2005 CSSA Mathematics Exam Solutions

Question 1

(a) |-6| - |-12| = 6 – 12 = -6

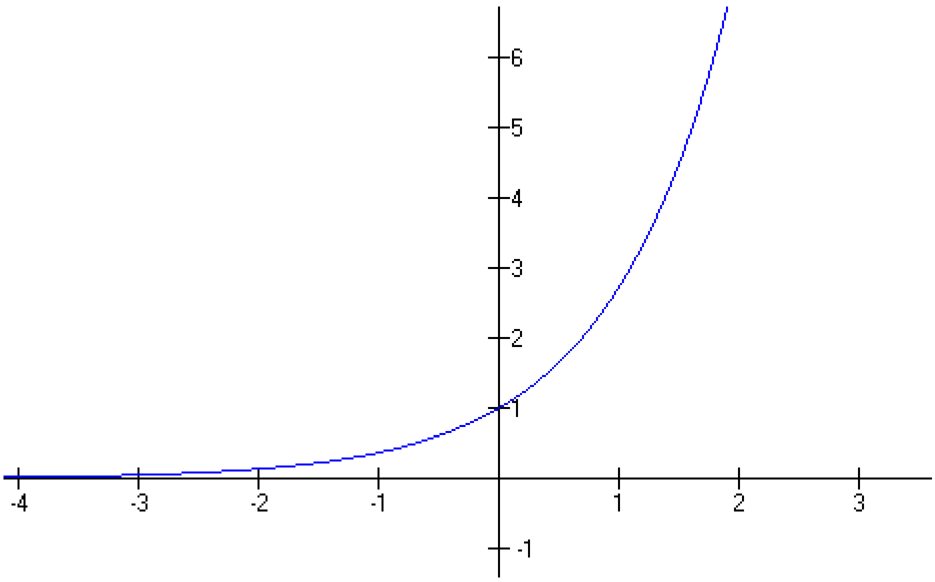
(b) $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$

When u = -5 and v = 7.5, $\frac{1}{f} = -\frac{1}{5} + \frac{1}{7.5}$, f = -15

(c)
$(x - 3)^2 = 9$
x – 3 = 3 or x – 3 = -3
x = 6 or x = 0

(d)
$$\frac{d}{dx}(x^5 + 4x^{-2}) = 5x^4 - 8x^{-3} = 5x^4 - \frac{8}{x^3}$$

(e)



Range: y > 0

(f)
$$\frac{1}{a} = \sqrt{10} - 3$$
$$a = \frac{1}{\sqrt{10} - 3}$$
$$a = \frac{1}{\sqrt{10} - 3} \text{ x } \frac{\sqrt{10} + 3}{\sqrt{10} + 3}$$
$$a = \frac{\sqrt{10} + 3}{10 - 9}$$
$$a = \sqrt{10} + 3$$

Question 2

(a)
$f(x) = x^5 - x^3$
$f(-x) = (-x)^5 - (-x)^3 = -x^5 + x^3 = -(x^5 - x^3) = -f(x)$

Hence $f(x) = x^5 - x^3$ is odd.

(b)
(i) $m_{BC} = \dfrac{0-(-1)}{2-0} = \dfrac{1}{2}$

(ii) $m_{AD} = m_{BC} = \dfrac{1}{2}$ (since AD||BC)

Equation of AD (using point A(0, 3)):

$y - 3 = \dfrac{1}{2}(x - 0)$
$2y - 6 = x$
$x - 2y + 6 = 0$

(iii)
$m_{CD} = -\dfrac{1}{m_{AD}} = -2$ (since CD and AD are perpendicular)

Equation of CD (using point C(0, -1)):

$y + 1 = -2(x - 0)$
$y + 1 = -2x$
$2x + y + 1 = 0$

(iv)

$2x + y = -1$ _________(1)
$x - 2y = -6$ _________(2)

(1) x 2

$4x + 2y = -2$ ________(3)

(3) + (2): $5x = -8$, $x = -\dfrac{8}{5}$

Sub x into either equation gives $y = \dfrac{11}{5}$

Hence $D(-\dfrac{8}{5}, \dfrac{11}{5})$

(v) ABCD is a trapezium.
$$\text{Area} = \frac{1}{2}CD(AD+BC) = \frac{1}{2}\left(\sqrt{(0+\frac{8}{5})^2+(-1-\frac{11}{5})^2}\right)\left(\sqrt{(0+\frac{8}{5})^2+(3-\frac{11}{5})^2}+\sqrt{(2+0)^2+(0+1)^2}\right) = \frac{36}{5}\text{units}^2$$

Question 3

(a)

$\tan\theta = \frac{3}{4}$

Opposite side = 3
Adjacent side = 4
Hypotenuse = $\sqrt{3^2 + 4^2} = 5$

$\sin\theta = \frac{3}{5}$

(b)

