Centre Number

Student Number

2008

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Afternoon Session
Thursday, 14 August 2008

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1-7
- All questions are of equal value

3201 - 1

Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Find the exact value of $\int_0^{\frac{\pi}{8}} \sec^2 2x\, dx$. **2**

(b) (i) Sketch the graph of $y = |2 - x|$. **1**

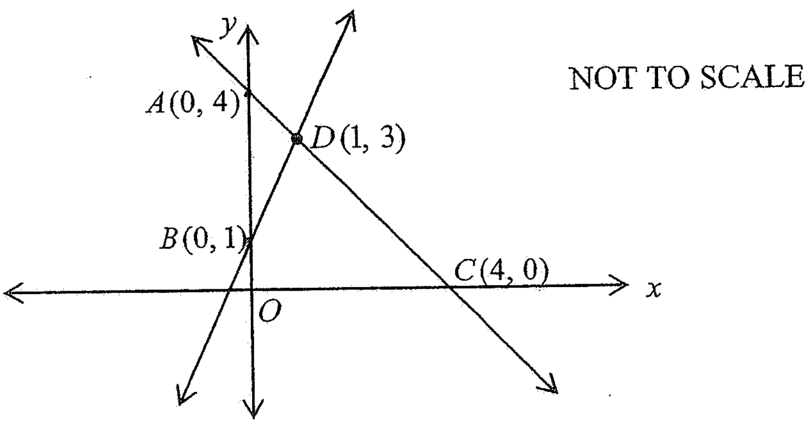
(ii) Using this graph, or otherwise, find the solution to $|2 - x| < x$. **2**

ie. consider the sketches of

$f(x) = |2 - x|$ and $g(x) = x$

(c) Find the value of k if $x + 2$ is a factor of $P(x) = x^2 + kx + 6$. **2**

(d) $A(0, 4)$, $B(0, 1)$, $C(4, 0)$ and $D(1, 3)$ are points in the plane where D is the point of intersection of the two lines shown. Find, correct to the nearest minute, the size of the acute angle, $\angle BDC$, between the two lines. **3**



(e) Jasi was trying to find the solution to the inequality $\frac{3}{x+1} < 2$. **2**

He stated that the solution is all values of x greater than $\frac{1}{2}$.

Solve the inequality $\frac{3}{x+1} < 2$ to determine if Jasi's solution is correct.

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) If α, β and γ are the roots of $2x^3 - 5x^2 + 3x - 5 = 0$ find the value of $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. **2**

(b) Let $f(x) = \dfrac{2x}{\sqrt{1-x^2}}$.

(i) For what values of x is $f(x)$ undefined? **2**

(ii) Find $\displaystyle\int_0^{\frac{1}{2}} \frac{2x}{\sqrt{1-x^2}}\,dx$, using the substitution $x = \sin u$. **3**

(c) (i) Find the derivative of $\sin^{-1} x + \cos^{-1} x$. **1**

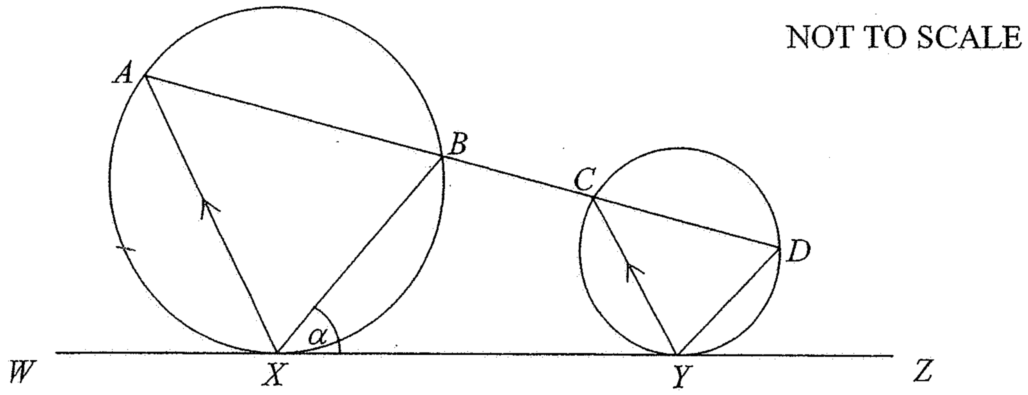
(ii) Explain why $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$. **2**

(d) Sketch the curve given by $y = 2\sin^{-1} x - 1$. 2

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) In the diagram below, WZ is a common tangent to the two circles and AX is parallel to CY. AD is a straight line through B and C on the circles as shown. Let $\angle BXY = \alpha$.



