CATHOLIC SECONDARY SCHOOLS ASSOCIATION

2010 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS EXTENSION 1

Question 1 (12 marks)
(a) (2 marks)
Outcomes assessed: PE2
Targeted Performance Bands: E2-E3

Criteria	Marks
• uses correct formula for division of interval or progress using other correct method	1
• finds correct coordinates from working	1

Sample Answer:

$A(x_1, y_1) = (-2, -1)$ and $B(x_2, y_2) = (1, 5)$; Q divides AB externally ie $m:n = 5:-2$

$$x_Q = \frac{-2\times -2 + 5\times 1}{5+(-2)} = \frac{9}{3} = 3 \qquad y_Q = \frac{-2\times -1 + 5\times 5}{5+(-2)} = \frac{27}{3} = 9$$

$\therefore Q$ has coordinates $(3, 9)$

(b) (2 marks)
Outcomes assessed: PE2
Targeted Performance Bands: E2-E3

Criteria	Marks
• correct trigonometric substitution	1
• completes the proof	1

Sample Answer:

$$\begin{aligned}
\text{LHS} &= \frac{\sin 2x}{1+\cos 2x} \\
&= \frac{2\sin x\cos x}{1+2\cos^2 x - 1} \\
&= \frac{2\sin x\cos x}{2\cos^2 x} \\
&= \frac{\sin x}{\cos x} \\
&= \tan x \\
&= \text{RHS}
\end{aligned}$$

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(c) (3 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• establishes correct quadratic or other correct significant step towards solution	1
• further progress towards solution	1
• finds correct solution	1

Sample Answer:

$$\frac{2x}{x-1} \geq 1 \quad \text{multiply by } (x-1)^2 \text{ with } x \neq 1$$
$$2x(x-1) \geq (x-1)^2$$
$$2x(x-1)-(x-1)^2 \geq 0$$
$$(x-1)\big(2x-(x-1)\big) \geq 0$$
$$(x-1)(x+1) \geq 0$$
Solution is $x \leq -1$ or $x > 1$

(d) (2 marks)
Outcomes assessed: HE4
Targeted Performance Bands: E2-E3

Criteria	Marks
• gives correct exact trigonometric value	1
• correctly evaluates exact inverse trigonometric value	1

Sample Answer:

$$\sin^{-1}\left(\sin\frac{7\pi}{6}\right) = \sin^{-1}\left(\frac{-1}{2}\right)$$
$$= \frac{-\pi}{6}$$

(e) (3 marks)
Outcomes assessed: HE6
Targeted Performance Bands: E2-E3

Criteria	Mark
• rewrites the integral using the substitution	1
• finds the correct primitive	1
• gives final result	1

Sample Answer:

$$\int \frac{dx}{x(\ln 3x)^2} = \int \frac{du}{u^2} \qquad u = \ln 3x$$
$$= -\frac{1}{u} + C \qquad \frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{dx}{x}$$
$$= \frac{-1}{\ln 3x} + C$$

Question 2 (12 marks)
(a) (i) (1 mark)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• gives correct result (correct numerical equivalence)	1

Sample Answer:

$$\begin{aligned}(n-1)! &= 9!\\ &= 362880\end{aligned}$$

(a) (ii) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• significant progress towards result	1
• gives correct result (correct numerical equivalence)	1

Sample Answer:

Number of arrangements without restrictions 9!
Number of arrangements if Gemma, Pasha and Ricky sit together is $3!\times 7!$
If Gemma, Pasha and Ricky sit separately then:
P(all 3 separate) = 1 − P(all together)

$$\begin{aligned}&= 1-\frac{3!\times 7!}{9!}\\ &= 1-\frac{6}{9\times 8}\\ &= \frac{11}{12}\end{aligned}$$

(b) (2 marks)
Outcomes assessed: PE5, HE7
Targeted Performance Bands: E3-E4

Criteria	Mark
• establishes correct differential relationship or progress toward result	1
• finds correct radius from working	1

Sample Answer

$$A = 4\pi r^2 \qquad \Rightarrow \frac{dA}{dr} = 8\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr}\times\frac{dr}{dt}$$

$$= 8\pi r\frac{dr}{dt} \qquad \text{but } \frac{dA}{dt} = \frac{dr}{dt}$$

$$\therefore 1 = 8\pi r$$

$$r = \frac{1}{8\pi}\text{ cm}$$

(c) (3 marks)

