

3/4 UNIT MATHEMATICS FORM VI

Time allowed: 2 hours (plus 5 minutes reading) Exam date: 13th August 2001

Instructions:

- All questions may be attempted.
- All questions are of equal value.
- Part marks are shown in boxes in the left margin.
- All necessary working must be shown.
- Marks may not be awarded for careless or badly arranged work.
- Approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

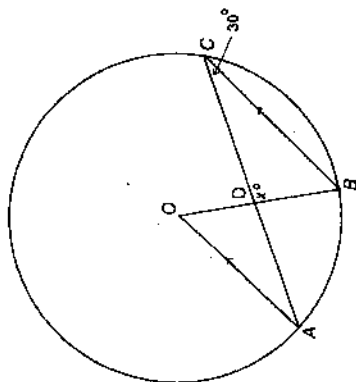
Collection:

- Each question will be collected separately.
- Start each question in a new answer booklet.
- If you use a second booklet for a question, place it inside the first. Don't staple.
- Write your candidate number on each answer booklet.

QUESTION ONE (Start a new answer booklet)

Marks

- 2 (a) Find the coordinates of the point that divides the interval joining the points $(-5, 6)$ and $(4, -3)$ in the ratio $3 : 1$.
- 3 (b) Find the acute angle between the lines $x + 2y = 5$ and $x - 3y = 0$.
- (c)

In the diagram above, O is the centre of the circle, $BC \parallel AO$ and $\angle ACB = 30^\circ$.

- 1 (i) Explain why $\angle AOB = 60^\circ$.
- 2 (ii) Find x , giving reasons.
- (d) Consider the polynomial $P(x) = x^3 - x^2 - 10x - 8$.
- 1 (i) Show that $x = -1$ is a zero of $P(x)$.
- 2 (ii) Express $P(x)$ as a product of three linear factors.
- 1 (iii) Solve $P(x) \leq 0$.

QUESTION TWO (Start a new answer booklet)

Marks

- 1 (a) Sketch the polynomial function $y = x^2(x^2 - 16)$, carefully showing all intercepts.
- 1 (b) (i) Write $x^2 + 4x + 5$ in the form $(x + a)^2 + b$.
- 2 (ii) Hence find $\int \frac{dx}{x^2 + 4x + 5}$.
- 3 (c) Find the general solution of $\cos 2x = \cos x$.
- 2 (d) (i) Sketch the parabola $f(x) = 9 - (x + 2)^2$, showing clearly any intercepts with the axes and the coordinates of the vertex.
- 1 (ii) What is the largest domain containing the value $x = 0$ for which the function has an inverse function?
- 2 (iii) On a separate diagram, sketch the graph of this inverse function, showing all intercepts with the axes.

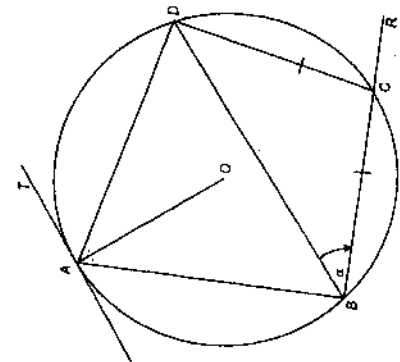
QUESTION THREE (Start a new answer booklet)

Marks

- 2 (a) Evaluate $\lim_{x \rightarrow 0} \left(\frac{\sin 4x}{\tan 2x} \right)$. You must show all working for full marks.
- 2 (b) Find the term independent of x in the expression $\left(x + \frac{1}{x^2} \right)^9$.
- 4 (c) A spherical balloon is expanding so that its volume $V \text{ m}^3$ increases at a constant rate of 72 m^3 per second. What is the rate of increase of the surface area when the radius is 12 metres? You may use the formulae $V = \frac{4}{3}\pi r^3$ for the volume of a sphere and $S = 4\pi r^2$ for its surface area.
- 1 (d) (i) Show that there is a root to the equation $\sin x = x - \frac{1}{2}$ between $x = 0.5$ and $x = 1.8$.
- 3 (ii) Taking $x = 1.2$ as a first approximation to this solution, apply Newton's method once to find a closer approximation to the solution. Give your answer correct to two decimal places.

