CSSA - NSW

4 Unit Mathematics

Trial HSC Examination 1988

1. **(i)** The graph of $f(x) = \frac{ax^2+bx+c}{x^2+qx+r}$ has the lines $x = 1, x = 3$ and $y = 2$ as asymptotes and a turning point at $(0, 1)$.
(a) Use this information to show that $f(x) = \frac{2x^2-4x+3}{x^2-4x+3}$.
(b) Sketch the graph of $y = f(x)$ showing clearly the coordinates of intersection with the x axis and the y axis, the coordinates of any turning points and the equations of any asymptotes. (There is no need to investigate points of inflection.)
(ii) **(a)** Show that $e^{-x} - 1 + x$ is never negative.
(b) If $f(x) = \frac{e^x-1}{x}$ for $x \neq 0$ and $f(0) = 1$ show that $f(x)$ increases with x for all $x \neq 0$.

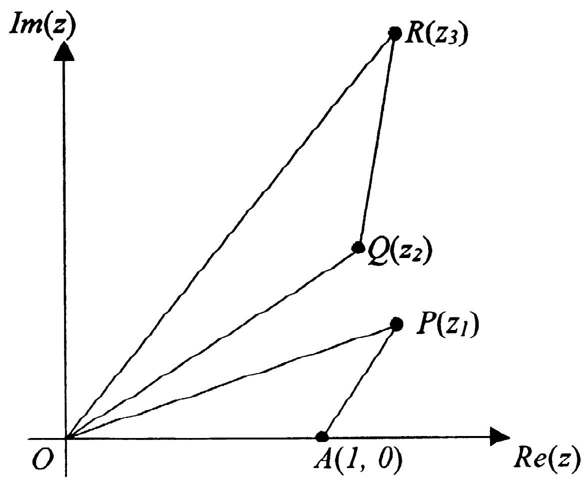
2. **(i)** Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{4+5\sin x}\,dx$.
(ii) Using the substitution $x = 4\sin^2\theta$ or otherwise show that $\int_0^2 \sqrt{x(4-x)}\,dx = \pi$.
(iii) Find $\int \frac{\ln x}{\sqrt{x}}\,dx$.
(iv) Use the substitution $u = \pi - x$ to show that $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$. Deduce that $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$.

3. **(i)** Obtain the solutions of the quadratic equation $(1-4i)z^2 - 4z + 1 = 0$ in the form $a + ib$.
(ii) Indicate on the Argand diagram the set of points $P(x, y)$ where $z = x + iy$ for which $0 \leq \arg(z+1) \leq \frac{\pi}{4}$ and $|z+i| < 2$ hold simultaneously.
(iii) Let $\tan\alpha = \frac{1}{3}$ where $0 < \alpha < \frac{\pi}{2}$.
(a) Show that $4\alpha = \tan^{-1}\frac{24}{7}$.
(b) Given that $0 < \alpha < \frac{\pi}{2}$ express $z = 7 + 24i$ in the form $r(\cos\theta + i\sin\theta)$ given all the possible values of θ in terms of α.
(c) Hence obtain in the form $a + ib$ the four fourth roots of z.

4. $P(2Ap, Ap^2)$ is a point on the parabola $x^2 = 4Ay$. $Q(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. In what follows you may use without proof the results that the tangent to $x^2 = 4Ay$ at P and the tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at Q have equations $px - y = Ap^2$ and $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ respectively.
(a) Using the fact that two lines are coincident if the corresponding coefficients are in proportion deduce that the tangent to $x^2 = 4Ay$ at P is also the tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at Q if $\cos\theta = \frac{a}{Ap}$ and $\sin\theta = -\frac{b}{Ap^2}$.

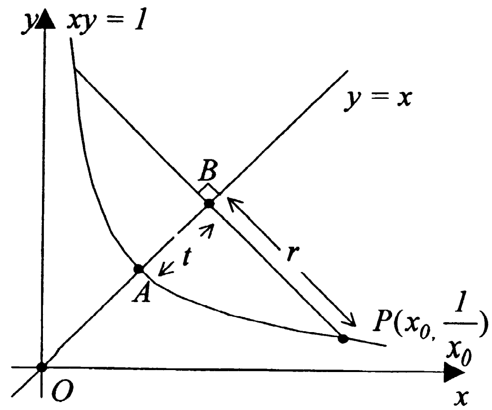
(b) Hence show that PQ is a common tangent to $x^2 = 4Ay$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $A^2p^4 - a^2p^2 - b^2 = 0$, and deduce that there are exactly two such common tangents.
(c) Let $p_0 > 0$ be the parameter of the point of contact of one of these common tangents with the parabola and let $A > 0$. Sketch the parabola, the ellipse and both common tangents showing, in terms of p_0 the coordinates of the points of contact of the tangents with both curves and the intercepts of the tangents on the coordinate axes.
(d) Using symmetry sketch on the same diagram the parabola $x^2 = -4Ay$ and the two common tangents to $x^2 = -4Ay$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. What is the nature of the quadrilateral formed by the four tangents on this diagram? Deduce that this quadrilateral is a square if $A^2 = a^2 + b^2$.
(e) Find the equation of the circle with centre $(0, 0)$ for which the quadrilateral formed by the four tangents common to the circle and the curve $x^2 = \pm 8y$ is a square.

