JAMES RUSE AGRICULTURAL HIGH SCHOOL
4 Unit Mathematics
1999 Trial HSC Examination

QUESTION 1

(a) Find $\int x\sqrt{x^2+16}\ dx$

(b) Find $\int \frac{x}{x+1}\ dx$

(c) Find $\int \frac{dx}{x^2+4x+13}$

(d) Using the substitution $u = \cos x$, or otherwise, find $\int \frac{\sin^3 x}{\cos^2 x}\ dx$

(e) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{3}} \frac{dx}{1+\cos x-\sin x}$.

QUESTION 2

(a) If $z = 2+3i$ and $w = 1-i$ express in the form $a+ib$

(i) $\overline{z}$ **(ii)** zw **(iii)** $\frac{z}{w}$

(b) **(i)** Express $1+i$ in mod/arg form.

(ii) Hence write $(1+i)^5$ in the form $x+iy$ where x and y are real.

(c) **(i)** Find both square roots of $-3+4i$

(ii) Hence solve $z^2-5z+(7-i)=0$ giving your answers in the form $z = p+iq$ where p and q are real.

QUESTION 3

(a) **(i)** Prove that $\sin(A+B)+\sin(A-B) = 2\sin A\sin B$.

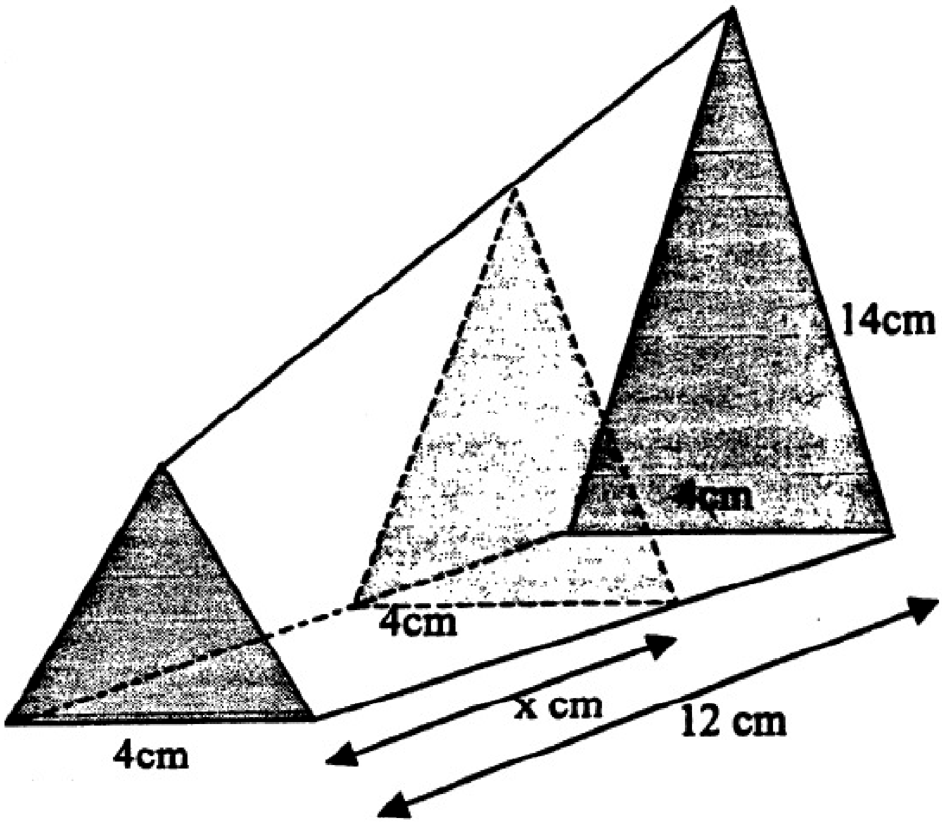
(ii) Hence solve $\sin 3\theta + \sin\theta = \sin 2\theta$ for $0 \le \theta \le \pi$.

(b) Find the volume of the solid formed when the region bounded by $y = \cos x$ and $y = \sin x$ for $0 \le x \le \frac{\pi}{4}$ is rotated one revolution about the x-axis.

(c) The front face of a solid is an equilateral triangle with sides 4 cm and the end face is an isosceles triangle with base 4 cm and equal sides 14 cm. The solid is 12 cm long and cross-sections parallel to the front face are isosceles triangles with base 4 cm. (See diagram).

