EXTENSION 1 2003 CSSA

Q1 (a) $\lim_{n\to\infty} \frac{5(10^n)+3}{2(10^n)+1} = \lim_{n\to\infty} \frac{5+3(10^{-n})}{2+1(10^{-n})}$

$= \frac{5+0}{2+0}$

$= \frac{5}{2}$

(b) (x_1,y_1) (x_2,y_2)

$A(-2,5)$ $B(7,-1)$ $m:n = 2:1$

$x = \frac{mx_2+nx_1}{m+n}$ $y = \frac{my_2+ny_1}{m+n}$

$x = \frac{2\times 7+1\times -2}{2+1}$ $y = \frac{2\times -1+1\times 5}{2+1}$

$x = \frac{12}{3}$ $y = \frac{3}{3}$

$\therefore (x,y) = (4,1)$

(c) $\frac{2}{x} > x-1$

$\frac{2}{x}\times x^2 > (x-1)x^2$

$2x > x^3 - x^2$

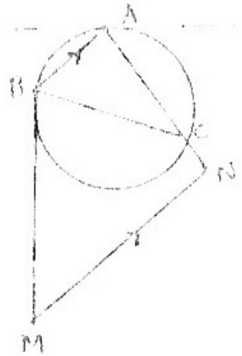
$x^3 - x^2 - 2x < 0$

$x(x^2-x-2) < 0$

$x(x-2)(x+1) < 0$

$x < -1$ $0 < x < 2$

(d)



(ii) $\angle MBC = \angle BAC$

Angle in Alt. Segment Th.

(iii) $\angle MBC = \angle BAC$ (ii)

$\angle BAC = 180° - \angle ANB$

(Coint. $\angle$s Supp. $BA \parallel MN$)

$\therefore \angle MBC = 180° - \angle ANB$

$\therefore$ MNCB is cyclic quad (opp. angles supp.)

Q2 (a) $y = (x^2+1)^5$

$y' = 5(x^2+1)^4 \cdot 2x$

$y' = 10x(x^2+1)^4$

$y'' = 40x(x^2+1)^3 \cdot 2x + 10(x^2+1)^4$

$= (x^2+1)^3 \cdot 80x^2 + (10x^2+10)(x^2+1)^3$

$y'' = (x^2+1)^3(90x^2+10)$

$y'' = 10(x^2+1)^3(9x^2+1)$

(b) $\sum_{n=2}^{5} {}^nC_2 = {}^2C_2 + {}^3C_2 + {}^4C_2 + {}^5C_2$

$= 1 + 3 + 6 + 10$

$= 20$

(c) (i) $(\sin A - \cos A)^2 = 1 - \sin 2A$

$RHS = 1 - \sin(A+A)$

$= 1 - (\sin A\cos A + \sin A\cos A)$

$= 1 - 2\sin A\cos A$

$LHS = \sin^2 A + \cos^2 A - 2\sin A\cos A$

$= 1 - 2\sin A\cos A$

$= LHS$

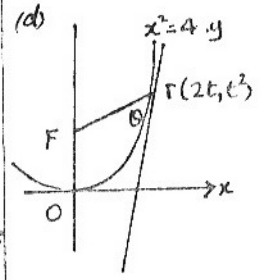
(c) (ii) $\sin 15° - \cos 15° \Rightarrow$

$(\sin 15° - \cos 15°) = -\sqrt{1 - \sin 30°}$ $(\sin 15° < \cos 15°)$

$= -\sqrt{1 - \frac{1}{2}}$

$= -\sqrt{\frac{1}{2}}$ or $-\frac{\sqrt{2}}{2}$

(d)



(i) $x^2 = 4y$

$y = \frac{x^2}{4}$

$m_T = y' = \frac{2x}{4}$

$= \frac{x}{2}$

$T(2t, t^2)$ $m_T = \frac{2t}{2}$

$m_T = t$

(ii) $F(0,1)$ $T(2t,t^2)$

$m_{FT} = \frac{t^2-1}{2t}$ $m_{TAN} = t$ (shown)

$\tan\theta = \left|\frac{m_T - m_{FT}}{1 + m_T \cdot m_{FT}}\right|$

$= \left|\frac{t - \frac{t^2-1}{2t}}{1 + t\left(\frac{t^2-1}{2t}\right)}\right|$

$= \left|\frac{\frac{2t^2-t^2+1}{2t}}{1+\frac{1}{2}t^2-\frac{1}{2}}\right|$

$= \left|\frac{2t^2-t^2+1}{2t+t^3-t}\right|$

$= \left|\frac{t^2+1}{t^3+t}\right|$

$= \left|\frac{t^2+1}{t(t^2+1)}\right|$

$= \frac{1}{|t|}$

(also accept $\frac{1}{t}$)

