CSSA
CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NSW

Centre Number

Student Number

2016
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Morning Session
Thursday, 4 August 2016

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A formula Reference Sheet is provided
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks – 100

Section I Pages 2 - 6

10 marks

- Attempt Questions 1 - 10
- Allow 15 minutes for this section

Section II Pages 7 - 16

90 marks

- Attempt Questions 11 - 16
- Allow about 2 hours and 45 minutes for this section

6400-1

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the Multiple-Choice Answer Sheet for Questions 1–10.

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1 Let $z = 1 - i$ and $\omega = -2 + i$. What is the value of $z\overline{\omega}$?

(A) $3 - i$

(B) $-3 - i$

(C) $-3 + i$

(D) $1 - 3i$

2 What is the primitive function of $\sin x\sqrt{\cos x}$?

(A) $\cos x\sqrt{\cos x} + C$

(B) $-\cos x\sqrt{\cos x} + C$

(C) $\frac{2}{3}\sqrt{\cos^3 x} + C$

(D) $-\frac{2}{3}\sqrt{\cos^3 x} + C$

3 What are the foci of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$?

(A) $(0, \pm 3)$

(B) $(\pm 5, 0)$

(C) $(\pm 3, 0)$

(D) $(0, \pm 5)$

4 The polynomial $P(x)$ with real coefficients has $x = 1$ as a root of multiplicity 2 and $x + i$ as a factor.

Which one of the following expressions could be a factorised form of $P(x)$?

(A) $(x^2 + 1)(x - 1)^2$

(B) $(x + i)^2(x - 1)^2$

(C) $(x - i)^2(x - 1)^2$

(D) $(x^2 + 1)(x - i)^2$

5 A particle of unit mass moves in a straight line against a resistance R, where $R = v(1 + v^2)$ and v ms^{-1} is the velocity of the particle at a distance x metres from the origin.

Given that the particle is initially at the origin with velocity Q ms^{-1}, which one of the following is correct?

(A) $x = -(\tan^{-1} Q + \tan^{-1} v)$

(B) $x = \tan^{-1} v + \tan^{-1} Q$

(C) $x = \tan^{-1} v - \tan^{-1} Q$

(D) $x = \tan^{-1} Q - \tan^{-1} v$

6 The polynomial equation $x^3 - 5x^2 + 6 = 0$ has roots α, β and γ.
Which one of the following polynomial equations has roots $\alpha - 1$, $\beta - 1$ and $\gamma - 1$?

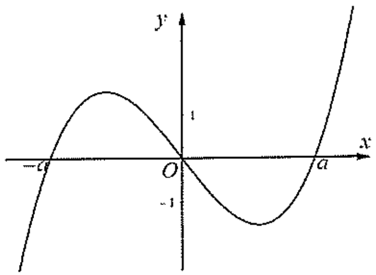
(A) $x^3 - 8x^2 + 13x = 0$

(B) $x^3 - 8x^2 - 7x = 0$

(C) $x^3 - 3x^2 - 7x + 2 = 0$

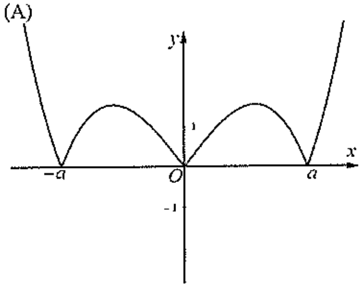
(D) $x^3 - 2x^2 - 7x + 2 = 0$

7 The graph of $y = f(x)$ is shown.

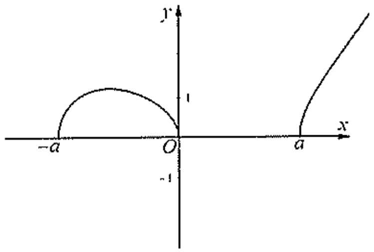


Which graph best represents $y = \sqrt{f(x)}$.

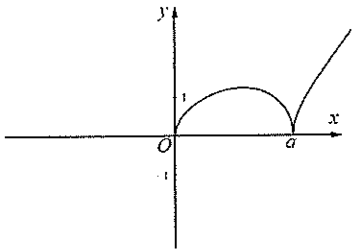
(A)



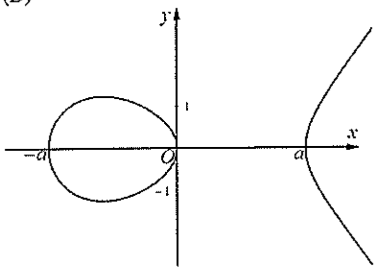
(B)



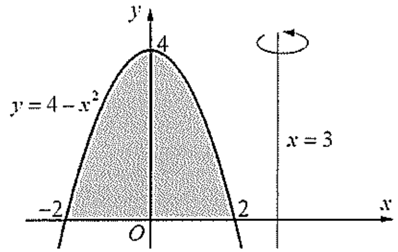
(C)



(D)



8 The region bounded by $y = 4 - x^2$ and the x-axis is rotated about the line $x = 3$ to form a solid.



Using slices perpendicular to $x = 3$, which one of the following integrals will give the volume of the solid?

(A) $12\pi\int_0^4 \sqrt{4-y}\,dy$

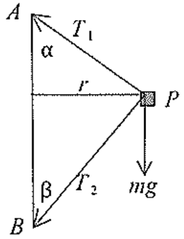
(B) $6\pi\int_0^4 \sqrt{4-y}\;dy$

(C) $12\pi\int_{-2}^2 \sqrt{4-y}\;dy$

(D) $6\pi\int_{-2}^2 \sqrt{4-y}\,dy$

9 A particle P of mass m is attached to the points A and B by two light, inextensible strings. The particle is moving in a horizontal circle of radius r with constant angular velocity ω. The strings make angles of α and β with the vertical.

The forces acting on the particle are the tensions T_1 and T_2 in the strings and the gravitational force mg.



Which one of the following gives the correct resolution of forces in the horizontal and vertical directions?

(A) $T_1 \sin\alpha - T_2 \sin\beta = mr\omega^2$ and $T_1 \cos\alpha - T_2 \cos\beta = mg$

(B) $T_1 \sin\alpha + T_2 \sin\beta = mr\omega^2$ and $T_1 \cos\alpha + T_2 \cos\beta = mg$

(C) $T_1 \sin\alpha + T_2 \sin\beta = mr\omega^2$ and $T_1 \cos\alpha - T_2 \cos\beta = mg$

(D) $T_1 \sin\alpha - T_2 \sin\beta = mr\omega^2$ and $T_1 \cos\alpha + T_2 \cos\beta = mg$

10 A bag contains 11 letters of the alphabet.

There are four different black letters, four different white letters and three different red letters.

How many different combinations of any number of letters can be chosen containing at least one black letter and at least one white letter?

(A) 38

(B) 40

(C) 1800

(D) 2048

End of Section I

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

Use a SEPARATE writing booklet.

(a) Let $z = \dfrac{3+i}{1+2i}$.

(i) Express z in the form $a+ib$ where a and b are real. **1**

(ii) Hence express z^7 in modulus-argument form. **2**

(b) Using the substitution $t = \tan\dfrac{\theta}{2}$, find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{3\sin\theta + 4\cos\theta + 5}$. **4**

(c) M, N and P are three points on a circle. **3**

The altitudes MB and NC in the acute-angled triangle MNP meet at the point Z. NC produced meets the circle at A as shown below.

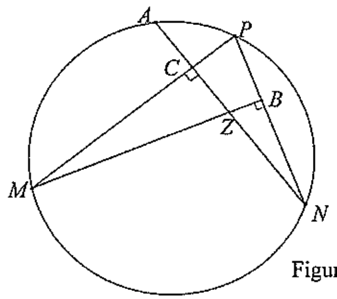


Figure not to scale

Prove that AC = CZ.

