Methods of Integration

3U97-1e)!

Using the substitution u = 2x + 1, or otherwise, find $\int_0^1 \frac{4x}{2x+1}\,dx$.¤

«→ 2 - Ln3 » 3U97-4b)!

By using the substitution x = sin t, or otherwise, evaluate $\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx$.¤

«→ $\frac{3\sqrt{3}+2\pi}{24}$ »

3U96-1f)!

Using the substitution u = e^x, find $\int \frac{e^x}{1+e^{2x}}\,dx$.¤

«→ $\tan^{-1} e^x + c$ »

3U96-2b)!

Use the table of standard integrals to show that $\int_6^{15} \frac{dx}{\sqrt{x^2+64}} = \text{Ln}\,2$.¤

«→ Proof »

3U95-1e)!

Use the subsitution u = 9 - x^2 to find $\int_0^1 6x\sqrt{9-x^2}\;dx$. ¤

«→ $2(27 - 16\sqrt{2})$ »

3U94-1b)!

Evaluate $\int_2^{10} \frac{x}{\sqrt{x-1}}\,dx$ using the substitution x = t^2 + 1. ¤

«→ $21\frac{1}{3}$ »

3U94-3b)!

Evaluate $\int_0^{\frac{\pi}{4}} 3\sin x\cos^2 x\;dx$. ¤

«→ $\frac{7}{8}$ »

3U93-1c)!

Evaluate $\int_{\frac{1}{2}}^{1} 4t(2t-1)^5\;dt$ by using the substitution u = 2t - 1. ¤

«→ $\frac{13}{42}$ »

3U92-1c)!

Find the exact value of $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\;dx$. ¤

«→ $\frac{\pi}{6}$ »

3U92-4a)!

Evaluate $\int_0^1 \frac{2x}{(2x+1)^2}\,dx$ by using the substitution u = 2x + 1. ¤

«→ $\frac{1}{2}\ln 3 - \frac{1}{3}$ »

3U91-1a)!

Evaluate:

i. $\int_0^1 \frac{x}{x^2+1}\,dx$

ii. $\int_{-1}^1 (1+5x)^4\,dx$, using the substitution u = 1 + 5x. ¤

«→ i) $\frac{1}{2}\ln 2$ ii) 352 »

3U90-1a)!

Evaluate:

i. $\int_0^1 \frac{1}{1+x^2}\,dx$

ii. $\int_0^1 \frac{x}{\sqrt{1+x}}\,dx$, using the substitution u = 1 + x. ¤

«→ i) $\frac{\pi}{4}$ ii) $\frac{2}{3}(2-\sqrt{2})$ »

3U90-4a)!

Find:

i. $\int \frac{\ln 2x}{x}\,dx$, using the substitution u = ln 2x

ii. $\int \cos^2 2x\,dx$. ¤

«→ i) $\frac{1}{2}$ $(\ln 2x)^2 + c$ ii) $\frac{1}{2}\left(x+\frac{\sin 4x}{4}\right)+c$ »

3U89-1b)!

Evaluate:

i. $\int_1^2 \frac{1}{\sqrt{4-x^2}}\,dx$,

ii. $\int_{-1}^0 x\sqrt{1+x}\,dx$, using the substitution u = 1 + x. ¤

«→ i) $\frac{\pi}{3}$ ii) $-\frac{4}{15}$ »

3U88-2b)!

Find $\int x\sqrt{2+x^2}\,dx$ using the substitution u = 2 + x^2. ¤

«→ $\frac{\sqrt{(2+x^2)^3}}{3}+c$ »

3U87-4i)!

Find the volume of the solid formed when the region bounded by the x-axis and the curve $y = x(8 - x^3)^4$ between x = 0 and x = 2 is rotated about the x-axis. (You may need to use the substitution $u = 8 - x^3$ to evaluate the integral involved.) ¤

«→ $\dfrac{\pi \times 2^{27}}{27}$ units^3 »

3U86-1iii)!

Use the substitution $u = x^2 + 2$ to evaluate $\displaystyle\int_0^1 \frac{x}{(x^2+2)^2}\,dx$. ¤

«→ $\dfrac{1}{12}$ »

3U85-1ii)!

Find $\displaystyle\int_0^1 x(1+x^2)^2\,dx$. ¤

«→ $\dfrac{7}{6}$ »

3U85-4i)!

Using the substitution $u = x^4$, or otherwise, show that $\displaystyle\int_0^1 \frac{x^3}{1+x^8}\,dx = \frac{\pi}{16}$. ¤

«→ Proof »

Primitive of $\sin^2 x$ and $\cos^2 x$

3U96-3bc)!

i. Show that $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 x \, dx = \frac{\pi}{8} - \frac{1}{4}$.¤

ii. The function g(x) is given by g(x) = 2 + cos x. The graph y = g(x) for $\frac{\pi}{4} \le x \le \frac{\pi}{2}$ is rotated about the x-axis. Find the volume of the solid generated. (You may use the result of part i.)¤

«→ i) Proof ii) $\frac{(9\pi + 30 - 16\sqrt{2})\pi}{8}$ units^3 »

3U95-5a)!

i. Solve the equation sin 2x = $2\sin^2 x$ for $0 < x < \pi$.

ii. Show that if $0 < x < \frac{\pi}{4}$, then $\sin 2x > 2\sin^2 x$.

iii. Find the area enclosed between the curves y = sin 2x and y = $2\sin^2 x$ for $0 \le x \le \frac{\pi}{4}$. ¤

«→ i) $x = \frac{\pi}{4}$ ii) Proof iii) $\left(1- \frac{\pi}{4}\right)$ units^2 »

3U91-3b)!

Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 3x \, dx$. ¤

«→ 0.25π »

3U90-4a)!

Find:

i. $\int \frac{\ln 2x}{x} dx$, using the substitution u = ln 2x

ii. $\int \cos^2 2x \, dx$. ¤

«→ i) $\frac{1}{2}$ $(\ln 2x)^2 + c$ ii) $\frac{1}{2}\left(x + \frac{\sin 4x}{4}\right) + c$ »

3U88-1c)!

Evaluate:

i. $\int_0^1 \frac{2x}{x^2+1} dx$,

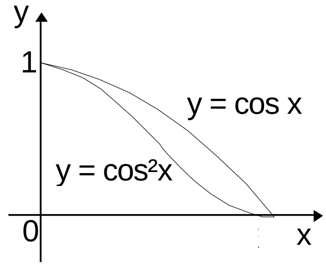
ii. $\int_0^{\pi} \sin^2 x \, dx$. ¤

«→ i) $\log_e 2$ ii) 0.5π »

3U84-4ii)!

a. Noting that $2\cos^2 x \equiv 1 + \cos 2x$, prove that $8\cos^4 x \equiv 3 + 4\cos 2x + \cos 4x$.

b. Sketch, on the same diagram, the curves y = cos x, y = $\cos^2 x$, for $0 \le x \le \frac{\pi}{2}$. Find the area enclosed between these curves and the volume generated when this area is rotated about the x-axis. ¤

«→ a) Proof b) 

Area = $1 - \frac{\pi}{4}$ units², Volume = $\frac{\pi^2}{16}$ units³ »

Equation $\frac{dN}{dt} = k$ (N - P)
Velocity and Acceleration as a Function of x
Projectile Motion
Simple Harmonic Motion

3U97-2b)!

A particle is moving in simple harmonic motion. Its displacement x at time t is given by $x = 3\sin(2t + 5)$.

i. Find the period of the motion.
ii. Find the maximum acceleration of the particle.
iii. Find the speed of the particle when x = 2.¤

«→ i) π ii) 12 iii) $2\sqrt{5}$ »

3U97-5a)!

A particle moves along the x axis, starting at x = 0.1 at time t = 0. The velocity of the particle is described by $v = \sqrt{2x}e^{-x^2}$, $x \geq 0.1$, where x is the displacement of the particle from the origin.

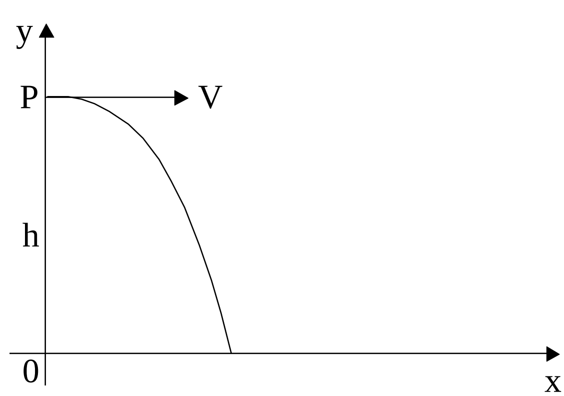
i. Show that the particle has acceleration given by $a = e^{-2x^2}\left(1 - 4x^2\right)$, $x \geq 0.1$.
ii. Hence find the fastest speed attained by the particle.
iii. Show that T, the time taken to travel from x = 1 to x = 2, can be expressed as

$$T = \int_1^2 \frac{1}{\sqrt{2x}} e^{x^2}\, dx.$$

iv. Use the trapezoidal rule with three function values to obtain an approximate value for T.¤

«→ i) Proof ii) $e^{-0.25}$ iii) Proof iv) 10 »

3U97-7a)!



A particle is projected horizontally from a point P, h metres above O, with a velocity of V metres per second. The equations of motion of the particle are $\ddot{x} = 0$ and $\ddot{y} = -g$.

i. Using calculus, show that the position of the particle at time t is given by x = Vt, $y = h - \frac{1}{2}gt^2$.

A canistor containing a life raft is dropped from a plane to a stranded sailor. The plane is travelling at a constant velocity of 216 km/h, at a height of 120 metres above sea level, along a path that passes above the sailor.

ii. How long will the canister take to hit the water? (Take $g = 10$ m/s^2.)
iii. A current is causing the sailor to drift at a speed of 3.6 km/h in the same direction as the plane is travelling. The canister is dropped from the plane when the horizontal distance from the plane to the sailor is D metres. What values can D take if the canister lands at most 50 metres from the stranded sailor?¤

«→ i) Proof ii) 4.9 sec. iii) 239.04 m or 339.04 m »

3U96-5a)!

A cup of hot coffee at temperature T°C loses heat when placed in a cooler environment. It cools according to the law

$$\frac{dT}{dt} = k(T - T_0),$$

where t is time elapsed in minutes, at T_0 is the temperature of the environment in degrees Celsius.

i. A cup of coffee at 100°C is placed in an environment at -20°C for 3 minutes, and cools to 70°C. Find k.

ii. The same cup of coffee, at 70°C, is then placed in an environment at 20°C. Assuming k remains the same, find the temperature of the coffee after a further 15 minutes.¤

«→ i) -0.09589 (to 5 d.p.) ii) 31.86° »

3U96-5b)!

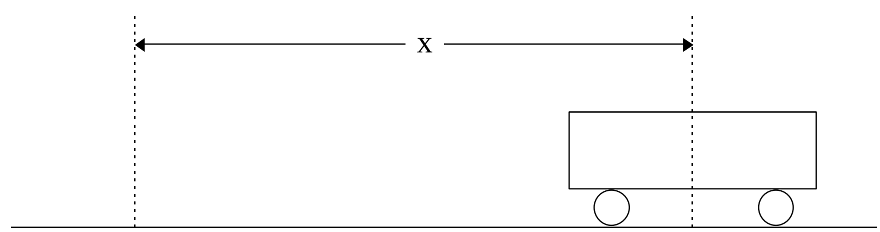
A particle is moving along the x-axis. Its velocity v at position x is given by

$$v = \sqrt{10x - x^2}.$$

Find the acceleration of the particle when x = 4.¤

«→ 1 »

3U96-6)!



A trolley is moving in simple harmonic motion about the origin O. The displacement, x metres, of the centre of the trolley form O at time t seconds is given by

$$x = 6\sin\left(2t + \frac{\pi}{4}\right).$$

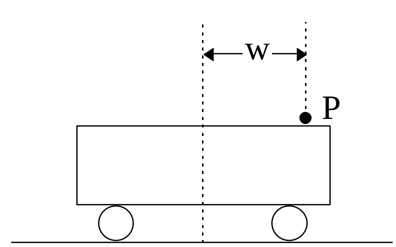
a. State the period and amplitute of the motion.

b. Sketch the graph of $x = 6\sin\left(2t + \frac{\pi}{4}\right)$ for $0 \le t \le 2\pi$.

c. Find the velocity of the trolley when t = 0.

d. Find the first time after t = 0 when the centre of the trolley is at x = 3.

e.



