2009.

Question 1.

a) $P(x) = x^4 - 3x^3 + ax^2 - 12$

$P(3) = 3^4 - 3.3^3 + a.3^2 - 12 = 0$ (1)

$\therefore 81 - 81 + 9a - 12 = 0$

$9a = 12$

$a = \frac{12}{9} = \frac{4}{3}$ (1)

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b) $y = x^3$ $\qquad y = 1 - \ln x$

$y' = 3x^2$ $\qquad y' = \frac{-1}{x}$ (2)

at $(1,1)$, $y' = 3.1 = 3$ $\qquad$ at $(1,1)$, $y' = -1$

$\therefore m_1 = 3$ $\qquad \therefore m_2 = -1$

$\tan\theta = \left|\frac{3 - -1}{1 - 3}\right| = \left|\frac{4}{-2}\right| = 2$

$\therefore \theta = \tan^{-1}(2) = 63^\circ$ (2)

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c) $\int \cos^2 4x\,dx$

$\cos^2 4x = \frac{\cos 8x}{2} + \frac{1}{2}$

$\therefore \int \cos^2 4x\,dx = \frac{1}{2}\int(\cos 8x + 1)\,dx$

$= \frac{1}{2}\left[\frac{\sin 8x}{8} + x\right] + c$

$= \frac{\sin 8x}{16} + \frac{x}{2} + C.$ (2)

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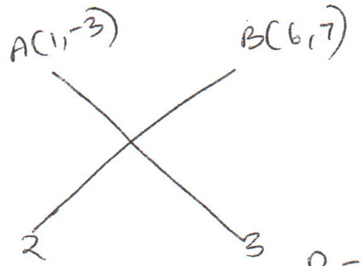
d)



$P = \left(\frac{3+12}{5}, \frac{-9+14}{5}\right)$

$= (3,1)$

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e) $\frac{d}{dx}(\cos^{-1}3x) = \frac{-1}{\sqrt{1-9x^2}} \times 3$

$= \frac{-3}{}$ (2)

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a) $u = x - 1$ $\quad \therefore \frac{du}{dx} = 1 \Rightarrow du = dx$

$\int 5x\sqrt{x-1}\,dx = \int 5.(u+1)\sqrt{u}\,du$

$= 5\int(u^{\frac{3}{2}} + u^{\frac{1}{2}})\,du$

$= 5\left[\frac{2}{5}.u^{\frac{5}{2}} + \frac{2}{3}.u^{\frac{3}{2}}\right] + c$

$= 2u^{\frac{5}{2}} + \frac{10}{3}u^{\frac{3}{2}} + c$

$= 2(x-1)^{\frac{5}{2}} + \frac{10}{3}(x-1)^{\frac{3}{2}} + c$ (3)

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b) (i) $4y = x^2$

$\therefore y = \frac{x^2}{4} \Rightarrow y' = \frac{2x}{4} = \frac{x}{2}$

at $(2t, t^2)$, $y' = \frac{2t}{2} = t$

$\therefore$ The equation is

$y - t^2 = t(x - 2t)$

$y = tx - t^2 \Rightarrow$

$tx - y - t^2 = 0$ (2)

(ii) Sub. $P(1,-2)$ in the equati

$tx - y - t^2 =$

$\therefore t + 2 - t^2 = 0 \Rightarrow t^2 - t - 2 = 0$

$(t-2)(t+1) = 0$

$\therefore t = 2$ or -1 (2)

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c) (i) $\cos x - \sqrt{3}\sin x = R\cos(x+\alpha)$

$= R\cos x\cos\alpha - R\sin x\sin\alpha$

$\therefore R\cos\alpha = 1$ and $R\sin\alpha = \sqrt{3}$

$\therefore \tan\alpha = \sqrt{3}$ $\qquad R^2 = 4$

$\therefore \alpha = \frac{\pi}{3}$ $\qquad R = 2$

$\therefore \cos x - \sqrt{3}\sin x = 2\cos(x + \frac{\pi}{3})$ (

(ii) $2\cos(x + \frac{\pi}{3}) = -2$

$\therefore \cos(x + \frac{\pi}{3}) = -1 \Rightarrow x + \frac{\pi}{3} = \cos^{-1}(-1)$

$(0 \le x \le 2\pi$

4) a) $\angle CAD = 90^\circ$ (angle at circumference in semicircle).

