Principle of Mathematical Induction*
Mathematics Extension 1

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Contents

Section **1**

Proof By Mathematical Induction

The statement $P(n)$ may be proved for all integers n greater than or equal to some starting value n_0 by doing the following steps:

1. Show that the statement is true for $n = n_0$, that is $P(n_0)$ is true.
2. Assume that the statement is true for $n = k$, where k is a positive integer such that $k \geq n_0$.
3. Prove that the statement is true for $n = k + 1$.
4. Write your Conclusion

Note. The conclusion that you need to write at the end of each proof would be almost identically the same conclusion. A good conclusion is something like:

> "Since the statement is true for $n = n_0$, then it must also be true for $n = n_0 + 1$ by induction. It follows by induction that the statement is also true for $n = n_0 + 1 + 1 = n_0 + 2$ and so on. Hence the statement is true for all the values of n."

Note. Generally in Mathematics Extension 1, we have three different kinds of statements that we want to prove by induction. These statements are:

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- Equalities
- Divisilbilities
- Inequalities

We give some examples of these statements in the following sections.

1.1 Equalities

Example 1.1.1. Prove the following statement for $n \geq 1$.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Proof. We first need to show that the statement is true for $n=1$.

$$P(1): \text{LHS}=\frac{1}{2}=\text{RHS}$$

Now we assume that the statement is true for $n=k$.

$$P(k): \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}$$

Now we should prove that the statement is true for $n=k+1$.

$$P(k+1): \underbrace{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}}_{\text{equals } \frac{k}{k+1} \text{ from the assumption}}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

therefore we only are required to show that

$$\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

$$\begin{aligned}
\text{LHS} &= \frac{k+1}{k+2}\left(\frac{k(k+2)}{(k+1)^2}+\frac{1}{(k+1)^2}\right)\\
&= \frac{k+1}{k+2}\left(\frac{k^2+2k+1}{(k+1)^2}\right)\\
&= \frac{k+1}{k+2}\left(\frac{(k+1)^2}{(k+1)^2}\right)\\
&= \frac{k+1}{k+2}\\
&= \text{RHS}
\end{aligned}$$

□

Example 1.1.2. Prove the following statement for $n \geq 1$ by mathematical induction.

$$\log 2+\log\left(\frac{3}{2}\right)+\log\left(\frac{4}{3}\right)+\cdots+\log\left(\frac{n+1}{n}\right)=\log(n+1)$$

Proof. We first need to show that the statement is true for $n = 1$.

$$P(1) : \text{LHS} = \log 2 = \text{RHS}$$

Now we assume that the statement is true for $n = k$.

$$P(k) : \log 2 + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \cdots + \log\left(\frac{k+1}{k}\right) = \log(k+1)$$

Now we should prove that the statement is true for $n = k + 1$.

$$P(k+1) : \underbrace{\log 2 + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \cdots + \log\left(\frac{k+1}{k}\right)}_{\text{equals } \log(k+1) \text{ from the assumption}} + \log\left(\frac{k+2}{k+1}\right) = \log(k+2)$$

So it is needed to show that

$$\log(k+1) + \log\left(\frac{k+2}{k+1}\right) = \log(k+2)$$

$$\begin{aligned} \text{LHS} &= \log(k+1) + \log(k+2) - \log(k+1) \\ &= \log(k+2) \\ &= \text{RHS} \end{aligned}$$

□

Example 1.1.3. Show that the following statement is tre for $n \geq 1$ by mathematical induction

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$$

Proof. First we need to show that the statement is true for $n = 1$.

$$P(1) : \text{LHS} = 1 = \text{RHS}$$

Now we assume that the statement is true for $n = k$.

$$P(k) : 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 = \frac{1}{4}k^2(k+1)^2$$

Now we should prove that the sttement is true for $n = k + 1$.

$$P(k+1) : \underbrace{1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3}_{\text{equals } \frac{1}{4}k^2(k+1)^2 \text{ from the assumption}} + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$$

So we really need to show that

$$\frac{1}{4}k^2(k+1)^2 + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$$

$$\begin{aligned} \text{LHS} &= \frac{1}{4}(k+1)^2\left(k^2 + 4k + 4\right) \\ &= \frac{1}{4}(k+1)^2(k+2)^2 \\ &= \text{RHS} \end{aligned}$$

□

1.2 Divisilbilities

Example 1.2.1. Prove by induction that $5^n + 2 \times 11^n$ is a multiple of 3 for $n \geq 1$.

Proof.

$$P(1) : 5^1 + 2 \times 11^1 = 27 = 3 \times 9$$

Hence $P(1)$ is true.

$$P(k) : 5^k + 2 \times 11^k = 3Q \qquad \text{Where } Q \text{ is some integer}$$

$$P(k+1) : 5^{k+1} + 2 \times 11^{k+1} = 3R \qquad \text{Where } R \text{ is some integer}$$

Now $P(k+1)$ can be written as:

$$\begin{aligned}
\text{LHS} = 5 \times 5^k + 2 \times 11 \times 11^k &= \\
&= 5\left(\underbrace{5^k + 2 \times 11^k}_{3Q}\right) + 12 \times 11^k \\
&= 5 \times 3Q + 12 \times 11^k \\
&= 3\left(\underbrace{5Q + 4 \times 11^k}_{R}\right) \\
&= 3R \\
&= \text{RHS}
\end{aligned}$$

□

Example 1.2.2. Prove by mathematical induction that $x^n - 1$ is divisible by $x - 1$ for $n \geq 1$.

