CSSA

CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW
2013 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION
MATHEMATICS EXTENSION 1

Section I
10 marks

Questions 1-10 (1 mark each)

Question 1 (1 mark)
Outcomes Assessed: HE6
Targeted Performance Bands: E2

Solution	Answer	Mark
$\int 2\sin^2 x\,dx = \int (1-\cos 2x)\,dx$ $= x - \frac{1}{2}\sin 2x + C$	B	1

Question 2 (1 mark)
Outcomes Assessed: PE3
Targeted Performance Bands: E2

Solution	Answer	Mark
$(x,y) = \left(\frac{3(9)+2(-1)}{3+2}, \frac{3(-6)+2(4)}{3+2}\right)$ $= (5,-2)$	D	1

Question 3 (1 mark)
Outcomes Assessed: PE3
Targeted Performance Bands: E2

Solution	Answer	Mark
$P\left(\frac{-1}{2}\right) = 0 \Rightarrow 8\left(\frac{-1}{2}\right)^3 + a\left(-\frac{1}{2}\right)^2 - 4\left(-\frac{1}{2}\right) + 1 = 0$ $\therefore a = -8$	A	1

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Question 4 (1 mark)
Outcomes Assessed: PE3
Targeted Performance Bands: E3

Solution	Answer	Mark
Total number of arrangements of TWITTER $=\dfrac{7!}{3!}$ Number of arrangements with TTT grouped is the same as the number of arrangements of TWIER $=5!$ $\therefore$ probability Ts are grouped together $=\dfrac{5!}{\left(\frac{7!}{3!}\right)}=\dfrac{1}{7}$	C	1

Question 5 (1 mark)
Outcomes Assessed: PE3
Targeted Performance Bands: E2

Solution	Answer	Mark
$\alpha+\beta+\gamma=-\dfrac{-5}{1}=5$ $\alpha\beta+\alpha\gamma+\beta\gamma=\dfrac{1}{1}=1$ $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$ $=5^2-2\times1$ $=23$	B	1

Question 6 (1 mark)
Outcomes Assessed: PE2, PE3
Targeted Performance Bands: E3

Solution	Answer	Mark
The coefficient of x^3 in $(1-3x)^7$ is $\binom{7}{3}1^4(-3)^3=-945$	A	1

Question 7 (1 mark)
Outcomes Assessed: PE2
Targeted Performance Bands: E2

Solution	Answer	Mark
$\sin x-\sqrt{3}\cos x\equiv 2\sin x\cos\alpha+2\cos x\sin\alpha$ $\therefore 2\cos\alpha=1, 2\sin\alpha=-\sqrt{3}\Rightarrow\alpha=-\dfrac{\pi}{3}$	B	1

Question 8 (1 mark)
Outcomes Assessed: HE4
Targeted Performance Bands: E3

Solution	Answer	Mark
$\frac{d}{dx}\sin^{-1}\left(\frac{3x}{4}\right)=\frac{1}{\sqrt{1-\left(\frac{3x}{4}\right)^2}}\times\frac{3}{4}$ $=\frac{3}{\sqrt{16-9x^2}}$	D	1

Question 9 (1 mark)
Outcomes Assessed: HE3
Targeted Performance Bands: E2

Solution	Answer	Mark
The vertical motion is the same for all three stones. ∴ All three stones reach the ground at the same time.	D	1

Question 10 (1 mark)
Outcomes Assessed: HE4
Targeted Performance Bands: E4

Solution	Answer	Mark
The graph of $y=f^{-1}(x)$ is the reflection of the graph of $y=f(x)$ about the line $y=x$. $\int_{-2}^{6} f^{-1}(x)dx=\text{area } A+\text{area of } 4\times 6 \text{ rectangle} - \text{area } B$ $=3+24-7$ $=20$ square units	C	1

Section II

60 marks

Question 11 (15 marks)
(a) (2 marks)
Outcomes assessed: HE6
Targeted Performance Bands: E2

Criteria	Mark
• Correct answer in terms of π	2
• Correct integration of $\dfrac{1}{\sqrt{4-x^2}}$	1

