

5

7 (b) (iii)

(cont'd) Also,  $\int_0^1 \frac{x}{(1+x)(1+x^2)} dx = \int_0^1 \left( \frac{-\frac{1}{2}}{2(x+1)} + \frac{\frac{1+x}{2}}{2(x^2+1)} \right) dx$  (partial fractions)  
 $= \left[ -\frac{1}{4} \log_e(1+x) \right]_0^1 + \left[ \frac{1}{4} \log_e(x^2+1) \right]_0^1 + \left[ \frac{1}{2} \tan^{-1} x \right]_0^1$   
 $= -\frac{1}{4} \log_e 2 + \frac{1}{4} \log_e 2 + \frac{1}{2} \tan^{-1} 1$   
 $= \frac{\pi}{8} - \frac{1}{4} \log_e 2$

Hence  $\frac{\pi}{8} - \frac{1}{4} \log_e 2 < \int_0^1 \frac{\log_e(1+x)}{1+x^2} dx < \frac{1}{2} \log_e 2$  for all  $x > 0$ . ✓

QUESTION 8

(a) (i) a. Number of arrangements when there are no restrictions = 11! ✓  
 = 39916800

β. The males and females are in alternate positions.  
 Sit a person down. There are 5! ways of seating the remaining members of the same sex.  
 Then there are 6! ways of seating the opposite sex.  
 So the total number of ways = 5! × 6! ways. ✓

(ii) Two cases:

- (1) If one state has two representatives, number of ways =  $\binom{6}{4} \times 2^4 = 480$  ✓  
 (2) If no state has two representatives, number of ways =  $2^6 = 64$  ✓  
 Hence total number of ways = 480 + 64 = 544

(b)

(i)	Criteria	Marks
(i) • one mark for answer		1
(ii) • one mark for replacing $a, b, c$ by $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ respectively		2
• one mark for final answer		
(iii) • one mark for use of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$		
• one mark for final answer		2

(i)

$\sqrt[3]{abc} \leq \frac{a+b+c}{3} = \frac{1}{3}$   
 $abc \leq \frac{1}{27}$   
 $\frac{1}{abc} \geq 27$   
 $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$   
 $\therefore \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$   
 $\therefore (1-a)(1-b)(1-c) \geq 8abc$

(ii)

$\frac{1}{3} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq \sqrt[3]{\frac{1}{a} \cdot \frac{1}{b} \cdot \frac{1}{c}}$   
 $= \frac{1}{3} (a+b+c) \geq \sqrt[3]{abc}$   
 $\geq 3 \sqrt[3]{27}$   
 $\geq abc(9-1)$   
 $\therefore (1-a)(1-b)(1-c) \geq 8abc$

8 (c)  $A_1 A_2 A_3 = x$

$A_1 A_2 = \sin x$   $0 A_1 - 0 A_2 = \cos x$

$A_1 A_3 = x \cos x$

$B_2 A_3 = 0 B_2 \sin x = \sin x \cos x$   $0 A_2 - 0 B_3 = \cos x \cos x = \cos^2 x$

$A_2 B_3 = 0 A_2 \cdot x = x \cos^2 x$

$B_3 A_3 = 0 B_3 \sin x = \sin x \cos^2 x$

$y = A_0 B_1 + A_1 B_2 + A_2 B_3 + A_3 B_4 + \dots$   
 $= x + \sin x + x \cos^2 x + \sin x \cos x + x \cos^3 x + \sin x \cos^2 x + \dots$   
 $= x(1 + \cos x + \cos^2 x + \dots) + \sin x(1 + \cos x + \cos^2 x + \dots)$   
 $= (x + \sin x)(1 + \cos x + \cos^2 x + \dots)$   
 $= \frac{x + \sin x}{1 - \cos x}$

1 mark for setting up the series

1 mark for recognising the infinite GP.

(ii)  $\frac{dy}{dx} = \frac{(1 + \cos x)(1 - \cos x) - (x + \sin x) \sin x}{(1 - \cos x)^2}$

$= \frac{1 - \cos^2 x - x \sin x - \sin^2 x}{(1 - \cos x)^2}$

$= \frac{\sin^2 x - x \sin x - \sin^2 x}{(1 - \cos x)^2}$

$= -\frac{x \sin x}{(1 - \cos x)^2}$

1 mark for derivative.

Since  $0 < x \leq \frac{\pi}{2}$ ,  $\sin x > 0$  &  $(1 - \cos x)^2 > 0$

$\therefore \frac{dy}{dx} = -\frac{x \sin x}{(1 - \cos x)^2} < 0$  for all  $0 < x \leq \frac{\pi}{2}$   
 explaining why  $\frac{dy}{dx} < 0$

1 mark for explaining why  $\frac{dy}{dx} < 0$

(iii) Since  $\frac{dy}{dx} < 0$  in  $0 < x \leq \frac{\pi}{2}$ ,  $y$  is a decreasing function in  $0 < x \leq \frac{\pi}{2}$  | mark

$\therefore$  Absolute min. value of  $y$  occurs at the end of  $x = \frac{\pi}{2}$ .

