**Solutions**

**Question 1 (12 marks)**

(a) $$\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{dx}{2\sqrt{\frac{9}{4}-x^2}} \checkmark$$
$$= \frac{1}{2}\int \frac{dx}{\sqrt{\frac{9}{4}-x^2}}$$
$$= \frac{1}{2}\sin^{-1}\frac{3x}{2}+C \checkmark$$

(b) $y = 3e^{\tan 3x}$
$$\frac{dy}{dx} = 3e^{\tan 3x} \times 3\sec^2 3x \checkmark\checkmark$$
$$= 9e^{\tan 3x}\sec^2 3x$$

(c) $\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$   $\theta = 45^\circ$, $m_1 = k$, $m_2 = -2$
$$\tan 45^\circ = \left|\frac{k+2}{1-2k}\right|$$
$$1 = \left|\frac{k+2}{1-2k}\right| \checkmark$$
$\therefore 1-2k = k+2$   or   $-(1-2k) = k+2$
$-1 = 3k$   $-1+2k = k+2$
$-\frac{1}{3} = k$   $k = 3$
$\therefore$ possible values of $k = -\frac{1}{3}$ or $3$ ✓

(d) $A(4, -1)$, $B(x, y)$, $C(10, -7)$ ratio $= -3 : 5$
$$P(X, Y) = \left(\frac{nx_1+mx_2}{m+n}, \frac{ny_1+my_2}{m+n}\right)$$
For $x$; $$10 = \frac{5\times 4+(-3)\times x}{-3+5}$$
$20 = 20 - 3x$
$x = 0$ ✓

For $y$; $$-7 = \frac{5\times(-1)+(-3)\times y}{-3+5}$$
$-14 = -5 - 3y$
$3y = 9$
$y = 3$
$\therefore B(0, 3)$ ✓

(e) horizontal asymptote $y = \lim_{x\to\infty} f(x)$
$$\therefore y = \lim_{x\to\infty}\frac{3x}{x-7}$$
$$= \lim_{x\to\infty}\frac{3}{1-\frac{7}{x}}$$
$$= \frac{3}{1-0}$$
$\therefore$ horizontal asymptote is $y = 3$ ✓

(f) $\frac{3}{x+1} \geq 4$,   Note $x \neq -1$
$3(x+1) \geq 4(x+1)^2$ ✓
$3x+3 \geq 4x^2+8x+4$
$0 \geq 4x^2+5x+1$
$0 \geq (4x+1)(x+1)$ but $x \neq -1$
$\therefore -1 < x \leq -\frac{1}{4}$ ✓

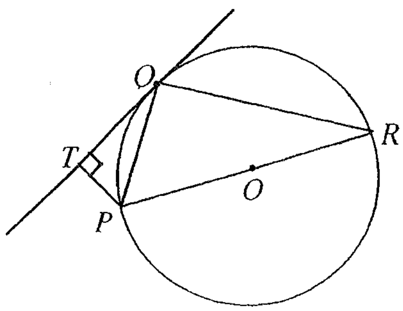
**Question 2 (12 marks)**

(a)(i) $\dfrac{7!}{3!2!} = 420$ ✓

(ii) $\dfrac{5!}{2!} \div \dfrac{7!}{3!2!} = \dfrac{1}{7}$ ✓

(iii) $\dfrac{\frac{5!}{3!} \times \frac{3!}{2!}}{\frac{7!}{2!3!}} = \dfrac{1}{7}$ ✓

(b)




Join $PQ$, $QO$ and $QR$.

Let $\angle QPR = \alpha$

$\angle PQR = 90^\circ$ (angle in a semi-circle) ✓

$\therefore \angle QRP = 180^\circ - 90^\circ - \alpha$ (angle sum of $\Delta\, PQR$)

$= 90^\circ - \alpha$

$\angle QRP = \angle TQP$ (angle in alternate segment)✓

$\therefore \angle TQP = 90^\circ - \alpha$

$\therefore \angle QPT = 180^\circ - 90^\circ - (90^\circ - \alpha)$

(angle sum of $\Delta\, QPT$)

$\therefore \angle QPT = \alpha = \angle QPR$

Hence $PQ$ bisects $\angle RPT$ ✓

(c) (i) $x = 3$ satisfies the equation

$3^3 - 5(3)^2 - 3 + k + 6 = 0$

$\therefore\ k = 15$ ✓

(ii) eqn becomes $x^3 - 5x^2 - x + 21 = 0$

sum of roots; $\alpha+\beta+\gamma = -\frac{b}{a}$

$\alpha+\beta+3 = 5$

$\alpha+\beta = 2$ ✓

product of roots; $\alpha\beta\gamma = -\frac{d}{a}$

$3\alpha\beta = -21$

$\alpha\beta = -7$ ✓

$\therefore$ the sum of the other two roots is 2

$\therefore$ the product of the other two roots is $-7$

(d) Prove $5^n + 3$ is divisible by 4

step 1 : Prove true for $n = 1$

$5^1 + 3 = 8$ which is divisible by 4

$\therefore$ true for $n = 1$. ✓

Step 2 : Assume true for $n = k$.

