THE KING'S SCHOOL

2005
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

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Mathematics Extension 2

Question	Complex Numbers	Functions	Integration	Conics	Mechanics	Harder Extension 1	Total
1		(d)	(a), (b), (c)				15
2	(b), (c), (d), (e)		(a)				15
3		(a), (b)(i)(ii)(iii)	(b)(iv)(v)				15
4		(b)	(a)	(c)			15
5			(b)		(a)		15
6		(a)				(b)	15
7						(a), (b)	15
8	(a)		(b)				15
Marks	20	24	37	9	9	21	120

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Express $\dfrac{2}{1-x^2}$ in partial fractions. **2**

(ii) Show that $\displaystyle\int_0^{\frac{1}{4}} \frac{2}{1-x^2}\,dx = \ln\left(\frac{5}{3}\right)$ **2**

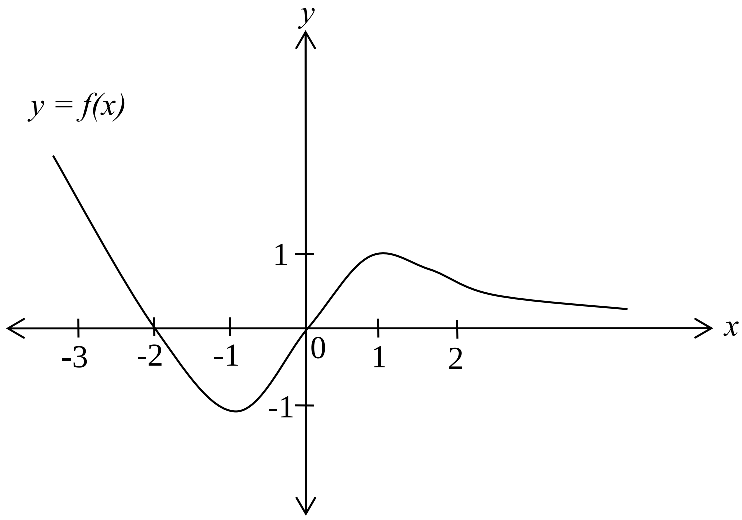
(iii) Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{2x}{1-x^4}\,dx$ **2**

(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{2}{1+\sin 2x+\cos 2x}\,dx$ **3**

(c) Use completion of square to prove that $\displaystyle\int_0^{1} \frac{4}{4x^2+4x+5}\,dx = \tan^{-1}\left(\frac{4}{7}\right)$ **3**

Question 1 is continued on the next page

(d)



On separate diagrams, sketch the graphs of:

(i) $y = \ln f(x)$ **2**

(ii) $y = e^{\ln f(x)}$ **1**

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Use integration by parts to show that

$$\int_0^1 (x-1)\, f'(x)\, dx = f(0) - \int_0^1 f(x)\, dx$$ **2**

(ii) Hence, or otherwise, evaluate $\displaystyle\int_0^1 \frac{x-1}{(x+1)^2}\, dx$ **2**

(b) Let $z = x + iy$, x, y real, where $\arg z = \dfrac{3\pi}{5}$

(i) Sketch the locus of z **1**

(ii) Find $\arg(-z)$ **1**

(c) Sketch the region in the complex plane where $|z - i| \le |z + 1|$ **2**

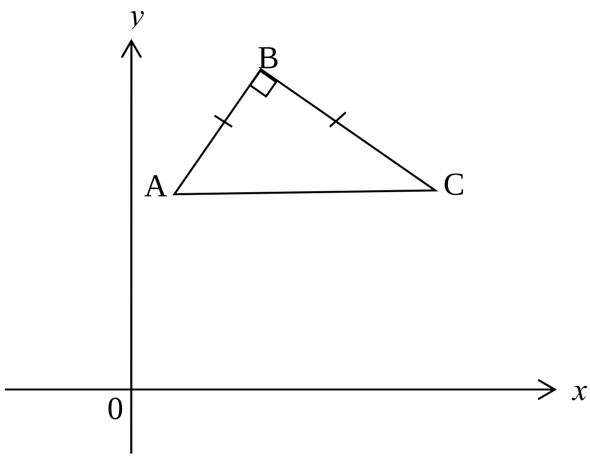
(d) $z = x + iy$, x, y real, is a complex number such that
$(z + \bar{z})^2 + (z - \bar{z})^2 = 4$

(i) Find the cartesian locus of z **2**

(ii) Sketch the locus of z in the complex plane showing any features necessary to indicate your diagram clearly. **2**

Question 2 is continued on the next page

(e)



In the Argand diagram, ΔABC is right-angled at B and isosceles.

A, B, C represent the complex numbers a, b, c respectively.

(i) Find the complex number $\overrightarrow{BA}$ in terms of a and b. **1**

(ii) Prove that $c = ai + b(1 - i)$ **2**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Sketch the parabola $y = \dfrac{1+x^2}{2}$ and use it to sketch the curve

$y = \dfrac{2}{1+x^2}$ on the same diagram. **2**

(ii) Hence, or otherwise, find the range of the function $y = \dfrac{2}{1+x^2} - 1$ **1**

(b) Consider the function $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

(i) By using (a), or otherwise, find the range of the function. **2**

(ii) Show that $\dfrac{d}{dx}\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \dfrac{2x}{(1+x^2)\sqrt{x^2}}$ and

give the simplest expressions for the derivative if

$(\alpha)\quad x > 0$ and $(\beta)\quad x < 0$ **3**

(iii) Sketch the curve $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ **2**

(iv) The region bounded by $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ and the line $y = \dfrac{\pi}{2}$ is

revolved about the y axis.

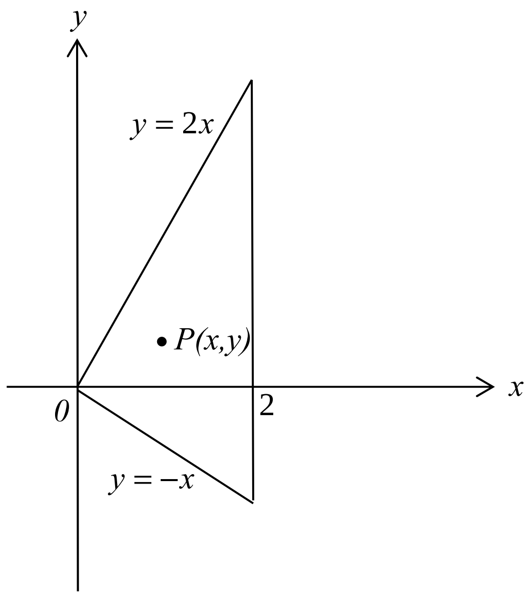
Show that the volume of the solid of revolution is given by

$$V = \pi \int_0^{\frac{\pi}{2}} \frac{1-\cos y}{1+\cos y}\, dy$$ **2**

(v) Find the volume V. **3**

End of Question 3

(a)



The base of a solid is the triangular region bounded by the lines $y = 2x$, $y = -x$ and $x = 2$.

