



2010 Trial Examination

FORM VI

MATHEMATICS EXTENSION 1

Wednesday 11th August 2010

General Instructions

- Reading time — 5 minutes
- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper

- Total marks — 84
- All seven questions may be attempted.
- All seven questions are of equal value.

Collection

- Write your candidate number clearly on each booklet.
- Hand in the seven questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question One.

Checklist

- SGS booklets — 7 per boy
- Candidature — 132 boys

Examiner

SJE

QUESTION ONE (12 marks) Use a separate writing booklet.**Marks**

- (a) The polynomial $P(x) = x^4 - x^3 + kx - 4$ has a factor $(x + 1)$. Find the value of k . **1**
- (b) Differentiate $y = \sin(\log_e x)$. **1**
- (c) Find, correct to the nearest degree, the acute angle between the lines **2**
 $x - 3y + 4 = 0$ and $2x + y - 1 = 0$.
- (d) Find the coordinates of the point that divides the interval from $(-3, 4)$ to $(5, -2)$ in the ratio $1 : 3$. **2**
- (e) Find the exact value of $\int_0^2 \frac{4}{4 + x^2} dx$. **2**
- (f) Find $\lim_{x \rightarrow 0} \left(\frac{\sin x \cos x}{x} \right)$. **1**
- (g) Find the term independent of x in the expansion of $\left(x^2 + \frac{2}{x} \right)^{15}$. **3**

QUESTION TWO (12 marks) Use a separate writing booklet.**Marks**

- (a) The equation $x^3 + bx^2 + cx + d = 0$ has roots $2 + \sqrt{3}$, $2 - \sqrt{3}$ and -3 . Use the sum and the product of the roots to find b , c and d . **3**
- (b) Consider the curve $f(x) = \sin^{-1}(2x)$. **2**
- (i) Sketch the curve. **2**
- (ii) Find the gradient of the tangent to the curve at the point where $x = \frac{1}{4}$. **2**
- (c) A particle is undergoing simple harmonic motion subject to the equation $\frac{d^2x}{dt^2} = -6x$. Initially it is at rest at $x = 2$. **1**
- (i) In which direction will it start to move? **1**
- (ii) Show that $v^2 = 6(4 - x^2)$. **2**
- (iii) State the period and the amplitude of the motion. **2**

QUESTION THREE (12 marks) Use a separate writing booklet.

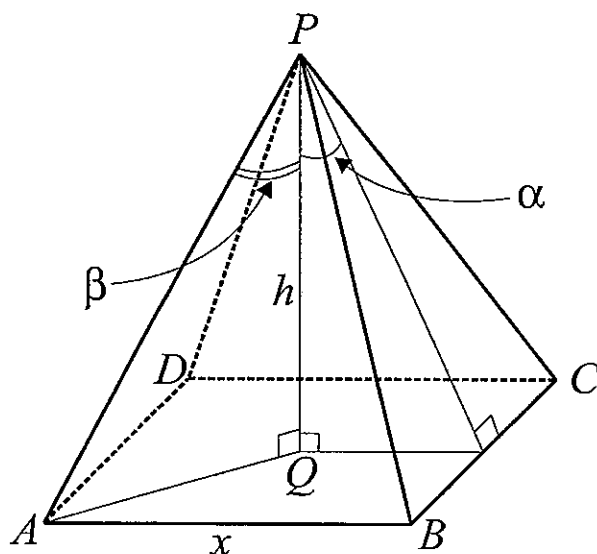
Marks

- (a) Use the substitution $u = x - 1$ to find $\int x(x - 1)^4 dx$. 2
- (b) (i) Express $\cos \theta - \sqrt{3} \sin \theta$ in the form $R \cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. 3
- (ii) Hence, or otherwise, solve $\cos \theta - \sqrt{3} \sin \theta = 1$, for $0 \leq \theta \leq 2\pi$. 2
- (c) Consider the function $f(x) = \frac{x-1}{x-2}$.
- (i) Show that $f^{-1}(x) = \frac{2x-1}{x-1}$. 1
- (ii) Find the vertical and horizontal asymptotes of $f^{-1}(x)$. 2
- (iii) Sketch $f^{-1}(x)$ showing the asymptotes and any x or y intercepts. 1
- (iv) Hence, or otherwise, solve $\frac{2x-1}{x-1} \geq 1$. 1

QUESTION FOUR (12 marks) Use a separate writing booklet.

Marks

(a)



A square pyramid has altitude PQ of length h and base $ABCD$ of side length x , as shown above. Each face makes an angle α with PQ and each edge makes an angle β with PQ . Assume that it is a right pyramid, so that Q lies in the centre of the base.

- (i) Show that $AQ = \frac{x}{\sqrt{2}}$. 1
- (ii) Hence express x in terms of h and β . 1
- (iii) Show that $\sqrt{2} \tan \alpha = \tan \beta$. 1

Exam continues overleaf ...

