Barker College

Trial HSC Examination

Mathematics Extension 2

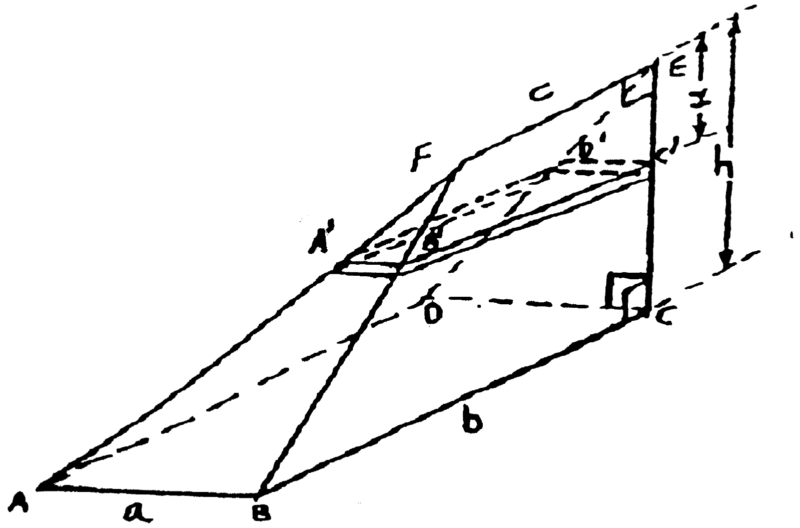
2002

1. **(a)** Find $\int_1^e \frac{dx}{x(1+(\ln x)^2)}$ by substituting $u = \ln x$.
(b) Find $\int \frac{x+1}{x^2+4}\, dx$
(c) Find $\int \frac{x^2+4}{x+1}\, dx$
(d) Evaluate $\int_0^{\frac{\pi}{4}} x^2 \sin x \; dx$
(e) Prove that $\int_0^{\frac{1}{4}} \sqrt{1-4x^2}\; dx = \frac{\pi}{24} + \frac{\sqrt{3}}{16}$

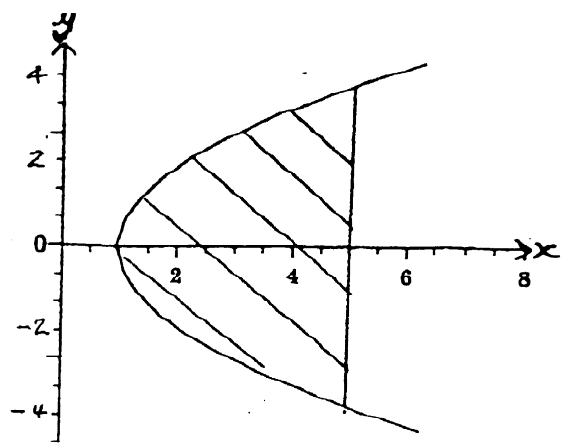
2. **(a)** Given that $f(x) = e^{-x}$, sketch the following showing the main features.
(i) $y = -f(x)$
(ii) $y = 1 - f(x)$
(iii) $y = \frac{1}{1-f(x)}$
(iv) $y = \left|\frac{1}{1-f(x)}\right|$
(b) Next to each graph state whether it is odd, even or neither.
(c) **(i)** For $x^2 + 2xy + y^5 = 4$, show that $\frac{dy}{dx} = \frac{-2x-2y}{2x+5y^4}$
(ii) A plane curve is defined implicitly by the equation $x^2 + 2xy + y^5 = 4$. This curve has a horizontal tangent at the point $P(x_1, y_1)$. Show that x_1 is a root of the equation $x^5 + x^2 + 4 = 0$.

3. **(a)** If $z_1 = 1 + 2i, z_2 = 2 - i$ and $z_3 = -1 + i\sqrt{3}$, find $\left|\frac{z_1 z_2}{i z_3}\right|$
(b) Simplify $\frac{(2\cos\theta + 2i\sin\theta)^5 (2\cos\theta + 2i\sin\theta)^{-3}}{(\cos 2\theta + i \sin 2\theta)}$
(c) Z is the point representing the complex number z on an Argand diagram.
(i) Describe in words the geometrical significance of the expressions $|z-2|$ and $\Re(z)$
(ii) Hence, or otherwise, sketch the locus of Z given that $|z-2| = \Re(z)$. Show all important features of this locus.
(d) Triangle OAB is an isosceles triangle with $AO = OB$ and $\angle OBA = 75°$. If O is the origin and A represents the complex number $-\sqrt{3}+i$, find **two** possible complex numbers represented by the point B, in the form $a + bi$.

4. **(a)** Consider solid $ABCDEF$ whose height is h, and whose base is a rectangle $ABCD$, where $AB = a, BC = b$ and the top edge $EF = c$.



Consider a rectangle slice $A'B'C'D'$ (parallel to the base $ABCD$) which is x units from the top edge with width Δx.
NOTE: $B'C'||BC$ and $A'B'||AB$
(i) Show that the volume Δv of the slice is given by $\Delta v = \left(\frac{x}{h}a\right)\left(c + \frac{b-c}{h}x\right)\Delta x$
(ii) Hence, show that the volume of the solid $ABCDEF$ is $\frac{ha}{6}(2b + c)$
(b) The diagram shows the region bounded by the curve $y^2 = 4(x - 1)$ and the line $x = 5$. By using the method of cylindrical shells, or otherwise, find the volume of the solid formed by rotating the given region about the y-axis.



