STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x > 0$

TRIAL 2003 MATHEMATICS (2U)

Q1.

(a) $1.0914\ldots = 1.1$ (2 sig figs) ②

(b) $\log(x+3) = \log x + \log 3$

$x+3 = 3x$ 1

$2x = 3$ ②

$x = \frac{3}{2}$ 1

(c) $\frac{d}{dx}\left(\frac{1}{e^x-1}\right) = \frac{d}{dx}\left(e^x-1\right)^{-1}$ 1

$= -\left(e^x-1\right)^{-2} \times e^x$ 1

$= \frac{-e^x}{\left(e^x-1\right)^2}$ ②

(d) $4 - \frac{2x}{3} \leq 6$

$4 - 6 \leq \frac{2x}{3}$ 1 ②

$2x \geq 3(-2)$

$x \geq -3$ 1

(e) $\frac{x^2+3x-4}{x^2+x-2} = 0$

$\frac{(x+4)(x-1)}{(x+2)(x-1)} = 0$ 1 ②

$x \neq -2,\ x \neq 1$

$\therefore x = -4$ only 1

(f) $5^m = 4$

$\therefore 5^{1-2m} = \frac{5}{5^{2m}}$ 1 ②

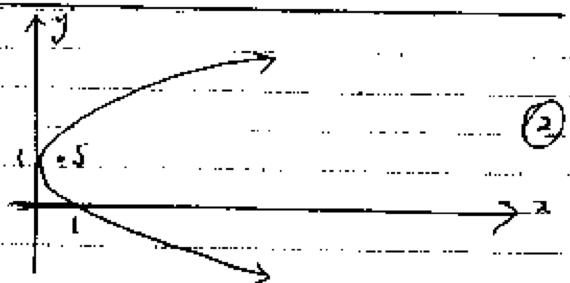
$= \frac{5}{(5^m)^2}$ 1

$= \frac{5}{16}$

Q2 (a) (i) $\int_0^2 f(x)\,dx = -1.3$ (1)

(ii) $\int_0^4 f(x)\,dx = -1.3 + 1.8$
$= 0.5$ (1)

(b) (i)



(2)

(ii) $x = (y-1)^2 \Rightarrow a = \frac{1}{4}$

so Focus is $\left(\frac{1}{4}, 1\right)$ (1)

(c) (i) $\parallel$ to $x + 2y = 2$ is
$x + 2y = k$
$(2,2)$ satisfies $\therefore 2 + 2(2) = k \Rightarrow k = 6$ (1)

$\therefore x + 2y = 6$ is eqn of DC

(ii) $\perp$ dist $(2,2)$ to $x + 2y = 2$

$d = \left| \frac{1(2) + 2(2) - 2}{\sqrt{1^2 + 2^2}} \right|$ (2)

$= \frac{4}{\sqrt{5}}$

(iii) $A = A_{\triangle ODC} - A_{\triangle OAB}$
$= \frac{1}{2}(6)(3) - \frac{1}{2}(2)(1)$ (2)
$= 9 - 1 = 8$

(iv) $2 \le x + 2y \le 6 \;\cap\; x \ge 0 \;\cap\; y \ge 0$

15

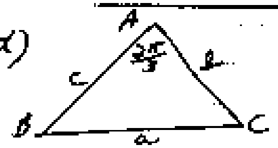
Q3 (a) (i) $\frac{d}{dx} \tan^3\left(\frac{x}{3}\right) = 3\tan^2\left(\frac{x}{3}\right)\sec^2\left(\frac{x}{3}\right) \cdot \frac{1}{3}$ (2)
$= \tan^2\left(\frac{x}{3}\right)\sec^2\left(\frac{x}{3}\right)$

(ii) $\frac{d}{dx} \frac{\ln\sqrt{x}}{x} = \frac{d}{dx} \frac{\frac{1}{2}\ln x}{x}$
$= \frac{d}{dx} \frac{\ln x}{2x}$ $\frac{1}{2}$
$= \frac{2x \cdot \frac{1}{x} - 2\ln x}{4x^2}$ 1 (2)
$= \frac{1 - \ln x}{2x^2}$ $\frac{1}{2}$

