



SYDNEY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT
TRIAL EXAMINATIONS 2004

FORM VI

MATHEMATICS EXTENSION 1

Examination date

Tuesday 10th August 2004

Time allowed

2 hours (plus 5 minutes reading time)

Instructions

- All seven questions may be attempted.
- All seven questions are of equal value.
- All necessary working must be shown.
- Marks may not be awarded for careless or badly arranged work.
- Approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Collection

- Write your candidate number clearly on each booklet.
- Hand in the seven questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

- SGS booklets: 7 per boy. A total of 1000 booklets should be sufficient.
- Candidature: 121 boys.

Examiner

MLS

QUESTION ONE (12 marks) Use a separate writing booklet.

Marks

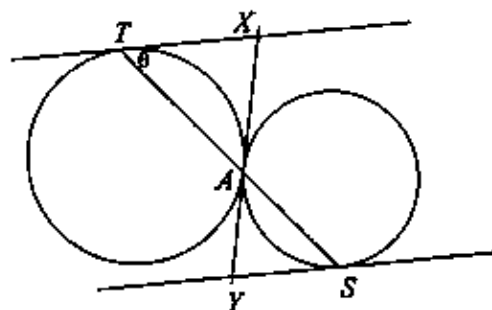
- (a) Solve the inequation $\frac{4}{5-x} \leq 1$. 3
- (b) For what value of p is the expression $4x^3 - x + p$ divisible by $x + 3$? 2
- (c) Expand $(a + \frac{1}{2})^5$, expressing each term in its simplest form. 2
- (d) Given the points $A(1, 4)$ and $B(5, 2)$, find the co-ordinates of the point that divides the interval AB externally in the ratio $1 : 3$. 2
- (e) Find $\int x(1-x^2)^5 dx$, using the substitution $u = 1 - x^2$, or otherwise. 3

QUESTION TWO (12 marks) Use a separate writing booklet.

Marks

- (a) Consider the parabola $x = 4t$, $y = 2t^2$.
- (i) Find the gradient of the parabola at the point where $t = 4$. 1
- (ii) Find the equation of the tangent to the parabola at $t = 4$. 2

(b)



In the diagram above, two circles touch one another externally at the point A . A straight line through A meets one of the circles at T and the other at S . The tangents at T and S meet the common tangent at A at X and Y respectively.

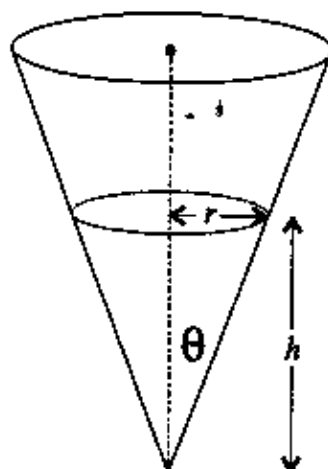
Let $\theta = \angle XTA$.

- (i) Explain why $\angle XAT$ is θ . 1
- (ii) Prove that $TX \parallel YS$. 2
- (c) (i) Write down the first three terms in the expansion of $(1 + mx)^n$. 1
- (ii) If $(1 + mx)^n \equiv 1 - 4x + 7x^2 - \dots$, find the values of m and n . 3
- (d) Evaluate $\lim_{x \rightarrow 0} \frac{5x \cos 2x}{\sin x}$, showing your reasoning. 2

QUESTION THREE (12 marks) Use a separate writing booklet.

Marks

(a)



The diagram above shows a container in the shape of a right circular cone. The semi-vertical angle $\theta = \tan^{-1} \frac{1}{2}$.

Water is poured in at the constant rate of 10 cm^3 per minute.

Let the height of the water at time t seconds be $h \text{ cm}$, let the radius of the water surface be $r \text{ cm}$, and let the volume of water be $V \text{ cm}^3$.

(i) Show that $r = \frac{1}{2}h$.

1

(ii) Show that $V = \frac{1}{12}\pi h^3$.

1

(iii) Find the exact rate at which h is increasing when the height of the water in the cone is 50 cm .

2

(b) Show that there is no term independent of x in the expansion of $\left(2x^2 - \frac{1}{4x}\right)^{11}$.

3

(c) Evaluate $\int_{-1}^0 x\sqrt{1+x} \, dx$, using the substitution $u = 1+x$.

4

(d) Find $\int \sin x \cos^3 x \, dx$.

1

QUESTION FOUR (12 marks) Use a separate writing booklet.

