SUGGESTED SOLUTIONS TO MATHEMATICS CSSA TRIAL

2001

P1/3

Question 1

(a)
$$ab - a - bx + x$$
$$= a(b-1) - x(b-1)$$
$$= (b-1)(a-x)$$

(b)
$$|2| + |-5| = 2+5$$
$$= 7$$

(c)
$$\frac{1}{\sqrt{3}+2} \times \frac{\sqrt{3}-2}{\sqrt{3}-2}$$
$$= \frac{\sqrt{3}-2}{3-4}$$
$$= -(\sqrt{3}-2)$$
$$= -\sqrt{3}+2$$
which is in the form $a\sqrt{3}+b$
where a $= -1$ and $b = 2$

(d)
$$\cos\frac{\pi}{8} = 0.9238795...$$
$= 0.924$ correct to 3 d.pl.

(e)
$$\theta = 180^\circ - 30^\circ \text{ or } 360^\circ - 30^\circ$$
$$= 150^\circ \text{ or } 330^\circ$$

(f) (i)
$$\Delta = b^2 - 4ac \quad a = 2$$
$$= 9 - 8k \quad b = -3$$
$$c = k$$

(ii) Solve $\Delta > 0$
$$9 - 8k > 0$$
$$9 > 8k$$
$$\Rightarrow k < \frac{9}{8}$$

Question 2

(a)
$x + 2y = 9$
Point A $(-3,6)$
test by substitution
$-3 + 2(6) = 9$
Point B $(5,2)$
test by substitution
$5 + 2(2) = 9$

(b)
$$AB = \sqrt{(5--3)^2 + (2-6)^2}$$
$$= \sqrt{64+16}$$
$$= \sqrt{80}$$
$$= \sqrt{16} \times \sqrt{5}$$
$= 4\sqrt{5}$ units

(c)
$$\frac{|1(0)+2(0)-9|}{\sqrt{1^2+2^2}}$$
$= \frac{9}{\sqrt{5}}$ units

(d)
$$\text{Area} = \frac{1}{2} \times 4\sqrt{5} \times \frac{9}{\sqrt{5}}$$
$= 18$ units^2

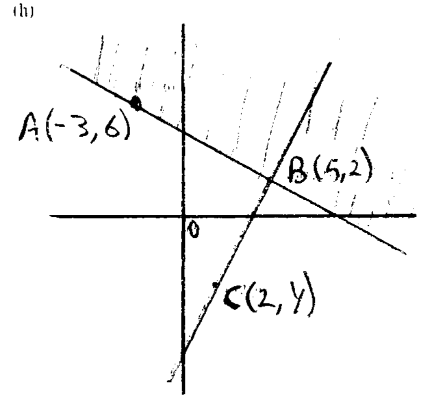
(e)
$$\text{gradient of } AB = \frac{2-6}{5+3}$$
$$= -\frac{1}{2}$$
gradient of BC is thus 2
since the product of the gradients of perpendicular lines is -1

$$\text{gradient of } BC = \frac{y-2}{2-5}$$
$$= \frac{y-2}{-3}$$
$$\text{Solve } \frac{y-2}{-3} = 2$$
$$y = -6+2$$
$$= -4$$

(f)
$$\text{gradient of } AO = \frac{6}{-3}$$
$$= -2$$
$$\text{gradient of } OC = \frac{-4}{2}$$
$$= -2$$
Hence AOC is a straight line and so AC passes through O

(g)
Let D have coordinates $(x,0)$
gradient of $AD \times$ gradient of $AB = -1$
$$\Rightarrow \frac{6-0}{-3-x} \times -\frac{1}{2} = -1$$
$$-6 = 6 + 2x$$
$$x = -6$$
$\Rightarrow D$ is the point $(-6,0)$

(h)



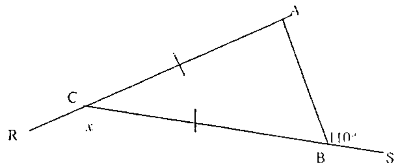
Question 3

(a) (i)
$$\int \sec^2 4x\,dx = \frac{1}{4}\tan dx = \frac{1}{4}\tan 4x + c$$

(ii)
$$\int (x^{-2} + e^{-2x})dx = \frac{x^{-1}}{-1} + \frac{e^{-2x}}{-2} + c$$
$$= \frac{-1}{x} - \frac{1}{2e^{2x}} + c$$

(b)
$$\int_0^3 \frac{1}{x+1}dx = [\log_e(x+1)]_0^3$$
$$= \log_e 4 - \log_e 1$$
$$= \log_e 4$$

(c)



$\angle ABC = 180^\circ - 110^\circ$ (straight angle)
$= 70^\circ$
$\angle BAC = \angle ABC$ (opposite equal sides)
$= 70^\circ$
$x \quad = \angle BAC + \angle ABC$ (exterior angle thm)
$= 140^\circ$

