H.S.C. TRIAL EXT 2 2004

QUESTION 1 **Marks**

(a) The complex number z is given by $z = -1 + i\sqrt{3}$. **2**

(i) Show that $z^2 = 2\bar{z}$.

(ii) Evaluate $|z|$ and Arg z. **2**

(iii) Show that z is a root of the equation $z^3 - 8 = 0$. **2**

(b) (i) Find $\int x\sec^2(x^2)dx$. **3**

(ii) Find $\int \frac{x^4}{x^2+1}dx$. **3**

(iii) Evaluate $\int_0^{\frac{\pi}{3}} \frac{\sin^3 x}{\cos^2 x}dx$. **3**

QUESTION 2 (Start a new page)

(a) (i) Sketch the graph of $y = \frac{x+3}{x+4}$ clearly showing all points of intersection with the x-axis and y-axis, and the equations of all the asymptotes. **2**

(ii) On separate axes, sketch the graphs of:

(α) $y = \left(\frac{x+3}{x+4}\right)^2$ **2**

(β) $y^2 = \left(\frac{x+3}{x+4}\right)$ **2**

Part (b) on the next page.

(b) A railway track has been constructed around a circular curve of radius 500 metres. The distance across the track between the rails is 1.5 metres and the outer rail is 0.1 metres above the inner rail. A train of mass m travels on the track at a speed of $v = v_0$ metres/second and no lateral forces.

(i) Draw a diagram showing all the forces on the train. **1**

(ii) Show that $v_o^2 = 500g \tan \theta$, where θ is the angle the track makes with the horizontal. **2**

The train now travels on the track at a speed of v metres/second, where $v > v_o$.

(iii) Draw a diagram showing all the forces on the train. **1**

(iv) Show that the lateral force, F, exerted by the rail on the wheel is given by $F = \frac{mv^2}{500}\cos\theta - mg\sin\theta$. **3**

(v) Deduce that F is one fifth of the weight of the train when $v = 2v_o$. **2**

QUESTION 3 (Start a new page)

(a) (i) If α is a double root of a polynomial $P(x)$, show that α is a zero of $P'(x)$. **2**

(ii) Find integers m and n such that $(x + 1)^2$ is a factor of $x^5 + 2x^2 + mx + n$. **4**

(b) Sketch the region on an Argand diagram whose points z satisfy both inequalities $|z - \bar{z}| \leq 4$ and $-\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{3}$. **3**

(c) The equation of motion of a particle moving x metres along a straight line after t seconds is given by $x = 2v - \tan^{-1} v$. Initially its velocity is 1 metre/second. Find the exact time when its velocity is 7 metres/second. **6**

QUESTION 4 (Start a new page)

(a) Beach volleyball is played with two teams where each team has two players.

(i) In how many ways can four players be grouped in pairs to play a game of beach volleyball. **2**

(ii) The eight members of a beach volleyball club meet to play two games at the same time on two separate courts. In how many different ways can the club members be selected to play these two games. **3**

(b) (i) Use the substitution $x = t - y$, where t is a constant, to show that $\int_0^t f(x)dx = \int_0^t f(t-x)dx$. **3**

(ii) Hence, or otherwise, evaluate $\int_0^1 x(1-x)^{2004}\, dx$. **2**

(c) In the diagram below, AX and OB are perpendicular to AB and OY bisects $\angle XOA$. If $XY = a$, $YA = b$, $AB = d$ and $OB = h$, show that

(i) $\frac{OX}{OA} = \frac{a}{b}$. **2**

(ii) $(a-b)d^2 = (a+b)b^2 - 2b^2h - (a-b)h^2$. **3**

QUESTION 5 (Start a new page)

(a) A tank contains *100* Litres of brine (salt water) whose concentration is *3* grams/Litre. Three Litres of brine whose concentration is *2* grams/Litre flow into the tank each minute, and at the same time *3* Litres of mixture flows out each minute. If Q is the quantity of salt in the mixture after a time t minutes,

(i) show that the rate of increase of the quantity of salt, $\frac{dQ}{dt}$, for $t > 0$ is given by $\frac{dQ}{dt} = \left(6 - \frac{3Q}{100}\right)$ grams/minute. **3**

(ii) Show that the quantity of salt in the tank is always between *200* grams and *300* grams. **4**

