QUESTION 1:

a) i) $$\int \frac{t^2-2}{t^3}\,dt = \int \frac{1}{t} - 2t^{-3}\,dt$$
$$= \ln|t| - \frac{2t^{-2}}{-2} + c$$
$$= \ln|t| + \frac{1}{t^2} + c$$

ii) $$\int xe^x\,dx = \int x\frac{d}{dx}(e^x)\,dx$$
$$= xe^x - \int e^x\,dx \quad \text{(by parts)}$$
$$= xe^x - e^x + c$$

iii) Let $$\int \frac{2x\,dx}{(x+1)(x+3)} = I = \int \frac{2x\,dx}{x^2+4x+3}$$

Let $$\frac{1}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$$

$$\therefore \quad 1 = A(x+3) + B(x+1)$$

Set $x=-3$: $1 = B(-2) \Rightarrow B = -\frac{1}{2}$

set $x=-1$: $1 = A(2) \Rightarrow A = \frac{1}{2}$

$$\therefore I = \int \frac{2x+4\,dx}{x^2+4x+3} - \int \frac{4\,dx}{(x+1)(x+3)}$$
$$= \ln|x^2+4x+3| - \frac{4}{2}\int \frac{1}{x+1} - \frac{1}{x+3}\,dx$$
$$= \ln|x^2+4x+3| - 2\ln|x+1| + 2\ln|x+3| + c$$
$$= \ln\left|\frac{(x+1)(x+3)(x+3)^2}{(x+1)^2}\right| + c$$
$$= \ln\left|\frac{(x+3)^3}{x+1}\right| + c$$

b) let $u = x-4$
$du = dx$

x	4	4.5
u	0	0.5

$$\therefore \int_4^{4.5} \frac{dx}{(x-3)(5-x)} = \int_0^{0.5} \frac{du}{(u+1)(1-u)}$$
$$= \frac{1}{2}\int_0^{0.5} \frac{1}{1-u} + \frac{1}{1+u}\,du$$
$$= \frac{1}{2}\Big[\ln|(1-u)(1+u)|\Big]_0^{0.5}$$
$$= \frac{1}{2}\left[\ln\left|\frac{\frac{1}{2}\cdot\frac{3}{2}}{1\cdot 1}\right|\right]$$
$$= \frac{1}{2}\ln\frac{3}{4}$$

c) i) $$u_n = \int_0^{\pi/2} x^n \sin x\,dx \quad , n \geq 2$$
$$= \Big[-\cos x . x^n\Big]_0^{\pi/2} - \int_0^{\pi/2} nx^{n-1}(-\cos x)\,dx \quad \text{(by parts)}$$
$$= 0 + n\int_0^{\pi/2} x^{n-1}\cos x\,dx$$
$$= \Big[n\,x^{n-1}\sin x\Big]_0^{\pi/2} - n\int_0^{\pi/2}(n-1)x^{n-2}\sin x\,dx$$
$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\,u_{n-2}$$

// as required

ii) $$u_2 = \int_0^{\pi/2} x^2 \sin x\,dx$$

$$u_0 = \int_0^{\pi/2}\sin x\,dx = \Big[-\cos x\Big]_0^{\pi/2} = 1$$

$$\therefore u_2 = 2\left(\frac{\pi}{2}\right)^1 - 2(1).1$$
$$= \pi - 2$$

PHOENIX MATHEMATICS
First Floor, 84-88 Bronte Road, Bondi Junction, 2022.
Ph: 9389-2699 Fax: 9386-1493
Web: www.maths.com.au Email: phoenix@maths.com.au

QUESTION 2:

a) i) $w = -1+\sqrt{3}i$

$w^2 = (-1+\sqrt{3}i)^2$

$= 1 - 3 - 2\sqrt{3}i$

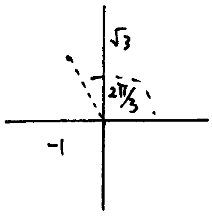
$= -2-2\sqrt{3}i$

$2\bar{w} = 2(-1-\sqrt{3}i)$

$= w^2$, as required.

