SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE
2001

MATHEMATICS
4 UNIT

Time allowed - Three hours
(Plus 5 minutes reading time)

Name: .... Class: ....
This test paper must be handed in with your answers

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Question 7	Question 8	TOTAL

DIRECTIONS TO CANDIDATES

- Attempt ALL questions
- ALL questions are of equal value
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed at the back of this test booklet.
- *Board-approved* calculators may be used.
- *Each* question is to be started on a new page clearly marked Question 1, Question 2, etc.. Each page must show your name.
- You may ask for more paper if you need it.

An academically selective school for boys

QUESTION 1:

(a) Find

1 (i) $\displaystyle\int \frac{dx}{x^2+2x+5}$

2 (ii) $\displaystyle\int_0^1 \frac{dx}{(x+1)\sqrt{x+1}}$

(b)

2 Prove that $\sec x = \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

and hence find $\displaystyle\int \sec x dx$

(c) (i)

2 Find the exact value of $\displaystyle\int_0^1 xe^{-x} dx$

(ii)

4 Find $\displaystyle\int \frac{5\, dx}{(x+1)(x^2+4)}$

(d)

2 Find $\displaystyle\int_k^1 \frac{dx}{x(x+1)}$ and hence prove that

2 $$\sum_{k=1}^{n} \int_k^1 \frac{dx}{x(x+1)} = \log_e(n+1) - n\log_e 2$$

QUESTION 2:

(a) If $z = 3\text{-}4i$ find

6 (i) $\bar{z}$ (ii) $|z|$ (iii) argz (iv) arg (iz) (v) $\sqrt{z}$

2 (b) The complex number $z = x + iy$ is such that $|z - i| = \text{Im}(z)$

Find, and describe geometrically, the locus of the point P representing z

4 (c) Sketch the locus on the Argand Diagram of the point Z representing the complex number z where $|z - 2i| = 1$

What is the least value of arg z ?

3 (d) A is the point representing the complex number z = 2+3i, while B represents the complex number *iz*.
The point C is such that AOBC is a square (where O is the origin)
Find the co-ordinates of C.

QUESTION 3:

3 (a) If one root of the polynomial equation $x^3 + ax^2 + bx + c = 0$ is the sum of the other two roots, show that

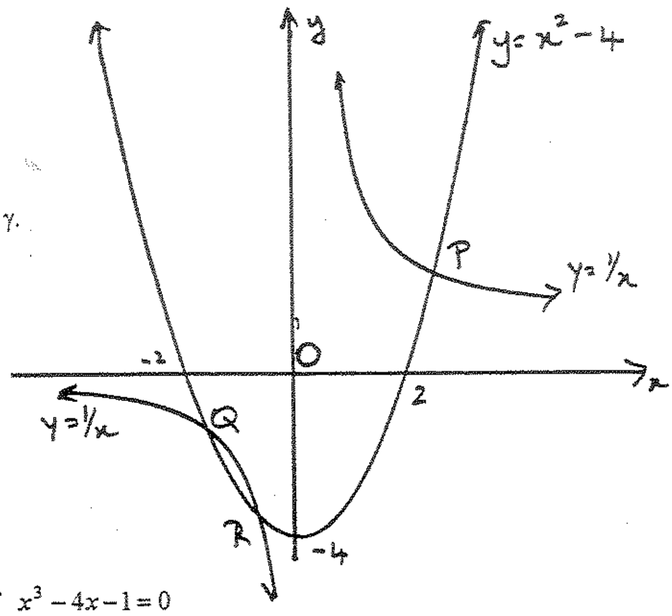
$$a^3 - 4ab + 8c = 0$$

3 (b) The polynomial $P(x) = x^3 + ax^2 + bx + 6$ where a and b are real numbers, has a zero of $1-i$.

Find a and b and express $P(x)$ as the product of two polynomials with real coefficients.

(c)

The curves $y = \dfrac{1}{x}$ and $y = x^2 - 4$ intersect at points P, Q, R as shown. P, Q and R have x-values α, β and γ. O is the origin.



1 (i) Show that α, β and γ are roots of $x^3 - 4x - 1 = 0$

2 (ii) Find a polynomial with numerical coefficients with roots α^2, β^2, and γ^2

3 (iii) Find an expression for $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2}$

3 (iv) Hence find the value of $OP^2 + OQ^2 + OR^2$

QUESTION 4:

(a) Given the hyperbola $9x^2 - 16y^2 = 144$ find

1 (i) the length of the major axis

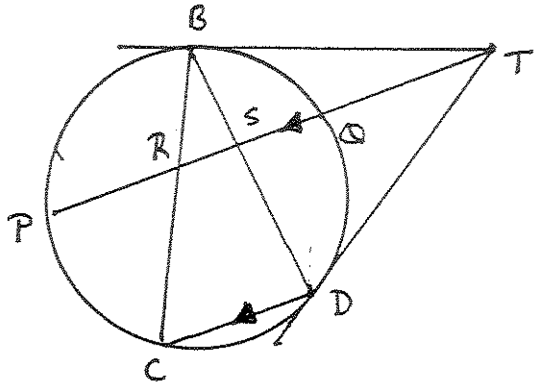
1 (ii) the eccentricity

1 (iii) the co-ordinates of the foci

1 (iv) the equations of the directrices

1 (v) the equations of the asymptotes

(b)



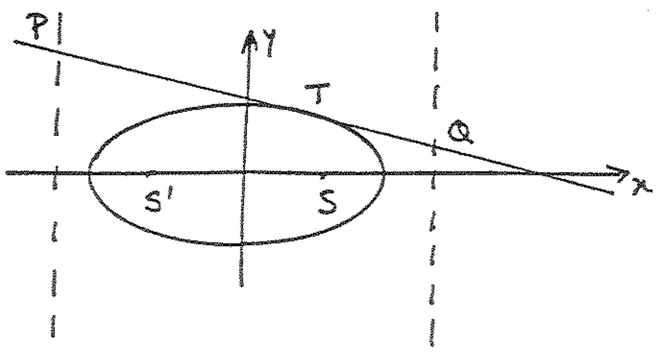
In the diagram at left, the chords PQ and [illegible] are parallel
The tangent at D cuts the chord PQ at T
The other point of contact from T is B and BC cuts PQ at R

(i) Copy the diagram onto your page

3 (ii) Prove that $\angle BDT = \angle BRT$ and state why B,T,D and R are concyclic

3 (iii) Show that ΔRCD is isosceles

4 (c)



The tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at the point $T(a\cos\theta, b\sin\theta)$ meets the directrices of the ellipse at P and Q.

