Solutions to 2005 T2 Yr 12 Ext 1.

Q1 a i) 2C 1A 2U 2L 1S Total 8 letters

$$\frac{8!}{2!\,2!\,2!} = \underline{\underline{5040}}$$

ii) No of arrangements begins with U and ends in U

$$= \frac{6!}{2!\,2!} = 180$$

$$\text{Prob} = \frac{180}{5040} = \underline{\underline{\frac{1}{28}}}$$

b) i) $T = 26 + Ae^{-kt}$

$$\frac{dT}{dt} = -kAe^{-kt}$$

$$= -k(T-26)e^{-kt} \quad (\text{since } A = T-26)$$

ii) $t=0$, $T = 90^\circ$

$$90 = 26 + A$$

$$\underline{A = 64}$$

$$\therefore T = 26 + 64e^{-kt}$$

when $t = 5$. $T = 70^\circ$

$$70 = 26 + 64e^{-5k}$$

$$\frac{44}{64} = e^{-5k}$$

$$-5k = \ln\left(\frac{44}{64}\right)$$

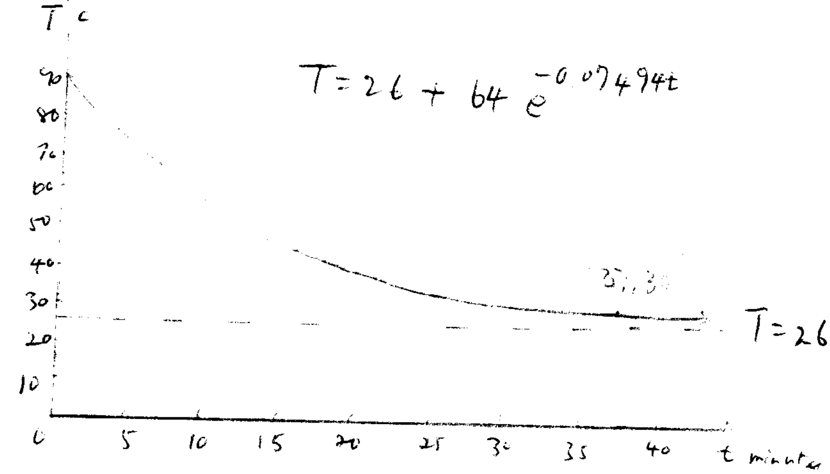
$$\underline{k = 0.07494 \ (4\text{ sf})}$$

iii) $30 = 26 + 64e^{-0.07494t}$

$$\frac{4}{64} = e^{-0.07494t}$$

$$\ln\left(\frac{1}{16}\right) = (-0.07494)t$$

$\underline{t = 37 \text{ min}}$ (nearest min) i b iv



Q 2

a) $x = \dfrac{3t^2}{4+t^3}$

i) $V = \dfrac{(4+t^3)6t - 3t^2(3t^2)}{(4+t^3)^2}$

$= \dfrac{24t + 6t^4 - 9t^4}{(4+t^3)^2}$

$= \dfrac{24t - 3t^4}{(4+t^3)^2}$

ii) $V = 0$ when $24t - 3t^4 = 0$

$8t - t^4 = 0$

$t(8 - t^3) = 0$

$t = 0$ or $t = 2$

$\therefore t = 2$

iii) $t_1 = 1, \quad x_1 = \dfrac{3}{4+1} = \dfrac{3}{5}$

$t_2 = 2 + 2\sqrt{2}, \quad x_2 = \dfrac{3(2+2\sqrt{2})^2}{4 + (2+2\sqrt{2})^3}$

$x_2 = \dfrac{3(4 + 8\sqrt{2} + 8)}{4 + 2^3 + 3(2^2 \cdot 2\sqrt{2}) + 3(2 \times 4 \cdot 2) + 8 \cdot 2\sqrt{2}}$

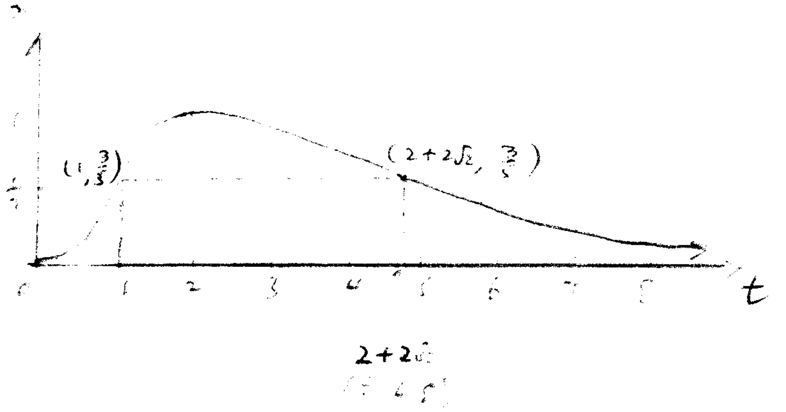
$= \dfrac{3(12 + 8\sqrt{2})}{12 + 24\sqrt{2} + 48 + 16\sqrt{2}}$