Marks

- (a) If $y = \frac{1}{200}te^{-t}$, show that $\frac{dy}{dt} = \frac{1}{200}(1-t)e^{-t}$. 1

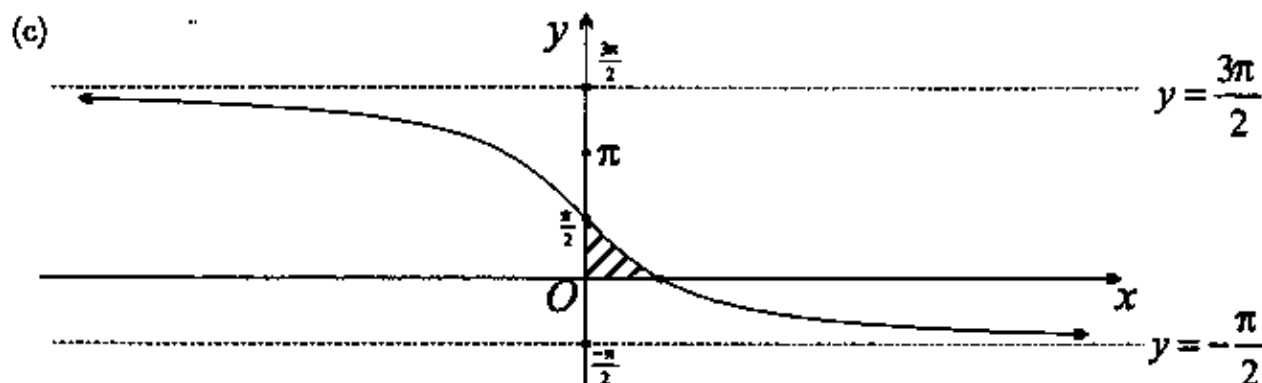
- (b) Fred has recently consumed three standard alcoholic drinks. Immediately after he has finished his last drink, his blood alcohol level is measured over a four-hour period.

Let his blood alcohol level at any time t be A , where t is the time in hours after his last drink.

It is found that the rate of change $\frac{dA}{dt}$ of his blood alcohol content is given by

$$\frac{dA}{dt} = \frac{1}{200}(1-t)e^{-t}, \text{ where } 0 \leq t \leq 4.$$

- (i) Show that his blood alcohol content increases during the first hour and decreases after the first hour. 2
- (ii) Initially his blood alcohol content was 0.0005. Find A as a function of t . You will need to use part (a). 2
- (iii) Determine his maximum alcohol content during the four-hour period. Give your answer correct to four decimal places. 1



The graph of the curve $y = \frac{\pi}{2} - 2 \tan^{-1} x$ is drawn above. It cuts the y -axis at $(0, \frac{\pi}{2})$.

- (i) Write down the domain of the inverse function of $y = \frac{\pi}{2} - 2 \tan^{-1} x$. 1
- (ii) Find the equation of the inverse function of $y = \frac{\pi}{2} - 2 \tan^{-1} x$. 1
- (iii) Find the volume generated when the shaded region is rotated about the y -axis. 4

QUESTION FIVE (12 marks) Use a separate writing booklet.

Marks

(a) Evaluate $\int_0^4 \frac{1}{3+\sqrt{x}} dx$, using the substitution $x = (u-3)^2$. 3

(b) (i) Write down the expansion of $(1+x)^n$ in ascending powers of x . Then differentiate both sides of your identity. 1

(ii) Make an appropriate substitution for x to show that 1

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + 4\binom{n}{4} + \dots + n\binom{n}{n} = n(2^{n-1}).$$

(iii) Hence find an expression for 1

$$2\binom{n}{1} + 3\binom{n}{2} + 4\binom{n}{3} + 5\binom{n}{4} + \dots + (n+1)\binom{n}{n}.$$

(c) Find values for R and α if $\sqrt{3}\sin\theta - \cos\theta = R\cos(\theta + \alpha)$, where R and α are positive constants and $0 < \alpha < 2\pi$. 2

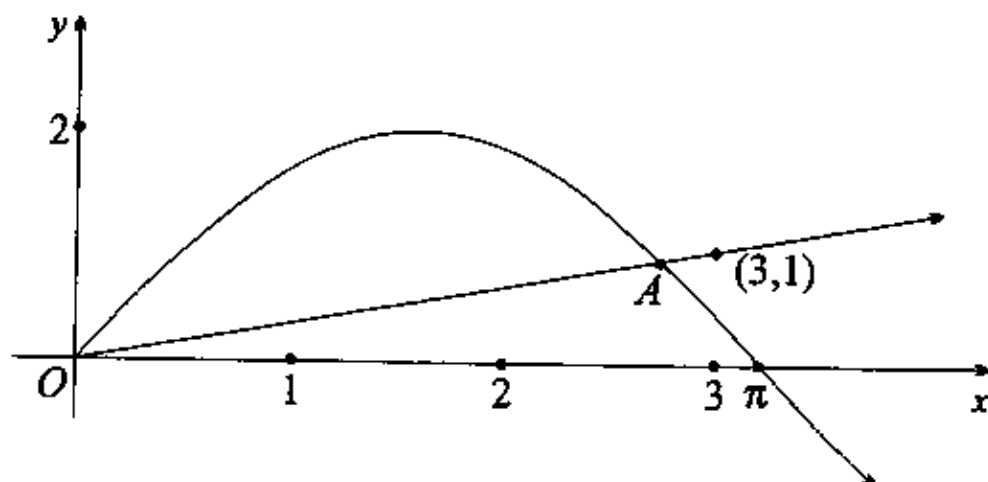
(d) Use the method of mathematical induction to prove that 4

