Name: ....

Form:........................

ASCHAM SCHOOL
MATHEMATICS EXAMINATION
FORM 6 - 3 UNIT
1999

July 1999 | Time allowed: 2 hours

* All questions should be attempted
* All necessary working must be shown
* All questions are of equal value
* Marks may not be awarded for careless or badly arranged work.
* Write your name on each booklet clearly marked:
  Question 1, Question 2, ... etc.
* Begin each question in a new booklet.
* Approved calculators may be used.
* Copies of diagrams for all questions are provided on pages 11-14 in order to save time. You may use them but **you must staple them into your booklets.**

Question 1 Marks:

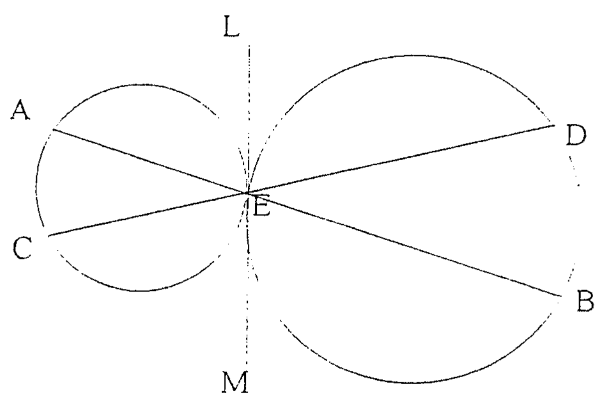
(a) Find the acute angle, to the nearest degree, between the lines $y = 3x + 1$ and $y = -x + 6$ **3**

(b) Solve the inequality $\frac{1}{x+1} < 3, \quad x \neq -1$ **3**

(c) Find the coordinates of the point P which divides the interval AB with end points A(-1, 2) and B(3, -5) internally in the ratio 2:3. **2**

(d) Use the substitution $u = t + 1$ to evaluate $\int_0^1 \frac{t}{\sqrt{t+1}}\,dt$ **3**

(e) Two circles touch externally at E. **3**



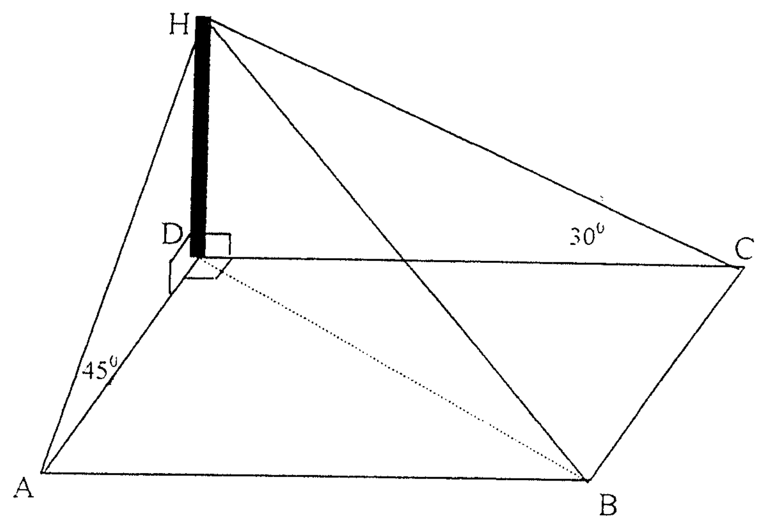
(A copy of the diagram above is on page 10.)

AB and CD intersect at E. LM is a common tangent at E

Prove that AC is parallel to DB.

Question 2

Marks:



(a) A post HD stands vertically at one corner of a rectangular field ABCD. The angles of elevation of the top H of the post from the nearest corners A and C respectively are 30^0 and 45^0.

(A copy of the diagram above is on page 13.)