$\angle BAC = x^\circ$ (alternate angles are equal, $AB \parallel CD$)

$\therefore \angle BAD = 90^\circ + x^\circ$ (1)

ABCD is a cyclic quadrilateral

$\therefore$ opposite angles are supplementary (1)

$\therefore \angle BCD = 180 - (90 + x) = 90 - x$

$\therefore \angle ACB = \angle BCD - \angle ACD = 90 - x^\circ - x^\circ$

$= 90 - 2x^\circ$ (1)

b) When $n = 2$, $T_2 = 9^2 - 16 - 1 = 64$ which is divisible by 64

$\therefore$ True for $n = 2$ (1)

Assume that it is true for $n = k$, $k \geq 2$

$\therefore 9^k - 8k - 1 = 64A$, and $A \in \mathbb{Z}$.

Then $9^{k+1} - 8(k+1) - 1 = 9 \cdot 9^k - 8k - 9$

$= 9(9^k - 1) - 8k$

$= 9(9^k - 8k - 1) + 64k$

$= 9 \times 64A + 64k$

$= 64(9A + 64)$ (2)

$\therefore$ true for $n = k+1$

$\therefore 9^n - 8n - 1$ is divisible by 64, $n \geq 2$

c) $\left(x - \frac{3}{x}\right)^8 = \sum_{r=0}^{8} {}^8C_r\, x^{8-r} \left(-\frac{3}{x}\right)^r$

$= \sum_{r=0}^{8} {}^8C_r (-3)^r \cdot x^{8-2r}$ (1)

$8 - 2r = 0 \Rightarrow r = 4$ (1)

$\therefore$ The term is ${}^8C_4 (-3)^4 = 5670$

a) ${}^6C_2 \times {}^8C_2 = 420$ (1)

b) (i) Number of words $= \frac{6!}{3!}$

$= 120$ (1)

(ii) C _ C _ C _

_ C _ C _ C

C _ _ C _ C

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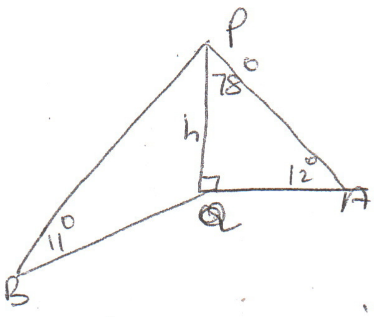
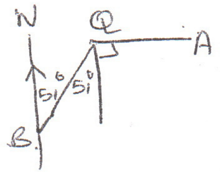
Total, if no C's are together

$= 4 \times 3! = 24$ (

c)





(i) $\angle AQB = 90^\circ + 51^\circ = 141^\circ$ (1)

(ii) In $\triangle APQ$, $\angle APQ = 78^\circ$

$\therefore \frac{AQ}{h} = \tan 78^\circ \Rightarrow AQ = h\tan 78^\circ$ (

(iii) In $\triangle PQB$, $\frac{BQ}{h} = \tan 79^\circ$

$\therefore BQ = h\tan 79^\circ$ (4)

(iv) In $\triangle ABQ$,

$AB^2 = AQ^2 + BQ^2 - 2AQ \cdot BQ\cos 141^\circ$

$\therefore 1000000 = h^2\tan^2 78^\circ + h^2\tan^2 79^\circ$

$- 2h^2 \tan 78 \tan 79 \cos$

$\therefore h^2 = \frac{1000000}{86.2188} \Rightarrow h = 107.69$

$= 108\,m$

(2)

Question 4 (cont)

d) (i) $\dfrac{2\tan x}{1+\tan^2 x} = \dfrac{2\dfrac{\sin x}{\cos x}}{\sec^2 x}$

$= 2\dfrac{\sin x}{\cos x} \times \cos^2 x$

$= 2\sin x.\cos x$

$= \sin 2x$ (1)

(ii) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\tan x}{1+\tan^2 x}\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin 2x\,dx$

$= -\frac{1}{4}\left[\cos 2x\right]_0^{\frac{\pi}{4}}$

$= -\frac{1}{4}\left[\cos\frac{\pi}{2} - \cos 0\right]$

$= -\frac{1}{4}\left[0-1\right] = \frac{1}{4}$ (2)

Question 5

a) (i) $\dfrac{dT}{dt} = k.Be^{kt}$

$= k(T-S)$ (1)

(ii) $\left.\begin{matrix}T=100\\ t=0\end{matrix}\right\}$ $100 = 25 + Be^{k\times 0} \Rightarrow B = 75$ (1)

