MATHEMATICAL INDUCTION QUESTIONS* Part 1

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1. Let a and r be real numbers with $r \neq 1$. prove by induction, that

$$a + ar + ar^2 + ar^3 + ... + ar^{n-1} = \frac{a(r^n - 1)}{r - 1} \qquad for\, n = 1, 2, 3, ...$$

2. Prove that

$$\sum_{r=n}^{2n-1} 2r + 1 = 3n^2 \qquad for\, n = 1, 2, 3, ...$$

3. Prove for $n \in \mathbb{N}$, that

$$\sqrt{n} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1$$

4. Show that

$$\sum_{r=1}^{n} \frac{1}{r^2} \leq 2 - \frac{1}{n} \qquad for\, n = 1, 2, 3, ...$$

5. Show that the sum of an arithmetic progression with initial value a, common difference d and n terms, is

$$\frac{n}{2}\left\{2a + (n-1)d\right\}$$

†In the case you need the answers to some of the exercises, you may email the author (vafa.khalighi@students.mq.edu.au).

6. Prove for $n \geq 2$ that,

$$\sum_{r=2}^{n} \frac{1}{r^2 - 1} = \frac{(n-1)(3n+2)}{4n(n+1)}$$

7. Let

$$S(n) = \sum_{r=0}^{n} r^2 \qquad for\, n \in \mathbb{N}$$

Show that there is a unique cubic $f(n) = an^3 + bn^2 + cn + d$, whose coefficients a, b, c, d you should determine, such that $f(n) = S(n)$ for $n = 0, 1, 2, 3$. Prove by induction that $f(n) = S(n)$ for $n \in \mathbb{N}$.

8. Show that

$$n + 3\sum_{r=1}^{n} r + 3\sum_{r=1}^{n} r^2 = \sum_{r=1}^{n} \left\{(r+1)^3 - r^3\right\} = (n+1)^3 - 1.$$

Hence, find an expression for $\sum_{r=1}^{n} r^2$.

9. Extend the method of last question to find expressions for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r^4$.

10. Use induction to show that

$$\sum_{k=1}^{n} \cos(2k-1)x = \frac{\sin 2nx}{2 \sin x}.$$

11. Use induction to show that

$$\sum_{k=1}^{n} \sin kx = \frac{\sin\left\{\frac{1}{2}(n+1)x\right\} \sin\left\{\frac{1}{2}nx\right\}}{sin\left\{\frac{1}{2}x\right\}}$$

12. Let k be a natural number. Deduce that

$$\sum_{r=1}^{n} r^k = \frac{n^{k+1}}{k+1} + E_k(n)$$

where $E_k(n)$ is a polynomial in n of degree at most k.

13. Prove **Bernoulli's Inequality** which states that

$$(1+x)^n \geq 1 + nx \qquad for\, x \geq -1\, and\, n \in \mathbb{N}$$

14. Show by induction that $n^2 + n \geq 42$ when $n \geq 6$ and $n \leq -7$.

15. Show by induction that there are $n!$ ways of ordering a set with n elements.

16. Show that there are 2^n subsets of the set $\{1, 2, ..., n\}$. [Be sure to include the empty set.]

17. Show for $n \geq 1$ and $0 \leq k \leq n$ that

$$\frac{n!}{k!(n-k)!} < 2^n$$

[**Hint:** you may find it useful to note that symmetry in the LHS which takes the same value for $k = k_0$ as it does at $k = n - k_0$.]

18. **Bertrand's Postulate** states that for $n \geq 3$ there is a prime number p satisfying

$$\frac{n}{2} < p < n.$$

Use this postulate and the strong form of induction to show that every positive integer can be written as a sum of prime numbers, all of which are distinct. (For the purpose of this question you will need to regard 1 as prime number.)

19. Assuming only the product rule of differentiation, show that

$$\frac{d}{dx}(x^n) = nx^{n-1} \qquad for\, n = 1, 2, 3, ...$$

20. Show that every natural number $n \geq 1$ can be written in the form $n = 2^k l$ where k, l are natural numbers and l is odd.

