Methods of Integration

3U96-2b)!

Use the substitution u = 1 - x to evaluate $\int_{-3}^{0} \frac{x}{\sqrt{1-x}}\,dx$.†

«→ $-\frac{8}{3}$ »

3U95-1b)!

Evaluate $\int_{0}^{4} x\sqrt{x^2+9}\,dx$ using the substitution $u = x^2 + 9$.†

«→ $32\frac{2}{3}$ »

3U94-2b)!

Use the substitution $u = \log_e x$ to evaluate $\int_{1}^{e} \frac{(\log_e x)}{x}\,dx$.†

«→ $\frac{1}{2}$ »

3U92-1b)!

Evaluate $\int_{3}^{5} x\sqrt{x^2-9}\,dx$ using the substitution $u = x^2 - 9$.†

«→ $21\frac{1}{3}$ »

3U92-3b)!

Use the substitution u = 2 - x, to evaluate $\int_{-1}^{2} x\sqrt{2-x}\,dx$.†

«→ $\frac{2\sqrt{3}}{5}$ »

3U91-3b)!

Evaluate the definite integral $\int_{0}^{\frac{1}{\sqrt{2}}} \frac{2x^3dx}{\sqrt{1-x^4}}$ by means of one of the substitutions $u = x^4$ or $x^2 = \sin\theta$.†

«→ $1 - \frac{\sqrt{3}}{2}$ »

3U89-3a)!

Find the value of $\int_{1}^{6} x\sqrt{x+3}\,dx$, by means of the substitution $u^2 = x + 3$.†

«→ $\frac{232}{5}$ »

3U87-2a)!

Using the substitution u = tan x, show that $\int \tan^2 x.\sec^2 x dx = \frac{\tan^3 x}{3} + C$. Hence evaluate

$\int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x \, dx$.†

«→ $\frac{1}{3}$ »

3U86-1iii)!

Use the substitution u = x^2 - 4 to find an expression for $\int \frac{2x}{\sqrt{x^2 - 4}} dx$.†

«→ $2\sqrt{x^2 - 4} + C$ »

3U86-1iv)!

Evaluate $\int_{\pi}^{\frac{3\pi}{2}} \sin x \cos x \, dx$.†

«→ $\frac{1}{2}$ »

3U85-3i)!

Evaluate

a. $\int_1^{\sqrt{2}} \frac{x}{\sqrt{4 - x^2}} dx$ using the substitution u = 4 - x^2.

b. $\int_0^1 \sqrt{1 - x^2} dx$ using the substitution x = sin θ.†

«→ a) $\sqrt{3} - \sqrt{2}$ b) $\frac{\pi}{4}$ »

3U84-3i)!

Evaluate $\int_1^9 \frac{dx}{x + \sqrt{x}}$ using the substitution x = u^2.

«→ 2Ln 2 »

Primitive of sin^{2}x and cos^{2}x

3U95-5a)!

Find $\int \sin^2 2x\,dx$.†

«→ $\frac{1}{2}x - \frac{1}{8}\sin 4x + C$ »

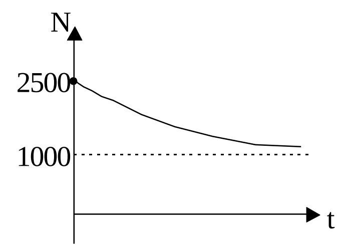
Equation $\frac{dN}{dt} = k$ (N - P)
Velocity and Acceleration as a Function of x
Projectile Motion
Simple Harmonic Motion

3U96-4a)!

N is the number of animals in a certain population at time t years. The population size satisfies the equation $\frac{dN}{dt} = -k(N - 1000)$, for some constant k.

i. Verify by differentiation that $N = 1000 + Ae^{-kt}$, A constant, is a solution of the equation.
ii. Initially there are 2500 animals but after 2 years there are only 2200 left. Find the values of A and k.
iii. Find when the number of animals has fallen to 1300.
iv. Sketch the graph of the population size against time.†

«→ i) Proof ii) A = 1500, $k = \frac{1}{2}\text{Ln}\left(\frac{5}{4}\right)$ iii) 14.4 years (to 1 d.p.) iv)



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3U96-6b)!

