CSSA
CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NSW

Centre Number

Student Number

2013
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Morning Session
Monday, 5 August 2013

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided on a separate sheet
- In Questions 11–16, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks – 100

Section I Pages 2–6

10 marks
- Attempt Questions 1–10
- Allow 15 minutes for this section

Section II Pages 7–14

90 marks
- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 If $z=2+3i$ and $w=-5-2i$, what is the value of zw?

(A) -4

(B) $-3+i$

(C) $-4-19i$

(D) $-16-19i$

2 What is the gradient of the tangent to the curve $-2x^2+y^2+y=0$ at the point $(1,1)$?

(A) 1

(B) $\frac{4}{3}$

(C) $\frac{3}{2}$

(D) 2

3 The point $P\left(ct,\frac{c}{t}\right)$ lies on the hyperbola $xy=c^2$.

What is the x-intercept of the tangent to the hyperbola at P?

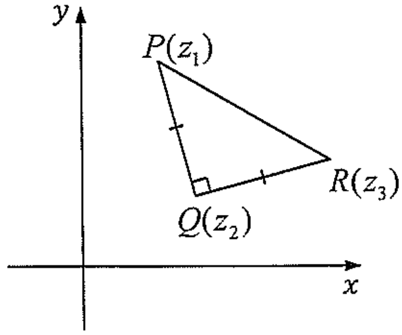
(A) $(2ct,0)$

(B) $\left(\frac{2c}{t},0\right)$

(C) $\left(ct-\frac{c}{t^3},0\right)$

(D) $\left(\frac{c}{t}-ct^3,0\right)$

4 The vertices of the triangle PQR are represented by the complex numbers z_1, z_2 and z_3 respectively. The triangle PQR is isosceles and right-angled at Q, as shown in the diagram.



Which of the following statements is true?

(A) $z_2 - z_1 = i(z_3 - z_2)$

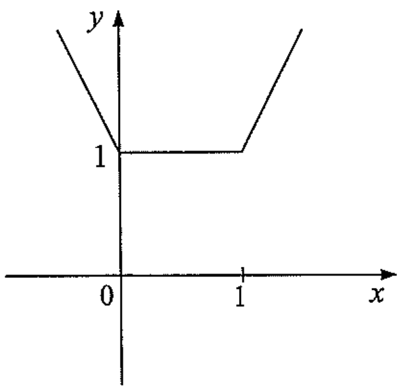
(B) $z_1 - z_2 = i(z_3 - z_2)$

(C) $z_2 - z_1 = i(z_1 - z_3)$

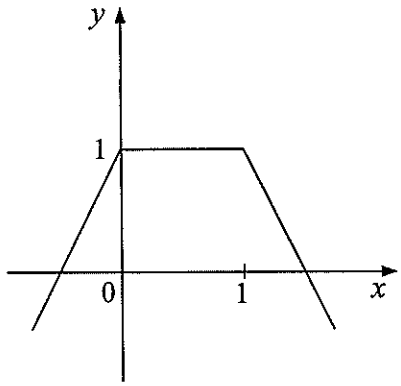
(D) $z_1 - z_2 = i(z_1 - z_3)$

5 Which of the following graphs could represent the graph of $y = |x| + |x-1|$?

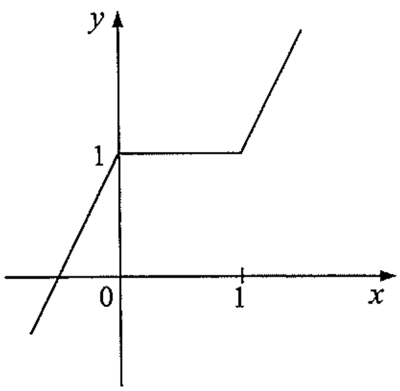
(A)



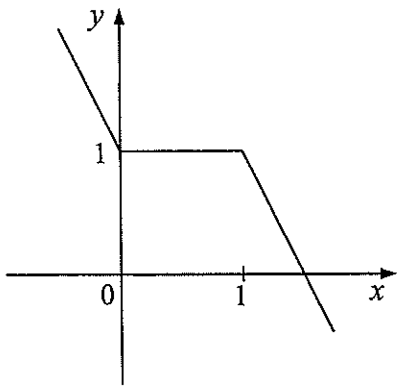
(B)



(C)



(D)



6 Which of the following, for $x > 0$, is an expression for $\int \frac{1}{x^3 + x} dx$?

(A) $\log_e\left(x\sqrt{x^2+1}\right)+C$

(B) $\log_e\left(x(x^2+1)\right)+C$

(C) $\log_e\left(\frac{x}{\sqrt{x^2+1}}\right)+C$

(D) $\log_e\left(\frac{x}{x^2+1}\right)+C$

7 A stone of mass m is dropped from rest and falls in a medium in which the resistance is directly proportional to the square of the velocity v. Suppose mk is the constant of proportionality and that the displacement downwards from the initial position is x at time t. The acceleration due to gravity is g.

Which of the following is true?

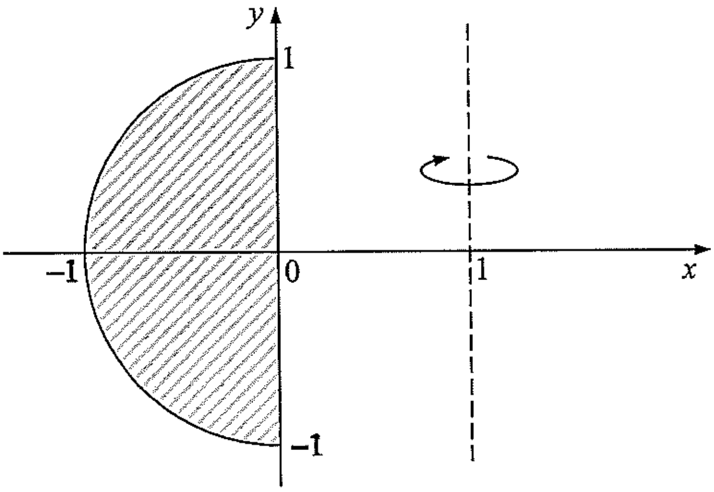
(A) The terminal velocity is $\frac{g}{k}$.

(B) As $t \to \infty$, $x \to L$ where L is a positive constant.

(C) The equation of motion is given by $v\frac{dv}{dx} = g - kv^2$.

(D) The time for the stone to reach velocity V is given by $\int_0^V g - kv^2 \, dv$.

8 The diagram shows the graph $x^2 + y^2 = 1$ for $-1 \le x \le 0$. The region bounded by the graph and the y-axis is rotated about the line $x = 1$ to form a solid.



Which integral represents the volume of the solid?

