

JAMES RUSSELL MHS

TRIM 2001

EX 1

Question 1:

(a)(i) Find the derivative of $x^2 \cos x$.

2

(ii) Evaluate $\int_1^6 \frac{x}{x^2 + 4} dx$.

2

(b)(i) Sketch $y = |x + 1|$.

2

(ii) Hence or otherwise solve $|x + 1| = 3x$.

1

(c) If $f(x) = 2 \sin^{-1}(3x)$, find

(i) the domain and range of $f(x)$,

2

(ii) $f\left(\frac{1}{6}\right)$,

1

(iii) $f'\left(\frac{1}{6}\right)$.

2

QUESTION 2:

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(a) P(-7,3), Q(9,15) and B(4,0) are three points and A divides the interval PQ in the ratio 3:1. Prove that PQ is perpendicular to AB.

3

(b) By using the substitution $u^2 = x + 1$ evaluate $\int_0^3 \frac{x+2}{\sqrt{x+1}} dx$.

3

(c) Water flows from a hole in the base of a cylindrical vessel at a rate given by

$$\frac{dh}{dt} = -k\sqrt{h}$$

6

where k is a constant and h mm is the depth of water at time t minutes. If the depth of water falls from 250mm to 90mm in 5 minutes, find how much longer it will take to empty the vessel.

QUESTION 3:

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(a) Find the value of the constant term in the expansion of $\left(3x + \frac{2}{\sqrt{x}}\right)^6$.

(b) Three boys (Adam, Bruce, Chris) and three girls (Debra, Emma, Fay) form a single queue at random in front of the school canteen window. Find the probability that:

(i) the first two to be served are Emma and Adam in that order,

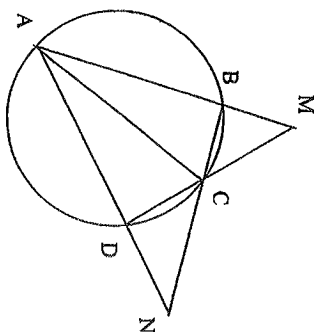
(ii) a boy is at each end of the queue,

(iii) no two girls stand next to each other.

(c) In the figure ABM , DCM , BCN and ADN are straight lines and $\angle AMD = \angle BNA$.

(i) Copy the diagram onto your answer sheet and prove that $\angle ABC = \angle ADC$.

(ii) Hence prove that AC is a diameter.



QUESTION 4:

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(a)(i) Given that $\sin^2 A + \cos^2 A = 1$, prove that $\tan^2 A = \sec^2 A - 1$.

(ii) Sketch the curve $y = 4 \tan^{-1} x$ clearly showing its range.

(iii) Find the volume of the solid formed when the area bounded by the curve $y = 4 \tan^{-1} x$, the y-axis and the line $y = \pi$ is rotated one revolution about the y-axis.

(b)(i) An object has velocity $v \text{ ms}^{-1}$ and acceleration $\ddot{x} \text{ ms}^{-2}$ at position $x \text{ m}$ from the origin, show that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$.

(ii) The acceleration (in ms^{-2}) of an object is given by $\ddot{x} = 2x^3 + 4x$.

(a) If the object is initially 2 m to the right of the origin traveling with velocity 6 ms^{-1} , find an expression for v^2 (the square of its velocity) in terms of x .

(b) What is the minimum speed of the object? (Give a reason for your answer)

QUESTION 5:

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- (a) The curves $y = e^{-2x}$ and $y = 3x + 1$ meet on the y-axis. Find the size of the acute angle between these curves at the point where they meet. 3
- (b)(i) Sketch the function $y = f(x)$ where $f(x) = (x-1)^2 - 4$ clearly showing all intercepts with the co-ordinate axes. (Use the same scale on both axes) 2
- (ii) What is the largest positive domain of f for which $f(x)$ has an inverse $f^{-1}(x)$? 1
- (iii) Sketch the graph of $y = f^{-1}(x)$ on the same axes as (i). 1
- (c) In tennis a player is allowed a maximum of two serves when attempting to win a point. If the first serve is not legal it is called a fault and the server is allowed a second serve. If the second serve is also illegal then it is called a double fault and the server loses the point. The probability that Pat Smash's first serve will be legal is 0.4. If Pat Smash needs to make a second serve then the probability that it will be legal is 0.7.
- (i) Find the probability that Pat Smash will serve a double fault when trying to win a point. 2
- (ii) If Pat Smash attempts to win six points, what is the probability that he will serve at least two double faults? (Give answer correct to 2 decimal places) 3

QUESTION 6:

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- (a) A spherical bubble is expanding so that its volume is increasing at $10 \text{ cm}^3 \text{ s}^{-1}$. Find the rate of increase of its radius when the surface area is 500 cm^2 . 3
(Volume = $\frac{4}{3}\pi r^3$, Surface area = $4\pi r^2$)
- (b) Prove by Mathematical Induction that:
 $2(1!) + 5(2!) + 10(3!) + \dots + (n^2 + 1)n! = n(n+1)!$ for positive integers $n \geq 1$. 4

- (c) If $y = \frac{\log_e x}{x}$ find $\frac{dy}{dx}$ and hence or otherwise show that
 $\int_e^x \frac{1 - \log_e x}{x \log_e x} dx = \log_e 2 - 1$. 5

QUESTION 7:

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- (i) By considering the expansion of $\sin(X+Y) - \sin(X-Y)$ prove that
 $\sin A - \sin B = 2 \cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right)$.
- (ii) Also given that $\cos A - \cos B = 2 \sin \left(\frac{A+B}{2} \right) \sin \left(\frac{B-A}{2} \right)$ prove that
 $\frac{\sin A - \sin B}{\cos A - \cos B} = -\cot \left(\frac{A+B}{2} \right)$.
- (iii) Prove that the position of a projectile t seconds after projection from ground level with initial horizontal and vertical velocity components of $V \cos \alpha$ and $V \sin \alpha$ respectively is given by $x = Vt \cos \alpha$ and $y = -\frac{1}{2}gt^2 + Vt \sin \alpha$. (Assume that there is no air resistance)
- (iv) Two objects P and Q are projected from the same ground position at the same time with initial speed $V \text{ ms}^{-1}$ at angles α and β respectively ($\beta > \alpha$).
- (a) If at time t seconds the line joining P and Q makes an acute angle θ with the horizontal prove that $\tan \theta = \left| \frac{\sin \beta - \sin \alpha}{\cos \beta - \cos \alpha} \right|$.
- (b) Hence show for $t > 0$, that $\theta = \frac{1}{2}(\pi - \alpha - \beta)$.

THIS IS THE END OF THE EXAMINATION PAPER