PYMBLE L.C. EXT 1 TRIAL 2001

Q1 (a) $\dfrac{d}{dx}\left(\dfrac{1}{1+x^2}\right) = \dfrac{-2x}{(1+x^2)^2}$ (1)

[1]

(b) $P(x) = 2x^3 - x + a$

$P(-2) = 0 \Rightarrow 2(-2)^3 - (-2) + a = 0$

$-16 + 2 + a = 0$

$\therefore a = 14$ (1)

[1]

c) A P B

x: -1 4 6

$\underbrace{\quad}_{5}\ \underbrace{\quad}_{2}$

$\dfrac{AP}{PB} = \dfrac{5}{2} = \dfrac{2\frac{1}{2}}{1}$ $\therefore k = 2\frac{1}{2}$ (2)

[2]

(d) $x - 1 = \sqrt{x+1}$ $\qquad x - 1 \geq 0$

$x \geq 1$

$(x-1)^2 = x+1$

$x^2 - 2x + 1 = x + 1$ (1)

$x^2 - 3x = 0$

$x(x-3) = 0$ (1)

$x = 0$ or 3

but since $x \geq 1$, only $x = 3$ (1)

[3]

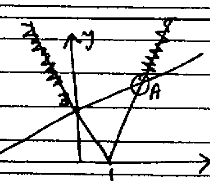
(e) $\displaystyle\int_1^{\sqrt{2}} \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\frac{x}{2}\right]_1^{\sqrt{2}}$ (1)

$= \sin^{-1}\dfrac{1}{\sqrt{2}} - \sin^{-1}\dfrac{1}{2}$

$= \dfrac{\pi}{4} - \dfrac{\pi}{3}$ (1)

$= -\dfrac{\pi}{12}$ (1)

[3]

(f) 

at A, $x + 3 = 3x - 3$

$2x = 6$

$x = 3$

so $x < 0$ (1) or $x > 3$ (1)

[2]

12

Q2 (a) $\cos\sin^{-1}\left(-\frac{1}{3}\right) = \cos\alpha$

$= \dfrac{2\sqrt{2}}{3}$

$\boxed{\begin{array}{l}\frac{1}{2}\ \text{quadrant} \\ \frac{1}{2}\ \text{sign} \\ \frac{1}{2}\ \sqrt{8} = 2\sqrt{2} \\ \frac{1}{2}\ \text{ratio}\end{array}}$ (2)

[2]

(b) $\log_b a = 2 \Rightarrow b^2 = a$ ($\frac{1}{2}$)

$\log_c b = 3 \Rightarrow c^3 = b$ ($\frac{1}{2}$)

$\therefore c^6 = b^2 = a \Rightarrow c = a^{\frac{1}{6}}$ ($\frac{1}{2}$)

$\therefore \log_a c = \frac{1}{6}$ ($\frac{1}{2}$) (2)

[2]

(c) $\displaystyle\int_0^3 \frac{t}{\sqrt{1+t}}\,dt$

$t = u^2 - 1$

$dt = 2u\,du$ (1)

$t = 0,\ u = 1$

$t = 3,\ u = 2$

$= \displaystyle\int_1^2 \frac{(u^2-1)\,2u\,du}{\sqrt{u^2}}$ (1)

$= 2\displaystyle\int_1^2 (u^2-1)\,du$

$= 2\left[\dfrac{u^3}{3} - u\right]_1^2$ (1)

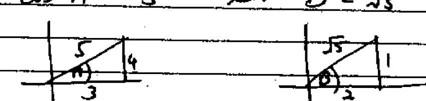
$= 2\left[\dfrac{8}{3} - 2 - \dfrac{1}{3} + 1\right]$

$= 2\left(\dfrac{4}{3}\right)$ (1)

$= \dfrac{8}{3}$

[4]

(d) $\cos A = \frac{3}{5}$ $\quad \sin B = \frac{1}{\sqrt{5}}$



$\cos 2B = 1 - 2\sin^2 B$

$= 1 - 2\left(\frac{1}{5}\right)$

$= \frac{3}{5} = \cos A \quad \therefore A = 2B$ (1)

