2000.

(i) $\int \frac{x+3}{x^2+6x-7}\,dx$

$= \frac{1}{2}\log_e(x^2+6x-7) + C$ ✓

(ii) $\int_0^{\pi/4} \cos^2 x\,dx$

$\cos 2x = 2\cos^2 x - 1$

$\therefore 2\cos^2 x = 1+\cos 2x$

$\therefore \cos^2 x = \frac{1+\cos 2x}{2}$

$= \int_0^{\pi/4} \frac{1+\cos 2x}{2}\,dx$ ✓

$= \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi/4}$ ✓

$= \left[\left(\frac{\pi}{8} + \frac{\sin 2\times\frac{\pi}{4}}{4}\right) - (0)\right]$ ✓

$= \frac{\pi}{8} + \frac{\sin\frac{\pi}{2}}{4}$

$= \frac{\pi}{8} + \frac{1}{4}$

$= \frac{1}{8}(\pi+2)$ ✓

(b). $\int \frac{4x}{\sqrt{1+x^2}}\,dx$

$u = 1+x^2$

$\therefore du = 2x\,dx$

$\therefore dx = \frac{1}{2x}\,du$

$= \int \frac{4x}{u^{1/2}} \cdot \frac{1}{2x}\,du$

$= 2\int u^{-1/2}\,du$

$= 2 \cdot \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + A$

$= 4\sqrt{u} + A$

$= 4\sqrt{1+x^2} + A$

(c). $\frac{dx}{dt} = 4t-7$

$\therefore \int dx = \int (4t-7)\,dt + A$

$\therefore x = 2t^2 - 7t + A$

When $t=0$, $x=3$

$\therefore 3 = A$

$\therefore \boxed{x = 2t^2 - 7t + 3}$

$$\int_0 x\sqrt{9-x^2}\,dx \quad , \quad x = 3\sin\theta$$

$$\therefore dx = 3\cos\theta\, d\theta.$$

When $x = 3$, $\sin\theta = 1$ $\therefore \theta = \frac{\pi}{2}$

When $x = 0$, $\sin\theta = 0$ $\therefore \theta = 0$

$$= \int_0^{\pi/2} 3\sin\theta \sqrt{9 - 9\sin^2\theta}\cdot 3\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} 3\sin\theta \cdot 3\cos\theta \cdot 3\cos\theta\, d\theta$$

$$= 27\int_0^{\pi/2} \cos^2\theta \cdot \sin\theta\, d\theta$$

$$= 27.\left[\frac{\cos^3\theta}{3}\right]_0^{\pi/2}$$

$$= 9\left[0 - 1\right]$$

$$= -9$$

(a) To Show: $\sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$$\sin 75^\circ = \sin(45^\circ + 30^\circ)$$

$$= \sin 45^\circ \cdot \cos 30^\circ + \cos 45^\circ \cdot \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

b. $\cos^2 x - 3\sin x + 3 = 0$

ie $1 - \sin^2 x - 3\sin x + 3 = 0$

ie $-\sin^2 x - 3\sin x + 4 = 0$

ie $\sin^2 x + 3\sin x - 4 = 0$

ie $(\sin x + 4)(\sin x - 1) = 0$

$\therefore \sin x = 1, \cancel{-4}$

$\therefore \quad x = n\pi + (-1)^n \frac{\pi}{2}$

$$\frac{\cos A+\sin A}{\cos A-\sin A}-\frac{\cos A-\sin A}{\cos A+\sin A}=2\tan 2A$$

$$LHS=\frac{\cos A+\sin A}{\cos A-\sin A}-\frac{\cos A-\sin A}{\cos A+\sin A}$$

$$=\frac{(\cos A+\sin A)^2-(\cos A-\sin A)^2}{\cos^2 A-\sin^2 A}$$

$$=\frac{\cos^2 A+2\sin A\cos A+\sin^2 A-\left[\cos^2 A-2\sin A\cos A+\sin^2 A\right]}{\cos 2A}$$

$$=\frac{\cancel{\cos^2 A}+2\sin A\cos A+\cancel{\sin^2 A}-\cancel{\cos^2 A}+2\sin A\cos A-\cancel{\sin^2 A}}{\cos 2A}$$

$$=\frac{4\sin A\cos A}{\cos 2A}$$

$$=\frac{2\sin 2A}{\cos 2A}$$

$$=2\tan 2A$$

$$=RHS$$

(d)

