SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2004

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Sample Solutions

Section	Marker
A	Mr Dunn
B	Ms Nesbitt
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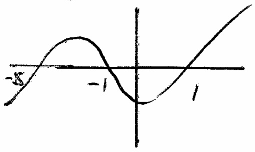
Section A

1 a) $(x^2-1)(x+5)>0$



$x>1$

AND
$-5<x<-1$ (2 marks)

b) $y=\ln\sqrt{x+1}$

$=\frac{1}{2}\ln(x+1)$

$y'=\frac{1}{2(x+1)}$ (2 marks)

c) $\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x \, dx$

$=\frac{1}{2}\sec 2x\Big]_0^{\pi/6}$

$=\frac{1}{2}\sec\frac{\pi}{3}-\frac{1}{2}\sec 0$

$=\frac{1}{2}\times 2-\frac{1}{2}\times 1$

$=\frac{1}{2}$ (2 marks)

d) $\int_0^{\sqrt{3}} \frac{dx}{9+x^2}=\frac{1}{3}\tan^{-1}\frac{x}{3}\Big]_0^{\sqrt{3}}$

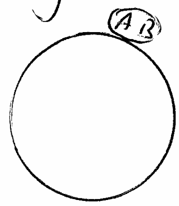
$=\left[\frac{1}{3}\tan^{-1}\frac{\sqrt{3}}{3}\right]$

$=\frac{1}{3}\tan^{-1}\frac{1}{\sqrt{3}}$

$=\frac{1}{3}\times\frac{\pi}{6}$

$=\frac{\pi}{18}$ (2 marks)

e) Total number of arrangements $=7!$



If A and B are together

Then $2\times 6!$

Hence not together

$=7!-2\times 6!$

$=6!(7-2)$

$=5\times 6!$

$=3600$ (2marks)

f) $LHS=\frac{1-\cos\theta}{\sin\theta}+\frac{\sin\theta}{1+\cos\theta}$

$=\frac{1-\cos^2\theta+\sin^2\theta}{\sin\theta(1+\cos\theta)}$

$=\frac{2\sin^2\theta}{\sin\theta(1+\cos\theta)}$

$=\frac{2\sin\theta}{1+\cos\theta}$

Let $t=\tan\frac{\theta}{2}$

$=\frac{2\times\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}$

$=\frac{\frac{4t}{1+t^2}}{1+t^2+1-t^2}=\frac{4t}{2}=2t$

$=2\tan\theta/2=RHS$ (2 marks)

QUESTION TWO

a) $y = \sin^{-1} 2x$

Let $u = 2x$

Then $y = \sin^{-1} u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-u^2}} \times 2$$

$$= \frac{2}{\sqrt{1-4x^2}} \quad \text{(2 marks)}$$

b) $y = 3 \sin^{-1} \sqrt{1-x^2}$

Consider $\sqrt{1-x^2}$

$-1 \leq x \leq 1$ ~~Range~~ Domain

Then

$y = 3\sin^{-1} 0$ to $3\sin^{-1} 1$

ie $0 \leq y \leq \frac{3\pi}{2}$ Range

(2marks)

c) $\sqrt{3} \cos x - \sin x = R\cos(x+\alpha)$

$= R\cos x \cos\alpha - R\sin x \sin\alpha$

$R\cos\alpha = \sqrt{3}$

$R\sin\alpha = 1$

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

$R^2(\cos^2\alpha + \sin^2\alpha) = 3+1$

$R = 2$

c) continued

$2\cos(x + \frac{\pi}{6}) = 1$ (2marks)

$\cos(x+\frac{\pi}{6}) = \frac{1}{2}$

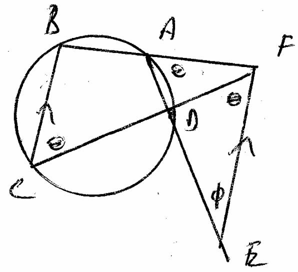
$x + \frac{\pi}{6} = \pm\frac{\pi}{3}$

$x = 2k\pi \pm \frac{\pi}{3} - \frac{\pi}{6}$

$x = 2k\pi + \frac{\pi}{6}$ (2marks)

