Mathematics Trial 2010 Solutions

Question 1

(a) $e^5 - 3\log_e 5 = 143.58...$

$= 140$ (2 sig. fig.)

(b) $64 - x^3 = 4^3 - x^3$

$= (4 - x)(16 + 4x + x^2)$

(c) $\displaystyle\int \left(\frac{1}{4x} + e^{2x} \right) dx = \frac{1}{4}\ln x + \frac{1}{2}e^{2x} + C$

(d) $$\frac{x}{x-2} - \frac{8}{x^2-4} = \frac{x}{x-2} - \frac{8}{(x-2)(x+2)}$$

$$= \frac{x(x+2)-8}{(x-2)(x+2)}$$

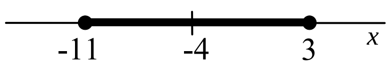
$$= \frac{x^2+2x-8}{(x-2)(x+2)}$$

$$= \frac{(x-2)(x+4)}{(x-2)(x+2)}$$

$$= \frac{x+4}{x+2}$$

(e) $|4 + x| \le 7$

$\therefore |x - (-4)| \le 7$



$\therefore -11 \le x \le 3$

(f) $y = x^2 + 4x - 3$

becomes $y = (x+2)^2 - 7$

$\therefore$ vertex is $(-2, -7)$

Alternatively:

Axis of symmetry is at $x = \dfrac{-4}{2} = -2$

Then $y = (-2)^2 + 4(-2) - 3 = -7$

$\therefore$ vertex is $(-2, -7)$

Question 2

(a) $y = \log_e(2x-1)$

$$y' = \frac{2}{2x-1}$$

At $x = 1$: $y' = \dfrac{2}{2(1)-1} = 2$

and $y = \ln(2(1)-1)$

$= \ln 1$

$= 0$

$\therefore$ tangent is $y - 0 = 2(x-1)$

$y = 2x - 2$

(b) $$\frac{d}{dx}\sqrt{5 + \log_e x} = \frac{d}{dx}(5 + \log_e x)^{\frac{1}{2}}$$

$$= \frac{1}{2}(5 + \log_e x)^{-\frac{1}{2}}\left(\frac{1}{x}\right)$$

$$= \frac{1}{2x\sqrt{5 + \log_e x}}$$

(c) $$y = \frac{1 + \sin x}{\cos x}$$

$$y' = \frac{\cos x(\cos x) - (1 + \sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1 + \sin x}{1 - \sin^2 x}$$

$$= \frac{1 + \sin x}{(1 - \sin x)(1 + \sin x)}$$

$$= \frac{1}{1 - \sin x}$$

(d) (i) $$\int 6e^{\frac{x}{2}}\,dx = 12\int \frac{1}{2}e^{\frac{x}{2}}\,dx$$

$$= 12e^{\frac{x}{2}} + C$$

(ii) $$\int \frac{x}{1-x^2}\,dx = -\frac{1}{2}\int \frac{-2x}{1-x^2}\,dx$$

$$= -\frac{1}{2}\ln(1-x^2) + C$$

(e) $$\int_0^{\frac{\pi}{6}}\left(1-\sec^2 2x\right)dx=\left[x-\frac{1}{2}\tan 2x\right]_0^{\frac{\pi}{6}}$$

$$=\frac{\pi}{6}-\frac{\tan\frac{\pi}{3}}{2}-\left(0-\frac{\tan 0}{2}\right)$$

$$=\frac{\pi}{6}-\frac{\sqrt{3}}{2}$$

(f) $4\sin^2\theta-3=0$ for $-\pi\le\theta\le\pi$

$$\sin^2\theta=\frac{3}{4}$$

$$\sin\theta=\pm\frac{\sqrt{3}}{2}$$

$$\theta=\frac{\pi}{3},\pi-\frac{\pi}{3},-\frac{\pi}{3},-\pi+\frac{\pi}{3}$$

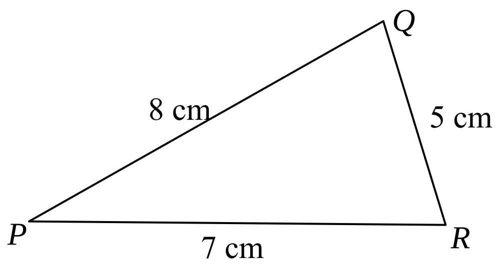
$$=\frac{\pi}{3},\frac{2\pi}{3},-\frac{\pi}{3},-\frac{2\pi}{3}$$

