SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2005

Mathematics Extension 2

General Instuctions

- Reading time - 5 minutes
- Working time - 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks - 120

- Attempt Questions 1 – 8
- All questions are of equal value

Name : ______________________________

Teacher : ______________________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Question 7	Question 8	**Total**

Question 1 (15 marks) **Marks**

(a) Find $\int \frac{2x}{x+1}\,dx$ 2

(b) Find $\int \frac{dx}{\sqrt{8+2x-x^2}}$ 2

(c) Use partial fractions to find $\int \frac{2}{x^2-x}\,dx$ 3

(d) Find $\int \sin 2x \cos^3 x \, dx$ 4

(e) Find $\int \frac{1}{(36+x^2)^{\frac{3}{2}}}\,dx$ using the substitution $x = 6\tan\theta$ 4

Question 2 (15 marks) (Start a new page) **Marks**

(a) Find the gradient of the curve $2x^3 - x^2 y + y^3 = 1$ 3

at the point $(2, -3)$.

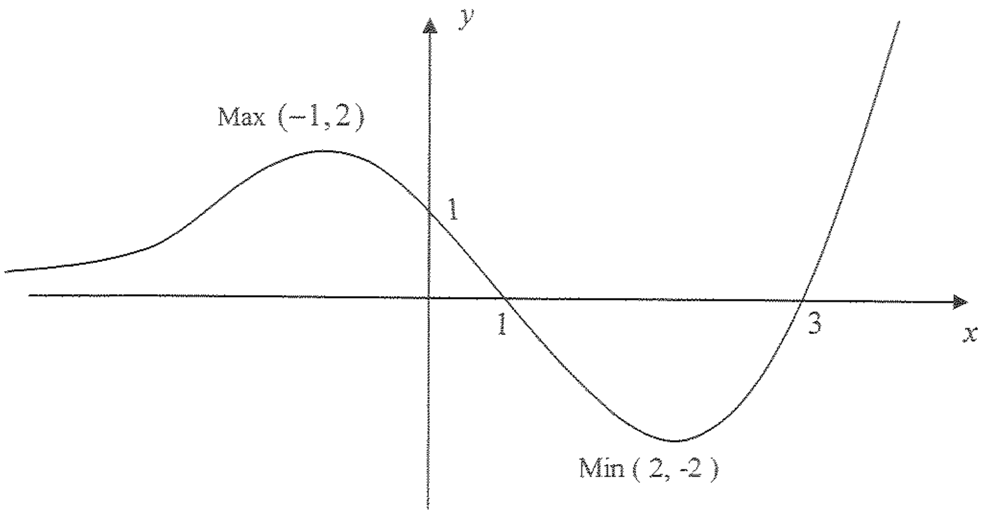
(b) Solve $|x-2| + |x+1| = 3$ 2

QUESTION 2 (Continued)

(c) The base of a solid is the area enclosed by the curve $y = x^2$, the line $x = 2$ and the x axis. Each cross-section of the solid by a plane perpendicular to the x axis is a regular hexagon with one side in the base of the solid. 4

Find the volume of the solid.

(d)



The sketch above shows the function $y = f(x)$.

Sketch possible graphs of the following

(i) $y = \dfrac{1}{f(x)}$ 2

(ii) $y = \int f(x)\,dx$ 2

(iii) $y^2 = f(x)$ 2

Question 3 (15 marks) (Start a new page) **Marks**

(a) Given $z = -3\sqrt{3} + 3i$

(i) express z in modulus argument form. 2

(ii) find the smallest positive integer n such that z^n is real. 1

(b) Evaluate $\text{Im}\left(\dfrac{4}{1-i}\right)$ 2

(c) Sketch the locus described by $|z+2| = |z-4i|$ 2

(d) (i) Sketch the intersection of the locus described by 3

$|z| \leq 3$ and $-\dfrac{\pi}{4} \leq \arg(z+3) \leq \dfrac{\pi}{4}$

(ii) If the complex number ω lies on the boundary of the region 2

sketched in part (i), find the minimum value of $|\omega|$.

(e) OABC is a rectangle on the Argand diagram in which side OC is twice the length of OA, where O is the origin.

(i) If A represents the complex number $1+2i$, find the complex numbers represented by B and C given that the argument of the complex number represented by the point C is negative.. 2

(ii) If this rectangle is rotated anticlockwise $\dfrac{\pi}{3}$ radians about O, find the complex number represented by the new position of A. 1

Question 4 (15 marks) (Start a new page) **Marks**

(a) For the hyperbola with equation $4x^2 - 9y^2 = 36$ find,

(i) the eccentricity 2

(ii) the equation of the asymptotes 1

(b) Given the point $P(a\sec\theta\ ,\ b\tan\theta)$ lies on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

(i) Show that the tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

at the point $P(a\sec\theta\ ,\ b\tan\theta)$ has equation $\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$ 3

(ii) If this tangent in part (i) meets the directrix of the hyperbola 4

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ corresponding to the focus at $S(ae, o)$ at the point Q,

show that $\angle PSQ$ is a right angle.

(c) (i) Show that $(1-\sqrt{x})^{n-1}\sqrt{x} = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$ 1

(ii) If $I_n = \int_0^1 (1-\sqrt{x})^n\, dx$ for $n \geq 0$ 4

show that $I_n = \dfrac{n}{n+2} I_{n-1}$ for $n \geq 1$.

Question 5 (15 marks) (Start a new page) **Marks**

(a) The region bounded by the curves $y = x^2$ and $y = x + 2$ is rotated about the line $x = 3$. 4

Use the method of cylindrical shells to find the volume of the solid of revolution formed.

(b) Solve the equation $8x^4 + 12x^3 - 30x^2 + 17x - 3 = 0$ given that it has a triple root. 4

(c) If α, β, δ are the roots of $x^3 + px^2 + qx + r = 0$, find the polynomial equation with roots $\alpha^2, \beta^2, \delta^2$. 2

(d) The acceleration of a particle moving in simple harmonic motion is given by $\ddot{x} = -n^2 x$ where x is the displacement of the particle from the origin and n is a constant.

