Ext 1 Trial 2004 SOLUTIONS

1. (a) $$\frac{xy^{-1}-yx^{-1}}{x-y} = \frac{\frac{x}{y}-\frac{y}{x}}{x-y}$$
$$= \frac{x^2-y^2}{xy(x-y)}$$
$$= \frac{x+y}{xy} \quad (2)$$

(b) External division, so let ratio be $-5:2$

$A(-5,12)$ $B(4,9)$

$$x = \frac{-5\times 2+4\times(-5)}{-5+2} \qquad y = \frac{12\times 2+9\times(-5)}{-5+2}$$
$$= 10 \qquad\qquad = 7$$

$\therefore$ P: $(10,7)$ (2)

(c) $f(x) = x^3+3x^2-10x-24$

$f(1) = 1+3-10-24 \neq 0$

$f(-2) = -8+12+20-24 = 0$

$\therefore (x+2)$ is a factor

$$\begin{array}{r|l} & x^2 + x - 12 \\ x+2 & x^3+3x^2-10x-24 \\ & x^3+2x^2 \\ \hline & x^2-10x \\ & x^2+2x \\ \hline & -12x-24 \\ & -12x-24 \\ \hline & 0 \end{array}$$

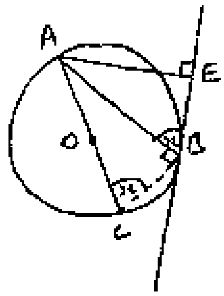
$\therefore f(x) = (x+2)(x^3+x-12)$
$= (x+2)(x+4)(x-3)$ (3)

(d) $$\int_0^{\pi/6} \sin^2 2x\,dx = \frac{1}{2}\int_0^{\pi/6}(1-\cos 4x)\,dx$$

$\left[\begin{array}{l}\cos 2x = 1-2\sin^2 x \\ \sin^2 x = \frac{1}{2}(1-\cos 2x)\end{array}\right]$

$$= \frac{1}{2}\left[x-\frac{1}{4}\sin 4x\right]_0^{\pi/6}$$
$$= \frac{1}{2}\left(\left[\frac{\pi}{6}-\frac{1}{4}\sin\frac{4\pi}{6}\right]-[0]\right)$$
$$= \frac{1}{2}\left(\frac{\pi}{6}-\frac{1}{4}\times\frac{\sqrt{3}}{2}\right)$$
$$= \frac{1}{2}\left(\frac{\pi}{6}-\frac{\sqrt{3}}{8}\right) \quad (3)$$

(e)



$\angle ABE = \angle ACB = x$ (angle in alt. segment equal)

$\angle ABC = 90°$ ($\angle$ in semi-circle)

$\therefore \angle CAB = 90 - x$ ($\angle$ sum of Δ)

and $\angle BAE = 90 - x$ ($\angle$ sum of Δ)

$\therefore \angle CAB = \angle BAE$ (2)

$\therefore$ AB bisects $\angle CAE$.

(Alternative proofs are possible)

2. (a) $$\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\frac{x}{2}\right]_1^{\sqrt{3}}$$
$$= \left[\sin^{-1}\frac{\sqrt{3}}{2}\right]-\left[\sin^{-1}\frac{1}{2}\right]$$
$$= \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6} \quad (2)$$

(b) $$\int_0^1 x\sqrt{1-x^2}\,dx = \int_1^0 u^{1/2}\cdot\frac{du}{-2}$$

Let $u = 1-x^2$
$\frac{du}{dx} = -2x$
$\frac{du}{-2} = x\,dx$
$x=1,\ u=0$
$x=0,\ u=1$

$$= \int_0^1 \frac{1}{2}u^{1/2}\,du$$
$$= \left[\frac{1}{3}u^{3/2}\right]_0^1$$
$$= \left[\frac{1}{3}\right]-[0]$$
$$= \frac{1}{3} \quad (3)$$