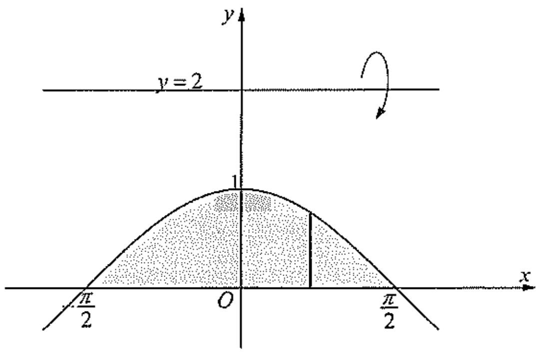
Question 11 continues on page 8

Question 11 continued

(d) Find the real numbers a, b and c such that **2**

$$\frac{x^3+5x^2+x+2}{x^2(x^2+1)} \equiv \frac{x+a}{x^2}+\frac{bx+c}{x^2+1}.$$

(e) A solid is formed by rotating the region bounded by $y=\cos x$, the x-axis, $x=-\frac{\pi}{2}$ and $x=\frac{\pi}{2}$ about the line $y=2$. **3**



Using slices perpendicular to the axis of rotation, find the volume of the solid that is formed.

End of Question 11

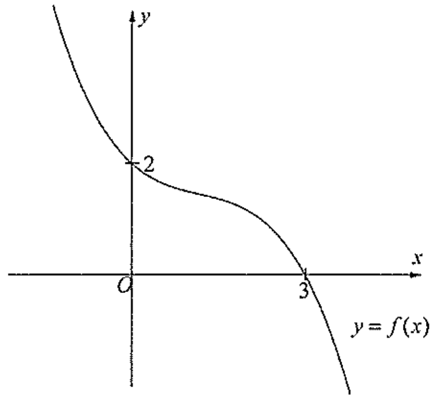
Question 12 (15 marks)

Use a SEPARATE writing booklet.

(a) Find the equation of an ellipse which has eccentricity $\frac{4}{5}$ and foci $(-5,0)$ and $(5,0)$. **2**

(b) Evaluate $\int_1^e x^2 \log_e x \, dx$. **3**

(c) The diagram shows the graph of $y=f(x)$.



Make neat, separate one-third page diagrams of each of the following graphs, showing all x and y intercepts and asymptotes.

(i) $y=[f(x)]^2$ **1**

(ii) $y=\dfrac{1}{f(x)}$ **1**

(iii) $y=f^{-1}(x)$ **1**

(iv) $y=f(x)-|f(x)|$ **2**

Question 12 continues on page 10

Question 12 continued.

(d) (i) Find the roots of the equation $x^4 - x^2 = -\frac{1}{8}$ using the substitution $x = \cos\theta$, given that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ and $0 \le \theta \le \pi$. **2**

(ii) Hence, or otherwise, show that $(\cos\frac{\pi}{8} + \cos\frac{3\pi}{8})^2 = \frac{2+\sqrt{2}}{2}$. **3**

End of Question 12

Question 13 (15 marks)

Use a SEPARATE writing booklet.

(a) Let $z = x + iy$. **3**

Sketch on an Argand Diagram the locus of the point P representing z given that $z\bar{z} - z(2+i) - \bar{z}(2-i) \le 4$.

(b) Using implicit differentiation, find the coordinates of the points on the curve whose equation is $x^2 + y^2 + xy = 3$ where the tangents are horizontal. **3**

(c) The cubic equation $x^3 + px + q = 0$ has three non-zero real roots α, β and γ.

(i) Find, in terms of the constants p and q, the values of $\alpha^2 + \beta^2 + \gamma^2$ and $\alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2$. **3**

(ii) Hence deduce that $p < 0$. **1**

(iii) Form the cubic equation with roots $\frac{\alpha\beta}{\gamma}, \frac{\beta\gamma}{\alpha}$ and $\frac{\alpha\gamma}{\beta}$. **2**

(d) A sequence $\{U_n\}$ is defined so that $U_1 = 1$, $U_2 = 2$, $U_3 = 3$ and $U_n = U_{n-1} + U_{n-2} + U_{n-3}$ for $n \ge 4$. **3**

Using mathematical induction, prove that $U_n < 2^n$ for integers $n \ge 1$.

End of Question 13

Question 14 (15 marks)

Use a SEPARATE writing booklet.

(a) For the complex number $z = \cos\theta + i\sin\theta$

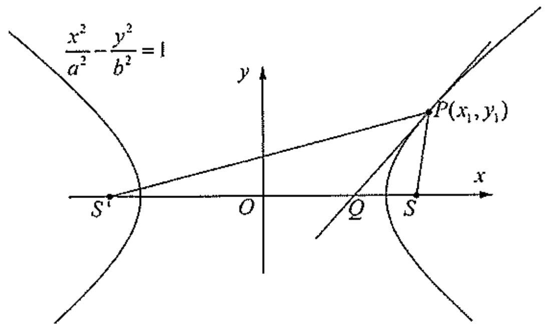
(i) Show that $z^n + z^{-n} = 2\cos n\theta$. **1**

(ii) Hence, or otherwise, solve $3z^4 - z^3 + 4z^2 - z + 3 = 0$. **3**

(b) Let $P(x_1, y_1)$ be a point on the right-hand branch of the hyperbola whose equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Let S and S' represent the foci of the hyperbola.

The tangent at P intersects the x-axis at Q.



(i) Show that $\frac{QS}{PS} = \frac{QS'}{PS'}$. **2**

(ii) Using the triangles PSQ and $PS'Q$, show that $\angle SPQ = \angle S'PQ$. **2**

Question 14 continues on page 13

Question 14 continued

(c) A particle of mass 4kg is projected vertically upwards.

The particle is subjected to a gravitational force of 40 Newtons and air resistance of $\frac{v^2}{10}$ Newtons.

The height of the particle at time t seconds is x metres and its velocity is v ms^{-1}.

(i) Given that $v^2 = 400\left(10e^{-\frac{x}{20}} - 1\right)$ until the particle reaches its maximum height, find the maximum height in exact form. **1**

(ii) After reaching maximum height the particle begins to fall. Show that the equation of motion as it falls is $\ddot{x} = \frac{400 - v^2}{40}$. **1**

(iii) How far has the particle fallen from its maximum height when the speed is 50% of its terminal velocity? **3**

(iv) Find the speed of the particle when it returns to its point of projection. **2**

End of Question 14

Question 15 (15 marks)

Use a SEPARATE writing booklet.

(a) The region bounded by $x^2 - y^2 = 1$, and the line $x = 2$ is rotated about the y-axis to form a solid. **3**

Use the method of cylindrical shells to find the volume of the solid.

(b) Consider the integral $I_n = \int_0^1 \frac{x^n}{\sqrt{1+x}}\, dx$, where n is a positive integer.

(i) Find I_0. **1**

(ii) Show that $I_{n-1} + I_n = \int_0^1 x^{n-1}\sqrt{1+x}\, dx$. **1**

(iii) Use integration by parts to show that $I_n = \frac{2\sqrt{2} - 2nI_{n-1}}{2n+1}$. **2**

(c) (i) Show that $\frac{1}{(1+t)^2} < \frac{1}{1+t} < 1$ given that t is a positive real number. **2**

(ii) Hence, show that for $u > 0$, $\frac{u}{1+u} < \log_e(1+u) < u$. **2**

(d) $P(2p, \frac{2}{p})$ and $Q(2q, \frac{2}{q})$ are two points on the rectangular hyperbola $xy = 4$. **4**
M is the midpoint of the chord PQ.

Find the equation of the locus of M if PQ is a tangent to the parabola $y^2 = 2x$.

End of Question 15

Question 16 (15 marks)

Use a SEPARATE writing booklet.

(a) In a multiple choice test there are x questions with a choice of five answers for each question, only one of which is correct. **2**
A student guesses the answers to all questions by choosing one of the five options.

Show that the probability that the student gets at least $(x-2)$ answers correct is given by $\frac{1}{5^x}(8x^2 - 4x + 1)$.

(b) (i) The diagram below shows a trapezium $ABCD$ whose parallel sides AB and DC are 9 m and 13 m respectively. **2**
The distance between these sides is 4m and $AD = BC$.
EF is parallel to AB at a distance of h m.

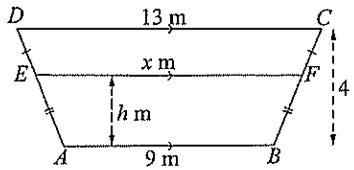


Figure not to scale

Show that $EF = (9+h)$ m.

