Name: ....

Form: ....

ASCHAM SCHOOL
MATHEMATICS EXAMINATION
FORM 6 - 3 UNIT
1999

July 1999 Time allowed: 2 hours

* All questions should be attempted
* All necessary working must be shown
* All questions are of equal value
* Marks may not be awarded for careless or badly arranged work.
* Write your name on each booklet clearly marked:
Question 1, Question 2, ... etc.
* Begin each question in a new booklet.
* Approved calculators may be used.
* Copies of diagrams for all questions are provided on pages 11-14 in order to save time. You may use them but **you must staple them into your booklets.**

Question 1 **Marks:**

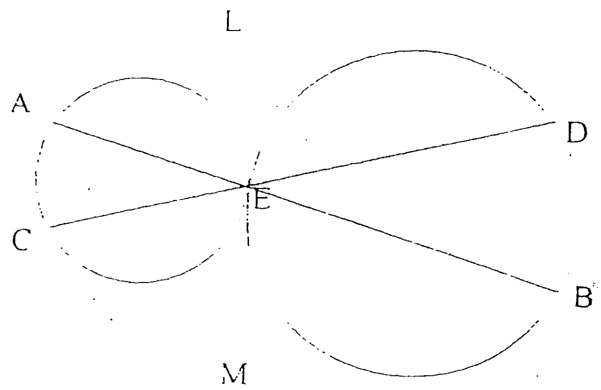
(a) Find the acute angle, to the nearest degree, between the lines $y = 3x + 1$ and $y = -x - 6$ 3

(b) Solve the inequality $\dfrac{1}{x-1} < 3, \quad x \neq 1$ 3

(c) Find the coordinates of the point P which divides the interval AB with end points A(-1, 2) and B(3, -5) internally in the ratio 2:3. 2

(d) Use the substitution $u = t - 1$ to evaluate $\displaystyle\int_0^1 \frac{t}{\sqrt{t+1}}\,dt$ 3

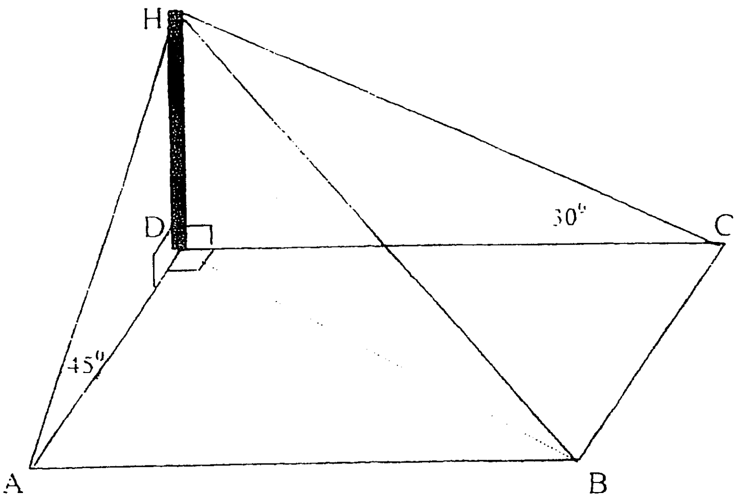
(e) Two circles touch externally at E. 3



(A copy of the diagram above is on page 10.)
AB and CD intersect at E. LM is a common tangent at E
Prove that AC is parallel to DB.

Question 2

Marks:



(a) A post HD stands vertically at one corner of a rectangular field ABCD. The angles of elevation of the top H of the post from the nearest corners A and C respectively are 30^{0} and 45^{0}.

(A copy of the diagram above is on page 13.)

(i) If $AD = a$ units, find the length of BD in terms of a. **2**

(ii) Hence find the angle of elevation of H from the corner B to the nearest minute. **1**

(b) Taking $x = -\dfrac{\pi}{6}$ as a first approximation to the root of the equation $2x + \cos x = 0$, use Newton's method once to show that a better approximation to the root of the equation is $\dfrac{-\pi - 6\sqrt{3}}{30}$ **4**

(c) (i) Find the domain and range of $f^{-1}(x) = \sin^{-1}(3x - 1)$ **2**

(ii) Sketch the graph of $y = f^{-1}(x)$. **2**

(iii) Find the equation representing the inverse function ... state the domain and range.