When  $x = \frac{\pi}{2}$ ,  $y = \frac{\pi + \sin \frac{\pi}{2}}{1 - \cos \frac{\pi}{2}} = \frac{\pi + 1}{1} = \pi + 1$  | mark

①

$$\begin{array}{c} \boxed{A} \\ \frac{1}{2} \\ \boxed{A} \\ \frac{1}{2} \\ \boxed{A} \\ \frac{1}{2} \\ \boxed{B} \\ - \\ \boxed{B} \end{array}$$

$$P(A \text{ wins}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} \quad \checkmark$$

has won the first game:

```

graph TD
    A1[A] -- 1/2 --> B1[B]
    A1 -- 1/2 --> A2[A]
    B1 -- 1/2 --> A3[A]
    B1 -- 1/2 --> B2[B]
    A2 -- 1/2 --> A4[A]
    A2 -- 1/2 --> B3[B]
    B2 -- 1/2 --> A5[A]
    B2 -- 1/2 --> B4[B]
    A3 -- 1/2 --> A6[A]
    A3 -- 1/2 --> B5[B]
    A4 -- 1/2 --> A7[A]
    A4 -- 1/2 --> B6[B]
    A5 -- 1/2 --> A8[A]
    A5 -- 1/2 --> B7[B]
    A6 -- 1/2 --> A9[A]
    A6 -- 1/2 --> B8[B]
    A7 -- 1/2 --> A10[A]
    A7 -- 1/2 --> B9[B]
    A8 -- 1/2 --> A11[A]
    A8 -- 1/2 --> B10[B]
    A9 -- 1/2 --> A12[A]
    A9 -- 1/2 --> B11[B]
    A10 -- 1/2 --> A13[A]
    A10 -- 1/2 --> B12[B]
    A11 -- 1/2 --> A14[A]
    A11 -- 1/2 --> B13[B]
    A12 -- 1/2 --> A15[A]
    A12 -- 1/2 --> B14[B]
    A13 -- 1/2 --> A16[A]
    A13 -- 1/2 --> B15[B]
    A14 -- 1/2 --> A17[A]
    A14 -- 1/2 --> B16[B]
    A15 -- 1/2 --> A18[A]
    A15 -- 1/2 --> B17[B]
    A16 -- 1/2 --> A19[A]
    A16 -- 1/2 --> B18[B]
    A17 -- 1/2 --> A20[A]
    A17 -- 1/2 --> B19[B]
    A18 -- 1/2 --> A21[A]
    A18 -- 1/2 --> B20[B]
    A19 -- 1/2 --> A22[A]
    A19 -- 1/2 --> B21[B]
    A20 -- 1/2 --> A23[A]
    A20 -- 1/2 --> B22[B]
    A21 -- 1/2 --> A24[A]
    A21 -- 1/2 --> B23[B]
    A22 -- 1/2 --> A25[A]
    A22 -- 1/2 --> B24[B]
    A23 -- 1/2 --> A26[A]
    A23 -- 1/2 --> B25[B]
    A24 -- 1/2 --> A27[A]
    A24 -- 1/2 --> B26[B]
    A25 -- 1/2 --> A28[A]
    A25 -- 1/2 --> B27[B]
    A26 -- 1/2 --> A29[A]
    A26 -- 1/2 --> B28[B]
    A27 -- 1/2 --> A30[A]
    A27 -- 1/2 --> B29[B]
    A28 -- 1/2 --> A31[A]
    A28 -- 1/2 --> B30[B]
    A29 -- 1/2 --> A32[A]
    A29 -- 1/2 --> B31[B]
    A30 -- 1/2 --> A33[A]
    A30 -- 1/2 --> B32[B]
    A31 -- 1/2 --> A34[A]
    A31 -- 1/2 --> B33[B]
    A32 -- 1/2 --> A35[A]
    A32 -- 1/2 --> B34[B]
    A33 -- 1/2 --> A36[A]
    A33 -- 1/2 --> B35[B]
    A34 -- 1/2 --> A37[A]
    A34 -- 1/2 --> B36[B]
    A35 -- 1/2 --> A38[A]
    A35 -- 1/2 --> B37[B]
    A36 -- 1/2 --> A39[A]
    A36 -- 1/2 --> B38[B]
    A37 -- 1/2 --> A40[A]
    A37 -- 1/2 --> B39[B]
    A38 -- 1/2 --> A41[A]
    A38 -- 1/2 --> B40[B]
    A39 -- 1/2 --> A42[A]
    A39 -- 1/2 --> B41[B]
    A40 -- 1/2 --> A43[A]
    A40 -- 1/2 --> B42[B]
    A41 -- 1/2 --> A44[A]
    A41 -- 1/2 --> B43[B]
    A42 -- 1/2 --> A45[A]
    A42 -- 1/2 --> B44[B]
    A43 -- 1/2 --> A46[A]
    A43 -- 1/2 --> B45[B]
    A44 -- 1/2 --> A47[A]
    A44 -- 1/2 --> B46[B]
    A45 -- 1/2 --> A48[A]
    A45 -- 1/2 --> B47[B]
    A46 -- 1/2 --> A49[A]
    A46 -- 1/2 --> B48[B]
    A47 -- 1/2 --> A50[A]
    A47 -- 1/2 --> B49[B]