2.(c) $y = \tan^{-1} x$ $\qquad x = \tan y$

$x^2 = \tan^2 y$

$$\text{Vol} = \pi \int_0^{\pi/4} \tan^2 y \, dy$$

$$= \pi \int_0^{\pi/4} (\sec^2 y - 1) \, dy$$

$$= \pi \Big[\tan y - y \Big]_0^{\pi/4}$$

$$= \pi \left(\left[\tan \pi/4 - \pi/4 \right] - [0] \right)$$

$$= \pi \left(1 - \pi/4 \right) \qquad 3$$

$$= \pi/4 \, (4 - \pi) = \frac{\pi(4-\pi)}{4} \text{ units}^3$$

(as reqd)

(d) Prove that $2^{3n} - 1$ is div. by 7.

For $n = 1$, $2^3 - 1 = 7$ which is div by 7, $\therefore$ true for $n = 1$.

Assume true for $n = k$, ie.

$2^{3k} - 1 = 7M$ (*), where M is an integer, $M > 0$.

If true for $n = k$, show true for $n = k+1$, ie. show that $2^{3(k+1)} - 1$ is div. by 7:

$$2^{3k+3} - 1 = 2^{3k} \cdot 2^3 - 1$$

$$= (7M+1) \cdot 8 - 1$$

(from (*))

$$= 56M + 7$$

$$= 7(8M+1),$$

$M > 0$ is an integer, $\therefore$ $8M+1$ is an integer, $\therefore$ $2^{3(k+1)} - 1$ is div. by 7.

Thus if true for $n = k$, it is true for $n = k+1$.

It is true for $n = 1$, $\therefore$ by the principle of mathematical induction, it is true for all n.

4

3. (a) $3\cos x + 4\sin x \equiv A\cos(x - \alpha)$

$\text{RHS} = A\cos(x - \alpha)$

$= A\cos x \cos\alpha + A\sin x \sin\alpha$

Equating coeffs:

$A\cos\alpha = 3 \qquad (1)$

$A\sin\alpha = 4 \qquad (2)$

$\frac{(2)}{(1)}$: $\tan\alpha = \frac{4}{3} \qquad \alpha = 53°8'$

$(1)^2 + (2)^2$:

$A^2\cos^2\alpha + A^2\sin^2\alpha = 9 + 16$

$A^2 = 25$

$\therefore A = 5$

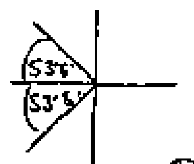
$\therefore 3\cos x + 4\sin x = 5\cos(x - 53°8')$

Solve $5\cos(x - 53°8') = -3$

$\cos(x - 53°8') = -3/5$

$x - 53°8' = 126°52'$

or $233°8'$



$x = \underline{\underline{180°}}$ or $\underline{\underline{286°16'}}$ $\qquad$ 4

(b) $(3 + 4x)^{16}$

$T_{k+1} = {}^{16}C_k \cdot 3^{16-k} \cdot (4x)^k$

$T_k = {}^{16}C_{k-1} \cdot 3^{16-(k-1)} \cdot (4x)^{k-1}$

$= {}^{16}C_{k-1} \cdot 3^{17-k} \cdot (4x)^{k-1}$

3(b) (cont.)

ratio of coeffs:

$$\frac{T_{k+1}}{T_k} = \frac{{}^{16}C_k \cdot 3^{16-k} \cdot 4^k}{{}^{16}C_{k-1} \cdot 3^{17-k} \cdot 4^{k-1}}$$

$$= \frac{{}^{16}C_k}{{}^{16}C_{k-1}} \cdot \frac{4}{3}$$

$$= \frac{\cancel{16\times15\times\cdots\times}(16-k+1)}{\cancel{1\times2\times\cdots}\times k} \times \frac{\cancel{1\times2\times\cdots\times(k-1)}}{\cancel{16\times15\times\cdots\times(16-(k-1)+1)}} \cdot \frac{4}{3}$$