(A) $2\pi \int_{-1}^{0} (1+x)\sqrt{1-x^2}\, dx$

(B) $2\pi \int_{-1}^{0} (1-x)\sqrt{1-x^2}\, dx$

(C) $4\pi \int_{-1}^{0} (1+x)\sqrt{1-x^2}\, dx$

(D) $4\pi \int_{-1}^{0} (1-x)\sqrt{1-x^2}\, dx$

9 The polynomial $p(x)$ of degree 4 has real coefficients.
$p(x)$ has roots α, β, γ and δ and it is known that $\alpha^2+\beta^2+\gamma^2+\delta^2=-8$.

Which of the following must be true?

(A) $p(x)$ has all its roots real.

(B) $p(x)$ has one real and three imaginary roots.

(C) $p(x)$ has two real and two imaginary roots.

(D) $p(x)$ has at least two imaginary roots.

10 If $f(x)$ is a non-zero odd function with period π, which of the following statements is false?

(A) $\int_0^{2\pi} f(x)\,dx=0$

(B) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx=2\int_0^{\frac{\pi}{2}} f(x)\,dx$

(C) $\int_0^{\pi} f(x)\,dx=-\int_0^{\pi} f(-x)\,dx$

(D) $\int_{\alpha}^{\alpha+\pi} f(x)\,dx=\int_0^{\pi} f(x)\,dx$ for any real number α.

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 2+3i$ and $w = -3-4i$.

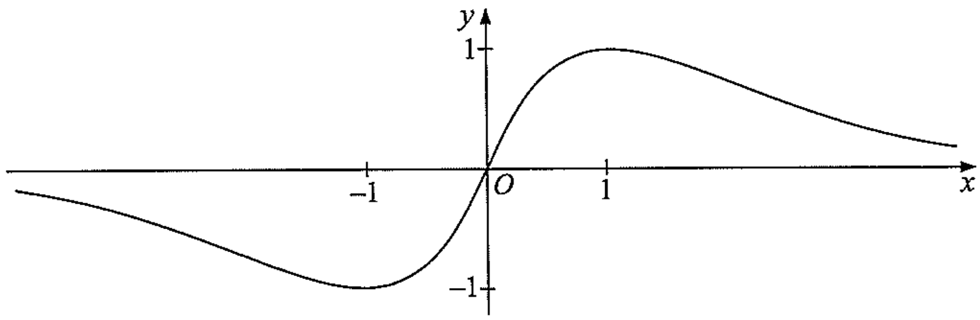
(i) Find $z+\overline{w}$ **1**

(ii) Express $\dfrac{w}{z}$ in the form $a+ib$, where a and b are real numbers. **2**

(b) Sketch the region in the complex plane which satisfies $\dfrac{\pi}{6} < \arg(z) < \dfrac{5\pi}{6}$ and $\dfrac{1}{2} \le \operatorname{Im}(z) \le 2$. **3**

(c) Find $\displaystyle\int \frac{dx}{\left(9-x^2\right)^{\frac{3}{2}}}$ using the substitution $x = 3\sin\theta$. Give your answer in terms of x. **4**

(d) The following diagram shows the graph of $f(x) = \dfrac{2x}{x^2+1}$.



Draw separate one-third page diagrams of the graphs of each of the following.

(i) $y = \left[f(x)\right]^2$ **2**

(ii) $y = \sqrt{f(x)}$ **1**

(iii) $y = \dfrac{1}{f(x)}$ **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Find the square roots of $-24-10i$. **2**

(ii) Hence, or otherwise, solve $x^2-(1-i)x+6+2i=0$. **2**

(b) Use integration by parts to find $\int \frac{\ln x}{x^2}\,dx$. **2**

(c) Using the substitution $t=\tan\frac{x}{2}$, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$. **4**

(d) An ellipse is defined by the parametric equations:

$$x=2\cos\theta$$
$$y=3\sin\theta$$

for $0\le\theta<2\pi$.

(i) Find the Cartesian equation of the ellipse. **1**

(ii) Find the eccentricity of the ellipse. **1**

(iii) Sketch the ellipse showing the intercepts, foci and directrices. **3**

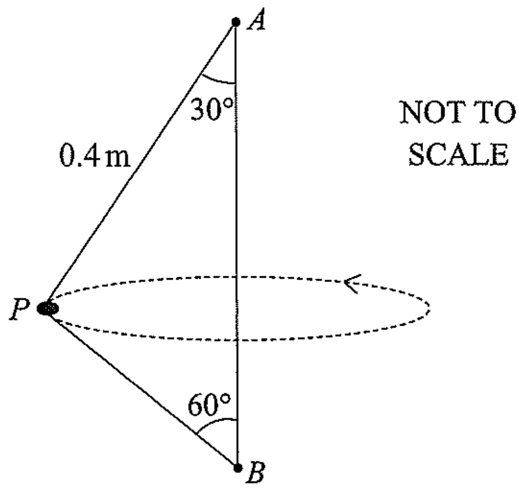
Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A group of 30 students is to be divided into three groups consisting of 7, 8 and 15 students. In how many ways can this be done? **1**

(b) (i) Find a and b such that $x = 2$ is a double root of $p(x) = x^4 + ax^3 + x^2 + b$. **3**

(ii) For the values of a and b above, factorise $p(x)$ over the real numbers. **1**

(c)



A particle P of mass 0.3 kg is attached to one end of each of two light inextensible strings of different lengths. The longer string is also attached to a fixed point A and the shorter string is also attached to a fixed point B, which is vertically below A.

AP makes an angle of 30° with the vertical and is 0.4 m long. PB makes an angle of 60° with the vertical. The particle moves in a horizontal circle with constant angular speed and with both strings taut. The tension in the string AP is 5 N. Assume the acceleration due to gravity is $10\,\text{ms}^{-2}$.

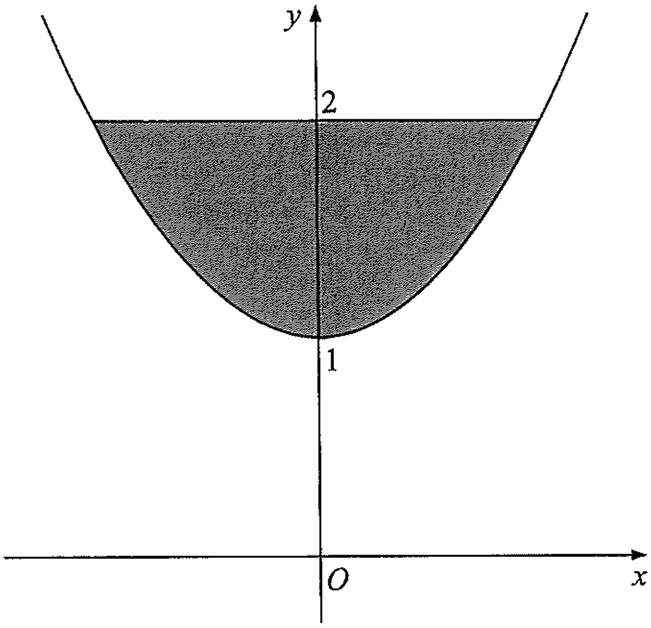
(i) Find the tension in the string PB. **2**

(ii) Calculate the angular velocity of the particle P correct to 1 decimal place. **3**

Question 13 continues of page 10

Question 13 (continued)

(d) The base of a solid, S, is the region enclosed by the parabola $y = x^2 + 1$ and the line $y = 2$.