(b) $x - 4 = 3\sqrt{x}$ Let $\sqrt{x} = u$
$u^2 - 4 = 3u$
$u^2 - 3u - 4 = 0$
$(u-4)(u+1) = 0$ 1
$u = 4$ or -1 (can't $= -1$) 1 (3)
$\therefore \sqrt{x} = 4$ 1
$x = 16$

(c) $\int_0^3 \frac{e^x}{e^x + 1}\,dx = \ln(e^x + 1)\Big]_0^3$ 1
$= \ln(e^3 + 1) - \ln(1 + 1)$ (2)
$= \ln\left(\frac{e^3 + 1}{2}\right)$ 1

(d)



$\cos\frac{2\pi}{3} = -\frac{1}{2}$

$a^2 = b^2 + c^2 - 2bc\cos\frac{2\pi}{3}$ 1
$= b^2 + c^2 + bc$
$\therefore a^2 - c^2 = b^2 + bc$ (3)

Also $\cos C = \frac{b^2 + a^2 - c^2}{2ab} = \frac{b^2 + b^2 + bc}{2ab}$
$= \frac{2b + c}{2a}$ 1

Q4 (a) $y = \frac{6x}{102-x}$

(i) $x = 90, \quad y = \frac{6 \times 90}{102-90}$

$= 45$

So cost $= \$45\,000$ (1)

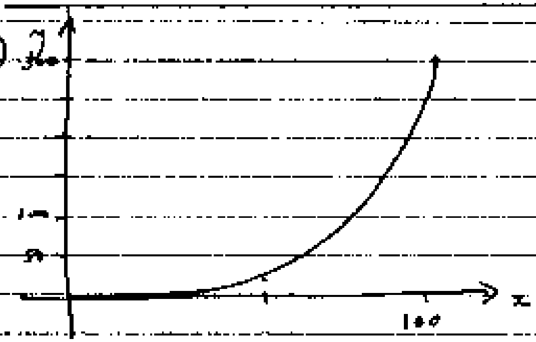
(ii) $y = 10, \quad 10 = \frac{6x}{102-x}$

$1020 - 10x = 6x$

$16x = 1020$

$x = 63\frac{3}{4}\,\%$ (2)

(iii)



$x=0,\ y=0$

$x=100,\ y=300$

$x=54,\ y=6$

(2)

(b)(i) $V'(t) = 150t^{-\frac{1}{4}} + 10$

$V(t) = 150 \cdot \frac{4}{3} t^{\frac{3}{4}} + 10t + C$ 1

$V(16) = 3000$

$\therefore 3000 = 200(16)^{\frac{3}{4}} + 160 + C$ 1

$= 1600 + 160 + C \quad \therefore C = 1240$ (2)

$V(t) = 200t^{\frac{3}{4}} + 10t + 1240$

(ii) 15 mins $\Rightarrow t = 900$

$V(900) = 200 \cdot 30\sqrt{30} + 9000 + 1240$ 1

So $V(900) - V(16) = 6000\sqrt{30} + 7240$ 1 (2)

$\approx 40\,100$ L.

4(c) $\frac{d^2y}{dx^2} = 12x - 2$

$\frac{dy}{dx} = 6x^2 - 2x + c$

when $x = 1, \ \frac{dy}{dx} = 1 \quad (\tan 45° = 1)$ 1

$1 = 6(1) - 2(1) + c \Rightarrow c = -3$

$\frac{dy}{dx} = 6x^2 - 2x - 3$ 1 (3)

$y = 2x^3 - x^2 - 3x + k$

$(1, -2)$: $-2 = 2(1) - 1 - 3(1) + k \Rightarrow k = 0$

$y = 2x^3 - x^2 - 3x$ 1

Q5 (a) $y = 2x^3 + ax^2 + bx + 3$

(i) $\frac{dy}{dx} = 6x^2 + 2ax + b$

$= 0$ when $x = 1$ — 1

$\therefore 6 + 2a + b = 0$ (1)

Also $6(-2)^2 + 2a(-2) + b = 0$ (2)