Outcomes assessed: HE4

Targeted Performance Bands: E2-E3

Criteria	Marks
• uses appropriate substitution or other progress towards solution	1
• further progress towards solution	1
• establishes correct expression	1

Sample Answer:

$$\tan^{-1} x = \tan^{-1} y + \frac{\pi}{4} \quad \Rightarrow \quad \tan^{-1} x - \tan^{-1} y = \frac{\pi}{4}$$

Let $\tan^{-1} x = A$ and $\tan^{-1} y = B \Rightarrow$ ie $x = \tan A$ and $y = \tan B$

$$\therefore A - B = \frac{\pi}{4}$$

Take the tan of both sides

$$\tan(A - B) = \tan\left(\frac{\pi}{4}\right)$$

$$\frac{\tan A - \tan B}{1 + \tan A \tan B} = \tan\left(\frac{\pi}{4}\right)$$

$$\frac{x - y}{1 + xy} = 1$$

$$x - y = 1 + xy$$

$$xy + y = x - 1$$

$$y(x + 1) = x - 1$$

$$y = \frac{x - 1}{x + 1}$$

(d) (i) (1 mark)

Outcomes assessed: HE3

Targeted Performance Bands: E2-E3

Criteria	Mark
• justifies the equation	1

Sample Answer:

$$T = 25 + 1315e^{-kt} \quad \Rightarrow \quad 1315e^{-kt} = T - 25$$

$$\frac{dT}{dt} = -1315ke^{-kt}$$

$$= -k(T - 25)$$

Also when $t = 0$, $T = 1340$ ie $1340 = 25 + 1315e^{0}$ which is true.

$\therefore$ $T = 25 + 1315e^{-kt}$ satisfies this information

(d) (ii) (3 marks)

Outcomes assessed: HE3

Targeted Performance Bands: E3-E4

Criteria	Marks
• uses information to establish value for k or other progress towards solution	1
• further progress towards solution	1
• finds the correct time	1

Sample Answer:

When $t = 12$, $T = 1010$ ie $1010 = 25 + 1315e^{-12k}$

$$985 = 1315e^{-12k}$$

$$\frac{985}{1315} = e^{-12k}$$

$$-12k = \ln\left(\frac{197}{263}\right)$$

$$k = \frac{-1}{12}\ln\left(\frac{197}{263}\right)$$

$$= 0.024079...$$

When $T = 60$; $60 = 25 + 1315e^{-kt}$

$$35 = 1315e^{-0.024t}$$

$$\frac{35}{1315} = e^{-0.024t}$$

$$-0.024t = \ln\left(\frac{7}{263}\right)$$

$$t = \frac{-1}{0.024}\ln\left(\frac{7}{263}\right)$$

$= 150.5965769...$ OR $t = 151.09349...$ if using $k = 0.024$

$\therefore t = 151$ minutes

Question 3 (12 marks)
(a) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• uses remainder theorem or other progress towards solution	1
• establishes correct conclusion	1

Sample Answer:

$P(x) = (2x^2 + x + 3)Q(x) + (4x - 1)$

$Q(x)$ has remainder 1 when divided by $(x+2)$

ie $Q(-2) = 1$

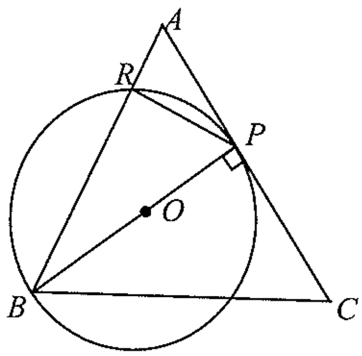
$$\begin{aligned}\therefore\ P(-2) &= \left(2\times(-2)^2 + (-2) + 3\right)\times 1 + \left(4\times(-2) - 1\right)\\ &= (9) + (-9)\\ &= 0\end{aligned}$$

Since $P(-2) = 0$ by the Factor Theorem $(x+2)$ is a factor of $P(x)$.

(b) (i) (1 mark)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• applies theorem correctly	1

Sample Answer:



AC is a tangent to the circle with diameter BP.

