MATHS EXTENSION 1 TRIAL, 2008

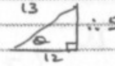
MATHEMATICS Extension 1 : Question 1

Suggested Solutions	Marks	Marker's Comments
Q1(a) $\lim_{x\to 0}\frac{3x}{\tan 5x} = \lim_{x\to 0}\frac{3x(5x)}{\tan(5x)\times 5}$ ✓	1	
$= \frac{3}{5}\lim_{x\to 0}\frac{5x}{\tan 5x}$		
$= \frac{3}{5}\times 1$ ✓	1	
$\boxed{2}$ $= \frac{3}{5}$		
(b) $x - y - 1 = 0 \quad m_1 = 1$ $2x + y - 1 = 0 \quad m_2 = -2$ $\boxed{2}$		
$\tan\theta = \frac{-2-1}{1+-2\times1} = \frac{-3}{1-2} = \frac{-3}{-1}$	1	
$\therefore \tan\theta = 3$		
$\therefore$ obtuse angle $= 180^\circ - \tan^{-1}3$ ✓		or $\tan^{-1}(-3)$
$= \underline{108^\circ 26'}$	1	
(c) $\sin\theta = \frac{\sqrt{3}}{2}$ $\boxed{2}$		
$\theta = n\pi + (-1)^n \sin^{-1}\frac{\sqrt{3}}{2}$ ✓	1	or $\theta = \begin{cases}\frac{\pi}{3} + 2n\pi \\ \pi - \frac{\pi}{3} + 2n\pi\end{cases}$
$\theta = n\pi + (-1)^n\frac{\pi}{3}$ where ✓ $n \in \mathbb{Z}$	1	Acc $\theta = 180^\circ n + (-1)^n.60^\circ$

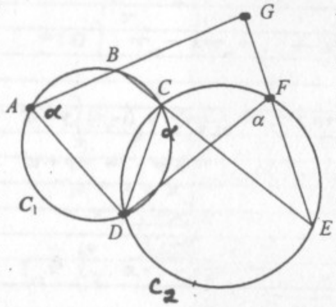
MATHEMATICS Extension 1 : Question 2

Suggested Solutions	Marks	Marker's Comments
Q2(a) $g(x) = \sqrt{x+2}$		$g^{-1}(x) = x^2 - 2$
$g^{-1}(5)$ is $g(x) = 5$ $\boxed{2}$	1	
$\therefore 5 = \sqrt{x+2}$		
$\therefore \underline{x = 23}$	1	
(b) (i) $\frac{2\tan x}{1+\tan^2 x} = \frac{2\frac{\sin x}{\cos x}}{\sec^2 x} = \frac{2\sin x}{\cos x}\times\cos^2 x$ ✓	1	
$= 2\sin x\cos x$ $\boxed{1}$		
$= \sin 2x$ qed.		
(ii) $\int_0^{\pi/4}\frac{\tan x}{1+\tan^2 x}\,dx = \frac{1}{2}\int_0^{\pi/4}\sin 2x\,dx$		
$= -\frac{1}{4}\Big[\cos 2x\Big]_0^{\pi/4}$ $\boxed{2}$	1	
$= -\frac{1}{4}\Big[\cos\frac{\pi}{2} - \cos 0\Big]$		
$= -\frac{1}{4}\Big[0 - 1\Big]$		
$= \underline{\frac{1}{4}}$	1	

