1994 Higher School Certificate Examination Paper
3 Unit (Additional) and 3/4 Unit (Common) Mathematics

QUESTION ONE

(a) Using the table of standard integrals, find the exact value of $\int_0^{\frac{\pi}{8}} \sec 2x \tan 2x \, dx$.

(b) Evaluate $\int_2^{10} \frac{x}{\sqrt{x-1}} \, dx$ using the substitution $x = t^2 + 1$.

(c) The interval AB has end-points $A(-2, 3)$ and $B(10, 11)$.
Find the coordinates of the point P which divides the interval AB in the ratio $3:1$.

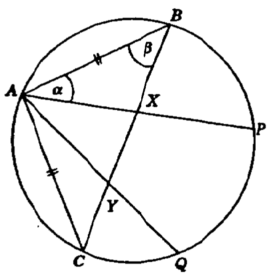
(d) The graphs of $y = x$ and $y = x^3$ intersect at $x = 1$. Find the size of the acute angle between these curves at $x = 1$.

QUESTION TWO

(a) Prove the following identity:

$$\frac{2\tan A}{1+\tan^2 A} = \sin 2A.$$

(b)



Let $ABPQC$ be a circle such that $AB = AC$, AP meets BC at X, and AQ meets BC at Y, as in the diagram. Let $\angle BAP = \alpha$ and $\angle ABC = \beta$.

(i) Copy the diagram and state why $\angle AXC = \alpha + \beta$.

(ii) Prove that $\angle BQP = \alpha$.

(iii) Prove that $\angle BQA = \beta$.

(iv) Prove that $PQYX$ is a cyclic quadrilateral.

(c) (i) Verify that $x = \frac{1}{3}$ and $x = 2$ satisfy the equation $7 - 3x = \frac{2}{x}$.

(ii) On the same set of axes, sketch the graphs of $y = 7 - 3x$ and $y = \frac{2}{x}$.

(iii) Using part (ii), or otherwise, write down all values of x for which $7 - 3x < \frac{2}{x}$.

QUESTION THREE

(a) New cars are subjected to a quality check, which 75% pass. Calculate the probability that of the next ten cars checked, more than seven will pass. Leave your answer in unsimplified form.

(b) Evaluate $\int_0^{\frac{\pi}{3}} 3 \sin x \cos^2 x \, dx$.

(c) Prove by mathematical induction that $n^3 + 2n$ is divisible by 3, for all positive integers n.

(d) Two points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ lie on the parabola $x^2 = 4ay$.

(i) Show that the equation of the tangent to the parabola at P is $y = px - ap^2$.

(ii) The tangent at P and the line through Q parallel to the y axis intersect at T. Find the coordinates of T.

(iii) Write down the coordinates of M, the midpoint of PT.

(iv) Determine the locus of M when $pq = -1$.

QUESTION FOUR

(a) When the polynomial $P(x)$ is divided by $(x+1)(x-4)$, the quotient is $Q(x)$ and the remainder is $R(x)$.

(i) Why is the most general form of $R(x)$ given by $R(x) = ax + b$?

(ii) Given that $P(4) = -5$, show that $R(4) = -5$.

(iii) Further, when $P(x)$ is divided by $(x+1)$, the remainder is 5. Find $R(x)$.

(b) Taking $x = 1{\cdot}0$ as the first approximation, use Newton's method to find a second approximation to the root of

$$x - 3 + e^{2x} = 0.$$

(c) The acceleration of a particle P moving in a straight line is given by

$$\frac{d^2x}{dt^2} = 3x(x-4),$$

where x metres is the displacement from the origin O and t is the time in seconds. Initially the particle is at O and its velocity is $4\sqrt{2}$ m/s.

(i) Using $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$, show that

$$v^2 = 2(x^3 - 6x^2 + 16),$$

where v is the velocity of P.

(ii) Calculate the velocity and acceleration of P at $x = 2$.

