MATHEMATICS REVISION OF FORMULAE AND RESULTS

Surds

- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
- $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
- $(\sqrt{a})^2 = a$

Absolute Value

$$|a| = a \text{ if } a \geq 0$$
$$|a| = -a \text{ if } a < 0$$

Geometrically:

$|x|$ is the distance of *x* from the origin on the number line
$|x - y|$ is the distance between *x* and *y* on the number line

$$|ab| = |a|.|b|$$

$$|a + b| \leq |a| + |b|$$

Factorisation

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

Real Functions

- A function is *even* if $f(-x) = f(x)$. The graph is symmetrical about the y-axis.
- A function is *odd* if $f(-x) = -f(x)$. The graph has point symmetry about the origin.

The Circle

The equation of a circle with:

- Centre the origin (0, 0) and radius *r* units is:

$$x^2 + y^2 = r^2$$

- Centre (*a*, *b*) and radius *r* units is:

$$(x - a)^2 + (y - b)^2 = r^2$$

Co-ordinate Geometry

- Distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- Gradient formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$ or $m = \tan\theta$
- Midpoint Formula: midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
- Perpendicular distance from a point to a line:

$$\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

- Acute angle between two lines (or tangents)

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

- Equations of a Line

  gradient-intercept form: $y = mx + b$

  point-gradient form: $y - y_1 = m(x - x_1)$

  two point formula: $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

  intercept formula: $\frac{x}{a} + \frac{y}{b} = 1$

  general equation: $ax + by + c = 0$

- Parallel lines: $m_1 = m_2$
- Perpendicular lines: $m_1.m_2 = -1$

Trigonometric Results

- $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ (SOH)

- $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ (CAH)

- $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$ (TOA)

- Complementary ratios:

$$\sin(90° - \theta) = \cos\theta$$

$$\cos(90° - \theta) = \sin\theta$$

$$\tan(90° - \theta) = \cot\theta$$

$$\sec(90° - \theta) = \operatorname{cosec}\theta$$

$$\operatorname{cosec}(90° - \theta) = \sec\theta$$

- Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{and} \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

- The Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

- The Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\operatorname{Cos}A$$

$$\operatorname{Cos}A = \frac{b^2 + c^2 - a^2}{2bc}$$

- The Area of a Triangle

$$\text{Area} = \tfrac{1}{2}ab\operatorname{Sin}C$$

The Quadratic Polynomial

- The general quadratics is: $y = ax^2 + bx + c$

- The quadratic formula is: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- The discriminant is: $\Delta = b^2 - 4ac$

  If $\Delta \geq 0$ the roots are real

  If $\Delta < 0$ the roots are not real

  If $\Delta = 0$ the roots are equal

  If Δ is a perfect square, the roots are rational

- If α and β are the roots of the quadratic equation

$$ax^2 + bx + c = 0$$

  then: $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$

- The axis of symmetry is: $x = -\dfrac{b}{2a}$

- If a quadratic function is positive for all values of x, it is *positive definite* i.e. $\Delta < 0$ and $a > 0$

- If a quadratic function is negative for all values of x, it is *negative definite* i.e. $\Delta < 0$ and $a < 0$

- If a function is sometimes positive and sometimes negative, it is *indefinite* i.e. $\Delta > 0$

The Parabola

- The parabola $x^2 = 4ay$ has vertex $(0,0)$, focus $(0,a)$, focal length ‘a’ units and directrix $y = -a$

- The parabola $(x - h)^2 = 4a(y - k)$ has vertex (h, k)

Differentiation

- First Principles:

$$f'(x) = \lim_{h \to \infty} \frac{f(x+h) - f(x)}{h} \quad \text{or}$$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{h}$$

- If $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$
- Chain Rule: $\frac{d}{dx}[f(u)] = f'(u)\,\frac{du}{dx}$
- Product Rule: If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
- Quotient Rule: If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} + u\frac{dv}{dx}}{v^2}$
- Trigonometric Functions:

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

- Exponential Functions: $\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$

$$\frac{d}{dx}(a^x) = a^x.\ln a$$

- Logarithmic Functions: $\frac{d}{dx}\left(\log_e f(x)\right) = \frac{f'(x)}{f(x)}$

Geometrical Applications of Differentiation

- Stationary points: $\frac{dy}{dx} = 0$
- Increasing function: $\frac{dy}{dx} > 0$
- Decreasing function: $\frac{dy}{dx} < 0$
- Concave up: $\frac{d^2y}{dx^2} < 0$
- Concave down: $\frac{d^2y}{dx^2} > 0$
- Minimum turning point: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$
- Maximum turning point: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$
- Points of inflexion: $\frac{d^2y}{dx^2} = 0$ and concavity changes about the point.
- Horizontal points of inflexion: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ and concavity changes about the point.

