National Educational Advancement Programs

16

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

1999

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

SOLUTIONS AND SUGGESTED MARKING SCHEME

Students are reminded that this is a trial examination only and cannot in any way guarantee the content or the format of the 1999 Mathematics 3 Unit (Additional) and 3/4 Unit (Common) Higher School Certificate Examination.

QUESTION 1.

(a) $\dfrac{x+1}{x} \geq 2 \quad x \neq 0$

$\therefore \dfrac{(x^2)(x+1)}{x} \geq 2(x^2)$ ✓

$\therefore x(x+1) \geq 2x^2$

$2x^2 - x^2 - x \leq 0$

$x^2 - x \leq 0$

$x(x-1) \leq 0$ ✓

but $x \neq 0$

$\therefore 0 < x \leq 1$ ✓

(b) $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$x + 3y = 4$

$2x - 5y = 0$

$m_1 = -\dfrac{1}{3}$

$m_2 = \dfrac{2}{5}$

$\tan\theta = \left|\dfrac{\left(-\frac{1}{3}\right) - \left(\frac{2}{5}\right)}{1 + \left(-\frac{1}{3}\right)\left(\frac{2}{5}\right)}\right|$ ✓

$= \left|\dfrac{\frac{-5-6}{15}}{1 - \frac{2}{15}}\right|$

$= \left|\dfrac{-11}{13}\right|$

$= \dfrac{11}{13}$ ✓

$\therefore \theta = \tan^{-1}\left(\dfrac{11}{13}\right)$

$= 40.236\ldots°$ (calculator)

$\therefore$ the angle between the lines is 40° (to the nearest degree). ✓

(c) $2\cos\theta - \sin\theta = -1$

$2\left(\dfrac{1-t^2}{1+t^2}\right) - \left(\dfrac{2t}{1+t^2}\right) = -1$ ✓

$2 - 2t^2 - 2t = -1 - t^2$

$t^2 + 2t - 3 = 0$

$(t+3)(t-1) = 0$

$\therefore t = -3 \quad \text{or} \quad t = 1$ ✓

$\tan\dfrac{\theta}{2} = -3 \quad \text{or} \quad \tan\dfrac{\theta}{2} = 1$

$\dfrac{\theta}{2} = \pi - 1.2490\ldots \qquad \dfrac{\theta}{2} = \dfrac{\pi}{4}$ ✓

$\therefore \theta = 3.79$ (3SF) or $\theta = 1.57$ (3SF) ✓

(d) $x = \dfrac{x_1 r_2 + x_2 r_1}{r_1 + r_2}$

$x_1 = 2 \quad x_2 = -6$

$y_1 = -7 \quad y_2 = 9$

$r_1 : r_2 = 3:5$

$= \dfrac{(2)(5) + (-6)(3)}{3+5}$

$= \dfrac{10 - 18}{8}$

$= -1$ ✓

$y = \dfrac{y_1 r_2 + y_2 r_1}{r_1 + r_2}$

$= \dfrac{(-7)(5) + (9)(3)}{3+5}$

$= \dfrac{-35 + 27}{8}$

$= -1$ ✓

$\therefore$ point P is $(-1, -1)$.

QUESTION 2.

(a) $\displaystyle\int \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int \frac{du}{\sqrt{1-u^2}}$ ✓

let $u = \ln x$

$du = \dfrac{1}{x}dx$

$= \sin^{-1} u + c$

$= \sin^{-1}(\ln x) + c$ ✓

(b) $r = \dfrac{T_2}{T_1} = -\tan^2 x$

(i) $-1 < r < 1$

$\therefore -1 < -\tan^2 x < 1$

$\therefore -1 < \tan^2 x < 1$

But $\tan^2 x \geq 0$, and $x > 0$, given.

