CSSA of NSW

1992 Trial HSC 4 Unit Mathematics

1. **(a)** **(i)** Sketch the graph of $f(x) = x^3 - 3x$ showing clearly the coordinates of any points of intersection with the x and the coordinates of any turning points.
(ii) Use the graph of $y = f(x)$ in part **(i)** to sketch the graph of $y = |f(x)|$ showing clearly the coordinates of any critical points (where $\frac{dy}{dx}$ is not defined) and the coordinates of any turning points.
(iii) Use the graph of $y = f(x)$ in part **(i)** to sketch the graph of $y = \frac{1}{f(x)}$ showing clearly the equations of any asymptotes and the coordinates of any turning points.
(b) **(i)** Show that the tangent from the origin to the curve $y - \log_e x$ has a gradient of $\frac{1}{e}$.
(ii) Hence find the set of values of the real number k for which the equation $\log_e x = kx$ has two distinct real roots.

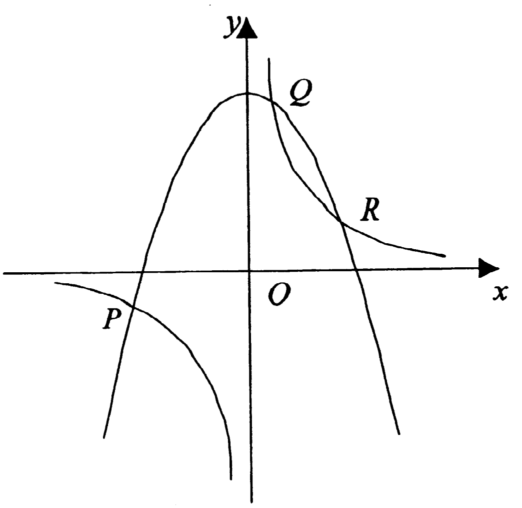
2. **(a)** Find $\int \frac{e^{\tan x}}{\cos^2 x}\ dx$.
(b) Evaluate $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\ dx$.
(c) **(i)** Use the substitution $u = -x$ to show that $\int_{-a}^{0} f(x)\ dx = \int_0^a f(-x)\ dx$. Deduce that $\int_{-a}^{a} f(x)\ dx = \int_0^a \{f(x) + f(-x)\}\ dx$.
(ii) Hence evaluate $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x^5 \cos x\ dx$.
(d) **(i)** Use the substitution $x = u^2 (u > 0)$ to show that $\int_4^9 \frac{\sqrt{x}}{x-1}\ dx = 2 + \log_e\left(\frac{3}{2}\right)$.
(ii) Hence use integration by parts to evaluate $\int_4^9 \frac{1}{\sqrt{x}} \log_e(x-1)\ dx$.

3. **(a)** It is given that $|z|^2 = z + \overline{z}$. On an Argand diagram sketch the locus of the point P representing z.
(b) **(i)** Expand $z = (1 + ic)^6$ in powers of c.
(ii) Hence find the five real values of c for which z is real.
(c) **(i)** Express $z = 2i$ and $w = -1 + \sqrt{3}i$ in modulus argument form. On an Argand diagram plot the points P and Q which represent z and w.
(ii) On the same diagram construct vectors which represent $z+w$ abd $z-w$. Deduce the exact values of $\arg(z + w)$ and $\arg(z - w)$.
(d) Solve the equation $z^5 + 16z = 0$, expressing each solution in the form $z = a + ib$ where a and b are real.

4. **(a)** Find the equation of the tangent to the curve $xy(x + y) + 16 = 0$ at the point on the curve where the gradient is -1.
(b) Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b > 0$.
(i) Show that the tangent to the ellipse at the point $P(a\cos\theta, b\sin\theta)$ has equation $bx\cos\theta + ay\sin\theta - ab = 0$.

(ii) R and R' are the feet of the perpendiculars from the foci S and S' on to the tangent at P. Show that $SR.S'R' = b^2$.

