Solutions

(1)(A)(i) $P(MF \text{ or } FM)$

$= \frac{5}{8} \times \frac{6}{10} + \frac{3}{8} \times \frac{4}{10}$ 1

$= \frac{3}{8} + \frac{3}{20}$

$= \frac{21}{40}$ 1

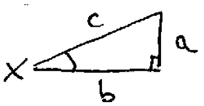
(ii) $P(A(M) \text{ or } B(M))$

$= \frac{1}{2} \times \frac{5}{8} + \frac{1}{2} \times \frac{4}{10}$ 1

$= \frac{5}{16} + \frac{1}{5}$

$= \frac{41}{80}$ 1

(B)



$\tan x = a/b$

$\sin x = a/c$

$\cos x = b/c$

Now $\frac{\sin x}{\cos x} = a/c \div b/c$ 1

$= \frac{a}{c} \times \frac{c}{b}$

$= a/b$

$= \tan x$ 1

$\therefore \int_0^h \tan x \, dx$

$= \int_0^k \frac{\sin x}{\cos x} \, dx$

$= -[\ln(\cos x)]_0^k$ 1

$= -\ln \cos k + \ln \cos 0$

$= -\ln \cos k + \ln 1$

$= -\ln \cos k + 0 = 1$

$\therefore e^{-\ln \cos k} = e^1$ 1

$e^{\ln \cos k} = e^{-1}$

$\cos k = 1/e$

$\therefore k = \cos^{-1}(\frac{1}{e})$

1 $\doteqdot 1.194$

(C) $\int_1^e \frac{\ln x}{x} dx$ $u = \ln x$

$\therefore du = \frac{1}{x} dx$ 1

$= \int_{\ln 1}^{\ln e} u \, du$

$= [\frac{1}{2}u^2]_0^1$ 1

$= \frac{1}{2}(1^2 - 0)$

$= \frac{1}{2}$ 1

(2)(A) $V = \pi \int_0^{\frac{3\pi}{4}} \sin^2 x \, dx$ 1

$= \pi [\frac{1}{2}x - \frac{1}{4}\sin 2x]_0^{\frac{3\pi}{4}}$ 1

$= \pi [\frac{1}{2} \times \frac{3\pi}{4} - \frac{1}{4} \times \sin 2 \times \frac{3\pi}{4}]$ 1

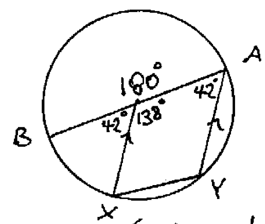
$- \pi [0 - \frac{1}{4}\sin 0]$

$= \pi [\frac{3\pi}{8} - \frac{1}{4}\sin\frac{3\pi}{2} + 0]$ 1

$= \pi(\frac{3\pi}{8} + \frac{1}{4})$

$= \frac{\pi}{4}(\frac{3\pi}{2} + 1)$ units^3 1

(B)



1 $\angle AOX = 138°$ (co-int. $\angle$'s, // lines)

$\angle BOX = 42°$ (corr. $\angle$'s, // lines)

$\angle AOX(\text{reflex}) = 2\angle AYX$ ($\angle$ at centre = double $\angle$ on circumference)

$\therefore 222° = 2\angle AYX$ 1

$\angle AYX = 111°$

$\therefore \angle OXY = 360 - 138 - 42 - 111$

1 $= 69°$ ($\angle$ sum quad.)

(C)(i) ${}^{12}C_4 = \frac{12 \times 11 \times 10 \times 9}{1 \times 2 \times 3 \times 4} = 495$

(ii) ${}^{10}C_2 = \frac{10 \times 9}{2 \times 1} = 45$ 1

(iii) ${}^3C_2 \times {}^{10}C_2 = \frac{3 \times 2}{2 \times 1} \times \frac{9 \times 8}{2 \times 1} = 108$ 1

