St Catharine's Trial HSC Solutions 1999

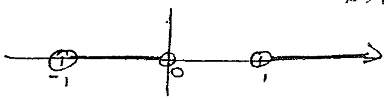
1 a) $\frac{x^2-1}{x} > 0$

Consider $\frac{x^2-1}{x} = 0$

$x \neq 0$ $\quad x^2-1=0$

$x = \pm 1$

$x > 1$ or $-1 < x < 0$ (3)



b) $\cos 2x = \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

$\therefore \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$\int_0^{\pi} \sin^2 x \, dx$

$= \int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$

$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi}$ $\quad$ (2,1 / 1,1,0)

$= \frac{\pi}{2} - \frac{1}{4}\sin 2\pi - 0 + \frac{1}{4}\sin 0$

$= \frac{\pi}{2} - \frac{1}{4}\times 0$

$= \frac{\pi}{2}$ (3)

) $\int \frac{t}{\sqrt{1+t}}\, dt$ where $t = u^2 - 1$

$\frac{dt}{du} = 2u$

$dt = 2u\, du$

$\therefore \int \frac{u^2-1}{\sqrt{1+u^2-1}} \times 2u\, du$

$= \int \frac{u^2-1}{\cancel{u}} \times 2\cancel{u}\, du$

$= 2\left[\frac{u^3}{3} - u\right] + c$ $\quad$ $u^2 = t+1$, $u = \sqrt{t+1}$ (3)

$= \frac{2(t+1)^{3/2}}{3} - 2(t+1)^{1/2} + c$

$t=0$ $\quad v = 20t - 5t^2$

$v=0$

$x=0$

(i) rest when $v=0$

$20t - 5t^2 = 0$

$5t(4-t) = 0$

$t=0 \quad t=4$ $\quad$ rest after 4 sec (1)

(ii) greatest velocity

when $\frac{dv}{dt} = 0$ $\quad \frac{dv}{dt} = 20 - 10t$ (2)

$\frac{dv}{dt} = 0$ when $20 - 10t = 0$

$t = 2$

2a) $A(x) = x - 4$ is a factor of $P(x)$

$\therefore P(4) = 0$

$P(x) = x^3 + 2x^2 + ax - 20$

$P(4) = 4^3 + 2(4)^2 + 4a - 20 = 0$ (2)

$\therefore 76 + 4a = 0$

$4a = -76$

$a = -19$

(16)

b) $12\cos\theta + 5\sin\theta$

$= 13\left(\frac{12}{13}\cos\theta + \frac{5}{13}\sin\theta\right)$

$= 13(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$= 13\cos(\theta - \alpha)$

$= 13\cos(\theta - 22^\circ 37')$

13, 5, 12, α (2)

$\tan\alpha = \frac{5}{12}$

$\alpha = 22°37'$

$\therefore 12\cos\theta + 5\sin\theta = 13$

$13\cos(\theta - 22^\circ 37') = 13$ $\quad$ (1,0)

$\cos(\theta - 22^\circ 37') = 1$

$\therefore \theta - 22^\circ 37' = -360^\circ, 0^\circ, 360^\circ \ldots$

$\theta = 22^\circ 37'$ for $0 \le \theta \le 360^\circ$ (3)

c) $e^x = x + 2$

$\therefore e^x - x - 2 = 0$

$P(x) = e^x - x - 2$

$P'(x) = e^x - 1$

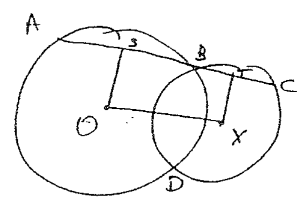
$P(1.2) = e^{1.2} - 1.2 - 2 = 0.1201$

$P'(1.2) = e^{1.2} - 1 = 2.3201$

2nd approx $x_2 = 1.2 - \frac{P(1.2)}{P'(1.2)}$

$= 1.15$ (2)

d)



S is midpt of AB $\quad$ data

$\angle OSA = \angle OSB = 90°$ $\quad$ join of centre to midpt of chord is perp to chord

