Q1 a) $\frac{x+1}{x} \geq 2$, $x \neq 0$

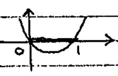
$x^2 \times \frac{(x+1)}{x} \geq 2x^2$ ①

$x(x+1) \geq 2x^2$

$-x^2 + x \geq 0$

$-x(x-1) \geq 0$

$x(x-1) \leq 0$ ①



Note $x \neq 0$

$\therefore 0 < x \leq 1$ ①

b) Line 1: $x + 3y = 4$

$3y = 4 - x$

$y = \frac{4}{3} - \frac{x}{3}$

$\therefore m_1 = -\frac{1}{3}$

Line 2: $2x - 5y = 0$

$5y = 2x$

$y = \frac{2}{5}x$

$\therefore m_2 = \frac{2}{5}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{(-\frac{1}{3}) - \frac{2}{5}}{1 + (-\frac{1}{3})\frac{2}{5}}\right|$ ①

$= \left|\frac{\frac{-5-6}{15}}{1 - \frac{2}{15}}\right|$

$= \left|-\frac{11}{13}\right|$

$\tan\theta = \frac{11}{13}$ ①

$\therefore \theta = 40.236\ldots°$ ①

$\therefore$ acute angle is $40°$ (nearest degree)

c) $\sqrt{3}\cos x - \sin x = R\cos(x + \theta)$

Since $R\cos(x+\theta)$

$= R\cos x\cos\theta - R\sin x\sin\theta$

$R\cos\theta = \sqrt{3}$ and $R\sin\theta = 1$

$\therefore \frac{R\sin\theta}{R\cos\theta} = \frac{1}{\sqrt{3}}$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = 30°$ ①

$R^2\sin^2\theta + R^2\cos^2\theta = 1 + 3$

$R^2 = 4$

$R = 2$ ①

OR

$R = \sqrt{(\sqrt{3})^2 + 1^2}$

$= \sqrt{4}$

$= 2$ ①

$\therefore \sqrt{3}\cos x - \sin x$

$= 2\cos(x + \theta)$

$\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$

$= \cos x\cos\theta - \sin x\sin\theta$

$\therefore \cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$

$\Rightarrow \theta = 30°$ ①

d) $\int_0^1 \frac{2x\,dx}{(2x+1)^2}$ $u = 2x + 1$

$\frac{du}{dx} = 2$

$\frac{du}{2} = dx$ ($\frac{1}{2}$)

When $x = 0$, $u = 1$

$x = 1$, $u = 3$ ($\frac{1}{2}$)

Also $2x = u - 1$

Q1 d) (cont)

$\therefore \int_1^3 \frac{u-1}{u^2}\,\frac{du}{2}$ ①

$= \frac{1}{2}\int_1^3 \left(\frac{u}{u^2} - \frac{1}{u^2}\right) du$

$= \frac{1}{2}\int_1^3 \left(\frac{1}{u} - \frac{1}{u^2}\right) du$

$= \frac{1}{2}\left[\ln u + \frac{1}{u}\right]_1^3$ ①

$= \frac{1}{2}\left[(\ln 3 + \frac{1}{3}) - (\ln 1 + \frac{1}{1})\right]$

$= \frac{1}{2}(\ln 3 + \frac{1}{3} - 0 - 1)$

$= \frac{1}{2}(\ln 3 - \frac{2}{3})$ ①

Question 2

a) Let $P(x) = x^3 + rx^2 - 4x + s$

also $x^2 + x - 2 = (x+2)(x-1)$

Since $x^2 + x - 2$ is a factor of $P(x)$, $P(-2) = 0$ and $P(1) = 0$

i) Since $P(1) = 0$

$1^3 + r \times 1^2 - 4 \times 1 + s = 0$

$r + s = 3$ Ⓐ ①

ii) Since $P(-2) = 0$

$(-2)^3 + r(-2)^2 - 4(-2) + s = 0$

$-8 + 4r + 8 + s = 0$

$4r + s = 0$ Ⓑ ①

$s = -4r$

Sub. Ⓑ into Ⓐ

$r + (-4r) = 3$

$-3r = 3$

$r = -1$

$\therefore s = 4$ ①

b) i) A geometric series $a + ar + ar^2 + \cdots$ has a limiting sum when $|r| < 1$ ①