Copy or trace this diagram into your writing booklet.

(i) Explain why BX is parallel to DY. 3

(ii) Show that $BCYX$ is a cyclic quadrilateral. 1

(b) If A and B are both reflex angles, and given $\cos A = \frac{3}{5}$ and $\tan B = \frac{12}{5}$, 3

find the exact value of $\sin(A - B)$.

(c) (i) Explain how you would know that 1

$x^3 + x^2 + x - 8$ has a root between $x = 1$ and $x = 2$.

(ii) Use the method of bisection once to find a closer approximation. 1

(d) Taking $x = 2$ as the first approximation, use one application of Newton's method to obtain a closer approximation to the solution to $x = \sqrt[3]{9}$. 2

Hence, find the approximate percentage error by using this method. 1

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) Use the method of Mathematical Induction to show that 3

$$2 + 5 + 8 + \dots + (3n - 1) = \frac{n(3n+1)}{2}.$$

(b) The acceleration of a particle P, moving in a straight line, is given by $\ddot{x} = 2x - 3$ where x metres is the displacement from the origin O. Initially the particle is at O and its velocity v is 2 metres per second.

(i) Show that the velocity v of the particle is $v^2 = 2x^2 - 6x + 4$. 2

(ii) Calculate the velocity and acceleration of P at $x = 1$ and briefly describe the motion of P after it moves from $x = 1$. 2

(c) The rate of change of the number of bees infected by a disease is given by the equation $\dfrac{dN}{dt} = N(200 - N)$, where N is the number of infected bees in the hive at time t years. There are 200 bees in the hive.

(i) If k is a constant, show that $N = \dfrac{200}{1 + ke^{-200t}}$ satisfies the above equation. 2

(ii) If at time $t = 0$ one bee was infected, after how many days will half the colony be infected? 2

(iii) Show that eventually all the bees will be infected. 1

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet

(a) Let $P(x) = -2x^3 + px^2 - qx + 5$.

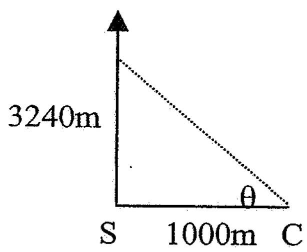
(i) Show that if $P(x)$ is to have any stationary points, then $p^2 - 6q \geq 0$. 2

(ii) Discuss the situation when $p^2 - 6q = 0$. 1

(b) A camera, one kilometre away in the horizontal direction from where the Space Shuttle is being launched, is tracking the ascent of the Shuttle. Assume the Shuttle ascends vertically. 3

Thirty seconds after the launch the Shuttle reaches a height, h, of 3240 metres and it is travelling at a speed of 230 metres per second.

The angle θ is the angle of elevation of the camera as it tracks the Shuttle. At what rate is θ increasing 30 seconds after the Shuttle is launched?



hint

asked to find
dθ /dt (consider $d\theta/dt$)

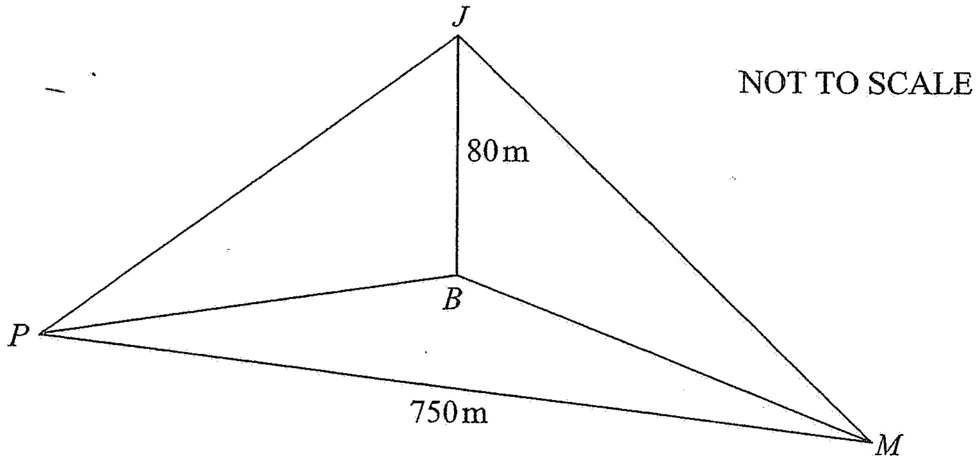
(c) Find the general solution to $2\cos 3x = 1$ 3

