SUGGESTED SOLUTIONS TO MATHEMATICS CSSA TRIAL 2004

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Question 1

(a) $x^5 = 5000 \therefore x = \sqrt[5]{5000} = 5.49$

(b) $0.3 + 0.\dot{3} = \frac{3}{10} + \frac{1}{3} = \frac{19}{30}$

(c) $\tan\alpha = 3 \therefore \alpha = 72^\circ$ or $\alpha = 252^\circ$ (to the nearest degree)

(d) $1 - \frac{a-b}{a+b} = \frac{a+b-(a-b)}{a+b} = \frac{2b}{a+b}$

(e) $8^x = 32 \therefore (2^3)^x = 2^5 \therefore 2^{3x} = 2^5 \therefore 3x = 5 \therefore x = \frac{5}{3}$

(f) $\frac{1}{2-\sqrt{3}} = \frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}} = 2+\sqrt{3} \therefore a = 2$ and $b = 1$

Question 2

(a) (i) $\frac{d}{dx}\left[(3x+4)^7\right] = 7(3x+7)^6 \times 3 = 21(3x+4)^6$

(ii) $\frac{d}{dx}\left(x^3e^x\right) = x^3e^x + 3x^2e^x = x^2e^x(x+3)$

(iii) $\frac{d}{dx}\left(\frac{\tan 5x}{5x}\right) = \frac{5\sec^2 5x \times 5x - \tan 5x \times 5}{25x^2} = \frac{5x\sec^2 5x - \tan 5x}{5x^2}$

(b) (i) $\int\left(e^{3x} + \sqrt{x}\right)dx = \frac{e^{3x}}{3} + \frac{2}{3}x^{\frac{3}{2}} + c$

(ii) $\int_1^2 \frac{x^4+1}{x}dx = \int_1^2\left(x^3 + \frac{1}{x}\right)dx = \left[\frac{x^4}{4} + \ln x\right]_1^2 = \left(\frac{2^4}{4} + \ln 2\right) - \left(\frac{1}{4} + \ln 1\right) = 3\frac{3}{4} + \ln 2$

(iii) $\frac{dy}{dx} = 2x - \sin x \therefore$ y=x²+cosx+c

when y=2 , x=0 ∴ c = 1 ∴ $y = x^2 + \cos x + 1$

Question 3

(a) $\Delta < 0$ and $a > 0$

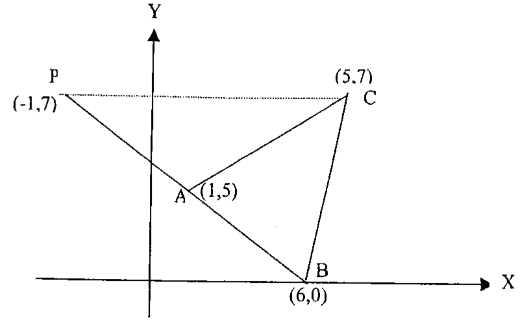
$\therefore 25 - 4a^2 < 0$ i.e. $(5-2a)(5+2a) < 0$

$a > \frac{5}{2}$ or $a < -\frac{5}{2}$, and for positive definite $\therefore a > \frac{5}{2}$

(b) $\int_1^k (x+1)dx = 6 \quad \therefore \left[\frac{x^2}{2} + x\right]_1^k = 6 \therefore \left[\frac{k^2}{2} + k\right] - \left[\frac{1}{2} + 1\right] = 6$

$\therefore k^2 + 2k - 15 = 0 \therefore (k+5)(k-3) = 0 \therefore k = 3$ or $k = -5$

3 c)



(i) AB= $\sqrt{(5-0)^2 + (1-6)^2} = \sqrt{50} = 5\sqrt{2}$

(ii) BC = = $\sqrt{(7-0)^2 + (5-6)^2} = \sqrt{50} = 5\sqrt{2}$ ∴Δ ABC is isosceles.

(iii) Gradient of AB = $\frac{5-0}{1-6} = -1$

∴equation of line AB is y – 0 = - 1 (x – 6) ∴ x + y = 6 (1)

(iv) Substitute y=7 into (1) ∴ x = - 1 ∴P is (- 1,7)

(v) PC = 5+1=6 units and the perpendicular distance from A to PC = 7-5=2 units.

∴Area of Δ PAC = ½ × 6 × 2 = 6 units²

Question 4

(a) (i) In Δ's ABC and CAD:

$\frac{AB}{AC} = \frac{9}{12} = \frac{3}{4}$ and $\frac{BC}{AD} = \frac{6}{8} = \frac{3}{4}$ and $\angle ABC = \angle DAC$ (Given)

∴Δ ABC ||| Δ CAD (two pairs of corresponding sides are proportional and their included angles are equal.)