(i) Show that the velocity v of the particle is given by $v^2 = n^2(a^2 - x^2)$ where a is the amplitude of the motion. 2

(ii) Given that the speed of the particle is $V\ m/s$ when it is d metres from the origin and that its speed is $\frac{V}{2}\ m/s$ when it is $2d$ metres from the origin, show that :

α) the particle's amplitude is $\sqrt{5}d$ metres. 2

β) the period of the motion is $\frac{4\pi d}{V}$ seconds. 1

Question 6 (15 marks) (Start a new page) **Marks**

(a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$ using the substitution $t = \tan \frac{x}{2}$ 4

(b) A particle of mass m is fired vertically upwards with initial velocity $V\ m/s$ and is subjected to air resistance equal to mkv Newtons where k is a constant and v is the velocity of the particle in metres per second as it moves through the air.

(i) Explain why the equation of motion of the particle is given by 1

$\ddot{x} = -g - kv$ where g is the acceleration due to gravity.

(ii) Show that the maximum height reached by the particle is given by 4

$$H = \frac{V}{k} - \frac{g}{k^2}\ln\left(1 + \frac{kV}{g}\right)$$

(c) (i) Find the seven complex roots of the equation $z^7 = 1$. 2

(ii) If ω is the complex root of $z^7 = 1$ 1

with smallest positive argument, find the value of

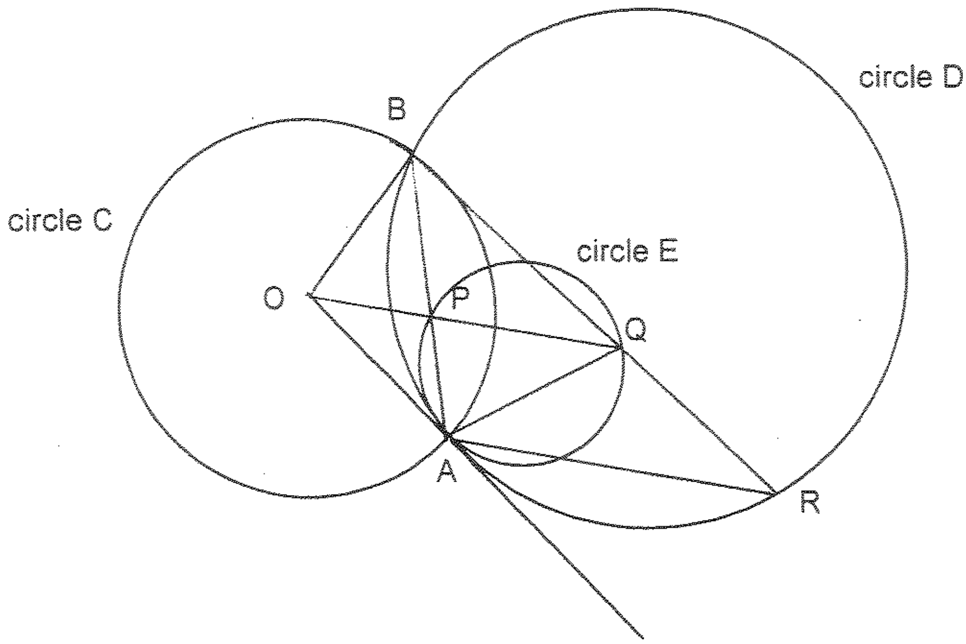
$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6$$

(iii) Find the cubic equation whose roots are 3

$$\omega + \omega^6\ ,\ \omega^2 + \omega^5\ ,\ \omega^3 + \omega^4$$

Question 7 (15 marks) (Start a new page) **Marks**

(a)



In the diagram above, OA is a radius of a circle C with centre O, and two circles D and E are drawn touching the line OA at A as shown. The larger circle D meets circle C again at B, and the line AB meets the smaller circle E again at P. The line OP meets circle E again at Q, and the line BQ meets the circle D again at R.

(i) Let $\angle OAP = \theta$. Explain why $\angle PQA = \theta$. 1

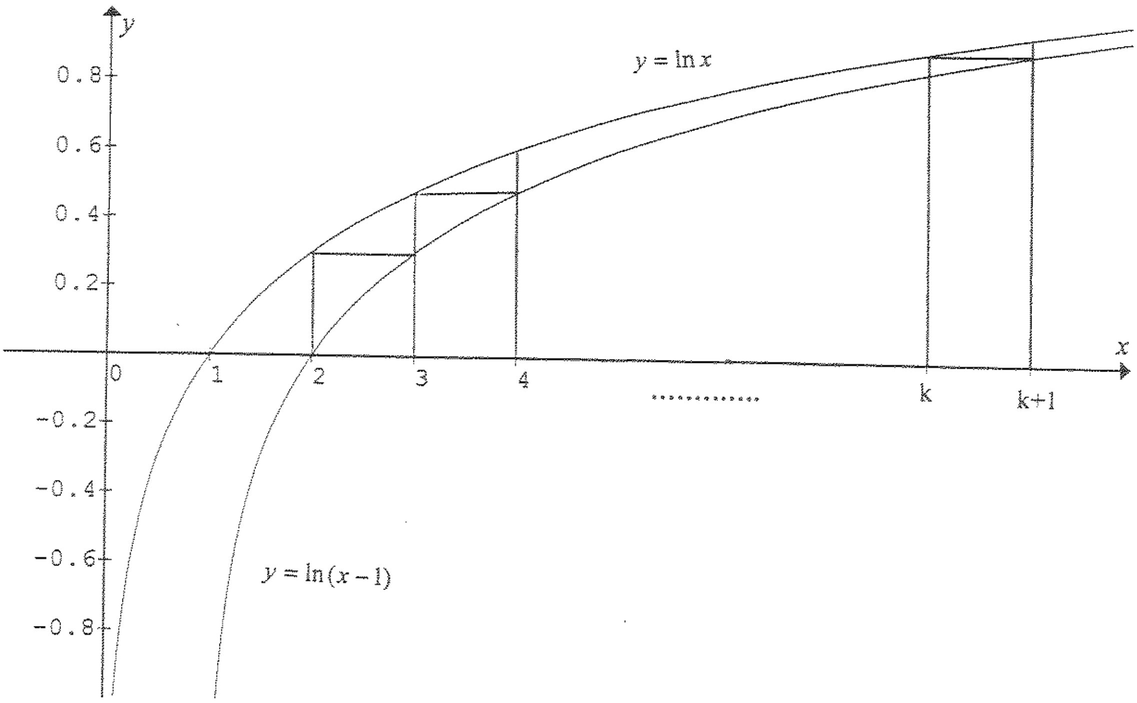
(ii) Prove that the points O , B , Q and A are concyclic. 2

