Question 1:

(a) Find the acute angle between the lines

$$2x + y = 17 \quad \text{and} \quad 3x - y = 3$$ **2**

(b) Differentiate $y = \tan^{-1}\sqrt{2x^2 - 1}$ **3**

(c) Evaluate $\int_0^3 \frac{y}{\sqrt{y+1}}\,dy$, using the substitution $y = u^2 - 1$ **3**

(d) Eight identical coins show 3 heads and 5 tails.

 (i) In how many ways can they be arranged in a straight line? **1**

 (ii) What is the probability that all the tails will be together? **1**

(e) Solve for x: $\frac{2x-3}{x-2} > 1$ **2**

Question 2: (START A NEW PAGE)

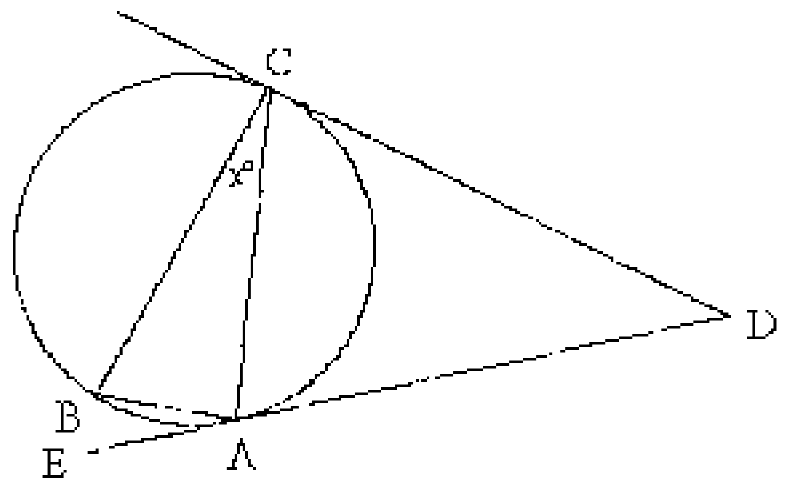
(a) 

Diagram not to scale **4**

AD and CD are tangents to a circle. B is a point on the circle such that $\angle CBA$ and $\angle CDA$ are equal and are both double $\angle BCA$. Prove that BC is a diameter of the circle.

(b) The roots of the equation $9x^2 + 6x + 1 = 4kx$ where k is a real constant, are α and β. Show that the equation with roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is **4**

$$x^2 + 6x + 9 = 4kx$$

(c) Prove by Mathematical Induction that

$1\times 2^0 + 2\times 2^1 + 3\times 2^2 + \dots + n\times 2^{n-1} = 1 + (n-1)2^n$ **4**

for all integers $n \geq 1$.

Question 3: (START A NEW PAGE)

(a) The angle of elevation of a tower PQ of height h metres at a point A due east of it is 15^0. From another point B, the bearing of the tower is 032^0T and the angle of elevation is 13^0. The points A and B are 500 metres apart and on the same level as the base Q of the tower.

(i) Draw a neat sketch showing all the information on your diagram **1**

(ii) Show that $\angle AQB = 122^0$. **1**

(iii) Calculate the height of the tower PQ to the nearest metre. **2**

(b) The speed v m/s of a particle moving in a straight line is given by

$$v^2 = 64 - 16x - 8x^2$$

where the displacement from a fixed point O is x metres.

(i) Find an expression for the acceleration and show the motion is simple harmonic. **2**

(ii) Find the period of the motion **1**

(iii) Find the amplitude of the motion **1**

(c) (i) Find the largest possible domain for which $f(x) = \sin^{-1}(2x+1)$ defines a function **1**

(ii) Hence find and sketch $f^{-1}(x)$, stating its domain and range. **3**

Question 4: (START A NEW PAGE)

(a) N is the number of kangaroos in a certain population at time t years.

The population size N satisfies the equation

$$\frac{dN}{dt} = -k(N - 500)\text{, for some constant } k.$$

(i) Verify that $N = 500 + Ae^{-kt}$ with A constant, is a solution of the equation **1**

(ii) Initially, there are 3500 kangaroos but after 3 years there are only 3300 left. Find the values of A and k. **2**

(iii) Find when the number of kangaroos begins to fall below 2300 **2**

(iv) Sketch the graph of the population size against time **2**

(b) An urn contains 6 cards numbered 1, 2, 3, 4, 5, 6. One card is drawn at random and a second card is drawn without the first card being replaced. Find the probability that: -

(i) the second number is 3 **1**

(ii) the larger number is 5 **2**

(iii) the larger number is even **2**

Question 5: (START A NEW PAGE)

(a) At an air show, a Harrier Jump Jet leaves the ground 200 metres from an observer and rises vertically at the rate of 25 m/sec. At what rate is the observer's angle of elevation of the aircraft changing when the jet is 500 metres above the ground? **3**

Question 5 continued over page.....