Question 5 (continued)

Marks

(d) Janus, J, is on top of an 80 metre cliff, watching the Sydney to Hobart yacht race. 3

From the base of the cliff, B, directly below Janus, *Poseidon, P,* is on a bearing of 202° and *Majorca, M,* is on a bearing of 140°. *Majorca* is 750 m from *Poseidon* on a bearing of 110°.



Copy or trace this diagram into your writing booklet.

Find the angle of depression of *Poseidon, P,* from Janus, J.

End of Question 5

Marks

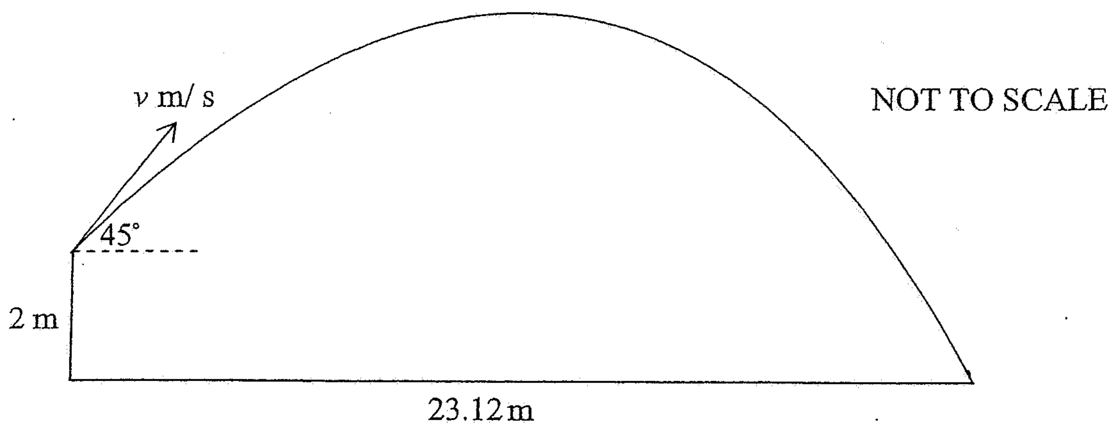
Question 6 (12 marks) Use a SEPARATE writing booklet

(a) Consider the function given by $f(x) = \dfrac{e^x}{x-1}$.

(i) Determine all vertical and horizontal asymptotes of the graph of $y = f(x)$. 2

(ii) Find any stationary point(s) and sketch the graph of $y = f(x)$ including any intercepts with the coordinate axes. 4

(b) The world record for men's shot-put is 23.12 metres.
You may assume that the shot-put is projected at an initial velocity of v m/s from a height of 2 metres at an angle of projection of 45°, there is no air resistance and that the acceleration due to gravity is 10m/s^2.



(i) Use integration to show that the equations of motion are 2

$$x = \frac{vt}{\sqrt{2}} \text{ and } y = -5t^2 + \frac{vt}{\sqrt{2}} + 2.$$

(ii) Find the minimum velocity v m/s at which the shot-put must be projected to achieve the world record distance. 4

Question 7 (12 Marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^{\pi/2} 2 \sin x \cos x \, dx.$ 3

(b) (i) Show that $d(x \sin^{-1} x)/dx = \sin^{-1} x + \dfrac{x}{\sqrt{1-x^2}}$. 2

(ii) Hence, evaluate $\int_0^1 \sin^{-1} x \; dx.$ 2

(c) A chord passes through points $P(2p, p^2)$ and $Q(2q, q^2)$ on the parabola $x^2 = 4y$.