Question 3 **Marks**

(a) (i) Express $3\sin x - \sqrt{3}\cos x$ in the form $A\sin(x - \alpha)$ where $A > 0$ and $0 \le \alpha \le \frac{\pi}{2}$ 3

(ii) Determine the minimum value of $3\sin x - \sqrt{3}\cos x$. 1

(iii) Solve $3\sin x - \sqrt{3}\cos x = \sqrt{3}$ for $0 \le x \le 2\pi$. 3

(b) Newton's Law of cooling states that the rate of cooling of a body is proportional to the excess of the temperature of a body above the surrounding temperature. This rate can be expressed by the differential equation:

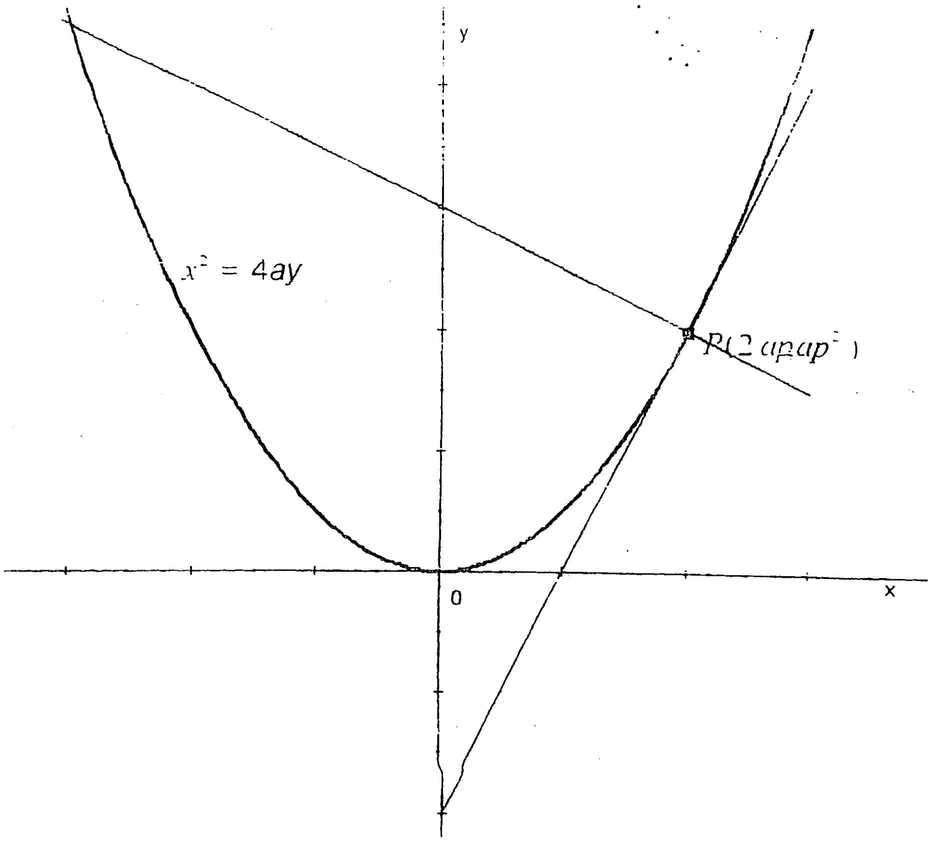
$$\frac{dT}{dt} = -k(T - T_o).$$

where T is the temperature of the body, T_o is the temperature of the surroundings, t the time in minutes and k is a constant.

