Sydney Grammar School
Mathematics Department
Trial Examinations 2008

FORM VI

MATHEMATICS EXTENSION 1

Examination date

Wednesday 13th August 2008

Time allowed

2 hours (plus 5 minutes reading time)

Instructions

All seven questions may be attempted.
All seven questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet.
Hand in the seven questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

SGS booklets: 7 per boy. A total of 1250 booklets should be sufficient.
Candidature: 125 boys.

Examiner

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QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Simplify $\dfrac{n!}{(n-1)!}$. **1**

(b) Write down the derivative of $y = \cos^{-1} x^2$. **1**

(c) Find $\displaystyle\int \frac{1}{40+x^2}\,dx$. **1**

(d) Simplify $\log_e \sqrt{e}$. **1**

(e) Write down a primitive of $2x\,e^{x^2}$. **1**

(f) Write $\cos 2\theta$ in terms of t, where $t = \tan\theta$. **1**

(g) A is the point $(-6, 2)$ and B is the point $(4, 10)$. Find the coordinates of the point P that divides the interval AB internally in the ratio $7:4$. **2**

(h) Sketch the graph of the polynomial function $y = x^3(3-x)$. (There is no need to find the coordinates of the turning point.) **2**

(i) Use the identity $(1+x)^n = \displaystyle\sum_{r=0}^{n} {}^n\mathrm{C}_r\, x^r$ to prove that **2**

$$ {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 + \cdots + {}^n\mathrm{C}_n = 2^n. $$

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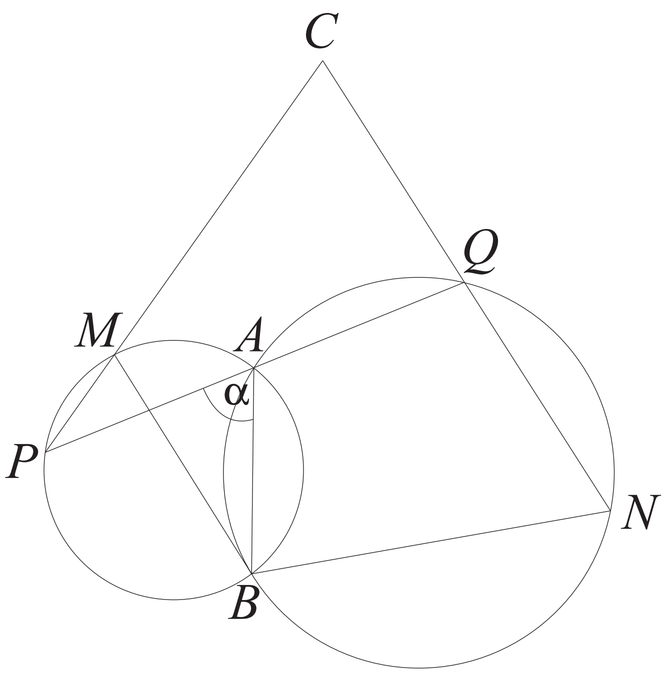
QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) Use the substitution $x = u - 2$ to find $\displaystyle\int \frac{x}{(x+2)^2}\,dx$. **3**

(b) Solve the inequation $\dfrac{x}{x+2} > 0$. **3**

(c) Show that $\tan\left(\tan^{-1} 2 - \tan^{-1}\sqrt{2}\right) = \dfrac{5\sqrt{2}-6}{7}$. **3**

(d)



The diagram above shows two circles intersecting at A and B. The points P, A and Q are collinear, and the chords PM and NQ, when produced, intersect at C. Let $\angle PAB = \alpha$.

(i) Give a reason why $\angle BNQ = \alpha$. **1**

(ii) Prove that the quadrilateral $CMBN$ is cyclic. **2**

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) An ice-cube is taken out of a freezer and begins to melt. Assume that it remains a cube as it does so. If its edge length is decreasing at the constant rate of 2 mm/min, find the rate at which its volume is decreasing at the instant when the edge length is 15 mm. **4**

(b) It is known that the polynomial equation $6x^3 - 17x^2 - 5x + 6 = 0$ has three real roots, and that two of them have a product of -2.

(i) Use the product of the roots to find one of the three roots. **1**

(ii) Use the sum of the roots, or any other suitable method, to find the other two roots. **3**

(c) Find the exact value of $\displaystyle\int_0^{\frac{\pi}{2}} \left(\cos x - \cos^2 x\right)\, dx$. **4**

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) Prove by mathematical induction that for all positive integer values of n, **4**

$$1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \cdots + n(n+1)^2 = \tfrac{1}{12}n(n+1)(n+2)(3n+5).$$

(b) Let α be the real root of the equation $\cos x = 2x$.

