CSSA

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NSW

Centre Number

Student Number

2014

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Morning Session
Thursday 31 July 2014

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided on a separate sheet
- In Questions 11–16, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks – 100

Section I Pages 2–5

10 marks

- Attempt Questions 1–10
- Allow 15 minutes for this section

Section II Pages 6–15

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

6400-1

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Write $\dfrac{40}{1-3i}$ in the form $a+ib$, where a and b are real.

(A) $4-12i$

(B) $4+12i$

(C) $-5-15i$

(D) $-5+15i$

2 What is the eccentricity of the hyperbola $16x^2-25y^2=400$?

(A) $\dfrac{3}{5}$

(B) $\dfrac{3}{4}$

(C) $\dfrac{\sqrt{41}}{5}$

(D) $\dfrac{\sqrt{41}}{4}$

3 The equation $y^3-xy+x^3=7$ implicitly defines y in terms of x.

Which of the following is an expression for $\dfrac{dy}{dx}$?

(A) $\dfrac{-3x^2}{3y^2-1}$

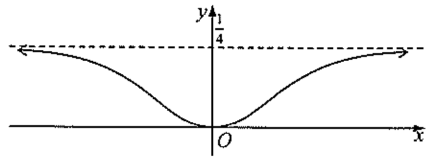
(B) $\dfrac{y-3x^2}{3y^2-x}$

(C) $\dfrac{y-3x^2+7}{3y^2-x}$

(D) $\dfrac{3y^2-y+3x^2}{x}$

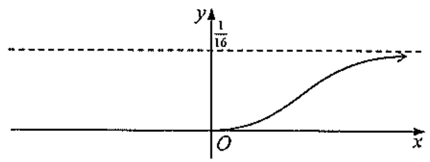
4 The diagram shows the graph of $y=f(x)$.

DIAGRAMS NOT TO SCALE.

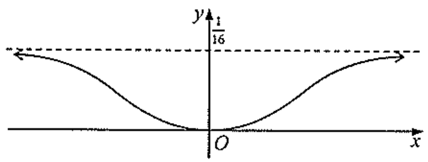


Which of the following best represents the graph of $y=\sqrt{f(x)}$?

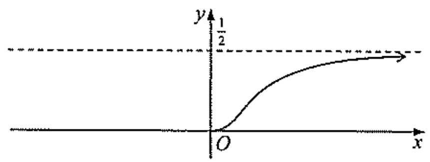
(A)



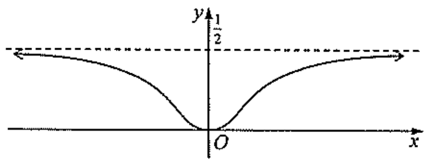
(B)



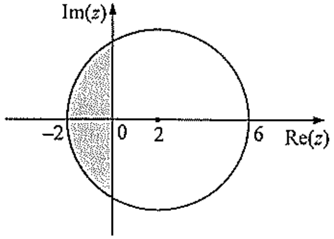
(C)



(D)



5 A circle with centre $(2, 0)$ and radius 4 units is shown on an Argand diagram below.



Which of the following inequalities represents the shaded region?

(A) $\text{Re}(z) \le 0$ and $|z-2| \le 4$

(B) $\text{Re}(z) \le 0$ and $|z-2| \le 16$

(C) $\text{Im}(z) \le 0$ and $|z-2| \le 4$

(D) $\text{Im}(z) \le 0$ and $|z-2| \le 16$

6 A particle moves in a circle of radius 40 cm with a constant angular speed of 15 revolutions per minute. What is the speed of the particle?

(A) $\frac{\pi}{5}$ ms^{-1}

(B) 6 ms^{-1}

(C) 12π ms^{-1}

(D) 20π ms^{-1}

7 The cube roots of unity are 1, ω and ω^2. Simplify $(1-\omega+\omega^2)(1+\omega-\omega^2)$.

(A) 0

(B) 1

(C) 2

(D) 4

8 Which integral is obtained when the substitution $t = \tan\frac{x}{2}$ is applied to $\int \frac{dx}{5+4\cos x}$?

(A) $\int \frac{2}{9-4t^2}\,dt$

(B) $\int \frac{2}{9+t^2}\,dt$

(C) $\int \frac{1+t^2}{9+t^2}\,dt$

(D) $\int \frac{2(1-t^2)}{(1+t^2)(9-t^2)}\,dt$

9 Given α, β and γ are the roots of the equation $x^3-4x+7=0$, find the cubic equation with roots α^2, β^2 and γ^2.

(A) $x^3-4\sqrt{x}+7=0$

(B) $x^3+16x+49=0$

(C) $x^3-4x^2+16x-49=0$

(D) $x^3-8x^2+16x-49=0$

10 Given z and w are non-zero complex numbers, $z \ne \pm w$, such that $z\bar{z} = w\bar{w}$, which of the following statements is true?

(A) $\arg\left(\frac{z+w}{z-w}\right)=0$

(B) $\arg\left(\frac{z+w}{z-w}\right)=\pi$

(C) $\arg\left(\frac{z+w}{z-w}\right)=\pm\frac{\pi}{2}$

(D) $\arg\left(\frac{z+w}{z-w}\right)$ cannot be determined

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11– 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $\omega = \sqrt{3} + i$.

(i) Express ω in modulus-argument form. **1**

(ii) Hence, or otherwise, express $z^3\omega$ in modulus-argument form. **2**

(b) By completing the square, find $\int \frac{9}{x^2+4x+13}\,dx$. **2**

(c) Evaluate $\int_0^1 xe^{4x}\,dx$. **3**

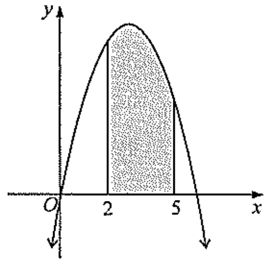
(d) (i) Find real numbers a and b such that **2**

$$\frac{3x}{(x-2)^2(x-3)} = \frac{a}{(x-2)^2} + \frac{b}{x-2} + \frac{9}{x-3}$$

(ii) Hence, or otherwise, find $\int \frac{3x}{(x-2)^2(x-3)}\,dx$. **2**

Question 11 continues on page 7

(e) The region enclosed between $y = 6x - x^2$, the x-axis and the lines $x = 2$ and $x = 5$ is shaded in the diagram below. **3**



The shaded region is rotated about the y-axis.

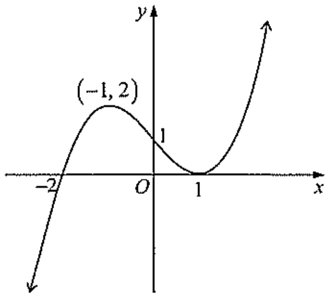
Using the method of cylindrical shells, find the volume of the solid generated.

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below is a sketch of the function $y = f(x)$,

where $f(x) = \frac{1}{2}(x+2)(x-1)^2$.



Draw separate one-third page diagrams of the graphs of each of the following.

(i) $y = |f(x)|$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

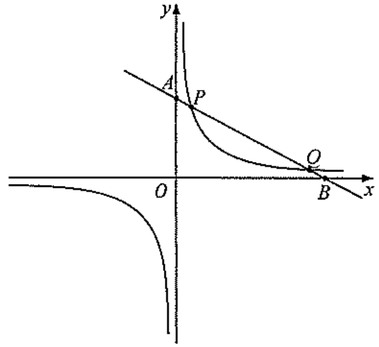
(iii) $y^2 = f(x)$ **2**

(b) It is given that $1+i$ is a root of $p(x) = x^4 - 2x^3 - 7x^2 + 18x - 18$. **3**

Express $p(x)$ as the product of quadratic and linear factors with real coefficients.

