SOLUTIONS TO 3/4 UNIT TRIAL H.S.C. 2000

Question 1

(a) $d = \left|\dfrac{3.1 - 4.4 + -2}{\sqrt{3^2+4^2}}\right|$ 1

$= \left|-\dfrac{15}{5}\right|$ (2)

$d = 3$ 1

(b) $2x^3 - 128$

$= 2(x^3 - 64)$ 1

$= 2(x-4)(x^2+4x+16)$ 1 (2)

(c) (i) $\dfrac{dy}{dx} = 2\cos 2x$ (1)

(ii) $y = \frac{1}{2}\left[\log_e(2x-1) - \log_e(3x+2)\right]$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{2}\left[\dfrac{2}{2x-1} - \dfrac{3}{3x+2}\right]$ 1

$= \dfrac{7}{2(2x-1)(3x+2)}$ 1 (2)

(d) If $(x-2)$ is a factor of $P(x)$ then $P(2) = 0$. 1

Now $P(2) = 2^4 - 2(2^3) - 3$

$= -3$

remainder is -3 1 (2)

(e) $\displaystyle\int_{\sqrt{2}}^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{2}}^{\sqrt{3}}$ 1

$= \sin^{-1}\dfrac{\sqrt{3}}{2} - \sin^{-1}\sqrt{2}$ 1

$= \dfrac{\pi}{3} - \dfrac{\pi}{4}$; $= \dfrac{\pi}{12}$ 1 (3)

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Question 2

(a) $P = \binom{7}{5}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^5$ 1

$= \dfrac{21875}{93312}$ 1 (2)

$P \doteq 0.2344$

(b) $A = \pi R^2$

$\dfrac{dA}{dt} = 5\ \text{cm}^2/\text{min}$

$\dfrac{dR}{dt} = ?$

$\dfrac{dA}{dR} = 2\pi R$

Now $\dfrac{dR}{dt} = \dfrac{dR}{dA}\cdot\dfrac{dA}{dt}$ 1

$= \dfrac{1}{2\pi R}\cdot 5$ 1

when $R = 10$

$\dfrac{dR}{dt} = \dfrac{5}{2\pi\times 10} = \dfrac{1}{4\pi}$ 1 cm/s.

$\doteq 0.08$ cm/s (3)

(c) $\displaystyle\int_0^{\frac{\pi}{3}} (\sec x \tan x)^2\,dx$ $\quad u = \tan x$ 1

$\therefore du = \sec^2 x\,dx$

$\displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x \tan^2 x\,dx$

$= \displaystyle\int_0^{\sqrt{3}} u^2\,du$ 1

$= \left.\dfrac{u^3}{3}\right]_0^{\sqrt{3}}$ 1

$= \sqrt{3}$ (3)

(d) Prove $\dfrac{\cos 2\theta}{\cos\theta - \sin\theta} = \cos\theta + \sin\theta$

$LHS = \dfrac{\cos 2\theta}{\cos\theta - \sin\theta}$

$= \dfrac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta}$ 1

$= \dfrac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{(\cos\theta - \sin\theta)}$ (2)

$= \cos\theta + \sin\theta$ 1

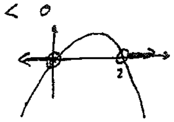
$= RHS$

(e) Since $\dfrac{1}{x(2-x)} < 0$

$\therefore \dfrac{x^2(2-x)^2}{x(2-x)} < 0$ 1 (2)

$\therefore x(2-x) < 0$

1



$\therefore x < 0,\ x > 2$

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Question 3

(a) $B\hat{D}C = B\hat{E}C$ (given)

Since equal angles at D, E are subtended on same side of interval BC, $\therefore$ the points B, D, E, C are concyclic. 2 (4)

$\therefore A\hat{D}E = B\hat{C}E$ (Exterior angle of cyclic quad BDEC equal to interior remote angle.) 2

(b) (i) $x(t) = 2 + 3\sin t + 4\cos t$

Let $3\sin t + 4\cos t \equiv A\sin(t+\alpha)$

$A = 5$, $\alpha = \tan^{-1}\frac{4}{3}$ 1

$\therefore x(t) = 2 + 5\sin(t+\alpha)$ (2)

$\therefore$ Amplitude is 5. 1

(ii) Time taken for 1 oscillation is 2π secs. Since amplitude is 5, 1

$\therefore$ it moves 20 metres in 2π secs.