(ii) The trench in the diagram below has a rectangular base with sides 9 m and 3 m. Its top is also rectangular with dimensions of 13 m and 5 m. The trench has a depth of 4 m and each of its four side faces is a symmetrical trapezium. **3**

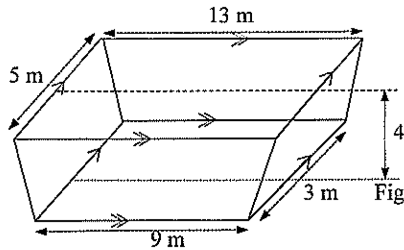


Figure not to scale

Find the volume of the trench.

Question 16 continues on page 16

Question 16 continued

(c) The series $1-x^2+x^4-.....+x^{4n}$ has $(2n+1)$ terms.

(i) Show that $1-x^2+x^4-.....+x^{4n}=\dfrac{1+x^{4n+2}}{1+x^2}$. Justify your answer. **2**

(ii) Hence show that $\dfrac{1}{1+x^2}\le 1-x^2+x^4-.....+x^{4n}\le\dfrac{1}{1+x^2}+x^{4n+2}$. **2**

(iii) Hence show that if $0\le a\le 1$, **2**

$$\tan^{-1}a\le a-\frac{a^3}{3}+\frac{a^5}{5}-.........+\frac{a^{4n+1}}{4n+1}\le\tan^{-1}a+\frac{1}{4n+3}.$$

(iv) Deduce that $0<(1-\dfrac{1}{3}+\dfrac{1}{5}-........+\dfrac{1}{101})-\dfrac{\pi}{4}<10^{-2}$. **2**

End of Paper

EXAMINERS

Vito Zurlo (Convenor) — St Scholastica's College, Glebe Point
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Paul Regan — Consultant

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CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW
2016 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION
MATHEMATICS EXTENSION 2 - MARKING GUIDELINES

Section I

10 marks

Questions 1-10 (1 mark each)

Question 1 (1 mark)
Outcomes Assessed: E3
Targeted Performance Bands: E2

Solution	Mark
$z\overline{w}=(1-i)(-2-i)$ $=-2-i+2i+i^2$ $=-3+i$ Hence (C)	1

Question 2 (1 mark)
Outcomes Assessed: E8
Targeted Performance Bands: E2-E3

Solution		Mark
$\int \sin x\sqrt{\cos x}\ dx$ $=-\int u^{\frac{1}{2}}du$ $=-(\frac{2}{3}u^{\frac{3}{2}})+C$ $=-\frac{2}{3}\sqrt{\cos^3 x}+C$ Hence (D)	$u=\cos x$ $du=-\sin x\ dx$ $\sin x\ dx=-du$	1

6400-2

Question 3 (1 mark)
Outcomes Assessed: E4
Targeted Performance Bands: E2

Solution	Mark
$\frac{x^2}{9}-\frac{y^2}{16}=1$ x-intercepts of the hyperbola are $(\pm 3,0)$. Foci are on the major axis. Foci must be $(\pm 5,0)$ Hence (B)	1

Question 4 (1 mark)
Outcomes Assessed: E4
Targeted Performance Bands: E2-E3

Solution	Mark
$x=1$ is a root of multiplicity 2 so $(x-1)^2$ is a factor of $P(x)$. Coefficients are real and $(x+i)$ is a factor so $(x-i)$ is also a factor. $\therefore P(x)=(x+i)(x-i)(x-1)^2$ $=(x^2-i^2)(x-1)^2$ $=(x^2+1)(x-1)^2$ Hence (A)	1

Question 5 (1 mark)
Outcomes Assessed: E5
Targeted Performance Bands: E3-E4

Solution	Mark
$\ddot{x}=-v(1+v^2)$ $v\frac{dv}{dx}=-v(1+v^2)$ $\frac{dv}{dx}=-(1+v^2)$ $\frac{dx}{dv}=-\frac{1}{1+v^2}$ $x=-\tan^{-1}v+C$ When $x=0, v=Q$ so $C=\tan^{-1}Q$ $\therefore x=\tan^{-1}Q-\tan^{-1}v$ Hence (D)	1

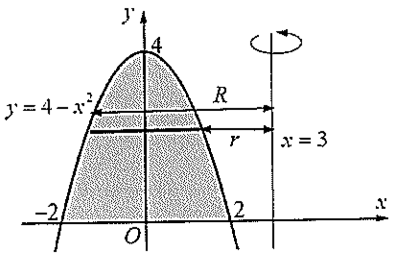
Question 6 (1 mark)
Outcomes Assessed: E6
Targeted Performance Bands: E3-E4

Solution	Mark
$(x+1)^3-5(x+1)^2+6=0$ $x^3+3x^2+3x+1-5x^2-10x-5+6=0$ $x^3-2x^2-7x+2=0$ is required equation Hence (D)	1

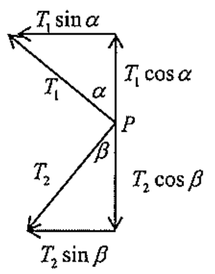
Question 7 (1 mark)
Outcomes Assessed: E6
Targeted Performance Bands: E2-E3

Solution	Mark
By inspection, (B)	1

Question 8 (1 mark)
Outcomes Assessed: E7
Targeted Performance Bands: E2-E3

Solution	Mark
	
$y=4-x^2$ $x=\pm\sqrt{4-y}$ Let R and r be outer and inner radii of a typical annular slice. $R=3+\sqrt{4-y}$ $r=3-\sqrt{4-y}$ $R+r=6$ $R-r=2\sqrt{4-y}$ Volume of slice $\approx \pi(R+r)(R-r)\delta y$ $=\pi\times 6\times 2\sqrt{4-y}\times\delta y$ $=12\pi\sqrt{4-y}\ \delta y$ $\therefore$ Volume of solid $=12\pi\int_0^4\sqrt{4-y}\ dy$ Hence (A)	1

Question 9 (1 mark)
Outcomes Assessed: E5
Targeted Performance Bands: E3-E4

Solution	Mark
	1
Horizontal forces: $T_1\sin\alpha+T_2\sin\beta=mr\omega^2$ Vertical forces : $mg+T_2\cos\beta-T_1\cos\alpha=0$ $T_1\cos\alpha-T_2\cos\beta=mg$ Hence (C)	

Question 10 (1 mark)
Outcomes Assessed: E9
Targeted Performance Bands: E4

Solution	Mark
At least one black letter can be chosen in ${}^4C_1+{}^4C_2+{}^4C_3+{}^4C_4 = 15$ ways. At least one white letter can be chosen in ${}^4C_1+{}^4C_2+{}^4C_3+{}^4C_4 = 15$ ways. Red letters can be chosen in ${}^3C_0+{}^3C_1+{}^3C_2+{}^3C_3=8$ ways. Hence required numbers of ways is $15\times15\times8=1800$ ways. Hence (C)	1

Section II

90 marks

Question 11 (15 marks)
(a) (i) (1 mark)
Outcomes Assessed: E3
Targeted Performance Bands: E2

Criteria	Mark
* Correct solution	1

Sample Answer:

$$\begin{aligned} z &= \frac{3+i}{1+2i} \\ &= \frac{(3+i)}{(1+2i)} \times \frac{(1-2i)}{(1-2i)} \\ &= \frac{3-6i+i-2i^2}{5} \\ &= \frac{5-5i}{5} \\ &= 1-i \end{aligned}$$

(a) (ii) (2 marks)
Outcomes Assessed: E3
Targeted Performance Bands: E2

Criteria	Marks
* Correct solution	2
* Writes z in modulus and argument form	1

Sample Answer:

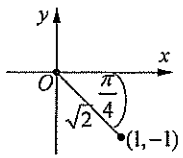
$z = 1-i$

$|z| = \sqrt{2}$, $\arg(z) = -\frac{\pi}{4}$

$\therefore z = \sqrt{2}(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}))$

$z^7 = (\sqrt{2})^7(\cos(-\frac{7\pi}{4}) + i\sin(-\frac{7\pi}{4}))$

$\quad = 8\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$



Question 11(continued)
(b) (4 marks)
Outcomes Assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
* Correct solution	4
* Demonstrates significant progress towards answer	3
* Simplifies integral correctly in terms of t	2
* Finds limit values for t and correctly substitutes expressions for $\sin\theta$ and $\cos\theta$	1

Sample Answer:

$t = \tan\frac{\theta}{2}$

$$\begin{aligned} \frac{dt}{d\theta} &= \frac{1}{2}\sec^2\frac{\theta}{2} \\ &= \frac{1}{2}(1+\tan^2\frac{\theta}{2}) \\ &= \frac{1+t^2}{2} \end{aligned}$$

