Suggested Marking Scheme as a result of markers meeting

ar 12 3 Unit Trial HSC Barker College 1999 - Solutions

estion 1

$\lim_{x\to 0} \frac{\sin 5x}{2x} = \lim_{x\to 0} \frac{\sin 5x}{5x} \times \frac{5}{2}$

$= 1 \times \frac{5}{2} = \frac{5}{2}$ 1

(i) $\int_0^1 \frac{e^{2x}}{e^{2x}+1}\,dx$

$\frac{1}{2}\int_0^1 \frac{2e^{2x}}{e^{2x}+1}\,dx$

$\frac{1}{2}\left[\ln(e^{2x}+1)\right]_0^1$ 1

$\frac{1}{2}\left[\ln(e^2+1)-\ln(e^0+1)\right]$

$\frac{1}{2}\left[\ln(e^2+1)-\ln(1+1)\right]$ 1

$\frac{1}{2}\left[\ln(e^2+1)-\ln 2\right] = \frac{1}{2}\ln\left(\frac{e^2+1}{2}\right)$

$\int_0^4 \frac{3}{\sqrt{16-x^2}}\,dx$

$3\left[\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4$ 1

$\left[\sin^{-1}(1)-\sin^{-1}(0)\right]$

$\left(\frac{\pi}{2}-0\right)$

$\frac{3\pi}{2}$ 1

$\frac{2x}{x-1} > 1$

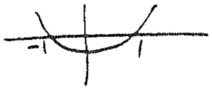
$(x-1)^2 \times \frac{2x}{(x-1)} > 1.(x-1)^2$

$x(x-1) > (x-1)^2$

$2x^2-2x > x^2-2x+1$ 1

$x^2-1>0$

$(x-1)(x+1)>0$



$x<-1$ or $x>1$ 1

Extend $\Rightarrow$ k : l = -3 : 5

(e) For $y=\log_e x$, $y' = \frac{1}{x}$

when $x=1$, $m_1 = 1$ 1

For $y = 1-x^2$, $y' = -2x$

When $x=1$, $m_2=-2$

$\therefore \tan\theta = \left|\frac{1+2}{1+1\times -2}\right| = \left|\frac{3}{-1}\right| = 3$ 1

$\therefore \theta \doteqdot 71^\circ 34'$ 1

Question 2

(a)(i) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ 1

(ii) $\cos 105^\circ = \cos(60^\circ+45^\circ)$

$= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$ 1

$= \frac{1}{2}\times\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$

$= \frac{1-\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$ 1

$= \frac{\sqrt{2}-\sqrt{6}}{4}$

(b)(i) $P(\text{no girls}) = \frac{{}^4C_3}{{}^{12}C_3} = \frac{4}{220} = \frac{1}{55}$ 1

(ii) $P(\text{exactly 1 girl}) = \frac{{}^8C_1 \times {}^4C_2}{{}^{12}C_3} = \frac{8\times 6}{220} = \frac{12}{55}$ 1

(iii) $P(\text{at least 2 girls}) = 1 - P(\text{No girls or 1 girl})$

$= 1 - \left(\frac{1}{55}+\frac{12}{55}\right)$ 1

$= \frac{42}{55}$

(c) $LHS = \frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1+(2\cos^2\theta-1)}$ 1

$= \frac{2\sin\theta\cos\theta}{2\cos^2\theta}$

$= \frac{\sin\theta}{\cos\theta}$ 1

$= \tan\theta = RHS$

$\int_0^1 x\sqrt{1-x}\,dx = \int_1^0 -(1-u)\sqrt{u}\,du$ (1)

$= \int_1^0 -u^{1/2}(1-u)\,du$

$= \int_1^0 -u^{1/2}+u^{3/2}\,du$

$= \left[-\frac{2u^{3/2}}{3}+\frac{2u^{5/2}}{5}\right]_1^0$ (1)

$= 0+0-\left(-\frac{2}{3}+\frac{2}{5}\right)$

$= \frac{2}{3}-\frac{2}{5}$

$= \frac{4}{15}$ (1)

uestion 3

i) 12 people

∴ No. of outcomes $= (12-1)!$

$= 11!$ (1)

