Induction

3U96-4b)!

Use Mathematical Induction to show that $\cos(x + n\pi) = (-1)^n \cos x$ for all positive integers $n \geq 1$.†

«→ Proof »

3U95-6c)!

i. Show that $(n+1)! = (n+1)! \times n!$

ii. Use the method of mathematical induction to show that $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n+1)! - 1$ for all positive integers n.†

«→ Proof »

3U93-6c)!

Use the method of mathematical induction to show that:

$2 \times 1! + 5 \times 2! + 10 \times 3! + \ldots + (n^2 + 1)n! = n(n+1)!$ for all positive integers $n \geq 1$†

«→ Proof »

3U92-6b)!

Use the Principle of Mathematical Induction to show that $9^{n+2} - 4^n$ is divisible by 5 for all positive integers n.†

«→ Proof »

3U91-6a)!

Using the principles of Mathematical Induction, show that $3^{3n} + 2^{n+2}$ is divisible by 5 for all positive integers n greater than or equal to 1.†

«→ Proof »

3U89-1d)!

Show by the principles of Mathematical Induction that:

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.†$$

«→ Proof »

3U87-3b)!

Use the principle of Mathematical Induction to prove that $7^n + 2$ is divisible by 3 for all positive integers n.†

«→ Proof »

3U85-5i)!

Prove by mathematical induction that

$2 \times 1! + 5 \times 2! + 10 \times 3! + \ldots + (n^2 + 1) \times n! = n \times (n+1)!$†

«→ Proof »

Binomial Theorem

3U96-7a)!

i. Write down the Binomial expansion of $(1 + x)^n$ in ascending powers of x. Hence show that ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$.

ii. Find how many groups of 1 or more digits can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 where repetition is not allowed.†

«→ i) ${}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... {}^nC_nx^n$ ii) 1023 »

3U95-6a)!

Find the term independent of x in the expansion of $\left(2x+\frac{1}{x^2}\right)^6$.†

«→ 240 »

3U94-4b)!

In the expansion of $(1 - 2x)(1 + ax)^{10}$ the coefficient of x^6 is 0. Find the value of a.†

«→ $\frac{12}{5}$ »

3U93-6a)!

Find the term independent of x in the expansion of $\left(2x+\frac{1}{x}\right)^{10}$.†

«→ 8064 »

3U92-6a)!

i. Write down the first three terms in the expansion of $(1 + ax)^n$ where a is a constant and n is a positive integer.

ii. If the first three terms of the expansion are $1 - 12x + 63x^2$

α. find the values of a and n

β. find the next term in the expansion.†

«→ i) $1 + nax + \frac{1}{2}n(n - 1)a^2x^2$ ii) α) 8, ± 1.5 β) ± $189x^3$ »

3U91-3c)!

When $(3 + 2x)^n$ is expanded in increasing powers of x, it is found that the coefficients of x^5 and x^6 have the same value. Find the value of n and show that the two coefficients mentioned are greater than all other coefficients in the expansion.†

«→ n = 14, Proof »

3U90-4c)!

Find the value of the term independent of x in the expansion of $\left(2x-\frac{1}{x^2}\right)^{12}$.†

«→ 126 720 »

3U89-4a)!

Find, expressed as a rational number, the term independent of x in the expansion of $(5x^2 + \frac{3}{x^3})^{10}$ and show also that it is the greatest coefficient in the expansion.†

«→ 265 781 250 »

3U88-3c)!

Find, as a rational number, the coefficient of x in the expansion of $(x^2 + \frac{1}{2x})^8$.†

«→ $\frac{7}{4}$ »

3U87-5a)!

In the expansion of $(x^2 + \frac{1}{2x})^{14}$ in powers of x, show that the terms involving x^{13} and x^{16} have the same numerical coefficient and state the value of this coefficient as a rational number.†

«→ Proof, $\frac{1001}{16}$ »

3U86-7ii)!

a. Find the greatest co-efficient in the expansion of $\left(\frac{1}{3}+2x\right)^{18}$.

b. Given that $x = \frac{2}{7}$ in the above expansion, show that there are two consecutive terms which are equal in value and greater than all other terms.†

«→ a) 17×2^{16} b) Proof »

3U85-7i)!

a. Write the expansion for $(1 + x)^3$.

