

**QUESTION ONE** (15 marks) Use a separate writing booklet.

Marks

(a) Use integration by parts to evaluate  $\int_0^1 x \tan^{-1} x \, dx$ .

3

(b) (i) Prove that  $\sqrt{\frac{1-x}{1+x}} = \frac{1-x}{\sqrt{1-x^2}}$ .

1

(ii) Hence or otherwise evaluate  $\int_0^{\frac{1}{2}} \sqrt{\frac{1-x}{1+x}} \, dx$ .

2

(c) (i) Express  $\frac{10+x-x^2}{(x+1)(x^2+3)}$  in the form  $\frac{A}{x+1} + \frac{Bx+C}{x^2+3}$ .

3

(ii) Hence find  $\int \frac{10+x-x^2}{(x+1)(x^2+3)} \, dx$ .

2

(d) (i) Prove that if  $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$ , then  $I_n = \frac{n-1}{n} I_{n-2}$ .

2

(ii) Hence evaluate  $\int_0^{\frac{\pi}{2}} \cos^7 x \, dx$ .

2

**QUESTION TWO** (15 marks) Use a separate writing booklet.

Marks

(a) Evaluate  $|3 - 2i|$ .

1

(b) Express in the form  $x + iy$ , where  $x$  and  $y$  are real:

(i)  $(7 + 3i)(4 - i)$

1

(ii)  $\frac{2-5i}{4-3i}$

2

(c) Find the real numbers  $a$  and  $b$  such that  $(a + bi)^2 = 9 + 40i$ .

3

(d) (i) Express  $1 + i$  in modulus-argument form.

1

(ii) Given that  $(1 + i)^n = x + iy$ , where  $x$  and  $y$  are real and  $n$  is an integer, prove that  $x^2 + y^2 = 2^n$ .

2

(e) (i) If  $\left| \frac{z-1}{z+1} \right| = 2$ , where  $z = x + iy$ , show that the equation of the locus of  $z$  is

2

$$\left(x + \frac{5}{3}\right)^2 + y^2 = \frac{16}{9}.$$

(ii) Represent this locus on an Argand diagram and shade the region for which the inequalities  $\left| \frac{z-1}{z+1} \right| \leq 2$  and  $0 \leq \arg z \leq \frac{3\pi}{4}$  are both satisfied.

3

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**QUESTION THREE** (15 marks) Use a separate writing booklet.

Marks

(a) Consider the equation  $z^3 + mz^2 + nz + 6 = 0$ , where  $m$  and  $n$  are real. It is known that  $1 - i$  is a root of the equation.

2

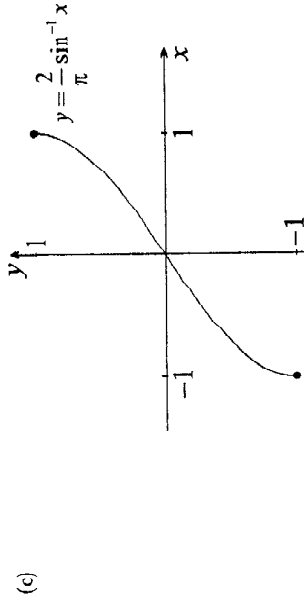
(i) Find the other two roots of the equation.

2

(ii) Find the values of  $m$  and  $n$ .

3

(b) Given that the roots of the equation  $4x^3 - 24x^2 + 45x - 26 = 0$  form an arithmetic sequence, solve the equation by using sums and products of the roots.



The function sketched above is  $f(x) = \frac{2}{\pi} \sin^{-1} x$ .

(i) Using at least one third of a page, sketch the function  $y = |f(x)|$ .

2

(ii) Using at least one third of a page, sketch the function  $y = \cos(f(x))$ .

2

(iii) Using at least one third of a page, sketch the graph of  $y^2 = f(x)$ .

2

(iv) Using at least one third of a page, sketch the function  $y = f'(x)$ , clearly marking and labelling any point where the graph cuts the axes.

2

Exam continues overleaf ...

**QUESTION FOUR** (15 marks) Use a separate writing booklet.

Marks

- (a) Let the points  $P(2ap, ap^2)$  and  $Q(2aq, aq^2)$  lie on the parabola  $x^2 = 4ay$ .

(i) Prove that the chord  $PQ$  has equation

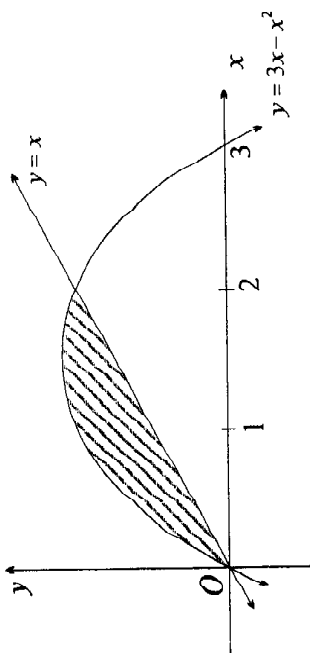
$$y - \frac{1}{2}(p+q)x + apq = 0.$$

(ii) Show that if the chord  $PQ$  passes through the point  $(0, -a)$ , then

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a},$$

where  $S(0, a)$  is the focus of the parabola.

