SYDNEY BOYS HIGH SCHOOL
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2003

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Sample Solutions

QUESTION 1

(a) (i) $\sin 3x \times 1 + x \times 3\cos 3x$

$\sin 3x + 3x\cos 3x$ 1

(ii) $e^{1-x^2} \times -2x = -2xe^{1-x^2}$ 1

(b) $y = \frac{2}{3}x + \frac{8}{3}$ $y = 5x - 9$

$m_1 = \frac{2}{3}$ $m_2 = 5$

$\tan\theta = \left|\dfrac{\frac{2}{3} - 5}{1 + \frac{2}{3}\times 5}\right|$

$\theta = 45^\circ$ 2

(c) (i) $\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2$

$\frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{8}$ 2

(ii) $\frac{1}{3}\int_0^1 \frac{3x^2}{x^3+2}$

$= \frac{1}{3}\left[\log(x^3+2)\right]_0^1$

$= \frac{1}{3}(\log 3 - \log 2)$

≈ 0.135 or $\frac{1}{3}\log\frac{3}{2}$ 2

(d) $\dfrac{5!}{8!}$ $\dfrac{1}{336}$ 2

(e) $\theta < 0$ or $\theta > 4$ 2

$1 - \frac{4}{\theta} > 0$ or $\dfrac{\theta^2(\theta-4)}{\theta} > 0$

$1 > \frac{4}{\theta}$ $\theta^2 - 4\theta > 0$

$\theta < 0$ or $\theta > 4$ $\theta(\theta - 4) > 0$

0 4

$\theta < 0$ or $\theta > 4$

QUESTION 2

(a) (i) $\alpha + \beta + \gamma = -\frac{b}{a} = \frac{5}{2}$

$\alpha\beta\gamma = -\frac{1}{2}$

(ii) $(\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$\frac{25}{4} - 2 \times \frac{-3}{2} = 9\frac{1}{4}$ 2

(b) $\int_0^{2\sqrt{3}} \frac{x}{\sqrt{U}} \times \frac{du}{2x}$ $U = x^2 + 4$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\frac{1}{2}\int U^{-\frac{1}{2}}\,du$ $x = 2\sqrt{3}$, $U = 16$

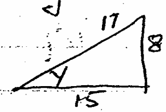
$x = 0$, $U = 4$

$= \frac{1}{2}\left[\frac{U^{\frac{1}{2}}}{\frac{1}{2}}\right]_4^{16}$

$= [4 - 2] = 2$

(c) $y = \tan^{-1}\frac{8}{15}$

$\tan y = \frac{8}{15}$



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$\cos y = \frac{15}{17}$

(d) (i) $A\left(\frac{1}{3}, 2\pi\right)$ $C\left(-\frac{1}{3}, 0\right)$

(ii) $y = \pi + 2\sin^{-1}3x$

$\frac{dy}{dx} = 2 \times \frac{3}{\sqrt{1-9x^2}}$ 1

$x = 0$ grad of tangent $= \frac{6}{\sqrt{1-0}}$

$= 6$ 1

QUESTION 3

(a) $y = f(x)$ $\quad y = 1 + e^{2x}$

$f^{-1}(x)$ $\quad x = 1 + e^{2y}$

$e^{2y} = x - 1$

$2y = \log(x-1)$

$y = \frac{1}{2}\log(x-1)$

Domain $x > 1$ Range All real y 2

(b) Rational roots when $\Delta = b^2 - 4ac = 0$ or has rational square root

$36 - 4(5k-4)(6k+3) = 0$

$36 - 120k^2 + 36k + 48 = 0$

$-120k^2 + 36k + 84 = 0$

$10k^2 - 3k - 7 = 0$

$(10k + 7)(k - 1) = 0$

rational roots when $k = -\frac{7}{10}$ or 1 3

multiple solutions when $-120k^2 + 36k + 84$ has rational roots

(c) (i) $\angle ABG = \angle BEG$ (angle in alternate segment)

$\angle BEG = CEH$ (vertically opposite)

$\angle CEH = \angle DCH$ (angle in alternate segment.

