CRANBROOK SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2001

MATHEMATICS

3 UNIT (Additional)
4 UNIT (First Paper)

Time allowed – Two hours

DIRECTIONS TO CANDIDATES

* Attempt all questions.
* ALL questions are of equal value.
* All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
* Standard integrals are printed on the back page.
* Board-approved calculators may be used.
* You may ask for extra Writing Booklets if you need them.

* Submit your work in four booklets :

(i) QUESTION 1 (4 page)

(ii) QUESTIONS 2 & 3 (8 page)

(iii) QUESTIONS 4 & 5 (8 page)

(iv) QUESTIONS 6 & 7 (8 page)

1. (4 page booklet)

(a) Evaluate $\int_0^{\pi/2} \cos^2 x\, dx$

[2 marks]

(b) (i) On the same set of axes, sketch the graphs of $y = 2|x|$ and $y = |x+3|$

(ii) Hence or otherwise solve for x $2|x| \le |x+3|$ **[4 marks]**

(c) In an Arithmetic Sequence, whose first term and common difference are both non-zero, T_n represents the n^{th} term and S_n represents the sum of the first n terms. Given that T_6, T_4, T_{10} form a Geometric Sequence

(i) show that $S_{10} = 0$

(ii) show that $S_6 + S_{12} = 0$

(iii) deduce that $T_7 + T_8 + T_9 + T_{10} = T_{11} + T_{12}$ **[6 marks]**

2. (new 8 page booklet please)

(a) Evaluate

(i) $sin^{-1}\left(\frac{1}{2}\right)$ (ii) $sin^{-1}\left(cos\frac{\pi}{3}\right)$ **[2 marks]**

(b) State the Domain and Range of $y = sin^{-1}(1-x^2)$ **[2 marks]**

(c) Sketch the graphs of (i) $y = sin^{-1}x + cos^{-1}x$

(ii) $y = sin^{-1}(1-x)$ **[4 marks]**

(d) Find the exact volume of the solid of rotation when the area bounded by the curve $y = \frac{1}{\sqrt{1+4x^2}}$ and the x-axis from $x = -\frac{1}{2}$ to $x = \frac{1}{2}$ is rotated about the x-axis.

[4 marks]

3.

(a) (i) Show that $(x-2)$ is a factor of $4x^3 - 8x^2 - 3x + 6$.

(ii) Find the general solution of $4\sin^3\theta - 8\sin^2\theta - 3\sin\theta + 6 = 0$. **[4 marks]**

(b) Given $\sin\theta = \frac{4}{5}$ and $\frac{\pi}{2} \le \theta \le \pi$ find $\sin 2\theta$. **[2 marks]**

(c) Show that $\frac{\sin 3\phi}{\sin\phi} - \frac{\cos 3\phi}{\cos\phi} = 2$. **[2 marks]**

4. (new 8 page booklet please)

(a) Find the locus of M(x, y) in cartesian form given :

$$x = p + q$$
$$y = \frac{1}{2}\left(p^2 + q^2 + 4\right)$$
and
$$pq = 2$$

[2 marks]

(b) A is the fixed point (- 4 , 8). P is a variable point on the parabola $x^2 = 8y$. Prove that the locus of M, the midpoint of AP, is a parabola with vertex $(-2,\ 4)$ and focal length 1 unit.

[5 marks]

(c) (i) Explain why $e^x - 2x - 1 = 0$ must have a root between 1.2 and 1.3

(ii) By using Newton's method (twice), and taking 1.3 as a first approximation, find a better approximation to the root, giving your answer correct to three decimal places.

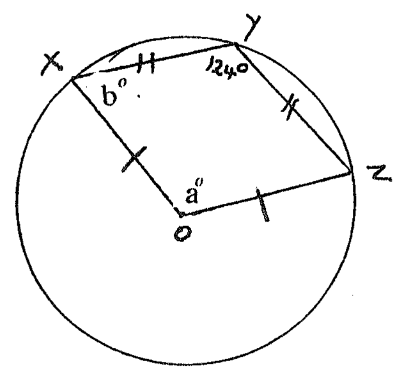
[5 marks]

5.

(a) In the diagram shown, $XY = YZ$ and O is the centre of the circle.

$\angle XYZ = 124^o$

Evaluate a and b, giving reasons for your answers.



[3 marks]

(b) Points A, B, C and D lie on a circle such that chords BC and … are equal and AD is a diameter of the circle (B and C are in the same half of the circle). BY… vn parallel to CD, meeting AD in X.

(i) Draw a neat and clear diagram representing the situation.

(ii) Let $\angle CDB = x^o$. Prove that ABX is an isosceles triangle.

[5 marks]

(c) Two of the roots of the equation $x^3 + ax^2 + b = 0$ are reciprocals of each other.

(i) Show that the third root is equal to $-b$.

6. (new 8 page booklet please)

(a) The daily growth rate of a population of a species of mosquito is proportional to the excess of the population over 5000

$$\text{i.e. } \frac{dP}{dt} = k(P - 5000).$$

(i) Show that $P = 5000 + Ae^{kt}$ is a solution of this differential equation. **[2 marks]**

(ii) If initially $P = 5002$ and after 6 days the population is 25000 find the values of A and k in exact form. **[2 marks]**

(iii) Find the mosquito population after 10 days (to the nearest whole number). **[2 marks]**

(b) On a certain day in July, 2001 the depth of water at high tide over a harbour bar in Auckland was $10\frac{2}{3}$ m and at low tide $6\frac{1}{4}$ hours earlier it was 7m. Hide tide occurred at 3.40 p.m. on this day.

(i) Assuming that the tide's motion is simple harmonic and of the form $\ddot{x} = -n^2(x-b)$, where $x = b$ is the centre of motion and $x = a$ is the amplitude, show that $x = b - a\cos nt$ satisfies this equation for simple harmonic motion. **[2 marks]**

(ii) Hence or otherwise find the earliest time before 3.40 p.m. on this day at which a ship requiring a $9\frac{1}{2}$ m depth of water could have crossed the bar (to the nearest minute). **[4 marks]**

7.

(a) Prove by mathematical induction that $3^n + 7^n$ is always even for n a positive integer. **[5 marks]**

(b) An executive borrows $\$P$ at $r\%$ per fortnight reducible interest and pays it off at $\$F$ per fortnight in n equal fortnightly instalments. (Assume that there are 26 fortnights in one year.)

(i) If D_n is the debt remaining after n fortnights prove that

$$D_n = P\left(1+\frac{r}{100}\right)^n - F\times\left[\frac{\left(1+\frac{r}{100}\right)^n - 1}{\frac{r}{100}}\right]$$ **[3 marks]**

(ii) If $D_n = 0$ prove that $n = \dfrac{\log_e\left[\dfrac{F}{F-\frac{rP}{100}}\right]}{\log_e\left(1+\frac{r}{100}\right)}$ **[2 marks]**

(iii) If the executive owed $47 000 at the beginning of July 2001 with interest payable at 7.8%

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1} \qquad (n \neq -1; x \neq 0 \; if \; n < 0)$$

$$\int \frac{1}{x} dx = \log_e x \qquad (x > 0)$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} \qquad (a \neq 0)$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax \qquad (a \neq 0)$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax \qquad (a \neq 0)$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax \qquad (a \neq 0)$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax \qquad (a \neq 0)$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \qquad (a \neq 0)$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} \qquad (a > 0, -a < x < a)$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log_e \left\{ x + \sqrt{x^2 - a^2} \right\} \qquad (|x| > |a|)$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \log_e \left\{ x + \sqrt{x^2 + a^2} \right\}$$