Student Number: ………..…………………….…

2008

HIGHER SCHOOL CERTIFICATE

Sample Examination Paper

MATHEMATICS EXTENSION 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Write your student number at the top of this page
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

Directions to school or college

ISBN: 978 1 44250 354 0

Total marks – 84
Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Sketch the region in the plane defined by $y \leq |3 - x|$. **2**

(b) State the domain and range of $y = \sin^{-1}\left(\dfrac{x}{3}\right)$. **2**

(c) Let C be the point (3, 1) and D the point (5, –1). **2**

Find the coordinates of the point Q which divides the interval CD externally in the ratio 2 : 3.

(d) Find $\displaystyle\int_0^2 \frac{dx}{4+x^2}$. **2**

(e) Using the difference of two cubes, simplify **2**

$$\frac{\sin^3\theta - \cos^3\theta}{\sin\theta - \cos\theta} - 1$$

for $0 < \theta < \dfrac{\pi}{2}$.

(f) Using the substitution $u = x^2 + 1$, or otherwise, find $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+1}}$. **2**

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Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Differentiate $\sin^{-1}(x^3)$ with respect to x. **2**

(b) Use the table of Standard Integrals to evaluate $\displaystyle\int_5^7 \frac{1}{\sqrt{x^2-9}}\,dx$. **2**

(c) (i) Write down an expression for $\cos 2\theta$. **1**

(ii) Given that $\sin 18° = \dfrac{\sqrt{5}-1}{4}$, find the exact value of cos 36°. **2**

(d) The polynomial $P(x) = x^3 + ax^2 + bx - 18$ has a zero at $x = -2$. When $P(x)$ is divided by $x - 1$, the remainder is –24. **3**

Find the values of a and b.

(e) A fair, six-sided die is thrown five times. What is the probability that a '3' occurs on exactly two of the five throws? **2**

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Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $P(x) = x^3 - x^2 - x - 1$ has a zero between 1 and 2. **1**

(ii) Take $x = 2$ as a first approximation and use Newton's method to calculate a second approximation. **2**

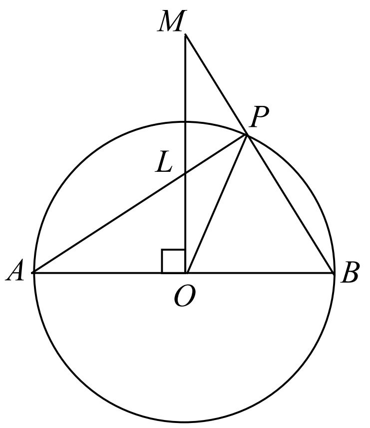
(iii) Explain why $x = 1$ was not a suitable first approximation in this case. **1**

(b) (i) If $\dfrac{2x+1}{1-x} = A + \dfrac{B}{1-x}$, find A and B. **1**

(ii) Hence find the vertical and horizontal asymptotes of $y = \dfrac{2x+1}{1-x}$. **2**

(iii) Hence, or otherwise, find the values of x for which $\dfrac{2x+1}{1-x} > -2$. **1**

(c)



NOT TO SCALE

O is the centre of the circle ABP. $MO \perp AB$. M, P and B are collinear.
MO intersects AP at L.

(i) Prove that A, O, P and M are concyclic. **2**

(ii) Prove that $\angle OPA = \angle OMB$. **2**

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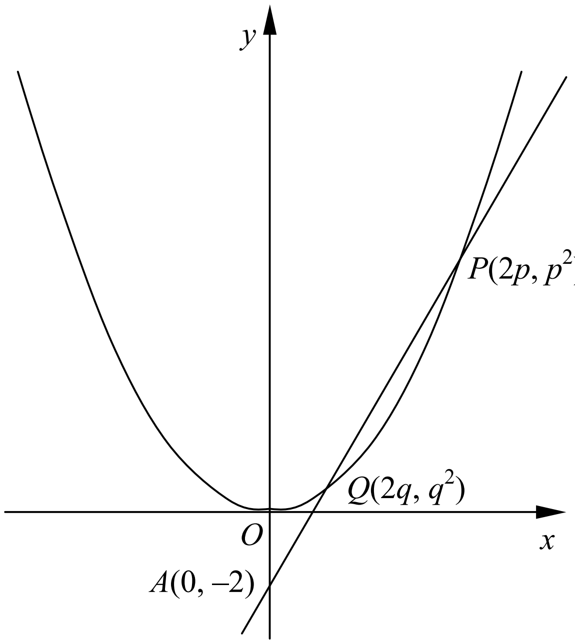
Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) Find the exact value of the volume of the solid of revolution formed when the region in the first quadrant bounded by the curve $y = \cos 2x$, the x axis and the line $x = \dfrac{\pi}{12}$ is rotated about the x-axis. **3**

(b) Use mathematical induction to prove that, for all integers $n \geq 1$: **3**

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

(c)



The diagram shows the points $P(2p, p^2)$ and $Q(2q, q^2)$ on the parabola $x = 2t$, $y = t^2$, where $p \neq q$.