$\sin 3B = \sin(B + 2B) = \sin(B + A)$

$= \sin B\cos A + \cos B\sin A$ (3)

$= \frac{1}{\sqrt{5}}\cdot\frac{3}{5} + \frac{2}{\sqrt{5}}\cdot\frac{4}{5} = \frac{11}{5\sqrt{5}}$

[4]

12

Q3 (a) $\sum_{1}^{3} \log_2 2n = \log_2 2 + \log_2 4 + \log_2 6$

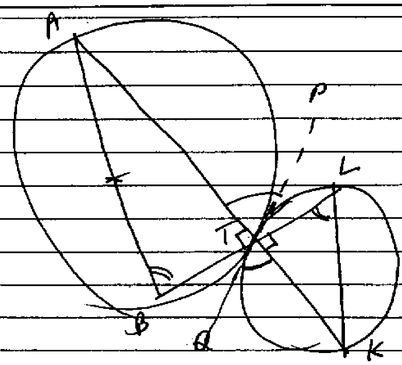
$= 1 + 2 + \log_2 6$

$\boxed{1}$ $= 1 + 2 + (\log_2 2 + \log_2 3)$

$= 4 + \log_2 3$

$\therefore k = 3$ (1)

(b)



(i) since AB is diameter

$\angle ATB = 90°$ ($\angle$ in semicircle)

$\therefore \angle LTK = 90°$ (vert opp $\angle$'s)

$\boxed{3}$ but $\angle LTK$ is also $\angle$ in a semicircle at circum.

so LK is diameter (1)

(ii) Construct tangent PTQ

$\angle PTA = \angle TBA$ ($\angle$ between tangent + chord = $\angle$ in alt. seg)

similarly $\angle QTK = \angle FLK$ (.. . . . .) (1)

but $\angle PTA = \angle QTK$ (vert. opp)

$\therefore \angle TBA = \angle TLK$ (1)

these are alternate equal $\angle$'s

$\therefore AB \parallel LK$

(c) $\int_0^{\frac{\pi}{4}} (\cos x + \sec x)^2 \, dx$ $\qquad \cos 2x = 2\cos^2 x - 1$

$= \int_0^{\frac{\pi}{4}} (\cos^2 x + 2 + \sec^2 x) \, dx$ ($\frac{1}{2}$)

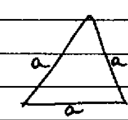
$= \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos 2x) + 2 + \sec^2 x \, dx$ ($\frac{1}{2}$)

$\boxed{4}$ $= \left[\frac{1}{4}\sin 2x + \frac{5x}{2} + \tan x\right]_0^{\frac{\pi}{4}}$ (2)

$= \left(\frac{1}{4} + \frac{5\pi}{8} + 1\right) - 0$ (1)

$= \frac{5}{4} + \frac{5\pi}{8}$

(d)



$P = 3a$

$\frac{dP}{dt} = 3\frac{da}{dt} = 6 \quad \therefore \frac{da}{dt} = 2$

$A = \frac{1}{2}a^2 \sin 60$

$= \frac{\sqrt{3}}{4}a^2$ (1)

$\boxed{4}$ $\frac{dA}{dt} = \frac{dA}{da} \cdot \frac{da}{dt}$ (1)

$= \frac{\sqrt{3}}{2}a \cdot 2$ (1)

$= \sqrt{3}\,a$

$= 8\sqrt{3}$ when $P = 24$ (1)

12

24 (a) $\dfrac{dN}{dt} = 0.5\,(N-100)$

(i) $N = 100 + Ae^{0.5t} \Longrightarrow Ae^{0.5t} = N - 100$

$\dfrac{dN}{dt} = 0.5Ae^{0.5t}$ (1)

$= 0.5\,(N-100)$ as required

when $t=0$, $N = 500$

$\therefore\ 500 = 100 + Ae^{0}$

so $A = 400$ (1)

[3]

(ii) $N = 100 + 400e^{0.5t}$

when $t = 10$,

$N = 100 + 400e^{5}$

$= 59465.26\ldots$ (1)