(i) Show that $T = T_o + Ae^{-kt}$, where A is a constant, is a solution of the differential equation $\frac{dT}{dt} = -k(T - T_o)$. 2

(ii) A cup of tea cools from 85°C to 80°C in 1 minute at a room temperature of 25°C. Find the temperature of the cup of tea after a further 4 minutes have elapsed. Answer to the nearest degree. 5

Question 4 **Marks:**



(a) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. Show the equation of the normal to the parabola at P is $x + py = 2ap + ap^3$. **3**

(b) Write down the equation of the normal to the parabola at Q. The normals intersect at N. Find the coordinates of N. **3**

(c) Show the equation of the chord PQ is $y - ap^2 = \left(\frac{p+q}{2}\right)(x - 2ap)$ and determine the condition necessary for PQ to be a focal chord. **3**

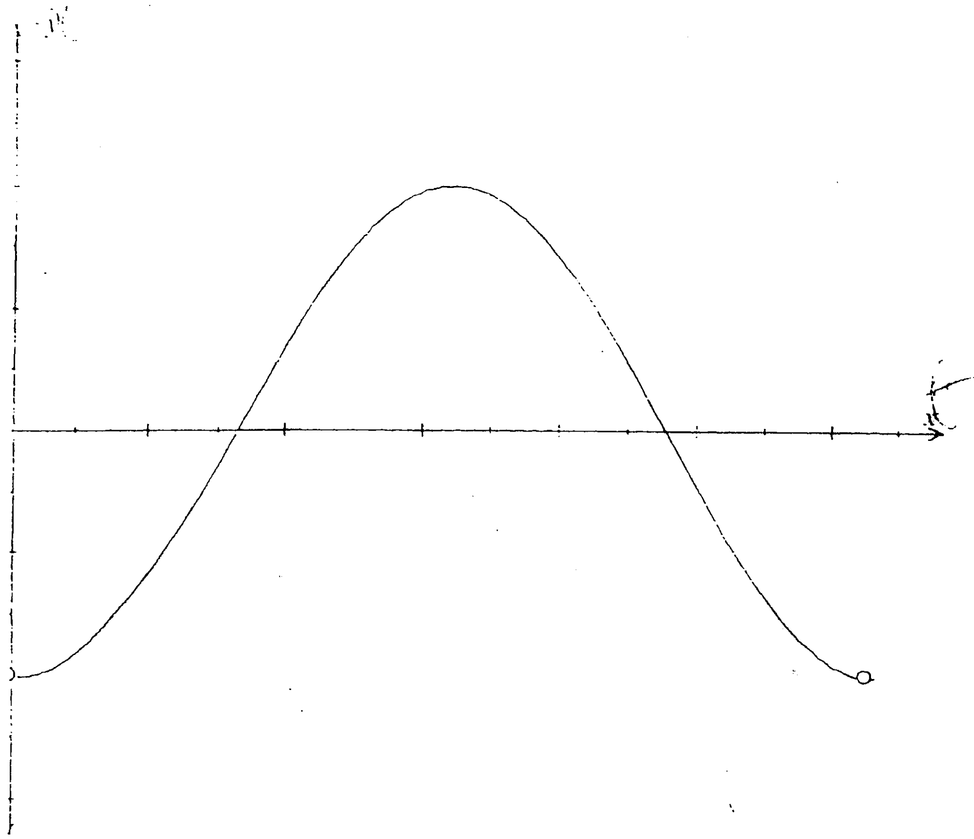
(d) If PQ is a focal chord and N is the intersection of the normals, find the equation of the locus of N. **4**

On the diagram above, the tangent and normal are drawn at P.
Mark clearly on your own diagram the points Q and N which correspond to P. **1**

Question 5

Marks

(a) The graph of $x = -a\cos nt$ for $0 \le t \le \frac{2\pi}{n}$ is drawn below. (A copy of the diagram above is on page 12.) Label axes and show intercepts accurately. 2