$\therefore 0 < \tan^2 x < 1$

$0 < \tan x < 1 \quad \left(\text{since } 0 < x < \dfrac{\pi}{2}\right)$

$0 < x < \dfrac{\pi}{4}$ ✓

(ii) $S_\infty = \dfrac{a}{1-r}$

$= \dfrac{1}{1-(-\tan^2 x)}$

$= \dfrac{1}{1+\tan^2 x}$ ✓

$= \dfrac{1}{\sec^2 x}$

$= \cos^2 x$ ✓

(c) Let $P(x) = x^3 + rx^2 - 4x + s$

and note $x^2 + x - 2 = (x+2)(x-1)$

If $x^2 + x - 2$ is a factor of $P(x)$

then $P(1) = 0$ and $P(-2) = 0$

(i) $P(1) = (1)^3 + r(1)^2 - 4(1) + s = 0$

$\therefore (1 + r - 4 + s) = 0$

$r + s = 3$ ① ✓

(ii) $P(-2) = (-2)^3 + (-2)^2 r - 4(-2) + s = 0$

$-8 + 4r + 8 + s = 0$

$\therefore 4r + s = 0$ ② ✓

from (i) $r + s = 3$ ①

② − ① $3r = -3$

$r = -1$

$s = 4$ ✓

(d) No. of tickets with direction $= 2 \times {}^{11}C_2$ (OR ${}^{11}P_2$)

$= 110$ ✓

(e) (i) No. of arrangements in a straight line $= 2! \times 5! \times 5!$ ✓

$= 28800$

(ii) No. of arrangements in a circle $= 4! \times 5!$ ✓

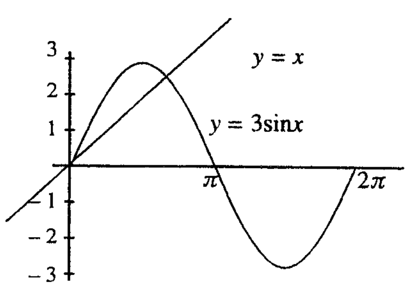
$= 2880$

(iii) P(m and w alternate) $= \dfrac{4! \times 5!}{9!}$

$= \dfrac{1}{126}$ ✓

QUESTION 3.

(a) (i)



✓

(ii) At $x = 2.2$ $3\sin(2.2) - (2.2) = 0.2254\ldots$

At $x = 2.4$ $3\sin(2.4) - (2.4) = -0.3736\ldots$

Since the sign of $3\sin x - x$ changes between $x = 2.2$ and $x = 2.4$ then a solution lies between 2.2 and 2.4. ✓

(iii) $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ $\qquad f(x) = 3\sin x - x$

$f'(x) = 3\cos x - 1$

$x_2 = (2.3) - \dfrac{3\sin(2.3) - 2.3}{3\cos(2.3) - 1}$ ✓

$= 2.27903\ldots$(calculator)

$= 2.279$ (correct to 3 dec. places) ✓

(b) (i) $\dfrac{d}{dx}(x\tan^{-1}x) = (\tan^{-1}x)(1) + (x)\left(\dfrac{1}{1+x^2}\right)$

$= \tan^{-1}x + \dfrac{x}{1+x^2}$ ✓

(ii) $\displaystyle\int_0^1 \tan^{-1}x\,dx = \int_0^1\left(\frac{d}{dx}(x\tan^{-1}x) - \frac{x}{1+x^2}\right)dx$ ✓

$= \left[x\tan^{-1}x - \frac{1}{2}\log_e(1+x^2)\right]_0^1$ ✓

$= \left[(1)\tan^{-1}(1) - \frac{1}{2}\log_e(1+(1)^2)\right]$

$-\left[0 - \frac{1}{2}\log_e 1\right]$

$= \dfrac{\pi}{4} - \dfrac{1}{2}\log_e 2$ ✓

(c) To prove $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$, $n \ge 1$.

When $n = 1$

$\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)} = \sum_{r=1}^{1}\frac{1}{r(r+1)}$

$= \dfrac{1}{1(2)}$

$= \dfrac{1}{2}$

and $\dfrac{n}{n+1} = \dfrac{1}{1+1}$

$= \dfrac{1}{2}$

$\therefore \displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = 1$. ✓

Assume $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = k$.

i.e. assume $\displaystyle\sum_{r=1}^{k}\frac{1}{r(r+1)} = \frac{k}{k+1}$.

When $n = k + 1$

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \sum_{r=1}^{k+1}\left(\frac{1}{r(r+1)}\right)$$

$$= \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+1+1)} \quad ✓$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \text{(from assumption)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

$$= \frac{(k+1)}{(k+1)+1} \quad ✓$$

$\therefore$ if $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = k$

then $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = k + 1$.

