Name:________________________________ Maths Class:______________

SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE

2008

EXTENSION 1 MATHEMATICS

Instructions:

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Start each question on a new page

Total Marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value

(For markers use only)

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total

Question 1

a) Differentiate:

i) $x^2 cos^{-1}x$ 2

ii) $log_{10}3x$ 2

b) There is a remainder of 1 when $P(x) = x^3 - 3x^2 + px - 14$ is divided by $x - 3$. Find the value of p. 2

c) Find the simultaneous solution of : $|x - 3| < 4$ and $|x - 1| > 1$ 3

d) The point P (3 ,5) divides the interval joining A (-1, 1) and B (5,7) <u>internally</u> in the ratio $m{:}n$. 2

Find $m{:}n$.

e) Find $\int cosx\, sinx\, dx$ 1

Question 2 (Start a new page)

a) Find $\lim\limits_{x \to \infty} \frac{3x^2-7x}{5+x^2}$ 1

b) Find the acute angle, to the nearest degree, between the curve $y = x^2$ and the line $5x - y - 6 = 0$ at the point of intersection (3, 9) 2

c) i) Solve $t^2 + 2t - 1 = 0$ 1

ii) Using your results from part i), and the expansion for $tan\ 2\theta$, find the exact value of $tan\ 22.5°$. Simplify your answer. 2

d) i) Express $3\cos x - 2\,sin\,x$ in the form $Acos(x{+}\propto)$ where $A > 0$ and $0^0 \leq \propto \leq 90^0$ **2**

ii) Hence find the smallest positive x degrees such that $3\,cos\,x - 2\sin x$ has a maximum value (do not use calculus). Give your answer correct to 1 decimal place. **1**

e) Express $\sin\,(tan^{-1}x + tan^{-1}y)$ in terms of x and y only. **3**

Question 3 **(Start a new page)**

a) Solve for $0 \leq \theta \leq 2\pi$: $cos\,2\theta = cos^2\theta$ **2**

b) Solve $\frac{x^2}{x-4} < 0$ **2**

c) Find $\int \frac{x+4}{x^2+4}\,dx$ **2**

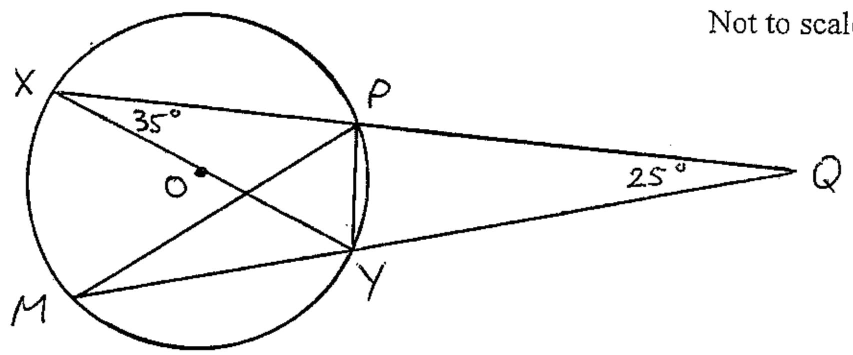
d) Use the substitution $u = e^x$ to find $\int \frac{e^x}{\sqrt{9-4e^{2x}}}\,dx$ **3**

e) $\propto, \beta, \gamma$ are the roots of the equation $2x^3 + 5x - 3 = 0$ **3**

Find the value of $\propto^2 + \beta^2 + \gamma^2$.

Question 4 (Start a new page)

a) 3



Not to scale

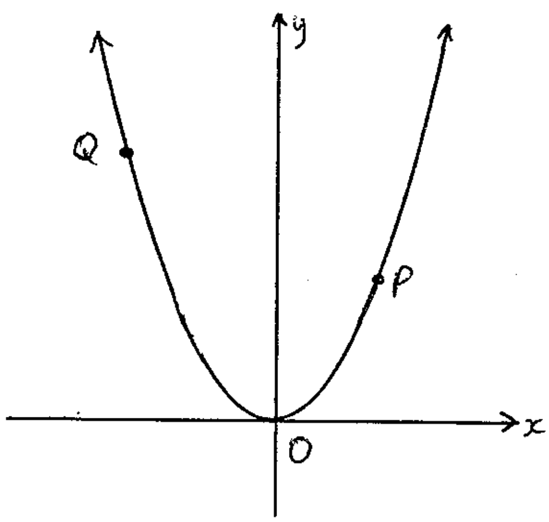
O is the centre of the circle

$\angle PXY = 35°$ and $\angle PQY = 25°$

i) Copy the diagram onto your answer paper

ii) Find $\angle MPY$ giving full reasons

b)



