Ext 1

NORTH SYDNEY BOYS HIGH SCHOOL

2007
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write on one side of the paper (with lines) in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

- Attempt all questions

Class Teacher:
(Please tick or highlight)
- O Mr Barrett
- O Mr Ee
- O Mr Lowe
- O Mr Rezcallah
- O Mr Trenwith
- O Mr Weiss

Student Number:____________________

(To be used by the exam markers only.)

Question No	1	2	3	4	5	6	7	Total	Total
Mark	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{84}$	$\overline{100}$

Question 1 — **Marks**

a) Find $\int 6\cos x\, e^{3\sin x}\, dx$ — 2

b) Find $\lim_{x \to 0} \frac{\sin 2x}{12x}$ — 2

c) Solve for x $\quad \frac{5x-7}{x} \leq 4$ — 3

d) Prove the identity $\quad \frac{\cos x + \sin x}{\cos x - \sin x} = \frac{\sin 2x + 1}{\cos 2x}$ — 2

e) Evaluate $\quad \int_0^{\ln 4} \frac{e^x\, dx}{e^x + 2}$

Optional: The substitution $v = e^x + 2$, may be of some use. — 3

Question 2

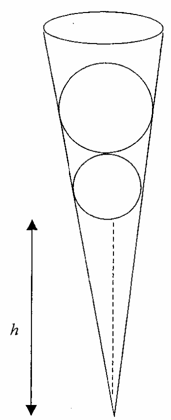
a) If $y = 2\cos^{-1}(\frac{x}{\pi})$

i) State the domain and range. **2**

ii) Sketch the curve. **1**

b) Show that $\tan^{-1}4 - \tan^{-1}(\frac{3}{5}) = \frac{\pi}{4}$ **2**

c) Two balls of radius 4cm and 8cm are placed in an inverted cone so that the balls touch each other and the sides of the cone.
Find the distance h, from the vertex of the cone to the smallest ball. **3**



d) i) Express $2\cos x + 2\sqrt{3}\sin x$ in the form $R\cos(x - \theta)$ **2**

ii) Find the two non-zero solutions to $2\cos x + 2\sqrt{3}\sin x = 2 \quad 0 \le x \le 2\pi$ **2**

Question 3

a) A particle moves in a straight line and its position at time t is given by $x = B\cos(4t + \alpha)$. The particle is initially at the origin moving with a velocity of 6m/s in a negative direction.

i) Show that the particle is undergoing simple harmonic motion. **2**

ii) Find the value of constants B and α. **2**

iii) Find the position of the particle after 4 seconds. **1**

b) Evaluate $\int_{-2}^{2} (2^x - 2)\,dx$ using Simpson's Rule and 3 function values. **3**

c) i) Show that the equation of a tangent to the parabola $x^2 = 4y$ at $(2p, p^2)$ is given by $y - px + p^2 = 0$. **2**

ii) This tangent meets the x-axis at R and the y-axis at Q.
Find the locus of M, the midpoint of QR. **2**

Question 4

a) Two roots of the equation $x^3 + 3x^2 - 4x + k = 0$ are opposites.

Find the value of k and the three roots. **3**

b) Consider the function $f(x) = 10x - 2\sin x - 5$

i) Show that the curve $y = 2\sin x + 5$ and the line $y = 10x$ meet at a point M whose x coordinate is approximately 0·6. **1**

ii) Use one application of Newton's method, starting at $x = 0{\cdot}6$ to find an approximation to the x coordinate of M. Give your answer correct to three decimal places. **2**

c) At time t, the displacement of a particle moving in a straight line is x.