(i) Find the co-ordinates of T, the midpoint of PQ. 1

(ii) The chord PQ subtends a right-angle at the vertex 0, of the parabola. Prove $pq = -4$. 1

(iii) Express in Cartesian form the locus of T as the points P and Q vary on the parabola. 3

Trial Maths 08 Ext 1

Q1

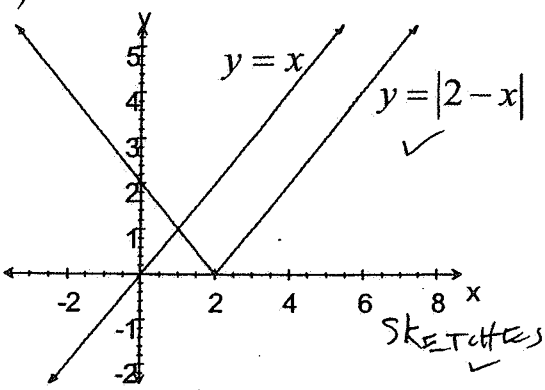
a) $\int_0^{\frac{\pi}{8}} \sec^2 2x \, dx$

$= \frac{1}{2} \tan 2x \Big|_0^{\frac{\pi}{8}}$

$= \frac{1}{2}\left(\tan\frac{\pi}{4} - \tan 0\right)$

$= \frac{1}{2}(1-0) = \frac{1}{2}$ 2

b)



SKETCHES ✓

$\therefore \; x > 1$ ✓ 3

c) $P(-2) \Rightarrow \; 4 + 2k + 6 = 0$

$k = -5$ 2

d) $m_1 = 2 \;\; \& \;\; m_2 = -1$ ✓

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{2+1}{1-2}\right|$ ✓

$= 3$

$\therefore \theta = 71^\circ 34'$ ✓ 3

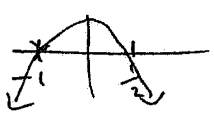
e) $(x+1)^2 \left[\frac{3}{x+1} - 2\right] < 0$

$3(x+1) - 2\left[x^2 + 2x + 1\right] < 0$

vs

$(x+1)\left[3 - 2x - 2\right] < 0$

$(x+1)(-2x+1) < 0$



$\therefore \; x < -1$

$x > \frac{1}{2}$ 2

Q2

$2x^3 - 5x^2 + 3x - 5 = 0$

a) $\Sigma = \frac{5}{2} \quad \text{prod} = \frac{5}{2}$

$\alpha\beta\gamma(\alpha + \beta + \gamma)$

$\frac{5}{2} \cdot \frac{5}{2} = \frac{25}{4}$ 2

b) $f(x) = \frac{2x}{\sqrt{1-x^2}}$

i) $1 - x^2 > 0$

$\therefore \; x = \pm 1$ $\quad x > 1$, $x < -1$ 2

ii) $\int_0^{\frac{1}{2}} \frac{2x}{\sqrt{1-x^2}} \, dx$

$x = \sin u$ $\quad \frac{1}{2} \to \frac{\pi}{6}$

$dx = \cos u \, du$ $\quad 0 \to 0$

$= 2\int_0^{\frac{\pi}{6}} \frac{\sin u \cos u \, du}{\cos u}$

$= -2 \cos u \Big|_0^{\frac{\pi}{6}}$

$= -2\left[\frac{\sqrt{3}}{2} - 1\right]$

$= -\sqrt{3} + 2$

c) $y = \sin^{-1} x + \cos^{-1} x$

i) $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}}$ ✓

$= 0$ 3

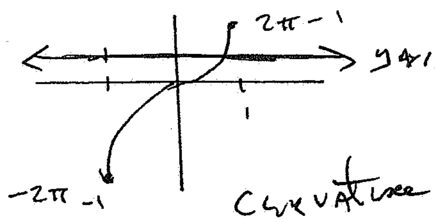
ii) statement ✓

d) $y = 2\sin^{-1} x - 1$

D: $-1 \le x \le 1$

at $(-1) \Rightarrow 2(-\pi) - 1 \Rightarrow -7.28$

at $(1) \Rightarrow 2\pi - 1 \Rightarrow 5.28$ 2

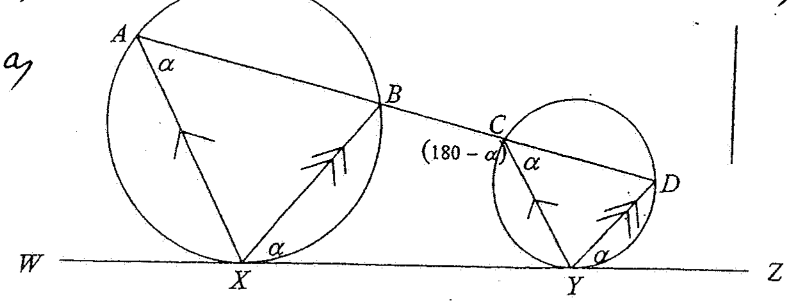


Curvature ✓

Trial Maths of Ext 1

Q3

a)



