2006 3U. HSC

Question One

(a) $\displaystyle\int \frac{1}{49+x^2}\,dx$

$= \frac{1}{7}\tan^{-1}\frac{x}{7} + C$

(b) $\displaystyle\int x^3\sqrt{x^4+8}\,dx$

$= \frac{1}{4}\left[\frac{2}{3}(x^4+8)^{\frac{3}{2}}\right] + C$

$= \frac{1}{6}(x^4+8)^{\frac{3}{2}} + C$

(c) $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{3x}$

$= \frac{5}{3}\displaystyle\lim_{x\to 0}\frac{\sin 5x}{5x}$

$= \frac{5}{3}(1)$

$= \frac{5}{3}$

(d) $LHS = \dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta\cos\theta} - 1$

$= \dfrac{(\sin\theta+\cos\theta)(\sin^2\theta+\cos^2\theta - \sin\theta\cos\theta)}{\sin\theta+\cos\theta} - 1$

$= \sin^2\theta + \cos^2\theta - \sin\theta\cos\theta - 1$

$= 1 - \sin\theta\cos\theta - 1$

$= -\sin\theta\cos\theta$

$= \frac{-1}{2}\sin 2\theta$

1(e) $y = 12x + b$ $\quad$ $y = x^3$

$dy/dx = 12$ $\quad$ $dy/dx = 3x^2$

Since they touch each other, their derivatives must be the same

$\therefore 12 = 3x^2$

$x = \pm\sqrt{12/3}$

$= \pm 2$

$\therefore y = \pm 8$

when $x = 2, y = 8$,

$8 = 12(2) + b$

$b = -16$

when $x = -2, y = -8$,

$-8 = 12(-2) + b$

$b = 16$

$\therefore b = 16, -16$.

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Question Two:

(a) $f(x) = \sin^{-1}(x+5)$

(i) Domain

$-1 \le x+5 \le 1$

$-6 \le x \le -4$

$\therefore$ domain is $[-6, -4]$

range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Range

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$-\frac{\pi}{2} \le f(x) \le \frac{\pi}{2}$

(ii)

$f'(x) = \dfrac{1}{\sqrt{1-(x+5)^2}}$

at $x = -5$

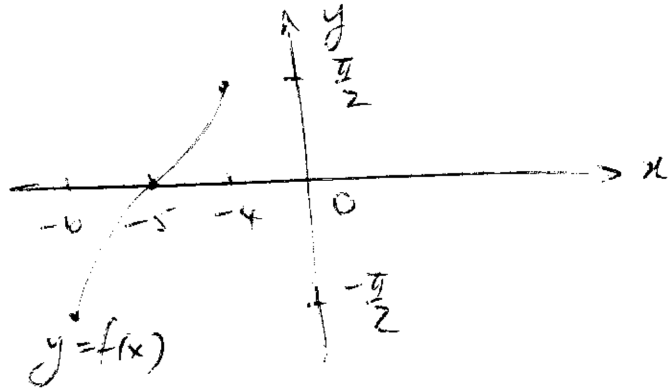
$f'(-5) = \dfrac{1}{\sqrt{1-(-5+5)^2}}$

$= \dfrac{1}{\sqrt{1}}$

$= 1$

$\therefore$ gradient as required $= 1$

(iii)



Question Two

(b)(i) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n}x^n$

differentiating wrt x

$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \ldots + r\binom{n}{r}x^{r-1} + \ldots$

$+ n\binom{n}{n}x^{n-1}$

(ii) $n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + \ldots + r\binom{n}{r}x^{r-1} + \ldots + n\binom{n}{n}x^{n-1}$

Replace x by 2

$n(1+2)^{n-1}$

$= n(3)^{n-1}$

$= \binom{n}{1} + 2\binom{n}{2}(2) + \ldots + r\binom{n}{r}2^{r-1} + \ldots + n\binom{n}{n}2^{n-1}$

$= \binom{n}{1} + \ldots + r\binom{n}{r}2^{r-1} + \ldots + n\binom{n}{n}2^{n-1}$.