If the acceleration is given by $\dfrac{d^2x}{dt^2} = 3 - 4x$ and the particle starts from rest at $x = 1$.

i) Find its velocity in terms of x. **2**

ii) At what point other than at $x = 1$, does the particle come to rest? **2**

d) Show that $x^2 - 3x + 2$ is a factor of

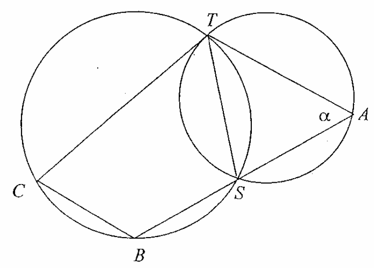
$$P(x) = x^n(2^m - 1) + x^m(1 - 2^n) + (2^n - 2^m)$$ **2**

Question 5

a) Use mathematical induction to prove that for all integers $n \geq 1$

$$\sum_{r=1}^{n} r \times 2^{r-1} = (n-1)2^n + 1$$ **4**

b) The line TC is a tangent. Prove that $TA \parallel CB$. **3**



c) A person drifting in a hot air balloon accidentally drops a water bottle from the basket and it falls from rest through the air. When both gravity and air resistance are taken into account, it is found that its velocity is given by $v = 160(1 - e^{-t/16})$ m/s and downwards has been taken as positive.

i) Show that $\dfrac{dv}{dt} = \dfrac{160 - v}{16}$. **2**

ii) What velocity does the bottle approach? **1**

iii) How long does it take to reach one eighth of this speed? **2**

Question 6

a) Consider the curve $f(x) = \dfrac{x^2 - 2x}{x^2 - 2x + 2}$

i) Find $\lim_{x \to \infty} f(x)$ **1**

ii) If $f'(x) = \dfrac{4(x-1)}{(x^2-2x+2)^2}$ and $f''(x) = \dfrac{8(2x-x^2)}{(x^2-2x+2)^3}$

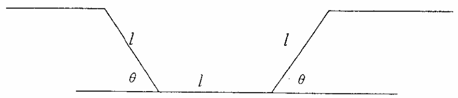
Sketch the graph of $f(x)$, showing any asymptotes, the coordinates of turning points, points of inflexion, the x and y intercepts. **3**

iii) The domain of $f(x)$ must be restricted if $f(x)$ is to have an inverse function. Find the domain for this to be possible which contains $x = 2$. **1**

iv) Sketch this inverse function. **1**

b) A large storm water channel is to have a cross-section in the shape of a trapezium as in the following diagram. The bottom and sides are each l metres long.

The sides of the channel make an angle of $\theta \le \frac{\pi}{2}$ with the horizontal.



i) Show that the cross-sectional area can be expressed as

$A = l^2(\sin\theta + \sin\theta\cos\theta)$. **3**

ii) For what angle θ is the area of the cross-section maximum? **3**

Question 7

a) From the top of a vertical cliff 15m above a beach, a stone is thrown with a speed of 35m/s at an angle of elevation α, to the horizontal.
The stone hits the sand at a point, which has horizontal displacement of 105m from the point of projection. Taking $g = 10\text{m/s}^2$.

i) Derive the expressions for vertical and horizontal displacement. **2**

ii) Find the time of flight in terms of α. **1**

iii) Hence or otherwise, show that $\tan\alpha = \frac{1}{3}$, or $\tan\alpha = 2$. **3**

b) A wine glass is formed by rotating $y = \dfrac{16x^2}{9}$ around the y axis.
The height of the liquid in the glass is h and the radius at the top of the wine is r.

i) Show the volume of the wine at height h cm is $\dfrac{8\pi r^4}{9}$ cm^3. **3**

ii) Wine is being added to the glass at a rate of $3(15 - h)$ mL/s.
Find the rate at which the radius of the surface is increasing when $h = 10$.
Express your answer to one decimal place. **3**

END OF EXAMINATION

Question 1

a) $\int 6\cos x\, e^{3\sin x}\, dx = 2e^{3\sin x} + c$ (2)

b) $\lim_{x \to 0} \frac{\sin 2x}{12x} = \frac{1}{6}$ (2)

c) $\frac{5x-7}{x} \leq 4$ (3)

