Question 1:

a) i) $\int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx$ ½

no ½ −1

$= \frac{1}{2}\left[\ln(x^2+1)\right]_0^1$ ½

$= \frac{1}{2}\left[\ln 2 - \ln 1\right]$

$= \frac{1}{2}\ln 2$ 1

ii) $\int_{-2}^{2\sqrt{3}} \frac{1}{x^2+4}\,dx = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-2}^{2\sqrt{3}}$ ½

$= \left(\frac{1}{2}\tan^{-1}\frac{2\sqrt{3}}{2}\right) - \left(\frac{1}{2}\tan^{-1}\frac{-2}{2}\right)$

$= \left(\frac{1}{2}\tan^{-1}\sqrt{3}\right) - \left(\frac{1}{2}\tan^{-1}(-1)\right)$

$= \left(\frac{1}{2}\cdot\frac{\pi}{3}\right) - \left(\frac{1}{2}\cdot\frac{-\pi}{4}\right)$ ½

$= \frac{\pi}{6} - -\frac{\pi}{8}$

$= \frac{7\pi}{24}$ 1

b) Find the gradient of the tangent to the curve $y = \tan^{-1}(\sin x)$ at $x = 0$

$\frac{dy}{dx} = \frac{1}{1+(\sin x)^2} \times (\cos x)$ −1 for no × $\cos x$.

$= \frac{\cos x}{1+\sin^2 x}$ 1

when $x = 0$

$\frac{dy}{dx} = \frac{\cos 0}{1+\sin^2 0}$

$= \frac{1}{1}$

$= 1$ 1

c) solve for x, $\frac{1}{x+1} < 3$

$x + 1 \neq 0$

$\therefore \quad x \neq -1$ ½

$\times$ by $(x+1)^2$

$\frac{(x+1)^{\cancel{2}}}{\cancel{x+1}} < 3(x+1)^2$ ½

$x+1 < 3x^2 + 6x + 3$

$0 < 3x^2 + 6x + 3 - (x+1)$

$0 < 3x^2 + 5x + 2$

$\frac{(3x+3)(3x+2)}{3}$

$0 < (x+1)(3x+2)$ ½

$= \frac{\cancel{3}(x+1)(3x+2)}{\cancel{3}}$

$x < -1, \quad x > -2/3$ ½

d) General solution for

$\cos(\theta + \pi/4) = 1/\sqrt{2}$

$\cos(\theta + \pi/4) = \cos \pi/4$ ½

$\theta + \pi/4 = 2n\pi \pm \pi/4$ 1

$\theta = 2n\pi \pm \pi/4 - \pi/4$ ½

e) $f(x) = 8x^3$ find inverse function $f^{-1}(x)$

$f(f^{-1}(x)) = x = f^{-1}(f(x))$ ½

$f(f^{-1}(x)) = x$

$\therefore \quad 8(f^{-1}(x))^3 = x$

$f^{-1}(x)^3 = x/8$

$\therefore \quad f^{-1}(x) = \sqrt[3]{\frac{x}{8}}$

$= \frac{\sqrt[3]{x}}{2}$ ½

$f^{-1}(f(x)) = \frac{\sqrt[3]{8x^3}}{2}$

$= \frac{2x}{2}$

$= x$ ½

$\therefore \quad f^{-1}(x) = \frac{\sqrt[3]{x}}{2}$ ½

-1 for $\sqrt[3]{8} = 2\sqrt{2}$

-1 for $\frac{\sqrt[3]{x}}{8}$

Question 2.

