SYDNEY GRAMMAR SCHOOL

2009 Trial Examination

FORM VI
MATHEMATICS EXTENSION 1

Tuesday 18th August 2009

General Instructions
- Reading time — 5 minutes
- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper
- Total marks — 84
- All seven questions may be attempted.
- All seven questions are of equal value.

Collection
- Write your candidate number clearly on each booklet.
- Hand in the seven questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question 1.

Checklist
- SGS booklets — 7 per boy
- Candidature — 111 boys

Examiner
BDD

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Find $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$. $\boxed{1}$

(b) (i) Without the use of calculus, sketch the polynomial $y = x(x-1)(x+3)$ showing all intercepts with the axes. $\boxed{2}$

(ii) Hence, or otherwise, solve the inequation $\dfrac{x(x-1)}{x+3} \geq 0$. $\boxed{3}$

(c) Differentiate $y = \tan^{-1} x^3$. $\boxed{1}$

(d) When the polynomial $P(x) = x^3 + ax^2 + 7$ is divided by $x + 2$, the remainder is 11. Find a. $\boxed{2}$

(e) Given that A is the point $(1, 2)$ and B is the point $(5, 4)$, find the coordinates of a third point P that divides AB externally in the ratio $1 : 5$. $\boxed{2}$

(f) Find the exact value of $\cos^{-1} \cos\left(\dfrac{7\pi}{6}\right)$. $\boxed{1}$

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Shade the region $y > |x+1|$. $\boxed{2}$

(ii) Write down any real x-values for which the function $f(x) = |x+1|$ is not differentiable. $\boxed{1}$

(b) (i) Shade the region bounded by the curve $y = \sin^{-1} x$, the y-axis and the line $y = \frac{\pi}{2}$. $\boxed{2}$

(ii) A solid is formed by rotating the region shaded in part (i) about the y-axis. Use the formula $V = \pi \displaystyle\int_a^b x^2\,dy$ to find the volume of this solid. $\boxed{3}$

(c) Simplify ${}^{n+1}C_r \div {}^{n}C_{r-1}$. $\boxed{2}$

(d) Let $y = \ln\sqrt{x^2 - 3}$.

(i) Use your knowledge of logarithms to simplify y. $\boxed{1}$

(ii) Hence find y'. $\boxed{1}$

Exam continues next page ...

QUESTION THREE (12 marks) Use a separate writing booklet. Marks

(a) Use the substitution $u = x^2 + 4x - 3$ to find **[3]**

$$\int_1^2 \frac{x+2}{\sqrt{x^2+4x-3}}\,dx\,.$$

(b) A particle moves in simple harmonic motion according to the equation $x = 6 + 3\cos 2t$.

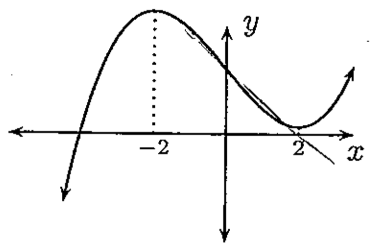
(i) Write down the period of the motion. **[1]**

(ii) In what interval is the particle moving? **[1]**

(iii) The particle begins at maximum displacement. At what time does it first pass through the centre of motion? **[1]**

(iv) Find the first time that the particle has speed $3\,\text{m/s}$. **[2]**

(c)



The graph of $y = x^3 - 12x + 17$ is shown above and a pupil is required to find the single x-intercept.

(i) Use Newton's method to find an approximate solution for the polynomial equation $x^3 - 12x + 17 = 0$. Use the initial value $x_0 = -5$ and record the results of two applications of Newton's Method. **[3]**

(ii) Copy the graph and use it to explain why $x_0 = -1$ would not be a good initial choice. **[1]**

Exam continues overleaf ...

