Question 8 CSSA 2007

a)

(i) $f(x) = (a_1 x - 1)^2 + (a_2 x - 1)^2 + \cdots + (a_n x - 1)^2$

$= x^2 (a_1^2 + a_2^2 + \cdots + a_n^2)$

$- x(2a_1 + 2a_2 + \cdots + 2a_n) + n$

$\equiv Ax^2 + Bx + C$

(ii) we note that

$(a_k x - 1)^2 > 0, \; k = 1, 2, \ldots, n$

Hence $f(x) = \sum_{k=1}^{n} (a_k x - 1)^2 > 0$

Thus, there exists no real roots for $f(x)$ such that $f(\alpha) = 0, \alpha \in R$

Thus for $f(x) = Ax^2 + Bx + C$

$\Delta < 0$

$\Rightarrow B^2 - 4AC < 0$

$4(a_1 + a_2 + \cdots + a_n)^2 < 4n(a_1^2 + a_2^2 + \cdots + a_n^2)$

$\Rightarrow (a_1^2 + a_2^2 + \cdots + a_n^2) > \frac{1}{n}(a_1 + a_2 + \cdots + a_n)^2$

$\Rightarrow \sum_{k=1}^{n} a_k^2 \geq \frac{1}{n}\left(\sum_{k=1}^{n} a_k\right)^2$

with equality if $n = 1$

(iii) Suppose that $a_k = 2k - 1, k \in Z$

then $a_1 = 1, a_2 = 3, \ldots, a_n = 2n - 1$

Hence, using (ii), we obtain

$(1)^2 + (3)^2 + (5)^2 + \cdots + (2n-1)^2$

$\geq \frac{1}{n}\left\{1 + 3 + 5 + \cdots + 2n - 1\right\}^2$

$= \frac{1}{n}\left\{\frac{n}{2} \times 2n\right\}^2$

$= \frac{1}{n} \times n^4 = n^3$

(using the sum of an arithmetic series, $S_n = \frac{n}{2}(a + e)$)

$\therefore (1)^2 + (3)^2 + \cdots + (2n-1)^2 \geq n^3$

Making the substitution $a_k \longrightarrow a_k^2$ gives

$\sum_{k=1}^{n} a_k^4 \geq \frac{1}{n}\left(\sum_{k=1}^{n} a_k^2\right)^2$

Hence using $a_k = 2k - 1, k \in Z$,

$(1)^4 + (3)^4 + \cdots + (2n-1)^4$

$\geq \frac{1}{n}\left((1)^2 + (3)^2 + \cdots + (2n-1)^2\right)^2$

$\geq \frac{1}{n}\left\{n^3\right\}^2 = n^5$

Hence

$1^4 + 3^4 + \cdots + (2n-1)^4 \geq n^5$

b) (i) $P(\text{Draw}) = \frac{1}{6}$

$\therefore \quad 1 - P(\text{Draw}) = \frac{5}{6}$

$= \frac{10}{12}$

Now A and B are equally likely to win,

$\therefore \quad P(A \text{ wins}) = \frac{10}{12} \times \frac{1}{2} = \frac{5}{12}$

(ii) Considering all possible cases,

$p_2 = \left(\frac{5}{12}\right)^2 + \left(\frac{5}{12}\right)^2$

$= \frac{25}{72}$ AND

$q_2 = \left(\frac{1}{6}\right)^2 + \left(4 \times \frac{5}{12} \times \frac{1}{6}\right) + \left(\frac{5}{12} \times \frac{5}{12}\right) \times 2$

$= \frac{47}{72}$

$\therefore \quad q_2 + p_2 = \frac{72}{72} = 1$

(iii) Clearly, $q_{n-1} - q_n$ will leave no other option other than for a win on the nth game $\Rightarrow p_n = q_{n-1} - q_n$

$\therefore \quad p_n + q_n = q_{n-1}$

Hence $\sum_{k=2}^{n} p_k = p_2 + p_3 + \cdots + p_n$

$= (1 - q_2) + (q_2 - q_3) + (q_3 - q_4) + \cdots + (q_{n-1} - q_n)$

$= 1 - q_n$ (using the established relation, $p_n + q_n = q_{n-1}$)

$\therefore \quad \sum_{k=2}^{n} p_k = 1 - q_n$

(iv) Considering all possible cases in n games,

$q_n = \left\{2 \times {}^nC_1 \times \frac{5}{12} \times \left(\frac{1}{6}\right)^{n-1}\right\} + \left(\frac{1}{6}\right)^n + {}^nC_2\left(\frac{5}{12}\right)^2\left(\frac{1}{6}\right)^{n-2}$

$= 5n\left(\frac{1}{6}\right)^n + \left(\frac{1}{6}\right)^n + \left\{n(n-1) \times 25 \times \frac{1}{4} \times \left(\frac{1}{6}\right)^n\right\}$

$= \frac{5n}{6^n} + \frac{1}{6^n} + \frac{25(n)(n-1)}{4 \times 6^n}$

$= \frac{20n + 4 + 25n^2 - 25n}{4 \times 6^n}$

$\therefore \quad q_n = \frac{25n^2 - 5n + 4}{4 \times 6^n}$

Hence $p_n = q_{n-1} - q_n$

$= \frac{25(n-1)^2 - 5(n-1) + 4}{4 \times 6^{n-1}} - \frac{(25n^2 - 5n + 4)}{4 \times 6^n}$

$$= \frac{150(n-1)^2 - 30(n-1) + 24 - 25n^2 + 5n - 4}{4 \times 6^n}$$

$$= \frac{125n^2 + 200 - 325}{4 \times 6^n} = \frac{25(n-1)(5n-8)}{4 \times 6^n}$$

as required.

(v) as $n \to \infty$, $q_n \to 0$, Hence the probability that game never ends $\to 0$

$\therefore$ P(never ends) $= 0$ as it must end eventually