ar 12 3 Unit Trial HSC Barker College 1999 - Solutions

Suggested Marking Scheme as a result of markers meeting

estion 1

$\lim_{x\to 0} \frac{\sin 5x}{2x} = \lim_{x\to 0} \frac{\sin 5x}{5x} \times \frac{5}{2}$

$= 1 \times \frac{5}{2} = \frac{5}{2}$ (1)

(i) $\int_0^1 \frac{e^{2x}}{e^{2x}+1}\,dx$

$\frac{1}{2}\int_0^1 \frac{2e^{2x}}{e^{2x}+1}\,dx$

$\frac{1}{2}\left[\ln(e^{2x}+1)\right]_0^1$ (1)

$\frac{1}{2}\left[\ln(e^2+1) - \ln(e^0+1)\right]$

$\frac{1}{2}\left[\ln(e^2+1) - \ln(1+1)\right]$

$\frac{1}{2}\left[\ln(e^2+1) - \ln 2\right] = \frac{1}{2}\ln\left(\frac{e^2+1}{2}\right)$ (1)

$\int_0^4 \frac{3}{\sqrt{16-x^2}}\,dx$

$3\left[\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4$ (1)

$\left[\sin^{-1}(1) - \sin^{-1}(0)\right]$

$\left(\frac{\pi}{2} - 0\right)$

$\frac{3\pi}{2}$ (1)

$\frac{2x}{x-1} > 1$

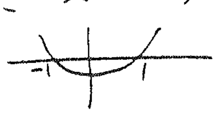
$-1)^2 \times \frac{2x}{(x-1)} > 1.(x-1)^2$

$x(x-1) > (x-1)^2$

$2x^2 - 2x > x^2 - 2x + 1$ (1)

$x^2 - 1 > 0$

$x-1)(x+1) > 0$



$x < -1$ or $x > 1$ (1)

Externd $\Rightarrow$ $k : l = -3 : 5$

(e) For $y = \log_e x$, $y' = \frac{1}{x}$

when $x = 1$, $m_1 = 1$

For $y = 1 - x^2$, $y' = -2x$ (1)

When $x = 1$, $m_2 = -2$

$\therefore \tan\theta = \left|\frac{1+2}{1+1\times -2}\right| = \left|\frac{3}{-1}\right| = 3$ (1)

$\therefore \theta \doteqdot 71^\circ 34'$ (1)

Question 2

(a)(i) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ (1)

(ii) $\cos 105^\circ = \cos(60^\circ + 45^\circ)$

$= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$ (1)

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{1-\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$ (1)

$= \frac{\sqrt{2}-\sqrt{6}}{4}$

(b)(i) $P(\text{no girls}) = \frac{{}^4C_3}{{}^{12}C_3} = \frac{4}{220} = \frac{1}{55}$ (1)

(ii) $P(\text{exactly 1 girl}) = \frac{{}^8C_1 \times {}^4C_2}{{}^{12}C_3} = \frac{8\times 6}{220} = \frac{12}{55}$ (1)

(iii) $P(\text{at least 2 girls}) = 1 - P(\text{No girls or 1 girl})$

$= 1 - \left(\frac{1}{55} + \frac{12}{55}\right)$ (1)

$= \frac{42}{55}$

(c) $LHS = \frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1+(2\cos^2\theta - 1)}$ (1)

$= \frac{2\sin\theta\cos\theta}{2\cos^2\theta}$

$= \frac{\sin\theta}{\cos\theta}$ (1)

$= \tan\theta = RHS$

$$\int_0^1 x\sqrt{1-x}\,dx = \int_1^0 -(1-u)\sqrt{u}\,du \quad (1)$$

$$= \int_1^0 -u^{1/2}(1-u)\,du$$

$$= \int_1^0 -u^{1/2}+u^{3/2}\,du$$

$$= \left[-\frac{2u^{3/2}}{3}+\frac{2u^{5/2}}{5}\right]_1^0 \quad (1)$$

$$= 0+0-\left(-\frac{2}{3}+\frac{2}{5}\right)$$

$$= \frac{2}{3}-\frac{2}{5}$$

$$= \frac{4}{15} \quad (1)$$

uestion 3

i) 12 people

No. of outcomes $= (12-1)!$

$= 11!$ (1)

