Ext 1 Trial 2007

Question 1

a) $\displaystyle\int \frac{dx}{\sqrt{9-25x^2}} = \int \frac{dx}{\sqrt{25(\frac{9}{25}-x^2)}}$

$= \frac{1}{5}\sin^{-1}\frac{5x}{3} + C$

(b) $m_1 = 3 \quad m_2 = -\frac{2}{3}$

$\tan\theta = \left|\dfrac{3 - -\frac{2}{3}}{1+3(-\frac{2}{3})}\right|$

$= \frac{11}{3}$

$\theta = 74^\circ 45'$

(c) $p(-1) = \cancel{0}\,2$

$2(-1)^3 - a(-1) + 1 = 2$

$-2 + a + 1 = 2$

$a = 3$

d) <u>To prove</u> $\cot\theta - \cot 2\theta = \operatorname{cosec} 2\theta$

<u>Proof</u>: $LHS = \dfrac{\cos\theta}{\sin\theta} - \dfrac{\cos 2\theta}{\sin 2\theta}$

$= \dfrac{\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}{\sin\theta\sin 2\theta}$

$= \dfrac{\sin(\cancel{2\theta - \theta})}{\cancel{\sin\theta}\sin 2\theta}$

$= \operatorname{cosec} 2\theta = RHS$

$\cot 15^\circ - \cot 30^\circ = \operatorname{cosec} 30^\circ$

$\therefore \cot 15^\circ = \cot 30^\circ + \operatorname{cosec} 30^\circ$

$= \sqrt{3} + 2$

(e) $\displaystyle\int \frac{u-1}{2}\cdot u^{10}\cdot\frac{du}{2}$ $\quad u = 2x+1$, $du = 2dx$, $x = \frac{u-1}{2}$

$= \frac{1}{4}\int u^{11} - u^{10}\,du$

$= \frac{1}{4}\left[\frac{u^{12}}{12} - \frac{u^{11}}{11}\right] + C$

$= \frac{1}{48}(2x+1)^{12} - \frac{1}{44}(2x+1)^{11} + C$

Question 2

(a) $\left(3x - \frac{1}{x^3}\right)^{12}$

$T_{k+1} = {}^{12}C_k\,(3x)^{12-k}\cdot\left(\frac{-1}{x^3}\right)^k$

$= {}^{12}C_k\,3^{12-k}(-1)^k . x^{12-4k}$

For term indep of x $\quad 12 - 4k = 0$, $k = 3$

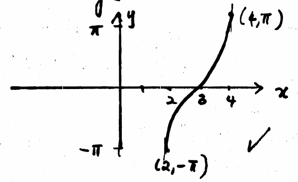
$\therefore T_4 = {}^{12}C_3\,3^9.(-1)^3$

$= -220(3)^9$
or -4330260

(b) (i) D: $-1 \le x - 3 \le 1$

$2 \le x \le 4$

R: $-\pi \le y \le \pi$



(ii) $y' = \dfrac{2}{\sqrt{1-(x-3)^2}}$

when $x = 3\frac{1}{2}$, $y' = \dfrac{2}{\sqrt{1-(\frac{1}{2})^2}}$

$= \dfrac{2}{\sqrt{3}/2} = \dfrac{4}{\sqrt{3}}$ or $\dfrac{4\sqrt{3}}{3}$

(c) $\frac{x}{3} = \tan 2\theta$, $y = \tan\theta$

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\dfrac{x}{3} = \dfrac{2y}{1-y^2}$

