Pymble Ladies College '02T

Q1

a) $\lim_{x \to 0} \frac{\sin 3x}{x} = 3 \lim_{x \to 0} \frac{\sin 3x}{3x} = 3$ ①

b) B is $(1, 3)$ ②

or

$(-2, 5) \quad (x, y)$

$-3 : 2$

$\frac{-4-3x}{-3+2} = 7$ $\qquad$ $\frac{10-3y}{-3+2} = -1$

$-4-3x = -7$ $\qquad$ $10-3y = 1$

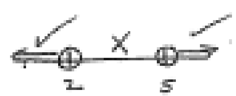
$-3x = -3$ $\qquad$ $-3y = -9$

$x = 1$ $\qquad$ $y = 3$

c) $\frac{x+1}{x-2} = 2$ $\qquad$ $x \neq 2$

$x+1 = 2x-4$

$x = 5$



$x > 5$ or $x < 2$

d) $y = \tan^{-1} 2x$ ②

$\frac{dy}{dx} = \frac{1}{1+(2x)^2}$

$= \frac{1}{1+4x^2}$

when $x = \frac{1}{2}$

$m_{tang} = \frac{1}{1+4 \cdot \frac{1}{4}}$

$= \frac{1}{2}$

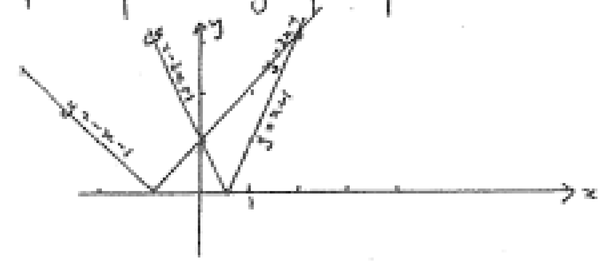
Q1

e) $\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx = \left[\sin^{-1}\frac{x}{3}\right]_0^3$

$= \sin^{-1}\frac{3}{3} - \sin^{-1} 0$

$= \frac{\pi}{2}$

f) $y = |2x-1|$ and $y = |x+1|$



$2x-1 = x+1$ $\qquad$ $-2x+1 = x+1$

$x = 2$ $\qquad$ $-3x = 0$

$x = 0$.

$\therefore |2x-1| \leq |x+1|$ when $0 \leq x \leq 2$.

Q2.

$$\frac{\sin 2\theta}{\sin\theta} - \sec\theta = \frac{\cos 2\theta}{\cos\theta}$$

a) $LHS = \dfrac{2\sin\theta\cos\theta}{\sin\theta} - \dfrac{1}{\cos\theta}$ (2)

$= 2\cos\theta - \dfrac{1}{\cos\theta}$

$= \dfrac{2\cos^2\theta - 1}{\cos\theta}$

$= \dfrac{\cos 2\theta}{\cos\theta}$.

b) $u = 2t-1$ when $t = 1$ $u = 1$ (4)

$\dfrac{du}{dt} = 2$ when $t = \frac{1}{2}$ $u = 0$.

$\displaystyle\int_{\frac{1}{2}}^{1} 4t(2t-1)\,dt = \int_0^1 2(u+1)u^5 \times \frac{1}{2}du$

$\displaystyle = \int_0^1 u^6 + u^5 \; du$

$\displaystyle = \left[\frac{u^7}{7} + \frac{u^6}{6}\right]_0^1$

$= \dfrac{1}{7} + \dfrac{1}{6}$

$= \dfrac{13}{42}$

c) $\tan\theta = \left|\dfrac{3-m}{1+3m}\right|$ (3)

$1 = \dfrac{3-m}{1+3m}$ $-1 = \dfrac{3-m}{1+3m}$

$1+3m = 3-m$ $-1-3m = 3-m$

$4m = 2$ $-2m = 4$

$m = \frac{1}{2}$ $m = -2$

d) $\tan 2\theta - \cot\theta = 0 \quad 0 \le \theta \le \pi$ (3)

