Sydney Grammar School
Mathematics Department
Trial Examinations 2005

FORM VI

MATHEMATICS

Examination date

Tuesday 2nd August 2005

Time allowed

3 hours (plus 5 minutes reading time)

Instructions

All ten questions may be attempted.
All ten questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet.
Hand in the ten questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

SGS booklets: 10 per boy. A total of 1250 booklets should be sufficient.
Candidature: 108 boys.

Examiner

TCW

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\dfrac{1}{15+5\times 3}$, correct to three significant figures. $\boxed{2}$

(b) Fully factorise $16x^3 - 64x$. $\boxed{2}$

(c) Solve $|x+3| = 8$. $\boxed{2}$

(d) Solve $x(x-9) = 0$. $\boxed{2}$

(e) Write down the supplement of $\dfrac{\pi}{6}$. $\boxed{1}$

(f) Differentiate $\cos x$. $\boxed{1}$

(g) Write down the coordinates of the focus of the parabola $x^2 = -4y$. $\boxed{1}$

(h) Write down a primitive of $\dfrac{1}{x}$. $\boxed{1}$

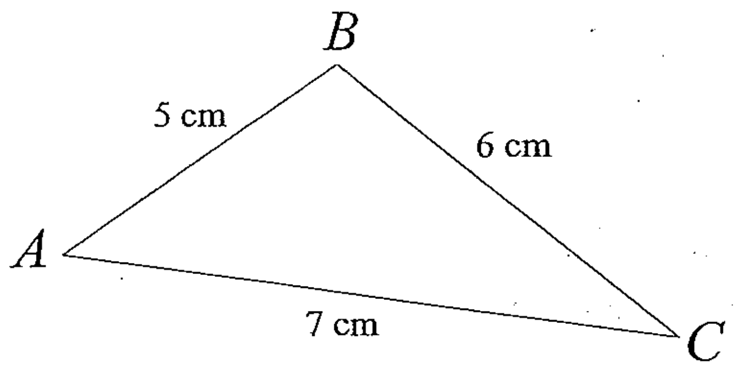
Exam continues next page ....

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) By rationalising the denominator, find a and b such that $\dfrac{3}{\sqrt{7}+2} = a + \sqrt{b}$. [2]

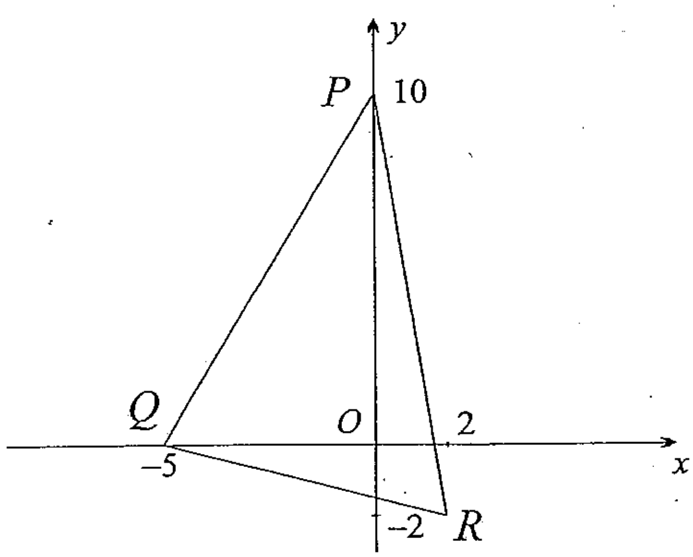
(b) Find the equation of the parabola with vertex $(-5, 0)$ and focus $(0, 0)$. [2]

(c) [2]



The diagram above shows $\triangle ABC$ where $AB = 5\,\text{cm}$, $BC = 6\,\text{cm}$ and $AC = 7\,\text{cm}$. Find the size of the largest angle in $\triangle ABC$, correct to the nearest degree.

(d)



In the diagram above $\triangle PQR$ has vertices $P(0, 10)$, $Q(-5, 0)$ and $R(2, -2)$. The origin is $O(0, 0)$.

(i) Show that PQ has length $5\sqrt{5}$ units. [1]

(ii) Show that PQ has equation $2x - y + 10 = 0$. [1]

(iii) Show that the perpendicular distance from R to PQ is $\frac{16}{\sqrt{5}}$ units. [1]

(iv) Find the coordinates of S such that $PQRS$ is a parallelogram. [1]

(v) Find the area of parallelogram $PQRS$. [1]

(vi) Find, correct to the nearest degree, the size of $\angle PQO$. [1]

Exam continues overleaf ...

