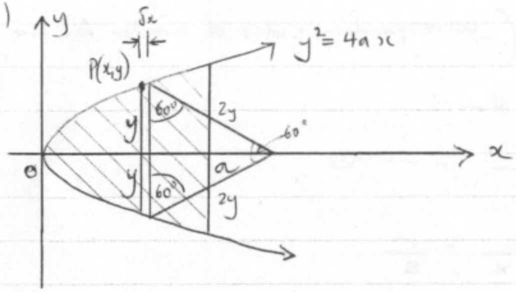
SOLUTIONS TO: YEAR 12 – Term 2 – ME 2 – 2008

Qu. 1 (a) (i)



Area of equilateral $\Delta = \frac{1}{2} \times 2y \times 2y \times \sin 60$

$= 2y^2 \times \frac{\sqrt{3}}{2}$

$= \sqrt{3}\,y^2$

$= \sqrt{3} \times 4ax \qquad (y^2 = 4ax)$

$\therefore \; A(x) = 4\sqrt{3}\,ax$ (1)

Volume of cross-sectional slice $= \delta V = 4\sqrt{3}\,ax\,\delta x$

Volume of Solid $= S_1 = \lim\limits_{\delta x \to 0} \sum\limits_{x=0}^{x=a} 4\sqrt{3}\,ax\,\delta x$

$= \int_0^a 4\sqrt{3}\,ax\,dx$ (1)

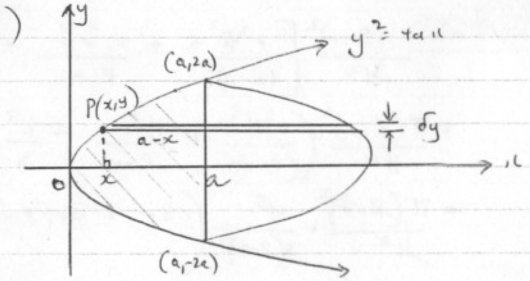
$= 4\sqrt{3}\,a\left[\frac{x^2}{2}\right]_0^a$

$= 4\sqrt{3}\,a\left(\frac{a^2}{2} - 0\right)$

$= \underline{2\sqrt{3}\,a^3}$ units³ (1)

$\boxed{3}$

(ii)



(α) A = Area of cross-section $= \pi r^2$

$= \pi(a-x)^2$ (1)

$= \pi\left(a - \frac{y^2}{4a}\right)^2 \qquad (\text{where } y^2 = 4ax)$

$= \pi\left(a^2 - 2.a.\frac{y^2}{4a} + \left(\frac{y}{4a}\right)^2\right)$ (1)

$\therefore \; A = \pi\left(a^2 - \frac{1}{2}y^2 + \frac{1}{16a^2}y^4\right)$

$\boxed{2}$

(β) Volume of cross-sectional disc $= \pi\left(a^2 - \frac{1}{2}y^2 + \frac{1}{16a^2}y^4\right)\delta y$

$V = \pi\int_{-2a}^{2a}\left(a^2 - \frac{1}{2}y^2 + \frac{1}{16a^2}y^4\right)dy$

$= 2\pi\int_0^{2a}\left(a^2 - \frac{1}{2}y^2 + \frac{1}{16a^2}y^4\right)dy$ (1)

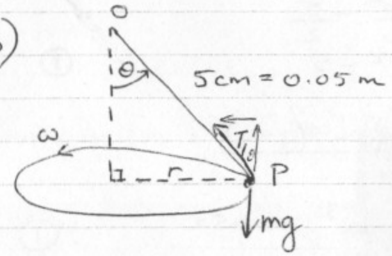
$= 2\pi\left[a^2y - \frac{1}{6}y^3 + \frac{y^5}{80a^2}\right]_0^{2a}$

$= 2\pi\left(2a^3 - \frac{1}{6} \times 8a^3 + \frac{32a^5}{80a^2} - 0\right)$

$= \frac{32}{15}\pi a^3$ (1)

$\boxed{2}$

1 (b)



(i) $\omega = 156$ revs/min.

