Suggested Solutions

Mathematics Extension 2 - Yr 12

TRIAL HSC KGS 2001

Question 1

(a) (i) $\displaystyle\int \frac{x}{\sqrt{9-4x^2}}\,dx$

by inspection,

$= -\frac{1}{4}\sqrt{9-4x^2} + c$

(or let $u = 9-4x^2$)

ie $\frac{du}{dx} = -8x$

separating variables,

$\therefore\ x\,dx = -\frac{1}{8}\,du$

$\displaystyle\int \frac{x}{\sqrt{9-4x^2}}\,dx = -\frac{1}{8}\int \frac{du}{\sqrt{u}}$

$= -\frac{1}{8}\int u^{-\frac{1}{2}}\,du$

$= -\frac{1}{8}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + c$

$= -\frac{1}{4}\sqrt{u} + c$

$= -\frac{1}{4}\sqrt{9-4x^2} + c$

(ii) $\displaystyle\int \frac{x^2}{x+1}\,dx$

$= \displaystyle\int \frac{(x-1)(x+1)+1}{(x+1)}$

$= \displaystyle\int (x-1) + \frac{1}{x+1}\,dx$

$= \frac{x^2}{2} - x + \ln|x+1| + c$

(or a substitution u=x+1)

(iii) $\displaystyle\int_0^{\ln 2} x e^x\,dx$

let $u = x$ $\quad dv = e^x$

$du = 1$ $\quad v = e^x$

$\therefore$ using $\int u\,dv = uv - \int v\,du$

$= [xe^x]_0^{\ln 2} - \displaystyle\int_0^{\ln 2} e^x\,dx$

$= [\ln 2(e^{\ln 2}) - 0] - [e^x]_0^{\ln 2}$

$= (\ln 2)(2) - e^{\ln 2} + e^0$

$= 2\ln 2 - 2 + 1$

$= 2\ln 2 - 1$

(b) (i) $\frac{2}{(t+1)(t^2+1)} \equiv \frac{A}{t+1} + \frac{Bt+C}{t^2+1}$

$\Rightarrow 2 \equiv A(t^2+1) + (t+1)(Bt+C)$

by substitution:

$t=-1$; $\quad 2 = 2A$

$\therefore A = 1$

$t = 0$; $\quad 2 = A + C$

$2 = 1 + C$

$\therefore C = 1$

$t = 1$; $\quad 2 = 2A + 2(B+C)$

$2 = 2 + 2(B+1)$

ie $B = -1$

(or use $t = \pm i$)

hence: $A = 1,\ B = -1,\ C = 1$

(ii) $\displaystyle\int_0^1 \frac{2}{(t+1)(t^2+1)}\,dt$

$= \displaystyle\int_0^1 \frac{1}{t+1} + \frac{1-t}{t^2+1}\,dt$

$= \displaystyle\int_0^1 \frac{1}{t+1} + \frac{1}{t^2+1} - \frac{t}{t^2+1}\,dt$

$= \left[\ln|t+1| + \tan^{-1}t - \frac{1}{2}\ln(t^2+1)\right]_0^1$

$= \left[\tan^{-1}t + \ln\left|\frac{t+1}{\sqrt{t^2+1}}\right|\right]_0^1$

$= \tan^{-1}1 + \ln\left|\frac{2}{\sqrt{2}}\right| - 0$

$= \frac{\pi}{4} + \ln\sqrt{2}$

$= \frac{\pi}{4} + \frac{1}{2}\ln 2$

(iii) $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}(\tan^2\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}(t^2+1)$

Separating vara

$\frac{2\,dt}{t^2+1} = dx$

also when $x = 0$,

$x = \frac{\pi}{2}$ t

$\sin x = \frac{2t}{1+t^2}$, cos

$\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x - \cos x}\,dx$

$= \displaystyle\int_0^1 \frac{\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2} - \left(\frac{1-t^2}{1+t^2}\right)}$

$= \displaystyle\int_0^1 \frac{4t}{(t^2+1)(2t^2+2t)}$

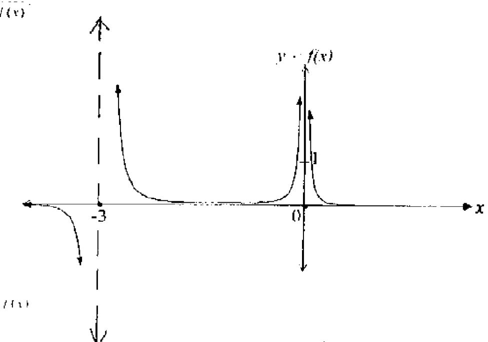
$= \displaystyle\int_0^1 \frac{2}{(t+1)(t^2+1)}$

from (b)(ii)