ii) For the series $1 - \tan^2 x + \tan^4 x + \ldots$

$r = -\tan^2 x$

$\therefore$ if $|r| < 1$

$|-\tan^2 x| < 1$

$\tan^2 x < 1$

But $\tan^2 x \geq 0$, and $0 < x < \frac{\pi}{2}$

$\therefore 0 < x < \frac{\pi}{4}$ ①

iii) $S_\infty = \frac{a}{1-r}$

$= \frac{1}{1 + \tan^2 x}$

$= \frac{1}{\sec^2 x}$

$= \cos^2 x$ ①

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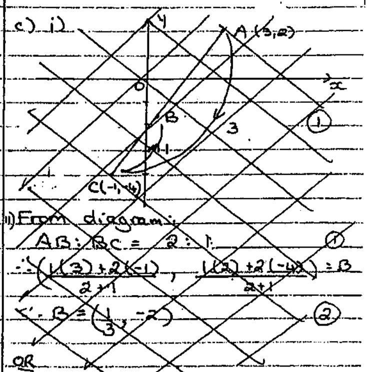
c) i)



①

ii) From diagram:

$AB : BC = 2 : 1$ ①

$\therefore \left(\frac{1(3) + 2(-1)}{2+1}, \frac{1(2) + 2(-4)}{2+1}\right) = B$

$\therefore B = \left(\frac{1}{3}, -2\right)$ ②

OR

Start a new page

Marks

the exact area enclosed between the curve $y = e^x$ and lines $x = 0$ and $x = 1$ — 5

d the volume of the solid of revolution formed by rotating the graph $y = e^x$ and the lines $x = 0$ and $x = 1$ about the x-axis. ve your answer in terms of π.

equation of the parabola whose focus is $(-1, -2)$ and directrix is — 2

le gives values for $f(x)$. — 3

x	1	1·2	1·4	1·6	1·8
$f(x)$	1·7	1·8	1·9	2·0	2·2

impson's rule to evaluate $\int_1^{1\cdot8} f(x)\ dx$, correct to 2 decimal places.

all real numbers which satisfy the equation : $x^4 = 72 - x^2$ — 2

$AC : CB = 3 : -1$

$A(3,2) \quad C(-1,-4)$

$m : n = 3 : -1$

If $B = (x, y)$

Then

$-1 = \frac{-1 \times 3 + 3 \times x}{3-1}$ ①

$-2 = -3 + 3x$

$\frac{1}{3} = x$

and

$-4 = \frac{-1 \times 2 + 3 \times y}{3-1}$

$-8 = 3y - 2$

$-6 = 3y$

$-2 = y$ ①

$\therefore B$ is $\left(\frac{1}{3}, -2\right)$

c) $\int_0^{\frac{\pi}{4}} \cos^2\left(\frac{1}{2}x\right) dx$

Since $\cos^2 A = \frac{1}{2}(\cos 2A + 1)$

$\cos^2 \frac{1}{2}x = \frac{1}{2}(\cos x + 1)$ ①

$\therefore \int_0^{\frac{\pi}{4}} \cos^2\left(\frac{1}{2}\right)x \, dx$

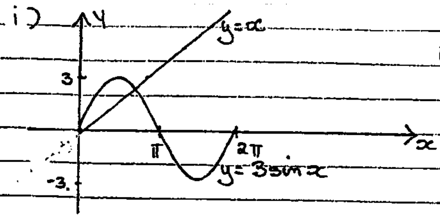
$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1 + \cos x) \, dx$

$= \frac{1}{2}\left[x + \sin x\right]_0^{\frac{\pi}{4}}$ ①

$= \frac{1}{2}\left(\frac{\pi}{4} + \sin\frac{\pi}{4}\right)$ ①

$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)$ ①

i)