(i) If $AD = a$ units, find the length of BD in terms of a. **2**

(ii) Hence find the angle of elevation of H from the corner B to the nearest minute. **1**

(b) Taking $x = -\dfrac{\pi}{6}$ as a first approximation to the root of the equation $2x + \cos x = 0$, use Newton's method once to show that a better approximation to the root of the equation is $\dfrac{-\pi - 6\sqrt{3}}{30}$ **4**

(c) (i) Find the domain and range of $f^{-1}(x) = \sin^{-1}(3x-1)$ **2**

(ii) Sketch the graph of $y = f^{-1}(x)$. **2**

(iii) Find the equation representing the inverse function $f(x)$ and state the domain and range. **3**

Question 3

Marks:

(a) (i) Express $3\sin x - \sqrt{3}\cos x$ in the form $A\sin(x-\alpha)$, where $A > 0$ and $0 \le \alpha \le \frac{\pi}{2}$. 3

(ii) Determine the minimum value of $3\sin x - \sqrt{3}\cos x$. 1

(iii) Solve $3\sin x - \sqrt{3}\cos x = \sqrt{3}$ for $0 \le x \le 2\pi$. 3

(b) Newton's Law of cooling states that the rate of cooling of a body is proportional to the excess of the temperature of a body above the surrounding temperature. This rate can be expressed by the differential equation:

$$\frac{dT}{dt} = -k(T - T_0),$$

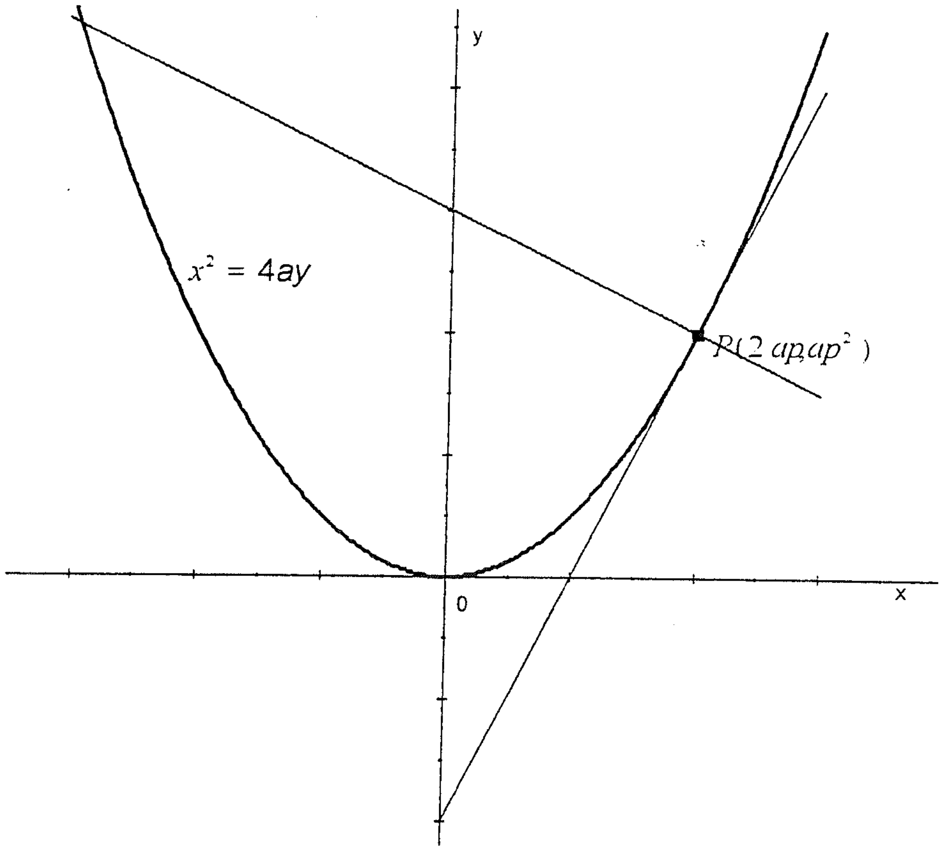
where T is the temperature of the body, T_0 is the temperature of the surroundings, t is the time in minutes and k is a constant.

(i) Show that $T = T_0 + Ae^{-kt}$, where A is a constant, is a solution of the differential equation $\frac{dT}{dt} = -k(T - T_0)$. 2

(ii) A cup of tea cools from 85°C to 80°C in 1 minute at a room temperature of 25°C. Find the temperature of the cup of tea after a further 4 minutes have elapsed. Answer to the nearest degree. 5

Question 4

Marks:



(a) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. Show the equation of the normal to the parabola at P is $x + py = 2ap + ap^3$. **3**

(b) Write down the equation of the normal to the parabola at Q. The normals intersect at N. Find the coordinates of N. **3**

(c) Show the equation of the chord PQ is $y - ap^2 = \left(\frac{p+q}{2}\right)(x - 2ap)$ **3**

and determine the condition necessary for PQ to be a focal chord.

