ext 2 trial Pymble 2001

Question 1

a) $\int \frac{\ln x}{x}\,dx = \int \ln x \; d\ln x$

$= \frac{(\ln x)^2}{2} + C$

b) $\int_0^1 t e^{-t}\,dt = \int_0^1 -t\; de^{-t}$

$= -\left[t e^{-t}\right]_0^1 + \int_0^1 e^{-t}\, d(t)$

$= -\frac{1}{e} + \left[-e^{-t}\right]_1^0$

$= -\frac{1}{e} + 1 - \frac{1}{e}$

$= 1 - \frac{2}{e}$

alternative method

Let $u = t \quad \frac{dv}{dt} = e^{-t}$

$\frac{du}{dt} = 1 \quad v = -e^{-t}$

c) (i) $\frac{1}{x(1+x^2)} = \frac{a}{x} + \frac{bx+c}{1+x^2}$

$1 = a(1+x^2) + bx^2 + cx$

$a+b = 0$

$c = 0$

$a = 1$

$\therefore b = -1$

$\frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$

(ii) $\int \frac{dx}{x(1+x^2)} = \ln|x| - \frac{1}{2}\ln(1+x^2) + C, \; x \neq 0$

d) $\int_0^4 \frac{x}{\sqrt{x+4}}\,dx = \int_0^4 \frac{x+4}{\sqrt{x+4}} - \frac{4}{\sqrt{x+4}}\,dx$

$= \int_0^4 (x+4)^{\frac{1}{2}}\,dx - \frac{4}{\sqrt{x+4}}\,dx$

$= \left[\frac{2}{3}(x+4)^{3/2}\right]_0^4 - \left[\frac{4(x+4)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^4$

$= \frac{2}{3}\left(8^{3/2} - 4^{3/2}\right) - 8\left(8^{\frac{1}{2}} - 4^{\frac{1}{2}}\right)$

$= \frac{16}{3}(2-\sqrt{2})$

alternative method

let $u = x+4$

$du = dx$

when $x = 0, u = 4$

$x = 4, u = 8$

$I = \int_4^8 \frac{u-4}{\sqrt{u}}\,du$

$= \int_4^8 \sqrt{u} - \frac{4}{\sqrt{u}}\,du$

$= \left[\frac{2}{3}u^{3/2} - 8u^{\frac{1}{2}}\right]_4^8$

$= \left[\frac{2}{3}(2\sqrt{2})^3 - 8(2\sqrt{2})\right] - \left[\frac{2}{3}(2)^3 - 8(2)\right]$

$= \frac{16}{3}(2-\sqrt{2})$

Question 1 e)

(i) $I_n = \int_0^{\pi/2} x^n \cos x\,dx$

$= \int_0^{\pi/2} x^n\, d\sin x$

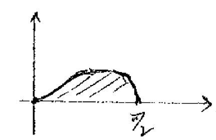
$= \left[x^n \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} \sin x \, n x^{n-1}\,dx$

$= \left(\frac{\pi}{2}\right)^n + n\int_0^{\pi/2} x^{n-1}\, d\cos x$

$= \left(\frac{\pi}{2}\right)^n + \underbrace{n\left[x^{n-1}\cdot\cos x\right]_0^{\pi/2}}_{=0} - n\int_0^{\pi/2} \cos x \cdot (n-1)x^{n-2}\,dx$

$= \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$

(ii)



$A = I_4$

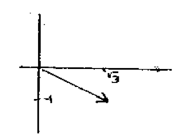
$= \left(\frac{\pi}{2}\right)^4 - 4\times 3\left[\left(\frac{\pi}{2}\right)^2 - 2\int_0^{\pi/2}\cos x\,dx\right]$

$= \left(\frac{\pi}{2}\right)^4 - 3\pi^2 + 24\left[\sin x\right]_0^{\pi/2}$

$= \left(\frac{\pi}{2}\right)^4 - 3\pi^2 + 24.$

Question 2 (a)

