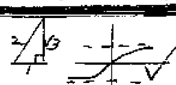
FKC

Solutions to Knox Grammar School Mathematics Extension I 2001 Trial HSC Examination

[✓ = one mark]

Question (1)



(a) $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

(b) $\int \frac{1}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2} + c$ ✓

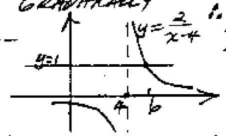
(c) $\frac{1+\cos 2\theta}{\sin 2\theta} \equiv \cot\theta$

$LHS = \frac{1+(2\cos^2\theta - 1)}{2\sin\theta\cos\theta}$ ✓

$= \frac{2\cos^2\theta}{2\sin\theta\cos\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta = RHS$ ✓

(d) $\frac{2}{(x-4)} \geq 1$ or Square both sides & Test

Graphically ✓ ✓



$y=\frac{2}{x-4}$ $\therefore 4 < x \leq 6$ ✓

(e) $\sin x - \cos x = 1$

Use "t" substitution or Alternative Methods – Graphically – $A\cos(x+\theta)$ type

$\frac{2t}{(1+t^2)} - \frac{(1-t^2)}{(1+t^2)} = 1$

$\therefore 2t - (1-t^2) = 1+t^2$ ✓

$2t - 1 + t^2 = 1 + t^2$

$2t = 2$

$t = 1$

$\therefore \tan\left(\frac{\theta}{2}\right) = 1$

$\therefore \frac{\theta}{2} = \frac{\pi}{4},$ ✓

$\theta = \frac{\pi}{2},$

Test $\theta = \pi$, TRUE ✓

$\therefore \theta = \frac{\pi}{2}, \pi$

(f) $y = 2x - 1 \quad \therefore m_1 = 2$

$3x - 2y = 5 \quad \therefore m_2 = \frac{3}{2}$

$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$

$= \left|\frac{2-(3/2)}{1+(2)(3/2)}\right| = \frac{\frac{1}{2}}{1+3} = \frac{1}{8}$ ✓

$\therefore \theta = \tan^{-1}\left(\frac{1}{8}\right) \doteqdot 0.12435$ ✓

$\doteqdot 0.12$

Question (2)

(a) $y = e^{x^2}$

$\frac{dy}{dx} = 2xe^{x^2}$ ✓

$\frac{d^2y}{dx^2} = 2x\left[e^{x^2}\cdot 2x\right] + \left[2\right]e^{x^2}$ ✓

$= 2e^{x^2}\{2x^2+1\}$ or equivalent

(b) (i) $\cos 2x = 1 - 2\sin^2 x$ ✓

(ii) $\int_0^{\frac{\pi}{2}} 2\sin^2 2x = \int_0^{\frac{\pi}{2}} (1-\cos 4x)\,dx$ ✓

$= \left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}}$

$= \left[\left(\frac{\pi}{2} - \frac{1}{4}\sin 2\pi\right)\right] - \left[0\right] = \frac{\pi}{2}$ ✓

(c) $\int \frac{x}{\sqrt{1-x}}\,dx$ let $\begin{cases} x = 1-u^2 \\ \frac{dx}{du} = -2u \end{cases}$

$\therefore I = \int \frac{1-u^2}{\sqrt{1-(1-u^2)}}(-2u\,du)$ ✓

$= \int \frac{1-u^2}{\sqrt{u^2}}(-2u\,du)$ ✓

$= -2\int(1-u^2)\,du = -2\left[u - \frac{1}{3}u^3\right] + c$

$= -2\left[(1-x)^{\frac{1}{2}} - \frac{1}{3}(1-x)^{3/2}\right] + c$ ✓

(d) $x^3 + 2x^2 - 5x - 4 = 0$

(i) $\alpha + \beta + \gamma = -\frac{b}{a} = -2$ ✓

(ii) $\alpha\beta\gamma = \frac{-d}{a} = 4$ ✓

(iv) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{-5}{4}$ ✓

(iv) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\Sigma\alpha\beta)$

$= (-2)^2 - 2(-5)$

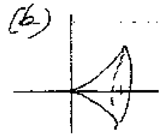
$= 4 + 10 = 14$ ✓

Question (3)

