St. Catherine's School

Trial HSC Examination

Mathematics Extension 2

2002

1. (a) Find $\int \sin^3 x \ dx$

(b) Evaluate $\int_1^2 \frac{dx}{\sqrt{3+2x-x^2}}$

(c) Find $\int x^2 \sin x \ dx$

(d) Find $\int \frac{2x^2}{x^2-4} \ dx$

(e) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \ dx$, $n \geq 3$ show that $I_n + I_{n-2} = \frac{1}{n-1}$

2. (a) $z = x + iy$ is a complex number which satisfies the equation $z\overline{z} + 2iz = 9 + 2i$, find the possible values of z, where $\overline{z}$ is the complex conjugate of z.

(b) Given that $z_1 = \frac{1-i}{1+i}$ and $z_2 = \frac{\sqrt{2}}{1+i}$

(i) Express z_1 and z_2 in the form $a + ib$ and also find their modulus and argument.

(ii) Plot z_1 and z_2 on an Argand Diagram and show that $z_1 + z_2$ on this diagram.

(iii) Find the modulus and argument of $(z_1 + z_2)$

(c) Point A represents the number 1 on an Argand diagram. O is the origin. Point P represents the complex number z such that $\arg(z-1) = 2\arg(z)$.

3. (a) (i) Show that the condition for the line $y = mx + c$ be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 + b^2$

(ii) Hence show that the pair of tangents from the point $(3, 4)$ to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ are at right angles to each other.

(b) Consider the rectangular hyperbola $xy = c^2$. Let $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ be two points on this hyperbola. The tangent at Q passes through the foot of the ordinate at P (i.e., the tangent passes through the point $(cp, 0)$)

(i) Show that $p = 2q$.

(ii) Show that the locus of the mid-pint M of the chord PQ is a hyperbola with the same asymptotes as the given hyperbola.

(c) Find the locus of z such that $\arg(2-z) = \frac{\pi}{4}$

4. (a) The circle $x^2 + (y-3)^2 = 1$ is rotated about the line $x = 5$.

(i) Use the method of cylindrical shells to show that the volume generated is given by $4\pi \int_{-1}^{1} (5-x)\sqrt{1-x^2} \ dx$

(ii) Hence find the volume.

(b) The base of a particular solid is the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Find the volume of this solid if every cross section perpendicular to the major axis is an equilateral triangle with one side on the base of the solid.
(c) The region between the curve $y = x^2 + 1$ and $y = 3 - x$ is rotated about the x-axis. By taking slices perpendicular to the x-axis, find the volume of the solid generated.

5. (a) $\sqrt{3} + i$ is a zero of the polynomial $x^4 + px^3 + q = 0$, where p and q are real numbers.
(i) Show that $p = -\sqrt{3}$ and $q = 8$.
(ii) Factorise $x^4 + px^3 + q$ into quadratic factors.
(b) If $u_1 = 3$ and $u_2 = 21$ and if $u_n = 7u_{n-1} - 10u_{n-2}$ show using Mathematical Induction that $u_n = 5^n - 2^n$ for $n \geq 1$
(c) (i) Solve $\tan^{-1} 2x - \tan^{-1} x = \tan^{-1} \frac{1}{3}$ for x.
(ii) Show that $\frac{d}{dx}(\tan^{-1} x + \tan^{-1} \frac{1}{x}) = 0$ and sketch the function
$f(x) = \tan^{-1} x + \tan^{-1} \frac{1}{x}$.

6. (a) The polynomial function $P(x) = x^4 - 4x^3 - 3x^2 + 50x - 52$ has a zero at $x = 3 - 2i$ Factorise $P(x)$ over the field of
(i) rationals
(ii) reals
(iii) complex numbers
(b) The equation $2x^3 - 9x^2 + 7 = 0$ has roots α, β, γ. Find the equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$.
(c) (i) Show that the solution of the equation $z^3 = 1$ in the complex number system are $z = \cos\theta + i\sin\theta$ for $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$
(ii) If $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ show that $\omega^2 + \omega + 1 = 0$ and $\omega - \omega^2 - \omega - 2 = 0$
(iii) Hence or otherwise solve the cubic equation $z^3 - z^2 - z - 2 = 0$.

7. (a) A body of mass 4kg is whirled in a horizontal circle of radius 0.6 metres. If the tension in the string is 540 Newtons, find the angular velocity of the body.
(b) An object of mass 20kg is dropped from rest through the atmosphere and there is air esistance of $2v$ Newtons at speed v m/s. Acceleration due to gravity is 10 m/s/s.
(i) Show that the acceleration, a, is given by $a = \frac{100-v}{10}$
(ii) Find an expression for velocity v m/s at any time t seconds.
(iii) Find the terminal velocity.
(iv) Show that the distance the object has travelled when the speed is v m/s is given by $x = 1000 \ln \frac{100}{100-v} - 10v$
(v) Hence find the distance the object has fallen before reaching half the terminalk velocity.
(b) A particle is projected from a point O with an initial velocity of $\frac{150}{7}$ m/s and the angle of projection is α, where $\tan\alpha$ is $\frac{4}{3}$. One second later another particle is projected with an initial velocity of $\frac{225}{7}$ m/s at an angle of elevation of β, where

$\tan\beta = \frac{3}{4}$ and in in the same vertical plane through O as the first particle. Show that the particles collide 2 seconds after the first particle is projected.

8. (a) (i) The number 11 (eleven) can be written as $11 = 1 + 1 \times 10$. Show that the number $\underbrace{111\cdots 11}_{n \text{ ones}} = \frac{10^n - 1}{9}$.

(ii) Hence or using Mathematical Induction show that
$1 + 11 + 111 + \cdots + \underbrace{111\cdots 11}_{n \text{ ones}} = \frac{1}{81}(10^{n+1} - 9n - 10)$

(b) $xy = 4$ is a rectangular hyperbola (note: $e = \sqrt{2}$). Find the coordinates of the foci and the equations of the directrices.
(c) (i) Given that $I_{2n+1} = \int_0^1 x^{2n+1} e^{x^2}\, dx$ where n is a positive integer, show that $I_{2n+1} = \frac{e}{2} - nI_{2n-1}$.
(iii) Hence evaluate I_5.