Each cross section of S perpendicular to the y-axis is a rectangle. The height of the rectangle that is y units from the origin is $\frac{1}{2}y$ units.

(i) Show that the area of the rectangle y units from the origin is given by $y\sqrt{y-1}$ square units. **2**

(ii) Hence find the volume of the solid S. **3**

End of Question 13

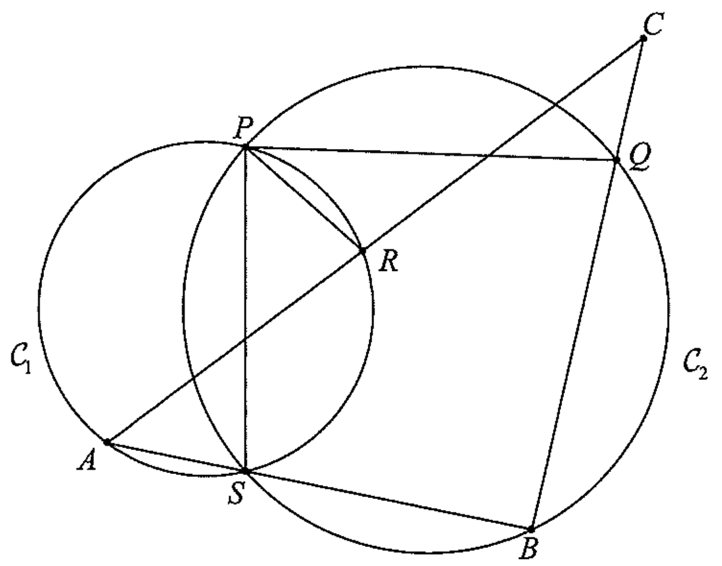
Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the polynomial $p(x)=x^3-x^2-21x+45$ with roots α, β and γ.

(i) Find the monic polynomial with roots $\alpha-3$, $\beta-3$, $\gamma-3$. **3**

(ii) Hence solve $p(x)=0$. **1**

(b) Two circles C_1 and C_2 meet at P and S. Points A and R lie on C_1 and points B and Q lie on C_2. AB passes through S and AR produced meets BQ produced at C, as shown in the diagram.



(i) Prove that $\angle PRA = \angle PQB$. **2**

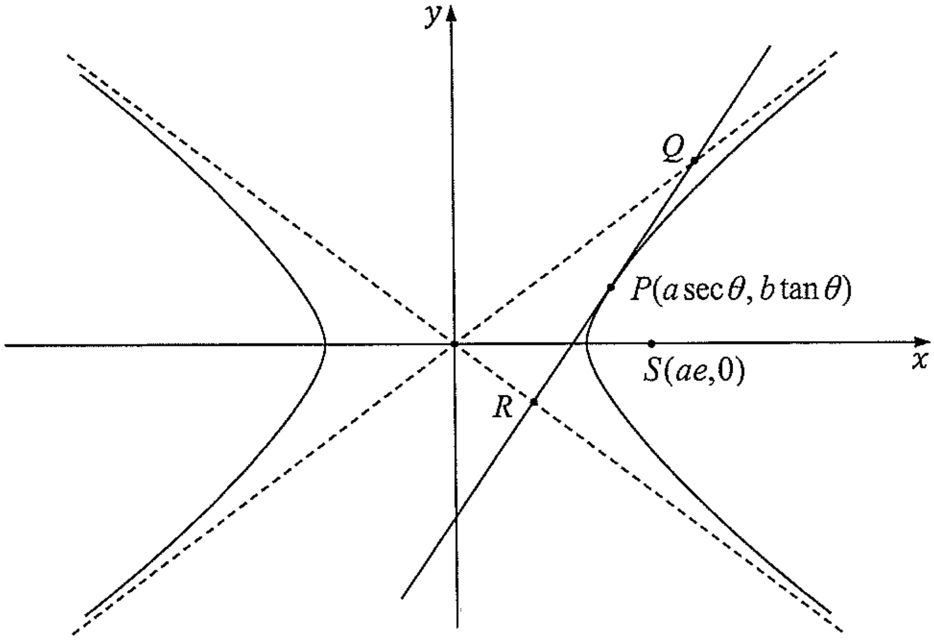
(ii) Prove that the points P, R, Q and C are concyclic. **2**

Question 14 continues on page 12

Question 14 (continued)

(c) The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with focus $S(ae,0)$.

The tangent to the hyperbola at P meets the asymptotes of the hyperbola at Q and R, as shown in the diagram.



(i) Show that the equation of the tangent to the hyperbola at P is given by **2**

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1.$$

(ii) Show that Q has coordinates $\left(\frac{a}{\sec\theta-\tan\theta},\frac{b}{\sec\theta-\tan\theta}\right)$. **2**

The coordinates of R are given by $\left(\frac{a}{\sec\theta+\tan\theta},\frac{-b}{\sec\theta+\tan\theta}\right)$. (Do NOT prove this).

(iii) Prove that $\left|\tan\angle QSR\right|=\frac{b}{a}$. **3**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan x \, dx$. **2**

(ii) Suppose $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x \, dx$. **2**

Given that $I_n = \dfrac{1}{n-1} - I_{n-2}$ for any integer $n \geq 2$, find the value of

$$\int_0^{\frac{\pi}{4}} \tan^5 x \, dx.$$

(b) (i) Find the five fifth roots of $z^5 = 1$ and plot these on an Argand diagram. **2**

(ii) Express $z^5 - 1$ as the product of real linear and quadratic factors. **2**

(iii) Prove that $\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5} = -\dfrac{1}{2}$. **2**

(c) A projectile starting from the origin has acceleration given by

$$\frac{d^2x}{dt^2} = -6\frac{dx}{dt} - 9x$$

where x is the displacement from the origin at time t.

The solution to this equation is known to be of the form, $x(t) = f(t)e^{-3t}$, for some function $f(t)$.

(i) Show that $f(t) = At + B$, where A and B are constants. **2**

(ii) Find the value of B. **1**

(iii) Assuming that A is positive, at what time is the displacement a maximum? **2**
You do not need to prove a maximum is attained.