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) Differentiate:

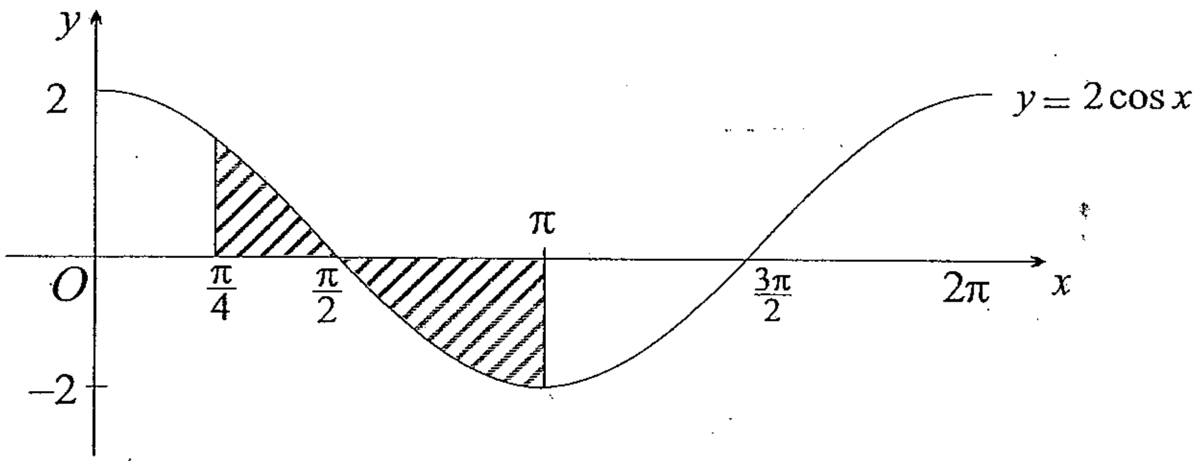
(i) $y = \dfrac{2}{e^x}$ **2**

(ii) $y = (x^2 - 1)^6$ **2**

(iii) $y = \dfrac{2x+1}{3x-1}$ **2**

(b) Evaluate $\displaystyle\int_0^2 \frac{2x}{x^2+4}\,dx$. **2**

(c) **4**



The diagram above shows the area between the curve $y = 2\cos x$ and the x-axis from $x = \frac{\pi}{4}$ to $x = \pi$. Show that this area is $4 - \sqrt{2}$ square units.

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) (i) How many terms are there in the arithmetic sequence $-5, 10, 25, \ldots, 955$? **2**

(ii) Find the limiting sum of the geometric series $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots$ **2**

(b) Consider the curve $y = 3x^2 - x^3$.

(i) Find the x-intercepts of the curve. **1**

(ii) Find the coordinates of any stationary points and determine their nature. **3**

(iii) Find the coordinates of the point of inflexion. **2**

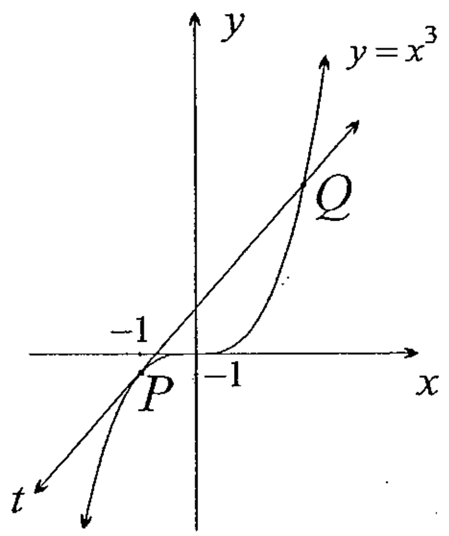
(iv) Sketch the curve, clearly showing all the stationary points, the inflexion and the intercepts. **1**

(v) Hence, or otherwise, solve $3x^2 - x^3 \leq 0$. **1**

Exam continues next page ...

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows the curve $y = x^3$. The point $P(-1,-1)$ lies on the curve. Line t is the tangent to the curve at P, which intersects the curve again at Q.

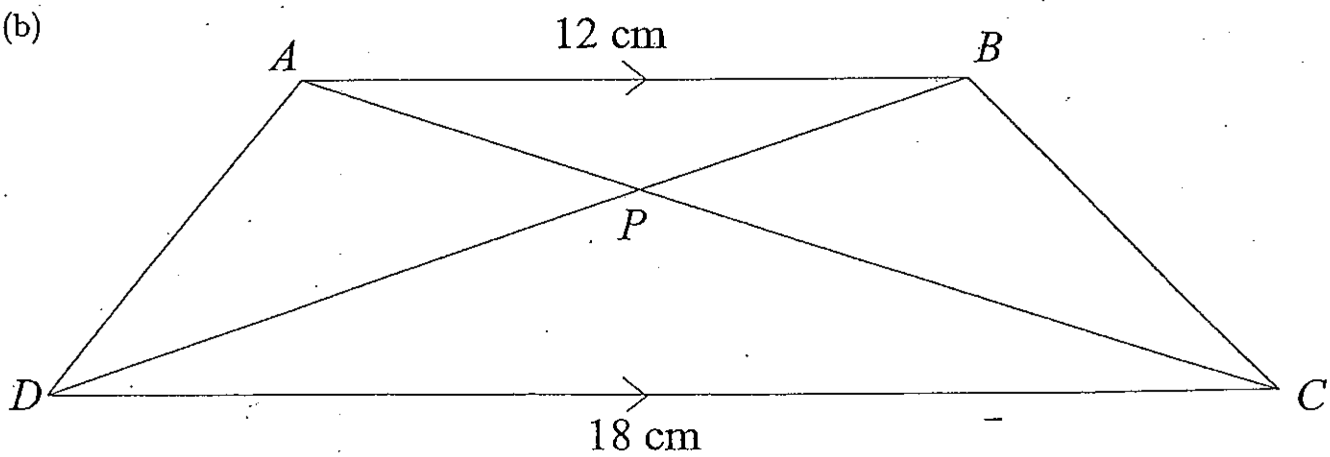
(i) Show that the tangent to the curve at P has equation $y = 3x + 2$. $\boxed{2}$