b. Given that $\binom{n}{r}$ and ${}^nC_r$ are different notations for the same idea, show that

$$\binom{n}{r}:\binom{n}{r-1} = (n-r+1):r .$$

c. Hence find the sum of $\dfrac{\binom{n}{1}}{\binom{n}{0}} + \dfrac{2\binom{n}{2}}{\binom{n}{1}} + \dfrac{3\binom{n}{3}}{\binom{n}{2}} + \ldots + \dfrac{n\binom{n}{n}}{\binom{n}{n-1}}$.†

«→ a) $1 + 3x + 3x^2 + x^3$ b) Proof c) $\frac{n(n+1)}{2}$ »

Further Probability

3U96-6a)!

A group consisting of 3 men and 6 women attends a prizegiving ceremony.

i. If the members of the group sit down at random in a straight line, find the probability that the three men sit next to each other.

ii. If 5 prizes are awarded at random to members of the group, find the probability that exactly 3 of the prizes are awarded to women if

α. there is a restriction of at most one prize per person.

β. there is no restriction on the number of prizes per person.†

«→ i) $\frac{1}{12}$ ii) α) $\frac{10}{21}$ β) $\frac{80}{243}$ »

3U95-6b)!

Each time a competitor shoots at a target he has a probability 0.2 of hitting the target. He has 5 shots at the target. Find the probability that:

i. he hits the target on the first 3 shots and misses on the other two;

ii. he hits the target on 3 out of the 5 shots.†

«→ i) 0.00512 ii) 0.0512 »

3U94-7a)!

An employer wishes to choose two people for a job. There are eight applicants, three of whom are women and five of whom are men.

i. If each applicant is interviewed separately and all of the women are interviewed before any of the men, find how many ways there are of carrying out the interviews.

ii. If the employer chooses two of the applicants at random, find the probability that at least one of those chosen is a woman.†

«→ i) 720 ii) $\frac{9}{14}$ »

3U93-6b)!

The letters of the word CALCULUS are arranged in a row.

i. How many different arrangements are there?

ii. If one of these arrangements is selected at random, what is the probability that it begins with 'U' and ends in 'U'?†

«→ i) 5040 ii) $\frac{1}{28}$ »

3U90-7b)!

A die is loaded in such a way that in 8 throws of the die, the probability of getting 3 even numbers is four times the probability of getting 2 even numbers. Find the probability that a single throw of the die results in an even number.†

«→ $\frac{2}{3}$ »

3U89-7a)!

i. The letters of the word PERSEVERE are arranged in a row. How many DIFFERENT arrangements would be possible?

ii. Out of all the different arrangements found in (i) above, one is chosen at random. Find the probability that this particular arrangement:

α. will have all the E's together in one group AND all the R's together in another group,

β. will have an 'E' at one end and an 'R' at the other end.†

«→ i) 7560 ii) α) $\frac{1}{63}$ β) $\frac{2}{9}$ »

3U87-7a)!

A machine is known to produce items of which 5% are too short, 5% are too long and the remaining 90% are satisfactory. A random sample of twenty items is taken from the production of the machine. Find the probability (correct to two decimal places) that:

i. none of these items is too short;
ii. at most, one of these items is too long;
iii. at least, eighteen of these items are satisfactory.†

«→ i) 0.36 ii) 0.74 iii) 0.68 »

3U86-7iii)!

Four families each have four children. What is the probability that exactly two of these families have two boys and two girls? (Assume that each child is equally likely to be a boy or a girl)†

«→ $\frac{675}{2048}$ »

3U84-6)!

i. Colour-blindness affects 5% of all men. What is the probability that any random sample of 20 men should contain:
   a. no colour-blind men;
   b. only one colour-blind man;
   c. two or more colour-blind men.†
ii. If we also know that 15% of all men are left-handed and further, that colour blindness and left-handedness are independent, find what percentage of men in the total population would:
   a. be left-handed and colour-blind;
   b. have only one of these characteristics.†

«→ i) a) 0.358 b) 0.377 c) 0.265 ii) a) 0.75% b) 19.3% »

Iterative Methods for Numerical Estimation of the Roots of a Polynomial Equation

3U96-5b)!