Similarly

T is midpt of BC $\quad$ data

$\angle XTB = \angle XTC = 90°$ $\quad$ join of centre etc

now $\angle OSB$ & $\angle XTB$ are cointerior and add to 180°

$\therefore OS \parallel XT$ (3)

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3 a) $f(x) = \frac{x}{x^2+1}$

(i) $f(-x) = \frac{-x}{(-x)^2+1}$

$= \frac{-x}{x^2+1}$

$= -\left(\frac{x}{x^2+1}\right)$

$= -f(x)$

$\therefore f(x)$ is odd fn (1)

(ii) $f'(x) = \frac{x^2+1\,(1) - x\,(2x)}{(x^2+1)^2}$

$= \frac{1-x^2}{(x^2+1)^2}$

St pt occur when $f'(x)=0$ ie $\frac{1-x^2}{(x^2+1)^2} = 0$

$x^2 = 1$

$x = \pm 1$

$y = \pm\frac{1}{2}$ (1)

St pts $(1, \frac{1}{2})$ & $(-1, -\frac{1}{2})$

Pts of inflex may occur when $f''(x) = 0$

$\frac{2x(x^2-3)}{(x^2+1)^3} = 0$

ie $x = 0$ $x = \pm\sqrt{3}$

check for concavity

$x = -2$ $f''(-2) : <0$ ∩

$x = -1$ $f''(-1) : >0$ ∪

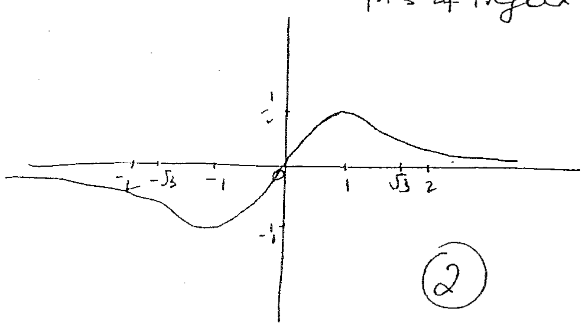
$x = 1$ $f''(1) < 0$ ∩

$x = 2$ $f''(2) > 0$ ∪ (2)

as there are changes of concavity pts of inflex are at $(0, 0)$

$(\sqrt{3}, \frac{\sqrt{3}}{4})$

$(-\sqrt{3}, -\frac{\sqrt{3}}{4})$



(2)

(iii) as $x \to \infty$

$f(x) \to 0^+$

as $x \to -\infty$

$f(x) \to 0^-$ (1)

3b) $\sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7)$

1. Prove for $n = 1$

LHS $= 1(1+2)$ $= 3$ RHS $= \frac{1}{6}(1)(2)9$ $= 3$ (1)

2. Assume true for $n = k$

ie $\sum_{r=1}^{k} r(r+2) = \frac{1}{6}k(k+1)(2k+7)$ (1)

3. Prove true for $n = k+1$

ie prove $\sum_{r=1}^{k+1} r(r+2) = \frac{1}{6}(k+1)(k+2)(2k+9)$

Proof: LHS $= \sum_{r=1}^{k+1} r(r+2)$

$= \sum_{r=1}^{k} r(r+2) + (k+1)(k+3)$

$= \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3)$

$= (k+1)\left[\frac{1}{6}(k(2k+7) + 6(k+3)\right]$ (2)

$= \frac{1}{6}(k+1)(2k^2+7k+6k+18)$

$= \frac{1}{6}(k+1)(2k^2+13k+18)$

$= \frac{1}{6}(k+1)(k+2)(2k+9)$

$=$ RHS

4. As it is true for $n=1$

then by Step 3 it is true for $n=2$

As it is true for $n=2$

then it is true for $n=3$ and so on

∴ therefore, by the principle of math Ind^n (1)

$\sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7)$

must say this for $\frac{1}{2}$m

4. a) (i) 2, 3, 4, 5, 6

$\boxed{\ }\ \boxed{\ }\ \boxed{\ }\ \boxed{2} = 2 \times {}^4P_3 = 48$ (1)