$\therefore \angle ABG = \angle DCH$ as required 2

(ii) $\angle CBH = \angle BGC$ (alternate segment)

$\angle BCE = \angle CHE$ " "

$\therefore \angle GBC = \angle HCB$ (angle sum of $\triangle$) 2

$\therefore \triangle BCG \parallel\!| \triangle BCH$ (equiangular)

(d) (i) $a = 2^N$ $\quad r = 2^{-1}$ $\quad S_n = \frac{a}{1-r}$

T_n $\quad ar^{n-1} = 2^{-N}$ $\quad = \frac{2^N}{1-\frac{1}{2}}$

$2^N(2^{-1})^{n-1} = 2^{-N}$ $\quad = 2 \cdot 2^N = 2^{N+1}$

$2^{-n+1} = 2^{-2N}$

$-n+1 = -2N$ 1

$n = 2N+1$ 2

4) (a) $\sin 2\theta = 2\sin\theta\cos\theta$

(i) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$.

$$\cos 2\theta = \begin{cases} \cos^2\theta - \sin^2\theta \\ 1 - 2\sin^2\theta \\ 2\cos^2\theta - 1 \end{cases}$$

so $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$1 - 2\sin^2\frac{\theta}{2}$

$2\cos^2\frac{\theta}{2} - 1$.

$$f(\theta) = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cancel{1} + 2\cos^2\frac{\theta}{2} - \cancel{1} + \cos\frac{\theta}{2}}$$

$$= \frac{\sin\frac{\theta}{2}\cancel{\left[2\cos\frac{\theta}{2} + 1\right]}}{\cos\frac{\theta}{2}\cancel{\left[2\cos\frac{\theta}{2} + 1\right]}}$$

$= \tan\frac{\theta}{2} = t$. 3

(ii) $f(\theta) = \tan\frac{\theta}{2} = 1$ general soln.

If $\tan\theta = a$, then $\theta = n\pi + \tan^{-1}(a)$

$\tan\frac{\theta}{2} = 1$, then $\frac{\theta}{2} = n\pi + \frac{\pi}{4}$

$\theta = 2n\pi + \frac{\pi}{2}$. 1

4 (b) (i) $t = 2x^2 - 5x + 3$

$\frac{dt}{dx} = 4x - 5$

$\frac{dx}{dt} = v = \frac{1}{4x-5}$ (1)

(ii) using $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2(4x-5)^2}\right)$

$= \frac{d}{dx}\left[\frac{1}{2}(4x-5)^{-2}\right]$

$= -(4x-5)^{-3} \times 4$

$= \frac{-4}{(4x-5)^3}$ (2)

(iii) (α) when $x = 2$, $v = \frac{1}{3}$ cm/s ($\frac{1}{2}$)

$a = -\frac{4}{27}$ cm/s^2 ($\frac{1}{2}$)

(β) when $t = 6$, $6 = 2x^2 - 5x + 3$.

$(2x+1)(x-3) = 0$

$x = -\frac{1}{2}, x = 3.$ (1)

take $x = 3$

at $x = 3$, $v = \frac{1}{7}$ cm/s ($\frac{1}{2}$)

$a = \frac{-4}{343}$ ($\frac{1}{2}$)

(iv) particle is travelling to the right but is slowing down

(2)

5 (a) (i) $\dfrac{\cos y - \cos(y+2\alpha)}{2\sin\alpha} = \sin(y+\alpha)$.

