

CRANBROOK SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2000

MATHEMATICS

3 UNIT (Additional)

4 UNIT (First Paper)

Time allowed – Two hours

DIRECTIONS TO CANDIDATES

- * **Attempt all questions.**
- * **ALL questions are of equal value.**
- * **All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.**
- * **Standard integrals are printed on the back page.**
- * **Board-approved calculators may be used.**
- * **You may ask for extra Writing Booklets if you need them.**

- * **Submit your work in five booklets :**
 - (i) **QUESTIONS 1 & 2 (8 page)**
 - (ii) **QUESTIONS 3 & 4 (8 page)**
 - (iii) **QUESTION 5 (4 page)**
 - (iv) **QUESTION 6 (4 page)**
 - (v) **QUESTION 7 (4 page)**

1. (8 page booklet)

- (a) If the equation $5x^3 - 6x^2 - 29x + 6 = 0$ has roots α, β, γ find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3 marks]

- (b) (i) Show that there exists one value of the constant b for which the polynomial $P(x) = x^4 + 2x^3 - x^2 - 8x - b$ is divisible by $Q(x) = x^2 - 4$. [2 marks]

- (ii) Hence or otherwise find the roots of $P(x)$ for this value of b . [2 marks]

- (c) (i) Find $\frac{d}{dx}(\operatorname{cosec} x \cot x)$ in terms of $\operatorname{cosec} x$. [3 marks]

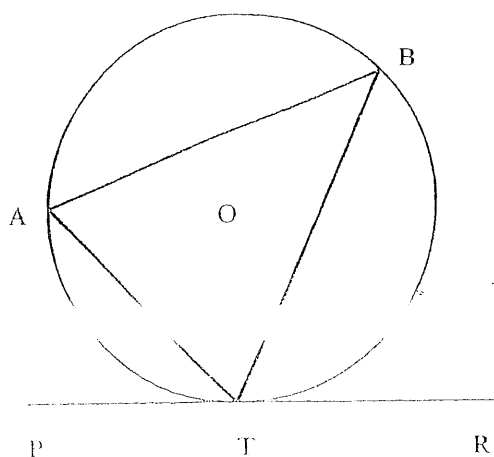
- (ii) Use your result in (i) to find the exact value of $\int_{\pi/6}^{\pi/3} \operatorname{cosec} x (\cot^2 x + \operatorname{cosec}^2 x) dx$. [2 marks]

2. (a) Find the general solutions of $\sin 2\theta + \cos \theta = 0$ in radian form. [3 marks]

- (b) Find the solutions of $3\sin \theta + 4\cos \theta = -4$ for $0 \leq \theta \leq 4\pi$, giving your answers in radians, correct (where necessary) to 3 decimal places. [4 marks]

- (c) PR is a tangent to the circle centre O, at the point T. Prove that $\angle ATP = \angle ABT$.
(Redraw the diagram below as part of your answer).

[5 marks]



3. (new 8 page booklet please)

- (a) Find the term independent of x in the expansion of $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$ [4 marks]
- (b) Twelve candidates for election to a committee of four include two well-known geniuses, Mr G.J. Baker and Mr S.K. Blazey. If all candidates have an equal chance of selection, what is the probability that the committee
- (i) includes Mr Baker but excludes Mr Blazey?
- (ii) includes at least one of these two geniuses? [4 marks]
- (c) A weather bureau finds that it predicts maximum temperatures with about 60% accuracy. What is the probability that, in a particular week, it is accurate
- (i) on every day but Saturday and Sunday?
- (ii) on exactly five days? [4 marks]

4.

