E.H

KNOX GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT

2001
TRIAL HSC EXAMINATION

Mathematics Extension 2

- **General Instructions**

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided on page 10
- All necessary working should be shown in every question

Total marks (120)

- Attempt Questions 1–8
- All questions are of equal value
- Use a SEPARATE writing booklet for each question

NAME:______________________ TEACHER:______________________

Total marks (120)
Attempt questions 1 – 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find:

(i) $\displaystyle\int \frac{x}{\sqrt{9-4x^2}}\,dx$ **2**

(ii) $\displaystyle\int \frac{x^2}{x+1}\,dx$ **2**

(iii) $\displaystyle\int_0^{\ln 2} xe^x\,dx$ **3**

(b) (i) Find real numbers A, B and C such that $\dfrac{2}{(t+1)(t^2+1)} \equiv \dfrac{A}{t+1} + \dfrac{Bt+C}{t^2+1}$. **3**

(ii) Hence, find $\displaystyle\int_0^1 \frac{2}{(t+1)(t^2+1)}\,dt$. **3**

(iii) By using the substitution $t = \tan\left(\dfrac{x}{2}\right)$ evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x-\cos x}\,dx$. **2**

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Suppose $z = 2 + 2i$ and $w = -1 + \sqrt{3}i$.

(i) Express z and w in modulus – argument form. **2**

(ii) Find $\left|\dfrac{z}{w}\right|^4$. **1**

(iii) Find the principal argument of $\left(\dfrac{z}{w}\right)^4$. **2**

(b) Sketch separately the following loci in an Argand plane and state the cartesian equations in each case given that:

(i) $|z-3i| = |z-4|$ **2**

(ii) $\mathrm{Re}\left(\dfrac{z-2}{2}\right) = 0$ **2**

(iii) $\arg(z+2) = -\dfrac{\pi}{6}$ **2**

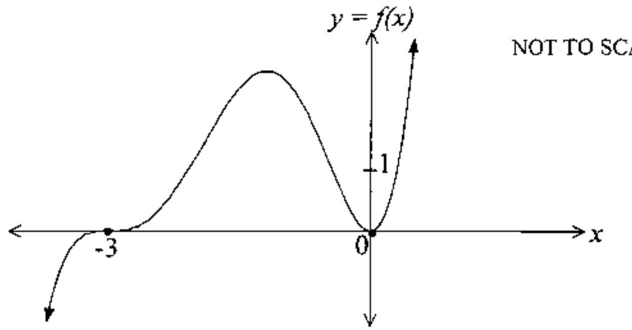
(c) (i) Show that if $z = x + iy$ then $|z|^2 = z\bar{z}$. **1**

(ii) Using the result of (c)(i), or otherwise, prove that for any two complex numbers z and w that:

$$|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2.$$ **2**

(iii) Interpret this result geometrically. A vector diagram may be useful. **1**

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**



(a) Consider the graph of $y = f(x)$ as shown above.

On the answer sheet provided on pages 11 & 12, use the graph of $y = f(x)$ to clearly sketch ***separately*** the graphs of:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y^2 = f(x)$ **2**

(iii) $y = f'(x)$. **1**

(b) Suggest a possible polynomial equation for the graph of $y = f(x)$ shown in ***part (a) of Question 3.*** **1**

(c) (i) Show that $x = 1$ is a zero of $x^3 + 3x^2 - 4$. **1**

(ii) Sketch the curve with the equation $y = x^3 + 3x^2 - 4$, giving the coordinates of any maximum or minimum points and the intercepts made on each axis. **3**

(iii) Use your results in (c)(ii) above to sketch the curves: **4**

(α) $y = \left| x^3 + 3x^2 - 4 \right|$

(β) $y = \ln\left| x^3 + 3x^2 - 4 \right|$

(iv) Hence, or otherwise, determine the value of m, where m is a constant such that the equation $2\ln|x+2| + \ln|x-1| = m$. **1**

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Draw a sketch graph of the hyperbola $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$ and shade clearly the region bounded by the lines $x = \pm a$ and the upper and lower branches of this hyperbola. **1**

(ii) Show $\dfrac{d}{d\theta}\ln(\sec\theta + \tan\theta) = \sec\theta$. **1**

(iii) Explain why the area, A, of the shaded region drawn in (a)(i) above can be by: **2**

$$A = \int_0^a \frac{4b}{a}\sqrt{a^2 + x^2}\, dx.$$

(iv) By using the substitution $x = a\tan\theta$ in (a)(iii), show that $A = 4ab\displaystyle\int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta$. **2**

(v) Show that the integral stated in (a)(iv) simplifies to $2ab\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)$. **3**