$$LHS = \frac{\cos y - (\cos y \cos 2\alpha - \sin y \sin 2\alpha)}{2\sin\alpha}$$

$$\frac{\cos y - (\cos y(1-2\sin^2\alpha) - \sin y\, 2\sin\alpha\cos\alpha)}{2\sin\alpha}$$

$$\frac{\cancel{\cos y} - \cancel{\cos y} + \cancel{2}\sin^2\alpha\cos y + \cancel{2}\sin\alpha\cos\alpha\sin y}{\cancel{2}\sin\alpha}$$

$$= \sin\alpha\cos y + \cos\alpha\sin y$$

$$= \sin(y+\alpha)$$

(2)

(ii) $\sin\alpha + \sin 3\alpha + \sin 5\alpha + \cdots + \sin(2n-1)\alpha = \dfrac{1-\cos 2n\alpha}{2\sin\alpha}$

step1 Prove true for $n=1$

$LHS = \sin\alpha$

$RHS = \dfrac{1-\cos 2\alpha}{2\sin\alpha} = \dfrac{1-(1-2\sin^2\alpha)}{2\sin\alpha} = \sin\alpha$

$= LHS$.

true for $n=1$.

step 2 Assume true for $n=k$ (a positive integer) so

$\sin\alpha + \sin 3\alpha + \sin 5\alpha + \cdots + \sin(2k-1)\alpha = \dfrac{1-\cos 2k\alpha}{2\sin\alpha}$

and we must prove it true for $n=k+1$, so

$\sin\alpha + \sin 3\alpha + \sin 5\alpha + \cdots + \sin(2k-1)\alpha + \sin(2k+1)\alpha = \dfrac{1-\cos 2(k+1)}{2\sin\alpha}$.

$LHS.\ \dfrac{1-\cos 2k\alpha}{2\sin\alpha} + \sin(2k+1)\alpha$.

$$\frac{1-\cos 2k\alpha}{2\sin\alpha} + \sin(2k\alpha+\alpha).$$

now using (a)(i) $\sin(y+\alpha) = \dfrac{\cos y - \cos(y+2\alpha)}{2\sin\alpha}$

then $\sin(2k\alpha+\alpha) = \dfrac{\cos 2k\alpha - \cos(2k\alpha+2\alpha)}{2\sin\alpha}$

now, $\dfrac{1-\cos \cancel{2}k\alpha}{2\sin\alpha} + \dfrac{\cos \cancel{2k}\alpha - \cos 2(k+1)\alpha}{2\sin\alpha}$

$$= \frac{1-\cos 2(k+1)\alpha}{2\sin\alpha}$$

$=$ RHS.

True for $n=k+1$.

step 3 If the statement is true for $n=k$, then it is also true for $n=k+1$. Since the statement is true for $n=1$, it follows that it must also be true for $n=2$ and so on. $\therefore$ the statement is true for all positive integers n.

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5) (b) (i) $y = \frac{x^3+4}{x^2} = \frac{x^3}{x^2} + \frac{4}{x^2} = x + 4x^{-2} = x + \frac{4}{x^2}$.

$y' = 1 - 8x^{-3} = 1 - \frac{8}{x^3}$

$y'' = 24x^{-4} = \frac{24}{x^4}$

Stat points exist when $y' = 0$, $1 - \frac{8}{x^3} = 0$

$\frac{8}{x^3} = 1 \Rightarrow x^3 = 8$

$\Rightarrow x = 2$.