Case 1 $x > 0$

$5x - 7 \leq 4x$

$x - 7 \leq 0$

$x \leq 7$

$\therefore 0 < x \leq 7$

Case 2

$x < 0$

$5x - 7 \geq 4x$

$x \geq 7$

No Solution

$\therefore 0 < x \leq 7$

Note inequality

d) $LHS = \frac{\cos x + \sin x}{\cos x - \sin x} \times \frac{\cos x + \sin x}{\cos x + \sin x}$ (2)

$= \frac{\cos^2 x + 2\sin x\cos x + \sin^2 x}{\cos^2 x - \sin^2 x}$

$= \frac{1 + \sin 2x}{\cos 2x} = RHS$

e) $\int_0^{\ln 4} \frac{e^x\, dx}{e^x + 2} = \left[\ln(e^x + 2) \right]_0^{\ln 4}$ (1) (3)

$= \ln(4+2) - \ln(1+2)$ (1)

$= \ln 2$ (1)

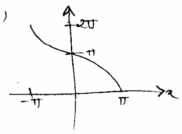
Question 2

a) i) D: $-1 \leq \frac{x}{\pi} \leq 1$ (1)

$-\pi \leq x \leq \pi$ (1)

R: $0 \leq y \leq 2\pi$

ii) (1)



b) Let $\tan \alpha = 4$, $\tan \beta = \frac{3}{5}$ (2)

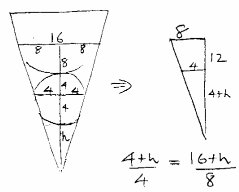
Consider $\tan(\alpha - \beta)$:

$\tan(\alpha - \beta) = \frac{T\alpha - T\beta}{1 + T\alpha T\beta}$

$\frac{4 - \frac{3}{5}}{1 + 4 \times \frac{3}{5}} = \frac{17}{17} = 1 = \tan \frac{\pi}{4}$

$\therefore LHS = RHS$

c) (3)



(1)

$\frac{4+h}{4} = \frac{16+h}{8}$ (1)

$8 + 2h = 16 + h$

$h = 8$ (1)

Question 2 d

i) Let $2\cos x + 2\sqrt{3}\sin x = R\cos x\cos\theta + R\sin x\sin\theta$

$\Rightarrow R\cos\theta = 2$ (2)

$R\sin\theta = 2\sqrt{3}$

$\therefore \tan\theta = \sqrt{3} \therefore \theta = \frac{\pi}{3}$ (1)

$R = \sqrt{4 + 12}$

$= 4$ (1)

$\Rightarrow 4\cos\left(x - \frac{\pi}{3}\right)$

ii) $4\cos\left(x - \frac{\pi}{3}\right) = 2$ (2)

$\cos\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$

$x - \frac{\pi}{3} = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

$\therefore x = \frac{2\pi}{3}$ or 2π (1)(1)

Question 3

i) $x = \beta\cos(4t + \alpha)$ ① (2)

$\dot{x} = -4\beta\sin(4t + \alpha)$ ②

$\ddot{x} = -16\beta\cos(4t + \alpha)$ ③

$\ddot{x} = -16x$ ④

$\therefore x$ is moving S.H.M

ii) $t = 0, x = 0, v = -6$ (2)

① $\Rightarrow 0 = \beta\cos\alpha$

$\therefore \alpha = \frac{\pi}{2}$

② $\Rightarrow -6 = -4\beta\sin\left(0 + \frac{\pi}{2}\right)$

$\therefore -6 = -4\beta \times 1$

iii) $x = \frac{3}{2}\cos\left(4t + \frac{\pi}{2}\right)$ (2)

$x(4) = \frac{3}{2}\cos\left(16 + \frac{\pi}{2}\right)$

$= 0.432$ (3DP)

Question 3

b)

x	-2	0	2
$f(x)$	$-\frac{7}{4}$	-1	2

(1) (3)

