Year 12 ... (header cut off)

tion A

on 1

$e^{2x} \cdot \sin x =$

$\cdot \cos x + \sin x \cdot 2e^{2x}$

$(\cos x + 2\sin x)$ ②

$2x + y = 4 \rightarrow y = 4 - 2x$, $m_1 = -2$

$x - y = 2 \rightarrow y = x - 2$, $m_2 = 1$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-2-1}{1+(-2)(1)}\right| = 3$$

$\theta = 72°$ (nearest degree) ②

Choose 3 from 8 and 4 from 6

$\therefore$ No of ways $= {}^8C_3 \times {}^6C_4$ ①
$= 56 \times 15$
$= 840$

$$\int_0^2 \frac{dx}{4+x^2} = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{2}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{0}{2}\right)$$
$$= \frac{1}{2}\times\frac{\pi}{4} - \frac{1}{2}\times 0$$
$$= \frac{\pi}{8}$$ ③

e) $\int_0^1 \frac{4x}{2x+1}\,dx$

$u = 2x+1$ also $2x = u - 1$

$\frac{du}{dx} = 2$

$du = 2\,.dx$

$x = 1 \quad u = 3$

$x = 0 \quad u = 1$

$$= \int \frac{4x}{2x+1}\,dx$$
$$= \int_1^3 \frac{u-1}{u}\,du$$
$$= \int_1^3 1 - \frac{1}{u}\,du$$
$$= \left[u - \log_e u\right]_1^3$$
$$= (3 - \log_e 3) - (1 - \log_e 1)$$
$$= 2 - \log_e 3$$ ④

Question 2

a) $x = 3\sin\left(4t + \frac{\pi}{4}\right)$

i) amplitude $= 3$ ($\frac{1}{2}$)

period $= \frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2}$ ($\frac{1}{2}$)

(ii) $v = \frac{dx}{dt}$

$= 3\cos\left(4t + \frac{\pi}{4}\right).4$

$= 12\cos\left(4t + \frac{\pi}{4}\right)$ ①

when $t = 0$

$v = 12\cos\left(\frac{\pi}{4}\right)$

$= 12 \times \frac{1}{\sqrt{2}}$

$= 12 \times \frac{\sqrt{2}}{2}$

$= 6\sqrt{2}$ ①

(iii) max acc occurs when $v = 0$

$v = 12\cos\left(4t + \frac{\pi}{4}\right)$

$0 = 12\cos\left(4t + \frac{\pi}{4}\right)$

$4t + \frac{\pi}{4} = \frac{\pi}{2}$

$4t = \frac{\pi}{4}$

$t = \frac{\pi}{16}$ ①

$a = \frac{dv}{dt}$

$= -48\sin\left(4t + \frac{\pi}{4}\right)$

at $t = \frac{\pi}{16}$

$a = -48 \left(4\left(\frac{\pi}{16}\right) + \frac{\pi}{4}\right)$

$= -48\sin\left(\frac{\pi}{2}\right)$

$= -48$ ①

$\therefore$ max acceleration is -48 m/

iv) $v^2 = n^2(a^2 - x^2)$ $n = 4$, $a = 3$, $x = 2$

when $x = 2$

$v = \pm\sqrt{80}$

$\therefore$ speed is $\sqrt{80}$ m/s or 4√

$x^3+bx^2+cx+d = x(x+3)(x-3)$

$P(0)=0 \qquad 0=d$

$P(3)=0 \qquad 0 = 27+9b+3c \quad (1)$

$P(-3)=0 \qquad 0 = -27+9b-3c \quad (2)$

$(1)+(2)$

$0 = 18b$

$b=0$ ✓

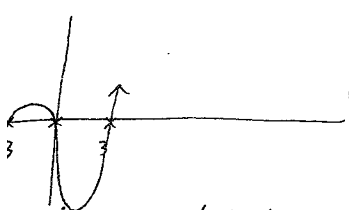
$\therefore 0 = 27+3c$

$27=-3c$

$c=-9$ ✓

$b=0, \quad c=-9, \quad d=0$ (2)

ii) $y = P(x)$
$= x^3-9x$



(1)

at $x=1 \quad y=-8$

x

$\frac{x^2-9}{x} = 0 \qquad x \neq 0$

$x^2-9=0$

$(x-3)(x+3)=0$

$x=3,-3$ are critical pts

$-3 \qquad 0 \qquad 3$

at $x=1 \quad \frac{x^2-9}{x} = -8$

etc

$-3 \leq x < 0$ or $x \geq 3$

(2)

or

$x(x^2-9) \qquad Q \geq 0$

$\frac{9}{5} \geq 0$

let $u = \sin x \qquad y = \tan^{-1} u$

$\frac{du}{dx} = \cos x \qquad dy/du = \frac{1}{1+u^2}$

$\frac{dy}{dx} = \cos x \times \frac{1}{1+\sin^2 x}$

$= \frac{\cos x}{1+\sin^2 x}$ (2)

ii) $\int_0^1 \frac{dx}{\sqrt{2-x^2}} = \left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1$