Sample answer:

$$\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_0^1$$
$$= \sin^{-1}\frac{1}{2} - \sin^{-1} 0$$
$$= \frac{\pi}{6}$$

(b) (3 marks)
Outcomes Assessed: PE3
Targeted Performance Bands: E3

Criteria	Mark
• Correct range of solutions given	3
• Significant progress towards solution (e.g. inequality if using solution presented below or points of intersection if using graphical techniques)	2
• Some progress towards answer (considering technique used e.g. graphical or algebraic)	1

Sample answer:

$$\frac{t-2}{t+3} > -2$$
$$(t+3)(t-2) > -2(t+3)^2$$
$$3t^2 + 13t + 12 > 0$$
$$(3t+4)(t+3) > 0$$
$$\therefore t < -3, t > \frac{-4}{3}$$

(c) (3 marks)
Outcomes Assessed: HE6
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	3
• Significant progress towards solution	2
• Correct integral after applying given substitution	1

Sample answer:

Let $u=\sqrt{x},\ du=\dfrac{dx}{2\sqrt{x}}$

$$\begin{aligned}\int\frac{e^{3\sqrt{x}}}{\sqrt{x}}\,dx &= 2\int\frac{e^{3\sqrt{x}}}{2\sqrt{x}}\,dx\\ &= 2\int e^{3u}\,du\\ &= \frac{2}{3}e^{3u}+C\\ &= \frac{2}{3}e^{3\sqrt{x}}+C\end{aligned}$$

(d)(i) (2 marks)
Outcomes Assessed: HE5
Targeted Performance Bands: E2

Criteria	Mark
• Correct derivative	2
• Significant progress applying the product rule	1

Sample answer:

$$\frac{d}{dx}(x\sin 2x)=2x\cos 2x+\sin 2x$$

(d)(ii) (2 marks)
Outcomes Assessed: PE5
Targeted Performance Bands: E3

Criteria	Mark
• Determines correct integral	2
• Significant progress towards the answer	1

Sample answer:

$$\int 2x\cos 2x\,dx+\int\sin 2x\,dx=x\sin 2x$$

$$\begin{aligned}\int x\cos 2x\,dx &= \frac{1}{2}x\sin 2x-\frac{1}{2}\int\sin 2x\,dx\\ &= \frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+C\end{aligned}$$

(e) (3 marks)

Outcomes Assessed: PE3

Targeted Performance Bands: E3

Criteria	Mark
• Correctly finds the height of the tower	3
• Makes significant progress towards finding the height of the tower	2
• Finds the correct expression for AC or BC in terms of h	1

Sample answer:

$$\tan 30° = \frac{h}{BC} \Rightarrow BC = \sqrt{3}h$$

$$\tan 45° = \frac{h}{AC} \Rightarrow AC = h$$

$$\left(\sqrt{3}h\right)^2 - h^2 = 400^2$$

$$2h^2 = 400^2$$

$$h = 200\sqrt{2}$$

$\therefore$ The height of the tower is $200\sqrt{2}$ metres.

Question 12 (15 marks)

(a)(i) (1 mark)

Outcomes assessed: HE3

Targeted Performance Bands: E3

Criteria	Mark
• Correct verification that T is a solution of the differential equation	1

Sample answer:

$$T = 20 + Ae^{-kt}$$

$$\frac{dT}{dt} = -kAe^{-kt}$$

$$= -k(T-20) \text{ since } Ae^{-kt} = T - 20$$

(a)(ii) (2 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E3

Criteria	Mark
• Correct values for A and k	2
• Significant progress towards solution	1

Sample answer:

$t = 0, T = 100 \Rightarrow A = 80$

$t = 5, T = 70 \Rightarrow 70 = 20 + 80e^{-5k}$

$e^{-5k} = \dfrac{5}{8}$

$\therefore k = \dfrac{-1}{5}\ln\left(\dfrac{5}{8}\right) \approx 0.094$ (3 d.p.)

