ABBOTSLEIGH

AUGUST 2003
YEAR 12
ASSESSMENT 4
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using blue or black pen.
- Board-approved calculators may be used
- A table of standard integrals is provided with this paper
- All necessary working should be shown in every question

Total marks - 84

- Attempt Questions 1-7
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve $\frac{4}{x-1} \geq 1$ **3**

(b) A is the point $(-2,-1)$ and B is the point $(1,5)$. Find the coordinates of the point Q which divides AB externally in the ratio $5:2$. **2**

(c) Given $f(x) = \tan^{-1}(\sin x)$ find $f'(\pi)$ **2**

(d) Prove $\frac{1+\sin x-\cos x}{1+\sin x+\cos x} = \tan\frac{x}{2}$ **2**

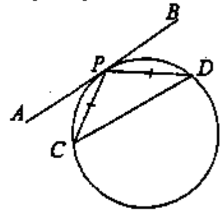
(e) Find the exact value of $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{(3-4x^2)}}$ **3**

End of Question 1

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve the equation $2\sin^2\theta = \sin 2\theta$ for $0 \le \theta \le 2\pi$ **2**

(b) PC and PD are equal chords of a circle.
A tangent to the circle, AB, is drawn at P.



Copy the diagram into your answer booklet and prove that AB is parallel to CD. **2**

(c) (i) Find $\int \frac{x}{x+9}\,dx$ **2**

(ii) Evaluate $\int_0^4 x\sqrt{x^2+9}\,dx$ using the substitution $u = x^2+9$ **3**

(d) (i) Sketch $y = |x+1|$ **1**

(ii) Using your graph, or otherwise, solve $|x+1| > -2x$ for x **2**

End of Question 2

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) For the polynomial $P(x) = x^3 - kx^2 - x + 2$

(i) Find the value of k if $x-1$ is a factor of $P(x)$ **1**

(ii) Hence factorise $P(x)$ completely. **2**

(b) Find the term which is independent of x in the expansion of $\left(x^2 + \frac{2}{x}\right)^9$ **2**

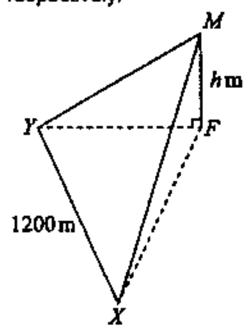
(c) For the function $f(x) = 4\sin^{-1}(x-2)$

(i) Evaluate $f\left(1\frac{1}{2}\right)$ **1**

(ii) Sketch $y = f(x)$ clearly indicating the domain and range. **2**

(iii) Find $\int_1^3 4\sin^{-1}(x-2)\,dx$ **1**

(d) In the diagram, Point X is due south and point Y is due west of the foot, F, of a mountain. From X and Y, the angles of elevation of the top of the mountain M are $35°$ and $43°$ respectively.



Not to scale

If X and Y are 1200 metres apart, show that the height, h metres, of the mountain is given by $h = 1200\left(\tan^2 55° + \tan^2 47°\right)^{-\frac{1}{2}}$ and evaluate h. **3**

End of Question 3

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Sketch the graph of $y = \cos x, -\pi \le x \le \pi$ and use this graph to show that $\cos x + x = 0$ has only one solution. **2**

(ii) Use Newton's method with a first approximation of $x = -1$ to find a second approximation to the root of $\cos x + x = 0$. **2**

(b) The inside of a vessel used for water has the shape of a solid of revolution obtained by the rotation of the parabola $9y = 8x^2$ about the y-axis. The depth of the vessel is $8\,\text{cm}$.

(i) Prove that the volume of water $h\,\text{cm}$ from its base is $\frac{9}{16}\pi h^2$ **1**

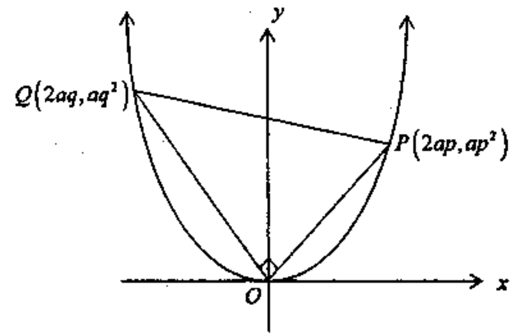
(ii) If water is poured into the vessel at a rate of $20\,\text{cm}^3/\text{sec}$, find the rate at which the level of water is rising when the vessel is half full. **3**

(c) Use the Principle of Mathematical Induction to prove that $2^{3n} - 3^n$ is divisible by 5 for all positive integers n. **4**

End of Question 4

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) In the diagram, PQ is a variable chord of the parabola $x^2 = 4ay$. It subtends a right angle at the vertex O. Let p and q be the parameters corresponding to the points P and Q respectively.



