Question 1

(a) $\dfrac{x^3+y^4}{y^2}$

$= \dfrac{\frac{2}{5}+\frac{9}{25}}{\frac{3}{5}} = 1\frac{4}{15}$

(b) Let $x = 0.23333\ldots$ ①
$10x = 2.33333\ldots$ ②

② − ①:
$9x = 2.1$
$x = \dfrac{21}{90} = \dfrac{7}{30}$

i.e. $0.2\dot{3} = \dfrac{7}{30}$

OR $0.2\dot{3} = 0.233333\ldots$
$= 0.2 + (0.03 + 0.003 + 0.0003 + 0.00003 + \ldots)$

$\therefore$ Infinite sum of a geometric progression,

where $a = 0.03$, $r = \dfrac{0.003}{0.03} = 0.1$

$\therefore \quad S = \dfrac{a}{1-r} = \dfrac{0.03}{1-0.1} = \dfrac{0.03}{0.9} = \dfrac{1}{30}$

$\therefore \quad 0.2\dot{3} = 0.2 + \dfrac{1}{30} = \dfrac{6}{30} + \dfrac{1}{30} = \dfrac{7}{30}$

(c) $40 - 5y^3$

$= 5(8 - y^3)$

$= 5(2-y)(4+2y+y^2)$

(d) $x^2 + 4x - 1 = 0$

$x = \dfrac{-4 \pm \sqrt{20}}{2}$

$x = \dfrac{-4 \pm 2\sqrt{5}}{2}$

$x = -2 \pm \sqrt{5}$

(e) (i) $4^x = 32$

$2^{2x} = 2^5$

$2x = 5$

$x = 2.5$

(ii) $\therefore \log_4 32 = 2.5$

Question 2

(a) (i) $\dfrac{d}{dx}\left(5x + \dfrac{3}{x^2}\right)$

$= \dfrac{d}{dx}\left(5x + 3x^{-2}\right)$

$= 5 - 6x^{-3}$

$= 5 - \dfrac{6}{x^3}$

(ii) $\dfrac{d}{dx}\left(e^{2x^2+3}\right)$

$= 4xe^{2x^2+3}$

(iii) $\dfrac{d}{dx}\left(\dfrac{3x}{\sin x}\right) = \dfrac{\sin x \times 3 - 3x \times \cos x}{\sin^2 x}$

$= \dfrac{3\sin x - 3x\cos x}{\sin^2 x}$

$= \dfrac{3(\sin x - x\cos x)}{\sin^2 x}$

(b) (i) Gradient of $AC = \dfrac{3-5}{1+5} = \dfrac{-2}{6} = -\dfrac{1}{3}$.

(ii) Midpoint of $AC = \left(\dfrac{-5+1}{2}, \dfrac{5+3}{2}\right) = (-2, 4)$.

(iii) Use $y - y_1 = m(x - x_1)$, with $(x_1, y_1) = (-2, 4)$ and $m = 3$.

$\therefore \quad y - 4 = 3(x+2)$

$\therefore \quad y = 3x + 10$

$\therefore \quad 3x - y + 10 = 0$.

(iv) Substitute $x = 0$ in equation $3x - y + 10 = 0$.
$3(0) - y + 10 = 0$
$y = 10$.
$\therefore$ B has coordinates (0,10).

(v) Substitute $D(-4, -2)$ into $3x - y + 10 = 0$.
LHS $= 3(-4) - (-2) + 10$
$= -12 + 2 + 10$
$= 0$
$=$ RHS.
$\therefore$ D lies on the line.

(vi) Midpoint of $BD = \left(\dfrac{0+(-4)}{2}, \dfrac{10+(-2)}{2}\right) = (-2, 4)$.

Since B and D both lie on the perpendicular bisector of AC and the midpoint of BD is equal to the midpoint of AC, then the diagonals AC and BD bisect each other at right angles.

$\therefore$ $ABCD$ is a rhombus.

Question 3

(a) $\displaystyle\int \frac{x}{x^2+5}\,dx = \frac{1}{2}\int \frac{2x}{x^2+5}\,dx$

$= \dfrac{1}{2}\ln\left(x^2+5\right) + C$

(or $\ln\sqrt{x^2+5} + C$).

(b) $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2 2x\,dx = \left[\frac{1}{2}\tan 2x\right]_0^{\frac{\pi}{8}}$

$= \dfrac{1}{2}\tan\dfrac{\pi}{4} - \dfrac{1}{2}\tan 0$

$= \dfrac{1}{2}$.

(c) $y = x\log_e x$

$\therefore \quad \dfrac{dy}{dx} = x \times \dfrac{1}{x} + \log_e x \times 1 = 1 + \log_e x$.