Approximation Methods

- The Trapezoidal Rule:

$$\int_a^b f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1})\right]$$

- Simpson's Rule:

$$\int_a^b f(x)dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)\right]$$

In both rules, $h = \frac{b-a}{n}$ where n is the number of strips.

Integration

- If $f(x) \geq 0$ for $a \leq x \leq b$, the area bounded by the curve $y = f(x)$, the x-axis and $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.
- The volume obtained by rotating the curve $y = f(x)$ about the x-axis between $x = a$ and $x = b$ is given by $\pi \int_a^b [f(x)]^2$
- If $\frac{dx}{dx} = x^n$ then $y = \frac{x^{n+1}}{n+1}$
- If $\frac{dx}{dx} = (ax+b)^n$ then $y = \frac{(ax+b)^n}{a(n+1)}$
- Trigonometric Functions:

$$\int \sin x\, dx = -\cos x + C$$

$$\int \cos x\, dx = \sin x + C$$

$$\int \sec^2 x\, dx = \tan x + C$$

- Exponential Functions:

$$\int e^{ax}\, dx = \frac{e^{ax}}{a} + C \quad \text{and} \quad \int a^x\, dx = \frac{1}{\ln a}.a^x$$

- Logarithmic Functions:

$$\int \frac{f'(x)}{f(x)} dx = \log_e x + C$$

Sequences and Series

- Arithmetic Progression

$$d = U_2 - U_1$$

$$U_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$S_n = \frac{n}{2}[a + l]$ where l is the last term

- Geometric Progression

$$r = \frac{U_2}{U_1}$$

$$U_n = ar^{n-1}$$

$$S_n = \frac{a(r^n - 1)}{r-1} = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r}$$

The Trigonometric Functions

- π radians = 180°
- Length of an arc: $l = r\theta$
- Area of a sector: $A = \frac{1}{2}r^2\theta$
- Area of a segment: $A = \frac{1}{2}r^2(\theta - \sin\theta)$
  [In these formulae, θ is measured in radians.]
- Small angle results:

$$\sin x \to 0$$
$$\cos x \to 1$$
$$\tan x \to 0$$
$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\tan x}{x} = 1$$

- For $y = \sin nx$ and $y = \cos nx$ the period is $\frac{2\pi}{n}$
- For $y = \sin nx$ the period is $\frac{\pi}{n}$

Logarithmic and Exponential Functions

- The Index Laws:

$$a^x \times a^y = a^{x+y}$$

$$a^x \div a^y = a^{x-y}$$

$$(a^x)^y = a^{xy}$$

$$a^{-x} = \frac{1}{a^x}$$

$$a^{\frac{x}{y}} = \sqrt[y]{a^x}$$

$$a^0 = 1$$

- The logarithmic Laws:

If $\log_a b = c$ then $a^c = b$

$$\log_a x + \log_a y = \log_a xy$$

$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$

$$\log_a x^n + n\log_a x$$

$\log_a a = 1$ and $\log_a 1 = 0$

- The Change of Base Result:

$$\log_a b = \frac{\log_e b}{\log_e a} = \frac{\log_{10} b}{\log_{10} a}$$

EXTENSION 1 REVISION OF FORMULAE AND RESULTS

Co-ordinate Geometry

- Dividing an interval in the ratio m:n

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right)$$

- Acute angle between two lines (or tangents)

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Trigonometric Ratios

- Sum and Difference Results

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

- Double Angle Results

$$\sin 2A = 2\sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\cos 2A = 1 - 2\sin^2 A$$
$$\cos 2A = 2\cos^2 A - 1$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

- The 't' Formulae where $t = \tan\frac{\theta}{2}$

$$\sin x = \frac{2t}{1 + t^2}$$

$$\cos x = \frac{1 - t^2}{1 + t^2}$$

$$\tan x = \frac{2t}{1 - t^2}$$

- Subsidiary Angle Method ($R\sin(\theta + \alpha)$)

  When solving $a\sin\theta + b\cos\theta = c$ we can solve by writing in the form $R\sin(\theta + \alpha) = c$ where:

$$R = \sqrt{a^2 + b^2} \quad \text{and} \quad \tan\alpha = \frac{b}{a}$$

Parameters

- The parametric equations for the parabola $x^2 = 4ay$ are $x = 2at$ and $y = at^2$

- All other formulae in this subject are not to be committed to memory but students must know how they are derived.