QUESTION FOUR (Continued)

(b) Solve for x and y :

3

$$\log_3 x + \log_3 y = 6$$

$$\log_2 x - \log_2 y = 4$$

(c) (i) Write $\cos^2 x$ in terms of $\cos 2x$.

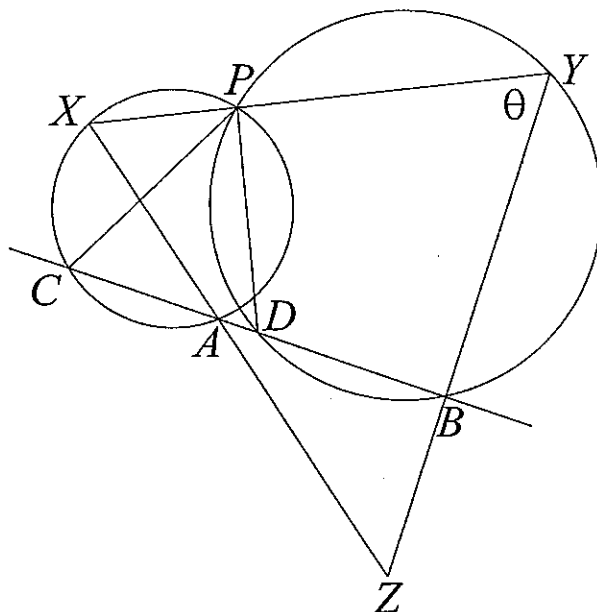
1

(ii) Hence evaluate $\int_0^{\frac{\pi}{3}} \sin 2x \cos^2 x \, dx$.

2

(d)

3



In the diagram above, two circles intersect and P is one of the points of intersection. A straight line is drawn through P cutting the two circles at X and Y . An isosceles triangle XYZ is constructed with $XZ = YZ$. Suppose that XZ cuts the smaller circle at A and YZ cuts the larger circle at B . Suppose also that the line AB cuts the circles at C and D . Let $\angle XYZ$ be θ .

Prove that $\triangle CPD$ is isosceles.

NOTE: You do not have to copy the diagram above. It has been reproduced for you on a tear-off sheet at the end of this paper. Insert this sheet into your answer booklet.

QUESTION FIVE (12 marks) Use a separate writing booklet.**Marks**

- (a) Consider the two points $P(4t, 2t^2)$ and $Q(8t, 8t^2)$ on the parabola $x^2 = 8y$. The tangents at P and Q intersect at R .

(i) Find the equations of the tangents at P and Q .

2

(ii) Find the coordinates of R .

2

(iii) Hence find the locus of R .

1

- (b) By substituting a suitable value for x in the expansion of $(1+x)^n$, show that

2

$$1 + 2 \binom{n}{1} + 4 \binom{n}{2} + \dots + 2^{n-1} \binom{n}{n-1} = 3^n - 2^n.$$

- (c) A forensic scientist is called upon to determine the time of death of a corpse found in a room which is maintained at a constant temperature of 20°C . The temperature T of the corpse was initially measured at midnight to be 29°C . The scientist measured the temperature of the corpse one hour later and it had fallen to 26°C . Assume that the temperature of the body at the time of death was 36.8°C and that the rate of temperature decrease obeys Newton's law of cooling. Let t be the number of hours after midnight.

(i) Show that $T = 20 + 9e^{-kt}$ satisfies the cooling equation $\frac{dT}{dt} = -k(T - 20)$.

1

(ii) Show that $k = \ln \frac{3}{2}$.

2

(iii) Hence estimate the time of death, to the nearest minute.

2

QUESTION SIX (12 marks) Use a separate writing booklet.**Marks**

- (a) Prove by mathematical induction that for all positive integer values of
- n
- ,

3

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

- (b) A film director must decide whether a stuntman is able to perform a dangerous stunt. The stuntman must leap from a building onto the centre of some erected scaffolding. The centre of the scaffolding is 5 m below his initial position and at a horizontal distance of 14 m. The stuntman jumps at an angle of 30° above the horizontal. Let the stuntman's initial velocity be V , and let x and y be his horizontal and vertical displacements respectively from his initial position. You may assume that the velocity and displacement equations are:

$$\dot{x} = V \cos 30^\circ \qquad \dot{y} = -10t + V \sin 30^\circ$$

$$x = Vt \cos 30^\circ \qquad y = -5t^2 + Vt \sin 30^\circ$$

- (i) Show that the Cartesian equation of the stuntman's path is

2

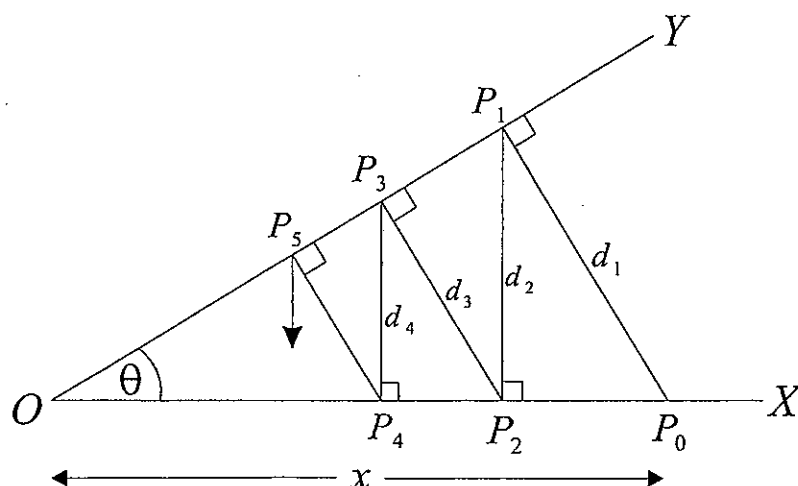
$$y = -\frac{20x^2}{3V^2} + \frac{x}{\sqrt{3}}.$$

- (ii) Hence determine the required initial velocity V so that he lands in the centre of the scaffolding. Write your answer to the nearest m/s.
- (iii) Safety requirements are such that if the impact velocity is greater than 15 m/s, then padding must be placed on the scaffolding. Assuming that the stuntman leaps at the required speed, determine whether or not padding is needed.

2**2**

QUESTION SIX. (Continued)

(c)



The diagram above shows two straight lines OX and OY . The points P_0, P_2, P_4, \dots lie on OX , while the points P_1, P_3, P_5, \dots lie on OY .

P_1 is the foot of the perpendicular from P_0 to OY ,
 P_2 is the foot of the perpendicular from P_1 to OX ,
 P_3 is the foot of the perpendicular from P_2 to OY , and so on.

Let $\angle XOY = \theta$, where $0^\circ < \theta < 90^\circ$, let $OP_0 = x$, and let the length of the line joining P_{r-1} to P_r be denoted by d_r , for $r = 1, 2, 3, \dots$

(i) Show that the lengths d_1, d_2, d_3, \dots form a geometric series.

2

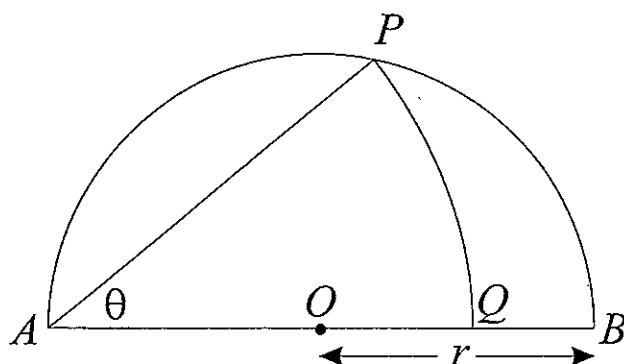
(ii) Hence prove that $\sum_{r=1}^{\infty} d_r = x \cot \frac{\theta}{2}$.

1

QUESTION SEVEN (12 marks) Use a separate writing booklet.

Marks

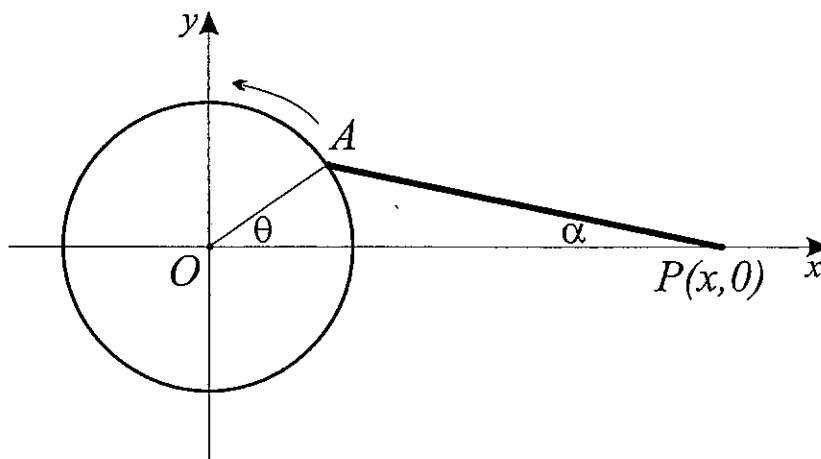
(a)



The diagram above shows a semi-circle with centre O , radius r and diameter AB . Let P be a point on arc AB . The arc PQ has centre A and Q lies on AB . Let $\angle PAQ = \theta$.

- (i) Show that $AP = 2r \cos \theta$. 1
- (ii) Prove that as θ varies, the arc PQ will have maximum length when $\theta \sin \theta = \cos \theta$. 3
- (iii) Taking $\theta = 1$ as a first approximation to the value of θ that maximises the arc PQ , use one application of Newton's method to find a better approximation. Round your answer to two decimal places. 2

(b)



The diagram above shows a rotating wheel with radius 40 cm and a connecting rod AP with length 120 cm. The pin P slides back and forth along the x -axis as the wheel rotates anticlockwise at a rate of 6 revolutions per second. In each part below you need only address the case where A is in the first quadrant.

