Kincoppal - Rosebay (2003)

Trial Extension 1 solutions 2003

Question 1:

(a) $4\lim_{x\to 0}\frac{\tan 4x}{4x}=4$ ✓

(b) $2x^3 3e^{3x}+6x^2e^{3x}=6x^2e^{3x}(x+1)$ ✓ ✓

(c) $(2-x)^2\times\frac{1}{2-x}\geq 3(2-x)^2 \qquad x\neq 2$ ✓

$2-x\geq 3(2-x)^2$

$2-x-3(2-x)^2\geq 0$

$(2-x)(1-3(2-x))\geq 0$

$(2-x)(3x-5)\geq 0$

$2<x\leq\frac{5}{3}$ never ever write like this

✓ ✓

(d) $-1\leq\frac{x}{3}\leq 1$ ✓ $\quad 0\leq f(x)\leq 2\pi$

$-3\leq x\leq 3$

✓

(e)

$m_1=3$ ✓

$m_2=-2$

For an acute angle

$\tan\theta=\left|\frac{m_1-m_2}{1+\ m_1m_2}\right|$

$\tan\theta=\left|\frac{3--2}{1-6}\right|$

$\tan\theta=\left|\frac{5}{-5}\right|=1$

$\theta=45°$ ✓

(f) $\frac{1}{2}\log(1+2\sin x)+C$

✓ ✓

Question 2

(a) $\frac{du}{dx}=4x \qquad x=2 \quad u=9$ ✓

$\qquad\qquad\qquad x=0 \quad u=1$

$dx=\frac{du}{4x}$

$\int_0^2\frac{8x}{\sqrt{1+2x^2}}\,dx=\int_1^9\frac{\cancel{8x}}{\sqrt{u}}\,\frac{du}{\cancel{4x}}=\int_1^9 2u^{-\frac{1}{2}}du=$ ✓

$\left[4u^{\frac{1}{2}}\right]_1^9=12-4=8$ ✓

(b) $\tan x=\frac{1}{\sqrt{3}}$ ✓

$x=n\pi+\tan^{-1}\frac{1}{\sqrt{3}}$

$x=n\pi+\frac{\pi}{6}$ ✓

(c) ✓

$P(2)=2(2)^4-4(2)^3+4(2)^2-15(2)+14$

$=0$

$\therefore (x-2)$ is a factor via the factor theorem

(d)

$\cos 2x=2\sin^2 x-1$

so $\cos 4x=2\sin^2 2x-1$

$\sin^2 2x=\frac{1}{2}(\cos 4x+1)$ ✓

$\int_0^{\frac{\pi}{4}}\sin^2 2x\ \ dx=\frac{1}{2}\int_0^{\frac{\pi}{4}}(\cos 4x+1)dx$

$=\frac{1}{2}\left[\frac{\sin 4x}{4}+x\right]_0^{\frac{\pi}{4}}$ ✓

$=\frac{1}{2}\left[\left(\frac{\sin\pi}{4}+\frac{\pi}{4}\right)-0\right]$

$=\frac{\pi}{8}$ ✓

(e) (i) Two tangents from an exterior point to a circle are equal. ✓

(ii)

Since $\angle BFE = \alpha$, then $\angle FBE = \alpha$ (isos $\triangle$)

Since $\angle BDE = \beta$, then $\angle DBE = \beta$ (isos $\triangle$) ✓

In $\triangle BFD$ $\quad \alpha+\alpha+\beta+\beta = 180$ ($\angle$ sum of $\triangle$)

$$2\alpha + 2\beta = 180$$

$$\alpha + \beta = 90 \quad ✓$$

$$\therefore \angle FBD = 90^\circ$$

Question 3.

(a) (i) $6! = 720$ ✓

(ii) $4! \times 2! = 48$ ✓ ✓

(b) (i)

$$f(2) = (2)^3 + 2(2) - 17$$
$$= 8 + 4 - 17$$
$$= -5$$
$$f(3) = (3)^3 + 2(3) - 17 \quad ✓$$
$$= 16$$

Since $f(x)$ changes sign between $x=2$ and $x=3$ and since $f(x)$ is continuous for all x, $f(x)$ must be zero somewhere between $x=2$ and $x=3$.

