7Q × 12 marks
✓ = 1 mark

[QUES]TION 1

$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ (1)

$\frac{d}{dx}\tan^{-1}(4x)$

$= \frac{\frac{1}{4}}{\frac{1}{16}+x^2} = \frac{4}{1+16x^2}$ (2)

$\int \frac{x}{\sqrt{x^2-3}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2-3}}\,dx$

$= \frac{1}{2}\int 2x\,(x^2-3)^{-\frac{1}{2}}\,dx$ ✓

$= \frac{1}{2}\cdot\frac{(x^2-3)^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= \sqrt{x^2-3} + C$ ✓ (2)

$P(-1) = 0 = (-1)^3 + 2(-1)^2 - k(-1) + 3$ ✓

$\therefore 0 = -1 + 2 + k + 3$

$\therefore k = -4$ ✓ (2)

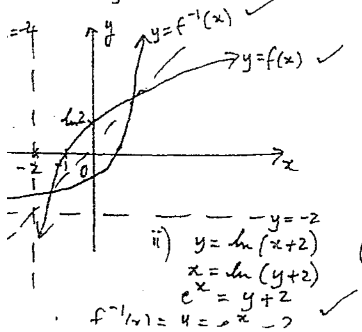
$10e^{-3t} = 2e^{2t}$

$5 = \frac{e^{2t}}{e^{-3t}}$

$5 = e^{5t}$ ✓

$\ln 5 = 5t$

$t = \frac{1}{5}\ln 5$ ✓ (2)



(1)

ii) $y = \ln(x+2)$

$x = \ln(y+2)$

$e^x = y+2$

$f^{-1}(x) = y = e^x - 2$ ✓ (2)

QUESTION 2

a) $\int_1^2 \frac{dx}{\sqrt{(x-1)^2+4}}$

let $u = x-1$, $du = 1\,dx$

If $x=2$ $u=1$; $x=1$ $u=0$ ✓

$= \int_0^1 \frac{du}{\sqrt{u^2+4}}$ ✓

$= \left[\ln\left(u+\sqrt{u^2+4}\right)\right]_0^1$ ✓

$= \ln(1+\sqrt{1+4}) - \ln(0+\sqrt{0+4})$

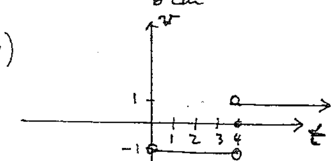
$= \ln\left(\frac{1+\sqrt{5}}{2}\right)$ (3)

b) i) P is at the origin O and changing direction, so velocity must be 0 cm/s! ✓ (1)

ii) velocity = −1 cm/s ie 1 cm/s to left. ✓ (1)

iii) Total d = 4 + 2 = 6 cm ✓✓ (2)

iv)



✓ (1)

c) i) $e^{x-2} - x = 0$ has 2 roots: ✓ (1)

ii) Let $f(x) = e^{x-2} - x$

$f'(x) = e^{x-2} - 1$ ✓ (3)

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$\therefore x_2 = 3.3 - \left(\frac{e^{3.3-2} - 3.3}{e^{3.3-2} - 1}\right)$ ✓

$= 3.3 - \frac{0.3692966\ldots}{2.6692966\ldots}$

$= 3.3 - 0.13834\ldots$

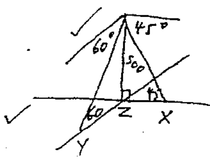
$= 3.161650\ldots$

$= 3.2$ to 2 d.p. ✓

QUESTION 3

a) i) $\tan 60° = \frac{500}{ZY}$

$ZY = \frac{500}{\sqrt{3}}$ m ✓



(2)

ii) $ZX = 500$ m

$\therefore XY^2 = ZX^2 + ZY^2$

$XY^2 = 500^2 + \left(\frac{500}{\sqrt{3}}\right)^2$ ✓ (2)

