2001 3 UNIT TRIAL MARKING SCHEME

QUESTION 1

(a) $x = \dfrac{3\times 4 + 1\times(-5)}{3+1}$

$= \dfrac{7}{4}$

$y = \dfrac{3\times(-3) + 1\times 6}{4}$

$= -\dfrac{3}{4}$

the point is $\left(\dfrac{7}{4}, -\dfrac{3}{4}\right)$

(b) $m_1 = -\frac{1}{2}$, $m_2 = \frac{1}{3}$

let θ be the acute angle

$\tan\theta = \left|\dfrac{-\frac{1}{2} - \frac{1}{3}}{1 + (-\frac{1}{2})(+\frac{1}{3})}\right|$

$\theta = 45^\circ$

(c) (i) the angle at the centre is equal to twice the angle at the circumference when they are subtended by the same arc.

(ii) $\angle OBC = 60^\circ$ (alternate angles, $AO \parallel BC$)

$x = 90$ (angle sum of $\triangle BCD$)

(d) $P(x) = x^3 - x^2 - 10x - 8$

(i) $P(-1) = -1 - 1 + 10 - 8 = 0$

so $x = -1$ is a zero of $P(x)$

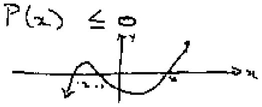
(ii) $(x+1)$ is a factor of $P(x)$

$$\begin{array}{r|l} & x^2 - 2x - 8 \\ x+1 & x^3 - x^2 - 10x - 8 \\ & x^3 + x^2 \\ & -2x^2 - 10x \\ & -2x^2 - 2x \\ & -8x - 8 \\ & -8x - 8 \\ & 0 \end{array}$$

$P(x) = (x+1)(x^2 - 2x - 8)$

$= (x+1)(x-4)(x+2)$

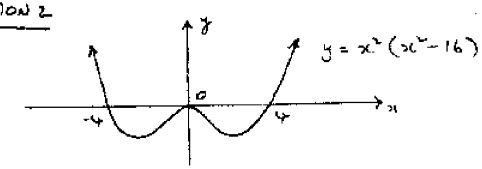
(iii) $P(x) \leq 0$



$x \leq -2$ or $-1 \leq x \leq 4$

QUESTION 2

(a)



(b) (i) $x^2 + 4x + 5 = (x+2)^2 + 1$

(ii) $\displaystyle\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{1+(x+2)^2}$

$= \tan^{-1}(x+2) + C$

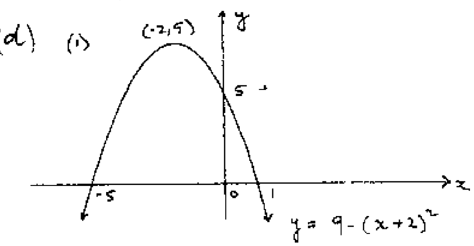
(c) $\cos 2x = \cos x$

$2x = 2n\pi \pm x$

$3x = 2n\pi$ or $x = 2n\pi$

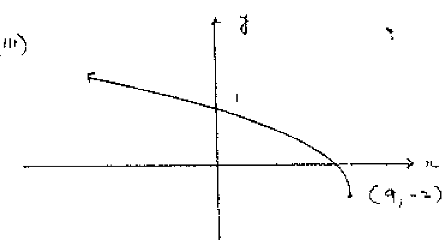
$x = \dfrac{2n\pi}{3}$ for any integer n

(d) (i)



(ii) $x \geq -2$

(iii)



QUESTION 3

(a) $\lim\limits_{x\to 0} \dfrac{\sin 4x}{\tan 2x} = \lim\limits_{x\to 0} \dfrac{\sin 4x}{4x} \times \lim\limits_{x\to 0} \dfrac{2x \times 2}{\tan 2x}$

$= 2$

(b) $\left(x + \frac{1}{x^2}\right)^9$

$T_r = {}^9C_r\, x^r \left(x^{-2}\right)^{9-r}$

$= {}^9C_r\, x^{3r-18}$

for the term independent of x

$3r - 18 = 0$

$r = 6$

Hence the term is ${}^9C_6 = 84$

(c) $\frac{dV}{dt} = 72$

$V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2$

$\frac{dV}{dr} = 4\pi r^2, \quad \frac{dS}{dr} = 8\pi r$

$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dV}{dt} \times \frac{dr}{dV}$

