QUESTION ONE

(a) $\sum_{n=1}^{4} n^2 = 1^2 + 2^2 + 3^2 + 4^2$

$= 30$

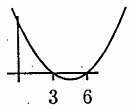
(b) Since $\dfrac{x}{x-3} > 2$

then $x(x-3) > 2(x-3)^2$

Thus $2(x-3)^2 - x(x-3) < 0$

$(x-3)(2x-6-x) < 0$

$(x-3)(x-6) < 0$



Hence from the graph $3 < x < 6$.

(c) (i) $y' = 7e^{7x}$.

(ii) At $x = 1$, $y = e^7$ and $y' = 7e^7$. Thus using the point-gradient form the required tangent is

$$y - e^7 = 7e^7(x-1)$$
$$y = 7e^7 x - 6e^7$$

(d) $$\frac{{}^nC_2}{{}^nC_1} = \frac{n!}{(n-2)!2!} \div \frac{n!}{(n-1)!1!}$$
$$= \frac{(n-1)!}{(n-2)!2}$$
$$= \tfrac{1}{2}(n-1)$$

(e) For any integer $n \in \mathbb{Z}$,

$$3x = \tfrac{\pi}{5} + 2n\pi \quad \text{or} \quad 3x = -\tfrac{\pi}{5} + 2n\pi$$
$$x = \tfrac{\pi}{15} + \tfrac{2}{3}n\pi \quad \text{or} \quad x = -\tfrac{\pi}{15} + \tfrac{2}{3}n\pi$$

QUESTION TWO

(a) (i) Now $A = \pi r^2$ and $\dfrac{dr}{dt} = 0{\cdot}1$ (given). Hence by the chain rule

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$$
$$= 2\pi r \times 0{\cdot}1.$$

(ii) If $\dfrac{dA}{dt} = 2\pi$, then using the result from part (i),

$$0.2\pi r = 2\pi$$

Thus $r = \frac{2}{0.2} = 10$ metres.

(b) The volume of revolution is

$$\begin{aligned} V &= \pi \int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} y^2\,dx \\ &= \pi \int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} \sec^2 3x\,dx \\ &= \Big[\tfrac{1}{3}\pi \tan 3x\Big]_{-\frac{\pi}{12}}^{\frac{\pi}{12}} \\ &= \tfrac{1}{3}\pi\big(\tan\tfrac{\pi}{4} - \tan(-\tfrac{\pi}{4})\big) \\ &= \tfrac{2}{3}\pi \text{ units}^3 \end{aligned}$$

(c) LHS $= \sin 2\theta(\tan\theta + \cot\theta)$

$$\begin{aligned} &= 2\sin\theta\cos\theta\Big(\frac{\sin\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\Big) \\ &= 2\sin\theta\cos\theta\Big(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\Big) \\ &= 2(\sin^2\theta + \cos^2\theta) \\ &= 2 \\ &= \text{RHS} \end{aligned}$$

(d) Since the acceleration is given by $\ddot{x} = \dfrac{d\frac{1}{2}v^2}{dx}$, then

$$\begin{aligned} \frac{d\frac{1}{2}v^2}{dx} &= -9x \\ \tfrac{1}{2}v^2 &= -\tfrac{9}{2}x^2 + C \end{aligned}$$

for some constant C. When $x = 4$, $v = 0$ so

$$\begin{aligned} C &= \tfrac{1}{2}v^2 + \tfrac{9}{2}x^2 \\ &= 0 + \tfrac{9}{2}(4)^2 \\ &= 72. \end{aligned}$$

Thus

$$\begin{aligned} v^2 &= -9x^2 + 144. \\ &= -9(x^2 - 16) \end{aligned}$$

(e) The function v^2 found in part (i) is a concave down parabola with a maximum of $v^2 = 9 \times 16$ at $x = 0$. Thus the maximum speed is $|v| = 12\,\text{m/s}$.