(iii) Prove that OQ bisects $\angle BQA$. 2

(iv) Prove that $OQ // AR$ 2

QUESTION 7 (Continued)

(b)



In the diagram above, the curves $y = \ln x$ and $y = \ln(x-1)$ are sketched

and $k-1$ rectangles are constructed between $x = 2$ and $x = k+1$ where $k \geq 2$.

Let $S = \ln 2 + \ln 3 + \ln 4 + \ldots\ldots + \ln k$.

(i) Explain why S represents the sum of the areas of the $k-1$ rectangles. 1

(ii) Use an appropriate integration method to show that

$$\int_2^{k+1} \ln(x-1)\, dx = k \ln k - k + 1$$ 4

(iii) Hence show that $k^k < k!e^{k-1} < \frac{1}{4}(k+1)^{k+1}$ where $k \geq 2$ 3

(note $n! = n(n-1)(n-2)\ldots\ldots\ldots 3 \times 2 \times 1$)

Question 8 (15 marks) (Start a new page) **Marks**

(a) Find all x such that $\sin x = \cos 5x$ and $0 < x < \pi$. 3

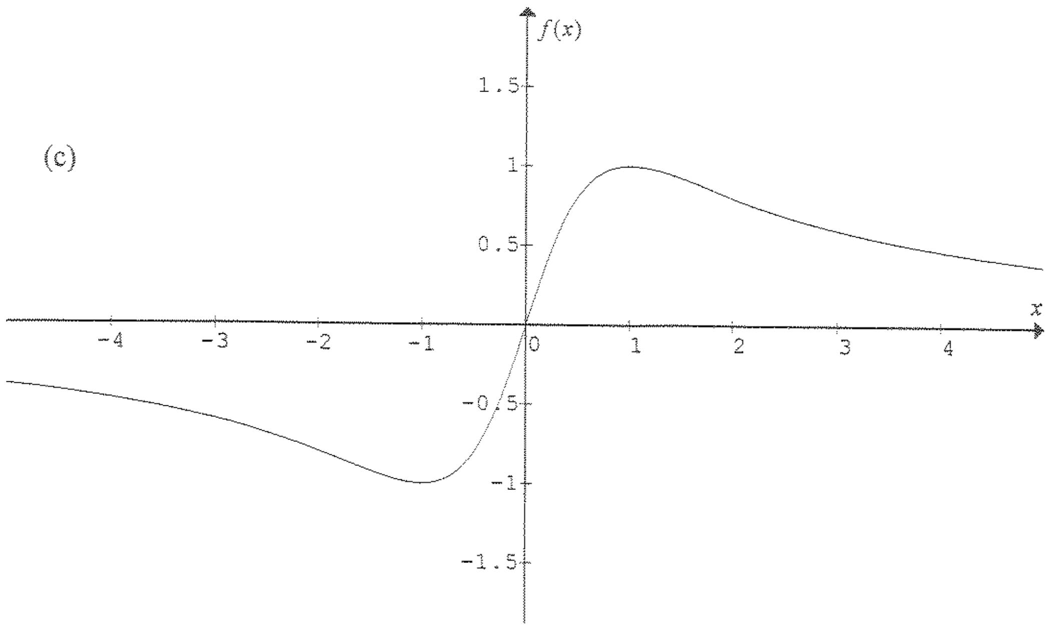
(b) If z is a complex number for which 2

$|z| = 1$ and $\arg(z) = \theta$, $0 \le \theta \le \dfrac{\pi}{2}$,

find the value of $\arg\left(\dfrac{2}{1-z^2}\right)$ in terms of θ.

Question 8 continued on next page.

(c)



The curve $f(x)=\dfrac{2x}{1+x^2}$ is sketched above. It has a maximum turning point at $(1,1)$ and minimum turning point at $(-1,-1)$.

(i) State the range of $f(x)=\dfrac{2x}{1+x^2}$ 1

(ii) Let x_0 be a real number not equal to 1 or -1 and consider the sequence

of real numbers defined by $x_{n+1}=f(x_n)$ for $n=0,1,2,\ldots\ldots$

(α) Given $x_1=g(r)$ and $g(x)=\dfrac{e^{2x}-1}{e^{2x}+1}$ express r in terms of x_1. 2

(β) Hence deduce that there exists a real number r such that $x_1=g(r)$ where $g(x)=\dfrac{e^{2x}-1}{e^{2x}+1}$. 1

(δ) Show that $\dfrac{2\,g(x)}{1+(g(x))^2}=g(2x)$. 2

(γ) Hence, using the above results and Mathematical Induction show that $x_n=g(2^{n-1}\,r)$ for $n=1,2,3,\ldots\ldots$ 4