The equation of the normal at P is $x + py - 2p - p^3 = 0$. (Do NOT prove this.)

(i) The normals at P and Q intersect at $R(X, Y)$. **2**
Show that $X = -pq(p + q)$ and $Y = (p + q)^2 - pq + 2$.

(ii) The chord PQ has gradient m and passes through the point $A(0, -2)$. **1**
Find, in terms of m, the equation of PQ and hence show that p and q are the roots of the equation $t^2 - 2mt + 2 = 0$.

(iii) By considering the sum and product of the roots of this quadratic equation, show that the point R lies on the original parabola. **3**

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Marks

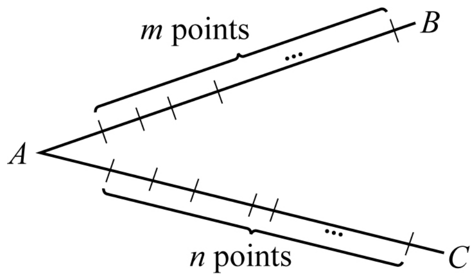
Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Write $4\sin x - 3\cos x$ in the form $A\sin(x - \alpha)$, where α is an acute angle in radians and $A > 0$. **1**

(ii) Find the abscissa of the first point of intersection of the curves $y = 3\cos x$ and $y = 4\sin x - 2.5$ over the domain $0 \le x \le 2\pi$. **1**

(iii) Hence, or otherwise, find the angle between the curves at this point. **3**

(b) There are m points marked on one straight line AB and n points marked on another straight line AC, none of them being the point A, as shown on the diagram.



(i) How many triangles can be formed with these points as vertices? Give your answer as an expression in terms of m and n. **3**

(ii) How many triangles can be formed if the point A can be one of the vertices? Give your answer as an expression in terms of m and n. **1**

(c) Find the exact values of x and y which satisfy the simultaneous equations **3**

$$\sin^{-1} x - \frac{1}{2}\cos^{-1} x = \frac{\pi}{3} \qquad \text{and}$$
$$2\sin^{-1} x + \frac{1}{2}\cos^{-1} x = \frac{7\pi}{6}.$$

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Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) Show that $y = 15e^{-0.6t} + 4$ is a solution of $\dfrac{dy}{dt} = -0.6(y-4)$. **2**

(b) A particle moves in a straight line so that its distance from the origin at time t seconds is x metres.

(i) If $\dfrac{d^2x}{dt^2} = 8x - 2x^3$, show that $v^2 + x^4 - 8x^2$ is a constant. **2**

(ii) If the initial conditions were $v = 3$ when $x = 0$, find the value of the constant in part (i). **1**

(iii) Hence, or otherwise, establish that the particle with this initial velocity remains at all times in the region $-3 \le x \le 3$. **2**

State, giving your reason, whether the motion is Simple Harmonic.

(c) Consider the function $f(x) = e^{-x} - e^{x}$.

(i) Show that $f(x)$ is decreasing for all values of x. **1**

(ii) Show that the inverse function is given by **3**

$$f^{-1}(x) = \log_e\left(\frac{\sqrt{x^2+4}-x}{2}\right).$$

(iii) Hence, or otherwise, solve $e^{x} - e^{-x} = 6$. **1**

Give your answer correct to two decimal places.

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Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) Let $f(\theta) = \cos\theta \sin^2\theta$.

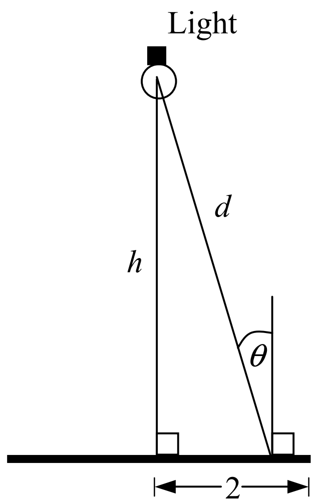
(i) Show that $f'(\theta) = \sin\theta(3\cos^2\theta - 1)$ for $0 < \theta \leq \dfrac{\pi}{2}$. **2**

(ii) Hence, or otherwise, find an expression for the exact value of θ for which $f(\theta)$ is a maximum. **1**

(b) A light hangs at a vertical distance h metres above the centre of a *circular* table of radius 2 metres.