$= 156 \times 2\pi$ radians/min

$= \frac{156 \times 2\pi}{60}$ radians/sec (1)

$\omega = \frac{26\pi}{5}$ radians/sec

$\boxed{1}$

(ii) Resolving forces at P:

Vertically: $0 = mg - T_1\cos\theta$

$mg = T_1\cos\theta$ ---- (1)

Normally: $mr\omega^2 = T_1\sin\theta$ --- (2)

but $\sin\theta = \frac{r}{0.05} \Longrightarrow r = 0.05\sin\theta$ --- (3)

(2) ÷ (1): $\frac{T\sin\theta}{T\cos\theta} = \frac{\cancel{m}r\omega^2}{\cancel{m}g}$

$\tan\theta = \frac{r\omega^2}{g}$

$\therefore \; \tan\theta = \frac{0.05\sin\theta}{9.8} \times \frac{26^2\pi^2}{5^2}$ (1) (From (3) and $\omega = \frac{26\pi}{5}$)

$\sec\theta = \frac{0.05 \times 26^2 \times \pi^2}{9.8 \times 5^2}$

or $\cos\theta = \frac{9.8 \times 5^2}{0.05 \times 26^2 \times \pi^2}$

$\underline{\theta = 43^\circ}$ (nearest degree) (1)

$\boxed{2}$

1.(c) (i) $\ddot{x} = -9x + \dfrac{5}{(2-x)^2}$.

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = -9x + 5(2-x)^{-2}$$

$$\tfrac{1}{2}v^2 = -\frac{9x^2}{2} + 5(2-x)^{-1} + c \qquad (1)$$

When $x=0, v=0$:

$$0 = 5(2)^{-1} + c$$

$$c = -\frac{5}{2}.$$

$$\tfrac{1}{2}v^2 = \frac{5}{2-x} - \frac{9x^2}{2} - \frac{5}{2}$$

$$v^2 = \frac{10}{2-x} - 9x^2 - 5 \qquad (1)$$

$$= \frac{10 - 9x^2(2-x) - 5(2-x)}{2-x}$$

$$= \frac{\cancel{10} - 18x^2 + 9x^3 - \cancel{10} + 5x}{2-x} \qquad (1)$$

$$= \frac{x(9x^2 - 18x + 5)}{2-x}$$

$$\therefore v^2 = \frac{x(3x-1)(3x-5)}{2-x}$$

$\boxed{3}$

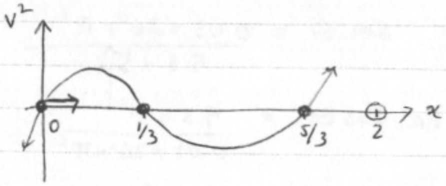
(ii) Particle is at rest $(v=0)$:

$$0 = \frac{x(3x-1)(3x-5)}{2-x}$$

$$0 = x(3x-1)(3x-5), \quad x \neq 2$$

$$x=0,\ x=\tfrac{1}{3},\ x=\tfrac{5}{3}$$

at $x=0$,
$\ddot{x} = \frac{5}{4}$
$\therefore$ particle moving right.



$v^2 \geq 0$ when $0 \leq x \leq \frac{1}{3}$ and $x \geq \frac{5}{3}, x \neq 2$ (1)

However, since particle was at $x=0$ initially, then the particle

Qu.2

(a) (i) $\dfrac{a}{a+H} = \dfrac{r}{R}$ (Corresponding sides of similar Δ's are in proportion) (1)

$$aR = ar + Hr$$

$$a = \frac{rH}{R-r} \quad ---(1) \qquad (1)$$

Similarly: $\dfrac{a}{a+x} = \dfrac{r}{s}$

$$as = r(a+x)$$

$$s = \frac{r}{a}(a+x) \quad ---(2) \qquad (1)$$

Sub. (1) into (2): $s = \cancel{r}\times\dfrac{(R-r)}{\cancel{r}H}\left(\dfrac{rH}{R-r} + x\right)$

$$\therefore s = \frac{R-r}{H}\left(\frac{rH}{R-r} + x\right) \text{ as required}$$

