St George Girls High School

Trial Higher School Certificate Examination

2005

Mathematics Extension 1

Total Marks – 84

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Attempt ALL questions.
- Begin each question on a new booklet
- Write your student number on each page
- All necessary working must be shown.
- Diagrams are not to scale.
- Board-approved calculators may be used.
- The mark allocated for each question is listed at the side of the question.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

SRV

Question 1 (12 marks) **Marks**

a) Find the coordinates of the point P that divides AB internally in the ratio 2 : 3 where A is $(-3, 5)$ and B is $(-6, -10)$ **2**

b) Find the possible values of a if the lines $2x + 3y - 5 = 0$ and $ax + 2y + 3 = 0$ are inclined to each other at 45° **4**

c) Solve for x: $\dfrac{2}{x-1} > 3$ **3**

d) Find $\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx$ using the substitution $x = u + 1$ **3**

Question 2 (12 marks)

a) (i) Express $\sqrt{3}\,\sin x + \cos x$ in the form $R\sin(x + \alpha)$ 2

(ii) Hence, sketch the graph of $y = \sqrt{3}\,\sin x + \cos x$ for $0 \leq x \leq 2\pi$ 2

b) (i) Show that $f(x) = 2\log_e x + 2x$ has a zero between $x = 0.5$ and $x = 1$ 1

(ii) Starting with $x = 0.5$, use one application of Newton's method to find a better approximation for this zero. Write your answer correct to three significant figures 3

c) Find $\int \frac{dx}{\sqrt{9-4x^2}}$ 2

d) Find $\int \cos^2 4x \; dx$ 2

Question 3 (12 marks) **Marks**

a) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $4ay = x^2$ such that the chord PQ subtends a right angle at the vertex O

(i) Show that $pq = -4$ 2

(ii) Find the locus of the mid-point of PQ 3

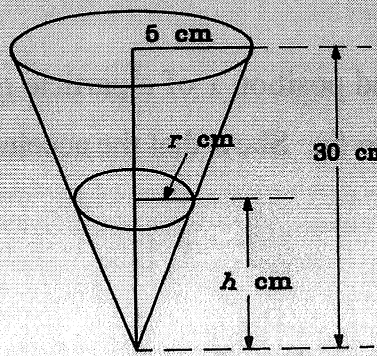
b) Show that $\displaystyle\int_0^3 \left(\frac{x}{x^2+9} + \frac{1}{x^2+9} \right) dx = \log_e \sqrt{2} + \frac{\pi}{12}$ 3

c) If the roots of the equation $x^3 + bx^2 + cx + d = 0$ are in geometric progression show that $c^3 = b^3 d$ 4

Question 4 (12 marks)

a) A container is in the shape of an inverted right circular cone of base radius 5cm and height 30cm. Water is poured into the container at a rate of $2\text{cm}^3/\text{min}$

(i) Show that $r = \dfrac{h}{6}$ 1



(ii) Find the rate at which the level of water is rising when the water is 10cm deep 3

b) (i) State the domain and range of $y = 2\cos^{-1}\left(\dfrac{x}{3}\right)$ 2

(ii) Hence sketch $y = 2\cos^{-1}\left(\dfrac{x}{3}\right)$ 1

c) Given $f(x) = \sqrt[3]{x-1}$ for $x > 1$

(i) Show that the function is monotonic increasing for all x in the given domain 2

(ii) State the domain and range of $f^{-1}(x)$ 1

(iii) Find $f^{-1}(x)$ and explain why the inverse is a function 2

Question 5 (12 marks) **Marks**

a) By induction show that $7^n - 3^n$ is divisible by 4 for all integers $n \geq 1$ 3

b) The velocity v and position x of a particle moving in a straight line are connected by the relation $v = 3 + 5x$. Show that the acceleration a of the particle is $5v$ 2

c) Find the term independent of x in the expansion of $(3-x)^4\left(1+\frac{2}{x}\right)^7$ 4

d) Evaluate $\cos\left(2\tan^{-1}\frac{3}{4}\right)$ without the use of a calculator 3

Question 6 (12 marks)

a) The cooling rate of a body is proportional to the difference between the temperature of the body and that of a surrounding medium ie. $\frac{dT}{dt} = -k(T - T_1)$ where T is the temperature of the cooling body and T_1 is the temperature of the surrounding medium

(i) Show that $T - T_1 = Ae^{-kt}$ satisfies this equation

(ii) A cup of coffee cools from 80° to 40° in 10 minutes when placed in a room with temperature 18°. How long will it take for the coffee's temperature to fall to 20°?

b) A particle is moving in a straight line such that its acceleration at time t seconds is $\ddot{x} = -4x$, where x is the displacement in metres from the origin. The particle is initially 6m to the right of the origin.

