1/(a) $\dfrac{\frac{1}{3}}{1+\frac{x^2}{9}} = \dfrac{3}{x^2+9}$

(b)(i) $\displaystyle\int_1^{\sqrt{2}} \frac{x}{\sqrt{4-x^2}}\,dx$

$u = 4 - x^2$

$\frac{du}{dx} = -2x$

$x\,dx = -\frac{du}{2}$

$x=1,\ u=3$

$x=\sqrt{2},\ u=2$

$= -\displaystyle\int_3^2 \frac{du}{2\sqrt{u}}$

$= -\displaystyle\int_3^2 \tfrac{1}{2}u^{-\frac{1}{2}}\,du$

$= \left[u^{\frac{1}{2}}\right]_2^3$

$= \sqrt{3}-\sqrt{2}$

(ii) $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$

$x = \sin\theta$

$dx = \cos\theta\,d\theta$

$x=0,\ \theta=0$

$x=1,\ \theta=\frac{\pi}{2}$

$= \displaystyle\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\,.\cos\theta\,d\theta$

$= \displaystyle\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$

$= \frac{1}{2}\displaystyle\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$

$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left(\frac{\pi}{2} + 0 - (0+0)\right)$

c) $5\left(\frac{3}{5}\sin\theta + \frac{4}{5}\cos\theta\right) = 2\cdot 5$

$\sin(\theta+\alpha) = \frac{1}{2}$, , $\alpha = \sin^{-1}\frac{4}{5}$

$\theta + \alpha = 30°, 150°, 390°$

$\theta = 30° - \alpha, 150° - \alpha, 390° - \alpha$

$= -23°8', 96°52', 336°52'$

$= 96°52', 336°52'$ (nearest minute)

2/

(a) (i) $\frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt}$

$= v\frac{dv}{dx}$

$= \frac{d}{dv}\left(\frac{1}{2}v^2\right)\cdot\frac{dv}{dx}$

$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

(ii) $\ddot{x} = -2e^{-x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -2e^{-x}$

$\frac{1}{2}v^2 = 2e^{-x} + c$

$x=0, v=2$

$2 = 2e^0 + c, \ c = 0$

$\frac{1}{2}v^2 = 2e^{-x}$

$v^2 = 4e^{-x}$

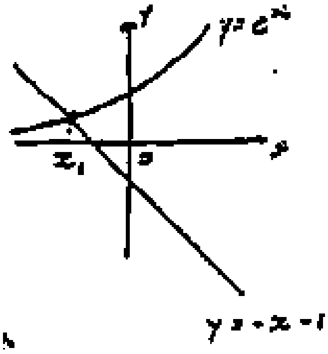
$v = \pm 2e^{-\frac{x}{2}}$

Initially $v > 0$ and $v^2 \neq 0$ $\therefore$ reject $-ve$ v

$v = 2e^{-\frac{x}{2}}$

(iii) as $x \to \infty$, $v \to 0$

(b) $y = e^x$
$y = -x - 1$



③

$f(x) = e^x + x + 1$
$f'(x) = e^x + 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= -1{\cdot}5 - \frac{e^{-1\cdot5} - 1{\cdot}5 + 1}{e^{-1\cdot5} + 1}$$
$$= -1{\cdot}27 \quad \text{(correct to 2 dec. pl)}$$

c) — V — V — V — —

No. of arrangements $= \frac{3!}{2!}\, 5!$
$= 360$

3(a)

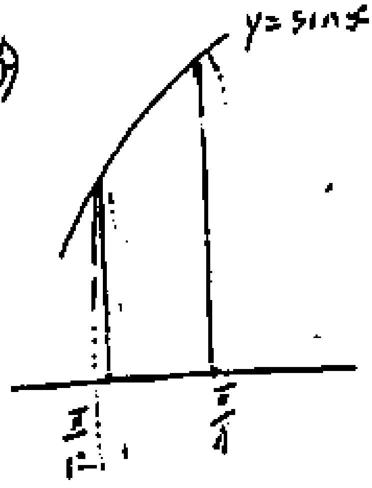
i) $\tan 2\theta = \frac{1}{\sqrt{3}}$

$2\theta = n\pi + \frac{\pi}{6}$

$\theta = n\frac{\pi}{2} + \frac{\pi}{12}$

Q3 cont. (4)

