N.S.W. DEPARTMENT OF EDUCATION

HIGHER SCHOOL CERTIFICATE EXAMINATION 1980

MATHEMATICS 3 UNIT (AND 4 UNIT - FIRST PAPER)

Instructions: Time allowed 3 hours. All questions may be attempted. All questions are of equal value. In every question, all necessary working should be shown. Marks will be deducted for careless or badly arranged work.

Mathematical tables will be supplied. Approved slide rules or calculators may be used.

QUESTION 1 (10 MARKS)

(i) Find the derivative of (a) $\frac{x}{1 + x^2}$; (b) $\sin(\cos^2 x + e^x)$.

(ii) Find the exact value of $\int_0^1 \left(e^{-x} + \frac{1}{1 + x} + \frac{1}{\sqrt{(1 - x^2)}}\right) dx$.

(iii) Let $f(x) = \sin^{-1}x + \cos^{-1}x \quad (0 \le x \le 1)$.

Find (a) $f'(x)$; (b) $\int_0^1 f(x)dx$.

QUESTION 2 (10 MARKS)

(i) The line $x = 1$ meets the curve $y = x^3 + 5$ at P. Write down the co-ordinates of P.

(ii) Find the equation of the tangent line ℓ to the curve $y = x^3 + 5$ at P.

(iii) Verify that $Q(-5, -12)$ lies on the line ℓ.

(iv) The curve $y = x^3 + 5$ meets the y-axis at R. Find the equation of the line QR.

(v) If θ is the acute angle between PQ and QR, find the exact value of $\tan\theta$.

QUESTION 3 (10 MARKS)

(i) Show that 404 is not a term of the arithmetic sequence 2, 6, 10, ...

(ii) The sum of the infinite geometric series $1 + 2^x + 2^{2x} + \ldots$ is 2. Find the value of x.

(iii) If $S_n = 1.2 + 2.3 + \ldots + n(n + 1)$, use the principle of mathematical induction to show that $S_n = \frac{1}{3}n(n + 1)(n + 2)$ for all positive integers n.

QUESTION 4 (10 MARKS)

(i) The function $f(x)$ is defined by the rule $f(x) = \begin{cases} 0 \text{ if } x \le 0, \\ 2x \text{ if } x > 0. \end{cases}$

(a) Sketch the function $f(x)$, from $x = -2$ to $x = 2$.

(b) Evaluate $\int_{-2}^{2} f(x)\,dx$.

(ii) The function $f(x)$ is defined by the rule $f(x) = x^3 - 3x^2$, in the domain $-1 \le x \le 3$.

(a) Draw a sketch of the graph of $y = f(x)$, showing clearly the turning points, the intercepts with the x- and y-axes, and the values at the extremes of the domain.

(b) Indicate on your sketch the region bounded entirely by parts of the graph of $y = f(x)$ and the x-axis.

Find the area of this region.

QUESTION 5 (10 MARKS)

(i) Find the indefinite integral $\int \frac{x+1}{x^2+4}\,dx$.

(ii) Given that $\frac{d^2x}{dt^2} = -9x$ and $\frac{dx}{dt} = 0$ at $x = 4$, find $\left|\frac{dx}{dt}\right|$ at $x = 2$.

(iii) Find all values of x in the range $0 \le x \le 2\pi$ for which $\sin 2x = \cos x$.

QUESTION 6 (10 MARKS)

Find the equation of the tangent line to the parabola $x^2 = 16y$ at any point $P(8t, 4t^2)$ on it.

Prove that for all values of t, this tangent is equally inclined to the axis of the parabola and to the focal chord through P. (The focal chord is the line joining P and the focus.)

Find the equation of the line ℓ through the focus S of the parabola and perpendicular to the focal chord SP.

Prove that the locus of the point of intersection of the line ℓ and the tangent line at P is a straight line whose equation is of the form $y = c$, and find the value of c.

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QUESTION 7 (10 MARKS)

The equations of the spheres S_1 and S_2 are given (in Cartesian (x, y, z) co-ordinates) respectively by $x^2 + y^2 + z^2 = 1$, $(x - 12)^2 + y^2 + z^2 = 25$. Q, R are the centres of S_1, S_2 respectively, and P is the point $(3, 2, 6)$. Find:

(i) the equations of the line PQ;

(ii) the equations of the line PR;

(iii) the exact value of the cosine of the angle QPR;

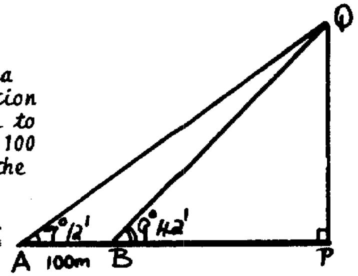
(iv) the equation of the plane PQR.

Show that a sphere S can be found such that its centre is P and such that it touches both S_1 and S_2.

Show that the locus of the centres of all spheres which touch both S_1 and S_2 and whose radius equals the radius of S is the set of points (X, Y, Z) satisfying the equations $X = 3$, $Y^2 + Z^2 = 40$.

QUESTION 8 (10 MARKS)

(i) The diagram given was sketched by a surveyor, who measured the angle of elevation of a tree top on the other side of a river to be $7°12'$ at the point A. At the point B, 100 metres directly towards the tree from A, the angle of elevation was $9°42'$.



(a) Derive an expression for the height of the tree.

(b) Calculate the height of the tree correct to three significant figures.

(ii) Two particles P and Q move along a given line ℓ, their displacements at time $t \ge 0$ with respect to a fixed point O on ℓ being $x(t)$ and $X(t)$ respectively.

(a) Given that $\frac{d^2x}{dt^2} = 6 + e^{-t}$, and that $\frac{dx}{dt} = -1$ at $t = 0$, and $x = 0$ at $t = 0$, find an expression for $x(t)$.

(b) If $X(t) = 2\sin 5t + 3t^2 + 2$, prove that $X(t) > x(t)$ for all $t \ge 0$. Explain this result in terms of the motions of the particles P and Q.

QUESTION 9 (10 MARKS)

In each of the following parts, use the information to obtain the required real polynomial $P(x)$ explicitly in the form $P(x) = p_0x^n + p_1x^{n-1} + \ldots + p_n$, where $n, p_0, p_1, \ldots, p_n$ are to be given numerical values.

(i) $P(x)$ is quadratic, $P(0) = 15$, and the minimum value of $P(x)$ is 3 when $x = 2$.

(ii) $P(x)$ is quadratic, $P(0) = 32$, and $P(2^t) = 0$ has roots $t = 1$ and $t = 3$.

(iii) $P(x)$ has degree 4, has factors $(x + 2)^2$ and $(x - 2)^2$, and has remainder 50 on division by $x - 3$.

(iv) $P(x)$ has degree 3, zeros at $x = 0$, $x = 1 + \sqrt{2}$ and at $x = 1 - \sqrt{2}$, and $P(1) = 3$.

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QUESTION 10 (10 MARKS)

(i) The probability that seeds from a certain species of plant are non-viable is 1/300.

(a) How many viable seeds would be expected to occur in a batch of one million seeds?

(b) A packet contains 100 seeds, chosen at random. Leaving your answers in unsimplified form, calculate the probability that:

(1) all seeds are viable;

(2) there is exactly one non-viable seed;

(3) there are at least two non-viable seeds.

(ii) Prove that for any positive integer n, the largest value of ${}^{2n}C_r$ for $0 \le r \le 2n$ (r integral) is ${}^{2n}C_n$ and that it occurs only when $r = n$.