The interior of a circle is divided into two segments with areas in the ratio 3:1 by a chord which subtends an angle θ radians at the centre of the circle.

i. Show that $\theta - \sin\theta = \frac{\pi}{2}$.

ii. Taking θ = 2.5 as a first approximation, use Newton's method twice to find a better approximation to θ, giving the answer correct to 2 decimal places.†

«→ i) Proof ii) 2.31 »

3U95-5c)!

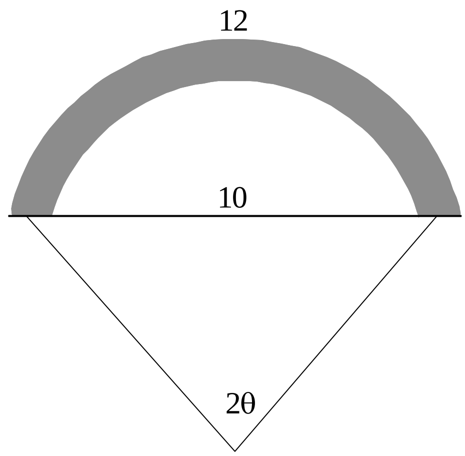
i. By considering the graph of $y = e^x$, show that the equation $e^x + x + 1 = 0$ has only one real root and that this root is negative.

ii. Taking x = -1.5 as a first approximation to this root, use one application of Newton's method to find a better approximation.†

«→ i) Proof ii) -1.27 »

3U94-5b)!

A pipe which is 12 metres long is bent into a circular arc which subtends an angle of 2θ radians at the centre of the circle. The chord of the circle joining the ends of the arc is 10 metres long.



i. Show that $6\sin\theta - 5\theta = 0$.

ii. Show that $\theta_0 = 1$ radian is a good first approximation to the value of θ.

iii. Use one application of Newton's method to find a better approximation θ_1 to the value of θ. Use this value of θ_1 to find an approximation to the length of the radius of the arc, rounding off this approximation correct to two decimal places.†

«→ i) ii) Proof iii) $\theta_1 = 1.0278$ (to 4 d.p.), radius of arc = 5.84 »

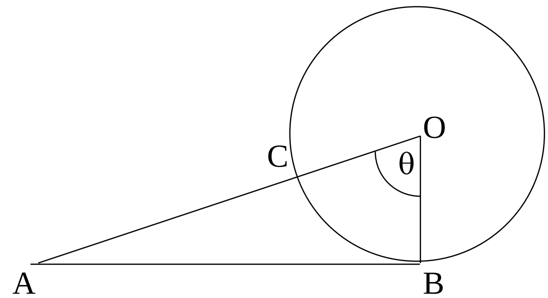
3U93-3b)!

i. Show that the equation $5x^4 - 4x^5 - 0.9 = 0$ has a root near x = 1.

ii. Starting with the approximation $x_0 = 1$ attempt to find an improved value for this root using Newton's Method. Explain why this attempt fails.†

«→ i) Proof ii) Using Newton's Method with $x_0 = 1$ gives $x_1 = 1 - \frac{0.1}{0}$ since $x_0 = 1$ is a stationary point of f(x). »

3U91-5b)!



In the above diagram, O is the centre of a circle and AB is a tangent to the circle, meeting it at point B. The line interval OA cuts the circumference of the circle at a point C.

i. If the arc of the circle CB divides the triangle AOB into two portions of equal area and if the angle AOB is denoted by θ, show that $\tan\theta = 2\theta$.

ii. If $\theta = 1.15$ radians is an approximate solution to the equation in (i) above, use one application of Newton's Method to find a better approximation, correct to three decimal places.†

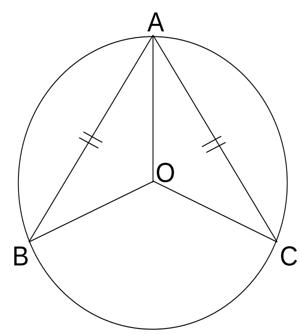
«→ i) Proof ii) 1.166 »

3U90-3b)!

The positive square root of 50 is approximately 7. Use one application of Newton's Method, to find a better approximation, correct to 2 decimal places.†

«→ 7.07 »

3U88-7)!