    A48 -- 1/2 --> A51[A]
    A48 -- 1/2 --> B50[B]
    A49 -- 1/2 --> A52[A]
    A49 -- 1/2 --> B51[B]
    A50 -- 1/2 --> A53[A]
    A50 -- 1/2 --> B52[B]
    A51 -- 1/2 --> A54[A]
    A51 -- 1/2 --> B53[B]
    A52 -- 1/2 --> A55[A]
    A52 -- 1/2 --> B54[B]
    A53 -- 1/2 --> A56[A]
    A53 -- 1/2 --> B55[B]
    A54 -- 1/2 --> A57[A]
    A54 -- 1/2 --> B56[B]
    A55 -- 1/2 --> A58[A]
    A55 -- 1/2 --> B57[B]
    A56 -- 1/2 --> A59[A]
    A56 -- 1/2 --> B58[B]
    A57 -- 1/2 --> A60[A]
    A57 -- 1/2 --> B59[B]
    A58 -- 1/2 --> A61[A]
    A58 -- 1/2 --> B60[B]
    A59 -- 1/2 --> A62[A]
    A59 -- 1/2 --> B61[B]
    A60 -- 1/2 --> A63[A]
    A60 -- 1/2 --> B62[B]
    A61 -- 1/2 --> A64[A]
    A61 -- 1/2 --> B63[B]
    A62 -- 1/2 --> A65[A]
    A62 -- 1/2 --> B64[B]
    A63 -- 1/2 --> A66[A]
    A63 -- 1/2 --> B65[B]
    A64 -- 1/2 --> A67[A]
    A64 -- 1/2 --> B66[B]
    A65 -- 1/2 --> A68[A]
    A65 -- 1/2 --> B67[B]
    A66 -- 1/2 --> A69[A]
    A66 -- 1/2 --> B68[B]
    A67 -- 1/2 --> A70[A]
    A67 -- 1/2 --> B69[B]
    A68 -- 1/2 --> A71[A]
    A68 -- 1/2 --> B70[B]
    A69 -- 1/2 --> A72[A]
    A69 -- 1/2 --> B71[B]
    A70 -- 1/2 --> A73[A]
    A70 -- 1/2 --> B72[B]
    A71 -- 1/2 --> A74[A]
    A71 -- 1/2 --> B73[B]
    A72 -- 1/2 --> A75[A]
    A72 -- 1/2 --> B74[B]
    A73 -- 1/2 --> A76[A]
    A73 -- 1/2 --> B75[B]
    A74 -- 1/2 --> A77[A]
    A74 -- 1/2 --> B76[B]
    A75 -- 1/2 --> A78[A]
    A75 -- 1/2 --> B77[B]
    A76 -- 1/2 --> A79[A]
    A76 -- 1/2 --> B78[B]
    A77 -- 1/2 --> A80[A]
    A77 -- 1/2 --> B79[B]
    A78 -- 1/2 --> A81[A]
    A78 -- 1/2 --> B80[B]
    A79 -- 1/2 --> A82[A]
    A79 -- 1/2 --> B81[B]
    A80 -- 1/2 --> A83[A]
    A80 -- 1/2 --> B82[B]
    A81 -- 1/2 --> A84[A]
    A81 -- 1/2 --> B83[B]
    A82 -- 1/2 --> A85[A]
    A82 -- 1/2 --> B84[B]
    A83 -- 1/2 --> A86[A]
    A83 -- 1/2 --> B85[B]
    A84 -- 1/2 --> A87[A]
    A84 -- 1/2 --> B86[B]
    A85 -- 1/2 --> A88[A]
    A85 -- 1/2 --> B87[B]
    A86 -- 1/2 --> A89[A]
    A86 -- 1/2 --> B88[B]
    A87 -- 1/2 --> A90[A]
    A87 -- 1/2 --> B89[B]
    A88 -- 1/2 --> A91[A]
    A88 -- 1/2 --> B90[B]
    A89 -- 1/2 --> A92[A]
    A89 -- 1/2 --> B91[B]
    A90 -- 1/2 --> A93[A]
    A90 -- 1/2 --> B92[B]
    A91 -- 1/2 --> A94[A]
    A91 -- 1/2 --> B93[B]
    A92 -- 1/2 --> A95[A]
    A92 -- 1/2 --> B94[B]
    A93 -- 1/2 --> A96[A]
    A93 -- 1/2 --> B95[B]
    A94 -- 1/2 --> A97[A]
    A94 -- 1/2 --> B96[B]
    A95 -- 1/2 --> A98[A]
    A95 -- 1/2 --> B97[B]
    A96 -- 1/2 --> A99[A]
    A96 -- 1/2 --> B98[B]
    A97 -- 1/2 --> A100[A]
    A97 -- 1/2 --> B99[B]
    A98 -- 1/2 --> A101[A]
    A98 -- 1/2 --> B100[B]
    A99 -- 1/2 --> A102[A]
    A99 -- 1/2 --> B101[B]
    A100 -- 1/2 --> A103[A]
    A100 -- 1/2 --> B102[B]
    A101 -- 1/2 --> A104[A]
    A101 -- 1/2 --> B103[B]
    A102 -- 1/2 --> A105[A]
    A102 -- 1/2 --> B104[B]
    A103 -- 1/2 --> A106[A]
    A103 -- 1/2 --> B105[B]
    A104
```