$x(1-y^2) = 6y$

(d) $\frac{1}{2}\int 1 + \cos 4x\,dx$

$= \frac{1}{2}\left[x + \frac{\sin 4x}{4}\right] + C$

Q 3

(a) $\dfrac{6^x + 4^x}{3^x + 2^x} = \dfrac{2^x.3^x + 2^x.2^x}{3^x + 2^x}$

$= \dfrac{2^x\cancel{(3^x + 2^x)}}{\cancel{3^x + 2^x}}$

$= 2^x$

(b) $P(x) = 2x^3 - 17x^2 + 40x - 16$

$P'(x) = 6x^2 - 34x + 40$

For double root $P'(x) = 0$, $P(x) = 0$

$2(3x^2 - 17x + 20) = 0$

$(3x-5)(x-4) = 0$

$x = 4$ or $5/3$

root is integer $\therefore x = 4$

$P(4) = 0$

$(x-4)^2(2x-1) = 0$

$\therefore x = 4, 4, \frac{1}{2}$

<u>OR</u> Let roots be α, α, β

$2\alpha + \beta = \frac{17}{2}$

$\alpha^2 + 2\alpha\beta = 20$

$\alpha^2 + 2\alpha\left(\frac{17}{2} - 2\alpha\right) = 20$

$\alpha^2 + 17\alpha - 4\alpha^2 = 20$

$3\alpha^2 - 17\alpha + 20 = 0$

$(3\alpha - 5)(\alpha - 4) = 0$

α integral $\therefore \alpha = 4$

$\therefore$ Roots are $4, 4, (\frac{17}{2} - 8)$

ie $4, 4, \frac{1}{2}$

(c) <u>To prove</u>

$2\times2^0 + 3\times2^1 + \ldots + (n+1)2^{n-1} = n.2^n$

<u>Proof</u>: test $n = 1$

$LHS = 2\times2^0 = 2$

$RHS = 1.2^1 = 2$

$\therefore$ true for $n = 1$

Assume true for $n = k$

ie $2\times2^0 + 3\times2^1 + \cdots + (k+1)2^{k-1} = k.2^k$

Prove true for $n = k+1$

ie. $2\times2^0 + 3\times2^1 + \ldots + (k+2).2^k$

$= k.2^k + (k+2).2^k$

$= 2^k(k + k + 2)$

$= 2^k(2k+2)$

$= 2^k.2(k+1)$

$= (k+1)2^{k+1}$

$\therefore$ If true for $n = k$, it will be true for $n = k+1$

Since true for $n = 1$, it will be true for $n = 2, 3, \ldots$ ie for all $n \geq 1$

(Lose 1 if conclusion not there)

(d) $r = \dfrac{3}{1+3x}$

For limiting sum $|r| < 1$

$\left|\dfrac{3}{1+3x}\right| < 1$

$3 < |1+3x|$

$1 + 3x > 3$ $\quad$ or $\quad -1 - 3x > 3$

$x > \frac{2}{3}$ $\quad\quad$ $x < -\frac{4}{3}$

Q 4

a) (i) $2\sqrt{3}\cos 2t - 2\sin 2t$

$\equiv R\cos(2t + \alpha)$

$\equiv R\cos 2t\cos\alpha - R\sin 2t\sin\alpha$

$R\cos\alpha = 2\sqrt{3}$

$R\sin\alpha = 2$

$\therefore R = \sqrt{(2\sqrt{3})^2 + 2^2} = 4$

$\tan\alpha = \frac{1}{\sqrt{3}}$, $\quad \alpha = \frac{\pi}{6}$

$\therefore 2\sqrt{3}\cos 2t - 2\sin 2t \equiv 4\cos(2t + \frac{\pi}{6})$

(ii) $x = 2\sqrt{3}\cos 2t - 2\sin 2t$

$= 4\cos\left(2t + \frac{\pi}{6}\right)$

$\dot{x} = -8\sin\left(2t + \frac{\pi}{6}\right)$

$\ddot{x} = -16\cos\left(2t + \frac{\pi}{6}\right)$

$= -4x$ ✓

of the form $-n^2 x$

$\therefore$ S.H.M

period $= \frac{2\pi}{2} = \pi$ ✓

amplitude $= 4$ ✓

(b) $f(x) = x + 2\tan x$

$f'(x) = 1 + 2\sec^2 x$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= \frac{3\pi}{4} - \frac{\frac{3\pi}{4} + 2\tan\frac{3\pi}{4}}{1 + 2\sec^2\frac{3\pi}{4}}$

$= \frac{3\pi}{4} - \frac{\frac{3\pi}{4} - 2}{1 + 2(-\sqrt{2})^2}$ ✓ ✓

$= \frac{3\pi}{4} - \frac{\frac{3\pi}{4} - 2}{5}$

$= \frac{3\pi}{5} + \frac{2}{5}$ ✓

(c) $\frac{d}{dx}\left(\frac{1}{3}\tan^3\theta\right)$

$= \frac{1}{\cancel{3}}\cdot\cancel{3}\tan^2\theta\sec^2\theta$ ✓

$= (\sec^2\theta - 1)\sec^2\theta$ ✓

$= \sec^4\theta - \sec^2\theta$

$\therefore \int_0^{\pi/4} \sec^4\theta \, d\theta$

$= \left[\frac{1}{3}\tan^3\theta\right]_0^{\frac{\pi}{4}} + \int_0^{\pi/4} \sec^2\theta \, d\theta$ ✓

$= \frac{1}{3}\cdot 1^3 + \left[\tan\theta\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{3} + 1 - 0$

$= \frac{4}{3}$ ✓

Q5

(a) $\frac{d}{dx}\log_{10}(x^2+1)$

$= \frac{d}{dx}\frac{\ln(x^2+1)}{\ln 10}$

$= \frac{1}{\ln 10}\cdot\frac{2x}{x^2+1}$ ✓ ✓

(b) $\ddot{x} = 6(1-x^2)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6 - 6x^2$

$\frac{v^2}{2} = 6x - 2x^3 + C$ ✓

$x = -3, v = 4 \quad \therefore 8 = -18 + 54 + C$

$-28 = C$ ✓

$\therefore v^2 = 12x - 4x^3 - 56$

(ii) If $x = 0$, $v^2 = -56$ not possible

$\therefore$ No ✓

(c) (i) OP: $y = \frac{p^2}{2p}x \quad \therefore y = \frac{p}{2}x$ ✓

(ii) $y - 0 = -\frac{2}{p}(x - 2p)$

$y = -\frac{2}{p}x + 4$ ✓

(iii) Sub (i) into (ii)

