TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2005

MATHEMATICS EXTENSION 2

Time Allowed – 3 Hours
(Plus 5 minutes Reading Time)

All questions may be attempted

All questions are of equal value

Department of Education approved calculators are permitted

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

No grid paper is to be used unless provided with the examination paper

The answers to all questions are to be returned in separate bundles clearly labeled Question 1, Question 2, etc. Each question must show your Candidate Number.

Question 1 (15 marks) **Marks**

(a) Find $\int \frac{dx}{x(\ln x)^2}$. **2**

(b) Find $\int \frac{x^2 - x - 21}{(2x-1)(x^2+4)} dx$. **4**

(c) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$ **4**

(d) (i) Prove $\int_o^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{1-\tan x}{1+\tan x}\,dx$. **3**

Question 2 (15 marks) (Start a new page)

(a) Express $z = \sqrt{3} + i$ in modulus-argument form. **3**
Hence show that $z^7 + 64z = 0$.

(b) On an argand diagram the point P representing the complex number z moves so that $|z-(1+i)| = 1$.

(i) Sketch the locus of P . **2**

(ii) Find the greatest value of $|z|$. **2**

(iii) Shade the region where $|z-(1+i)| \le 1$ and $0 < \arg(z-1) < \frac{\pi}{4}$ and find the area of this region. **3**

(c) If w is one of the complex roots of $z^3 = 1$.

(i) Show that w^2 is also a root. **1**

(ii) Show that $1 + w + w^2 = 0$. **1**

(iii) Evaluate $(1-w)(1-w^2)(1-w^4)(1-w^8)$. **3**

Question 3 (15 marks) (Start a new page) **Marks**

(a) Given that $x + i$ is a factor of $P(x)= x^4+ 3x^3 + 6x^2 + 3x+5$, factorize $P(x)$ over the complex field. **4**

(b) Given that the equation $x^4-5x^3 - 9x^2 + 81x - 108=0$ has a root of multiplicity 3, find all the roots of this equation. **3**

(c) If α, β, γ are the roots of $x^3-3x^2+2x-1=0$, find the equation whose roots are

(i) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ **2**

(ii) $\alpha^2, \beta^2, \gamma^2$. **3**

(d)

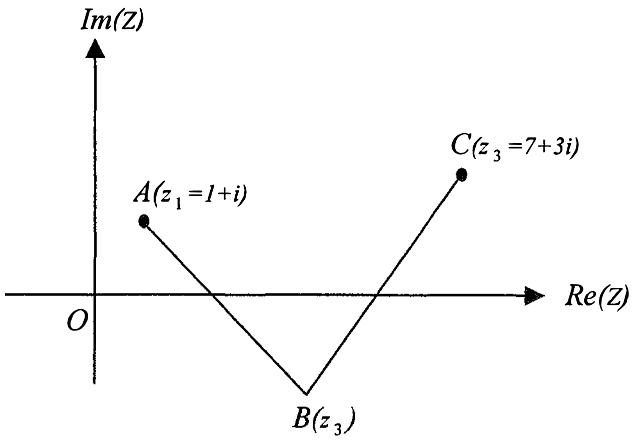


diagram not to scale

The points A and C represent the complex numbers $z_1 = 1+i$ and $z_2 = 7+3i$. **3**

Find the complex number z_2 represented by B such that ΔABC is isosceles and right angled at B.

Question 4 (15 marks) (Start a new page) **Marks**

(a) An ellipse has the equation $\frac{x^2}{16}+\frac{y^2}{9}=1$.

(i) Sketch the ellipse showing the foci S and S' and the directrices. **4**

(ii) Prove that the tangent at the point $P(4\cos\theta,\ 3\sin\theta)$ to the ellipse has the equation $\frac{x\cos\theta}{4}+\frac{y\sin\theta}{3}=1$. **3**

(iii) The ellipse meets the y-axis at B and B'. The tangents at B and B' meet the tangent at P at the points Q and Q' respectively. **3**

Prove that $BQ.BQ' = 16$.

(b)

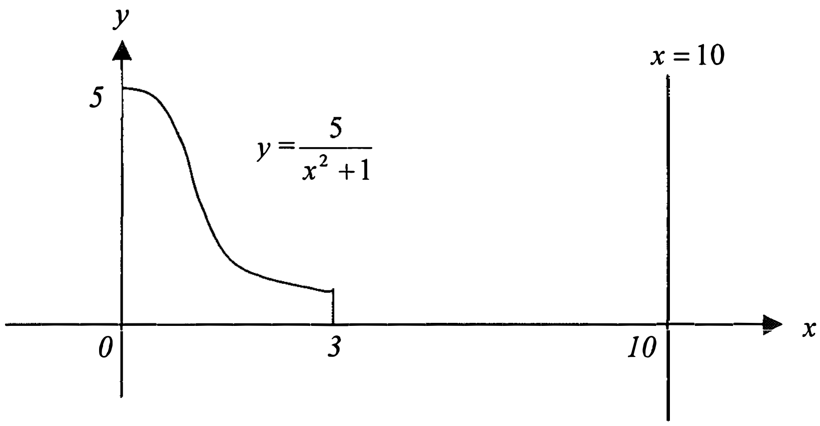


diagram not to scale

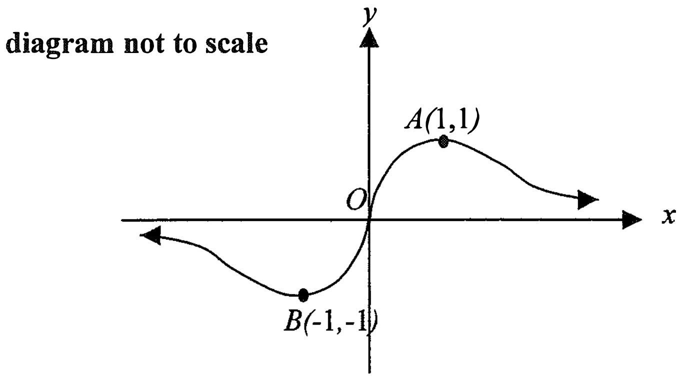
A circular flange is formed by rotating the region bounded by the curve $y=\frac{5}{x^2+1}$, the x axis and the lines $x = 0$ and $x = 3$, through one complete revolution about the line $x = 10$. (All measurements are in centimetres).

(i) Use the method of cylindrical shells to show that the volume V cm^3 of the flange is given by $V=\int_0^3 \frac{(100\pi-10\pi x)}{x^2+1}\,dx$. **2**

(ii) Find the volume of the flange correct to the nearest cm^3. **3**

Question 5 (15 marks) (Start a new page) **Marks**

(a)



In the diagram above, the curve $y = \dfrac{2x}{1+x^2}$ is sketched with turning points $A(1,1)$ and $B(-1,-1)$.

