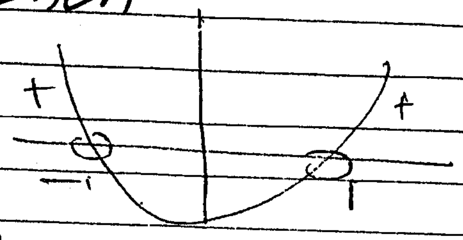
year 12 3U THSC 2000 (PSJ)

KAMBALA

Q1 (a) $\frac{x^2-1}{x} > 0$



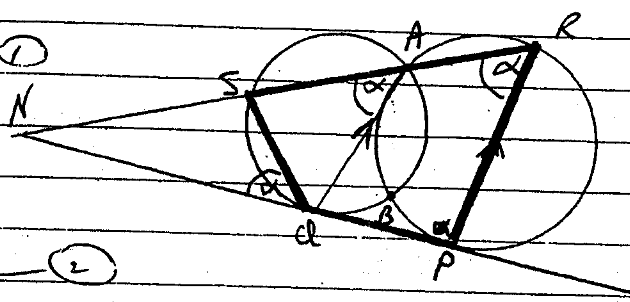
$\times x^2$: $x^2 - 1 > 0$

$(x-1)(x+1) > 0$

2 $\quad$ $x > 1$, $x < -1$ — 2

(b) $\angle SAQ = \angle ARP = \alpha$ — (1)

(corresp. $\angle$s = in $\parallel$ line PR, AQ)



$\angle SAQ = \angle SQN = \alpha$ — (2)

3 $\quad$ (angle between a tangent and a chord = angle in the alternate segment)

$\therefore \angle SQN = \angle SRP$

$\therefore$ PQRS is a cyclic quad (ext $\angle$ = inter. opp $\angle$)

(c) $\displaystyle\int_0^{2/3} \frac{dx}{4+9x^2} = \frac{1}{9}\int_0^{2/3} \frac{dx}{\frac{4}{9}+x^2} = \frac{1}{9}\cdot\frac{3}{2}\left[\tan^{-1}\left(\frac{3x}{2}\right)\right]_0^{2/3}$

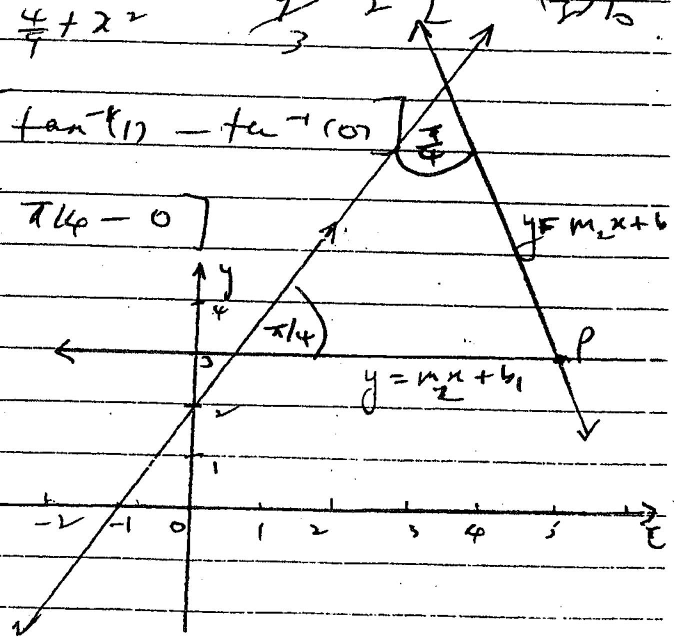
3 $\quad = \frac{1}{6}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$ $\frac{\pi}{4}$

$= \frac{1}{6}\left[\pi/4 - 0\right]$

$= \pi/24$

(d) $2x - y + 2 = 0$

x	0	-1
y	2	0



$m_1 = 2$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| \Rightarrow$

Q1(d)

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan \pi/4 = \left|\frac{2 - m_2}{1 + 2m_2}\right|$$