AB and AC are two equal chords of a circle, whose centre is the point O and whose radius is r. The angle BAC is denoted by θ.

i. Show that the triangles AOB and AOC are congruent, and write down an expression for the area of each triangle in terms of r and θ.

ii. Find an expression for the area of the sector BOC in terms of r and θ.

iii. If the area bounded by the two chords AB, AC and the minor arc BC is equal to half of the area of the circle, show that $\theta + \sin\theta = \dfrac{\pi}{2}$.

iv. Show that $\theta = 0.8$ radians is an approximate solution to the equation in (iii) above, and use one application of Newton's Method to find a better approximation, correct to 2 decimal places.†

«→ i) $\frac{1}{2}r^2\sin\theta$ ii) $r^2\theta$ iii) Proof iv) 0.83 »

3U86-4ii)!

Show that the equation $x^3 + 2x - 4 = 0$ has only one real root, and that this root lies between 1 and 1.5. Taking 1.2 as a first approximation to this root, use Newton's Method to obtain a second approximation.†

«→ 1.18 »

3U85-5iii)!

It is known that $\log_e x + \sin x = 0$ has a root close to $x = 0.5$. Use one application of Newton's Method to obtain a better approximation of the root.†

«→ 0.57 »

3U84-5i)!

Show that $x = 0.7$ in an approximate solution of the equation $\cos x = x$ and use one application of Newton's Method to find a better approximation correct to two decimal places.†

«→ 0.74 »

Harder Applications of HSC 2 Unit Topics

3U96-1b)!

Consider the function $y = x \operatorname{Ln} x - x$.

i. Solve the equation $y = 0$.

ii. Find $\frac{d^2y}{dx^2}$ and hence show that the function is concave up for all values of x in its domain.†

«→ i) $x = e$ ii) $\frac{d^2y}{dx^2} = \frac{1}{x}$, Proof »

3U96-2a)!

i. Find $\frac{d}{dx}\left(e^{\tan x}\right)$.

ii. Hence find $\int \frac{e^{\tan x}}{\cos^2 x} dx$.†

«→ i) $\sec^2 x\, e^{\tan x}$ ii) $e^{\tan x} + C$ »

3U95-1a)!

A geometric series is given by $1 - \tan^2 x + \tan^4 x - \tan^6 x + \ldots$ for $0° < x < 45°$.

i. Show that the limiting sum exists and is given by $S = \cos^2 x$.

ii. Find the set of possible values of S.†

«→ i) Proof ii) $\frac{1}{2} < S < 1$ »

3U95-3b)!

Solve the equation $2\operatorname{Ln}(3x + 1) - \operatorname{Ln}(x + 1) = \operatorname{Ln}(7x + 4)$.†

«→ $x = 3$ »

3U95-4a)!

The surface area of a cube is increasing at a rate of 10 cm^2 per second. Find the rate of increase of the volume of the cube when the edge of the cube has length 12cm.†

«→ 30 cm^2s^{-1} »

3U94-1a)!

If the positive numbers a, b, c are three consecutive terms in a geometric sequence show that $\log_e a$, $\log_e b$, $\log_e c$ are three consecutive terms in an arithmetic sequence.†

«→ Proof »

3U94-2a)!

i. Find $\frac{d}{dx} e^{3x^2}$.

ii. Hence find $\int x e^{3x^2} dx$.†

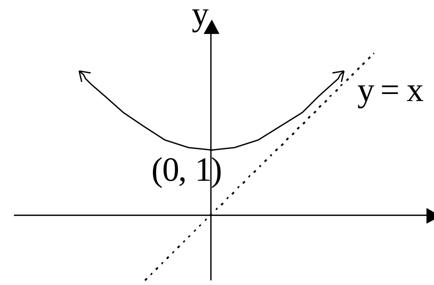
«→ i) $6xe^{3x^2}$ ii) $\frac{1}{6}e^{3x^2} + C$ »

3U94-3b)!

For the function $y = x + e^{-x}$

i. find the co-ordinates and the nature of any stationary points on the graph of $y = f(x)$ and show that the graph is concave upwards for all values of x.

ii. sketch the graph of $y = f(x)$ showing clearly the co-ordinates of any turning points and the equations of any asymptotes.†

«→ i) (0, 1) is a minimum turning point. ii) 

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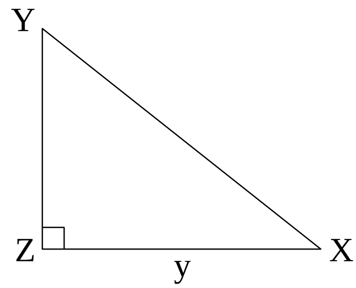
3U94-6b)!