Question 12 continues on page 9

(c) The points $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(cq, \dfrac{c}{q}\right)$ lie on the same branch of the rectangular hyperbola $xy = c^2$. The line PQ intersects the asymptotes at A and B as shown in the diagram.

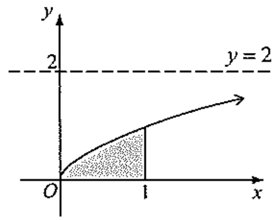


(i) Show that the equation of PQ is given by $x + pqy = c(p+q)$. **2**

(ii) The midpoint M of PQ is $\left(\dfrac{c(p+q)}{2}, \dfrac{c(p+q)}{2pq}\right)$. (Do NOT prove this.) **2**

Using this given information, or otherwise, show that $AP = BQ$.

(d) The area under the curve $y = \sqrt{x}$ from $x = 0$ to $x = 1$ is rotated about the line $y = 2$. **3**

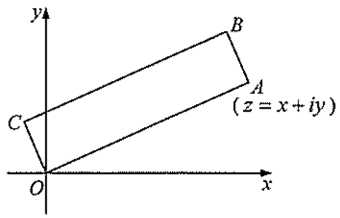


By taking slices perpendicular to the line $y = 2$, find the volume of the solid generated.

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) In the Argand diagram below, $OABC$ is a rectangle. O is the origin and the distance OA is four times the distance AB. The vertex A is represented by the complex number $z = x + iy$. **2**

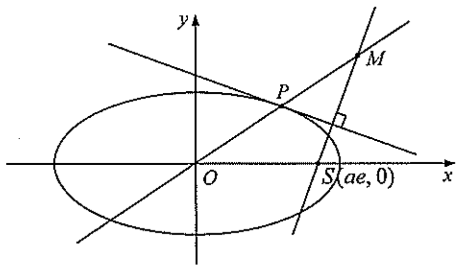


Find an expression for the complex number that represents the vertex B. Leave your answer in the form $a + ib$.

(b) (i) Show that if α is a zero of multiplicity 2 of a polynomial $f(x)$, then $f(\alpha) = f'(\alpha) = 0$. **2**

(ii) The polynomial $g(x) = px^3 - 3qx + r$ has a zero of multiplicity 2. Show that $4q^3 = pr^2$. **3**

(c) The diagram shows the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with focus $S(ae, 0)$ and origin O. $P(a\cos\theta, b\sin\theta)$ is any point on the ellipse. The line through S perpendicular to the tangent at P and the line OP produced meet at M.



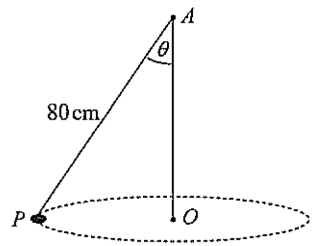
(i) Show that the gradient of the tangent to the ellipse at P is given by $-\frac{b\cos\theta}{a\sin\theta}$. **1**

(ii) Show that M lies on the corresponding directrix to S. **4**

Question 13 continues on page 11

(d) A particle P of mass 3 kg is attached by a string of length 80 cm to a point A. The particle moves with constant angular velocity ω in a horizontal circle with centre O which lies directly below A. The angle the string makes with OA is θ. **3**

The forces acting on the particle are the tension, T, in the string and the force due to gravity. The greatest tension that can safely be allowed in the string is 200 Newtons.



By considering the forces acting on the particle in the horizontal direction, find the maximum angular velocity ω of the particle. Give your answer correct to 1 decimal place.

End of Question 13

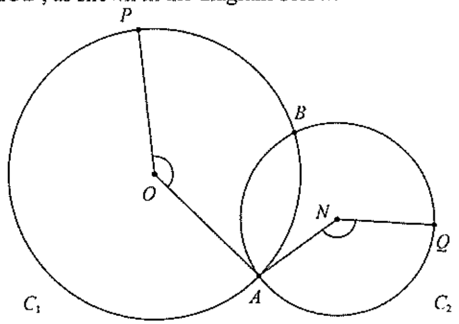
Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Let $I_n = \int_1^e (\ln x)^n \, dx$, where $n \geq 0$.

(i) Show that $I_n = e - nI_{n-1}$ for $n \geq 1$. **2**

(ii) Hence evaluate $\int_1^e (\ln x)^3 \, dx$. **2**

(b) Two circles C_1 and C_2 with centres O and N respectively intersect at A and B. P lies on C_1 and Q lies on C_2 such that $\angle AOP = \angle ANQ$ and $\angle AOP > \angle AOB$, as shown in the diagram below. **3**



Prove that the points P, B and Q are collinear.

(c) (i) Given $z^9 - 1 = (z^3 - 1)(z^6 + z^3 + 1)$, plot the roots of $z^6 + z^3 + 1 = 0$ on an Argand diagram. **2**

(ii) Show that **2**

$$z^6 + z^3 + 1 = \left(z^2 - 2z\cos\frac{2\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{8\pi}{9} + 1\right)$$

(iii) Show that $\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} + \cos\frac{2\pi}{9}\cos\frac{8\pi}{9} + \cos\frac{4\pi}{9}\cos\frac{8\pi}{9} = -\frac{3}{4}$ **1**

(d) The inequality $x > \ln(1+x)$ holds for all real $x > 0$. (Do NOT prove this.) **3**

Use this result and the method of mathematical induction to prove that for all positive integers n,

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(n+1).$$

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$. **2**

(ii) Hence, or otherwise, find $\int_0^{\frac{\pi}{2}} x \sin 2x \, dx$. **3**

(b) An object of mass 70 kg, initially at rest, is pulled along a horizontal surface by a constant force of 140 N. It experiences a resistance proportional to its speed.
When the speed is 10 ms^{-1}, the acceleration is 1 ms^{-2}.
Let x represent the displacement in metres from the initial position of the object.

(i) Show that the equation of motion is $\ddot{x} = 2 - \frac{1}{10}v$. **2**

(ii) Find an expression for x as a function of v. **3**

(iii) Show that the object's speed cannot exceed 20 ms^{-1}. **1**

(c) A nine letter arrangement consists of 3 A's, 3 B's and 3 C's such that there are:

- no A's in the first three letters
- no B's in the next three letters
- no C's in the last three letters

(i) Find the number of nine letter arrangements if the first three letters are 2 B's and 1 C in some order. **2**

(ii) Find the total number of nine letter arrangements. **2**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

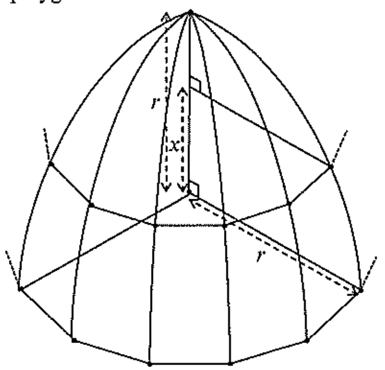
(a) (i) Prove that $x^2+y^2+z^2 \geq xy+yz+xz$ for x, y and z positive real numbers. **2**

(ii) The inequality $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{9}{x+y+z}$ holds for x, y and z positive real numbers. (Do NOT prove this). **2**

Given x, y and z are positive real numbers with $x^2+y^2+z^2=9$, prove that

$$\frac{1}{1+xy}+\frac{1}{1+yz}+\frac{1}{1+xz} \geq \frac{3}{4}.$$

(b) The diagram below shows part of a polygonal dome. Each cross-section is a regular n-sided polygon.