$= 39916800$

i) S R leaving 10 people

i.e. no. of outcomes $= 10!$

But can have SR or RS, thus (1) for method

no. of outcomes $= 2\times 10!$

$$P(\text{S and R together}) = \frac{2\times 10!}{11!}$$

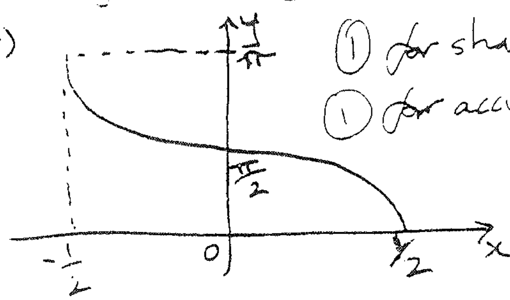
$$= \frac{2}{11} \quad (1)$$

(i) Domain $= -1 \le 2x \le 1$

$\therefore -\frac{1}{2} \le x \le \frac{1}{2}$ (1)

Range $= 0 \le y \le \pi$

ii)



(1) for shape

(1) for accuracy

)(i) $\tan^{-1}(\sqrt{3}) - \tan^{-1}(-1) = \frac{\pi}{3} - \left(-\frac{\pi}{4}\right)$

$= 7\pi$ (1)

$$\therefore \text{Area} = \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2\sqrt{3}}$$

$$= \frac{1}{2}\left[\tan^{-1}(\sqrt{3}) - \tan^{-1}(-1)\right]$$

$$= \frac{1}{2}\times\frac{7\pi}{12}$$

$$= \frac{7\pi}{24} \text{ units}^2 \quad (1)$$

(d) If $n=1$, $7^1-1=6$ which is divisible by 6

$\therefore$ Statement is true for $n=1$ (1)

Assume statement is true for $n=k$

i.e. $\frac{7^k-1}{6} = M$ (where M is an integer)

i.e. $7^k-1 = 6M$

i.e. $7^k = 6M+1$

Now, $7^{k+1}-1 = 7^k \times 7^1 - 1$

$= (6M+1)7-1$

$= 42M+7-1$

$= 42M+6$

$= 6(7M+1)$ which is divisible by 6. (1)

$\therefore$ If statement is true for $n=k$, then statement is true for $n=k+1$.

Thus, since statement is true for $n=1$, it is true for $n=2,3,4$, etc.

Thus, statement is true for all $n \ge 1$ (where n is an integer) (1)

Question 4

(a) $y = \sin^{-1}(\cos x)$

$$\therefore \frac{dy}{dx} = \frac{-\sin x}{\sqrt{1-\cos^2 x}} \quad (1)$$

$$= \frac{-\sin x}{\sqrt{\sin^2 x}}$$

$$= \frac{-\sin x}{\sin x}$$

$$= -1 \text{ (if } \sin x > 0) \quad (1)$$

(b) General term $= {}^6C_r\, x^{6-r}\left(\frac{1}{2x^2}\right)^r$

$= {}^6C_r\, x^{6-r}\left(\frac{1}{2}\right)^r (x^{-2})^r$

...rm independent of x occurs when $6-3r=0$

$\therefore 3r=6$ i.e. when $r=2$ (1)

$\text{Term} = {}^6C_2 \times \left(\frac{1}{2}\right)^2$

$= 15 \times \frac{1}{4}$

$= \frac{15}{4}$ (1)

OR

(2)

(1)

$3\sin x + 4\cos x = A\sin(x+y)$

$= A\sin x\cos y + A\sin y\cos x$

$A\cos y = 3$ and $A\sin y = 4$

$\cos y = \frac{3}{A}$ and $\sin y = \frac{4}{A}$

$\therefore A = \sqrt{3^2+4^2}$

$= 5$ (1)

$y = \cos^{-1}\left(\frac{3}{5}\right) \doteqdot 0.9273$

$3\sin x + 4\cos x = 5\sin(x+0.9273)$

$\therefore 5\sin(x+0.9273) = 2$ (1)

$\sin(x+0.9273) = 2/5$

$x+0.9273 \doteqdot 0.4115, 2.7301, 6.6947$

$\therefore x \doteqdot 1.8028, 5.7674$ (1) for two solns

OR using t-method
(1) Subst.
(1) Getting quad
(1) Answers

(i) $f(x) = x - \sin x - 2$

$f'(x) = 1 - \cos x$

$f'(x)=0 \Rightarrow \cos x = 1$

$\therefore x = \ldots, \cancel{-2\pi}, 0, 2\pi, 4\pi, \ldots$ (1)