a) $A(-4,-6)$ $B(6,-1)$ divide externally in ratio $3:1$

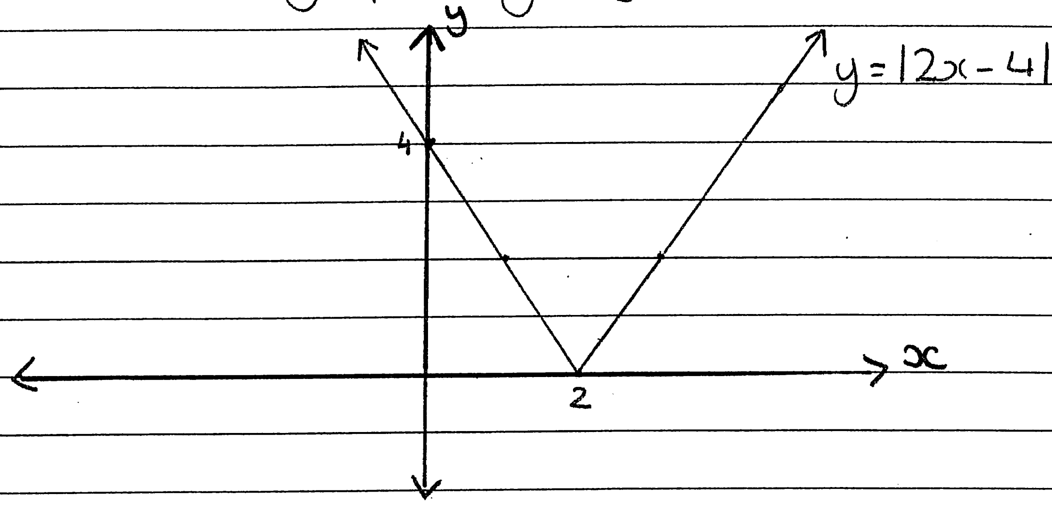
x - co-ordinate $= \dfrac{(3\times 6)-(1\times -4)}{3-1} = 11$ ½

y - co-ordinate $= \dfrac{(3\times -1)-(1\times -6)}{3-1} = \dfrac{3}{2}$ ½

$\therefore$ The co-ordinates of P are $(11, 3/2)$ 1

1½ if x_2, x_1 y_2 & y_1 are switched.

b) i) sketch the graph of $y=|2x-4|$



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ii) solve $|2x-4|>x$

$\sqrt{(2x-4)^2} > x$

$(2x-4)^2 > x^2$

$4x^2-16x+16 > x^2$

$3x^2-16x+16 > 0$

$\dfrac{(3x-12)(3x-4)}{3} > 0$

$\dfrac{\cancel{3}(x-4)(3x-4)}{\cancel{3}} > 0$

$x>4, \; x<4/3$ 2

c) Use $u \doteq 1+x$ to evaluate $\int_{-1}^{3} x\sqrt{1+x}\, dx$

$u \doteq 1+x$ limits:

$\therefore \frac{du}{dx} = 1$ $\therefore x = u-1$ $x=3 \therefore u = 1+3 \Rightarrow u = 4$ 1/2

$x=-1 \therefore u = 1-1 \Rightarrow u = 0$

$du = dx$

$\int_0^4 (u-1)\cdot\sqrt{u}\, du$ 1/2

$= \int_0^4 (u-1)\cdot u^{1/2}\, du$

$= \int_0^4 \left(u^{3/2} - u^{1/2}\right) du$

$= \left[\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right]_0^4$ 1/2

$= \left[\frac{2}{5}\cdot 4^{5/2} - \frac{2}{3}\cdot 4^{3/2}\right] - \left[\frac{2}{5}\cdot 0^{5/2} - \frac{2}{3}\cdot 0^{3/2}\right]$

$= \left[\frac{2\times 32}{5} - \frac{2\times 8}{3}\right]$

$= \frac{64}{5} - \frac{16}{3}$

$= \frac{112}{15}$ OR $7\frac{7}{15}$ 1/2

d) solve for n, $2\times {}^nC_4 = 5\times {}^nC_2$ −1/2 for n = −3

$\frac{2\times n!}{(n-4)!\,4!} = \frac{5\times n!}{(n-2)!\times 2!}$ 1

÷ by n! $\frac{2}{24}\cdot\frac{1}{(n-4)!} = \frac{5}{2}\cdot\frac{1}{(n-2)!}$

× by (n−4)! $\frac{2}{24} = \frac{5}{2}\cdot\frac{1}{(n-2)(n-3)}$

× by 2/5 $\frac{1}{30} = \frac{1}{(n-2)(n-3)}$

$\therefore n = 8.$ 1

Check:
$2\times {}^8C_4 = 140 = 5\times {}^8C_2$

✓

2e) circle $x^2+y^2+2x+4y=1$

$\therefore \quad (x+1)^2+(y+2)^2=6$

$\therefore$ centre $(-1,-2)$ radius $\sqrt{6}$ ½

line $3x+4y=6 \implies 3x+4y-6=0$

least distance between circle & line is the distance between the line & centre of the circle less the radius. ½

