N.S.W. DEPARTMENT OF EDUCATION
HIGHER SCHOOL CERTIFICATE EXAMINATION 1984
MATHEMATICS - 3UNIT/4UNIT COMMON PAPER

(i.e. 3Unit Course - Additional Paper; 4Unit Course - First Paper)

INSTRUCTIONS. Time allowed - Two hours.

All questions may be attempted. All questions are of equal value. In every question, all necessary working should be shown.

Marks may not be awarded for careless or badly arranged work.

Standard integrals given may be used; approved slide-rules or silent calculators may be used.

QUESTION 1.

(i) Find the derivative (with respect to x) of

(a) $\sin^{-1} 2x$, for $|x| < \frac{1}{2}$, (b) $\dfrac{e^{2x}}{x^2 + 3}$.

(ii) Define the function f by $f(x) = x^3 + 3x^2 - 9x - 27$.

(a) Show that $(x - 3)$ is a factor of $x^3 + 3x^2 - 9x - 27$, and factorise this expression completely.

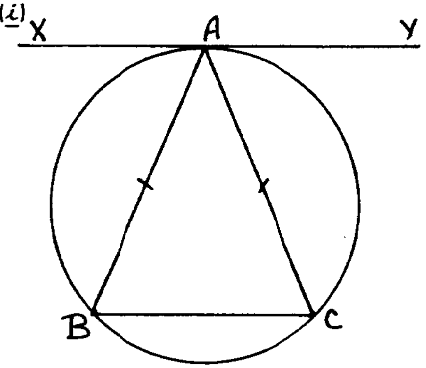
(b) Find where the graph of $y = f(x)$ meets the axes.

(c) Find the stationary points of f and determine their nature.

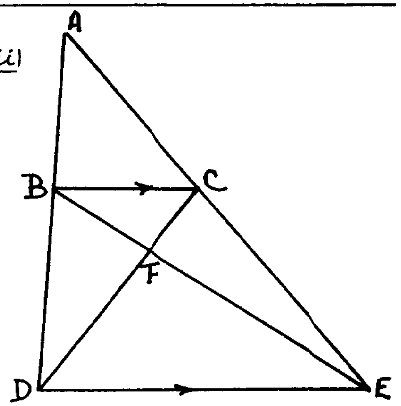
(d) Find the point(s) of inflexion (if any) of f.

QUESTION 2.

(i)



(ii)



(i) In the figure, it is given that $AB = AC$ and XY is tangent to circle ABC at A, prove that XY is parallel to BC.

(ii) In the figure, $BC || DE$ and $AB:BD = 3:5$.

Show that

(a) $\Delta ABC ||| \Delta ADE$, (b) $\Delta BFC ||| \Delta EFD$, (c) $DF:FC = 8:3$.

(iii) A particle executes simple harmonic motion with period T seconds and amplitude A cm. What is its maximum velocity?

QUESTION 3.

(i) The carbon isotope C^{14} decays at a rate proportional to its mass. Tree ring experiments suggest that 50% decay takes 5580 years. A fossil contains 30% of the amount of C^{14} in a similarly sized living organism. Estimate the age of the fossil.

(ii) A box contains ten tennis balls of which four have never been used. For the first game two balls are selected at random and, after play, are returned to the box. For the second game two balls are also selected at random from the box. Find the probability of each of the following events:

(a) precisely one of the balls selected for the first game has been used before;

(b) neither ball selected for the first game has been used before, but both balls selected for the second game have been used before the second game.

QUESTION 4.

(i) A spherical bubble is expanding so that its volume is increasing at the constant rate of 10 mm³ per second. What is the rate of increase of the radius when the surface area is 500 mm²? ($V = \frac{4}{3}\pi R^3$, $S = 4\pi R^2$.)

(ii) (a) Noting that $2\cos^2 x \equiv 1 + \cos 2x$,

prove that $8\cos^4 x \equiv 3 + 4\cos 2x + \cos 4x$.

(b) Sketch, on the same diagram, the curves

$y = \cos x$, $y = \cos^2 x$, for $0 \le x \le \frac{\pi}{2}$.

Find the area enclosed between these curves and the volume generated when this area is rotated about the x-axis.

QUESTION 5.

The straight line $y = mx + b$ meets the parabola $x = 2At$, $y = At^2$ in real distinct points P, Q which correspond respectively to the values $t = p, t = q$.

(a) Prove that $pq = -b/A$.

(b) Prove that $p^2 + q^2 = 4m^2 + (2b/A)$.

(c) Show that the equation of the normal to the parabola at P is

$$x + py = 2Ap + Ap^3.$$

(d) The point N is the point of intersection of the normals to the parabola at P and Q. Prove that the coordinates of N are $(-Apq(p + q),$ $A(2 + p^2 + pq + q^2))$, and express these coordinates in terms of $A, m,$ and b.

(e) Suppose that the chord PQ is free to move while maintaining a fixed gradient. Show that the locus of N is a straight line and verify that this straight line is a normal to the parabola.

QUESTION 6.

(i) Write down a formula for calculating $\frac{d}{dx}F(u)$ when u is a function of x.

Differentiate (with respect to x) $\{\tan^{-1}(\frac{x}{3})\}^2$

and hence find the exact value of $\frac{1}{\pi}\int_0^{\sqrt{3}} \dfrac{\tan^{-1}(\frac{x}{3})}{x^2 + 9}\,dx$.

(ii) (a) Consider the statement $\cos(\frac{\pi}{2} + A) = \pm\sin A$.
For which sign is this statement true for all A?
For which A is the statement true for both signs?

(b) Taking $A = 5\theta$ in (a) write down a value of B such that $-\sin 5\theta = \cos B$.
Hence find the least value of θ between 0 and 2π such that $\sin 5\theta + \cos 8\theta = 0$.

QUESTION 7.

(i) Assume that, for all real numbers x and all positive integers n,

$$(1 + x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

Show that (a) $0 = \sum_{r=0}^{n} (-1)^r \binom{n}{r}$,

and find simple expressions for (b) $\sum_{r=0}^{n} 2^r \binom{n}{r}$, (c) $\sum_{r=0}^{n} r\binom{n}{r}$.

(ii) It is given that $A > 0$, $B > 0$ and n is a positive integer.
(a) Divide $A^{n+1} - A^nB + B^{n+1} - B^nA$ by $A - B$, and deduce that
$$A^{n+1} + B^{n+1} \geq A^nB + B^nA.$$
(b) Using (a), show by mathematical induction that
$$\left(\frac{A + B}{2}\right)^n \leq \frac{A^n + B^n}{2}.$$

STANDARD INTEGRALS

$\int x^n dx = \frac{1}{n+1} x^{n+1}$, $n \neq -1$; $x \neq 0$, if $n < 0$.

$\int \frac{1}{x} dx = \log_e x$, $x > 0$. $\int e^{ax} dx = \frac{1}{a} e^{ax}$, $a \neq 0$. $\int \cos ax\, dx = \frac{1}{a} \sin ax$, $a \neq 0$.

$\int \sin ax\, dx = -\frac{1}{a} \cos ax$, $a \neq 0$. $\int \sec^2 ax\, dx = \frac{1}{a} \tan ax$, $a \neq 0$.

$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax$, $a \neq 0$.

$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$, $a \neq 0$. $\int \frac{1}{\sqrt{(a^2 - x^2)}} dx = \sin^{-1} \frac{x}{a}$, $a > 0$, $-a < x < a$.

$\int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \log_e\{x + \sqrt{(x^2 - a^2)}\}$, $|x| > |a|$.

$\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log_e\{x + \sqrt{(x^2 + a^2)}\}$