(b) A cricketer is capable of catching a ball with equal ease at any height from level ground between y_1 and y_2 where $y_1 > y_2$ as shown in the diagram below. For a hit which gives a ball a range R and greatest height h, show that he should estimate his position on the field in the plane within an interval of length $\frac{R}{2}\left[\left(\sqrt{1-\frac{y_2}{h}}\right)-\left(\sqrt{1-\frac{y_1}{h}}\right)\right]$. **8**

(You may assume the equation $y = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha$)

QUESTION 6 (Start a new page)

(a) Find $\int \frac{5}{16 + 9\cos^2 x} dx$ **5**

(b) The complex number z is a function of the real number t, given that $z = \frac{t - i}{t + i}$ for $0 \le t \le 1$. Evaluate $|z|$ and hence describe the locus of z in an Argand diagram. **5**

(c) Find the equation of the tangent to the curve $xy\,(x + y) + 16 = 0$ at point on the curve where the gradient of the tangent is -1. **5**

QUESTION 7 (Start a new page)

(a) (i) Show that $\tan\left(A + \frac{\pi}{2}\right) = -\cot A$. **2**

(ii) Use mathematical induction to prove that $\tan\left[(2n+1)\frac{\pi}{4}\right] = (-1)^n$ for n a positive integer. **4**

(b) A polynomial $P(x)$ is divided by $x^2 - a^2$ where $a \neq 0$, and the remainder is $px + q$.

(i) Show that $p = \frac{1}{2a}[P(a) - P(-a)]$ and $q = \frac{1}{2}[P(a) + P(-a)]$. **2**

(ii) Find the remainder when $P(x) = x^n - a^n$, for n a positive integer, is divided by $x^2 - a^2$. **3**

(c) A particle moving with a speed of v metres/second experiences air resistance of kv^2 per unit mass, where k is a constant. Falling from rest in a vertical line through a distance d, prove that it will acquire a speed of $v = V\sqrt{1 - e^{-2kd}}$ metres/second, where $V = \sqrt{\frac{g}{k}}$ and g the constant acceleration due to gravity. **4**

QUESTION 8 (Start a new page)

(a) In the diagram below, AB is a common tangent and XY is a common chord. Extend BX to meet AY at Q and extend AX to meet BY at P.

(i) Copy the diagram onto your answer sheet showing all the information given. **1**

(ii) Prove that $PXQY$ is a cyclic quadrilateral. **3**

(iii) Prove that AB is parallel to PQ. **2**

(iv) Prove that XY bisects PQ. **3**

(b) (i) If k is an integer where $k \geq 3$ and $(k-1)(k+1) < k^2$, show that $\dfrac{1}{(k-1)k(k+1)} > \dfrac{1}{k^3}$. **1**

(ii) Given that $S_n = \dfrac{1}{3^3} + \dfrac{1}{4^3} + \dfrac{1}{5^3} + \ldots\ldots + \dfrac{1}{n^3} = \sum_{3}^{n} \dfrac{1}{k^3}$, use partial fractions in part (i) or otherwise to prove that $S_n < \dfrac{1}{12}$. **5**

END of PAPER

EULOGY

I met Wally about 35 years ago during my years at Sydney University with his son Ross. At that time our conversations were minimal and it was difficult to appreciate the true Wally. Spending many years as a foundry worker and floor sander he was a true labour man and a devout believer of truth and respect to his friends and fellow workers. He was indeed a man of good faith and trust. Hidden in the shadows of this apparently average person was a family man who made sure that his children Ross and Katrina were well educated at selective schools, and was prepared to suffer financial losses in return for a successful future for his children. He was eternally devoted to his loving wife May, and to all his family members.

After several years, it became clear that Wally was a far greater man than originally thought. We became very good friends and spent many happy trips overseas together, including trips to England, Bali, Greece and the Philippines. On these occasions one could never have expected a better friend. There were many times during these trips where, over a bottle of ouzo or port, we spoke about many things both personal and informative. Wally loved a good conversation and had the ability and general knowledge to talk about anything. If you mentioned politics, royalty, Jews or religion, you were sure to have a full blown discussion or argument where Wally would strongly enforce his ideas and opinions which were always witty, humorous and generally respected.

You were a great guy Wall and your friendship throughout these years is priceless. You will be sadly missed by all of us, and your memories will live with us forever. Goodbye,Wal.