S and S′ are the foci.

Show that $\angle TSQ = 90°$

QUESTION 5:

(a) Sketch, on separate axes, the following graphs, <u>showing all important features</u>
(DO NOT use Calculus)

2 (i) $y = \sin^2 x \qquad -2\pi \le x \le 2\pi$

2 (ii) $y = \ln(\frac{1}{x}) \qquad x > 0$

2 (iii) $y = \frac{\sin x}{x} \qquad x > 0$

2 (iv) y=max(x,1-x) where max(a,b) } = a when a≥b
} = b when a<b

2 (b) (i) Use De Moivre's Theorem to show that

$$(1+i\tan\theta)^n + (1-i\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}$$

where n is an integer

$(\cos\theta \ne 0)$

3 (ii) Use this result to show that the equation

$(1+z)^4 + (1-z)^4 = 0$ has roots of $\pm i\tan\frac{\pi}{8}, \quad \pm i\tan\frac{3\pi}{8}$

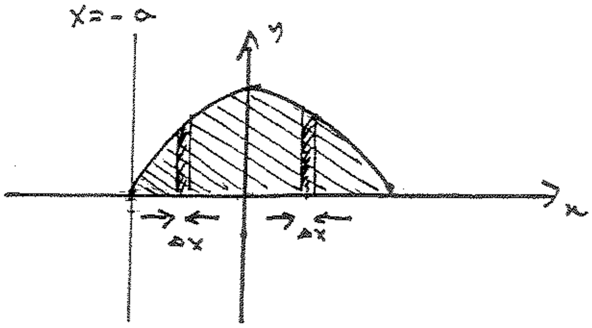
2 (iii) Hence, or otherwise, show that $\tan^2\frac{\pi}{8} = 3 - 2\sqrt{2}$

QUESTION 6:

5 (a) Find $\int_0^1 \sqrt{4-(1+x)^2}\,dx$

(b) The curve $y=f(x)$ is reflected in the y-axis to give the shape shown

The strips shown both have width Δx and are equidistant from the y-axis.

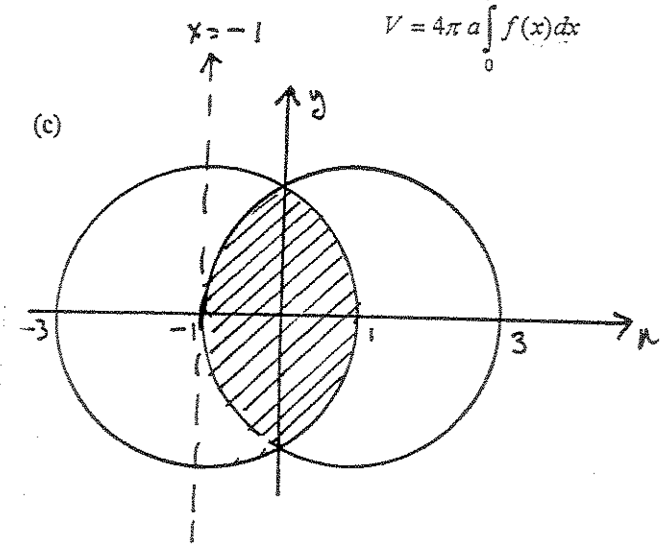


3 (i) The shaded area is rotated around the line ~~x=a~~ $x=-a$
Find each of the volumes of the two cylindrical shells as the <u>two strips</u> are rotated.
(Δx is small)

3 (ii) Show that the volume of the solid so formed is given by

$$V = 4\pi a\int_0^a f(x)\,dx$$

(c)



Two circles, centres (-1,0) and (1,0) and of radii 2 units have a common region as shown, and this region is rotated about ~~x=1~~ $x=-1$

2 (i) Show that the volume of the solid formed is given by

$$V = 8\pi\int_0^1 \sqrt{4-(x+1)^2}\,dx$$

2 (ii) By using your answer to part (a) of this question above, find the exact volume of the solid.

QUESTION 7:

(a) A particle moves in a straight line so that its distance from the origin at any time t is given by x and its velocity by v.

3 (i) The acceleration of the particle at a distance x is given by the equation

$$a = n^2(3-x) \quad \text{where } n \text{ is a constant.}$$

If the particle moves from rest from the origin (x=0), show that

$$\frac{1}{2}v^2 - n^2(3x - \frac{1}{2}x^2) = 0$$

2 (ii) Hence show that the particle never moves outside a certain interval and give that interval.

5 (b) (i) Let $I_n = \int_1^e x(\ln x)^n dx$ where n=0,1,2,3,..............

Using integration by parts, show that

$$I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1} \qquad n=1,2,3,\ldots\ldots\ldots.$$

5 (ii) The area bounded by the curve $y = \sqrt{x}(\ln x)$ $\quad x \geq 1$

the x-axis and the line x=e is rotated about the x-axis through 2π radians.

Find the exact value of the volume of the solid of revolution so formed.