A particle P, on top of the trolley, is moving in simple harmonic motion about the centre of the trolley. Its displacement, w metres, from the centre of the trolley at time t seconds, is given by

$$w = \sin(2t).$$

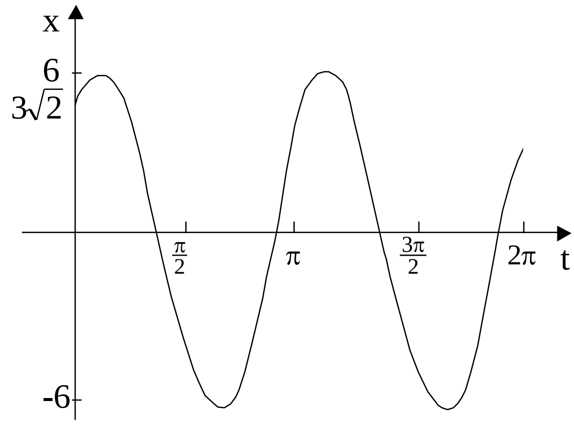
The displacement, y metres, of P from the origin is the sum of the two displacements x and w, so that

$$y = 6\sin\left(2t + \frac{\pi}{4}\right) + \sin(2t).$$

i. Show that P is moving in simple harmonic motion about O.

ii. Find the amplitute of this motion.¤

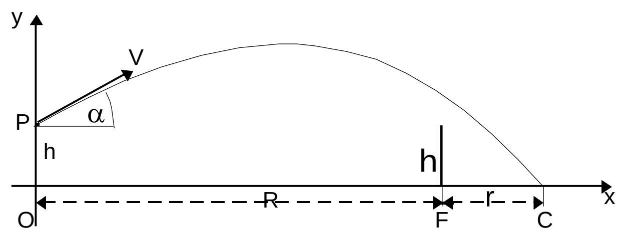
«→ a) Period = π, Amplitude = 6 b)



c) $6\sqrt{2}$ m/s

d) $\dfrac{7\pi}{24}$ sec e) i) Proof ii) 6.74 m »

3U95-7)!



A cap C is lying outside a softball field, r metres from the fence F, which is h metres high. The fence is R metres from the point O, and the point P is h metres above O. Axes are based at O, as shown. At time t = 0, a ball is hit from P at a speed V metres per second and at an angle α to the horizontal, towards the cap.

a. The equations of motion of the ball are
$\ddot{x}$ 1 = 0, $\ddot{y} = -g$.
Using calculus, show that the position of the ball at time t is given by
$x = Vt\cos\alpha$
$y = Vt\sin\alpha - \frac{1}{2}gt^2 + h$.

b. Hence show that the trajectory of the ball is given by
$$y = h + x\tan\alpha - x^2\frac{g}{2V^2\cos^2\alpha}.$$

c. The ball clears the fence. Show that
$$V^2 \geq \frac{gR}{2\sin\alpha\cos\alpha}.$$

d. After clearing the fence, the ball hits the cap C. Show that
$$\tan\alpha \geq \frac{Rh}{(R+r)r}.$$

e. Suppose that the ball clears the fence, and that $V \leq 50$, $g = 10$, $R = 80$, and $h = 1$. What is the closest point to the fence where the ball can land? ¤

«→ a) b) c) d) Proof e) 0.16m (to 2 d.p.) »

3U94-4c)!

The acceleration of a particle P moving in a straight line is given by $\frac{d^2x}{dt^2} = 3x(x-4)$, where x metres is the displacement from the origin O and t is the time in seconds. Initially the particle is at O and its velocity is $4\sqrt{2}$ m/s.

i. Using $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$, show that $v^2 = 2\left(x^3 - 6x^2 + 16\right)$, where v is the velocity of P.

ii. Calculate the velocity and acceleration of P at x = 2.

iii. In which direction does P move from x = 2? Give a reason for your answer.

iv. Briefly describe the motion of P after it moves from x = 2. ¤

«→ i) Proof ii) v = 0, $\ddot{x} = -12ms^{-2}$ iii) Towards O. When x = 2, the velocity is O and the acceleration is towards O. iv) The particle accelerates to the origin and then slows to a stop. It then moves off in the opposite direction. »

3U93-3c)!

The velocity v ms^{-1} of a particle moving in simple harmonic motion along the x axis is given by $v^2 = 8 + 2x - x^2$.

i. Between which two points is the particle oscillating?

ii. What is the amplitude of the motion?

iii. Find the acceleration of the particle in terms of x.

iv. Find the period of the oscillation. ¤

«→ i) x = -2 and 4 ii) 3 metres iii) $\ddot{x} = (1 - x)ms^{-2}$ iv) 2π seconds »

3U93-6a)!

Let T be the temperature inside a room at time t and let A be the constant outside air temperature. Newton's law of cooling states that the rate of change of the temperature T is proportional to (T - A).

i. Show that $T = A + Ce^{kt}$ (where C and k are constants) satisfies Newton's law of cooling.

ii. The outside air temperature is 5°C and a heating system breakdown causes the inside room temperature to drop from 20°C to 17°C in half an hour. After how many hours is the inside room temperature equal to 10°C? ¤

«→ i) Proof ii) 2 hours and 28 minutes »

3U93-7b)!

A projectile is fired from the origin O with velocity V and with angle of elevation θ, where $\theta \neq \frac{\pi}{2}$.

You may assume that x = Vt cosθ and $y = -\frac{1}{2}gt^2 + Vt\sin\theta$, where x and y are the horizontal and vertical displacements of the projectile in metres from O at time t seconds after firing.

i. Show that the equation of flight of the projectile can be written as

$$y = x\tan\theta - \frac{1}{4h}x^2\left(1+\tan^2\theta\right), \text{ where } \frac{V^2}{2g} = h.$$

ii. Show that the point (X, Y), where X ≠ 0, can be hit by firing at two different angles θ_1 and θ_2 provided $X^2 < 4h(h - Y)$.

iii. Show that no point above the x axis can be hit by firing at two different angles θ_1 and θ_2, satisfying $\theta_1 < \frac{\pi}{4}$ and $\theta_2 < \frac{\pi}{4}$. ¤

«→ Proof »

3U92-2b)!

The displacement x metres of a particle moving in simple harmonic motion is given by x = 3cos πt, where the time t is in seconds.

i. What is the period of oscillation?

ii. What is the speed v of the particle as it moves through the equilibrium position?

iii. Show that the acceleration of the particle is proportional to the displacement from the equilibrium position. ¤

«→ i) 2 seconds ii) 3π ms^{-1} iii) Proof »

3U92-5b)!

In a flock of 1000 chickens, the number P infected with a disease at time t years is given by $P = \dfrac{1000}{1+ce^{-1000t}}$, where c is a constant.

i. Show that, eventually, all the chickens will be infected.

ii. Suppose that when time t = 0, exactly one chicken was infected. After how many days will 500 chickens be infected?

iii. Show that $\dfrac{dP}{dt} = P(1000-P)$. ¤

«→ i) Proof ii) Approximately $2\frac{1}{2}$ days iii) Proof »

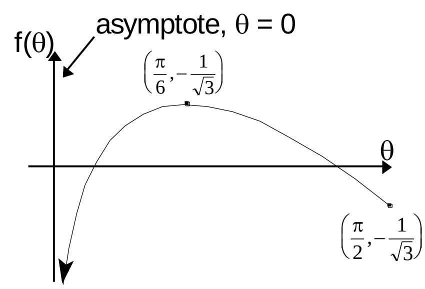
3U92-7a)!

Consider the function y = f(θ), where $f(\theta) = \cos\theta - \dfrac{1}{4\sqrt{3}\sin\theta}$.

i. Verify that $f'(\frac{\pi}{6}) = 0$.

ii. Sketch the curve y = f(θ) for $0 < \theta \le \dfrac{\pi}{2}$ given that $f''(\theta) < 0$. On your sketch, write the coordinates of the turning point in exact form and label the asymptote.¤

«→ i) Proof ii)



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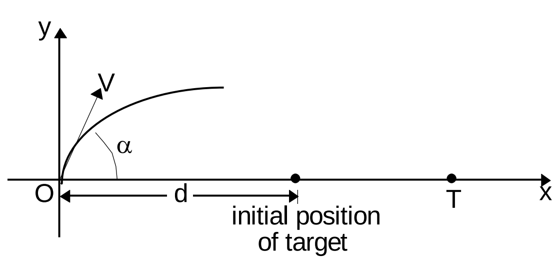
3U92-7b)!



A projectile, of initial speed V m/s, is fired at an angle of elevation α from the origin O towards a target T, which is moving away from O along the x axis. You may assume that the projectile's trajectory is defined by the equations $x = Vt\cos\alpha$ and $y = -\frac{1}{2}gt^2 + Vt\sin\alpha$, where x and y are the horizontal and vertical displacements of the projectile in metres at time t seconds after firing, and where g is the acceleration due to gravity.

i. Show that the projectile is above the x axis for a total of $\dfrac{2V\sin\alpha}{g}$ seconds.

ii. Show that the horizontal range of the projectile is $\dfrac{2V^2\sin\alpha\cos\alpha}{g}$ metres.

iii. At the instant the projectile is fired, the target T is d metres from O and it is moving away at a constant speed of u m/s. Suppose the projectile hits the target when fired at an angle of elevation α. Show that $u = V\cos\alpha - \dfrac{gd}{2V\sin\alpha}$.

In parts (iv) and (v), assume that $gd = \dfrac{V^2}{2\sqrt{3}}$.

iv. By using (iii) and the graph of part (a), show that if $u > \dfrac{V}{\sqrt{3}}$ the target cannot be hit by the projectile, no matter at what angle of elevation α the projectile is fired.

v. Suppose that $u < \dfrac{V}{\sqrt{3}}$. Show that the target can be hit when it is at precisely two distances from O. ¤

«→ Proof »

3U91-4b)!

The acceleration of a particle moving in a straight line is given by $\dfrac{d^2x}{dt^2} = 2x - 3$ where x is the displacement, in metres, from the origin O and t is the time in seconds. Initially the particle is at rest at x = 4.

i. If the velocity of the particle is v m/s, show that $V^2 = 2\left(x^2 - 3x - 4\right)$.

ii. Show that the particle does not pass through the origin.

iii. Determine the position of the particle when v = 10. Justify your answer. ¤

«→ i) Proof ii) Proof iii) x = 9 is the only solution since the particle does not pass through the origin »

3U91-6b)!

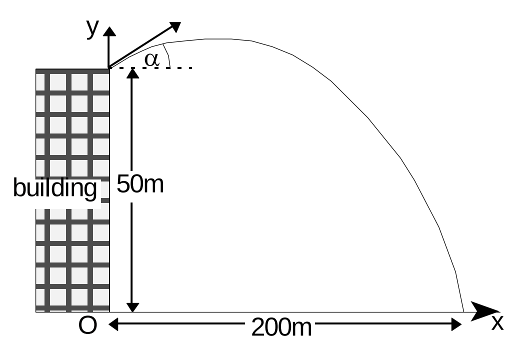


FIGURE NOT TO SCALE.

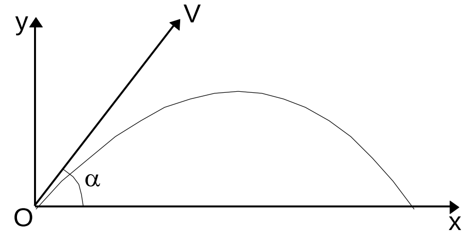
The diagram shows the path of a projectile launched at an angle of elevation α, with an initial speed of 40 m/s, from the top of a 50 metre high building. The acceleration due to gravity is assumed to be 10 m/s².

i. Given that $\dfrac{d^2x}{dt^2} = 0$ and $\dfrac{d^2y}{dt^2} = -10$, show that x = 40t cos α and y = -5t² + 40t sin α + 50, where x and y are the horizontal and vertical displacements of the projectile in metres from O at time t seconds after launching.

ii. The projectile lands on the ground 200 metres from the base of the building. Find the two possible values for α. Give your answers to the nearest degree. ¤

«→ i) Proof ii) α = 31°, 45° »

3U90-2c)!



The path of a projectile fired from the origin O is given by $x = Vt\cos\alpha$, $y = Vt\sin\alpha - 5t^2$ where V is the initial speed in metres per second and α is the angle of projection as in the diagram and t is the time in seconds.

i. Find the maximum height reached by the projectile in terms of V and α.
ii. Find the range in terms of V and α.
iii. Prove that the range is maximum when $\alpha = 45°$. ¤

«→ i) $\dfrac{V^2\sin^2\alpha}{20}$ m ii) $\dfrac{V^2\sin 2\alpha}{10}$ m iii) Proof »

3U90-3b)!

The velocity v ms^{-1} of a particle moving in simple harmonic motion along the x axis is given by $v^2 = -5 + 6x - x^2$, where x is in metres.

i. Between which two points is the particle oscillating?
ii. Find the centre of motion of the particle.
iii. Find the maximum speed of the particle.
iv. Find the acceleration of the particle in terms of x. ¤

«→ i) x = 1 and 5 ii) x = 3 iii) $2ms^{-1}$ iv) $a = (3 - x)ms^{-2}$ »

3U90-6b)!