$\therefore$ it takes 10π secs to move 100 metres. 1 (2)

(c) Prove $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \dots + \dfrac{n}{(n+1)!} = 1 - \dfrac{1}{(n+1)!}$

Test for $n=1$:

$LHS = RHS = \frac{1}{2}$ $\quad\therefore$ proven true for $n=1$. 1

Assume true for $n=K$:

$\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \dots + \dfrac{K}{(K+1)!} = 1 - \dfrac{1}{(K+1)!}$

Prove true for $n=K+1$:

RTP that $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \dots + \dfrac{K}{(K+1)!} + \dfrac{(K+1)}{(K+2)!} = 1 - \dfrac{1}{(K+2)!}$

Now $LHS = 1 - \dfrac{1}{(K+1)!} + \dfrac{(K+1)}{(K+2)!}$ $\quad$ by assumption

$= 1 - \dfrac{1}{(K+1)!}\left[1 - \dfrac{(K+1)}{(K+2)}\right]$ 2

$= 1 - \dfrac{1}{(K+1)!}\left[\dfrac{1}{(K+2)}\right]$ (4)

$= 1 - \dfrac{1}{(K+2)!}$ 1

$= RHS$

Hence proven true for $n = 1, 2, 3, \dots$ by mathematical induction.

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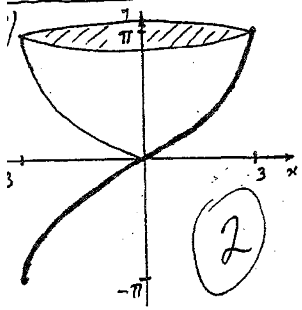
Question 4



(2)

(ii) Now $V = \pi \int_0^{\pi} x^2\, dy$ $\qquad \frac{y}{2} = \sin^{-1}\left(\frac{x}{3}\right)$

$\therefore \frac{x}{3} = \sin\frac{y}{2}$ 1

$\therefore V = 9\pi \int_0^{\pi} \sin^2\left(\frac{y}{2}\right) dy$ 1 $\qquad \therefore x^2 = 9\sin^2\left(\frac{y}{2}\right)$

$= 18\pi \int_0^{\pi} \sin^2\left(\frac{y}{2}\right) d\left(\frac{y}{2}\right)$

$= 9\pi \int_0^{\pi} \left[1 - \cos 2\left(\frac{y}{2}\right)\right] d\left(\frac{y}{2}\right)$ 1

$= 9\pi \left[\frac{y}{2} - 2\sin y\right]_0^{\pi}$

$\therefore V = \frac{9\pi^2}{2}$ units^3 1

(4)

(b) $\frac{dy}{dx} = 1 + y$

$\therefore \frac{dy}{1+y} = dx$

$\ln(1+y) = x + C$

When $x = 1$, $y = 2$ $\therefore C = \ln 3 - 1$

$\ln(1+y) = x + \ln 3 - 1$

$\ln\frac{(1+y)}{3} = x - 1$

$\frac{1+y}{3} = e^{x-1}$ 1

$\therefore y = 3e^{x-1} - 1$, and RANGE is $\{y : y > -1\}$ 1

[Note: Since $\frac{dy}{dx} = y + 1$, then $y = -1 + Ae^{x}$ by definition of exponential growth. At $(1, 2)$ we find $A = 3e^{-1}$ and $y = 3e^{x-1} - 1$.]

(3)

(c) Solve $\cos^2\theta = \sin\theta\cos\theta$

Now $\cos^2\theta - \sin\theta\cos\theta = 0$

$\therefore \cos\theta(\cos\theta - \sin\theta) = 0$ 1

$\therefore \cos\theta = 0$ or $\tan\theta = 1$

$\therefore \theta = 2n\pi \pm \cos^{-1} 0$ or $\theta = n\pi + \tan^{-1} 1$

$\therefore \theta = 2n\pi \pm \frac{\pi}{2}$ 1 or $n\pi + \frac{\pi}{4}$ 1

(3)

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Question 5

(a) $x = \sin y$

$\therefore \frac{dx}{dy} = \cos y$

$\therefore \frac{dy}{dx} = \frac{1}{\cos y}$ 1

Now $x^2 = \sin^2 y = 1 - \cos^2 y$

$\therefore \cos^2 y = 1 - x^2$ 1

$\therefore \cos y = \pm\sqrt{1 - x^2}$

But $\frac{\pi}{2} \le y \le \pi$ for $0 \le x \le 1$

$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$ 1

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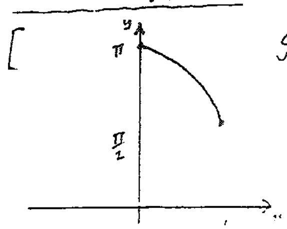
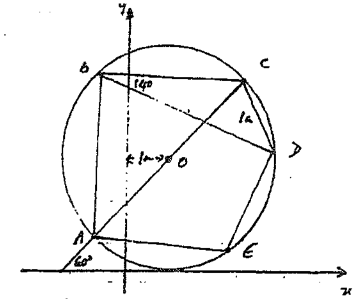


Graph of $x = \sin y$

(4)



(i) Since $\sin 14^{\circ} = \frac{1}{2R}$ 1

$\therefore R = \frac{1}{2\sin 14^{\circ}} \doteq 2$ cm.

