) (i) Tangent at P is $y = tx - at^2$

Tangent at Q is $y = ux - au^2$

Solve these equations

Subtract $0 = x(t-u) - a(t^2-u^2)$

$\underline{a(t+u) = x}$

Sub into first equation

$y = at(t+u) - at^2$

$\underline{\underline{y = atu}}$ $\boxed{2}$

ii) Referring to part 1,

$u + t = 1$

$ut = -6$

Solving, $u = 3,\ t = -2$

(or vice-versa)

$\therefore$ P is $(-4a, 4a)$

and Q is $(6a, 9a)$

R is $(a, -6a)$

$PQ = \sqrt{25a^2 + 100a^2} = a5\sqrt{5}$

$PR = \sqrt{25a^2 + 100a^2} = a5\sqrt{5}$.

$\therefore \underline{\underline{\triangle PQR \text{ is isosceles}}}$ $\boxed{3}$

c) i) $(5+2x)^{12} = 5^{12} + {}^{12}C_1\,5^{11}(2x)^1 + \ldots$

$= \displaystyle\sum_0^{12} a_k x^k$ $\boxed{2}$

where $a_k = \underline{\underline{{}^{12}C_k\,2^k\,5^{12-k}}}$

ii) $\dfrac{a_{k+1}}{a_k} = \dfrac{{}^{12}C_{k+1}\,2^{k+1}\,5^{12-k-1}}{{}^{12}C_k\,2^k\,5^{12-k}}$

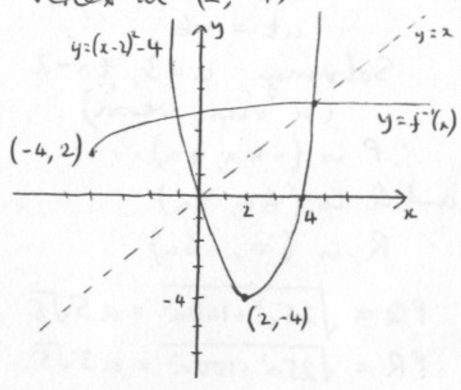
$= \dfrac{\dfrac{12!}{(k+1)!\,(12-(k+1))!} \times 2}{\dfrac{12!}{k!\,(12-k)!} \times 5}$

$= \dfrac{k!\,(12-k)! \times 2}{(k+1)!\,(12-(k+1))! \times 5}$

$= \dfrac{2\,(12-k)}{5\,(k+1)}$ $\boxed{2}$

$= \underline{\underline{\dfrac{24-2k}{5k+5}}}$

(4) a) i) This is a parabola with vertex at $(2, -4)$



ii) Largest domain will be $\{x : x \geq 2\}$

iii) See diagram. [4]

b) i) Let $x = R\sin(2t-\alpha)$

$= R\sin 2t\cos\alpha - R\cos 2t\sin\alpha$

$\therefore R\sin\alpha = 2\sqrt{3}$, $R\cos\alpha = 6$ } comparing coefficients

Divide equations

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \pi/6$

Square and add equations

$R^2 = 12 + 36$

$R = \sqrt{48} = 4\sqrt{3}$ [2]

$x = 4\sqrt{3}\sin(2t - \pi/6)$

ii) Using this form of x (the other works just as well)

$\dot{x} = 8\sqrt{3}\cos(2t - \pi/6)$

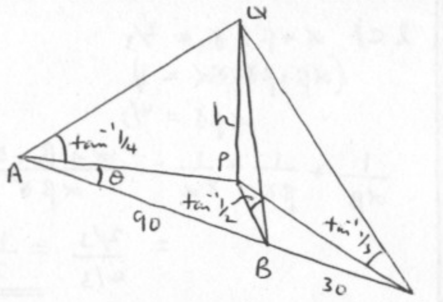
$\ddot{x} = -16\sqrt{3}\sin(2t-\pi/6)$

$\ddot{x} = -4x$ [1]

$\ddot{x} = -n^2 x$ where $n = 2$

Thus the particle moves in S.H.M.

c)



Let the flagpole be PQ with height h as in the diagram

$\frac{h}{AP} = \frac{1}{4}$ (Normal trig on ΔAPQ)

$\therefore AP = 4h$

Similarly $BP = 2h$, $CP = 3h$.

