Mathematics Extension I CSSA HSC Trial Examination 2002
Marking Guidelines

estion 1

Outcomes Assessed: H5, PE5

Marking Guidelines

Criteria	Marks
• finding first derivative	1
• finding second derivative in form $\dfrac{e^x}{(e^x+1)^2}$	1

swer

$$\frac{d}{dx}\ln(e^x+1) = \frac{e^x}{e^x+1} \qquad \frac{d^2}{dx^2}\ln(e^x+1) = \frac{e^x.(e^x+1)-e^x.e^x}{(e^x+1)^2} = \frac{e^x}{(e^x+1)^2}$$

Outcomes Assessed: H5, PE3, PE6

Marking Guidelines

Criteria	Marks
• interpreting Σ notation to write sum of terms in expanded form	1
• calculating value of sum as $-\frac{5}{8}$	1

wer

$$\sum_{k=1}^{4}\frac{(-1)^k}{k!} = -\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!} = \frac{-24+12-4+1}{24} = -\frac{5}{8}$$

Outcomes Assessed: (i) P3 (ii) P3, PE2, HE7

Marking Guidelines

Criteria	Marks
(i) • writing expressions for $1 \pm \cos 2x$ in terms of $\cos^2 x$, $\sin^2 x$	1
• simplifying to obtain final result	1
(ii) • substituting $x = 22\frac{1}{2}°$ and $\cos 45° = \frac{1}{\sqrt{2}}$ to find expression for $\tan^2 22\frac{1}{2}°$	1
• using expression for $\tan^2 22\frac{1}{2}°$ to show $\tan 22\frac{1}{2}° = \sqrt{2}-1$	1

wer

) $\dfrac{1-\cos 2x}{1+\cos 2x} = \dfrac{2\sin^2 x}{2\cos^2 x} = \tan^2 x$

(ii) $\tan^2 22\frac{1}{2}° = \dfrac{1-\cos 45°}{1+\cos 45°} = \dfrac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \dfrac{\sqrt{2}-1}{\sqrt{2}+1}$

$\tan^2 22\frac{1}{2}° = \dfrac{(\sqrt{2}-1)(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} = \dfrac{(\sqrt{2}-1)^2}{2-1}$

$\therefore \tan 22\frac{1}{2}° = (\sqrt{2}-1)$, since $\tan 22\frac{1}{2}° > 0$

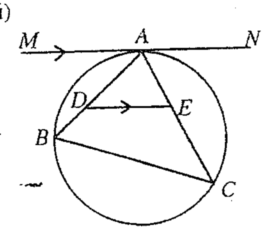
1(d) Outcomes Assessed: (i) (ii) PE3 (iii) H5, PE2, PE3

Marking Guidelines

Criteria	Mar
(i) • copying diagram	0
(ii) • using alternate segment theorem	1
(iii) • using equal alternate angles with parallel lines to deduce $A\hat{D}E = M\hat{A}D$	1
• deducing $A\hat{D}E = E\hat{C}B$ with explanation	1
• deducing $BCED$ is cyclic by applying appropriate test	1

Answer

(i)



(ii)

$M\hat{A}B = A\hat{C}B$ (*angle between tangent MAN*
AB equal to angle in alternate

(iii)

$A\hat{D}E = M\hat{A}D$ (*Alternate angles equal, DE*

$A\hat{D}E = E\hat{C}B$ (*Both equal to* $M\hat{A}D$)

$\therefore$ $BCED$ is a cyclic quadrilateral

(*Exterior angle* $A\hat{D}E$ = *opposite interior an*

Question 2

2(a) Outcomes Assessed: P4

Marking Guidelines

Criteria	Ma
• finding the x coordinate of P	
• finding the y coordinate of P	

Answer

$$x = \frac{4\times 4+1\times(-2)}{4+1} = 2\cdot 8, \quad y = \frac{4\times(-5)+1\times 3}{4+1} = -3\cdot 4 \quad \therefore P(2\cdot 8, -3\cdot 4)$$