- (a) Solve $\frac{3x+2}{x-1} > 2$ [3 marks]
- (b) Prove by Mathematical Induction that

$$2 \times 1! + 5 \times 2! + 10 \times 3! + \dots + (n^2 + 1) \times n! = n \times (n+1)!$$
 [5 marks]
- (c) (i) Show that ${}^nC_r : {}^nC_{r-1} = (n-r+1) : r$



Hence evaluate $\frac{{}^nC_1}{{}^nC_0} + \frac{2 \times {}^nC_2}{{}^nC_1} + \frac{3 \times {}^nC_3}{{}^nC_2} + \dots + \frac{n \times {}^nC_n}{{}^nC_{n-1}}$ [4 marks]

5. (new 4 page booklet please)

- (a) Find the derivative of $\cos^{-1}(2x+1)$, stating the values of x for which it is defined. [2 marks]
- (b) Differentiate $\sin^{-1}(e^{2x})$ and hence find $\int_{-\ln 2}^0 \frac{e^{2x}}{\sqrt{1-e^{4x}}} dx$ correct to two decimal places. [4 marks]
- (c) The rate of emission E , in tonnes per year, of chloro-fluorocarbons (CFC's) in Australia from 18th July 2000 will be given by $E = 80 + \left(\frac{t}{1+t}\right)$, where t is the time in years.
- (i) What is the rate of emission E on 18th July 2000? [1 mark]
- (ii) What is the rate of emission E on 18th July 2005? [1 mark]
- (iii) Draw a sketch of E as a function of t . [2 marks]
- (iv) Calculate the total amount of CFCs emitted in Australia during the years 2000 to 2005. [2 marks]

6. (new 4 page booklet please)

- (a) Evaluate $\int_0^{\pi} 2 \sin x \cos^2 x \, dx$. [2 marks]
- (b) Integrate the following using the substitutions given
- (i) $\int \frac{x^4}{(x^5 + 1)^3} dx$ ($u = x^5 + 1$) (ii) $\int_{\frac{1}{2}}^1 \frac{\sqrt{1-x^2}}{x^2} dx$ ($x = \cos \theta$) [6 marks]
- (c) Two roads intersect, making an angle of 30° between them. After an argument at the intersection, George storms off at 6 km/h along one of the roads, and Jerry walks off calmly at 2 km/h along the other. Show that the rate at which the distance between them is increasing is constant. Find this rate of increase correct to three significant figures. [4 marks]

7. (new 4 page booklet please)

- (a) The rate of change of the volume of water (V kL) in a dam at any given time t (in hours) is given by $\frac{dV}{dt} = k(V - 5000)$, where k is a constant.
- (i) Show that $V = 5000 + Ae^{kt}$ is a solution of this differential equation. [2 marks]
- (ii) If the initial volume is 87 000 kL, and after 10 hours the volume is 129 000 kL, find the exact values of A and k . [3 marks]
- (iii) Determine how long it will take the volume to reach 4.2 million kL. [Give your answer in days and hours, correct to the nearest hour.] [2 marks]
- (b) The inner and outer radii of a cylindrical tube of constant length change in such a way that the volume of the material forming the tube remains constant. Find the rate of increase of the outer radius at the instant when the radii are 3 cm and 5 cm, and the rate of increase of the inner radius is $3\frac{1}{3}$ cm/s. [5 marks]

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} \quad (n \neq -1; x \neq 0 \text{ if } n < 0)$$

$$\int \frac{1}{x} \, dx = \log_e x \quad (x > 0)$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax \quad (a \neq 0)$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax \quad (a \neq 0)$$

$$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x \quad (a \neq 0)$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} \quad (a > 0, -a < x < a)$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \log_e \left\{ x + \sqrt{x^2 - a^2} \right\} \quad (|x| > |a|)$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \log_e \left\{ x + \sqrt{x^2 + a^2} \right\}$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} \quad (a \neq 0)$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax \quad (a \neq 0)$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax \quad (a \neq 0)$$

3 UNIT TRIAL CRANBROOK 2000.

(a) $5x^3 - 6x^2 - 29x + 6 = 0$, has roots

α, β, γ .

$$\therefore \alpha + \beta + \gamma = -\frac{-6}{5} = \frac{6}{5}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{-29}{5}$$

$$\alpha\beta\gamma = -\frac{6}{5}$$

$$\begin{aligned} \text{Now } \alpha^4 + \beta^4 + \gamma^4 &= (\alpha + \beta + \gamma)^4 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)(\alpha + \beta + \gamma) \\ &= \left(\frac{6}{5}\right)^4 - 2\left(\frac{-29}{5}\right)\left(\frac{6}{5}\right) \\ &= \frac{36}{25} + \frac{58}{5} \\ &= \frac{326}{25} \end{aligned}$$