Solve for $\cos PAB$ in ΔPAB and PAC:

$\cos\theta = \frac{(4h)^2 + (90)^2 - (2h)^2}{2.4h.90}$ (ΔPAB)

$= \frac{(4h)^2 + (120)^2 - (3h)^2}{2.4h.120}$ (ΔPAC)

$\therefore \frac{16h^2 + 8100 - 4h^2}{3} = \frac{16h^2 + 14400 - 9h^2}{4}$

$4(12h^2 + 8100) = 3(7h^2 + 14400)$

$27h^2 = 10800$

$h^2 = 400$

$h = 20$ [5]

$\therefore$ Flagpole is 20m in height.

(5) a) Acceleration $= \frac{d}{dx}\left(\frac{v^2}{2}\right)$

$= \frac{d}{dx}\left(\frac{5x}{2} - \frac{x^2}{2}\right)$

$= \frac{5}{2} - x$ [2]

At $x = 2$, acc^n is $\frac{1}{2}\text{ m s}^{-2}$

(5) b) Assume that, for some pos^{ve} integer k, that

$(k+1)(k+2)\dots(k+k) = 2^k M$

where M is an odd integer (since product NOT divisible by 2^{k+1})

Now consider the product for $k+1$

$((k+1)+1)((k+1)+2)\dots((k+1)+(k+1))$

$= (k+2)(k+3)\dots 2k(2k+1)(2k+2)$

$= 2(k+1)(k+2)\dots(k+k)(2k+1)$

$= 2 \,.\, 2^k M \,.\, (2k+1)$ from assumption

$= 2^{k+1} M(2k+1)$

This is divisible by 2^{k+1} but NOT by 2^{k+2} since both M and $(2k+1)$ are odd.

Thus, if true for k, the result is also true for $k+1$.

But, if $k=1$, $(1+1) = 2$ is divisible by 2^1 but not by 2^2.

Since true for $k=1$, it will be true for $k=2$, hence for $k=3$ etc.

Thus proved that result true for all positive integers. [5]

c) i) CPQB is a cyclic quadrilateral because the angle subtended at P by the interval CB is equal to the angle CB subtends at Q.

Also PAQT is a cyclic quadrilateral because the opposite angles are supplementary — witness the 90° angles at $\angle APT$ and $\angle AQT$

$\angle TAQ = \angle TPQ$ (Angles at the circumference from the chord TQ of cyclic quad^l PAQT are equal)

$\angle TPQ = \angle QCB$ (Angles at the circumference from the chord BQ of cyclic quad^l CPQB are equal)

$\therefore \angle TAQ = \angle QCB$ [3]

ii) Consider the sums of the angles in ΔAQT and ΔTCR.

$\angle TAQ = \angle TCR$ (Proved above)

$\angle ATQ = \angle CTR$ (Vertically opposite angles are equal)

$\therefore \angle TCR = \angle TQA$ (Sum of the angles of each triangle adds to 180°)

But $\angle TQA = 90^\circ$

$\therefore \angle TCR = 90^\circ$ [2]

$\therefore AR \perp CB$.

(6) a) i) LHS $= \frac{d}{dt}(S + Ae^{-kt})$

$= -kAe^{-kt}$

RHS $= k(S - (S + Ae^{-kt}))$

$= k(-Ae^{-kt})$

$= -kAe^{-kt}$ [1]

LHS = RHS

$\therefore M = S + Ae^{-kt}$ satisfies the equation.

ii) When $t = 0$, $M = 0 \Rightarrow A = -S$

When $t = 10$, $M = 0.4S$

$0.4S = S(1 - e^{-10k})$

$0.6 = e^{-10k}$

$5/3 = e^{10k}$ [2]

$\frac{1}{10}\ln(5/3) = k$

iii) Find t so that $M = 0.99S$

$0.99 = 1 - e^{-kt}$ (Dividing by S)

$e^{kt} = 100$

$t = 10\ln 100/\ln(5/3)$ [2]

6b) i) B will follow path

$$\underline{\underline{y = -\frac{gx^2}{2V^2}\sec^2\beta + x\tan\beta}} \quad \boxed{1}$$

Collision will occur when y values same and x values same.