Let m_1 = gradient of tangent at (e, e)
and m_2 = gradient of normal at (e, e).

$\therefore \quad m_1 = 1 + \log_e e = 1 + 1 = 2$.

$m_2 = -\dfrac{1}{m_1} = -\dfrac{1}{2}$

$\therefore$ Equation of normal is
$y - e = -\dfrac{1}{2}(x - e)$

$2y - 2e = -x + e$

$\therefore \quad x + 2y - 3e = 0$

3 continued

$a^2 = b^2 + d^2 - 2bd\cos A$
$a^2 = 38^2 + 32^2 - 2(38)(32)\cos 30^\circ$
$\therefore \quad a = 19.02173\ldots$

$BD = 19$ cm (2 s.f.)

$\frac{\sin C}{c} = \frac{\sin B}{b}$

$\frac{\sin C}{19.02173\ldots} = \frac{\sin 42^\circ}{26}$

$\angle BCD = 29.31\ldots^\circ$

$\therefore \angle BCD = 29^\circ$ (nearest degree).

Question 4

(a) (i) $ar = 120$
$ar^4 = 50.625$

$\frac{ar^4}{ar} = \frac{50.625}{120}$

$r^3 = \frac{27}{64}$

$\therefore$ the common ratio, $r = \frac{3}{4} = 0.75$

$a = \frac{120}{r} = \frac{120}{0.75} = 160$

$\therefore$ the first term, $a = 160$.

(ii) $S_\infty = \frac{a}{1-r}$

$= \frac{160}{1-0.75}$

$= \frac{160}{0.25} = 640$

$\therefore$ the limiting sum, $S_\infty = 640$

(iii) $S_{40} = \frac{160\left(1-0.75^{40}\right)}{1-0.75} = 639.9935\ldots$

$S_\infty - S_{40} = 0.0064362\ldots$

$= 0.0064$ (2 s.f.)

$= 6.4 \times 10^{-3}$ (2 s.f.)

(b) (i) $x^2 + kx - 3x + 2 - k = 0$

$x^2 + x(k-3) + (2-k) = 0$

$a = 1,\ b = k-3,\ c = 2-k$

$\Delta = b^2 - 4ac$

$= (k-3)^2 - 4(1)(2-k)$

$= k^2 - 6k + 9 - 8 + 4k$

$= k^2 - 2k + 1$

$\therefore \Delta = k^2 - 2k + 1$

(ii) For real roots, $b^2 - 4ac \geq 0$ for all k

i.e. $k^2 - 2k + 1 \geq 0$ for all k

Now $k^2 - 2k + 1 = (k-1)^2$

and $(k-1)^2 \geq 0$ for all k

$\therefore$ the roots are real for all values of k.

(c) (i) For a regular polygon, interior angle $= 180^\circ -$

$\therefore$ for a regular octagon, interior angle $= 180^\circ$

$= 180^\circ$

$= 135^\circ$

$\therefore \angle ABC = 135^\circ$.

(ii) $\angle GAH = \frac{180^\circ - 135^\circ}{2}$

$\therefore \angle GAH = 22.5^\circ$.

Question 4 continued

(c) (iii) $\angle CGF = \frac{135^\circ}{2}$

$\therefore \angle CGF = 67.5^\circ$.

(iv) $\angle AGC = 135^\circ - (67.5^\circ + 22.5^\circ)$

$\therefore \angle AGC = 45^\circ$.

Question 5

(a) (i) Area of a sector $= \frac{1}{2}r^2\theta = \frac{1}{2} \times 8.2^2 \times \frac{\pi}{3}$

$= 35.2067\ldots$ cm^2

$\therefore$ Area of sector $COD = 35$ cm^2 (nearest cm^2)

(ii) Area $AOB = \frac{1}{2}r^2\theta = 18.4$

$r^2 = \frac{18.4 \times 2}{\theta} = 36.8 \div \frac{\pi}{3} = 35.1414\ldots$

$\therefore r = 5.928\ldots$ cm

$\therefore$ radius of sector $AOB = 5.9$ cm (2 s.f.)

(iii) Area $\Delta COB = \frac{1}{2}(8.2)(5.9)\sin\frac{\pi}{3}$

$= 21.0486\ldots$ cm^2

$\therefore$ Area of $\Delta COB = 21$ cm^2 (nearest cm^2)

(b) (i)

x	0	0.2	0.4	0.6	0.8	1
$f(x)$	1	**1.12**	**1.48**	**2.08**	**2.92**	4

(ii) $h = \frac{1}{5}$

Trapezoidal rule:

$$\int_0^1 (3x^2+1)\,dx \approx \frac{h}{2}\left[f(0) + 2\{f(0.2) + f(0.4) + f(0.6) + f(0.8)\} + f(1)\right]$$

$$= \frac{1}{10}\left[1 + 2\{1.12 + 1.48 + 2.08 + 2.92\} + 4\right]$$

$= 2.02$.

