ST IGNATIUS COLLEGE RIVERVIEW

TASK 4

YEAR 12

2004

EXTENSION 2

TRIAL HSC EXAMINATION

Time allowed: 3 hours + 5 minutes reading time.

Instructions to Candidates

- Attempt **all** questions
- Show all necessary working.
- Marks may be deducted for missing or poorly arranged work.
- Board approved calculators may be used.
- Each question attempted must be returned in a ***separate*** writing booklet clearly marked Question 1, Question 2 etc, on the cover
- **Each booklet must have your name and the name of your mathematics teacher written on the cover.**

Question 1 (15 marks) Use a SEPARATE writing booklet. Marks

a If $Z_1 = 1+2i$, $Z_2 = 2-i$ and $Z_3 = 1-\sqrt{3}\,i$,

Express in the form $(a+bi)$ where a and b are real.

(i) $Z_1 + Z_2$ 1

(ii) $\dfrac{1}{Z_2}$ 1

(iii) $(Z_3)^3$ 2

b Express $\dfrac{4+3i}{3+i}$ in the form $(a+bi)$ where a and b are real numbers. 2

c (i) Express $Z = \sqrt{3}+i$ in modulus- argument form. 1

(ii) Hence, show that $Z^7 + 64Z = 0$. 3

d (i) Find the square root(s) of $(-8+6i)$. 3

(ii) Hence, solve the equation 2

$2Z^2 - (3+i)Z + 2 = 0$, expressing Z in the form $(a+bi)$ where a and b are real.

Question 2 (15 marks) Use a SEPARATE writing booklet. Marks

a Evaluate

(i) $\int_0^{\frac{\pi}{4}} x\sin 2x\, dx$. 3

(ii) $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$. 2

(iii) $\int_0^{\frac{\pi}{3}} \frac{\tan x}{1+\cos x}\, dx$. (using $t = \tan\frac{x}{2}$). 4

b Show that, if $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx$. 6

Then $I_n + I_{n-2} = \frac{1}{n-1}$, where n is an integer and $n \geq 3$

Hence evaluate I_7.

Question 3 (15 marks) Use a SEPARATE writing booklet. Marks

a The point A $(a\cos\alpha, b\sin\alpha)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

B is the foot of the perpendicular from A to the x-axis. The normal at A cuts the x-axis at C.

(i) Represent this information with a suitable diagram. 1

(ii) Derive the equation of the normal AC. 3

(iii) Show that the length of CB is $\left|\frac{b^2\cos\alpha}{a}\right|$. 3

b Consider the hyperbola H with equation $4x^2-9y^2=36$. The point $R(x_1, y_1)$ is an arbitrary point on H.

(i) Prove that the equation of the tangent l at R is $4x_1x-9y_1y=36$. 3

(ii) Find the co-ordinates of the point K at which l cuts the x-axis. 1

(iii) Hence, prove that $\frac{SR}{PR}=\frac{SK}{PK}$ where S and P are the foci of H. 4

Question 4 (15 marks) Use a SEPARATE writing booklet. Marks

a The equation $x^3 - 3x + 3 = 0$ has roots which are α, β and γ. Find the equation in x where the roots are α^2, β^2 and γ^2. 4

b The base of a solid is a circle of radius 2 units. A diameter runs through the centre of the base. Any cross section of the solid formed by a plane perpendicular to the given diameter is an equilateral triangle. 5

Show that the volume of the solid is $\frac{32\sqrt{3}}{3}$ units^3.

c The region bounded by the curve $y = \log_e x$, the straight lines $y = 1$ and $x = 3$ is rotated about the y-axis. Find the volume of the resulting solid using the method of cylindrical shells. 6

Question 5 (15 marks) Use a SEPARATE writing booklet. Marks

a Find the four fourth roots of -16 in the form $(a + bi)$. 4

b A function is defined by $f(x) = \dfrac{\log_e x}{x}$ for $x > 0$.

(i) Find the x intercept. 1

(ii) Find the turning point. 2

(iii) Find the point of inflection. 2

(iv) Sketch the graph of $y = f(x)$. 2

c Consider the function in part (b) sketch

i $y = |f(x)|$. 2

ii $y = \dfrac{1}{f(x)}$. 2

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

a Consider the polynomial $Q(x) = ax^4 + bx^3 + cx^2 + dx + e$ where a, b, c, d and e are integers. Suppose α is an integer such that $Q(\alpha) = 0$.

