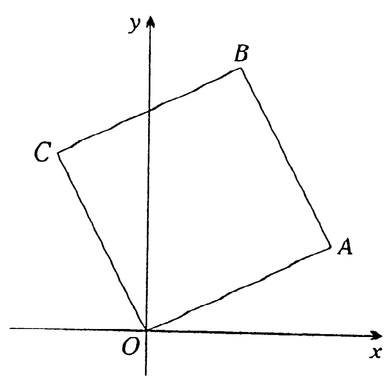
Sydney Grammar School

4 unit mathematics

Trial HSC Examination 2000

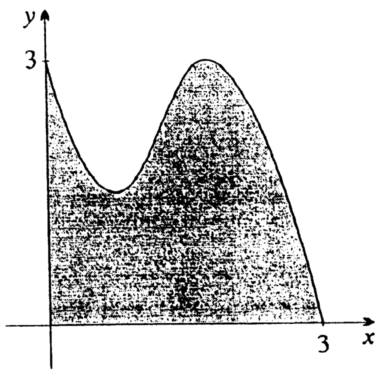
1. **(a)** Evaluate $\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\, dx$.
(b) The integral I_n is defined by $I_n = \int_0^1 x^n e^{-x}\, dx$.
(i) Show that $I_n = nI_{n-1} - e^{-1}$.
(ii) Hence show that $I_3 = 6 - 16e^{-1}$.
(c) Use partial fractions to find $\int \frac{2y+3}{(y-2)(y^2+3)}\, dy$.
(d) Use integration by parts to find $\int \tan^{-1} x\, dx$.
(e) **(i)** Find $\int \frac{dx}{x^2+2x+5}$.
(ii) Hence find $\int \frac{x^2}{x^2+2x+5}\, dx$.

2. **(a)** Simplify $(2-3i)^2$.
(b) On an Argand diagram, sketch the region specified by both the conditions $|z+3-4i| \le 5$ and $\Re(z) \le 1$. You must show intercepts with the axes, but you do not need to find other points of intersection.
(c) **(i)** Determine the modulus and argument of $-1+i$.
(ii) Hence find the least positive value of n for which $(-1+i)^n$ is real.
(d) **(i)** Let $z = r(\cos\theta + i\sin\theta)$ be a complex number in the Argand diagram. Show that multiplication of z by i rotates z by $\frac{\pi}{2}$ anticlockwise about the origin.
(ii)



In the square $OABC$, shown above, the point A represents $2+i$. What complex numbers do the points B and C represents?
(e) Let $z = a+ib$, where a and b are both real.
(i) For what values of a and b will $z + \frac{1}{z}$ be purely real?
(ii) Is it possible for $z + \frac{1}{z}$ to be purely imaginary?

3. (a)



In the diagram above, the region under the curve $y = 3 - 4x + 4x^2 - x^3$ in the first quadrant is shaded. A solid of revolution is formed by rotating this region about the y-axis. Use the method of cylindrical shells to find the volume of this solid.

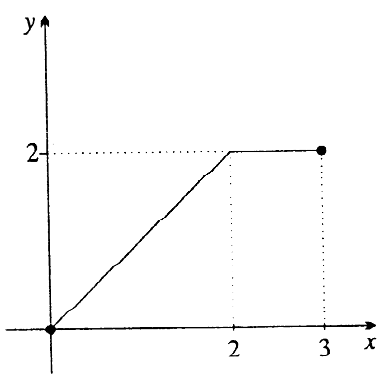
(b) Suppose the cubic equation $x^3 - 2x + 4 = 0$ has roots α, β and γ.

(i) Use the substitution $x^2 = y$ to show that a cubic equation which has roots α^2, β^2 and γ^2 is $y^3 - 4y^2 + 4y - 16 = 0$.

(ii) Factorise this new cubic into linear factors by initially grouping in pairs.

(iii) Hence show that the original equation has only one real root.

