Trial HSC Solutions 2002 - Mathematics Extension 1

Solutions	Marks	Comments
Question 1.		
a) C is external to AB so use ratio $8 : -3$		Could also use B divides AC internally in ratio 6:3 & solve to find (p, q)
$p = \frac{8(7)-3(2)}{8-3}$ $\quad q = \frac{8(7)-3(-3)}{8-3}$	1	
$= \frac{56-6}{5}$ $\quad = \frac{56+9}{5}$ $= 10$ $\quad = 13$	1	
$C(p,q)$ is $(10,13)$		
(b) $\sum_{n=3}^{16} 3n-1 = 8+11+14+\ldots\ldots+47$ This Arithmetic Series with $d=3, a=8, l=47, n=14$ $S_n = \frac{n}{2}(a+l)$ or $S_n = \frac{n}{2}(2a+(n-1)d)$ $= \frac{14}{2}(8+47)$ $\quad = \frac{14}{2}(2(8)+13(3))$ $= 7\times 55$ $= 385$	1.	Mark is for interpreting Σ notation and doing substitution into series. Still give 1 mark if an arithmetic error is made.
(c) $f(x) = e^t - 4t^2$ $\quad f(4) = e^4 - 64$ $f'(x) = e^t - 8t$ $\quad f'(4) = e^4 - 32$	1.	Finding correct values for $f(4)$ and $f'(4)$
$a_1 = a_0 - \frac{f(a_0)}{f'(a_0)}$ $= 4 - \frac{e^4-64}{e^4-32}$ $= 4 - (-0.416)$	1	Sub correctly into formula and evaluate.
$= 4.4$ (to 1 decimal place)		1 mark if recall formula incorrectly but sub correctly.

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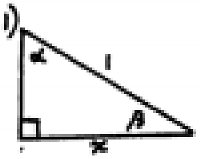
Solutions	Marks	Comments
1 continued		
(d) $\frac{d}{dx}(1+x^2)\tan^{-1}x = (1+x^2).\frac{1}{1+x^2}$ $+ \tan^{-1}x.(2x)$	1	Use product rule correctly
$= 1 + 2x\tan^{-1}x$	1	Simplify.
(e) $\int_5^8 \frac{dx}{\sqrt{x^2-16}} = \left[\ln\left(x+\sqrt{x^2-16}\right)\right]_5^8$	1.	Use standard integral correctly.
$= \ln\left(8+\sqrt{64-16}\right)$ $- \ln\left(5+\sqrt{25-16}\right)$ $= \ln(8+\sqrt{48}) - \ln(5+\sqrt{9})$ $= \ln(8+4\sqrt{3}) - \ln 8$ $= \ln\left(\frac{8+4\sqrt{3}}{8}\right)$ $= \ln\left(\frac{2+\sqrt{3}}{2}\right)$	1	Substitute correctly.
(f) $\frac{x+3}{x-1} \leq 2$ Multiply by $(x-1)^2$ since it is +ve. $(x-1)^{\cancel{2}}\frac{(x+3)}{\cancel{(x-1)}} \leq 2(x-1)^2$.		3marks for correct result by any method.
$x^2+2x-3 \leq 2x^2-4x+2$ $0 \leq x^2-6x+5$	1	obtaining quadratic
$(x-5)(x-1) \geq 0$	1	finding boundary points.
Test  $x < 1$ or $x \geq 5$	1	correct inequality signs.

