Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2001
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Morning Session
Friday 17 August 2001

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided separately
- All necessary working should be shown in every question

Total marks **(120)**
- Attempt Questions 1 – 8
- All questions are of equal value

2604 - 1

EXAMINERS

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Marks

Question 1 **Begin a new page**

(a) $P(x)=(x+2)(x-1)(x-3)$

(i) Sketch $y=P(x)$ showing the intercepts on the coordinate axes. **1**

(ii) On separate diagrams, sketch the graphs of $y=|P(x)|$, $y=P(|x|)$, $y=\dfrac{1}{P(x)}$ showing the intercepts on the coordinate axes and the equations of any asymptotes. **4**

(b) (i) $P(x_1, y_1)$ is a point on the curve $y=e^{-x}$. The tangent to the curve at P passes through the origin. Find the coordinates of P. **3**

(ii) Find the set of values of the real number k such that the equation $e^{-x}=kx$ has two real and distinct solutions. **2**

(c) Consider the function $f(x)=\ln(1+\cos x)$, $-2\pi \le x \le 2\pi$, where $x \ne \pi$, $x \ne -\pi$.

(i) Show that the function f is even and the curve $y=f(x)$ is concave down for all values of x in its domain. **3**

(ii) Sketch the graph of the curve $y=f(x)$. **2**

Question 2 **Begin a new page**

(a) Find all the complex numbers $z=a+ib$, a, b real, such that $|z|^2-iz = 16-2i$. **3**

(b) (i) Find $\displaystyle\int \frac{e^x+1}{e^x}\,dx$ **1**

(ii) Find $\displaystyle\int \frac{x^2+x+1}{x(x^2+1)}\,dx$ **2**

(c) (i) Use the substitution $t=\tan\frac{x}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x+\sin x}\,dx$. **2**

(ii) Hence use the substitution $u=\frac{\pi}{2}-x$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x+\sin x}\,dx$. **2**

(d) (i) If $I_n=\displaystyle\int_0^1 \left(1+x^2\right)^n\,dx$, $n=0, 1, 2, \ldots$ show that $(2n+1)\,I_n = 2^n+2n\,I_{n-1}$ for $n=1, 2, 3, \ldots$ **3**

(ii) Hence find a reduction formula for $J_m=\displaystyle\int_0^{\frac{\pi}{4}} \sec^{2m} x\;dx$ **2**

Marks

Question 3 **Begin a new page**

(a) In an Argand Diagram, the point P representing the complex number z moves so that $|z-(1+i)|=1$.

(i) Sketch the locus of P. **1**

(ii) Shade the region where $|z-(1+i)|\le 1$ and $0<\arg(z-i)<\frac{\pi}{4}$ **1**

(b) In an Argand Diagram, a regular hexagon $ABCDEF$, with the vertices taken in anticlockwise order, has its centre at the origin O and vertex A at $z=2$.

(i) Find the set of values of $\operatorname{Im}(z)$ for points z on the hexagon. **1**

(ii) Find the set of values of $|z|$ for points z on the hexagon. **1**

(iii) If the hexagon is rotated in a clockwise direction about the origin through an angle of $45°$, find the value in modulus / argument form of the complex number which is represented by the new position of the vertex C. **1**

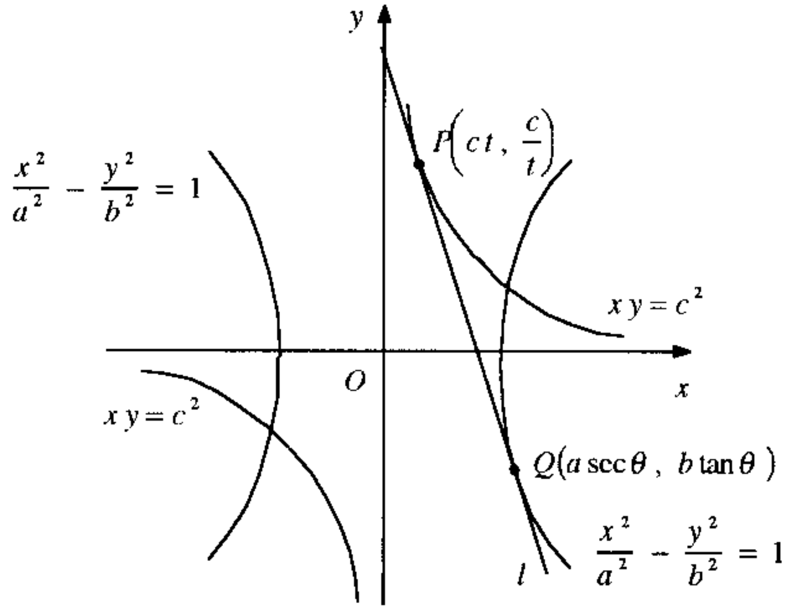
(c) (i) If $z=\cos\theta + i\sin\theta$, show that for positive integers n, $z^n+\dfrac{1}{z^n} = 2\cos n\theta$ and $z^n-\dfrac{1}{z^n} = 2i\sin n\theta$. Hence expand $\left(z+\dfrac{1}{z}\right)^4+\left(z-\dfrac{1}{z}\right)^4$ to show that $\cos^4\theta + \sin^4\theta = \frac{1}{4}(\cos 4\theta+3)$. **3**