At $x = 2$, $y = 2 + \frac{4}{2^2} = 3$ $(2, 3)$ (1) (min stat pt) $y'' > 0$

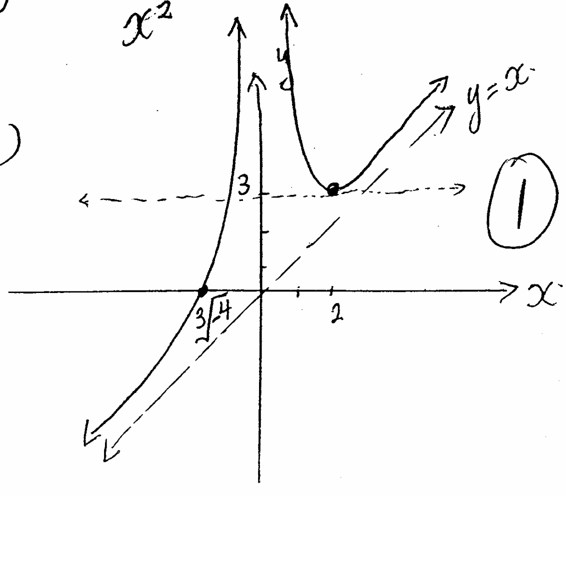
Inflexions occur when $y'' = 0$ and $\exists$ a sign change

$\frac{24}{x^4} = 0 \Rightarrow 24 = 0x^4$ does not exist. (1)

(ii) $y = \frac{x^3+4}{x^2} \Rightarrow x \neq 0$ (y axis) $\left(\frac{1}{2}\right)$ vertical asymptote

$y = \frac{x^3\left(1 + \frac{4}{x^3}\right)}{x^2} = x\left(1 + \frac{4}{x^3}\right)$ and as $x \to \infty$ $y \to x$. $\left(\frac{1}{2}\right)$ oblique asymptote

(iv)



(1)

when $y = 0$,

$0 = \frac{x^3+4}{x^2}$

so $x^3 + 4 = 0$

$x^3 = -4$

$x = \sqrt[3]{-4}$.

5 (b) (iv) $x^3 - kx^2 + 4 = 0$

$$x^3 + 4 = kx^2$$

So $\dfrac{x^3+4}{x^2} = k.$

$\Rightarrow y = \dfrac{x^3+4}{x^2} = k$

3 intersections will occur between $y = k$ and $y = \dfrac{x^3+4}{x^2}$ if

$k > 3.$

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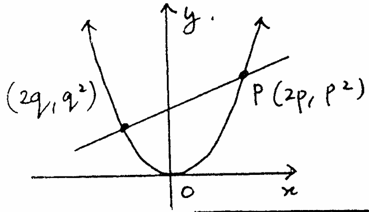
$\boxed{\text{Solution — Section C}}$

Question (6) [12]

$x = 2t,\ y = t^2 \quad \therefore \quad y = \frac{x^2}{4}.$



Solve:

$$\begin{cases} y = \frac{x^2}{4} \\ y = mx + c \end{cases}$$

$\therefore \boxed{x^2 - 4mx - 4c = 0} \quad — (1)$

The roots to (1) are: $2p, 2q$.

$\therefore \sum \alpha_i : 2p + 2q = 4m$

i.e $p + q = 2m. \quad — (2)$

Product of roots: $4pq = -4c$

(i) $\therefore \boxed{pq = -c} \quad — (3)$ **[2]**

(ii) Now, $p^2 + q^2 = (p+q)^2 - 2pq$

$= 4m^2 - 2(-c)$

$\therefore \boxed{p^2 + q^2 = 4m^2 + 2c}$ **[2]**

(iii) gradient of tgt. $= p$

$\therefore$ gradient of normal $= -\frac{1}{p}.$

$\therefore$ equation of normal: $y - p^2 = -\frac{1}{p}(x - 2p)$

$\therefore \boxed{x + py = p^3 + 2q}$ [–]

(iv) The equation of normal at Q:

$x + qy = q^3 + 2q \quad — (5)$

$\therefore$ (4) − (5) we have:

$(p-q)\, y = (p^3 - q^3) + 2(p-q)$

$\therefore y = 2 + p^2 + pq + q^2 \quad — (6)$

Substitute (6) into (4) we have:

$x + \cancel{2p} + \cancel{p^3} + p^2q + pq^2 = \cancel{p^3} + \cancel{2p}$

$\therefore x = -p^2q - pq^2 = -pq(p+q)$

$\therefore \boxed{N\left(-pq(p+q),\ (2 + p^2 + pq + q^2)\right)}$

[2].