(i) Find its displacement in terms of time

(ii) Find the position and time when the particle first obtains a velocity of 6m/s

Question 7 (12 marks) **Marks**

a) (i) Differentiate $x(1+x)^n$ 1

(ii) Write the binomial expansion for $x(1+x)^n$ 1

(iii) Hence show that $\sum_{r=0}^{n} (r+1)\,{}^nC_r = (n+2)\,2^{n-1}$ 3

b) A particle is projected from a point O with an initial velocity of 60m/s at an angle of 30° to the horizontal. At the same instant a second particle is projected in the opposite direction with an initial velocity of 50m/s from a point level with O and 100m from O.

(i) Show that the horizontal and vertical displacement equations of the first particle are given by: 2

$x = 60\cos 30° t$ and $y = 60\sin 30° t - \frac{1}{2} g t^2$
where g is acceleration due to gravity

(ii) Find the angle of projection of the second particle if they collide 3

(iii) Find the time at which the two particles collide 2

End of Paper

Extension 1 – Trial HSC Solutions 2005

1 a) $A(-3,5)$ $B(-6,-10)$ Ratio $2:3$

$x = \dfrac{3\times -3 + 2\times -6}{2+3}$ $\quad y = \dfrac{3\times 5 + 2\times -10}{2+3}$

$= -\dfrac{21}{5}$ $\quad = -1$

$\therefore P$ is $(-4\frac{1}{5}, -1)$

b) $2x+3y-5=0$ $\quad ax+2y+3=0$

$m_1 = -\frac{2}{3}$ $\quad m_2 = -\frac{a}{2}$

$\therefore \tan 45 = 1 = \left| \dfrac{-\frac{2}{3} + \frac{a}{2}}{1 + \frac{-2}{3} \times \frac{-a}{2}} \right|$

$1 = \left| \dfrac{\frac{-4+3a}{6}}{\frac{6+2a}{6}} \right|$

$\therefore |6+2a| = |3a-4|$

$\therefore 6+2a = 3a-4$ or $6+2a = 4-3a$

$10 = a$ $\quad 5a = -2$

$a = -\frac{2}{5}$

c) $\dfrac{2}{x-1} > 3$ $\quad x \neq 1$

$2(x-1) > 3(x-1)^2$

$2(x-1) - 3(x-1)^2 > 0$

$(x-1)[2-3(x-1)] > 0$

$(x-1)(5-3x) > 0$

$\therefore 1 < x < \frac{5}{3}$

d) $\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx$ $\quad x = u+1$, $dx = du$