$= 39916800$

i) SR leaving 10 people

ie. no. of outcomes $= 10!$

But can have SR or RS, thus (1) for method

no. of outcomes $= 2\times 10!$

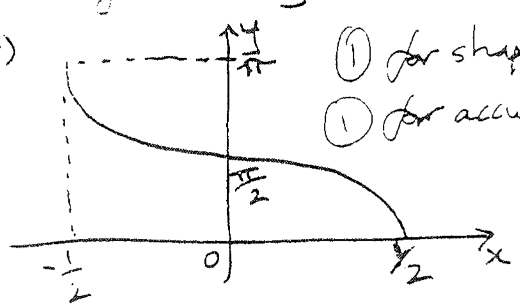
$P(\text{S and R together}) = \frac{2\times 10!}{11!}$

$= \frac{2}{11}$ (1)

(i) Domain $= -1 \le 2x \le 1$

$\therefore -\frac{1}{2} \le x \le \frac{1}{2}$ (1)

Range $= 0 \le y \le \pi$

ii) 

(1) for shape

(1) for accuracy

)(i) $\tan^{-1}(\sqrt{3}) - \tan^{-1}(-1) = \frac{\pi}{3} - \left(-\frac{\pi}{4}\right)$

$= \frac{7\pi}{}$ (1)

$\therefore \text{Area} = \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2\sqrt{3}}$

$= \frac{1}{2}\left[\tan^{-1}(\sqrt{3}) - \tan^{-1}(-1)\right]$

$= \frac{1}{2}\times\frac{7\pi}{12}$

$= \frac{7\pi}{24}$ units^2 (1)

(d) If $n=1$, $7^1-1=6$ which is divisible by 6

∴ Statement is true for $n=1$ (1)

Assume statement is true for $n=k$

ie. $\frac{7^k-1}{6} = M$ (where M is an integer)

ie. $7^k-1 = 6M$

ie. $7^k = 6M+1$

Now, $7^{k+1}-1 = 7^k\times 7^1 - 1$

$=(6M+1)7-1$

$= 42M+7-1$

$= 42M+6$

$= 6(7M+1)$ which is divisible by 6. (1)

∴ If statement is true for $n=k$, then statement is true for $n=k+1$.

Thus, since statement is true for $n=1$, it is true for $n=2,3,4$, etc. (1)

Thus, statement is true for all $n\ge 1$ (where n is an integer)

Question 4

(a) $y = \sin^{-1}(\cos x)$

$\therefore \frac{dy}{dx} = \frac{-\sin x}{\sqrt{1-\cos^2 x}}$ (1)

$= \frac{-\sin x}{\sqrt{\sin^2 x}}$

$= \frac{-\sin x}{\sin x}$

$= -1$ (if $\sin x > 0$) (1)

(b) General term $= {}^6C_r\, x^{6-r}\left(\frac{1}{2x^2}\right)^r$

$= {}^6C_r\, x^{6-r}\left(\frac{1}{2}\right)^r\left(x^{-2}\right)^r$

...rm independent of x occurs when
$6-3r=0$
$\therefore 3r=6$ i.e. when $r=2$ (1)

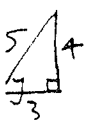
Term $= {}^6C_2 \times \left(\frac{1}{2}\right)^2$
$= 15 \times \frac{1}{4}$
$= \frac{15}{4}$ (1)

OR
(2)
(1)

