NEWINGTON COLLEGE

Trial Higher School Certificate Examination 1999

12 MATHEMATICS
3 UNIT ADDITIONAL

Time allowed : *Two Hours*

(plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES :

All questions are of equal value.

All questions may be attempted.

In every question, show all necessary working.

Marks may not be awarded for careless or badly arranged work.

Approved silent calculators may be used.

A table of standard integrals is provided for your convenience.

The answers to the seven questions in this paper are to be returned in separate bundles clearly marked Question 1, Question 2 etc.

Each bundle must show the candidate's computer number.

The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.

Unless otherwise stated candidates should leave their answers in simplest exact form.

Question 1 (12 Marks)

a) Solve for x: $\dfrac{x+4}{x-2} \geq 3$.

b) Sketch the function $y = \dfrac{1}{2}\sin^{-1}\left(\dfrac{x}{2}\right)$

c) Find the acute angle between the lines whose equations are:

$$y = 2\sqrt{3}x - \sqrt{6}$$
$$7y = \sqrt{3}x + \sqrt{2}$$

d)

(i) Find the equation of the tangent to the curve $y = -(x-2)^3$ at the point P(1,1) and find the coordinates of the point A where the tangent cuts the y-axis.

(ii) If P divides the interval AB internally in the ratio 1:3, find the coordinates of B.

(iii) Show that B also lies on the curve.

Question 2 (12 Marks) *Start a new page*

a)

(i) Express $\sin^2 x$ in terms of $\cos 2x$.

(ii) Hence find the volume of the solid of revolution formed when $y = \sin x$ is rotated about the x-axis between the ordinates $x = \dfrac{\pi}{4}$ and $x = \dfrac{\pi}{2}$.

b)

(i) Express $5\cos x - 12\sin x$ in the form $A\cos(x + \alpha)$, w' >0 and $0 \leq \alpha < 2\pi$.

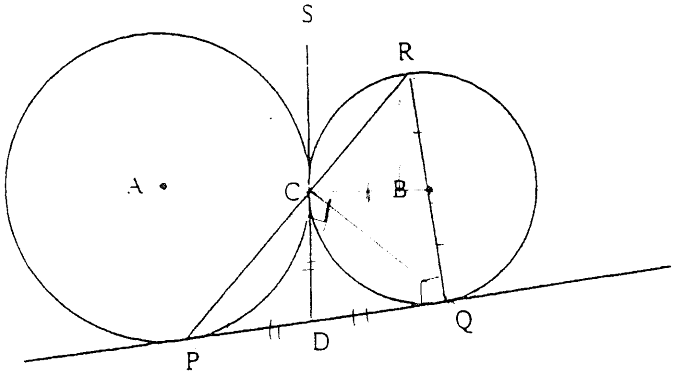
(ii) Find the maximum value of $5\cos x - 12\sin x$ and state st positive value of x for which this maximum value occurs.

c) A kite, 50 metres high, is being carried horizontally by the wind at of $4\ ms^{-1}$. How fast is the string being let out, when the length of the string is 100 metres.

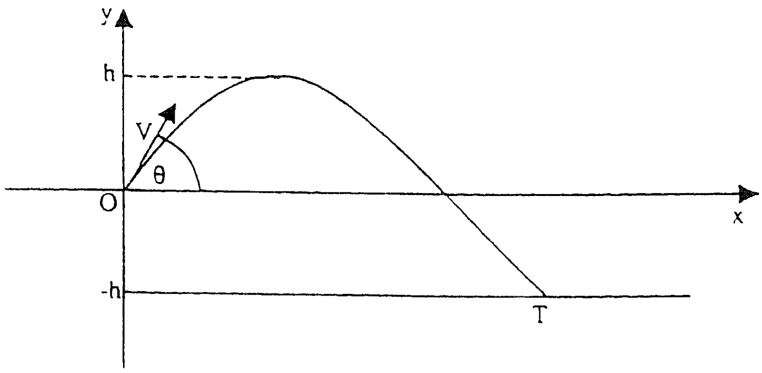
d) Calculate the coefficient of x^8 in the expansion of $\left(2x^3 - \dfrac{1}{x}\right)^{12}$.