End of Paper

Solutions Ext II 2005 Trial HSC

QUESTION 1

a) $\displaystyle\int \frac{2x}{x+1}\,dx$

$= 2\displaystyle\int 1 - \frac{1}{x+1}\,dx$

$= 2x - 2\ln(x+1) + c$

b) $\displaystyle\int \frac{dx}{\sqrt{8+2x-x^2}}$

$= \displaystyle\int \frac{dx}{\sqrt{9-(x-1)^2}}$

$= \sin^{-1}\dfrac{x-1}{3} + c$

c) $\displaystyle\int \frac{2}{x^2-x}\,dx$

$\dfrac{2}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1}$

$\therefore\ 2 = A(x-1) + B(x)$

$\therefore\ A = -2\ ,\ B = 2$

$\therefore\ \displaystyle\int \frac{2}{x^2-x}\,dx$

$= \displaystyle\int \frac{2}{x-1} - \frac{2}{x}\,dx$

$= 2\ln(x-1) - 2\ln x + c$

$= 2\ln\left(\dfrac{x-1}{x}\right) + c$

d) $\displaystyle\int \sin 2x \cos^3 x\,dx$

$= 2\displaystyle\int \sin x \cos^4 x\,dx$

$= -\dfrac{2}{5}\cos^5 x\,dx$

e) $x = 6\tan\theta$

$dx = 6\sec^2\theta\,d\theta$

$\displaystyle\int \frac{1}{(36+x^2)^{\frac{3}{2}}}\,dx$

$= \displaystyle\int \frac{6\sec^2\theta\,d\theta}{(36+36\tan^2\theta)^{\frac{3}{2}}}$

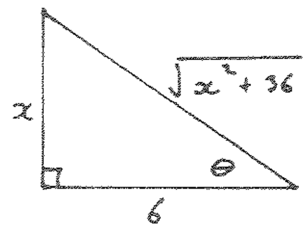
$= \dfrac{1}{36}\displaystyle\int \frac{\sec^2\theta\,d\theta}{(1+\tan^2\theta)^{\frac{3}{2}}}$

$= \dfrac{1}{36}\displaystyle\int \frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^{\frac{3}{2}}}$

$= \dfrac{1}{36}\displaystyle\int \frac{d\theta}{\sec\theta}$

$= \dfrac{1}{36}\displaystyle\int \cos\theta\,d\theta$

$= \dfrac{1}{36}\sin\theta + c$



$= \dfrac{x}{36\sqrt{x^2+36}} + c$

QUESTION 2

a) $2x^3 - x^2 y + y^3 = 1$

$6x^2 - \left[2xy + x^2 \frac{dy}{dx}\right] + 3y^2 \frac{dy}{dx} = 0$

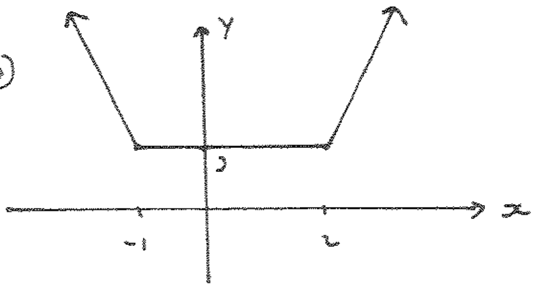
$\frac{dy}{dx} = \frac{2xy - 6x^2}{-x^2 + 3y^2}$

when $x = 2$, $y = -3$

$\therefore m_T = \frac{-12 - 24}{-4 + 27}$

$= \frac{-36}{23}$

b)



$\therefore \quad -1 \leq x \leq 2$

c) hexagon is 6 equilateral Δ's with side length y.

$A(x) = 6 \times \frac{1}{2} \times y \times y \times \sin 60°$

$= \frac{3\sqrt{3}}{2} y^2$

but $y = x^2$

$\therefore A(x) = \frac{3\sqrt{3}}{2} x^4$

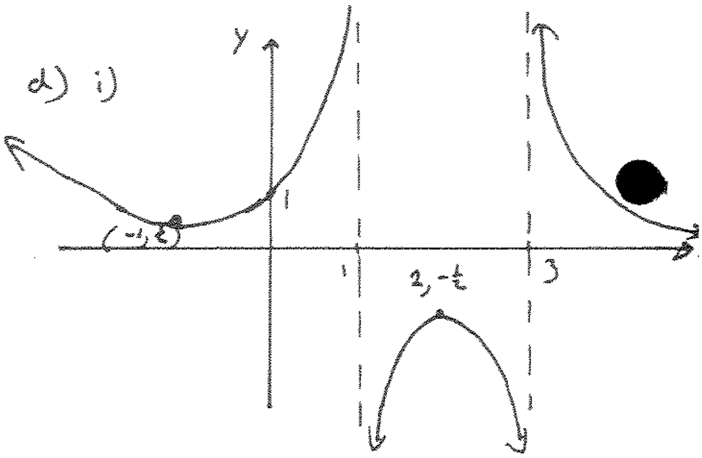
$\therefore \Delta V(x) = \frac{3\sqrt{3}}{2} x^4 \, dx$

$\therefore V = \int_0^2 \frac{3\sqrt{3}}{2} x^4 \, dx$

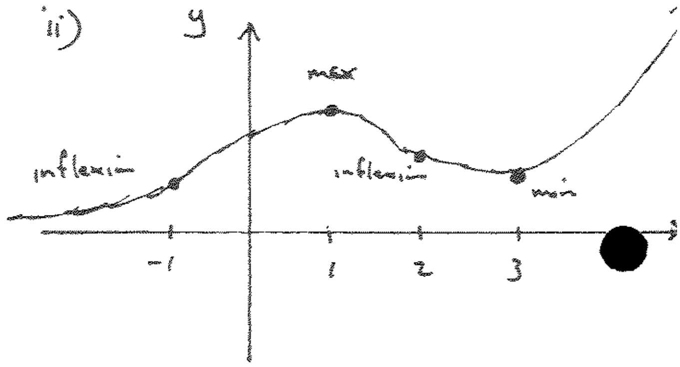
$= \frac{3\sqrt{3}}{10} x^5 \Big]_0^2$

$= \frac{48\sqrt{3}}{5}$ cubic units

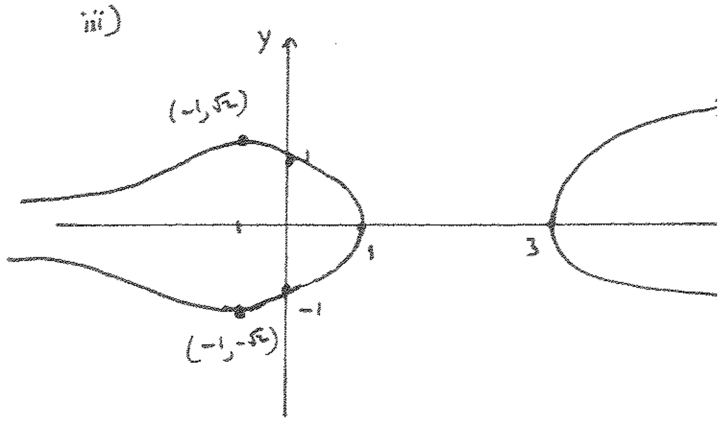
d) i)



ii)