①

ii) When $x = 2.2$

$3\sin x - x = 3\sin 2.2 - 2.2$

$= 0.2254\ldots$

When $x = 2.4$

$3\sin x - x = 3\sin 2.4 - 2.4$

$= -0.3736\ldots$

Since the sign of $3\sin x - x$ changes from $x = 2.2$ to $x = 2.4$, then the solution lies between $x = 2.2$ and $x = 2.4$. ①

iii) $f(x) = 3\sin x - x$

$f'(x) = 3\cos x - 1$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 2.3 - \frac{3\sin(2.3) - (2.3)}{3\cos(2.3) - 1}$ ①

$= 2.27903\ldots$

$= 2.279$ (3dp) ①

b) i) $\frac{d}{dx}(2x \tan^{-1} x)$

$= (\tan^{-1} x) \times 2 + 2x \times \frac{1}{1+x^2}$

$= 2\tan^{-1} x + \frac{2x}{1+x^2}$ ①

$\therefore 2\tan^{-1} x = \frac{d}{dx}(2x \tan^{-1} x) - \frac{2x}{1+x^2}$

Q3 b) (cont)

ii) $2\int_0^1 \tan^{-1} x \, dx$

$= \int_0^1 \frac{d}{dx}(2x \tan^{-1} x) \, dx$

$- \int_0^1 \frac{2x}{1+x^2} \, dx$ ①

$= \left[2x \tan^{-1} x\right]_0^1$

$- \left[\log_e(1+x^2)\right]_0^1$ ①

$= \left[2\tan^{-1} 1 - 0\right]$

$- \left[\log_e 2 - \log_e 1\right]$

$= 2 \times \frac{\pi}{4} - \log_e 2$

$\therefore \int_0^1 \tan^{-1} x \, dx$

$= \frac{\pi}{4} - \frac{1}{2}\log_e 2$ ①

c) Step 1: Let $n = 1$

LHS $= 1 \times 2 = 2$

RHS $= (1-1) \times 2^{2(1+1)} + 2$

$= 2$

$\therefore$ true when $n = 1$ ①

Step 2: Assume formula true when $n = k$

i.e.

$1 \times 2 + 2 \times 2^2 + \ldots + k \times 2^k$

$= (k-1) \times 2^{k+1} + 2$

Step 3: When $n = k+1$

$1 \times 2 + 2 \times 2^2 + \ldots + k \times 2^k + (k+1)2^{k+1}$

$= (k-1) \times 2^{k+1} + 2 + (k+1) \times 2^{k+1}$ ①

Q3(c) (cont)

$= 2^{k+1}(k - 1 + k + 1) + 2$

$= 2k \times 2^{k+1} + 2$

$= k \times 2^{k+2} + 2$

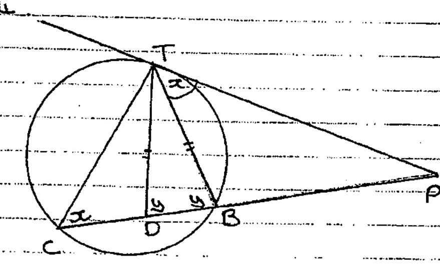
$= (k+1-1) \times 2^{(k+1+1)} + 2$ ①

$\therefore$ If true when $n = k$, then formula true when $n = k+1$

Step 4: But, formula is true when $n = 1$ $\therefore$ true when $n = 1+1$ or $n = 2$ $\therefore$ true when $n = 2+1$ or $n = 3$ etc

$\therefore$ Formula is true for all n ①

Q4



let $\angle TCB = x°$

In Δ TDB, $\angle D = \angle B = y°$ ①

(base angles of isosceles Δ)

Also $\angle PTB = \angle TCB$

$= x°$ ①

(Alternate segment theorem)

In Δ TCD

$\angle T + \angle C = \angle TDP$ (ext. $\angle$ of Δ Theorem) ①

$\therefore \angle CTD = y° - x°$

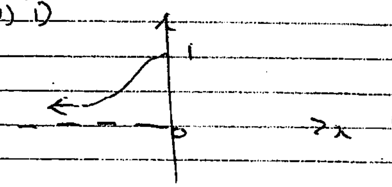
Similarly in Δ TBP, $\angle P = y° - x°$