Examiners

Gerry Sozio (Convenor)	Catholic Education Office, Wollongong
Peter Brown	University of New South Wales, Kensington
Robert Muscatello	Mount Carmel Catholic High School, Varroville
Frank Reid	University of New South Wales, Australian Catholic University
Thanom Shaw	SCEGGS, Darlinghurst

CSSA

CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW
2013 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION
MATHEMATICS EXTENSION 2

Section I
10 marks

Questions 1-10 (1 mark each)

Question 1 (1 mark)
Outcomes Assessed: E3
Targeted Performance Bands: E2

Solution	Answer	Mark
$(2+3i)(-5-2i)$ $=-10-4i-15i-6i^2$ $=-4-19i$	C	1

Question 2 (1 mark)
Outcomes Assessed: E6
Targeted Performance Bands: E3

Solution	Answer	Mark
Using implicit differentiation $-4x+2y\frac{dy}{dx}+\frac{dy}{dx}=0$ $\frac{dy}{dx}=\frac{4x}{2y+1}$ Substituting (1, 1), $\frac{dy}{dx}=\frac{4}{3}$	B	1

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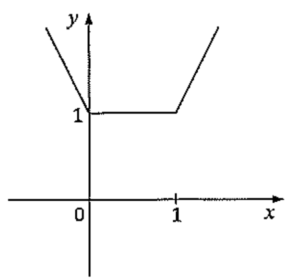
Question 3 (1 mark)
Outcomes Assessed: E4
Targeted Performance Bands: E2

Solution	Answer	Mark
$\frac{dy}{dx}=\frac{-c^2}{x^2}$. At $P\left(ct,\frac{c}{t}\right)$, $\frac{dy}{dx}=\frac{-c^2}{c^2t^2}=\frac{-1}{t^2}$. Equation of the tangent at P: $y-\frac{c}{t}=\frac{-1}{t^2}(x-ct)$ $x+t^2y=2ct$ The x-intercept is $(2ct,0)$.	A	1

Question 4 (1 mark)
Outcomes Assessed: E3
Targeted Performance Bands: E3

Solution	Answer	Mark
$\overrightarrow{QP}=i\times\overrightarrow{QR}$ $z_1-z_2=i(z_3-z_2)$	B	1

Question 5 (1 mark)
Outcomes Assessed: E6
Targeted Performance Bands: E2

Solution	Answer	Mark
	A	1

Question 6 (1 mark)
Outcomes Assessed: E8
Targeted Performance Bands: E3

Solution	Answer	Mark
$\int\frac{1}{x^3+x}dx=\int\left(\frac{1}{x}+\frac{-x}{x^2+1}\right)dx$ $=\log_e x-\frac{1}{2}\log_e(x^2+1)+C$ $=\log_e\frac{x}{\sqrt{x^2+1}}+C$	C	1

Question 7 (1 mark)
Outcomes Assessed: E5
Targeted Performance Bands: E3

Solution	Answer	Mark
The equation of motion is $m\ddot{x}=mg-mkv^2$ $\ddot{x}=g-kv^2$ Since $\ddot{x}=v\frac{dv}{dx}$, $v\frac{dv}{dx}=g-kv^2$.	C	1

Question 8 (1 mark)
Outcomes Assessed: E7
Targeted Performance Bands: E3

Solution	Answer	Mark
$V=2\pi\int_{-1}^{0}(1-x)\times2\sqrt{1-x^2}\,dx$ $=4\pi\int_{-1}^{0}(1-x)\sqrt{1-x^2}\,dx$	D	1

Question 9 (1 mark)
Outcomes Assessed: E4
Targeted Performance Bands: E4

Solution	Answer	Mark
$p(x)$ has at least 2 imaginary roots.	D	1

Question 10 (1 mark)
Outcomes Assessed: E8
Targeted Performance Bands: E4

Solution	Answer	Mark
$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx \neq 2\int_0^{\frac{\pi}{2}} f(x)\,dx$	B	1

Section II
60 marks

Question 11 (15 marks)
(a) (i) (1 mark)
Outcomes assessed: E3
Targeted Performance Bands: E2

Criteria	Mark
• Evaluates $z+\overline{w}$ correctly	1

Sample answer:

$$z+\overline{w} = 2+3i+(-3+4i)$$
$$= -1+7i$$

(a) (ii) (2 marks)
Outcomes assessed: E3
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	2
• Attempts to realize the denominator	1

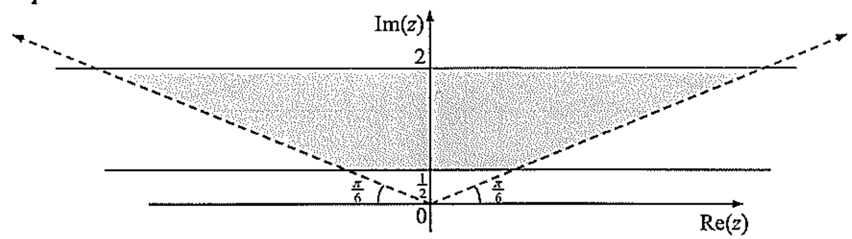
Sample answer:

$$\frac{-3-4i}{2+3i} = \frac{-3-4i}{2+3i}\times\frac{2-3i}{2-3i}$$
$$= \frac{-18+i}{13}$$
$$= \frac{-18}{13}+\frac{i}{13}$$

(b) (3 marks)
Outcomes assessed: E6
Targeted Performance Bands: E3

Criteria	Mark
• Correct solution	3
• Both graphs correct	2
• ONE correct graph	1

Sample answer:



(c) (4 marks)
Outcomes assessed: E8
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer in terms of x	4
• Correct answer in terms of θ	3
• Significant working towards answer	2
• Correctly substitutes $x = 3\sin\theta$	1

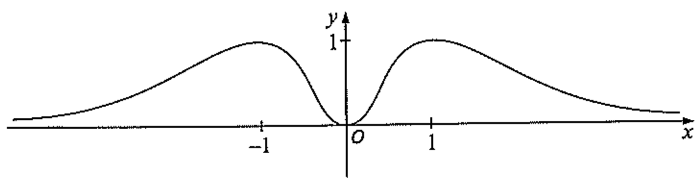
Sample answer:

$$\int \frac{dx}{\left(9-x^2\right)^{\frac{3}{2}}} = \int \frac{3\cos\theta\, d\theta}{\left(9-9\sin^2\theta\right)^{\frac{3}{2}}}$$
$$= \int \frac{\cos\theta\, d\theta}{9\cos^3\theta}$$
$$= \frac{1}{9}\int \sec^2\theta\, d\theta$$
$$= \frac{1}{9}\tan\theta + C$$
$$= \frac{x}{9\sqrt{9-x^2}} + C$$

(d) (i) (2 marks)
Outcomes assessed: E6
Targeted Performance Bands: E3

Criteria	Mark
• Correct graph, including turning point at origin and $(\pm 1, 1)$ and asymptotic features at $\pm\infty$	2
• Basic shape	1

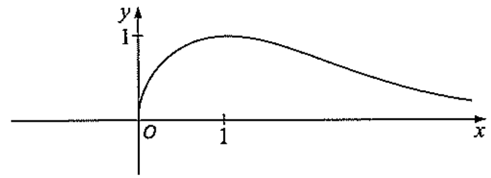
Sample answer:



(d) (ii) (1 mark)
Outcomes assessed: E6
Targeted Performance Bands: E2

Criteria	Mark
• Correct graph, including maximum turning point at (1,1) and vertical tangent at the origin	1

