2007
Higher School Certificate
Trial Examination

Mathematics
Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- Write using black or blue pen
- Write your student number and/or name at the top of every page
- All necessary working should be shown in every question
- A table of standard integrals is provided

Total marks – 120

Attempt Questions 1 – 8

All questions are of equal value

This paper MUST NOT be removed from the examination room

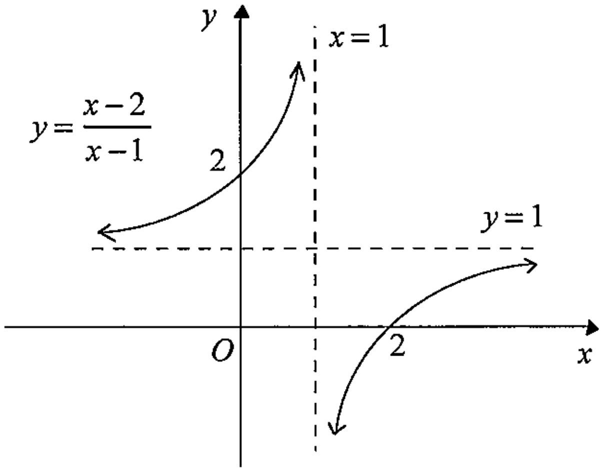
STUDENT NUMBER/NAME....................................

Student name / number ....

Marks

Question 1 **Begin a new booklet**

(a) The diagram below shows the graph of the function $f(x) = \dfrac{x-2}{x-1}$.



On separate diagrams, sketch the following graphs, showing clearly any intercepts on the axes and the equations of any asymptotes:

(i) $y = f(-x)$. **1**

(ii) $y = |f(x)|$. **1**

(iii) $y = f(|x|)$. **2**

(iv) $y = e^{f(x)}$. **2**

(b) The line $y = mx$ through the origin $O(0, 0)$ is tangent to the curve $y = \dfrac{x-2}{x-1}$, touching it at the point $P(x_1, y_1)$.

(i) By considering the gradient of *OP* in two different ways, show that $x_1^2 - 4x_1 + 2 = 0$. **2**

(ii) Hence find the two possible values of *m*. **2**

(c) Consider the function $y = e^{-2x}\tan x$ for $0 \le x < \frac{\pi}{2}$.

(i) Show that $\dfrac{dy}{dx} = e^{-2x}(1 - \tan x)^2$. **2**

(ii) Sketch the graph of the function showing the coordinates of the endpoint, the equation of the asymptote and the coordinates of the stationary point. **3**

Student name / number ....

Marks

Question 2 **Begin a new booklet**

(a)(i) Find $\int \frac{1+e^{x}}{1+e^{-x}}\,dx$. **1**

(ii) Find $\int \frac{1}{\sqrt{1+x}+\sqrt{x}}\,dx$. **2**

(b) Use the substitution $x=\sin\theta$ to find $\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$. **3**

(c) Evaluate $\int_0^{\sqrt{3}} \frac{x^3-8x^2+9x}{(1+x^2)(9+x^2)}\,dx$. **4**

(d)(i) Use the substitution $t=\tan\frac{x}{2}$ to show that $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx = 1$. **2**

(ii) Show that $\int_0^{a} f(x)\,dx = \int_0^{\frac{a}{2}} \{f(x)+f(a-x)\}\,dx$. **2**

(iii) Hence evaluate $\int_0^{\pi} \frac{x}{1+\sin x}\,dx$. **1**

Student name / number ....

Marks

Question 3 **Begin a new booklet**

(a)(i) Write down the expansion of $(1+ia)^4$ in ascending powers of a. **1**

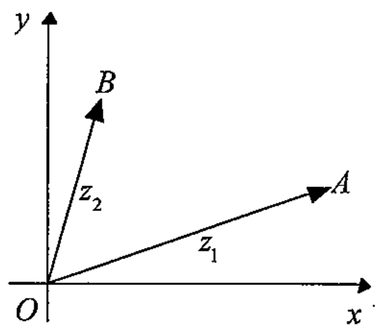
(ii) Hence find the values of a such that $(1+ia)^4$ is real. **2**

(b) The equation $\left(\sin^2\theta\right)z^2 - \left(\sin 2\theta\right)z + 1 = 0$, where $0 < \theta < \frac{\pi}{2}$, has roots α and β.

(i) Show that the roots of the equation are $\cot\theta + i$ and $\cot\theta - i$. **2**

(ii) Hence show that $\alpha^n + \beta^n = \dfrac{2\cos n\theta}{\sin^n\theta}$. **2**

(c) In the Argand diagram below vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the complex numbers $z_1 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $z_2 = \sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$ respectively.



(i) Show that $|z_2 - z_1| = \sqrt{2}$. **2**

(ii) Show that $z_2 - z_1 = i\,z_2$. **2**

(d)(i) On an Argand diagram shade the region where both $|z-2| \le 1$ and $\text{Re}(z) \le \frac{3}{2}$. **2**

(ii) Find the set of values of $\text{Arg}\, z$ for points in the shaded region. **2**

Student name / number ....