QUESTION 8:

4 (a) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x}$ using the substitution $t=\tan\frac{x}{2}$

4 (b) A plane curve is defined by $x^2+2xy+y^5=4$

This curve has a horizontal tangent at the point $P(X,Y)$

By using Implicit Differentiation (or otherwise), show that X is the unique real root of

$$X^5+X^2+4=0$$

3 (c) (i) If $x_1>1$ and $x_2>1$ show that $x_1+x_2>\sqrt{x_1x_2}$

4 (ii) Use the Principle of Mathematical Induction to show that, for $n\geq 2$, if $x_j>1$ where $j=1,2,3,\ldots,n$ then

$$\ln(x_1+x_2+\ldots\ldots+x_n)>\frac{1}{2^{n-1}}(\ln x_1+\ln x_2+\ldots\ldots+\ln x_n)$$

STHS. 2001 EXT 2 TRIAL HSC - SOLUTIONS & COMMENTS.

QUESTION 1

a) i) $\int \frac{dx}{(x+1)^2+4} = \frac{1}{2}\tan^{-1}\frac{x+1}{2} + k$ ✓

i) All students OK on this

ii) $\int_0^1 (x+1)^{-3/2}\,dx = \left[-2(x+1)^{-\frac{1}{2}}\right]_0^1$ ✓

$= 2-\sqrt{2}$ ✓

Many students unable to do the arithmetic to get correct answer. Need to write it out in detail - not carry signs in their head.

b) $\frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}$

Cancel common factor

$= \frac{\cancel{\sec x}\,(\tan x + \sec x)}{(\tan x + \sec x)} = \sec x$ ✓

$\int \sec x\,dx = \int \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}\,dx$

Numerator is derivative of denominator

$= \ln(\sec x + \tan x) + k.$ ✓

c) i) $\int_0^1 x e^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx$ ✓

$= \left[-xe^{-x} - e^{-x}\right]_0^1$

$= \left[-\frac{1}{e} - \frac{1}{e}\right] - \left[0 - 1\right]$

$= 1 - \frac{2}{e}$ ✓

• must put terminals on $-xe^{-x}$

• Again, many students lost track of minus signs here

← Set it out properly

i) $\frac{5}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4}$

$= \frac{ax^2+4a+bx^2+cx+bx+c}{(x+1)(x^2+4)}$

← Setting up correct numerators is basic (but important)

$\therefore\ 4a+c = 5$

$\left.\begin{array}{l} a+b=0 \\ b+c=0 \end{array}\right\}\ \begin{array}{l} a=-b \\ c=-b \end{array}$

$\therefore a = c$

$\therefore 4a + a = 5$

$\therefore a = 1$

$\therefore c = 1$

$\therefore b = -1$

Q1 c) (ii) (cont)

$$\therefore \int \frac{5dx}{(x+1)(x^2+4)} = \int \frac{1}{x+1} + \frac{1-x}{(x^2+4)} \; dx \checkmark$$

$$= \int \frac{dx}{x+1} + \int \frac{dx}{x^2+4} - \frac{1}{2}\int \frac{2x}{x^2+4}dx$$

$$= \ln(x+1) + \frac{1}{2}\tan^{-1}\frac{x}{2} - \frac{1}{2}\ln(x^2+4) + c$$

$\checkmark \quad \checkmark \quad \checkmark$

This question generally handled well

1) $$\int_k^1 \frac{dx}{x(x+1)} = \int_k^1 \frac{1}{x} - \frac{1}{x+1} \; dx \checkmark$$

Most students could generate these partial fractions easily.

$$= \Big[\ln x - \ln x+1\Big]_k^1$$

$$= (\ln 1 - \ln 2) - (\ln k - \ln(k+1))$$

$$= \ln(k+1) - \ln k - \ln 2 \checkmark$$

Many forms of answer possible eg. $\left[\ln\frac{(k+1)}{k} - \ln 2\right]$ Hint: look below at next part of question & leave $\ln 2$ separate.

$$\text{Now } \sum_1^n \int_k^1 \frac{dx}{x(x+1)} = \sum_{k=1}^n \left(\ln\frac{k+1}{k} - \ln 2\right)$$

$$= \ln\frac{2}{1} - \ln 2 + \ln\frac{3}{2} - \ln 2 + \cdots + \ln\frac{n}{n-1} - \ln 2 + \ln\frac{n+1}{n} - \ln 2 \checkmark$$

*Write out a few terms of the series, 1st, 2nd ... penultimate, last. Look for the pattern.

$$= \left(\ln 2 + \ln\frac{3}{2} + \cdots + \ln\frac{n}{n-1} + \ln\frac{n+1}{n}\right) - n\ln 2$$

Intermediate steps must be shown

$$= \ln\left(\cancel{2}\cdot\frac{\cancel{3}}{\cancel{2}}\cdot\frac{4}{\cancel{3}}\cdot \cdots \cdot\frac{\cancel{n}}{\cancel{n-1}}\cdot\frac{n+1}{\cancel{n}}\right) - n\ln 2 \checkmark$$

Show cancelling

$$= \ln(n+1) - n\ln 2$$

No marks for last line

QUESTION 2:

(a) $z = 3 - 4i$

(i) $\bar{z} = 3 + 4i$ (ii) $|z| = 5$ (iii) $\arg z = \tan^{-1}\left(-\frac{4}{3}\right)$
$= -53^\circ 8'$

(iv) Let $a + ib = \sqrt{3-4i}$

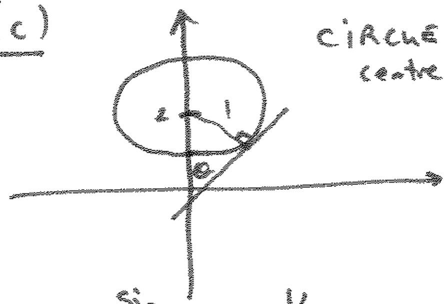
$$a^2 - b^2 = 3$$
$$2ab = -4$$
$$b = -\frac{2}{a}$$
$$a^2 - \frac{4}{a^2} = 3$$
$$a^4 - 4 = 3a^2$$
$$(a^2+1)(a^2-4) = 0$$
$$a = \pm 2 \text{ or } a = \pm i$$
$$\therefore \quad b = \mp 1$$
$$\therefore \quad \sqrt{z} = \pm(2 - i)$$