(i) Show that the height (h cm) of a triangular cross-section x cm from the front face is given by $h = \frac{\sqrt{3}}{2}(x + 4)$.

(ii) Hence find the volume of the solid.



QUESTION 4

(a) **(i)** Prove that $\int_{-a}^{a} f(x)\ dx = \int_0^a (f(x) + f(-x))\ dx$.

(ii) Using the result of **(i)** and the definition of odd and even functions prove

(α) if $f(x)$ is even then $\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx$.

(β) if $f(x)$ is odd then $\int_{-a}^{a} f(x)\ dx = 0$.

(iii) Hence evaluate $\int_{-\pi}^{\pi} x \cos x\ dx$.

(b) A sequence is defined by the formula $a_n = 3 + 33 + 333 + \cdots + \overbrace{333\ldots3}^{n \text{ digits}}$ where the last term contains n 3's. Use the principle of mathematical induction to prove that $a_n = \frac{1}{27}(10^{n+1} - 9n - 10)$ for integer $n \geq 1$.

QUESTION 5

(a) The point $T\left(ct, \frac{c}{t}\right)$ lies on the hypebola $xy = c^2$. The tangent at T meets the x-axis at P and the y-axis at Q. The normal at T meets the line $y = x$ at R.

(i) Prove that the tangent at T has equation $x + t^2y = 2ct$.

(ii) Find the co-ordinates of P and Q.

(iii) Write down the equation of the normal at T.

(iv) Show that the x co-ordinate of R is $x = \frac{c}{t}(t^2 + 1)$.

(v) Prove that $\triangle PQR$ is isosceles.

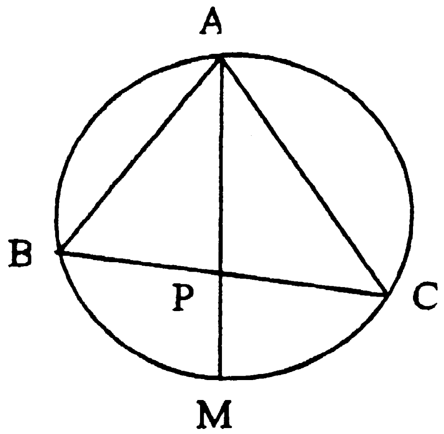
(b) A circle is drawn to pass through the vertices of $\triangle ABC$. AM bisects $\angle BAC$ and meets BC at P. (see diagram)

(i) Prove that $\triangle ABM$ and $\triangle ACP$ are similar.

(ii) Prove that $AB.AC = AP.AM$.

(iii) Explain why $BP.PC = AP.PM$.

(iv) Hence prove that $AB.AC - BP.PC = AP^2$.



QUESTION 6

(a) (i) Sketch $y = \sin(x^2)$ for $-\sqrt{2\pi} \leq x \leq \sqrt{2\pi}$ showing all intercepts with the co-ordinate axes and turning points.

(ii) The region bounded by $y = \sin(x^2)$ and the x-axis for $0 \leq x \leq \sqrt{\pi}$ is rotated one revolution about the y-axis. Find the volume of the solid.

(b) A conical pendulum is constructed using a 200 gram mass attached to a light string. When the mass moves in a horizontal circle with speed 1.5 m.s^{-1} the string makes an angle of magnitude $\tan^{-1}\frac{3}{4}$ with the vertical. Taking $g = 10\ m.s^{-1}$, find

(i) the tension in the string

(ii) the length of the string.

(c) The locus of a point is defined by the equation $|z-2| = 2\Re\left(z-\frac{1}{2}\right)$.

(i) If $z = x+iy$ explain why $x \geq \frac{1}{2}$.

(ii) Show that the locus is a branch of the hyperbola $3x^2 - y^2 = 3$.

(iii) Sketch the locus showing its asymptotes and vertex.

(iv) Find the largest set of possible values for each of $|z|$ and $\arg z$.

QUESTION 7

(a) A plane of mass M kg on landing experiences a variable resistive force (due to air resistance) of magnitude Bv^2 Newtons, where v is the speed of the plane, i.e., $M\ddot{x} = -Bv^2$. After the brakes are applied the plane experiences a constant resistive force A Newtons (due to the brakes) as well as the variable resistive force Bv^2, i.e., $M\ddot{x} = -(A+Bv^2)$.

(i) Show that the distance (D_1) travelled in slowing from speed V to speed U under the effect of air resistance only is given by: $D_1 = \frac{M}{B}\ln\left(\frac{V}{U}\right)$.