Given $\angle BXY = \alpha$

$\therefore \angle BAX = \alpha$ (angle between tangent and chord at the point of contact is equal to the angle in the alternate segment)

$\angle DCY = \alpha$ (corresponding to $\angle BAX$, $AX \parallel CY$)

$\angle DYZ = \alpha$ (angle between tangent and chord at the point of contact is equal to the angle in the alternate segment)

$\therefore \angle DYZ = \angle BXY$

$\therefore BX \parallel DY$ (corresponding angles are equal)

3 ✓ proper defn 1

Answer:

$\angle BCY = 180 - \alpha$ (BCD is a straight line)

$\therefore \angle BCY + \angle BXY = 180^\circ$

$\therefore BCYX$ is a cyclic quadrilateral as one pair of opposite angles are supplementary

b) $\cos A = \frac{3}{5}$, $\tan B = \frac{12}{5}$

A: 5, 4, 3   B: 13, 12, 5   But Reflex

$\sin(A - B)$

$= \sin A \cos B - \cos A \sin B$

$= -\frac{4}{5}\left(\frac{-5}{13}\right) - \frac{3}{5}\left(\frac{-12}{13}\right)$

reflex   NB

$= 56/65$ rule ✓ reflex ✓ ans ✓   3

c) $f(1) = -5$ & $f(2) = 6$

i) change of signs ✓

ii) $f(1.5) = -0.875$ ✓   2

d) $f(x) = x^3 - 9$ $f(2) = -1$

$f'(x) = 3x^2$ $f'(2) = 12$

$\therefore x_2 = 2 - \frac{-1}{12} = 2\frac{1}{12}$ 2

$\boxed{\frac{\text{Error}}{\sqrt[3]{9}}} \times 100\%$ 1

0.17 %

Q4

a) $n = 1$

$2 = \frac{1(3+1)}{2}$ ✓ True

Assume true

$2 + 5 \ldots (3n-1) \Rightarrow \frac{n(3n+1)}{2} + 3n+2$

$+ (3(n+1)-1)$

$\Rightarrow \frac{n(3n+1) + 6n + 4}{2}$

$\Rightarrow (3n^2 + n + 6n + 4)/2$

$\Rightarrow (3n^2 + 7n + 4)/2$

$\Rightarrow (3n + 4)(n+1)/2$

ie $\frac{(n+1)[3(n+1)+1]}{2}$ 3

statement

b) $\ddot{x} = 2x - 3$

$\frac{d\frac{1}{2}v^2}{dx} = 2x - 3$

i) $\frac{1}{2}v^2 = x^2 - 3x + c$

$x = 0$, $v = 2$ $\Rightarrow c = 2$

$\therefore v^2 = 2x^2 - 6x + 4$ 2

ii) $x = 1$ $v = 0$

$\ddot{x} = -1\ m/s$ 2

discuss

c) i) $N = \frac{200}{1 + Ke^{-200t}}$

$\boxed{\frac{N}{200} = \frac{1}{1 + Ke^{-200t}}}$

$\frac{dN}{dt} = -200(1 + Ke^{-200t})^{-2}(-200Ke^{-200t})$

$= \frac{(200)^2 Ke^{-200t}}{(1 + Ke^{-200t})^2}$ 2

$= (200)^2 Ke^{-200t}\left[\frac{1}{(1 + Ke^{-200t})}\right]^2$

$= 200^2 Ke^{-200t} \frac{N^2}{(200)^2}$

$= N^2 Ke^{-200t}$

$= N^2\left[\frac{200 - N}{N}\right]$

Trial Ext1 '08

Q4 (continued)

