QUESTION ONE

(a)
$$\frac{1}{x-3} < 3,\ x \neq 3$$
$$\frac{1}{x-3} \times (x-3)^2 < 3(x-3)^2$$
$$x-3 < 3(x-3)^2 \quad \boxed{\surd}$$
$$3(x-3)^2 - (x-3) > 0$$
$$(x-3)(3(x-3)-1) > 0$$
$$(x-3)(3x-10) > 0$$
$$x < 3 \text{ or } x > \tfrac{10}{3}. \quad \boxed{\surd}$$

(b)
$$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\tfrac{x}{3}\right]_0^3 \quad \boxed{\surd}$$
$$= \sin^{-1} 1 - \sin^{-1} 0$$
$$= \tfrac{\pi}{2}. \quad \boxed{\surd}$$

(c) (i) $y = \tan^{-1} 2x$
$$\frac{dy}{dx} = \frac{2}{1+4x^2}. \quad \boxed{\surd}$$

(ii) $y = \log_e \cos x$
$$\frac{dy}{dx} = -\frac{\sin x}{\cos x} \quad \boxed{\surd \text{ for } -\sin x \ \surd \text{ for quotient}}$$

(d) $\tan\theta = |-\tfrac{5}{3}|$ $\boxed{\surd}$

$\theta \doteqdot 59^\circ$ $\boxed{\surd}$

(e)
$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} \quad \boxed{\surd}$$
$$= \frac{3}{2} \div 1$$
$$= \frac{3}{2} \quad \boxed{\surd\surd \text{ any correct method}}$$

QUESTION TWO

(a) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{1+\tan x}}\,dx = \int_1^2 \frac{du}{u^{\frac{1}{2}}}$ $\boxed{\surd}$

$\displaystyle = \int_1^2 u^{-\frac{1}{2}}\,du$

$\displaystyle = \left[2u^{\frac{1}{2}}\right]_1^2$ $\boxed{\surd}$

$= 2\sqrt{2} - 2$ $\boxed{\surd}$

Let $u = 1 + \tan x$

$du = \sec^2 x\,dx$

When $x = 0$, $u = 1$,

When $x = \frac{\pi}{4}$, $u = 2$.

(b) General term $= {}^6\mathrm{C}_r\,(x^2)^{6-r}\,(-1)^r\,(3x^{-2})^r$

$= {}^6\mathrm{C}_r\,(x)^{12-2r}\,(-1)^r\,(3)^r\,(x)^{-2r}$

$= {}^6\mathrm{C}_r\,(-1)^r\,(3)^r\,(x)^{12-4r}$ $\boxed{\surd}$

Let $12 - 4r = 0$

$r = 3$ $\boxed{\surd}$

Term independent of $x$ $= {}^6\mathrm{C}_3\,(-1)^3\,(3)^3$

$= -540.$ $\boxed{\surd}$

(c) $\displaystyle LHS = \frac{\tan 2\theta - \tan\theta}{\tan 2\theta + \cot\theta}$

Let $t = \tan\theta$

$\displaystyle LHS = \left(\frac{2t}{1-t^2} - t\right) \div \left(\frac{2t}{1-t^2} + \frac{1}{t}\right)$ $\boxed{\surd}$

$\displaystyle = \frac{2t - t + t^3}{1-t^2} \times \frac{t(1-t^2)}{2t^2 + 1 - t^2}$

$\displaystyle = \frac{t(1+t^2}{1-t^2} \times \frac{t(1-t^2)}{t^2+1}$

$\boxed{\surd \text{ correct method of simplification of the algebraic fractions}}$

$= t^2$ $\boxed{\surd}$

$= \tan^2\theta$

$= RHS$

(d) (i) $V = \frac{4}{3}\pi r^3$

$\displaystyle \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$ $\boxed{\surd}$