Polynomials

- A real polynomial is in the form:

$$P(x) = p_n x^2 + p_{n-1} x^{n-1} + \ldots p_2 x^2 + p_1 x + p_0$$

- $p_1, p_2, p_3, \ldots, p_n$ are *coefficients* and are real numbers, usually integers.

- The *degree* of the polynomial is the highest power of x with non-zero coefficient.

- A polynomial of degree n has at most n real roots but may have less.

- The result of a long division can be written in the form $P(x) = A(x) \,.\, Q(x) + R(x)$

- The *remainder theorem* states that when $P(x)$ is divided by $(x - a)$ the remainder is $P(a)$.

- The *factor theorem* states that if $x = a$ is a factor of $P(x)$ then $P(a) = 0$.

- If $\alpha, \beta, \gamma, \delta, \ldots$ are the roots of a polynomial then

$$\Sigma\alpha = -\frac{b}{a}, \ \Sigma\alpha\beta = \frac{c}{a}, \ \Sigma\alpha\beta\gamma = -\frac{d}{a}, \ \Sigma\alpha\beta\gamma\delta = \frac{e}{a}$$

Numerical Estimation of the Roots of an Equation

- Halving the Interval Method
- Newton's Method

  If $x = x_0$ is an approximation to a root of $P(x) = 0$ then $x_1 = x_0 - \frac{P(x_0)}{P'(x_0)}$ is generally a better approximation.

  Be familiar with the conditions under which this method fails.

Mathematical Induction

- Step 1: Prove result true for $n = 1$ (It is sometimes necessary to have a different first step.)
- Step 2: Assume it is true for $n = k$ and then prove true for $n = k + 1$
- Step 3: Conclusion as given in class

Integration

- $\int \sin^2\theta \, d\theta$ and $\int \cos^2\theta \, d\theta$ can be solve using the substitutions:

  $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

  $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$
- Integration by first making a substitution.
- Table of Standard Integrals as provided in HSC

Inverse Trigonometric Functions

$y = \sin^{-1}x$	Domain:	$-1 \le x \le 1$
	Range:	$-\frac{\pi}{2} \le y \le -\frac{\pi}{2}$
$y = \cos^{-1}x$	Domain:	$-1 \le x \le 1$
	Range:	$0 \le y \le \pi$
$y = \tan^{-1}x$	Domain:	all real x
	Range:	$-\frac{\pi}{2} \le y \le -\frac{\pi}{2}$

- Properties:

  $\sin^{-1}(-x) = -\sin^{-1}x$
  $\cos^{-1}(-x) = \pi - \cos^{-1}x$
  $\tan^{-1}(-x) = -\tan^{-1}x$
  $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$
  $\sin(\sin^{-1}x) = x$
  $\cos(\cos^{-1}x) = x$
  $\tan(\tan^{-1}x) = x$

- General Solutions of Trigonometric Equations:

  if $\sin\theta = q$, then $\theta = n\pi + (-1)^n \sin^{-1}q$
  if $\cos\theta = q$, then $\theta = 2n\pi \pm \cos^{-1}q$
  if $\tan\theta = q$, then $\theta = n\pi + \tan^{-1}q$

- Derivatives:

  $\frac{d}{dx}\left[\sin^{-1}x\right] = \frac{1}{\sqrt{1-x^2}}$
  $\frac{d}{dx}\left[\cos^{-1}x\right] = \frac{-1}{\sqrt{1-x^2}}$
  $\frac{d}{dx}\left[\tan^{-1}x\right] = \frac{1}{1+x^2}$

- Integrals:

  $$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) = -\cos^{-1}\left(\frac{x}{a}\right)$$
  $$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$