Solutions	Marks	Comments
Question 2 (a) (i) 12! arrangements of beads = ${}^{12}P_{12}$ 4! arrangements of 4 red beads = ${}^{4}P_{4}$ ∴ Number of distinct arrangements = $\frac{{}^{12}P_{12}}{{}^{4}P_{4}}$ $= \frac{12!}{4!}$ $= 19\,958\,400$	1.	Answer correct if left as factorials or permutations.
ii) Arrangements in circle = ${}^{11}P_{11} = 11!$ Arrang's of red beads = ${}^{4}P_{4} = 4!$ 2 ways of viewing bracelet. ∴ Number of distinct bracelets = $\frac{{}^{11}P_{11}}{{}^{4}P_{4} \times 2}$ $= \frac{11!}{4! \times 2} = 831600$	1	"
iii) If the 4 red are together we have 9 units to arrange in a circle in $8! = {}^{8}P_{8}$ ways. ∴ Number of distinct bracelets = $\frac{8!}{2} = 20160$	1	"
(b) (i) let $x = 1$. $(1+1)^n = \binom{n}{0} + \binom{n}{1}1 + \binom{n}{2}1^2 + \binom{n}{3}1^3 + \dots \binom{n}{n}1^n$ ∴ $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots \binom{n}{n}$ as required.	1	
(ii) let $x = -1$ $(1+(-1))^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots \binom{n}{n}$ $0 = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} - [\binom{n}{1} + \binom{n}{3} + \dots]$ ∴ $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots$ From (i) $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots + \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots = 2^n$ ∴ $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots + \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots = 2^n$ $2[\binom{n}{1} + \binom{n}{3} + \binom{n}{5} \dots] = 2^n$ $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots = \frac{2^n}{2} = 2^{n-1}$ as required.		2 marks for any valid proof. 1 mark if sub correctly $x=-1$ but don't follow on.

Solutions	Marks	Comments
2 continued. (c) $P(1) = (1)^3 + 4(1)^2 + (1) - 6$ $= 1 + 4 + 1 - 6$ $= 0.$ ∴ $x - 1$ is a factor of $P(x)$	1	Factor Theor
$x-1 \overline{) x^3 + 4x^2 + x - 6}$ quotient $x^2 + 5x + 6 = (x+3)(x+2)$ $x^3 - x^2$ $5x^2 + x$ $5x^2 - 5x$ $6x - 6$ $6x - 6$ 0	1	Division
$P(x) = (x-1)(x+3)(x+2)$	1	Factorisation.
(d) $u = 2 + x^4$ $\frac{du}{dx} = 4x^3$ $du = 4x^3dx$ $x = 1$ $u = 2 + 1^4$ $= 3$ $x = 0$ $u = 2 + 0^4$ $= 2$ $\int_0^1 \frac{x^3dx}{(2+x^4)^2} = \frac{1}{4}\int_0^1 \frac{4x^3dx}{(2+x^4)^2}$ $= \frac{1}{4}\int_2^3 \frac{du}{u^2}$ $= \frac{1}{4}\int_2^3 u^{-2}du$ $= \frac{1}{4}\left[-u^{-1}\right]_2^3$	2	1 for limits 1 for substituti of variables
$= \frac{1}{4}\left[-3^{-1} - (-2^{-1})\right]$ $= \frac{1}{4}\left[-\frac{1}{3} + \frac{1}{2}\right]$ $= \frac{1}{4} \cdot \frac{1}{6}$ $= \frac{1}{24}$	1	For evaluation of integral