$\therefore \angle CTD = \angle P$ ①

(87)

Q4 (cont)

b) i)



①

ii) For inverse:

$x = \frac{1}{1+y^2}$, $y \le 0$

$1+y^2 = \frac{1}{x}$ ①

$y^2 = \frac{1}{x} - 1$

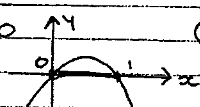
$y = \pm\sqrt{\frac{1}{x}-1}$, $y \le 0$

$\therefore y = -\sqrt{\frac{1-x}{x}}$ ①

iii) Domain of inverse function

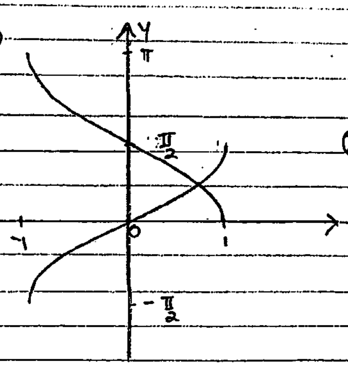
$\frac{1-x}{x} \ge 0$ and $x \ne 0$

$x(1-x) \ge 0$ ①



$0 < x \le 1$

c) i)



②

Q4 c) (cont)

ii) By adding ordinates at some key points on the graph and by noting the symmetry of the graphs it can be seen that

$f(x) = \sin^{-1}x + \cos^{-1}x$

$= \text{constant}$ ①

$(= \frac{\pi}{2})$

OR $f(x) = \sin^{-1}x + \cos^{-1}x$

$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$

$= 0$

$\therefore$ ...

Note $f(0) = \sin^{-1}(0) + \cos^{-1}(0)$

$= \frac{\pi}{2}$ ①

iii) $\int_0^1 f(x)\,dx$

$= \int_0^1 \frac{\pi}{2}\,dx$

$= \left[\frac{\pi}{2}x\right]_0^1$ ①

$= \frac{\pi}{2}$

OR

From the graph:

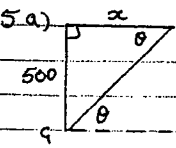
$\int_0^1 (\sin^{-1}x + \cos^{-1}x)\,dx$

= area of rectangle with width 1 and height $\frac{\pi}{2}$

$\therefore$ Area $= 1 \times \frac{\pi}{2}$

$= \frac{\pi}{2}.$ ②

Q5 a)



$\tan\theta = \frac{500}{x}$

$\therefore x = \frac{500}{\tan\theta}$

i) $\frac{dx}{d\theta} = \frac{(\tan\theta)\,0 - 500\sec^2\theta}{\tan^2\theta}$ ①

$= \frac{-500}{\cos^2\theta} \times \frac{\cos^2\theta}{\sin^2\theta}$

$= \frac{-500}{\sin^2\theta}$ ①

$(= -500\,\text{cosec}^2\theta)$

ii) $\frac{dx}{dt} = 20$

$\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$

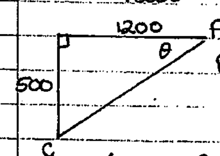
$= \frac{1}{-500\,\text{cosec}^2\theta} \times 20$

$= -\frac{1}{25}\sin^2\theta$ ①

iii) At 8:01 am , $t = 60$

$x = 20 \times 60$

$= 1200$



PQ = 1300

(Pyth. thm)

$\therefore \sin\theta = \frac{500}{1300}$ ①

$\therefore \frac{d\theta}{dt} = -\frac{1}{25} \times \sin^2\theta$

$= -\frac{1}{25} \times \left(\frac{5}{13}\right)^2$

$= -\frac{1}{169}$ radians/sec ①

b) $P(2ap, ap^2)$ $Q(2aq, aq^2)$

i) $M = \left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$= \left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$

ii) Let m = Grad. of PQ

$m = \frac{aq^2 - ap^2}{2aq - 2ap}$

$= \frac{q^2-p^2}{2(q-p)}$

$= \frac{(q-p)(q+p)}{2(q-p)}$

$= \frac{q+p}{2}$ ①

Now if m is constant, then $\frac{q+p}{2} = k$

or $q+p = 2k$ ①

$\therefore$ x-co-ord of midpoint, M, is $x = a(p+q)$

$= 2ak$

$= \text{constant}$ ①

$\therefore$ Locus of M is a line parallel to the y-axis.