(b) $f^{-1}(x) = \dfrac{2x-2}{x-2} = y$

so $f(x)$: $x = \dfrac{2y-2}{y-2}$ (1)

$xy - 2x = 2y - 2$

$y(x-2) = 2x - 2$

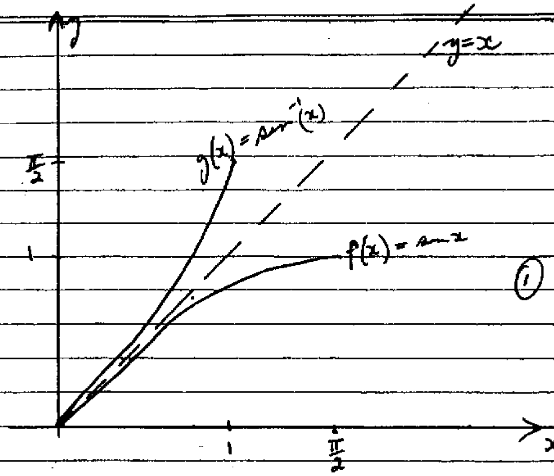
$y = \dfrac{2x-2}{x-2} = f(x)$ (1)

[3]

Since $f(x) = f^{-1}(x)$ then the function is symmetric about $y = x$ (1)

or the fn is the inverse of itself

(c) (i)



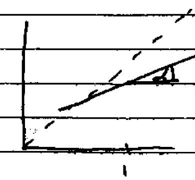
(1)

[1]

(ii) $f(x) = \sin x$

$f'(x) = \cos x$

$= \cos 1$ at $x = 1$ ($\frac{1}{2}$)



$\therefore\ \tan\alpha = \cos 1$

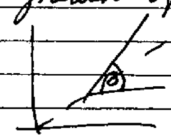
$\alpha = 0.49536\ldots^{c}$ (1)

[4]

so angle between tangent & $y = x$ is $\frac{\pi}{4} - 0.49\ldots$

$= 0.29^{c}$ ($\frac{1}{2}$)

gradient of tangent at $y = 1$ on $g(x)$ is $\frac{1}{\cos 1}$ (1)



so $\tan\beta = \frac{1}{\cos 1}$

$\beta = 1.0754\ldots^{c}$ ($\frac{1}{2}$)

$\therefore$ angle between tangent & $y = x$ is

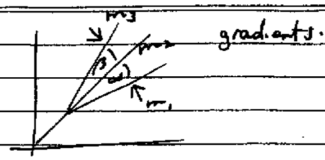
$1.0754 - \frac{\pi}{4} = 0.29^{c}$ ($\frac{1}{2}$)

[1]

(iii) Required angle $= 0.29 \times 2 = 0.58^{c}$ (1)

12

(c) <u>Alternative:</u> for (ii)



$f(x) = \sin x$

$f'(x) = \cos x$ (½)

$= \cos 1$ when $x = 1$

$= m_1$

So $\tan \alpha = \left| \dfrac{m_2 - m_1}{1 + m_1 m_2} \right|$

$= \left| \dfrac{1 - \cos 1}{1 + \cos 1} \right|$ (1)

for $g(x) = \sin^{-1} x$ $\qquad g(x) = y = 1$

$g'(x) = \dfrac{1}{\sqrt{1-x^2}}$ $\qquad x = \sin 1$

$= \dfrac{1}{\sqrt{1 - \sin^2 1}}$ at $y = 1$

$= \dfrac{1}{\cos 1}$ since $1 - \sin^2 1 > 0$ (1)

$= m_3$

So $\tan \beta = \left| \dfrac{\frac{1}{\cos 1} - 1}{1 + \frac{1}{\cos 1}} \right|$ (1)

$= \left| \dfrac{1 - \cos 1}{\cos 1 + 1} \right| = \tan \alpha$

$\therefore \alpha = \beta$ as required. (½)

Since α & β are both acute

Q5 (a) (i) $x = a \cos nt$

$v = \dfrac{dx}{dt} = -an \sin nt$

$v^2 = a^2 n^2 \sin^2 nt$

$= a^2 n^2 \left[1 - \cos^2 nt \right]$ (1)