(i) On separate diagrams, draw sketches of :

(α) $y = \dfrac{1+x^2}{2x}$ **2**

(β) $y^2 = \dfrac{2x}{1+x^2}$ **2**

(γ) $y = \ln\left(\dfrac{2x}{1+x^2}\right)$. **2**

(ii) Show that the equation $kx^3 + (k-2)x = 0$ can be written in the form of $\dfrac{2x}{1+x^2} = kx$. **1**

(iii) Using a graphical approach based on the curve $y = \dfrac{2x}{1+x^2}$, or otherwise, find the real values of k for which the equation $kx^3 + (k-2)x = 0$ has exactly one real root. **2**

(b) (i) A particle of mass m travels with constant speed v in a horizontal circle of radius R, centre C around a track banked at an angle θ to the horizontal. The acceleration due to gravity is g. **1**

(α) Draw a diagram showing the forces acting on the particle. **2**

(β) Show that if there is not tendency for the particle to slip sideways then $v = \sqrt{Rg\tan\theta}$.

(ii) One particle travels in a horizontal circle of radius 1 metre around the lower half of a track where the angle of banking is $\tan^{-1}\frac{5}{18}$. Another particle travels in a horizontal circle of radius 1.2 metres around the upper half of the track where the angle of banking is $\tan^{-1}\frac{16}{27}$. Each particle travels with constant speed so that it has no tendency to slip sideways. The particles are initially observed to be along side each other.

3

Taking $g = 10$ m s^{-2}, find the time that elapses before the particles are next to be alongside each other.

(Question 6 is continued on the next page)

Question 6 (15 marks) (Start a new page) **Marks**

(a) The line through O perpendicular to the tangent at $P\left(cp, \frac{c}{p}\right)$ on the rectangular hyperbola $xy = c^2$ meets the tangent at N.

(i) Find the coordinates of N. **4**

(ii) Show that as p varies, the locus of N is $\left(x^2+y^2\right)^2 = 4c^2xy$. **3**

(b) A drinking glass having the form of a right circular cylinder of radius a and height h, is filled with water. The glass is slowly tilted over, spilling water out of it, until it reaches the position where the water's surface bisects the base of the glass. Figure 1 shows this position.

diagram not to scale

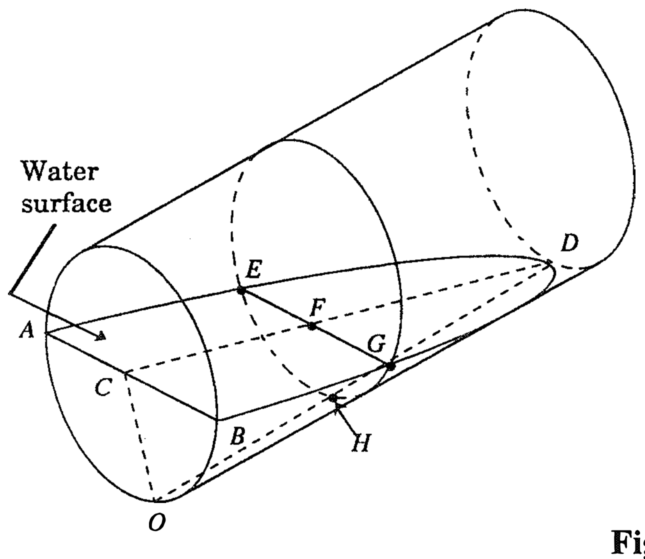


Figure 1

In Figure 1, AB is a diameter of the circular base with centre C, O is the lowest point on the base, and D is the point where the water's surface touches the rim of the glass.

Figure 2 shows a cross-section of the tilted glass parallel to its base. The centre of this circular section is *C'* and *EFG* shows the water level. The section cuts the lines *CD* and *OD* of Figure 1 in *F* and *H* respectively.

Note: $EG \perp C'H$ at F

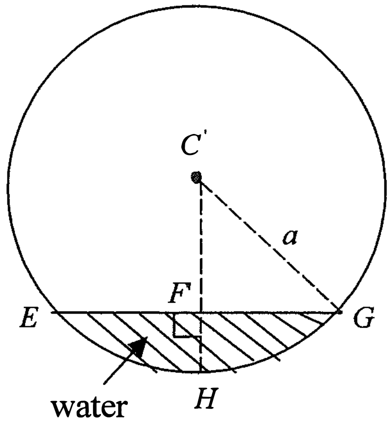


Figure 2

Figure 3 shows the section *COD* of the tilted glass.

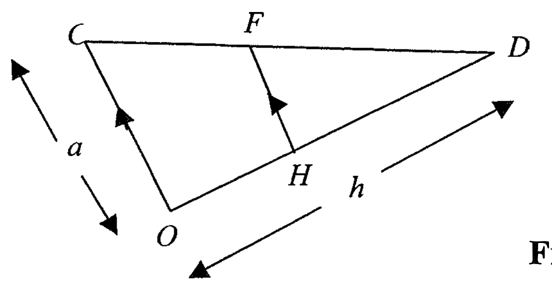


Figure 3

Note: $FH \parallel CO$, $CO = a$, and $OD = h$

diagrams not to scale

(i) Use Figure 3 to show that $FH = \frac{a}{h}(h - x)$, where $OH = x$. **1**

(ii) Use Figure 2 to show that $C'F = \frac{ax}{h}$ and $\angle HCG = \cos^{-1}(\frac{x}{h})$. **2**

(iii) Use (ii) to show that the area of the shaded segment *EGH* is **2**

$$a^2\left[\cos^{-1}\left(\frac{x}{h}\right) - \left(\frac{x}{h}\right)\sqrt{1 - \left(\frac{x}{h}\right)^2}\,\right].$$

(iv) Given that $\int \cos^{-1}\theta \, d\theta = \theta\cos^{-1}\theta - \sqrt{1-\theta^2}$, **3**
find the volume of the water in the tilted glass in Figure 1.

Question 7 (15 marks) (Start a new page) **Marks**

(a) A body of unit mass falls under gravity through a resisting medium. The body falls from rest from a height of 50 metres above the ground. The resistance to its motion is $\frac{1}{100}v^2$ where v metres per second is the speed of the body when it has fallen a distance x metres. The acceleration due to gravity is g m s^{-2}.