(b)



The diagram above shows the parabola  $y = 3x - x^2$  and the line  $y = x$ . Use the method of cylindrical shells to find the volume of the solid formed when the shaded region is rotated about the  $y$ -axis.

- (c) When a person dies, the temperature of their body will gradually decrease from  $37^\circ\text{C}$ , normal body temperature, to the temperature of the surroundings. The situation is modelled by Newton's law of cooling, which states that the temperature of the cooling body changes at a rate proportional to the difference between the temperature of the body and the temperature of its surroundings. That is

$$\frac{d\theta}{dt} = -K(\theta - \mathcal{R}) \quad (1)$$

where  $K$  is a positive constant,  $\theta$  is the temperature of the body after  $t$  hours, and  $\mathcal{R}$  is the temperature of the surroundings.

A person was found murdered in his home. Police arrived on the scene at 10 : 56pm. The temperature of the body at that time was  $31^\circ\text{C}$ , and 1 hour later it was  $30^\circ\text{C}$ . The temperature  $\mathcal{R}$  of the room in which the body was found was  $22^\circ\text{C}$ .

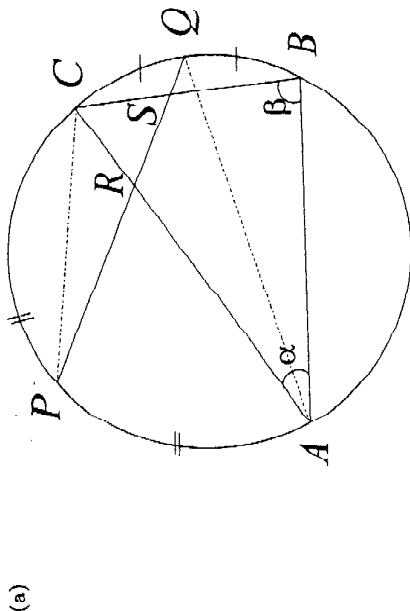
- Show that  $\theta = 22 + Ae^{-Kt}$  is a solution of equation (1), where  $A$  is a constant.
- Find the exact values of  $A$  and  $K$ .
- Determine when the murder was committed, correct to the nearest minute.

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**QUESTION FIVE** (15 marks) Use a separate writing booklet.

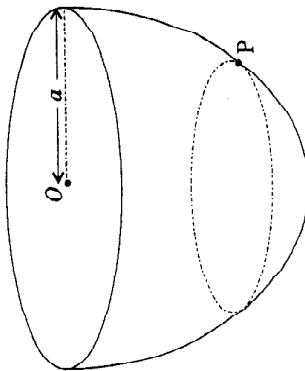
Marks



In the diagram above,  $\triangle ABC$  is inscribed in a circle. The midpoints of the arcs  $AC$  and  $BC$  are  $P$  and  $Q$  respectively. The line  $PQ$  intersects  $AC$  and  $BC$  at  $R$  and  $S$  respectively. Let  $\angle BAC = \alpha$  and  $\angle ABC = \beta$ .

- Copy the diagram into your answer booklet and show that  $\angle QPC = \frac{1}{2}\alpha$  and  $\angle AQP = \frac{1}{2}\beta$ .
- Hence prove that  $CR = CS$ .

(b)



In the diagram above, a particle  $P$  is moving with constant speed  $v$  in a horizontal circle on the smooth inner surface of a hemisphere of radius  $OP = a$ . The force exerted by the hemisphere on the particle equals twice the weight of the particle. Let the acceleration due to gravity be  $g$ .

- Draw a diagram showing all the forces on the particle.
- Prove that  $v^2 = \frac{3ag}{2}$ .

Exam continues overleaf ...

1  
3

- (c) Onur uses a bowling machine to project a cricket ball of mass  $m$  vertically upwards with a velocity of  $120\text{m/s}$ . It is known that the air resistance on the ball when its velocity is  $v$  is  $3mv$  newtons.

- (i) Show that if the acceleration due to gravity is  $10\text{m/s}^2$ , then the equation of motion of the ball is  $\ddot{x} = -(10 + 3v)$ . [1]  
 (ii) Find the greatest height attained by the ball. Give your answer correct to the nearest metre. [3]  
 (iii) Find the time the ball took to reach the maximum height. Give your answer to the nearest  $\frac{1}{10}$  of a second. [2]

**QUESTION SIX** (15 marks) Use a separate writing booklet.

Marks

- (a) The depth  $x$  metres of the water in a certain South Coast harbour is found to vary in approximate accordance with the equation

$$\ddot{x} = -\frac{x}{4},$$

where  $t$  is the time in hours. It is known that the difference between high and low tide is 4 metres.