At any point on the table where the angle of incidence is θ and the distance from the light is d, as shown in the diagram, assume that the illumination I is given by

$$I = \frac{k\cos\theta}{d^2}, \text{ where } k \text{ is a positive constant.}$$



(i) Show that, at the edge of the table, $I = \dfrac{k\cos\theta\sin^2\theta}{4}$. **1**

(ii) The vertical height of the light above the table is varied. Using part (a), or otherwise, find the value of h that gives the maximum illumination at the edge of the table. **1**

(iii) If the light is raised vertically at 0.2 ms^{-1}, find an expression for $\dfrac{d\theta}{dt}$. **1**

(iv) Hence, or otherwise, find $\dfrac{dI}{dt}$ at the edge of the table when the light is 2 metres above the table. **2**

Question 7 continues on page 9

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Marks

Question 7 (continued)

(c) (i) By applying the binomial theorem to $(1 + x)^n$ and differentiating, show that **1**

$$\binom{n}{1}+2\binom{n}{2}x+\ldots+r\binom{n}{r}x^{r-1}+\ldots+n\binom{n}{n}x^{n-1}=n(1+x)^{n-1}.$$

(ii) Hence find an expression for **1**

$$\binom{n}{1}x+2\binom{n}{2}x^2+\ldots+r\binom{n}{r}x^r+\ldots+n\binom{n}{n}x^n.$$

(iii) Hence, or otherwise, if $n > 2$, find the value of **2**

$$\binom{n}{1}-2^2\binom{n}{2}+3^2\binom{n}{3}-\ldots+(-1)^{n-1}n^2\binom{n}{n}.$$

End of paper

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STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

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Mapping grid

Question	Mark	Content	Outcome	Band
1(a)	2	Regions and absolute value	P4	E2
1(b)	2	Inverse functions	HE4	E2
1(c)	2	External division	PE2	E2
1(d)	2	Integration with inverse functions	HE4	E2
1(e)	2	Trigonometric identities, algebra	PE2	E2
1(f)	2	Integration by substitution	HE6	E2
2(a)	2	Differentiation, inverse functions	HE4, HE5	E2/3
2(b)	2	Techniques of integration	HE6	E2
2(c)	3	Trigonometry, double angle results	PE2	E2
2(d)	3	Polynomial results	PE3	E2
2(e)	2	Binomial probability	HE3	E2
3(a)	4	Newton's method with polynomials	HE1, HE3	E3
3(b)	4	Asymptotes and inequalities	HE3	E2/3
3(c)	4	Circle geometry	PE3	E3
4(a)	3	Integration of $\cos^2 2x$	HE6	E3
4(b)	3	Mathematical induction	HE2	E3
4(c)	6	Parabola and parametric equations	PE3, PE4	E3
5(a)	5	Auxiliary angle method, angle between curves	PE2, HE1, HE7	E3
5(b)	4	Combinations and counting techniques	PE3, HE7	E3/4
5(c)	3	Inverse trigonometry and simultaneous equations	HE4	E3/4
6(a)	2	Exponential growth and decay	HE3	E2
6(b)	5	Equations of motion	HE3, HE5	E3/4
6(c)	5	Application of inverse functions	HE4	E4
7(a)	3	Harder Mathematics, differentiation	PE5, HE7	E3
7(b)	5	Application of calculus	HE5	E4
7(c)	4	Binomial theorem	HE3	E4

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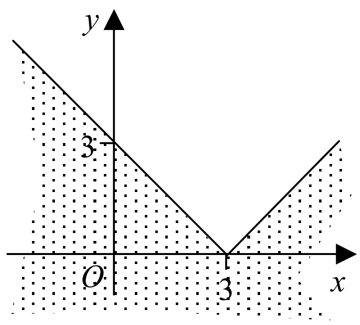
Solutions and marking guidelines

