Unit Maths

Trial HSC 1998

Solutions

Question 1

a) $(3x-6)-(5-4x)$

$= 3x-6-5+4x$

$= 7x-11$

b) $\dfrac{23.1}{53.6 \times \sqrt{25.04}} = 0.081994821\ldots$

$= 0.0820$ (3 sig. fig.)

c) $\dfrac{d}{dx}\tan\left(\frac{1}{2}x\right) = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$

d) $2-3p<7$

$-3p<5$

$p > -\frac{5}{3}$

e) $126^\circ + T\hat{Q}R = 180^\circ$

$T\hat{Q}R = 54^\circ$

$y^\circ = 54^\circ + 90^\circ$

$= 144^\circ$

f) $\dfrac{x-2}{2} + \dfrac{x+1}{5} = 2$

$5(x-2)+2(x+1) = 2\times 2\times 5$

$5x-10+2x+2 = 20$

$7x-8 = 20$

$7x = 28$

$\therefore x = 4$

Question 2

a) $\sqrt{27}-\sqrt{3}+\sqrt{18}$

$= \sqrt{9\times 3}-\sqrt{3}+\sqrt{9\times 2}$

$= 3\sqrt{3}-\sqrt{3}+3\sqrt{2}$

$= 2\sqrt{3}+3\sqrt{2}$

b) $|x-1| = 2x-1$

$\therefore x-1 = 2x-1$ or $-(x-1) = 2x-1$

$0 = x$ or $-x+1 = 2x-1$

$2 = 3x$

$x = \frac{2}{3}$

c) (i) since L_1 is at 45° to the x-axis then

$m = \tan 45^\circ = 1$

(ii) L_1 : $m = 1$, $A(4,3)$

$y-y_1 = m(x-x_1)$

$y-3 = 1(x-4)$

$y-3 = x-4$

$\therefore 0 = x-y-1$

(iii) sub $y=1$ into L_1

$\therefore 0 = x-1-1$

$\therefore x = 2$

B is $(2,1)$

(iv) if $L_1 \perp L_2$ and $m_1 = 1$

$\therefore m_1 m_2 = -1$ gives $m_2 = -1$

through $A(4,3)$

$\therefore y-y_1 = m(x-x_1)$

$y-3 = -1(x-4)$

$y-3 = -x+4$

(Q2 cont'd...)

(v) ΔABC is isosceles if exactly 2 sides are equal.

L_2 : $x+y-7=0$

sub. $y=1$

$\therefore x+1-7=0$

$x = -6$

$\therefore$ C is $(-6,1)$

$A(4,3)$ $B(2,1)$ $C(-6,1)$

$d_{AB} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$= \sqrt{(2-4)^2+(1-3)^2}$

$= \sqrt{4+4}$

$= \sqrt{8}$

$d_{BC} = \sqrt{(-6-2)^2+(1-1)^2}$

$= \sqrt{8^2+0^2}$

$= 8$

$d_{AC} = \sqrt{(4--6)^2+(3-1)^2}$

$= \sqrt{2^2+2^2}$

$= \sqrt{8}$

$\therefore d_{AB} = d_{AC} \neq d_{BC}$

$\therefore$ isosceles Δ.

Question 3

a) (i) $\dfrac{d}{dx}(4-3x)^6 = 6(4-3x$

$= -18(4-3$

(ii) $\dfrac{d}{dx}\, x^2 e^{2x} = u'v+uv$

$u = x^2$

$u' = 2x$

$v = e^{2x}$

$v' = 2e^{2x}$

$= 2xe^{2x} + x^2$

$= 2xe^{2x}(1+$

(iii) $\dfrac{d}{dx}\,\dfrac{\sin 2x}{x} = \dfrac{u'v-}{v^2}$

$u = \sin 2x$

$u' = 2\cos 2x$

$v = x$

$v' = 1$

$= \dfrac{2\cos 2x \cdot x - \sin}{x^2}$

b) $0.\dot{3}\dot{4} = 0.3434343434$

$= \dfrac{34}{100} + \dfrac{34}{10^4} + \dfrac{34}{10^6}$

(i) $a = \dfrac{34}{100}$ $r = \dfrac{1}{100}$

(ii) $S_\infty = \dfrac{a}{1-r}$

$= \dfrac{34}{100} \div \left(1-\frac{1}{100}\right.$

$= \dfrac{34}{100} \times \dfrac{100}{99}$

$\therefore 0.\dot{3}\dot{4} = \dfrac{34}{99}$

(Q3 cont'd ...)

c) $y = \ln x$

$y' = \frac{1}{x}$

at $x=1$, $y' = \frac{1}{1}$

$= 1$

$\therefore m_t = 1$

$\therefore m_n = -1$

at $x=1$, $y = \ln 1$

$= 0$

$\therefore y - y_1 = m(x - x_1)$

$y - 0 = -1(x-1)$

$y = -x + 1$

$\therefore x + y - 1 = 0$ is the normal.