Question 3

(a) $$\sum_{n=5}^{11}\left(2n-5\right)=5+7+9+11+13+15+17$$

$$=77$$

(b)



(i) $\cos\angle PQR=\dfrac{8^2+5^2-7^2}{2(8)(5)}$ (cos rule)

$$=\frac{40}{80}$$

$$=\frac{1}{2}$$

$$\therefore\angle PQR=60^\circ$$

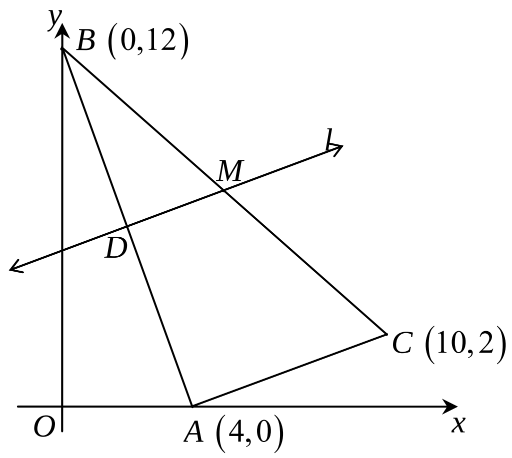
(ii) $$A=\frac{1}{2}(8)(5)\sin 60^\circ$$

$$=20\times\frac{\sqrt{3}}{2}$$

$$=10\sqrt{3}$$

$\therefore$ the area of $\Delta\, PQR$ is $10\sqrt{3}\text{ cm}^2$

(c)



(i) $$m_{AC}=\frac{2-0}{10-4}$$

$$=\frac{1}{3}$$

(ii) $D=(2,6)$

(iii) $l:\ y-6=\frac{1}{3}(x-2)$

$$3y-18=x-2$$

$$x-3y+16=0$$

(iv) $AC\parallel DM$ (same gradient)

$\therefore\dfrac{AD}{DB}=\dfrac{CM}{MB}$ (parallel lines preserve ratios)

$$1=\frac{CM}{MB}$$

$$CM=MB$$

$\therefore M$ is the midpoint of CB

(v) Centre is $(5,7)$

$$d_{MB}=\sqrt{5^2+5^2}$$

$$=\sqrt{50}$$

$$=5\sqrt{2}$$

$\therefore$ the circle is $(x-5)^2+(y-7)^2=50$

(vi) Substitute $(4,0)$ into

$$(x-5)^2+(y-7)^2=50$$

$$(4-5)^2+(0-7)^2=1+49$$

$$=50$$

The point A does lie on the circle

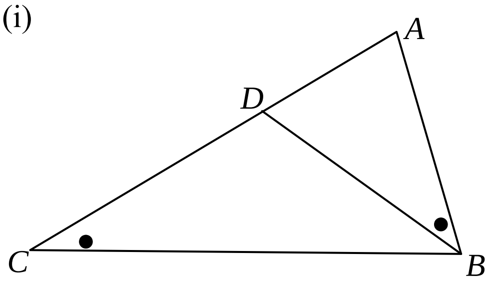
Question 4

(a) $x^2 - 8x + 5 = 0$

$\therefore\ \alpha + \beta = 8$ and $\alpha\beta = 5$

$$\begin{aligned}(\alpha-\beta)^2 &= \alpha^2 - 2\alpha\beta + \beta^2 \\ &= \alpha^2 + \beta^2 - 2\alpha\beta \\ &= (\alpha+\beta)^2 - 2\alpha\beta - 2\alpha\beta \\ &= (\alpha+\beta)^2 - 4\alpha\beta \\ &= 8^2 - 4(5) \\ &= 44\end{aligned}$$

