Student name/number: ____________________

Advanced Mathematical Publications

(Place your crest here)

2001
PRELIMINARY FINAL EXAM

Mathematics Extension 1

Total marks (72)

- Attempt Questions 1 – 6
- All questions are of equal value

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- All necessary working should be shown in every question

Question 1 (12 marks) **Start a NEW page.** **Marks**

(a) Find $\displaystyle\int \frac{dx}{\sqrt{9-4x^2}}$ **2**

(b) Differentiate $y = 3e^{\tan 3x}$ **2**

(c) Find all possible values of k if the lines $2x + y + 3 = 0$ and $kx - y + 4 = 0$ intersect at $45°$. **2**

(d) The point $C(10, -7)$ divides the interval AB externally in the ratio 3:5. Find the coordinates of B if A has coordinates $(4, -1)$. **2**

(e) Write down the equation of the horizontal asymptote of $f(x) = \dfrac{3x}{x-7}$ **1**

(f) Solve $\dfrac{3}{x+1} \geq 4$ **3**

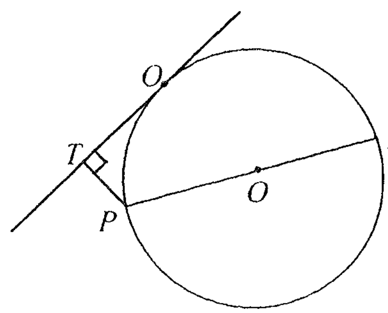
Question 2 (12 marks) **Start a NEW page.** **Marks**

(a) (i) In how many ways can the letters of the word **NONAGON** be arranged? **3**

(ii) Find the probability that the N's are together?

(iii) Find the probability that the vowels are together?

(b) **3**



In the diagram P, Q and R are points on a circle centre O, with PR being a diameter. PT is the perpendicular from P to the tangent at Q.

Copy the diagram into your Writing Booklet

Prove that PQ bisects $\angle RPT$.

(c) Given that one root of $x^3 - 5x^2 - x + k + 6 = 0$ is 3, Find **3**

(i) the value of k

(ii) the sum and the product of the other two roots.

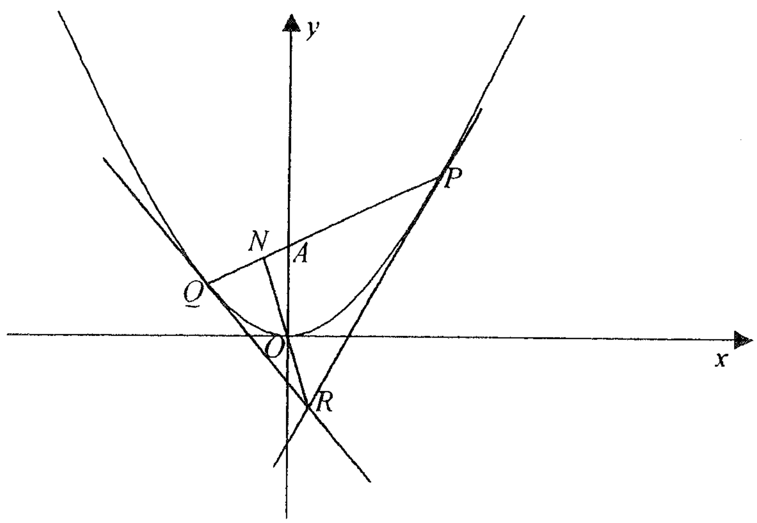
(d) Prove by induction that $5^n + 3$ is divisible by 4 for any integer $n \geq 1$. **3**

Question 3 (12 marks) **Start a NEW page.** **Marks**

(a) Evaluate $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\cos^{-1}\left(-\frac{1}{2}\right)$ **2**

(b) Out of 8 letters of which some are **E**'s and others are different, 6720 different eight-letter words can be formed. How many **E**'s are there? **2**

(c)



A parabola is defined by the parametric equations

$$x = 2t, \qquad y = t^2.$$

(i) Find the equations of the tangents at the points $P(2p, p^2)$ and $Q(2q, q^2)$. **2**

