SCGS SOLUTIONS TO 3/4 UNIT TRIAL HSC, 1998

TOTAL = 84

(1) (a) $\overset{x_1\ y_1}{(5,2)}$ $\quad$ $\overset{x_2\ y_2}{(-1,8)}$

2 : 1

$P = \left(\frac{-2+5}{3}, \frac{16+2}{3}\right)$ ✓

$= (1,6)$ ✓

(b) $\frac{d}{dx}(\cos^{-1}3x) = \frac{-1}{\sqrt{1-(3x)^2}} \cdot 3$ ✓

$= \frac{-3}{\sqrt{1-9x^2}}$ ✓

(c)(i) $\tan^{-1}\frac{x}{2} + c$ ✓

(ii) $\log_e(4+x^2) + c$ ✓

(no penalty for omission of c)

(d) $\frac{2\sin x\cos x}{\cos^2 x - \sin^2 x} = \frac{\sin 2x}{\cos 2x}$ ✓

$= \tan 2x$ ✓

(e) $T_{r+1} = {}^5C_r \cdot 2^{5-r} \cdot x^r$ ✓

The coefficient of x^3 is

${}^5C_3 \cdot 2^2 = 40$. ✓

(f) $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$. ✓

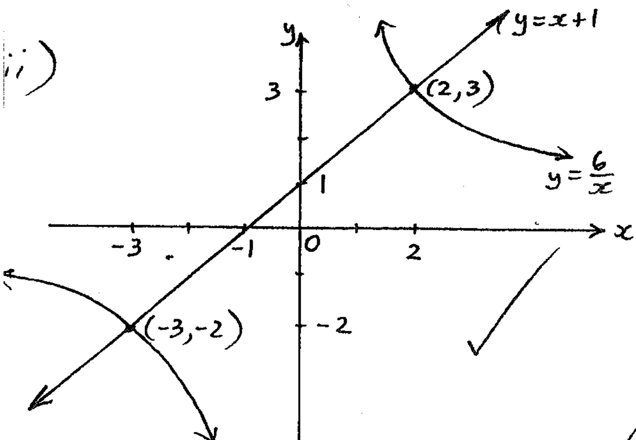
So $\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$. ✓

(i) Solving simultaneously,

$$\frac{6}{x} = x + 1$$
$$x^2 + x - 6 = 0$$
$$(x+3)(x-2) = 0$$
$$x = -3 \text{ or } x = 2$$

When $x = -3$, $y = -2$.
When $x = 2$, $y = 3$.
So the points of intersection are $(-3,-2)$ and $(2,3)$.

(ii)



(iii) $-3 \leq x < 0$ or $x \geq 2$

b) When $n=1$,
$LHS = 1(5) = 5$,
$RHS = \frac{1}{6}(1)(2)(15)$
$= 5$.
So the result is true for $n=1$.
Assume the result is true for $n=k$, where k is a positive integer.
i.e. assume that

$$1\times 5 + 2\times 6 + \dots + k(k+4) = \frac{1}{6}k(k+1)(2k+13).$$

Prove the result is true for $n=k+1$ if it is true for $n=k$.
i.e. prove that

$$1\times 5 + 2\times 6 + \dots + k(k+4) + (k+1)(k+5) = \frac{1}{6}(k+1)(k+2)(2k+15)$$

$$LHS = \frac{1}{6}k(k+1)(2k+13) + (k+1)(k+5)$$
$$= \frac{1}{6}(k+1)\{k(2k+13) + 6(k+5)\}$$
$$= \frac{1}{6}(k+1)(2k^2 + 19k + 30)$$
$$= \frac{1}{6}(k+1)(k+2)(2k+15)$$
$$= RHS.$$

So the result is true for $n=k+1$ if it is true for $n=k$.
But it is true for $n=1$, so, by induction, it is true for all positive integer values of n.

(c)(i) $P(-2) = (-2)^3 - (-2)^2 - (-2) + 10$
$= -8 - 4 + 2 + 10$
$= 0$.
So -2 is a zero of $P(x)$.

(ii) Sum of zeros $= -\frac{b}{a}$
So $-2 + \alpha + \beta = 1$.
So $\alpha + \beta = 3$.
Product of zeros $= \frac{-d}{a}$
So $-2\alpha\beta = -10$.
So $\alpha\beta = 5$.