(b) A chord joining the points $P(2p, p^2)$ and $Q(2q, q^2)$ on the parabola $x^2 = 4y$ passes through the point (0,-1)

(i) Find the coordinates of M, the midpoint of PQ, as a function of m, the gradient of the chord **3**

(ii) Show that the cartesian equation of the locus of M is

$$x^2 = 2(y+1) \text{ for } |x| \geq 2.$$ **2**

(c) (i) Express $\sin x + \sqrt{3}\cos x$ in the form $A\cos(x+\alpha)$. **2**

(iii) Hence solve $\sin x + \sqrt{3}\cos x = 1$ for $0 \leq x \leq 2\pi$. **2**

<u>Question 6:</u> (START A NEW PAGE)

(a) The deck of a ship was $1 \cdot 4\,m$ below the level of a wharf at low tide and $0 \cdot 6\,m$ above wharf level at high tide. Low tide was at 8:24 am and hightide at 2:40pm. If tide's motion is simple harmonic, find the first time after low tide that the deck was level with the wharf. **4**

(b) Steven borrows \$50 000 to pay for a new car. He plans to repay the loan by making 60 equal monthly instalments. Interest is charged at the rate of $0 \cdot 6$ % per month on the balance owing.

(i) Show that immediately after making two monthly instalments of \$$P$, the balance owing is given by

$$\$(50\,601 \cdot 80 - 2 \cdot 006P)$$ **2**

(ii) Calculate the value of each monthly instalment **2**

(c) A particle is projected with an initial velocity of 60 m/s at an angle of 45° to the horizontal. (use $g = 10ms^{-2}$)

(i) Calculate the greatest height reached by the particle. **3**

(ii) What is the speed of the particle at the greatest height? **1**

Question 7: (START A NEW PAGE)

(a) In a box, there are 10 black counters (each marked with the digit "2") and 5 white counters (each marked with digits "3"). 4 counters are withdrawn one at a time, the first being replaced before the second is drawn. Find the probability that

(i) 2 blacks and 2 white counters are drawn in any order **2**

(ii) The sum of digits on the counters drawn is greater than 9 **3**

(b) (i) Show that $(1+x)^m(1-\frac{1}{x})^m = (x-\frac{1}{x})^m$ **1**

(ii) By considering the term(s) independent of x in the expansion of the result from part (b) (i), justify the result: **3**

$$\binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \dots - \binom{2002}{2002}^2 = -1\binom{2002}{1001}$$

(iii) Hence, or otherwise, show that: **3**

$$\sum_{k=0}^{1001}(-1)^k\binom{2002}{k}^2 = -\frac{1}{2}\binom{2002}{1001}\left[1+\binom{2002}{1001}\right].$$

END OF PAPER

RIAL H.S.C. EXTENSION 1 2002

QU 1:

(a) $2x+y=17 \quad m_1=-2$

$3x-y=3 \quad m_2=3$

$\tan\theta = \left|\frac{-2-3}{1+-2\times3}\right|$

$=1$

$\therefore \theta = 45^\circ$ (2)

(b) $y = \tan^{-1}\sqrt{2x^2-1}$

$\frac{dy}{dx} = \frac{1}{(2x^2-1)+1} \times \frac{4x}{2\sqrt{2x^2-1}}$

$= \frac{1}{2x^2} \times \frac{2x}{\sqrt{2x^2-1}}$

$= \frac{1}{x\sqrt{2x^2-1}}$ (3)

(c) $\int_0^3 \frac{y}{\sqrt{y+1}}dy = \int_1^2 \frac{u^2-1}{u}\cdot 2u\,du$

$= \int_1^2 (2u^2-2)\,du$

$= \left[\frac{2u^3}{3} - 2u\right]_1^2$

$= 2\frac{2}{3}$

$y=u^2-1$

$\frac{dy}{du}=2u$

$dy = 2u\,du$

When $y=0$ $u=1$

When $y=3$ $u=2$ (3)