(ii) Show that the point of the intersection of the two tangents is at $R(p+q, pq)$. **2**

(iii) Show that the equation of the chord PQ is $(p+q)x - 2y - 2pq = 0$. **2**

(iv) If the points P and Q move on the parabola in such a way that pq remains constant and equal to -2, prove that the chord PQ always passes through the point $A(0, 2)$. **1**

(v) Show that RN which passes through O is perpendicular to PQ. **1**

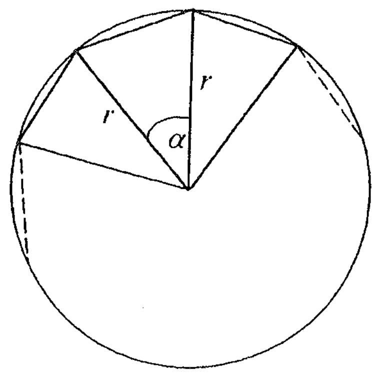
Question 4 (12 marks) **Start a NEW page.** **Marks**

(a) Using the substitution $u = 1 - 2x$, find 3

$$\int 4x\sqrt{1-2x}\,dx$$

(b) The expression $x^2 + 9x + 4$ has the same remainder whether divided by $x - a$ or $x + b$, where $a \neq -b$. Find the value of $a - b$. 2

(c)



The diagram above shows a regular n-sided polygon inscribed in a circle of radius r units. Each side of the polygon subtends an angle of α radians at the centre of the circle.

(i) Show that the perimeter of the polygon is $2nr\sin\dfrac{\pi}{n}$ 2

(ii) Show that the area of the polygon is $\dfrac{1}{2}nr^2\sin\dfrac{2\pi}{n}$. 1

(iii) Also show that the corresponding area of the circumscribed polygon is 2

$$nr^2\tan\frac{\pi}{n}.$$

(iv) Deduce the area of the circle by using the inequality, 2

Area of inscribed polygon < Area of circle < Area of circumscribed polygon

Question 5 (12 marks) **Start a NEW page.** **Marks**

(a) (i) Find the coefficient of x^7 in the expansion of $\left(px^2 + \dfrac{1}{qx}\right)^{11}$. **3**

(ii) If this coefficient is equal to the coefficient of x^{-7} in the expansion of $\left(px - \dfrac{1}{qx^2}\right)^{11}$, prove that $pq = 1$. **3**

(b) (i) By expanding $\cos(2\theta + 2\theta)$ or otherwise, show that **3**

$$\cos 4\theta = 1 - 8\sin^2\theta + 8\sin^4\theta.$$

(ii) Hence evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin^2\theta - \sin^4\theta)\, d\theta$ **3**

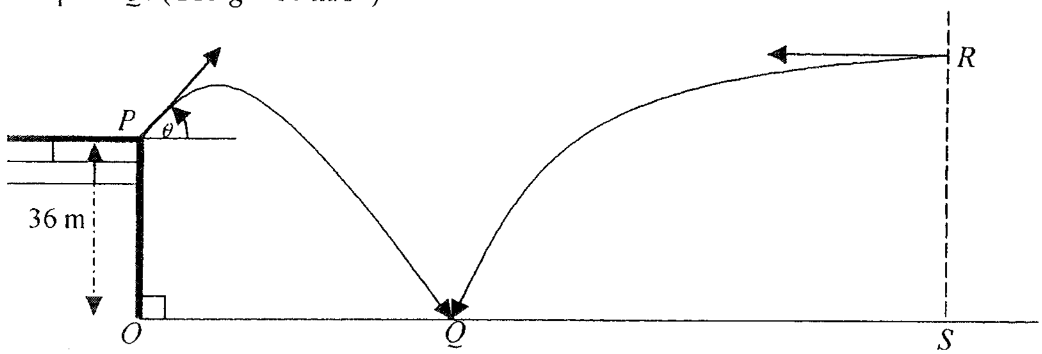
Question 6 (12 marks) **Start a NEW page.** **Marks**

(a) (i) By inserting n arithmetic means between n and $n+1$, show that the arithmetic sequence is **2**

$$n,\ n+\frac{1}{n+1},\ n+\frac{1}{n+2},\ \ldots,\ n+\frac{n+1}{n+1}$$

(ii) Hence show $n+\left(n+\frac{1}{n+1}\right)+\ldots+(n+1)=\frac{(n+2)(2n+1)}{2}$ **2**

(b) A projectile is fired from a point P with a speed $V = 40$ m/s at an angle θ to the horizontal where $\tan\theta = \frac{3}{4}$. The projectile strikes the ground 36 metres below P, at a point Q. (Use $g = 10$ m/s^2)



You may assume the following equations of motion.

