2003 MATHEMATICS CSSA

Q1 (a) $\frac{2.1^2 \times 4.5^2}{2.1^2 + 4.5^2} = \underline{3.6 \text{ (1DP)}}$ ✓ ✓

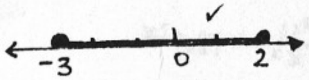
(b) $128x - 16x^4 = 16x(8 - x^3)$ ✓

$= \underline{16x(2-x)(4+2x+x^2)}$ ✓

(c) $|2x+1| \le 5$ $\quad -5 \le 2x+1 \le 5$

$-6 \le 2x \le 4$

$-3 \le x \le 2$ ✓

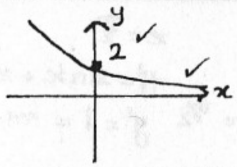


(d) $\frac{\sqrt{5}}{3\sqrt{2}-1} = \frac{\sqrt{5}}{3\sqrt{2}-1} \times \frac{3\sqrt{2}+1}{3\sqrt{2}+1}$ ✓

$= \frac{\sqrt{5}(3\sqrt{2}+1)}{17}$ ✓ $\left(\frac{3\sqrt{10}+\sqrt{5}}{17}\right)$

(e) $\tan\frac{\pi}{3} + \text{cosec}\,\frac{\pi}{4} = \underline{\sqrt{3}+\sqrt{2}}$ ✓ ✓

(f) $y = 2e^{-x}$



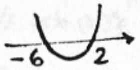
Q2 (i) $x^2 - (k+2)x + 4 = 0$ $\quad$ Real Roots $\Delta \ge 0$ ✓

$\Delta = [(k+2)]^2 - 4 \times 1 \times 4$

$= k^2 + 4k - 12$

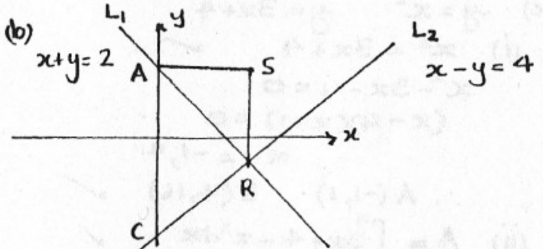
≥ 0

$(k+6)(k-2) \ge 0$



$\Rightarrow \underline{x \le -6,\ x \ge 2}$ ✓

(b)



(i) $L_1 : x+y = 2$ $\qquad L_2 : x - y = 4$

$x = 0 \quad y = 2$ $\qquad x = 0 \quad y = -4$

$\therefore \underline{A(0,2)}$ ✓ $\qquad \therefore \underline{C(0,-4)}$ ✓

(ii) $L_2 : x - y = 4$

$L_1 : x + y = 2$

$L_2 + L_1 : \quad 2x = 6$

$x = 3$

Sub into L_1: $3 + y = 2$ ✓

$y = -1$

$\therefore \underline{R : (3,-1)}$ ✓

(iii) $\underline{SR : x = 3}$ ✓

(iv) $L_1 : y = -x + 2$

$\therefore \underline{m_{L_1} = -1}$ ✓

(v) $AR^2 = (3-0)^2 + (-1-2)^2$

$= 9 + 9$

$AR = \sqrt{18}$

$\underline{AR = 3\sqrt{2}}$ ✓

(vi) $L_2 : y = x - 4$ $\qquad CR^2 = (3-0)^2 + (-4+1)^2$

$m_{L_2} = 1$ ✓ $\qquad = 18$

$CR = 3\sqrt{2}$ ✓

Since $m_{L_2} \cdot m_{L_1} = -1$ & $CR = AR$

ΔARC is $\perp$ Δ.