(d)(i) No. of ways $= \frac{8!}{5!\,3!}$

$= 56$ (1)

(ii) Pr(all tails tog.) $= \frac{4}{56}$

$= \frac{1}{14}$

TTTTTHHH

HTTTTTHH (1)

HHTTTTTH

HHHTTTTT

(e) $\frac{2x-3}{x-2} \geq 1$

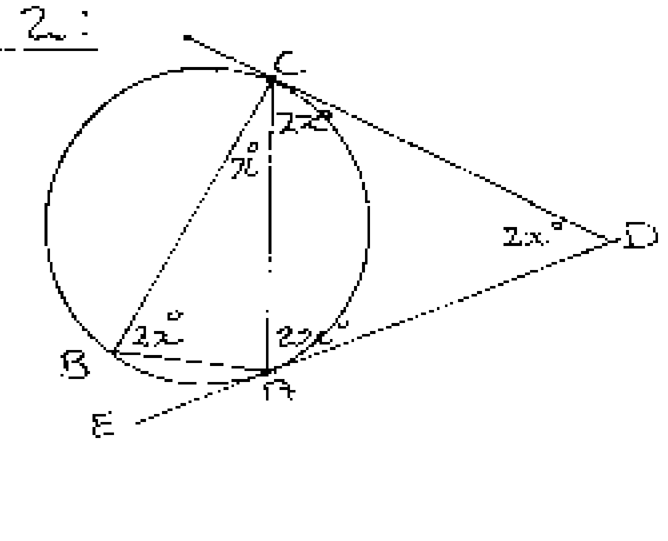
$\frac{2x-3}{x-2} - \frac{x-2}{x-2} \geq 0$

$\frac{x-1}{x-2} \geq 0$

$\therefore x \leq 1$ and $x > 2$ (2)

Qu 2:

(a)



$\angle CBA = \angle CDA = 2x^\circ$ (given)

$\angle DAC = \angle CBA$ (angle between a tangent & a chord equals angle in the alternate segment)

Similarly (1)

$\angle DCA = \angle CBA = 2x^\circ$

$\therefore$ In $\triangle CDA$ $\quad 2x+2x+2x=180^\circ$ (angle sum $\triangle$)

$\therefore x = 30^\circ$ (1)

In $\triangle BAC$

$x^\circ + 2x^\circ + \angle BAC = 180^\circ$

$30^\circ + 60^\circ + \angle BAC = 180^\circ$ (1)

$\therefore \angle BAC = 90^\circ$

$\therefore$ BC is a diameter (angle in semi-circle is 90°) (1)

(b) $9x^2+6x+1 = 4kx$

$9x^2 + (6-4k)x + 1 = 0$

$\alpha+\beta = \frac{4k-6}{9}$ (1)

$\alpha\beta = \frac{1}{9}$

$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{4k-6}{1} = \frac{-b}{a}$ (1)

$\frac{1}{\alpha}\times\frac{1}{\beta} = \frac{9}{1} = \frac{c}{a}$

$\therefore a=1$

$b = 6-4k$ (1)

$c = 9$

$\therefore$ Eqn. is $x^2 + (6-4k)x + 9 = 0$ (1)

$x^2+6x+9 = 4kx$

(c) Let P(n) be proposition

$1\times2^0 + 2\times2^1 + 3\times2^2 + \ldots + n\times2^{n-1} = 1+(n-1)2^n$

Step 1: For P(1)

$1\times2^0 = 1+(1-1)2^0$

$1 = 1$ (1)

$\therefore$ P(1) is true

Step 2: Assume that P(k) is true for some integer $k\geq1$

i.e. P(k): $1\times2^0+2\times2^1+3\times2^2+\ldots+k\times2^{k-1} = 1+(k-1)2^k$

and RTS P(k+1) is true. (1)

Proof: For P(k+1)

$1\times2^0+2\times2^1+3\times2^2+\ldots+k\times2^{k-1}+(k+1)2^k$

$= 1+(k-1)2^k + (k+1)2^k$

$= 1+((k-1)+(k+1))2^k$

$= 1+2k\cdot2^k$ (2)