Conclusion:

Since $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = 1$

then $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = 2, 3, ...$

$$\therefore \sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}. \quad ✓$$

QUESTION 4.

(a) $A = \pi r^2$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

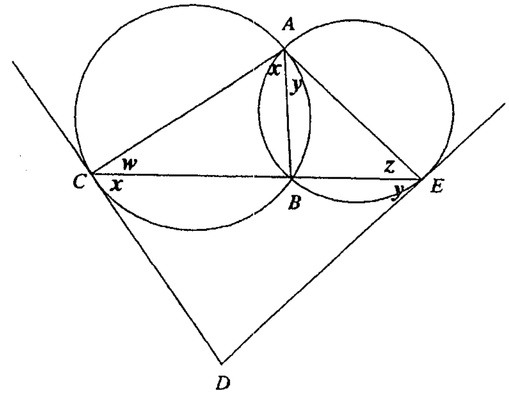
$$\frac{dr}{dt} = \frac{dA}{dt} \div 2\pi r$$

$$= -3000 \times \frac{1}{2\pi \times 2100} \quad ✓$$

$$= -0.22736\ldots \text{(calculator)}$$

$\therefore$ radius is decreasing at 227 metres per year (to the nearest metre) ✓

(b)



Let $x = \angle DCB$.

$\angle DCB = \angle CAB$ (Alternate segment theorem)

$\therefore \angle CAB = x$. ✓

Similarly, if $y = \angle DEB$,

then $\angle EAB = y$.

Let $\angle ACB = w$ and let $\angle AEB = z$.

In ΔAEC, $w + x + y + z = 180°$ (Angle sum of triangle) ✓

In quadrilateral $AEDC$,

$ACD + DEA = w + x + y + z$

$= 180$ as shown above ✓

$\therefore$ opposite angles are supplementary

$\therefore$ quadrilateral $AEDC$ is cyclic. ✓

OR

Proof:

$\angle BCD = \angle CAB$ (angle between tangent & chord = angle in alternate segement) ✓

$\angle BED = \angle EAB$ (same reason)

$\angle BCD + \angle BED + \angle D = 180°$ (angle sum of Δ) ✓

$\therefore \angle CAB + \angle EAB + \angle D = 180°$

$\therefore \angle CAE + \angle D = 180°$ ✓

$\therefore$ sum of opposite angles of $ACDE = 180°$ ✓

$\therefore$ $ACDE$ is cyclic.

(c) (i) $v^2 = 108 + 36x - 9x^2$

$$\frac{1}{2}v^2 = \frac{1}{2}(108 + 36x - 9x^2)$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 18 - 9x \quad ✓$$

$$\therefore \ddot{x} = -9x + 18$$

$$\ddot{x} = -9(x-2)$$

$\therefore$ motion is simple harmonic of the form $\ddot{x} = -n^2(x-2)$ ✓

(ii) Period of motion $= \frac{2\pi}{n} = \frac{2\pi}{3}$ seconds ✓

(iii) To find the amplitude let $v = 0$ i.e. $v^2 = 0$.

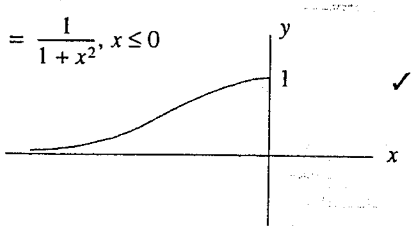
$$\begin{aligned}
\therefore 9x^2 - 36x - 108 &= 0 \\
9(x^2 - 4x - 12) &= 0 \quad ✓ \\
9(x-6)(x+2) &= 0 \\
\therefore x &= 6 \text{ or } x = -2 \\
\therefore 2a &= 6 - (-2) \\
&= 8 \\
a &= 4 \quad ✓
\end{aligned}$$

$\therefore$ amplitude is 4 metres.

(iv) Maximum speed occurs at centre of motion, ✓ at $x = 2$.

QUESTION 5.