$d\theta = \frac{2\,dt}{1+t^2}$

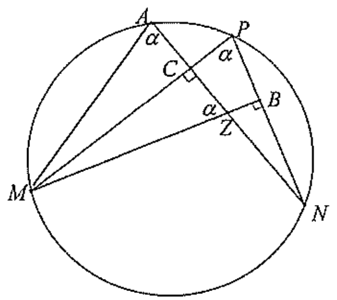
$\theta = \frac{\pi}{2} \;\therefore t = 1$, $\theta = 0 \therefore t = 0$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{d\theta}{3\sin\theta + 4\cos\theta + 5} &= \int_0^1 \frac{1}{3(\frac{2t}{1+t^2}) + 4(\frac{1-t^2}{1+t^2}) + 5(\frac{1+t^2}{1+t^2})} \times \frac{2\,dt}{(1+t^2)} \\ &= \int_0^1 \frac{2\,dt}{6t+4-4t^2+5+5t^2} \\ &= \int_0^1 \frac{2\,dt}{t^2+6t+9} \\ &= 2\int_0^1 \frac{dt}{(t+3)^2} \\ &= 2\left[\frac{-1}{t+3}\right]_0^1 \\ &= -2(\frac{1}{4} - \frac{1}{3}) \\ &= \frac{1}{6} \end{aligned}$$

Question 11(continued)
(c) (3 marks)
Outcomes Assessed: E9
Targeted Performance Bands: E2-E3

Criteria	Marks
* Correct solution	3
* Recognises that $CZPB$ is a cyclic quadrilateral	2
* Shows that $\angle MAN = \angle MPN$	1

Sample Answer:



Join A to M

$\angle MAN = \angle MPN (= \alpha)$ (Angles at the circumference subtended by the same chord are equal.)

$CPBZ$ is a cyclic quadrilateral (opposite angles at C and B are supplementary.)

$\angle MZC = \angle CPB$ $(= \alpha)$ (Exterior angle of a cyclic quadrialteral is equal to interior opposite angle.)

$\therefore \angle MAZ = \angle MZA$

$\therefore \Delta MAZ$ is isosceles.

MC is perpendicular to AZ and hence bisects AZ.

$\therefore AC = CZ$

Question 11(continued)
(d) (2 marks)
Outcomes assessed: E4
Targeted Performance Bands: E2-E3

Criteria	Marks
* Correct solution	2
* Demonstrates significant progress towards answer	1

Sample Answer:

$$\frac{x^3+5x^2+x+2}{x^2(x^2+1)} = \frac{x+a}{x^2} + \frac{bx+c}{x^2+1}$$

$(x^2+1)(x+a) + x^2(bx+c) = x^3+5x^2+x+2$

Put $x = 0 \quad \rightarrow \quad a = 2$

$\therefore\ (x^2+1)(x+2) + x^2(bx+c) = x^3+5x^2+x+2$

Coefficient of x^3: $\quad 1+b=1$

$\therefore b = 0$

Coefficient of x^2: $\quad 2+c=5$

$\therefore c = 3$

$\therefore$ Required values are $a = 2,\ b = 0$ and $c = 3$.

Question 11(continued)

(e) (3 marks)

Outcomes assessed: E7

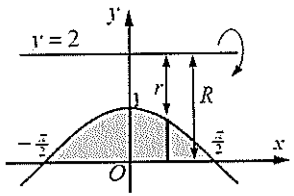
Targeted Performance Bands: E2-E3

Criteria	Marks
* Correct solution	3
* Substantial progress towards solution	2
* Correct integral	1

Sample Answer:

When the strip is rotated about $y = 2$ it will form an annulus.

Let R and r be the outer and inner radii of the annulus.



$R = 2$; $r = 2 - \cos x$

Area of annulus = $\pi(R+r)(R-r)$

$R + r = 4 - \cos x$; $R - r = \cos x$

Area of annulus $= \pi(4-\cos x)\cos x$

$= \pi(4\cos x - \cos^2 x)$

Let thickness of slice be δx

Volume of slice = $\pi(4\cos x - \cos^2 x)\delta x$

Volume of solid $= \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(4\cos x - \cos^2 x)dx$

$= \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(4\cos x - \frac{1}{2}\cos 2x - \frac{1}{2})dx$

$= \pi\left[4\sin x - \frac{1}{4}\sin 2x - \frac{x}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \pi\left[(4\sin\frac{\pi}{2} - \frac{1}{4}\sin\pi - \frac{\pi}{4}) - (4\sin\frac{-\pi}{2} - \frac{1}{4}\sin(-\pi) - \frac{-\pi}{4})\right]$

$= \pi\left[(4 - \frac{\pi}{4}) - (-4 + \frac{\pi}{4})\right]$

$= \pi(8 - \frac{\pi}{2})$ units^3

Question 12 (15 marks)

(a) (2 marks)

Outcomes Assessed: E4

Targeted Performance Bands: E2

Criteria	Marks
* Correct solution	2
* Finds correct value of a	1

Sample Answer:

$e = \frac{4}{5}$

$ae = 5$

$a \times \frac{4}{5} = 5$

$a = \frac{25}{4}$

$b^2 = a^2(1-e^2)$

$= \frac{625}{16}(1 - \frac{16}{25})$

$= \frac{625}{16} \times \frac{9}{25}$

$= \frac{225}{16}$

Equation of ellipse is $\frac{x^2}{\frac{625}{16}} + \frac{y^2}{\frac{225}{16}} = 1$

$\frac{16x^2}{625} + \frac{16y^2}{225} = 1$

Question 12(continued)
(b) (3 marks)
Outcomes Assessed: E8
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	3
* Significant progress towards answer	2
* Applies integration by parts correctly	1

Sample Answer:

$$\int v\,du = uv - \int u\,dv$$

$v = \log_e x \qquad du = x^2$

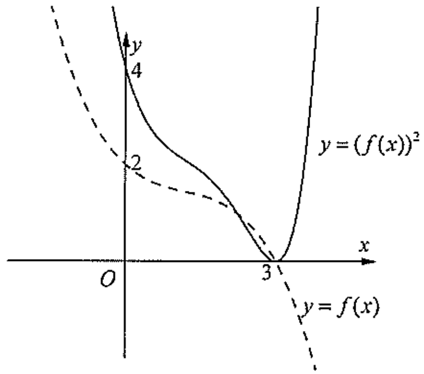
$dv = \dfrac{1}{x} \qquad u = \dfrac{x^3}{3}$

$$\int_1^e x^2 \log_e x\,dx = \left[\frac{x^3}{3}\log_e x\right]_1^e - \int_1^e \frac{x^3}{3}\frac{1}{x}\,dx$$
$$= \left[\frac{x^3}{3}\log_e x\right]_1^e - \int_1^e \frac{x^2}{3}\,dx$$
$$= \left[\frac{x^3}{3}\log_e x - \frac{x^3}{9}\right]_1^e$$
$$= \left(\frac{e^3}{3}\log_e e - \frac{e^3}{9}\right) - \left(\frac{1}{3}\log_e 1 - \frac{1}{9}\right)$$
$$= \frac{e^3}{3} - \frac{e^3}{9} + \frac{1}{9}$$
$$= \frac{2e^3+1}{9}$$

Question 12(continued) (c) (i) (1 mark)
Outcomes Assessed: E9
Targeted Performance Bands: E2

Criteria	Mark
* Correct diagram	1

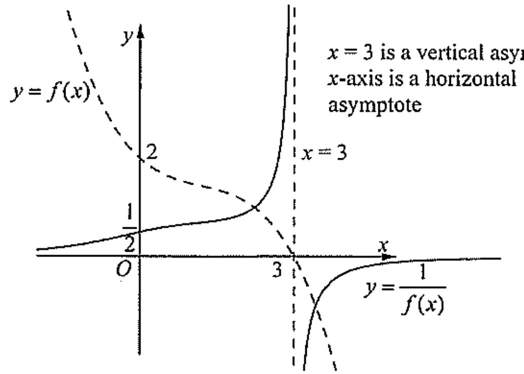
Sample Answer:



(c) (ii) (1 mark)
Outcomes Assessed: E9
Targeted Performance Bands: E2

Criteria	Mark
* Correct diagram	1

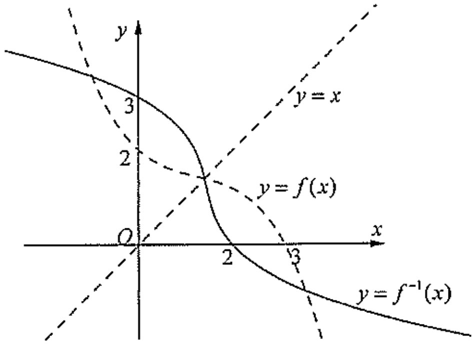
Sample Answer:



Question 12(continued)
(c) (iii) (1 mark)
Outcomes Assessed: E9
Targeted Performance Bands: E2-E3

Criteria	Mark
* Correct diagram	1

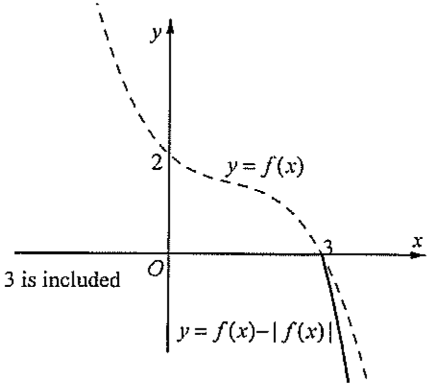
Sample Answer:



Question 12(continued)
(c) (iv) (2 marks)
Outcomes Assessed: E9
Targeted Performance Bands: E3

Criteria	Mark
* Correct diagram	2
* Shows some correct part of diagram	1

Sample Answer:



x-axis for $x \le 3$ is included in graph

Question 12(continued)
(d) (i) (2 marks)
Outcomes Assessed: E4
Targeted Performance Bands: E3-E4

Criteria	Marks
* Correct solution	2
* Significant progress towards answer	1

Sample Answer:

$x^4 - x^2 = -\frac{1}{8}$

$8x^4 - 8x^2 + 1 = 0$

Put $x = \cos\theta$

$8\cos^4\theta - 8\cos^2\theta + 1 = 0$

But $8\cos^4\theta - 8\cos^2\theta + 1 = \cos 4\theta$ (given)

$\therefore$ Equation becomes $\cos 4\theta = 0$

$4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$

$\therefore$ Roots are $\cos\frac{\pi}{8}, \cos\frac{3\pi}{8}, \cos\frac{5\pi}{8}, \cos\frac{7\pi}{8}$

Question 12(continued)
(d) (ii) (3 marks)
Outcomes Assessed: E4
Targeted Performance Bands: E4

Criteria	Marks
* Correct solution	3
* Significant progress towards answer	2
* Correctly finds product of roots	1

Sample Answer:

$\cos\frac{5\pi}{8} = -\cos\frac{3\pi}{8}$ and $\cos\frac{7\pi}{8} = -\cos\frac{\pi}{8}$

Hence roots are $\pm\cos\frac{\pi}{8}$ and $\pm\cos\frac{3\pi}{8}$

$(\cos\frac{\pi}{8} + \cos\frac{3\pi}{8})^2 = \cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + 2\cos\frac{\pi}{8}\cos\frac{3\pi}{8}$

Using product of roots, $-\cos^2\frac{\pi}{8} \times -\cos^2\frac{3\pi}{8} = \frac{1}{8}$

$\therefore \cos\frac{\pi}{8}\cos\frac{3\pi}{8} = \frac{1}{2\sqrt{2}}$

Using sum of roots, two at a time,

$-\cos^2\frac{\pi}{8} + \cos\frac{\pi}{8}\cos\frac{3\pi}{8} + (-\cos\frac{\pi}{8}\cos\frac{3\pi}{8}) + (-\cos\frac{\pi}{8}\cos\frac{3\pi}{8}) + \cos\frac{\pi}{8}\cos\frac{3\pi}{8} - \cos^2\frac{3\pi}{8} = -1$

$\therefore \cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} = 1$

Hence $(\cos\frac{\pi}{8} + \cos\frac{3\pi}{8})^2 = 1 + 2\times\frac{1}{2\sqrt{2}}$

$= 1 + \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{2}+1}{\sqrt{2}}$

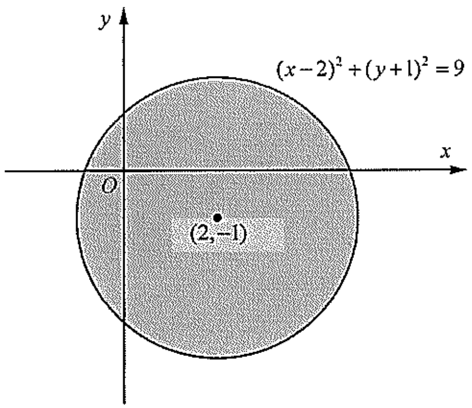
$= \frac{2+\sqrt{2}}{2}$, as required.

Question 13 (15 marks)
(a) (3 marks)
Outcomes Assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Marks
* Correct graph of locus	3
* Obtains correct simplified form of locus	2
* Correctly substitutes for $z\overline{z}, z$ *and* $\overline{z}$	1

Sample Answer:

Let $z = x + iy$
$z\overline{z} - z(2+i) - \overline{z}(2-i) \le 4$
$x^2 + y^2 - (x+iy)(2+i) - (x-iy)(2-i) \le 4$
$x^2 + y^2 - 2x - ix - 2iy - i^2 y - 2x + ix + 2iy - i^2 y \le 4$
$x^2 + y^2 - 2x - ix - 2iy + y - 2x + ix + 2iy + y \le 4$
$x^2 + y^2 - 4x + 2y \le 4$
$x^2 - 4x + 4 + y^2 + 2y + 1 \le 4 + 4 + 1$
$(x-2)^2 + (y+1)^2 \le 9$
This describes the circumference and interior of a circle, centre $(2,-1)$ and radius 3.



Question 13(continued)
(b) (3 marks)
Outcomes Assessed: E6
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	3
* Significant progress towards answer	2
* Finds correct expression for $\frac{dy}{dx}$	1

Sample Answer:

$x^2 + y^2 + xy = 3 \qquad (*)$
Differentiate each term with respect to x.
$2x + 2y\dfrac{dy}{dx} + y + x\dfrac{dy}{dx} = 0$
$(2x+y) + (2y+x)\dfrac{dy}{dx} = 0$
$\dfrac{dy}{dx} = \dfrac{-(2x+y)}{2y+x}$
For horizontal tangents, $\dfrac{dy}{dx} = 0$
$\therefore$ Tangents are horizontal when $y = -2x$
Sub in (*)
$x^2 + (-2x)^2 + x(-2x) = 3$
$x^2 + 4x^2 - 2x^2 = 3$
$3x^2 = 3$
$x^2 = 1$
$x = \pm 1$
When $x = 1,\ y = -2$
When $x = -1,\ y = 2$
$\therefore$ Tangents are horizontal at $(1,-2)$ and $(-1,2)$.

Question 13(continued)
(c) (i) (3 marks)
Outcomes Assessed: E4
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	3
* Significant progress towards the answer	2
* Finds correct expression for $\alpha^2+\beta^2+\gamma^2$	1

Sample Answer:

$$\begin{aligned}\alpha^2+\beta^2+\gamma^2 &= (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)\\ &= 0-2p\\ &= -2p\end{aligned}$$

$$\begin{aligned}\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 &= (\alpha\beta+\beta\gamma+\alpha\gamma)^2-2(\alpha\beta^2\gamma+\beta\gamma^2\alpha+\alpha^2\beta\gamma)\\ &= p^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)\\ &= p^2-2\times p\times 0\\ &= p^2\end{aligned}$$

(c) (ii) (1 mark)
Outcomes Assessed: E4
Targeted Performance Bands: E3

Criteria	Mark
* Correct deduction	1

Sample Answer:

α,β and γ are real non-zero roots.
Hence the sum of their squares must be positive.
From (i) the sum of their squares is $-2p$.
Hence $-2p>0$ ie $p<0$.