(i) Show that the equation of the motion of the body is: **2**

$$x = g - \frac{1}{100}v^2 .$$

(ii) Show that the terminal velocity V of the body is given by: **1**

$$V = \sqrt{100g} .$$

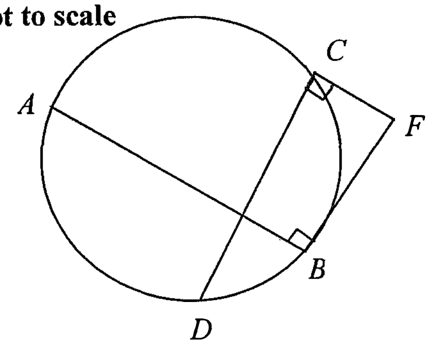
(iii) Hence show that $v^2 = V^2(1 - e^{-\frac{x}{50}})$. **3**

(iv) Find the distance fallen in metres until the body reaches a velocity equal to 50% that of the terminal velocity. **2**

(v) Find the velocity reached as a percentage of the terminal velocity when the body hits the ground. **2**

(b) In the following figure, AB and CD are two chords of the circles. AB and CD intersect at E. F is a point such that ABF and DCF are right angles.

diagram not to scale

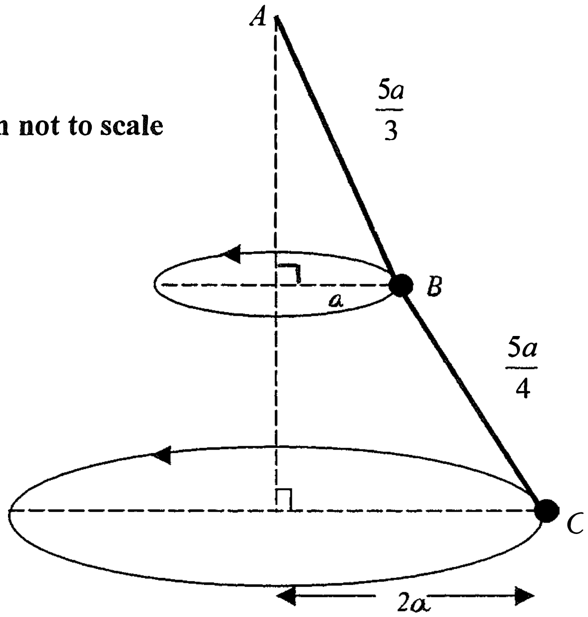


Copy the diagram onto the your answer sheet. **5**
Prove that FE produced is perpendicular to AD.

Question 8 (15 marks) (Start a new page) **Marks**

(a) A light inextensible string ABC is such that $AB = \frac{5a}{3}$ and $BC = \frac{5a}{4}$. A particle of mass m is attached to the string at C and a mass of $7m$ is attached at B. The end A is tied to a fixed point and the whole system rotates steadily with uniform angular velocity about the vertical through A in such a way that B and C describe horizontal circles of radii a and $2a$ respectively. The acceleration due to gravity is g.

diagram not to scale



(i) Show that the tension in BC is $\frac{5mg}{3}$. **3**

(ii) Find the tension in AB. **2**

(iii) Find the speeds of B and C. **4**

(b) (i) Show that if $I_n = \int \frac{dx}{\left(x^2 + a^2\right)^n}$, then **4**

$$I_n = \frac{1}{2a^2(n-1)}\left(\frac{x}{(x^2+a^2)^{n-1}} + (2n-3)\right)I_{n-1}\Bigg).$$

(ii) Hence evaluate $I_2 = \int_0^1 \frac{dx}{(x^2+a^2)^2}$. **2**

☺ **END of Paper** ☺

2005 ME II TRIAL JRAHS

Q1 a) $\int \frac{dx}{x(\ln x)^2} = \int u^{-2}\,du$ $\quad u = \ln x$, $du = \frac{dx}{x}$ $\quad$ 1

$= -\frac{1}{u} + c$

$= \underline{\underline{-\frac{1}{\ln x} + c}}$ $\quad$ 1

b) $\frac{x^2 - x - 21}{(2x-1)(x^2+4)} = \frac{A}{2x-1} + \frac{Bx+c}{x^2+4}$

$A(x^2+4) + (Bx+c)(2x-1) = x^2 - x - 21$

when $x = \frac{1}{2}$ $\quad A(\frac{1}{4}+4) = \frac{1}{4} - \frac{1}{2} - 21$ $A = -5$	when $x = 0$ $4A - c = -21$ $-5\times 4 - c = -21$ $c = 1$	when $x = 1$ $-5(5) + B + 1 = 1 - 1 - 21$ $B = 3$

2

$\therefore \int \frac{-5}{2x-1} + \frac{3x+1}{x^2+4}\,dx = -\frac{5}{2}\ln(2x-1) + \frac{3}{2}\int \frac{2x\,dx}{x^2+4} + \int \frac{dx}{x^2+4} + c$

2 $\quad = \underline{\underline{-\frac{5}{2}\ln(2x-1) + \frac{3}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} + c}}$

c) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x + \cos x} = \int_0^1 \frac{\frac{2\,dt}{1+t^2}}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}$ $\quad$ 1

$t = \tan\frac{x}{2}$

$dx = \frac{2\,dt}{1+t^2}$ $\quad$ 1

$x = 0,\ t = 0$

$x = \frac{\pi}{2},\ t = 1$

$= \int_0^1 \frac{2\,dt}{1 + \cancel{t^2} + 2t - 1 + 1 - \cancel{t^2}}$

$= \int_0^1 \frac{dt}{t+1}$ $\quad$ 1

$= \left[\ln(t+1)\right]_0^1$ $\quad$ 1

$= \underline{\underline{\ln 2}}$

d i) $\int_0^a f(x)\,dx = -\int_a^0 f(a-u)\,du = \int_0^a f(a-u)\,du = \underline{\underline{\int_0^a f(a-x)\,dx}}$

1 $\quad$ 1

$x = a - u$

$dx = -du$

$x = 0,\ u = a$

$x = a,\ u = 0$

1 diii) $$\int_0^{\frac{\pi}{4}} \frac{1-\tan x}{1+\tan x}\,dx = \int_0^{\frac{\pi}{4}} \frac{1-\tan\left(\frac{\pi}{4}-x\right)}{1+\tan\left(\frac{\pi}{4}-x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\dfrac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\cdot\tan x}}{1+\dfrac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\cdot\tan x}}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\dfrac{1-\tan x}{1+\tan x}}{1+\dfrac{1-\tan x}{1+\tan x}}\,dx$$ 1

$$= \int_0^{\frac{\pi}{4}} \frac{\cancel{1}+\tan x-\cancel{1}+\tan x}{1+\cancel{\tan x}+1-\cancel{\tan x}}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \tan x\,dx$$ 1

$$= \int_0^{\frac{\pi}{4}} \frac{\sin x\,dx}{\cos x}$$

$$= \Big[-\ln(\cos x)\Big]_0^{\frac{\pi}{4}}$$ $\frac{1}{2}$

$$= -\ln\left(\cos\frac{\pi}{4}\right) + \ln(\cos 0)$$

$$= -\ln\left(\frac{1}{\sqrt{2}}\right) + 0$$

$$= \ln\sqrt{2}$$

$$= \underline{\underline{\frac{1}{2}\ln 2}}$$ $\frac{1}{2}$

Q2

(a) $z = \sqrt{3} + i = 2\,\text{cis}\,\frac{\pi}{6}$ 1 $\quad r = \sqrt{3+1} = 2$

