SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

AUGUST 2005

Trial Higher School Certificate Examination

YEAR 12

Mathematics Extension 1

Sample Solutions

Section	Marker
A	RD
B	RB
C	FN
D	AMG

Section A

Q1 (a) $\dfrac{3^n}{3^{n+1} - 3^n} = \dfrac{3^n}{3^n(3-1)}$

$= \dfrac{1}{2}$ 1

(b) $\lim\limits_{x \to 0} \dfrac{\sin 5x}{4x} = \lim\limits_{5x \to 0} \dfrac{\sin 5x}{5x} \times \dfrac{5}{4}$

$= \dfrac{5}{4}$ 1

(c) $P(3) = 27 - 27 + 3p - 14 = 1$

$\therefore 3p = 15$

$p = 5$ 2

(d) $\log_a 2a = \log_a 2 + \log_a a$

$= x + 1$ 2

(e) $P \equiv \left(\dfrac{-3 \times 6 + 2 \times -1}{-3+2}, \dfrac{-3 \times \cancel{-4} + 2 \times 5}{-3+2} \right)$

$\equiv \left(\dfrac{-20}{-1}, \dfrac{22}{-1} \right)$

$\equiv (20, -22)$ 2

(f) $\tan \theta = \left| \dfrac{m_1 - m_2}{1 + m_1 m_2} \right|$

$= \left| \dfrac{-\frac{3}{2} - \frac{1}{5}}{1 + -\frac{3}{2} \times \frac{1}{5}} \right|$

$= \left| \dfrac{-\frac{15}{10} - \frac{2}{10}}{1 - \frac{3}{10}} \right|$

$= \left| -\dfrac{17}{7} \right|$

$= \dfrac{17}{7}$

$\therefore \theta = 67^\circ 37'$ 2

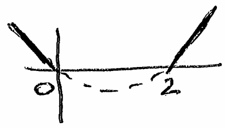
(g) $\dfrac{2}{x} \leq 1$

$2x \leq x^2$

$0 \leq x^2 - 2x$

$x(x-2) \geq 0$

$x < 0$ or $x \geq 2$



2

12

Q2

(a) (i) $y = \tan^3(5x+4)$

$y' = 3\tan^2(5x+4) \cdot \sec^2(5x+4) \cdot 5$

$= 15\tan^2(5x+4) \cdot \sec^2(5x+4)$

(ii) $y = \ln\left(\frac{2x+3}{3x+4}\right)$

$= \ln(2x+3) - \ln(3x+4)$

$y' = \frac{1}{2x+3} \times 2 - \frac{1}{3x+4} \times 3$

$= \frac{2}{2x+3} - \frac{3}{3x+4}$

(iii) $y = \cos(e^{1-5x})$

$y' = -\sin(e^{1-5x}) \cdot e^{1-5x} \cdot -5$

$= 5\sin(e^{1-5x}) \cdot e^{1-5x}$

(b) (i) $30 \times 29 \times 28 \times 27 \times 26 \times 25$

$= 427\ 518\ 000$

(ii) Choose the six finalists and then consider where they may be placed:

${}^1C_1 \times {}^{29}C_5 \times 6!$

$= 118755 \times 270$

$= 85\ 503\ 600$

(c) (i) $f(\tfrac{1}{2}) = \tan^{-1} 1$

$= \frac{\pi}{4}$

(ii) $f'(x) = \frac{1}{1+(2x)^2} \times 2$

$= \frac{2}{1+4x^2}$

$f'(\tfrac{1}{2}) = \frac{2}{1+4\times\frac{1}{4}}$

$= 1$

Section B

3 (a) (i) $y = 3\cos^{-1}(x-2)$

Domain $-1 \leq x-2 \leq 1$

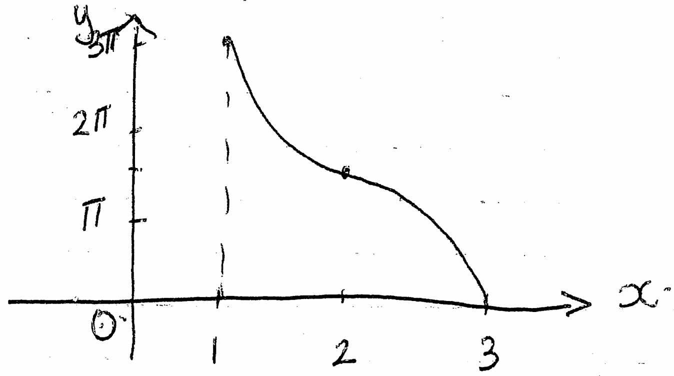
$+2 \quad +2 \quad +2$

$1 \leq x \leq 3.$

Range $0 \leq \cos^{-1}(x-2) \leq \pi$

$0 \leq 3\cos^{-1}(x-2) \leq 3\pi$

(ii) $y = 3\cos^{-1}(x-2)$



(b) $\int x \cdot \sqrt{16+x^2}\, dx$ using $u = 16+x^2$

$u = 16+x^2$

$\frac{du}{dx} = 2x$

$du = 2x\, dx$

$\frac{du}{2} = x\, dx$

becomes $\int \frac{du}{2} \cdot u^{\frac{1}{2}}$

$\frac{1}{2}\int u^{\frac{1}{2}}\, du$

$= \frac{1}{\cancel{2}} \cdot u^{\frac{3}{2}} \cdot \frac{\cancel{2}}{3} + c$

$= \frac{1}{3} u^{1\frac{1}{2}} + c$

$= \frac{1}{3} \cdot (16+x^2)\sqrt{16+x^2} + c$

or $\frac{1}{3} \cdot \left[\sqrt{16+x^2}\right]^3 + c$.

(c) general soln to $\sin 2\theta = \sqrt{3}\cos 2\theta$

$$\frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta = \frac{\sqrt{3}}{1}$$

So $2\theta = \frac{\pi}{3} + k\pi$

$$\theta = \frac{\pi}{6} + \frac{k}{2}\pi \quad \text{where } k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

(d) $4x^3 + 6x^2 + c = 0$

$c \neq 0$, roots are $\alpha, \beta, \alpha\beta$.

$a = 4$
$b = 6$
$c = 0$
$d = c$.

(i) sum of roots $\alpha + \beta + \alpha\beta = -\frac{b}{a} = -\frac{6}{4}$

product $\alpha\beta\alpha\beta = (\alpha\beta)^2 = -\frac{d}{a} = -\frac{c}{4}$

product in twos $\alpha\beta + \alpha^2\beta + \alpha\beta^2 = \frac{c}{a} = \frac{0}{4} = 0$

now since $(\alpha\beta)^2 = -\frac{c}{4}$ and $c \neq 0$

then $\alpha\beta \neq 0$ //

(ii) From above, since $\alpha\beta + \alpha^2\beta + \alpha\beta^2 = \frac{0}{4} = 0$

then $\alpha\beta(1 + \alpha + \beta) = 0$

So $\alpha\beta = 0$, but it cannot from (i)

So $1 + \alpha + \beta = 0$

$\alpha + \beta = -1$ //

(iii) since $\alpha + \beta + \alpha\beta = -\frac{6}{4} = -\frac{3}{2}$ from above

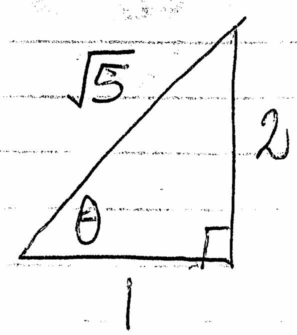
and $\alpha + \beta = -1$.