Assume that the rate at which a body warms in air is proportional to the difference between its temperature T and the constant temperature A of the surrounding air. This rate can be expressed by the differential equation $\dfrac{dT}{dt} = k(T - A)$ where t is the time in minutes and k is a constant.

i. Show that $T = A + Ce^{kt}$, where C is a constant, is a solution of the differential equation.
ii. A cooled body warms from 5°C to 10°C in 20 minutes. The air temperature around the body is 25°C. Find the temperature of the body after a further 40 minutes have elapsed. Give your answer to the nearest degree.
iii. By referring to the equation for T, explain the behaviour of T as t becomes large. ¤

«→ i) Proof ii) 17° iii) T approaches 25° »

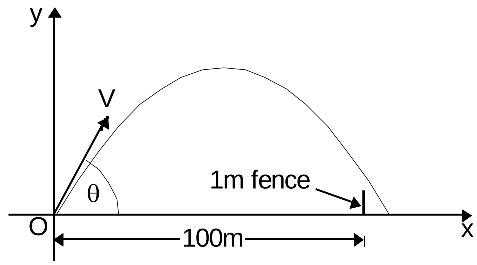
3U89-2c)!

i. Show that $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = \dfrac{dv}{dt}$.
ii. The acceleration of a particle moving in a straight line is given by $\ddot{x} = -2e^{-x}$ where x metres is the displacement from the origin. Initially, the particle is at the origin with velocity 2 ms^{-1}. Prove that $v = 2e^{-\frac{x}{2}}$.
iii. What happens to v as x increases without bound? ¤

«→ i) Proof ii) Proof iii) The velocity of the particle approaches zero »

3U89-7a)!

A "six" is scored in a cricket game when the ball is hit over the boundary fence on the full as in the diagram. A ball is hit from O with velocity $V = 32ms^{-1}$ at an angle θ to the horizontal and towards the 1 metre high boundary fence 100 metres away.



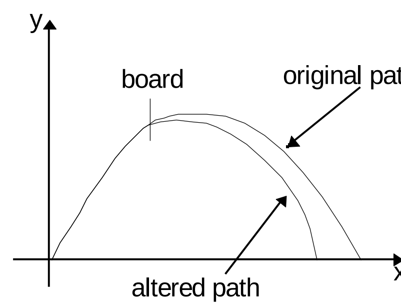
i. Derive the equations of motion for the ball in flight using axes as in the diagram. (Air resistance is to be neglected and the acceleration due to gravity is taken as 10 ms^{-2}.)

ii. Show that the ball just clears the boundary fence when
$50000\tan^2\theta - 102400\tan\theta + 51024 = 0$.

iii. In what range must θ lie for a "six" to be scored?

iv. If, during the flight of the ball, its velocity is reduced by piercing an extremely thin "board", show by a sketch how the path is altered. Without further calculation, discuss qualitatively the effect of air resistance on your answer in (iii). ¤

«→ i) Vertical Motion: $\ddot{y} = -10$, $\dot{y} = -10t + 32\sin\theta$, $y = -5t^2 + 32t\sin\theta$. Horizontal Motion: $\ddot{x} = 0$,



$\dot{x} = 32\cos\theta$, $x = 32t\cos\theta$. ii) Proof iii) $40°35' \le \theta \le 50°$ iv) The altered path is parabolic in shape, however the horizontal velocity of the ball is reduced. Under air resistance, both the horizontal and vertical velocities are reduced. As a result, the time of flight, horizontal distance and vertical height are less than those of the same hit and no air resistance. »

3U88-3a)!

A particle undergoes simple harmonic motion about the origin O. Its displacement x centimetres from O at time t seconds, is given by $x = 3\cos\left(2t + \frac{\pi}{3}\right)$.

i. Express the acceleration as a function of displacement.

ii. Write down the amplitude of the motion.

iii. Find the value of x for which the speed is a maximum and determine this speed. ¤

«→ i) $\ddot{x} = -4x$ ii) 3cm iii) x = 0, max. speed = 6cm/s »

3U87-4ii)!

A pebble is projected from the top of a vertical cliff with velocity 20ms^{-1} at an angle of elevation of 30°. The cliff is 40 metres high and overlooks a lake.

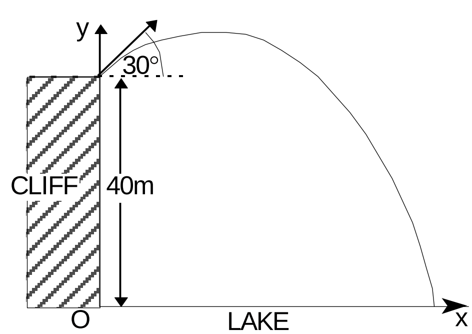


FIGURE NOT TO SCALE

a. Take the origin O to be the point at the base of the cliff immediately below the point of projection. Derive expressions for the horizontal component x(t) and vertical component y(t) of the pebble's displacement from O after t seconds. (Air resistance is to be neglected.)

b. Calculate the time which elapses before the pebble hits the lake and the distance of the point of impact from the foot of the cliff. [Assume the acceleration due to gravity is $10ms^{-2}$.] ¤

«→ a) $x(t) = 10\sqrt{3}t$ metres, $y(t) = -\frac{1}{2}gt^2 + 10t + 40$ metres b) 4 seconds, $40\sqrt{3}$ metres from the origin. »

3U87-6i)!

The rate at which a body cools in air is assumed to be proportional to the difference between its temperature T and the constant temperature S of the surrounding air. This can be expressed by the differential equation $\frac{dT}{dt} = k(T - S)$ where t is the time in hours and k is a constant.

a. Show that $T = S + Be^{kt}$, where B is a constant, is a solution of the differential equation.

b. A heated body cools from 80°C to 40°C in 2 hours. The air temperature S around the body is 20°C. Find the temperature of the body after one further hour has elapsed. Give your answer correct to the nearest degree. ¤

«→ a) Proof b) 32°C »

3U86-5ii)!

A particle is oscillating in simple harmonic motion such that its displacement x metres from a given origin O satisfies the equation $\frac{d^2x}{dt^2} = -4x$, where t is the time in seconds.

a. Show that x = a cos(2t + ß) is a possible equation of motion for this particle, where a and ß are constants.

b. The particle is observed at time t = 0 to have a velocity of 2 metres per second and a displacement from the origin of 4 metres. Show that the amplitude of oscillation is $\sqrt{17}$ metres.

c. Determine the maximum velocity of the particle. ¤

«→ a) Proof b) Proof c) $\pm 2\sqrt{17}\ ms^{-1}$ »

3U85-5i)!

Firefighters are forced to stay 60m away from a dangerous fire burning in a low open tank on horizontal ground. They have two pumps. One, which can eject water in any direction at $30ms^{-1}$, is on the ground, while the other, which can eject water at $40ms^{-1}$ but only horizontally, is on a vertical stand 5m high. Can both pumps reach the fire? Justify your answer. (Assume that $g = 10ms^{-2}$, and that all frictional forces, including air resistance, can be neglected.) ¤

«→ Only the first pump can reach the fire. The first pump has a maximum range of 90m whilst the range of the second pump is fixed at 40m. »

3U85-5ii)!

A scientist found that the amount, Q(t), of a substance present in a mineral at time t ≥ 0 satisfies the equation $4\frac{d^2Q}{dt^2} + 4\frac{dQ}{dt} + Q = 0$.

a. Verify that $Q(t) = A(1 + t)e^{-0.5t}$ satisfies this equation for any constant A > 0.

b. If Q(0) = 10mg, find the maximum value of Q(t) and the time at which this occurs.

c. Describe what happens to Q(t) is t increases indefinitely. ¤

«→ a) Proof b) The maximum value of $Q(t) = 20e^{-0.5}$, at t = 1 c) Q(t) approaches zero. »

3U84-2iii)!

A particle executes simple harmonic motion with period T seconds and amplitude A cm. What is its maximum velocity? ¤

«→ $\frac{2\pi A}{T}$ cm/s »

Inverse Functions and Inverse Trigonometric Functions

3U97-1d)!

Evaluate $\int_0^2 \frac{dx}{4+x^2}$.¤

«→ $\frac{\pi}{8}$ »

3U97-6a)!

The function f(x) = sec x for $0 \le x \le \frac{\pi}{2}$, and is not defined for other values of x.

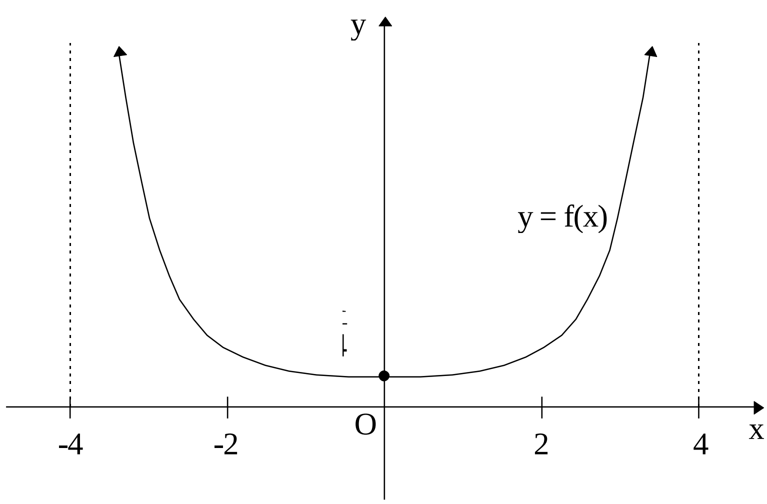
i. State the domain of the inverse function $f^{-1}(x)$.

ii. Show that $f^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$.

iii. Hence find $\frac{d}{dx} f^{-1}(x)$.¤

«→ i) $x \ge 1$ ii) Proof iii) $\frac{1}{x\sqrt{x^2-1}}$ »

3U96-3a)!



Let $f(x) = \frac{1}{\sqrt{16-x^2}}$. The graph of y = f(x) is sketched above.

i. Show that f(x) is an even function.

ii. Find the area enclosed by y = f(x), the x-axis, x = -2 and x = 2.¤

«→ i) Proof ii) $\frac{\pi}{3}$ units^2 »

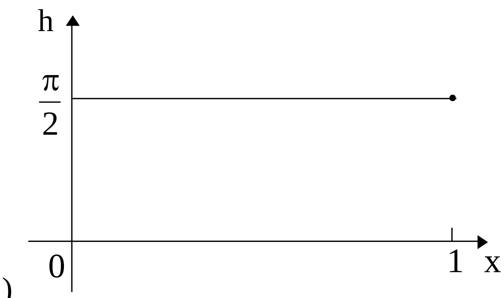
3U96-3d)!

The function h(x) is given by $h(x) = \sin^{-1}x + \cos^{-1}x$, $0 \le x \le 1$.

i. Find $h'(x)$.

ii. Sketch the graph of y = h(x).¤

«→ i) 0 ii)



»

3U96-7b)!

Consider the function $f(x) = \frac{1}{4}[(x-1)^2 + 7]$.

i. Sketch the parabola y = f (x), showing clearly any intercepts with the axes, and the coordinates of its vertex. Use the same scale on both axes.

ii. What is the largest domain containing the value x = 3, for which the function has an inverse function $f^{-1}(x)$?

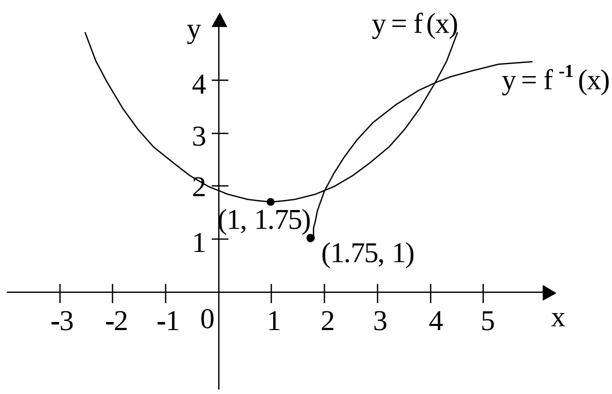
iii. Sketch the graph of $y = f^{-1}(x)$ on the same set of axes as your graph in part (i). Label the two graphs clearly.

iv. What is the domain of the inverse function?

v. Let a be a real number not in the domain found in part (ii). Find $f^{-1}(f(a))$.

vi. Find the coordinates of any points of intersection of the two curves y = f(x) and $y = f^{-1}(x)$.¤

«→ i) iii)



ii) $x \geq 1$ iv) $x \geq 1.75$ v) $f^{-1}(f(a)) = 2 - a$
vi) (2, 2), (4, 4) »

3U95-4)!