(a) (i) $y = \frac{1}{1+x^2}, x \le 0$ ✓



(ii) Inverse $x = \frac{1}{1+y^2} \quad y \le 0$

$$\begin{aligned}
1 + y^2 &= \frac{1}{x} \quad ✓ \\
y &= \pm\sqrt{\frac{1}{x} - 1}, \ y \le 0 \\
\therefore f^{-1}(x) &= -\sqrt{\frac{1}{x} - 1} \quad ✓
\end{aligned}$$

(iii) Domain of $f^{-1}(x)$: $0 < x \le 1$. ✓

(b) Let p be the probability of selecting a male and q be the probability of selecting a female.

(i) Binomial expansion for ten fish:

$(p+q)^{10} = {}^{10}C_0 p^{10} + {}^{10}C_1 p^9 q^1 + {}^{10}C_2 p^8 q^2 \ldots$

$$\begin{aligned}
P(4 \text{ males}) &= {}^{10}C_6 p^4 q^6 \\
&= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \times \left(\frac{1}{4}\right)^4 \times \left(\frac{3}{4}\right)^6 \\
&= \frac{10 \times 3 \times 7 \times 3^6}{4^{10}} \\
&= \frac{76545}{524288} \\
&= 0.146 \text{ (3 dp)} \quad ✓
\end{aligned}$$

(ii) $1 - P(\text{no males}) > \frac{99}{100}$ ✓

$$\begin{aligned}
1 - \left(\frac{3}{4}\right)^n &> \frac{99}{100} \\
\left(\frac{3}{4}\right)^n &< 1 - \frac{99}{100} \\
(0.75)^n &< 0.01 \\
n \log 0.75 &< \log 0.01 \\
n &> \frac{\log 0.01}{\log 0.75} \quad ✓
\end{aligned}$$

(N.B. $\log 0.75$ is negative)

$> 16.0078\ldots$ (calculator)

$n = 17$

$\therefore$ 17 fish have to be taken. ✓

(c) (i)

$$\begin{aligned}
\ddot{x} &= 0 \\
\boxed{\dot{x} = 30} \\
x &= \int 30\,dt \\
x &= 30t + c \text{ (where } c \text{ is a constant)}
\end{aligned}$$

when $t = 0, x = 0 \quad \therefore c = 0$

$\boxed{x = 30t}$ ✓

$$\begin{aligned}
\ddot{y} &= -10 \\
\dot{y} &= \int -10\,dt \\
&= -10t + c
\end{aligned}$$

when $t = 0, \dot{y} = 0 \quad \therefore c = 0$

$$\begin{aligned}
\boxed{\dot{y} = -10t} \\
y &= \int -10t\,dt \\
&= -5t^2 + c
\end{aligned}$$

when $t = 0, y = 100 \quad \therefore c = 100$

$\boxed{y = -5t^2 + 100}$ ✓

(ii) Water bomb hits the ground when $y = 0$.

$$\begin{aligned}
\therefore 5t^2 &= 100 \\
t^2 &= 20 \\
t &= 2\sqrt{5} \text{ sec.} \quad ✓ \\
\therefore x &= 30(2\sqrt{5}) \\
&= 134.16\ldots \text{ (calculator)} \quad ✓ \\
\therefore D &= 134 \text{ metres (to the nearest metre)}
\end{aligned}$$

QUESTION 6.

(a) $|2x-1| < 3x+2$

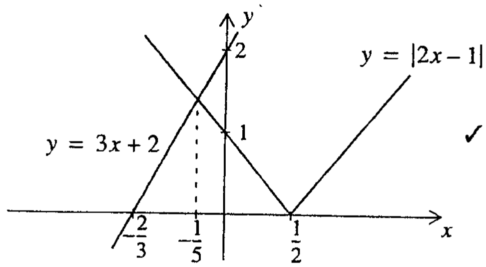
Solve $2x-1 = 3x+2$ and $2x-1 = -3x-2$

$x = -3$ $\qquad$ $x = -\frac{1}{5}$

But if $x = -3$, $3x+2 < 0$ ✓

$\therefore$ $x = -3$ is not a solution.