QUESTION THREE

(a) By examining the zeros a possible polynomial is

$$y = a(x+1)^3(x-1)(x-3)^2,$$

for some constant $a \neq 0$. Now the polynomial passes through $(0, -0{\cdot}5)$, so

$$\begin{aligned} -\tfrac{1}{2} &= a \times 1^3 \times -1 \times (-3)^2 \\ -\tfrac{1}{2} &= -9a \\ a &= \tfrac{1}{18} \end{aligned}$$

Thus a possible polynomial is

$$y = \tfrac{1}{18}(x+1)^3(x-1)(x-3)^2,$$

(b) We have the expansion

$$(x^2 + \tfrac{3}{x})^{12} = \sum_{k=0}^{12} {}^{12}C_k (x^2)^{12-k}(\tfrac{3}{x})^k$$

Thus the general term is

$$\begin{aligned} {}^{12}\mathrm{C}_k(x^2)^{12-k}(\tfrac{3}{x})^k &= {}^{12}\mathrm{C}_k x^{24-2k}3^k x^{-k} \\ &= {}^{12}\mathrm{C}_k x^{24-3k}3^k \end{aligned}$$

The term independent of x occurs when $24-3k=0$, that is when $k=8$. Hence the term independent of x is

$$\begin{aligned} {}^{12}\mathrm{C}_8 3^8 &= \frac{12!}{8!4!}3^8 \\ &= 495 \times 3^8 \\ &= 3\,247\,695 \end{aligned}$$

(c) (i) The interval $5-3 \le x \le 5+3$, i.e. $2 \le x \le 8$.

(ii) The period is

$$T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi.$$

(iii) That is, find the two smallest positive solutions of

$$\begin{aligned} \sin(2t + \tfrac{\pi}{4}) &= 1 \\ 2t + \tfrac{\pi}{4} &= \tfrac{\pi}{2} \text{ or } \tfrac{5\pi}{2} \\ 2t &= \tfrac{\pi}{4} \text{ or } \tfrac{9\pi}{4} \\ t &= \tfrac{\pi}{8} \text{ or } \tfrac{9\pi}{8} \end{aligned}$$

Thus the first two times are after $\frac{\pi}{8}$ and $\frac{9\pi}{8}$ seconds.

(d) (i)

$$\begin{aligned} (1+2x)^n &= \sum_{k=0}^{n} {}^n\mathrm{C}_k(2x)^k \\ &= {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1(2x) + {}^n\mathrm{C}_2(2x)^2 + \cdots + {}^n\mathrm{C}_n(2x)^n \end{aligned}$$

(ii) Now if we let $x=1$ in the identity in part (i);

$$\begin{aligned} (1+2)^n &= {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1(2) + {}^n\mathrm{C}_2(2)^2 + \cdots + {}^n\mathrm{C}_n(2)^n \\ 3^n &= {}^n\mathrm{C}_0 + 2 \times {}^n\mathrm{C}_1 + 2^2 \times {}^n\mathrm{C}_2 + \cdots + 2^n \times {}^n\mathrm{C}_n. \end{aligned}$$

QUESTION FOUR

(a) (i) Let $u = 1-x$. Then $du = -dx$. If $x=0$, $u=1$. If $x=1$, $u=0$. Thus

$$\begin{aligned} \int_0^1 x(1-x)^7\,dx &= -\int_1^0 (1-u)u^7\,du \\ &= \int_0^1 u^7 - u^8\,du \\ &= \Big[\tfrac{1}{8}u^8 - \tfrac{1}{9}u^9\Big]_0^1 \\ &= \tfrac{1}{8} - \tfrac{1}{9} \\ &= \tfrac{1}{72} \end{aligned}$$

(b) (i) Let $f(x) = x^2 + 2x + 2 - x^3$ and let $x_0 = 2$ be the initial approximation to the root of $y = f(x)$. Then Newton's method says that if x_n is an approximation to the root then a better approximation is

