Mathematics Extension I CSSA HSC Trial Examination 2003
Marking Guidelines

Question 1

1(a) Outcomes Assessed : H3

Marking Guidelines

Criteria	Marks
• rearranging limit	1
• finding answer	1

Answer

$$\lim_{n \to \infty} \frac{5(10^n)+3}{2(10^n)+1} = \lim_{n \to \infty} \frac{5+3(10^{-n})}{2+1(10^{-n})} = \frac{5+0}{2+0} = {}^5/_2$$

1(b) Outcomes Assessed : H5

Marking Guidelines

Criteria	Marks
• finding x coordinate	1
• finding y coordinate	1

Answer

$$x = \frac{2(7)+1(-2)}{2+1} = 4 \quad , \quad y = \frac{2(-1)+1(5)}{2+1} = 1$$

1(c) Outcomes Assessed : PE3

Marking Guidelines

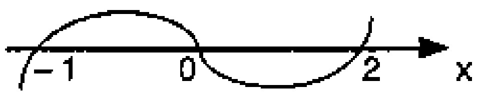
Criteria	Marks
• finding cubic inequality	1
• factoring cubic expression	1
• using diagram	1
• finding answer	1

Answer

$$\frac{2}{x} > x-1 \quad , \quad \frac{2}{x} \times x^2 > (x-1) \times x^2 \quad , \quad 2x > x^3 - x^2 \; , \; x^3 - x^2 - 2x < 0 \; ,$$

$$x(x^2-x-2) < 0 \quad , \quad x(x+1)(x-2) < 0$$

$y = x(x+1)(x-2)$



$x < -1$ or $0 < x < 2$

1(d) Outcomes Assessed : (i) / (ii) PE3 (iii) PE2

Marking Guidelines

Criteria	Marks
(i) copying diagram	0
• (ii) stating alternate segment theorem	1
• (iii) showing that $B\hat{A}C + M\hat{N}C = 180^\circ$	1
• showing that $M\hat{B}C + M\hat{N}C = 180^\circ$	1
• giving a reason why MNCB is cyclic	1

Answer

(i) /

(ii) The angle between the tangent BM and the chord BC is equal to the angle in the alternate segment.

(iii) $B\hat{A}C + M\hat{N}C = 180^\circ$ (cointerior angles supplementary BA parallel to MN)

$M\hat{B}C + M\hat{N}C = 180^\circ$ ($M\hat{B}C = B\hat{A}C$)

MNCB is cyclic (a pair of opposite interior angles is supplementary)

Question 2

2(a) Outcomes Assessed : PE5

Marking Guidelines

Criteria	Marks
• finding first derivative	1
• finding second derivative	1

Answer

$dy/dx = 5(x^2+1)^4 \times 2x = 10x(x^2+1)^4$

$d^2y/dx^2 = 10x \times 4(x^2+1)^3 \times 2x + (x^2+1)^4 \times 10 = 10(x^2+1)^3(9x^2+1)$

2(b) Outcomes Assessed : PE3 , PE6

Marking Guidelines

Criteria	Marks
• writing as a sum of four binomial coefficients	1
• finding answer	1

Answer

$\sum_{n=2}^{n=5} {}^nC_2 = {}^2C_2 + {}^3C_2 + {}^4C_2 + {}^5C_2 = 1 + 3 + 6 + 10 = 20$

2(c) Outcomes Assessed : (i) P4 , H5 (ii) P4 , H5

Marking Guidelines

Criteria	Marks
• (i) expanding LHS	1
• showing answer	1
• (ii) using $A = 15^\circ$ to find value of RHS	1
• finding answer	1

Answer

(i) $(\sin A - \cos A)^2 = \sin^2 A + \cos^2 A - 2\sin A\cos A = 1 - \sin 2A$

(ii) $(\sin 15^\circ - \cos 15^\circ)^2 = 1 - \sin 30^\circ = 1 - {}^1/_2 = {}^1/_2$

$\sin 15^\circ - \cos 15^\circ = {}^{-1}/\sqrt{2}$ (since $\sin 15^\circ < \cos 15^\circ$)

2(d) Outcomes Assessed : (i) PE4 (ii) H5 , PE3

Marking Guidelines

Criteria	Marks
• (i) finding gradient of tangent	1
• (ii) finding gradient of FT	1
• finding initial expression for $\tan\theta$	1
• finding final expression for $\tan\theta$	1

