YEAR 12 EXT. 1 MATHS TRIAL 2003, SOLUTIONS

QUESTION 1:

(a) $f[g(9)] - g[f(9)] = (-\sqrt{9})^2 - \sqrt{9^2}$ 1
$= 9 + 9 = 18$ 1

(b)(i) $y = \frac{1}{2}x + b$
Inv. $x = \frac{1}{2}y + b$ 1
$2x = y + c$
$\therefore y = 2x - c$ 1
(ii) $m_{inv.} = 2$ 1

(c) $\int \frac{2}{3\sqrt{16 - x^2}}\,dx = \frac{2}{3}\int \frac{1}{\sqrt{4^2 - x^2}}\,dx$
$= \frac{2}{3}\sin^{-1}\frac{x}{4} + C$ 1

(d) $A(-1,3)$ $B(2,8)$
$-3 : 2$
Point $= \left(\frac{2\times -1 + -3\times 2}{-3+2}, \frac{2\times 3 + -3\times 8}{-3+2}\right)$ 1
$= (8, 18)$ 1

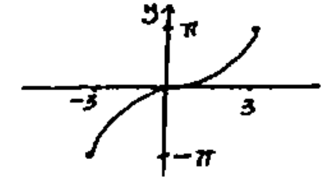
(e) $\frac{1}{\sin A \cos A} = \frac{1}{\frac{1}{2}\sin 2A}$ 1
$= \frac{1}{\frac{1}{2}\times\frac{1}{2}}$
$= 4$ 1

(f) $y = t^2$, $x = \sqrt{t}$
$\therefore t = x^2$ 1
$\therefore y = (x^2)^2 = x^4$ 1

QUESTION 2:

(a)(i) $y = 2\sin^{-1}\frac{x}{3}$
Dom. $-1 \le \frac{x}{3} \le 1$
$\therefore -3 \le x \le 3$ 1
Range: $-\frac{\pi}{2} \le \sin^{-1}\frac{x}{3} \le \frac{\pi}{2}$
$\therefore -\pi \le y \le \pi$ 1

(ii)



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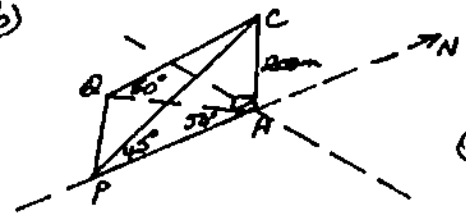
(b)



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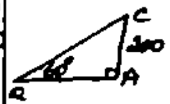
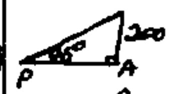


$QA = \frac{200}{\tan 60^\circ}$
$\doteqdot 115.47$

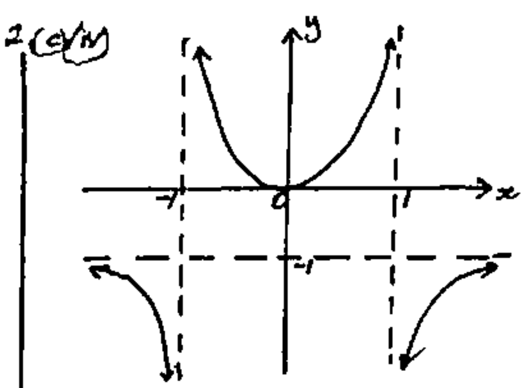


$PA = 200$ (isos. $\triangle$)

$QP^2 = (115.47)^2 + 200^2 - 2\times 115.47\times 200\times\cos 50$
$= 23644.24678$
$\therefore QP \doteqdot 154$ m (nearest m) 1

(c) $y = \frac{x^2}{1 - x^2}$
(i) Vert. asym. $x \ne \pm 1$ 1
(ii) $\lim_{x\to\pm\infty}\frac{x^2}{1-x^2} = \lim_{x\to\pm\infty}\frac{1}{\frac{1}{x^2} - 1}$
$= -1$ 1
(iii) $f(x) = \frac{x^2}{1-x^2}$
$f(-x) = \frac{(-x)^2}{1-(-x)^2}$ 1
$= \frac{x^2}{1-x^2} = f(x)$.
$\therefore$ Even

2(c)(iv)

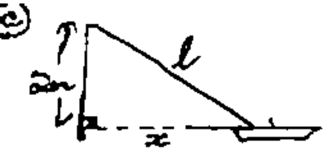


QUESTION 3:

(a) Let $y = x\cos^{-1}x$ 1 rule 1 ans.
$\frac{dy}{dx} = \cos^{-1}x + x\times\frac{-1}{\sqrt{1-x^2}}$
$= \cos^{-1}x - \frac{x}{\sqrt{1-x^2}}$

(b) $\sin 3x = 3\sin x - 4\sin^3 x$
$\therefore \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$ 1
$\int_0^{\pi}\sin^3 x\,dx = \frac{1}{4}\int_0^{\pi}(3\sin x - \sin 3x)\,dx$ 1
$= \frac{1}{4}\left[-3\cos x + \frac{1}{3}\cos 3x\right]_0^{\pi}$
$= \frac{1}{4}\left\{\left(-3\cos\pi + \frac{1}{3}\cos 3\pi\right) - \left(-3\cos 0 + \frac{1}{3}\cos 0\right)\right\}$
$= \frac{1}{4}\left\{\left(3 - \frac{1}{3}\right) - \left(-3 + \frac{1}{3}\right)\right\}$
$= \frac{4}{3}$ 1

(c)



$l = \sqrt{x^2 + 4}$, $\frac{dl}{dt} = -1$, $l = 3$, $x = \sqrt{5}$ 1
$\frac{dl}{dx} = \frac{1}{2}(x^2+4)^{-\frac{1}{2}}2x$
$= \frac{x}{\sqrt{x^2+4}}$ 1

$\frac{dl}{dt} = \frac{dl}{dx}\times\frac{dx}{dt}$
$-1 = \frac{\sqrt{5}}{\sqrt{5+4}}\frac{dx}{dt}$ 1
$\therefore \frac{dx}{dt} = -\frac{3}{\sqrt{5}}$
$\doteqdot -1.3416$
$\therefore$ Boat app. jetty at $1.3\,ms^{-1}$

(d)(i) $x^2 + x + 1 = x^2 + x + \left(\frac{1}{2}\right)^2 + 1 - \left(\frac{1}{2}\right)^2$
$= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$
OR
$x^2 + x + 1 \equiv x^2 - 2Ax + A^2 + B$
$\therefore -2A = -1 \Rightarrow A = -\frac{1}{2}$
$\therefore \left(-\frac{1}{2}\right)^2 + B = 1 \Rightarrow B = \frac{3}{4}$
$\therefore x^2 + x + 1 \equiv \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

(ii) $\int\frac{dx}{x^2+x+1} = \int\frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$
$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{3}} +$

QUESTION 4:

(a) At $x = 1$, $y = -4\sqrt{x}$
$\frac{dy}{dx} = -2x^{-\frac{1}{2}}$
$\therefore m_1 = -2$ (at $x = 1$)
$y = 2x - 6$
$m_2 = 2$
$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$
$= \left|\frac{-2-2}{1 + -2\times 2}\right|$
$= \left|\frac{-4}{-3}\right| = \frac{4}{3}$
$\therefore \theta \doteqdot 53^\circ 8'$ 1

4.(b) $\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}} = \sqrt{\dfrac{1-(1-2\sin^2\theta)}{1+(1-2\sin^2\theta)}}$ (1)

$= \sqrt{\dfrac{2\sin^2\theta}{2-2\sin^2\theta}}$

$= \sqrt{\dfrac{2\sin^2\theta}{2\cos^2\theta}}$ (1)

$= \tan\theta$

$= \dfrac{2t}{1-t^2}$ (1)

(c) $\cos 105° = \cos(60°+45°)$ (1)

$= \cos 60° \cos 45° - \sin 60° \sin 45°$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$

$= \dfrac{1-\sqrt{3}}{2\sqrt{2}}$ (1)

(d) $\sqrt{2}\cos x - \sin x \equiv R\cos(x+\alpha)$

$R = \sqrt{(\sqrt{2})^2+1^2} = \sqrt{3}$ (1)

$\cos\alpha = \sqrt{2}/\sqrt{3}$

$\therefore \alpha = 35°16'$ (1)

$\therefore \sqrt{3}\cos(x+35°16') = \frac{3}{2}$

$\cos(x+35°16') = \frac{\sqrt{3}}{2}$ (1)

$\therefore x+35°16' = 30°, 330°, 390°, \ldots$

$\therefore x = 294°44'$ or $354°44'$ (1)