A particle moving in a straight line is performing Simple Harmonic Motion about a fixed point O on the line. At time t seconds the displacement x metres of the particle from O is given by:

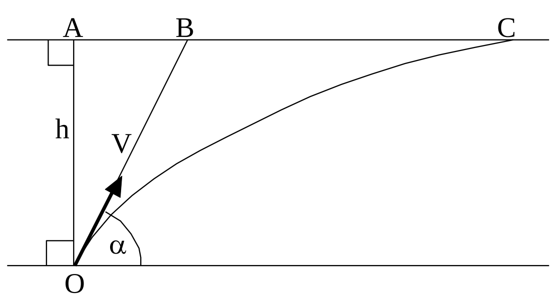
$$x = a \cos nt, \qquad \text{where } a > 0 \text{ and } 0 < n < \pi.$$

After 1 second the particle is 1 metre to the right of O, and after 2 seconds the particle is 1 metre to the left of O.

i. Find the values of n and a.
ii. Find the amplitude and period of the motion.†

«→ i) $n = \frac{\pi}{3}$, a = 2 ii) amplitude = 2 metres, period = 6 seconds »

3U96-7b)!



In the diagram an aircraft is flying with constant velocity U at a constant height h above horizontal ground. When the plane is at A it is directly over a gun at O. When the plane is at B a shell is fired from the gun at the aircraft along OB. The shell is fired with initial velocity V at an angle of elevation α.

i. If x and y are the horizontal and vertical displacements of the shell from O at time t seconds, show that if g is the acceleration due to gravity,

$$x = Vt \cos \alpha \text{ and } y = Vt \sin \alpha - \frac{1}{2}gt^2.$$

ii. Show that if the shell hits the aircraft at time T at point C, then VTcos $\alpha = \dfrac{h}{\tan\alpha} + UT$.

iii. Show that if the shell hits the aircraft then $2U(V\cos\alpha - U)\tan^2\alpha = gh$.†

«→ Proof »

3U95-3c)!

A particle moves in a straight line so that its displacement x metres from an origin O at time t seconds is given by $x = 10\sin\dfrac{\pi t}{2}$.

i. Show that $\dfrac{d^2x}{dt^2} = -\dfrac{\pi^2}{4}x$.

ii. State the amplitude and the period of the motion.

iii. Find the maximum speed of the particle.†

«→ i) Proof ii) amplitude = 10m, period = 4s iii) 5π ms^{-1} »

3U95-5b)!

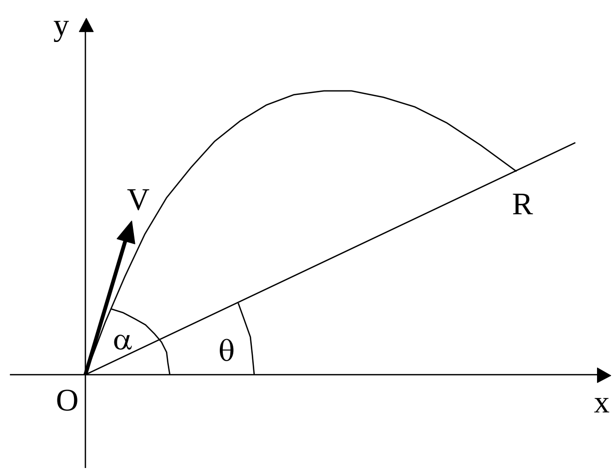
At time t the temperature T° of a body in a room of constant temperature 20° is decreasing according to the equation $\dfrac{dT}{dt} = -k(T-20)$ for some constant $k > 0$.

i. Verify that $T = 20 + Ae^{-kt}$, A constant, is a solution of the equation.

ii. The initial temperature of the body is 90° and it falls to 70° after 10 minutes. Find the temperature of the body after a further 5 minutes.†

«→ i) Proof ii) 62° (to nearest degree) »

3U95-7)!