In ΔXYZ, ZX = y and $\angle YZX = 90°$.

i. Show that the area A and perimeter P of the triangle are given by $A = \frac{1}{2}y^2 \tan X$ and $P = y(1 + \tan X + \sec X)$ respectively.

ii. α. If $X = \frac{\pi}{4}$ radians and y is increasing at a constant rate of 0.1cms^{-1} find the rate at which the area of the triangle is increasing at the instant when y = 20cm.

β. If y = 10cm and X is increasing at a constant rate of 0.2 radians s^{-1} find the rate at which the perimeter of the triangle is increasing when $X = \frac{\pi}{6}$ radians.†

«→ i) Proof ii) α) 2 cm^2s^{-1} β) 4 cms^{-1} »

3U92-5b)!

A sector of a circle with centre O and radius r cm is bounded by the radii OP and OQ, and by the arc PQ. The angle POQ is θ radians.

i. Given that r and θ vary in such a way that the area of the sector POQ has a constant value of 100cm^2, show that $\theta = \frac{200}{r^2}$.

ii. Given also that the radius is increasing at a constant rate of 0.5 cm/sec, find the rate at which the angle POQ is decreasing when r = 10cm.†

«→ i) Proof ii) -0.2 radians/sec »

3U91-1b)!

If $2^x = 5^y = 10^z$, using logarithms to base 10 , or otherwise, show that $\frac{1}{z} = \frac{1}{x} + \frac{1}{y}$.†

«→ Proof »

3U91-1d)!

In an Arithmetic Progression, whose first term and common difference are both non-zero, U_n denotes the nth term and S_n denotes the sum of n terms. If U_6, U_4, U_{10} form a Geometric Progression:

i. show that $S_{10} = 0$;

ii. show that $S_6 + S_{12} = 0$;

iii. deduce that $U_7 + U_8 + U_9 + U_{10} = U_{11} + U_{12}$.†

«→ Proof »

3U91-2b)!

The lengths of the sides of a scalene triangle are in Arithmetic Progression. If the largest angle in such a triangle measures 120°, show that the smallest angle measures approximately 22°.†

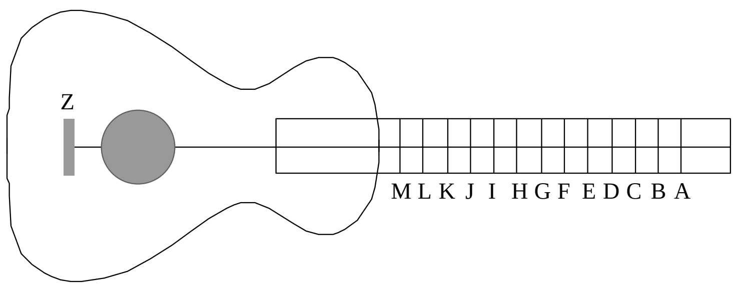
«→ Proof »

3U90-1d)!

Find the first derivative of the function $\log_e\left[\dfrac{1}{\sqrt{\sin x}}\right]$.†

«→ $-\dfrac{1}{2}\cot x$ »

3U90-4d)!



On the keyboard of a guitar, the mark M is exactly half-way between A and Z. The 13 marks lettered A to M are such that their distances from Z are in Geometric Progression. The length AZ is 52cm. Find correct to 1 decimal place:

i. the distance AB;

ii. the distance FG.†

«→ i) 2.9 cm ii) 2.2 cm »

3U90-5b)!

Find the largest possible domain of the function $f(x) = \log_e\left(\dfrac{x-2}{x}\right)$.†

«→ $x < 0$ or $x > 2$ »

3U90-6b)!

The curve of the function $y = \mathrm{Ln}(x - 1)$ meets the line $y = 2$ at P and the x axis at Q. From P, perpendiculars are drawn to the x-axis and the y-axis, meeting them at R and S respectively.

i. Show that the normal to the curve at Q passes through S.

ii. Show that the arc QP of the curve divides the rectangle OSPR into two portions of equal area, where O is the origin.

iii. Show that the area enclosed between the arc QP and the straight line interval QP equals the area of triangle OSQ.†

«→ Proof »

3U89-2c)!