4

$$1 = \left|\frac{2 - m_2}{1 + 2m_2}\right|$$

$$1 + 2m_2 = \pm(2 - m_2)$$

$1 + 2m_2 = 2 - m_2$

$3m_2 = 1$

$m_2 = 1/3$

$1 + 2m_2 = -2 + m_2$

$m_2 = -3$

$m_2 = \frac{1}{3}$, $(5, 3)$

$y - y_1 = m(x - x_1)$

$y - 3 = \frac{1}{3}(x - 5)$

$3y - 9 = x - 5$

$x - 3y + 4 = 0$

$m_2 = -3$, $(5, 3)$

$y - 3 = -3(x - 5)$

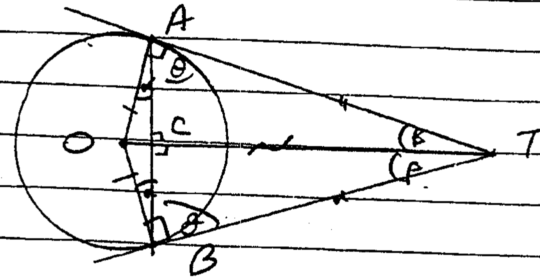
$y - 3 = -3x + 15$

$3x + y - 18 = 0$

Q2 (a) Let $\angle OAB = \alpha$

and $\angle ATB = 2\beta$.

and $\angle BAT = \theta$



$\Delta OAT \equiv \Delta OBT$ (RHS)

$\therefore \angle ATO = \angle BTO = \beta$

(Corresp. $\angle$s of $\equiv\Delta$s are $=$)

$\Delta ACT \equiv \Delta BCT$ (SAS)

$\therefore \angle ACT = \angle BCT = 90°$ ($=$ supp. $\angle$s of a st. angle)

4

In ΔOAT, $\alpha + \theta = 90°$ (compl. $\angle$s of a L)

In ΔCAT, $\beta + \theta = 90°$ ($\angle$ sum of a Δ)

$\therefore \alpha = \beta$

$\therefore \angle ATB = 2\beta = 2\alpha = 2 \times \angle OAB$.

$\therefore \dfrac{\angle OAB}{\angle ATB} = \dfrac{1}{2}$

(b) $\sin 2\theta = \sin\theta$

$2\sin\theta\cos\theta = \sin\theta$

$\sin\theta(2\cos\theta - 1) = 0$.

3

$\therefore \sin\theta = 0$ or $\cos\theta = \frac{1}{2}$

$\theta = n\pi$ or $\theta = 2n\pi \pm \pi/3$

(c) $3^{3n} + 2^{n+2}$ is a multiple of 5.

S1 $n=1$ $3^{3n} + 2^{n+2} = 3^3 + 2^3 = 27 + 8 = 35 = 5 \times 7$

$\therefore$ True for $n=1$

S2 $n=k$: $3^{3k} + 2^{k+2} = 5P$

$\therefore 3^{3k} = 5P - 2^{k+2}$ —— (A)

3

S3: $n = k+1$ $3^{3(k+1)} + 2^{(k+1)+2} = 5Q$ —— (B)

(B). LHS $= 3^{3k+3} + 2^{k+3}$

$= 27(3^{3k}) + 2(2^{k+2})$

$= 27(5P - 2^{k+2}) + 2(2^{k+2})$

$= 5(27P) - 2^{k+2}(27 - 2)$

Q2 (c) $LHS = 5(27p) - 2^{k+1}(25)$

$= 5[27p - 5(2^{k+1})]$

$= 5Q$

$= RHS$

$\therefore$ By Induction true for all +ve integral n.

Q3 (a) $A(7,0)$ $B(3,4)$

$+m:n$

$P = \left(\frac{+3m+7n}{+m+n}, \frac{4m+0.n}{m+n}\right)$

$\therefore P = (1, 6)$

$\therefore \frac{3m+7n}{m+n} = 1$

3

$3m+7n = m+n$

$2m = -6n$

$m = -3n$

$\therefore \frac{m}{n} = \frac{-3}{1}$

$\therefore m:n = -3:1$

(b) $\frac{d}{dx}\left(x \cdot \sin^{-1}2x + \frac{1}{2}\sqrt{1-4x^2}\right)$ $(1-4x^2)^{1/2}$

$= \sin^{-1}2x \cdot 1 + x \cdot \frac{2}{\sqrt{1-4x^2}} + \frac{1}{2} \cdot \left(\frac{1}{2}\right) \cdot (1-4x^2)^{-\frac{1}{2}} \cdot -8x$