$= \displaystyle\int \frac{u+1}{\sqrt{u}}\,du$

$= \displaystyle\int \sqrt{u} + \frac{1}{\sqrt{u}}\,du$

$= \displaystyle\int u^{\frac{1}{2}} + u^{-\frac{1}{2}}$

$= \dfrac{2u^{3/2}}{3} + 2u^{1/2} + c$

$= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + c$

2 a) i) $\sqrt{3}\sin x + \cos x \equiv R\sin(x+\alpha)$

$\sqrt{3}\sin x + \cos x \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

$\therefore R\cos\alpha = \sqrt{3}$ $\quad R\sin\alpha = 1.$

$\therefore R^2(\cos^2\alpha + \sin^2\alpha) = 3+1$

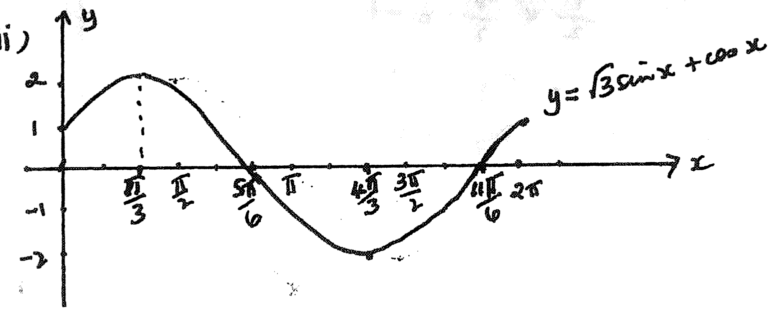
$\therefore R = 2$ $\quad R > 0.$

and $\dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{1}{\sqrt{3}}$

$\therefore \tan\alpha = \frac{\pi}{6}.$

$\therefore \sqrt{3}\sin x + \cos x = 2\sin(x + \frac{\pi}{6})$

(ii)



b) i) $f(x) = 2\log_e x + 2x.$

$f(0.5) \doteqdot -0.386$

$f(1) = 2.$

$\therefore$ Since sign change a zero lies between $\frac{1}{2}$ and 1.

ii) $f'(x) = \frac{2}{x} + 2.$

If $z_1 = 0.5$

then $z_2 = 0.5 - \dfrac{f(0.5)}{f'(0.5)}$

$= 0.5 - \dfrac{(2\ln 0.5 + 1)}{6}$

$\doteqdot 0.56438\ldots$

$\doteqdot 0.564$ (to 3 sig. fig)

c) $\displaystyle\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\sin^{-1}\frac{2x}{3} + c$

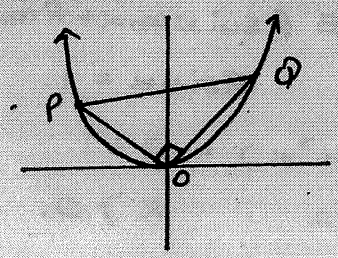
d) $\displaystyle\int \cos^2 4x\,dx = \frac{1}{2}\int 1 + \cos 8x\,dx$

$= \frac{1}{2}\left(x + \dfrac{\sin 8x}{8}\right) + c$

$= \dfrac{x}{2} + \dfrac{\sin 8x}{16} + c$

SRV

③ a) $P(2ap, ap^2)$ $Q(2aq, aq^2)$



i) m of $OP = \frac{ap^2-0}{2ap-0}$ m of $OQ = \frac{aq^2-0}{2aq-0}$

$= \frac{p}{2}$ $= \frac{q}{2}$

Since $P\hat{O}Q = 90^0$

$\frac{p}{2} \times \frac{q}{2} = -1$

$\therefore pq = -4$

ii) midpt $PQ = \left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$= \left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$

$x = a(p+q)$

$\therefore p+q = \frac{x}{a}$

$y = \frac{a(p^2+q^2)}{2}$

$\frac{2y}{a} = (p+q)^2 - 2pq$

$= \left(\frac{x}{a}\right)^2 - 2 \times -4$

$\frac{2y}{a} = \frac{x^2}{a^2} + 8$

$2ay = x^2 + 8a^2$

$x^2 = 2a(y-4a)$

(b) $\int_0^3 \frac{x}{x^2+9} + \frac{1}{x^2+9} \, dx$

$= \left[\frac{1}{2}\ln(x^2+9) + \frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3$

$= \left(\frac{1}{2}\ln 18 + \frac{1}{3}\tan^{-1} 1\right) - \left(\frac{1}{2}\ln 9 + 0\right)$

$= \frac{1}{2}\ln 2 + \frac{1}{3} \times \frac{\pi}{4}$

$= \ln\sqrt{2} + \frac{\pi}{12}$ as req.

c) $x^3 + bx^2 + cx + d = 0$

Let roots be $\frac{\alpha}{t}, \alpha, \alpha t$

$\therefore \frac{\alpha}{t} + \alpha + \alpha t = -b$ —(1)

$\frac{\alpha^2}{t^2} + \alpha^2 + \alpha^2 t = c$ —(2)

$\frac{\alpha}{t} \times \alpha \times \alpha t = -d$ —(3)