(a)(iii) (1 mark)
Outcomes assessed: HE3
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	1

Sample answer:

$T = 20 + 80e^{\frac{1}{5}\ln\left(\frac{5}{8}\right)\times 15}$

$= 39.531\ldots$

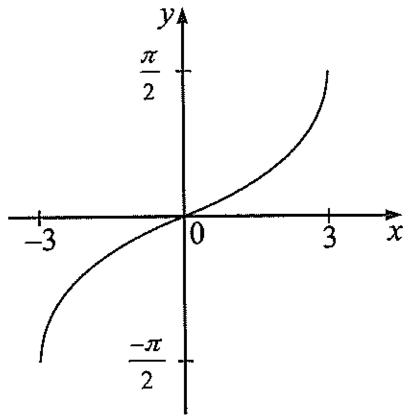
≈ 40 (nearest degree)

$\therefore$ The temperature of the soup after 15 minutes is 40°C (to the nearest degree)

(b) (2 marks)
Outcomes assessed: HE4
Targeted Performance Bands: E3

Criteria	Mark
• Correct sketch	2
• Correct shape with either correct domain or range	1

Sample answer:



(c) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	2
• Some progress towards the correct answer	1

Sample answer:
If Jack is seated first, there are 4 seats (not next to Jack) in which Jill can sit. There are then 5! ways to seat the remaining students.

$\therefore$ The total number of ways is $1 \times 4 \times 5! = 480$.

(d)(i) (2 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E3

Criteria	Mark
• Correct solution, including mentioning $f(x)$ is continuous	2
• Evaluates $f(0.7)$ and $f(0.8)$	1

Sample answer:

$f(0.7) = \ln 0.7 - \sin 0.7 + 1 = -0.00089\ldots < 0$

$f(0.8) = \ln 0.8 - \sin 0.8 + 1 = 0.05950\ldots > 0$

Since $f(x)$ is continuous for $x > 0$ and changes sign from $x = 0.7$ to $x = 0.8$, $f(x)$ has a zero between 0.7 and 0.8.

(d)(ii) (2 marks)
Outcomes assessed: PE5
Targeted Performance Bands: E3

Criteria	Mark
• Correct solution	2
• Correct application using halving-the-interval method	1

Sample answer:

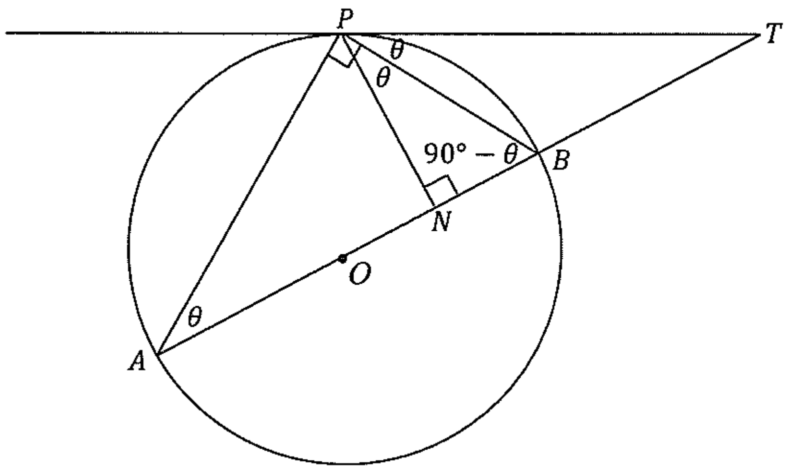
$f(0.75) = \ln 0.75 - \sin 0.75 + 1 = 0.030679\ldots > 0$

$\therefore f(x)$ has a zero between 0.7 and 0.75, hence the zero is 0.7 correct to one decimal place.

(e) (3 marks)
Outcomes assessed: PE3, HE7
Targeted Performance Bands: E3

Criteria	Mark
• Correct proof	3
• Significant progress towards required result	2
• Establishes one correct pair of equal or complementary angles leading to result	1

Sample answer:



Let $\angle TPB = \theta$

Hence, $\angle PAB = \theta$ (The angle between a tangent and a chord equals the angle in the alternate segment).