(ii) Show that Q has coordinates $(2, 8)$. $\boxed{2}$

(iii) Find the area of the region enclosed by the curve and the tangent from P to Q. $\boxed{2}$

(iv) Use inequalities to describe the region in part (iii). You may assume the boundaries are included. $\boxed{1}$

(b)



The diagram above shows the trapezium $ABCD$.
AB is parallel to CD. AC and BD intersect at P.
$AB = 12\,\text{cm}$ and $CD = 18\,\text{cm}$.

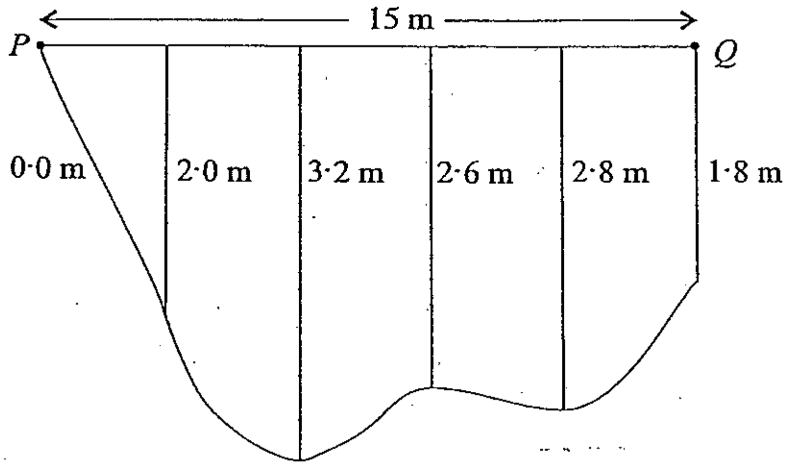
(i) Prove $\triangle ABP \,|||\, \triangle CDP$. $\boxed{3}$

(ii) Given that $AC = 15\,\text{cm}$ find the length of AP. $\boxed{2}$

Exam continues overleaf ...

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

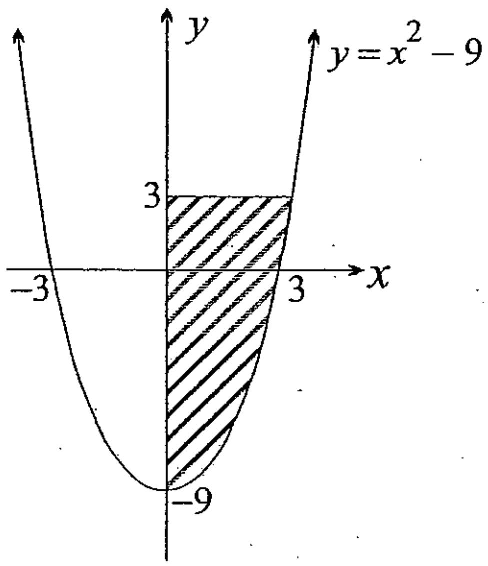
(a) A surveyor measures the depth of a river at equal intervals across its 15 metre width PQ. The sketch below shows the measurements and the cross-section.



(i) Use the trapezoidal rule to find an approximation for the cross-sectional area of the river at PQ. **2**

(ii) Given that the river flows at an average speed of 2 metres/second find the approximate volume of water passing PQ every hour in cubic metres. **2**

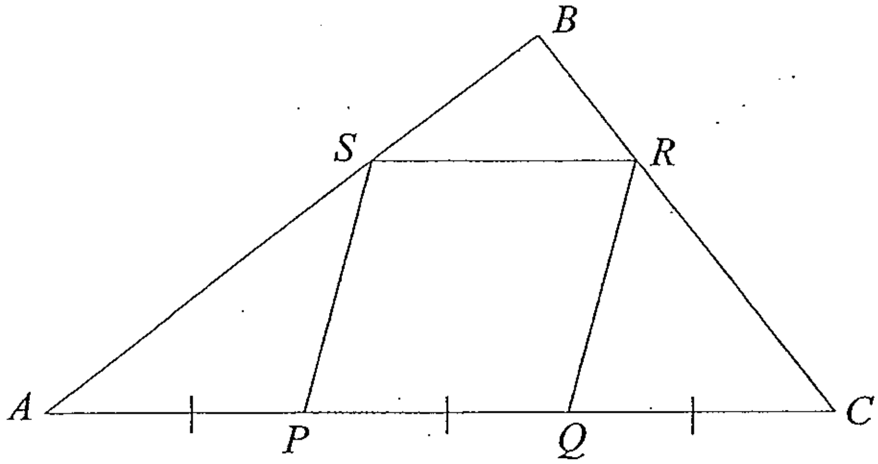
(b) **3**



The diagram above shows the region to the right of the y-axis bounded by the curve $y = x^2 - 9$, the y-axis and the line $y = 3$. Find the volume of the solid formed when this region is rotated about the y-axis.