Answer

(i) $y = {}^{x^2}/_4$, ${}^{dy}/_{dx} = {}^{2x}/_4 = {}^{x}/_2$

When $x = 2t$, ${}^{dy}/_{dx} = {}^{2t}/_2 = t$. The tangent at T has gradient t

(ii) F is the point (0, 1) . FT has gradient ${}^{(t^2-1)}/_{2t}$

$$\tan\theta = \left|\frac{t - {}^{(t^2-1)}/_{2t}}{1 + t\,{}^{(t^2-1)}/_{2t}}\right| = \left|\frac{2t^2 - t^2 + 1}{2t + t^3 - t}\right| = \left|\frac{t^2+1}{t^3+t}\right| = \frac{1}{|t|}$$

Question 3

3(a) Outcomes Assessed : (i) H6 (ii) PE3 (iii) HE4

Marking Guidelines

Criteria	Marks
• (i) showing function increasing	1
• showing graph concave down	1
• (ii) finding numerical expression for x intercept	1
• finding answer	1
• (iii) showing intercepts and asymptotes on graph	1
• showing shape of graph	1
• sketching graph of inverse function	1

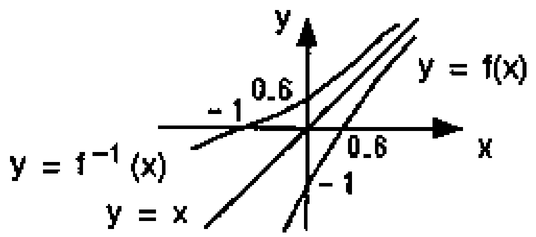
Answer

(i) $f'(x) = 1 + e^{-x} > 0$ for all x , function increasing

$f''(x) = -e^{-x} < 0$ for all x , graph concave down

(ii) $x = 0.5 - \dfrac{f(0.5)}{f'(0.5)} = 0.5 - \dfrac{0.5 - e^{-0.5}}{1 + e^{-0.5}} = 0.6$

(iii)



3(b) Outcomes Assessed : (i) HE5 (ii) HE4 (iii) HE7

Marking Guidelines

	Criteria	Marks
•	(i) finding a in terms of x	1
•	finding x in terms of t	1
•	(ii) sketching graph	1
•	(iii) describing motion	1
•	finding limiting position	1

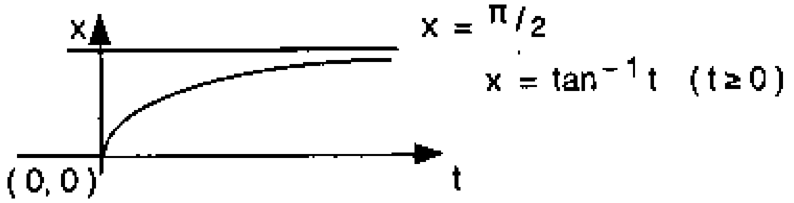
Answer

(i) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{1}{2}\cos^4 x\right) = -2\cos^3 x \sin x$

$\frac{dx}{dt} = \cos^2 x$, $\frac{dt}{dx} = \sec^2 x$, $t = \int \sec^2 x \, dx$

$t = \tan x + c$, $t = 0, x = 0, c = 0$, $t = \tan x$, $x = \tan^{-1} t$

(ii)



(iii) The particle starts at O moving to the right ($v > 0$ for $0 \le x < \pi/2$) and slowing down ($v > 0$ and $a < 0$ for $0 < x < \pi/2$). It approaches its limiting position of $\pi/2$ metres to the right of O .

Question 4

4(a) Outcomes Assessed : (i) HE4 (ii) H8 (iii) H8

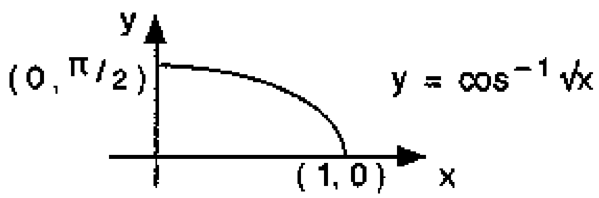
Marking Guidelines

	Criteria	Marks
•	(i) finding domain	1
•	finding range	1
•	sketching graph	1
•	(ii) finding numerical expression for area	1
•	finding answer	1
•	(iii) finding primitive	1
•	finding answer	1

Answer

(i) domain : $-1 \le \sqrt{x} \le 1$, $0 \le \sqrt{x} \le 1$, $0 \le x \le 1$

range : $\cos^{-1} 1 \le y \le \cos^{-1} 0$, $0 \le y \le \pi/2$



(ii)

x	0	$^1/_2$	1
y	$^{\pi}/_2$	$^{\pi}/_4$	0

Area $= \int_0^1 y\,dx$

Area $= \dfrac{^1/_2}{3}\{\,^{\pi}/_2 + 4(^{\pi}/_4) + 0\,\} = {}^{\pi}/_4$ units^2

