SOLUTIONS

Question 1

(a) $(2x-y)^5 = 1(2x)^5(-y)^0 + 5(2x)^4(-y)^1 + 10(2x)^3(-y)^2 + 10(2x)^2(-y)^3 + 5(2x)^1(-y)^4 + 1(2x)^0(-y)^5$ ①

$= 32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$ ①

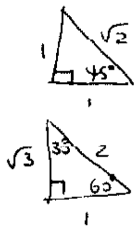
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$\boxed{2}$

(b) (i) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ ①

(ii) $\cos 45^\circ = \frac{1}{\sqrt{2}}$

$\cos 30^\circ = \frac{\sqrt{3}}{2}$



① need both

(iii) $\cos 15^\circ = \cos(45-30)^\circ$

$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$ ①

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

$\boxed{3}$

(c)(i) $1^\circ = \frac{\pi}{180}$

$72^\circ = \frac{72\pi}{180}$

$= \frac{8\pi}{20}$

$= \frac{2\pi}{5}$ ①

(ii) Area of minor segment $= \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2} \cdot 20^2 \cdot \left(\frac{2\pi}{5} - \sin\frac{2\pi}{5}\right)$ ①

$= 61.11610\ldots$

$= 61.1 \text{ mm}^2$ (3 sig. fig.) ①

$\boxed{3}$

(d) $\sin 2x = \sqrt{3}\cos 2x$

$\frac{\sin 2x}{\cos 2x} = \sqrt{3}$

$\tan 2x = \sqrt{3}$

$2x = \frac{\pi}{3}, \pi+\frac{\pi}{3}, 2\pi+\frac{\pi}{3}, 3\pi+\frac{\pi}{3}, \ldots$ ①

$x = \pi/6, 2\pi/3, 7\pi/6, 5\pi/3$ ①

$\boxed{2}$

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e)(i) $\frac{d}{dx}\sqrt[3]{4x-1} = \frac{d}{dx}(4x-1)^{\frac{1}{3}}$

$= \frac{1}{3}\cdot(4x-1)^{-2/3}\cdot 4$

$= \frac{4}{3}(4x-1)^{-2/3}$ or $\frac{4}{3\sqrt[3]{4x-1}^{\,2}}$ ①

(ii) $\frac{d}{dx}\left(\frac{x}{\cot x}\right) = \frac{d}{dx} x\tan x$

$= x \cdot \sec^2 x + \tan x \cdot 1$

$= x\sec^2 x + \tan x$ ①

$\boxed{2}$

Question 2

(a) (i) $= \int_0^1 f(t)\,dt + \int_1^3 f(t)\,dt + \int_1^3 1\,dt$

$= \int_0^3 f(t)\,dt + \left[t\right]_1^3$ ①

$= 6 + 3 - 1$

$= 8$ ①

(ii) $= \int_3^0 f(t)\,dt + \int_3^0 t\,dt$

$= -\int_0^3 f(t)\,dt + \left(\frac{t^2}{2}\right)_3^0$ ①

$= -6 + 0 - \frac{9}{2}$

$= -10\frac{1}{2}$ ①

$\boxed{4}$

(b) (i) $\int \frac{x^4+2x^3+3}{x^2}\,dx$

$= \int (x^2 + 2x + 3x^{-2})\,dx$

$= \frac{x^3}{3} + x^2 - 3x^{-1} + c$ ①

or $\frac{x^3}{3} + x^2 - \frac{3}{x} + c$

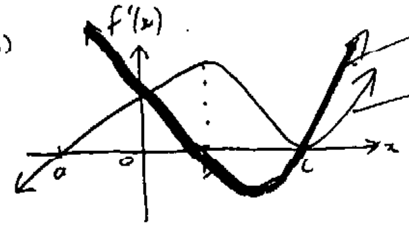
(ii) $\int \frac{dt}{(3-t)^2} = \int (3-t)^{-2}\,dt$

$= \frac{(3-t)^{-1}}{-1 \times -1}$

$= \frac{1}{3-t} + c$ ①

$\boxed{2}$

(c) (i)(ii)



y = f''(x) to get this

y = f'(x) copy in ①

(i)
(ii) } 1
(iii) 1
(iv) 1+1

(iii) $b < x < c$ ①

(iv) $x = b$ ①

x	b^-	b	b^+
$f''(x)$	+	0	−

There is a change in concavity at $x=b$

$x = c$

x	c^-	c	c^+
$f''(x)$	−	0	+

there is a change in concavity at $x=c$

∴ Point of inflection at $x=b$, $x=c$.