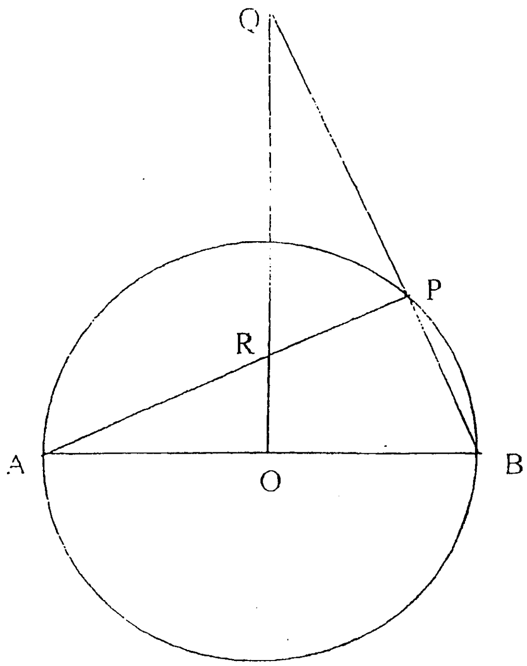
(b) On a certain day the depth of water in a harbour at low tide at 4:30 am is 5 metres. At the following high tide at 10:45 am the depth is 15 metres. Assuming the rise and fall of the surface of the water to be simple harmonic, find between what times during the morning a ship may safely enter the harbour if the minimum depth of $12\frac{1}{2}$ metres of water is required. 6

(c) Given that $\sin^{-1}x$, $\cos^{-1}x$ and $\sin^{-1}(2-x)$ have values for $0 \le x \le \frac{\pi}{2}$

(ii) Hence, or otherwise, solve the equation $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(2-x)$ 3

Question 6 **Marks:**

(a) O is the centre of the circle. BPQ is a straight line. ORQ is perpendicular to AOB as shown below.



(A copy of the diagram above is on page 14.)

Prove that:

(i) A, O, P, Q are concyclic, and **3**

(ii) $\angle OPA = \angle OQB$. **2**

(b) Prove by using mathematical induction that $5^n \geq 1+4n$, for $n > 1$, $n \in J^+$. **4**

(c) The cubic equation $2x^3 - x^2 + x - 1 = 0$ has roots α, β, and γ. Evaluate

(i) $\alpha\beta + \beta\gamma + \alpha\gamma$ **1**

(ii) $\alpha\beta\gamma$ **1**

(iii) $\alpha^2\beta^2\gamma + \beta^2\gamma^2\alpha + \alpha^2\gamma^2\beta$ **1**

(d) The equation $2\cos^3\theta - \cos^2\theta + \cos\theta - 1 = 0$ has roots $\cos a$, $\cos b$ and $\cos c$. **2**

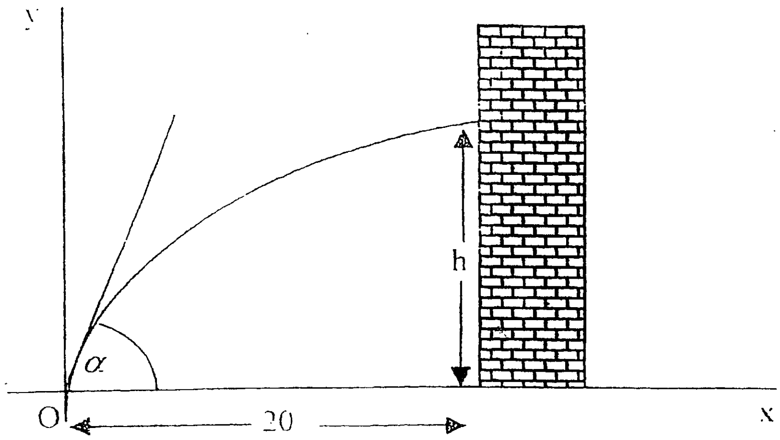
Using appropriate information from (c) above prove that

$$\sec a + \sec b + \sec c = 1$$

Question 7 **Marks**

A softball player hits the ball from ground level with a speed of 20 ms^{-1} and an angle of elevation α. It flies toward a high wall 20 m away on level ground.

(a) Taking the origin at the point where the ball is hit, derive expressions for **3**



the horizontal and vertical components x and y of displacement at time t seconds. Take g = 10ms^{-2}.

(b) Hence find the equation of the path of the ball in flight in terms of x, y and α. **1**

(c) Show that the height h at which the ball hits the wall is given by **2**

$$h = 20\tan\alpha - 5(1 + \tan^2\alpha).$$

(d) Using part (c) above, show that the maximum value of h occurs when $\tan\alpha = 2$. **2**

(e) Find **6**

(i) this maximum height h.