$\boxed{3}$

(ii) Volume of cross-sectional slice $= \pi s^2\,\delta x$

$$= \pi\frac{(R-r)^2}{H^2}\left(\frac{rH}{R-r} + x\right)^2 \delta x$$

Volume of solid $= V = \dfrac{\pi(R-r)^2}{H^2}\displaystyle\int_0^H \left(\frac{rH}{R-r} + x\right)^2 dx$ (1)

$$= \frac{\pi(R-r)^2}{H^2}\int_0^H \left(\frac{r^2H^2}{(R-r)^2} + \frac{2xrH}{R-r} + x^2\right) dx$$

$$= \frac{\pi(R-r)^2}{H^2}\left[\frac{r^2H^2x}{(R-r)^2} + \frac{x^2rH}{R-r} + \frac{x^3}{3}\right]_0^H$$

$$= \frac{\pi(R-r)^2}{H^2}\left(\frac{r^2H^3}{(R-r)^2} + \frac{H^3r}{R-r} + \frac{H^3}{3}\right) \qquad (1)$$

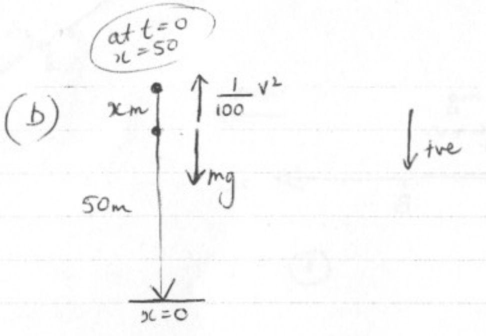
$$= \frac{\pi\cancel{(R-r)^2}}{\cancel{H^2}}\cdot\frac{H^3}{3\cancel{(R-r)^2}}\left(3r^2 + 3r(R-r) + (R-r)^2\right) \qquad (1)$$

$$= \frac{\pi H}{3}\left(\cancel{3r^2} + 3rR - \cancel{3r^2} + R^2 - 2Rr + r^2\right)$$

$$= \frac{\pi H}{3}\left(R^2 + Rr + r^2\right)$$

$\boxed{3}$

(b)



(i) $m\ddot{x} = mg - \frac{1}{100}v^2$ (1)

since $m = 1$:

$\ddot{x} = g - \frac{v^2}{100}$ as required $\boxed{1}$

(ii) Terminal velocity $(\ddot{x} = 0)$

$0 = g - \frac{1}{100}u^2$ (1)

$u^2 = 100g$

$u = \sqrt{100g}$ $\boxed{1}$

(iii) $\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$ (1)

$\ddot{x} = \frac{dv}{dx} \cdot v$ as required. $\boxed{1}$

(iv) $v\frac{dv}{dx} = g - \frac{v^2}{100}$

$\frac{dv}{dx} = \frac{100g - v^2}{100v}$

$\frac{dx}{dv} = \frac{100v}{100g - v^2}$

$= 50\left(\frac{2v}{100g - v^2}\right)$

$x = -50\ln(100g - v^2) + c$ (1)

When $x = 0$, $v = 0$, $100g = u^2 \Longrightarrow$

$0 = -50\ln u^2 + c$

$c = 50\ln u^2$.

