Newington College

4 unit mathematics

Trial HSC Examination 1989

1. (a) Solve the equation $\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta = 0$ for $0° < \theta < 360°$
(b) An urn contains 3 balls marked "6" and 5 balls marked "4". A succession of 4 drawings of a ball from the urn is made and after each drawing the ball is replaced and the balls remixed.
(i) What is the probability of drawing two balls marked "6" and two balls marked "4" (in any order)
(ii) Prove that the probability that the sum of the numbers on the four balls drawn should be greater than 20 is between 15 and 16 percent.
(c) In a triangle ABC the altitudes AD, BE and CF meet in a point H. The altitude AD also intersects the circumcircle of triangle ABC in X
(i) Explain why $HDCE$ and $AEDB$ are cyclic quadrilaterals
(ii) Prove that the triangles BDH and BDX are congruent.

2. (a) Find the following integrals:
(i) $\int \sqrt{\frac{3+x}{3-x}}\, dx$ **(ii)** $\int \sin^{-1} x\, dx$
(b) Evaluate the following
(i) $\int_0^{\frac{\pi}{8}} \sin 5\theta \cos 5\theta\, d\theta$ **(ii)** $\int_0^2 \frac{dx}{(4+x^2)^{\frac{3}{2}}}$ **(iii)** $\int_0^1 \frac{x}{(x+1)(x+3)^2}$

3. (a) (i) Show that the equation of the tangent at $P(x_1, y_1)$ on the hyperbola $x^2 - y^2 = 1$ is $xx_1 - yy_1 = 1$ and that the line perpendicular to this tangent and which passes through the origin O is $y = \frac{-y_1 x}{x_1}$.
(ii) The two lines in **(i)** meet in $T(X, Y)$. Find the co-ordinates of T in terms of x_1 and y_1 and show that the equation of the locus of T as P moves on the hyperbola is $(X^2 - Y^2) = (X^2 + Y^2)^2$.
(b) A point $Q_1(x_1, y_1)$ moves on the line $y = x\tan a$ and another point $Q_2(x_2, y_2)$ moves on the line $y = -x\tan a$. Express the co-ordinates of the modpoint P of Q_1Q_2 in terms of x_1, x_2 and a. Show that if Q_1 and Q_2 move in such a way that the length Q_1Q_2 remains equal to a constant $2k$, then the locus of P is an ellipse.

4. (a) If the equation $ax^3 + bx^2 + cx + d = 0$ has a pair of reciprocal roots, α and $\frac{1}{\alpha}$, prove that $a^2 - d^2 = ac - bd$. Verify that the condition is satisfied is satisfied for the equation $6x^3 + 11x^2 - 24x - 9 = 0$ and solve the equation.
(b) Given that one root of the equation $z^4 - 4z^3 + 12z^2 + 4z - 13 = 0$ is $2 - 3i$, find the other three roots.
(c) If α, β, γ are the roots of the polynomial $x^3 - x^2 + 5x - 3$ in the field of complex numbers find the values of $\alpha^2 + \beta^2 + \gamma^2$ and $\alpha^3 + \beta^3 + \gamma^3$.

5. Consider the curve $y = x^2(1 - x^2)$
(a) Find the x-intercepts and verify that the curve has a maximum turning point at $(\frac{1}{\sqrt{2}}, \frac{1}{4})$. Sketch the graphs for $x \geq 0$.
(b) The area between the curve and the x-axis in the first quadrant is now rotated around the y-axis. Find the volume generated using the method of "slices".
(c) Verify your answer using the method of "shells".
(d) Find also the volume by direct integration.

6. **(a)** Sketch the following on the Argand diagram and describe in geometric terms the locus represented by: **(i)** $|\frac{z-4}{z+3i}| = 1$ **(ii)** $\arg(z + 1 - i) = \frac{\pi}{3}$
(b) **(i)** State de Moivre's Theorem
(ii) Hence, prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$
(iii) Solve the equation $\cos 5\theta = 1$ for $0 \leq \theta \leq \pi$ and hence show that the roots of the equation $16x^5 - 20x^3 + 5x - 1 = 0$ are $x = \cos\frac{2k\pi}{5}$ for $k = 0, 1, 2, 3, 4$.
(iv) Hence prove that $\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}$ and $\cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = \frac{1}{2}$
(c) Solve the equation $z^6 - 1 = 0$, giving the roots in the form $a + ib$. Show these roots on an Argand diagram.

7. **(a)** Prove that when a body is moving in a circle of radius r the normal component of acceleration is $r\omega^2$ (where ω is the angular velocity)
(b) A particle of mass 2kg at the end of a string $2\frac{1}{2}$ metres long is suspended from a point vertically above the highest point of a smooth sphere of radius $2\frac{1}{2}$ metres. It describes a horizontal circle of radius $1\frac{1}{2}$ metres on the surface of the sphere. If the angular velocity is 2 radians/second, find the tension in the string and the force exerted on the sphere.
(c) What least angular velocity will ensure there is no force on the sphere?

8. **(a)** The solution of the equation $x^4 + x^3 + x^2 + x = 5$ is known to be $x = 1 + h$ where h is small. Neglecting powers of h above the first and using the binomial theorem show that the solution of the equation is $x = 1.1$ approximately.
(b) A square $A_1B_1C_1D_1$ is of side $2a$. The midpoints of the sides are joined to form a second square $A_2B_2C_2D_2$, the midpoints of the sides of this square are joined to form a third square $A_3B_3C_3D_3$ and so on. Prove that the lengths of the sides A_1B_1, A_2B_2, A_3B_3, ... form a geometric progression and determine the length of the side A_nB_n of the nth square. Show that the sum of the areas of the first six squares is $\frac{63}{32}$ times as large as the first square $A_1B_1C_1D_1$.