CSSA - NSW

4 Unit Mathematics

Trial HSC Examination 1983

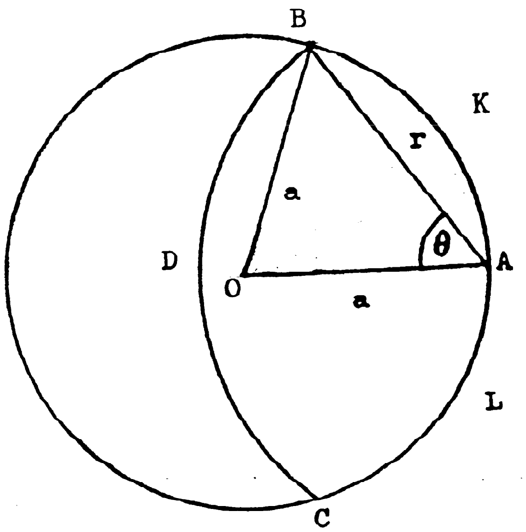
1. The values of the parameter t at the points P, Q, R, S on the hyperbola, given parametrically by $x = t, y = \frac{1}{t}$, are p, q, r, s respectively. Given that P, Q, R, S also lie on the circle whose equation is $x^2 + y^2 + 2gx + 2fy + c = 0$, find the quartic equation whose roots are p, q, r, s expressing the coefficients of this equation in terms of g, f and c. Show that
(i) $p^2 + q^2 + r^2 + s^2 = 4g^2 - 2c$
(ii) $\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} + \frac{1}{s^2} = 4f^2 - 2c$
Hence prove that the sum of the squares of the distances of P, Q, R, S from the origin is equal to the square of the diameter of the circle.

2. **(i)** Find the indefinite integrals
(a) $\int x^3(1 + x^4)^{-2}\ dx$
(b) $\int \frac{x+3}{(x+1)(x^2+1)}\ dx$
(ii) Evaluate $\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\ dx$
(iii) If $I_0 = \int_0^1 x^n(1 - x)^{\frac{1}{2}}\ dx$ show that for $n > 0,\ I_n = (\frac{2n}{2n+3})I_{n-1}$ where n is an integer. Hence, or otherwise evaluate $\int_0^1 x^3(1 - x)^{\frac{1}{2}} dx$

3. **(i)** For the complex number $z = x + iy$, find the locus of z if
(a) $\arg(z - 4) = \frac{\pi}{4}$
(b) $|z| = z + \overline{z} + 1$
(ii) Given that $1 + i$ is a root of the equation $z^2 + (a + 2i)z + (5 + ib) = 0$, where a, b are real, determine the values of a and b.
(iii) If q is real and $z = \frac{3+iq}{3-iq}$ show that as q varies the point in the complex plane which represents z lies on a circle. Find the centre and radius of this circle.
(iv) The complex numbers $z = x + iy$ and $w = u + iv$ are such that $w = z + \frac{1}{z}$. Show that $u = x + \frac{x}{x^2+y^2},\ v = y - \frac{y}{x^2+y^2}$. Find the locus in the $u - v$ plane of a point which, in the $x - y$ plane,
(a) Traces out the circle $|z| = 1$
(b) Traces out the circle $|z| = 2$
and describe each locus geometrically.

4. Show that the area bounded by the upper and lower branches of the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ and the lines $x = \pm a$ is $2ab[\sqrt{2} + \ln(\sqrt{2} + 1)]$ square units. Also show that the volume generated by revolving this area about the y axis is $\frac{4\pi ba^2}{3}[2\sqrt{2} - 1]$ cubic units.

5. (i) On the same axes, sketch the curves $y = xe^{-\frac{1}{2}x^2}$ and $y = e^{-\frac{1}{2}x^2}$ showing clearly the coordinates of any stationary points and any points of inflexion. Find the coordinates of the point of intersection of the two curves and verify that this point is a stationary point of one curve and a point of inflexion of the other.
(ii) In the diagram below, O is the centre of the circle with radius a, BDC is an arc of a circle centre A and radius r.



(a) Show that $r = 2a\cos\theta$
(b) If the area between the two arcs BDC and $BKALC$ is exactly half the area of the circle, centre O and radius a, show that $2\theta\cos 2\theta - \sin 2\theta + \frac{\pi}{2} = 0$. Verify that $\theta = 54°36'$ is a good approximate solution to this equation. Hence show that $r \doteqdot 1.16a$.

6. A particle moves under gravity in a medium for which the resistance to its motion per unit mass is k times its speed, where k is a constant.
(i) If the particle falls vertically from rest that its terminal velocity is given by $V = \frac{g}{k}$.
(ii) If the particle is projected vertically upwards with speed V show that after time t its speed v and height x are given by $v = V(2e^{-kt} - 1)$, $x = \frac{V}{k}(2 - 2e^{-kt} - kt)$. Hence show that the greatest height H that the particle can attain is given by $H = \frac{V}{k}(1 - \ln 2)$.

7. (i) Find all x such that $\cos 2x - \sin 2x = \cos x - \sin x$ and $0 \le x \le 2\pi$
(ii) Given that $z = \cos\theta + i\sin\theta$, use De Moivre's theorem to show that $z^n + z^{-n} = 2\cos n\theta$. Hence or otherwise solve the equation $2z^4 + 3z^3 + 5z^2 + 3z + 2 = 0$.
(iii) Find the limiting sum of the series $\frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \cdots$.

8. (i) Given $S_t = \sum_1^r k^2$, prove by mathematical induction or otherwise that $nS_n - \sum_1^{n-1} S_r = \sum_1^n r^3$ for integral $n > 1$.

(ii) A sequence of numbers $a_1, a_2, a_3, \ldots$ is such that $a_{n+1} - a_n = br^n$ when $r \neq 0, 1$. Given that a_n can be expressed in the form $p + qr^n$, where p and q are independent of n, find the values of p and q in terms of a, b and r. Verify that the numbers 1,4,10,22,... begin a sequence of the above type. Obtain a formula for the nth term of this sequence and find the sum of the first n terms of the sequence.