Catholic Secondary Schools' Association of New South Wales

4 Unit Mathematics Trial HSC Papers

1990

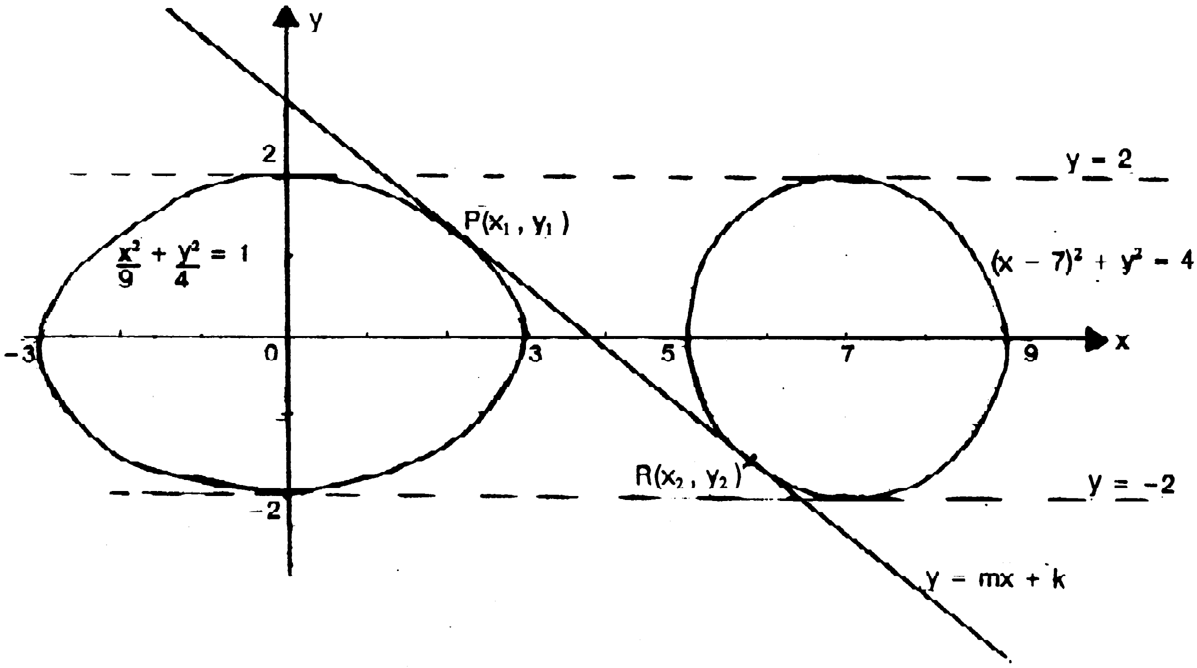
1. (a) (i) If $y = [f(x)]^n$ write down an expression for $\frac{dy}{dx}$. Hence show that all x co-ordinates of stationary points and points of intersection with the x axis on the graph of $y = f(x)$ are x co-ordinates of stationary points on the graph of $y = [f(x)]^n$, where $n > 1$ is a positive integer.
(ii) Sketch the graph of $y = 4 - x^2$ showing clearly the co-ordinates of any points of intersection with the x axis and the y axis and the co-ordinates and the nature of any stationary points. Hence, or otherwise, sketch on separate axes the graphs of $y = (4 - x^2)^2$ and $y = (4 - x^2)^3$, in each case showing clearly the co-ordinates of any points of intersection with the x axis and the y axis and the co-ordinates and the nature of any stationary points.
(b) (i) State the domain and the range of the function $y = \cos^{-1}(e^x)$.
(ii) Sketch the graph of the function $y = \cos^{-1}(e^x)$ showing clearly the co-ordinates of any points of intersection with the x axis and the y axis and the equations of any asymptotes.