(b) (i)



(ii) In ΔABC and ΔADB

1. $\angle A$ is common
2. $\angle ACB = \angle ABD$ (given)

$\therefore \Delta ABC \,|||\, \Delta ADB$ (equiangular)

(iii) $\dfrac{AB}{AD} = \dfrac{BC}{DB} = \dfrac{AC}{AB}$

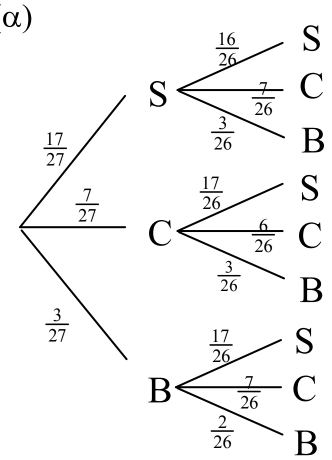
(matching sides of similar Δ's)

$$\frac{12}{9} = \frac{AC}{12}$$

$$AC = \frac{12 \times 12}{9} = 16$$

(c) (i) $P(2\text{ symbols}) = \dfrac{3}{27} = \dfrac{1}{9}$

(ii) (α)



(β)(1) $\text{P}(\text{exactly 1 star})$

$$\begin{aligned}&= \text{P}(\text{SC}) + \text{P}(\text{CS}) + \text{P}(\text{CB}) + \text{P}(\text{BC}) \\ &= \frac{17}{27}\times\frac{7}{26} + \frac{7}{27}\times\frac{17}{26} + \frac{7}{27}\times\frac{3}{26} + \frac{3}{27}\times\frac{7}{26} \\ &= \frac{140}{351}\end{aligned}$$

(2) $\text{P}(2\text{stars})$

$$\begin{aligned}&= \text{P}(\text{SS}) + \text{P}(\text{SB}) + \text{P}(\text{BS}) + \text{P}(\text{BB}) \\ &= \frac{17}{27}\times\frac{16}{26} + \frac{17}{27}\times\frac{3}{26} + \frac{3}{27}\times\frac{17}{26} + \frac{3}{27}\times\frac{2}{26} \\ &= \frac{190}{351}\end{aligned}$$

Question 5

(a) (i) Each face has one less orange than in the row below. There are 4 faces. $\therefore$ there are 4 fewer oranges than the row on which it rests.

(ii) $56 + 52 + 48 + \ldots + 4$

$n = ?$
$a = 56$
$T_n = 4$
$d = -4$

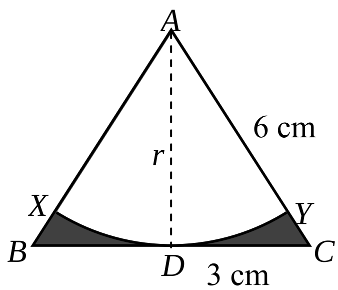
$$\begin{aligned}T_n &= a + (n-1)d \\ 4 &= 56 + (n-1)(-4) \\ 4 &= 56 - 4n + 4 \\ 4n &= 56 \\ n &= 14\end{aligned}$$

$\therefore$ there are 14 rows

(iii)
$$\begin{aligned}S_n &= \frac{n}{2}[a+l] + 1 \\ &= \frac{14}{2}[56+4] + 1 \\ &= 421\end{aligned}$$

$\therefore$ he will use 421 oranges

(b)



(i)
$$\begin{aligned}6^2 &= r^2 + 3^2 \\ r^2 &= 36 - 9 \\ &= 27 \\ r &= 3\sqrt{3}\end{aligned}$$

$\therefore$ the radius is $3\sqrt{3}$ cm.

