Shore - Sydney Church of England Grammar School

4 unit mathematics

Trial HSC Examination 1993

1. Evaluate the following integrals. Give your answers correct to 3 significant figures.
(a) $\int_3^4 \frac{4}{x^2-3x+2}\,dx$ **(b)** $\int_1^2 2^x\,dx$ **(c)** $\int_0^1 \sin^{-1} x\,dx$ **(d)** $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x}$ **(e)** $\int_0^{1.5} \sqrt{9-x^2}\,dx$

2. **(a)** For the hyperbola $\frac{x^2}{144} - \frac{y^2}{25} = 1$ find:
(i) the eccentricity
(ii) the coordinates of the foci
(iii) the equations of the asymptotes
(iv) the equations of the directrices
Sketch the graph showing the above information.
(b) $P(a\cos\theta, b\sin\theta)$ is any point on the ellipse whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and S is the focus of the ellipse. Prove that the line through S perpendicular to the tangent at P, and the line OP produced, meet on the directrix corresponding to the focus S.

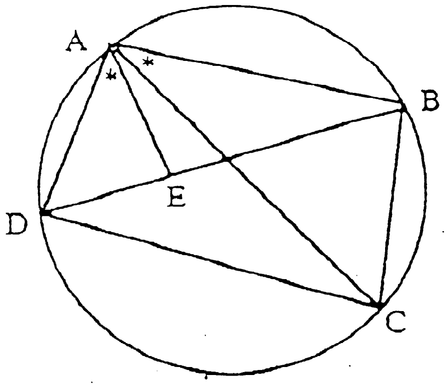
3. **(a)** Solve for z: $z + \frac{2+8i}{z} = 4 + i$
(b) Express $w = \frac{(-1+i\sqrt{3})(1+i)}{\sqrt{3}-i}$ in modulus-argument form.
(c) Given that $(2-i)$ is a zero of $2x^3 + mx^2 + nx + 15$, determine m and n, where m and n are real. Hence factorise $2x^3 + mx^2 + nx + 15$ in the real field.

4. **(a)** Determine all the roots of $8x^4 - 25x^3 + 27x^2 - 11x + 1 = 0$ given that it has a root of multiplicity 3.
(b) α, β, γ are the roots of $x^3 + 2x^2 - 3x + 4 = 0$
(i) Evaluate $\alpha^2 + \beta^2 + \gamma^2$
(ii) Evaluate $\alpha^3 + \beta^3 + \gamma^3$
(iii) Find the equation whose roots are $\frac{\alpha\beta}{\gamma}, \frac{\alpha\gamma}{\beta}, \frac{\beta\gamma}{\alpha}$.

5. **(a)** Let $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x\,dx$, where n is an integer. Show that $I_n = \frac{1}{n-1} - I_{n-2}$ and hence evaluate I_7.
(b) **(i)** Write down the general solution of $\tan 4\theta = 1$.
(ii) Use De Moivre's theorem to express $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$. Hence show that $\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$.
(iii) Find the roots of the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ in trigonometric form. Hence show that $\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16} = 28$.

6. (a) A solid has a base in the form of a circle with centre the origin and radius 6 units. If every section perpendicular to the x-axis is an equilateral triangle, show that the volume of the solid is $288\sqrt{3}$ cubic units.
(b) The region bounded by the curve $y = x^3$, the line $y = 1$ and the y-axis is rotated about the line $y = -1$. By noticing that rectangular strips taken parallel to the axis of rotation give rise to cylindrical shells, find the volume of the solid of revolution.

7. (a) A quiz consists of twenty True-False questions. Find the chance that someone who knows the correct answers to ten of the questions, but answers the remaining ones by tossing a coin, will obtain a score of at least 85% on the quiz.
(b) By using mathematical induction prove that: $\sin(n\pi + x) = (-1)^n \sin x$ for all positive integral values of n.
(c)



$ABCD$ is a cyclic quadrilateral. E is a point on diagonal BD, such that $\angle DAE = \angle BAC$. Prove that:
(i) $AB.CD = AC.BE$ **(ii)** $BC.DA = AC.DE$ **(iii)** $AB.CD + BC.DA = AC.BD$

8. (a) Prove that $\frac{a^2+b^2}{2} > (\frac{a+b}{2})^2$, where a and b are positive, real and unequal.
(b) Sketch $y = 1 + x^2$ and hence sketch on separate diagrams: (do not use calculus)
(i) $y = \frac{1}{x^2+1}$ **(ii)** $y = \frac{x}{x^2+1}$ **(iii)** $y = |\frac{x}{x^2+1}|$ **(iv)** $y = \pm\sqrt{\frac{x}{x^2+1}}$
(c) For the rational function $F(x) = \frac{x^4}{x^2-1}$
(i) Find if $F(x)$ is odd or even or neither.
(ii) Show algebraically that the range of $F(x)$ is: $y \leq 0$ or $y \geq 4$. Hence calculate the coordinates of its three turning points.
(iii) Considering large values of $|x|$ and any discontinuities, sketch the graph of $y = F(x)$. Show also the curved asymptotes by dotted lines.