17th Aug 2001, 2001 ~~Examin~~ SOLUTIONS YR 12 EXTENS 1. p. 1

YR11 3U TRIAL HSC SOLUTIONS (EXTENSION 1)

QUESTION 1

i) $\lim_{x\to 0}\left(\frac{\sin 3x}{2x}\right)$

$= \lim_{x\to 0} \frac{3}{2} \times \frac{\sin 3x}{3x}$

$= \frac{3}{2}\times 1$

$= \frac{3}{2}$

) $\int_0^2 \frac{4}{\sqrt{x^2+16}}\,dx$

$= 4\left[\ln\{x+\sqrt{x^2+4^2}\}\right]_0^2$

$= 4\left[\ln(2+\sqrt{20}) - \ln(0+4)\right]$

$\doteq 1.9248473$

) $\frac{d}{dx}\left(e^{2x}\sin x\right)$

$= e^{2x}\times\cos x + \sin x \times 2e^{2x}$

$= e^{2x}\cos x + 2e^{2x}\sin x$

$e^{2x}(\cos x + 2\sin x)$

d) $A(-1,3)$, $B(2,-3)$

$k:l = 1:-2$.

$x = \frac{kx_2+lx_1}{k+l}$, $y = \frac{ky_2+ly_1}{k+l}$

$x = \frac{1\times 2 - 2\times -1}{-1}$, $y = \frac{1\times -3 - 2\times 3}{-1}$

$x = \frac{+4}{-1}$, $y = \frac{-3-6}{-1}$

$x = -4$, $y = 9$

$\therefore$ The point is $(-4,9)$

e) $x-y+3=0 \to m_1 = 1$

$2x+y+1=0 \to m_2 = -2$.

$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$= \left|\frac{1+2}{1-2}\right|$

$= |-3|$

$\tan\theta = 3$

$\therefore \theta = 71°34'$

f) $\frac{x^2-4}{x} < 3$

$x(x^2-4) < 3x^2$

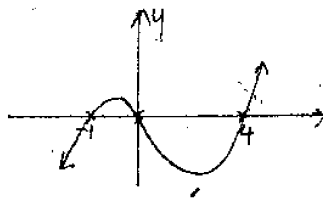
$x^3-4x < 3x^2$

$x^3-3x^2-4x < 0$

$x(x^2-3x-4) < 0$

$\therefore x(x-4)(x+1) < 0$

f) cont...



$\therefore x < -1$ or $0 < x < 4$

QUESTION 2

a) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \tan^{-1}(1)$

$= \frac{5\pi}{6} - \frac{\pi}{4}$

$= \frac{7\pi}{12}$

b) $u^2 = x+4$

$\frac{dx}{du} = 2u$

$dx = 2u\,du$

$x=-3$, $u=1$

$x=0$, $u=2$.

$\therefore I = \int_1^2 \frac{u^2-4}{\sqrt{u^2}}\,2u\,du$

$= \int_1^2 \frac{u^2-4}{\cancel{u}} \times 2\cancel{u}\,du$

$= 2\left[\frac{u^3}{3}-4u\right]_1^2$

p. 2

$= 2\left[\left(\frac{8}{3}-8\right)-\left(\frac{1}{3}-4\right)\right]$

$I = \frac{-10}{3}$

c) $\left(2x+\frac{1}{x^2}\right)^6$

$(k+1)$th term $= \binom{6}{k}(2x)^{6-k}(x^{-2})^k$

$\therefore x^{6-k}.x^{-2k} = x^0$

$6-k-2k=0$

$6-3k=0$

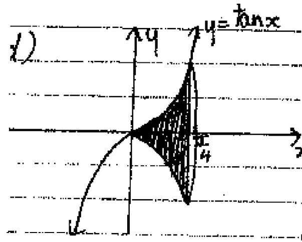
$k=2$

$\therefore$ the third term

Its value is

$\binom{6}{2}2^4 = 240$.

d)



$V = \pi\int_0^{\frac{\pi}{4}} y^2\,dx$

$= \pi\int_0^{\frac{\pi}{4}} \tan^2 x\,dx$

$= \pi\int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\,dx$

$= \pi\left[\tan x - x\right]_0^{\frac{\pi}{4}}$

$= \pi\left[\left(\tan\frac{\pi}{4}-\frac{\pi}{4}\right)-(\tan 0 - 0)\right]$

$= \pi\left[1-\frac{\pi}{4}\right]$

$V = \frac{\pi(4-\pi)}{4}\,u^3$

QUESTION 3

a) $\frac{\cos 2\theta}{\sin\theta} + \frac{\sin 2\theta}{\cos\theta} = \operatorname{cosec}\theta$