MATHEMATICS Extension 1 : Question 3

Suggested Solutions	Marks	Marker's Comments
Q3(a) $\tan\left(2\cos^{-1}\frac{12}{13}\right)$ Let $\theta = \cos^{-1}\frac{12}{13} \Rightarrow \cos\theta = \frac{12}{13}$  ∴5 ✓ $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\times\frac{5}{12}}{1-\frac{25}{144}}$ ✓ $\boxed{2}$ $= \frac{2\times5\times12}{144-25} = \frac{120}{119}$.		$\frac{1}{2}$ For $\cos\theta = \frac{12}{13}$ $\frac{1}{2}$ For $\tan\theta = \frac{5}{12}$ 1 For $\frac{2\times\frac{5}{12}}{1-\frac{25}{144}}$ or $\frac{120}{119}$
(b) (i) $x^2 = 8y$ $\therefore y = \frac{x^2}{8}$ P$(4p, 2p^2)$ ① $\frac{dy}{dx} = \frac{2x}{8} = \frac{x}{4}$ Gradient of tangent at P : $m_T = \frac{4p}{4} = p$ Equ. of tangent at P: $y - 2p^2 = p(x - 4p)$ ✓ $y - 2p^2 = px - 4p^2$ $\therefore y = px - 2p^2$.	1	1 For getting to ✓
(ii) $C = (0, -2p^2)$ For Q $(0, -2p^2)$ 1:3 P$(4p, 2p^2)$ $Q = \left(\frac{4p+0}{4}, \frac{2p^2-6p^2}{4}\right) = (p, -p^2)$. ✓ ③ Let Q (x, y) be the general point on the required locus $\therefore x \equiv p$ — (1) $y \equiv -p^2$ — (2) (1) ⇒ $p = x$ in (2) $y = -(x)^2$ ∴ locus of Q $x^2 = -y$ ✓	1+1 1	
(c) $v = x^3 - x$ $\boxed{2}$ $\ddot{x} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ $\ddot{x} = (x^3 - x)(3x^2 - 1)$	1 1	
(d) For two 1s $1\,\underline{1}\,\_\,\_$ No of ways $= 3\times 9.8 =$ For not having two 1s $1\,\underline{X}\,\underline{X}\,\underline{Y}$ No of ways $= 9.3\times8 =$ TOTAL or ${}^4C_2 \times 1 \times 9.8 = 432$ $\boxed{2}$	216 216 432	1 1
(e) $I = \int \cos^2\frac{x}{2}\,dx = \frac{1}{2}\int(1+\cos x)\,dx$ $= \frac{1}{2}[x + \sin x] + c$ $\boxed{2}$	1 1	1 For $\cos^2 A = \frac{1}{2}(1+\cos 2A)$ or equiv.

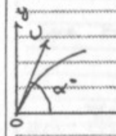
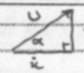
MATHEMATICS Extension 1 : Question 4

Suggested Solutions	Marks	Marker's Comments
Q4(a) $\boxed{2}$ $\left(2x^2 - \frac{3}{x}\right)^9$ General term $T_{r+1} = {}^9C_r(2x^2)^{9-r} \times \left(-\frac{3}{x}\right)^r \equiv Ax^0$ $\therefore {}^9C_r\, 2^{9-r}(-3)^r . x^{18-2r} \times x^{-r} = Ax^0$. $\Rightarrow 18 - 3r = 0$ $r = 6$ ✓ ∴ Term is the seventh ✓ $T_7 = {}^9C_6\, 2^3.3^6$ (=489888) 1	1	
(b) $\boxed{3}$  1. ∠DCE = α (Angles in same segment standing on arc DE are equal) ✓ 2. ∠DAB = α (Exterior angle of cyclic Quad ABCD equals interior opposite angle) ✓ 3. As ∠DFE = ∠DAB = α ∴ AGED is a cyclic Quad ✓ (Exterior angle equals interior opposite angle [converse])		
(c) (i) No of ways = 1 ODD + 3 ODD + 5 ODD $= {}^6C_1\times{}^5C_5 + {}^6C_3\times{}^5C_3 + {}^6C_5\times{}^5C_1$ $= 6 + 20\times10 + 6\times5$ $= 236$. $\boxed{2}$ $\boxed{3}$	1 1	Note: O+O = E E+E = E Need Odd No of Odd Nos for sum to be ODD
(ii) $P(E) = \frac{236}{{}^{11}C_6}$ ✓ $= \frac{236}{462} = \frac{118}{231}$ ①	1	
(d) Let $P = 2000$, int rate $= 0{\cdot}08$, n is... $R = 1{\cdot}08$, $M = 200$ ③ (i) After 1st prize: $B_1 = P\times R - 200$ After 2nd prize awarded: $B_2 = B_1R - 200 = (PR - 200)R - 200$ $= PR^2 - 200(1+R)$ After 3rd: $B_3 = B_2R - 200 = (PR^2 - 200(1+R))R - 200$ $= PR^3 - 200(1+R+R^2)$ ∴ After nth : $B_n = PR^n - 200(1+R+R^2+\ldots+R^{n-1})$ $= PR^n - 200\left(\frac{R^n-1}{R-1}\right)$ $= 2000R^n - \frac{200}{0{\cdot}08}(R^n-1)$ $= 2000R^n - 2500(R^n-1)$ $= -500R^n + 2500 = 500[5 - 1{\cdot}08^n]$ req. $\boxed{4}$	$\frac{1}{2}$ $\frac{1}{2}$ 1 1	(ii) ① Set $B_n = 0$ ⇒ $1{\cdot}08^n = 5$ $n = \frac{\log 5}{\log 1{\cdot}08} = 20{\cdot}912$ ∴ No of years is 20