(vii) $R(3,-1)$ radius $= CR = 3\sqrt{2}$ ✓

Circle: $(x-3)^2 + (y+1)^2 = (3\sqrt{2})^2$ ✓

$\underline{(x-3)^2 + (y+1)^2 = 18}$ ✓

Q3. (a) (i) $\frac{d}{dx}\sqrt{x} = \frac{d}{dx}(x)^{1/2}$

$= \frac{1}{2}x^{-1/2}$

$= \frac{1}{2\sqrt{x}}$ ✓ ✓

(ii) $\frac{d}{dx}\, x^3 e^{-3x} = -3x^3 e^{-3x} + 3x^2 e^{-3x}$

$u = x^3 \quad v = e^{-3x}$

$u' = 3x^2 \quad v' = -3e^{-3x}$

$\underline{= 3x^2 e^{-3x}(1-x)}$ ✓

(iii) $\frac{d}{dx}\,\frac{\tan x}{2x+1} = \frac{(2x+1)\sec^2 x - 2\tan x}{(2x+1)^2}$ ✓

$u = \tan x \quad v = 2x+1$

$u' = \sec^2 x \quad v' = 2$ ✓

(b) $\int \frac{e^{2x}}{e^{2x}+4}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{e^{2x}+4}\,dx$

$= \underline{\frac{1}{2}\log_e(e^{2x}+4) + C}$ ✓ ✓

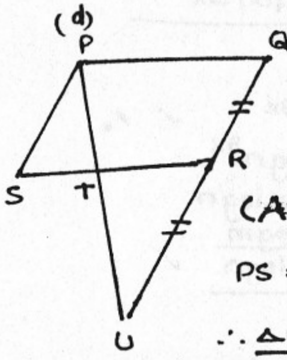
(c) $\int_0^{\pi/4} \left(\frac{1}{2}x + \cos 2x\right) dx$

$= \left[\frac{x^2}{4} + \frac{1}{2}\sin 2x\right]_0^{\pi/4}$ ✓

$= \frac{\pi^2}{64} + \frac{1}{2}\times 1 - 0 - \frac{1}{2}\times 0$

$= \underline{\frac{\pi^2}{64} + \frac{1}{2}}$ ✓

(d)



(i) In Δs PST & URT

$\angle STP = \angle RTU$

(Vert. Opp. $\angle$ =)

$\angle TPS = \angle TUR$

(Alt. $\angle$s = PS∥QU – opp. sides parm) ✓

PS = QR (opp. sides parm)

= RU (given) ✓

$\therefore$ $\underline{\Delta PST \equiv \Delta URT}$ (AAS Rule)

(ii) ST = TR (Corres. sides $\equiv \Delta$s)

$\therefore$ T midpoint SR. ✓

Q4 (a) $\sum_{k=4}^{20} 2k-5 = 3+5+7+\ldots+35$

AP $a=3$ $d=2$ $n=17$ $t_{17}=35$ ✓

$S_{17} = \frac{17}{2}(3+35)$

$\underline{S_{17} = 323}$ ✓

(b) GP $t_3 = \frac{3}{4}$ $t_7 = 12$ $t_{14} = ?$

$\underbrace{ar^2 = \frac{3}{4} \qquad ar^6 = 12}_{✓}$ $\therefore \frac{ar^6}{ar^2} = 16$

$r^4 = 16$

$r = \pm 2$

$r = 2$ $\frac{3}{4} = 4a$

$a = \frac{3}{16}$

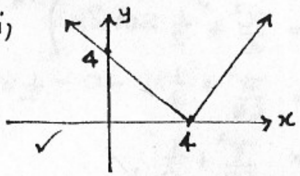
$t_{14} = ar^{13}$

$= \frac{3}{16} \times (\pm 2)^{13}$

$\underline{t_{14} = \pm 1536}$ ✓

(c) $f(x) = |4-x|$

(i)