Consider the function $f(x) = \frac{e^x}{3+e^x}$

Note that e^x is always positive, and that f(x) is defined for all real x.

a. Show that f(x) has no stationary points.

b. Find the coordinates of the point of inflexion, given that $f''(x) = \frac{3e^x(3-e^x)}{(3+e^x)^3}$

c. Show that 0 < f(x) < 1 for all x.

d. Describe the behaviour of f(x) for very large positive and very large negative values of x, i.e. as x tends to ∞ and x tends to -∞.

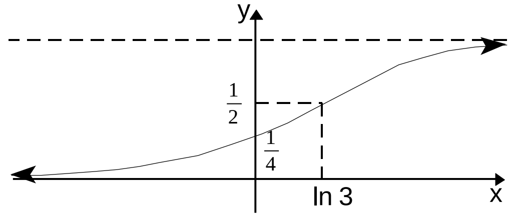
e. Sketch the curve y = f(x).

f. Explain why f(x) has an inverse function.

g. Find the inverse function $y = f^{-1}(x)$. ¤

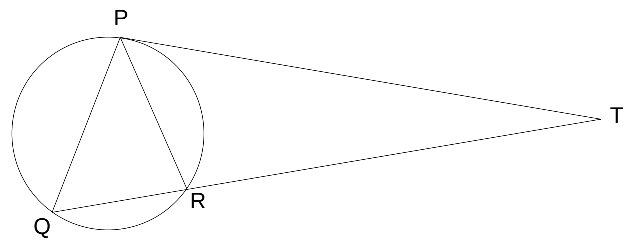
«→ a) Proof b) (ln 3, 0.5 1) c) Proof d) As x tends to ∞, f(x) tends to 1 and as x tends to $-\infty$, f(x) tends to.

e)



f) f(x) is an increasing function and so any horizontal line will cut it in one point only. g) $y = \ln\left(\frac{3x}{1-x}\right)$ »

3U95-6a)!

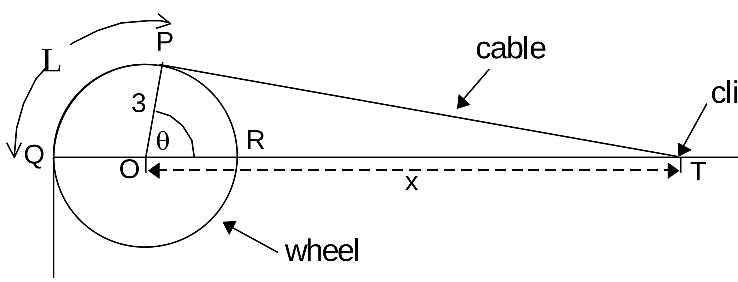


PT is a tangent to the circle PRQ, and QR is a secant intersecting the circle in Q and R. The line QR intersects PT at T.
Copy or trace the diagram into your Writing Booklet.

i. Prove that the triangles PRT and QPT are similar.
ii. Hence prove that $PT^2 = QT \times RT$. ¤

«→ Proof »

3U95-6b)!



A long cable is wrapped over a wheel of radius 3 metres and one end is attached to a clip at T. The centre of the wheel is at O, and QR is a diameter. The point T lies on the line OR at a distance x metres from O.
The cable is tangential to the wheel at P and Q as shown. Let $\angle POR = \theta$ (in radians).
The length of cable in contact with the wheel is L metres; that is, the length of the arc between P and Q is L metres.

i. Explain why $\cos\theta = \frac{3}{x}$.

ii. Show that $L = 3\left[\pi - \cos^{-1}\left(\frac{3}{x}\right)\right]$.

iii. Show that $\frac{dL}{dx} = \frac{-9}{x\sqrt{x^2 - 9}}$.

What is the significance of the fact that $\frac{dL}{dx}$ is negative?

iv. Let s = L + PT.
Using part (a), or otherwise, express s in terms of x.

v. The clip at T is moved away from O along the line OR at a constant speed of 2 metres per second. Find the rate at which s changes when x = 10. ¤

«→ i) PT is a tangent $\therefore \angle OPT = 90°$ and $\cos\theta = \dfrac{OP}{OT}$ ii) Proof iii) Proof. As x increases, the length of the cable decreases. iv) $S = 3\left(\pi - \cos^{-1}\dfrac{3}{x}\right) + \sqrt{x^2 - 9}$ v) $\dfrac{\sqrt{91}}{5}$ m/sec »

3U94-6a)!

Consider the function $f(x) = 3x - x^3$.

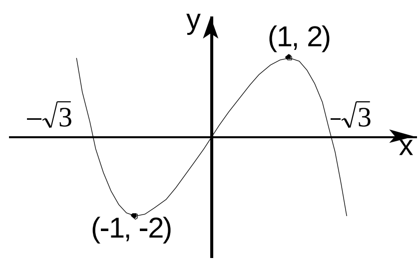
i. Sketch y = f(x), showing the x and y intercepts and the coordinates of the stationary points.

ii. Find the largest domain containing the origin for which f(x) has an inverse function, $f^{-1}(x)$.

iii. State the domain of $f^{-1}(x)$.

iv. Find the gradient of the inverse function at x = 0. ¤

«→ i)



ii) $-1 \le x \le 1$ iii) $-2 \le x \le 2$ iv) 1/3 »

3U93-3a)!

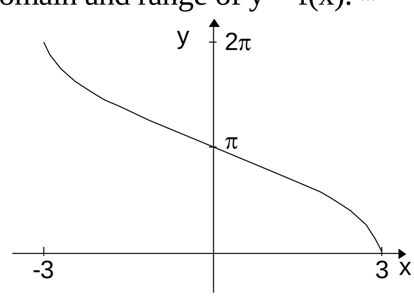
Consider the function $f(x) = 2\cos^{-1}\dfrac{x}{3}$.

i. Evaluate f(0).

ii. Draw the graph of y = f(x).

iii. State the domain and range of y = f(x). ¤

«→ i) π ii)



iii) Domain: $-3 \le x \le 3$, Range: $0 \le f(x) \le 2\pi$ »

3U92-1c)!

Find the exact value of $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4 - x^2}}\,dx$. ¤

«→ π/6 »

3U92-3b)!

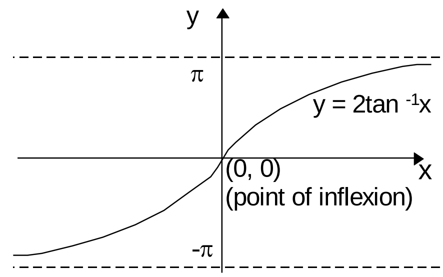
Consider the function $f(x) = 2\tan^{-1} x$.

i. Evaluate $f(\sqrt{3})$.

ii. Draw the graph of y = f(x), labelling any key features.

iii. Find the slope of the curve at the point where it cuts the y axis. ¤

«→ i) $\frac{2\pi}{3}$ ii)



iii) 2 »

3U91-5a)!

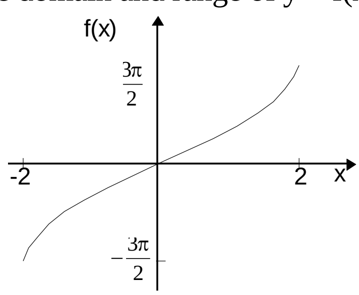
Consider the function $f(x) = 3\sin^{-1}\frac{x}{2}$.

i. Evaluate f(2).

ii. Draw the graph of y = f(x).

iii. State the domain and range of y = f(x). ¤

«→ i) $\frac{3\pi}{2}$ ii)



iii) Domain: $-2 \le x \le 2$ Range: $-\frac{3\pi}{2} \le f(x) \le \frac{3\pi}{2}$ »

3U91-7b)!

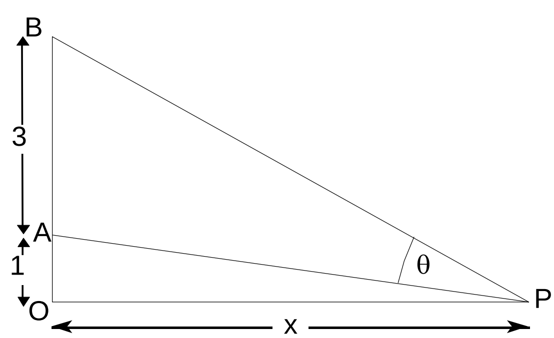


FIGURE NOT TO SCALE.

In the diagram, a vertical pole AB, 3 metres high, is placed on top of a support 1 metre high. The pole subtends an angle of θ radians at the point P, which is x metres from the base O of the support.

i. Show that $\theta = \tan^{-1}\frac{4}{x} - \tan^{-1}\frac{1}{x}$.

ii. Show that θ is a maximum when x = 2.

iii. Deduce that the maximum angle subtended at P is $\theta = \tan^{-1}\frac{3}{4}$. ¤

«→ Proof »

3U90-1a)! Evaluate:

i. $\int_0^1 \frac{1}{1+x^2}\,dx$

ii. $\int_0^1 \frac{x}{\sqrt{1+x}}\,dx$, using the substitution u = 1 + x. ¤

«→ i) π/4 ii) $\frac{2}{3}(2-\sqrt{2})$ »

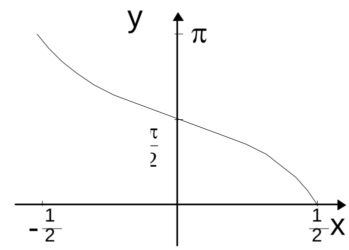
3U90-4c)!

i. State the domain and range of the function given by $y = \cos^{-1}2x$.

ii. Sketch the graph of the function given by $y = \cos^{-1}2x$.

iii. Find the slope of the curve at the point where it cuts the y axis. ¤

«→ i) Domain: $-0.5 \le x \le 0.5$, Range: $0 \le y \le \pi$ ii)



iii) -2 »

3U89-1b)! Evaluate:

i. $\int_1^2 \frac{1}{\sqrt{4-x^2}}\,dx$,

ii. $\int_{-1}^0 x\sqrt{1+x}\,dx$, using the substitution u = 1 + x. ¤

«→ i) π/3 ii) -4/15 »

3U89-6a)!

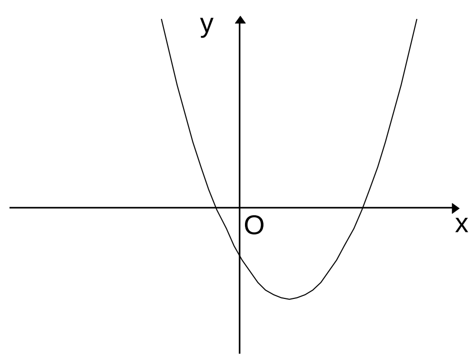


FIGURE NOT TO SCALE

The figure shows a sketch of the curve $y = (x - 1)^2 - 3$.

i. Find the largest positive domain such that the graph defines a function f(x) which has an inverse.

ii. Find this inverse function and state its domain.

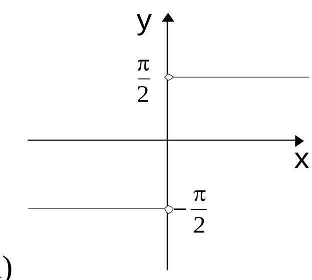
iii. State a domain for which the function does not have an inverse. Give a brief reason for your answer. ¤

«→ i) $x \ge 1$ ii) $y = 1 + \sqrt{x+3}$, Domain: $x \ge -3$ ii) Any domain which includes values of x on both sides of the vertex (1, -3). The corresponding inverse relation would be the parabola $(y - 1)^2 = x + 3$, which is not a function. »

3U89-7b)!

i. Differentiate $y = \tan^{-1}\frac{1}{x}$, $x \neq 0$, and hence show that $\frac{d}{dx}\left(\tan^{-1}x + \tan^{-1}\frac{1}{x}\right) = 0$.

ii. Sketch the curve $y = \tan^{-1} x + \tan^{-1}\frac{1}{x}$. ¤

«→ i) $-\frac{1}{x^2+1}$, Proof ii) 

»

3U88-1a)!

Differentiate:

i. x cos x

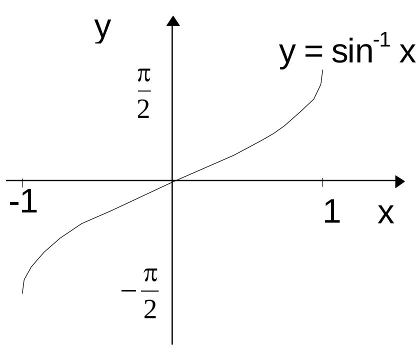
ii. $\tan^{-1}3x$. ¤

«→ i) cos x - x sin x ii) $\frac{3}{1+9x^2}$ »

3U88-2a)!

i. Draw a sketch of $y = \sin^{-1}x$. State the domain and range.

ii. A region R is bounded by the curve $y = \sin^{-1}x$, the x-axis and the line x = 1. Use Simpson's rule with three function values to find an approximation for the area of R. Give your answer correct to 2 decimal places. ¤



«→ i) Domain: $-1 \le x \le 1$, Range: $-0.5\pi \le y \le 0.5\pi$ ii) 0.61 units^2 »

3U87-1ii)!

