uestion 1

Scots College
2000 3/4 U Trial
Solns.

$$\int_1^2 \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \sin^{-1}(1) - \sin^{-1}\left(\tfrac{1}{2}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{6}$$

$$= \frac{\pi}{3}$$

3

$y = -3\sin^{-1}\frac{x}{2}$

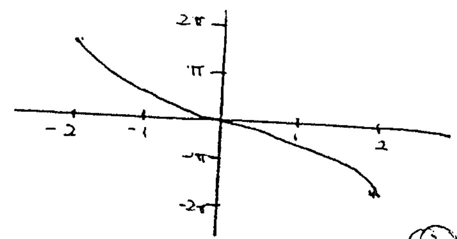
i) Domain: $-1 \leq \frac{x}{2} \leq 1$

$-2 \leq x \leq 2$

Range: $+3\left(-\frac{\pi}{2}\right) \leq y \leq 3\left(\frac{\pi}{2}\right)$

$-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}$

ii)



3

$$1 - \frac{1}{2}\left(\frac{2t}{1+t^2}\right)(t)$$

$$= 1 - \frac{2t^2}{2(1+t^2)}$$

$$= \frac{2+2t^2-2t^2}{2(1+t^2)}$$

$$= \frac{1}{(1+t^2)}$$

3

$x^2 - 2x - 3 = (x-3)(x+1)$

$\therefore (x-3)$ is a factor ie $f(3) = 0$

$(x+1)$ is a factor ie $f(-1) = 0$

3

$f(x) = x^3 - 3x^2 + ax + b$

$f(3) = 27 - 27 + 3a + b = 0$

$3a + b = 0 \Rightarrow b = -3a$

$f(1) = -1 - 3 - a + b$

$= -4 - a + b = 0$

sub $b = -3a$

$-4 - a - 3a = 0$

Solution $a = -1$

$b = 3$

Question 2

) $\int \frac{x}{1+2x}\,dx$ let $u = 1+2x \Rightarrow x = \frac{u-1}{2}$

$du = 2\,dx$

$dx = \frac{1}{2}du$

$= \int \frac{\frac{u-1}{2}}{u}\,\frac{1}{2}du$

$= \int \frac{u-1}{4u}\,du$

$= \int \frac{u}{4u}du - \int \frac{1}{4u}du$

$= \int \frac{1}{4}du - \int \frac{1}{4u}du$

$= \frac{1}{4}u - \frac{1}{4}\ln u + c$ but $u = (1+2x)$

$= \frac{1+2x}{4} - \frac{\ln(1+2x)}{4} + c$ 3

i) $\log_e x - \cos x = 0$

$x=1$ $\log_e 1 - \cos 1 = -0.54$

$x=2$ $\log_e 2 - \cos 2 = 1.11$

$\therefore$ a root lies between $x=1$ (-ve answer) & $x=2$ (+ve answer)

ii) $z_1 = 1.2$ $f(x) = \log_e x - \cos x$

$f'(x) = \frac{1}{x} + \sin x$

$z_2 = z_1 - \frac{f(z_1)}{f'(z_1)}$

$= 1.2 - \frac{f(1.2)}{f'(1.2)}$

$= 1.2 - \left(\frac{\log_e 1.2 - \cos 1.2}{\frac{1}{1.2} + \sin 1.2}\right)$ 3

$= 1.30$ (to 2 dec pl)

$3\cos x + 4\sin x$

$(\sqrt{9+16})\cos(x-\alpha)$

$5\cos(x-\alpha)$ $\tan\alpha = \frac{4}{3}$

$5\cos(x - 53°8')$ $\alpha = 53°8'$

$\cos x + 4\sin x = -3$

$5\cos(x-53°8') = -3$

$\cos(x-53°8') = \frac{-3}{5}$ $233°08'$

$x - 53°8' = 126°52'$ or 4

$\int_0 \cos^2 2x \, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 4x)\,dx$

$= \frac{1}{2}\left[x + \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{1}{4}\sin 2\pi\right) - \left(0 + \frac{1}{4}\sin 0\right)\right]$

$= \frac{\pi}{4}$

(2)

uestion 3

$2\ln(3x+1) - \ln(x+1) = \ln(7x+4)$

$\ln(3x+1)^2 - \ln(x+1) = \ln(7x+4)$

$\ln\left(\frac{(3x+1)^2}{(x+1)}\right) = \ln(7x+4)$

$\frac{(3x+1)^2}{(x+1)} = (7x+4)$

$(3x+1)^2 = (7x+4)(x+1)$

$9x^2+6x+1 = 7x^2+4x+7x+4$

$2x^2-5x-3=0$

$(2x+1)(x-3)=0$

$2x+1=0$ or $x-3=0$

$2x=-1$ $\quad x=3$

$x=-\frac{1}{2}$

But $\ln(3x+1)$ is not defined for $x=-\frac{1}{2}$

$\therefore$ Solution $x=3$ (3)