$\therefore x = -50\ln(u^2 - v^2) + 50\ln u^2$ (1)

$-\frac{x}{50} = \ln\left(\frac{u^2 - v^2}{u^2}\right)$

$-\frac{x}{50} = \ln\left(1 - \frac{v^2}{u^2}\right)$

$1 - \frac{v^2}{u^2} = e^{-\frac{x}{50}}$ (1)

$\frac{v^2}{u^2} = 1 - e^{-\frac{x}{50}}$, as required. $\boxed{3}$

(v) $v = \frac{1}{2}u$

$\frac{v^2}{u^2} = \frac{1}{4}$

$1 - e^{-\frac{x}{50}} = \frac{1}{4}$ (1)

$e^{-\frac{x}{50}} = \frac{3}{4}$

$-\frac{x}{50} = \ln\frac{3}{4}$

$x = -50\ln\frac{3}{4}$

$\doteqdot 14.38410362$

$\therefore \underline{x = 14.4 \text{ (1 dp)}}$ (1)

$\boxed{2}$

(vi) Body hits ground, when $x = 50$:

$\frac{v^2}{u^2} = 1 - e^{-1}$

$\frac{v}{u} = \sqrt{1 - e^{-1}}$

$= 0.795060097$

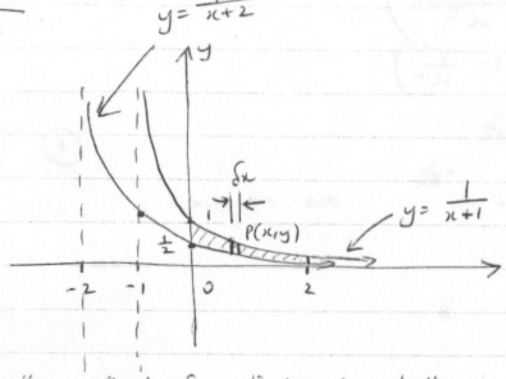
$\frac{v}{u} = 79.5\%$ (1 dp) (1)

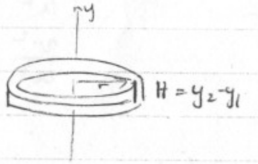
$v = 79.5\% \times u$

$\therefore$ Speed of particle when it hits the ground is 79.5% of terminal speed. $\boxed{1}$

Qu 3

(a)





(i) By the method of cylindrical shells:

Volume of cylindrical shell $= \delta V = 2\pi r H$

$$= 2\pi x\left(\frac{1}{x+1} - \frac{1}{x+2}\right) \quad (1)$$

$$= 2\pi x\left(\frac{x+2-(x+1)}{(x+1)(x+2)}\right)$$

$$= 2\pi x \times \frac{1}{(x+1)(x+2)} \quad (1)$$

$$\delta V = \frac{2\pi x}{(x+1)(x+2)}$$

Volume of solid $= V = 2\pi\int_0^2 \frac{x}{(x+1)(x+2)}$ $\boxed{2}$

(ii) Let $\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$

$$x = A(x+2) + B(x+1)$$

Let $x=-1$: $-1 = A$

Let $x=-2$: $-2 = -B$

$\underline{B = 2}$ $\Big\}$ (1)

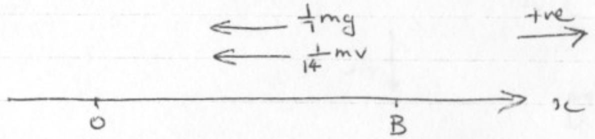
$$\therefore V = 2\pi\int_0^2\left(\frac{2}{x+2} - \frac{1}{x+1}\right)dx$$

$$= 2\pi\Big[2\ln(x+2) - \ln(x+1)\Big]_0^2 \quad (1)$$

$$= 2\pi\left(2\ln 4 - \ln 3 - 2\ln 2 + \cancel{\ln 1}\right)$$

$$= 2\pi(2\ln 2 - \ln 3) = \underline{\underline{1.81}}\ (3\text{ sf})\ u^3 \quad (1) \quad \boxed{3}$$

(b)