(b)



$$V = \pi \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \sin^2 x \, dx$$

$$= \frac{\pi}{2} \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} (1 - \cos 2x)\, dx$$

$$= \frac{\pi}{2} \left[x - \frac{1}{2} \sin 2x \right]_{\frac{\pi}{12}}^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2} \left(\frac{\pi}{4} - \frac{1}{2} - \left(\frac{\pi}{12} - \frac{1}{2} \cdot \frac{1}{2} \right) \right)$$

$$= \frac{\pi}{2} \left(\frac{\pi}{6} - \frac{1}{4} \right)$$

$$= \frac{\pi}{24}(2\pi - 3) \text{ units}^3$$

(c) (i) $\frac{dy}{dx} = \frac{x}{2y}$ at $P(2p, p^2)$

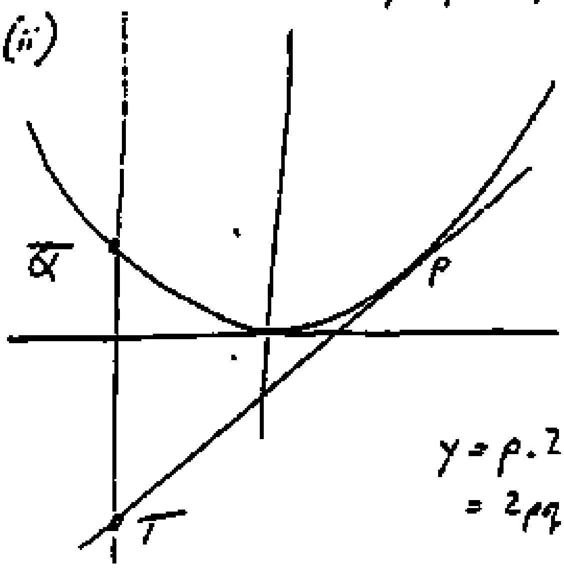
$= p$

equired equation: $y - p^2 = p(x - 2p)$

$= px - 2p^2$

$y = px - p^2$

(ii)



$y = p \cdot 2q - p^2$

$= 2pq - p^2$

$\therefore T(2q, 2pq - p^2)$

3/ cont.

(c)

(iii) $M(p+q,\ pq)$

(iv) $y=-1$

4/ (a) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

$\tan A+\tan B=\frac{5}{3}$

$\tan A\tan B=-\frac{1}{3}$

$\therefore \tan(A+B)=\dfrac{\frac{5}{3}}{1--\frac{1}{3}}$

$=\frac{5}{4}$

(b) $\ddot{x}=-n^2x$

$\frac{1}{2}v^2=-\frac{n^2x^2}{2}+C$

$v=V,\ x=d$

$\therefore C=\frac{1}{2}V^2+\frac{n^2d^2}{2}$

$\frac{1}{2}v^2=-\frac{n^2x^2}{2}+\frac{1}{2}V^2+\frac{n^2d^2}{2}$

$v^2=V^2+n^2(d^2-x^2)$

$v=\frac{V}{2}\quad x=2d$

$\left(\frac{V}{2}\right)^2=V^2+n^2(d^2-4d^2)$

$n^2=\left(V^2-\frac{V^2}{4}\right)\div 3d^2$

$=\frac{3V^2}{4}\times\frac{1}{3d^2}$

$=\frac{V^2}{}$

(6)

Q4 cont

(b)

$$\text{Period} = \frac{2\pi}{n}$$
$$= 2\pi \cdot \frac{2d}{v}$$
$$= \frac{4\pi d}{v}$$

