Independent Trial HSC 2007 Mathematics Extension 2 Marking Guidelines

Question 1

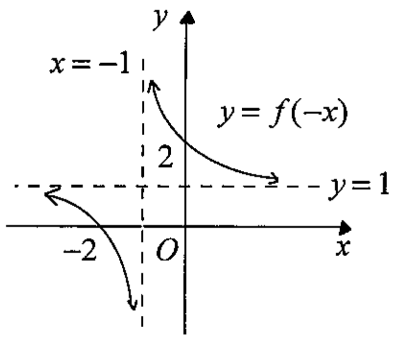
a. Outcomes assessed : E6

Marking Guidelines

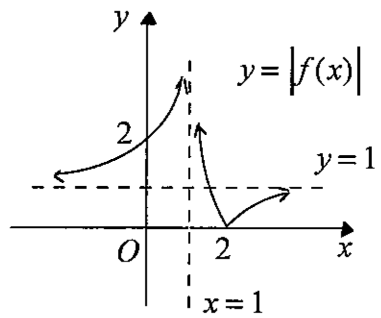
Criteria	Marks
i • sketches reflection in the y-axis showing intercepts on the axes and asymptotes	1
ii • reflects section of curve lying below the x-axis in x-axis, retaining asymptotes and intercepts	1
iii • sketches $y = f(x)$, $x \geq 0$ and its reflection in the y-axis	1
• shows all intercepts and asymptotes	1
iv • sketches left hand branch of curve showing asymptotes and intercept on y-axis	1
• sketches right hand branch of curve showing asymptote and nature of curve near $(1, 0)$	1

Answer

i.



ii.



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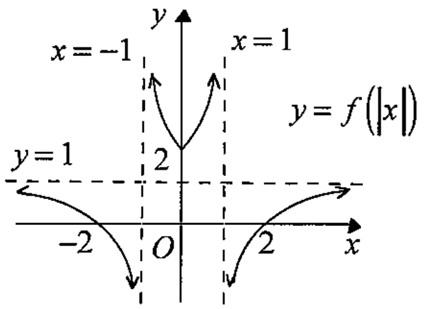

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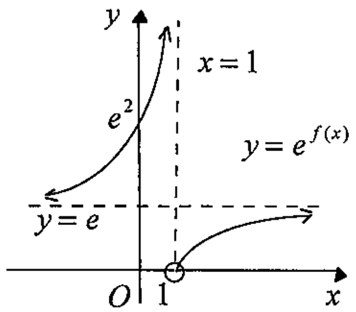


b. Outcomes assessed : E6

Marking Guidelines

Criteria	Marks
i • uses differentiation to find one expression for gradient of tangent OP	1
• uses coordinates of O and P to find a second expression for gradient of interval OP	1
ii • solves the quadratic equation for x_1	1
• substitutes for x_1 to find the two values for m	1

Answer

i. $y=\dfrac{x-2}{x-1}$

$y=1-\dfrac{1}{x-1}$

$\dfrac{dy}{dx}=\dfrac{1}{(x-1)^2}$

Since OP is tangent to the parabola at $P(x_1\,,y_1)$, gradient of OP is $\dfrac{1}{(x_1-1)^2}$.

Also gradient of OP is $\dfrac{y_1}{x_1}=\dfrac{x_1-2}{x_1(x_1-1)}$.

Hence $\dfrac{1}{(x_1-1)^2}=\dfrac{x_1-2}{x_1(x_1-1)}$

$(x_1-1)(x_1-2)=x_1$

$x_1^{\,2}-4x_1+2=0$

ii.

$x_1^{\,2}-4x_1+2=0$

$(x_1-2)^2=2$

$x_1-2=\pm\sqrt{2}$

$x_1-1=1\pm\sqrt{2}$

$\therefore m=\dfrac{1}{\left(1+\sqrt{2}\right)^2}=\dfrac{\left(1-\sqrt{2}\right)^2}{1^2}=3-2\sqrt{2}$

or $m=\dfrac{1}{\left(1-\sqrt{2}\right)^2}=\dfrac{\left(1+\sqrt{2}\right)^2}{1^2}=3+2\sqrt{2}$

c. Outcomes assessed : E6

Marking Guidelines

Criteria	Marks
i • applies product rule	1
• uses trigonometric identity to simplify derivative	1
ii • sketches a rising curve with endpoint at $(0\,,0)$	1
• shows vertical asymptote at $x=\frac{\pi}{2}$	1
• shows coordinates of point of horizontal inflexion	1

Answer

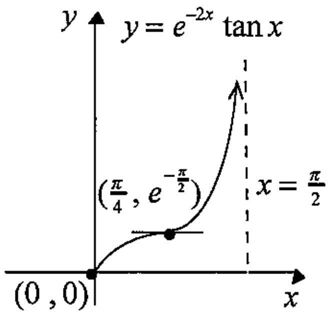
i. $y=e^{-2x}\tan x\,,\quad 0\le x<\frac{\pi}{2}$

$\dfrac{dy}{dx}=-2e^{-2x}\tan x+e^{-2x}\sec^2 x$

$=e^{-2x}\left\{-2\tan x+(1+\tan^2 x)\right\}$

$=e^{-2x}(1-\tan x)^2$

ii. $\dfrac{dy}{dx}=0$ for $x=\dfrac{\pi}{4}$ and $\dfrac{dy}{dx}>0$ for all other x values in the domain. Hence $\left(\frac{\pi}{4}\,,e^{-\frac{\pi}{2}}\right)$ is a point of horizontal inflexion on a rising curve.