The vertex of the dome is r units directly above the centre of the polygonal base, which is r units from each vertex. A circular arc joins the top of the dome to each vertex of the base.

(i) Show that the area of the horizontal cross-section x units from the base is given by $\frac{n}{2}\sin\left(\frac{2\pi}{n}\right)\times\left(r^2-x^2\right)$. **2**

(ii) Hence show that the volume of the dome is given by **2**

$$\frac{nr^3}{3}\sin\left(\frac{2\pi}{n}\right).$$

(iii) Show that as $n \to \infty$, the volume of the dome approaches that of a hemisphere. **1**

Question 16 continues on page 15

Question 16 (continued)

(c) (i) Show that $\frac{x^{2^n}}{1-x^{2^{n+1}}} = \frac{1}{1-x^{2^n}} - \frac{1}{1-x^{2^{n+1}}}$. **2**

(Note that $x^{2^n} = x^{(2^n)}$).

(ii) Using the result in part (i), **1**

show that $\sum_{n=0}^{N} \frac{x^{2^n}}{1-x^{2^{n+1}}} = \frac{1}{1-x} - \frac{1}{1-x^{2^{N+1}}}$.

(iii) Let x be a real number with $-1<x<1$. **1**

Given $\lim_{N\to\infty} \sum_{n=0}^{N} \frac{x^{2^n}}{1-x^{2^{n+1}}} = \sum_{n=0}^{\infty} \frac{x^{2^n}}{1-x^{2^{n+1}}}$,

show that $\sum_{n=0}^{\infty} \frac{x^{2^n}}{1-x^{2^{n+1}}} = \frac{x}{1-x}$.

(iv) Hence find $\sum_{n=0}^{\infty} \frac{1}{2014^{2^n}-2014^{-2^n}}$. **2**

End of Paper

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CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW
2014 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION
MATHEMATICS EXTENSION 2 – MARKING GUIDELINES

Section I
10 marks

Questions 1-10 (1 mark each)

Question 1 (1 mark)
Outcomes Assessed: E3
Targeted Performance Bands: E2

Solution	Answer	Mark
$\frac{40}{1-3i}=\frac{40}{1-3i}\times\frac{1+3i}{1+3i}$ $=\frac{40(1+3i)}{1+9}$ $=4+12i$	B	1

Question 2 (1 mark)
Outcomes Assessed: E3
Targeted Performance Bands: E3

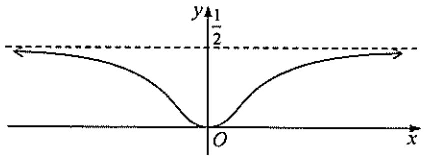
Solution	Answer	Mark
$16x^2-25y^2=400\Rightarrow\frac{x^2}{25}-\frac{y^2}{16}=1$ $\therefore a=5, b=4$ $b^2=a^2(e^2-1)$ $16=25(e^2-1)$ $e^2=\frac{16}{25}+1$ $e=\frac{\sqrt{41}}{5}\quad(e>0)$	C	1

6400-2

Question 3 (1 mark)
Outcomes Assessed: E6
Targeted Performance Bands: E3

Solution	Answer	Mark
$y^3 - xy + x^3 = 7$ $3y^2\frac{dy}{dx} - \left(x\frac{dy}{dx} + y\right) + 3x^2 = 0$ $\frac{dy}{dx}\left(3y^2 - x\right) = y - 3x^2$ $\frac{dy}{dx} = \frac{y-3x^2}{3y^2-x}$	B	1

Question 4 (1 mark)
Outcomes Assessed: E6
Targeted Performance Bands: E3

Solution	Answer	Mark
	D	1

Question 5 (1 mark)
Outcomes Assessed: E3
Targeted Performance Bands: E3

Solution	Answer	Mark
$x \le 0 \Rightarrow \text{Re}(z) \le 0$ Inside region of a circle with centre (2,0) and radius 4 $\Rightarrow \lvert z-2\rvert \le 4$ $\therefore \text{Re}(z) \le 0$ and $\lvert z-2\rvert \le 4$ defines the shaded region.	A	1

Question 6 (1 mark)
Outcomes Assessed: E5
Targeted Performance Bands: E3

Solution	Answer	Mark
$r = 0.4\,\text{m}$ $\omega = \frac{15\times 2\pi}{60} = \frac{\pi}{2}$ radians per second. $\therefore$ Speed of the particle $= r\omega = 0.4\times\frac{\pi}{2} = \frac{\pi}{5}$ ms^{-1}	A	1

Question 7 (1 mark)
Outcomes Assessed: E3
Targeted Performance Bands: E3

Solution	Answer	Mark
1, ω and ω^2 are the roots of the equation $z^3 - 1 = 0$. $\therefore \omega^3 = 1$ and $1 + \omega + \omega^2 = 0$. $(1-\omega+\omega^2)(1+\omega-\omega^2)$ $=(-2\omega)(-2\omega^2)$ $=4\omega^3$ $=4$	D	1

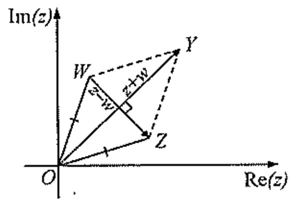
Question 8 (1 mark)
Outcomes Assessed: E8
Targeted Performance Bands: E2

Solution	Answer	Mark
$t = \tan\frac{x}{2} \Rightarrow \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} \Rightarrow dx = \frac{2}{1+t^2}dt$; $\cos x = \frac{1-t^2}{1+t^2}$ $\int\frac{dx}{5+4\cos x} = \int\frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)}\times\frac{2}{(1+t^2)}dt$ $= \int\frac{2}{5(1+t^2)+4(1-t^2)}dt$ $= \int\frac{2}{9+t^2}dt$	B	1

Question 9 (1 mark)
Outcomes Assessed: E4
Targeted Performance Bands: E3

Solution	Answer	Mark
$x^3-4x+7=(x-\alpha)(x-\beta)(x-\gamma)=0$ has roots α, β and γ. $\therefore$ the polynomial equation with roots α^2, β^2 and γ^2 is given by $\left(\sqrt{x}-\alpha\right)\left(\sqrt{x}-\beta\right)\left(\sqrt{x}-\gamma\right)=0$ $\left(\sqrt{x}\right)^3-4\left(\sqrt{x}\right)+7=0$ $\sqrt{x}\left(x-4\right)=-7$ $x\left(x-4\right)^2=49$ $x^3-8x^2+16x-49=0$	D	1

Question 10 (1 mark)
Outcomes Assessed: E3
Targeted Performance Bands: E4

Solution	Answer	Mark
$z\overline{z}=w\overline{w}\Rightarrow\lvert z\rvert^2=\lvert w\rvert^2\Rightarrow\lvert z\rvert=\lvert w\rvert$ Let O be the origin, Z be the point representing the complex number z, W be the point representing the complex number w and Y be the point representing $z+w$. Then $OZYW$ is a rhombus, with diagonals OY and WZ meeting at right angles.  Hence $\arg(z+w)-\arg(z-w)=\pm\frac{\pi}{2}$. i.e. $\arg\left(\frac{z+w}{z-w}\right)=\pm\frac{\pi}{2}$.	C	1