Exam continues next page ...

(c)



The diagram above shows $\triangle ABC$ where $AP = PQ = QC$ and $PQRS$ is a rhombus.

(i) Prove that $\angle SPQ = 2\angle SAP$. $\boxed{2}$

(ii) Prove that $\angle ABC = 90°$. $\boxed{3}$

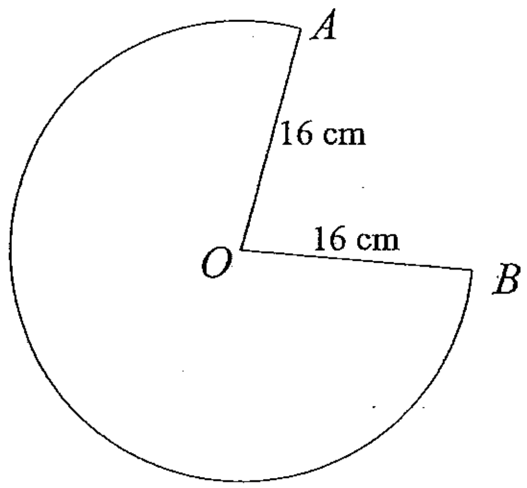
QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) Solve $2\log_e x = \log_e(2x+3)$. $\boxed{3}$

(b) (i) Solve $e^{x-2} - 1 = 0$. $\boxed{1}$

(ii) Sketch $y = e^{x-2} - 1$, clearly showing the x and y-intercepts and the horizontal asymptote. $\boxed{2}$

(c) $\boxed{3}$



In the diagram above, major sector AOB has a radius of 16 cm and an area of 624 cm^2. Find the perimeter of major sector AOB.

(d) Consider the quadratic equation $x^2 + (k+1)x + \frac{k+1}{2} = 0$.

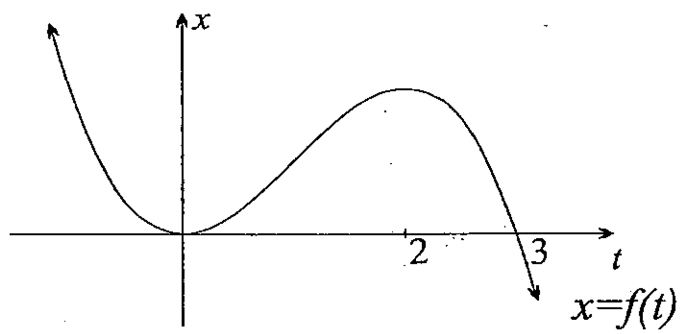
(i) Write down an expression for the discriminant of the quadratic. $\boxed{1}$

(ii) For what values of k does the equation have no real roots? $\boxed{2}$

Exam continues overleaf ...

QUESTION EIGHT (12 marks) Use a separate writing booklet. **Marks**

(a) [1]



The diagram above shows the graph of a body's displacement function $x = f(t)$. Sketch a possible graph of the body's velocity function.

(b) A particle moves in a straight line such that after t seconds its acceleration function is $\ddot{x} = (6t - 2)\,\text{ms}^{-2}$. Initially the velocity of the particle is $-1\,\text{ms}^{-1}$.

(i) Find the particle's velocity after 2 seconds. [2]

(ii) Find the time at which the particle is stationary. [1]

(iii) Find the distance travelled by the particle in the third second of motion. [2]

(c) A 2 kilogram block of ice is removed from the freezer. The block of ice begins to melt so that its mass I grams after t minutes is given by the equation $I = I_o e^{-kt}$. After 45 minutes half of the block remains.

(i) Find the value of I_o. [1]

(ii) Show that the value of k is $\frac{1}{45}\ln 2$. [2]

(iii) For how long has the block been out of the freezer when only 10% of the block remains? Give your answer to the nearest minute. [2]

(iv) At what rate is the block melting when only 10% remains? [1]

Exam continues next page ...

QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Use the derivative to show that $y = \tan x$ is increasing for all x in its domain. [1]

(ii) Graph $y = \tan x$, $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$. Clearly show all intercepts and asymptotes on your diagram. [2]

(iii) Show $\frac{1}{y}\left(\frac{d^2y}{dx^2}\right) \div \frac{dy}{dx} = 2$ when $y = \tan x$. [2]

(b) Heidi's father deposits \$100 into an account on each of her birthdays from her first to her eighteenth. The money earns 6 % per annum with interest compounded annually. Lucky Heidi receives the full account on her eighteenth birthday.