When $v=0$, $v^2 + n^2(d^2 - x^2) = 0$
$$v^2 + \frac{v^2}{4d^2}(d^2 - x^2) = 0$$
$$x^2 - d^2 = v^2 \cdot \frac{4d^2}{v^2}$$
$$x^2 = 5d^2$$
$$\therefore \text{amplitude} = \sqrt{5d^2}$$
$$= d\sqrt{5}$$

(c)

(i) $\frac{dT}{dt} = kBe^{kt}$
$$= k(S + Be^{kt} - S)$$
$$= k(T - S)$$

(ii) $t=0$, $T = 80°$
$$80° = 20° + Be^{0}$$
$$B = 60$$
$$T = 20 + 60e^{kt}$$
$$40 = 20 + 60e^{2k}$$
$$e^{2k} = \frac{1}{3}$$
$$2k = \ln\left(\frac{1}{3}\right)$$
$$k = \frac{1}{2}\ln\left(\frac{1}{3}\right)$$

$t = 3$, $T = 20 + 60\,e^{\frac{1}{2}\ln\left(\frac{1}{3}\right)\cdot 3}$
$$= 20 + 60\,e^{\ln\left(\frac{1}{3}\right)^{3/2}}$$
$$= 20 + 60 \cdot 3^{-3/2}$$

5/(a) (i) $^9C_3 = 84$

(ii) $\frac{^5C_1\ ^3C_1}{84} = \frac{15}{84}$

$= \frac{5}{28}$

(iii) $\frac{^5C_3 + 1}{84} = \frac{11}{84}$

(b) $x^3 - 12x^2 + 12x + 80 = 0$

Let roots be $\alpha - m, \alpha, \alpha + m$

sum of roots: $3\alpha = 12$

$\alpha = 4$

product of roots: $(4-m)4(4+m) = -80$

$16 - m^2 = -20$

$m = \pm 6$

$\therefore$ roots are $-2, 4, 10$

(c) $1 + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n = (1+x)^n$

(i) $x = -1$, $\quad 1 - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^n\binom{n}{n} = 0$

(ii) Integrate both sides wrt x

$$x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \dots + \frac{1}{n+1}\binom{n}{n}x^{n+1} = \frac{1}{n+1}(1+x)^{n+1} + c$$

Let $x = 0$ $\quad \therefore \frac{1}{n+1} + c = 0, \quad c = -\frac{1}{n+1}$

$$x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \dots + \frac{1}{n+1}\binom{n}{n}x^{n+1} = \frac{1}{n+1}(1+x)^{n+1} - \frac{1}{n+1}$$

Let $x = -1$,

$$-1 + \frac{1}{2}\binom{n}{1} - \frac{1}{3}\binom{n}{2} + \dots + (-1)^{n+1}\frac{1}{n+1}\binom{n}{n} = -\frac{1}{n+1}$$

$$1 - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \dots + (-1)^n\frac{1}{n+1}\binom{n}{n} = \frac{1}{n+1}$$

(a) $p = \frac{1}{20}$, $q = \frac{19}{20}$

(i) $(q+p)^{20} = \sum_{r=0}^{20} \binom{20}{r} q^{20-r} p^r$

$P(\text{no colour-blind}) = \binom{20}{0}\left(\frac{19}{20}\right)^{20}$

$= \left(\frac{19}{20}\right)^{20}$

(ii) $P(\text{one colour-blind}) = \binom{20}{1} q^{19} p$

$= \cancel{20} \cdot \left(\frac{19}{20}\right)^{19} \cdot \frac{1}{\cancel{20}}$

$= \left(\frac{19}{20}\right)^{19}$

(iii) $P(\text{at least 2 colour-blind})$

$= 1 - \left(\left(\frac{19}{20}\right)^{20} + \left(\frac{19}{20}\right)^{19}\right)$

(b) $(3+2x)^n = \sum_{r=0}^{n} \binom{n}{r} 3^{n-r} (2x)^r$

$\binom{n}{5} 3^{n-5} (2\cancel{x})^5 = \binom{n}{6} 3^{n-6} (2\cancel{x})^6$

$\binom{n}{5} \div \binom{n}{6} = \frac{3^{n-6}}{3^{n-5}} \cdot \frac{2^6}{2^5}$

$\frac{\cancel{n!}}{(n-5)!\,5!} \cdot \frac{(n-6)!\,6!}{\cancel{n!}} = \frac{2}{3}$

