Sydney Grammar School
Mathematics Department
Trial Examinations 2007

FORM VI

MATHEMATICS EXTENSION 1

Examination date

Monday 6th August 2007

Time allowed

2 hours (plus 5 minutes reading time)

Instructions

All seven questions may be attempted.
All seven questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet.
Hand in the seven questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

SGS booklets: 7 per boy. A total of 1000 booklets should be sufficient.
Candidature: 117 boys.

Examiner

MLS

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) If $(x-2)$ is a factor of the polynomial **1**

$$P(x) = 2x^3 + x + a,$$

find the value of a.

(b) Given that $\log_a b = 2{\cdot}8$ and $\log_a c = 4{\cdot}1$, find $\log_a \frac{b}{c}$. **1**

(c) Shade the region on the number plane satisfied by $y \ge |x+2|$. **2**

(d) Solve the inequality $\dfrac{5}{x-4} \ge 1$. **3**

(e) State the domain and range of $y = \cos^{-1} \dfrac{x}{4}$. **2**

(f) Evaluate $\displaystyle\int_0^3 \frac{dx}{9+x^2}$. **3**

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\displaystyle\lim_{x\to 0} \frac{\sin \frac{x}{3}}{2x}$. **1**

(b) The point A has coordinates $(-2, 1)$ and the point B has coordinates $(b, -3)$. The point $P(13, -9)$ divides the interval AB externally in the ratio $5:3$. Find the value of b. **2**

(c) Using the substitution $u = e^x$, find **3**

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx.$$

(d) (i) Write down an expression for $\tan 2x$ in terms of $\tan x$. **1**

(ii) Hence show that if $f(x) = x \cot x$, then $f(2x) = (1 - \tan^2 x) f(x)$. **3**

(e) Find the coefficient of x^3 in the expansion of $(2-5x)^6$. **2**

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) Differentiate $\cos^{-1} x^2$. **2**

(b) Show that $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{\pi}{4}$. **2**

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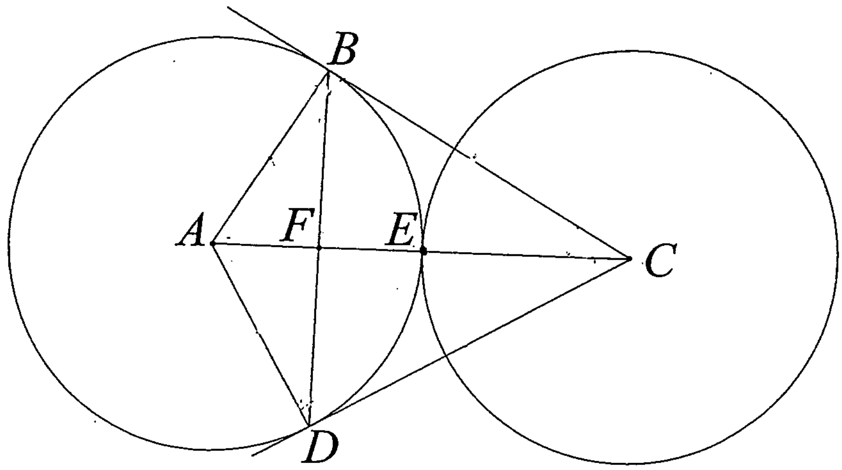
(c)



Two circles with equal radii and centres A and C touch externally at E as shown in the diagram. The lines BC and DC are tangents from C to the circle with centre A.

(i) Explain why $ABCD$ is a cyclic quadrilateral. $\boxed{2}$

(ii) Show that E is the centre of the circle that passes through A, B, C and D. $\boxed{2}$

(iii) Show that $\angle BCA = \angle DCA = 30°$. $\boxed{2}$

(iv) Deduce that $\triangle BCD$ is equilateral. $\boxed{2}$

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) The region between the curve $y = \dfrac{1}{\sqrt{1+4x^2}}$ and the x-axis is rotated about the x-axis. Find the volume of the solid enclosed between $x = \dfrac{2}{\sqrt{3}}$ and $x = 2\sqrt{3}$. $\boxed{3}$

(b) Use the substition $u = x - 3$ to evaluate $\displaystyle\int_3^4 x\sqrt{x-3}\,dx$. $\boxed{4}$

(c) A metal rod is taken from a freezer at $-8°$C into a room where the air temperature is $22°$C. The rate at which the rod warms follows Newton's law, that is

$$\frac{dT}{dt} = -k(T-22)$$

where k is a positive integer, time t is measured in minutes, and temperature T is measured in degrees centigrade.

(i) Show that $T = 22 - Ae^{-kt}$ is a solution of the equation $\dfrac{dT}{dt} = -k(T-22)$, and find the value of A, $\boxed{2}$

(ii) The temperature of the rod reaches $4°$C in 90 minutes. Find the exact value of k. $\boxed{2}$

(iii) Find the temperature of the rod after another 90 minutes. $\boxed{1}$

Exam continues overleaf ...

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a) A particle is moving in a straight line so that its displacement x at time t seconds is given by $x = \sqrt{3}\cos 2t - \sin 2t$ metres.

