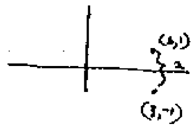
1993 - CSSA TRIAL SOLUTIONS

1. (a) $\dfrac{\sqrt{3\cdot 2+4\cdot 66}}{2\cdot 33^2} = 0\cdot 5$

(b) $x^2-11x+10 = (x-10)(x-1)$

(c) $(x-3)^2 = 8(y+1)$

Vertex $(3,-1)$ $a=2$.



$\therefore$ focus $(3,1)$

(d). $3y = 2x+6$

$\therefore y = \frac{2}{3}x+2$ $\therefore m = \frac{2}{3}$

(e) solve $(x+2)^2 = 9$

$x+2 = \pm 3$

$\therefore x = 1$ or -5

(f) $\dfrac{3}{\sqrt{5}+2} \times \dfrac{\sqrt{5}-2}{\sqrt{5}-2} = \dfrac{3\sqrt{5}-6}{5-4} = 3\sqrt{5}-6$

(g) $V = V_0(1-R)^n$ $V_0 = 15000$, $V = 9000$ $n=3$

$\therefore 9000 = 15000(1-R)^3$

$(1-R)^3 = \frac{9}{15} = \frac{3}{5}$

$\therefore 1-R = \sqrt[3]{3/5} = 0\cdot 8434$

$-R = 0\cdot 8434 - 1$

$R = 0\cdot 156\ldots$

$\therefore R = 16\%$

Q2.

(a) $3x-2y+3=0$.

let $y=0$ $\therefore 3x+3=0$

$x=-1$

$\therefore R(-1,0)$

(b) $m = \dfrac{y_2-y_1}{x_2-x_1}$ $P(0,8)$; $Q(12,0)$

$\therefore m_{PQ} = \dfrac{0-8}{12-0} = \dfrac{-8}{12} = -\frac{2}{3}$

(c). $m_{L_2} = -\frac{2}{3}$; $m_{L_1} = \frac{-3}{-2} = \frac{3}{2}$

Now $m_{L_2}\cdot m_{L_1} = -\frac{2}{3}\times\frac{3}{2} = -1$

$\therefore L_1$ is $\perp$ to L_2.

(d). Eq^n L_2.

$m_{L_2} = -\frac{2}{3}$ pt is $(0,8)$.

$\therefore y-8 = -\frac{2}{3}(x-0)$

$3y-24 = -2x$.

Eq^n L_2 $2x+3y-24=0$

(e) $3x-2y+3=0 \quad \ldots (1)$

$2x+3y-24=0 \quad \ldots (2)$

(1)$\times 2$ and (2)$\times 3$.

$6x-4y+6=0 \quad \ldots (A)$

$6x+9y-72=0 \quad \ldots (B)$

A − B $-13y = -78$

$y = 6$.

when $y=6$ $3x-12+3=0$

$3x = 9$

$x=3$.

$\therefore$ Pt S is $(3,6)$

Q3.

(a) Dwrtx

(i) $y = 4x^3+7$

$y' = 12x^2$

(ii) $y = xe^{2x}$

$y' = x\cdot e^{2x}\cdot 2 + e^{2x}\cdot 1$

$= e^{2x}(2x+1)$

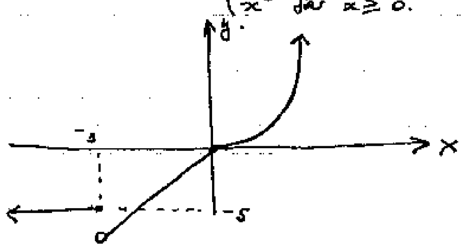
(iii) $y = \dfrac{\sin x}{x}$

$y' = \dfrac{x\cdot\cos x - \sin x\cdot 1}{x^2}$

$= \dfrac{x\cos x - \sin x}{x^2}$

(b) Sketch $f(x) = \begin{cases} -5 & \text{for } x \le -3 \\ 2x & \text{for } -3 < x < 0 \\ x^2 & \text{for } x \ge 0. \end{cases}$

i)



ii) $f(-3) = -5$ $f(3) = 9$

$\therefore f(-3)+f(3) = -5+9$

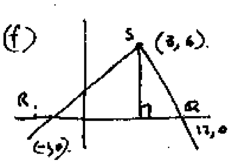
$= 4$

(c) $AB^2 = 7^2+9^2-2\times7\times9\cos 48 = 45\cdot 689\ldots$

$AB \doteqdot 6\cdot 7$

$AB \doteqdot 7m$.