Question 2

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • rearranges integrand and finds primitive	1
ii • rationalises denominator	1
• finds primitive	1

Answer

i.

$$\int \frac{1+e^x}{1+e^{-x}}\,dx = \int \frac{e^x(e^{-x}+1)}{1+e^{-x}}\,dx$$
$$= \int e^x\,dx$$
$$= e^x + c$$

ii.

$$\int \frac{1}{\sqrt{1+x}+\sqrt{x}}\,dx = \int \frac{\sqrt{1+x}-\sqrt{x}}{\left(\sqrt{1+x}+\sqrt{x}\right)\left(\sqrt{1+x}-\sqrt{x}\right)}\,dx$$
$$= \int \frac{\sqrt{1+x}-\sqrt{x}}{(1+x)-x}\,dx$$
$$= \tfrac{2}{3}(1+x)^{\frac{3}{2}} - \tfrac{2}{3}x^{\frac{3}{2}} + c$$
$$= \tfrac{2}{3}\left\{(1+x)\sqrt{1+x} - x\sqrt{x}\right\} + c$$

b. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• writes dx in terms of $d\theta$ simplifies integrand in terms of θ	1
• finds primitive in terms of θ	1
• finds primitive in terms of x	1

Answer

$x = \sin\theta,\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = \cos\theta\; d\theta$

$$\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx = \int \frac{1}{(\cos^2\theta)^{\frac{3}{2}}}\cos\theta\,d\theta = \int \sec^2\theta\,d\theta$$
$$= \tan\theta + c$$
$$= \frac{x}{\sqrt{1-x^2}} + c$$

c. Outcomes assessed : E8

Marking Guidelines

Criteria	Marks
• expresses integrand as a sum of partial fractions	1
• finds logarithm part of primitive	1
• finds both inverse tan parts of primitive	1
• evaluates by substitution of limits	1

Answer

$$\frac{x^3-8x^2+9x}{(1+x^2)(9+x^2)} \equiv \frac{ax+b}{(1+x^2)} + \frac{cx+d}{(9+x^2)}$$
$$x^3-8x^2+9x \equiv (ax+b)(9+x^2)+(cx+d)(1+x^2)$$

Equating coefficients of x^3: $a+c=1$

Equating coefficients of x: $9a+c=9$

$\therefore a=1,\ c=0$

Equating coefficients of x^2: $b+d=-8$

Putting $x=0$: $9b+d=0$

$\therefore b=1,\ d=-9$

$$\therefore \frac{x^3-8x^2+9x}{(1+x^2)(9+x^2)} \equiv \frac{x+1}{1+x^2} + \frac{-9}{9+x^2}$$

$$\begin{aligned}\int_0^{\sqrt{3}} \frac{x^3-8x^2+9x}{(1+x^2)(9+x^2)}\,dx &= \int_0^{\sqrt{3}} \left\{\frac{1}{2}\left(\frac{2x}{1+x^2}\right) + \frac{1}{1+x^2} - 3\left(\frac{3}{9+x^2}\right)\right\} dx \\ &= \left[\tfrac{1}{2}\ln(1+x^2) + \tan^{-1}x - 3\tan^{-1}\tfrac{x}{3}\right]_0^{\sqrt{3}} \\ &= \tfrac{1}{2}(\ln 4 - \ln 1) + (\tan^{-1}\sqrt{3} - \tan^{-1}0) - 3(\tan^{-1}\tfrac{1}{\sqrt{3}} - \tan^{-1}0) \\ &= \ln 2 + \tfrac{\pi}{3} - 3\times\tfrac{\pi}{6} \\ &= \ln 2 - \tfrac{\pi}{6}\end{aligned}$$

d. Outcomes assessed : HE6, E8

Marking Guidelines

Criteria	**Marks**
i • converts dx to dt, x limits to t limits and writes $\sin x$ in terms of t	1
• finds primitive in terms of t and substitutes limits	1
ii • converts integral of $f(x)$ between $\frac{1}{2}a$ and a into integral of $f(a-x)$ between 0 and $\frac{1}{2}a$	1
• completes proof of required result	1
iii • applies result to evaluate given definite integral	1

Answer

i.

$$t = \tan\tfrac{x}{2}$$
$$dt = \tfrac{1}{2}\sec^2\tfrac{x}{2}\,dx$$
$$2\,dt = (1+t^2)\,dx$$
$$dx = \frac{2}{1+t^2}\,dt$$

$$x = 0 \Rightarrow t = 0$$
$$x = \tfrac{\pi}{2} \Rightarrow t = 1$$
$$1+\sin x = 1 + \frac{2t}{1+t^2} = \frac{1+t^2+2t}{1+t^2}$$
$$\frac{1}{1+\sin x} = \frac{1+t^2}{(1+t)^2}$$

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx &= \int_0^1 \frac{1+t^2}{(1+t)^2}\cdot\frac{2}{1+t^2}\,dt \\ &= -2\left[\frac{1}{1+t}\right]_0^1 \\ &= -2\left(\tfrac{1}{2}-1\right) \\ &= 1\end{aligned}$$

ii.

$$u = a - x$$
$$du = -dx$$
$$x = \tfrac{a}{2} \Rightarrow u = \tfrac{a}{2}$$
$$x = a \Rightarrow u = 0$$

$$\begin{aligned}\int_{\frac{a}{2}}^{a} f(x)\,dx &= \int_{\frac{a}{2}}^{0} f(a-u)\,.-du \\ &= \int_0^{\frac{a}{2}} f(a-u)\,du \\ &= \int_0^{\frac{a}{2}} f(a-x)\,dx\end{aligned}$$

$$\begin{aligned}\therefore \int_0^a f(x)\,dx &= \int_0^{\frac{a}{2}} f(x)\,dx + \int_{\frac{a}{2}}^{a} f(x)\,dx \\ &= \int_0^{\frac{a}{2}} \{f(x) + f(a-x)\}\,dx\end{aligned}$$

iii. $$\int_0^{\pi} \frac{x}{1+\sin x}\,dx = \int_0^{\frac{\pi}{2}} \left\{\frac{x}{1+\sin x} + \frac{\pi - x}{1+\sin(\pi - x)}\right\} dx = \int_0^{\frac{\pi}{2}} \left\{\frac{x}{1+\sin x} + \frac{\pi - x}{1+\sin x}\right\} dx$$

$$\therefore \int_0^{\pi} \frac{x}{1+\sin x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\pi}{1+\sin x}\,dx = \pi\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx = \pi$$

Question 3

a. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • writes expansion, simplifying powers of i	1
ii • writes equation for a	1
• writes three solutions for a	1

Answer

i. $(1+ia)^4 = 1+4ia-6a^2-4ia^3+a^4$

ii. $(1+ia)^4$ is real if $4a-4a^3=0$. Then $a(1-a^2)=0$. $\therefore a=0,\ 1,\ -1$

b. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • uses trigonometric identities to complete the square or find the discriminant	1
• solves for z using the completed square or the quadratic formula	1
ii • uses de Moivre's theorem to write an expression for $(\cot\theta+i)^n$	1
• writes a similar expression for $(\cot\theta-i)^n$ to obtain the required result by addition	1