$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}\frac{0}{\sqrt{2}}$

$\doteq \frac{\pi}{4}$ (2)

3b) $\frac{dT}{dt} = k(T-T_0) \quad \therefore T = T_0 + Ae^{kt}$

i) $T_0 = -20$

At $t=0, \quad T=100$

$100 = -20+A$

$A = 120$

at $t=3 \quad T=70$

$70 = -20+120e^{3k}$

$e^{3k} = \frac{9}{12}$

$k = \frac{1}{3}\ln\frac{3}{4}$

$\doteq -0.095894...$ (2)

ii) let $t=0$ when ap placed in environment $20°C$

$T = 20 + Be^{kt}$

at $t=0 \quad T=70°$

$70 = 20 + Be^0$

$B = 50$

$T = 20+50e^{kt}$

where $k \doteq 0.095894...$

at $t=15 \quad T = 20+50e^{15k}$

$\doteq 31.86°$ (3)

3c)

$h(x) = \sin^{-1} x + \cos^{-1} x \quad 0 \le x \le 1$

$h'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1+x^2}}$

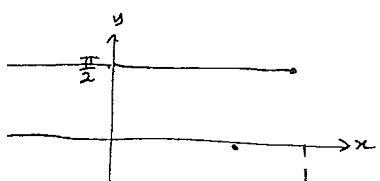
$= 0$ ①

$\therefore$ The function $h(x)$ has a gradient of zero

$\therefore$ The fn is a straight line of the form $y = a$ where a is a constant.

when $x = 0$ $h(x) = \frac{\pi}{2}$

$\therefore h(x) = \frac{\pi}{2}$ for $0 \le x \le 1$



②

Question 4

$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi R^3\right)$

$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) \times \frac{dR}{dt}$

$= 4\pi R^2 \frac{dR}{dt}$

hence $\frac{dV}{dt} = 10$ and $4\pi R^2 = 500$

$\frac{dR}{dt} = \frac{10}{500} = \frac{1}{50}$

SECTION B

4) b) The constant term of

$\left(3x^2 - \frac{1}{2x}\right)^9$

$U_{r+1} = {}^9C_r \, (3x^2)^{9-r} \left(-\frac{1}{2x}\right)^r$

$= A\,(x^2)^{9-r}\,(x^{-1})^r$

$= A\, x^{18-2r} \cdot x^{-r}$

$= A x^{18-3r}$

where A is the num. coeff of U_{r+1}

now $18 - 3r = 0 \therefore r = 6$

$\therefore u_7 = {}^9C_6\,(3x^2)^{9-6}\left(-\frac{1}{2x}\right)^6$

$= \frac{9!}{6!\,3!}\,(3x^2)^3\left(-\frac{1}{2x}\right)^6$

$= \frac{567}{16}$ ④

4c) Chord PQ $y - ap^2 = \frac{p+q}{2}(x - 2ap)$

will be satisfied by $(0, a)$

$a - ap^2 = \frac{p+q}{2}(0 - 2ap)$

$a - ap^2 = (p+q)(-ap)$

$1 - p^2 = -p^2 - pq$

$1 = -pq$

$\therefore pq = -1$ ①

ii) Midpt $pq = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$

$= \left(a(p+q), \frac{a}{2}(p^2+q^2)\right)$

①

4c)(iii) $x = a(p+q)$

$p + q = \frac{x}{a}$

$pq = -1$

$y = \frac{a}{2}(p^2+q^2)$

$\frac{2y}{a} = (p+q)^2 - 2pq$

$\frac{2y}{a} = \left(\frac{x}{a}\right)^2 - 2$

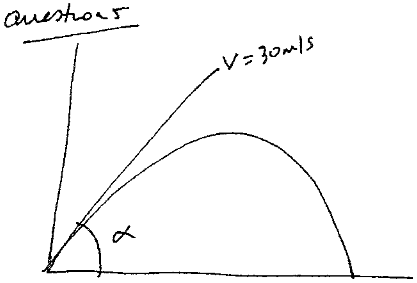
$2y = \frac{x^2}{a} - 2a$

$2ay = x^2 - 2$

$x^2 = 2ay + 2$

$x^2 = 2(ay + 1)$

Question 5



a) $\ddot{x} = 0$

$\dot{x} = t + c_1$ but when $t = 0$ $\dot{x} = V\cos\alpha$ $\therefore c_1 = V\cos\alpha$