$d = \dfrac{|(3\times-1)+(4\times-2)-6|}{\sqrt{3^2+4^2}}$

$= \dfrac{|-3-8-6|}{5}$

$= \dfrac{|-17|}{5}$ ½

$\therefore$ minimum distance $= \dfrac{17}{5} - \sqrt{6}$ ½

17/5 1 mark only.

SECTION B QUESTION 3

a) $\sum_{\substack{1 \\ i\neq j\neq k}}^{4} (t_i t_j t_k)^{-1} = \frac{1}{t_1t_2t_3} + \frac{1}{t_2t_3t_4} + \frac{1}{t_3t_4t_1} + \frac{1}{t_4t_1t_2}$

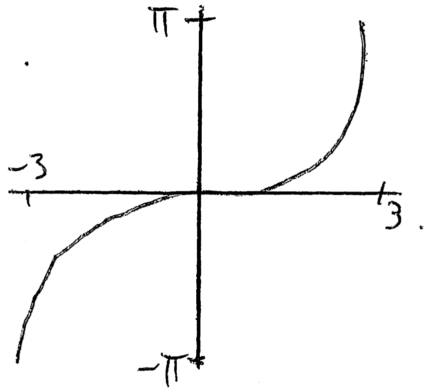
$$= \frac{t_1+t_2+t_3+t_4}{t_1t_2t_3t_4}$$

$t_1+t_2+t_3+t_4 = -\frac{b}{a}$
$= 2$

$t_1t_2t_3t_4 = \frac{e}{a}$
$= 1$

So $\sum_{\substack{1 \\ i\neq j\neq k}}^{4} (t_i t_j t_k)^{-1} = 2$.

b).



Domain: $-1 \le \frac{x}{3} \le 1$
$-3 \le x \le 3$.

Range: $-\frac{\pi}{2} \le \sin^{-1}\left(\frac{x}{3}\right) \le \frac{\pi}{2}$.

$-\pi \le 2\sin^{-1}\left(\frac{x}{3}\right) \le \pi$

$-\pi \le y \le \pi$.

c i) $LHS = \frac{dT}{dt}$ $\qquad$ $RHS = -k(T+5)$

$$= -kAe^{-kt}$$

$$= -k(Ae^{-kt} - 5 + 5)$$
$$= -kAe^{-kt}$$
$$= LHS.$$

Initial conditions

$$100 = Ae^{-k0} - 5$$
$$A = 105.$$

ii) After 20 minutes

$$40 = 105e^{-20k} - 5.$$

$$\frac{45}{105} = e^{-20k}$$

$$-20k = \ln\frac{3}{7}$$

$$k = \frac{\ln\frac{3}{7}}{-20}.$$

At 0°C.

$$0 = 105e^{-kt} - 5.$$

$$e^{-kt} = \frac{5}{105}$$

$$t = \frac{\ln\frac{1}{21}}{-k}.$$

$t = 72$ minutes.

d i) Since $f(x) = \ln x + x^2 - 4x$ is a continuous function and

$$f(3) = \ln 3 + 3^2 - 4 \times 3$$
$$\approx -1.9 < 0$$

and
$$f(4) = \ln 4 + 4^2 - 4 \times 4.$$
$$\approx 1.4 > 0$$

Therefore $f(x) = \ln x - x^2 - 4x$ must have a root between $x = 3$, $x = 4$.

ii) $f'(x) = \frac{1}{x} + 2x - 4.$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

$$x_0 = 4$$

$$x_1 = 4 - \frac{\ln 4 + 16 - 16}{\frac{1}{4} + 8 - 4}$$

$$\simeq 3.67.$$

Yes, since we know $f(x)$ has a root between 3 and 4 and this approximation is close to 3 than than the first approximation of 4.

QUESTION 4.

i). $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$.