(iii) Solving the equations in (ii) simultaneously,
$$\alpha(3-\alpha) = 5$$
$$\alpha^2 - 3\alpha + 5 = 0$$
$$\Delta = -11 < 0.$$
So α is not real.
Similarly, $\beta^2 - 3\beta + 5 = 0$, and so β is not real either.

$\therefore du = dx$

x	1	2
u	2	3

$$\int_1^2 \frac{1-x}{(1+x)^3}\,dx = \int_2^3 \frac{2-u}{u^3}\,du$$

$$= \int_2^3 (2u^{-3} - u^{-2})\,du$$

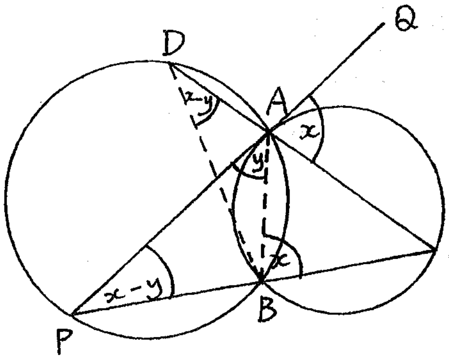
$$= \left[\frac{2u^{-2}}{-2} - \frac{u^{-1}}{-1}\right]_2^3$$

$$= \left[-\frac{1}{u^2} + \frac{1}{u}\right]_2^3$$

$$= \left(-\frac{1}{9} + \frac{1}{3}\right) - \left(-\frac{1}{4} + \frac{1}{2}\right)$$

$$= -\frac{1}{36}.$$

(b)



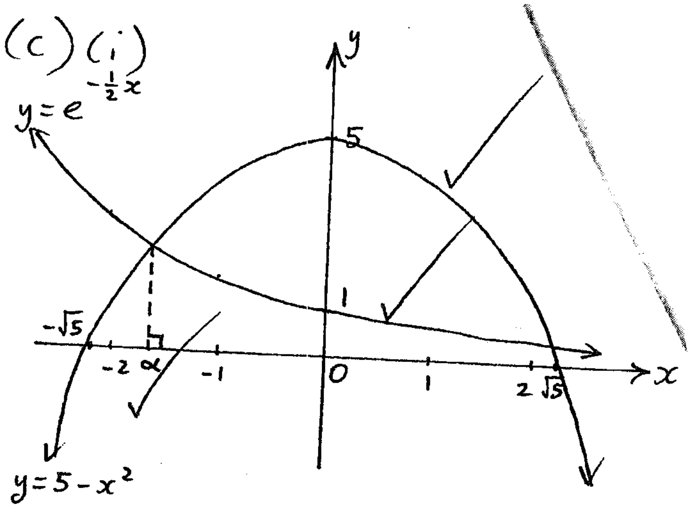
Join BA and BD.

$\angle ABC = \angle QAC = x$ (alternate segment theorem).

So $\angle BPA = x - y$ (exterior angle of ΔABP).

So $\angle BDA = x - y$ (angles at circumference standing on the same arc).

(c) (i)



(ii) See above.

(iii) Let $f(x) = x^2 + e^{-\frac{1}{2}x} - 5$.

$f(-2) = e - 1 > 0$

and $f(-1) = e^{\frac{1}{2}} - 4 < 0$.

It follows that $-2 < \alpha < -1$, since $f(x)$ is continuous for all x.

(iv) $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= -2 - \frac{e-1}{f'(-2)}.$$

Now, $f'(x) = 2x - \frac{1}{2}e^{-\frac{1}{2}x}$.

So $f'(-2) = -4 - \frac{1}{2}e$.

So $$x_2 = -2 - \frac{e-1}{-4-\frac{1}{2}e} \cdot \frac{2}{2}$$

$$= -2 - \frac{2e-2}{-8-e}$$

$$= -2 + \frac{2e-2}{e+8}$$

$$= \frac{-2(e+8) + 2e - 2}{e+8}$$

$$= \frac{-18}{e+8}.$$

(ii) $A = \pi r^2$,

so $\dfrac{dA}{dr} = 2\pi r = C$.