$\frac{p}{2}x = -\frac{2}{p}x + 4$

$p^2 x = -4x + 8p$ ✓

$x(p^2+4) = 8p$

$x = \frac{8p}{p^2+4}$

$y = \frac{p}{2}\cdot\frac{8p}{p^2+4} = \frac{4p^2}{p^2+4}$ ✓

iv) $\frac{x}{y} = \frac{2}{p} \quad \therefore p = \frac{2y}{x}$ ✓

Sub in x: $\cancel{x} = \frac{8\left(\frac{2y}{x}\right)}{\frac{4y^2}{x^2} + 4} = \frac{16y\cancel{x}}{4y^2 + 4x^2}$

Locus is $\left.\begin{array}{l} 4x^2 + 4y^2 = 16y \\ x^2 + (y-2)^2 = 4 \end{array}\right\}$ ✓

centre $(0, 2)$, $r = 2$ ✓

Q6

a) $e^x - \frac{1}{e^x} = 1$

$e^{2x} - e^x - 1 = 0$ ✓

$u = e^x \quad u^2 - u - 1 = 0$

$e^x = u = \frac{1 \pm \sqrt{1 - 4.1.(-1)}}{2}$

$e^x > 0 \quad \therefore e^x = \frac{1+\sqrt{5}}{2}$ ✓

$x = \ln\left(\frac{1+\sqrt{5}}{2}\right)$ ✓

(b) $(1+ax)^7 + (1+bx)^7$

$= 1 + {}^7C_1 ax + {}^7C_2 a^2x^2 \ldots$

$+ 1 + {}^7C_1 bx + {}^7C_2 b^2x^2 \ldots$

$\therefore 7a + 7b = 21$ or $a + b = 3$ ✓

${}^7C_2 a^2 + {}^7C_2 b^2 = 609$ or $a^2 + b^2 = 29$ ✓

$a = 3 - b$

$9 - 6b + b^2 + b^2 = 29$

$2b^2 - 6b - 20 = 0$

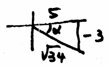
$b^2 - 3b - 10 = 0$

$(b-5)(b+2) = 0$

$\therefore b = 5$ or -2 any pair

$a = -2$ or 5 ✓✓

(c) $\sin(2\underbrace{\tan^{-1} -\tfrac{3}{5}}_{\alpha})$



$= 2\sin\alpha\cos\alpha$

$= 2\left(-\frac{3}{\sqrt{34}}\right)\cdot\frac{5}{\sqrt{34}} = -\frac{30}{34} = -\frac{15}{17}$ ✓ ✓

(d) Join AB, QB, BT

Let $\angle APQ = \alpha$

✓ $\angle ABQ = 180° - \alpha$ (opp. $\angle$s of cyclic quad. supp.)

✓ $\angle STA = \angle APQ = \alpha$ (alt. $\angle$s on // lines)

✓ $\angle ABT = \angle STA = \alpha$ ($\angle$ between tangent and chord $= \angle$ in alt seg.)

$\therefore \angle ABQ + \angle ABT$

$= 180° - \alpha + \alpha$

$= 180°$

$\therefore$ QBT is a straight line

ie Q, B, T are collinear

Q7

(a) Let $x = \ln(2y+3)$

$e^x = 2y + 3$ ✓

$f^{-1}(x) = y = \frac{e^x - 3}{2}$ ✓

(b) (i) Area of OBPC

= area of ΔOPB + area of ΔOPC

$= \frac{1}{2}.30^2\sin\theta + \frac{1}{2}30^2\sin(90° - \theta)$ ✓

$= 450(\sin\theta + \cos\theta)$ ✓

(ii) $\frac{dA}{d\theta} = 450(\cos\theta - \sin\theta)$

$\frac{d\theta}{dt} = \frac{\pi}{60}$ rad/min ✓

$\therefore \frac{dA}{dt} = 450\left(\cos\frac{\pi}{6} - \sin\frac{\pi}{6}\right) \times \frac{\pi}{60}$ cm²/min

$= 450\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) \times \frac{\pi}{60}$ cm²/min

$= \frac{15\pi}{4}(\sqrt{3} - 1)$ or 8.624 cm²/min

either one ✓

(c) (i) Time of flight when $y = 0$

$t\left(V\sin\alpha - \frac{gt}{2}\right) = 0$

$\therefore t = \frac{2V\sin\alpha}{g}$ ✓

(ii) $\tan 45° = \frac{\dot{y}}{\dot{x}}$

$1 = \frac{V\sin\alpha - gT}{V\cos\alpha}$ ✓

$V\cos\alpha = V\sin\alpha - gT$ ✓

$\therefore T = \frac{V\sin\alpha - V\cos\alpha}{g}$

(iii) $\frac{V\sin\alpha - V\cos\alpha}{\cancel{g}} = \frac{1}{3}\cdot\frac{2V\sin\alpha}{\cancel{g}}$ ✓

$3\sin\alpha - 3\cos\alpha = 2\sin\alpha$

$\tan\alpha = 3$ ✓

$\alpha = 72°$ ✓