Question (6).

(v). $\boxed{\begin{array}{l} pq = -c, \quad p+q = 2m. \\ p^2+q^2 = 4m^2+2c. \end{array}}$

The x-coord. of N becomes $c(2m)$
The y-coord. of N becomes
$\{2+(4m^2+2c)-c\}$

$\therefore \boxed{N = (2mc, \ 4m^2+c+2)}$

(α) Chord PQ, whose equation is $y = mx+c$, is free to move whilst maintaining a fixed grad. i.e $m_{PQ} = m$ (a constant), but c is a variable.

Now $x = 2mc, \implies c = \dfrac{x}{2m}$.

$y = 2+4m^2+\dfrac{x}{2m}$ **[2]**

$\therefore \boxed{y = \dfrac{x}{2m} + 2(1+2m^2)}$

i.e Equation of locus of N is a straight line with gradient $1/2m$ and y-intercept $2(1+m^2)$.

The points of intersection of the locus of N and $x^2 = 4y$ are form by solving

$$\begin{cases} y = \dfrac{x}{2m} + 2(1+2m^2) \\ x = 2t, \ y = t^2 \end{cases}$$

i.e $t^2 = \dfrac{\cancel{2}t}{\cancel{2}m} + 2(1+2m^2)$. **(4)**

$mt^2 - t - 2m(1+2m^2) = 0$.

$\therefore t = \dfrac{1 \pm \sqrt{1+8m^2(1+2m^2)}}{2m}$

$= \dfrac{1 \pm (1+4m^2)}{2m}$

$\therefore \boxed{t = \dfrac{1+2m^2}{m}, \text{ or } -2m}$

$\therefore$ locus of N cut parabola in 2 pt say U, V with parameters

$$t = \begin{cases} \dfrac{1+2m^2}{m} \\ -2m. \end{cases}$$ **[2]**

From gradient of tgt, $(=t) \implies$ the gradients of tgts at U, V are $\dfrac{1+2m^2}{m}$, and $-2m$. In particular, the tgt at V has gradient $-2m$ while the locus of N has gradient $\dfrac{1}{2m}$. Hence the locus of N is perp to tgt at V $\implies$ normal at V

Question (7). [12]

(a) $P(x) = (x+4)\,m(x) + 5$

$= (x-1)\,n(x) + 9.$ [3]

$\therefore\ P(-4) = 5,\quad P(1) = 9.$ —— (1)

$P(x) = (x-1)(x+4)\,q(x) + (ax+b).$

From (1) $\begin{cases} -4a + b = 5 \\ a + b = 9 \end{cases}$ —— (2)

$\therefore\ 5a = 4 \implies a = 4/5$

$\therefore\ b = 9 - 4/5 = 41/5$

i.e $\boxed{\dfrac{4x}{5} + \dfrac{41}{5}}$

(b) To find the range, set $y = 0$.

(i) i.e $x\left(\tan\theta - \dfrac{gx}{2V^2\cos^2\theta}\right) = 0.$

i.e $x = 0$, or $x = \dfrac{2V^2\cos^2\theta \times \tan\theta}{g}$

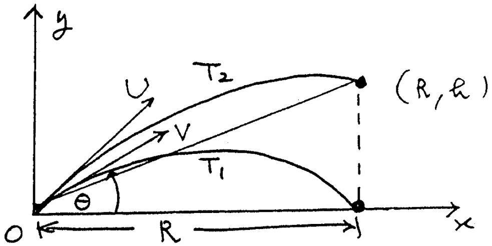
i.e range $= \dfrac{V^2(2\sin\theta\cos\theta)}{g}$

$= \dfrac{V^2\sin 2\theta}{g}$ [2]

$\therefore$ Maximum range occurs when $\sin 2\theta = 1 \implies \boxed{R = \dfrac{V^2}{\ }}$

(b)

(ii)



Equation of higher trajectory (T_2) is

(1) — $\boxed{h = R\tan\theta - \dfrac{gR^2}{2U^2\cos^2\theta}}$ (velocity U)

When the speed of projectile was V, the range was :

$\boxed{R = \dfrac{V^2\sin 2\theta}{g}}$ —— (2)

Substitute (2) into (1) we have.