$$P(4 \text{ wins}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$
$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{4}{16}$$

(a)

-

 $OB = OB$  (common) $OB = OB$  (common)
$$OP = OR \text{ (radii)}$$
 $\angle OPB = \angle ORB = 90^\circ$  (radius and tangent) $\Delta POB = \Delta ROB \text{ (RHS)} \quad \checkmark$ 

Hence  $\angle OBA = \angle OBC$  (corresponding angles of congruent triangles)

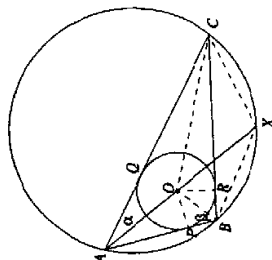
then by a similar proof to (ii),  $\angle BAX = \alpha$ .  $\checkmark$

hence  $\angle BOX = \alpha + \beta$  (exterior angle of  $\triangle ABO$ ). ✓

but  $\angle O B X = \alpha + \beta$  (adjacent angles),

$\angle BXY = \angle OX$  (opposite angles in  $\triangle OBX$  are equal).

ence  $BX = CX$ . ✓



The graph shows a Cartesian coordinate system with x and y axes. A straight line  $y = x$  passes through the origin  $O$ . A curve  $y = \log_e(1+x)$  also passes through the origin and is concave down. For  $x > 0$ , the curve is below the line, illustrating the inequality  $y = \log_e(1+x) < y = x$ . A dashed horizontal line is drawn at  $y = -1$ .

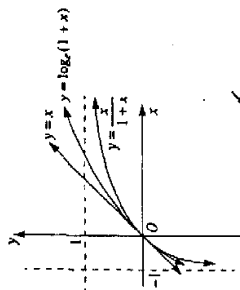
$$\frac{dy}{dx} = \frac{1}{1+x}$$

β. When  $x = 0$ ,  $\frac{dy}{dx} = 1$ , so  $y = x$  is a tangent at  $(0, 0)$ .

Since  $y = \log_e(1+x)$  is concave down, it follows that its graph is below the line  $y = x$  for  $x > 0$ . ✓

(ii)  $\alpha, \beta = \frac{x}{1+x}$

Using the quotient rule,  $\frac{dy}{dx} = \frac{1}{(1+x)^2}$ .



$\beta$ . When  $x = 0$ ,  $\frac{dy}{dx} = 1$ , so  $y = x$  is a tangent to both curves at  $(0, 0)$ .

But for  $x > 0$ , the gradient function of  $y = \frac{x}{1+x}$  is less than the gradient function of

$$y = \log_r(1+x), \text{ because } \frac{1}{(1+x)^2} < \frac{1}{1+x} \text{ for } x > 0.$$

Hence the graph of  $y = \frac{x}{1+x}$  is always below the graph of  $y = \log_e(1+x)$  for  $x > 0$ . ✓

(iii) From (i) and (ii),  $\frac{x}{1+r} < \log_e(1+x) < x$  for all  $x > 0$ .

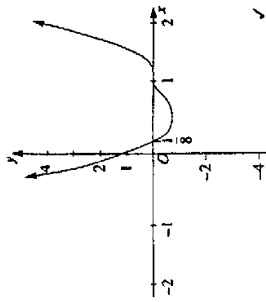
Hence  $\frac{x}{\log_e(1+x)} < \frac{\log_e(1+x)}{2}$ , for all  $x > 0$

and so  $\int_0^1 \frac{x}{(1+x)(1+x^2)} dx < \int_0^1 \frac{\log_2(1+x)}{1+x^2} dx < \int_0^1 \frac{x}{1+x^2} dx$  for all  $x > 0$ . ✓

$$\begin{aligned} \text{Now} \int_0^1 \frac{x}{1+x^2} dx &= \left[ \frac{1}{2} \log_e(x^2 + 1) \right]_0^1 \\ &= \frac{1}{2} \log_e 2 \quad \checkmark \end{aligned}$$