$= \sin^{-1}(2x) + \cancel{\frac{2x}{\sqrt{1-4x^2}}} - \cancel{\frac{2x}{\sqrt{1-4x^2}}}$

$= \sin^{-1}(2x)$

4

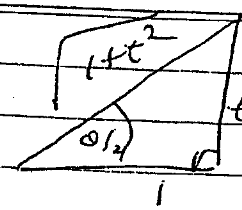
$\therefore \int_0^{1/2} \sin^{-1}(2x)\,dx = \left[x \sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2}\right]_0^{1/2}$

$= \left[\frac{1}{2}\sin^{-1}(1) + 0\right] - \left[0 + \frac{1}{2}\right]$

$= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}$

$= \frac{1}{2}\left(\frac{\pi}{2} - 1\right)$

Q3(c)(i) $t = \tan\frac{\theta}{2}$



$\tan\frac{\theta}{2} = \frac{t}{1}$

$\sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}}$

$\cos\frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}$

$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$= \frac{1}{1+t^2} - \frac{t^2}{1+t^2}$

$\cos\theta = \frac{1-t^2}{1+t^2}$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= 2\frac{t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}}$

$\sin\theta = \frac{2t}{1+t^2}$

2

(ii) $3\sin\theta + 4\cos\theta = 5$

$3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right) = 3$

$6t + 4 - 4t^2 = 5 + 5t^2$

$9t^2 - 6t + 1 = 0$

$(3t-1)^2 = 0$

$t = 1/3$

$\tan\frac{\theta}{2} = \frac{1}{3}$

$\frac{\theta}{2} = 18°26'$

$\therefore \theta = 36°52'$

3

Q4 (a) (i) $\int_0^3 \frac{x}{\sqrt{4-x}}\,dx = I$

$u = 4 - x$ $\quad$ x: 0, 3

$du = -dx$ $\quad$ u: 4, 1

$\therefore I = \int_4^1 \frac{4-u}{\sqrt{u}} - du$

$= + \int_1^4 \frac{4}{\sqrt{u}} - \sqrt{u}\, du$

$= + \int_1^4 4u^{-\frac{1}{2}} - u^{1/2} du$

$= \left[\frac{4u^{\frac{1}{2}}}{1/2} - \frac{u^{3/2}}{3/2}\right]_1^4$

$= \left[8\sqrt{u} - \frac{2}{3}\sqrt{u^3}\right]_1^4$

$= \left(8\sqrt{4} - \frac{2}{3}\sqrt{4^3}\right) - \left(8\sqrt{1} - \frac{2}{3}\sqrt{1^3}\right)$

$= \left(16 - \frac{16}{3}\right) - \left(8 - \frac{2}{3}\right)$

$= 8 - \frac{14}{3}$

$= 10/3$

$I = 3\frac{1}{3}$

Q1(a),(ii) $\quad I = \displaystyle\int_0^2 \frac{dx}{(4+x^2)^{\frac{3}{2}}}$

$x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$

x	0	2
θ	0	$\pi/4$

$$I = \int_0^{\pi/4} \frac{2\sec^2\theta \, d\theta}{\cancel{4}\,8\sec^3\theta}$$