$-1 + \alpha\beta = -1\frac{1}{2}$

$\alpha\beta = -1\frac{1}{2} + 1 = -\frac{1}{2}$ //

4 (a) $\tan\theta = 2$, $0 < \theta < \frac{\pi}{2}$

evaluate $\sin\left(\theta + \frac{\pi}{4}\right)$



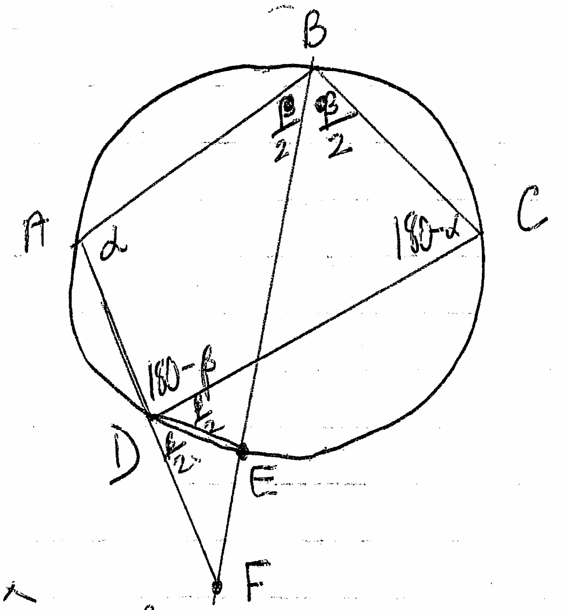
$$\sin\left(\theta+\frac{\pi}{4}\right) = \sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}$$

$$= \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} \times \frac{1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{10}} + \frac{1}{\sqrt{10}} = \frac{3}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$$

$(0.94868\ldots)$

(b) (i)



cyclic quad $\alpha + (180-\alpha) = 180°$

$\frac{\beta}{2} + \frac{\beta}{2} + (180-\beta) = 180°$

(ii) $C\hat{B}E = \frac{\beta}{2}$ stands on minor arc CE.

$C\hat{D}E = \frac{\beta}{2}$ also since it stands on minor arc CE

(iii) now $A\hat{D}C + C\hat{D}E + E\hat{D}F = 180°$ straight line angle

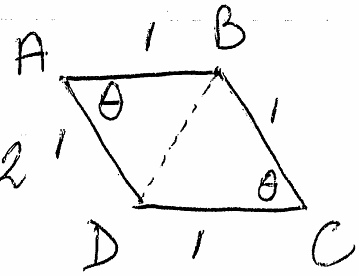
$(180-\beta) + \frac{\beta}{2} + E\hat{D}F = 180°$

$-\frac{\beta}{2} + E\hat{D}F = 0 \Rightarrow E\hat{D}F = \frac{\beta}{2}$

$\Rightarrow DE$ bisects $C\hat{D}F$

4 (c) Square ABCD side 1 unit

$$\frac{d\theta}{dt} = -0\cdot 1 \text{ radians/sec.}$$



(i) area rhombus $A = \frac{1}{2} \times 1 \times 1 \times \sin\theta \times 2$

$$A = \sin\theta$$

Now $$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$$

$$= \cos\theta \times -0\cdot 1$$

When $\theta = \frac{\pi}{6}$, $$\frac{dA}{dt} = -0\cdot 1 \times \frac{\sqrt{3}}{2} = -\frac{1}{10} \times \frac{\sqrt{3}}{2}$$

$$= -\frac{\sqrt{3}}{20} \text{ units}^2/\text{sec.}$$

$(-0\cdot 0866\ldots)$.

area is decreasing at a rate of $\frac{\sqrt{3}}{20}$ u^2/s.