$= a^2 n^2 \left[1 - \dfrac{x^2}{a^2} \right]$

$= a^2 n^2 - n^2 x^2$ (1)

$\boxed{3}$ so $v^2 = n^2 (a^2 - x^2)$

(ii) $\ddot{x} = \dfrac{d}{dx}\left(\tfrac{1}{2} v^2\right)$

$= \dfrac{d}{dx}\left[\dfrac{n^2}{2}(a^2 - x^2) \right]$

$= \dfrac{n^2}{2} \cdot -2x$ (1)

$\ddot{x} = -n^2 x$

(b) $x = a \cos nt$

when $t = 0$, $x = 2$ $\therefore$ $2 = a \cos 0$

$\therefore a = 2$ (1)

so $x = 2 \cos nt$

when $x = 0$, $v^2 = 4$

$\boxed{3}$ $\therefore 4 = n^2 (4 - 0)$

$\therefore n^2 = 1$ (1)

So period $= \dfrac{2\pi}{n} = 2\pi$

so takes $\frac{1}{4}(2\pi)$ secs to get to 0 (1)

ie $\dfrac{\pi}{2}$ secs.

<u>OR</u> $x = 2 \cos t$

when $x = 0$ $\cos t = 0$

$\therefore t = \dfrac{\pi}{2}$

(c) $y = \dfrac{x+4}{x(x+8)}$

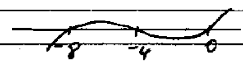
when $y=0$, $x=-4$

$x \neq 0$

Vert. asymp. $x=0$ & $x=-8$ (1)

horiz. asymp. $y=0$

Sign:



$$\frac{dy}{dx} = \frac{(x^2+8x)1 - (x+4)(2x+8)}{x^2(x+8)^2}$$ (2)

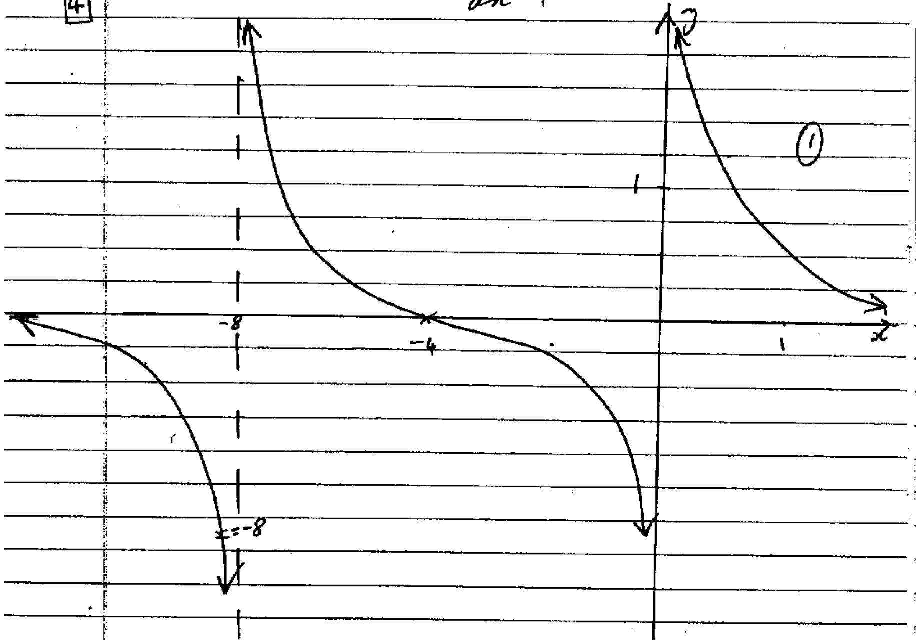
$= 0$ when $x^2+8x-2x^2-16x-32=0$

$x^2+8x+32=0$

but $(x+4)^2+16 \neq 0$

$\therefore \frac{dy}{dx} \neq 0$ so no S.P's.

