Mathematics Extension 1: Question

Suggested Solutions	Marks Awarded	Marker's Comments

1(a) $y = 2x - 5 \qquad y = 6 - 3x$

$$\tan\theta = \left|\frac{2-(-3)}{1+2\times(-3)}\right| \qquad \left(\frac{m_1 - m_2}{1+m_1 m_2}\right)$$

$$= 1$$

$$\theta = 45^\circ$$

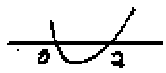
(2)

(b) $$\frac{x+4}{x} < 3$$

$$x(x+4) < 3x^2$$

$$2x^2 - 4x > 0$$

$$2x(x-2) > 0$$



$$x < 0, \; x > 2$$

(3)

(c) $$\sin 2\theta = \sin^2\theta$$

$$2\sin\theta\cos\theta - \sin^2\theta = 0$$

$$\sin\theta(2\cos\theta - \sin\theta) = 0$$

$\sin\theta = 0$ or $\sin\theta = 2\cos\theta$

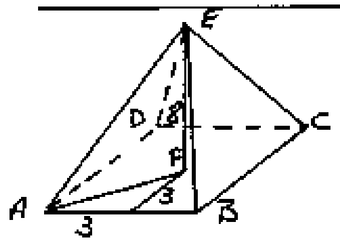
$$\tan\theta = 2.$$

$\theta = n\pi, \quad \theta = n\pi + \tan^{-1}2$

or $n\pi + 1.11$ (2dp)

(4)

(d).



$$AF = \sqrt{3^2+3^2} = \sqrt{18}$$

$$\tan\theta = \frac{EF}{AF} = \frac{8}{\sqrt{18}}$$

$$\theta = 62^\circ 04'$$

or 62° (nearest degree)

(3)

Mathematics Extension 1: Question 2		
Suggested Solutions	Marks Awarded	Marker's Comments

2(a) $\begin{matrix} x_1 & y_1 \\ (-2, & 5) \end{matrix}$ $\begin{matrix} x_2 & y_2 \\ (a, & b) \end{matrix}$ $\begin{matrix} m_1 & m_2 \\ 3 & : 2 \end{matrix}$

$$\frac{3a + 2(-2)}{3+2} = 2 \ ; \quad \frac{3\times b + 2\times 5}{3+2} = 2$$

$$a = 4\tfrac{2}{3}, \quad b = 0$$ (2)

(b) $2x^3 - 6x^2 + 5x - 1 = 0$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$$

$$= \frac{\frac{5}{2}}{\frac{1}{2}}$$

$$= 5$$ (3)

(c) $x^3 - 11x^2 + px + q = 0$

Roots are $\alpha, \alpha, \alpha + 2$.

Sum of roots: $\alpha + \alpha + (\alpha + 2) = 11$

$\alpha = 3$.

$\therefore$ Roots are $3, 3, 5$.

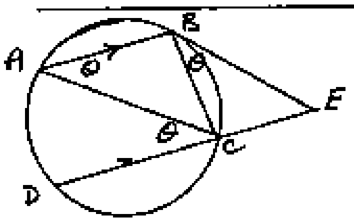
$\alpha\beta + \alpha\gamma + \beta\gamma$: $3\times 3 + 3\times 5 + 3\times 5 = p$

$p = 39$.

$\alpha\beta\gamma$: $3 \times 3 \times 5 = -q$

$q = -45$. (4)

(d)



$\angle CBE = \angle BAC = \theta$ (alt. segment thm) (A)

$\angle BAC = \angle ACD = \theta$ (alt. angles $AB \parallel DC$)

$\angle BCD = \angle CBE + \angle BEC$ (ext. angle of $\triangle BCE$)

$\theta + \angle ACB = \theta + \angle BEC$

$\therefore \angle ACB = \angle BEC$ (B)

From (A), (B), two pairs of angles are equal

$\therefore \triangle ABC \ ||| \ \triangle BCE$ (3)

Marker's comments: Alternative: $\angle ABC = \angle BCE$ (alternate $\angle$s in parallel lines)

Mathematics Extension 1: Question 3

Suggested Solutions	Marks Awarded	Marker's Comments

3.(a) Let $y = \frac{5-2x}{3}$

Inverse is: $x = \frac{5-2y}{3}$

$3x = 5-2y$

$y = \frac{5-3x}{2}$

$\therefore f^{-1}(x) = \frac{5-3x}{2}$ (2)

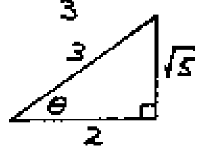
(b) $\cos^{-1}\left(\frac{1}{2}\tan\frac{2\pi}{3}\right) = \cos^{-1}\left(\frac{1}{2}\times -\sqrt{3}\right)$

$= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{5\pi}{6}$ (2)

(c). $\sin\left(2\cos^{-1}\frac{2}{3}\right)$ Let $\theta = \cos^{-1}\frac{2}{3}$

$\cos\theta = \frac{2}{3}$



$= \sin 2\theta$

$= 2\sin\theta\cos\theta$

$= 2\times\frac{\sqrt{5}}{3}\times\frac{2}{3}$

$= \frac{4\sqrt{5}}{9}$ (3)