$LHS = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta} + \frac{\sin 2\theta}{\cos\theta}$

$= \frac{1-2\sin^2\theta}{\sin\theta} + \frac{2\sin\theta\cos\theta}{\cos\theta}$

$= \frac{1-2\sin^2\theta + 2\sin^2\theta}{\sin\theta}$

$= \frac{1}{\sin\theta}$

$= \operatorname{cosec}\theta$

$= RHS$

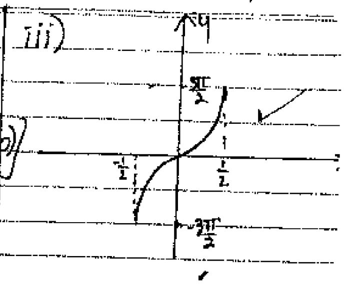
b) $f(x) = 3\sin^{-1}(2x)$

i) $-1 \le 2x \le 1$

$\therefore -\frac{1}{2} \le x \le \frac{1}{2}$.

ii) $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

iii)



iv) $f'(x) = 3\times\frac{1}{\sqrt{1-(2x)^2}}\times 2$

$= \frac{6}{\sqrt{1-4x^2}}$

Crosses x-axis at $(0,0)$

$f'(0) = \frac{6}{\sqrt{1}} = 6$

$\therefore$ eqn tangent is

$y-y_1 = m(x-x_1)$

$y-0 = 6(x-0)$

$\therefore y = 6x$

c) i) $RHS = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

$= \frac{1}{2}(1-\cos 2\theta)$

$= \frac{1}{2}(1-(1-2\sin^2\theta))$

$= \frac{1}{2}\times 2\sin^2\theta$

$= \sin^2\theta$

$= LHS$

i) $\int_0^{\frac{\pi}{3}} \sin^2\theta \, d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{3}} 1-\cos 2\theta \, d\theta$

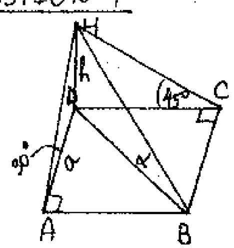
$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3}\right) - \left(0 - \frac{1}{2}\sin 0\right)\right]$

$= \frac{1}{2}\left(\frac{\pi}{3} - \frac{1}{2}\times\frac{\sqrt{3}}{2}\right) = \frac{1}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$

QUESTION 4

i)



$\tan 30° = \frac{h}{a}$

$h = a\tan 30°$

$h = \frac{a}{\sqrt{3}}$

$\therefore a = h\sqrt{3}$

$\tan 45° = \frac{h}{CD}$

$CD = \frac{h}{1} = h$

$\therefore (BD)^2 = (h\sqrt{3})^2 + (h)^2$

$= h^2 \times 3 + h^2$

$= 4h^2$

$\therefore BD = 2h$ units

ii) $\tan\alpha = \frac{h}{2h} = \frac{1}{2}$

$\therefore \alpha = 26°34'$

b) Step 1

Prove true for $n=1$

$7^1 + 5 = 12$, which is divisible by 3

$\therefore$ true for $n=1$

Step 2

Assume true for $n=k$.

ie $\frac{7^k+5}{3} = M$ where M is an integer

ie assume $7^k + 5 = 3M$.

Prove $7^{k+1} + 5$ is divis. by 3.

$7^{k+1} + 5$

$= 7.7^k + 5$

$= 7(7^k + 5) - 30$

$= 7.3M - 30$

$= 3(7M - 10)$, which is divis. by 3.

$\therefore$ if true for $n=k$, it's true for $n=k+1$.

