N.S.W. DEPARTMENT OF EDUCATION
HIGHER SCHOOL CERTIFICATE EXAMINATION 1978
MATHEMATICS - 3 UNIT (AND 4 UNIT - FIRST PAPER)

Instructions: Time allowed 3 hours. All questions may be attempted. In every question, all necessary working should be shown. Marks will be deducted for careless or badly arranged work.
Mathematical tables will be supplied. Slide rules (or calculators) may be used.

QUESTION 1

(i) Find the exact value (as a rational number in its lowest terms) of:

(a) $\frac{1}{2} + \frac{2}{3} - \frac{4}{7} - \frac{5}{14}$

(b) $\frac{A^4C}{B^4}$ where $A = (\frac{2}{3})^2$, $B = (\frac{4}{3})^4$, $C = (\frac{8}{3})^7$

(ii) If the volume of one litre of a certain liquid decreases to V litres after n days according to the formula $V = (1 - r)^n$, find the value of r (to three significant figures) if $V = 0.31$ after five days.

(iii) In the triangle ABC the **side** BC has length 4 cm. and the angles at B and C are 60^0, 70^0 respectively. Determine (to three significant figures):

(a) the length of AB

(b) the length of AP where P is the midpoint of BC.

QUESTION 2

(i) Differentiate with respect to : (a) $\frac{x}{1 + x}$ (b) $x^2 \sin x$

(ii) Write down primitives (indefinite integrals) of

(a) $\sin 2x$ (b) $\frac{e^x}{1 + e^x}$ (c) $\frac{1}{\sqrt{4 - x^2}}$

(iii) Express $\sin 40^0 \cos 25^0$ in the form $\frac{1}{2}(\sin A + \sin B)$.

(iv) Evaluate (a) $\int_0^{\pi} \cos \frac{1}{2}x \, dx$ (b) $\int_1^3 \frac{1}{2x} \, dx$

QUESTION 3

(i) For what values of the constant B are the graphs of the linear equations: $x - 4y + 4 = 0$, $2x + By - 2 = 0$

(a) parallel (b) perpendicular ?

(ii) Indicate, using a sketch (not on graph paper) the region of the Cartesian plane for which the following inequalities hold simultaneously:

$x - 4y \leq -4$; $4x + y \leq +4$.

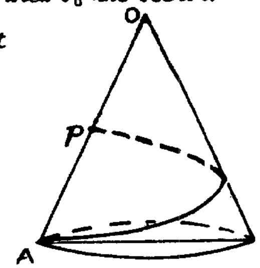
(iii) Sketch (not on graph paper) the graph of $y = \sin^{-1} x$.

QUESTION 4

A thin sheet of smooth metal is in the shape of a sector of a circle with OA, OB as bounding radii each of length 10 cm, and the angle AOB is 60^0.

(i) Find the length of the arc AB and the area of the sector.

(ii) The sheet is now bent to form a right cicular cone by welding the bounding radii OA, OB together (and inserting a circular disc to close in the cone at the base).



(a) Find the volume of the cone. (Note: The volume of a right circular cone is, in the usual notation, $\frac{1}{3}\pi r^2 h$.)

(b) On the surface of this cone a thin string is pulled tight starting with one end fixed at the point A and passing once round the cone to the other end P which is at the midpoint of AO (as illustrated in the diagram). Find the length of this string.

QUESTION 5

(i) For the sequence 2, - 1.5, . . . write down expressions for the nth term and the sum of n terms for the two cases:

(a) the sequence is arithmetic;

(b) the sequence is geometric.

Evaluate each of these four expressions when $n = 4$ and, in those cases where it exists, the limit as $n \to \infty$.

(ii) Write down the formula for the sum of the first n positive odd integers. Explain the method of mathematical induction and use it to prove this formula.

QUESTION 6

(i) For the curve $y = f(x)$ it is given that $f'(x) = 4x - 2$ and the curve passes through the point $(1, 0)$. Deduce the equation of this curve and show that it also passes through the origin.

(ii) Show that this curve is a parabola; find its focal length, the coordinates of its focus and the equation of its directrix.

(iii) Sketch the region in the Cartesian plane bounded by this parabola, its tangent at $(1, 0)$ and the line $x = 0$. Find the equation of this tangent and hence calculate the area of the region.

QUESTION 7

(i) Find the maximum and minimum values of $2 \sec \theta - \tan \theta$, for $0 \le \theta \le \pi/4$.

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(ii) Write down equations or inequalities which describe the following:

(a) the plane through the point $(2, 3, -2)$ parallel to the plane $x + y + z = 0$;

(b) the interior and surface of the sphere whose centre is at the origin and whose radius is $\sqrt{3}$;

(c) the interior and surface of the cube whose faces are portions of the planes:

$x = 1,\ x = -1,\ y = 1,\ y = -1,\ z = 1,\ z = -1$.

Find the distance from the origin of the plane described by the answer to (a) and hence, or otherwise, determine the set of points common to (a), (b) and (c).

QUESTION 8

(i) Two cars, represented by points A and B, are travelling due east and due north respectively along two roads represented by two straight lines intersecting at O. At a certain instant, car A is 2 km west of O and car B is 1 km south of O, the former travelling at a constant speed of 1 km/minute and the latter at constant speed of V km/minute.

(a) What value of V will cause a collision?

(b) Prove that, in the course of motion, the minimum distance between the cars is $|2V - 1|/\sqrt{V^2 + 1}$ km.

(ii) Determine the range of values of V which will cause collision in the above if the problem is made more realistic by representing the cars, not by points A and B, but by rectangles each of length a km and width b km, with their centres at A and B.

QUESTION 9

(i) (a) Write down the value of ${}^nC_j - {}^nC_{n-j}$

(b) By comparing the coefficients of x^8 on both sides of the identity $(1 + x)^8(1 + x)^8 \equiv (1 + x)^{16}$, or otherwise, show that

$$\sum_{j=0}^{8} (C_j)^2 = {}^{16}C_8$$

(ii) A tosses a coin 16 times. What is the probability that it falls heads uppermost on 8 occasions?

(iii) A tosses a coin 8 times and, quite independently, B also tosses a coin 8 times. Calculate the probabilities:

(a) the number of heads obtained by A equals the number of heads obtained by B.

(b) the number of heads obtained by A plus the number obtained by B is 8.

(iv) Describe experiments similar to those in (ii) and (iii) which would lead to the conclusion $\sum_{j=0}^{n} ({}^nC_j)^2 = {}^{2n}C_n$.

QUESTION 10

(i) Calculate the volume of the solid generated when the region in the first quadrant bounded by the curve $y = e^x$, the line $x = \log_e 2$ and the coordinate axes, is rotated about the x- axis.

(ii) The motion of a pendulum may be approximately represented by the equation $\frac{d^2x}{dt^2} = -\frac{g}{L}x$, where x is the displacement at time t. The constants g and L represent the value of gravity at the earth's surface and the length of the pendulum respectively. Consider the case $L = g/\pi^2$.

(a) Let $v = \frac{dx}{dt}$; show that $\frac{dv}{dt} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

(b) The pendulum is swinging in accordance with the initial conditions $t = 0,\ x = 0,\ v = \pi$.

Using the approximate equation of motion above find the first value of x where $v = 0$.

(c) The motion of the pendulum may more accurately be represented by

$$\frac{d^2x}{dt^2} = -\frac{g}{L}\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right).$$

Using the equation of motion and one application of Newton's method find a more accurate answer to (b).