c) Possible outcomes

(1,5) (2,5) (3,5) (4,5) (5,5),
(6,5), (5,1), (5,2), (5,3) (5,4)
(5,6)

Sample space = 11

Suitable outcomes = 2

P(sum is 8) $= \frac{2}{11}$

⑧⑧

Q5 d)

Since ABCD is a cyclic quad.

$\angle A + \angle C = 180$ (1)

$\therefore \angle C = 180 - \angle A$

$\therefore \Rightarrow \tan C = -\tan A$ (1)

Similarly $\tan D = -\tan B$ (1)

$\therefore \tan A + \tan B + \tan C + \tan D$

$= \tan A + \tan B - \tan A - \tan B$

$= 0$

Q6

a) $\tan x = \frac{1}{\sqrt{3}}$

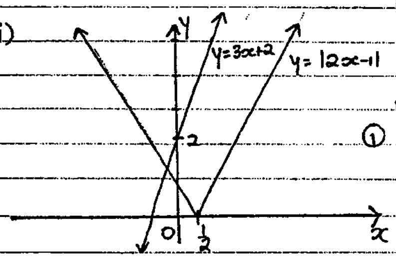
$x = \frac{\pi}{6}, \pi + \frac{\pi}{6}$ (1) for $0 \le x \le 2\pi$

$\therefore$ General solution:

$x = \frac{\pi}{6} \pm 2n\pi, \pi + \frac{\pi}{6} \pm 2n\pi$

$= \frac{\pi}{6} \pm n\pi$ for all integral n. (1)

b) i)



(1)

ii) Pt. of intersection:

$y = 3x+2$ and $y = 1-2x$

$3x + 2 = 1 - 2x$

$5x = -1$

$x = -\frac{1}{5}$ (1)

Q6 b) ii) (cont)

$\therefore$ From the graph

$|2x-1| < 3x+2$ for $x > -\frac{1}{5}$ (1)

OR $|2x-1| < 3x+2$

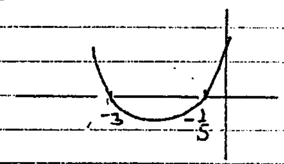
$(2x-1)^2 < (3x+2)^2$

$(2x-1)^2 - (3x+2)^2 < 0.$

$(2x-1+3x+2)(2x-1-(3x+2)) < 0$

$(5x+1)(-x-3) < 0$

$(5x+1)(x+3) > 0$



$\therefore x < -3, x > \frac{1}{5}$

BUT: when $x < -3$, $3x+2 < 0$

$\therefore x > -\frac{1}{5}$ is only soln

c) i)

$T = S + Be^{kt}$ (A)

$\frac{dT}{dt} = 0 + B \times ke^{kt}$

$= Bke^{kt}$ (1)

but: $Be^{kt} = T - S$ from (A)

$\therefore \frac{dT}{dt} = k(T-S)$ (1)

$\therefore T = S + Be^{kt}$ is a solution

Q6 c) (cont)

ii) When $t = 0$ $T = 150$

$S = 30$

$T = S + Be^{kt}$

$150 = 30 + Be^0$

$\therefore B = 120$ (1)

When $t = 3$ $T = 90$

$\therefore 90 = 30 + 120e^{3k}$

$60 = 120e^{3k}$

$0.5 = e^{3k}$

$k = \frac{1}{3}\log_e 0.5$ (1)

$= -0.231$ (3dp) (1)

iii) When $t = 6$

$T = 30° + 120e^{6 \times \frac{1}{3}\log_e(\frac{1}{2})}$ (1)

$= 30° + 120e^{\frac{1}{4}}$

$= 30 + 30$

$= 60$ (1)

$\therefore$ Temperature is 60° after a further 3 hours.