(ii)

$$f'(x) = 3x^2 + 2$$
$$f'(2\cdot4) = 3(2\cdot4)^2 + 2 = 19\cdot24 \quad ✓$$
$$f(2\cdot4) = (2\cdot4)^3 + 2(2\cdot4) - 17 = 1\cdot624$$
$$x_2 = x_1 - \frac{f(x)}{f'(x)}$$
$$= 2\cdot4 - \frac{1\cdot624}{19\cdot24} \quad ✓$$
$$= 2\cdot3155925.....$$
$$= 2\cdot32 \quad (2d.p) \quad ✓$$

(c) $\ln(x + \sqrt{x^2+9}) + C$ ✓ ✓

(d)

$$\left[\frac{1}{16}\times\frac{4}{3}\tan^{-1}\frac{4x}{3}\right]_0^{\frac{3}{4}} \quad ✓ ✓$$
$$= \frac{1}{16}\times\frac{4}{3}\tan^{-1}1 - \frac{1}{16}\times\frac{4}{3}\tan^{-1}0$$
$$= \frac{1}{16}\times\frac{4}{3}\times\frac{\pi}{4}$$
$$= \frac{\pi}{48} \quad ✓$$

Question 4.

(a) (i) $-4:1$ ✓

(ii)

$$x = \frac{kx_2 + lx_1}{k+l}$$

$$2 = \frac{7k-l}{k+l} \quad ✓$$
$$2k + 2l = 7k - l$$
$$-5k = -3l$$
$$\frac{k}{l} = \frac{3}{5}$$
$$k:l = 3:5 \quad ✓$$

bad question. use $y = \frac{ky_2 + ly_1}{k+l}$ and you get different answer

This suggests that... XWY is not a line

(b) (i) $\alpha+\beta+\gamma = 3$ ✓

(ii) $\alpha\beta\gamma = 2$ ✓

(iii)

$$\alpha \times \frac{1}{\alpha} \times \gamma = 2$$
$$\gamma = 2 \quad ✓$$
$$\alpha + \beta + 2 = 3$$
$$\alpha + \beta = 1$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = -k$$
$$1 + 2\beta + 2\alpha = -k$$
$$1 + 2(\beta + \alpha) = -k$$
$$1 + 2\times 1 = -k$$
$$k = -3 \quad ✓$$

(c) (i)

$T = A + Ce^{kt}$ ✓

$\frac{dT}{dt} = kCe^{kt}$

$\frac{dT}{dt} = k(T - A)$ as $Ce^{kt} = T - A$ ✓

(ii)

When $t = 0$ $T = 100$

$100 = 20 + Ae^{0}$

$A = 80$ ✓

$T = 20 + 80e^{kt}$

when $t = 2$ $T = 95$

$95 = 20 + 80e^{k2}$

$e^{2k} = \frac{15}{16}$

$2k = \ln\frac{15}{16}$

$k = \frac{1}{2}\ln\frac{15}{16}$ ✓

when $t = 10$

$T = 20 + 80e^{\frac{1}{2}\ln\frac{15}{16}.10}$

$T = 77.93571472$

$T = 78°$ ✓

Question 5:

(a) Let $7^n + 13^n = 10M$ where M is any integer.

For $n=1$ $7^1 + 13^1 = 20 = 10 \times 2$ which is divisible by 10. ✓

Assume $7^k + 13^k = 10M$ is true for $n=k$
Prove true for $n=k+2$

$13^k = 10M - 7^k$

$7^{k+2} + 13^{k+2} =$

$7^2 7^k + 13^2 13^k = 7^2 7^k + 13^2(10M - 7^k)$ ✓

$= 49.7^k + 1690M - 169.7^k$

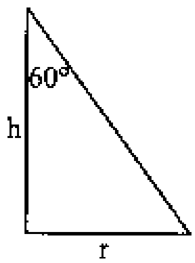
$= 1690M - 120.7^k$

$= 10(169 - 12.7^k)$ ✓

which is a multiple of 10, therefore true for $n=k+2$.

Since it is true for n=1, it is true for n=1+2
And so on, so it is true for all positive odd integers.

(b)



(i) $\tan 60° = \frac{r}{h}$

$h \tan 60° = r$ ✓

$r = \sqrt{3}h$

(ii) $\frac{dV}{dt} = 12$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi(3h^2)h$

$V = \pi h^3$ ✓

$\frac{dV}{dh} = 3\pi h^2$

$\frac{dV}{dh} = 432\pi$ when h=12 ✓

$\frac{dh}{dt} = \frac{dh}{dV}.\frac{dV}{dt}$

$= \frac{1}{432\pi}.12$

$= \frac{1}{36\pi} cm/s$ ✓

(c) (i) $x \neq 0$ ✓

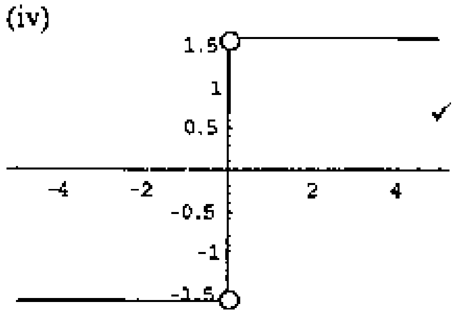
$$f(1) = \tan^{-1} 1 + \tan^{-1}\left(\frac{1}{1}\right)$$