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) Find the term independent of u in the expansion of $\left(3u + \dfrac{9}{u}\right)^{12}$. Leave your answer in the form ${}^{12}\mathrm{C}_p\, 3^q$. $\boxed{3}$

(b) Use induction to prove that $4^n + 14$ is divisible by 6 for all positive integers n. $\boxed{3}$

(c) A projectile is launched across a level plain at 30° to the horizontal and at an initial speed of $60\,\mathrm{m/s}$. Take the origin as the launching point and take $g \doteqdot 10\,\mathrm{m/s^2}$.

(i) Show that the equations of motion are $\boxed{2}$

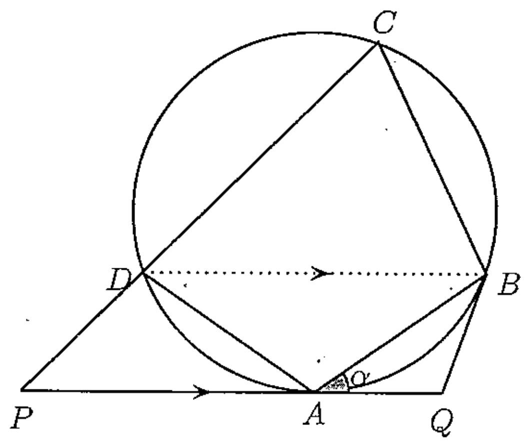
$$x = 30\sqrt{3}t$$
$$y = 30t - 5t^2\,.$$

(ii) Find the maximum height of the projectile. $\boxed{2}$

(iii) Find the exact speed of the projectile one second after launch. $\boxed{2}$

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a)



Let $ABCD$ be a cyclic quadrilateral. The tangents from Q touch the circle at A and B. The diagonal DB is parallel to the tangent AQ and QA produced intersects with CD produced at P. Let $\angle QAB = \alpha$.

Copy the diagram into your answer booklet.

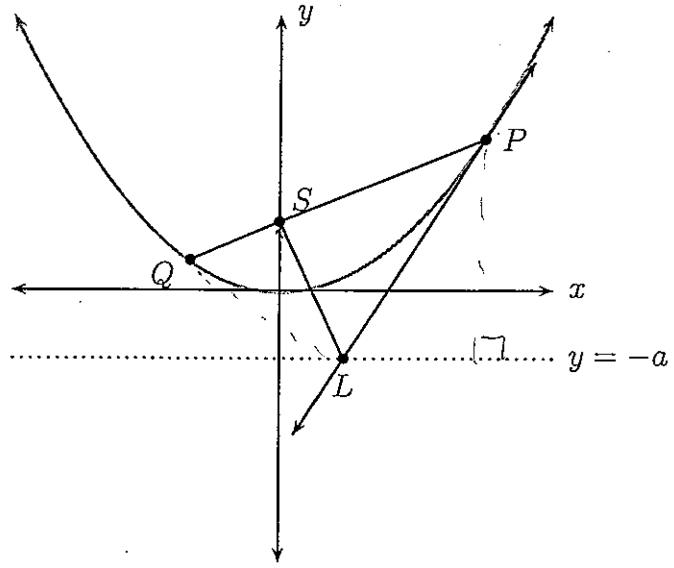
(i) Prove that $\triangle BAD$ is isosceles. $\boxed{2}$

(ii) Find $\angle BCD$ in terms of α. $\boxed{2}$

(iii) Show that P, Q, B and C are concyclic points. $\boxed{2}$

Exam continues next page ...

(b)



Let $x^2 = 4ay$ be a parabola with focus $S(0, a)$.

Let $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ be two points on the parabola and let L be the intersection of the tangent at P and the directrix. Let PQ be a focal chord.

You may assume without proof that $pq = -1$ and that the tangent at P has equation $y = px - ap^2$.

(i) Show that $SP = a(p^2 + 1)$. $\boxed{1}$

(ii) Show that L has coordinates $L(ap - \frac{a}{p}, -a)$. $\boxed{2}$

(iii) Show that $PS \times SQ = SL^2$. $\boxed{3}$

Exam continues overleaf ...