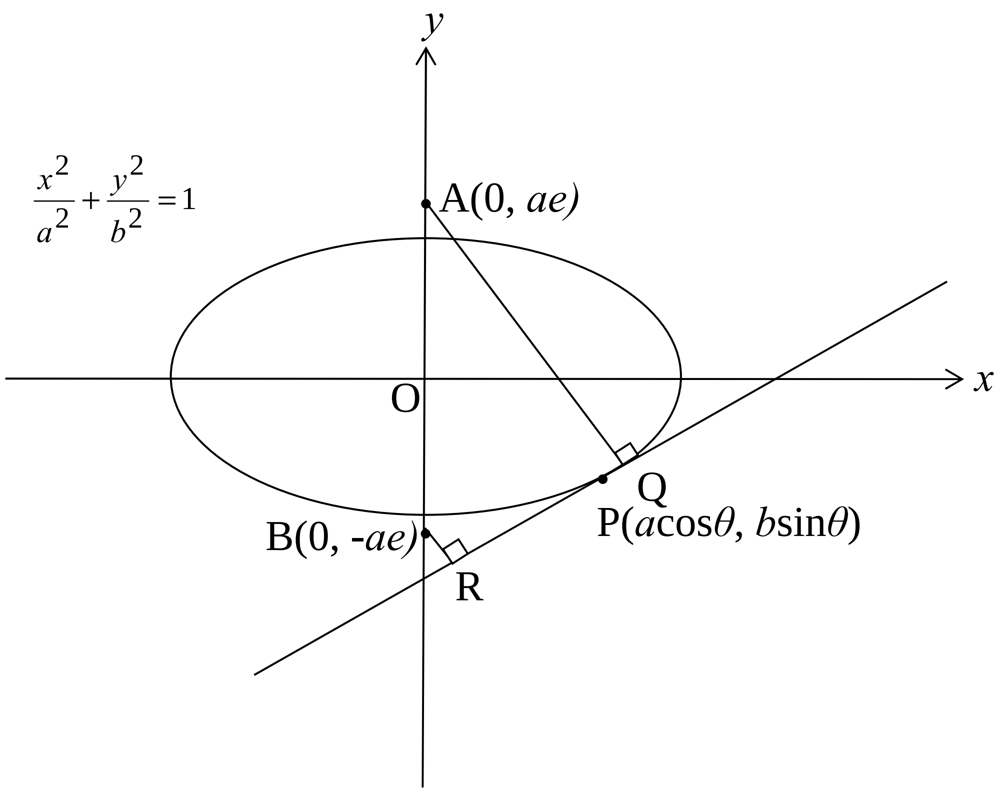
At each point $P(x, y)$ in the base the height of the solid is $4x^2 + x$

Find the volume of the solid. **4**

(b) If $xy^2 + 1 = x^2$, $y \neq 0$, show that $\dfrac{dy}{dx} = \dfrac{1}{y} - \dfrac{y}{2x}$ **2**

Question 4 is continued on the next page

(c)



P($a\cos\theta$, $b\sin\theta$) is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a>b>0$, where e is the eccentricity of the ellipse.

From A(0, ae) and B(0, -ae) perpendiculars are drawn to meet the tangent at P($a\cos\theta$, $b\sin\theta$) at Q and R, respectively.

(i) Prove that the equation of the tangent at P is $\frac{\cos\theta}{a}x+\frac{\sin\theta}{b}y=1$ **3**

(ii) Hence, or otherwise, show that the line $x\cos\alpha+y\sin\alpha=k$ is a tangent to the ellipse if $a^2\cos^2\alpha+b^2\sin^2\alpha=k^2$ **2**

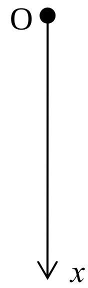
(iii) Hence, or otherwise, prove that $AQ^2+BR^2=2a^2$ **4**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A particle of mass m moving with speed v experiences air resistance mkv^2, where k is a positive constant. g is the constant acceleration due to gravity.

(i) The particle of mass m falls from rest from a point O.



Taking the positive x axis as vertically downward, show that $\ddot{x} = k(V^2 - v^2)$, where V is the terminal speed. **2**

(ii) Another particle of mass m is projected vertically upward from ground level with a speed V^2, where V is the terminal speed as in (i).

Prove that the particle will reach a maximum height of $\dfrac{1}{2k}\ln(1+V^2)$ **3**

(iii) Prove that the particle in (ii) will return to the ground with speed U where $U^{-2} = V^{-2} + V^{-4}$ **4**

(b) The ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ is revolved about the line $x = 4$.

(i) Use the method of cylindrical shells to show that the volume of the solid of revolution is given by

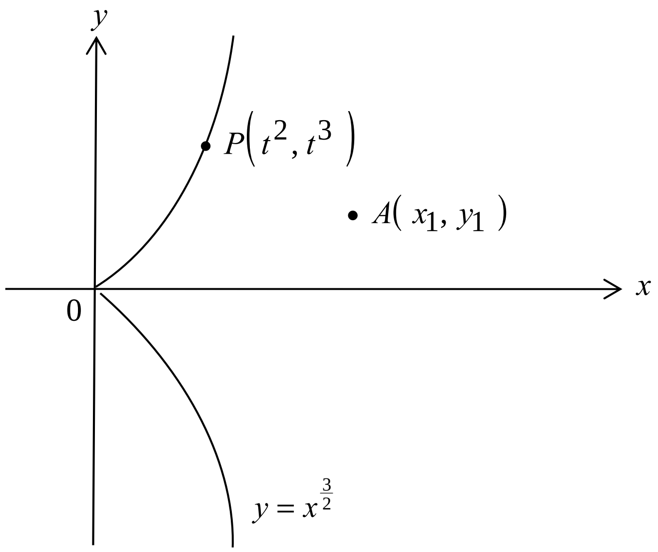
$$V = 8\sqrt{3}\,\pi \int_{-2}^{2} \sqrt{4-x^2}\,dx - 2\sqrt{3}\,\pi \int_{-2}^{2} x\sqrt{4-x^2}\,dx$$ **4**