(ii) shorter diagonal BD.

$$(BD)^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos\theta$$

$$= 2 - 2\cos\theta$$

$$BD = \sqrt{2(1-\cos\theta)} = \sqrt{2}.(1-\cos\theta)^{\frac{1}{2}}$$

$$\frac{dBD}{dt} = \frac{dBD}{d\theta} \times \frac{d\theta}{dt}$$

$$= \sqrt{2} \times \frac{1}{2}(1-\cos\theta)^{-\frac{1}{2}} \times \sin\theta \times -0\cdot 1$$

$$= \frac{\sqrt{2} \times \sin\theta \times -0\cdot 1}{2.\sqrt{1-\cos\theta}}$$

at $\theta = \frac{\pi}{3}$,

$$\frac{dBD}{dt} = \frac{\sqrt{2} \times \frac{\sqrt{3}}{2} \times -0.1}{2.\sqrt{1-\frac{1}{2}}}$$

$$= \frac{\frac{\sqrt{6}}{2} \times -\frac{1}{10}}{2 \times \sqrt{\frac{1}{2}}}$$

$$= \frac{\frac{\sqrt{6}}{2} \times -\frac{1}{10}}{\frac{2}{\cancel{\sqrt{2}}}} \times \frac{\sqrt{2}}{2}$$

$$= -\frac{\sqrt{12}}{40} = \frac{-2\sqrt{3}}{40} = \frac{-\sqrt{3}}{20} \text{ u/s}$$

shortes diagonal decreasing at $\frac{\sqrt{3}}{20}$ u/s

Section C

QUESTION 5

(a)

(i) $\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}=\frac{25}{216}$

(ii) $\frac{1}{6}+\left(\frac{5}{6}\right)^2\times\frac{1}{6}+\left(\frac{5}{6}\right)^4\times\frac{1}{6}$............ geometric series

$$S_\infty=\frac{\frac{1}{6}}{1-\left(\frac{5}{6}\right)^2}=\frac{6}{11}$$

(b)(i) $y=\frac{x^2}{4a}, y^1=\frac{x}{2a}=\frac{2at}{2a}=t$ = gradient of tangent

gradient of normal $=-\frac{1}{t}$

eqn. of normal is $y-at^2=-\frac{1}{t}(x-2at)$

$yt-at^3=-x+2at$

$x+ty-2at-at^3=0$ as required.

(ii) $when\ y=0, x=2at+at^3$ X $\left(2at+at^3,0\right)$

$when\ x=0, y=\frac{2at+at^3}{t}=2a+at^2$ Y $\left(0,2a+at^2\right)$

(iii) Midpoint, P is $\left(at+\frac{at^3}{2}, a+\frac{at^2}{2}\right)$

$2at=at+\frac{at^3}{2}$ $\quad$ $at^2=a+\frac{at^2}{2}$

$4at=2at+at^3$ $\quad$ $2at^2=2a+at^2$

$4=2+t^2$ $\quad$ $2t^2=2+t^2$

$t=\pm\sqrt{2}$ $\quad$ $t=\sqrt{2},\ t>0$

(c)(i) $\angle TOP=\pi-\phi$

$\tan\angle TOP=\frac{PT}{r}=-\tan\phi,\ PT=-r\tan\phi$

area ΔTOP = area sector TOA (given)

$\frac{1}{2}r\times PT=\frac{1}{2}r^2\phi$

$-r\tan\phi=r\phi$

$-\tan\phi=\phi$

$\phi+\tan\phi=0$ as required.

(ii) $a_1=a-\frac{f(a)}{f^1(a)}$ = $a_1=2-\frac{f(2)}{f^1(2)}$

$2-\frac{2+\tan 2}{1+\sec^2 2}$ = $2-\frac{-0\bullet 185}{6\bullet 774}$

$2\cdot 03\ (2d.p.)$

QUESTION 6

(a)(i) if $x=\alpha\cos(2t+\beta)$

$\frac{dx}{dt}=-2\alpha\sin(2t+\beta)$

$\frac{d^2x}{dt^2}=-4\alpha\cos(2t+\beta)=-4x$ (a possible equation)