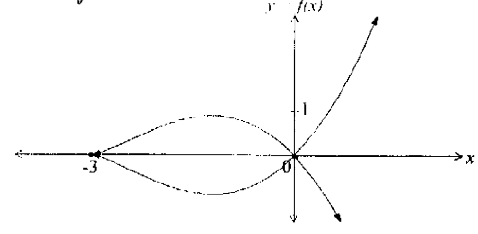
$= \frac{\pi}{4} + \frac{\ln 2}{2}$

Question 3(a)

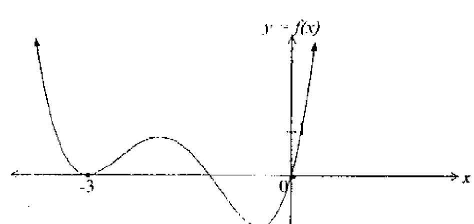
(i) $y = \frac{1}{f(x)}$



(ii) $y^2 = f(x)$



(iii) $y = f'(x)$



(b) Possible equation: $y = ax^2(x+3)^3$ $(a \neq 0)$

Question 3

(c) (i) $1^3 + 3(1)^2 - 4$

$= 1 + 3 - 4$

$= 4 - 4$

$= 0$

$\therefore x = 1$ is a zero of $x^3 + 3x^2 - 4$.

(ii) other intercepts:

when $x = 0$, $y = -4 \Rightarrow (0,-4)$

also $x = 1$.

N.B. $y = (x-1)(x+2)^2$

$\therefore$ x-intercepts are $x = 1$

$x = -2$ (double root)

$\frac{dy}{dx} = 3x^2 + 6x$

TP's when $y' = 0$

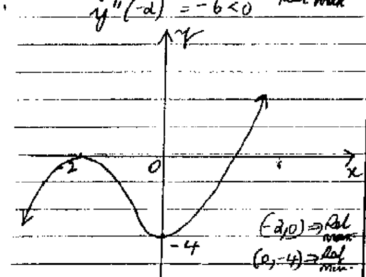
ie when $3x(x+2) = 0$

$x = 0,\ x = -2$

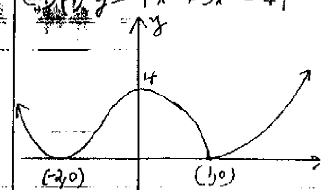
$y'' = 6x + 6$

$y''(0) = 6 > 0$ Rel min

$y''(-2) = -6 < 0$ Rel max



(iii) (α) $y = |x^3 + 3x^2 - 4|$



(β) $y = \ln|x^3 + 3x^2 - 4|$

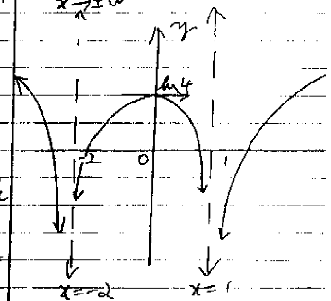
This is defined only wh $y = |x^3+3x^2-4| > 0$

However, we use the graph we observe $y = \ln|x^3+3x^2-4|$ is undefined when $x = -2, 1$

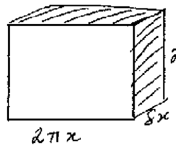
These are asymptotes.

Also, when $x = 0$, $y = \ln 4$ a max T.P as well.

