STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \quad \text{if } n < 0$

$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$

HSC TRIAL EXAMINATION PAPER 2001 SOLUTIONS + MAPPING GRID
MATHEMATICS – EXTENSION I

QUESTION 1

(a) $\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{x\to 0} \frac{3\sin 3x}{3x} = 3$

(1 mark)

(b) Since $x-2$ is a factor $\therefore P(2)=0$

$\therefore P(2) = 8p+20-3p=0 \quad \therefore p=-4$ (2 marks)

(c) $y = x\tan^{-1}x$

Using productrule:

Let $u=x \quad v=\tan^{-1}x$

$\therefore u'=1 \quad v'=\frac{1}{1+x^2}$

$\therefore \frac{dy}{dx} = \tan^{-1}x + \frac{x}{1+x^2}$ (2 marks)

(d) $y=\ln(2x+1) \quad \therefore \frac{dy}{dx} = \frac{2}{2x+1}$

At $x=0$, $\frac{dy}{dx}=2$

At $x=\frac{1}{2}$, $\frac{dy}{dx}=1$

$\therefore \tan\alpha = \left|\frac{2-1}{1+2\times 1}\right| = \frac{1}{3}$

$\therefore \alpha = 18°26'$ (to nearest minute).

(2 marks)

(e) $\int_0^{1/6} \frac{9\,dx}{\sqrt{1-9x^2}} = \frac{1}{3}\int_0^{1/6} \frac{9\,dx}{\sqrt{\frac{1}{9}-x^2}}$

$= \int_0^{1/6} \frac{3\,dx}{\sqrt{\frac{1}{9}-x^2}} = 3\left[\sin^{-1}3x\right]_0^{1/6}$

$= 3\left[\frac{\pi}{6}-0\right] = \pi/2$

(3 marks)

(f) $\frac{1}{x} > \frac{1}{x+2}$

$\therefore \frac{1}{x} - \frac{1}{x+2} > 0 \quad \therefore \frac{2}{x(x+2)} > 0$

x		-2		0	
$x(x+2)$	$+$	‖	$-$	‖	$+$

$\therefore$ Solution is $x<-2$ or $x>0$.

QUESTION 2

(a) Arrangements $= \frac{8!2!}{3!2!4!} = 280$

(we divide by $3!$, $2!$ & $4!$ because the red, blue & green are identical). (2 marks)

(b) $\sin 2\theta = 2\cos^2\theta, \quad 0 \leq \theta \leq 2\pi$

$2\sin\theta\cos\theta = 2\cos^2\theta$

$\cos\theta(\sin\theta - \cos\theta) = 0$

$\therefore \cos\theta = 0, \quad \sin\theta = \cos\theta$

$\therefore \cos\theta = \cos\frac{\pi}{2}$

$\therefore \theta = \frac{\pi}{2} + 2k\pi \quad \text{or} \quad \theta = -\frac{\pi}{2} + 2k\pi$

for $k=0$, $\theta = \frac{\pi}{2}$ for $k=1$, $\theta = \frac{3\pi}{2}$

$\sin\theta = \cos\theta$

$\therefore \tan\theta = \tan\frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4} + k\pi$

for $k=0$, $\theta = \pi/4$

for $k=1$, $\theta = 5\pi/4$

$\therefore$ Solutions are $\theta = \pi/4, \pi/2, 5\pi/4, \frac{3\pi}{2}$

in the domain $0 \leq \theta \leq 2\pi$ (3 marks).