(d) (i)
$$x^3\cos x + 3x^2 \sin x$$

(ii)
$$\frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot -2x$$
$$= \frac{-x}{\sqrt{1-x^2}}$$

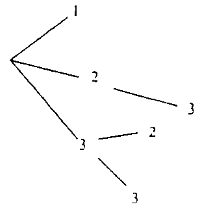
Question 4

(a) (i)
$$T_{12} = 3(12) + 4$$
$$= 40$$

(ii)
$$S_n = \frac{n}{2}(T_1 + T_n)$$
$$S_{20} = \frac{20}{2}(7 + 60 + 4)$$
$$= 710$$

(b) (i)
$$P(11) = \frac{2}{7} \cdot \frac{1}{6}$$
$$= \frac{1}{21}$$

(ii)



$$P(23) = \frac{1}{7} \cdot \frac{4}{6}$$
$$P(32) = \frac{4}{7} \cdot \frac{1}{6}$$
$$P(33) = \frac{4}{7} \cdot \frac{3}{6}$$

$$P(\text{sum greater than } 4) = \frac{4}{42} + \frac{4}{42} + \frac{12}{42}$$
$$= \frac{10}{21}$$

(c)
$$\frac{\sin\theta}{\sqrt{12}} = \frac{\sin 45^\circ}{\sqrt{8}}$$
$$\sin\theta = \frac{\frac{1}{\sqrt{2}} \cdot \sqrt{12}}{\sqrt{8}}$$
$$= \frac{\sqrt{3}}{2}$$
$\Rightarrow \theta = 180^\circ - 60^\circ$, since $90^\circ \le \theta \le 180^\circ$
$= 120^\circ$

(d) (i)
$$S\infty = 1$$
$$\Rightarrow \frac{a}{1-r} = 1 \quad 1$$
$$\Rightarrow a = 1 - r$$

$$T_2 = ar$$
$$\text{But } T_2 = \frac{1}{4}$$
$$\Rightarrow ar = \frac{1}{4} \quad 2$$
Substituting (1) into (2)

(ii)
$$(1-r)(r) = \frac{1}{4}$$
$$r - r^2 = \frac{1}{4}$$
$$4r^2 - 4r + 1 = 0$$
$$(2r-1)^2 = 0$$
$$r = \frac{1}{2}$$

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Question 5

(a) (i) $\frac{dy}{dx} = 6x^2 - 6x - 12$

(ii)
$6x^2 - 6x - 12 = 0$ stationary points occur where $\frac{dy}{dx}=0$

$\Rightarrow x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$x = -1 \quad x = 2$

$y = 7 \quad y = -20$

The stationary points are
$(-1,7), (2,-20)$

(iii) $\frac{d^2y}{dx^2} = 12x - 6$

At $(-1,7)\ \frac{d^2y}{dx^2} = -12 - 6 < 0$

$\Rightarrow (-1,7)$ is a maximum t.pt.

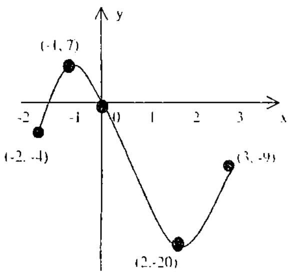
At $(2,-20)\ \frac{d^2y}{dx^2} = 24 - 6 > 0$

$\Rightarrow (2,-20)$ is a minimum t.pt.

(iv) When $x = 0, \ y = 0$

$x = -2, \ y = -4$

$x = 3, \ y = -9$



5(b) (i)
$y = x^2 + 1$

$y = 7 - x$

$x^2 = 1 = 7 - x$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = -3$ or $x = 2$

At B $x = 2$, $y = 2^2 + 1 = 5$

(ii)

x	0	1	2	3	4
value	1	2	5	4	3

Area $\approx \frac{1}{3}[1 + 4(2) + 2(5) + 4(4) + 3]$

$= \frac{38}{3}u^2$

Question 6

(a) Volume $= \pi \int_0^5 x^2 dy$

$= \pi \int_0^5 \left|\frac{y^4}{25}\right| dy$

$= \frac{\pi}{25}\int_0^5 y^4 dy$

$= \frac{\pi}{25}\left[\frac{y^5}{5}\right]_0^5$

$= \frac{\pi}{25}\cdot 5^4$

$= 25\pi$ units^3

(b) (i)
$N = 2N_o$ when $t = 0.5$

Solve $2N_o = N_o e^{0.5k}$

$\Rightarrow e^{0.5k} = 2$

$\Rightarrow 0.5k = \ln 2$

$\Rightarrow k = \frac{\ln 2}{0.5} = 1.38629...$

(ii)
$600 = 3e^{1.386...t}$

$\ln 200 = 1.386...t$

$\Rightarrow t = \frac{\ln 200}{1.386...}$

$= 3.8219...$h

(iii)
when $t = 0, N = N_o$

$t = 1, N = N_o e^k$

$t = 2, N = N_o e^{2k}$

$t = 3, N = N_o e^{3k}$

$t = 4, N = N_o e^{4k}$

$\frac{N_o e^{2k}}{N_o e^k} = \frac{N_o e^{3k}}{N_o e^{2k}} = \frac{N_o e^{4k}}{N_o e^{3k}} = \frac{N_o e^k}{N_o} = e^k$