(ii) the speed and the angle at which the ball hits the wall in this case.

19c 3U TRIAL (Ascham)

Q.1. (a) $y = 3x + 2 \quad m_1 = 3 \qquad y = 1 - x \quad m_2 = -1$

$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ ✓

$= \left|\frac{3 - (-1)}{1 + 3(-1)}\right|$

$= 2$ ✓

$\therefore \alpha = 63°26'$

$= 63°$ ✓ (nearest deg.)

(b) $\frac{1}{x+1} < 3$

$\Rightarrow x + 1 < 3(x+1)^2$ ✓

$= 3(x^2 + 2x + 1)$

$3x^2 + 5x + 2 > 0$

$(3x+2)(x+1) > 0$ ✓

$x < -1$ or $x > -\frac{2}{3}$ ✓ ½

(c) $(-1, 2) \quad (3, -5) \quad 2 : 3$

$P\left(\frac{2\times 3 - 3\times 1}{5}, \frac{2\times -5 + 3\times 2}{5}\right)$ ✓

$P\left(\frac{3}{5}, -\frac{4}{5}\right)$ ✓

(d) $\int_0^1 \frac{t}{\sqrt{t+1}}\,dt$

$u = t + 1 \qquad$ if $t = 0 \quad u = 1$

$t = u - 1 \qquad$ if $t = 1 \quad u = 2$

$du = dt$

$= \int_1^2 \frac{u-1}{\sqrt{u}}\,du$ ✓

$= \int_1^2 \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du$

$= \left[\frac{2u^{3/2}}{3} - 2u^{\frac{1}{2}}\right]_1^2$ ✓

$= \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2}\right) - \left(\frac{2}{3} - 2\right)$

$= -\frac{2\sqrt{2}}{3} + \frac{4}{3}$

$= \frac{4 - 2\sqrt{2}}{3}$

Q1. (e) $\angle HEL = \angle ALE$ ($\angle$ in alt. segm.) ½

$\angle AEL = \angle MEB$ (vert. opp. $\angle$s) ½

$\angle BDE = \angle MEB$ ($\angle$ in alt. segm) ½

$\therefore \angle ACE = \angle BDE$ ½

But $\angle ACE$ is alternate to $\angle BDE$ $\therefore AC \parallel DB$ ½

Q.2. (a)(i) In $\triangle ADH \quad AD = DH = a$

In $\triangle HDC \quad \tan 30° = \frac{a}{DC}$

$\therefore DC = a\sqrt{3}$ ✓

In $\triangle BDC \quad BD^2 = DC^2 + CB^2$

$= (a\sqrt{3})^2 + a^2$

$= 4a^2$

$\therefore BD = 2a$ ✓

(ii) In $\triangle HDB \quad \frac{HD}{BD} = \tan H\hat{B}D$

$\frac{a}{2a} =$

$\frac{1}{2} = \tan H\hat{B}D$

$\therefore \angle HBD = 26°34'$ (to nearest minute) ✓

(b) $\frac{d}{dx}(2x + \cos x) = 2 - \sin x$ ✓

$f'\left(\frac{\pi}{6}\right) = 2 + \sin\frac{\pi}{6}$

$= 2 + \frac{1}{2}$

$= \frac{5}{2}$ ½

$f\left(\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{2} - \frac{\pi}{3}$

$= \frac{3\sqrt{3} - 2\pi}{6}$ ½

$x_1 = -\frac{\pi}{6} - \frac{3\sqrt{3} - 2\pi}{6} \times \frac{2}{5}$ ✓

$= -\frac{\pi}{6} - \frac{3\sqrt{3}}{15} + \frac{2\pi}{15}$

$= \frac{-\pi - 6\sqrt{3}}{30}$ ✓

(2)

Q 2(c) (i) $y = \sin^{-1}(3x - 1)$

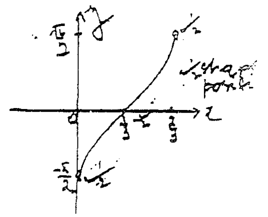
Domain of f: $-1 \le 3x-1 \le 1$

$0 \le 3x \le 2$

$0 \le x \le \frac{2}{3}$

Range of f: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(ii)