Now consider the graphs



✓

From graph $|2x-1| < 3x+2$ for $x > -\frac{1}{5}$. ✓

OR

$|2x-1| < 3x+2$

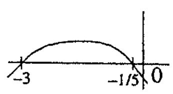
$(2x-1)^2 < (3x+2)^2$ (since $|2x-1| \geq 0$) ✓

$(2x-1)^2 - (3x+2)^2 < 0$

$(2x-1-(3x+2))(2x-1+(3x+2)) < 0$

$(-x-3)(5x+1) < 0$

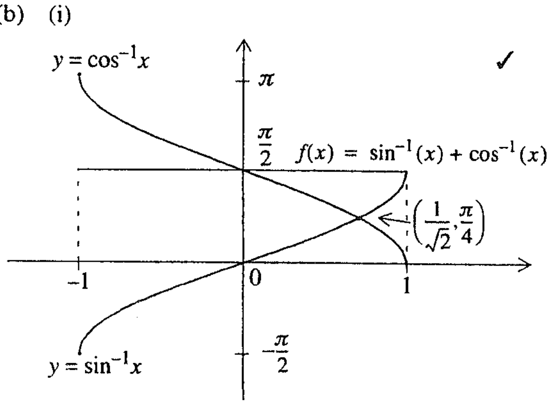
$\therefore x < -3$ or $x > -\frac{1}{5}$ ✓



But if $x < -3$, $3x+2 < 0$. ✓

$\therefore x > -\frac{1}{5}$

(b) (i)



✓

(ii) By adding ordinates at some key points on the graphs, and by noting the symmetry of the graphs, it can be seen that

$f(x) = \sin^{-1}x + \cos^{-1}x$ ✓

$= \text{constant}$

$= \frac{\pi}{2}$ ✓

OR

$f(x) = \sin^{-1}x + \cos^{-1}x$

$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}}$

$= 0$

But $f(0) = \frac{\pi}{2}$, and from graph $f(x)$ has one value only.

$\therefore f(x) = \text{constant function}$

$= \frac{\pi}{2}$

(iii) $\int_{-1}^{1} (\sin^{-1}x + \cos^{-1}x)\ dx$

$= \int_{-1}^{1} \left(\frac{\pi}{2}\right) dx$

$= \left[\frac{\pi}{2}x\right]_{-1}^{1}$

$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$

$= \pi$ ✓

OR

From graph, value of integral is given by area of rectangle width 2 and height $\frac{\pi}{2}$.

Area $= 2 \times \frac{\pi}{2}$

$= \pi$

$\therefore \int_{-1}^{1} (\sin^{-1}x + \cos^{-1}x)\ dx = \pi$

(c) (i) $v = 4 + Ae^{-kt}$ (given)

$\therefore \frac{dv}{dt} = -k \times Ae^{-kt}$.

Now $k(4-v) = k(4-(4+Ae^{-kt}))$

$= k \times (-Ae^{-kt})$

$= -k \times Ae^{-kt}$ ✓

$\therefore \frac{dv}{dt} = k(4-v)$,

$\therefore v = 4 + Ae^{-kt}$ satisfies $\frac{dv}{dt} = k(4-v)$

(ii) $v = 4 + Ae^{-kt}$

At $t = 0$, $v = 25$

$25 = 4 + Ae^{0}$

$\therefore A = 21$ ✓

(iii) $v = 4 + 21e^{-kt}$

At $t = 2$, $v = 12$

$$12 = 4 + 21e^{-2k}$$
$$8 = 21e^{-2k}$$
$$\frac{8}{21} = e^{-2k}$$
$$-2k = \ln\left(\frac{8}{21}\right)$$
$$k = 0.4825\ldots \quad ✓$$
$$= 0.483 \text{ (3 significant figures)}$$

(iv) $v = 4 + 21e^{-0.483t}$

$$= 4 + \frac{21}{e^{0.483t}}$$
$$\to 4 \text{ as } t \to \infty \quad ✓$$

At $t = 20$

$$v = 4 + 21e^{-0.483 \times 20}$$
$$= 4.0013 \text{ (4 decimal places)}$$

$$\% \text{ difference} = \frac{0.0013}{4} \times \frac{100}{1}\%$$
$$\doteqdot 0.03\% < 0.1\% \quad ✓$$

$\therefore$ speed differs from 4 by less than 0.1%.

QUESTION 7.