Question 3 (12 Marks) *Start a new page* Mar

a) In the diagram below, two circles with centres A and B touch externally at the point C. The common tangent at C intersects the common tangent PQ at D. The line QB produced meets the circumference of the smaller circle in R. Show that P, C and R are collinear, giving reasons.



b) The diagram below shows the path of a projectile fired from the top O of a cliff. Its initial velocity is V m/s, its initial angle of elevation is θ and it rises to a maximum height h metres above O. It strikes a target T situated on a horizontal plane h metres below O.



(i) Given that $\ddot{y} = -g$ and $\ddot{x} = 0$ derive equations for y and x as functions of time.

(ii) Prove that $h = \dfrac{V^2 \sin^2\theta}{2g}$, where g is the acceleration due to gravity.

(iii) Prove that the time taken for the projectile to reach its target is $\dfrac{V\sin\theta\left(1+\sqrt{2}\right)}{g}$ seconds.

(iv) Show that the distance from the target to the base of the cliff is $\dfrac{V^2\left(1+\sqrt{2}\right)\sin 2\theta}{2g}$ metres.

Question 6 (12 Marks) *Start a new page*

a) The speed $v\ ms^{-1}$ of a particle moving in simple harmonic motion is given by

$$v^2 = 6 + 4x - 2x^2.$$

(i) Show that $\ddot{x} = -2(x-1)$.

(ii) Find the centre, period and amplitude of the motion.

b) Evaluate $\cos\left(\sin^{-1}\frac{5}{13} + \sin^{-1}\frac{4}{5}\right)$ without the use of a calculator.

c)

(i) Show that $\frac{d}{dx}\left(\frac{x}{\sqrt{1-x^2}}\right) = \frac{1}{(1-x^2)^{\frac{3}{2}}}$.

(ii) Hence find $\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right)$.

Question 7 (12 Marks) *Start a new page*

a) The numbers 1, 3, 5, 6 and 8 are written on cards and placed in a bag. Cards are drawn without replacement from the bag to form numbers with one or more digits, all of which are less than 6000. 4

(i) How many numbers can be formed in this way?

(ii) If one of these numbers is selected at random, find the probability that it is even.

b) 8

(i) Prove by induction that

$$1 + \frac{x}{1!} + \frac{x(x+1)}{2!} + \ldots + \frac{x(x+1)(x+2)\cdots(x+n-1)}{n!} = \frac{(x+1)(x+2)\cdots(x+n)}{n!}$$ for n a positive integer.

(ii) Deduce that $1 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + (-1)^n\ {}^nC_n = 0$

END OF PAPER

Marks

Question 4 (12 Marks) *Start a new page*

a) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $x^2 = 4ay$. The chord PQ varies in such a way as to always pass through the point $(0, -3a)$.

(i) Show that the equations of the tangents to the parabola at P and Q are given by

$y = px - ap^2$ and $y = qx - aq^2$ respectively.

(ii) Show that pq = 3.

(iii) The tangents at P and Q meet in T. Find the locus of T.

b) Find $\int \frac{dx}{\sqrt{x}\sqrt{1-\sqrt{x}}}$ using the substitution $u = \sqrt{x}$.

Question 5 (12 Marks) *Start a new page*

a) The rate of change of a quantity I with respect to t is given by $\frac{dI}{dt} = \frac{V}{L} - \frac{R}{L}I$, where R, L and V are constants.

(i) Show that $I = \frac{V}{R} + Ae^{-\frac{R}{L}t}$ satisfies this equation where A is a constant.

(ii) Find a relationship between I, V and R as t increases without bound.

(iii) Initially $I = 0$ and it is known that $V = 5$, $R = 2.2 \times 10^3$ and $L = 6.5 \times 10^{-3}$. Find the value of I correct to two significant figures when $t = 2 \times 10^{-6}$.

b) $x^4 + x^2 - 80 = 0$ has a root near $x = 3$. Use Newton's Method twice to find an approximation to the root.

c) Prove by induction that $5^n + 3$ is divisible by 4, where n is a positive integer.

