St Catherine's School Trial Examination

Year: 12
Subject: Mathematics
Time Allowed: 3 hours
Date: August 2003

Exam number:________________

Directions to candidates:

- All questions are to be attempted.
- All questions are of equal value
- All necessary **working** must be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Each question attempted should be started on a **new page**.
- This page is a cover sheet for **Question 1**
- **Write your exam number on every section.**
- Hand in your work in 3 bundles:
  - Bundle 1 - Questions 1,2,3 ,4
  - Bundle 2 - Questions 5, 6, 7
  - Bundle 3 - Questions 8, 9, 10

* Standard integrals are attached

TEACHER'S USE ONLY Total Marks	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
TOTAL	

Question 1

(a) Simplify, giving your answer as a rational number: **2**

(i) $\sqrt{7}\times\sqrt{63}$,

(ii) $49^{-\frac{1}{2}}$.

(b) Solve $3^x = 0.0568$, giving your answer correct to 3 significant figures. **2**

(c) Express $\dfrac{x^{-1}-1}{x-1}$ in simplest form. **2**

(d) Solve the equation $\tan x = 1$ for $0 \le x \le 2\pi$. **2**

(e) Consider the parabola $8x-1 = y^2+6y$. Find: **4**

(i) the coordinates of the vertex,

(ii) the coordinates of the focus,

(iii) the equation of the directrix.

Question 2 Begin a new page.

(a) Differentiate with respect to x: **3**

(i) $y = \dfrac{e^{3x}}{x}$,

(ii) $f(x) = \sin 4x$.

(b) **6**

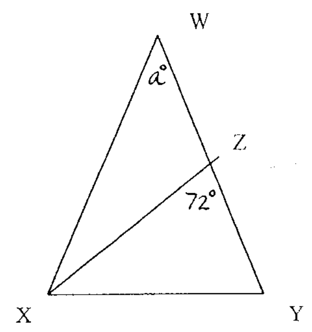


Diagram not to scale.

In the diagram above, WX=WY and ZX=XY.

(i) Copy the diagram onto your answer sheet.

(ii) Find the value of a.

(iii) Show that ZW=ZX.

(c) Solve $2^{2x} - 15 \times (2^x) - 16 = 0$. **3**

Question 3 Begin a new page.

(a) Sketch the following graphs, showing features: **4**

(i) $y = |x+2|$,

(ii) $y = \dfrac{1}{x-3}$.

(b) Find: **2**

(i) the primitive of $\dfrac{2}{5+2x}$,

(ii) $\int \sec^2 \pi x \; dx$.

(c) The area of a sector of a circle with radius 6 cm is 3π cm^2. Find the perimeter, leaving your answer in exact form. **3**

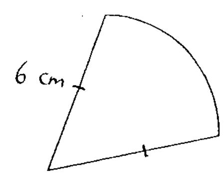


Diagram not to scale.

(d) Show that the sum of the following series, **3**

$\sqrt{12} + \sqrt{6} + \sqrt{3} + \ldots$ is $2\sqrt{6}(\sqrt{2}+1)$.

Question 4 Begin a new page.

(a) (i) Simplify $(1-\cos^2\theta)(\cot^2\theta+1)$. **4**

(ii) Write down the exact value of $\cos^2\frac{5\pi}{6}+\tan\frac{4\pi}{3}$.

(b) (i) Factorise a^3-b^3. **3**

(ii) Hence or otherwise simplify $(2-\sqrt{3})^3-(2+\sqrt{3})^3$.

(c) **5**

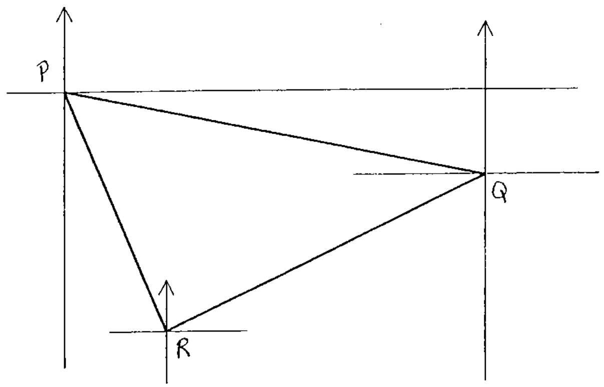


Diagram not to scale.