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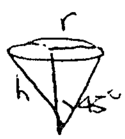
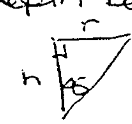


let the depth be 'h' cm

$r=h$



$\tan 45° = \frac{r}{h}$

$1 = \frac{r}{h}$

$r=h$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi h^3$

$\frac{dV}{dh} = \pi h^2$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$8 = \pi h^2 \times \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{8}{\pi h^2}$

$= \frac{8}{\pi \times 12^2}$

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$ but as $r=h$ $\frac{dr}{dt} = \frac{dh}{dt}$

$= \frac{dA}{dh} \times \frac{dh}{dt}$

$= 2\pi r \times \frac{1}{18\pi}$

$= 24\pi \times \frac{1}{18\pi}$

$= 1\frac{1}{3}$ cm²/s

) $\dfrac{P\sin A}{\tan B} = P\cos A + Q$

$P\sin A \times \dfrac{\cos B}{\sin B} = P\cos A + Q$

$P\sin A\cos B = P\cos A\sin B + Q\sin B$

$P\sin A\cos B - P\cos A\sin B = Q\sin B$

$P(\sin A\cos B - \cos A\sin B) = Q\sin B$

$P(\sin(A-B)) = Q\sin B$

$P = \dfrac{Q\sin B}{\sin(A-B)}$ 3

$y = \cos^{-1}(5x-4)$

$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-(5x-4)^2}} \times 5$

$= \dfrac{5}{\sqrt{1-(25x^2-40x+16)}}$

$= \dfrac{5}{\sqrt{40x-25x^2-15}}$ 1

uestion 4

$\ddot{x} = 3-4x$

$\dfrac{d}{dx}\left(\frac{1}{2}v^2\right) = 3-4x$

$\frac{1}{2}v^2 = \int(3-4x)\,dx$

$= 3x-2x^2+c$

hen $t=0$, $x=1$, $v=0$

$0 = 3-2+c$

$c=-1$

$\frac{1}{2}v^2 = 3x - 2x^2 - 1$

$v^2 = 2(3x-2x^2-1)$

hen the particle comes to rest $v=0$

$0 = -2(2x^2-3x+1)$

$2x^2-3x+1=0$

$(2x-1)(x-1)=0$

$x=\frac{1}{2}$ or $x=1$

he particle will come to rest again hen $x=\frac{1}{2}$m. 6

$\ddot{x} = 3-4x$ when $x=\frac{1}{2}$

$\ddot{x} = 3-2$

$$\frac{d^2x}{dt^2} = 0$$

$\ddot{x} = 0$

$\dot{x} = c$

when $t=0$ $\dot{x} = V\cos\theta$

$\therefore c = V\cos\theta$

$\dot{x} = V\cos\theta$

$x = Vt\cos\theta + c$

when $t=0$, $x=0$

$\therefore c=0$

$x = Vt\cos\theta$

$x = (40\cos 30°)t$

$= 20\sqrt{3}t$

$\ddot{y} = -10$

$\dot{y} = -10t + c$

when $t=0$ $\dot{y} = V\sin\theta$

$\therefore c = V\sin\theta$

$\dot{y} = V\sin\theta - 10t$

$y = -5t^2 + Vt\sin\theta + c$

when $t=0$, $y=0$

$\therefore c=0$

$y = -5t^2 + 40\sin 30° t$