$\sqrt{3} - i = 2\,\text{cis}\left(-\frac{\pi}{6}\right)$



$(\sqrt{3}-i)^9 = 2^9\,\text{cis}\left(-\frac{9\pi}{6}\right)$

$= 512\,\text{cis}\,-\frac{3\pi}{2}$

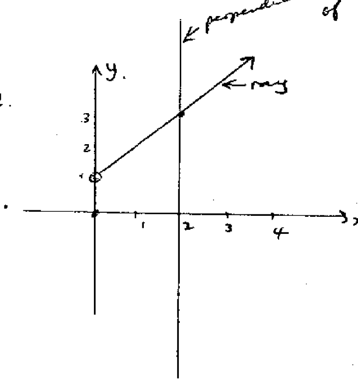
$= 512\,\text{cis}\,\frac{\pi}{2}$

$= 512i$

$= ci \quad$ where $c = 512.$

b)

$a+ib$ that satisfies simultaneously is $(2+3i)$.



Question 2 (c)

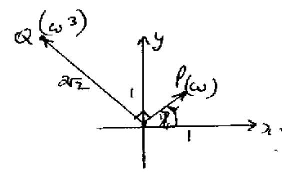
(i) $\omega = \frac{3+i}{2-i} \times \frac{2+i}{2+i}$

$= \frac{5+5i}{5}$

$= 1+i$

$\arg \omega = \pi/4$

$|\omega| = \sqrt{2}.$



(ii) see opposite ↗

(iii) $ki\omega$ represents ω^3 where $k=2$

(because the vector is longer by a factor of 2, and it is rotated anticlockwise 90° hence multiplied by i)

(d)(i) $z^n = \cos n\theta + i \sin n\theta$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$; but $\cos(-n\theta) = \cos n\theta$ and $\sin(-n\theta) = -\sin n\theta$.

$\boxed{z^n + z^{-n} = 2\cos n\theta}$ *

(ii) $3z^4 - z^3 + 2z^2 - z + 3 = 0$

divide by z^2 for $z \neq 0$

$3z^2 - z + 2 - \frac{1}{z} + \frac{3}{z^2} = 0$

$3\left(z^2 + \frac{1}{z^2}\right) - \left(z + \frac{1}{z}\right) + 2 = 0$

$3.(2\cos 2\theta) - (2\cos\theta) + 2 = 0$ (using the result above *)

$3\cos 2\theta - \cos\theta + 1 = 0$

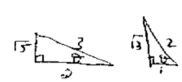
$3(2\cos^2\theta - 1) - \cos\theta + 1 = 0$

$6\cos^2\theta - \cos\theta - 2 = 0$

$(3\cos\theta - 2)(2\cos\theta + 1) = 0$

$\cos\theta = \frac{2}{3}$, $\cos\theta = -\frac{1}{2}$

$\sin\theta = \pm\frac{\sqrt{5}}{3}$, $\sin\theta = \pm\frac{\sqrt{3}}{2}$



$\therefore z = \frac{2 \pm i\sqrt{5}}{3}$ or $z = \frac{-1 \pm \sqrt{3}}{2}$

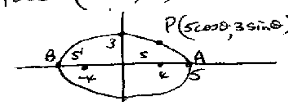
Question 3 (a)

$\frac{x^2}{25} + \frac{y^2}{9} = 1$

$9 = 25(1-e^2)$

(i) $e = 4/5$

(ii) foci $(\pm 4, 0)$



$y = 3\sin\theta$, $x = 5\cos\theta$

(iii) $\therefore \frac{dy}{dx} = -\frac{3\cos\theta}{5\sin\theta}$

Equation of tangent at P

$\therefore \boxed{y - 3\sin\theta = \frac{-3\cos\theta}{5\sin\theta}(x - 5\cos\theta)}$ ①