$$= \frac{1}{4}\int_0^{\pi/4} \cos\theta \, d\theta$$

$$= \frac{1}{4}\Big[\sin\theta\Big]_0^{\pi/4}$$

$$= \frac{1}{4}\left(\sin\pi/4 - \sin 0\right)$$

$$= \frac{1}{4}\cdot\left(\frac{1}{\sqrt{2}} - 0\right)$$

3

$$= \frac{1}{4\sqrt{2}}$$

$$I = \frac{\sqrt{2}}{8}$$

Q4 (b) $f(x) = x^3 - 4x^2 + 3x + 2 = 0$

$\alpha + \beta + \gamma = -b/a = 4$

$\alpha\beta + \beta\gamma + \gamma\alpha = c/a = 3$

$\alpha\beta\gamma = -d/a = -2$

3

(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{3}{-2} = -\frac{3}{2}$

(ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= (4)^2 - 2(3)$

$= 16 - 6$

$= 10$

Q4 (b)(i) $f(x) = ax^3 + bx^2 - 8x + 3$

$(x-1)$ is a factor of $f(x)$, $\therefore f(1) = 0$

$f(1) = a(1)^3 + b(1)^2 - 8(1) + 3$

Q4 (b) (ii) :- $p(x) = 2x^3 + 3x^2 - 8x + 3$

$= (x-1)(2x^2 + 5x - 3)$

$2x^2 + 5x - 3 = 2x^2 + 6x - 1x - 3$

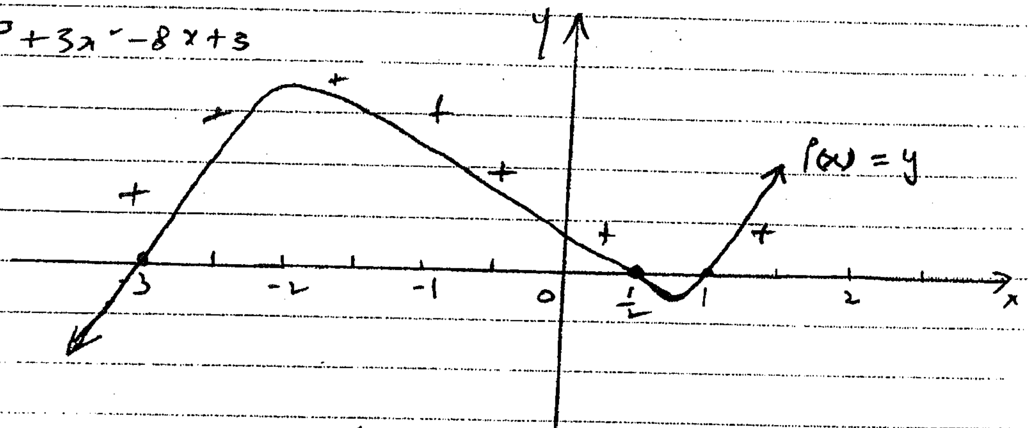
$= 2x(x+3) - 1(x+3)$

$= (x+3)(2x-1)$

$\therefore p(x) = (x+3)(2x-1)(x-1)$

2

$f(x) = 2x^3 + 3x^2 - 8x + 3$



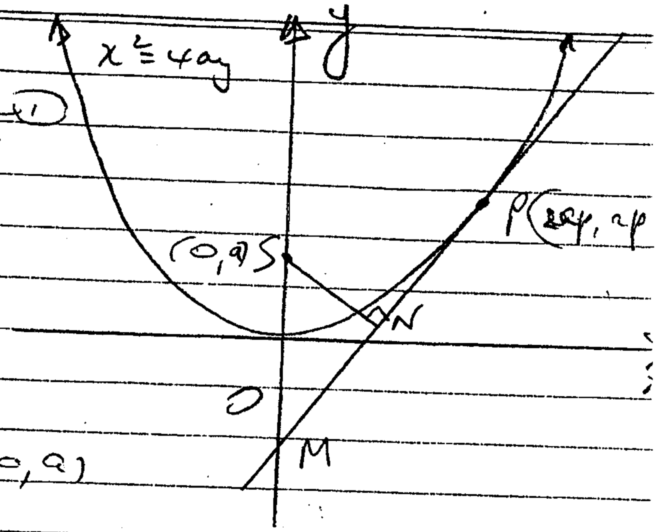
(iii) $p(x) > 0$, from the graph occurs when

$-3 < x < \frac{1}{2}$ & $x > 1$

1

Q5 (a) Tangent at P:

(i) $y - px + ap^2 = 0$ —— (1)



At M, $x = 0$

$y = -ap^2$

2

$\therefore M = (0, -ap^2)$

SN: $m = -\frac{1}{p}$, $S = (0, a)$

$y - a = -\frac{1}{p}(x - 0)$

$py - pa = -x$

$x + py = pa$ —— (2)

To find N, solve (1) & (2) simultaneously

(1) $\Rightarrow$ $y - px = -ap^2$

$y = px - ap^2$ —— (3)

Sub into (2)

$x + p(px - ap^2) = pa$

[redacted]