MATHEMATICS Extension 1 : Question 5

Suggested Solutions	Marks	Marker's Comments
Q 5(a) (i) $(9+5x)^{29} = {}^{29}C_0 9^{29} + {}^{29}C_1 9^{28}(5x) + {}^{29}C_2 9^{27}(5x)^2 + \dots$ $\therefore p_k = \underline{{}^{29}C_k 9^{29-k} . 5^k}$ ✓ $k = 0, 1, 2, \dots, n$	①	
(ii) $\frac{p_{k+1}}{p_k} = \frac{{}^{29}C_{k+1} \cdot 9^{29-(k+1)} \cdot 5^{k+1}}{{}^{29}C_k \cdot 9^{29-k} \cdot 5^k}$ ② $= \frac{29!}{(k+1)!(28-k)!} \times \frac{k!(29-k)!}{29!} \times 9^{-1} \times 5^1$ $= \frac{(29-k)}{(k+1)} \times \frac{1}{9} \times 5 = \frac{5(29-k)}{9(k+1)}$ ✓	 1 1	1 For showing how to get the result
(iii) Find the least positive integer k such that $\frac{p_{k+1}}{p_k} = \frac{5(29-k)}{9(k+1)} \leq 1$ ② so $145 - 5k \leq 9k + 9$ as $k > 0$ $136 \leq 14k$ $\therefore k \geq \frac{136}{14} = 9.714\dots$ $\therefore \underline{k = 10}$ ✓ $\therefore$ Largest coefft. is $p_{10} = {}^{29}C_{10} \cdot 9^{19} \cdot 5^{10}$ $= {}^{29}C_{10} 3^{38} . 5^{10}$	 1 1	$\underline{\underline{\text{If}}}$ do $\frac{p_{k+1}}{p_k} \geq 1$ $k = 9$ but still $\underline{p_{9+1} = p_{10}}$
(b) (i) $\frac{d}{dt}(We^{kt}) = \frac{dW}{dt} . e^{kt} + W . ke^{kt}$ ② $= -k(W+15)e^{kt} + kWe^{kt}$ $= -kWe^{kt} - 15ke^{kt} + kWe^{kt}$ $\therefore \frac{d}{dt}(We^{kt}) = -15ke^{kt}$ q.e.d.	1 1	
(ii) As $\frac{d}{dt}(We^{kt}) = -15ke^{kt}$ ② $\therefore We^{kt} = -15e^{kt} + C$ when $t = 0$ $W = 20$ so $20 = -15 + C$ ie $C = 35$ $\therefore We^{kt} = -15e^{kt} + 35$ so $W = -15 + 35e^{-kt}$ q.e.d.	1 1	
(iii) (α) As $t = 5$ $W = 6$ ② $\therefore 6 = -15 + 35e^{-5k}$ so $e^{-5k} = \frac{21}{35} = \frac{3}{5} = 0.6$ $-5k = \ln 0.6$; $k = -\frac{\ln 0.6}{5}$ ✓ $\therefore$ Rate $= -\left(-\frac{\ln 0.6}{5}\right)(6+15) = \frac{21 \ln 0.6}{5}$ using (i) $= -2.145\dots$ $\underline{\text{Rate} = -2.1\,°C/\text{min}}$ ✓	 1 1	
(β) $-15 + 35e^{-kt} = -10$ ① $e^{-kt} = \frac{5}{35} = \frac{1}{7} \Rightarrow t = \frac{\ln(\frac{1}{7})}{-k} = \dots 19.0467\dots$		

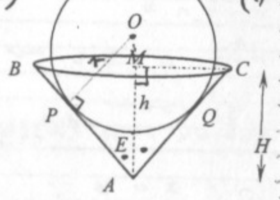
MATHEMATICS Extension 1 : Question 6.

