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Pymble Ladies' College
Mathematics Department

7 (b) (iii)
(cont'd) Also, $\int_0^1 \frac{x}{(1+x)(1+x^2)}\,dx = \int_0^1 \left(-\frac{1}{2(x+1)} + \frac{1+x}{2(x^2+1)}\right)dx$ (partial fractions)

$$= \left[-\frac{1}{2}\log_e(1+x)\right]_0^1 + \left[\frac{1}{4}\log_e(x^2+1)\right]_0^1 + \left[\frac{1}{2}\tan^{-1}x\right]_0^1 \quad ✓$$

$$= -\frac{1}{2}\log_e 2 + \frac{1}{4}\log_e 2 + \frac{1}{2}\tan^{-1}1$$

$$= \frac{\pi}{8} - \frac{1}{4}\log_e 2$$

Hence $\frac{\pi}{8} - \frac{1}{4}\log_e 2 < \int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx < \frac{1}{2}\log_e 2$ for all $x > 0$. ✓

QUESTION 8

(a) (i) α. Number of arrangements when there are no restrictions $= 11!$ ✓
$= 39916800$

β. The males and females are in alternate positions.
Sit a person down. There are $5!$ ways of seating the remaining members of the same sex.
Then there are $6!$ ways of seating the opposite sex.
So the total number of ways $= 5! \times 6!$ ways. ✓

(ii) Two cases:

(1) If one state has two representatives, number of ways $= \binom{6}{4} \times 2^5 = 480$ ✓

(2) If no state has two representatives, number of ways $= 2^6 = 64$ ✓

Hence total number of ways $= 480 + 64 = 544$

(b)

Criteria	Marks
(i) • one mark for answer	1
(ii) • one mark for replacing a, b, c by $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ respectively • one mark for final answer	2
(iii) • one mark for use of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$ • one mark for final answer	2

(i)

$\sqrt[3]{abc} \leq \frac{a+b+c}{3} = \frac{1}{3}$

$abc \leq \frac{1}{27}$

$\frac{1}{abc} \geq 27$

(ii)

$\frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq \sqrt[3]{\left(\frac{1}{a}\right)\left(\frac{1}{b}\right)\left(\frac{1}{c}\right)}$

$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3\sqrt[3]{\frac{1}{abc}}$

$\geq 3\sqrt[3]{27}$

$\therefore \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$

(iii)

$(1-a)(1-b)(1-c)$

$= 1 - (a+b+c) + (bc+ca+ab) - abc$

$= (bc+ca+ab) - abc$

$= abc\left\{\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 1\right\}$

$\geq abc\,(9-1)$

$\therefore (1-a)(1-b)(1-c) \geq 8\,abc$

8 (c) (i) $A_0B_1 = x$

$A_1B_1 = \sin x$ $\qquad OA_1 = OB_2 = \cos x$

$A_1B_2 = x\cos x$

$B_2A_2 = OB_2 \sin x = \sin x\cos x$ $\qquad OA_2 = OB_3 = OB_2 \cos x = \cos^2 x$

$A_2B_3 = OA_2 \cdot x = x\cos^2 x$

$B_3A_3 = OB_3 \sin x = \sin x\cos^2 x$

$y = A_0B_1 + A_1B_1 + A_1B_2 + B_2A_2 + A_2B_3 + B_3A_3 + \ldots$

$= x + \sin x + x\cos x + \sin x\cos x + x\cos^2 x + \sin x\cos^2 x + \ldots$

$= x(1 + \cos x + \cos^2 x + \ldots) + \sin x(1 + \cos x + \cos^2 x + \ldots)$ — 1 mark for setting up the series

$= (x + \sin x)(1 + \cos x + \cos^2 x + \ldots)$

$= \frac{x + \sin x}{1 - \cos x}$

1 mark for recognis. the infinite G.P.

(ii) $\frac{dy}{dx} = \frac{(1+\cos x)(1-\cos x) - (x + \sin x)\sin x}{(1-\cos x)^2}$

$= \frac{1 - \cos^2 x - x\sin x - \sin^2 x}{(1-\cos x)^2}$

$= \frac{\sin^2 x - x\sin x - \sin^2 x}{(1-\cos x)^2}$

$= -\frac{x\sin x}{(1-\cos x)^2}$ — 1 mark for derivative.