$y = \ldots, \times, -2, 2\pi-2, 4\pi-2, \ldots$

x	-1	0	1
y'	+	0	+

$\therefore$ Horizontal pt of inflexion at $(0,-2)$

x	6	2π	7
y'	+	0	+

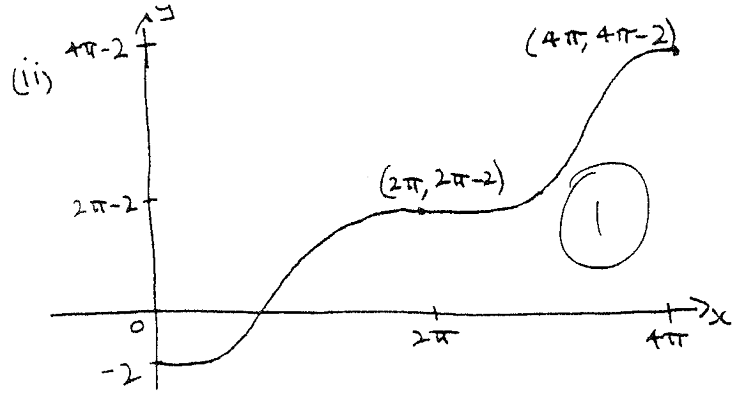
$\therefore$ Horizontal pt of inflexion at $(2\pi, 2\pi-2)$

x	12	4π	13
y'	+	0	+

$\therefore$ Horizontal pt of inflexion at $(4\pi, 4\pi-2)$ (1)

$f''(x) = \sin x$

(ii)



(1)

Question 5

(a)(i) If $T = T_0 + Ae^{-kt}$, then $T - T_0 = Ae^{-kt}$

Now, $\frac{dT}{dt} = 0 - kAe^{-kt}$

$= -k(Ae^{-kt})$

$= -k(T-T_0)$ (1)

(ii) $T_0 = 70$ and thus $T = 70 + Ae^{-kt}$

When $t=0$, $T=150$

$\therefore 150 = 70 + Ae^0$

$\therefore 150 = 70 + A \quad \therefore A = 80$ (1)

When $t=40$, $T=90$

$\therefore 90 = 70 + 80e^{-40k}$

$\therefore 20 = 80e^{-40k}$

$\therefore e^{-40k} = \frac{20}{80} = \frac{1}{4}$ (1)

$\therefore -40k = \log_e(1/4)$

$\therefore k = \frac{\ln(1/4)}{-40} = \frac{\ln 4^{-1}}{-40} = \frac{\ln 4}{40} \doteqdot 0.0347$ (1)

$T = 76 \Rightarrow 76 = 70 + 80e^{-kt}$

$\therefore 6 = 80e^{-kt}$

$\therefore e^{-kt} = \frac{6}{80} = \frac{3}{40}$

$\therefore -kt = \log_e(3/40)$

$\therefore t = \frac{\ln(3/40)}{-\frac{\ln 4}{40}} \doteqdot 74.74$ minutes (1)

(b)(i) $\frac{dV}{dt} = 1000e^{-2t}$

$\therefore V = \frac{1000e^{-2t}}{-2} + c$

$\therefore V = -500e^{-2t} + c$ (1)

But, when $t=0$, $V=0$

$\therefore 0 = -500e^0 + c \quad \therefore c = 500$

$\therefore V = 500 - 500e^{-2t}$ (1)

$\therefore e^{-2t} = 1 - \frac{4}{5} = \frac{1}{5}$

$\therefore -2t = \log_e(\frac{1}{5})$ 1

$t = \frac{\ln(1/5)}{-2} \doteqdot 0.8047$ seconds

) As $t \to \infty$, $e^{-2t} \to 0$

$\therefore (1-e^{-2t}) \to 1$

$\therefore 500(1-e^{-2t}) \to 500$ 1

Max possible volume $= 500$ units^3

Assuming sphere $\Rightarrow V = \frac{4}{3}\pi r^3$

when $V = 400$,

$400 = \frac{4}{3}\pi r^3$

$\therefore 300 = \pi r^3$

$\therefore r^3 = \frac{300}{\pi}$ $\quad \therefore r = \sqrt[3]{\frac{300}{\pi}} \doteqdot 4.5708$ 1

ow, if $V = \frac{4}{3}\pi r^3$, then

$\frac{dV}{dr} = 4\pi r^2$

$\therefore \frac{dr}{dV} = \frac{1}{4\pi r^2}$

en $V = 400$, $r = \sqrt[3]{\frac{300}{\pi}}$, $t = \frac{\ln(1/5)}{-2}$

$\therefore \frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$ 1

$\frac{dr}{dt} = \frac{1}{4\pi r^2} \times 1000e^{-2t}$

$= \frac{1}{4\pi\left(\sqrt[3]{\frac{300}{\pi}}\right)^2} \times 1000e^{-2 \times \frac{\ln(1/5)}{-2}}$

