SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2004

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK # 3

Mathematics Extension 1

Sample Solutions

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Section A

Question 1

(a) $$\int_0^2 \frac{1}{\sqrt{16-x^2}}\,dx = \int_0^2 \frac{1}{\sqrt{4^2-x^2}}\,dx$$
$$= \left[\sin^{-1}\frac{x}{4}\right]_0^2$$
$$= \sin^{-1}\frac{1}{2}$$
$$= \frac{\pi}{6}$$

(b) (i) $$\lim_{x\to\infty}\frac{\sin 3x}{4x} = \frac{3}{4}\lim_{x\to\infty}\frac{\sin 3x}{3x}$$
$$= \frac{3}{4}\times 1$$
$$= \frac{3}{4}$$

(ii) $$\lim_{x\to\infty}\frac{\sin 3x}{\sin 7x} = \lim_{x\to\infty}\frac{\sin 3x}{3x}\times\frac{7x}{\sin 7x}$$
$$= \frac{3}{7}\lim_{x\to\infty}\frac{\sin 3x}{3x}\times\frac{7x}{\sin 7x}$$
$$= \frac{3}{7}\times 1$$
$$= \frac{3}{7}$$

(c) $$\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$$

Let $u = \ln x$
$$\frac{du}{dx} = \frac{1}{x}$$
$$du = \frac{1}{x}\,dx$$

$$\int \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int \frac{1}{\sqrt{1-(u)^2}}\,du$$
$$= \sin^{-1} u + C$$
$$= \sin^{-1}(\ln x) + C$$

(d) $\log_e(\sin^3 x)$

Let $u = \sin^3 x$
$$\frac{du}{dx} = 3\sin^2 x\cos x$$

Let $y = \log_e u$
$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$$
$$= \frac{1}{u}\times 3\sin^2 x\cos x$$
$$= \frac{1}{\sin^3 x}\times 3\sin^2 x\cos x$$
$$= \frac{3\cos x}{\sin x}$$
$$\therefore \frac{dy}{dx} = 3\cot x$$

(e) $\dfrac{d}{dx}(\tan^{-1}x)^2$

Let $u=\tan^{-1}x$

$$\frac{du}{dx}=\frac{1}{1+x^2}$$

Let $y=u^2$

$$\frac{dy}{du}=2u$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$=2u\times\frac{1}{1+x^2}$$

$$\therefore\frac{dy}{dx}=\frac{2\tan^{-1}x}{1+x^2}$$

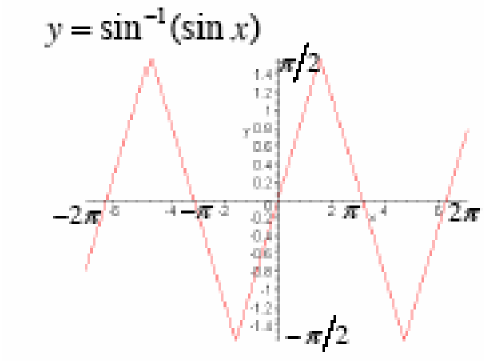
Question 2

(a) (i) $y=\sin^{-1}(\sin x)$

Domain $\{x:x\in\mathbb{R}\}$

Range $\left\{y:-\dfrac{\pi}{2}\le y\le\dfrac{\pi}{2}\right\}$

(ii)



(b) $y=\sin^{-1}\left(\sqrt{x}\right)$

$$\sin y=\sqrt{x}$$

$$\sin^2 y=x$$

$$\therefore x=\sin^2 y$$

$$\frac{dx}{dy}=2\sin y\cos y$$

$$=\sin 2y$$

$$\therefore\frac{dy}{dx}=\frac{1}{\sin 2y}$$

(c) (i) $y = x\tan x - \ln(\sec x)$

Now $\dfrac{d}{dx} x\tan x$

Let $u = x \qquad v = \tan x$

$\dfrac{du}{dx} = 1 \qquad \dfrac{dv}{dx} = \sec^2 x$

$$\therefore \frac{d}{dx}\left(x\tan x\right) = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= (x)(\sec^2 x) + (\tan x)(1)$$
$$= x\sec^2 x + \tan x$$