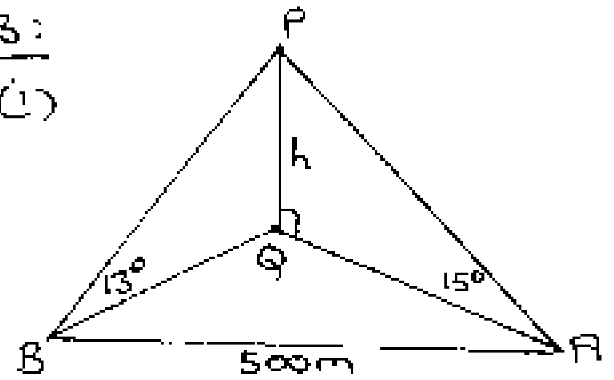
$= 1+k\cdot2^{k+1}$

$\therefore$ P(k+1) is true.

Step 3: If the result is true for P(1), assumed true for P(k) and proven true for P(k+1) then it is true for all positive integral values of n.

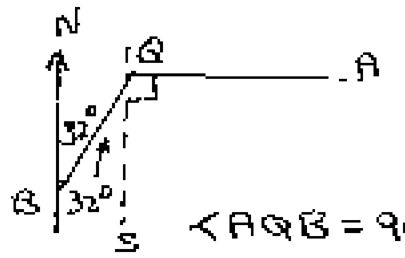
Qu 3:

(a) (i)



(1)

(ii)



$\angle AQB = 90^\circ + 32^\circ$ (1)

$\angle AQB = 122^\circ$ as $QA \perp QS$ + $\angle NBQ = 32^\circ$

(iii) In $\triangle APQ$ In $\triangle PQB$

$h = AQ \cot 15^\circ$ $h = BQ \cot 13^\circ$

In $\triangle ABQ$

$500^2 = h^2\cot^2 13^\circ + h^2\cot^2 15^\circ - 2h^2\cot 13^\circ\cot 15^\circ\cos 122^\circ$

$h^2 = \dfrac{500^2}{\cot^2 13^\circ + \cot^2 15^\circ + 2\cot 13^\circ\cot 15^\circ\cos 58^\circ}$

$h = \dfrac{500}{\sqrt{\cot^2 13^\circ + \cot^2 15^\circ + 2\cot 13^\circ\cot 15^\circ\cos 58^\circ}}$ (2)

$\therefore h = 71$ m (nearest m)

(b) $v^2 = 64 - 16x - 8x^2$

(i) $\frac{1}{2}v^2 = 32 - 8x - 4x^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -8 - 8x$

$\therefore \ddot{x} = -8(x+1)$ (2)

$\therefore$ Motion is SHM centre at $x = -1$

(ii) Period $= \dfrac{2\pi}{2\sqrt{2}} = \dfrac{\pi}{\sqrt{2}}$ sec as $n = 2\sqrt{2}$ (1)

(iii) For motion to exist $v^2 \geq 0$

$8(4+x)(2-x) \geq 0$

$-4 \leq x \leq 2$ (1)

$-4 \quad -1 \quad 2$

$\therefore$ Amplitude $= 3$ m

(c) (i) $D: -1 \leq 2x + 1 \leq 1$

$-2 \leq 2x \leq 0$

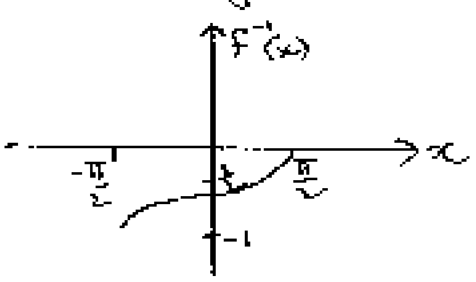
$-1 \leq x \leq 0$ (1)

(ii) $x = \sin^{-1}(2y+1)$

$y = \frac{1}{2}(\sin x - 1)$

$D_{f^{-1}} = -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ (2)

$R_{f^{-1}} = -1 \leq y \leq 0$



(1)

Qu 4:

(a) (i) $N = 500 + Ae^{-kt}$

$\frac{dN}{dt} = -kAe^{-kt}$ (1)

$\frac{dN}{dt} = -k(N - 500)$ as $Ae^{-kt} = N - 500$

(ii) $A = 3000$ when $t = 0$

When $t = 3$ $N = 3300$

$3300 = 500 + 3000e^{-3k}$

$\ln\frac{14}{15} = -3k$ (2)