Use Simpson's Rule, with three function values, to find an approximate value for $\int_0^1 e^{-x^2}\,dx$, to three significant figures.†

«→ 0.747 »

3U89-5b)!

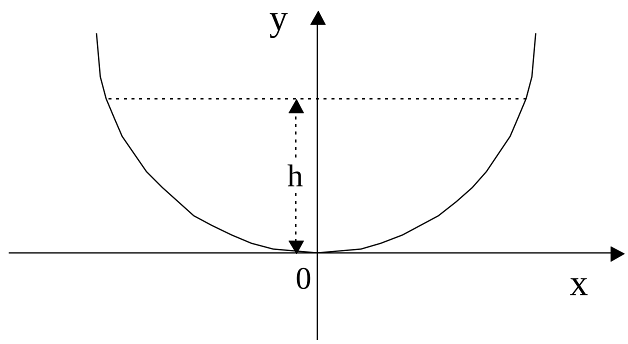
A prize fund is set up with a single investment of \$2000 to provide an annual prize of \$250. The fund accrues interest at the rate of 10% per annum, compounded yearly. The first prize is awarded one year after the investment is initially set up.

i. If $\$P_n$ denotes the value of the fund at the end of n years (and after the n^{th} prize has been awarded), show that: $P_n = 2500 - 500(1.1)^n$.

ii. Hence find the number of years for which the full amount of the prize can be awarded.†

«→ i) Proof ii) 16 »

3U89-6)!



The diagram above shows the lower half of the circle whose equation is $x^2 + y^2 - 20y = 0$.

A hemispherical bowl is obtained by rotating the semi-circle about the y-axis.

i. Show that when the depth of the water in the bowl is h cm, the volume of water, V, in the bowl is given by: $V = \dfrac{\pi h^2}{3}(30 - h)$ cm^3, and the surface area of the water, S, exposed to the open air is given by: $S = \pi h(20 - h)$ cm^2.

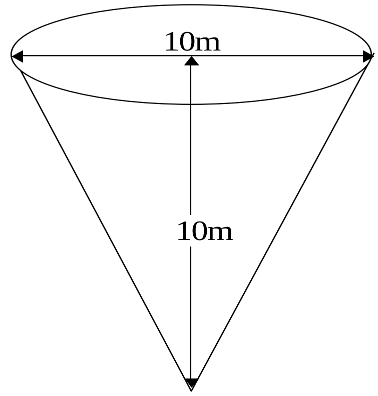
ii. If the water is poured into the bowl at the rate of $50cm^3$/sec, determine, at the instant when the depth of water is 5cm:

α. the rate of increase of the depth of water in cm/sec. (correct to 2 decimal places);

β. the rate of increase of the open surface area of water in cm^2/sec. (correct to 1 decimal place).†

«→ i) Proof ii) α) 0.21 cms^{-1} β) 6.7 cm^2s^{-1} »

3U87-3a)!



A large grain storage container is in the shape of an inverted cone, in which the diameter of the top is 10 metres and the vertical height is 10 metres.

i. If the height (in metres) of the grain in the container at any given time is denoted by h, show that the volume V (in cubic metres) of grain present at the time is given by $V = \frac{1}{12}\pi h^3$.

iii. If grain runs out of the bottom of the container at the rate of 2 cubic metres per second, find the rate of change of the height of grain in the container at the instant when this height is 5 metres. (Give answer in exact form.)†

«→ i) Proof ii) $\frac{8}{25\pi}$ ms^{-1} »

3U86-5i)!

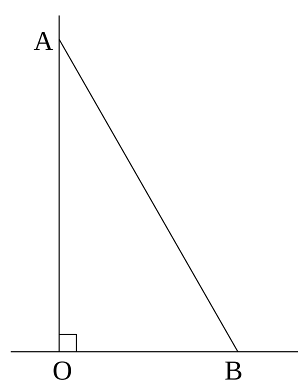
An electrical condenser discharges at a rate proportional to the charge such that $Q = Q_0e^{-kt}$, where Q is the charge at time t in minutes.

a. Prove that $\frac{dQ}{dt}$ is proportional to Q.

b. If it takes 8 minutes for the original charge of 1 unit to reduce to half, at what rate is the condenser discharging when the charge has been reduced to a quarter?†

«→ a) Proof b) 0.02166 units/min »

3U86-5ii)!



