Neap:

HSC Trial Examination 2009

Mathematics Extension 1

This paper must be kept under strict security and may only be used on or after the afternoon of Thursday 13 August, 2009 as specified in the Neap Examination Timetable.

General Instructions

Reading time – 5 minutes

Working time – 2 hours

Write using black or blue pen

Board-approved calculators may be used

A table of standard integrals is provided at the back of this paper

All necessary working should be shown in every question

Total marks – 84

Attempt questions 1–7

All questions are of equal value

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2009 HSC Mathematics Extension 1 Examination.

Total marks 84
Attempt Questions 1–7
All questions are of equal value
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 Marks) Use a SEPARATE writing booklet.

(a) Solve the inequality $\frac{x}{x-3} \geq 2$. **2**

(b) Determine the coordinates of the point that divides the interval joining $(6, 2)$ to $(4, 9)$ externally in the ratio of $2 : 1$. **2**

(c) Determine the general solution to the equation $2\sin 2x = 1$. **2**

(d) (i) Sketch the graph of $y = 4\sin^{-1} 3x$. **1**

(ii) What is the domain and range of $y = 4\sin^{-1} 3x$? **2**

(e) $(x + 2)$ is a factor of the polynomial $P(x) = 3x^3 + kx^2 - 5x + 10$.

(i) Determine the value of k. **1**

(ii) The roots of $P(x) = 0$ are -2, α and β. Determine the value of $\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{2}$. **2**

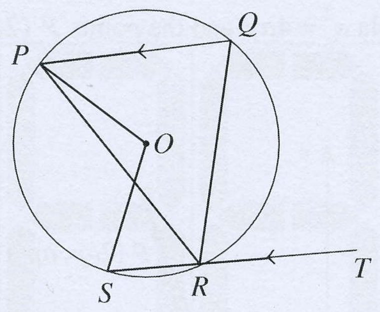
Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) The line $y = 12 - x$ crosses the parabola $y = x^2$ at the point $(3, 9)$. Calculate the size of the acute angle between the line and the parabola at $(3, 9)$. **2**

(b) Use the substitution $u = \sqrt{x}$ to evaluate $\displaystyle\int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$. **3**

(c) In the diagram below, PQ is parallel to ST, O is the centre of the circle, $\angle PQR = 80°$ and $\angle POS = 110°$. **3**



Find the value of $\angle PRQ$ giving reasons.

(d) Use mathematical induction to prove that, when n is a positive integer, **4**

$$\frac{1}{4} + \frac{1}{28} + \frac{1}{77} + \ldots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}.$$

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

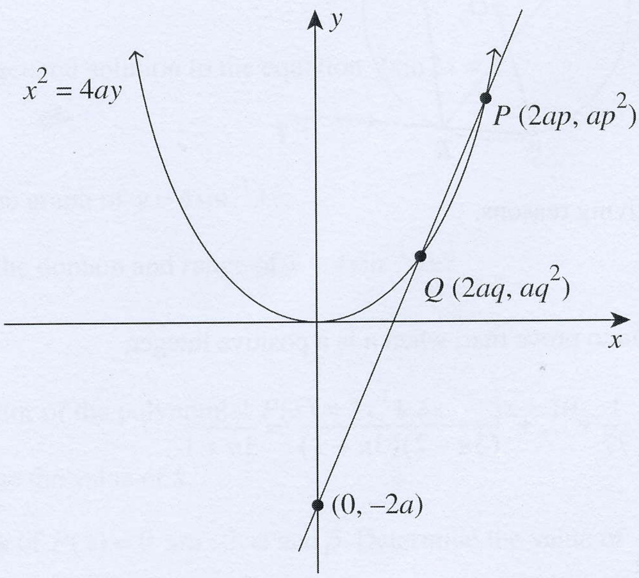
(a) Differentiate $y = x\tan^{-1} x$. **2**

(b) Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{4 - x^2}}$. **2**

(c) (i) Prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. **2**

(ii) Hence find $\displaystyle\int 2\sin^3\theta \, d\theta$. **1**

(d) The diagram below shows the parabola $x^2 = 4ay$ and the points $P\ (2ap, ap^2)$ and $Q\ (2aq, aq^2)$.



