Neap:

HSC Trial Examination 2008

Mathematics Extension 1

This paper must be kept under strict security and may only be used on or after the afternoon of Thursday 14 August, 2008 as specified in the Neap Examination Timetable.

General Instructions

Reading time – 5 minutes

Working time – 2 hours

Write using black or blue pen

Board-approved calculators may be used

A table of standard integrals is provided at the back of this paper

All necessary working should be shown in every question

Total marks – 84

Attempt questions 1–7

All questions are of equal value

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2008 HSC Mathematics Extension 1 Examination.

Total marks 84
Attempt Questions 1–7
All questions are of equal value
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 Marks) Use a SEPARATE writing booklet.

(a) Express $(\sqrt{2}-1)^4$ in the form of $a\sqrt{2}+b$, where a and b are integers. **2**

(b) Evaluate $\lim\limits_{x\to 0} \dfrac{\sin 3x}{2x}$. **2**

(c) The point (6, 4) divides the interval joining (4, 2) to (9, 7) in the ratio of $1{:}k$. **2**

Calculate the value of k.

(d) Determine the exact value of $\sin^{-1}\left(\sin\dfrac{5\pi}{4}\right)$. **2**

(e) Find the term independent of x in the expansion of $\left(x+\dfrac{1}{2x}\right)^8$. **2**

(f) Determine the exact value of $\displaystyle\int_0^{\frac{\pi}{24}} \sin^2 6x\,dx$. **2**

TENME1_QA_08.FM

Marks

Question 2 (12 Marks) Use a SEPARATE writing booklet.

(a) Use the substitution $x = \cos\theta$ to evaluate $\displaystyle\int_{\frac{1}{2}}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx$. **3**

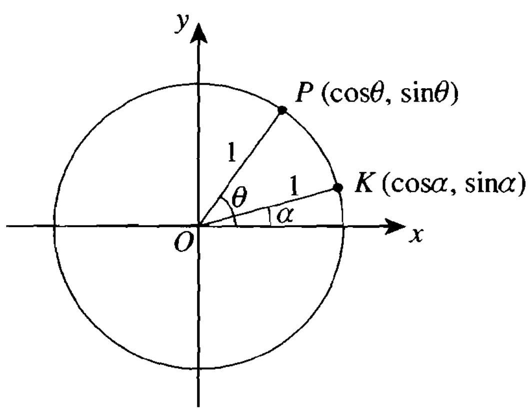
(b) The graphs $y = x^2$ and $y = x + 6$ intersect at (3, 9). **3**

Determine the size of the acute angle between the line and the curve at (3, 9).
Give your answer in radians correct to two decimal places.

(c) (i) Determine the domain and range of $y = 1 + 2\sin^{-1}3x$. **2**

(ii) Sketch the graph of $y = 1 + 2\sin^{-1}3x$. **1**

(d)



The diagram shows unit circle centre O. Points $P(\cos\theta, \sin\theta)$ and $K(\cos\alpha, \sin\alpha)$ are on the circumference of the circle.

(i) Use the cosine rule in ΔPKO to find an expression for $(PK)^2$. **1**

(ii) By using Pythagoras' theorem, the distance formula, or otherwise, determine a different expression for $(PK)^2$ than the expression in part (i). **1**

(iii) Hence show that $\cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha$. **1**

Marks

Question 3 (12 Marks) Use a SEPARATE writing booklet.

(a) Solve the inequality $\frac{x}{x-1} \geq 2$. **2**

(b) The probability that a woman has a height greater than 175 cm is 0.2.

(i) What is the probability that neither of two randomly selected women will be taller than 175 cm? **1**

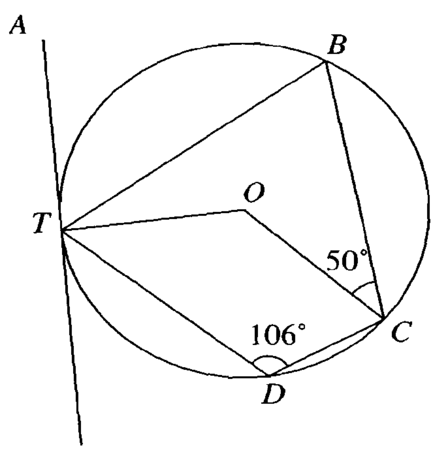
(ii) Five women are selected at random. **1**

Determine the probability that exactly three of the women will have a height greater than 175 cm.

(iii) Determine the probability that no more than two women in a randomly selected group of 12 women will have a height over 175 cm. Give your answer correct to three decimal places. **2**

(c) Use mathematical induction to prove that $3^{2n-1} + 5$ is divisible by 8, for all integers n, $n \geq 1$. **3**

(d) **3**



NOT TO SCALE

In the diagram AT is a tangent at T to the circle centre O. Points B, C and D lie on the circumference of the circle. $\angle BCO = 50°$ and $\angle TDC = 106°$.

Copy or trace this diagram into your writing booklet.

Determine the size of $\angle ATB$ and the obtuse $\angle TOC$.