(c) $\int_0^{\pi/2} \frac{\cos x\,dx}{\sqrt{1+3\sin x}}$ Let $u = 3\sin x$

$\therefore \cos x\,dx = du/3$

for $x=0$, $u=0$

$x=\frac{\pi}{2}$, $u=3$

$= \frac{1}{3}\int_0^3 \frac{du}{\sqrt{1+u}}$

$= \frac{1}{3}\int_0^3 (1+u)^{-1/2}\,du = \frac{1}{3}\left[2\sqrt{1+u}\right]_0^3$

$= \frac{2}{3}\left[2-1\right] = \frac{2}{3}$

(3 marks)

(d)(i) $\frac{dy}{dx} = 2(x-1)$ (gradient function)

at $x = t+1$, $m_{tan} = 2t$

Using gradient point formula

$y - t^2 = 2t(x-t-1)$

$\therefore y - t^2 = 2tx - 2t^2 - 2t$

$\therefore y = 2tx - t^2 - 2t$ ① (2 marks)

(ii) Let $x=1$ in ① to find C

$y = 2t - t^2 - 2t = -t^2$ $\therefore C(1, -t^2)$

Let $y=0$ in ① to find B

$\therefore 2tx = t^2 + 2t$ $\therefore x = \frac{t+2}{2}$

$\therefore$ B is $\left(\frac{t+2}{2}, 0\right)$

$\therefore M_{AC} = \left(\frac{t+1+1}{2}, \frac{-t^2+t^2}{2}\right)$

$= \left(\frac{t+2}{2}, 0\right)$

$\therefore$ B is mid-point of AC (2 marks).

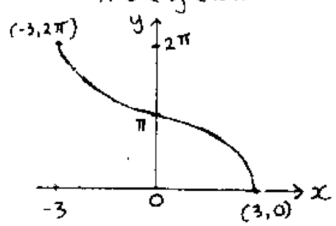
QUESTION 3

(a)(i) $y = 2\cos^{-1}\frac{x}{3}$

Domain: $-1 \le x/3 \le 1$ $\therefore -3 \le x \le 3$ (1 mark)

(ii) Range: $0 \le \cos^{-1}\frac{x}{3} \le \pi$

$\therefore 0 \le y \le 2\pi$



(2 marks)

(iii) $y = 2\cos^{-1} x/3$

$\therefore dy/dx = \dfrac{2/3}{\sqrt{1 - \frac{x^2}{9}}}$

At $x=0$, $m_{tan} = 2/3$ (1 mark)

(b)(i) $v^2 = -7 + 8x - x^2$

$\therefore \frac{1}{2}v^2 = -\frac{7}{2} + 4x - \frac{x^2}{2}$

$\therefore \frac{d(\frac{1}{2}v^2)}{dx} = 4 - x$

$\therefore$ Acceleration: $a = 4 - x$ (2 marks)

(ii) $a = -(x-4)$

$\therefore$ Acceleration is proportional to displacement but negative (i.e. directed towards the centre)

$\therefore$ Motion is simple harmonic, centred at $x=4$.

To find amplitude, let $v=0$.

$\therefore x^2 - 8x + 7 = 0$ $\therefore x=7$ or $x=1$

$\therefore$ Particle is oscillating between $x=1$ & $x=7$.

$\therefore$ Amplitude $=3$ (2 marks)

(iii) Maximum speed occurs when $a=0$ (i.e. when $x=4$)

$\therefore v^2 = -7+32-16 = 9$ $\therefore v = 3$m/s (1 mark)

(c) Considering term T_{r+1}

$\therefore T_{r+1} = {}^8C_r \left(\frac{x^4}{2}\right)^{8-r} \left(\frac{2}{x^2}\right)^r$

$= {}^8C_r \cdot \frac{x^{32-4r}}{2^{8-r}} \cdot \frac{2^r}{x^{2r}}$

$= {}^8C_r \cdot 2^{2r-8} \cdot x^{32-6r}$

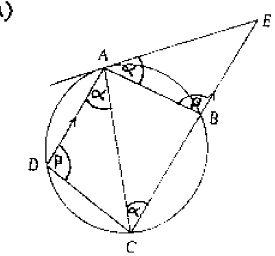
To get coefficient of x^2 we let

$32 - 6r = 2$. $\therefore r = 5$

$\therefore$ Coefficient of $x^2 = {}^8C_5 \cdot 2^2 = 224$ (3 marks)

QUESTION 4

(a)



Data: ABCD is a cyclic quadrilateral

AD ∥ BC

Aim: Prove that: (i) $\Delta ABE \,|||\, \Delta ADC$

(ii) $AE \times DC = AC \times BE$

Construction: Figure

Proof: (i) Let $\angle EAB = \alpha$

$\therefore \angle ACB = \alpha$ (angle in alternate segment).