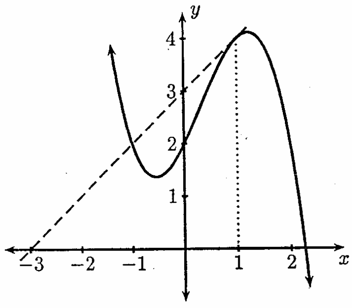
$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^2 + 2x_n + 2 - x_n^3}{2x_n + 2 - 3x_n^2}. \end{aligned}$$

From a calculator,

$$x_1 = 2{\cdot}\dot{3} \qquad x_2 \doteqdot 2{\cdot}2720 \qquad x_3 \doteqdot 2{\cdot}2695$$

Since x_2 and x_3 agree to 2 decimal places, to 2 decimal places the root is $2{\cdot}27$.

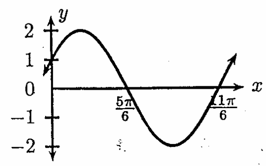
(ii) A tangent to the graph at $x = 1$ does not intersect the x-axis anywhere near the root. This occurs since $(1, 4)$ is on the other side of the turning point from the root. (See the graph below, the tangent is the dashed line).



(c) (i) The easy method is to expand $y = 2\sin(x + \frac{\pi}{6})$. We have

$$\begin{aligned} y &= 2\sin(x + \tfrac{\pi}{6}) \\ &= 2(\sin x \cos \tfrac{\pi}{6} + \cos x \sin \tfrac{\pi}{6}) \\ &= 2(\sin x \times \tfrac{\sqrt{3}}{2} + \cos x \times \tfrac{1}{2}) \\ &= \sqrt{3}\sin x + \cos x \end{aligned}$$

(ii)



QUESTION FIVE

(a) (i) Since $\angle ACD = \angle ABD$ (angles standing on the arc AD), then

$$\begin{aligned} \angle BCD &= \angle ACD + \angle BCA \\ &= \angle ABD + \angle BCA \\ &= 90^\circ \text{(given)} \end{aligned}$$

Hence BD must be a diameter (since $\angle BCD = 90^\circ$).

(ii) Notice $\quad \angle BAC = \angle BCA \quad$ (equal angles opposite equal sides)

Thus $\quad \angle ABD + \angle BAC = \angle ABD + \angle BCA$

$= 90^\circ \quad$ (given)

But $\quad \angle ABD + \angle BAC + \angle BIA = 180^\circ \quad$ (angle sum of triangle)

Hence $\quad \angle BIA = 90^\circ$

But $\quad \angle IDY = 90^\circ \quad$ (radius and tangent)

It follows that $AC \parallel YX$, since the corresponding angles $\angle BIA$ and $\angle IDY$ are equal.

(b) At 6% per annum, the investment grows by a factor of 1·06 each year.
Harry's first contribution is invested for n-years and grows to $M \times 1{\cdot}06^n$.
Harry's second contribution is invested for $(n-1)$-years and grows to $M \times 1{\cdot}06^{n-1}$
His last contribution is invested for 1 year and grows to $M \times 1{\cdot}06$.
following this pattern,the total value of the investment after n-years is

$$M \times 1{\cdot}06 + M \times 1{\cdot}06^2 + \cdots + M \times 1{\cdot}06^n.$$

(c) This is a geometric progression with first term $a = M \times 1{\cdot}06$, common ratio $r = 1{\cdot}06$ and we are required to sum n terms. The sum is

$$\frac{a(r^n - 1)}{r-1} = \frac{M \times 1{\cdot}06 \times (1{\cdot}06^n - 1)}{1{\cdot}06 - 1}$$

If this sum is \$500 000 then

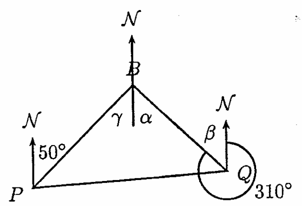
$$\frac{M \times 1{\cdot}06 \times (1{\cdot}06^n - 1)}{1{\cdot}06 - 1} = 500\,000.$$

Thus $M = 500\,000 \times 0{\cdot}06 \div 1{\cdot}06 \div (1{\cdot}06^n - 1)$
and when $n = 20$ years,

$$\begin{aligned} M &= 500\,000 \times 0{\cdot}06 \div 1{\cdot}06 \div (1{\cdot}06^{21} - 1) \\ &\doteqdot \$12823 \end{aligned}$$

(to the nearest dollar).