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) Find the horizontal asymptote of the function $y = \dfrac{x^2 \sin \frac{1}{x}}{x+1}$. $\boxed{2}$

(b) By differentiating twice the identity $\boxed{3}$

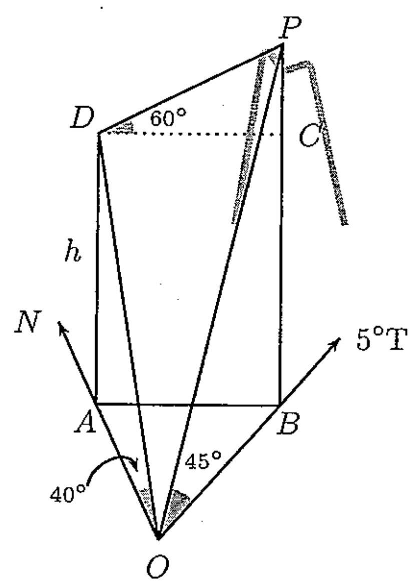
$$(1+x)^n = \sum_{k=0}^{n} {}^n\mathrm{C}_k x^k ,$$

prove that

$${}^2\mathrm{C}_2\,{}^n\mathrm{C}_2 + {}^3\mathrm{C}_2\,{}^n\mathrm{C}_3 + {}^4\mathrm{C}_2\,{}^n\mathrm{C}_4 + \cdots + {}^n\mathrm{C}_2\,{}^n\mathrm{C}_n \;=\; n(n-1)2^{n-3}$$

for any integer $n \geq 2$.

(c)



Flight 451, emerging from a thick bank of cloud, finds itself confronted by the sheer bulk of Mount Massiff. Immediately the pilot takes action, pointing the nose of the plane to climb at 60° to the horizontal at a speed of $180\sqrt{3}$ km/h.

An observer at O, viewing the imminent catastrophe, sees the plane due north at D at height h and angle of inclination 40°. He also sees the mountain peak P at an angle of inclination of 45° on a true bearing of 5°.

The plane needs to reach a height of 5000 metres after 10 seconds to clear the peak at P.

(i) Find the lengths of CD and PC in metres. $\boxed{2}$

(ii) Find expressions for OA and OB in terms of h. $\boxed{2}$

(iii) Prove that $\boxed{2}$

$$h^2(1 + \cot^2 40° - 2\cot 40° \cos 5°) + h(1500 - 1500 \cot 40° \cos 5°) + 375\,000 = 0\,.$$

(iv) Hence determine whether or not the plane will clear the mountain. $\boxed{1}$

Exam continues next page ...

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Use the substitution $t = \tan \frac{1}{2}\theta$ to prove the identity $\boxed{3}$

$$\tfrac{1}{2}(3\cos\theta + 4\sin\theta + 5) = (\sin\tfrac{1}{2}\theta + 2\cos\tfrac{1}{2}\theta)^2$$

where θ is acute.

(ii) Hence use the substitution $\theta = \frac{\pi}{3}$ to find the two square roots of $\frac{1}{4}(13 + 4\sqrt{3})$. $\boxed{1}$

(b) Use the substitution $u = \frac{1}{\sqrt{3}}\tan x$ to evaluate $\boxed{4}$

$$\int_0^{\frac{\pi}{4}} \frac{dx}{3 - 2\sin^2 x}\,.$$

(c) Consider the pair of simultaneous equations

$$y = \sin x \cos x$$
$$y = kx\,.$$

(i) Suppose k is positive. Find any restriction on k so that the equations will have a unique simultaneous solution. $\boxed{1}$

(ii) Suppose k is negative. Show that the pair of equations have a unique simultaneous solution if $k < \cos u$, where u satisfies the equation $\tan u = u$ for $\pi < u < \frac{3\pi}{2}$. $\boxed{3}$

END OF EXAMINATION

BLANK PAGE

BLANK PAGE

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1},\ \ n \neq -1;\ \ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ \ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax},\ \ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