(f)

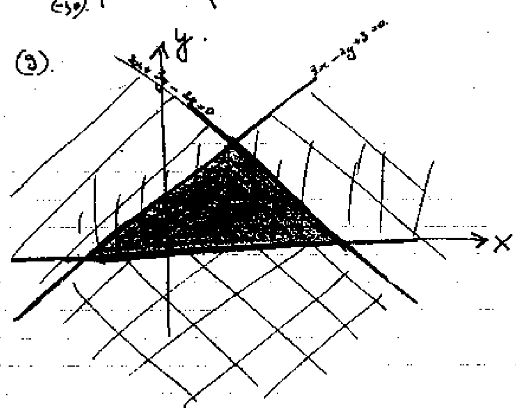


Area $\triangle QRS = \frac{1}{2} RQ \times \perp h$.

$= \frac{1}{2}\times 13\times 6$

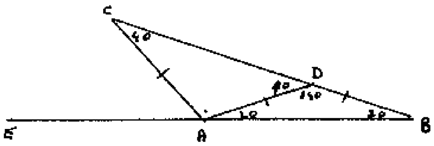
$= 39$ u.

(g).



4.

(a)



(i) To show that $\angle ADC = 40°$ giving reasons.

$\angle DAB = 20°$ ($\triangle ABD$ isosceles $\triangle$, $AD = DB$) base angles of isosceles $\triangle$ are equal.

$\angle ADC = 40$ (exterior angle of a triangle equals sum of two opposite interior angles)

(ii) Find size of $\angle CAE$.

$\angle DCA = 40°$ ($\triangle ACD$ is isosceles, ~~base angles are equal~~)

$\angle CAD = 100°$ ($180°$ in a triangle).

$\therefore \angle CAE = 180 - (100 + 20) = 60°$ ($180°$ in st line)

(b) (i) $\displaystyle\int \frac{dx}{x+5} = \log(x+5) + C$

(ii) $\displaystyle\int \sec^2 3x\,dx = \tfrac{1}{3}\tan 3x + C$

c)(i) For intersection solve simult. $y = 2x^2$ and $y = 12 - x^2$.

$\therefore 2x^2 = 12 - x^2$

$3x^2 = 12 \qquad \therefore x^2 = 4$

hence $x = \pm 2$.

when $x = -2$, $y = 8$; $x = +2$ $y = 8$

$\therefore$ pts are $(-2, 8)$ and $(2, 8)$

(ii) $A = 2 \times \left[\int_0^2 12 - x^2\,dx - \int_0^2 2x^2\,dx\right]$

$= 2\int_0^2 12 - 3x^2\,dx$

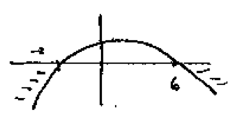
$= 2\left[12x - \frac{3x^3}{3}\right]_0^2$

$= 2\left[24 - 8\right]$

$= 32u^2$.

Q5. (a) $y = 12 + 4x - x^2$

$y = (6 - x)(2 + x)$



for $x < -2$ or $x > 6$

(b). i) P(opened) $= \frac{1}{3}$

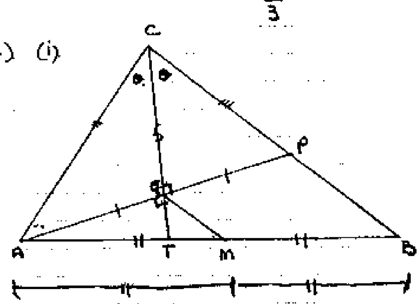
(ii) P(all three keys) = P(not opened and not opened and opened)

$= \frac{2}{3} \times \frac{1}{2} \times 1$

$= \frac{2}{6}$

$= \frac{1}{3}$

(c) (i)



(ii) In $\triangle ACE$ and CEP

(i) $\angle ACE = \angle ECP$ (given)

(ii) $\angle CEA = \angle CEP = 90°$ ($AE \perp$ to CT) ($AP \perp$ to CT)

(iii) CE is a common side and corresponding side in $\triangle$'s.

$\therefore \triangle ACE \equiv \triangle CEP$ (ASA)

(iii) $AE = EP$. Corresponding sides opposite equal angles in congruent triangle

(iv) $AE = EP$ and $AM = MB$. $\therefore$ in $\triangle ABP$,

$\therefore EM \parallel PB$ (The line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.)

Q6

(a) i) Since $\angle OPQ = \angle OQP$ $\triangle OQP$ is isosceles

$\therefore OQ = OP$.

Now slope $= \frac{OQ}{OP} = \frac{-1}{1} = -1$.