Q7. a) $P(x) = ax^3 + bx^2 + cx + d$

i) $P(0) = 4$

$\therefore d = 4$

$\Rightarrow$ y-intercept is 4 (1)

$P(2) = 0$

$\therefore 8a + 4b + 2c + 4 = 0$ (A)

$4a + 2b + c + 2 = 0$

$P(-2) = 0$

$-8a + 4b - 2c + 4 = 0$ (B)

$-4a + 2b - c + 2 = 0$

Q7 a) (cont)

Since $x = -2$ is a turning point,

$P'(-2) = 0$

$P'(x) = 3ax^2 + 2bx + c$.

$\therefore 12a - 4b + c = 0.$ (C)

3(A) − C $12a + 6b + 3c + 6 = 0$

$12a - 4b + c = 0$

$10b + 2c + 6 = 0$

$5b + c + 3 = 0$ (D)

(C) + 3(B) $12a - 4b + c = 0$

$-12a + 6b - 3c + 6 = 0$

$2b - 2c + 6 = 0$

$b + c + 3 = 0.$ (E)

(D) − (E)

$5b + c + 3 = 0$ $b = -1$

$b + c + 3 = 0$ $c = 2$

$4b = 0$

$b = 0.$ (1)

$\therefore c = -3.$ (1)

Sub. into (C)

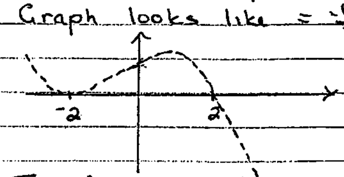
$12a - 0 + 3 = 0$ $a = -\frac{1}{2}$

$12a = -3$

$a = -\frac{1}{4}$ (1)

$\therefore P(x) = -\frac{x^3}{4} - 3x + 4 = 0$

$\therefore$ Graph looks like $= -\frac{1}{2}(x-2)(x+$



$\therefore$ Turning point at $x = -2$ is a relative minimum (1)

Q7 b)

i) At P $y = 2$

$\therefore \log_e(x-1) = 2$

$x - 1 = e^2$

$x = e^2 + 1$ (1)

$\therefore$ P is $(e^2+1, 2)$

ii) $y = \log_e(x-1)$

$\frac{dy}{dx} = \frac{1}{x-1}$

at Q $x = 2$, $y = 0$

$\therefore \frac{dy}{dx} = \frac{1}{2-1} = 1$

$\therefore$ gradient of normal is

$m = -1$ (1)

Eqn. of normal:

$y - y_1 = m(x - x_1)$

$y - 0 = -1(x - 2)$

$y = 2 - x$ (1)

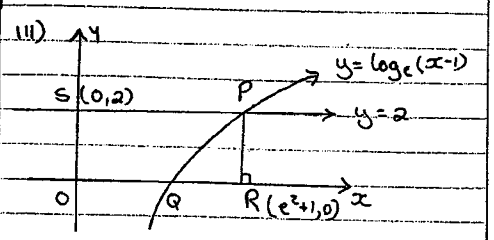
At S $x = 0$, $y = 2$

$\therefore$ on the line $y = 2 - x$

when $x = 0$, $y = 2$

$\therefore$ S lies on the normal at Q (1)

iii)



Area of OSPR

$= (e^2+1) \times 2$

$= 2(e^2+1)$ (1)

$y = \log_e(x-1)$

$e^y = x - 1$

$e^y + 1 = x$

Area OSPQ

$= \int_0^2 g(y)\, dy$

$= \int_0^2 (e^y + 1)\, dy$ (1)

$= \left[e^y + y\right]_0^2$

$= (e^2 + 2) - (e^0 + 0)$

$= e^2 + 1$ ($\frac{1}{2}$)

$\therefore$ Area of OSPQ $= \frac{1}{2}$ area of rectangle OSPR. ($\frac{1}{2}$)