SYDNEY BOYS HIGH SCHOOL
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2001

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Sample Solutions

Q1 a) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

gradients $m_1 = -\frac{3}{2}$, $m_2 = \frac{4}{3}$

2

$\tan\theta = \frac{-\frac{3}{2} - \frac{4}{3}}{1 + -\frac{3}{2} \times \frac{4}{3}}$

$= \frac{17}{6}$

$\theta = 70^\circ 34'$

(b) $\tan(30-45) = \frac{\tan 30 - \tan 45}{1 + \tan 30 \tan 45}$

2

$= \frac{\frac{\sqrt{3}}{3} - 1}{1 + \frac{\sqrt{3}}{3} \times 1} = \frac{\sqrt{3} - 3}{3 + \sqrt{3}}$

$= \frac{6\sqrt{3} - 12}{6} = \sqrt{3} - 2$

(c) $\lim_{x \to 0} \frac{\sin 4x}{x} + \lim_{x \to 0} \frac{\tan x}{x}$

$4 \lim_{x \to 0} \frac{\sin 4x}{4x} + \lim_{x \to 0} \frac{\tan x}{x}$

2

$= 5.$

(d) (i) $y = \ln(\cos x)$

$\frac{dy}{dx} = \frac{1}{\cos x} \times \frac{-\sin x}{1}$

1

$= -\tan x$

(ii) $y = \tan^{-1} 3x$

1

$\frac{dy}{dx} = \frac{3}{1 + 9x^2}$

(e) $2(1 - \sin^2 x) + 3\sin x - 3 = 0$

$2 - 2\sin^2 x + 3\sin x - 3 = 0$

$2\sin^2 x - 3\sin x + 1 = 0$

2

$(2\sin x - 1)(\sin x - 1) = 0$

$\sin x = \frac{1}{2}$ $\sin x = 1$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ or $\frac{\pi}{2}$

(f) $k = 4$, $l = -3$

$(5, 12)$ 2

Question 2

(a) $\tan x = \sqrt{3}$

$\therefore x = 180n + \tan^{-1}(\sqrt{3})$

$\boxed{x = 180n + 60^\circ}$ or $\boxed{x = n\pi + \frac{\pi}{3}}$

b) $\dfrac{9!}{2!\,3!}$ = $\begin{cases} \text{repetition of S} & (\times 3) \\ \text{repetition of E} & (\times 2) \end{cases}$

$\boxed{30240}$

c) domain: $-1 \leq 1 - \sqrt{x} \leq 1$

$\therefore -2 \leq -\sqrt{x} \leq 0$

$\therefore 0 \leq \sqrt{x} \leq 2$

$\therefore \boxed{0 \leq x \leq 4}$ $(\frac{1}{2})$

range. $-\frac{\pi}{2} \leq \sin^{-1}u \leq \frac{\pi}{2}$

$\therefore \boxed{-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}}$ $(\frac{1}{2})$

d) $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{3-x^2}} = \sin^{-1}\left(\frac{x}{\sqrt{3}}\right)\Big]_0^{\sqrt{3}}$

$= \sin^{-1}(1) - \sin^{-1}(0)$

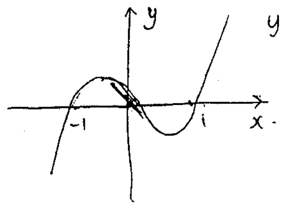
$= \frac{\pi}{2}$

(e) $\dfrac{x}{x^2-1} > 0$ $\boxed{x \neq \pm 1}$

$\therefore \dfrac{x}{(x+1)(x-1)} > 0$

$[\times \ (x+1)^2 (x-1)^2]$

$\therefore (x+1)(x-1)x > 0$



$\therefore \boxed{-1 < x < 0, \ x > 1}$

(f) (i) $\because AB = AC$

$\therefore \angle ABC = \angle ACB$ (base angles of isos. Δ)

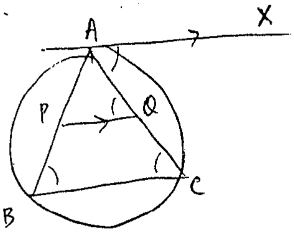
$X\hat{A}C = A\hat{B}C$ (alternate segment theorem)