The points $P(2p, p^2)$ and $Q(2q, q^2)$ move on the parabola $x^2 = 4y$ such that the chord PQ subtends a right angle at the origin O

i) Show that $pq = -4$ 2

ii) M is the midpoint of PQ. Derive the locus of M and show that it is the parabola $y = \frac{x^2+8}{2}$ 2

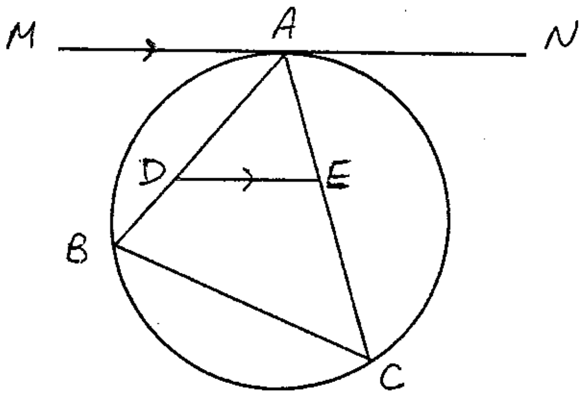
iii) Find the focus of the parabola for M. 1

c) Prove by mathematical induction, that

$1 \times 1 + 2 \times 2^1 + 3 \times 2^2 + \ldots\ldots + n \times 2^{n-1} = 1 + (n-1)2^n$ where n is a positive integer 4

Question 5 (**Start a new page**)

a)



ΔABC is inscribed in the circle.

MAN is tangent to the circle at A and $DE \| MN$

i) Copy the diagram onto your answer page

ii) Prove that $BCED$ is a cyclic quadrilateral 3

iii) Describe how to find the centre of the circle passing through B, C, E, D. 1

b) Given $f(x) = \frac{2}{x+1}$ for $x > -1$:

i) Find the equation of the inverse function $f^{-1}(x)$ 1

ii) On the same diagram, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$. 3

Clearly show the coordinates of any points of intersection, intercepts on the coordinates axes and equations of any asymptotes.

c) i) Sketch the curve $y = sin^{-1}(\frac{x}{2})$ 1

ii) The area between the curve $y = sin^{-1}(\frac{x}{2})$ and the y axis is rotated about the y axis. 3

Find the volume thus generated.

Question 6 (Start a new page)

a) Differentiate $y = tan^{-1}(sin\ 3x)$ 2

b) In a population study, the population P of a town after t years is given by

$P = 200 + Ae^{kt}$.

The initial population was 300 and increased to 400 over 3 years.

i) Find the population after a further 2 years (nearest whole person) 3

ii) Find the rate of population growth after 10 years. 1

c) Kramer hits a golf ball from the top of the edge of a vertical cliff 25 metres above the sea. He hits it with an initial velocity of 50 m/s at a 30° angle of elevation.

The cliff top is taken as the point of origin.

i) Given $\ddot{x} = 0$ and $\ddot{y} = -10$, derive the equations of the horizontal and vertical components of the motion for the golf ball. 2

ii) Find the maximum height of the golf ball above the cliff. 2

iii) Find the angle at which the golf ball hits the water (nearest degree). 2

Question 7 (Start a new page)

a) A particle is moving according to the velocity equation $v = 4 - 2t$ m/s. Find the total distance it travels in the first 5 seconds of its motion. 2

b) A particle is moving with simple harmonic motion in a straight line with velocity

$v^2 = 108 + 36x - 9x^2$ where x cm is its displacement from a point O.