Write down primitive functions of:

a. $(3x + 2)^{10}$

b. $\frac{5}{2 + x^2}$. ¤

«→ a) $\frac{(3x+2)^{11}}{33} + c$ b) $\frac{5}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$ »

3U87-2iii)!

a. Differentiate $x\sin^{-1}x + \sqrt{1-x^2}$.

b. Hence evaluate $\int_0^1 \sin^{-1} x \, dx$. ¤

«→ a) $\sin^{-1}x$ b) $0.5\pi - 1$ »

3U87-2iv)!

State the domain and range of $y = 2\sin^{-1}(3x)$. ¤

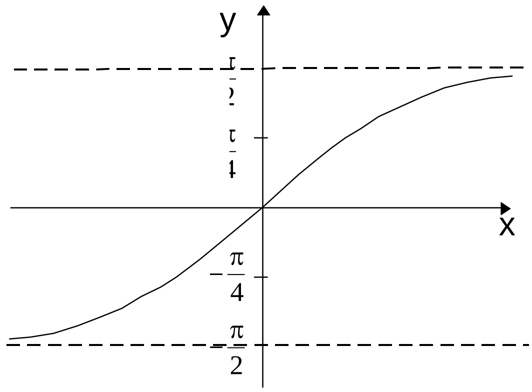
«→ Domain: $-1/3 \le x \le 1/3$, Range $-\pi \le y \le \pi$ »

3U86-6i)!

a. Sketch the graph of the function $y = \tan^{-1}x$ stating clearly the range and domain.

b. Given that $x^2 + 4x + 5 \equiv (x + a)^2 + b^2$, find a, b.

c. Using the result in (b), find $\int \frac{1}{x^2+4x+5}\,dx$. ¤



«→ a) Range: $-0.5\pi < y < 0.5\pi$, Domain: All real x b) a = 2, b = ±1 c) $\tan^{-1}(x + 2) + c$ »

3U85-4i)!

Using the substitution $u = x^4$, or otherwise, show that $\int_0^1 \frac{x^3}{1+x^8}\,dx = \frac{\pi}{16}$. ¤

«→ Proof »

3U84-1i)!

Find the derivative (with respect to x) of

a. $\sin^{-1} 2x$, for $|x| < \frac{1}{2}$,

b. $\frac{e^{2x}}{x^2+3}$. ¤

«→ a) $\frac{2}{\sqrt{1-4x^2}}$ b) $\frac{2e^{2x}(x^2-x+3)}{(x^2+3)^2}$ »

3U84-6i)!

Write down a formula for calculating $\frac{d}{dx}F(u)$ when u is a function of x.

Differentiate (with respect to x) $\left(\tan^{-1}\frac{x}{3}\right)^2$ and hence find the exact value of $\frac{1}{\pi}\int_0^{\sqrt{3}} \frac{\tan^{-1}\frac{x}{3}}{x^2+9}\,dx$. ¤

«→ $\frac{d}{dx}F(u) = \frac{d}{du}F(u) \times \frac{du}{dx}$, $\frac{6\tan^{-1}\left(\frac{x}{3}\right)}{x^2+9}$, $\frac{\pi}{216}$ »

Induction

3U97-5b)!

i. For positive integers n and r, with r < n, show that $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$, where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. Do not use induction.

ii. Use mathematical induction to prove that, for $n \geq 3$, $\sum_{j=3}^{n} \binom{j-1}{2} = \binom{n}{3}$.¤

«→ Proof »

3U94-3c)!

Prove by mathematical induction that $n^3 + 2n$ is divisible by 3, for all positive integers n. ¤

«→ Proof »

3U93-5a)!

For n = 1, 2, 3, . . ., let $S_n = 1^2 + 2^2 + ... + n^2$.

i. Use mathematical induction to prove that, for n = 1, 2, 3, . . ., $S_n = \frac{1}{6}n(n+1)(2n+1)$

ii. By using the result of (i) estimate the least n such that $S_n \geq 10^9$. ¤

«→ i) Proof ii) 1442 »

3U92-4b)!

Let $S_n = 1 \times 2 + 2 \times 3 + . . . + (n - 1) \times n$. Use mathematical induction to prove that, for all integers n with $n \geq 2$, $S_n = \frac{1}{3}(n-1)n(n+1)$. ¤

«→ Proof »

3U91-4a)!

Use mathematical induction to prove that, for all positive integers n, $1 + 2 + 4 + ... + 2^{n-1} = 2^n - 1$. ¤

«→ Proof »

3U90-7a)!

Use mathematical induction to prove that, for every positive integer n, $13 \times 6^n + 2$ is divisible by 5. ¤

«→ Proof »

3U89-5b)!

i. By considering the sum of the terms of an arithmetic series, show that

$$(1+2+\ldots+n)^2 = \frac{1}{4}n^2(n+1)^2.$$

ii. By using the Principle of Mathematical Induction prove that $1^3 + 2^3 + ... + n^3 = (1 + 2 + ... + n)^2$, for all $n \geq 1$. ¤

«→ Proof »

3U88-3b)!

Prove by mathematical induction that for $n \geq 1$, $1^2 + 3^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$. ¤

«→ Proof »

3U86-7i)!

Prove by mathematical induction that

$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + . . . + n \times 2^{n-1} = 1 + (n - 1)2^n$

for all integers $n \geq 1$. ¤

«→ Proof »

3U85-4iii)!

Use the Principle of Mathematical Induction to prove that $5^n + 2(11^n)$ is a multiple of 3 for all positive integers n. ¤

«→ Proof »

3U84-7ii)!

It is given that $A > 0$, $B > 0$ and n is a positive integer.

a. Divide $A^{n+1} - A^nB + B^{n+1} - B^nA$ by $A - B$, and deduce that $A^{n+1} + B^{n+1} \geq A^nB + B^nA$.

b. Using (a), show by mathematical induction that $\left(\frac{A+B}{2}\right)^n \leq \frac{A^n + B^n}{2}$. ¤

«→ a) $A^n - B^n$ b) Proof »

Binomial Theorem

3U97-5b)!

i. For positive integers n and r, with r < n, show that $\binom{n}{r}+\binom{n}{r+1}=\binom{n+1}{r+1}$, where $\binom{n}{r}=\frac{n!}{r!(n-r)!}$. Do not use induction.

ii. Use mathematical induction to prove that, for $n\geq 3$, $\sum_{j=3}^{n}\binom{j-1}{2}=\binom{n}{3}$.¤

«→ Proof »

3U97-7b)!

i. Simplify $n\binom{n-1}{1}+n\binom{n-1}{2}+\ldots+n\binom{n-1}{n-2}$.

ii. Find the smallest positive integer n such that $n\binom{n-1}{1}+n\binom{n-1}{2}+\ldots+n\binom{n-1}{n-2}>20\,000$.¤

«→ i) $2n(2^{n-2}-1)$ ii) 12 »

3U96-7a)!

Using the fact that $(1+x)^4(1+x)^9=(1+x)^{13}$, show that
${}^4C_0{}^9C_4+{}^4C_1{}^9C_3+{}^4C_2{}^9C_2+{}^4C_3{}^9C_1+{}^4C_4{}^9C_0={}^{13}C_4$.¤

«→ Proof »

3U95-3b)!

Find the value of the term that does not depend on x in the expansion of
$\left(x^2+\frac{3}{x}\right)^6$.¤

«→ 1215 »

3U92-6c)!

Consider the binomial expansion $1+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n=(1+x)^n$.

i. Show that $1-\binom{n}{1}+\binom{n}{2}-\ldots+(-1)^n\binom{n}{n}=0$.

ii. Show that $1-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\ldots+(-1)^n\frac{1}{n+1}\binom{n}{n}=\frac{1}{n+1}$.¤

«→ Proof »

3U90-6c)!

i. Show that $x^n(1+x)^n\left(1+\frac{1}{x}\right)^n=(1+x)^{2n}$.

ii. Hence prove that $1+\binom{n}{1}^2+\binom{n}{2}^2+\ldots+\binom{n}{n}^2=\binom{2n}{n}$.¤

«→ Proof »

3U89-3b)!

Find the constant term in the expansion of $\left(x-\frac{1}{2x^3}\right)^{20}$.¤

«→ $-484\frac{1}{2}$ »

3U88-6b)!

Suppose $(7+3x)^{25} = \sum_{k=0}^{25} t_k x^k$.

i. Use the Binomial Theorem to write an expression for t_k, $0 \le k \le 25$.

ii. Show that $\frac{t_{k+1}}{t_k} = \frac{3(25-k)}{7(k+1)}$.

iii. Hence or otherwise find the largest coefficient t_k. You may leave your answer in the form $\binom{25}{k} 7^c 3^d$. ¤

«→ i) $t_k = \binom{25}{k} 7^{25-k} 3^k$ ii) Proof iii) $t_7 = \binom{25}{7} 7^{18} 3^7$ »

3U87-3ii)!

Find the coefficient of x^3 in the expansion of $\left(2x - \frac{1}{x}\right)^{11}$. ¤

«→ 42240 »

3U86-4ii)!

Factorise $a^2 + 3a + 2$ and hence or otherwise find the coefficient of a^4 in $(a^2 + 3a + 2)^6$. ¤

«→ $a^2 + 3a + 2 = (a + 1)(a + 2)$, 9420 »

3U85-6i)!

When $(3 + 2x)^n$ is written out as a polynomial in x, the coefficients of x^5 and x^6 have the same value. Find n. ¤

«→ 14 »

3U85-6ii)!

Prove that $1 + \binom{10}{2} 3^2 + \binom{10}{4} 3^4 + \binom{10}{6} 3^6 + \binom{10}{8} 3^8 + 3^{10} = 2^9(2^{10}+1)$. ¤

«→ Proof »

3U84-7i)!

Assume that, for all real numbers x and all positive integers n, $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$.

Show that

a. $0 = \sum_{r=0}^{n} (-1)^r \binom{n}{r}$, and find simple expressions for

b. $\sum_{r=0}^{n} 2^r \binom{n}{r}$,

c. $\sum_{r=0}^{n} r \binom{n}{r}$. ¤

«→ a) Proof b) 3^n c) $n \times 2^{n-1}$ »

Further Probability

3U97-3c)!

In each game of Sic Bo, three regular, six-sided dice are thrown once.

i. In a single game, what is the probability that all three dice show 2?

ii. What is the probability that exactly two of the dice show 2?

iii. What is the probability that exactly two of the dice show the same number?

iv. A player claims that you expect to see three different numbers on the dice in at least half of the games. Is the player correct? Justify your answer.¤

«→ i) $\frac{1}{216}$ ii) $\frac{5}{72}$ iii) $\frac{5}{12}$ iv) Yes, since the probability of three different numbers on the dice is $\frac{5}{9}$ which is greater than $\frac{1}{2}$. »

3U96-5c)!

Mice are placed in the centre of a maze which has five exits. Each mouse is equally likely to leave the maze through any one of the five exits. Thus, the probability of any given mouse leaving by a particular exit is $\frac{1}{5}$.

Four mice, A, B, C and D, are put into the maze and behave independently.

i. What is the probability that A, B, C and D all come out the same exit?

ii. What is the probability that A, B and C come out the same exit, and D comes out a different exit?

iii. What is the probability that any three of the four mice come out the same exit, and the other comes out a different exit?

iv. What is the probability that no more than two mice come out the same exit?¤

«→ i) $\frac{1}{125}$ ii) $\frac{4}{125}$ iii) $\frac{16}{125}$ iv) $\frac{108}{125}$ »

3U95-5b)!

In a Jackpot Lottery, 1500 numbers are drawn from a barrel containing the 100 000 ticket numbers available.

After all the 1500 prize-winning numbers are drawn, they are returned to the barrel and a jackpot number is drawn. If the jackpot number is the same as one of the 1500 numbers that have already been selected, then the additional jackpot prize is won.

The probability that the jackpot prize is won in a given game is thus

$p = \frac{1500}{100\ 000} = 0.015$.

i. Calculate the probability that the jackpot prize will be won exactly once in 10 independent lottery games.

ii. Calculate the probability that the jackpot prize will be won at least once in 10 independent lottery games.

iii. The jackpot prize is initially \$8000, and it increases by \$8000 each time the prize in NOT won.

Calculate the probability that the jackpot prize will exceed \$200 000 when it is finally won. ¤

«→ i) 0.131 (to 3 d.p.) ii) 0.140 (to 3 d.p.) iii) 0.685 (to 3 d.p.) »

3U94-3a)!