(a) $\int \frac{1}{(4+9x^2)}\,dx = \frac{1}{9}\int \frac{dx}{\left(\frac{4}{9}+x^2\right)}$ ✓

$= \frac{1}{9}\left[\frac{1}{\left(\frac{2}{3}\right)}\tan^{-1}\frac{x}{\left(\frac{2}{3}\right)}\right] = \frac{1}{6}\tan^{-1}\frac{3x}{2} + C$ ✓

(b)



$V = \pi\int_a^b y^2\,dx$

$= \pi\int_0^{\frac{\pi}{4}} \tan^2 x\,dx$ ✓

$\therefore V = \pi\int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\,dx$ ✓

$= \pi\left[\tan x - x\right]_0^{\frac{\pi}{4}}$ ✓

$= \pi\left[\left(1 - \frac{\pi}{4}\right) - (0-0)\right] = \left(\pi - \frac{\pi^2}{4}\right)$ cubic units

(c) $Q(x) = x^3 + 2x^2 + ax + b$

$Q(-2) = (-2)^3 + 2(-2)^2 + a(-2) + b = 0$ — (i)

$-8 + 8 - 2a + b = 0$

$\therefore -2a + b = 0$ ✓

$Q(2) = (2)^3 + 2(2)^2 + a(2) + b = 12$ — (ii)

$8 + 8 + 2a + b = 12$

$\therefore 2a + b = -4$ ✓

Solve (i) & (ii) $\quad 2b = -4, \; b = -2$

$\therefore a = -1$

(d) $f(x) = e^x + 4x$

(i) $f'(x) = e^x + 4$ ✓

(ii) $f'(x) > 0$ for all values of x as $e^x > 0$ ✓

$\therefore f(x)$ is increasing for all x

(iii) $f(-1) = e^{-1} + 4(-1) \doteqdot \frac{1}{e} - 4 < 0$

$f(0) = e^0 + 4(0) = 1 \longrightarrow > 0$ ✓

$\therefore$ There is a root between -1 and 0

(iv) Let $x = -0.5$ be a_1

$\therefore a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$

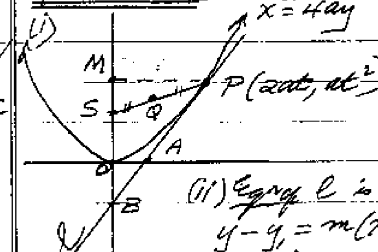
$= -0.5 - \left[\frac{e^{-.5} + 4(-.5)}{e^{-.5} + 4}\right]$ ✓

$= -0.5 - [\;]$

$\doteqdot -0.20$ (To 2dp) ✓

Question (4)

(i)



(ii) Eqn of l is

$y - y_1 = m(x - x_1)$ ✓

$y - at^2 = t(x - 2at)$

$\therefore y = tx - at^2$ ✓

(iii) A is $(at, 0)$ ✓

B is $(0, -at^2)$ ✓

(iv) $PA : PB = MO : MB = at^2 : 2at^2 = 1:2$

Externally ✓

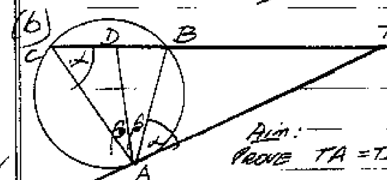
(v) Let Q be (x, y) where S is $(0, a)$

$\therefore \begin{cases} x = \frac{0+2at}{2} & \text{— (i)} \\ y = \frac{a+at^2}{2} & \text{— (ii)} \end{cases}$ $\quad \therefore x = at$ ✓ $\quad \therefore y = \frac{a(1+t^2)}{2}$ $\quad \therefore 2y = a(1+t^2)$ ✓

Hence $t = \frac{x}{a}$

$\therefore 2y = a\left[1 + \left(\frac{x}{a}\right)^2\right]$ or equivalent ✓

(b)