$\therefore\ \angle RPA = \angle RBP$ (angle between tangent and chord is equal to the angle in the alternate segment)

(b) (ii) (3 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• correctly identifies one pair of angles	1
• correctly identifies second pair of angles or other progress towards the proof	1
• completes the proof	1

Sample Answer:

In ΔBRP and ΔBPC

$\angle BPC = 90^\circ$ (tangent is perpendicular to the radius drawn from the point of contact)

$\angle BRP = 90^\circ$ (angle in a semi-circle is a right angle)

$\therefore\ \angle BRP = \angle BPC$

$\angle RBP = \angle PBC$ (PB bisects $\angle RBC$ given ΔABC is isosceles and $BP \perp AC$)

$\therefore\ \Delta BRP$ is similar to ΔBPC (equiangular)

(c) (i) (2 marks)
Outcomes assessed: PE3, PE4
Targeted Performance Bands: E2-E3

Criteria	Marks
• progress towards finding the coordinates	1
• derives correct coordinates	1

Sample Answer:

Equation of tangent at P is $y = px - ap^2$ and equation of tangent at Q is $y = qx - aq^2$

solve simultaneously $\quad (p-q)x = a(p^2 - q^2)$

$$x = \frac{a(p-q)(p+q)}{(p-q)}$$

$$x = a(p+q)$$

Substitute for x: $y = p\big(a(p+q)\big) - ap^2$

$y = apq$

$\therefore\ T$ is $\big(a(p+q), apq\big)$

(c) (ii) (2 marks)
Outcomes assessed: PE3, PE4
Targeted Performance Bands: E3-E4

Criteria	Marks
• states the gradient of the normal	1
• equates gradients to show the result	1

Sample Answer:

The gradient of the tangent at P is p $\therefore$ gradient of normal is $\dfrac{-1}{p}$.

The gradient of the chord PQ is $\dfrac{p+q}{2}$ $\therefore$ $\dfrac{p+q}{2} = -\dfrac{1}{p}$

ie $p + q + \dfrac{2}{p} = 0$

(c) (iii) (2 marks)

Outcomes assessed: PE3, PE4

Targeted Performance Bands: E3-E4

Criteria	Marks
• progress towards result	1
• establishes the correct locus	1

Sample Answer:

At T $x = a(p+q), y = apq$ and from (ii) $p+q = \frac{-2}{p}$

$$\therefore x = \frac{-2a}{p} \text{ ie } p = \frac{-2a}{x}$$

also $q = \frac{x}{a} - p$ ie $q = \frac{x}{a} - \frac{-2a}{x} = \frac{x^2+2a^2}{ax}$

$$y = a \times \frac{-2a}{x} \times \frac{x^2+2a^2}{ax}$$

$$= \frac{-2a^2x^2 - 4a^4}{ax^2}$$

$$\therefore y = \frac{-4a^3}{x^2} - 2a$$

Question 4 (12 marks)

(a) (3 marks)

Outcomes assessed: HE3, HE7

Targeted Performance Bands: E3-E4

Criteria	Marks
• progress towards solution	1
• further progress towards solution	1
• finds correct answer (correct numerical equivalence)	1

Sample Answer:

Total of 10 games – Harry wins 5 out of the first 9 and the last game

For the first 9 games consider the binomial probability of winning 5 from 9 with $p = \frac{2}{3}$

$$P(\text{winning } 5) = {}^9C_5\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^5$$

$$= \frac{9!}{5!4!} \times \frac{1}{3^4} \times \frac{2^5}{3^5}$$

$$= \frac{9 \times 7 \times 2^6}{3^9}$$

$$= \frac{448}{2187}$$

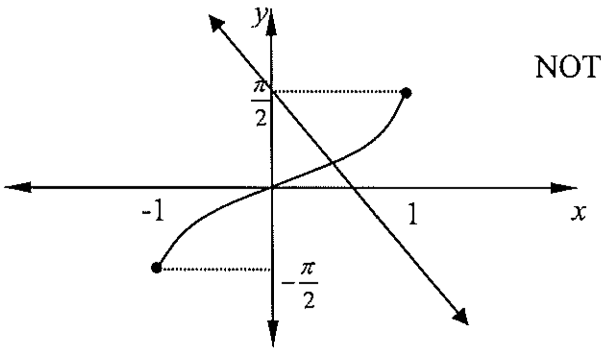
Harry wins the last game $\therefore$ probability of winning 6 games to 4 is $\frac{448}{2187} \times \frac{2}{3} = \frac{896}{6561}$

(b) (i) (2 marks)
Outcomes assessed: PE2, HE7
Targeted Performance Bands: E2-E3

Criteria	Marks
• draws correct graph or other progress towards solution	1
• explains the conclusion	1

Sample Answer:

To solve $\sin^{-1} x + x - \dfrac{\pi}{2} = 0$ consider the graphs of $y = \sin^{-1} x$ and $y = -x + \dfrac{\pi}{2}$



NOT TO SCALE

There is only one point of intersection of the two graphs at a point where x is positive.