5 continued

$A = 100\ 000$

) When $t = 20$, $P = 150\ 000$

By substitution into $P = Ae^{kt}$.

$$150000 = 100000e^{20k}$$

$$e^{20k} = 1.5$$

$$20k = \ln 1.5$$

$$k = \frac{\ln 1.5}{20} = 0.02027\ldots$$

$\therefore k = 0.0203$ (3 s.f.)

) When $t = 40$, $P = 100000e^{40k}$

$$= 100000e^{2\ln 1.5}$$

$$= 225\ 000$$

$\therefore$ population that will be present 20 years from now is 225 000.

n 6

$= x^3 + 3x^2 - 9x - 5$

$$\frac{dy}{dx} = 3x^2 + 6x - 9$$

$$= 3(x^2 + 2x - 3).$$

Stationary points when $\frac{dy}{dx} = 0$.

That is, $(x^2 + 2x - 3) = 0$

$$(x+3)(x-1) = 0.$$

$\therefore x = -3$ or $x = 1$

Stationary points are $(-3, 22)$ and $(1, -10)$.

$$\frac{d^2y}{dx^2} = 6x + 6$$

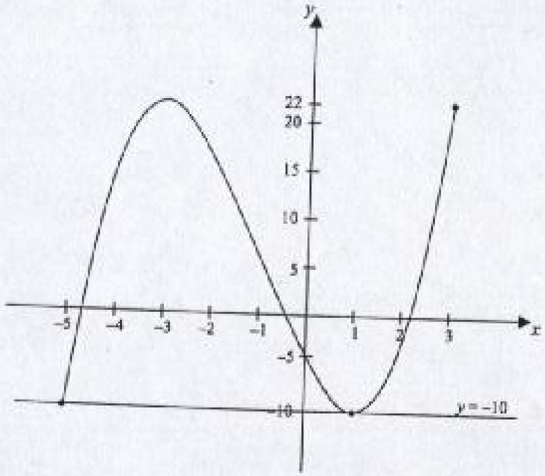
When $x = -3$, $\frac{d^2y}{dx^2} = -18 + 6 < 0$,

$\therefore$ the curve is concave down and $(-3, 22)$ is a relative maximum.

When $x = 1$, $\frac{d^2y}{dx^2} = 6 + 6 > 0$,

$\therefore$ the curve is concave up and $(1, -10)$ is a relative minimum.

(a) (iv)



(v) $x^3 + 3x^2 - 9x + 5 = 0$

when $x^3 + 3x^2 - 9x - 5 = -10$

$\therefore$ by drawing the line $y = -10$ on the graph,

Solutions are $x = -5$ and $x = 1$.

(b) $V = \pi\int_0^1 y^2\,dx \quad = \pi\int_0^1 \left(1 + 2e^{-x}\right)^2 dx$

$$= \pi\int_0^1 \left(1 + 4e^{-x} + 4e^{-2x}\right) dx$$

$$= \pi\left[x - 4e^{-x} - 2e^{-2x}\right]_0^1$$

$$= \pi\left[\left(1 - 4e^{-1} - 2e^{-2}\right) - \left(0 - 4e^{-0} - 2e^{-0}\right)\right]$$

$\therefore$ Volume $= \pi\left(7 - 4e^{-1} - 2e^{-2}\right)$ unit^3.

Question 7

(a) (i) $P(BBB) = \frac{5}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{1}{22}$

(ii) $P(BBB) + P(RRR) = \frac{1}{22} + \left(\frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}\right) = \frac{1}{22} + \frac{1}{55} = \frac{7}{110}$

(iii) $P(B,B,NB) + P(NB,B,B) + P(B,NB,B) = 3\left(\frac{5}{12} \times \frac{4}{11} \times \frac{7}{10}\right) = \frac{7}{22}$

(b) (i) $x = 24.5t - 4.9t^2$

$$\frac{dx}{dt} = v = 24.5 - 9.8t$$

(ii) $v = 24.5 - 9.8t$

Particle comes to rest when $v = 0$,

$0 = 24.5 - 9.8t$

$9.8t = 24.5$

$t = 2.5$ seconds.

$\therefore$ particle comes to rest after 2.5 seconds.