Aim: Prove $TA = TD$

Proof: Let $B\hat{A}T = \alpha$

$\therefore B\hat{C}A = \alpha$ [Angle in alternate segment of a circle theorem] ✓

Let $C\hat{A}D = B\hat{A}D = \beta$

$\therefore B\hat{D}A = \alpha + \beta$ [Exterior angle of ΔCDA] ✓

$\therefore B\hat{D}A = D\hat{A}T$ (Both $(\alpha+\beta)$)

$\therefore \Delta TDA$ is isosceles [2 equal angles] ✓

$\therefore TA = TD$ [opposite equal angles in an isosceles Δ]

QUESTION 5

(a) $f(x) = (x-1)^2$

(i) Sketch

(ii) It does not have an inverse because {ONE-TO-ONE or ...}

OR EQUIVALENT REASON

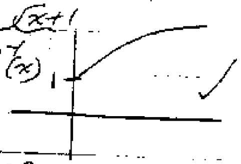
(iii) To have an inverse: $f(x)$ has $\begin{cases} D: x \geq 1 \\ R: y \geq 0 \end{cases}$

(iv) Inverse is $x = (y-1)^2$

$\sqrt{x} = y - 1$

$f^{-1}(x)$: $y = \sqrt{x} + 1$

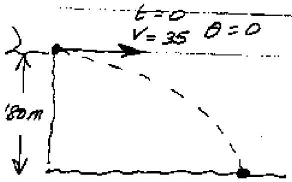
(v) Sketch $y = f^{-1}(x)$



$t=0$, $V=35$, $\theta=0$

$g = 10$

80m



(b)(i) $\ddot{x} = 0$

$\dot{x} = C_1$

Initially $t=0$, $\dot{x}=35$

$\therefore \dot{x} = 35$

Hence $x = 35t + C_2$

Initially $x=0$, $t=0$

$\therefore x = 35t$

$\ddot{y} = -10$

$\dot{y} = -10t + C_3$

Initially $t=0$, $\dot{y}=0$

$\therefore \dot{y} = -10t$

Hence $y = -\frac{10t^2}{2} + C_4$

Initially $t=0$, $y=0$

$\therefore y = -5t^2$

(ii) $\therefore$ At the ocean, $y = -180$

$\therefore 5t^2 = 180$ $\therefore t^2 = 36$ $\therefore t = 6$

$\therefore$ 6 seconds

(iii) DISTANCE: $x = 35t = 35(6) = 210$ metres

(iv) $\therefore \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{-10t}{35} = \frac{-10\times 6}{35} = -1\frac{5}{7}$

$\therefore \theta = 59.74 = 60^\circ$ or 120°

QUESTION 6

(a) $\frac{dT}{dt} \propto (T - A)$

$= k(T-A)$

(i) If $T = A + Be^{kt}$

$\frac{dT}{dt} = Bke^{kt} = k[Be^{kt}]$

$= k[T-A]$

$\therefore$ it satisfies the LAW

(ii) If $A=4$, $t=0$, $T=25$

$\therefore 25 = 4 + Be^{0}$

$\therefore B = 21$ $\therefore T = 4 + 21e^{kt}$

Then $T=15$, $t=45$

$\therefore 15 = 4 + 21e^{45k}$

$11 = 21e^{45k}$

$\ln\left(\frac{11}{21}\right) = 45k$

$k = \frac{1}{45}\ln\left(\frac{11}{21}\right) \doteqdot -0.0144$

Then If $T = 8$

in $T = 4 + 21e^{kt}$

$\therefore 8 = 4 + 21e^{kt}$

$4 = 21e^{kt}$

$\ln\left(\frac{4}{21}\right) = kt$

$-1.6582 = -0.0144t$

$\therefore t \doteqdot 115$ minutes

(b) $x = 5\cos(4\pi t)$

(i) $\dot{x} = 5[-4\pi\sin 4\pi t] = -20\pi[\sin 4\pi t]$

$\ddot{x} = -20\pi[4\pi\cos 4\pi t] = -80\pi^2\cos 4\pi t$

$\therefore \ddot{x} = -16\pi^2[5\cos 4\pi t] = -16\pi^2 x$

which is in the FORM

$\ddot{x} = -n^2 x$

(ii) Period $T = \frac{2\pi}{n} = \frac{2\pi}{4\pi} = \frac{1}{2}$ second

QUESTION 6 CTD.