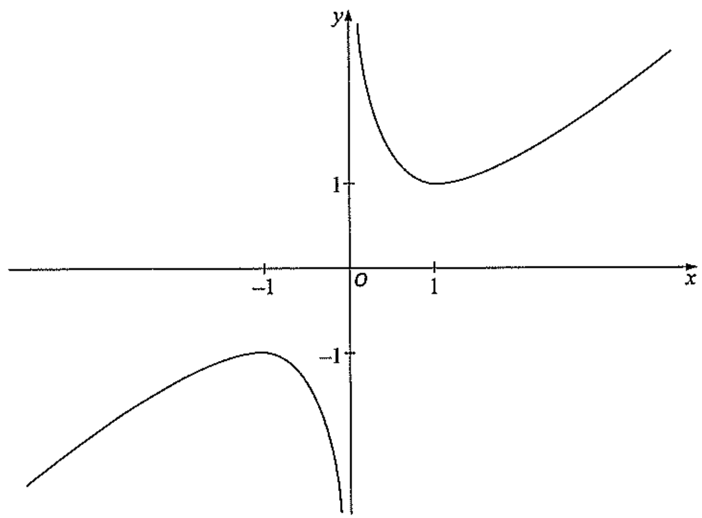
Sample answer:



(d) (iii) (2 marks)
Outcomes assessed: E6
Targeted Performance Bands: E2

Criteria	Mark
• Correct graph, including vertical asymptote $x = 0$ and turning points at $(1,1)$ and $(-1,-1)$.	2
• Basic shape	1

Sample answer:



Question 12 (15 marks)
(a) (i) (2 marks)
Outcomes assessed: E3
Targeted Performance Bands: E2

Criteria	Mark
• Correct answer	2
• Significant progress solving simultaneously for the real and imaginary parts	1

Sample answer:
Let $-24-10i=(a+ib)^2$ where a and b are real.
Then, $a^2-b^2+2abi=-24-10i$

Equating real and imaginary parts: $a^2-b^2=-24$ and $2ab=-10$

Substituting $b=\frac{-5}{a}$ into $a^2-b^2=-24$, gives

$$a^4+24a^2-25=0$$
$$\left(a^2+25\right)\left(a^2-1\right)=0$$
$a=\pm 1$ since a is real.
$\therefore b=\mp 5$

$\therefore$ The square roots of $-24-10i$ are $1-5i, -1+5i$.

(a) (ii) (2 marks)
Outcomes assessed: E3
Targeted Performance Bands: E2

Criteria	Mark
• Correct answer	2
• Significant progress solving the quadratic equation e.g. applies the quadratic formula	1

Sample answer:
$$x=\frac{1-i\pm\sqrt{(1-i)^2-4(6+2i)}}{2}$$
$$=\frac{1-i\pm\sqrt{-24-10i}}{2}$$
$$=\frac{1-i\pm(1-5i)}{2}$$
$$=1-3i,\ 2i$$

(b) (2 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2

Criteria	Mark
• Correct answer	2
• Correctly applies the method of integration by parts	1

Sample answer:
$$\int\frac{\ln x}{x^2}dx=\int \ln x.x^{-2}dx$$
$$=\ln x\left(-x^{-1}\right)-\int\left(-x^{-1}\right)\frac{1}{x}dx$$
$$=-\frac{\ln x}{x}+\int x^{-2}\,dx$$
$$=-\frac{\ln x}{x}-\frac{1}{x}+C$$

(c) (4 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2

Criteria	Mark
• Correct answer	4
• Correct integration and correct treatment of the limits either by substitution or rewriting the integrated expression in terms of x	3
• Correct expression for the integrand using the substitution $t=\tan\frac{x}{2}$	2
• Significant progress towards writing the integrand using the substitution $t=\tan\frac{x}{2}$	1

Sample answer:
$$\int_0^{\frac{\pi}{2}}\frac{dx}{2+\cos x}=\int_0^1\frac{1}{2+\frac{1-t^2}{1+t^2}}\times\frac{2dt}{1+t^2}$$
$$=\int_0^1\frac{2dt}{t^2+3}$$
$$=\left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$$
$$=\frac{\sqrt{3}\pi}{9}$$

(d) (i) (1 mark)
Outcomes assessed: E4
Targeted Performance Bands: E2

Criteria	Mark
• Correct answer	1

Sample answer:

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

(d) (ii) (1 mark)
Outcomes assessed: E4
Targeted Performance Bands: E2

Criteria	Mark
• Correct answer	1

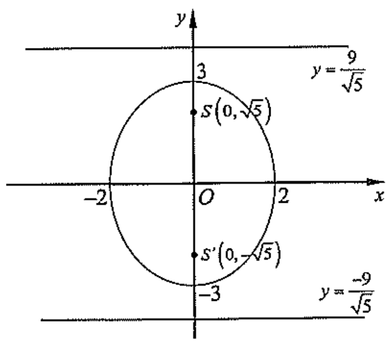
Sample answer:

$$4=9(1-e^2)$$
$$e=\frac{\sqrt{5}}{3}$$

(d) (iii) (3 marks)
Outcomes assessed: E4
Targeted Performance Bands: E3

Criteria	Mark
• Entire correct diagram	3
• Diagram of an ellipse with TWO of the following shown: intercepts, foci, directrices	2
• Diagram of an ellipse with ONE of the following shown: intercepts, foci, directrices	1

Sample answer:



Question 13 (15 marks)
(a) (1 mark)
Outcomes assessed: HE3
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	1

Sample answer:

$$\frac{30!}{7!8!15!} \quad \text{or} \quad \binom{30}{7}\binom{23}{8}\binom{15}{15}$$

(b) (i) (3 marks)
Outcomes assessed: E4
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	3
• Correctly formulates simultaneous equations	2
• Substitutes $x=2$ into either $p(x)$ or $p'(x)$	1

Sample answer:
$p(x)=x^4+ax^3+x^2+b$
$p'(x)=4x^3+3ax^2+2x$

$p'(2)=0\Rightarrow 32+12a+4=0\Rightarrow a=-3$
$p(2)=0\Rightarrow 16-24+4+b=0\Rightarrow b=4$

$\therefore a=-3$ and $b=4$

(b) (ii) (1 mark)
Outcomes assessed: E4
Targeted Performance Bands: E3

Criteria	Mark
• Correct factorisation	1

Sample answer:
$p(x)=x^4-3x^3+x^2+4$
$(x^4-3x^3+x^2+4)\div(x^2-4x+4)=x^2+x+1$
$\therefore p(x)=(x-2)^2(x^2+x+1)$

(c) (i) (2 marks)
Outcomes assessed: E5
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	2
• Significant progress towards resolving forces vertically	1

Sample answer: Let T be the tension in the string PB.

$$5\cos 30° = T\cos 60° + mg$$
$$\frac{5\sqrt{3}}{2} = \frac{T}{2} + 0.3\times 10$$
$$T = 5\sqrt{3} - 6 \text{ Newtons}$$
$$(\approx 2.66 \text{ Newtons})$$

(c) (ii) (3 marks)
Outcomes assessed: E5
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	3
• Resolves forces horizontally	2
• Calculates horizontal forces on P	1

Sample answer: Let P move in a horizontal circle of radius r metres at ω radians/second.

$$r = 0.4\sin 30° = 0.2 \text{ metres}$$

Resolving forces horizontally

$$5\sin 30° + T\sin 60° = m\omega^2 r$$
$$\frac{5}{2} + \left(5\sqrt{3}-6\right)\frac{\sqrt{3}}{2} = 0.3\times\omega^2\times 0.2$$
$$\omega = \sqrt{\frac{50\left(10-3\sqrt{3}\right)}{3}}$$
$$\approx 8.9 \text{ radians / second (1 d.p.)}$$

(d) (i) (2 marks)
Outcomes assessed: E7
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	2
• Make progress towards finding the length of the base of the rectangle	1

Sample answer:
Parabola has equation $y = x^2 + 1 \Rightarrow x = \pm\sqrt{y-1}$

$\therefore$ y units from the origin, the base of the rectangular cross-section has length $2\sqrt{y-1}$.
The height of the rectangular cross-section is $\frac{1}{2}y$ (given).