Question 4

a)(i) $b^2 = c^2 + a^2 - 2ac\cos B$

$= 3{\cdot}31^2 + 4{\cdot}56^2 - 2 \times 3{\cdot}31 \times 4{\cdot}56 \times \cos 46°27'$

$= 31{\cdot}7497 - 20{\cdot}798598$

$\therefore b = 3{\cdot}309244916$

$= 3{\cdot}31$ (2 d.p.)

(ii) $A = \frac{1}{2} ac \sin B$

$= \frac{1}{2} \times 4{\cdot}56 \times 3{\cdot}31 \times \sin 46°27'$

$= 5{\cdot}4697\ldots$

$\therefore A = 5m^2$ (to nearest m^2)

b) $y' = 3x^2 - 6x - 9$ $\quad (1,-2)$

i) $y = \int 3x^2 - 6x - 9\,dx$

$= x^3 - 3x^2 - 9x + c$

through $(1,-2)$

$\therefore -2 = (1)^3 - 3(1)^2 - 9(1) + c$

$-2 = 1 - 3 - 9 + c$

$\therefore c = 9$

$\therefore y = x^3 - 3x^2 - 9x + 9$ is the curve.

ii) stat. pts. when $y' = 0$

$\therefore 0 = 3x^2 - 6x - 9$

$0 = x^2 - 2x - 3$

$0 = (x-3)(x+1)$

$\therefore x = 3$ or $x = -1$

sub. into $y = x^3 - 3x^2 - 9x + 9$

$\therefore (3,-18)$ and $(-1,14)$

(Q4 cont'd ...)

if $y' = 3x^2 - 6x - 9$

then $y'' = 6x - 6$

at $x = 3$, $y'' = 6 \times 3 - 6$

> 0

$\therefore (3,-18)$ is concave up

min.

at $x = -1$, $y'' = 6 \times (-1) - 6$

< 0

$\therefore (-1,14)$ is concave down

max.

(iii) inflexion pts. at $y'' = 0$

$\therefore 0 = 6x - 6$

$\therefore x = 1$

$\therefore y = x^3 - 3x^2 - 9x + 9$

$= 1^3 - 3(1)^2 - 9(1) + 9$

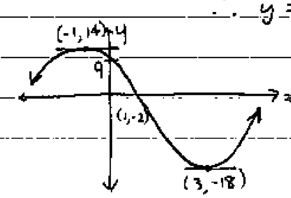
$= -2$

$\therefore$ infl. pt. at $(1,-2)$

ie. curve changes from concave down to concave up at $(1,-2)$.

(iv) y-intercept at $x = 0$

$\therefore y = 9$



as $x \to \infty^+$, $y \to \infty^+$

as $x - 0^-$, $y \to \infty^-$

Question 5

a) 5 function values $\therefore$ 4 strip

$h = \frac{b-a}{n}$

$= \frac{5-1}{4}$

$= 1$

$A \doteqdot \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 +$

$= \frac{1}{3}(0 + 4 \times 1{\cdot}386 + 2 \times 3{\cdot}2$

$+ 4 \times 5{\cdot}545 + 8{\cdot}04$

$= 14{\cdot}121$

$= 14{\cdot}12$ units^2 (2 d.p.)

b) $y = 2x^2 - 4x + 1$

(i) vertex is on axis of symmetry or rearrange the equation.

$\therefore 2x^2 - 4x = y - 1$

$x^2 - 2x = \frac{1}{2}y - \frac{1}{2}$

$x^2 - 2x + 1 = \frac{1}{2}y - \frac{1}{2} + 1$

$(x-1)^2 = \frac{1}{2}y + \frac{1}{2}$

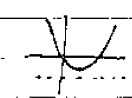
$(x-1)^2 = \frac{1}{2}(y+1)$

$\therefore$ vertex is at $(1,-1)$

(ii) $4a = \frac{1}{2}$

$\therefore a = \frac{1}{8}$

(iii)

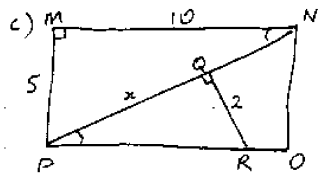


$y = -1 - \frac{1}{8}$

$= -1\frac{1}{8}$

(Q5 cont'd...)