(c) (i) chord PR: $y = \frac{1}{2}(p+r)x - apr$

U when $x = 0$,

$y_U = \frac{1}{2}(p+r)(0) - apr$

$= -apr$

$\therefore U(0, -apr)$

(ii) $\left.\begin{array}{ll} y = px - ap^2 & \text{(tangent at P)} \\ y = qx - aq^2 & \text{(tangent at Q)} \end{array}\right\}$ These intersect at T.

equating the y's,

$px - ap^2 = qx - aq^2$

$(p-q)x = a(p^2 - q^2)$

Question Two

(c) (ii)

$$x = \frac{a(p^2-q^2)}{p-q}$$
$$= \frac{a(p-q)(p+q)}{p-q}$$
$$= a(p+q)$$

subtracting the two above equations, after multiplying $y = px - ap^2$ by q and multiplying $y = qx - aq^2$ by p:

$$qy - py = (pqx - ap^2q) - (pqx - apq^2)$$
$$(q-p)y = -apq(p-q)$$
$$y = apq$$

$\therefore T(a(p+q), apq)$

(iii) $T(a(p+q), apq)$
$U(0, -apr)$

m_{TU} (gradient of TU)

$$= \frac{apq - (-apr)}{a(p+q) - 0}$$
$$= \frac{p(q+r)}{(p+q)}$$

Using the evenness of the parabola, the parameters for points R and Q will have the same magnitude but opposite sign. ie. $q = -r$

Question Two

(c)(iii) $\therefore m_{TU} = \frac{p(-r+r)}{p+q}$
$$= 0$$

Since the axis of the parabola is vertical and $m_{TU} = 0$, ie horizontal. TU is $\perp$ to the axis

Question Three

(a) $\int_0^{\frac{\pi}{4}} \sin^2 x \, dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} 1 - \cos 2x \, dx$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2} - 0 + 0\right]$$
$$= \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$
$$= \frac{1}{8}(\pi - 2)$$

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(b)(i) For $y = 3\ln x$ and $y = x$ to meet,

$$3\ln x = x$$
$$3\ln x - x = 0$$

Let $f(x) = 3\ln x - x$.

$$f(1.5) = 3\ln 1.5 - 1.5$$
$$= -0.28 < 0$$
$$f(2) = 3\ln 2 - 2$$
$$= 0.079 > 0$$

Since $f(x)$ is continuous in the interval $(1.5, 2)$ and there is a sign change across that interval, there is a point $x = x_p$ in $(1.5, 2)$ such that $f(x_p) = 0$

$$\therefore f(x_p) = 3\ln x_p - x_p$$
$$= 0$$

i.e. $3\ln x_p = x_p$

$\therefore$ $y = 3\ln x$ and $y = x$ meets at a point P.

(ii) $f(x) = 3\ln x - x$

$$f'(x) = \frac{3}{x} - 1$$

By Newton's method, using first approximation $x_0 = 1.5$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 1.5 - \frac{f(1.5)}{f'(1.5)}$$

Question 3

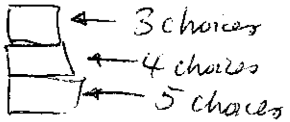
(b)(ii) $x_1 = 1.5 - \dfrac{3\ln 1.5 - 1.5}{\frac{3}{1.5} - 1}$

$$= 1.7836\ldots$$
$$= 1.78 \text{ (2dp, as required)}$$

3(c) 5 blocks

1R, 1B, 1G, 1Y, 1W.

(i)



$\therefore$ # of ways $= 3 \times 4 \times 5$
$= 60$.