From P the bearing of a point Q 30 km away is 114°T. The bearing from Q of a point R 20 km away is 230°T.

(i) Copy the diagram and show this information on it.

(ii) Show that $\angle PQR = 64°$.

(iii) Find the distance from P to R, correct to the nearest 0.1 km.

(iv) Find the bearing of R from P, correct to the nearest degree.

Question 5 Begin a new page.

(a) Evaluate $\cos\left(\frac{1}{\sqrt{2}}\right)$, correct to 4 significant figures. **1**

(b) Consider the points S(-2, -1), T(4, 1) and U(6, 7).

Plot the points on a number plane. **2**

(i) Find the midpoint M of SU. **3**

(ii) Show that TM is perpendicular to SU. **2**

(iii) Show that ΔSTU is isosceles. **2**

(iv) Find the coordinates of V such that STUV is a rhombus, giving reasons for your answer. **2**

(v) Calculate the area of rhombus STUV.

Question 6 Begin a new page.

(a) Differentiate with respect to x: $y=\log_e\left(\frac{\sqrt{x}}{3x-2}\right)$. **2**

(b) The sum of the first n terms of a series is given by $S_n = 3n-n^2$. **3**

(i) Find the sum of the first 10 terms.

(ii) Find the tenth term.

(c) (i) Show that the equation of the normal to the curve $y=x^2$ at the point R where x=2 is given by $x+4y-18=0$. **7**

(ii) The normal intersects the parabola again at Q. Find the coordinates of Q.

Question 7 **Begin a new page.**

(a) (i) Show that $\frac{d}{dx}(x\ln x - x) = \ln x$. **3**

(ii) Hence evaluate $\int_1^{e^2} \ln x \, dx$, leaving your answer in exact form.

(b) Use Simpson's Rule with 4 sub-intervals to find an approximation to the area under the curve $y = \tan x$ between $x = 0$ and $x = 1$. **3**

(c) An Australian airline restricts the size of cabin baggage allowed onto aircraft. For bags in the recommended shape of a rectangular prism, the sum of the length, breadth and height should be equal to 115 cm. **6**

Sandy buys a bag in the recommended shape, with a square base of length x cm and height h cm, to take onto the aircraft.

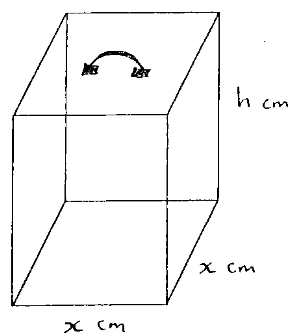


Diagram not to scale.

(i) Show that the volume in cubic centimetres is given by $V = 115x^2 - 2x^3$.

(ii) Hence find the maximum volume.

Question 8 **Begin a new page.**

(a) **6**

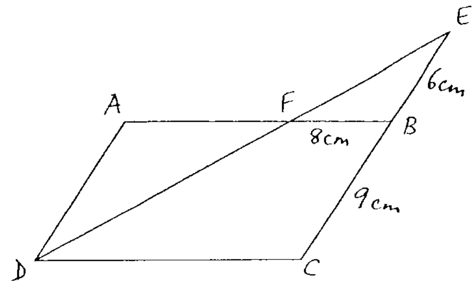


Diagram not to scale.

ABCD is a parallelogram. CB is produced to E. ED intersects AB at F. FB = 8 cm, EB = 6 cm and BC = 9 cm.

Copy the diagram onto your answer sheet.

(i) Prove $\Delta FAD \,|||\, \Delta FBE$.

(ii) Hence find the length of AF.

(b) A square chess board contains alternating black and white squares with eight squares along each side. Polly places one grain of rice on the first square, two grains of rice on the second square, four grains on the third, eight on the fourth, and so on, until every square is covered. **6**

(i) Find the number of grains she places on the last square. Leave your answer in index form.