$\dfrac{2\tan\theta}{1-\tan^2\theta} - \dfrac{1}{\tan\theta} = 0$

$2\tan^2\theta = 1 - \tan^2\theta$

$3\tan^2\theta = 1$

$\tan\theta = \pm\dfrac{1}{\sqrt{3}}$

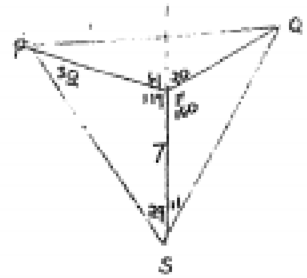
$\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

Check $\theta = \dfrac{\pi}{2}$ $0 - 0 = 0$ True

$\therefore$ Solns $\theta = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$

Q4

c.



(i) Consider $\triangle PFS$.

$$\frac{PF}{\sin 29} = \frac{7}{\sin 32}$$

$$PF = \sin 29 \times \frac{7}{\sin 32}$$

(ii) Similarly considering $\triangle QFS$

$$QF = \sin 11 \times \frac{7}{\sin 9}$$

Now considering $\triangle PQF$

$$PQ^2 = PF^2 + QF^2 - 2\,PF \times QF \cos 81^\circ$$

is. $$d^2 = \sin^2 29 \times \frac{49}{\sin^2 32} + \sin^2 11 \times \frac{49}{\sin^2 9} - 2 \times \sin 29 \times \frac{7}{\sin 32} \times \sin 11 \times \frac{7}{\sin 9} \times \cos 81$$

$$= 49\left[\frac{\sin^2 29}{\sin^2 32} + \frac{\sin^2 11}{\sin^2 9} - \frac{2\sin 29 \sin 11 \cos 81}{\sin 32 \sin 9}\right.$$

Katie

I like to write 2

3

Q4

$$x^3 + 2x^2 - 3x + 5 = 0$$

(i) $\alpha + \beta + \gamma = -\frac{b}{a} = -2$ ②

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = -3$

(ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2\alpha\beta - 2\alpha\gamma - 2\beta\gamma$ ②

$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 4 - 2 \times -3$

$= 10$

b) $P(x) = Q(x)(x+1)(2x+1) + ax + b$ ③

$P(-1) = -a + b = 5$ ①

$P(-\frac{1}{2}) = -\frac{1}{2}a + b = 3$ ②

①-② $-\frac{1}{2}a = 2$

$a = -4$

$b = 1$

$\therefore$ Remainder $= -4x + 1$

Katie

Kate is the neatest writer

Q 3.

(ii) $\lim_{n\to\infty}\left(\frac{1^3+2^3+3^3+\dots+n^3}{n^4}\right) = \lim_{n\to\infty} \frac{n^2(n+1)^2}{4n^4}$ ①

$= \lim_{n\to\infty} \frac{n^4+2n^3+n^2}{4n^4}$

$= \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4}$

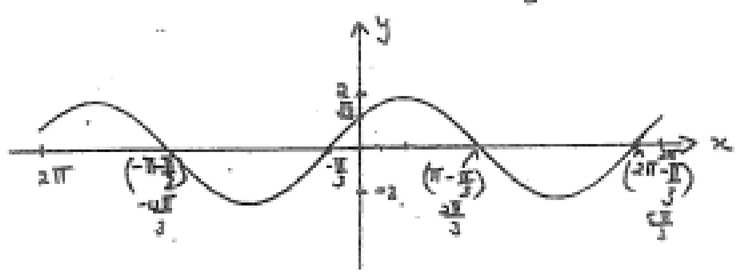
$= \frac{1}{4}$

c let $\sin x + \sqrt{3}\cos x = R\sin(x+\alpha)$

$R\cos\alpha = 1$ ①
$R\sin\alpha = \sqrt{3}$ ②

②/① $\tan\alpha = \sqrt{3}$ $\quad R^2 = 4$
$\therefore \alpha = \frac{\pi}{3}$ $\quad R = 2$.