Hence, the area of the rectangle y units from the origin is $2\sqrt{y-1}\times\frac{1}{2}y = y\sqrt{y-1}$ square units.

(d) (ii) (3 marks)
Outcomes assessed: E8
Targeted Performance Bands: E3

Criteria	Mark
• Correct evaluation of integral	3
• Some progress using substitution or integration by parts	2
• Correct integral	1

Sample answer:
$V = \int_1^2 y\sqrt{y-1}\,dy$

Using the substitution $y = u + 1$

$$V = \int_0^1 (u+1)\sqrt{u}\,du$$
$$= \int_0^1 \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$$
$$= \left[\frac{2u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3}\right]_0^1$$
$$= \frac{16}{15}$$

$\therefore$ The volume of the solid S is $\frac{16}{15}$ cubic units.

Question 14 (15 marks)
(a) (i) (3 marks)
Outcomes assessed: E4
Targeted Performance Bands: E3

Criteria	Mark
• Correct simplified polynomial	3
• Correct polynomial equation $(x+3)^3-(x+3)^2-21(x+3)+45=0$ or correctly finds the sums and products of roots	2
• Correct transformation $x \mapsto x+3$ or attempts to find sums and products of roots	1

Sample answer:
$p(x)=x^3-x^2-21x+45=(x-\alpha)(x-\beta)(x-\gamma)$ has roots α, β and γ.
$\therefore$ The monic polynomial with roots $\alpha-3$, $\beta-3$ and $\gamma-3$ is given by

$$\begin{aligned}&((x+3)-\alpha)((x+3)-\beta)((x+3)-\gamma)\\ &=(x+3)^3-(x+3)^2-21(x+3)+45\\ &=x^3+8x^2.\end{aligned}$$

(a) (ii) (1 mark)
Outcomes assessed: E4
Targeted Performance Bands: E2

Criteria	Mark
• Correct solution	1

Sample answer:
$x^3+8x^2=x^2(x+8)$ has roots $\alpha-3=0, \beta-3=0, \gamma-3=-8$
$\therefore \alpha=3, \beta=3, \gamma=-5$.

$\therefore x=3,-5$ are the solutions to $p(x)=0$.

(b) (i) (2 marks)
Outcomes assessed: H5
Targeted Performance Bands: E3

Criteria	Mark
• Complete proof	2
• Significant progress towards solution	1

Sample answer:
$\angle PRA=\angle PSA$ (angles in the same segment are equal)
$\angle PSA=\angle PQB$ (exterior angle of cyclic quadrilateral $PSBQ$ equals the opposite interior angle)
$\therefore \angle PRA=\angle PQB$

(b) (ii) (2 marks)
Outcomes assessed: H5, PE3
Targeted Performance Bands: E3

Criteria	Mark
• Complete answer	2
• Significant progress towards the solution	1

Sample answer:
$\angle PRA=\angle PQB$ (from part i)
$180°-\angle PRA=180°-\angle PQB$
Therefore, $\angle PRC=\angle PQC$
$\therefore$ P, R, Q and C are concyclic as PC subtends equal angles at R and Q (on the same side of PC)

(c) (i) (2 marks)
Outcomes assessed: E2
Targeted Performance Bands: E2

Criteria	Mark
• Correct solution	2
• Correct expression for the gradient of the tangent at P	1

Sample answer:
$x=a\sec\theta \Rightarrow \dfrac{dx}{d\theta}=a\sec\theta\tan\theta$

$y=b\tan\theta \Rightarrow \dfrac{dy}{d\theta}=b\sec^2\theta$

$\therefore \dfrac{dy}{dx}=\dfrac{b\sec\theta}{a\tan\theta}$

The equation of the tangent at P:

$$\begin{aligned}y-b\tan\theta&=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta)\\ ay\tan\theta-ab\tan^2\theta&=bx\sec\theta-ab\sec^2\theta\\ bx\sec\theta-ay\tan\theta&=ab(\sec^2\theta-\tan^2\theta)\\ \frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta&=1\end{aligned}$$

(c) (ii) (2 marks)

Outcomes assessed: E2

Targeted Performance Bands: E2

Criteria	Mark
• Correct solution	2
• Significant progress towards solving tangent equation simultaneously with $y=\frac{b}{a}x$	1

Sample answer:

For the coordinates of Q solve $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$ simultaneously with the equation of the asymptote $y=\frac{b}{a}x$.

$$\frac{x}{a}\sec\theta-\frac{\frac{b}{a}x}{b}\tan\theta=1$$

$$x\left(\frac{\sec\theta}{a}-\frac{\tan\theta}{a}\right)=1$$

$$x=\frac{a}{\sec\theta-\tan\theta}$$

Substituting into $y=\frac{b}{a}x$:

$$y=\frac{b}{a}\left(\frac{a}{\sec\theta-\tan\theta}\right)$$
$$=\frac{b}{\sec\theta-\tan\theta}$$

Hence, the coordinates of Q are $\left(\frac{a}{\sec\theta-\tan\theta},\frac{b}{\sec\theta-\tan\theta}\right)$.