$= \frac{4}{3}\cdot 500^2$

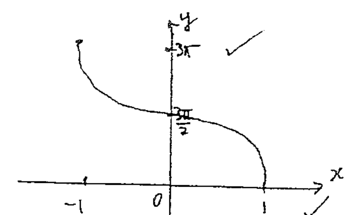
$\therefore XY = \frac{2\times 500}{\sqrt{3}}$

$XY = \frac{1000}{\sqrt{3}}$ m or $\frac{1000\sqrt{3}}{3}$ m ✓

b) $f(x) = 3\cos^{-1}x$

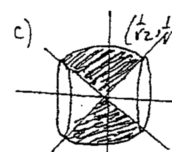
i) $f(-1) = 3\cos^{-1}(-1) = 3\pi$ ✓ (1)

ii)



✓ (2)

c)



i) $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$ (1)

and $y = \frac{1}{\sqrt{2}}$ if $x = \frac{1}{\sqrt{2}}$

$\therefore \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ satisfies. ✓

ii) Volume $= 2\pi\times\left[\int_0^{\frac{1}{\sqrt{2}}} (1-x^2)\,dx - \int_0^{\frac{1}{\sqrt{2}}} x^2\,dx\right]$ ✓

$= 2\pi\left[\int_0^{\frac{1}{\sqrt{2}}} 1 - 2x^2\,dx\right]$ (4)

$= 2\pi\left[x - \frac{2x^3}{3}\right]_0^{\frac{1}{\sqrt{2}}}$ ✓

$= 2\pi\left[\frac{1}{\sqrt{2}} - \frac{2\cdot\left(\frac{1}{\sqrt{2}}\right)^3}{3}\right] - 0$ ✓

$= 2\pi\left[\frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}}\right] = \frac{2\sqrt{2}\,\pi}{3}\,u^3$.

QUESTION 5

a) i) $P(\text{no pearls}) = \left(\frac{19999}{20000}\right)^3$ ✓ (1)

ii) $P(\text{a pearl}) = P(\text{at least 1 pearl})$

$= 1 - P(\text{no pearls})$ ✓

$= 1 - \left(\frac{19999}{20000}\right)^3$ ✓ (2)

iii) $P(\text{a pearl}) = 1 - \left(\frac{19999}{20000}\right)^n$ ✓ (1)

iv) $\therefore P(\text{a pearl}) \geq 50\%$

$\therefore 1 - \left(\frac{19999}{20000}\right)^n \geq \frac{1}{2}$ ✓ (2)

$\therefore \left(\frac{19999}{20000}\right)^n \leq \frac{1}{2}$

$n\log\left(\frac{19999}{20000}\right) \leq \log\left(\frac{1}{2}\right)$

$\therefore n \geq \frac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{19999}{20000}\right)}$

$n \geq 13862.59\ldots$ ✓

$\therefore$ Need to open at least 13863 pea[rls]

b) i) $x = Vt\cos\theta \Rightarrow t = \frac{x}{V\cos\theta}$

$\therefore y = V\left(\frac{x\sin\theta}{V\cos\theta}\right) - \frac{1}{2}g\left(\frac{x^2}{V^2\cos^2\theta}\right)$ ✓

$\therefore y = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$. (1)

ii) Bell must reach point (66, 8) $\theta = 6[0°]$

$\therefore 8 = 66\sqrt{3} - \frac{10\times 66^2}{2V^2\times 0.25}$ ✓

$\therefore \frac{87120}{V^2} = 66\sqrt{3} - 8$

$\therefore V^2 = \frac{87120}{66\sqrt{3}-8}$ (2)

$\therefore V = \sqrt{819.44\ldots}$

$= 28.626\ldots$ m/s ✓

$\doteqdot 29$ m/s

c) i) No. of ways $= {}^{25}P_{11}$ ✓

ii) " " " $= {}^{5}P_4 \times {}^{21}P_7$ ✓

iii) " " " $= 3!\times 2!\times {}^{20}P_6$ ✓ (3)