(b)(i) $v^2 = -kx^2 + c$

$\frac{1}{2}v^2 = -\frac{k}{2}x^2 + c_1$ (1)

$\therefore \frac{d}{dx}\left(\frac{v^2}{2}\right) = -kx$

(ii) $\dfrac{d^2x}{dt^2} = -\dfrac{x}{9}$

i.e. $\dfrac{d(\frac{1}{2}v^2)}{dx} = -\dfrac{x}{9}$

$\therefore \frac{1}{2}v^2 = -\frac{x^2}{18} + c$ (1)

$v=0$, $x=2$ $\therefore 0 = -\frac{4}{18} + c$ (1)

$\therefore c = \frac{4}{18}$

$\therefore \frac{1}{2}v^2 = -\frac{x^2}{18} + \frac{4}{18}$

$\therefore v^2 = -\frac{x^2}{9} + \frac{4}{9}$ (1) meth.

$= \frac{1}{9}(4-x^2)$

Max v when $x=0$ $\therefore v^2 = \frac{4}{9}$

$v = \pm\frac{2}{3}$

$\therefore$ Max speed $\frac{2}{3}$ cms^{-1} (1)

(c) $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x\,dx$

$u = \cos x$
$du = -\sin x\,dx$
$x=\frac{\pi}{3}, u=\frac{1}{2}$
$x=\frac{\pi}{2}, u=0$ (1)

$= \displaystyle\int_{\frac{1}{2}}^{0} -u^2\,du$

$= \left[-\dfrac{u^3}{3}\right]_{\frac{1}{2}}^{0}$ (1)

$= 0 - -\dfrac{\frac{1}{8}}{3} = \dfrac{1}{24}$ (1)

QUESTION 5:

(a) $T = 15 + Ae^{-kt}$

$t=0$, $T=75$ $\therefore 75 = 15 + Ae^0$

$\therefore A = 60$ (1)

$t=10$, $T=55$ $\therefore 55 = 15 + 60e^{-10k}$ 47.1

$\therefore \frac{2}{3} = e^{-10k}$

$k = -\frac{1}{10}\ln\frac{2}{3}$ (1)

(or $\frac{1}{10}\ln\frac{3}{2}$)

$T = 15 + 60e^{\frac{t}{10}\ln\frac{2}{3}}$

$t=15$, $T = 15 + 60e^{\frac{15}{10}\ln\frac{2}{3}}$ (1)

$= 47.659 \doteq 47.7°$ (1)

QUESTION 6:

(a) ${}^nC_j - {}^nC_{n-j} = 0$ (1)

(b) $\left(x^3 + \frac{5}{x}\right)^8$

$T_{r+1} = {}^8C_r\, x^{3(8-r)}\left(\frac{5}{x}\right)^r$ (1)

$= \binom{8}{r}5^r x^{24-4r}$ (1)

$\therefore 24-4r = 0$ for T indep. of x

$\therefore r = 6$

$\therefore T_7 = \binom{8}{6}5^6$ (1)

$= 437\,500$

6.(c) $(1+x)^{2n} = \displaystyle\sum_{k=0}^{2n}\binom{2n}{k}x^k$

Differentiating both sides wrt x:

$2n(1+x)^{2n-1} = \displaystyle\sum_{k=1}^{2n} k\binom{2n}{k}x^{k-1}$ (2) 1 each side

Let $x=1$, $2n \times 2^{2n-1} = \displaystyle\sum_{k=1}^{2n} k\binom{2n}{k}$ (1)

$\therefore n2^{2n} = \displaystyle\sum_{k=1}^{2n} k\binom{2n}{k}$

$\therefore \displaystyle\sum_{k=1}^{2n} k\binom{2n}{k} = n4^n$ (1)

(d) $(2+3x)^{30}$

$\dfrac{\text{coeff. } T_{r+1}}{\text{coeff. } T_r} = \dfrac{n-r+1}{r} \times \dfrac{b}{a}$

$= \dfrac{31-r}{r} \times \dfrac{3}{2}$ (1)

If $T_{r+1} > T_r$ then:

$93-3r > 2r$

$5r < 93$

$r < 18\frac{3}{5}$

$\therefore r = 18, 17, 16, \ldots$

$\therefore T_{19} > T_{18} > T_{17} > \ldots$

(2) method

If $T_{r+1} < T_r$ then:

$93-3r < 2r$

$5r > 93$

$r > 18\frac{3}{5}$

$\therefore r = 19, 20, \ldots$

$\therefore T_{20} < T_{19}, T_{21} < T_{20}, \ldots$

$\therefore T_{19}$ coeff. is greatest.

coeff. $T_{19} = \binom{30}{18}2^{30-18}3^{18}$

$= \binom{30}{18}2^{12}3^{18}$ (1)

QUESTION 7:

(a) STEP 1: Prove true for $n=1$

$y = \sin x$

$\dfrac{dy}{dx} = \cos x = \sin\left(x+\frac{\pi}{2}\right)$ (1)

$\therefore$ True for $n=1$

STEP 2: Assume true for $n=k$

i.e. assume $\dfrac{d^ky}{dx^k} = \sin\left(x+\dfrac{k\pi}{2}\right)$ (1)

7. (a) cont. STEP 3: Prove true for $n=k+1$

i.e. Prove $\dfrac{d^{k+1}y}{dx^{k+1}} = \sin\left[x + \dfrac{(k+1)\pi}{2}\right]$

$$\text{Now } \frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$$

$$= \frac{d}{dx}\sin\left(x + \frac{k\pi}{2}\right) \text{ from assump.}$$

$$= \cos\left(x + \frac{k\pi}{2}\right)$$

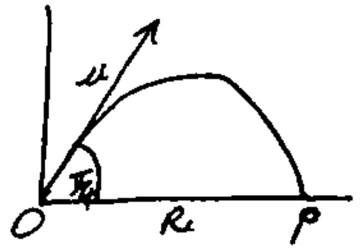
$$= \sin\left(x + \frac{k\pi}{2} + \frac{\pi}{2}\right)$$

$$= \sin\left[x + \frac{(k+1)\pi}{2}\right]$$

(3) including 1 for conclusion if it follows from proof

Hence if true for $n=k$, then true for $n=k+1$. But true for $n=1$ ∴ true for $n=2$, $n=3$ and all positive integers n.

(b) (i)



Initially: $\dot{x} = u\cos\frac{\pi}{4} = \frac{u}{\sqrt{2}}$, $\dot{y} = u\sin\frac{\pi}{4} = \frac{u}{\sqrt{2}}$

$x = 0$, $y = 0$

Horiz.

$\ddot{x} = 0$

$\dot{x} = C = \frac{u}{\sqrt{2}}$

$x = \frac{u}{\sqrt{2}}t + C_1$

$t=0, x=0$ ∴ $C = 0$

∴ $x = \frac{u}{\sqrt{2}}t$ (1)

Vert.

$\ddot{y} = -g$

$\dot{y} = -gt + K$

$t=0, \dot{y}=\frac{u}{\sqrt{2}}$ ∴ $K = \frac{u}{\sqrt{2}}$

∴ $\dot{y} = -gt + \frac{u}{\sqrt{2}}$

$y = -\frac{gt^2}{2} + \frac{u}{\sqrt{2}}t + K_1$

$t=0, y=0$ ∴ $K=0$ ∴ $y = -\frac{gt^2}{2} + \frac{u}{\sqrt{2}}t$ (1)

Now: $t = \frac{\sqrt{2}x}{u}$

∴ $y = -\frac{g}{2}\left(\frac{\sqrt{2}x}{u}\right)^2 + \frac{u}{\sqrt{2}} \times \frac{\sqrt{2}x}{u}$ (1)

$= x - g\frac{x^2}{u^2}$

(ii) At P, $y=0$ ∴ $0 = x\left(1 - \frac{gx}{u^2}\right)$

∴ $x = 0$ or $\frac{u^2}{g}$

∴ $OP = R = \frac{u^2}{g}$

OR (1) for method

$R = \frac{V^2\sin 2\alpha}{g} = \frac{u^2\sin\frac{\pi}{2}}{g} = \frac{u^2}{g}$

(iii) From (i) $15 = \frac{30}{\sqrt{2}}t$

∴ $t = \frac{\sqrt{2}}{2}$

$\dot{x} = \frac{30}{\sqrt{2}} = 15\sqrt{2}$

$\dot{y} = -10t + \frac{30}{\sqrt{2}}$

$= -5\sqrt{2} + 15\sqrt{2} = 10\sqrt{2}$

∴ Speed $= \sqrt{(15\sqrt{2})^2 + (10\sqrt{2})^2}$

$= \sqrt{450 + 200}$

$= \sqrt{650}$

$\doteqdot 25.5\ ms^{-1}$