$= \dfrac{8\pi r \times 72}{4\pi r^2}$

$= \dfrac{2 \times 72}{r}$

when $r = 12 \quad \frac{dS}{dt} = 12\ m^2/s$

(d)(i) Consider $f(x) = \sin x - x + \frac{1}{2}$

$f(.5) > 0$

$f(1.8) < 0$

so there is a root between $x = 0.5$ and $x = 1.8$

(ii) $f'(x) = \cos x - 1$

$x = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 1.2 - \dfrac{f(1.2)}{f'(1.2)}$

$= 1.56$ (2 decimal places)

QUESTION 4

(a) $3\sin x + \sqrt{3}\cos x = R\sin(x + \alpha)$

$= R\sin x\cos\alpha + R\cos x\sin\alpha$

$R\sin\alpha = \sqrt{3}$

$R\cos\alpha = 3$

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

$R = \sqrt{3^2 + \sqrt{3}^2}$

$= 2\sqrt{3}$

$3\sin x + \sqrt{3}\cos x = 2\sqrt{3}\sin\left(x + \frac{\pi}{6}\right)$

(b) (i) $\angle BDC = \alpha$ (base angles of isosceles Δ)

$\angle DCR = 2\alpha$ (exterior angle of ΔBCD)

(ii) $\angle BAD = 2\alpha$ (exterior angle of cyclic quad. $ABCD$)

$\therefore \angle OAD = \alpha$ (OA bisects $\angle BAD$)

(iii) $OA \perp AT$ (radius is perpendicular to the tangent at the point of contact)

so, $\angle TAD = 90° - \alpha$

$\angle ABD = \angle TAD$ (alternate segment theorem)

so, $\angle ABC = (90° - \alpha) + \alpha$

$= 90°$

(c)(i) In ΔLMP: $\tan 20° = \frac{LM}{PM}$

$PM = 50\cot 20°$ metres

(ii) $PQ^2 = PM^2 + QM^2 - 2.PM.QM.\cos PMQ$ (cosine rule)

$= 50^2\cot^2 20° + 50^2\cot^2 12° - 2.50\cot 20°.\cot 12°.\cos 65°$

so, $PQ = 50\sqrt{\cot^2 20° + \cot^2 12° - 2\cot 20°\cot 12°.\cos 65°}$

(iii) Speed $= \dfrac{PQ}{10 \times 60}$

$= 0.36\ m/s$ (2 sig. fig.)

QUESTION 5

a) (i) $\frac{d}{dx}\left(x\cos^{-1}x - \sqrt{1-x^2}\right)$

$= \cos^{-1}x - \frac{x}{\sqrt{1-x^2}} - \frac{-\frac{1}{2}2x}{\sqrt{1-x^2}}$ ✓✓

$= \cos^{-1}x$

(ii) $\int \cos^{-1}x\,dx = \left[x\cos^{-1}x - \sqrt{1-x^2}\right]_0^1$

$= 1$ ✓

(b) $u = 1-x \Rightarrow x = 1-u$

$du = -dx$

when $x=-3 \quad u=4$

when $x=0 \quad u=1$ ✓

$I = \int_4^1 \frac{1-u}{\sqrt{u}}\cdot -du$ ✓✓

$= \int_1^4 u^{-1/2} - u^{1/2}\,du$ ✓

$= \left[2u^{1/2} - \frac{2}{3}u^{3/2}\right]_1^4$

$= (4 - \frac{2}{3}\times 4\times 2) - (2 - \frac{2}{3})$

$= -\frac{8}{3}$ ✓

(c) for $(1+x)^{2n}$

the coefficient of x^n is $\binom{2n}{n}$ ✓

$(1+x)^{2n} = (1+x)^n(1+x)^n$

$= \left[\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n\right]\left[\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n\right]$ ✓

the coefficient of x^n is: $\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\cdots+\binom{n}{n}\binom{n}{0}$ ✓

since $\binom{n}{r} = \binom{n}{n-r}$ then the coefficient of x^n is

$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2$ ✓

Equating the co-efficients of x^n gives

$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2 = \binom{2n}{n}$

QUESTION 6

(a) (i) $(3+2x)^{20} = \sum_{r=0}^{20} {}^{20}C_r\, 3^{20-r}(2x)^r$

so, $a_r = {}^{20}C_r\, 3^{20-r}\, 2^r$ ✓

(ii) $\frac{a_{r+1}}{a_r} = \frac{{}^{20}C_{r+1}\, 3^{19-r}\, 2^{r+1}}{{}^{20}C_r\, 3^{20-r}\, 2^r}$