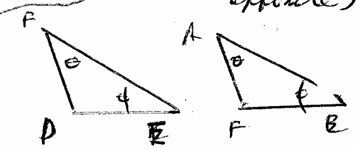
or $2k\pi - \frac{\pi}{2}$

d)



$F\hat{A}E = F\hat{B}C$ (angle in alternate segment)

$B\hat{C}F = C\hat{F}E$ (alternate opposite)



Hence $\Delta DEF \ ||| \ \Delta FEA$ (2marks)

$$\frac{EF}{EA} = \frac{ED}{EF}$$

$EF^2 = EA \times ED$

(2marks)

QUESTION THREE

a) Prove $2^{3n}-1$ is divisible by 7 for $n \geq 1$ (integers)

Let $n=1$ Then $2^3-1=7$

ie True for $n=1$

Assume

$2^{3k}-1 = 7K$ where K is an integer

Try to prove

$2^{3k+3}-1 = 7N$ where N is an integer

$LHS = 2^3 \cdot 2^{3k}-1$

$= 8(7K+1)-1$ from assumption.

$= 56K+7$

$= 7(8K+1)$

$= 7N$

True for $n=1$

$n=1+1=2$

$n=2+1=3$

$\vdots$

All integers $n \geq 1$ (3 marks)

b) i) $y = 1+2\cos x - 2\cos^2 x$

$y' = -2\sin x + 4\cos x \sin x$

$= 2\sin x\,(2\cos x - 1)$ (1 mark)

ii) $y'=0$ when $\sin x = 0$

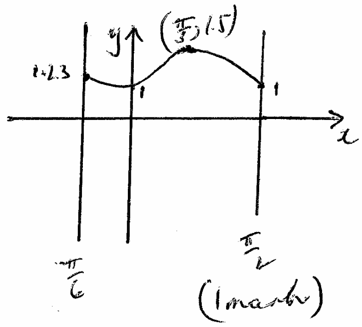
$\cos x = 1/2$

ie $x = 0, \frac{\pi}{3}$

When $x=0$, $y=1$

$x=\frac{\pi}{3}$, $y=\frac{3}{2}$ } 2 marks

iii)



(1 mark)

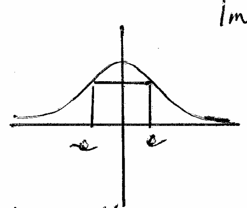
Max of 1.5 at $x=\frac{\pi}{3}$

Minimum of 1 at

$x=0$ or $x=\frac{\pi}{2}$

1 mark

c)

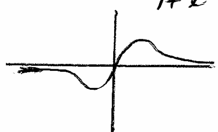


Area = lb

$= 2a \times \frac{1}{1+a^2} = \frac{2a}{1+a^2}$

(1 mark)

Consider $y = \frac{2x}{1+x^2}$



$y' = \frac{(x^2+1)2 - 2.2x}{(1+x^2)^2}$

$= \frac{2-2x^2}{(1+x^2)^2}$

$y'=0$ when $x=\pm 1$

THREE

c ii) continued.

When $x = 1+\epsilon \quad y' < 0$

$x = 1-\epsilon \quad y' > 0$

Hence $x = 1$ produces maximum

$$\text{Area} = \frac{2}{1+1} = 2 \text{ square units.} \quad (3 \text{ marks})$$

or

$$y'' = \frac{(1+x^2)^2(-4x) - (2-2x^2)\,4x(1+x^2)}{(1+x^2)^4}$$

$$= \frac{-4x(1+x^2)\left[1+x^2+(2-2x^2)\right]}{(1+x^2)^4}$$

$$= \frac{-4x(1+x^2)(3-x^2)}{(1+x^2)^4}$$

When $x = 1$ $\quad y'' = \dfrac{-4 \times 2 \times 2}{2^4}$

$y'' < 0$ Hence maximum.