- (i) Show that $\alpha = \sin^{-1} \left(\frac{\sin \theta}{3} \right)$. 1
- (ii) Use the chain rule to show that $\frac{d\alpha}{dt} = \frac{12\pi \cos \theta}{\sqrt{9 - \sin^2 \theta}}$ radians per second. 1
- (iii) Show that $x = 40 \left(\cos \theta + \sqrt{9 - \sin^2 \theta} \right)$. 2
- (iv) Find an expression for the velocity of the pin P in terms of θ . 2

END OF EXAMINATION

Question 1

$$(a) \quad p(1) = (-1)^4 - (-1)^3 + kx - 4 = 0$$

$$1 + 1 - k - 4 = 0$$

$$-k - 2 = 0$$

$$k = -2$$

(b) $y = \sin(\log x)$

$$\frac{dy}{dx} = \frac{\cos(\log x)}{x}$$

(c) $\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$

$$= \left| \frac{\frac{1}{3} - (-2)}{1 + \frac{1}{3}(-2)} \right|$$

$$= \left| \frac{\frac{7}{3}}{\frac{1}{3}} \right|$$

$$= 7$$

$$\therefore \theta \div 82^\circ \text{ (nearest degree)}$$

(d) $x = \frac{1(5) + 3(-3)}{1+3}$

$$= -1$$

$$\therefore \text{Coordinates are } (-1, \frac{5}{2})$$

$$y = \frac{1(-2) + 3(4)}{1+3}$$

$$= \frac{5}{2}$$

(e)

$$\int_0^2 \frac{4}{4+x^2} dx = 4 \left[\frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) \right]_0^2$$

$$= 2 \left[\tan^{-1}(1) - \tan^{-1}(0) \right]$$

$$= 2 \cdot \frac{\pi}{4}$$

$$= \frac{\pi}{2}$$

(f)

$$\lim_{x \rightarrow 0} \left(\frac{\sin x \cos x}{x} \right) = \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) \times \lim_{x \rightarrow 0} (\cos x)$$

$$= 1 \times 1$$

$$= 1$$

(g) $(x^2 + \frac{2}{x})^{15}$

General Term: ${}^{15}C_r (x^2)^r \left(\frac{2}{x}\right)^{15-r}$

$$= {}^{15}C_r x^{2r-15+r} 2^{15-r}$$

$$= {}^{15}C_r x^{3r-15} 2^{15-r}$$

So Term independent of x has

$$3r - 15 = 0$$

$$r = 5$$

$$\therefore \text{Term is } {}^{15}C_5 2^{10} = 3075072$$

Question 2

(a) Sum of roots: $2 + \sqrt{3} + 2 - \sqrt{3} - 3 = -\frac{b}{a}$

$$1 = -b$$

$$b = -1$$

Product of roots: $(2 + \sqrt{3})(2 - \sqrt{3})(-3) = -d$

$$(4 - 3)(-3) = -d$$

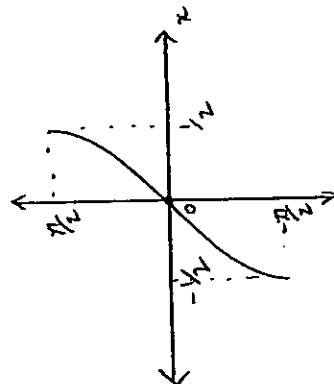
$$d = 3$$

$$A_{60}, (2 + \sqrt{3})(2 - \sqrt{3}) + (2 - \sqrt{3})(-3) + (-3)(2 + \sqrt{3}) = c$$

$$4 - 3 + -6 + 3\sqrt{3} - 6 - 3\sqrt{3} = c$$

$$-11 = c \quad \text{--- (3)}$$

(b) (i)



(ii) $f'(x) = \frac{2}{\sqrt{1 - (2x)^2}}$

$$f'\left(\frac{1}{4}\right) = \frac{2}{\sqrt{1 - \frac{1}{4}}}$$

$$= \frac{2}{\frac{\sqrt{3}}{2}}$$

$$= \frac{4}{\sqrt{3}}$$

(4)

(c) $\frac{d^2x}{dt^2} = -6x$

$$t = 0$$

$$x = 2$$

$$v = 0$$

(i) Negative direction towards the origin

(ii) $\frac{d}{dt}\left(\frac{1}{2}v^2\right) = -6x$

$$\frac{1}{2}v^2 = -3x^2 + c_1$$

$$v^2 = -6x^2 + c_2$$

Now $v = 0, x = 2$

$$0 = -24 + c_2$$

$$\therefore c_2 = 24$$

So $v^2 = -6x^2 + 24$

$$= 6(4 - x^2)$$

as required

(iii) $\frac{d^2x}{dt^2} = -n^2x$

$$\therefore n^2 = \frac{6}{\sqrt{6}}$$

Period = $\frac{2\pi}{n}$

$$= \frac{2\pi}{\sqrt{\frac{6}{\sqrt{6}}}}$$

$$= \frac{\pi\sqrt{6}}{3}$$

Amplitude = 2

(5)