Q1

(a) $\frac{x+4}{x-2} \geq 3$ $\quad x(x-2)^2$

$(x+4)(x-2) = 3(x-2)^2$

$x^2+2x-8 = 3x^2-12x+12$

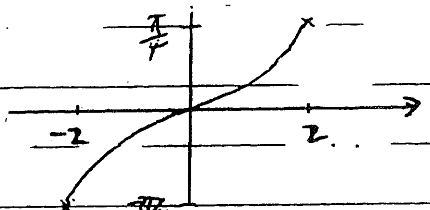
$2x^2-14x+20 = 0$

$x^2-7x+10 = 0$

$(x-2)(x-5) = 0$.

$x = 2, 5$. ✓✓✓

check region $x > 2$, $x \leq 5$.



✓✓

(b) gradients of $y = 2\sqrt{3}x - \sqrt{6}$ and

$y = \frac{\sqrt{3}}{7}x + \frac{\sqrt{2}}{7}$ are respectively

$m_1 = 2\sqrt{3}$, $m_2 = \frac{\sqrt{3}}{7}$

then $\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \frac{2\sqrt{3} - \frac{\sqrt{3}}{7}}{1 + 2\sqrt{3}\cdot\frac{\sqrt{3}}{7}}$

$= \frac{14\sqrt{3} - \sqrt{3}}{7 + 2.3}$

$= \frac{\cancel{13}\sqrt{3}}{\cancel{13}}$

$\alpha = \tan^{-1}\sqrt{3} = 60^\circ$ or $\pi/3^c$ ✓✓

(c)(i) gradient $\frac{dy}{dx} = -3(x-2)^2$

when $x = 1$ $\frac{dy}{dx} = -3$.

equation $y - 1 = -3(x-1)$ ✓✓

$y = -3x + 4$.

y-axis where $x = 0$. $\Rightarrow y = 4$

ie $(0,4)$ ✓

(ii) if P is the point $(1,1)$ A is $(0,4)$

B is (x_1, y_1) ratio $1:3$

then $(1,1) = \left(\frac{1\times x_1 + 3\times 0}{1+3}, \frac{1\times y_1 + 3\times 4}{1+3}\right)$

ie $\frac{x_1}{+4} = 1$ $\quad \frac{y_1+12}{+4} = 1$

$x_1 = +4$ $\quad y_1 = -8$.

$\therefore$ B is the point $(4,-8)$. ✓

(iii) $(+4,-8)$ must satisfy $y = -(x-2)^3$

if $x = +4$ $y = -(4-2)^3 = -8$ ✓

Q2 (a) $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ ✓

Vol. $= \pi\int_{\pi/4}^{\pi/2} y^2\,dx$

$= \pi\int_{\pi/4}^{\pi/2} \sin^2 x\,dx$

$= \pi\int_{\pi/4}^{\pi/2} \frac{1}{2}(1-\cos 2x)\,dx$

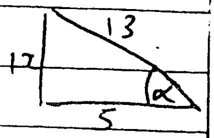
$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\pi/4}^{\pi/2}$ ✓

$= \frac{\pi}{2}\left[\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right) - \left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right)\right]$

$= \frac{\pi}{2}\left(\frac{\pi}{2} - \frac{\pi}{4} + \frac{1}{2}\right)$ $\left(\frac{\pi^2}{8} + \frac{\pi}{4}\right)$

$= \frac{\pi}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) u^3$ ✓

(b)(i) $5\cos x - 12\sin x$



$13\left(\frac{5}{13}\cos x - \frac{12}{13}\sin x\right)$

~~13(cos α sin x − sin α sin x)~~ $\alpha = \tan^{-1}\frac{12}{5}$

$= 67^\circ 23'$

$= 13\cos\left(x + \tan^{-1}\frac{12}{5}\right)$ ✓ $(= 1.18^c)$

(ii) find max value of $13\cos\left(x + \tan^{-1}\frac{12}{5}\right)$ is 13 ✓

ie where $13\cos\left(x + \tan^{-1}\frac{12}{5}\right) = 13$

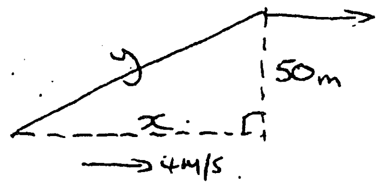
$\cos\left(x + \tan^{-1}\frac{12}{5}\right) = 1$