QUESTION 4

i) $x=-2t$, $t=-\frac{x}{2}$

$y=\frac{1}{4}x^2$

$y'=\frac{x}{2} = -t$

eqn of tangent $y-t^2=-t(x+2t)$

$y-t^2+tx+2t^2=0$

$tx+y+t^2=0$

ii) $tx+y+t^2=0$

at A , $y=0$

$tx+t^2=0$

$t(x+t)=0$, $x=-t$

$A(-t,0)$ $\quad T(-2t,t^2)$

Midpoint M $\frac{-t-2t}{2}$, $\left(\frac{0+t^2}{2}\right)$

$M=\left(-\frac{3t}{2},\frac{t^2}{2}\right)$

$x=-\frac{3t}{2}$, $t=-\frac{2x}{3}$

$y=\frac{t^2}{2}$

$=\frac{1}{2}\left(-\frac{2x}{3}\right)^2$

$y=\frac{2x^2}{9}$

Locus of M $\quad x^2=\frac{9}{2}y$

$4x^3-12x^2+11x-3=0$

roots $\alpha-d, \alpha, \alpha+d$ (arith series)

Sum of roots $=3\alpha=-b/a=3$

$\alpha=3$

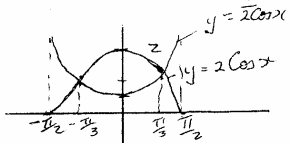
product $1(1-d)+1(1+d)+(1-d)(1+d)=c/a$

$3-d^2=11/4$

$d^2=1/4$

$d=\pm\frac{1}{2}$

roots $\frac{1}{2}, 1, 1\frac{1}{2}$.



$2\cos x=\frac{1}{2\cos x}$

$4\cos^2 x=1$

$\cos x=\frac{1}{2}$ or $\cos x=-\frac{1}{2}$

$x=-\frac{\pi}{3}, \frac{\pi}{3}$ or no sol'n in domain

$V=\pi\int_{-\pi/3}^{\pi/3}\left(4\cos^2 x-\frac{1}{4}\sec^2 x\right)dx$

$2\cos^2 x=\cos 2x+1$

$V=2\pi\int_0^{\pi/3}\left(2\cos 2x+2-\frac{1}{4}\sec^2 x\right)dx$

$=2\pi\left[\sin 2x+2x-\frac{1}{4}\tan x\right]_0^{\pi/3}$

$=2\pi\left(\frac{\sqrt3}{2}+\frac{2\pi}{3}-\frac{\sqrt3}{4}\right)-0$

$V=\left(\frac{4\pi^2}{3}+\frac{\sqrt3}{2}\right)u^3$

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5 (a) Find $\frac{dV}{dt}=\frac{dV}{dr}\cdot\frac{dr}{dt}$

$\frac{dr}{dt}=-5$ cm/s $\quad V=\frac{4}{3}\pi r^3$

$\frac{dV}{dr}=4\pi r^2$

$r=10$ cm

$\frac{dV}{dt}=-5\times4\times\pi\times100$

$=-2000\pi\ cm^3/s$

(b) $x=2\cos\left(t+\frac{\pi}{6}\right)$

$\dot{x}=-2\sin\left(t+\frac{\pi}{6}\right)$

$\ddot{x}=-2\cos\left(t+\frac{\pi}{6}\right)$

$\ddot{x}=-1^2x$, in the form $-n^2x$, $n=1$

∴ motion is SHM

(ii) Period $=\frac{2\pi}{n}=2\pi$

(iii) $x=2\cos\left(t+\frac{\pi}{6}\right)=0$

$t+\frac{\pi}{6}=\frac{\pi}{2}+2n\pi$

$t=\frac{\pi}{3}$ sec (1st osc.)

(iv) $2\cos\left(t+\frac{\pi}{6}\right)=1$

$t+\frac{\pi}{6}=\pm\frac{\pi}{3}+2n\pi$

$t=\frac{\pi}{6}$ (1st osc.)

