HIGHER SCHOOL CERTIFICATE EXAMINATION 1986

MATHEMATICS - 3/4 UNIT

Directions to Candidates

Time allowed - Two hours (includes reading time)

All questions may be attempted. All questions are of equal value.

All necessary working should be shown in every question. Marks may not be awarded for careless or badly arranged work.

Standard integrals are provided; approved slide-rules or silent calculators may be used.

QUESTION 1

(i) Write $\dfrac{1+\sqrt{7}}{3-\sqrt{7}}$ in the form $a + b\sqrt{7}$ where a and b are rational.

(ii) (a) Write down, in surd form, the values of

$\sin 45°$, $\cos 45°$, $\sin 30°$, $\cos 30°$.

(b) Hence show that $\sin 75° = \dfrac{1+\sqrt{3}}{2\sqrt{2}}$

(iii) Use the substitution $u = x^2 + 2$ to evaluate $\displaystyle\int_0^1 \frac{x}{(x^2+2)^2}\,dx$.

(iv) Find all real numbers x such that $x^2 + 4x > 5$.

QUESTION 2

(i) Find the coordinates of the point P which divides the interval AB internally in the ratio $2:3$ where A and B have coordinates $(1,-3)$ and $(6,7)$ respectively.

(ii) Two circles with centres X and Y intersect at two points A and B.

(a) Draw a neat sketch joining XA, XB, YA, YB, XY, AB. Let P be the point where XY meets AB.

(b) Prove that the triangles AXY and BXY are congruent.

(c) Prove that $AP = BP$.

(d) Given that XA is also a tangent to the circle with centre Y, prove that $XAYB$ is a cyclic quadrilateral.

QUESTION 3

(i) Find the volume of the solid obtained when the region between the curves $y = x^3$ and $y = x^2$, from $x = 0$ to $x = 1$, is rotated about the x axis.

(ii) Consider the curve $y = x^4 + 4x^3 - 16x + 1$.

(a) Verify that the curve has a minimum at $x = 1$.

(b) Factorise dy/dx completely, and hence determine the location and nature of any other stationary points of the curve.

QUESTION 4

(i)(a) Find the coordinates of the vertex and the focus and the equation of the directrix of the parabola $y = x^2 - 4x$.

Draw a sketch of the curve.

(b) A line whose equation is $y = mx - 4$ passes through the point $(0,-4)$ and is a tangent to the parabola $y = x^2 - 4x$. Find the two possible values of m.

(ii) Factorise $a^2 + 3a + 2$ and hence or otherwise find the coefficient of a^4 in $(a^2 + 3a + 2)^6$.

QUESTION 5

(*i*) Find all values θ with $0 \le \theta \le \pi$ such that $2 \sin \theta + \cos \theta = 1$.

(*ii*) A particle is oscillating in simple harmonic motion such that its displacement x metres from a given origin O satisfies the equation

$\frac{d^2x}{dt^2} = -4x$, where t is the time in seconds.

(*a*) Show that $x = a \cos(2t + \beta)$ is a possible equation of motion for this particle, where a and β are constants.

(*b*) The particle is observed at time $t = 0$ to have a velocity of 2 metres per second and a displacement from the origin of 4 metres. Show that the amplitude of oscillation is $\sqrt{17}$ metres.

(*c*) Determine the maximum velocity of the particle.

QUESTION 6

(*i*)(*a*) Sketch the graph of the function $y = \tan^{-1}x$ stating clearly the range and domain.

(*b*) Given that $x^2 + 4x + 5 \equiv (x + a)^2 + b^2$, find a, b.

(*c*) Using the result in (*b*), find $\int \frac{1}{x^2 + 4x + 5}\,dx$.

(*ii*) A given school in a certain State has 3 mathematics teachers. The probability in that State that a mathematics teacher is female is 0.4.

(*a*) What is the probability that in the given school there is at least one female mathematics teacher?

(*b*) In the same State the probability that a mathematics teacher (male or female) is a graduate is 0.7. What is the probability that in the given school none of the three mathematics teachers is a female graduate?
(Give your answer correct to 3 decimal places.)

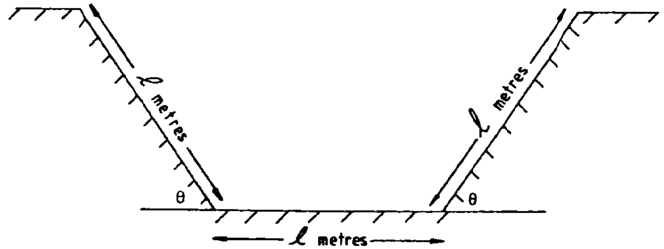
QUESTION 7

(*i*) Prove by mathematical induction that

$$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + n \times 2^{n-1} = 1 + (n - 1)2^n$$

for all integers $n \ge 1$.

(*ii*) An irrigation channel is to have a cross-section in the shape of a trapezium as in the accompanying figure. The bottom and sides are each ℓ metres long. Suppose that the sides of the channel make an angle $\theta \le \pi/2$ with the horizontal.



(*a*) Find the area of the cross-section of the channel as a function of θ.

(*b*) For what angle θ is the area of the cross-section a maximum?