2(b) Outcomes Assessed: PE3

Marking Guidelines

Criteria	M
• using at least one of the factors ${}^7C_2$, 3^5	
• completing the calculation ${}^7C_2 \times 3^5 = 5103$	

Answer

Choose the 2 questions to be answered correctly ${}^7C_2$ ways

Outcomes Assessed: (i) PE3 (ii) PE3, PE6

Marking Guidelines

Criteria	Marks
(i) • partial factorisation $P(x)=(x-1)(x^2+x-2)$	1
• completing factorisation $P(x)=(x+2)(x-1)^2$	1
(ii) • deducing $x\le -2$	1
• including $x=1$	1

Answer

(i)

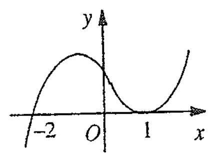
$(x-1)$ is a factor of $P(x)$

$x^3-3x+2=(x-1)(x^2+x-2)$

$=(x-1)(x-1)(x+2)$

$\therefore P(x)=(x+2)(x-1)^2$

(ii)



$y=(x+2)(x-1)^2$

By inspection of the graph,

$x^3-3x+2\le 0$ when

$x\le -2$ or $x=1$

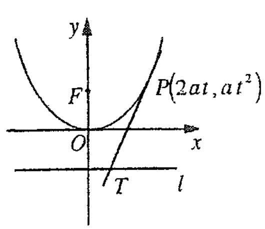
2(d) **Outcomes Assessed:** (i) PE3 (ii) PE3, PE4

Marking Guidelines

Criteria	Marks
(i) • finding the x coordinate of T	1
(ii) • finding the gradient of PF	1
• finding the gradient of TF	1
• showing the product of the gradients is -1 to prove $TF\perp PF$	1

Answer

(i)



At T, $\left.\begin{array}{l} y=-a \\ tx-y-at^2=0 \end{array}\right\} \Rightarrow \begin{array}{l} tx=a(t^2-1) \\ x=a\left(t-\frac{1}{t}\right) \end{array}$

$\therefore T\left(a\left(t-\frac{1}{t}\right), -a\right)$

(ii) $F(0,a)\Rightarrow$ *gradient* $PF=\frac{a(t^2-1)}{2at}=\frac{1}{2}\left(t-\frac{1}{t}\right)$ and *gradient* $TF=\frac{-2a}{a\left(t-\frac{1}{t}\right)}=-\frac{2}{\left(t-\frac{1}{t}\right)}$

$\therefore$ *gradient* PF . *gradient* $TF=-1$ and hence $TF\perp PF$.

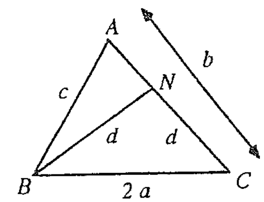
Question 3

(a) **Outcomes Assessed:** (i) H5 (ii) P4

Marking Guidelines

Criteria
(i) • using similarity and sides in proportion to deduce $\frac{d}{2a}=\frac{c}{b}=\frac{b-d}{c}$
• selecting the appropriate relationships to show $bd=2ac$, $c^2=b(b-d)$
• using these simultaneously to show $c^2=b^2-2ac$
(ii) • substitution in expansion of $(a+c)^2$ to show $(a+c)^2=a^2+b^2$

Answer



(i)

$\Delta ABN \mathrel{|||} \Delta ACB$ (*given*)

$\frac{BN}{CB}=\frac{AB}{AC}=\frac{AN}{AB}$ (*corresponding sides* *Δ's are in prop*

$\frac{d}{2a}=\frac{c}{b}=\frac{b-d}{c} \Rightarrow \begin{cases} bd=2ac \\ c^2=b(b-d)= \end{cases}$

$\therefore c^2=b^2-2ac$

(ii)

$\left.\begin{array}{l} b^2=c^2+2ac \quad (\textit{from (i)}) \\ (a+c)^2=a^2+c^2+2ac \end{array}\right\} \Rightarrow (a+c)^2$