(ii) 4 types of towers are possible

- 2 block high
  $= {}^5C_2 \times 2! = 20$
- 3 block high
  $= {}^5C_3 \times 3! = 60$
- 4 block high
  $= {}^5C_4 \times 4! = 120$
- 5 block high
  $= {}^5C_5 \times 5! = 120$

Total # of different towers is 320

Question Three

(d) (i) $\angle QKT = 90^\circ$ ($QP \perp KT$)

$\angle QMT = 90^\circ$ ($QM \perp MN$)

$\therefore \angle QKT + \angle QMT = 180^\circ$

$\therefore$ QKTM is a cyclic quadrilateral (opposite $\angle$s Σ 180°)

(ii) $\angle KMT = \angle KQT$

(angles subtended by same arc KT at the circumference are equal for circle QKTM)

(iii) $\angle PTN = \angle PQT$ (angle in the alternate segment)

$\angle PQT = \angle KQT$ (common)

$= \angle KMT$ (from ii)

$\therefore \angle PTN = \angle KMT$

$\therefore$ $KM \parallel PT$ (corresponding angles are equal in $\parallel$ lines)

Question Four

(a)(i) $P(x) = x^3 + rx^2 + sx + t$

$\Sigma roots = 1 + \alpha - \alpha$

$= 1$

$\Sigma roots = \frac{-b}{a}$

$= -\frac{r}{1}$

$= -r$

$\therefore -r = 1$

$r = -1$

Question 4

(a)(ii) $P(1) = 0$ since 1 is a root of $P(x)$

$= 1^3 + r(1)^2 + s(1) + t$

$= 1 + r + s + t$

$= 1 - 1 + s + t$ since $r = -1$

$= s + t$

$\therefore s + t = 0$

(b)(i) if $x = A\sin(nt + \alpha)$ then,

$\dot{x} = nA\cos(nt + \alpha)$

$\dot{x} = 0,\ t = 0$

$\therefore 0 = nA\cos(\alpha)$

now given amplitude $A = 18$

$\therefore 0 = 18n\cos\alpha \Rightarrow \alpha = \frac{\pi}{2}$

Since period $= \frac{2\pi}{n} = 5$

$n = \frac{2}{5}\pi$

$\therefore x = 18\sin\left(\frac{2\pi}{5}t + \frac{\pi}{2}\right)$

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Question Four

(b)(ii) $x = 18\sin\left(\frac{2\pi}{5}t + \frac{\pi}{2}\right)$

Rest position = amplitude $x = 18$

equilibrium position $x = 0$

$\therefore$ half way $\Rightarrow x = 9$ or -9.

$\therefore 9 = 18\sin\left(\frac{2\pi}{5}t + \frac{\pi}{2}\right)$

$\frac{1}{2} = \sin\left(\frac{2\pi}{5}t + \frac{\pi}{2}\right)$

$\frac{2\pi}{5}t + \frac{\pi}{2} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \ldots$

$\frac{2\pi}{5}t = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \ldots$

$t = -\frac{5}{6}, \frac{5}{6}, \frac{25}{6}, \frac{35}{6}, \ldots$

$-9 = 18\sin\left(\frac{2\pi}{5}t + \frac{\pi}{2}\right)$

$-\frac{1}{2} = \sin\left(\frac{2\pi}{5}t + \frac{\pi}{2}\right)$

$\frac{2\pi}{5}t + \frac{\pi}{2} = \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$

$\frac{2\pi}{5}t = \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots$

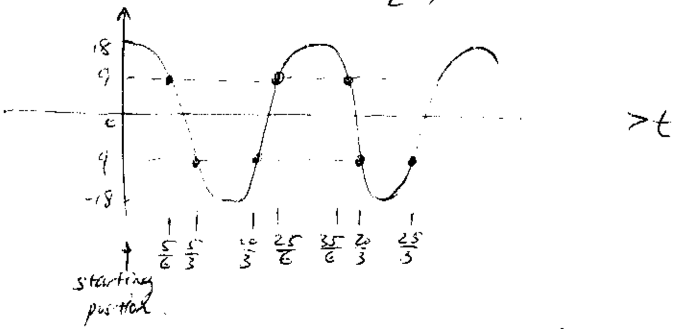
$t = \frac{5}{3}, \frac{10}{3}$

Taking the difference between these times, it can be seen that the time taken to get to the half way point fluctuates between the two values $\frac{5}{3}$ and $\frac{5}{6}$. These means that

it takes the particle $\frac{5}{6}$ seconds to move from its starting position to the half way point and then $\frac{5}{6}$ seconds to the half way point after reaching the equilibrium position then another $\frac{5}{3}$ seconds to reach the ~~to~~ next half way point, and it takes another $\frac{5}{6}$ s to arrive at the next half way point.