Since $0 < x \leq \frac{\pi}{2}$, $\sin x > 0$ & $(1-\cos x)^2 > 0$

$\therefore \frac{dy}{dx} = -\frac{x\sin x}{(1-\cos x)^2} < 0$ for all $0 < x \leq \frac{\pi}{2}$ — 1 mark for explaining why $\frac{dy}{dx} < 0$

(iii) Since $\frac{dy}{dx} < 0$ in $0 < x \leq \frac{\pi}{2}$ $\therefore y$ is a decreasing function in $0 < x \leq \frac{\pi}{2}$ — 1 mark

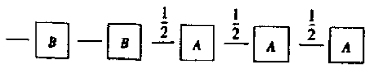
$\therefore$ Absolute min. value of y occurs at the end-pt $x = \frac{\pi}{2}$

When $x = \frac{\pi}{2}$, $y = \frac{\frac{\pi}{2} + \sin\frac{\pi}{2}}{1 - \cos\frac{\pi}{2}} = \frac{\pi}{2} + 1$ — 1 mark

(9)

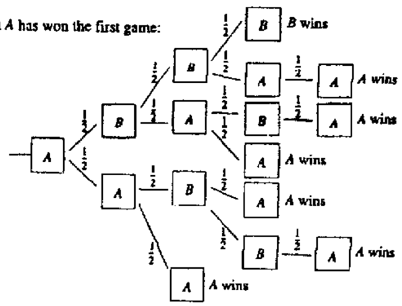
6 (c) The probability that A wins any game is $\frac{1}{2}$.

(i) If team B wins the first two games:



$P(A \text{ wins}) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8}$ ✓

(ii) If team A has won the first game:

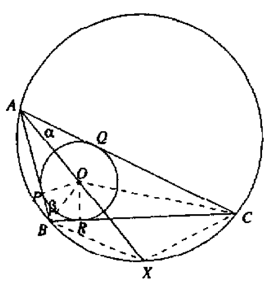


$P(A \text{ wins}) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}$ ✓

$= \frac{1}{16}+\frac{1}{16}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{4} = \frac{11}{16}$ ✓

QUESTION 7

(a)



(i) $\therefore \angle CBX = \angle CAX$ (angles on the same arc CX) ✓

(ii) In the triangles POB and ROB:

1. $OB = OB$ (common)
2. $OP = OR$ (radii)
3. $\angle OPB = \angle ORB = 90°$ (radius and tangent)

so $\Delta POB \equiv \Delta ROB$ (RHS) ✓

Hence $\angle OBA = \angle OBC$ (corresponding angles of congruent triangles) ✓

(iii) Let $\angle OBA = \angle OBC = \beta$ and $\angle CAX = \angle CBX = \alpha$.

Then by a similar proof to (ii), $\angle BAX = \alpha$. ✓

Hence $\angle BOX = \alpha + \beta$ (*exterior angle of* ΔABO). ✓

But $\angle OBX = \alpha + \beta$ (adjacent angles),

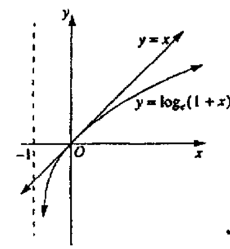
so $BX = OX$ (opposite angles in ΔOBX are equal). ✓

(iv) Similarly, $CX = OX$.

Hence $BX = CX$. ✓

7 (b) (i) α. $y = \log_e(1+x)$

$\frac{dy}{dx} = \frac{1}{1+x}$



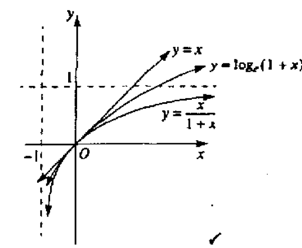
✓

β. When $x = 0$, $\frac{dy}{dx} = 1$, so $y = x$ is a tangent at $(0, 0)$.

Since $y = \log_e(1+x)$ is concave down, it follows that its graph is below the line $y = x$ for $x > 0$. ✓

(ii) α. $y = \frac{x}{1+x}$

Using the quotient rule, $\frac{dy}{dx} = \frac{1}{(1+x)^2}$.