$P(x)=x^3+x^2+kx-4$

$x=2$ is a solution

$\therefore$ ~~$x=2$~~ $P(2)=0$

$8+4+2k-4=0$

$2k=-8$

$k=-4$

$P(x)=x^3+x^2-4x-4$

$$\begin{array}{r} x^2+3x+2 \\ x-2\,\overline{)\,x^3+x^2-4x-4} \\ \underline{x^3-2x^2\qquad\quad} \\ 3x^2-4x\quad \\ \underline{3x^2-6x\quad} \\ 2x-4 \\ \underline{2x-4} \end{array}$$

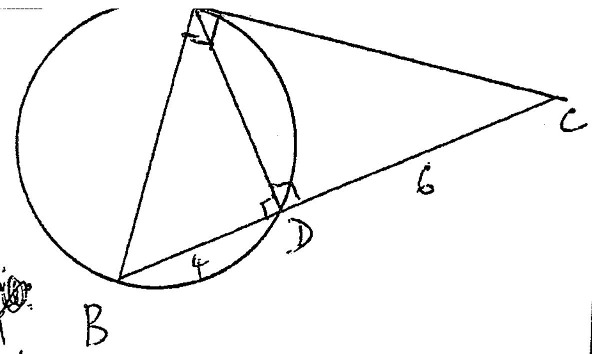
$x^2+3x+2=(x+2)(x+1)$

$\therefore P(x)=(x-2)(x+2)(x+1)$

Data : AC is a tangent at A

$\angle BAC = 90^\circ$

$BD = 4cm$, $DC = 6cm$.



PROVE : (i) $\Delta ABD \parallel\!| \Delta CAD$

(ii) Calculate length of radius

PROOF :

$\angle BAC = 90^\circ$ —— data

$\therefore$ AB is the diameter

$\therefore \angle ADB = 90^\circ$ —— (angle in a semicircle)

In Δ^s ABD, CAD

$\angle ABD = \angle CAD$ — $\angle$s in the alternate segment

$\angle ADB = \angle ADC = 90^\circ$ —— (adj $\angle$s on str line)

$\therefore$ $\underline{\Delta ABD \parallel\!| \Delta CAD}$ (A, A, $\therefore$A)

$AC^2 = CB \cdot CD$ (tangent^2 = product of intercepts of chord)

$= 10 \times 6$

$\therefore AC = \sqrt{60}$

$AD^2 = AC^2 - CD^2$

$= 60 - 36$

$= 24$

$\therefore AD = 2\sqrt{6}$ cm

$AB^2 = AD^2 + BD^2$

$= 24 + 16$

$= 40$

$\therefore AB = 2\sqrt{10}$

$\therefore$ radius $= \frac{1}{2} AB = \underline{\sqrt{10}\ cm}$

$\boxed{5}$

(b) Given: $\frac{dr}{dt} = 1$ cm/s when $r = 40$cm.

(i)

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$V = \frac{4}{3}\pi r^3$

$\therefore \frac{dV}{dr} = \frac{4}{3} \times \pi \times 3r^2$

$= \underline{4\pi r^2}$

$\therefore \frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$

$\therefore$ When $r = 40$cm.

$\left(\frac{dV}{dt}\right)_{r=40} = 4 \times \pi \times 40^2 \times 1$

$= \underline{6400\pi\ cm^3/s}$

$3\frac{1}{2}$

11) $S = 4\pi r^2$ $\therefore \frac{dS}{dr} = 8\pi r$

$\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$

$= 8\pi r \cdot \frac{dr}{dt}$

$\left(\frac{dS}{dt}\right)_{r=40cm} = 8 \times \pi \times 40 \times 1$

$= \underline{320\pi \ cm^2/s}$

Question No. 4

(a) $y = \cos^{-1}(\sin x)$

$\therefore \cos y = \sin x$

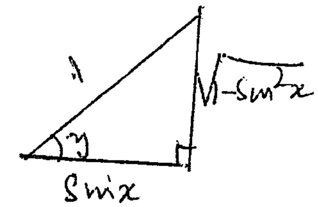
$\therefore -\sin y \frac{dy}{dx} = \cos x$