$\angle APB = 90°$ (angle subtended at the circumference by a diameter is 90°)

$\angle PBA = 90° - \theta$ (angle sum of ΔAPB is 180°)

$\angle BPN = \theta$ (angle sum of ΔNPB is 180°)

$\therefore \angle TPB = \angle BPN$, hence BP bisects $\angle NPT$.

Question 13 (15 marks)
(a)(i) (1 mark)
Outcomes assessed: HE3, HE5
Targeted Performance Bands: E3

Criteria	Mark
• Correct expression for acceleration in terms of x	1

Sample answer:

$$\begin{aligned}\ddot{x} &= \frac{d}{dx}\left(\frac{1}{2}v^2\right)\\ &= \frac{d}{dx}\left(-\frac{1}{2}x^2 - 2x + 6\right)\\ &= -x - 2\end{aligned}$$

(a)(ii) (2 marks)
Outcomes assessed: HE3, HE5
Targeted Performance Bands: E3

Criteria	Mark
• Correct centre and period of motion	2
• Correct centre or period of motion	1

Sample answer:
Since $\ddot{x} = -1^2(x+2)$,
Centre of motion is $x = -2$.
Period of motion is $\dfrac{2\pi}{n} = \dfrac{2\pi}{1} = 2\pi$ seconds.

(a)(iii) (1 mark)
Outcomes assessed: HE3, HE5
Targeted Performance Bands: E3

Criteria	Mark
• Correct maximum speed	1

Sample answer:
Maximum speed occurs at the centre of motion, $x = -2$.

$$v^2 = -(-2)^2 - 4(-2) + 12 = 16$$
$$v = \pm 4$$

$\therefore$ The maximum speed of the particle is $4\,\text{ms}^{-1}$.

(b) (3 marks)

Outcomes assessed: PE3

Targeted Performance Bands: E4

Criteria	Mark
• Correct co-ordinates for P	3
• Correct solutions for θ	2
• Significant progress towards solving $2\cos\theta - \cos 2\theta = 1$	1

Sample answer:

Solve $x = 1$ and $x = 2\cos\theta - \cos 2\theta$ simultaneously for points of intersection:

$$2\cos\theta - \cos 2\theta = 1$$

$$2\cos\theta - 2\cos^2\theta + 1 = 1$$

$$2\cos\theta(1 - \cos\theta) = 0$$

$$\cos\theta = 0,\ \cos\theta = 1$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, 0 \text{ for } 0 \le \theta < 2\pi$$

$\theta = \frac{\pi}{2} \Rightarrow y = 2\sin\frac{\pi}{2} - \sin\pi = 2$ which is in the first quadrant,

Hence P has coordinates $(1, 2)$.

Note: $\theta = \frac{3\pi}{2} \Rightarrow y = 2\sin\frac{3\pi}{2} - \sin 3\pi = -2$ which is the point of intersection in the 4th quadrant and $\theta = 0 \Rightarrow y = 2\sin 0 - \sin 0 = 0$ which is the point of intersection on the x-axis.

(c)(i) (2 marks)

Outcomes assessed: PE3

Targeted Performance Bands: E2

Criteria	Mark
• Correct proof	2
• Some progress towards the proof	1

Sample answer:

$$m_{OP} = \frac{ap^2}{2ap} = \frac{p}{2}$$

$$m_{OQ} = \frac{aq^2}{2aq} = \frac{q}{2}$$

$$OP \perp OQ \Rightarrow \frac{p}{2} \times \frac{q}{2} = -1.$$

$$\therefore pq = -4.$$

(c)(ii) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E3

Criteria	Mark
• Correct equation of the parabola	2
• Some attempt to eliminate p and q	1

Sample answer:

$$x = 2a(p+q) \Rightarrow p+q = \frac{x}{2a}$$

$$y = a\left(p^2+q^2\right) \Rightarrow p^2+q^2 = \frac{y}{a}$$

$pq = -4$ (from part i)

$$(p+q)^2 = p^2+q^2+2pq$$
$$\Rightarrow \left(\frac{x}{2a}\right)^2 = \frac{y}{a}+2(-4)$$
$$\Rightarrow \frac{x^2}{4a^2} = \frac{y}{a}-8$$
$$\Rightarrow x^2 = 4a(y-8a)$$

$\therefore$ The equation of the locus of R is a parabola.