Q3.(a) $f(x) = x - e^{-x}$

(i) $f'(x) = 1 + e^{-x} > 0 \ \forall x$

$\therefore f^n$ increasing

$f''(x) = -e^{-x} < 0 \ \forall x$

$\therefore$ concave down

(ii) $x = 0.5$ $\quad f(0.5) = 0.5 - e^{-0.5}$

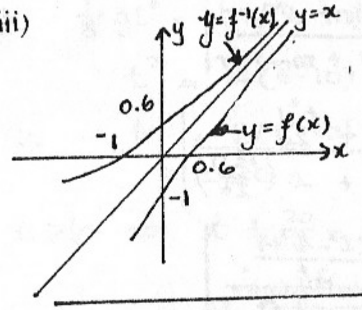
$f'(0.5) = 1 + e^{-0.5}$

$\Rightarrow \quad x \doteqdot 0.5 - \dfrac{0.5 - e^{-0.5}}{1 + e^{-0.5}}$

$x \doteqdot 0.56631\ldots$

$\underline{x \doteqdot 0.6 \text{ (1 DP)}}$

(iii)



✓ intercepts

✓ asymptotes

✓ inverse

(b) $v = \cos^2 x \quad t_0 = 0$

(i) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ $\qquad \frac{dx}{dt} = v$

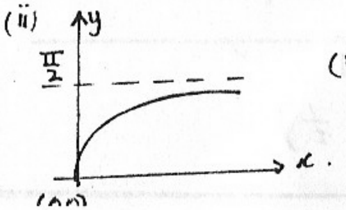
$a = \frac{d}{dx}\left(\frac{1}{2}\cos^4 x\right)$ $\qquad \Rightarrow \frac{dx}{dt} = \cos^2 x$

$\underline{a = -2\cos^3 x \sin x}$ $\qquad \frac{dt}{dx} = \sec^2 x$

$t = \tan x$

$\underline{x = \tan^{-1} t}$

(ii)



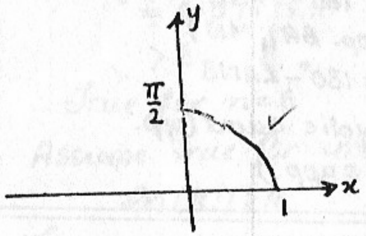
(iii) Particle moves to right, slowing down. It approaches its limiting pos^n

Q4.(a) $f(x) = \cos^{-1}\sqrt{x}$

(i) $D: \ -1 \le \sqrt{x} \le 1$ $\qquad R: \ x = 0 \ \ f(0) = \frac{\pi}{2}$

$0 \le \sqrt{x} \le 1$ $\qquad x = 1 \ \ f(1) = 0$

$\underline{D: \ 0 \le x \le 1}$ $\qquad \therefore \ \underline{R: \ 0 \le y \le \frac{\pi}{2}}$



(ii)

x	0	$\frac{1}{2}$	1
y	$\frac{\pi}{2}$	$\frac{\pi}{4}$	0

$A \doteqdot \frac{1}{6}\left[y_0 + 4y_{\frac{1}{2}} + y_1\right]$

$= \frac{1}{6}\left(\frac{\pi}{2} + \pi + 0\right)$

$A = \frac{\pi}{4}$ units^2

$\underline{(A \doteqdot 0.785 \text{ (3DP)})}$

(iii) $A = \displaystyle\int_0^{\pi/2} \cos^2 y \, dy$ $\qquad \left(\begin{array}{l} y = \cos^{-1}\sqrt{x} \\ \sqrt{x} = \cos y \\ x = \cos^2 y \end{array}\right)$

$= \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{2}(\cos 2y + 1)\, dy$

$= \left.\frac{1}{2}y + \frac{1}{4}\sin 2y\right)_0^{\frac{\pi}{2}}$

$= \frac{\pi}{4} + \frac{1}{4}\sin\pi - 0 - \frac{1}{4}\sin 0$

$\underline{A = \frac{\pi}{4} \text{ units}^2}$

(b) $\ddot{x} = -4(x-2) \quad t = 0 \ \ x = 0$

(i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\therefore v^2 = 2\int \ddot{x}\, dx$

$= -8\int x - 2\, dx$

$= -4x^2 + 16x + c$

$x = 0, v = 0 \Rightarrow c = 0$

$\therefore \ \underline{v^2 = -4x^2 + 16x}$

(ii) $\ddot{x} = -(2)^2(x-2) \ \therefore \ n = 2$ $\quad$ $\underline{\text{Period } \frac{2\pi}{2} = \pi \text{ sec}}$