A stone is projected from O with velocity V at an angle α above the horizontal. A straight road goes through O at an angle θ above the horizontal, where θ < α. The stone strikes the road at R. Air resistance is to be ignored, and the acceleration due to gravity is g.

i. If the stone is at the point (x, y) at time t, find expressions for x and y in terms of t. Hence show that the equation of the path of the stone is $y = x\tan\alpha - \dfrac{gx^2\sec^2\alpha}{2V^2}$.

ii. If R is the point (X,Y), express X and Y in terms of OR and θ. Hence show that the range OR of the stone up the road is given by $OR = \dfrac{2V^2\cos\alpha\sin(\alpha-\theta)}{g\cos^2\theta}$.

iii. Show that OR is a maximum when $\alpha = \dfrac{1}{2}\left(\theta + \dfrac{\pi}{2}\right)$, and interpret this result geometrically.

iv. Hence show that the maximum value of OR is $\dfrac{V^2}{g(1+\sin\theta)}$.†

«→ i) $x = Vt\cos\alpha$, $y = Vt\sin\alpha - \dfrac{gt^2}{2}$ ii) $X = 0R\cos\theta$, $Y = 0R\sin\theta$ iii) Proof. For maximum range, the angle of projection α of the stone bisects the angle $\theta + \dfrac{\pi}{2}$ iv) Proof »

3U94-5a)!

A body is moving in a straight line. At time t seconds its displacement is x metres from a fixed point O on the line and its velocity is v ms^{-1}. If $v = \dfrac{1}{x}$ find its acceleration when $x = 0.5$.†

«→ $-8\ ms^{-2}$ »

3U93-2c)!

A particle is moving in a straight line with Simple Harmonic Motion. If the amplitude of the motion is 4cm and the period of the motion is 3 seconds, calculate:

i. the maximum velocity of the particle;
ii. the maximum acceleration of the particle;
iii. the speed of the particle when it is 2cm from the centre of the motion.†

«→ i) $\dfrac{8\pi}{3}cms^{-1}$ ii) $\dfrac{16\pi^2}{9}cms^{-2}$ iii) $\dfrac{4\pi\sqrt{3}}{3}cms^{-1}$ »

3U93-3c)!

i. At any time t the rate of cooling of the temperature T of a body, when the surrounding temperature is P, is given by the equation $\dfrac{dT}{dt} = -k(T-P)$, for some constant k. Show that the solution $T = P + Ae^{-kt}$, for some constant A, satisfies this equation.
ii. A metal bar has a temperature of 1340°C and cools to 1010°C in 12 minutes, when the surrounding temperature is 25°C. Find how much longer it will take the bar to cool to 60°C, giving your answer correct to the nearest minute.†

«→ i) Proof ii) 139 minutes »

3U93-7b)!

A particle moves in a straight line. At time t its displacement from a fixed point O on the line is x, its velocity is v and its acceleration is a.

i. Show that $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

ii. If $a = 4x - 4$ and when $t = 0$, $x = 6$ and $|v| = 8$ show that $v^2 = 4x^2 - 8x - 32$.

iii. Use the expression for v^2 to find the set of possible values of x.

iv. Describe the motion of the particle in each of the cases

α. when $t = 0$, $x = 6$ and $v = 8$.