(d)



(i) In the diagram,

$$\begin{aligned} \beta &= 50^\circ && \text{(revolution)} \\ \alpha &= 50^\circ && \text{(alternate angles, parallel lines)} \\ \gamma &= 50^\circ && \text{(alternate angles, parallel lines)} \end{aligned}$$

Hence $\angle PBQ = 100^\circ$.

(ii) In $\triangle TBP$, $\dfrac{h}{PB} = \tan 30^\circ$, thus $PB = h \cot 30^\circ$.

In $\triangle TBQ$, $\dfrac{h}{BQ} = \tan 45^\circ$, thus $BQ = h \cot 45^\circ$.

(iii) By the cosine rule,

$$\begin{aligned} PQ^2 &= PB^2 + BQ^2 - 2PB \times BQ \cos \angle PBQ. \\ 1000^2 &= h^2 \cot^2 30^\circ + h^2 \cot^2 45^\circ - 2h^2 \cot 30^\circ \cot 45^\circ \cos 100^\circ \\ 1000^2 &= \left(\cot^2 30^\circ + \cot^2 45^\circ - 2 \cot 30^\circ \cot 45^\circ \cos 100^\circ\right) h^2. \end{aligned}$$

Now dividing by the coefficient of h^2 gives the required equation.

(iv) The height of the cliff is about 466 metres.

QUESTION SIX

(a) (i) The gradient is

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tfrac{1}{8}x^2\right) = \tfrac{1}{4}x$$

Thus $\dfrac{dy}{dx} = p$ at $P(4p, 2p^2)$.

Using the point-gradient form, the tangent has equation

$$\begin{aligned} y - 2p^2 &= p(x - 4p) \\ y &= px - 2p^2. \end{aligned}$$

(ii) If $A(3, -2)$ lies on the tangent then

$$\begin{aligned} -2 &= 3p - 2p^2 \\ 2p^2 - 3p - 2 &= 0 \\ (2p+1)(p-2) &= 0 \end{aligned}$$

Hence the points of contact are defined by $p = -0{\cdot}5$ or $p = 2$. The points are $(-2, 0{\cdot}5)$ and $P(8, 8)$.

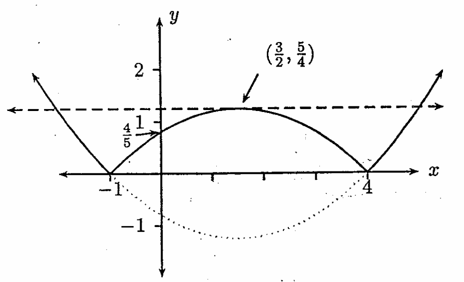
(b) (i) Let $P(x, y)$ be any point on the parabola $y = \frac{1}{4}x^2$. Then the distance d from P to the line $3x - 4y + 4 = 0$ is

$$\begin{aligned} d &= \frac{|3x - 4y + 4|}{\sqrt{3^2 + 4^2}} \\ &= \tfrac{1}{5}|3x - 4(\tfrac{1}{4}x^2) + 4| \\ &= \tfrac{1}{5}|x^2 - 3x - 4| \end{aligned}$$

(ii) The quadratic $\frac{1}{5}|x^2 - 3x - 4| = \frac{1}{5}|(x-4)(x+1)$ has zeros at $x = -1$ and $x = 4$. The vertex is half-way between the zeros, at $x = 1.5$ and

$$\begin{aligned} y &= \tfrac{1}{5}|(1{\cdot}5 - 4)(1{\cdot}5 + 1)| \\ &= -1.25 \end{aligned}$$

The graph is below (the solid dark line).