1

(ii) $\int_0^{\frac{1}{3}} \frac{1}{\sqrt{4 - 9x^2}} dx$

$= \int_0^{\frac{1}{3}} \frac{4}{\sqrt{9(\frac{4}{9} - x^2)}} dx$

$= \int_0^{\frac{1}{3}} \frac{4}{3} \frac{1}{\sqrt{(\frac{2}{3})^2 - x^2}} dx$ 1

$= \frac{4}{3}[\sin^{-1}\frac{x}{\frac{2}{3}}]_0^{1/3}$

$= \frac{4}{3}[\sin^{-1}\frac{3x}{2}]_0^{1/3}$

$= \frac{4}{3}[\sin^{-1}\frac{1}{2} - \sin^{-1} 0]$ 1

$= \frac{4}{3}(\frac{\pi}{6} - 0)$

$= \frac{2}{9}\pi$ 1

(C) $\sqrt{3}\sin x - \cos x$

$a\sin\theta + b\cos\theta$

$= r\sin(\theta + \alpha)$

where $r = \sqrt{a^2 + b^2}$

$\tan \alpha = b/a$ 1

$\therefore = \sqrt{(\sqrt{3})^2 + 1^2} \times \sin(x \mp \alpha)$

$= 2\sin(x - \alpha)$

where $\tan \alpha = \frac{1}{\sqrt{3}}$

$\therefore \alpha = \frac{\pi}{6}$

$2\sin(x - \frac{\pi}{6}) = 1$ 1

$\sin(x - \frac{\pi}{6}) = \frac{1}{2}$

$\therefore x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$

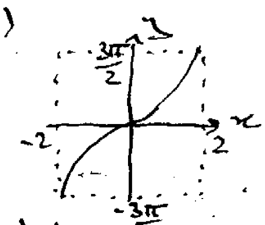
$\therefore x = \frac{\pi}{3}, \pi$ 1

(3)(A)(i) $3\sin^{-1}(\frac{2}{2})$

$= 3 \times \frac{\pi}{2}$ 1

$= \frac{3\pi}{2}$

(ii)



1

(iii) $d: -2 \leq x \leq 2$ 1

$r: -\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}$ 1

(iv) $y = 3\sin^{-1}(\frac{x}{2})$

$\therefore y' = 3 \times \frac{1}{\sqrt{2^2 - x^2}}$ 1

$= \frac{3}{\sqrt{4 - x^2}}$

$\therefore$ at $x = 0$, $m = \frac{3}{\sqrt{4}}$

1 $= 3/2$ 1

(7)(A)(i) $x = 2at, y = at^2$

$t = \frac{x}{2a} = \sqrt{\frac{y}{a}}$

$\frac{x^2}{4a^2} = \frac{y}{a}$

$y = \frac{x^2}{4a}$ 1

$\therefore y' = \frac{2x}{4a} = \frac{x}{2a} = \frac{2at}{2a} = t$

$\therefore y - at^2 = t(x - 2at)$

$y - at^2 = tx - 2at^2$

$y - tx + at^2 = 0$ 1

(ii) Use $y - px + ap^2 = 0$

$y - qx + aq^2 = 0$

subtract: $0 - px + qx + ap^2 - aq^2 = 0$

$x(q - p) + a(p^2 - q^2) = 0$

$x = \frac{-a(p^2 - q^2)}{-(p - q)}$

$= a(p + q)$ 1

$\therefore y - pa(p+q) + ap^2 = 0$

$y - apq - ap^2 + ap^2 = 0$

$y = apq$

$\therefore$ T is $(a(p+q), apq)$

iii) $x + y + 5a = 0$

$a(p+q) + apq + 5a = 0$

$p + q + pq + 5 = 0$

$pq = -(p+q) - 5$

.... use later !?#—

$P(2ap, ap^2)$, $Q(2aq, aq^2)$

Midpoint is

$\left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$x = a(p+q)$

$y = \frac{a(p^2+q^2)}{2}$

$= \frac{1}{2}a[(p+q)^2 - 2pq]$

$= \frac{1}{2}a\left[\left(\frac{x}{a}\right)^2 - 2pq\right]$

$2y = \frac{x^2}{a} - 2apq$

$2y = \frac{x^2}{a} - 2a[-(p+q) - 5]$

$2y = \frac{x^2}{a} + 2a(p+q) + 10a$

$= \frac{x^2}{a} + 2a\frac{x}{a} + 10a$

$y = \frac{x^2}{2a} + x + 5$

B) $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$= 4\pi r^2 \times 3$

$= 4\pi(5)^2 \times 3$

$= 300\pi$ cm³/min

c) $v = \frac{10}{\sqrt{1-t^2}} + \frac{1}{(1-t)^2}$

(i) $\therefore x = 10\sin^{-1}t + (1-t)^{-1} + c$

$t=0$, $x = 0 + 1 + c$ m

$t=\frac{1}{2}$, $x = 10 \times \frac{\pi}{6} + 2 + c$

$\therefore$ distance $= \frac{10\pi}{6} + 2 + c - (1+c)$

$= \frac{5\pi}{3} + 1$ m

(ii) $v = 10(1-t^2)^{-\frac{1}{2}} + (1-t)^{-2}$

$\therefore a = -5(1-t^2)^{-\frac{3}{2}} \times -2t - 2(1-t)^{-3} \times -1$

$= \frac{10t}{\sqrt{(1-t^2)^3}} - \frac{4t}{(1-t)^3} = 0$ ~~for a max.~~

and solve or ——

$v' = 10t(1-t^2)^{-\frac{3}{2}} + 2(1-t)^{-3}$

$\therefore v'' = (1-t^2)^{-3/2} \times 10 + 10t \times -\frac{3}{2}(1-t^2)^{-\frac{5}{2}} \times -2t - 6(1-t)^{-4} \times -1$

