Question 1

(a) (3 marks)

Find the acute angle between the lines $2x - y = 0$ and $x + 3y = 0$, giving the answer correct to the nearest minute.

(b) (4 marks)

Consider the function $y = x\ln x - x$.

(i) Solve the equation $y = 0$.

(ii) Find $\frac{d^2y}{dx^2}$ and hence show that the function is concave up for all values of x in its domain.

(c) (5 marks)

Consider the polynomial $P(x) = 6x^3 - 5x^2 - 2x + 1$

(i) Show that 1 is a zero of $P(x)$.

(ii) Express $P(x)$ as a product of 3 linear factors.

(iii) Solve the inequality $P(x) \le 0$.

Question 2

(a) (3 marks)

(i) Find $\frac{d}{dx}\left(e^{\tan x}\right)$ (ii) Hence find $\int \frac{e^{\tan x}}{\cos^2 x}\,dx$

(b) Use the substitution $u = 1 - x$ to evaluate $\int_{-3}^{0} \frac{x}{\sqrt{1-x}}\,dx$. (4 marks)

(c) (5 marks)

(i) Find the value of x such that $\sin^{-1} x = \cos^{-1} x$.

(ii) On the same axes sketch the graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$.

(iii) On the same diagram as the graphs in (ii), draw the graph of $y = \sin^{-1} x + \cos^{-1} x$.

Question 3

(a) (8 marks)

$$f(x) = \frac{8}{4 + x^2}$$

(i) Show that f is an even function, and the x axis is a horizontal asymptote to the curve $y = f(x)$.

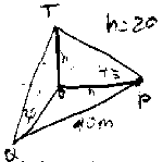
(ii) Find the coordinates and nature of the stationary point on the curve $y = f(x)$.

(iii) Sketch the graph of the curve showing the above features.

(iv) Find the exact area of the region in the first quadrant bounded by the curve $y = f(x)$ and the line $x = 2$.

(b) (4 marks)

A vertical tower of height h metres stands on horizontal ground. From a point P on the ground due east of the tower the angle of elevation of the top of the tower is $45°$. From a point Q on the ground due south of the tower the angle of elevation of the top of the tower is $30°$. If the distance PQ is 40 metres, find the exact height of the tower.



$h = 20$ $\tan 30 = \frac{h}{OQ}$ $OQ = \frac{h}{\tan 30}$

$h^2 + \frac{h^2}{\tan 30^2} = 40^2$

Question 4

(a) (8 marks)

N is the number of animals in a certain population at time t years. The population size N satisfies the equation $\frac{dN}{dt} = -k(N - 1000)$, for some constant k.

(i) Verify by differentiation that $N = 1000 + Ae^{-kt}$, A constant, is a solution of the equation.

(ii) Initially there are 2500 animals but after 2 years there are only 2200 left. Find the values of A and k.

(iii) Find when the number of animals has fallen to 1300.

(iv) Sketch the graph of the population size against time.

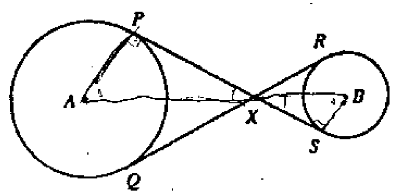
(b) (4 marks)

Use Mathematical Induction to show that $\cos(x + n\pi) = (-1)^n \cos x$ for all positive integers $n \ge 1$.

1995

Question 5

(a) (7 marks)



In the diagram PS and QR are tangents to each of the circles with centres A and B. The tangents intersect at X and A, X, B are collinear.

(i) Copy the diagram and show that $\Delta APX \parallel \Delta BSX$.

(ii) Suppose that the diagram represents two circles of radii 5 cm and 3 cm that are placed in the same plane with their centres 16 cm apart. A taut string surrounds the circles and crosses itself between them. Find the exact length of the string.

(b) (5 marks)

The interior of a circle is divided into two segments with areas in the ratio $3:1$ by a chord which subtends an angle θ radians at the centre of the circle.

(i) Show that $\theta - \sin\theta = \frac{\pi}{2}$.

(ii) Taking $\theta = 2 \cdot 5$ as a first approximation, use Newton's method twice to find a better approximation to θ, giving the answer correct to 2 decimal places.

Question 6

(a) (7 marks)

A group consisting of 3 men and 6 women attends a prizegiving ceremony.

(i) If the members of the group sit down at random in a straight line, find the probability that the 3 men sit next to each other. $3 \times 6!$

(ii) If 5 prizes are awarded at random to members of the group, find the probability that exactly 3 of the prizes are awarded to women if

for women for men

(α) there is a restriction of at most one prize per person. $6 \times 5 \times 4 \times 3 \times 2$

(β) there is no restriction on the number of prizes per person. $6 \times 6 \times 6 \times 3 \times 3$

$\boxed{6}$ $\boxed{6}$ $\boxed{4}$ $\boxed{3}$ $\boxed{2}$

Question 6 (cont.)

(b) (5 marks)

A particle moving in a straight line is performing Simple Harmonic Motion about a fixed point O on the line. At time t seconds the displacement x metres of the particle from O is given by

$$x = a\cos nt \quad , \quad \text{where } a > 0 \text{ and } 0 < n < \pi .$$

After 1 second the particle is 1 metre to the right of O, and after 2 seconds the particle is 1 metre to the left of O.

(i) Find the values of n and a.

(ii) Find the amplitude and period of the motion.

Question 7

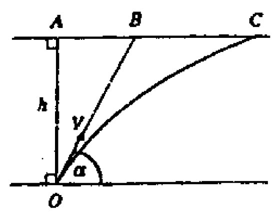
(a) (4 marks)

(i) Write down the Binomial expansion of $(1+x)^n$ in ascending powers of x.

Hence show that $\quad {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$.

(ii) Find how many groups of 1 or more digits can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 where repetition is not allowed.

(b) (8 marks)



In the diagram an aircraft is flying with constant velocity U at a constant height h above horizontal ground. When the plane is at A it is directly over a gun at O. When the plane is at B a shell is fired from the gun at the aircraft along OB. The shell is fired with initial velocity V at an angle of elevation α.

(i) If x and y are the horizontal and vertical displacements of the shell from O at time t seconds, show that if g is the acceleration due to gravity,

$$x = Vt\cos\alpha \quad \text{and} \quad y = Vt\sin\alpha - \tfrac{1}{2}gt^2 .$$

(ii) Show that if the shell hits the aircraft at time T at point C, then

$$VT\cos\alpha = \frac{h}{\tan\alpha} + UT .$$

(iii) Show that if the shell hits the aircraft then $2U(V\cos\alpha - U)\tan^2\alpha = gh$.