$X\hat{A}C = A\hat{Q}P$ (alternate angles)

$\therefore A\hat{Q}P = A\hat{C}B$

$\therefore PQ \parallel BC$

(corresponding angles equal)



$\therefore A\hat{P}Q = A\hat{Q}P$

$\therefore AP = AQ$ (isos Δ)

(ii) $BC \parallel PQ$ & $PQ \parallel AX$

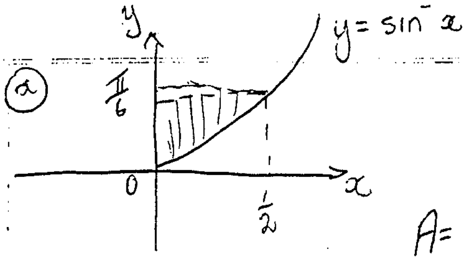
$\therefore BC \parallel AX$

(iii) $\angle A\hat{P}Q = A\hat{C}B$ (from (i))

$\therefore$ exterior angle equals opposite interior angle

$\therefore PQCB$ is cyclic quad.

③ ⓐ



$\sin^{-1}\frac{1}{2} = 30^\circ = \frac{\pi}{6}$

If $y = \sin^{-1} x$ then $\sin y = x$

$$A = -\int_0^{\frac{\pi}{6}} -\sin y \, dy$$

$$= -\cos y \Big]_0^{\frac{\pi}{6}}$$

$$= -\cos\frac{\pi}{6} + \cos 0$$

$$= -\frac{\sqrt{3}}{2} + 1$$

$$= 1 - \frac{\sqrt{3}}{2}$$

Now area rectangle is $\frac{1}{2} \times \frac{\pi}{6} = \frac{\pi}{12}$. (exact)

area required is $\frac{\pi}{12} - \left(1 - \frac{\sqrt{3}}{2}\right) = \frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}$ u^2

④

ⓑ In $\triangle BAQ$, $\tan 22^\circ = \frac{h}{AQ}$.

$h = AQ \tan 22^\circ$

$AQ = \frac{h}{\tan 22^\circ} = h\cot 22^\circ$

In $\triangle BAP$ $\tan 43^\circ = \frac{h}{AP}$

$h = AP \tan 43^\circ$

$AP = \frac{h}{\tan 43^\circ} = h\cot 43^\circ$

④

Now In $\triangle APQ$, $AP^2 + PQ^2 = AQ^2$

$AP^2 - AQ^2 = -\cancel{16}160000$

$h^2\cot^2 43^\circ - h^2\cot^2 22^\circ = -160000$

$$h^2 = \frac{-160000}{\cot^2 43^\circ - \cot^2 22^\circ} = \frac{160000}{\cot^2 22^\circ - \cot^2 43^\circ}$$

3 (c) $\int \sec^2 x \tan^2 x \, dx$

Let $u = \tan x$

$\frac{du}{dx} = \sec^2 x$.

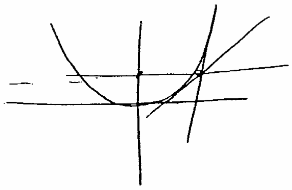
$du = \sec^2 x . dx$

$= \int u^2 . du$

$= \frac{u^3}{3} + c$.