(i) Show that Heidi's account will grow to \$3090·57 after the last payment on her eighteenth birthday. [2]

(ii) Henry's mother makes a similar arrangement for her son by investing \$100 on each of his birthdays from his first to his eighteenth. The money earns 5·75 % per annum with interest compounded monthly. Show that Henry's first \$100 investment is worth \$105·90 after 12 months. [1]

(iii) Who holds the larger balance in their account on their eighteenth birthday and by how much? [4]

QUESTION TEN (12 marks) Use a separate writing booklet. **Marks**

(a) Differentiate $y = x \log_e x$ and hence find $\int \log_e x \, dx$. [2]

(b) The line $y = mx$ is a tangent to the curve $y = e^{4x}$. Find m.
Clearly show your working. [3]

(c) An object is dropped from a point P which is $20 \ln 2$ metres above the horizontal ground below. The object's motion as it falls is governed by the differential equation

$$\frac{dx}{dv} = \frac{40v}{400 - v^2}$$

where v m/s is the velocity of the object after it has fallen x metres from P.

(i) Integrate both sides of the differential equation with respect to v to show that the displacement is given by $x = 20 \ln \frac{400}{400 - v^2}$. [2]

(ii) Hence show that $v^2 = 400\left(1 - e^{-\frac{1}{20}x}\right)$. [2]

(iii) Hence find the speed at which the object strikes the ground. [1]

(iv) The object approaches its limiting velocity as the distance travelled gets larger and has no restriction. What percentage of its limiting velocity has the object reached when it strikes the ground? Give your answer correct to the nearest percentage. [2]

END OF EXAMINATION

2005

SGS TRIAL EXAMINATION - MATHEMATICS

12 marks / question — TOTAL = 120 marks

QUESTION 1

(a) $\dfrac{1}{15+5\times 3} = \dfrac{1}{30}$ ✓

$= 0\cdot 0333$ ✓ (3 sig figs)

(b) $16x^3 - 64x = 16x(x^2-4)$ ✓

$= 16x(x-2)(x+2)$ ✓

(c) $|x+3| = 8$

$x+3 = 8$ or $x+3 = -8$ ✓

$x = 5$ or $x = -11$ ✓

(d) $x(x-9) = 0$

$x = 0$ or $x = 9$ ✓✓

(e) $\pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$ ✓

(f) $\dfrac{d}{dx}(\cos x) = -\sin x$ ✓

(g) focus $= (0,-1)$ ✓

(h) $\displaystyle\int \frac{1}{x}\,dx = \log_e x + c$ ✓

$\boxed{12}$

QUESTION 2

(a) $\dfrac{3}{\sqrt{7}+2} \times \dfrac{\sqrt{7}-2}{\sqrt{7}-2} = \dfrac{3(\sqrt{7}-2)}{7-4}$ ✓

$= \dfrac{\cancel{3}(\sqrt{7}-2)}{\cancel{3}}$

$\therefore\ a + \sqrt{b} = -2 + \sqrt{7}$

$a = -2$

$b = 7$ ✓

(b) $(y-k)^2 = 4a(x-h)$

$(y-0)^2 = 4\times 5(x+5)$ ✓

$y^2 = 20(x+5)$ ✓

(c) $\cos\angle B = \dfrac{5^2+6^2-7^2}{2\times 5\times 6}$ ✓

$\cos\angle B = \dfrac{1}{5}$

$\angle B \doteqdot 78^\circ$ (nearest degree) ✓

(d) (i) $PQ^2 = 10^2 + 5^2$

$PQ = \sqrt{125}$ ✓

$PQ = 5\sqrt{5}$

(ii) $m_{PQ} = \dfrac{10}{5} = 2$ ✓

$y = mx + b$

$y = 2x + 10$

(iii) $d = \dfrac{|2(2) - (-2) + 10|}{\sqrt{2^2+1^2}}$ ✓

$= \dfrac{16}{\sqrt{5}}$

(iv) $S = (0+7,\ 10-2) = (7,8)$ ✓

(v) $A = bh$

$= 5\sqrt{5} \times \dfrac{16}{\sqrt{5}}$

$= 80$ units^2 ✓

(vi) Let $\angle PQO = \theta$

$\tan\theta = 2$

$\theta \doteqdot 63^\circ$ (nearest degree) ✓

$\boxed{12}$

QUESTION 3

(a) (i) $y = \frac{2}{e^x}$

$y = 2e^{-x}$ ✓

$y' = -2e^{-x}$ ✓

$= -\frac{2}{e^x}$

(ii) $y = (x^2-1)^6$

$y' = 6(x^2-1)^5 \times 2x$ ✓

$= 12x(x^2-1)^5$ ✓

(iii) $y = \frac{2x+1}{3x-1}$

$y' = \frac{2(3x-1) - 3(2x+1)}{(3x-1)^2}$ ✓

$= \frac{6x - 2 - 6x - 3}{(3x-1)^2}$

$= \frac{-5}{(3x-1)^2}$ ✓

(b) $\int_0^2 \frac{2x}{x^2+4}\,dx = \left[\log_e(x^2+4)\right]_0^2$ ✓

$= \log_e 8 - \log_e 4$

$= \log_e 2$ ✓

(c) $A_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cos x\,dx$

$= 2\left[\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= 2\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right)$