$\displaystyle 8 = 64\pi \frac{dr}{dt}$

$\displaystyle \frac{dr}{dt} = \frac{1}{8\pi}$ m/min $\boxed{\surd}$

(ii) $S = 4\pi r^2$

$\displaystyle \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

$\displaystyle = 8\pi r \times \frac{1}{8\pi}$

$= 4\,\mathrm{m}^2/\mathrm{min}.$ $\boxed{\surd}$

QUESTION THREE

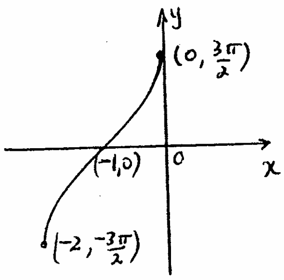
(a) (i) $$f(x) = 3\sin^{-1}(x+1)$$

Domain: $-1 \le x+1 \le 1$

$-2 \le x \le 0$ $\boxed{\surd}$

Range: $-\dfrac{3\pi}{2} \le y \le \dfrac{3\pi}{2}$. $\boxed{\surd}$

(ii)




$\boxed{\surd \text{ Shape}}$
$\boxed{\surd \text{ Labels}}$

(b) (i) $$v^2 = 2x(6-x)$$

$2x(6-x) \ge 0$

$0 \le x \le 6$ $\boxed{\surd}$

(ii) $x = 3$ $\boxed{\surd}$

(iii) Maximum speed when $x = 3$.

$v^2 = 6 \times 3$

$|v| = 3\sqrt{2}$ $\boxed{\surd}$

(iv) $$v^2 = 2x(6-x)$$

$$\tfrac{1}{2}v^2 = 6x - x^2$$

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 6 - 2x$$

$$\ddot{x} = 6 - 2x \quad \boxed{\surd}$$

(c) Given $\left(2 + \dfrac{x}{3}\right)^n$:

term in $x^6 = {}^nC_6 \times 2^{n-6} \times \left(\frac{x}{3}\right)^6$

term in $x^7 = {}^nC_7 \times 2^{n-7} \times \left(\frac{x}{3}\right)^7$ $\boxed{\surd \text{ 1 mark for both answers}}$

$$\text{Ratio of coefficients} = \frac{\dfrac{n!}{6!(n-6)!} \times 2^{n-6} \times (\dfrac{1}{3})^6}{\dfrac{n!}{7!(n-7)!} \times 2^{n-7} \times (\dfrac{1}{3})^7}$$

$$= \frac{n!}{6!(n-6)!} \times \frac{7!(n-7)!}{n!} \times 3 \times 2 \quad \boxed{\surd}$$

$$= \frac{42}{n-6} \quad \boxed{\surd}$$

$$\text{so } \frac{7}{8} = \frac{42}{n-6}$$

$$n - 6 = 48$$

$$n = 54. \quad \boxed{\surd}$$

QUESTION FOUR

(a)

$$\begin{aligned}
\text{Let } P(x) &= 2x^3 + ax^2 + bx + 6 \\
P(1) &= 2 + a + b + 6 \\
0 &= a + b + 8 \\
a + b &= -8 && \cdots(1) \quad \boxed{\checkmark \text{ for any correct form}} \\
P(-2) &= -16 + 4a - 2b + 6 \\
-12 &= 4a - 2b - 10 \\
4a - 2b &= -2 \\
2a - b &= -1 && \cdots(2) \quad \boxed{\checkmark \text{ for any correct form}} \\
(1) + (2) \qquad 3a &= -9 \\
a &= -3 \quad \boxed{\checkmark} \\
b &= -5 \quad \boxed{\checkmark}
\end{aligned}$$

(b) $x^3 + px^2 + qx + r = 0$

$$\begin{aligned}
3\alpha &= -p && \cdots(1) \quad \boxed{\checkmark} \\
3\alpha^2 &= q && \cdots(2) \quad \boxed{\checkmark} \\
\alpha^3 &= -r && \cdots(3) \quad \boxed{\checkmark} \\
(1) \times (2) \qquad 9\alpha^3 &= -pq \\
-9r &= -pq \\
pq &= 9r \quad \boxed{\checkmark}
\end{aligned}$$