$\therefore \sin x + \sqrt{3}\cos x = 2\sin\left(x+\frac{\pi}{3}\right)$ ②



②

(iii) $2\sin\left(x+\frac{\pi}{3}\right) = \sqrt{2}$ ②

$\sin\left(x+\frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$

$x+\frac{\pi}{3} = n\pi + (-1)^n\frac{\pi}{4}$

$\therefore \quad x = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{3}$

Q 3

a) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\left(\frac{x}{2}\right) dx$ ②

$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\cos x + 1)\, dx$

$= \frac{1}{2}\left[\sin x + x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[1 + \frac{\pi}{2} - \left(\frac{1}{2}+\frac{\pi}{6}\right)\right]$

$= \frac{1}{2}\left[\frac{1}{2}+\frac{\pi}{3}\right]$

$= \frac{1}{4} + \frac{\pi}{6}$

b) let $S_n = 4(1^3+2^3+\dots+n^3)$ ③

Required to prove $S_n = n^2(n+1)^2$

For $n=1$ $\quad$ LHS $= 4(1^3) = 4$

RHS $= 1^2(1+1)^2 = 4$

Statement is true for $n=1$.

Assume statement is true for $n=k$

$S_k = 4(1^3+2^3+\dots+k^3) = k^2(k+1)^2$

when $n=k+1$

$S_{k+1} = 4(1^3+2^3+\dots+k^3+(k+1)^3)$

$= 4(1^3+2^3+\dots+k^3) + 4(k+1)^3$

$= k^2(k+1)^2 + 4(k+1)^3$

$= (k+1)^2(k^2+4(k+1))$

$= (k+1)^2(k+2)^2$

Thus if it is true for $n=k$ it is true for $n=k+1$

It is true for $n=1$ & hence it is true for $n=2$ & so on

" " " "

Q6.

$$\frac{d^2x}{dt^2} = -16x$$

(i) $x = a\cos(4t+\alpha)$ ①

$\dot{x} = -4a\sin(4t+\alpha)$

$\ddot{x} = -16a\cos(4t+\alpha)$

$= -16x$

$\therefore\ x = a\cos(4t+\alpha)$ is a soln

(ii) When $t=0$ $v=4$ ②

$4 = -4a\sin\alpha$

i.e. $-1 = a\sin\alpha$ ①

when $t=0$ $x=5$

$5 = a\cos\alpha$ ②

i.e. ①^2+②^2 $(\sin^2\alpha+\cos^2\alpha=1)$

$a^2 = (-1)^2+(5)^2$

$a^2 = 26$

$a = \sqrt{26}$.

OR

using $v^2 = n^2(a^2-x^2)$

$16 = 16(a^2-25)$

$1 = a^2-25$

$a^2 = 26$

$a = \sqrt{26}$

(iii) Max speed occurs when $\sin(4t+\alpha)=1$. ①

$\dot{x} = -4\times\sqrt{26}\sin(4t+\alpha)$

$= -4\sqrt{26}$

$\text{speed}_{max} = \left|-4\sqrt{26}\right|$

$= 4\sqrt{26}$

Q6

b $x^2 = 4ay$

$y = \frac{1}{4a}x^2$

(i) $\frac{dy}{dx} = \frac{1}{2a}x$ ②

$m_{tang_P} = \frac{1}{2a}\times 2ap$

$= p$

Eq^n of tangent at P

$y-ap^2 = p(x-2ap)$

$y = px - ap^2$ ①

(ii) $y = qx - aq^2$ ② ①

(iii) Solving eqns for tangents simult. ①

$0 = (p-q)x - a(p^2-q^2)$

$x = \frac{a(p^2-q^2)}{p-q} = \frac{a(p+q)(p-q)}{p-q}$

$x = a(p+q)$

① $y = ap(p+q) - ap^2$

$= apq$

$\therefore\ T$ is $(a(p+q), apq)$

(iv) T lies on $x^2=-4ay$ ①

i.e. $a^2(p+q)^2 = -4a^2pq$

$(p+q)^2 = -4pq$

$p^2+q^2+2pq = -4pq$

$p^2+q^2 = -6pq$

(v) M is $\left(\frac{2a(p+q)}{2}, \frac{a(p^2+q^2)}{2}\right)$ ①

$= \left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$

Q 5

$f(x) = \frac{x-1}{x^2}$

(i) $f'(x) = \frac{x^2 \cdot 1 - 2x(x-1)}{x^4}$ ③

$= \frac{x^2 - 2x^2 + 2x}{x^4}$

$= \frac{x(2-x)}{x^4}$

$= \frac{2-x}{x^3}$

$f'(x) = 0$ when $x = 2$.