$\therefore\ \sin^{-1} x + x - \dfrac{\pi}{2} = 0$ has only one real positive root.

(b) (ii) (3 marks)
Outcomes assessed: PE3, HE7
Targeted Performance Bands: E2-E3

Criteria	Marks
• progress towards solution using Newton's Method	1
• further progress towards solution	1
• finds correct approximation (correct numerical equivalence)	1

Sample Answer:

$$f(x) = \sin^{-1} x + x - \frac{\pi}{2} \quad \therefore f'(x) = \frac{1}{\sqrt{1-x^2}} + 1$$

$$\text{For } x_1 = 0.7 \quad f(x_1) = \sin^{-1} 0.7 + 0.7 - \frac{\pi}{2} = -0.09539883...$$

$$f'(x_1) = \frac{1}{\sqrt{1-0.7^2}} + 1 = 2.40028008...$$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.7 - \frac{-0.09539883...}{2.40028008...}$$

$$= 0.73974...$$

$$= 0.74$$

(c) (i) (2 marks)
Outcomes assessed: PE2, PE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes correct relationship	1
• finds correct values (correct numerical equivalence)	1

Sample Answer:

Series is geometric with $r = -\tan^2 x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

For a limiting sum $|r| < 1$, ie consider $-1 < -\tan^2 x < 1$

If $-1 < -\tan^2 x < 1$ then $1 > \tan^2 x > -1$, ie $-1 < \tan^2 x < 1$

Since $\tan^2 x \ge 0$, solve $0 \le \tan^2 x < 1$ for x

$\tan x$ is an increasing function for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

and $\tan\left(\frac{-\pi}{4}\right) = -1,\ \tan(0) = 0,\ \tan\left(\frac{\pi}{4}\right) = 1$

ie for $-\frac{\pi}{4} < x < \frac{\pi}{4},\ 0 \le \tan^2 x < 1$

$\therefore$ for a limiting sum $-\frac{\pi}{4} < x < \frac{\pi}{4}$

(c) (ii) (2 marks)
Outcomes assessed: PE2, HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• applies correct formula	1
• correctly simplifies the expression	1

Sample Answer:

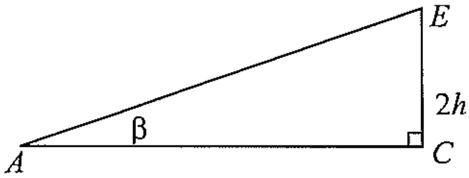
$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{\tan^2 x}{1-\left(-\tan^2 x\right)} \\ &= \frac{\tan^2 x}{1+\tan^2 x} \\ &= \frac{\tan^2 x}{\sec^2 x} \\ &= \frac{\sin^2 x}{\cos^2 x} \times \cos^2 x \\ &= \sin^2 x \end{aligned}$$

(a) (i) (1 mark)
Outcomes assessed: PE2, HE7
Targeted Performance Bands: E2-E3

Criteria	Marks
• derives correct result	1

Sample Answer:

In $\Delta ACE,\ \tan\beta = \dfrac{2h}{AC} \quad \Rightarrow \quad AC = 2h\cot\beta$



(a) (ii) (2 marks)
Outcomes assessed: PE2, HE7
Targeted Performance Bands: E2-E3

Criteria	Marks
• derives correct result for AB	1
• derives correct result for BC	1

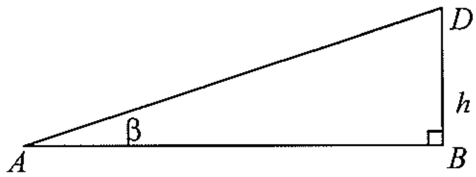
Sample Answer:

In $\Delta ABD,\ \tan\beta = \dfrac{h}{AB} \quad \Rightarrow \quad AB = h\cot\beta$

Also $BC = DF$

In $\Delta DEF,\ \tan\alpha = \dfrac{h}{DF} \quad \Rightarrow \quad DF = h\cot\alpha$