(i) On the same diagram, sketch the graphs of the functions $y = \cos x$ and $y = 2x$. **1**

(ii) Show α on your diagram. **1**

(iii) Use one application of Newton's method with starting value $\frac{1}{2}$ to estimate α. Write your answer correct to two decimal places. **3**

(c) Use the identity $(1+x)^4(1+x)^{96} = (1+x)^{100}$ to prove that **3**

$$\binom{96}{4} + \binom{4}{1}\binom{96}{3} + \binom{4}{2}\binom{96}{2} + \binom{4}{3}\binom{96}{1} = \binom{100}{4} - 1.$$

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QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a) Find the term independent of x in the expansion of $\left(ax^3 + \dfrac{b}{x^2}\right)^{5n}$, where n is a positive integer. **4**

(b) Newton's law of cooling states that the rate of decrease of the temperature of a heated body is proportional to the excess of the temperature of the body over that of its surroundings. Using t for time (in minutes), H for the temperature of the body (in °C), and S for the constant temperature of the surroundings (also in °C), the law of cooling can be modelled by the differential equation $\dfrac{dH}{dt} = -k\,(H - S)$, where k is a positive constant.

(i) Show that the function $H = Ae^{-kt} + S$ satisfies the differential equation, where A is a constant. **1**

(ii) Suppose that a body is heated to 80 °C in a room whose temperature is 20 °C, and that after 5 minutes the temperature of the body is 70 °C.

(α) Show that, at any time $t \geq 0$, $H = 20 + 60\left(\dfrac{5}{6}\right)^{\frac{t}{5}}$. **3**

(β) Find, correct to one decimal place, the temperature of the body after one hour. **1**

(c) Let $P(a) = a^2(b+c) + b^2(c+a) + c^2(a+b) + 2abc$.

(i) Use the factor theorem to show that $a + b$ is a factor of $P(a)$. **2**

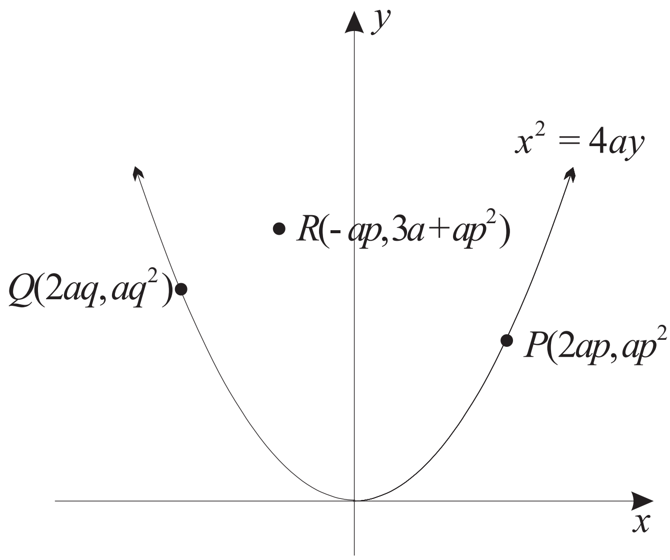
(ii) Hence, or otherwise, factorise $P(a)$. **1**

Exam continues overleaf ...

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) A particle moves along the x-axis. It starts from rest at the point $x = 1$. Its acceleration is given by $\ddot{x} = -4\left(x + \dfrac{1}{x^3}\right)$. Find its velocity when it is half-way from its starting point to the origin. $\boxed{4}$

(b)



In the diagram above, $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are distinct points on the parabola $x^2 = 4ay$, and R is the point $(-ap, 3a + ap^2)$.

(i) Show that the normal to the parabola at P has equation $x + py = 2ap + ap^3$. $\boxed{2}$

