JAMES RUSE AGRICULTURAL HIGH SCHOOL
TERM 1 ASSESSMENT 1998
YEAR 12 3/4 UNIT

Time allowed: 85 minutes.

All questions are to be attempted

Each question is to be handed in separately.

Question 1 : START A NEW PAGE

(a) Evaluate exactly : $\sin^{-1}\left(\frac{-1}{2}\right)$

(b) Evaluate : $\lim\limits_{x \to 0}\left(\frac{\sin 4x}{2x}\right)$

(c) Differentiate with respect to x :

(i) $y = 2\tan(x^3)$

(ii) $y = x\sin^{-1} x$

(d) Find the exact value of the gradient to the curve $y = \operatorname{cosec} x$ at $x = \frac{7\pi}{6}$.

(e) Find $\frac{d}{dx}\cos^{-1}\left(\frac{1}{x}\right)$ in simplest terms.

Question 2 : START A NEW PAGE

(a) Evaluate : $\displaystyle\int_{-4}^{4} \frac{6\,dx}{x^2+16}$

(b) Find : $\displaystyle\int \frac{2\,dx}{\sqrt{3-4x^2}}$

(c) Find : $\displaystyle\int \frac{4x+5\,dx}{\sqrt{9-x^2}}$

(d) (i) Express $143\cos\theta - 24\sin\theta$ in the form $R\cos(\theta+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give α correct to four decimal places.

(ii) Solve the equation $143\cos\theta - 24\sin\theta = 100$, for $0 < \theta < 2\pi$, giving answer correct to four decimal places.

Question 3 : START A NEW PAGE

(a) Find the sum of : $3^n + 3^{n-1} + \dots\dots\dots + 3^{-2n}$

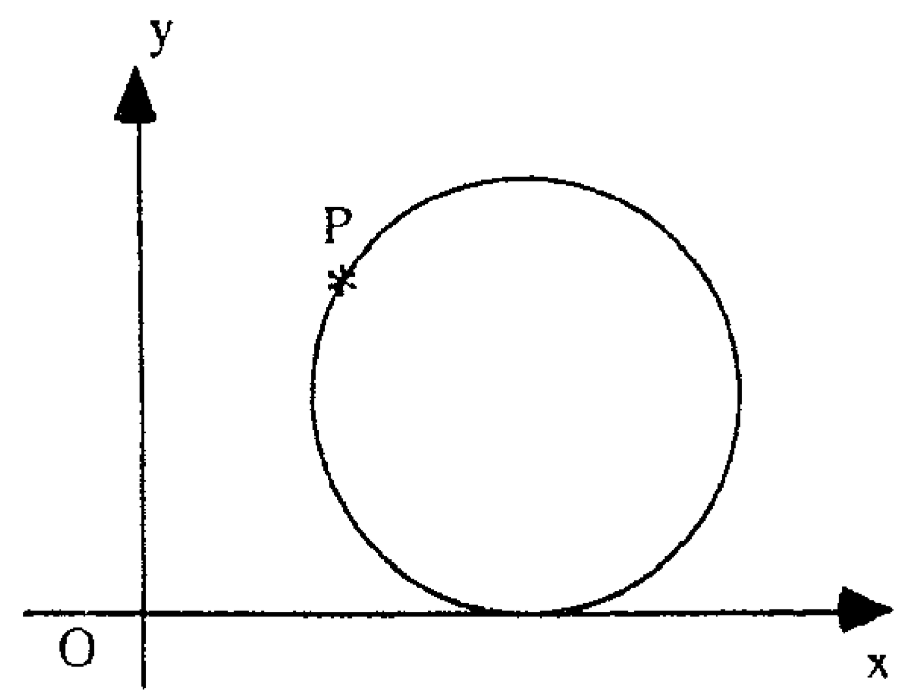
(b) A person borrows \$ 90, 000 at an interest rate of $7\frac{1}{2}$% per annum compounded monthly. It is to be paid with equal monthly instalments over 15 years. Find:

(i) the value of each monthly repayment.

(ii) the amount remaining on the loan after the 120th payment.

(iii) how many repayments are needed if the monthly instalment is increased to \$ 1000 after the 120th payment.

Question 4 : START A NEW PAGE



The cycloid curve is described by the locus of a fixed point P on the circumference of a circle as the circle moves horizontally along the x-axis . The parametric equations of a cycloid are given by :

$$x(\theta) = \theta - \sin\theta \text{ and } y(\theta) = 1 - \cos\theta .$$

(i) Copy and fill in the following table .