c)



(i) aim: $\triangle PQR \ ||| \ \triangle NMP$

$N\hat{M}P = 90^\circ$ (vertex angle of a rectangle)

$\therefore N\hat{M}P = P\hat{Q}R$

$MN \parallel PO$ (opp. sides of a rectangle)

$\therefore M\hat{N}P = Q\hat{P}R$ (alternate $\angle$'s, $MN \parallel PO$)

$\therefore \triangle PQR \ ||| \ \triangle NMP$ (2 corresp. $\angle$'s equal)

(ii) $\frac{MN}{MP} = \frac{QP}{QR}$ (corresp. sides of similar $\triangle$'s)

$\therefore \frac{10}{5} = \frac{QP}{2}$

$\therefore QP = 4$ cm

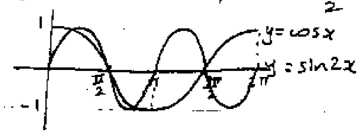
PQ = 4 cm.

Question 6

a) $y = \sin x$ looks like

$\therefore y = \sin 2x$ has period $\frac{2\pi}{2} = \pi$

(ii)



iii) there are 4 solutions since they meet (are equal) 4 times on the graph.

b) i) $x = -5$ at $t = 0$

ii) $x = 0$ when $t = 1, 6, 10$ sec

iii) $v = \frac{dx}{dt}$ $\therefore$ at rest when $\frac{dx}{dt} = 0$

ie. $v = 0$ when $t = 4, 8$ sec.

iv) $a = \frac{d^2x}{dt^2}$ ie. $a = 0$ at the inflexion point

$\therefore a = 0$ at $t \doteqdot 5$ seconds

v) from $t = 0$ to 3 it travelled from $x = -5$ to 4 ie. 9 metres

from $t = 3$ to $t = 8$ it travelled from $x = 3$ to -1 ie. 4 metres

from $t = 8$ to 10 it travelled 1m.

$\therefore$ total distance $= 9 + 4 + 1$

(Q6 cont'd...)

c) \$60 000

$r = \frac{1}{2}\%$ per month

$n = 5 \times 12 = 60$ months

M = monthly repayment

i) After 1 month Bruce owes $= \$60000 \times (1 + \frac{0.5}{100})^1 - M$

$\therefore A_1 = \$60300 - M$

ii) After 2 months Bruces owes $A_2 = A_1 \times 1.005 - M$

$= (\$60000 \times 1.005 - M) \times 1.005 - M$

$\therefore A_2 = \$60000 \times 1.005^2 - 1.005M - M$

$A_3 = A_2 \times 1.005 - M$

$= (\$60000 \times 1.005^2 - 1.005M - M) \times 1.005 - M$

$\therefore A_3 = \$60000 \times 1.005^3 - 1.005^2 M - 1.005M - M$

$= \$[60000(1.005)^3 - M(1.005^2 + 1.005 + 1)]$

iii) $A_n = \$60000 \times 1.005^n - M(1 + 1.005 + 1.005^2 + \ldots + 1.005^{n-1})$

but $n = 60$ months

and $A_{60} = 0$ left to repay

$\therefore 0 = \$6000 \times 1.005^{60} - M(1 + 1.005 + \ldots + 1.005^{59})$

$\therefore M = \frac{\$60\,000 \times 1.005^{60}}{1 + 1.005 + \ldots + 1.005^{59}}$

(Q6 cont'd...)

$1 + 1.005 + \ldots + 1.005^{59}$ is a G.S. with $a = 1$, $r = 1.005$,

$\therefore S_{60} = \frac{a(r^n - 1)}{r - 1}$

$= \frac{1(1.005^{60} - 1)}{1.005 - 1}$

$\therefore M = \frac{\$60\,000 \times 1.005^{60}}{\frac{1.005^{60} - 1}{0.005}}$

