EDMUND RICE COLLEGE

Extension 2 Mathematics
Trial HSC Examination 2001

QUESTION 1

a) **(i)** Without evaluating the integral, explain why $\int_{-1}^{1} xe^{-x^2}\ dx = 0$.

(ii) By using a suitable substitution or otherwise evaluate the integral $\int_0^1 xe^{-x^2}\ dx$.

b) Use the method of partial fractions to evaluate $\int_0^1 \frac{6}{9-x^2}\ dx$.

c) Show that $\int_2^4 \frac{dx}{x\sqrt{x-1}} = \frac{\pi}{6}$.

d) Use the substitution $t = \tan\frac{x}{2}$ to evaluate the integral $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$

e) Using integration by parts show that $\int e^{ax}\cos(bx)\ dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx)+b\sin(bx)\right]+C$.

QUESTION 2

a) Given the complex number $\omega = 1-\sqrt{3}i$:

(i) Find the exact value of $|\omega|$ and $\arg\omega$.

(ii) Find the value of ω^5 in the form $a+ib$.

(iii) Find the two square roots of ω expressing each in the form $a+ib$ where a, b are both real.

b) **(i)** Find and describe the cartesian equation of the locus represented by $|z|^2 = z+\overline{z}$.

(ii) Sketch this locus on an argand diagram.

c) **(i)** Mark clearly on an Argand diagram the region satisfied simultaneously by $|z+2| < 2$ and $0 < \arg z < \frac{3\pi}{4}$.

(ii) Solve simulataneously $|z+2| = 2$ and $\arg z = \frac{3\pi}{4}$ writing your answer in the form $a+ib$.

d) The point A represents the real number $z_1 = 1$. The point B represents the complex number $z_2 = (1+\sqrt{3})+i$. If $ABCD$ is a square in anti-clockwise rotational order, find:

(i) The complex number z_3 represented by the point C.

(ii) The complex number z_4 represented by the point D.

QUESTION 3

a) If $f(x)$ is an odd function and $g(x)$ is an even function, determine whether the following are odd, even or neither.

(i) $f[g(x)]$

(ii) $g[f(x)]$

b) **(i)** Show that $\frac{1}{x-2} - \frac{4}{x+3} + 3 = \frac{3x^2-7}{x^2+x-6}$

(ii) Find the vertical and horizontal asymptotes of $f(x) = \frac{3x^2-7}{x^2+x-6}$

(iii) Find the turning points and determine their nature.

(iv) Sketch the curve showing all important points. You may omit the point of infexion.

c) Find $\int \sin^2 x \cos^2 x \; dx$.

QUESTION 4

a) Given that the quartic polynomial $P(x) = x^4 - 5x^3 - 9x^2 + 81x - 108$ has a zero of multiplicity three (i.e., a treble root), then completely factorise the polynomial and find all the zeros.

b) A solid has a base in the shape of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If every cross-section perpendicular to the base is a semi-circle, with its diameter at right-angles to the major axis of the ellipse, find the volume of the solid by *slicing*.

c) A cubic polynomial is given by $P(x) = x^3 + ax + b$ where a, b are constants. It is given that the polynomial equation $P(x) = 0$ has three roots α, β and γ.

(i) Find the value of $\alpha + \beta + \delta$

(ii) Show that $\alpha^2 + \beta^2 + \delta^2 = -2a$

(iii) If the polynomial has a positive double zero, show that this double zero is $\frac{-3b}{2a}$

(iv) If the polynomial has three distinct zeros show that $4a^3 + 27b^2 < 0$

QUESTION 5

a) **(i)** Find (as trigonometrical quantities) the five fifth roots of unity, and represent them on the Argand diagram.

(ii) Show that the points representing the five roots on this Argand diagram represent the vertices of a pentagon. Find the area of this pentagon.

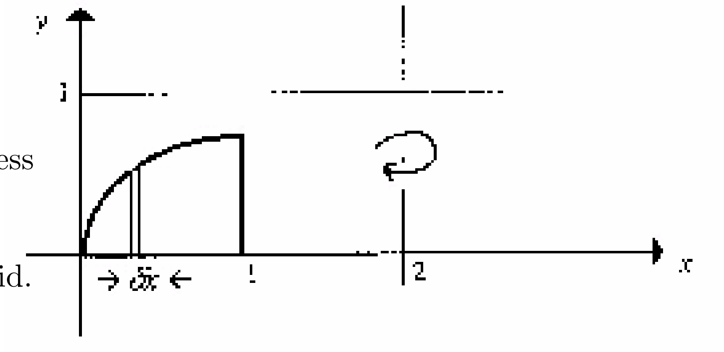
(iii) If α is one of the complex roots, show that the other complex roots can be expressed as $\alpha^2, \alpha^{-1}, \alpha^{-2}$ and find the value of $\alpha^2 + \alpha^{-2}$.