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x					
y					

(ii) Show that the y–axis is tangent to the cycloid at $\theta = 0$.

(iii) On a set of x and y axes, graph from the above table the path of P for $0 \le \theta \le 2\pi$.

(iv) The area bounded by the path of P for $0 \le \theta \le 2\pi$ and the x-axis is given by :

$$\text{Area} = \int_0^{2\pi} (1-\cos\theta)^2\, d\theta$$

Evaluate the exact area.

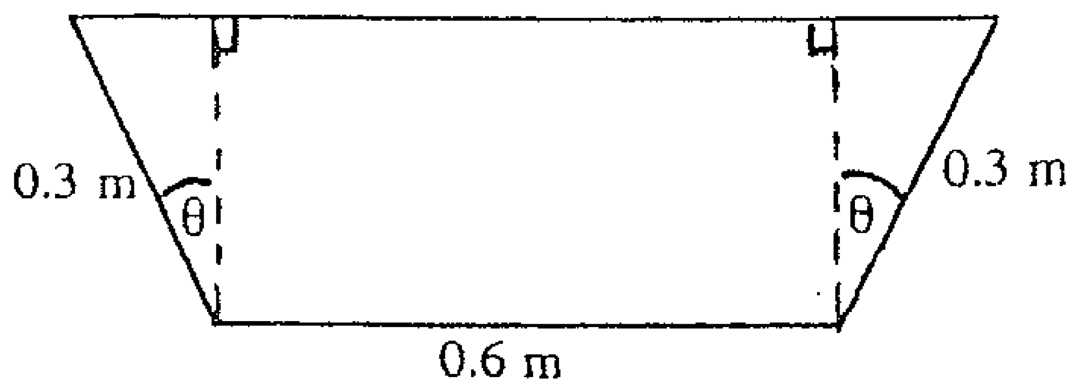
Question 5: START A NEW PAGE

(a) (i) Graph, on the same axes: $y = \sec x$ and $y = \tan x$ for $0 \le x < \frac{\pi}{2}$.

(ii) The area bounded by the positive y-axis, the curves $y = \sec x$ and $y = \tan x$, and the line $x = N$, where $0 < N < \frac{\pi}{2}$, is rotated about the x-axis. Find the volume of revolution in terms of N.

(iii) Find the volume as N approaches $\frac{\pi}{2}$.

(b)



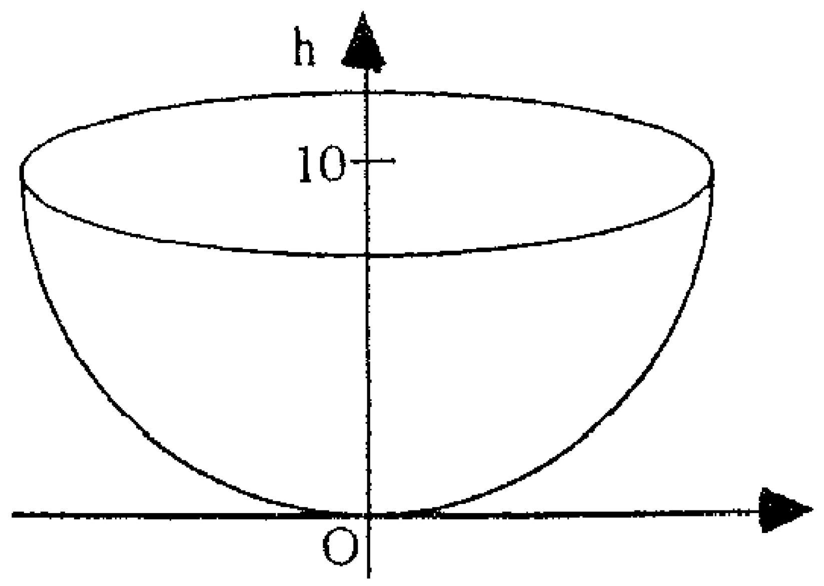
A farmer is to make a discharge chute with a trapezoidal cross-section as shown.

(i) Show the cross-sectional area A is given by: $A = 0.09 \cos\theta \,(2 + \sin\theta)\ \text{m}^2$

(ii) Hence find the angle θ (to the nearest degree for $0 < \theta < 90^0$) which will maximise the cross-sectional area A.