$\boxed{4}$



(1)

$\boxed{2}$ 12

(d) $A = \displaystyle\int_1^2 \frac{x+4}{x^2+8x}\,dx$

$= \frac{1}{2}\ln(x^2+8x)\Big]_1^2$ (1)

$= \frac{1}{2}(\ln 20 - \ln 9)$

$= \frac{1}{2}\ln\frac{20}{9}$ (1)

Q.6 (a) (i) $f(x) = 3x - 4x^3$

$f'(x) = 3 - 12x^2$

$f'(a) = 3 - 12a^2$ and $f(a) = 3a - 4a^3$ (1)

[2] $\therefore$ eqn is: $y - (3a - 4a^3) = (3 - 12a^2)(x - a)$

$y = (3 - 12a^2)x + 3a - 4a^3 - 3a + 12a^3$ (1)

$y = (3 - 12a^2)x + 8a^3$

(ii) From $(1,0)$:

$0 = (3 - 12a^2)1 + 8a^3$

How many 'a' values satisfy $8a^3 - 12a^2 + 3 = 0$ ($\frac{1}{2}$)

Let $P(a) = 8a^3 - 12a^2 + 3$

$P'(a) = 24a^2 - 24a$

$= 24a(a-1)$

$= 0$ when $a = 0$ or 1

[3] $P''(a) = 48a - 24$

$P''(0) = -24$

$< 0 \therefore$ max at $(0,3)$ (1)

$P''(1) = 24$

$> 0 \therefore$ min at $(1,-1)$

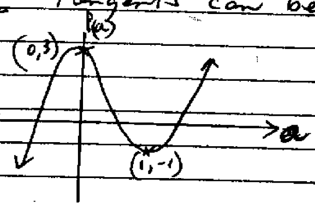
since $P(0) > 0$

and $P(1) < 0$ (1)

there must be a root between 0 & 1

$\therefore P(a) = 0$ has 3 real roots

so 3 tangents can be drawn.

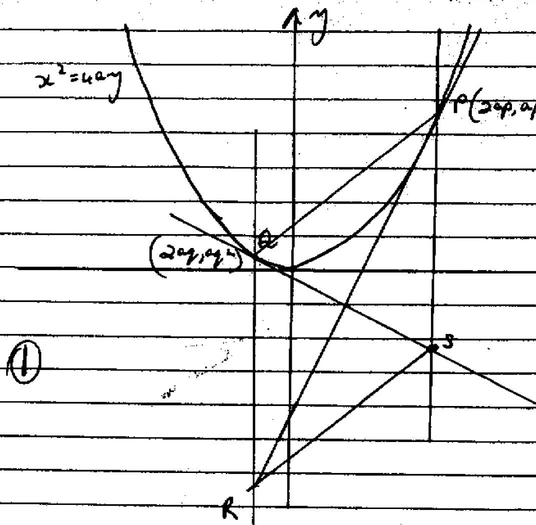


($\frac{1}{2}$)

(b)

(i)

Award ½ as P is not vertex.



No coord deduct ½

[1] (1)

(ii) at P: $\frac{dy}{dp} = 2ap$ $\therefore \frac{dy}{dx} = 2ap \cdot \frac{1}{2a}$

$\frac{dx}{dp} = 2a$ $= p$ ($1\frac{1}{2}$)

[2] $\therefore$ Eqn PR is

$y - ap^2 = p(x - 2ap)$ ($\frac{1}{2}$)

$y = px - 2ap^2 + ap^2$

$y = px - ap^2$

(iii) PS // QR given equation of QS is $y = qx - aq^2$

at S, $x = 2ap$ $\therefore$ $y = q(2ap) - aq^2$ ($\frac{1}{2}$)

$\therefore PS = ap^2 - (2apq - aq^2)$

$= a(p^2 - 2pq + q^2) = a(p-q)^2$ (1)

[2] at R, $x = 2aq$ $\therefore y = p(2aq) - ap^2$ ($\frac{1}{2}$)

$\therefore QR = aq^2 - (2apq - ap^2)$

$= a(q^2 - 2pq + p^2) = a(q-p)^2$

since $a(p-q)^2 = a(q-p)^2$ ($\frac{1}{2}$)

then $PS = QR$ so PQRS is a parallelogram

(one pr of opp. sides equal and parallel).