Section II
90 marks

Question 11 (15 marks)

(a) (i) (1 Mark)
Outcomes assessed: E3
Targeted Performance Bands: E2

Criteria	Mark
• Correct expression for ω	1

Sample answer:

$$\omega=\sqrt{3}+i=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

(a) (ii) (2 Marks)
Outcomes assessed: E3
Targeted Performance Bands: E2

Criteria	Mark
• Correct expression for $z^3\omega$	2
• Makes progress towards finding $z^3\omega$ by applying De Moivre's theorem	1

Sample answer:

$$\begin{aligned}z^3\omega&=2^3\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^3\times2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\\&=8\left(\cos\pi+i\sin\pi\right)\times2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\\&=16\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)\end{aligned}$$

(b) (2 Marks)
Outcomes assessed: E8
Targeted Performance Bands: E2

Criteria	Mark
• Correct integral	2
• Simplifies the integral by completing the square	1

Sample answer:

$$\begin{aligned}\int\frac{9}{x^2+4x+13}\,dx&=\int\frac{9}{(x+2)^2+9}\,dx\\&=9\times\frac{1}{3}\tan^{-1}\frac{x+2}{3}+C\\&=3\tan^{-1}\frac{x+2}{3}+C\end{aligned}$$

Question 11 (continued)

(c) (3 Marks)
Outcomes assessed: E8
Targeted Performance Bands: E2

Criteria	Mark
• Correct solution	3
• Significant progress towards the correct solution	2
• Correctly applies the method of integration by parts	1

Sample answer:
Applying the method of integration by parts:

$$\int_0^1 xe^{4x}\,dx = \left[x\times\frac{e^{4x}}{4}\right]_0^1 - \int_0^1 \frac{e^{4x}}{4}\,dx$$

$$= \left(\frac{e^4}{4}-0\right)-\left[\frac{e^{4x}}{16}\right]_0^1$$

$$= \left(\frac{e^4}{4}\right)-\left(\frac{e^4}{16}-\frac{1}{16}\right)$$

$$= \frac{3e^4+1}{16}$$

(d) (i) (2 Marks)
Outcomes assessed: E4
Targeted Performance Bands: E2

Criteria	Mark
• Correct values for a and b	2
• Correct value for a or b	1

Sample answer:

$$\frac{3x}{(x-2)^2(x-3)} = \frac{a}{(x-2)^2}+\frac{b}{x-2}+\frac{9}{x-3}$$

$$\therefore 3x \equiv a(x-3)+b(x-2)(x-3)+9(x-2)^2$$

Substituting $x=2 \Rightarrow 6=-a \Rightarrow a=-6$
Comparing the coefficient of $x^2 \Rightarrow 0=b+9 \Rightarrow b=-9$

$\therefore a=-6, b=-9$

Question 11 (continued)

(d) (ii) (2 Marks)
Outcomes assessed: E8
Targeted Performance Bands: E2

Criteria	Mark
• Correct solution	2
• Significant progress towards finding the integral of the expression	1

Sample answer:

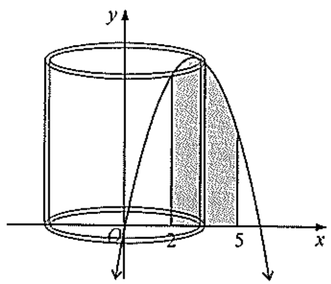
$$\int \frac{3x}{(x-2)^2(x-3)}\,dx = \int \frac{-6}{(x-2)^2}+\frac{-9}{(x-2)}+\frac{9}{(x-3)}\,dx$$

$$= \frac{6}{x-2}-9\ln|x-2|+9\ln|x-3|+C$$

(e) (3 Marks)
Outcomes assessed: E8
Targeted Performance Bands: E3

Criteria	Mark
• Correct volume	3
• Significant progress towards calculating the volume	2
• Correctly applies the method of cylindrical shells to find an expression for the volume	1

Sample answer:
Let ΔV represent the volume of a cylindrical shell.



$$\Delta V \approx 2\pi xy\,\Delta x$$

$$V \approx \sum 2\pi xy\,\Delta x$$

$$V = \lim_{\Delta x\to 0}\sum 2\pi xy\,\Delta x$$

$$= \int_2^5 2\pi x\left(6x-x^2\right)dx$$

$$= 2\pi\int_2^5 6x^2-x^3\,dx$$

$$= 2\pi\left[2x^3-\frac{x^4}{4}\right]_2^5$$

$$= \frac{327\pi}{2}$$

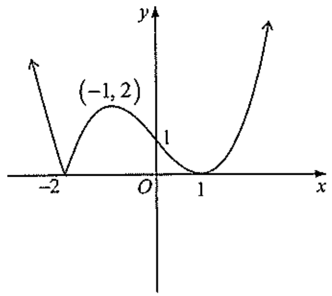
$\therefore$ The volume of the solid is $\frac{327\pi}{2}$ units^3.

Question 12 (15 marks)

(a) (i) (1 Mark)
Outcomes assessed: E6
Targeted Performance Bands: E2

Criteria	Mark
• Correct sketch	1

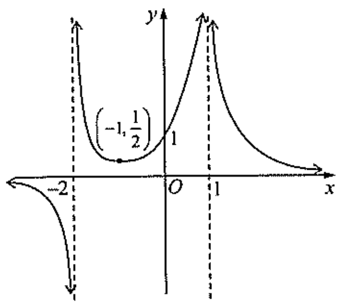
Sample answer:



(a) (ii) (2 Marks)
Outcomes assessed: E6
Targeted Performance Bands: E2

Criteria	Mark
• Correct sketch clearly showing asymptotes, y intercept and local minimum at $x=-1$	2
• Progress towards the correct sketch	1

Sample answer:

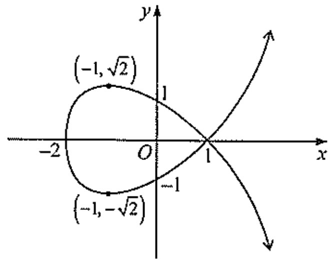


Question 12 (continued)

(a) (iii) (2 Marks)
Outcomes assessed: E6
Targeted Performance Bands: E2

Criteria	Mark
• Correct sketch, showing intercepts and key points	2
• Progress towards the correct sketch	1

Sample answer:



(b) (3 Marks)
Outcomes assessed: E4
Targeted Performance Bands: E3

Criteria	Mark
• Correctly factorises $p(x)$	3
• Significant progress towards factorising $p(x)$	2
• Identifies the complex conjugate of $1+i$ as a root of $p(x)$.	1

Sample answer:
Since the coefficients of $P(x)$ are real, complex roots occur in conjugate pairs.
Hence, if $1+i$ is a root, $1-i$ is also a root.