Question 6 : START A NEW PAGE

(a)



An empty hemi-spherical bowl 20 cm. in diameter rests on a flat surface. Water is poured into the bowl at a constant rate of $20\ \text{cm}^3$ / minute. After t minutes, the water level is h cm. above the base of the bowl. Find the rate at which the water level is rising when the deepest water in the bowl is 6 cm. [You may use $V = \frac{1}{3}\pi h^2 (30-h)$]

(b) (i) Draw a neat sketch of $y = f(x)$, where $f(x) = 2\sin^{-1}x$.

(ii) On the same axes, graph and **clearly label** $y = f^{-1}(x)$, the inverse of $y = f(x)$.

(iii) Find the exact value of the area bounded by the curves $y = f(x)$ and $y = f^{-1}(x)$, and the line $x = 1$.

THIS IS THE END OF THE PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \ne -1; \quad x \ne 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \ne 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \ne 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \ne 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \ne 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \ne 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \ne 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

1

(a) $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ (1)

(b) $\lim_{x\to 0} \frac{\sin 4x}{2x} = \lim_{x\to 0} \frac{\sin 4x}{4x} \cdot 2$

$= 2$ (2)

(c) (i) $\frac{d}{dx}\, 2\tan x^3 = 6x^2 \sec^2(x^3)$ (2)

(ii) $\frac{d}{dx}\, x \sin^{-1} x = \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x$

(d) $y = \operatorname{cosec} x$

$y' = -\operatorname{cosec} x \cot x$

$x = \frac{7\pi}{6}$ $\quad m = -\operatorname{cosec}\frac{7\pi}{6} \cot\frac{7\pi}{6}$ (2)

$= -(-2)\cdot\sqrt{3}$

$= 2\sqrt{3}$

e) $\frac{d}{dx} \cos^{-1}\left(\frac{1}{x}\right) = \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \cdot \frac{-1}{x^2}$

$= \frac{1}{\frac{\sqrt{x^2-1}}{\sqrt{x^2}}} \cdot \frac{1}{x^2}$

$= \frac{1}{|x|\sqrt{x^2-1}}$ (3)

2 (a) $\int_{-4}^{4} \frac{6\,dx}{x^2+16} = 6 \cdot \frac{1}{4}\left[\tan^{-1}\frac{x}{4}\right]_{-4}^{4}$

$= \frac{3}{2}\left\{\tan^{-1}1 - \tan^{-1}(-1)\right\}$

$= \frac{3}{2}\left\{\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right\}$

$= \frac{3\pi}{4}$ (2)

b) $\int \frac{2\,dx}{\sqrt{3-4x^2}} = \int \frac{2\,dx}{\sqrt{4\left(\frac{3}{4}-x^2\right)}}$

$= \sin^{-1}\frac{2x}{\sqrt{3}} + C$ (2)

c) $\int \frac{\ldots}{\sqrt{9-x^2}} \ldots = \int \left(4x(9-x^2)^{\ldots} + \frac{\ldots}{\sqrt{9-x^2}}\right) dx$

$= -4\sqrt{9-x^2} + 5\sin^{-1}\frac{x}{3} + C$ (3)

d(i) $143\cos\theta - 24\sin\theta = R\cos(\theta+\alpha)$

$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$\therefore$ $R\sin\alpha = 24$ $\quad R>0 \quad 0<\alpha<\frac{\pi}{2}$

$R\cos\alpha = 143$

$R = \sqrt{24^2 + 143^2}$

$= 145$

$\tan\alpha = \frac{24}{143}$

$\alpha = 0.1663$

$\therefore$ $143\cos\theta - 24\sin\theta = 145\cos(\theta + 0.1663)$ (2)

ii) $143\cos\theta - 24\sin\theta = 100$

$145\cos(\theta + 0.1663) = 100$

$\cos(\theta + 0.1663) = \frac{100}{145}$

$\theta + 0.1663 = 0.8098, \; 5.4734$ (3)

$\theta = 0.6435, \; 5.3071$ $\quad 0<\theta<2\pi$

3 (a) $S_n = 3^n + 3^{n-1} + \ldots + 3^{-2n}$

$\frac{S_n}{3} = 3^{n-1} + 3^{n-2} + 3^{-2n} + 3^{-2n-1}$

$\therefore \frac{2}{3} S_n = 3^n - 3^{-2n-1}$ (3)

