Year 12 3u THSC 2000 PSJ

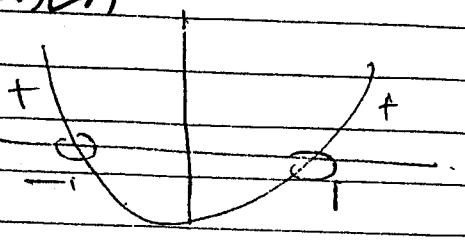
KAMBALA

Q1 (a) $\frac{x^2-1}{x} > 0$

$x \neq 0$: $x^2 - 1 > 0$

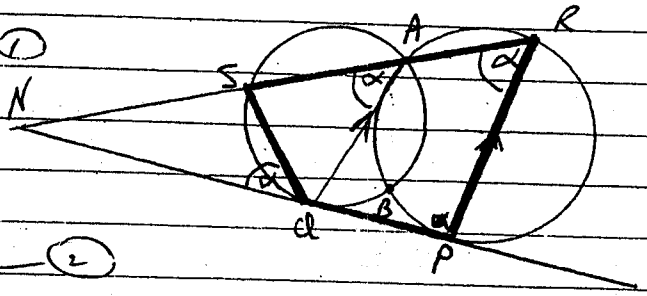
$(x-1)(x+1) > 0$

$x > 1$, $x < -1$ — 2



(b) $\angle SAQ = \angle ARP = \alpha$ — (1)

(corresp. $\angle$s = in // lines PR, AQ)



$\angle SAQ = \angle SQN = \alpha$ — (2)

(angle between a tangent and a chord = angle in the alternate segment)

$\therefore \angle SQN = \angle SRP$

$\therefore$ PQRS is a cyclic quad (ext $\angle$ = interior opp $\angle$)

(c) $\int_0^{2/3} \frac{dx}{4+9x^2} = \frac{1}{9}\int_0^{2/3} \frac{dx}{\frac{4}{9}+x^2} = \frac{1}{9} \cdot \frac{3}{2}\left[\tan^{-1}\left(\frac{3x}{2}\right)\right]_0^{2/3}$

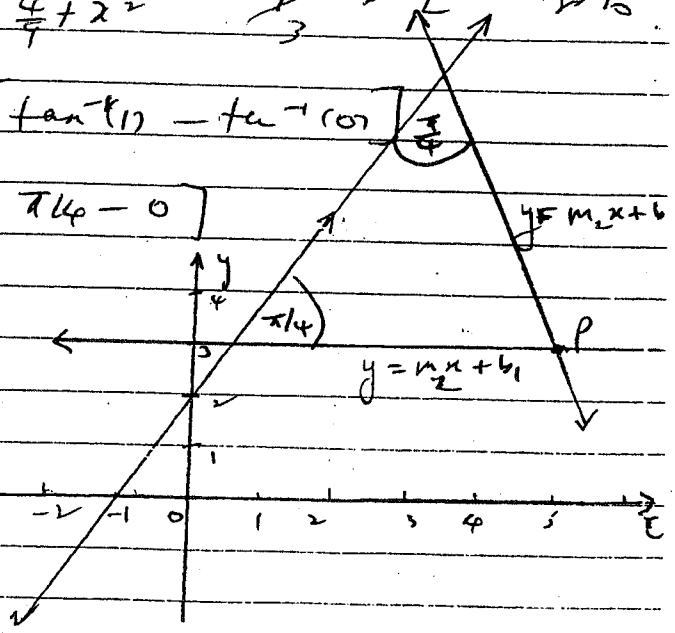
$= \frac{1}{6}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$

$= \frac{1}{6}\left[\frac{\pi}{4} - 0\right]$

$= \frac{\pi}{24}$

(d) $2x - y + 2 = 0$

x	0	−1
y	2	0



$m_1 = 2$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \Rightarrow$

Q1(d) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$m_2 = \frac{1}{3}$, (5, 3)

$y - y_1 = m(x - x_1)$

$y - 3 = \frac{1}{3}(x - 5)$

$3y - 9 = x - 5$

$x - 3y + 4 = 0$

$\tan\pi/4 = \left|\frac{2 - m_2}{1 + 2m_2}\right|$

$1 = \left|\frac{2 - m_2}{1 + 2m_2}\right|$

$m_2 = -3$, (5, 3)