$= \frac{\tan^3 x}{3} + C$

4



Q4 (a) (i) $y = px - ap^2$ —①

$y = qx - aq^2$ —②

$0 = (p-q)x - a(p^2-q^2)$

$x = \dfrac{a(p^2-q^2)}{p-q}$

$\boxed{x = a(p+q)}$ ↗ $y = pa(p+q) - ap^2$

$= ap^2 + apq - ap^2$

$\boxed{y = apq}$

$\therefore \boxed{A \text{ is } (a(p+q), apq)}$ 3

(ii) If $y = a$ then $apq = a$

$\boxed{pq = 1}$ 1

(iii) Chord $\boxed{y = \tfrac{1}{2}(p+q)x - apq}$

Show $\dfrac{y-ap^2}{x-2ap} = \dfrac{aq^2-ap^2}{2aq-2ap} = \dfrac{q+p}{2}$

$\therefore y - ap^2 = \left(\dfrac{p+q}{2}\right)x - \dfrac{2ap}{2}(p+q)$

$y - ap^2 = \dfrac{p+q}{2}x - ap^2 - apq$

$\boxed{y = \dfrac{p+q}{2}x - apq}$ 2

now if $x = 0$ $y = -apq$ 2

$\boxed{y = -a}$ because $pq = 1$

$-8 - 8 + 3$

(b) $y = x^3 - 2x^2 + 3$

$\dfrac{dy}{dx} = 3x^2 - 4x$ ✓✓

$m = 3\times 4 - 4\times 2$ ✓ 1

$\underline{m = 4}$

$\dfrac{y-3}{x-2} = 4$

$y - 3 = 4x - 8$

$\boxed{y = 4x - 5}$

2

Solve $x^3 - 2x^2 + 3 = 4x - 5$

$x^3 - 2x^2 - 4x - 8 = 0$ ✓✓

roots are $2, 2, \alpha$

$2 + 2 + \alpha = 2$

$\alpha = -2$

$\therefore (-2, -13)$ 2

5(a) Test for $n=1$, $S_1 = 3^3 + 2^3$

$= 27 + 8$

$= 35$ which is divisible by 5.

Assume true for $n = k$, ie $S_k = 3^{3k} + 2^{k+2}$

$= 5P$ where $P \in J$.

Now test for $n = k+1$, ie $S_{k+1} = 3^{3k+3} + 2^{k+3}$

$= 5Q$ where $Q \in J$.

$S_{k+1} = 27(5P - 2^{k+2}) + 2.2^{k+2}$

$= 5.27P - (27-2)2^{k+2}$

$= 5\{27P - 5.2^{k+2}\}$

$= 5Q$.

$\therefore$ True for $n = k+1$ if true for $n = k$.

Now true for $n = 1$ so true for $n = 2$ and so on for all integer n.

(b) $f(x) = x^3 - 7$, $f'(x) = 3x^2$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$= 2 - \dfrac{8-7}{3.4}$

$= 23/12$.

(c) $\dfrac{v^2}{2} = 45 - 6x - 3x^2$

$\ddot{x} = \dfrac{d}{dx}\left(\dfrac{v^2}{2}\right)$

$= -6 - 6x$

$= -6(x+1)$

$= -(\sqrt{6})^2 X$ where $X = x+1$

$\therefore$ Motion is SHM with centre of motion $= -1$.

$n = \sqrt{6}$ so period $= 2\pi/\sqrt{6} = \sqrt{6}\pi/3$

$v^2 = -6(x^2 + 2x + 1) + 90 + 6$

$= 96 - 6(x+1)^2$

$= 6\{4^2 - (x+1)^2\}$

Amplitude = 4

5 (d) (i) $RHS = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} \times \dfrac{\cos^2\theta}{\cos^2\theta}$

$= \dfrac{\sin^2\theta - \cos^2\theta}{\sin^2\theta + \cos^2\theta}$

$= \cos 2\theta$ ✓

$= LHS$

(ii) If $\theta = \pi/8$, $\cos 2\theta = \dfrac{1}{\sqrt{2}} = \dfrac{1-x^2}{1+x^2}$

$1 + x^2 = \sqrt{2} - \sqrt{2}\,x^2$

$x^2(1+\sqrt{2}) = \sqrt{2} - 1$

$x^2 = \dfrac{\sqrt{2}-1}{\sqrt{2}+1} \times \dfrac{\sqrt{2}-1}{\sqrt{2}-1}$

$= \dfrac{(\sqrt{2}-1)^2}{2-1}$ ✓

$x = \sqrt{2} - 1$ as $\tan \pi/8$ is in 1st quadrant.