✓

(ii) $\int_0^6 f(x)\,dx$

= Area under curve

$= \frac{1}{2}\times 4\times 4 + \frac{1}{2}\times 2\times 2$

$\underline{\int_0^6 f(x)\,dx = 10}$ ✓

(d) $\sqrt[x]{m} = n^3$

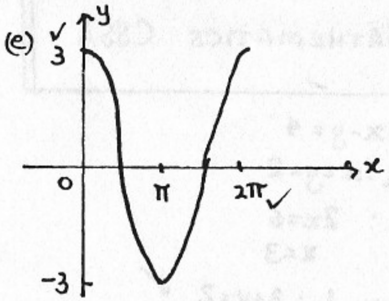
$m = n^{3x}$ ✓

$\log m = \log n^{3x}$

$= 3x \log n$

$\underline{x = \frac{\log m}{3\log n}}$ ✓

(e) ✓

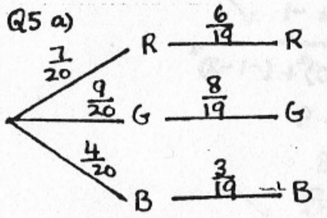


✓

(f) $\frac{dP}{dt} > 0$ f^n increasing

$\frac{d^2P}{dt^2} < 0$ increasing at decreasing rate.

Q5 a)



(i) $P(GG) = \frac{9}{20}\times\frac{8}{19}$

$= \frac{18}{95}$ ✓

✓

(ii) $1 - P(RR, GG, BB) = 1 - \left(\frac{7}{20}\times\frac{6}{19}\right) - \frac{18}{95} - \left(\frac{4}{20}\times\frac{3}{19}\right)$

$= \underline{\frac{127}{190}}$ ✓

(b) $y = 6x^2 - x^3$

(i) SP when $y' = 0$

$\therefore 12x - 3x^2 = 0$ ✓

$3x(4-x) = 0$

$x = 0, 4$

$\therefore$ $\underline{\text{SP } (0,0) \quad (4,32)}$ ✓

(ii) $y'' = 12 - 6x$

$x = 0$ $y'' > 0$ $\therefore$ $\underline{\text{Rel Min } (0,0)}$ ✓

$x = 4$ $y'' < 0$ $\therefore$ $\underline{\text{Rel Max } (4,32)}$ ✓

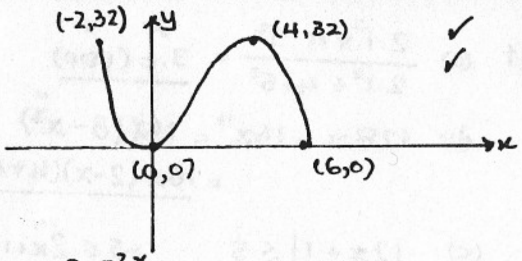
(iii) Pts of inflexion when $y'' = 0$

$12 - 6x = 0$

$x = 2$

$\left.\begin{array}{l} x = 2 \quad y = 16 \quad \therefore (2,16) \\ x < 2 \quad y'' > 0 \ ; \ x > 2 \quad y'' < 0 \end{array}\right\}$ ✓

(IV)



✓ ✓

(c) $y = 3e^{-2x}$

$\frac{dy}{dx} = -6e^{-2x}$ $\frac{d^2y}{dx^2} = 12e^{-2x}$

$3y' = -18e^{-2x}$ ✓ $2y' = 24e^{-2x}$ ✓ $2y = 6e^{-2x}$

$\therefore 2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 24e^{-2x} - 18e^{-2x} - 6e^{-2x}$

$= \underline{0}$

Q6 (a) $y = x\sin x$ $x = \frac{\pi}{2}$

$x = \frac{\pi}{2}$ $y = \frac{\pi}{2}$ $y' = \sin x + x\cos x$ ✓

$x = \pi/2$ $y' = 1 = m_T$

$m_{\text{Normal}} = \frac{-1}{m_T}$

$\therefore m = -1$ ✓ $(\pi/2, \pi/2)$

Normal : $y - \frac{\pi}{2} = -(x - \frac{\pi}{2})$

$\underline{x + y - \pi = 0}$ ✓

(b) $\int_0^4 f(x)\,dx \doteqdot \frac{1}{3}\left[2 + 4(3+35) + 2(12) + 80\right]$ ✓