4

$1 + 2m_2 = \pm(2 - m_2)$

$y - 3 = -3(x - 5)$

$y - 3 = -3x + 15$

$3x + y - 18 = 0$

$1 + 2m_2 = 2 - m_2$

$3m_2 = 1$

$m_2 = 1/3$

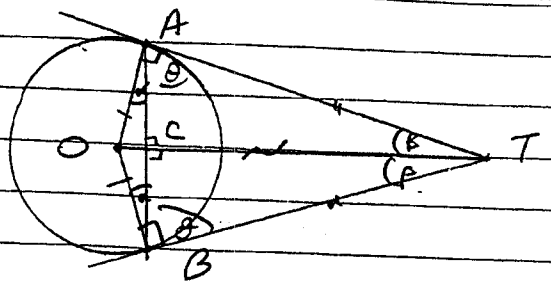
$1 + 2m_2 = -2 + m_2$

$m_2 = -3$

Q2 (a) Let $\angle OAB = \alpha$

and $\angle ATB = 2\beta$.

and $\angle BAT = \theta$



$\Delta OAT \equiv \Delta OBT$ (RHS)

$\therefore \angle ATO = \angle BTO = \beta$

(Corresp. $\angle$s of $\equiv \Delta$s are =)

$\Delta ACT \equiv \Delta BCT$ (SAS)

$\therefore \angle ACT = \angle BCT = 90°$ (= supp. $\angle$s of a St. angle)

4

In ΔOAT, $\alpha + \theta = 90°$ (compl. $\angle$'s of a $\angle$)

In ΔCAT, $\beta + \theta = 90°$ ($\angle$ Sum of a Δ)

$\therefore \alpha = \beta$.

$\therefore \angle ATB = 2\beta = 2\alpha = 2 \times \angle OAB$.

$\therefore \boxed{\frac{\angle OAB}{\angle ATB} = \frac{1}{2}}$

(b) $\sin 2\theta = \cos\theta$

$2\sin\theta\cos\theta = \cos\theta$

$\cos\theta(2\sin\theta - 1) = 0$.

$\therefore \cos\theta = 0$ or $\sin\theta = \frac{1}{2}$

$\boxed{\theta = n\pi}$ or $\boxed{\theta = 2n\pi \pm \pi/3}$

(c) $3^{3n} + 2^{n+2}$ is a multiple of 5.

S1 $n = 1$ $\quad 3^{3n} + 2^{n+2} = 3^3 + 2^3 = 27 + 8 = 35 = 5 \times 7$

$\therefore$ True for $n = 1$

S2 $n = k$: $\quad 3^{3k} + 2^{k+2} = 5P$

$\therefore 3^{3k} = 5P - 2^{k+2}$ ——— (A)

S3: $n = k+1$ $\quad 3^{3(k+1)} + 2^{(k+1)+2} = 5Q$ ——— (B)

(B): LHS $= 3^{3k+3} + 2^{k+3}$

$= 27(3^{3k}) + 2(2^{k+2})$

$= 27(5P - 2^{k+2}) + 2(2^{k+2})$

$= 5(27P) - 2^{k+2}(27 - 2)$

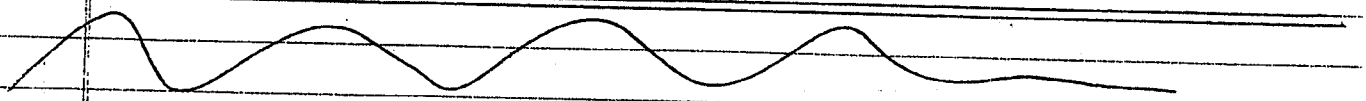
Q2 (c) LHS = $5(2^{7p}) - 2^{k+1}(25)$

$= 5\left[2^{7p} - 5(2^{k+1})\right]$

$= 5Q$

$= RHS$

$\therefore$ By Induction true for all the integral n.