(c) (iii) (3 marks)

Outcomes assessed: E2, E4

Targeted Performance Bands: E4

Criteria	Mark
• Correct proof	3
• Significant progress towards finding $\tan(\angle QRS)$	2
• Significant progress towards finding expressions for the gradients of QS and RS	1

Sample answer:

$$m_{QS}=\frac{\frac{b}{\sec\theta-\tan\theta}}{\frac{a}{\sec\theta-\tan\theta}-ae}=\frac{b}{a(1-e\sec\theta+e\tan\theta)}$$

$$m_{RS}=\frac{\frac{-b}{\sec\theta+\tan\theta}}{\frac{a}{\sec\theta+\tan\theta}-ae}=\frac{-b}{a(1-e\sec\theta-e\tan\theta)}$$

$$|\tan(\angle QRS)|=\left|\frac{m_{QS}-m_{RS}}{1+m_{QS}m_{RS}}\right|$$

$$=\left|\frac{\frac{b}{a(1-e\sec\theta+e\tan\theta)}-\frac{-b}{a(1-e\sec\theta-e\tan\theta)}}{1+\frac{b}{a(1-e\sec\theta+e\tan\theta)}\times\frac{-b}{a(1-e\sec\theta-e\tan\theta)}}\right|$$

$$=\left|\frac{ab(1-e\sec\theta-e\tan\theta+1-e\sec\theta+e\tan\theta)}{a^2(1-e\sec\theta+e\tan\theta)(1-e\sec\theta-e\tan\theta)-b^2}\right|$$

$$=\left|\frac{2ab(1-e\sec\theta)}{a^2-2a^2e\sec\theta+a^2e^2\sec^2\theta-a^2e^2\tan^2\theta-b^2}\right|$$

$$=\left|\frac{2ab(1-e\sec\theta)}{a^2-2a^2e\sec\theta+a^2e^2(\sec^2\theta-\tan^2\theta)-(a^2e^2-a^2)}\right|$$

$$=\left|\frac{2ab(1-e\sec\theta)}{2a^2(1-e\sec\theta)}\right|$$

$$=\left|\frac{b}{a}\right|$$

$$=\frac{b}{a}\text{ since }a,b>0$$

Question 15 (15 marks)
(a) (i) (2 marks)
Outcomes assessed: E8
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	2
• Correctly integrates $\tan x$	1

Sample answer:

$$\int_0^{\frac{\pi}{4}} \tan x \, dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} dx$$
$$= \left[-\ln(\cos x)\right]_0^{\frac{\pi}{4}}$$
$$= \ln\sqrt{2}$$

(a) (ii) (2 marks)
Outcomes assessed: E8
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	2
• Some progress towards the correct answer	1

Sample answer:

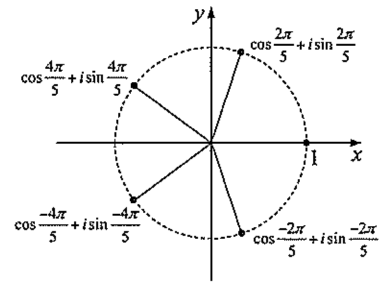
$$I_5 = \frac{1}{4} - I_3$$
$$= \frac{1}{4} - \left(\frac{1}{2} - I_1\right)$$
$$= \frac{-1}{4} + \ln\sqrt{2}$$

(b) (i) (2 marks)
Outcomes assessed: E3, E4
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	2
• Correct roots or correct diagram	1

Sample answer:

$$z = 1, \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}, \cos\frac{-2\pi}{5} + i\sin\frac{-2\pi}{5}, \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}, \cos\frac{-4\pi}{5} + i\sin\frac{-4\pi}{5}$$



(b) (ii) (2 marks)
Outcomes assessed: E3, E4
Targeted Performance Bands: E3

Criteria	Mark
• Expression written as a product of linear and quadratic factor	2
• Progress towards answer	1

Sample answer:

The quadratic with roots $\alpha = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$, $\beta = \cos\frac{-2\pi}{5} + i\sin\frac{-2\pi}{5} = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$

is $z^2 - (\alpha + \beta)z + \alpha\beta = z^2 - \left(2\cos\frac{2\pi}{5}\right)z + 1$.

Similarly, the quadratic with roots $\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}, \cos\frac{-4\pi}{5} + i\sin\frac{-4\pi}{5}$ is $z^2 - \left(2\cos\frac{4\pi}{5}\right)z + 1$.

Hence, $z^5 - 1 = (z-1)\left(z^2 - \left(2\cos\frac{2\pi}{5}\right)z + 1\right)\left(z^2 - \left(2\cos\frac{4\pi}{5}\right)z + 1\right)$.

(b) (iii) (2 marks)
Outcomes assessed: E3, E4
Targeted Performance Bands: E4

Criteria	Mark
• Correct proof	2
• Significant progress applying sum of roots	1

Sample answer:

sum of roots $= \dfrac{-b}{a} = 0$

$1+\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+\cos\frac{-2\pi}{5}+i\sin\frac{-2\pi}{5}+\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}+\cos\frac{-4\pi}{5}+i\sin\frac{-4\pi}{5}=0$

$1+\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}+\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}+\cos\frac{4\pi}{5}-i\sin\frac{4\pi}{5}=0$

$1+2\cos\frac{2\pi}{5}+2\cos\frac{4\pi}{5}=0$

$\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=\frac{-1}{2}$

(c) (i) (2 marks)
Outcomes assessed: E5, E9
Targeted Performance Bands: E3

Criteria	Mark
• Complete proof	2
• Significant progress towards finding $f'(t)$ and $f''(t)$	1

Sample answer:

$x(t)=f(t)e^{-3t}$

$x'(t)=\left(-3f(t)+f'(t)\right)e^{-3t}$

$x''(t)=\left(9f(t)-3f'(t)\right)e^{-3t}+\left(-3f'(t)+f''(t)\right)e^{-3t}$

$\quad =\left(9f(t)-6f'(t)+f''(t)\right)e^{-3t}$

Since $\dfrac{d^2x}{dt^2}=-6\dfrac{dx}{dt}-9x$,

$\left(9f(t)-6f'(t)+f''(t)\right)e^{-3t}=-6\left(\left(-3f(t)+f'(t)\right)e^{-3t}\right)-9\left(f(t)e^{-3t}\right)$

$\left(9f(t)-6f'(t)+f''(t)\right)e^{-3t}=\left(9f(t)-6f'(t)\right)e^{-3t}$

Hence, $f''(t)=0$

$\therefore f(t)$ is linear, i.e. $f(t)=At+B$ where A and B are constants

(c) (ii) (1 mark)
Outcomes assessed: HE3
Targeted Performance Bands: E2

Criteria	Mark
• Correct answer	1

Sample answer:

$x(t)=(At+B)e^{-3t}$

When $t=0, x=0$ (as the projectile is starting from the origin)

$(A\times 0+B)e^{-3\times 0}=0$

$B=0$

(c) (iii) (2 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	2
• Correct expression for $x'(t)$ for calculated value of B	1

Sample answer:

$x(t)=Ate^{-3t}$

$x'(t)=Ae^{-3t}-3Ate^{-3t}$

$\quad =Ae^{-3t}(1-3t)$

Displacement is a maximum when $x'(t)=0$, i.e. $t=\frac{1}{3}$

Therefore, the particle's displacement is at maximum after $\frac{1}{3}$ seconds.

Note: since $x''\left(\frac{1}{3}\right)=-3Ae^{-1}<0$ since $A>0$, a maximum is attained.