(i) $\frac{d}{dx}(\sin x \log_e x) = \cos x \log_e x + \frac{\sin x}{x}$

(ii) $\frac{d}{dx}(2\tan\frac{\pi x}{3}) = \frac{2\pi}{3}\sec^2\frac{\pi x}{3}$

(c)

(i) $\int \sin(e - x)dx = \cos(e - x) + c$

(ii) $\int_0^1 \frac{2x}{x^2+1}dx = [\ln(x^2+1)]_0^1 = \ln(2) - \ln(1) = \ln(2)$

(d)

$y = e^{4x} - 1$
$y' = 4e^{4x}$

When x = 0, y = 0, $m_T = 4$, so $m_N = -\frac{1}{4}$

Equation of normal:

$y - 0 = -\frac{1}{4}(x - 0)$
$x + 4y = 0$

Question 4

(a)

$$x^2-(2+\sqrt{3}+2-\sqrt{3})x+(2+\sqrt{3})(2-\sqrt{3})=0$$

$$x^2-4x+1=0$$

(b)

(i)

$T_1 = a = 7$

$T_{13} = a + 12d = 7 + 12d = 1$

$12d = -6$

$d = -\dfrac{1}{2}$

(ii)

$$S_n=\frac{n}{2}(2a+(n-1)d)$$

$$S_n=0$$

$$\frac{n}{2}(14-\frac{1}{2}(n-1))=0$$

$$n(14-\frac{1}{2}n+\frac{1}{2})=0$$

$$14n-\frac{1}{2}n^2+\frac{1}{2}n=0$$

$$n^2-29n=0$$

$$n=0 \text{ or } 29$$

(c)

(i)

$QS^2 = 2^2 + 2^2$ (Pythagoras' theorem)

$QS = 2\sqrt{2}$

(ii)

$\tan 60° = \dfrac{PR}{QR} = \dfrac{PR}{2}$, so $PR = 2\tan 60° = 2\sqrt{3}$

$PS = PR – SR = 2\sqrt{3} - 2$

(iii)

$\angle QPR = 180 – (60 + 90) = 30°$ (angle sum in triangle QPR = 180°)

$$\frac{\sin 15}{2\sqrt{3}-2}=\frac{\sin 30}{2\sqrt{2}}\rightarrow\frac{\sin 15}{2\sqrt{3}-2}=\frac{1}{4\sqrt{2}}\rightarrow\sin 15=\frac{2\sqrt{3}-2}{4\sqrt{2}}\rightarrow\sin 15=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

Question 5

(a)

(i)
$y = x^3 - 6x^2 + 9x + 4$
$y' = 3x^2 - 12x + 9$
$y' = 0$ for stationary points
$3x^2 - 12x + 9 = 0$
$x^2 - 4x + 3 = 0$
$x = 1$ or $x = 3$
$y = 8$ or $y = 4$

Stationary points are (1,8) and (3,4)

$y'' = 6x - 12$

At $x = 1$, $y'' = -6 < 0$. So (1,8) is a maximum turning point
At $x = 3$, $y'' = 6 > 0$. So (3,4) is a minimum turning point

(ii)
$y'' = 0$ for inflexion points
$6x - 12 = 0$
$x = 2$
$y = 6$

Inflexion point at (2,6)

(iii)

(iv)
$y' = 3x^2 - 12x + 9$
$x^2 - 4x + 3 < 0$

$1 < x < 3$

(b)
(i)

x	1	1.5	2	2.5	3
$y = 3^{x-1}$	1	1.732	3	5.196	9

(ii)

$$A = \frac{0.5}{3}[1 + 4(1.73 + 5.196) + 2(3) + 9] = 7.29 \text{ units}^2$$

Question 6

(a)
(i) $2a^2 - 7a + 3 = (a - 3)(2a - 1)$

(ii)
$2(\log_2 x)^2 - 7(\log_2 x) + 3 = 0$
Let $a = \log_2 x$

$2a^2 - 7a + 3 = 0$
$a = 3$ or $a = 0.5$
$\log_2 x = 3$ or $\log_2 x = 0.5$
$x = 2^3$ or $x = 2^{0.5}$
$x = 8$ or $x = \sqrt{2}$

(b)

(i)
In ΔABM and ΔAPD

$\angle ABM = \angle APD = 90°$
$\angle BMA = \angle PAD$ (alternate angles, BM||AD, AM transversal)
$\therefore \angle BAM = \angle PDA$ (since two angles are equal, third must also be equal for angle sum to be 180°)
$\therefore \Delta ABM \,|||\, \Delta APD$ (equiangular)

(ii)
$\frac{PD}{AB} = \frac{AD}{AM}$ (corresponding sides of similar triangles are in ratio)