(***Hint***: Write $\sec^3\theta$ in the form $\sec\theta.\sec^2\theta$ and then use integration by parts)

(vi) Use the ***method of cylindrical shells*** to show that the volume (in cubic units) of the solid generated by revolving this area about the y-axis is given by: **3**

$$V = \frac{4\pi ba^2}{3}\left(2\sqrt{2} - 1\right).$$

(b) A solid has a base, which is the *standard ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with major axis of length $2a$ units and minor axis of length $2b$ units ($a > b$). In the vertical plane, the cross–sections of the solid are always isosceles triangles with perpendicular height h and whose base is parallel to the major axis. **3**

Use the ***method of slicing*** to find the volume of the solid.

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The point T with coordinates $(at^2, 2at)$, $t \neq 0$, $a > 0$, lies on the parabola with equation $y^2 = 4ax$. The tangent to the parabola at T meets the axis of the parabola at R. The normal at T meets the axis of the parabola at Q and the parabola again at P. The coordinates of P are $(ap^2, 2ap)$.

(i) Represent this information on a clear and well-labelled diagram. **1**

(ii) Derive the equations of the tangent and normal to the parabola at T. **2**

(iii) Show that the length of RQ is $2a(1+t^2)$ units. **1**

(iv) Show that the values of t for which R will lie on the directrix of the parabola satisfy $t^2 = 1$. **2**

(v) Show that if $t \neq p$, then $p = -\left(t + \frac{2}{t}\right)$. **2**

(vi) Find TP, in terms of a and simplify your expression as far a possible. **1**

(vii) Hence, or otherwise, prove that the area of ΔTPR is $16a^2$ square units. **1**
(You may assume R lies on the directrix)

(b) The equation of a rectangular hyperbola in cartesian form is given by $xy = c^2$ where $c > 0$.

(i) Verify that the point $P\left(cp, \frac{c}{p}\right)$ lies on $xy = c^2$, where p is a non-zero real number. **1**

(ii) Q has coordinates $\left(cq, \frac{c}{q}\right)$ where q is a non-zero real number. **2**
Show that the equation of the chord PQ is given by $x + pqy = c(p+q)$.

(iii) Find the equation of the locus of the midpoint of the chord PQ if it is known that the chord must always pass through the point $(0, 2)$. **2**

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) A particle of mass m units is projected vertically upward from the ground with initial speed u. The air resistance at any instance is proportional to the velocity v at that instant. For this question you may assume $R = kmv$ where k is a constant.

(i) With the aid of a suitable diagram show that $\frac{dv}{dt} = -(g + kv)$? **1**

(ii) Show at any time t, that $t = \frac{1}{k}\ln\left|\frac{g+ku}{g+kv}\right|$ seconds. **3**

(iii) Prove that the particle reaches it highest point in time T seconds when: **1**

$$T = \frac{1}{k}\ln\left(\frac{ku}{g} + 1\right)$$

(iv) The highest point reached by the particle is at H metres above the ground.

(α) Prove that $x = \frac{1}{k^2}(g + ku)(1 - e^{-kt}) - \frac{gt}{k}$. **3**

(β) Prove that $H = \frac{1}{k}(u - gT)$. **2**

(b) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$ where n is a positive integer such that $n \geq 2$.

(i) By replacing $\sin^n \theta$ with $\sin^{n-1}\theta . \sin\theta$, and using integration by parts or otherwise, show that $I_n = \frac{n-1}{n} I_{n-2}$. **3**

(ii) Hence, or otherwise, evaluate I_{10}. **2**

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Let α, β, and γ be the non – zero roots of the equation $x^3 + rx + s = 0$.

(i) Find in terms of r, the simplified value of $\alpha^2 + \beta^2 + \gamma^2$. 2

(ii) Find in terms of r and s, the simplified value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$. 2

(iii) Find in terms of r and s, the cubic equations (in general form) whose roots are

(A) $\frac{1}{\alpha}, \frac{1}{\beta}$, and $\frac{1}{\gamma}$; 2

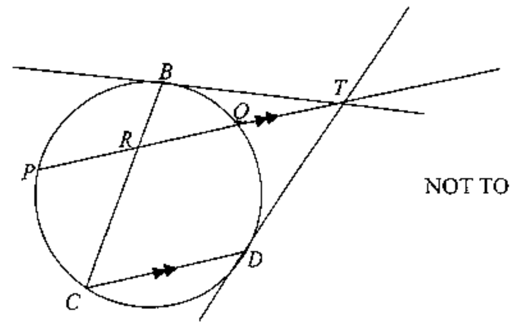
(B) $\alpha + \beta - \gamma, \beta + \gamma - \alpha$ and $\gamma + \alpha - \beta$ 3

(b) Suppose $x^3 + rx + s = 0$ (with r and s being non–zero and real) has a double root. 2

Show that $x = -\frac{3s}{2r}$.

