Solutions Trial Ext 1 Maths 2007

Q1.

(a) $P(x) = 2x^3 + x + a$

$P(2) = 16 + 2 + a = 0$

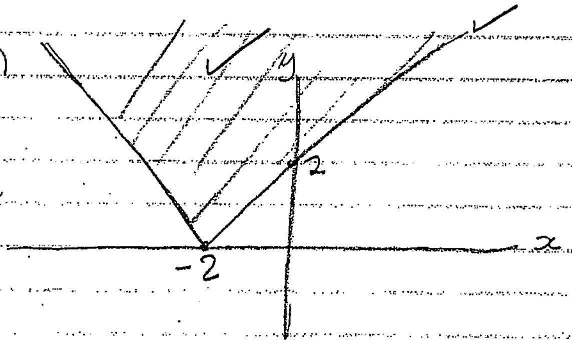
$a = -18$

(b) $\log \frac{b}{c} = \log b - \log c$

$= 2.8 - 4.1$

$= -1.3$

(c)



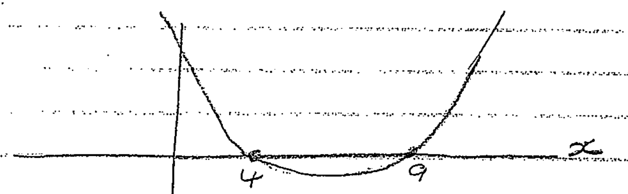
(d) $\frac{5}{x-4} \geq 1, \quad x \neq 4$

$5(x-4) \geq (x-4)^2$

$(x-4)^2 - 5(x-4) \leq 0$

$(x-4)(x-4-5) \leq 0$

$(x-4)(x-9) \leq 0$



$4 < x \leq 9$

(e)

Range: $0 \leq y \leq \pi$

Domain $-1 \leq \frac{x}{4} \leq 1$

$-4 \leq x \leq 4$

f. $\int_0^3 \frac{dx}{9+x^2} = \left[\frac{1}{3} \tan^{-1} \frac{x}{3}\right]_0^3$

$= \frac{1}{3}(\tan^{-1} 1 - \tan^{-1} 0)$

$= \frac{1}{3}\left(\frac{\pi}{4} - 0\right)$

$= \frac{\pi}{12}$

Q2.

(a) $\lim_{x\to 0} \frac{\sin\frac{x}{3}}{2x} = \frac{1}{6} \lim_{x\to 0} \frac{\sin \frac{x}{3}}{\frac{x}{3}}$

$= \frac{1}{6}$

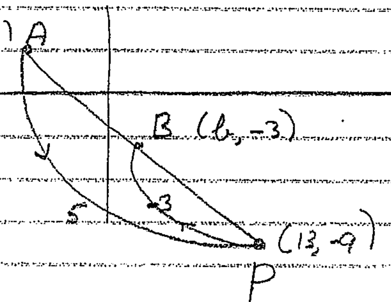
✓ correct intermediate step is necessary here.

(b) (-3,1) A

B (b,-3)

(13,-9)

P



$\frac{(-3)\times(-2)+5b}{5-3} = 13$ ✓

$\frac{6+5b}{2} = 13$

$6+5b = 26$

$5b = 20$

$b = 4$ ✓

(c) $u = e^x$

$du = e^x\,dx$

$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{du}{\sqrt{1-u^2}}$ ✓

$= \sin^{-1} u + c$ ✓

$= \sin^{-1} e^x + c$ ✓

(ignore +c)

(d)

(i) $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$ ✓

(ii) $f(x) = x\cot x$

$f(2x) = 2x \cot 2x$ ✓

$= \cancel{2}x \times \frac{1-\tan^2 x}{\cancel{2}\tan x}$ ✓

$= \frac{x(1-\tan^2 x)}{\tan x}$

$= x(1-\tan^2 x)\cot x$ ✓

$= (1-\tan^2 x)\, f(x)$

(e) $(2-5x)^6$

general term is ${}^6C_r\, 2^r (-5x)^{6-r}$

need $6-r = 3$ so $r=3$ ✓✓

so coefficient is ${}^6C_3\, 2^3 (-5)^3 = -20000$

✓ for ${}^6C_3$, ✓ for $2^3 \times (-5)^3$, ✓ for $r=3$,

~20

Q3.