$\therefore$ Diameter is approx. 4 cm 1

(2)

(ii) Equation of the circle is:

$(x-1)^2 + (y-2)^2 = 4$

ie; $x^2 + y^2 - 2x - 4y + 1 = 0$

(1)

(iii) Join AB, AE, DE

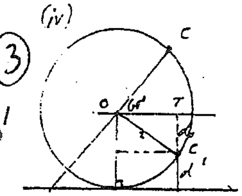
$A\hat{B}C = 90^{\circ}$ (Angle in semi-circle) 1

$\therefore A\hat{B}D = 76^{\circ}$ (by subtraction) 1

$\therefore A\hat{E}D = 104^{\circ}$ (Opposite angles of cyclic quad. ABDE supplementary) 1

(3)

(iv)



Since $l = R\theta$

$\therefore 3 = 2\theta$ $\therefore \theta = \frac{3}{2}^{c}$

$\therefore \theta \doteq 85^{\circ}57'$

$\therefore TOC_1 \doteq 25^{\circ}57'$ 1

Now $\sin 25^{\circ}57' = \frac{d_0}{2}$

$\therefore d_0 \doteq 0.875$ cm

$\therefore$ distance required is $(2 - 0.875)$ cm

ie; $d \doteq$ 1.12 cm. 1

(2)

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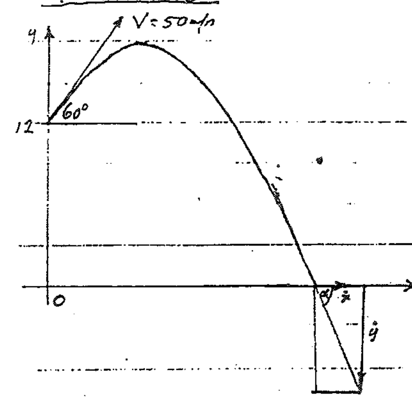
Question 6



(a)

$\ddot{x} = 0$ $\qquad \ddot{y} = -g$

$\dot{x} = C_1 = V\cos 60^{\circ}$ $\qquad \dot{y} = -gt + V\sin 60^{\circ}$

$\therefore \dot{x} = 50\cos 60$ $\qquad = -10t + 50 \cdot \frac{\sqrt{3}}{2}$

$\dot{x} = 25$ $\qquad \therefore \dot{y} = -10t + 25\sqrt{3}$

$x = 25t + C_3$ $\qquad y = -5t^2 + 25\sqrt{3}t + C_4$

When $t = 0$, $C_3 = 0$, $C_4 = 12$

$\therefore x = 25t$ 1 $\qquad y = -5t^2 + 25\sqrt{3}t + 12$ 1

When $y = 0$ $\quad 5t^2 - 25\sqrt{3}t - 12 = 0$

$t = \frac{25\sqrt{3} \pm \sqrt{(25\sqrt{3})^2 + 240}}{10}$

$\therefore t \doteq 8.93$ $(t > 0)$

$\therefore \dot{x} = 25$, $\dot{y} \doteq -46$

Now $v = \sqrt{v_x^2 + v_y^2}$ $\left(= \sqrt{\dot{x}^2 + \dot{y}^2}\right)$

$= \sqrt{25^2 + (46)^2}$

$\therefore v \doteq 52.35$ m/s

$\tan\alpha = \frac{\dot{y}}{\dot{x}}$

$= -\frac{46}{25}$

$\therefore \alpha = 61^{\circ}29'$ 1

$\therefore$ Velocity of particle is 52.35 m/s at an angle of $61^{\circ}29'$ to horizontal ... or $118^{\circ}31'$

(5)

6(b) PROPRIETY (PPRROIETY)

(i) $\frac{9!}{2!2!} = 90720$ (1) (ii) $\frac{7!3!}{2!2!} = 7560$ (1) (iii) $P = \frac{4!}{7.6.5.4} = \frac{1}{35}$ (1)

(c)(i) $\ddot{x} = v\frac{dv}{dx} = 10x - 4x^3$

$\therefore v\,dv = (10x - 4x^3)\,dx$

$\therefore \frac{1}{2}v^2 = 5x^2 - x^4 + C$ 1

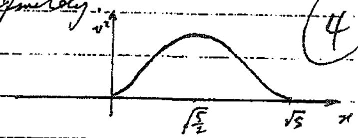
When $x = \sqrt{5}$, $v = 0$

$\therefore 0 = 25 - 25 + C \quad \therefore C = 0$ 1

$\therefore v^2 = 2x^2(5 - x^2)$ 1

(ii) The particle starts at $x = \sqrt{5}$ m and moves towards 0 with increasing speed. After $x = \sqrt{\frac{5}{2}}$ m it continues towards 0 decreasing its speed indefinitely. 1