$3\sin x + 4\cos x = A\sin(x+y)$
$= A\sin x\cos y + A\sin y\cos x$

$A\cos y = 3$ and $A\sin y = 4$

$\cos y = \frac{3}{A}$ and $\sin y = \frac{4}{A}$



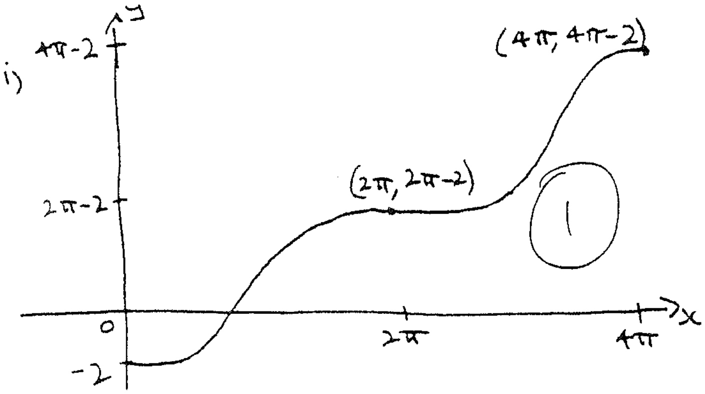
$\therefore A = \sqrt{3^2+4^2}$
$= 5$ (1)

$y = \cos^{-1}\left(\frac{3}{5}\right) \doteqdot 0.9273$

$3\sin x + 4\cos x = 5\sin(x + 0.9273)$

OR using t-method
(1) Subst.
(2) Getting quad
(1) Answers

(ii)



(1)

Question 5

(a)(i) If $T = T_0 + Ae^{-kt}$, then $T - T_0 = Ae^{-kt}$

Now, $\frac{dT}{dt} = 0 - kAe^{-kt}$
$= -k(Ae^{-kt})$
$= -k(T-T_0)$ (1)

(ii) $T_0 = 70$ and thus $T = 70 + Ae^{-kt}$
when $t=0$, $T=150$
$\therefore 150 = 70 + Ae^0$
$\therefore 150 = 70 + A$ $\therefore A = 80$ (1)

When $t=40$ $T=90$

$\therefore e^{-2t} = 1 - 4/5 = 1/5$

$\therefore -2t = \log_e(1/5)$ 1

$t = \frac{\ln(1/5)}{-2} \doteqdot 0.8047$ seconds

) As $t \to \infty$, $e^{-2t} \to 0$

$\therefore (1 - e^{-2t}) \to 1$

$\therefore 500(1 - e^{-2t}) \to 500$ 1

Max possible volume $= 500$ units^3

Assuming sphere $\Rightarrow V = \frac{4}{3}\pi r^3$

$\therefore$ when $V = 400$,

$400 = \frac{4}{3}\pi r^3$

$\therefore 300 = \pi r^3$

$\therefore r^3 = \frac{300}{\pi}$ $\quad \therefore r = \sqrt[3]{\frac{300}{\pi}} \doteqdot 4.5708$ 1

...ow, if $V = \frac{4}{3}\pi r^3$, then

$\frac{dV}{dr} = 4\pi r^2$

$\therefore \frac{dr}{dV} = \frac{1}{4\pi r^2}$

...en $V = 400$, $r = \sqrt[3]{\frac{300}{\pi}}$, $t = \frac{\ln(1/5)}{-2}$

$\therefore \frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$ 1

$\frac{dr}{dt} = \frac{1}{4\pi r^2} \times 1000e^{-2t}$

$= \frac{1}{4\pi\left(\sqrt[3]{\frac{300}{\pi}}\right)^2} \times 1000e^{-2\times\frac{\ln(1/5)}{-2}}$

$\doteqdot 0.7618$ 1

...stion 6

$(1+x)^{m+n} = 1 + {}^{m+n}C_1 x + {}^{m+n}C_2 x^2 + \ldots$

...efficient of $x^2 = {}^{m+n}C_2 = \binom{m+n}{2}$ 1

... $(1+x)^m(1+x)^n$

$(1 + {}^mC_1 x + {}^mC_2 x^2 + \ldots)(1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots)$

...s containing x^2 will be 1

$\ldots x^2 + {}^mC_1 x \times {}^nC_1 x + {}^mC_2 x^2 \times 1$

1

$\therefore$ Comparing coefficients of x^2 on both sides

$$\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + \binom{m}{1}\binom{n}{1}$$

(b)(i) $x = \sqrt{3}\cos 3t - \sin 3t$

$\dot{x} = -3\sqrt{3}\sin 3t - 3\cos 3t$

$\ddot{x} = -9\sqrt{3}\cos 3t + 9\sin 3t$

$= -9(\sqrt{3}\cos 3t - \sin 3t)$ 1

$\therefore \ddot{x} = -9x$ which is the form $\ddot{x} = -n^2 x$

$\therefore$ Motion is SHM.