$\therefore \frac{dy}{dx} = -\frac{\cos x}{\sin y}$

$= -\frac{\cos x}{\sqrt{1-\sin^2 x}}$

$= \frac{-\cos x}{\cos x}$

$= -1$



or: $\cos y = \sin x$

$\therefore x + y = 90^\circ$

$\therefore 1 + \frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -1$

(b) $y = 2\cos^{-1} 3x$

$\therefore \frac{y}{2} = \cos^{-1} 3x$

D: $-1 \le 3x \le 1$

ie $\underline{-\frac{1}{3} \le x \le \frac{1}{3}}$

R: $0 \le \frac{y}{2} \le \pi$

$\therefore \underline{0 \le y \le 2\pi}$

$$\int_0 \frac{}{9+4x^2}$$

$$= \int_0^{\frac{\sqrt3}{2}} \frac{dx}{4\left(\frac{9}{4}+x^2\right)}$$

$$= \frac{1}{4}\int_0^{\sqrt3/2} \frac{dx}{\left(\frac{3}{2}\right)^2+x^2}$$

$$= \frac{1}{4}\cdot\frac{1}{\frac{3}{2}}\left[\tan^{-1}\frac{x}{\frac{3}{2}}\right]_0^{\frac{\sqrt3}{2}}$$

$$= \frac{1}{6}\left[\tan^{-1}\frac{2x}{3}\right]_0^{\frac{\sqrt3}{2}}$$

$$= \frac{1}{6}\tan^{-1}\frac{1}{\sqrt3}]$$

$$= \frac{1}{6}\times\frac{\pi}{6}$$

$$= \frac{\pi}{36}$$

$\cos^{-1}(-x) = \pi - \cos^{-1}x$

Let $y = \cos^{-1}(-x)$

$\therefore \cos y = -x$

$\therefore x = -\cos y$

ie $x = \cos(\pi - y)$

$\therefore \cos^{-1}x = \pi - y$

$\therefore y = \pi - \cos^{-1}x$

ie $\cos^{-1}(-x) = \pi - \cos^{-1}x$

(ii) $\sin^{-1}\frac{1}{2} + \cos^{-1}\left(-\frac{\sqrt3}{2}\right)$

$= \sin^{-1}\frac{1}{2} + \pi - \cos^{-1}\frac{\sqrt3}{2}$

$= \frac{\pi}{6} + \pi - \frac{\pi}{6}$

$= \pi$

(a) $x = 3\sin 2t + 4\cos 2t$

$= 5\left[\frac{3}{\sqrt{3^2+4^2}}\cdot\sin 2t + \frac{4}{\sqrt{3^2+4^2}}\cos 2t\right]$

ie $x = 5\left[\frac{3}{5}\sin 2t + \frac{4}{5}\cos 2t\right]$

$= 5\left[\sin 2t\cos\alpha + \cos 2t\sin\alpha\right]$

when $\cos\alpha = \frac{3}{5}$ 

ie $x = 5\sin(2t+\alpha)$.

$\dot{x} = 10\cos(2t+\alpha)$

$\ddot{x} = -20\sin(2t+\alpha)$

$= -4\times 5\sin(2t+\alpha)$

$\therefore \ddot{x} = -4x$

This is of the form $\ddot{x} = -n^2x$

$\therefore$ The motion is Simple Harmonic.

Period $= \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$ seconds.

Max. displacement $=$ Amplitude $= 5$ cm

b) To show that.

$$1+4+7+\cdots+(3n-2) = \frac{n(3n-1)}{2}$$

Let this be true for $n=K$. when $K \geq 1$

$$\therefore 1+4+7+\cdots+(3K-2) = \frac{K(3K-1)}{2} \quad \text{—— (i)}$$

Adding the next, $(K+1)^{th}$ term:

$$1+4+7+\cdots+(3K-2)+(3K+1) = \frac{K(3K-1)}{2}+(3K+1)$$

$$= \frac{K(3K-1)+2(3K+1)}{2}$$

$$= \frac{3K^2-K+6K+2}{2}$$

$$= \frac{3K^2+5K+2}{2}$$

$$= \frac{(K+1)(3K+2)}{2} \quad \text{—— (2)}$$

This is of the same form as (1) when K is replaced by $K+1$.

$\therefore$ If Statement (1) is true for $n=K$, it is true for $n=K+1$ also.

When $K=1$, LHS in (1) $=1$, RHS in (1) $= \frac{1(3-1)}{2} = 1$.

$\therefore$ By the Principle of Induction: $1+4+7+\cdots+(3n-2) = \frac{n(3n-1)}{2}$

Question No 6

(a)

$$\frac{dT}{dt} = -k(T-T_0)$$

$$\therefore \frac{dT}{T-T_0} = -k\,dt$$

$$\therefore \int \frac{dT}{T-T_0} = -k\int dt + C$$

$$\therefore \log_e(T-T_0) = -kt+C$$

$$\therefore T-T_0 = e^{-kt+c}$$

$$\therefore \boxed{T = T_0 + Ae^{-kt}}$$

Alternative method

$T = T_0 + Ae^{-kt}$

$\frac{dT}{dt} = -kAe^{-kt}$

But from above $Ae^{-kt} = T-T_0$

$\therefore \frac{dT}{dt} = -k(T-T_0)$

(b) $T=85^\circ$ when $t=0$.