(ii) Prove that the volume $V = 16\sqrt{3}\,\pi^2$ **2**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



$P(t^2, t^3)$ is any point in the curve $y = x^{\frac{3}{2}}$

(i) Show that the equation of the tangent at $P(t^2, t^3)$ is $3tx - 2y - t^3 = 0$ **2**

(ii) $A(x_1, y_1)$ is a point not on the curve $y = x^{\frac{3}{2}}$

Deduce that at most three tangents to the curve pass through A. **1**

(iii) If the tangents with parameters t_1, t_2, t_3 do pass through $A(x_1, y_1)$, show that

(α) $t_1^3 + t_2^3 + t_3^3 = -6y_1$ **2**

(β) $(t_1t_2)^2 + (t_2t_3)^2 + (t_3t_1)^2 = 9x_1^2$ **2**

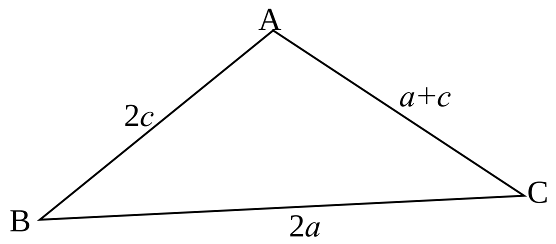
(iv) Find a cubic equation with roots $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}$ **2**

Question 6 is continued on the next page

(b) (i) Given that $\sin(X+Y)+\sin(X-Y)=2\sin X\cos Y$, show that

$$\sin A+\sin C=2\sin\frac{A+C}{2}\cos\frac{A-C}{2}$$ **1**

(ii) Consider ΔABC where



(α) Use the sine rule to show that $\sin A+\sin C=2\sin B$ **2**

(β) Deduce that $\sin\frac{B}{2}=\frac{1}{2}\cos\frac{A-C}{2}$ **3**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $f(n) = (n+1)^3 + (n+2)^3 + \ldots + (2n-1)^3 + (2n)^3$, $n = 1, 2, 3, \ldots$

(i) Show that $f(n+1) - f(n) = (2n+1)^3 + 7(n+1)^3$ **2**

(ii) Show that

$$(2n+1)^3 - \frac{2n+1}{4}(3n+1)(5n+3) = \frac{2n+1}{4}(n+1)^2$$ **1**

(iii) Use mathematical induction for integers $n = 1, 2, 3, \ldots$ to prove that

$$f(n) = (n+1)^3 + (n+2)^3 + \ldots + (2n)^3 = \frac{n^2}{4}(3n+1)(5n+3)$$ **4**

(iv) Given that $1^3 + 2^3 + \ldots + n^3 = \left[\frac{n}{2}(n+1)\right]^2$, prove that

$(n+1)^3 + (n+2)^3 + \ldots + (2n)^3 = \frac{n^2}{4}(3n+1)(5n+3)$ without induction. **2**

(b) (i) Show that $\dfrac{\binom{n}{k}}{n^k} = \dfrac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)}{k!}$, $2 \le k \le n$ **2**

(ii) Deduce that $\dfrac{\binom{n+1}{k}}{(n+1)^k} > \dfrac{\binom{n}{k}}{n^k}$, $2 \le k \le n$ **2**

(iii) Deduce that, if n is a positive integer, $\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$ **2**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the equation $z^7 - 1 = (z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$

(i) Show that $v = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ is a complex root of $z^7 - 1 = 0$ **1**

(ii) Show that the other five complex roots of $z^7 - 1 = 0$ are

$$v^k \text{ for } k = 2, 3, 4, 5, 6$$ **2**

(iii) Show that $\left(\overline{v^{7-k}}\right) = v^k$ for $k = 1, 2, \ldots, 6$

i.e. show that the conjugate of v^{7-k} is v^k **2**

(iv) Deduce that $v + v^2 + v^4$ and $v^3 + v^5 + v^6$ are conjugate complex numbers. **1**

(v) Deduce that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$ **3**

Question 8 is continued on the next page

(b) (i) Use a suitable substitution to show that

$$\int_0^{\frac{\pi}{2}} \cos x \sin^{n-1} x \, dx = \frac{1}{n}, \quad n = 1, 2, 3, \ldots$$ **1**

(ii) Show by integration that

$$\int x \sin x \, dx = -x \cos x + \sin x$$ **1**

(iii) Let $t_n = \displaystyle\int_0^{\frac{\pi}{2}} x \sin^n x \, dx, \quad n = 0, 1, 2, \ldots$

Use integration by parts to prove that

$$t_n = \frac{1}{n^2} + \frac{n-1}{n} t_{n-2}, \quad n = 2, \quad 3, \quad 4, \quad \ldots$$ **4**

End of Examination

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

TKS MATHEMATICS EXTENSION 2 TRIAL HSC 2005

Qu 1

(a) (i) Put $\frac{2}{1-x^2} = \frac{2}{(1-x)(1+x)} \equiv \frac{A}{1-x} + \frac{B}{1+x}$

$$: A(1+x) + B(1-x) \equiv 2$$

For $x=1$, $2A = 2$, $A = 1 \Rightarrow B = 1$

$$\therefore \frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$

(ii) From (i), $\int_0^{\frac{1}{4}} \frac{2}{1-x^2}\,dx = \int_0^{\frac{1}{4}} \frac{1}{1-x} + \frac{1}{1+x}\,dx$

$$= \left[\ln(1+x) - \ln(1-x)\right]_0^{\frac{1}{4}}$$

$$= \ln\frac{5}{4} - \ln\frac{3}{4} = \ln\left(\frac{5}{3}\right)$$

(iii) Put $u = x^2$; $x=0,\ u=0$

$\frac{du}{dx} = 2x$ $\quad x = \frac{1}{2},\ u = \frac{1}{4}$

$$\therefore I = \int_0^{\frac{1}{4}} \frac{du}{1-u^2} = \ln\left(\frac{5}{3}\right) \text{, from (ii)}$$