✓

β. When $x = 0$, $\frac{dy}{dx} = 1$, so $y = x$ is a tangent to both curves at $(0, 0)$.

But for $x > 0$, the gradient function of $y = \frac{x}{1+x}$ is less than the gradient function of $y = \log_e(1+x)$, because $\frac{1}{(1+x)^2} < \frac{1}{1+x}$ for $x > 0$.

Hence the graph of $y = \frac{x}{1+x}$ is always below the graph of $y = \log_e(1+x)$ for $x > 0$. ✓

(iii) From (i) and (ii), $\frac{x}{1+x} < \log_e(1+x) < x$ for all $x > 0$.

Hence $\frac{x}{(1+x)(1+x^2)} < \frac{\log_e(1+x)}{1+x^2} < \frac{x}{1+x^2}$ for all $x > 0$

and so $\int_0^1 \frac{x}{(1+x)(1+x^2)}\,dx < \int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx < \int_0^1 \frac{x}{1+x^2}\,dx$ for all $x > 0$. ✓

Now $\int_0^1 \frac{x}{1+x^2}\,dx = \left[\frac{1}{2}\log_e(x^2+1)\right]_0^1$

$= \frac{1}{2}\log_e 2$ ✓

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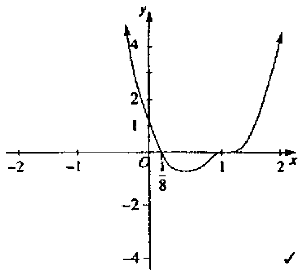
QUESTION 5

(a) (iii)



✓

5(b)

Marking Guidelines

Criteria	Marks
(i) • one mark for general solution • one mark for particular solution	2
(ii) • one mark for expression for $\operatorname{Re}(\cos\theta + i\sin\theta)^5$ in terms of $\cos\theta$, $\sin\theta$ • one mark for expression for $\operatorname{Re}(\cos\theta + i\sin\theta)^5$ in terms of $\cos\theta$ • one mark for final answer	3
(iii) • one mark for noting that $x = \cos\theta$ where $\cos 5\theta = -1$ • one mark for solution	2
(iv) • one mark for value of $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ • one mark for value of $\cos\frac{\pi}{5}.\cos\frac{3\pi}{5}$ • one mark for factorisation	3

Answer

(i)

$\cos 5\theta = -1 \Rightarrow 5\theta = (2n+1)\pi$

$\theta = (2n+1)\frac{\pi}{5}$, $n = 0, \pm 1, \pm 2, \dots$

$0 \le \theta \le 2\pi \Rightarrow \theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$

(ii) Using the binomial expansion,

$\operatorname{Re}\left\{(\cos\theta + i\sin\theta)^5\right\}$

$= \cos^5\theta + 10\cos^3\theta\,(i\sin\theta)^2 + 5\cos\theta\,(i\sin\theta)^4$

$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta\,(1-\cos^2\theta) + 5\cos\theta\,(1-\cos^2\theta)^2$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

Using De Moivre's Theorem,

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

Hence $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(iii) $16x^5 - 20x^3 + 5x + 1 = 0$

has solutions $x = \cos\theta$ where $\cos 5\theta = -1$.

$x = \cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, \cos\pi, \cos\frac{7\pi}{5}, \cos\frac{9\pi}{5}$

$x = \cos\frac{\pi}{5}, \cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, \cos\frac{3\pi}{5}, -1$

(iv)

$\sum\alpha = 0 \Rightarrow 2\left(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}\right) - 1 = 0$

$\therefore \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$

Product of roots is $-\frac{1}{16}$

$\therefore -\left(\cos\frac{\pi}{5}.\cos\frac{3\pi}{5}\right)^2 = -\frac{1}{16}$

$\therefore \cos\frac{\pi}{5}.\cos\frac{3\pi}{5} = -\frac{1}{4}$

$\left(\text{since } \cos\frac{\pi}{5} > 0,\ \cos\frac{3\pi}{5} < 0\right)$

Then $\cos\frac{\pi}{5}$, $\cos\frac{3\pi}{5}$ are roots of the equation $4x^2 - 2x - 1 = 0$. Hence