As seen from the curve below representing

$x = 18\sin\left(\frac{2\pi}{5}t + \frac{\pi}{2}\right)$



Hence, from my interpretation of the question, to reach the half way point the FIRST TIME after leaving the extreme position is $\frac{5}{6}$ seconds

Question 4

(c) $\ddot{x} = 18x^3 + 27x^2 + 9x$

$x = -2 \quad v = -6 \quad t = 0$

(i) $\ddot{x} = \frac{d(\frac{1}{2}v^2)}{dx} = 18x^3 + 27x^2 + 9x$

$$\int_{-6}^{v} d(\tfrac{1}{2}v^2) = \int_{-2}^{x} 18x^3 + 27x^2 + 9x \, dx$$

$$\tfrac{1}{2}\left[v^2\right]_{-6}^{v} = \left[\tfrac{9}{2}x^4 + 9x^3 + \tfrac{9}{2}x^2\right]_{-2}^{x}$$

$$\tfrac{1}{2}(v^2 - 36) = \tfrac{9}{2}x^4 + 9x^3 + \tfrac{9}{2}x^2 - \tfrac{18}{4}(16) - 9(-2)^3 - \tfrac{9}{2}(-2)^2$$

$$v^2 - 36 = 9x^4 + 18x^3 + 9x^2 - 144 + 144 - 36$$

$$v^2 = 9x^4 + 18x^3 + 9x^2$$

$$= 9x^2(x^2 + 2x + 1)$$

$$= 9x^2(x+1)^2$$

(ii) $v^2 = 9x^2(x+1)^2$

$v = \pm 3x(x+1)$

since at $x = -2$, $v = -6$

$\therefore v = -3x(1+x)$.

$$\frac{dx}{dt} = -3x(1+x)$$

$$\int \frac{dx}{x(1+x)} = -3\int dt$$

$$\int \frac{dx}{x(1+x)} = -3t$$

(iii) $\ln(1 + \frac{1}{x}) = 3t + C$

$x = 2,\ v = -6,\ t = 0$

$\ln(1 - \frac{1}{2}) = 3(0) + C$

$-\ln 2 = C$

$\therefore \ln(1 + \frac{1}{x}) = 3t - \ln 2$

$\ln(2 + \frac{2}{x}) = 3t$

$2 + \frac{2}{x} = e^{3t}$

$\frac{2}{x} = e^{3t} - 2$

$$x = \frac{2}{e^{3t} - 2}$$

Question Five

(a) $dy/dt = -0.7(y-3)$

$$\int \frac{dy}{y-3} = -0.7\int dt$$

$\ln|y-3| = -0.7t + C$ assuming $y > 3$

$y - 3 = e^{-0.7t + C}$

$y - 3 = Ae^{-0.7t}$

since A is an arbitrary constant which is subject to change given conditions

$\therefore$ A can take any real values ie. A can be 10

$\therefore y = 10e^{-0.7t} + 3$.

is a possible solution

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Question Five

(b) $f(x) = \ln(1+e^x)$

$f'(x) = \dfrac{e^x}{1+e^x}$

$= 1 - \dfrac{1}{1+e^x}$

Now, $1+e^x > 1$ for all x

$\dfrac{1}{1+e^x} < 1$ for all x

$\dfrac{-1}{1+e^x} > -1$

$1 - \dfrac{1}{1+e^x} > 0$

$= f'(x)$

$\therefore f'(x) > 0$ for all values of x

i.e. $f(x)$ cannot have a turning point.

$\therefore$ no horizontal line can cut the curve more than once.