(i) Show that the equation of the tangent to $x^2 = 4ay$ at P is $y - px + ap^2 = 0$ **1**

(ii) Hence, write down the equation of the tangent at Q, and then find R, the point of intersection of the two tangents drawn at P and Q. **2**

(iii) Find the gradients of OP and OQ and hence prove $pq = -4$ **2**

(iv) Show that the locus of R, the point of intersection of the two tangents drawn at P and Q is $y = -4a$ **1**

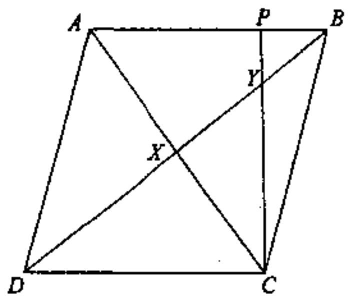
(b) By considering $f(x) = (1+x)^n$ in $\int_0^1 f(x)\,dx$, prove that

$$\sum_{r=0}^{n} \frac{1}{r+1}\binom{n}{r} = \frac{2^{n+1}-1}{n+1}$$ **3**

Question 5 continues on page 7

Question 5 (continued) **Marks**

(c) $ABCD$ is a rhombus whose diagonals intersect at X. The perpendicular CP from C to AB cuts BD at Y.



Not to scale

Copy the diagram into your writing booklet and prove that B, P, X, C are concyclic. **3**

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \sin^2 x \cos^2 x \, dx$ **2**

(b) A particle moves in a straight line so that its velocity after t seconds is $v \text{ ms}^{-1}$ and its displacement is x.

(i) Given that $\frac{d^2x}{dt^2} = 2x^3 - 10x$ and that initially $v = 0$ when $x = -1$, find v in terms of x. **3**

(ii) Explain why this motion can only exist between $x = -1$ and $x = 1$. **2**

(iii) Describe briefly what would have happened if the initial conditions were $v = 0$ when $x = 0$. **1**

(c) In a colony of 400 ants the number, N, diseased at time, t, is given by $N = \frac{400}{1 + ke^{-400t}}$ where k is a constant and t is time in years. (Assume one year is 365 days.)

(i) If at time $t = 0$ one ant was infected, after how many days will half the colony be infected? **3**

(ii) Show that eventually all the ants will be infected. **1**

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) A particle is projected from a point on level ground with a speed of $V\ \mathrm{ms}^{-1}$ and angle of projection, α. Assume that acceleration due to gravity is $g\ \mathrm{ms}^{-2}$ and that there is no air resistance.

(i) Show that the horizontal and vertical displacements, x and y, of the particle in metres from the point of projection at time t seconds after projection are given by

$x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$ **2**

(ii) Show that the greatest height of the particle is $\dfrac{V^2\sin^2\alpha}{2g}$ **1**

(iii) Show that the range of the particle is $\dfrac{V^2\sin 2\alpha}{g}$ **1**

(iv) Two particles are projected from the same point on level ground with the same speed $V\ \mathrm{ms}^{-1}$ and with angles of projection α and $90° - \alpha$ respectively.

The greatest heights the two particles reach are h_1 and h_2 respectively.

Show that, for any α, $h_1 + h_2 = \frac{1}{2}R$ where R is the maximum range. **3**

(b) A_n and B_n are two series given by

$A_n = 1^2 + 5^2 + 9^2 + 13^2 + \ldots + (4n-3)^2$

$B_n = 3^2 + 7^2 + 11^2 + 15^2 + \ldots$ for $n = 1, 2, 3, \ldots$

(i) Find the nth term of B_n. **1**

(ii) If $S_{2n} = A_n - B_n$, prove that $S_{2n} = -8n^2$. **2**

(iii) Hence, or otherwise, evaluate

$101^2 - 103^2 + 105^2 - 107^2 + \ldots + 2001^2 - 2003^2$ **2**

End of Paper