(v) 1+1

(v) $f'(a) = 0$ } tests
$f''(a) > 0$

∴ $x = a$ is a rel. min. turning pt

①

$f'(c) = 0$

x	c^-	c	c^+
$f'(x)$	+	0	+
slope	/	—	/

① ∴ $x = c$ is a stationary point of inflection HPOI

$\boxed{6}$

Question 3

(a) $y = \dfrac{x^2+x+1}{x}$

$y = x+1 + \dfrac{1}{x}$

vertical asymptote $x=0$ ①

diagonal asymptote $y=x+1$ ①

$\boxed{2}$

(b)(i) $WP^2 = 5^2 + x^2$

$WP = \sqrt{25+x^2}$ ①

Cost $= 125\,000 \times WP + 75\,000 \times PB$ ①

$C = 125\sqrt{25+x^2} + 75(8-x)$ ① (C is in thousands of dollars)

$\boxed{3}$

(ii) $x^2+25>0$ and $8-x \geq 0$ and $x \geq 0$

$x^2 > -25$ $\quad -x \geq -8$

$\therefore x \in \mathbb{R}$ $\quad x \leq 8$

$\therefore 0 \leq x \leq 8$ ①

$\boxed{1}$

(iii) $\dfrac{dc}{dx} = 125 \cdot \frac{1}{2}(25+x^2)^{-\frac{1}{2}} \times 2x - 75$ ①

$= \dfrac{125x}{\sqrt{25+x^2}} - 75$

$\dfrac{dc}{dx} = 0$ for max/min

$\dfrac{125x}{\sqrt{25+x^2}} - 75 = 0$ ①

$125x = 75\sqrt{25+x^2}$

$5x = 3\sqrt{25+x^2}$ only if $x \geq 0$ Yes $0 \leq x \leq 8$

Square both sides;

$25x^2 = 9(25+x^2)$ ①

$25x^2 = 225 + 9x^2$

$16x^2 = 225$

$x^2 = \dfrac{225}{16}$

$x = \pm \dfrac{15}{4}$

Since $0 \leq x \leq 8$

$x = \dfrac{15}{4}$

(this ① allocated at end of question after testing)

x	$\frac{15}{4}^-$ (3)	$\frac{15}{4}$	$\frac{15}{4}^+$ (4)
$\frac{dc}{dx}$	$-$	0	$+$
slope	\	—	/

①

$\therefore$ rel. min when $x = \dfrac{15}{4}$ and Cost $= \left(125\sqrt{(\frac{15}{4})^2+25} + 75(8-\frac{15}{4})\right) \times 1000$

$\therefore$ cost $= \$1\,100\,000$

test end points of $0 \leq x \leq 8$

$x=0$

Cost $= (5 \times 125 + 8 \times 75) \times 1000$

$= \$1\,225\,000$

$x=8$

$WP = \sqrt{8^2+5^2}$

$= \sqrt{89}$

$PB = 0$

$\therefore$ Cost $= (\sqrt{89} \times 125) \times 1000$ ①

$= \$1179247.642\ldots$

$= \$1179247.64$

$\therefore$ The point P should be located $\frac{15}{4} = 3\frac{3}{4} = 3.75$ km

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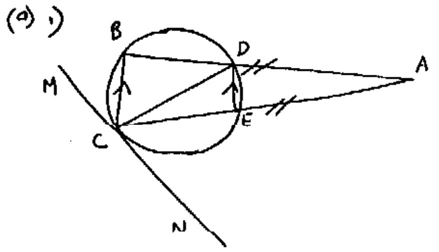
④

Question 4

(a) i)



ii) $\triangle ABC$ is isosceles $(AB = AC)$

Let $\angle ABC = \alpha$

$\therefore \angle ABC = \angle BCA = \alpha$ (equal base $\angle$'s, isos $\triangle$) ①