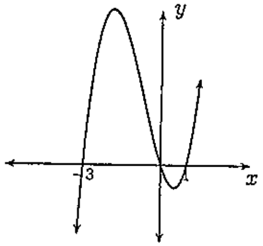
NOTE : $\ln x = \log_e x,\ \ x > 0$

QUESTION ONE (12 marks)

Marks

(a) $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\frac{x}{2} + C$ 1

(b) (i) 2



(ii) 3

$$\frac{x(x-1)}{x+3} \geq 0,\ x \neq -3$$

Multiplying by $(x+3)^2$ gives

$$x(x-1)(x+3) \geq 0$$

Hence $-3 < x \leq 0$ or $x \geq 1$

(c) 1

$$\begin{aligned} y' &= \frac{3x^2}{1+(x^3)^2} \\ &= \frac{3x^2}{1+x^6} \end{aligned}$$

(d) 2

By the remainder theorem, $P(-2) = 11.$

Hence

$$\begin{aligned} (-2)^3 + a(-2)^2 + 7 &= 11 \\ -8 + 4a + 7 &= 11 \\ 4a &= 12 \\ a &= 3 \end{aligned}$$

(e) By the ratio division formula, $P(x, y)$ has coordinates 2

$$\begin{aligned} x &= \frac{lx_1 + kx_2}{k+l} \\ &= \frac{-5\times 1 + 5\times 1}{1-5} \\ &= 0 \end{aligned} \qquad \begin{aligned} y &= \frac{ly_1 + ky_2}{k+l} \\ &= \frac{-5\times 2 + 1\times 4}{1-5} \\ &= 1{\cdot}5 \end{aligned}$$

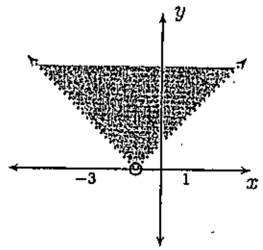
(f) 1

$$\begin{aligned} \theta &= \cos^{-1}\cos\left(\frac{7\pi}{6}\right), \quad 0 \leq \theta \leq \pi \\ &= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) \\ &= \frac{5\pi}{6} \end{aligned}$$

QUESTION TWO (12 marks)

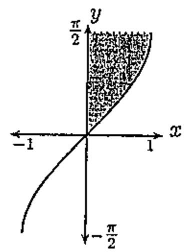
Marks

(a) (i) 2



(ii) It is not differentiable at $x = -1$. 1

(b) (i) 2



(ii) 3

$$\begin{aligned} V &= \pi\int_a^b x^2\,dy \\ &= \pi\int_0^{\frac{\pi}{2}} \sin^2 y\,dy \\ &= \pi\int_0^{\frac{\pi}{2}} \frac{1-\cos 2y}{2}\,dy \\ &= \tfrac{1}{2}\pi\int_0^{\frac{\pi}{2}} 1-\cos 2y\,dy \\ &= \tfrac{1}{2}\pi\Big[y - \tfrac{1}{2}\sin 2y\Big]_0^{\frac{\pi}{2}} \\ &= \tfrac{1}{2}\pi\Big(\big(\tfrac{\pi}{2} - \tfrac{1}{2}\sin\pi\big) - \big(0 - \tfrac{1}{2}\sin 0\big)\Big) \\ &= \tfrac{1}{4}\pi^2 \end{aligned}$$

(c) [2]

$$\begin{aligned} {}^{n+1}C_r \div {}^{n}C_{r-1} &= \frac{(n+1)!}{r!(n+1-r)!} \div \frac{n!}{(r-1)!(n-(r-1))!} \\ &= \frac{(n+1)!}{r!(n+1-r)!} \times \frac{(r-1)!(n-r+1)!}{n!} \\ &= \frac{(n+1)!}{r!} \times \frac{(r-1)!}{n!} \\ &= \frac{n+1}{r} \end{aligned}$$

(d) Let $y = \ln\sqrt{x^2-3}$.

