SOLUTIONS EXT 2 · 2007 TRIAL

Q1.(a) $\int_0^{\frac{\pi}{2}} \sin^n x \cos x \, dx$ Let $u = \sin x$ For $x = 0$, $u = 0$

$\frac{du}{dx} = \cos x$ For $x = \frac{\pi}{2}$, $u = 1$

$du = \cos x \, dx$

$= \int_0^1 u^n \, du$

$= \left[\frac{1}{n+1} u^{n+1}\right]_0^1$

$= \frac{1}{n+1}\left[(1) - (0)\right] = \frac{1}{n+1}$.

(b) (i) $\int_0^1 \frac{dx}{(2x+1)(x+2)} = \int_0^1 \frac{1}{3} \cdot \frac{2}{2x+1} - \frac{1}{3} \cdot \frac{1}{x+2} \, dx$

$= \frac{1}{3}\left[\ln(2x+1) - \ln(x+2)\right]_0^1$

$= \frac{1}{3}\left[\ln \frac{2x+1}{x+2}\right]_0^1$

$= \frac{1}{3}\left[\ln 1 - \ln \frac{1}{2}\right] = \frac{1}{3}\ln 2$.

(ii) $\int_0^{\frac{\pi}{2}} \frac{3 \, dx}{4 + 5\sin x} = \int_0^1 \frac{3 \cdot \frac{2\,dt}{1+t^2}}{4 + 5 \cdot \frac{2t}{1+t^2}}$ For $t = \tan\frac{x}{2}$

$x = 0$, $t = 0$

$x = \frac{\pi}{2}$, $t = 1$

$= \int_0^1 \frac{6\,dt}{4(1+t^2) + 10t}$

$= \int_0^1 \frac{3\,dt}{2(1+t^2) + 5t}$

$= \int_0^1 \frac{3\,dt}{2t^2 + 5t + 2}$

$= 3\int_0^1 \frac{dt}{(2t+1)(t+2)}$

$= 3 \cdot \frac{1}{3}\ln 2$

$= \ln 2$.

over

SOLUTIONS EXT 2. SCOTS 2007.

Q1. (c)(i) $I_0 = \int_0^1 x^0 e^x \, dx = \int_0^1 e^x \, dx = [e^x]_0^1 = e - 1$. ✓

(ii) $I_n = \int_0^1 x^n \frac{de^x}{dx} dx = [x^n e^x]_0^1 - \int_0^1 n x^{n-1} e^x \, dx$ ✓

so $I_n = [e - 0] - n \int_0^1 x^{n-1} e^x \, dx$ ✓

$I_n = e - n I_{n-1}$. ✓

(iii) $I_3 = e - 3 I_2$ ✓

$= e - 3[e - 2 I_1]$

$= e - 3[e - 2(e - I_0)]$ ✓

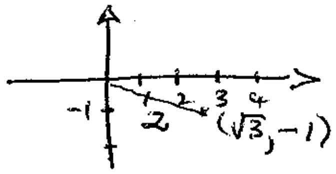
$= e - 3e + 6(e - (e-1))$

$= -2e + 6e - 6e + 6$

$= 6 - 2e$. ✓

Q2. (a) (i) $z = \sqrt{3} - i$

$z = 2\operatorname{cis}\left(-\frac{\pi}{6}\right)$



✓

(ii) $z^8 = 2^8\left(\operatorname{cis}\left(-\frac{\pi}{6}\right)\right)^8$

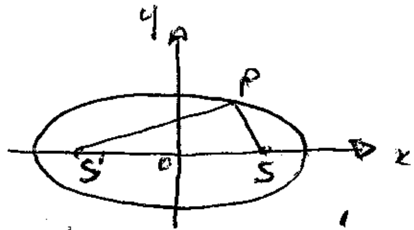
$= 256\operatorname{cis}\left(-\frac{4\pi}{3}\right)$

$= 256\operatorname{cis}\left(\frac{2\pi}{3}\right)$ ✓

$= 256\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= -128 + 128\sqrt{3}\,i$ ✓

(b) (i) $|z-2| + |z+2| = 5$



SINCE $2a = 5$

$a = \frac{5}{2}$

NOTE: $PS + PS' = 2a$

A PARABOLA WITH FOCII $S(2,0)$ AND $S'(-2,0)$.

SO $ae = 2$

$e = \frac{2}{\frac{5}{2}}$ ✓

$e = \frac{4}{5}$

NOW $a^2e^2 = a^2 - b^2$

$b^2 = a^2 - a^2e^2$

$= a^2(1-e^2)$

$b^2 = \frac{25}{4}\left(1 - \frac{16}{25}\right)$

$b^2 = \frac{9}{4}$

AND $b = \frac{3}{2}$ ✓

$\left.\begin{array}{l} \text{A PARABOLA, FOCII } (2,0) \text{ AND } (-2,0) \\ \text{SEMI MAJOR AXIS } \frac{5}{2} \text{ UNITS} \\ \text{SEMI MINOR AXIS } \frac{3}{2} \text{ UNITS.} \quad e = \frac{4}{5} \end{array}\right\}$ ✓

(ii) $\left.\begin{array}{l} \text{MAXIMUM VALUE OF } |z| = \frac{5}{2} \\ \text{MINIMUM } |z| = \frac{3}{2} \end{array}\right\}$ ✓

Q(2) (c) (i) $z^n + \frac{1}{z^n} = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$ ✓

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$ ✓

$= 2\cos n\theta$.