(ii) By letting $x=\cos\theta$, show that the equation $8x^4+8\left(1-x^2\right)^2=7$ has roots $\pm\cos\frac{\pi}{12}$, $\pm\cos\frac{5\pi}{12}$. **2**

(iii) Deduce that $\cos\frac{\pi}{12}$, $\cos\frac{5\pi}{12}$ have a product of $\frac{1}{4}$ and a sum of $\sqrt{\frac{3}{2}}$. **3**

(iv) Hence or otherwise find a surd expression for $\cos\frac{\pi}{12}$. **2**

Question 4 **Begin a new page** **Marks**



The line l is a common tangent to the hyperbolas $xy = c^2$, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with points of contact P and Q respectively.

(i) Considering l as a tangent to $xy = c^2$ at $P\left(ct, \frac{c}{t}\right)$, show l has equation $x + t^2 y = 2ct$. **2**

(ii) Considering l as a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $Q(a\sec\theta, b\tan\theta)$, show l has equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$. **2**

(iii) Deduce that $\frac{\sec\theta}{a} = \frac{-\tan\theta}{bt^2} = \frac{1}{2ct}$. **1**

(iv) Write the coordinates of Q in terms of t, a, b and c, and show that $b^2t^4 + 4c^2t^2 - a^2 = 0$. Deduce that there are exactly two such common tangents to the hyperbolas. **3**

(v) Copy the diagram and use the symmetry in the graphs to draw in the second common tangent with points of contact R on $xy = c^2$ and S on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Write the coordinates of R and S in terms of t, a, b and c. **2**

(vi) Show that if $PQRS$ is a rhombus, then $b^2 = a^2$ and deduce that $t^2 < 1$. **2**

(vii) Show that if $PQRS$ is a square, then $t^4 + 2t^2 - 1 = 0$ and deduce that $2c^2 = a^2$. What is the relationship between the two hyperbolas if $PQRS$ is a square? **3**

Marks

Question 5 **Begin a new page**

(a) $I = \int_0^\pi x e^x \cos x \; dx$ and $J = \int_0^\pi e^x \cos x \; dx$

(i) Use integration by parts to show that $I - J = -\int_0^\pi x e^x \sin x \; dx$. **2**

(ii) Differentiate xe^x and hence find $\int (x+1) e^x \; dx$. Hence or otherwise show that **2**

$$I + J = -\pi \, e^\pi + \int_0^\pi x e^x \sin x \; dx \; .$$

(iii) Evaluate I. **1**

(b) (i) On the same diagram and without using calculus, sketch the graphs of $y = e^{-x}$, $y = -e^{-x}$ and $y = e^{-x} \cos x$, $0 \le x \le 2\pi$. Shade the region bounded by $y = e^{-x}$, $y = e^{-x} \cos x$ and $x = \pi$ for $x \ge 0$. **3**

(ii) The region shaded in (i) is rotated through one revolution about the line $x = \pi$. Use the method of cylindrical shells to show that the volume of the solid of revolution is given by **2**

$$V = 2\pi \int_0^\pi (\pi - x) \, e^{-x} \, (1 - \cos x) \; dx \, .$$

(iii) Use the substitution $u = \pi - x$ to show $V = 2\pi e^{-\pi} \left\{ \int_0^\pi u e^u \; du \; + I \right\}$, where I is as defined in (a). **2**

(iv) Hence find the volume of the solid. **3**

Question 6 **Begin a new page**

An object of mass m kg is dropped from rest from the top of a cliff 40 m above the water. Before the object reaches the water, the resistance to its motion has magnitude $\frac{1}{10} mv$ when the object has speed v ms^{-1}. After the object enters the water, the resistance to its motion has magnitude $\frac{1}{10} mv^2$. Take $g = 10 \, \text{ms}^{-2}$.