$\doteqdot 0.7618$ 1

stion 6

$=(1+x)^{m+n} = 1 + {}^{m+n}C_1 x + {}^{m+n}C_2 x^2 + \ldots$

oefficient of $x^2 = {}^{m+n}C_2 = \binom{m+n}{2}$ 1

$\therefore (1+x)^m (1+x)^n$

$(1 + {}^mC_1 x + {}^mC_2 x^2 + \ldots)(1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots)$

containing x^2 will be 1

$\ldots x^2 + {}^mC_1 x \times {}^nC_1 x + {}^mC_2 x^2 \times 1$

1

$\therefore$ Comparing coefficients of x^2 on both sides

$$\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + \binom{m}{1}\binom{n}{1}$$

(b)(i) $x = \sqrt{3}\cos 3t - \sin 3t$

$\dot{x} = -3\sqrt{3}\sin 3t - 3\cos 3t$

$\ddot{x} = -9\sqrt{3}\cos 3t + 9\sin 3t$

$= -9(\sqrt{3}\cos 3t - \sin 3t)$ 1

$\therefore \ddot{x} = -9x$ which in the form $\ddot{x} = -n^2x$

$\therefore$ Motion is SHM.

(ii) Period $= \frac{2\pi}{3}$ 1

(iii) When $x = 0$,

$0 = \sqrt{3}\cos 3t - \sin 3t$ 1

$\therefore \sin 3t = \sqrt{3}\cos 3t$

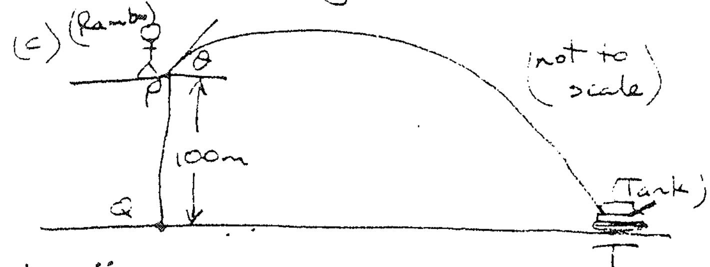
$\therefore \tan 3t = \sqrt{3}$

$\therefore 3t = \frac{\pi}{3}, \frac{4\pi}{3}, \ldots$

$\therefore t = \frac{\pi}{9}, \frac{4\pi}{9}, \ldots$ 1

$\therefore$ First passes origin at $t = \frac{\pi}{9}$ seconds

(c)



(i) $\ddot{x} = 0$ $\qquad \ddot{y} = -10$

$\therefore \dot{x} = c_1$ $\qquad \dot{y} = -10t + c_2$

when $t = 0$, $\dot{x} = V\cos\theta$ and $\dot{y} = V\sin\theta$

$\therefore \dot{x} = V\cos\theta$ $\qquad \dot{y} = -10t + V\sin\theta$

$\therefore x = Vt\cos\theta + c_3$ $\quad y = -5t^2 + Vt\sin\theta + c_4$

Let P be origin, thus when $t = 0$, $x = 0$ and $y = 0$

$\therefore c_3 = c_4 = 0$

$\therefore x = Vt\cos\theta$ and $y = -5t^2 + Vt\sin\theta$ 1

Now, $V = \frac{190}{\sqrt{3}}$ and when $t = 20$, $y = -100$

$\therefore -100 = -5 \times 20^2 + \frac{190}{\sqrt{3}} \times 20 \times \sin\theta$ 1

$\therefore -100 = -2000 + \frac{3800\sin\theta}{\sqrt{3}}$

(iii) Max height occurs when $\dot{y} = 0$

$\therefore 0 = -10t + \frac{190}{\sqrt{3}}\sin 60°$ (1)

$\therefore 10t = \frac{190}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = 95$