Question 1

Solution	Marks
(a) $y \le \|3-x\|$ 	**2** correct region 1 for correct boundary 1 correct region for wrong boundaries
(b) $y=\sin^{-1}\left(\frac{x}{3}\right)$ $\quad -1\le \frac{x}{3}\le 1$, Domain: $-3\le x\le 3$ Range: $\frac{-\pi}{2}\le y\le \frac{\pi}{2}$	**2** both answers 1 domain 1 range
(c) $Q\left(\frac{3\times 3-2\times 5}{-2+3}, \frac{3\times 1-2\times(-1)}{1}\right)$ i.e. Q is $(-1, 5)$	**2** correct answer 1 evidence of correct use of formula
(d) $\int_0^2 \frac{dx}{4+x^2}=\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2=\frac{1}{2}\left(\tan^{-1}1-0\right) \quad =\frac{\pi}{8}$	**2** correct answer 1 correct primitive 1 correctly evaluate their primitive
(e) $\frac{\sin^3\theta-\cos^3\theta}{\sin\theta-\cos\theta}-1$ $=\frac{(\sin\theta-\cos\theta)\left(\sin^2\theta+\sin\theta\cos\theta+\cos^2\theta\right)}{\sin\theta-\cos\theta}-1$ $=1+\sin\theta\cos\theta-1=\sin\theta\cos\theta \quad \left\{=\frac{\sin 2\theta}{2}\right\}$	**2** correct answer 1 correct factorisation 1 correct simplification of their factorisation
(f) $u=x^2+1,\quad du=2x\,dx,\quad xdx=\frac{du}{2}$ $\int\frac{xdx}{\sqrt{x^2+1}}=\int\frac{\frac{du}{2}}{\sqrt{u}}=\frac{1}{2}\int u^{\frac{-1}{2}}du=u^{\frac{1}{2}}+C=\sqrt{x^2+1}+C$	**2** correct answer 1 correct manipulation of substitution 1 correct x from their incorrect substitution

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Question 2

Solution	Marks
(a) $\frac{d}{dx}\left[\sin^{-1}\left(x^3\right)\right]=\frac{1}{\sqrt{1-\left(x^3\right)^2}}\times 3x^2=\frac{3x^2}{\sqrt{1-x^6}}$	**2** correct answer 1 evidence of chain rule 1 a correct derivative
(b) $\int_5^7 \frac{1}{\sqrt{x^2-9}}\,dx=\left[\ln\left(x+\sqrt{x^2-9}\right)\right]_5^7$ $=\ln\left(7+\sqrt{40}\right)-\ln\left(5+\sqrt{16}\right)\quad\left\{=\ln\left(\frac{7+2\sqrt{10}}{9}\right)\right\}$	**2** correct answer 1 correct primitive 1 correct substitution into their primitive
(c)(i) $\cos 2\theta = 1-2\sin^2\theta\quad\{=2\cos^2\theta-1\}$	**1** correct expression
(c)(ii) $\cos 36° = 1-2\sin^2 18°$ $=1-2\times\left(\frac{\sqrt{5}-1}{4}\right)^2=1-\frac{2(5-2\sqrt{5}+1)}{16}=\frac{8-6+2\sqrt{5}}{8}=\frac{1+\sqrt{5}}{4}$	**2** correct answer 1 correct substitution 1 simplifying
(d) $P(x)=x^3+ax^2+bx-18$ $P(-2)=-8+4a-2b-18=0,\quad 4a-2b=26,\quad 2a-b=13$ $P(1)=1+a+b-18=-24\quad a+b=-7$ $3a=6$ so $a=2$ and $b=-9$	**3** correct answer 1 for each equation 1 both *a* and *b*
(e) $p=\frac{1}{6},\ q=\frac{5}{6},\ n=5$ $P(X=2)=\binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3\quad\left\{=10\times\frac{5^3}{6^5}=0.16075\right\}$	**2** correct solution 1 setting up *p*, *q*, *n* 1 binomial expression for their *p*, *q*, *n*.

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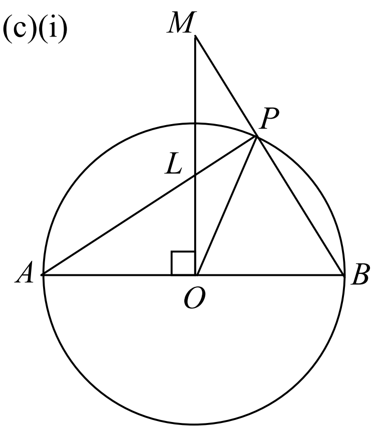
Question 3

