Trial Exam — Solutions

1 a) Term in $x^5 = {}^7C_5\, 2^2 (-x)^5$ (1)

$= -84x^5$ (1)

b) $y = e^{2x}\sin x$

$\frac{dy}{dx} = 2e^{2x}\sin x + e^{2x}\cos x$ (1) + (1)

$= e^{2x}(2\sin x + \cos x)$

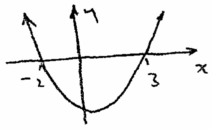
c) $\frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$ (1)

so gradient at $x=1$ is $\frac{1}{\sqrt{4-1}} = \frac{1}{\sqrt{3}}$ (1)

d) $(x-3)(x+2) > 0$ (1)

from the graph (1)

$x < -2$ or $x > 3$



e) Let ϕ be the angle

$\tan\phi = \left|\frac{1-(-2)}{1+1\times -2}\right|$ (1)

$= 3$

so $\phi = 71°34'$ (to nearest minute) (1)

f) (i) $\int \frac{1+e^x}{e^x}\,dx = \int e^{-x} + 1\,dx$

$= -e^{-x} + x + c$ (1)

(ii) $\int \frac{e^x}{1+e^x}\,dx = \log(1+e^x) + c.$ (1)

2. a) $\cos x = -\frac{1}{2}$

so x is in 2nd or 3rd quadrants (1)

thus $x = \frac{2\pi}{3} + 2n\pi$ or $-\frac{2\pi}{3} + 2n\pi$ (1)

b) vertex is $(-3, 1)$, focal length $= 2$, axis vertical (1)

so focus is $(-3, 3)$ (1)

c) (i) gradient $OP = \frac{ap^2}{2ap} = \frac{p}{2}$ (1)

(ii) $\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)}$

$= \frac{2at}{2a}$ (1)

$= t$

(iii) thus at A parameter $t = \frac{p}{2}$

so $A = \left(ap, \frac{ap^2}{4}\right)$ (1)

and $M = \left(ap, \frac{ap^2}{2}\right)$

clearly y-coord of M is twice y-coord of A, as required (1)

d) (i) Expand RHS or

$(\alpha+\beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$ (1)

$= \alpha^3 + \beta^3 + 3\alpha\beta(\alpha+\beta)$

so $\alpha^3 + \beta^3 = (\alpha+\beta)\left[(\alpha+\beta)^2 - 3\alpha\beta\right]$ (1)

or ...

(ii) here $\alpha+\beta = -3$ and $\alpha\beta = -2$

so $\alpha^3 + \beta^3 = -3\left[(-3)^2 - 3\times -2\right]$ (1)

$= -45$ (1)

3 a) (i) $RHS = \frac{1}{2}(1-\cos 2\theta)$ (1)

$= \frac{1}{2}(1-\cos^2\theta+\sin^2\theta)$

$= \frac{1}{2}\cdot 2\sin^2\theta$ (1)

$= LHS$ #

(ii) $\int_0^{\pi}\sin^2\theta\,d\theta = \int_0^{\pi}\frac{1}{2}(1-\cos 2\theta)\,d\theta$

$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$ (1)

$= \frac{\pi}{2}$ (1)

b) $u = 1-x$

at $x=0$ $u=1$ and at $x=1$ $u=0$ (1)

$x = 1-u$

$dx = -du$

so $\int_0^1 (1+3x)(1-x)^7\,dx = \int_1^0 (4-3u)\,u^7\,.(-du)$ (1)

$= \int_0^1 4u^7 - 3u^8\,du$ (1)

$= \left[\frac{u^8}{2} - \frac{u^9}{3}\right]_0^1$

$= \frac{1}{6}$ (1)

c) (i) when $1+\sin\theta = 0$

ie $\theta = \frac{3\pi}{2} + 2n\pi$. (1)

(ii) $LHS = \frac{1}{1+\sin\theta}\cdot\frac{1-\sin\theta}{1-\sin\theta}$ (1)

$= \frac{1-\sin\theta}{\cos^2\theta}$ (1)

$= \sec^2\theta - \sec\theta\tan\theta$

$= RHS$ #

(iii) $\int \frac{1}{1+\sin\theta}\,d\theta = \int \sec^2\theta - \sec\theta\tan\theta\,d\theta$

