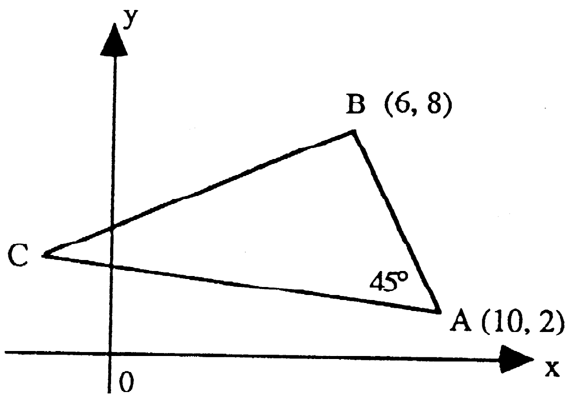
The King's School

4 unit mathematics

Trial HSC Examination 1994

1. **(a)** Evaluate $\int_0^1 \frac{x^3}{(x^2+1)^2}\,dx$ by using the substitution $u = x^2 + 1$.
(b) Evaluate $\int_0^1 2x \tan^{-1} x\; dx$
(c) **(i)** Show that $\int_0^a f(x)\; dx = \int_0^a f(a-x)\; dx$
(ii) Prove that $I = \int_0^{\frac{\pi}{2}} \frac{A\sin x + B\cos x}{\sin x + \cos x}\; dx = \frac{\pi}{4}(A+B)$, A, B constants.
(d) Find $\int \frac{2\tan x}{\tan 2x + \sin 2x}\; dx$ by using the substitution $t = \tan x$.

2. **(a)** **(i)** Find the square roots of $-35 + 12i$
(ii) Solve $z^2 - (5+4i)z + 11 + 7i = 0$
(b) Describe in the Argand diagram the locus of the complex numbers z if
(i) $|z-1| = 4$
(ii) $|z-1| = |z+1|$
(iii) $|z-1| + |z+1| = 4$
(iv) $\Re(\frac{i}{z}) = \frac{1}{2}$, $z \neq 0$.
(c)



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$\triangle ABC$ is drawn in the Argand diagram. $B\hat{A}C = 45°$, $A = (10, 2)$, $B = (6, 8)$. The length of side AC is twice the length of side AB.
Find **(i)** the complex number that the vector AB represents.
(ii) the complex number that point C represents.

3. **(a)** Sketch carefully the hyperbola $3x^2 - y^2 = 12$, showing on your diagram the foci, the directrices and the asymptotes in their correct positions.

(b) A tangent to the parabola $y^2 = 2ax$ meets the hyperbola $xy = c^2$ in the points P, Q.
(i) Show that the equation of the tangent at $R(x_1, y_1)$ on the parabola is $y_1 y = a(x + x_1)$
(ii) Show that the x coordinates of P and Q are given by the equation
$ax^2 + ax_1 x - c^2 y_1 = 0$
(iii) Deduce the cartesian equation of the locus of the midpoint M of the interval PQ.

4. (a) (i) Sketch the line $y = x - 1$ and the rectangular hyperbola $y = \frac{1}{x-1}$ on the same axes, showing their points of intersection.
(ii) On separate diagrams and using **(i)**, sketch the graphs of the following functions and relations. For each graph label any asymptote.
(α) $y = x - 1 + \frac{1}{x-1}$
(β) $y = |x - 1 + \frac{1}{x-1}|$
(γ) $y^2 = x - 1 + \frac{1}{x-1}$
(δ) $y = x - 1 - \frac{1}{x-1}$
(b) Consider two functions f and g for which we know the following facts:
$f(c) = g(c) = 0$, $f'(c)$ and $g'(c)$ exist, $g'(c) \neq 0$.
By considering $f'(c) = \lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c}$ and a similar result for $g'(c)$, show that
$\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$. Hence, or otherwise, show that
(i) $\lim\limits_{x \to 1} \frac{\ln x}{x^2 - 1} = \frac{1}{2}$
(ii) $\lim\limits_{x \to 0} \frac{1 - \cos^7 x}{x^2} = \frac{7}{2}$

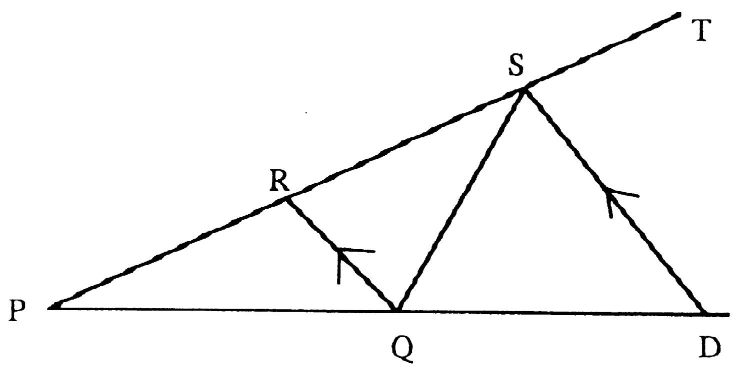
5. (a) A particle of mass m moves in the x axis under the influence of a force $\frac{mn^2}{x^3}$, n a positive constant, directed **away** from the origin, O. Initially, the particle is at rest at $x = a > 0$.
(i) Prove that the velocity v is given by $v^2 = n^2(\frac{1}{a^2} - \frac{1}{x^2})$
(ii) Deduce that $ax = \sqrt{n^2 t^2 + a^4}$
(b) A particle of mass m is projected from a point O on horizontal ground with speed u at an angle of elevation α. It hits the ground again at a distance $2a$ from O and in its flight reaches a maximum height of b. The acceleration due to gravity is g and no forces other than the gravitational force act on the particle. At time t, the horizontal and vertical displacements from O are x and y, respectively
(i) Prove that $\dot{y}^2 = (u \sin \alpha)^2 - 2gy$
(ii) Deduce that $\tan \alpha = \frac{2b}{a}$ and $u^2 = \frac{g}{2b}(4b^2 + a^2)$