$\therefore \angle DAC = \alpha$ (alternate angles, AD ∥ BC)

Let $\angle ABE = \beta$

$\therefore \angle CDA = \beta$ (exterior angle of cyclic quadrilateral equals opposite interior angle).

$\angle ACD = \angle AEB$ (remaining angles)

$\therefore \Delta ACD \,|||\, \Delta AEB$ (equiangular). (3 marks)

(ii) Since Δ's ADC & ABE are similar, their corresponding sides are in the same ratio.

Ratio of sides: $\frac{AE}{AC} = \frac{BE}{DC}$

$\therefore AE \times DC = BE \times AC$ (1 mark)

(b)(i) Product of roots: $\sqrt{p} \times \frac{1}{\sqrt{p}} \times \alpha = -\frac{C}{A}$

$\therefore \alpha = -\frac{C}{A}$ ① (1 mark)

(ii) Sum of roots: $\frac{1}{\sqrt{p}} + \sqrt{p} + \alpha = -\frac{B}{A}$

$\therefore \sqrt{p} + \frac{1}{\sqrt{p}} = \frac{C-B}{A}$ ②

Sum of roots 2 at a time:

$(\sqrt{p} \times \frac{1}{\sqrt{p}}) + \alpha\sqrt{p} + \frac{\alpha}{\sqrt{p}} = 2$

$\therefore 1 + \alpha(\sqrt{p} + \frac{1}{\sqrt{p}}) = 2$

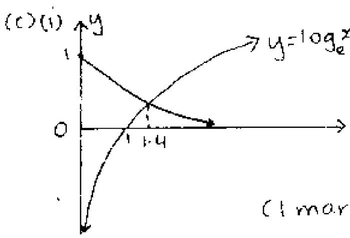
$\therefore \alpha(\sqrt{p} + \frac{1}{\sqrt{p}}) = 1$ ③

Sub ② & ① in ③

$\therefore -\frac{C}{A} \cdot \frac{C-B}{A} = 1$

$\therefore -C^2 + BC = A^2$ $\therefore A^2 + C^2 = BC$ (2 marks)

(c)(i)



(1 mark)

(ii) From the graph, we can see that the curves $y=e^{-x}$ & $y=\log_e x$ intersect near $x=1\cdot4$.

∴ The equation $e^{-x}=\log_e x$ (i.e. $e^{-x}-\log_e x=0$) has a root close to $x=1\cdot4$. (1 mark)

(iii) Let $h(x)=e^{-x}-\log_e x$

$\therefore h'(x)=-e^{-x}-\frac{1}{x}$

$\therefore h(1\cdot4)=e^{-1\cdot4}-\log_e 1\cdot4 \doteq -0\cdot089875272$

$h'(1\cdot4)=-e^{-1\cdot4}-\frac{1}{1\cdot4}\doteq -0\cdot960882678$

$\therefore x_2=1\cdot4-\frac{h(1\cdot4)}{h'(1\cdot4)}\doteq 1\cdot306465925$

$\therefore h(1\cdot306465925)=3\cdot4495834\times10^{-3}$

$\therefore x=1\cdot306465925$ is a better approximation of the root. (2 marks)

QUESTION 5

(a)(i) All real numbers except $x=0$ (1 mark)

(ii) $f'(x)=\frac{e^x(e^x-1)-e^{2x}}{(e^x-1)^2}$

$=\frac{-e^x}{(e^x-1)^2}$ (gradient function).