(c)



The graph of $y = f(x)$ is shown above. Sketch graphs of:

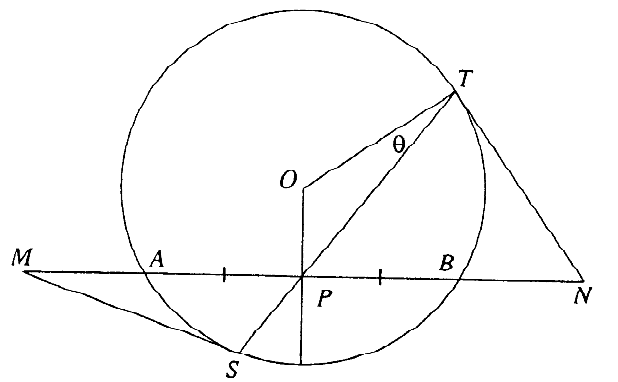
(i) $y = f(3 - x)$,

(ii) $y = f(|x|)$,

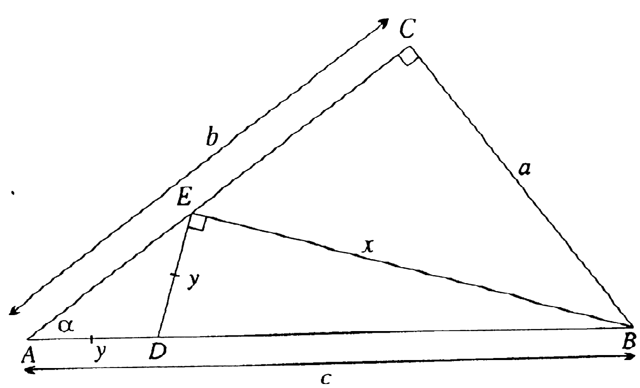
(iii) $y = \frac{1}{f(x)}$,

(iv) $y^2 = f(x)$.

4. (a)



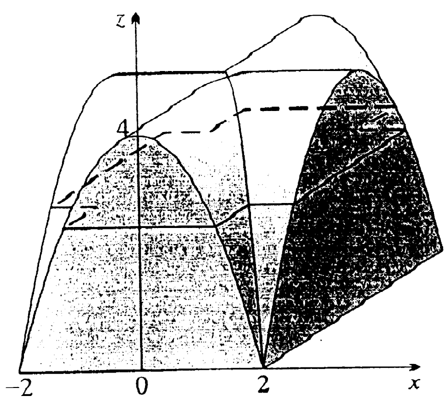
In the diagram above, P is the midpoint of the chord AB in the circle with centre O. A second chord ST passes through P, and the tangents at the endpoints meet AB produced at M and N respectively.
(i) Explain why $OPNT$ is a cyclic quadrilateral.
(ii) Explain why $OPSM$ is also cyclic.
(iii) Let $\angle OTS = \theta$. Show that $\angle ONP = \angle OMP = \theta$.
(iv) Hence prove that $AM = BN$.
(b)

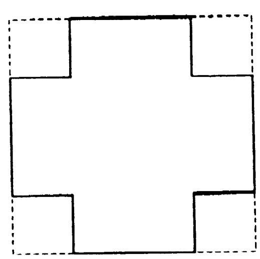


In the diagram above, $\triangle ABC$ is right-angled at C, with $a < b < c$. The points E on AC and D on AB are constructed so that $\angle BED$ is a right angle and $\triangle ADE$ is isosceles with $AD = DE$. Let $EB = x$, let $AD = DE = y$, and let $\angle CAB = \alpha$.
(i) Prove that $\triangle ABC ||| \triangle BEC$.
(ii) Show that $x = \frac{ca}{b}$.
(iii) Explain why $\angle BDE = 2\alpha$.
(iv) Hence show that $y = \frac{c(b^2-a^2)}{2b^2}$.
(c) Suppose the function $f(x)$ may be written as $f(x) = g(x) + h(x)$, where $g(x)$ is even and $h(x)$ is odd.
(i) Find an expression for $g(x)$ in terms of $f(x)$ alone.
(ii) Hence write down $g(x)$ for the function $f(x) = e^x$.

