Sydney Grammar School

4 unit mathematics

Trial HSC Examination 1999

1. (a) Let $z = \frac{1-i}{2+i}$.
(i) Show that $z + \frac{1}{z} = \frac{3+2i}{3-i}$.
(ii) Hence find **($\boldsymbol{\alpha}$)** $\overline{(z + \frac{1}{z})}$, in the form $a + ib$, where a and b are real,
($\boldsymbol{\beta}$) $\Im(z + \frac{1}{z})$.
(b) (i) Express $-\sqrt{27} - 3i$ in modulus-argument form.
(ii) Hence find $(-\sqrt{27} - 3i)^6$, giving your answer in the form $a + bi$, where a and b are real.
(c) Sketch on separate Argand diagrams the locus of z defined as follows:
(i) $\arg(z - 1) = \frac{3\pi}{4}$ **(ii)** $\Re\big(z(\overline{z} + 2)\big) = 3$.
(d) If z is a complex number such that $z = k(\cos\phi + i\sin\phi)$, where k is real, show that $\arg(z + k) = \frac{1}{2}\phi$.

2. (a) Find $\int x^2 e^{-2x}\, dx$.
(b) (i) Resolve $\frac{9+x-2x^2}{(1-x)(3+x^2)}$ into partial fractions.
(ii) Hence find $\int \frac{9+x-2x^2}{(1-x)(3+x^2)}\, dx$.
(c) Evaluate each of the following:
(i) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$,
(ii) $\int_0^{\frac{\pi}{3}} \sec^4\theta \tan\theta\, d\theta$,
(iii) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta+\cos\theta}\, d\theta$. (Hint: Use the substitution $t = \tan\frac{\theta}{2}$.)

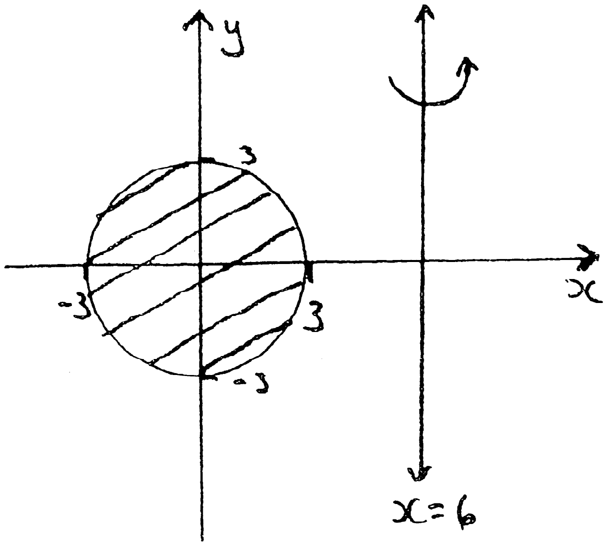
3. (a) Consider the function $y = \ln(\ln x)$.
(i) State the domain of the function.
(ii) Prove that the function is increasing at all points in its domain.
(iii) On separate number planes, sketch the following, clearly labelling all axial intercepts and asymptotes:
($\boldsymbol{\alpha}$) $y = \ln(\ln x)$, **($\boldsymbol{\beta}$)** $y = \ln(\ln|x|)$, **($\boldsymbol{\gamma}$)** $\ln|\ln x|$.
(b) Find a cubic equation with roots α, β and γ such that:

$$\begin{aligned} \alpha\beta\gamma &= 5, \text{ and} \\ \alpha + \beta + \gamma &= 7, \text{ and} \\ \alpha^2 + \beta^2 + \gamma^2 &= 29. \end{aligned}$$

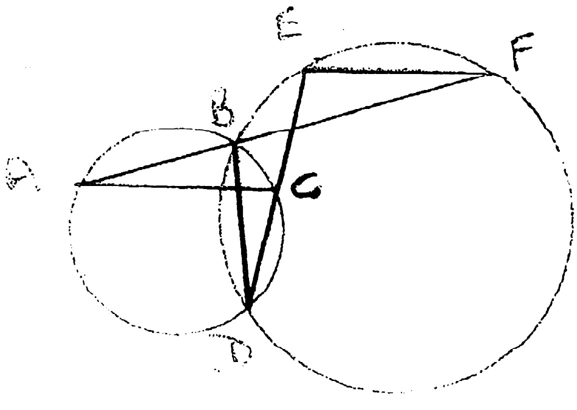
(c) If α, β and γ are the roots of the equation $8x^3 - 4x^2 + 6x - 1 = 0$, find the equation whose roots are $\frac{1}{1-\alpha}$, $\frac{1}{1-\beta}$ and $\frac{1}{1-\gamma}$.

(d) If the equation $x^3 + 3kx + \ell = 0$ has a double root, where k and ℓ are real, prove that $\ell^2 = -4k^3$.