(ii) Since the two triangles are similar

$\therefore \frac{AB}{AC} = \frac{AC}{CD}$ (corresponding sides are proportional)

$\therefore \frac{9}{12} = \frac{12}{CD} \quad \therefore CD = 16$ cm.

(b) (i) $y = ax^2$ and $y = 12x + 3$

$ax^2 = 12x + 3 \therefore ax^2 - 12x - 3 = 0$ (*)

(ii) Since the line is a tangent to the parabola (one point of contact)∴the roots are equal.

$\therefore \Delta = 0 \therefore (-12)^2 - 4 \times a \times (-3) = 0 \therefore a = -12$

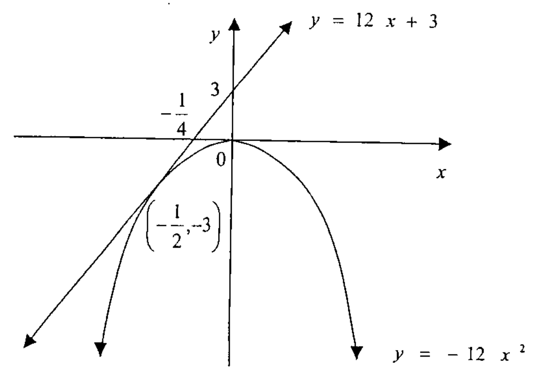
(iii) To find the point of contact, substitute a = - 12 into equation (*)

$-12x^2 - 12x - 3 = 0 \therefore 4x^2 + 4x + 1 = 0 \therefore (2x+1)^2 = 0$

$x = -\frac{1}{2} \therefore y = 12 \times \left(-\frac{1}{2}\right) + 3 \therefore y = -3$

∴ the point of contact is $\left(-\frac{1}{2}, -3\right)$

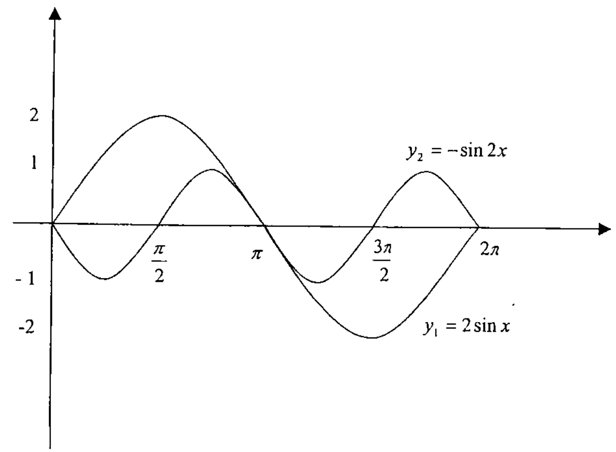
4 b (iv)



Question 5

(a) Base angle is 30° ∴angles are: $30^\circ, 180^\circ - 30^\circ = 150^\circ, -210^\circ, -360^\circ + 30^\circ = -330^\circ$

(b) (i)



(ii) 0, π and 2π.

(c) (i) $M = M_0e^{-kt} \quad \therefore \frac{dM}{dt} = -kM_0e^{-kt} \quad \therefore \frac{dM}{dt} = -kM$

(ii) (α) $80 = 100\ e^{-20t} \quad \therefore 0.8 = e^{-20k} \quad \therefore k = \frac{\ln 0.8}{-20} = 0.011157$

(β) M= $100\ e^{-30\times 0.011157}$ = 72 grams (to the nearest gram)

(iii) $50 = 100e^{-0.011157t} \therefore t = \frac{\ln 0.5}{-0.011157}$ = 62 hours (to the nearest hour)

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Question 6

(a) (i) $y = x^3+ax^2+7x-5 \therefore \frac{dy}{dx} = 3x^2 + 2ax + 7$

At $x = 1$, $\frac{dy}{dx} = 0 \therefore 3+2a+7=0 \therefore a = -5$

(ii) $y = x^3 - 5x^2 + 7x - 5 \therefore \frac{dy}{dx} = 3x^2 - 10x + 7$

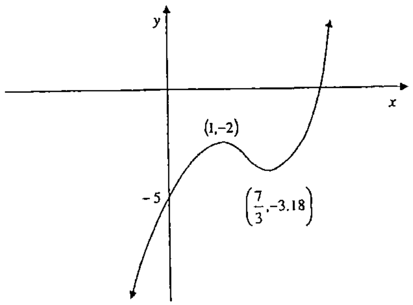
$\therefore \frac{dy}{dx} = 0 \therefore (x-1)(3x-7) = 0 \therefore x = 1$ or $x = \frac{7}{3}$

$\therefore$ stationary points are: $(1,-2)$ and $\left(\frac{7}{3},-3.18\right)$

(iii) $\therefore \frac{d^2y}{dx^2} = 6x - 10$, for $x = 1 \therefore \frac{d^2y}{dx^2} = -4 < 0 \therefore (1,-2)$ is a local max.

for $x = \frac{7}{3} \therefore \frac{d^2y}{dx^2} = 4 > 0 \therefore \left(\frac{7}{3},-3.18\right)$ is a local min.