i.e. $5^k + 3 = 4p$ for some integer $p$

Step 3: Prove true for $n = k + 1$.

$$\begin{aligned} 5^{k+1} + 3 &= 5^{k+1} + 15 - 12 \\ &= 5(5^n + 3) - 12 \\ &= 5 \times 4p - 12 \\ &= 4(5p - 3) \end{aligned}$$

which is divisible by 4 ✓

$\therefore$ true for $n = k + 1$.

Hence if it is true for $n = k$, then it is true for $k + 1$. We have proved that it is true for $n = 1$, it must be true for $n = 2$. If it is true for $n = 2$, then it must be true for $n = 3$, and so on. Henc is true for all $n \geq 1$. ✓

**Question 3 (12 marks)**

(a) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\cos^{-1}\left(-\frac{1}{2}\right)=\frac{\pi}{4}+\left(\pi-\frac{\pi}{3}\right)$ ✓

$$=\frac{\pi}{4}+\frac{2\pi}{3}=\frac{11\pi}{12}. ✓$$

(b) Let the number of e's be $n$.

$\therefore \frac{8!}{n!}=6720$ ✓

$\therefore\ n!=\frac{8!}{6720}=6$

$\therefore\ n=3$ ✓

(c) (i) $\frac{dy}{dx}=\frac{dy}{dp}\times\frac{dp}{dx}$

$=2p\times\frac{1}{2}=p$ ✓

$\therefore$ the equation of the tangent at $P$ is

$y-p^2=p(x-2p)$

$y-p^2=px-2p^2$

$\therefore\ px-y-p^2=0$ ✓

$\therefore$ similarly, the equation of the tangent at $Q$ is

$\therefore\ qx-y-q^2=0$

(ii) Solving simultaneously the equations of tangents at $P$ and $Q$

$px-y-p^2=0 \ldots\ldots (1)$

$qx-y-q^2=0 \ldots\ldots (2)$

(1) $-$ (2) gives

$(p-q)x-(p^2-q^2)=0$

$x=\frac{(p+q)\cancel{(p-q)}}{\cancel{(p-q)}}=p+q$ ✓

Substituting into (1) gives

$y=p(p+q)-p^2=pq$ ✓

(iii) Gradient of chord $PQ$ is

$$m_{PQ}=\frac{p^2-q^2}{2p-2q}=\frac{(p+q)\cancel{(p-q)}}{2\cancel{(p-q)}}$$

$$=\frac{p+q}{2} ✓$$

$\therefore$ the equation of the chord $PQ$ is

$y-p^2=\frac{p+q}{2}(x-2p)$

$2(y-p^2)=(p+q)(x-2p)$

$2y-2p^2=(p+q)x-2p^2-2pq$ ✓

$\therefore\ (p+q)x-2y-2pq=0$ as required.

(iv) If $pq=-2$ then

$(p+q)x-2y+4=0$

At $x=0,\ 2y=4$

$\therefore\ y=2$ ✓

$\therefore$ the coordinates of $A$ is $(0,2)$ as required

(v) The gradient of $RN$ is

$$m_{RN}=\frac{pq-0}{(p+q)-0}=\frac{-2}{p+q}$$

Since $m_{PQ}\times m_{RN}=\frac{\cancel{p+q}}{2}\times\frac{-2}{\cancel{p+q}}=-1$ ✓

$\therefore PQ$ is perpendicular to $RN$.

**Question 4 (12 marks)**

(a) $u = 1 - 2x \qquad du = -2dx$

$2x = 1 - u \qquad dx = -\dfrac{du}{2}$ ✓

$$\int 4x\sqrt{1-2x}\,dx = \int 2(1-u)\sqrt{u}\,\frac{du}{-2}$$

$$= -\int (u^{\frac{1}{2}} - u^{\frac{3}{2}})du \text{ ✓}$$

$$= -\left(\tfrac{2}{3}u^{\frac{3}{2}} - \tfrac{2}{5}u^{\frac{5}{2}}\right) + C$$

$$= \tfrac{2}{5}(1-2x)^{\frac{5}{2}} - \tfrac{2}{3}(1-2x)^{\frac{3}{2}} + C \text{ ✓}$$

