YEAR 12 TRIAL HSC 2001 - EXTENSION 1 SOLUTIONS

Question 1

$y = 2x + 3 \qquad m_1 = 2$

$y = 4x + 1 \qquad m_2 = 4$

$\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\dfrac{2-4}{1+2\times 4}\right|$

$= \dfrac{2}{9}$

$\theta \doteq 13^\circ$

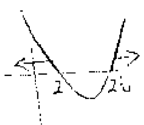
$\dfrac{1}{x-2} \leq 4 \qquad \boxed{x \neq 2}$

$(x-2) \leq 4(x-2)^2$

$4(x-2)^2 - (x-2) \geq 0$

$(x-2)(4x-9) \geq 0$

$x < 2, \; x \geq 2\frac{1}{4}$



A$(-1, 4)$ B(x, y) $3 : -2$

$8 = \dfrac{-2\times -1 + 3x}{3-2} \qquad -5 = \dfrac{-2\times 4 + 3y}{3-2}$

$8 = 2 + 3x \qquad -5 = -8 + 3y$

$x = 2 \qquad y = 1$

B $(2, 1)$

$\displaystyle\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx = \left[\sin^{-1}\frac{x}{3}\right]_0^3$

$= \sin^{-1} 1 - \sin^{-1} 0$

$= \dfrac{\pi}{2}$

(e) $\sin\theta = \cos 2\theta$

$\sin\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\sin\theta = \frac{1}{2}, \; \sin\theta = -1$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

Question 2

(a) $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx$

$u = \ln x$

$\dfrac{du}{dx} = \dfrac{1}{x}$

$x = e \quad u = \ln e = 1$

$x = e^2 \quad u = \ln e^2 = 2\ln e = 2$

$= \displaystyle\int_1^2 \frac{1}{u}\frac{du}{dx}\,dx$

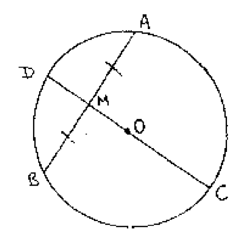
$= \displaystyle\int_1^2 \frac{1}{u}\,du$

$= \left[\ln u\right]_1^2$

$= \ln 2 - \ln 1$

$= \ln 2$

(b)



(i) AM = BM given

Line through the centre that bisects the chord is perpendicular to the chord.

$\therefore$ AB $\perp$ CD.

(ii) AM × BM = CM × DM

product of intercepts of intersecting chords.

$5 \times 5 = CM \times 2$

$CM = 12\frac{1}{2}$

$\therefore DC = 14\frac{1}{2}$ (diameter)

radius $OC = 7\frac{1}{4}$ cm

(iii) Area $= 2 \times$ Area $\triangle ADC$

$= 2 \times \frac{1}{2} \times 14\frac{1}{2} \times 5$

$= 72\frac{1}{2}$ cm^2

(c) $f(x) = e^x + x - 5$

(i) $f(x)$ is a continuous function

$f(1) = e^1 + 1 - 5$

$= -1.28 < 0$

$f(2) = e^2 + 2 - 5$

$= 4.39 > 0$

Since $f(x)$ is continuous and there is a sign change between $x=1$ and $x=2$ there is a root in the interval $1 < x < 2$.

(ii) $f(x) = e^x + x - 5$

$f'(x) = e^x + 1$

$x_1 = 1.5 - \dfrac{f(1.5)}{f'(1.5)}$

$= 1.5 - \dfrac{0.981689\ldots}{5.481689\ldots}$

$= 1.3$

Question 3

(a) $f(x) = 2\cos^{-1}\left(\frac{x}{4}\right)$

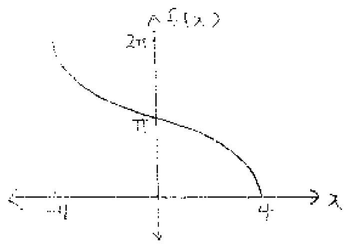
(i) $-1 \le \frac{x}{4} \le 1$

Domain $-4 \le x \le 4$

$0 \le \cos^{-1}\left(\frac{x}{4}\right) \le \pi$

Range $0 \le 2\cos^{-1}\left(\frac{x}{4}\right) \le 2\pi$

(ii)



(b) $x = 5\cos 6t$

(i) $\dot{x} = -30\sin 6t$

$\ddot{x} = -180\cos 6t$

$= -36(5\cos 6t)$

$= -6^2 x$

Since $\ddot{x}$ is of the form $-n^2 x$

the motion is simple harmonic.