Answer

i. $\left(\sin^2\theta\right)z^2-\left(\sin 2\theta\right)z+1=0,\quad 0<\theta<\frac{\pi}{2}$

$$\left(\sin^2\theta\right)z^2-2\left(\sin\theta\cos\theta\right)z+\cos^2\theta=\cos^2\theta-1$$

$$\left\{(\sin\theta)\,z-\cos\theta\right\}^2=-\sin^2\theta$$

$$(\sin\theta)\,z-\cos\theta=\pm\, i\sin\theta$$

$$\therefore\ z-\frac{\cos\theta}{\sin\theta}=\pm i$$

$$z=\cot\theta\pm i$$

$\therefore$ equation has roots $\cot\theta+i$, $\cot\theta-i$

ii. $\alpha=\cot\theta+i=\dfrac{1}{\sin\theta}\left(\cos\theta+i\sin\theta\right)\ \Rightarrow\ \alpha^n=\dfrac{1}{\sin^n\theta}\left(\cos n\theta+i\sin n\theta\right)$ (by De Moivre's theorem)

Then $\beta=\cot\theta-i=\bar{\alpha}\ \Rightarrow\ \beta^n=\overline{\alpha^n}=\dfrac{1}{\sin^n\theta}\left(\cos n\theta-i\sin n\theta\right)$

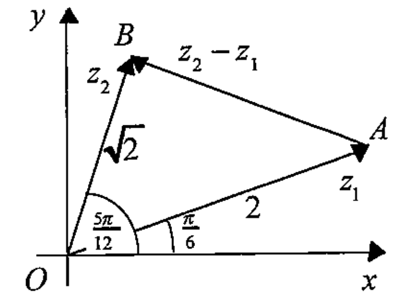
Hence $\alpha^n+\beta^n=\dfrac{2\cos n\theta}{\sin^n\theta}$

c. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • writes an expression for the square of AB using the cosine rule in triangle AOB	1
• deduces the value of $\lvert z_2-z_1\rvert$	1
ii • shows that $\angle OBA=\frac{\pi}{2}$	1
• uses rotation of vectors to deduce required result.	1

Answer



i. In ΔAOB, $AB^2 = 2+4-2\times\sqrt{2}\times 2\cos\frac{\pi}{4} = 2$

$\therefore |z_2 - z_1| = AB = \sqrt{2}$

ii. ΔAOB is isosceles, since $OB = AB = \sqrt{2}$

$\therefore \angle OAB = \angle AOB = \frac{\pi}{4}$, and hence $\angle OBA = \frac{\pi}{2}$.

$\therefore \overrightarrow{AB}$ is an anticlockwise rotation of $\overrightarrow{OB}$ by $\frac{\pi}{2}$.

But $\overrightarrow{AB}$ represents $z_2 - z_1$. $\therefore z_2 - z_1 = i\, z_2$

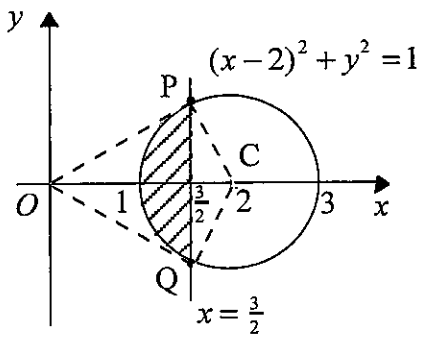
d. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • shades a region inside the appropriate circle	1
• shades region that also lies to the left of appropriate vertical line	1
ii • shows that tangents to circle from O have points of contact that lie in the shaded region	1
• deduces set of values of Arg z from the angles of inclination of these tangents	1

Answer

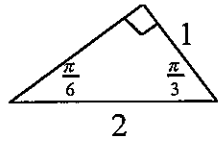
i.



ii.

$\cos\angle OCP = \dfrac{(\frac{1}{2})}{1} = \frac{1}{2}$ $\therefore \angle OCP = \frac{\pi}{3}$

Hence ΔOCP, ΔOCQ are congruent SAS to



Hence OP and OQ are tangents to the circle and P, Q represent z with max, min values of $\text{Arg}\, z$.

The set of values of $\text{Arg}\, z$ is $\left\{\theta : -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}\right\}$.

Question 4

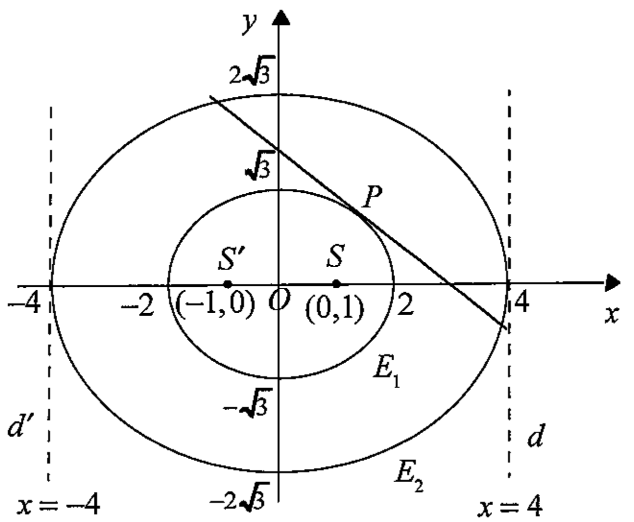
a. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • sketches E_1 with correct intercepts on axes	1
• sketches E_2 with correct intercepts on axes	1
• shows foci for E_1	1
• shows directrices for E_1	1
ii • finds gradient of tangent by differentiation	1
• writes expression for equation of tangent	1
• uses trig. identity to simplify this equation into required form	1
iii • writes equation for parameter t for point on E_2 where tangent to E_1 at P cuts E_2	1
• writes parameters at Q, R in terms of p to deduce result	1

Answer

i.



ii.

$$x = 2\cos p \qquad y = \sqrt{3}\sin p$$

$$\frac{dx}{dp} = -2\sin p \qquad \frac{dy}{dp} = \sqrt{3}\cos p$$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{3}\cos p}{-2\sin p}$$

Tangent at P has gradient $-\dfrac{\sqrt{3}\cos p}{2\sin p}$

and equation $(\sqrt{3}\cos p)x + (2\sin p)y = k$ for some constant k.