(b) let $t = \tan x$ $\quad x=0,\ t=0$

$\frac{dt}{dx} = \sec^2 x = 1+t^2$ $\quad x = \frac{\pi}{4},\ t=1$

$$\therefore I = \int_0^1 \frac{2\,dt}{(1+t^2)\left[1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}\right]}$$

$$= \int_0^1 \frac{2\,dt}{1+t^2+2t+1-t^2} = \int_0^1 \frac{1}{t+1}\,dt$$

$$= \left[\ln(t+1)\right]_0^1 = \ln 2$$

OR $I = \int_0^{\frac{\pi}{4}} \frac{2\,dx}{2\cos^2 x + 2\sin x\cos x} = \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1+\tan x}\,dx$

$$= \left[\ln(1+\tan x)\right]_0^{\pi/4}$$

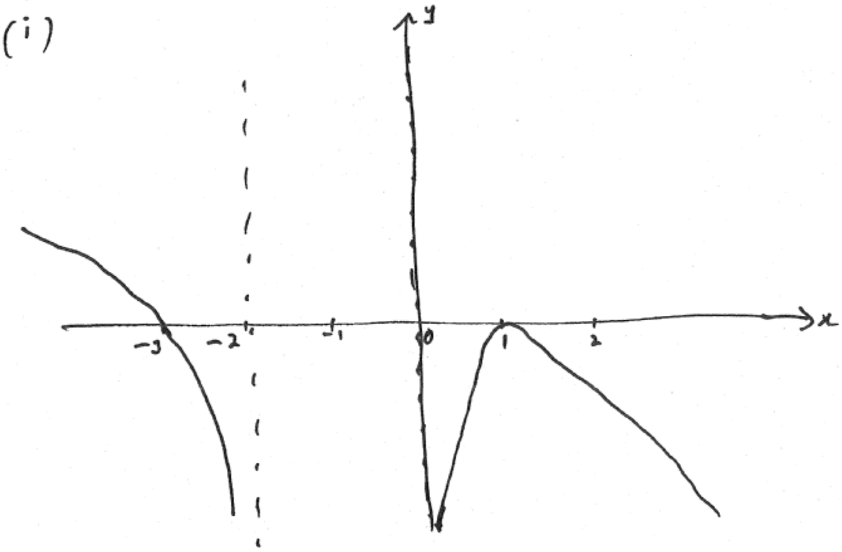
$$= \ln 2$$

(c) $$I = \int_0^1 \frac{4}{(2x+1)^2+4}\,dx = 4 \cdot \frac{1}{2}\left[\tan^{-1}\frac{2x+1}{2}\right]_0^1 \frac{1}{2}$$

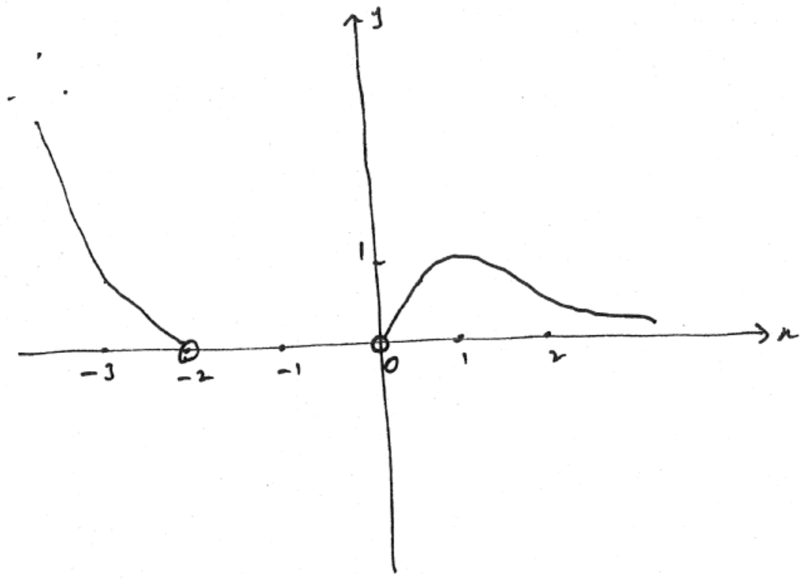
$$= \tan^{-1}\frac{3}{2} - \tan^{-1}\frac{1}{2}$$

$$= \tan^{-1}\left(\frac{\frac{3}{2}-\frac{1}{2}}{1+\frac{3}{2}\cdot\frac{1}{2}}\right) = \tan^{-1}\left(\frac{4}{7}\right)$$

(d) (i)



(ii) $y = e^{\ln f(x)} = f(x)$ if $f(x) > 0$



Qu 2

(a) (i) put $u = x-1 \qquad \frac{dv}{dx} = f'(x)$

$\therefore \frac{du}{dx} = 1, \qquad v = f(x)$

$$\therefore \int_0^1 (x-1)f'(x)\,dx = \Big[(x-1)f(x)\Big]_0^1 - \int_0^1 f(x)\,dx$$

$$= 0 - (-f(0)) - \int_0^1 f(x)\,dx$$

$$= f(0) - \int_0^1 f(x)\,dx$$

(ii) Hence .... $\int_0^1 \frac{x-1}{(x+1)^2}\,dx \Rightarrow f'(x) = \frac{1}{(x+1)^2}, \; f(x) = -\frac{1}{x+1}$

$$\therefore I = -1 + \int_0^1 \frac{1}{x+1}\,dx = -1 + \Big[\ln(x+1)\Big]_0^1$$

$$= \ln 2 - 1$$

or, Otherwise ....