(iii) Area $= \int_0^{\pi/2} x\,dy = \int_0^{\pi/2} \cos^2 y\,dy = \int_0^{\pi/2} {}^1/_2(1+\cos 2y)\,dy$

$= \left[\,^1/_2\, y + {}^1/_4 \sin 2y\,\right]_0^{\pi/2} = {}^{\pi}/_4$ units^2

4(b) Outcomes Assessed : (i) HE3 (ii) HE3 (iii) HE3

Marking Guidelines

Criteria	Marks
• (i) finding primitive	1
• finding answer	1
• (ii) finding period	1
• finding amplitude	1
• (iii) finding answer	1

Answer

(i) $v^2 = \int 2\ddot{x}\,dx = \int(-8x+16)\,dx = -4x^2+16x+c$

$x=0,\ v=0,\ c=0\ ,\ v^2 = -4x^2+16x$

(ii) $\ddot{x} = -(2)^2(x-2)\ ,\ n=2\ ,$ period $= {}^{2\pi}/_2 = \pi$ seconds.

When $v=0\ ,\ -4x^2+16x=0\ ,\ -4x(x-4)=0\ ,\ x=0$ or $x=4$

Centre at $x=2$. Amplitude $=2$ metres.

(iii) In π seconds the particle travels 8 metres.

In 1 minute the particle travels $60 \times {}^8/\pi = 153$ metres.

Question 5

5(a) Outcomes Assessed : PE3

Marking Guidelines

Criteria	Marks
• finding numerical coefficient of x^5	1
• finding numerical coefficient of x^6	1
• finding equation for a	1
• finding answer	1

Answer

(i) $(1+ax)^9 = 1 + \ldots + {}^9C_5\,(ax)^5 + {}^9C_6\,(ax)^6 + \ldots + (ax)^9$

$= 1 + \ldots + 126\,a^5x^5 + 84\,a^6x^6 + \ldots + a^9x^9$

$126\,a^5 = 2 \times 84\,a^6\ ,\ a = {}^3/_4$

5(b) **Outcomes Assessed : HE6**

Marking Guidelines

Criteria	Marks
• finding new integrand	1
• finding new limits	1
• finding primitive	1
• finding answer	1

Answer

$$\int_1^{49} \frac{1}{\sqrt{1+\sqrt{x}}}\,\frac{1}{\sqrt{x}}\,dx = \int_2^{8} \frac{1}{\sqrt{u}}\,2\,du = [\,4\sqrt{u}\,]_2^8 = 8\sqrt{2} - 4\sqrt{2} = 4\sqrt{2} = \sqrt{32}$$

5(c) **Outcomes Assessed : (i) HE3 (ii) HE3**

Marking Guidelines

Criteria	Marks
• (i) differentiating	1
• showing answer	1
• (ii) finding expression for k	1
• finding answer	1

Answer

(i) $V = A - Ae^{-kt}$, $Ae^{-kt} = A - V$

$dV/dt = 0 - A(-ke^{-kt}) = Ake^{-kt} = k(Ae^{-kt}) = k(A - V)$

(ii) When $t = 2$, $V = A/4$ $(= 4A/16)$

$A/4 = A(1 - e^{-2k})$, $1 - e^{-2k} = 1/4$, $e^{-2k} = 3/4$

When $t = 4$, $V = A(1 - e^{-4k}) = A(1 - (e^{-2k})^2) = A(1 - (3/4)^2)$

$= A(1 - 9/16) = 7A/16$. $3/16$ of the container is filled in the next 2 minutes.

Question 6

6(a) **Outcomes Assessed : H5**

Marking Guidelines

Criteria	Marks
• finding AC and BC in terms of h	1
• using Pythagoras Theorem in Δ ABC	1
• finding expression for h	1
• finding answer	1

Answer

(i) In ΔACD , $\tan 20^\circ = h/AC$, $AC = h\cot 20^\circ$

In ΔBCD , $\tan 10^\circ = h/BC$, $BC = h\cot 10^\circ$

In ΔCAB , $BC^2 = AB^2 + AC^2$, $(h\cot 10^\circ)^2 = 40^2 + (h\cot 20^\circ)^2$

$h^2(\cot^2 10^\circ - \cot^2 20^\circ) = 40^2$, $h = \dfrac{40}{\sqrt{(\cot^2 10^\circ - \cot^2 20^\circ)}} = 8$