(iv)



y = f(x) is increasing for $x < 1$ or $x > \frac{7}{3}$

(b) (i) Since $T_n = a+(n-1)d \therefore T_k = a+(K-1)d \therefore L = a+(K-1)d$ (1)

(ii) Similarly $T_L = a+(L-1)d \therefore K = a+(L-1)d$ (2)

(iii) (1) – (2) $\therefore L - K = (K - L)d \therefore d = -1$

(iv) Substitute $d = -1$ into equation (1)
$\therefore L = a+(k-1)(-1) \therefore L = a - K+1 \therefore a = L+K-1$

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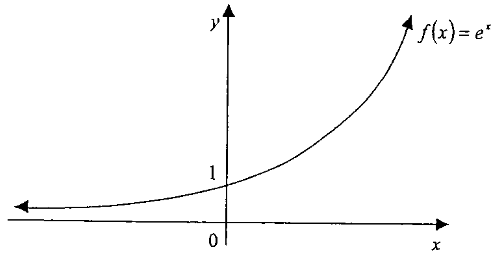
Question 7

(a) (i) $A=\int_1^3 \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^3 = -\frac{1}{3}+1 = \frac{2}{3}$ square units

(ii) $V_x = \pi\int_1^3 y^2 dx = \pi\int_1^3 \frac{1}{x^4}dx = \pi\int_1^3 x^{-4}dx = \frac{\pi}{-3}\left[x^{-3}\right]_1^3 = \frac{26}{81}\pi$ cubic units

(b) (i)



Range: $\{y : y > 0\}$

(ii) The volume of the solid obtained by the rotation of the curve y=f(x) about y – axis between $y = 3$ and $y = 5$ is given by:

$V_y = \pi\int_3^5 x^2 dy$, and making x the subject from $y = e^x \therefore x = \ln y$

$\therefore V_y = \pi\int_3^5 (\ln y)^2 dy$, as required.

(iii) let $f(y) = (\ln y)^2$

y	3	3.5	4	4.5	5
$f(y)$	1.2069...	1.5694...	1.9218...	2.2622...	2.5902...

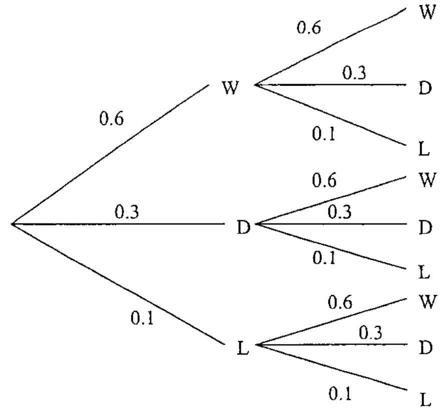
$h = \frac{5-3}{4} = 0.5$

$\therefore V_y = \pi\frac{0.5}{3}[1.2069... + 4\times(1.5694... + 2.2622) + 2\times 1.9218... + 2.5902...]$

$\therefore V_y = 12$ cubic units (to 2 sign. fig.)

Question 8

(a) (i)



(ii) P(winning at least one match) = [P(WW) + P(WD) + P(WL)] + P(DW) + P(LW) = $[0.36 + 0.18 + 0.06] + 0.18 + 0.06 = 0.6 + 0.18 + 0.06 = 0.84$

(iii) P(not win either match) = 1 – P(winning at least one match)
$= 1 - 0.84 = 0.16$

(b)

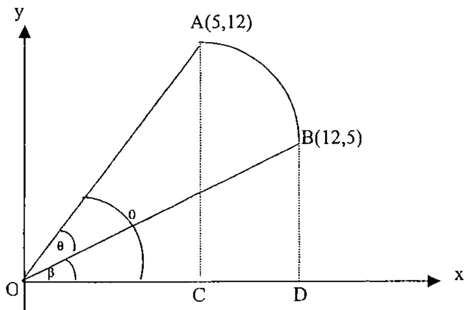


Fig.1

(i) $1 \text{ rad} = \frac{180^\circ}{\pi} = 57^\circ 18'$

(ii) From Δ OAC, $\tan\alpha = \frac{12}{5} \therefore \alpha = 1.12$ radians.

and from ΔOBD, $\tan\beta = \frac{5}{12} \therefore \beta = 0.34$ radians.