P on tangent $\Rightarrow 2\sqrt{3}\cos^2 p + 2\sqrt{3}\sin^2 p = k$

$$\therefore k = 2\sqrt{3}$$

$\therefore$ Tangent at P is $\dfrac{x\cos p}{2} + \dfrac{y\sin p}{\sqrt{3}} = 1$

iii. Tangent to E_1 at P meets E_2 at point $(4\cos t, 2\sqrt{3}\sin t)$ where

$$\frac{4\cos t\cos p}{2} + \frac{2\sqrt{3}\sin t\sin p}{\sqrt{3}} = 1$$

$$2(\cos t\cos p + \sin t\sin p) = 1$$

$$\cos(t-p) = \tfrac{1}{2}$$

Also $0 < p < \frac{\pi}{2}$ and $-\pi < t < \pi$ $\qquad \therefore t - p = \pm\frac{\pi}{3}$

Hence one of Q, R has parameter $p + \frac{\pi}{3}$, and the other has parameter $p - \frac{\pi}{3}$.

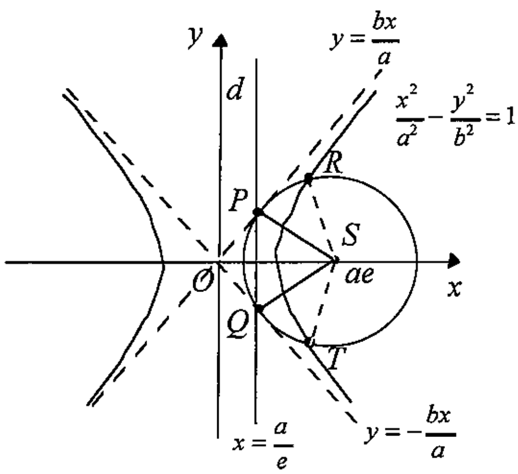
Hence q and r differ by $\frac{2\pi}{3}$.

b. Outcomes assessed : E3, E4

Marking Guidelines

Criteria	**Marks**
i • finds the gradient of PS in terms of a, b, e	1
• uses the relationship between a, b, e to deduce that PS and OP are perpendicular	1
• uses either distance formula to show that $PS = b$	1
ii • uses a geometric argument to deduce required result	1
iii • finds the x coordinates of R, T in terms of a, b, e	1
• shows that if $a = b$ then R, S, T each have x coordinate $a\sqrt{2}$ to deduce required result	1

Answer



i. S has coordinates $(ae\,,0)$ and d has equation $x=\frac{a}{e}$. At P,

$x=\frac{a}{e},\ y=\frac{b}{a}x \qquad \therefore P\left(\frac{a}{e},\frac{b}{e}\right)$ and

$$gradient\ PS=\frac{\left(\frac{b}{e}\right)}{\left(\frac{a}{e}-ae\right)}=\frac{b}{a(1-e^2)}$$

$\therefore$ product of gradients of PS and OP is $\dfrac{b^2}{a^2(1-e^2)}=\dfrac{b^2}{-b^2}=-1$.

Hence $PS\perp OP$.

Also
$$\begin{aligned}PS^2&=\left(\tfrac{a}{e}-ae\right)^2+\left(\tfrac{b}{e}\right)^2\\&=\tfrac{1}{e^2}\left\{a^2(1-e^2)^2+b^2\right\}\\&=\tfrac{1}{e^2}\left\{-b^2(1-e^2)+b^2\right\}\\&=b^2\end{aligned}$$
$\therefore PS=b$

ii. Since the perpendicular distance of S from the asymptote OP is b, the circle with centre S that touches this asymptote has radius b and point of contact P, since $PS=b$ and $PS\perp OP$.
By symmetry, $QS\perp OQ$ and $QS=b$.
Hence Q is the point of contact of this same circle (centre S, radius b) with the asymptote OQ.

iii. Since $SR=ST=b$, at R, T the locus definition of the hyperbola gives $b=e\left(x-\frac{a}{e}\right)$. $\quad\therefore x=\frac{a+b}{e}$

If $a=b$, $e^2=\dfrac{b^2}{a^2}+1=2$.

Then S has x coordinate $a\sqrt{2}$, and at R and T, $x=2\frac{a}{e}=a\sqrt{2}$. Hence if $a=b$, R, S and T are collinear and RT is the diameter of the circle.

Question 5
a. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
i • writes division transformation and expressions for $P(i)$, $P(-i)$ in terms of A and B	1
• solves simultaneous equations to find A and B in terms of $P(i)$ and $P(-i)$	1
ii • deduces B is zero	1
• deduces value of A and hence writes down remainder	1

Answer

i. $P(x)\equiv(x^2+1)\,.\,Q(x)+Ax+B$ for some polynomial $Q(x)$.

$P(i)=0\,.\,Q(i)+Ai+B \quad\Rightarrow\quad Ai+B=P(i) \quad (1)$

$P(-i)=0\,.\,Q(-i)-Ai+B \quad\Rightarrow\quad -Ai+B=P(-i) \quad (2)$

$(1)-(2)\Rightarrow\ 2Ai=P(i)-P(-i) \qquad \therefore A=\dfrac{P(i)-P(-i)}{2i}$

$(1)+(2)\Rightarrow\ 2B=P(i)+P(-i) \qquad \therefore B=\dfrac{P(i)+P(-i)}{2}$

ii. $P(x)$ odd $\Rightarrow P(-i)=-P(i) \qquad \therefore A=\dfrac{P(i)}{i}$ and $B=0$. $\quad$ Hence remainder is $\dfrac{P(i)}{i}x$

b. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
i • notes that expression is continuous and shows it changes sign between $x=0$ and $x=1$	1
ii • finds required monic polynomial equation	1
• deduces the value of the sum of squares of $\alpha,\ \beta,\ \gamma,\ \delta$ from the coefficient of x^3	1
iii • deduces that not all the roots are real	1
• explains why there are exactly two non-real roots	1

Answer

i. Let $f(x)=x^4-5x+2$
Then $f(0)=2>0$ and $f(1)=-2<0$.
Since f is continuous, $f(x)=0$ for some real x such that $0<x<1$.
Hence $x^4-5x+2=0$ has a real root between $x=0$ and $x=1$.

ii. $\alpha^2,\ \beta^2,\ \gamma^2,\ \delta^2$ are roots of $(x^{\frac{1}{2}})^4-5(x^{\frac{1}{2}})+2=0$

$$x^2+2=5x^{\frac{1}{2}}$$
$$(x^2+2)^2=25x$$
$$x^4+4x^2-25x+4=0$$

Since for this equation the coefficient of x^3 is 0, $\alpha^2+\beta^2+\gamma^2+\delta^2=0$.

iii. Let α be the real root which satisfies $0<\alpha<1$.
Then $\beta^2+\gamma^2+\delta^2=-\alpha^2<0$. Hence not all of $\beta,\ \gamma,\ \delta$ are real.
Since the coefficients of $x^4-5x+2=0$ are real, the complex conjugate of any non-real root is also a root of the equation. Hence two of $\beta,\ \gamma,\ \delta$ are non-real complex conjugates, while he remaining root must then be real. Hence the equation has exactly two non-real roots.