At $P(2ap, ap^2)$

$m_P = \frac{2ap}{2a}$

$= p$.

ii) Similarly to part (i) the tangent at Q is $m_Q = q$.

Thus the gradient of the normal will be $m = -\frac{1}{q}$.

Given that the chord goes through the focus $(0, a)$, then

$a = \left(\frac{p+q}{2}\right)0 - apq$.

$pq = -1$

$\therefore q = -\frac{1}{p}$.

Thus the gradient of the normal at Q will be $-\frac{1}{q} = -\frac{1}{\left(-\frac{1}{p}\right)}$

$= p.$

$\therefore$ Tangent at P is parallel to the normal at Q.

b) i) $\binom{4}{3}\binom{3}{1}\binom{2}{1} = 24.$

ii) $n(\text{Sample space}) = \binom{9}{5} = 126.$

$n(e) = n(4\ \text{l.b}) + n(3\ \text{l.b})$

$= \binom{4}{4}\binom{5}{1} + \binom{4}{3}\binom{5}{2}.$

$= 5 + 40$

$= 45.$

$P(e) = \frac{n(e)}{n(S)}.$

$= \frac{45}{126}$

$= \frac{5}{14}.$

c i) $R = \sqrt{49+1}$

$= 5\sqrt{2}$

$\tan\alpha = \frac{1}{7}$

$\alpha = 8^\circ 8'$

So $7\cos\theta - \sin\theta = 5\sqrt{2}\cos(\theta + 8^\circ 8')$.

ii) $5\sqrt{2}\cos(\theta + 8°8') = 5$

$\cos(\theta + 8°8') = \frac{1}{\sqrt{2}}$

$\theta + 8°8' = 45°, 315°$

$\theta \approx 37°, 307°$

d) $P(-1) = (-1)^3 - 3(-1)^2 + a(-1) + b$

$= -4 - a + b$

$P(3) = (3)^3 - 3(3)^2 + (3)a + b$

$= 3a + b$

So $-4 - a + b = 0$ (A)

and $3a + b = 0$ (B)

from (A) $a = b - 4$

sub into (B)

$3b - 12 + b = 0$

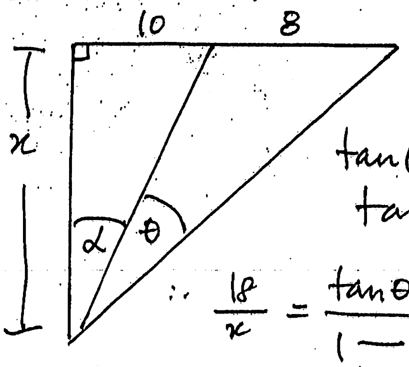
$b = 3$

so $a = -1$

$\therefore a = -1, b = 3$

Question (5)

(i)



$\tan(\theta+\alpha) = \frac{18}{x}$

$\tan\alpha = \frac{10}{x}$

$\therefore \frac{18}{x} = \frac{\tan\theta + \frac{10}{x}}{1 - \tan\theta \cdot \frac{10}{x}}$

$(\tan\theta)x^2 - 8x + 180\tan\theta = 0$ 2

$\tan\theta = \frac{8x}{180+x^2}$ [2]

$\therefore \theta = \tan^{-1}\left(\frac{8x}{180+x^2}\right)$

$\frac{d\theta}{dx} = \frac{(180+x^2)8 - 8x(2x)}{(x^2+180)^2}$ 1

i.e. $\frac{1440-8x^2}{(x^2+180)^2} = 0.$ [2]

$8(x^2-180) = 0, \quad x^2 = 180$

$x = 6\sqrt{5}.$

Test: 1

θ	13	$6\sqrt{5}$	14
$\frac{d\theta}{dx}$	+	0	$\underline{-1568}$

(b) $v = 2(2x-1)^{\frac{1}{2}}$.

(i) $v^2/2 = 4x - 2$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4.$ [1] 1

(ii) $\frac{dx}{dt} = 2(2x-1)^{\frac{1}{2}}$

$\frac{dt}{dx} = \frac{1}{2\sqrt{2x-1}}$

$\therefore t = \frac{1}{2}(2x-1)^{\frac{1}{2}} + c$

When $t=0$, $x = \frac{1}{2}$. $\Rightarrow c=0$.