(iv) Coords of T: from ①

$\left(0, \frac{3}{\sin\theta}\right)$

Equation of AP gradient $= -\frac{3\sin\theta}{5-5\cos\theta}$

$y = -\frac{3\sin\theta}{5(1-\cos\theta)}(x-5)$ ②

$\therefore$ Coords of Q: from ②

$\left(0, \frac{3\sin\theta}{1-\cos\theta}\right)$

Equation of BP gradient $= \frac{3\sin\theta}{5+5\cos\theta}$

$y = \frac{3\sin\theta}{5(1+\cos\theta)}(x+5)$ ③

$\therefore$ Coords of R: from ③

$\left(0, \frac{3\sin\theta}{1+\cos\theta}\right)$

Midpt of QR

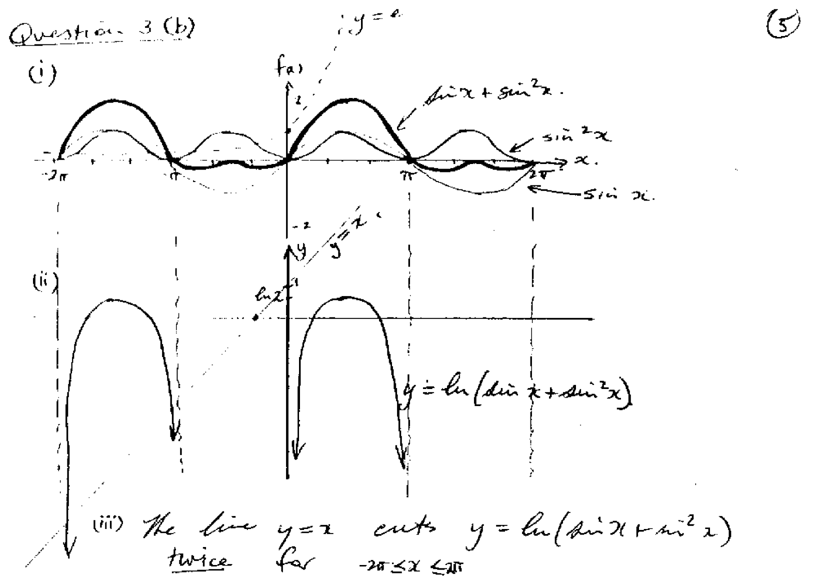
$\left(0, \frac{1}{2}\frac{3\sin\theta(1+\cos\theta+1-\cos\theta)}{1-\cos^2\theta}\right)$

$= \left(0, \frac{3}{\sin\theta}\right)$

= coords of T as required.

Question 3 (b)

(i)

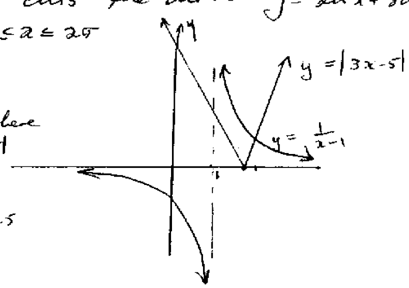


(iii) The line $y = x$ cuts $y = \ln(\sin x + \sin^2 x)$ twice for $-2\pi \le x \le 2\pi$

or

the curve $y = e^x$ cuts the curve $y = \sin x + \sin^2 x$ twice for $-2\pi \le x \le 2\pi$

Question 4 (a)



from the sketch opposite there is one solution of $\frac{1}{x-1} = |3x-5|$

that is for $\frac{1}{x-1} = 3x-5$

$1 = 3x^2 - 8x + 5$

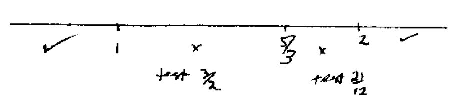
$3x^2 - 8x + 4 = 0$

$(3x-2)(x-2) = 0$

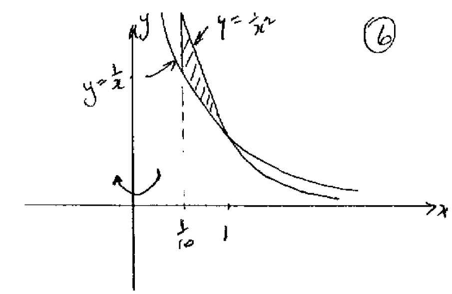
$x = \frac{2}{3}, x = 2$ the solution $(x > 1)$ is $x = 2$.

and so the solution of $\frac{1}{x-1} < |3x-5|$ is $x < 1$, $x > 2$.

a critical values approach, testing



Question 4(b).



