3 UNIT MATHEMATICS

2000 TRIAL SOLUTIONS

[Ques]TION 1 : 12 marks

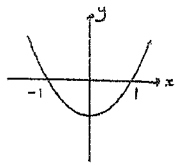
$\frac{2x}{x+1} \le 1 \quad (x \neq -1)$

$2x(x+1) \leqslant (x+1)^2$

$2x^2 + 2x \leqslant x^2 + 2x + 1$

$\therefore x^2 - 1 \leqslant 0$

$\therefore -1 \leqslant x \leqslant 1$



$x \neq -1$

$\therefore -1 < x \leqslant 1$

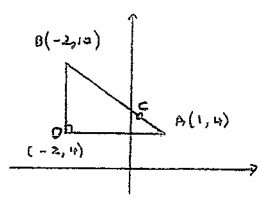
i) INFINITE

Number of different words $= \frac{8!}{3!\,2!}$

$= 3360$

i) Number of possibilities with I's together $= \frac{6!}{2!} = 360$

$\therefore$ Probability $= \frac{360}{3360} = \frac{3}{28}$



w right angled triangle.

d $\frac{1}{6}$ of length AD and subtract this distance from the x-value of A - x value

d $\frac{1}{6}$ of length BD and add this distance to the y-value of A. - y value.

$... - y\ \overline{3}$

(e) $LHS = n! + (n-1)! + (n-2)!$

$= (n-2)!\left[n(n-1) + (n-1) + 1\right]$

$= (n-2)!\,(n^2 - n + n - 1 + 1)$

$= n^2 (n-2)!$

$= RHS$

QUESTION 2 : 12 marks

(a) $\int \cos^2 5x \, dx$

$= \frac{1}{2}\int \cos 10x + 1 \, dx$

$= \frac{1}{2}\left[\frac{\sin 10x}{10} + x\right] + C$

$= \frac{1}{20}\sin 10x + \frac{1}{2}x + C$

(b) $V = \pi \int_0^{3\sqrt{3}} \frac{1}{x^2+9}\, dx$

$= \left[\frac{\pi}{3} \tan^{-1}\frac{x}{3}\right]_0^{3\sqrt{3}}$

$= \frac{\pi}{3}\tan^{-1}\sqrt{3} - \frac{\pi}{3}\tan^{-1} 0$

$= \frac{\pi^2}{9}$

(c) $\int_0^1 \frac{4x}{(4x+1)^2}\, dx \qquad u = 4x+1$

$\frac{du}{dx} = 4$

when $x = 0,\ u = 1$

$x = 1,\ u = 5$

$= \int_1^5 \frac{u-1}{u^2} \cdot \frac{1}{4}\, du$

$= \frac{1}{4}\int_1^5 \frac{1}{u} - \frac{1}{u^2}\, du$

$= \frac{1}{4}\left[\ln u + \frac{1}{u}\right]_1^5$

$= \frac{1}{4}\left[(\ln 5 + \frac{1}{5}) - (\ln 1 + 1)\right]$

$= \frac{1}{4}(\ln 5 - \frac{4}{5})$

$= \frac{1}{4}\ln 5 - \frac{1}{5}$

(d) $\lim_{x\to 0} \frac{\sin 4x}{\tan 2x} = \lim_{x \to 0} \frac{2\sin 2x \cos 2x}{\frac{\sin 2x}{\cos 2x}}$

$= \lim_{x\to 0} 2\cos^2 2x$

$= 2$

QUESTION 3 : 12 marks

(a) $f(x) = x^3 + 3x^2 - 10x - 24$

(i) $f(-2) = -8 + 12 + 20 - 24 = 0$

(ii) $\therefore f(x) = (x+2)(x^2 + x - 12)$

$= (x+2)(x+4)(x-3)$

(b) $x^3 - 3x + 5 = 0$

(i) $\alpha + \beta + \gamma = 0$

(ii) $\alpha\beta\gamma = -5$

(iii) $(\alpha-1)(\beta-1)(\gamma-1)$

$= (\alpha\beta - \alpha - \beta + 1)(\gamma - 1)$

$= \alpha\beta\gamma - \alpha\gamma - \beta\gamma + \gamma - \alpha\beta + \alpha + \beta - 1$

$= \alpha\beta\gamma - (\alpha\beta + \beta\gamma + \alpha\gamma) + (\alpha+\beta+\gamma) - 1$

$= -5 + 3 + 0 - 1$

$= -3$

(c) (i) $x^2 = 4ay$

$\therefore y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

At P, $\frac{dy}{dx} = \frac{2ap}{2a} = p$

$\therefore$ Grad norm $= -\frac{1}{p}$

$\therefore$ Eq^n normal:

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$\therefore x + py = ap^3 + 2ap$

(ii) Q lies on normal:

$2aq + paq^2 = ap^3 + 2ap$

$\therefore 2q + pq^2 = p^3 + 2p$

$p^3 - pq^2 + 2p - 2q = 0$

$p(p^2 - q^2) + 2(p-q) = 0$

$(p-q)\left(p(p+q) + 2\right) = 0$

but $p \neq q$ since P and Q are distinct points

$\therefore (p-q) \neq 0$

$\therefore p^2 + pq + 2 = 0$

QUESTION 4 : 12 marks

(a) $\left(3x^2 + \frac{1}{x}\right)^9$

$T_{k+1} = {}^9C_k\,(3x^2)^{9-k} \cdot \left(\frac{1}{x}\right)^k$

$= {}^9C_k \cdot 3^{9-k} \cdot x^{18-2k} \cdot x^{-k}$

$= {}^9C_k \cdot 3^{9-k} \cdot x^{18-3k}$

$\therefore$ Coefficient of x^3:

$\Rightarrow 18 - 3k = 3$

$3k = 15$

$k = 5$

$\therefore$ Coefficient $= {}^9C_5 \times 3^{9-5}$

$= 126 \times 3^4$

$= 10206$

(b) $f(x) = 1 + e^{2x}$

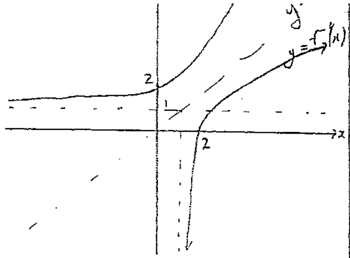
(i) $y > 1$

(ii) $x = 1 + e^{2y}$

$\therefore x - 1 = e^{2y}$

$\ln(x-1) = 2y$

$\therefore y = \frac{1}{2}\ln(x-1)$



(iv) Normal at $(2,0)$ is

$2x+y-4=0$

ie. $y=4-2x$

$\therefore 4-2x = 1+e^{2x}$

$\therefore e^{2x}+2x = 3.$

(v) $f(x) = e^{2x}+2x-3$

$f'(x) = 2e^{2x}+2$

$f(0.4) = 0.025\ldots$

$f'(0.4) = 6.451\ldots$

$\therefore a_1 = 0.4 - \frac{0.025}{6.451}$

$\doteq 0.396$ (correct to 3 significant figures).

(a) $2^{2x+1} - 5(2^x)+2 = 0$

$2(2^{2x}) - 5(2^x)+2 = 0$

let $u = 2^x$

$2u^2 - 5u + 2 = 0$

$(2u-1)(u-2) = 0$

$\therefore u = \frac{1}{2}$ or 2

$\therefore x = -1$ or 1

(b) $n=0$:

LHS: $2^3+3 = 11$ which is divisible by 11.

Assume true for $n=k$:

ie $2^{10k+3}+3 = 11M$ for some $M \in \mathbb{Z}^+$

When $n=k+1$:

$2^{10(k+1)+3}+3$

$= 2^{10k+10+3}+3$

$= 2^{10}.2^{10k+3}+3$

$= 2^{10}(11M-3)+3$ by induction hypothesis

$= 11M.2^{10} - 3.2^{10}+3$

$= 11M.2^{10} - 3069$

$= 11(2^{10}.M - 279)$

which is divisible by 11 since $2^{10}.M > 279$ for all $M \in \mathbb{Z}^+$.

$\therefore$ If hypothesis is true for $n=k$, it is also true for $n=k+1$.

Since it is true for $n=0$, it is also true for $n=1,2,\ldots$ and hence all non-negative integers by mathematical induction.

(i) $\dot{x} = \frac{5\pi}{2}\cos\frac{\pi}{2}(t+\frac{1}{3})$

$\ddot{x} = -\frac{5\pi^2}{4}\sin\frac{\pi}{2}(t+\frac{1}{3})$

$= -\frac{\pi^2}{4}x$

(ii) Amp $= 5$

Period $= \frac{2\pi}{\frac{\pi}{2}} = 4$

(iii) $\ddot{x} = -\frac{\pi^2}{4}\times\frac{5}{2}$

$= \frac{-5\pi^2}{8}$

QUESTION 6 : 12 marks

(a) (i) $N = 1000 + Ae^{-kt}$

$\frac{dN}{dt} = -k.\,Ae^{-kt}$

$= -k(N-1000)$

(ii) when $t=0$, $N=2500$

$2500 = 1000 + Ae^0$

$\therefore A = 1500$

when $t=2$, $N=2200$

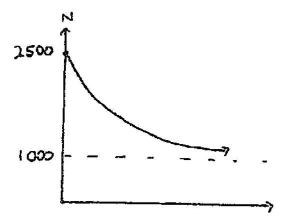
$2200 = 1000 + 1500e^{-2k}$

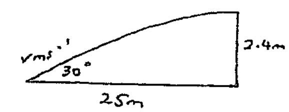
$0.8 = e^{-2k}$

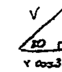
$\therefore k = -\frac{1}{2}\ln 0.8$

$\doteq 0.11157$

(iii) $N = 1000 + 1500e^{-0.1t}$







$V\sin 30 = \frac{v}{2}$

$v\cos 30 = \frac{\sqrt{3}v}{2}$

(i) Horizontal:

$\ddot{x} = 0$

$\therefore \dot{x} = c$

when $t=0$, $\dot{x} = \frac{\sqrt{3}v}{2}$

$\therefore \dot{x} = \frac{\sqrt{3}v}{2}$

$x = \frac{\sqrt{3}}{2}vt + c$

when $t=0$, $x=0$

$\therefore x = \frac{\sqrt{3}}{2}vt$

Vertical:

$\ddot{y} = -10$

$\dot{y} = -10t + c$

when $t=0$, $\dot{y} = \frac{v}{2}$

$\therefore \dot{y} = -10t + \frac{v}{2}$

$\therefore y = -5t^2 + \frac{vt}{2} + c$

when $t=0$, $y=0$

$\therefore y = -5t^2 + \frac{vt}{2}$

(ii) $t = \frac{2x}{\sqrt{3}v}$

$\therefore y = -5\left(\frac{2x}{\sqrt{3}v}\right)^2 + \frac{\cancel{v}}{\cancel{2}}\left(\frac{2x}{\sqrt{3}\cancel{v}}\right)$

$y = -\frac{20x^2}{3v^2} + \frac{x.\sqrt{3}}{3}$

(iii) when $x=25$, $y=2.4$

$2.4 = -\frac{20\times 25^2}{3v^2} + \frac{25\sqrt{3}}{3}$

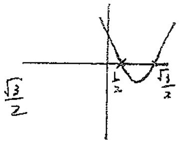
$7.2v^2 = -12500 + 25\sqrt{3}v^2$

$\therefore v^2 = \frac{12500}{25\sqrt{3}-7.2}$

$= 346.248\ldots$

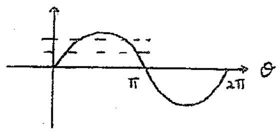
$\therefore v = 18.6\,m/s$

(i) $(2x-1)(2x-\sqrt{3}) < 0$



$\therefore \frac{1}{2} < x < \frac{\sqrt{3}}{2}$

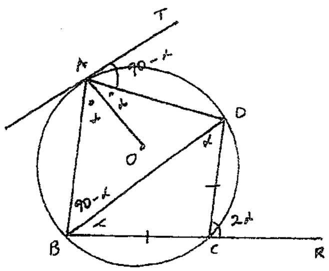
ii) $\therefore \frac{1}{2} < \sin\theta < \frac{\sqrt{3}}{2}$



$\sin\theta = \frac{1}{2}$ $\quad$ $\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ $\quad$ $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

$\therefore \frac{\pi}{6} < \theta < \frac{\pi}{3}$ or $\frac{2\pi}{3} < \theta < \frac{5\pi}{6}$



$\angle BDC = \alpha$ ($\angle$ opp = sides in isos. Δ are =)

$\therefore \angle DCR = 2\alpha$ (exterior $\angle$ in Δ = 2 interior opp $\angle$)

$\therefore \angle BAD = 2\alpha$ (exterior $\angle$ in cyclic quad = opp $\angle$)

$\therefore \angle OAD = \alpha$ (OA bisects BAD - given).

$\therefore \angle OAT = 90°$ ($\angle$ between tangent + radius = $90°$)

$\therefore \angle DAT = 90 - \alpha$

$\therefore \angle ABD = 90 - \alpha$ ($\angle$ between chord + tangent = $\angle$ in the alt. segment)

$\therefore \angle ABC = 90 - \alpha + \alpha$
$= 90°$.

$\therefore \angle ABC$ is a right angle.

(c) (i) Coefficient $= {}^nC_r$

$= \frac{n!}{(n-r)!\,r!}$

(ii) $5^{2n} + 2nt - 1$

$= (t-1)^{2n} + 2nt - 1$

$= \left(t^{2n} + {}^{2n}C_1 t^{2n-1}(-1)^1 + {}^{2n}C_2 t^{2n-2}(-1)^2 + \dots + {}^{2n}C_{2n-1} t(-1)^{2n-1} + (-1)^{2n}\right) + 2nt - 1$

$= \left(t^{2n} - 2nt^{2n-1} + {}^{2n}C_2 t^{2n-2} - \dots {}^{2n}C_2 t^2 - \cancel{2nt} + \cancel{1}\right) + \cancel{2nt} - \cancel{1}$

$= t^{2n} - 2nt^{2n-1} + {}^{2n}C_2 t^{2n-2} + \dots + {}^{2n}C_2 t^2$

$= t^2\left(t^{2n-2} - 2nt^{2n-3} + \dots + {}^{2n}C_2\right)$

$\therefore 5^{2n} + 2nt - 1$ is divisible by t^2

(iii) $5^{20} + 119$

$s = 5 \quad \therefore t = 6$

$n = 10$

$\therefore$ By (ii) $5^{20} + 120 - 1$ is divisible by $t^2 = 36$.

END OF SOLUTIONS.