$k = -\frac{1}{3}\ln\frac{14}{15}$

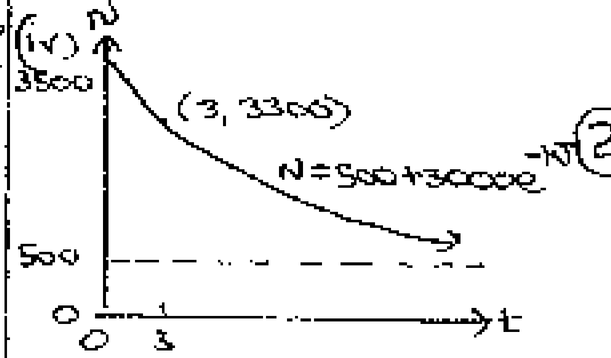
(iii) $500 + 3000e^{-kt} \leq 2300$

$e^{-kt} \leq \frac{1800}{3000}$ (2)

$\therefore t \geq 22.21$ yrs

$\therefore$ No. falls below 2300 when $t \doteq 22.21$ yrs.

(iv)



(2)

(b) (i) Sample space $= {}^6P_2 = 30$

No. of favourable events $= 5$

i.e. (1,3) (2,3) (4,3) (5,3) (6,3)

$P(\text{2nd. no. is } 3) = \frac{5}{30} = \frac{1}{6}$ (1)

(ii) Sample space $= {}^6C_2 = 15$

No. of favourable events $= 4$ (2)

$P(\text{larger no. is a } 5) = \frac{4}{15}$

(iii) $n(S) = {}^6C_2 = 15$

5 has 2 larger even no's.

3 " 4 " " "

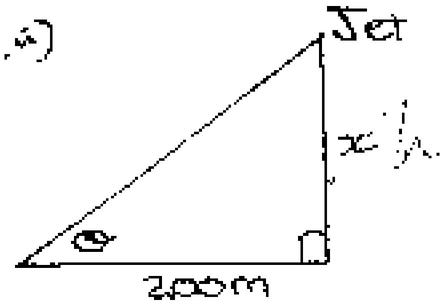
1 " 6 " " "

9 $\therefore n(E) = 9$

$P(\text{larger no. even}) = \frac{9}{15} = \frac{3}{5}$ (2)

5:

(a)



Given $\frac{dh}{dt} = 25$

$\tan\theta = \frac{h}{200}$

$\theta = \tan^{-1}\frac{h}{200}$ (1)

$\frac{d\theta}{dh} = \frac{200}{200^2 + h^2}$

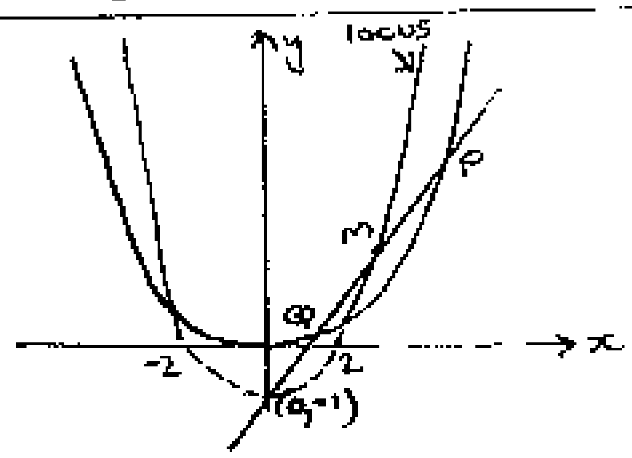
$\left[\frac{d\theta}{dh}\right]_{h=500} = \frac{200}{290000}$ (1)

Now $\frac{d\theta}{dt} = \frac{d\theta}{dh} \times \frac{dh}{dt}$

$= \frac{2}{2900} \times 25$ (1)

$= \frac{1}{58}$ rads/sec.

(b)



(i) $y + 1 = mx$ .... (1)

$x^2 = 4y$ .... (2)

$\therefore x^2 = 4(mx - 1)$ on subst.