(b) $\sqrt{x^2 + (y-1)^2} = y$

$$x^2 + y^2 - 2y + 1 = y^2$$
$$x^2 - 2y + 1 = 0$$
$$y = \tfrac{1}{2}(x^2+1)$$

A parabola, vertex $(0, \frac{1}{2})$

(c)



CIRCLE
centre $(0,2)$ $r = 1$

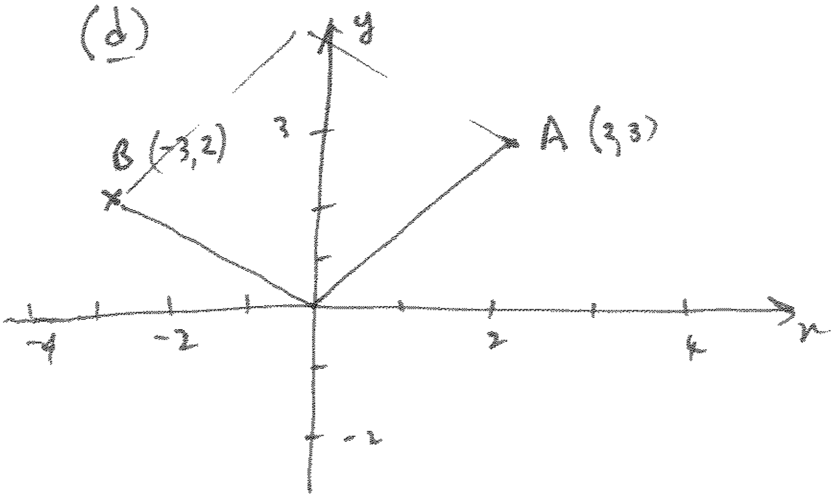
$\therefore \sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

$\therefore$ Smallest angle $= \frac{\pi}{2} - \frac{\pi}{6}$
$= \frac{\pi}{3}$

note that the tangent to the circle gives the most extreme point of the circle. ie. the least value of $\arg z$ is the angle the tangent makes with x-axis

(d)



By inspection C is $(-1, 5)$

Question 3.

(a) $x^3+ax^2+bx+c=0$

Let the roots be $\alpha, \beta, \alpha+\beta$.

Sum $\alpha+\beta+\alpha+\beta=-a$

$\alpha+\beta=-a/2$.

Product (x2) $\alpha\beta+\alpha(\alpha+\beta)+\beta(\alpha+\beta)=b$

$\therefore \alpha\beta+(\alpha+\beta)(\alpha+\beta)=b$

$\therefore \alpha\beta+a^2/4=b \Rightarrow \alpha\beta=b-a^2/4$

Product $\alpha\beta(\alpha+\beta)=-c$

$\therefore (b-a^2/4)(-a/2)=-c$

$\therefore -ab/2+a^3/8=-c$

$\therefore a^3-4ab+8c=0$

Alternatively

$\therefore \gamma=-a/2$.

Now γ is a root so

$P(\gamma)=0$

$\therefore (-a/2)^3+a(-a/2)^2+b(-a/2)+c=0$

$\therefore -a^3/8+a^3/4-\frac{ab}{2}+c=0$

$\therefore -a^3+2a^3-4ab+8c=0$

$\therefore a^3-4ab+8c=0$

(b) $P(x)=x^3+ax^2+bx+6$

$1-i$ is a root.

$\therefore$ So is $1+i$

$\therefore (x-(1-i))(x-(1+i))$ is a factor.

$\therefore (x-1+i)(x-1-i)$ " "

ie. $(x-1)^2-i^2$ " "

$\therefore P(x)=x^3+ax^2+bx+6=(x^2-2x+2)Q(x)$

By inspection $Q(x)=(x+3)$

and so $a=+1$

$b=-4$

and $P(x)=(x^2-2x+2)(x+3)$

Alternatively

Product of roots

$\Rightarrow (1-i)(1+i)\alpha=-6$

$\therefore \alpha=-3$ (*)

Sum of Roots $=-a$

$\therefore a=1$.

Product (x2)

$= 2-3+3i-3-3i=b$

$b=-4$.

Result (*) gives $P(x)=(x+3)Q(x)$

division giving $Q(x)$ as

x^2-2x+2

ie. $P(x)=(x^2-2x+2)(x+3)$

Some very "shoddy" proofs here. The most popular was to find $P(1-i)$ and $P(1+i)$ and solve simultaneously. The other was to perform a long division using $x-1+i$. These methods show little appreciation of Polynomial Theory outside the 2/3 unit factor theorems. Chances are, in 4 unit, we are always going to use Sums of Roots, etc...

(c) (i) $y = 1/x$ and $y = x^2 - 4$

$1/x = x^2 - 4$

$x^3 - 4x - 1 = 0$ Comment: Easy mark

(ii) In above, $\alpha + \beta + \gamma = 0$

$\alpha\beta + \beta\gamma + \alpha\gamma = -4$

$\alpha\beta\gamma = 1$

This is the more complicated method but the easiest to understand when in doubt, it's this easier than half-knowing a formula

Sum of new roots) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= 0 + 8 = 8$.

Product of new roots ×2) $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 = (\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$

$= 16 - 2.1.0$

$= 16$.

Product of new roots) $\alpha^2\beta^2\gamma^2 = 1$

$\therefore$ New polynomial is $x^3 - 8x^2 + 16x - 1 = 0$ *

ALTERNATIVELY (if you know it)

Let $y = x^2$ $\therefore \sqrt{y} = x$.