$\therefore f(x)$ is an one to one function

$\therefore f(x)$ has an inverse

Question Five

(c) (i) $\dfrac{dx}{dt} = \dfrac{dV}{dt} \cdot \dfrac{dx}{dV}$

Now, $V = \dfrac{\pi}{3} x^2 (3r - x)$

$= \dfrac{\pi}{3}(3rx^2 - x^3)$

$\dfrac{dV}{dx} = \dfrac{\pi}{3}(6rx - 3x^2)$

$\dfrac{dx}{dV} = \dfrac{3}{\pi(6rx - 3x^2)}$

$\therefore \dfrac{dx}{dt} = k \cdot \dfrac{3}{\pi(6rx - 3x^2)}$

$= \dfrac{3k}{\pi x(6r - 3x^2)}$

$= \dfrac{k}{\pi x(2r - x^2)}$

(ii) $\dfrac{dx}{dt} = \dfrac{k}{\pi x(2r - x^2)}$

$\int_0^x \pi x(2r - x^2)\,dx = \int_0^t k\,dt$

$\pi\left[rx^2 - \frac{1}{3}x^3\right]_0^x = k(t-0)$

$\pi\left[rx^2 - \frac{1}{3}x^3\right] = kt$

at $x_A = \frac{2}{3}r$, t_A?

$\pi\left(r\left(\frac{2}{3}r\right)^2 - \frac{1}{3}\left(\frac{2}{3}r\right)^3\right) = kt_A$

$\pi\left(\frac{4}{9}r^3 - \frac{8}{81}r^3\right) = kt_A$

$t_A = \frac{\pi}{k}\left(\frac{28}{81}\right)r^3$

Question 5

(c) (ii) at $x_B = \frac{1}{3}r$, t_B ?

$$\pi\left[r\left(\tfrac{1}{3}r\right)^2 - \tfrac{1}{3}\left(\tfrac{1}{3}r\right)^3\right] = kt_B$$

$$\frac{\pi}{k}\left(\frac{1}{9}r^3 - \frac{1}{81}r^3\right) = kt_B$$

$$t_B = \frac{8}{81}\frac{\pi}{k}r^3$$

$$\frac{t_A}{t_B} = \frac{\frac{\pi}{k}\left(\frac{28}{81}\right)r^3}{\frac{\pi}{k}\left(\frac{8}{81}\right)r^3}$$

$$= \frac{28}{8}$$

$$= 3\cdot 5$$

$\therefore t_A = 3\cdot 5\, t_B$

$\therefore$ It takes 3.5 times as long to fill the bowl to the point $x = \frac{2}{3}r$ as it does to $x = \frac{1}{3}r$.

Question Five

(d)(i) $$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha.\tan\beta}$$

$$\tan(\alpha - \beta).(1 + \tan\alpha\tan\beta) = \tan\alpha - \tan\beta$$

Put $\alpha = (n+1)\theta$ $\beta = n\theta$

$$\therefore \{\tan[(n+1)\theta - n\theta]\}[1 + \tan(n+1)\theta.\tan(n\theta)] = \tan(n+1)\theta - \tan(n\theta)$$

$$\tan\theta[1 + \tan(n+1)\theta\tan(n\theta)] = \tan(n+1)\theta - \tan(n\theta)$$

$$1 + \tan(n+1)\theta\tan(n\theta) = \cot\theta[\tan(n+1)\theta - \tan(n\theta)]$$

(ii) For $n=1$

LHS $= \tan\theta . \tan 2\theta$

RHS $= -2 + \cot\theta . \tan 2\theta$

From part (i)

$$1 + \tan(n\theta)\tan(n+1)\theta = \cot\theta[\tan(n+1)\theta - \tan(n\theta)]$$

Put $n=1$

$$1 + \tan\theta\tan 2\theta = \cot\theta(\tan 2\theta - \tan\theta)$$

$$\tan\theta + \tan^2\theta . \tan 2\theta = \tan 2\theta - \tan\theta$$

$$\tan\theta + \tan^2\theta . \frac{2\tan\theta}{1 - 2\tan^2\theta} = \frac{2\tan\theta}{1 - 2\tan^2\theta} - \tan\theta$$

$$1 + \frac{2\tan^2\theta}{1 - 2\tan^2\theta} = \frac{2}{1 - 2\tan^2\theta} - 1$$

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Question 5 (d) (iii)