Since $e^x>0$ for all x & denominator is a perfect square greater than 0

$\therefore f'(x)<0$ for all x (2 marks)

(iii) When $x\to+\infty$, $y\simeq\frac{e^x}{e^x}\to1$

∴ $y=1$ is a horizontal asymptote

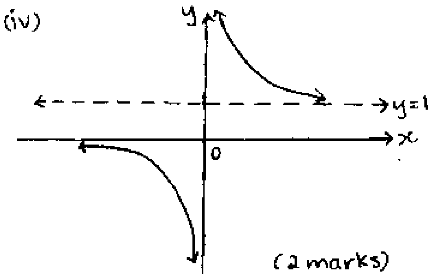
When $x\to-\infty$, $y\simeq\frac{0}{-1}\to0$ (n.b. $e^{-\infty}\to0$)

∴ $y=0$ is a horizontal asymptote

When $x\to0$, $y\simeq\frac{1}{0}\to\pm\infty$

∴ $x=0$ is a vertical asymptote (2 marks)

(iv)



(2 marks)

(v) Since $f(x)$ is a one-one function (i.e. for every x, there is only one y-value & vice versa).

∴ It has an inverse function (1 mark)

(vi) By interchanging x & y:

$x=\frac{e^y}{e^y-1}\longrightarrow \therefore e^y-x=e^y$

$\therefore e^y=\frac{x}{x-1}$

$\therefore \log_e e^y=\log_e\frac{x}{x-1}$

$\therefore y=\log_e\frac{x}{x-1}$ (1 mark)

(b)(i) $P(ace)=3/10$ $P(no\ ace)=7/10$

$\therefore P(one\ ace)={}^6C_1(3/10)(7/10)^5$

$=0\cdot302526$ (1 mark)

(ii) $P(at\ least\ 2\ aces)=1-P(no\ ace)-P(ace)$

$=1-{}^6C_0(3/10)^0(\frac{7}{10})^6-{}^6C_1(\frac{3}{10})(\frac{7}{10})^5$

$=1-0\cdot117649-0\cdot302526$

$=0\cdot579825$ (1 mark)

(iii) He has to serve ace, no ace, no ace, no ace, no ace, ace in this order.

$P=(\frac{3}{10})^2(\frac{7}{10})^4=0\cdot021609$ (1 mark)

QUESTION 6

(a) Step 1: For $n=1$, $3.2!=1(1+2)!$

∴ $6=6$ Hence statement is true for $n=1$.

Step 2: Assume that the statement is true for $n=k$

$3.2!+7.3!+\ldots+(k^2+k+1)(k+1)!$

$=k(k+2)!$ ①

Our aim is to prove it true for $n=k+1$

i.e. $3.2!+7.3!+\ldots+[(k+1)^2+k+2](k+2)!$

$=(k+1)(k+3)!$

Starting from ① and adding $[(k+1)^2+k+2](k+2)!$ to both sides:

$3.2!+7.3!+\ldots+[(k+1)^2+k+2](k+2)!$

$=k(k+2)!+[(k+1)^2+k+2](k+2)!$

∴ LHS $=[k^2+4k+3](k+2)!$ (factorizing)

$=(k+1)(k+3)(k+2)!$

$=(k+1)(k+3)!$ (Since $(k+3)!=(k+2)!\times(k+3)$).

Hence if the statement is true for $n=k$, it is also true for $n=k+1$.

Step 3: If the statement is true for $n=1$ & so it is true for $n=2$ & so on. Hence it is true for all $n\geq1$. (3 marks)

(b)(i) $\frac{d(\frac{1}{2}v^2)}{dx}=-e^{-x}-e^{-2x}$

$\therefore \frac{1}{2}v^2=\int(-e^{-x}-e^{-2x})dx$

$\therefore \frac{1}{2}v^2=e^{-x}+\frac{1}{2}e^{-2x}+c$

when $x=0$, $v=2$

$\therefore 2=1+\frac{1}{2}+c$ $\therefore c=\frac{1}{2}$

$\therefore \frac{1}{2}v^2=\frac{1}{2}e^{-2x}+e^{-x}+\frac{1}{2}$

$\therefore v^2=e^{-2x}+2e^{-x}+1$

$\therefore v^2=(e^{-x}+1)^2$ $\therefore v=\pm(e^{-x}+1)$

Since when $x=0$, $v=2$ (positive)

∴ Positive solution only is accepted.