$$V = 2\pi\int_{\frac{1}{10}}^{1} xy\,dx$$

$$= 2\pi\int_{\frac{1}{10}}^{1} x\left(\frac{1}{x^2} - \frac{1}{x}\right)dx.$$

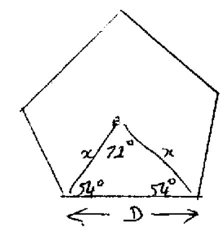
$$= 2\pi\int_{\frac{1}{10}}^{1} \frac{1}{x} - 1\,dx.$$

$$= 2\pi\left[\ln x - x\right]_{\frac{1}{10}}^{1}$$

$$= 2\pi\left\{(-1) - \left(\ln\tfrac{1}{10} - \tfrac{1}{10}\right)\right\}$$

$$= 2\pi\left(\ln 10 - \frac{9}{10}\right) \quad u^3.$$

Question 4 (c)



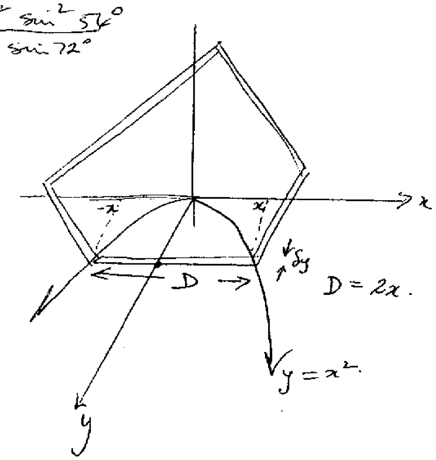
(i) Area of pentagon

$= 5 \times \frac{1}{2} \times x^2 \sin 72.$

Now $\frac{x}{\sin 54} = \frac{D}{\sin 72}.$

$\therefore$ $\text{Area}_{\text{pentagon}} = \frac{5}{2} \cdot \frac{D^2 \sin^2 54°}{\sin 72°}$

(ii)



$$\delta V = \frac{5}{2}(2x)^2\frac{\sin^2 54°}{\sin 72°}\,\delta y$$

$$= 10\,y\,\frac{\sin^2 54°}{\sin 72°}\,\delta y$$

$$V = 10.\frac{\sin^2 54°}{\sin 72°}\int_0^3 y\,dy$$

$$= 5\,\frac{\sin^2 54°}{\sin 72°}\left[y^2\right]_0^3$$

$$= 45\,\frac{\sin^2 54°}{\sin 72°} \quad u^3.$$

/5

Question 5 (a)

in $\triangle EAB$, $\triangle ECD$

① $\hat{YEB} = \hat{EAB}$ (angle between chord and tangent = angle in alternate segment)

$\hat{YEB} = \hat{ECD}$ (" ").

$\hat{AEB} = \hat{CED}$ (common angle)

$\therefore \triangle EAB \parallel\parallel \triangle ECD$ (equiangular)

$\therefore \frac{EB}{ED} = \frac{AB}{CD}$ ② (corresponding sides of similar triangles)

Join FB, GD

In $\triangle EFB$, $\triangle EGD$.

$\hat{EBF} = 90^\circ$ (angle in a semicircle)

$\hat{EDG} = 90^\circ$ (" ")

$\hat{BEF} = \hat{DEG}$ (common angle)

$\therefore \triangle EFB \parallel\parallel \triangle EGD$ (equiangular)

$\therefore \frac{EB}{ED} = \frac{EF}{EG}$ ③ (corresponding sides of similar $\triangle$s).

(i) from ① $AB \parallel CD$ (($\hat{EAB} = \hat{ECD}$ proved above); corresponding angles equal)

(ii) from ② & ③.

$\frac{AB}{CD} = \frac{EF}{EG}$.

$= \frac{2}{3}$ (given)

$\therefore$ Since $AB = 1\cdot 8$, $CD = 1\cdot 8 \times \frac{3}{2}$

ie $CD = 2\cdot 7$ cm.

Question 5(b)

(i) $\ddot{x} = 0$; $t=0, \dot{x} = 4$ $\quad$ $\ddot{y} = -10$; $t=0$, $\dot{y} = 4\sqrt{3}$

$\therefore \dot{x} = 4$; $t=0, x=0$ $\quad$ $\therefore \dot{y} = -10t + 4\sqrt{3}$; $t=0$, $y=0$

$\therefore x = 4t + 0$ $\quad$ $\therefore y = -5t^2 + 4t\sqrt{3} + 0$.