(iii) This is clear if we add the horizontal line $y = 1\frac{1}{4}$ onto our graph (the dashed line)- it touches or cuts the graph in part (ii) three times.

(c) **Part A:** If $n = 1$, then
RHS$= \frac{1}{12} \times 1 \times 2 \times 3 \times 8$
$= 4$
$= 1 \times 2^2$
$=$ LHS
Hence the statement is true for $n = 1$.
Part B: Suppose k is a positive integer for which the result is true.
That is, suppose that

$$1 \times 2^2 \ + \ \cdots \ + \ k(k+1)^2 = \tfrac{1}{12}k(k+1)(k+2)(3k+5) \qquad ---(*)$$

We shal prove the statement is then true for $n = k + 1$. That is, we shall prove that :

$$1 \times 2^2 \ + \ \cdots \ + \ (k+1)(k+2)^2 = \tfrac{1}{12}(k+1)(k+2)(k+3)(3k+8).$$

Here is the proof;
LHS$= 1 \times 2^2 \ + \ \cdots \ + \ k(k+1)^2 \ + \ (k+1)(k+2)^2$
$= \frac{1}{12}k(k+1)(k+2)(3k+5) \ + \ (k+1)(k+2)^2,$
(by the inductive hypothesis (*).)

Thus LHS$= \frac{1}{12}k(k+1)(k+2)(3k+5) \ + \ \frac{1}{12} \times 12(k+1)(k+2)^2,$

$= \frac{1}{12}(k+1)(k+2)\big(k(3k+5)+12(k+2)\big)$

$= \frac{1}{12}(k+1)(k+2)\big(3k^2+17k+24\big)$

$= \frac{1}{12}(k+1)(k+2)\big((3k+8)(k+3)\big)$

$=$ RHS

Part C: It follows from Parts A and B, and the principle of mathematical induction, that the statement is true for all integers $n \geq 1$.

QUESTION SEVEN

(a) Join OA, OB, OC, OP, OQ, OR. Then the radii OP, OQ and OR are perpendicular to the tangents CB, AC and AB respectively.
The total area of the triangle is

$$\begin{aligned}\text{Area } \triangle ABC &= \text{Area } \triangle ABO \ + \ \text{Area } \triangle ACO \ + \ \text{Area } \triangle BCO\\ &= \tfrac{1}{2} \times AB \times OR \ + \ \tfrac{1}{2} \times AC \times OQ \ + \ \tfrac{1}{2} \times BC \times OP\\ &= \tfrac{1}{2}r(AB \ + \ AC \ + \ BC)\\ &= \tfrac{1}{2}rp\end{aligned}$$

(b) (i) Let $u = \dfrac{1-x}{1+x}$ and $y = \tan^{-1} x + \tan^{-1} u$. By the quotient rule,

$$\begin{aligned}\frac{du}{dx} &= \frac{-1(1+x)-1(1-x)}{(1+x)^2}.\\ &= \frac{-2}{(1+x)^2}\end{aligned}$$

Now using the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1+x^2} + \frac{1}{1+u^2} \times \frac{du}{dx}\\ &= \frac{1}{1+x^2} + \frac{1}{1+u^2} \times \frac{-2}{(1+x)^2}\end{aligned}$$

Since $u \times (1+x) = 1-x$, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1+x^2} + \frac{-2}{(1+x)^2+(1-x)^2}\\ &= \frac{1}{1+x^2} + \frac{-2}{2+2x^2}\\ &= 0\end{aligned}$$

Now the function y is differentiable on the restricted domain $x > 1$, and since it has derivative zero there it must be a constant. Thus $\tan^{-1} x + \tan^{-1} \frac{1-x}{1+x} = C$, for some constant C. To find the value of this constant, substitute any value of $x > -1$, say $x = 0$. Thus

$$\begin{aligned}\tan^{-1} 0 + \tan^{-1} 1 &= C\\ 0 + \frac{\pi}{4} &= C\end{aligned}$$

