TRIAL EXAMINATION 2000. 84 Pg.1

1. (a) (i) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ Aw-1

(ii) $\tan 105^\circ = \tan(45^\circ + 60^\circ)$ A1-1

$= \dfrac{\tan 45 + \tan 60}{1 - \tan 45 \tan 60}$

$= \dfrac{1 + \sqrt{3}}{1 - \sqrt{3}}$ A1-1

$= \dfrac{1+\sqrt{3}}{1-\sqrt{3}} \times \dfrac{1+\sqrt{3}}{1+\sqrt{3}}$

$= -2-\sqrt{3}$ Aw-3

(b) $\dfrac{2x+1}{2x-1} \geqslant 2$

Critical Value when $2x - 1 = 0$

ie. $x = \frac{1}{2}$ A1-1

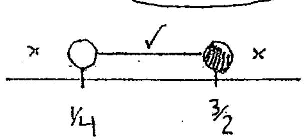
Now, $2x+1 \geqslant 2(2x-1)$

$2x+1 \geqslant 4x-2$

$3 \geqslant 2x$

$\therefore x \leqslant \frac{3}{2}$ A1-1

TEST:



$\therefore \frac{1}{2} < x \leqslant 1\frac{1}{2}$ Aw-3

Q1 – continued...

(c) $\displaystyle\int_0^{\pi/4} \cos x \cdot \sin^2 x \, dx = \left[\tfrac{1}{3}\sin^3 x\right]_0^{\pi/4}$ A1-

$= \frac{1}{3}(\sin^3 \frac{\pi}{4} - \sin^3 0)$

$= \frac{1}{3}(\frac{1}{\sqrt{2}})^3 - 0$

$= \dfrac{1}{6\sqrt{2}} = \dfrac{\sqrt{2}}{12} \approx 0.1178.$ Aw-

(d) $x^6 - 9x^3 + 8 = 0$

let $a = x^3$

$\therefore a^2 - 9a + 8 = 0$ A1-

$(a-8)(a-1) = 0$

$\therefore a = 8, 1.$ A1-

Now, $x^3 = 8 \longrightarrow x = 2$ Aw-

$x^3 = 1 \longrightarrow x = 1$

Q2. (a)(i) $\frac{d}{dx}(x\log x - x) = (x\cdot \frac{1}{x} + \log x \cdot 1) - 1$
$= 1 + \log x - 1$
$= \log x$ — A1-1, Aw-②

(ii) $\int_2^e \log x \, dx = \left[x\log x - x \right]_2^e$
$= (e\log e - e) - (2\log 2 - 2)$
$= (e - e) - 2(\log 2 - 1)$
$= 2(1 - \log 2)$ — A1-1, Aw-②

(b)(i) $CF = CE$ (tangents from an external point are equal).
$\therefore \Delta ECF$ is isosceles.
$\therefore \angle EFC = \angle CEF = 65°$ (base ∠'s isosc. Δ) — A1-1., Aw-②

(ii) $\angle EDF = \angle CEF = 65°$ (alt. segment theorem) — Aw-①

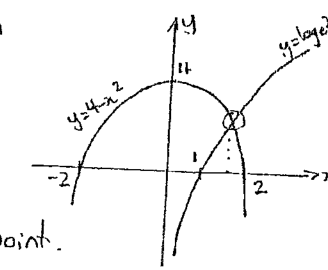
(c)(i) $\cos 2\alpha = 2\cos^2\alpha - 1$
let $\alpha = 3x$
$\therefore \cos 6x = 2\cos^2 3x - 1$ — Aw-①

(ii) $V = \pi \int y^2 \, dx$
$= \pi \int_0^{\pi/6} \cos^2 3x \, dx$ — A1-1
$= \pi \int_0^{\pi/6} \frac{1}{2}(\cos 6x + 1) \, dx$ — A1-1
$= \frac{\pi}{2}\left[x - \frac{1}{6}\sin 6x \right]_0^{\pi/6}$ — A1-1
$= \frac{\pi}{2}\left[(\frac{\pi}{6} - \frac{1}{6}\sin\pi) - (0 - \frac{1}{6}\sin 0) \right]$ ↓
$= \frac{\pi}{2}(\frac{\pi}{6})$
$= \frac{\pi^2}{12}$ units³ — Au

Q3. (a)(i) Let $f(x) = x^2 - 4 + \log_e x$
$\therefore f(1) = 1^2 - 4 + \log_e(1) = -3$
$f(2) = 2^2 - 4 + \log_e(2) = 0.693$
Since $f(x)$ changes sign, the root lies between 1 and 2. — A

(ii) Half the interval of 1 and 2 is $1\frac{1}{2}$.
$f(1\frac{1}{2}) = 1.5^2 - 4 + \log_e(1.5) = -1.34$
Since $f(1\frac{1}{2}) < 0$, the root lies between 1.5 and 2 — A
Half the interval of 1.5 and 2 is 1.75
$f(1.75) = 1.75^2 - 4 + \log_e(1.75) = -0.378$
Since $f(1.75) < 0$, the root lies between 1.75 and 2 — Au

(iii) The root is the intersection of $y = \log_e x$ and $y = 4 - x^2$.
ie. $\log_e x = 4 - x^2$



There is only one such point. — Aw-

Q3. continued...