(b) (i) If $P(x)$ is divisible by $Q(x)$

$$\text{then } P(2) = P(-2) = 0$$

$$\text{as } Q(x) = x^2 - 4$$

$$= (x-2)(x+2).$$

$$\text{Now } P(2) = 16 + 16 - 4 - 16 - b$$

$$= 12 - b$$

$$\therefore \text{If } P(2) = 0 \text{ then } b = 12$$

$$\text{Also } P(-2) = 16 - 16 - 4 + 16 - b$$

$$= 12 - b$$

$$\therefore \text{if } P(-2) = 0 \text{ then } b = 12 \text{ again.}$$

\therefore there exists only 1 value of the constant b if $P(x)$ is divisible by $Q(x)$.

(ii) $P(x) = x^4 + 2x^3 - x^2 - 8x - 12$

$$(x^2 - 4)(x^2 + 2x + 3)$$

$$= (x-2)(x+2)(x^2 + 2x + 3)$$

$$\therefore \text{Roots are } x = 2, -2$$

$$(x^2 + 2x + 3 = 0 \text{ has no real roots})$$

(i) Let $y = \operatorname{cosec} x \cot x$

$$\therefore \frac{dy}{dx} = \operatorname{cosec} x \cdot -\operatorname{cosec}^2 x$$

$$+ \cot x \cdot -\operatorname{cosec} x \cot x$$

$$= -\operatorname{cosec}^3 x - \operatorname{cosec} x (\cot^2 x)$$

$$= -\operatorname{cosec}^3 x - \operatorname{cosec} x (\operatorname{cosec}^2 x - 1)$$

$$= -2\operatorname{cosec}^3 x + \operatorname{cosec} x$$

$$\therefore \frac{d}{dx}(\operatorname{cosec} x \cot x) = -2\operatorname{cosec}^3 x + \operatorname{cosec} x$$

or let $y = \operatorname{cosec} x \cot x$

$$= \frac{1}{\sin x \tan x}$$

$$= \frac{\cos x}{\sin^2 x}$$

$$\therefore \frac{dy}{dx} = \frac{2\sin^2 x \cdot -\sin x - \cos x \cdot 2\sin x \cos x}{\sin^4 x}$$

$$= \frac{-\sin^3 x - 2\sin x (1 - \sin^2 x)}{\sin^4 x}$$

$$= \frac{-\sin^3 x - 2\sin x + 2\sin^3 x}{\sin^4 x}$$

$$= \frac{\sin^3 x - 2\sin x}{\sin^4 x}$$

$$= \frac{1}{\sin x} - \frac{2}{\sin^3 x}$$

$$= \operatorname{cosec} x - 2\operatorname{cosec}^3 x$$

$$\therefore \frac{d}{dx}(\operatorname{cosec} x \cot x) = \operatorname{cosec} x - 2\operatorname{cosec}^3 x$$

(ii) $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{cosec} x (\cot^2 x + \operatorname{cosec}^2 x) dx$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{cosec} x (\operatorname{cosec}^2 x - 1 + \operatorname{cosec}^2 x) dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{cosec} x (2\operatorname{cosec}^2 x - 1) dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\operatorname{cosec}^3 x - \operatorname{cosec} x dx$$

$$= - \left[\operatorname{cosec} x \cot x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \quad \left(\text{using result of part (i)} \right)$$

$$= - \left[\frac{1}{\sin x \tan x} \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= - \left[\frac{1}{\frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{3}}} - \frac{1}{\frac{1}{2} \cdot \frac{1}{1}} \right]$$