$v = 0 \quad -4x^2 + 16x = 0$

$-4x(x - 4) = 0$

$x = 0, 4$

Centre $x = 2$ $\quad \therefore$ $\underline{\text{Amplitude } 2}$

(iii) $t = \pi \quad x = 8 \quad \therefore \ 1\text{sec} \ \ \frac{8}{\pi}$ metres

$1\text{min} \ \ \frac{8}{\pi} \times 60 = \underline{153\text{m}}$

Q5. (a) $(1 + ax)^9 =$

$1 + {}^9C_1 + {}^9C_2(ax)^2 + \ldots + {}^9C_5(ax)^5 + {}^9C_6(ax)^6 + \ldots + (ax)^9$

${}^9C_5\, a^5 = 2\, {}^9C_6\, a^6$

$126a^5 = 168a^6$

$a = \frac{126}{168}$

$\underline{a = \frac{3}{4}}$

(b) $\displaystyle\int_1^{49} \frac{1}{\sqrt{x}\sqrt{1 + \sqrt{x}}}\, dx = I$

$u = 1 + \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$2du = \frac{dx}{\sqrt{x}}$

$\left.\begin{array}{ll} x = 49 & u = 8 \\ x = 1 & u = 2 \end{array}\right\}$

$I = \displaystyle\int_2^8 \frac{2\, du}{\sqrt{u}}$

$= 2\displaystyle\int_2^8 u^{-\frac{1}{2}}\, du$

$= \left.2 \cdot 2u^{\frac{1}{2}}\right)_2^8$

$= 4\sqrt{8} - 4\sqrt{2}$

$= 8\sqrt{2} - 4\sqrt{2} = 4\sqrt{2}$

$= \sqrt{32}$

Q5 Cont.

(c) $V = A(1 - e^{-kt}) \quad k > 0$

(i) $\frac{dV}{dt} = d/dt\,(A - Ae^{-kt})$

$= Ake^{-kt}$ ✓

$A - V = A - (A - Ae^{-kt})$ ✓

$= Ae^{-kt}$

$\therefore \frac{dV}{dt} = k(A - V)$

(ii) $t = 2 \quad V = \frac{A}{4}$

$\frac{A}{4} = A(1 - e^{-2k})$

$1 - e^{-2k} = \frac{1}{4}$

$-e^{-2k} = -\frac{3}{4}$

$e^{-2k} = \frac{3}{4}$ ① ✓

$t = 4 \quad V = A(1 - e^{-4t})$

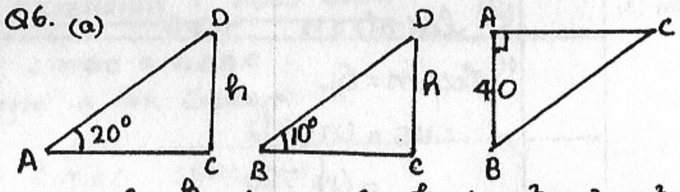
$= A(1 - (e^{-2k})^2)$

$= A(1 - (\frac{3}{4})^2)$ from ①

$= \frac{7A}{16}$

$\therefore \frac{7A}{16} - \frac{4A}{16} \Rightarrow \frac{3}{16}$ ✓ of container is filled in next 2 minutes.

Q6. (a)



$\tan 20° = \frac{h}{AC}$ | $\tan 10° = \frac{h}{BC}$ | $AC^2 + AB^2 = BC^2$ (Pyth.)

$AC = \frac{h}{\tan 20°}$ | $BC = \frac{h}{\tan 10°}$ | $\left(\frac{h^2}{\tan^2 20}\right) + 40^2 = \frac{h^2}{\tan^2 10}$ ✓

$\Rightarrow \frac{h^2 + 1600\tan^2 20}{\tan^2 20} = \frac{h^2}{\tan^2 10}$

$h^2 \tan^2 20 = h^2 \tan^2 10 + 1600\tan^2 20 \tan^2 10$

$h^2 = \frac{1600\tan^2 20 \tan^2 10}{\tan^2 20 - \tan^2 10}$ ✓

$h = 8.06\ldots$

$h \doteqdot 8\text{m}$ ✓

(b) $P(O) = p \quad 0 < p < 1 \quad p \neq 0.5$

(i) Six throws:

$P(\text{At Most } 1E) = P(5O \text{ or } 6O)$ ✓

$= {}^6C_5 p^5(1-p) + {}^6C_6 p^6$

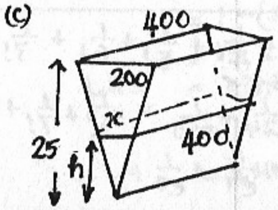
$= 6p^5(1-p) + p^6$

$= 6p^5 - 5p^6$ ✓

(ii) $P(\text{Product Even}) = 1 - P(\text{Product Odd})$ ✓

$= 1 - p^6$ ✓

(c)



$V = 16$ L/sec

(Note: all measurements changed to cm)

(i) By similar △s:

$\frac{x}{200} = \frac{h}{25}$

$25x = 200h$

$x = 8h$ ✓

$V = \frac{1}{2} \times x \times h \times 400$

$= \frac{1}{2} \times 8h^2 \times 400$ ✓

$V = 1600h^2$

(ii) $\frac{dh}{dt} = ?$ when $h = 10$

$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$ ① ✓

$\frac{dV}{dt} = -16$ L/sec

$= 16000\text{cm}^3/\text{sec}$

$\frac{dV}{dh} = 3200h \Rightarrow \frac{dh}{dV} = \frac{1}{3200h}$

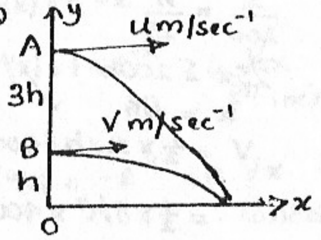
$h = 10 \quad \frac{dV}{dh} = \frac{1}{32000}$

$\therefore$ ① $\frac{dh}{dt} = \frac{-16000}{32000}$

$= \frac{-1}{2}$ ✓

$\therefore$ Water level falling 0.5cm/sec^{-1}

Q7. (a)



(i) $x(A) = Ut$ $\quad x(B) = V(t-10)$

$y(A) = 4h - \frac{1}{2}gt^2$ $\quad y(B) = h - \frac{1}{2}g(t-10)^2$

(ii) When particle hits ground:

$y(A) = y(B) = 0$

$\therefore\ 4h = \frac{1}{2}gt^2 \qquad h = \frac{1}{2}g(t-10)^2$

$h = \frac{1}{8}gt^2$

$\Rightarrow\ \frac{1}{8}gt^2 = \frac{1}{2}g(t-10)^2$

$t^2 = 4(t-10)^2$

$\therefore\ t = 2(t-10) \quad \text{or} \quad -t = 2(t-10)$

$t = 20 \qquad\qquad t = 20/3$

$t > 10 \Rightarrow t \neq \frac{20}{3} \quad t = 20$

$\therefore$ particle A takes 20 sec

& particle B takes 10 sec

(iii) On impact $x(A) = x(B)$

$\therefore\ 20U = V(20-10)$

$20U = 10V$

$V = 2U$

(b) $\ln(n!) > n \quad n \geq 6$

(i) Let $n = 6$

LHS $= \ln(6!)$

$= \ln 720$

$\doteqdot 6.57925\ldots$

> 6

$\therefore$ True for $n = 6$

Assume true for $n = k$, $k \geq 6$

$\ln(k!) > k$

To prove true for $n = k+1$:

$\ln((k+1)!) = \ln[k!(k+1)]$

$= \ln(k+1) + \ln k!$

$> \ln(k+1) + k$

$\ln e = 1 \quad$ for $k \geq 6$ $\quad k+1 > e$

$\ln(k+1) > \ln e$

$\therefore\ \ln(k+1) > 1$

$\therefore\ \ln[(k+1)!] \geq 1+k$

Since shown true for $n=6$ & $n=k+1$, whe assuma true for $n=k$ then it must hold true for all k.

(ii) $\ln(n!) > n \qquad n \geq 6$

$\therefore\ n! > e^n$

$\therefore\ \frac{1}{n!} < \frac{1}{e^n} \qquad n \geq 6$

(iii) $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \ldots$

$< 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{6!} + \frac{1}{7!} + \ldots$

$< 1\frac{43}{60} + \frac{1}{e^6} + \frac{1}{e^7} + \frac{1}{e^8} + \ldots$

$< \frac{103}{60} + \frac{1}{e^6}\underbrace{\left(1 + \frac{1}{e} + \frac{1}{e^2} + \ldots\right)}_{GP\ a=1\ r=\frac{1}{e}}$

$< \frac{103}{60} + \frac{1}{e^6}\left(\frac{1}{1-\frac{1}{e}}\right)$

$< \frac{103}{60} + \frac{\frac{1}{e^6}}{1-\frac{1}{e}}$

$< \frac{103}{60} + \frac{1}{e^6 - e^5}$

$< \frac{103}{60} + \frac{1}{e^5(e-1)}$