12

Question 3

(a) $\int x(x-1)^4 dx$ $u = x-1, x = u+1$
 $du = dx$

$$= \int (u+1)u^4 du$$

$$= \int u^5 du + \int u^4 du$$

$$= \frac{u^6}{6} + \frac{u^5}{5} + C$$

$$= \frac{(x-1)^6}{6} + \frac{(x-1)^5}{5} + C \quad \textcircled{2}$$

(b) (i) $\cos \theta - \sqrt{3} \sin \theta = R \cos(\theta + \alpha)$
 $= R [\cos \theta \cos \alpha - \sin \theta \sin \alpha]$

Equating coefficients:

$$R \cos \alpha = 1$$

$$R \sin \alpha = \sqrt{3}$$

Squaring and adding

$$R^2 = 4$$

$$\therefore R = 2$$

$$\cos \alpha = \frac{1}{2} \quad (\alpha \text{ acute})$$

$$\therefore \alpha = \frac{\pi}{3}$$

$$\text{So } \cos \theta - \sqrt{3} \sin \theta = 2 \cos\left(\theta + \frac{\pi}{3}\right)$$

(ii) $2 \cos\left(\theta + \frac{\pi}{3}\right) = 1$
 $\cos\left(\theta + \frac{\pi}{3}\right) = \frac{1}{2}$

$$\theta + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$$

$$\therefore \theta = 0, \frac{4\pi}{3}, 2\pi$$

✓ 5

(c) (i) $f(x) = y = \frac{x-1}{x-2}$
 Interchange x and y

$$x = \frac{y-1}{y-2}$$

$$x(y-2) = y-1$$

$$xy - y = -1 + 2x$$

$$y(x-1) = 2x-1$$

$$y = \frac{2x-1}{x-1}$$

$$\therefore f^{-1}(x) = \frac{2x-1}{x-1} \text{ as required.}$$

(ii) Vertical asymptote: $x = 1$

$$f^{-1}(x) = \frac{2 - \frac{1}{x}}{1 - \frac{1}{x}}$$

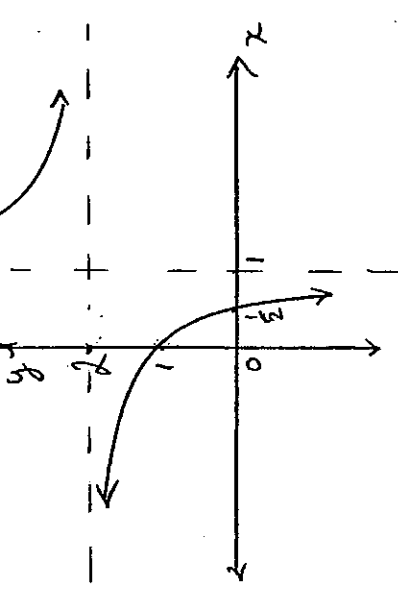
as $x \rightarrow \pm \infty$ $f^{-1}(x) \rightarrow 2$

\therefore Horizontal asymptote: $y = 2$

(iii) y -intercept ($x=0$)

x -intercept ($y=0$)

$$\left. \begin{matrix} y = 1 \\ x = \frac{1}{2} \end{matrix} \right\}$$



(iv) $\frac{2x-1}{x-1} \geq 1$ when $x \leq 0$ or $x > 1$

12

Question 4

(a) (i) $AC^2 = x^2 + x^2$
 $= 2x^2$

$AC = \sqrt{2}x$

$AQ = \frac{\sqrt{2}x}{2}$

$= \frac{x}{\sqrt{2}}$ as required

(ii) $\tan \beta = \frac{AQ}{h}$

$= \frac{\frac{x}{\sqrt{2}}}{h}$

$\therefore x = \sqrt{2}h \tan \beta$

(iii) $\tan \alpha = \frac{x}{\frac{x}{2}}$

$\therefore 2 \tan \alpha = \frac{x}{h}$ and $\sqrt{2} \tan \beta = \frac{x}{h}$

$\therefore 2 \tan \alpha = \sqrt{2} \tan \beta$

$\sqrt{2} \tan \alpha = \tan \beta$

(b) $\log_3 x + \log_3 y = 6$
 $\log_2 x - \log_2 y = 4$

$xy = 3^6$
 $\frac{x}{y} = 2^4$

Solving simultaneously

$x^2 = 11664$

$x = 108$ ($x > 0$)

$y = \frac{3^6}{108}$
 $= 6^{3/4}$

(3)

(c) (i) $\cos 2x = \frac{1}{2} \cos 2x + \frac{1}{2}$

(ii) $\int_0^{\pi/3} \sin 2x \cos^2 x \, dx = \frac{1}{2} \int_0^{\pi/3} (\sin 2x \cos 2x + \sin 2x) \, dx$