(or slope is 1 but line goes \ $\therefore$ is negative slope)

ii) $y = \frac{4}{x^2} = 4x^{-2}$

$\frac{dy}{dx} = \frac{-8}{x^3}$ when $x = 2$;

$\frac{dy}{dx} = \frac{-8}{8} = -1$ which is slope of PQ $\therefore PQ$ is a tangent at $(2, 1)$

or $\frac{dy}{dx} = -\frac{8}{x^3}$

when $\frac{dy}{dx} = -1$

$x^3 = +8$

$\therefore x = 2$.

when $x = 2$

$y = 1$

$\therefore$ pt is $(2, 1)$

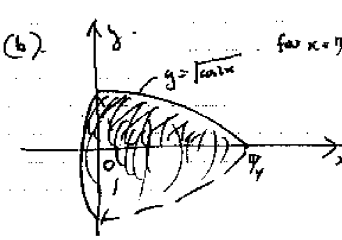
$\therefore PQ$ is a tangent at $(2,1)$ to curve.

iii) $\therefore$ Eqn of PQ is $y - 1 = -1(x - 2)$

$y - 1 = -x + 2$

$x + y = 3$.

(b)



for $x = \frac{\pi}{4}$ $y = \sqrt{\cos\frac{\pi}{2}} = 0$.

$V = \pi\int_a^b [f(x)]^2\,dx$

$= \pi\int_0^{\pi/4} \cos 2x\,dx$.

$= \pi\left[\frac{1}{2}\sin 2x\right]_0^{\pi/4}$

$= \pi\left[\frac{1}{2} \times 1 - 0\right]$

$= \frac{\pi}{2}\ u^3$.

$$A \doteqdot \frac{h}{2}\left\{(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right\}$$

where $h = \frac{b-a}{n}$

$h = 10$

$y_0 = 12$
$y_1 = 15$
$y_2 = 19.5$
$y_3 = 22$
$y_4 = 17$
$y_5 = 15$
$y_n = 13$

$$\therefore A \doteqdot \frac{10}{2}\left\{(12+13)+2(15+19.5+22+17+15)\right\}$$

$$\doteqdot 5 \times \{25+177\}$$

$$\doteqdot 1010\ m^2$$

Q7.

(a). $\frac{16}{2^{3x}\times 8^{1-x}} = \frac{2^4}{2^{3x}\cdot 2^{3-3x}}$

$= \frac{2^4}{2^3}$

$= 2.$

(b). $3^{2x} - 10(3^x)+9=0.$

let $u = 3^x \quad \therefore \quad u^2-10u+9=0$

$(u-1)(u-9)=0$

$u=+1$ or $+9$

$\therefore 3^x = 1$ or 9

$\therefore x = 0$ or $2.$

(c). $6x^2-11 \equiv A(x+2)^2+Bx+C.$

RHS $= A(x^2+4x+4)+Bx+C$

$= Ax^2+4Ax+4A+Bx+C$

$= Ax^2+(4A+B)x+4A+C$

LHS $= 6x^2+0x-11$

cf LHS = RHS.

$\therefore A=6 \quad 4A+B=0$ and $4A+C=-11.$

$B=-24 \qquad C=-11-24$

$= -35$

$\therefore A=6, \quad B=-24$ and $C=-35.$

(d). i) $V = 20\ ms^{-1}$ after 1 sec.

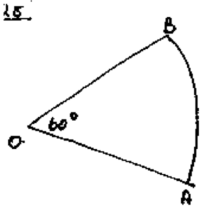
ii) $a = 0\ ms^{-2}$ (since $\frac{dV}{dt}=0$ at $(3,-10)$).

[or $a = \frac{\Delta v}{\Delta t} = \frac{2+30}{4-3} = 10\ ms^{-2}$ ← average answer as after 3 s not instantaneous]

(iii) Changes direction after 2 sec.

(iv) Area represents "displacement" after 2sec (or position from origin after 2sec. or $A = \int_0^2 V(t)\,dt$ = displacement

Q5.



$AB = 2\pi$ cm.

(i) Show OA is 6 cm.

$l = r\theta$; θ in radians

$\therefore 60° = \frac{\pi}{3}$ rad.

$2\pi = r\cdot\frac{\pi}{3}$

$\therefore r = 6\ cm = OA.$

(ii) Area of sector AOB $= \frac{1}{2}r^2\theta$

$= \frac{1}{2}\cdot 6^2 \times \frac{\pi}{3}$

$= \frac{1}{6}\times 18\pi\ u^2.$

$= 6\pi\ u^2.$

Alternatives for (i) (ii) exist. eg.