$\therefore 2t = \sqrt{2x-1}$ 2

$1 + 4t^2 = 2x.$

$\therefore x = \frac{1+4t^2}{2}$ [2]

$= 2t^2 + \frac{1}{2}.$

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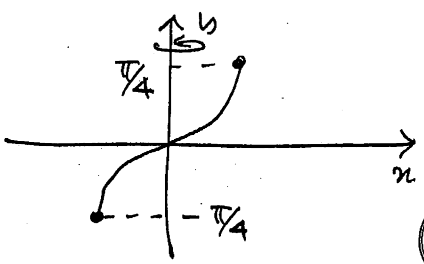
(c)



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$\left(\sin^2 y = \frac{1-\cos 2y}{2}\right)$ 1

$V = 2\pi \int_0^{\pi/4} \sin^2 y \, dy.$

$= \pi \int_0^{\pi/4} (1 - \cos 2y)\, dy$

$= \pi \left[y - \frac{\sin 2y}{2}\right]_0^{\pi/4}$

$= \pi\left(\frac{\pi}{4} - \frac{1}{2}\right)$ [3]

(d) $\frac{dp}{dt} = 3, \quad p = 2\pi r$

$\therefore r = \frac{p}{2\pi}.$

$A = \pi\left(\frac{p^2}{4\pi^2}\right) = \frac{p^2}{4\pi}$

$\frac{dA}{dt} = \frac{dA}{dp} \cdot \frac{dp}{dt} = \frac{p}{2\pi} \times 3.$

$p = 100$ 2

$\therefore \frac{dA}{dt} = \frac{150}{\pi}$ cm²/s

$= \frac{0.015}{\pi}$ m²/s. [2]

Question (6)

6 (a) $n=1$, $6 \nmid 8$.

(i) [1]

(ii) For $n=1$, $7+5=12$ and $6 \mid 12$. 1

Assume $S(k)$ is true

i.e. $7^k + 5 = 6M$, $M \in \mathbb{N}$

Consider $n = k+1$

$7^{k+1} + 5$ 2

$= 7 \cdot 7^k + 5$ [2]

$= 7(7^k+5) - 30$

$= 42M - 30 = 6(7M-5)$

$= 6N$ where $N = 7M - 5 \in \mathbb{N}$.

$\because S(k+1)$ is true when $S(k)$ is true and $S(1)$ is true

$\therefore$ by mathematical induction $6 \mid (7^n + 5) \quad \forall n \in \mathbb{N}$

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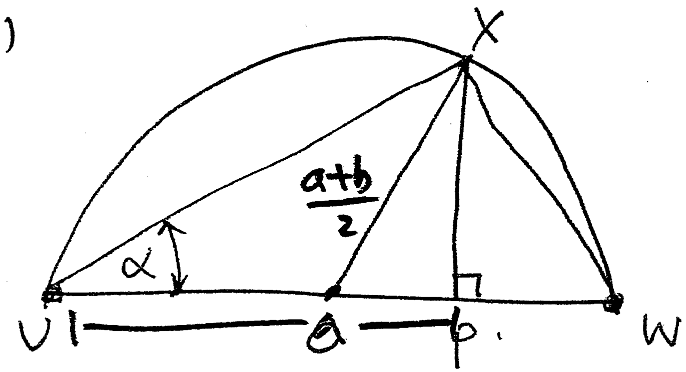
(b)



$OX = \frac{1}{2} UW = a+b.$

(radius = ½ diameter)

$\therefore OX = \frac{a+b}{2}$ —— (i) [1]

(ii) Let $\angle XUP = \alpha.$

$\angle UXW = 90^\circ$ (Angle in a semi circle)

$\therefore \angle XWP = 90^\circ - \alpha.$

In ΔUXP, ΔXWP.

$\angle XWP = 90 - \alpha \Rightarrow \angle XWP = \alpha$

(Angle sum of ΔXPW.