$= \frac{1}{2} \left[\frac{\sin^2 2x}{4} \right]_0^{\pi/3} + \frac{1}{2} \left[-\frac{\cos 2x}{2} \right]_0^{\pi/3}$

$= \frac{1}{8} \left[\left(\frac{\sqrt{3}}{2} \right)^2 - 0 \right] + \frac{1}{4} \left[-\frac{1}{2} - -1 \right]$

$= \frac{3}{32} + \frac{3}{8}$

$= \frac{15}{32}$

(3)

(c) Alternate Solution

$\int_0^{\pi/3} \sin 2x \cos^2 x \, dx = \frac{1}{2} \int_0^{\pi/3} \sin 2x (1 + \cos 2x) \, dx$

$= \frac{1}{2} \int_0^{\pi/3} \sin 2x \, dx + \frac{1}{2} \int_0^{\pi/3} \sin 2x \cos 2x \, dx$

$= -\frac{1}{4} \left[\cos 2x \right]_0^{\pi/3} + \frac{1}{4} \int_0^{\pi/3} \sin 4x \, dx$

$= \left[-\frac{\cos 2x}{4} \right]_0^{\pi/3} + \left[-\frac{\cos 4x}{16} \right]_0^{\pi/3}$

$= \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16}$

$= \frac{15}{32}$

(c) Alternate Solution.

$$\begin{aligned} \int_0^{\pi/3} \sin 2x \cos^2 x \, dx &= 2 \int_0^{\pi/3} \cos^3 x \sin x \, dx \\ &= -2 \left[\frac{\cos^4 x}{4} \right]_0^{\pi/3} \\ &= -2 \left(\frac{1}{64} - \frac{1}{4} \right) = \frac{15}{32} \end{aligned}$$

(d) $\angle YXZ = \theta$ (base angles of isosceles triangle XYZ) ✓
 $\angle PXA = \angle PCA$ (angles subtended at the circumference by arc PA) ✓
 $= \theta$
 $\angle PDC = \theta$ (exterior angle of a cyclic quadrilateral PYBD) ✓

$$\therefore \angle PCD = \angle PDC = \theta$$

$\therefore \triangle CPD$ is isosceles

12

Question 5

(a) $x^2 = 8y$ $\therefore a = 2$

(i) gradient of tangent at $P(4t, 2t^2)$ is t
 Equation of tangent at P : $y - 2t^2 = t(x - 4t)$ ✓
 $y = tx - 2t^2$ ✓

gradient of tangent at $Q(8t, 8t^2)$ is $2t$
 Equation of tangent at Q : $y - 8t^2 = 2t(x - 8t)$ ✓
 $y = 2tx - 8t^2$ ✓

(ii) Solving the equations of the tangents simultaneously

$$tx - 2t^2 = 2tx - 8t^2$$

$$6t^2 = tx$$

$$x = 6t$$

$$y = t(6t) - 2t^2$$

$$= 4t^2$$

\therefore Coordinates of R are $(6t, 4t^2)$

(iii) locus of R : $t = \frac{x}{6}$

$$y = 4\left(\frac{x}{6}\right)^2$$

$$= \frac{4x^2}{36}$$

$$x^2 = 9y$$

5

$$b) (1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$$

Substitute $x=2$

$$3^n = 1 + \binom{n}{1}2 + \binom{n}{2}4 + \dots + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n$$

$$\text{Now } \binom{n}{n} = 1$$

$$\therefore 3^n = 1 + 2\binom{n}{1} + 4\binom{n}{2} + \dots + 2^{n-1}\binom{n}{n-1} + 2^n$$

$$3^n - 2^n = 1 + 2\binom{n}{1} + 4\binom{n}{2} + \dots + 2^{n-1}\binom{n}{n-1}$$

as required (2)

(c) At midnight: $t=0$, $T=29^\circ\text{C}$

$$t=1, T=26^\circ\text{C}$$

$$i) T = 20 + 9e^{-kt} \Rightarrow T-20 = 9e^{-kt}$$

$$\frac{dT}{dt} = -k \cdot 9e^{-kt}$$

$$= -k(T-20) \text{ as required.}$$

$$(ii) T = 20 + 9e^{-kt}$$

$$t=1, T=26$$

$$26 = 20 + 9e^{-k}$$

$$6 = 9e^{-k}$$

$$\frac{2}{3} = e^{-k}$$

$$\ln\left(\frac{2}{3}\right) = -k$$

$$\therefore k = \ln\left(\frac{3}{2}\right)$$

Question 5. (cont.)