ii) $|w| = \sqrt{(-1)^2+(\sqrt{3})^2}$

$= 2$



$\arg w = \frac{2\pi}{3}$

iii) $w = 2\,\text{cis}\,\frac{2\pi}{3}$

$\therefore LHS = w^3 - 8 = 8\,\text{cis}\,3\left(\frac{2\pi}{3}\right) - 8$

$= 8.1 - 8$

$= 0 = RHS$, as required

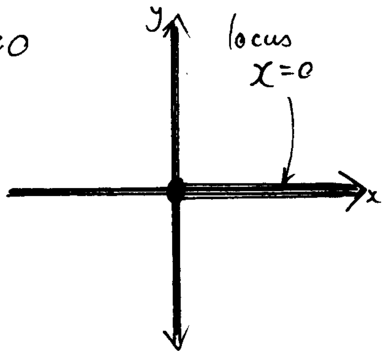
b) i) $\text{Re}(z) = |z|$, $\therefore x \geq 0$

$\therefore x = \sqrt{x^2+y^2}$

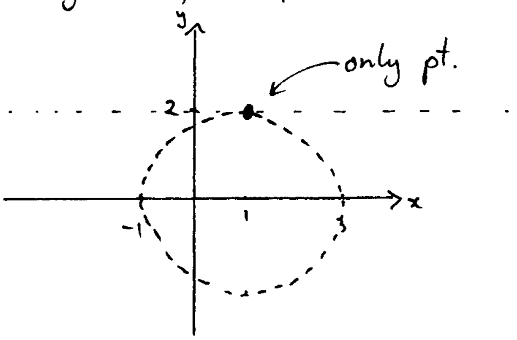
$x^2 = x^2+y^2$

$y^2 = 0$

$y = 0,\ x \geq 0$



ii) $y \geq 2$, $|z-1| \leq 2$



c) $\dfrac{a}{1+i} + \dfrac{b}{1+2i} = 1$

$a + 2ai + b + bi = (1+i)(1+2i)$

$(a+b) + (2a+b)i = 1 - 2 + 3i$

equating real and imaginary parts.

$a+b = -1$ ①

$2a+b = 3$ ②

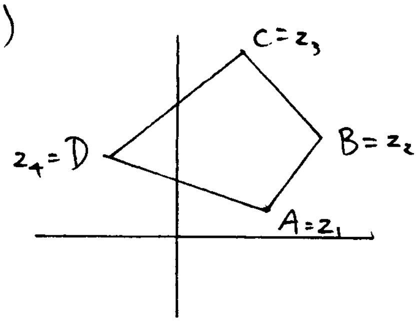
② − ①: $a = 4$ ③

③ ⇒ ①: $4+b = -1$

$b = -5$

$\therefore \underline{a = 4,\ b = -5}$

d)



Now $z_1 + z_3 = z_2 + z_4$

$\therefore z_1 - z_2 = z_4 - z_3$

ie $|z_1 - z_2| = |z_4 - z_3|$ ie $AB = CD$

$\arg(z_1 - z_2) = \arg(z_4 - z_3)$ $\therefore AB \parallel CD$

$\therefore$ since $AB = CD$ and $AB \parallel CD$

ABCD is a parallelogram. // as required

QUESTION 3

a) i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= (-b)^2 - 2(1)$

$= \underline{b^2-2}$

ii) all real if $\alpha^2+\beta^2+\gamma^2 \geq 0$

$\therefore b^2 - 2 \geq 0$

$\therefore \underline{-\sqrt{2} \leq b \leq \sqrt{2}}$

iii) if $2\alpha, 2\beta, 2\gamma$ are roots of new equation y, then $\alpha = \frac{y}{2}$ is a root of the original equation.

$\therefore \left(\frac{y}{2}\right)^3 + b\left(\frac{y}{2}\right)^2 + \frac{y}{2} + 2 = 0$

$y^3 + 2by^2 + 4y + 16 = 0$

b) let α be the root of multiplicity 3.

