CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE FINAL TESTS 2000

MATHEMATICS
2/3 UNIT

Morning session — Wednesday 9 August 2000

Time allowed – three hours
(Plus five minutes reading time)

EXAMINERS
P Rockett (co-ordinator)
J Mann
R Pantua
A Kollias
E Rainert
C Reichel
K Breen

DIRECTIONS TO CANDIDATES:

- **ALL** questions may be attempted.
- **ALL** questions are of equal value.
- All necessary working should be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Approved slide rules or calculators may be used.
- Table of standard integrals is printed at the end of the paper.
- The answers to the ten questions in this paper are to be returned in separate writing sheets clearly marked **QUESTION 1, QUESTION 2** etc. on the top of the sheet.
- If required, additional writing sheets may be obtained from the examinations supervisor upon request.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

2602-3

QUESTION 1 *Use a SEPARATE writing sheet.* Marks

(a) Find the value of $\dfrac{\sqrt{12{\cdot}35 - 8{\cdot}66}}{6{\cdot}5}$ correct to two decimal places. **2**

(b) Solve the equation $\dfrac{x - 6}{3} = \dfrac{4x}{5}$ **2**

(c) **2**

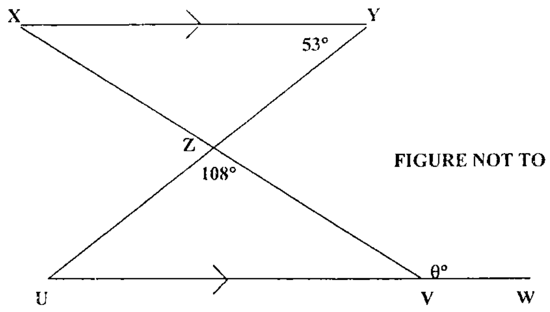

FIGURE NOT TO SCALE

The diagram shows XY parallel to UW, $\angle XYU = 53°$, $\angle UZV = 108°$ and $\angle ZVW = \theta°$.

Find the value of θ. Give reasons.

(d) Find a primitive function for $x^3 + 4$. **2**

(e) A function $g(x)$ is defined as: **2**

$$g(x) = \begin{cases} 2x - 1 & \text{when } x < 2 \\ -3 & \text{when } x \geq 2 \end{cases}$$

Evaluate $g(-4) + g(2)$.

(f) Graph the solution of $|2x + 3| \geq 2$ on the number line. **2**

QUESTION 2 *Use a SEPARATE writing sheet.* Marks

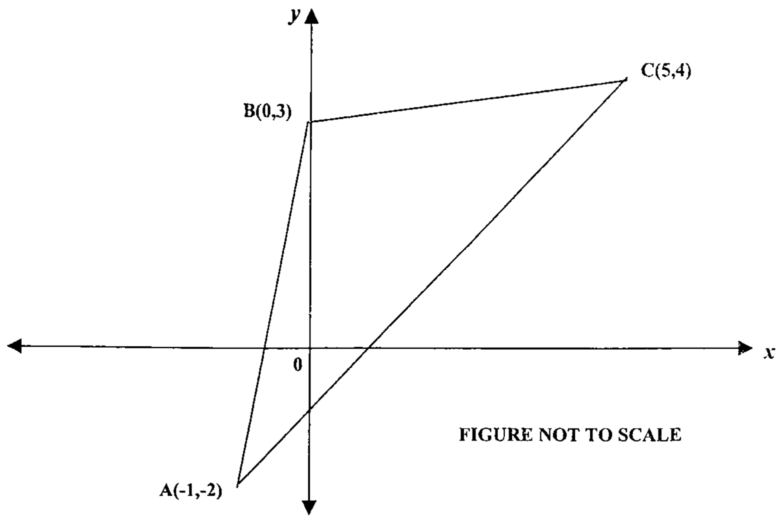

FIGURE NOT TO SCALE

The diagram shows ΔABC with vertices A(-1,-2), B(0,3) and C(5,4).

Copy the diagram onto your answer sheet.

(a) E is the midpoint of AC. Show that the coordinates of E are (2,1). **1**

(b) Show that the gradient of AC is 1. **1**

(c) A line L is drawn through B, perpendicular to AC. Show the equation of line L is $y = 3 - x$. **2**

(d) Show that E lies on line L. **1**

(e) On your diagram, draw line L and plot point E. Prove ΔBEC is congruent to ΔBEA. **3**

(f) AC is a diameter of a circle.

