HIGHER SCHOOL CERTIFICATE EXAMINATION 1989

MATHEMATICS - 3/4 UNIT

Directions to Candidates

Time allowed - Two hours (includes reading time).

ALL questions may be attempted. All questions are of equal value.

All necessary working should be shown in every question. Marks may not be awarded for careless or badly arranged work.

Standard integrals are provided (see page 187); approved slide-rules or silent calculators may be used.

QUESTION 1

(a) Factorize $2m^3 - 128$.

(b) Evaluate:

(i) $\int_1^2 \frac{1}{\sqrt{4-x^2}}\,dx$ (ii) $\int_{-1}^{0} x\sqrt{1+x}\,dx$, using the substitution $u = 1+x$.

(c) Find the co-ordinates of the point which divides the interval AB with A(1,4) and B(5,2) externally in the ratio 1 : 3.

(d) Solve for x : $\frac{4}{5-x} \geq 1$.

QUESTION 2

(a)

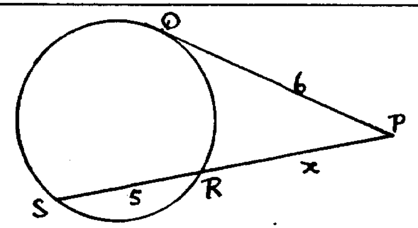


Figure not to scale

PQ is a tangent to a circle QRS, while PRS is a secant intersecting the circle in R and S, as in the diagram. Given that PQ = 6, RS = 5, PR = x, find x.

(b) Find all angles θ with $0 \leq \theta \leq 2\pi$ for which $\sin 2\theta = \sin\theta$.

(c) (i) Show that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{dv}{dt}$.

(ii) The acceleration of a particle moving in a straight line is given by $\ddot{x} = -2e^{-x}$ where x metres is the displacement from the origin. Initially, the particle is at the origin with velocity 2ms^{-1}.

Prove that $v = 2e^{-x/2}$.

(iii) What happens to v as x increases without bound?

QUESTION 3

(a) A committee of 3 is to be elected from a club of 8 members.

(i) How many different committees can be formed?

(ii) If there are 4 Queenslanders in the club, what is the probability that a randomly selected committee of 3 contains only Queenslanders?

(b) Find the constant term in the expansion of $\left(x - \frac{1}{2x^3}\right)^{20}$.

(c)

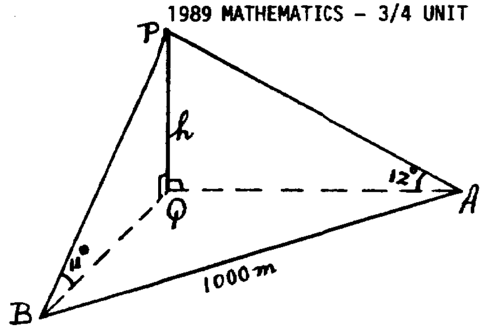


Figure not to scale

The angle of elevation of a tower PQ of height h metres at a point A due east of it is 12°. From another point B, the bearing of the tower is 051°T and the angle of elevation is 11°. The points A and B are 1000 metres apart and on the same level as the base Q of the tower.

(i) Show that $\angle AQB = 141°$.

(ii) Consider the triangle APQ and show that $AQ = h\tan 78°$.

(iii) Find a similar expression for BQ.

(iv) Use the cosine rule in the triangle AQB to calculate h to the nearest metre.

QUESTION 4

(a) A circular plate of radius r is heated so that the area of the plate expands at a constant rate of $3.2\text{cm}^2\text{min}^{-1}$. At what rate does r increase when $r = 10$cm?

(b)(i) The polynomial equation $P(x) = 0$ has a double root at $x = a$. By writing $P(x) = (x-a)^2 Q(x)$, where $Q(x)$ is a polynomial, show that $P'(a) = 0$.