QUESTION 5

(a) (iii)



5(b)

Marking Guidelines

Criteria	Marks
(i) • one mark for general solution • one mark for particular solution	2
(ii) • one mark for expression for $\operatorname{Re}(\cos \theta + i \sin \theta)^5$ in terms of $\cos \theta$ , $\sin \theta$ • one mark for expression for $\operatorname{Re}(\cos \theta + i \sin \theta)^5$ in terms of $\cos \theta$ • one mark for final answer	3
(iii) • one mark for noting that $x = \cos \theta$ where $\cos 5\theta = -1$ • one mark for solution	2
(iv) • one mark for value of $\cos \frac{\pi}{3} + \cos \frac{3\pi}{3}$ • one mark for value of $\cos \frac{\pi}{3} \cdot \cos \frac{3\pi}{3}$ • one mark for factorisation	3

Answer

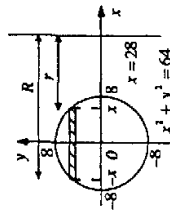
- (i)  $\cos 5\theta = -1 \Rightarrow 5\theta = (2n+1)\pi$   
 $\theta = (2n+1)\frac{\pi}{5}$ ,  $n = 0, \pm 1, \pm 2, \dots$   
 $0 \leq \theta \leq 2\pi \Rightarrow \theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$
- (ii) Using the binomial expansion,  
 $\operatorname{Re}\{(\cos \theta + i \sin \theta)^5\}$   
 $= \cos^5 \theta + 10 \cos^3 \theta (\sin \theta)^2 + 5 \cos \theta (\sin \theta)^4$   
 $= \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta$   
 $= \cos^5 \theta - 10 \cos^3 \theta (1 - \cos^2 \theta) + 5 \cos \theta (1 - \cos^2 \theta)^2$   
 $= 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$
- Using De Moivre's Theorem,  
 $(\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta$
- Hence  $\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$
- (iii)  $16x^5 - 20x^3 + 5x + 1 = 0$   
 has solutions  $x = \cos \theta$  where  $\cos 5\theta = -1$ .  
 $x = \cos \frac{\pi}{5}, \cos \frac{3\pi}{5}, \cos \pi, \cos \frac{7\pi}{5}, \cos \frac{9\pi}{5}$   
 $x = \cos \frac{\pi}{5}, \cos \frac{3\pi}{5}, \cos \frac{7\pi}{5}, \cos \frac{9\pi}{5}, -1$
- (iv)  $\sum \alpha = 0 \Rightarrow 2(\cos \frac{\pi}{5} + \cos \frac{3\pi}{5}) - 1 = 0$   
 $\therefore \cos \frac{\pi}{5} + \cos \frac{3\pi}{5} = \frac{1}{2}$   
 Product of roots is  $-\frac{1}{16}$   
 $\therefore -(\cos \frac{\pi}{5} \cdot \cos \frac{3\pi}{5}) = -\frac{1}{16}$   
 $\therefore \cos \frac{\pi}{5} \cdot \cos \frac{3\pi}{5} = \frac{1}{16}$   
 (since  $\cos \frac{\pi}{5} > 0$ ,  $\cos \frac{3\pi}{5} < 0$ )  
 Then  $\cos \frac{\pi}{5}, \cos \frac{3\pi}{5}$  are roots of  
 the equation  $4x^2 - 2x - 1 = 0$ . Hence  
 $16x^4 - 20x^3 + 5x + 1 = (x+1)(4x^2 - 2x - 1)^2$

Question 6

Marking Guidelines

Criteria	Marks
(i) • one mark for identifying slice as annular prism, thickness $\delta y$ • one mark for inner radius $r$ in terms of $y$ • one mark for outer radius $R$ in terms of $y$ • one mark for simplified value of $\delta V$ in terms of $y$	5
(ii) • one mark for expression for $V$ • one mark for using area of semi circle, or appropriate integration process • one mark for final answer	2

(i)



Volume of slice is

$$\delta V = \pi(R^2 - r^2)\delta y$$

$$= \pi(R + r)(R - r)\delta y$$

$$= \pi \cdot 56 \cdot 2 \sqrt{64 - y^2} \cdot \delta y$$

$$V = \lim_{\delta y \rightarrow 0} \sum_{y=-8}^8 112 \pi \sqrt{64 - y^2} \cdot \delta y$$

$$= 112 \pi \int_{-8}^8 \sqrt{64 - y^2} dy$$

$$\int_{-8}^8 \sqrt{64 - y^2} dy = \frac{1}{2} \pi \cdot 8^2 = 32 \pi \quad (\text{Area of semicircle radius 8}) \Rightarrow V = 3584 \pi^2$$