$$= \frac{17-k}{k} \cdot \frac{4}{3}$$

For $T_{k+1} > T_k$, $\frac{T_{k+1}}{T_k} > 1$

$$\therefore \frac{4(17-k)}{3k} > 1 \qquad (4)$$

$68 - 4k > 3k$

$68 > 7k$

$k < 9.7 \quad \therefore k = 9$

$\therefore$ greatest coeff. is $\underline{{}^{16}C_9 \cdot 3^7 \cdot 4^9}$

(c) $v^2 = 16x - 4x^2 + 20$

(i) $\frac{1}{2}v^2 = 8x - 2x^2 + 10$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 8 - 4x$ (1)

$= -4(x-2)$, since $\ddot{x}$ is in the form $\ddot{x} = -n^2 X$, where $X = x - 2$, the motion is SHM.

(ii) Centre: $\underline{x = 2}$ (1)

(iii) Particle is at rest when $v = 0$.

$\therefore 16x - 4x^2 + 20 = 0$

$x^2 - 4x - 5 = 0$

$(x+1)(x-5) = 0$

$\therefore x = -1, \ x = 5$

$\therefore$ amplitude $= 5 - (-1)$

$= \underline{6m}$ (2)

4. (a) $2x^3 + 3x^2 - 4 = 0$

(i) $\alpha + \beta + \gamma = \underline{-3/2}$ (1)

(ii) $\alpha\beta\gamma = 4/2 = \underline{2}$ (1)

(iii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= (-3/2)^2 - 2\times 0$

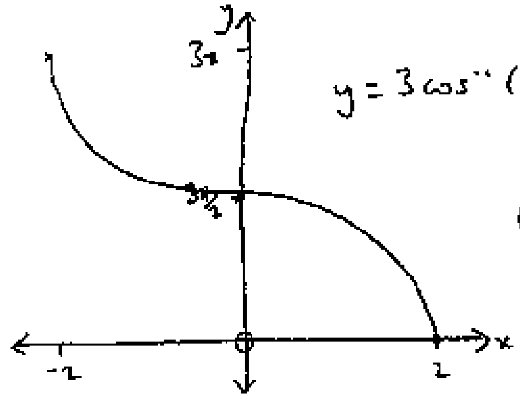
$= \underline{9/4}$ (1)

(b) $y = 3\cos^{-1}(x/2)$

(i) D: $-2 \le x \le 2$ (2)

R: $0 \le y \le 3\pi$

(ii)



(1)

(c) Let $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

If $x_1 = 0.5$ is a close approx. to a root, then $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ is a better approx.

$x_2 = 0.5 - \frac{f(0.5)}{f'(0.5)}$

$= 0.5 - \frac{(-0.375)}{1.75}$ (3)

$= \underline{0.71}$ (to 2 d.p.'s)

(d) $\frac{2}{\tan A + \cot A} = \sin 2A$

LHS $= \frac{2}{\tan A + \cot A} = \frac{2}{\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}}$

$= \frac{2\sin A\cos A}{\sin^2 A + \cos^2 A} = 2\sin A\cos A$

$= \sin 2A$

5(a) $2x - y + 5 = 0$ $\quad y = -3x + 7$

$2x + 5 = y$

$m_1 = 2 \qquad m_2 = -3$

$$\tan\theta = \left|\frac{2+3}{1+2\times(-3)}\right| = \left|\frac{5}{-5}\right|$$

$\tan\theta = 1 \qquad \therefore \theta = \underline{\underline{45^\circ}}$ (3)

(b) $N = A(1 + e^{-kt})$

(i) $t = 0, \; N = 5$

$5 = A(2) \qquad A = \underline{\underline{5/2}}$ (1)

(ii) $t = 3, \; N = 80$

$80 = 5/2(1 + e^{-3k})$

$e^{-3k} = 31$

$-3k = \ln 31$ (2)

$k = \underline{\underline{-\tfrac{1}{3}\ln 31}} = -1.14466$ (to 5 d.p.'s)