 TENME1_QA_08.FM

Marks

Question 4 (12 Marks) Use a SEPARATE writing booklet.

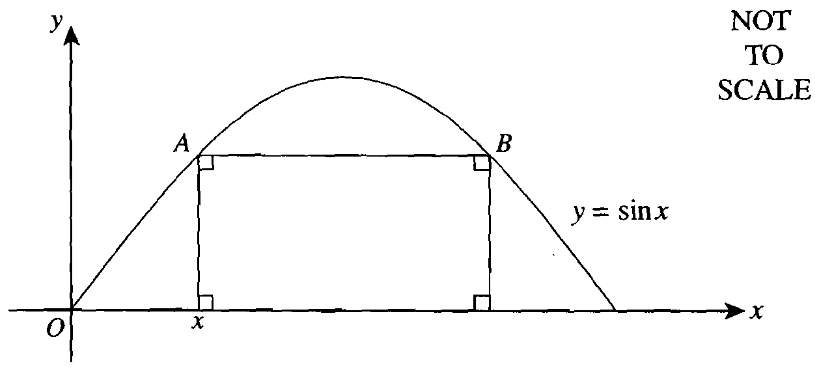
(a) (i) Prove that $\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1 + \cos\alpha}$. **2**

(ii) Hence find the value of $\tan\frac{\alpha}{2}$, $\frac{\pi}{2} < \alpha < 2\pi$, when $\sin\alpha = \frac{3}{5}$. **2**

(b) (i) Show that there is a root to the equation $2\tan x + 2x - \pi = 0$ between $x = 0.6$ and $x = 0.75$. **1**

(ii) Start with $x = 0.6$ and use one application of Newton's method to approximate the root to $2\tan x + 2x - \pi = 0$ in $0 < x < \frac{\pi}{2}$. **2**

(c)



The diagram shows a rectangle inscribed under one arch of the curve $y = \sin x$ in $0 < x < 2\pi$.

(i) The coordinates of point A are $(x, \sin x)$. **1**

Explain why the coordinates of point B are $(\pi - x, \sin x)$.

(ii) Show that the area $A(x)$ of the rectangle is given by $A(x) = (\pi - 2x)\sin x$. **1**

(iii) Hence determine the dimensions of the rectangle with the largest area that can be inscribed under one arch of the graph of $y = \sin x$. **3**

Marks

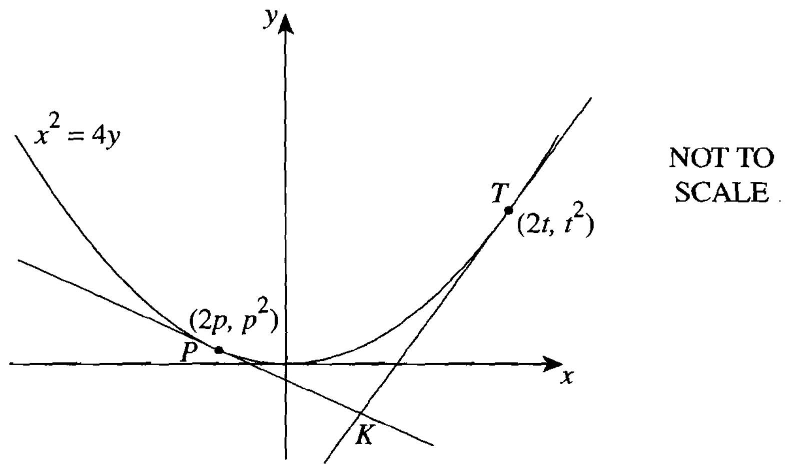
Question 5 (12 Marks) Use a SEPARATE writing booklet.

(a) One of the factors of $P(x) = ax^3 - 7x^2 + kx + 4$ is $(x-4)$ and the remainder when $P(x)$ is divided by $(x-1)$ is -6.

(i) Determine the values of a and k. **2**

(ii) Calculate the sum of the roots of $P(x)$. **1**

(b)



The diagram shows the graph of the parabola $x^2 = 4y$ and the tangent at $T(2t, t^2)$ and $P(2p, p^2)$. The tangents intersect at point K.

(i) Prove that the equation of the tangent at T is $y = tx - t^2$. **2**

(ii) Show that the coordinates of point K, the point where the tangents at T and P intersect are $(p + t, pt)$. **2**

(iii) The angle TKP is a right angle. **1**

Show that the locus of K is a straight line.

Question 5 continues on page 7

Marks

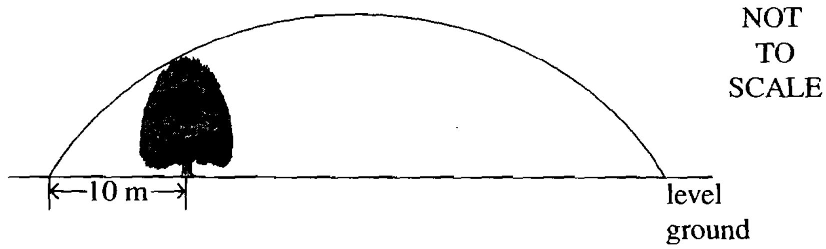
Question 5 (continued)

(c) Andrew hit a golf ball with a velocity of 15 m/s at an angle of 50° to the ground. The ball just cleared a tree 10 m horizontally away from Andrew, as shown in the diagram below.