(c) (iii) For time of death solve for t
when $T=36.8$

$$36.8 = 20 + 9e^{-kt} \quad (k = \ln \frac{3}{2})$$

$$\frac{16.8}{9} = e^{-kt}$$

$$\ln\left(\frac{16.8}{9}\right) = -kt$$

$$t = -\frac{\ln\left(\frac{16.8}{9}\right)}{\ln\left(\frac{3}{2}\right)}$$

$$\doteq -1.54 \text{ hours}$$

$$\doteq -1 \text{ hour } 32 \text{ mins.}$$

\therefore Time of death is approximately 10:28 p.m.
(1 hour 32 minutes before midnight)

(5)

12

Question 6

(a) Step 1: $n=1$

$$LHS = \frac{1}{2!}$$

$$= \frac{1}{2}$$

$$RHS = 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

$$= LHS$$

Hence, the result is true for $n=1$.

Step 2: Suppose the result is true for $n=k$.

$$\text{i.e. } \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!} (*)$$

We need to show that the result is true for $n=k+1$

$$\text{i.e. } \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$$

$$LHS = 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$$

$$= 1 - \frac{(k+2) - (k+1)}{(k+2)!}$$

$$= 1 - \frac{k+2-k-1}{(k+2)!}$$

$$= 1 - \frac{1}{(k+2)!}$$

$$= RHS$$

Step 3 It follows from Step 1 and Step 2 by mathematical induction that it is true for all positive integers.

(3)

Question 6 (cont.)

(b) (i)

$$x = Vt \cos 30^\circ$$

$$= \frac{V\sqrt{3}}{2} t$$

$$\therefore t = \frac{2x}{\sqrt{3}V}$$

$$\text{So } y = -5\left(\frac{2x}{\sqrt{3}V}\right)^2 + \frac{2x}{\sqrt{3}} \cdot \frac{1}{2}$$

$$= -\frac{20x^2}{3V^2} + \frac{x}{\sqrt{3}} \text{ as required.}$$

(ii) Sub. $x=14$, $y=-5$ and solve for V

$$-5 = -\frac{20(14)^2}{3V^2} + \frac{14}{\sqrt{3}}$$

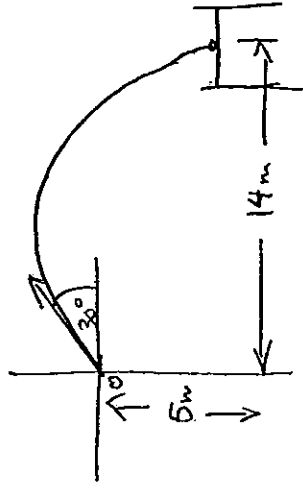
$$-15V^2 = -20 \times 196 + 14\sqrt{3}V^2$$

$$V^2(14\sqrt{3}+15) = 20 \times 196$$

$$V^2 = \frac{3920}{14\sqrt{3}+15}$$

$$\div 99.87589\dots$$

$$\therefore V \div 10 \text{ m/s}$$



iii)

$$\text{Impact Velocity} = \sqrt{x^2 + y^2}$$

First, calculate time of flight, $x = 14$, $v = 10$

$$t = \frac{2(14)}{\sqrt{3}(10)}$$

$$= \frac{14}{5\sqrt{3}} \text{ s}$$

$$\dot{x} = \frac{10\sqrt{3}}{2}$$

$$= 5\sqrt{3}$$

$$\dot{y} = -10 \frac{14}{5\sqrt{3}} + 10 \cdot \frac{1}{2}$$

$$= -\frac{28}{\sqrt{3}} + 5$$

$$= \frac{-28\sqrt{3} + 15}{3}$$

$$V = \sqrt{(5\sqrt{3})^2 + \left(\frac{15 - 28\sqrt{3}}{3}\right)^2}$$

$$= \sqrt{199.99 \dots}$$

$$\approx 14.14 \text{ m/s}$$

\therefore Padding is not required

Question 6. (cont.)

- (c) (i) $\angle OP_1P_2 = 90^\circ - \theta$ (angle sum of $\triangle OP_1P_2$)
 $\angle P_2P_1P_0 = \theta$ (angle sum of $\triangle P_2P_1P_0$)

$$\text{Now } \cos \theta = \frac{d_2}{d_1}$$

$$\text{Similarly } \angle P_3P_2P_1 = \theta$$

$$\text{and } \cos \theta = \frac{d_3}{d_2}$$

$$\text{Geometric series as } \frac{d_3}{d_2} = \frac{d_2}{d_1} = \cos \theta$$

$$a = d_1 = x \sin \theta \quad (\text{from } \triangle P_1OP_0)$$

$$r = \cos \theta$$

$$(ii) \sum_{n=1}^{\infty} dr = \frac{x \sin \theta}{1 - \cos \theta} \quad \text{since } |\cos \theta| < 1$$

$$= x \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}}$$

$$= x \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}}$$

$$= x \cot \frac{\theta}{2} \quad \text{as required}$$

(3)