(i) Show that the equation of the chord PQ is $2y = (p + q)x - 2apq$. **2**

(ii) The line joining P and Q passes through the point $(0, -2a)$. Show that $pq = 2$. **1**

(iii) The normals to the parabola $x^2 = 4ay$ at points $P\ (2ap, ap^2)$ and $Q\ (2aq, aq^2)$ intersect at K. The coordinates of K are $\left(-apq(p + q),\ a(p^2 + q^2 + pq + 2)\right)$. Do *not* prove this. **2**

Prove that the locus of K is the parabola $x^2 = 4ay$.

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Marks

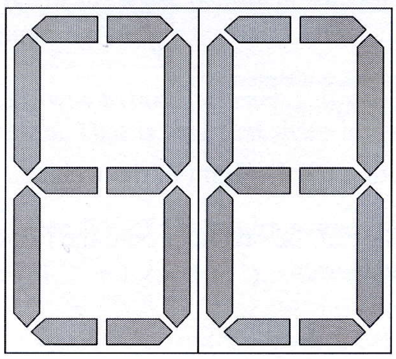
Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Find the coordinates of the stationary points on the curve $y = \dfrac{5x^2 + 5}{x}$ and determine their nature. **2**

(ii) Sketch the graph of $y = \dfrac{5x^2 + 5}{x}$. **1**

(iii) Hence determine the range of the function $y = \dfrac{5x^2 + 5}{x}$. **1**

(b) The 10 segments on each digital display panel below are lights that can be switched on or off. The panels are connected to a computer chip that randomly switches on 14 segments.



(i) How many sets of 14 lights can be selected from the 20 lights on the two display panels? **1**

(ii) Determine the total number of combinations possible if 4 vertical and 10 horizontal segments are to be lit. **1**

(iii) How many of the different 14-light displays show exactly 5 vertical segments lit up? **2**

(c) (i) Sketch the graph of $y = x^3 - 5x^2$. **1**

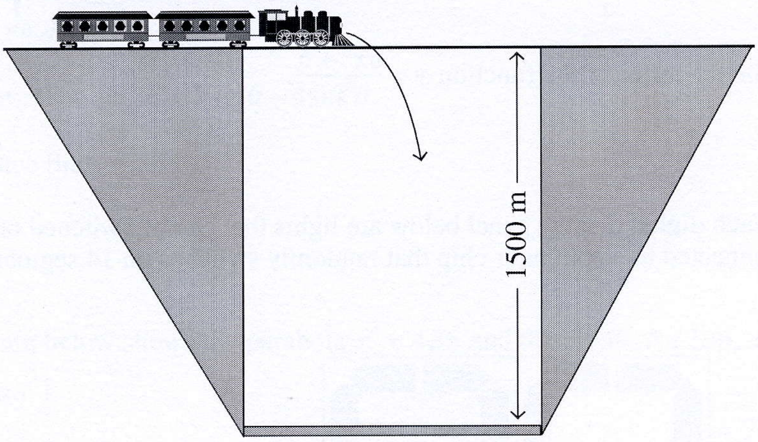
(ii) Using a diagram, or otherwise, explain how you know that $y = x^3 - 5x^2 - 1$ has only one root. **1**

(iii) Use one iteration of Newton's method to approximate the root of $y = x^3 - 5x^2 - 1$ starting with $x_1 = 5$. Express the root correct to two decimal places. **2**

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) At the exact time a train travelling at 24 m/s begins to cross a 600 m–long bridge, a weight falls off the front of the train and becomes a projectile as it falls into the ravine as shown on the diagram below. The ravine is 1 500 m deep. Use $g \approx 10$ m/s^2.