5. (i)

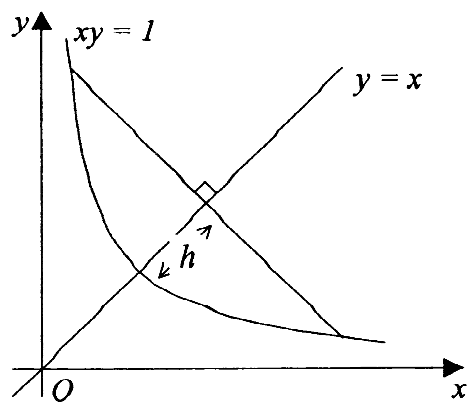


In the Argand diagram above, P is the point representing the complex number z_1, Q is the point representing the complex number z_2 and A is the point $(1, 0)$. The triangle OQR is constructed similar to triangle OAP. Let the point R represent the complex number z_3.
(a) Show that:
(α) $|z_3| = |z_1||z_2|$;
(β) $\arg z_3 = \arg z_1 + \arg z_2$.
(b) What is the significance of these results?

(ii)



(a) In the diagram above:
(α) Show that $2r^2 = (x_0 - \frac{1}{x_0})^2$.
(β) Hence show that $OP^2 = 2(1 + r^2)$.
(γ) Hence or otherwise show that $r^2 = (t + \sqrt{2})^2 - 2$.
(b)



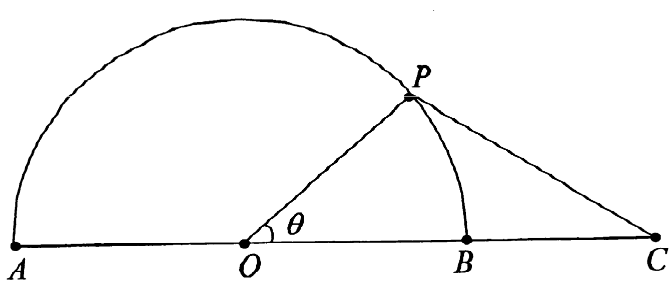
Hence find the volume of the solid formed by rotating the shaded area above through 2π radians about the line $y = x$.

6. The ends of a light string are fixed to two points A and B in the same vertical line with A above B and the string passes through a small smooth ring of mass m. The ring is fastened to the string at a point P and when the string is taut the angle APB is a right angle, the angle BAP is θ and the distance of P from AB is r. The ring revolves in a horizontal circle with constant angular velocity w and with the string taut.
(a) Draw a diagram to show the forces acting on the ring.
(b) Show that the tensions T_1 and T_2 in the parts AP and PB respectively of the string are $T_1 = m(rw^2 \sin\theta + g\cos\theta)$ and $T_2 = m(rw^2\cos\theta - g\sin\theta)$.
(c) Given that $AB = 5a$ and $AP = 4a$ show that $16aw^2 > 5g$.
(d) If the ring is free to move on the string instead of being fastened show that if it remains in the same position on the string as before, revolving in a horizontal circle with constant angular velocity Ω, then Ω satisfies the equation $12a\Omega^2 = 35g$.

7. **(i)** Find the integers m and n such that $(x+1)^2$ is a factor of $x^5 + 3x^2 + mx + n$.

(ii) None of the roots α, β and γ of the equation $x^3 + 3px + q = 0$ is zero.
(a) Obtain the monic equation whose roots are $\frac{\beta\gamma}{\alpha}, \frac{\alpha\gamma}{\beta}, \frac{\alpha\beta}{\gamma}$ expressing its coefficients in terms of p and q.
(b) Deduce that $\gamma = \alpha\beta$ if and only if $(3p - q)^2 + q = 0$.
(iii) (a) Show that the equation $x^3 - 6x^2 + 9x - 5 = 0$ has only one real root $x = \alpha$.
(b) Determine the two consecutive intergers between which α lies.
(c) By considering the product of the roots of the equation express the modulus of each of the complex roots in terms of α and deduce that the value of this modulus lies between 1 and $\frac{\sqrt{5}}{2}$.

8. (i) The equation of a curve is $x^2y^2 - x^2 + y = 0$
(a) Show that the numerical value of y is never greater than the corresponding value of x.
(b) Show that the numerical value of y is always less than unity.
(c) Find the equation of the asymptotes.
(d) Find the equations of the tangents at the origin.
(e) Sketch the curve.
(ii)



In the diagram above the fixed points A, O, B and C are on a straight line such that $AO = OB = BC = 1$ unit. The points A and B are also joined by a semicircle and P is a variable point on this semicircle such that the angle POC is θ. R is the region bounded by the arc AP of the semicircle and the straight lines AC and PC.
(a) Show that the area S of R is given by $S = \frac{\pi}{2} - \frac{\theta}{2} + \sin\theta$. Find the value of θ for which S is a maximum.
(b) Show that the perimeter L of R is given by $L = 3 + \pi - \theta + \sqrt{5 - 4\cos\theta}$. Show that L has just one stationary point and that occurs at the same value of θ for which S is a maximum. Find the least value of L and the greatest value of L in the interval $0 \leq \theta \leq \pi$.