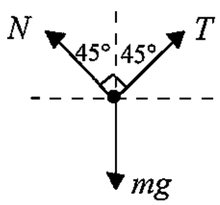
c. Outcomes assessed : E5

Marking Guidelines

Criteria	Marks
i • resolves forces on P, using zero vertical component of resultant to obtain first equation	1
• uses horizontal component of magnitude $mr\omega^2$ to obtain second equation	1
ii • writes required expression for N	1
• writes required expression for $l\omega^2$	1
iii • uses $N\ge 0$ to deduce upper limit	1
• uses $l\omega^2\ge 0$ to deduce lower limit	1

Answer

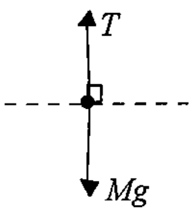
Forces on P



i.

Since P is travelling with constant angular velocity in a horizontal circle, the resultant force on P is directed horizontally toward the centre of the circle with magnitude $mr\omega^2$. Hence resolving horizontally and vertically,

$$T\cos 45°+N\cos 45°=mg \quad\Rightarrow\quad T+N=mg\sqrt{2}$$
$$T\sin 45°-N\sin 45°=mr\omega^2 \quad\Rightarrow\quad T-N=ml\omega^2 \qquad (since\ \ r=l\sin 45°)$$



ii.

Since Q is in equilibrium, $T = Mg$

$\therefore N = mg\sqrt{2} - Mg$ and $ml\omega^2 = Mg - N \Rightarrow l\omega^2 = 2g\dfrac{M}{m} - g\sqrt{2}$

iii. $N = mg\left(\sqrt{2} - \dfrac{M}{m}\right)$ and $l\omega^2 = 2g\left(\dfrac{M}{m} - \dfrac{\sqrt{2}}{2}\right)$

But $N \geq 0$ and $l\omega^2 \geq 0$. Hence $\dfrac{\sqrt{2}}{2} \leq \dfrac{M}{m} \leq \sqrt{2}$.

Question 6 H5, E7

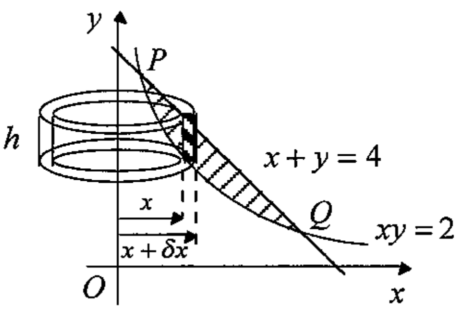
a. Outcomes assessed :

Marking Guidelines

Criteria	Marks
i • finds x coordinates at P and Q	1
• finds volume of cylindrical shell in terms of x and δx	1
• writes V as limiting sum and hence as integral	1
ii • finds a primitive function (with or without an appropriate substitution)	1
• substitutes limits	1
• simplifies exact numerical value	1

Answer

i.



At P, Q $\quad x + y = 4$ and $xy = 2$

$$\therefore \quad x^2 + xy = 4x$$
$$x^2 + 2 = 4x$$
$$x^2 - 4x = -2$$
$$(x-2)^2 = 2$$
$$\therefore \quad x = 2 \pm \sqrt{2}$$

Cylindrical shell has volume

$$\delta V = \pi\left\{(x+\delta x)^2 - x^2\right\}h$$
$$= \pi\,(2x+\delta x)\,\delta x\left\{(4-x) - \frac{2}{x}\right\}$$
$$= \pi\,\frac{(2x+\delta x)\,\delta x}{x}\,(4x - x^2 - 2)$$

Then, ignoring second order terms in $(\delta x)^2$,

$$V = \lim_{\delta x \to 0} \sum_{2-\sqrt{2}}^{2+\sqrt{2}} 2\pi\,(4x - x^2 - 2)\,\delta x$$
$$= 2\pi \int_{2-\sqrt{2}}^{2+\sqrt{2}} (4x - x^2 - 2)\,dx$$

ii. $V = 2\pi \int_{2-\sqrt{2}}^{2+\sqrt{2}} \left\{ 2-(x-2)^2 \right\} dx$.

Making the substitution $u = x-2$,

$du = dx$

$x = 2-\sqrt{2} \Rightarrow u = -\sqrt{2}$

$x = 2+\sqrt{2} \Rightarrow u = \sqrt{2}$

$$\begin{aligned} V &= 2\pi \int_{-\sqrt{2}}^{\sqrt{2}} (2-u^2)\, du \\ &= 4\pi \int_0^{\sqrt{2}} (2-u^2)\, du \\ &= 4\pi \left[2u - \tfrac{1}{3}u^3 \right]_0^{\sqrt{2}} \\ &= 4\pi \left\{ 2\sqrt{2} - \tfrac{1}{3}\left(2\sqrt{2}\right) \right\} \\ &= \frac{16\pi\sqrt{2}}{3} \end{aligned}$$

b. Outcomes assessed : HE6, E8

Marking Guidelines

Criteria	Marks
i • makes substitution	1
• uses integration by parts	1
• rearranges new integrand into powers of $(\frac{1}{4}-u^2)$	1
• obtains required reduction formula	1
ii • uses reduction formula to find expression for I_5 in terms of I_0	1
• evaluates and rearranges into required form	1
iii • expresses dx in terms of dt, and converts t limits to x limits	1
• uses trigonometric identities to convert integrand into required form	1
• generalises expression for I_5 to obtain similar expression for I_n and hence deduce result	1

Answer

i. $I_n = \int_0^1 x^n (1-x)^n \, dx \ , \quad n = 0, 1, 2, \ldots$

$u = \frac{1}{2} - x$

$du = -dx$

$x = 0 \Rightarrow u = \frac{1}{2}$

$x = 1 \Rightarrow u = -\frac{1}{2}$

$$\begin{aligned} x(1-x) &= (\tfrac{1}{2}-u)(\tfrac{1}{2}+u) \\ &= \tfrac{1}{4} - u^2 \end{aligned}$$

$$\begin{aligned} \therefore \ I_n &= \int_{\frac{1}{2}}^{-\frac{1}{2}} \left(\tfrac{1}{4}-u^2\right)^n . -du \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\tfrac{1}{4}-u^2\right)^n du \end{aligned}$$