$= \underline{86}$ ✓

(c) $y = x^2$ $y = 3x + 4$

(i) $x^2 = 3x + 4$ ✓

$x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

$x = -1, 4$

$\therefore A(-1,1)$ $B(4,16)$ ✓

(ii) $A = \int_{-1}^{4} 3x + 4 - x^2\,dx$ ✓

$= \frac{3}{2}x^2 + 4x - \frac{1}{3}x^3 \Big]_{-1}^{4}$

$= 18\frac{2}{3} + 2\frac{1}{6}$

$\underline{A = 20\tfrac{5}{6} \text{ units}^2}$ ✓

Q6 Cont. (d) $V = \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sqrt{\cot x})^2 \, dx$ ✓

$= \pi \int_{\pi/4}^{\pi/3} \frac{\cos x}{\sin x} \, dx$

$= \pi \left(\log_e (\sin x) \right)_{\pi/4}^{\pi/3}$ ✓

$= \pi \left[\left(\log_e (\sin \frac{\pi}{3}) - \log_e (\sin \frac{\pi}{4}) \right) \right]$

$= \pi \left(\log_e \frac{\sqrt{3}}{2} - \log_e \frac{1}{\sqrt{2}} \right)$

$= \pi \left(\log_e \frac{\sqrt{3} \times \sqrt{2}}{2} \right)$

$V = \pi \left(\log_e \frac{\sqrt{6}}{2} \right)$ units^3 ✓

Q7. (a) $y = \log_e \left(\frac{2x+1}{3x-7} \right)$

$y' = \frac{3x-7}{2x+1} \times \frac{d}{dx} \left(\frac{2x+1}{3x-7} \right)$ ✓

$u = 2x+1 \quad v = 3x-7$

$u' = 2 \quad v' = 3$

$y' = \frac{3x-7}{2x+1} \cdot \frac{2(3x-7) - 3(2x+1)}{(3x-7)^2}$

$y' = \frac{6x - 14 - 6x - 3}{(2x+1)(3x-7)}$

$y' = \frac{-17}{(2x+1)(3x-7)}$ ✓

(OR $y = \log_e (2x+1) - \log_e (3x-7)$ ✓

$y' = \frac{2}{2x+1} - \frac{3}{3x-7}$ ✓)

Q7 b) $v = 3 - \frac{2}{1+t}$

$x_0 = 1$

(i) $x = \int 3 - \frac{2}{1+t} \, dt$

$x = 3t - 2\log_e (1+t) + C$

$x = 1 \quad t = 0 \Rightarrow C = 1$ ✓

$\therefore x = 3t - 2\log_e (1+t) + 1$ ✓

(ii) As $t \to \infty \quad \frac{2}{1+t} \to 0$

$\therefore v \to 3$ but never reaches ✓

(iii) $a = v'$

$= + \frac{2}{(1+t)^2}$ ✓

$t = 2 \quad a = \frac{2}{9}$ ✓ m/sec^2

(c)(i) $(\operatorname{cosec}^2 A - 1) \sin^2 A = \cos^2 A$

LHS $= (\operatorname{cosec}^2 A - 1) \sin^2 A$

$= \cot^2 A \cdot \sin^2 A$ ✓

$(\operatorname{cosec}^2 A - 1) = \cot^2 A$

LHS $= \frac{\cos^2 A}{\sin^2 A} \cdot \sin^2 A$

LHS $= \cos^2 A$

$=$ RHS ✓

(ii) $(\operatorname{cosec}^2 A - 1) \sin^2 A = \frac{3}{4}$

$-\pi \le A \le \pi$

$\therefore \cos^2 A = \frac{3}{4}$

$\cos A = \pm \frac{\sqrt{3}}{2}$ ✓

$A = \pm \frac{\pi}{6}, \pm \frac{5\pi}{6}$ ✓

Q8. (a) \$480 000 6% p.a quarterly. 20 years

(i) 6% p.a = 1.5% per quarter ✓

$A_1 = 480\,000 \times 1.015 - \$P$

$A_1 = \$487\,200 - \$P$

(ii) $A_2 = (487\,200 - P) \times 1.015 - P$

$= 494\,508 - 1.05P - P$

$A_3 = 501\,925.62 - P(1.05^2 + 1.05 + 1)$ ✓

(iii) 20 yr = 80 quarters

$A_{80} = 480\,000 \times 1.015^{80} - P(1 + 1.05 + 1.05^2 + \ldots + 1.05^{79})$