Solution	Marks
(a)(i) $P(x) = x^3 - x^2 - x - 1$ $P(1) = 1 - 1 - 1 - 1 = -2$ $P(2) = 8 - 4 - 2 - 1 = 1$ $P(x)$ is continuous and changes sign between $x = 1$ and $x = 2$, hence zero there.	**1** correct answer
(a)(ii) $x = a - \dfrac{P(a)}{P'(a)}$, $a = 2$ $P(x) = x^3 - x^2 - x - 1$ $\quad$ $P'(x) = 3x^2 - 2x - 1$ $P(2) = 1$ $\quad$ $P'(2) = 12 - 4 - 1 = 7$ $x = 2 - \dfrac{1}{7} = 1\dfrac{6}{7} \approx 1.857$	**2** correct answer 1 finding and evaluating $P'(x)$ 1 using correct formula for their $P'(x)$
(a)(iii) $P'(1) = 0$ so Newton's method would involve dividing by zero. **OR** Tangent at $x = 1$ is horizontal, so will not cut x axis again to provide the next approximation.	**1** correct answer
(b)(i) $\dfrac{2x+1}{1-x} = A + \dfrac{B}{1-x}$ $\quad$ $2x + 1 = A(1 - x) + B$ $x = 1$: $\quad 3 = B$ $\quad$ $x = 0$: $\quad 1 = A + B$, $A = -2$ **OR** $2x + 1 = A - Ax + B$ $2 = -A \quad 1 = A + B$ $A = -2$, $B = 3$	**1** correct answer
(b)(ii) $y = -2 + \dfrac{3}{1-x}$ $\quad$ Horizontal asymptote is: $\quad y = -2$ Vertical asymptote is: $\quad x = 1$	**2** correct answer 1 for each correct asymptote
(b)(iii) $\dfrac{2x+1}{1-x} > -2$ when $-2 + \dfrac{3}{1-x} > -2$ or $\dfrac{3}{1-x} > 0$, so $x < 1$	**1** correct answer
(c)(i)  $\angle APB = 90°$ (angle in a semi-circle, AB a diameter) $\angle APM = 90°$ (MPB a straight angle) $\therefore\ \angle AOM = \angle APM = 90°$ ($MO \perp AB$) $\angle$s AOP and APM are a pair of equal angles standing on the same side of AM so the points A, O, P and M are concyclic.	**2** correct answer 1 all correct statements without reasons 1 two correct statements including reasons

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Question 3 (continued)

Solution	Marks
(c)(ii) Let $\angle OMB = x°$ $(= \angle OMP)$ $\angle OAP = \angle OMP = x°$ (angles in the same segment standing on AM) $OA = OP$, radii of circle, so ΔOAP is isosceles $\angle OPA = \angle OAP = x°$ (angles opposite equal sides of isosceles triangle) $\angle OPA = \angle OMB$ (both $x°$)	**2** correct answer 1 all correct statements without reasons 1 correct statement including reasons

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Question 4

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<tr><th>Solution</th><th>Marks</th></tr>
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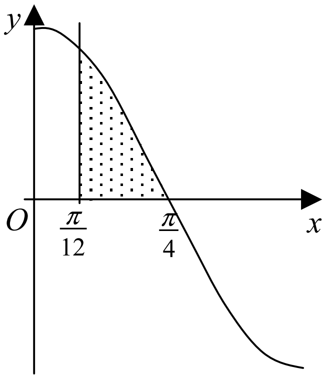
(a)



$y = \cos 2x, y = 0,\ x = \frac{\pi}{12},\ x = \frac{\pi}{4}$

$V = \pi \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} y^2 dx = \pi \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \cos^2 2x\, dx$

$V = \frac{\pi}{2} \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} (1 + \cos 4x)\, dx$

$V = \frac{\pi}{2}\left[x + \frac{\sin 4x}{4}\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}} = \frac{\pi}{2}\left(\frac{\pi}{4} + \frac{\sin \pi}{4} - \left(\frac{\pi}{12} + \frac{\sin \frac{\pi}{3}}{4}\right)\right)$

$V = \frac{\pi}{2}\left(\frac{\pi}{4} + 0 - \left(\frac{\pi}{12} + \frac{1}{4} \times \frac{\sqrt{3}}{2}\right)\right) \quad \left\{= \frac{\pi}{2}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)\right\}$

</td><td>

3 correct solution

1 correct integral

1 correct primitive

1 correct substitution and evaluation of trig functions

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(b) $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Prove true for $n = 1$: LHS $= \frac{1}{1\times 2} = \frac{1}{2}$, RHS $= \frac{1}{1+1} = \frac{1}{2} =$ LHS

Result is true for $n = 1$.

Assume result is true for $n = k$, i.e. assume:

$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + ... + \frac{1}{k(k+1)} = \frac{k}{k+1}$

Prove true for $n = k + 1$, i.e. prove:

$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + ... + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$

LHS $= \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + ... + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)}$

$= \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)}$

$= \frac{k+1}{k+2} =$ RHS

Hence result is true for $n = k$ if it is true for $n = k + 1$.

But result is true for $n = 1$, so it is true for $n = 1 + 1 = 2$ and, by the PMI, it is true for all integers $n \geq 1$.