$S_n = \frac{1}{2}\left[3^{n+1} - 3^{-2n}\right]$

3(b) Principal $P = \$90000$ Interest Rate $= 7\frac{1}{2}\%$ pa

$= 0.625\%$ per month

i) Let R be repayment per month

$\therefore$ Amount owing after 1st Payment $= P(1 + 0.00625) - R$

$A_1 = 1.00625P - R$

Amount owing after 2nd Payment $= (1.00625P - R)1.00625 - R$

$A_2 = 1.00625^2 P - R[1 + 1.00625]$

Amount owing after 3rd Payment $= (1.00625^2 P - R[1 + 1.00625])1.00625 - R$

$A_3 = 1.00625^3 P - R[1 + 1.00625 + 1.00625^2]$

$\vdots$

Amount owing after n Payments $= 1.00625^n P - R[1 + 1.00625 + \ldots 1.00625^{n}]$

$$A_n = 1.00625^n . P - R\left[\frac{1.00625^n - 1}{0.00625}\right] \quad - ⊗$$

For 15 years $n = 180$ $A_{180} = 0$ $P = 90000$

$$\therefore R\left[\frac{1.00625^{180} - 1}{0.00625}\right] = 90000 . 1.00625^{180}$$

$$R = \frac{90000 . 1.00625^{180} . 0.00625}{1.00625^{180} - 1}$$

4

Monthly Repayment $= \$834.31$

ii) $A_{120} = 90000 . 1.00625^{120} - 834.31\left(\frac{1.00625^{120} - 1}{0.00625}\right)$

Amount owing end 120th Payment $= \$41636.75$

2

iii) $R = 1000$ $P = 41636.75$, $A = 0$. by ⊗

$$1.00625^n . 41636.75 = 1000\left(\frac{1.00625^n - 1}{0.00625}\right)$$

$= 160000 . 1.00625^n - 160000$

$$1.00625^n = \frac{160000}{118363.25}$$

$= 1.35$

$$n = \frac{\ln 1.35}{\ln 1.00625} = 48.38 \Longrightarrow 49 \text{ payment}$$

3

4(i)

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	0	$\frac{\pi}{2} - 1$	π	$\frac{3\pi}{2} + 1$	2π
y	0	1	2	1	0

3

(ii) $y = 1 - \cos\theta$ $x = \theta - \sin\theta$

$\frac{dy}{d\theta} = \sin\theta$ $\frac{dx}{d\theta} = 1 - \cos\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta} \quad 1$$

At $\theta = 0$ $\frac{dy}{dx} = \lim_{\theta \to 0}\left(\frac{\sin\theta}{1 - \cos\theta} . \frac{1 + \cos\theta}{1 + \cos\theta}\right)$

$$= \lim_{\theta \to 0} \frac{\sin\theta . (1 + \cos\theta)}{1 - \cos^2\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta(1 + \cos\theta)}{\sin^2\theta}$$