$h = \dfrac{V^2\sin 2\theta\tan\theta}{g} - \dfrac{V^4\sin^2 2\theta}{gU^2\cos^2\theta}.$

Note: $\boxed{\sin 2\theta = 2\sin\theta\cos\theta,\quad \tan\theta = \sin\theta/\cos\theta}$

2 $\therefore\ h = \dfrac{2V^2\sin^2\theta}{g} - \dfrac{2V^4\sin^2\theta}{gU^2}.$

When $v = V$, Range is R_{max} $\therefore\ \theta = 4$

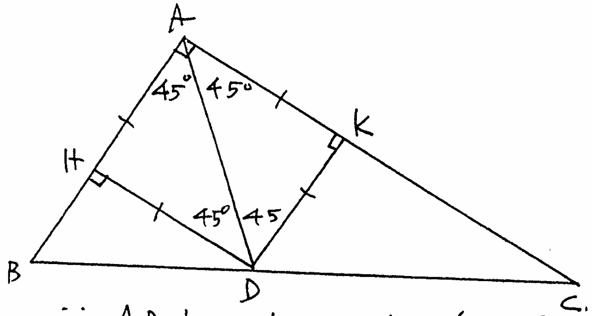
2 $\therefore\ h = \dfrac{V^2}{g} - \dfrac{V^4}{gU^2}.$ [4]

$\therefore\ U^2gh = V^2U^2 - V^4,\quad U^2(V^2 - gh) =$

$\therefore\ U^2 = \dfrac{V^4}{V^2 - gh}\qquad \therefore\ \boxed{U = \dfrac{V^2}{\sqrt{V^2 - gh}}}$

Question

7(c)



$\because$ AD bisects $\angle BAC$ $(=90°)$

$\therefore \angle BAD = \angle DAC = 45°$

$\Rightarrow \angle HDA = \angle KDA = 45°$

(Angle sum of a $\triangle$).

i.e $\triangle AHD$ is isos. $\Rightarrow AH = DH$.

In $\triangle AHD$, $AD^2 = AH^2 + DH^2$

(Pythagoras) $= 2DH^2$

$\therefore \left(\frac{AD}{DH}\right)^2 = 2$

(i) $\Rightarrow \frac{AD}{DH} = \sqrt{2}$. [1].

$\therefore$ $\boxed{\frac{1}{DH} = \frac{\sqrt{2}}{AD}}$ —— (1)

(ii)

$\triangle AHD \equiv \triangle AKD$. (AAS).

$\therefore DH = DK$. —— (2)

Area of $\triangle ABC$

$= \frac{1}{2} AB \cdot AC$.

but area of $\triangle ABC$

$=$ area of $\triangle ABD$

$+$ area of $\triangle ACD$.

Area of $\triangle ABD = \frac{1}{2} AB \cdot DH$

Area of $\triangle ACD = \frac{1}{2} AC \cdot DK$.

from (2) $\because DK = DH$

$\therefore$ area of $\triangle ACD = \frac{1}{2} AC \cdot DH$.

$\therefore \frac{1}{2} AB \cdot AC = \frac{1}{2}(AB \cdot DH + AC \cdot DH)$

$\therefore DH(AB + AC) = AB \cdot AC$. [2].

$\therefore \frac{AB + AC}{AB \cdot AC} = \frac{1}{DH}$ from (1)

$\therefore$ $\boxed{\frac{1}{AC} + \frac{1}{AB} = \frac{\sqrt{2}}{AD}}$.