$24 - 4a + b = 0$ (2) — 1

(2) − (1): $18 - 6a = 0$

$a = 3$

$b = -12$

(ii) $\frac{d^2y}{dx^2} = 12x + 6$

when $x = 1$, $\frac{d^2y}{dx^2} > 0$ $\therefore$ min at $(1, -4)$ — 1

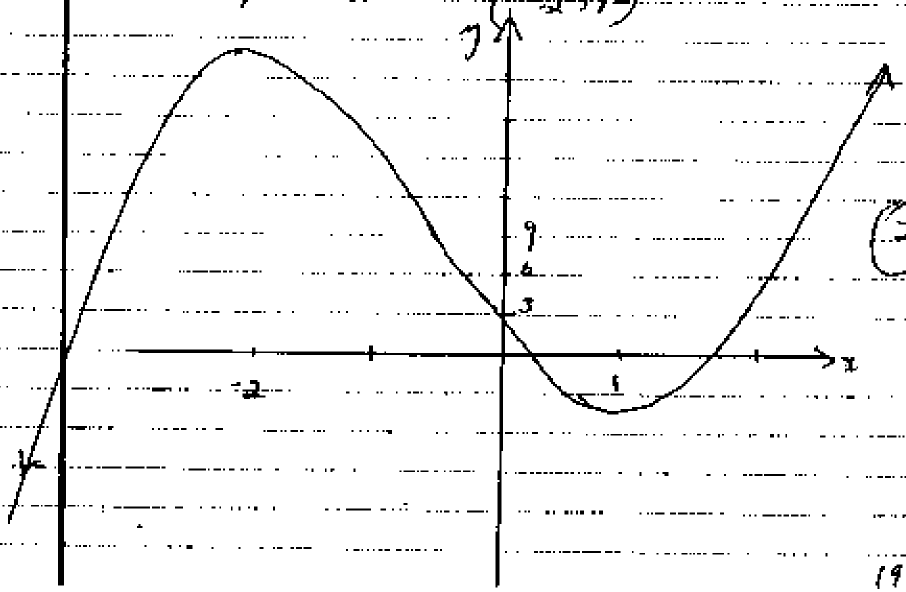
when $x = -2$, $\frac{d^2y}{dx^2} < 0$ $\therefore$ max at $(-2, 23)$ — 1

$\frac{d^2y}{dx^2} = 0$ when $x = -\frac{1}{2}$

> 0 when $x = -\frac{1}{2}^+$

< 0 when $x = -\frac{1}{2}^-$ — 1 (3)

So inflexion at $(-\frac{1}{2}, 9\frac{1}{2})$



(2)

5 (b) $h(x) = e^{-2x} + e^{x}$ $\quad 0 \le x \le 2$

(i) $h'(x) = -2e^{-2x} + e^{x}$

$= 0$ when $e^{-2x}(-2 + e^{3x}) = 0$ — 1

$e^{3x} = 2$

$3x = \ln 2$

$x = \frac{1}{3}\ln 2$ — 1

$h''(x) = 4e^{-2x} + e^{x}$

> 0 for all x (3)

$\therefore$ minimum h when $x = \frac{1}{3}\ln 2$ — 1

(ii) $h(\frac{1}{3}\ln 2) = e^{-2\cdot\frac{1}{3}\ln 2} + e^{\frac{1}{3}\ln 2}$

$= (e^{\ln 2})^{-\frac{2}{3}} + (e^{\ln 2})^{\frac{1}{3}}$ — 1

$= 2^{-\frac{2}{3}} + 2^{\frac{1}{3}}$

$= 2^{-\frac{2}{3}}(1 + 2)$ (2)

$= 3 \cdot 2^{-\frac{2}{3}}$ — 1

Q6 (a) $y = \sin^2 t - \sin t + 2$

(i) $\sin^2 t - \sin t + 2 = \left(\sin t - \frac{1}{2}\right)^2 + \frac{7}{4}$ — (1)

(ii) $y = \left(\sin t - \frac{1}{2}\right)^2 + \frac{7}{4}$

least y is $\frac{7}{4}$ — (1)

(iii) $\sin t - \frac{1}{2} = 0$

$\sin t = \frac{1}{2}$

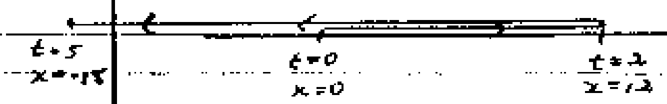
$t = \frac{\pi}{6}$ — (1)

(b) $x = 12t - 3t^2$ (1)

(i) $t=3$, $x = 36 - 27 = 9$ m to right of O.