(iv) $A_{PQRS} = PS \times \text{perp dist from } P \text{ to } Q$

$= a(p-q)^2 \times |2ap - 2aq|$ (1)

$= a(p-q)^2 \times 2a|p-q|$

$= 2a^2(p-q)^2|p-q|$

$= 2a^2|(p-q)^3|$ (1)

$= 2a^2|p-q|^3$

$\boxed{2}$

$\cancel{\ }\ 12$

Aw 1/2 for (2ap(

(iii) alternative method.

$M_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq}$ $\quad M_{SR} = \frac{2apq - ap^2 - 2apq + aq^2}{2aq - 2ap}$

$= \frac{p+q}{2}$ $\quad = \frac{p+q}{2}$

$\therefore$ $SR \parallel PQ$

$QR \parallel PS$

PQRS is a parallelogram.

* if all that's given is $QR \parallel PS \parallel$ y-axis - Aw $\frac{1}{2}$.

Q7 (a) (i) $\ddot{x} = -\frac{4}{x^3}$ $x \neq 0$ $x = 2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{4}{x^3}$

$\frac{1}{2}v^2 = 2x^{-2} + C$ (1/2)

when $t = 0$, $v = 0$ & $x = 2$

$\therefore\ 0 = \frac{1}{2} + C \Rightarrow C = -\frac{1}{2}$ (1/2)

$\boxed{3}$ $\frac{1}{2}v^2 = \frac{2}{x^2} - \frac{1}{2}$

$v^2 = \frac{4}{x^2} - 1$ (1)

Now at $t=0$, $v = 0$ & $\ddot{x} < 0$ $\therefore$ moves left

But since motion is not defined for $x = 0$, particle can not ever stop

so $v = -\sqrt{\frac{4}{x^2} - 1}$ (1)

$= \frac{-\sqrt{4-x^2}}{x}$ since $x > 0$

(ii) $\frac{dx}{dt} = \frac{-\sqrt{4-x^2}}{x}$ (1/2)

$\frac{dt}{dx} = \frac{-x}{\sqrt{4-x^2}}$ (1/2)

$\boxed{3}$

$t = \cancel{+}\sqrt{4-x^2} + C$ (1)

when $t = 0$, $x = 2$ so $C = 0$

$t = \sqrt{4-x^2}$ (1/2)

when $x = 1$, $t = \sqrt{3}$ (1/2)

(b) (i) Let $P(n)$ be the proposition that for integral $n \geq 1$

$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Test $P(1)$: $LHS = 1^2 = 1$

$RHS = \frac{1(2)(3)}{6} = 1 = RHS$ (1/2)

$\therefore\ P(1)$ is true

Let k be a value of n for which $P(k)$ is true

ie $1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$

Then for $P(k+1)$

$1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$ (1/2)

$\boxed{3}$

$= \frac{k+1}{6}\left[k(2k+1) + 6(k+1)\right]$

$= \frac{k+1}{6}\left[2k^2 + 7k + 6\right]$ (1)

$= \frac{k+1}{6}(2k+3)(k+2)$ (1/2)

$= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$

as required

So provided $P(k)$ is true, it is established that $P(k+1)$ is true.

$\therefore$ Since $P(1)$ is true, then $P(2)$ is true, hence $P(3)$ is true etc (1/2)

$\therefore\ P(n)$ is true for all integers $n \geq 1$.

(ii) $2^2 + 4^2 + 6^2 + \cdots + 100^2 = 2^2\left[1^2 + 2^2 + \cdots + 50^2\right]$ (1) so using $n = 50$,

$\boxed{2}$ $= \frac{4 \times 50 \times 51 \times 101}{6} = 171\ 700$ (1)

$\boxed{1}$ (iii) $1^2 + 3^2 + \cdots + 99^2 = (1^2 + 2^2 + \cdots + 100^2) - (2^2 + 4^2 + 6^2 + \cdots + 100^2)$

$= \frac{100 \times 101 \times 201}{6} - 171\ 700 = 166\ 650$ (1)

12