(ii) Show that the normal at P passes through R. $\boxed{1}$

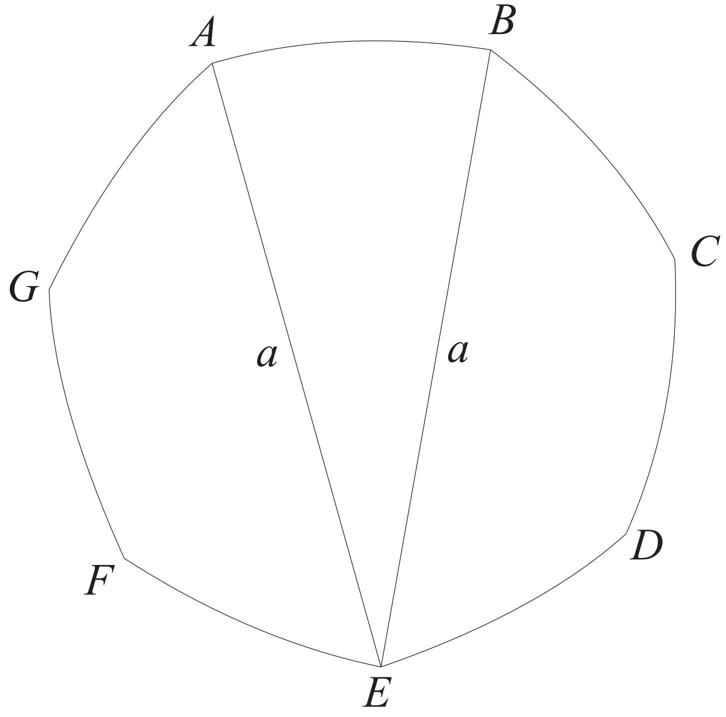
(iii) If the normal at Q also passes through R, show that $q^2 + pq - 1 = 0$. $\boxed{2}$

(iv) Show that there are always two real values of q satisfying the equation in part (iii). $\boxed{1}$

(v) Deduce that three normals to the parabola, two of which are perpendicular to each other, pass through the point R. (You may assume that $p^2 \neq \frac{1}{2}$.) $\boxed{2}$

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows a British 50 pence coin. The seven circular arcs AB, BC, ..., GA are of equal length and their centres are E, F, ..., D respectively. Each arc has radius a.

(i) Show that the sector AEB has area $\frac{1}{14}\pi a^2$. **2**

(ii) Hence, or otherwise, show that the face of the coin has area $\frac{1}{2}a^2\left(\pi - 7\tan\frac{\pi}{14}\right)$. **2**

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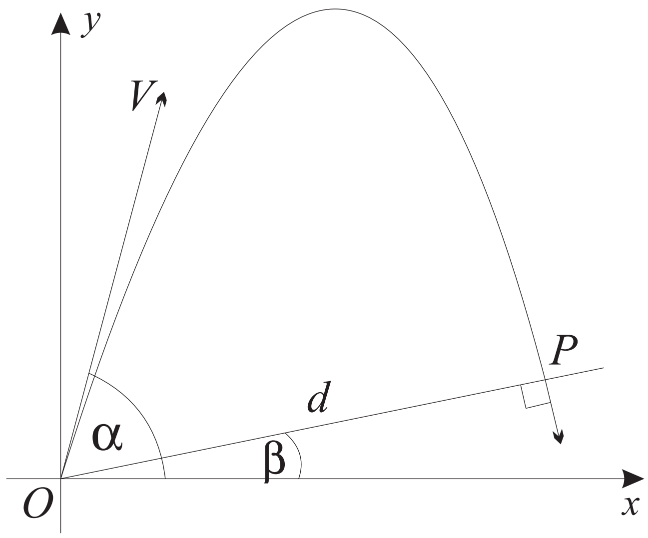
(b)



The diagram above shows the parabolic path of a particle that is projected from the origin O with velocity V at an angle of α to the horizontal. It lands at the point P, which lies on a plane inclined at an angle of β to the horizontal. When the particle strikes the plane, it is travelling at 90° to the plane.

Let $OP = d$, and assume that the horizontal and vertical components of the displacement of the particle from O while it is moving on its parabolic path are given by

$$x = Vt\cos\alpha \qquad \text{and} \qquad y = Vt\sin\alpha - \tfrac{1}{2}gt^2,$$

where t is the time elapsed, and g is acceleration due to gravity.

(i) Find the coordinates of P in terms of d and β. **1**

(ii) By substituting the coordinates of P found in part (i) into the displacement equations, show that **2**

$$d = \frac{2V^2\cos^2\alpha}{g\cos^2\beta}\Big(\tan\alpha\cos\beta - \sin\beta\Big).$$

(iii) By resolving the horizontal and vertical components of the velocity at P, show that **3**

$$\cot\beta = \frac{gd\cos\beta}{V^2\cos^2\alpha} - \tan\alpha.$$

(iv) Hence show that $\tan\alpha = \cot\beta + 2\tan\beta$. **2**

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1}, \;\; n \neq -1; \;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \;\; x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax}, \;\; a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \;\; a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \;\; a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \;\; a > 0, \;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

$$\text{NOTE}: \;\; \ln x = \log_e x, \;\; x > 0$$