$\therefore \big(x-(1+i)\big)\big(x-(1-i)\big)=x^2-2x+2$ is a factor

By long division,

$$\begin{aligned} p(x) &= x^4-2x^3-7x^2+18x-18 \\ &= (x^2-2x+2)(x^2-9) \\ &= (x^2-2x+2)(x-3)(x+3) \end{aligned}$$

Question 12 (continued)

(c) (i) (2 Marks)
Outcomes assessed: E3
Targeted Performance Bands: E3

Criteria	Mark
• Shows the equation of PQ is given by $x+pqy=c(p+q)$	2
• Finds the gradient of PQ or shows either P or Q lie on the given line	1

Sample answer:

$$m_{PQ}=\frac{\frac{c}{q}-\frac{c}{p}}{cq-cp}=\frac{c(p-q)}{pq\times c(q-p)}=\frac{-1}{pq}$$

$\therefore$ equation of PQ is

$$y-\frac{c}{p}=\frac{-1}{pq}(x-cp)$$
$$pqy-cq=-x+cp$$
$$x+pqy=c(p+q)$$

(c) (ii) (2 Marks)
Outcomes assessed: E3
Targeted Performance Bands: E3

Criteria	Mark
• Correctly proves that $AP=BQ$	2
• Progress towards proving the result	1

Sample answer:

A is the y-intercept of the tangent, $\therefore$ coordinates of A are $\left(0,\dfrac{c(p+q)}{pq}\right)$.

B is the x-intercept of the tangent, $\therefore$ coordinates of B are $(c(p+q),0)$.

$\therefore$ the midpoint of AB is $\left(\dfrac{c(p+q)}{2},\dfrac{c(p+q)}{2pq}\right)$ which is the same as the given midpoint of PQ.

Since M is the midpoint of AB and of PQ, $AM=BM$ and $PM=QM$.
$\therefore AM-PM=BM-QM$
$\Rightarrow AP=BQ$ as required.

Question 12 (continued)

(d) (3 Marks)
Outcomes assessed: E7
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	3
• Correct expression for the volume	2
• Some progress towards the correct expression for the volume	1

Sample answer:

At a point x, where $0<x<1$, take a slice of thickness Δx.

Let ΔV represent the volume of the cross-sectional slice.

$$\Delta V\approx\pi\left(2^2-(2-y)^2\right)\Delta x=\pi\left(4y-y^2\right)\Delta x=\pi\left(4\sqrt{x}-x\right)\Delta x$$
$$V\approx\sum\pi\left(4\sqrt{x}-x\right)\Delta x$$
$$V=\lim_{\Delta x\to 0}\sum\pi\left(4\sqrt{x}-x\right)\Delta x$$
$$=\int_0^1\pi\left(4\sqrt{x}-x\right)dx$$
$$=\pi\left[\frac{8x^{\frac{3}{2}}}{3}-\frac{x^2}{2}\right]_0^1$$
$$=\frac{13\pi}{6}$$

$\therefore$ The volume of the solid is $\dfrac{13\pi}{6}$ units^3.

Question 13 (15 marks)

(a) (2 Marks)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Mark
• Correct answer	2
• Correct expression for the complex number represented by vertex C	1

Sample answer:

$\overrightarrow{OC}$ is $\frac{1}{4}$ of the length of $\overrightarrow{OA}$ and rotated $\frac{\pi}{2}$ (counter-clockwise)

$\therefore \overrightarrow{OC} = \frac{1}{4} i \times \overrightarrow{OA} = \frac{i(x+iy)}{4}$

$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{OC} \\ &= x + iy + \frac{i(x+iy)}{4} \\ &= \left(x - \frac{y}{4}\right) + i\left(y + \frac{x}{4}\right)\end{aligned}$$

$\therefore$ the complex number that represents the vertex B is $\left(x - \frac{y}{4}\right) + i\left(y + \frac{x}{4}\right)$.

(b) (i) (2 Marks)
Outcomes assessed: E4
Targeted Performance Bands: E3

Criteria	Mark
• Correct proof	2
• Writes an expression for $f(x)$ given α is a zero of multiplicity 2 and makes some progress towards proving the result	1

Sample answer:

Given α is a zero of multiplicity 2, $f(x) = (x-\alpha)^2 q(x)$ for some polynomial $q(x)$.

$$\begin{aligned}f'(x) &= (x-\alpha)^2 q'(x) + 2(x-\alpha)q(x) \\ &= (x-\alpha)\left((x-\alpha)q'(x) + 2q(x)\right)\end{aligned}$$

$\therefore f(\alpha) = (\alpha-\alpha)^2 q(\alpha) = 0 \times q(\alpha) = 0$

and $f'(\alpha) = (\alpha-\alpha)\left((\alpha-\alpha)q'(\alpha) + 2q(\alpha)\right) = 0 \times \left(0 + 2q(\alpha)\right) = 0$

Hence if α is a zero of multiplicity 2 of a polynomial $f(x)$, then $f(\alpha) = f'(\alpha) = 0$.

Question 13 (continued)

(b) (ii) (3 Marks)
Outcomes assessed: E4
Targeted Performance Bands: E3

Criteria	Mark
• Correctly shows that $4q^3 = pr^2$	3
• Some progress towards the correct expression through substitution	2
• Determines $g'(x)$ and applies the result from part (i)	1

Sample answer:

$g(x) = px^3 - 3qx + r \Rightarrow g'(x) = 3px^2 - 3q$

Let α be the root of multiplicity 2.

Then $g(\alpha) = p\alpha^3 - 3q\alpha + r = 0$ and $g'(\alpha) = 3p\alpha^2 - 3q = 0$

$3p\alpha^2 - 3q = 0 \Rightarrow \alpha^2 = \frac{q}{p}$

$$\begin{aligned}p\alpha^3 - 3q\alpha + r = 0 &\Rightarrow r = \alpha(3q - p\alpha^2) \\ &\Rightarrow r^2 = \alpha^2(3q - p\alpha^2)^2 \\ &= \frac{q}{p}\left(3q - p\frac{q}{p}\right)^2 \\ &= \frac{q}{p}(2q)^2 \\ &= \frac{4q^3}{p}\end{aligned}$$

$\therefore 4q^3 = pr^2$

Question 13 (continued)

(c) (i) (1 Mark)
Outcomes assessed: E2, E3
Targeted Performance Bands: E2

Criteria	Mark
• Correctly shows that the gradient of the tangent to the ellipse at P is $-\dfrac{b\cos\theta}{a\sin\theta}$	1

Sample answer:

$$x = a\cos\theta \Rightarrow \frac{dx}{d\theta} = -a\sin\theta$$

$$y = b\sin\theta \Rightarrow \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = b\cos\theta \div -a\sin\theta = -\frac{b\cos\theta}{a\sin\theta}$$

(c) (ii) (3 Marks)
Outcomes assessed: E2, E3
Targeted Performance Bands: E3

Criteria	Mark
• Correct proof	4
• Makes progress solving simultaneously to obtain the x coordinate of M and uses the identity $b^2 = a^2(1-e^2)$	3
• Finds both the equation of the line OP and the equation of the line through S perpendicular to the tangent at P	2
• Finds either the equation of the line OP or the equation of the line through S perpendicular to the tangent at P	1

Sample answer:

The line through S perpendicular to the tangent at P has gradient $\dfrac{a\sin\theta}{b\cos\theta}$ and passes through $S(ae, 0)$. $\therefore$ Equation of the line SM is $y = \dfrac{a\sin\theta}{b\cos\theta}(x - ae)$ (Eqn 1)

Equation of the line OP is $y = \dfrac{b\sin\theta}{a\cos\theta}x$ (Eqn 2)

Question 13(c) (ii) (continued)

Solving Eqn 1 and Eqn 2 simultaneously for the point of intersection M,

$$\frac{b\sin\theta}{a\cos\theta}x = \frac{a\sin\theta}{b\cos\theta}(x - ae)$$

$$b^2x = a^2(x - ae)$$

$$a^2(1-e^2)x = a^2(x-ae) \quad \text{since } b^2 = a^2(1-e^2)$$

$$-e^2x = -ae$$

$$x = \frac{a}{e}$$

$\therefore$ M lies on the corresponding directrix to S, $x = \dfrac{a}{e}$.