$\left.\begin{matrix}T=80\\ t=30\end{matrix}\right\}$ $80 = 25 + 75e^{30k}$

$\dfrac{55}{75} = e^{30k}$

$\dfrac{11}{15} = e^{30k}$

$\log_e\left(\frac{11}{15}\right) = 30k$

$\therefore k = \frac{1}{30}\log_e\left(\frac{11}{15}\right)$ (2)

$t=60, T=?$

$T = 25 + 75e^{60\times k}$

$= 65.33^\circ$ (1)

$= 65^\circ$

b) (i) $V = 2-x$

$a = v\dfrac{dv}{dx} = (2-x)(-1)$

$= x-2$ (2)

(ii) $V = \dfrac{dx}{dt} = 2-x$

$\therefore \dfrac{dt}{dx} = \dfrac{1}{2-x}$

$dt = \dfrac{dx}{2-x}$ (1)

$\int dt = \int \dfrac{dx}{2-x}$

$t = -\ln(2-x) + c$

$t=0, x=-4$

$\therefore 0 = -\ln 6 + c$

$\Rightarrow c = \ln 6$

$\therefore t = -\ln(2-x) + \ln 6$ (1)

$\therefore t = \ln\dfrac{6}{2-x}$

$\therefore e^t = \dfrac{6}{2-x}$

$2e^t - xe^t = 6$

$\therefore xe^t = 2e^t - 6$

$x = 2 - 6e^{-t}$ (1)

(iii) When $t=0$, $x=-4$

After the particle has travelled 4m from its starting position,

$x=0 \Rightarrow 0 = 2-6e^{-t}$

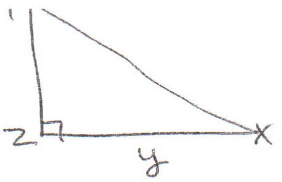
$\therefore 6e^{-t} = 2$

$e^{-t} = \frac{2}{6} = \frac{1}{3}$

$\therefore t = \ln 3.$

(2)

Questions
a) (i)



$\tan x = \frac{ZY}{y} \Rightarrow ZY = y\tan x$

$\therefore$ Area $= \frac{1}{2}\times y\tan x \times y$

$= \frac{1}{2}y^2\tan x$ (1)

$\frac{y}{XY} = \cos x \Rightarrow XY = y\sec x$

$\therefore$ Perimeter $P = y + y\tan x + y\sec x$

$= y[1 + \tan x + \sec x]$ (1)

(ii) $\frac{dy}{dt} = 0.1\,cms^{-1}$.

$\frac{dA}{dt} = \frac{dA}{dy} \times \frac{dy}{dt}$

$A = \frac{1}{2}y\tan^2 x$

$\therefore \frac{dA}{dy} = y\tan x$.

$= y\tan\frac{\pi}{4}$ when $x = \frac{\pi}{4}$

$= y$ (1)

$\therefore \frac{dA}{dt} = y \times 0.1 = 20\times 0.1$

$= 2\,cm^2/sec$. (1)

(iii) $\frac{dx}{dt} = 0.2$ radians/sec.

$\frac{dP}{dt} = \frac{dP}{dx} \times \frac{dx}{dt}$

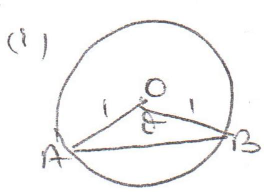
$P = y(1 + \tan x + \sec^2 x)$

$\therefore \frac{dP}{dx} = y(\sec^2 x + \sec x\tan x)$ (1)

$= 10(\sec^2\frac{\pi}{6} + \sec\frac{\pi}{6}\cdot\tan\frac{\pi}{6})$

when $y = 10, x = \frac{\pi}{6}$

(i)



Using cosine rule,

$AB^2 = 1^2 + 1^2 - 2\cos\theta$

$= 2(1-\cos\theta)$

$= 4\cdot\sin^2\frac{1}{2}\theta$

$\therefore AB = 2\cdot\sin\frac{1}{2}\theta$

Arc $AB = r\theta = \theta$. $(r=1)$

$\therefore$ Perimeter = diameter $\Rightarrow$

$\theta + 2\sin\frac{1}{2}\theta = 2$

$\therefore \theta + 2\sin\frac{1}{2}\theta - 2 = 0$. (2)