$\therefore \alpha^3 = -d$

From (1) : $\alpha\left(\frac{1}{t} + 1 + t\right) = -b$

From (2) : $\alpha^2\left(\frac{1}{t} + 1 + t\right) = c$

$\therefore \frac{1}{\alpha} = -\frac{b}{c}$

$\alpha = \frac{c}{-b}$

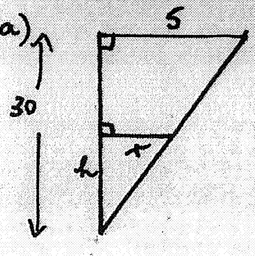
$\therefore \left(\frac{c}{-b}\right)^3 = -d$

$\frac{c^3}{-b^3} = -d$

$\therefore c^3 = b^3 d$

as req.

④ a) i) Since Δ's are similar



$$\frac{r}{5} = \frac{h}{30}$$

$$\therefore r = \frac{5h}{30} = \frac{h}{6}$$

ii) vol of cone $= \frac{1}{3}\pi r^2 h$

$$\therefore V = \frac{1}{3}\pi \times \left(\frac{h}{6}\right)^2 \times h$$

$$= \frac{\pi h^3}{108}$$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$2 = \frac{3\pi h^2}{108} \times \frac{dh}{dt}$$

$\therefore$ when $h = 10$: $2 = \dfrac{300\pi}{108} \times \dfrac{dh}{dt}$

$$\therefore \frac{dh}{dt} = \frac{216}{300\pi}$$

$$= \frac{18}{25\pi}$$

$\therefore$ water is rising at $\frac{18}{25\pi}$ cm/min

(b) i) $y = 2\cos^{-1}\frac{x}{3}$

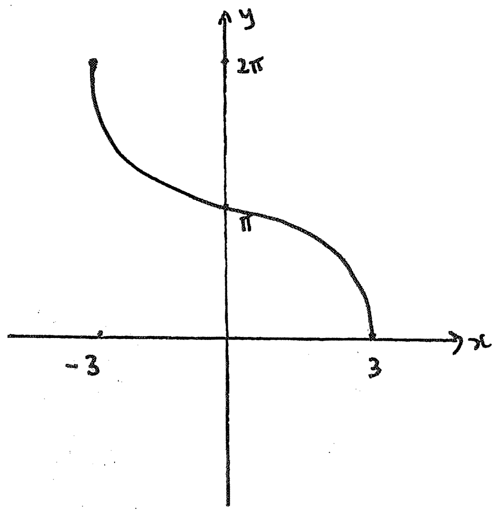
D: $-1 \leq \frac{x}{3} \leq 1$

$-3 \leq x \leq 3$

R: $0 \leq \frac{y}{2} \leq \pi$

$0 \leq y \leq 2\pi$

ii)



c) $f(x) = \sqrt[3]{x-1}$ $\quad x > 1$

i) $f(x) = (x-1)^{1/3}$

$$f'(x) = \frac{1}{3}(x-1)^{-2/3}$$

$$= \frac{1}{3\sqrt[3]{(x-1)^2}}$$

Since $(x-1)^2$ is positive for all x

$$\sqrt[3]{(x-1)^2} > 0$$

$\therefore \dfrac{1}{3\sqrt[3]{(x-1)^2}} > 0$ for all x

$\therefore f(x)$ is montonic increasing

(ii) For $f(x)$: D: $x > 1$

R : $y > 0$

$\therefore$ For $f^{-1}(x)$: D: $x > 0$

R : $y > 1$

(iii) $y = (x-1)^{1/3}$

For inverse : $x = (y-1)^{1/3}$

$$x^3 = y - 1$$

$$\therefore y = x^3 + 1$$

Since $f(x)$ is montonic increasing and it passes horizontal line test inverse will also be a function

5. a) Assertion: that $7^n - 3^n$ is divisible by

For $n=1$: $7^1 - 3^1 = 4$ which is divisib

$\therefore$ Assertion is true for $n=1$.