(c) Find all the roots of $p(x) = x^4 - 8x^3 + 39x^2 - 122x + 170$ given that $3 - i$ is one of the roots. 4

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**



NOT TO SCALE

(a) In the diagram, PQ and CD are parallel chords of a circle. The tangent at D meets PQ produced externally at T. B is the point of contact of the other tangent from the circle. BC meets PQ internally at R.

Copy or trace this diagram into your writing booklet

(i) Explain why $\angle BDT = \angle BRT$? 2

(ii) Show that B, T, D and R are concylic points. 2

(iii) Prove that $\angle BRT = \angle DRT$. 2

(iv) Show that ΔRCD is isosceles. 2

(v) Show that BC bisects PQ. 2

(b) (i) Show that $\cos x = \sin\left(x + \frac{\pi}{2}\right)$. 1

(ii) Given that $y = 3\sin x + 4\cos x$, prove by the Principle of Mathematical Induction that $\frac{d^n y}{dx^n} = 5\sin\left(x + \alpha + \frac{n\pi}{2}\right)$ where $\frac{d^n y}{dx^n}$ means the nth derivative of y with respect to x and $n = 1, 2, 3, \ldots\ldots$ 4

You are advised to first express $y = 3\sin x + 4\cos x$ in the form $R\sin(x + \alpha)$.

End of Paper

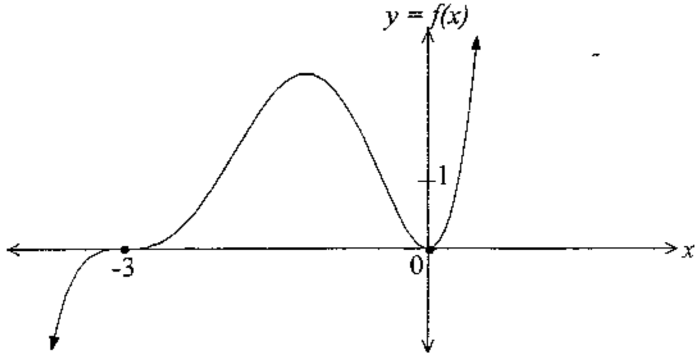
Year 12 Trial HSC Mathematics Extension 2 – 2001
Answer Sheet for Question 3 (a) only!

Detach and submit this page with your solutions to Question 3

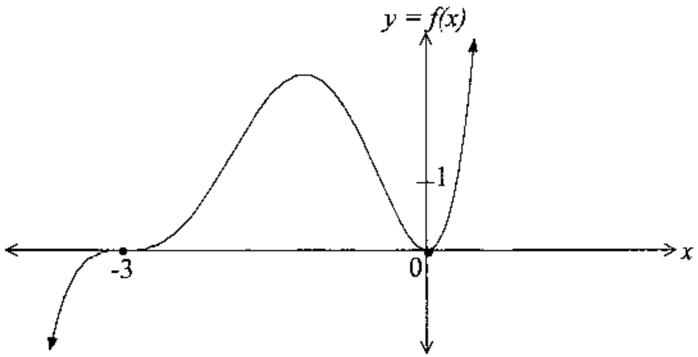
Student Name: ____________________

Question 3 (a) *In each case use the graph of y = f(x) to clearly sketch the following:*

(i) $y = \frac{1}{f(x)}$.

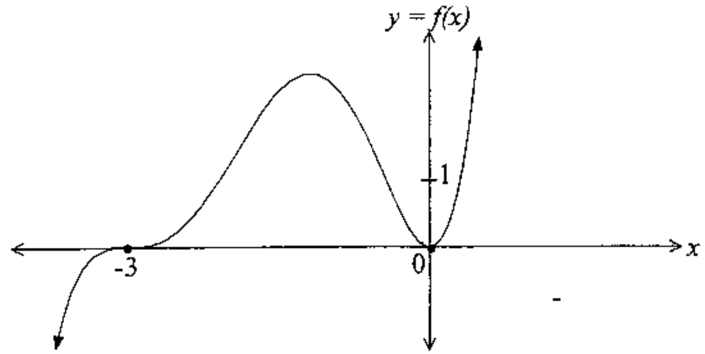


(ii) $y^2 = f(x)$.



Please turn over for part (a)(iii).

(iii) $y = f'(x)$



Question 3 (b):

Possible polynomial equation for y = f(x):____________________