(a) (i) Write an expression for $\ddot{x}$ before the object enters the water, where x metres is the distance the object has fallen in t seconds. **1**

(ii) Show $10 \dfrac{dv}{dx} = \dfrac{100 - v}{v}$, and show that the speed of the object as it enters the water is V ms^{-1} where V satisfies $\dfrac{V}{100} + \ln\left(1 - \dfrac{V}{100}\right) + 0 \cdot 04 = 0$. **3**

(iii) Show this equation has a solution for V between 20 and 30, and taking 25 as a first approximation, use Newton's Method to show that $V \approx 25 \cdot 7$ to one decimal place. **3**

Question 6 continued Marks

(b) (i) Write an expression for $\ddot{x}$ after the object enters the water. Deduce the object slows on entry to the water, and find its terminal velocity in the water. **3**

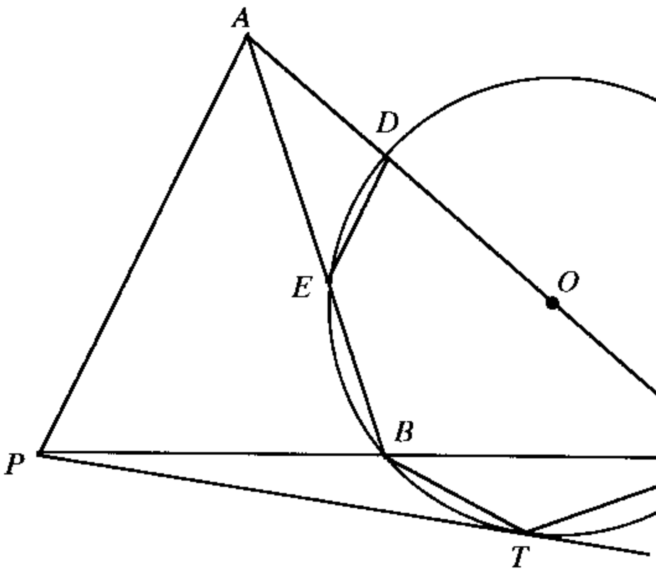
(ii) Show that t seconds after entering the water $10\dfrac{dv}{dt} = 100 - v^2$, and the velocity **3**

v ms^{-1} of the object is given by $2t = \ln\left\{\dfrac{(v+10)(V-10)}{(v-10)(V+10)}\right\}$, where V is the velocity on entry to the water calculated in (a).

(iii) How long after it enters the water will the body slow to 105% of its terminal velocity? **2**

Question 7 **Begin a new page**

(a)



A is a point outside a circle with centre O. P is a second point outside the circle such that $PT = PA$ where PT is a tangent to the circle at T. AO cuts the circle at D and C. PC cuts the circle at B. AB cuts the circle at E.

(i) Copy the diagram.
(ii) Show that $\Delta PBT \parallel\!| \Delta PTC$. **2**
(iii) Show that $\Delta APB \parallel\!| \Delta CPA$. **3**
(iv) Hence show that DE is parallel to AP. **3**

(b) A sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 2$, $u_2 = 12$ and $u_n = 6u_{n-1} - 8u_{n-2}$ for $n \geq 3$.

(i) Use Mathematical Induction to show that $u_n = 4^n - 2^n$ for $n \geq 1$. **4**

(ii) If $S_n = u_1 + u_2 + u_3 + \ldots + u_n$, find an expression for S_n in the form $S_n = a\,2^{2n+2} + b\,2^{n+1} + c$ where a, b, c are numerical constants. **3**

Marks

Question 8 **Begin a new page**

(a) (i) Given that $y = x - \ln(\sec x + \tan x)$, $0 \leq x < \frac{\pi}{2}$, show that $\dfrac{dy}{dx} = 1 - \sec x$. **2**

(ii) Hence show that $x < \ln(\sec x + \tan x)$ for $0 < x < \frac{\pi}{2}$. **3**

(b) (i) Show that $\dfrac{\sin(A+B) - \sin(A-B)}{2\sin B} = \cos A$. **1**

(ii) Hence show that **2**

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos(2n-3)x + \cos(2n-1)x = \frac{\sin 2nx}{2\sin x}$$

(iii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 8x}{\sin x}\,dx$. **2**

(c) (i) Find the values of the constants A and B such that **2**

$$4x^4 + 1 = (2x^2 + Ax + 1)(2x^2 + Bx + 1)$$

(ii) Hence find the prime factors of the integer $2^{14} + 1$. **3**