1992 Higher School Certificate Examination Paper

3/4 Unit Mathematics

QUESTION ONE

(a) Solve $x^2 - x - 2 > 0$.

(b) Differentiate $\dfrac{1}{\sqrt{1+x^2}}$.

(c) Find the exact value of $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$.

(d) The probability that any one of the thirty-one days in December is rainy is 0·2.

What is the probability that December has exactly ten rainy days? Leave your answer in index form.

(e)

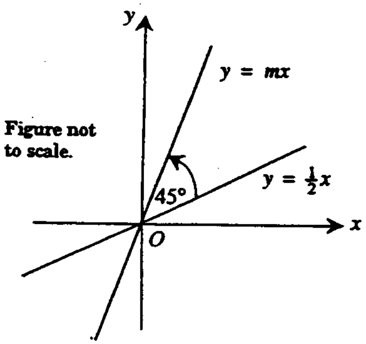


Figure not to scale.

The angle between the lines $y = mx$ and $y = \frac{1}{2}x$ is 45° as shown in the diagram.

Find the exact value of m.

QUESTION TWO

(a) Solve the equation

$$2\sin^2\theta = \sin 2\theta \quad \text{for} \quad 0 \le \theta \le 2\pi.$$

(b) The displacement x metres of a particle moving in simple harmonic motion is given by

$$x = 3\cos \pi t,$$

where the time t is in seconds.

(i) What is the period of the oscillation?

(ii) What is the speed v of the particle as it moves through the equilibrium position?

(iii) Show that the acceleration of the particle is proportional to the displacement from the equilibrium position.

(c) Use Newton's method to find a second approximation to the positive root of $x - 2\sin x = 0$. Take $x = 1{\cdot}7$ as the first approximation.

QUESTION THREE

(a)

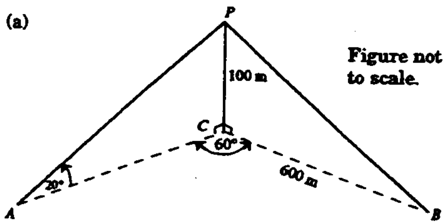


Figure not to scale.

Two yachts A and B subtend an angle of 60° at the base C of a cliff. From yacht A the angle of elevation of the point P, 100 metres vertically above C, is 20°. Yacht B is 600 metres from C.

(i) Calculate the length AC.

(ii) Calculate the distance between the two yachts.

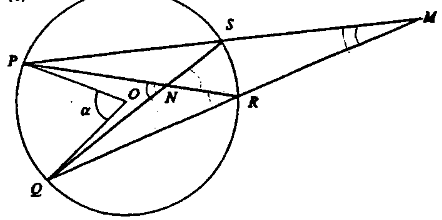
(b) Consider the function $f(x) = 2\tan^{-1}x$.

(i) Evaluate $f(\sqrt{3})$.

(ii) Draw the graph of $y = f(x)$, labelling any key features.

(iii) Find the slope of the curve at the point where it cuts the y-axis.

(c)



In the diagram P, Q, R, and S are points on a circle centre O, and $\angle POQ = \alpha$. The lines PS and QR intersect at M and the lines QS and PR intersect at N.

(i) Explain why $\angle PRM = \pi - \frac{1}{2}\alpha$.

(ii) Show that $\angle PNQ + \angle PMQ = \alpha$.

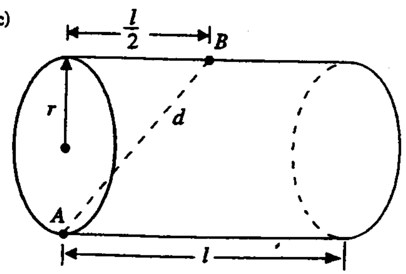
QUESTION FOUR

(a) Evaluate $\displaystyle\int_0^1 \frac{2x}{(2x+1)^2}\,dx$ by using the substitution $u = 2x + 1$.

(b) Let $S_n = 1\times 2 + 2\times 3 + \cdots + (n-1)\times n$.
Use mathematical induction to prove that, for all integers n with $n \ge 2$,

$$S_n = \frac{1}{3}(n-1)n(n+1).$$

(c)



The diagram shows a cylindrical barrel of length l and radius r. The point A is at one end of the barrel, at the very bottom of the rim. The point B is at the very top of the barrel, half-way along its length.

The length of AB is d.