(ii) $\boxed{\ }\ \boxed{\ }\ \boxed{\ }\ \boxed{2} = 5^3 \times 2 \quad \therefore P(\text{odd}) = \frac{5^3 \times 2}{5^4} = \frac{2}{5}$ (2)

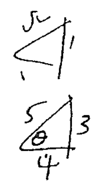
any number $= 5^4$

) (i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ (1)

(ii) $\cos\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)$

$= \cos(-\theta)$

$= +\frac{4}{5}$ (2)



$\frac{S \mid A}{T \mid C}$

) $\dfrac{\cot\frac{\theta}{2} + \tan\frac{\theta}{2}}{\cot\frac{\theta}{2} - \tan\frac{\theta}{2}} = \sec\theta$

let $t = \tan\frac{\theta}{2}$

$\therefore \frac{1}{t} = \cot\frac{\theta}{2}$

LHS $= \dfrac{\frac{1}{t} + t}{\frac{1}{t} - t} \quad \begin{matrix}\times t \\ \times t\end{matrix}$

$= \dfrac{1 + t^2}{1 - t^2}$

$= \dfrac{1}{\cos\theta}$ as $\cos\theta = \dfrac{1-t^2}{1+t^2}$

$= \sec\theta$

$=$ RHS (3)

) $\displaystyle\int_0^{1/4} \frac{dx}{1+16x^2} = \int_0^{1/4} \frac{dx}{16\left(\frac{1}{16}+x^2\right)}$

$= \frac{1}{16} \displaystyle\int_0^{1/4} \frac{\frac{1}{4}}{\frac{1}{16}+x^2}\,dx$

$= \frac{1}{16} \times 4 \left[\tan^{-1}\frac{x}{1/4}\right]_0^{1/4}$

$= \frac{1}{4}\left[\tan^{-1} 4x\right]_0^{1/4}$

$= \frac{1}{4}\left[\tan^{-1}1 - \tan^{-1}0\right] = \frac{1}{4}\cdot\frac{\pi}{4} = \frac{\pi}{16}$ (3)

5 a) $\alpha + \beta + \gamma = 45^\circ$

$\tan\alpha = \frac{1}{2}$ & $\tan\beta = \frac{1}{4}$

find $\tan\gamma$

$\tan\gamma = \tan(45 - \beta - \alpha)^\circ$

$= \tan(45 - (\alpha+\beta))^\circ$

now $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \dfrac{\frac{1}{2} + \frac{1}{4}}{1 - \frac{1}{2}\cdot\frac{1}{4}}$

$= \dfrac{\frac{3}{4}}{\frac{7}{8}}$

$= \frac{3}{4} \times \frac{8}{7}$

$= \frac{6}{7}$

$\therefore \tan(45 - (\alpha+\beta))^\circ = \dfrac{\tan 45 - \tan(\alpha+\beta)}{1 + \tan 45 \cdot \tan(\alpha+\beta)}$

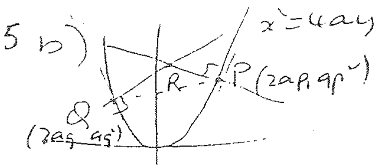
$= \dfrac{1 - \frac{6}{7}}{1 + 1\cdot\frac{6}{7}}$

$= \dfrac{\frac{1}{7}}{\frac{13}{7}}$

$\tan(45 - (\alpha+\beta))^\circ = \frac{1}{13}$

$\tan\gamma = \frac{1}{13}$ 4

5 b)



i)

$$x^2 = 4ay$$
$$\therefore y = \frac{1}{4a}x^2$$
$$y' = \frac{1}{2a}x$$

when $x = 2ap$ grad of tang $= \frac{2ap}{2a}$
$= p$

$\therefore$ grad of normal $= -\frac{1}{p}$

$\therefore$ eqn of normal $y - ap^2 = -\frac{1}{p}(x - 2ap)$
$$py - ap^3 = -x + 2ap$$
$$\therefore x + py = 2ap + ap^3$$
$$x + py = a(2p + p^3)$$