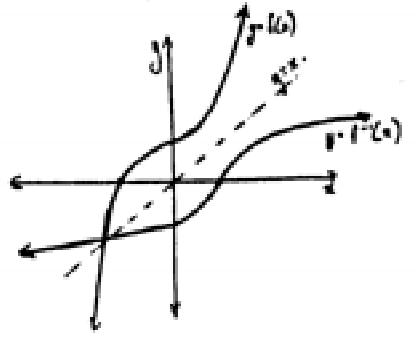
Solutions	Marks	Comments
Question 3) (i) $f'(x) = \lim\limits_{h\to 0} \dfrac{2^{x+h} - 2^x}{h}$ $= \lim\limits_{h\to 0} \dfrac{2^x . 2^h - 2^x}{h}$ $= \lim\limits_{h\to 0} \dfrac{2^x(2^h - 1)}{h}$	1	Factorising
$= 2^x . \lim\limits_{h\to 0} \dfrac{2^h - 1}{h}$ $= c . 2^x$ since $\lim\limits_{h\to 0} \dfrac{2^h-1}{h} =$ constant	1	Taking 2^x out of limit.
ii) By calculator. $h = 0{\cdot}1$ $\dfrac{2^h - 1}{h} = 0{\cdot}717$ $h = 0{\cdot}01$ $\dfrac{2^h - 1}{h} = 0{\cdot}695$ $h = 0{\cdot}001$ $\dfrac{2^h - 1}{h} = 0{\cdot}693$ $\therefore \lim\limits_{h\to 0} \dfrac{2^h - 1}{h} = 0{\cdot}69$ (2 dec places)	1.	
b) i)  $\sin\alpha = x$ $\cos\beta = x$ $\alpha = \sin^{-1}x$ $\beta = \cos^{-1}x$	1	Writing α & β in terms of x
$\alpha + \beta = \frac{\pi}{2}$ (angle sum Δ) $\therefore \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$	1.	Using angle sum
ii) $\int_2^5 \sin^{-1}x + \cos^{-1}x \, dx = \int_2^5 \frac{\pi}{2} dx$ $= \left[\frac{\pi x}{2}\right]_2^5$ $= \frac{5\pi}{2} - \frac{4\pi}{2}$ $= \frac{3\pi}{2}$	1.	

Solutions	Marks	Comments
3 continued. (c) i) End points occur when $v = 0$. $0^2 = 2 - x - x^2$ $x^2 + x - 2 = 0$ $(x+2)(x-1) = 0$ $\therefore$ Endpoints are at $x = -2$ and $x = 1$.	1.	
ii) Max velocity occurs at centre of the motion. i.e at $x = \frac{-2+1}{2} = -\frac{1}{2}$. $v^2 = 2 - (-\frac{1}{2}) - (-\frac{1}{2})^2$ $= 2 + \frac{1}{2} - \frac{1}{4} = \frac{9}{4}$ $\therefore v = \pm \frac{3}{2}$ Max velocity is $1\frac{1}{2}$ ms^{-1}	1.	
d) i) In Δ APH $\tan 60° = \frac{x}{AP}$ $\therefore AP = \frac{x}{\tan 60°}$ In Δ BPH $\tan 30° = \frac{x}{BP}$ $\therefore BP = \frac{x}{\tan 30°}$	1.	
ii) In Δ APB $AP^2 + AB^2 = BP^2$ $\left(\frac{x}{\tan 60}\right)^2 + 1^2 = \left(\frac{x}{\tan 30}\right)^2$	1	Using Pythagoras,
$\frac{x^2}{\tan^2 60} + 1 = \frac{x^2}{\tan^2 30}$ $\frac{x^2}{\tan^2 60} - \frac{x^2}{\tan^2 30} = -1$ $x^2\left[\left(\frac{1}{\sqrt{3}}\right)^2 - \left(\frac{1}{1/\sqrt{3}}\right)^2\right] = -1$ OR $x^2[-2{\cdot}6] = -1$ $x^2 = \frac{1}{2{\cdot}6}$ $x^2 = 0{\cdot}375$ $x = \sqrt{0{\cdot}375}$ $= 0{\cdot}610km$ $= 610m.$	1	Substitution of values.
$x^2\left(\frac{1}{3} - 3\right) = -1$ $x^2\left(-\frac{8}{3}\right) = -1$ $x^2 = -1\left(-\frac{3}{8}\right)$ $= \frac{3}{8}$ $x = \sqrt{\frac{3}{8}}$ $= 0{\cdot}610$ km $= 610m.$	1.	Evaluating answer.