(c)(iii) (2 marks)
Outcomes assessed: HE5
Targeted Performance Bands: E3

Criteria	Mark
• Correct solution	2
• Progress towards correct solution	1

Sample answer:

$$pq = -4 \Rightarrow q = \frac{-4}{p} \Rightarrow \frac{dq}{dp} = \frac{4}{p^2}$$

$$\frac{dq}{dt} = \frac{dq}{dp}\times\frac{dp}{dt}$$
$$= \frac{4}{p^2}\times\frac{dp}{dt}$$
$$= \frac{-q}{p}\times\frac{dp}{dt} \text{(since } pq=-4, \frac{4}{p} = -q\text{)}.$$

(c)(iv) (2 marks)
Outcomes assessed: HE5
Targeted Performance Bands: E4

Criteria	Mark
• Correct solution	2
• Progress towards finding an expression for $\frac{dy_Q}{dt}$ by applying the chain rule	1

Sample answer:

Given $\frac{dy_P}{dt} = 1$, we are required to prove $\frac{dy_Q}{dt} = \frac{-q^2}{p^2}$

$$y_P = ap^2 \Rightarrow \frac{dy_P}{dy_p} = 2ap$$

$$y_Q = aq^2 \Rightarrow \frac{dy_Q}{dy_q} = 2aq$$

$$\frac{dy_P}{dt} = \frac{dy_P}{dp} \times \frac{dp}{dt}$$

$$1 = 2ap \times \frac{dp}{dt}$$

$$\frac{dp}{dt} = \frac{1}{2ap}$$

$$\frac{dy_Q}{dt} = \frac{dy_Q}{dq} \times \frac{dq}{dt}$$

$$= 2aq \times \frac{-q}{p} \times \frac{dp}{dt} \text{ (using part (i))}$$

$$=2aq \times \frac{-q}{p} \times \frac{1}{2ap} \text{ (from above)}$$

$$= \frac{-q^2}{p^2}$$

$\therefore$ The y-coordinate of Q is decreasing at the rate $\frac{q^2}{p^2}$ units per second.

Question 14 (15 marks)
(a) (4 marks)
Outcomes assessed: HE2
Targeted Performance Bands: E4

Criteria	Mark
• Complete proof	4
• Significant progress towards the correct proof	3
• Makes use of the assumption	2
• Proof for $P(1)$	1

Sample answer:
Let $P(n)$ be the given proposition. $P(1)$ is true since $1^3 + 5(1) = 6$ which is divisible by 6.

Assume $P(k)$ is true for some positive integer k.
i.e. $k^3 + 5k = 6M$ for some integer M.

Prove $P(k+1)$ is true:

$$\begin{aligned}(k+1)^3 + 5(k+1) &= k^3 + 3k^2 + 8k + 6 \\ &= 6M - 5k + 3k^2 + 8k + 6 \text{ (using assumption)} \\ &= 6M + 6 + 3k^2 + 3k \\ &= 6(M+1) + 3k(k+1).\end{aligned}$$

Since either k or $k+1$ is even, $k(k+1)$ is divisible by 2. $\therefore 3k(k+1)$ is divisible by 6.
Also, as M is an integer, $6(M+1)$ is divisible by 6.
Hence, the above expression is divisible by 6.

$\therefore$ By the Principle of Mathematical Induction, $P(n)$ is true for integers $n \geq 1$.