$= \dfrac{36 + 24\sqrt{2}}{60 + 40\sqrt{2}}$

At t_2 $\quad x_2 = \dfrac{12(3+2\sqrt{2})}{20(3+2\sqrt{2})} = \dfrac{3}{5}$

$\therefore x_1 = x_2$

iv)



2

2b i) $\ddot{x} = 6 + e^{-t}$ $\qquad \dot{x} = \int 6 + e^{-t}\,dt$

$\dot{x} = 6t - e^{-t} + k_1$

$t=0, \quad \dot{x} = -1 \quad \therefore -1 = -e^{-0} + k_1$

$\therefore k_1 = 0$

$\therefore \dot{x} = 6t - e^{-t}$ $\qquad 1$

$x = \int 6t - e^{-t}\,dt$

$x = 3t^2 + e^{-t} + k_2$

$t=0, \quad x = 0, \qquad 0 = 1 + k_2$

$-1 = k_2$

$\therefore \underline{\underline{x = 3t^2 + e^{-t} - 1}}$ $\qquad 1$

ii) $X(t) - x(t) = 2\sin 5t + \cancel{3t^2} + 2 - (\cancel{3t^2} + e^{-t} - 1)$

$= 3 + 2\sin 5t - e^{-t}$

$\left.\begin{array}{l} \min \sin 5t = -1 \\ \min 2\sin 5t = -2 \end{array}\right\} \quad 3 + 2\sin 5t \geq 1$ $\qquad \frac{1}{2}$

$\max e^{-t} = 1$ for $t \geq 0$ because e^{-t} is a decreasing function and $e^0 = 1$ and $e^{-t} < 1$ for $t > 0$ $\qquad 1$

At $t = 0$ $\qquad X(0) - x(0) = 3 + 0 - 1 = 2$ $\qquad \frac{1}{2}$

$\therefore X(t) - x(t) > 0$ for $t \geq 0$

Q 3

a) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx}$

$$= v\frac{dv}{dx}$$

$$= \frac{dx}{dt} \cdot \frac{dv}{dx}$$

$$= \frac{dv}{dt}$$

$$= \frac{d^2x}{dt^2}$$ #

/

b) i) $\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 4x-4$

$v^2 = 2\int 4x-4\,dx$

$v^2 = 4x^2 - 8x + k$ /

$t=0, \quad x=6, \quad v=-8$

$64 = 4(36) - 48 + k$

$-32 = k$

$\therefore \underline{\underline{v^2 = 4x^2 - 8x - 32}}$ /

ii) $v^2 = 4(x-4)(x+2) \geq 0$

$\therefore \underline{x \geq 4 \text{ or } x \leq -2}$ for motion to exist /

In this case initially $x = 6$ $v=-8$ $\ddot{x} = 20$

The particle starts at 6 m to the right of O, moving to the left, slowing down ($\ddot{x} = 20 > 0$) until it reaches 4 m to the right of O. It stops there momentarily, turn around and move to the right, speeding up forever and never return. 2

3c) Period $= \pi$

$\frac{2\pi}{n} = 2 \Rightarrow n = 2$ |

max $|V| = 12$ at $x = 0$

$v^2 = n^2(a^2 - x^2)$

$144 = 4(a^2 - 0)$

$a = 6$ |

$t = 0$, $x = 3$ moving to the right

$x = 6\sin(2t + \alpha)$

$3 = 6\sin(\alpha)$

$\frac{1}{2} = \sin\alpha$

$\alpha = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ | for rejecting $\frac{5\pi}{6}$

But $\dot{x} = 12\cos(2t + \alpha)$

At $t = 0$, $\dot{x} = 12\cos\frac{\pi}{6} = 6$ (moving to the right)

$\dot{x} = 12\cos\frac{5\pi}{6} = -6$ (moving to the left)