$\therefore t = 9.5$ sec

when $t = 9.5$,

$y = 100 + (-5 \times 9.5^2 + \frac{190}{\sqrt{3}} \times 9.5 \times \sin 60°)$

$y = 100 - 451.25 + 902.5$

Max height $= 551.25$ m (1)

estion 7

(i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

P, $m = \frac{2ap}{2a} = p$

n of ~~tangent at~~ P is

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

Q, $m = \frac{2aq}{2a} = q$

n of tangent at Q is

$y - aq^2 = q(x - 2aq)$

$y - aq^2 = qx - 2aq^2$

$\therefore y = qx - aq^2$ (1)

$\left.\begin{array}{l} y = px - ap^2 \\ y = qx - aq^2 \end{array}\right\} \therefore px - ap^2 = qx - aq^2$

$\therefore px - qx = ap^2 - aq^2$

$\therefore x(p - q) = a(p - q)(p + q)$

$\therefore x = a(p + q)$

$y = ap(p+q) - ap^2$

$= ap^2 + apq - ap^2$

$= apq$

$\equiv (a(p+q), apq)$ (1)

T lies on parabola $x^2 = -4ay$

$a^2(p+q)^2 = -4a^2pq$

(iv) Midpt of $PQ = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$

$= \left(a(p+q), \frac{a}{2}(p^2 + q^2)\right)$

$\therefore x = a(p+q)$ and $y = \frac{a}{2}(p^2 + q^2)$ (1)

$\therefore \frac{x}{a} = p + q \qquad y = \frac{a}{2} \times -6pq = -3apq$

$\therefore \frac{y}{-3a} = pq$

$\left.\begin{array}{l} \text{Now, if } p^2 + q^2 = -6pq\text{, then} \\ p^2 + 2pq + q^2 = -4pq \\ \therefore (p+q)^2 = -4pq \end{array}\right)$ using these (1)

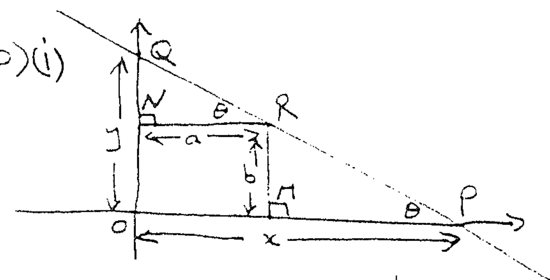
Thus, $\left(\frac{x}{a}\right)^2 = -4 \times \frac{y}{-3a}$

$\therefore \frac{x^2}{a^2} = \frac{4y}{3a}$

$\therefore$ Eqn of locus of midpt of PQ is

$y = \frac{3x^2}{4a}$ (1)

(b)(i)



From ΔRPM, $\tan\theta = \frac{b}{x - a}$

$\therefore \cot\theta = \frac{x - a}{b}$

$\therefore b\cot\theta = x - a$

$\therefore x = a + b\cot\theta$

From ΔQRN, $\tan\theta = \frac{y - b}{a}$

$\therefore a\tan\theta = y - b$

$\therefore y = b + a\tan\theta$

Now, length of $PQ = \sqrt{x^2 + y^2}$

$\therefore l^2 = x^2 + y^2$

$= (a + b\cot\theta)^2 + (b + a\tan\theta)^2$

$= a^2 + 2ab\cot\theta + b^2\cot^2\theta + b^2 + 2ab\tan\theta + a^2\tan^2\theta$

$= a^2 + a^2\tan^2\theta + b^2 + b^2\cot^2\theta + 2ab(\tan\theta + \cot\theta)$

$= a^2(1 + \tan^2\theta) + b^2(1 + \cot^2\theta) + 2ab\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$

(2) for method

$l^2 = a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta + 2ab\left(\frac{1}{\sin\theta\cos\theta}\right)$

$= a^2\sec^2\theta + 2ab\sec\theta\operatorname{cosec}\theta + b^2\operatorname{cosec}^2\theta$

$l^2 = (a\sec\theta + b\operatorname{cosec}\theta)^2$

$l = a\sec\theta + b\operatorname{cosec}\theta$ (since $l > 0$)

$l = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$

$l = a(\cos\theta)^{-1} + b(\sin\theta)^{-1}$

$l' = -a(\cos\theta)^{-2} \times -\sin\theta - b(\sin\theta)^{-2}\cos\theta$

$= \frac{a\sin\theta}{\cos^2\theta} - \frac{b\cos\theta}{\sin^2\theta}$ ← 1

$l' = 0 \Rightarrow \frac{a\sin\theta}{\cos^2\theta} - \frac{b\cos\theta}{\sin^2\theta} = 0$