(i) $m\ddot{x} = -\frac{mg}{7} - \frac{mv}{14}$ (1)

$$\ddot{x} = -\left(\frac{v}{14} + \frac{g}{7}\right)$$

$$\ddot{x} = -\left(\frac{v+2g}{14}\right) \quad \boxed{1}$$

(ii) Take $g=10$: $\ddot{x} = -\left(\frac{v+20}{14}\right)$

$$\frac{dv}{dt} = -\left(\frac{v+20}{14}\right)$$

$$\frac{dt}{dv} = -\frac{14}{v+20}$$

$$t = -14\int_{42}^{v}\frac{1}{v+20}\,dv$$

$$= -14\Big[\ln(v+20)\Big]_{42}^{v}$$

$$= -14\Big[\ln(v+20) - \ln 62\Big]$$

$$t = 14\ln\frac{62}{v+20} \quad \text{--- (A)} \quad (1)$$

Time taken for particle to come to rest $(v=0)$

$$t = 14\ln\frac{62}{20}$$

OR $\underline{t = 14\ln 3.1}$ ($\doteqdot 15.84$ sec) (1)

From (A): $\frac{t}{14} = \ln\frac{62}{v+20}$

$$\frac{62}{v+20} = e^{t/14}$$

$$62e^{-t/14} = v+20$$

$$v = 62e^{-t/14} - 20 \quad (1)$$

$$\frac{dx}{dt} = 62e^{-t/14} - 20$$

$$x = \int_0^{14\ln 3.1}\left(62e^{-t/14} - 20\right)dt$$

$$= \Big[-14\times 62e^{-t/14} - 20t\Big]_0^{14\ln 3.1}$$

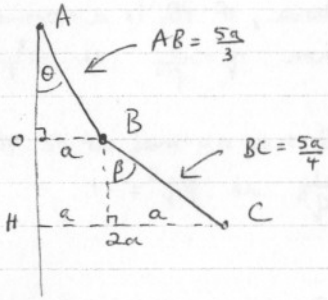
$$\therefore x = -868e^{-\ln 3.1} - 20 \times 14 \ln 3.1 + 14 \times 62$$
$$= -868 \times \frac{10}{31} - 280 \ln 3.1 + 868$$

$$\doteqdot 271.2074088$$
$$x = \underline{271\,m \text{ (nearest metre)}} \quad (1)$$

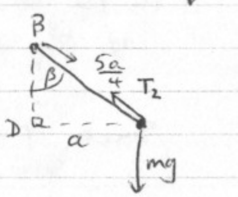
$\therefore$ It takes $\underline{14 \ln 3.1}$ sec and $\underline{271\,m}$ for the car to completely come to rest.

$\boxed{4}$

(c)



(i) At C, resolving forces: $\downarrow +ve$



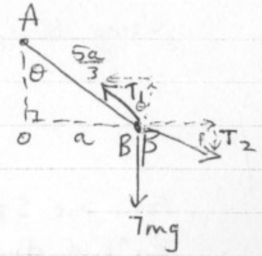
Vertically:
$$0 = mg - T_2 \cos\beta$$
$$T_2 = \frac{mg}{\cos\beta} \quad (1)$$
$$= \frac{mg}{\frac{3}{5}}$$
$$T_2 = \frac{5mg}{3}$$

$$BD = \sqrt{\left(\frac{5a}{4}\right)^2 - a^2}$$
$$= \sqrt{\frac{25a^2}{16} - a^2}$$
$$BD = \frac{3a}{4}$$

$$\therefore \cos B = \frac{3a}{4} \div \frac{5a}{4}$$
$$\cos B = \frac{3}{5} \quad (1)$$

$\boxed{2}$

(ii) Resolving forces at B:



Vertically:
$$0 = 7mg - T_1 \cos\theta + T_2 \cos\beta$$
$$-7mg = T_2 \cos\beta - T_1 \cos\theta \quad \text{--- (A)}$$

$$-7mg = \frac{5mg}{3} \times \frac{3}{5} - T_1 \times \frac{4}{5} \quad (1)$$

$$\underline{\underline{T_1 = 10mg}}$$

$$AO = \sqrt{\left(\frac{5a}{3}\right)^2 - a^2}$$
$$= \sqrt{\frac{25a^2}{9} - a^2}$$
$$= \sqrt{\frac{16a^2}{9}}$$
$$AO = \frac{4a}{3}$$
$$\therefore \cos\theta = \frac{4\cancel{a}}{\cancel{3}} \div \frac{5\cancel{a}}{\cancel{3}}$$
$$\therefore \cos\theta = \frac{4}{5}$$

(iii) Need to find ω:

At C: $T_2 \cos\beta = mg$ (Vertically)
$T_2 \sin\beta = 2am\omega^2$ (normally)

$$\frac{\cancel{T_2} \sin\beta}{\cancel{T_2} \cos\beta} = \frac{2a\cancel{m}\omega^2}{\cancel{m}g}$$
$$\tan\beta = \frac{2a\omega^2}{g}$$
$$\omega^2 = \frac{\tan\beta \times g}{2a}$$
$$= \frac{4/3\, g}{2a}$$
$$\omega = \sqrt{\frac{2g}{3a}} \quad (1)$$

At B, $v = r\omega = a\sqrt{\frac{2g}{3a}} = \sqrt{\frac{2ag}{3}}$ (1)

$\boxed{2}$

Qu 4

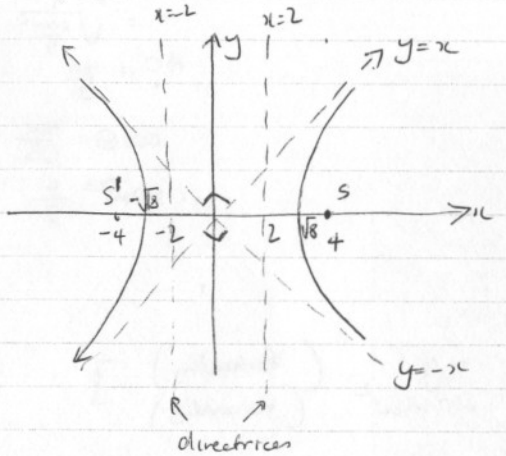
(a) $x^2-y^2=8 \implies \frac{x^2}{8}-\frac{y^2}{8}=1 \quad \therefore a=\sqrt{8},\ b=\sqrt{8}$

Using $b^2=a^2(e^2-1)$ gives
$$8=8(e^2-1)$$
$$1=e^2-1$$
$$\underline{e=\sqrt{2}} \quad (1)$$

Foci are S & S', using $(\pm ae, 0) = \underline{(\pm 4, 0)}$. (1)

Eq^{n}s of directrices are $x=\pm\frac{a}{e} \implies x=\pm\frac{\sqrt{8}}{\sqrt{2}}$

ie $\underline{x=\pm 2}$ (1)



Asymptotes ($\frac{1}{2}$)
x-intercepts ($\frac{1}{2}$)
directrices ($\frac{1}{2}$)
Foci ($\frac{1}{2}$)

$\boxed{5}$

(b) (i) $xy=4 \implies y=4x^{-1}$
$$\frac{dy}{dx}=-4x^{-2}$$
at $P(2p, \frac{2}{p})$, $\frac{dy}{dx}=\frac{-4}{(2p)^2}=\frac{-4}{4p^2}=\frac{-1}{p^2}$ (1)

(Alternative: $x\frac{dy}{dx}+y.1=0 \implies \frac{dy}{dx}=-\frac{y}{x}$
at $P(2p,\frac{2}{p})$, $\frac{dy}{dx}=-\frac{2/p}{2p}=\frac{-1}{p^2}$)

$\therefore$ Gradient of normal $=p^2$.