Question 13(continued)
(c) (iii) (2 marks)
Outcomes Assessed: E4
Targeted Performance Bands: E4

Criteria	Marks
* Correct solution	2
* Finds correctly either sum or product of roots	1

Sample Answer:

$$\text{Sum of roots (one at a time)} = \frac{\alpha\beta}{\gamma}+\frac{\beta\gamma}{\alpha}+\frac{\gamma\alpha}{\beta} = \frac{\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2}{\alpha\beta\gamma} = -\frac{p^2}{q}$$

$$\text{Sum of roots (two at a time)} = \frac{\alpha\beta}{\gamma}.\frac{\beta\gamma}{\alpha}+\frac{\beta\gamma}{\alpha}.\frac{\gamma\alpha}{\beta}+\frac{\gamma\alpha}{\beta}.\frac{\alpha\beta}{\gamma} = \beta^2+\gamma^2+\alpha^2 = -2p$$

$$\text{Product of roots} = \frac{\alpha\beta}{\gamma}.\frac{\beta\gamma}{\alpha}.\frac{\gamma\alpha}{\beta} = \alpha\beta\gamma = -q$$

Hence required cubic equation is $x^3+\frac{p^2}{q}x^2-2px+q=0$

Question 13(continued)
(d) (3 marks)
Outcomes Assessed: E9
Targeted Performance Bands: E3-E4

Criteria	Marks
* Correct solution	3
* Significant progress towards	2
* Shows that $n = 4$ is true	1

Sample Answer:

Clearly, $U_n < 2^n$ for $n = 1, 2$ *and* 3.

When $n = 4$, $U_n = 1 + 2 + 3 = 6$

$2^4 = 16$

$\therefore U_n < 2^n$ when $n = 4$.

Assume $U_n < 2^n$ for all integers $n = 5, 6, 7, ...k$.

Need to show that $U_{k+1} < 2^{k+1}$.

From definition of U_n,

$$\begin{aligned}
U_{k+1} &= U_k + U_{k-1} + U_{k-2} \\
&< 2^k + 2^{k-1} + 2^{k-2} && \text{(from assumption)} \\
&< \frac{2}{2}2^k + \frac{2^2}{2^2}2^{k-1} + \frac{2^3}{2^3}2^{k-2} && \text{(Multiply each term by 1 to get } 2^{k+1}\text{)} \\
&< \frac{1}{2}2^{k+1} + \frac{1}{4}2^{k+1} + \frac{1}{8}2^{k+1} \\
&< \frac{7}{8}2^{k+1} \\
&< 2^{k+1}, \text{ as required.}
\end{aligned}$$

$\therefore$ By mathematical induction $U_n < 2^n$ for all integers $n \geq 1$.

Question 14 (15 marks)
(a) (i) (1 mark)
Outcomes Assessed: E3
Targeted Performance Bands: E2

Criteria	Mark
* Correct solution	1

Sample Answer:

$z = \cos\theta + i\sin\theta$

$$\begin{aligned}
z^n + z^{-n} &= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta) \\
&= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta \\
&= 2\cos n\theta, \text{ as required.}
\end{aligned}$$

Question 14(continued)
(a) (ii) (3 marks)
Outcomes Assessed: E3
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	3
* Solves correctly for $\cos\theta$	2
* Obtains correctly $6\cos^2\theta - \cos\theta - 1 = 0$	1

Sample Answer:

$3z^4 - z^3 + 4z^2 - z + 3 = 0$

Take out a factor of z^2.

$z^2(3z^2 - z + 4 - z^{-1} + 3z^{-2}) = 0$

$z \neq 0$ so $3z^2 - z + 4 - z^{-1} + 3z^{-2} = 0$

$3(z^2 + z^{-2}) - (z + z^{-1}) + 4 = 0$

$3(2\cos 2\theta) - (2\cos\theta) + 4 = 0$

$6(2\cos^2\theta - 1) - 2\cos\theta + 4 = 0$

$12\cos^2\theta - 2\cos\theta - 2 = 0$

$6\cos^2\theta - \cos\theta - 1 = 0$

$(2\cos\theta - 1)(3\cos + 1) = 0$

$\therefore \cos\theta = \frac{1}{2}$, $\cos\theta = -\frac{1}{3}$

If $\cos\theta = \frac{1}{2}$, $\sin\theta = \pm\frac{\sqrt{3}}{2}$

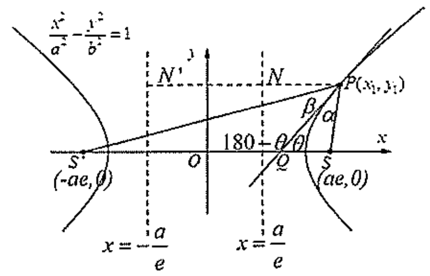
If $\cos\theta = -\frac{1}{3}$, $\sin\theta = \pm\frac{\sqrt{8}}{3}$

$\therefore$ Solutions are $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ and $-\frac{1}{3} \pm \frac{\sqrt{8}}{3}i$

Question 14(continued)
(b) (i) (2 marks)
Outcomes Assessed: E4
Targeted Performance Bands: E3-E4

Criteria	Marks
* Correct solution	2
* Obtains correct expressions for PS and SQ	1

Sample Answer:



Equation of tangent at P is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

At Q, $x = \frac{a^2}{x_1}$

$PS = ePN$ $\quad$ $PS' = ePN'$

$= e(x_1 - \frac{a}{e})$ $\quad$ $= e(x_1 + \frac{a}{e})$

$= ex_1 - a$ $\quad$ $= ex_1 + a$

$SQ = ae - \frac{a^2}{x_1}$ $\quad$ $QS' = \frac{a^2}{x_1} + ae$

$= \frac{a(ex_1 - a)}{x_1}$ $\quad$ $= \frac{a(ex_1 + a)}{x_1}$

$\frac{SQ}{PS} = \frac{\frac{a(ex_1 - a)}{x_1}}{ex_1 - a}$ $\quad$ $\frac{S'Q}{PS'} = \frac{\frac{a(ex_1 + a)}{x_1}}{ex_1 + a}$

$= \frac{a}{x_1}$ $\quad$ $= \frac{a}{x_1}$

$\therefore \frac{SQ}{PS} = \frac{S'Q}{PS'}$, as required.

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Question 14(continued)
(b) (ii) (2 marks)
Outcomes Assessed: E4
Targeted Performance Bands: E3-E4

Criteria	Marks
* Correct solution	2
* Applies sine rule correctly	1

Sample Answer:

Let $\angle SPQ = \alpha$, $\angle S'PQ = \beta$
Let $\angle SQP = \theta, \angle S'QP = 180 - \theta$

In ΔPSQ, $\dfrac{\sin\alpha}{SQ} = \dfrac{\sin\theta}{PS}$

$\therefore \sin\alpha = \dfrac{SQ}{PS}\sin\theta$

In $\Delta PS'Q$, $\dfrac{\sin\beta}{QS'} = \dfrac{\sin(180-\theta)}{PS'}$

$\therefore \sin\beta = \dfrac{QS'}{PS'}\sin(180-\theta)$
But $\sin(180-\theta) = \sin\theta$
$\therefore \sin\alpha = \sin\beta$
$\therefore \angle SPQ = \angle S'PQ$,as required.

Question 14(continued)
(c) (i) (1 mark)
Outcomes Assessed: E5
Targeted Performance Bands:

Criteria	Mark
* Correct solution for maximum height	1

Sample Answer:

$v^2 = 400(10e^{-\frac{x}{20}} - 1)$
At maximum height $v = 0$

ie $10e^{-\frac{x}{20}} - 1 = 0$

$e^{-\frac{x}{20}} = \dfrac{1}{10}$

$e^{\frac{x}{20}} = 10$

$\dfrac{x}{20} = \log_e 10$

$x = 20\log_e 10$

$\therefore$ Maximum height reached is $20\log_e 10$ metres.

(c) (ii) (1 mark)
Outcomes Assessed: E5
Targeted Performance Bands: E2-E3

Criteria	Mark
* Correct solution	1

Sample Answer:

$4\ddot{x} = 40 - \dfrac{v^2}{10}$

$= \dfrac{400 - v^2}{10}$

$\ddot{x} = \dfrac{400 - v^2}{40}$, as required.