BM = 30 cm (since BC = AD = 60 and M is the midpoint of BC)

$AM^2 = BM^2 + AB^2$
$AM = \sqrt{30^2 + 40^2} = 50$

$\frac{PD}{40} = \frac{60}{50}$

PD = 48 cm

(iii)

$AD^2 = AP^2 + PD^2$
$AP = \sqrt{60^2 - 48^2} = 36\,\text{cm}$

(iv)
Area of rectangle = 60 x 40 = 2400 cm^2
Area of triangle ABM = $\frac{1}{2}$ x 40 x 30 = 600 cm^2
Area of triangle ADP = $\frac{1}{2}$ x 36 x 48 = 864 cm^2
Area of quad
= Area of rectangle – (Area of triangle ABM + Area of triangle ADP)
= 2400 – (600 + 864)
= 936 cm^2

Question 7

(a)

Note: N = Nicole wins set, M = Mariana wins set

(i) Game will last two sets if Nicole wins the first two or if Mariana wins the first two
P(two sets) = P(N)xP(N) + P(M)xP(M) = 0.7x0.7 + 0.3x0.3 = 0.49 + 0.09 = 0.58

(ii)
P(Nicole wins)
= P(N)P(N) + P(M)P(N)P(N) + P(N)P(M)P(N)
= (0.7 x 0.7) + (0.3 x 0.7 x 0.7) + (0.7 x 0.3 x 0.7)
= 0.784

(iii)
P(Mariana wins) = 1 – P(Nicole wins) = 1 – 0.784 = 0.216

(b)
(i) $N = 20000e^{0.003t}$

When $t = 0$, $N = 20000e^{0} = 20000$ bacteria

(ii)

When $t = 20$, $N = 20000e^{0.003 \times 20} = 21236$ bacteria

(iii) Bacteria has doubled when N = 40000

$40000 = 20000e^{0.003t}$
$e^{0.003t} = 2$
$0.003t = \ln 2$
$t = 231.05$ seconds

(iv)

$N = 20000e^{0.003t}$

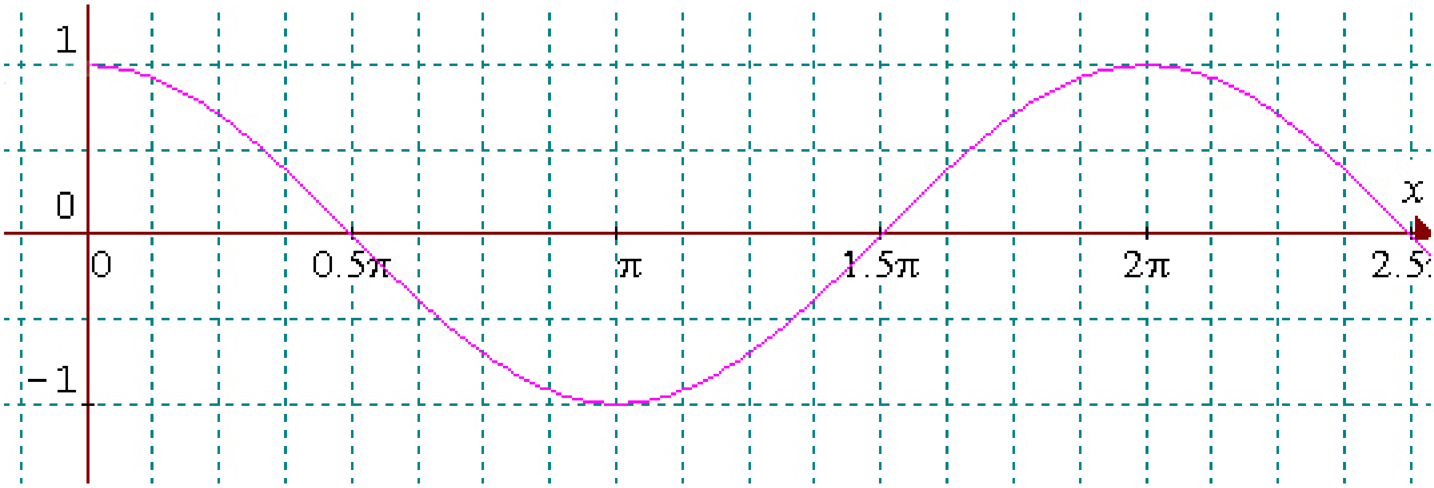
$$\frac{dN}{dt} = 60e^{0.003t}$$

When $t = 20$, $\dfrac{dN}{dt} = 60e^{0.003 \times 20} = 63.71$

The rate the number of bacteria is increasing by when t = 20 seconds is 63.71 bacteria/second

Question 8

(a) (i)



(ii)

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

(iii)