(ii)
$$= \frac{\pi}{4} + \frac{\pi}{4} \quad ✓$$
$$= \frac{\pi}{2}$$

(iii)

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \frac{d}{dx}(x^{-1})$$
$$= \frac{1}{1+x^2} + \frac{1}{\frac{x^2+1}{x^2}} \cdot -x^{-2} \quad ✓$$
$$= \frac{1}{1+x^2} + \frac{x^2}{x^2+1} \cdot -\frac{1}{x^2}$$
$$= \frac{1}{1+x^2} - \frac{1}{x^2+1}$$
$$= 0 \quad ✓$$

(iv)



✓

Question 6 (12 marks) (Start a new booklet)

(i) $\ddot{x} = -4x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x$ ☑

$\frac{1}{2}v^2 = -\frac{4x^2}{2} + C \qquad (v = 0, x = a)$

$0 = -2a^2 + C$

$C = 2a^2$ ☑

$\frac{1}{2}v^2 = -2x^2 + 2a^2$

$v^2 = 4\left(a^2 - x^2\right)$ ☑

(ii) $v^2 = 4\left(a^2 - x^2\right) \qquad x = 2, v = 4$

$16 = 4\left(a^2 - 4\right)$

$a^2 - 4 = 4$

$a^2 = 8$

$a = 2\sqrt{2} \qquad (a > 0)$ ☑

(iii) $v^2 = 4\left(8 - x^2\right)$

$v = \pm 2\sqrt{8 - x^2}$ ☑

but if $x = 2, v = 4 > 0$

$v = 2\sqrt{8 - x^2}$ ☑

↳ nope.
SHM so it's ±

(iv) $v = 2\sqrt{8 - x^2}$

$\frac{dx}{dt} = 2\sqrt{8 - x^2}$

$\frac{dt}{dx} = \frac{1}{2\sqrt{8 - x^2}}$ ☑

$t = \frac{1}{2}\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right) + C, \left(t = \frac{\pi}{4}, x = 2\sqrt{2}\right)$ ☑

$\frac{\pi}{4} = \frac{1}{2}\sin^{-1}\left(\frac{2\sqrt{2}}{2\sqrt{2}}\right) + C$

$\frac{\pi}{4} = \frac{1}{2}\sin^{-1}(1) + C$

$\frac{\pi}{4} = \frac{\pi}{4} + C$

$C = 0$

$t = \frac{1}{2}\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right)$ ☑

$\sin(2t) = \frac{x}{2\sqrt{2}}$

$x = 2\sqrt{2}\sin(2t)$

(v) at the origin (the centre of the motion. ☑

(vi) $v = 4\sqrt{2}\cos(2t)$ $\boxed{\checkmark}$

$\text{max speed} = 4\sqrt{2} \times 1$

$= 4\sqrt{2}\ m/s$ $\boxed{\checkmark}$

Question 7 (12 marks) (Start a new booklet)

(i) $x = 2e^{-t}(\cos t + \sin t)$

$\dot{x} = (\cos t + \sin t) \times -2e^{-t} + 2e^{-t}(-\sin t + \cos t)$

$= -2e^{-t} \times 2\sin t$ $\boxed{\checkmark}$

$= -4e^{-t}\sin t$

$\ddot{x} = \sin t \times 4e^{-t} + -4e^{-t}\cos t$

$= 4e^{-t}(\sin t - \cos t)$ $\boxed{\checkmark}$

(ii) As $t \to \infty$, $x \to 0$ since $e^{-t} \to 0$. $\boxed{\checkmark}$

(iii) $0 = 2e^{-t}(\cos t + \sin t)$ but $e^{-t} \neq 0$

$\cos t + \sin t = 0$

$\sin t = -\cos t$ $\boxed{\checkmark}$

$\tan t = -1$

$t = \frac{3\pi}{4}, \frac{7\pi}{4}$ $\boxed{\checkmark}$

(iv) $\dot{x} = -4e^{-t}\sin t$ moving in a positive direction $\dot{x} > 0$, $\boxed{\checkmark}$

$-4e^{-t}\sin t > 0$

$\sin t < 0$

$\pi < t < 2\pi$

(v) Stationary when $\dot{x} = 0 \to t = 0, \pi, 2\pi$ $\boxed{\checkmark}\boxed{\checkmark}$

(vi) $x = 2e^{-t}(\cos t + \sin t)$ $t = 0, \pi, 2\pi$

$x = 2(\cos 0 + \sin 0) = 2$

$x = 2e^{-\pi}(\cos \pi + \sin \pi) = -0.086$

$x = 2e^{-2\pi}(\cos 2\pi + \sin 2\pi) = 0.004$ $\boxed{\checkmark}\boxed{\checkmark}$

(vii)