(b) $y = \dfrac{x^2-2x-3}{x-1}$

(i) When $x=0$: $y = \frac{-3}{-1}$ $\rightarrow$ $y=3$

When $y=0$: $x^2-2x-3=0$

$(x-3)(x+1)=0$ $\rightarrow$ $x = 3, -1$ | Aw-(1)

(ii) Vertical Asymptotes : when $x-1=0$

$\rightarrow$ $x=1$

Oblique Asymptotes : when $y = x-1$ | Aw-(2)

DIVIDE : $\dfrac{x^2-2x+1}{x-1} - \dfrac{4}{x-1}$

$(x-1) - \dfrac{4}{x-1}$.

(iii) $\dfrac{dy}{dx} = \dfrac{(x-1)(2x-2)-(x^2-2x-3)}{(x-1)^2}$

$= \dfrac{2x^2-2x-2x+2-x^2+2x+3}{(x-1)^2}$

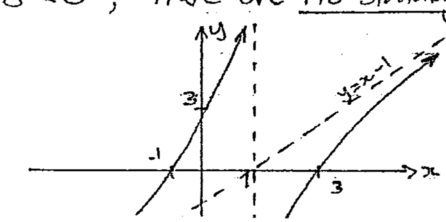
$= \dfrac{x^2-2x+5}{(x-1)^2}$ | A1 - 1

TP when $\frac{dy}{dx} = 0$

ie. $x^2-2x+5=0$

Since $\Delta < 0$, there are no stationary points. | Aw-(2)

(iv)



Aw-(2)

Q4. (a) (i) AP: $\beta - \alpha = \gamma - \beta$

$\therefore\ 2\beta = \alpha + \gamma$ | Aw

(ii) $\alpha+\beta+\gamma = -\frac{b}{a} = \frac{36}{8} = 4.5$ | Aw

(iii) $(\alpha+\gamma)+\beta = 4.5$

$2\beta + \beta = 4.5$

$3\beta = 4.5$ $\rightarrow$ $\beta = 1.5$ | A1 -

Now, $\alpha + \gamma = \quad = 3$ —— (1)

Also, $\alpha\beta\gamma = -\frac{d}{a} = -\frac{21}{8}$

$\therefore\ \alpha\gamma = -\frac{7}{4}$ —— (2)

Eq.(1): $\gamma = 3-\alpha$

Sub into (2): $\alpha(3-\alpha) = -\frac{7}{4}$

$4\alpha^2 - 12\alpha - 7 = 0$

$(2\alpha+1)(2\alpha-7) = 0$

$\therefore\ \alpha = -\frac{1}{2}, \frac{7}{2}$.

When $\left.\begin{aligned}\alpha = -\tfrac{1}{2},\ \gamma = \tfrac{7}{2}\\ \alpha = \tfrac{7}{2},\ \gamma = -\tfrac{1}{2}\end{aligned}\right\}$

$\therefore$ The roots are $-\frac{1}{2}, \frac{3}{2}, \frac{7}{2}$ | Aw.

(b) (i) $\sum_{r=1}^{n}(5r-4) = 1+6+11+\cdots+(5n-4)$

(ii) Prove: $1+6+11+\cdots+(5n-4) = \frac{1}{2}n(5n-3)$

* Prove true for $n=1$:

LHS = $5-4=1$ ... | A1 -

Q4. continued---

* Assume true for $n=k$:

ie. $1+6+11+\cdots+5k-4 = \frac{1}{2}k(5k-3)$

* Prove true for $n=k+1$:

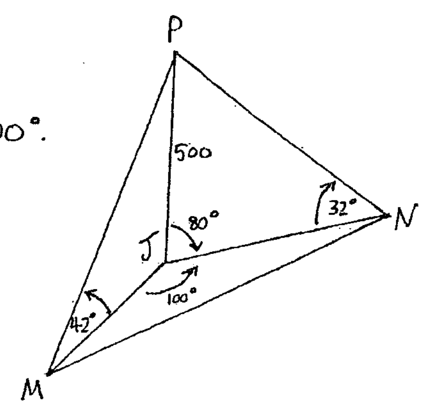
RTP: $[1+6+11+\cdots+5k-4] + 5(k+1)-4 = \frac{1}{2}(k+1)(5(k+1)-3)$

ie. $1+6+11+\cdots+(5k-4)+(5k+1) = \frac{1}{2}(k+1)(5k+2)$ A1 - 1

$$
\begin{aligned}
LHS &= \tfrac{1}{2}k(5k-3)+(5k+1) \\
&= \tfrac{1}{2}[5k^2-3k+10k+2] \\
&= \tfrac{1}{2}[5k^2+7k+2] \\
&= \tfrac{1}{2}(k+1)(5k+2) \\
&= RHS.
\end{aligned}
$$

Since proved true for $n=k+1$ and show true for $n=1$; then the result must be true for $n=2,3,$ etc. by the process of Mathematical Induction. Aw - 4

(c) $\angle MJN = 100°$.