Proof.

$$P(1) : x^1 - 1 = 1 \times (x - 1)$$

Hence the statement is true for $n = 1$.

$$P(k) : x^k - 1 = Q\,(x - 1) \qquad \text{Where } Q \text{ is some integer}$$

$$P(k+1) : x^{k+1} - 1 = R\,(x - 1) \qquad \text{Where } R \text{ is some integer}$$

Now in order to prove $P(k+1)$, let's multiply both sides of $P(k)$ by x.

$$x^{k+1} - x = Qx\,(x - 1) \Rightarrow x^{k+1} - 1 = Qx\,(x - 1) + x - 1$$

$$\therefore x^{k+1} - 1 = (x - 1)\left(\underbrace{Qx + 1}_{R}\right) \Rightarrow x^{k+1} - 1 = R\,(x - 1)$$

□

Example 1.2.3. Prove by induction for even n that $n^3 + 2n$ is divisible by 12.

Proof.

$$P(2): 2^3 + 2 \times 2 = 8 + 4 = 12 = 12 \times 1$$

Hence the statement is true for $n = 2$.

$$P(k): k^3 + 2k = 12Q \qquad \text{Where } Q \text{ is some integer}$$

$$P(k+2): (k+2)^3 + 2(k+2) = 12R \qquad \text{Where } R \text{ is some integer}$$

Now $P(k+2)$ can be written as

$$\begin{aligned}
\text{LHS} &= k^3 + 6k^2 + 12k + 8 + 2k + 4 \\
&= \underbrace{k^3 + 2k}_{12Q} + 6k^2 + 12k + 12 \\
&= 12\left(\underbrace{Q + \frac{1}{2}k^2 + k + 1}_{R}\right) \\
&= 12R \\
&= \text{RHS}
\end{aligned}$$

□

1.3 Inequalities

Example 1.3.1. By using induction show that $(1+p)^n \geq 1 + np$, where $p > -1$.

Proof.

$$P(1): 1 + p = 1 + p$$

Therefore the statement is true for $n = 1$.

$$P(k): (1+p)^k \geq 1 + kp$$

$$P(k+1): (1+p)^{k+1} \geq 1 + p(k+1)$$

By multiplying both sides of $P(k)$ by $(1+p)$, we get

$$(1+p)^{k+1} \geq (1+kp)(1+p) \tag{1.3.1}$$

$$\begin{aligned}
(1+kp)(1+p) &= 1 + p + kp + kp^2 \\
&= (1 + p(k+1)) + kp^2 > 1 + p(k+1)
\end{aligned} \tag{1.3.2}$$

By comparing equations 1.3.1 and 1.3.2, it becomes clear that $(1+p)^{k+1} \geq 1 + p(k+1)$.

□

Example 1.3.2. Prove by mathematical induction that for all integers $n \geq 5$, $n^2 < 2^n$.

Proof.

$$P(5): 5^2 < 2^5 \Rightarrow 25 < 32$$

Thus the statement is true for $n = 5$.

$$P(k): k^2 < 2^k$$

$$P(k+1) : (k+1)^2 < 2^{k+1}$$

By multiplying both sides of $P(k)$ by 2, we have

$$2k^2 < 2^{k+1} \tag{1.3.3}$$

$$(k+1)^2 < 2k^2 \tag{1.3.4}$$

By comparing equation 1.3.3 and 1.3.4, it becomes clear that $(k+1)^2 < 2^{k+1}$. □

Example 1.3.3. Prove by induction that $\displaystyle\sum_{r=1}^{n} \frac{1}{r^2} \leq 2 - \frac{1}{n}$, for $n \geq 1$.

Proof.

$$P(1) : 1 \leq 2 - 1 = 1$$

Hence the statement is true for $n = 1$.

$$P(k) : \sum_{r=1}^{k} \frac{1}{r^2} \leq 2 - \frac{1}{k}$$

$$P(k+1) : \sum_{r=1}^{k+1} \frac{1}{r^2} \leq 2 - \frac{1}{k+1}$$

Now by adding $\dfrac{1}{(k+1)^2}$ to both sides of $P(k)$, we get

$$\sum_{r=1}^{k+1} \frac{1}{r^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \tag{1.3.5}$$

$$\begin{aligned}
2 - \frac{1}{k} + \frac{1}{(k+1)^2} &= 2 + \frac{-k^2 - 2k - 1 + k}{(k+1)^2} \\
&= 2 + \frac{-k^2 - k - 1}{(k+1)^2} \\
&= 2 - \left(\frac{k^2 + k + 1}{(k+1)^2}\right) \\
&= 2 - \left(\frac{k(k+1) + 1}{(k+1)^2}\right) \\
&= 2 - \left(\frac{k}{k+1} + \frac{1}{(k+1)^2}\right) < 2 - \frac{1}{k+1} \qquad (1.3.6)
\end{aligned}$$

By comparing equations 1.3.5 and 1.3.6, it becomes clear that $\displaystyle\sum_{r=1}^{k+1} \frac{1}{r^2} \leq 2 - \frac{1}{k+1}$. □