(iii) $y = \sin^{-1}(3x - 1)$ is the inverse function of $f(x)$

$\sin y = 3x - 1$ is the inverse of $f(x)$

$\sin x = 3y - 1$

$y = \frac{1}{3} + \frac{1}{3}\sin x$ is $f(x)$

So $f(x) = \frac{1}{3} + \frac{1}{3}\sin x$. It has domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and range $0 \le y \le \frac{2}{3}$

Q. 3(a)(i) $3\sin x - \sqrt{3}\cos x = A\sin(x - \alpha)$

$= A\sin x\cos\alpha - A\cos x\sin\alpha$

$\frac{A\sin\alpha}{A\cos\alpha} = \frac{\sqrt{3}}{3}$ ∴ $\tan\alpha = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$ ∴ $\alpha = \frac{\pi}{6}$

$A^2\sin^2\alpha + A^2\cos^2\alpha = 9 + 3$ ∴ $A^2 = 12$, $A = 2\sqrt{3}$

∴ $3\sin x - \sqrt{3}\cos x = 2\sqrt{3}\sin(x - \frac{\pi}{6})$

(ii) Min. value $-2\sqrt{3}$

(iii) $2\sqrt{3}\sin(x - \frac{\pi}{6}) = \sqrt{3}$

$\sin(x - \frac{\pi}{6}) = \frac{1}{2}$

$x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$ ∴ $x = \frac{\pi}{3}$ or $x = \pi$ for $0 \le x \le \pi$

(4)

Q3 (b)(i) $T = T_0 + Ae^{-kt}$ i.e. $T - T_0 = Ae^{-kt}$

$\frac{dT}{dt} = -kAe^{-kt}$

$= -k(T - T_0)$

(ii) $85 = 25 + Ae^0$ ∴ $A = 60$.

$T = 25 + 60e^{-kt}$

$80 = 25 + 60e^{-k}$

$e^{-k} = \frac{55}{60}$

$-k = \ln\frac{11}{12}$ (or $k = \ln\frac{12}{11}$)

∴ $T = 25 + 60e^{(\ln\frac{11}{12})t}$

when $t = 5$ $T = 25 + 60e^{-5\ln\frac{12}{11}}$

$\doteqdot 64$

$T \doteqdot 64°$ after a further 4 minutes.

Q.4 (a) $x^2 = 4ay$ $\frac{dy}{dx} = \frac{x}{2a}$ at $x = 2ap$

$\frac{dy}{dx} = p$

gradient of normal $= -\frac{1}{p}$

Eqn: $y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = 2ap + ap^3$

(3)

4 (b) Normal at

$x + py = 2ap + aq^3$ (1)

$x + qy = 2aq + aq^3$ (2)

(1) − (2)

$y(p-q) = 2a(p-q) + a(p^3-q^3)$

$y = 2a + a(p^2+q^2+pq)$

$y = a(p^2+q^2+pq+2)$

(1) × q: $xq + pqy = 2apq + aqp^3$

(2) × p: $-xp - pqy = -2apq - apq^3$

$x(q-p) = apq(p-q)(p+q)$

$= -apq(p+q)$

$N[-apq(p+q),\ a(p^2+q^2+pq+2)] = N(X, Y)$

(c) $m_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq} \Rightarrow m_{PQ} = \frac{a(p+q)\cancel{(p-q)}}{2a\cancel{(p-q)}}$

$= \frac{p+q}{2}$

Chord PQ: Sub. S(0, a)

$-ap^2 = \frac{p+q}{2}(x - 2ap)$

$-ap^2 = \frac{p+q}{2}(-2ap)$

$-\cancel{ap^2} = -\cancel{ap^2} - apq$

∴ $pq = -1$ if PQ is through S.