$\angle BDE = 180^\circ - \angle BCE$ (opp $\angle$'s cyclic quad are supp) ①

$= 180^\circ - \alpha$

$\angle CBD + \angle BDE = 180^\circ$ ①

and these are cointerior angles ①

$\therefore BC \parallel DE$

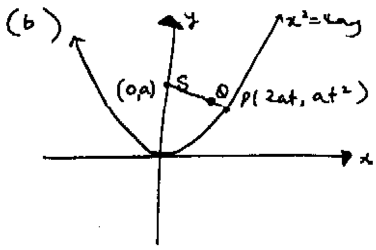
iii) $\angle ACN = \angle CDE$ (angle in the alternate segment thm) ①

$\angle CDE = \angle BCD$ (alt $\angle$'s $=$, $BC \parallel DE$) ①

$\therefore \angle ACN = \angle BCD$

$\boxed{6}$

(b)



(i) $Q = \left(\dfrac{nx_1 + mx_2}{m+n}, \dfrac{ny_1 + my_2}{m+n}\right)$

$= \left(\dfrac{1 \times 2at + t^2 \times 0}{t^2+1}, \dfrac{1 \times at^2 + t^2 \times a}{t^2+1}\right)$ ①

$= \left(\dfrac{2at}{t^2+1}, \dfrac{2at^2}{t^2+1}\right)$ ①

(ii) $\dfrac{y}{x} = \dfrac{\frac{2at^2}{t^2+1}}{\frac{2at}{t^2+1}}$

$= \dfrac{2at^2}{2at}$

$\therefore \dfrac{y}{x} = t$ ①

(iii) $x = \dfrac{2at}{t^2+1}$

$= \dfrac{2a\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)^2 + 1}$

$= \dfrac{\frac{2ay}{x}}{\frac{y^2+x^2}{x^2}}$

$\therefore x = \dfrac{2axy}{x^2+y^2}$ ①

$x^2 + y^2 = 2ay$

$x^2 + y^2 - 2ay = 0$

$x^2 + (y-a)^2 = a^2$ ①

circle centre $(0, a)$
radius a units $\Big\}$ ①

$\boxed{6}$

Question 5

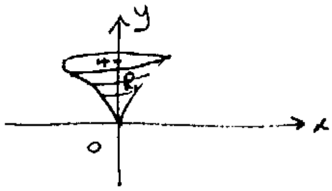
(a)

(i) $\text{Volume} = \pi \int_0^4 y^3\,dx$

$= \pi \left[\frac{y^4}{4} \right]_0^4$ ①

$= \pi \left[\frac{4^4}{4} - 0 \right]$

$= 64\pi$ cubic units ①



(ii) $\text{Volume} = \pi \int_0^8 x^{4/3}\,dx$ ①

$y^3 = x^2$

$y = x^{2/3}$

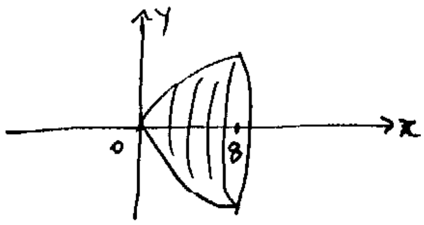
$y^2 = x^{4/3}$

$= \pi \left[\frac{3}{7} x^{7/3} \right]_0^8$ ①

$= \pi \left[\frac{3}{7} 8^{7/3} - 0 \right]$

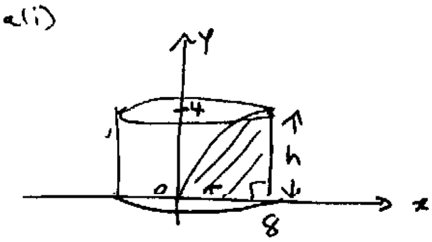
$= \pi \cdot \frac{3}{7} \cdot 2^7$

$= \frac{384}{7}\pi$ cubic units ①



(iii) $\text{Volume} = \pi \times 8^2 \times 4 - 64\pi$ ① ⟵ Q5a(i)

$= 192\pi$ cubic units ①



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(b) (i) $\sin x - \cos x \equiv A\sin(x - \alpha)$

$= A\sin x\cos\alpha - A\cos x\sin\alpha$ ①

$\therefore A\cos\alpha = 1$ . . . . ①

$A\sin\alpha = 1$ . . . . ②

①^2 + ②^2:

$A^2\sin^2\alpha + A^2\cos^2\alpha = 1 + 1 = 2$

$A^2(\sin^2\alpha + \cos^2\alpha) = 2$

$A^2 = 2$

$A = \sqrt{2}$ ①

from ①, ② $\left.\begin{array}{l}\cos\alpha = \frac{1}{\sqrt{2}} \\ \sin\alpha = \frac{1}{\sqrt{2}}\end{array}\right\}$ quad ①

$\therefore \alpha = \frac{\pi}{4}$ ①

$\sin x - \cos x = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

(ii) $\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \pi/4} = \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)}{x - \pi/4}$ ①

$= \sqrt{2}$ ①

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~~QUESTION 6~~

(a) $m_{OP} = \dfrac{\frac{1}{K}-0}{K-0}$

$= \dfrac{1}{K^2}$

∴ eqn OP is $y = \dfrac{1}{K^2}x$ $\qquad x - k^2y = 0$ ① $\boxed{1}$

(b) Solve simultaneously $y = \dfrac{1}{K^2}x$

and $y = \dfrac{1}{x}$

$\therefore \dfrac{1}{K^2}x = \dfrac{1}{x}$ ①

$x^2 = K^2$

$x = K$ or $x = -K$

Since P is $(K, \frac{1}{K})$

R is $(-K, -\frac{1}{K})$ ① $\boxed{2}$

(c) $y = x^{-1}$

$\dfrac{dy}{dx} = -x^{-2}$

$= -\dfrac{1}{x^2}$

at $x = K$, $\dfrac{dy}{dx} = -\dfrac{1}{K^2}$

∴ gradient of tangent $= -\dfrac{1}{K^2}$ ①

equation of tangent is $y - \dfrac{1}{K} = -\dfrac{1}{K^2}(x-K)$

$K^2y - K = -x + K$

$x + K^2y = 2K$ as required ① $\boxed{2}$

(d) Slope of tangent $= -\frac{1}{K^2}$

∴ Slope of normal $= K^2$ $\quad (m_1m_2 = -1)$ ①

Equation of normal $y - \dfrac{1}{K} = K^2(x - K)$

$Ky - 1 = K^3x - K^4$ $\qquad y = k^2x + \frac{1}{k} - k^3$

$K^3x - Ky = K^4 - 1$ ① $\boxed{2}$

(e) Solve simultaneously

$y = \dfrac{1}{x}$ and $K^3x - Ky = K^4 - 1$

eqn ① $\qquad$ eqn ②

Sub eqn ① into eqn ②

$K^3x - K\left(\dfrac{1}{x}\right) = K^4 - 1$

$K^3x^2 - K = (K^4-1)x$

$K^3x^2 - (K^4-1)x - K = 0$ ①

Since $P(k, \frac{1}{k})$ lies on the normal

$x = k$ is a root of the equation.

or use $\alpha + \beta = \frac{k^4 ...}{k^3}$

method ①

$$\begin{array}{r} k^3x + 1 \\ x-k \,\overline{)\, k^3x^2 - (k^4-1)x - k} \\ k^3x^2 - k^4x \\ \hline x - k \\ x - k \\ \hline 0 \end{array}$$

$\therefore (x-k)(k^3x+1) = 0$

$x = k$ or $x = -\frac{1}{k^3}$

Q is $\left(-\frac{1}{k^3}, -k^3\right)$

② or formula

method ②

product of roots

$\alpha\beta = -\frac{k}{k^3}$

$= -\frac{1}{k^2}$

since $\alpha = k$ is one root

$k\beta = -\frac{1}{k^2}$

$\beta = -\frac{1}{k^3}$

$\therefore Q$ is $\left(-\frac{1}{k^3}, -k^3\right)$

1 method
1 answer

$\boxed{3}$

(f) Slope $QR = \dfrac{-\frac{1}{k} - -k^3}{-k - -\frac{1}{k^3}}$

$= \dfrac{\frac{(k^4-1)}{\cancel{k}}}{\frac{-(k^4-1)}{\cancel{k^3}\, k^2}}$ ①

$= -k^2$

Slope $PR = \frac{1}{k^2}$

Slope $QR \times$ slope $PR = -k^2 \times \frac{1}{k^2}$ ①

$= -1$

$\therefore QR \perp PR$

$\boxed{2}$