QUESTION FOUR (Start a new answer booklet)

- (a) Write $3 \sin x + \sqrt{3} \cos x$ in the form $R \sin(x + \alpha)$, where $0 \leq \alpha \leq \frac{\pi}{2}$.

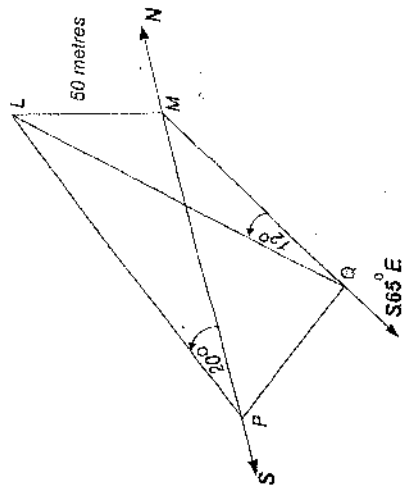


In the diagram above, the points A, B, C and D lie on a circle with centre O . The line TA is a tangent to the circle. The chord BC is produced to R . The interval AO bisects $\angle BAD$ and $BC = CD$. Let $\angle DBC = \alpha$.

Copy the diagram onto your answer paper.

- (i) Prove that $\angle DCR = 2\alpha$.
 (ii) Show that $\angle OAD = \alpha$.
 (iii) Prove that $\angle ABC$ is a right angle.

(c)



From the top L of a lighthouse 50 metres high a boat is observed at a point P due south at an angle of depression of 20° , as shown in the diagram above. The boat drifts at a constant speed and in a constant direction. After 10 minutes it is again observed from the top of the lighthouse at the point Q at an angle of depression of 12° . The base M of the lighthouse is at sea-level, and the bearing of Q from M is $S65^\circ E$.

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- (i) Find an expression for PM .

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- (ii) Show that the distance PQ is given by

$$PQ = 50\sqrt{\cot^2 20^\circ + \cot^2 12^\circ - 2 \cot 20^\circ \cot 12^\circ \cos 65^\circ}.$$

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- (iii) How fast was the boat drifting? Give your answer in metres per second, correct to two significant figures.

QUESTION FIVE (Start a new answer booklet)

Marks 3/20
2 (a) (i) Differentiate $x \cos^{-1} x - \sqrt{1-x^2}$.

1 (ii) Hence evaluate $\int_0^1 \cos^{-1} x \, dx$.

5 (b) Use the substitution $u = 1 - x$ to evaluate $\int_{-3}^0 \frac{x}{\sqrt{1-x}} \, dx$.

4 (c) By considering the expansion of $(1+x)^{2n} = (1+x)^n(1+x)^n$ in two different ways, show that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}.$$

QUESTION SIX (Start a new answer booklet)

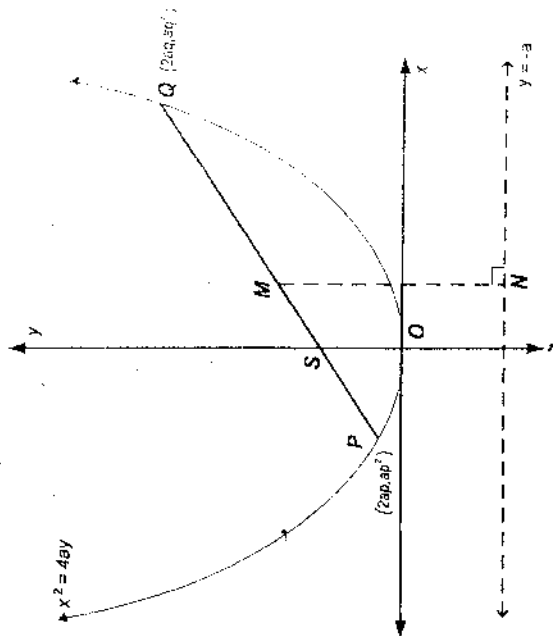
(a) Let $(3+2x)^{20} = \sum_{r=0}^{20} a_r x^r$.