β. when $t = 0$, $x = 6$ and $v = -8$.†

«→ i) ii) Proof iii) $x \geq 4$ iv) α) The particle starts 6 units to the right of O. It accelerates to the right. β) The particle starts 6 units to the right of O. It moves to the left, slows to a stop 4 units to the right of O, the accelerates to the right. »

3U92-5a)!

i. A ball is thrown from a point O on the edge of a cliff which is 20 metres above a beach. The ball is thrown with speed $15\sqrt{2}$ ms^{-1} at an angle of 45° above the horizontal. Taking $g = 10ms^{-2}$ show that the ball hits the beach at a point 60 metres along the beach.

ii. A second ball is thrown horizontally from 0 and hits the beach at the same point as the first ball. Taking $g = 10ms^{-2}$ find the speed of projection of the second ball. (Standard results about projectile motion can be quoted without proof.)†

«→ i) Proof ii) 60 ms^{-1} »

3U91-4a)!

O is a fixed point on a given straight line. A particle moves along this line and its displacement x cms, from O at a given time, t secs, after its start of motion is given by: $x = 2 + \cos^2 t$.

i. Show that the acceleration is given by: $\ddot{x} = 10 - 4x$.

ii. State the centre of motion.

iii. State the first two occasions when the particle is at rest and the displacements on these occasions.

iv. State the amplitude and period of motion.†

«→ i) Proof ii) $x = \frac{5}{2}$ iii) $t = 0$, $x = 3$ and $t = \frac{\pi}{2}$, $x = 2$ iv) Amplitude = $\frac{1}{2}$ cm, Period = π secs »

3U91-5a)!

A stone is thrown from a point O which is at the top of a cliff 20 metres above a horizontal beach. The stone is thrown at an angle of elevation θ° above the horizontal and with a speed of $35ms^{-1}$. The stone hits the beach at a point which is distant 140 metres horizontally from the point O.

i. Taking g, the gravitational constant, as $10ms^{-2}$, show that $\tan\theta = \frac{3}{4}$ or $\tan\theta = 1$.

ii. Hence find the two possible times for which the stone is in the air, giving your answers in exact form.†

«→ i) Proof ii) $4\sqrt{2}$ seconds and 5 seconds »

3U90-5d)!

A certain particle moves along the x-axis in accordance with the law $t = 2x^2 - 5x + 3$ where x is measured in cm and t in seconds. Initially, the particle is 1.5 cm to the right of O and moving away from O.

i. Prove that the velocity, v cm/sec, is given by $v = \dfrac{1}{4x-5}$.

ii. Find an expression for the acceleration, a cm/sec^2, in terms of x.

iii. Find the velocity and acceleration of the particle when:

α. x = 2 cm.

β. t = 6 sec.

iv. Describe carefully in words the motion of the particle.†

«→ i) Proof ii) $\dfrac{-4}{(4x-5)^3}$ iii) α) $v = \dfrac{1}{3}cms^{-1}$, $a = -\dfrac{4}{27}cms^{-2}$ β) $v = \dfrac{1}{7}cms^{-1}$, $a = -\dfrac{4}{343}cms^{-2}$ iv) The particle moves in the postive direction with a negative acceleration retarding its motion.»

3U89-3b)!

A particle moves in a straight line and at time t seconds, its distance x cm from a fixed origin point O, on the line is given by: $x = 1 + \dfrac{1}{2}\cos 2t$.

i. Sketch a graph of x as a function of t in the domain $0 \le t \le 2\pi$

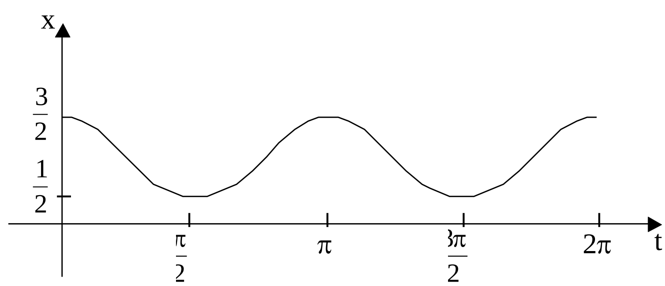
ii. Show that the motion of the particle is Simple Harmonic Motion.

iii. State the centre of motion of the particle.

iv. Find the displacements of the particle when it is at rest and thus determine the length of its path.

v. State the period of motion for the particle.†

«→ i)



ii) Proof iii) x = 1 iv) $x = \dfrac{3}{2}, \dfrac{1}{2}, \dfrac{3}{2}, \dfrac{1}{2},\ldots$ Length of path = 1 cm v) π secs »

3U88-5)!