Place the origin at the position where the ball was hit.

You may assume the equations of motion, i.e. $y = vt\sin\theta - \frac{1}{2}gt^2$ and $x = vt\cos\theta$.

Assume the acceleration due to gravity is 10 m/s^2.



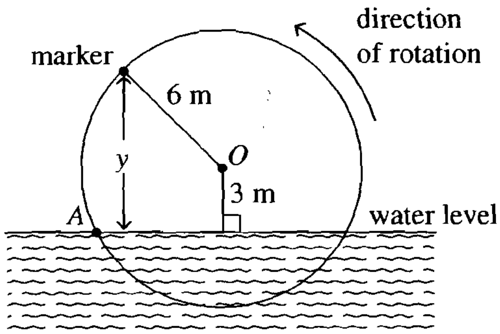
(i) Calculate the height of the tree in metres. Give your answer correct to one decimal place. **2**

(ii) How far beyond the tree did the ball hit the ground? **2**

Marks

Question 6 (12 Marks) Use a SEPARATE writing booklet.

(a) **4**



NOT
TO
SCALE

The diagram shows a water wheel that is rotating once every four minutes in an anticlockwise direction. As the wheel rotates the marker point moves up and down. The distance y represents the height of the marker point above the water. The marker is moving in simple harmonic motion.

The acceleration of the marker point is given by $\ddot{y} = -\frac{\pi^2}{4}y \ m/s^2$.

Initially the marker is at position A going into the water.

How far is the marker below the surface of the water at $t = 1$ minute?

(b) (i) Show that $f(x) = \frac{e^x}{1 + e^x}$ is a monotonic increasing function. **1**

(ii) Determine the value of $\lim_{x \to \infty} \frac{e^x}{1 + e^x}$ and $\lim_{x \to -\infty} \frac{e^x}{1 + e^x}$. **2**

(iii) For what value of x does $f^{-1}(x)$, the inverse function of $f(x)$, exist? **1**

(iv) Determine the equation of the inverse function of $f(x) = \frac{e^x}{1 + e^x}$. **2**

(v) Without making any further calculations, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate plane. **2**

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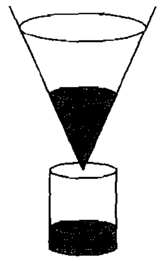
Marks

Question 7 (12 Marks) Use a SEPARATE writing booklet.

(a)



NOT TO SCALE

Cooking oil is being filtered. The oil is in a container that is in the shape of a cone and it is dripping into a cylindrical container at a rate of $288\pi\ \text{cm}^3/\text{min}$.

The height and radius of the cone are equal. The radius of the cylinder is 10 cm.

(i) At what rate is the depth of the oil in the cone decreasing when the depth is 12 cm? **3**

(ii) At what rate is the depth of oil in the cylinder increasing when the depth of oil in the cone is 12 cm? **1**

(b) Show algebraically that ${}^{n}C_3 + {}^{n}C_4 = {}^{n+1}C_4$. **3**

(c) The velocity of a particle, v m/s, when it is x m from the origin is given by $v = e^{-x}$. Initially, the particle is at the origin and has a speed of 1 m/s.

(i) Prove that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$. **1**

(ii) Prove that the acceleration of the particle at time t seconds is given by $\ddot{x} = \frac{-1}{(t+1)^2}$. **4**

End of paper

HSC Trial Examination 2008

Mathematics Extension 1

Solutions and marking guidelines

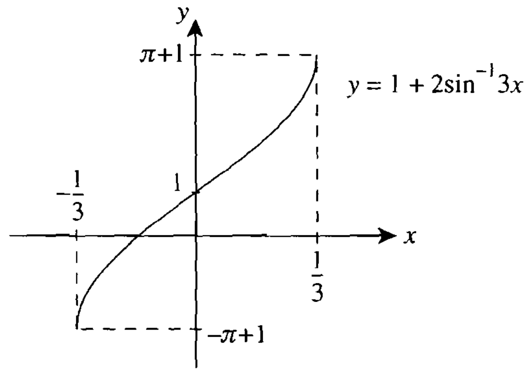
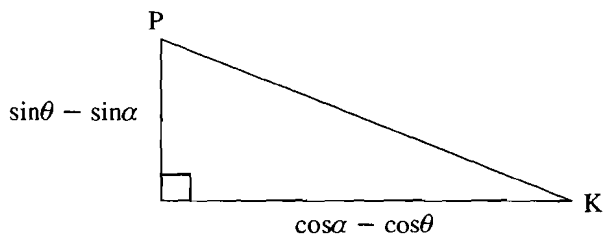
Question 1