$\therefore \dot{x} = V\cos\alpha$

$= 30\cos\alpha$

$x = 30t\cos\alpha + c_2$ but when $t = 0$ $\therefore c_2 = 0$

$\boxed{\therefore x = 30t\cos\alpha}$

b) $\ddot{y} = -g$

$= -10$

$\dot{y} = -10t + c_3$

now when $t = 0$ $\dot{y} = V\sin\alpha$

$= 30\sin\alpha$

$\therefore c_3 = 30\sin\alpha$

$\dot{y} = -10t + 30\sin\alpha$

$y = -5t^2 + 30t\sin\alpha + c_4$

when $t = 0$ $y = 0$ $\therefore$

$\therefore y = -5t^2 + 30t\sin\alpha$

$-5t^2+30t\sin\alpha$

$= +t(30\sin\alpha - 5t)$

0 or $0 = 30\sin\alpha - 5t$

$5t = 30\sin\alpha$

$t = 6\sin\alpha$

; at $t = 6\sin\alpha$

$= (30\cos\alpha)(6\sin\alpha)$

$= 2(90\cos\alpha.\sin\alpha)$

$= 90\sin 2\alpha$ (2)

max dist, x, th
mp can reach will
ur when $\sin 2\alpha = 1$

: the max dist is 90m.

(1)

$= 30t\cos\alpha$

$= \frac{x}{30\cos\alpha}$

$= -5\left(\frac{x}{30\cos^2\alpha}\right)^2 + 30\left(\frac{x}{30\cos\alpha}\right)\sin\alpha$

now find α when $y=0$, $x=60$

$= -5\left(\frac{60}{30\cos\alpha}\right)^2 + 30\left(\frac{60}{30\cos\alpha}\right)\sin\alpha$

$= -5\left(\frac{4}{\cos^2\alpha}\right) + 60\,\overset{\tan}{\cancel{\sin}}\,\alpha$

$0 = -20(\tan^2\alpha+1) + 60\tan\alpha$

$= \tan^2\alpha - 3\tan\alpha + 1$

$\tan\alpha = \frac{3 \pm \sqrt{9-4(1)(1)}}{2}$

$= \frac{3 \pm \sqrt{5}}{2}$

$\alpha = 69°05', 20°54'$

OR

$x = 30t\cos\alpha$

$60 = 30t\cos\alpha$

$2 = t\cos\alpha$

$t = \frac{2}{\cos\alpha}$

$\frac{2}{\cos\alpha} = 6\sin\alpha$

$2 = 6\sin\alpha\cos\alpha$

$2 = 3(2\sin\alpha\cos\alpha)$

$\frac{2}{3} = \sin 2\alpha$

$2\alpha = \sin^{-1}\left(\frac{2}{3}\right)$

$2\alpha = 41.8, 138.2$

$\alpha = \frac{1}{2}\sin^{-1}\left(\frac{2}{3}\right)$

$\alpha = 20°54', 69°05'$

$V = 40$ m/s $\alpha = 0$ $g = 10$

$y = -\frac{1}{2}gt^2 + Vt\sin\alpha$

$y = -\frac{1}{2}(10)t^2 + 40t\sin 0$

$y = -5t^2$

now when $t=0$ $y=5$

$\therefore$ $y = -5t^2 + 5$

$x = Vt\cos\alpha$

$x = 40t\cos 0$

$x = 40t$

now when $y=0$ the pumped water reaches the ground

$0 = -5t^2+5$

$t = \pm 1$ $(t \geq 0)$

$\therefore t = 1$

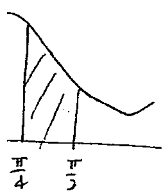
at $t=1$ $x = 40$

$\therefore$ The water will hit the ground 40m away. It will not reach the fire.