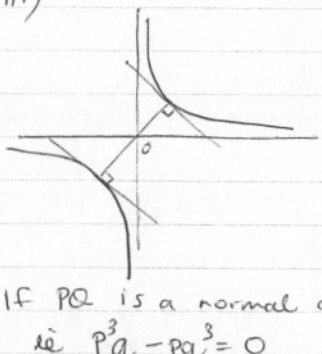
Eq^n of normal:
$$y-\frac{2}{p}=p^2(x-2p) \quad (1)$$
$$py-2=p^3(x-2p)$$
$$py-p^3x=2(1-p^4)$$

$\boxed{2}$

(ii) If normal passes through $Q(2q, \frac{2}{q})$, then it satisfies equation:
$$p\times\frac{2}{q}-p^3\times 2q = 2(1-p^4)$$
$$\cancel{2}p-\cancel{2}p^3q^2=\cancel{2}q-\cancel{2}p^4q \quad (1)$$
$$p-q=p^3q^2-p^4q$$
$$p-q=p^3q(q-p) \quad (1)$$
$$-1=p^3q \quad (p\neq q)$$
$$\therefore q=\frac{-1}{p^3}$$

$\boxed{2}$

(iii)



From above, if PQ is a normal at P then $q=\frac{-1}{p^3}$ ie $p^3q=-1$.

Also, PQ is a normal at Q then $p=-\frac{1}{q^3}$ ie $pq^3=-1$

If PQ is a normal at both P and Q then $p^3q=pq^3$

ie $p^3q-pq^3=0$ ie $pq(p^2-q^2)=0$

ie $pq(p-q)(p+q)=0$.

Since $p\neq 0, q\neq 0$ and $p\neq q$ then $\underline{p+q=0}$ only.

ie $\underline{q=-p}$.

Since $p^3q=-1$, then
$$p^3\times -p=-1$$
$$p^4=1$$
$$\underline{p=\pm 1}$$

If $p=1$, Eq^n of normal is $1y-1^3x=2(1-1^4)$

ie $\underline{y=x}$

If $p=-1$, Eq^n of normal is $-y+x=2(1-1)$

$\underline{y=x}$

Thus, there is only one chord of the hyperbola where where the gradients of the normal, at both ends, are equal. Its equation is $\underline{\underline{y=x}}$

(iv) Midpoint of PQ = R

$\therefore R = \left(\frac{2p+2q}{2}, \frac{2/p + 2/q}{2}\right)$

$R = \left(p+q, \frac{p+q}{pq}\right)$

$\therefore \quad x = p+q \quad \xRightarrow{\text{but}} \quad q = \frac{-1}{p^3}$

1 $\quad y = \frac{p+q}{pq} \qquad pq = \frac{-1}{p^2} \quad$ --- A

$\frac{x}{y} = \frac{p+q}{\frac{p+q}{pq}}$

$= pq$

$\frac{x}{y} = -\frac{1}{p^2}$ (from A)

$y = -p^2x \quad$ OR $\quad -\frac{y}{x} = p^2 \qquad$ 1

Since R lies on the normal, it satisfies eq^n.

ie $\quad py - p^3x = 2(1-p^4)$

$y - p^2x = \frac{2}{p}(1-p^4)$

$-p^2x - p^2x = \frac{2}{p}(1-p^4)$

$-2p^2x = \frac{2}{p}(1-p^4)$

$-2x\left(\frac{-y}{x}\right) = \frac{2}{p}\left(1 - \frac{y^2}{x^2}\right)$

$2y = \frac{2}{p}\left(\frac{x^2-y^2}{x^2}\right)$

$4y^2 = \frac{4}{p^2}\left(\frac{x^2-y^2}{x^2}\right)^2 \qquad$ (squaring)

$\cancel{4}y^2 = \cancel{4} \times -\frac{x}{y}\left(\frac{x^2-y^2}{x^2}\right)^2 \qquad$ 1

$y^3 = -\frac{(x^2-y^2)^2}{x^3}$

$x^3y^3 + (x^2-y^2)^2 = 0$

$\boxed{3}$