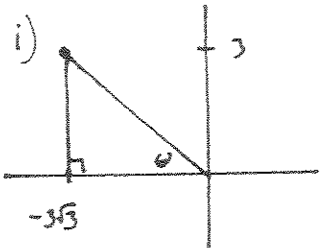


iii)



QUESTION 3

a) i)



$|z| = \sqrt{(3\sqrt{3})^2 + 3^2}$

$= \sqrt{36}$

$= 6$

$\tan\theta = \frac{3}{3\sqrt{3}}$

$\theta = \frac{\pi}{6}$

$\therefore \arg(z) = \frac{5\pi}{6}$

$\therefore z = 6\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

ii) to be real $\frac{5n\pi}{6} = m\pi$ (m is integer)

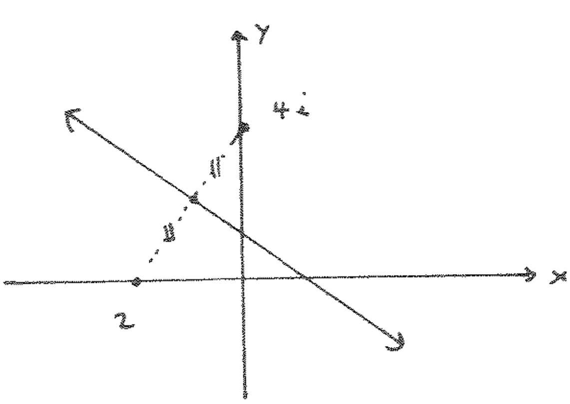
$\therefore$ n equals 6.

b) $\frac{4}{1-i} = \frac{4}{1-i} \times \frac{1+i}{1+i}$

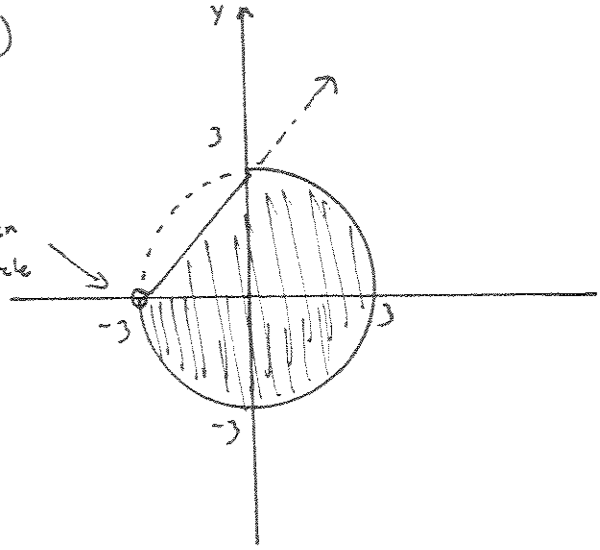
$= \frac{4+4i}{2}$

$\therefore Im\left(\frac{4}{1-i}\right) = 2$

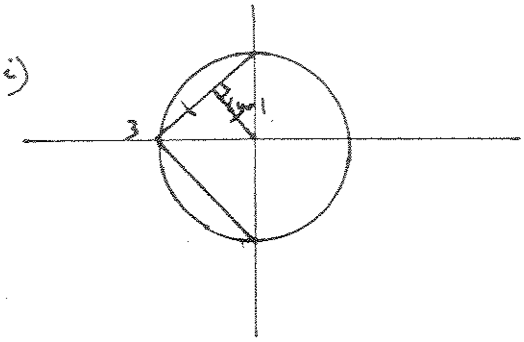
c)



d) i)



ii)



let $|w| = x$

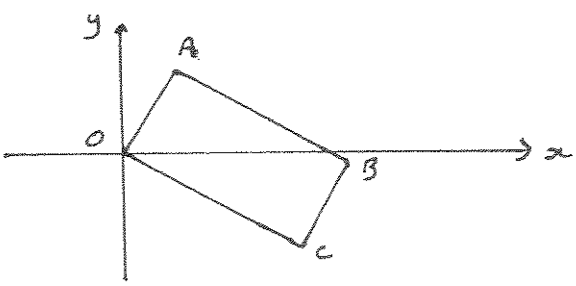
$\therefore x^2 + x^2 = 3^2$

$2x^2 = 9$

$x = \frac{3}{\sqrt{2}}$

$\therefore \min |w| = \frac{3}{\sqrt{2}}$

e)



i) $C = (-i)(2)(1+2i)$

$= 4 - 2i$

$B = (1+2i) + (4-2i)$

$= 5$

ii) multiply A by $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

or $\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$\therefore A' = (1+2i)\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2} - \sqrt{3} + i\left(\frac{\sqrt{3}}{2} + 1\right)$

QUESTION 4

a) $\frac{x^2}{9} - \frac{y^2}{4} = 1$

i) $b^2 = a^2(e^2 - 1)$

$4 = 9(e^2 - 1)$

$e = \frac{\sqrt{13}}{3}$

ii) $y = \pm\frac{2x}{3}$

b) i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} - \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{b^2 x}{a^2 y}$

$\therefore m_T = \frac{b^2 a \sec\theta}{a^2 b \tan\theta}$

$= \frac{b\sec\theta}{a\tan\theta}$

$\therefore$ using $y - y_1 = m(x - x_1)$

$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$bx\sec\theta - ay\tan\theta = ab(\sec^2\theta - \tan^2\theta)$