2

ii)

$$x + py = 2ap + ap^3 \quad \text{——— (1)}$$
$$x + qy = 2aq + aq^3 \quad \text{——— (2)}$$

eqn (1) − (2)
$$(p-q)y = 2a(p-q) + a(p^3 - q^3)$$
$$\cancel{(p-q)}\, y = 2a\cancel{(p-q)} + a\cancel{(p-q)}(p^2+pq+q^2) \qquad p \neq q$$
$$\therefore y = 2a + a(p^2+pq+q^2) \quad \text{——— (3)}$$

Sub (3) into (1)
$$x + p(2a + a(p^2+pq+q^2)) = 2ap + ap^3$$
$$x + \cancel{2ap} + \cancel{ap^3} + ap^2q + apq^2 = \cancel{2ap} + \cancel{ap^3}$$
$$x = -apq(p+q)$$

2

$$\therefore R\left(-apq(p+q),\ 2a + a(p^2+pq+q^2)\right)$$

iii)

$P(2ap, ap^2)$ $\quad Q(2aq, aq^2)$

$$\text{grad of } PQ = \frac{aq^2 - ap^2}{2aq - 2ap}$$
$$= \frac{\cancel{a}\cancel{(q-p)}(p+q)}{2\cancel{a}\cancel{(q-p)}}$$
$$= \frac{p+q}{2}$$

2

$$y - ap^2 = \frac{p+q}{2}(x - 2ap)$$
$$y - ap^2 = \left(\frac{p+q}{2}\right)x - ap^2 - apq$$
$$y = \left(\frac{p+q}{2}\right)x - apq$$

(iv) PQ: $y = \left(\frac{p+q}{2}\right)x - apq$

passes through $(0, 2a)$
$$\therefore 2a = \left(\frac{p+q}{2}\right)0 - apq$$
$$pq = -2 \quad \text{——— (1)}$$

Locus of R:
$$x = -apq(p+q)$$
$$x = 2a(p+q) \quad \text{from (1)}$$
$$p+q = \frac{x}{2a}$$

$$y = 2a + a(p^2+pq+q^2)$$
$$y = 2a + a((p+q)^2 - pq)$$
$$y = 2a + a\left(\left(\frac{x}{2a}\right)^2 + 2\right)$$
$$y = 2a + \frac{x^2}{4a} + 2a$$
$$y = 4a + \frac{x^2}{4a}$$
$$4ay = x^2 + 16a^2$$

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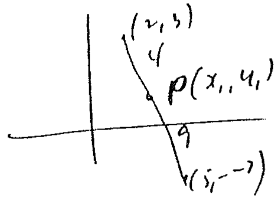
6 a)



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$x_1 = \frac{4\times 5 + 9\times 2}{4+9}$

$= 2\frac{12}{13}$

$y_1 = \frac{4\times -7 + 9\times 3}{4+9}$

$= -\frac{1}{13}$

$\therefore P(2\frac{12}{13}, -\frac{1}{13})$

b) (i) $\frac{ds}{dt} = 0.01$ 1

$S = 4\pi r^2$

$\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$ 1

$\frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt}$

when $r = 5$

$0.01 = 8\pi . 5 \cdot \frac{dr}{dt}$ 1

$\frac{dr}{dt} = \frac{0.01}{40\pi}$

$= 7.96\times 10^{-5}$

rate of change of radius is

(ii) $V = \frac{4}{3}\pi r^3$ 1

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$= 4\pi r^2 \cdot \frac{dr}{dt}$

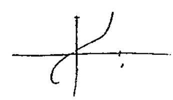
when $r=5$ $\frac{dV}{dt} = 4\pi(25) \cdot 7.96\times 10^{-5}$ 1

$= 0.025$

6 c) $\frac{d}{dx} \sin^{-1}(e^{2x}) = \frac{1}{\sqrt{1-(e^{2x})^2}} \cdot 2e^{2x}$