Suggested Solutions	Marks	Marker's Comments
Q6(a) (i)  $t = 0 \begin{cases} x = 0 \\ y = 0 \end{cases}$  $\dot{y} = U\sin\alpha$ ② $\ddot{y} = -g$ $\dot{y} = \int -g\,dt$ $\dot{y} = -gt + C$ but $t = 0$ $\dot{y} = U\sin\alpha$ $\therefore C = U\sin\alpha$ ✓ $\dot{y} = U\sin\alpha - gt$ $y = \int(U\sin\alpha - gt)\,dt$ $y = Ut\sin\alpha - \frac{1}{2}gt^2 + D$ $t = 0$ $y = 0$ so $D = 0$ ✓ $\Rightarrow y = Ut\sin\alpha - \frac{1}{2}gt^2$	 1 1	
(ii) For the range : $y = 0$ so $t(U\sin\alpha - \frac{1}{2}gt) = 0$ ② $\therefore t = 0$ or $t = \frac{2U\sin\alpha}{g}$ $\therefore R = x = U . \frac{2U\sin\alpha}{g} . \cos\alpha = U^2 \frac{\sin 2\alpha}{g}$ ✓	 1 1	
(iii) (α) Max. height is 3.5m when $t = \frac{1}{2} \times \frac{2U\sin\alpha}{g} = \frac{U\sin\alpha}{g}$ so $3.5 = U . \frac{U\sin\alpha}{g} . \sin\alpha - \frac{1}{2}g \times U^2\frac{\sin^2\alpha}{g^2}$ $= U^2\frac{\sin^2\alpha}{g} - \frac{U^2\sin^2\alpha}{2g}$ ② $3.5 = \frac{U^2\sin^2\alpha}{2g}$ so $U^2 = \frac{3.5 \times 2g}{\sin^2\alpha} = 7g\,\mathrm{cosec}^2\alpha$	 1 1	or $\dot{y} = U\sin\alpha - gt = 0$
(β) Max R will then be $R = U^2\frac{\sin 2\alpha}{g}$ ① $= \frac{7g}{\sin^2\alpha} . \frac{2\sin\alpha\cos\alpha}{g}$ $= 14\frac{\cos\alpha}{\sin\alpha}$ $\therefore$ max $R = 14\cot\alpha$ q.e.d.	 1	1 For subst (iii)(α) into (ii) and showing how $14\frac{\cos\alpha}{\sin\alpha}$

MATHEMATICS Extension 1 : Question 6

Suggested Solutions	Marks	Marker's Comments
Q6(b) (i) $f_2(x) = 1 + \frac{x}{1!} + \frac{x(x+1)}{2!}$		
$= \frac{2 + 2x + x(x+1)}{2} = \frac{2+2x+x^2+x}{2}$		1 For getting to $\frac{x^2+3x+2}{2}$
$= \frac{x^2+3x+2}{2}$ ✓	1	
$= \frac{1}{2!}(x+1)(x+2)$ ②		
and the zeros are -1 and -2	1	
(ii) Let $P(n)$ be the proposition that:		
$f_n(x) = 1 + \frac{x}{1!} + \frac{x(x+1)}{2!} + \dots + \frac{x(x+1)x\dots(x+n-1)}{n!} = \frac{1}{n!}(x+1)(x+2)\dots(x+n)$		
• Now $P(1)$ was given		
$P(2)$ was shown true in part (i)		
* Assume $P(n)$ is true for some integer $k \geq 1$		
i.e. $f_k(x) = 1 + \frac{x}{1!} + \frac{x(x+1)}{2!} + \dots + \frac{x(x+1)\dots(x+k-1)}{k!} = \frac{1}{k!}(x+1)(x+2)\dots(x+k)$ —(*)	1	
RTP : $P(k+1)$ is true		
i.e. $f_{k+1}(x) = \frac{1}{(k+1)!}(x+1)(x+2)\dots(x+k)$		
PROOF : For $P(k+1)$		
$f_{k+1}(x) = 1 + \frac{x}{1!} + \frac{x(x+1)}{2!} + \dots + \frac{x(x+1)\dots(x+k-1)}{k!} + \frac{x(x+1)(x+2)\dots(x+k)}{(k+1)!}$		
$= \frac{1}{k!}(x+1)(x+2)\dots(x+k) + \frac{x(x+1)\dots(x+k-1)(x+k)}{(k+1)!}$ using assumption (*)	1	1 For using/ substituting assumption
$= \frac{(x+1)(x+2)\dots(x+k)}{k!}\left\{1 + \frac{x}{k+1}\right\}$		
$= \frac{1}{k!}(x+1)(x+2)\dots(x+k)\left\{\frac{k+1+x}{k+1}\right\}$	1	
$= \frac{1}{(k+1)!}(x+1)(x+2)\dots(x+k+1)$ ③		
∴ $P(k+1)$ is true		
* ∴ by the PMI $P(n)$ is true for $n=1,2,3,\dots$		