(i) $\frac{60}{360} = \frac{1}{6}$ of circle.

$\therefore$ if $AB = 2\pi$ then full circle $= 6\times 2\pi = 12\pi.$

now circum $= \pi D = 2\pi r.$

$\therefore 2\pi r = 12\pi$

$r = 6$ cm $= OA$

(ii) Area of circle $= \pi r^2 = 36\pi$

$\therefore$ Area of sector AOB $= \frac{1}{6}\times 36\pi$

$= 6\pi\ u^2.$

Question 8.

(b) $PM = 2\ PN \qquad P \equiv (x,y) \quad M(3,0) \quad N(0,3)$

$\therefore PM^2 = 4PN^2$

$PM^2 = (x-3)^2+(y-0)^2 = x^2-6x+9+y^2$

$PN^2 = (x-0)^2+(y-3)^2 = x^2+y^2-6y+9.$

Now $x^2-6x+9+y^2 = 4(x^2+y^2-6y+9)$

$x^2-6x+9+y^2 = 4x^2+4y^2-24y+36.$

$3x^2+6x+3y^2-24y = -27$

$\div 3. \quad x^2+2x+y^2-8y+9=0.$

(ii). $x^2+2x+1+y^2-8y+16 = 1+16-9$

$(x+1)^2+(y-4)^2 = 8.$

Circle centre $(-1,4)$; $r = \sqrt{8}$

(c). $x = 20(1-e^{-kt}).$

$t=3, \quad x=10$

$\therefore 10 = 20(1-e^{-k\times 3}).$

$\frac{1}{2} = 1-e^{-3k}.$

$\therefore e^{-3k} = \frac{1}{2}.$

take logs on both sides $\quad -3k\log_e e = \log_e \frac{1}{2}.$

$-3k = \frac{\log_e \frac{1}{2}}{\log_e e} = \log_e \frac{1}{2}$

$\therefore k = \frac{\log_e \frac{1}{2}}{-3} = 0.23104906$

$= 0.231$ (3 sig fig)

(ii) $x = 20\left(1 - e^{-kt}\right) = 20 - 20e^{-kt}$

$\frac{dx}{dt} = -20e^{-kt} \cdot -k$

$= 20k\, e^{-kt}$ where $k = 0.231$ and $t = 5$

$\therefore \frac{dx}{dt} = 20 \times 0.231\, e^{-0.231 \times 5}$

$= 1.45 \ldots$

≈ 1 gm min^{-1}

Q9.

$f(x) = x^3 + 3x^2 - 9x - 1$ for $x: -4 \le x \le 2$

(i) $f(x) = x^3 + 3x^2 - 9x - 1$

$f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x+3)(x-1)$

$f''(x) = 6x + 6 = 6(x+1)$.

Stat. pts, $f'(x) = 0$ $\therefore x = -3$ or $+1$

If $x = -3$, $f(-3) = (-3)^3 + 3(-3)^2 - 9(-3) - 1 = 26$.

$x = 1$ $f(1) = 1^3 + 3(1)^2 - 9(1) - 1 = -6$

pts are $(-3, 26)$ and $(1, -6)$

Test nature of points: for $x = -3$.

$f''(-3) = -18 + 6 < 0$ $\therefore$ is a max at $(-3, 26)$

for $x = +1$

$f''(1) = 6 + 6 > 0$ $\therefore$ is a min at $(1, -6)$.

(ii) For an inflexion point $f''(x) = 0$ and must show change in concavity.

$6x + 6 = 0$

$\therefore x = -1$; test $x = -1$

$f''(-1-\epsilon) < 0$ concave down
$f''(-1+\epsilon) > 0$ concave up. } change in concavity

$\therefore x = -1$ is an inflexion pt.

when $x = -1$, $f(-1) = -1 + 3 + 9 - 1 = 10$ pt is $(-1, 10)$

ii) For sketch.