Now, $\dfrac{dA}{dt} = \dfrac{dA}{dr} \cdot \dfrac{dr}{dt}$,

so $\dfrac{dA}{dt} = C \cdot 18$.

So when $C = 30$ cm,
the area is increasing at
$30 \times 18 = 540$ cm^2/s.

(b)(i) $\dot{x} = -2C\sin 2t + 2D\cos 2t$

$\ddot{x} = -4C\cos 2t - 4D\sin 2t$
$= -4(C\cos 2t + D\sin 2t)$
$= -4x$.

So $\ddot{x}$ has the form $-n^2x$, where $n = 2$.

So the motion is SHM.

(ii) When $t = \frac{\pi}{3}$, $x = \frac{\sqrt{3}}{2}$.

So $\frac{\sqrt{3}}{2} = C\cos\frac{2\pi}{3} + D\sin\frac{2\pi}{3}$
$= C\left(-\frac{1}{2}\right) + D\left(\frac{\sqrt{3}}{2}\right)$.

So $-C + \sqrt{3}D = \sqrt{3}$ (1)

When $t = \frac{\pi}{3}$, $\dot{x} = -5$.

So $-5 = -2C\left(\frac{\sqrt{3}}{2}\right) + 2D\left(-\frac{1}{2}\right)$.

So $\sqrt{3}C + D = 5$ (2)

(1)$\times\sqrt{3}$: $-\sqrt{3}C + 3D = 3$ (3)

(3) + (2) : $4D = 8$

So $C = \sqrt{3}$, $D = 2$.

One for the equations, one for solving

(iii) $x = \sqrt{3}\cos 2t + 2\sin 2t$
$= A\cos(2t - \theta)$,
where $A = \sqrt{(\sqrt{3})^2 + 2^2}$
$= \sqrt{7}$.

So the amplitude is $\sqrt{7}$ cm.

(c)(i) $a + b = -1$ (1)
$a + d + ba = -2$ (2)
$a + 2d + ba^2 = 6$ (3)

From (1), $b = -1 - a$.

Sub. into (2):
$a + d + a(-1 - a) = -2$.

so $a + d - a - a^2 = -2$,
so $d = a^2 - 2$.

So (3) becomes
$a + 2(a^2 - 2) + a^2(-1 - a) = 6$
$a + 2a^2 - 4 - a^2 - a^3 = 6$
i.e. $a^3 - a^2 - a + 10 = 0$.

(ii) From (2)(c)(i), $a = -2$.

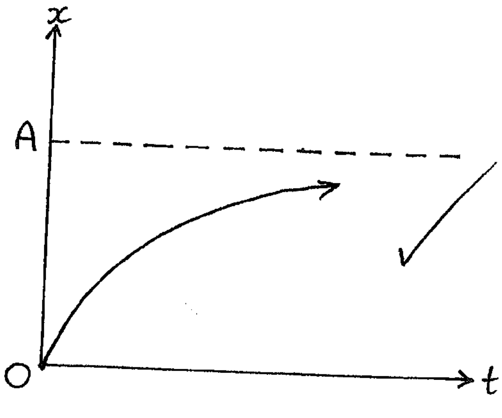
(iii) $a = -2$, $b = 1$, $d = 2$.

$T_n = a + (n-1)d + ba^{n-1}$
$= -2 + 2(n-1) + 1(-2)^{n-1}$
$= 2n - 4 + (-2)^{n-1}$.

(5)(a)(i) $x = A - Ae^{-kt}$

$$\frac{dx}{dt} = kAe^{-kt} = k(A - x).$$

(ii)



(iii) When $x = \frac{3}{4}A$,

$$\frac{3}{4} = 1 - e^{-kt}$$

$$e^{-kt} = \frac{1}{4}$$

$$-kt = \ln 2^{-2}$$

$$kt = 2\ln 2$$

$$t = \frac{2}{k}\ln 2.$$

(b)(i) $\dfrac{d(\frac{1}{2}v^2)}{dx} = 4x - 2$

$$\frac{1}{2}v^2 = 2x^2 - 2x + c_1$$

When $x = 0$, $v = 1$,

so $c_1 = \frac{1}{2}$.