MATHEMATICS Extension 1 : Question 7

Suggested Solutions	Marks	Marker's Comments
(a) ∴ $V = \pi\int_{r-w}^{r}(r^2-x^2)\,dx$ $\boxed{2}$		
$= \pi\left[r^2x - \frac{1}{3}x^3\right]_{r-w}^{r}$		
$= \pi[(r^3 - \frac{1}{3}r^3) - (r^2(r-w) - \frac{1}{3}(r-w)^3]$	1	
$= \pi[\frac{2}{3}r^3 - \frac{(r-w)}{3}(3r^2 - (r-w)^2)]$		
$= \frac{\pi}{3}[2r^3 - (r-w)(3r^2 - r^2 + 2rw - w^2)]$		
$= \frac{\pi}{3}[2r^3 - (r-w)(2r^2 + 2rw - w^2)]$		
$= \frac{\pi}{3}[2r^3 - (2r^3 + \cancel{2r^2w} - rw^2 - \cancel{2r^2w} - 2rw^2 + w^3)]$	1	
$= \frac{\pi}{3}[3rw^2 - w^3] = \frac{\pi}{3}(3r-w)w^2$		
(b)  (i) As $\Delta OPA \,\vert\vert\vert\, \Delta CMA$ (equiangular)		
$\frac{r}{R} = \frac{OA}{AC}$ ✓ (corresponding sides in similar Δs are in the same ratio)	1	
$\frac{r}{R} = \frac{H+(r-h)}{L}$		
$rL = HR + rR - hR$		
$r(L-R) = (H-h)R$ ✓	1	
② ∴ $r = \frac{(H-h)R}{L-R}$		
(ii) Using (a) where $h = w$, $r = \frac{(H-h)R}{L-R}$		
∴ $V = \frac{\pi}{3}\left(3.\frac{(H-h)R}{L-R} - h\right)h^2$		
$= \frac{\pi}{3(L-R)}[3HR - 3hR - hL + hR]h^2$ ①	1	1 For subst and simplifying to
$= \frac{\pi}{3(L-R)}[3RHh^2 - (L+2R)h^3]$		
(iii) $\frac{dV}{dh} = \frac{\pi}{3(L-R)}[6RHh - 3(L+2R)h^2]$	[illegible]	
$= \frac{\pi}{L-R}[2RHh - (L+2R)h^2]$		
For possible max/min values of V to occur $\frac{dV}{dr} = 0$		
∴ $h(2RH - (L+2R)h) = 0$	1	
④ ∴ $h = 0$ or $h = \frac{2HR}{L+2R}$	1	
but $h \neq 0$		
TEST : $\frac{d^2V}{dh^2} = \frac{\pi}{L-R}[2RH - 2(L+2R)h]$		
at $h = \frac{2RH}{L+2R}$ $\frac{d^2V}{dh^2} = \frac{\pi}{L-R}[2RH - 4RH] = \frac{-2RH}{L-R}$	1 + 1	< 0 as $L > R$

∴ a Relative max T.P. at $h = \frac{2RH}{L+2R}$. $r = \frac{RHL}{(L-R)(L+2R)}$

9.

MATHEMATICS Extension 1 : Question 7

Suggested Solutions	Marks	Marker's Comments
(c) (i) ①		
$(1-x)^{2n} = \binom{2n}{0} - \binom{2n}{1}x + \binom{2n}{2}x^2 - \binom{2n}{3}x^3 + \dots + \binom{2n}{2n}x^{2n}$		1
(ii) By differentiating both sides wrt x		
$-2n(1-x)^{2n-1} = -\binom{2n}{1} + 2\binom{2n}{2}x - 3\binom{2n}{3}x^2 + \dots + 2n\binom{2n}{2n}x^{2n-1}$ ✓	1	
put $x = 1$ ②		
$0 = -\binom{2n}{1} + 2\binom{2n}{2} - 3\binom{2n}{3} + \dots - (2n-1)\binom{2n}{2n-1} + 2n\binom{2n}{2n}$ ✓	1	1 For Differe... 1 For subst $x=1$ and
so $\binom{2n}{1} + 3\binom{2n}{3} + \dots + (2n-1)\binom{2n}{2n-1} =$		
$= 2\binom{2n}{2} + 4\binom{2n}{4} + \dots + 2n\binom{2n}{2n}$ q.e.d.		