$\therefore \frac{a\sin\theta}{\cos^2\theta} = \frac{b\cos\theta}{\sin^2\theta}$

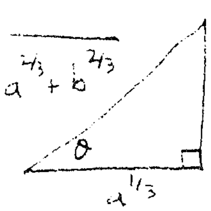
$\therefore a\sin^3\theta = b\cos^3\theta$

$\therefore \tan^3\theta = \frac{b}{a}$

$\therefore \tan\theta = \left(\frac{b}{a}\right)^{1/3}$

$\therefore \theta = \tan^{-1}\left(\frac{b}{a}\right)^{1/3}$ ← 1

if $\tan\theta = \frac{b^{1/3}}{a^{1/3}}$, Thus



$\cos\theta = \frac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$

and $\sin\theta = \frac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$

$l = \frac{a}{\frac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}} + \frac{b}{\frac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}}$

$= a^{2/3}\sqrt{a^{2/3}+b^{2/3}} + b^{2/3}\sqrt{a^{2/3}+b^{2/3}}$

$= \sqrt{a^{2/3}+b^{2/3}}\,(a^{2/3}+b^{2/3})$

$= (a^{2/3}+b^{2/3})^{3/2}$ 1

Alternative soln

(i) From ΔRPM, $\cos\theta = \frac{x-a}{PR}$ $\therefore PR = \frac{x-a}{\cos\theta}$

From ΔQRN, $\sin\theta = \frac{y-b}{QR}$ $\therefore QR = \frac{y-b}{\sin\theta}$

Now $PQ = PR + QR$

$\therefore PQ = \frac{x-a}{\cos\theta} + \frac{y-b}{\sin\theta}$

$\therefore PQ = \frac{x}{\cos\theta} - \frac{a}{\cos\theta} + \frac{y}{\sin\theta} - \frac{b}{\sin\theta}$

(Now, from ΔOPQ, $\sin\theta = \frac{y}{PQ}$ and $\cos\theta = \frac{x}{PQ}$

$\therefore PQ = \frac{y}{\sin\theta}$ and $PQ = \frac{x}{\cos\theta}$)

$\therefore PQ = PQ - \frac{a}{\cos\theta} + PQ - \frac{b}{\sin\theta}$

$\therefore PQ = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$ 2 for part (i) method

OR

(ii) $l' = \frac{a\sin^3\theta - b\cos^3\theta}{\sin^2\theta\cos^2\theta}$

$= \frac{(a^{1/3}\sin\theta - b^{1/3}\cos\theta)(a^{2/3}\sin^2\theta + (ab)^{1/3}\sin\theta\cos\theta + b^{2/3}\cos^2\theta)}{\sin^2\theta\cos^2\theta}$

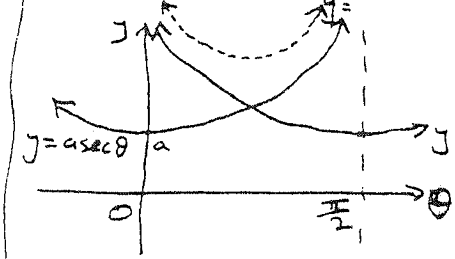
$l' = 0 \Rightarrow a^{1/3}\sin\theta - b^{1/3}\cos\theta = 0$

(since $0 < \theta < \frac{\pi}{2}$ and thus $(a^{2/3}\sin^2\theta + (ab)^{1/3}\sin\theta\cos\theta + b^{2/3}\cos^2\theta) > 0$)

$\therefore a^{1/3}\sin\theta = b^{1/3}\cos\theta$

$\therefore \tan\theta = \frac{b^{1/3}}{a^{1/3}}$ $\therefore \theta = \tan^{-1}\left(\frac{b}{a}\right)^{1/3}$

To prove minimum value, investigate the graphs of $y = a\sec\theta$ and $y = b\operatorname{cosec}\theta$ (where $a > 0$ and $b > 0$) for $0 < \theta < \frac{\pi}{2}$.



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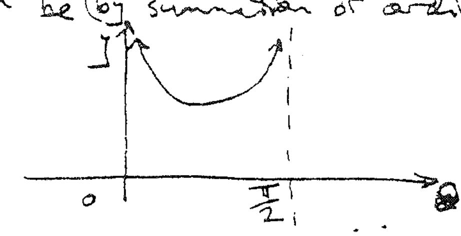
thus the graph of $y = a\sec\theta + b\operatorname{cosec}\theta$ will be (by summation of ordinates)



$\therefore$ Minimum turning pt exists in domain $0 < \theta < \frac{\pi}{2}$