QUESTION 1 (15 marks) Scots 2005 Ext 2 trial

a) Evaluate $|3+2i|$ **1**

b) i) If $v=\dfrac{1+i\sqrt{3}}{2}$ show that $v^3=-1$. **2**

ii) Hence calculate v^{10}. **2**

c) If z is a complex number so that $|z|=2$ and $\arg z=\dfrac{\pi}{6}$, mark clearly on the same Argand diagram the points representing the complex numbers:

i) z ii) iz iii) $\bar{z}$ iv) $\dfrac{1}{z}$ v) $z\bar{z}$ vi) z^2 vii) z^2+z viii) z^2-z **10**

QUESTION 2 (15 marks)

a) Find $\displaystyle\int\frac{dx}{x^2-6x+13}$ **2**

b) Find $\displaystyle\int\tan x\sec^2 x\,dx$ **2**

c) i) Show that $f(x)=\sin^{-1}x$ is an odd function. **2**

ii) Hence or otherwise find $\displaystyle\int_{-1}^{1}(\sin^{-1}x)^3dx$ **1**

d) $\displaystyle\int_0^{\sqrt{2}}\sqrt{4-x^2}\,dx$ **4**

e) $\displaystyle\int e^x\cos x\,dx$ **4**

QUESTION 3 (10 marks)

a) Use the method of cylindrical shells to find the volume of the solid (paraboloid) obtained when the region between the curve $y = \frac{1}{2}\sqrt{x-2}$, the x-axis and the line x = 6 is rotated about the x axis. **4**

b) Prove $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$, where x denotes displacement, and v denotes velocity. **2**

c) The acceleration of a particle moving in a straight line is given by $\ddot{x} = xe^x$,

where x is the displacement from 0. The particle is initially at rest.

The particle starts at $x = 0$.

i) Prove that $v^2 = 2e^x(x\text{-}1) + 2$ **3**

ii) Describe the subsequent motion of the particle after it leaves the origin and explain why the particle can only move in one direction **1**

QUESTION 4 (18 marks)

a) The equation $x^3 - x^2 - 3x+2 = 0$ has roots α, β, γ. Find the monic polynomial equation with roots $\alpha^2, \beta^2, \gamma^2$. **4**

b) If $x = \alpha$ is a double root of the equation P(x) = 0, show that $x = \alpha$ is a root of the equation $P'(x) = 0$. **4**

c) i) Show that $1+i$ is a root of the polynomial $Q(x) = x^3 + x^2 - 4x + 6$ **2**

ii) hence resolve Q(x) into irreducible factors over the complex number field. **3**

d) If α , β , γ are the roots of the cubic equation x^3 + qx + r = 0, prove that $(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2 = -6q$. **5**

QUESTION 5 (18 marks)

The ellipse $\mathcal{E}$ has the cartesian equation $\frac{x^2}{25}+\frac{y^2}{16}=1$.

i) Write down the eccentricity **1**

ii) Write down the coordinates of the foci S and S' **1**

iii) Write down the equations of the directrices. **1**

iv) Sketch the ellipse $\mathcal{E}$. **1**

v) Show that any point P on $\mathcal{E}$ can be represented by the coordinates $(5\cos\theta, 4\sin\theta)$ **1**

vi) Prove that $PS + PS'$ is independent of the position of P on the ellipse $\mathcal{E}$. **3**

vii) Show that the equation of the normal N at the point P on the ellipse $\mathcal{E}$ is $5\sin\theta\, x - 4\cos\theta\, y = 9\sin\theta\cos\theta$ **2**

viii) If this normal meets the major axis of the ellipse in M and the minor axis in N, prove that $\frac{PM}{PN}=\frac{16}{25}$. **3**

ix) Also show that the line PN bisects the angle $S'PS$. **5**

QUESTION 6 (14 marks)

i) By considering the curve $g(x) = x^6 - 4x^5 + 4x^4$, sketch the graph of $f(x) = x^6 - 4x^5 + 4x^4 - 1$ showing that it has 4 real zeroes. **4**