$= -5t^2 + 20t$

) $y=0$ $-5t^2+20t=0$

$-5t(t-4)=0$

$t=0$ or $t=4$ sec $\therefore$ object lands after 4 sec

when $t=4$ $x = 20\sqrt{3} \times 4$

$= 80\sqrt{3}$ m $\therefore OA = 80\sqrt{3}$ m

6

uestion 5

Let $n^3+2n = 3P$

$n=1$ $1+2=3$ $\therefore$ true for $n=1$

assume it true for $n=k$

ie $k^3+2k = 3P$

rove true for $n=(k+1)$

so $(k+1)^3+2(k+1)$

$= k^3+3k^2+3k+1 +2k+2$

$= k^3+2k + 3k^2+3k+3$

$= 3P + 3(k^2+k+1)$

$= 3[P+(k^2+k+1)]$

$= 3m$ for $m = [P+(k^2+k+1)]$

5

$\therefore$ divisible by 3

o since true for $n=1$, & $n=k$ & proven for $n=k+1$

hen true for all $n > 0$

i) $y = 2x^2$

$\frac{dy}{dx} = 4x$ at $x = t$

$m = 4t$

$m = \frac{y - y_1}{x - x_1}$

$\frac{4t}{1} = \frac{y - 2t^2}{x - t}$

$y - 2t^2 = 4t(x - t)$

$y - 2t^2 = 4tx - 4t^2$

$y = 4tx - 2t^2$

i $y = x^2 + 1$

$\frac{dy}{dx} = 2x$ at $x = t$

$m = 2t$

if $x = t$, $y = (t^2 + 1)$

$m = \frac{y - y_1}{x - x_1}$

$\frac{2t}{1} = \frac{y - (t^2 + 1)}{x - t}$

$y - (t^2 + 1) = 2t(x - t)$

$y - (t^2 + 1) = 2tx - 2t^2$

$y = 2tx - t^2 + 1$

$y = 2tx + (1 - t^2)$

i) $\left.\begin{array}{l} y = 4tx - 2t^2 \\ y = 2tx + (1 - t^2) \end{array}\right]$ Solve simultaneously

$4tx - 2t^2 = 2tx + (1 - t^2)$

$2tx = 2t^2 - t^2 + 1$

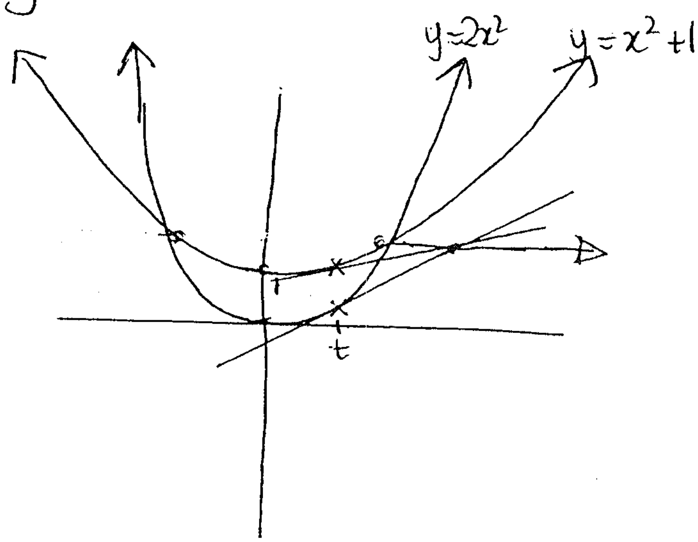
$2tx = t^2 + 1$

$x = \frac{t^2 + 1}{2t}$

$y = 4t\left(\frac{t^2 + 1}{2t}\right) - 2t^2$

$y = 2(t^2 + 1) - 2t^2$

$y = 2$



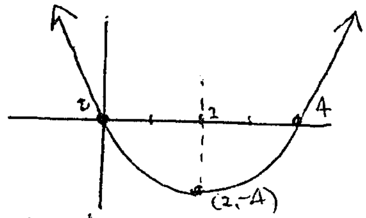
required locus is $y = 2$

where $x > 1$ & $x < -1$

as x must be positive only $x > 1$

Question 6

) $f(x) = x^2 - 4x$
$= x(x-4)$



i) for every y there is one & only one x.

ii) Domain $x \geq -4$

Range $y \geq 2$

iii) $y = x^2 - 4x$ $\qquad x = y^2 - 4y$

$y = (y^2-4y)^2 - 4(y^2-4y)$
$y = y^4 - 8y^3 + 16y^2 - 4y^2 + 16y$
$0 = y^4 - 8y^3 + 12y^2 + 15y$
$y(y^3 - 8y^2 + 12y + 15) = 0$
$P(x) = y^3 - 8y^2 + 12y + 15$
$P(3) = 27 - 72 + 36 + 15$
$= 6$
$P(5) = 5^3 - 8\times5^2 + 12\times5 + 15$
$= 0$ $\quad \therefore (y-5)$ is a factor

$$\begin{array}{r|l} & y^2 - 3y - 3 \\ y-5 & y^3 - 8y^2 + 12y + 15 \\ & y^3 - 5y \\ & -3y^2 + 12 \\ & -3y^2 + 15y \\ & -3y + 15 \\ & -3y + 15 \\ & \cdot\ \cdot \end{array}$$

