N.S.W. DEPARTMENT OF EDUCATION

HIGHER SCHOOL CERTIFICATE EXAMINATION 1982

MATHEMATICS 3 UNIT and 4 UNIT - FIRST PAPER

Instructions: Time allowed 3 hours. All questions may be attempted. All questions are of equal value. In every question, all necessary working should be shown.
Marks may not be awarded for careless or badly arranged work. Mathematical tables will be supplied; approved slide-rules or calculators may be used.

QUESTION 1

(i) Rationalise the denominator of $\frac{1+3\sqrt{3}}{5-2\sqrt{3}}$

(ii) Given that $\log_8 2 = \log_x 5$, find x.

(iii) Sketch the region in the cartesian plane determined by the points (x,y) satisfying $0 \le y-2x \le 2$.

(iv) Find the exact value of: (a) $\int_0^1 \frac{2x}{1+x^2}\,dx$; (b) $\int_{\pi/4}^{\pi/3} \cos(x+\pi)\,dx$

(v) Show that the acute angle θ between the tangents at the point $(1,1)$ to the curves $y=x^2$ and $y=x^{\frac{1}{2}}$ is given by $\theta = \tan^{-1}(3/4)$.

QUESTION 2

(i) Differentiate $\cos(1+x^3)$ with respect to x.

(ii) Find the area between the curve $y=x^4$, the x-axis, and the ordinates $x=1$ and $x=2$

(iii) The function $f(x)$ is defined by the rule $f(x)=4xe^{-2x}$, in the domain $-\frac{1}{2} \le x \le 2$.

(a) Draw up a table of values of $f(x)$, correct to one decimal place, for each of the values $x = -\frac{1}{2}, 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2$.

(b) Use the derivative of $f(x)$ to find the coordinates of the turning point of $f(x)$, and determine whether it is a maximum or minimum.

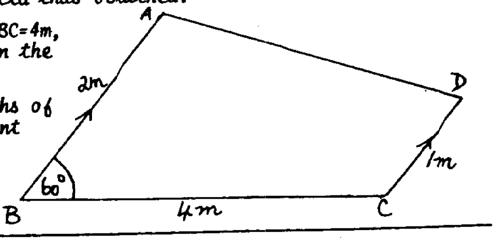
(c) Hence draw a sketch of the graph of $f(x)$, showing clearly the turning point and the values at the end points of the domain.

QUESTION 3

(i) The area between the curve $y=\sin x$ and the x-axis, for $0 \le x \le \pi$, is rotated about the x-axis. Find the volume of the solid thus obtained.

(ii) A quadrilateral ABCD has AB=2m, BC=4m, CD=1m, and AB is parallel to DC, as in the figure (not drawn to scale).

Given that $\hat{B}=60$, calculate the lengths of AC and AD, correct to three significant figures.



QUESTION 4

(i) (a) Derive the formula for the sum $S(n)$ of n terms of a geometric series with first term a and common ratio r.

(b) The sum $C(n)$ of the first n terms of a certain series is given by
$C(n) = 1000(0.9)^n + 10\,000\{1+0.9+(0.9)^2+ \dots + (0.9)^{n-1}\}$

Find the limiting sum C of this series as the number of terms increases indefinitely.

(ii) (a) Sketch the functions $\sin^{-1}(x)$ and $\cos^{-1}(x)$, illustrating clearly their range and domain.

(b) Evaluate $\int_0^1 \frac{dx}{\sqrt{(2-x^2)}}$ (c) Show that the function $\cos^{-1}(-x)+\cos^{-1}(x)$ is constant, and find its value.

QUESTION 5

(i) Given that $\cos x = 4/5$, and $0 < x < \frac{\pi}{2}$, find exact values for:

(a) $\sin x$; (b) $\cos 2x$.

(ii) (a) The coordinates of P are $(2,1)$. Show that P lies on both the parabolas $4y=x^2$, and $4y=(x-4)^2$. Show also that P is the only point of intersection of the two curves.

(b) Find the equation of the tangent at P to the parabola $4y=(x-4)^2$. Also find the coordinates of the other point Q at which this tangent intersects the parabola $4y=x^2$.