(ii) How many grains does she need altogether? Leave your answer in index form.

Question 9 **Begin a new page.**

(a) Consider the equation $(k+1)x^2 - 2kx + 3k - 2 = 0$, where k is a constant. **2**

Find the value of k if the roots are reciprocals of one another.

(b) Find the equation for $f(x)$ given that $f''(x) = 6x - 2$, and $f'(1) = 4$ and $f(-1) = 3$. **4**

(c) Consider the circle $x^2 + y^2 = 9$ and the line $y = mx + b$. **6**

(i) Write down an expression for the perpendicular distance from the centre of the circle to the line.

(ii) Hence or otherwise show that if the line is a tangent to the circle then $b^2 = 9m^2 + 9$.

(iii) Hence find the equations of the tangents which are parallel to the line $y = x$.

Question 10 **Begin a new page.**

(a) (i) Sketch the graph of $y = 2\sin x$ for $-\pi \le x \le \pi$. **7**

(ii) Evaluate. $\int_0^{\frac{\pi}{2}} 2\sin x\,dx$.

(iii) Without integrating, evaluate the following giving reasons:

(α) $\int_{-\pi}^{\pi} 2\sin x\,dx$,

(β) the area bounded by the curve $y = 2\sin x$ for $-\pi \le x \le \pi$ and the x-axis.

(iv) On the same axes, sketch the curve $y = x^2 - 1$. Hence find the number of solutions to the equation $2\sin x - x^2 + 1 = 0$.

(b) **5**

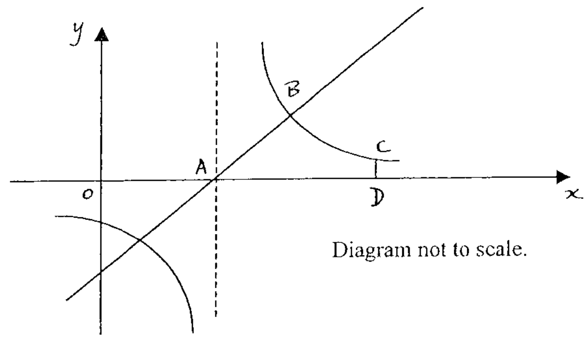


Diagram not to scale.

The graphs of $y = \frac{1}{x-2}$ and $y = x - 2$ are sketched above. The point D has coordinates $(4, 0)$.

(i) Show that the point $(3,1)$ satisfies both equations.

(ii) Find the volume generated when the area $ABCD$ is rotated about the x-axis.

✓ = 1 mark 10 QUESTIONS 12 marks each

YEAR 12 MATHEMATICS TRIAL EXAM 2003 ST. CATHERINE'S SOLUTIONS

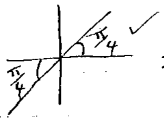
1a) i) $\sqrt{7}\times\sqrt{63} = \sqrt{7}\times\sqrt{9}\times\sqrt{7}$
$= 21$

ii) $49^{-\frac{1}{2}} = \frac{1}{\sqrt{49}}$
$= \frac{1}{7}$ (2)

b) $3^x = 0.0568$
$x \ln 3 = \ln 0.0568$
$x = \frac{\ln 0.0568}{\ln 3}$ (2)
$x = -2.61$

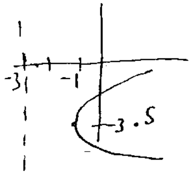
c) $\frac{x^{-1}-1}{x-1} = \frac{\frac{1}{x}-\frac{1}{1}}{x-1}$
$= \frac{1-x}{x} \times \frac{1}{x-1}$
$= \frac{-1}{x}$ (2)

d) $\tan x = 1 \quad 0 \le x \le 2\pi$ (2)



$x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

e) $8x-1 = y^2+6y$
$8x-1+9 = y^2+6y+9$
$8x+8 = (y+3)^2$
$8(x+1) = (y+3)^2$ (4)
i) Vertex $(-1,-3)$