$\ddot{x} = 0,$ $\quad\quad \ddot{y} = -g$

$\dot{x} = V\cos\theta,$ $\quad\quad \dot{y} = -gt + V\sin\theta$

Find;

(i) The time taken by the projectile to reach Q. **2**

(ii) The distance OQ. **1**

(iii) The magnitude and direction of the velocity of the projectile immediately before hitting the ground at Q. **2**

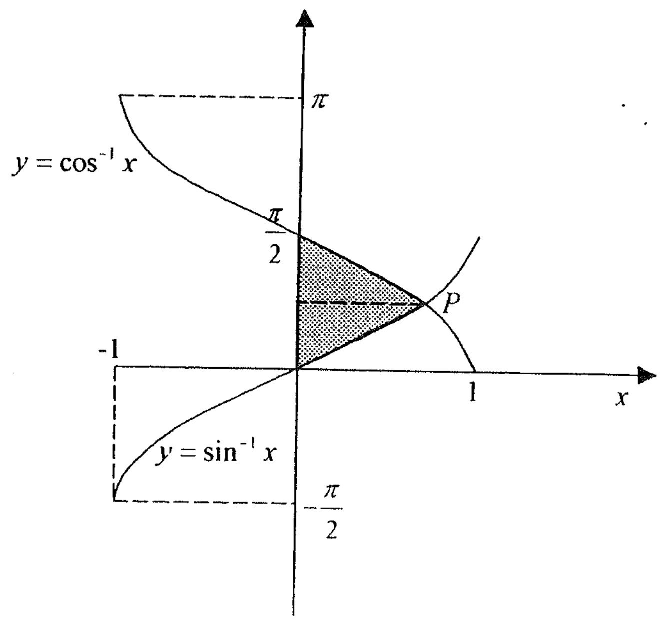
A second projectile is fired horizontally from R at 100 m/s at the same instance as the first projectile is fired. Given that both projectiles land on the same spot Q simultaneously, calculate

(iv) The vertical height SR. **2**

(v) The horizontal distance QS. **1**

Question 7 (12 marks) **Start a NEW page.** **Marks**

(a)



The diagram above shows the shaded area between the curves $y = \sin^{-1} x$, $y = \cos^{-1} x$ and the y-axis.

(i) Show that the point of intersection P is $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$. **1**

(ii) Hence, show that the shaded area is equal to $\left(2 - \sqrt{2}\right)$ sq. units. **3**

(iii) Show that the volume of solid generated by rotating this area about the y-axis is given by $\frac{\pi}{2}\left[\frac{\pi}{2} - 1\right]$ cubic units. **3**

(b) The rate at which a metal block cools in air is assumed to be proportional to the difference between its temperature T and the constant temperature A of the surrounding air. This can be expressed by the differential equation;

$$\frac{dT}{dt} = k(T\text{-}A)$$

where t is the time in hours and k is a constant.

(i) Show that $T = A - Be^{kt}$ is a solution to the differential equation, given that B is a constant. **2**

(ii) A metal block which has been heated to 80°C cools to 40°C in two hours. If the air temperature around the metal block is 20°C, find the temperature of the metal block after one further hour has elapsed. Give your answer correct to the nearest degree. **3**

End of paper

STANDARD INTEGRALS

$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; n \neq 1; \; x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x} \, dx = \ln x, \; x > 0$

$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$

$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$

$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$

$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$

$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$

$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$

$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$

$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$

$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE: $\ln x = \log_e x, \quad x > 0$