(ii) Hence or otherwise find the values of a and b if $x = 1$ is a double root of $x^4 + ax^3 + bx^2 - 5x + 1 = 0$.

(c) Let each different arrangement of all the letters of D E L E T E D be called a word.

(i) How many words are possible?

(ii) In how many of these words will the D's be separated?

QUESTION 5

(a) AB and CD are two intersecting chords of a circle and CD is parallel to the tangent to the circle at B.

(i) Draw a neat sketch of the above information.

(ii) Prove that AB bisects $\angle CAD$.

(b)(i) By considering the sum of the terms of an arithmetic series, show that $(1 + 2 + \dots + n)^2 = \frac{1}{4}n^2(n+1)^2$.

(ii) By using the Principle of Mathematical Induction prove that $1^3 + 2^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$, for all $n \geq 1$.

QUESTION 6

(a) The figure shows a sketch of the curve $y = (x-1)^2 - 3$.

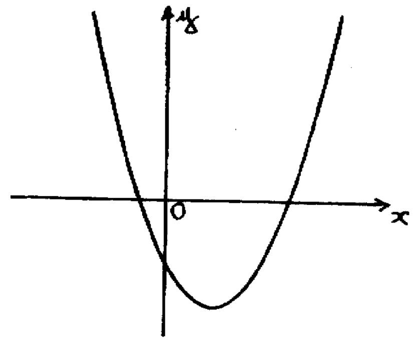


Figure not to scale.

(i) Find the largest positive domain such that the graph defines a function $f(x)$ which has an inverse.

(ii) Find this inverse function and state its domain.

(iii) State a domain for which the function does not have an inverse. Give a brief reason for your answer.

(b) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

(i) Derive the equation of the tangent to the parabola at P.

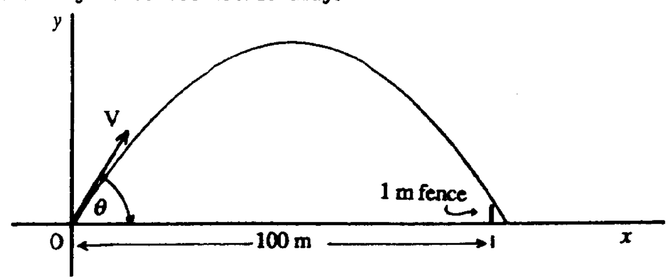
(ii) Find the co-ordinates of the point of intersection T of the tangents to the parabola at P and Q.

(iii) You are given that the tangents at P and Q in (ii) intersect at an angle of 45°. Show that $p - q = 1 + pq$.

(iv) By evaluating the expression $x^2 - 4ay$ at T, or otherwise, find the locus of T when the tangents at P and Q intersect as given in (iii).

QUESTION 7

(a) A "six" is scored in a cricket game when the ball is hit over the boundary fence on the full as in the diagram. A ball is hit from O with velocity $V = 32\text{ms}^{-1}$ at an angle θ to the horizontal and towards the 1 metre high boundary fence 100 metres away.



(i) Derive the equations of motion for the ball in flight using axes as in the diagram. (Air resistance is to be neglected and the acceleration due to gravity is taken as 10ms^{-2}.)

(ii) Show that the ball just clears the boundary fence when $50\,000\tan^2\theta - 102\,400\tan\theta + 51\,024 = 0$.

(iii) In what range must θ lie for a "six" to be scored?

(iv) If, during the flight of the ball, its velocity is reduced by piercing an extremely thin "board", show by a sketch how the path is altered.
Without further calculation, discuss qualitatively the effect of air resistance on your answer in (iii).

(b)(i) Differentiate $y = \tan^{-1}\frac{1}{x}$, $x \neq 0$, and hence show that $\frac{d}{dx}\left(\tan^{-1}x+\tan^{-1}\frac{1}{x}\right)=0$.

(ii) Sketch the curve $y = \tan^{-1}x + \tan^{-1}\frac{1}{x}$.