	Sample answer	Syllabus outcomes and marking guide
(a)	$(\sqrt{2})^4 - 4(\sqrt{2})^3 + 6(\sqrt{2})^2 - 4(\sqrt{2}) + 1$ $= 4 - 8\sqrt{2} + 12 - 4\sqrt{2} + 1$ $= 17 - 12\sqrt{2}$	HE3 • Gives correct answer .... 2 • Shows the first line in the worked solution. 1
(b)	$\lim\limits_{x \to 0} \dfrac{\sin Ax}{Ax} = 1$ $\lim\limits_{x \to 0} \dfrac{\sin 3x}{2x} = \dfrac{3}{2} \lim\limits_{x \to 0} \dfrac{\sin 3x}{3x}$ $= \dfrac{3}{2} \times 1 = \dfrac{3}{2}$	HE4 • Gives correct answer .... 2 • Uses $\lim\limits_{x \to 0} \dfrac{\sin(Ax)}{Ax} = 1$ 1
(c)	$\dfrac{4k+9}{k+1} = 6$ and $\dfrac{2k+7}{1+k} = 4$ $4k + 9 = 6k + 6$ $k = 1.5$	P4 • Gives correct answer .... 2 • Determines a correct equation in k 1
(d)	$\sin^{-1}\left(\sin\dfrac{5\pi}{4}\right) = \sin^{-1}\left(\dfrac{-1}{\sqrt{2}}\right)$ $= -\dfrac{\pi}{4}$	HE4 • Gives correct answer .... 2 • Obtains $\dfrac{5\pi}{4}$ or $\dfrac{\pi}{4}$ 1
(e)	${}^8C_k(x)^{8-k}\left(\dfrac{1}{2x}\right)^k = Ax^0$ $x^{8-k} \times x^{-k} = x^0$ $x^{8-2k} = x^0$ $k = 4$ ${}^8C_4(x)^4\left(\dfrac{1}{2x}\right)^4$ $= \dfrac{35}{8}$	HE3 • Gives correct answer .... 2 • Shows or implies that $k = 4$ 1
(f)	$\displaystyle\int_0^{\frac{\pi}{24}} \sin^2 6x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{24}} (1 - \cos 12x)\,dx$ $= \dfrac{1}{2}\left[x - \dfrac{1}{12}\sin 12x\right]_0^{\frac{\pi}{24}}$ $= \dfrac{\pi - 2}{48}$	H5 • Gives correct answer .... 2 • Obtains an expression involving $\int(1 - \cos 12x)\,dx$ 1

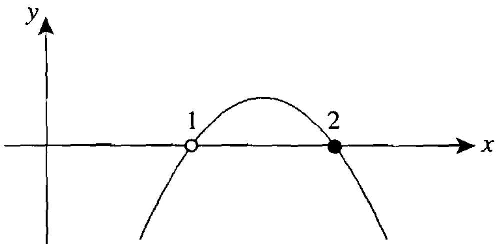
Question 2

	Sample answer	Syllabus outcomes and marking guide
(a)	$x = \cos\theta$ $\frac{dx}{d\theta} = -\sin\theta$ $dx = -\sin\theta d\theta$ When $x = 1$, $\theta = 0$. When $x = \frac{1}{2}$, $\theta = \frac{\pi}{3}$. $\int_{\frac{\pi}{3}}^{0} \frac{\sqrt{1-\cos^2\theta}}{\cos^2\theta} \times -\sin\theta d\theta$ $= \int_0^{\frac{\pi}{3}} \tan^2\theta d\theta$ $= \int_0^{\frac{\pi}{3}} (\sec^2\theta - 1)d\theta$ $= [\tan\theta - \theta]_0^{\frac{\pi}{3}}$ $= \sqrt{3} - \frac{\pi}{3}$	HE6 • Gives correct answer. . . . . 3 • Makes significant progress . . . . 2 • Makes some progress, e.g. determines the value of the limits in terms of θ or shows that $\frac{dx}{d\theta} = -\sin\theta$ 1
(b)	Gradient of the line $y = x + 6$ is 1. For $y = x^2$ $y' = 2x$ $m_T = 3 \times 2$ $= 6$ $\tan\theta = \left\|\frac{6-1}{1+6\times 1}\right\|$ $= \frac{5}{7}$ $\theta = 0.62$	P3 • Gives correct answer, ignore rounding . . 3 • Gives $\theta = \tan^{-1}\left(\frac{5}{7}\right)$ . . . . 2 • Determines that $m_1 = 6$ and $m_2 = 1$ 1

Question 2 **(Continued)**

	Sample answer	Syllabus outcomes and marking guide
(c) (i)	Domain: $-\frac{1}{3} \le x \le \frac{1}{3}$ Range: $1-\pi \le y \le 1+\pi$	HE4 • Both domain and range correct . . . . 2 • Either domain or range correct 1
(ii)		HE4 • Draws correct graph. 1
(d) (i)	$\angle POK = \theta - \alpha$ $(PK)^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos(\theta - \alpha)$ $= 2 - 2\cos(\theta - \alpha)$	PE2 • Gives the correct answer 1
(ii)	 $(PK)^2 = (\cos\alpha - \cos\theta)^2 + (\sin\theta - \sin\alpha)^2$ $= \cos^2\alpha + \cos^2\theta - 2\cos\alpha\cos\theta$ $+ \sin^2\theta + \sin^2\alpha - 2\sin\theta\sin\alpha$ $= 2 - 2\cos\alpha\cos\theta - 2\sin\theta\sin\alpha$	PE2 • Gives the correct answer 1
(iii)	$2 - 2\cos(\theta - \alpha) = 2 - 2\cos\alpha\cos\theta - 2\sin\theta\sin\alpha$ $2\cos(\theta - \alpha) = 2\cos\alpha\cos\theta + 2\sin\theta\sin\alpha$ $\cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha$	PE2 • Gives a correct proof 1