(ii) Period $= \frac{2\pi}{3}$ 1

(iii) When $x = 0$,

$0 = \sqrt{3}\cos 3t - \sin 3t$ 1

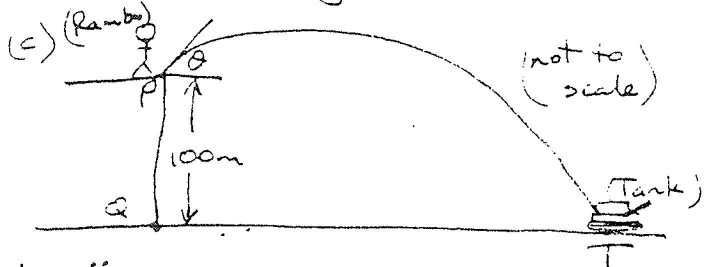
$\therefore \sin 3t = \sqrt{3}\cos 3t$

$\therefore \tan 3t = \sqrt{3}$

$\therefore 3t = \frac{\pi}{3}, \frac{4\pi}{3}, \ldots$

$\therefore t = \frac{\pi}{9}, \frac{4\pi}{9}, \ldots$ 1

$\therefore$ First passes origin at $t = \frac{\pi}{9}$ seconds

(c)


(i) $\ddot{x} = 0$ $\qquad \ddot{y} = -10$

$\therefore \dot{x} = c_1$ $\qquad \dot{y} = -10t + c_2$

when $t = 0$, $\dot{x} = V\cos\theta$ and $\dot{y} = V\sin\theta$

$\therefore \dot{x} = V\cos\theta$ $\qquad \dot{y} = -10t + V\sin\theta$

$\therefore x = Vt\cos\theta + c_3$ $\qquad y = -5t^2 + Vt\sin\theta + c_4$

Let P be origin, thus when $t = 0$, $x = 0$ and $y = 0$

$\therefore c_3 = c_4 = 0$

$\therefore x = Vt\cos\theta$ and $y = -5t^2 + Vt\sin\theta$ 1

Now, $V = \frac{190}{\sqrt{3}}$ and when $t = 20$, $y = -100$

$\therefore -100 = -5 \times 20^2 + \frac{190}{\sqrt{3}} \times 20 \times \sin\theta$ 1

$\therefore -100 = -2000 + \frac{3800\sin\theta}{\sqrt{3}}$

(iii) Max height occurs when $\dot{y} = 0$

$\therefore 0 = -10t + \frac{190}{\sqrt{3}}\sin 60°$ 1

$\therefore 10t = \frac{190}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = 95$

$\therefore t = 9.5$ sec

when $t = 9.5$,

$y = 100 + \left(-5 \times 9.5^2 + \frac{190}{\sqrt{3}} \times 9.5 \times \sin 60°\right)$

$y = 100 - 451.25 + 902.5$

Max height = 551.25 m 1

estion 7

(i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

P, $m = \frac{2ap}{2a} = p$

n of tangent at P is

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

Q, $m = \frac{2aq}{2a} = q$

n of tangent at Q is

$y - aq^2 = q(x - 2aq)$

$y - aq^2 = qx - 2aq^2$

$\therefore y = qx - aq^2$ 1

$\left.\begin{array}{l} y = px - ap^2 \\ y = qx - aq^2 \end{array}\right\} \therefore px - ap^2 = qx - aq^2$