$\therefore P(\alpha) = P'(\alpha) = P''(\alpha) = 0$

$P'(x) = 4x^3 + 3x^2 - 6x - 5$

$P''(x) = 12x^2 + 6x - 6$

let $P''(x) = 0$ $\quad \therefore 2x^2 + x - 1 = 0$

$(2x-1)(x+1) = 0$

$\therefore x = \frac{1}{2}$, or -1

$P'(-1) = -4+3+6-5 = 0$

$\therefore -1$ is the triple root

since $P(x)$ is a quartic there are at most four roots. $\prod\alpha = -2$

$= (-1)^3\beta$, β is the other root

$\therefore \beta = 2$

$\therefore$ zeros of $P(x)$ are $-1, -1, -1, 2$.

c) i) $z^5 - 1 = 0$

$\therefore (z-1)(z^4+z^3+z^2+z+1) = 0$

but $z \neq 1$ $\therefore z^4+z^3+z^2+z+1 = 0$

but $z \neq 0$ $\therefore z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0$

ii) let $x = z + \frac{1}{z}$

$\therefore x^2 = z^2 + 2 + \frac{1}{z^2}$

$\therefore$ equation in (i) becomes

$x^2 + x - 1 = 0$

iii) now $x = z + \frac{1}{z}$, but $|z| = 1$

$\therefore x = z + \bar{z}$

$= 2\,\mathrm{Re}(z)$

from (i) $z = \text{cis}\,\frac{\pm 2\pi}{5}$ or $\text{cis}\,\frac{\pm 4\pi}{5}$.

$\therefore x = 2\cos\frac{2\pi}{5}$ or $2\cos\frac{4\pi}{5}$

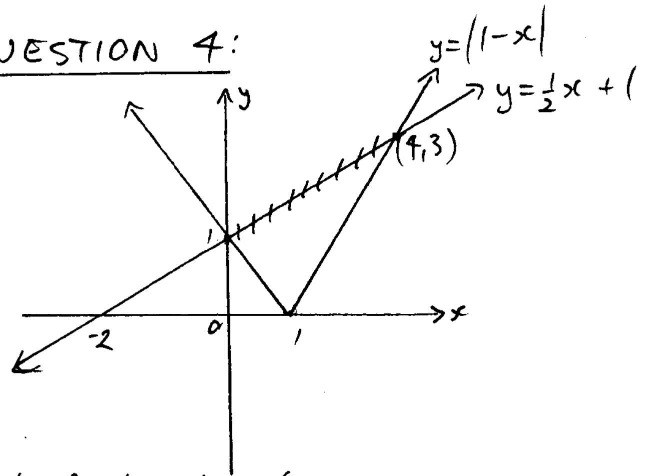
$\therefore$ product of roots from eqn. in (ii) gives

$2\cos\frac{2\pi}{5} \cdot 2\cos\frac{4\pi}{5} = -1$

$\therefore \cos\frac{2\pi}{5}\cos\frac{4\pi}{5} = -\frac{1}{4}$ // as required.

QUESTION 4:

a)(i)



ii) pts of intersection $(0,1)$,

$x-1=\frac{1}{2}x+1$

$\frac{1}{2}x=2$

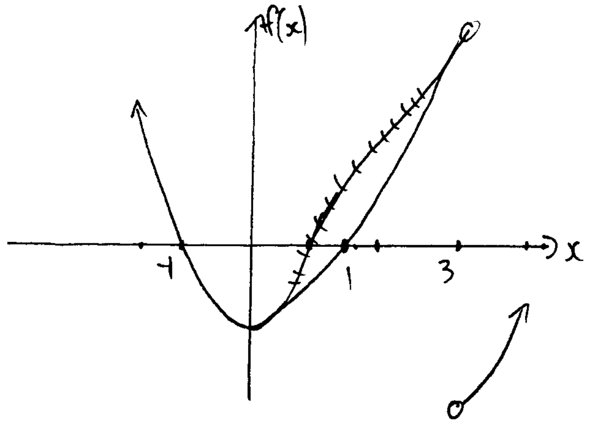
$x=4$

$\therefore (0,1),(4,3)$.

$\therefore \frac{1}{2}x+1 \geq |1-x|$

for $\underline{0 \leq x \leq 4}$.

b)



c)(i) $y=\dfrac{x^3+4}{x^2} = x+4x^{-2}$

$y' = 1-\dfrac{2.4}{x^3}$

$= 1-\dfrac{8}{x^3}$

Let $y'=0$ for stationary point.