5. **(a)** The base of a particular solid is the region bounded by the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$ between its vertex $(2, 0)$ and its corresponding latus rectum. Every cross section of the solid perpendicular to the major axis of the hyperbola is an isosceles right angled triangle with hypotenuse in the base of the solid.
(i) Show that the latus rectum has equation $x = 4$.
(ii) Find the volume of the solid.
(b)

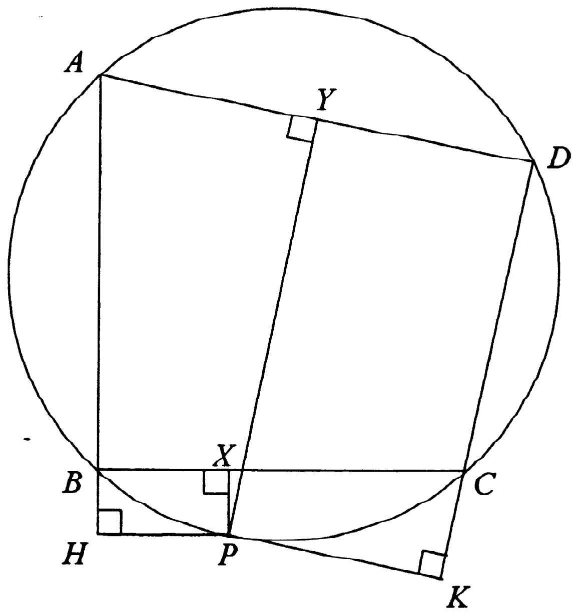


The curves $y = k - x^2$, for some real number k, and $y = \frac{1}{x}$ intersect at the points P, Q and R where $x = \alpha, x = \beta$ and $x = \gamma$.
(i) Show that the monic cubic equation with coefficients in terms of k whose roots are α^2, β^2 and γ^2 is given by $x^3 - 2kx^2 + k^2x - 1 = 0$.
(ii) Find the monic cubic equation with coefficients in terms of k whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$.
(iii) Hence show that $OP^2 + OQ^2 + OR^2 = k^2 + 2k$, where O is the origin.

6. A railway track has been constructed around a circular curve of radius 500 metres. The distance across the track between the rails is 15 metres and the outer rail $0{\cdot}1$ metres above the inner rail.
(a) If the train travels on the track at a speed v_0 which eliminates any sideways force on the wheels
(i) Draw a diagram showing all the forces acting on the train.
(ii) Show that $v_0^2 = 500g\tan\theta$, where θ is the angle the track makes with the horizontal. Taking $g = 9{\cdot}8$ m.s^{-2} calculate v_0.
(b) If the train travels on the track at a speed $v > v_0$
(i) State which rail exerts a lateral force on the wheel at the point of contact.
(ii) Draw a diagram showing all the forces on the train.
(iii) Show that the lateral force F exerted by the rail on the wheel is given by $F = \frac{mv^2}{500}\cos\theta - mg\sin\theta$, where m is the mass of the train. Deduce that F is one

fifth the weight of the train when $v = 2v_0$.

7.



$ABCD$ is a cyclic quadrilateral. P is a point on the circle through A, B, C and D. PH, PX, PK and PY are the perpendiculars from P to AB produced, BC, DC produced and DA respectively.
(a) Copy the diagram.
(b) **(i)** Explain why $XPKC$ an d $AHPY$ are cyclic quadrilaterals.
(ii) Hence show that $\angle XPK = \angle HPY$ and $\angle PKX = \angle PYH$.
(c) **(i)** Deduce that $\triangle XPK ||| \triangle HPY$.
(ii) Hence show that $PX.PY = PH.PK$ and $\frac{PX.PK}{PH.PY} = \frac{XK^2}{HY^2}$.

8. **(a)** Four distinct parabolas and three distinct circles are drawn on a large sheet of paper. Find the maximum number of points of intersection of these curves.
(b) A sequence of terms (u_n) is such that $u_1 = 1$ and $u_n = \sqrt{3u_{n-1}}$ for $n \geq 2$. Show that: **(i)** $u_n < 3$ for $n \geq 1$; **(ii)** $u_{n+1} > u_n$ for $n \geq 1$.
(c) Let $f(x) = \cos x - \left(1 - \frac{x^2}{2}\right)$
(i) Show that $f(x)$ is an even function.
(ii) Find expressions for $f'(x)$ and $f''(x)$.
(iii) Deduce that $f'(x) \geq 0$ for $x \geq 0$.
(iv) Hence show that $\cos x \geq 1 - \frac{x^2}{2}$ for all x.