(b) **Outcomes Assessed:** (i) PE2, PE3 (ii) PE3

Marking Guidelines

Criteria
(i) • establishing that $P(0)$, $P(1)$ have opposite signs
• noting that $P(x)$ is continuous to deduce existence of root α, $0<\alpha<1$
(ii) • quoting correct expression for approximate value of α using Newton's method
• calculating approximate value of α correct to 2 decimal places

Answer

(i) $P(x)=x^3+3x^2+6x-5 \Rightarrow \begin{cases} P(0)=-5<0 \\ P(1)=5>0 \end{cases}$ and $P(x)$ is continuous

$\therefore P(x)=0$ has a root α, $0<\alpha<1$.

(ii) $P'(x)=3x^2+6x+6$ $\qquad \alpha \approx 0\cdot5-\frac{P(0\cdot5)}{P'(0\cdot5)}=0\cdot5-\frac{(-1\cdot125)}{9\cdot75}\approx 0\cdot62$ (to 2 dec

(c) **Outcomes Assessed:** HE6

Marking Guidelines

Criteria
• using substitution process correctly to obtain new integrand in terms of u
• finding the new limits for the integral in terms of u
• obtaining the primitive function $2\sin^{-1}u$
• evaluating the definite integral by substitution of the limits

$u > 0$

du

$u = \frac{1}{2}$

$u = \frac{1}{\sqrt{2}}$

$$I = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx = \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{1}{u\sqrt{1-u^2}}\,2u\,du$$

$$I = 2\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-u^2}}\,du = 2\left[\sin^{-1}u\right]_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}}$$

$$I = 2\left(\sin^{-1}\tfrac{1}{\sqrt{2}} - \sin^{-1}\tfrac{1}{2}\right) = 2\left(\tfrac{\pi}{4} - \tfrac{\pi}{6}\right) = \tfrac{\pi}{6}$$

n 4

utcomes Assessed: (i) **H5** (ii) **H8**

Marking Guidelines

Criteria	Marks
• obtaining the primitive function $\frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)$	1
• evaluation of the definite integral by substitution of the limits	1
) • using the pattern for Simpson's rule with correct x values, h value and multipliers.	1
• calculation of 3 function values and final approximation for definite integral	1

(ii)

$$= \int_0^{\frac{\pi}{2}} \sin^2 x\ dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2x)\ dx$$

$$= \tfrac{1}{2}\left[x - \tfrac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{2}\left(\tfrac{\pi}{2} - 0\right) = \tfrac{\pi}{4}$$

$f(x) = \sin^2 x$, $h = \frac{\pi}{4}$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
f	0	$\frac{1}{2}$	1
	$\times 1$	$\times 4$	$\times 1$

$$I \approx \tfrac{h}{3}\{f_0 + 4f_1 + f_2\}$$
$$= \tfrac{\pi}{12}\{0 + 2 + 1\}$$
$$= \tfrac{\pi}{4}$$

utcomes Assessed: (i) **PE3** (ii) **PE3**

Marking Guidelines

Criteria	Marks
• determining that there are 3 appropriate sets of three cards for a sum of 9	1
• calculating $\frac{3}{{}^9C_3} = \frac{1}{28}$ as the required probability	1
) • realising that there are now ${}^8C_2$ possible sets of three cards given 2 is selected	1
• calculating $\frac{2}{{}^8C_2} = \frac{1}{14}$ as the required probability	1

r

Exactly 3 sets of cards have a sum of 9: 1+2+6, 1+3+5, 2+3+4

$$P(\text{sum is } 9) = \frac{3}{{}^9C_3} = \frac{3.2.1}{9.8.7} = \frac{1}{28}$$

If the set of cards contains the number 2, exactly two such sets have a sum of 9.
The two cards chosen to complete the set of 3 are selected from the remaining 8 cards.