Marks

Question 4 **Begin a new booklet**

(a)(i) On the same diagram sketch the graphs of the ellipses $E_1 : \dfrac{x^2}{4}+\dfrac{y^2}{3}=1$ **4**

and $E_2 : \dfrac{x^2}{16}+\dfrac{y^2}{12}=1$, showing clearly the intercepts on the axes. Show the coordinates of the foci and the equations of the directrices of the ellipse E_1.

(ii) $P(2\cos p, \sqrt{3}\sin p)$, where $0 < p < \frac{\pi}{2}$, is a point on the ellipse E_1. Use differentiation to show that the tangent to the ellipse E_1 at P has equation **3**

$\dfrac{x\cos p}{2}+\dfrac{y\sin p}{\sqrt{3}}=1$.

(iii) The tangent to the ellipse E_1 at P meets the ellipse E_2 at the points $Q(4\cos q, 2\sqrt{3}\sin q)$ and $R(4\cos r, 2\sqrt{3}\sin r)$, where $-\pi < q < \pi$ and $-\pi < r < \pi$. Show that q and r differ by $\frac{2\pi}{3}$. **2**

(b) The hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ with eccentricity e, has one focus S on the positive x-axis and the corresponding directrix d cuts the asymptotes to the hyperbola at points P and Q in the first and fourth quadrants respectively.

(i) Show that PS is perpendicular to the asymptote through P and that $PS = b$. **3**

(ii) A circle with centre S touches the asymptotes of the hyperbola. Deduce that the points of contact are the points P and Q. **1**

(iii) The circle with centre S which touches the asymptotes of the hyperbola cuts the hyperbola at points R and T. If $b = a$, show that RT is a diameter of the circle. **2**

Student name / number ....

Marks

Question 5 **Begin a new booklet**

(a) When the polynomial $P(x)$ is divided by (x^2+1) the remainder is $Ax+B$.

(i) Show that $A=\dfrac{P(i)-P(-i)}{2i}$ and $B=\dfrac{P(i)+P(-i)}{2}$. **2**

(ii) If $P(x)$ is odd, find the remainder when $P(x)$ is divided by (x^2+1). **2**

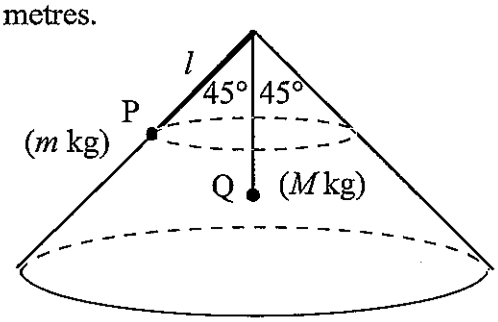
(b) The equation $x^4-5x+2=0$ has roots α, β, γ and δ.

(i) Show that the equation $x^4-5x+2=0$ has a real root between $x=0$ and $x=1$. **1**

(ii) Find the monic equation with roots α^2, β^2, γ^2 and δ^2. Hence or otherwise show that $\alpha^2+\beta^2+\gamma^2+\delta^2=0$. **2**

(iii) Find the number of non-real roots of $x^4-5x+2=0$, giving full reasons for your answer. **2**

(c) A right circular inverted cone has semi-vertical angle $45°$. There is a smooth hole in the top of the cone and a light, inextensible string passes through this hole. A particle Q of mass M kg is attached to one end of this string and hangs at rest inside the cone. A second particle P of mass m kg, attached to the other end of the string, travels in a horizontal circle around the smooth outside surface of the cone with constant angular velocity ω radians per second. The length of string between P and the hole is l metres.



(i) If the tension in the string is T Newtons, the force exerted by the surface on P is N Newtons, and the acceleration due to gravity is g ms^{-2}, explain why $T+N=\sqrt{2}mg$ and $T-N=ml\omega^2$. **2**

(ii) Find expressions for the product $l\omega^2$ and N in terms of M, m and g. **2**

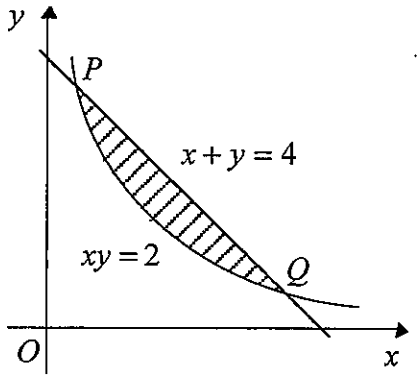
(iii) Deduce that $\dfrac{\sqrt{2}}{2}\le\dfrac{M}{m}\le\sqrt{2}$. **2**

Student name / number ....

Marks

Question 6 **Begin a new booklet**

(a)



The region in the first quadrant bounded by the line $x + y = 4$ and the rectangular hyperbola $xy = 2$ is rotated through one complete revolution about the y-axis.