Now $\dfrac{d}{dx}\ln(\sec x)$

Let $u = \sec x \qquad y = \ln u$

$= (\cos^{-1} x) \qquad \dfrac{dy}{du} = \dfrac{1}{u}$

$$\frac{du}{dx} = -(\cos x)^{-2}(-\sin x)$$
$$= \frac{\sin x}{\cos^2 x}$$
$$= \tan x \sec x$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{u} \times \tan x \sec x$$
$$= \frac{1}{\sec x} \times \tan x \sec x$$
$$= \tan x$$

$$\therefore y = x\tan x - \ln(\sec x)$$
$$\frac{dy}{dx} = x\sec^2 x + \tan x - \tan x$$
$$= x\sec^2 x$$

(c)

(ii) $\int x\sec^2 x\,dx = [x\tan x - \ln(\sec x)]_0^{\frac{\pi}{4}}$

$$= \left\{\frac{\pi}{4}\tan\frac{\pi}{4} - \ln(\sec\frac{\pi}{4})\right\} - \{0\tan 0 - \mathrm{l}$$

$$= \left\{\frac{\pi}{4}(1) - \ln(\sqrt{2})\right\} - \{-\ln(1)\}$$

$$= \frac{\pi}{4} - \ln\sqrt{2}$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

$$= \frac{\pi - 2\ln 2}{4}$$

(d) $y = 10^x$

$$\log_{10} y = \log_{10} 10^x$$

$$\log_{10} y = x\log_{10} 10$$

$$x = \log_{10} y$$

$$x = \frac{\log_e y}{\log_{e10}}$$

$$x = \frac{1}{\log_e 10} \times \log_e y$$

$$x\log_e 10 = \log_e y$$

$$\therefore y = e^{x\log_e 10}$$

$$\therefore \frac{dy}{dx} = \log_e 10 \times e^{x\log_e 10}$$

when $x = 1$

$$\frac{dy}{dx} = \log_e 10 \times e^{(1)\log_e 10}$$

$$= \log_e 10 \times 10$$

$$= 10\log_e 10$$

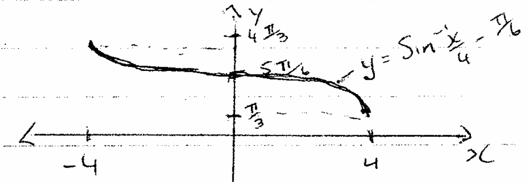
QUESTION 3

(a) Inverse $x = 4\sin(y + \frac{\pi}{6})$ $\frac{\pi}{3} \le y \le \frac{4\pi}{3}$

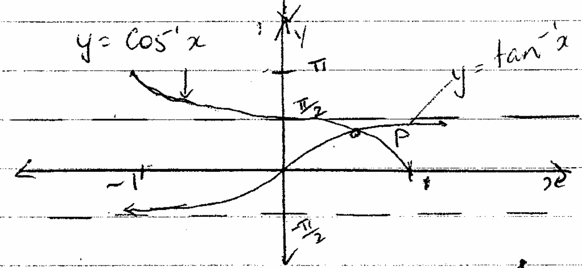
(i) $y + \frac{\pi}{6} = \sin^{-1}\frac{x}{4}$

$y = \sin^{-1}\frac{x}{4} - \frac{\pi}{6}$

Domain $-4 \le x \le 4$

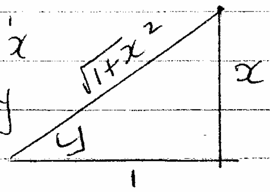


b)(i)



(ii) $\tan^{-1}x = y = \cos^{-1}x$

$\tan y = x = \cos y$



$x = \frac{1}{\sqrt{1+x^2}}$

$x^2(1+x^2) = 1$

$x^4 + x^2 - 1 = 0$

$x^2 = \frac{-1 \pm \sqrt{1+4}}{2}$

$x^2 = 0.618 \quad (x^2 > 0)$

$x = 0.79, \quad y = 0.67 \text{ (2 d.p.)}$

(c) $x = \tan^{-1}(\frac{1}{4})$ $y = \tan^{-1}(\frac{3}{5})$

$\tan x = \frac{1}{4}$ $\tan y = \frac{3}{5}$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

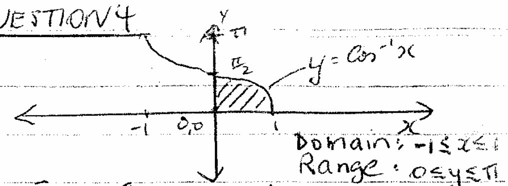
$\tan(x+y) = \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{1}{4}\cdot\frac{3}{5}} = 1$