M = \$1159.97 (to nearest cen

Question 7

a) $x^2 + (k+3)x - k = 0$

$\Delta = b^2 - 4ac$

$= (k+3)^2 - 4 \times 1 \times (-k)$

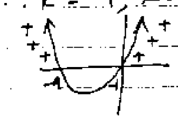
$= k^2 + 10k + 9$

real roots when $\Delta \geq 0$

$\therefore k^2 + 10k + 9 \geq 0$

$(k+9)(k+1) \geq 0$

$\therefore k = -9, k = -1$

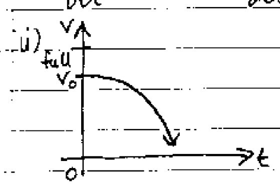


$\therefore k \leq -9$ or $k \geq -1$

b) if V is dec. at an inc. rate

i) $\frac{dV}{dt} < 0$ and $\frac{d^2V}{dt^2} < 0$

ii)



c) $x = 5 + \ln(1+2t)$

i) $v = \frac{dx}{dt}$

$= \frac{d}{dt}(5 + \ln(1+2t))$

$= \frac{2}{1+2t}$

$a = \frac{dv}{dt}$

$= \frac{d}{dt} \frac{2}{1+2t}$

$= -2 \times 2 = -4$

ii) $x = 10$

$\therefore 10 = 5 + \ln(1+2t)$

$5 = \ln(1+2t)$

$e^5 = 1 + 2t$

$\therefore t = \frac{e^5 - 1}{2}$

≈ 73.7 seconds

iii) $v = \frac{2}{1+2t}$

$= \frac{2}{1+e^5-1}$

$= \frac{2}{e^5}$

$\approx 0.013475894\ldots$

$a = \frac{-4}{1+e^5-1}$

$= \frac{-4}{e^5}$

$\approx 1.82 \times 10^{-4}$

iv) since $v = \frac{2}{1+2t}$ and $t \geq 0$

then $v \geq 0$ for all t

similarly $a \geq 0$ for all t

$\therefore$ always moving in a positive direction

$\therefore$ it does not change direction

Question 8

a) 15, 18, 21, 24

A.S. $d = 21 - 18$ $a = 15$

$= 18 - 15$ $n = 52$

$= 3$

i) $u_n = a + (n-1)d$

$u_{52} = 15 + 51 \times 3$

$= 168$ cars

ii) $S_n = \frac{n}{2}(a + l)$

$= \frac{52}{2}(15 + 168)$

$= 4758$ cars

b) $\int_1^3 \frac{x}{x^2+1}\,dx = \left[\frac{1}{2}\ln(x^2+1)\right]_1^3$

$= \frac{1}{2}\ln(3^2+1) - \frac{1}{2}\ln(1^2+1)$

$= \frac{1}{2}\ln 10 - \frac{1}{2}\ln 2$

$\approx 0.804718956\ldots$

c) $y = \tan x$, $y = 2\sin x$

i) $\tan x = \tan\frac{\pi}{3}$

$= \sqrt{3}$

$2\sin x = 2 \times \sin\frac{\pi}{3}$

$= 2 \times \frac{\sqrt{3}}{2}$

$= \sqrt{3}$

$\therefore \tan x = 2\sin x$

for $x = \frac{\pi}{3}$ and $y = \sqrt{3}$

ii) $\frac{d}{dx}(\ln\cos x) = \frac{d}{dx}\cos x \cdot \frac{1}{\cos x}$

$= -\sin x \cdot \frac{1}{\cos x}$

$= -\frac{\sin x}{\cos x}$

(Q8 cont'd...)

iii) $Area = \int_0^{\frac{\pi}{3}} \tan x\,dx - \int_0^{\frac{\pi}{3}} 2\sin x$

$\therefore A = \left[-\ln\cos x\right]_0^{\frac{\pi}{3}} - \left[2\cos x\right]_0^{\frac{\pi}{3}}$

$\therefore A = \left(-\ln\cos\frac{\pi}{3} - -\ln\cos 0\right)$

$- \left(-2\cos\frac{\pi}{3} - -2\cos 0\right)$

$= -\ln\frac{1}{2} + \ln 1 + 2 \times \frac{1}{2} + 2 \times 1$

$= -\ln\frac{1}{2} + 0 + 1 + 2$

$= 3 - \ln\frac{1}{2}$

Question 9

a) $r = 10\text{cm} \quad \theta = \frac{3\pi}{8}$

$\therefore A = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 10^2 \times \frac{3\pi}{8}$

$= \frac{75\pi}{2}\ \text{cm}^2$

b) $d_{1990} = \$50\,000$

$d_{1991} = \$50\,000 \times 75\%$

$= \$37\,500$

$d_{1992} = d_{1991} \times 75\%$

etc.

i) $a = 50\,000$ G.S.