$\lim\limits_{x \to \pm\infty} \ln|x^3+3x^2-4| = \infty$



(vi)



$V = 4\pi \int_0^a xy\,dx$

$= 4\pi\int_0^a x\left(\frac{b}{a}\sqrt{a^2+x^2}\right)dx$ $(y>0)$

$V = \frac{4\pi b}{a}\int_0^a x\sqrt{a^2+x^2}\,dx$

$= \frac{4\pi b}{a}\left[\frac{(a^2+x^2)^{3/2}}{3}\right]_0^a$

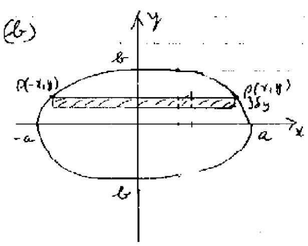
$= \frac{4\pi b}{3a}\left[(a^2+a^2)^{3/2} - (a^2)^{3/2}\right]$

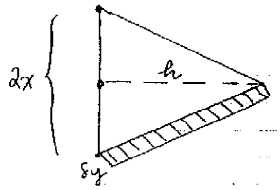
$= \frac{4\pi b}{3a}\left[(2a^2)^{3/2} - a^3\right]$

$= \frac{4\pi b}{3a}\left[2^{3/2}\cdot a^3 - a^3\right]$

$= \frac{4\pi b a^2}{3}\left[2\sqrt{2} - 1\right]\ u^3.$

(b)





Area of a typical crosssectional slice

$A(y) = \frac12\cdot 2x\cdot h$

$A(y) = xh$

Volume of a typical slice
$\delta V = xh\,\delta y$

∴ Total Volume $= \lim_{\delta y\to 0}\sum_{y=-b}^{b} xh\,\delta y$

ie $V = 2h\int_0^b x\,dy$

Since $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$

$x^2 = \frac{a^2(b^2-y^2)}{b^2}$

∴ $x = \frac{a}{b}\sqrt{b^2-y^2}$

∴ $V = \frac{2ha}{b}\int_0^b\sqrt{b^2-y^2}\,dy$

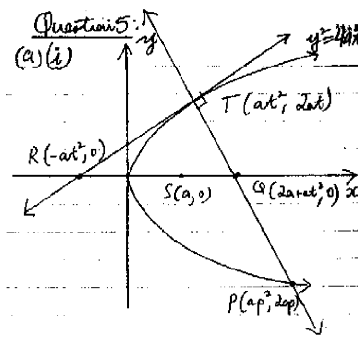
$= \frac{2ha}{b}\left[\frac14\cdot\pi\cdot b^2\right]$

$= \frac{2ah\pi b^2}{4b}$

$V = \frac{\pi abh}{2}$ cubic units

Question 5:

(a)(i)



(ii) $y^2 = 4ax$

$2y\frac{dy}{dx} = 4a$

∴ $\frac{dy}{dx} = \frac{2a}{y}$

when $y = 2at$, at T

then $\frac{dy}{dx} = \frac{2a}{2at} = \frac1t$

Also, the slope of the normal at T is $-t$

Equation of tangent:

$y - 2at = \frac1t(x-at^2)$

$x - ty = -at^2$ ①

Equation of normal:

$y - 2at = -t(x-at^2)$

$tx + y = 2at + at^3$ ②

(iii) To find the length of RQ we need the coordinates of R & Q.

Equation of tangent meets x-axis at R ie when y

∴ in ① $x = -at^2$

∴ $R(-at^2, 0)$

Equation of normal meets x-axis at Q ie when

∴ in ② $tx = 2at + a$

$(t\neq 0)$ ∴ $x = 2a + at^2$

∴ $Q(2a+at^2, 0)$

∴ $RQ = |-at^2| + |2a+a$

$= at^2 + 2a + at^2$

$= 2at^2 + 2a$

$= 2a(t^2+1)$

(iv) R will lie on the directrix if when $x=$ $y=0$ (by definition y

in ① $x - ty = -at^2$

$-a = -at^2$

∴ $t^2 = 1$

$\Rightarrow t = \pm 1$

(v) Since $P(ap^2, 2ap)$ sat equation of normal at T then: in ②

$t(ap^2) + 2ap = 2at + a$

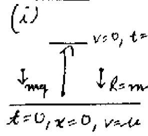
ie $p^2t + 2p = 2t + t^3$

ie $2(p-t) = t(t^2-p^2)$

$-2 = t(t+p)$ (t

Question 6:

(a) (i)



$m\ddot{x} = \sum F_x$

$m\ddot{x} = -mg - mkv$

∴ $\ddot{x} = -g - kv$

ie $\frac{dv}{dt} = -(g+kv)$

(ii) $\frac{dt}{dv} = -\frac{1}{g+kv}$

$t = -\frac1k\ln|g+kv| + c_1$

when $t=0$, $v=u$

∴ $0 = -\frac1k\ln|g+ku| + c_1$

ie $c_1 = \frac1k\ln|g+ku|$

hence $t = \frac1k\ln|g+ku| - \frac1k\ln|g+kv|$

ie $t = \frac1k\ln\left|\frac{g+ku}{g+kv}\right|$.