$\therefore v=e^{-x}+1$ (3 marks)

(ii) $\frac{dx}{dt}=e^{-x}+1=\frac{1+e^x}{e^x}$

$\therefore \int dt=\int\frac{e^x dx}{1+e^x}$ $\therefore t=\ln(e^x+1)+d$

When $t=0$, $x=0$

$\therefore 0=\ln2+d$ $\therefore d=-\ln2$

$\therefore t=\ln(e^x+1)-\ln2=\ln(\frac{e^x+1}{2})$ ①

$v=1\frac{1}{2}$, $e^{-x}+1=1\frac{1}{2}$ $\therefore \frac{1}{e^x}=\frac{1}{2}$

$\therefore e^x = 2$ Sub in ①

$\therefore t = \ln \frac{3}{2}$ seconds

$\therefore$ It will take the particle $\ln \frac{3}{2}$ seconds to drop its velocity to $1{\cdot}5$ m/s. (2 marks)

(c)(i) $V = \pi \int x^2 dy \quad y = \sin^{-1} x$

$\therefore \sin y = x \quad \therefore x^2 = \sin^2 y$

as $\cos 2y = 1 - 2\sin^2 y \quad \therefore \sin^2 y = \frac{1}{2}(1 - \cos 2y)$

$\therefore x^2 = \frac{1}{2}(1 - \cos 2y)$

$\therefore V = \frac{\pi}{2} \int_0^h (1 - \cos 2y)\, dy$

$= \frac{\pi}{2} \left[y - \frac{1}{2} \sin 2y \right]_0^h$

$= \frac{\pi}{2} \left[(h - \frac{1}{2} \sin 2h) - 0 \right]$

$= \frac{\pi}{4} (2h - \sin 2h)$ (2 marks)

(ii) $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$V = \frac{\pi}{4}(2h - \sin 2h) \quad \therefore \frac{dV}{dh} = \frac{\pi}{4}(2 - 2\cos 2h)$

$= \frac{\pi}{2}(1 - \cos 2h)$

$\therefore 2 = \frac{\pi}{2}(1 - \cos 2h) \cdot \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{4}{\pi(1 - \cos 2h)}$ (rate at any depth)

when $h = \frac{\pi}{4}$, $\frac{dh}{dt} = \frac{4}{\pi(1-0)} = \frac{4}{\pi}$ cm/s

(2 marks)

QUESTION 7

(a)(i) $\dot{y} = -gt + v\sin\alpha$

At maximum height, $\dot{y} = 0$ (vertical component) $\therefore gt = v\sin\alpha \quad \therefore t = \frac{v\sin\alpha}{g}$

Substitute in y, we get:

$\therefore y_{max} = -\frac{1}{2} g \times \left(\frac{v\sin\alpha}{g}\right)^2 + v\sin\alpha \times \frac{v\sin\alpha}{g}$

$\therefore 3h = -\frac{v^2\sin^2\alpha}{2g} + \frac{v^2\sin^2\alpha}{g}$

$\therefore 3h = \frac{v^2\sin^2\alpha}{2g} \quad \therefore v^2\sin^2\alpha = 6gh$

$\therefore v\sin\alpha = \sqrt{6gh}$ (since initial vertical component is positive). (2 marks)

(ii) Let $y = 0 \quad \therefore -\frac{1}{2}gt^2 + v\sin\alpha\, t = 0$

$\therefore t\left(-\frac{gt}{2} + v\sin\alpha\right) = 0 \quad \therefore t = 0$ (initial time) or $t = \frac{2v\sin\alpha}{g}$ (time to return to x-axis if it didn't strike plane at Q).