$$\int_0^1 \frac{x-1}{(x+1)^2}\,dx = \int_0^1 \frac{x+1-2}{(x+1)^2}\,dx$$

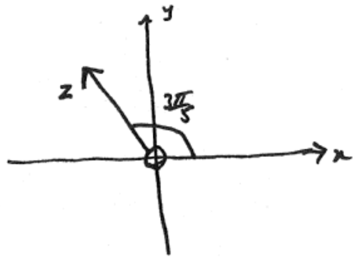
$$= \int_0^1 \frac{1}{x+1} - \frac{2}{(x+1)^2}\,dx$$

$$= \Big[\ln(x+1) + \frac{2}{x+1}\Big]_0^1$$

$$= \ln 2 + 1 - (0 + 2)$$

$$= \ln 2 - 1$$

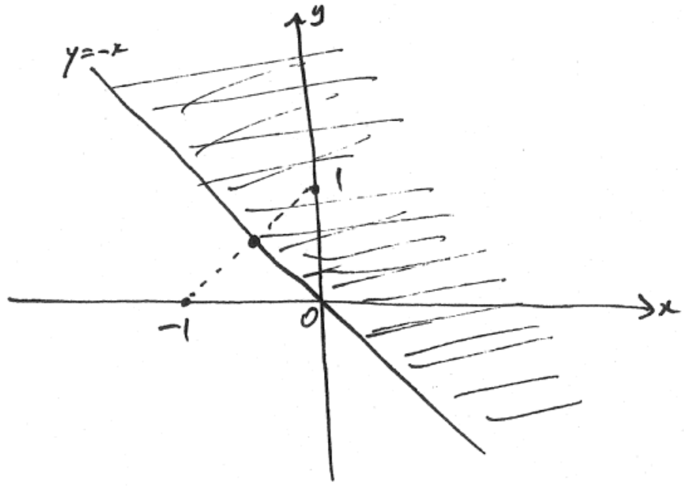
(b) (i)



(ii) $\arg(-z) = \pi + \frac{3\pi}{5}$

$$= \frac{8\pi}{5} \text{ or } -\frac{2\pi}{5}$$

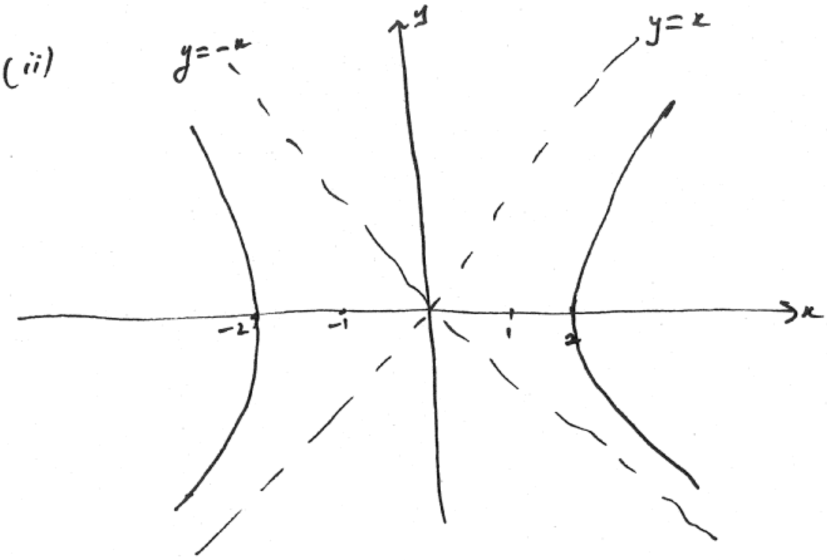
(c)



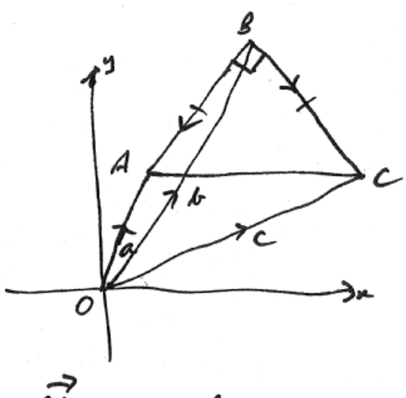
(d) (i) $\therefore\ (2x)^2 + (2iy)^2 = 4$

$\Rightarrow x^2 - y^2 = 4$ [rectangular hyperbola]

(ii)



(e) (i)



$\vec{BA} = a - b$

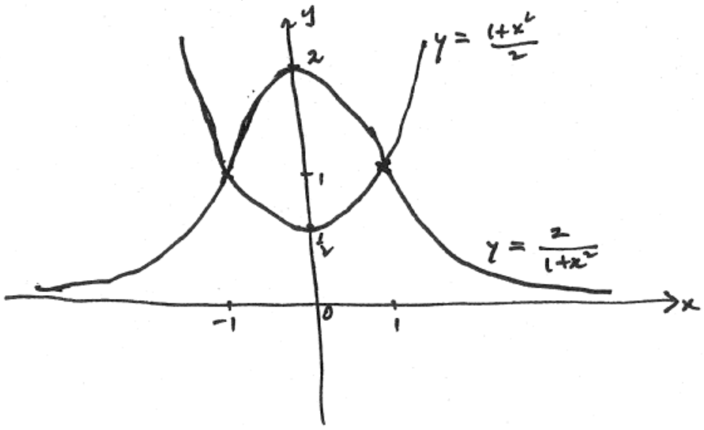
(ii) $\vec{BC} = i\,\vec{BA}$

$\therefore\ c - b = i(a - b)$

or $c = ai + b(1 - i)$

Qn 3

(a) (i)



(ii) From (i), $\quad 0 < \dfrac{2}{1+x^2} \le 2$

$$\therefore \quad -1 < \frac{2}{1+x^2} - 1 \le 1$$

ie range is $\quad -1 < y \le 1$

(b) (i) $\quad \dfrac{2}{1+x^2} - 1 = \dfrac{2-(1+x^2)}{1+x^2} = \dfrac{1-x^2}{1+x^2}$

$\therefore$ from (a)(ii), range is $\quad 0 \le y < \pi$

(ii) $$\frac{d}{dx}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = -\frac{1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \cdot -2(1+x^2)^{-2} \cdot 2x$$

$$= \frac{4x}{\sqrt{(1+x^2)^2-(1-x^2)^2}\ (1+x^2)}$$

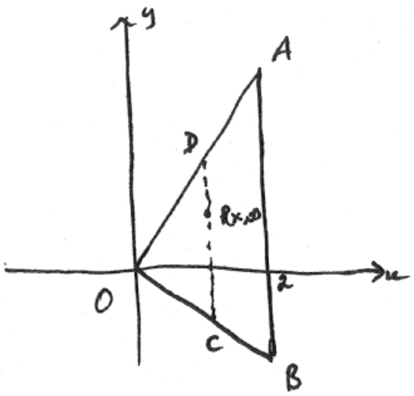
$$= \frac{4x}{(1+x^2)\sqrt{4x^2}} = \frac{2x}{(1+x^2)\sqrt{x^2}}$$