Initially it is at rest at $x = 6$ cm.

i) Use differentiation to find its acceleration in terms of x and find its maximum acceleration. 2

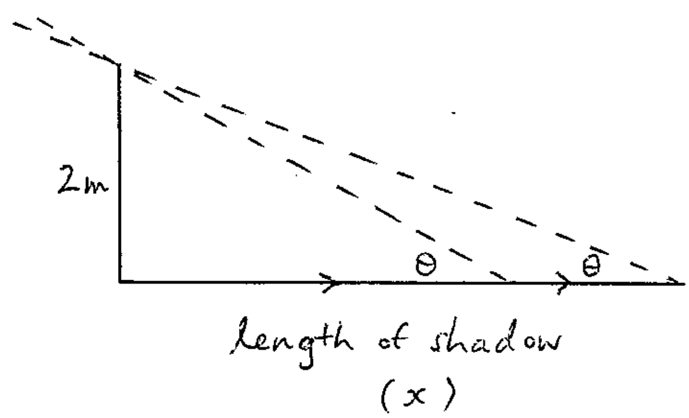
ii) Find the maximum speed of the particle and the time when this first occurs. 3

iii) Write an expression for the particle's displacement in terms of time t. 1

c) A vertical pole, 2 metres high, casts a lengthening shadow as the sun sets. 4

At a particular instant, the shadow's length, x, is increasing by 0.3m/min.

Simultaneously, the angle of the Sun, θ, is decreasing by 0.05 radians/min.



Find the angle θ (to the nearest degree) when this is occurring.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

2008 Extension Solutions

1

a) i) $\frac{dy}{dx} = 2x\cos^{-1}x - \frac{x^2}{\sqrt{1-x^2}}$

ii) $y = \frac{\log 3x}{\log 10}$

$\frac{dy}{dx} = \frac{1}{x \log 10}$

b) $P(3) = 1$

$\therefore 27 - 27 + 3p - 14 = 1$

$\therefore 3p = 15$

$\therefore p = 5$

c) $|x-3| < 4 \Rightarrow -4 < x - 3 < 4$

$-1 < x < 7$

$|x-1| > 1 \Rightarrow x - 1 > 1$ or $x - 1 < -1$

$x > 2$ or $x < 0$

$\therefore$ simultaneous sol. is

$2 < x < 7$ or $-1 < x < 0$

d) $3 = \frac{-n + 5m}{m+n}$

$3m + 3n = -n + 5m$

$-2m = -4n$

$m = 2n$

$\frac{m}{n} = 2$

$\therefore m : n = 2 : 1$

e) $\frac{\sin^{-2} x}{2} + c$

2

a) $\lim_{x \to \infty} \frac{3 - \frac{7}{x}}{\frac{5}{x^2} + 1} = \frac{3 - 0}{0 + 1}$

$= 3$

b) $\frac{dy}{dx} = 2x \Rightarrow m_1 = 6$

$m_2 = 5$

$\tan\theta = \left|\frac{6-5}{1+30}\right|$

$= \frac{1}{31}$

$\therefore \theta \doteqdot 2°$

c) i) $t = \frac{-2 \pm \sqrt{4+4}}{2}$

$= \frac{-2 \pm 2\sqrt{2}}{2}$

$= -1 \pm \sqrt{2}$

ii) $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

$\tan 45° = \frac{2\tan 22.5°}{1 - \tan^2 22.5°}$

$\therefore 1 - \tan^2 22.5° = 2\tan 22.5°$

$\therefore \tan^2 22.5° + 2\tan 22.5° - 1 = 0$

$\therefore \tan 22.5° = -1 + \sqrt{2}$ (>0)

(from i) above)

d) i) $3\cos x - 2\sin x = A\cos(x+\alpha)$

$(A = \sqrt{13})$

$= \sqrt{13}\cos(x+\alpha)$

$\therefore \frac{3}{\sqrt{13}}\cos x - \frac{2}{\sqrt{13}}\sin x = \cos(x+\alpha)$

$= \cos x\cos\alpha - \sin x \sin\alpha$

$\therefore \left.\begin{array}{l}\cos\alpha = 3/\sqrt{13}\\ \sin\alpha = 2/\sqrt{13}\end{array}\right\} \therefore \alpha = 33\cdot7^\circ$