12

Question 7

a) i) Join PB $\therefore \angle APB = 90^\circ$ (angle in a semi-circle)

$$S_o \quad \frac{AP}{2r} = \cos \theta$$

$$AP = 2r \cos \theta$$

(ii) Let l be arc length PQ

$$l = AP \times \theta = 2r \cos \theta \times \theta \quad (\theta \text{ in radians})$$

$$\frac{dl}{d\theta} = 2r \cos \theta + 2r \theta (-\sin \theta)$$

$$= 2r (\cos \theta - \theta \sin \theta)$$

Stationary point when $\cos \theta - \theta \sin \theta = 0$
 $\theta \sin \theta = \cos \theta$

Now $\frac{d^2l}{d\theta^2} < 0$ for a maximum

$$\begin{aligned} \frac{d^2l}{d\theta^2} &= 2r(-\sin \theta) - 2r[\sin \theta + \theta \cos \theta] \\ &= 2r(-2\sin \theta - \cos \theta) \\ &= -2r(2\sin \theta + \cos \theta) < 0 \end{aligned}$$

since $0 < \theta < \frac{\pi}{2}$ (θ in a semi-circle)

(iii) let $f(\theta) = \theta \sin \theta - \cos \theta$

$$f'(\theta) = \theta \cos \theta + 2\sin \theta$$

$$\theta_0 = 1$$

Question 7 (cont.)

(a) (ii) cont. $\theta_1 = 1 - \frac{f(\theta_0)}{f'(\theta_0)}$

$$= 1 - \frac{1 \times \sin 1 - \cos 1}{1 \times \cos 1 + 2 \sin 1}$$

$$= \frac{\cos 1 + 2 \sin 1 - \sin 1 + \cos 1}{\cos 1 + 2 \sin 1}$$

$$= \frac{2 \cos 1 + \sin 1}{\cos 1 + 2 \sin 1}$$

$$\approx 0.86 \quad (2 \text{ dec. pl.})$$

(b) i) Using sine rule:

$$\frac{\sin \alpha}{40} = \frac{\sin \theta}{120}$$

$$\sin \alpha = \frac{\sin \theta}{3}$$

$$\alpha = \sin^{-1} \left(\frac{\sin \theta}{3} \right) \text{ as required}$$

$$0 \leq \alpha < \frac{\pi}{2}$$

(ii) $\frac{d\alpha}{dt} = \frac{d\alpha}{d\theta} \cdot \frac{d\theta}{dt}$

Now $\frac{d\theta}{dt} = 6 \text{ rev/s}$

$$= 12\pi \text{ radians/sec}$$

$$\frac{d\alpha}{d\theta} = \frac{1}{3} \cos \theta \cdot \frac{1}{\sqrt{1 - \frac{\sin^2 \theta}{9}}}$$

$$\frac{dx}{d\theta} = \frac{1}{3} \cos \theta \cdot \frac{3}{\sqrt{9 - \sin^2 \theta}}$$

$$= \frac{\cos \theta}{\sqrt{9 - \sin^2 \theta}}$$

$$\text{So } \frac{dx}{dt} = \frac{\cos \theta}{\sqrt{9 - \sin^2 \theta}} \cdot 12\pi$$

$$= \frac{12\pi \cos \theta}{\sqrt{9 - \sin^2 \theta}} \quad \text{as required}$$

(iii) Drop a perpendicular AX from A intersecting OP at X

$$x = OX + XP$$

$$\text{Now } OX = 40 \cos \theta$$

$$XP = 120 \cos \alpha$$

$$\begin{aligned} \therefore x &= 40 \cos \theta + 120 \sqrt{1 - \sin^2 \alpha} \\ &= 40 \cos \theta + 120 \sqrt{1 - \frac{\sin^2 \theta}{9}} \quad \text{from (i)} \\ &= 40 (\cos \theta + 3 \frac{\sqrt{9 - \sin^2 \theta}}{3}) \\ &= 40 (\cos \theta + \sqrt{9 - \sin^2 \theta}) \end{aligned}$$

Alternate solution to (iii) using cosine rule

$$\cos \theta = \frac{40^2 + x^2 - 120^2}{2(40)x}$$

$$x^2 - 80x \cos \theta + (40^2 - 120^2) = 0$$

Using the quadratic formula

$$x = \frac{80 \cos \theta \pm \sqrt{80^2 \cos^2 \theta - 4(40^2 - 120^2)}}{2}$$

$$= 40 \cos \theta \pm 40 \sqrt{\cos^2 \theta + 8}$$

$$= 40 (\cos \theta + \sqrt{\cos^2 \theta + 8}), \quad x > 0$$

$$= 40 (\cos \theta + \sqrt{9 - \sin^2 \theta}) \quad \text{as required}$$

$$(iv) \quad \frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt}$$

$$= 40 (-\sin \theta) + \frac{1}{2} \frac{1}{\sqrt{9 - \sin^2 \theta}} \cdot -2 \sin \theta \cos \theta$$

$$\times 12\pi$$

$$= 480\pi \left[-\sin \theta - \frac{\cos \theta \sin \theta}{\sqrt{9 - \sin^2 \theta}} \right]$$

(6)