(ii) amplitude 5, centre of motion 0

endpoints $x = \pm 5$

period $= \frac{2\pi}{6}$

$= \frac{\pi}{3}$ seconds

(iii) $\frac{1}{2}$ way $\Rightarrow x = 2.5$

$2.5 = 5\cos 6t$

$\cos 6t = \frac{1}{2}$

$6t = \frac{\pi}{3}$ (first time)

$t = \frac{\pi}{18}$ seconds

(c)

Area = A(rectangle) − A(to y-axis)

$= \frac{\pi}{2}\times 1 - \int_0^{\frac{\pi}{2}} \sin y \, dy$

$= \frac{\pi}{2} - \left[-\cos y\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} - \left(-\cos\frac{\pi}{2} + \cos 0\right)$

$= \frac{\pi}{2} - 1$ sq. units

Question 4

(a) $t = \tan\frac{x}{2} \Rightarrow \cos x = \frac{1-t^2}{1+t^2}$

$\frac{1+\cos x}{1-\cos x} = \frac{1+\frac{1-t^2}{1+t^2}}{1-\frac{1-t^2}{1+t^2}}$

$= \frac{\frac{1+t^2+(1-t^2)}{1+t^2}}{\frac{1+t^2-(1-t^2)}{1+t^2}}$

$= \frac{2}{2t^2}$

$= \frac{1}{\tan^2\frac{x}{2}}$

$= \cot^2\frac{x}{2}$

(b) Initially $\dot{x} = 20$, $x = 0$

$\dot{y} = 0$, $y = 80$

(i)

Horizontal

$\ddot{x} = 0$

$\dot{x} = c_1$

when $t=0$ $\dot{x}=20$

$\therefore c_1 = 20$

$\dot{x} = 20$

$x = 20t + c_2$

when $t=0$, $x=0$

$\therefore c_2 = 0$

$\therefore x = 20t$

Vertical

$\ddot{y} = -10$

$\dot{y} = -10t + c_3$

when $t=0$ $\dot{y}=0$

$\therefore c_3 = 0$

$\dot{y} = -10t$

$y = -5t^2 + c_4$

when $t=0$, $y=80$

$\therefore c_4 = 80$

$y = 80 - 5t^2$

(ii) when $y = 35$

$35 = 80 - 5t^2$

$5t^2 = 45$

$t^2 = 9$

$t = 3$ $(t > 0)$

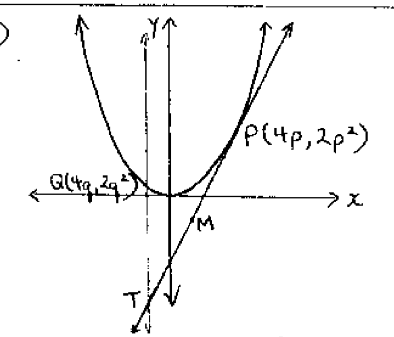
when $t = 3$

$x = 20t$

$= 60$

60 m from cliff to tower

(c)