$x^2 - 4mx + 4 = 0$

$\frac{x_p + x_q}{2} = x_m$ where x_p, x_q are roots

$2x_m = x_p + x_q = \alpha + \beta = 4m$

$\therefore x_m = 2m$ .... (3)

$y_m = m(2m) - 1$ subst. in (1)

$M(2m, 2m^2 - 1)$ ... (3)

(ii) For locus $x^2 - 4mx + 4 = 0$

$16m^2 - 16 \geqslant 0$

$|m| \geqslant 1$ (1)

Subst. in (3)

$|x| \geqslant 2$

$x = 2m$ .... (4)

$y = 2m^2 - 1$ .... (5)

Square (4) Sub in (5) i.e. $x^2 = 2(2m^2)$

$\therefore y + 1 = \frac{x^2}{2}$

$x^2 = 2(y+1)$ (1)

(c)(i) $\sin x + \sqrt{3}\cos x = A\cos(x + \alpha)$

$A = \sqrt{(1)^2 + (\sqrt{3})^2}$

$\therefore A = 2$

$\sin x + \sqrt{3}\cos x = 2\cos x\cos\alpha - 2\sin x\sin\alpha$

$\therefore \sin\alpha = -\frac{1}{2}$ 3rd & 4th quad.

$\cos\alpha = \frac{\sqrt{3}}{2}$ 1st + 4th quad.

$\tan\alpha = -\frac{1}{\sqrt{3}}$ (2)

$\therefore \sqrt{3}\sin x + \cos x = 2\cos\left(x - \frac{\pi}{6}\right)$

OR $= 2\cos\left(x + \frac{11\pi}{6}\right)$

(ii) $\sin x + \sqrt{3}\cos x = 1 \quad 0 \leqslant x \leqslant 2\pi$

$2\cos\left(x - \frac{\pi}{6}\right) = 1$

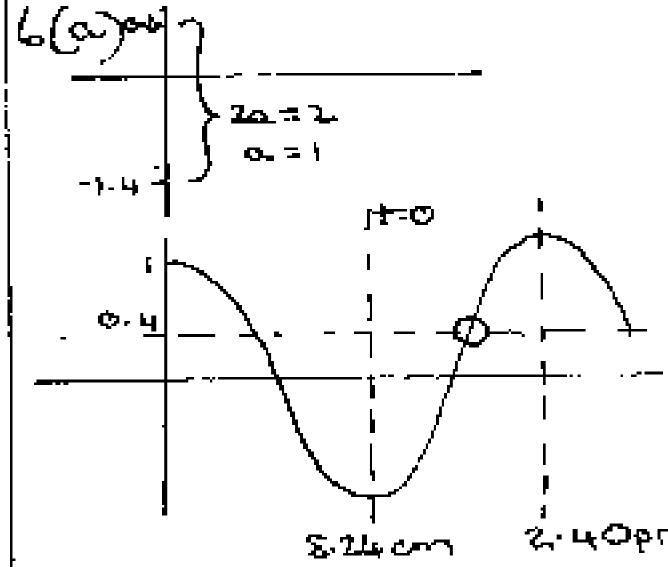
$\cos\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$

$x - \frac{\pi}{6} = \frac{\pi}{3} \Rightarrow x = \frac{\pi}{2}$ (2)

$x - \frac{\pi}{6} = \frac{5\pi}{3} \Rightarrow x = \frac{11\pi}{6}$

$x = \frac{\pi}{2}, \frac{11\pi}{6}$

6(a)(i)



(4)

$\frac{T}{2} = 6\text{hrs }16\text{min} = 6\frac{4}{15}$

$\frac{T}{2} = \frac{2\pi}{2n}$

$n = \frac{15\pi}{94}$

$\therefore 0.4 = -\cos\frac{15\pi}{94}t$

$\therefore t = \frac{94}{15\pi}\left[2n\pi \pm \cos^{-1}\left(-\frac{2}{5}\right)\right]$

$= 3\text{hrs } 57\text{ min}$

$\therefore$ First time after low tide deck is level with wharf is 12.21pm.

(b)(i) Money owing after 1st payment

$= \$50000 \times 1.006 - P$

Money owing after 2nd payment

$= \$50000 \times 1.006^2 - P(1 + 1.006)$

Balance $= \$50601.80 - 2.006P$ (2)

(ii) $50000 \times 1.006^{60} - \frac{P(1.006^{60} - 1)}{0.006} = 0$

$\therefore P = \frac{50000 \times 1.006^{60}}{\frac{1.006^{60} - 1}{0.006}}$

$\therefore P = \$994.78$

6(c)(i) $\ddot{y} = -10$

$\dot{y} = -10t + 30\sqrt{2}$ sub $t=0$, $\dot{y} = 60\sin 45^\circ = 30\sqrt{2}$