Question 6

(a)

$y = \sin^{-1}(\cos x)$

(i)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\cos^2 x}} \cdot -\sin x$$

$$= \frac{-\sin x}{\sqrt{\sin^2 x}} \qquad 2$$

$$= -\frac{\sin x}{|\sin x|}$$

$= -1$ for $0 < x < \pi$

$= 1$ for $-\pi < x < 0$

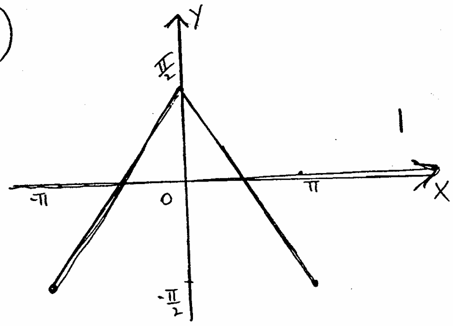
(ii) $y = \sin^{-1}[\cos \pi]$

$= \sin^{-1}[-1] \qquad 1$

$= -\sin^{-1}[1]$

$= -\frac{\pi}{2}$

(iii)



1

(b) $\dot{x} = V\cos\theta = 35\cos 0 = 35$

$\dot{y} = 35\sin 0 - 10t = -10t$

$y = 1.8 - 5t^2$

$x = Vt\cos\theta = 35t$

(i) Strikes ground when $y = 0$

$\therefore\ 0 = 1.8 - 5t^2$

$t = 3/5$ sec. $\quad 2$

(ii) $x = 35 \times \frac{3}{5} = 21$ m $\quad 2$

(iii) When $x = 14$, $\quad 14 = 35t$

ie $t = \frac{2}{5}$

When $t = \frac{2}{5}$, $y = 1.8 - 5\left(\frac{2}{5}\right)^2 \quad 2$

ie $y = 1$ m

$\therefore$ Clears net by $1 - 0.95\text{m} = 5\text{cm}$.

(c)

(i) $\frac{d}{dx}(xe^x) = xe^x + 1.e^x = e^x(x+1)$ ½

(ii) $\frac{d}{dx} xe^x - e^x = xe^x$

ie $\frac{d}{dx}[xe^x - e^x] = xe^x$ ½

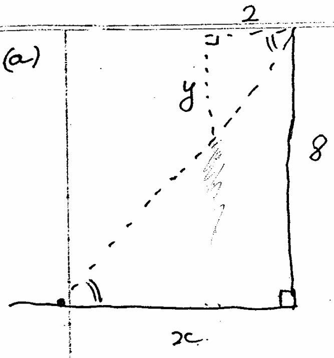
$\Rightarrow \int xe^x\,dx = xe^x - e^x$ ½ $\quad 1$

$\therefore \int_0^1 xe^x\,dx = \left[xe^x - e^x\right]_0^1$ ½

$= [(e - e) - (0 - e^0)]$

$= 1$

7 (a)



(i) Triangles are similar

$$\therefore \quad \frac{x}{2} = \frac{8}{y}$$

$$x = \frac{16}{y}$$ 2

(ii) $$\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt}$$

$$= -\frac{16}{y^2} \cdot 10$$ 2

When $y = 6$, $\frac{dx}{dt} = \frac{-16}{36} \cdot 10$

$$= -\frac{40}{9} \text{ m/s.}$$

(iii) $$\frac{dy}{dt} = \frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dx}{dt}$$

$$\frac{dy}{dx} = 1$$

$$-\frac{16}{y^2} = -1 \text{ (speed)}$$

$$y = 4 .$$ 2

(b) (i) $f(x) = (e^x - 1)\ln x$

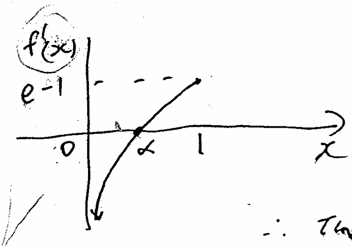
$$f'(x) = \frac{e^x - 1}{x} + e^x \ln x.$$ 2

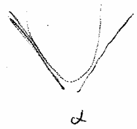
$$f'(1) = e - 1 + 0 = e - 1$$

(ii) As $x \to 0$, $f'(x) \to -\infty$ 1

since $\ln x \to -\infty$ as $x \to 0$

(iii)





1

$f'(x) = 0$ for $0 < x < 1$

Let this root be α.

For $x < \alpha$ $f'(x) < 0$

$x > \alpha$ $f'(x) > 0$. 2

$\therefore$ The stationary point is a local minimum.