$\theta = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$z^7 = \left(2\,\text{cis}\,\frac{\pi}{6}\right)^7 = 2^7\,\text{cis}\,\frac{7\pi}{6}$

$z^7 + 64z = 2^7\,\text{cis}\,\frac{7\pi}{6} + 64\left(2\,\text{cis}\,\frac{\pi}{6}\right)$ 1

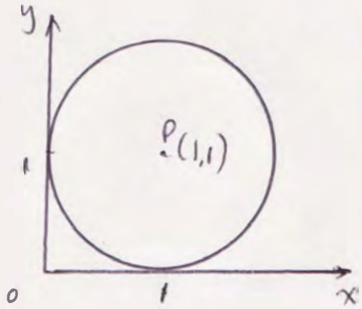
$= 128\,\text{cis}\,\frac{7\pi}{6} + 128\,\text{cis}\,\frac{\pi}{6}$

$= 128\left[\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} + \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$

$= 128\left[-\cancel{\cos\frac{\pi}{6}} - \cancel{i\sin\frac{\pi}{6}} + \cancel{\cos\frac{\pi}{6}} + \cancel{i\sin\frac{\pi}{6}}\right]$

$= \underline{\underline{0}}$ 1

(2b i)



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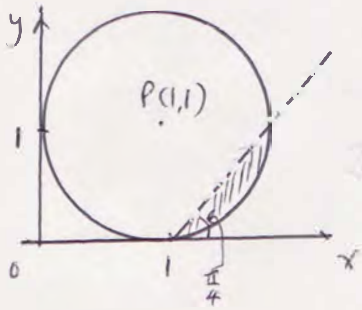
ii) $OP = \sqrt{1^2 + 1^2} = \sqrt{2}$ 1

$\max |z| = \underline{\underline{1 + \sqrt{2}}}$ 1

iii)



1

Shaded Area $= \frac{1}{2} \times \left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right)$

$= \underline{\underline{\left(\frac{\pi}{4} - \frac{1}{2}\right)u^2}}$ 2

2c i) $(w^2)^3 = w^6 = (w^3)^2 = 1$ (since $w^3 = 1$) 1

ii) The roots for $z^3 - 1 = 0$ are $1, w, w^2$

$1 + w + w^2 = \frac{-b}{a} = \underline{\underline{0}}$ 1

Alternatively
$z^3 - 1 = (z-1)(z^2+z+1) = 0$
$z \neq 1$, $z^2 + z + 1 = 0$ i.e. $w^2 + w + 1 = 0$
Since w is a complex root of $z^3 = 1$

iii) $(1-w)(1-w^2)(1-w^4)(1-w^8) = (1-w)(1-w^2)(1-w\cdot w^3)(1-w^2\cdot w^6)$

$= \left[(1-w)(1-w^2)\right]^2$ (Since $w^3 = w^6 = 1$) 1

$= (1 - w - w^2 + \overset{1}{\cancel{w^3}})^2$

$= \left[2 - (w + w^2)\right]^2$

$= \left[2 - (-1)\right]^2$ 1

$= 3^2$

$= \underline{\underline{9}}$ 1

Q 3

a) Since all coeff of $P(x)$ are real, $(x-i)$ is also a factor of $P(x)$

$(x-i)(x+i) = x^2+1$ 1

$$\begin{array}{r|l} & x^2 + 3x + 5 \\ \hline x^2+1 & x^4 + 3x^3 + 6x^2 + 3x + 5 \\ & x^4 + x^2 \\ \hline & 3x^3 + 5x^2 + 3x \\ & 3x^3 + 3x \\ \hline & 5x^2 + 5 \\ & 5x^2 + 5 \\ \hline \end{array}$$ 1

$x^2 + 3x + 5 = 0$

$x = \dfrac{-3 \pm \sqrt{9 - 4 \times 5}}{2}$

$x = \dfrac{-3 \pm \sqrt{11}\, i}{2}$ 1

$\therefore P(x) = (x-i)(x+i)\left(x + \dfrac{3+\sqrt{11}\,i}{2}\right)\left(x + \dfrac{3-\sqrt{11}\,i}{2}\right)$ 1

b) $P(x) = x^4 - 5x^3 - 9x^2 + 81x - 108$

$P'(x) = 4x^3 - 15x^2 - 18x + 81$

$P''(x) = 12x^2 - 30x - 18 = 6(2x^2 - 5x - 3)$

$P''(x) = 6(2x+1)(x-3) = 0$ when $x = -\frac{1}{2}, 3$ 1

But $P'(-\frac{1}{2}) = 4(-\frac{1}{8}) - 15(\frac{1}{4}) + 9 + 81 = 85\frac{3}{4} \neq 0$

$P'(3) = 108 - 135 - 54 + 81 = 0$

$P(3) = \cancel{81} - 5 \times 27 - \cancel{81} + 243 - 108 = 0$

$\therefore x = 3$ is a treble root of $P(x)$ 1

Let γ be the remaining root $3 + 3 + 3 + \gamma = 5 \quad \therefore \gamma = -4$ 1

Alternatively by inspection $\gamma = -4$

$\therefore$ The roots are $3, 3, 3, -4$

c) $x^3 - 3x^2 + 2x - 1 = 0$

Let $y = \frac{1}{\alpha}$, $\alpha = \frac{1}{y}$

$\therefore \left(\frac{1}{y}\right)^3 - 3\left(\frac{1}{y}\right)^2 + 2\left(\frac{1}{y}\right) - 1 = 0$ 1