</td><td>

3 correct solution

1 prove true for $n = 1$ and write assumption for $n = k$

1 setting up first line of prove

1 correct simplification to $\frac{(k+1)^2}{(k+1)(k+2)}$

Final statements not needed for full marks

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Question 4 (continued)

Solution	Marks
(c)(i) Normal at P: $x + py - 2p - p^3 = 0$ (i) Normal at Q: $x + qy - 2q - q^3 = 0$ (ii) For R, (i) – (ii): $(p - q)y - 2(p - q) - (p^3 - q^3) = 0$ $(p - q)y - 2(p - q) - (p - q)(p^2 + pq + q^2) = 0$ $p \neq q$: $y - 2 - (p^2 + pq + q^2) = 0$ $y = (p^2 + 2pq + q^2) - pq + 2$ $= (p + q)^2 - pq + 2$ Substitute in (i): $x + p(p^2 + pq + q^2 + 2) - 2p - p^3 = 0$ $x + p^3 + p^2q + pq^2 + 2p - 2p - p^3 = 0$ $x = -(p^2q + pq^2) = -pq(p + q)$	**2** correct answer 1 for each answer, CPE
(c)(ii) Equation of PQ: $y = mx - 2$ Chord cuts parabola $x = 2t$, $y = t^2$ at P, Q so: $t^2 = m \times 2t - 2$ Hence p and q are the roots of: $t^2 - 2mt + 2 = 0$	**1** correct answer
(c)(iii) $p + q = 2m$ $pq = 2$ $X = -pq(p + q)$ $Y = (p + q)^2 - pq + 2$ Hence: $X = -2 \times 2m$ $Y = (2m)^2 - 2 + 2$ $X = -4m$ $Y = 4m^2$ Locus of R is $4Y = X^2$, which is the original parabola.	**3** correct answer 1 sum and product of roots 1 X and Y 1 equation of locus

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Question 5

Solution	Marks
(a)(i) $4\sin x - 3\cos x = 5\left(\frac{4}{5}\sin x - \frac{3}{5}\cos x\right) = 5\sin(x-\alpha)$ $\alpha = \sin^{-1}\left(\frac{3}{5}\right) \approx 0.6435$ so $4\sin x - 3\cos x = 5\sin(x - 0.6435)$	**1** correct answer
(a)(ii) Intersection when: $3\cos x = 4\sin x - 2.5$ $4\sin x - 3\cos x = 2.5$ $5\sin(x - 0.6435) = 2.5$ $\sin(x - 0.6435) = 0.5$ $x - 0.6435 = \frac{\pi}{6}$ $\quad x = 1.1671$	**1** correct solution
(a)(iii) $y = 3\cos x$ $\quad \frac{dy}{dx} = -3\sin x$ $\quad x = 1.1671,\ m_1 = -2.7588$ $y = 4\sin x - 2.5$ $\quad \frac{dy}{dx} = 4\cos x$ $\quad x = 1.1671,\ m_2 = 1.5713$ $\tan\theta = \left\lvert\frac{m_1 - m_2}{1 + m_1 m_2}\right\rvert = \left\lvert\frac{-2.7588 - 1.5713}{1 - 2.7588 \times 1.5713}\right\rvert = 1.2984$ $\theta = 0.9145$ rad $\quad \{= 52°\ 24'\}$	**3** correct answer 1 each correct gradient 1 correct use of formula
(b)(i) Number of triangles $= m\binom{n}{2} + n\binom{m}{2} = \frac{mn(n-1)}{2} + \frac{nm(m-1)}{2}$ $= \frac{mn}{2}(n - 1 + m - 1)$ $= \frac{mn(m+n-2)}{2}$	**3** correct answer 1 for $m\binom{n}{2}$ 1 for $n\binom{m}{2}$ 1 simplification
(b)(ii) If A is a vertex then you can form an additional mn triangles. Number of triangles $= \frac{mn(m+n-2)}{2} + mn = \frac{mn(m+n)}{2}$	**1** correct answer
(c) Let $a = \sin^{-1}x$ and $b = \cos^{-1}x$. $a - \frac{b}{2} = \frac{\pi}{3}$ $\quad$ (i) $2a + \frac{b}{2} = \frac{7\pi}{6}$ $\quad$ (ii) (i) + (ii) $\quad 3a = \frac{9\pi}{6}$ $\quad a = \frac{\pi}{2}$ Subst in (ii) $\quad \pi + \frac{b}{2} = \frac{7\pi}{6}$ $\quad b = \frac{\pi}{3}$ $x = \sin\frac{\pi}{2} = 1$ $\quad y = \cos\frac{\pi}{3} = \frac{1}{2}$	**3** correct answers 1 setting up equations 1 solving equations 1 finding x and y