$A_{80} = 0$

$1 + 1.05 + 1.05^2 + \ldots + 1.05^{79}$ is a GP $a = 1 \quad r = 1.05 \quad n = 80$

$S_{80} = \frac{1(1.015^{80} - 1)}{1.015 - 1}$

$\therefore 480\,000 \times 1.015^{80} = \frac{P(1.015^{80} - 1)}{0.015}$ ✓

$P = \frac{480\,000 \times 1.015^{80} \times 0.015}{1.015^{80} - 1}$

$\$P = \$10\,343.20$ ✓

(b) $\frac{dV}{dt} = kt \quad V_0 = 1000L \quad t = 40\text{min} \quad V = 800L$

(i) $V = 1000e^{-kt}$

$800 = 1000e^{-40k}$

$e^{-40k} = 0.8$

$-40k = \log_e 0.8$ ✓

$k = -\frac{1}{40} \log_e 0.8$ ✓

$t = 60 \quad V = 1000e^{+\frac{60}{40} \log_e 0.8}$

$V = 715.54\ldots$ ✓

$V = 716L$ (nearest L)

(ii) $V = 1L \quad t = ?$ ✓

$1 = 1000e^{\frac{t}{40} \log_e 0.8}$

$0.001 = e^{t/40 \log_e 0.8}$

$\log_e 0.001 = \frac{t}{40} \log_e 0.8$

$t = \frac{40 \log_e 0.001}{\log_e 0.8}$

$= 1238.26..$ min

$t = 20h\ 38min$ ✓

and tank will be empty

Q8 Cont (c)

$\sin^2x+\sin^4x+\sin^6x+\dots \quad 0<x<\frac{\pi}{2}$

(i) GP $a=\sin^2x$ $r=\sin^2x$

$|r|=|\sin^2x|$

<1 ✓ $0<x<\frac{\pi}{2}$

∴ Lim Sum exists

(ii) $\lim S=\frac{a}{1-r}$

$=\frac{\sin^2x}{1-\sin^2x}$ ✓

$=\frac{\sin^2x}{\cos^2x}$

$\lim S=\tan^2x$ ✓

Q9. $P=375m$

$P=2r+\frac{\theta}{2\pi}\cdot2\pi r$ ✓

$375=2r+\theta r$

$\theta=\frac{375-2r}{r}$ (1)

(i) $A=\frac{\theta}{2\pi}\cdot\pi r^2$ (2)

$=\frac{375-2r}{2\pi r}\cdot\pi r^2$

$=\frac{375r}{2}-r^2$

$A=\frac{r}{2}(375-2r)$ ✓

(ii) Max A when $A'=0$

$A'=\frac12(375-2r)+\frac r2(-2)$

$=\frac{375}{2}-r-r$

$=\frac{375-4r}{2}$ ✓

$=0$

$r=93.75$

$A''=-2<0$ ∴ Max A when $r=93.75$

$r=93.75$ $A_{MAX}=\frac{93.75}{2}(375-2\times93.75)$ ✓

$A_{MAX}=8789.06\,m^2$ (2DP)

(iii) $A=\frac12r^2\theta$

$8789.06=\frac12\times93.75^2\times\theta$

$\theta=2^c$ ✓

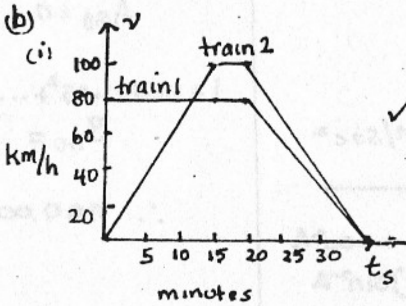
$\theta=\frac{2\times180^\circ}{\pi}$

$\theta\doteqdot115^\circ$ (nearest deg.) ✓

∴ For Max A requires at least 115°

ie Not possible

(b) (i)



Let t_s be the time when both trains arrive at the 2nd station.