$\therefore \theta = 1.116 - 0.335 = 0.78$ rad. $\therefore \angle AOB = 0.78$ radians.

(iii) $OA^2 = OC^2 + AC^2 = 25 + 144 = 169$
$\therefore OA = 13 = OB$
The length of the arc AB = $r\,\theta = 13 \times 0.78 = 10.14$
$\therefore$ The perimeter of sector OAB = 13 + 13 + 10.14 = 36.14

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Question 9

(a) (i) $V = (50-2x)(20-2x)x = 4x^3 - 140x^2 + 1000x$ (cm^3)

(ii) $\frac{dV}{dx} = 12x^2 - 280x^2 + 1000 \therefore \frac{dV}{dx} = 0 \therefore x = 4.4$ (cm), correct to one decimal place.

$\frac{d^2V}{dx^2} = 24x - 560x$, and for $x = 4.4$ $\frac{d^2V}{dx^2} = -2359.39 < 0 \therefore$ Volume is maximum.

(iii) For $x = 4.4\ldots \therefore V = 2030.34$ cm^3

(b) (i) $M_n = \left(1+\frac{r}{100}\right)M_{n-1}$ When n =2

$M_2 = \left(1+\frac{r}{100}\right)M_1$ $M_2 = \left(1+\frac{12}{100}\right)500$

$M_2 = 500(1.12)$ $\therefore M_2 = \$560$

(ii) $M_3 = 1.12M_2$ $\therefore M_3 = 500(1.12)^2$.

$\therefore M_4 = 500(1.12)^3$.

$\therefore M_5 = 500(1.12)^4$.

....

....

$\therefore M_{20} = 500(1.12)^{19} = \$\ 4306.38$

(iii) The total value is given by:

$500 + 500(1.12) + 500(1.12)^2 + 500(1.12)^3 + \ldots + 500(1.12)^{19}$

$\therefore S_n = \frac{a(r^n-1)}{r-1}$ $\therefore S_{20} = \frac{500(1.12^{20}-1)}{1.12-1} = \$36026.22$

Question 10

(a) (i) $\frac{dv}{dt} = k \therefore v = \int k.dt \therefore v = kt + c_1$ (1)

(ii) $\frac{dx}{dt} = kt + c_1 \therefore x = \int (kt + c_1)dt \therefore x = \frac{kt^2}{2} + c_1 t + c_2$ (2)

When t = 0, x = 1 $\therefore$ 1 = 0 + 0 + c_2 $\therefore$ c_2 = 1.

When t = 1, x = 2 $\therefore$ 2 = ½ k + c_1+1 $\therefore$ k+2c_1 =2 (3)

When t = 2, x = 9 $\therefore$ 9 = 2k +2 c_1 +1 $\therefore$ k+c_1 = 4 (4)

(3) – (4) $\therefore c_1$ = -2 sub. into (3) $\therefore$ k= 6

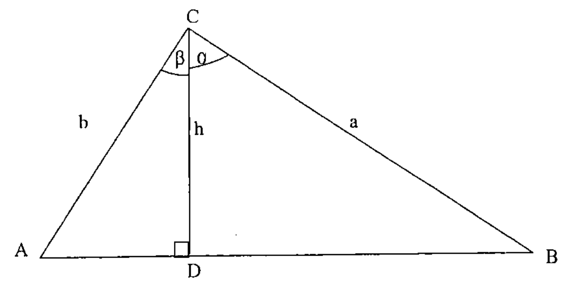
$\therefore$ sub. into eq. (2) $\therefore x = 3t^2 - 2t + 1$

(iii) The particle at rest when v=0 $\therefore$ from (1)

$v = 6t - 2 \therefore 0 = 6t - 2 \therefore t = \frac{1}{3}$

$\therefore$ the particle come to the rest at $t = \frac{1}{3}$ sec.

(b)



(i) In Δ ADC, $\cos\beta = \frac{h}{b}$ $\therefore$ h = b cosβ

In Δ BCD, $\cos\alpha = \frac{h}{a}$ $\therefore$ h = a cosα

$\therefore$ h = b cosβ = b cosα

(ii) Area of Δ ACD = ½ × AC × CD sinβ = ½ × b × h sinβ

= ½ × b × a cosα × sinβ

= ½ a b sinβ cosα

(iii) Area of Δ BCD = ½ × a × h sinα

= ½ × a × b cosβ sinα

= ½ a b cosβ sinα

(iv) Area of Δ ACB = ½ × AC × BC sin(α + β)

= ½ a b sin(α + β)

(v) Area of Δ ACB = Area of Δ ACD + Area of Δ BCD

½ a b sin(α + β) = ½ a b sinβ cosα + ½ a b cosβ sinα

$\therefore$ sin(α + β)= sinα cosβ + cosα sinβ.

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