Q5 (a) (i) $x + p^2x - ap^3 = ap$

$(1+p^2)x = ap + ap^3$

$\cancel{(1+p^2)}x = ap\cancel{(1+p^2)}$

$x = ap.$

Sub $x = a.p$ into (3)

4

$\therefore y = p(ap) - ap^2$

$\therefore y = 0.$

$\therefore \boxed{N = (ap, 0)}$

(ii) K = midpt of MN

$N = (ap, 0)$
$M = (0, -ap^2)$

1

$\therefore K = \left(\frac{ap + 0}{2}, \frac{0 - ap^2}{2}\right)$

$\boxed{K = \left(\frac{ap}{2}, \frac{-ap^2}{2}\right)}$

(iii) $x = \frac{ap}{2}$ — (1) $\Rightarrow$ $p = \frac{2x}{a}$ — (3)

$y = -\frac{ap^2}{2}$ — (2)

Sub (3) into (2):

2

$y = -\frac{a}{2}\left(\frac{2x}{a}\right)^2$

$= -\frac{\cancel{a}}{\cancel{2}}\left(\frac{\cancel{4}x^2}{\cancel{a^2}}\right)$

$y = \frac{-2x^2}{a}$

$\therefore x^2 = -\frac{1}{2}ay$

$$\frac{\cos A - \cos(A+2B)}{2\sin B} = \sin(A+B)$$

$$LHS = \frac{\cos A - \cos A\cos 2B + \sin A\sin 2B}{2\sin B}$$

$$= \frac{\cos A(1-\cos 2B) + \sin A(2\sin B\cos B)}{2\sin B}$$

$$= \frac{\cos A(2\sin^2 B) + \sin A(2\sin B\cos B)}{2\sin B}$$

$$= \frac{\cancel{2\sin B}(\sin B\cos A + \cos B\sin A)}{\cancel{2\sin B}}$$

$$= \sin(B+A)$$

$$= \sin(A+B)$$

$$= RHS$$

5 (ii)

$$\frac{dA}{dt} = 1500\,m^2/h$$

$$r = 1250\,m$$

$$C = 2\pi r$$

$$\frac{dC}{dt} = 2\pi \times \frac{dr}{dt}$$

$$= 2\cancel{\pi} \times \frac{3}{5\cancel{\pi}}$$

$$\frac{dC}{dt} = \frac{6}{5}\,m/h$$

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

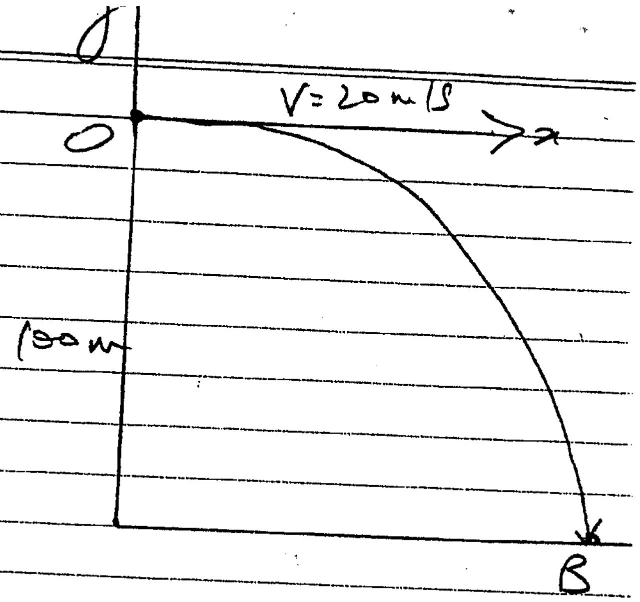
$$1500 = 2\pi \times 1250 \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{\cancel{1500}^{3}}{\cancel{2500}_{5}\pi}$$