$= \frac{10}{(1-t^2)^{3/2}} + \frac{30t^2}{(1-t^2)^{5/2}} + \frac{6}{(1-t)^4}$

Now since $0 \le t \le \frac{1}{2}$, all denominators have pos. values $\therefore v'' > 0$ $\therefore$ only has minimum between 0 and $\frac{1}{2}$. So,

at $t=0$, $v = 10 + 1 = 11$ m/s

at $t=\frac{1}{2}$, $v = \frac{10}{\sqrt{1-\frac{1}{4}}} + \frac{1}{\frac{1}{4}}$

$= \frac{10}{\sqrt{3}/2} + 4$

$= \frac{20}{\sqrt{3}} + 4$

$\doteqdot 15.55$ (2dp)

$\therefore$ Max vel $\doteqdot 15.55$ m s⁻¹

4 (A) $a^2 + b^2 + c^2$

$= (a+b+c)^2 - 2(ab+bc+ac)$

$= (0)^2 - 2\left(\frac{3}{1}\right)$

$= 6$

(B) $x = 2t^3 - 9t^2 + 12t$

$v = 6t^2 - 18t + 12 = 0$

$t^2 - 3t + 2 = 0$

$(t-2)(t-1) = 0$

$\therefore t = 1, 2$

$\therefore$ first comes to rest when $t = 1$ sec

$\therefore x = 2 - 9 + 12 = 5$ m

(C) $\left(2x - \frac{1}{2x^2}\right)^9$

$T_{k+1} = {}^nC_k\, a^{n-k} \times b^k$

$= {}^9C_k (2x)^{9-k}\left(-\frac{1}{2x^2}\right)^k$

$= {}^9C_k\, 2^{9-k} . x^{9-k} . (-1)^k . 2^{-k} . x^{-2k}$

$= {}^9C_k\, 2^{9-k} . 2^{-k} (-1)^k . x^{9-k} . x^{-2k}$

$= {}^9C_k\, 2^{9-2k} . (-1)^k . x^{9-3k}$

x^{-3}

$x^{9-3k} = x^{-3}$

$-3k = -12$

$k = 4$

$\therefore {}^9C_4\, 2^{9-8} . (-1)^4$

$= 126 \times 2 \times 1$

$= 252$

(D) $y = \sin^{-1}x$

$\therefore x = \sin y$

$\frac{dx}{dy} = \cos y$

$\frac{dy}{dx} = \frac{1}{\cos y}$

$= \frac{1}{\sqrt{1-\sin^2 y}}$

$= \frac{1}{\sqrt{1-x^2}}$

5 (A)

For $n = 1$.

$\frac{1}{1\times4} = \frac{1}{3\times1+1}$

$\frac{1}{4} = \frac{1}{4}$ $\therefore$ True

Assume $n = k$.

$\frac{1}{1\times4} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}$

Prove $n = k+1$.

ie $S_k + T_{k+1} = S_{k+1}$

LHS $= \frac{k}{3k+1} + \frac{1}{(3(k+1)-2)(3(k+1)+1)}$

$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$

$= \frac{k(3k+4)+1}{(3k+1)(3k+4)}$

$= \frac{3k^2+4k+1}{(3k+1)(3k+4)}$

$= \frac{(3k+1)(k+1)}{(3k+4)(3k+1)}$

$= \frac{k+1}{3k+4}$ = RHS

$\therefore$ holds for $n=1$ then for $n=2$ etc $\therefore$ holds for all integers $n \ge 1$.

(E) $\lim_{x\to0} \frac{\sin 3x}{\tan\frac{1}{2}x}$

$= \lim_{x\to0} \frac{\sin 3x}{3x} \times \frac{\frac{1}{2}x}{\tan\frac{1}{2}x} \times \frac{3}{\frac{1}{2}}$

$= 1 \times 1 \times 6 = 6$

(B)(i) ${}^nC_k = \frac{n!}{(n-k)!\,k!}$ 1

$\therefore {}^nC_k - {}^nC_{n-k}$

$= \frac{n!}{(n-k)!\,k!} - \frac{n!}{(n-(n-k))!\,(n-k)!}$ 1

$= \frac{n!}{(n-k)!\,k!} - \frac{n!}{k!\,(n-k)!}$

$= 0$ 1

(ii) Coefficient of x^6 in $(1+x)^{12}$ is ${}^{12}C_6$.

Coefficient of x^6 in $(1+x)^6(1+x)^6$ is 1

${}^6C_0\,{}^6C_6 + {}^6C_1\,{}^6C_5 + {}^6C_2\,{}^6C_4 + {}^6C_3\,{}^6C_3 + {}^6C_4\,{}^6C_2 + {}^6C_5\,{}^6C_1 + {}^6C_6\,{}^6C_0$

$= ({}^6C_0)^2 + ({}^6C_1)^2 + ({}^6C_2)^2 + ({}^6C_3)^2$ 1

$+ ({}^6C_4)^2 + ({}^6C_5)^2 + ({}^6C_6)^2$

— since ${}^6C_6 = {}^6C_0$, etc!