2 2.2.1 1

4(c) **Outcomes Assessed:** **PE2, HE5**

Marking Guidelines

Criteria	Marks
• finding the relationship between $\frac{dV}{dt}$ and $\frac{dr}{dt}$	1
• finding the relationship between $\frac{dL}{dt}$ and $\frac{dV}{dt}$, where the equator has length L cm	1
• using the numerical values of $\frac{dV}{dt}$ and r to show $\frac{dL}{dt} = 0\cdot 125$	1
• interpreting this to deduce that length of equator is increasing at a rate of $0\cdot 125\,\text{cm s}^{-1}$	1

Answer

$$V = \tfrac{4}{3}\pi r^3 \qquad L = 2\pi r$$

$$\frac{dV}{dt} = \tfrac{4}{3}\pi . 3r^2\frac{dr}{dt} \qquad \frac{dL}{dt} = 2\pi\frac{dr}{dt} = 2\pi . \frac{1}{4\pi r^2}\frac{dV}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \qquad \therefore \frac{dL}{dt} = \frac{1}{2r^2}\frac{dV}{dt} = \frac{25}{2\times 10^2} = 0\cdot 125 \quad \text{when } r = 10$$

Length of equator is increasing at a rate of $0\cdot 125\,\text{cm s}^{-1}$ when the radius is 10 cm

Question 5

(a) **Outcomes Assessed:** (i) **HE4** (ii) **P5, HE4**

Marking Guidelines

Criteria	Marks
(i) • finding the equation of the inverse function $f^{-1}(x)$	1
(ii) • showing intercepts on the coordinate axes and asymptotes for both curves	1
• showing intersection point $(1, 1)$	1
• correct shapes with curves as reflections in $y = x$	1

Answer

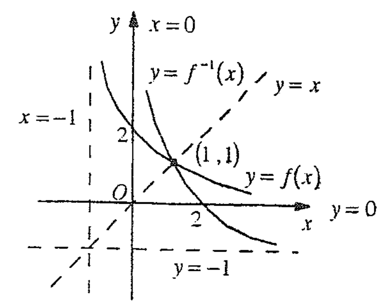
(i)

$$y = \frac{2}{x+1} \Rightarrow (x+1) = \frac{2}{y} \Rightarrow x = \frac{2}{y} - 1$$

$$f(x) = \frac{2}{x+1},\ x > -1 \Rightarrow f^{-1}(x) = \frac{2}{x} - 1,\ y > -1$$

$$\therefore f \text{ has inverse } \quad f^{-1}(x) = \frac{2}{x} - 1,\ \ x > 0$$

Curves are reflections in $y = x$ and hence intersect on $y = x$ where $\frac{2}{x+1} = x \Rightarrow x = 1$



) **Outcomes Assessed:** (i) HE5 (ii) H5, PE2

Marking Guidelines

Criteria	Marks
(i) • obtaining expression for a in terms of x	1
(ii) • integrating expression for $\frac{dt}{dx}$ to obtain primitive function (even if $+c$ omitted)	1
• including and evaluating the constant of integration to find t in terms of x	1
• finding x in terms of t by rearrangement.	1

Answer

(i) $v = -x^2 \Rightarrow a = v\dfrac{dv}{dx} = -x^2 . (-2x) = 2x^3$

(ii)

$$\frac{dx}{dt} = -x^2 \Rightarrow \frac{dt}{dx} = -\frac{1}{x^2} \Rightarrow t = \frac{1}{x} + c, \quad c \text{ constant}$$

$$\left.\begin{array}{l} t = 0 \\ x = 1 \end{array}\right\} \Rightarrow 0 = 1 + c \Rightarrow c = -1 \Rightarrow t = \frac{1}{x} - 1 \quad \therefore x = \frac{1}{t+1}$$

5(c) **Outcomes Assessed:** PE3

Marking Guidelines

Criteria	Marks
• writing general term with appropriate binomial coefficient and powers of x^2 and $\frac{a}{x}$	1
• showing term independent of x is ${}^6C_4 a^4$ or ${}^6C_2 a^4$	1
• deducing ${}^6C_4 a^4 = 240$ or ${}^6C_2 a^4 = 240$ and hence $a^4 = 16$	1
• stating both solutions $a = \pm 2$	1