(i) $y = \frac{1}{2} \times \ln(x^2-3)$ [1]

(ii) [1]

$$\begin{aligned} y' &= \frac{1}{2} \times \frac{2x}{x^2-3} \\ &= \frac{x}{x^2-3} \end{aligned}$$

QUESTION THREE (12 marks) Marks

(a) [3]

$$\begin{aligned} \int_1^2 \frac{x+2}{\sqrt{x^2+4x-3}}\,dx &= \int_2^9 \frac{1}{\sqrt{u}}\,\tfrac{1}{2}\,du \\ &= \int_2^9 \tfrac{1}{2}u^{-\frac{1}{2}}\,du \\ &= \left[u^{\frac{1}{2}}\right]_2^9 \\ &= 3-\sqrt{2} \end{aligned}$$

(b) (i) π [1]

(ii) $3 \le x \le 9$ [1]

(iii) After quarter of a period i.e. at $t = \frac{\pi}{4}$ seconds. [1]

(iv) The velocity is $\dot{x} = -6\sin 2t$. [2]

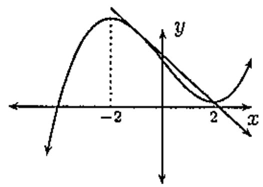
$$\begin{aligned} \text{If } |\dot{x}| = 3,\ 6\sin 2t &= \pm 3 \\ \sin 2t &= \pm\tfrac{1}{2} \\ 2t &= \frac{\pi}{6} \quad \text{(and later times)} \\ t &= \frac{\pi}{12} \end{aligned}$$

(c) (i) [3]

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^3 - 12x_n + 17}{3x_n^2 - 12} \\ &= \frac{2x_n^3 - 17}{3x_n^2 - 12} \end{aligned}$$

Using a calculator, with $x_0 = -5$, we get $x_1 = -\frac{89}{21}$ and $x_2 = -4{\cdot}0408$.

(ii) [1]



The tangent to the graph at $x_0 = -1$ cuts the x-axis near the other turning point. At least initially, the sequence of approximations is not moving towards the root.

QUESTION FOUR (12 marks) Marks

(a) The general term in the expansion of $\left(3u + \frac{9}{u}\right)^{12}$ is [3]

$${}^{12}C_r(3u)^{12-r}\left(\tfrac{9}{u}\right)^r = {}^{12}C_r 3^{12-r} 9^r u^{12-2r}$$

The constant term is indexed by $r = 6$ and is thus

$${}^{12}C_r 3^{12+r} = {}^{12}C_6 3^{18}$$

(b) When $n = 1$, $4^n + 14 = 18 = 6 \times 3$. Hence the result is true for $n = 1$. [3]

Suppose the result is true for some positive integer $n = k$. That is, suppose that $4^k + 14 = 6M$ for some integer M.

We need to show that $4^{k+1} + 14$ is divisible by 6. Now

$$\begin{aligned} 4^{k+1} + 14 &= 4^k \times 4 + 14 \\ &= (6M - 14) \times 4 + 14 \\ &= 24M - 56 + 14 \\ &= 24M - 42 \\ &= 6(4M - 7) \end{aligned}$$

Which is divisible by 6, hence the result holds for $n = k + 1$.

It follows that the result holds for all positive integers, by the principle of mathematical induction.

(c) 2

(i)

Horizontally

$\ddot{x} = 0$

$\dot{x} = C$ (for some constant C)
But $\dot{x} = 60\cos 30°$ when $t = 0$.
Hence $C = 60\cos 30°$.
$\dot{x} = 60\cos 30$

$\dot{x} = 30\sqrt{3}$

$x = 30\sqrt{3}t + D$ (for some constant D)
But $x = 0$ when $t = 0$, since the origin is taken as the point of launch.
Hence $D = 0$ and $x = 30\sqrt{3}$.