$\therefore 3\cos x - 2\sin x = \sqrt{13}\cos(x+33\cdot7^\circ)$

ii) max. value of $3\cos x - 2\sin x$

= max. value of $\sqrt{13}\cos(x+33\cdot7^\circ)$

$= \sqrt{13}$, and this occurs

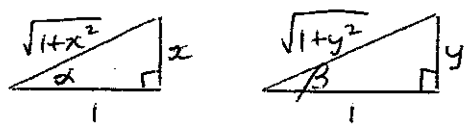
when $\cos(x+33\cdot7^\circ) = 1$

$\therefore x + 33\cdot7^\circ = 360^\circ$ (not 0°)

$\therefore x = 326\cdot3^\circ$

e) Let $\tan^{-1}x = \alpha$, $\tan^{-1}y = \beta$

$x = \tan\alpha$, $y = \tan\beta$



$\therefore \sin(\tan^{-1}x + \tan^{-1}y)$

$= \sin(\alpha+\beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+x^2}}\cdot\frac{y}{\sqrt{1+y^2}}$

$= \frac{x+y}{\sqrt{(1+x^2)(1+y^2)}}$

3 a) $2\cos^2\theta - 1 = \cos^2\theta$

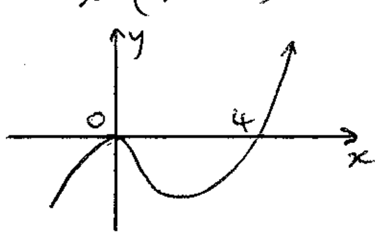
$\cos^2\theta = 1$

$\cos\theta = \pm 1$

$\theta = 0, \pi, 2\pi$.

b) $\frac{x^2}{\cancel{x-4}} \times (x-4)^{\cancel{2}} < 0$

$x^2(x-4) < 0$



$\therefore x < 0$ or $0 < x < 4$

or $x < 4$ $(x \neq 0)$

c) $\int \frac{x}{x^2+4}\,dx + \int \frac{4}{x^2+4}\,dx$

$= \frac{1}{2}\log(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) + c$

d) $\int \frac{e^x}{\sqrt{9-4e^{2x}}}\,dx = \int \frac{\cancel{u}}{\sqrt{9-4u^2}}\cdot\frac{du}{\cancel{u}}$

$u = e^x$

$\frac{du}{dx} = e^x$

$dx = \frac{du}{e^x}$

$= \frac{du}{u}$

$= \int \frac{1}{2\sqrt{9/4 - u^2}}\,du$

$= \frac{1}{2}\sin^{-1}\left(\frac{u}{3/2}\right) + c$

$= \frac{1}{2}\sin^{-1}\left(\frac{2e^x}{3}\right) + c$

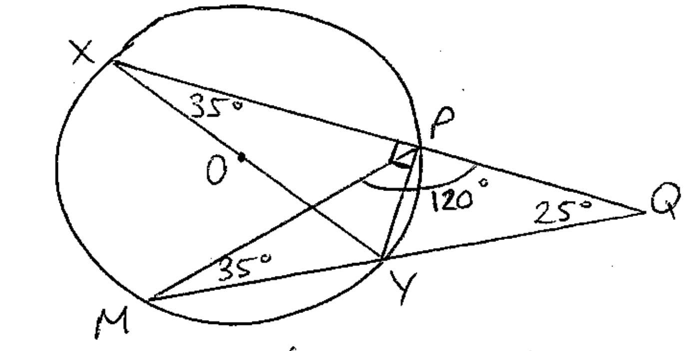
e) $(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\alpha\gamma$

$\therefore \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$

$= 0 - 2\left(\frac{5}{2}\right) = -5$

4) a) i)



$\angle PMQ = 35°$ (angles standing on same chord PY)

$\angle XPY = 90°$ (angle in a semi circle)

$\angle MPQ = 120°$ (angle sum of ΔMPQ)