(i) $x^2 = 8y \Rightarrow y = \frac{x^2}{8}$

$\frac{dy}{dx} = \frac{2x}{8} = \frac{x}{4}$

$P(4p, 2p^2)$ $m_T = \frac{4p}{4} = p$

equation $y - 2p^2 = p(x - 4p)$

$y - 2p^2 = px - 4p^2$

$y = px - 2p^2$

(ii) through Q, ∥ to y axis

$\therefore x = 4q$

sub into eq. tangent

$y = 4pq - 2p^2$

$T\left(4q, 4pq - 2p^2\right)$

(iii) $M = \left(\frac{4p+4q}{2}, \frac{2p^2+4pq-2p^2}{2}\right)$

$= (2p+2q, 2pq)$

(iv) when $pq = -1$

$y = 2pq$

$= -2$

$\therefore$ locus of M is $y = -2$

tion 5

(i) $x^2 + 8x + 20 = (x+4)^2 + 4$

(ii) $\int \frac{dx}{x^2+8x+20} = \int \frac{dx}{(x+4)^2+4}$

$= \frac{1}{2}\tan^{-1}\left(\frac{x+4}{2}\right) + C$

(i) $P = 2000 + Ae^{kt}$

$\frac{dP}{dt} = Ak e^{kt}$

$= k(Ae^{kt})$

$= k(P - 2000)$

) $t = 0 \quad P = 2500$

$2500 = 2000 + Ae^{0}$

$A = 500$

$t = 2 \quad P = 5000$

$5000 = 2000 + 500e^{2k}$

$e^{2k} = 6$

$k = \frac{1}{2}\ln 6$

$\doteqdot 0.8959\ldots$

) $P = 2000 + 500e^{4\ln 6}$

$= 650\,000$

) ( ) Show true for $n = 1$

$S_1 = \frac{1}{1\times 2\times 3} = \frac{1}{6}$

$\frac{1(1+3)}{4(1+1)(1+2)} = \frac{1(1+3)}{4(1+1)(1+2)} = \frac{4}{4\times 2\times 3} = \frac{1}{6}$

true for $n = 1$

Assume true for $n = k$

$S_k = \frac{1}{1\times 2\times 3} + \frac{1}{2\times 3\times 4} + \ldots + \frac{1}{k(k+1)(k+2)}$

$= \frac{k(k+3)}{4(k+1)(k+2)}$

Show true for $n = k+1$

$S_{k+1} = \frac{1}{1\times 2\times 3} + \ldots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$

$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$

$S_{k+1} = S_k + \frac{1}{(k+1)(k+2)(k+3)}$

$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$

$= \frac{1}{(k+1)(k+2)}\left[\frac{k(k+3)}{4} + \frac{1}{k+3}\right]$

$= \frac{1}{(k+1)(k+2)}\left(\frac{k(k+3)^2+4}{4(k+3)}\right)$

$= \frac{1}{(k+1)(k+2)}\left(\frac{k^3+6k^2+9k+4}{4(k+3)}\right)$

$= \frac{1}{(k+1)(k+2)}\left(\frac{(k+1)(k^2+5k+4)}{4(k+3)}\right)$

$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$ as required

$\therefore$ true for $n = k+1$ if true for $n = k$

Since true for $n=1$ also true for $n = 1+1 = 2$, thus true for $n = 3$ and for $n = 4$ and so on for all positive integers.

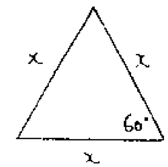
(ii) $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n^2+3n}{4n^2+12n+8}$

$= \lim_{n\to\infty} \frac{1+\frac{3}{n}}{4+\frac{12}{n}+\frac{8}{n^2}}$

$= \frac{1}{4}$

Question 6

(a)



(i) $A = \frac{1}{2}\times x\times x\times \sin 60°$

$= \frac{\sqrt{3}\,x^2}{4}$

(ii) $\frac{dx}{dt} = \frac{1}{6} \qquad \frac{dA}{dx} = \frac{\sqrt{3}\,x}{2}$

$\frac{dA}{dt} = \frac{dA}{dx}\times\frac{dx}{dt}$

$= \frac{\sqrt{3}\times 12}{2}\times\frac{1}{6}$

$= \sqrt{3}\ \text{cm}^2\,\text{s}^{-1}$

(b) (i) In ΔPVT

$\tan\alpha = \frac{h}{PV}$

$PV = \frac{h}{\tan\alpha}$

(ii) Similarly for ΔQVT

$QV = \frac{h}{\tan\beta}$

$PV^2 + QV^2 = 1000^2$

$\frac{h^2}{\tan^2\alpha} + \frac{h^2}{\tan^2\beta} = 1000^2$

$h^2\left(\frac{\tan^2\beta+\tan^2\alpha}{\tan^2\alpha\times\tan^2\beta}\right) = 1000^2$