2. (a) (i) By expressing $x^2 - 2x - 1$ as the difference of two squares, or otherwise, show that $x^2 - 2x - 1 = (x - 1 - \sqrt{2})(x - 1 + \sqrt{2})$.
(ii) Hence express $\frac{1}{x^2-2x-1}$ in the form $\frac{A}{(x-1-\sqrt{2})} + \frac{B}{(x-1+\sqrt{2})}$, where A and B are real numbers, and find $\int \frac{1}{x^2-2x-1}\, dx$.
(b) Use the substitution $u = -x$ to show that $\int_{-2}^{2} \frac{x^2}{e^x+1}\, dx = \int_{-2}^{2} \frac{x^2 e^x}{e^x+1}\, dx$. Hence evaluate $\int_{-2}^{2} \frac{x^2}{e^x+1}\, dx$.
(c) (i) If $I_n = \int_0^1 (1 - x^2)^n\, dx$ show that $I_n = \frac{2n}{2n+1} I_{n-1}$ for all positive integers $n \geq 1$.
(ii) Hence, or otherwise, show that $I_n = \frac{2^{2n}(n!)^2}{(2n+1)!}$ for all positive integers $n \geq 1$.

3. (a) Express the complex number $z = 1 + i\sqrt{3}$ in modulus argument form. Hence, or otherwise, express each of the complex numbers $\frac{1}{z}$ and iz in modulus argument form.
(b) $2 + i$ and $1 - 3i$ are two roots of the equation $x^4 + bx^3 + cx^2 + dx + e = 0$, where b, c, d and e are real numbers. Write down the other two roots of the equation and hence find the values of b and e.
(c) (i) Show that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$ for any complex numbers z_1 and z_2.
(ii) In an Argand diagram P and Q are the points representing the complex numbers z_1 and z_2 respectively. By considering the parallelogram $OPRQ$, where O is the

origin, interpret this result geometrically.
(d) (i) The complex number z satisfies both $|z-1| \leq |z-i|$ and $|z-2-2i| \leq 1$. In an Argand diagram indicate the region which contains the point P representing z.
(ii) If P moves on the boundary of this region and $\arg(z-1) = \frac{\pi}{4}$, find the value of z in the form $x+iy$ where x and y are real.

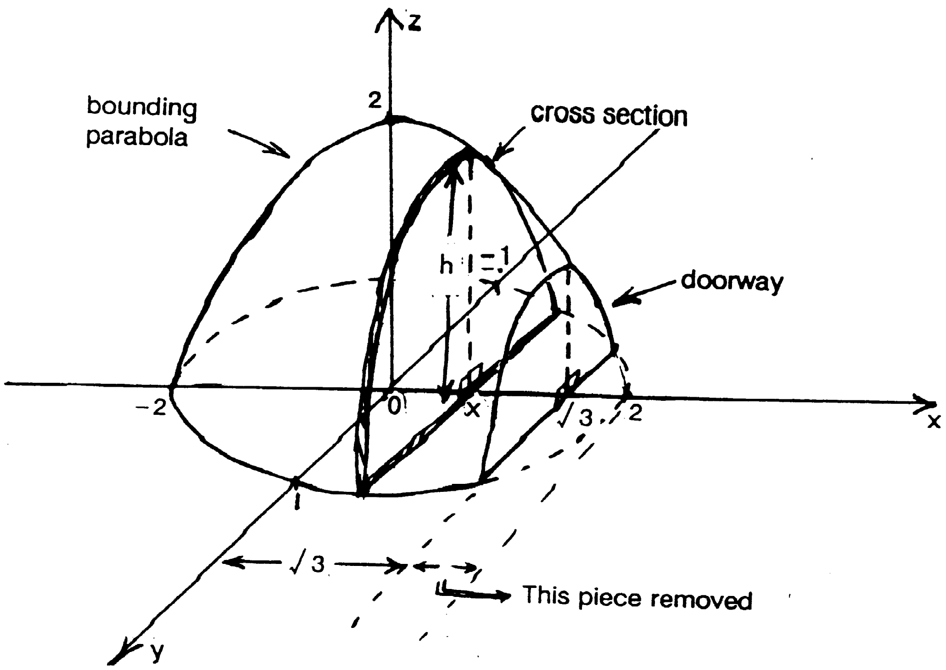
4. (a) Show that the tangent to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ at the point $P(x_1, y_1)$ has cartesian equation $\frac{xx_1}{9} + \frac{yy_1}{4} = 1$.
(b) Show that if tangents are drawn from a point $W(x_0, y_0)$ external to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, touching the ellipse at P, Q respectively, then the equation of the chord of contact PQ is $\frac{xx_0}{9} + \frac{yy_0}{4} = 1$.
(c)