$\therefore\ BC = h\cot\alpha$



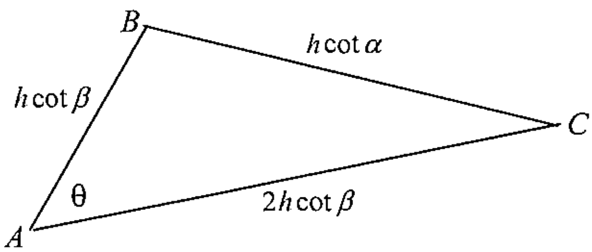
(a) (iii) (2 marks)
Outcomes assessed: PE2, HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• applies the Cosine Rule to correct triangle	1
• shows correct result	1

Sample Answer:

In ΔABC

$$\begin{aligned}\cos\theta &= \frac{AC^2 + AB^2 - BC^2}{2AC \times AB}\\ &= \frac{4h^2\cot^2\beta + h^2\cot^2\beta - h^2\cot^2\alpha}{4h^2\cot^2\beta}\\ &= \frac{5h^2\cot^2\beta - h^2\cot^2\alpha}{4h^2\cot^2\beta}\\ &= \frac{h^2(5\cot^2\beta - \cot^2\alpha)}{4h^2\cot^2\beta}\\ &= \frac{(5\cot^2\beta - \cot^2\alpha)}{4\cot^2\beta}\end{aligned}$$



(b) (3 marks)

Outcomes assessed: PE2, HE3

Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes correct terms or coefficients for comparison or other progress towards result	1
• significant progress toward the result	1
• finds correct values (correct numerical equivalence)	1

***Sample Answer*:**

Consider the 6th, 7th and 8th terms of the expansion of $(2+bx)^{11}$

$T_6 = {}^{11}C_5 \times 2^6 \times (bx)^5$

$T_7 = {}^{11}C_6 \times 2^5 \times (bx)^6$

$T_8 = {}^{11}C_7 \times 2^4 \times (bx)^7$

Take coefficients of T_6 and T_8, and compare to T_7

Consider $T_7 > T_6$ ie ${}^{11}C_6 \times 2^5 \times b^6 > {}^{11}C_5 \times 2^6 \times b^5$

$$\therefore b > \frac{{}^{11}C_5 \times 2}{{}^{11}C_6} = 2$$

Similarly for $T_7 > T_8$ ie ${}^{11}C_6 \times 2^5 \times b^6 > {}^{11}C_7 \times 2^4 \times b^7$

$$\therefore b < \frac{{}^{11}C_6 \times 2}{{}^{11}C_7} = 2.8$$

$\therefore$ seventh term has the largest coefficient for $2 < b < 2.8$

(c) (i) (2 marks)

Outcomes assessed: HE3

Targeted Performance Bands: E2-E3

Criteria	Mark
• uses correct formula or progress using other correct method	1
• establishes correct result	1

Sample Answer:

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2 x$$

$$\frac{1}{2}v^2 = \frac{-n^2x^2}{2} + C$$

at $x = a,\ v = 0$ since velocity is zero at the extremities

$$0 = \frac{-n^2a^2}{2} + C \quad \Rightarrow \quad C = \frac{n^2a^2}{2}$$

$$\frac{1}{2}v^2 = \frac{-n^2x^2}{2} + \frac{n^2a^2}{2} \qquad \text{ie } v^2 = n^2a^2 - n^2x^2$$

$$\therefore v^2 = n^2(a^2 - x^2)$$

(c) (ii) (2 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• uses correct substitutions or other progress towards result	1
• finds correct values	1

Sample Answer:

$v = 6$ when $x = 4 \quad \Rightarrow \quad 36 = n^2(a^2 - 16)$

maximum velocity is at the centre of the motion

$\therefore \; v = 10$ when $x = 0 \quad \Rightarrow \quad 100 = n^2a^2$

solving simultaneously $\Rightarrow \quad 36 = 100 - 16n^2$

$$16n^2 = 64$$

$$n^2 = 4$$

hence $a^2 = 25$

$\therefore$ extremities of motion are $a = 5$ and $a = -5$

Question 6 (12 marks)
(a) (i) (1 mark)
Outcomes assessed: HE7
Targeted Performance Bands: E2-E3