$h^2 = \frac{1000^2\tan^2\alpha\tan^2\beta}{\tan^2\alpha+\tan^2\beta}$

$h = \frac{1000\tan\alpha\tan\beta}{\sqrt{\tan^2\alpha+\tan^2\beta}}$

(iii) $\alpha = 36° \quad \beta = 45°$

$h = \frac{1000\tan 36\tan 45}{\sqrt{\tan^2 36+\tan^2 45}}$

$= 588$ m (nearest metre)

(c) (i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6x^2$

$\frac{1}{2}v^2 = 2x^3 + C$

$x = 1, \quad v = -2$

$\frac{1}{2}(-2)^2 = 2(1)^3 + C$

$2 = 2 + C$

$C = 0$

$\frac{1}{2}v^2 = 2x^3$

$v^2 = 4x^3$

(ii) Initially particle moving left, velocity negative. Particle moves from $x = 1$ towards $x = 0$

If $x = 0 \qquad \ddot{x} = 0$

$v^2 = 0 \quad (v = 0)$

no acceleration $\therefore$ if particle reaches origin it stops and no acceleration for it to start again

$\therefore$ velocity won't change from negative to positive.

[$x = 0$ is limiting position of the particle

Question 7

a) If ... factor $P(2) = 0$

$P(2) = 2^3 + k(2^2) - 2k - 6$

$10 - 2k = 0$

$k = 5$

b) (i) $x \geq 0$

(ii) Inverse $x = \frac{1}{1+y^2}$

$1 + y^2 = \frac{1}{x}$

$y^2 = \frac{1}{x} - 1$

$y = \pm\sqrt{\frac{1-x}{x}}$

positive case, from (i) $y \geq 0$

$f^{-1}(x) = \sqrt{\frac{1-x}{x}}$

domain: $0 < x \leq 1$

c) (i) for max ht. $\dot{y} = 0$

$V \sin\alpha - gt = 0$

$gt = V\sin\alpha$

$t = \frac{V\sin\alpha}{g}$

$y = V \cdot \frac{V\sin\alpha}{g} \cdot \sin\alpha - \frac{1}{2}g \cdot \frac{V^2\sin^2\alpha}{g^2}$

$= \frac{V^2\sin^2\alpha}{g} - \frac{V^2\sin^2\alpha}{2g}$

$= \frac{V^2\sin^2\alpha}{2g}$

(ii) max. ht. of Q is

$y = \frac{(\sqrt{\frac{5}{2}}V)^2 \sin^2\frac{\alpha}{2}}{2g}$

$y = \frac{5V^2\sin^2\frac{\alpha}{2}}{4g}$

Both reach same max ht.

$\frac{5V^2\sin^2\frac{\alpha}{2}}{4g} = \frac{V^2\sin^2\alpha}{2g}$

$5\sin^2\frac{\alpha}{2} = 2\sin^2\alpha$

$5\left(\frac{1}{2}(1-\cos\alpha)\right) = 2(1-\cos^2\alpha)$

$5 - 5\cos\alpha = 4 - 4\cos^2\alpha$

$4\cos^2\alpha - 5\cos\alpha + 1 = 0$

$(4\cos\alpha - 1)(\cos\alpha - 1) = 0$

$\cos\alpha = \frac{1}{4}$, $\cos\alpha = 1$

$\alpha = \cos^{-1}\frac{1}{4}$ no soln ($\alpha = 0$ not possible)