So $v^2 = 4x^2 - 4x + 1$.

So $v^2 = (1 - 2x)^2$.

(ii) $\ddot{x} = 4x - 2 = -2(1 - 2x) = -2v.$

(iii) $v = 1 - 2x$

$$\frac{dx}{dt} = 1 - 2x$$

$$\frac{dt}{dx} = \frac{1}{1 - 2x}$$

$$t = -\frac{1}{2}\ln(1 - 2x) + c_2$$

When $t = 0$, $x = 0$.

So $c_2 = 0$.

So $t = -\frac{1}{2}\ln(1 - 2x)$,

so $e^{-2t} = 1 - 2x$,

so $x = \frac{1}{2}(1 - e^{-2t})$

and $v = e^{-2t}$.

(iv) As $t \to \infty$, $e^{-2t} \to 0$.

So as $t \to \infty$,

$x \to \frac{1}{2}^-$

and $v \to 0^+$.

So the particle approaches, but never reaches, $x = \frac{1}{2}$.

(i) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

$= \dfrac{6t}{6}$

$= t.$

So at P, gradient $= p$.

So equation of tangent is

$y - 3p^2 = p(x - 6p)$

$y - 3p^2 = px - 6p^2$

$y = px - 3p^2.$

(ii) The tangent at Q has equation $y = (1-p)x - 3(1-p)^2$

Solving simultaneously,

$px - 3p^2 = (1-p)x - 3(1-p)^2$

$(2p-1)x = -3 + 6p$

$(2p-1)x = 3(2p-1)$

$p \neq \frac{1}{2}$, since P, Q distinct,

so $x = 3$ and $y = 3p - 3p^2$.

So T is the point $(3, 3p - 3p^2)$.

(iii) The point T can only lie <u>outside</u> the parabola.

When $x = 3$, $t = \frac{3}{6}$

$= \frac{1}{2}$,

and so $y = 3\left(\frac{1}{2}\right)^2$

$= \frac{3}{4}.$

So the locus of T is specified by $x = 3$ <u>and</u> $y < \frac{3}{4}$.

(iv) When $p = \frac{1}{2}$, the points P and Q coincide, in which case T is not uniquely defined.

(b)(i) The heights of P_1 and P_2 at time t are given by

$-\frac{1}{2}gt^2 + v_1 t \sin\theta_1$ and

$-\frac{1}{2}gt^2 + v_2 t \sin\theta_2$.

If the particles collide when $t = T$, their heights are equal at this time.

So $-\frac{1}{2}gT^2 + v_1 T \sin\theta_1 = -\frac{1}{2}gT^2 + v_2 T \sin\theta_2$,

and so $v_1 \sin\theta_1 = v_2 \sin\theta_2$.

(ii)(α) $v_1 \sin\theta_1 = v_2 \sin\theta_2$.

So $30\left(\frac{4}{5}\right) = v_2\left(\frac{3}{5}\right)$,

so $120 = 3v_2$,

so $v_2 = 40$ m/s.

(β) At the instant they collide, sum of horizontal displacements is 200m.

So $v_1 T \cos\theta_1 + v_2 T \cos\theta_2 = 200$,

so $30T\left(\frac{3}{5}\right) + 40T\left(\frac{4}{5}\right) = 200$,

so $18T + 32T = 200$,

so $50T = 200$,

so $T = 4$ seconds.

(γ) Height $= -\frac{1}{2}gT^2 + v_1 T \sin\theta_1$

$= -\frac{1}{2}(10)(16) + 30(4)\left(\frac{4}{5}\right)$

$= -80 + 96$

$= 16$ metres.

(δ) When $t = 4$:

For P_1:

$\dot{x} = 30\left(\frac{3}{5}\right) = 18$

$\dot{y} = -10(4) + 30\left(\frac{4}{5}\right)$

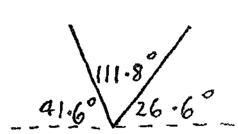
$= -16$

For P_2:

$\dot{x} = 40\left(\frac{4}{5}\right) = 32$

$\dot{y} = -10(4) + 40\left(\frac{3}{5}\right)$

$= -16$



The required angle is approximately $112°$.