$$-\frac{gx^2}{2V^2}\sec^2\beta + x\tan\beta = -\frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha$$

$$\frac{gx^2}{2V^2}(\sec^2\alpha - \sec^2\beta) = x(\tan\alpha - \tan\beta)$$

Neglect the zero at $x=0$ (start point)

$$x = \frac{2V^2(\tan\alpha - \tan\beta)}{g(\sec^2\alpha - \sec^2\beta)}$$

$$= \frac{2V^2(\tan\alpha - \tan\beta)}{g(\tan^2\alpha - \tan^2\beta)} \quad (\sec^2\alpha = 1+\tan^2\alpha)$$

$$= \frac{2V^2}{g(\tan\alpha + \tan\beta)} \quad \boxed{3}$$

$$= \frac{2V^2\cos\alpha\cos\beta}{g(2\sin\alpha\cos\beta)} = \underline{\underline{\frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}}}$$

ii) For particle B, $\underline{x_B = V(t-T)\cos\beta}$ $\boxed{1}$
(t measured from when A fired)

Collision occurs when both x values are equal, at some time t, to the form from part (ii)

$$\therefore V t\cos\alpha = \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)} \Rightarrow t = \frac{2V\cos\beta}{g\sin(\alpha+\beta)}$$

Also $$V(t-T)\cos\beta = \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)} \Rightarrow t-T = \frac{2V\cos\alpha}{g\sin(\alpha+\beta)}$$

$$\therefore T = \{t-(t-T)\} = \underline{\underline{\frac{2V(\cos\beta - \cos\alpha)}{g\sin(\alpha+\beta)}}} \quad \boxed{2}$$

(7) a) This can be achieved in three ways, the individual probabilities of which must be added.

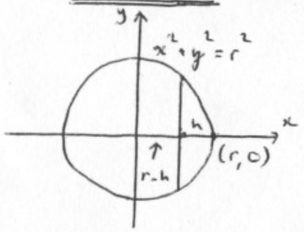
$$\text{Prob}(2 \text{ from A}, 0 \text{ from B}) = {}^{10}C_2(0.05)^2(0.95)^8(0.93)^{10}$$

$$\text{Prob}(1 \text{ from A}, 1 \text{ from B}) = {}^{10}C_1(0.05)^1(0.95)^9\,{}^{10}C_1(0.07)^1(0.93)^9$$

$$\text{Prob}(0 \text{ from A}, 2 \text{ from B}) = (0.95)^{10}\ {}^{10}C_2(0.07)^2(0.93)^8$$

$$\therefore \text{Total Prob} = 0.036 + 0.115 + 0.074$$
$$= \underline{0.225} \text{ appx.} \quad \boxed{4}$$

b) i)



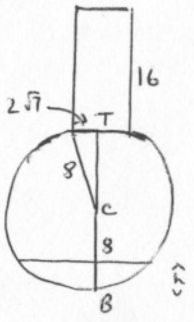
$$V\Omega = \pi\int_{r-h}^{r} y^2\,dx$$

$$= \pi\int_{r-h}^{r} r^2 - x^2\,dx = \pi\left[r^2x - \frac{x^3}{3}\right]_{r-h}^{r}$$

$$= \pi\left(\frac{2r^3}{3} - \left\{r^2(r-h) - \frac{(r-h)^3}{3}\right\}\right)$$

$$= \underline{\underline{\frac{\pi h^2}{3}(3r-h)}} \quad \boxed{3}$$

(7) b) ii)



$$CT = \sqrt{64-28}$$
$$= 6 \text{ cm}$$

$\therefore$ Total height

$\therefore 8 + 6 + 16$

$= \underline{\underline{30 \text{ cms.}}}$ $\boxed{1}$

iii) $$V = 8\pi h^2 - \frac{\pi h^3}{3}$$

$$\therefore \frac{dV}{dt} = 16\pi h - \pi h^2\ \frac{dh}{dt}$$

But $\frac{dV}{dt} = 100$

$$\therefore \frac{dh}{dt} = \underline{\underline{\frac{100}{\pi h(16-h)}}} \text{ cm/sec} \quad \boxed{2}$$

iv) When $h = 14$, $\frac{dh}{dt} = \frac{100}{28\pi}$ $\boxed{1}$

$$\doteq \underline{\underline{1.14 \text{ cm/sec}}}$$

At this rate it will take

$\frac{16}{1.14}$ secs $\doteq$ $\underline{\underline{14 \text{ seconds}}}$ $\boxed{1}$