c ii $N = \dfrac{200}{1+Ke^{-200t}}$

$t=0$

$1 = \dfrac{200}{1+Ke^{-200(0)}}$

$1 = \dfrac{200}{1+K}$

$K = 199$

$100 = \dfrac{200}{1+199e^{-200t}}$

$1+199e^{-200t} = 2$

$e^{-200t} = \dfrac{1}{199}$

$\ln \dfrac{1}{199} = -200t$

$t = -\dfrac{1}{200} \ln \dfrac{1}{199}$

$\doteqdot 0.0265$ yr

9.66 days 2

---

iv) $t \to \infty$

$N \to \dfrac{200}{1+0} = 200$ 1

Q5

a i) $P(x) = -2x^3 + px^2 - qx + 5$

ii) $P'(x) = -6x^2 + 2px - q$ ✓

$x = \dfrac{\sqrt{\Delta}}{}$

$\Delta = 4p^2 - 36q$

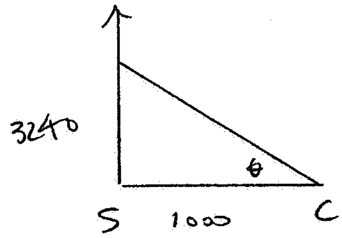
$2\sqrt{p^2 - 6q}$

$\therefore p^2 - 6q \geq 0$ ✓ 2

iii) $\therefore$ 1 root

& horizontal pt of inflection ✓

b)



Asking $\dfrac{d\theta}{dt} \Rightarrow \dfrac{d\theta}{dh} \dfrac{dh}{dt}$

$\tan\theta = \dfrac{h}{1000}$

$h = 1000\tan\theta$

$\dfrac{dh}{d\theta} = 1000\sec^2\theta$

$t = 30,\ h = 3240$

$\tan\theta = \dfrac{3240}{1000}$

$\theta \approx 1.271$ radians

$\dfrac{d\theta}{dt} = \dfrac{d\theta}{dh} \dfrac{dh}{dt}$

$= \dfrac{\cos^2 \boxed{1.271}}{1000} \times 230$

≈ 0.02 rad/sec 3

c) $2\cos 3x = 1$ ✓

$\cos 3x = \frac{1}{2}$

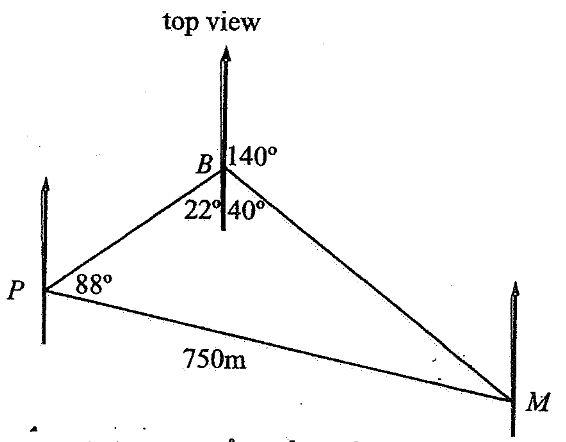
$3x = 2n\pi \pm \dfrac{\pi}{3}$ ✓

$x = \dfrac{2n\pi}{3} \pm \dfrac{\pi}{9}$ ✓ 3

$x = \dfrac{6n\pi \pm \pi}{9}$

Q5 continued Trial Ext 1 08

d)



$\angle BPM = 110^\circ - 22^\circ = 88^\circ$
$\angle PBM = 202^\circ - 140^\circ = 62^\circ$
$\therefore \angle PMB = 30^\circ$

$\therefore \dfrac{BP}{\sin 30^\circ} = \dfrac{750}{\sin 62^\circ}$

$\therefore BP = \dfrac{750 \sin 30^\circ}{\sin 62^\circ}$

$= 424.71$ (2 d.p.)

individual sketch (correct) ✓
Rules ✓

3

angle of elevation $= \tan^{-1}\left(\dfrac{80}{BP}\right)$

$= 10^\circ 40'$

$\therefore$ angle of depression is $10^\circ 40'$

Q6

a)