The common ratio is e^k

(c) (i) $t = 3$ or $t = 5$

(ii) The shaded region represents the distance travelled during the third second.

(iii) The particle changes direction at $t = 3$ (after it has come to rest) and begins to move back towards its initial position. Hence, the particle is further from its initial position at $t = 3$.

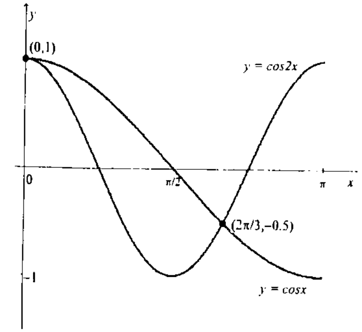
Question 7

(a) (i) $\cos\frac{2\pi}{3} = -\frac{1}{2}$

$\cos 2\left(\frac{2\pi}{3}\right) = \cos\frac{4\pi}{3}$

$= -\frac{1}{2}$

(ii)



(iii) Area $\int_0^{\frac{2\pi}{3}} (\cos x - \cos 2x)dx$

$= \left[\sin x - \frac{1}{2}\sin 2x\right]_0^{\frac{2\pi}{3}}$

$= \sin\frac{2\pi}{3} - \frac{1}{2}\sin\frac{4\pi}{3} - (0 - 0)$

$= \frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{-\sqrt{3}}{2}$

$= \frac{3\sqrt{3}}{4}$ units^2

(b) (i)
$NS^2 = 1^2 + 2^2 - 2\cos 30^\circ$

$= 5 - 2\sqrt{3}$

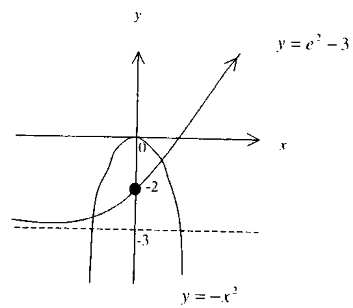
$\Rightarrow NS = \sqrt{5 - 2\sqrt{3}} \quad (NS > 0)$

(ii) $ArcMN = 2 \times \frac{\pi}{6}$

$= \frac{\pi}{3}$

Perimeter $= \frac{\pi}{3} + \sqrt{5 - 2\sqrt{3}} + 1$

(c) (i)



(ii) From the diagram, it is clear that the curves $y = e^x - 3$ and $y = -x^2$ have two points of intersection hence the equation $e^x - 3 = -x^2$ **has two solutions**

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Question 8

(a) $y = \log_2 x$

$= \frac{\log_e x}{\log_e 2}$ **(by change of base rule)**

$\frac{dy}{dx} = \frac{1}{\log_e 2}\cdot\frac{1}{x}$

(b) (i) $AB = 2x$

$BC = 6 - \frac{x^2}{4}$

Area of ABCD $= 2x\left(6 - \frac{x^2}{4}\right)$

$= 12x - \frac{x^3}{2}$

(ii) $A = 12x - \frac{x^3}{2}\ 0 < x < 2\sqrt{6}$

$\frac{dA}{dx} = 12 - \frac{3}{2}x^2$

Solve $\frac{dA}{dx} = 0$

$12 - \frac{3}{2}x^2 = 0$

$x^2 = 8$

$x = \pm 2\sqrt{2}$

$= 2\sqrt{2}$ **(since $x > 0$)**

$\frac{d^2A}{dx^2} = -3x$

when $x = 2\sqrt{2}, \frac{d^2A}{dx^2} < 0$

$\Rightarrow A$ **is maximised when** $x = 2\sqrt{2}$

Dimensions of rectangle $4\sqrt{2}$ **by 4**

(c) (i) $\frac{dv}{dt} = -1.92t\ (t \geq 0)$

$v = \frac{-1.92t^2}{2} + C$

$= -0.96t^2 + C$

when $t = 0, V = 25000$

$\Rightarrow 25000 = C$

$\Rightarrow V = 25000 - 0.96t^2$

(ii) When 40% **full** the container holds 0.4 x 25000 = 10000 litres

Solve $25000 - 0.96t^2 = 10000$

$\Rightarrow 0.96t^2 = 15000$

$\Rightarrow t^2 = 15625$

$t = 125s\ (t > 0)$

Question 9

(a) (i) Following the first withdrawal of \$E, Mia has \$3000 (1.005) - \$E
Following the second withdrawal of \$E, she has
$[\$3000(1.005) - \$E)]1.005 + \$3000(1.005) - \$E$