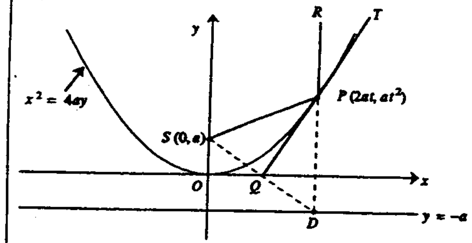
(i) Show that the volume of the barrel is

$$V = \frac{\pi l}{4}\left(d^2 - \frac{l^2}{4}\right).$$

(ii) Find l in terms of d if the barrel has maximum volume for the given d.

QUESTION FIVE

(a)



The diagram shows the parabola $x^2 = 4ay$ with focus $S(0,\ a)$ and directrix $y = -a$.
The point $P(2at,\ at^2)$ is an arbitrary point on the parabola and the line RP is drawn parallel to the y-axis, meeting the directrix at D. The tangent QPT to the parabola at P intersects SD at Q.

(i) Explain why $SP = PD$.

(ii) Find the gradient m_1 of the tangent at P.

(iii) Find the gradient m_2 of the line SD.

(iv) Prove that PQ is perpendicular to SD.

(v) Prove that $\angle RPT = \angle SPQ$.

(b) In a flock of 1000 chickens, the number P infected with a disease at time t years is given by

$$P = \frac{1000}{1 + ce^{-1000t}}$$

where c is a constant.

(i) Show that, eventually, all the chickens will be infected.

(ii) Suppose that when time $t = 0$, exactly one chicken was infected.
After how many days will 500 chickens be infected?

(iii) Show that $\dfrac{dP}{dt} = P(1000 - P)$.

QUESTION SIX

(a) Show that $(x-1)(x-2)$ is a factor of

$$P(x)=x^n(2^m-1)+x^m(1-2^n)+(2^n-2^m)$$

where m and n are positive integers.

(b) A total of five players is selected at random from four sporting teams. Each of the teams consists of ten players numbered from 1 to 10.

(i) What is the probability that of the five selected players, three are numbered '6' and two are numbered '8'?

(ii) What is the probability that the five selected players contain at least four players from the same team?

(c) Consider the binomial expansion

$$1+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n=(1+x)^n.$$

(i) Show that

$$1-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}=0.$$

(ii) Show that

$$1-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\cdots+(-1)^n\frac{1}{n+1}\binom{n}{n}=\frac{1}{n+1}.$$

QUESTION SEVEN

(a) Consider the function $y=f(\theta)$, where

$$f(\theta)=\cos\theta-\frac{1}{4\sqrt{3}\sin\theta}.$$

(i) Verify that $f'\left(\frac{\pi}{6}\right)=0$.

(ii) Sketch the curve $y=f(\theta)$ for $0<\theta\le\frac{\pi}{2}$ given that $f''(\theta)<0$.

On your sketch, write the coordinates of the turning point in exact form and label the asymptote.

(b)

A projectile, of initial speed V m/s, is fired at an angle of elevation α from the origin O towards a target T, which is moving away from O along the x-axis.

You may assume that the projectile's trajectory is defined by the equations

$$x=Vt\cos\alpha \quad\text{and}\quad y=-\frac{1}{2}gt^2+Vt\sin\alpha,$$

where x and y are the horizontal and vertical displacements of the projectile in metres at time t seconds after firing, and where g is the acceleration due to gravity.

(i) Show that the projectile is above the x-axis for a total of $\dfrac{2V\sin\alpha}{g}$ seconds.

(ii) Show that the horizontal range of the projectile is $\dfrac{2V^2\sin\alpha\cos\alpha}{g}$ metres.

(iii) At the instant the projectile is fired, the target T is d metres from O and it is moving away at a constant speed of u m/s.

Suppose that the projectile hits the target when fired at an angle of elevation α.

Show that $u=V\cos\alpha-\dfrac{gd}{2V\sin\alpha}$.

In parts (iv) and (v), assume that

$$gd=\frac{V^2}{2\sqrt{3}}.$$

(iv) By using (iii) and the graph of part (a), show that if $u>\dfrac{V}{\sqrt{3}}$ the target cannot be hit by the projectile, no matter at what angle of elevation α the projectile is fired.

(v) Suppose $u<\dfrac{V}{\sqrt{3}}$.

Show that the target can be hit when it is at precisely two distances from O.