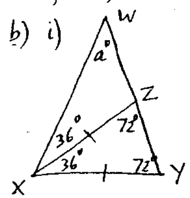
$4a = 8$
$a = 2$

ii) Focus $(1,-3)$
iii) Directrix $(x=-3)$

2a) i) $y = \frac{e^{3x}}{x}$ $u = e^{3x}$, $u' = 3e^{3x}$, $v = x$, $v' = 1$
$y' = \frac{u'v - vu'}{v^2}$
$= \frac{3e^{3x}.x - 1.e^{3x}}{x^2}$
$= \frac{e^{3x}(3x-1)}{x^2}$ (3)

ii) $f(x) = \sin 4x$
$f'(x) = 4\cos 4x$

b) i)



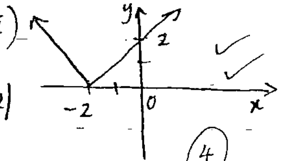
(6)

ii) In ΔXZY,
$\angle XYZ = 72^\circ$ (base $\angle$s of isos Δ equal)
In ΔWXY,
$\angle WXY = 72^\circ$ (base $\angle$s of isos Δ equal)
$\therefore a = 36$ ($\angle$ sum of $\Delta WXY = 180^\circ$)

iii) $\angle WXZ = 36^\circ$ (ext $\angle$ of ΔWXZ = sum of 2 int. opp)
$\therefore WZ = ZX$ (sides opp equal $\angle$s are equal)

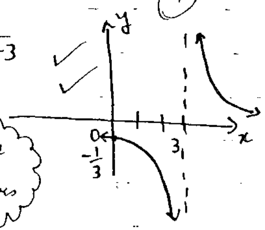
c) $2^{2x} - 15(2^x) - 16 = 0$ (3)
let $A = 2^x$
$\therefore A^2 - 15A - 16 = 0$
$(A-16)(A+1) = 0$
$A = 16$ or $A = -1$
$\therefore 2^x = 16$ or $2^x = -1$
$\therefore x = 4$ (no solution)

3. a) i)



$y = |x+2|$ (4)

ii) $y = \frac{1}{-x-3}$



1 for shape
1 for features

b) i) $\int \frac{2\,dx}{5+2x} = \ln(5+2x) + C$ (2)

ii) $\int \sec^2 \pi x\, dx = \frac{1}{\pi}\tan \pi x + C$

c) Area $= \frac{1}{2}r^2\theta$ (3)
$3\pi = \frac{1}{2}\times 36\times\theta$
$\frac{1}{6}\frac{3\pi}{18} = \theta$ $l = r\theta$
$\therefore \theta = \frac{\pi}{6}$
$\therefore$ Perimeter $= r + r + l$
$= 6 + 6 + 6\times\frac{\pi}{6}$
$= (12+\pi)$ cm

d) $\sqrt{12}+\sqrt{6}+\sqrt{3}+\ldots$
$\frac{T_3}{T_2} = \frac{\sqrt{3}}{\sqrt{6}} = \frac{1}{\sqrt{2}}$ $\frac{T_2}{T_1} = \frac{\sqrt{6}}{\sqrt{12}} = \frac{1}{\sqrt{2}}$ $\therefore$ G.P $r = \frac{1}{\sqrt{2}}$
$a = \sqrt{12}$
$\therefore S = \frac{a}{1-r}$ (3)
$= \frac{\sqrt{12}}{1-\frac{1}{\sqrt{2}}}$
$= \frac{\sqrt{12}}{\frac{\sqrt{2}-1}{\sqrt{2}}}$
$= \frac{\sqrt{12}\times\sqrt{2}}{\sqrt{2}-1}$
$= \frac{2\sqrt{6}}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}$
$= 2\sqrt{6}(\sqrt{2}+1)$

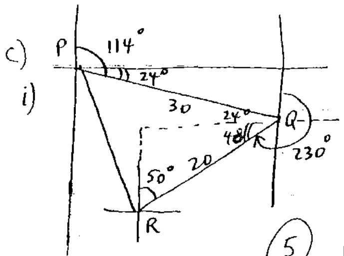
YR 12 TRIAL 2003 MATHEMATICS ST. CATHERINE'S cont'd SOLUTIONS