Q3 (a)

A (7, 0) B (3, 4)

$+m : n$

$$P = \left(\frac{+3m+7n}{+m+n}, \frac{4m+0.n}{m+n} \right)$$

$$\therefore P = (1, 6)$$

$$\therefore \frac{3m+7n}{m+n} = 1$$

3

$$3m + 7n = m + n$$

$$2m = -6n$$

$$m = -3n$$

$$\therefore \frac{m}{n} = \frac{-3}{1}$$

$$\therefore m : n = -3 : 1$$

(b) $\frac{d}{dx}\left(x \cdot \sin^{-1} 2x + \frac{1}{2}\sqrt{1-4x^2} \right)$

$$= \sin^{-1} 2x \cdot 1 + x \cdot \frac{2}{\sqrt{1-4x^2}} + \frac{1}{2} \cdot \left(\frac{1}{2}\right) \cdot (1-4x^2)^{-\frac{1}{2}} \cdot (-8x)$$

$$= \sin^{-1}(2x) + \frac{2x}{\sqrt{1-4x^2}} - \frac{2x}{\sqrt{1-4x^2}}$$

$$= \sin^{-1}(2x)$$

4

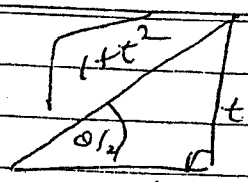
$$\therefore \int_0^{1/2} \sin^{-1}(2x)\, dx = \left[x \sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} \right]_0^{1/2}$$

$$= \left[\frac{1}{2}\sin^{-1}(1) + 0 \right] - \left[0 + \frac{1}{2} \right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 1\right)$$

Q3 (i) $t = \tan\frac{\theta}{2}$



$\tan\frac{\theta}{2} = \frac{t}{1}$

$\sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}}$

$\cos\frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}$

$\cos\theta = \cos\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$= \frac{1}{1+t^2} - \frac{t^2}{1+t^2}$

$\therefore \cos\theta = \frac{1-t^2}{1+t^2}$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= 2\frac{t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}}$

$\therefore \sin\theta = \frac{2t}{1+t^2}$

2

(ii) $3\sin\theta + 4\cos\theta = 3$

$3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right) = 3$

$6t + 4 - 4t^2 = 3 + 3t^2$

$9t^2 - 6t + 1 = 0$

3 $(3t-1)^2 = 0$

$t = 1/3$

$\tan\frac{\theta}{2} = \frac{1}{3}$

$\frac{\theta}{2} = 18°26'$

$\therefore \theta = 36°52'$

Q4 (a) (i)

$$\int_0^3 \frac{x}{\sqrt{4+x}}\,dx = I$$

$u = 4 + x$ $\quad$ $dx = du$

x	0	3
u	4	7

$$\therefore I = \int_4^7 \frac{4-u}{\sqrt{u}}\,du$$

$$= \int_4^7 \frac{4}{\sqrt{u}} - \sqrt{u}\,du$$

$$= \int_4^7 4u^{-1/2} - u^{1/2}\,du$$

$$= \left[\frac{4u^{1/2}}{1/2} - \frac{u^{3/2}}{3/2}\right]_4^7$$

$$= \left[8\sqrt{u} - \frac{2}{3}\sqrt{u^3}\right]_4^7$$

$$= \left(8\sqrt{4} - \frac{2}{3}\sqrt{4^3}\right) - \left(8\sqrt{1} - \frac{2}{3}\sqrt{1^3}\right)$$

$$= \left(16 - \frac{16}{3}\right) - \left(8 - \frac{2}{3}\right)$$

$$= 8 - \frac{14}{3}$$

$$= 10/3$$

$$I = 3\,1/3$$

Q vi d. (ii) $I = \int_0^2 \frac{dx}{(4+x^2)^{\frac{3}{2}}}$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

x	0	2
θ	0	$\pi/4$

$$I = \int_0^{\pi/4} \frac{2\sec^2\theta\, d\theta}{4\sec^3\theta}$$

$$= \frac{1}{2}\int_0^{\pi/4} \cos\theta\, d\theta$$

$$= \frac{1}{2}\left[\sin\theta\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left(\sin\pi/4 - \sin 0\right)$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}} - 0\right)$$

$$= \frac{1}{2\sqrt{2}}$$

3

$$\bar{x} = \frac{x}{\theta}$$

Q4 (b) $f(x) = x^3 - 4x^2 + 3x + 2 = 0$

$\alpha + \beta + \gamma = -b/a = 4$

$\alpha\beta + \beta\gamma + \gamma\alpha = c/a = 3$ } 3

$\alpha\beta\gamma = -d/a = -2$

(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{3}{-2} = -\frac{3}{2}$

(ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= (4)^2 - 2(3)$

$= 16 - 6$

$= 10$

Q4 (b) (i) $f(x) = ax^3 + bx^2 - 8x + 3$

$(x-1)$ is a factor of $f(x)$, $\therefore f(1) = 0$

$f(1) = a(1)^3 + b(1)^2 - 8(1) + 3$

$0 = a + b - 8 + 3$

$a + b = 5$ ——— (1)

remainder = 15 when $\div$ by $(x+2)$, $\therefore f(-2) = 15$.