$\therefore \angle MPY = 30°$

b) i) $m_{OP} = \frac{p^2}{2p} = \frac{p}{2}$, $m_{OQ} = \frac{q}{2}$

$m_{OP} \times m_{OQ} = -1$ for perpend. lines

$\therefore \frac{p}{2} \times \frac{q}{2} = -1$

$\therefore pq = -4$ as reqd.

ii) M has coords $\left(\frac{2p+2q}{2}, \frac{p^2+q^2}{2}\right)$

ie. $x = p+q$, $y = \frac{p^2+q^2}{2}$
$= \frac{(p+q)^2 - 2pq}{2}$
$= \frac{x^2 - 2(-4)}{2}$

$\therefore y = \frac{x^2+8}{2}$ is the locus of M as reqd.

iii) $2y = x^2 + 8$
$x^2 = 2y - 8$
$\therefore (x-0)^2 = 2(y-4)$
$\therefore$ vertex at $(0,4)$ and $4a = 2$
$\therefore a = \frac{1}{2}$

$\therefore$ focus at $(0, 4\frac{1}{2})$.

c) Test $n=1 \Rightarrow LHS = 1 \times 2^0 = 1$, $RHS = 1 + 0 \times 2^1 = 1$

$\therefore$ result is true for $n = 1$

Assume result is true for $n=k$,
ie assume that $S_k = 1 + (k-1)2^k$

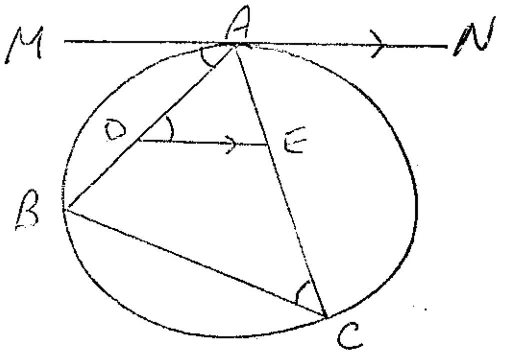
Prove true for $n = k+1$,
ie. prove that $S_{k+1} = 1 + k.2^{k+1}$

Now, $S_{k+1} = S_k + T_{k+1}$
$= 1 + (k-1)2^k + (k+1)2^k$
$= 1 + 2^k(k-1+k+1)$
$= 1 + 2^k . 2k$
$= 1 + k . 2^k . 2$
$= 1 + k . 2^{k+1}$
(shown)

So, if the result is true for $n = k$, then it has been proved true for $n = k+1$.

Since the result is true for $n = 1$, then from above it must be true for $n = 1+1 = 2$ and so on for all positive integral n.

5 a) i)



ii) $\angle MAN = \angle ADE$ (alt. angles $MN \parallel DE$)

$\angle MAN = \angle BCA$ (angle in alt. segment)

$\therefore \angle ADE = \angle BCA$

$\therefore$ BCED is a cyc. quad. since exterior angle equals interior opposite angle.

iii) Perpendicular bisectors of at least 2 sides of BCED meet at the centre of the circle.