(d) (3 Marks)
Outcomes assessed: E5
Targeted Performance Bands: E3

Criteria	Mark
• Correct solution	3
• Identifies the correct radius of the circular motion and makes progress towards determining the maximum angular velocity	2
• Resolves the forces on P horizontally	1

Sample answer:
Resolving the forces on P horizontally: $T\sin\theta = m\omega^2 r$, where $m = 3$, $r = 0.8\sin\theta$

$$\therefore T\sin\theta = 3\omega^2 \times 0.8\sin\theta$$

$$\Rightarrow T = 2.4 \times \omega^2$$

Since $T \le 200$, and taking ω to be positive, $\omega^2 \le \dfrac{200}{2.4} \Rightarrow \omega \le \sqrt{\dfrac{200}{2.4}} \Rightarrow \omega \le 9.1287...$

$\therefore$ the maximum angular velocity ω of the particle is 9.1 radians per second, correct to 1 decimal place.

Question 14 (15 marks)

(a) (i) (2 Marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Mark
• Correct solution	2
• Applies the method of integration by parts	1

Sample answer:

$$I_n = \int_1^e 1\times(\ln x)^n \, dx$$

$$= \left[x\times(\ln x)^n\right]_1^e - \int_1^e x\times n(\ln x)^{n-1}\times\frac{1}{x}\, dx$$

$$= (e-0) - n\int_1^e (\ln x)^{n-1}\, dx$$

$$= e - nI_{n-1}$$

(a) (ii) (2 Marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Mark
• Correct solution	2
• Applies the recurrence formula in part (i) or evaluates I_0	1

Sample answer:

$$I_0 = \int_1^e 1\, dx = [x]_1^e = e-1$$

$$I_1 = e - I_0 = e-(e-1) = 1$$

$$I_2 = e - 2I_1 = e-2$$

$$I_3 = e - 3I_2 = e - 3(e-2) = 6-2e$$

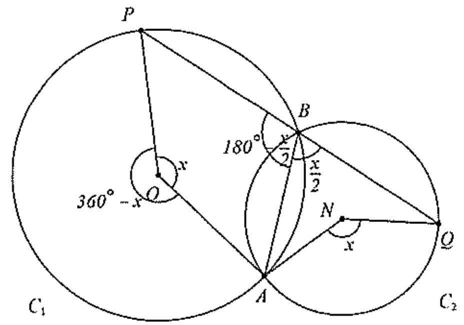
$$\int_1^e (\ln x)^3\, dx = 6-2e.$$

Question 14 (continued)

(b) (3 marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E3

Criteria	Mark
• Correct proof	3
• Substantial progress towards proving P, B and Q are collinear	2
• Identifies the size of an angle in relation to $\angle AOP$ or $\angle ANQ$ with correct reasoning, and which would lead to a correct proof	1

Sample answer:



Let $\angle AOP = \angle ANQ = x$.

$\angle ABQ = \frac{1}{2}\times\angle ANQ = \frac{x}{2}$ (the angle at the circumference is half the angle at the centre when subtended by the same arc)

Reflex $\angle AOP = 360° - x$ (angles in a revolution add to 360°)

$\angle ABP = \frac{1}{2}\times$ reflex $\angle AOP = 180° - \frac{x}{2}$ (the angle at the circumference is half the angle at the centre when subtended by the same arc)

$$\therefore \angle ABP + \angle ABQ = \left(180° - \frac{x}{2}\right) + \frac{x}{2} = 180°$$

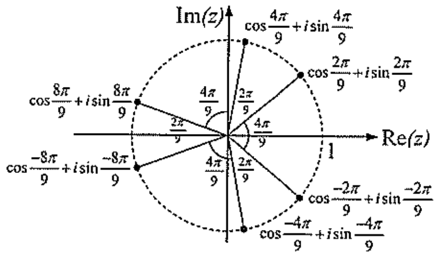
Since a straight angle is 180°, the points P, B and Q lie on a straight line, i.e. are collinear.

Question 14 (continued)

(c) (i) (2 Marks)
Outcomes assessed: E3, E4
Targeted Performance Bands: E3-E4

Criteria	Mark
• Correct plot of the roots on an Argand diagram.	2
• Recognises the roots of $z^6+z^3+1=0$ are a subset of the roots of $z^9=1$ (which include 1 and are evenly spaced around the unit circle)	1

Sample answer:
The roots $z^6+z^3+1=0$ are the roots of $z^9=1$ less the roots of $z^3=1$.



(c) (ii) (2 Marks)
Outcomes assessed: E3, E4
Targeted Performance Bands: E3-E4

Criteria	Mark
• Correct solution	2
• Identifies the roots occur in conjugate pairs and makes progress towards finding a quadratic factor	1

Sample answer:
Since $\cos\frac{-2\pi}{9}+i\sin\frac{-2\pi}{9}=\cos\frac{2\pi}{9}-i\sin\frac{2\pi}{9}$ and similarly for $\cos\frac{-4\pi}{9}+i\sin\frac{-4\pi}{9}$ and $\cos\frac{-8\pi}{9}+i\sin\frac{-8\pi}{9}$,

$$\begin{aligned}z^6+z^3+1&=\left(z-\left(\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}\right)\right)\left(z-\left(\cos\frac{2\pi}{9}-i\sin\frac{2\pi}{9}\right)\right)\\&\times\left(z-\left(\cos\frac{4\pi}{9}+i\sin\frac{4\pi}{9}\right)\right)\left(z-\left(\cos\frac{4\pi}{9}-i\sin\frac{4\pi}{9}\right)\right)\\&\times\left(z-\left(\cos\frac{8\pi}{9}+i\sin\frac{8\pi}{9}\right)\right)\left(z-\left(\cos\frac{8\pi}{9}-i\sin\frac{8\pi}{9}\right)\right)\end{aligned}$$

$$\therefore z^6+z^3+1=\left(z^2-2z\cos\frac{2\pi}{9}+1\right)\left(z^2-2z\cos\frac{4\pi}{9}+1\right)\left(z^2-2z\cos\frac{8\pi}{9}+1\right)$$

Question 14 (continued)

(c) (iii) (1 Mark)
Outcomes assessed: E3, E4
Targeted Performance Bands: E3-E4

Criteria	Mark
• Correct solution	1

Sample answer:
Equating the coefficient of z^2 on the left- and right- hand side of the expression in part (ii),

$$0=1+1+1+4\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}+4\cos\frac{2\pi}{9}\cos\frac{8\pi}{9}+4\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}$$

$$\therefore \cos\frac{2\pi}{9}\cos\frac{4\pi}{9}+\cos\frac{2\pi}{9}\cos\frac{8\pi}{9}+\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}=-\frac{3}{4}$$

(d) (3 Marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E3–E4

Criteria	Mark
• Complete proof	3
• Makes use of the assumption	2
• Proof for $P(1)$	1

Sample answer:
Let $P(n)$ be the given proposition. $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}>\ln(n+1)$.