Criteria	Marks
• correctly justifies the result	1

Sample Answer:

$${}^nC_k = \frac{n!}{k!(n-k)!} \qquad {}^nC_{n-k} = \frac{n!}{(n-k)!(n-(n-k))!}$$

$$= \frac{n!}{(n-k)!k!}$$

$$\therefore \; {}^nC_k = {}^nC_{n-k}$$

(a) (ii) (2 marks)
Outcomes assessed: HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• consider terms in binomial expansion or other progress towards solution	1
• establishes the result	1

Sample Answer:

Consider terms in the expansion of $(1+x)^{2n}$

$$T_{k+1} = {}^{2n}C_k 1^{2n-k} x^k$$
$$= {}^{2n}C_k x^k$$

coefficient of x^n is: ${}^{2n}C_n = \dfrac{(2n)!}{n!(2n-n)!}$

$$= \frac{(2n)!}{(n!)^2}$$

Consider the expansion of $(1+x)^n(1+x)^n$

$$(1+x)^n(1+x)^n = \left[{}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_{n-1}x^{n-1} + {}^nC_nx^n\right] \times \left[{}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_{n-1}x^{n-1} + {}^nC_nx^n\right]$$

coefficients of x^n are: ${}^nC_0\,{}^nC_n + {}^nC_1\,{}^nC_{n-1} + {}^nC_2\,{}^nC_{n-2} + \ldots\ldots + {}^nC_{n-1}\,{}^nC_1 + {}^nC_n\,{}^nC_0$

$$= \left({}^nC_0\right)^2 + \left({}^nC_1\right)^2 + \left({}^nC_2\right)^2 + \ldots + \left({}^nC_{n-1}\right)^2 + \left({}^nC_n\right)^2 \qquad \text{using (i)}$$

$$= \sum_{k=0}^{n} \left({}^nC_k\right)^2$$

$\therefore$ Since $(1+x)^n(1+x)^n = (1+x)^{2n}$ equating coefficients gives $\displaystyle\sum_{k=0}^{n}\left({}^nC_k\right)^2 = \frac{(2n)!}{(n!)^2}$

(b) (i) (1 mark)
Outcomes assessed: HE4
Targeted Performance Bands: E3-E4

Criteria	Marks
• differentiates or other method to explain correct conclusion	1

Sample Answer:

$f(x) = x^3 + x + 1$

$f'(x) = 3x^2 + 1 > 0$ for all x since $x^2 \geq 0$

$\therefore$ $f(x)$ is monotonic increasing and thus has an inverse function, $f^{-1}(x)$, for all x.

(b) (ii) (2 marks)
Outcomes assessed: HE4
Targeted Performance Bands: E3-E4

Criteria	Marks
• identifies that curves intersect on $y = x$ or other progress towards result	1
• finds correct point of intersection	1

Sample Answer:

$f(x)$ and $f^{-1}(x)$ intersect on $y = x$

$\therefore$ solve $x^3 + x + 1 = x$

$x^3 + 1 = 0$

$x^3 = -1$

$x = -1$

$\therefore$ Point of intersection is $(-1, -1)$

(c) (i) (2 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• progress towards result	1
• shows correct result	1

Sample Answer:

$x = vt\cos\theta$

$\therefore t = \dfrac{x}{v\cos\theta}$

substitute into $y = vt\sin\theta - \dfrac{1}{2}gt^2$

$$y = \frac{x}{v\cos\theta}(v\sin\theta) - \frac{1}{2}g\left(\frac{x}{v\cos\theta}\right)^2$$

$$= x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta}$$

$$= x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta$$

(c) (ii) (2 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• progress toward solution	1
• substitutes and simplifies to obtain desired result	1

Sample Answer:

At the point P, $x=10$, $y=15$ and given $g=9.8$, $v=7\sqrt{10}$

Using (i) $$15=10\times\tan\theta-\frac{9.8(10)^2}{2\left(7\sqrt{10}\right)^2}\sec^2\theta$$

$$15=10\tan\theta-\frac{9.8\times100}{2\times49\times10}\left(1+\tan^2\theta\right)$$

$$15=10\tan\theta-1-\tan^2\theta$$

$$\tan^2\theta-10\tan\theta+16=0$$

$$(\tan\theta-8)(\tan\theta-2)=0$$

$$\therefore\tan\theta=8\text{ or }\tan\theta=2$$

since $\alpha<\beta$, $\tan\beta=8$ and $\tan\alpha=2$

(c) (iii) (2 marks)
Outcomes assessed: HE3, HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• significant progress towards solutions	1
• shows correct solution	1