$$\frac{dr}{dt} = \frac{3}{5\pi}\,m/h$$

3

Q1 $g \doteq 10\,m/s^2$



(i) $\ddot{x} = 0$

$\dot{x} = V\cos\theta$

$\dot{x} = 20 \times \cos 0$

$\dot{x} = 20$

$x = 20t$

$\ddot{y} = -10$

$\dot{y} = -10t + V\sin\theta$

$\dot{y} = -10t + 20 \times 0$

$\dot{y} = -10t$

$y = -5t^2$

2

(ii) From (i) $x = 20t \Rightarrow t = \frac{x}{20}$

& (ii) $y = -5t^2$

$\therefore y = -5\left(\frac{x}{20}\right)^2$

$= \frac{-5 \cdot x^2}{400}$

$y = \frac{-x^2}{80}$

$\therefore x^2 = -80y$

2

(iii) $y = -100$

$-5t^2 = -100$

$t^2 = 20$

$t = \sqrt{20}$ sec

2

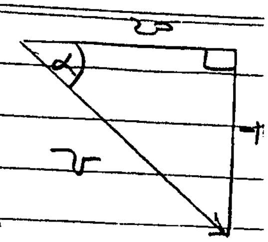
(iv) $t = \sqrt{20}$ $x = 20t$

$R = 20\sqrt{20}$ m

✓

Q6 (v) $t = \sqrt{20}$ $\dot{x} = 20$

$\dot{y} = -10(\sqrt{20})$



$v^2 = \dot{x}^2 + \dot{y}^2$

$= (20)^2 + (-10\sqrt{20})^2$

$= 400 + 2000$

$v^2 = 2400$

$\therefore v = \sqrt{2400}$

$v = 20\sqrt{6}$ m/s

$\tan\alpha = \frac{\dot{y}}{\dot{x}}$

$= \frac{-10\sqrt{20}}{20}$

$\tan\alpha = -\frac{\sqrt{20}}{2}$

$= \frac{-2\sqrt{5}}{2}$

$\tan\alpha = -\sqrt{5}$

$\therefore \alpha = 180 - 65°54'$

$\alpha = 114°5'$

2 ✓

Q7 (a) A particle is moving in SHM if
2 (1) it is travelling in a straight line and
(2) if it experiences an acceleration of the $\ddot{x} = -n^2x$

(b) (i) When $t = 0$, $x = 0$, $v = 2p$ m/s

$\ddot{x} = -9x$

$\frac{d}{dx}(\frac{1}{2}v^2) = -9x$

$\therefore \frac{1}{2}v^2 = -9\frac{x^2}{2} + c$

$\left.\begin{array}{l} x=0 \\ v=2p \end{array}\right\}$ $\frac{1}{2}(2p)^2 = -9\frac{(0)^2}{2} + c$

$\therefore c = 2p^2$

$\therefore \frac{1}{2}v^2 = -\frac{9x^2}{2} + 2p^2$

$\therefore v^2 = 4p^2 - 9x^2$

3 When it comes to rest, $v = 0$.

$\therefore 0 = 4p^2 - 9x^2$

$\therefore x^2 = \frac{4p^2}{9}$

$\therefore \boxed{x = \frac{2p}{\sqrt{9}} \text{ m}}$

07 (b)(ii) $x = a\cos(nt + \alpha)$

when $t = 0$, $x = 0$ ∴

∴ $0 = a\cos\alpha$

∴ $\alpha = \pi/2$

∴ $x = a\cos(nt + \pi/2)$

$= a\left[\cos nt \cdot \cos\frac{\pi}{2} - \sin nt \cdot \sin\frac{\pi}{2}\right]$

2

$x = -a\sin nt$

~~At rest~~, $x = \frac{2p}{\sqrt{q}} = a$

∴ $\sin nt = -1$ when $n^2 = q$

2

$\Rightarrow n = \sqrt{q}$

∴ $\sin(\sqrt{q}\,t) = -1$

∴ $\sqrt{q}\,t = \frac{3\pi}{2}$

∴ $t = \frac{3\pi}{2\sqrt{q}}$ sec

(iii) $t = \frac{\pi}{2\sqrt{4q}} = \frac{\pi}{4\sqrt{q}}$,

$x = -\frac{2p}{\sqrt{q}} \cdot \sin(\sqrt{q}\,t)$

3

$= -\frac{2p}{\sqrt{q}}\left(\sin \sqrt{q} \cdot \frac{\pi}{4\sqrt{q}}\right)$

$= -\frac{2p}{\sqrt{q}} \cdot \sin\pi/4 = -\frac{\cancel{2}^{\sqrt{2}}p}{\sqrt{q}} \cdot \frac{1}{\cancel{\sqrt{2}}}$

$x = -\frac{\sqrt{2}}{\sqrt{q}}p$ m