N.S.W. DEPARTMENT OF EDUCATION

HIGHER SCHOOL CERTIFICATE EXAMINATION 1968

MATHEMATICS - PAPER 8 (2F) - EQUIVALENT TO 3U AND 4U - 1st PAPER

Instructions: Time allowed 3 hours. All questions may be attempted. In every question, all necessary working should be shown. Marks will be deducted for careless or badly arranged work. Mathematical tables will be supplied.

QUESTION 1 (12 Marks)

(i) Find the area under the curve $y = (x - 3)^3$ between $x = 3$ and $x = 4$.

(ii) Find a primitive function of $1/(x^2 + 4)$.

(iii) Find the second derivative of $f(x) = \tan(x^2)$.

(iv) P is a variable point of the curve $y = x^2/4a$ and ℓ is the line $x = 1$. P' is the point such that $PP' \parallel 0x$, $PP' \cap \ell = N$ and $*PP' = 2 *NP$. Find the equation of the locus of P'.

QUESTION 2 (9 Marks)

(i) Determine the numerical values of a and b such that $\cos 3\theta = a \cos^3\theta + b \cos\theta$ is an identity in θ.

(ii) Three children have their birthdays in the same week. What is the probability that two and only two have their birthdays on the Tuesday?

(iii) Use Simpson's rule with 3 function values to approximate the integral $\int_0^1 \sin \pi x \, dx$. Find an expression for the true value of the integral.

QUESTION 3 (9 Marks)

(i) In a plane $\{P\}$ is the set of points satisfying the equation $y = x$ referred to a given pair of rectangular axes. The axes are displaced parallel to themselves so that the new origin has coordinates $(1, -2)$ referred to the original axes; what is the equation satisfied by the points of the set $\{P\}$ in relation to the new axes?

(ii) (a) For an arbitrary positive integer n find the sum
$$3^{-n} + 3^{-n-2} + 3^{-n-4} + \ldots + 3^{-3n}.$$

(b) State whether this sum has a limit as n tends to infinity.

(iii) The coefficients b and c are such that the polynomial $x^4 + bx + c$ is divisible by $x - 1$ and $x - 2$. Find b and c.

QUESTION 4 (10 Marks)

(i) State the domain and range of the function $x \sin^{-1}(x^2)$.

(*ii*) Determine the derivative of $x \sin^{-1}(x^2)$ and describe the behaviour of the function in the neighbourhood of:

(a) $x = 0$; and

(b) $x = 1$.

(*iii*) Determine the greatest and least values of $x \sin^{-1}(x^2)$ and the local maxima and minima (if any).

QUESTION 5 (10 Marks)

(*i*) Use mathematical induction to prove the formula

$$1^3 + 3^3 + 5^3 + \ldots + (2n - 1)^3 = n^2(2n^2 - 1).$$

(*ii*) Defining the integral as the limit of a sum, and using the result of part (*i*), find the value of $\int_0^2 x^3 dx$.

QUESTION 6 (10 Marks)

(*i*) State the binomial expansion of $(1 + x)^n$ and state the "Pascal triangle" relation in terms of the coefficients ${}^nC_r$.

(*ii*) By evaluating the integral $\int_0^2 (1 - x)^{2n+1} dx$ in two different ways, or otherwise, prove that the following identity holds for all odd values of m:

$$\sum_{r=0}^{m} (-1)^r \frac{2^{r+1}}{r + 1} {}^mC_r = 0.$$

QUESTION 7 (10 MARKS)

(*i*) State Newton's method for determining successive approximations to a root of an equation $f(x) = 0$. (You need not comment on the precautions to be observed in applying the method.)

(*ii*) By application of Newton's method to the equation $x^3 - a = 0$, or otherwise, derive an iterative method for approximating cube roots.

(*iii*) Test your method on the value $a = 9$. Use $x_1 = 2$ as a first approximation, carry out one step in the iteration, and, by comparing your approximation with that given for $\sqrt[3]{9}$ in the tables, state (to one significant figure) the percentage error in your approximation.

(OMIT)

QUESTION 8 (10 MARKS)

(*i*) Describe, in geometrical terms, the set G of points in three-dimensional space defined by the equation $(x - 2)^2 + y^2 = 1$.

(*ii*) Describe, in geometrical terms, the set H of points defined by the inequality $(y - 2)^2 + z^2 < 1$.

(*iii*) Describe the set of points $G \cap H$.

(*iv*) A and B are the sets of points satisfying respectively the equations $z = 0$ and $z = 2$. Describe in geometrical terms the sets:

(a) $A \cap G$. (b) $B \cap G$. (c) $A \cap H$. (d) $B \cap H$.

QUESTION 9 (10 Marks)

A projectile is fired from the origin with an initial velocity with x-component V_1 and y-component V_2. (The x-axis is in a horizontal plane at ground level and the y-axis points vertically upwards. Ignore air-friction on the projectile.)

(*i*) Write down the equations of motion for the projectile, and the initial conditions which must be satisfied by the solution of these equations.

(*ii*) Prove that the time of flight T is independent of V_1, and derive a formula for T. Discuss this result in qualitative terms.

(*iii*) Allowance is now to be made for the fact that the acceleration of gravity, g, decreases as the projectile ascends:

(a) Show that the time of flight T under these altered conditions is still independent of V_1.

(b) Using a purely qualitative argument, state whether T is increased or decreased as a result of this alteration in the conditions.

QUESTION 10 (10 Marks)

(*i*) An urn contains four balls, each with a number painted on its surface. These numbers are 2, 4, 6 and 8, respectively. A man draws a ball at random, and scores the number on the ball. Determine the expected value of his score. Is the expected value a possible result of the drawing? Comment.

(*ii*) An urn contains 3 balls each marked "6" and 5 balls each marked "4". A succession of 4 drawings of a ball from the urn is made; after each drawing the ball is replaced in the urn and the balls remixed.

(a) What is the probability of drawing 2 balls marked "6" and 2 balls marked "4" (in any order)?

(b) Prove that the probability that the sum of the numbers on the four balls drawn should be greater than 20 is between 15 per cent and 16 per cent.