(b) Let $P(x) = x^2 + 9x + 4$

$P(a) = a^2 + 9a + 4$ .......... (i)

$P(-b) = (-b)^2 - 9b + 4$ ....... (ii)

Since $P(a) = P(-b)$

(i) – (ii) $\qquad a^2 - b^2 + 9(a+b) = 0$ ✓

$(a+b)(a-b) + 9(a+b) = 0$

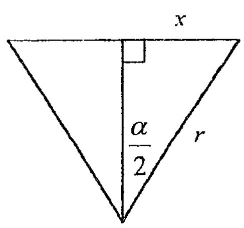
$(a+b)[a-b+9] = 0$

Since $a + b \neq 0 \Rightarrow a \neq -b$

$\therefore\ a - b + 9 = 0$

$\therefore\ a - b = -9$ ✓

(c) (i) Since $\alpha = \dfrac{2\pi}{n}$




$\sin\left(\dfrac{\alpha}{2}\right) = \dfrac{x}{r} \quad \therefore\ x = r\sin\left(\dfrac{\pi}{n}\right)$ ✓

Each side of the polygon $= 2r\sin\left(\dfrac{\pi}{n}\right)$

Perimeter of the polygon is

$n \times 2r\sin\left(\dfrac{\pi}{n}\right) = 2nr\sin\left(\dfrac{\pi}{n}\right)$ ✓

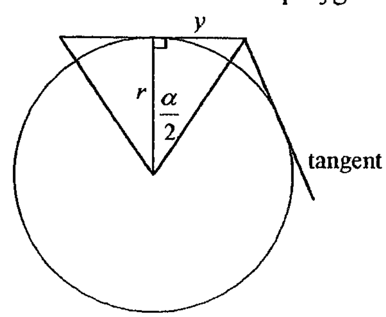
(ii) Area of each triangle is $\dfrac{1}{2}r^2\sin\alpha$

$= \dfrac{1}{2}r^2\sin\left(\dfrac{2\pi}{n}\right)$

$\therefore$ Area of the polygon is $n \times$ Area of each triangle

$= \dfrac{1}{2}nr^2\sin\left(\dfrac{2\pi}{n}\right)$ sq. units. ✓

(iii) For the circumscribed polygon




$\tan\dfrac{\alpha}{2} = \dfrac{y}{r}$

$y = r\tan\dfrac{\pi}{n}$ ✓

Area of the each triangle is

$\dfrac{1}{2}bh = \dfrac{1}{2} \times 2r\tan\dfrac{\pi}{n} \times r \ = r^2\tan\dfrac{\pi}{n}$ ✓

Area of the circumscribed polygon is

$n \times r^2\tan\dfrac{\pi}{n}$ as required.

(iv) Using the inequality

$$A_{\text{insribed polygon}} < A_{\text{circle}} < A_{\text{circumscribed polygon}}$$

$$\frac{1}{2}nr^2\sin\frac{2\pi}{n} < A_{\text{circle}} < nr^2\tan\frac{\pi}{n}$$

taking the limit as $n$ approaches infinity

$$\lim_{n\to\infty}\frac{1}{2}nr^2\sin\frac{2\pi}{n} = A_{\text{circle}} = \lim_{n\to\infty} nr^2\tan\frac{\pi}{n}$$

$$\lim_{n\to\infty}\frac{1}{2}r^2 \times \frac{\sin\frac{2\pi}{n}}{\frac{1}{n}} = A_{\text{circle}} = \lim_{n\to\infty} r^2\frac{\tan\frac{\pi}{n}}{\frac{1}{n}} \text{ ✓}$$

But note that $n \to \infty$, then $\dfrac{1}{n} \to 0$

Let $h = \frac{1}{n}$, $\therefore \lim\limits_{h\to 0} \frac{1}{2} r^2 \frac{\sin 2\pi h}{h}$

$$= \lim_{h\to 0} \frac{1}{2} r^2 \times \frac{\sin 2\pi h}{2\pi h} \times 2\pi \ ✓$$

(Note: $\lim\limits_{h\to 0} \frac{\sin 2\pi h}{2\pi h} = 1$)

$$= \frac{1}{2} r^2 \times 1 \times 2\pi = \pi r^2$$

Hence, the area of the circle is $\pi r^2$.

Or alternately use,

$$\lim_{h\to 0} r^2 \frac{\tan \frac{\pi}{n}}{\frac{\pi}{n}} \times \pi = \pi r^2 . ✓✓$$

**Question 5 (12 marks)**

(a)(i) $T_{r+1} = {}^{11}C_r \left(px^2\right)^{11-r}\left(\dfrac{1}{qx}\right)^r = Kx^r$

where $K$ is a constant.