HSC 3UNIT TRIAL 1999 RIVERVIEW SOLUTIONS

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QUESTION 4

a) i) $f(-x) = \sqrt[3]{(-x)^2}$ ✓ ①
$= \sqrt[3]{x^2}$
$= f(x) \therefore f(x)$ is even

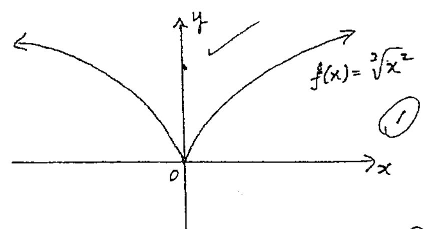
ii) $f(0) = \sqrt[3]{0^2}$ ✓ ①
$= 0$

iii) $f(x) = x^{2/3}$
$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$ ✓ , $f''(x) = -\frac{2}{9}x^{-\frac{4}{3}}$ ✓
$= \frac{2}{3\sqrt[3]{x}}$ $\quad = \frac{-2}{9\sqrt[3]{x^4}}$ ②

iv) $0 = \frac{2}{3\sqrt[3]{x}}$
$\therefore 0 = 2$ No solution. $\frac{dy}{dx} \neq 0$ ✓ ①
$\therefore$ No stat. points.

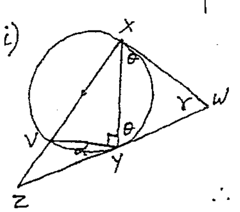
) $f'' = \frac{-2}{9\sqrt[3]{x^4}} < 0$ for all $x > 0$ since $x^4 > 0$ ①
$\therefore f(x)$ is concave down.

i)



✓ ①

) i)



$\angle XYV = 90°$ ✓ ③
($\angle$ in semicircle)
$\angle XYW = \theta$
(isos $\triangle$ XW = YW) ✓
tangents from ext. pt
$\therefore \alpha + 90° + \theta = 180°$ ✓
(straight line ZYW)
$\therefore \alpha + \theta = 90°$ QED

ii) $\gamma + 2\theta = 180°$ ($\angle$ sum $\triangle XWY$) ✓ ②
$\therefore \gamma + 2(90° - \alpha) = 180°$ from (i) ✓
$\gamma + 180 - 2\alpha = 180$
QED.

QUESTION 6

a) i) $T = A + Be^{-kt} \Rightarrow Be^{-kt} = T - A$ ✓
$\frac{dT}{dt} = 0 - kBe^{-kt}$ ✓ ②
$\therefore \frac{dT}{dt} = -k(T-A)$ on substitution.

ii) $t \to \infty \quad e^{-kt} \to 0 \therefore T \to A$. ✓
From graph, asymptote $= 22 \therefore A = 22$.
$t = 0 \quad 100 = 22 + Be^0$ ✓ ②
$\therefore B = 78$.

iii) $\therefore T = 22 + 78e^{-kt}$
$50 = 22 + 78e^{-90t}$ ✓ ②
$\therefore \frac{28}{78} = e^{-90t}$
$\therefore \ln\left(\frac{14}{39}\right) = -90t$
$\therefore t = \frac{-1}{90}\ln\left(\frac{14}{39}\right)$. ✓

iv) $\frac{dT}{dt} = -\frac{-1}{90}\ln\left(\frac{14}{39}\right)[50-22]$ ✓ ①
$= -0.3187\ldots$
$\therefore$ Cooling $\approx 0.32°C$ per minute.