$\therefore 0 = 1-\dfrac{8}{x^3}$

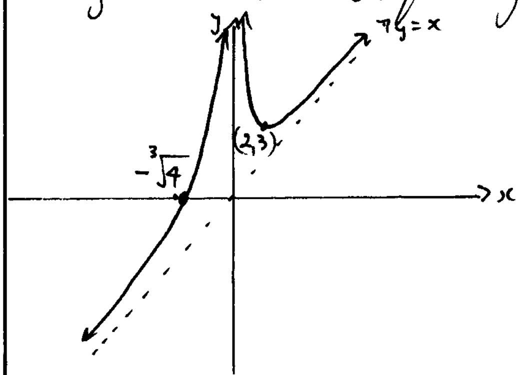
$\therefore x=2, y=3$

$y'' = \dfrac{24}{x^4} > 0$ for all x

$\therefore$ there is no point of inflexion, it is always concave up and $(2,3)$ is a minimum turning point.

ii) $x=0$ is a vertical asymptote

$y=x$ is an oblique asymptote.



Let $y=0$ for x-intercept. $\therefore x^3+4=0$

$x=-\sqrt[3]{4}$

iv) $x^3-kx^2+4=0$

$\dfrac{x^3+4}{x^2}=k$

ie ~~for~~ what horizontal lines pass through the graph 3 times. ie $\underline{k>3}$.

QUESTION 5:

a) Volume of a typical slice,

$$\delta V = (2x)^2 \delta y = 4x^2 \delta y$$

$$\therefore V \doteqdot \sum_{y=1}^{4} \delta V$$

$$= \sum_{y=1}^{4} 4x^2 \delta y$$

$$\therefore V = \lim_{\delta y \to 0} \sum_{y=1}^{4} 4 \cdot \frac{1}{y^2} \delta y$$

$$= \int_1^4 \frac{4}{y^2}\, dy$$

$$= \left[\frac{-4}{y}\right]_1^4$$

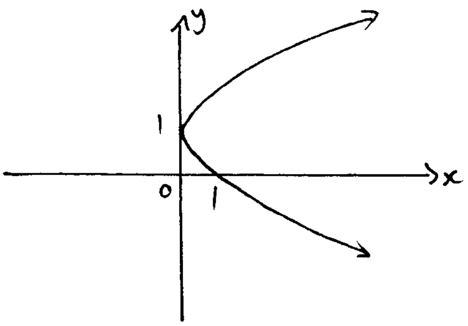
$$= 4\left(1 - \frac{1}{4}\right)$$

$$= 3 \quad u^3$$

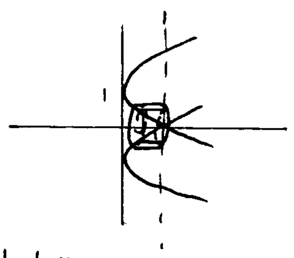
b) (i) $2y = y^2 - x + 1$

$x = y^2 - 2x + 1$

$x = (y-1)^2$



(ii)



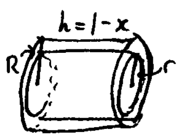
$\pm\sqrt{x} = y - 1$

$y = 1 \pm \sqrt{x}$

$h = 1 - x = 2y - y^2$

$r = y$

$R = y + \delta y$



typical shell volume $\delta V = \pi(R^2 - r^2)h$

$$\doteqdot \pi\left((y+\delta y)^2 - y^2\right)(1-x)$$

$$= \pi(y+\delta y+y)(y+\delta y-y)(1-x)$$

$$= \pi(2y+\delta y)\delta y(1-x)$$

$$= \pi(1-x)(2y\delta y + \delta y^2)$$

$$\doteqdot \pi(1-x)2y\delta y,$$

since δy^2 is negligible.