6(b) Outcomes Assessed : (i) HE3 (ii) HE3

Marking Guidelines

Criteria	Marks
• (i) using binomial probabilities	1
• showing answer	1
• (ii) using complementary probability	1
• finding answer	1

Answer

(i) P(at most one even score) = P(5 or 6 odd scores) = ${}^6C_5\ p^5 (1 - p) + p^6$

$= 6 p^5 (1 - p) + p^6 = 6 p^5 - 6p^6 + p^6 = 6 p^5 - 5p^6$

(ii) P(product of scores even) = 1 – P(product of scores odd)

= 1 – P(6 odd scores) = $1 - p^6$

6(c) Outcomes Assessed : (i) H5 (ii) HE5

Marking Guidelines

Criteria	Marks
• (i) using similar triangles to find expression for x	1
• showing answer	1
• (ii) using chain rule to find expression for dh/dt	1
• finding answer	1

Answer

(i) Using similar triangles $x/200 = h/25$, $x = 8h$

$V = 1/2 \times h \times 400 = 1600h^2$

(ii) $dh/dt = dV/dt \div dV/dh = -16000 \div 3200h = -5/h$

When $h = 10$, $dh/dt = -5/10 = -0.5$.

The water level is falling at 0.5 cm s^{-1}.

Question 7

7(a) Outcomes Assessed : (i) HE3 (ii) HE3 (iii) HE3

Marking Guidelines

Criteria	Marks
• (i) writing answers for particle from A	1
• writing answers for particle from B	1
• (ii) eliminating h	1
• finding time of flight for particle from A	1
• finding time of flight for particle from B	1
• (iii) showing answer	1

Answer

(i) $x(A) = Ut$, $y(A) = 4h - 1/2\ g t^2$

$x(B) = V (t - 10)$, $y(B) = h - 1/2\ g (t - 10)^2$

(ii) At impact $y(A) = y(B) = 0$

$4h = {}^1/_2\, g t^2$ and $h = {}^1/_2\, g(t-10)^2$, $4(t-10)^2 = t^2$

$2(t-10) = t$ or $2(t-10) = -t$, $t = 20$ or $t = {}^{20}/_3$ (but $t > 10$)

Time of flight for particle from A = 20 seconds

Time of flight for particle from B = 10 seconds

(iii) At impact $x(A) = x(B)$

$20U = 10V$, $V = 2U$

7(b) Outcomes Assessed : (i) HE2 (ii) H3 (iii) H5

Marking Guidelines

Criteria	Marks
• (i) showing $S(1)$ is true	1
• using logarithm law	1
• showing $S(k)$ true implies $S(k+1)$ true	1
• (ii) showing answer	1
• (iii) using limiting sum	1
• showing answer	1

Answer

(i) $S(n): \ln n! > n$ for all $n \ge 6$

When $n = 6$, $\ln 6! = 6.58 > 6$. $S(1)$ is true

If $S(k)$ is true for some $k \ge 6$, i.e. if $\ln k! > k$ for some $k \ge 6$

then $\ln(k+1)! = \ln\{(k+1) \times k!\} = \ln(k+1) + \ln k!$

$> \ln e + \ln k!$ (since $k+1 > e$) $> 1 + k$

and so $S(k+1)$ is also true

It follows that $S(n)$ is true for all $n \ge 6$

(ii) $\ln n! > n$ for all $n \ge 6$, $n! > e^n$ for all $n \ge 6$, $\dfrac{1}{n!} < \dfrac{1}{e^n}$ for all $n \ge 6$

(iii) $\dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \dfrac{1}{5!} + \dfrac{1}{6!} + \dfrac{1}{7!} + \dfrac{1}{8!} + \dfrac{1}{9!} + \ldots$

$< \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \dfrac{1}{5!} + \dfrac{1}{e^6} + \dfrac{1}{e^7} + \dfrac{1}{e^8} + \dfrac{1}{e^9} + \ldots$

$< \dfrac{206}{120} + \dfrac{1}{e^6}\left(1 + \dfrac{1}{e} + \dfrac{1}{e^2} + \dfrac{1}{e^3} + \ldots\right)$

(limiting sum of geometric series in brackets exists since $-1 < r = {}^1/_e < 1$)

$< \dfrac{103}{60} + \dfrac{1}{e^6}\,\dfrac{1}{1 - {}^1/_e}$

$< \dfrac{103}{60} + \dfrac{1}{e^5(e-1)}$