A particle is projected from a point, O, on ground level with the velocity of 20 metres per second at an angle of 60° to the horizontal. After a time T seconds, it reaches a point P, on its upward path, where the direction of the flight is at 30° to the horizontal. Taking the acceleration due to gravity, g, to be 10m/s,

i. show that $T = \dfrac{2\sqrt{3}}{3}$.

ii. find the height of P above ground level.

iii. find the greatest height reached by the particle.†

«→ i) Proof ii) $\dfrac{40}{3}$m iii) 15m »

3U87-7b)!

A particle moves in a straight line and its displacement, x cm, from a fixed origin point after t seconds is determined by the function: x = sin t - sin t cos t - 2t.

i. Find the initial displacement and velocity of the particle.

ii. Show that the particle never comes to rest and always moves in one particular direction, stating what this direction is.

iii. Show that the particle initially has zero acceleration and find the first occasion after this when zero acceleration occurs again.†

«→ i) $x = 0$ cm, $v = -2$ cms^{-1} ii) Proof, negative direction iii) 1.31 seconds »

3U86-5iii)!

The speed v centimetres/second of a particle moving with simple harmonic motion in a straight line is given by $v^2 = 6 + 4x - 2x^2$, where x cm is the magnitude of the displacement from a fixed point O.

a. Show that $\ddot{x} = -2(x - 1)$.

b. Find the centre of the motion.

c. Find the period of the motion.

d. Find the amplitude of the motion.†

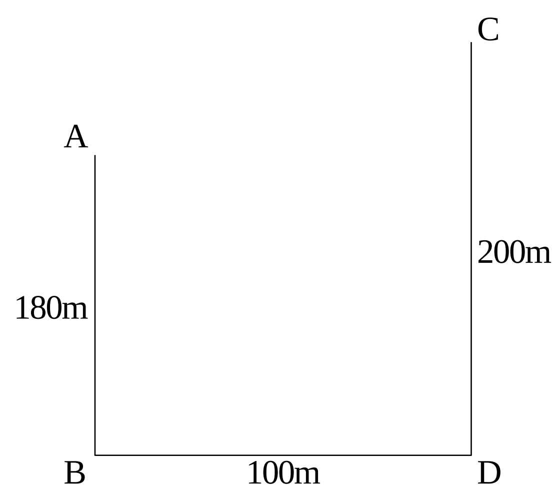
«→ a) Proof b) $x = 1$ c) $\sqrt{2}\,\pi$ secs d) 2 cm »

3U85-5ii)!

A body is moving with simple harmonic motion in a straight line. It has an amplitude of 10 metres and a period of 10 seconds. How long would it take for the body to travel from one of the extremities of its path of motion to a point 4 metres away?†

«→ 1.5 seconds »

3U85-6ii)!



AB and CD are two buildings situated 100 metres apart on level ground. Their heights are 180m and 200m respectively. An object is projected from point A at an angle of 45° to the horizontal, and this object strikes point C. Take the acceleration due to gravity, g, as 10m/sec^2. Show that the time taken for the object to get from A to C is 4 seconds, and find the value of the initial velocity of projection.†

«→ Proof, $25\sqrt{2}$ ms^{-1} »

Inverse Functions and Inverse Trigonometric Functions

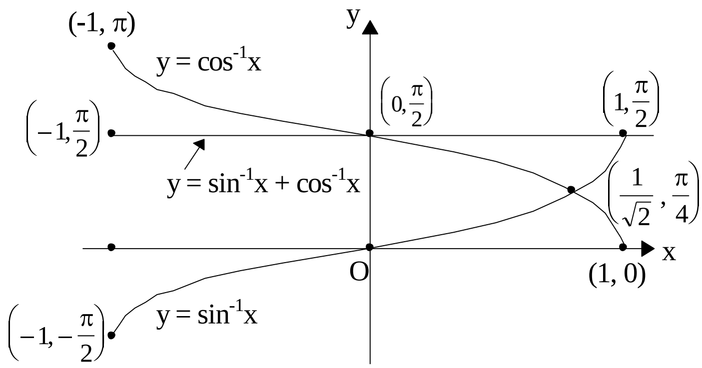
3U96-2c)!