Now $V = \lim_{\delta y \to 0} \sum_{y=0}^{2} \pi(1-x)2y\,\delta y$

$$= \int_0^2 \pi(2y - y^2)2y\, dy$$

$$= 2\pi\int_0^2 2y^2 - y^3\, dy$$

$$= 2\pi\left[\frac{2}{3}y^3 - \frac{1}{4}y^4\right]_0^2$$

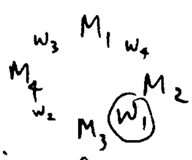
$$= 2\pi\left[\frac{16}{3} - 4 - 0\right]$$

$$= 2\pi \cdot \frac{4}{3}$$

$$= \frac{8\pi}{3}$$

// as required.

c) fix one combo of 4 men, only 2 places for W_1, choose 1. Then only 1 place for remaing women.



orderings of men.

$\therefore$ No. of combos $= 3! \times 2$

$= 12$

QUESTION 6:

a) i) $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{2at}{2a} = t$

$\therefore$ equation of tgt at P is

$y - at^2 = t(x - 2at)$

$\frac{y}{t} - at = x - 2at$

$\frac{y}{t} + at = x$ // as required.

ii) directrix is $y = -a$

$\therefore$ for Q, let $y = -a$

$\therefore \frac{-a}{t} + at = x$

$\therefore$ Q is $\left(at - \frac{a}{t}, -a\right)$

$M = \left(\frac{at - \frac{a}{t} + 2at}{2}, \frac{-a + at^2}{2}\right)$

$= \left(\frac{3at}{2} - \frac{a}{2t}, \frac{at^2}{2} - \frac{a}{2}\right)$

iii) $LHS = x^2(2y + a)$

$= \left(\frac{3at}{2} - \frac{a}{2t}\right)^2 (at^2 - a + a)$

$= at^2 \cdot \frac{a^2}{4}\left(3t - \frac{1}{t}\right)^2$

$= \frac{a^3t^2}{4}\left(9t^2 - 6a + \frac{a^2}{t^2}\right)$

$RHS = a(3y + a)^2$

$= a\left(\frac{3at^2}{2} - \frac{3a}{2} + a\right)^2$

$= a^3\left(\frac{3t^2}{2} - \frac{1}{2}\right)^2$

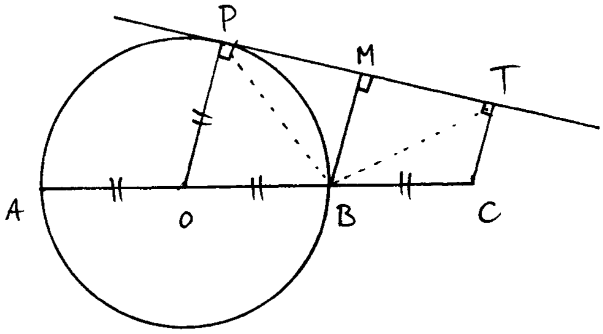
$= \frac{a^3}{4}\left(9t^4 - 6t^2 + 1\right)$

$= \frac{a^3t^2}{4}\left(9t^2 - 6 + \frac{1}{t^2}\right)$

$= LHS$

// as required

b) (i)



(ii) $\angle OPT + \angle OCT = 90° + 90°$ (angle made by tgt and radius)

$= 180°$

$\therefore OP \parallel TC$ (converse of cointerior angles are supplementary)

iii) construct perpendicular from B to PT at M. Similarly to (ii) $OP \parallel BM \parallel CT$.

$\therefore \frac{MT}{PM} = \frac{BC}{BO} = \frac{1}{1}$ (parallel lines cut lines in same proportions)

$\therefore MT = PM$

also in Δs PMB + TMB,

MB is common,

$\angle PMB = \angle TMB = 90°$

$\therefore \Delta PMB \equiv \Delta TMB$ (SAS)

$\therefore BP = BT$ (corresponding sides of congruent triangles)

QED.