$x + \tan^{-1}\frac{12}{5} = 0, 360,$

$x = 292^\circ 37'$ ✓

$(= 5.1^c)$

(c) Height of kite is constant at 50m.



y is length of string, x is horizontal distance

then $y^2 = x^2 + 50^2$

* $y = \sqrt{x^2 + 2500}$ $\left(= (x^2+2500)^{\frac{1}{2}}\right)$

Now $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$\frac{dy}{dx} = \frac{1}{2}(x^2+2500)^{-\frac{1}{2}} \times 2x$

$= \frac{x}{(x^2+2500)^{\frac{1}{2}}}$ ✓

$\frac{dx}{dt} = 4$

$\frac{dy}{dt} = \frac{4x}{x^2+2500}$

when $y = 100$ $x = \sqrt{7500}$ from * ✓

$\frac{dy}{dt} = \frac{4\sqrt{7500}}{(7500+2500)^{\frac{1}{2}}} = 2\sqrt{3}$ m/s (3.46 m/s 2 d.p)

(d) Term is ${}^{12}C_7(2x^3)^5\left(-\frac{1}{x}\right)^7$

coefficient is ${}^{12}C_7\, 2^5 (-1)^7$

$= -25344$ ✓✓

or $T_{r+1} = {}^{12}C_r (2x^3)^{12-r}(-x^{-1})^r$

$= {}^{12}C_r\, 2^{12-r}(-1)^r x^{3(12-r)} x^{-r}$

∴ $3(12-r) - r = 8$

$36 - 3r - r = 8$

$4r = 28$

$r = 7$

Q3

a) $PO = CO$ (tangents meeting at an external point equal)
$OQ = OC$ (")

$\therefore$ $PO = CO = OQ$ ①

$\therefore$ O is the centre of a circle passing through PCQ

$\therefore$ $\angle PCQ = 90^\circ$ (angle in semicircle) ①

In the circle QCR centre B

$\angle QCR = 90^\circ$ (angle in semicircle)

Since $\angle PCQ = 90$ and $\angle PCQ = 90^\circ$

$\angle PCR = 180^\circ$ ①

$\therefore$ PCR is a straight line

$\therefore$ PCR is collinear.

b) i) Initially

$t=0,\ x=0,\ y=0 \quad V_x = V\cos\theta \quad V_y = V\sin\theta$

$\ddot{x} = 0$

$\dot{x} = \int 0\,dt$

$= c$

$t=0 \quad V_x = V\cos\theta$

$\dot{x} = V\cos\theta$

$x = \int V\cos\theta\,dt$

$= V\cos\theta\, t + c$

$t=0 \quad x=0 \quad \therefore c=0$

$x = V\cos\theta\, t$ ①

$\ddot{y} = -g$

$\dot{y} = \int -g\,dt$

$= -gt + c$

$t=0 \quad \dot{y} = V\sin\theta$

$\therefore \dot{y} = -gt + V\sin\theta$

$y = \int -gt + V\sin\theta$

$= \frac{-gt^2}{2} + Vt\sin\theta + c$

$t=0 \quad y=0 \quad \therefore c=0$

① $y = \frac{-gt^2}{2} + Vt\sin\theta$

ii)

Max height when $\dot{y}=0$

$0 = -gt + V\sin\theta$

$t = \frac{V\sin\theta}{g}$ ①

$\therefore h = -\frac{1}{2}g\left(\frac{V\sin\theta}{g}\right)^2 + V\left(\frac{V\sin\theta}{g}\right)\sin\theta$