(iii) when $t=T$, $v=0$ (at max height)

∴ $T = \frac1k\ln\left(\frac{g+ku}{g}\right)$

$T = \frac1k\ln\left(1+\frac{ku}{g}\right)$ from (ii).

(iv) Considering $kt = \ln\left[\frac{g+ku}{g+kv}\right]$

and solving for v.

$e^{kt} = \frac{g+ku}{g+kv}$

ie $kv = (g+ku)e^{-kt} - g$

$v = \frac1k\left[(g+ku)e^{-kt} - g\right]$

let $v = \frac{dx}{dt}$

thence: $\frac{dx}{dt} = \frac1k\left[(g+ku)e^{-kt} - g\right]$

hence: $x = \frac1k\left[-\frac1k(g+ku)e^{-kt} - gt\right] +$

when $t=0$, $x=0$ ∴ $c_2 = \frac{1}{k^2}(g+ku)$

hence: $x = \frac{1}{k^2}(g+ku)\left[1-e^{-kt}\right] - \frac{gt}{k}$

at $x=H$, $t=T$

∴ $H = \frac{1}{k^2}(g+ku)\left[1-e^{-kT}\right] - \frac{gT}{k}$

since $T = \frac1k\ln\left(1+\frac{ku}{g}\right)$

then $kT = \ln\left(1+\frac{ku}{g}\right)$

$e^{kT} = \quad (g+ku)$

$H = \frac{1}{k^2}(g+ku)\left[1-e^{-\ln\left(\frac{g+ku}{g}\right)}\right] - \frac{gT}{k}$

$= \frac{1}{k^2}(g+ku)\left[1-\frac{g}{g+ku}\right] - \frac{gT}{k}$

$= \frac{g+ku}{k^2} - \frac{g}{k^2} - \frac{gT}{k}$

$H = \frac{g+ku-g-gkT}{k^2}$

$H = \frac{ku - gkT}{k^2}$

$H = \frac{u - gT}{k}$

ie $H = \frac1k(u-gT)$.

(b) $I_n = \int_0^{\pi/2}\sin^n\theta\,d\theta$

$= \int_0^{\pi/2}\sin\theta\,\sin^{n-1}\theta\,d\theta$

let $u = \sin^{n-1}\theta$ $\quad dv = \sin\theta$

$du = (n-1)\cos\theta\sin^{n-2}\theta$ $\quad v = -\cos\theta$

∴ $I_n = uv - \int v\,du$

$= \left[-\cos\theta\sin^{n-1}\theta\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2}\cos^2\theta\sin^{n-2}\theta\,d\theta$

$I_n = 0 + (n-1)\int_0^{\pi/2}(1-\sin^2\theta)\sin^{n-2}\theta\,d\theta$

$I_n = (n-1)\left[\int_0^{\pi/2}\sin^{n-2}\theta - \sin^n\theta\,d\theta\right]$

$I_n = (n-1)\left[I_{n-2} - I_n\right]$

ie $I_n = (n-1)I_{n-2} - (n-1)$

ie $nI_n = (n-1)I_{n-2}$

$I_n = \left(\frac{n-1}{n}\right)I_{n-2}$

(n

(ii) $I_{10} = \frac{9}{10}I_8$

$= \frac{9}{10}\times\frac78 I_6$

$= \frac{9}{10}\times\frac78\times\frac56 I_4$

$= \frac{9}{10}\times\frac78\times\frac56\times\frac34 I_2$

$= \frac{9}{10}\times\frac78\times\frac56\times\frac34\times\frac12 I_0$

$I_0 = \int_0^{\pi/2}1\,d\theta$

$= \pi/2$

∴ $I_{10} = \frac{9}{10}\times\frac78\times\frac56\times\frac34\times\frac12$

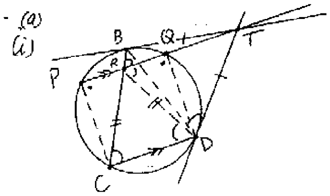
$= \frac{945\pi}{7680}$

$= \frac{63\pi}{512}$

Question 8:

(a)

(i)



(ii) Join B to D – see diagram.