$\dot{x}=-2\sin\frac{\pi}{3}$

$V=-2\times\frac{\sqrt3}{2}$

$V=-\sqrt3$ cm/s

QUESTION 5 (c)

$$\int \sqrt{16-x^2}\ dx$$

$x = 4\sin\theta$

$\frac{dx}{d\theta} = 4\cos\theta$

$dx = 4\cos\theta\ d\theta$

$$= \int \sqrt{16-16\sin^2\theta}\ .\ 4\cos\theta\ d\theta$$

$$\int \sqrt{16\cos^2\theta}\ .\ 4\cos\theta\ d\theta$$

$$\int 4\cos\theta\ .\ 4\cos\theta\ d\theta$$

$$16\int \cos^2\theta\ d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

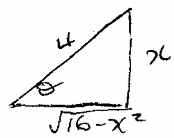
$2\cos^2\theta = \cos 2\theta + 1$

$$8\int (\cos 2\theta + 1)\ d\theta$$

$$8\left(\tfrac{1}{2}\sin 2\theta + \theta\right)$$

$$4\sin 2\theta + 8\theta + c$$

$$4 . 2\sin\theta\cos\theta + 8\theta$$



$$4 \times 2 . \frac{x}{4} \frac{\sqrt{16-x^2}}{4} + 8\sin^{-1}\frac{x}{4}$$

$\theta = \sin^{-1} \frac{x}{4}$

$$= \frac{x}{2}\sqrt{16-x^2} + 8\sin^{-1}\frac{x}{4} + C$$

QUESTION 6.

(a) If $y' = \frac{3x}{4+x^2}$

$\boxed{y = \frac{3}{2}\ln(4+x^2)+c.}$

(b). $P(x) = 8x^3-12x^2+6x+13$

$P'(x) = 24x^2-24x+6.$

$= 6(2x-1)^2$

(i) $P(x)$ is increasing when $P'(x)>0.$

ie $6(2x-1)^2>0$

$\therefore$ all Reals, except $x=\frac{1}{2}$.

(ii) Since $P(x)\to-\infty$ as $x\to-\infty$, $P(0)=13.$
and $P(x)$ is increasing for all $x\neq\frac{1}{2}$.
it follows that there must be a root x_1 where $x_1<0$.

(iii) $a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$

if $a_1=-1$ then $a_2 = -1 - \frac{-8-12-6+13}{24+24+6.}$

$= -1 - \frac{-13}{54}$

$= \frac{-41}{54}$

$= \boxed{-0.76}$ (2.D.P).

(c) (i) $T = S + Ae^{-kt}$ — (A)

$\therefore \dfrac{dT}{dt} = -kAe^{-kt}$

$= -k(T-S)$ from (A)

(ii) when $t=0$, $T=1390$ and $S=30$ (constant)

$\therefore 1390 = 30 + Ae^{0}$

$\therefore A = 1360$

$\therefore T = 30 + 1360e^{-kt}$

when $t=10$, $T = 1060$

$\therefore 1060 = 30 + 1360e^{-10k}$

$\dfrac{1030}{1360} = e^{-10k}$

$-10k = \ln \dfrac{103}{136}$

$k \doteqdot 0.0278$

$\therefore T = 30 + 1360e^{-0.0278t}$

Let $T = 110$

$110 = 30 + 1360e^{-0.0278t}$

$\dfrac{80}{1360} = e^{-0.0278t}$

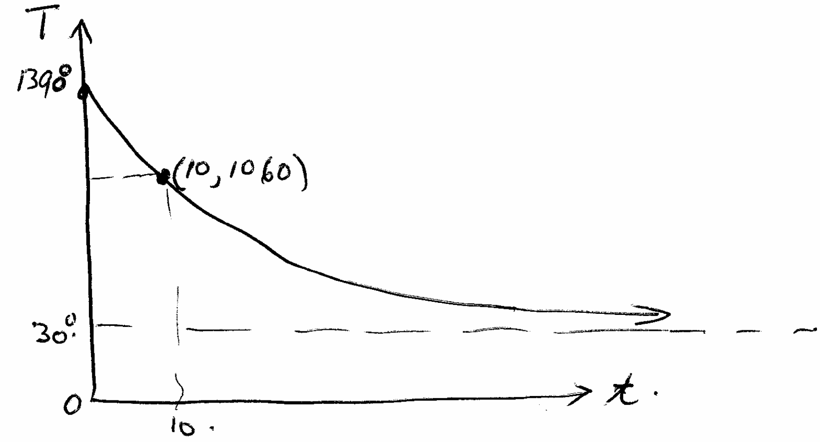
$\therefore \ln \dfrac{8}{136} = -0.0278t$

$t = \dfrac{\ln \frac{1}{17}}{-0.0278}$

$\doteqdot 102$ mins

$\therefore$ $\boxed{\text{it takes 92 mins longer.}}$

(iii)



QUESTION 7.