$= -\frac{1}{2}g\frac{V^2\sin^2\theta}{g^2} + \frac{V^2\sin^2\theta}{g}$

$= \frac{1}{2}\frac{V^2\sin^2\theta}{g}$ ①

Q3 cont

iii) $y = -h$

$$\therefore \quad -\frac{V^2\sin^2\theta}{2g} = -\tfrac{1}{2}gt^2 + Vt\sin\theta \quad (1)$$

$$0 = gt^2 - 2Vt\sin\theta - \frac{V^2\sin^2\theta}{g}$$

$$t = \frac{2V\sin\theta \pm \sqrt{(-2V\sin\theta)^2 - 4\times g\left(-V^2\sin^2\theta/g\right)}}{2g}$$

$$= \frac{2V\sin\theta \pm \sqrt{4V^2\sin^2\theta + 4V^2\sin^2\theta}}{2g}$$

$$= \frac{2V\sin\theta \pm 2\sqrt{2}V\sin\theta}{2g}$$

$$= \frac{V\sin\theta\,(1\pm\sqrt{2})}{g}$$

Cannot have negative time ($\sqrt{2} > 1$)

$$\therefore \quad t = \frac{V\sin\theta\,(1+\sqrt{2})}{g} \quad (1)$$

iv) $$x = V\cos\theta\, t \qquad t = \frac{V\sin\theta(1+\sqrt{2})}{g}$$

$$= V\cos\theta \times \frac{V\sin\theta(1+\sqrt{2})}{g} \quad (1)$$

$$= \frac{V^2(1+\sqrt{2}) \times 2\sin\theta\cos\theta}{2g}$$

$$= \frac{V^2(1+\sqrt{2})\sin 2\theta}{2g} \quad (1)$$

Q4

a) $y = \frac{x^2}{4a}$

i) $y' = \frac{2x}{4a} = \frac{x}{2a}$ (1)

Tangent to $P(2ap, ap^2)$

$m = p$

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

Similarly

$y = qx - aq^2$ (1)

ii) PQ

$$\frac{y - ap^2}{x - 2ap} = \frac{aq^2 - ap^2}{2aq - 2ap} \quad (1)$$

$$\frac{y - ap^2}{x - 2ap} = \frac{q+p}{2}$$

$2y - 2ap^2 = (q+p)x - 2apq - 2ap^2$

$2y - (q+p)x + 2apq = 0$ (1)

passes through $(0, -3a)$

$\therefore -6a + 2apq = 0$ (1)

$\therefore pq = 3$

iii) $y = px - ap^2$ $\boxed{1}$

$y = qx - aq^2$ $\boxed{2}$

$\boxed{1} - \boxed{2}$

$0 = (p-q)x - a(p^2 - q^2)$

$x = a(p+q)$ (1)

Sub $x = a(p+q)$ in $\boxed{1}$

$y = a(p+q)p - ap^2$

$= ap^2 + apq - ap^2$

$= apq$ (1)

$a = \frac{x}{p+q}$ $\quad y = apq$ $\quad pq = 3$

$\therefore y = \frac{3x}{p+q}$ (1)

b) $$\int \frac{dx}{\sqrt{x}\sqrt{1-\sqrt{x}}} = \int \frac{2\,du}{\sqrt{1-u}} \quad (1)$$

let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$2du = \frac{dx}{\sqrt{x}}$ (1)

$= \int 2(1-u)^{-\frac{1}{2}}du$

$= -2\int -1(1-u)^{-\frac{1}{2}}du$ (1)

$= -2\frac{(1-u)^{\frac{1}{2}}}{\frac{1}{2}} + C$ (1)

$= -4(1-u)^{\frac{1}{2}} + C$

1999 3 UNIT TRIAL PAPER SOLUTIONS

5)a i) Let $I = \frac{V}{R} + Ae^{-\frac{Rt}{L}}$

Then $\frac{dI}{dt} = -\frac{RA}{L}e^{-\frac{Rt}{L}} = \frac{-R}{L}\left(I - \frac{V}{R}\right)$

$= \frac{V}{L} - \frac{R}{L}I$ //

ii) as $t \to \infty$, $I \to \frac{V}{R}$

iii) at $t = 0$, $I = \frac{V}{R} + A = 0 \Rightarrow A = \frac{-V}{R}$

$\therefore$ at $t = 2 \times 10^{-6}$, $I = 1.1 \times 10^{-3}$ (correct to 2 significant figures)

b) Let $f(x) = x^4 + x^2 - 80$

$\therefore f'(x) = 4x^3 + 2x$

Let $x_0 = 3$

$\therefore x_1 = 3 - \frac{10}{114} = \frac{166}{57}$

$\therefore x_2 \approx 2.908312632$

c) Let $n = 1$. Then $LHS = 5 + 3 = 8$ which is divisible by 4

Now Assume $4 | 5^k + 3$

Then $5^{k+1} + 3 = 5(5^k + 3) - 12$ which is divisible by 4.