(ii) $A = \frac{1}{2}bh - \frac{1}{2}r^2\theta$ where $\theta = \frac{\pi}{3}$

$$A = \frac{1}{2}(6)\left(3\sqrt{3}\right) - \frac{1}{2}\left(3\sqrt{3}\right)^2\left(\frac{\pi}{3}\right)$$

$$= 9\sqrt{3} - \frac{9\pi}{2}$$

$\therefore$ the area is $\left(9\sqrt{3} - \frac{9\pi}{2}\right)$ cm^2

(c) (i) For $5x^2 - 2kx + k$

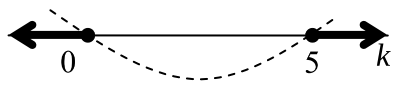
$$\Delta = (-2k)^2 - 4(5)(k)$$

$$= 4k^2 - 20k$$

(ii) For real roots: $\Delta \ge 0$

$$4k^2 - 20k \ge 0$$

$$4k(k-5) \ge 0$$



$k \le 0$ or $k \ge 5$

Question 6

(a) $2\ln x = \ln(5+4x)$

$$2\ln x = \ln(5+4x)$$

$$\ln x^2 = \ln(5+4x)$$

$$x^2 = 5+4x$$

$$x^2 - 4x - 5 = 0$$

$$(x-5)(x+1) = 0$$

$$x = -1, 5$$

But $x > 0$ and $4x+5>0$ for the logs to exist

$\therefore$ $x = 5$ is the only solution

(b) $A = \int_{-1}^{4}\left[x+1-\left(x^2-2x-3\right)\right]dx$

$$= \int_{-1}^{4}\left(3x+4-x^2\right)dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3} + 4x\right]_{-1}^{4}$$

$$= 24 - \frac{64}{3} + 16 - \left(\frac{3}{2} + \frac{1}{3} - 4\right)$$

$$= 20\tfrac{5}{6}$$

$\therefore$ the area is $20\tfrac{5}{6}$ unit^2

(b) For $y = e^{3x}$, $y' = 3e^{3x}$

For $y = 6x$, $m = 6$

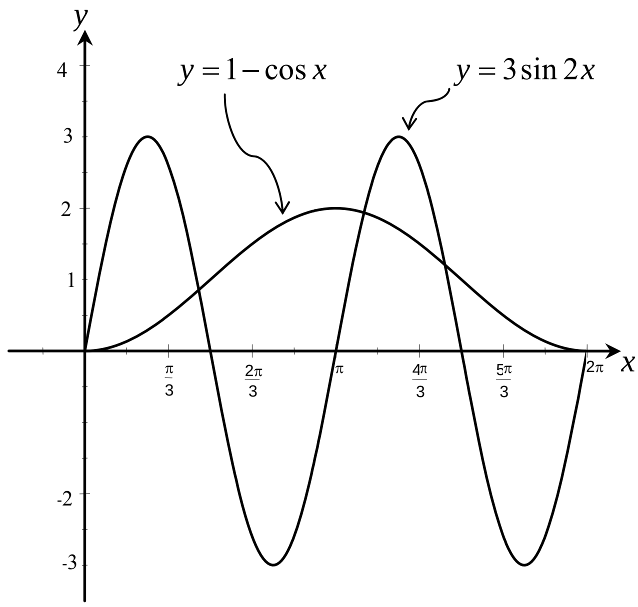
$$\therefore\ 3e^{3x} = 6$$

$$e^{3x} = 2$$

$$3x = \ln 2$$

$$x = \frac{1}{3}\ln 2$$

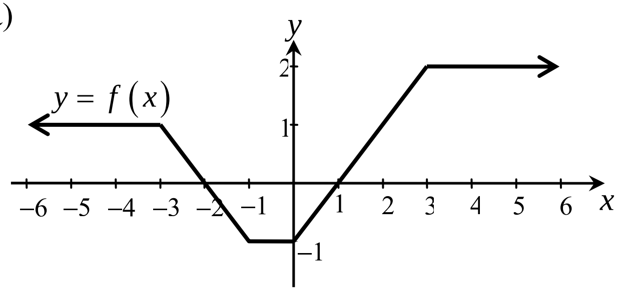
(c) (i) and (ii)



(iii) There are 5 solutions because the curves intersect in 5 different points.