$\frac{1}{y^3} - \frac{3}{y^2} + \frac{2}{y} - 1 = 0$

$1 - 3y + 2y^2 - y^3 = 0$

This is the same as polynomial in x

$1 - 3x + 2x^2 - x^3 = 0$ 1

3c ii) $y = \alpha^2, \quad \alpha = \sqrt{y}$

$(\sqrt{y})^3 - 3(\sqrt{y})^2 + 2\sqrt{y} - 1 = 0$ 1

$y^{3/2} + 2y^{\frac{1}{2}} = 1 + 3y$

$y^{1/2}(y+2) = 1 + 3y$

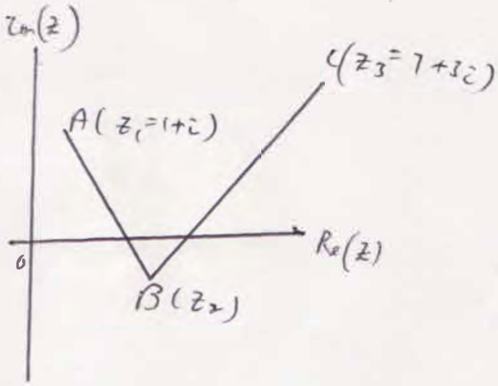
Squaring both sides

$y(y^2 + 4y + 4) = 1 + 6y + 9y^2$ 1

$y^3 + 4y^2 + 4y = 1 + 6y + 9y^2$

$\underline{\underline{y^3 - 5y^2 - 2y - 1 = 0}}$ 1

d) Let $z_2 = x + iy$



$i\overrightarrow{BC} = \overrightarrow{BA}$

$i[(7-x) + (3-y)i] = (1-x) + i(1-y)$ 1

$-(3-y) + i(7-x) = (1-x) + i(1-y)$

$-(3-y) = 1-x$ and $7-x = 1-y$

$y - 3 = 1-x$ and $7-x = 1-y$

$x + y = 4$ ①

$x - y = 6$ ② 1

① + ② $x = 5$

$\therefore y = -1$

$z_2 = \underline{\underline{5-i}}$ 1

Q.4

a i) $a = 4 \quad b = 3$ $\quad \therefore 9 = 16(1-e^2)$

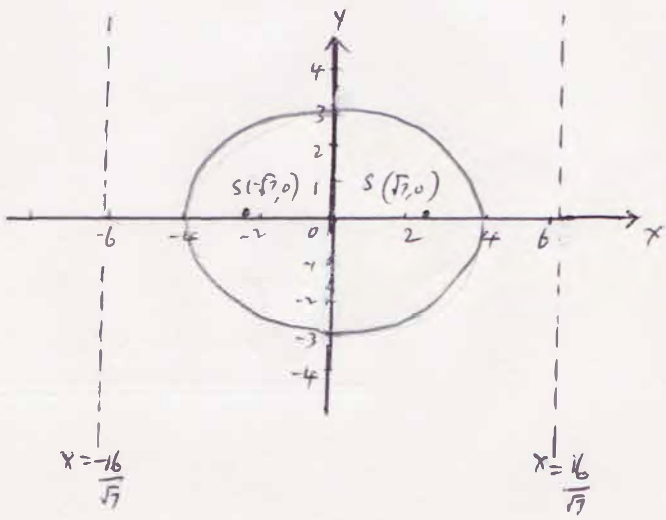
$e^2 = \frac{7}{16}$

$e = \frac{\sqrt{7}}{4}$

$S = (\sqrt{7}, 0)$

$S' = (-\sqrt{7}, 0)$ 1

Directrix : $x = \pm \frac{16}{\sqrt{7}}$ 1



2

4a ii) $x = 4\cos\theta \quad y = 3\sin\theta$

$\frac{dx}{d\theta} = -4\sin\theta \quad \frac{dy}{d\theta} = 3\cos\theta$

$\frac{dy}{dx} = -\frac{3\cos\theta}{4\sin\theta}$ 1

$\therefore y - 3\sin\theta = \frac{-3\cos\theta}{4\sin\theta}(x - 4\cos\theta)$

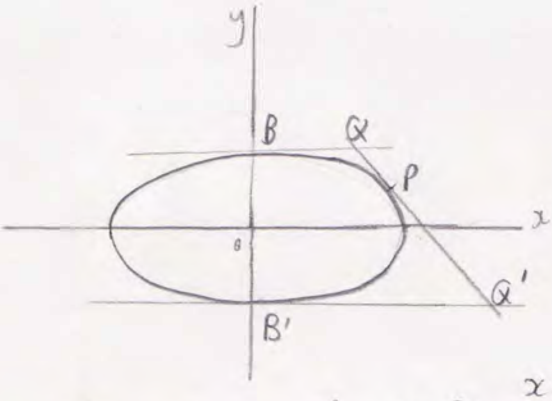
$(4\sin\theta)y - 12\sin^2\theta = -3\cos\theta\, x + 12\cos^2\theta$ 1

$(3\cos\theta)x + (4\sin\theta)y = 12(\sin^2\theta + \cos^2\theta)$

$= 12$

$\frac{x\cos\theta}{4} + \frac{y\sin\theta}{3} = 1$ 1

iii)



$B(0,3) \quad B'(0,-3)$ when $y = 3$, $\frac{x\cos\theta}{4} = 1 - \sin\theta$

$x = \frac{4(1-\sin\theta)}{\cos\theta}$

when $y = -3$, $\frac{x\cos\theta}{4} = 1 + \sin\theta$

$x = \frac{4}{\cos\theta}(1+\sin\theta)$, 1

$\therefore Q\left[\frac{4}{\cos\theta}(1-\sin\theta), 3\right]$ and $Q' = \left[\frac{4}{\cos\theta}(1+\sin\theta, -3\right]$

$BQ = \frac{4}{\cos\theta}(1-\sin\theta)$ and $B'Q' = \frac{4}{\cos\theta}(1+\sin\theta)$ 1

$BQ \times B'Q' = \frac{16}{\cos^2\theta}(1-\sin^2\theta) = \frac{16}{\cos^2\theta}\cos^2\theta = 16$

$\therefore BQ \times B'Q' = \underline{\underline{16}}$ 1

Q 4b

bi) Take strips of thickness Δx parallel to the y-axis
$\therefore$ volume of resulting shell is given by

$$\Delta V = 2\pi(10-x)y\,\Delta x$$

$$V = \lim_{\Delta x \to 0} \sum \Delta V$$

$$= \lim_{\Delta x \to 0} \sum 2\pi(10-x)y\,\Delta x \qquad 1$$

$$V = \int_0^3 2\pi(10-x)\frac{5}{x^2+1}\,dx$$

$$= \int_0^3 \frac{100\pi - 10\pi x}{x^2+1}\,dx \qquad 1$$

ii) $$V = 100\pi\int_0^3 \frac{dx}{x^2+1} - 5\pi\int_0^3 \frac{2x\,dx}{x^2+1}$$

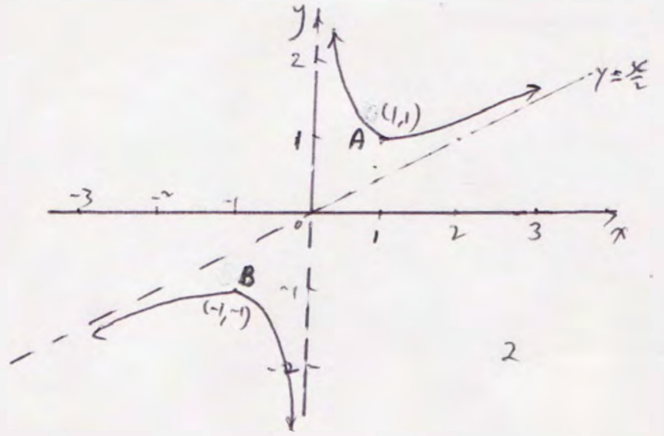
$$= 100\pi\left[\tan^{-1}x\right]_0^3 - 5\pi\left[\ln(x^2+1)\right]_0^3 \qquad 1$$

$$= 100\pi\tan^{-1}(3) - 5\pi \times \ln(10) \qquad 1$$

$$= 356\text{ cm}^3 \text{ (nearest cm}^3\text{)} \qquad 1$$

Q 5a)