New cars are subjected to a quality check, which 75% pass. Calculate the probability that of the next ten cars checked, more than seven will pass. Leave your answer in unsimplified form. ¤

«→ $\binom{10}{8}(0.25)^2(0.75)^8+\binom{10}{9}(0.25)(0.75)^9+\binom{10}{10}(0.75)^{10}$ »

3U92-1d)!

The probability that any one of the thirty-one days in December is rainy is 0.2. What is the probability that December has exactly ten rainy days? Leave your answer in index form. ¤

«→ $\binom{31}{10}(0.2)^{10}(0.8)^{21}$ »

3U92-6b)!

A total of five players is selected at random from four sporting teams. Each of the teams consists of ten players numbered from 1 to 10.

i. What is the probability that of the five selected players, three are numbered '6' and two are numbered '8'?

ii. What is the probability that the five selected players contain at least four players from the same team? ¤

«→ i) $\frac{1}{27417}$ ii) $\frac{28}{703}$ »

3U91-2b)!

When Mendel crossed a tall strain of pea with a dwarf strain of pea, he found that $\frac{3}{4}$ of the offspring were tall and $\frac{1}{4}$ were dwarf. Suppose five such offspring were selected at random. Find the probability that:

i. all of these offspring were tall

ii. at least three of these offspring were tall.

Leave your answers in index form. ¤

«→ i) $\left(\frac{3}{4}\right)^5$ ii) $34 \times \frac{3^3}{4^5}$ »

3U90-6a)!

Sam sits for a multiple choice examination which has 10 questions, each with four possible answers only one of which is correct. What is the probability that Sam answers exactly six questions correctly by chance alone? ¤

«→ $\frac{8505}{524288}$ »

3U89-3a)!

A committee of 3 is to be elected from a club of 8 members.

i. How many different committees can be formed?

ii. If there are 4 Queenslanders in the club, what is the probability that a randomly selected committee of 3 contains only Queenslanders? ¤

«→ i) 56 ii) $\frac{1}{14}$ »

3U88-5b)!

A meeting room contains a round table surrounded by ten chairs. These chairs are indistinguishable and equally spaced around the table.

i. A committee of ten people includes three teenagers. How many seating arrangements are there in which all three sit together? Give brief reasons for your answer.

ii. Elections are held for the position of Chairperson and Secretary in a second committee of ten people seated around this table. What is the probability that the two people elected are sitting directly opposite each other? Give brief reasons for your answer. ¤

«→ i) 30240 ii) $\frac{1}{9}$ »

3U87-3iii)!

One fifth of all jellybeans are black. A random sample of ten jellybeans is chosen.

a. What is the probability that this sample contains exactly two black jellybeans? Give your answer correct to 3 decimal places.

b. What is the probability that the sample contains fewer than two black jellybeans? Give your answer correct to 3 decimal places.

c. Which is more likely: the sample contains fewer than 2 black jellybeans, or the sample contains more than 2 black jellybeans? Give reasons for your answer. ¤

«→ a) 0.302 b) 0.376 c) The probability that the sample contains more than 2 black jellybeans is 0.322. ∴ It is more likely the sample contains less than 2 black jellybeans. »

3U86-6ii)!

A given school in a certain State has 3 mathematics teachers. The probability in that State that a mathematics teacher is female is 0.4.

a. What is the probability that in the given school there is at least one female mathematics teacher?

b. In the same State the probability that a mathematics teacher (male or female) is a graduate is 0.7. What is the probability that in the given school none of the three mathematics teachers is a female graduate? (Give your answer correct to 3 decimal places.) ¤

«→ a) 0.784 b) 0.373 »

3U85-6iii)!

David plays a game in which the probability that his score is s is

$$\begin{cases} \frac{1}{8}, \text{ for } s = -4, -3, -2, -1; \\ \frac{1}{2}\binom{4}{s}(0.3)^s(0.7)^{4-s}, \text{ for } s = 0, 1, 2, 3, 4; \\ 0, \text{ for all other values of s.} \end{cases}$$

What is

a. his most likely score?

b. the probability (expressed as a decimal correct to three places) that his score is positive?

c. the probability (expressed as a decimal correct to three places) that, after playing the game twice, his total score is -3? ¤

«→ a) 1 b) 0.380 c) 0.113 »

3U84-3ii)!

A box contains ten tennis balls of which four have never been used. For the first game two balls are selected at random and, after play, are returned to the box. For the second game two balls are also selected at random from the box. Find the probability of each of the following events:

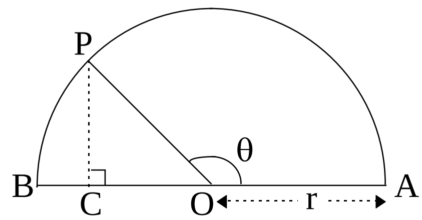
a. precisely one of the balls selected for the first game has been used before;

b. neither ball selected for the first game has been used before, but both balls selected for the second game have been used before the second game. ¤

«→ a) $\frac{8}{15}$ b) $\frac{56}{675}$ »

Iterative Methods for Numerical Estimation of the Roots of a Polynomial Equation

3U97-3b)!



NOT TO SCALE

The point P lies on the circumference of a semicircle of radius r and diameter AB, as shown. The point C lies on AB and PC is perpendicular to AB.

The arc AP subtends an angle θ at the centre O, and the length of the arc AP is twice the length of PC.

i. Show that $2\sin\theta = \theta$.

ii. Taking $\theta = 1.8$ as an approximation for the solution to the equation $2\sin\theta = \theta$ between $\frac{\pi}{2}$ and π, use one application of Newton's method to give a better approximation.¤

«→ i) Proof ii) 1.9 »

3U96-2a)!

The function $f(x) = x^3 - \text{Ln}(x + 1)$ has one root between 0.5 and 1.

i. Show that the root lies between 0.8 and 0.9.

ii. Hence use the halving-the-interval method to find the value of the root, correct to one decimal place.¤

«→ i) Proof ii) 0.9 »

3U95-2a)!

Let $f(x) = x^3 + 5x^2 + 17x - 10$. The equation $f(x) = 0$ has only one real root.

i. Show that the root lies between 0 and 2.

ii. Use one application of the `halving the interval' method to find a smaller interval containing the root.

iii. Which end of the smaller interval found in part (ii) is closer to the root? Briefly justify your answer. ¤

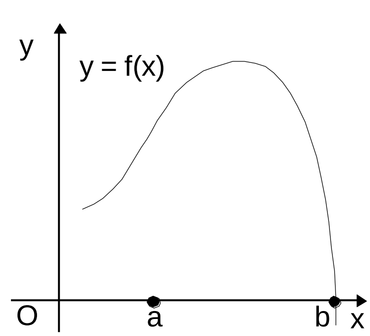
«→ i) Proof ii) The root lies between 0 and 1 iii) $f(0.5) = -0.125$, ∴ the root lies between 0.5 and 1. The root is closer to 1 than 0. »

3U94-4b)!

Taking $x = 1.0$ as the first approximation, use Newton's method to find a second approximation to the root of $x - 3 + e^{2x} = 0$. ¤

«→ 0.7 (to 1 d.p.) »

3U93-5c)!



Consider the above graph of $y = f(x)$. The value a shown on the axis is taken as the first approximation to the solution b of $f(x) = 0$. Is the second approximation obtained by Newton's method a better approximation to b than a is? Give a reason for your answer. ¤

«→ No. There is a stationary point between the first approximation and the solution to $f(x) = 0$. »

3U92-2c)!

Use Newton's method to find a second approximation to the positive root of $x - 2\sin x = 0$. Take $x = 1.7$ as the first approximation. ¤

«→ 1.93 (to 2 d.p.) »

3U91-3a)!

Taking $x = 0.5$ as a first approximation to the root of $x + \ln x = 0$, use Newton's method to find a second approximation. ¤

«→ 0.56 (to 2 d.p.) »

3U90-3c)!

You are given that 3.5 is an approximate root of the equation $x^3 - 50 = 0$. Using one application of Newton's method, find a better approximation. ¤

«→ 3.69 (to 2 d.p.) »

3U87-5)!

The polynomial equation $f(x) = 8x^3 + 12x^2 - 18x - 20 = 0$ has a root at $x = -2$.

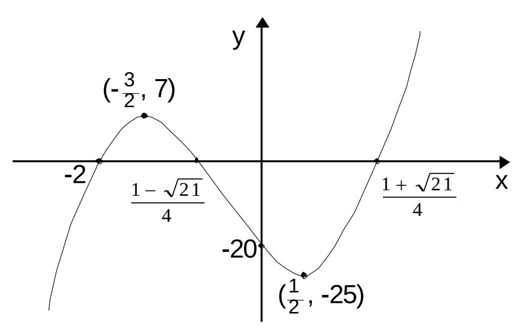
a. Find all roots of $f(x) = 0$.

b. Draw a sketch of the graph of $y = f(x)$ showing the coordinates of its points of intersection with the axes and all stationary points.

c. Apply Newton's method once to approximate a root of $f(x) = 0$ beginning with an initial approximation $x_1 = 1$.

d. Willy chose an initial approximation of $x_1 = 0.49$ and used Newton's method a number of times in order to approximate a root of $f(x) = 0$. State, giving reasons, the root of $f(x) = 0$ to which Willy's approximations are getting closer. (It is not necessary to do additional calculations.) ¤

«→ a) $x = -2, \dfrac{1 \pm \sqrt{21}}{4}$ b)



c) 1.6 d) $x = -2$ »

Harder Applications of HSC 2 Unit Topics

3U97-1a)!

Differentiate $e^{3x}\cos x$.¤

«→ $e^{3x}(3\cos x - \sin x)$ »

3U97-1c)!

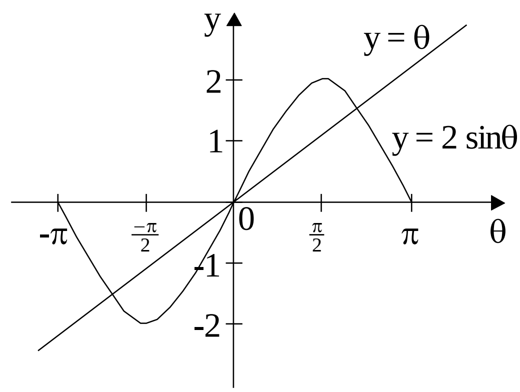
Given that $\log_a b = 2.8$ and $\log_a c = 4.1$, find $\log_a\left(\frac{b}{c}\right)$.¤

«→ -1.3 »

3U97-3a)!

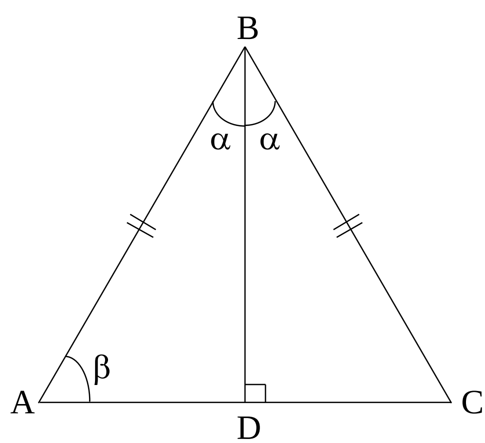
i. On the same set of axes, sketch the graphs of $y = 2\sin\theta$ and $y = \theta$ for $-\pi \le \theta \le \pi$.

ii. Use your sketch to find the number of solutions of the equation $2\sin\theta = \theta$ for $-\pi \le \theta \le \pi$.¤



«→ i) ii) 3 solutions »

3U97-4a)!



The triangle ABC is isosceles, with AB = BC, and BD is perpendicular to AC. Let $\angle ABD = \angle CBD = \alpha$ and $\angle BAD = \beta$, as shown in the diagram.

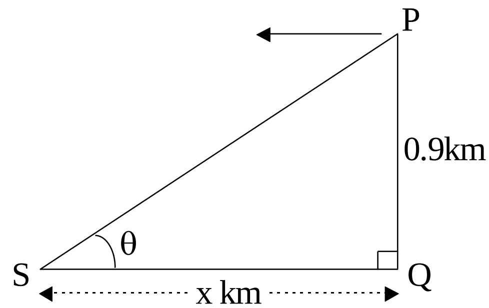
i. Show that $\sin\beta = \cos\alpha$.

ii. By applying the sine rule in ΔABC, show that $\sin 2\alpha = 2\sin\alpha\cos\alpha$.

iii. Given that $0 < \alpha < \frac{\pi}{2}$, show that the limiting sum of the geometric series $\sin 2\alpha + \sin 2\alpha\cos^2\alpha + \sin 2\alpha\cos^4\alpha + \sin 2\alpha\cos^6\alpha + \ldots$ is equal to $2\cot\alpha$.¤

«→ Proof »

3U97-4c)!