(a)
$$(1 - ax)^n = 1 - nax + \frac{n(n-1)}{2 \times 1}(ax)^2 - \ldots$$
$$= 1 - 4x + \frac{20}{3}x^2 - \ldots \text{(given)}$$

$$\therefore na = 4 \quad ①$$

and $\frac{n(n-1)}{2}a^2 = \frac{20}{3}$ ② ✓

From ① $a = \frac{4}{n}$ ③

Substituting in ② :

$$\frac{n(n-1)}{2} \times \left(\frac{4}{n}\right)^2 = \frac{20}{3} \quad ✓$$
$$48n(n-1) = 40n^2$$
$$6n^2 - 6n = 5n^2$$
$$n^2 - 6n = 0$$
$$n(n-6) = 0$$
$$n = 0 \text{ or } 6.$$

But $n \neq 0$, since n is a positive integer.

$$\therefore n = 6 \quad ✓$$

From ③ $a = \frac{4}{6}$

$$= \frac{2}{3} \quad ✓$$

$$\therefore a = \frac{2}{3},\ n = 6.$$

(b) (i) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$\therefore \cos 3x = \cos(2x + x)$$
$$= \cos 2x \cos x - \sin 2x \sin x \quad ✓$$
$$= (2\cos^2 x - 1)\cos x - 2\sin^2 x \cos x$$
$$= (2\cos^2 x - 1)\cos x - 2(1 - \cos^2 x)\cos x$$
$$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x \quad ✓$$
$$= 4\cos^3 x - 3\cos x$$
$$\therefore \cos 3x = 4\cos^3 x - 3\cos x.$$

(ii) $\cos 3x - \sin 2x = 0 \qquad 0 < x < \frac{\pi}{2}$

$$4\cos^3 x - 3\cos x - 2\sin x \cos x = 0$$
$$\cos x(4\cos^2 x - 3 - 2\sin x) = 0$$
$$\cos x(4(1 - \sin^2 x) - 3 - 2\sin x) = 0$$
$$\cos x(4 - 4\sin^2 x - 3 - 2\sin x) = 0$$
$$\cos x(4\sin^2 x + 2\sin x - 1) = 0 \quad ✓$$

$\cos x = 0 \Rightarrow x = \frac{\pi}{2}$ which is not in required range.

$$\therefore 4\sin^2 x + 2\sin x - 1 = 0$$
$$\sin x = \frac{-2 \pm \sqrt{4 + 16}}{8}$$
$$= \frac{\pm\sqrt{5} - 1}{4}$$

$\sin x = \frac{-\sqrt{5} - 1}{4} \Rightarrow x$ is outside required range. ✓

$$\therefore \sin x = \frac{\sqrt{5} - 1}{4}.$$

(iii) $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$ ✓

Now if $\theta = 3 \times \frac{\pi}{10}$,

then $\frac{\pi}{2} - \theta = \frac{\pi}{2} - 3 \times \frac{\pi}{10} = 2 \times \frac{\pi}{10}$.

$\therefore x = \frac{\pi}{10}$ is a solution of $\cos 3x = \sin 2x$.

OR $3x + 2x = \frac{\pi}{2}$

$5x = \frac{\pi}{2}$

$\therefore x = \frac{\pi}{10}$.

(iv) $\sin\frac{\pi}{5}\cos\frac{\pi}{10} = \sin\left(2 \times \frac{\pi}{10}\right)\cos\frac{\pi}{10}$

$= \left(2\sin\frac{\pi}{10}\cos\frac{\pi}{10}\right)\cos\frac{\pi}{10}$ ✓

$= 2\sin\frac{\pi}{10}\cos^2\frac{\pi}{10}$

$= 2\sin\frac{\pi}{10}\left(1 - \sin^2\frac{\pi}{10}\right)$

$= 2 \times \frac{\sqrt{5}-1}{4} \times \left(1 - \left(\frac{\sqrt{5}-1}{4}\right)^2\right)$ ✓

$= \frac{\sqrt{5}-1}{2} \times \left(\frac{16 - (5 - 2\sqrt{5} + 1)}{16}\right)$

$= \frac{\sqrt{5}-1}{2} \times \frac{10 + 2\sqrt{5}}{16}$

$= \frac{\sqrt{5}-1}{2} \times \frac{5 + \sqrt{5}}{8}$

$= \frac{5\sqrt{5} + 5 - 5 - \sqrt{5}}{16}$ ✓

$= \frac{4\sqrt{5}}{16}$

$= \frac{\sqrt{5}}{4}$.