(b)(iii) Maximum Velocity

$\dot{x} = -20\pi[\sin 4\pi t]$

is when $\sin 4\pi t = \pm 1$

$\therefore \dot{x} = 20\pi$ is MAXIMUM

(iv) $v^2 = n^2(a^2 - x^2)$

$= 16\pi^2(25 - x^2)$

OR EQUIVALENT

(c) MATHEMATICAL INDUCTION

Step 1: Test for $n = 1$

$\therefore (2^{10+3} + 3) = (8195) \div 11 = 745$

which is DIVISIBLE BY 11

Step 2: Assume True for $(n = k)$

$\therefore$ Assume

$2^{10k+3} + 3 = 11P$

when P is a Positive Integer.

Step 3 PROVE TRUE for $n = (k+1)$

Consider

$2^{10(k+1)+3} + 3 = 2^{10k+13} + 3$

$= (2^{10k+3})2^{10} + 3 = [11P - 3]2^{10} + 3$

$= 11P(2^{10}) - 3(2^{10}) + 3 = 11P(2^{10}) - 3069$

which is DIVISIBLE BY 11

Hence TRUE for $n = (k+1)$

Step 4 Hence since it is True for $n=1$, it is True for $n = 2, 3, \ldots$

Hence $(2^{10n+3} + 3)$ is divisible by 11 for all positive integers [OR EQUIVALENT STATEMENT]

QUESTION 7

(a) Using THE SQUARE on the TANGENT EQUIVALENT CONCEPT STATED OR USE SIMILAR TRIANGLES

$(PT)^2 = (PA)(PB)$

$\therefore x^2 = (6+a)a$

$\therefore x = \sqrt{a(6+a)}$

(b) $\theta = (\beta - \alpha)$

(i) $\therefore \tan\theta = \tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$

$= \frac{\left(\frac{x}{a}\right) - \left(\frac{x}{a+6}\right)}{1 + \left(\frac{x}{a}\right)\left(\frac{x}{a+6}\right)} = \frac{x(a+6) - x(a)}{a(a+6) + x^2}$

$= \frac{6x}{a^2 + 6a + x^2}$

(ii) $\frac{dT}{dx} = \frac{(a^2+6a+x^2)[6] - 6x[2x]}{(a^2+6a+x^2)^2}$

$= \frac{6a^2 + 36a - 6x^2}{(\ \ )^2} = 0$

$\therefore a^2 + 6a - x^2 = 0$

$a^2 + 6a = x^2$; $x = \sqrt{a(a+6)}$

* TEST (FOR MAXIMUM)

(iii) $T = \tan\theta = \frac{6\sqrt{a(a+6)}}{a^2 + 6a + (a^2+6a)}$

$\therefore = \frac{6\sqrt{a(a+6)}}{2(a^2+6a)} = \frac{3}{\sqrt{a^2+6a}}$

$\therefore \theta = \tan^{-1}\left(\frac{3}{\sqrt{a^2+6a}}\right)$

(iv) $x = 12.64911064 \doteqdot \sqrt{160}$

$\doteqdot 12.65\text{m} \doteqdot 4\sqrt{10}$

$\therefore \theta = 13^\circ 20' 33'' \doteqdot 13^\circ 21'$

(v) Reason: Maximum θ occurs when $x = \sqrt{a(a+6)}$ ~~using the Result (a) we see that~~ the GOAL POSTS AND P all lie on a circle with PT as Tangent

(OR A REASON CONNECTING (a) GEOMETRY WITH THE CONCEPT OF MAXIMUM AS IN (b))