$= \frac{2e^{2x}}{\sqrt{1-e^{4x}}}$ 2

$\int_{-\ln\sqrt{2}}^{0} \frac{2e^{2x}}{\sqrt{1-e^{4x}}}\,dx = \left[\sin^{-1}(e^{2x})\right]_{-\ln\sqrt{2}}^{0}$



$= \sin^{-1}(e^0) - \sin^{-1}(e^{2(-\ln\sqrt{2})})$

$= \sin^{-1}(1) - \sin^{-1}(e^{-2\ln\sqrt{2}})$

$= \sin^{-1}(1) - \sin^{-1} e^{\ln(\sqrt{2})^{-2}}$

$= \frac{\pi}{2} - \sin^{-1}\frac{1}{2}$ as $(\sqrt{2})^{-2} = \frac{1}{(\sqrt{2})^2}$

$= \frac{\pi}{2} - \frac{\pi}{6}$

$= \frac{\pi}{3}$ 3

7 a) (i)

$\xrightarrow{25 L/m}$ 500L $\xrightarrow{25 L/m}$

Amount of salt in tank initially is 0

$t = 0$
$A = 0$

Brine going in:

Concentration of salt $= \frac{1}{10}$ in solution

$\therefore$ Rate of salt going in $= 25 \times \frac{1}{10}$ L/min (1)

$= \frac{25}{10}$

$= 2.5$ L/min

Running out:

Concentration of salt $= \frac{A}{500} \times 25$ L/min (1)

$= \frac{\cancel{25}^{5} A}{\cancel{500}_{100}}$

$= \frac{A}{20}$ L/min

$\frac{dA}{dt} = 2.5 - \frac{A}{20}$ (1)

$= \frac{50 - A}{20}$

$= \frac{1}{20}(50 - A)$

$\frac{dA}{dt} = -\frac{1}{20}(A - 50)$

(ii) if $\frac{dA}{dt} = k(A - B)$ (1)

then $A = B + A_0 e^{kt}$

So $A = 50 + A_0 e^{-\frac{1}{20}t}$

when $T = 0$, $A = 0$: $0 = 50 + A_0 e^{0}$

$A_0 = -50$

$\therefore A = 50 - 50 e^{-\frac{1}{20}t}$

after 100 min: $A = 50 - 50 e^{-\frac{1}{20} \times 100}$ (1)

$= 50(1 - e^{-5})$

$= 49.66$ kg

b) $\ddot{x} = mx + b$ (1) page 6

$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = mx + b$

$t = 0$, $x = 0$, $v = 0$, $a = 3$; max speed $x = 160$

$\frac{1}{2}v^2 = \frac{m}{2}x^2 + bx + C$

$v^2 = mx^2 + 2bx + C_1$

$0 = 0 + C_1$ (1)

$v^2 = mx^2 + 2bx$.

now $\ddot{x} = mx + b$

$a = 3$, $x = 0$: $3 = 0 + b$ (1)

$\boxed{\ddot{x} = mx + 3}$

It reaches max speed when $x = 160$

i.e. $\frac{dv}{dt} = 0$

$\therefore \ddot{x} = 0$ when $x = 160$

$0 = 160m + 3$

$m = -\frac{3}{160}$ (1)

$v^2 = \frac{-3}{160}x^2 + 6x$

(i) max speed when $x = 160$ (1)

$v^2 = \frac{-3}{160} \cdot 160^2 + 6 \cdot 160$

$= 3 \cdot 160$

$= 480$

$v = \sqrt{480}$

$= \pm 4\sqrt{30}$ m/s

$\therefore$ max speed is $4\sqrt{30}$ m/s

21.9

(ii) speed when $x = 80$ m (1)

$v^2 = \frac{-3}{160}(80)^2 + 6 \cdot 80$

$= 360$

$v = \pm\sqrt{360} = \pm 6\sqrt{10}$

$\therefore$ speed is $6\sqrt{10}$ m/s