(ii) $x = 4t$ $\quad$ $y = -5\left(\frac{\sqrt{27}}{5}\right)^2 + 4\sqrt{3}\cdot\frac{\sqrt{27}}{5}$

$= 4\frac{\sqrt{27}}{5}$ $\quad$ $= \frac{9}{5}$

$= \frac{12\sqrt{3}}{5}$

(iii) When $t = \frac{\sqrt{27}}{5}$ $\quad$ $\dot{x} = 4$ $\quad$ $\dot{y} = 2\sqrt{3}$

$\tan\alpha = \frac{2\sqrt{3}}{4}$

$= \frac{\sqrt{3}}{2}$

(iv) $\dot{x} = 4$ $\quad$ $\dot{y} = 2\sqrt{3}$

Velocity of projection $= \sqrt{16+12}$

$= 2\sqrt{7}$.

Question 5(b) continued

(v) $\ddot{x} = 0$ $\quad$ $\ddot{y} = -10$

$\dot{x} = 2\sqrt{7}\cos\alpha$ $\quad$ $\dot{y} = -10t + \frac{\sqrt{3}}{\sqrt{7}}\cdot 2\sqrt{7}$

$= 4$ as before $\quad$ $= -10t + 2\sqrt{3}$.

$x = 4t + \frac{12\sqrt{3}}{5}$ ② $\quad$ $y = -5t^2 + 2\sqrt{3} + 1\cdot 8$ ①

(vi) When $y = 0$ in equation ① above

$t = \frac{-2\sqrt{3} \pm \sqrt{12+36}}{-10}$

$= \frac{-6\sqrt{3}}{-10}$ since $t > 0$

$= \frac{3\sqrt{3}}{5}$

from ② $x\left(\frac{3\sqrt{3}}{5}\right) = \frac{12\sqrt{3}}{5} + \frac{12\sqrt{3}}{5}$

$= \frac{24\sqrt{3}}{5}$ m from the original point of projection

Question 6(a)

(i) $P(x) = (x^2+6x+8)\cdot Q(x) + (2x-11)$ ① (same quotient)

$F(x) = (x^2+6x+8)\cdot Q(x) + (x+4)$ ②

$\deg P(x) = 2 + \deg Q$

$\deg F(x) = 2 + \deg Q$ $\quad$ $\therefore \deg P = \deg F$.

(ii) $P(x) - F(x) = (2x-11) - (x+4)$ from ① & ② above

$= x - 15$

(iii) $P(4) = (16 - 24 + 8)\cdot Q(x) + (2\times 4 - 11)$

$= 0 - 19$

$= -19$

Question 6(b)

(i) $P(x) = x^3 + mx^2 - n$

$P'(x) = 3x^2 + 2mx$

$P''(x) = 6x + 2m$

$= 0$ for $x = -\frac{m}{3}$

now $P'\left(-\frac{m}{3}\right) = \frac{m^2}{3} - \frac{2m^2}{3}$

$\neq 0$ since $m \neq 0$ given

$\therefore$ no triple root.

(ii) $P'(x) = 0$ for $x(3x+2m) = 0$

$x = 0$ or $x = -\frac{2m}{3}$

but $P(0) \neq 0$ since $n \neq 0$ given

Sub $x = -\frac{2m}{3}$ into $P(x) \Rightarrow$

$-\frac{8m^3}{27} + \frac{4m^3}{9} - n = 0$

$n = \frac{4m^3}{27}$

alternative method, Q 6(b)

(i) Let roots be α, α, α.

then $3\alpha = -m$ ①

$3\alpha^2 = 0$ ②

$\alpha^3 = n$ ③

from ② $\alpha^2 = 0$ $\therefore \alpha^3 = 0$

but $n \neq 0$ given $\therefore$ no triple root

(ii) Let roots be α, α, β

then $2\alpha + \beta = -m$ ①

$\alpha^2 + 2\alpha\beta = 0$ ②

$\alpha^2\beta = n$ ③

from ② $\alpha(\alpha + 2\beta) = 0$ $\quad$ $\alpha = 0$ or $\alpha + 2\beta = 0$

but $P(0) \neq 0$ $\because n \neq 0$

$\therefore \alpha \neq 0$

ie $\alpha + 2\beta = 0$

$2\alpha + 4\beta = 0$ ④

$3\beta = m$

$\beta = \frac{m}{3}$

$\alpha = -2\beta$

$\alpha^2 = \frac{4m^2}{9}$

$\therefore \frac{4m^2}{9}\cdot\frac{m}{3} = n$ $\quad$ $\therefore 4m^3 = 27n$.