i) Show that $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 x \, dx = \frac{\pi}{8} - \frac{1}{4}$

$$\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos 2x + 1)\,dx = \frac{1}{2}\left[\frac{1}{2}\sin 2x + x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}\sin\pi + \frac{\pi}{2}\right) - \left(\frac{1}{2}\sin\frac{\pi}{2} + \frac{\pi}{4}\right)\right]$$



$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2} - \frac{\pi}{4}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \frac{\pi}{8} - \frac{1}{4} \text{ units}^2 \quad (2)$$

$$V = \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2 + \cos x)^2 \, dx \quad (1)$$

$$= \pi\int 4 + 4\cos x + \cos^2 x \, dx$$

$$= \pi\left[4x + 4\sin x + \int \cos^2 x \, dx\right]$$

$$= \pi\left[4x + 4\sin x + \int \cos(2x+1)\,dx\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \pi\left[4x + 4\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \pi\left[\frac{\pi}{8} - \frac{1}{4}\right]$$

$$= \pi\left(\left[\frac{4\pi}{2} + 4\sin\frac{\pi}{2}\right] - \left[\frac{4\pi}{4} + 4\sin\frac{\pi}{4}\right] + \pi\left[\frac{\pi}{8} - \frac{1}{4}\right]\right)$$

=

$$= \pi\left[\frac{9\pi}{8} + \frac{15}{4} + 2\sqrt{2}\right] u^3$$

6b) i) $v^2 = 16x - 4x^2 + 20$

$$a = \frac{d}{dx}\left[\frac{1}{2}v^2\right]$$

$$= \frac{d}{dx}\left[8x - 2x^2 + 10\right]$$

$$a = 8 - 4x$$

$$a = 4(2 - x)$$
$$= -4(x - 2) \quad (2)$$

ii) centre of motion is $x = 2$

iii) when $v = 0$

$$0 = 16x - 4x^2 + 20$$
$$= -4(x^2 - 4x - 5)$$
$$= -4(x - 5)(x + 1) \quad (2)$$
$$x = 5 \text{ or } x = -1$$

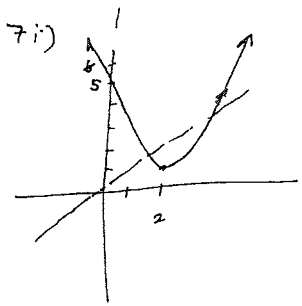
$\therefore$ The amplitude $= 3$

(iv) $n^2 = 4 \quad \therefore n = 2 \; (n > 0) \quad (1)$

v) Max speed when $x = 1$ (centre of motion)

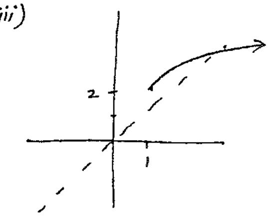
$$v^2 = 4(3 + 2 - 1)$$
$$= \pm 4 \quad (2)$$

7 i)



ii) $x \geq 2$ for $f^{-1}(x)$ (1)

iii)



(1.5)

iv) domain of inverse fn is $x \geq 1$ (1)

7a) $(1+x)^n = {}^nC_0 + {}^nC_1 x^1 + {}^nC_2 x^2 + \dots + {}^nC_n x^n$

$n(1+x)^{n-1} = {}^nC_1 + 2\,{}^nC_2 x + \dots + n\,{}^nC_n x^{n-1}$ (1)

$x = 1$

$n \cdot 2^{n-1} = {}^nC_1 + 2\,{}^nC_2 + \dots + n\,{}^nC_n$ (1)

$= \sum_{r=1}^{n} r \cdot {}^nC_r$ (1)

v) At $x = a$, $a < 2$

$$f(a) = (a-2)^2 + 1$$
$$= A$$

$$f^{-1}(f(a)) = f^{-1}(A)$$
$$= 2 + \sqrt{A - 1}$$
$$= 2 + \sqrt{(a-2)^2 + 1 - 1}$$

$$= 2 + |(a-2)| \quad \text{since } a <$$
$$= 2 + 2 - a$$
$$= 4 - a$$

$\therefore f^{-1}[f(a)] = 4 - a$

vi) Since the inverse fn is the reflection of $f(x)$ about the line $y = x$, pts of int. occurs when $y = x$

from $y = (x-2)^2 + 1$ & $y = x$

$$x = (x-2)^2 + 1$$
$$x = (x^2 - 4x + 4) + 1$$
$$x = x^2 - 4x + 5$$
$$0 = x^2 - 5x + 5$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$