$\therefore x^3 - 4x - 1 = 0$ becomes

$(\sqrt{y})^3 - 4(\sqrt{y}) - 1 = 0$

ie. $y^{3/2} - 4y^{1/2} = 1$

A lot of people "half" know this method. A lot left the polynomial as this line, but this is NOT a polynomial (powers of x are not integral)

Squaring both sides gives,

$y^3 + 16y - 8y^2 = 1$

ie. $y^3 - 8y^2 + 16y - 1 = 0$ *

(iii) $1/\alpha^2 + 1/\beta^2 + 1/\gamma^2 = \dfrac{\beta^2\gamma^2 + \beta^2\alpha^2 + \alpha^2\gamma^2}{\alpha^2\beta^2\gamma^2}$

$= 16/1$ if you used method I above.

OR, from the polynomial marked (*) above,

$= \dfrac{\text{Sum of roots taken 2 at a time}}{\text{Product of roots}}$

$= +16/-(-1) = 16$

(iv) $OP^2 = (\alpha - 0)^2 + (1/\alpha - 0)^2$ by distance formula
$= \alpha^2 + 1/\alpha^2$

Similarly, $OQ^2 = \beta^2 + 1/\beta^2$ and $OR^2 = \gamma^2 + 1/\gamma^2$.

Using previous 2 answers

$OP^2 + OQ^2 + OR^2 = \alpha^2 + \beta^2 + \gamma^2 + (1/\alpha^2 + 1/\beta^2 + 1/\gamma^2)$
$= 8 + 16 = 24$

QUESTION 4:

(a) $9x^2 - 16y^2 = 144$

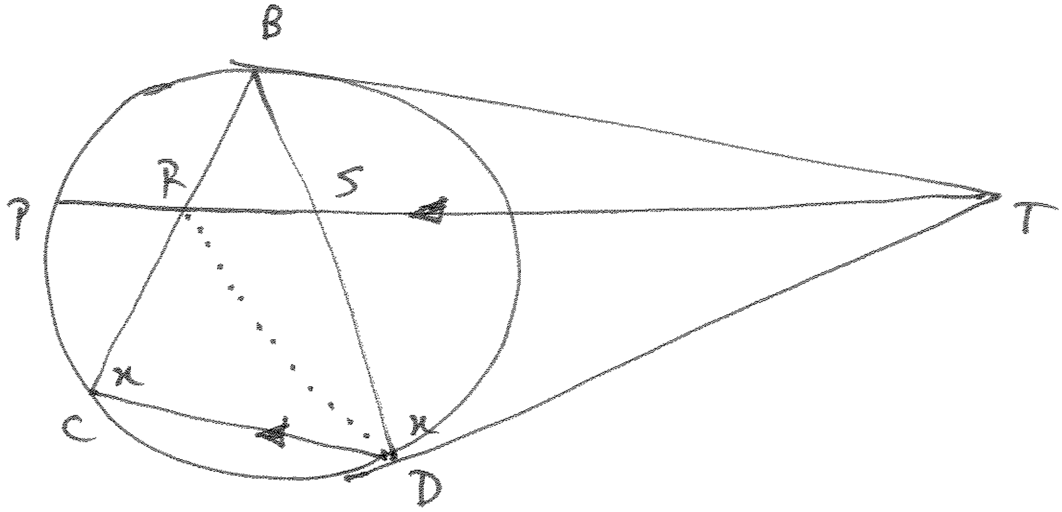
$\frac{x^2}{16} - \frac{y^2}{9} = 1$

$a = 4 \quad b = 3$.

(i) 8 (ii) $\frac{5}{4}$ (iii) Foci are $(\pm 5, 0)$ (iv) $\mathscr{D}$: $x = \pm \frac{16}{5}$

(v) Assymptotes are $y = \pm \frac{3x}{4}$

(b)



(i) Let $\angle BDT = x°$

$\therefore \angle BCD = x°$ (angle in the alternate segment is the same as angle made by tangent striking a chord DB)

$\angle BCD = \angle BRT = x°$ (corresponding angles, $PT \parallel CD$)

(ii) Since. $\angle BRT = \angle BDT$ they can be considered as angles standing on arc BT. ie. Circle goes through B, T, D, R

(iii) $\angle TBD = x°$ (tangents striking a circle make the same angle with the chord of contact) OR (use alt seg theorem with $\angle BCD$.)

$\angle TBD = \angle TRD = x°$ (angles on circumference on arc TD of circle touching B, T, D, R)

$\angle TRD = \angle RDC = x°$ (alternate angles, $PT \parallel CD$)

$\therefore \angle BCD = \angle RDC = x°$

$\therefore \triangle RCD$ is isosceles

(c) $\frac{dy}{dx} = -\frac{2x}{a^2} \times \frac{b^2}{2y}$

$= -\frac{b^2}{a^2} \cdot \frac{x}{y}$

At $T(a\cos\theta, b\sin\theta)$

$m_T = -\frac{b^2}{a^2} \frac{a\cos\theta}{b\sin\theta}$

$= \frac{-b\cos\theta}{a\sin\theta}$

Tangent is:

$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$\therefore ay\sin\theta - ab\sin^2\theta$

$= -bx\cos\theta + ab\cos^2\theta$

$\therefore ay\sin\theta + bx\cos\theta = ab$

OR $\frac{y\sin\theta}{b} + \frac{x\cos\theta}{a} = 1$

At Q $x = + a/e$

$\therefore \frac{y\sin\theta}{b} + \frac{a/e \cos\theta}{a} = 1$

$\therefore y\sin\theta/b = 1 - \frac{a/e\cos\theta}{a}$

$\therefore y\sin\theta = \frac{ab - \frac{ab\cos\theta}{e}}{a}$

$= \frac{b(e-\cos\theta)}{e}$

$\therefore y = \frac{b(e-\cos\theta)}{e\sin\theta}$ $\therefore Q$ is $\left(a/e, \frac{b(e-\cos\theta)}{e\sin\theta}\right)$