Sample Answer:

Consider the two paths and find time travelled to reach P

Pebble 1: $v=7\sqrt{10}$, $\theta=\beta$, $\tan\beta=8$ and $x=10$

$$t_1=\frac{10}{7\sqrt{10}\cos\beta}=\frac{10\sec\beta}{7\sqrt{10}}\text{ and }\sec^2\beta=1+\tan^2\beta=65$$

$$\therefore t_1=\frac{10\sqrt{65}}{7\sqrt{10}}$$

$$=\frac{\sqrt{650}}{7}$$

Pebble 2: $v=7\sqrt{10}$, $\theta=\alpha$, $\tan\alpha=2$ and $x=10$

$$t_2=\frac{10}{7\sqrt{10}\cos\alpha}=\frac{10\sec\alpha}{7\sqrt{10}}\text{ and }\sec^2\alpha=1+\tan^2\alpha=5$$

$$\therefore t_2=\frac{\sqrt{50}}{7}$$

$$\therefore t_1-t_2=\frac{\sqrt{650}-\sqrt{50}}{7}\text{ seconds}$$

Question 7 (12 marks)
(a) (3 marks)
Outcomes assessed: HE2
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes the truth of $S(2)$	1
• establishes the correct relationship between $S(k)$ and $S(k+1)$	1
• deduces the required result	1

Sample Answer:

Let $S(n)$ be the statement $2n^2 > n^2 + n + 1$ for $n > 1$

Consider $S(2)$: $2 \times 2^2 = 8$ and $2^2 + 2 + 1 = 7$

$\therefore 2n^2 > n^2 + n + 1$ for $n = 2$ and hence $S(2)$ is true

Assume $S(k)$ is true: $2k^2 > k^2 + k + 1$ $\quad *$

RTP: $S(k+1)$ is true, ie prove $2(k+1)^2 > (k+1)^2 + (k+1) + 1$

$$
\begin{aligned}
2(k+1)^2 &= 2k^2 + 4k + 2 \\
&> k^2 + k + 1 + 4k + 2 \qquad \text{if } S(k) \text{ is true using } * \\
&= k^2 + 2k + 1 + 3k + 2 \\
&= (k+1)^2 + (k+1) + 1 + 2k
\end{aligned}
$$

$\therefore$ $2(k+1)^2 > (k+1)^2 + (k+1) + 1$ since $k > 0$

Hence if $S(k)$ is true then $S(k+1)$ is also true. Thus since $S(2)$ is true it follows by induction that $S(n)$ is true for positive integers $n > 1$.

(b) (i) (2 marks)
Outcomes assessed: PE2, HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• substitutes correctly	1
• shows correct result	1

Sample Answer:

$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$ and $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$

$$
\begin{aligned}
\text{LHS} &= 2\sinh x \cosh x \\
&= 2 \times \frac{1}{2}\left(e^x + e^{-x}\right) \times \frac{1}{2}\left(e^x - e^{-x}\right) \\
&= \frac{1}{2}\left(\left(e^x\right)^2 - \left(e^{-x}\right)^2\right) \\
&= \frac{1}{2}\left(e^{2x} - e^{-2x}\right) \\
&= \sinh(2x) \\
&= \text{RHS}
\end{aligned}
$$

(b) (ii) (2 marks)
Outcomes assessed: PE2, HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes correct equation using given substitutions	1
• shows correct result	1

Sample Answer:

$$p\cosh x + q\sinh x = r \text{ and } \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) \text{ and } \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

$$p \times \frac{1}{2}\left(e^x + e^{-x}\right) + q \times \frac{1}{2}\left(e^x - e^{-x}\right) = r$$

$$\frac{pe^x}{2} + \frac{pe^{-x}}{2} + \frac{qe^x}{2} - \frac{qe^{-x}}{2} = r$$

$$\frac{e^x}{2}(p+q) + \frac{1}{2e^x}(p-q) = r$$

$$e^{2x}(p+q) + (p-q) = 2re^x$$

$$(p+q)e^{2x} - 2re^x + (p-q) = 0$$

(b) (iii) (3 marks)
Outcomes assessed: PE2, HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• recognises that the equation is a quadratic or other progress towards the solution	1
• uses the discriminant or other progress towards the solution	1
• establishes correct conclusion	1

Sample Answer:

From (ii) the equation $p\cosh x + q\sinh x = r$ is equivalent to

$(p+q)e^{2x} - 2re^x + (p-q) = 0$, which is a quadratic in e^x

$$\begin{aligned}\Delta &= 4r^2 - 4(p+q)(p-q) \\ &= 4r^2 - 4(p^2 - q^2) \\ &= 4(r^2 - p^2 + q^2) \\ &= 0 \quad \text{since } p^2 = q^2 + r^2\end{aligned}$$

$\therefore$ the equation has only one solution

(b) (iv) (2 marks)

Outcomes assessed: PE2, HE7

Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes the correct equation or other progress towards solution	1
• finds the correct solution	1

Sample Answer:

For the equation $13\cosh x + 5\sinh x = 12 \Rightarrow p = 13, \quad q = 5, \quad r = 12$

$\therefore (p+q)e^{2x} - 2re^{x} + (p-q) = 0$ becomes $18e^{2x} - 24e^{x} + 8 = 0$

ie solve $9e^{2x} - 12e^{x} + 4 = 0$

$$(3e^{x} - 2)^{2} = 0$$

$$3e^{x} = 2$$

$$e^{x} = \frac{2}{3}$$

$$\therefore \quad x = \ln\left(\frac{2}{3}\right)$$

OR

Let $e^{2x} = y$ ie solve $9y^{2} - 12y + 4 = 0$

$$(3y-2)^{2} = 0$$

$$y = \frac{2}{3} \quad \Rightarrow \quad e^{x} = \frac{2}{3}$$

$$\therefore \quad x = \ln\left(\frac{2}{3}\right)$$

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Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Ten students are seated around a circular table.

(i) In how many ways can they be arranged? **1**

(ii) What is the probability that three particular students, Gemma, Pasha and Ricky, are not sitting together when the seats are randomly assigned. **2**

(b) A ball in the shape of a sphere has radius r centimetres at time t seconds. The surface area is changing as the radius changes over time. At a particular time, t seconds, the rate of change of the surface area is equal to the rate of change of the radius. **2**

Find the exact radius at this time.

(c) Find an expression for y in terms of x if for $x > 0$ and $y > 0$, **3**

$\tan^{-1} x = \tan^{-1} y + \dfrac{\pi}{4}$.

(d) A heated metal bar has a temperature of 1340°C when it is removed from a furnace. Its temperature T after t minutes in a room with a constant temperature of 25°C satisfies the equation $\dfrac{dT}{dt} = -k(T - 25)$, where k is a constant.

(i) Show that the equation $T = 25 + 1315e^{-kt}$ satisfies this information. **1**

(ii) The metal bar cools to 1010°C after 12 minutes. **3**

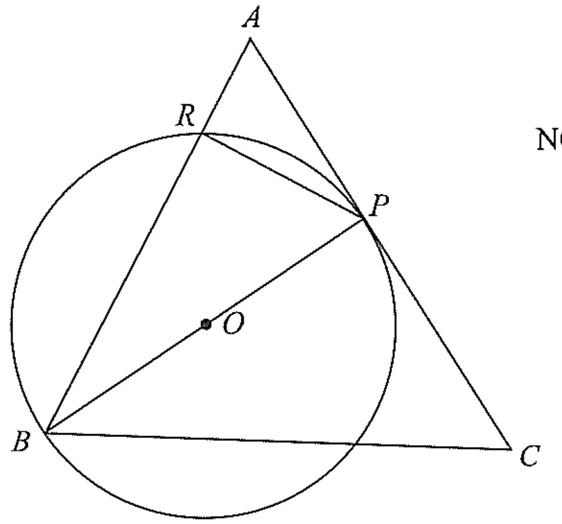
Find how long it will take for the bar to cool to 60°C, giving your answer correct to the nearest minute.

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) The polynomial $P(x)$ is expressed as $P(x) = (2x^2 + x + 3)Q(x) + (4x - 1)$. **2**

If $Q(x)$ leaves a remainder of 1 when divided by $(x+2)$, show that $(x+2)$ is a factor of $P(x)$.

(b) The diagram shows an isosceles triangle ABC, with $AB = BC$. The point P lies on AC and the point O lies on BP. A circle with centre O passes through B and P and cuts AB at R.



Copy or trace the diagram into your writing booklet.

(i) Explain why $\angle RPA = \angle RBP$. **1**

(ii) Hence, or otherwise, prove that ΔBRP is similar to ΔBPC. **3**

Question 3 continues on page 5