(i) Determine the equations of motion for the flight of the weight. **2**

(ii) Will the weight crash into the side of the ravine or onto the floor of the ravine? Use calculations to justify your answer. **2**

(b) The acceleration of a particle moving in simple harmonic motion is given by $\ddot{x} = -8x$. Initially the particle is stationary 2 m to the right of the origin.

(i) Show that $v^2 = 8(4 - x^2)$. **2**

(ii) Calculate the time taken for the particle to first pass the origin. **2**

(c) The rate of change of a cool item placed in a hot environment is proportional to the difference between the temperature of the cool item (T) and the temperature of the hot environment (S). That is, $\frac{dT}{dt} \propto (S - T)$.

(i) Prove that $T = S - Ae^{kt}$ is a solution to the differential equation $\frac{dT}{dt} \propto (S - T)$. **1**

(ii) Jamie is cooking a large roast in an oven set to 160 °C. The roast will be cooked when the thermometer shows that the temperature of the centre of the meat is 150 °C. **3**

When Jamie started cooking, the temperature of the centre of the meat was 4 °C and 30 minutes later the temperature was 60 °C.

How long will it take for the roast to be cooked?

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Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) What is the value of the term in x^4 in the expansion of $(1 + 2x)^{15}$? **1**

(b) In a large pile of chips, the ratio of red to black chips is 3 : 5.
Gemma is going to select 12 chips at random from the pile.

(i) What is the probability that one quarter of the chips Gemma selects will be black? Express your answer correct to 4 decimal places. **2**

(ii) What is the most likely number of red and black chips Gemma will select? Use a calculation to justify your answer. **4**

(c) At the beginning of each year Cassandra invests in superannuation. Her account pays 5.6% p.a. annually compounding interest.

Cassandra's first investment was \$1000 and each following year her investment was 20% more than the previous year's. That is, her first three investments were \$1000, \$1200 and \$1440.

(i) Show that the total value, $\$V$, of Cassandra's investment at the end of three years is given by $V = 1000M(1.2^2 + 1.2M + M^2)$, where $M = 1.056$. **1**

(ii) Show that the total value of Cassandra's investment after n years is given by $V = 8800 \times (1.2)^{n-1} \times (1 - 0.88^n)$. **2**

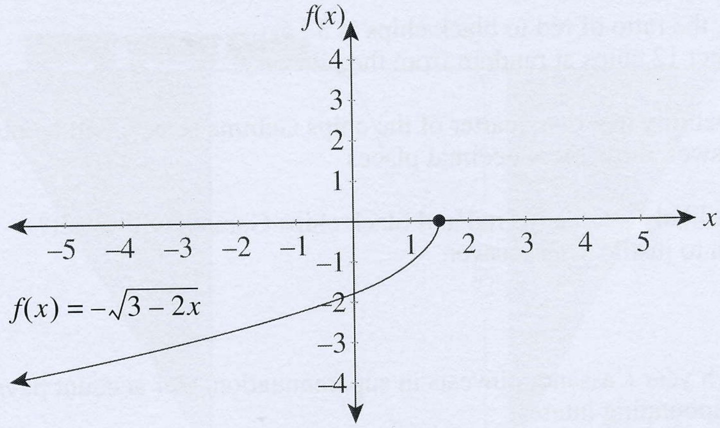
(iii) Cassandra continued her investment for 30 years. That is, until the end of the year following her 30^{th} deposit. **2**

What percentage of the total value of the account at the end of 30 years was interest? Express your answer correct to the nearest whole percentage.

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph of $f(x) = -\sqrt{3-2x}$.



