HIGHER SCHOOL CERTIFICATE EXAMINATION 1988

MATHEMATICS - 3/4 UNIT

Directions to Candidates

Time allowed - Two hours (includes reading time).

All questions may be attempted. All questions are of equal value.

All necessary working should be shown in every question. Marks may not be awarded for careless or badly arranged work.

Standard integrals are provided (See page 74); approved slide-rules or silent calculators may be used.

QUESTION 1

(a) Differentiate: (i) $x \cos x$; (ii) $\tan^{-1} 3x$.

(b) Find the co-ordinates of the point P which divides the interval AB with end points A(2,3) and B(5,-7) internally in the ratio 4 : 9.

(c) Evaluate: (i) $\int_0^1 \frac{2x}{x^2+1}\,dx$; (ii) $\int_0^{\pi} \sin^2 x\,dx$.

(d) Find the number of six-letter arrangements that can be made from the letters in the word SYDNEY.

QUESTION 2

(a) (i) Draw a sketch of $y = \sin^{-1} x$. State the domain and range.

(ii) A region R is bounded by the curve $y = \sin^{-1} x$, the x-axis and the line $x = 1$. Use Simpson's rule with three function values to find an approximation for the area of R. Give your answer correct to 2 decimal places.

(b) Find $\int x\sqrt{2 + x^2}\,dx$ using the substitution $u = 2 + x^2$.

(c) If α, β and γ are the roots of $x^3 - 3x + 1 = 0$ find:

(i) $\alpha + \beta + \gamma$;

(ii) $\alpha\beta\gamma$;

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$.

QUESTION 3

(a) A particle undergoes simple harmonic motion about the origin O. Its displacement x centimetres from O at time t seconds, is given by

$x = 3 \cos (2t + \frac{\pi}{3})$.

(i) Express the acceleration as a function of displacement.

(ii) Write down the amplitude of the motion.

(iii) Find the value of x for which the speed is a maximum and determine this speed.

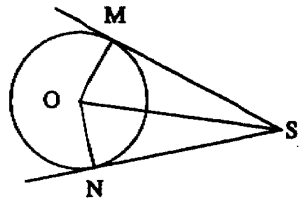
(b) Prove by mathematical induction that for $n \geq 1$,

$$1^2 + 3^2 + \ldots + (2n - 1)^2 = \frac{1}{3}n(2n - 1)(2n + 1).$$

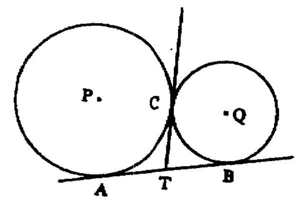
(c) Given that $0 < x < \frac{\pi}{4}$, prove that $\tan (\frac{\pi}{4} + x) = \frac{\cos x + \sin x}{\cos x - \sin x}$

QUESTION 4

(a) (i)



(ii)



(i) SM and SN are tangents drawn from an external point S to a circle with centre O. The points of contact of these tangents with the circle are M and N.

Copy this diagram into your writing booklet. By proving triangles OMS and ONS are congruent show that SM = SN.

(ii) Two circles touch externally at C. The circles, which have centres P and Q, are touched by a common tangent at A and B respectively. The common tangent at C meets AB in T.

(α) Copy this diagram in your writing booklet. Using the result from (i) prove that AT = TB.

(β) Show that ACB is a right angle.

(b) (i) Divide the polynomial $f(x) = 2x^4 - 10x^3 + 12x^2 + 2x - 3$ by $g(x) = x^2 - 3x + 1$.

(ii) Hence write $f(x) = g(x)\,q(x) + r(x)$ where $q(x)$ and $r(x)$ are polynomials and $r(x)$ has degree less than 2.

(iii) Hence show that $f(x)$ and $g(x)$ have no zeros in common.

QUESTION 5

(a) (i) Find the stationary points for the curve $y = x - 2 \sin x$ for $0 \leq x \leq 2\pi$.

Determine whether they are relative maxima or minima.

(ii) Find the co-ordinates of those points on the curve corresponding to $x = 0, \pi$ and 2π.

(iii) Hence draw a careful sketch of the curve.

(b) A meeting room contains a round table surrounded by ten chairs. These chairs are indistinguishable and equally spaced around the table.

(i) A committee of ten people includes three teenagers. How many seating arrangements are there in which all three sit together? Give brief reasons for your answer.

(ii) Elections are held for the positions of Chairperson and Secretary in a second committee of ten people seated around this table. What is the probability that the two people elected are sitting directly opposite each other? Give brief reasons for your answer.

QUESTION 6

(a) (i) Show that the equation of the tangent to the parabola $x^2 = 4Ay$ at any point $P(2Ap, Ap^2)$ is given by $px - y - Ap^2 = 0$.

(ii) S is the focus of the parabola and T the point of intersection of the tangent and the y-axis. Prove that SP = ST.

(*iii*) *Hence show that $S\hat{P}T$ is equal to the acute angle between the tangent and the line through P parallel to the axis of the parabola.*

(*b*) *Suppose* $(7 + 3x)^{25} = \sum_{k=0}^{25} t_k x^k$

(*i*) *Use the Binomial Theorem to write an expression for* t_k, $0 \le k \le 25$.

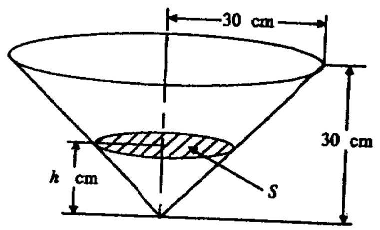
(*ii*) *Show that* $\dfrac{t_{k+1}}{t_k} = \dfrac{3(25 - k)}{7(k + 1)}$.

(*iii*) *Hence or otherwise find the largest coefficient* t_k.

You may leave your answer in the form $\binom{25}{k} 7^c 3^d$.

QUESTION 7

(*a*) *Water is poured into a conical vessel at a constant rate of 24 cm³ per second. The depth of water is h cm at any time t seconds. What is the rate of increase of the area of the surface S of the liquid when the depth is 16 cm?*



(*b*) *A parcel, in the shape of a rectangular prism, has sides xcm, xcm and ycm. The girth is the smallest distance around the parcel. A Courier Company will only deliver parcels for which the longest side ℓ cm and the girth g cm satisfy $\ell + g \le 100$.*

Find the dimensions of the parcel of largest volume, for which $\ell + g = 100$, that the Courier Company will deliver.