$= \tan\theta - \sec\theta + c$ (1)

4/ a) (i) $(1+x)^{2n} = \sum_{r=0}^{2n} {}^{2n}C_r\, x^r$ (1)

(ii) at $x = -\frac{1}{2}$ (1)

$$\left(\tfrac{1}{2}\right)^{2n} = \sum_{r=0}^{2n} {}^{2n}C_r \left(-\tfrac{1}{2}\right)^r$$

(iii) $\left(\tfrac{1}{2}\right)^{2n} = \sum_{r=0}^{2n-1} {}^{2n}C_r \left(-\tfrac{1}{2}\right)^r + \left(\tfrac{1}{2}\right)^{2n}$ from part (ii)

thus $\sum_{r=0}^{2n-1} {}^{2n}C_r \left(-\tfrac{1}{2}\right)^r = 0$ (1)

b) (i) $x^2 - 2 + \frac{1}{x^2}$ (1)

(ii) $\left(x^2 + \frac{1}{x^2}\right)^{14} = \sum_{r=0}^{14} {}^{14}C_r (x^2)^{14-r}(x^{-2})^r$ (1)

$$= \sum_{r=0}^{14} {}^{14}C_r\, x^{28-4r}$$

(iii) $\left(x - \frac{1}{x}\right)^2\left(x^2 + \frac{1}{x^2}\right)^{14} = \left(x^2 - 2 + \frac{1}{x^2}\right)\sum_{r=0}^{14} {}^{14}C_r\, x^{28-4r}$

$$= \sum_{r=0}^{14} {}^{14}C_r\, x^{30-4r} - 2\sum_{r=0}^{14} {}^{14}C_r\, x^{28-4r} + \sum_{r=0}^{14} {}^{14}C_r\, x^{26-4r}$$

x^6 term comes from $r=6$ in 1st sum
and $r=5$ in last sum (1)

so coeff of $x^6 = {}^{14}C_6 + {}^{14}C_5$ (1)

$= {}^{15}C_6$ by the recurrence relation (Pascal's Δ)

c) (i) $A_0 = P$

$A_1 = P(1+R) - M$ (1)

$A_2 = P(1+R)^2 - M(1+R) - M$

$\vdots$

$A_n = P(1+R)^n - M\left[(1+R)^{n-1} + \dots + (1+R) + 1\right]$ (1)

$= P(1+R)^n - \dfrac{M\left[(1+R)^n - 1\right]}{R}$

(ii) Here $A_n = 0$ so $P = \dfrac{M\left[(1+R)^n - 1\right]}{R(1+R)}$ (1)

and $M = 450$, $R = 0.012$, $n = 60$ (1)

for which $P \doteqdot 19168 < 20000$

The bank will not give them the loan as they cannot pay it back. (1)

[there are other methods!]

5. a) (i) $\dot{x} = -3\sin 3t + 6\cos 3t$

$\ddot{x} = -9\cos 3t - 18\sin 3t$

$= -3^2(\cos 3t + 2\sin 3t)$. 1

and $n = 3$

(ii) $r^2 = 1^2 + 2^2 = 5$

so $\sin\alpha = \frac{1}{\sqrt{5}}$ and $\cos\alpha = \frac{2}{\sqrt{5}}$ 1

thus $r = \sqrt{5}$, $\alpha = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$. 1

$(\doteqdot 0.46 \text{ rads})$

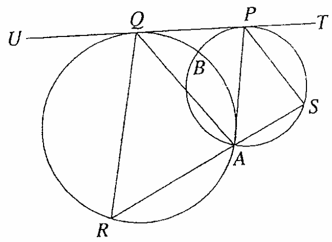
(iii) $r\sin(3t + \alpha) = 2$ 1

so $3t = \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$

$t = \frac{1}{3}\left[\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)\right]$ 1

$\doteqdot 0.2$ to 1 dec. pl.

b)



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(i) exterior angle of cyclic quadrilateral.