Vertically

$\ddot{y} = -g$

$\dot{y} = -10t + E$ (for some constant E)
But $\dot{y} = 60\sin 30°$ when $t = 0$.
Hence $C = 60\sin 30°$.
$\dot{y} = -10t + 60\sin 30$

$\dot{y} = -10t + 30$

$y = -5t^2 + 30t + F$ (for some constant F)
But $y = 0$ when $t = 0$, since the origin is taken as the point of launch.
Hence $F = 0$ and $y = 30t - 5t^2$.

(ii) The maximum height is reached when $\dot{y} = 0$, thus when $t = 3$ seconds. The maximum height is $y = 30(3) - 5(3)^2 = -45 + 90 = 45$ metres. 2

(iii) When $t = 1$, $\dot{x} = 30\sqrt{3}$ and $\dot{y} = -10(1) + 30 = 20$. From a velocity resolution diagram, the speed is 2

$$\sqrt{(30\sqrt{3})^2 + 20^2} = \sqrt{3100} = 10\sqrt{31} \quad (\doteqdot 55{\cdot}7\,\text{m/s})$$

QUESTION FIVE (12 marks)

Marks

(a) (i) 2

$\angle ABD = \alpha$ (alternate angles, $DB \| PQ$)

$\angle BDA = \alpha$ (angle in the alternate segment)

Hence $\triangle ABD$ is isosceles (two equal angles).

(ii) 2

$\angle DAP = \alpha$ (alternate angles, $DB \| PQ$)

$\angle DAB = 180 - 2\alpha$ (straight angle)

$\angle DCB = 2\alpha$ (opposite angles of cyclic quadrilateral)

(iii) 2

$AQ = BQ$ (tangents from the external point Q)
Hence $\triangle ABQ$ is isosceles.
Thus $\angle ABQ = \alpha$ (base angles of isosceles $\triangle ABQ$)

$\angle Q = 180 - 2\alpha$ (angle sum of $\triangle ABQ$)

So $PQBC$ is a cyclic quadrilateral (since the opposite angles are supplementary).

(b) (i) 1

$$\begin{aligned} SP^2 &= (2ap)^2 + (ap^2 - a)^2 \\ &= 4a^2p^2 + a^2(p^2 - 1)^2 \\ &= a^2(p^2 + 1)^2 \end{aligned}$$

Thus $SP = a(p^2 + 1)$.

(ii) We need to intersect the tangent and directrix, that is solve simultaneously the equations 2

$$y = px - ap^2$$
$$y = -a$$

Equating y's gives;

$$\begin{aligned} -a &= px - ap^2 \\ px &= ap^2 - a \\ x &= ap - ap^{-1} \end{aligned}$$

Hence L has coordinates $(ap - ap^{-1}, -a)$.

(iii) Show that $PS \times SQ = SL^2$. 3

$$\begin{aligned} PS \times SQ &= a^2(p^2 + 1)(q^2 + 1) \\ &= a^2(p^2 + 1)\left(\frac{1}{p^2} + 1\right) \\ &= a^2(p^2 + 1)\left(\frac{1 + p^2}{p^2}\right) \\ &= a^2(p + p^{-1})(p + p^{-1}) \\ &= a^2(p + p^{-1})^2 \end{aligned}$$

QUESTION SIX (12 marks)

Marks

(a) 2

$$\begin{aligned} \lim_{x \to \pm\infty} \frac{x^2 \sin\frac{1}{x}}{x + 1} &= \lim_{1/x \to 0} \frac{\frac{\sin 1/x}{1/x}}{1 + 1/x} \\ &= \frac{1}{1 + 0} \\ &= 1 \end{aligned}$$

Thus $y = 1$ is the horizontal asymptote.