(i) Calculate the radius of the circle. **2**

(ii) Hence find the equation of the circle. **2**

QUESTION 3 *Use a SEPARATE writing sheet.*

Marks

(a) Differentiate the following expressions with respect to x:

(i) $(3x^2 + 2)^3$ **2**

(ii) $3x \cos 2x$ **2**

(iii) $\dfrac{e^{3x}}{x}$ **2**

(b) Evaluate the following definite integrals:

(i) $\displaystyle\int_0^{\pi/4} \sin 3x \, dx$ **2**

(ii) $\displaystyle\int_0^3 e^{2x+3} \, dx$ **2**

(c) Find $\displaystyle\int \frac{x}{x^2+1} \, dx$ **2**

QUESTION 4 *Use a SEPARATE writing sheet.*

Marks

(a) The third term and the tenth term of an arithmetic series are 7 and 42 respectively. Find the: **3**

(i) first term and the common difference.

(ii) sum of the first ten terms of the series.

(b) A box contains five blue, three yellow and eight red beads. Two beads are selected at random from the box without replacement. Find the probability that: **3**

(i) both beads are blue.

(ii) at most one of the beads is blue.

(c) **6**

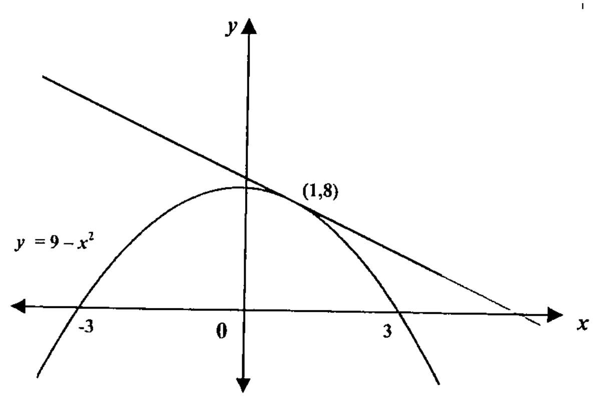

FIGURE NOT TO SCALE

The diagram shows thr graph of the parabola $y = 9 - x^2$. A tangent is drawn to the parabola at the point (1,8).

(i) Show that the equation of the tangent at (1,8) is $2x + y = 10$.

(ii) Explain how you know the tangent crosses the x axis at (5,0).

(iii) Calculate the area bounded by the parabola, the tangent and the x axis.

QUESTION 5 *Use a SEPARATE writing sheet*

Marks

(a) **4**

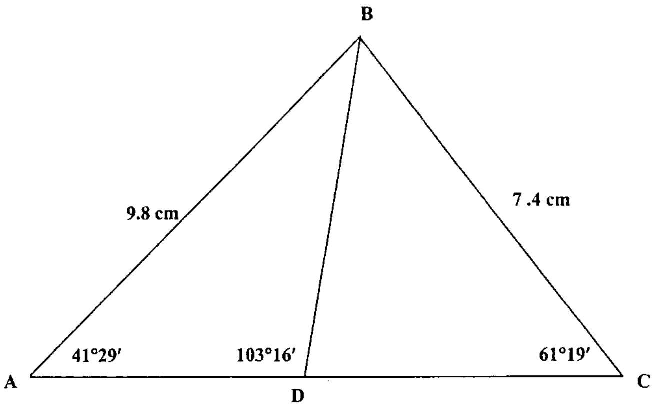

FIGURE NOT TO SCALE

In the diagram, AB = 9.8 cm, BC = 7.4 cm, ∠BCD = 61°19′, ∠BDA = 103°16′ and ∠BAD = 41°29′.

(i) Find the length of BD correct to the nearest mm.

(ii) Calculate the area of ΔABC correct to the nearest cm^2.

(b) Consider the curve $y = 4x^3 + 6x^2$. **8**

(i) Find the coordinates of any turning points and determine their nature.

(ii) Find the x coordinate of any points of inflexion.

(iii) Sketch the curve for the domain $-2 \le x \le 1$.

(iv) What is the maximum value of $4x^3 + 6x^2$ in the domain $-2 \le x \le 1$?