(i) Copy the diagram onto your answer page and show the graph of $y = f^{-1}(x)$, the inverse function of $y = f(x)$, on it. **1**

(ii) Write down the equation for the inverse function of $f(x) = -\sqrt{3-2x}$. Include the domain of the inverse function in your answer. **1**

(b) (i) By determining the coefficient of x^2 on both sides of the equation $(1+x)^n \times (1+x)^{2n} = (1+x)^{3n}$, or otherwise, show that **2**

$$\binom{n}{0}\binom{2n}{2} + \binom{n}{1}\binom{2n}{1} + \binom{n}{2}\binom{2n}{0} \equiv \binom{3n}{2}.$$

(ii) Hence, or otherwise, show $\sum_{k=0}^{n} \binom{n}{k}\binom{2n}{n-k} = \binom{3n}{n}$. **2**

Question 7 continues on page 9

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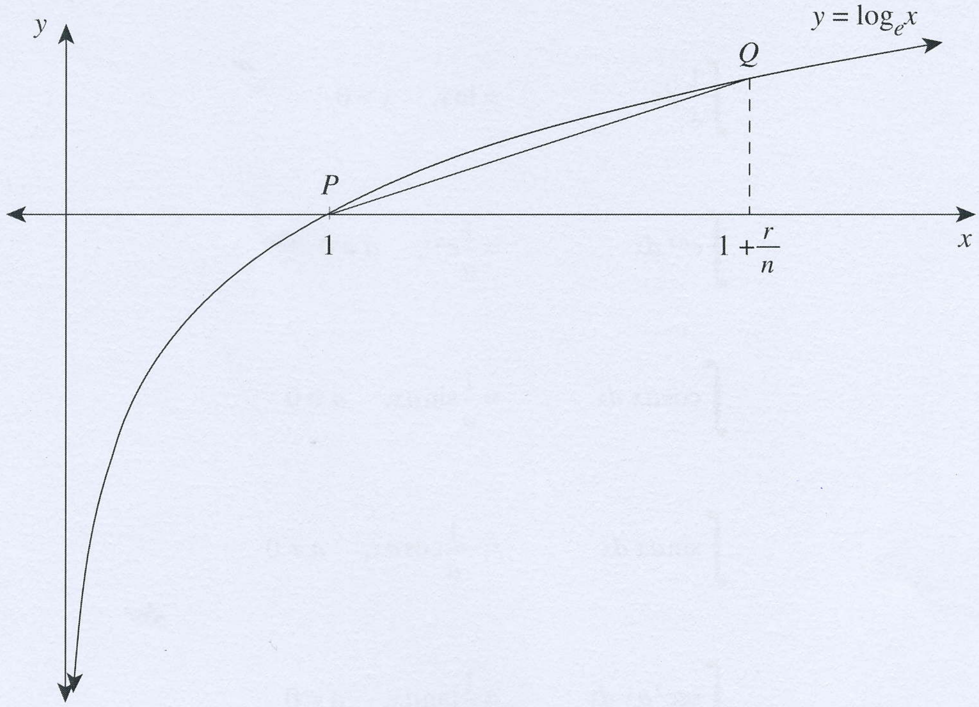
Marks

Question 7 (continued)

(c) (i) Use the compound interest formula to show that a rate of 6% p.a. is effectively 6.168% p.a. when the investment is compounded monthly. **1**

(ii)



The diagram above shows the graph of $y = \log_e x$ and the secant joining points P and Q on the curve. P is at $x = 1$ and Q is at $x = 1 + \frac{r}{n}$.

(α) Show that the gradient of the secant PQ is $\frac{1}{r}\log_e\left(1 + \frac{r}{n}\right)^n$. **1**

(β) Use $\frac{d}{dx}\log_e x = \frac{1}{x}$ to show that $\lim_{n \to \infty}\left(1 + \frac{r}{n}\right)^n = e^r$. **3**

(γ) Hence or otherwise determine an expression for the effective annual rate of interest when an annual rate of 6% p.a. is compounded continually, that is, compounded an infinite number of times per year. **1**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

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