S.R $\Rightarrow \int_{-2}^{2} (2^x - 2)\, dx \doteqdot \frac{2 - -2}{6}\left[-\frac{7}{4} + 4 \times -1 + 2\right]$ (1)

$= -\frac{5}{2}$ (1)

c) i) $y = \frac{x^2}{4}$ (2)

$\frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2}$

at $x = 2p$

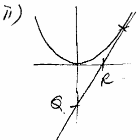
$m = p$

$\therefore y - p^2 = p(x - 2p)$

$y - p^2 = px - 2p^2$

$y - px + p^2 = 0$

ii)



$R \Rightarrow y = 0 \therefore x = p$

$R(p, 0)$

$Q \Rightarrow x = 0 \quad y = -p^2$

$(0, -p^2)$

Mid Point $M\left(\frac{p}{2}, \frac{-p^2}{2}\right)$ (1)

$\boxed{(2)}$

$\Rightarrow x = \frac{p}{2} \Rightarrow p = 2x$

$y = -\frac{(2x)^2}{2}$

$y = -2x^2$ (1)

Question 4 (3)

a) Let the roots be $\alpha, -\alpha + \beta$

$\therefore \alpha - \alpha + \beta = -3$

$\therefore \beta = -3$

Subst $x = -3 \Rightarrow$ $-27 + 27 + 12 + K = 0$

$\therefore K = -12$

$x^3 + 3x^2 - 4x - 12 = 0$

Product of roots $-\alpha^2\beta = 12$

$-\alpha^2 \times -3 = 12$

$\alpha^2 = 4$

$\alpha = \pm 2$

Roots ± 2 and -3

b) i) Subst 0.6 (1)

$2\sin 0.6 + 5 = 6.13$

$10 \times .6 = 6$

$\therefore$ Meet somewhere near $x = 0.6$

ii) $f(x) = 10x - 2\sin x - 5$ (2)

$f'(x) = 10 - 2\cos x$

Newtons Method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = .6 - \frac{(10 \times .6 - 2\sin .6 - 5)}{(10 - 2\cos(.6))}$$

$$= .6 - \frac{(6 - 2\sin .6 - 5)}{10 - 2\cos(.)}$$

$$= .6 - \frac{-0.129}{8.349}$$

$$= .6 + .054$$

$$= .615$$

Question 5 (4)

a) Step 1 1

let $n = 1$ 2

$LHS = 1 \times 2^0 = 1$ 1

$RHS = (1-1) \times 2^1 + 1$

$= 1$

$\therefore$ True for $n = 1$

Step 2 Assume true for $n = k$

ie $1 \times 2^0 + 2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \ldots K \times 2^{K-1} = (K-1)2^K + 1$

$T_K = K2^{K-1}$

Need to Show

$$S_K + T_{K+1} = K \times 2^{K+1} + 1$$

$LHS = (K-1) \times 2^K + 1 + (K+1) \times 2^K$

$= K \times 2^K - 2^K + 1 + K \times 2^K + 2^K$

$= 2K \times 2^K + 1$

$= K \times 2^{K+1}$

$= S_{K+1}$

If true for $n = K$ then it is true for $n = K+1$

Step 3

It is true for $n = 1$, so by the process of Mathematical induction it is true for $n = 2, 3, \ldots$ [illegible]

Question 4

c)

i)

$$\frac{d(\frac{1}{2}v^2)}{dx} = 3-4x$$

$$\frac{1}{2}v^2 = \int (3-4x)\,dx$$

$$= 3x-2x^2+c$$

$x=1 \quad v=0$

$$\Rightarrow 3-2+c=0$$

$$1+c=0$$

$$c=-1$$

$$\therefore \frac{1}{2}v^2 = 3x-2x^2-1$$

$$v^2 = 6x-4x^2+2$$

$$v = \pm\sqrt{-2(2x^2-3x+1)}$$

ii) $2x^2-3x+1=0$

$$(x-1)(2x-1)=0$$

$$x=1 \text{ or } x=\frac{1}{2}$$

d)