$= {}^{11}C_r p^{11-r} \times \dfrac{1}{q^r}\left(x^{22-2r}.x^{-r}\right) = Kx^7$ ✓

Equating the powers of $x$ gives

$22 - 2r - r = 7$

$\therefore\ 3r = 15 \Rightarrow r = 5$ ✓

$\therefore\ T_6 = {}^{11}C_5 \left(px^2\right)^6\left(\dfrac{1}{qx}\right)^5$

$= {}^{11}C_5 p^6 q^{-5} x^7$

The coefficient is ${}^{11}C_5 p^6 q^{-5}$ ✓

(ii) For the expansion $\left(px - \dfrac{1}{qx^2}\right)^{11}$

$T_{s+1} = {}^{11}C_s (px)^{11-s}\left(-\dfrac{1}{qx^2}\right)^s = Kx^{-7}$ ✓

where $K$ is a constant.

Similarly, comparing powers of $x$ gives

$11 - s - 2s = -7$

$\therefore\ 3s = 18 \Rightarrow s = 6$ ✓

Since the coefficients are equal, then

${}^{11}C_5 p^6 q^{-5} = {}^{11}C_6 p^5 q^{-6}$

$\therefore\ \dfrac{p^6 q^{-5}}{p^5 q^{-6}} = \dfrac{{}^{11}C_6}{{}^{11}C_5}$

$\therefore\ \dfrac{p}{q^{-1}} = 1\ \left(\because\ {}^nC_r = {}^nC_{r-1}\right)$ ✓

$\therefore\ pq = 1$ as required.

(b) (i) $\cos(2\theta + 2\theta) = \cos^2 2\theta - \sin^2 2\theta$

$= 1 - \sin^2 2\theta - \sin^2 2\theta$

$= 1 - 2\sin^2 2\theta$ ✓

$= 1 - 2(2\sin\theta\cos\theta)^2$ ✓

$= 1 - 2\left(4\sin^2\theta\left(1 - \sin^2\theta\right)\right)$

$= 1 - 2\left(4\sin^2\theta - 4\sin^4\theta\right)$ ✓

$= 1 - 8\sin^2\theta + 8\sin^4\theta$ as requir

(ii) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(1 - 8\sin^2\theta + 8\sin^4\theta\right)\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos 4\theta\,d\theta$

$\therefore 8\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\sin^2\theta - \sin^4\theta\right)d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1 - \cos 4\theta)\,d\theta$ ✓

$\therefore\ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\sin^2\theta - \sin^4\theta\right)d\theta$

$= \dfrac{1}{8}\left[\theta - \dfrac{\sin 4\theta}{4}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ ✓

$= \dfrac{1}{8}\left[\left(\dfrac{\pi}{2} - \dfrac{\sin 2\pi}{4}\right) - \left(\dfrac{\pi}{4} - \dfrac{\sin\pi}{4}\right)\right]$

$= \dfrac{1}{8}\left[\left(\dfrac{\pi}{2} - 0\right) - \left(\dfrac{\pi}{4} - 0\right)\right]$

$= \dfrac{1}{8}\left[\dfrac{\pi}{4}\right] = \dfrac{\pi}{32}$. ✓

**Question 6 (12 marks)**

(a) (i) Let $a = n$ and $l = T_N = n+1$
where $N = n+2$

$$T_N = n + ((n+2)-1)d = n+1$$
$$(n+1)d = 1$$
$$\therefore\ d = \frac{1}{n+1}\ \checkmark$$

Hence, the arithmetic sequence is

$$n,\ n+\frac{1}{n+1},\ n+\frac{2}{n+1},\ \ldots,\ n+\frac{n+1}{n+1}\ \checkmark$$

(ii) $$n+\left(n+\frac{1}{n+1}\right)+\left(n+\frac{2}{n+1}\right)+\ldots+(n+1)$$

(Using $S_N = \frac{N}{2}(a+l)$) ✓

$$= \frac{n+2}{2}(n+(n+1))$$
$$= \frac{(n+2)(2n+1)}{2}\ \checkmark$$

(b) (i) $x_P = Vt\cos\theta,\ y_P = -\frac{gt^2}{2} + Vt\sin\theta + 36$

When $y = 0$, then $-5t^2 + 40\times\frac{3}{5}t + 36 = 0$ ✓

$$5t^2 - 24t - 36 = 0$$
$$(5t+6)(t-6) = 0$$
$$t = -\frac{6}{5}\text{ or }6\text{ s}$$

