Name : ______________________

Number : ______________________

HURLSTONE AGRICULTURAL HIGH SCHOOL

YEAR 12 2008

MATHEMATICS EXTENSION 1

TRIAL HIGHER SCHOOL CERTIFICATE

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General Instructions

- Reading time : 5 minutes
- **Working time : 2 hours**
- Attempt **all** questions
- **Start a new sheet of paper for each question**
- All necessary working should be shown
- This paper contains 8 questions worth 10 marks each. Total Marks: **80 marks**
- Marks may not be awarded for careless or badly arranged work
- Board approved calculators and mathematical templates may be used
- This examination paper must **not** be removed from the examination room

Marks

QUESTION 1. *Start a new sheet of paper.*

a) Let A $(-2,3)$ and B $(-5,-3)$ be points on the number plane. Find the coordinates of the point P which divides the interval AB externally in the ratio 3:1. **2**

b) Find the acute angle between the lines $2x-3y+1=0$ and $y=2x-1$ to the nearest degree. **2**

c) Solve the inequality $\dfrac{3}{x+2} \leq 1$ **3**

d) Solve $|x+1| > |x-2|$ **3**

QUESTION 2. *Start a new sheet of paper.*

a) Use mathematical induction to prove, for all positive integers n, that **3**

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

b) Find the Cartesian equation of the curve defined by the parametric equations $x=\sin\theta$ and $y=\cos^2\theta-3$. **2**

c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2=4ay$.
The equation of the normal to the parabola at P is $x+py=2ap+ap^3$ and the equation of the normal at Q is similarly given by $x+qy=2aq+aq^3$.

(i) Show that the normals at P and Q intersect at the point R whose coordinates are $\left(-apq[p+q],\, a[p^2+pq+q^2+2]\right)$. **2**

(ii) The equation of the chord PQ is $y=\frac{1}{2}(p+q)x-apq$. (Do NOT show this.) **1**

If the chord PQ passes through $(0, a)$, show that $pq=-1$.

(iii) Find the equation of the locus of R if the chord PQ passes through $(0, a)$. **2**

Marks

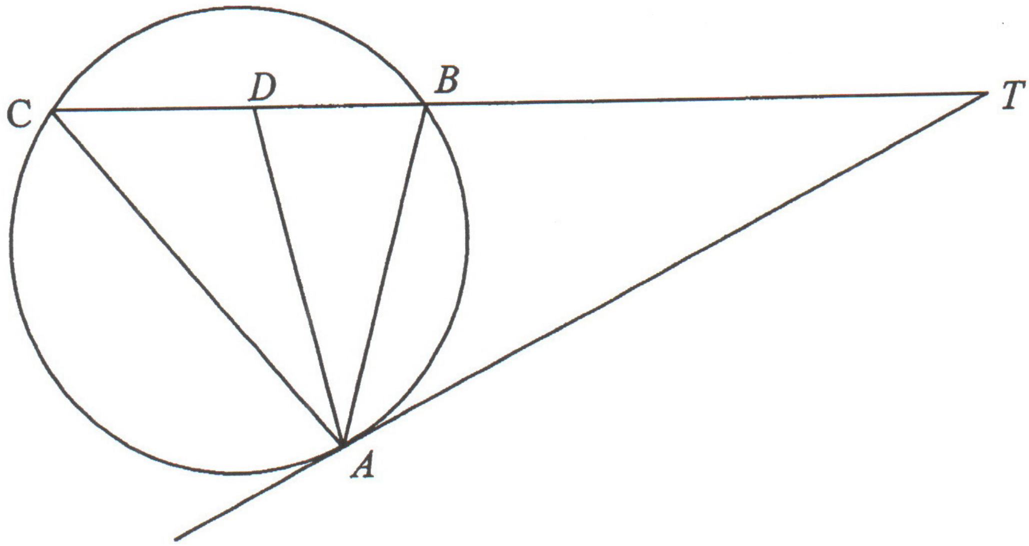
QUESTION 3. *Start a new sheet of paper.*

a) If $y = e^{kx}$ is a solution of the equation **2**

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

determine the value(s) of k.

b) **3**



TA is a tangent to the circle above. AD bisects $\angle BAC$.
Prove that $TA = TD$.

c) On a warm summer afternoon, two cans of drink, initially at 30°C, are put into a refrigerator whose inside temperature is maintained at 4°C. The cooling rate of the soft drink is proportional to the difference between the temperature of the refrigerator and the temperature, T, of the can of drink. This means that T satisfies the equation

$$\frac{dT}{dt} = -k(T-4)$$

where t is the number of minutes after the can of drink is placed in the refrigerator