$r = 0.75$

$n = 6$

$u_n = ar^{n-1}$

$u_6 = 50\,000 \times (0.75)^5$

$\therefore d_{1995} = \$11\,865.23$

ii) $S_\infty = \frac{a}{1-r}$

$= \frac{\$50\,000}{1-0.75}$

$= \$200\,000$

c) i) Perimeter = rectangle + circle

$\therefore L = 2 \times \text{length} + 2 \times \text{width} + \frac{1}{4}\text{circle}$

$= 2 \times y + 2 \times 2x + \frac{1}{4} r\theta$

$= 2y + 4x + \frac{1}{4} \times x \times \frac{\pi}{2}$

$\therefore = 2y + 4x + \frac{\pi x}{2}$

but Area = length × width

$\therefore 10000 = y \times 2x$

$\therefore y = \frac{10\,000}{2x}$

(Q9 cont'd...)

hence, $L = \frac{10000}{x} + 4x + \frac{\pi x}{2}$

ii) L is a minimum when $L' = 0$

$\therefore L = 4x + \frac{\pi x}{2} + 10000x^{-1}$

$L' = 4 + \frac{\pi}{2} - \frac{10000}{x^2}$

$\therefore 0 = 4 + \frac{\pi}{2} - \frac{10000}{x^2}$

$\frac{10000}{x^2} = 4 + \frac{\pi}{2}$

$\therefore x^2 = \frac{10\,000}{4+\frac{\pi}{2}}$

$\therefore x = +\sqrt{\frac{20\,000}{9\pi}}$ +ve only since it is a length.

$\approx 42.4\text{m}$

$L'' = +10000x^{-3}$

when $x = 42.4\text{m}$

$\therefore L'' > 0$

$\therefore \cup \therefore$ minimum

Question 10

a) i) $-1 < r < 1$

where $S_\infty = \frac{a}{1-r}$

ii) $S_\infty = \frac{a}{1-r}$

$\therefore \frac{1}{1-\omega} = \frac{a}{1-\frac{1}{\omega}}$

$\frac{1}{1-\omega} = \frac{a}{\frac{\omega-1}{\omega}}$

$\therefore a = \frac{1}{1-\omega} \times \frac{\omega-1}{\omega}$

$= \frac{-(1-\omega)}{\omega(1-\omega)}$

$= \frac{-1}{\omega}$

$\therefore$ G.S. is

$-\frac{1}{\omega}, -\frac{1}{\omega^2}, -\frac{1}{\omega^3}, -\frac{1}{\omega^4}, \ldots$

b) $y = 4x \rightarrow x = \frac{y}{4}$

$y = 5 - x^2 \rightarrow x^2 = 5 - y$

i) $V = \pi \int x^2\, dy$

$\therefore V = \pi \int_0^4 \left(\frac{1}{4}y\right)^2 dy$

$+ \pi \int_0^4 (5-y)\, dy$

$= \pi \left[\frac{1}{48}y^3\right]_0^4$

$+ \pi \left[5y - \frac{1}{2}y^2\right]_0^4$

$= \pi \left\{\frac{1}{48} \times 4^3 - 0\right\}$

$+ \pi \left\{5 \times 5 - \frac{5^2}{2} - 5 \times 4 + \frac{4^2}{2}\right\}$

$= 11\pi$

(Q10 cont'd...)

$\therefore V \approx 5.7596$ units^3

(ii) 1 Litre $\approx 1000\text{cm}^3$

$1000 \div 5.76 \approx 173.6$ icecre

$\therefore$ 173 icecreams can be made

c) i) $S = S_0 e^{-kt}$

$\frac{dS}{dt} = \frac{d}{dt} S_0 e^{-kt}$

$= -k \cdot S_0 e^{-kt}$

$= -kS$

ii) $t = 30$, $S = \frac{1}{2}S_0$

$\therefore \frac{1}{2}S_0 = S_0 e^{-k.30}$

$\frac{1}{2} = e^{-k.30}$

$\ln \frac{1}{2} = \ln e^{-k.30}$

$\ln \frac{1}{2} = -k.30$

$\therefore k = \frac{\ln \frac{1}{2}}{-30}$

iii) $S = 15\% \times S_0$

$= \frac{3}{20} S_0$

$\therefore \frac{3}{20} S_0 = S_0 e^{\frac{1}{30}\ln\frac{1}{2}.t}$

$\frac{3}{20} = e^{\frac{1}{30}\ln\frac{1}{2}.t}$

$\therefore \ln \frac{3}{20} = \ln e^{\frac{1}{30}\ln\frac{1}{2}.t}$

$\ln \frac{3}{20} = \frac{1}{30}\ln\frac{1}{2}.t$

$\therefore t = 30.\frac{\ln\frac{3}{20}}{\ln\frac{1}{2}}$

$\doteqdot 82.1089672$

$\therefore$ fell to 15% after 82 days