$bx\sec\theta - ay\tan\theta = ab$

$\therefore \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

ii) directrix $x = \frac{a}{e}$

sub. into tangent

$\frac{a}{e} \cdot \frac{\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

$y = \frac{b(\sec\theta - e)}{e\tan\theta}$

$\therefore Q\left(\frac{a}{e}, \frac{b(\sec\theta - e)}{e\tan\theta}\right)$

$P(a\sec\theta, b\tan\theta)$

$S(ae, 0)$

$\therefore m_{QS} = \frac{\frac{b(\sec\theta - e)}{e\tan\theta}}{\frac{a}{e} - ae}$

$= \frac{b(\sec\theta - e)}{a(1 - e^2)\tan\theta}$

$m_{PS} = \frac{b\tan\theta}{a(\sec\theta - e)}$

$\therefore m_{QS} \times m_{PS} =$

$\frac{b\tan\theta}{a(\sec\theta - e)} \cdot \frac{b(\sec\theta - e)}{a(1 - e^2)\tan\theta}$

$= \frac{b^2}{a^2(1 - e^2)}$

$= \frac{b^2}{-b^2}$ $\quad b^2 = a^2(e^2 - 1)$

$= -1$

$\therefore \angle PSQ$ is right angle.

c) i) $RHS = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$

$= (1-\sqrt{x})^{n-1}\left[1 - (1-\sqrt{x})\right]$

$= (1-\sqrt{x})^{n-1}\sqrt{x}$

$= LHS$

ii) $I_n = \int_0^1 (1-\sqrt{x})^n \, dx$

$u = (1-\sqrt{x})^n \qquad u' = n(1-\sqrt{x})^{n-1} \cdot -\frac{1}{2}x^{-\frac{1}{2}}$

$v = x \qquad v' = 1$

$\therefore$ using integration by parts

$I_n = x(1-\sqrt{x})^n\Big]_0^1 + n\int_0^1 x(1-\sqrt{x})^{n-1}\frac{1}{2}x^{-\frac{1}{2}}\,dx$

$= \frac{n}{2}\int_0^1 \sqrt{x}(1-\sqrt{x})^{n-1}\,dx$

$= \frac{n}{2}\int_0^1 (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n \, dx$

$= \frac{n}{2}\left[I_{n-1} - I_n\right]$

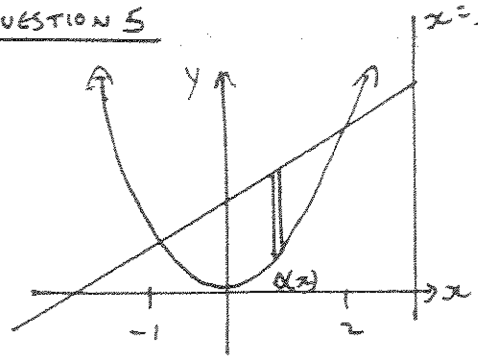
$\left(\frac{n}{2}+1\right)I_n = \frac{n}{2}I_{n-1}$

$(n+2)I_n = nI_{n-1}$

$I_n = \frac{n}{n+2}I_{n-1}$

QUESTION 5

a)



$y = x^2$ need pt. of

$y = x + 2$ intersection

by inspection $x = -1, 2$.

$$\Delta V = 2\pi \times \text{radius} \times \text{height} \times \text{thickness}$$
$$= 2\pi \times (3-x) \times (x+2-x^2) \times \Delta x$$
$$= 2\pi \left[6 + x - 4x^2 + x^3 \right] \Delta x$$

$$\therefore V = 2\pi \int_{-1}^{2} 6 + x - 4x^2 + x^3 \, dx$$

$$= 2\pi \left[6x + \tfrac{1}{2}x^2 - \tfrac{4}{3}x^3 + \tfrac{1}{4}x^4 \right]_{-1}^{2}$$

$$= \frac{45\pi}{2} \text{ cubic units}$$

b)

$$P(x) = 8x^4 + 12x^3 - 30x^2 + 17x - 3$$
$$P'(x) = 32x^3 + 36x^2 - 60x + 17$$
$$P''(x) = 96x^2 + 72x - 60$$

a root of $P''(x) = 0$ is triple root of $P(x) = 0$

$$96x^2 + 72x - 60 = 0$$
$$12(8x^2 + 6x - 5) = 0$$
$$12(4x+5)(2x-1) = 0$$
$$x = -\tfrac{5}{4}, \tfrac{1}{2}$$

$P'(-\tfrac{5}{4}) \neq 0$

$P'(\tfrac{1}{2}) = 0$

$\therefore x = \tfrac{1}{2}$ is triple root

$\therefore \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \alpha = \dfrac{-12}{8}$

$\alpha = -3$

$\therefore x = -3, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}$

QUESTION 5 (cont.)

c) required polynomial is $P(\sqrt{x})=0$

$$(\sqrt{x})^3 + p(\sqrt{x})^2 + q\sqrt{x} + r = 0$$

$$x\sqrt{x} + q\sqrt{x} = -px - r$$

$$(x\sqrt{x} + q\sqrt{x})^2 = (-px-r)^2$$

$$x^3 + 2qx^2 + q^2x = p^2x^2 + 2prx + r^2$$

$$x^3 + (2q-p^2)x^2 + (q^2-2pr)x - r^2 = 0$$

d) i)

$$\ddot{x} = -n^2x$$

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = -n^2x$$

$$\tfrac{1}{2}v^2 = -\frac{n^2}{2}x^2 + c$$

when $v=0$ $x=a$

$$\therefore c = \frac{n^2a^2}{2}$$

$$\therefore \tfrac{1}{2}v^2 = \frac{n^2a^2}{2} - \frac{n^2x^2}{2}$$

$$v^2 = n^2(a^2-x^2)$$

ii) from information

$$V^2 = n^2(a^2-d^2)$$

$$\frac{V^2}{4} = n^2(a^2-4d^2)$$

α) Solve for a

$$\therefore \frac{n^2(a^2-d^2)}{4} = n^2(a^2-4d^2)$$

$$a^2 - d^2 = 4a^2 - 16d^2$$

$$3a^2 = 15d^2$$

$$a^2 = 5d^2$$

$$a = \sqrt{5}\,d$$

$\therefore$ amplitude $= \sqrt{5}\,d$

β) using part α)

$$V^2 = n^2(5d^2-d^2)$$

$$V^2 = n^2\,4d^2$$

$$n^2 = \frac{V^2}{4d^2}$$

$$n = \frac{V}{2d}$$

$$\therefore \text{period} = \frac{2\pi}{n}$$

$$= \frac{4\pi d}{V}$$

QUESTION 6

a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$

$t = \tan\frac{x}{2}$

$dx = \dfrac{2dt}{1+t^2}$

$\cos x = \dfrac{1-t^2}{1+t^2}$

$\therefore \displaystyle\int_0^1 \frac{\frac{2dt}{1+t^2}}{2+\frac{1-t^2}{1+t^2}}$