On different diagrams sketch the curves:

ii) $y = |f(x)|$ **2**

iii) $y = f(|x|)$ **2**

iv) $y^2 = f(x)$ **3**

v) Calculate the slope of the curve $y^2 = f(x)$ at any point x and describe the nature of the curve at a zero of $f(x)$. **3**

QUESTION 7 (15 marks)

a) A parachutist of M kilograms is dropped from a stationary helicopter of height H metres above the ground. The parachutist experiences air resistance during its fall equal to MkV^2, where V is its velocity in metres per second and k is a positive constant. Let x be the distance in metres of the parachutist from the helicopter, measured positively as it falls.

i) Show that the equation of motion of the parachutist is $\ddot{x} = g - kV^2$, where g is the acceleration due to gravity. **1**

ii) Find V^2 as a function of x. **4**

iii) Find the velocity U of the parachutist as he hits the ground in terms of g, k and H. **1**

iv) Find the velocity of the parachutist as he hits the ground if air resistance is neglected. **2**

b)

i) Prove the identity $\cos 3A = 4\cos^3 A - 3\cos A$ **2**

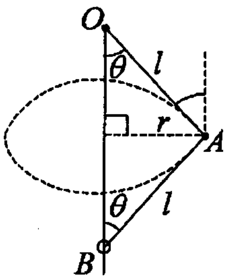
ii) Show that $x = 2\sqrt{2}\cos A$ is a root of the equation $x^3 - 6x + 2 = 0$ provided that $\cos 3A = -\dfrac{1}{2\sqrt{2}}$ **2**

iii) Find the three roots of the equation $x^3 - 6x + 2 = 0$, using the results from part (ii) above. Give your answer to three decimal places. **3**

QUESTION 8 (15 marks)

a) A particle ***A*** of mass ***2m*** is attached by a light inextensible string of length ***l*** to a fixed point ***O*** and is also attached by another light inextensible string of the same length to a small ring ***B*** of mass ***3m*** which can slide on a fixed smooth vertical wire passing through ***O***. The particle ***A*** describes a horizontal circle of radius ***r***, and ***OA*** is inclined at an angle $\theta = \frac{\pi}{3}$ with the downward vertical.

Dimension diagram



$\theta = \dfrac{\pi}{3}$

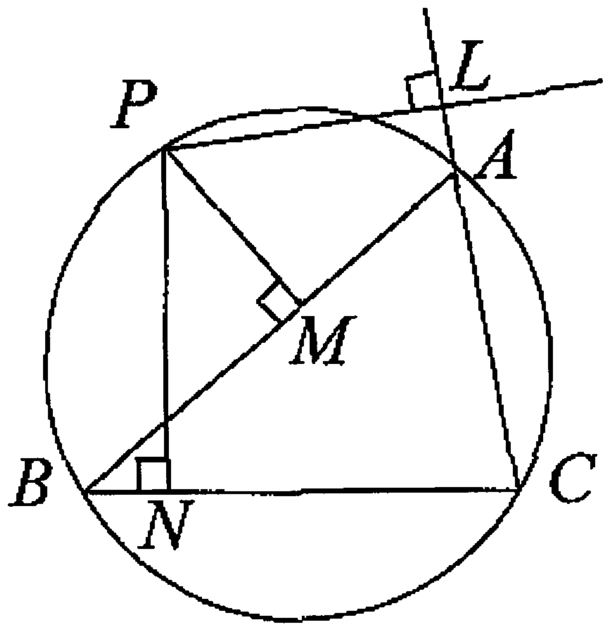
i) Find the tension in the strings ***OA*** and ***AB*** **5**

ii) Find the angular velocity of ***A***. **3**

iii) Describe what happens to the system as the angular velocity increases. **1**

b) **ABC** is a triangle inscribed in the circle. ***P*** is a point on the minor arc ***AB***. The points ***L***, ***M***, and ***N*** are the feet of the perpendiculars from ***P*** to ***CA*** produced, ***AB***, and ***BC*** respectively.

Copy the diagram into your answer booklet and show that ***L***, ***M*** and ***N*** are collinear.



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