(b)(i) (2 marks)

Outcomes assessed: HE3

Targeted Performance Bands: E3

Criteria	Mark
• Correct simplified expression	2
• Correct expansion of $(a+b)^n$ or $(a-b)^n$	1

Sample answer:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \binom{n}{4}a^{n-4}b^4 + \ldots + \binom{n}{n}b^n$$

$$(a-b)^n = \binom{n}{0}a^n - \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 - \binom{n}{3}a^{n-3}b^3 + \binom{n}{4}a^{n-4}b^4 + \ldots + \binom{n}{n}b^n$$

$$(a+b)^n + (a-b)^n = 2\binom{n}{0}a^n + 2\binom{n}{2}a^{n-2}b^2 + 2\binom{n}{4}a^{n-4}b^4 + \ldots + 2\binom{n}{n}b^n$$

(b)(ii) (1 mark)

Outcomes assessed: HE3

Targeted Performance Bands: E3

Criteria	Mark
• Correct expression	1

Sample answer:

The probability of obtaining exactly two sixes is given by $\binom{n}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{n-2}$.

(b)(iii) (2 marks)

Outcomes assessed: HE3

Targeted Performance Bands: E4

Criteria	Mark
• Complete proof	2
• Some progress towards an expression for the correct probability	1

Sample answer:

$$P(\text{even number of sixes}) = \binom{n}{0}\left(\frac{5}{6}\right)^n + \binom{n}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{n-2} + \binom{n}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^{n-4} + \ldots + \binom{n}{n}\left(\frac{1}{6}\right)^n$$

$$= \frac{1}{2}\left(\left(\frac{5}{6}+\frac{1}{6}\right)^n + \left(\frac{5}{6}-\frac{1}{6}\right)^n\right) \text{ using part i}$$

$$= \frac{1}{2}\left(1+\left(\frac{2}{3}\right)^n\right)$$

(c)(i) (1 mark)
Outcomes assessed: HE7
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	1

Sample answer:
$0 \le t \le 1$

(c)(ii) (3 marks)
Outcomes assessed: HE7
Targeted Performance Bands: E4

Criteria	Mark
• Correct proof and expression for y-coordinate	3
• Significant progress towards the correct proof and expression for y-coordinate	2
• Progress towards finding the equation of AP and solving simultaneously for P	1

Sample answer:
The equation of the line through $A(-1,0)$ with gradient t is $y = t(x+1)$.

For the coordinates of P solve $y = t(x+1)$ and $x^2 + y^2 = 1$ simultaneously:

$$x^2 + \left(t\left(x+1\right)\right)^2 = 1$$
$$\left(1+t^2\right)x^2 + 2xt^2 + \left(t^2 - 1\right) = 0$$

Since one of the roots of this quadratic is $x = -1$ (x-coordinate of A) and the product of roots is $\dfrac{t^2-1}{1+t^2}$, the other root is $\dfrac{1-t^2}{1+t^2}$.

Hence the x coordinate of P is $\dfrac{1-t^2}{1+t^2}$.

Substituting into $y = t(x+1)$ gives $y = t\left(\dfrac{1-t^2}{1+t^2} + 1\right) = \dfrac{2t}{1+t^2}$

Hence the y coordinate of P is $\dfrac{2t}{1+t^2}$.

(c)(iii) (1 mark)
Outcomes assessed: HE7
Targeted Performance Bands: E3

Criteria	Mark
• Correct solution	1

Sample answer:

$$\tan\theta = m_{OP}$$
$$= \frac{2t}{1+t^2} \div \frac{1-t^2}{1+t^2}$$
$$= \frac{2t}{1-t^2},\ t \neq 1$$

(c)(iv) (1 mark)
Outcomes assessed: HE7
Targeted Performance Bands: E4

Criteria	Mark
• Correct value for t	1

Sample answer:

$\theta = 2\alpha$ since the angle at the centre of a circle is twice the angle at the circumference, subtended by the same arc.

$\alpha = 22\frac{1}{2}° \Rightarrow \theta = 45°$

$\therefore \tan\theta = 1$

Hence, $\dfrac{2t}{1-t^2} = 1$

$$t^2 + 2t - 1 = 0$$
$$t = \frac{-2 \pm \sqrt{8}}{2}$$
$$t = -1 \pm \sqrt{2}$$
$$t = -1 + \sqrt{2},\ \text{since } 0 \le t \le 1$$