$\therefore \alpha = \frac{\pi}{6}$

$\therefore x = 6\sin(2t + \frac{\pi}{6})$ |

Note, there are other alternative solutions

Q 4 a i) $x = 2\cos^2 t = 1 + \cos 2t$

$x - 1 = \cos 2t$

$\frac{dx}{dt} = -2\sin 2t$ |

$\frac{d^2x}{dt^2} = -4\cos 2t$

$= -2^2(x - 1)$ which is in the form of $\ddot{x} = -n^2(x - b)$ |

$\therefore$ SHM

ii) amplitude $= 1$ m |

Q4 b i) $3!\,5! = 720$

ii) $1 - P(\text{All men}) = 1 - \frac{5}{8} \times \frac{4}{7} = \frac{9}{14}$

c) i) Parents 2 ways

Students as a group $4!$

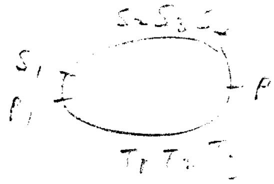
teachers as a group $3!$

Total no. of ways $= 2 \times 4! \times 3! = 288$

ii) $S_1 P_1$ can swap with P

Total no of ways $= 2 \times 3! \times 3! = 72$

Prob $= \frac{72}{288} = \frac{1}{4}$



5a) 10 different digits. Each can be in or out (2 choices)
but at least 1 digit needs to be included (ie exclude all digits out)

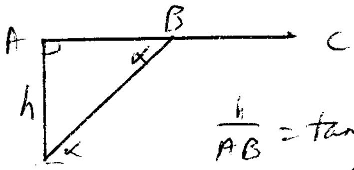
$2^{10} - 1 = \underline{\underline{1023}}$

b) $x = VT\cos\alpha$ horizontal distance travelled by shell from A to C

Horizontal distance travelled by plane from A to C

$= AB + BC$

$= \frac{h}{\tan\alpha} + uT$



From BC
Speed is u, constant
time is T

$\frac{h}{AB} = \tan\alpha$

$AB = \frac{h}{\tan\alpha}$

$$\therefore VT\cos\alpha = \frac{h}{\tan\alpha} + uT$$

5 biii) $V T \cos\alpha = \dfrac{h}{\tan\alpha} + uT$ (from part ii)

Solve for T : $VT\cos\alpha - uT = \dfrac{h}{\tan\alpha}$

$$T(V\cos\alpha - u) = \frac{h}{\tan\alpha}$$

$$T = \frac{h}{\tan\alpha\,(V\cos\alpha - u)}$$

$$y = Vt\sin\alpha - \tfrac{1}{2}gt^2 \quad \text{(from part i)}$$

At c, $t = T$, $y = h$

$$h = VT\sin\alpha - \tfrac{1}{2}gT^2$$

$$h = V\frac{h\sin\alpha}{\tan\alpha\,(V\cos\alpha - u)} - \frac{1}{2}g\frac{h^2}{(\tan\alpha)^2(V\cos\alpha - u)^2}$$

$$1 = \frac{V\cos\alpha}{(V\cos\alpha - u)} - \frac{1}{2}\cdot\frac{gh}{(\tan\alpha)^2(V\cos\alpha - u)^2}$$

$$1 = \frac{2(V\cos\alpha)(V\cos\alpha - u)(\tan\alpha)^2 - gh}{2(\tan\alpha)^2(V\cos\alpha - u)^2}$$

$$2(\tan\alpha)^2(V\cos\alpha - u)^2 - 2V\cos\alpha\,(V\cos\alpha - u)(\tan\alpha)^2 = -gh$$

$$2(V\cos\alpha - u)(\tan\alpha)^2[\cancel{V\cos\alpha} - u - \cancel{V\cos\alpha}] = -gh$$

$$-2u(V\cos\alpha - u)(\tan\alpha)^2 = -gh$$

$$\underline{2u(V\cos\alpha - u)(\tan\alpha)^2 = gh} \quad \text{QED.}$$

5 c) i) $\ddot{x} = \dfrac{d(\frac{1}{2}v^2)}{dx} = \dfrac{d}{dx}\left[\frac{1}{2} \times \left(\frac{4}{x}\right)^2\right] = \dfrac{d}{dx}\left(\frac{8}{x^2}\right) = \dfrac{-16}{x^3}$ /