(a) $y = \cos^{-1} x^2$

$\frac{dy}{dx} = \frac{-2x}{\sqrt{1-x^4}}$ ✓ ✓

($\frac{2x}{\sqrt{1-x^4}}$ is worth one mark.) ✓

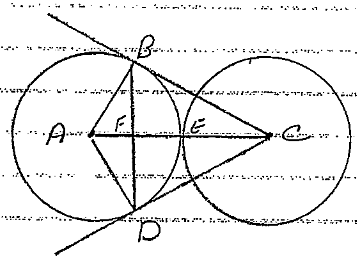
(b) $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2x)\, dx$ ✓

$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$ ✓

$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin \pi\right) - \left(0 + \frac{1}{2}\sin 0\right)\right]$

$= \frac{\pi}{4}$

(c)



(i) The angle between a tangent and chord is a right angle
so $\angle ABC$ and $\angle ADC$ are right angles ✓
so the opposite angles of ABCD are supplementary
so ABCD is cyclic ✓

(ii) Right angles are subtended at the circumference by diameters ✓
So AC is a diameter of ABCD ✓
But AE = EC, since these are radii of equal circles ✓
So E is the centre of the circle

(iii) $\sin \angle BCA = \frac{AB}{AC} = \frac{1}{2}$

so $\angle BCA = 30°$ ✓

Similarly $\sin \angle ACD = \frac{AD}{AC} = \frac{1}{2}$

so $\angle ACD = 30°$ ✓

(iv) Tangents from an external point are equal
So BC = DC and $\triangle BCD$ is isosceles ✓
So $\angle CBD = \angle BDC$
$\angle CBD + \angle BDC = 180° - 60°$, angle sum of triangle ✓
$= 120°$
So $\angle CBD = \angle BDC = 60°$
and $\triangle BCD$ is equilateral

(There are many other correct ways to do this question)

Q4.

(a) $y = \frac{1}{\sqrt{1+4x^2}} = \frac{1}{\sqrt{4(\frac{1}{4}+x^2)}} = \frac{\frac{1}{2}}{\sqrt{\frac{1}{4}+x^2}}$

$V = \frac{\pi}{4}\int_{\frac{2}{\sqrt{3}}}^{2\sqrt{3}} \frac{1}{\frac{1}{4}+x^2}\,dx$ ✓ $a=\frac{1}{2}$

$= \frac{\pi}{2}\left[\tan^{-1}2x\right]_{\frac{2}{\sqrt{3}}}^{2\sqrt{3}}$ ✓

$= \frac{\pi}{2}\left(\tan^{-1}4\sqrt{3} - \tan^{-1}\frac{4}{\sqrt{3}}\right)$ ✓

(b) $u = x-3$

$du = dx$

limits: when $x=4$, $u=1$

$x=3$ $u=0$

$\int_3^4 x\sqrt{x-3}\,dx = \int_0^1 (u+3)\sqrt{u}\,du$ ✓ ✓

$= \int_0^1 (u^{3/2} + 3u^{\frac{1}{2}})\,du$ ✓

$= \left[\frac{2u^{5/2}}{5} + 3u^{\frac{3}{2}}\times\frac{2}{3}\right]_0^1$

$= \left[\frac{2}{5}u^{5/2} + 2u^{3/2}\right]_0^1$

$= 2\frac{2}{5}$ ✓

c.

(i) $T = 22 - Ae^{-kt}$

$\frac{dT}{dt} = kAe^{-kt}$ ✓

$= k(22-T)$

$= -k(T-22)$

when $t=0$, $T=-8$

$-8 = 22-A$

$A = 30$ ✓

ii) $T=4$ when $t=90$

$4 = 22 - 30e^{-90k}$ ✓

$30e^{-90k} = 18$

$e^{-90k} = \frac{18}{30}$

$-90k = \ln\frac{3}{5}$

$k = -\frac{1}{90}\ln\frac{3}{5}$ ✓

iii) Find T when $t = 180$

$T = 22 - 30e^{-180k}$

$= 22 - 30e^{-180\times(-\frac{1}{90}\ln\frac{3}{5})}$

$= 22 - 30e^{2\ln\frac{3}{5}}$

$= 22 - \cancel{30}^{6}\times\frac{9}{\cancel{25}_{5}}$

$= 22 - \frac{54}{5}$

$= 11\frac{1}{5}$ ✓

Q5

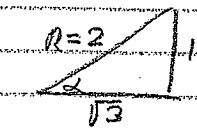
(a)