Assume assertion is true for $n=k$

i.e. that $7^k - 3^k$ is divis

i.e. $7^k - 3^k = 4M$ (where

We need to prove that:

$7^{k+1} - 3^{k+1}$ is also divisibl

$7^{k+1} - 3^{k+1} = 7^k \cdot 7 - 3^k \cdot 3$

$= (8-1) \cdot 7^k - (4-1) \cdot 3^k$

$= 8 \cdot 7^k - 7^k - 4 \cdot 3^k + 3^k$

$= 8 \cdot 7^k - 4 \cdot 3^k - (7^k - 3^k)$

$= 8 \cdot 7^k - 4 \cdot 3^k - 4M$ usi

$= 4(2 \cdot 7^k - 3^k - M)$

$= 4J$ where J is

$\therefore 7^{k+1} - 3^{k+1}$ is divisible by 4

$\therefore$ If statement is true for $n=k$,

$\therefore$ Since statement is true for

$n=2$ and by induction it is tru

b) $v = 3 + 5x$

Since $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$

$\frac{d}{dx}\left(\frac{1}{2}(3+5x)^2\right) = 2 \times \frac{1}{2}(3+5$

$= 5(3+5x)$

$= 5v$

$\therefore$ acceleration =

c) $(3-x)^4\left(1+\frac{2}{x}\right)^7$

$(3-x)^4 = {}^4C_0 3^4 - {}^4C_1 3^3 x + {}^4C_2 3^2 x^2 - {}^4C_3$

$\left(1+\frac{2}{x}\right)^7 = {}^7C_0 + {}^7C_1 \frac{2}{x} + {}^7C_2 \frac{4}{x^2} + {}^7C_3$

$\therefore$ Term Independent of $x = 3^4 \times {}^7C_0 - {}^4C_1 {}^7C_1 27 \times 2 +$

$= 81 - 1512 + 4536 -$

$= 305$

d) next page

6 a) i) $\frac{dT}{dt} = -k(T - T_1)$

$T - T_1 = Ae^{-kt}$

$\therefore T = T_1 + Ae^{-kt}$

LHS $= \frac{dT}{dt}$ 1

$= -k \times Ae^{-kt}$

RHS $= -k(T - T_1)$

$= -k(\cancel{T_1} + Ae^{-kt} - \cancel{T_1})$

$= -k . Ae^{-kt}$ 1

$=$ LHS

$\therefore T - T_1 = Ae^{-kt}$ satifies eqn. 2

ii) $T_1 = 18$

$t = 0$: $T = 80$

$\therefore 80 = 18 + A \times 1$

$\therefore A = 62$ 1

$\therefore T = 18 + 62e^{-kt}$

$t = 10$, $T = 40$

$40 = 18 + 62e^{-10k}$

$\frac{22}{62} = e^{-10}$

$\therefore k = \frac{\ln \frac{11}{31}}{-10.}$ 1

$T = 20$:

$20 = 18 + 62e^{-kt}$ 1

$\frac{2}{62} = e^{-kt}$

$t = \frac{\ln \frac{1}{31}}{-\left(\frac{\ln \frac{11}{31}}{-10}\right)}$ 1

$= 33.14$ mins (2dp).

4

b) i) Since $\ddot{x} = -4x$ particle is moving in SHM about origin

$\therefore x = a\cos(nt + \alpha)$ 1

$= 6\cos(2t + \alpha)$

$t = 0$, $x = 6$:

$\therefore 6 = 6\cos\alpha$ 1

$\cos\alpha = 1$

$\therefore \alpha = 0$ 3

$\therefore x = 6\cos 2t$ 1

ii) $v = -12\sin 2t$ 1

$v = 6$: $6 = -12\sin 2t$

$\therefore \sin 2t = -\frac{1}{2}$

$2t = \frac{7\pi}{6}$

$t = \frac{7\pi}{12}$ 1 1.83

$t = \frac{7\pi}{12}$: $x = 6\cos 2 \times \frac{7\pi}{12}$

$= 6\cos\frac{7\pi}{6}$ 1

$= 6 \times -\frac{\sqrt{3}}{2}$ 5.196

$= -3\sqrt{3}$

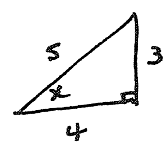
$\therefore$ Particle first reaches 6m/sec after $\frac{7\pi}{12}$ secs., $3\sqrt{3}$ metres to left of origin. 3