$$= \lim_{\theta \to 0}\left(\frac{1 + \cos\theta}{\sin\theta}\right)$$

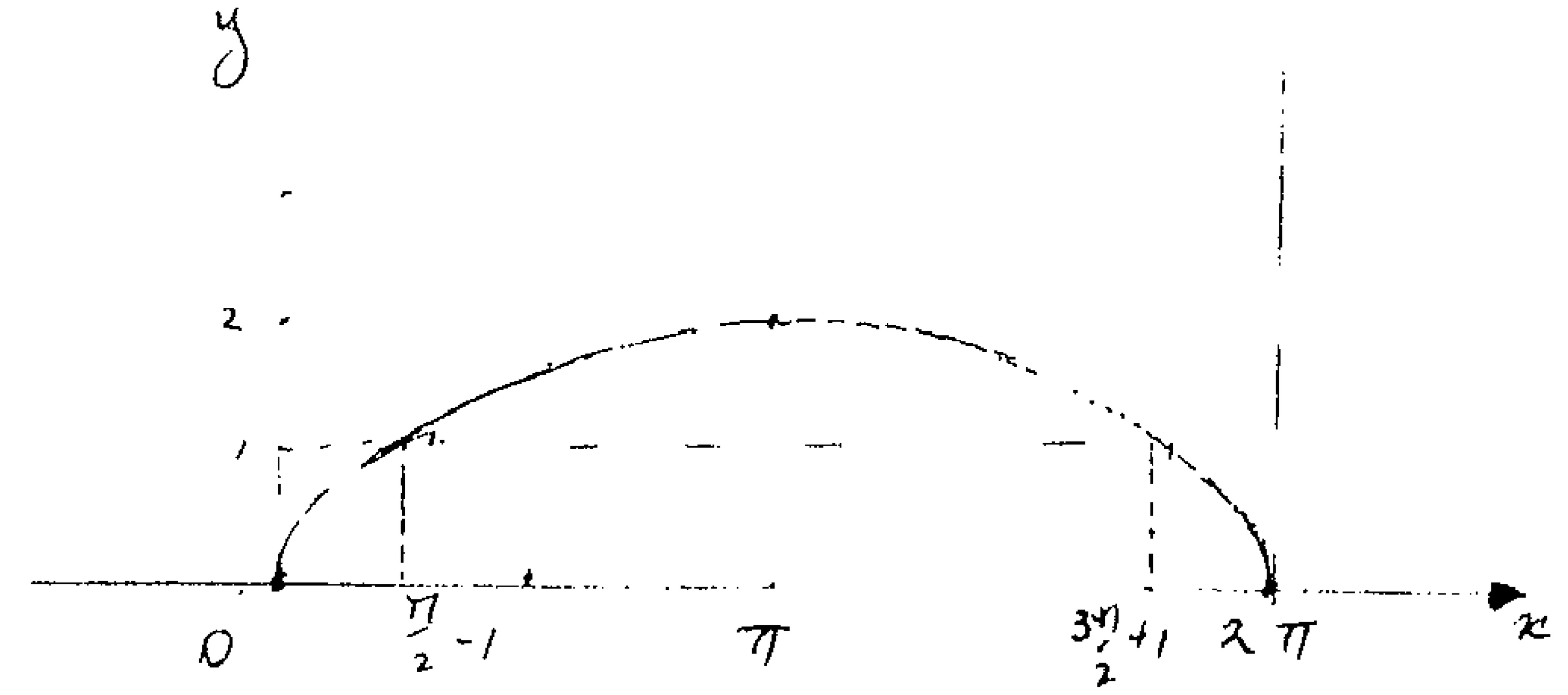
$= \infty$.

ie gradient is vertical when $\theta = 0$ 1

ie Equation Tangent is $x = 0$

$\therefore$ y axis is a tangent to the cycloid

iii)



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(iv) $A = \int_0 (1-\cos\theta)^2 \, d\theta$

$= \int_0^{2\pi} \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta$

$= \int_0^{2\pi} 1 - 2\cos\theta + \dfrac{\cos 2\theta + 1}{2} \, d\theta$

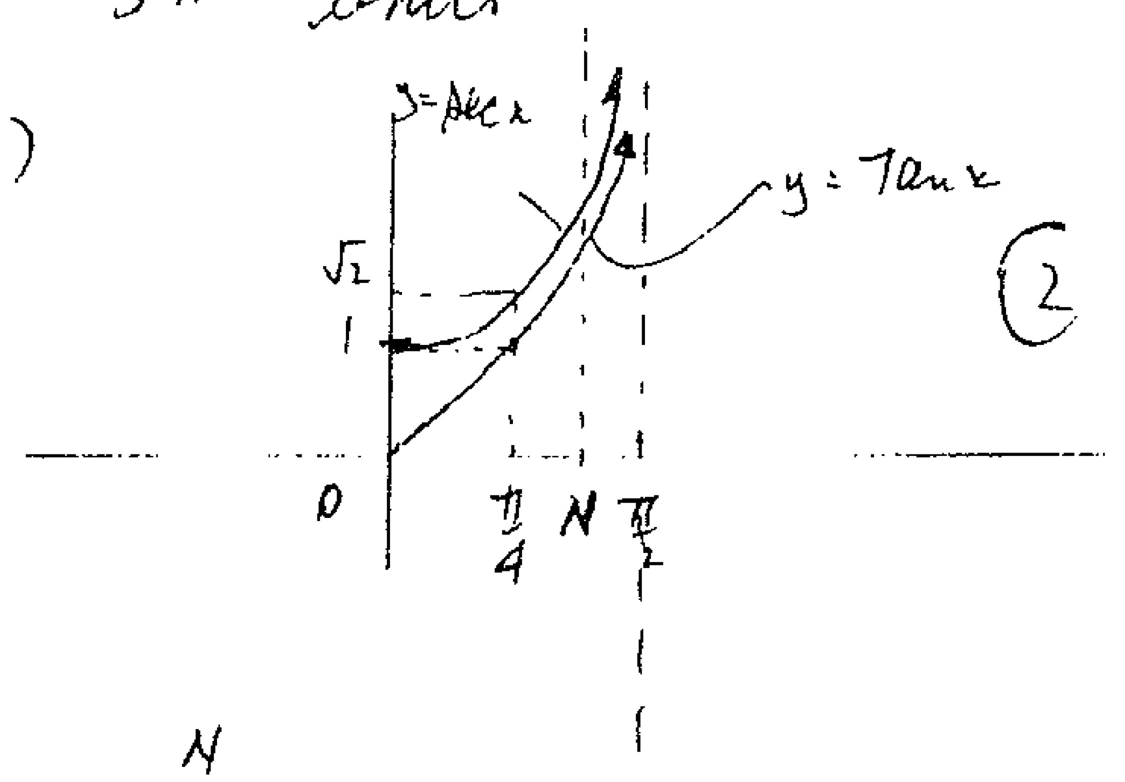
$= \int_0^{2\pi} \dfrac{3}{2} - 2\cos\theta + \dfrac{1}{2}\cos 2\theta \, d\theta$