$f''(x) = \frac{x^3 \cdot -1 - 3x^2(2-x)}{x^6}$

$= \frac{-x^3 - 6x^2 + 3x^3}{x^6}$

$= \frac{2x^2(x-3)}{x^6}$

$= \frac{2(x-3)}{x^4}$

when $x = 2$ $f''(x) < 0$

$\therefore$ Only stationary pt $(2, \frac{1}{4})$ which is a max

(ii) $f''(x) = \frac{2(x-3)}{x^4} = 0$ when $x = 3$ ①

x	3^-	3	3^+
$f''(x)$	$-$	0	$+$

Change in concavity

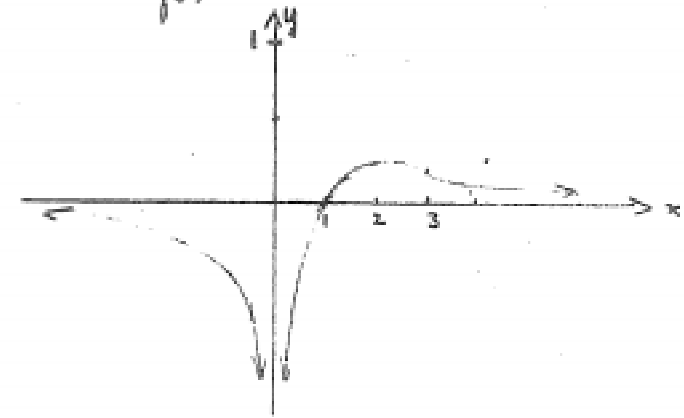
$\therefore$ there is a pt of inflexion at $(3, \frac{2}{9})$

Q5.

(iii) As $x \to \infty$ $f(x) \to 0$ ①

As $x \to -\infty$ $f(x) \to 0$

(iv) As $x \to 0$ $f(x) \to -\infty$ ①



②

b) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx}$

$= v \cdot \frac{dv}{dx}$ ②

$= \frac{dx}{dt} \cdot \frac{dv}{dx}$

$= \frac{dv}{dt}$

$= \ddot{x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 8x - 3x^2$ ②

$\frac{1}{2}v^2 = \int 8x - 3x^2 \, dx$

$\frac{1}{2}v^2 = 4x^2 - x^3 + c$

when $x = 0$ $v = 4$ $\therefore c = 8$

$v^2 = 8x^2 - 2x^3 + 16$

when $x = 1$ $v^2 = 8 - 2 + 16$

Q6

For M

$x = a(p+q)$

$x^2 = a^2(p^2+q^2+2pq)$

$= a^2(-6pq+2pq)$

$x^2 = -4a^2pq$ (1)

$y = \frac{a}{2}(p^2+q^2)$

$= \frac{a}{2} \times -6pq$

$y = -3apq$

$\therefore \frac{y}{-3a} = pq$

Sub in (1)

$x^2 = -4a^2 \times \frac{y}{-3a}$

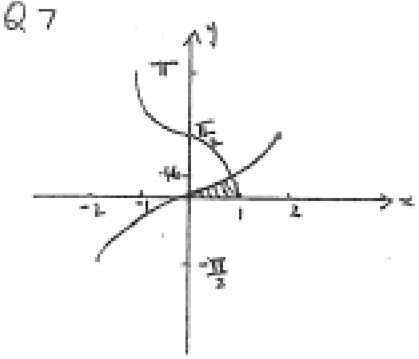
$x^2 = \frac{4}{3}ay$

$3x^2 = 4ay$

(2)

Q 7

(i)



②

(ii) $y = \cos^{-1} x$

when $x = \frac{2}{\sqrt{5}}$

$y = \cos^{-1} \frac{2}{\sqrt{5}}$

i.e. $\cos y = \frac{2}{\sqrt{5}}$.