$= \sum_{k=0}^{6} ({}^6C_k)^2 = {}^{12}C_6$ 1

(C) $y = \operatorname{cosec} 3x$

$= \frac{1}{\sin 3x} = (\sin 3x)^{-1}$

$\therefore y' = -(\sin 3x)^{-2} \times \cos 3x \times 3$

$= -3\frac{\cos 3x}{(\sin 3x)^2}$ 1

So, at $x = \frac{\pi}{4}$, $= \frac{-3\cos 3\pi/4}{(\sin \frac{3\pi}{4})^2}$

$= \frac{-3\times -1/\sqrt{2}}{(\frac{1}{\sqrt{2}})^2}$

$= \frac{3}{\sqrt{2}} \times 2 = \frac{6}{\sqrt{2}}$ 1

$\perp$: Use $-\sqrt{2}/6$

$\therefore y - \sqrt{2} = -\frac{\sqrt{2}}{6}(x - \frac{\pi}{4})$

$6y - 6\sqrt{2} = -\sqrt{2}x + \frac{\sqrt{2}\pi}{4}$

$\sqrt{2}x - 6y + 6\sqrt{2} - \frac{\sqrt{2}\pi}{4} = 0$

or $4\sqrt{2}x + 24y - 24\sqrt{2} - \sqrt{2}\pi = 0$ 1

(D)(A) $y = \frac{x+2}{(x+1)(x-3)} = \frac{x+2}{x^2-2x-3}$

(i) At $x=0$, $y = \frac{2}{1\times -3} = -\frac{2}{3}$ 1

At $y=0$, $0 = x+2$, $x = -2$ 1

(ii) vertical Asymptotes at $x = -1, 3$. 1

(iii) $y' = \frac{(x+1)(x-3)\times 1 - (x+2)(2x-2)}{(x+1)^2(x-3)^2} = 0$ for st. pt's 1

$x^2 - 2x - 3 - 2x^2 - 2x + 4 = 0$

$-x^2 - 4x + 1 = 0$

$x^2 + 4x - 1 = 0$ 1

$x = \frac{-4 \pm \sqrt{16+4}}{2}$

$= -2 \pm \frac{1}{2}\sqrt{20}$

$= -2 \pm \sqrt{5}$

$\doteq 0.236, -4.236$ 1

test:

x	-5	-4.2	0
y'	$-$	0	$+$

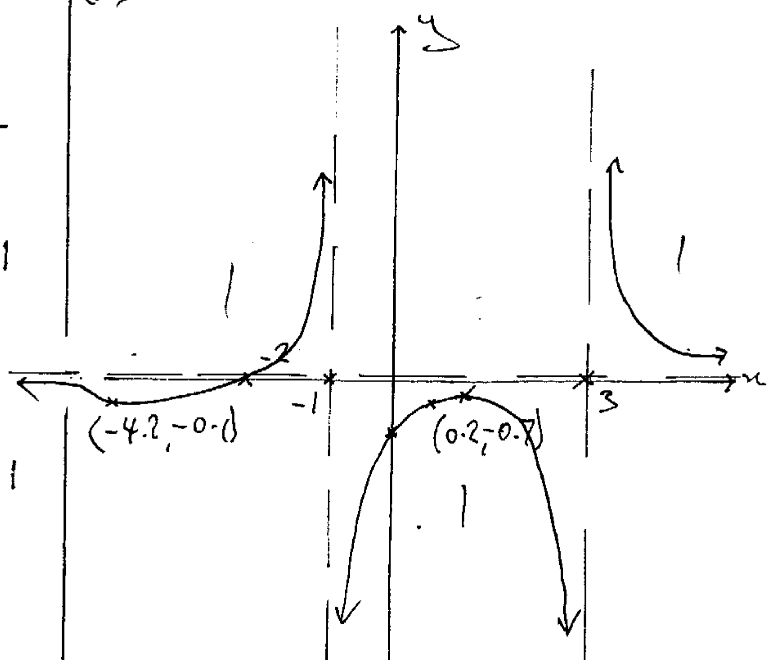
x	0	0.2	1
y'	$+$	0	$-$

1

$\therefore$ Min at $(-4.236, -0.095)$

Max at $(0.236, -0.655)$ 1

(iv)



Horizontal asymptotes:

$\lim_{x\to\infty} \frac{x+2}{x^2-2x-3}$

$= \lim_{x\to\infty\,(-\infty)} \frac{1/x + 2/x^2}{1 - 2/x - 3/x^2} = 0^+ (0^-$