(d) If PQ is a focal chord and N is the intersection of the normals, find the equation of the locus of N. **4**

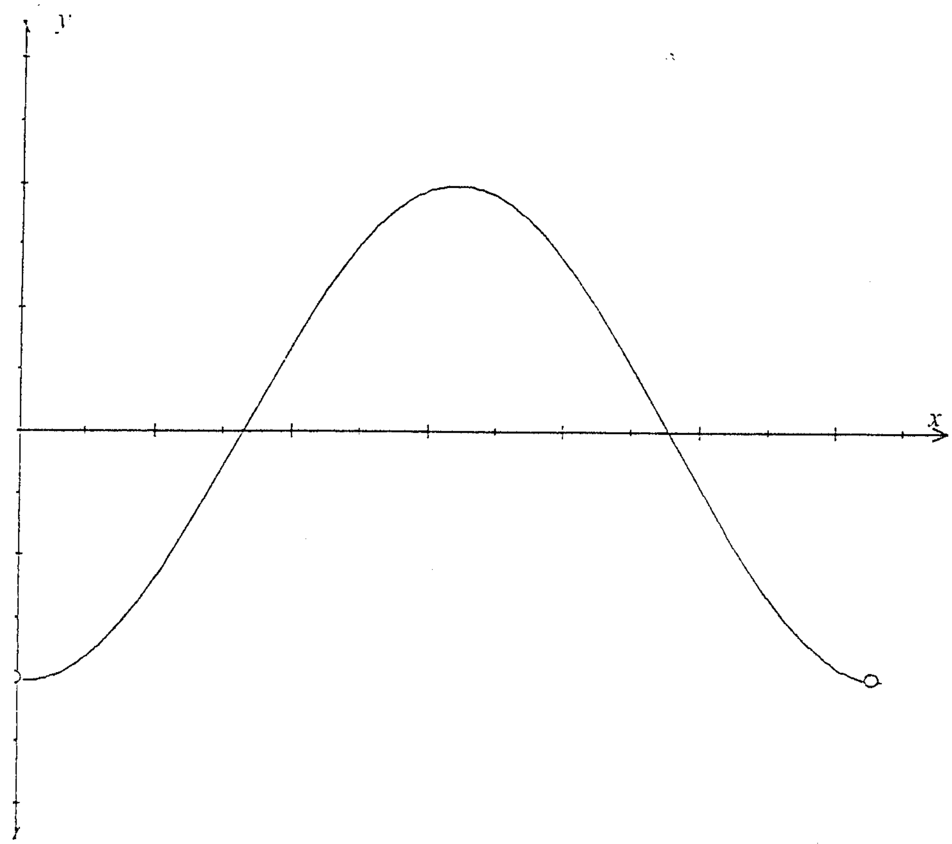
(e) (A copy of the diagram above is on page 11.) **1**

On the diagram above, the tangent and normal are drawn at P.

Mark clearly on your own diagram the points Q and N which correspond to P.

Question 5 **Marks:**

(a) The graph of $x = -a\cos nt$ for $0 \le t \le \dfrac{2\pi}{n}$ is drawn below. (A copy of the diagram above is on page 12.) Label axes and show intercepts accurately. **2**



(b) On a certain day the depth of water in a harbour at low tide at 4:30 am is 5 metres. At the following high tide at 10:45 am the depth is 15 metres. Assuming the rise and fall of the surface of the water to be simple harmonic, find between what times during the morning a ship may safely enter the harbour if the minimum depth of $12\frac{1}{2}$ metres of water is required. **6**

(c) Given that $\sin^{-1} x$, $\cos^{-1} x$ and $\sin^{-1}(2-x)$ have values for $0 \le x \le \dfrac{\pi}{2}$