$\therefore$ the projectile takes 6 seconds to reach Q ✓

(ii) $x = 40\times 6\times\frac{4}{5} = 192\text{ m}$ ✓

(iii) At $t = 6$, $\dot{x} = 40\times\frac{4}{5} = 32\text{ m/s}$

$$\dot{y} = -10\times 6 + 40\times\frac{3}{5} = -36\text{ m/s}$$

Angle of impact $= \tan^{-1}\left|\frac{-36}{32}\right| = 48^0 22'$. ✓

Magnitude is

$$\sqrt{\left(\dot{x}\right)^2+\left(\dot{y}\right)^2} = \sqrt{32^2+(-36)^2} = 48.17\text{ m/}$$

(to 2 d.p.)

(iv) For the particle $Q$

$$\ddot{x} = 0,\quad \ddot{y} = -g$$
$$\dot{x} = 100,\quad \dot{y} = -gt$$
$$x = 100t,\quad y = -\frac{gt^2}{2} + SR\ \checkmark$$
$$\therefore\ 0 = -\frac{10\times 6^2}{2} + SR$$
$$SR = 180\text{ m}\ \checkmark$$

(v) $\therefore\ x = 100\times 6 = 600\text{ m}$ ✓

**Question 7 (12 marks)**

(a) (i) Solving $\sin y = x$ and $\cos y = x$

$$\sin y = \cos y$$

$$\tan y = 1$$

$$\therefore\ y = \tan^{-1} 1 = \frac{\pi}{4}$$

$$\therefore\ x = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

The point of intersection is $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$. ✓

(ii) The shaded area is equal to

$$\int_0^{\frac{\pi}{4}} \sin y\, dy + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos y\, dy \ ✓$$

$$= \left[-\cos y\right]_0^{\frac{\pi}{4}} + \left[\sin y\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left[\left(-\cos\frac{\pi}{4}\right) - \left(-\cos 0\right)\right] + \left[\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right] ✓$$

$$= -\frac{1}{\sqrt{2}} + 1 + 1 - \frac{1}{\sqrt{2}}$$

$$= 2 - \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} ✓$$

$$= \left(2 - \sqrt{2}\right) \text{ sq. units. as required.}$$

(iii) The volume of solid of revolution about the $y$-axis is given by

$$V_y = \pi\int_0^{\frac{\pi}{4}} \sin^2 y\, dy + \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 y\, dy \ ✓$$

$$= \pi\int_0^{\frac{\pi}{4}} \frac{1}{2}\left(1 - \cos 2y\right) dy + \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}\left(\cos 2y + 1\right)$$

$$= \frac{\pi}{2}\left[\left(y - \frac{\sin 2y}{2}\right)\right]_0^{\frac{\pi}{4}} + \frac{\pi}{2}\left[\frac{\sin 2y}{2} + y\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} ✓$$

$$= \frac{\pi}{2}\left[\left(\frac{\pi}{4} - \frac{\sin\frac{\pi}{2}}{2}\right) - 0\right] + \frac{\pi}{2}\left[\left(\frac{\sin\pi}{2} + \frac{\pi}{2}\right) - \left(\frac{\sin\frac{\pi}{2}}{2} + \frac{\pi}{4}\right)\right]$$

$$= \frac{\pi}{2}\left[\frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{2} - \frac{1}{2} - \frac{\pi}{4}\right] ✓$$

$$= \frac{\pi}{2}\left[\frac{\pi}{2} - 1\right] \text{ cubic units.}$$

(b)(i) given $T = A + Be^{kt}$

differentiating gives $\frac{dT}{dt} = kBe^{kt}$ ✓

but $\frac{dT}{dt} = k(T - A)$

$= k(A + Be^{kt} - A)$ ✓

$= kBe^{kt}$

$\therefore\ T = A + Be^{kt}$ is a solution

(ii) when $t = 0$, $A = 20°\text{C}$, $T = 80\,°\text{C}$

$\therefore 80 = 20 + Be^0$

$B = 60$ ✓

$\therefore\ T = 20 + 60e^{kt}$

when $t = 2$, $T = 40\,°\text{C}$

$\therefore 40 = 20 + 60e^{2k}$

$20 = 60\,e^{2k}$

$\frac{1}{3} = e^{2k}$

$\therefore k = \frac{1}{2}\ln\frac{1}{3}$ ✓

when $t = 3$,

$T = 20 + 60e^{(3 \times \frac{1}{2}\ln\frac{1}{3})}$

$T = 31{\cdot}547\ldots$

$\therefore T = 32\,°\text{C}$ ✓