(i) Prove that α is a factor of e. 2

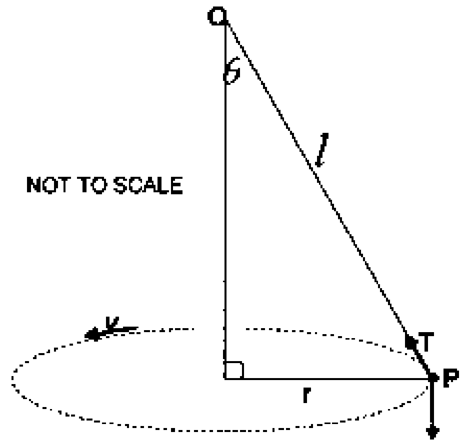
(ii) Prove that the polynomial equation $P(x) = 0$, 2
where $P(x) = 4x^4 - x^3 + 3x^2 + 2x - 3$ does not have an integer root.

b It is estimated that the probability that a torpedo will hit its target is $\frac{1}{3}$.

(i) If 5 torpedoes are fired, what is the probability of 3 successes. 2

(ii) How many torpedoes must be fired so that the probability of at least one success should be greater than 0.9? 2

c



The above diagram shows a light string of length l, fixed at O, and making an angle θ with the vertical as shown in the above diagram. A particle is attached at P. The particle moves with uniform speed v metres / second in a horizontal circle of radius r. The centre of the circle is directly below O.

If the particle is to maintain its motion in a horizontal circle, show by resolving forces vertically and horizontally, that the particle's velocity is given by $v = \sqrt{rg \tan\theta}$. (Note: g is the acceleration due to gravity) 4

d When a polynomial $P(x)$ is divided by $(x-3)$ the remainder is 5 and when it is divided by $(x-4)$ the remainder is 9. Find the remainder when $P(x)$ is divided by $(x-4)(x-3)$. 3

Question 7 (15 marks) Use a SEPARATE writing booklet. Marks

a If $\sin^{-1} x$, $\cos^{-1} x$ and $\sin^{-1}(1-x)$ are acute, show that 6

$\sin\left(\sin^{-1} x - \cos^{-1} x\right) = 2x^2 - 1$.

Hence, solve the equation

$\sin^{-1} x - \cos^{-1} x = \sin^{-1}(1-x)$.

b Find the general solution of the equation $3\tan^2 x = 2\sin x$. 5

c Each of the following statements is either true or false. Write 'True' or 'False' for each statement giving a brief reason for your answers. (You are not required to evaluate the integrals).

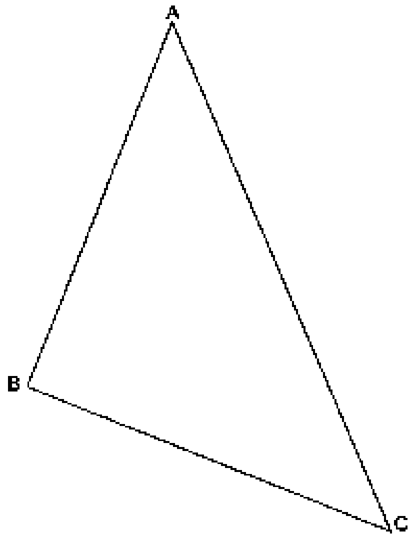
(i) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x \, dx = 0$. 2

(ii) $\int_{-1}^{1} e^{-x^2} \cos^{-1} x \, dx = 0$. 2

Question 8 (15 marks) Use a SEPARATE writing booklet. Marks

a In the Argand diagram, the points A, B and C represent the complex numbers Z_1, Z_2 and Z_3 respectively.

What can you say about triangle ABC if $i(Z_3 - Z_2) = (Z_1 - Z_2)$. 2



b Solve for x if $|3x+3| + |x-1| \le 4x+3$. 5

c A particle, projected vertically upward with initial speed u is subjected to forces which create a constant vertical downward acceleration of magnitude g and an acceleration, directed against the motion, of magnitude kv when the speed is v. 8

(i) Show that the acceleration function is given by $\ddot{x} = -g - kv$.

(ii) Prove that the maximum height reached by the particle after a time T is given by $T = \frac{1}{k}\log_e\left(\frac{g+ku}{g}\right)$.

(iii) Prove that the maximum height reached is $\frac{1}{k}(u - gT)$.