(d) $\frac{\cos A - \sin A}{\cos A + \sin A} = \frac{\cos A - \sin A}{\cos A + \sin A}\times\frac{\cos A + \sin A}{\cos A + \sin A}$

$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A + 2\sin A\cos A}$

$= \frac{\cos 2A}{1+\sin 2A}$ (2)

(e) $y = x + \cos^{-1}x$

$\frac{dy}{dx} = 1 - \frac{1}{\sqrt{1-x^2}} = 1-(1-x^2)^{-\frac{1}{2}}$

For stat. point: $1 = \frac{1}{\sqrt{1-x^2}}$ $\therefore \sqrt{(1-x^2)} = 1$

$\therefore x = 0$

$\frac{d^2y}{dx^2} = \frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x) = \frac{-x}{(1-x^2)^{3/2}}$

When $x=0$, $\frac{d^2y}{dx^2} = 0$.

If $x<0$, $\frac{d^2y}{dx^2} > 0$; If $x>0$, $\frac{d^2y}{dx^2} < 0$

Concavity changes $\therefore$ one stationary point is a horizontal point of inflexion. (3)

2003

Mathematics Extension 1: Question 4

Suggested Solutions	Marks Awarded	Marker's Comments

4.(a)(i) No. of different hands $= \binom{52}{4}$

$= 270\,725$ (1)

(ii) $P(2\text{ aces}) = \dfrac{\binom{4}{2}\binom{48}{2}}{\binom{52}{4}}$

$= 0{\cdot}025$ (2)

(b) (i) No. of arrangements $= \dfrac{8!}{2!\,2!\,2!}$

$= 5040$ (2)

(ii) No. arrgts. with U at ends $= \dfrac{6!}{2!\,2!}$

$= 180$ (1)

(c)(i) PQ: $\dfrac{y-q^2}{x-2q} = \dfrac{p^2-q^2}{2p-2q} = \dfrac{p+q}{2}$

$2y-2q^2 = (p+q)x - 2q(p+q)$

$2y-2q^2 = (p+q)x - 2pq - 2q^2$

$(p+q)x - 2y - 2pq = 0$ (2)

(ii) M: $\left(\dfrac{2p+2q}{2}, \dfrac{p^2+q^2}{2}\right)$

i.e. $\left(p+q, \dfrac{p^2+q^2}{2}\right)$ (1)

(iii) If PQ passes through $(0,2)$

Subst. $x=0, y=2$: $0 - 2\times 2 - 2pq = 0$

$pq = -2$

$\therefore\ x = p+q, \quad y = \dfrac{p^2+q^2}{2}$

$(p+q)^2 = p^2+q^2+2pq$

$x^2 = 2y + 2\times(-2)$

$x^2 = 2y-4$

Locus of M is $x^2 = 2y-4$. (3)

Mathematics Extension 1: Question 5		
Suggested Solutions	Marks Awarded	Marker's Comments

5. (a)(i) $\int \frac{1}{4+x^2}\,dx = \frac{1}{2}\tan^{-1}\frac{x}{2} + C$ (1)

(ii) $\int \frac{x}{4+x^2}\,dx = \frac{1}{2}\int \frac{2x}{4+x^2}\,dx$

$= \frac{1}{2}\log_e(4+x^2) + C$ (1)

(b) $\int_0^{\frac{\pi}{6}} \sin^2 x\,dx = \int_0^{\frac{\pi}{6}} \frac{1}{2}(1-\cos 2x)\,dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{6} - \frac{1}{2}\times\frac{\sqrt{3}}{2}\right) - (0-0)\right]$

$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$ or $\frac{2\pi - 3\sqrt{3}}{24}$ (3)

(c) $\int \frac{e^{2x}}{e^x - 2}\,dx$

$u = e^x - 2$

$\frac{du}{dx} = e^x$

$du = e^x\,dx$

$= \int \frac{e^x e^x\,dx}{e^x - 2}$

$= \int \frac{(u+2)\,du}{u}$

$= \int \left(1 + \frac{2}{u}\right)du$

$= u + 2\ln u + c$

$= e^x - 2 + 2\ln(e^x - 2) + C$ (3)

(d) $\ddot{x} = \frac{-6}{(x+1)^2}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -6(x+1)^{-2}$

$\frac{1}{2}v^2 = 6(x+1)^{-1} + C$

When $x=0, v=4$: $8 = 6 + C$

$C = 2$

$\therefore \frac{1}{2}v^2 = \frac{6}{x+1} + 2$

$= \frac{6 + 2(x+1)}{x+1}$

$= \frac{2x+8}{x+1}$

$= 2\left(\frac{x+4}{x+1}\right)$

$v^2 = 4\left(\frac{x+4}{x+1}\right)$

$v = \pm 2\sqrt{\frac{x+4}{x+1}}$ (4)

Mathematics Extension 1: Question 6		
Suggested Solutions	Marks Awarded	Marker's Comments

6(a) $\int \frac{1}{\sqrt{x^2+16}}\,dx = \log_e\left(x+\sqrt{x^2+16}\right)+C$ (1)

(b) Prove $\frac{1}{1.3}+\frac{1}{3.5}+\dots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

When $n=1$, $LHS=\frac{1}{1.3}=\frac{1}{3}$, $RHS=\frac{1}{2+1}=\frac{1}{3}$

$\therefore$ it is true for $n=1$.