$PQ \parallel CD \therefore \angle BRT = \angle BCD$ (corresp. $\angle$'s in $\parallel$ lines are equal)

but $\angle TDB = \angle DCB$ (angle between tangent TD and chord BD is equal to the $\angle$ in the alt. segment)

hence $\angle BDT = \angle BRT$

(iii) Join DR – not necessary though

since $\angle BDT = \angle BRT$ then BRDT is a cyclic quadrilateral because ~~the~~ the chord BT subtends equal angles at R & D – see diagram

(i.e. 2 $\angle$'s standing on the same chord are equal! in this case ~~they~~ are)

(iv) $BT = TD$ (two tangents drawn from the same ext. pt are =)

since BRDT is a cyclic quad. and since equal chords BT & TD subtend equal angles on the circumference then

$\angle TRD = \angle BRT$ as well.

(v) $\angle DRT = \angle RDC$ (alt. $\angle$'s in parallel lines are equal)

$\angle BRT = \angle BCD$ from (ii)

$\angle BRT = \angle DRT$ (from iv)

$\therefore \angle RDC = \angle BCD = \angle RCD$

$\therefore \angle RCD$ is isosceles.

($RC = RD$ ∴)

(vi) Join PC & QD

∴ PQDC is a cyclic quad now

$\angle RPC = 180 - \angle PCD$ (cointerior $\angle$'s in $\parallel$ lines).

$\angle PQD = 180 - \angle PCD$ (opp. $\angle$'s of a cyclic quad add to 180)

$\therefore \angle RPC = \angle PQD$

In ΔPRC & ΔRQD

$RC = RD$ (see (v))

& $\angle RPC = \angle RQD$ (from above)

& $\angle QRD = \angle PRC$ (alt $\angle$'s in $\parallel$ lines & isosceles Δ).

$\therefore \Delta PRC \equiv \Delta QRD$ (AAS)

$\therefore PR = RQ$ (corresp. sides in $\equiv$ Δ's are equal)

∴ BC bisects PQ as required

(b)(i) $\cos x = \cos(-x)$

$= \sin\left(\frac{\pi}{2} - (-x)\right)$

$= \sin\left(\frac{\pi}{2} + x\right)$

or: $\sin\left(x + \frac{\pi}{2}\right) = \sin x \cos\frac{\pi}{2} + \sin\frac{\pi}{2}\cos x$

$= \cos x$

(ii) $y = 3\sin x + 4\cos x$

$y = \sqrt{3^2+4^2}\,\sin(x+\alpha)$

$y = 5\sin(x+\alpha)$

By induction

Prove the statement $\frac{d^n y}{dx^n} = 5\sin\left(x+\alpha+\frac{n\pi}{2}\right)$ is true for $n=1$

$LHS = \frac{dy}{dx} = 5\cos(x+\alpha)$

$RHS = 5\sin\left(x+\alpha+\frac{\pi}{2}\right)$

$= 5\sin\left(\left(x+\frac{\pi}{2}\right)+\alpha\right)$

$= 5\sin\left((x+\alpha)+\frac{\pi}{2}\right)$

$= 5\cos(x+\alpha)$ from (i).

just replace x with $(x+\alpha)$

Assume the statement is true for $n=k$ [$1 \le k \le n$, $(k,n) \in \mathbb{Z}^+$]

RTP the statement is true for $n = k+1$.

Assume:

$\frac{d^k y}{dx^k} = 5\sin\left[(x+\alpha)+\frac{\pi}{2}k\right]$

RTP:

$\frac{d^{k+1} y}{dx^{k+1}} = 5\sin\left[(x+\alpha)+(k+1)\frac{\pi}{2}\right]$

From (2)

$LHS = \frac{d^{k+1} y}{dx^{k+1}}$

$= \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$

$= \frac{d}{dx}\left[5\sin\left[(x+\alpha)+\frac{k\pi}{2}\right]\right]$ from

$= 5\cos\left[(x+\alpha)+\frac{k\pi}{2}\right]$

$= 5\sin\left[\frac{\pi}{2} + (x+\alpha ...\right.$

[just replace $x \to x+$ ... in (i)]

$= 5\sin\left[\frac{\pi}{2}(k+1) + (x+\alpha)\right]$

The statement is true for Whenever it is true for n it is also true for $n = k+1$ and for all positive integer values for $n \geq 1$.