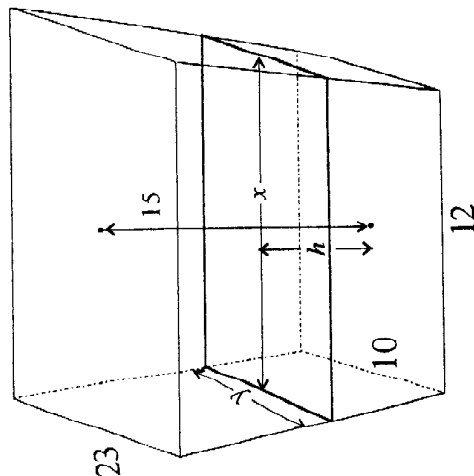
- (i) Prove that the time between successive high tides is 12.6 hours, correct to the nearest  $\frac{1}{10}$  of an hour. [2]  
 (ii) Find the rise in the water level during the first hour after low tide. Give your answer in metres, correct to two decimal places. [3]  
 (iii) Find the rate at which the level is falling two hours after high tide. Give your answer in metres per hour, correct to two decimal places. [2]  
 (b) (i) If  $z = \cos \theta + i \sin \theta$ , show that  $z - \frac{1}{z} = 2i \sin \theta$  and  $z^n - \frac{1}{z^n} = 2i \sin n\theta$ . [2]  
 (ii) Hence show that  $\sin^5 \theta = \frac{1}{16} \sin 5\theta - \frac{5}{16} \sin 3\theta + \frac{5}{16} \sin \theta$ . [2]  
 (iii) Hence solve the equation  $16 \sin^5 \theta = \sin 5\theta$ , for  $0 \leq \theta \leq 2\pi$ . [4]

Exam continues next page ...

**QUESTION SEVEN** (15 marks) Use a separate writing booklet.

Marks

(a) 29



It is proposed to construct an underground reservoir to store storm water in order for it to be used on council parks and gardens. The plans are for the reservoir to be 15 metres deep and each horizontal cross-section to be rectangular. The base is to measure 12 metres by 10 metres and the top is to measure 29 metres by 23 metres.

- (i) Consider a cross-section of the reservoir  $h$  metres above the base with dimensions  $x$  metres by  $y$  metres. Show that the area of this cross-section is given by [3]  

$$A = \frac{221}{25}h^2 + \frac{326}{15}h + 120 \text{ m}^2.$$
  
 (ii) Hence find the volume of the reservoir. [2]  
 (b) Consider the Fibonacci sequence defined by  

$$u_1 = 1$$
  

$$u_2 = 1$$
  

$$u_{n+2} = u_{n+1} + u_n, \text{ for all positive integers } n \geq 1.$$
  
 (i) (a) Show that for all positive integers  $n$  [2]  

$$u_{3n+3} = 2u_{3n+1} + u_{3n}.$$
  
 (b) Prove by mathematical induction that  $u_{3n}$  is always even. [3]  
 (ii) Prove by mathematical induction that  $u_n < \left(\frac{7}{4}\right)^n$ , for all positive integers  $n$ . [5]

Exam continues overleaf ...

**QUESTION EIGHT** (15 marks) Use a separate writing booklet. Marks

- (a) The variables  $x$  and  $y$  are positive and related by 4

$$x^a y^b = (x + y)^{a+b}$$

where  $a$  and  $b$  are positive constants. By taking logarithms of both sides, show that

$$\frac{dy}{dx} = \frac{y}{x}, \text{ provided that } bx \neq ay.$$

- (b) Consider the function  $f_n(x) = \frac{x^n(1-x)^n}{n!}$ , where  $n$  is a positive integer.

- (i) ( $\alpha$ ) Show that  $0 < f_n(x) \sin \pi x < \frac{1}{n!}$ , for  $0 < x < 1$ . 1

- ( $\beta$ ) Show that  $0 < \int_0^1 f_n(x) \sin \pi x \, dx < \frac{1}{n!}$ . 1

- (ii) Suppose that  $\pi^2$  were rational, and let  $a$  and  $b$  be two positive integers such that

$$\pi^2 = \frac{a}{b}.$$

Define the function  $F_n(x)$  by

$$F_n(x) = b^n \left( \pi^{2n} f_n(x) - \pi^{2n-2} f_n^{(2)}(x) + \pi^{2n-4} f_n^{(4)}(x) - \dots + (-1)^n f_n^{(2n)}(x) \right),$$

where  $f_n^{(k)}(x)$  denotes the  $k$ th derivative of  $f_n(x)$ .

- ( $\alpha$ ) Show that  $\frac{d^2}{dx^2} F_n(x) = \pi^2 a^n f_n(x) - \pi^2 F_n(x)$ . 2

- ( $\beta$ ) Hence show that  $\frac{d}{dx} (F_n'(x) \sin \pi x - \pi F_n(x) \cos \pi x) = \pi^2 a^n f_n(x) \sin \pi x$ . 2

- ( $\gamma$ ) Deduce from ( $\beta$ ) that  $\pi a^n \int_0^1 f_n(x) \sin \pi x \, dx = F_n(1) + F_n(0)$ . 3

- (iii) Assume now, without proof, that the function  $F_n(x)$  has the property that  $F_n(0)$  and  $F_n(1)$  are integers. 2

Using parts (i) and (ii), plus the fact that  $\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0$  for any real number  $a$ , prove that  $\pi^2$  is an irrational number.

END OF EXAMINATION