4 a) i) $(1-\cos^2\theta)(\cot^2\theta+1)$
$= \sin^2\theta \times \operatorname{cosec}^2\theta$
$= 1$

ii) $\cos^2\frac{5\pi}{6} + \tan\frac{4\pi}{3}$
$= \left(\frac{-\sqrt{3}}{2}\right)^2 + \sqrt{3}$ (4)
$= \frac{3}{4} + \sqrt{3}$

b) i) $a^3 - b^3 = (a-b)(a^2+ab+b^2)$
ii) $(2-\sqrt{3})^3 - (2+\sqrt{3})^3$
$= (2-\sqrt{3}-2-\sqrt{3})$
$\times[(2-\sqrt{3})^2 + (2-\sqrt{3})(2+\sqrt{3}) + (2+\sqrt{3})^2]$
$= -2\sqrt{3}[4-4\sqrt{3}+3+4-3+4+4\sqrt{3}+3]$
$= -2\sqrt{3}\times 15$ (3)
$= -30\sqrt{3}$

c) i)



(5)

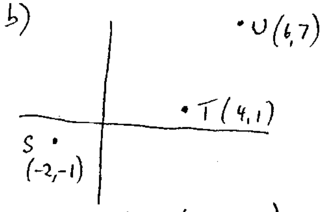
ii) Using alternate $\angle$s,
$\angle PQR = 24^\circ + 40^\circ$
$= 64^\circ$.

iii) $PR^2 = 30^2 + 20^2 - 2.30.20.\cos 64$
$= 773.95\ldots$
$PR = 27.8$ km (0.1 km)

v) $\cos\angle PRQ = \frac{27.8^2+20^2-30^2}{2\times 27.8\times 20}$ (2)
$\therefore \angle PRQ \doteqdot 75^\circ 48'$
$\therefore$ bearing $= 180^\circ - (75^\circ 48' - 50^\circ)$
$\doteqdot 154^\circ T$

5a) $\cos\left(\frac{1}{\sqrt{2}}\right) = 0.7602$ (4s.f.) (1)

b)



i) $M_{SU} = \left(\frac{-2+6}{2}, \frac{-1+7}{2}\right)$
$= (2, 3)$ (2)

ii) $m_{SU} = \frac{7--1}{6--2}$, $m_{TM} = \frac{1-3}{4-2}$
$= 1$, $= -1$ (3)
$\therefore m_{SU}\times m_{TM} = 1\times -1$
$= -1$
$\therefore$ Perpendicular.

iii) $SU = \sqrt{(4--2)^2+(1--1)^2}$
$= \sqrt{40}$
$TU = \sqrt{(6-4)^2+(7-1)^2}$ (2)
$= \sqrt{40}$
$\therefore ST = TU \therefore$ isosceles.

iv) STUV is rhombus $\therefore$
M is midpt of TV also
$\therefore (2,3) = \left(\frac{4+x}{2}, \frac{1+y}{2}\right)$
$\therefore 2 = \frac{4+x}{2}, \quad 3 = \frac{1+y}{2}$
$\therefore x = 0, \quad y = 5$ (2)
$\therefore V(0,5)$.

v) Area $= \frac{1}{2}SU\times TV$
$= \frac{1}{2}\sqrt{(6+2)^2+(7+1)^2}\times\sqrt{(4-0)^2+(1-5)^2}$
$= \frac{1}{2}\times\sqrt{128}\times\sqrt{32}$
$= 32$ units^2

Q6 a) $y = \log\left(\frac{\sqrt{x}}{3x-2}\right)$ (2)
$= \frac{1}{2}\log x - \log(3x-2)$
$y' = \frac{1}{2x} - \frac{3}{3x-2}$

b) $S_n = 3n - n^2$
i) $S_{10} = 3(10) - 10^2$
$= -70$ (3)
ii) $T_{10} = S_{10} - S_9$
$= -70 - (3(9) - 9^2)$
$= -70 - (-54)$
$= -16$

c) i) $y = x^2$ (7)
$y' = 2x$
$x = 2 \quad y' = 2(2) \quad y = 2^2$
$= 4 \quad = 4$
Normal: $m = -\frac{1}{4}$ $(2,4)$
$y - 4 = -\frac{1}{4}(x-2)$
$4y - 16 = -x + 2$
$x + 4y - 18 = 0$ as req'd.