Solutions	Marks	Comments
Question 4 (a) $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \dots n^2 = \frac{1}{6}n(n+1)(2n+1)$ When $n = 1$ $LHS = 1^2 \quad RHS = \frac{1}{6}1(1+1)(2(1)+1)$ $= \frac{1}{6}.1.2.3$ $= 1.$ ∴ Statement is true for $n = 1$.	1	Showing true for $n = 1$.
Assume true for $n = k$. $1^2 + 2^2 + 3^2 \dots + k^2 = \frac{1}{6}k(k+1)(2k+1)$ Show for $n = k+1$ $1^2 + 2^2 + 3^2, \dots (k+1)^2 = \frac{1}{6}(k+1)(k+2)(2k+3)$ $LHS = 1^2 + 2^2 + 3^2 + \dots k^2 + (k+1)^2$ $= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2.$ $= \frac{1}{6}\left[k(k+1)(2k+1) + 6(k+1)^2\right]$ $= \frac{k+1}{6}\left[k(2k+1) + 6(k+1)\right]$ $= \frac{k+1}{6}\left[2k^2 + k + 6k + 6\right]$ $= \frac{k+1}{6}\left[2k^2 + 7k + 6\right]$ $= \frac{k+1}{6}\left[(k+2)(2k+3)\right]$ $= \frac{1}{6}(k+1)(k+2)(2k+3)$ $= \frac{1}{6}(k+1)\left[(k+1)+1\right]\left[2(k+1)+1\right].$ ∴ If true for $n = k$, it is also true for $n = k+1$. But since it is true for $n = 1$, by induction it is true for all $n \geq 1$ $n \in J$.	2	2 Completing case for $n = k+1$ and conclusion 1 mark if any step incomplete.

Solutions	Marks	Comments
4 continued (b) $\sqrt{3}\sin x + \cos x = 1.$ Let $\sqrt{3}\sin x + \cos x = R\sin(\theta + \alpha)$ $R = \sqrt{\sqrt{3}^2 + 1^2} \quad \alpha = \tan^{-1}\frac{1}{\sqrt{3}}$ $R = 2 \quad \alpha = \frac{\pi}{6}$	1	Full marks if results obtained by other methods, e.g t results.
∴ $2\sin\left(\theta + \frac{\pi}{6}\right) = 1$ for $\frac{\pi}{6} \leq \theta + \frac{\pi}{6} \leq \frac{13\pi}{6}$ $\sin\left(\theta + \frac{\pi}{6}\right) = \frac{1}{2}.$ $\theta + \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$	1.	
$\theta = 0, \frac{4\pi}{6} \quad \frac{12\pi}{6}$ $\theta = 0, \frac{2\pi}{3}, 2\pi.$	1.	
(c) $(1+x)^2 = (1+2x+x^2)$ so in the expansion of $(1+x)^2(1+x)^8$, terms in x^2 come from $1 . x^2$ term $2x . x$ term x^2. constant. $kx^2 = 1. {}^8C_2 x^2 + 2x \; {}^8C_1 x \; + x^2 . {}^8C_0$ $= 28x^2 + 16x^2 + x^2$ $= 45x^2$ Coefficient of $x^2 = 45.$	3	2 marks if correct terms but errors made in calculations 1 mark if not all terms included.
(d) $\int_0^{\pi} \sin^2 3x - x \; dx = \int_0^{\pi} \frac{1}{2}\left(1 - \cos 2(3x)\right) - x \; dx$ $= \int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos 6x - x \; dx$ $= \left[\frac{1}{2}x - \frac{1}{12}\cos 6x - \frac{x^2}{2}\right]_0^{\pi}$ $= \left(\frac{\pi}{2} - \frac{1}{12}\sin 6\pi - \frac{\pi^2}{2}\right) - \left(0 - \frac{1}{2}\sin 0 - 0\right)$ $= \frac{\pi}{2} - \frac{\pi^2}{2}$	3	2marks if errors made in substitution or minor error in changing $\sin^2 3x$ 1 mark if an errors in both aspects.