(iii) Greatest height occurs when velocity is zero,

At $t = 2.5$, $\quad x = 24.5(2.5) - 4.9(2.5)^2$

$x = 30.625$

$\frac{d^2x}{dt^2} = -9.8 < 0$ for all t

$\therefore$ the curve is concave down and $(2.5, 30.625)$ is a absolute maximum.

However, if the particle is projected from 2 metres above the ground then greatest height is 32.625 metres.

on 7 continued

(iv) For particle to be at least 21.6 metres above the ground,

$\therefore x = 21.6 - 2 = 19.6$ metres

and $24.5t - 4.9t^2 \geq 19.6$

$5t - t^2 \geq 4$

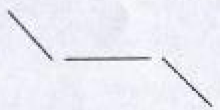
$t^2 - 5t + 4 \leq 0$

$\therefore 1 \leq t \leq 4$ seconds.

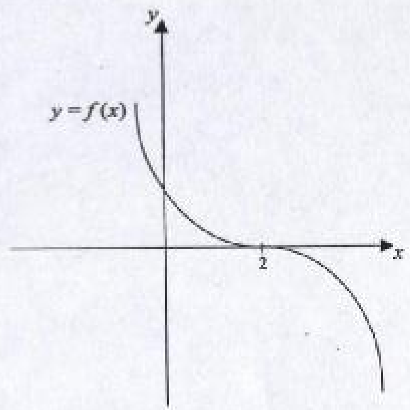
$\therefore$ the particle is at least 21.6 metres above the ground for 3 seconds.

ion 8

x	<2	2	>2
$f'(x)$	Decreasing	Stationary point	Decreasing
$f''(x)$	Concave Up	Point of inflexion	Concave down



Also $f(2) = 0$,



Question 8 continued

(b) (i) $y \leq 4 - x^2, \quad y \geq x^2 - 2x$

(ii) Solving simultaneously,

$x^2 - 2x = 4 - x^2$

$2x^2 - 2x - 4 = 0$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$x = -1$ or $x = 2$.

(iii) Area $= \int_{-1}^{2} (4 - x^2 - x^2 + 2x)\, dx$

$= \int_{-1}^{2} (4 - 2x^2 + 2x)\, dx$

$= \left[4x - \frac{2x^3}{3} + x^2\right]_{-1}^{2}$

$= \left(8 - \frac{16}{3} + 4\right) - \left(-4 + \frac{2}{3} + 1\right)$

$= 9$ square units.

(c) (i) For the first 10 seconds, $\frac{dV}{dt} = \frac{6t}{50}$

$\therefore V = \frac{3t^2}{50} + C$

When $t = 0$, $V = 0$

$\therefore V = \frac{3t^2}{50}, t \leq 10.$

When $t = 10$ seconds, $V = \frac{3(10)^2}{50} = 6$ Litres

After 10 seconds, rate of flow remains constant

and so, $\frac{dV}{dt} = \frac{6(10)}{50} = \frac{6}{5}$ L / sec

$\therefore V = \frac{6t}{5} + C$

When $t = 10$, $V = 6$

$\therefore 6 = \frac{6(10)}{5} + C$

$C = -6$

$\therefore V = \frac{6t}{5} - 6 = \frac{6t - 30}{5} = \frac{6}{5}(t-5).$

(ii) Volume that flows into container while tap is closing is 6 Litres.

$\therefore$ Volume required $= 120 - 6 = 114$ Litres

$\frac{6}{5}(t-5) = 114$

$t - 5 = 95$

$t = 100$ seconds

$\therefore$ tap must remain fully open for 90 seconds.

Question 9

(a) (i) $\angle AED = \angle BCD = 90^\circ$ ($AE \perp BD$ and $ABCD$ is a rectangle)

$\angle ADE = \angle DBC$ (Alternate angles on parallel lines, $AD \parallel BC$)

$\therefore \Delta AED ||| \Delta BCD$ (equiangular).

(ii) $\Delta AED ||| \Delta BCD$.

$\therefore \frac{AD}{BD} = \frac{DE}{BC}$

Now $BC = AD$ (opposite sides of rectangle are equal)

$\therefore \frac{AD}{BD} = \frac{DE}{AD}$

$\therefore AD^2 = BD.DE.$

(iii) $AD = \sqrt{25+4} = \sqrt{29}$ cm

$\therefore BD.DE = 29$

$BD \times 2 = 29$

$BD = 14.5$ cm

$\therefore$ Area $ABCD = 14.5 \times 5 = 72.5 \text{ cm}^2$.

) (i) Surface Area $= 2\pi r^2 + 2\pi rh$.