(ii) $\frac{dx}{dt} = 12 - 6t$

$= 0$ when $t = 2$ so stops after 2 secs.



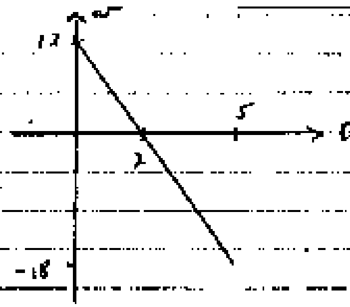
— 1

when $t=2$, $x = 12$ — 1

when $t=5$, $x = -15$ — 1

so travels $12 + 12 + 15 = 39$ m (3)

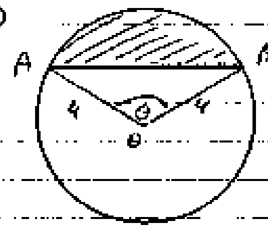
(iii)



when $t=5$, $v = 12 - 30$
$= -18$

so max speed is 18 m/sec (10)

6(c)



(i) $A_{segment} = \frac{1}{2} r^2 (\theta - \sin\theta)$

$= \frac{1}{2}(4^2)(\theta - \sin\theta)$ (1)

$= 8(\theta - \sin\theta)$

(ii) as $\theta \to 0$, Area gets closer to 0.

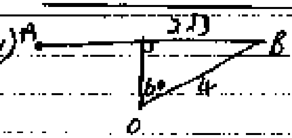
ie $\theta - \sin\theta \to 0$

$\theta \to \sin\theta$

$\frac{\sin\theta}{\theta} \to 1$ (1)

Smaller θ gets, closer $\frac{\sin\theta}{\theta}$ gets to 1

(iii)



$AB = 4\sqrt{3}$ — 1

arc AB: $l = r\theta$

$= 4 \times \frac{2\pi}{3}$ — 1

$\therefore P = 4\sqrt{3} + \frac{8\pi}{3}$ (2)

Q7 (a) $\frac{dV}{dt} = kV$

(i) $V = V_0 e^{kt}$

$\frac{dV}{dt} = V_0 e^{kt} . k$

$= kV$ as required (1)

(ii) when $t=0$, $V = 150\,000$

$150\,000 = V_0 e^0 \therefore V_0 = 150\,000$

$V = 150\,000 e^{kt}$

when $t=20$, $V = 150\,000 e^{20k} = 600\,000$ — 1

$e^{20k} = 4$

$20k = \ln 4$

$k = \frac{1}{20}\ln 4$

$= \frac{1}{20} \cdot 2\ln 2$

$= \frac{1}{10}\ln 2$ — 1 (2)

(iii) when $t=10$, $V = 150\,000 \cdot e^{10 \cdot \frac{1}{10}\ln 2}$

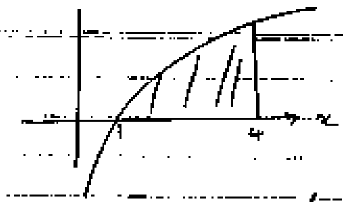
$= 150\,000 \cdot e^{\ln 2}$ — 1

$= 150\,000(2)$

$= 300\,000$ — 1 (2)

So apartment is worth \$50 000 more than anticipated

7 (b)



$y = 4 - \frac{4}{x}$

$V = \pi \int_1^4 y^2 \, dx = \pi \int_1^4 \left(4 - \frac{4}{x}\right)^2 dx$ (1)