$P(1)$ is true since $\frac{1}{1}>\ln(1+1)$ (Note: $e>2\Rightarrow \ln e>\ln 2\Rightarrow 1>\ln 2$)

Assume $P(k)$ is true for some positive integer k.

i.e. $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}>\ln(k+1)$

Prove $P(k+1)$ is true:

$$\begin{aligned}\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}+\frac{1}{k+1}&>\ln(k+1)+\frac{1}{k+1}\quad\text{(using the assumption)}\\&>\ln(k+1)+\ln\left(1+\frac{1}{k+1}\right)\quad\text{(using the inequality given)}\\&=\ln\left((k+1)\left(1+\frac{1}{k+1}\right)\right)\\&=\ln\left((k+1)+1\right)\end{aligned}$$

$\therefore$ By the Principle of Mathematical Induction, $P(n)$ is true for all positive integers n.

Question 15 (15 marks)

(a) (i) (2 Marks)
Outcomes assessed: E8
Targeted Performance Bands: E3

Criteria	Mark
• Correctly shows the result	2
• Correct initial substitutions	1

Sample answer:
Let $x = a - u$.

Then $dx = -1\,du$

$x = 0 \Rightarrow u = a$
$x = a \Rightarrow u = 0$

$$\int_0^a f(x)\,dx = \int_a^0 f(a-u) \times -1\,du$$
$$= \int_0^a f(a-u)\,du$$
$$= \int_0^a f(a-x)\,dx$$

(a) (ii) (3 Marks)
Outcomes assessed: E8
Targeted Performance Bands: E3

Criteria	Mark
• Correctly evaluates the integral	3
• Significant progress towards evaluating the integral	2
• Uses the result of part (i) to set up an integral, or equivalent merit	1

Sample answer:

$$\int_0^{\frac{\pi}{2}} x\sin 2x\,dx = \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)\sin\left(2\left(\frac{\pi}{2} - x\right)\right)dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\pi}{2}\sin(\pi - 2x) - x\sin(\pi - 2x)\,dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\pi}{2}\sin 2x - x\sin 2x\,dx$$

Question 15(a) (ii) (continued)

$$2\int_0^{\frac{\pi}{2}} x\sin 2x\,dx = \int_0^{\frac{\pi}{2}} \frac{\pi}{2}\sin 2x\,dx$$

$$\int_0^{\frac{\pi}{2}} x\sin 2x\,dx = \frac{\pi}{4}\left[\frac{-\cos 2x}{2}\right]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{4}\left(\frac{-\cos\pi}{2} - \frac{-\cos 0}{2}\right)$$
$$= \frac{\pi}{4}$$

(b) (i) (2 Mark)
Outcomes assessed: E5, E9
Targeted Performance Bands: E3

Criteria	Mark
• Correctly shows that the equation of motion is given by $\ddot{x} = 2 - \frac{1}{10}v$	2
• Resolves forces correctly	1

Sample answer:
Resolving forces:

$kv \longleftarrow$ $\boxed{\text{object}}$ $\longrightarrow 140$ N

$70 \times \ddot{x} = 140 - kv$
Given $v = 10$ when $\ddot{x} = 1$
$70 \times 1 = 140 - k \times 10$
$70 = 140 - 10k$
$10k = 70$
$k = 7$
$\therefore 70\ddot{x} = 140 - 7v$
$\ddot{x} = 2 - \frac{1}{10}v$

Question 15 (continued)
(b) (ii) (3 Marks)
Outcomes assessed: E5, E8
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	3
• Correct integration	2
• Correct integral for x	1

Sample answer:

$$\ddot{x} = 2 - \frac{1}{10}v$$

$$v\frac{dv}{dx} = \frac{20-v}{10}$$

$$\int dx = \int \frac{10v}{20-v}\,dv$$

$$x = \int \frac{-10(20-v)}{20-v} + \frac{200}{20-v}\,dv$$

$$= \int -10 + \frac{200}{20-v}\,dv$$

$$= -10v - 200\ln(20-v) + C$$

$$x = 0, v = 0 \Rightarrow 0 = 0 - 200\ln 20 + C \Rightarrow C = 200\ln 20$$

$$\therefore x = -10v - 200\ln(20-v) + 200\ln 20$$

$$\therefore x = 200\left[\ln\left(\frac{20}{20-v}\right)\right] - 10v$$

Question 15 (continued)
(b) (iii) (1 Mark)
Outcomes assessed: E5, E9
Targeted Performance Bands: E3

Criteria	Mark
• Correctly explains why the speed cannot exceed 20 ms^{-1}	1

Sample answer:
The terminal velocity can be found from the equation of motion by finding v when $\ddot{x} = 0$.

$\ddot{x} = 0 \;\Rightarrow 2 - \frac{1}{10}v = 0 \;\Rightarrow v = 20$.

Since $20\,\text{ms}^{-1}$ is the terminal velocity of the object, the object's speed cannot exceed 20 ms^{-1}.

Note that solving $\frac{dv}{dt} = 2 - \frac{1}{10}v$ gives $v = 20 - Ae^{\frac{-1}{10}t}$.

Therefore, as $t \to \infty$, $v \to 20$. Hence the object's speed cannot exceed 20 ms^{-1}.

(c) (i) (2 Marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E4

Criteria	Mark
• Correct answer	2
• Progress towards the correct answer	1

Sample answer:

The first three letters are 2B's and 1C in some order. Therefore, since there are no C's in the last three letters and no B's in middle three letters, the middle three letters must contain the remaining 2C's and 1A in some order. Hence the last three letters are 2A's and 1B in some order.

The number of arrangements of (2B's and 1C) followed by (2C's and 1A) followed by (2A's and 1B) is $\frac{3!}{2!} \times \frac{3!}{2!} \times \frac{3!}{2!} = 27$.

Hence there are 27 nine letter arrangements if the first three letters are 2B's and 1C in some order.

Question 15 (continued)

(c) (ii) (2 Marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E4

Criteria	Mark
• Correct number of arrangements	2
• Some progress towards the correct answer	1

Sample answer:
There are 4 possible cases for the nine letter arrangements outlined in the table below

First 3 letters (no A's)	Middle 3 letters (no B's)	Last 3 letters (no C's)	# arrangements
BBB	CCC	AAA	1
BBC	CCA	AAB	27
BCC	CAA	ABB	27
CCC	AAA	BBB	1
		Total	56

Hence, there are a total of 56 nine letter arrangements.

Question 16 (15 marks)

(a) (i) (2 Marks)
Outcomes assessed: E2
Targeted Performance Bands: E2

Criteria	Mark
• Correct proof	2
• Progress towards the correct proof	1

Sample answer:

$(x-y)^2 \geq 0 \Rightarrow x^2+y^2 \geq 2xy$

$(y-z)^2 \geq 0 \Rightarrow y^2+z^2 \geq 2yz$

$(x-z)^2 \geq 0 \Rightarrow x^2+z^2 \geq 2xz$

Therefore,

$$2x^2+2y^2+2z^2 \geq 2xy+2yz+2xz$$
$$x^2+y^2+z^2 \geq xy+yz+xz$$

(a) (ii) (2 Marks)
Outcomes assessed: E2
Targeted Performance Bands: E3

Criteria	Mark
• Correct proof	2
• Makes progress using the given inequality or the result in part (i)	1