$\therefore \Delta UXP \ ||| \ \Delta XWP$ [1]

(equiangular

In ΔXPU

$XP^2 + a^2 = XU^2$ $\quad \frac{XP}{PW} = \frac{UP}{XP}$

$(XP)^2 = a.b.$

In ΔXWP

$XP^2 + b^2 = XW^2.$

In ΔUXW

$(a+b)^2 = XU^2 + XW^2$

$a^2+b^2+2ab = (a^2+b^2) + 2XP^2$

$\therefore XP^2 = ab$ [1]

$\Longrightarrow XP = \sqrt{ab}.$

In ΔOXP $|OX| > XP$

(OX is the hypotenuse) [1]

$\therefore \frac{a+b}{2} > \sqrt{ab}.$

(c) $\dot{x} = -15 \sin(3t - \frac{\pi}{6})$

(i) $\ddot{x} = -45 \cos(3t - \frac{\pi}{6})$

$= -9 [5 \cos(3t - \frac{\pi}{6})]$

$= -9x$ [1]

(ii) $T = \frac{2\pi}{n} = \frac{2\pi}{3}.$ [1]

(d) (i) $a_k = \binom{12}{k} 5^{12-k} 2^k.$ [1]

(ii) $\frac{a_{k+1}}{a_k} = \frac{\binom{12}{k+1} 5^{11-k} 2^{k+1}}{\binom{12}{k} 5^{12-k} 2^k} = \frac{2}{5} \frac{(12-k)}{k+1}$ [2]

QUESTION 7

(i) $t = \dfrac{x}{v\cos\vartheta}$

$$y = -\frac{gx^2}{2v^2\cos^2\vartheta} + \frac{vx\sin\vartheta}{v\cos\vartheta}$$

$$y = x\tan\vartheta - \frac{gx^2}{2v^2\cos^2\vartheta}$$

(ii)
At P, $y = k = h\tan\vartheta, x = h$

$$h\tan\beta = h\tan\vartheta - \frac{gh^2}{2v^2\cos^2\vartheta} \text{ from (i)}$$

$$\frac{gh^2}{2v^2\cos^2\vartheta} = h(\tan\vartheta - \tan\beta)$$

$$h = \frac{(\tan\vartheta - \tan\beta)2v^2\cos^2\vartheta}{g}$$

(iii)

$$OP = \frac{h}{\cos\beta}$$

$$= \frac{(\tan\vartheta - \tan\beta)2v^2\cos^2\vartheta}{g\cos\beta} \text{[from (ii)]}$$

$$= \frac{\left(\frac{\sin\vartheta}{\cos\vartheta} - \frac{\sin\beta}{\cos\beta}\right)2v^2\cos^2\vartheta}{g\cos\beta}$$

$$= \frac{(\sin\vartheta\cos\beta - \sin\beta\cos\vartheta)2v^2\cos\vartheta}{g\cos^2\beta}$$

$$= \frac{2v^2\sin(\vartheta - \beta)\cos\vartheta}{g\cos^2\beta}$$

(iv)

$$\text{OP} = \frac{[\sin(2\vartheta - \beta) - \sin\beta]v^2}{g\cos^2\beta} \text{ (given)}$$

$$\frac{d(OP)}{(d\vartheta)} = \frac{2v^2}{g\cos^2\beta}[2\cos(2\vartheta - \beta)]$$

OP max/min $\cos(2\vartheta - \beta) = 0$

$2\vartheta - \beta = 90^0$

$$\vartheta = \frac{90^0 + \beta}{2}$$

$$OP'' = \frac{4v^2}{g\cos^2\beta} \times -2\sin(2\vartheta - \beta)$$

always < 0 as $(2\vartheta - \beta) < 180^0$

$\therefore$ max val OP when $\vartheta = \dfrac{90^0 + \beta}{2}$

$$\text{Max val. OP} = \frac{v^2(\sin 90^0 - \sin\beta)}{g(1 - \sin^2\beta)}$$

$$= \frac{v^2(1 - \sin\beta)}{g(1 - \sin^2\beta)}$$

$$= \frac{v^2}{g(1 + \sin\beta)}$$

(v)

max val OP when $\vartheta = \dfrac{90^0 + \beta}{2}$ [from (iv)]

$$\vartheta = \frac{90^0 + 14^0}{2}$$

$$\vartheta = 52^0$$