$= 2\left(1 - \frac{1}{\sqrt{2}}\right)$ ✓

$= 2 - \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= 2 - \sqrt{2}$

$A_2 = -\int_{\frac{\pi}{2}}^{\pi} 2\cos x\,dx$ ✓

$= -2\left[\sin x\right]_{\frac{\pi}{2}}^{\pi}$

$= -2\left(\sin\pi - \sin\frac{\pi}{2}\right)$

$= -2(0-1)$

$= 2$ ✓

Area $= A_1 + A_2$

$= 2 - \sqrt{2} + 2$ ✓

$= 4 - \sqrt{2}$ units^2

$\boxed{12}$

QUESTION 4

(a) (i) $-5, 10, 25, \ldots, 955$

$a = -5, \quad d = 15$

$T_n = a + (n-1)d$

$955 = -5 + 15(n-1)$ ✓

$n - 1 = \frac{960}{15}$

$n = 65$ ✓

So there are 65 terms in the sequence.

(ii) $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots$

GP: $a = \frac{1}{2}, \quad r = -\frac{1}{2}$

$S_\infty = \frac{a}{1-r}$

$= \frac{\frac{1}{2}}{1+\frac{1}{2}}$ ✓

$= \frac{1}{2} \times \frac{2}{3}$

$= \frac{1}{3}$ ✓

(b) (i) when $y=0$,

$3x^2 - x^3 = 0$

$x^2(3-x) = 0$

$x = 0$ or 3 ✓

(ii) when $y' = 0$

$6x - 3x^2 = 0$

$3x(2-x) = 0$

$x = 0$ or 2 ✓

$y = 0$ or 4 ✓

So the stationary points are $(0,0)$ and $(2,4)$.

x	-1	0	1	2	3
y'	-9	0	3	0	-9
	\	—	/	—	\

✓

so $(0,0)$ is a minimum turning point and $(2,4)$ is a maximum turning point.

(iii) when $y'' = 0$

$6 - 6x = 0$

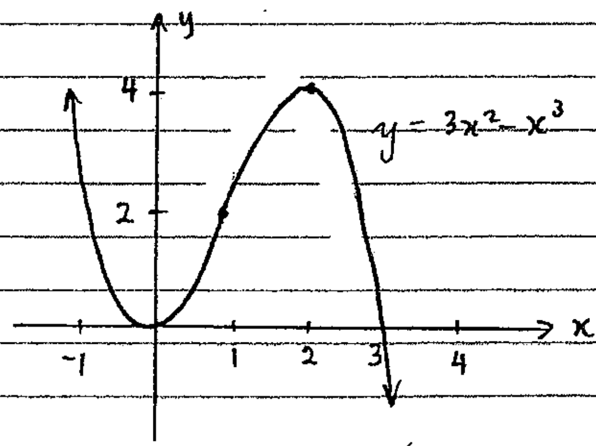
$\left.\begin{array}{l} x = 1 \\ y = 2 \end{array}\right\}$ ✓

x	0	1	2
y''	6	0	-6

✓

There is a change in concavity at $x=1$ so $(1,2)$ is the point of inflexion.

(iv)



✓ (turning points only)

(v) $3x^2 - x^3 \leq 0$

$x = 0$ or $x \geq 3$ ✓

12

QUESTION 5

(a) (i) $y = x^3$

$y' = 3x^2$

At $P(-1,-1)$ $y' = 3(-1)^2$

$= 3$ ✓

Tangent at P:

$y + 1 = 3(x+1)$ ✓

$y + 1 = 3x + 3$

$y = 3x + 2$

(ii) Substituting $Q(2,8)$ into both equations:

$y = x^3$

$RHS = 2^3$

$= 8$

$= y$

$= LHS$ ✓

$y = 3x + 2$

$RHS = 3(2) + 2$

$= 6 + 2$

$= 8$

$= LHS$ ✓

so $Q(2,8)$ is common to the curve and the tangent.

OR

Solving simultaneously $x^3 = 3x + 2$	when $x = 2$, $y = 2^3$ $= 8$
$x^3 - 3x - 2 = 0$	So $Q = (2,8)$
$(x-2)(x+1)^2 = 0$	
$x = 2$ or -1	

(iii) $A = \int_{-1}^{2} 3x + 2 - x^3 \, dx$ ✓

$= \left[\frac{3x^2}{2} + 2x - \frac{x^4}{4}\right]_{-1}^{2}$

$= (6 + 4 - 4) - \left(\frac{3}{2} - 2 - \frac{1}{4}\right)$

$= 6\frac{3}{4}$ units^2 ✓

(iv) $y \leq 3x + 2$ and $y \geq x^3$ and $x \geq -1$ ✓

(b) (i) In Δs ABP and CDP

$\angle BAP = \angle DCP$ (alternate $\angle$'s, $AB \parallel CD$) ✓

$\angle APB = \angle CPD$ (vertically opposite $\angle$'s) ✓

$\therefore \Delta ABP \parallel\!\mid \Delta CDP$ (AA) ✓

(ii) $\frac{AP}{CP} = \frac{AB}{CD}$ (matching sides of similar Δs in the same ratio)