$$= -\left[\frac{2}{3} - 2\sqrt{3}\right]$$

$$= 2\sqrt{3} - \frac{2}{3}$$

2 (a) $\sin 2\theta + \cos \theta = 0$

$$\therefore 2\sin \theta \cos \theta + \cos \theta = 0$$

$$\therefore \cos \theta (2\sin \theta + 1) = 0 \quad \checkmark$$

$$\therefore \cos \theta = 0 \text{ or } \sin \theta = -\frac{1}{2}$$

$$\therefore \cos \theta = \cos \frac{\pi}{2} \text{ or } \sin \theta = \sin\left(-\frac{\pi}{6}\right)$$

$$\therefore \theta = 2n\pi \pm \frac{\pi}{2} \text{ or } \theta = n\pi + (-1)^n \cdot \left(-\frac{\pi}{6}\right)$$

for $n \in \mathbb{Z}$.

(b) $3\sin \theta + 4\cos \theta = -4, 0 \leq \theta \leq 4\pi$

Let $t = \tan \frac{\theta}{2} \quad (\theta \neq \pi)$

$$\therefore \sin \theta = \frac{2t}{1+t^2}, \cos \theta = \frac{1-t^2}{1+t^2}$$

$$\therefore 3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right) = -4 \quad \checkmark$$

$$\therefore 6t + 4 - 4t^2 = -4 - 4t^2$$

$$\therefore 6t = -8$$

$$\therefore t = -\frac{4}{3}$$

$$\therefore \tan \frac{\theta}{2} = -\frac{4}{3} \quad \checkmark$$

$$\therefore \text{basic angle } \frac{\theta}{2} = \tan^{-1} \frac{4}{3} \text{ (negate 2nd, 4th quadrants)}$$

$$\therefore \frac{\theta}{2} = \pi - \tan^{-1} \frac{4}{3} \text{ or } 2\pi - \tan^{-1} \frac{4}{3}$$

$$\therefore \angle \theta = 2\pi - 2\tan^{-1} \frac{4}{3} \text{ or } 4\pi - 2\tan^{-1} \frac{4}{3}$$

$$\therefore \angle \theta = 4.429 \text{ or } 10.712 \text{ (3 d.p.)}$$

Now LHS = -4 = RHS

$$\therefore \angle \theta = \pi, 3\pi, 4.429 \text{ or } 10.712 \text{ (3 d.p.)}$$

on $3\sin \theta + 4\cos \theta = -4$

$$\therefore 5\left(\frac{3}{5}\sin \theta + \frac{4}{5}\cos \theta\right) = -4$$

$$\therefore \sin(\theta + \alpha) = -\frac{4}{5} \quad \checkmark$$

where $\cos \alpha = \frac{3}{5}, \sin \alpha = \frac{4}{5}$

$$\therefore \tan \alpha = \frac{4}{3} \text{ i.e. } \alpha = \tan^{-1} \frac{4}{3}$$

$$\therefore \sin(\theta + \tan^{-1} \frac{4}{3}) = -\frac{4}{5}$$

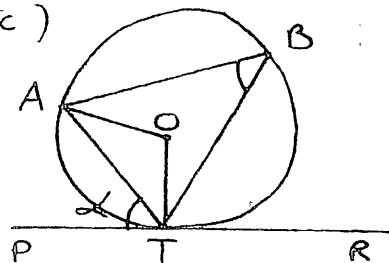
$$\therefore \text{basic angle } (\theta + \tan^{-1} \frac{4}{3}) = \sin^{-1} \frac{4}{5} \text{ (negate 3rd, 4th quadrants)}$$

$$\therefore \angle \theta = \pi + \sin^{-1} \frac{4}{5} - \tan^{-1} \frac{4}{3} - 2\pi - \sin^{-1} \frac{4}{5} - \tan^{-1} \frac{4}{3},$$

$$3\pi + \sin^{-1} \frac{4}{5} - \tan^{-1} \frac{4}{3}, 4\pi - \sin^{-1} \frac{4}{5} - \tan^{-1} \frac{4}{3}$$

$$= \pi, 4.429 \text{ (3 d.p.)}, 3\pi, 10.712$$

(c)