Question 7

(a)



(i) $\int_0^5 f(x)\,dx = -\frac{1}{2} + 2 + 4$

$= 5\tfrac{1}{2}$

(ii) If $\int_a^5 f(x)\,dx = 4$ we need values of a for which the signed area gives a result of 4. This occurs when a = 3 or −2

(b) (i) $\frac{d\left(\sin^2 x - \cos 4x\right)}{dx}$

$= 2\sin x\cos x + 4\sin 4x$

(ii) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\sin x\cos x + 2\sin 4x\right) dx$

$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(2\sin x\cos x + 4\sin 4x\right) dx$

$= \frac{1}{2}\left[\sin^2 x - \cos 4x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\sin^2\frac{\pi}{2} - \cos 2\pi - \left(\sin^2\frac{\pi}{4} - \cos\pi\right)\right]$

$= \frac{1}{2}\left[1 - 1 - \left(\frac{1}{2} - (-1)\right)\right]$

$= -\frac{3}{4}$

(c) (i) 8% pa = 2% every ¼ year;
investment periods = 4 × 15 −1= 59
(no interest for the 1st 3 months)

$A = 500(1+0.02)^{59}$

$= 1608.348426$

$\therefore$ The amount = \$1608.35

(ii)(α) 1st payment grows to $500(1.02)^3$

2nd payment grows to $500(1.02)^2$

3rd payment grows to $500(1.02)^1$

Last payment remains as 500

$\therefore$ Value on the day after 1st birthday is

$A = 500(1.02)^3 + 500(1.02)^2 + 500(1.02)^1 + 500$

$= 500\left[1 + 1.02 + 1.02^2 + 1.02^3\right]$

(β) 1st payment now grows to $500(1.02)^{59}$

$A = 500(1.02)^{59} + 500(1.02)^{58} + \ldots + 500$

$= 500\left[1 + 1.02 + \ldots + 1.02^{58} + 1.02^{59}\right]$

$= 500\left[\frac{a\left(r^n - 1\right)}{r-1}\right]$ where $a = 1;\ r = 1.02;\ n = 60$

$= 500\left[\frac{1\left((1.02)^{60} - 1\right)}{1.02 - 1}\right]$

$= 57025.769\ldots$

$\therefore$ Total in the account on Emily's 16th birthday is \$57025.77.

(iii) During the year, interest is paid 4 times

$A = 57025.76971 \times (1.02)^4$

$= 61726.527\ldots$

$\therefore$ Amount = \$61726.53 (to nearest cent)

(iv) Interest earned = \$61726.53 − \$57025.77
= \$4700.76

If she withdraws \$4 000, the account will continue to grow.

If she withdraws \$5 000, the money will eventually run out.

Question 8

(a) (i) $f(x) = x^3 - x^2 - 5x + 1$

$f'(x) = 3x^2 - 2x - 5$

$f''(x) = 6x - 2$

Stat points if $f'(x) = 0$

i.e. $3x^2 - 2x - 5 = 0$

$(3x-5)(x+1) = 0$

$\therefore x = -1$ or $\frac{5}{3}$

If $x = -1$: $f''(-1) = 6(-1) - 2$

< 0

$f(-1) = (-1)^3 - (-1)^2 - 5(-1) + 1$

$= 4$

$\therefore$ a maximum at $(-1, 4)$

If $x = \frac{5}{3}$: $f''\left(\frac{5}{3}\right) = 6\left(\frac{5}{3}\right) - 2$

> 0

$f\left(\frac{5}{3}\right) = \left(\frac{5}{3}\right)^3 - \left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right) + 1$