(i) Write $x = \sqrt{3}\cos 2t - \sin 2t$ in the form $x = R\cos(2t + \alpha)$, where $R > 0$ and $0 \le \alpha < 2\pi$. $\boxed{2}$

(ii) When is the particle first at $x = 1$? $\boxed{1}$

(iii) What is the maximum velocity of the particle and when does it first occur? $\boxed{2}$

(b) (i) Show that $x^3 - x - 2 = 0$ has a root between $x = 1$ and $x = 2$. $\boxed{1}$

(ii) Given that $x = 1{\cdot}5$ is your first approximation to a root of $x^3 - x - 2 = 0$, use one application of Newton's method to find another approximation. Give your answer correct to one decimal place. $\boxed{2}$

(c) A particle is moving in simple harmonic motion on a straight line. Its velocity v is given by $v^2 = 4(2x - x^2)$, where x is its displacement from a fixed point O on the line.

(i) Show that its acceleration is given by $\ddot{x} = -4(x - 1)$. $\boxed{1}$

(ii) Find the centre of the motion. $\boxed{1}$

(iii) Find the displacement of the particle when its speed is half the maximum speed. $\boxed{2}$

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) The length of a rectangle is increasing at $6\,\text{cm s}^{-1}$, while the breadth is decreasing so that the area of the rectangle remains constant at $50\,\text{cm}^2$. Find the rate of change of the breadth when the length is $10\,\text{cm}$. $\boxed{3}$

(b) (i) Use the method of mathematical induction to show that if x is a positive integer, then $(1 + x)^n - 1$ is divisible by x, for all positive integers $n \ge 1$. $\boxed{3}$

(ii) Write $12^n - 4^n - 3^n + 1$ as a product of two factors. $\boxed{1}$

(iii) Use parts (i) and (ii) to deduce that $12^n - 4^n - 3^n + 1$ is divisible by 6 for all integers $n \ge 1$. $\boxed{1}$

(c) The quadratic equation $ax^2 + bx + c = 0$ has roots $x = \tan\alpha$ and $x = \tan\beta$.

(i) Show that $\tan(\alpha + \beta) = \dfrac{b}{c - a}$. $\boxed{2}$

(ii) Show that $\tan^2(\alpha - \beta) = \dfrac{b^2 - 4ac}{(a + c)^2}$. $\boxed{2}$

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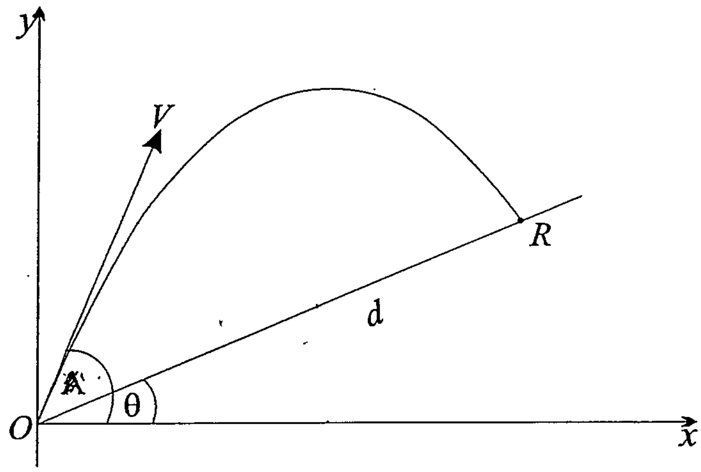
QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) Find the value of n if $\boxed{2}$

$$^{n}\mathrm{C}_2 + {}^{n}\mathrm{C}_1 + {}^{n}\mathrm{C}_0 = 37.$$

(b)



In the diagram above, a particle is projected at an angle of elevation α with velocity V from a point O which is at the bottom of an inclined plane. The plane is inclined to the horizontal at an angle θ, where $\theta < \alpha$. The particle meets the inclined plane again at R. The acceleration due to gravity is g, and $0^\circ < \alpha < 90^\circ$. Let $OR = d$.

(i) Given that $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$, where t is the time elapsed, show that the Cartesian equation of the path of the particle is $\boxed{1}$

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V^2}.$$

(ii) Find expressions for the coordinates of R in terms of θ and d. $\boxed{1}$

(iii) Show that the range of the particle up the inclined plane is given by $\boxed{3}$

$$d = \frac{2V^2\cos\alpha\sin(\alpha-\theta)}{g\cos^2\theta}.$$

(c) Consider the identity $(1+x)^n = 1 + {}^{n}\mathrm{C}_1\,x + {}^{n}\mathrm{C}_2\,x^2 + {}^{n}\mathrm{C}_3\,x^3 + \cdots + {}^{n}\mathrm{C}_n\,x^n$, where n is a positive integer.

(i) Use the formula for the sum of a GP to simplify $\boxed{1}$

$$1 + (1+x) + (1+x)^2 + (1+x)^3 + \cdots + (1+x)^{n-1}.$$

(ii) Use part(i) to show that $\boxed{1}$

$$1+(1+x)+(1+x)^2+(1+x)^3+\cdots+(1+x)^{n-1} = {}^{n}\mathrm{C}_1 + {}^{n}\mathrm{C}_2\,x + {}^{n}\mathrm{C}_3\,x^2 + \cdots + {}^{n}\mathrm{C}_n\,x^{n-1}.$$

(iii) Find $\displaystyle\int_{-1}^{0} {}^{n}\mathrm{C}_1 + {}^{n}\mathrm{C}_2\,x + {}^{n}\mathrm{C}_3\,x^2 + \cdots + {}^{n}\mathrm{C}_n\,x^{n-1}\,dx$. $\boxed{1}$

(iv) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{(-1)^{r+1}}{r}\,{}^{n}\mathrm{C}_r = \sum_{r=1}^{n}\frac{1}{r}$. $\boxed{2}$

END OF EXAMINATION

The following list of standard integrals may be used:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

$$\text{NOTE}: \ \ln x = \log_e x, \quad x > 0$$