(i) $x = \sqrt{3}\cos 2t - \sin 2t = R\cos(2t+\alpha)$

$= R\cos 2t\cos\alpha - R\sin 2t\sin\alpha$

$R\cos\alpha = \sqrt{3}$ and $R\sin\alpha = 1$

$\cos\alpha = \frac{\sqrt{3}}{R}$ $\quad$ $\sin\alpha = \frac{1}{R}$



so $R = \sqrt{3+1} = 2$ ✓

and $\alpha = \frac{\pi}{6}$ ✓

So $x = 2\cos(2t + \frac{\pi}{6})$

(ii) Find t when $x = 1$

$2\cos(2t + \frac{\pi}{6}) = 1$

$\cos(2t + \frac{\pi}{6}) = \frac{1}{2}$

$2t + \frac{\pi}{6} = \frac{\pi}{3}$

$2t = \frac{\pi}{6}$

$t = \frac{\pi}{12}$ seconds ✓

(iii) $x = 2\cos(2t + \frac{\pi}{6})$

$\dot{x} = -4\sin(2t + \frac{\pi}{6})$ ✓

maximum value of $\dot{x}$ is 4 ms^{-1}, it occurs when

$\sin(2t + \frac{\pi}{6}) = -1$

$2t + \frac{\pi}{6} = \frac{3\pi}{2}$

$2t = \frac{3\pi}{2} - \frac{\pi}{6}$

$= \frac{4\pi}{3}$

$t = \frac{2}{3}\pi$ ✓

it reaches maximum velocity after $\frac{2\pi}{3}$ s.

(b) (i)

$f(x) = x^3 - x - 2$

$f(1) = 1 - 1 - 2 = -2$

$f(2) = 8 - 2 - 2 = 4$ $\}$ ✓

$f(x)$ changes sign between $x=1$ and 2, it is continuous, so there is a root between $x = 1$ and 2.

(ii) $f'(x) = 3x^2 - 1$

$f(1.5) = 1.5^3 - 1.5 - 2$

$f'(1.5) = 3(1.5)^2 - 1$

$x_1 = 1.5 - \frac{1.5^3 - 1.5 - 2}{3(1.5)^2 - 1}$ ✓

$\doteqdot 1.5$ ✓

($\frac{35}{23}$ is exact) also accept

(c) i) $\frac{1}{2}v^2 = 2(2x - x^2) = 4x - 2x^2$

$\ddot{x} = \frac{d(\frac{1}{2}v^2)}{dx}$ $\}$ ✓

$= 4 - 4x$

$= -4(x-1)$

ii) Centre is when $\ddot{x} = 0$, $x = 1$ ✓

iii) Maximum speed is when $\ddot{x} = 0$, $x = 1$

$x = 1$, $v^2 = 4(2-1) = 4$

so maximum speed is 2, and half is 1. ✓

so $2\sqrt{2x - x^2} = 1$

$2x - x^2 = \frac{1}{4}$

$4x^2 - 8x + 1 = 0$

$$x = \frac{8 \pm \sqrt{64-16}}{8}$$

$$= \frac{8 \pm \sqrt{48}}{8} \qquad 48 = 16 \times 3$$

$$= \frac{8 \pm 4\sqrt{3}}{8}$$

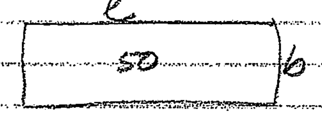
$$= \frac{2 \pm \sqrt{3}}{2}$$

$$= 1 \pm \frac{\sqrt{3}}{2} \text{ m}$$

So when the particle is $1+\frac{\sqrt{3}}{2}$ or $1-\frac{\sqrt{3}}{2}$ m from O it has half the maximum speed

6.

(a)



$lb = 50$ so $b = \frac{50}{l} = 50l^{-1}$

$$\frac{db}{dl} = -50l^{-2} = \frac{-50}{l^2}$$

$$\frac{db}{dt} = \frac{db}{dl}\,\frac{dl}{dt}$$

$$= \frac{-50}{l^2} \times 6$$

$$= \frac{-50}{100} \times 6 \quad \text{when } l = 10$$

$$= -3 \text{ cm s}^{-1}$$

The breadth is decreasing at 3 cm s^{-1}

(b)

A. For $n=1$, we have $(1+x)^1 - 1 = x$ which is divisible by x.

B. Assume the expression is divisible by x for some integer k

then $(1+x)^k - 1 = xM$, M an integer

Now prove that $(1+x)^{k+1} - 1$ is divisible by x.

Now, $(1+x)^{k+1} - 1 = (1+x)(1+x)^k - 1$

$= (1+x)(xM+1) - 1$, using the induction hypothesis

$= xM + 1 + x^2M + x - 1$

$= xM + x^2M + x$

$= xM + x^2M + x$

$= x(M + xM + 1)$ which is divisible by x

So, from parts A & B using the principle of mathematical induction the expression is divisible by x for all $n \geq 1$.