$= \$3000(1.005)^2 - \$E(1.005) + \$3000(1.005) - \$E$

$= \$3000(1.005^2 + 1.005) - \$E(1.005 + 1)$

(ii) After 4 years or 48 months, Mia has
$\$3000(1.005^{48} + 1.0005^{47} + ... + 1.005) - \$E(1.005^{47} + ... + 1.005 + 1)$
But she has saved \$60000 after 4 years
Solve $\$3000(1.005 + 1.005^2 + ... + 1.005^{48}) - \$E(1 + 1.005 + ... + 1.005^{47}) = \$60000$

$\Rightarrow E = \dfrac{3000 \times 1.005\frac{(1.005^{48} - 1)}{0.005} - 60000}{1\frac{(1.005^{48} - 1)}{0.005}}$

$= 1905.898...$

(b) $x = 60t + 100e^{\frac{-t}{5}}$

When $t = 0, x = 100e$

(i) $= 100$

Initially, the particle is 100 units to the right of the origin

(ii) $\frac{dx}{dt} = 60 - \frac{1}{5}.100e^{\frac{-t}{5}}$

$= 60 - 20e^{\frac{-t}{5}}$

> 0 (for all $t \geq 0$)

hence particle is always moving to the right

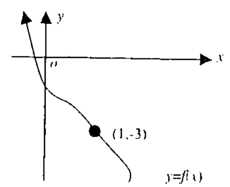
(iii) $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\frac{1}{5}\cdot -20e^{-\frac{t}{5}}$

$= 4e^{-\frac{t}{5}}$

As $t \to \infty\ \frac{d^2x}{dt^2} \to 0$

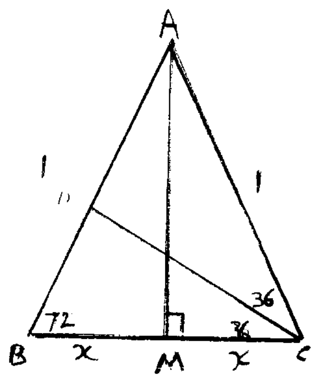
(c) (i) From the graph, $f'(1) = 0$, hence $y = f(x)$ has a stationary point at $x = 1$.
Also from the graph,
$f'(x) < 0$ for $x \neq 1$, hence $y = f(x)$ is decreasing everywhere but at $x = 1$.
This means $x = 1$ is a stationary point of inflexion.

ii)



Question 10

(a)



(i) $\angle BDC = 180° - (72° + 36°)$ (angle sum of ΔBCD)

$= 72°$

DC = BC (opposite equal angles)

$= 2x$

$\angle BAC = 180° - 72° - 72°$ (angle sum of ΔABC)

$= 36°$

$AD = DC$ (opposite equal angles)

$= 2x$

(ii) In
Δ's ABC, CBD
∠BAC = ∠BCD = 36°
∠ABC = ∠CBD = 72°
$\Rightarrow \Delta ABC ||| \Delta CBD$ (equiangular)

(iii) $\frac{AB}{BC} = \frac{BC}{BD}$ (corresponding sides of similar triangles are in proportion)

$\Rightarrow \frac{1}{2x} = \frac{2x}{1 - 2x}$

$\Rightarrow 1 - 2x = 4x^2$

$\Rightarrow 4x^2 + 2x - 1 = 0$

$\Rightarrow x = \frac{-2 \pm \sqrt{4 + 16}}{8}$

$= \frac{-2 \pm 2\sqrt{5}}{8}$

$= \frac{-1 \pm \sqrt{5}}{4}$

But $x > 0$ and so

$x = \frac{-1 + \sqrt{5}}{4}$

(iv) $\angle CAM = 180° - 90° - 72°$ (angle sum of ΔAMC)

$= 18°$

In ΔAMC, $\sin 18° = \frac{x}{1}$

$= \frac{-1 + \sqrt{5}}{4}$

(b) (i) $P(AA) = \frac{1}{5}\cdot\frac{1}{5}$

P (any letter twice) $= 5 \times P(AA)$

$= \frac{1}{5}$

(ii) $P(\overline{E}) = 1 - \frac{1}{5} = \frac{4}{5}$

Solve $1 - \left(\frac{4}{5}\right)^n = \frac{99}{100}$

OR $1 - 0.8^n = 0.99$

$\Rightarrow 0.8^n = 0.01$

$n = \frac{\log_e 0.01}{\log_e 0.8}$

$= 20.63...$

$= 21$ (n is an integer)

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