(i) Use the method of cylindrical shells to show that the volume V of the solid formed is given by **3**

$$V = 2\pi \int_{2-\sqrt{2}}^{2+\sqrt{2}} (4x - x^2 - 2)\, dx .$$

(ii) Hence find the simplest exact numerical value of the volume of the solid formed. **3**

(b) $I_n = \int_0^1 x^n (1-x)^n \, dx$, $n = 0, 1, 2, \ldots$

(i) Using the substitution $u = \frac{1}{2} - x$, show that $I_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} (\frac{1}{4} - u^2)^n \, du$ and hence **4**

$I_n = \dfrac{n}{2(2n+1)} I_{n-1}$, $n = 1, 2, 3, \ldots$.

(ii) Show that $\int_0^1 x^5 (1-x)^5 \, dx = \dfrac{(5!)^2}{11!}$. **2**

(iii) Use the substitution $x = \sin^2(\frac{1}{2}t)$ to show that $\int_0^\pi \sin^{2n+1} t \, dt = 2^{2n+1} I_n$. **3**

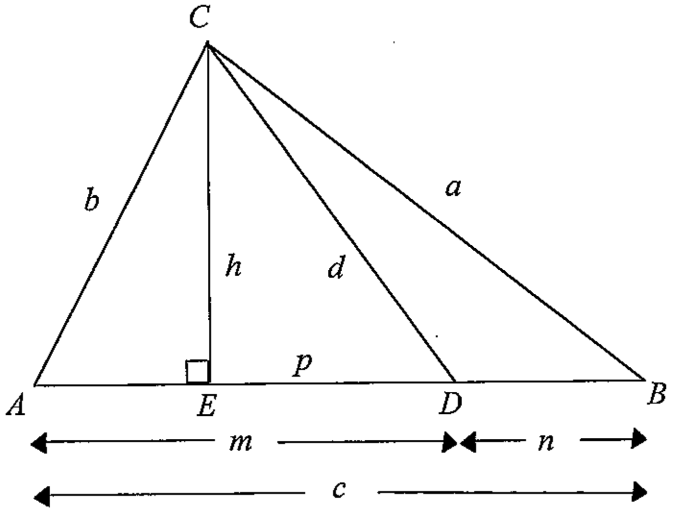
Hence deduce that $\int_0^\pi \sin^{2n+1} t \, dt = \dfrac{2^{2n+1}(n!)^2}{(2n+1)!}$, $n = 0, 1, 2, \ldots$

Student name / number ....

Marks

Question 7 **Begin a new booklet**

(a)



In ΔABC above, $BC = a$, $CA = b$ and $AB = c$. D is a point on AB such that $DA = m$, $DB = n$ and $DC = d$. CE is an altitude of the triangle with $CE = h$ and $ED = p$.

(i) Use Pythagoras' theorem in ΔCEA and ΔCED to show that $b^2 = d^2 + m^2 - 2mp$. **2**

(ii) Show similarly that $a^2 = d^2 + n^2 + 2np$. **2**

(iii) Hence show that $a^2 m + b^2 n = c(d^2 + mn)$. **1**

(iv) In the case where CD bisects $\angle BCA$, use the sine rule in ΔCDA and ΔCDB to show that $am = bn$. Hence show that in this case, $d^2 = ab - mn$. **4**

(b) In any single play of a game, n people throw a fair coin, where $n \geq 3$. The play of the game results in an '*odd one out*' if all but one of the coins show heads or all but one of the coins show tails.

(i) Show that in any single play of the game, the probability of an '*odd one out*' is $\dfrac{n}{2^{n-1}}$. **2**

(ii) Find the probability that there is at least one '*odd one out*' in N plays of the game. **1**

(iii) Find the probability that the first '*odd one out*' occurs on the N^{th} play of the game. **1**

(iv) Find the probability that the second '*odd one out*' occurs on the N^{th} play of the game. **2**

Student name / number ....

Marks

Question 8 **Begin a new booklet**

(a) A sequence of numbers x_n, $n = 1, 2, 3, \ldots$ is given by $x_1 = 1$ and

$$x_{n+1} = \frac{2x_n^{\ 3} + 8}{3x_n^{\ 2}}, \quad n = 1, 2, 3, \ldots$$

(i) Use Mathematical Induction to show that $x_n > 2$ for all positive integers $n \geq 2$. **3**

(ii) Hence show that $x_{n+1} < x_n$ for all positive integers $n \geq 2$. **2**

(b) Let $f(x) = \ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1} \frac{x^n}{n} \right)$ where n is a positive integer.

(i) Show that $f(x)$ is stationary at $x = 0$, and show that for $x > 0$, $f(x)$ is monotonic increasing if n is even, or $f(x)$ is monotonic decreasing if n is odd. **2**

(ii) Hence show that if n is a positive integer, then for all $x > 0$, **2**

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots - \frac{x^{2n}}{2n} < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{x^{2n-1}}{2n-1}.$$

(iii) Hence find $\ln(1 \cdot 2)$ correct to 2 decimal places. **1**

(c) The number e is given by the value of the limiting sum

$$e = \sum_{r=0}^{\infty} \frac{1}{r!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots.$$

(i) If n is a positive integer, and $a = n! \left\{ e - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} \right) \right\}$, show that $0 < a < \frac{1}{n}$. **3**

(ii) Hence deduce that e is irrational. **2**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$