$\therefore d = v\cos\alpha \times \frac{2v\sin\alpha}{g} = v\cos\alpha \times \frac{2\sqrt{6gh}}{g}$

$\therefore v\cos\alpha = \frac{gd}{2\sqrt{6gh}}$ (2 marks)

(iii) $x = v\cos\alpha\, t \quad \therefore t = \frac{x}{v\cos\alpha} = \frac{2x\sqrt{6gh}}{gd}$

$\therefore y = -\frac{1}{2} g \times \left(\frac{2x\sqrt{6gh}}{gd}\right)^2 + \sqrt{6gh} \cdot \frac{2x\sqrt{6gh}}{gd}$

$\therefore y = -\frac{1}{2} g \times 4x^2 \times \frac{6gh}{g^2 d^2} + \frac{12xgh}{gd}$

$\therefore y = -\frac{12x^2 h}{d^2} + \frac{12xh}{d} = \frac{12xh}{d}\left(1 - \frac{x}{d}\right)$

(2 marks)

(iv) The rocket will strike the plane at Q when $y = h$.

$\therefore h = \frac{12xh}{d}\left(1 - \frac{x}{d}\right)$

$\therefore 1 = \frac{12x}{d}\left(1 - \frac{x}{d}\right)$

$\therefore d = 12x - \frac{12x^2}{d}$

$\therefore d^2 - 12xd + 12x^2 = 0$

$\therefore 12x^2 - 12dx + d^2 = 0$

$\therefore x = \frac{12d \pm \sqrt{144d^2 - 48d^2}}{24}$

$\therefore x = \frac{3d \pm d\sqrt{6}}{6} \quad \therefore x_Q = \frac{3d + d\sqrt{6}}{6}$

Time taken by rocket to reach Q is:

$x = v\cos\alpha\, t$

$\therefore \frac{3d + d\sqrt{6}}{6} = 100(3 + \sqrt{6})t$

$\therefore \frac{d(3 + \sqrt{6})}{6} = 100(3 + \sqrt{6})t$

$\therefore t = d/600$ (2 marks)

(v) The distance travelled by the plane from P to Q is:

$\frac{3d + d\sqrt{6}}{6} - \frac{3d - d\sqrt{6}}{6} = \frac{d\sqrt{6}}{3}$

Time taken for plane to travel from P to Q is the same time taken by rocket to reach Q. $\therefore t = d/600$

$\therefore u = \frac{d\sqrt{6}}{3} \times \frac{600}{d} = 200\sqrt{6}$ m/s (1 mark)

(b)(i) $(1+x)^{2n+1} = {}^{2n+1}C_0 + {}^{2n+1}C_1 x^1 + \ldots + {}^{2n+1}C_n x^n + {}^{2n+1}C_{n+1} x^{n+1} + \ldots + {}^{2n+1}C_{2n} x^{2n} + {}^{2n+1}C_{2n+1} x^{2n+1}$

For $x = 1$:

$2^{2n+1} = {}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + \ldots + {}^{2n+1}C_{2n} + {}^{2n+1}C_{2n+1}$

Using ${}^nC_r = {}^nC_{n-r}$

$\therefore {}^{2n+1}C_n = {}^{2n+1}C_{n+1}$

Also, ${}^{2n+1}C_0 = {}^{2n+1}C_{2n+1}$

${}^{2n+1}C_1 = {}^{2n+1}C_{2n}$

$\therefore 2^{2n+1} = 2({}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots + {}^{2n+1}C_n)$

$\therefore 2^{2n} - 1 = {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n$

$\therefore (n-2)!(2^{2n} - 1) = (n-2)!\,{}^{2n+1}C_1 + \ldots + (n-2)!\,{}^{2n+1}C_n$

(2 marks)

(ii) $(n-2)!\,{}^{2n+1}C_1 + \ldots + (n-2)!\,{}^{2n+1}C_n > 1000000$

$\therefore (n-2)!(2^{2n} - 1) > 1000000$

By calculator, for $n = 6$: 98280

for $n = 7$: 1965960

$\therefore n = 7$ is the smallest positive integer.

(1 mark)