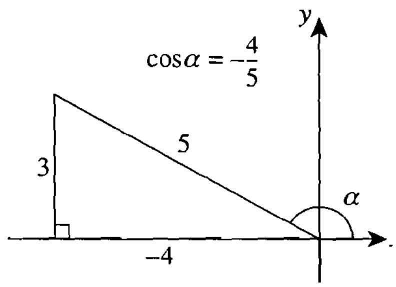
Question 3

	Sample answer	Syllabus outcomes and marking guide
(a)	$x \neq 1$ Multiply both sides by $(x-1)^2$. $x(x-1) \geq 2(x-1)^2$ $x(x-1) - 2(x-1)^2 \geq 0$ $(x-1)(2-x) \geq 0$  $\therefore 1 < x \leq 2$	PE3 • Gives correct solution .... 2 • Indicates that $x \neq 1$ 1
(b)	(i) $0.8 \times 0.8 = 0.64$	HE3 • Gives correct answer 1
	(ii) ${}^5C_3(0.8)^2(0.2)^3 = 0.0512$	HE3 • Gives correct answer, ignore rounding. .. 1
	(iii) Need zero, one or two women. ${}^{12}C_0(0.8)^{12} + {}^{12}C_1(0.8)^{11}(0.2) + {}^{12}C_2(0.8)^{10}(0.2)^2$ $= 0.558$	HE3 • Gives correct answer, ignore rounding .. 2 • Uses the correct terms in the appropriate binomial expansion 1
(c)	Test for $n = 1$: $3^{(2\times 1)-1} + 5 = 8$ which is divisible by 8. Thus it is true for $n = 1$. Assume 8 divides $3^{2k-1} + 5$. $\therefore 3^{2k-1} + 5 = 8\lambda$ where λ is an integer i.e. $3^{2k-1} = 8\lambda - 5$ Test for $n = k + 1$: $3^{2(k+1)-1} + 5 = 3^2 \times 3^{2k-1} + 5$ $= 9 \times (8\lambda - 5) + 5$ $= 72\lambda - 40$ $= 8(9\lambda - 5)$ $8(9\lambda - 5)$ is divisible by 8 as $9\lambda - 5$ is an integer. Thus if $3^{2n-1} + 5$ is divisible by 8 for an integer value of n, then it is divisible by 8 for the following integer value of n. Since it is divisible by 8 for $n = 1$ and $n = 2$, it is divisible by 8 for all positive integers n.	H5, HE7 • Gives a complete proof............... 3 • Provides a proof that lacks a minor component .... 2 • Makes some progress towards a proof, e.g. shows that $3^{2n-1} + 5$ is divisible by 8 for $n = 1$, and begins to test divisibility for $n = k + 1$ on the assumption that the expression is divisible by 8 for $n = k$ ... 1

Question 3 **(Continued)**

	Sample answer	Syllabus outcomes and marking guide
(d)	$\angle TBC = 74°$ (Opposite angles in a cyclic quadrilateral add up to 180°.) $\angle TOC = 148°$ (The angle at the centre is twice the angle at the circumference, standing on the same arc.) Join TC in ΔTOC. ΔTOC is isosceles as $OT = OC$ (radii). $\therefore \angle OCT = 16°$ (Base angles of an isosceles triangle are equal) $\therefore \angle TCB = 66°$ $\angle ATB = \angle TCB$ (The angle between a tangent and a chord is equal to the angle in the alternate segment.) $\therefore \angle ATB = 66°$	PE2 • Gives both angles correctly with supporting reasons . . . . 3 • Gives one angle found with supporting reasons and the other angle without supporting reasons . . . . 2 • Both angles found without supporting reasons. OR • One angle found with supporting reasons 1

Question 4

Sample answer	Syllabus outcomes and marking guide
(a) (i) $\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$ $\text{RHS} = \dfrac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{1+\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}$ $= \dfrac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}$ $= \dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$ $= \tan\frac{\alpha}{2}$ $= \text{LHS}$ *Note: an alternate approach is to use the t results.*	P3, P4 • Gives a complete proof .... 2 • Uses two different half angle results appropriately 1
(ii) $\sin\alpha > 0$ and $\frac{\pi}{2} < \alpha < 2\pi$. $\therefore \alpha$ is in the second quadrant.  Using the identity proven in part (a) (i): $\tan\frac{\alpha}{2} = \dfrac{\frac{3}{5}}{1-\frac{4}{5}}$ $= 3$	P3, P4 • Gives correct answer .... 2 • Implies that $\cos\alpha = -\frac{4}{5}$ 1