(ii)

x	1	2	3	4
y	0	2	$\frac{8}{3}$	3
y^2	0	4	$\frac{64}{9}$	9

— 1

$V = \pi \cdot \frac{h}{2}\left[0 + 2\left(4 + \frac{64}{9}\right) + 9\right]$ — 1

$= \frac{\pi}{2}\left[0 + 8 + \frac{128}{9} + 9\right]$ — 1 (3)

$= 35.1$ units^3 (to 1 dec pl)

(iii) $V = \pi \int_1^4 \left(4 - \frac{4}{x}\right)^2 dx$

$= \pi \int_1^4 \left(16 - \frac{32}{x} + \frac{16}{x^2}\right) dx$

$= \pi \left[16x - 32\ln x - \frac{16}{x}\right]_1^4$ — 1

$= \pi\left[64 - 32\ln 4 - 4 - (16 - 0 - 16)\right]$

$= \pi\left[60 - 32\ln 4\right]$ — 1

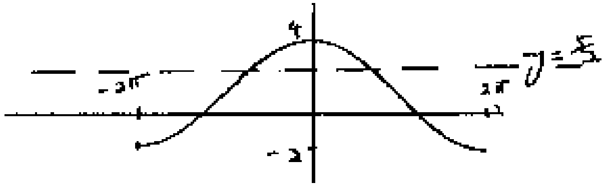
$= 4\pi\left[15 - 8\ln 4\right]$ — 1 (3)

$= 4\pi\left[15 - 16\ln 2\right]$ as required.

Q8 (a) $y = 1 + 3\cos\frac{x}{2}$

(i) Amp = 3 Period = $\frac{2\pi}{\frac{1}{2}} = 4\pi$ (1)

(ii) Range. $\left.\begin{array}{l}1+3=4\\1-3=-2\end{array}\right\}$



(2)

(iii) $3\cos\frac{x}{2} = \frac{3}{2}$

$1 + 3\cos\frac{x}{2} = \frac{5}{2}$ (1)

no draw $y = \frac{5}{2}$

2 points of intersection so 2 sol'ns.

(b)(i) $1 + 0.4 + 0.16 + \cdots$

$a = 1,\ r = 0.4$

$S_\infty = \frac{a}{1-r}$ $\qquad S_n = \frac{a(1-r^n)}{1-r}$

$= \frac{1}{1-0.4}$ $\qquad = \frac{1-(0.4)^n}{1-0.4}$

$= \frac{5}{3}$ $\qquad = \frac{5}{3}(1-0.4^n)$

$S_\infty - S_n = \frac{5}{3}\left[1-(1-0.4^n)\right]$ (2)

$= \frac{5}{3}(0.4^n)$

(ii) $\frac{5}{3}(0.4)^n < 10^{-6}$

$0.4^n < \frac{3 \cdot 10^{-6}}{5}$ (1)

$n \log 0.4 < \log 3 - 6\log 10 - \log 5$

$n > \frac{\log 3 - 6\log 10 - \log 5}{\log 0.4}$ $\boxed{16}$

$> 15 \cdots$

8 (c)(i) $\cos 2x = \cos x$

$\cos^2 x - \sin^2 x = \cos x$

$\cos^2 x - (1-\cos^2 x) - \cos x = 0$

$2\cos^2 x - \cos x - 1 = 0$ —— 1

$(2\cos x + 1)(\cos x - 1) = 0$ (2)

$\cos x = -\frac{1}{2}$ or $\cos x = 1$

$x = \frac{2\pi}{3}$ or 0 —— 1

(ii) $A = \int_0^{\frac{2\pi}{3}} (\cos x - \cos 2x)\,dx$

$= \left[\sin x - \frac{1}{2}\sin 2x\right]_0^{\frac{2\pi}{3}}$ —— 1

$= \sin\frac{2\pi}{3} - \frac{1}{2}\sin\frac{4\pi}{3} - 0$ —— 1

$= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)$ —— 1 (3)