$x + y = \frac{\pi}{4}$

$\tan^{-1}(\frac{1}{4}) + \tan^{-1}(\frac{3}{5}) = \frac{\pi}{4}$

QUESTION 4

a(i)



$A = \int_0^{\frac{\pi}{2}} \cos y \, dy$

$= [\sin y]_0^{\frac{\pi}{2}}$

$= 1 - 0 = 1 \, u^2$

(b) $y = \log_e 2x - 2\log_e(x-1)$

$= \frac{1}{x} - \frac{2}{x-1}$ or $-\frac{x+1}{x(x-1)}$

(c) $\frac{dT}{dt} = k(T-16)$

(i) $\frac{dt}{dT} = \frac{1}{k(T-16)}$

$t = \frac{1}{k}\log_e(T-16) + C$

$k(t-c) = \log_e(T-16)$

$T - 16 = e^{kt-kc}$

$T - 16 = Pe^{kt}$ ($P = e^{-kc}$ = constant)

$T = 16 + Pe^{kt}$ as required

(ii) $T = 16 + Pe^{kt}$

$T=0, t=0$ $P = -16$

$T = 16 - 16e^{kt}$

$T=12,$ $12 = 16 - 16e^{kt}$

$-4 = -16e^{10k}$

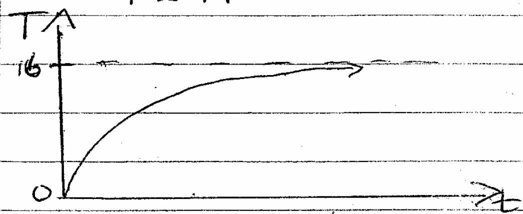
$\frac{1}{4} = e^{10k}$, $k = \frac{1}{10}\log_e\frac{1}{4}$

$k \approx -0.1386$

(iii) $t = 15$ $T = 16 - 16e^{-0.1386\times 15}$

$T = 14°$

(iv)



as $t \to \infty$, $16e^{kt} \to 0$

$T \to 16$

QUESTION 5.

(a). Let $f(x) = \ln x + \sin x$.
$f'(x) = \frac{1}{x} + \cos x$. ($\frac{1}{2}$ for diff)

If $x_1 = 0.5$.
then $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$
$= 0.5 - \frac{\ln 0.5 + \sin 0.5}{\frac{1}{0.5} + \cos 0.5}$ ⇓ ① NB If calculator is in degree mode 0.73 ($\frac{1}{2}$ MARK)
$= 0.5 - -0.07427\ldots$
$= \boxed{0.57}$ (2.D.P.). $\boxed{2}$

(b)(i). $\ddot{x} = 8x(x^2+1)$ and when $t=0$, $x=0$, $v=2$.
$\frac{d}{dx}(\frac{1}{2}v^2) = 8x^3+8x$
$\frac{1}{2}v^2 = 2x^4+4x^2+c$.
now $v=2$ when $x=0$.
$\therefore \frac{1}{2}2^2 = 0+0+c$
$c = 2$.
$\therefore \frac{1}{2}v^2 = 2x^4+4x^2+2$
$v^2 = 4x^4+8x^2+4$
$v^2 = 4(x^4+2x^2+1)$
$= 4(x^2+1)^2$
$v = \pm 2(x^2+1)$ (now $v=2$ when $x=0$. $\therefore v \neq -2(x^2+1)$)
$\therefore \boxed{v = 2(x^2+1)}$ $\boxed{2}$

must show this.

(ii) $\frac{dx}{dt} = 2(x^2+1)$
$\frac{dt}{dx} = \frac{1}{2(x^2+1)}$
$t = \frac{1}{2}\tan^{-1}x + c$

now $t=0$, when $x=0$.
$\therefore 0 = \frac{1}{2}\tan^{-1}0 + c$
$c = 0$. $\boxed{2}$
$\therefore t = \frac{1}{2}\tan^{-1}x$
$\therefore 2t = \tan^{-1}x \Rightarrow \boxed{x = \tan 2t}$