Q4 continued...

(i) In ΔJMP: $\tan 42° = \frac{500}{JM}$ $\therefore JM = \frac{500}{\tan 42°}$ A

In ΔJNP: $\tan 32° = \frac{500}{JN}$ $\therefore JN = \frac{500}{\tan 32°}$ A

(ii) In ΔJMN: By the Cosine Rule:

$MN^2 = \left(\frac{500}{\tan 42}\right)^2 + \left(\frac{500}{\tan 32}\right)^2 - 2\left(\frac{500}{\tan 42}\right)\left(\frac{500}{\tan 32}\right)\cos 100°$

$\therefore MN = 1050.2$ metres Aw

Q5. (a) (i) $M_{PQ} = \frac{ap^2-aq^2}{2ap-2aq} = \frac{a(p-q)(p+q)}{2a(p-q)} = \frac{p+q}{2}$

PQ parallel to $y=x$, ie. $m=1$. $\Big\}$ $\therefore \frac{p+q}{2}=1$

$p+q=2$ Aw

(ii) $m = -\frac{1}{p}$ at $P(2ap, ap^2)$

$\boxed{y-y_1 = m(x-x_1)}$

$\therefore y-ap^2 = -\frac{1}{p}(x-2ap)$

$yp - ap^3 = -x+2ap$

$\therefore x+py = 2ap^2+ap^3$ ——— (1) Aw - (

(iii) Similarly, eqn. of normal at Q is:

$x+qy = 2aq^2+aq^3$ ——— (2)

Eq. (1)-(2): $y(p-q) = 2a(p-q)+a(p^3-q^3)$

$y = 2a + a(p^2+q^2+pq)$ A1 -

Q5. Continued...

Sub into Eq.①: $x + p(2a+ap^2+aq^2+apq) = 2ap+ap^3$

$\therefore x = -2ap-ap^3-apq^2-ap^2q+2ap+ap^3$

$x = -apq(p+q)$

But $p+q = 2 \implies \therefore x = -2apq$ Aw-②

$\therefore$ Coordinates of R: $x = -2apq$ ——③

$y = 2a+a(p^2+q^2+pq)$ —④

(iv) Eq.④: $y = 2a + a\left[(p+q)^2 - pq\right]$

$y = 2a + a(4-pq)$

$y = 6a - apq.$

Eq.③: $pq = \frac{-x}{2a}$

$\therefore y = 6a + \frac{x}{2}$

$2y = 12a + x$

$\therefore x - 2y + 12a = 0$ is the eqn of the locus of R. Aw③

(b)(i) $T = T_0 + Ae^{kt}$

$\frac{dT}{dt} = 0 + Ae^{kt} \times k$

$= (T - T_0) \times k$

$\therefore \frac{dT}{dt} = k(T - T_0)$ Aw-①

Q5. Continued...

(ii) $T_0 = -40°$, $T = 19°$

(α) When $t=0$: $24 = -40 + Ae^0$

$\therefore A = 64$

(β) When $t=5$: $19 = -40 + 64e^{5k}$

$59 = 64e^{5k}$

$e^{5k} = \frac{59}{64}$

$5k = \ln\left(\frac{59}{64}\right)$

$\therefore k = \frac{1}{5}\ln\left(\frac{59}{64}\right)$

ie. $k = -0.0163$ (3sf) A

(γ) For $T=0$: $0 = -40 + 64e^{-0.0163t}$

$e^{-0.0163t} = \frac{40}{64}$

$-0.0163t = \ln\left(\frac{40}{64}\right)$

$\therefore t = \frac{\ln(40/64)}{-0.0163}$

ie. $t = 28.9$ seconds Aw

Q6. (a) $\tan 2x = 1$

$2x = n\pi + \pi/4$

$\therefore x = \frac{n\pi}{2} + \pi/8$ — Aw-2

(b) $\ddot{x} = 0$ $\qquad \ddot{y} = -10$

$\dot{x} = V\cos\theta$ $\qquad \dot{y} = -10t + V\sin\theta$

$x = Vt\cos\theta$ $\qquad y = -5t^2 + Vt\sin\theta$

(assuming originally at origin (0,0).)