(i) Show that $T = 4 + Ae^{-kt}$ satisfies the above equation. **1**

(ii) After 20 minutes, one of the cans is removed. It is found that its temperature is now 15°C. Find the values of A and k. **2**

(iii) How long will it take for the other can to cool down to 8°C? **2**
{Time is measured from when it was put in the refrigerator}

Marks

QUESTION 4. *Start a new sheet of paper.*

a) State the domain of $y = \cos^{-1} 3x$. Sketch the graph of $y = \cos^{-1} 3x$ **2**

b) Evaluate $\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right)$ **2**

c) By writing an expression for $\tan(\alpha - \beta)$, show that **3**

$$\tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{4}\right) = \tan^{-1}\left(\frac{2}{9}\right)$$

d) If $f(x) = \tan^{-1}\left(\dfrac{2}{x}\right)$ find the equation of the tangent to the curve at the point where $x = 2$. **3**

QUESTION 5. *Start a new booklet.*

a) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 x\, dx$ **2**

b) Use the substitution $u = 1 - 2x$ to evaluate $\displaystyle\int_0^{\frac{1}{2}} 2x\sqrt{1-2x}\, dx$ **3**

c) If $\displaystyle\int_0^{t} \frac{1}{1+x^2}\, dx = 0.9$, find t correct to two decimal places **2**

d) Show that $\displaystyle\int_0^{\frac{\pi}{4}} \tan x\, dx = \frac{1}{2}\ln 2$ **3**

Marks

QUESTION 6. *Start a new sheet of paper.*

a) i) Noting that $2\cos^2 x \equiv 1+\cos 2x$, prove that $8\cos^4 x \equiv 3+4\cos 2x+\cos 4x$ **2**

ii) Hence, find the volume generated when the area bounded by the curve $y=\cos^2 x$, the x axis and $x=0$ and $x=\dfrac{\pi}{4}$ is rotated about the x axis **2**

b) Prove the identity: $\dfrac{\sin A}{\cos A+\sin A}+\dfrac{\sin A}{\cos A-\sin A}\equiv \tan 2A$ **2**

c) Solve the equation $2\sin^2\theta=\sin 2\theta$.
Give your answer in general form. **2**

d) Solve $\sin x+\cos x=1$ for x where $0\le x\le 2\pi$ **2**

QUESTION 7. *Start a new sheet of paper.*

The function $f(x)$ is defined by $f(x)=\dfrac{1}{2}\left(e^{x}-e^{-x}\right)$

a) State the domain of $f(x)$ **1**

b) Prove that $f(x)$ is an increasing function and hence state its range **3**

c) Explain why the function has an inverse function $f^{-1}(x)$. **1**

d) By interchanging x and y, show that $f^{-1}(x)=\ln\left(x+\sqrt{x^2+1}\right)$ **3**

e) Find $\dfrac{d}{dx}\left(f^{-1}(x)\right)$ and hence find $\displaystyle\int\frac{dx}{\sqrt{1+x^2}}$ **2**

Marks

QUESTION 8. *Start a new sheet of paper.*

a) Let each different arrangement of all the letters of D E L E T E D be called a word.

i) How many words are possible? **1**

ii) In how many of these words will the D's be separated? **1**

b) Use polynomial division to find the remainder when $x^4 - 3x^2 + 4x - 8$ is divided by $x^2 + 1$ **2**

c) i) Show that the function $f(x) = x^3 - x^2 - x - 1$ has a zero between 1 and 2 **1**

ii) Taking $x = 2$ as a first approximation to this zero, use Newton's method to calculate a second approximation **1**

iii) Give a geometrical interpretation of the process used in ii). Why is $x = 1$ unsuitable as a first approximation to this zero? **2**

d) In each of the following, use the given information to obtain the real polynomial $P(x)$ in the form $P(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots.. + p_n$ where $n, p_0, p_1, p_2, \ldots\ldots, p_n$ are to be given numerical values.

i) $P(x)$ is quadratic, $P(0) = 32$, and $P(2^t) = 0$ has roots $t = 1$ and $t = 3$ **1**

ii) $P(x)$ has degree 4, has factors $(x-2)^2$ and $(x+2)^2$, and has a remainder of 50 when divided by $(x-3)$. **1**