$16x^5 - 20x^3 + 5x + 1 = (x+1)(4x^2 - 2x - 1)^2$

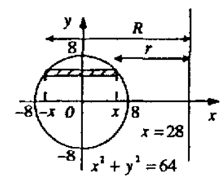
Question 6

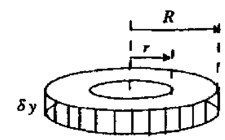
Marking Guidelines

(a)

Criteria	Marks
(i) • one mark for identifying slice as annular prism, thickness δy • one mark for inner radius r in terms of y • one mark for outer radius R in terms of y • one mark for simplified value of δV in terms of y • one mark for expression for V	5
(ii) • one mark for using area of semi circle, or appropriate integration process • one mark for final answer	2

(i)





$R = 28 + \sqrt{64 - y^2}$

$r = 28 - \sqrt{64 - y^2}$

Volume of slice is

$\delta V = \pi\left(R^2 - r^2\right)\delta y$

$= \pi(R+r)(R-r)\delta y$

$= \pi.56.2\sqrt{64-y^2}.\delta y$

$V = \lim_{\delta y \to 0} \sum_{y=-8}^{8} 112\pi\sqrt{64-y^2}.\delta y$

$= 112\pi \int_{-8}^{8} \sqrt{64-y^2}\ dy$

(ii) $\int_{-8}^{8} \sqrt{64-y^2}\ dy = \frac{1}{2}\pi.8^2 = 32\pi$ (*Area of semicircle radius 8*) $\Rightarrow V = 3584\pi^2$

Exact volume of lifebelt is $3584\pi^2\ \text{cm}^3$

(b) (i) $\text{RHS} = t^{n-2} - \frac{t^{n-2}}{1+t^2}$

$= \frac{(1+t^2)t^{n-2} - t^{n-2}}{1+t^2}$

$= \frac{t^{n-2} + t^n - t^{n-2}}{1+t^2}$

$= \frac{t^n}{1+t^2}$

$= \text{LHS}$ ✓

(ii) $I_n = \int \frac{t^n}{1+t^2}dt$

$= \int\left(t^{n-2} - \frac{t^{n-2}}{1+t^2}\right)dt$

$= \frac{t^{n-1}}{n-1} - \int \frac{t^{n-2}}{1+t^2}dt$

$= \frac{t^{n-1}}{n-1} - I_{n-2}$ ✓

(iii) Let $J_n = \int_0^1 \frac{t^n}{1+t^2}dt$

Then $J_n = \left[\frac{t^{n-1}}{n-1}\right]_0^1 - J_{n-2}$

$= \frac{1}{n-1} - J_{n-2}$ ✓

Hence $J_6 = \frac{1}{5} - J_4$

$= \frac{1}{5} - \frac{1}{3} + J_2$

$= \frac{1}{5} - \frac{1}{3} + 1 - J_0$ ✓

But $J_0 = \int_0^1 \frac{1}{1+t^2}dt$

$= \left[\tan^{-1}t\right]_0^1 = \frac{\pi}{4}$

Hence $J_6 = \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}$

$= \frac{13}{15} - \frac{\pi}{4}$ ✓

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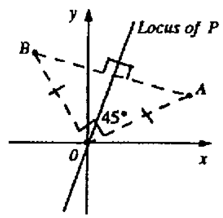
3(c)

Marking Guidelines

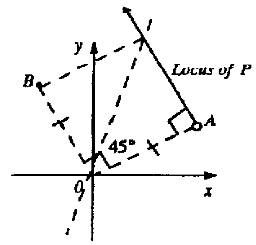
Criteria	Marks
(i) • one mark for answer	1
(ii) • one mark for answer	1
(iii) • one mark for answer	1

Let $z = x + iy$, x, y real

(i) Locus of P is perpendicular bisector of AB.



(ii) Locus is ray from A parallel to $\overrightarrow{OB}$



(iii) If P is the point of intersection of these loci, $OAPB$ is a square and the diagonal $\overrightarrow{OP}$ represents the sum of α and $i\alpha$. Hence P represents $(1+i)\alpha$.