$= \frac{20-r}{r+1}\times\frac{2}{3}$ ✓

$= \frac{40-2r}{3r+3}$

(iii) let $\frac{a_{r+1}}{a_r} > 1$ ✓

then, $\frac{40-2r}{3r+3} > 1$

$40-2r > 3r+3$

$5r < 37$

$r < 7\frac{2}{5}$ ✓

when $r=7$: $a_8 > a_7$

$r=6$: $a_7 > a_6$

$\vdots$

$r=0$: $a_1 > a_0$

i.e. $a_8 > a_7 > a_6 > \cdots > a_0$

if $\frac{a_{r+1}}{a_r} < 1$ then $a_8 > a_9 > \cdots > a_{20}$ ✓

So the greatest co-efficient is $a_8 = {}^{20}C_8\, 3^{12}\, 2^8$ ✓

(b)(i) $m_{PQ} = \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$= \dfrac{p+q}{2}$

equation of PQ: $y - ap^2 = \frac{p+q}{2}(x - 2ap)$

so, $y = \frac{p+q}{2}x - apq$

(ii) If $S \in PQ$ then when $x = 0$, $y = a$

i.e. $a = 0 - apq$

so, $pq = -1$

(iii) M is $\left(a(p+q), \dfrac{ap^2 + aq^2}{2}\right)$

N is $(a(p+q), -a)$

so T is $\left(a(p+q), \dfrac{ap^2 + aq^2 - 2a}{4}\right)$

The locus of T is

$x = a(p+q)$ — (1)

$y = \frac{a}{4}(p^2 + q^2 - 2)$ — (2)

from (ii) $pq = -1$, $y = \frac{a}{4}(p^2 + q^2 + 2pq)$

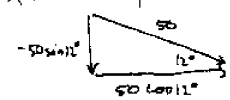
i.e. $y = \frac{a}{4}(p+q)^2$

so, $y = \frac{a}{4}\frac{x^2}{a^2}$ from (1)

i.e. $x^2 = 4ay$

QUESTION 7

(a)(i) 180 km/h $= 50$ m/s



$\dot{x} = 50\cos 12°$

$x = 50t\cos 12° + c_1$

when $t = 0$, $x = 0$

so, $x = 50t\cos 12°$

$\ddot{y} = -10$

$\dot{y} = -10t + c_2$

when $t = 0$ $\dot{y} = -50\sin 12°$

so, $\dot{y} = -10t - 50\sin 12°$

$y = -5t^2 - 50t\sin 12° + c_3$

when $t = 0$ $y = 2.5$

so, $y = -5t^2 - 50t\sin 12° + 2.5$

(ii) when $x = 6$ $t = \dfrac{6}{50\cos 12°}$

when $t = \dfrac{6}{50\cos 12°}$ $y = 1.149\ldots$

so the ball clears the net by 15cm.

(iii) when $y = 0$, $5t^2 + 50t\sin 12° - 2.5 = 0$

$t = \dfrac{-50\sin 12° \pm \sqrt{(50\sin 12°)^2 + 50}}{10}$

when $t = \dfrac{-50\sin 12° + \sqrt{(50\sin 12°)^2 + 50}}{10}$

$x = 10.6468\ldots$

So it lands 7.35 metres from the base line

(b) (i) $4 \text{ revs/min} = 8\pi \text{ rad/min}$

so, $\frac{d\theta}{dt} = 8\pi$

(ii) $\tan\theta = \frac{x}{3}$

$x = 3\tan\theta$

$\frac{dx}{d\theta} = 3\sec^2\theta$

$\frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt}$

$= 3\sec^2\theta \cdot 8\pi$

$= 24\pi\sec^2\theta$.

at P $\theta = 0$

so $\frac{dx}{dt} = 24\pi$ km/min.

(iii) when $x = 2$, $\cos\theta = \frac{3}{\sqrt{13}}$

so, $\frac{dx}{dt} = \frac{24\pi}{\left(\frac{3}{\sqrt{13}}\right)^2}$

$= \frac{104\pi}{3}$ km/min.