A searchlight on the ground at S detects and tracks a plane P that is due east of the searchlight. The plane is flying due west at a constant velocity of 240 kilometres per hour and maintains a constant height of 900 metres above ground level. Let $\theta(t)$ radians be the angle of elevation of the plane at time t seconds and let $x(t)$ kilometres be the distance from S to the point Q on the ground directly below P.

i. Show that $\frac{dx}{d\theta} = -\frac{0.9}{\sin^2\theta}$.

ii. Show that the rate of change of the angle of elevation of the plane when $\theta = \frac{\pi}{4}$ is equal to $\frac{1}{27}$ radians per second.¤

«→ Proof »

3U97-6b)!

An amount \$A is borrowed at r% per annum reducible interest, calculated monthly. The loan is to be repaid in equal monthly instalments of \$M. Let $R = \left(1 + \frac{r}{1200}\right)$ and let $\$B_n$ be the amount owing after n monthly repayments have been made.

i. Show that $B_n = AR^n - M\left(\frac{R^n - 1}{R - 1}\right)$.

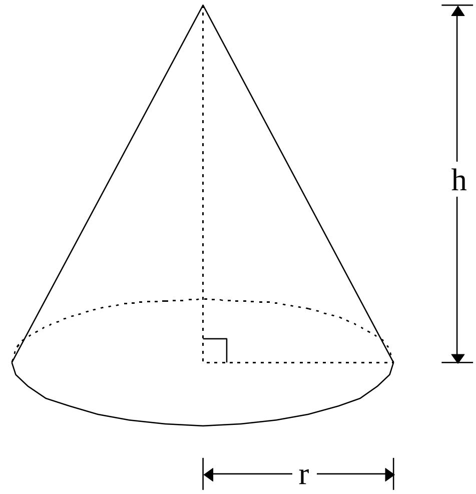
Pat borrows \$30 000 at 9% per annum reducible interest, calculated monthly. The loan is to be repaid in 60 equal monthly instalments.

ii. Show that the monthly repayments should be \$622.75.

iii. With the twelfth repayment, Pat pays an additional \$5000, so this payment is \$5622.75. After this, repayments continue at \$622.75 per month. How many more repayments will be needed?¤

«→ i) ii) Proof iii) 37 instalments »

3U96-4b)!



Grain is poured at a constant rate of 0.5 cubic metres per second. It forms a conical pile, with the angle at the apex of the cone equal to 60°. The height of the pile is h metres, and the radius of the base is r metres.

i. Show that $r = \dfrac{h}{\sqrt{3}}$.

ii. Show that V, the volume of the pile, is given by $V = \dfrac{\pi h^3}{9}$.

iii. Hence find the rate at which the height of the pile is increasing when the height of the pile is 3 metres.¤

«→ i) Proof ii) Proof iii) $\dfrac{1}{6\pi}$ m/s »

3U95-1b)!

Evaluate $\displaystyle\int_1^4 y\,dx$ if xy = 1. ¤

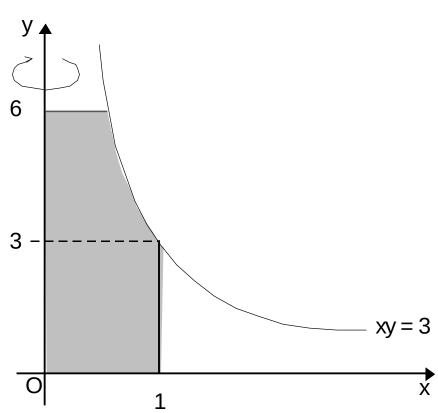
«→ 2ln 2 »

3U95-1c)!

Find $\displaystyle\lim_{x\to 0} \frac{\sin x}{5x}$. ¤

«→ $\dfrac{1}{5}$ »

3U95-2b)!



The shaded area is bounded by the curve xy = 3, the lines x = 1 and y = 6, and the two axes. A solid is formed by rotating the shaded area about the y axis.
Find the volume of this solid by considering separately the regions above and below y = 3. ¤

«→ $\dfrac{9\pi}{2}$ units^3 »

3U95-4)!

Consider the function $f(x) = \dfrac{e^x}{3+e^x}$.

Note that e^x is always positive, and that f(x) is defined for all real x.

a. Show that f(x) has no stationary points.

b. Find the coordinates of the point of inflexion, given that $f''(x) = \dfrac{3e^x(3-e^x)}{(3+e^x)^3}$.

c. Show that $0 < f(x) < 1$ for all x.

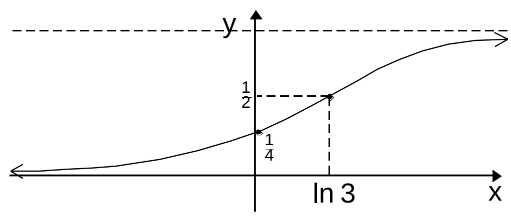
d. Describe the behaviour of f(x) for very large positive and very large negative values of x, i.e. as x approaches ∞ and x approaches $-\infty$.

e. Sketch the curve y = f(x).

f. Explain why f(x) has an inverse function.

g. Find the inverse function $y = f^{-1}(x)$. ¤

«→ a) Proof b) (ln 3, $\frac{1}{2}$) c) Proof d) As x tends to ∞, f(x) tends to 1 and as x tends to $-\infty$, f(x) tends to 0.

e) 

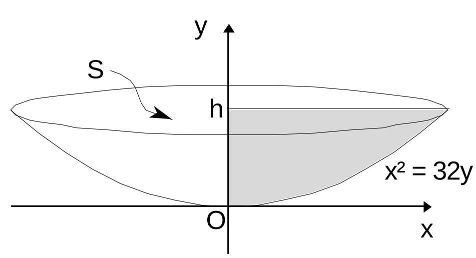
f) f(x) is an increasing function and so any horizontal line will cut it in one point only. g) $y = \ln\left(\dfrac{3x}{1-x}\right)$ »

3U94-1a)!

Using the table of standard integrals, find the exact value of $\int_0^{\frac{\pi}{8}} \sec 2x \tan 2x \, dx$. ¤

«→ $\frac{1}{2}(\sqrt{2}-1)$ »

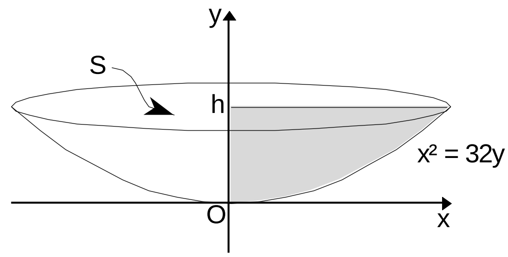
3U94-5b)!



The part of the curve $x^2 = 32y$ between y = 0 and y = h is rotated about the y axis. Show that the volume enclosed is given by $V = 16\pi h^2$. ¤

«→ Proof »

3U94-5c)!



The diagram represents the water in a dam on a farm. The depth of the water is h metres, the volume of water in the dam is V m^3, and the area of the surface of the water is S m^2. The water in the dam evaporates according to the rule $\frac{dV}{dt} = -kS$, where k is a positive constant, and t is the time in hours.

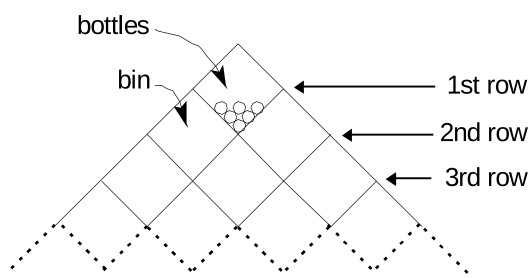
i. Describe in words what the rule says about the rate of evaporation.

ii. Show that $\frac{dh}{dt} = -k$

iii. Initially the dam contains 64π m^3 of water. Calculate how long it will take for the dam to empty by evaporation when k = 0.001. ¤

«→ i) The rate of evaporation is directly proportional to the surface area of the water. ii) Proof iii) 2000 hours »

3U94-6b)!



The figure shows a bottle-storage rack. It consists of n rows of 'bins' stacked in such a way that the number of bins in the r th row is r, counting from the top.

i. Show that the total number of bins in the storage rack is $\frac{n(n+1)}{2}$.

ii. Each bin in the r th row contains c + r bottles, where c is a constant. (For example, each bin in the third row contains c + 3 bottles). Find an expression for the total number of bottles in the storage rack. [You may assume that $1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$).]

iii. Enzo notices that c = 5 and that the average number of bottles per bin in the storage rack is 10. Calculate the number of rows in the storage rack. ¤

«→ i) Proof ii) $\frac{1}{6}n(n + 1)(3c + 2n + 1)$ iii) 7 »

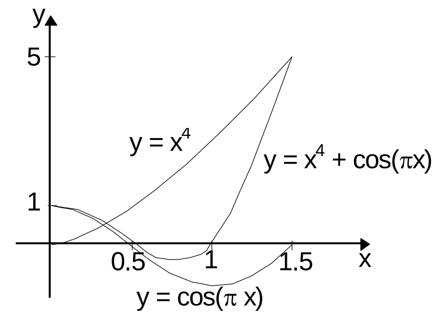
3U94-7a)!

i. Sketch carefully on the same set of axes the graphs of $y = x^4$ and $y = \cos(\pi x)$ for $0 \le x \le 1.5$ (Your diagram should be at least half a page in size.)

ii. On the same diagram, sketch the graph of $y = x^4 + \cos(\pi x)$. Label clearly the three curves on your diagram.

iii. Using the graph, determine the number of positive real roots of the equation $x^4 + \cos(\pi x) = 0$. ¤

«→ i) ii)



iii) 2 »

3U93-1b)!

Find the exact value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sec^2 x\,dx$. ¤

«→ $\dfrac{2}{\sqrt{3}}$ or $\dfrac{2\sqrt{3}}{3}$ »

3U93-1d)!

If y = cos(Ln x) find:

i. $\dfrac{dy}{dx}$

ii. $\dfrac{d^2y}{dx^2}$ ¤

«→ i) $\dfrac{-\sin(\ln x)}{x}$ ii) $\dfrac{\sin(\ln x) - \cos(\ln x)}{x^2}$ »

3U93-2c)!

Suppose the cubic $f(x) = x^3 + ax^2 + bx + c$ has a relative maximum at $x = \alpha$ and a relative minimum at $x = ß$.

i. Prove that $\alpha + ß = -\frac{2}{3}a$.

ii. Deduce that the point of inflexion occurs at $x = \dfrac{\alpha + ß}{2}$. ¤

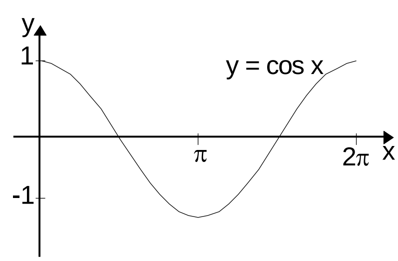
«→ Proof »

3U93-4c)!

i. Sketch the graph of $y = \cos x$ for $0 \le x \le 2\pi$.

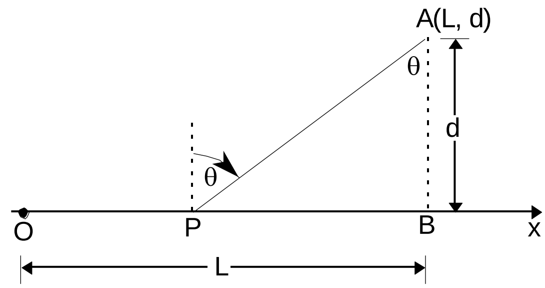
ii. By using (i), or otherwise, find those values of x satisfying $0 \le x \le 2\pi$ for which the geometrical series $1 + 2\cos x + 4\cos^2 x + 8\cos^3 x + \ldots$ has a limiting sum. ¤

«→ i)



ii) $\dfrac{\pi}{3} < x < \dfrac{2\pi}{3}$ or $\dfrac{4\pi}{3} < x < \dfrac{5\pi}{3}$ »

3U93-6b)!



In the diagram, the x axis represents a major blood vessel, whilst the line PA represents a minor blood vessel that joins the major blood vessel at P. The point A has coordinates (L, d) and PA makes an angle $\theta < \frac{\pi}{2}$ with the normal to the x axis at P, as shown in the diagram.

It is known that the resistance to flow in a blood vessel is proportional to its length, where the constant of proportionality depends upon the particular blood vessel.

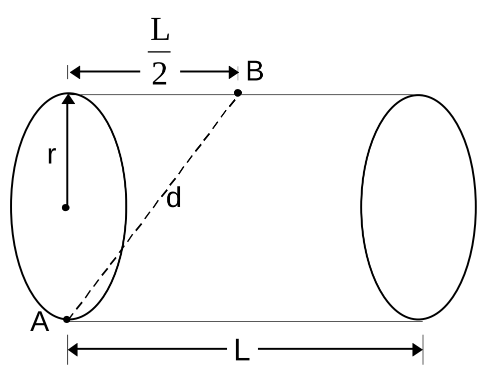
Let R be the sum of the resistances to flow in OP and PA.

i. Show that $R = c_1(L - d\tan\theta) + c_2 d\sec\theta$, where c_1 and c_2 are constants.

ii. The blood vessel PA is joined to the blood vessel Ox in such a way that R is minimized. If $\frac{c_2}{c_1} = 2$ find the angle θ that minimizes R. (You may assume that L is large compared to d.) ¤

«→ i) Proof ii) $\frac{\pi}{6}$ »

3U92-4c)!