$\therefore$ (α), if $x > 0$, $\quad \dfrac{dy}{dx} = \dfrac{2}{1+x^2}$

& (β), if $x < 0$, $\quad \dfrac{dy}{dx} = \dfrac{-2}{1+x^2}$

Qu 14

(a)



$\Rightarrow CD = 2x + x = 3x$

$\therefore \delta V \approx 3x(4x^2 + x)\,\delta x$

$$\therefore \quad V = \int_0^2 12x^3 + 3x^2 \, dx$$

$$= \left[3x^4 + x^3\right]_0^2 = 56 \ u^2$$

(b) $\therefore \ x\, 2y \dfrac{dy}{dx} + y^2 = 2x$

$$\Rightarrow \frac{dy}{dx} = \frac{2x - y^2}{2xy} = \frac{1}{y} - \frac{y}{2x}$$

(c)(i) $\dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{b^2 x}{a^2 y}$

$$= -\frac{b^2}{a^2} \cdot \frac{a\cos\theta}{b\sin\theta} \quad \text{at } P$$

$$= -\frac{b\cos\theta}{a\sin\theta}$$

$\therefore$ tangent at P is $y - b\sin\theta = -\dfrac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

ie $\dfrac{\sin\theta}{b} y - \sin^2\theta = -\dfrac{\cos\theta}{a} x + \cos^2\theta$

ie $\dfrac{\cos\theta}{a} x + \dfrac{\sin\theta}{b} y = \cos^2\theta + \sin^2\theta = 1$

(ii) Rewrite as $\frac{\cos\alpha}{k}x + \frac{\sin\alpha}{k}y = 1$

Then, from (ii), we need $\frac{\cos\alpha}{k} = \frac{\cos\theta}{a}$ and $\frac{\sin\alpha}{k} = \frac{\sin\theta}{b}$

$$\Rightarrow (a\cos\alpha)^2 + (b\sin\alpha)^2 = k^2\cos^2\theta + k^2\sin^2\theta$$
$$= k^2(\cos^2\theta + \sin^2\theta)$$

i.e. $a^2\cos^2\alpha + b^2\sin^2\alpha = k^2$

(iii)

From (ii) & (iii),

$$AQ^2 + BR^2 = \frac{(ae\sin\alpha - k)^2 + (ae\sin\alpha + k)^2}{\cos^2\alpha + \sin^2\alpha}$$

$$= 2\left(a^2e^2\sin^2\alpha + k^2\right)$$

$$= 2\left((a^2 - b^2)\sin^2\alpha + a^2\cos^2\alpha + b^2\sin^2\alpha\right), \text{ from (i), (iii)}$$

$$= 2\left(a^2(\sin^2\alpha + \cos^2\alpha)\right)$$

$$= 2a^2$$

Qu 5

(a) (i) $m\ddot{x} = mg - mkv^2$

$$\Rightarrow \ddot{x} = g - kv^2 \Rightarrow g - kV^2 = 0 \quad \text{or} \quad V^2 = \frac{g}{k}$$

$$\therefore \ddot{x} = k\left(\frac{g}{k} - v^2\right) = k\left(V^2 - v^2\right)$$

(ii) $m\ddot{x} = -mg - mkv^2$

+ve ↑ x=0

$$\therefore \ddot{x} = -k\left(V^2 + v^2\right)$$

$$\therefore v\frac{dv}{dx} = -k\left(V^2 + v^2\right)$$

$$\text{or} \quad \frac{dx}{dv} = -\frac{1}{k} \cdot \frac{v}{V^2+v^2}$$

$$\therefore \text{max ht} = -\frac{1}{k}\int_{V^2}^{0} \frac{v}{V^2+v^2}\,dv$$

$$= -\frac{1}{2k}\left[\ln\left(V^2+v^2\right)\right]_{V^2}^{0}$$

$$= \frac{1}{2k}\left(\ln\left(V^2+V^4\right) - \ln V^2\right)$$

$$= \frac{1}{2k}\ln\left(1+V^2\right)$$

(iii) $v\frac{dv}{dx} = k\left(V^2 - v^2\right) \Rightarrow \frac{dx}{dv} = \frac{1}{k}\cdot\frac{v}{V^2-v^2}$

$$\therefore \text{from (ii)}, \quad \frac{1}{2k}\ln\left(1+V^2\right) = \frac{1}{k}\int_0^U \frac{v}{V^2-v^2}\,dx$$

$$= -\frac{1}{2k}\left[\ln\left(V^2-v^2\right)\right]_0^U$$

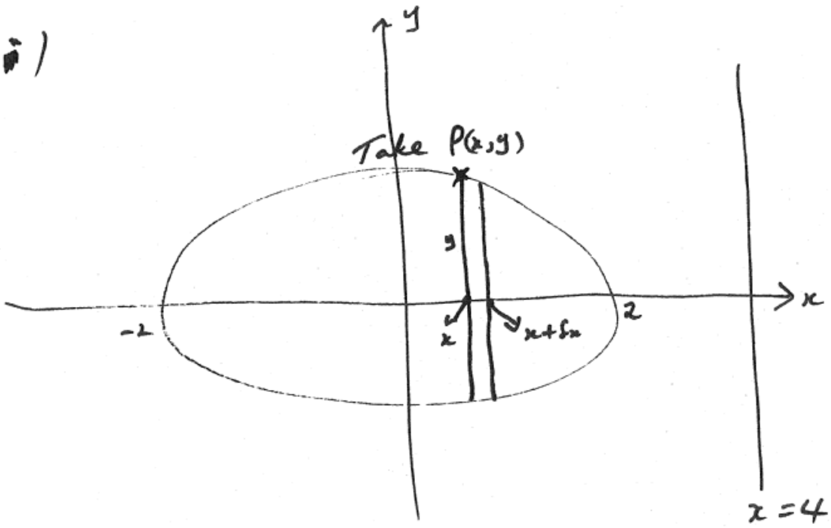
$$= \frac{1}{2k}\left(\ln V^2 - \ln\left(V^2-U^2\right)\right) = \frac{1}{2k}\ln\left(\frac{V^2}{V^2-U^2}\right)$$