$= \displaystyle\int_0^1 \frac{2\,dt}{3+t^2}$

$= \dfrac{2}{\sqrt3}\tan^{-1}\dfrac{t}{\sqrt3}\Big]_0^1$

$= \dfrac{2}{\sqrt3}\cdot\dfrac{\pi}{6}$

$= \dfrac{\pi}{3\sqrt3}$

b) i) direction of motion +ve

two forces (mg and mkv) acting against motion

$\therefore\ ma = -mg - mkv$

$\therefore\ \ddot{x} = -g - kv$

ii) $\ddot{x} = -g - kv$

$\therefore\ v\dfrac{dv}{dx} = -(g+kv)$

$\displaystyle\int \frac{v\,dv}{g+kv} = -\int dx$

$\dfrac{1}{k}\displaystyle\int \frac{g+kv}{g+kv} - \frac{g}{g+kv}\,dv = -x + c$

$\dfrac{1}{k}\displaystyle\int 1 - \frac{g}{g+kv}\,dv = -x + c$

$\dfrac{1}{k}\left[v - \dfrac{g}{k}\ln(g+kv)\right] = -x + c$

when $x = 0$ $v = V$

$\therefore\ c = \dfrac{1}{k}\left[V - \dfrac{g}{k}\ln(g+kV)\right]$

$v = 0$ at greatest height ($x = H$)

$\therefore\ \dfrac{1}{k}\left[-\dfrac{g}{k}\ln g\right] = -H + \dfrac{1}{k}\left[V - \dfrac{g}{k}\ln(g+kV)\right]$

$H = \dfrac{1}{k}\left[V - \dfrac{g}{k}\ln(g+kV) + \dfrac{g}{k}\ln g\right]$

$= \dfrac{1}{k}\left[V - \dfrac{g}{k}\ln\left(\dfrac{g+kV}{g}\right)\right]$

$= \dfrac{V}{k} - \dfrac{g}{k^2}\ln\left(1+\dfrac{kV}{g}\right)$

QUESTION 6 (CONT)

c) i) roots are

$$z_1 = 1$$
$$z_2 = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} \qquad (= w)$$
$$z_3 = \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7} \qquad (= w^2)$$
$$z_4 = \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7} \qquad (= w^3)$$
$$z_5 = \cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7} \qquad (= w^4)$$
$$z_6 = \cos\frac{10\pi}{7} + i\sin\frac{10\pi}{7} \qquad (= w^5)$$
$$z_7 = \cos\frac{12\pi}{7} + i\sin\frac{12\pi}{7} \qquad (= w^6)$$

(or equivalent expressions)

ii) roots can be expressed as $1, w, w^2, w^3, w^4, w^5, w^6$

$\therefore$ sum of roots $\left(-\frac{b}{a}\right)$ is

$$1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$$

iii) roots $w + w^6,\ w^2 + w^5,\ w^3 + w^4$

Sum of roots 1 at a time: $w + w^6 + w^2 + w^5 + w^3 + w^4 = -1$

Sum of roots 2 at a time: $(w+w^6)(w^2+w^5) + (w+w^6)(w^3+w^4) + (w^2+w^5)(w^3+w^4)$

$$= w^3 + w^6 + w^8 + w^{11} + w^4 + w^5 + w^9 + w^{10} + w^5 + w^6 + w^8 + w^9$$
$$= 2(w + w^2 + w^3 + w^4 + w^5 + w^6)$$
$$= -2$$

note $w^8 = w^1$
$w^9 = w^2$
etc.

Sum of roots 3 at a time:

$$(w + w^6)(w^2 + w^5)(w^3 + w^4)$$
$$= (w + w^6)(w^5 + w^6 + w^8 + w^9)$$
$$= w^6 + w^7 + w^9 + w^{10} + w^{11} + w^{12} + w^{14} + w^{15}$$
$$= w^6 + 1 + w^2 + w^3 + w^4 + w^5 + 1 + w$$
$$= 2 + -1$$
$$= 1$$

$\therefore$ required polynomial

$$x^3 - (-1)x^2 + (-2)x - (1) = 0$$
$$x^3 + x^2 - 2x - 1 = 0$$

QUESTION 7

a) i) $\angle PQA = \theta$ (angle between tangent and chord equals angle in the alternate segment)

ii) $\angle OBA = \theta$ (angles opposite equal sides of triangle are equal)
(OA = OB radii of circle)

$\therefore$ O, B, Q, A concyclic as OA subtends equal angles at B and Q.

iii) $\angle BQO = \angle BAO = \theta$ (chord OB subtends equal angles at Q and A)
(O, B, Q, A concyclic)

$\therefore \angle BQO = \angle PQA = \theta$

$\therefore$ OQ bisects $\angle BQA$.

iv) $\angle BRA = \angle OAB = \theta$ (angle between tangent and chord equals angle in alternate segment)

$\therefore \angle BRA = \angle BQO = \theta$

$\therefore OQ \parallel AR$ (corresponding angles equal)

b) i) each rectangle has width 1 unit and heights
$\ln 2, \ln 3, \ln 4, \ldots, \ln k$ and there is $(k-1)$ rectangles.