(9)

Question 6(c)

(i) -20° to -4° = rise of 16° in 400 sec

ie rise 1° per 25 sec

$\therefore$ from -20° to 0° will take 20×25 sec

$= 500$ sec.

Alternately

$\frac{dT}{dt} = k$, k a constant.

$T = kt + c$ but $T(0) = -20$

$\therefore T = kt - 20$ and $T(400) = -4$

$\therefore -4 = 400k - 20$ $\therefore k = \frac{1}{25}$

$\therefore$ When $T = 0^\circ$ $0 = \frac{t}{25} - 20$

$\therefore t = 500$ sec.

(ii) $\ln(60-\theta) = c - kt$ (θ is temperature)

$60 - \theta = e^{c-kt}$ — ①

$\theta = 60 - e^{c-kt}$

$\frac{d\theta}{dt} = k e^{c-kt}$

$= k(60-\theta)$ from ①

$\propto (60-\theta)$ since k is a constant.

(iii) from $\ln(60-\theta) = c - kt$

when $t = 0$ $\theta = -20$

$\therefore \boxed{\ln 80 = c}$

and when $t = 400$, $\theta = -4$

$\therefore \ln 64 = \ln 80 - 400k$

$400k = \ln \frac{80}{64}$

$\boxed{k = \frac{1}{400} \ln(5/4)}$.

Now thawing ends and melting begins at 0°

$\therefore$ when $\theta = 0^\circ$.

$\ln 60 = \ln 80 - \frac{1}{400}\ln(5/4) \cdot t$

$\frac{1}{400}\ln(5/4)\, t = \ln \frac{80}{60}$

$t = 400 \cdot \ln(\frac{4}{3}) \div \ln(5/4)$

$= 515.68\ldots$

≈ 516 seconds.

Question 7(a)

When $n = 1$ LHS $= \cos\theta$, RHS $= \cos\theta$

$\therefore$ result is true for $n = 1$

consider $n = k$, $k \geq 1$ an integer

assume $\cos\theta + \cos 3\theta + \ldots + \cos(2k-1)\theta = \sin k\theta . \cos k\theta / \sin\theta$

To show $\cos\theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2k-1)\theta + \cos(2k+1)\theta$

$= \frac{\sin(k+1)\theta \cos(k+1)\theta}{\sin\theta}$.

Now $\cos\theta + \cos 3\theta + 5\theta + \ldots + \cos(2k+1)\theta = \frac{\sin k\theta \cos k\theta}{\sin\theta} + \cos(2k+1)\theta \times \frac{\sin\theta}{\sin\theta}$

$= \frac{\sin k\theta \cos k\theta + \cos(2k+1)\theta \sin\theta}{\sin\theta}$ ①

But $\sin(2k+2)\theta = \sin(2k+1)\theta \cos\theta + \cos(2k+1)\theta \sin\theta$

$\therefore \cos(2k+1)\theta \sin\theta = \sin(2k+2)\theta - \sin(2k+1)\theta \cos\theta$

And line ① becomes: $\frac{\sin k\theta \cos k\theta + (\sin(2k+2)\theta - \sin(2k+1)\theta\cos\theta)}{\sin\theta}$

$= \frac{\sin k\theta \cos k\theta + \sin(2k+2)\theta - (\sin 2k\theta \cos\theta + \cos 2k\theta \sin\theta)\cos\theta}{\sin\theta}$

$= \frac{\sin k\theta \cos k\theta + \sin(2k+2)\theta - (\sin 2k\theta \cos^2\theta + \cos 2k\theta \sin\theta\cos\theta)}{\sin\theta}$

$= \frac{\frac{1}{2}(\cancel{\sin 2k\theta}) + \sin(2k+2)\theta - (\cancel{\sin 2k\theta}(\frac{1}{2}(1+\cos 2\theta)) + \cos 2k\theta . \frac{1}{2}\sin 2\theta)}{\sin\theta}$

$= \frac{\sin(2k+2)\theta}{\sin\theta} - \frac{\frac{1}{2}\sin(2k+2)\theta}{\sin\theta}$

$= \frac{1}{2}\frac{\sin 2(k+1)\theta}{\sin\theta}$

$= \frac{\sin(k+1)\theta . \cos(k+1)\theta}{\sin\theta}$ as required

$\therefore$ If result is true for $n = k$, then it is true for $n = k+1$

But the result is true for $n = 1$, $\therefore$ it is true for $n = 1+1$

$= 2$ and so on for all positive integers n.