Answer

General term in expansion of $\left(x^2 + \dfrac{a}{x}\right)^6$ is ${}^6C_r \left(\dfrac{a}{x}\right)^r \left(x^2\right)^{6-r} = {}^6C_r\, a^r x^{12-3r}$, $r = 0, 1, 2, \ldots, 6$

Then term independent of x is ${}^6C_4\, a^4 x^0 = 15a^4 \Rightarrow 15a^4 = 240 \Rightarrow a^4 = 16 \quad \therefore a = \pm 2$

Question 6

6(a) **Outcomes Assessed:** (i) H5, HE4 (ii) P4, HE7

Marking Guidelines

Criteria	Marks
(i) • showing $\tan\theta = \dfrac{A+B}{1-AB}$	1
(ii) • showing $6x^2 + 5x - 1 = 0$	1
• solving this quadratic equation	1
• rejecting the solution $x = -1$ with explanation	1

Answer

(i) Let $x = \tan^{-1} A$ and $y = \tan^{-1} B$. Then $\theta = x + y$, $\tan x = A$, $\tan y = B$ and hence

$$\tan\theta = \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x . \tan y} = \frac{A+B}{1-AB}$$

(ii) $\tan^{-1} 3x + \tan^{-1} 2x = \frac{\pi}{4} \Rightarrow \dfrac{3x+2x}{1-3x.2x} = \tan\frac{\pi}{4}$, using $A = 3x$, $B = 2x$ in (i)

$$\frac{5x}{1-6x^2} = 1 \Rightarrow \begin{array}{l} 6x^2 + 5x - 1 = 0 \\ (6x-1)(x+1) = 0 \\ x = \frac{1}{6} \text{ or } x = -1 \end{array}$$

But $x = -1 \Rightarrow \begin{cases} \tan^{-1} 3x < 0 \text{ and } \tan \\ \therefore \tan^{-1} 3x + \tan^{-1} 2 \end{cases}$

Hence $x \neq -1$. $\therefore x = \frac{1}{6}$

6(b) **Outcomes Assessed:** (i) H3, HE3 (ii) H3, HE3

Marking Guidelines

Criteria
(i) • finding value of A
• finding exact value of k
(ii) • showing $t = \dfrac{\ln 2}{k}$
• finding the further time 5 min 38 s

Answer

(i)

$$T = 20 + Ae^{-kt}$$

$$\left.\begin{array}{l} t = 0 \\ T = 100 \end{array}\right\} \Rightarrow \begin{array}{l} 100 = 20 + Ae^0 \\ 100 = 20 + A \end{array} \quad \therefore A = 80 \text{ and } T = 20 + 80e^{-kt}$$

$$\left.\begin{array}{l} t = 4 \\ T = 80 \end{array}\right\} \Rightarrow \begin{array}{l} 80 = 20 + 80e^{-4k} \\ e^{-4k} = \frac{60}{80} = \frac{3}{4} \end{array} \quad \therefore \begin{array}{l} -4k = \ln\frac{3}{4} \\ k = -\frac{1}{4}\ln\frac{3}{4} = \frac{1}{4}\ln\frac{4}{3} \end{array}$$

T

(ii)

$$\left.\begin{array}{l} T = 20 + 80e^{-kt} \\ T = 60 \end{array}\right\} \Rightarrow \begin{array}{l} e^{-kt} = \frac{40}{80} = \frac{1}{2} \\ -kt = \ln\frac{1}{2} = -\ln 2 \end{array} \quad \therefore t = \frac{\ln 2}{\left(\frac{1}{4}\ln\frac{4}{3}\right)} \approx 9 \cdot 6377$$

Hence it falls to 60°C after 9 min 38 sec, that is after a further 5 min 38 sec.