Area under both graphs are the same, since they travel the same distance and, $\int v\,dt=x$

∴ $A_{t_1}=\frac12(20+t_s)\times80$

$=800+40t_s$ ✓

$A_{t_2}=\frac12(5+t_s)\times100$

$=250+50t_s$

$A_{t_1}=A_{t_2}\Rightarrow 800+40t_s=250+50t_s$

$10t_s=550$

$t_s=55$

∴ 55min. to 2nd station. ✓

(ii) d = Area under curve/line ✓

35min $=\frac{35}{60}=\frac{7}{12}h$

20min $=\frac{20}{60}=\frac13h$

∴ Area between stations $=80\times\frac13+\frac12\times\frac{7}{12}\times80$

$=50km$ ✓

Q10. (a) (i) $\frac{d}{dx}\,x\ln x-x=1.\ln x+x.\frac1x-1$

$=\ln x+1-1$

$=\ln x$ ✓

(ii) $\int\ln x^2dx=2\int\ln x\,dx$ ✓

$=2(x\ln x-x)+C$

(iii) $x=5$ $y=2\ln5$ $A_{under\ curve}=\int_1^5\ln x^2dx$

∴ $A_{RECT}=5.2\ln5$

$A_R=10\ln5$ units²

$=2(x\ln x-x)\Big|_1^5$

$=(10\ln5-10)-(\ln1-2)$

$A_U=10\ln5-8$ ✓

∴ $A=A_R-A_U$

$=10\ln5-10\ln5+8$

$A=8$ units² ✓

(b) $f(x)=e^{-x}\cos x$ $0\le x\le2\pi$

(i) S.P $f'(x)=0$ $f'(x)=-e^{-x}\cos x-e^{-x}\sin x$

$=0$

$-e^{-x}(\cos x+\sin x)=0$ ✓

$\Rightarrow \sin x=-\cos x$ ie $\tan x=-1$ ✓ ∴ $x=\frac{3\pi}{4},\frac{7\pi}{4}$

(ii) $x<\frac{3\pi}{4}^-$ $f'(x)<0$ $x>\frac{3\pi}{4}^+$ $f'(x)>0$

∴ Rel. Min at $x=\frac{3\pi}{4}$ ✓

$\frac{3\pi}{4}<x<\frac{7\pi}{4}^-$ $f'(x)>0$ $x>\frac{7\pi}{4}^+$ $f'(x)<0$

∴ Rel Max at $x=\frac{7\pi}{4}$ ✓

(iii) $x=\frac{3\pi}{4}$ $y=e^{-\frac{3\pi}{4}}\cos\frac{3\pi}{4}$

$=\frac{1}{\sqrt2}e^{-3\pi/4}$

∴ Min $\left(\frac{3\pi}{4},\frac{-e^{-3\pi/4}}{\sqrt2}\right)$

$x=\frac{7\pi}{4}$ $y=e^{-\frac{7\pi}{4}}\cos\frac{7\pi}{4}$

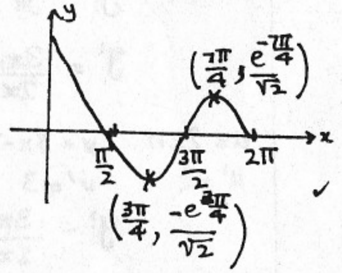
$=y=e^{-\frac{7\pi}{4}}\cdot\frac{1}{\sqrt2}$

∴ Max $\left(\frac{7\pi}{4},\frac{e^{-\frac{7\pi}{4}}}{\sqrt2}\right)$

$x=0$ $y=1$ $(0,1)$

$y=0$ $x=\frac{\pi}{2},\frac{3\pi}{2},2\pi$

$(\frac{\pi}{2},0)$ $(\frac{3\pi}{2},0)$ $(2\pi,0)$



(iv) If the line $y=\frac12x$ ✓ was sketched on the number plane above it would cut the curve exactly ONCE.

∴ $e^{-x}\cos x-\frac12x=0$ has only one solⁿ $(0\le x\le2\pi)$