Marks 1 (i) Write an expression for a_r .

1 (ii) Show that $\frac{a_{r+1}}{a_r} = \frac{40-2r}{3r+3}$.

4 (iii) Hence find the greatest coefficient in the expansion of $(3+2x)^{20}$.

(b)



Let $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ be points on the parabola $x^2 = 4ay$, as shown in the above diagram.

1 (i) Show that the equation of the chord PQ is $y = \frac{p+q}{2}x - apq$.

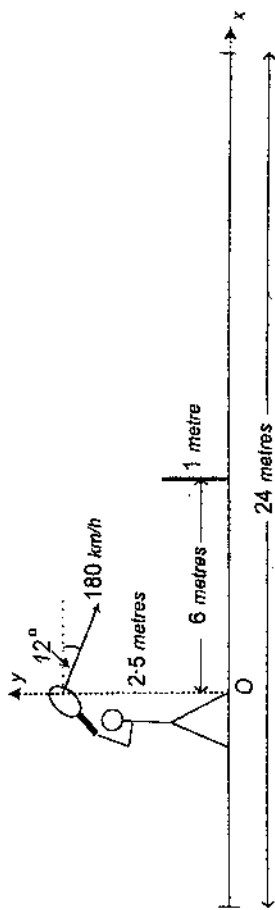
1 (ii) Show that if the chord PQ passes through the focus $S(0, a)$, then $pq = -1$.

4 (iii) M is the midpoint of the focal chord PQ . N lies on the directrix such that MN is perpendicular to the directrix. T is the midpoint of MN . Find the locus of T .

THE EXAMINATION PAPER CONTINUES ON THE NEXT PAGE

QUESTION SEVEN (Start a new answer booklet)

(a)



In the diagram above, a tennis court is 24 metres long and has a net one metre high positioned in the middle.

During a match a player standing 6 metres from the net smashes a ball into the opposing court with an initial speed of 180 km/h. The ball is hit parallel to the sideline and is projected with an angle of depression of 12° from a height of 2.5 metres above the ground. Let $g = 10 \text{ m/s}^2$.

Marks

3

(i) Taking the axes as given on the diagram, show that the horizontal and vertical components of the displacement are given by

$$x = 50t \cos 12^\circ \quad \text{and} \quad y = -5t^2 - 50t \sin 12^\circ + 2.5$$

respectively, where t is the time in seconds and both x and y are measured in metres.

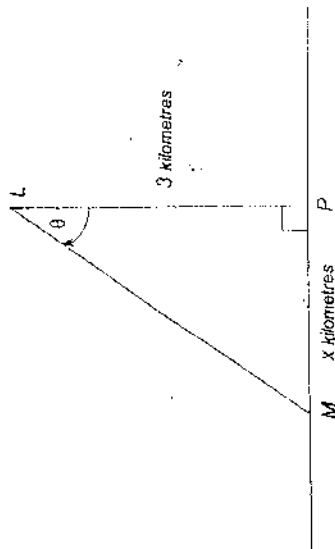
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(ii) By what margin does the ball clear the net? Give your answer correct to the nearest centimetre.

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(iii) How far from the opposing court's baseline does the ball land? Give your answer correct to the nearest centimetre.

(b)



In the diagram above, a lighthouse L containing a revolving beacon is located out at sea, 3 kilometres from P , the nearest point on a straight shoreline. The beacon rotates clockwise with a constant rotation rate of 4 revolutions per minute and throws a spot of light onto the shoreline.
When the spot of light is at M , x km from P , the angle at L is θ .

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(i) Explain why $\frac{d\theta}{dt} = 8\pi$, where t is the time measured in minutes.

2

(ii) How fast is the spot moving when it is at P ?

2

(iii) How fast is the spot moving when it is at a point on the shoreline 2 km from P ?

JCM