$$\Rightarrow \frac{V^2}{V^2-U^2} = 1+V^2 \quad \text{or} \quad V^2 - U^2 = \frac{V^2}{1+V^2}$$

$$\text{ie } U^2 = V^2 - \frac{V^2}{1+V^2} = \frac{V^4}{1+V^2}$$

$$\therefore U^{-2} = \frac{1+V^2}{V^4} = V^{-2} + V^{-4}$$

b (ii)



$$\therefore \delta V \approx \pi \left[(4-x)^2 - (4-x-\delta x)^2 \right] 2y$$

$$\approx 2\pi \left[2(4-x)\,\delta x \right] y \quad \text{, ignoring } \delta x^2 \text{ term}$$

$$= 4\pi\,(4-x)\,y\,\delta x \qquad : \; y^2 = 3\left(1-\frac{x^2}{4}\right) = \frac{3}{4}(4-x^2)$$

$$\therefore V = 4\pi \int_{-2}^{2} (4-x)\,\frac{\sqrt{3}}{2}\sqrt{4-x^2}\,dx$$

$$= 2\pi\sqrt{3} \int_{-2}^{2} (4-x)\sqrt{4-x^2}\,dx$$

$$= 8\sqrt{3}\pi \int_{-2}^{2} \sqrt{4-x^2}\,dx - 2\sqrt{3}\pi \int_{-2}^{2} x\sqrt{4-x^2}\,dx$$

(iii) $V = 8\sqrt{3}\pi \int_{-2}^{2} \sqrt{4-x^2}\,dx$ since $x\sqrt{4-x^2}$ is an odd function

$$= 8\sqrt{3}\pi \cdot \frac{1}{2}\pi \cdot 2^2 \; u^3 \qquad [\text{semi-circle}]$$

$$= 16\sqrt{3}\pi^2 \; u^3$$

Qn 6

(a) (i) $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}t$ at P

$\therefore$ tangent at P is $y - t^3 = \frac{3}{2}t(x - t^2)$

or $3tx - 2y + 2t^3 - 3t^3 = 0$

i.e. $3tx - 2y - t^3 = 0$

(ii) The tangent at $P(t^2, t^3)$ is a cubic equation in t

$\Rightarrow$ at most 3 values for t from $3tx_1 - 2y_1 - t^3 = 0$

$\Rightarrow$ at most 3 tangents

(iii) Now, $t^3 - 3x_1 t + 2y_1 = 0$ has roots t_1, t_2, t_3

$\therefore$ (α) $\Sigma t_1^3 - 3x_1 \Sigma t_1 + 3(2y_1) = 0$ where $\Sigma t_1 = 0$

$\therefore \Sigma t_1^3 = -6y_1$

& (β) $\Sigma (t_1 t_2)^2 = (t_1 t_2 + t_2 t_3 + t_3 t_1)^2 - 2(t_1 t_2 t_2 t_3 + t_2 t_3 t_3 t_1 + t_3 t_1 t_1 t_2)$

$= (-3x_1)^2 - 2t_1 t_2 t_3 (t_1 + t_2 + t_3)$

$= 9x_1^2$ since $\Sigma t_1 = 0$

(iv) It is $\left(\frac{1}{t}\right)^3 - 3x_1\left(\frac{1}{t}\right) + 2y_1 = 0$

i.e. $2y_1 t^3 - 3x_1 t^2 + 1 = 0$

(b) (i) put $X+Y=A$ $\Rightarrow$ $2X = A+C$

$X-Y=C$ $\quad$ $2Y = A-C$

$\therefore \sin A + \sin C = 2\sin\frac{A+C}{2}\cos\frac{A-C}{2}$ from data

(ii) (α) $\frac{\sin A}{2a} = \frac{\sin B}{a+c} = \frac{\sin C}{2c}$

$\therefore \sin A + \sin C = \left(\frac{2a}{a+c} + \frac{2c}{a+c}\right)\sin B$

$= \frac{2(a+c)}{a+c}\sin B = 2\sin B$

(β) $2\sin B = 2\sin\frac{A+C}{2}\cos\frac{A-C}{2}$ from (i)

$\Rightarrow 2\sin\frac{B}{2}\cos\frac{B}{2} = \sin\frac{A+C}{2}\cos\frac{A-C}{2}$

$= \sin\left(\frac{\pi}{2}-\frac{B}{2}\right)\cos\frac{A-C}{2}$ since $A+B+C=\pi$

$= \cos\frac{B}{2}\cos\frac{A-C}{2}$

$\therefore \sin\frac{B}{2} = \frac{1}{2}\cos\frac{A-C}{2}$

Qn 7

(a) (i) $f(n+1) - f(n) = (n+2)^3 + \cdots + (2n)^3 + (2n+1)^3 + (2n+2)^3 - \left((n+1)^3 + \cdots + (2n)^3\right)$

$$= (2n+1)^3 + (2n+2)^3 - (n+1)^3$$
$$= (2n+1)^3 + 8(n+1)^3 - (n+1)^3$$
$$= (2n+1)^3 + 7(n+1)^3$$

(ii) $LS = \frac{2n+1}{4}\left(4(2n+1)^2 - (3n+1)(5n+3)\right)$

$$= \frac{2n+1}{4}\left(16n^2 + 16n + 4 - 15n^2 - 14n - 3\right)$$
$$= \frac{2n+1}{4}\left(n^2 + 2n + 1\right) = \frac{2n+1}{4}(n+1)^2$$

(iii) $f(1) = 2^3 = 8$ and $\frac{1^2}{4}(4)(8) = 8$

$\therefore$ Assume $f(n) = (n+1)^3 + \cdots + (2n)^3 = \frac{n^2}{4}(3n+1)(5n+3)$ for some integer $n \geq 1$

Then, $f(n+1) = f(n) + (2n+1)^3 + 7(n+1)^3$ from (i)

$$= \frac{n^2}{4}(3n+1)(5n+3) + (2n+1)^3 + 7(n+1)^3, \text{ using the assumption}$$

$$= \frac{(n+1)^2}{4}(3n+1)(5n+3) - \frac{2n+1}{4}(3n+1)(5n+3) + (2n+1)^3 + 7(n+1)^3$$

$$= \frac{(n+1)^2}{4}(3n+1)(5n+3) + \frac{2n+1}{4}(n+1)^2 + 7(n+1)^3, \text{ from (ii)}$$

$$= \frac{(n+1)^2}{4}\left[(3n+1)(5n+3) + 2n+1 + 28(n+1)\right]$$

$$= \frac{(n+1)^2}{4}\left(15n^2 + 44n + 32\right)$$

$$= \frac{(n+1)^2}{4}(3n+4)(5n+8) = \frac{(n+1)^2}{4}\left(3(n+1)+1\right)\left(5(n+1)+3\right)$$

$\therefore$ if the result is true for n it is also true for $n+1$.