The diagram shows a cylindrical barrel of length L and radius r. The point A is at one end of the barrel, at the very bottom of the rim. The point B is at the very top of the barrel, half-way along its length. The length of AB is d.

i. Show that the volume of the barrel is $V = \frac{\pi L}{4}\left(d^2 - \frac{L^2}{4}\right)$.

ii. Find L in terms of d if the barrel has maximum volume for the given d. ¤

«→ i) Proof ii) $L = \frac{2d}{\sqrt{3}}$ »

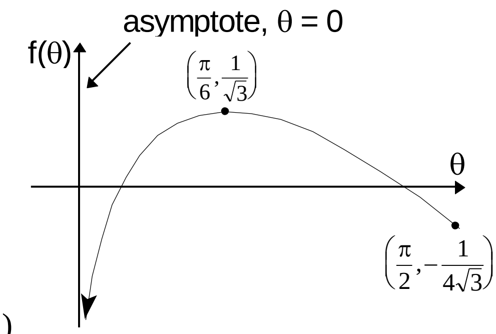
3U92-7a)!

Consider the function y = f(θ), where $f(\theta) = \cos\theta - \frac{1}{4\sqrt{3}\sin\theta}$.

i. Verify that $f'(\frac{\pi}{6}) = 0$.

ii. Sketch the curve $y = f(\theta)$ for $0 < \theta \le \frac{\pi}{2}$ given that $f''(\theta) < 0$. On your sketch, write the coordinates of the turning point in exact form and label the asymptote.¤

«→ i) Proof ii)



»

3U91-2a)!

Consider $y = e^{kx}$ where k is a constant.

i. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

ii. Determine the values of k for which $y = e^{kx}$ satisfies the equation $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0$. ¤

«→ i) $\frac{dy}{dx} = ke^{kx}$, $\frac{d^2y}{dx^2} = k^2e^{kx}$ ii) k = -3, -4 »

3U91-3c)!

If $y = 10^x$, find $\frac{dy}{dx}$ when x = 1. ¤

«→ 10 Ln10 »

3U91-3d)!

The volume, V, of a sphere of radius r mm is increasing at a constant rate of 200 mm^3 per second.

i. Find $\frac{dr}{dt}$ in terms of r.

ii. Determine the rate of increase of the surface area, S, of the sphere when the radius is 50mm. $(V = \frac{4}{3}\pi r^3,\ S = 4\pi r^2)$¤

«→ i) $\frac{dr}{dt} = \frac{50}{\pi r^2}$ ii) 8 mm^2s^{-1} »

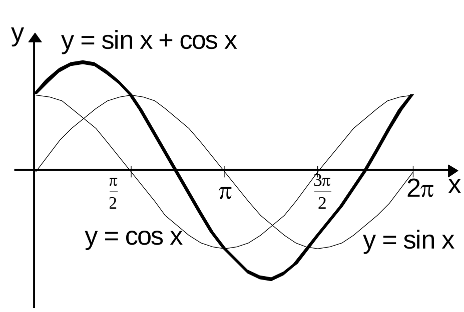
3U91-6a)!

I. On the same axes, sketch the curves y = sin x, y = cos x, and y = sin x + cos x, for $0 \le x \le 2\pi$.

ii. From your graph, determine the number of values of x in the interval $0 \le x \le 2\pi$ for which sin x + cos x = 1.

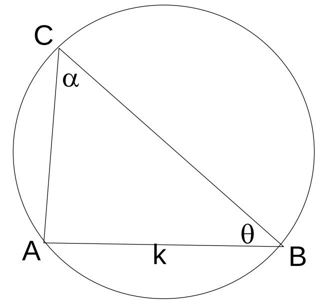
iii. For what values of the constant k does sin x + cos x = k have exactly two solutions in the interval $0 \le x \le 2\pi$? ¤

«→ i)



ii) 3 iii) $-\sqrt{2} < k < \sqrt{2}$, $k \ne 1$ »

3U90-7b)!



Points A, B and C lie on a circle. The length of the chord AB is a constant k. The radian measures of $\angle ABC$ and $\angle BCA$ are θ and α respectively.

i. Let L equal the sum of the lengths of chords CA and CB. Show that L is given by

$$L = \frac{k}{\sin\alpha}[\sin\theta + \sin(\theta+\alpha)].$$

ii. Why is α a constant?

iii. Evaluate $\frac{dL}{d\theta}$ when $\theta = \frac{\pi}{2} - \frac{\alpha}{2}$.

iv. Hence show that L is a maximum when $\theta = \frac{\pi}{2} - \frac{\alpha}{2}$.

¤

«→ i) Proof ii) α is subtended by a chord of fixed length, k. This is independent of the position on the circumference of C. iii) 0 iv) Proof »

3U89-4a)!

A circular plate of radius r is heated so that the area of the plate expands at a constant rate of 3.2 cm² min^{-1}. At what rate does r increase when r = 10cm? ¤

«→ 0.051cm min^{-1} (to 2 significant figures) »

3U88-1a)!

Differentiate:

i. x cos x

ii. $\tan^{-1}3x$. ¤

«→ i) cos x - x sin x ii) $\frac{3}{1+9x^2}$ »

3U88-1c)!

Evaluate:

i. $\int_0^1 \frac{2x}{x^2+1}\,dx$,

ii. $\int_0^{\pi} \sin^2 x\,dx$. ¤

«→ i) $\log_e 2$ ii) $\frac{\pi}{2}$ »

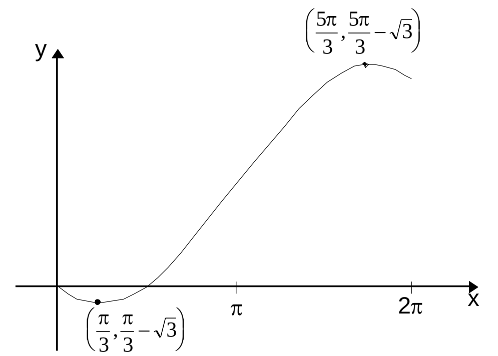
3U88-5a)!

i. Find the stationary points for the curve y = x - 2sin x for $0 \le x \le 2\pi$. Determine whether they are relative maxima or minima.

ii. Find the co-ordinates of those points on the curve corresponding to x = 0, π and 2π.

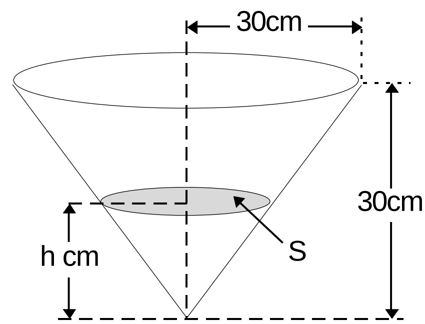
iii. Hence draw a careful sketch of the curve. ¤

«→ i) $(\frac{\pi}{3}, \frac{\pi}{3} - \sqrt{3})$ is a relative minimum turning point, $(\frac{5\pi}{3}, \frac{5\pi}{3} + \sqrt{3})$ is a relative maximum turning point



ii) (0, 0), (π, π), (2π, 2π) iii) »

3U88-7a)!



Water is poured into a conical vessel at a constant rate of 24cm^3 per second. The depth of water is h cm at any time t seconds. What is the rate of increase of the area of the surface S of the liquid when the depth is 16cm? ¤

«→ 3cm^2/s »

3U88-7b)!

A parcel, in the shape of a rectangular prism, has sides x cm, x cm and y cm. The girth is the smallest distance around the parcel. A Courier Company will only deliver parcels for which the longest side L cm and the girth g cm satisfy L + g ≤ 100. Find the dimensions of the parcel of largest volume, for which L + g = 100, that the Courier Company will deliver. ¤

«→ $16\frac{2}{3}$ cm × $16\frac{2}{3}$ cm × $33\frac{1}{3}$ cm »

3U87-1i)!

Differentiate:

a. $\dfrac{1}{3+x^2}$

b. $e^x \log_e(2x)$. ¤

«→ a) $\dfrac{-2x}{(3+x^2)^2}$ b) $e^x\{\dfrac{1}{x} + \ln(2x)\}$ »

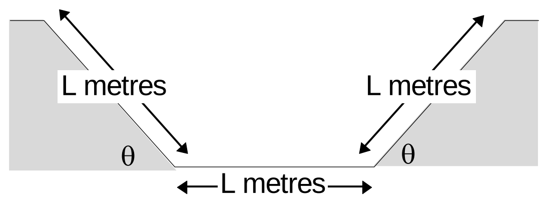
3U86-3i)!

Find the volume of the solid obtained when the region between the curves $y = x^3$ and $y = x^2$, from $x = 0$ to $x = 1$, is rotated about the x axis. ¤

«→ $\dfrac{2\pi}{35}$ units^3 »

3U86-7ii)!

An irrigation channel is to have a cross-section in the shape of a trapezium as in the accompanying figure. The bottom and sides are each L metres long. Suppose that the sides of the channel make an angle $\theta \le \frac{\pi}{2}$ with the horizontal.



a. Find the area of the cross-section of the channel as a function of θ.
b. For what angle θ is the area of the cross-section a maximum? ¤

«→ a) $L^2\sin\theta(1 + \cos\theta)$ m² b) $\frac{\pi}{3}$ »

3U85-1i)!

Find the value of the derivative of $\tan^2 x$ at $x = \frac{\pi}{4}$. ¤

«→ 4 »

3U85-1ii)!

Find $\int_0^1 x(1+x^2)^2\ dx$. ¤

«→ $\frac{7}{6}$ »

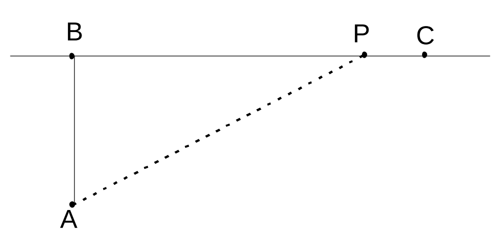
3U85-3ii)!

The function x(t) is given by $x(t) = 4 - 60\sin\left(\frac{t}{15}\right)$. Find

a. M, the maximum value of x(t);
b. the least positive value of t for which x(t) = M;
b. the values of x(t) for which $|\dot{x}(t)| = 2$. ¤

«→ a) 64 b) $\frac{45\pi}{2}$ c) $4 \pm 30\sqrt{3}$ »

3U85-7)!



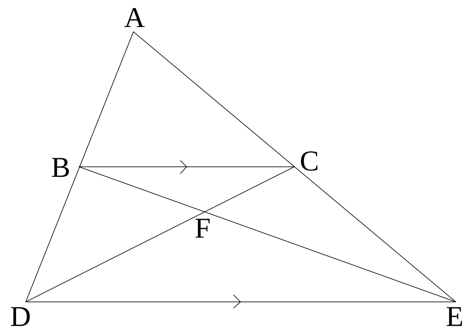
The diagram shows a straight road BC running due East. A four-wheel drive ambulance is in open country at A, 3km due South of B. It must reach C, 9km due East of B, as quickly as possible. The driver knows that she can travel at 80km per hour in open country and at 100km per hour along the road. She intends to proceed in a straight line to some point P on the road and then to continue along the road to C. She wishes to choose P so that the total time for the journey APC is a minimum.

a. If the distance BP is x km, derive an expression for t(x), the total journey time from A to C via P, in terms of x.

b. Show that the minimum time for the total journey APC is $6\frac{3}{4}$ minutes. ¤

«→ a) $t(x) = \frac{\sqrt{9+x^2}}{80} + \frac{9-x}{100}$ b) Proof »

3U84-2ii)!



In the figure, BC ∥ DE and AB:BD = 3:5. Show that

a. ΔABC is similar to ΔADE,
b. ΔBFC is similar to ΔEFD,
c. DF : FC = 8 : 3. ¤

«→ Proof »

3U84-3i)!

The carbon isotope C^{14} decays at a rate proportional to its mass. Tree ring experiments suggest that 50% decay takes 5580 years. A fossil contains 30% of the amount of C^{14} in a similarly sized living organism. Estimate the age of the fossil. ¤

«→ 9690 years (to the nearest 10 years) »

3U84-4i)!

A spherical bubble is expanding so that its volume is increasing at the constant rate of 10 mm^3 per second. What is the rate of increase of the radius when the surface area is 500 mm²?
($V = \frac{4}{3}\pi R^3$, $S = 4\pi R^2$.)¤

«→ $\frac{1}{50}$ mm/s »