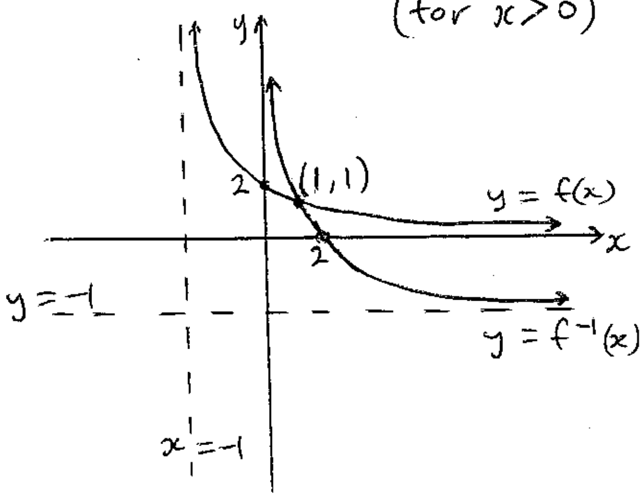
b) i) $x = \dfrac{2}{y+1}$

$xy + x = 2$

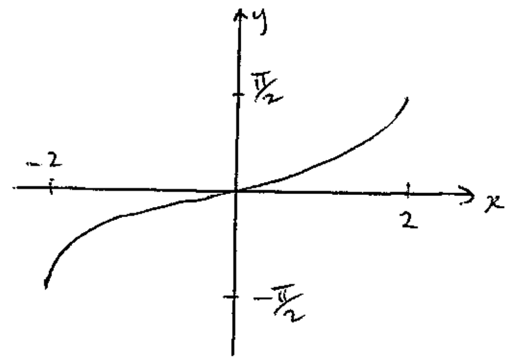
$xy = 2 - x$

$\therefore f^{-1}(x) \Rightarrow y = \dfrac{2-x}{x}$ or $\dfrac{2}{x} - 1$ (for $x > 0$)

ii)



c) i)



ii) $V = 2\pi \int_0^{\pi/2} (2\sin y)^2 \, dy$

$= 8\pi \int_0^{\pi/2} \sin^2 y \, dy$

$= 8\pi \int_0^{\pi/2} \tfrac{1}{2}(1 - \cos 2y) \, dy$

$= 4\pi \left[y - \dfrac{\sin 2y}{2} \right]_0^{\pi/2}$

$= 4\pi \left[\tfrac{\pi}{2} - 0 - (0 - 0) \right]$

$= 2\pi^2 \ u^3.$

6 a) $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ $(u = \sin 3x)$

$= \dfrac{1}{1+u^2} \times 3\cos 3x$

$= \dfrac{3\cos 3x}{1 + \sin^2 3x}$

b) i) $P = 300$, $t = 0$:

$300 = 200 + A$ $(A = 100)$

$\therefore P = 200 + 100e^{kt}$

$P = 400$, $t = 3$:

$400 = 200 + 100e^{3k}$

$200 = 100e^{3k}$

$\therefore e^{3k} = 2$

$3k = \log 2$

$k = \dfrac{\log 2}{3}$

$\therefore P = 200 + 100e^{\frac{t\log 2}{3}}$

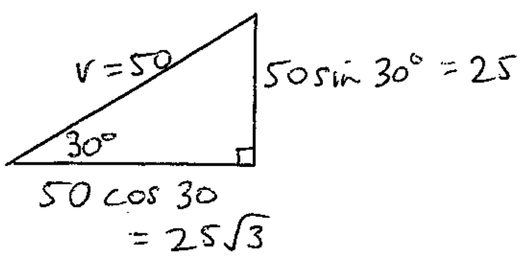
When $t = 5$, $P = 200 + 100e^{\frac{5\log 2}{3}}$

$\doteqdot 517$ people

ii) $\dfrac{dP}{dt} = 100e^{\frac{t\log 2}{3}} \times \dfrac{\log 2}{3}$

When $t = 10$, $\dfrac{dP}{dt} = 233$ people per year.

c)



i)

$\ddot{x} = 0$

$\dot{x} = c$

When $t=0$, $\underline{\underline{\dot{x} = 25\sqrt{3}}}$

$x = 25\sqrt{3}\,t + k$

When $t = 0$, $x = 0$

$(k=0)$

$\therefore \underline{\underline{x = 25\sqrt{3}\,t}}$

$\ddot{y} = -10$

$\dot{y} = -10t + c$

When $t = 0$, $\dot{y} = 25$

$\therefore 25 = 0 + c$

$(c = 25)$

$\therefore \underline{\underline{\dot{y} = -10t + 25}}$

$\therefore y = -5t^2 + 25t + k$

When $t=0$, $y = 0$

$(k=0)$

$\therefore \underline{\underline{y = -5t^2 + 25t}}$

ii) max. height when $\dot{y} = 0$ (or $y = \frac{-b}{2a}$)

$\therefore -10t + 25 = 0$

$\therefore t = 2.5$ seconds

$\therefore y_{max} = -5(2.5)^2 + 25(2.5)$

$= 31.25$ m above cliff.