$= -5\frac{13}{27}$

$\therefore$ a minimum at $\left(1\frac{2}{3}, -5\frac{13}{27}\right)$

(ii) Points of inflexion when

$f''(x) = 0$ and concavity changes

i.e. $6x - 2 = 0$

$$x = \frac{1}{3}$$

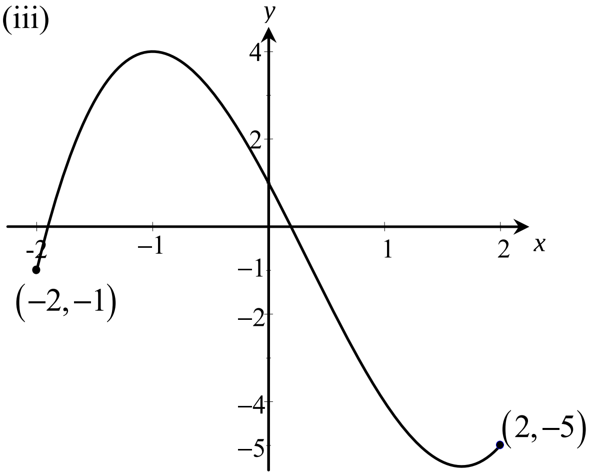
x	$\frac{1}{3}^-$	$\frac{1}{3}$	$\frac{1}{3}^+$
$f''(x)$	$-$	0	$+$

If $x = \frac{1}{3}$: $f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 - 5\left(\frac{1}{3}\right) + 1$

$= -\frac{20}{27}$

$\therefore$ an inflection at $\left(\frac{1}{3}, -\frac{20}{27}\right)$

(iii)



(iv) $y = f(x)$ decreasing but concave up when $\frac{1}{3} < x < \frac{5}{3}$

(b) (i) $y = e^{1-x^2}$

$\log_e y = 1 - x^2$

$\therefore \quad x^2 = 1 - \log_e y$

(ii) $V = \pi \int_1^e (1 - \ln y)\, dy$

(iii)

y	1	$\frac{1+e}{2}$	e
$1 - \ln y$	$1 - \ln 1$ $= 1$	$1 - \ln\left(\frac{1+e}{2}\right)$	$1 - \ln e$ $= 0$

$$V = \pi \int_1^e (1 - \ln y)\, dy$$

$$\approx \pi \frac{(e-1)}{6}\left[1 + 4\left(1 - \ln\left(\frac{1+e}{2}\right) + 0\right)\right]$$

$= 2.2668...$

$\therefore$ the volume is 2.3 unit^3 (2 sig. fig.)

Question 9

(a) (i) $1000\,000+1000\,000(0.8)+1000\,000(0.8)^2+\ldots$ to 7 prizes

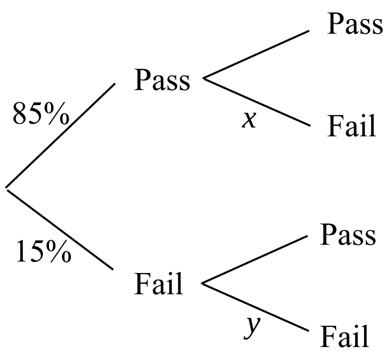
$$7^{\text{th}}\ \text{prize}=\$1000\,000(0.8)^6$$
$$=\$262144$$

(ii) $$20^{\text{th}}\ \text{prize}=\$262144-13\times\$20\,000$$
$$=\$2144$$

(iii) $$\text{Total}=\underbrace{\frac{a\left(1-r^n\right)}{1-r}}_{\text{for the first 7 terms}}+\underbrace{\frac{N}{2}[A+L]}_{\text{For the next 13 terms}}$$
$$=\frac{\$1000\,000\left[1-(0.8)^7\right]}{1-0.8}+\frac{13}{2}[\$242144+\$2144]$$
$$=\$5\,539\,296$$

(b) Let $P(F\text{ second test when passed first test})=x$

and $P(F\text{ second test when failed first test})=y$



$$P(\text{at least one }P)=P(PP)+P(PF)+P(FP)$$
$$=1-P(FF)$$
$$=97\%$$

$$\therefore\ P(FF)=3\%$$
$$\therefore 15\%\times y=3\%$$
$$y=\frac{3\%}{15\%}$$
$$=20\%$$

$$P(P\text{ only one test})=P(PF)+P(FP)$$
$$=85\%\times x+15\%\times 20\%$$
$$=0.85x+0.03$$

$$\therefore 17.1\%=85\%\times x+15\%\times 20\%$$
$$0.171=0.85x+0.03$$
$$x=0.06$$
$$=6\%$$