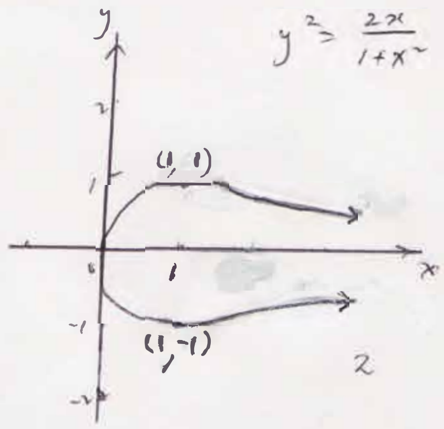
i α) $y = \dfrac{1+x^2}{2x}$



2

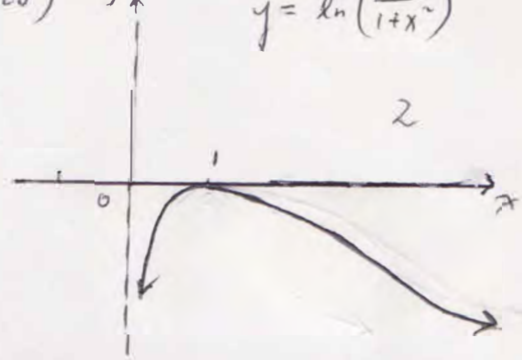
Note $\dfrac{1+x^2}{2x} = \dfrac{1}{2}\left[x + \dfrac{1}{x}\right]$

i β) $y^2 = \dfrac{2x}{1+x^2}$



2

i γ) $y = \ln\left(\dfrac{2x}{1+x^2}\right)$



2

Don't penalize concavity as $x \to \infty$

5 a ii) $kx^3 + (k-2)x = 0$

$2x = kx^3 + kx$

$2x = kx(1+x^2)$

$\therefore kx = \dfrac{2x}{1+x^2}$ #   1

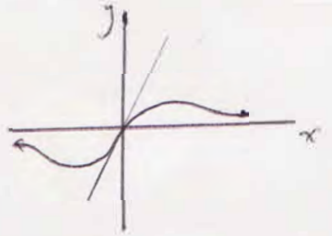
iii) $y = \dfrac{2x}{1+x^2}$

$y' = \dfrac{2(1+x^2) - 2x(2x)}{(1+x^2)^2}$

$y' = \dfrac{-2x^2+2}{(1+x^2)^2}$

At $x=0$, $y' = 2$

$\therefore$ y has gradient 2 at $(0,0)$ 1



Alternatively

$x(kx^2 + k - 2) = 0$

$x = 0$ or $x^2 = \dfrac{2-k}{k}$ $(k \neq 0)$

For one real root $\dfrac{2-k}{k} \leq 0$ 1

$\dfrac{k-2}{k} \geq 0$

$k(k-2) \geq 0$

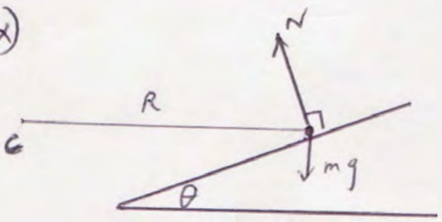
$k < 0$ or $k \geq 2$

$\frac{1}{2}$

$\therefore y = kx$ and $y = \dfrac{2x}{1+x^2}$ will intersect exactly once for $k \geq 2$ or $k < 0$ $\frac{1}{2}$

When $k=0$, $y = 0.x = 0$ $\therefore k=0$ is also a solution $\therefore$ $\underline{\underline{k \geq 2 \text{ or } k \leq 0}}$ $\frac{1}{2}$

5 b i α)



1

β) Considering the forces acting on the particle

Vertically $N\cos\theta = mg$ ①

Horizontally $N\sin\theta = \dfrac{mv^2}{R}$ ②

② ÷ ① $\dfrac{N\sin\theta}{N\cos\theta} = \dfrac{\frac{mv^2}{R}}{mg}$ 1

$\therefore \tan\theta = \dfrac{v^2}{Rg}$

$\underline{\underline{v = \sqrt{Rg\tan\theta}}}$ 1

ii) For the first particle

speed $= \sqrt{1 \times 10 \times \frac{5}{18}} = 5/3$ m/s $\therefore$ first particle completes its circuit in $2\pi \div 5/3 = 1.2\pi$ sec   1

For the second particle

speed $= \sqrt{1.2 \times 10 \times 16/27} = 8/3$ m/s $\therefore$ 2nd particle completes its circuit in $1.2 \times 2\pi \div 8/3$

1 $= 0.9\pi$ sec

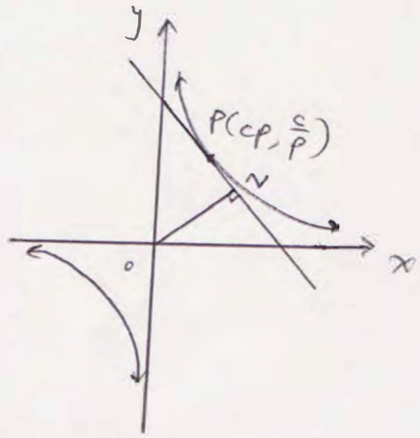
$\therefore$ the particles are next observed to be alongside each other after $\underline{\underline{3.6\pi \text{ sec}}}$

1

Q 6 a i)

$$y = \frac{c^2}{x}$$

$$y' = -\frac{c^2}{x^2}$$



m = slope at $P(cp, \frac{c}{p}) = -\frac{c^2}{(cp)^2} = -\frac{1}{p^2}$

Eq of tangent at P:

$$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$$

$$yp^2 - cp = -x + cp$$

$$x + yp^2 = 2cp \quad ①$$ /

Eq of No is:

$$y - 0 = p^2(x - 0)$$

$$y = p^2 x \quad ②$$ /

solving ① ② simultaneously to find coord. of N:

$$x = 2cp - p^2 y \quad \text{and} \quad x = \frac{y}{p^2}$$

$$2cp - p^2 y = \frac{y}{p^2}$$

$$2cp = y\left[\frac{1}{p^2} + p^2\right]$$

$$y = \frac{2cp^3}{1+p^4}$$ /

$$x = \frac{2cp^3}{1+p^4} \times \frac{1}{p^2} = \frac{2cp}{1+p^4}$$

$\therefore$ coordinates of N is $\underline{\underline{\left(\frac{2cp}{1+p^4}, \frac{2cp^3}{1+p^4}\right)}}$ /

aii) Since $x(1+p^4) = 2cp$ and $\frac{y}{x} = p^2$

$$x^2(1+p^4)^2 = (2cp)^2$$ /

$$x^2\left(1 + \frac{y^2}{x^2}\right)^2 = 4c^2\left(\frac{y}{x}\right)$$

$$x^2\left(1 + \frac{2y^2}{x^2} + \frac{y^4}{x^4}\right) = 4c^2\left(\frac{y}{x}\right)$$ /