$T=80$ when $t=1$ mt

$T=?$ when $t=4$ mts

$T_0 = 25^\circ$

$$T = T_0 + Ae^{-kt}$$

i.e. $T = 25 + Ae^{-kt}$

When $t=0$, $T=85$

$$85 = 25 + A$$

$$\therefore A = 60$$

When $t=1$, $T = 80^{\circ}$

$\therefore\ 80 = 25 + 60 \cdot e^{-k}$

ie $60e^{-k} = 55$

$\therefore\ e^{-k} = \frac{55}{60}$

ie $-k = \log_e \frac{55}{60}$

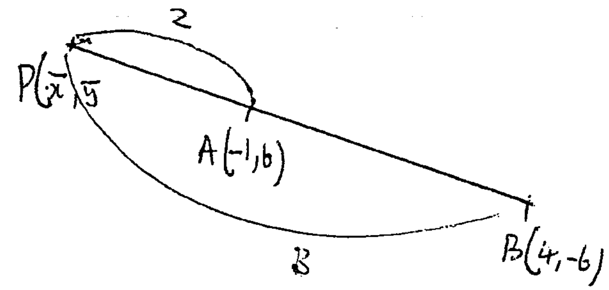
$\therefore\ k = 0.087011376$

When $t = 4$ mts $T = ?$

$T = 25 + 60 \cdot e^{-4k}$

$= 67.36^{\circ}$ ~~mts~~

$= 67^{\circ}$



$$\left.\begin{array}{l} A\overset{x_1\ y_1}{(-1,6)} \\ B\overset{x_2\ y_2}{(4,-6)} \\ m = 2 \\ n = 3 \end{array}\right\}$$

$$\bar{x} = \frac{mx_2 - nx_1}{m-n} = \frac{2\times 4 - 3\times(-1)}{2-3} = \frac{8+3}{-1} = -11$$

$$\bar{y} = \frac{my_2 - ny_1}{m-n} = \frac{2(-6) - 3\times 6}{2-3} = \frac{-12-18}{-1} = 30$$

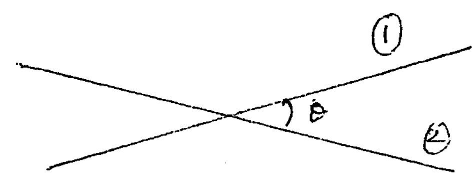
$\therefore$ P is the point $(-11, 30)$.

(e) $2x - y - 3 = 0$

$x - 3y - 7 = 0$

ie $y = 2x \mp 3$ ——— (1)

$y = \frac{x}{3} \mp \frac{7}{3}$ ——— (2)

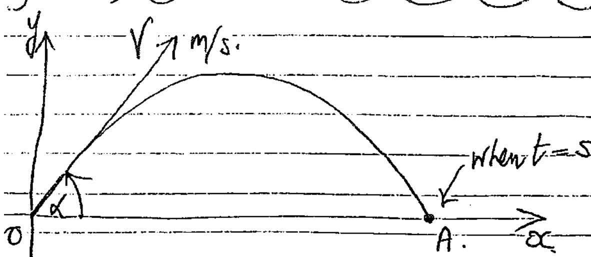


$$\left.\begin{array}{l} m_1 = 2 \\ m_2 = \frac{1}{3} \end{array}\right\}$$

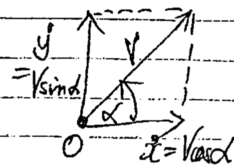
$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 \cdot m_2} = \frac{2 - \frac{1}{3}}{1 + \frac{2\times 1}{3}} = \frac{\frac{5}{3}}{\frac{5}{3}} = 1$$

$\therefore$ The Angle between the lines is 45°

PROJECTILE MOTION SOLUTION — MR. SEABORN.