(d) $X = -apq(p+q) = X = a(p+q) \Rightarrow p+q = \frac{X}{a}$ (3)

$Y = a(p^2+q^2+pq+2) \Rightarrow Y = a(p^2+q^2+1) \Rightarrow p^2+q^2 = \frac{Y}{a} - 1$ (4)

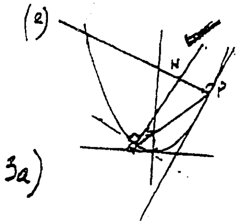
(5) $p^2+q^2 = (p+q)^2 - 2$

(3)&(4) → (5): $\frac{Y}{a} - 1 = \frac{X^2}{a^2} - 2$

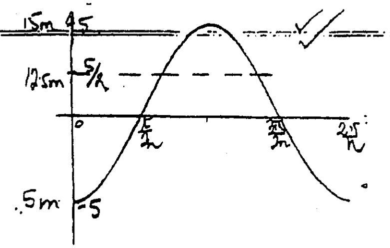
$\frac{Y}{a} = \frac{X^2}{a^2} - 3$

$Y = \frac{X^2}{a} + 3a$ or $X^2 = a(Y - 3a)$

(e)



Q5 (a)



∴ $x = -5\cos\frac{4\pi}{25}t$

when $x = \frac{5}{2}$

$\frac{5}{2} = -5\cos\frac{4\pi}{25}t$

$-\frac{1}{2} = \cos\frac{4\pi}{25}t$

$\frac{4\pi}{25}t = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \ldots$

$t = \frac{25}{6}, \frac{25}{3}, \ldots$

∴ the times between which the ship may enter the harbour are 8:10 am and 12:50 pm.

(b) $a = 5$ ½

$T = \frac{2\pi}{n}$ ½

$\frac{25}{2} = \frac{2\pi}{n}$ ½

$n = \frac{4\pi}{25}$ ½

(6)(i) LHS $= \sin(\sin^{-1}x - \cos^{-1}x)$

$= \sin(\alpha - \beta)$

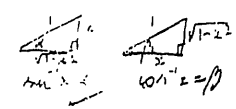
$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= x \cdot x - \sqrt{1-x^2}\cdot\sqrt{1-x^2}$

$= 2x^2 - 1$

= RHS

where



$$5^{k+1} = 5 \times 5^k$$
$$\geq 5(1+4k)$$
$$= 5+20k$$
$$= 4k+5+16k$$
$$> 4k+5 \quad \text{since } k \geq 1 \therefore 16k > 0$$
$$\therefore 5^{k+1} \geq 1+4(k+1) \quad \text{when } n = k+1$$

If the statement is true when $n=1$ and if it is true when $n=k+1$ then true for $n=2,3,\ldots$ & for all positive integral value of n.

Q.5(c) $2x^3 - x^2 + x - 1 = 0$

(i) $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$ $\quad \therefore \alpha\beta + \alpha\gamma + \beta\gamma = \frac{1}{2}$

(ii) $\alpha\beta\gamma = -\frac{d}{a}$ $\quad \therefore \alpha\beta\gamma = \frac{1}{2}$

(iii) $\alpha^2\beta^2\gamma + \alpha\beta^2\gamma^2 + \alpha^2\gamma^2\beta = \alpha\beta\gamma(\alpha\beta + \beta\gamma + \alpha\gamma)$ $\frac{1}{2}$

$= \frac{1}{4}$ $\frac{1}{2}$

(d) $2\cos^3\theta - \cos^2\theta + \cos\theta - 1 = 0$ has roots $\alpha = \cos a$, $\beta = \cos b$, $\gamma = \cos c$

$$\sec a + \sec b + \sec c = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$$
$$= \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$$
$$= 1 \quad \text{using (c)(i),(ii)}$$

QED.

Q5(c)(ii) $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(2-x)$

$\sin(\sin^{-1}x - \cos^{-1}x) = 2-x$

$2x^2 - 1 = 2-x$ from (i)

$2x^2 + x - 3 = 0$

$(2x+3)(x-1) = 0$

$\therefore x = 1$ or $x = -1\frac{1}{2}$ (non admissible) $\frac{1}{2}$

$\therefore x = 1$ $\frac{1}{2}$

Q.6.(a)(i) $\angle AOQ = 90^\circ$ (given)

$\angle APB = 90^\circ$ ($\angle$ in semicircle)

$\angle APQ = 90^\circ$ (suppl. adjacent $\angle$)

$\therefore A, O, P, Q$ are concyclic as AQ subtends two right angles $\angle AOQ$ and $\angle APQ$ thus AQ is the diameter of $\odot$ through A, O, P, Q.