Solutions	Marks	Comments
uestion 5) i) $\angle ADB = \underline{\angle QAB}$ (angle between a tangent and a chord equal to the angle in the alternate segment)	1.	
ii) $\angle BAD = \angle BCD$ (opposite angles in kite) (or by showing $\Delta ABD \equiv \Delta BCD$) $\angle BAD = 180° - \angle BCD$ (opposite angles in cyclic quadrilateral) $\angle BAD = 180° - \angle BAD$ $(\angle BCD = \angle BAD)$ $2\angle BAD = 180°$ $\angle BAD = 90°$ low on the line PQ $x° + y° + \angle BAD = 180°$ $x° + y° + 90° = 180°$ $x° + y° = 90°$	3	2 marks for result with a simple single error 1 mark if give some of necessary statements.
iii) $\angle BAD = 90°$ (from above) $\therefore$ BD is diameter (angle in a semi circle is 90°)	1.	
i) $P(10 \text{ hits}) = {}^{10}C_{10}(0.9)^{10}(0.1)^0$ $= (0.9)^{10}$ $= 0.35$ (2 dec places)	1	
ii) $P(\text{at least } 8) = P(8) + P(9) + P(10)$ $= {}^{10}C_8(0.9)^8(0.1)^2 + {}^{10}C_9(0.9)^9(0.1)$ $+ {}^{10}C_{10}(0.9)^{10}$	1	
$= 0.194 + 0.387 + 0.349$ $= 0.93$ (2 dec p)	1.	ignore rounding in awarding mark.

Solutions	Marks	Commen
5(c) i) $f(x) = x^3 + 1$ $f'(x) = 3x^2$ $3x^2 \geqslant 0$ Since $f'(x) \geqslant 0$ $f(x)$ is monotonic increasing and hence an inverse exists. OR $y = f(x) = x^3 + 1$ forms a graph for which each y value has a unique x value. Hence it has an inverse function. OR. for $y = f(x)$ any horizontal line cuts the curve only once. Hence it has an inverse function.	1	
ii) $f(x) = y = x^3 + 1$. For $f^{-1}(x)$ $x = y^3 + 1$ $x - 1 = y^3$ $y = \sqrt[3]{x - 1}$	1	equation of $f^{-1}(x)$
	1	graph $f(x)$
	1	graph $f^{-1}(x)$

Solutions	Marks	C
Question 6 a) i) $P(2ap, ap^2)$ $Q(2aq, aq^2)$ PQ: $\frac{y-ap^2}{x-2ap} = \frac{ap^2-aq^2}{2ap-2aq}$ $\frac{y-ap^2}{x-2ap} = \frac{a(p-q)(p+q)}{2a(p-q)}$ $2(y-ap^2) = (x-2ap)(p+q)$ $2y-2ap^2 = (p+q)x - 2ap(p+q)$ $2y-2ap^2 = (p+q)x - 2ap^2 - 2apq$ $0 = (p+q)x - 2y - 2apq = 0$	1	* can ⁿCr obt fo gv
ii) (0,a) satisfies $(p+q)\cdot 0 - 2a - 2apq = 0$ $-2a = 2apq$ $pq = -1$	1	ea x
iii) $y = \frac{x^2}{4a}$ $\frac{dy}{dx} = \frac{x}{2a}$ grad at $P(2ap, ap^2) = \frac{2ap}{2a} = p$ grad normal $= -\frac{1}{p}$ Eqn of normal at $P(2ap, ap^2)$ $m = -\frac{1}{p}$ $y - ap^2 = -\frac{1}{p}(x-2ap)$ $-py + ap^3 = x - 2ap$ $x + py = ap^3 + 2ap$ ①	2	
iv) Q normal $x + qy = ap^3 + 2aq$ ② ①−②: $(p-q)y = a(p^3-q^3) + 2a(p-q)$ $(p-q)y = a(p-q)(p^2+pq+q^2) + 2a(p-q)$ $y = a[p^2+pq+q^2+2]$ $x = ap^3+2ap - p\cdot a(p^2+pq+q^2+2)$ $= ap^3+2ap - ap^3 - ap^2q - apq^2 - 2ap$ $= -ap^2q - apq^2$ $\therefore x = -apq(p+q)$ $\therefore R$ is $[-apq(p+q), a(p^2+pq+q^2+2)]$	2	