$= \left[\dfrac{3}{2}\theta - 2\sin\theta + \dfrac{\sin 2\theta}{4}\right]_0^{2\pi}$

$= 3\pi - 0 + 0 - 0$ (5)

$= 3\pi$ units^2

5 (a) (i)



(2)

(ii) $V = \pi \int_0^N \left(y_1^2 - y_2^2\right) dx$

$= \pi \int_0^N \sec^2 x - \tan^2 x \, dx$

$= \pi \int_0^N 1 \, dx$

$= \pi [x]_0^N$

$= \pi N$ units^3 (2)

(iii) $\lim\limits_{N \to \frac{\pi}{2}} V = \dfrac{\pi^2}{2}$ units^3 (1)

(b) Area Trapezium $= \dfrac{h}{2}(x+y)$

$= \dfrac{1}{2} \cdot 0.3\cos\theta \left[0.6 + 0.6 + 2 \times 0.3 \sin\theta\right]$

(7)

$A = 0.09\cos\theta\left[2 + \sin\theta\right]$

$A' = 0.09\left\{-\sin\theta(2 + \sin\theta) + \cos^2\theta\right\}$

$= 0.09\left\{\cos^2\theta - \sin^2\theta - 2\sin\theta\right\}$

$= 0.09\left\{\cos 2\theta - 2\sin\theta\right\}$

$A'' = 0.09\left\{-2\sin 2\theta - 2\cos\theta\right\}$

For maximum area

$A' = 0$

$\cos^2\theta - \sin^2\theta - 2\sin\theta = 0$

$1 - 2\sin^2\theta - 2\sin\theta = 0$

$2\sin^2\theta + 2\sin\theta - 1 = 0$

$\sin\theta = \dfrac{-2 \pm \sqrt{4+8}}{4}$

$= \dfrac{\sqrt{3} - 1}{2}$ only $\sin\theta > 0$ $0 < \theta < 90^\circ$

$\theta = 0.37$ rad

$\theta = 21^\circ$ (nearest degree)

we will test A'' for sign concavity, since A, A', A'' are continuous in domain $0 < x < \frac{\pi}{2}$

$A''(0.37) = -0.288$

< 0

$\therefore$ there is a relative maximum at $\theta = 21^\circ$, but since there is only one turning point in domain $0 < \theta < \frac{\pi}{2}$ there is an absolute maximum cross-sectional area when $\theta = 21^\circ$.