$m_{TS} = \frac{b\sin\theta}{a\cos\theta - ae}$

$m_{SQ} = \frac{b(e-\cos\theta)}{e\sin\theta} \Big/ a/e - ae$

$= \frac{eb(e-\cos\theta)}{e\sin\theta} \Big/ a - ae^2$ (MULTIPLYING TOP & BOTTOM of $\frac{a}{e}$ by e)

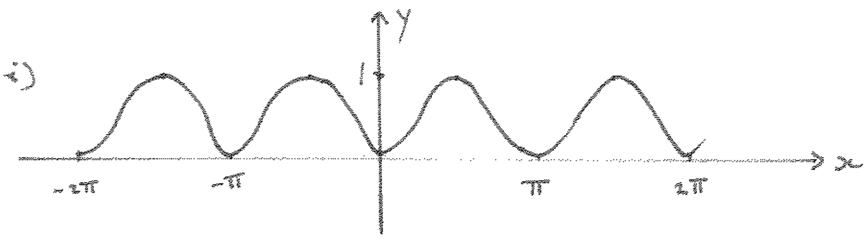
$= \frac{b(e-\cos\theta)}{\sin\theta} \Big/ a - ae^2$

$m_{TS} \times m_{SQ} = \frac{b\sin\theta}{a(\cos\theta - e)} \times \frac{b(e-\cos\theta)}{\sin\theta} \times \frac{1}{a - ae^2}$

$= \frac{b^2(e-\cos\theta)}{a^2(\cos\theta - e)(1-e^2)} = -1$ because $a^2(1-e^2) = b^2$

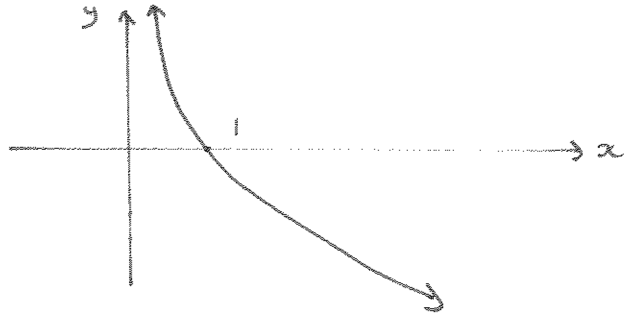
QUESTION 5

a) i)

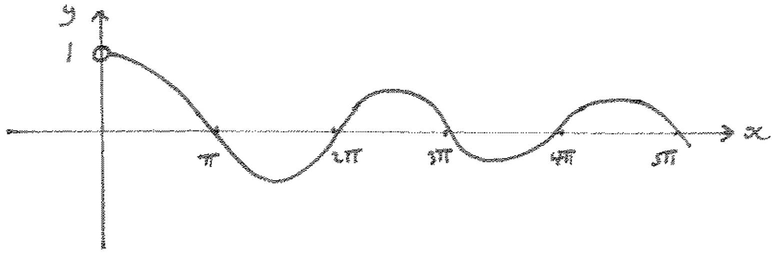


question clearly said to show all important features – x intercepts, etc.
curves should be smooth except for sharp corner in (iv).

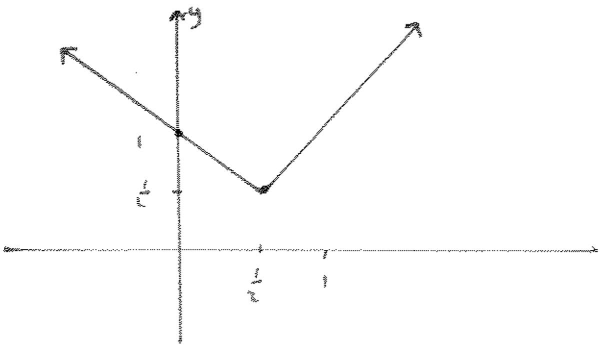
ii)



iii)



iv)



b) i)

$$LHS = (1 + i\tan\theta)^n + (1 - i\tan\theta)^n$$

$$= \left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n + \left(\frac{\cos\theta - i\sin\theta}{\cos\theta}\right)^n$$

$$= \frac{\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta}{\cos^n\theta}$$

$$= \frac{2\cos n\theta}{\cos^n\theta}$$

$$= RHS$$

make sure each step clearly follows on from previous step.

ii) $(1+z)^4 + (1-z)^4 = 0$

let $z = i\tan\theta$

$\therefore (1+i\tan\theta)^4 + (1-i\tan\theta)^4 = 0$

$\therefore \dfrac{2\cos 4\theta}{\cos^4\theta} = 0$ from (i)

$\therefore \cos 4\theta = 0$

$\therefore 4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}, -\frac{3\pi}{2}, \ldots$

$\theta = \pm\frac{\pi}{8}, \pm\frac{3\pi}{8}$

$\therefore z = i\tan\left(\pm\frac{\pi}{8}\right), i\tan\left(\pm\frac{3\pi}{8}\right)$

$z = \pm i\tan\frac{\pi}{8}, \pm i\tan\frac{3\pi}{8}$

iii) $(1+z)^4 + (1-z)^4 = 0$

$1 + 4z + 6z^2 + 4z^3 + z^4 + 1 - 4z + 6z^2 - 4z^3 + z^4 = 0$

$2 + 12z^2 + 2z^4 = 0$

$z^4 + 6z^2 + 1 = 0$

$z^2 = \dfrac{-6 \pm \sqrt{32}}{2}$

$= -3 \pm \sqrt{8}$

very few got this far

$\therefore \left(\pm i\tan\frac{\pi}{8}\right)^2 = -3 + \sqrt{8}$ from part ii)

$-\tan\frac{\pi}{8} = -3 + 2\sqrt{2}$

$\tan\frac{\pi}{8} = 3 - 2\sqrt{2}$

QUESTION 6:

(a) $\int\sqrt{4-(1+x)^2}\,dx$.