21. Show that every integer n can be written as a sum $3a + 5b$ where a and b are integers.

22. Show that $3^{3n} + 5^{4n+2}$ is divisible by 13 for all natural numbers n.

23. (a) Show that $7^{m+3} - 7^m$ and $11^{m+3} - 11^m$ are both divisible by 19 for all $m \geq 0$.
    (b) Calculate the remainder when $7^m - 11^n$ is divided by 19, for the cases $0 \leq m \leq 2$ and $0 \leq n \leq 2$.
    (c) Deduce that $7^m - 11^n$ is divisible by 19, precisely when $m + n$ is a multiple of 3.

24. By setting up an identity between I_n and I_{n-2} show that

$$I_n = \int_0^\pi \frac{\sin nx}{\sin x} dx$$

equals π when n is odd. What value does I_n take when n is even?

25. Show that

$$\int_0^{\pi/2} \cos^{2n+1} x \, dx = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

26. **Euler's Gamma** function $\Gamma(\alpha)$ is defined for all $a > 0$ by the integral

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx.$$

Show that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ for $a > 0$, and deduce that

$$\Gamma(n+1) = n! \qquad for\, n \in \mathbb{N}$$

27. **Euler's Beta** function $\beta(a, b)$ is defined for all positive a, b by the integral

$$\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} \, dx.$$

Set up a reduction formula involving β, and reduce that if m and n are natural numbers then

$$\beta(m+1, n+1) = \frac{m!n!}{(m+n+1)!}$$

28. **The Hermite** polynomials $H_n(x)$ for $n = 0, 1, 2, ...$ are defined recursively as

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \qquad for\, n \geq 1,$$

with $H_0(x) = 1$ and $H_1(x) = 2x$.

(a) Calculate $H_n(x)$ for $n = 2, 3, 4, 5$.

(b) Show by induction that

$$H_{2k}(0) = (-1)^k \frac{(2k)!}{k!} \, and \, H_{2k+1}(0) = 0.$$

(c) Show by induction that

$$\frac{dH_n}{dx} = 2nH_{n-1}.$$

(d) Deduce that $H_n(x)$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0.$$

(e) Use **Leibiniz's** rule for differentiating a product to show that the polynomials

$$(-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$$

satisfy the same recursion as $H_n(x)$ with the same initial conditions and deduce that

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) \qquad for\, n = 0, 1, 2, ...$$

29. What is wrong with the following "proof" that all people are of the same height?
"Let $P(n)$ be the statement that n persons must be of the same height. Clearly $P(1)$ is true as a person is the same height as him/herself. Suppose now that $P(k)$ is true for some natural number k and we shall prove that $P(k+1)$ is also true. If we have a crowd of $k+1$ people then we can invite one person to briefly leave so that k remains- from $P(k)$ we know that these people must all be equally tall. If we invite back the missing person and someone else leaves, then these k persons are also of the same height. Hence $k+1$ persons were all of equal height and so $P(k+1)$ follows. By induction everyone is of the same height."

30. Below are certain families of statements $P(n)$ (included by $n \in \mathbb{N}$), which satisfy rules that are similar (but not identical) to the hypotheses required for induction. In each case say for which $n \in \mathbb{N}$ the truth of $P(n)$ must follow from the given rules.

(a) $P(0)$ is true; for $n \in \mathbb{N}$ if $P(n)$ is true then $P(n+2)$ is true;

(b) $P(1)$ is true; for $n \in \mathbb{N}$ if $P(n)$ is true then $P(2n)$ is true;

(c) $P(0)$ and $P(1)$ are true; for $n \in \mathbb{N}$ if $P(n)$ is true then $P(n+2)$ is true;

(d) $P(0)$ and $P(1)$ are true; for $n \in \mathbb{N}$ if $P(n)$ and $P(n+1)$ are true then $P(n+2)$ is true;

(e) $P(0)$ is true; for $n \in \mathbb{N}$ if $P(n)$ is true then $P(n+2)$ and $P(n+3)$ are both true;

(f) $P(0)$ is true; for $n \geq 1$ if $P(n)$ is true then $P(n+1)$ is true.