$\therefore px - qx = ap^2 - aq^2$

$\therefore x(p - q) = a(p - q)(p + q)$

$\therefore x = a(p + q)$

$y = ap(p + q) - ap^2$

$= ap^2 + apq - ap^2$

$= apq$

$\equiv (a(p + q), apq)$ 1

T lies on parabola $x^2 = -4ay$

$a^2(p + q)^2 = -4a^2pq$

(iv) Midpt of PQ = $\left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$

$= \left(a(p + q), \frac{a}{2}(p^2 + q^2)\right)$

$\therefore x = a(p + q)$ and $y = \frac{a}{2}(p^2 + q^2)$ 1

$\therefore \frac{x}{a} = p + q$ $\quad y = \frac{a}{2} \times -6pq = -3apq$

$\therefore \frac{y}{-3a} = pq$

Now, if $p^2 + q^2 = -6pq$, then

$p^2 + 2pq + q^2 = -4pq$

$\therefore (p + q)^2 = -4pq$

using these 1

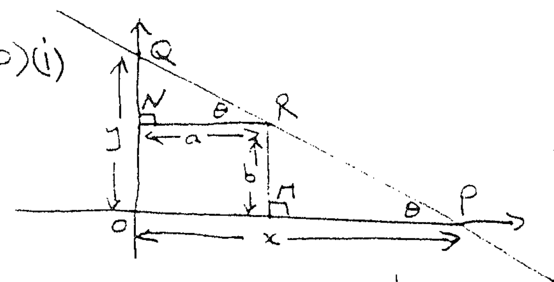
Thus, $\left(\frac{x}{a}\right)^2 = -4 \times \frac{y}{-3a}$

$\therefore \frac{x^2}{a^2} = \frac{4y}{3a}$

$\therefore$ Eqn of locus of midpt of PQ is

$y = \frac{3x^2}{4a}$ 1

(b)(i)



From ΔRPM, $\tan\theta = \frac{b}{x - a}$

$\therefore \cot\theta = \frac{x - a}{b}$

$\therefore b\cot\theta = x - a$

$\therefore x = a + b\cot\theta$

From ΔQRN, $\tan\theta = \frac{y - b}{a}$

$\therefore a\tan\theta = y - b$

$\therefore y = b + a\tan\theta$

Now, length of PQ = $\sqrt{x^2 + y^2}$

$\therefore l^2 = x^2 + y^2$

$= (a + b\cot\theta)^2 + (b + a\tan\theta)^2$

$= a^2 + 2ab\cot\theta + b^2\cot^2\theta + b^2 + 2ab\tan\theta + a^2\tan^2\theta$

$= a^2 + a^2\tan^2\theta + b^2 + b^2\cot^2\theta + 2ab(\tan\theta + \cot\theta)$

$= a^2(1 + \tan^2\theta) + b^2(1 + \cot^2\theta) + 2ab\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$

2 for method

$l^2 = a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta + 2ab\left(\frac{1}{\sin\theta\cos\theta}\right)$

$= a^2\sec^2\theta + 2ab\sec\theta\operatorname{cosec}\theta + b^2\operatorname{cosec}^2\theta$

$l^2 = (a\sec\theta + b\operatorname{cosec}\theta)^2$

$l = a\sec\theta + b\operatorname{cosec}\theta$ (since $l>0$)

$l = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$

$l = a(\cos\theta)^{-1} + b(\sin\theta)^{-1}$

$l' = -a(\cos\theta)^{-2}\times -\sin\theta - b(\sin\theta)^{-2}\cos\theta$

$= \frac{a\sin\theta}{\cos^2\theta} - \frac{b\cos\theta}{\sin^2\theta}$ $\Longleftarrow$ 1

$\therefore l' = 0 \Rightarrow \frac{a\sin\theta}{\cos^2\theta} - \frac{b\cos\theta}{\sin^2\theta} = 0$

$\therefore \frac{a\sin\theta}{\cos^2\theta} = \frac{b\cos\theta}{\sin^2\theta}$

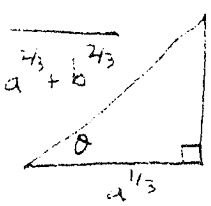
$\therefore a\sin^3\theta = b\cos^3\theta$

$\therefore \tan^3\theta = \frac{b}{a}$

$\therefore \tan\theta = \left(\frac{b}{a}\right)^{1/3}$

$\therefore \theta = \tan^{-1}\left(\frac{b}{a}\right)^{1/3}$ $\Longleftarrow$ 1

$\therefore$ if $\tan\theta = \frac{b^{1/3}}{a^{1/3}}$, Thus



$\cos\theta = \frac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$

and $\sin\theta = \frac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$

$l = \frac{a}{\frac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}} + \frac{b}{\frac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}}$