(iii) In which direction does P move from $x = 2$? Give a reason for your answer.

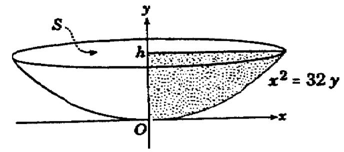
(iv) *Briefly* describe the motion of P after it moves from $x = 2$.

QUESTION FIVE

(a) Find all angles θ, where $0 \le \theta \le 2\pi$, for which

$$\sqrt{3}\sin\theta - \cos\theta = 1.$$

Use the following diagram to answer parts (b) and (c):



(b) The part of the curve $x^2 = 32y$ between $y = 0$ and $y = h$ is rotated about the y axis. Show that the volume enclosed is given by $V = 16\pi h^2$.

(c) The diagram represents the water in a dam on a farm. The depth of the water is h metres, the volume of water in the dam is V m^3, and the area of the surface of the water is S m^2. The water in the dam evaporates according to the rule

$$\frac{dV}{dt} = -kS,$$

where k is a positive constant, and t is the time in hours.

(i) Describe in words what the rule says about the rate of evaporation.

(ii) Show that $\frac{dh}{dt} = -k$.

(iii) Initially the dam contains 64π m^3 of water. Calculate how long it will take for the dam to empty by evaporation when $k = 0{\cdot}001$.

QUESTION SIX

(a) Consider the function $f(x) = 3x - x^3$.

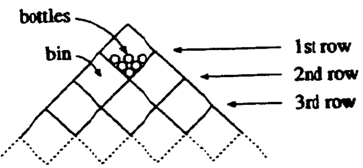
(i) Sketch $y = f(x)$, showing the x and y intercepts and the coordinates of the stationary points.

(ii) Find the largest domain containing the origin for which $f(x)$ has an inverse function, $f^{-1}(x)$.

(iii) State the domain of $f^{-1}(x)$.

(iv) Find the gradient of the inverse function at $x = 0$.

(b)



The figure shows a bottle-storage rack. It consists of n rows of 'bins' stacked in such a way that the number of bins in the rth row is r, counting from the top.

(i) Show that the total number of bins in the storage rack is $\dfrac{n(n+1)}{2}$.

(ii) Each bin in the r th row contains $c+r$ bottles, where c is a constant.
(For example, each bin in the third row contains $c+3$ bottles.)

Find an expression for the total number of bottles in the storage rack.

[You may assume that

$$1^2+2^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1).]$$

(iii) Enzo notices that $c=5$ and that the average number of bottles per bin in the storage rack is 10.
Calculate the number of rows in the storage rack.

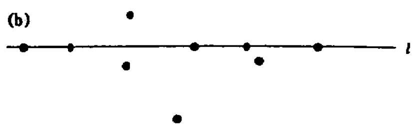
QUESTION SEVEN

(a) (i) Sketch carefully on the same set of axes the graphs of $y=x^4$ and $y=\cos(\pi x)$ for $0 \le x \le 1{\cdot}5$. (Your diagram should be at least half a page in size.)

(ii) On the same diagram, sketch the graph of $y=x^4+\cos(\pi x)$. Label clearly the three curves on your diagram.

(iii) Using the graph, determine the number of positive real roots of the equation $x^4+\cos(\pi x)=0$.

(b)



The figure shows 9 points lying in the plane, 5 of which lie on the line l. No other set of 3 of these points is collinear.

(i) How many sets of 3 points can be chosen from the 5 points lying on l?

(ii) How many different triangles can be formed using the 9 points as vertices?

(c) Uluru is a large rock on flat ground in Central Australia. Three tourists A, B, and C are observing Uluru from the ground. A is due north of Uluru, C is due east of Uluru, and B is on the line-of-sight from A to C and between them. The angles of elevation to the summit of Uluru from A, B, and C are 26°, 28°, and 30°, respectively.

Determine the bearing of B from Uluru.