TO PROVE:

$$\angle ATP = \angle ABT$$

PROOF: Let $\angle ATP = \alpha$

Join OA and OT. \checkmark

$\angle OTP = 90^\circ$ (\angle between tangent and radius at pt. contact = 90°)

$$\therefore \angle OTA = 90^\circ - \alpha$$

Now as $OA = OT$ (equal radii) \checkmark

$\therefore \triangle AOT$ is isosceles.

$$\therefore \angle OAT = 90^\circ - \alpha \text{ (base } \angle \text{ s of isos. } \triangle \text{ are equal)}$$

$$= 2\alpha$$

$$\therefore \angle ABT = \angle \left(\begin{array}{l} \angle \text{ at centre} = 2\angle \\ \text{circum. standing on a} \\ \text{common arc} \end{array} \right)$$

$$\therefore \angle ATP = \angle ABT. \quad \checkmark$$

$$3 \quad (a) \quad \left(\frac{3x^2}{2} - \frac{1}{3x} \right)^9 = \left(\frac{3x^2}{2} \right)^9 \left(1 - \frac{2}{9x^3} \right)^9$$

$$\frac{18}{2x}$$

$$\frac{-18}{2x} = \left(\frac{-2}{9x^3} \right)^6$$

$$T_{\text{em}} = \frac{3^9 x^{18}}{2^9} \cdot {}^9C_6 \left(-\frac{2}{9x^3} \right)^6$$

$$= \frac{3^9}{2^9} \cdot 84 \cdot \frac{2^6}{3^{12}} = \frac{7}{18} \quad 4$$

$$(b) \quad (i) \quad \text{Prob} = \frac{{}^{12}C_4}{{}^{12}C_4} = \frac{8}{33} \quad 2$$

$$(ii) \quad \text{Prob} = 1 - \text{Prob}(\text{both excluded})$$

$$= 1 - \frac{{}^{10}C_4}{{}^{12}C_4} = \frac{19}{33} \quad 2$$

$$(c) \quad (i) \quad \text{Prob} = \left(\frac{3}{5} \right)^5 \left(\frac{2}{5} \right)^2$$

$$= \frac{3^5 \times 2^2}{5^7} = 0.012 \quad \checkmark \quad \left(\frac{972}{78125} \right)$$

$$(ii) \quad \text{Prob} = {}^7C_5 \left(\frac{3}{5} \right)^5 \left(\frac{2}{5} \right)^2$$

$$= 21 \left(\frac{3}{5} \right)^5 \left(\frac{2}{5} \right)^2$$

$$= \frac{1 \times 5 \times 2}{5^7} = 0.261 \quad \checkmark$$

4. (a) $\frac{3x+2}{x-1} > 2$

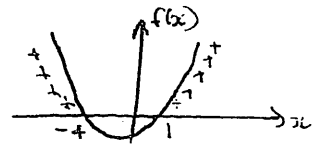
$$\therefore (3x+2)(x-1) > 2(x-1)^2$$

$$3x^2 - x - 2 > 2x^2 - 4x + 2$$

$$\therefore x^2 + 3x - 4 > 0$$

$$(x+4)(x-1) > 0$$

$$\therefore x > 1 \text{ or } x < -4$$



3

(b) Let S be the set of +ve integers n for which
 $2 \times 1! + 5 \times 2! + \dots + (n^2+1) \times n! = n \times (n+1)!$

If $n=1$

$$\text{L.H.S.} = 2 \times 1! = 2$$

$$\text{R.H.S.} = 1 \times 2! = 2$$

$$\therefore 1 \in S$$

Suppose $h \in S$

$$\therefore 2 \times 1! + 5 \times 2! + \dots + (h^2+1) \times h! = h \times (h+1)!$$

If $n=h+1$

$$\text{L.H.S.} = 2 \times 1! + \dots + (h^2+1) \times h! + ((h+1)^2+1) \times (h+1)!$$

$$= h \times (h+1)! + (h^2+2h+2) \times (h+1)!$$

$$= (h+1)! (h^2+3h+2)$$

$$= (h+1)! (h+1)(h+2)$$

$$= (h+1) \times (h+2)! = \text{R.H.S.}$$

$$\therefore h \in S \Rightarrow (h+1) \in S$$

$\therefore S$ is the set of all +ve integers

(c) (i) ${}^nC_r : {}^nC_{n-1} = \frac{n!}{r!(n-r)!} \cdot \frac{(n-r+1)!}{(n-r)!} = \frac{n-r+1}{1}$