(b) Direction of motion +ve $\downarrow$ $\quad$ $\downarrow mg$ $\quad$ $\uparrow R = \frac{3}{10}v$

(i) Taking origin at top of library,

$$m\ddot{x} = mg - \frac{3}{10}v$$

$$\ddot{x} = g - \frac{3}{10m}v$$

Now, $m = 12$ grams $= 0.012$ kg

$$\therefore \ddot{x} = g - \frac{3}{10 \times 0.012}v \quad \boxed{1}$$

$$\therefore \ddot{x} = 10 - 25v$$

(ii) $$\frac{dv}{dt} = 10 - 25v$$

$$\frac{dt}{dv} = \frac{1}{10-25v}$$

$$t = \int_0^v \frac{1}{10-25v}\,dv$$

$$= -\frac{1}{25}\Big[\ln(10-25v)\Big]_0^v$$

$$= -\frac{1}{25}\Big[\ln(10-25v) - \ln 10\Big]$$

$$= -\frac{1}{25}\ln\left(\frac{10-25v}{10}\right)$$

$$\therefore -25t = \ln\left(\frac{10-25v}{10}\right)$$

$$\frac{10-25v}{10} = e^{-25t}$$

$$10 - 25v = 10e^{-25t}$$

$$25v = 10 - 10e^{-25t} \quad \boxed{3}$$

$$\therefore v = \frac{10}{25}\left(1 - e^{-25t}\right)$$

$$\therefore v(t) = \frac{2}{5}\left(1 - e^{-25t}\right)$$

(iii) As $t \to \infty$, $e^{-25t} \to 0$ $\quad \boxed{1}$

$\therefore$ Terminal velocity $v_T = \frac{2}{5}$

(iv) When velocity has reached 99% of v_T, $(= 0.396)$

then $$t = \int_0^{0.396} \frac{1}{10-25v}\,dv$$

$$= -\frac{1}{25}\Big[\ln(10-25v)\Big]_0^{0.396}$$

$$= -\frac{1}{25}\left[\ln\left(\frac{10-25\times0.396}{10}\right)\right] \quad \boxed{2}$$

$$= 0.184\ldots$$

$$= 0.18 \text{ seconds (to 2 d.p.)}$$

(v) $$v(t) = \frac{2}{5}\left(1 - e^{-25t}\right)$$

$$\frac{dx}{dt} = \frac{2}{5} - \frac{2}{5}e^{-25t}$$

$\therefore x = \int_0^t \left(\frac{2}{5} - \frac{2}{5}e^{-25t}\right) dt$ $\boxed{2}$

$= \left[\frac{2}{5}t + \frac{2}{125}e^{-25t}\right]_0^t$

$\therefore x(t) = \frac{2}{5}t + \frac{2}{125}e^{-25t} - \frac{2}{125}$

(vi) When $t = 1$,

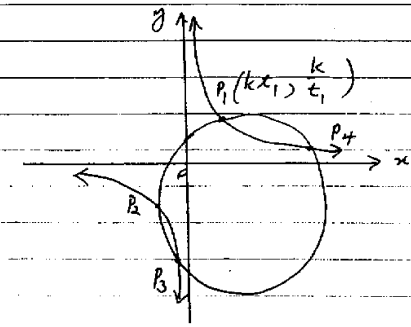
$x = \frac{2}{5} + \frac{2}{125}e^{-25} - \frac{2}{125}$ $\boxed{1}$

$= 0.384$ m

$\doteqdot 38$ cm

Question 8

(a)



(i) As points $\left(kt, \frac{k}{t}\right)$ lie on circle, coordinates satisfy equation

$\therefore (kt)^2 + \left(\frac{k}{t}\right)^2 + 2g(kt) + 2f\left(\frac{k}{t}\right) + c = 0$

$k^2t^2 + \frac{k^2}{t^2} + 2gkt + \frac{2fk}{t} + c = 0$

$k^2t^4 + 2gkt^3 + 2fkt + ct^2 + k^2 = 0$ $\boxed{2}$

Roots of this equation are parameters t_1, t_2, t_3, t_4.