i. Find the value of x such that $\sin^{-1}x = \cos^{-1}x$.

ii. On the same axes sketch the graphs of $y = \sin^{-1}x$ and $y = \cos^{-1}x$.

iii. On the same diagram as the graphs in (ii), draw the graph of $y = \sin^{-1}x + \cos^{-1}x$.†

«→ i) $x = \dfrac{1}{\sqrt{2}}$ ii) iii)



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3U96-3a)!

$$f(x) = \frac{8}{4 + x^2}$$

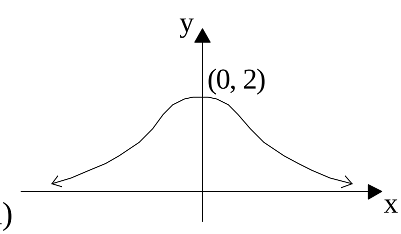
i. Show that f is an even function, and the x axis is a horizontal asymptote to the curve y = f(x).

ii. Find the co-ordinates and nature of the stationary point on the curve y = f(x).

iii. Sketch the graph of the curve showing the above features.

iv. Find the exact area of the region in the first quadrant bounded by the curve y = f(x) and the line x = 2.†

«→ i) Proof ii) (0, 2) is a maximum turning point iii)



iv) π units^2 »

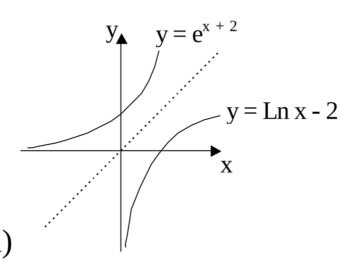
3U95-2b)!

i. If $f(x) = e^{x + 2}$, find the inverse function $f^{-1}(x)$.

ii. State the domain and range of $f^{-1}(x)$.

iii. On one diagram sketch the graphs of f(x) and $f^{-1}(x)$.†

«→ i) $f^{-1}(x) = \text{Ln } x - 2$ ii) Domain: $x > 0$, Range: All real y iii)



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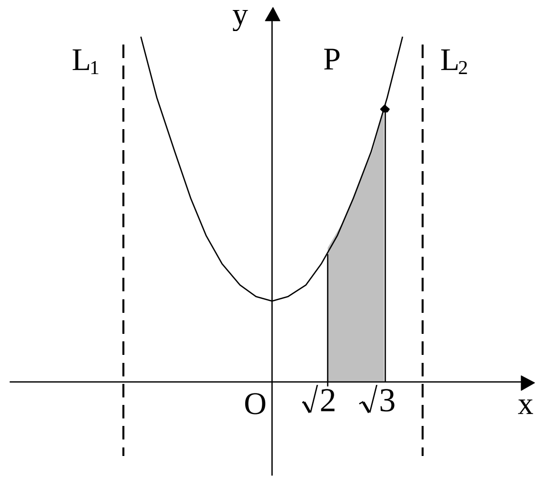
3U95-4b)!