Step 3

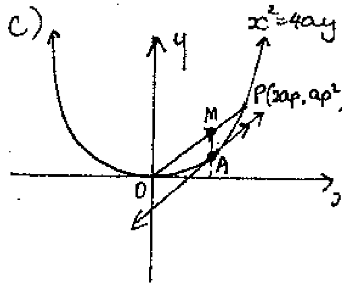
But it's true for $n=1$

$\therefore$ true for $n=1+1=2$;

it's true for $n=2+1=3$

and so on for all +ve integer n.

c)



i) $M = \left(\frac{2ap+0}{2}, \frac{ap^2+0}{2}\right)$

$= \left(ap, \frac{ap^2}{2}\right)$

ii) m OP $= \frac{ap^2-0}{2ap-0}$

$= \frac{p}{2}$.

iii) $y = \frac{x^2}{2a}$

$\frac{dy}{dx} = \frac{2x}{2a} = \frac{x}{a}$

$\frac{x}{2a} = \frac{p}{2}$ $\therefore x = ap$

$\therefore x = ap$, $y = \frac{a^2p^2}{4a} = \frac{ap^2}{4}$

$\therefore A = \left(ap, \frac{ap^2}{4}\right)$

iv)

dist $MA = \frac{ap^2}{2} - \frac{ap^2}{4} = \frac{ap^2}{4}$

dist A to x-axis $= \frac{ap^2}{4}$

$\therefore$ equidistant.

QUESTION 5

a) $100e^{-3t} = 20e^{2t}$

$\frac{100}{20} = \frac{e^{2t}}{e^{-3t}}$

$5 = e^{5t}$

$5t \ln e = \ln 5$

$\therefore t = \frac{\ln 5}{5}$

b)

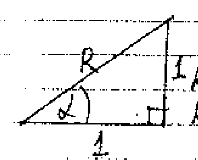
i)

$\cos 3\theta + \sin 3\theta = R\cos(3\theta - \alpha)$

$= R\cos 3\theta\cos\alpha + R\sin 3\theta\sin\alpha$

$\therefore R\cos\alpha = 1 \rightarrow \cos\alpha = \frac{1}{R}$

$R\sin\alpha = 1 \rightarrow \sin\alpha = \frac{1}{R}$



$R^2 = 2$

$R = \sqrt{2}$

$\therefore \tan\alpha = 1$

$\therefore \alpha = \frac{\pi}{4}$

$\therefore R\cos(3\theta - \alpha)$

$= \sqrt{2}\cos\left(3\theta - \frac{\pi}{4}\right)$

ii)

$\sqrt{2}\cos\left(3\theta - \frac{\pi}{4}\right) = 1$

$\cos\left(3\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\therefore 3\theta - \frac{\pi}{4} = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \ldots$

$\therefore 3\theta = 0, \frac{\pi}{2}, 2\pi, \frac{5\pi}{2}, \ldots$

$\therefore \theta = 0, \frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}$

c) $x = \cos 3t + \sin 3t$

i) $\dot{x} = -3\sin 3t + 3\cos 3t$

$\ddot{x} = -9\cos 3t - 9\sin 3t$

$= -9(\cos 3t + \sin 3t)$

$\ddot{x} = -9x$

$\ddot{x} = -n^2 x$ $\therefore$ SHM

ii) Period $= \frac{2\pi}{3}$

iii)

$t=0$, $x = \cos 0 + \sin 0$

$x = 1$ unit right of O.

iv) Returns to initial position when $x=1$

$\therefore \cos 3t + \sin 3t = 1$

From b)

$\sqrt{2}\cos\left(3t - \frac{\pi}{4}\right) = 1$

$\therefore t = \frac{\pi}{6}$ sec

$t = \frac{\pi}{6}$, $\dot{x} = -3\sin 3\times\frac{\pi}{6} + 3\cos 3\times\frac{\pi}{6}$

$\dot{x} = -3\sin\frac{\pi}{2} + 3\cos\frac{\pi}{2}$

$= -3\times 1 + 3\times 0$

$\dot{x} = -3$ cm/s

QUESTION 6.

a) $f(x) = \frac{x+1}{x^2+3}$

i) Crosses y-axis at $\frac{1}{3}$

Crosses x-axis when

$\frac{x+1}{x^2+3} = 0$

$x + 1 = 0 \rightarrow x = -1$

ii) $f'(x) = \frac{(x^2+3)\times 1 - (x+1)\times 2x}{(x^2+3)^2}$

$= \frac{x^2+3-2x^2-2x}{(x^2+3)^2}$

$= \frac{3-2x-x^2}{(x^2+3)^2}$

$= \frac{(3+x)(1-x)}{(x^2+3)^2}$

For stat pts $f'(x) = 0$.