(ii) $\frac{{}^nC_1}{{}^nC_0} + \frac{2 \cdot {}^nC_2}{{}^nC_1} + \frac{3 \cdot {}^nC_3}{{}^nC_2} + \dots + \frac{n \cdot {}^nC_n}{{}^nC_{n-1}} = \frac{n}{1} + 2 \cdot \frac{n-1}{2} + 3 \cdot \frac{n-2}{3} + \dots + n$

$$= n + (n-1) + (n-2) + \dots + 1$$

$$= \underline{n(n+1)}$$

(A) $\cos^{-1}(2x+1)$ is defined for $-1 \leq 2x+1 \leq 1$
 ie. $-2 \leq 2x \leq 0$
 $-1 \leq x \leq 0$

Hence $\cos^{-1}(2x+1)$ is defined for $-1 \leq x \leq 0$.

Let $y = \cos^{-1}(2x+1)$

$y = \cos^{-1}(u)$ where $u = 2x+1$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-u^2}} \times 2$$

$$y = \cos^{-1} u$$

$$\frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}}$$

$$= -\frac{2}{\sqrt{1-(2x+1)^2}} \quad \text{provided } -1 \leq x \leq 0 \quad (1)$$

$$= -\frac{2}{\sqrt{1-(4x^2+4x+1)}}$$

optional

$$= -\frac{2}{\sqrt{1-4x^2-4x-1}}$$

$$= -\frac{2}{\sqrt{-4x(x+1)}}$$

Q5. (b) Let $y = \sin^{-1} u$ where $u =$

$$u = e^{2x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \times 2e^{2x}$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-u^2}} \times 2e^{2x}$$

$$= \frac{2e^{2x}}{\sqrt{1-e^{4x}}}$$

$$\int_{-\ln 2}^0 \frac{2e^{2x}}{\sqrt{1-e^{4x}}} dx = \left[\sin^{-1} \left(\frac{e^{2x}}{\sqrt{1-e^{4x}}} \right) \right]_{-\ln 2}^0 \quad (2)$$

$$= \frac{1}{2} \left[\sin^{-1} \left(\frac{e^{2x}}{\sqrt{1-e^{4x}}} \right) \right]_{-\ln 2}^0$$

$$= \frac{1}{2} \left(\sin^{-1} \left(\frac{e^{2 \ln 2}}{\sqrt{1-e^{4 \ln 2}}} \right) - \sin^{-1} \left(\frac{e^{2(-\ln 2)}}{\sqrt{1-e^{4(-\ln 2)}}} \right) \right)$$

$$= \frac{1}{2} \left(\frac{\pi}{2} - \sin^{-1} \left(\frac{1}{4} \right) \right)$$

$$= \frac{1}{2} (1.5708 - 0.252680255)$$

$$= 0.66 \quad (2)$$

$$-2 \ln 2 = -\ln 4$$

$$e^{-\ln 4} = \frac{1}{e^{\ln 4}} = \frac{1}{4}$$

Q5. (c)

$$E = 80 + \left(\frac{30}{1+t}\right)^2$$

(i) On 18th July 2000, $t=0$

$$E = 80 + \left(\frac{30}{1+0}\right)^2$$

$$E = 80 + (30)^2$$

(1)

$$E = 980 \text{ tonnes/year}$$

(ii) On 18th July 2005, $t=5$

$$E = 80 + \left(\frac{30}{1+5}\right)^2$$

$$E = 80 + \left(\frac{30}{6}\right)^2$$

$$E = 80 + 5^2$$

$$E = 105 \text{ tonnes/year}$$

(1)