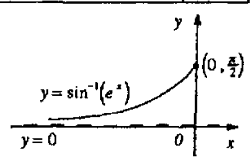
(d)

Criteria	Marks
(i) • one mark for domain • one mark for range	2
(ii) • one mark for coordinates of endpoint and equation of asymptote • one mark for graph	2

(i) $y = \sin^{-1}(e^x)$

Domain	Range
$-1 \le e^x \le 1$	$0 < e^x \le 1$
$0 < e^x \le 1$	$0 < \sin^{-1}(e^x) \le \frac{\pi}{2}$
$\{x : x \le 0\}$	$\{y : 0 < y \le \frac{\pi}{2}\}$

(ii)



Question 4

(a)

Marking Guidelines

Criteria	Marks
(i) • one mark for eccentricity	1
(ii) • one mark for coordinates of foci • one mark for equations of directrices	2
(iii) • one mark for graph with intercepts • one mark for showing foci and directrices	2

(i)

$\frac{x^2}{16} + \frac{y^2}{12} = 1$

$12 = 16(1 - e^2)$

$e^2 = \frac{1}{4}$, $e = \frac{1}{2}$

(ii)

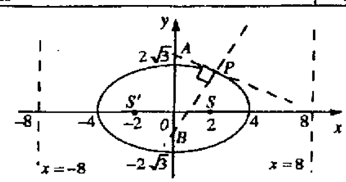
foci $(\pm ae, 0)$

$S'(-2, 0)$, $S(2, 0)$

directrices $x = \pm\frac{a}{e}$

$x = -8$, $x = 8$

(iii)



4 (b)

Marking Guidelines

Criteria	Marks
(i) • one mark for expression $\frac{dy}{dx} = -\frac{3x}{4y}$ • one mark for equation of tangent • one mark for equation of normal	3
(ii) • one mark for coordinates of A and B	1

(i)

$\frac{x^2}{16} + \frac{y^2}{12} = 1 \Rightarrow \frac{2x}{16} + \frac{2y}{12}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x}{8} \div \frac{y}{6} = -\frac{3x}{4y}$

at $P(2, 3)$ $\frac{dy}{dx} = -\frac{1}{2}$

(ii)

Tangent at $P(2, 3)$ has gradient $-\frac{1}{2}$ and equation

$y - 3 = -\frac{1}{2}(x - 2) \Rightarrow x + 2y - 8 = 0 \Rightarrow A(0, 4)$

Normal at $P(2, 3)$ has gradient 2 and equation

$y - 3 = 2(x - 2) \Rightarrow 2x - y - 1 = 0 \Rightarrow B(0, -1)$

(iii)

4(c)

Criteria	Marks
(i) • one mark for the gradients of AS and BS • one mark for showing $AS \perp BS$	2
(ii) • one mark for showing points A, P, S and B are concyclic	1
(iii) • one mark for noting AB is diameter • one mark for centre of circle • one mark for radius of circle	3

(i)

$S(2, 0)$ $A(0, 4)$ $B(0, -1)$

$grad\ AS \cdot grad\ BS = -2 \times \frac{1}{2} = -1$

$\therefore A\hat{S}B = 90°$

(ii)

$A\hat{P}B = 90°$ (*tangent* $\perp$ *normal*)

AB subtends equal angles of 90° at P and S.

$\therefore A, P, S, B$ are concyclic

(iii)

Diameter AB

centre $\left(0, \frac{3}{2}\right)$

radius $\frac{5}{2}$

QUESTION 5

(a) (i) For $P(x)$ to have a zero with multiplicity of 3, we can write $P(x)$ as follows:

$P(x) = (x - \alpha)^3 Q(x)$, where $Q(\alpha) \ne 0$ ✓

Differentiating, $P'(x) = (x - \alpha)^3 Q'(x) + 3(x - \alpha)^2 Q(x)$ (product rule)

$= (x - \alpha)^2[(x - \alpha)Q'(x) + 3Q(x)]$

$= (x - \alpha)^2 R(x)$, where $R(\alpha) = 3Q(\alpha) \ne 0$ ✓

So $P'(x)$ has a zero of multiplicity 2.

(ii) Let $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$ and let $x = \alpha$ be the zero of multiplicity 3.

Differentiating, $P'(x) = 32x^3 - 75x^2 + 54x - 11$

and $P''(x) = 96x^2 - 150x^2 + 54$

$= 6(16x^2 - 25x^2 + 9)$

$= 6(x - 1)(16x - 9)$

So the zeros of $P''(x)$ are $x = 1$ and $x = \frac{9}{16}$. ✓

Testing $x = 1$, $P(1) = 0$ and $P'(1) = 0$, so $P(x) = (x - 1)^3 Q(x)$

Let $x = \beta$ be the other zero.