iii) $y = -25 \Longrightarrow -25 = -5t^2 + 25t$

$5t^2 - 25t - 25 = 0$

$t^2 - 5t - 5 = 0$

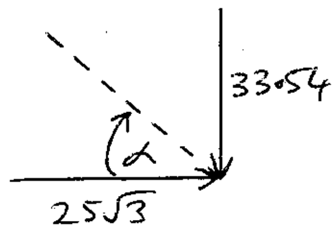
$t = \dfrac{5 \pm \sqrt{25+20}}{2}$

$= \dfrac{5 \pm \sqrt{45}}{2}$

$= \dfrac{5 + 3\sqrt{5}}{2}$ $(t > 0)$

$\therefore \dot{y} = -10\left(\dfrac{5+3\sqrt{5}}{2}\right) + 25$

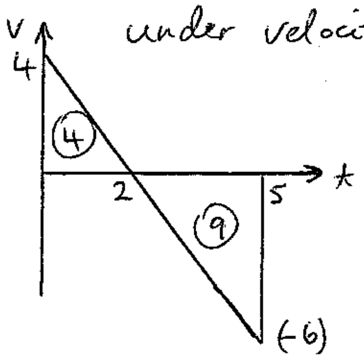
$\doteqdot -33.54$ and $\dot{x} = 25\sqrt{3}$



$\tan\alpha = \dfrac{33.54}{25\sqrt{3}}$

$\therefore \alpha \doteqdot 38°$

7 a) dist. travelled = total area under velocity graph.



$\therefore$ total dist. travelled $= 13$ metres

b) i) $\ddot{x} = \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right)$

$= \dfrac{d}{dx}\left(54 + 18x - \tfrac{9}{2}x^2\right)$

$= 18 - 9x$ or $-9(x-2)$

$\ddot{x}_{max}$ $(x = 6$ or $-2)$ $= -9 \times (\pm 4)$

$= -36$ (or 36) cm/s²

ii) V_{max} when x = centre of oscillation ($x = 2$)

$\therefore v^2 = 108 + 72 - 36$
$= 144$

$\therefore$ max. speed $= 12$ cm/s.

and time taken $= \frac{1}{4}$ of period
$= \frac{1}{4} \times \frac{2\pi}{3}$
$= \frac{\pi}{6}$ seconds

iii) $x = b + a\cos(nt + \alpha)$

$\therefore x = 2 + 4\cos 3t$

c) $\frac{dx}{dt} = 0.3$, $\frac{d\theta}{dt} = -0.05$

$\tan\theta = \frac{2}{x}$

$\therefore x = \frac{2}{\tan\theta}$

$\frac{dx}{d\theta} = \frac{-2\sec^2\theta}{\tan^2\theta}$
$= \frac{-2}{\sin^2\theta}$

$\frac{dx}{d\theta} = \frac{dx}{dt} \times \frac{dt}{d\theta}$

$\frac{-2}{\sin^2\theta} = 0.3 \times \frac{1}{-0.05}$
$= -6$

$\therefore \sin^2\theta = \frac{1}{3}$

$\therefore \sin\theta = +\frac{1}{\sqrt{3}}$ (θ is acute)

$\therefore \theta \doteqdot 35^\circ$

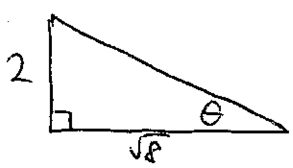
OR $\theta = \tan^{-1}\left(\frac{2}{x}\right)$

$\frac{d\theta}{dx} = \frac{1}{1 + \left(\frac{2}{x}\right)^2} \times (-2x^{-2})$
$= \frac{1}{1 + \frac{4}{x^2}} \times \frac{-2}{x^2}$
$= \frac{-2}{x^2 + 4}$

$\frac{d\theta}{dx} = \frac{d\theta}{dt} \times \frac{dt}{dx}$

$\frac{-2}{x^2+4} = -0.05 \times \frac{1}{0.3}$
$= -\frac{1}{6}$

$\therefore 12 = x^2 + 4$
$x^2 = 8$
$x = +\sqrt{8}$ ($x > 0$)



$\tan\theta = \frac{2}{\sqrt{8}}$

$\therefore \theta \doteqdot 35^\circ$