$\therefore \frac{(3+x)(1-x)}{(x^2+3)^2} = 0$

$\therefore x = -3$ OR 1. ✓

When $x = -3$, $y = \frac{1}{6}$ $(-3, \frac{1}{6})$

$x = 1$, $y = \frac{1}{2}$ $(1, \frac{1}{2})$

Test:

x	-4	-3	0	1	2
$f'(x)$	$\frac{-5}{361}$	0	$\frac{1}{3}$	0	$\frac{-5}{49}$

\ — / ‾ \

$\therefore$ min. turning pt at $(-3, \frac{1}{6})$ ✓

Max. tur. pt at $(1, \frac{1}{2})$

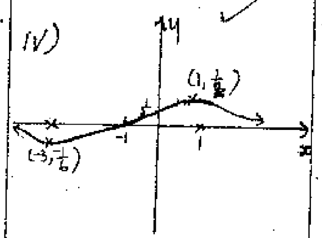
i) $\lim_{x \to \pm\infty} \frac{x+1}{x^2+3}$

$= \lim_{x \to \pm\infty} \frac{\frac{x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{3}{x^2}}$

$\lim_{x \to \pm\infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 + \frac{3}{x^2}}$

$= 0$ ✓

iv)



✓

b)

i) $T = A + Be^{-kt}$

$Be^{-kt} = T - A$.

$\frac{dT}{dt} = -kBe^{-kt}$

$\therefore \frac{dT}{dt} = -k(T - A)$ ✓

ii) $t = 0$, $T = 100°$.

As $t \to \infty$, $T \to 22°$.

$\therefore 100 = A + Be^{0}$

$\therefore A + B = 100$.

As $t \to \infty$, $e^{-kt} \to 0$

$\therefore T \to A$.

ie $A \to 22$. ✓

$\therefore 22 + B = 100$

$B = 78$. ✓

iii)

$T = 50°$, $t = 90$.

$T = A + Be^{-kt}$

$T = 22 + 78e^{-kt}$

$50 = 22 + 78e^{-90k}$

$78e^{-90k} = 28$

$e^{-90k} = \frac{28}{78} = \frac{14}{39}$ ✓

$-90k \ln e = \ln\left(\frac{14}{39}\right)$ ✓

$\therefore k = \frac{-1}{90} \ln\left(\frac{14}{39}\right)$

iv) $\frac{dT}{dt} = -k(T - A)$

from i).

$\therefore \frac{dT}{dt} = -\frac{1}{90}\ln\left(\frac{14}{39}\right)(50 - 22)$ ✓

$\frac{dT}{dt} = -0.3187°C$ /min

QUESTION 7

2) $(1+x)^{2n}$

i) $(1+x)^{2n} = \binom{2n}{0}x^0 + \binom{2n}{1}x^1 + \binom{2n}{2}x^2 + \dots + \binom{2n}{2n}x^{2n}$ *

Let $x = -2$ on B.S. of *

$\therefore (-1)^{2n} = \binom{2n}{0}x^0 + \binom{2n}{1}(-2) + \binom{2n}{2}(-2)^2 + \dots + \binom{2n}{2n}(-2)^{2n}$ ✓

ie $[(-1)^2]^n = 1 - 2\binom{2n}{1} + 4\binom{2n}{2} \dots + (4)^n\binom{2n}{2n}$ ✓

$\therefore 1 = 1 - 2\binom{2n}{1} + 4\binom{2n}{2} + \dots + 4^n\binom{2n}{2n}$ as required.

ii) Differentiate B.S. of * wrt x

$\therefore$

$2n(1+x)^{2n-1} = 0 + \binom{2n}{1} + 2x\binom{2n}{2} + 3x^2\binom{2n}{3} + \dots (2n)\binom{2n}{2n}x^{2n-1}$ ✓

Now let $x = -2$

$\therefore$

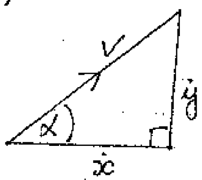
$2n(-1)^{2n-1} = \binom{2n}{1} - 4\binom{2n}{2} + 12\binom{2n}{3} + \dots + (2n)\binom{2n}{2n}(-2)^{2n-1}$

$n[[(-1)^2]^n \cdot (-1)^{-1}] = \binom{2n}{1} - 4\binom{2n}{2} + 12\binom{2n}{3} + \dots + (2n)\binom{2n}{2n}[[(-2)^2]^n \cdot (-2)^{-1}]$

$\therefore$

$2n(-1) = \binom{2n}{1} - 4\binom{2n}{2} + 12\binom{2n}{3} + \dots + (2n)\binom{2n}{2n}4^n \cdot \frac{-1}{2}$ ✓

$\therefore -2n = \binom{2n}{1} - 4\binom{2n}{2} + 12\binom{2n}{3} + \dots - n\binom{2n}{2n}4^n$ as required b)



$\cos\alpha = \frac{\dot{x}}{V}$

$\dot{x} = V\cos\alpha$
$\dot{y} = V\sin\alpha$.