Hence for $n = 1, 2, 3, \ldots$

$$\begin{aligned} I_n &= \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 . \left(\tfrac{1}{4}-u^2\right)^n du \\ &= \left[u\left(\tfrac{1}{4}-u^2\right)^n \right]_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} u \,.\, n\left(\tfrac{1}{4}-u^2\right)^{n-1} (-2u)\, du \\ &= 0 - 2n \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\tfrac{1}{4}-u^2-\tfrac{1}{4}\right) . \left(\tfrac{1}{4}-u^2\right)^{n-1} du \\ &= -2n \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\tfrac{1}{4}-u^2\right)^n du - \tfrac{1}{4} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\tfrac{1}{4}-u^2\right)^{n-1} du \right) \\ &= -2n I_n + \tfrac{1}{2} n I_{n-1} \end{aligned}$$

$$\begin{aligned} \therefore \ (2n+1) I_n &= \tfrac{1}{2} n I_{n-1} \\ I_n &= \frac{n}{2(2n+1)} I_{n-1} \end{aligned}$$

ii. $I_0 = \int_0^1 1 \, dx = 1$

$$I_5 = \frac{5}{2\times 11} I_4 = \frac{5}{2\times 11} . \frac{4}{2\times 9} I_3 = \ \ldots \ = \frac{5}{2\times 11} . \frac{4}{2\times 9} . \frac{3}{2\times 7} . \frac{2}{2\times 5} . \frac{1}{2\times 3} I_0$$

$$\therefore I_5 = \frac{5^2}{11\times(2\times 5)} . \frac{4^2}{9\times(2\times 4)} . \frac{3^3}{7\times(2\times 3)} . \frac{2^2}{5\times(2\times 2)} . \frac{1^2}{3\times 2\times 1} = \frac{(5\,!)^2}{11\,!}$$

iii.

$x = \sin^2(\frac{1}{2}t)$

$dx = \sin(\frac{1}{2}t)\cos(\frac{1}{2}t)\ dt$

$2dx = \sin t\ dt$

$t = 0 \Rightarrow x = 0$

$t = \pi \Rightarrow x = 1$

$$\begin{aligned}\sin^2 t &= 2^2\sin^2(\tfrac{1}{2}t)\cos^2(\tfrac{1}{2}t)\\ &= 2^2\sin^2(\tfrac{1}{2}t)\left\{1-\sin^2(\tfrac{1}{2}t)\right\}\\ &= 2^2\ x(1-x)\end{aligned}$$

Hence for $n = 0, 1, 2, ...$

$$\begin{aligned}\int_0^\pi \sin^{2n+1} t\ dt &= \int_0^\pi (\sin^2 t)^n \sin t\ dt\\ &= 2^{2n}\int_0^1 x^n(1-x)^n . 2dx\\ &= 2^{2n+1} I_n\end{aligned}$$

But for $n = 1, 2, ...$

$$\begin{aligned}I_n &= \frac{n}{2(2n+1)} \cdot \frac{n-1}{2(2n-1)} \cdot \frac{n-2}{2(2n-3)} \cdot \ \cdots \ \cdot \frac{1}{2(3)} I_0\\ &= \frac{n^2}{(2n+1)2n} \cdot \frac{(n-1)^2}{(2n-1)(2n-2)} \cdot \frac{(n-2)^2}{(2n-3)(2n-4)} \cdot \ \cdots \ \cdot \frac{1^2}{3\times 2\times 1}\\ &= \frac{(n!)^2}{(2n+1)!}\end{aligned}$$

and for $n = 0$

$$\begin{aligned}I_0 &= 1\\ &= \frac{(0!)^2}{(2\times 0+1)!}\end{aligned}$$

$$\therefore \int_0^\pi \sin^{2n+1} t\ dt = \frac{2^{2n+1}(n!)^2}{(2n+1)!}\ ,\quad n = 0, 1, 2, ...$$

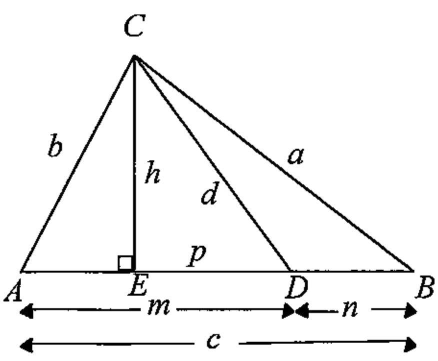
Question 7

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • writes appropriate expression from Pythagoras' theorem in each triangle	1
• eliminates h to obtain require result	1
ii • writes expression from applying Pythagoras' theorem to an appropriate third triangle	1
• eliminates h to obtain required result	1
iii • combines expressions from i. and ii., using $c = m + n$	1
iv • writes appropriate pair of equal ratios derived from sine rule for each specified triangle	1
• uses fact that $\angle CDB$, $\angle CDA$ are supplementary to deduce $\sin\angle CDB = \sin\angle CDA$	1
• deduces that $am = bn$ if CD bisects $\angle BCA$	1
• uses this equality and iii. to deduce required result	1

Answer



i. In ΔCEA, $b^2 = h^2 + (m-p)^2$. In ΔCED, $h^2 = d^2 - p^2$

$\therefore b^2 = (d^2 - p^2) + (m^2 - 2mp + p^2) = d^2 + m^2 - 2mp$

ii. In ΔCEB, $a^2 = h^2 + (p+n)^2$

$\therefore a^2 = (d^2 - p^2) + (n^2 + 2np + p^2) = d^2 + n^2 + 2np$

iii. $a^2 m + b^2 n = d^2(m+n) + n^2 m + m^2 n$

$\therefore a^2 m + b^2 n = (m+n)(d^2 + mn) = c(d^2 + mn)$

iv. In ΔCDA, $\dfrac{\sin\angle ACD}{m}=\dfrac{\sin\angle CDA}{b} \quad \Rightarrow \quad \dfrac{b}{m}=\dfrac{\sin\angle CDA}{\sin\angle ACD}$