Let $1+x = 2\sin\theta$ $\quad\begin{cases} x=0 & \theta=\pi/6 \\ x=1 & \theta=\pi/2\end{cases}$

$dx/d\theta = 2\cos\theta$

$\Rightarrow \quad dx = 2\cos\theta\, d\theta$

$\therefore \int_0^1 \sqrt{4-(1+x)^2}\,dx = \int_{\pi/6}^{\pi/2} 2\sqrt{1-\sin^2\theta}\; 2\cos\theta\, d\theta$

$= 4\int_{\pi/6}^{\pi/2} \cos\theta\cos\theta\, d\theta$

$= 4\int_{\pi/6}^{\pi/2} \frac{\cos 2\theta + 1}{2}\, d\theta$

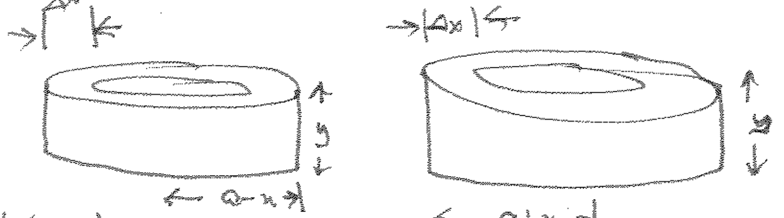
$= 2\left[\frac{1}{2}\sin 2\theta + \theta\right]_{\pi/6}^{\pi/2}$

$= (\sin\pi + \pi) - (\sin\pi/3 + \pi/3)$

$= 2\pi/3 - \frac{\sqrt{3}}{2}$

If you can do all sorts of integration (and nothing else) you can go very close to passing 4 unit.

(b)



Volume 1 $= 2\pi(a-x)y\Delta x$ $\qquad$ $VOL_2 = 2\pi(a+x)y\Delta x$

Note: x is a variable! In these 2 cases x is different and will travel over different limits. The Question asked for each of the volumes.

(ii) VOL of the RHS $= 2\pi\int_0^a (a+x)\,y\,dx$

VOL of the LHS $= 2\pi\int_{-a}^{0} (a-x)\,y\,dx$

(changing limits around) $= -2\pi\int_0^{-a}(a-x)y\,dx$

This is true because $f(x)$ is an even function $= 2\pi\int_0^a (a-x)y\,dx$

$\therefore VOL = 2\pi\int_0^a \big((a-x)+(a+x)\big)y\,dx$

$= 4a\pi\int_0^a y\,dx$

NOTE the different limits here. Most students now just added these 2 integrals and (surreptitiously) ignored the limits. You need to explain how this can be done.

(c) (i) Comparing this diagram with the first.

— $f(x) = \sqrt{4-(x+1)^2}$ (since $y^2+(x+1)^2 = 4$ and we are using $f(x)$ as the RHS in the first diagram

— $a = 1$

— The second diagram will have twice the volume of the first (due to part below the x-axis)

not many students saw this subtle point and "fudged" the sum of the integrals.

$\therefore$ From part (b),

$$V = 2 \times \left[4\pi(1)\int_0^1 \sqrt{4-(x+1)^2}\,dx\right]$$

$$= 8\pi \int_0^1 \sqrt{4-(x+1)^2}\,dx$$

a lot of people proved the whole formula again here. Use part (b).

(ii) From part (a), $\int_0^1 \sqrt{4-(x+1)^2}\,dx$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

$$\therefore \text{VOL} = 8\pi\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$= \frac{16\pi^2}{3} - 4\pi\sqrt{3}.$$

QUESTION 7

a) i) $a = n^2(3 - x)$

$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = n^2(3-x)$

$\frac{1}{2}v^2 = n^2\left(3x - \frac{1}{2}x^2 + c\right)$

when $x = 0$ $v = 0$ $\Rightarrow$ $c = 0$

$\therefore \frac{1}{2}v^2 = n^2\left(3x - \frac{1}{2}x^2\right)$

$\therefore \frac{1}{2}v^2 - n^2\left(3x - \frac{1}{2}x^2\right) = 0$

need to evaluate $c=0$.

ii) $3x - \frac{1}{2}x^2 \geq 0$ as $\frac{1}{2}v^2 \geq 0$

$6x - x^2 \geq 0$

$0 \leq x \leq 6$

b) i) $I_n = \int_1^e x(\ln x)^n \, dx$ $u = (\ln x)^n$ $v = \frac{1}{2}x^2$

$u' = \frac{n(\ln x)^{n-1}}{x}$ $v' = x$

$= \left[\frac{1}{2}x^2(\ln x)^n\right]_1^e - \frac{1}{2}\int_1^e x^2 \cdot \frac{n(\ln x)^{n-1}}{x} \, dx$

$= \left[\frac{e^2}{2} - 0\right] - \frac{n}{2}\int_1^e x(\ln x)^{n-1} \, dx$

$= \frac{e^2}{2} - \frac{n}{2} I_{n-1}$

limits and dx should be written where appropriate

ii) Volume $= \pi \int_1^e x(\ln x)^2 \, dx$

$= \pi\left(\frac{e^2}{2} - I_1\right)$

$= \pi\left(\frac{e^2}{2} - \left(\frac{e^2}{2} - \frac{1}{2} I_0\right)\right)$

$= \pi\left(\frac{e^2}{2} - \left(\frac{e^2}{2} - \frac{1}{2}\left(\frac{e^2}{2} - \frac{1}{2}\right)\right)\right)$

$= \frac{\pi}{4}\left(e^2 - 1\right)$ cubic units

a lot of careless errors here.

QUESTION 8

a) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\cos x+\sin x}$ where $t = \tan\frac{x}{2}$

Now $dx = \dfrac{2\,dt}{1+t^2}$ *

$\cos x = \dfrac{1-t^2}{1+t^2}$

$\sin x = \dfrac{2t}{1+t^2}$

LEARN THESE (incl. $\underline{\underline{dx}}$) to save time.