QUESTION 7

a)(i) $\ddot{x} = 0$ $\qquad$ $\ddot{y} = -g$

$\dot{x} = V\cos\theta$ $\qquad$ $\dot{y} = -gt + V\sin\theta$

$x = Vt\cos\theta + c$ $\qquad$ $y = \frac{-gt^2}{2} + Vt\sin\theta + d$

at $t=0$, $(x,y) = (0,0)$ $\therefore c = d = 0$

$\therefore t = \frac{x}{V\cos\theta}$

$$\therefore y = \frac{-g}{2}\left(\frac{x}{V\cos\theta}\right)^2 + V\left(\frac{x}{V\cos\theta}\right)\sin\theta$$

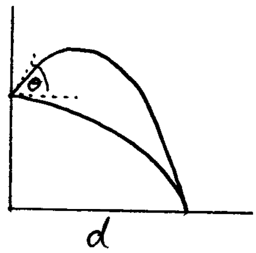
$$= -\frac{gx^2\sec^2\theta}{2V^2} + x\tan\theta$$

as required

ii) when $\theta = 0$, $t = T$, let distance from point to target be d.

$$\therefore T = \frac{d}{V\cos 0}$$

$$\therefore V = \frac{d}{T}$$



$$y_{\theta=0} = \frac{-gx^2}{2V^2}$$

$$y_{\theta=\theta} = x\tan\theta - \frac{gx^2\sec^2\theta}{2V^2}$$

Now d satisfies when $y_{\theta=0} = y_{\theta=\theta}$

$$\therefore \frac{-gd^2}{2V^2} = d\tan\theta - \frac{gd^2\sec^2\theta}{2V^2}$$

$$0 = d\tan\theta + \frac{gd^2}{2V^2}(1 - \sec^2\theta)$$

$$= d\tan\theta - \frac{gd^2\tan^2\theta}{2V^2}$$

$$= d\tan\theta\left(1 - \frac{gd\tan\theta}{2V^2}\right)$$

$$\therefore d = 0 \text{ or } d = \frac{2V^2}{g\tan\theta}$$

but $d > 0$

$$\therefore d = \frac{2\left(\frac{d}{T}\right)^2}{g\tan\theta}$$

$$\therefore d = \tfrac{1}{2}gT^2\tan\theta$$

as required.

b)i) $m = 1$ $\therefore \ddot{x} = F - kv$

ii) let $t = \alpha$ at beginning of time interval.

$$\frac{dv}{dt} = F - kv$$

$$\frac{dt}{dv} = \frac{1}{F - kv}$$

$$\int_{\alpha}^{\alpha+T} dt = \int_{u}^{2u} \frac{1}{F - kv}\,dv$$

$$\left[t\right]_{\alpha}^{\alpha+T} = -\frac{1}{k}\left[\ln(F - kv)\right]_{u}^{2u}$$

$$T = -\frac{1}{k}\ln\left|\frac{F - 2ku}{F - ku}\right|$$

$$e^{-kT} = \frac{F - 2ku}{F - ku}$$

$$F - ku = Fe^{kT} - 2kue^{kT}$$

$$F(e^{kT} - 1) = 2kue^{kT} - ku$$

$$F = \frac{ku(2e^{kT} - 1)}{e^{kT} - 1}$$

as required

7. b) iii) $x = \int_{\alpha}^{\alpha+T} v\, dt$

$\ddot{x} = F - kv$

$\frac{dv}{dt} = F - kv$

$t = \int \frac{1}{F-kv}\, dv$

$= -\frac{1}{k} \ln|F-kv| + c$

$\alpha = -\frac{1}{k} \ln|F-ku| + c$

$c = \alpha + \frac{1}{k} \ln|F-ku|$

$\therefore t = \alpha + \frac{1}{k} \ln\left(\frac{F-ku}{F-kv}\right)$

$e^{k(t-\alpha)} = \frac{F-ku}{F-kv}$

$F-kv = (F-ku)e^{-k(t-\alpha)}$

$v = \frac{1}{k}\left(F - \frac{(F-ku)e^{-k(t-\alpha)}}{\cancel{k}}\right)$

$x = \int_{\alpha}^{\alpha+T} v\, dt$

$= \frac{1}{k}\left[\left(Ft + \frac{(F-ku)}{k} e^{-k(t-\alpha)}\right)\right]_{\alpha}^{\alpha+T}$

$= \frac{1}{k}\left(FT + \frac{F-ku}{k}\left(e^{-kT} - 1\right)\right)$

$= \frac{1}{k}\left(FT + ku\left(\frac{2e^{kT}-1-e^{kT}+1}{k(e^{kT}-1)} \cdot \frac{(1-e^{kT})}{e^{kT}}\right)\right)$

$= \frac{1}{k}\left(FT + \frac{u(-e^{kT})}{e^{kT}}\right)$

$= \frac{1}{k}\left(FT - u\right)$

$= \frac{1}{k}\left(Tku\left[\frac{2e^{kT}-1}{e^{kT}-1}\right] - u\right)$

QUESTION 8.