$= \frac{3\sqrt{3}}{4}$ units^2

Q9 (a) (i) $750 \times 3000 = \$2\,250\,000$ — ①

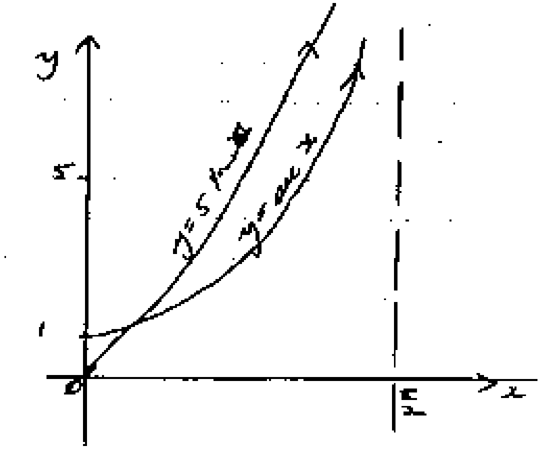
(ii) $S_n = \frac{n}{2}(a+l)$ — ②

$750 \times 3000 = \frac{750}{2}(1000+l)$

$6000 = 1000 + l$

So seat costs $\$5000$

(b) (i)



②

(ii) $5\tan x = \sec x$

$\frac{5\sin x}{\cos x} = \frac{1}{\cos x}$

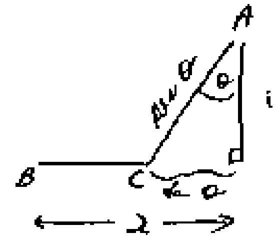
$5\sin x = 1$ ②

$\sin x = 0.2$

$x = 0.2013\ldots$

≈ 0.2

9 (c)



(i) $AC = \sec\theta$, $BC = 2 - \tan\theta$

$C = 5h\sec\theta + h(2 - \tan\theta)$ ②

(ii) $\frac{dC}{d\theta} = 5h\sec\theta\tan\theta - h\sec^2\theta$

$= h\sec\theta\left[5\tan\theta - \sec\theta\right]$ — 1

$= 0$ when $5\tan\theta - \sec\theta = 0$

From (b) $\theta = 0.2$ — 1

From (b) when $\theta > 0.2$, $5\tan\theta > \sec\theta$

$5\tan\theta - \sec\theta > 0$

i.e. $\frac{dC}{d\theta} > 0$ ③

when $\theta < 0.2$, $5\tan\theta < \sec\theta$

$5\tan\theta - \sec\theta < 0$

i.e. $\frac{dC}{d\theta} < 0$ — 1

$\therefore$ a local min occurs when $\underline{\underline{\theta = 0.2}}$

Q10 (a) $x^2+4x+2=0$

(i) $\alpha+\beta=-4$

$\alpha\beta=2$ — 1

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$= 16-4$ — 1

$= 12$ (2)

(ii) Roots $\frac{\alpha^2}{\beta}$ & $\frac{\beta^2}{\alpha}$

Σ roots $= \frac{\alpha^3+\beta^3}{\alpha\beta}$

$= \frac{(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)}{\alpha\beta}$ ---→ 3

$= \frac{(-4)(12-2)}{2} = -20$ (4)

prod. roots $= \frac{\alpha^2}{\beta}\cdot\frac{\beta^2}{\alpha} = \alpha\beta = 2$, 1

So eqn is $x^2 - \Sigma$ roots x + prod. roots $= 0$

$x^2+20x+2=0$

10 (b) (i) $f(x)=a^x$

$f'(x)=a^x . \ln a$

$g(x)=\log_a x$

$= \frac{\log x}{\log a}$

$g'(x) = \frac{1}{x \ln a}$ (2)

(ii) $a^x = \frac{\ln x}{\ln a}$ $(= x)$ (1)

(iii) Since they touch on $y=x$,

$f'(x)=g'(x)=1$ at point of contact.

i.e. $a^x \ln a = 1$

$a^x = \frac{1}{\ln a}$

So from (ii) $\frac{1}{\ln a} = \frac{\ln x}{\ln a}$ (2)

$\ln x = 1$

$x=e$, $y=e$

(iv) So $\frac{1}{e \ln a} = 1$ (1)

$\ln a = \frac{1}{e}$

$a = e^{\frac{1}{e}}$