6(c) **Outcomes Assessed:** (i) PE2, HE3 (ii) H5, HE3

Marking Guidelines

Criteria
(i) • finding values of v and a when $t = 0$
• interpreting these values to deduce particle is moving right and slowing down
(ii) • showing if particle is at O at time t, then $\tan 2t = -3$
• solving this equation to find the first such time.

Answer

(i)

$x = 3\cos 2t + \sin 2t$ $\qquad \therefore t = 0 \Rightarrow x = 3,\ v = 2,\ a = -12$

$v = -6\sin 2t + 2\cos 2t$ $\qquad$ Hence particle is initially 3 m to the right of O,

$a = -12\cos 2t - 4\sin 2t$ $\qquad$ moving to the right (since $v > 0$) and

$\tan^{-1}3x+\tan^{-1}2x=\frac{\pi}{4} \Rightarrow \frac{3x+2x}{1-3x.2x}=\tan\frac{\pi}{4}$, using $A=3x$, $B=2x$ in (i)

$$\frac{5x}{1-6x^2}=1 \Rightarrow \begin{array}{l} 6x^2+5x-1=0 \\ (6x-1)(x+1)=0 \\ x=\frac{1}{6} \text{ or } x=-1 \end{array}$$

But $x=-1 \Rightarrow \begin{cases} \tan^{-1}3x<0 \text{ and } \tan^{-1}2x<0 \\ \therefore \tan^{-1}3x+\tan^{-1}2x \neq \frac{\pi}{4} \end{cases}$

Hence $x\neq -1$. $\therefore x=\frac{1}{6}$

Outcomes Assessed: (i) **H3, HE3** (ii) **H3, HE3**

Marking Guidelines

Criteria	Marks
(i) • finding value of A	1
• finding exact value of k	1
(ii) • showing $t=\frac{\ln 2}{k}$	1
• finding the further time 5 min 38 s	1

er

$T=20+Ae^{-kt}$

$\left.\begin{array}{l} t=0 \\ T=100 \end{array}\right\} \Rightarrow \begin{array}{l} 100=20+Ae^{0} \\ 100=20+A \end{array}$ $\therefore A=80$ and $T=20+80e^{-kt}$ Then

$\left.\begin{array}{l} t=4 \\ T=80 \end{array}\right\} \Rightarrow \begin{array}{l} 80=20+80e^{-4k} \\ e^{-4k}=\frac{60}{80}=\frac{3}{4} \end{array}$ $\therefore \begin{array}{l} -4k=\ln\frac{3}{4} \\ k=-\frac{1}{4}\ln\frac{3}{4}=\frac{1}{4}\ln\frac{4}{3} \end{array}$

$\left.\begin{array}{l} T=20+80e^{-kt} \\ T=60 \end{array}\right\} \Rightarrow \begin{array}{l} e^{-kt}=\frac{40}{80}=\frac{1}{2} \\ -kt=\ln\frac{1}{2}=-\ln 2 \end{array}$ $\therefore t=\frac{\ln 2}{\left(\frac{1}{4}\ln\frac{4}{3}\right)}\approx 9\cdot 6377$

Hence it falls to 60°C after 9 min 38 sec, that is after a further 5 min 38 sec.

utcomes Assessed: (i) **PE2, HE3** (ii) **H5, HE3**

Marking Guidelines

Criteria	Marks
) • finding values of v and a when $t=0$	1
• interpreting these values to deduce particle is moving right and slowing down	1
i) • showing if particle is at O at time t, then $\tan 2t=-3$	1
• solving this equation to find the first such time.	1

r

$=3\cos 2t+\sin 2t$

$=-6\sin 2t+2\cos 2t$

$=-12\cos 2t-4\sin 2t$

$\therefore t=0 \Rightarrow x=3,\ v=2,\ a=-12$

Hence particle is initially 3 m to the right of O, moving to the right (since $v>0$) and

(ii) At O, $x=0$

$3\cos 2t+\sin 2t=0$

$\sin 2t=-3\cos 2t$

$\tan 2t=-3$

smallest positive such t is given by

$2t=\pi-\tan^{-1}3 \Rightarrow t=\frac{1}{2}\left(\pi-\tan^{-1}3\right)\approx 0\cdot 95$