(ii) After the breaks are applied with the plane travelling at speed U, show that the distance (D_2) required to come to rest is given by: $D_2 = \frac{M}{2B}\ln\left(1+\frac{B}{A}U^2\right)$.

(iii) Use the above information to estimate the stopping distance for a 100 tonne plane if it slows from 90 m.s^{-1} to 60 m.s^{-1} under a resistive force of magnitude $125v^2$ Newtons and is finally brought to rest with the assistance of constant braking force of magnitude 75000 Newtons.

(b) (i) Prove that $\frac{x^2}{(x^2+1)^{n+1}} = \frac{1}{(x^2+1)^n} - \frac{1}{(x^2+1)^{n+1}}$

(ii) Given $I_n = \int_0^1 \frac{1}{(x^2+1)^n}\,dx$, prove that $2nI_{n+1} = 2^{-n} + (2n-1)I_n$.

(iii) Hence evaluate $\int_0^1 \frac{1}{(x^2+1)^3}\,dx$.

QUESTION 8

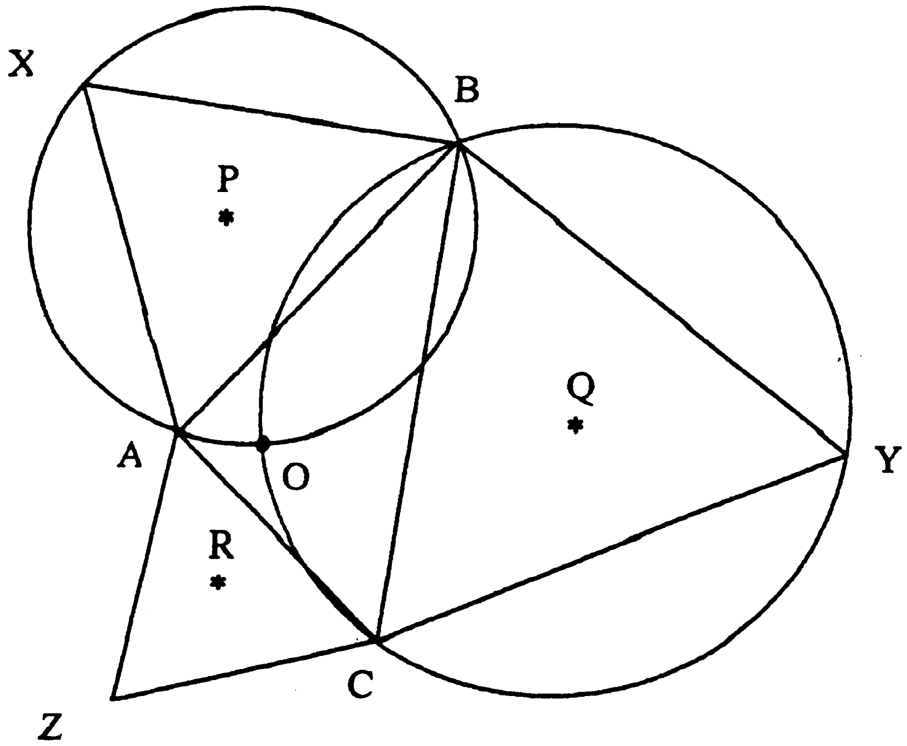
(a) **(i)** Use the substitution $u = 1 + x$ to evaluate $\int_0^1 x(1+x)^n \, dx$.

(ii) Use the binomial theorem to write an expansion of $x(1+x)^n$.

(iii) Prove that $\sum_{r=0}^{r=n} \frac{1}{r+2} \cdot {}^nC_r = \frac{n.2^{n+1}+1}{(n+1)(n+2)}$.

(iv) Find the largest integer value of n such that $\sum_{r=0}^{r=n} \frac{1}{r+2} \cdot {}^nC_r < 50$.

(b) ABC is any triangle. Equilateral triangles ABX, BCY and ACZ are constructed on the sides of $\triangle ABC$. Circles with centres P and Q are drawn to pass through the vertices of $\triangle ABX$ and $\triangle BCY$. The circles meet at B and O. (see diagram)



(i) Find the size of $\angle AOB, \angle BOC$ and $\angle AOC$, giving reasons.

(ii) Prove that $AOCZ$ is a cyclic quadrilateral.

(iii) If R is the centre of the circle $AOCZ$, prove that $\triangle PRQ$ is equilateral.