Exact volume of lifebelt is  $3584 \pi^2 \text{ cm}^3$

(b)

$$(i) \text{ RHS} = \frac{r^{n-2}}{1+r^2}$$

$$= \frac{(1+t^2)^{n-2} \cdot t^{n-2}}{1+t^2}$$

$$= \frac{t^{n-2} \cdot t^{n-2}}{1+t^2}$$

$$= \frac{t^n}{1+t^2}$$

$$= \text{LHS} \quad \checkmark$$

$$(ii) I_n = \int \frac{t^n}{1+t^2} dt$$

$$= \int \left( t^{n-2} - \frac{t^{n-2}}{1+t^2} \right) dt$$

$$= \frac{t^{n-1}}{n-1} - \int \frac{t^{n-2}}{1+t^2} dt$$

$$= \frac{t^{n-1}}{n-1} - I_{n-2} \quad \checkmark$$

(iii)

$$\text{Let } J_n = \int_0^1 \frac{t^n}{1+t^2} dt$$

$$\text{Then } J_n = \left[ \frac{t^{n-1}}{n-1} \right]_0^1 - J_{n-2}$$

$$= \frac{1}{n-1} - J_{n-2} \quad \checkmark$$

$$\text{Hence } J_0 = \frac{1}{5} - J_4$$

$$= \frac{1}{5} - \frac{1}{3} + J_2$$

$$= \frac{1}{5} - \frac{1}{3} + 1 - J_0 \quad \checkmark$$

$$\text{But } J_0 = \int_0^1 \frac{1}{1+t^2} dt$$

$$= \left[ \tan^{-1} t \right]_0^1 = \frac{\pi}{4}$$

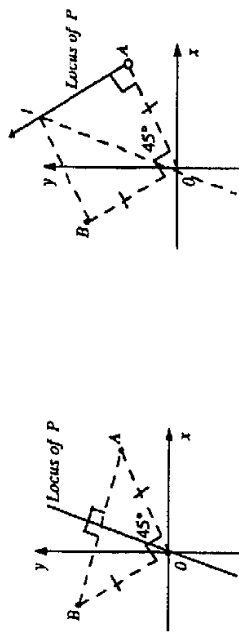
$$\text{Hence } J_0 = \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}$$

$$= \frac{13}{15} - \frac{\pi}{4} \quad \checkmark$$

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Marking Guidelines	
Criteria	Marks
(i) • one mark for answer	1
(ii) • one mark for answer	1
(iii) • one mark for answer	1

- 3(c) (i) Locus of  $P$  is perpendicular bisector of  $AB$ . (ii) Locus is ray from  $A$  parallel to  $OB$ .  
Let  $z = x + iy$ ,  $x, y$  real

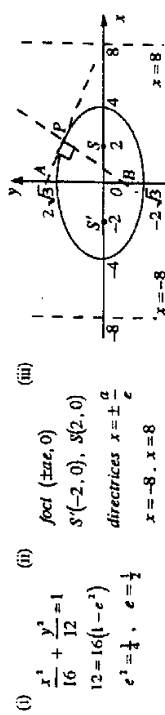


- (iii) If  $P$  is the point of intersection of these loci,  $OAPB$  is a square and the diagonal  $OP$  represents the sum of  $\alpha$  and  $i\alpha$ . Hence  $P$  represents  $(1+i)\alpha$ .

Marking Guidelines	
Criteria	Marks
(i) • one mark for domain	2
• one mark for range	
(ii) • one mark for coordinates of endpoint and equation of asymptote	2
• one mark for graph	



Marking Guidelines	
Criteria	Marks
(i) • one mark for eccentricity	1
(ii) • one mark for coordinates of foci	2
• one mark for equations of directrices	
(iii) • one mark for graph with intercepts	2
• one mark for showing foci and directrices	



#### 4 (b) Marking Guidelines

Criteria	Marks
(i) • one mark for expression $\frac{dy}{dx} = -\frac{3x}{4y}$	3
• one mark for equation of tangent	
• one mark for equation of normal	
(ii) • one mark for coordinates of $A$ and $B$	1

- (i)  $\frac{x^2}{16} + \frac{y^2}{12} = 1 \Rightarrow \frac{2x}{16} + \frac{2y}{12} \frac{dy}{dx} = 0$   
Tangent at  $P(2, 3)$  has gradient  $-\frac{3}{2}$  and equation  $y - 3 = -\frac{3}{2}(x - 2) \Rightarrow x + 2y - 8 = 0 \Rightarrow A(0, 4)$   
Normal at  $P(2, 3)$  has gradient 2 and equation  $y - 3 = 2(x - 2) \Rightarrow 2x - y - 1 = 0 \Rightarrow B(0, -1)$

Criteria	Marks
(i) • one mark for the gradients of $AS$ and $BS$	2
• one mark for showing $AS \perp BS$	
(ii) • one mark for showing points $A, P, S$ and $B$ are concyclic	1
• one mark for noting $AB$ is diameter	
• one mark for centre of circle	
• one mark for radius of circle	3