$\therefore$ By the principle of mathematical induction $4 | 5^k + 3 \quad \forall n \in \mathbb{Z}^+$

1999 3 UNIT TRIAL PAPER SOLUTIONS

a) i) $\ddot{x} = \dfrac{d(\frac{1}{2}v^2)}{dx} = 2 - 2x = -2(x-1)$

centre of motion is at $x = 1$

period is $\sqrt{2}\pi$

now at $v = 0$, $x = -1, 3$

$\therefore$ Amplitude is 2

) $\cos\left(\sin^{-1}\frac{5}{13} + \sin^{-1}\frac{4}{5}\right) = \cos\left(\cos^{-1}\frac{12}{13}\right)\cos\left(\cos^{-1}\frac{3}{5}\right) - \sin\left(\sin^{-1}\frac{5}{13}\right)\sin\left(\sin^{-1}\right.$

$= \dfrac{16}{65}$

) i) $\dfrac{d}{dx}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \dfrac{\sqrt{1-x^2} - x \times \frac{1}{2} \times {}^{-}2x\,(1-x^2)^{-\frac{1}{2}}}{1-x^2}$

$= \dfrac{1 - x^2 + x^2}{(1-x^2)^{3/2}}$

$= \dfrac{1}{(1-x)^{3/2}}$

) $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)\right) = \dfrac{1}{1 + \left(\frac{x}{\sqrt{1-x^2}}\right)^2} \cdot \dfrac{1}{(1-x^2)^{3/2}}$

$= \dfrac{1-x^2}{(1-x^2)^{3/2}}$

$= \dfrac{1}{\sqrt{1-x^2}}$

1999 3 UNIT TRIAL PAPER SOLUTIONS

) Number of possible numbers is $(5+5\times4+5\times4\times3)+(3\times4\times3\times2)$

$= 85+72 = 157$

The number of even numbers is $\frac{2}{5}\times 85+\frac{1}{2}\times 72 = 70$

$\therefore P(\text{even})$ is $\frac{70}{157}$

) Let $n=1$, then $LHS = 1+\frac{x}{1}$

$RHS = \frac{x+1}{1} = LHS$

Now assume $1+\frac{x}{1!}+\frac{x(x+1)}{2!}+\dots+\frac{x(x+1)(x+2)\dots(x+k-1)}{k!} = \frac{(x+1)(x+2)\cdots(x+k)}{k!}$

Then at $n=k+1$, $LHS = 1+\frac{x}{1!}+\dots+\frac{x(x+1)(x+2)\dots(x+k-1)}{k!}+\frac{x(x+1)(x+2)\dots(x+k)}{(k+1)!}$

$$= \frac{(x+1)(x+2)\dots(x+k)}{k!}+\frac{x(x+1)(x+2)\cdots(x+k)}{(k+1)!}$$

$$= \frac{(k+1)(x+1)(x+2)\dots(x+k)+x(x+1)(x+2)\dots(x+k)}{(k+1)!}$$

$$= \frac{(x+1)(x+2)\dots(x+k)[x+k+1]}{(k+1)!} = RHS$$

By the principle of mathematical induction the statement is true $\forall n \in \mathbb{Z}$

) Let $x=-n$ in the above

Then $1-\frac{n}{1!}+\frac{n(n-1)}{2!}-\frac{n(n-1)(n-2)}{3!}+\dots+{}^{n}\frac{n(n-1)\dots(n-n+1)}{n!} = \frac{(-n+1)(-n+2)\dots(-n}{n!}$

$\therefore 1-{}^nC_1+{}^nC_2-{}^nC_3+\dots+(-1)^n\,{}^nC_n =$

ALTERNATIVELY $(1-x)^n = 1-{}^nC_1x+{}^nC_2x^2+\dots+(-1)\quad C_nx^n$

$\therefore 0=(1-1)^n = 1-{}^nC_1+{}^nC_2+\dots+(-1)^n\,{}^nC_n$