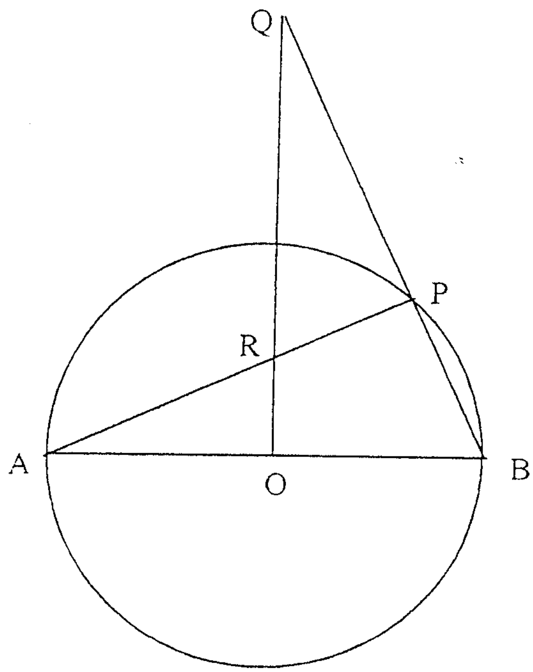
(i) show that $\sin(\sin^{-1} x - \cos^{-1} x) = 2x^2 - 1$ **3**

(ii) Hence, or otherwise, solve the equation $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(2-x)$ **3**

Question 6

Marks:

(a) O is the centre of the circle. BPQ is a straight line ORQ is perpendicular to AOB as shown below.



(A copy of the diagram above is on page 14.)

Prove that:

(i) A, O, P, Q are concyclic, and **3**

(ii) $\angle OPA = \angle OQB$. **2**

(b) Prove by using mathematical induction that $5^n \geq 1 + 4n$, for $n > 1$, $n \in J^+$. **4**

(c) The cubic equation $2x^3 - x^2 + x - 1 = 0$ has roots α, β, and γ. Evaluate

(i) $\alpha\beta + \beta\gamma + \alpha\gamma$ **1**

(ii) $\alpha\beta\gamma$ **1**

(iii) $\alpha^2\beta^2\gamma + \beta^2\gamma^2\alpha + \alpha^2\gamma^2\beta$ **1**

(d) The equation $2\cos^3\theta - \cos^2\theta + \cos\theta - 1 = 0$ has roots $\cos a$, $\cos b$ and $\cos c$. **2**

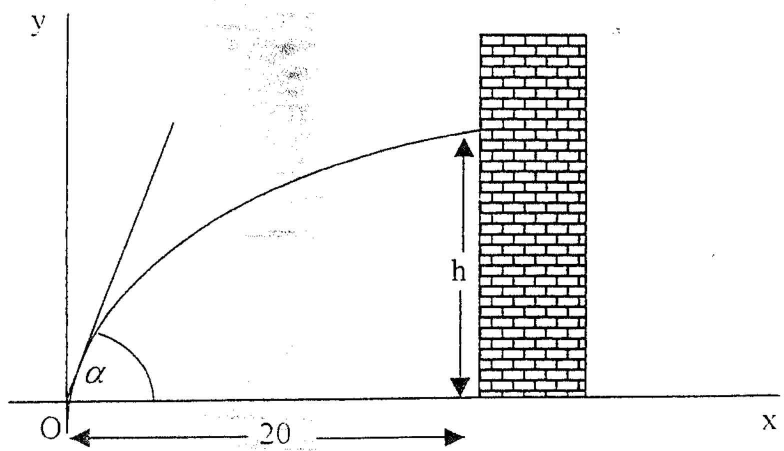
Using appropriate information from (c) above prove that

$$\sec a + \sec b + \sec c = 1$$

Question 7 **Marks:**

A softball player hits the ball from ground level with a speed of 20 ms^{-1} and an angle of elevation α. It flies toward a high wall 20 m away on level ground.

(a) Taking the origin at the point where the ball is hit. derive expressions for **3**



the horizontal and vertical components x and y of displacement at time t seconds. Take g = $10ms^{-2}$.

(b) Hence find the equation of the path of the ball in flight in terms of x, y and. α. **1**

(c) Show that the height h at which the ball hits the wall is given by **2**

$$h = 20\tan\alpha - 5(1 + \tan^2\alpha).$$

(d) Using part (c) above, show that the maximum value of h occurs when $\tan\alpha = 2$. **2**

(e) Find **6**

(i) this maximum height h.

(ii) the speed and the angle at which the ball hits the wall in this case.