ii) $\dfrac{dx}{dt} = \dfrac{4}{x}$

$\int x\,dx = \int 4\,dt$

$\dfrac{x^2}{2} = 4t + k$

$t=0,\ x=8 \quad \dfrac{64}{2} = k \quad \therefore k = 32$

$\therefore x^2 = 2(4t + 32)$

$\underline{\underline{x^2 = 8t + 64}}$ /

6 a) $^nC_k + {}^nC_{k+1} = \dfrac{n!}{(n-k)!\,k!} + \dfrac{n!}{(n-k-1)!(k+1)!}$

$= \dfrac{n!(\cancel{k}+1 + n-\cancel{k})}{(n-k)!\,(k+1)!}$ /

$= \dfrac{(n+1)\,n!}{(n-k)!(k+1)!}$

$= \dfrac{n!}{(n-k)!(k+1)!}$ /

$= {}^{n+1}C_{k+1}$

b) $P = \dfrac{AMe^{Mkt}}{Ae^{Mkt}+1}$

$\dfrac{dP}{dt} = \dfrac{(Ae^{Mkt}+1)AM^2ke^{Mkt} - AMe^{Mkt}(AMke^{Mkt})}{(Ae^{Mkt}+1)^2}$ /

$\dfrac{dP}{dt} = \dfrac{A^2M^2\cancel{k}\cancel{e^{Mkt}} + AM^2ke^{Mkt} - A^2M^2\cancel{k}e^{2Mkt}}{(Ae^{Mkt}+1)^2} = \dfrac{AM^2ke^{Mkt}}{(Ae^{Mkt}+1)^2}$

$$kP(M-P) = \frac{k \cdot AMe^{Mkt}}{Ae^{Mkt}+1}\left(M - \frac{AMe^{Mkt}}{Ae^{Mkt}+1}\right)$$

$$= \frac{kAMe^{Mkt}}{Ae^{Mkt}+1}\left(\frac{\cancel{AMe^{Mkt}} + M - \cancel{AMe^{Mkt}}}{Ae^{Mkt}+1}\right)$$

$$kP(M-P) = \frac{kAM^2e^{Mkt}}{(Ae^{Mkt}+1)^2}$$

$$\therefore \quad \underline{\underline{\frac{dP}{dt} = kP(M-P)}} \quad \text{QED}$$

ii) $M = 860 \times 10^6 \quad \rightarrow \quad P = \dfrac{A \times 860 \times 10^6 \times e^{860 \times 10^6 \times kt}}{Ae^{860 \times 10^6 \times kt} + 1}$

(Yr 1790) $t = 0 \qquad P = 4 \times \cancel{10^6} = \dfrac{860 \times \cancel{10^6} A}{A+1}$

$$4(A+1) = 860A$$

$$4 = 856A$$

$$\therefore A = \underline{\frac{1}{214}}$$

(Yr 1800) $t = 10 \qquad P = 6 \times \cancel{10^6} = \dfrac{\frac{1}{214} \times 860 \times \cancel{10^6} \times e^{860 \times 10^6 \times 10k}}{\frac{1}{214}e^{860 \times 10^6 \times 10k} + 1}$

$$\frac{6}{214}e^{860 \times 10^6 \times 10k} + 6 = \frac{1}{214} \times 860 \times e^{860 \times 10^6 \times 10k}$$

$$e^{860 \times 10^6 \times 10k} \cdot \frac{856}{214} = 6$$

$$e^{860 \times 10^6 \times 10 k} = \frac{6 \times 214}{856} = \frac{3}{2}$$

$$860 \times 10^6 \times 10k = \ln\left(\frac{3}{2}\right)$$

$$k = \left(\ln \frac{3}{2}\right) \div (860 \times 10^6 \times 10)$$

$$k = 4.7419 \times 10^{-11} \text{ (5 s.f)}$$

Half of $M = 430$

$$4\cancel{3}0 \times \cancel{10^6} = \frac{\frac{1}{214} \times \overset{2}{\cancel{860}} \times \cancel{10^6} \times e^{860 \times 10^6 \times 4.7419 \times 10^{-11} t}}{\frac{1}{214} e^{860 \times 10^6 \times 4.7419 \times 10^{-11} t} + 1}$$

$$\frac{1}{214} e^{860 \times 4.7419 \times 10^{-5} t} + 1 = \frac{2}{214} e^{860 \times 10^{-5} \times 4.7419 t}$$

$$1 = \frac{1}{214} e^{860 \times 10^{-5} \times 4.7419 t}$$

$$\ln(214) = (860 \times 10^{-5} \times 4.7419) t$$

$$t = \frac{\ln(214)}{860 \times 10^{-5} \times 4.7419}$$

$$t \doteqdot 132$$

i.e. $1790 + 132 = 1922$

<u>Yr 1922</u>

iii) Since $\frac{dP}{dt} = kP(M-P)$ as $P \to M$, $\frac{dP}{dt} \to 0$