The above diagram shows the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the circle $(x-7)^2 + y^2 = 4$. Clearly $y = 2$ and $y = -2$ are common tangents to the ellipse and the circle. Suppose the line $y = mx + k, m \neq 0$, is also a common tangent, touching the ellipse at $P(x_1, y_1)$ and the circle at $R(x_2, y_2)$ as shown.
(i) Copy the diagram and use symmetry to draw a fourth common tangent, touching the ellipse at Q and the circle at T, and write down the co-ordinates of Q and T on your diagram. Deduce that the equation of QT is $y = -mx - k$.
(ii) PR and QT intersect at V. Show V has co-ordinates $\left(-\frac{k}{m}, 0\right)$
(iii) Use the fact that PQ is the chord of contact of tangents from V to the ellipse to show that $x_1 = -\frac{9m}{k}$
(iv) Deduce that $x_1 = -\frac{9m}{k}$ is a double root of the equation $\frac{x^2}{9} + \frac{(mx+k)^2}{4} = 1$, and hence show that $9m^2 - k^2 + 4 = 0$.

(v) Show that if $y = mx + k$ is a tangent to the circle $(x-7)^2 + y^2 = 4$, then $45m^2 + k^2 + 14mk - 4 = 0$.
(vi) Show that $\frac{m}{k} = -\frac{7}{27}$, and find the co-ordinates of P, Q and V, and the equations of the two oblique common tangents.

5. **(a)** Use $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ to show $\cos^4\theta = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$.
(b) A parabola passes through the points $(-a, 0)$, $(0, h)$ and $(a, 0)$ where $a > 0, h > 0$. Show that the area enclosed by this parabola and the x-axis is $\frac{4}{3}ah$.
(c) A mathematically inclined Eskimo decides to build himself an igloo based on conic sections. The base of the interior of the igloo is an ellipse with semi-axes 2 metres and 1 metre. Vertical cross sections taken at right angles to the major axis are bounded by parabolic arcs, the axis of the parabola being vertical and passing through the major axis of the ellipse. The cross section taken vertically through the centre of the ellipse along the major axis is also bounded by a parabolic arc with axis vertical through the centre of the ellipse.

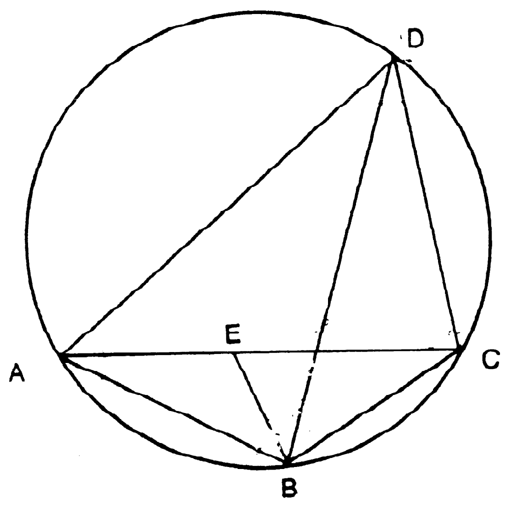


The igloo is shown in the diagram, with co-ordinate axes taken through the centre of the elliptical base. The maximum height of the interior is to be 2 metres as shown, and the internal dimensions are indicated on the diagram. The Eskimo forms the entrance to the igloo by slicing the structure vertically at right angles to the major axis of the ellipse at a distance $\sqrt{3}$ metres from the centre and removing the material from this point outwards to the end of the major axis, as shown in the diagram.
(i) Find the equation of the ellipse which bounds the floor of the interior of the igloo.
(ii) By finding the equation of the bounding parabola indicated in the diagram, show that the height h of the vertical cross section, drawn at a distance x from the

centre of the ellipse is given by $h = \frac{1}{2}(4 - x^2)$.
(iii) Find the maximum height and width of the doorway.
(iv) By finding the area of the indicated cross section of height h, show that the volume of air in the igloo is given by $V = \frac{1}{3}\int_{-2}^{\sqrt{3}}(4 - x^2)^{3/2}\, dx$.
(v) Calculate the volume of air in the igloo by using the substitution $x = 2\sin\theta$ in the above integral for V.