Then $\alpha + \alpha + \alpha + \beta = \frac{25}{8}$

$\beta = \frac{25}{8} - 3$

$= \frac{1}{8}$

So the zeroes of $8x^4 - 25x^3 + 27x^2 - 11x + 1$ are $x = 1, 1, 1, \frac{1}{8}$ ✓

(13)

QUESTION 2

(b) $-1-i\sqrt{3}=2\text{cis}\left(-\frac{2\pi}{3}\right)$ ✓

Hence $\left(2\text{cis}\left(-\frac{2\pi}{3}\right)\right)^{-10}=2^{-10}\text{cis}\left(\frac{20\pi}{3}\right)$ ✓

$=2^{-10}\text{cis}\left(\frac{2\pi}{3}\right)$

$=\frac{1}{1024}\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$

$=-\frac{1}{2048}+\frac{i\sqrt{3}}{2048}$ ✓

(c)

$(a+ib)^2=5-12i \Rightarrow (a^2-b^2)+2ab\,i=5-12i$

$\therefore a^2-b^2=5$ and $ab=-6$

$a^4-a^2b^2=5a^2 \Rightarrow a^4-5a^2-36=0$

$(a^2+4)(a^2-9)=0 \quad \therefore a^2>0 \Rightarrow a^2=9$

$\therefore \begin{cases} a=3 \\ b=-2 \end{cases}$ or $\begin{cases} a=-3 \\ b=2 \end{cases}$

Criteria
• one mark for equating real and imaginary parts • one mark for values of a and b

(d) To rotate $\overrightarrow{OA}$ by $-60°$, we need to multiply by $\text{cis}\left(-\frac{\pi}{3}\right)$ ✓.

Thus $\overrightarrow{OC}=2\times\overrightarrow{OA}\times\text{cis}\left(-\frac{\pi}{3}\right)$

$=2\times\text{cis}\left(\frac{2\pi}{3}\right)\times\text{cis}\left(-\frac{\pi}{3}\right)$

$=2\text{cis}\left(\frac{\pi}{3}\right)$ ✓

$=1+i\sqrt{3}$ ✓

(e)

Criteria	Marks
(i) • one mark for answer	1
(ii) • one mark for answer	1
(iii) • one mark for choice of z_1, z_2 • one mark for answer	2

(i)

$z_1=a+ib,\ z_2=c+id \Rightarrow z_1z_2=(ac-bd)+i(ad+bc)$

$|z_1z_2|^2=(ac-bd)^2+(ad+bc)^2=a^2c^2-2acbd+b^2d^2+a^2d^2+2adbc+b^2c^2$

$\therefore |z_1z_2|^2=(a^2+b^2)(c^2+d^2)=|z_1|^2.|z_2|^2 \qquad \therefore |z_1|.|z_2|=|z_1z_2|$

(ii)

$z_1=2+3i \qquad \Rightarrow |z_1|^2=4+9=13$

$z_2=4+5i \qquad \Rightarrow |z_2|^2=16+25=41$

$z_1.z_2=-7+22i \Rightarrow |z_1.z_2|^2=7^2+22^2$

$\therefore 533=13\times 41=7^2+22^2$

(iii)

For example :

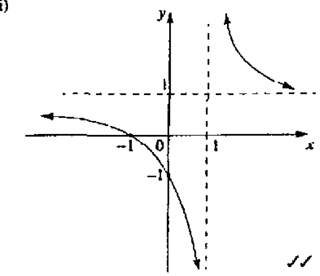
$z_1=3+2i,\ z_2=5-4i,\ z_1.z_2=23-2i$

$|z_1|^2=13,\ |z_2|^2=41,\ |z_1.z_2|^2=23^2+2^2$

$\therefore 533=13\times41=23^2+2^2$

QUESTION 3

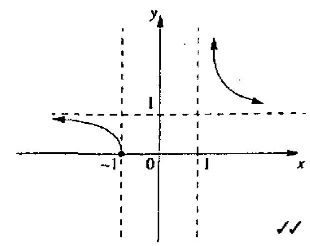
(a) (i)