Also $\sin y = \frac{1}{\sqrt{5}}$.

Consider $y = \sin^{-1} x$

$y = \sin^{-1}\left(\frac{\frac{2}{\sqrt{5}}}{2}\right)$

i.e. $\sin y = \frac{1}{\sqrt{5}}$ True

$\therefore$ Curves intersect at $x = \frac{2}{\sqrt{5}}$.

(iii) $y = \sin^{-1} \frac{x}{2}$

Inv. fn is $\sin y = \frac{x}{2}$

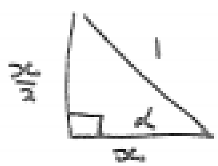
i.e. $x = 2\sin y$

or ....

②

let $\cos^{-1} x = \sin^{-1} \frac{x}{2} = \alpha$.

$\cos\alpha = x \qquad \sin\alpha = \frac{x}{2}$



$x^2 + \frac{x^2}{4} = 1$

$5x^2 = 4$

$x = \mp \frac{2}{\sqrt{5}}$ but

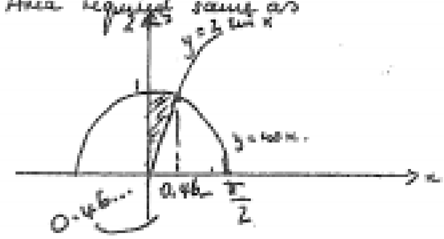
$x = \frac{2}{\sqrt{5}} \qquad x > 0$ from graph

Q 7.

(iv) $y = \cos^{-1} x$

Inv $x = \cos y$.

Area required same as



$$\text{Area} = \int_0^{0.46\ldots} (\cos x - 2\sin x)\, dx$$

$$\left[\sin x + 2\cos x\right]_0^{0.46\ldots}$$

$$[0.443.. + 1.792.. - 0 - 2]$$

$$= 0.24 \text{ units}^2.$$

17 b
(i) To maximise hectares irrigated,
we need to minimise Water per hectare (W)

$$\frac{dW}{dg} = 2Cg - \frac{D}{g^2}$$ $\frac{1}{2}$

$= 0$ for stationary pts.

ie $g^3 = \frac{D}{2C}$ $\frac{1}{2}$

$$\boxed{g = \left(\frac{D}{2C}\right)^{\frac{1}{3}}}$$ 1

$$\frac{d^2W}{dg^2} = 2C + \frac{2D}{g^3}$$ $\frac{1}{2}$

> 0 since $C, D, g > 0$ $\therefore$ min^m TP. $\frac{1}{2}$

(ii) Let G = tonnes of grain per kL water
We need to maximise G
G = tonnes of grain per hectare × hectares per kL water $\frac{1}{2}$

$$= g \times \frac{1}{Cg^2 + \frac{D}{g}}$$

$$\frac{dG}{dg} = \frac{Cg^2 + \frac{D}{g} - g\left(2Cg - \frac{D}{g^2}\right)}{\left(Cg^2 + \frac{D}{g}\right)^2}$$

$= 0$ for stationary pts $\frac{1}{2}$

ie $g^3 = \frac{2D}{C}$

$$\boxed{g = \left(\frac{2D}{C}\right)^{\frac{1}{3}}}$$ 2

$$\frac{dG}{dg} = -\frac{1}{g}\left(Cg^3 - 2D\right) \times \frac{1}{\left(Cg^2 + \frac{D}{g}\right)^2}$$

g	$\sqrt[3]{\frac{2D}{C}} - \epsilon$	$\sqrt[3]{\frac{2D}{C}}$	$\sqrt[3]{\frac{2D}{C}} + \epsilon$
$\frac{dG}{dg}$	neg × neg × pos = pos	0	neg × pos × pos = neg

$\therefore$ Max^m T.P. $\frac{1}{2}$

Now from 1 & 2 above
comparing results

$$\frac{\sqrt[3]{2D/C}}{\sqrt[3]{\frac{D}{2C}}} = \sqrt[3]{4}$$ $\frac{1}{2}$

$= 1.587\ldots$

$\approx 59\%$ more than before