(c) (i) $(1+x)^4(1+x)^4 = ({}^4\mathrm{C}_0 + {}^4\mathrm{C}_1 x + {}^4\mathrm{C}_2 x^2 + {}^4\mathrm{C}_3 x^3 + {}^4\mathrm{C}_4 x^4)$
$\times ({}^4\mathrm{C}_0 + {}^4\mathrm{C}_1 x + {}^4\mathrm{C}_2 x^2 + {}^4\mathrm{C}_3 x^3 + {}^4\mathrm{C}_4 x^4)$ $\boxed{\checkmark}$

Term in $x^5 = {}^4\mathrm{C}_1 x \times {}^4\mathrm{C}_4 x^4 + {}^4\mathrm{C}_2 x^2 \times {}^4\mathrm{C}_3 x^3 + {}^4\mathrm{C}_3 x^3 \times {}^4\mathrm{C}_2 x^2 + {}^4\mathrm{C}_4 x^4 \times {}^4\mathrm{C}$

Coefficient $= {}^4\mathrm{C}_1 \times {}^4\mathrm{C}_4 + {}^4\mathrm{C}_2 \times {}^4\mathrm{C}_3 + {}^4\mathrm{C}_3 \times {}^4\mathrm{C}_2 + {}^4\mathrm{C}_4 \times {}^4\mathrm{C}_1$
$= {}^4\mathrm{C}_0 \times {}^4\mathrm{C}_1 + {}^4\mathrm{C}_1 \times {}^4\mathrm{C}_2 + {}^4\mathrm{C}_2 \times {}^4\mathrm{C}_3 + {}^4\mathrm{C}_3 \times {}^4\mathrm{C}_4$, by symmetr

(ii) Coefficient of $x^5$ in $(1+x)^8$ $= {}^8\mathrm{C}_5$

$$= \frac{8!}{3! \times 5!} \quad \boxed{\checkmark}$$

Now $(1+x)^4(1+x)^4 = (1+x)^8$,

so ${}^4\mathrm{C}_0 \times {}^4\mathrm{C}_1 + {}^4\mathrm{C}_1 \times {}^4\mathrm{C}_2 + {}^4\mathrm{C}_2 \times {}^4\mathrm{C}_3 + {}^4\mathrm{C}_3 \times {}^4\mathrm{C}_4 = \dfrac{8!}{3! \times 5!}$.

QUESTION FIVE

(a) (i) Given $T = 20 + Ae^{-kt}$

$$\frac{dT}{dt} = -kAe^{-kt}$$
$$= -k(T - 20). \quad \boxed{\checkmark}$$

So $T = 20 + Ae^{-kt}$ is a solution.

(ii) When $t = 0,\ T = 36$

$$\text{so } 36 = 20 + Ae^0$$
$$A = 16. \quad \boxed{\checkmark}$$

When $t = 5,\ T = 35$

$$\text{so } 35 = 20 + 16e^{-5k}$$
$$15 = 16e^{-5k}$$
$$e^{-5k} = \frac{15}{16} \quad \boxed{\checkmark}$$
$$-5k = \log_e \frac{15}{16}$$
$$k = -\frac{1}{5}\log_e \frac{15}{16}. \quad \boxed{\checkmark}$$

(iii) When $T = 27$,

$$27 = 20 + 16e^{-kt}$$
$$e^{-kt} = \frac{7}{16} \quad \boxed{\checkmark}$$
$$t = \frac{\log_e \frac{7}{16}}{-k}$$
$$= 64{\cdot}045.....$$

It will take 64 minutes. $\boxed{\checkmark}$

(iv) As $t \to \infty$, $T \to 20$ from above.
The temperature does not drop below 20°C and so will never reach 18°C. $\boxed{\checkmark}$

(b) (i)

$$y = \frac{x^2}{4a}$$
$$\frac{dy}{dx} = \frac{x}{2a}$$

At $T$,

$$\frac{dy}{dx} = \frac{2at}{2a}$$
$$= t. \quad \boxed{\checkmark}$$

Now

$$y - at^2 = t(x - 2at)$$
$$y - at^2 = tx - 2at^2$$

so

$$y - tx + at^2 = 0. \quad \boxed{\checkmark}$$

(ii) Let $x = 0$

so $y = -at^2$

$R$ is the point $(0, -at^2)$. $\boxed{\surd}$

(iii) $R$ lies on $PQ$.