$1 + \tan 2\theta \tan\theta = 2\cot\theta . \tan 2\theta - 1$

$-2 + 2\cot\theta . \tan 2\theta = \tan 2\theta \tan\theta$

$\therefore$ RHS = LHS

$\therefore$ Statement is true for $n=1$

Assume statement true for $n=k$, ie. $\tan\theta\tan 2\theta + \dots + \tan k\theta \tan(k+1)\theta = -(k+1) + \cot\theta\tan(k+1)\theta$

Now prove that it's true for $n=k+1$

ie. $\tan\theta\tan 2\theta + \dots + \tan(k+1)\theta\tan(k+2)\theta = -(k+2) + \cot\theta\tan(k+2)\theta$

$LHS = \tan\theta\tan 2\theta + \dots + \tan k\theta \tan(k+1)\theta + \tan(k+1)\theta . \tan(k+2)\theta$

$= -(k+1) + \cot\theta\tan(k+1)\theta + \tan(k+1)\theta\tan(k+2)\theta$ using assumption that '$n=k$' is true.

$\rightarrow$ replacing expression from part (i) by $n=k+1$, we obtain

$1 + \tan(k+1)\theta\tan(k+2)\theta = \cot\theta\left[\tan(k+2)\theta - \tan(k+1)\theta\right]$

$= \cot\theta . \tan(k+2)\theta - \cot\theta . \tan(k+1)\theta$

$\tan(k+1)\theta\tan(k+2)\theta + \cot\theta\tan(k+1)\theta = -1 + \cot\theta . \tan(k+2)\theta$

$\therefore LHS = -(k+1) + \cot\theta . \tan(k+1)\theta + \tan(k+1)\theta . \tan(k+2)\theta$

$= -(k+1) - 1 + \cot\theta . \tan(k+2)\theta$

$= -(k+2) + \cot\theta\tan(k+2)\theta$

$=$ RHS for expression "$n=k+1$"

$\therefore$ Statement is true for $n=k+1$, if its true for $n=k$. But '$n=1$' is true, then '$n=2$' is true ... By Induction, statement is true for all values of n, $n \geq 1$

Question 6

(i) let particle 1 take coordinate (x_1, y_1)
particle 2 take coordinate (x_2, y_2)

Apply distance formula,

$L^2 = (y_2 - y_1)^2 + (x_2 - x_1)^2$

$= (Vt - \frac{1}{2}gt^2 - Vt\sin\theta + \frac{1}{2}gt^2)^2 + (a - Vt\cos\theta)^2$

$= (Vt - Vt\sin\theta)^2 + (a - Vt\cos\theta)^2$

$= V^2t^2(1 + \sin^2\theta - 2\sin\theta) + a^2 - 2aVt\cos\theta + V^2t^2\cos^2\theta$

$= 2V^2t^2 + a^2 - 2aVt\cos\theta - 2V^2t^2\sin\theta$

$= 2V^2t^2(1-\sin\theta) - 2aVt\cos\theta + a^2$

(ii) Differentiating expression (i) implicitly wrt 't'. where θ is a constant, same for a, V.

$2L\frac{dL}{dt} = 4V^2t(1-\sin\theta) - 2aV\cos\theta$

max/min value when $dL/dt = 0$

$\therefore 4V^2t(1-\sin\theta) = 2aV\cos\theta$

$t = \dfrac{a\cos\theta}{2V(1-\sin\theta)}$

$2L\frac{dL}{dt} = 4V^2t(1-\sin\theta) - 2aV\cos\theta$

$= 4V^2(1-\sin\theta)\left[t - \dfrac{a\cos\theta}{2V(1-\sin\theta)}\right]$

Testing minima/maxima,

t	α^-	α	α^+
$\frac{dL}{dt}$	$-$	0	$+$

where $\alpha = \dfrac{a\cos\theta}{2V(1-\sin\theta)}$

$\therefore$ minima at $t = \alpha$

Question 6

(ii) At $t = \dfrac{a\cos\theta}{2V(1-\sin\theta)}$, L?