Solutions	Marks	Comments
6. v) If PQ is a focal chord. $pq = -1$ R becomes $[-a(-1)(p+q), a(p^2+(-1)+q^2+2)]$ $[a(p+q), a(p^2+q^2+1)]$ $x = a(p+q)$, $y = a(p^2+q^2+1)$ $p+q = \frac{x}{a}$ $= a(p^2+2pq+q^2-2pq+1)$ $= a((p+q)^2-2(-1)+1)$ $y = a((p+q)^2+3)$ $y = a((\frac{x}{a})^2+3)$ * $y = a(\frac{x^2}{a^2}+3)$ * $y = \frac{x^2}{a}+3a$ * or $x^2 = a(y-3a)$ *	3 ↗	1 mark if only state $pq=-1$ 2marks if sub $pq=-1$ int R and manipulating 3 marks for obtaining one of the equati linking x an y. *
Q(b) i) $P(x) = x^4+x^3+x^2+x-2.$ $\alpha\beta\gamma\delta = \frac{e}{a}$ $= \frac{-2}{1}$ $= -2.$	1.	
ii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta} = \frac{\alpha\beta\gamma+\alpha\gamma\delta+\alpha\beta\delta+\beta\gamma\delta}{\alpha\beta\gamma\delta}$ $= \left(-\frac{d}{a}\right) \div (-2)$ $= -1 \div -2$ $= \frac{1}{2}.$	1	getting valu $\frac{1}{2}$ for $\frac{1}{\alpha}+\frac{1}{\beta}+$
$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta} = \frac{1}{2}.$ $\alpha = 1$, so $1+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta} = \frac{1}{2}$ $\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta} = -\frac{1}{2}.$	1	Reaching fi result.

Solutions	Marks	Comments
Question 7 a) Volume of Grain remaining is given by $V = \frac{1}{3}\pi r^2 h$ The triangles shown are similar.  $\frac{r}{8} = \frac{h}{10}$ $r = \frac{4h}{5}$ $\quad h = \frac{5r}{4}$	1	Using similar Δ's
$V = \frac{1}{3}\pi r^2\left(\frac{5r}{4}\right)$ $= \frac{5\pi r^3}{12}$	1	Expression for V in terms of r
$\frac{dV}{dr} = \frac{5\pi r^2}{4}$ $\quad \therefore \frac{dr}{dV} = \frac{4}{5\pi r^2}$ $\frac{dV}{dt} = -35$ since grain released at 35 m³/s.	1	Obtaining derivatives required.
$\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$ $= \frac{4}{5\pi r^2} \cdot (-35)$ When $h = 6.5$, $r = \frac{4(6.5)}{5}$ $= 5.2$.	1	Product of derivatives
$= \frac{4}{5\pi(5.2)^2} \cdot (-35)$ $= -0.3296\ldots$ Radius is decreasing at 0.33 m/s at the required time.	1	Evaluating

Solutions	Marks/Comments
Question 7 b) i) $\frac{dR}{dt} = \frac{d}{dt}\left(Ae^{-kt}\right)$ $= Ae^{-kt} \cdot -k$ $= -kAe^{-kt}$ $= -kR$. as required.	1.
(ii) $t = 10$ $\quad R = \frac{A}{2}$. $\frac{A}{2} = Ae^{-10k}$ $\frac{1}{2} = e^{-10k}$. $\ln\frac{1}{2} = -10k$ $k = \frac{-\ln\frac{1}{2}}{10} \doteqdot 0.69$. When $t = 4$ $\quad R = Ae^{-0.69(4)}$ $= A(0.757\ldots)$ ∴ 76% remains.	3 correct answer or 2 if single error in calculation of R or 1 for multiple errors or wrong method.

Solutions	Marks	Comments
c) i) No ways $= {}^{40}C_4 \overset{✓}{=} 91\,390$ ways *	1	* All answers can be left in ${}^nC_r$ or ! form.
ii) 2 girls can be chosen in ${}^{22}C_2$ ways 2 boys " " " " ${}^{18}C_2$ ways.	1	obtaining values for 2 boys & 2 girls
Total number of ways $= {}^{22}C_2 \times {}^{18}C_2$ * $= 231 \times 153$ $= 35\,343$ ways *	1	reaching final result. *