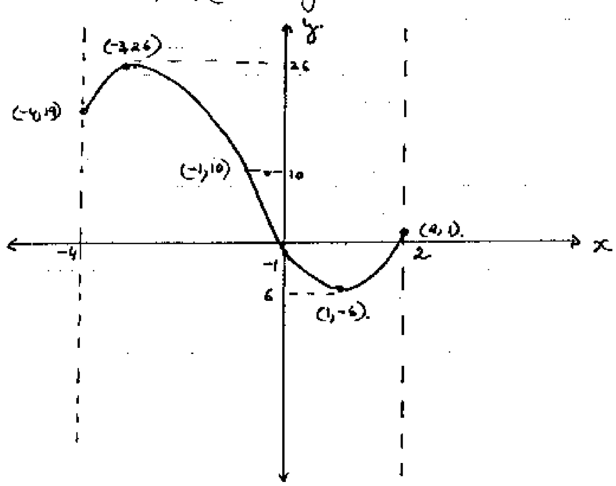
$(1, -6)$ is a min; $(-3, 26)$ is a max

$(-1, 10)$ is a pt of inflexion

Boundaries: $x = -4$ $y = (-4)^3 + 3(-4)^2 - 9 \times -4 - 1 = 19$

$x = 2$ $y = (2)^3 + 3(2)^2 - 9 \times 2 - 1 = 1$

ie $(-4, 19)$ and $(2, 1)$ are boundaries and also when $x = 0$, $f(x)$ cuts y axis at -1.



Q9.

(b) $P = 30{,}000$ $n = 48$ months $r = 1.1\%$ per month

(i) After the first month amount owing is

$A_1 = [P + P \times 0.011] - M$

$A_1 = P(1.011) - M = 1.011P - M$

After the second month the amount owing is

$A_2 = [A_1 + A_1 \times 0.011] - M$

$= A_1(1.011) - M = 1.011A_1 - M$

Now subs. A_1 into above.

$A_2 = 1.011[1.011P - M] - M$

$= 1.011^2 P - 1.011M - M$ .... ①

$= 1.011^2 P - 2.011M$

Subs for $P = 30\,000$

$A_2 = 1.011^2 \times 30000 - 2.011M$

$= 30663.63 - 2.011M$.

(ii) From ① above

$A_2 = 1.011^2 P - [1 + 1.011]M$

$A_3 = 1.011^3 P - [1 + 1.011 + 1.011^2]M$.

$\vdots$

$A_{48} = 0 = 1.011^{48} P - [1 + 1.011 + 1.011^2 + 1.011^3 + \ldots + 1.011^{47}]M$

$\therefore M = \frac{1.011^{48} \times 30\,000}{[1 + 1.011 + 1.011^2 + \cdots + 1.011^{47}]}$

Denominator is a GP with $a = 1$, $r = 1.011$ $n = 48$

$$\therefore S_{48} = \frac{1(1.011^{48} - 1)}{1.011 - 1} = 62.78698633$$

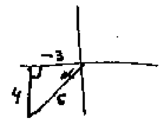
$$\therefore M = \frac{1.011^{48} \times 30000}{62.78698633} = \$807.81$$

$\therefore$ monthly repayment is \$807.81

Q10

(a) $\cos\alpha = -\frac{3}{5}$ $\pi < \alpha <$ [illegible]

α is in 3rd quadrant.



$\therefore \tan\alpha = \frac{4}{3}$

(b) $x, x^2, 5x$ in arithmetic series.

(i) $x^2 - x = 5x - x^2$

$\therefore 2x^2 - 6x = 0.$

(ii) $2x(x-3) = 0$

$\therefore x = 0$ or 3.

$\therefore$ Terms are 3, 9, 15 in A.P.

$\therefore d = 9 - 3 = 15 - 9$

$d = 6.$

(c) $V = 4\text{ m}^3$

Now $V = x \cdot 2x \cdot y = 4$

$\therefore y = \frac{4}{2x^2} = \frac{2}{x^2}$

Cost for base + top.   Area $= 2 \times (x \cdot 2x) = 4x^2 \text{ m}^2$

at \$15 pm^2

$\therefore$ costs $4x^2 \times 15 = 60x^2.$

Cost for Sides   Area $= 2xy \times 2 + xy \times 2$

$= 4x \cdot \frac{2}{x^2} + 2x \cdot \frac{2}{x^2}$

$= \frac{8}{x} + \frac{4}{x} = \frac{12}{x}$

at \$10 $\therefore$ cost $= \frac{120}{x}$

Total Cost for box $= C = 60x^2 + \frac{120}{x}$

(i) For cheapest box find min value of C.

$$C = 60x^2 + 120x^{-1}$$

$$\frac{dC}{dx} = 120x - \frac{120}{x^2}$$

Let $\frac{dC}{dx} = 0$. $\therefore 120x - \frac{120}{x^2} = 0$ $x \neq 0$.

$120x^3 = 120.$

$x^3 = 1$

$\therefore x = 1$

To check min. $\frac{d^2C}{dx^2}$ must be positive

Now $\frac{d^2C}{dx^2} = 120 + 120x^{-3}$

which is positive for all $x > 0$.

Thus $x = 1$ $\therefore$ width is 1 m.