(a) now

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{r}x^r + \cdots$$
$$\cdots + \binom{n}{n}x^n \quad \text{— (A)}$$

(i) differentiate both sides of (A) above.

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + r\binom{n}{r}x^{r-1} + \cdots$$
$$\cdots + n\binom{n}{n}x^{n-1}.$$

let $x = 1$

$$n \cdot 2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + r\binom{n}{r} + \cdots + n\binom{n}{n}$$

ie. $\boxed{\sum_{r=1}^{n} r\binom{n}{r} = n2^{n-1}}$ ✓✓ (NB this is equivalent to $\sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1}$.)

(ii) R.T.P. $\sum_{r=0}^{n} (r+1)\binom{n}{r} = 2^{n-1}(n+2)$

$$LHS = \sum_{r=0}^{n} r\binom{n}{r} + \sum_{r=0}^{n}\binom{n}{r}$$

$$= n2^{n-1} + 2^n$$ (if we let $x=1$ in (A) $2^n = \sum_{r=0}^{n}\binom{n}{r}$)

$$= \boxed{2^{n-1}(n+2)}$$ ✓✓

$= RHS.$

(b) (i) $x = Vt \Rightarrow t = \frac{x}{V}$.

$\therefore y = -\frac{1}{2}gt^2 + h$. becomes.

$y = -\frac{1}{2}g\left(\frac{x}{V}\right)^2 + h$.

$\boxed{y = h - \frac{1}{2}\frac{g x^2}{V^2}}$ ✓

(ii) $x = Vt\cos\alpha \Rightarrow t = \frac{x}{V\cos\alpha}$ $\therefore y = -\frac{1}{2}gt^2 + Vt\sin\alpha + h$.

becomes $y = -\frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2 + V\frac{x}{V\cos\alpha}\sin\alpha + h$.

ie $\boxed{y = x\tan\alpha - \frac{g x^2 \sec^2\alpha}{2V^2} + h}$ ✓

(iii) Substitute $(d, 0)$ in (i) $0 = h - \frac{gd^2}{2V^2}$

$\therefore \boxed{h = \frac{gd^2}{2V^2}}$ ✓

(iv) Substitute $(d, 0)$ in (ii)

$0 = d\tan\alpha - \frac{gd^2}{2V^2}\sec^2\alpha + h$.

$0 = d\tan\alpha - h(1 + \tan^2\alpha) + h$ $\left(h = \frac{gd^2}{2V^2}\right)$

$\therefore h\tan^2\alpha - d\tan\alpha = 0$

$\tan\alpha(h\tan\alpha - d) = 0$

$\therefore \tan\alpha = 0$ or $\tan\alpha = \frac{d}{h}$

Clearly $\tan\alpha \neq 0$ $\therefore \boxed{\tan\alpha = \frac{d}{h}.}$ ✓✓

(v). Substitute $(2d, 0)$ into (ii).

$$2d\tan\alpha - \frac{g.4d^2}{2v^2}\sec^2\alpha + h = 0.$$

$$2d\tan\alpha - 4h\sec^2\alpha + h = 0.$$

$$2d\tan\alpha - 4h(1+\tan^2\alpha) + h = 0$$

$$2d\tan\alpha - 4h - 4h\tan^2\alpha + h = 0$$

$$4h\tan^2\alpha - 2d\tan\alpha + 3h = 0$$

For $\tan\alpha$ to be real $4d^2 - 4\times 4h\times 3h \geqslant 0.$

ie $4d^2 - 48h^2 \geqslant 0.$

$$4d^2 \geqslant 48h^2$$

$$d^2 \geqslant 12h^2$$

$$\boxed{d \geqslant 2h\sqrt{3}}$$