$$\frac{1}{2!} + \frac{2}{3!} + \dots + \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}, \text{ for all positive integers } n.$$

QUESTION SIX (12 marks) Use a separate writing booklet.

Marks

(a)



The sketch above shows the curve $y = 2\sin x$ and the line $x - 3y = 0$. The graphs meet at the point A in the first quadrant.

(i) Write down an equation whose solution gives the x -coordinate of A . 1

(ii) An approximate value for the x -coordinate of A is $x = 3$. Apply Newton's method once to find a closer approximation for this value. Give your answer correct to one decimal place. 2

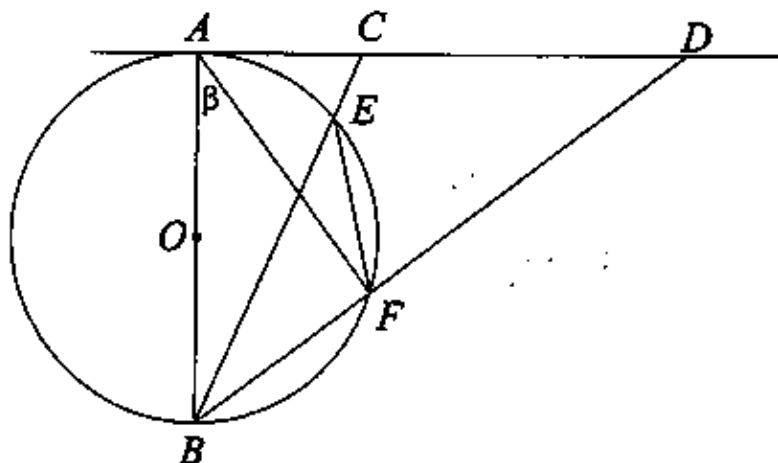
- (b) Newton's law of cooling states that a body cools according to the equation

$$\frac{dT}{dt} = -k(T - S),$$

where T is the temperature of the body at time t , S is the temperature of the surroundings and k is a constant.

- (i) Show that $T = S + Ae^{-kt}$ satisfies the equation, where A is a constant. 1
- (ii) A metal rod has an initial temperature of 470°C and cools to 250°C in 10 minutes. The surrounding temperature is 30°C .
 - (a) Find the value of A and show that $k = \frac{1}{10} \log_e 2$. 2
 - (b) Find how much longer it will take the rod to cool to 70°C , giving your answer correct to the nearest minute. 2

(c)



In the diagram above, the straight line ACD is a tangent at A to the circle with centre O . The interval AOB is a diameter of the circle. The intervals BC and BD meet the circle at E and F respectively.

Let $\angle BAF = \beta$.

Copy or trace this diagram into your answer booklet.

- (i) Explain why $\angle ABF = \frac{\pi}{2} - \beta$. 1
- (ii) Prove that the quadrilateral $CDFE$ is cyclic. 3

QUESTION SEVEN (12 marks) Use a separate writing booklet.

Marks

- (a) Car A and car B are travelling along a straight level road at constant speeds V_A and V_B respectively. Car A is behind car B , but is travelling faster.

When car A is exactly D metres behind car B , car A applies its brakes, producing a constant deceleration of $k \text{ m/s}^2$.

- (i) Using calculus, find the speed of car A after it has travelled a distance x metres under braking. 2

- (ii) Prove that the cars will collide if $V_A - V_B > \sqrt{2kD}$. 4

- (b) A particle is moving in simple harmonic motion of period T about a centre O . Its displacement at any time t is given by $x = a \sin nt$, where a is the amplitude.

- (i) Draw a neat sketch of one period of this displacement-time equation, showing all intercepts. 1

- (ii) Show that $\dot{x} = \frac{2\pi a}{T} \cos \frac{2\pi t}{T}$. 1

- (iii) The point P lies D units on the positive side of O . Let V be the velocity of the particle when it first passes through P . 4

Show that the time between the first two occasions when the particle passes through P is $\frac{T}{\pi} \tan^{-1} \frac{VT}{2\pi D}$.

END OF EXAMINATION