(b) 3

Differentiating $\quad (1 + x)^n = \sum_{k=0}^{n} {}^nC_k x^k$

once: $\quad n(1 + x)^{n-1} = \sum_{k=0}^{n} {}^nC_k k x^{k-1}$

twice: $\quad n(n - 1)(1 + x)^{n-2} = \sum_{k=0}^{n} {}^nC_k k(k - 1)x^{k-2}$

$$= 2\sum_{k=0}^{n} {}^nC_k \tfrac{1}{2}k(k - 1)x^{k-2}$$

$$= 2\sum_{k=0}^{n} {}^nC_k {}^kC_2 x^{k-2}$$

let $x=1$: $n(n-1)(2)^{n-2} = 2\sum_{k=0}^{n} {}^{n}C_k\,{}^{k}C_2$ ✓

$\frac{1}{2}n(n-1)(2)^{n-2} = \sum_{k=0}^{n} {}^{n}C_k\,{}^{k}C_2$

$n(n-1)(2)^{n-3} = \sum_{k=0}^{n} {}^{n}C_k\,{}^{k}C_2$ ✓

(c) (i) The speed of the plane is $180\sqrt{3}$ km/h $= 50\sqrt{3}$ m/s. $\boxed{2}$

The distance travelled in 10 seconds is $PD = 500\sqrt{3}$ m. ✓
Hence the horizontal distance travelled $CD = 500\sqrt{3}\cos 60°$
$= 250\sqrt{3}$ m

The vertical distance travelled $PC = 500\sqrt{3}\sin 60°$
$= 750$ m ✓

(ii) $\dfrac{h}{OA} = \tan 40°$ hence $OA = h\cot 40°$. ✓ $\boxed{2}$

$\dfrac{h+750}{OB} = \tan 45°$ hence $OB = (h+750)\cot 45° = h + 750$. ✓

(iii) By the cosine rule in $\triangle OAB$, $\boxed{2}$

$$(250\sqrt{3})^2 = h^2\cot^2 40° + (h+750)^2 - 2h\cot 40°(h+750)\cos 5°$$ ✓✓

$$187500 = h^2(\cot^2 40° + 1 - 2\cot 40°\cos 5°) + h(1500 - 1500\cot 40°\cos 5°) + 562500 = 0$$

Hence

$$h^2(1+\cot^2 40° - 2\cot 40°\cos 5°) + h(1500 - 1500\cot 40°\cos 5°) + 375000 = 0$$

(iv) The two solutions of this equation are $h \doteqdot 4160$ m and $h \doteqdot 1967$ m. In either situation $h + 750$ is less than 5000, so the plane will NOT clear the mountain peak. $\boxed{1}$ ✓

Note: The two solutions for h represent two distant possible situations given the data in this question. In one situation the plane is travelling in a plane tilted slightly towards the observer, and in the other the plane of travel is tilted away from the observer. To explore this further, use the cosine rule to find the angle OAB.

QUESTION SEVEN (12 marks)

Marks

(a) (i) With the substitution $t = \tan\frac{1}{2}\theta$, $\boxed{3}$

$$\begin{aligned} \text{LHS} &= \frac{1}{2}\left(3\times\frac{1-t^2}{1+t^2} + 4\frac{2t}{1+t^2} + 5\right) \checkmark \\ &= \frac{1}{2(1+t^2)}(3 - 3t^2 + 8t + 5 + 5t^2) \\ &= \frac{1}{2(1+t^2)}(8 + 8t + 2t^2) \\ &= \frac{1}{(1+t^2)}(2+t)^2. \checkmark \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \left(\frac{t}{\sqrt{1+t^2}} + \frac{2}{\sqrt{1+t^2}}\right)^2 \\ &= \frac{(t+2)^2}{1+t^2} \checkmark \\ &= \text{LHS} \end{aligned}$$

(ii) Making the substitution $\theta = \frac{\pi}{3}$ gives $\boxed{1}$

$$\tfrac{1}{2}(3\cos\tfrac{\pi}{3} + 4\sin\tfrac{\pi}{3} + 5) = (\sin\tfrac{\pi}{6} + 2\cos\tfrac{\pi}{6})^2$$
$$\tfrac{1}{2}(\tfrac{3}{2} + 2\sqrt{3} + 5) = (\tfrac{1}{2} + \sqrt{3})^2$$ ✓
$$\tfrac{1}{4}(13 + 4\sqrt{3}) = \tfrac{1}{4}(1 + 2\sqrt{3})^2$$