4. (a) (i) Prove that the function $f(x) = x\sqrt{a^2 - x^2}$ is odd.



(ii) The diagram shows the region $x^2+y^2 \leq 9$ and the line $x = 6$. Copy the diagram.
(iii) Use the method of cylindrical shells to show that if the region $x^2 + y^2 \leq 9$ is rotated about the line $x = 6$ the volume V of the torus formed is given by $V = 24\pi \int_{-3}^{3} \sqrt{9 - x^2}\, dx - 4\pi \int_{-3}^{3} x\sqrt{9 - x^2}\, dx$.
(iv) Hence find the volume of the torus.
(b)

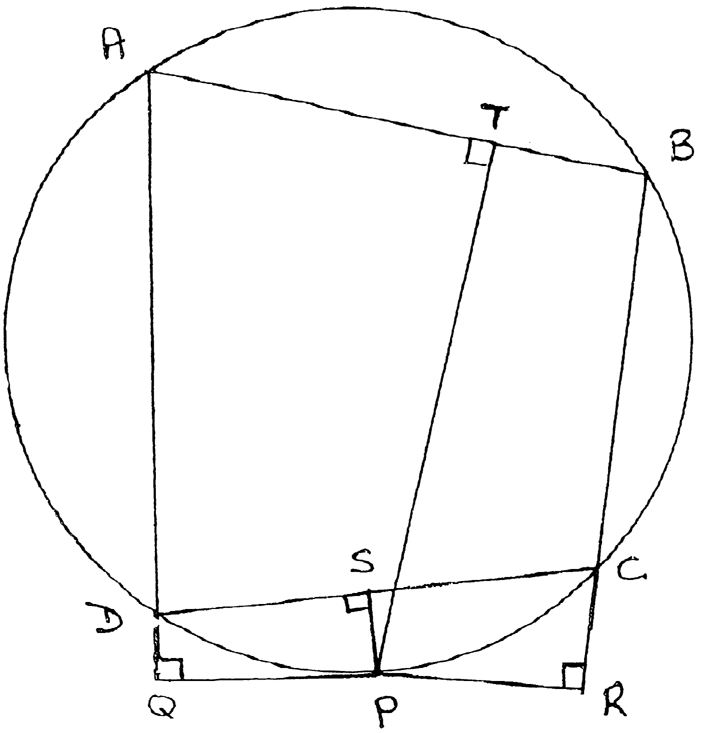


In the diagram above, ABF and DCE are straight lines.
(i) Copy the diagram.
(ii) Prove that AC is parallel to EF.
(c) A car of mass 1.8 tonnes negotiates a curve of radius 130 metres, banked at an angle of 9° to the horizontal, at a constant speed of 70 km/h. Take the acceleration due to gravity to be 10m/s^2.

(i) Draw a diagram showing all the forces acting on the car.
(ii) By resolving forces vertically and horizontally, calculate the frictional force between the road surface and the wheels, to the nearest Newton.
(iii) What speed (to the nearest km/h) must the driver maintain in order for the car to experience no sideways frictional force?

5. **(a)** A particle of mass m projected vertically upwards with initial speed u metres per second experiences a resistance of magnitude kmv Newtons when the speed is v metres per second where k is a positive constant. After T seconds the particle attains its maximum height h. Let the acceleration due to gravity be g m/s^2.
(i) Show that the acceleration of the particle is given by $\ddot{x} = -(g + kv)$, where x is the height of the particle t seconds after the launch.
(ii) Prove that T is given by $T = \frac{1}{k}\ln(\frac{g+ku}{g})$ seconds.
(iii) Prove that h is given by $h = \frac{u-gT}{k}$ metres.
(b) Let A and B be the points $(1, 1)$ and $(b, \frac{1}{b})$ respectively, where $b > 1$.
(i) The tangents to the curve $y = \frac{1}{x}$ at A and B intersect at $C(\alpha, \beta)$. Show that $\alpha = \frac{2b}{b+1}$ and $\beta = \frac{2}{b+1}$.
(ii) Let A', B' and C' be the points $(1, 0), (b, 0)$ and $(\alpha, 0)$ respectively.
(α) Draw a diagram that represents the information above.
(β) Obtain an expression for the sum of the areas of the quadrilaterals $ACC'A'$ and $CBB'C'$.
(γ) Hence or otherwise prove that for $u > 0$, $\frac{2u}{2+u} < \ln(1 + u) < u$.

6. (a)



In the diagram above, $ABCD$ is a cyclic quadrilateral. P is a point on the circle through A, B, C and D. PQ, PR, PS and PT are the perpendiculars from P to AD produced, BC produced, CD and AB respectively.

(i) Copy the diagram.

(ii) Explain why $SPRC$ and $AQPT$ are cyclic quadrilaterals.