Assume it is true for $n=k$.

i.e. assume $\frac{1}{1.3}+\frac{1}{3.5}+\dots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$

When $n=k+1$,

$LHS=\frac{1}{1.3}+\frac{1}{3.5}+\dots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$

$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$ by assumption

$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$

$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$

$=\frac{(k+1)(2k+1)}{(2k+1)(2k+3}$

$=\frac{k+1}{2(k+1)+1}=\frac{n}{2n+1}$ where $n=k+1$.

$\therefore$ if it is true for $n=k$, it is true for $n=k+1$.

Since it is true for $n=1$, it is true for $n=2$, $n=3$, ... (4)

(c) Let $f(x)=2x-4\sin 3x$

$f'(x)=2-12\cos 3x$.

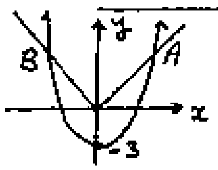
$f(1)=2-4\sin 3=1.4355$

$f'(1)=2-12\cos 3=13.880$

Approx'n $=1-\frac{f(1)}{f'(1)}=1-\frac{1.4355}{13.880}$

$=0.90$ (2dP) (3)

(d)(i)



(2)

(ii) $x^2-3=2x$

$x^2-2x-3=0$

$(x-3)(x+1)=0$

$x=3,-1$

$A(3,6)$ $B(-3,6)$

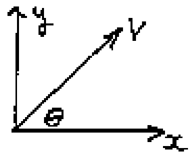
$\therefore |2x|>x^2-3$ for $-3<x<3$. (2)

Mathematics Extension 1: Question 7(a)

Suggested Solutions	Marks Awarded	Marker's Comments

7(a) (i) $\ddot{x} = 0$

$\dot{x} = c$



When $t=0$, $\dot{x} = V\cos\theta$

$\therefore c = V\cos\theta$

$\therefore \dot{x} = V\cos\theta$

$x = V\cos\theta\, t + c'$

When $t=0$, $x=0$ $\therefore c'=0$

$\therefore x = V\cos\theta\, t$

$\ddot{y} = -10$

$\dot{y} = -10t + k$

When $t=0$, $\dot{y} = V\sin\theta$ $\therefore k = V\sin\theta$

$\therefore \dot{y} = V\sin\theta - 10t$

$y = V\sin\theta\, t - 5t^2 + k'$

When $t=0$, $y=0$ $\therefore k'=0$

$\therefore y = V\sin\theta\, t - 5t^2$ (2)

(ii) When $t=4$, $y=0$, $x=100$

$100 = 4V\cos\theta$ $\qquad 0 = 4V\sin\theta - 80$

$V\cos\theta = 25$ $\qquad V\sin\theta = 20$

$\frac{V\sin\theta}{V\cos\theta} = \frac{20}{25}$ $\therefore \tan\theta = 0\cdot8$

$\theta = 38°40'$.

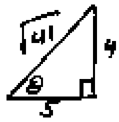
Also, $V^2\cos^2\theta + V^2\sin^2\theta = 25^2 + 20^2$

$V^2(\cos^2\theta + \sin^2\theta) = 1025$

$V = \sqrt{1025}$ or $32\cdot0$ m/s (2)

$= 5\sqrt{41}$

(iii) Maximum height when $\dot{y} = 0$



$5\sqrt{41}\sin 38°40' - 10t = 0$

$t = 2$

When $t=2$, $y = 5\sqrt{41} \times \frac{4}{\sqrt{41}} \times 2 - 5\times 2^2$

$= 20$

Maximum height is 20 m.

(2)

Mathematics Extension 1: Question 7(b)		
Suggested Solutions	Marks Awarded	Marker's Comments
7.(b) $\frac{dT}{dt} = -k(T - T_0)$ (i) $T = T_0 + Ae^{-kt}$ $\frac{dT}{dt} = -kAe^{-kt}$ $= -k(T - T_0)$ ① (ii) When $t = 0$, $T = 150$, $T_0 = 25$ $150 = 25 + A$ $\therefore A = 125$ ✓ $\therefore T = 25 + 125e^{-kt}$ When $t = 1$, $T = 100$ $100 = 25 + 125e^{-k}$ $75 = 125e^{-k}$ $e^{k} = \frac{125}{75}$ $k = \ln\left(\frac{125}{75}\right)$ $= 0.5108$ (4dp) ✓✓ $\therefore T = 25 + 125e^{-0.5108t}$ When $T = 50$, $50 = 25 + 125e^{-0.5108t}$ $25 = 125e^{-0.5108t}$ $e^{0.5108t} = \frac{125}{25} = 5$ $0.5108t = \ln 5$ $t = \frac{\ln 5}{0.5108}$ $= 3.15$ (2dp) It takes 3.15 minutes to reach 50°. ✓✓ ⑤		