ISBN: 978 1 44250 354 0

Question 6

Solution	Marks
(a) $y = 15e^{-0.6t} + 4$ $\frac{dy}{dx} = 15\times(-0.6)e^{-0.6t} = -0.6\left(15e^{-0.6t}\right) = -0.6\left(15e^{-0.6t} + 4 - 4\right)$ $\frac{dy}{dx} = -0.6(y-4)$ so $y = 15e^{-0.6t} + 4$ is a solution.	**2** correct answer 1 correct differentiation 1 correct rearranging
(b)(i) $\frac{d^2x}{dt^2} = 8x - 2x^3$ $\quad\quad \frac{d}{dx}\left(\frac{v^2}{2}\right) = 8x - 2x^3$ $\frac{v^2}{2} = 4x^2 - \frac{x^4}{2} + C$ $v^2 = 8x^2 - x^4 +$ constant $\quad \therefore \quad v^2 + x^4 - 8x^2 =$ constant	**2** correct solution 1 correct use of $\ddot{x} = \frac{d}{dx}\left(\frac{v^2}{2}\right)$ 1 correct primitive
(b)(ii) $v = 3$, $x = 0$ $\quad\quad$ constant = 9	**1** correct answer
(b)(iii) $v^2 + x^4 - 8x^2 = 9$ When $v = 0$, $x^4 - 8x^2 - 9 = 0$, $(x^2 - 9)(x^2 + 1) = 0$ $x^2 = 9, -1$ so $x = \pm 3$. $v = 0$, $x = \pm 3$ so particle only moves on the interval $-3 \le x \le 3$ **OR** $v^2 = (9 - x^2)(x^2 + 1)$ so $v^2 \ge 0$ when $9 - x^2 \ge 0$, i.e. $-3 \le x \le 3$ $\ddot{x} = 8x - 2x^3 = -2x\left(4 - x^2\right)$. This can't be written in the form $\ddot{x} = -n^2(x - a)$ so motion is not simple harmonic.	**2** correct answers 1 mark for each part
(c)(i) $f(x) = e^{-x} - e^x$ $\quad\quad f'(x) = -e^{-x} - e^x = -(e^{-x} + e^x) < 0$ for all x	**1** correct answer
(c)(ii) $x = e^{-y} - e^y$ $x = \frac{1}{e^y} - e^y$ or $e^{2y} + xe^y - 1 = 0$ $e^y = \frac{-x \pm \sqrt{x^2 + 4}}{2}$ but $e^y > 0$ so $e^y = \frac{-x + \sqrt{x^2 + 4}}{2}$ $y = \log_e\left(\frac{\sqrt{x^2 + 4} - x}{2}\right)$, i.e. $f^{-1}(x) = \log_e\left(\frac{\sqrt{x^2 + 4} - x}{2}\right)$	**3** correct answer 1 for forming equation 1 solving for e^y 1 finding $f^{-1}(x)$
(c)(iii) $e^{-x} - e^x = -6$ $f^{-1}(-6) = \log_e\left(\frac{\sqrt{36+4} + 6}{2}\right) = \log_e\left(\frac{\sqrt{40} + 6}{2}\right) = 1.82$ Solution is $x = 1.82$	**1** correct answer