(i) At $t = 1.5$, $x = 60$:

$60 = V(1.5)\cos\theta$

$V\cos\theta = 40$ —— ①

At $t = 1.5$, $y = 2.25$:

$2.25 = -5(1.5)^2 + V(1.5)\sin\theta$

$V\sin\theta = 9$ —— ② — A1 - 1

Eq. ② ÷ ①: $\tan\theta = 9/40 \rightarrow \theta = 12.68°$ — A1 - 1

Sub into Eq. ①: $V\cos(12.68°) = 40 \rightarrow V = 41\,ms^{-1}$ — Aw-3

(ii) When $y = 0$: $-5t^2 + Vt\sin\theta = 0$

$t(-5t + V\sin\theta) = 0$

$\therefore t = 0$ or $t = \frac{V\sin\theta}{5}$

ie. $t = 0, 9/5$. — A1 - 1

When $t = 9/5$: $x = Vt\cos\theta$

$x = 41 \times 9/5 + \cos 12.68°$

$\therefore x = 72$ metres. — Aw-2

Q6. Continued...

(c) (i) $\frac{r}{7} = \frac{h}{22}$

$22r = 7h \rightarrow \therefore r = \frac{7h}{22}$ — Aw

(ii) $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \cdot \frac{22}{7} \cdot \left(\frac{7h}{22}\right)^2 \cdot h$

$= \frac{22}{21} \cdot \frac{49h^2}{484} \cdot h$

$\therefore V = \frac{1078h^3}{10164} = \frac{7}{66}h^3$ — Aw-

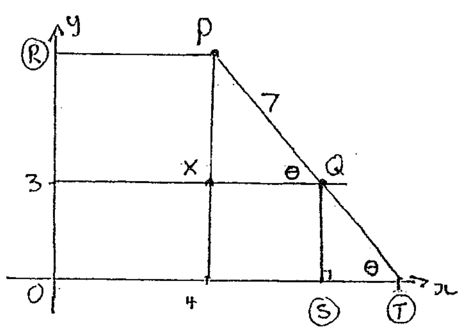
(iii) Now, $\frac{dV}{dh} = \frac{7}{22}h^2$. and $\frac{dV}{dt} = 7$ cm/min. — A1 -

$\therefore \boxed{\frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{dh}{dV}}$

$= 7 \cdot \frac{22}{7h^2}$

$\therefore \frac{dh}{dt} = \frac{22}{h^2}$ — A1 - 1

When $h = 11$, $\frac{dh}{dt} = \frac{22}{121} = \frac{2}{11}$ cm/min — Aw-(

Q7.		Marks
	(a) (i) $\frac{1}{2}v^2 = 24 - \frac{3}{2}x^2$ $\frac{d}{dx}(\frac{1}{2}v^2) = -3x$ $\therefore \ddot{x} = -3x$ Since the acceleration is proportional to its distance from the origin, it is moving in SHM.	Aw-1
	(ii) $v^2 = 3(16-x^2)$ ie. $\boxed{v^2 = n^2(a^2-x^2)}$ $\rightarrow \therefore a = 4$ metres	Aw-1
	(iii) Max. Speed when $x=0$: ie. $v^2 = 3(16-0)$ $\therefore v = \sqrt{48}\ ms^{-1}$ is the max. speed.	Aw-1
	(iv) Max. Acceleration when $x = \pm 4$ ie. $\ddot{x} = -3 \times -4$ $\therefore \ddot{x} = 12\ ms^{-2}$ is the max. acceleration	Aw-1
	(b)  (i) $OR = 3 + PX = 3 + 7\sin\theta$ $OS = 4 + QX = 4 + 7\cos\theta$	Aw-1

Q7.	Continued...	Marks
	(ii) $RS^2 = OR^2 + OS^2$ (Pythagoras' Theorem) $= (3+7\sin\theta)^2 + (4+7\cos\theta)^2$ $= 9 + 42\sin\theta + 49\sin^2\theta + 16 + 56\cos\theta + 49\sin^2\theta.$ $= 74 + 42\sin\theta + 56\cos\theta$	Au
	(iii) Let $42\sin\theta + 56\cos\theta = r\cos(\theta - \alpha)$ where $r = \sqrt{42^2+56^2} = 70$ $\alpha = \tan^{-1}(\frac{42}{56}) = 36°52'$	A1-
	$\therefore 42\sin\theta + 56\cos\theta = 70\cos(\theta - 36°52')$ $\therefore RS^2 = 74 + 70\cos(\theta - 36°52')$	Aw-
	(iv) Maximum value of RS occurs when: $\cos(\theta - 36°52') = 1$ ie. when $\theta = 36°52'$.	A1-
	$\therefore RS^2 = 74 + 70 = 144$ ie. $RS = 12$ units	Aw-(