Product of roots $= t_1t_2t_3t_4 = \frac{k^2}{k^2}$

$\therefore t_1t_2t_3t_4 = 1$

(ii) Eqn of chord P_1P_2 is

$$\frac{y-\frac{k}{t_1}}{x-kt_1} = \frac{\frac{k}{t_2}-\frac{k}{t_1}}{kt_2-kt_1} = \frac{\frac{kt_1-kt_2}{t_1t_2}}{kt_2-kt_1}$$

$$\therefore \frac{y-\frac{k}{t_1}}{x-kt_1} = \frac{-1}{t_1t_2}$$

$\boxed{2}$

$$t_1t_2y - kt_2 = -x + kt_1$$

$$\therefore x + t_1t_2y = k(t_1+t_2)$$

(iii) If P_1P_2 passes through origin, coords. satisfy equation

$$\therefore 0 = k(t_1+t_2)$$

$$\therefore t_1 = -t_2$$

If P_3P_4 is a diameter of circle, then $P_1P_4 \perp P_1P_3$

$$\text{grad. of } P_1P_4 = \frac{\frac{k}{t_4}-\frac{k}{t_1}}{kt_4-kt_1}$$

$$= \frac{\frac{kt_1-kt_4}{t_4t_1}}{kt_4-kt_1}$$

$$= -\frac{1}{t_1t_4}$$

Similarly, grad. of $P_1P_3 = \dfrac{-1}{t_1t_3}$

Now, if $P_1P_4 \perp P_1P_3$, product of gradients $= -1$

$$\therefore \frac{-1}{t_1t_4} \cdot \frac{-1}{t_1t_3} = \frac{1}{t_1t_1t_3t_4}$$

$$= \frac{1}{t_1 \cdot -t_2t_3t_4} \text{, as } t_1 = -t_2$$

$\boxed{4}$

$$= \frac{-1}{t_1t_2t_3t_4}$$

$$= -1$$

$\therefore P_3P_4$ is diameter of circle

(i) Unshaded area $= \int_0^k \frac{e^{2x}-1}{e^{2x}+1}\,dx$

Let $u = e^{2x}$

$du = 2e^{2x}\,dx$

$\therefore dx = \frac{1}{2u}\,du$

When $x=0$, $u=1$

When $x=k$, $u=e^{2k}$

$\therefore$ Unshaded area $= \int_1^{e^{2k}} \frac{u-1}{u+1}\cdot\frac{1}{2u}\,du$

$= \frac{1}{2}\int_1^{e^{2k}} \frac{u-1}{u(u+1)}\,du$ $\boxed{5}$

$= \frac{1}{2}\int_1^{e^{2k}} \frac{u}{u(u+1)} - \frac{1}{u(u+1)}\,du$

$= \frac{1}{2}\int_1^{e^{2k}} \left(\frac{1}{u+1} - \frac{1}{u} + \frac{1}{u+1}\right)du$

$= \frac{1}{2}\Big[2\ln(u+1) - \ln u\Big]_1^{e^{2k}}$

$= \frac{1}{2}\Big[2\ln(e^{2k}+1) - \ln(e^{2k}) - 2\ln 2\Big]$

$= \ln(e^{2k}+1) - k - \ln 2$

$\therefore$ Shaded area $= k\times 1 - \Big[\ln(e^{2k}+1) - k - \ln 2\Big]$

$= 2k - \ln(e^{2k}+1) + \ln 2$

(ii) Now, $e^{2k}+1 > e^{2k}$ for all $k>0$

$\therefore \ln(e^{2k}+1) > \ln e^{2k}$ $\boxed{2}$

$\ln(e^{2k}+1) > 2k$

$\therefore 2k - \ln(e^{2k}+1) < 0$

So, area is always less than $\ln 2$