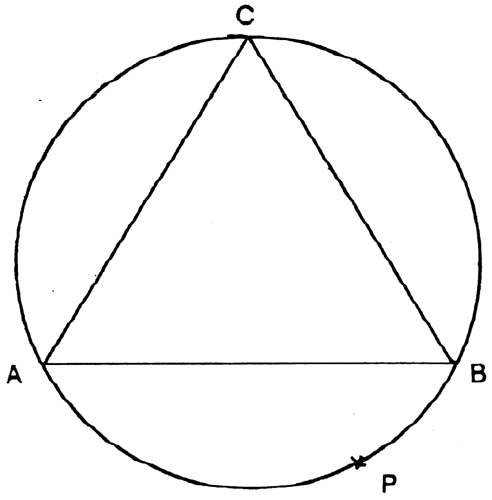
6. A body of unit mass falls under gravity through a resisting medium. The body falls from rest from a height of 50 metres above the ground. The resistance to its motion is $\frac{1}{100}v^2$ where v metres per second is the speed of the body when it has fallen a distance x metres.
(i) Draw a diagram to show the forces acting on the body. Show that the equation of motion of the body is $\ddot{x} = g - \frac{1}{100}v^2$.
(ii) Show that the terminal velocity V of the body is given by $V = \sqrt{100g}$.
(iii) Show that $v^2 = V^2(1 - e^{-x/50})$.
(iv) Find the distance fallen in metres until the body reaches a velocity equal to 50% that of the terminal velocity.
(v) Find the velocity reached as a percentage of the terminal velocity when the body hits the ground.
(vi) If $v = v_1$ when $x = d$ and $v = v_2$ when $x = 2d$ show that $v_2^2 = v_1^2\left(2 - \frac{v_1^2}{V^2}\right)$.

7. (a) (i)



In the diagram $ABCD$ is a cyclic quadrilateral. E is the point on AC such that $\angle ABE = \angle DBC$.
(α) Show that $\triangle ABE|||\triangle DBC$ and $\triangle ABD|||\triangle EBC$.
(β) Hence show that $AB.DC + AD.BC = AC.DB$.

(ii)



In the diagram ABC is an equilateral triangle inscribed in a circle. P is a point on the minor arc AB of the circle. Use the result above to show that $PC = PA + PB$.
(b) The equation $x^3 + bx^2 + x + 2 = 0$, where b is a real number, has roots α, β and γ.
(i) Obtain an expression, in terms of b, for $\alpha^2 + \beta^2 + \gamma^2$.
(ii) Hence determine the set of possible values of b if the roots are all real and non-zero.

8. (a) (i) Show that $\operatorname{cosec} 2\theta + \cot 2\theta = \cot \theta$ for all real values of θ.
(ii) Use the above result
(α) to find in surd form the values of $\cot \frac{\pi}{8}$ and $\cot \frac{\pi}{12}$.
(β) to show without using a calculator that

$$\operatorname{cosec} \tfrac{4\pi}{15} + \operatorname{cosec} \tfrac{8\pi}{15} + \operatorname{cosec} \tfrac{16\pi}{15} + \operatorname{cosec} \tfrac{32\pi}{15} = 0.$$

(b) An unbiased die is thrown six times. Find the probabilities that the six scores obtained will:
(i) be 1,2,3,4,5,6 in some order,
(ii) have a product which is an even number,
(iii) consist of exactly two 6's and four odd numbers.
(iv) be such that a 6 occurs only on the last throw and exactly three of the first five throws result in odd numbers.