$$x^2 + 2y^2 + \frac{y^4}{x^2} = 4c^2\frac{y}{x}$$

$$x^4 + 2y^2x^2 + y^4 = 4c^2xy$$

$$\underline{\underline{(x^2+y^2)^2 = 4c^2xy}}$$ /

6 b i) $\triangle COD \parallel\!| \triangle FHD$

$\therefore \frac{FH}{CO} = \frac{HD}{OD}$

$\frac{FH}{a} = \frac{h-x}{h}$

$FH = \frac{a}{h}(h-x)$ 1

ii) $C'F = a - FH$

$= a - \frac{a}{h}(h-x)$

$= \frac{ax}{h}$ 1

$\cos \angle HC'G = \frac{C'F}{a} = \frac{x}{h}$

$\therefore \angle HC'G = \cos^{-1}\left(\frac{x}{h}\right)$ 1

iii) Shaded area = area sector $C'EHG$ − 2 × area $\triangle C'FG$

$= \frac{1}{2}a^2 \cdot 2\cos^{-1}\left(\frac{x}{h}\right) - 2 \cdot \frac{1}{2} C'F \cdot FG$ 1

$= a^2 \cos^{-1}\left(\frac{x}{h}\right) - \frac{ax}{h}\sqrt{a^2 - \frac{a^2x^2}{h^2}}$

$= a^2\cos^{-1}\left(\frac{x}{h}\right) - \frac{a^2x}{h}\sqrt{1-\left(\frac{x}{h}\right)^2}$

$= a^2\left[\cos^{-1}\frac{x}{h} - \frac{x}{h}\sqrt{1-\left(\frac{x}{h}\right)^2}\right]$ 1

iv) Volume $= \int_0^h a^2\left[\cos^{-1}\left(\frac{x}{h}\right) - \frac{x}{h}\sqrt{1-\left(\frac{x}{h}\right)^2}\right] dx$

$= a^2h\int_0^1 \cos^{-1}\theta - \theta\sqrt{1-\theta^2}\, d\theta \qquad \left(\begin{array}{c}\theta = \frac{x}{h} \\ h\, d\theta = dx\end{array}\right)$ 1

$= a^2h\int_0^1 \cos^{-1}\theta\, d\theta - \frac{a^2h}{2}\int_0^1 2\theta\sqrt{1-\theta^2}\, d\theta$

$= a^2h\left[\theta\cos^{-1}\theta - \sqrt{1-\theta^2}\right]_0^1 + \left[\frac{a^2h\,\cancel{2}}{\cancel{2}\cdot 3}(1-\theta^2)^{3/2}\right]_0^1$ 1

$= a^2h - \frac{a^2h}{3}$

$= \frac{2a^2h}{3}\ u^3$ # 1

Q. 7

(i) $\Downarrow +ve$ $\quad \uparrow \frac{1}{100}V^2$ $\quad \downarrow mg$

$m\ddot{x} = mg - \frac{1}{100}V^2$

$m=1,$

$\ddot{x} = g - \frac{V^2}{100}$ 1

* without indicating positive direction of motion will lose $\frac{1}{2}$ mark

(ii) $\ddot{x} = 0$ when $g = \frac{V^2}{100}$

$\therefore$ terminal velocity $V^2 = 100g$

$V = \sqrt{100g}$ (motion going down $V>0$) 1

(iii) $v\frac{dv}{dx} = g - \frac{v^2}{100}$

$v\frac{dv}{dx} = \frac{100g - v^2}{100}$

1 $-\frac{1}{2}\int \frac{-2v\,dv}{100g - v^2} = \int \frac{dx}{100}$

1 $-\frac{1}{2}\ln(100g - v^2) + c = \frac{x}{100}$

Put $c = \frac{\ln A}{2}$, $-\frac{1}{2}\left[\ln(A(100g - v^2))\right] = \frac{x}{100}$

$-\ln\left[A(100g - v^2)\right] = \frac{x}{50}$

$x=0,\ v=0 \therefore -\ln\left[A(100g - 0)\right] = 0$

$\ln\left[A(100g)\right] = 0$

$A(100g) = 1$

$A = \frac{1}{100g}$

1 $\therefore A = \frac{1}{V^2}$

$\therefore \frac{x}{50} = -\ln\left[\frac{100g - v^2}{V^2}\right]$

$e^{-\frac{x}{50}} = \frac{100g - v^2}{V^2} = 1 - \frac{v^2}{V^2}$ because $V^2 = 100g$

$\frac{v^2}{V^2} = 1 - e^{-\frac{x}{50}}$

1 $v^2 = V^2\left[1 - e^{-\frac{x}{50}}\right]$

Alternatively

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g - \frac{v^2}{100}$

$\frac{2\,dx}{d(v^2)} = \frac{100}{100g - v^2}$ 1

$\int \frac{dx}{50\,d(v^2)} = \int \frac{1}{V^2 - v^2}$

$\frac{x}{50} = -\ln\left(A(V^2 - v^2)\right)$ A is a constant 1

$x=0,\ v=0,\ A = \frac{1}{V^2}$ 1

$0 = -\ln A(V^2 - 0)$

$AV^2 = 1$

$-\frac{x}{50} = \ln\left(1 - \frac{v^2}{V^2}\right)$

$1 - \left(\frac{v}{V}\right)^2 = e^{-\frac{x}{50}}$

$v^2 = V^2\left(1 - e^{-\frac{x}{50}}\right)$ 1

1

iv) $\left(\frac{v}{V}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$$\frac{-x}{50} = \ln\left(1 - \frac{1}{4}\right) = \ln\frac{3}{4}$$

$$x = 50 \ln\frac{4}{3} \doteqdot 14.4\,m \text{ (1dp)}$$

distance fallen is 14.4m (1dp) 1

v) $x = 50$ $\left(\frac{v}{V}\right)^2 = 1 - e^{-\frac{50}{50}} = 1 - e^{-1}$ 1

$$\frac{v}{V} = \sqrt{1 - e^{-1}} = \sqrt{\frac{e-1}{e}} \qquad (v > 0,\ V > 0)$$

$$v = \sqrt{\frac{e-1}{e}} \times 100\% \text{ of } V$$

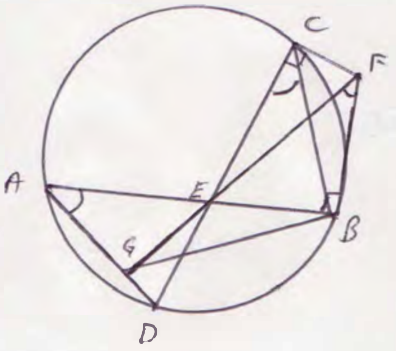
$$v \doteqdot \underline{\underline{79.5\% \text{ of } V}}$$ 1

Q 8a)

To prove FF produced is perpendicular to AD

Proof : Join AD, Extend FE to meet AD at G.