$$P(x) = x^n(2^m-1)+x^m(1-2^n)+(2^n-2^m)$$

$$(x^2-3x+2) = (x-1)(x-2)$$

$$P(1) = 1(2^m-1)+1(1-2^n)+2^n-2^m = 0$$

$$P(2) = 2^n(2^m-1)+2^m(1-2^n)+2^n-2^m$$

$$= 2^{m+n}-2^n+2^m-2^{m+n}+2^n-2^m$$

$$=0$$

$\therefore (x-1)$ & $(x-2)$ are factors

$\therefore (x^2-3x+2)$ is a factor of $P(x)$

Question 5b (3)

Aim to Prove $TA \parallel CB$

$\angle CTS = \angle TAS = \alpha$ ($\angle$ between tgt + chord = $\angle$ in alt segmt)

$\angle CBS = 180 - \alpha$ (opp $\angle$s cyclic quad supplementary)

$\therefore CB \parallel TA$ (co-int $\angle$s sum to $180°$)

c) $v = 160 - 160e^{-t/16}$ (2)

i) $\frac{dv}{dt} = 10e^{-t/16}$ ①

$\frac{dv}{dt} = \frac{160 - (160 - 160e^{-t/16})}{16}$

$= \frac{160 - 160 + 160e^{-t/16}}{16}$

$= 10e^{-t/16}$ ②

Now ① = ② $\therefore v = 160(1 - e^{-t/16})$ satisfies the eqn

ii) $\lim_{t \to \infty} v = 160(1-0) = 160$ m/s (1)

iii) $20 = 160 - 160e^{-t/16}$ (2)

$\frac{-140}{-160} = e^{-t/16}$

$\frac{7}{8} = e^{-t/16}$

$\ln\left(\frac{7}{8}\right) = -\frac{t}{16}$

$-16 \times \ln\left(\frac{7}{8}\right) = t$

$t \doteqdot 2.14$ secs

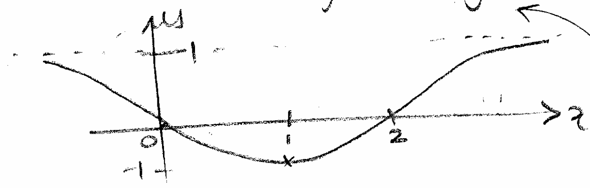
Question 6

a) i) $\lim_{x \to +\infty} = 1.$ (1)

ii) $f'(x) = 0 \Rightarrow x = 1$

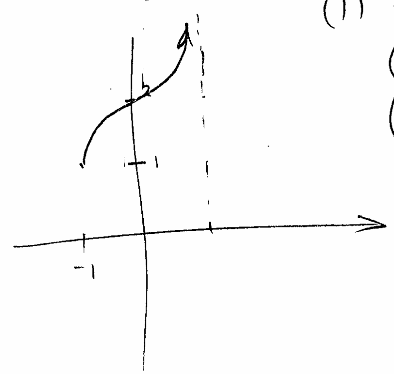
$f''(1) = \frac{8}{1} > 0 \therefore$ Min at $x = 1$, $y = -1$ $(1, -1)$

$f''(x) = 0 \Rightarrow x = 0$ or $x = 2$
$y = 0$ or $y = 0$ (3)



iii) $x \geq 1$ (1)

iv)

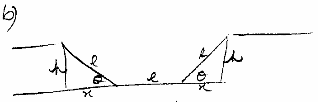


(1)

As the curve is rising from a minimum to a horizontal asymptote $x = 0$ and 2 have to be pts of inflexion

Question 6

b)



i) $\frac{x}{l} = \cos\theta$ $\quad \frac{h}{l} = \sin\theta$,

$x = l\cos\theta$ $\quad h = l\sin\theta$

$A = (l + 2x + l) \times \frac{h}{2}$,

$= (x + l)h$

$= (l\cos\theta + l) \times l\sin\theta$

$= l^2(\sin\theta + \sin\theta\cos\theta)$,

$\frac{\sin 2\theta}{2}$

(3)

ii) $\frac{dA}{d\theta} = l^2(\cos\theta + \frac{2\cos 2\theta}{2})$,

(3)