But it is correct for $n=1$

$\therefore$ by induction, $(n+1)^3 + \cdots + (2n)^3 = \frac{n^2}{4}(3n+1)(5n+3) \quad \forall n \geq 1$

(iv) $(n+1)^3 + \cdots + (2n)^3 = 1^3 + \cdots + n^3 + \cdots + (2n)^3 - (1^3 + \cdots + n^3)$

$$= \left(\frac{2n}{2}(2n+1)\right)^2 - \left(\frac{n}{2}(n+1)\right)^2$$

$$= \frac{n^2}{4}\left(4(2n+1)^2 - (n+1)^2\right)$$

$$= \frac{n^2}{4}(4n+2-n-1)(4n+2+n+1)$$

$$= \frac{n^2}{4}(3n+1)(5n+3)$$

(b) (i) $$\frac{\binom{n}{k}}{n^k} = \frac{n!}{(n-k)!\,k!\,n^k}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!\,n^k}$$

$$= \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}{k!}$$

(ii) From (i), $$\frac{\binom{n+1}{k}}{(n+1)^k} = \frac{\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right)}{k!}$$

$$> \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}{k!} \quad \text{since } \frac{1}{n+1} < \frac{1}{n}$$

$$= \frac{\binom{n}{k}}{n^k}$$

(iii) $$\left(1+\frac{1}{n+1}\right)^{n+1} = 1 + (n+1)\frac{1}{n+1} + \frac{\binom{n+1}{2}}{(n+1)^2} + \cdots + \frac{\binom{n+1}{k}}{(n+1)^k} + \cdots + \frac{\binom{n+1}{n}}{(n+1)^n} + \frac{1}{(n+1)^{n+1}}$$

$$> \left(1 + 1 + \frac{\binom{n}{2}}{n^2} + \cdots + \frac{\binom{n}{k}}{n^k} + \cdots + \frac{\binom{n}{n}}{n^n}\right) + \frac{1}{(n+1)^{n+1}}$$

$$> 1 + 1 + \cdots + \frac{\binom{n}{k}}{n^k} + \cdots + \frac{\binom{n}{n}}{n^n}$$

$$= \left(1+\frac{1}{n}\right)^n$$

Qu 8

(a) (i) $v^7 - 1 = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7 - 1$

$= \cos 2\pi + i\sin 2\pi - 1 = 1 - 1 = 0$

ie v is a root of $z^7 - 1 = 0$

(ii) $v^k = \cos\frac{2\pi k}{7} + i\sin\frac{2\pi k}{7}$

$\therefore (v^k)^7 = \cos 2\pi k + i\sin 2\pi k$

$= 1$ if $k = 2, 3, \ldots, 6$

$\Rightarrow v^2, v^3, \ldots, v^6$ are also roots of $z^7 = 1$

(iii) $\left(\overline{v^{7-k}}\right) = \cos\frac{2\pi}{7}(7-k) - i\sin\frac{2\pi}{7}(7-k)$

$= \cos\left(-\frac{2\pi k}{7}\right) - i\sin\left(-\frac{2\pi k}{7}\right)$

$= \cos\frac{2\pi k}{7} + i\sin\frac{2\pi k}{7} = v^k$

(iv) $\overline{v + v^2 + v^4} = \bar{v} + \bar{v}^2 + \bar{v}^4$

$= v^6 + v^5 + v^3$ from (iii)

ie $v + v^2 + v^4$ and $v^3 + v^5 + v^6$ are conjugates

(v) From (iv), $(v + v^2 + v^4) + (v^3 + v^5 + v^6)$

$= 2\,\mathrm{Re}\,(v + v^2 + v^4)$

$= 2\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}\right)$

$= 2\left(\cos\frac{2\pi}{7} - \cos\frac{3\pi}{7} - \cos\frac{\pi}{7}\right)$

$= -1$ since $v^6 + v^5 + \cdots + v^2 + v + 1 = 0$

$\therefore \cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$

(b) (i) put $u = \sin x$; $x=0$, $u=0$

$\frac{du}{dx} = \cos x$; $x = \frac{\pi}{2}$, $u=1$

$$\therefore \quad I = \int_0^1 u^{n-1}\,du = \left[\frac{u^n}{n}\right]_0^1 = \frac{1}{n}$$

(ii) put $u = x$, $\frac{dv}{dx} = \sin x$

$$\therefore \frac{du}{dx} = 1, \quad v = -\cos x$$

$$\therefore \int x\sin x\,dx = x(-\cos x) - \int (-\cos x)\,dx$$

$$= -x\cos x + \sin x$$

(iii) As suggested from (ii),

put $u = \sin^{n-1} x$, $\frac{dv}{dx} = x\sin x$

$$\therefore \frac{du}{dx} = (n-1)\sin^{n-2} x \cos x \quad , \quad v = -x\cos x + \sin x$$

$$\therefore t_n = \left[\sin^{n-1} x\,(-x\cos x + \sin x)\right]_0^{\frac{\pi}{2}} - (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos x\,(-x\cos x + \sin x)\,dx$$

$$= 1 - (n-1)\int_0^{\frac{\pi}{2}} -x\sin^{n-2} x\,(1-\sin^2 x) + \cos x \sin^{n-1} x \quad dx$$

$$= 1 + (n-1)\,t_{n-2} - (n-1)\,t_n - (n-1)\int_0^{\frac{\pi}{2}} \cos x \sin^{n-1} x \quad dx$$

$$= 1 + (n-1)\,t_{n-2} - (n-1)\,t_n - \frac{n-1}{n} \quad , \text{from (i)}$$

$$\therefore \quad n t_n = 1 - 1 + \frac{1}{n} + (n-1)\,t_{n-2}$$

$$\text{or,} \quad t_n = \frac{1}{n^2} + \frac{n-1}{n}\,t_{n-2} \quad , n = 2, 3, \ldots$$