In ΔCDB, $\dfrac{\sin\angle BCD}{n}=\dfrac{\sin\angle CDB}{a} \quad \Rightarrow \quad \dfrac{a}{n}=\dfrac{\sin\angle CDB}{\sin\angle BCD}$

But $\sin\angle CDB=\sin(180°-\angle CDA)=\sin\angle CDA$.
If CD bisects $\angle BCA$, $\angle BCD=\angle ACD$.
Then

$$\frac{a}{n}=\frac{b}{m}$$
$$\therefore am=bn$$

$$\begin{aligned} c(d^2+mn) &= a^2m+b^2n \\ &= a(am)+b(bn) \\ &= a(bn)+b(am) \\ &= ab(m+n) \\ &= abc \end{aligned}$$

$$\therefore d^2+mn=ab$$
$$d^2=ab-mn$$

b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • finds probability of either 1 head and $(n-1)$ tails, or of 1 tail and $(n-1)$ heads	1
• adds these to find the required probability	1
ii • uses the complementary event to find an expression for this probability	1
iii • finds the required probability	1
iv • states probability is zero for $N=1$, and considers how this event can occur if $N\ge 2$	1
• writes probability for $N\ge 2$	1

Answer

i. For $n=3,4,5,\ldots$, $P(odd\ one\ out)=P(1\ tail\ and\ n-1\ heads)+P(1\ head\ and\ n-1\ tails)$

$$\therefore P(odd\ one\ out)=n\times\tfrac{1}{2}\times(\tfrac{1}{2})^{n-1}+n\times\tfrac{1}{2}\times(\tfrac{1}{2})^{n-1}=\frac{n}{2^{n-1}}$$

ii. $P(at\ least\ one\ 'odd\ one\ out')=1-P(none)$

$$=1-\left(1-\frac{n}{2^{n-1}}\right)^N$$

iii. Require probability that '*odd one out*' does not occur during first $(N-1)$ plays, then does occur on N^{th} play. Hence probability is $\dfrac{n}{2^{n-1}}\left(1-\dfrac{n}{2^{n-1}}\right)^{N-1}$

iv. For $N=1$, this probability is clearly 0.
For $N\ge 2$, require exactly one '*odd one out*' occurs during first $(N-1)$ plays, then '*odd one out*' occurs again on N^{th} play.

Hence probability is

$$(N-1)\left(\frac{n}{2^{n-1}}\right)\left(1-\frac{n}{2^{n-1}}\right)^{N-2}\left(\frac{n}{2^{n-1}}\right)=(N-1)\left(\frac{n}{2^{n-1}}\right)^2\left(1-\frac{n}{2^{n-1}}\right)^{N-2}$$

Question 8

a. Outcomes assessed : HE2, HE3

Marking Guidelines

Criteria	Marks
i • defines a sequence of statements and shows the first is true	1
• writes $x_{k+1}-2$ in terms of x_k, factoring the cubic numerator	1
• deduces the truth of $S(k+1)$ conditional on the truth of $S(k)$, then completes the induction	1
ii • writes expression for $x_{n+1} \div x_n$	1
• shows $x_{n+1} \div x_n < 1$ for $x_n > 2$ to deduce result	1

Answer

i. $x_1 = 1$ and $x_{n+1} = \dfrac{2x_n^{\ 3}+8}{3x_n^{\ 2}}$, $n = 1, 2, 3, \ldots$

Define a sequence of statements $S(n)$, $n = 2, 3, 4, \ldots$ by $S(n)$: $x_n > 2$

Consider $S(2)$: $\quad x_2 = \dfrac{2x_1^{\ 3}+8}{3x_1^{\ 2}} = \dfrac{2+8}{3} > 2 \quad \therefore S(2)$ is true.

If $S(k)$ is true : $\quad x_k > 2 \quad$ **

Consider $S(k+1)$: $\quad x_{k+1} = \dfrac{2x_k^{\ 3}+8}{3x_k^{\ 2}}$

$$x_{k+1} - 2 = \frac{2x_k^{\ 3}+8-6x_k^{\ 2}}{3x_k^{\ 2}}$$

$$= \frac{2(x_k^{\ 3}-3x_k^{\ 2}+4)}{3x_k^{\ 2}}$$

$$= \frac{2(x_k+1)(x_k^{\ 2}-4x_k+4)}{3x_k^{\ 2}}$$

$$= \frac{2(x_k+1)(x_k-2)^2}{3x_k^{\ 2}}$$

>0 if $S(k)$ is true, using **

Hence if S(k) is true, then $S(k+1)$ is true. But $S(2)$ is true, hence S(3) is true, and then S(4) is true and so on. Hence $x_n > 2$ for $n = 2, 3, 4, \ldots$

ii. $x_{n+1} = \dfrac{2x_n^{\ 3}+8}{3x_n^{\ 2}}$, $n = 1, 2, 3, \ldots$

$$\frac{x_{n+1}}{x_n} = \frac{2x_n^{\ 3}+8}{3x_n^{\ 3}}$$

$$= \frac{2}{3} + \frac{1}{3} \times \left(\frac{2}{x_n}\right)^3$$

But $x_n > 2 \Rightarrow \dfrac{2}{x_n} < 1$ for $n = 2, 3, 4, \ldots$

Hence $\dfrac{x_{n+1}}{x_n} < \dfrac{2}{3} + \dfrac{1}{3} \times 1^3 = 1$ for $n = 2, 3, 4, \ldots$

$\therefore x_{n+1} < x_n$ for all positive integers $n \geq 2$

b. Outcomes assessed : H5, HE3

Marking Guidelines

Criteria	Marks
i • uses sum formula for geometric progression to simplify expression for $f'(x)$	1
• deduces required properties of $f(x)$	1
ii • explains why for $x>0$, $f(x)>0$ for even n and $f(x)<0$ for n odd	1
• identifies $2n$ as even and $2n-1$ as odd to deduce required result	1
iii • selects an appropriate value of n to evaluate $\ln(1\cdot 2)$ correct to two decimal places	1

Answer

i. $f(x)=\ln(1+x)-\left(x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\ldots+(-1)^{n-1}\dfrac{x^n}{n}\right)$

$$\begin{aligned} f'(x) &= \frac{1}{1+x} - \left(1-x+x^2-x^3\ldots+(-1)^{n-1}x^{n-1}\right)\\ &= \frac{1}{1+x} - \frac{\{1-(-x)^n\}}{1+x}\\ &= \frac{(-x)^n}{1+x}\end{aligned}$$

$\therefore f'(0)=0$ and $f(x)$ is stationary at $x=0$.