5d

$\cos\left(2\tan^{-1}\frac{3}{4}\right) = \cos 2x$

where $x = \tan^{-1}\frac{3}{4}$

$\therefore \tan x = \frac{3}{4}$



$\therefore \cos 2x = 2\cos^2 x - 1$

$= 2\left(\frac{4}{5}\right)^2 - 1$

$= \frac{32}{25} - 1$

$= \frac{7}{25}$

(7) a) i) $\frac{d}{dx}\left(x(1+x)^n\right) = (1+x)^n \times 1 + x \times n(1+x)^{n-1}$

$= (1+x)^n + nx(1+x)^{n-1}.$

ii) $x(1+x)^n = x\,{}^nC_0 + x\,{}^nC_1 x + x\,{}^nC_2 x^2 + \ldots + x\,{}^nC_n x^n$

$= {}^nC_0 x + {}^nC_1 x^2 + {}^nC_2 x^3 + \ldots + {}^nC_n x^{n+1}$

iii) $\sum_{r=0}^{n} (r+1)\,{}^nC_r = {}^nC_0 + 2\,{}^nC_2 + 3\,{}^nC_3 + \ldots + (n+1)\,{}^nC_r.$

from (ii): $\frac{d}{dx} x(1+x)^n = {}^nC_0 + 2\,{}^nC_1 x + 3\,{}^nC_2 x^2 + \ldots + (n+1)\,{}^nC_n x^n$

$\therefore$ from(i): $(1+x)^n + nx(1+x)^{n-1} = {}^nC_0 + 2\,{}^nC_1 x + 3\,{}^nC_2 x^2 + \ldots + (n+1)\,{}^nC_n x^n$

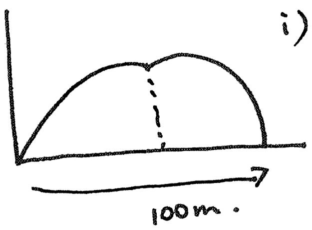
Let $x = 1$:

$2^n + n(2)^{n-1} = {}^nC_0 + 2\,{}^nC_1 + 3\,{}^nC_2 + \ldots + (n+1)\,{}^nC_n$

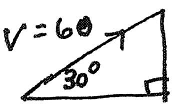
$\therefore 2^{n-1}(2+n) = {}^nC_0 + 2\,{}^nC_1 + 3\,{}^nC_2 + \ldots + (n+1)\,{}^nC_n.$

$\therefore \sum_{r=0}^{n} (r+1)\,{}^nC_r = 2^{n-1}(n+2)$

b)



i) First particle.



$\ddot{x} = 0$

$\dot{x} = C = 60\cos 30$

$x = \int 60\cos 30\, dt$

$= 60\cos 30 t + k$

$t=0 \quad x=0 \;\therefore k=0$

$\therefore x = 60\cos 30 t$

$\ddot{y} = -g$

$\dot{y} = -gt + M$

$t=0 \quad \dot{y} = 60\sin 30$

$\therefore \dot{y} = -gt + 60\sin 30$

$y = \int \dot{y}\, dt$

$= -\frac{gt^2}{2} + 60\sin 30 t + N$

$t=0 \quad y=0 \;\therefore N=0$

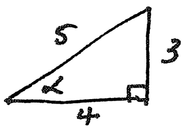
$\therefore y = -\frac{gt^2}{2} + 60\sin 30 t$

ii) When particles collide height above ground is same

$\therefore 60\sin 30 t - \cancel{\tfrac{1}{2}gt^2} = 50\sin\alpha t - \cancel{\tfrac{1}{2}gt^2}$

$30t = 50\sin\alpha t$

$\therefore \sin\alpha = \frac{3}{5}$



$\therefore \alpha = 36°52'$

iii) x values add to 100 m.

$\therefore 60\cos 30 t + 50\cos 36°52' t = 100$

$30\sqrt{3}\,t + 50 \times \frac{4}{5} t = 100$

$\frac{100}{} = 1.087$ secs (to 3dp)