$\therefore$ sum of areas $= 1 \times \ln 2 + 1 \times \ln 3 + 1 \times \ln 4 + \ldots + 1 \times \ln k$
$= \ln 2 + \ln 3 + \ln 4 + \ldots + \ln k$

ii) $\int_2^{k+1} \ln(x-1)\,dx$ $\qquad u = \ln(x-1) \qquad u' = \frac{1}{x-1}$

using parts $\qquad v = x \qquad v' = 1$

$= x\ln(x-1)\Big]_2^{k+1} - \int_2^{k+1} \frac{x}{x-1}\,dx$

$= (k+1)\ln k - 2\ln 1 - \left[\int_2^{k+1} 1 + \frac{1}{x-1}\,dx\right]$

$= (k+1)\ln k - \left[x + \ln(x-1) \right]_2^{k+1}$

$= (k+1)\ln k - \left[(k+1+\ln k) - (2+\ln 1) \right]$

$= (k+1)\ln k - k - 1 - \ln k + 2$

$= k\ln k + \ln k - k - \ln k + 1$

$= k\ln k - k + 1$

iii) from diagram we can see

$$\int_2^{k+1} \ln(x-1)\,dx < S < \int_2^{k+1} \ln x\,dx$$

now $\int_2^{k+1} \ln x\,dx = x\ln x\Big]_2^{k+1} - \int_2^{k+1} dx$

$= \left[(k+1)\ln(k+1) - 2\ln 2 \right] - \left[x \right]_2^{k+1}$

$= (k+1)\ln(k+1) - \ln 4 - k + 1$

and $S = \ln 2 + \ln 3 + \ln 4 + \cdots + \ln k$

$= \ln k!$

$\therefore\ k\ln k - k + 1 < \ln k! < (k+1)\ln(k+1) - \ln 4 - k + 1$

$k\ln k < \ln k! + k - 1 < (k+1)\ln(k+1) - \ln 4$

$$\ln k^k < \ln k! + k - 1 < \ln \frac{(k+1)^{k+1}}{4}$$

$$k^k < e^{\ln k! + k - 1} < \frac{(k+1)^{k+1}}{4}$$

$$k^k < k!\, e^{k-1} < \frac{1}{4}(k+1)^{k+1}$$

QUESTION 8

a) $\sin x = \cos 5x$

$$\cos\left(\frac{\pi}{2} - x\right) = \cos 5x$$

$$5x = 2n\pi \pm \left(\frac{\pi}{2} - x\right)$$

$\therefore\ 5x = 2n\pi + \left(\frac{\pi}{2} - x\right)$ $\qquad$ $5x = 2n\pi - \left(\frac{\pi}{2} - x\right)$

$x = \frac{1}{6}\left(2n\pi + \frac{\pi}{2}\right)$ $\qquad$ $x = \frac{1}{4}\left(2n\pi - \frac{\pi}{2}\right)$

when $n=0$ $\qquad$ $n=1$

$x = \frac{\pi}{12}$ $\qquad$ $x = \frac{3\pi}{8}$

$n=1$ $\qquad$ $n=2$

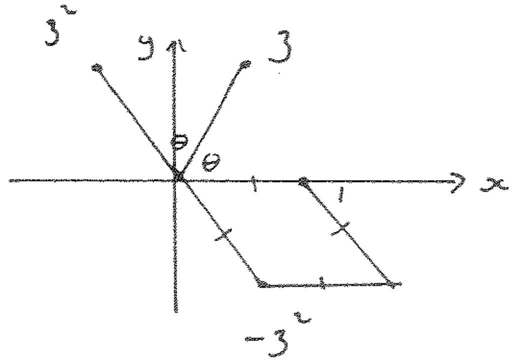
$x = \frac{5\pi}{12}$ $\qquad$ $x = \frac{7\pi}{8}$

$n=2$

$x = \frac{3\pi}{4}$

$\therefore$ Solutions $\quad x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{3\pi}{8}, \frac{7\pi}{8}$

b) $\operatorname{Arg}\left(\frac{2}{1-z^2}\right)$



$= \operatorname{Arg} 2 - \operatorname{Arg}(1-z^2)$

$= 0 - -\frac{1}{2}(\pi - 2\theta)$

$= \frac{\pi}{2} - \theta$

c) i) $\qquad -1 \le f(x) \le 1$

ii) α) $\qquad x_1 = g(r)$

$$x_1 = \frac{e^{2r}-1}{e^{2r}+1}$$

$$x_1 e^{2r} + x_1 = e^{2r} - 1$$

$$e^{2r} = \frac{1+x_1}{1-x_1}$$

$$r = \frac{1}{2}\ln\left(\frac{1+x_1}{1-x_1}\right)$$

(β) $1 + x_1$ and $1 - x_1$ are both positive as $-1 < x_1 < 1$

$\therefore \dfrac{1 + x_1}{1 - x_1}$ is positive

$\therefore \ln\left(\dfrac{1 + x_1}{1 - x_1}\right)$ exists

$\therefore r$ exists

(δ)
$$\frac{2\,g(x)}{1 + (g(x))^2} = \frac{2\,(e^{2x} - 1)}{e^{2x} + 1} \div \left(1 + \left(\frac{e^{2x} - 1}{e^{2x} + 1}\right)^2\right)$$

$$= \frac{2\,(e^{2x} - 1)}{e^{2x} + 1} \times \frac{(e^{2x} + 1)^2}{(e^{2x} + 1)^2 + (e^{2x} - 1)^2}$$

$$= \frac{2\,(e^{2x} - 1)\,(e^{2x} + 1)}{2\,(e^{4x} + 1)}$$

$$= \frac{e^{4x} - 1}{e^{4x} + 1}$$

$$= g(2x)$$

(γ) from part β) the result is true for $n = 1$

assume true for $n = k$ i.e. $x_k = g(2^{k-1} . r)$

test for $n = k+1$

$$\therefore \quad x_{k+1} = f(x_k)$$

$$= \frac{2x_k}{1 + (x_k)^2}$$

$$= \frac{2\,g(2^{k-1} . r)}{1 + (g(2^{k-1} . r))^2}$$

$$= g(2 . 2^{k-1} . r)$$

$$= g(2^k . r)$$

which is the required result

etc, etc.