(ii) $12^n - 4^n - 3^n + 1$

$= 4^n(3^n - 1) - 1(3^n - 1)$

$= (3^n - 1)(4^n - 1)$

(iii) From (i), $3^n - 1 = (1+2)^n - 1$ is divisible by 2

$4^n - 1 = (1+3)^n - 1$ is divisible by 3

so the product is divisible by 6 for $n \geq 1$

c)

(i) Sum of roots $= \tan\alpha + \tan\beta = -\frac{b}{a}$

product of roots $= \tan\alpha \tan\beta = \frac{c}{a}$

$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \dfrac{-\frac{b}{a}}{1 - \frac{c}{a}}$

$= \dfrac{-b}{a-c}$

$= \dfrac{b}{c-a}$

ii) $\tan^2(\alpha-\beta) = \left(\dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}\right)^2$

$= \dfrac{\tan^2\alpha - 2\tan\alpha\tan\beta + \tan^2\beta}{(1 + \tan\alpha\tan\beta)^2}$

$= \dfrac{(\tan\alpha + \tan\beta)^2 - 4\tan\alpha\tan\beta}{(1 + \tan\alpha\tan\beta)^2}$

$= \dfrac{\frac{b^2}{a^2} - \frac{4c}{a}}{(1 + \frac{c}{a})^2}$

$= \dfrac{b^2 - 4ac}{(a+c)^2}$

Q7

a) ${}^nC_2 + {}^nC_1 + {}^nC_0 = 37$

$\frac{n(n-1)}{2} + n + 1 = 37$ ✓

$n^2 - n + 2n + 2 = 74$

$n^2 + n - 72 = 0$

$(n+8)(n+9) = 0, \quad n \geq 2$

so $n = 8$ ✓

b) (i) $t = \frac{x}{V\cos\alpha}$

$y = V\frac{x}{V\cos\alpha}\sin\alpha - \frac{1}{2}g\frac{x^2}{V^2\cos^2\alpha}$ ✓

$= x\tan\alpha - \frac{1}{2}\frac{g}{V^2}x^2\sec^2\alpha$

(ii) $R = (d\cos\theta, d\sin\theta)$ ✓

(iii) from (i) and (ii) ✓

$d\sin\theta = d\cos\theta\tan\alpha - \frac{1}{2}\frac{g}{V^2}d^2\cos^2\theta\sec^2\alpha$

$\sin\theta = \cos\theta\tan\alpha - \frac{gd\cos^2\theta\sec^2\alpha}{2V^2}$

$\frac{dg\cos^2\theta}{2V^2\cos^2\alpha} = \cos\theta\tan\alpha - \sin\theta$

$d = \left(\cos\theta\frac{\sin\alpha}{\cos\alpha} - \sin\theta\right)\frac{2V^2\cos^2\alpha}{g\cos^2\theta}$ ✓

$= \frac{2V^2(\cos\theta\sin\alpha\cos\alpha - \sin\theta\cos^2\alpha)}{g\cos^2\theta}$

$= \frac{2V^2\cos\alpha(\cos\theta\sin\alpha - \sin\theta\cos\alpha)}{g\cos^2\theta}$ ✓

recognises $\sin(\alpha-\theta)$

$= \frac{2V^2\cos\alpha\sin(\alpha-\theta)}{g\cos^2\theta}$

c) next page

C

i) $1+(1+x)+(1+x)^2+\dots+(1+x)^{n-1} = \frac{1\left((1+x)^n-1\right)}{(1+x)-1}$

$= \frac{(1+x)^n-1}{x}$

ii)

$1+(1+x)+(1+x)^2+\dots+(1+x)^{n-1} = \frac{(1+x)^n-1}{x}$

$= \frac{\left(1+{}^nC_1x+{}^nC_2x^2+\dots{}^nC_nx^n\right)-1}{x}$

$= \frac{{}^nC_1x+{}^nC_2x^2+\dots{}^nC_nx^n}{x}$

$= {}^nC_1+{}^nC_2x+\dots{}^nC_nx^{n-1}$

iii) $\int_{-1}^{0} {}^nC_1+{}^nC_2x+\dots{}^nC_nx^{n-1}\,dx$

$= \left[{}^nC_1x+\frac{1}{2}{}^nC_2x^2+\frac{1}{3}{}^nC_3x^3+\dots\frac{1}{n}{}^nC_nx^n\right]_{-1}^{0}$

$= {}^nC_1-\frac{1}{2}{}^nC_2+\frac{1}{3}{}^nC_3-\dots+\frac{(-1)^{n+1}}{n}{}^nC_n$

iv)

Now $\sum_{r=1}^{n}\frac{(-1)^{r+1}}{r}{}^nC_r = {}^nC_1-\frac{1}{2}{}^nC_2+\frac{1}{3}{}^nC_3-\dots\frac{(-1)^{n+1}}{n}{}^nC_n$

$= \int_{-1}^{0} {}^nC_1+{}^nC_2x+\dots{}^nC_nx^{n-1}\,dx$

(over)

$= \int_{-1}^{0} 1+(1+x)+(1+x)^2+\dots(1+x)^{n-1}\,dx$

$= \left[x+\frac{(1+x)^2}{2}+\frac{(1+x)^3}{3}+\dots\frac{(1+x)^n}{n}\right]_{-1}^{0}$

$= \left(\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{n}\right)-(-1)$

$= 1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{n}$

$= \sum_{r=1}^{n}\frac{1}{r}$