$\frac{AP}{CP} = \frac{12}{18}$

$AP : CP = 2 : 3$ ✓

Given that $AC = 15$ cm

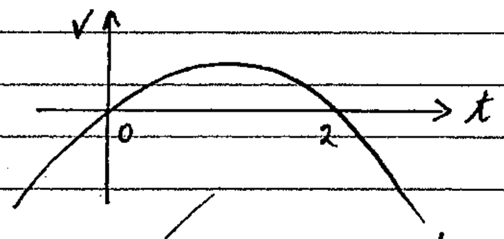
$AP = \frac{2}{5} \times 15$

$= 6$ cm ✓

$\boxed{12}$

QUESTION 8

(a)



$v = \frac{dx}{dt}$

(b) (i) $\ddot{x} = 6t - 2$

$\frac{dv}{dt} = 6t - 2$

$v = 3t^2 - 2t + c$

when $t = 0$, $v = -1$:

$-1 = 0 - 0 + c$

$c = -1$

$v = 3t^2 - 2t - 1$

when $t = 2$,

$v = 3(4) - 2(2) - 1$

$= 7$ m/s

(ii) stationary $v = 0$,

$3t^2 - 2t - 1 = 0$

$(3t+1)(t-1) = 0$

$t = 1$ or $-\frac{1}{3}$

and $t \geq 0$

so the particle is stationary after 1 second.

(iii)

Distance $= \int_2^3 3t^2 - 2t - 1 \, dt$

$= \left[t^3 - t^2 - t\right]_2^3$

$= 27 - 9 - 3 - 8 + 4 + 2$

$= 13$ metres

(c) $I = I_0 e^{-kt}$

(i) when $t = 0$, $I = 2000 e^0$

$= 2000$

$\therefore I_0 = 2000$

(ii) $I = 2000 e^{-kt}$

when $t = 45$, $I = 1000$

$1000 = 2000 e^{-45k}$

$e^{-45k} = \frac{1}{2}$

$-45k = \log_e \frac{1}{2}$

$k = -\frac{1}{45} \log_e \frac{1}{2}$

$k = \frac{1}{45} \log_e 2$

(iii) $0.1 I_0 = I_0 e^{-kt}$

$-kt = \log_e 0.1$

$t = \dfrac{\log_e 0.1}{-\frac{1}{45}\log_e 2}$

$t \doteqdot 149$ minutes

(iv) $\frac{dI}{dt} = -kI$

$= -\frac{1}{45} \log_e 2 \times 200$

$= -3.08$ g/min

$\boxed{12}$

QUESTION 9

(a) (i) $y = \tan x$

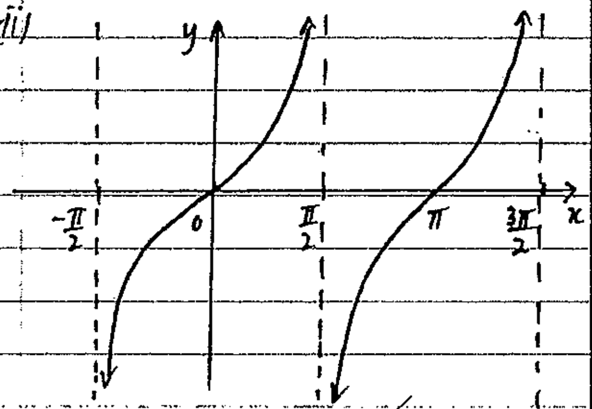
$y' = \sec^2 x$

> 0 for all x,

$x \neq \ldots -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

so $y = \tan x$ is increasing for all x in its domain.

(ii)



intercepts

asymptotes and shape

(iii)

$\frac{dy}{dx} = \sec^2 x$

$= (\cos x)^{-2}$

$\frac{d^2y}{dx^2} = -2 \times -\sin x \times (\cos x)^{-3}$

$= \frac{2\sin x}{\cos^3 x}$

[OR $2\sec^2 x \tan x$]

$\text{LHS} = \frac{1}{y}\left(\frac{d^2y}{dx^2}\right) \div \frac{dy}{dx}$

$= \frac{1}{\tan x} \times \frac{2\sin x}{\cos^3 x} \div \sec^2 x$

$= \frac{\cos x}{\cancel{\sin x}} \times \frac{2\cancel{\sin x}}{\cos^3 x} \times \cos^2 x$

$= 2$

$= \text{RHS}$

(b)(i) Let $\$A_n$ be the amount in Heidi's account after n birthdays

$A_1 = 100$

$A_2 = 100 \times 1{\cdot}06 + 100$

$A_3 = 100 \times 1{\cdot}06^2 + 100 \times 1{\cdot}06 + 100$

$= 100(1 + 1{\cdot}06 + 1{\cdot}06^2)$

$A_{18} = 100(1 + 1{\cdot}06 + 1{\cdot}06^2 + \ldots + 1{\cdot}06^{17})$

$= 100 \times \frac{1(1{\cdot}06^{18} - 1)}{1{\cdot}06 - 1}$

$\doteqdot 3090{\cdot}57$

so Heidi's account grows to \$3090.57 after the 18th and final payment.