(iv) Fatorise $x^5 - 1$ as the product of a linear and two real quadratic factors.

b) The region shown bounded by the portion of the curve $y = \frac{x}{x+1}$ the x-axis and the line $x = 1$ is rotated about the line $x = 2$.

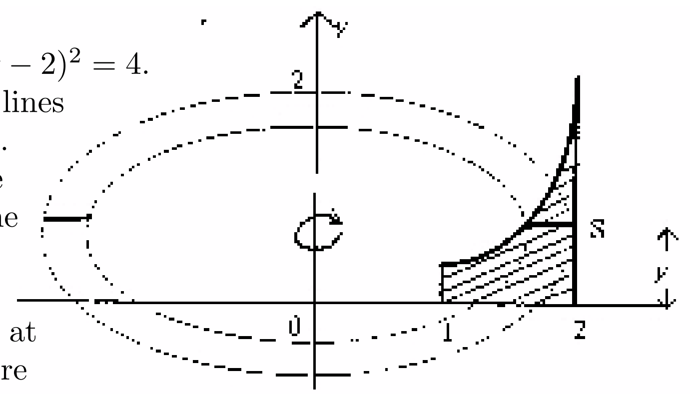
(i) Using the method of cylindrical shells, show that the volume δV of a typical shell at a distance x from the origin and with thickness δx is given by
$\delta V = 2\pi(2 - x)\frac{x}{1+x}\delta x$

(ii) Hence find the volume of the solid.



QUESTION 6

a) The curve is part of the circle $x^2 + (y - 2)^2 = 4$. The shaded region is bounded by the lines $x = 1, x = 2$, the curve and the x axis. This region is to be rotated about the y axis. When the region is rotated, the line segment at S, (height y), sweeps out an annulus.



(i) Show that the area of the annulus at height y is given by $\pi(y - 2)^2$ where $2 - \sqrt{3} \leq y \leq 2$.

(ii) Hence find the exact volume of the solid when the entire shaded region is rotated about the y axis (the cylindrical pipe portion of the solid has $V = \pi(6 - 3\sqrt{3})$).

b) **(i)** Given the equation of the hyperbola $xy = c^2$, show that the equation of the tangent at $T\left(ct, \frac{c}{t}\right)$ is given by $x + t^2y - 2ct = 0$.

(ii) P, Q and R are three points on one branch of this hyperbola, with parameters p, q and r respectively. The tangents at P and Q intersect at U. Find the co-ordinates of U. Hence determine the equation of the line OU, where O is the origin.

(iii) If O, U and R are collinear, show that $r^2 = pq$.

c) The equation $x^3 + px^2 + r = 0$ has roots α, β and γ. Find the equation with roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$.

QUESTION 7

a) **(i)** Show that the point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with asymptotes $y = \pm\frac{b}{a}x$.

(ii) Find the equation of the tangent to the hyperbola at P.

(iii) Show that the equation of the normal to the hyperbola at P is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$.

(iv) The tangent meets the asymptote $y = \frac{b}{a}x$ at A. The normal meets the x axis at B. The line $x = a\sec\theta$ meets the asymptote at C. Find the coordinates of B and C and prove that $\angle ACB$ is a right-angle.

(v) Give a geometrical description of the quadrilateral $ACPB$.

b) $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. A line through the origin and parallel to the tangent at P, meets the ellipse at R.

(i) Draw a diagram and mark the given information.

(ii) Prove that the area of triangle OPR is equal to $\frac{1}{2}ab$ for all positions of P.

QUESTION 8

a) Consider the function $y = x^x$ for $x > 0$.
Show that the derivative is given by $(\log_e x + 1)x^x$.

b) **(i)** Show that the sum of n terms of $1 + x + x^2 + x^3 + \cdots = \frac{1-x^n}{1-x}$ where $x < 1$.

(ii) Show that $1 + 2x + 3x^2 + \cdots + (n-1)x^{n-2} = \frac{(n-1)x^n - nx^{n-1} + 1}{(1-x)^2}$.

(iii) Hence find an expression for $1 + 1 + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \cdots + \frac{n-1}{2^{n-2}}$ and show that this sum is always less than 4.

c) **(i)** Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ (let $u = a - x$).

(ii) Consider $f(x) = \frac{1}{1+\tan x}$ where $0 \leq x \leq \frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right) = 0$.
Show that $f(x) = f\left(\frac{\pi}{2} - x\right) = 1$ (note: $\tan\left(\frac{\pi}{2} - x\right) = \cot x$)

(iii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x}\,dx$.