6(a) $V = \frac{1}{3}\pi h^2(30-h)$

$$V = \frac{\pi}{3}(30h^2 - h^3)$$

$$\frac{dV}{dh} = \frac{\pi}{3}(60h - 3h^2)$$

$$= \pi(20h - h^2)$$

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$$

$$\frac{dV}{dt} = \pi(20h - h^2)\frac{dh}{dt}$$

$h = 6\text{cm}$ $\quad \frac{dV}{dt} = 20 \text{ cm}^3/\text{minute}$

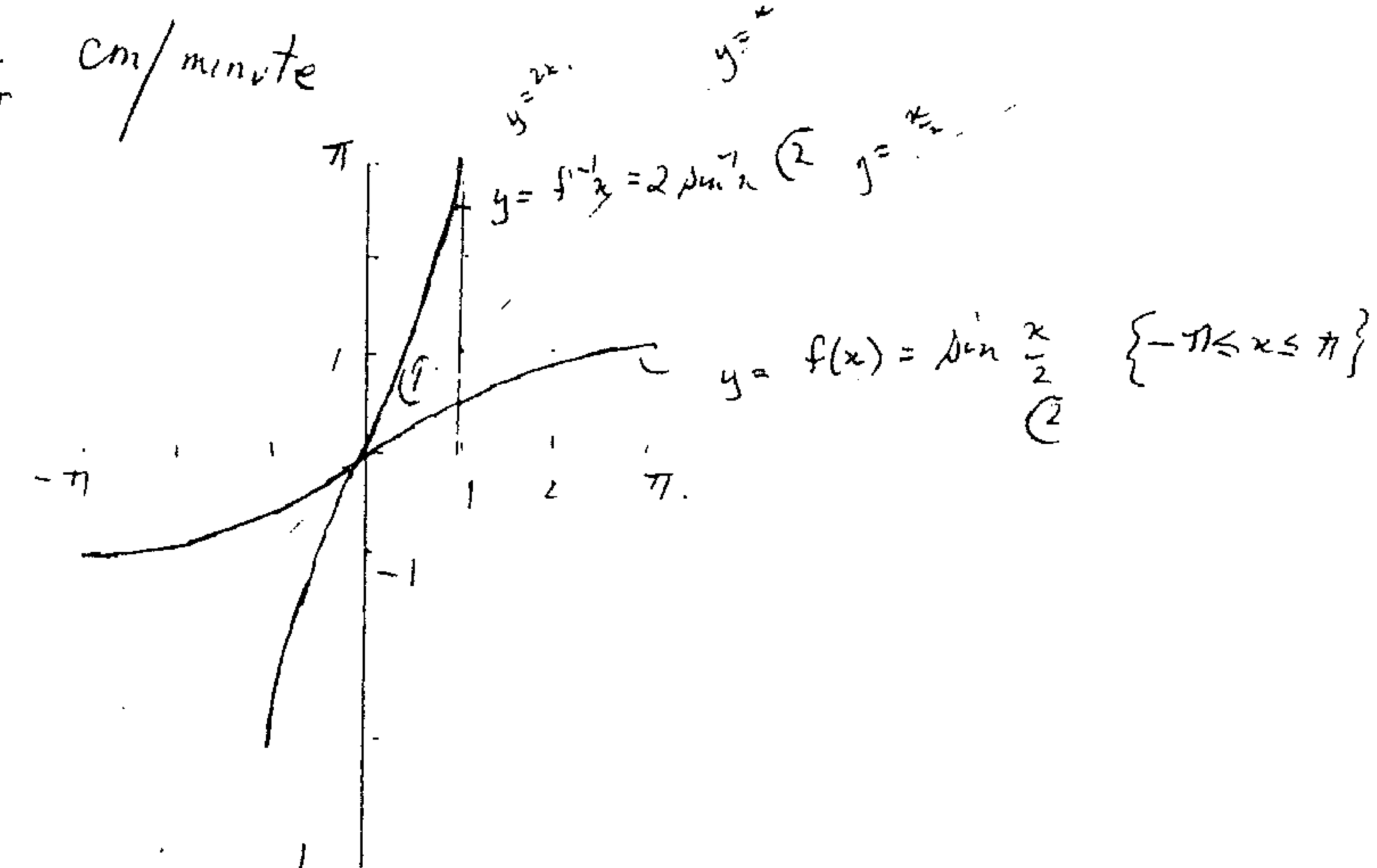
$$20 = \pi(20\times 6 - 6^2) \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{20}{84\pi}$$

3

rate rise $= \frac{5}{21\pi}$ cm/minute

b (i)



$$A = \int_0^1 2\sin^{-1}x\,dx - \int_0^1 \sin\frac{x}{2}\,dx$$

$$= \pi - \int_0^{\pi} \sin\frac{x}{2}\,dx - \int_0^1 \sin\frac{x}{2}\,dx$$

$$= \pi + \left[2\cos\frac{x}{2}\right]_0^{\pi} + \left[2\cos\frac{x}{2}\right]_0^1$$

4

$$= \pi + 0 - 2 + 2\cos\tfrac{1}{2} - 2$$

$$= \pi + 2\cos\tfrac{1}{2} - 4 \text{ units}^2$$