3/4 UNIT MATHEMATICS FORM VI

Time allowed: 2 hours (plus 5 minutes reading) **Exam date:** 19th August, 1998

Instructions:

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the left margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

Each question will be collected separately.
Start each question in a new answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet.
Write your name, class, and master's initials on each answer booklet:

QUESTION ONE (Start a new answer booklet)

Marks

2 (a) If A is the point $(5,2)$ and B is the point $(-1,8)$, find the coordinates of the point P which divides the interval AB internally in the ratio $2:1$.

2 (b) Differentiate $\cos^{-1} 3x$ with respect to x.

2 (c) Find:

(i) $\displaystyle\int \frac{2}{4+x^2}\,dx$,

(ii) $\displaystyle\int \frac{2x}{4+x^2}\,dx$.

2 (d) Simplify $\dfrac{2\sin x\cos x}{\cos^2 x - \sin^2 x}$.

2 (e) Find the coefficient of x^3 in the expansion of $(2+x)^5$.

2 (f) Find $\displaystyle\int \cos^2 x\,dx$.

QUESTION TWO (Start a new answer booklet)

Marks

3 (a) (i) By solving simultaneously, show that the points of intersection of the graphs of the functions $y = \dfrac{6}{x}$ and $y = x+1$ are $(-3,-2)$ and $(2,3)$.

(ii) Sketch the graphs of the two functions on the same diagram.

(iii) Hence, or otherwise, find the values of x for which $\dfrac{6}{x} \le x+1$

4 (b) Prove, by mathematical induction, that for all positive integer values of n:

$$1\times 5 + 2\times 6 + 3\times 7 + \cdots + n(n+4) = \tfrac{1}{6}n(n+1)(2n+13).$$

5 (c) Consider the polynomial $P(x) = x^3 - x^2 - x + 10$:

(i) Show that -2 is a zero of $P(x)$.

(ii) Given that the zeros of $P(x)$ are -2, α and β, show that $\alpha+\beta = 3$ and $\alpha\beta = 5$.

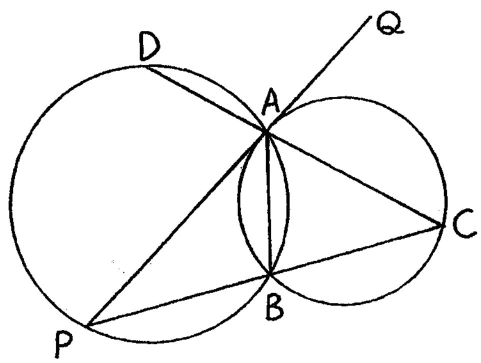
(iii) Show that α and β are not real numbers.

QUESTION THREE (Start a new answer booklet)

Marks

3 (a) Use the substitution $u = 1 + x$ to find $\displaystyle\int_1^2 \frac{1-x}{(1+x)^3}\,dx$.

3 (b)



The diagram shows two unequal circles intersecting at A and B. The tangent to the smaller circle at A cuts the larger circle at P, PB cuts the smaller circle at C and CA cuts the larger circle at D. If $\angle QAC = x$ and $\angle PAB = y$, show, giving reasons, that $\angle BDA = x - y$.

6 (c) (i) On the same diagram, sketch the graphs of $y = e^{-\frac{1}{2}x}$ and $y = 5 - x^2$, showing all intercepts with the x and y axes.

(ii) On your diagram, indicate the negative root α of the equation $x^2 + e^{-\frac{1}{2}x} = 5$.

(iii) Show that $-2 < \alpha < -1$.

(iv) Use one iteration of Newton's method with starting value $x_1 = -2$ to show that α is approximately $\dfrac{-18}{e+8}$.

QUESTION FOUR (Start a new answer booklet)

Marks

2 (a) A stone was dropped into a smooth pond. The radius of a particular circular ripple is increasing at the constant rate of 18 cm per second. Find the rate at which the area of the circular ripple is increasing at the instant when the circumference is 30 cm.

5 (b) A particle moves in a straight line with displacement in centimetres from the point $x = 0$ at time t seconds given by $x = C\cos 2t + D\sin 2t$, where C and D are constants.

(i) Show that the motion is simple harmonic by showing that the acceleration has the form $-n^2x$, where n is a constant.

(ii) It is known that when $t = \dfrac{\pi}{3}$, $x = \dfrac{\sqrt{3}}{2}$ and $\dot{x} = -5$.

(α) Find C and D.

(β) Find the amplitude of the motion.

QUESTION FOUR (Continued)

5 (c) Each term of the arithmetic sequence a, $a+d$, $a+2d$, ... is added to the corresponding term of the geometric sequence b, ba, ba^2, ... to form a third sequence S, whose first three terms are -1, -2 and 6. (Note that the common ratio of the geometric sequence is equal to the first term of the arithmetic sequence.)

(i) Show that $a^3 - a^2 - a + 10 = 0$.

(ii) Use question 2 part (c) to find the value of a, assuming that it is real.