(ii) $f(\theta) = \theta + 2\sin\frac{1}{2}\theta - 2$

$f(1) = 1 + 2\sin\frac{1}{2} - 2 \simeq -0.04$

$f(2) = 2 + 2\sin 1 - 2 \simeq 1.68 > 0$

Since $f(\theta)$ is continuous,

$f(\theta) = 0$ for some $1 < \theta < 2$

(2)

(iii) $f(\theta) = \theta + 2\sin\frac{\theta}{2} - 2$

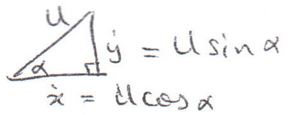
$f'(\theta) = 1 + \cos\frac{\theta}{2}$

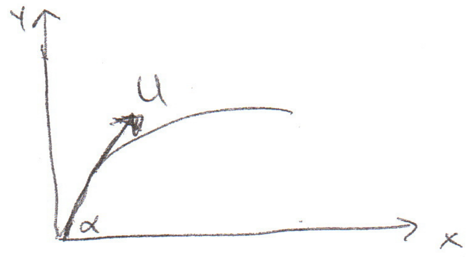
$\therefore \theta_1 = \theta_0 - \frac{f(\theta_0)}{f'(\theta_0)}$

$= 1 - \frac{-1 + 2\sin\frac{1}{2}}{1 + \cos\frac{1}{2}}$

$\simeq 1.0$ (to one deci

(i) $t=0$, $x=0$ and $y=0$





$$\ddot{y} = -g$$

$$\dot{y} = -\int g\,dt = -gt + c$$

when $t=0$, $\dot{y} = u\sin\alpha \Rightarrow C = u\sin\alpha$

$$\therefore \dot{y} = -gt + u\sin\alpha$$

$$\therefore y = \int(-gt + u\sin\alpha)\,dt$$

$$= ut\sin\alpha - \frac{gt^2}{2} + D$$

$t=0$, $y=0 \Rightarrow D=0$ (2)

$$\therefore y = ut\sin\alpha - \frac{gt^2}{2} \quad \ldots (i)$$

(ii) For the range, $y=0$

$$\therefore t\left(u\sin\alpha - \frac{gt}{2}\right) = 0$$

$$\therefore t = 0 \text{ or } t = \frac{2u\sin\alpha}{g}$$

$$\therefore \text{Range} = x = ut\cos\alpha$$

$$= u \cdot \frac{2u\sin\alpha}{g} \cdot \cos\alpha$$

$$= \frac{u^2\sin 2\alpha}{g}$$ (2)

(iii) At maximum height, $\dot{y} = 0$

$$\therefore u\sin\alpha - gt = 0$$

$$\therefore t = \frac{u\sin\alpha}{g}$$

Sub. $t = \frac{u\sin\alpha}{g}$ in (i)

$$\text{maximum height} = 3.5 = \frac{u \cdot u\sin\alpha}{g} - \frac{g}{2}\cdot\frac{u^2\sin^2\alpha}{g}$$

(iv) Maximum range $R = \dfrac{u^2\sin 2\alpha}{g}$

$$= \frac{7g}{\sin^2\alpha} \times \frac{2\sin\alpha\cos\alpha}{g}$$

$$= \frac{14\cos\alpha}{\sin\alpha}$$

$$= 14\cot\alpha$$ (2)

b) (i)

$$(1-x)^{2n} = {}^{2n}C_0 - {}^{2n}C_1 x + {}^{2n}C_2 x^2 - {}^{2n}C_3 x^3 + \cdots + \cdots + {}^{2n}C_{2n-1}(-x)^{2n-1} + {}^{2n}C_{2n} x^{2n}$$ (1)

(ii) By differentiating both sides w.r.t

$$-2n(1-x)^{2n-1} = -{}^{2n}C_1 + 2\cdot{}^{2n}C_2 - 3\cdot{}^{2n}C_3 + \cdots + 2n\cdot{}^{2n}C_{2n}\cdot x^{2n-1}$$ (1)

Sub. $x=1$ both sides,

$$0 = -{}^{2n}C_1 + 2\cdot{}^{2n}C_2 - 3\cdot{}^{2n}C_3 + \cdots - (2n-1)\,{}^{2n}C_{2n-1} + 2n\cdot{}^{2n}C_{2n}$$

$$\therefore {}^{2n}C_1 + 3\cdot{}^{2n}C_3 + \cdots + (2n-1)\,{}^{2n}C_{2n-1} = 2\cdot{}^{2n}C_2 + 4\,{}^{2n}C_4 + \cdots + 2n\cdot{}^{2n}C_{2n}$$ (1)