$$y - \tfrac{1}{2}(p+q)x + apq = 0$$

$$-at^2 + apq = 0 \quad \boxed{\surd}$$

$$t^2 = pq, \quad a \neq 0$$

$$\frac{t}{p} = \frac{q}{t}$$

So $p, t,$ and $q$ form a geometric sequence. $\boxed{\surd}$

QUESTION SIX

(a) (i) Area of minor segment $= \frac{1}{2}r^2(\theta - \sin\theta)$

Area of major segment $= \pi r^2 - \frac{1}{2}r^2(\theta - \sin\theta)$

$$\text{Ratio of areas} = \frac{\pi r^2 - \frac{1}{2}r^2(\theta - \sin\theta)}{\frac{1}{2}r^2(\theta - \sin\theta)} \quad \boxed{\surd}$$

$$= \frac{2\pi - \theta + \sin\theta}{\theta - \sin\theta} \quad \boxed{\surd}$$

(ii) ($\alpha$)

$$\frac{2\pi - \theta + \sin\theta}{\theta - \sin\theta} = \frac{\pi - 1}{1}$$

$$\pi\theta - \pi\sin\theta - \theta + \sin\theta = 2\pi - \theta + \sin\theta$$

$$\theta - 2 - \sin\theta = 0 \quad \boxed{\surd}$$

($\beta$) Let $f(\theta) = \theta - 2 - \sin\theta$

$$f(2) = -\sin 2$$
$$\doteqdot -0{\cdot}909$$
$$< 0$$
$$f(3) = 1 - \sin 3$$
$$\doteqdot 0{\cdot}859$$
$$> 0.$$

So the root lies between $\theta = 2$ and $\theta = 3$, $\boxed{\surd}$

($\gamma$) $f(\theta) = \theta - 2 - \sin\theta$

$f'(\theta) = 1 - \cos\theta$.

Let $\theta_0$ be the first approximation.

$$\theta_1 = \theta_0 - \frac{\theta_0 - 2 - \sin\theta_0}{1 - \cos\theta_0}$$

$$\theta_1 = 2{\cdot}5 - \frac{2{\cdot}5 - 2 - \sin 2{\cdot}5}{1 - \cos 2{\cdot}5} \quad \boxed{\surd}$$

$$\doteqdot 2{\cdot}55$$

($\delta$) When $\theta = 2{\cdot}5$,
$|\theta - 2 - \sin\theta| \doteqdot 0{\cdot}09847$.

($\varepsilon$) When $\theta = 2{\cdot}55$,
$|\theta - 2 - \sin\theta| \doteqdot 0{\cdot}00768$. So $\theta = 2{\cdot}55$ yields a smaller value. $\boxed{\surd}$

(b) (i) $\angle PAF = \angle PBF$ angles at circumference standing on the same arc $\boxed{\surd}$
$\angle PAF = \alpha$.

(ii) $\angle ANB = \angle AMB$ (both given as rightangles)..
These lie on the same interval $AB$ and so A,N,M and B are concyclic. $\boxed{\surd}$

(iii) $\angle NBM = \angle MAN$ (angles standing on the same arc of circle $ANMB$) $\boxed{\surd}$
$\angle NBM = \alpha$.

(iv) $\triangle BHM \equiv \triangle BFM$ ($AAS$ test) $\boxed{\surd}$
$HM = MF$ (matching sides of congruent triangles) $\boxed{\surd}$

(v) $\angle APB$ stands on fixed chord $AB$ and its size is independent of the position of $P$ (angles at circumference standing on the same chord). So $\alpha$ is independent of the position of $P$. $\boxed{\surd}$

QUESTION SEVEN

(a) (i) For $A$: $y = -\dfrac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha \cdots (1)$

For $B$: $y = -\dfrac{gx^2}{2V^2}\sec^2\beta + x\tan\beta \cdots (2)$

At $R$ the coordinates are identical, so substitute (1) in (2).