Question 16 (15 marks)
(a) (3 marks)
Outcomes assessed: E3, E4
Targeted Performance Bands: E4

Criteria	Mark
• Correct diagram	3
• Makes use of $\|z\|=1$	2
• Attempts to make z the subject	1

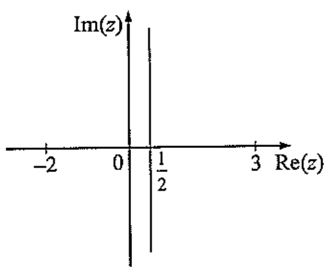
Sample answer:

$$w=\frac{3z+2}{z-1}$$
$$wz-w=3z+2$$
$$z=\frac{w+2}{w-3}$$

Since $|z|=1$,

$$\left|\frac{w+2}{w-3}\right|=1\Rightarrow|w+2|=|w-3|$$

Thus w is equidistant from -2 and 3.



(b) (i) (1 mark)
Outcomes assessed: E9
Targeted Performance Bands: E3

Criteria	Mark
• Correct proof	1

Sample answer:

$$\frac{1}{2^k+1}+\frac{1}{2^k+2}+\frac{1}{2^k+3}+\ldots+\frac{1}{2^k+2^k}\geq\frac{1}{2^k+2^k}+\frac{1}{2^k+2^k}+\frac{1}{2^k+2^k}+\ldots+\frac{1}{2^k+2^k}$$
$$=2^k\times\frac{1}{2\left(2^k\right)}$$
$$=\frac{1}{2}$$

(b) (ii) (3 mark)
Outcomes assessed: HE2, E9
Targeted Performance Bands: E3, E4

Criteria	Mark
• Complete proof	3
• Makes use of the assumption or the result in part i	2
• Proof for $P(1)$	1

Sample answer:
Let $P(n)$ be the given proposition. $P(1)$ is true since $1+\frac{1}{2}=\frac{3}{2}\geq\frac{1}{2}(1+1)=1$.

Assume $P(k)$ is true for some positive integer k.
i.e. $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^k}\geq\frac{1}{2}(k+1)$

Prove $P(k+1)$ is true:

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^{k+1}}=\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^k}\right)+\left(\frac{1}{2^k+1}+\frac{1}{2^k+2}+\frac{1}{2^k+3}+\ldots+\frac{1}{2^k+2^k}\right)$$
$$\geq\frac{1}{2}(k+1)+\frac{1}{2}\text{ using the assumption and the result in part i}$$
$$=\frac{1}{2}((k+1)+1)$$

$\therefore$ By the Principle of Mathematical Induction, $P(n)$ is true for integers $n\geq1$.

(c) (i) (2 marks)
Outcomes assessed: E3
Targeted Performance Bands: E3

Criteria	Mark
• Complete proof	2
• Attempts to apply DeMoivre's Theorem	1

Sample answer:
By DeMoivre's theorem,

$$x^k = (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$$
$$x^{-k} = (\cos\theta + i\sin\theta)^{-k} = \cos(-k\theta) + i\sin(-k\theta) = \cos k\theta - i\sin k\theta$$

$$\begin{aligned} x^k + x^{-k} &= \cos k\theta + i\sin k\theta + \cos k\theta - i\sin k\theta \\ &= 2\cos k\theta \end{aligned}$$

(c) (ii) (3 marks)
Outcomes assessed: HE3, E4, E9
Targeted Performance Bands: E4

Criteria	Mark
• Correct proof	3
• Significant progress towards the correct proof	2
• Correct binomial expansion of $\left(x+\frac{1}{x}\right)^{2n}$	1

Sample answer:

$$\begin{aligned} \left(x+\frac{1}{x}\right)^{2n} &= \binom{2n}{0}x^{2n} + \binom{2n}{1}x^{2n-2} + \binom{2n}{2}x^{2n-4} + \ldots + \binom{2n}{n} + \ldots + \binom{2n}{2n-2}\frac{1}{x^{2n-4}} + \binom{2n}{2n-1}\frac{1}{x^{2n-2}} + \binom{2n}{2n}\frac{1}{x^{2n}} \\ &= \binom{2n}{0}x^{2n} + \binom{2n}{1}x^{2n-2} + \binom{2n}{2}x^{2n-4} + \ldots + \binom{2n}{n} + \ldots + \binom{2n}{2}\frac{1}{x^{2n-4}} + \binom{2n}{1}\frac{1}{x^{2n-2}} + \binom{2n}{0}\frac{1}{x^{2n}} \end{aligned}$$

since $\binom{2n}{2n-k} = \binom{2n}{k}$

$$\left(x+\frac{1}{x}\right)^{2n} = \binom{2n}{0}\left(x^{2n}+\frac{1}{x^{2n}}\right) + \binom{2n}{1}\left(x^{2n-2}+\frac{1}{x^{2n-2}}\right) + \binom{2n}{2}\left(x^{2n-4}+\frac{1}{x^{2n-4}}\right) + \ldots + \binom{2n}{n-1}\left(x^2+\frac{1}{x^2}\right) + \binom{2n}{n}$$

Note $\binom{2n}{0} = 1$

(c) (iii) (1 mark)
Outcomes assessed: E9
Targeted Performance Bands: E4

Criteria	Mark
• Correct proof	1

Sample answer:

Using the result from part i, $x^k + \frac{1}{x^k} = 2\cos k\theta$ in the identity from part ii:

$$(2\cos\theta)^{2n} = \binom{2n}{0}(2\cos 2n\theta) + \binom{2n}{1}(2\cos(2n-2)\theta) + \binom{2n}{2}(2\cos(2n-4)\theta) + \ldots + \binom{2n}{n-1}(2\cos 2\theta) + \binom{2n}{n}$$

Dividing both sides by 2:

$$2^{2n-1}\cos^{2n}\theta = \cos 2n\theta + \binom{2n}{1}\cos(2n-2)\theta + \binom{2n}{2}\cos(2n-4)\theta + \ldots + \binom{2n}{n-1}\cos 2\theta + \frac{1}{2}\binom{2n}{n}$$

(c) (iv) (2 marks)
Outcomes assessed: E8
Targeted Performance Bands: E4

Criteria	Mark
• Complete correct proof	2
• Integration of an expression progressing towards the correct proof	1

Sample answer:

$$\int_0^{2\pi} 2^{2n-1}\cos^{2n}\theta\, d\theta = \int_0^{2\pi} \cos 2n\theta + \binom{2n}{1}\cos(2n-2)\theta + \binom{2n}{2}\cos(2n-4)\theta + \ldots + \binom{2n}{n-1}\cos 2\theta + \frac{1}{2}\binom{2n}{n}\, d\theta$$

Since $\int_0^{2\pi}\cos k\theta\, d\theta = 0$ for all even integers k, all the integrals on the right hand side are zero except for the constant term.

$$\begin{aligned} \int_0^{2\pi} 2^{2n-1}\cos^{2n}\theta\, d\theta &= \int_0^{2\pi}\frac{1}{2}\binom{2n}{n}d\theta \\ &= \left[\frac{1}{2}\binom{2n}{n}\theta\right]_0^{2\pi} \\ &= \pi\binom{2n}{n} \end{aligned}$$

Dividing both sides by 2^{2n-1},

$$\int_0^{2\pi}\cos^{2n}\theta\, d\theta = \frac{\pi}{2^{2n-1}}\binom{2n}{n}.$$