QUESTION 6

(i) A biased coin is found to have probability 0.7 of showing a head when tossed.

(a) What is the probability that exactly one head occurs in a sequence of five tosses?

(b) What is the most likely combination of heads and tails to occur in a sequence of five tosses? What is the probability that this event occurs?

(ii) A company assumes that the proportion P of viewers who will buy a new product after it is advertised n times on television satisfies a relation $P = 1-e^{kn}$, where k is a constant.
If 50 per cent of viewers buy the product after 10 advertisements appear, how many times should the company advertise the product if it wants at least 90 per cent of its viewers to buy it?

QUESTION 7

(i) (a) Plot (not on squared paper) the points $A(1,0)$, $B(4,0)$ and $C(0,2)$ on a sketch diagram.

(b) Show that the length CB is twice that of CA.

(c) A point $P(X,Y)$ moves such that the length of PB is twice that of PA. Show that the locus of P is a circle, and determine its centre and radius. Draw a sketch of the circle on your diagram.

(ii) Five similar discs are taken and each marked with one and only one letter. Two are marked with A, one with H, one with L, and the final one with O. They are then placed in a box. The five discs are drawn at random one at a time from the box without replacement. What is the probability that:

(a) the first disc drawn is marked A?

(b) the order of selection of the five discs spells the word ALOHA?

QUESTION 8

OMIT

A triangle OPQ has vertices $O(0,0,0)$, $P(2,2,1)$ and $Q(-2,9,5)$. (i) Calculate:

(a) the length of PQ;

(b) the cosine of the angle OPQ;

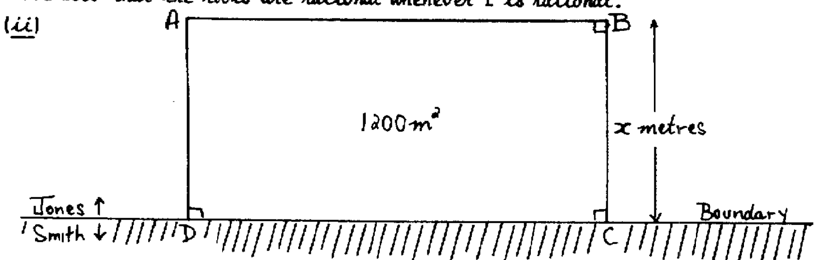
(c) the area of the triangle, correct to one decimal place.

(ii) Find the equation of the plane OPQ, and the equations of the line through O perpendicular to this plane.

QUESTION 9

(i) The quadratic equation $x^2+Lx+M=0$ has one root which is twice the other. Prove that $2L^2 = 9M$.
Prove also that the roots are rational whenever L is rational.

(ii)



Farmer Jones wishes to fence off a rectangular yard ABCD of area 1200m² from a paddock, as in the figure, with the side CD against the property of Farmer Smith. Fencing costs \$3 per metre, and Smith has agreed to pay half the cost of fencing the side CD.
Let \$y be the cost to Jones of fencing the yard, and let x metres be the length of BC. Obtain a formula for y as a function of x, and hence find the minimum cost to Jones of fencing the yard, assuming Smith meets his share of the expense.

QUESTION 10

A particle P is projected from a point O on a horizontal plane with initial velocity Vm s^{-1} in a direction inclined at an angle α upwards from the horizontal. At time t seconds after the instant of projection its horizontal and vertical distances from O are x metres, y metres, respectively. Air resistance may be neglected.

(i) Write down expressions for x and y as functions of t.

(ii) Show that the time of flight T seconds and the range R metres are given by
$T = (2V/g)\sin\alpha$, $R = (V^2/g)\sin 2\alpha$,
and derive a similar expression for the maximum height reached by P.

(iii) A ball is thrown from a height 1 metre from the ground and is caught, without bouncing, 2 seconds later 50m away, also at a height of 1m. Assuming no air resistance, and that g has the approximate value 10 m s^{-2}, find:

(a) the velocity and angle of projection of the ball;

(b) the maximum height of the ball above the ground during its flight.