Question 14(continued)
(c) (iii) (3 marks)
Outcomes Assessed: E5
Targeted Performance Bands: E2-E3

Criteria	Marks
* Correct solution	3
* Substantial progress towards solution	2
* Obtains correct expression for $\frac{dx}{dv}$ or finds 50% of terminal velocity	1

Sample Answer:

$$\ddot{x} = \frac{400 - v^2}{40}$$
$$v\frac{dv}{dx} = \frac{400 - v^2}{40}$$
$$\frac{dv}{dx} = \frac{400 - v^2}{40v}$$
$$\frac{dx}{dv} = \frac{40v}{400 - v^2}$$
$$x = -20\log_e(400 - v^2) + C$$
When $x = 0, v = 0$ so $C = 20\log_e 400$
$$\therefore x = 20\log_e \frac{400}{400 - v^2}$$

From (ii), $\ddot{x} \to 0$ as $v^2 \to 400$
$\therefore$ Terminal velocity is 20 ms^{-1}.

When $v = 10,\ x = 20\log_e \frac{400}{400 - 100}$
$$= 20\log_e \frac{4}{3}$$
$\therefore$ Body has fallen $20\log_e \frac{4}{3}$ metres when its speed is 50% of terminal velocity.

Question 14(continued)
(c) (iv) (2 marks)
Outcomes Assessed: E5
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	2
* Demonstrates some correct working	1

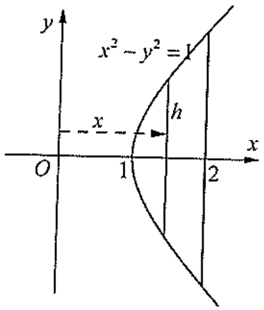
Sample Answer:

$$x = 20\log_e \frac{400}{400 - v^2}$$
From (i), $x = 20\log_e 10$ when body returns to point of projection.
$$\therefore 20\log_e 10 = 20\log_e \frac{400}{400 - v^2}$$
$$\frac{400}{400 - v^2} = 10$$
$$400 - v^2 = 40$$
$$v^2 = 360$$
$$v = 6\sqrt{10} \text{ ms}^{-1}$$

Question 15 (15 marks)
(a) (3 marks)
Outcomes Assessed: E7
Targeted Performance Bands: E2-E3

Criteria	Marks
* Correct solution	3
* Writes correct integral for volume	2
* Finds correct expression for h	1

Sample Answer:



$x^2 - y^2 = 1$

$y = \pm\sqrt{x^2 - 1}$

$\therefore h = 2\sqrt{x^2 - 1}$

$\text{Volume} = 2\pi\int_1^2 2x\sqrt{x^2 - 1}\ dx$

$= 2\pi\int_0^3 u^{\frac{1}{2}}\ du$

$= 2\pi\left[\frac{2}{3}(u)^{\frac{3}{2}}\right]_0^3$

$= \frac{4\pi}{3}\left[(3)^{\frac{3}{2}}\right]$

$= \frac{4\pi}{3} \times 3\sqrt{3}$

$= 4\sqrt{3}\,\pi\ \text{units}^3$

Put $u = x^2 - 1$

$\frac{du}{dx} = 2x$

$2x\,dx = du$

When $x = 2, u = 3$

$x = 1, u = 0$

Question 15(continued)
(b) (i) (1 mark)
Outcomes Assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Mark
* Correct solution	1

Sample Answer:

$I_0 = \int_0^1 \frac{dx}{\sqrt{1+x}}$

$= 2\left[\sqrt{1+x}\right]_0^1$

$= 2\sqrt{2} - 2$

(b) (ii) (1 mark)
Outcomes Assessed: E8
Targeted Performance Bands: E3-E4

Criteria	Mark
* Correct solution	1

Sample Answer:

$I_{n-1} + I_n = \int_0^1 \frac{x^{n-1}}{\sqrt{1+x}}dx + \int_0^1 \frac{x^n}{\sqrt{1+x}}dx$

$= \int_0^1 \frac{x^{n-1}(1+x)}{\sqrt{1+x}}dx$

$= \int_0^1 x^{n-1}\sqrt{1+x}\ dx$, as required.

Question 15(continued)
(b) (iii) (2 marks)
Outcomes Assessed: E8
Targeted Performance Bands: E4

Criteria	Marks
* Correct solution	2
* Demonstrates some correct working	1

Sample Answer:

$$I_n = \int_0^1 \frac{x^n}{\sqrt{1+x}}dx$$
$$= \left[2x^n\sqrt{1+x}\right]_0^1 - 2n\int_0^1 x^{n-1}\sqrt{1+x}\ dx \qquad \text{(using integration by parts)}$$
$$= 2\sqrt{2} - 2n(I_{n-1} + I_n) \qquad \text{(using part (ii))}$$
$$\therefore 2nI_n + I_n = 2\sqrt{2} - 2nI_{n-1}$$
$$(2n+1)I_n = 2\sqrt{2} - 2nI_{n-1}$$
$$I_n = \frac{2\sqrt{2} - 2nI_{n-1}}{2n+1}, \text{ as required.}$$

Question 15(continued)
(c) (i) (2 marks)
Outcomes Assessed: E3
Targeted Performance Bands: E4

Criteria	Marks
* Correct solution	2
* Shows $\frac{1}{(1+t)^2} < \frac{1}{1+t}$	1

Sample Answer:

$$\frac{1}{1+t} - \frac{1}{(1+t)^2} = \frac{1+t-1}{(1+t)^2}$$
$$= \frac{t}{(1+t)^2}$$
$$> 0 \text{ if } t > 0$$
$$\therefore \frac{1}{(1+t)^2} < \frac{1}{1+t}$$
$$\text{Also } 1 - \frac{1}{1+t} = \frac{1+t-1}{1+t}$$
$$= \frac{t}{1+t}$$
$$> 0 \text{ if } t > 0$$
$$\therefore \frac{1}{1+t} < 1$$
$$\text{Hence } \frac{1}{(1+t)^2} < \frac{1}{1+t} < 1, \text{ as required.}$$

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Question 15(continued)
(c) (ii) (2 marks)
Outcomes Assessed: E3
Targeted Performance Bands: E4

Criteria	Marks
* Correct solution	2
* Integrates expressions from correctly from 0 to u	1

Sample Answer:

From (i)

$$\frac{1}{(1+t)^2} < \frac{1}{1+t} < 1$$

$$\int_0^u \frac{dt}{(1+t)^2} < \int_0^u \frac{dt}{1+t} < \int_0^u 1\, dt$$

$$\left[\frac{-1}{1+t}\right]_0^u < \left[\log_e(1+t)\right]_0^u < \left[t\right]_0^u$$

$$\therefore \left(\frac{-1}{1+u}\right) - \left(\frac{-1}{1}\right) < \log_e(1+u) - \log_e 1 < u - 0$$

Hence $1 - \frac{1}{1+u} < \log_e(1+u) < u$

$\therefore \frac{u}{1+u} < \log_e(1+u) < u$, as required.

Question 15(continued)
(d) (4 marks)
Outcomes Assessed: E4
Targeted Performance Bands: E4

Criteria	Marks
* Correct solution	4
* Substantial progress towards solution	3
* Finds equation of chord PQ	2
* Finds midpoint of PQ	1

Sample Answer:

Gradient of $PQ = \dfrac{\frac{2}{p} - \frac{2}{q}}{2p - 2q} = -\dfrac{1}{pq}$

Equation of PQ : $y - \frac{2}{p} = -\frac{1}{pq}(x - 2p)$

$$pqy - 2q = -x + 2p$$

$$x + pqy = 2(p+q) \qquad (1)$$

Midpoint of PQ: $M = (p+q, \frac{p+q}{pq})$

Let $T(\frac{t^2}{2}, t)$ be a point on $y^2 = 2x$

$$2y\frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = \frac{1}{y}$$

Equation of tangent at T: $y - t = \frac{1}{t}(x - \frac{t^2}{2})$

$$ty - t^2 = x - \frac{t^2}{2}$$

$$x - ty = -\frac{t^2}{2} \qquad (2)$$

Equate coefficients in (1) and (2) $pq = -t$, $2(p+q) = -\frac{t^2}{2}$

At M, $x = -\frac{t^2}{4}$, $\quad y = \dfrac{-\frac{t^2}{4}}{-t} = \frac{t}{4}$

$t^2 = -4x \qquad t^2 = 16y^2$

$\therefore$ At M, $16y^2 = -4x$

$$y^2 = -\frac{x}{4}$$

Question 16 (15 marks)
(a) (2 marks)
Outcomes Assessed: E4
Targeted Performance Bands: E3-E4

Criteria	Marks
* Correct solution	2
* Demonstrates some correct working	1

Sample Answer:

Let p = probability of a correct answer = $\frac{1}{5}$

Let q = probability of an incorrect answer = $\frac{4}{5}$

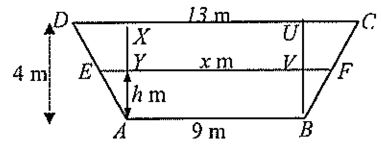
Let n be the number of correct answers where $0 \le n \le x$

$$\begin{aligned}P(n \ge x-2) &= P(n = x-2) + P(n = x-1) + P(n = x)\\
&= {}^{x}C_{x-2}q^2p^{x-2} + {}^{x}C_{x-1}qp^{x-1} + p^x\\
&= p^{x-2}({}^{x}C_{x-2}q^2 + {}^{x}C_{x-1}qp + p^2)\\
&= (\frac{1}{5})^{x-2}(\frac{x(x-1)}{2}q^2 + xqp + p^2)\\
&= (\frac{1}{5})^{x-2}(\frac{x(x-1)}{2}(\frac{16}{25}) + x(\frac{4}{25}) + (\frac{1}{25}))\\
&= (\frac{1}{5})^{x}(\frac{x(x-1)}{2}(16) + x(4) + (1))\\
&= \frac{1}{5^x}(8x^2 - 8x + 4x + 1)\\
&= \frac{1}{5^x}(8x^2 - 4x + 1), \text{ as required.}\end{aligned}$$

Question 16(continued)
(b) (i) (2 marks)
Outcomes Assessed: E7
Targeted Performance Bands: E2-E3

Criteria	Marks
* Correct solution	2
* Establishes expression EY correctly	1

Sample Answer:



Draw AX and BU perpendicular to DC.
Triangles AXD and BUC are congruent (RHS)
$DX = CU = 2$

By similar triangles,

$$\frac{EY}{AY} = \frac{DX}{AX}$$

$$\frac{EY}{h} = \frac{2}{4}$$

$$EY = \frac{h}{2}$$

Similarly, $VF = \frac{h}{2}$

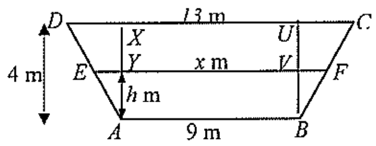
$$\begin{aligned}EF &= EY + 9 + VF\\ &= 9 + h\end{aligned}$$

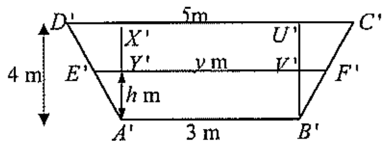
Question 16(continued)
(b) (ii) (3 marks)
Outcomes Assessed: E7
Targeted Performance Bands: E3-E4

Criteria	Marks
* Correct solution	3
* Establishes correct integral	2
* Finds correct expression for $E'Y'$	1

Sample Answer:

Vertical cross-sections through centre of the base and parallel to the sides of the base will be symmetric trapeziums.





Cross-sections parallel to the base of the trench will be rectangles with sides x and y as shown in diagrams above.

From (i), $x = 9 + h$

From second diagram, $D'X' = C'U' = 1$.

Using similar triangles again, $\frac{E'Y'}{h} = \frac{1}{4}$

$$\therefore y = 3 + 2 \times \frac{h}{4}$$
$$= 3 + \frac{h}{2}$$

$$\therefore \text{Area of cross-section} = (9+h)(3+\frac{h}{2}) = \frac{h^2}{2} + \frac{15h}{2} + 27$$

$$\therefore \text{Volume} = \int_0^4 (\frac{h^2}{2} + \frac{15h}{2} + 27)\,dh$$
$$= \left[\frac{h^3}{6} + \frac{15h^2}{4} + 27h\right]_0^4$$
$$= 178\frac{2}{3}\text{ m}^3$$

Question 16(continued)
(c) (i) (2 marks)
Outcomes Assessed: E4
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution with justification	2
* Shows correct sum of geometric series	1

Sample Answer:

The series $1 - x^2 + x^4 - \ldots\ldots + x^{4n}$ is a geometric series with $a = 1$, $r = -x^2$ and $(2n+1)$ terms.

$$\therefore 1 - x^2 + x^4 - \ldots\ldots + x^{4n} = \frac{1(1-(-x^2)^{2n+1})}{1-(-x^2)}$$
$$= \frac{1+x^{4n+2}}{1+x^2}, \quad \text{since } (-1)^{2n+1} = -1$$

(c) (ii) (2 marks)
Outcomes Assessed: E4
Targeted Performance Bands: E3

Criteria	Marks
* Correct solution	2
* Shows some correct working	1

Sample Answer:

$$\frac{1+x^{4n+2}}{1+x^2} = \frac{1}{1+x^2} + \frac{x^{4n+2}}{1+x^2}$$
$$\geq \frac{1}{1+x^2}, \text{ since } \frac{x^{4n+2}}{1+x^2} \geq 0 \text{ for all } x$$
$$\therefore \frac{1}{1+x^2} \leq \frac{1+x^{4n+2}}{1+x^2}$$
$$\frac{1}{1+x^2} \leq 1 - x^2 + x^4 - \ldots\ldots + x^{4n}$$

Also $\frac{1}{1+x^2} + \frac{x^{4n+2}}{1+x^2} < \frac{1}{1+x^2} + x^{4n+2}$ since $x^2 > 0$ and $1 + x^2 \geq 1$

$$\therefore 1 - x^2 + x^4 - \ldots + x^{4n} \leq \frac{1}{1+x^2} + x^{4n+2}$$

Hence $\frac{1}{1+x^2} \leq 1 - x^2 + x^4 - \ldots\ldots + x^{4n} \leq \frac{1}{1+x^2} + x^{4n+2}$.

Question 16(continued)
(c) (iii) (2 marks)
Outcomes Assessed: E4
Targeted Performance Bands: E4

Criteria	Marks
* Correct solution	2
* Finds three correct integrals of each expression in the inequality	1

Sample Answer:

$$\frac{1}{1+x^2} \le 1 - x^2 + x^4 - \dots\dots + x^{4n} \le \frac{1}{1+x^2} + x^{4n+2}$$

$$\therefore \int_0^a \frac{1}{1+x^2}\,dx \le \int_0^a (1 - x^2 + x^4 - \dots\dots + x^{4n})dx \le \int_0^a (\frac{1}{1+x^2} + x^{4n+2})dx$$

$$\left[\tan^{-1} x\right]_0^a \le \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\dots + \frac{x^{4n+1}}{4n+1}\right]_0^a \le \left[\tan^{-1} x + \frac{x^{4n+3}}{4n+3}\right]_0^a$$

$$\therefore \tan^{-1} a \le a - \frac{a^3}{3} + \frac{a^5}{5} - \dots\dots + \frac{a^{4n+1}}{4n+1} \le \tan^{-1} a + \frac{a^{4n+3}}{4n+3}$$

Since $0 \le a \le 1$, $\dfrac{a^{4n+3}}{4n+3} \le \dfrac{1}{4n+3}$

$$\therefore \tan^{-1} a \le a - \frac{a^3}{3} + \frac{a^5}{5} - \dots\dots + \frac{a^{4n+1}}{4n+1} \le \tan^{-1} a + \frac{1}{4n+3}.$$

(c) (iv) (2 marks)
Outcomes Assessed: E4
Targeted Performance Bands: E4

Criteria	Marks
* Correct solution	2
* substitutes correctly for a and n	1

Sample Answer:

From (iii), let $a = 1$ and $n = 25$.

Hence $\tan^{-1} 1 < (1 - \frac{1}{3} + \frac{1}{5} - \dots\dots + \frac{1^{101}}{101}) < \tan^{-1} 1 + \frac{1}{103}$

$\therefore 0 < (1 - \frac{1}{3} + \frac{1}{5} - \dots\dots + \frac{1^{101}}{101}) - \frac{\pi}{4} < \frac{1}{103}$ (after subtracting $\frac{\pi}{4}$ from each expression)

$0 < (1 - \frac{1}{3} + \frac{1}{5} - \dots\dots + \frac{1^{101}}{101}) - \frac{\pi}{4} < \frac{1}{100}$

$0 < (1 - \frac{1}{3} + \frac{1}{5} - \dots\dots + \frac{1}{101}) - \frac{\pi}{4} < 10^{-2}$, as required.