(iii) $560 = \frac{5}{2}(1 + e^{\frac{1}{3}\ln 31 t})$

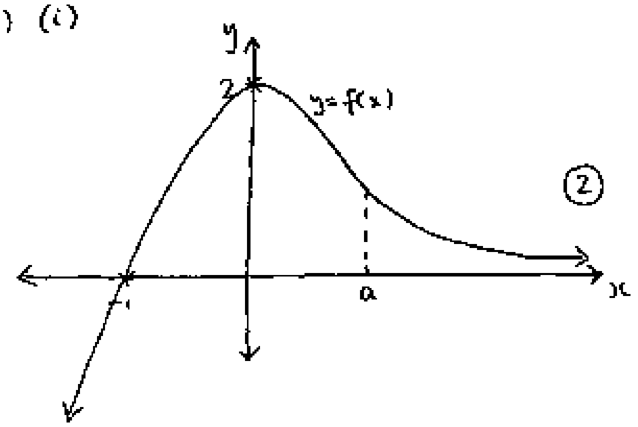
$e^{\frac{1}{3}\ln 31 t} = 223$

$\frac{1}{3}\ln 31 t = \ln 223$

$t = \underline{\underline{4.7 \text{ days}}}$ (2)

$\therefore$ all students have heard the rumour within 5 days.

(c) (i)



(2)

(ii) For $x < a$, $f(x)$ is concave down.

For $x > a$, $f(x)$ is concave up.

Hence $f(x)$ changes concavity and there is an inflexion point. (2)

6. (a) Given $\frac{dV}{dt} = 50$

Need to find $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$

$A = 4\pi r^2 \qquad V = \frac{4}{3}\pi r^3$ (1)

$\frac{dA}{dr} = 8\pi r \qquad \frac{dV}{dr} = 4\pi r^2$

Find $\frac{dr}{dt}$ from:

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$50 = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{50}{4\pi r^2} = \frac{25}{2\pi r^2}$$ (2)

$$\therefore \frac{dA}{dt} = 8\cancel{\pi}\cancel{r} \cdot \frac{25}{2\cancel{\pi} r^{\cancel{2}}} = \frac{100}{r}$$ (1)

When $r = 20$mm,

$$\frac{dA}{dt} = \frac{100}{20} = \underline{\underline{5\text{mm}^2/\text{sec}}}$$ (1)

(b) (i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

At $x = 2ap$, $\frac{dy}{dx} = \frac{2ap}{2a} = p$

Eqn. of tangent at $(2ap, ap^2)$:

$y - ap^2 = p(x - 2ap)$ (2)

$y - ap^2 = px - 2ap^2$

$y - px + ap^2 = 0$ ① (as reqd)

(ii) Tangent at Q:

$y - qx + aq^2 = 0$ ②

Point of int:

① − ②:

$-px + qx + ap^2 - aq^2 = 0$

$x(q - p) = a(q^2 - p^2)$

6 (a) (ii) (cont)

$x = a(q+p)$

$y - ap(q+p) + ap^2 = 0$

$y = apq + ap^2 - ap^2$

$y = apq$

$\therefore R: \left(a(q+p),\ apq\right)$ ②

(iii) grad of $PO = \dfrac{ap^2 - 0}{2ap - 0}$

$= \dfrac{p}{2}$

grad. of $QO = \dfrac{aq^2 - 0}{2aq - 0}$

$= \dfrac{q}{2}$ ①

PO and QO are perp.,

$\therefore \dfrac{p}{2} \cdot \dfrac{q}{2} = -1$ ①

$\therefore pq = -4$ (as reqd.)

(iv) $x = a(p+q)$

$y = apq$, but $pq = -4$.