$15 = a(-2)^3 + b(-2)^2 - 8(-2) + 3$

$15 = -8a + 4b + 16 + 3$

$8a - 4b = 4$

$2a - b = 1$ ——— (2)

Sub $a = \frac{1}{2}$ into (2):

$2(2) - b = 1$

$4 - b = 1$

$b = 3$

(1) + (2):

$3a = 6$

$a = 2$

(ii) $f(x) = 2x^3 + 3x^2 - 8x + 3 = (x-1)(2x^2 + 5x - 3)$

$p = -6$

$s = 5$

$F = 6, -1$

$$
\begin{array}{r}
2x^2 + 5x - 3 \\
x - 1 \overline{)\,2x^3 + 3x^2 - 8x + 3} \\
2x^3 - 2x^2 \\
\hline
5x^2 - 8x \\
5x^2 - 5x \\
\hline
-3x + 3 \\
-3x + 3 \\
\hline
0
\end{array}
$$

Q4 (b) (ii) :- $P(x) = 2x^3 + 3x^2 - 8x + 3$

$= (x-1)(2x^2 + 5x - 3)$

$2x^2 + 5x - 3 = 2x^2 + 6x - 1x - 3$

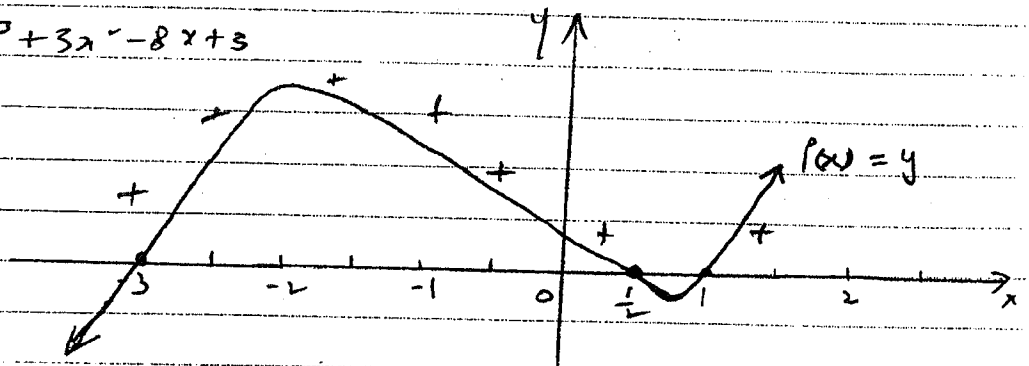
$= 2x(x+3) - 1(x+3)$

$= (x+3)(2x-1)$

$\therefore P(x) = (x+3)(2x-1)(x-1)$

2

$P(x) = 2x^3 + 3x^2 - 8x + 3$



(iii) $P(x) > 0$, from the graph occurs when

$-3 < x < \frac{1}{2}$ & $x > 1$

1

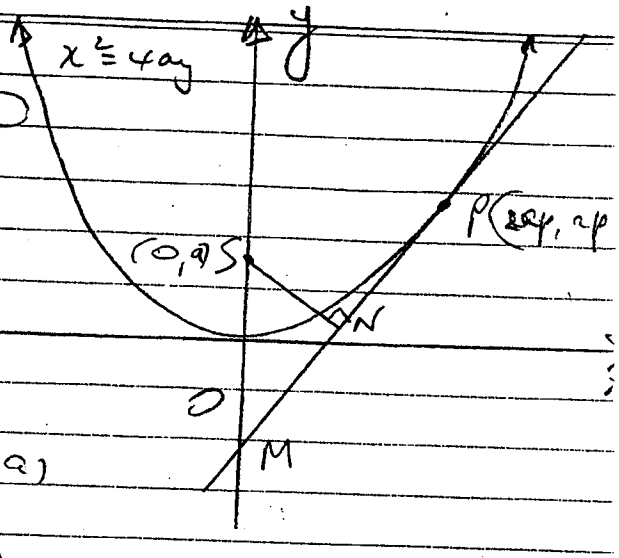
Q5 (a) Tangent at P:

(i) $y - px + ap^2 = 0$ ——— (1)

At M, $x = 0$

$y = -ap^2$

$\therefore M = (0, -ap^2)$



SN: $m = -\frac{1}{p}$, $S = (0, a)$

$y - a = -\frac{1}{p}(x - 0)$

$py - pa = -x$

$x + py = pa$ ——— (2)

To find N, solve (1) & (2) simultaneously

(1) $\Rightarrow$ $y - px = -ap^2$

$y = px - ap^2$ ——— (3)

Sub into (2)

$x + p(px - ap^2) = pa$

[redacted]

Q5 (a) (i) $x + p^2 x - ap^3 = ap$

$(1+p^2)x = ap + ap^3$

$(1+p^2)x = ap(1+p^2)$

$x = ap.$

Sub $x = ap$ into (3)

4 $y = p(ap) - ap^2$

$\therefore y = 0.$

$\therefore \boxed{N = (ap, 0)}$

(ii) K = midpt of MN

$M = (0, -ap^2)$, $N = (ap, 0)$

$\therefore K = \left(\frac{ap+0}{2}, \frac{0-ap^2}{2}\right)$

$\boxed{K = \left(\frac{ap}{2}, -\frac{ap^2}{2}\right)}$

(iii) $x = ap$ — (1) $\Rightarrow$ $\boxed{p = \frac{2x}{a}}$ — (3)

$y = -ap^2$ — (2)

Sub (3) into (2):

2 $y = -a\left(\frac{2x}{a}\right)^2$

$= -a\left(\frac{4x^2}{a^2}\right)$

$y = -\frac{2x^2}{a}$

$\therefore \boxed{x^2 = -\frac{1}{2}ay}$

$$\frac{\cos A - \cos(A+2B)}{2\sin B} = \sin(A+B)$$

$$LHS = \frac{\cos A - \cos A\cos 2B + \sin A \sin 2B}{2\sin B}$$

$$= \frac{\cos A(1-\cos 2B) + \sin A(2\sin B\cos B)}{2\sin B}$$

$$= \frac{\cos A(2\sin^2 B) + \sin A(2\sin B\cos B)}{2\sin B}$$

$$= \frac{2\sin B(\sin B\cos A + \cos B\sin A)}{2\sin B}$$

$$= \sin(B+A)$$

$$= \sin(A+B)$$

$$= RHS$$

5 (h)

$\frac{dA}{dt} = 1500\ m^2/h$

$r = 1250\ m$

$C = 2\pi r$

$\frac{dC}{dt} = 2\pi \times \frac{dr}{dt}$

$= 2\pi \times \frac{3}{5\pi}$

$\frac{dC}{dt} = \frac{6}{5}\ m/h$

3

$A = \pi r^2$

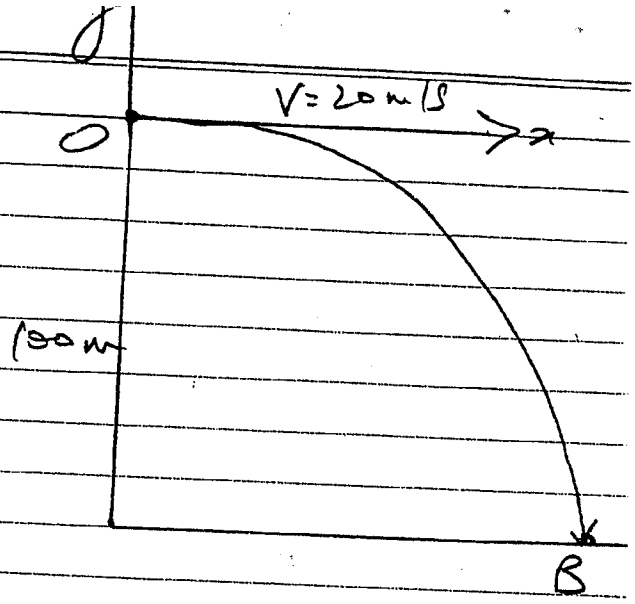
$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$1500 = 2\pi \times 1250 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1500}{2500\pi}$