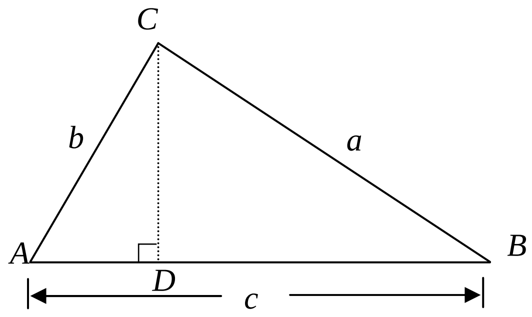
$$P(\text{passes first and fails second})=P(PF)$$
$$=85\%\times 6\%$$
$$=5.1\%$$

(c)

(i) In $\Delta ADC : CD = b\sin A$

In $\Delta CDB : CD = a\sin B$

$\therefore b\sin A = a\sin B$



(ii) In $\Delta ADC : AD = b\cos A$

In $\Delta CDB : DB = a\cos B$

$c = AB$

$= AD + DB$

$= a\cos B + b\cos A$

(iii) $c = a\cos B + b\cos A \Rightarrow c^2 = \left(a\cos B + b\cos A\right)^2$

But $c^2 = a^2\cos^2 B + 2ab\cos A\cos B + b^2\cos^2 A$

$\therefore a^2\cos^2 B + 2ab\cos A\cos B + b^2\cos^2 A = 4ab\cos A\cos B$

$a^2\cos^2 B + 2ab\cos A\cos B + b^2\cos^2 A - 4ab\cos A\cos B = 0$

$a^2\cos^2 B - 2ab\cos A\cos B + b^2\cos^2 A = 0$

$\left(a\cos B - b\cos A\right)^2 = 0$

$\therefore\ a\cos B - b\cos A = 0$

$\therefore a\cos B = b\cos A$ *

$\therefore AD = DB$

$\therefore \Delta ABC$ isosceles [CD perpendicular bisector of AB]

$\therefore a = b$

<u>Alternatively:</u>

$a\cos B = b\cos A$ from *

$a\sin B = b\sin A$ from (i)

$\therefore\ \dfrac{a\sin B}{a\cos B} = \dfrac{b\sin A}{b\cos A}$

$\therefore\ \tan B = \tan A$

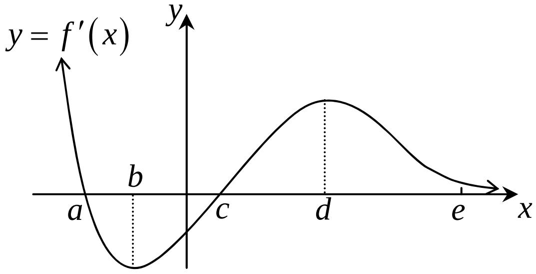
but both angles are acute as they are in ΔABC

$\therefore\ A = B$

$\therefore\ a = b$ (opposite equal angles in ΔABC)

Question 10

(a) The graph below represents $y = f'(x)$. Specific x-values a, b, c, d and e are as indicated in the diagram.



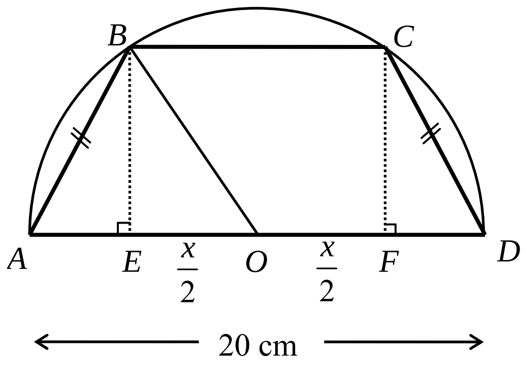
(i) The graph of $y = f(x)$ have a stationary point when $x = a$ or c.

(ii) The graph of $y = f(x)$ is increasing when $x < a$ or $x > c$.

(iii) The graph of $y = f(x)$ is concave up when $f''(x) > 0$
i.e. when the gradient graph is increasing.
This is when $b < x < d$.