(iii) Hence show that the nth term of S is given by:

$$T_n = 2n - 4 + (-2)^{n-1}.$$

QUESTION FIVE (Start a new answer booklet)

Marks

4 (a) The rate at which a substance evaporates is proportional to the amount of the substance which has not yet evaporated. That is, $\dfrac{dx}{dt} = k(A - x)$, where A is the initial amount of the substance, x is the amount which has evaporated at time t and k is a constant.

(i) Show that the function $x = A(1 - e^{-kt})$ satisfies the differential equation.

(ii) Sketch the graph of x against t.

(iii) Show that the time it takes for 75% of the substance to evaporate is $\dfrac{2}{k}\ln 2$.

8 (b) A particle moves in a straight line with acceleration given by $\ddot{x} = (4x - 2)\,\text{ms}^{-2}$, where x is the displacement. Initially the particle is at the origin with velocity $1\,\text{ms}^{-1}$.

(i) If the velocity at time t seconds is $v\,\text{ms}^{-1}$, show that $v^2 = (1 - 2x)^2$.

In the next three parts, you may assume that $v = 1 - 2x$ throughout the motion.

(ii) Hence show that $\ddot{x} = -2v$.

(iii) Find expressions for x and v in terms of t.

(iv) Show that the particle approaches, but never reaches, $x = \frac{1}{2}$.

QUESTION SIX (Start a new answer booklet)

Marks

6 (a) A certain parabola has parametric equations $x = 6t$ and $y = 3t^2$. P is the point on the parabola where $t = p$.

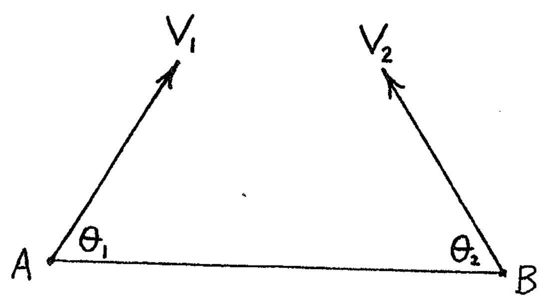
(i) Show that the tangent to the parabola at P has equation $y = px - 3p^2$.

(ii) If Q is the point on the parabola where $t = 1 - p$, and P and Q are distinct, show that the tangents at P and Q meet at the point $T(3, 3p - 3p^2)$.

(iii) Specify algebraically the locus of T.

(iv) Comment on the points P, Q and T in the case where $p = \frac{1}{2}$.

6 (b)



Two particles P_1 and P_2 are projected simultaneously from the points A and B, where AB is horizontal. The motion takes place in the vertical plane through A and B. The initial velocity of P_1 is V_1 at an angle θ_1 to the horizontal, and the initial velocity of P_2 is V_2 at an angle θ_2 to the horizontal.

You may assume that the equations of motion of a particle projected with initial velocity V at an angle θ to the horizontal are:

$$x = Vt\cos\theta \quad \text{and} \quad y = -\tfrac{1}{2}gt^2 + Vt\sin\theta.$$

(i) If the particles are to collide, show that the condition $V_1 \sin\theta_1 = V_2 \sin\theta_2$ must be satisfied.

(ii) Suppose $AB = 200\,\text{m}$, $V_1 = 30\,\text{m/s}$, $\theta_1 = \sin^{-1}\frac{4}{5}$, $\theta_2 = \sin^{-1}\frac{3}{5}$, $g = 10\,\text{m/s}^2$ and the particles collide with each other.

(α) Show that $V_2 = 40\,\text{m/s}$.

(β) Show that the particles collide 4 seconds after they were projected.

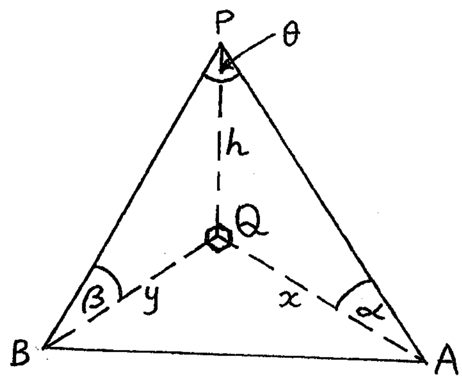
(γ) Find the height of the point of collision above AB.

(δ) Find, correct to the nearest degree, the angle between the directions of motion of the particles at the instant they collide.

QUESTION SEVEN (Start a new answer booklet)

Marks

5 (a)



In the above diagram of a triangular pyramid, $AQ = x$, $BQ = y$, $PQ = h$, $\angle APB = \theta$, $\angle PAQ = \alpha$ and $\angle PBQ = \beta$. Also, there are three right-angles at Q.

(i) Show that $x = h \cot \alpha$ and obtain a similar expression for y.

(ii) Show that $\cos\theta = \dfrac{h^2}{\sqrt{(x^2+h^2)(y^2+h^2)}}$.

(iii) Hence show that $\sin\alpha \sin\beta = \cos\theta$.

7 (b) Suppose a and b are positive integers and $a = b - 1$.

(i) Write down, using sigma notation or otherwise, the expansion of $(b-1)^{2n}$, where n is a positive integer.

(ii) Hence show that $a^{2n} + 2bn - 1$ is exactly divisible by b^2.

(iii) Use part (ii) to show that $2^{159} + 1$ is a multiple of 9.

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