$\frac{dr}{dt} = \frac{3}{5\pi}\ m/h$

Q6



$g \doteq 10 m/s^2$

(i) $\ddot{x} = 0$

$\dot{x} = V\cos\theta$

$\dot{x} = 20 \times \cos 0$

$\dot{x} = 20$

$x = 20t$

$\ddot{y} = -10$

$\dot{y} = -10t + V\sin\theta$

$\dot{y} = -10t + 20 \times 0$

$\dot{y} = -10t$

$y = -5t^2$

(ii) From (i) $x = 20t \Rightarrow t = \frac{x}{20}$

& (ii) $y = -5t^2$

$\therefore y = -5\left(\frac{x}{20}\right)^2$

$= -5 \cdot \frac{x^2}{400}$

$y = -\frac{x^2}{80}$

$\therefore x^2 = -80y$

(iii) $y = -100$

$-5t^2 = -100$

$t^2 = 20$

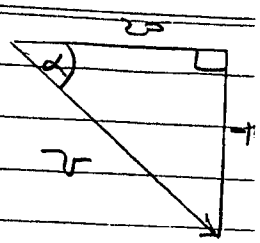
$t = \sqrt{20}$ sec

(iv) $t = \sqrt{20}$ $x = 20t$

$R = 20\sqrt{20}$ m

Q6 (V2) $t = \sqrt{20}$ $\dot{x} = 20$

$\dot{y} = -10(\sqrt{20})$



$v^2 = \dot{x}^2 + \dot{y}^2$

$= (20)^2 + (-10\sqrt{20})^2$

$= 400 + 2000$

$v^2 = 2400$

$\therefore v = \sqrt{2400}$

$v = 20\sqrt{6}$ m/s

2 ✓

$\tan\alpha = \frac{\dot{y}}{\dot{x}}$

$= \frac{-10\sqrt{20}}{20}$

$\tan\alpha = -\frac{\sqrt{20}}{2}$

$= -\frac{2\sqrt{5}}{2}$

$\tan\alpha = -\sqrt{5}$

$\therefore \alpha = 180 - 65°54'$

$\alpha = 114°5'$ ✓

Q7 (a) A particle is executing simple harmonic motion if

(i) it is travelling in a straight line and

(ii) it experiences an acceleration of the form $\ddot{x} = -n^2 x$

(b) (i) When $t = 0$, $x = 0$, $v = 2p$ m/s

$$\ddot{x} = -9x$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9x$$

$$\therefore \frac{1}{2}v^2 = -\frac{9x^2}{2} + c$$

$x = 0$, $v = 2p$:

$$\frac{1}{2}(2p)^2 = -\frac{9(0)^2}{2} + c$$

$$\therefore c = 2p^2$$

$$\therefore \frac{1}{2}v^2 = -\frac{9x^2}{2} + 2p^2$$

$$\therefore v^2 = 4p^2 - 9x^2$$

(ii) When it comes to rest, $v = 0$.

$$\therefore 0 = 4p^2 - 9x^2$$

$$\therefore x^2 = \frac{4p^2}{9}$$

$$\therefore x = \frac{2p}{3} \text{ m}$$

Q7 (b) (ii) $x = a\cos(nt + \alpha)$

when $t = 0$, $x = 0$ ∴

$\therefore 0 = a\cos\alpha$

$\therefore \alpha = \pi/2$

$\therefore x = a\cos(nt + \pi/2)$

$= a\left[\cos nt \cdot \cos\frac{\pi}{2} - \sin nt \cdot \sin\frac{\pi}{2}\right]$

$x = -a\sin nt$

At $t = \frac{2\pi}{\sqrt{9}}$, $x = a$

$\therefore \sin nt = -1$ when $n^2 = 9$

$\therefore \sin(\sqrt{9}t) = -1$ $\Rightarrow n = \sqrt{9}$

$\therefore \sqrt{9}t = \frac{3\pi}{2}$,

$\therefore t = \frac{3\pi}{2\sqrt{9}}$ sec

(iii) $t = \frac{\pi}{2\sqrt{(cq)}} = \frac{\pi}{4\sqrt{q}}$,

$v = -2p \cdot \sin(\sqrt{q}t)$

$= -\frac{2p}{\sqrt{q}}\left(\sin\sqrt{q} \cdot \frac{\pi}{4\sqrt{q}}\right)$

$= -\frac{2p}{\sqrt{q}} \cdot \sin\pi/4 = -\frac{2p}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$

$v = -\frac{\sqrt{2}}{2}p$ m