(ii) $v^2=n^2\left(\alpha^2-x^2\right)$, $n=2$ and $x=4$ when $v=2$

$4=4(\alpha^2-16)$

$\alpha=\sqrt{17}\ m$

(ii) Max velocity when displacement = 0

$v^2=4(17-0)$

$v=2\sqrt{17}m/s$

(b) When $n=1$, $1^3=\frac{1}{4}\times1^2\times2^2$- P(1) is true

Assume P(k) is true $1^3+2^3.....+k^3=\frac{1}{4}k^2(k+1)^2$

if $n=k+1$,

$1^3+2^3.....+k^3+(k+1)^3=\frac{1}{4}(k+1)^2(k+2)^2$

LHS = $\frac{1}{4}k^2(k+1)^2+(k+1)^3$ (using assumption)

$=(k+1)^2\left(\frac{1}{4}k^2+k+1\right)$

$=(k+1)^2\frac{1}{4}\left(k^2+4k+4\right)$

$=\frac{1}{4}(k+1)^2(k+2)^2$

= RHS

P$(k+1)$ is true if P(k) is true. P(1) is true.

$\therefore$, by Mathematical Induction, P(n) is true for any integer n $\geq$ 1

(c)(i) $1-x^2>0$ $\quad -1<x<1$

(ii) If $y=f(x)$, the inverse function is

$x=\frac{y}{\sqrt{1-y^2}}$

$x^2=\frac{y^2}{1-y^2}$

$x^2-x^2y^2=y^2$

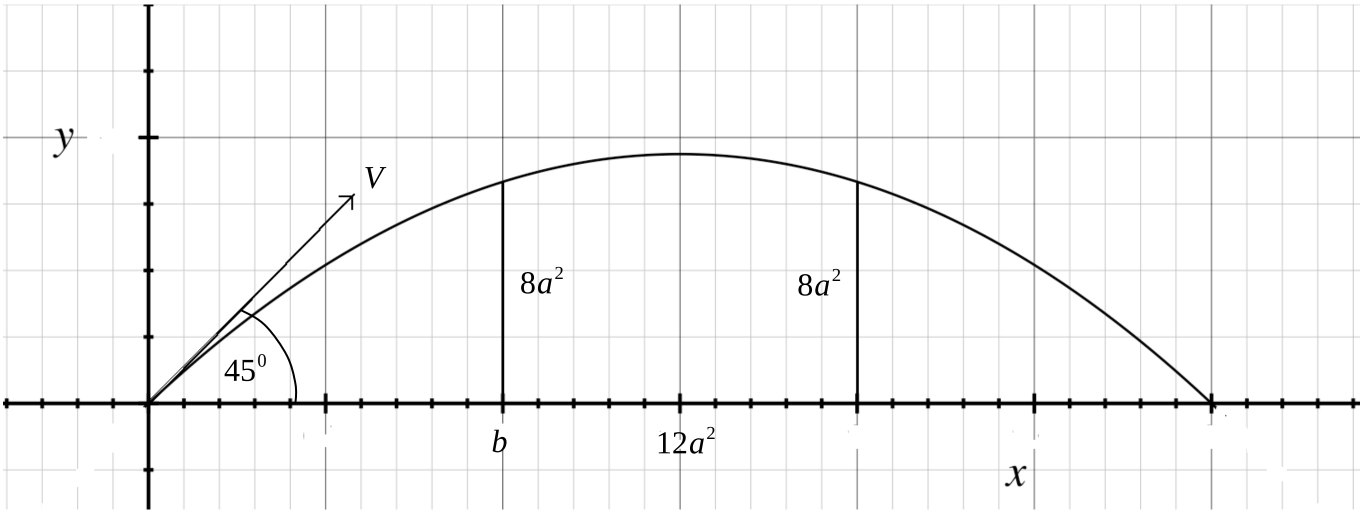
$y^2\left(1+x^2\right)=x^2$

$y^2=\frac{x^2}{1+x^2}$

$f^{-1}(x)=\frac{x}{\sqrt{1+x^2}}$ (odd function)

Section D

(7) (a)



(i)