(iii) Hence show that $\angle SPR = \angle QPT$ and $\angle PRS = \angle PTQ$.

(iv) Prove that $\triangle SPR$ is similar to $\triangle QPT$.

(v) Hence show that **($\boldsymbol{\alpha}$)** $PS \times PT = PQ \times PR$, **($\boldsymbol{\beta}$)** $\frac{PS \times PR}{PQ \times PT} = \frac{SR^2}{QT^2}$.

(b) The function f is given by $f(x) = e^{x/(1+kx)}$, where k is a positive constant.

(i) Find $f'(x)$ and $f''(x)$.

(ii) Show $f(x)$ has a point of inflexion at $(\frac{1}{2k^2} - \frac{1}{k}, e^{\frac{1}{k}-2})$.

(iii) Show that the tangent to $y = f(x)$ at $x = a$ passes through the origin if and only if $k^2a^2 + (2k-1)a + 1 = 0$.

(iv) Hence show that no tangents to $y = f(x)$ pass through the origin if and only if $k > \frac{1}{4}$.

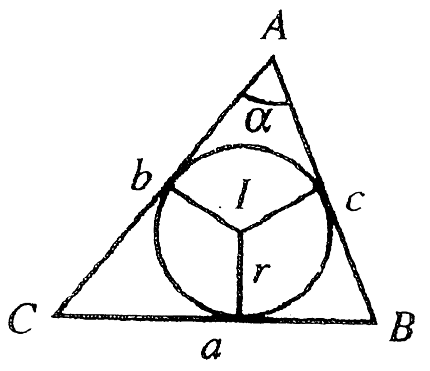
7. (a) Let $P(z) = z^7 - 1$.

(i) Solve the equation $P(z) = 0$, displaying your seven solutions on an Argand diagram.

(ii) Show that $P(z) = z^3(z-1)\big((z+\frac{1}{z})^3 + (z+\frac{1}{z})^2 - 2(z+\frac{1}{z}) - 1\big)$.

(iii) Hence solve the equation $x^3 + x^2 - 2x - 1 = 0$.

(iv) Hence prove that $\operatorname{cosec}\frac{\pi}{14}\operatorname{cosec}\frac{3\pi}{14}\operatorname{cosec}\frac{5\pi}{14} = 8$.
(b)



The diagram above shows a circle, centre I and radius r, touching the three sides of a triangle ABC. Denote AB by c, BC by a and AC by b. Let $\angle BAC = \alpha$, $s = \frac{1}{2}(a+b+c)$ and $\Delta =$ the area of triangle ABC.
(i) By considering the area of the triangles AIB, BIC and CIA, or otherwise, show that $\Delta = rs$.
(ii) By using the formula $\Delta = \frac{1}{2}bc\sin\alpha$, show that $\Delta^2 = \frac{1}{16}(4b^2c^2 - (2bc\cos\alpha)^2)$.
(iii) Use the cosine rule to show that $\Delta^2 = \frac{1}{16}(a^2-(b-c)^2)((b+c)^2-a^2)$. Hence deduce that $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.
(iv) A hole in the shape of the triangle ABC is cut in the top of a level table. A sphere of radius R rests in the hole. Find the height of the centre of the sphere above the level of the table-top, expressing your answer in terms of a, b, c, s and R.

8. **(a)** For $n = 0, 1, 2, 3, \ldots$, define $I_n = \int_0^{\frac{\pi}{4}} \frac{1-\cos 2nx}{\sin 2x}\,dx$.
(i) Evaluate I_1.
(ii) Show that, for $r \geq 1$:
(α) $I_{2r+1} - I_{2r-1} = \frac{1-(-1)^r}{2r}$.
(β) $I_{2r} - I_{2r-2} = \frac{1}{2r-1}$.
(iii) Hence evaluate I_8 and I_9.
(b) The Bernoulli polynomials $B_n(x)$, are defined by $B_0(x) = 1$ and, for $n = 1, 2, 3, \ldots$, $\frac{dB_n(x)}{dx} = nB_{n-1}(x)$, and $\int_0^1 B_n(x)\,dx = 0$. Thus

$$B_1(x) = x - \tfrac{1}{2},$$
$$B_2(x) = x^2 - x + \tfrac{1}{6},$$
$$B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x.$$

(i) Show that $B_4(x) = x^2(x-1)^2 - \frac{1}{30}$.
(ii) Show that, for $n \geq 2$, $B_n(1) - B_n(0) = 0$.
(iii) Show, by mathmatical induction, that for $n \geq 1$: $B_n(x+1) - B_n(x) = nx^{n-1}$.
(iv) Hence show that for $n \geq 1$ and any positive integer k:
$n\sum_{m=0}^{k} m^{n-1} = B_n(k+1) - B_n(0)$.
(v) Hence deduce that $\sum_{m=0}^{135} m^4 = 9\ 134\ 962\ 308$.