$\therefore$ particle first reaches O after $0\cdot 95$ s (to 2 de

Question 7

7(a) **Outcomes Assessed:** (i) **P5, PE6** (ii) **P5, PE6, HE2** (iii) **P5, PE2, PE6**

Marking Guidelines

Criteria	Mar
(i) • showing $f(0)=1$	1
• showing $f(-x)=\frac{1}{f(x)}$	1
(ii) • noting that $S(1)$ is true	1
• showing that if $S(k)$ is true, then $S(k+1)$ is true	1
• deducing the truth of $S(n)$ for all positive integers	1
(iii) • using (i) and (ii) to deduce that $f(-nx)=[f(x)]^{-n}$	1

Answer

(i)

$f(0+0)=f(0).f(0)$

$f(0)-f(0).f(0)=0$

$f(0)\,[1-f(0)]=0$

$\therefore f(0)>0 \Rightarrow f(0)=1$

$f(x+[-x])=f(x).f(-x)$

$\therefore f(x).f(-x)=f(0)=1$

$\therefore \quad f(-x)=\frac{1}{f(x)}$

(ii) Let $S(n)$ be the statement $f(nx)=[f(x)]^n$, $n=1,2,3,\ldots$

Clearly $S(1)$ is true, since $f(1.x)=[f(x)]^1$.

If $S(k)$ is true for some positive integer k, then $f(kx)=[f(x)]^k$ **

Consider $S(k+1)$: $f([k+1]x)=f(kx+x)=f(kx).f(x)$

$=[f(x)]^k.f(x)$

$=[f(x)]^{k+1}$ if $S(k)$ is true, using

Hence if $S(k)$ is true for some positive integer k, then $S(k+1)$ is true. But $S(1)$ is true. Hence $S(2$ true, and then $S(3)$ is true and so on. Hence $S(n)$ is true for all positive integers n.

(iii) If n is a positive integer, $f(-nx)=\frac{1}{f(nx)}=\frac{1}{[f(x)]^n}$, using (i) and (ii), and hence $f(-nx)=[f(x)]^{-n}$.

Outcomes Assessed: (i) **HE3** (ii) **PE2, HE3**

Marking Guidelines

Criteria	Marks
(i) • writing expressions for horizontal displacements of both particles	1
• writing expressions for vertical displacements of both particles	1
(ii) • showing $U\cos\alpha = V\cos\beta$	1
• showing $UT\sin\alpha = h + VT\sin\beta$	1
• eliminating V from this relationship	1
• rearrangement to obtain T in required form	1

swer

(i) For particle projected from O

$$x_O = Ut\cos\alpha$$
$$y_O = Ut\sin\alpha - \tfrac{1}{2}gt^2$$

For particle projected from A

$$x_A = Vt\cos\beta$$
$$y_A = h + Vt\sin\beta - \tfrac{1}{2}gt^2$$

(ii) Particles collide at time T, having equal horizontal displacements and equal vertical displacements.

$$UT\cos\alpha = VT\cos\beta \quad \Rightarrow \quad U\cos\alpha = V\cos\beta \qquad (1)$$
$$UT\sin\alpha - \tfrac{1}{2}gT^2 = h + VT\sin\beta - \tfrac{1}{2}gT^2 \quad \Rightarrow \quad UT\sin\alpha = h + VT\sin\beta \qquad (2)$$

From (2):
$$T(U\sin\alpha - V\sin\beta) = h$$
$$T(U\sin\alpha\cos\beta - V\cos\beta\sin\beta) = h\cos\beta$$

Using (1):
$$T(U\sin\alpha\cos\beta - U\cos\alpha\sin\beta) = h\cos\beta$$
$$UT\sin(\alpha-\beta) = h\cos\beta$$
$$\therefore \quad T = \frac{h\cos\beta}{U\sin(\alpha-\beta)}$$