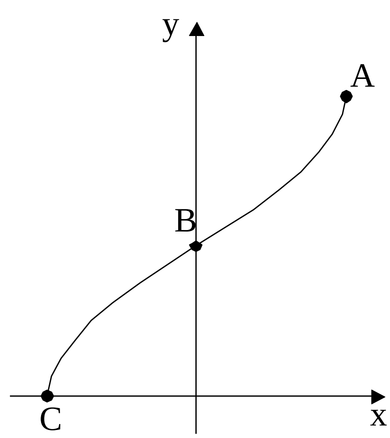
The diagram shows the graph of the function $f(x) = \dfrac{1}{\sqrt{4-x^2}}$.

i. Find the equations of the asymptotes L_1 and L_2.
ii. By comparing the values of f(-x) and f(x) show that f is an even function. What is the geometrical significance of this result?
iii. Find the exact equation of the tangent to the curve at the point P where $x = \sqrt{3}$.
iv. Find the exact area of the shaded region.†

«→ i) x = -2, x = 2 ii) The function is symmetrical about the y-axis iii) $y = x\sqrt{3} - 2$ iv) $\dfrac{\pi}{12}$ units^2 »

3U94-2c)!

The diagram below shows the graph of $y = \pi + 2\sin^{-1}3x$.



i. Write down the co-ordinates of the endpoints A and C.
ii. Write down the co-ordinates of the point B.
iii. Find the equation of the tangent to the curve $y = \pi + 2\sin^{-1}3x$ at the point B.†

«→ i) $A(\frac{1}{3}, 2\pi)$, $C(-\frac{1}{3}, 0)$ ii) $B(0, \pi)$ iii) $6x - y + \pi = 0$ »

3U93-1a)!

Find $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$.†

«→ $\sin^{-1}\left(\dfrac{x}{3}\right) + C$ »

3U93-2a)!

Given that $\int_0^1 \frac{1}{x^2+3}dx = k\pi$, find the value of the constant k.†

«→ $\frac{\sqrt{3}}{18}$ »

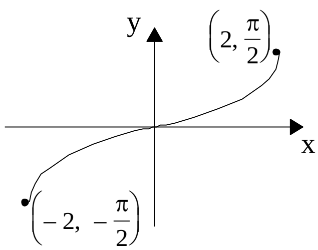
3U92-1c)!

i. Sketch the graph of the function $y = \sin^{-1}\left(\frac{x}{2}\right)$.

ii. State the domain and the range of the function.

iii. Find the exact equation of the tangent to the curve $y = \sin^{-1}\left(\frac{x}{2}\right)$ at the point where x = 1.†

«→ i)



ii) Domain: $-2 \le x \le 2$, Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ iii) $2\sqrt{3}x - 6y + \pi - 2\sqrt{3} = 0$ »

3U92-3a)!

Find the exact value of $\sin(2\tan^{-1}\frac{1}{2})$.†

«→ $\frac{4}{5}$ »

3U91-2a)!

Show that $\int_1^{\sqrt{2}} \frac{dx}{\sqrt{4-2x^2}} = 6 \times \int_{\sqrt{2}}^{\sqrt{6}} \frac{dx}{4+2x^2}$.†

«→ Proof »

3U90-1a)!

Evaluate $\int_0^3 \frac{dx}{x^2+9}$, leaving the answer in exact form.†

«→ $\frac{\pi}{12}$ »

3U90-1e)!

Find the value of the expression $\cos^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$ in terms of π.†

«→ $\frac{5\pi}{6}$ »

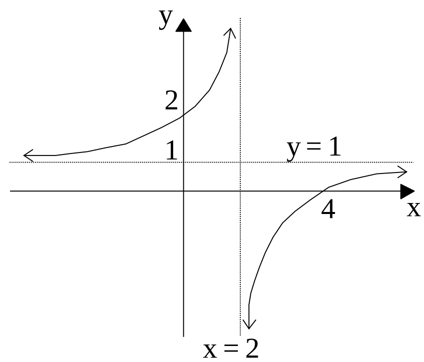
3U90-3d)!

i. Show that the function $f(x) = \dfrac{x-4}{x-2}$ $(x \neq 2)$ is an increasing function for all values of x in its domain.

ii. Sketch the graph of the function, showing clearly the co-ordinates of any points of intersection with the x-axis and the y-axis, and also the equations of any asymptotes.

iii. Find the inverse function, $f^{-1}(x)$, and state its range.†

«→ i) Proof ii)



ii) $f^{-1}(x) = \dfrac{2x-4}{x-1}$, Range: All real y, except y = 2 »

3U90-5c)!