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Question 7

Solution	**Marks**
(a)(i) $f(\theta) = \cos\theta\sin^2\theta$ $f'(\theta) = -\sin\theta\sin^2\theta + \cos\theta \times 2\sin\theta\cos\theta$ $= \sin\theta(2\cos^2\theta - \sin^2\theta)$ $= \sin\theta(2\cos^2\theta - 1 + \cos^2\theta)$ $= \sin\theta(3\cos^2\theta - 1)$	**2** correct solution 1 correct differentiation 1 correct simplification
(a)(ii) $f'(\theta) = 0,\ \sin\theta(3\cos^2\theta - 1) = 0,\ 0 < \theta \le \frac{\pi}{2}$ $\sin\theta = 0,\ \cos\theta = \pm\frac{1}{\sqrt{3}}$. Since $0 < \theta \le \frac{\pi}{2},\ \theta = \cos^{-1}\frac{1}{\sqrt{3}}$ For $0 < \theta \le \frac{\pi}{2}$, $\sin\theta > 0$ so $f'(\theta) > 0$ when $\theta < \cos^{-1}\frac{1}{\sqrt{3}}$, $f'(\theta) < 0$ when $\theta > \cos^{-1}\frac{1}{\sqrt{3}}$, so maximum $f(\theta)$ when $\theta = \cos^{-1}\frac{1}{\sqrt{3}}$ **OR** $\theta = 0.9,\ f'(\theta) = 0.125 \quad \theta = 1,\ f'(\theta) = -0.105$ Sign changes, so maximum	**1** correct answer Accept testing derivative either side of $\theta \approx 0.955$ and making conclusion
(b)(i) At edge of table, $\frac{2}{d} = \sin\theta$ so $\frac{1}{d} = \frac{\sin\theta}{2}$ Hence $I = k\cos\theta \times \frac{\sin^2\theta}{4} = \frac{k\cos\theta\sin^2\theta}{4}$	**1** evidence of using $\frac{1}{d} = \frac{\sin\theta}{2}$ correctly
(b)(ii) $I = \frac{k}{4}f(\theta)$ has a maximum value when $\theta = \cos^{-1}\frac{1}{\sqrt{3}}$ $h = 2\cot\theta$, $\cos\theta = \frac{1}{\sqrt{3}}$ and $\sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$ $h = 2 \times \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{2}} = \sqrt{2}$	**1** correct answer
(b)(iii) $\frac{dh}{dt} = 0.2 \quad h = 2\cot\theta \quad \frac{dh}{d\theta} = -2\operatorname{cosec}^2\theta$ $\frac{dh}{dt} = \frac{dh}{d\theta} \times \frac{d\theta}{dt}$ $0.2 = -2\operatorname{cosec}^2\theta \times \frac{d\theta}{dt}$ $\frac{d\theta}{dt} = \frac{-\sin^2\theta}{10}$	**1** correct answer

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Question 7 (continued)

Solution	Marks
(b)(iv) $h = 2$, $\tan\theta = 1$ so $\sin\theta = \cos\theta = \dfrac{1}{\sqrt{2}}$ $I = \dfrac{k\cos\theta\sin^2\theta}{4}$ $\quad \dfrac{dI}{d\theta} = \dfrac{k\sin\theta}{4}\left(3\cos^2\theta - 1\right)$ $\dfrac{dI}{dt} = \dfrac{dI}{d\theta}\times\dfrac{d\theta}{dt}$ $\quad \dfrac{dI}{dt} = \dfrac{k\sin\theta}{4}\left(3\cos^2\theta - 1\right)\times\left(\dfrac{-\sin^2\theta}{10}\right)$ $\sin\theta = \cos\theta = \dfrac{1}{\sqrt{2}}$: $\dfrac{dI}{dt} = \dfrac{k}{4}\times\dfrac{1}{\sqrt{2}}\left(\dfrac{3}{2} - 1\right)\times\left(\dfrac{-1}{20}\right) = \dfrac{-k}{160\sqrt{2}}$ ms^{-1}	**2** correct answer 1 setting up result 1 evaluating result
(c)(i) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n}x^n$ Differentiate wrt x $n(1+x)^{n-1} = 0 + 1\binom{n}{1} + 2\binom{n}{2}x + \ldots + r\binom{n}{r}x^{r-1} + \ldots + n\binom{n}{n}x^{n-1}$ $\binom{n}{1} + 2\binom{n}{2}x + \ldots + r\binom{n}{r}x^{r-1} + \ldots + n\binom{n}{n}x^{n-1} = n(1+x)^{n-1}$	**1** correct answer
(c)(ii) Multiply the expression in (i) by x: $\binom{n}{1}x + 2\binom{n}{2}x^2 + \ldots + r\binom{n}{r}x^r + \ldots + n\binom{n}{n}x^n = nx(1+x)^{n-1}$	**1** correct answer
(c)(iii) Differentiate the expression in (ii) wrt x: $\binom{n}{1} + 2\binom{n}{2}2x + 3\binom{n}{3}3x^2 \ldots + r\binom{n}{r}rx^{r-1} + \ldots + n\binom{n}{n}nx^{n-1}$ $= n\left[(1+x)^{n-1} + x(n-1)(1+x)^{n-2}\right]$ $\binom{n}{1} + 2^2\binom{n}{2}x + 3^2\binom{n}{3}x^2 \ldots + r^2\binom{n}{r}x^{r-1} + \ldots + n^2\binom{n}{n}x^{n-1}$ $= n(1+x)^{n-2}\left[1 + x + nx - x\right] = n(1+x)^{n-2}(1+nx)$ Let $x = -1$ $\binom{n}{1} - 2^2\binom{n}{2} + 3^2\binom{n}{3} \ldots + (-1)^{r-1}r^2\binom{n}{r} + \ldots + (-1)^{n-1}n^2\binom{n}{n} = 0$	**2** correct answer 1 correct differentiation 1 substitution of $x = -1$

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