A ladder, AB, 5 metres long, is leaning against a vertical wall, with its foot on horizontal ground OB. The foot of the ladder begins to slide along the ground away from the wall at a constant speed of 1 metre/second. Find the speed at which the top of the ladder A is moving down the wall at the time when the ladder foot is 3 metres from the wall.†

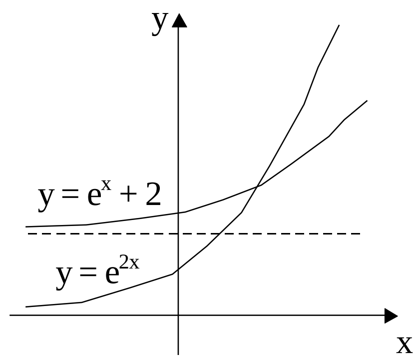
«→ $\frac{3}{4}$ ms^{-1} »

3U85-1ii)!

a. Draw a neat sketch showing the graphs of $y = e^{2x}$ and $y = e^x + 2$ on the same diagram.

b. Find the coordinates of the point(s) of intersection of $y = e^{2x}$ and $y = e^x + 2$.

c. Find the area bounded by the y axis and the two curves $y = e^{2x}$ and $y = e^x + 2$. Give your final answer correct to 2 decimal places.†



«→ a) b) (Ln 2, 4) c) 0.89 »

3U85-6i)!

A circular plate is being expanded by heating. When the radius just reaches a value of 20cm, it (the radius) is increasing at the rate of 0.01cm/sec. Find the rate of increase in the area at this moment.†

«→ 1.26 cm^2s^{-1} »

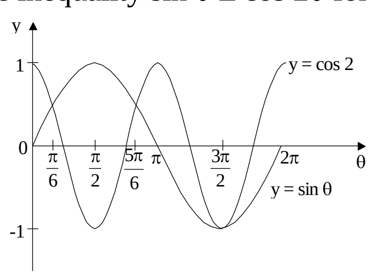
3U84-1i)!

a. One the same diagram, sketch the graphs of both the functions $y = \sin\theta$ and $y = \cos 2\theta$ for $0 \le \theta \le 2\pi$.

b. Find all the values of θ in the range $0 \le \theta \le 2\pi$ which satisfy the equation $\sin\theta = \cos 2\theta$.

c. Solve the inequality $\sin\theta \ge \cos 2\theta$ for $0 \le \theta \le 2\pi$.†

«→ a)



b) $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ or $\frac{3\pi}{2}$ c) $\frac{\pi}{6} \le \theta \le \frac{5\pi}{6}$ or $\theta = \frac{3\pi}{2}$ »

3U84-2i)!

Air is being pumped into a spherical balloon at the rate of 20 cubic centimetres per second. Find the rate of increase of the surface area of the balloon when the radius is 5cm.

(Volume of sphere = $\frac{4}{3}\pi r^3$ and surface area of a sphere = $4\pi r^2$)†

«→ 8 cm^2s^{-1} »

3U84-3ii)!

A solid of revolution is formed by rotating the area under the curve y = tan x between the ordinates at x = 0 and x = $\frac{\pi}{4}$ around the x-axis. Find the volume of this solid, leaving your answer in exact form.†

«→ $\pi\left(1-\frac{\pi}{4}\right)$ units^3 »

3U84-5ii)!

Ordinates are drawn from P(1, 0) and Q(2, 0) to intersect the curve y = $\frac{1}{x}$ at R and S respectively.

a. Find the exact value for the area bounded by the curve y = $\frac{1}{x}$, the x axis and the lines PR and QS.
b. Use the trapezoidal rule with one interval to find a rational approximation for this area.
c. A tangent is drawn to touch the curve y = $\frac{1}{x}$ at the point T on it where x = $1\frac{1}{2}$. This tangent cuts PR and QS at L and M respectively. Find the area of the trapezium PQML.
d. Hence show that $\frac{2}{3} < \log_e 2 < \frac{3}{4}$.†

«→ a) Ln 2 units^2 b) $\frac{3}{4}$ units^2 c) $\frac{2}{3}$ units^2 d) Proof »

3U84-7i)!

Find the roots of the equation $x^3 - 12x^2 + 12x + 80 = 0$ given that they are three consecutive terms in an Arithmetic Series.†

«→ -2, 4, 10 »