1 (a) $(a-b)^2 \geqslant 0$

$a^2+b^2-2ab \geqslant 0$

$\frac{a^2+b^2}{2} \geqslant ab$

substitute a for a^2, b for b^2

$\therefore \frac{a+b}{2} \geqslant \sqrt{ab}$ // as required.

ii) $(a+b)^2 = a^2+b^2+2ab$

~~but~~ $a+b=1$ $\therefore$ $a^2+b^2 = 1-2ab$

but $ab \leq \left(\frac{a+b}{2}\right)^2 = \frac{1}{4}$

$\therefore$ $a^2+b^2 \geqslant 1-2\left(\frac{1}{4}\right)$

$= \frac{1}{2}$ // as required

b) $(\sqrt{a}+\sqrt{b})^2 = a+b+2\sqrt{ab}$

$\leq a+a+2\sqrt{aa}$

$=4a$

also $a+b+2\sqrt{ab} \geqslant b+b+2\sqrt{b.b}$

$=4b$

result follows.

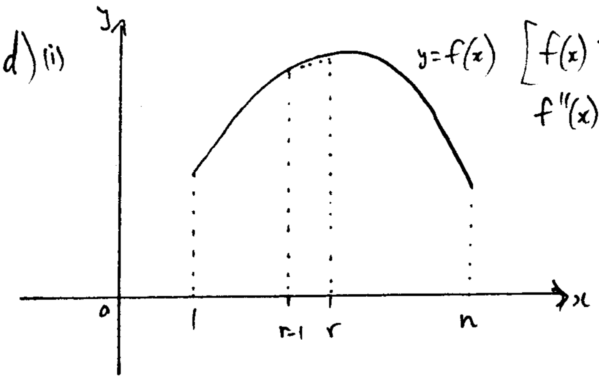
c) $P(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)$

$P'(x) = 1.(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n) +$
$(x-\alpha_1).1.(x-\alpha_3)\ldots(x-\alpha_n) +$
$\vdots$
$(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_{n-1}).1.$

$= \frac{P(x)}{x-\alpha_1} + \frac{P(x)}{x-\alpha_2} + \ldots + \frac{P(x)}{x-\alpha_n}$

// as required.

d) (i)



$\left[f(x) \geqslant 0,\ f''(x) < 0\right]$

Because $f(x) \geqslant 0$ and $f''(x)<0$, the curve always lies above the trapezium.
Now area of trapezium from $x=r-1$ to $x=r$

$= \frac{1}{2}\left(f(r-1)+f(r)\right).\left(r-(r-1)\right)$

$= \frac{1}{2}\left(f(r-1)+f(r)\right)$

Because the curve always lies above the trapezia the area under the curve is greater than the area of the trapezia.

$\therefore \int_1^n f(x)\,dx > \sum_{r=2}^{n}\left(\frac{1}{2}(f(r-1)+f(r))\right)$

$= \left[\frac{1}{2}f(1)+\frac{1}{2}f(2)\right]+\left[\frac{1}{2}f(2)+\frac{1}{2}f(3)\right]+\ldots$

$+\left[\frac{1}{2}f(n-1)+\frac{1}{2}f(n)\right]$

$= \frac{1}{2}f(1)+\frac{1}{2}f(n)+f(2)+f(3)+\ldots+f(n-1)$

$= \sum_{r=2}^{n-1}f(r) + \frac{1}{2}f(1)+\frac{1}{2}f(n)$

// as required.

(ii) $f(x) = \log_e x \geqslant 0$ for $x \geqslant 1$.

$f'(x) = \frac{1}{x}$

$f''(x) = \frac{-1}{x^2} < 0$ for $x>1$

$\therefore$ from part (i)

$\int_1^n \log_e x\,dx > \sum_{r=2}^{n-1}\log_e r + \frac{1}{2}\log_e 1 + \frac{1}{2}\log_e n$