$f(x) = \dfrac{e^x}{x-1}$

i) $x = 1$ ✓

$x \to -\infty$

$y \to 0^-$ ✓

$\dfrac{\frac{e^x}{x}}{\frac{x}{x} - \frac{1}{x}}$

2

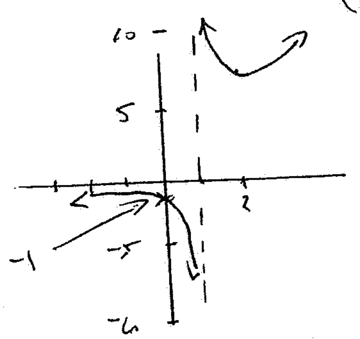
ii) $f'(x) = \dfrac{(x-1)e^x - e^x}{(x-1)^2}$

s.v

$0 = \dfrac{e^x[x-2]}{(x-1)^2}$

x	$\leftarrow$	2	$\rightarrow$
y'	neg	0	pos

min $(2, e^2)$



4

b) hor vert

i) $\ddot{x} = 0$ $\ddot{y} = -10$

$\dot{x} = v\cos 45$ $\dot{y} = -10t + \dfrac{v}{\sqrt{2}}$

$= \dfrac{v}{\sqrt{2}}$

$x = \dfrac{vt}{\sqrt{2}}$ $\cancel{+c}$ $y = -5t^2 + \dfrac{vt}{\sqrt{2}} + 2$

2

ii) $\dfrac{\sqrt{2}x}{v} = t$

$y = -5t^2 + \dfrac{vt}{\sqrt{2}} + 2$

$y = -5\left[\dfrac{2x^2}{v^2}\right] + \dfrac{v}{\sqrt{2}}\dfrac{\sqrt{2}x}{v} + 2$

$y = \dfrac{-10x^2}{v^2} + x + 2$

$0 = \dfrac{-10[23.12]^2}{v^2} + 23.12 + 2$

$\dfrac{10[23.12]^2}{v^2} = 24.12$

$v^2 = 221.6119867$

$v = 14.59$ m/s

4

TRIAL EXT 1 -0

Q7

a) $\int_0^{\frac{\pi}{2}} 2\sin x \cos x \; dx$

$x = \sin x$
$dx = \cos x \, dx$

or $\int \sin 2x$

$2\int_2^{\frac{\pi}{2}} \sin x \; dx$

$= -2\cos x \Big|_0^{\frac{\pi}{2}}$

$= -2\cos\frac{\pi}{2} - -2\cos 0$

$= 2$ 3

b) $y = x\sin^{-1}x$

$\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x$

i $\frac{d\, x\sin^{-1}x}{dx} = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x$

$x\sin^{-1}x = \int \frac{x}{\sqrt{1-x^2}}dx + \int \sin^{-1}x\,dx$

$\int \sin^{-1}x \, dx$
$= x\sin^{-1}x - \int \frac{x\,dx}{\sqrt{1-x^2}}$ 2

ii $\int_0^1 \sin^{-1}x \, dx$

$= \left[1\sin^{-1}1 - \int_0^1 \frac{x\,dx}{\sqrt{1-x^2}}\right]$

$- \left[0\sin^{-1}0 - \uparrow\right]$

$\Rightarrow \sin^{-1}1 = -\frac{1}{2}\int_0^1 \frac{2x}{\sqrt{1-x^2}}dx$

$u = 1-x^2$
$du = -2x\,dx$ 2

$+\frac{1}{2}\int_1^0 (u)^{-1/2}du$

$+\frac{1}{2}u^{1/2}\,2\Big|_1^0$

$\sin^{-1}1 + u^{1/2}\Big|_1^0$

$\frac{\pi}{2} + [0-1]$

$\frac{\pi}{2} - 1 \Rightarrow \frac{\pi-2}{2}$

c) $P(2p, p^2) \mid Q(2q, q^2)$

i) $T\left[\frac{2p+2q}{2}, \frac{p^2+q^2}{2}\right]$ 1

$T\left[p+q, \frac{1}{2}(p^2+q^2)\right]$

ii) $m_{PO} = \frac{p^2-0}{0-2p} = -\frac{p}{2}$

$m_{QO} = -\frac{q}{2}$ ↑ As above

$\perp$ $-\frac{p}{2}\left(-\frac{q}{2}\right) = -1$

$pq = -4$

iii) $T\left[p+q, \frac{1}{2}(p^2+q^2)\right]$

$\left[\left(p + \frac{-4}{p}\right), \frac{1}{2}\left[p^2 + \frac{16}{p^2}\right]\right]$

$x = p - \frac{4}{p}$

$y = \frac{1}{2}\left[p^2 + \frac{16}{p^2}\right]$

$y = \frac{1}{2}\left[\left(p - \frac{4}{p}\right)^2 + 8\right]$

$y = \frac{1}{2}\left[x^2 + 8\right]$

NB $x^2 = 2y - 8$