ii) $y = x^2$ & $x + 4y - 18 = 0$
$x + 4x^2 - 18 = 0$
$4x^2 + x - 18 = 0$
$(4x+9)(x-2) = 0$
$x = \frac{-9}{4}$ or $x = 2$
$y = \frac{81}{16}$
$\therefore$ Coords of Q are $\left(\frac{-9}{4}, \frac{81}{16}\right)$.

ST. CATHERINE'S HSC TRIAL MATHEMATICS 2003 SOLUTIONS Cont'd.

. a) i) $\frac{d}{dx}(x\ln x - x)$ $\quad u = x,\ u' = 1,\ v = \ln x,\ v' = \frac{1}{x}$

$= (u'v + v'u) - 1$

$= 1.\ln x + \frac{1}{x}.x - 1$ ✓

$= \ln x + 1 - 1$

$= \ln x$

ii) $\int_1^{e^2} \ln x\, dx = \left[x\ln x - x\right]_1^{e^2}$ ✓

$= e^2\ln e^2 - e^2 - (1\ln 1 - 1)$

$= 2e^2\ln e - e^2 - 0 + 1$

3

$= 2e^2 - e^2 + 1$

$= e^2 + 1$ ✓

) $h = \frac{1-0}{4}$ $\quad x$ in radians

$= \frac{1}{4}$

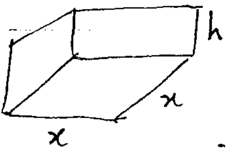
x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
y	tan0	tan$\frac{1}{4}$	tan$\frac{1}{2}$	tan$\frac{3}{4}$	tan1

✓

rea $\doteqdot \frac{\frac{1}{4}}{3}\left[\tan 0 + \tan 1\right.$ ✓

$\left. + 4\tan\frac{1}{4} + 2\tan\frac{1}{2} + 4\tan\frac{3}{4}\right]$

$\doteqdot 0.61648\ldots$ unit^2 ✓



i) $V = x^2h$

$2x + h = 115$

$\therefore h = 115 - 2x$ ✓

$\therefore V = x^2(115 - 2x)$ ✓

$V = 115x^2 - 2x^3$.

Max volume when $V' = 0$ and $V'' < 0$.

6

$V' = 230x - 6x^2$

$0 = 2x(115 - 3x)$ ✓

$x = 0$ (no sol'n) or $x = \frac{115}{3}$

st: $V'' = 230 - 12x$

$= \frac{115}{3}$ $\quad V'' = 230 - 12\left(\frac{115}{3}\right)$

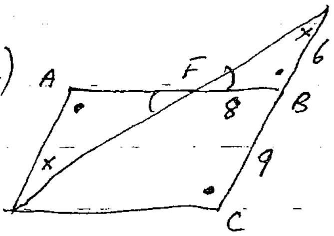
$= -230 < 0$ ✓

Max volume when $x = \frac{115}{3}$

$V = \left(\frac{115}{3}\right)^2\left(115 - 2\left(\frac{115}{3}\right)\right)$ ✓

$\doteqdot 56329$ cm^3.

8 a)



6

i) In $\triangle DAF$ and $\triangle FBE$,

1. $\angle AFD = \angle EFB$ (vert. opp $\angle$s equal) ✓

2. $\angle ADF = \angle FEB$ (alternate $\angle$s equal, AD∥EC)

$\therefore \triangle FAD \mid\mid\mid FBE$ (two angles equal) ✓

ii) $\therefore \frac{FA}{FB} = \frac{AD}{BE} = \frac{FD}{FE}$ (3 matching sides in proportion) ✓

$\therefore \frac{FA}{8} = \frac{9}{6}$ ✓ (AD = BC opp sides of par^m)