(iv) As $x \to \infty$ the graph of $y = f(x)$ approaches a horizontal tangent.

(b)



(i) In $\Delta OBE : OB^2 = BE^2 + OE^2$ (Pythagoras)

$$\therefore 10^2 = BE^2 + \left(\frac{x}{2}\right)^2$$

$$BE^2 = 100 - \frac{x^2}{4}$$

$$= \frac{1}{4}\left(400 - x^2\right)$$

$$\therefore \quad BE = \frac{1}{2}\sqrt{400 - x^2} \qquad \text{(length positive)}$$

(ii) $A = \frac{1}{2}h\left[a+b\right]$

$$A = \frac{1}{2}\left(BE\right)\left[BC+AD\right]$$
$$= \frac{1}{2}\cdot\frac{1}{2}\sqrt{400-x^2}\left[x+20\right]$$
$$= \frac{1}{4}\left(x+20\right)\sqrt{400-x^2}$$

(iii) $A = \frac{1}{4}\left(x+20\right)\sqrt{400-x^2}$

$$A = \frac{1}{4}\left(x+20\right)\left(400-x^2\right)^{\frac{1}{2}}$$
$$A' = \frac{1}{4}\left(x+20\right)\cdot\frac{1}{2}\left(400-x^2\right)^{-\frac{1}{2}}\left(-2x\right)+\left(400-x^2\right)^{\frac{1}{2}}\cdot\frac{1}{4}$$
$$= -\frac{x}{4}\left(x+20\right)\left(400-x^2\right)^{-\frac{1}{2}}+\frac{1}{4}\left(400-x^2\right)^{\frac{1}{2}}$$

Max/min occurs when $A' = 0$

i.e. $-\frac{x}{4}\left(x+20\right)\left(400-x^2\right)^{-\frac{1}{2}}+\frac{1}{4}\left(400-x^2\right)^{\frac{1}{2}} = 0$

$$x\left(x+20\right)\left(400-x^2\right)^{-\frac{1}{2}}-\left(400-x^2\right)^{\frac{1}{2}} = 0$$
$$\frac{x\left(x+20\right)}{\left(400-x^2\right)^{\frac{1}{2}}}-\left(400-x^2\right)^{\frac{1}{2}} = 0$$
$$x\left(x+20\right)-\left(400-x^2\right) = 0$$
$$x^2+20x-400+x^2 = 0$$
$$2x^2+20x-400 = 0$$
$$x^2+10x-200 = 0$$
$$\left(x-10\right)\left(x+20\right) = 0$$
$$x = 10, -20$$

But $x > 0$ $\therefore x = 10$

x	10^-	10	10^+
A'	$+$	0	$-$

$\therefore$ the maximum occurs when $x = 10$
i.e. when $BC = 10$ cm

Alternatively:

$$A' = -\frac{x}{4}(x+20)\left(400-x^2\right)^{-\frac{1}{2}} + \frac{1}{4}\left(400-x^2\right)^{\frac{1}{2}}$$

$$= -\frac{1}{4}\left(400-x^2\right)^{-\frac{1}{2}}\left[x^2+20x+400-x^2\right]$$

$$= -\frac{1}{4}\left(400-x^2\right)^{-\frac{1}{2}}\left[20x+400\right]$$

$$= -\left(400-x^2\right)^{-\frac{1}{2}}(5x+100)$$

$$A'' = -\left(400-x^2\right)^{-\frac{1}{2}}[5] + (5x+100)\left[\frac{1}{2}\left(400-x^2\right)^{-\frac{3}{2}}(-2x)\right]$$

$$= -5\left(400-x^2\right)^{-\frac{1}{2}} - x(5x+100)\left(400-x^2\right)^{-\frac{3}{2}}$$

If $x = 10$:

$$A'' = -5\left(400-10^2\right)^{-\frac{1}{2}} - 10(50+100)\left(400-10^2\right)^{-\frac{3}{2}}$$

$$< 0$$

$\therefore$ the maximum occurs when $x = 10$

i.e. when $BC = 10$ cm

End of solutions