Also for $x>0$, when n is even, $f'(x)>0$ and f is monotonic increasing

when n is odd, $f'(x)<0$ and f is monotonic decreasing

ii. $f(0)=\ln 1-0=0$

Hence for $x>0$, if n is even, $f(x)>0$ since f is monotonic increasing

if n is odd, $f(x)<0$ since f is monotonic decreasing

Hence for $x>0$, if n is even, $\ln(1+x)>x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\ldots+(-1)^{n-1}\dfrac{x^n}{n}$

if n is odd, $\ln(1+x)<x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\ldots+(-1)^{n-1}\dfrac{x^n}{n}$

But for positive integers n, $2n$ is even and $(2n-1)$ is odd.

$$\therefore\ x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots-\frac{x^{2n}}{2n} < \ln(1+x) < x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+\frac{x^{2n-1}}{2n-1}$$

iii.

x	$x-\dfrac{x^2}{2}$	$x-\dfrac{x^2}{2}+\dfrac{x^3}{3}$
$0\cdot 2$	$0\cdot 18$	$0\cdot 182\dot{6}$

$\therefore 0\cdot 18<\ln(1\cdot 2)<0\cdot 182\dot{6}$

$\therefore\ \ln(1\cdot 2)\approx 0\cdot 18$ (correct to 2 decimal places)

c. Outcomes assessed : HE3, E9

Marking Guidelines

Criteria	**Marks**
i • writes a as infinite sum and deduces $a > 0$	1
• compares this sum with the sum of a geometric progression	1
• finds the limiting sum of this G.P. to deduce the required inequality for a	1
ii • if e rational, uses definition of a rational number to select a value of n for which a is integral	1
• argues by contradiction that e must be irrational	1

Answer

i. $e = \sum_{r=0}^{\infty} \frac{1}{r!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$

$$a = n!\left\{ e - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} \right) \right\}$$

$$= n!\left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \dots \right) \qquad *$$

$$= \frac{1}{n+1} + \frac{1}{(n+2)(n+1)} + \frac{1}{(n+3)(n+2)(n+1)} + \dots$$

$$< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \dots \qquad \text{(limiting sum of a G.P)}$$

$$= \frac{\left(\frac{1}{n+1}\right)}{1 - \left(\frac{1}{n+1}\right)} \qquad \text{(since } \left|\tfrac{1}{n+1}\right| < 1 \text{ for } n = 1, 2, 3, \dots)$$

$$= \frac{1}{(n+1) - 1}$$

$$= \frac{1}{n}$$

Clearly from * above, $a > 0$. $\therefore\ 0 < a < \frac{1}{n}$ for $n = 1, 2, 3, \dots$

ii. Suppose e is rational. Then $e = \frac{p}{q}$ for some positive integers p, q with no common factor.

Let $n = q$. Then $a = q!\left\{ \frac{p}{q} - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{q!} \right) \right\}$

Now each of $q,\ 2!,\ 3!, \dots, q!$ is a factor of $q!$.

Hence a must be an integer.

But there is no integer a such that $0 < a < \frac{1}{q}$

Hence e must be irrational.

Independent HSC Trial Examination 2007 Mathematics Extension 2 Mapping Grid

Question	Marks	Content	Syllabus Outcomes	Targeted Performance Bands
1 a i	1	Graphs	E6	E2-E3
a ii	1	Graphs	E6	E2-E3
a iii	2	Graphs	E6	E2-E3
a iv	2	Graphs	E6	E2-E3
b i	2	Graphs	E6	E2-E3
b ii	2	Graphs	E6	E2-E3
c i	2	Graphs	E6	E2-E3
c ii	3	Graphs	E6	E2-E3
2 a i	1	Integration	H5	E2-E3
a ii	2	Integration	H5	E2-E3
b	3	Integration	HE6	E2-E3
c	4	Integration	E8	E2-E3
d i	2	Integration	HE6	E2-E3
d ii	2	Integration	E8	E2-E3
d iii	1	Integration	E8	E2-E3
3 a i	1	Complex numbers	E3	E2-E3
a ii	2	Complex numbers	E3	E2-E3
b i	2	Complex numbers	E3	E2-E3
b ii	2	Complex numbers	E3	E2-E3
c i	2	Complex numbers	E3	E3-E4
c ii	2	Complex numbers	E3	E3-E4
d i	2	Complex numbers	E3	E2-E3
d ii	2	Complex numbers	E3	E2-E3
4 a i	4	Conics	E3	E2-E3
a ii	3	Conics	E3	E2-E3
a iii	2	Conics	E3	E2-E3
b i	3	Conics	E3, E4	E2-E3
b ii	1	Conics	E3, E4	E2-E3
b iii	2	Conics	E3, E4	E2-E3
5 a i	2	Polynomials	E4	E2-E3
a ii	2	Polynomials	E4	E2-E3
b i	1	Polynomials	E4	E2-E3
b ii	2	Polynomials	E4	E2-E3
b iii	2	Polynomials	E4	E2-E3
c i	2	Mechanics	E5	E3-E4
c ii	2	Mechanics	E5	E3-E4
c iii	2	Mechanics	E5	E3-E4
6 a i	3	Volumes	E7	E3-E4
a ii	3	Integration	H8	E2-E3
b i	4	Integration	HE6, E8	E3-E4
b ii	2	Integration	E8	E3-E4
b iii	3	Integration	HE6, E8	E3-E4

Question	Marks	Content	Syllabus Outcomes	Targeted Performance Bands
7 a i	2	Plane geometry	H5	E2-E3
a ii	2	Plane geometry	H5	E2-E3
a iii	1	Plane geometry	H5	E2-E3
a iv	4	Plane geometry	H5	E2-E3
b i	2	Probability	HE3	E3-E4
b ii	1	Probability	HE3	E3-E4
b iii	1	Probability	HE3	E3-E4
b iv	2	Probability	HE3	E3-E4
8 a i	3	Induction	HE2	E3-E4
a ii	2	Inequalities	HE3	E3-E4
b i	2	Differentiation	H5	E3-E4
b ii	2	Inequalities	HE3	E3-E4
b iii	1	Inequalities	HE3	E3-E4
c i	3	Inequalities	HE3	E3-E4
c ii	2	Inequalities	E9	E3-E4