Join BG, BC



$\angle ABF = \angle DCF = 90°$ (given)

$\therefore \angle ECF + \angle EBF = 180°$

1 $\therefore$ ECFB is a cyclic quadrilateral

(opposite angles ~~supplementary~~)

$\angle BAD = \angle ECB$ (angles in same segment of the given circle)

$\angle ECB = \angle EFB$ (angles in same segment of circle ECFB

1 $\therefore \angle BAD = \angle EFB$

1 $\therefore$ AFBG is a cyclic quadrilateral

(line interval BG subtends equal angles at 2 points on the same side of it, the end points of the intervals are the 2 points are concyclic)

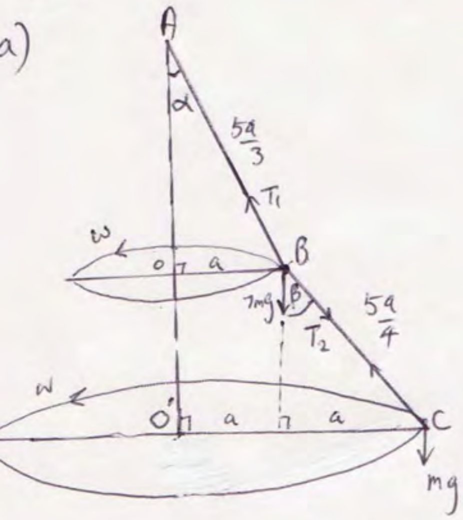
1 $\angle AGF = \angle ABF = 90°$ (angles in same segments of circle AFBG)

ie. $FG \perp AD$

1 ie FE produced is perpendicular to AD

Q 8

a)



The particle at B, C rotate about the vertical AO' with the same angular velocity w, in horizontal circles of radii a and $2a$ respectively. The strings AB, BC are inclined at fixed angles α, β to the vertical.

$$AO = \sqrt{\left(\frac{5a}{3}\right)^2 - a^2} = \frac{4a}{3}$$
$$BD = \sqrt{\left(\frac{5a}{4}\right)^2 - a^2} = \frac{3a}{4}$$
(Pythagoras' Th.)

$$\cos\beta = \frac{BD}{BC} = \frac{3a/4}{5a/4} = \frac{3}{5}$$

1 $$\tan\beta = \frac{a}{\frac{3a}{4}} = \frac{4}{3}$$

$$\cos\alpha = \frac{4a/3}{5a/3} = \frac{4}{5}$$

i) Forces acting on particle C are its weight and tension T_2

1 vertically : $mg = T_2 \cos\beta$ (1)

$$mg = T_2 \times \frac{3}{5}$$

$$\text{Tension in } BC = T_2 = \frac{5mg}{3}$$

1

(ii) Horizontally along CO'

$$m \cdot 2a \cdot w^2 = T_2 \sin\beta \quad (2)$$

(2) ÷ (1) $$\frac{2aw^2\cancel{m}}{\cancel{m}g} = \tan\beta$$

1

$$w^2 = \frac{\tan\beta}{2a} \cdot g$$

$$w^2 = \frac{\cancel{4}}{3} \times \frac{g}{\cancel{2}a} = \frac{2g}{3a}$$

1 $$w = \sqrt{\frac{2g}{3a}}.$$

∴ the angular velocity of the particle at B and C is given by $\omega = \sqrt{\frac{2g}{3a}}$

The forces acting on particle B are its weight and tension T_1 in string AB

Vertically : $7mg + T_2\cos\beta = T_1\cos\alpha$ 1

$$7mg + \frac{5mg}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{5}} = T_1 \cdot \frac{4}{5}$$

$$8mg = T_1 \frac{4}{5}$$

$$\text{Tension in } AB = T_1 = 10\,mg$$ 1

iii) Since particle B moves with angular velocity ω in a circle with radius a, its speed $= |V| = a\sqrt{\frac{2g}{3a}} = \sqrt{\frac{2ag}{3}}$ 1

Particle C moves with angular velocity ω in a circle with radius $2a$ its speed $= |V| = 2a\sqrt{\frac{2ag}{3}} = \sqrt{\frac{8ag}{3}}$ 1

Alternative way to find T_1:

Horizontally along BO :

$$7m a \cdot w^2 = T_1 \sin\alpha - T_2 \sin\beta$$

$$7m\cancel{a} \cdot \frac{2g}{3\cancel{a}} = T_1 \frac{3}{5} - \frac{\cancel{5}mg}{3} \times \frac{4}{\cancel{5}}$$

$$\frac{14mg + 4mg}{3} = T_1 \times \frac{3}{5}$$

$$T_1 = 10mg.$$

8 bi) $I_n = \int \frac{dx}{(x^2+a^2)^n}$

Let $u = \frac{1}{(x^2+a^2)^n}$ $\quad du = \frac{-2nx}{(x^2+a^2)^{n+1}}$

$V = x \qquad dv = dx$

$$I_n = u \cdot V - \int u'v = \frac{x}{(x^2+a^2)^n} + \int \frac{2nx^2\,dx}{(x^2+a^2)^{n+1}}$$ /

But $\int \frac{x^2dx}{(x^2+a^2)^{n+1}} = \int \frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}} dx = \int \frac{dx}{(x^2+a^2)^n} - a^2\int\frac{dx}{(x^2+a^2)^{n+1}}$

$= I_n - a^2 I_{n+1}$

$$\therefore \quad I_n = \frac{x}{(x^2+a^2)^n} + 2n\, I_n - 2a^2n\, I_{n+1}$$ /

we have a higher order integral on the RHS, we solve this difficulty by changing the subject of the equations adjusting the order in the last step by replacing n by n-1

$$2a^2n\, I_{n+1} = \frac{x}{(x^2+a^2)^n} + (2n-1)I_n$$ /

$$I_{n+1} = \frac{1}{2a^2(n)}\left(\frac{x}{(x^2+a^2)^n} + (2n-1)\, I_n\right)$$

$$I_n = \frac{1}{2a^2(n-1)}\left[\frac{x}{(x^2+a^2)^{n-1}} + (2n-3)\, I_{n-1}\right]$$ /

bii) Put $n=2$

$$I_2 = \frac{1}{2a^2}\left[\frac{x}{(x^2+a^2)} + I_1\right]_0^1 = \frac{1}{2a^2}\left[\frac{x}{x^2+a^2} + \int_0^1 \frac{dx}{a^2+x^2}\right]$$ /

$$I_2 = \frac{\left[\frac{x}{x^2+a^2}\right]_0^1 + \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^1}{2a^2} = \underline{\underline{\left[\frac{1}{1+a^2} + \frac{1}{a}\tan^{-1}\frac{1}{a}\right]\frac{1}{2a^2}}}$$ /