- (i)  $S(2, 0)$   $A(0, 4)$   $B(0, -1)$   
 $\text{grad } AS \cdot \text{grad } BS = -2 \times \frac{1}{2} = -1$   
 $\therefore ASB = 90^\circ$   
 $\therefore A, P, S, B$  are concyclic
- (ii)  $AB$  is diameter  
Diameter  $AB$   
centre  $(0, \frac{3}{2})$   
radius  $\frac{5}{2}$

#### QUESTION 5

- (a) (i) For  $P(x)$  to have a zero with multiplicity of 3, we can write  $P(x)$  as follows:  
 $P(x) = (x - \alpha)^3 Q(x)$ , where  $Q(\alpha) \neq 0$  ✓  
Differentiating,  $P'(x) = (x - \alpha)^3 Q'(x) + 3(x - \alpha)^2 Q(x)$  (product rule)  
 $= (x - \alpha)^2 [(x - \alpha) Q'(x) + 3Q(x)]$   
 $= (x - \alpha)^2 R(x)$ , where  $R(\alpha) = 3Q(\alpha) \neq 0$  ✓  
So  $P'(x)$  has a zero of multiplicity 2.
- (ii) Let  $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$  and let  $x = \alpha$  be the zero of multiplicity 3.  
Differentiating,  $P'(x) = 32x^3 - 75x^2 + 54x - 11$   
and  $P''(x) = 96x^2 - 150x + 54$   
 $= 6(16x^2 - 25x + 9)$   
 $= 6(x - 1)(16x - 9)$

- So the zeros of  $P''(x)$  are  $x = 1$  and  $x = \frac{9}{16}$ . ✓  
Testing  $x = 1$ ,  $P(1) = 0$  and  $P'(1) = 0$ , so  $P(x) = (x - 1)^3 Q(x)$   
Let  $x = \beta$  be the other zero.  $\beta = \frac{25}{8} - 3$   
Then  $\alpha + \alpha + \alpha + \beta = \frac{25}{8}$   
 $= \frac{1}{8}$   
So the zeros of  $8x^4 - 25x^3 + 27x^2 - 11x + 1$  are  $x = 1, 1, 1, \frac{1}{8}$  ✓

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# QUESTION 2

(b)  $-1 - i\sqrt{3} = 2\text{cis}\left(-\frac{2\pi}{3}\right)$  ✓  
Hence  $\left(2\text{cis}\left(-\frac{2\pi}{3}\right)\right)^{-10} = 2^{-10}\text{cis}\left(\frac{20\pi}{3}\right)$  ✓  
 $= 2^{-10}\text{cis}\left(\frac{2\pi}{3}\right)$  ✓  
 $= \frac{1}{1024}\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$  ✓  
 $= -\frac{1}{2048} + \frac{i\sqrt{3}}{2048}$  ✓

(c)  $(a+ib)^2 = 5-12i \Rightarrow (a^2-b^2) + 2abi = 5-12i$

$\therefore a^2 - b^2 = 5$  and  $ab = -6$

$a^4 - a^2b^2 = 5a^2 \Rightarrow a^4 - 5a^2 - 36 = 0$

$(a^2+4)(a^2-9) = 0 \Rightarrow a^2 > 0 \Rightarrow a^2 = 9$

$\therefore \begin{cases} a=3 \\ b=-2 \end{cases} \text{ or } \begin{cases} a=-3 \\ b=2 \end{cases}$

(d) To rotate  $\vec{OA}$  by  $-60^\circ$ , we need to multiply by  $\text{cis}\left(-\frac{\pi}{3}\right)$  ✓.

Thus  $\vec{OC} = 2 \times \vec{OA} \times \text{cis}\left(-\frac{\pi}{3}\right)$   
 $= 2 \times \text{cis}\left(\frac{2\pi}{3}\right) \times \text{cis}\left(-\frac{\pi}{3}\right)$   
 $= 2\text{cis}\left(\frac{\pi}{3}\right)$  ✓  
 $= 1 + i\sqrt{3}$  ✓

(e)

Criteria	Marks
(i) • one mark for answer	1
(ii) • one mark for answer	1
(iii) • one mark for choice of $z_1, z_2$	2
• one mark for answer	

(i)  $z_1 = a+ib, z_2 = c+id \Rightarrow z_1z_2 = (ac-bd) + i(ad+bc)$

$|z_1z_2|^2 = (ac-bd)^2 + (ad+bc)^2 = a^2c^2 - 2acbd + b^2d^2 + a^2d^2 + 2adbc + b^2c^2$

$\therefore |z_1z_2|^2 = (a^2+b^2)(c^2+d^2) = |z_1|^2 |z_2|^2$

(ii)  $z_1 = 2+3i \Rightarrow |z_1|^2 = 4+9=13$

(iii) For example :