Hence the roots of $\frac{1}{4}(13+4\sqrt{3})$ are

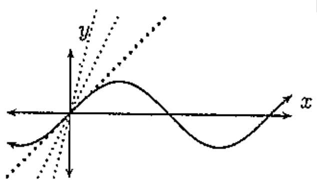
$$\pm\sqrt{\tfrac{1}{4}(13+4\sqrt{3})} = \pm\tfrac{1}{2}(1+2\sqrt{3})$$

(b) With $u = \frac{1}{\sqrt{3}} \tan x$, then $\dfrac{du}{dx} = \frac{1}{\sqrt{3}} \sec^2 x$. Hence $\boxed{4}$

$$
\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{dx}{3 - 2\sin^2 x} &= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{3 - 2\sin^2 x} \frac{\sqrt{3}}{\sec^2 x}\, du \\
&= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{3\sec^2 x - 2\tan^2 x} \sqrt{3}\, du \\
&= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{3(1 + \tan^2 x) - 2\tan^2 x} \sqrt{3}\, du \\
&= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{3 + \tan^2 x} \sqrt{3}\, du \\
&= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{3 + 3u^2} \sqrt{3}\, du \\
&= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{\sqrt{3}} \frac{1}{1 + u^2}\, du \\
&= \left[\frac{1}{\sqrt{3}} \tan^{-1} u \right]_0^{\frac{1}{\sqrt{3}}} \\
&= \frac{\pi}{6\sqrt{3}}
\end{aligned}
$$

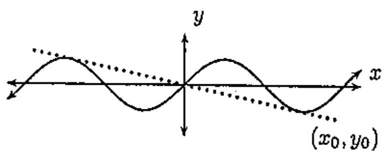
(c) (i) $\boxed{1}$

The graph $y = \sin x \cos x$ may be written as $y = \frac{1}{2}\sin 2x$. The line $y = x$ is a tangent to the curve at $(0, 0)$. It is clear from a graph (and the concavity of the graph) that the tangent meets the curve only once, and that a steeper line $y = kx$, $k > 1$ will only intersect it once, while $y = kx$, $0 < k < 1$ will intersect more than once.



(ii) $\boxed{3}$

There is a line $y = kx$, $k < 0$ that meets the curve in three places, being tangent to two of them. Any steeper line will only meet the curve at the origin, and any line less steep will cut more than three times.



The line that meets the curve in three places is a tangent, hence at some point (x_0, y_0) it intersects with the curve and has the same gradient. This leads to the equations

$$
\begin{aligned}
\tfrac{1}{2}\sin 2x_0 &= kx_0 \quad \text{(by equating } y_0\text{-values)} \\
\cos 2x_0 &= k \quad \text{(by equating gradients)}
\end{aligned}
$$

Dividing these equations gives $\frac{1}{2}\tan 2x_0 = 2x_0$, i.e. $\tan 2x_0 = 2x_0$. Further, the point of tangency is clearly between the root and stationary point, i.e. $\frac{\pi}{2} < x_0 < \frac{3\pi}{4}$. Letting $u = 2x_0$ leads to the equation $\tan u = u$, $\pi < u < \frac{3\pi}{2}$. The value of

k is then found from the equation

$$
\begin{aligned}
k &= \frac{y_0}{x_0} \\
&= \frac{\frac{1}{2}\sin 2x_0}{\frac{1}{2}\tan 2x_0} \\
&= \cos 2x_0 \\
&= \cos u
\end{aligned}
$$

The question doesn't require it, but this equation may be solved using Newton's method. With an initial value of $u = 4{\cdot}6$, we find that $u = 4.4934$ and $k = -0.217$. Hence the line $y = kx$ will cut only once if $k < -0{\cdot}217$.

BDD