Integral becomes $\displaystyle\int_0^1 \frac{\frac{2\,dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}$ ✓ ✓

$$= \int_0^1 \frac{\frac{2\,dt}{1+t^2} \times (1+t^2)}{\left(1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}\right)\times(1+t^2)}$$

Learn how to simplify compound fractions. Hint: best method is to multiply top & bottom by highest denominator. $(1+t^2)$ in this case

$$= \int_0^1 \frac{2\,dt}{1+t^2+1-t^2+2t}$$

$$= \int_0^1 \frac{2\,dt}{2+2t}$$

$$= \int_0^1 \frac{dt}{1+t} = \Big[\ln(1+t)\Big]_0^1$$

$$= \ln 2 \quad ✓$$

b) Now $x^2 + 2xy + y^5 = 4$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(2xy) + \frac{d}{dx}(y^5) = \frac{d}{dx}(4)$$

Spend time to do this step. This $\underline{\text{is}}$ implicit differentiation

ie $2x + \underline{2y + 2x\frac{dy}{dx}} + 5y^4\cdot\frac{dy}{dx} = 0$ ✓

Use product rule for $2xy$. Don't forget RHS $= 0$

$$\therefore \frac{dy}{dx} = \frac{-2(x+y)}{2x+5y^4}$$

$= 0$ when $x = X$, $y = Y$ ✓

Horizontal tangent (stat. point)

$\therefore\ -2(x+y) = 0$

$\therefore\ x = -y$

or $y = -x$ ✓

Substituting into original equation gives

$x^2 + 2x(-x) + (-x)^5 = 4$ ✓

ie $x^2 - 2x^2 - x^5 = 4$

or $x^2 + x^5 + 4 = 0$ as req^d.

c)(i) $x_1 + x_2 > \sqrt{x_1 x_2}$

Now $(\sqrt{x_1} - \sqrt{x_2})^2 > 0$ ✓

ie $x_1 - 2\sqrt{x_1 x_2} + x_2 > 0$ ✓

ie $x_1 + x_2 > 2\sqrt{x_1 x_2}$ ✓

$\therefore\ x_1 + x_2 > \sqrt{x_1 x_2}$

<u>OR</u>

Now if $x_1 + x_2 > \sqrt{x_1 x_2}$, by squaring both sides we obtain

$x_1^2 + 2x_1x_2 + x_2^2 > x_1 x_2$

ie $x_1^2 + x_1x_2 + x_2^2 > 0$ ✓ which must be true since both x_1 & x_2 are greater than 1. ✓

$\therefore$ the original statement $x_1 + x_2 > \sqrt{x_1 x_2}$ must have been true. ✓

i) To prove $\ln(x_1 + x_2 \cdots x_n) > \frac{1}{2^{n-1}}(\ln x_1 + \ln x_2 + \cdots + \ln x_n)$

for $n \geq 2$.

Now when $n = 2$, we know

$x_1 + x_2 > \sqrt{x_1 x_2}$ from above

$\therefore\ \ln(x_1 + x_2) > \ln\sqrt{x_1 x_2}$ since $x_1 + x_2$ & x_1x_2 are both > 1.

<u>NEVER</u> START WITH THE STATEMENT YOU ARE REQUIRED TO <u>PROVE</u>

You cannot get full marks by working on the statement you have to prove (unless you are very clever).

* Use the word "Now" to signal the 1st line of your argument. (ALWAYS!)
* use "ie" to write the same thing but in a different form
* "$\therefore$" means something new based on what came before.

* If you try to do the 2nd version of this proof and don't use the words to explain your reasoning you WILL LOSE MARKS!

ie $\ln(x_1+x_2) > \ln(x_1x_2)^{\frac{1}{2}}$

ie $\ln(x_1+x_2) > \frac{1}{2}\ln(x_1x_2)$

ie $\ln(x_1+x_2) > \frac{1}{2}(\ln x_1 + \ln x_2)$

as req^d

$\therefore$ true for $n=2$. ✓

Assume statement true when $n=k$

ie

$\ln(x_1+x_2+\cdots+x_k) > \frac{1}{2^{k-1}}(\ln x_1 + \cdots + \ln x_k)$

when $n=k+1$,

LHS $= \ln(x_1+x_2+\cdots x_k + x_{k+1})$

$> \frac{1}{2}\left(\ln(x_1+x_2+\cdots+x_k) + \ln x_{k+1}\right)$ ✓

from result proved for $n=2$.

$> \frac{1}{2}\left(\frac{1}{2^{k-1}}(\ln x_1 + \ln x_2 + \cdots + \ln x_k) + \ln x_{k+1}\right)$

$> \frac{1}{2}\left(\qquad " \qquad + \frac{1}{2^{k-1}}\ln x_{k+1}\right)$

since $\frac{1}{2^{k-1}} < 1$ ✓

$= \frac{1}{2^k}\left(\ln x_1 + \ln x_2 + \cdots + \ln x_k + \ln x_{k+1}\right)$

which is correct form for RHS when $n=k+1$.

$\therefore$ By theory of Mathematical Induction the statement is true for all $n \geq 2$.

Proof for $n=2$ must refer back to statement proved in (i)

DO NOT WRITE THE STATEMENT YOU'RE TRYING TO PROVE AND THEN JUST WORK ON IT. MUST USE LHS = RHS = etc.

$\Leftarrow$ using the assumption. This must be used somewhere in your proof.

* Many students invented their own log laws!

eg $\ln(x_1+x_2+x_3)$
$= \ln(x_1+x_2) \cdot \ln x_3$ (try with $x_3=1$)

* Note correct use of $=$ & $>$, each refers to line above.

* Don't waste time in a lengthy conclusion. No marks for it (usually)!