$\frac{6}{n-5} = \frac{2}{3}$

$2n - 10 = 18$

$n = 14$

6(b) cont

$$\frac{t_{r+1}}{t_r} = \frac{{}^{14}C_r\, 3^{14-r} (2x)^r}{{}^{14}C_{r-1}\, 3^{15-r} (2x)^{r-1}}$$ where t_r = rth term in the expansion

$$= \frac{14!}{(14-r)!\,r!} \cdot \frac{(15-r)!\,(r-1)!}{14!} \cdot \frac{2x}{3}$$

Let c_r = coefficient of the rth term

$$\frac{c_{r+1}}{c_r} = \frac{15-r}{r} \cdot \frac{2}{3}$$

$$= \frac{30-2r}{3r}$$

$\frac{c_{r+1}}{c_r} = 1$ when $r = 6$ $\therefore c_6 = c_7$

$\frac{c_{r+1}}{c_r} > 1$ when $r < 6$ $\therefore c_1 < c_2 < c_3 < c_4 < c_5 < c_6$

$\frac{c_{r+1}}{c_r} < 1$ when $r > 6$ $\therefore c_7 > c_8 > c_9 > c_{10} > c_{11} > c_{12} > c$

c_6 & c_7 are greatest coefficients

7(a) Step 1 Let $n=1$

$LHS = (2\times1)^3$
$= 8$ $\therefore$ true for $n=1$

$RHS = 2\times1^2(1+1)^2$
$= 8$

Step 2 Assume result true for $n=k$, k is a positive integer

ie. $2^3 + 4^3 + 6^3 + \ldots + (2k)^3 = 2k^2(k+1)^2$

Step 3 Prove result true for $n=k+1$

ie. prove $2^3 + 4^3 + 6^3 + \ldots + (2k)^3 + (2(k+1))^3 = 2(k+1)^2((k+1)+1)^2$

7(a) cont.

$$LHS = 2k^2(k+1)^2 + (2(k+1))^3 \quad \text{from assumption}$$
$$= 2(k+1)^2(k^2+4(k+1))$$
$$= 2(k+1)^2(k+2)^2$$
$$= RHS$$

Step 4 Result is true for $n=1$. Hence it is true for $n=1+1=2$, $n=2+1=3$ etc. ∴ The result is true for all positive integers

(b) (i) $x = vt\cos\alpha$
$y = vt\sin\alpha - \frac{1}{2}gt^2$

(ii) $p = vt\cos\alpha$
$t = \frac{p}{v\cos\alpha}$

$$h = v \cdot \frac{p}{v\cos\alpha} \cdot \sin\alpha - \frac{1}{2}g\frac{p^2}{v^2\cos^2\alpha}$$

$$= p\tan\alpha - \frac{5p^2}{v^2\cos^2\alpha}$$

$$= p\tan\alpha - \frac{5p^2(\tan^2\alpha+1)}{v^2}$$

$$5p^2(\tan^2\alpha+1) = v^2(p\tan\alpha - h)$$

$$v^2 = \frac{5p^2(\tan^2\alpha+1)}{p\tan\alpha - h}$$

(iii) $$v^2 = \frac{5q^2(\tan^2\alpha+1)}{q\tan\alpha - h}$$

$$\therefore \frac{\cancel{5}p^2(\cancel{\tan^2\alpha+1})}{p\tan\alpha - h} = \frac{\cancel{5}q^2(\cancel{\tan^2\alpha+1})}{q\tan\alpha - h}$$

$$p^2(q\tan\alpha - h) = q^2(p\tan\alpha - h)$$

$$\tan\alpha(p^2q - q^2p) = p^2h - q^2h$$

$$\tan\alpha = \frac{h\cancel{(p-q)}(p+q)}{pq\cancel{(p-q)}} = \frac{h(p+q)}{pq}$$