QUESTION 6 *Use a SEPARATE writing sheet*

Marks

(a) Find all the values of x for which $\ln x = 2 \ln x$ **2**

(b) **5**

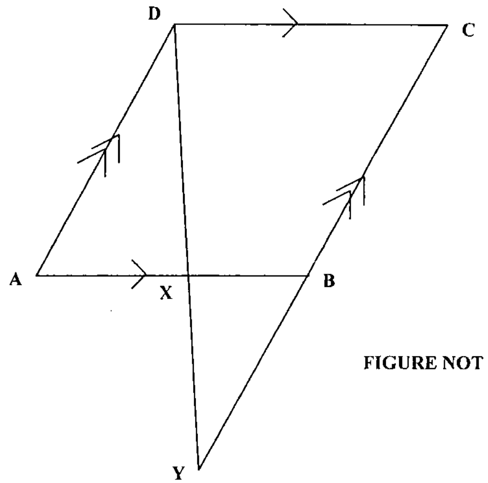

FIGURE NOT TO SCALE

In the diagram, ABCD is a parallelogram. X is a point on AB. DX and CB are both produced to Y.

(i) Copy this diagram onto your answer sheet.

(ii) Prove that ΔADX is similar to ΔCYD.

(iii) Hence find the length of XY given AX = 8 cm, DC = 12 cm and DX = 10 cm.

QUESTION 6 ***(Continued)*** **Marks**

(c) **5**

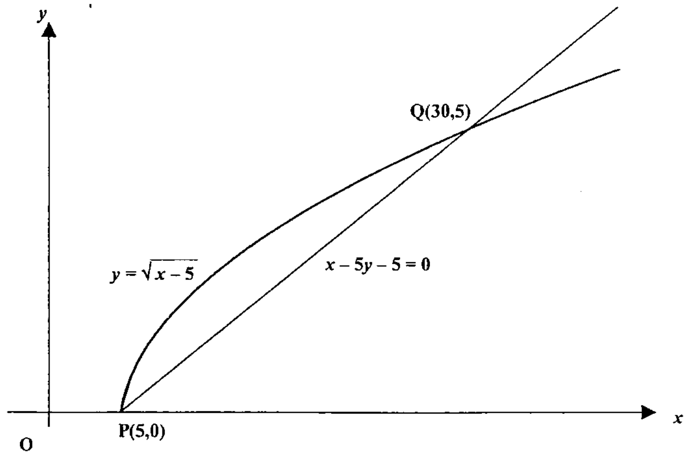

FIGURE NOT TO SCALE

The diagram shows the graphs of the curve $y = \sqrt{x-5}$ and the line $x - 5y - 5 = 0$. The curve and the line intersect at the points P(5,0) and Q(30,5). The region bounded by the curve and the line is rotated about the y axis.
Find the volume of the solid generated.

QUESTION 7 ***Use a SEPARATE writing sheet*** **Marks**

(a) An experimental vaccine was injected into a cat. The amount, M millilitres, of vaccine present in the bloodstream of the cat, t hours later was given by $M = e^{-2t} + 3$. **5**

(i) How much vaccine was initially injected into the cat?

(ii) At what rate was the amount of vaccine decreasing at the end of 3 hours?

(iii) Show that there will always be more than 3 millilitres of vaccine present in the cat's bloodstream.

(iv) Sketch the curve of $M = e^{-2t} + 3$ to show how the amount of vaccine present in the cat's bloodstream changes over time.

(b) A particle moves in a straight line. At time t seconds, its displacement, x metres from a fixed point O on the line is given by **7**

$$x = 1 - \cos \pi t$$

(i) What is the initial displacement of the particle?

(ii) Sketch the graph of x as a function of t.

(iii) Find an expression for the velocity of the particle at any time t.

(iv) What is the velocity of the particle at time $t = \frac{1}{6}$?

(v) At what time does the particle first reach its maximum speed?

QUESTION 8 *Use a SEPARATE writing sheet* **Marks**

(a) When Jack left school, he borrowed \$15 000 to buy his first car. The interest rate on the loan was 18% p.a. and Jack planned to pay back the loan in 60 equal monthly instalments of \$M. **6**

(i) Show that immediately after making his first monthly instalment, Jack owed

$$\$[15\,000 \times 1{\cdot}015 - M]$$

(ii) Show that immediately after making his third monthly instalment, Jack owed

$$\$[15\,000 \times 1{\cdot}015^3 - M(1 + 1{\cdot}015 + 1{\cdot}015^2)]$$

(iii) Calculate the value of M.

(b) The table shows the values of a function $f(x)$ for five x values. **3**

x	0	2	4	6	8
$f(x)$	0·9	1·4	1·8	2·1	1·7

Approximate the value of $\int_0^8 f(x)\,dx$ using the five function values and Simpson's rule.