5. (a) An object of mass m kg is projected vertically upwards from ground level

with an initial speed U m/s. Its characteristic shape results in air resistance which is proportional to the square of its velocity, that is, mkv^2 for some constant k. The only other force acting on the body is that due to gravity. Take upwards as the positive direction for displacement x. Take ground level as the origin of displacement.
(i) **(α)** Show that $\ddot{x} = -(g + kv^2)$.
(β) Use $\ddot{x} = \frac{dv}{dt}$ to show that the time T_u taken to reach the highest point of its flight is $T_u = \frac{1}{\sqrt{gk}} \tan^{-1}\left(U\sqrt{\frac{k}{g}}\right)$.
(ii) Let T_d be the time taken for the object to fall back down to the ground, and for convenience let $w = U\sqrt{\frac{k}{g}}$. It can be shown that $\sqrt{gk}(T_d - T_u)$ simplifies to the function $f(w) = \ln(w + \sqrt{w^2+1}) - \tan^{-1} w$.
(α) Evaluate $f(0)$.
(β) Determine $f'(w)$, and show that $f'(w) > 0$ for $w > 0$.
(γ) Hence show that it takes longer for the object to fall back to the ground than it does to reach its highest point.
(b)



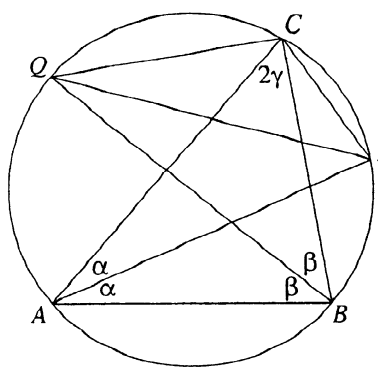


A sandstone cap on the corner of a fence is shown above, formed in the shape of two intersecting parabolic cylinders. On the front face, the equation of the parabola is $z = 4 - x^2$, where x is the horizontal distance measured from the mid-point of the base of the front face, and z is the height. The shape of a horizontal slice of thickness dz taken at height z is also shown. It is a square with four smaller squares removed, one from each corner.
(i) Find x in terms of z.
(ii) Show that $V = \int_0^4 (4^2 - 4(2 - \sqrt{4-z})^2)\, dz$.
(iii) Hence find the volume of stone in the cap.

6. **(a)** Let $P(z) = z^7 - 1$.
(i) Use de Moivre's theorem to find the roots of $P(z)$.
(ii) Hence write $P(z)$ as a product of:
(α) real and complex linear factors,
(β) real linear and irreducible quadratic factors.
(iii) Show that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$.

(iv) **(α)** Write down the quotient when $P(z)$ is divided by $z-1$.
(β) Hence show that $(1-\cos\frac{2\pi}{7})(1-\cos\frac{4\pi}{7})(1-\cos\frac{6\pi}{7})=\frac{7}{8}$.
(b) For $n=0,1,2,\ldots$, the integral G_n is defined by $G_n=\int_0^{\pi}\frac{\sin nx}{3-2\cos x}\,dx$.
(i) Find G_0 and show that $G_1=\frac{1}{2}\ln 5$.
(ii) Show that $G_{n+1}+G_{n-1}-3G_n=\frac{1}{n}((-1)^n-1)$. [Hint: You may assume that $\sin A+\sin B=2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$.]
(iii) Calculate G_3.