$y = 30\sqrt{2}t - 5t^2$

Greatest height is when $\dot{y} = 0$

$-10t + 30\sqrt{2} = 0$

$t = 3\sqrt{2}$ sec

Subst. in y for greatest height

$y = 30\sqrt{2}(3\sqrt{2}) - 5(3\sqrt{2})^2$

$y = 90$m (3)

(ii) At greatest height entire speed is horizontal

$\dot{x} = 60\cos 45^\circ$

$= 30\sqrt{2}$ m/s (1)

Qu7:

(a) 4 chosen (2B, 2W) $= \frac{4!}{2!2!}$ ways

(i) $P(2B, 2W) = 6\left(\frac{10}{15}\right)^2\left(\frac{5}{15}\right)^2$

$= 6\left(\frac{4}{9}\right)\left(\frac{1}{9}\right)$ (2)

$= \frac{8}{27}$

(ii) Listing

$P(2W, 2B) = 6\left(\frac{10}{15}\right)^2\left(\frac{5}{15}\right)^2$

$P(3W, 1B) = \frac{4!}{3!}\left(\frac{10}{15}\right)\left(\frac{5}{15}\right)^3$

$P(4W) = \frac{4!}{4!}\left(\frac{5}{15}\right)^4$

$P(\text{sum} > 9) = \frac{5^2}{15^4}\left(6\times10^2 + 4\times10\times5 + 25\right)$

$= \frac{11}{27}$ (3)

(b)(i) $(1+x)^m\left(1-\frac{1}{x}\right)^m = \left[(1+x)\left(1-\frac{1}{x}\right)\right]^m$

$= \left[1 - \frac{1}{x} + x - 1\right]^m$

$= \left[x - \frac{1}{x}\right]^m$ (1)

(ii) Letting $m = 2002$

LHS $= (1+x)^{2002}\left(1-\frac{1}{x}\right)^{2002}$

$= \left[\binom{2002}{0} + \binom{2002}{1}x + \cdots \binom{2002}{r}x^r + \cdots \binom{2002}{2002}x^{2002}\right]$

$\times\left[\binom{2002}{0} - \binom{2002}{1}\frac{1}{x} + \cdots + (-1)^r\binom{2002}{r}\frac{1}{x^r} + \cdots + \binom{2002}{2002}\frac{1}{x^{2002}}\right]$ (1)

Coeff. of x^0 in LHS is

$\binom{2002}{0}\times\binom{2002}{0} + \binom{2002}{1}\times -\binom{2002}{1} + \cdots + (-1)^r\binom{2002}{r}^2 + \cdots + \binom{2002}{2002}^2$ (1)

i.e. $\binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \cdots + (-1)^r\binom{2002}{r}^2 + \cdots + \binom{2002}{2002}^2$

RHS $= \left(x - \frac{1}{x}\right)^{2002}$

General term is $\binom{2002}{r}x^{2002-r}\left(\frac{-1}{x}\right)^r$

$= (-1)^r\binom{2002}{r}x^{2002-2r}$

$\therefore$ coeff. of x^0 occurs when $2002 - 2r = 0$ (1)

i.e. $r = 1001$

$\therefore$ coeff. is $(-1)^{1001}\binom{2002}{1001} = -1\binom{2002}{1001}$

$\therefore$ LHS = RHS

(iii) L.H.S. $= \binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \cdots + \binom{2002}{1000}^2 - \binom{2002}{1001}^2 + \binom{2002}{1002}^2 - \cdots + \binom{2002}{2000}^2 - \binom{2002}{2001}^2 + \binom{2002}{2002}^2$

$= 2\left[\binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \cdots + \binom{2002}{1000}^2\right] - \binom{2002}{1001}^2$ as ${}^nC_r = {}^nC_{n-r}$ (1)

$= 2\left[\binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \cdots + \binom{2002}{1000}^2 - \binom{2002}{1001}^2\right] + \binom{2002}{1001}^2$ (1)

$\therefore -1\binom{2002}{1001} = 2\Sigma + \binom{2002}{1001}^2$

$2\Sigma = -1\binom{2002}{1001} - \binom{2002}{1001}^2$

$\therefore \sum_{k=0}^{1001}(-1)^k\binom{2002}{k}^2 = -\frac{1}{2}\binom{2002}{1001}\left[1 + \binom{2002}{1001}\right]$ (1)