(iv) Total amount of CFCs emitted

$$= \int_0^5 \left(80 + \left(\frac{30}{1+t}\right)^2\right) dt$$

$$= \int_0^5 80 dt + \int_0^5 \frac{900}{(1+t)^2} dt$$

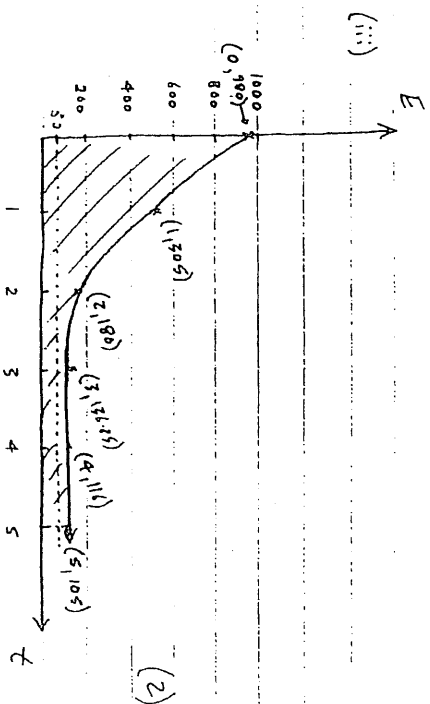
$$\left[80t\right]_0^5 + \left[-\frac{900}{1+t}\right]_0^5$$

$$\left[400 - 0\right] + \left[-\frac{900}{6} - \left(-\frac{900}{1}\right)\right]$$

$$400 + \left(900 - \frac{900}{6}\right)$$

$$1150 \text{ tonnes}$$

(2)



t	0	1	2	3	4	5
E	980	705	180	131.25	111	105

(a) $\int_0^{\pi} 2 \sin x \cos^2 x \, dx$.

Let $\cos x = u$ when $x = \pi$ $u = -1$
 $-\sin x = \frac{du}{dx}$ $x = 0$ $u = 1$

$\sin x \, dx = -du$.

$$\int_1^{-1} 2u^2 \, du = -\left[\frac{2u^3}{3}\right]_1^{-1}$$

$$= -\left[\left[-\frac{2}{3}\right] - \left[\frac{2}{3}\right]\right]$$

$$= 1\frac{1}{3} \text{ units.}$$

(b) (i) $\int \frac{x^4}{(x^5+1)^3} \, dx \Rightarrow$ Given $u = x^5+1$
 $\frac{du}{dx} = 5x^4$
 $du = 5x^4$

$$\therefore \frac{1}{5} \int \frac{du}{u^3} = \frac{1}{5} \left(-\frac{1}{2u} \right) + C$$

$$= -\frac{1}{10u} + C$$

$$= -\frac{1}{10(x^5+1)} + C$$

(ii) $\int_{\pi/3}^1 \frac{\sqrt{1-x^2}}{x^2} \, dx$ $x = \cos \theta$
 $\frac{dx}{d\theta} = -\sin \theta$

$$\int_{\pi/3}^0 \left[\frac{\sqrt{1-\cos^2 \theta}}{\cos^2 \theta} - \sin \theta \right] d\theta$$

$$= \int_{\pi/3}^0 \frac{-\sin^2 \theta}{\cos^2 \theta} d\theta$$

$$= - \int \tan^2 \theta \, d\theta$$

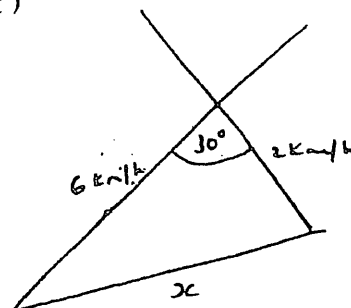
$$= - \int \sec^2 \theta - 1 \, d\theta$$

$$= - \left[\tan \theta - \theta \right]_{\pi/3}^0$$

$$= - \left[(0 - 0) - (\tan \pi/3 - \pi/3) \right]$$

$$= -(-\sqrt{3} + \pi/3)$$

(c)



(i) Let x be the distance between them.

$$x^2 = 6^2 + 2^2 - 2 \cdot 6 \cdot 2 \cdot \cos 30$$

$$x^2 = 36 + 4 - 24 \times \frac{\sqrt{3}}{2}$$

$$x^2 = 40 - 12\sqrt{3}$$

$$\therefore x = \sqrt{40 - 12\sqrt{3}} \quad [\text{NB. Only two solutions since length cannot be negative}]$$

Distance $\therefore x = (2\sqrt{10 - 3\sqrt{3}})t$

Rate $\therefore \frac{dx}{dt} = 2\sqrt{10 - 3\sqrt{3}} \therefore$ constant value.