$\therefore y = -4a$ (as reqd.) ①

7. (a) (i) $\sin\alpha = \dfrac{h}{2}$

$h = 2\sin\alpha$ ①

(ii) Δ's APD and AQC are similar (AA)

$\therefore \dfrac{QC}{PD} = \dfrac{AC}{AD}$ $AD = 3 + BD$

$BD = 2\cos\alpha$

$\dfrac{x}{h} = \dfrac{6}{3+2\cos\alpha}$, but $h = 2\sin\alpha$

$\therefore x = \dfrac{12\sin\alpha}{3+2\cos\alpha}$ (as reqd.)

②

(iii) $\dfrac{dx}{d\alpha} = \dfrac{(3+2\cos\alpha)(12\cos\alpha) - 12\sin\alpha(-2\sin\alpha)}{(3+2\cos\alpha)^2}$

$= \dfrac{36\cos\alpha + 24\cos^2\alpha + 24\sin^2\alpha}{(3+2\cos\alpha)^2}$

$= \dfrac{36\cos\alpha + 24(\cos^2\alpha + \sin^2\alpha)}{(3+2\cos\alpha)^2}$

$= \dfrac{36\cos\alpha + 24}{(3+2\cos\alpha)^2}$ ②

Turning points occur when

$36\cos\alpha + 24 = 0$

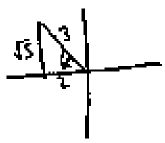
$\cos\alpha = -\dfrac{24}{36}$

$\cos\alpha = -\dfrac{2}{3}$ ($\alpha \doteqdot 2.3^c$)

$\alpha = 2.2^c$, $\dfrac{dx}{d\alpha} = 0.8467\ldots > 0$

$\alpha = 2.4^c$, $\dfrac{dx}{d\alpha} = -1.0945\ldots < 0$ ①

$\therefore$ max. turning pt at $\alpha = 2.3^c$



$\cos\alpha = -\dfrac{2}{3}$

$\sin\alpha = \dfrac{\sqrt{5}}{3}$

$\therefore$ max. value of $x = \dfrac{12 \times \frac{\sqrt{5}}{3}}{3 + 2(-\frac{2}{3})}$

$= \dfrac{4\sqrt{5}}{\frac{5}{3}} = \dfrac{12\sqrt{5}}{5}$ ①

(b) Horizontal

$\ddot{x} = 0$

$\dot{x} = C$, when $t = 0$, $\dot{x} = 25\cos\alpha$

$\therefore \dot{x} = 25\cos\alpha$

$x = 25t\cos\alpha + d$, when $t = 0, x = 0$

$\therefore x = 25t\cos\alpha$ ①

$t = \dfrac{x}{25\cos\alpha}$ (*)

7(b) (cont.)

Vertical

$\ddot{y} = -10$

$\dot{y} = -10t + e$, when $t=0$, $\dot{y} = 25\sin\alpha$

$\therefore \dot{y} = -10t + 25\sin\alpha$

$y = -5t^2 + 25t\sin\alpha + f$,

when $t=0$, $y=2$

$\therefore y = -5t^2 + 25t\sin\alpha + 2$ (1)

Subs. ⊛ into y:

$$y = -5\left(\frac{x}{25\cos\alpha}\right)^2 + 25\left(\frac{x}{25\cos\alpha}\right)\cdot\sin\alpha + 2$$

$$= \frac{-x^2}{125}\sec^2\alpha + x\tan\alpha + 2$$ (1)

When $x = 20$, $y = 15$

$$15 = -\frac{400}{125}\sec^2\alpha + 20\tan\alpha + 2$$

$$13 = \frac{-16}{5}(1+\tan^2\alpha) + 20\tan\alpha$$

$$65 = -16 - 16\tan^2\alpha + 100\tan\alpha$$

$16\tan^2\alpha - 100\tan\alpha + 81 = 0$ (1)

$$\tan\alpha = \frac{100 \pm \sqrt{100^2 - 4\times16\times81}}{32}$$

$\tan\alpha = 0.956\ldots$, or $5.29\ldots$

$\alpha = 44°$ or $79°$

$\therefore$ $\underline{\underline{44° \le \alpha \le 79°}}$ (1)