$\therefore FA = 8 \times \frac{9}{6}$

$= 12$ cm ✓

b) i) $1, 2, 2^2, 2^3, \ldots$ G.P. ✓

$T_n = ar^{n-1}$ where $a = 1$, $r = 2$ ✓

$T_{64} = 1 \times 2^{63}$

$= 2^{63}$ ✓

6

ii) $S_n = \frac{a(r^n - 1)}{r - 1}$

$S_{64} = \frac{1(2^{64} - 1)}{2 - 1}$ ✓

Total Grains $= 2^{64} - 1$. ✓

9 a) $\alpha\beta = \frac{c}{a}$ $\quad$ Roots reciprocals

$\therefore 1 = \frac{3k-2}{k+1}$ ✓

2

$k + 1 = 3k - 2$

$3 = 2k$

$\therefore k = 1\frac{1}{2}$ ✓

b) $f''(x) = 6x - 2$

4

$f'(x) = 3x^2 - 2x + C_1$ ✓

$4 = 3(1)^2 - 2(1) + C_1$

$\therefore C_1 = 3 \quad \therefore f'(x) = 3x^2 - 2x + 3$ ✓

$\therefore f(x) = x^3 - x^2 + 3x + C_2$ ✓

$3 = (-1)^3 - (-1)^2 + 3(-1) + C_2$

$\Rightarrow C_2 = 8$ ✓ $\therefore f(x) = x^3 - x^2 + 3x + 8$

Q9 Cont'd

c) i) Perp dist (0,0) $\quad mx - y + b = 0$ ✓

$\therefore d = \left|\frac{0 - 0 + b}{\sqrt{m^2+1}}\right|$ ✓

ii) If tangent then $r = d$

$\therefore 3 = \left|\frac{b}{\sqrt{m^2+1}}\right|$ ✓

6

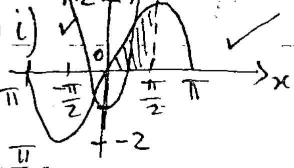
$\therefore 9 = \frac{b^2}{m^2+1}$ ✓ $\Rightarrow 9m^2 + 9 = b^2$ #

iii) parallel to $y = x \Rightarrow m = 1$

$\therefore 9(1)^2 + 9 = b^2$ ✓ $\therefore b = \pm 3\sqrt{2}$.

$\therefore$ Equations: $y = x \pm 3\sqrt{2}$ ✓

Q10 a) i)



✓

7

ii) $\int_0^{\frac{\pi}{2}} 2\sin x\, dx = \left[-2\cos x\right]_0^{\frac{\pi}{2}}$ ✓

$= -2\cos\frac{\pi}{2} + 2\cos 0$

$= 2$. ✓

iii) α) $\int_{-\pi}^{\pi} 2\sin x\, dx = 0$ ✓ areas cancel.

β) Area $= 4 \times \int_0^{\frac{\pi}{2}} 2\sin x\, dx$

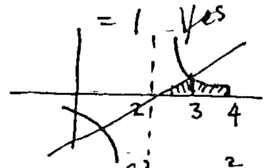
$= 8$ units^2 (symmetry) ✓

iv) 2 solutions. ✓

b) i) (3,1) $\quad y = \frac{1}{3-2}$ $\quad$, $y = 3 - 2$

$= 1$ yes $\quad = 1$ yes ✓

ii)



Volume $= \pi\int_2^3 (x-2)^2 dx + \pi\int_3^4 \frac{1}{(x-2)^2} dx$ ✓

$= \pi\left[\int_2^3 (x-2)^2 dx + \int_3^4 (x-2)^{-2} dx\right]$

$= \pi\left(\left[\frac{(x-2)^3}{3}\right]_2^3 + \left[\frac{(x-2)^{-1}}{-1}\right]_3^4\right)$ ✓

$= \pi\left(\frac{(3-2)^3}{3} - 0 - \left(\frac{1}{4-2} - \frac{1}{3-2}\right)\right)$

$= \pi\left[\frac{1}{3} - \left(\frac{1}{2} - 1\right)\right]$

$= \pi\left[\frac{1}{3} + \frac{1}{2}\right]$

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$= \frac{5\pi}{6}$ units^3. ✓