(ii) $\angle OAP = \angle OQB$ (angles at the circumference standing on the same arc)

$\angle OAP = \angle OPA$ (base angles of isosceles $\triangle AOP$) $\frac{1}{2}$

$\therefore \angle OQB = \angle OPA$ (both equal to $\angle OAP$) $\frac{1}{2}$

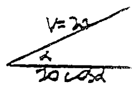
(b) $5^n \geq 1 + 4n \quad n \geq 1$

when $n = 1$ $\quad LHS = 5$
$\quad RHS = 5 \quad \therefore$ Statement is true.

Assume $5^k \geq 1 + 4k$ is true i.e. when $n = k$.
Try to show that it is true when $n = k+1$, i.e. that $5^{k+1} \geq 1 + (k+1)4$
$= 4k + 5$

(5)

Q7 (a)



$\ddot{x} = 0$

$\dot{x} = 20\cos\alpha$

$x = 20t\cos\alpha$ ①

at $t = 0$

$\ddot{y} = -10$

$\dot{x} = 20\cos\theta$ $\quad x = 0$

$\dot{y} = 20\sin\theta$ $\quad y = 0$

$\ddot{y} = -10$

$\dot{y} = -10t + 20\sin\alpha$

$y = -5t^2 + 20t\sin\alpha$ ②

(b) Sub $t = \frac{x}{20\cos\alpha} \to$ ②

$y = \frac{x\,20\sin\alpha}{20\cos\alpha} - 5\frac{x^2}{400\cos^2\alpha}$

$y = x\tan\alpha - \frac{x^2}{80}\sec^2\alpha$

(c) when $x = 20$, h

$\therefore h = 20\tan\alpha - \frac{400}{80}\sec^2\alpha$

$h = 20\tan\alpha - 5\sec^2\alpha$

(d) $\frac{dh}{d\alpha} = 20\sec^2\alpha - 10\sec^2\alpha\tan\alpha$

$10\sec^2\alpha(2 - \tan\alpha) = 0$ for max

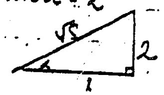
$\sec\alpha = 0$ $\quad \tan\alpha = 2$

$\alpha = 63°$ (nearest deg)

(e)(i)

α	60°	63°	70°
$\frac{dh}{d\alpha}$	+	0	−

$\therefore$ max. when $\tan\alpha = 2$



$h_{max} = 20 \times 2 - 5 \times \sqrt{5}^2$

$= 15$ metres

(ii) $V = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{(20\cos\alpha)^2 + (20\sin\alpha - 10t)^2}$

$= \sqrt{\left(\frac{20}{\sqrt{5}}\right)^2 + \left(\frac{20}{1} \cdot \frac{2}{\sqrt{5}} - 10 \times \sqrt{5}\right)^2}$

$= \sqrt{80 + (8\sqrt{5} - 10\sqrt{5})^2}$

$= \sqrt{80 + 20}$

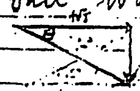
$= 10$ m/s is the speed of ball when hits the wall on its way ↓.

To find t:

$20 = Vt\cos\alpha$

$20 = 20t \times \frac{1}{\sqrt{5}}$

$t = \sqrt{5}$



$\frac{dy}{dt} = 20\sin\alpha - 10t$

$= -2\sqrt{5}$

$\frac{dx}{dt} = 20\cos\alpha$

$= \frac{20}{\sqrt{5}}$

$= 4\sqrt{5}$

$\tan\theta = \frac{2\sqrt{5}}{4\sqrt{5}}$

$\theta = \tan^{-1}\left(\frac{1}{2}\right)$

hits wall at

$\alpha = \tan^{-1}\left(\frac{4\sqrt{5}}{2\sqrt{5}}\right)$

$= \tan^{-1} 2$