$y(y-5)(y^2-3y-3) = 0$
$y = 0$ or $y = 5$ or $y^2 - 3y - 3 = 0$
$\therefore x = 25 - 20$
$x = 5$
$(5, 5)$

6

$f^{-1}(f(a))$
$= 2 + \sqrt{f(a)+4}$
$= 2 + \sqrt{a^2 - 4a + 4}$
$= 2 + \sqrt{(a-2)^2}$
$= 2 + (\sqrt{a-2})^2$
$= 2 + |a-2|$

$AP^2 = AB.AT$ (square of tangent equals product of secant segments
$BQ^2 = AB.BT$ (" " " ")
$P^2 + BQ^2 = AB.AT + AB.BT$
$= AB(AT + BT)$
$= AB.AB$
$= AB^2$

3

Let $\tan^{-1} \frac{1}{4} = \alpha$

$\tan\alpha = \frac{1}{4} \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

Let $\tan^{-1} \frac{3}{5} = \beta$

$\tan\beta = \frac{3}{5} \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

$\tan\left[\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5}\right]$

$= \tan(\alpha+\beta)$

$= \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \dfrac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{1}{4}\times\frac{3}{5}}$

$= \dfrac{\frac{17}{20}}{\frac{17}{20}}$

$= 1$

$\tan\left[\tan^{-1}(\frac{1}{4}) + \tan^{-1}(\frac{3}{5})\right] = 1$

$\tan^{-1}(1) = \tan^{-1}(\frac{1}{4}) + \tan^{-1}(\frac{3}{5})$

$\tan^{-1}(\frac{1}{4}) + \tan^{-1}(\frac{3}{4}) = \frac{\pi}{4}$ 3

Question 7

$x^3 - 4x + 1 = 0 \quad a = 1, \; b = 0, \; c = -4, \; d = 1$

$\alpha+\beta+\gamma = -\frac{b}{a}$

$= 0$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$= \frac{-4}{1}$

$= -4$

$\alpha\beta\gamma = \frac{-d}{a}$

$= \frac{-1}{1}$

$= -1$

ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \dfrac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{-4}{-1}$

$= 4$

3

$\frac{dQ}{dt} = -0.02(Q-30)$

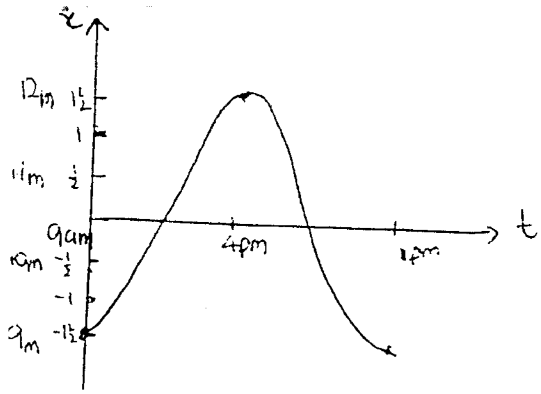
$Q = 30 + Ae^{-0.02t}$

$t = 0, \quad Q = 70$

$70 = 30 + A$ $\quad Q = 30 + 40e^{-0.02t}$

ii) $t = 45 \quad Q = 30 + 40e^{-0.02\times45}$

$\doteqdot 46g$ 3



Let 12m $\Rightarrow$ $1\frac{1}{2}$ m or x axis
9m $\Rightarrow$ $-1\frac{1}{2}$ m or x axis

i) Period $= \frac{2\pi}{n}$

$14 = \frac{2\pi}{n}$

$n = \frac{\pi}{7}$

Amplitude $= 1\frac{1}{2}$m

$\therefore$ Period $=$ 14hrs

$\therefore x = a\cos(nt+\alpha)$

$x = -1\frac{1}{2}\cos(\frac{\pi}{7}t+\alpha)$

when $t=0$, $x=1\frac{1}{2}$

$-1\frac{1}{2} = -1\frac{1}{2}\cos\alpha$

$\cos\alpha = 1$

$\alpha = 0$

when $x = -\frac{1}{2}$

$-\frac{1}{2} = -1\frac{1}{2}\cos(\frac{\pi}{7}t)$

$\cos(\frac{\pi}{7}t) = \frac{1}{3}$

$\frac{\pi}{7}t = 1.2309....$

$t = 2.74$

$= 2$hrs 45min

$\therefore$ 9am + 2hrs 45min

= 11.45 am

11 − 2hr 45min

= 8.15 pm

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