So $\tan^{-1} x + \tan^{-1} \frac{1-x}{1+x} = \frac{\pi}{4}$, for $x > -1$

(ii) Let $x = \sqrt{2} - 1$. Then $u = \dfrac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1$.
Thus using part (i),

$$\begin{aligned}\tan^{-1}(\sqrt{2}-1) + \tan^{-1}(\sqrt{2}-1) &= \frac{\pi}{4}\\ \tan^{-1}(\sqrt{2}-1) &= \frac{\pi}{8}\end{aligned}$$

(c) (i) The cardinal n represents the number of times this dissection process has been carried out. Initially $n = 0$ and we have the whole square of 1×1 units. Thus $r(0,0) = 1$. At each stage of the division process, rectangles of size $\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{n-k}$ may arise in only one of two ways;

- if one takes $\frac{1}{3}$ rectangle of size $\left(\frac{1}{3}\right)^{k-1}\left(\frac{2}{3}\right)^{n-k}$;
- or if one takes $\frac{2}{3}$ of a rectangle of size $\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{n-k-1}$.

Thus $r(n,k) =$ number of rectangles of size $\left(\frac{1}{3}\right)^{k-1}\left(\frac{2}{3}\right)^{n-k}$

$$+ \text{ number of rectangles of size } \left(\tfrac{1}{3}\right)^k\left(\tfrac{2}{3}\right)^{n-k-1}$$

$$= r(n-1,k-1) + r(n-1,k)$$

(ii) It is clear also that $r(n,0) = 1$, hence the numbers $r(n,k)$ form the entries of Pascal's triangle, that is $r(n,k) = {}^n\mathrm{C}_k$.
The number of rectangles of size $\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{n-k}$ is the term in $\left(\frac{1}{3}\right)^k$ in the expansion of $\left(\frac{1}{3}+\frac{2}{3}\right)^n$. Since this has 12 terms when $n = 11$, the artist will use 12 different colours, and he has plenty of colours in his pack of pencils. The greatest term will corresponding to the colour he uses the most of. Thus to complete the question we need to find the greatest term in this expansion.
To simplify the algebra, let $x = \frac{1}{3}$ and $y = \frac{2}{3}$. Then

$$(x+y)^n = \sum_{k=0}^{11} T_k$$

where $T_k = {}^{11}\mathrm{C}_k x^k y^{11-k}$. Hence

$$\begin{aligned}
\frac{T_{k+1}}{T_k} &= \frac{11!}{(k+1)!(10-k)!} \times \frac{k!(11-k)!}{11!} \times \frac{x^{k+1}y^{10-k}}{x^k y^{11-k}} \\
&= \frac{(11-k)x}{(k+1)y} \\
&= \frac{(11-k)\times\frac{1}{3}}{(k+1)\times\frac{2}{3}} \\
&= \frac{11-k}{2k+2}
\end{aligned}$$

The terms will be increasing if $\dfrac{T_{k+1}}{T_k} > 1$, that is if

$$\begin{aligned}
\frac{11-k}{2k+2} &> 1 \\
11-k &> 2k+2 \qquad \text{(since } (2k+2) > 0\text{)} \\
9 &> 3k
\end{aligned}$$

That is, if $k < 3$. So $T_0 < T_1 < T_2 < T_3 = T_4 > T_5 > \cdots > T_{11}$.
Hence the greatest area coloured by one colour is of size

$$\begin{aligned}
T_3 &= {}^{11}\mathrm{C}_3 \times \left(\tfrac{1}{3}\right)^3 \times \left(\tfrac{2}{3}\right)^8 \\
&= \frac{11\times10\times9}{1\times2\times3} \times \frac{2^8}{3^{11}} \\
&= \frac{2^8\times5\times11}{3^{10}}
\end{aligned}$$

(Since $T_3 = T_4$ the same answer is obtained for the fourth term).

BDD