Use the substitution $u = \sqrt{x}$ to evaluate $\displaystyle\int_1^3 \frac{dx}{(1+x)\sqrt{x}}$, giving the answer in exact form.†

«→ $\dfrac{\pi}{6}$ »

3U89-1a)!

Evaluate $\displaystyle\int_0^{1.5} \frac{dx}{\sqrt{9-2x^2}}$, leaving your answer in exact form.†

«→ $\dfrac{\sqrt{2}\pi}{8}$ »

3U89-1b)!

State the domain and range for the function: $y = 2\cos^{-1}(2x)$.†

«→ Domain: $-\dfrac{1}{2} \le x \le \dfrac{1}{2}$, Range: $0 \le y \le 2\pi$ »

3U89-2a)!

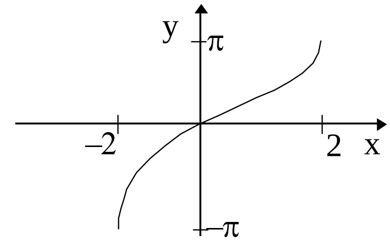
If $f(x) = 2\cos^{-1}\left(\dfrac{x}{\sqrt{2}}\right) - \sin^{-1}(1 - x^2)$, find $f'(x)$. Hence, or otherwise, show that:

$2\cos^{-1}\left(\dfrac{x}{\sqrt{2}}\right) - \sin^{-1}(1 - x^2) = \dfrac{\pi}{2}$.†

«→ $f'(x) = 0$ »

3U88-1b)!

State the range and domain of the function $y = 2\sin^{-1}(\dfrac{x}{2})$ and draw a sketch of the function, carefully labelling the extremities of both the range and the domain.†



«→ Range: $-\pi \le y \le \pi$, Domain: $-2 \le x \le 2$ »

3U88-2c)!

Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \frac{2\cos x}{1+4\sin^2 x}\,dx$ using the substitution $u = 2\sin x$.†

«→ $\dfrac{\pi}{4}$ »

3U87-1c)!

Show that the two curves $y = \cos^{-1}x$ and $y = 2\tan^{-1}(1 - x)$ cut the y-axis at the same point and have a common tangent at this point.†

«→ Proof »

3U86- 2ii)!

Given that $y = \sin^{-1}\sqrt{x}$, show that $\dfrac{dy}{dx} = \dfrac{1}{\sin 2y}$.†

«→ Proof »

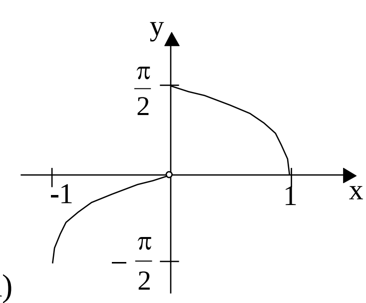
3U85-4iii)!

A function is defined by the rules $f(x) = \begin{cases} \sin^{-1} x & \text{if } \; 1 \le x < 0 \\ \cos^{-1} x & \text{if } \; 0 \le x \le 1 \end{cases}$.

a. Sketch the function for $-1 \le x \le 1$

b. Evaluate $f(-\frac{1}{2}) + 2f(0) - f(\frac{1}{2})$.†

«→ a)



b) $\dfrac{\pi}{2}$ »

3U84-1ii)!

a. Prove that $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$.

b. Find the exact values of x and y which satisfy the simultaneous equations

$\sin^{-1}x - \cos^{-1}y = \dfrac{\pi}{12}$; $\qquad \cos^{-1}x + \sin^{-1} y = \dfrac{5\pi}{12}$.†

«→ a) Proof b) $x = \dfrac{\sqrt{3}}{2}$, $y = \dfrac{1}{\sqrt{2}}$ »