$A' = 0 \Rightarrow \cos\theta = \sin^2\theta - \cos^2\theta$

$= 1 - \cos^2\theta - \cos^2\theta$

$2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1)$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

$\theta = \frac{\pi}{3}$, , out of the domain

$\frac{d^2A}{d\theta^2} = l^2(-\sin\theta - 2\sin 2\theta)$

at $x = \frac{\pi}{3}$

$= l^2(-\sin\frac{\pi}{3} - 2\sin\frac{2\pi}{3})$

$< 0 \therefore$ Max at $x = \frac{\pi}{3}$,

Question 7

a) i) $\ddot{x} = 0$ $\quad \ddot{y} = -10$

$\dot{x} = 35\cos\alpha$ $\quad \dot{y} = -10t + 35\sin\alpha$ (2)

$x = 35t\cos\alpha$ $\quad$ 1? $y = -5t^2 + 35t\sin\alpha + 15$,

ii) $x = 105 = 35t\cos\alpha$

$t = \frac{3}{\cos\alpha} = 3\sec\alpha$ $\quad$ 1 $\quad$ (1)

iii) at $t = 3\sec\alpha$ $y = 0$

$\Rightarrow -5(3\sec\alpha)^2 + 35(3\sec\alpha)\sin\alpha + 15 = 0$ (3)

$-45(1 + \tan^2\alpha) + 105\tan\alpha + 15 = 0$

$-45 - 45\tan^2\alpha + 105\tan\alpha + 15 = 0$

$-45(3\tan^2\alpha - 7\tan\alpha + 2) = 0$ 1

$(3\tan\alpha - 1)(\tan\alpha - 2) = 0$

$\therefore \tan\alpha = \frac{1}{3}$ $\quad \tan\alpha = 2$. 1

OR

i) $\ddot{x} = 0$ $\quad \ddot{y} = -10$

$\dot{x} = 35\cos\alpha$ $\quad \dot{y} = -10t + 35\sin\alpha$

$x = 35t\cos s$ $\quad y = -5t^2 + 35t\sin\alpha$

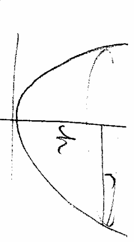
iii) $t = 3\sec\alpha$ $\quad y = -15$

hence

eqn $= -5(3\sec\alpha)^2 + 35(3\sec\alpha)\sin\alpha = -15$ etc.

Question 7 b

$y = \frac{16x^2}{9}$ $\quad x^2 = \frac{9y}{16}$



i) $V = \pi\int_0^h x^2\,dy = \pi\int_0^h \frac{9y}{16}$

$= \pi\left[\frac{9y^2}{32}\right]_0^h$

$= \pi \times \frac{9h^2}{32}$ ① 1 $\quad$ (3)

$y = h$ $\quad x = r$

$h = \frac{16r^2}{9}$ ② 1

$\frac{9h}{16} = r^2$

② into ①

$\pi \times \cancel{9} \times \frac{256 r^4}{32 \times 81}$

$V = \frac{8\pi r^4}{9}$ $\quad \frac{dV}{dr} = \frac{32\pi r^3}{9}$ 1

ii) $h = 10$ $\quad$ (3)

$r = \sqrt{\frac{90}{16}}$

$\doteqdot 2.371$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

@ $h = 10$

$\frac{dV}{dt} = 15$

$15 = \frac{32\pi r^3}{9} \times \frac{dr}{dt}$ 1

$15 = 148.9\,\frac{dr}{dt}$

$\frac{15}{148.9} = \frac{dr}{dt}$ 1

$.1 = \frac{dr}{dt}$ 1