$\frac{1}{2} > \cos x$

ie. $\cos x < \frac{1}{2}$

$\frac{\pi}{3} < x < \frac{5\pi}{3}$ $(0 \le x \le 2\pi)$

(b)

(i)

$x = at^2 + bt$

$v = \frac{dx}{dt} = 2at + b$

(ii)

At $t = 0$, $v = 16$

$b = 16$

At $t = 8$ $x = 0$

$0 = a(8)^2 + 16(8)$

$-128 = 64a$

$a = -2$

(iii)

Now $v = -4t + 16$. When the object is at "rest", then $v = 0$

$0 = -4t + 16$

$\therefore t = 4$ seconds when object at rest

(iv)

When it is at rest then $t = 4$

Position would be $x = -2(4)^2 + 16(4) = 32$ cm from O.

Question 9

(a)

(i) Area = $\frac{1}{2}$ x 3^2 x $\frac{\pi}{3}$ = $\frac{3\pi}{2}$ units^2

(ii) Area = $\frac{1}{2}$ x r^2 x $\frac{\pi}{3}$ = $\frac{r^2\pi}{6}$ units^2

(iii)

Shaded area = Area(OSR) – Area(OPQ) = $\frac{r^2\pi}{6}$ - $\frac{3\pi}{2}$

$\frac{r^2\pi}{6} - \frac{3\pi}{2} = \frac{27\pi}{6}$

$r^2\pi - 9\pi = 27\pi$
$r^2 = 36$
$r = 6$ (ignoring negative since $r > 0$)

PS = r – 3 = 6 – 3 = 3 cm

(b)
(i)

$A_1 = 12000 + 12000(0.06) = 12000(1.06)$
$A_2 = 12000(1.06) + 12000(1.06)(0.06) = 12000(1.06)^2$
.
.
.
$A_{10} = 12000(1.06)^{10} = \$21490.17$

(ii)
$A_1 = 12000 + 12000(0.06) + 1000 = 12000(1.06) + 1000$
$A_2 = (12000(1.06) + 1000) + (12000(1.06) + 1000)(0.06) + 1000 = (12000(1.06) + 1000)(1.06) + 1000 = 12000(1.06)^2 + 1000(1.06 + 1)$
.
.
.
$A_{10} = 12000(1.06)^{10} + 1000(1.06^9 + 1.06^8 + \ldots + 1) = 12000(1.06)^{10} + 1000\frac{1.06^{10} - 1}{1.06 - 1} = \$34670.97$

(iii)

$35639.36 = 12000(1 + r)^{10}$
$(1 + r) = 1.115$

$r = 0.115$

Rate of interest = 11.5%

Question 10

(a) (i)
$\log_e e^{2ax}$
$=2ax \log_e e$ ($\log_b c^d = d\log_b c$)
$=2ax(1)$ ($\log_a a=1$)
$=2ax$

(ii)

$$\int_0^a \log_e e^{2ax}\,dx$$

$$= \int_0^a 2ax\,dx$$

$=[2ax^2/2]^a{}_0$
$=[ax^2]^a{}_0$
$=a^3-a(0)$
$=a^3$

(b) (i)

$D=\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
$D^2=(x-1)^2+(y-4)^2$ --- (1)

But $y^2 = 2x$
$\therefore x = \dfrac{y^2}{2}$

Subbing $x = \dfrac{y^2}{2}$ in (1)

$$D^2 = (\frac{1}{2}y^2-1)^2 + (y-4)^2$$

As required.

(ii) Now D is a distance and is always positive, $\therefore$ the point of minimum of D^2 is also the point of minimum of D

$$\frac{dD^2}{dy} = 2(\frac{1}{2}y^2-1)y + 2(y-4)$$
$= 2y(\frac{1}{2}y^2-1) + 2y - 8$
$= y^3 - 2y + 2y - 8$
$= y^3 - 8$
$= (y - 2)(y^2 + 2y + 4)$ (Difference of two cubes)

For stationary points $\dfrac{dD^2}{dy} = 0$

$\therefore$ $(y - 2)(y^2 + 2y + 4) = 0$ for stationary points

$y = 2$ for stationary point

check concavity

$$\frac{d^2D^2}{dy^2} = 3y^2$$

at $y=2$, $\frac{d^2D^2}{dy^2} = 12 > 0 \therefore$ it is a minimum.

Hence the minimum distance occurs at $y=2$.

(iii)
Minimum distance occurs at $y=2$
Sub $y = 2$ into $D^2=(\frac{1}{2}y^2-1)^2+(y-4)^2$

$D^2= (\frac{1}{2}4-1)^2 + (2 - 4)^2$
$= 1 + 4$
$= 5$

$\therefore D=\sqrt{5}$ units (distance must be > 0)

As required.