$$-\frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha = -\frac{gx^2}{2V^2}\sec^2\beta + x\tan\beta \quad \boxed{\surd}$$

$$\frac{gx^2}{2V^2}\left(\sec^2\alpha - \sec^2\beta\right) = x\left(\tan\alpha - \tan\beta\right)$$

$$\frac{gx}{2V^2}\left(\tan^2\alpha - \tan^2\beta\right) = \left(\tan\alpha - \tan\beta\right),\ x \neq 0 \quad \boxed{\surd}$$

$$\frac{gx}{2V^2} = \frac{(\tan\alpha - \tan\beta)}{(\tan^2\alpha - \tan^2\beta)}$$

$$x = \frac{2V^2}{g} \times \frac{1}{\tan\alpha + \tan\beta},\ \tan\alpha \neq \tan\beta$$

$$= \frac{2V^2}{g} \times \frac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

$$= \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)} \quad \boxed{\surd}$$

(ii) ($\alpha$) $x = V(t-T)\cos\beta$. $\boxed{\surd}$

($\beta$) When $A$ is at $R$:

$$Vt\cos\alpha = \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}$$

$$t = \frac{2V\cos\beta}{g\sin(\alpha+\beta)} \quad \cdots (3) \quad \boxed{\surd}$$

When $B$ is at $R$:

$$V(t-T)\cos\beta = \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}$$

$$t - T = \frac{2V\cos\alpha}{g\sin(\alpha+\beta)}$$

$$T = t - \frac{2V\cos\alpha}{g\sin(\alpha+\beta)}$$

$$= \frac{2V\cos\beta}{g\sin(\alpha+\beta)} - \frac{2V\cos\alpha}{g\sin(\alpha+\beta)},\ \text{from (3)}$$

$$= \frac{2V(\cos\beta - \cos\alpha)}{g\sin(\alpha+\beta)} \quad \boxed{\surd}$$

(b) (i) Prove by mathematical induction the proposition that for all positive integers $n$,

$$\sin(n\pi + x) = (-1)^n \sin x, \text{ for } 0 < x < \tfrac{\pi}{2}.$$

A. When $n = 1$,

$$\begin{aligned} LHS &= \sin(\pi + x) \\ &= -\sin x \\ &= RHS. \end{aligned}$$

The proposition is true for $n = 1$. $\boxed{\surd}$

$B$. Assume the proposition is true for some positive integer $k$ so that
$\sin(k\pi + x) = (-1)^k \sin x \ \cdots (*)$
We are required to prove the proposition true for $n = k + 1$.
That is, $\sin[(k+1)\pi + x] = (-1)^{k+1} \sin x$. $\boxed{\surd}$

Now
$$\begin{aligned} LHS &= \sin[(k+1)\pi + x] \\ &= \sin[\pi + (k\pi + x)] \quad \boxed{\surd} \\ &= \sin\pi\cos(k\pi + x) + \cos\pi\sin(k\pi + x) \\ &= -1 \times \sin(k\pi + x) \\ &= -1 \times (-1)^k \sin x, \text{ from } (*) \quad \boxed{\surd} \\ &= (-1)^{k+1} \sin x \\ &= RHS \end{aligned}$$

It follows from $A$ and $B$ by mathematical induction that for all positive integers $n$, $\sin(n\pi + x) = (-1)^n \sin x$, for $0 < x < \frac{\pi}{2}$.

(ii)
$$\begin{aligned} S &= \sin(\pi + x) + \sin(2\pi + x) + \sin(3\pi + x) + \cdots + \sin(n\pi + x) \\ &= -\sin x + \sin x - \sin x + \cdots + \sin(n\pi + x) \end{aligned}$$

When $n$ is odd $\quad S = -\sin x$

so $\quad -1 < S < 0$, for $0 < x < \frac{\pi}{2}$. $\boxed{\surd}$

When $n$ is even $\quad S = 0$.

So $\quad -1 < S \leq 0$. $\boxed{\surd}$

GJ