b) i) Period $= 2 \times$ (Time between low & high)
$= 2 \times (6h\ 42min)$ ✓ ①
$= 13h\ 24min$

ii) Period $= \frac{2\pi}{n} \quad \therefore n = \frac{2\pi}{13h24min}$ ✓
$\therefore n = \frac{2\pi}{13\frac{2}{5}} = \frac{10\pi}{67}$. ①

iii) Sub. 5:14am and 3.3m ②
ie. $t = 2h\ 14min$, $D = 3.3$ ✓
$3.3 = -a\cos\left(\frac{10\pi}{67} \times 2h\ 14min\right) + 6$
$-2.7 = -a \times 0.5$ (in RADIAN MODE) ✓
$\therefore a = 5.4$

iv) $\therefore$ 9:42am = 6h 42min after 3am
$\therefore D = -5.4\cos\left(\frac{10\pi}{67} \times 6h42min\right) + 6$
High Tide $= 11.4$ m ✓ ①

QUESTION 7

a) $\frac{(n+9)^2(n+10)^2 - n^2(n-1)^2}{4}$
$= \frac{[(n+9)(n+10) - n(n-1)][(n+9)(n+10) + n(n-1)]}{4}$ ✓
$= \frac{(\cancel{n^2} + 19n + 90 - \cancel{n^2} + n)(n^2 + 19n + 90 + n^2 - n)}{4}$
$= \frac{(20n+90)(2n^2+18n+90)}{4}$ ②
$= \frac{5\cancel{10}(2n+9)(\cancel{2}(n^2+9n+45))}{\cancel{4}}$ ✓
$= 5(2n+9)(n^2+9n+45)$ QED.

b) i) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2}{4}(n+1)^2 \quad n \geq 1$
Proof:
$n = 1 \quad$ LHS $= 1^3$, RHS $= \frac{1}{4}(1+1)^2$
$= 1 \qquad = 1 \quad \therefore$ True ✓

Assume True for $n = k$
ie. $1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2}{4}(k+1)^2$ ✓ *

Required to Prove true for $n = k+1$ ✓
ie. $1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \frac{(k+1)^2}{4}(k+2)^2$

Proof: Consider the LHS of $n = k+1$ statement
$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \frac{k^2}{4}(k+1)^2 + (k+1)^3$ by *
$= (k+1)^2\left[\frac{k^2}{4} + \frac{4}{4}(k+1)\right]$ ④
$= (k+1)^2\left[\frac{k^2+4k+4}{4}\right]$
$= \frac{(k+1)^2(k+2)^2}{4}$ ✓ ✓
$=$ RHS

$\therefore$ Truth of $n = k$ statement implies truth of $n = k+1$ statement but $n = 1$ also true $\therefore$ true for all $n \geq 1$, $n \in N$.

ii) $1^3 + 2^3 + 3^3 + \ldots + (n+k)^3 = \frac{(n+k)^2}{4}(n+k+1)^2$ ① ✓

iii) $n^3 + (n+1)^3 + (n+2)^3 + \ldots + (n+9)^3$ ②
$= \sum_{k=n}^{n+9} k^3$
$= \sum_{k=1}^{n+9} k^3 - \sum_{k=1}^{n-1} k^3$ ✓

b) iii) cont'd
$= \frac{(n+9)^2}{4}(n+10)^2 - \frac{(n-1)^2}{4}(n-1+1)^2$
$= \frac{(n+9)^2(n+10)^2 - n^2(n-1)^2}{4}$ ✓
$= 5(2n+9)(n^2+9n+45)$ QED.

iv) Total mass of dolls
$= c \times 4^3 + c \times 5^3 + c \times 6^3 + \ldots + c \times 13^3$
$= c(4^3 + 5^3 + 6^3 + \ldots + 13^3)$
$= c(n^3 + (n+1)^3 + (n+2)^3 + \ldots + (n+9)^3)$, where $n = 4$
$= c[5(2n+9)(n^2+9n+45)]$, $n = 4$
$= c[5(2 \times 4 + 9)(4^2 + 9 \times 4 + 45)]$
$= c \times 8245$
$= 8245c$ grams ✓
$\therefore$ Total mass is $8245c$ grams

③