(c) (i) On the same number plane, sketch the curves $y = \sin x$ and $y = \tan\frac{x}{2}$ in the domain $0 \le x \le 2\pi$. **3**

(ii) Hence find the number of real solutions to the equation $\sin x = \tan\frac{x}{2}$ for $0 \le x \le 2\pi$.

QUESTION 9 *Use a SEPARATE writing sheet* **Marks**

(a) **4**

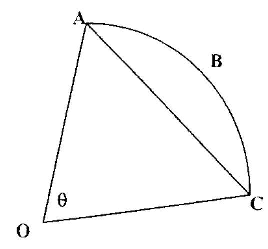

FIGURE NOT TO SCALE

The diagram shows a sector OAC with area 90π cm^2 and OA = 15 cm

(i) Find the size of θ in radians.

(ii) Find the perimeter of the segment ABC. Give your answer correct to the nearest cm.

(b) **8**

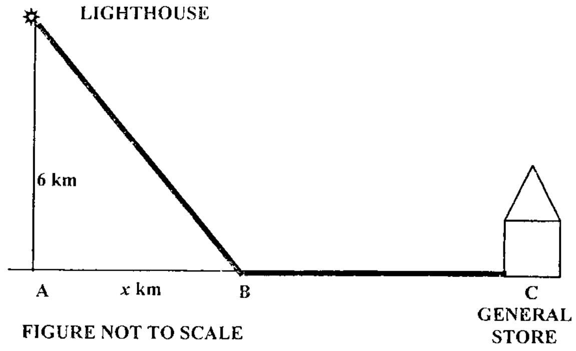

FIGURE NOT TO SCALE

The water's edge is a straight line ABC which runs East-West. A lighthouse is 6 km due north of A. 10 km due east of A is the general store. To get to the general store as quickly as possible, the lighthouse keeper rows to a point B, x km from A, and then jogs to the general store. The lighthouse keeper's rowing speed is 6 km/h and his jogging speed is 10 km/h.

(i) Show that it takes the lighthouse keeper $\frac{\sqrt{36 + x^2}}{6}$ hours to row from the lighthouse to B.

(ii) Show that the total time taken for the lighthouse keeper to reach the general store is given by:

$$T = \frac{\sqrt{x^2 + 36}}{6} + \frac{10 - x}{10} \text{ hours}$$

(iii) Hence show that when $x = 4\frac{1}{2}$ km, the time it takes for the lighthouse keeper to travel from the lighthouse to the general store is a minimum.

(iv) Hence find the quickest time it takes the lighthouse keeper to go to the general store from the lighthouse. Give your answer correct to the nearest minute.

QUESTION 10 *Use a SEPARATE writing sheet*

Marks

(a) 7

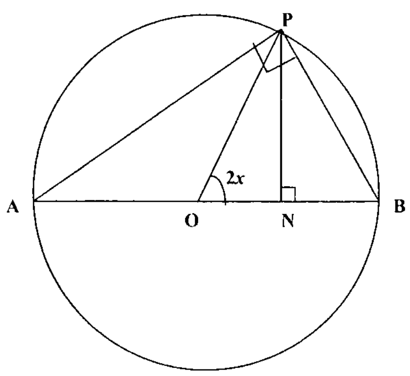

FIGURE NOT TO SCALE

The diagram shows a circle with centre O and diameter AB. P is a point on the circumference of the circle. PN is drawn perpendicular to AB and AP is perpendicular to PB. Let $\angle POB = 2x$.

(i) Explain why ΔAPO is isosceles.

(ii) Explain why $\angle OAP = \angle OPA = x$

(iii) Show that $\sin 2x = \dfrac{2PN}{AB}$

(iv) Use ΔAPN and ΔPAB to show that $2 \sin x \cos x = \sin 2x$

(b) Kellie and Lachlan play a game where they each take turns at throwing two ordinary dice. The winner is the first person to throw a double. Kellie throws first. 5

(i) Show that the probability that Lachlan wins the game on his first throw is $\dfrac{5}{36}$.

(ii) Show that the probability Lachlan wins the game on his first or second throw is given by $\dfrac{5}{36} + \dfrac{5^3}{6^4}$.

(iii) Calculate the probability that Lachlan wins the game.

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NOTE : $\ln x = \log_e x, \quad x > 0$