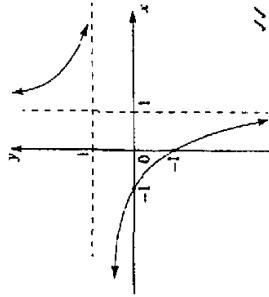
$z_1 = 3+2i, z_2 = 5-4i, z_1z_2 = 23-2i$

$|z_1|^2 = 13, |z_2|^2 = 41, |z_1z_2|^2 = 23^2 + 2^2$

$\therefore 533 = 13 \times 41 = 23^2 + 2^2$

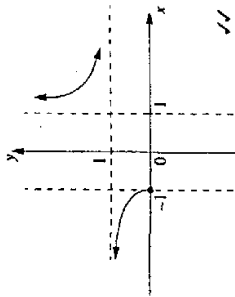
# QUESTION 3

(a) (i)

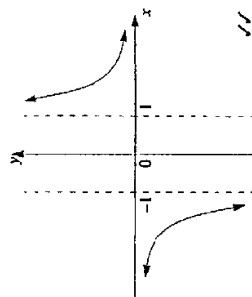


- vertical asymptote at  $x=1$
- horizontal asymptote at  $y=1$
- x intercept at  $x=-1$
- y intercept at  $y=-1$

(ii)



(iii)



- (b) The locus is the parabola with focus at (0, 2) and the x axis as directrix.  
or give the equation of the parabola as  $x^2 = 4(y-1)$

QUESTION 1

(a)  $\int_0^1 \frac{x dx}{\sqrt{16+x^2}} = \int_{16}^{25} \frac{1}{2} \frac{du}{\sqrt{u}}$   
 $= \frac{1}{2} \int_{16}^{25} u^{-\frac{1}{2}} du$   
 $= \frac{1}{2} \left[ 2u^{\frac{1}{2}} \right]_{16}^{25}$   
 $= \left[ \sqrt{u} \right]_{16}^{25}$   
 $= \sqrt{25} - \sqrt{16}$   
 $= 5 - 4$   
 $= 1$  ✓

(b)  $\int \frac{dx}{x^2+6x+13} = \int \frac{dx}{(x+3)^2+4}$   
 $= \frac{1}{2} \tan^{-1} \frac{x+3}{2} + c$  ✓

(c)  $\int x e^{-x} dx = \int x \frac{d}{dx} (-e^{-x}) dx$   
 $= -x e^{-x} - \int 1(-e^{-x}) dx$   
 $= -x e^{-x} + \int e^{-x} dx$   
 $= -x e^{-x} - e^{-x} + c$  ✓

(d)  $\int \cos^3 \theta d\theta = \int \cos^2 \theta \cos \theta d\theta$  ✓  
 $= \int (1 - \sin^2 \theta) \cos \theta d\theta$   
 $= \int (1 - u^2) du$  ✓  
 $= u - \frac{1}{3} u^3 + c$   
 $= \sin \theta - \frac{1}{3} \sin^3 \theta + c$  ✓

(e) (i) Let  $\frac{x^2-4x-1}{(1+2x)(1+x^2)} = \frac{A}{1+2x} + \frac{Bx+C}{1+x^2}$   
 Then  $x^2-4x-1 = A(1+x^2) + (1+2x)(Bx+C)$  ✓  
 Equating coefficients of like terms,  
 $1 = A + 2B$  [Eq. 1]  
 $-4 = B + 2C$  [Eq. 2]  
 $-1 = A + C$  [Eq. 3]  
 Multiply Eq. 2 by -2:  
 $8 = 2B - 4C$  [Eq. 2a]  
 Eq. 1 + Eq. 2a:  
 $9 = A - 4C$  [Eq. 4]  
 Eq. 3 - Eq. 4:  
 $-10 = 5C$   
 $C = -2$   
 Substitute C into Eq. 3:  
 $-1 = A - 2$   
 $A = 1$   
 Substitute A into Eq. 1:  
 $1 = 1 + 2B$   
 $B = 0$  ✓✓

(ii)  $\int \frac{x^2-4x-1}{(1+2x)(1+x^2)} dx = \int \left( \frac{1}{1+2x} + \frac{-2}{1+x^2} \right) dx$   
 $= \frac{1}{2} \ln|1+2x| - 2 \tan^{-1} x + c$  ✓✓

QUESTION 2

(a) (i)  $wz^2 = -3(1+i)^2$   
 $= -3(1-1+2i)$   
 $= -6i$  ✓  
 (ii)  $\frac{z}{z+w} = \frac{1+i}{-2+i} \times \frac{-2-i}{-2-i}$   
 $= \frac{-(1+i)(2+i)}{5}$   
 $= \frac{-(1+3i)}{5}$   
 $= -\frac{1}{5} - \frac{3i}{5}$  ✓

Let  $u = 16+x^2$   
 $du = 2x dx$  ✓