$= a^{2/3}\sqrt{a^{2/3}+b^{2/3}} + b^{2/3}\sqrt{a^{2/3}+b^{2/3}}$

$= \sqrt{a^{2/3}+b^{2/3}}\,(a^{2/3}+b^{2/3})$

$= (a^{2/3}+b^{2/3})^{3/2}$ 1

Alternative soln

(i) From ΔRPM, $\cos\theta = \frac{x-a}{PR}$ $\therefore PR = \frac{x-a}{\cos\theta}$

From ΔQRN, $\sin\theta = \frac{y-b}{QR}$ $\therefore QR = \frac{y-b}{\sin\theta}$

Now $PQ = PR + QR$

$\therefore PQ = \frac{x-a}{\cos\theta} + \frac{y-b}{\sin\theta}$

$\therefore PQ = \frac{x}{\cos\theta} - \frac{a}{\cos\theta} + \frac{y}{\sin\theta} - \frac{b}{\sin\theta}$

(Now, from ΔOPQ, $\sin\theta = \frac{y}{PQ}$ and $\cos\theta = \frac{x}{PQ}$

$\therefore PQ = \frac{y}{\sin\theta}$ and $PQ = \frac{x}{\cos\theta}$)

$\therefore PQ = PQ - \frac{a}{\cos\theta} + PQ - \frac{b}{\sin\theta}$

$\therefore PQ = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$ 2 for Part (i) method

OR

(ii) $l' = \frac{a\sin^3\theta - b\cos^3\theta}{\sin^2\theta\cos^2\theta}$

$= \frac{(a^{1/3}\sin\theta - b^{1/3}\cos\theta)(a^{2/3}\sin^2\theta + (ab)^{1/3}\sin\theta\cos\theta + b^{2/3}\cos^2\theta)}{\sin^2\theta\cos^2\theta}$

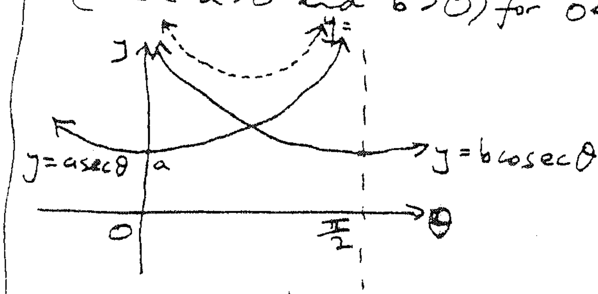
$l' = 0 \Rightarrow a^{1/3}\sin\theta - b^{1/3}\cos\theta = 0$

(since $0<\theta<\frac{\pi}{2}$ and thus $(a^{2/3}\sin^2\theta + (ab)^{1/3}\sin\theta\cos\theta + b^{2/3}\cos^2\theta) > 0$)

$\therefore a^{1/3}\sin\theta = b^{1/3}\cos\theta$

$\therefore \tan\theta = \frac{b^{1/3}}{a^{1/3}}$ $\therefore \theta = \tan^{-1}\left(\frac{b}{a}\right)^{1/3}$

To prove minimum value, investigate the graphs of $y = a\sec\theta$ and $y = b\operatorname{cosec}\theta$ (where $a>0$ and $b>0$) for $0<\theta<\frac{\pi}{2}$.



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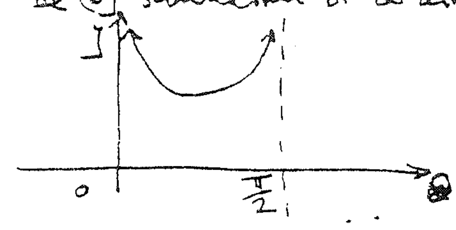
thus the graph of $y = a\sec\theta + b\operatorname{cosec}\theta$ will be (by summation of ordinates)



$\therefore$ Minimum turning pt exists in domain $0<\theta<\frac{\pi}{2}$