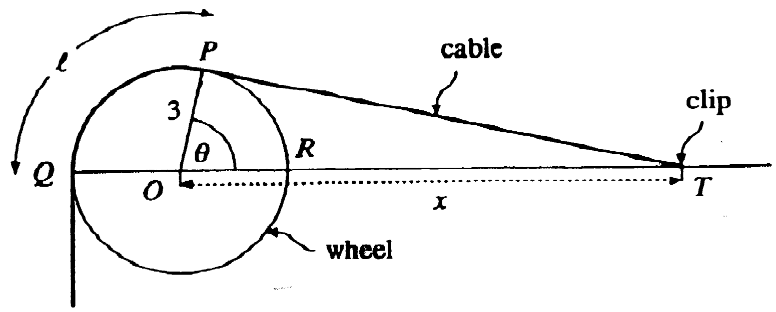
- 5 -

5. The normal at $P(ct, \frac{c}{t})$ to the hyperbola $xy = c^2$ meets the curve again at Q.
(a) Prove that the equation of the normal is $t^3x - ty = ct^4 - c$
(b) Find the coordinates of Q.
(c) A line from P through the origin meets the hyperbola again at R. Prove that PR is perpendicular to QR.

(d) If M is the midpoint of PQ, find the equation of the locus M.

6. (a) α and β are the complex roots of $iz^2 + \sqrt{3}z - 1 = 0$.
(i) Find α and β in $a + ib$ form.
(ii) Show that $\alpha^2\beta^2 + 1 = 0$.
(b) Solve the equation $4x^3 - 12x^2 + 11x - 3 = 0$ given that the roots are in arithmetic sequence.
(c) (i) Prove, by calculus if you wish, that the polynomial equation
$\frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x + c = 0$ has no real roots if $c > 9\frac{1}{3}$
(ii) Find an approximation for the **largest** root of the polynomial equation in **(i)** above. if $c = -2$, using one application of Newton's Method.

7. (a) Let n be a positive integer where $I_n = \int_1^2 (\ln x)^n \, dx$
(i) Prove that $I_n = 2(\ln 2)^2 - nI_{n-1}$
(ii) Hence, evaluate $\int_1^2 (\ln x)^4 \, dx$
(b)



A long cable is wrapped over a wheel of radius 3 metres and one end is attached to a clip at T. The centre of the wheel is at O and QR is a diameter. The point T lies on the line OR at a distance x metres from O. The cable is tangential to the wheel at P and Q as shown. Let $\angle POR = \theta$ (in radians). The length of cable in contact with the wheel is l metres; that is, the length of the arc between P and Q is l metres.
(i) Explain why $\cos\theta = \frac{3}{x}$
(ii) Show that $l = 3(\pi - \cos^{-1}(\frac{3}{x}))$
(iii) Show that $\frac{dl}{dx} = \frac{-9}{x\sqrt{x^2-9}}$
(iv) What is the significance of the fact that $\frac{dl}{dx}$ is negative?
(v) Let $s = l + PT$. Given that $PT^2 = QT \times RT$, or otherwise, express s in terms of x.
(vi) The clip at T is moved away from O along the line OR at a constant speed of 2 metres per second. Find the rate at which s changes when $x = 10$.

8. (a) It is given that the equation $ax^4 + 4bx + c = 0$ has a double root. If α is the double root, show that $a\alpha^3 + b = 0$ and deduce that $ac^3 = 27b^4$

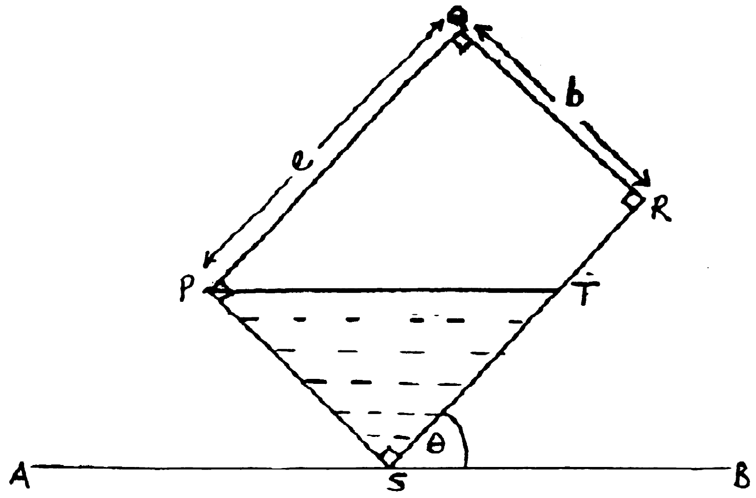
(b) $P(x)$ is divided by $(x-a)(x-b)$ so that a remainder $R(x)$ is obtained. Show that the remainder is given by $R(x) = \left(\frac{P(a)-P(b)}{a-b}\right)x + \frac{aP(b)-bP(a)}{a-b}$.

(c) Using the fact that $\cos\theta = \sin(\frac{\pi}{2} - \theta)$, or otherwise,

(i) find a general solution of the equation $\sin 3x = -\cos 2x$

(ii) find the smallest positive solution of the equation $\sin 3x = -\cos 2x$

(d) A rectangular fish tank $PQRS$ is tilted at an angle of θ to the horizontal surface AB. The surface of the water is PT, $QR = b$ and $RS = e$.



If the fish tank is lowered so that SR lies on AB, prove that the height, h, of the water in the tank is given by $h = \frac{b^2 \cot\theta}{2e}$