6. (a) u, v, w are the roots of the equation $P(x) = 8x^3 + 28x^2 + 14x - 15 = 0$
(i) Form the equation with roots $2u + 3, 2v + 3$ and $2w + 3$
(ii) Hence, or otherwise, solve $P(x) = 0$.
(b) Consider the polynomial equation $f(x) = x^n + nkx + (n - 1) = 0$, $n > 1$. For what values of k will $f(x) = 0$ have a double root if
(i) n is odd

(ii) n is even?
(c) If $a > b > 0$, show that $P(x) = x^3 + x^2 - ax - b$ always has
(i) two distinct stationary points, and
(ii) 3 distinct real zeros.

7. **(a)** Consider the region between the line $y = x$ and the curve $y = x^3$ in the first quadrant. Take $P(x, x^3)$ as any point on the curve $y = x^3$.
(i) The region is rotated about the line $x = 2$. Use the method of cylindrical shells to find the volume of the solid of revolution.
(ii) The region is rotated about the line $y = x$. By taking a slice in the region perpendicular to $y = x$, find the volume of the solid of revolution.
(b) **(i)** Find a, b, c if $\frac{4n-2}{n(n+1)(n+2)} \equiv \frac{a}{n} + \frac{b}{n+1} + \frac{c}{n+2}$
(ii) Use **(i)** to deduce that the sum to infinity of the series $\frac{2}{1\times2\times3} + \frac{6}{2\times3\times4} + \frac{10}{3\times4\times5} + \cdots$ is $1\frac{1}{2}$.

8. **(a)** **(i)**



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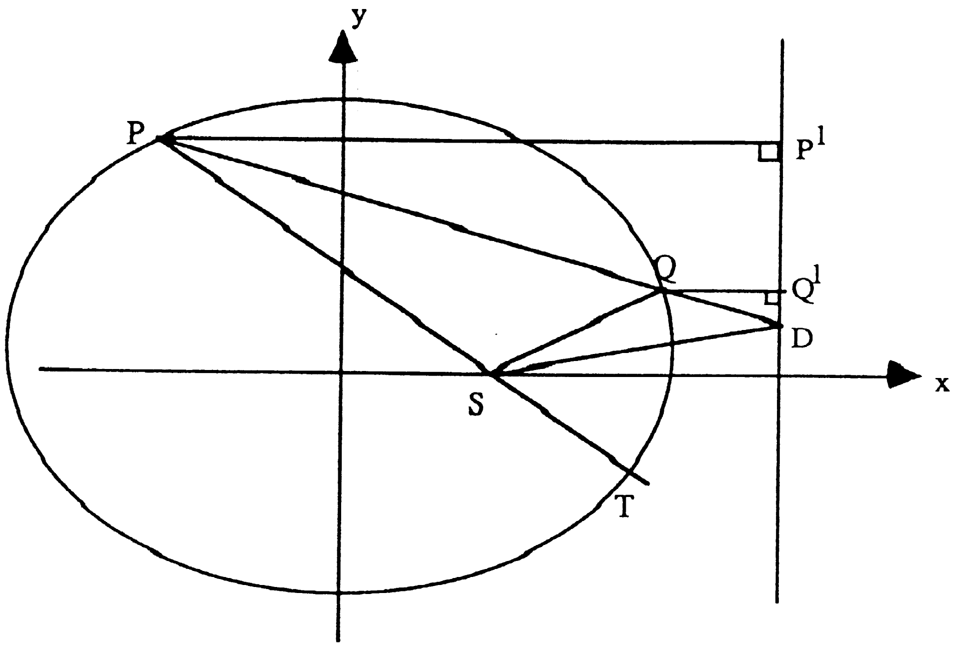
In the diagram, $QR || DS$ and $\frac{PD}{DQ} = \frac{PS}{SQ}$. Copy the diagram. Prove that
(α) $SQ = SR$
(β) $Q\hat{S}D = T\hat{S}D$

(ii)



In the diagram, PQ is a chord of an ellipse with eccentricity e. PQ produced meets a directrix at D and PP', QQ' are drawn perpendicular to this directrix. S is the corresponding focus of the ellipse. Copy the diagram.

(α) Prove that $\frac{PP'}{QQ'} = \frac{PD}{QD}$

(β) Deduce that DS bisects $Q\hat{S}T$.

(b) Consider the series of n terms

$S_n = 1 + \frac{2n-2}{2n-3} + \frac{(2n-2)(2n-4)}{(2n-3)(2n-5)} + \cdots + \frac{(2n-2)(2n-4)\cdots\times4\times2}{(2n-3)(2n-5)\cdots\times3\times1}$

(i) Show that $S_3 = 5$

(ii) Prove by induction, for $n \geq 1$, that $S_n = 2n - 1$.