(ii) $\angle QAR = \angle UQR$ (angle in alternate segment) 1

$= \angle PSA$

hence $QA \parallel PS$ (corresponding angles equal) 1

(iii) $\angle PAS = \angle TPS$ (angle in alt segment)

$= \angle QRA$ (exterior angle of cyclic quadrilateral) 1

hence in ΔQRA and ΔPAS

$\angle QAR = \angle PSA$ proven

$\angle PAS = \angle QRA$ proven 1

thus $\Delta QRA \;|||\; \Delta PAS$ (AA)

1

c) (i) $4a$

(ii) in the right hand graph, the focal length is longer but the latus rectum is shorter. 1

6. a) (i) $2w$ g/min (1)

(ii) $\frac{Q}{1000}$ g/L (1)

(iii) $\frac{Qw}{1000}$ g/min (1)

(iv) $\frac{dQ}{dt}$ = inflow − outflow

$= 2w - \frac{Qw}{1000}$ (1)

$= -\frac{w}{1000}(Q - 2000)$

(v) $\text{LHS} = -\frac{w}{1000} \cdot Ae^{-wt/1000}$

$\text{RHS} = \frac{-w}{1000}\left(\cancel{2000} + Ae^{-wt/1000} - \cancel{2000}\right)$ (1)

$= \frac{-w}{1000} Ae^{-wt/1000}$

$= \text{LHS}$. #

(vi) at $t = 0$ $Q = 0$ so $A = -2000$ (1)

and $Q = 2000(1 - e^{-wt/1000})$

(vii) as $t \to \infty$, $e^{-wt/1000} \to 0$ hence $Q \to 2000$ (1)

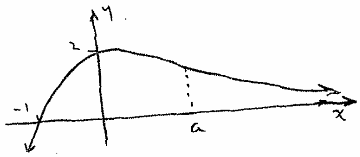
(viii) $1000 = 2000(1 - e^{-w345/1000})$

$e^{w345/1000} = 2$

$w = \frac{1000}{345} \log 2$ (1)

$(\doteqdot 2\ \text{L/min}.)$

b) (i)



(other graphs are possible) (1) + (1)

(ii) for $x < a$ $f(x)$ is concave down (1)

for $x > a$ $f(x)$ is concave up

hence $f(x)$ changes concavity and there is an inflexion point. (1)

More precisely, for the curve to rise from & return to the x-axis it must be concave down over some domain. Also $f(x)$ must be decreasing over some range. As $x \to \infty$ $f'(x)$ increases to zero, so $f(x)$ is concave up, and there is a change in concavity

7 a) (i) $PQ = rx$

$QR = r\sin x$ ①

(ii) along each tooth of radius r the ant travels $r(x+\sin x)$ ①

each successive tooth has radius $\cos x$ times the previous

so $y = (x+\sin x) + \cos x\,(x+\sin x) + \cos^2 x\,(x+\sin x) + \ldots$ ①

$= \dfrac{x+\sin x}{1-\cos x}$ ①

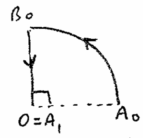
(iii) $y' = \dfrac{(1-\cos x)(1+\cos x) - (x+\sin x)(\sin x)}{(1-\cos x)^2}$

$= \dfrac{\sin^2 x - x\sin x - \sin^2 x}{(1-\cos x)^2}$

$= \dfrac{-x\sin x}{(1-\cos x)^2} < 0$ for $0 < x \leq \frac{\pi}{2}$ ①

ie y is decreasing so min is at right hand end pt ①

$y\left(\frac{\pi}{2}\right) = \dfrac{\frac{\pi}{2}+1}{1-0} = \frac{\pi}{2}+1$ ①



b) (i) (α) the square of the tangent is equal to the product of the intercepts of the secant ①

(β) $a^2 + 2ar - t^2 = 0$

$a^2 + 2ar + r^2 = t^2 + r^2$

$(a+r)^2 = t^2 + r^2$

$a + r = \sqrt{t^2 + r^2}$

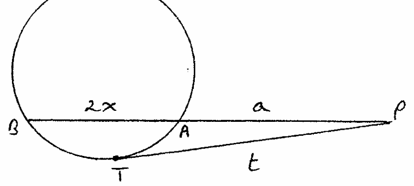
$\geqslant t$ ①

with equality when $r = 0$.

LHS is AM of PA & PB

RHS is GM of PA & PB. ①

(ii)



$t^2 = a(a+2x)$ so t is constant

so locus is the circle centre P radius t ①

less the points where the line through PB intersects the circle. ①