$$L^2 = 2V^2(1-\sin\theta)\,\frac{a^2\cos^2\theta}{4V^2(1-\sin\theta)^2} - \frac{2aV\cos\theta\cdot a\cos\theta}{2V(1-\sin\theta)} + a^2$$

$$= \frac{a^2\cos^2\theta}{2(1-\sin\theta)} - \frac{a^2\cos^2\theta}{1-\sin\theta} + a^2$$

$$= \frac{a^2}{1-\sin\theta}\left(\frac{\cos^2\theta}{2} - \cos^2\theta + 1 - \sin\theta\right)$$

$$= \frac{a^2}{1-\sin\theta}\left(\frac{-\cos^2\theta + 2 - 2\sin\theta}{2}\right)$$

$$= \frac{a^2}{1-\sin\theta}\left(\frac{1+\sin^2\theta + 2 - 2\sin\theta}{2}\right)$$

$$= \frac{a^2}{1-\sin\theta}\cdot\frac{(1-\sin\theta)^2}{2}$$

$$= \frac{a^2(1-\sin\theta)}{2}$$

$$L = a\sqrt{\frac{1-\sin\theta}{2}} \quad \text{since } L \geq 0$$

$\therefore$ minimum distance $= a\sqrt{\dfrac{1-\sin\theta}{2}}$

(iii) at $t = \dfrac{a\cos\theta}{2V(1-\sin\theta)}$ $y = ?$

$$y = V\sin\theta\left(\frac{a\cos\theta}{2V(1-\sin\theta)}\right) - \frac{1}{2}g\left(\frac{a\cos\theta}{2V(1-\sin\theta)}\right)^2$$

$$= \frac{a\cos\theta\sin\theta}{2(1-\sin\theta)} - \frac{1}{2}\cdot\frac{a^2g\cos^2\theta}{4V^2(1-\sin\theta)^2}$$

$$= \frac{a\cos\theta}{2(1-\sin\theta)}\left[\sin\theta - \frac{ag\cos\theta}{2V^2(1-\sin\theta)}\right]$$

Q 6 (iii)

If $V > \sqrt{\dfrac{ag\cos\theta}{2\sin\theta(1-\sin\theta)}}$

$$V^2 > \frac{ag\cos\theta}{2\sin\theta(1-\sin\theta)}$$

$$\sin\theta > \frac{ag\cos\theta}{2V^2(1-\sin\theta)}$$

$$\sin\theta - \frac{ag\cos\theta}{2V^2(1-\sin\theta)} > 0$$

$\therefore y > 0$

$\therefore$ the smallest distance occurs between the two particles in flight when particle 1 is ascending if $V > \sqrt{\dfrac{ag\cos\theta}{2\sin\theta(1-\sin\theta)}}$

Question 6(b) see after Q7. I left it out whilst copying

Question Seven

(a) Area $= \frac{1}{2}r^2(2\theta) - \frac{1}{2}r^2\sin 2\theta$

$$A = r^2\theta - \frac{1}{2}r^2\cdot 2\sin\theta\cos\theta$$

$$A = r^2(\theta - \sin\theta\cos\theta)$$

(b) $r \cdot 2\theta = \omega$

$$r = \frac{\omega}{2\theta} \rightarrow r^2 = \frac{\omega^2}{4\theta^2}$$

$$\therefore A = \frac{\omega^2}{4\theta^2}(\theta - \sin\theta\cos\theta)$$

$$= \frac{\omega^2}{4\theta} - \frac{\omega^2\sin 2\theta}{8\theta^2}$$

$$\frac{dA}{d\theta} = \frac{-\omega^2}{4\theta^2} - \frac{2\omega^2\cos 2\theta \cdot 8\theta^2 - 16\theta\cdot\omega^2\sin 2\theta}{64\theta^4}$$

$$= \frac{-16\omega^2\theta - 2\omega^2\cos 2\theta\cdot 8\theta + 16\omega^2\sin 2\theta}{64\theta^3}$$