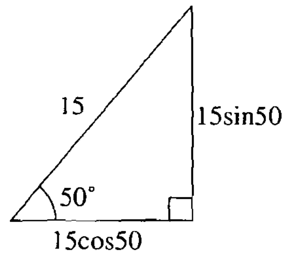
Question 4 **(Continued)**

		Sample answer	Syllabus outcomes and marking guide
(b)	(i)	$f(x) = 2\tan x + 2x - \pi$ $f(0.6) = -0.57$ $f(0.75) = 0.22$ As there is a change in sign on a continuous curve there is a root between $x = 0.6$ and $x = 0.75$.	PE3, P6 • Gives a correct demonstration 1
	(ii)	$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ $f'(x) = 2\sec^2 x + 2$ $f'(0.6) = 4.93609$ $x_2 = 0.6 - \frac{-0.573319}{4.93609}$ $= 0.716148$ $= 0.71$ or 0.72	PE3, P6 • Obtains either 0.71, 0.7161... or 0.72.... 2 • Makes progress towards a solution, e.g. finds the value of $f'(0.6)$ 1
(c)	(i)	The value where $y = \sin x$ crosses the x-axis is π. By symmetry, the length of the line joining π to the right side of the rectangle is x. Thus the x-coordinate is $\pi - x$. The y-coordinate is $\sin(\pi - x)$ and $\sin(\pi - x) = \sin x$.	P4 • Gives a correct demonstration 1
	(ii)	The length of AB is $\pi - 2x$. The height of the rectangle is $\sin x$. area = length × breadth $= (\pi - 2x) \times \sin x$	H5 • Gives a correct demonstration 1
	(iii)	$\frac{dA}{dx} = -2\sin x + (\pi - 2x)\cos x$ $-2\sin x + \pi\cos x - 2x\cos x = 0$ $\cos x = 0$ is not a solution, so divide through by $\cos x$. $-2\tan x + \pi - 2x = 0$ $2\tan x + 2x - \pi = 0$ (the solution from part (b) (ii)) $x = 0.71$ Test: $\frac{d^2A}{dx^2} = -2\cos x - \pi\sin x - 2\cos x + 2x\sin x$ $= -4\cos x + (2x - \pi)\sin x$ $= -4.16$ when $x = 0.71$ The stationary point is a maximum. The dimensions of the rectangle are 1.72 by 0.65.	H5, PE3 • Provides the correct solution OR • Provides a solution correct with respect to part (b) (ii) .... 3 • Provides solution with minor errors or omissions, e.g. omits a test for maximum .... 2 • Makes some progress toward a correct solution, e.g. uses the product rule to correctly differentiate the equation and equates the derivative to zero. 1

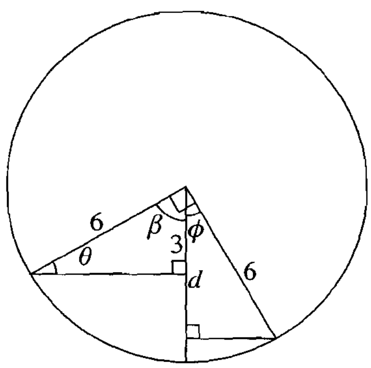
Question 5

		Sample answer	Syllabus outcomes and marking guide
(a)	(i)	$P(x) = ax^3 - 7x^2 + kx + 4$ $P(4) = 64a - 112 + 4k + 4 = 0$ $64a + 4k = 108$ $16a + k = 27$ $P(1) = a - 7 + k + 4 = -6$ $a + k = -3$ $16a + k = 27$ $a + k = -3$ $15a = 30$ $a = 2,\ k = -5$	PE3 • Obtains the correct values for a and b . . . 2 • Determines one correct equation relating a and b. 1
	(ii)	sum of the roots $= \frac{7}{2}$ (Formula for the sum of roots $= -\frac{b}{a}$.)	PE3 • Gives a correct answer with respect to part (a) (i). 1
(b)	(i)	$y = \frac{1}{4}x^2$ $y' = \frac{1}{2}x$ at $(2t, t^2)$, $y' = \frac{1}{2} \times 2t$ $y' = t$ $y - t^2 = t(x - 2t)$ $y = tx - 2t^2 + t^2$ $y = tx - t^2$	PE4 • Gives a correct proof . . . . 2 • Proves $y' = t$. OR • Derives the equation without proving $y' = t$ 1
	(ii)	solving $y = tx - t^2$ and $y = px - p^2$ simultaneously $px - p^2 = tx - t^2$ $px - tx = p^2 - t^2$ $x(p - t) = (p - t)(p + t)$ $p \neq t$ $x = p + t$ $y = t(p + t) - t^2$ $= tp + t^2 - t^2$ $= tp$ $K = (p + t, tp)$	PE4 • Gives correct answer . . . . 2 • Correctly demonstrates either the x coordinate or the y coordinate. OR • Makes an error in determining one of the coordinates then uses the wrong value to correctly determine the other value.. 1
	(iii)	as $\angle TKP = 90°$ $p \times t = -1$ $\therefore K$ is $(p + t, -1)$, which is a point on the line $y = -1$	PE4 • Gives a correct demonstration 1