7. **(a)** Let ω be a non-real cube root of unity.
(i) Show that $1+\omega+\omega^2=0$.
(ii) Hence simplify $(1+\omega)^2$.
(iii) Show that $(1+\omega)^3=-1$.
(iv) Use part **(iii)** to simplify $(1+\omega)^{3n}$ and hence show that
${}^{3n}C_0-\frac{1}{2}({}^{3n}C_1+{}^{3n}C_2)+{}^{3n}C_3-\frac{1}{2}({}^{3n}C_4+{}^{3n}C_5)+{}^{3n}C_6-\cdots+{}^{3n}C_{3n}=(-1)^n$.
[Hint: You may assume that $\Re(\omega)=-\frac{1}{2}$ and that $\Re(\omega^2)=-\frac{1}{2}$.]
(b)



In the diagram above, AB is a fixed chord of a circle and C is a variable point on the major arc AB. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the circle again at P and Q respectively. Let $\angle CAB=2\alpha, \angle ABC=2\beta$ and $\angle BCA=2\gamma$.
(i) Show that $\angle PCQ=\alpha+\beta+2\gamma$.
(ii) Hence explain why the length of PQ is constant.
(iii) Use the sine rule to show that $\frac{AB}{PQ}=2\sin\gamma$.
(c) **(i)** Show that $\tan^2\theta=\frac{1-\cos 2\theta}{1+\cos 2\theta}$.
(ii) Hence show that $\tan^2(\frac{\alpha+\beta}{2})-\tan\alpha\tan\beta=\frac{\cos(\alpha+\beta)(1-\cos(\alpha-\beta))}{\cos\alpha\cos\beta(1+\cos(\alpha+\beta))}$.
(iii) Hence show that for $0<\alpha<\frac{\pi}{4}$ and $0<\beta<\frac{\pi}{4}$, $\sqrt{\tan\alpha\tan\beta}\leq\tan(\frac{\alpha+\beta}{2})$.

8. **(a)** Let $y=uv$ be the product of u and v, where u and v are functions of x.
(i) Show that $y''=uv''+2u'v'+u''v$.
(ii) Develop similar expressions for y''', y'''' and y'''''.
(iii) Hence, or otherwise, find and simplify $\frac{d^5}{dx^5}((1-x^2)e^{-x})$,
(b) The *Bernstein polynomial* $B_{n,k}(t)$ of degree n and order k is defined by:

$$B_{n,k}(t)={}^{n}C_k t^k(1-t)^{n-k},\ \text{for } 0\leq k\leq n.$$

(i) Write doen the three Bernstein polynomials of degree 2, namely $B_{2,0}(t), B_{2,1}(t)$ and $B_{2,2}(t)$.
(ii) The three fixed complex numbers α, β and γ are represented on the Argand diagram by the points A, B and C respectively. Three other complex numbers p, q and r are represented by the points P, Q and R respectively.

The point P divides the interval AB in the ratio $t : (1-t)$.
The point Q also divides the interval BC in the ratio $t : (1-t)$.
Likewise the point R divides the interval PQ in the ratio $t : (1-t)$.
($\boldsymbol{\alpha}$) Use the ratio division formula to find p and q in terms of α, β and γ.
($\boldsymbol{\beta}$) Hence show that $r = \alpha B_{2,0}(t) + \beta B_{2,1}(t) + \gamma B_{2,2}(t)$.
($\boldsymbol{\gamma}$) Given that $\alpha = 1 + i, \beta = 2 + 3i$ and $\gamma = 3 + i$, find the Cartesian equation of the locus of R as t varies.
(iii) ($\boldsymbol{\alpha}$) Show that $\sum_{k=0}^{n} B_{n,k}(t) = 1$.
($\boldsymbol{\beta}$) Show that for $r \leq k \leq n$, $\frac{{}^kC_r}{{}^nC_r} B_{n,k}(t) = {}^{n-r}C_{k-r} t^k (1-t)^{n-k}$.
($\boldsymbol{\gamma}$) Using the previous two parts, or otherwise, show that $\sum_{k=2}^{5} \frac{{}^kC_2}{{}^5C_2} B_{5,k}(t) = t^2$.