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Question Seven (b)

$$\frac{dA}{d\theta} = \frac{-w^2\theta - w^2\theta\cos 2\theta + w^2 \sin 2\theta}{4\theta^3}$$

$$= \frac{-w^2\theta - w^2\theta(1-\sin^2\theta . 2) + 2w^2\sin\theta\cos\theta}{4\theta^3}$$

$$= \frac{-2w^2\theta + 2w^2\theta\sin^2\theta + 2w^2\sin\theta\cos\theta}{4\theta^3}$$

$$= \frac{-w^2\theta(1-\sin^2\theta) + w^2\sin\theta\cos\theta}{2\theta^3}$$

$$= \frac{w^2\cos\theta(\sin\theta - \theta\cos\theta)}{2\theta^3}$$

(c) $g(\theta) = \sin\theta - \theta\cos\theta$

$g'(\theta) = \cos\theta - \cos\theta + \theta\sin\theta$

$= \theta\sin\theta$

for $0 < \theta < \pi$

$$\left.\begin{array}{l}\sin\theta > 0\\ \theta > 0\end{array}\right\} \theta\sin\theta > 0$$

$\therefore g'(\theta) > 0$

(d) $\frac{dA}{d\theta} = \frac{w^2\cos\theta(\sin\theta - \theta\cos\theta)}{2\theta^3}$

from (c) $\sin\theta - \theta\cos\theta > 0$ and $\theta \neq 0$

then the only 'θ' complement remains is $\cos\theta$

now for $0 < \theta < \pi$, $\cos\theta$ equals 0

only for one value of θ, namely $\theta = \frac{\pi}{2}$

$\therefore$ There is only one value of θ in $(0, \pi)$

for which $dA/d\theta = 0$

Question Seven (e)

$$\frac{dA}{d\theta} = \frac{w^2\cos\theta(\sin\theta - \theta\cos\theta)}{2\theta^3}$$

now since $\sin\theta - \theta\cos\theta > 0$, $\theta > 0$ $w^2 > 0$

$\therefore$ the only factor that changes the sign of $dA/d\theta$ is $\cos\theta$

From (d), $\theta = \frac{\pi}{2}$ gives $dA/d\theta = 0$

Analyzing the behaviour of $\cos\theta$ around $\theta = \frac{\pi}{2}$,

$\cos\theta > 0$ for $0 < \theta < \frac{\pi}{2}$

$\cos\theta < 0$ for $\frac{\pi}{2} < \theta < \pi$

$\therefore \theta = \frac{\pi}{2}$ gives maximum cross sectional area.

$$\therefore A_{max} = \frac{w^2}{4(\frac{\pi}{2})^2}\left(\frac{\pi}{2} - \sin\frac{\pi}{2}\cos\frac{\pi}{2}\right)$$

$$= \frac{w^2}{\pi^2}\left(\frac{\pi}{2}\right)$$

$$= \frac{w^2}{2\pi} \text{ units}^2.$$

Question Six: (b)

(i) 4 member team

$P(\geq 3 \text{ not complete})$

$= P(3 \text{ not complete}) + P(4 \text{ not complete})$

$= \binom{4}{3} q^3 p + \binom{4}{4} q^4$

$= 4pq^3 + q^4$

(ii) To score, at least 2 members must complete

$P(\text{at least 2 complete})$

$= 1 - P(\geq 3 \text{ not complete})$

$= 1 - (4pq^3 + q^4)$

$= 1 - 4pq^3 - q^4$

$= 1 - 4(1-q)q^3 - q^4$ since $p+q=1$

$= 1 - 4q^3 + 3q^4$

(iii) Two member score

$\therefore$ At least 1 must complete

Probability $= \binom{2}{1} pq + \binom{2}{2} p^2$

$= 2(1-q)q + (1-q)^2$

$= 2q - 2q^2 + 1 - 2q + q^2$

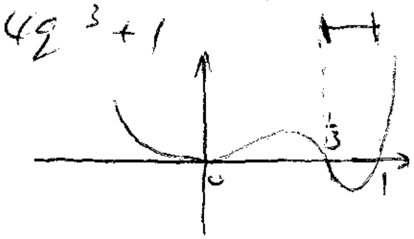
$= 1 - q^2$

(iv) $P(2 \text{ member score}) > P(4 \text{ member score})$

$1 - q^2 > 3q^4 - 4q^3 + 1$

$3q^4 - 4q^3 + q^2 < 0$

$q^2(3q-1)(q-1) < 0$



$\therefore \frac{1}{3} < q < 1$.

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