(ii) $5{\cdot}75\%$ p.a. $= \frac{5{\cdot}75}{12}\%$ per month

$= 0{\cdot}004791\dot{6}$ per month

After 12 months the first \$100 will be compounded 12 times:

Amount $= \$100 \times (1{\cdot}004791\dot{6})^{12}$

$= \$105{\cdot}90$

(iii) Let $\$S_n$ be the amount in Henry's account after n birthdays

$R = (1{\cdot}004791\dot{6})^{12}$

$S_1 = 100$

$S_2 = 100 \times R + 100$

$S_3 = 100 \times R^2 + 100 \times R + 100$

$= 100(1 + R + R^2)$

$S_{18} = 100(1 + R + R^2 + \ldots + R^{17})$

$= 100 \times \frac{1(R^{18} - 1)}{R - 1}$

$= \$3062{\cdot}60$

So Heidi holds the larger balance by \$27.97.

$\boxed{12}$

QUESTION 10

(a) $y = x\log_e x$

$$\frac{dy}{dx} = 1\times\log_e x + x\times\frac{1}{x}$$

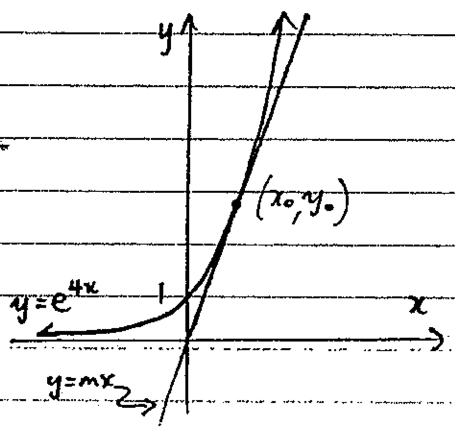
$$= \log_e x + 1$$

$$\therefore \int(\log_e x + 1)\,dx = x\log_e x + c_1$$

$$\int \log_e x\,dx + x = x\log_e x + c$$

$$\underline{\int \log_e x\,dx = x\log_e x - x + c}$$

(b)



Let (x_0, y_0) be the point of tangency

$$y = e^{4x}$$

$$y' = 4e^{4x}$$

$$m = 4e^{4x_0} \quad \text{——— ①}$$

Solving $y = mx$ and $y = e^{4x}$ simultaneous,

$$mx = e^{4x}$$

so $mx_0 = e^{4x_0}$

$$4e^{4x_0}\times x_0 = e^{4x_0} \quad \text{from ①}$$

$$4x_0\left(e^{4x_0}\right) = 1\left(e^{4x_0}\right)$$

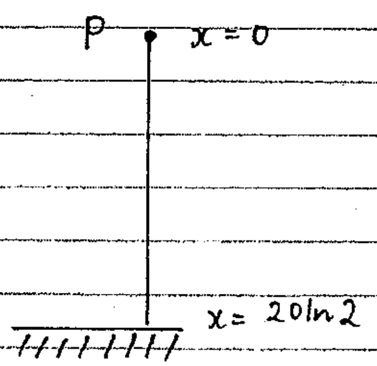
$$4x_0 = 1 \quad \text{since } e^{4x_0} \neq 0$$

$$x_0 = \frac{1}{4}$$

$$\therefore m = 4e^{4\times\frac{1}{4}}$$

$$\underline{m = 4e}$$

(c)



(i) $$\frac{dx}{dv} = \frac{40v}{400-v^2}$$

$$x = \int \frac{40v}{400-v^2}\,dv$$

$$x = -20\int \frac{-2v}{400-v^2}\,dv$$

$$x = -20\ln(400-v^2) + c$$

QUESTION 10 (cont.)

(i) continued when $t=0,\ v=0,\ x=0$:

$$0 = -20\ln 400 + c$$
$$c = 20\ln 400$$

$$x = 20\ln 400 - 20\ln(400-v^2)$$
$$x = 20\left(\ln 400 - \ln(400-v^2)\right)$$
$$x = 20\ln\frac{400}{400-v^2}$$

(ii)

$$\frac{x}{20} = \ln\frac{400}{400-v^2}$$
$$e^{\frac{x}{20}} = \frac{400}{400-v^2}$$
$$400-v^2 = \frac{400}{e^{\frac{x}{20}}}$$
$$v^2 = 400 - 400e^{-\frac{x}{20}}$$
$$v^2 = 400\left(1-e^{-\frac{x}{20}}\right)$$

(iii) when $x = 20\ln 2$, $v^2 = 400\left(1-e^{-\ln 2}\right)$

$$v^2 = 400\left(1-e^{\ln\frac{1}{2}}\right)$$
$$v^2 = 400\left(1-\tfrac{1}{2}\right)$$
$$v^2 = 200$$
$$|v| = 10\sqrt{2}\ \text{m/s}$$

(iv) The object's limiting velocity occurs when $x\to\infty$

as $x\to\infty$, $e^{-\frac{x}{20}}\to 0$

so $v^2 \to 400(1-0)$

limiting velocity $= 20$ m/s

Percentage of limiting velocity reached upon striking the ground $= \frac{10\sqrt{2}}{20}\times 100\%$

$= 50\sqrt{2}\,\%$

$\doteqdot 71\%$

$\boxed{12}$