$\therefore t=0, x=0, y=0, \dot{x}=V\cos\alpha, \dot{y}=V\sin\alpha$.

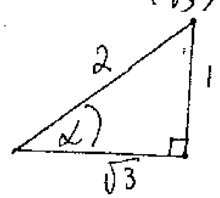
Horiz.

$\ddot{x} = 0$
$\dot{x} = C_1$
say $\boxed{\dot{x} = V\cos\alpha}$
$x = \int V\cos\alpha \, dt$
$x = Vt\cos\alpha + C_2$
$t=0, x=0 \therefore C_2 = 0$
$\boxed{x = Vt\cos\alpha}$

Vertic.

$\ddot{y} = -10$
$\dot{y} = -10t + C_3$
$t=0, \dot{y} = V\sin\alpha$
$\therefore C_3 = V\sin\alpha$
$\boxed{\dot{y} = -10t + V\sin\alpha}$
$y = \int -10t + V\sin\alpha \, dt$
$y = -5t^2 + Vt\sin\alpha + C_4$
$t=0, y=0 \therefore C_4 = 0$
$\boxed{y = -5t^2 + Vt\sin\alpha}$

) $\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$



$\therefore \cos\alpha = \frac{\sqrt{3}}{2}$

$\sin\alpha = \frac{1}{2}$.

$x = Vt\cos\alpha \longrightarrow t = \frac{x}{V\cos\alpha}$

sub. into $y = -5t^2 + Vt\sin\alpha$
$= -5 \times \frac{x^2}{V^2\cos^2\alpha} + V \times \frac{x}{V\cos\alpha} \times \sin\alpha$

$\therefore y = \frac{-5x^2\sec^2\alpha}{V^2} + x\tan\alpha$

$y = \frac{-5x^2}{V^2}(1+\tan^2\alpha) + x\tan\alpha$ *

Using $x = 158.5$, $y = 5$, $\tan\alpha = \frac{1}{\sqrt{3}}$

then

$5 = \frac{-5 \times 158.5^2}{V^2}\left(1 + \frac{1}{3}\right) + 158.5 \times \frac{1}{\sqrt{3}}$

$5V^2 = -5 \times 158.5^2 \times \frac{4}{3} + \frac{158.5V^2}{\sqrt{3}}$

$15\sqrt{3}V^2 = -5 \times 158.5^2 \times 4 \times \sqrt{3} + 158.5V^2 \times 3$

$\therefore V^2(158.5 \times 3 - 15\sqrt{3}) = 5 \times 158.5^2 \times 4 \times \sqrt{3}$

$\therefore V^2 = \frac{20\sqrt{3} \times 158.5^2}{158.5 \times 3 - 15\sqrt{3}}$

$V^2 = 1935.980031$

for some working

$\therefore V = 43.999$ m/s
ie $V \doteqdot 44$ m/s.

iii) Using * in part ii) above

$y = \frac{-5x^2}{V^2}(1+\tan^2\alpha) + x\tan\alpha$

$5 = \frac{-5 \times 158.5^2}{44^2}(1+\tan^2\alpha) + 158.5\tan$

iii) Cont...

$5 \times 44^2 = -5 \times 158.5^2(1+\tan^2\alpha) + 158.5 \times 44^2 \times \tan\alpha$

$5 \times 44^2 = -5 \times 158.5^2 - 5 \times 158.5^2\tan^2\alpha + 158.5 \times 44^2 \times \tan\alpha$

$\therefore$ ~~158.5 × 44² tan α − 5 × 158.5²~~

$5 \times 158.5^2\tan^2\alpha - 158.5 \times 44^2\tan\alpha + (5 \times 44^2 + 5 \times 158.5^2) = 0$

$\therefore \tan\alpha = \frac{158.5 \times 44^2 \pm \sqrt{158.5^2 \times 44^4 - 4 \times 5 \times 158.5^2 \times (5 \times 44^2 + 5 \times 158.5^2)}}{2 \times 158.5^2 \times 5}$

$\tan\alpha = 1.865562615$ OR 0.577339593

$\therefore \alpha = 61.8^\circ$ OR 30°

$\therefore$ the alternative α is 61.8°.

THE END!