Question 5 **(Continued)**

	Sample answer	Syllabus outcomes and marking guide
(c) (i)	 $y = 15\sin 50t - 5t^2$ $x = 15\cos 50t$ need the value of y when $x = 10$ $\therefore\ 10 = 15\cos 50t$ $t = \dfrac{2}{3\cos 50}$ $y = 15\sin 50 \times \dfrac{2}{3\cos 50} - 5 \times \left(\dfrac{2}{3\cos 50}\right)^2$ $= 10\tan 50 - 5\left(\dfrac{2}{3\cos 50}\right)^2$ $= 6.5391$ m $= 6.5$ m to 1 decimal place	HE3 • Obtains a height of 6.5 m, ignore rounding. . . . . 2 • Determines the equation $10 = 15t\cos 50°$ or equivalent. 1
(ii)	Need the value of x when $y = 0$. When $y = 0$, $-5t(t - 3\sin 50°) = 0$. $t = 3\sin 50°$ is required. Calculating x gives $x = 15\cos 50° \times 3\sin 50°$ $= 22.16$ m The ball landed 12.16 m past the tree.	HE3 • Gives correct answer, ignore rounding . . 2 • Determines $x = 22.16$, ignore rounding. OR • Correctly determines the distance past the tree using the range calculated by a valid method but including a minor error 1

Question 6

	Sample answer	Syllabus outcomes and marking guide
(a)	**SHM approach:** period $= 4$ minutes $\frac{2\pi}{n} = 4$ $n = \frac{\pi}{2}$ $\ddot{y} = -\left(\frac{\pi}{2}\right)^2 y$ $y = a\cos(nt + \alpha) + k$ $= 6\cos\left(\frac{\pi}{2}t + \alpha\right) + 3$ At A, $t = 0$ and $d = 0$. $0 = 6\cos\alpha + 3$ $\cos\alpha = -\frac{1}{2}$ $\alpha = \frac{2\pi}{3}$ $\therefore y = 6\cos\left(\frac{\pi}{2}t + \frac{2\pi}{3}\right) + 3$ The value of y when $t = 1$ is required. $y = 6\cos\left(\frac{\pi}{2} + \frac{2\pi}{3}\right) + 3$ $= -2.196$ It is 2.2 metres below the water at $t = 1$ minute.	HE3 **SHM approach:** • Gives correct solution .... 4 • Obtains the equation $y = 6\cos\left(\frac{\pi}{2}t + \frac{2\pi}{3}\right) + 3$ .... 3 • Determines any three of the values of a, n, k and α .... 2 • Makes a start e.g. determines the value of n 1
	Alternative solution: In 1 minute the wheel will have rotated $90°$ in an anticlockwise direction. $\sin\theta = \frac{3}{6}$ $\theta = 30°$ $\beta = 60°$ $\phi = 30°$ $\frac{d}{6} = \cos 30°$ $d = 5.196$ $d - 3 \approx 2.2$ metres 	**Alternative solution approach:** • Gives correct solution .... 4 • Determines that point A is approximately 5.2 metres below the centre of the wheel at $t = 1$ minute...................... 3 • Determines the values of θ, β and ϕ .... 2 • Determines that the wheel has rotated through $90°$ at $t = 1$ minute 1

Question 6 **(Continued)**

Sample answer | **Syllabus outcomes and marking guide**

(b) (i) $f'(x) = \dfrac{(1 + e^x)e^x - e^x \times e^x}{(1 + e^x)^2}$

$= \dfrac{e^x}{(1 + e^x)^2}$

Both the numerator and the denominator are positive.

$\therefore\ f'(x) > 0$

$\therefore$ the function is monotonic increasing.

H5, H3, HE7
- Provides the correct solution 1

(ii) $\lim\limits_{x \to \infty} \dfrac{e^x}{1 + e^x}$

Divide the numerator and denominator by e^x.

$= \lim\limits_{x \to \infty} \dfrac{1}{\frac{1}{e^x} + 1} = 1 \qquad \left[\dfrac{1}{e^x} \to 0 \text{ as } x \to \infty\right]$

$\lim\limits_{x \to -\infty} \dfrac{e^x}{1 + e^x}$

$= \dfrac{0}{1 + 0} = 0 \qquad \left[e^x \to 0 \text{ as } x \to -\infty\right]$

H3
- Gives both correct limits 2
- Gives one correct limit............... 1

(iii) The inverse exists for all real values of x because the curve is monotonic increasing.

HE7
- Provides the correct answer (reason not required)......................... 1

(iv) The inverse is $x = \dfrac{e^y}{1 + e^y}$.