✓✓

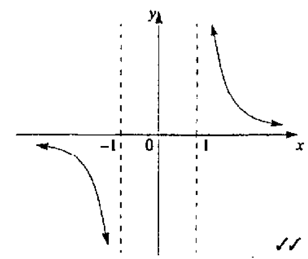
- vertical asymptote at $x=1$
- horizontal asymptote at $y=1$
- x intercept at $x=-1$
- y intercept at $y=-1$

(ii)



✓✓

(iii)



✓✓

(b) The locus is the parabola with focus at (0, 2) ✓ and the x axis as directrix. ✓

or give the equation of the parabola as $x^2=4(y-1)$

QUESTION 1

(a) $\displaystyle\int_0^3 \frac{x\,dx}{\sqrt{16+x^2}} = \int_{16}^{25} \frac{\frac{1}{2}du}{\sqrt{u}}$ Let $u = 16 + x^2$, $du = 2x\,dx$ ✓

$= \frac{1}{2}\int_{16}^{25} u^{-\frac{1}{2}}\,du$

$= \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_{16}^{25}$ ✓

$= \left[\sqrt{u}\right]_{16}^{25}$

$= \sqrt{25} - \sqrt{16}$

$= 5 - 4$

$= 1$ ✓

(b) $\displaystyle\int \frac{dx}{x^2+6x+13} = \int \frac{dx}{(x+3)^2+4}$ ✓

$= \frac{1}{2}\tan^{-1}\frac{x+3}{2} + c$ ✓

(c) $\displaystyle\int xe^{-x}dx = \int x\frac{d}{dx}(-e^{-x})dx$

$= -xe^{-x} - \int 1(-e^{-x})dx$ ✓

$= -xe^{-x} + \int e^{-x}dx$

$= -xe^{-x} - e^{-x} + c$ ✓

(d) $\displaystyle\int \cos^3\theta\,d\theta = \int \cos^2\theta\cos\theta\,d\theta$ ✓   Let $u = \sin\theta$, $du = \cos\theta\,d\theta$

$= \int (1-\sin^2\theta)\cos\theta\,d\theta$

$= \int (1-u^2)\,du$ ✓

$= u - \frac{1}{3}u^3 + c$

$= \sin\theta - \frac{1}{3}\sin^3\theta + c$ ✓

(e) (i) Let $\dfrac{x^2-4x-1}{(1+2x)(1+x^2)} \equiv \dfrac{A}{1+2x} + \dfrac{Bx+C}{1+x^2}$.

Then $x^2-4x-1 \equiv A(1+x^2) + (1+2x)(Bx+C)$ ✓

$x^2-4x-1 \equiv A + Ax^2 + Bx + C + 2Bx^2 + 2Cx$

Equating coefficients of like terms,

$1 = A + 2B$ [Eq. 1]

$-4 = B + 2C$ [Eq. 2]

$-1 = A + C$ [Eq. 3]

Multiply Eq. 2 by -2:

$8 = -2B - 4C$ [Eq. 2a]

Eq. 1 + Eq. 2a:

$9 = A - 4C$ [Eq. 4]

Eq. 3 − Eq. 4:

$-10 = 5C$

$C = -2$

Substitute C into Eq. 3:

$-1 = A - 2$

$A = 1$

Substitute A into Eq. 1:

$1 = 1 + 2B$

$B = 0$ ✓✓

(ii) $\displaystyle\int \frac{x^2-4x-1}{(1+2x)(1+x^2)}\,dx = \int\left(\frac{1}{1+2x} + \frac{-2}{1+x^2}\right)dx$

$= \frac{1}{2}\ln|1+2x| - 2\tan^{-1}x + c$ ✓✓

QUESTION 2

(a) (i) $wz^2 = -3(1+i)^2$

$= -3(1-1+2i)$

$= -6i$ ✓

(ii) $\dfrac{z}{z+w} = \dfrac{1+i}{-2+i} \times \dfrac{-2-i}{-2-i}$

$= \dfrac{-(1+i)(2+i)}{5}$ ✓

$= \dfrac{-(1+3i)}{5}$

$= -\frac{1}{5} - \frac{3i}{5}$ ✓