$54\pi = 2\pi r^2 + 2\pi rh$

$h = \dfrac{54\pi - 2\pi r^2}{2\pi r}$

$\therefore h = \dfrac{27}{r} - r$

i) $V = \pi r^2 h$

$V = \pi r^2\left(\dfrac{27}{r} - r\right)$

$\therefore V = 27\pi r - \pi r^3$.

i) Greatest possible volume V occurs when $\dfrac{dV}{dr} = 0$ and

$V = 27\pi r - \pi r^3$.

$\dfrac{dV}{dr} = 27\pi - 3\pi r^2$

$\dfrac{d^2V}{dr^2} = -6\pi r$

$\dfrac{dV}{dr} = 0, \quad \therefore \quad 27\pi - 3\pi r^2 = 0$

$r^2 = \dfrac{27\pi}{3\pi} = 9$

$r = \pm 3$.

But $r > 0$, so $r = 3$ cm.

When $r = 3$, $\dfrac{d^2V}{dr^2} = -6\pi(3) < 0$.

The value of r that ...

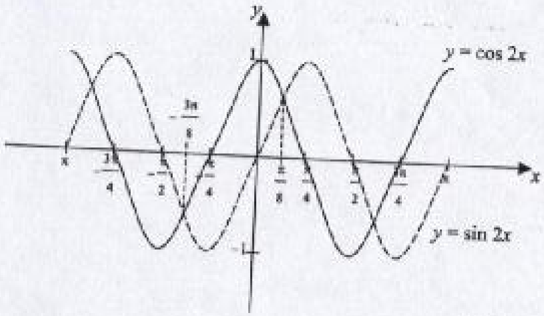
Question 10

(a) (i) LHS: $\sin 2x = \sin\dfrac{2\pi}{8} = \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$.

RHS: $\cos 2x = \cos\dfrac{2\pi}{8} = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} = \text{LHS}$

That is, $\sin 2x = \cos 2x$ when $x = \dfrac{\pi}{8}$.

(ii) Period $= \dfrac{2\pi}{n} = \dfrac{2\pi}{2} = \pi$



(iii) $\tan 2x = 1$ when $\dfrac{\sin 2x}{\cos 2x} = 1$

That is, when $\sin 2x = \cos 2x$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

From the diagram, it can be seen that the curves have two points of intersection for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

Therefore, the equation $\tan 2x = 1$ has **two** solutions for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

(iv) $\tan 2x \le 1$ when $\sin 2x \le \cos 2x$ for $-\dfrac{3\pi}{8} < x < \dfrac{\pi}{8}$.

(b) (i) $A_1 = (250000 \times 1.00505) - M$

$A_2 = [(250000 \times 1.00505) - M] \times 1.00505 - M$

$= 250000 \times 1.00505^2 - M(1 + 1.00505)$

Continuing the pattern

$A_{60} = 250000 \times 1.00505^{60} - M(1 + 1.00505 + \ldots + 1.00505^{59})$

$\therefore A_{60} = 250000 \times 1.00505^{60} - M \times \dfrac{(1.00505^{60} - 1)}{(1.00505 - 1)}$.

(ii) If the loan is to be repaid at the end of 15 years then $A_{180} = 0$.

$\therefore 250000 \times 1.00505^{180} - M \times \dfrac{(1.00505^{180} - 1)}{(1.00505 - 1)} = 0$

$\therefore M = \dfrac{(250000 \times 1.00505^{180}) \times 0.00505}{(1.00505^{180} - 1)}$

$\therefore M = 2117.7545571$

$\therefore$ The monthly repayment is \$2117.75 to the nearest cent.

(iii) Amount still owing after 5 years,

$A_{60} = 250000 \times 1.00505^{60} - 2117.7545571 \times \dfrac{(1.00505^{60} - 1)}{(1.00505 - 1)}$

$\therefore A_{60} = 190236.7605$

$\therefore$ The amount still owing after 5 years is \$190236.76 to the nearest cent.

iv) After 5 years, number of months needed to pay off remainder of loan at interest rate of 7.2% per annum with monthly repayments of \$1800,

$190236.7605 \times 1.006^n = 1800 \times \dfrac{(1.006^n - 1)}{0.006}$

$190236.7605 \times 1.006^n = 300000 \times (1.006^n - 1)$

$1.006^n = \dfrac{300000}{300000 - 190236.7605}$

$n = \dfrac{\ln\left(\dfrac{300000}{300000 - 190236.7605}\right)}{\ln 1.006}$

$n = 168.07836$

$\therefore$ Approximately 169 months are needed to pay off the remainder of the loan.