Changing the subject to y:

$x + xe^y = e^y$

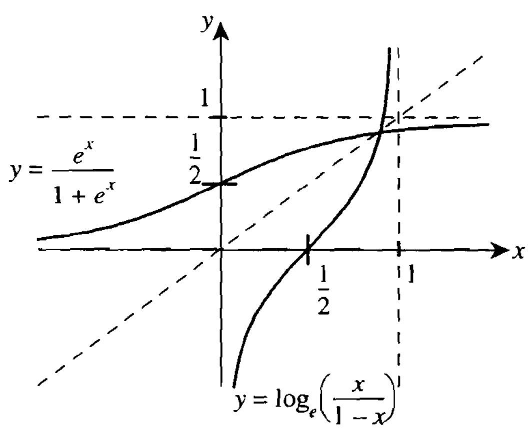
$x = e^y(1 - x)$

$e^y = \dfrac{x}{1 - x}$

$y = \log_e\left(\dfrac{x}{1 - x}\right), \quad x < 1$

HE4, H3
- Obtains the correct answer. 2
- Obtains the inverse function with x as the subject 1

(v)



HE4, HE7
- Both graphs correct 2
- One graph correct.

OR
- A pair of non-trivial graphs that are reflections of each other in the line $y = x$ 1

Question 7

Sample answer | **Syllabus outcomes and marking guide**

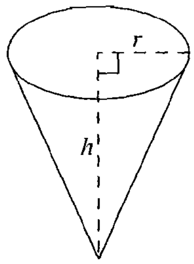
(a) (i) $\dfrac{dV}{dt} = 288\pi \text{ cm}^3/\text{min}$

$r = h$

$V = \dfrac{1}{3}\pi r^2 h$

$ = \dfrac{1}{3}\pi h^3$

$\dfrac{dV}{dh} = \pi h^2$



The value of $\dfrac{dh}{dt}$ when $h = 12$ is required.

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$
$$= \frac{1}{\pi h^2} \times 288\pi$$
$$= \frac{288}{144}$$
$$= 2 \text{ cm/min}$$

HE5

- Gives correct answer 3
- Makes significant progress towards the solution. 2
- Obtains $\dfrac{dV}{dt} = 288\pi$.

OR

- Obtains $\dfrac{dV}{dh} = \pi h^2$.

OR

- Obtains $\dfrac{dh}{dt} = \dfrac{dh}{dV} \times \dfrac{dV}{dt}$ 1

(ii) $\dfrac{dh}{dt}$ is constant.

$V = \pi \times 10^2 \times h$

In 1 minute:

$288\pi = \pi \times 10^2 \times h$

$h = 2.88$

The height is increasing at a rate of 2.88 cm/min.

HE3

- Gives correct solution 1

(b) ${}^nC_3 + {}^nC_4$

$$= \frac{n!}{3!(n-3)!} + \frac{n!}{4!(n-4)!}$$
$$= \frac{4 \times n!}{4!(n-3)!} + \frac{n!(n-3)}{4!(n-3)!}$$
$$= \frac{n!(4 + n - 3)}{4!(n-3)!}$$
$$= \frac{n!(n+1)}{4!(n+1-4)!}$$
$$= \frac{(n+1)!}{4!(n+1-4)!}$$
$$= {}^{n+1}C_4$$

PE3

- Gives a correct proof 3
- Makes significant progress towards a proof 2
- Demonstrates one algebraic factorial fact, e.g. $n!(n+1) = (n+1)!$ 1

Question 7

Sample answer	Syllabus outcomes and marking guide
(c) (i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2 \times \frac{1}{2}v^1 \times \frac{dv}{dx}$ $= v\frac{dv}{dx}$ $= \frac{dx}{dt} \cdot \frac{dv}{dx}$ $= \frac{dv}{dt}$ $= \ddot{x}$	HE3 • Gives a correct proof 1
(ii) $v = e^{-x}$ $\quad t = 0,\ x = 0,\ \lvert v \rvert = 1$ $\frac{1}{2}v^2 = \frac{1}{2}e^{-2x}$ $\ddot{x} = -2 \times \frac{1}{2}e^{-2x}$ $= -e^{-2x}$ $\frac{dx}{dt} = e^{-x}$ $\frac{dt}{dx} = e^{x}$ $\therefore\ t = e^{x} + K$ when $t = 0,\ x = 0$ $\therefore\ 0 = 1 + K$ $K = -1$ $t = e^{x} - 1$ $\therefore\ e^{x} = t + 1$ $\therefore\ x = \log_e(t+1)$ Now $\ddot{x} = -e^{-2x}$ $= -e^{-2\log_e(t+1)}$ $= -e^{\log_e(t+1)^{-2}}$ $= -(t+1)^{-2}$ $= \frac{-1}{(t+1)^2}$	• Gives a correct proof . . . . 4 • Makes substantial progress, e.g. determines $\ddot{x} = -e^{-2x}$ and $x = \log_e(t+1)$, or equivalent merit . . . . 3 • Makes progress towards a solution, e.g. determines $x = \log_e(t+1)$ or $\ddot{x} = -e^{-2x}$, or makes equivalent progress . . . . 2 • Uses $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$ or $\frac{dx}{dt} = e^{-x}$ 1