(iii) If $x = \tan 2t$

$v = 2\sec^2 2t$ $\downarrow$ $\boxed{1}$

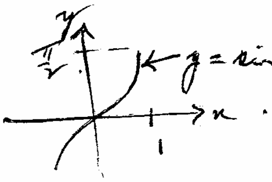
At $t = \frac{\pi}{8}$

$v = 2\times \sec^2\frac{\pi}{4}$

$= 2\times(\sqrt{2})^2$

$= \boxed{4\text{ m s}^{-1}}$ $\boxed{2}$

(c)



$V = \pi\int_0^1 [\sin^{-1}x]^2\,dx$

$\doteqdot \pi\times\frac{\frac{1}{2}}{3}\left[(\sin^{-1}0)^2+4\left[\sin^{-1}\frac{1}{2}\right]^2+(\sin^{-1}1)^2\right]$

$=\frac{\pi}{6}\left[0^2+4\times\left(\frac{\pi}{6}\right)^2+\left(\frac{\pi}{2}\right)^2\right]$

$=\frac{\pi}{6}\left[0+\frac{\pi^2}{9}+\frac{\pi^2}{4}\right]$

$=\frac{\pi}{6}\times\frac{13\pi^2}{36}$

$=\frac{13\pi^3}{216}$

$\doteqdot \boxed{1.87}\,u^3$ (2 DP) $\boxed{3}$

QUESTION 6

(a) (i) $v^2 = 28 + 24x - 4x^2$

$= 4(7 + 6x - x^2)$

$= 4(7 - x)(1 + x).$

Clearly $v^2 \geq 0$

$\therefore 4(7-x)(1+x) \geq 0$

$\therefore \boxed{-1 \leq x \leq 7}$ $\boxed{1}$

(ii) Amplitude $= \frac{7 - -1}{2} = \boxed{4}$ $\boxed{1}$

(iii) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}(14 + 12x - 2x^2)$

$= 12 - 4x$ $\boxed{1}$

$= \boxed{-4(x-3)}$ (NB $n^2 = 4$, $n = 2$ and centre of motion is $x = 3$)

(iv) $T = \frac{2\pi}{n} = \frac{2\pi}{2}$

$= \boxed{\pi \text{ secs}}$ $\boxed{1}$

(v) $x = 3 + 4\cos(2t + \varepsilon)$

if $x = 7$ when $t = 0$.

$7 = 3 + 4\cos\varepsilon$.

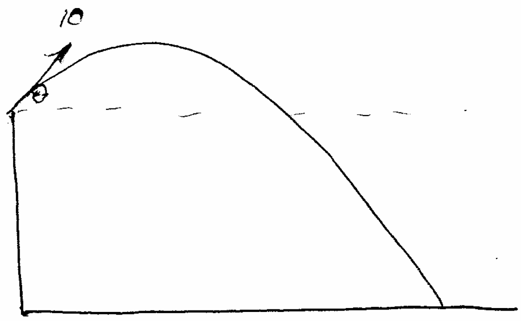
$4 = 4\cos\varepsilon$

$\cos\varepsilon = 1$

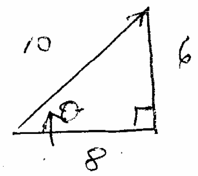
$\varepsilon = 0$ $\boxed{2}$

$\therefore \boxed{x = 3 + 4\cos 2t}$

(b).



$t=0, \quad x=0, \ y=8$



NB $\quad \theta = \tan^{-1}\left(\frac{6}{8}\right)$

(i)

$\ddot{x} = 0$

$\dot{x} = 8$

$x = 8t + c_1$

when $t=0, x=0 \therefore c_1 = 0$

$\boxed{x = 8t}$

$\ddot{y} = -g$

$\dot{y} = -gt + c_2$

Clearly $\dot{y} = 6$ when $t=0$

$\therefore c_2 = 6$

$\dot{y} = -gt + 6$

$\therefore y = -\frac{gt^2}{2} + 6t + c_3$

when $t=0, \ y = 8 \ \therefore c_3 = 8$

$\therefore \boxed{y = -\frac{1}{2}gt^2 + 6t + 8}$

$\boxed{3}$

(ii) If $y = 0$

$-5t^2 + 6t + 8 = 0 \Rightarrow -(5t^2 - 6t - 8) = 0$

$-(5t+4)(t-2) = 0$

$t = 2, -\frac{4}{5}$

$\therefore$ 2 secs have elapsed.

$\boxed{1}$

and $\boxed{x = 16}$ $\quad \boxed{1}$