Sample answer:

$$\frac{1}{1+xy}+\frac{1}{1+yz}+\frac{1}{1+xz} \geq \frac{9}{1+xy+1+yz+1+xz} \quad \text{(using the given inequality)}$$
$$= \frac{9}{3+xy+yz+xz}$$
$$\geq \frac{9}{3+x^2+y^2+z^2} \quad \text{(since } xy+yz+xz \leq x^2+y^2+z^2 \text{ from part (i))}$$
$$= \frac{9}{3+9}$$
$$= \frac{3}{4}$$

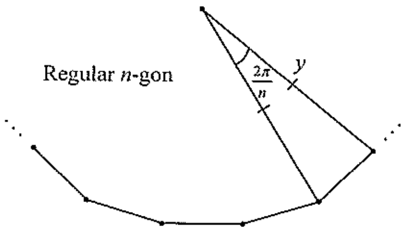
$$\therefore \frac{1}{1+xy}+\frac{1}{1+yz}+\frac{1}{1+xz} \geq \frac{3}{4}.$$

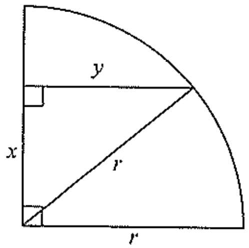
Question 16 (continued)

(b) (i) (2 Marks)
Outcomes assessed: E2, E8
Targeted Performance Bands: E3-E4

Criteria	Mark
• Correct solution	2
• Progress towards obtaining the required area	1

Sample answer:
Let y be the 'radius' of the polygonal cross-section x units from the base.





$y=\sqrt{r^2-x^2}$ since circular arcs join the top of the dome to each vertex of the base.

The horizontal cross section x units from the base is a regular n-gon which consists of n isosceles triangles (two side lengths y and included angle $\frac{2\pi}{n}$).

$$\begin{aligned}\text{Area of horizontal cross-section} &= n\times\left(\frac{1}{2}\times y^2\times\sin\left(\frac{2\pi}{n}\right)\right)\\ &= n\times\left(\frac{1}{2}\times\left(r^2-x^2\right)\times\sin\left(\frac{2\pi}{n}\right)\right)\\ &= \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)\left(r^2-x^2\right)\end{aligned}$$

Question 16 (continued)

(b) (ii) (2 Marks)
Outcomes assessed: E2, E8
Targeted Performance Bands: E3-E4

Criteria	Mark
• Correct solution	2
• Progress towards finding the volume of the dome	1

Sample answer:

$$\begin{aligned}V &= \int_0^r \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)(r^2-x^2)\,dx\\ &= \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)\left[r^2x-\frac{x^3}{3}\right]_0^r\\ &= \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)\left(\left(r^3-\frac{r^3}{3}\right)-(0-0)\right)\\ &= \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)\times\frac{2r^3}{3}\\ &= \frac{nr^3}{3}\sin\left(\frac{2\pi}{n}\right)\end{aligned}$$

(b) (iii) (1 Mark)
Outcomes assessed: E2, E8
Targeted Performance Bands: E3-E4

Criteria	Mark
• Correct solution	1

Sample answer:

$$\begin{aligned}\lim_{n\to\infty}\frac{nr^3}{3}\sin\left(\frac{2\pi}{n}\right) &= \lim_{n\to\infty}\frac{2\pi r^3}{3}\times\frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}\\ &= \frac{2\pi r^3}{3}\lim_{\frac{2\pi}{n}\to 0}\frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}\\ &= \frac{2\pi r^3}{3}\times 1 = \text{half the volume of a sphere}\end{aligned}$$

$\therefore$ as $n\to\infty$ the volume of the dome approaches that of a hemisphere.

Question 16 (continued)

(c) (i) (2 Marks)
Outcomes assessed: E4
Targeted Performance Bands: E3

Criteria	Mark
• Correct proof	2
• Makes progress simplifying the expression	1

Sample answer:

$$\frac{1}{1-x^{2^n}}-\frac{1}{1-x^{2^{n+1}}}=\frac{1-x^{2^{n+1}}-1+x^{2^n}}{\left(1-x^{2^n}\right)\left(1-x^{2^{n+1}}\right)}$$
$$=\frac{x^{2^n}-x^{2\times 2^n}}{\left(1-x^{2^n}\right)\left(1-x^{2^{n+1}}\right)}$$
$$=\frac{x^{2^n}\left(1-x^{2^n}\right)}{\left(1-x^{2^n}\right)\left(1-x^{2^{n+1}}\right)}$$
$$=\frac{x^{2^n}}{1-x^{2^{n+1}}}$$

(c) (ii) (1 Mark)
Outcomes assessed: E2, E9
Targeted Performance Bands: E3

Criteria	Mark
• Correct proof	1

Sample answer:

$$\sum_{n=0}^{N}\frac{x^{2^n}}{1-x^{2^{n+1}}}$$
$$=\left(\frac{x}{1-x^2}\right)+\left(\frac{x^2}{1-x^{2^2}}\right)+\cdots+\left(\frac{x^{2^N}}{1-x^{2^{N+1}}}\right)$$
$$=\left(\frac{1}{1-x}-\frac{1}{1-x^2}\right)+\left(\frac{1}{1-x^2}-\frac{1}{1-x^{2^2}}\right)+\cdots+\left(\frac{1}{1-x^{2^N}}-\frac{1}{1-x^{2^{N+1}}}\right)$$
$$=\frac{1}{1-x}+\left(-\frac{1}{1-x^2}+\frac{1}{1-x^2}\right)+\left(-\frac{1}{1-x^{2^2}}+\frac{1}{1-x^{2^2}}\right)+\cdots+\left(-\frac{1}{1-x^{2^N}}+\frac{1}{1-x^{2^N}}\right)-\frac{1}{1-x^{2^{N+1}}}$$
$$=\frac{1}{1-x}-\frac{1}{1-x^{2^{N+1}}}$$

Question 16 (continued)

(c) (iii) (1 Mark)
Outcomes assessed: E2, E9
Targeted Performance Bands: E3-E4

Criteria	Mark
• Correct proof	1

Sample answer:

$$\sum_{n=0}^{\infty}\frac{x^{2^n}}{1-x^{2^{n+1}}}=\lim_{N\to\infty}\left(\sum_{n=0}^{N}\frac{x^{2^n}}{1-x^{2^{n+1}}}\right)$$
$$=\lim_{N\to\infty}\left(\frac{1}{1-x}-\frac{1}{1-x^{2^{N+1}}}\right)\quad\text{(from part (ii))}$$
$$=\frac{1}{1-x}-\frac{1}{1}\ \left(\text{as } N\to\infty,\ x^{2^{N+1}}\to 0 \text{ since } -1<x<1\right)$$
$$=\frac{x}{1-x}$$

Question 16 (continued)
(c) (iv) (2 Marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E4

Criteria	Mark
• Correct solution	2
• Some progress towards the correct solution	1

Sample answer:

$$\sum_{n=0}^{\infty}\frac{1}{2014^{2^n}-2014^{-2^n}}=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2014^{2^n}}\right)}{1-\left(\frac{2014^{-2^n}}{2014^{2^n}}\right)}$$

$$=\sum_{n=0}^{\infty}\frac{\left(2014^{-2^n}\right)}{1-\left(2014^{-2\times2^n}\right)}$$

$$=\sum_{n=0}^{\infty}\frac{\left(2014^{-1}\right)^{2^n}}{1-\left(2014^{-1}\right)^{2^{n+1}}}$$

$$=\frac{2014^{-1}}{1-2014^{-1}}\quad(\text{since } -1<2014^{-1}<1)$$

$$=\frac{1}{2013}$$

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