\therefore The rate of distance increasing

is $2\sqrt{10 - 3\sqrt{3}} \text{ km/h.}$

$$\therefore 4.383536279 \dots$$

$$4.38 \text{ km/h (3 s.f.)}$$

(4)

P2 SU TRIAL EXAMINATION 2000 SOLUTIONS (87)

$$(a) \quad \frac{dv}{dt} = k(v - 5000)$$

$$\text{If } v = 5000 + Ae^{kt}$$

$$\frac{dv}{dt} = kAe^{kt}$$

$$= k(v - 5000)$$

$$v = 5000 + Ae^{kt} \text{ is a solution}$$

$$(ii) \quad \text{when } t=0, \quad v=87000$$

$$87000 = 5000 + Ae^0$$

$$A = 82000$$

$$\text{when } t=10, \quad v=129000$$

$$129000 = 5000 + 82000e^{10k}$$

$$\frac{124000}{82000} = e^{10k}$$

$$10k = \ln \frac{124}{82}$$

$$k = \frac{1}{10} \ln \frac{62}{41}$$

$$(iii) \quad v = 5000 + 82000e^{t \ln \frac{62}{41}}$$

$$420000 = 5000 + 82000e^{t \ln \frac{62}{41}}$$

$$e^{t \ln \frac{62}{41}} = \frac{419500}{82000}$$

$$t = 10 \ln \frac{4195}{82}$$

$$\ln \frac{62}{41}$$

$$= 3 \text{ days } 23 \text{ h}$$

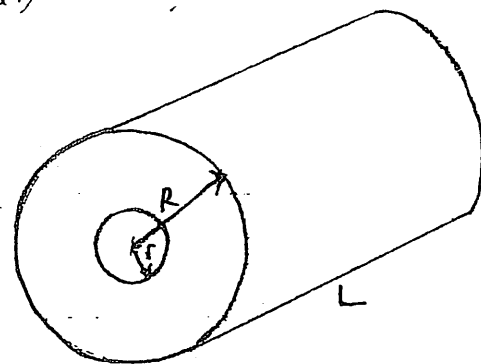
(to nearest hour)

(3 days 24h accepted since

theorically, 420000 L would not

have been reached by 3 days 23h)

(c)



$$\text{Volume of material, } V = (\pi R^2 - \pi r^2)L$$

$$\therefore \frac{V}{L} = \pi R^2 - \pi r^2$$

constant

$$\frac{dR}{dt} = \frac{dR}{dr} \times \frac{dr}{dt}$$

$$= \frac{r}{\sqrt{\frac{V}{\pi} + r^2}} \times \frac{dr}{dt}$$

$$\pi R^2 = \frac{V}{L} + \pi r^2$$

$$R^2 = \frac{V}{\pi L} + r^2$$

$$R = \sqrt{\frac{V}{\pi L} + r^2}$$

$$\text{when } R=5, \quad r=3$$

$$\frac{V}{L} = \pi \times 5^2 - \pi \times 3^2$$

$$= 16\pi$$

$$\frac{dR}{dt} = \frac{1}{2} \left(\frac{V}{\pi L} + r^2 \right)^{-\frac{1}{2}}$$

$$= \frac{r}{\sqrt{\frac{V}{\pi L} + r^2}}$$

$$\therefore \frac{dR}{dt} = \frac{3}{\sqrt{16+3^2}} \times \frac{16}{3}$$

$$= \frac{16}{\sqrt{25}}$$

$$= 2$$

outer radius is increasing at
rate of 2 cm/s