(4)



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Question 7

(a)(i)

	B	E
1	120 (ie; $\frac{4800 \times 0.065}{26}$)	$120 + \frac{1}{26}.\frac{5}{100}.120 \left(= 120\left(\frac{2605}{2600}\right)\right)$
2	$120\left(\frac{2605}{2600}\right) + 120$	$120\left(\frac{2605}{2600}\right)^2 + 120\left(\frac{2605}{2600}\right)$ 1

At the end of 10 years (260 fortnights), the investment A, is worth

$A = 120\left[\left(\frac{2605}{2600}\right) + \left(\frac{2605}{2600}\right)^2 + \cdots + \left(\frac{2605}{2600}\right)^{260}\right]$

$= 120 \times \left(\frac{2605}{2600}\right)\left[\dfrac{\left(\frac{2605}{2600}\right)^{260} - 1}{\left(\frac{5}{2600}\right)}\right]$ 1

(3)

$\therefore A = \$40{,}508$ 1 which is the amount Ross will receive.

(ii) At the end of 8 years (208 fortnights), the investment B is

$B = 120 \times \left(\frac{2605}{2600}\right)\left[\dfrac{\left(\frac{2605}{2600}\right)^{208} - 1}{\left(\frac{5}{2600}\right)}\right]$

$\therefore B = \$30{,}713$ 1

For the next 2 years, at 12% PA, compounded monthly, Evelyn will receive $\$30{,}713\,(1.01)^{24}$

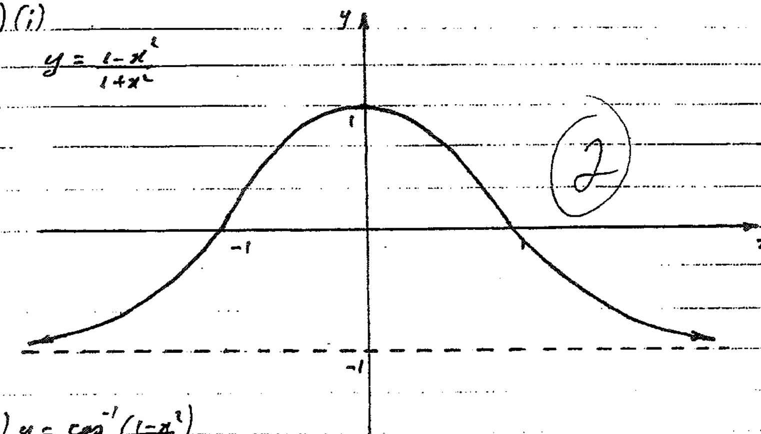
$= \$38{,}997$ 1

(3)

The difference is $\$40{,}508 - \$38{,}997 = \$1511$ 1 in favour of ROSS $\therefore$ he pays for Sherry's holiday.

7(b)(i)

$y = \frac{1 - x^2}{1 + x^2}$



(2)

(ii) $y = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$

$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - \left(\frac{1-x^2}{1+x^2}\right)^2}} \times \left[\frac{(1+x^2).-2x - (1-x^2).2x}{(1+x^2)^2}\right]$ 1

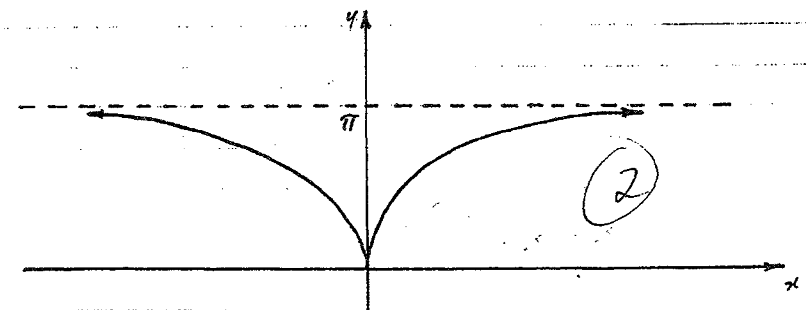
$= -\frac{(1+x^2)}{\sqrt{(1+x^2)^2 - (1-x^2)^2}}\left[\frac{-2x - \cancel{2x^3} - 2x + \cancel{2x^3}}{(1+x^2)^2}\right]$

$= \frac{(1+x^2)}{\sqrt{4x^2}} \cdot \frac{4x}{(1+x^2)^2}$

$\therefore \frac{dy}{dx} = \frac{2x}{|x|\,(1+x^2)}$ 1

(2)

(iii)



(2)

Since $f(-x) = f(x)$ $\therefore$ $f(x)$ is EVEN

For $x > 0$, $\frac{dy}{dx} > 0$ and for $x < 0$, $\frac{dy}{dx} < 0$

As $x \to \pm\infty$, $f(x) \to \pi$.

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