$\ddot{x} = 0$

Integrate w.r.t. t

$\dot{x} = K$

When $t = 0,\ \dot{x} = \dfrac{V}{\sqrt{2}}$

$\therefore K = \dfrac{V}{\sqrt{2}}$

$\therefore \dot{x} = \dfrac{V}{\sqrt{2}}$

Integrate w.r.t. t

$x = \dfrac{Vt}{\sqrt{2}} + M$

When $t = 0,\ x = 0$

$\therefore M = 0$

$\therefore x = \dfrac{Vt}{\sqrt{2}}$

$\ddot{y} = -g$

Integrate w.r.t. t

$\dot{y} = -gt + L$

When $t = 0,\ \dot{y} = \dfrac{V}{\sqrt{2}}$

$\therefore L = \dfrac{V}{\sqrt{2}}$

$\therefore \dot{y} = \dfrac{V}{\sqrt{2}} - gt$

Integrate w.r.t. t

$y = \dfrac{Vt}{\sqrt{2}} - \dfrac{1}{2}gt^2 + N$

When $t = 0,\ y = 0$

$\therefore N = 0$

$\therefore y = \dfrac{Vt}{\sqrt{2}} - \dfrac{1}{2}gt^2$

(ii) From the equation for x:

$t = \dfrac{\sqrt{2}x}{V} \qquad \therefore y = \dfrac{V}{\sqrt{2}}\dfrac{\sqrt{2}x}{V} - \dfrac{1}{2}g\left(\dfrac{\sqrt{2}x}{V}\right)^2$

$y = x - \dfrac{gx^2}{V^2}$

(iii) The range is achieved when $y = 0$

$$\therefore x - \frac{gx^2}{V^2} = 0$$

$$x\left(1 - \frac{gx}{V^2}\right) = 0$$

$$\therefore 1 - \frac{gx}{V^2} = 0$$

$$x = \frac{V^2}{g} \qquad \text{(Range)}$$

(iv) (α) By symmetry the second post is b units from point of impact

$$\therefore (x_R =)\ \frac{V^2}{g} = 2b + 12a^2$$

(β) When $x = b,\ \ y = 8a^2$, in the equation from (ii):

$$8a^2 = b - \frac{gb^2}{V^2}$$

(v) From (α):

$$2b = \frac{V^2}{g} - 12a^2$$

$$\therefore b = \frac{V^2}{2g} - 6a^2$$

$$\therefore \frac{V^2}{2g} = b + 6a^2$$

$$\therefore V^2 = 2g\left(b + 6a^2\right)$$

$$= g\left(2b + 12a^2\right)$$

$$\therefore \ V = \sqrt{g}\sqrt{2b + 12a^2} \qquad \text{———— (*)}$$

Hence it remains to prove that $2b = 24a^2$.

Now $\dfrac{g}{V^2} = \dfrac{1}{2b+12a^2}$

So $$\begin{aligned} 8a^2 &= b - \frac{gb^2}{V^2} \\ &= b - \frac{b^2}{2b+12a^2} \\ &= \frac{2b^2+12a^2b-b^2}{2b+12a^2} \end{aligned}$$

$$\begin{aligned} \therefore 16a^2b + 96a^4 &= 2b^2 + 12a^2b - b^2 \\ &= b^2 + 12a^2b \end{aligned}$$

$$\therefore b^2 - 4a^2b - 96a^4 = 0$$

$$\begin{aligned} \therefore b &= \frac{4a^2 \pm \sqrt{16a^4 + 4\times 96a^4}}{2} \\ &= \frac{4a^2 \pm 4\sqrt{a^4+24a^4}}{2} \\ &= \frac{4a^2 \pm 4\times 5a^2}{2} \\ &= 12a^2 \quad \text{(Neg result extraneous)} \end{aligned}$$

$\therefore$ In equation (*)

$$\begin{aligned} V &= \sqrt{g}\sqrt{36a^2} \\ &= 6a\sqrt{g} \qquad \text{As required.} \end{aligned}$$