INITIAL VELOCITY COMPONENTS (when t=0)



Given $\ddot{x} = 0$
$\dot{x} = \int 0\,dt$
$= C_1$

INITIALLY when $t = 0$ $\dot{x} = V\cos\alpha \Rightarrow C_1 = V\cos\alpha$

$\Rightarrow \dot{x} = V\cos\alpha$

$x = \int (V\cos\alpha)\,dt$
$= (V\cos\alpha)t + C_2$

When $t = 0$ $x = 0 \Rightarrow C_2 = 0$

$\Rightarrow x = (V\cos\alpha)t$ ——— (1)

$\ddot{y} = -10$
$\dot{y} = \int (-10)\,dt$
$= C_3 - 10t$

and $\dot{y} = V\sin\alpha$ when $t = 0$
$\Rightarrow C_3 = V\sin\alpha$

$\Rightarrow \dot{y} = V\sin\alpha - 10t$

$y = \int (V\sin\alpha - 10t)\,dt$
$= (V\sin\alpha)t - 5t^2 + C_4$

When $t = 0$ $y = 0 \Rightarrow C_4 = 0$

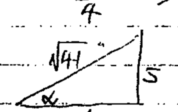
$\Rightarrow y = (V\sin\alpha)t - 5t^2$ ——— (2)

) At A $x = 100$, $y = 0$, $t = 5$

$\Rightarrow 100 = (V\cos\alpha)\times 5$ and $0 = (V\sin\alpha)\times 5 - 5\times 5^2$

$\Rightarrow V\cos\alpha = 20$ ——— (3) $\Rightarrow V\sin\alpha = 25$ ——— (4)

$\Rightarrow \dfrac{25}{20} = \dfrac{V\sin\alpha}{V\cos\alpha} \Rightarrow \tan\alpha = \dfrac{5}{4} \Rightarrow \alpha = \tan^{-1}\left(\dfrac{5}{4}\right)$



$= 51°20'$ (nearest minute).

MARKS

$\Rightarrow \cos\alpha = \dfrac{4}{\sqrt{41}}$ and $\sin\alpha = \dfrac{5}{\sqrt{41}}$

Sub in (3) (or (1)) to find $V \Rightarrow V = 20 \div \dfrac{4}{\sqrt{41}} = 5\sqrt{41}$ m/s.

1 MARK

(iv) Particle is at maximum height when $\dot{y} = 0$

$\Rightarrow 0 = V\sin\alpha - 10t$
$= 25 - 10t$

$\Rightarrow t = 2\cdot 5$ s.

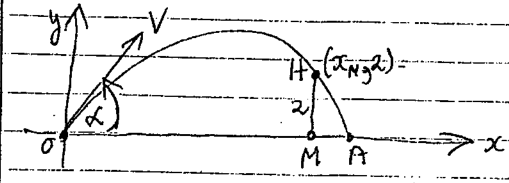
To find maximum height $y_{MAX} = (V\sin\alpha)t - 5t^2$ when $t = 2$

1 MARK

$= 25\times 2\cdot 5 - 5\times 2\cdot 5^2$
$= 31\cdot 25$ m (exactly)

To the nearest metre the maximum height is 31 m.

(b) When the man is hit (at point H on the trajectory, say)



METHOD 1

$y = (V\sin\alpha)t - 5t^2$
$2 = 25t - 5t^2$
$5t^2 - 25t + 2 = 0$
$t = \dfrac{25 \pm \sqrt{625 - 40}}{10}$
$= \dfrac{25 \pm \sqrt{585}}{10}$
$= 0\cdot 08$ s or $4\cdot 92$ s (2d

"A few seconds later" means he is hit when $t \approx 4\cdot 92$ s $= \dfrac{25 + \sqrt{585}}{10}$

$x = (V\cos\alpha)t$

$x_M = 20 \times \dfrac{25 + \sqrt{585}}{10}$

3 MARKS

$= 50 + 2\sqrt{585}$
$= 98\cdot 3735$ m (4dp)
≈ 98 m

$\Rightarrow$ The man is hit approxima 98 m from O.

METHOD 2. Equation of projectile is

$y = (\tan\alpha)x - \left(\dfrac{g\sec^2\alpha}{2V^2}\right)x^2$
$= \dfrac{5}{4} - \dfrac{10\left(\frac{\sqrt{41}}{4}\right)^2}{2\times(5\sqrt{41})^2}\cdot x^2$
$= \dfrac{5}{4} - \dfrac{1}{80}x^2$

$\therefore 80y = 100x - x^2$

At H: ($y = 2$) $160 = 100x - x^2$

$x^2 - 100x + 160 = 0$

$\Rightarrow x = \dfrac{100 \pm \sqrt{10000 - 640}}{2}$
$= 50 \pm \sqrt{2340}$
$\approx 50 \pm 48\cdot 3735 \approx 98$ m

since projectile hits man after a few seconds, not right away b.) $f(x) = 4x^2 - 11x + 7$

$f'(x) = 8x - 11$

$f(0.73) = 1.016$

$f'(0.73) = -5.16$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1}$$

$$= 0.73 - \frac{1.016}{-5.16}$$

$$= 0.73 + \frac{1.016}{5.16}$$

$$= 0.9435 \text{ (4 d.p)}.$$