(ii) Consider $z^5 = 1$ Let $z = \cos\theta + i\sin\theta$

$(\cos\theta + i\sin\theta)^5 = 1$

$\cos 5\theta + i\sin 5\theta = 1$ and equating real parts

$\cos 5\theta = 1$.

$5\theta = 0, 2\pi, 4\pi, 6\pi$ and 8π

$\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$ ✓✓

The roots of $z^5 = 1$ are : $\text{cis}\,0, \text{cis}\frac{2\pi}{5}, \text{cis}\frac{4\pi}{5}, \text{cis}\frac{6\pi}{5}, \text{cis}\frac{8\pi}{5}$

The factors of $z^5 - 1$ are : $(z - \text{cis}\,0)(z - \text{cis}\frac{2\pi}{5})(z - \text{cis}\frac{8\pi}{5})(z - \text{cis}\frac{4\pi}{5})(z - \text{cis}\frac{6\pi}{5})$ ✓

So $z^5 - 1 = (z-1)(z - \text{cis}\frac{2\pi}{5})(z - \text{cis}(-\frac{2\pi}{5}))(z - \text{cis}\frac{4\pi}{5})(z - \text{cis}(-\frac{4\pi}{5}))$

$z^5 - 1 = (z-1)(z^2 - 2\cos\frac{2\pi}{5}z + 1)(z^2 - 2\cos\frac{4\pi}{5} + 1)$ ✓

(iii) Now $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$

So $z^4 + z^3 + z^2 + z + 1 = (z^2 - 2\cos\frac{2\pi}{5}z + 1)(z^2 - 2\cos\frac{4\pi}{5} + 1)$ ✓

For $z = 1$, $1+1+1+1+1 = (1 - 2\cos\frac{2\pi}{5} + 1)(1 - 2\cos\frac{4\pi}{5} + 1)$

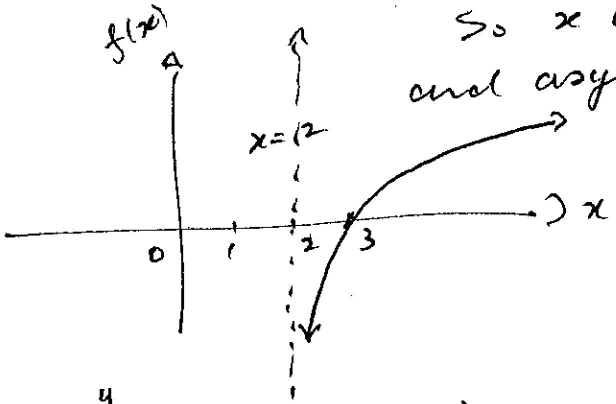
$5 = 2(1 - \cos\frac{2\pi}{5})\,2(1 - \cos\frac{4\pi}{5})$ ✓

and $\frac{5}{4} = (1 - \cos\frac{2\pi}{5})(1 - \cos\frac{4\pi}{5})$ as required

Q3.(a) $f(x) = \ln(x-2)$

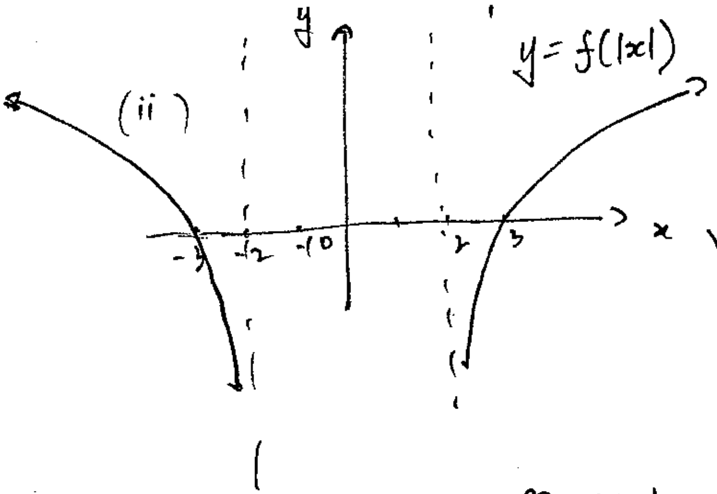
$x - 2 = 1$ where $x = 3$.
$x - 2 = 0$ where $x = 2$

So x intercept is $(3, 0)$
and asymptote is $x = 2$.



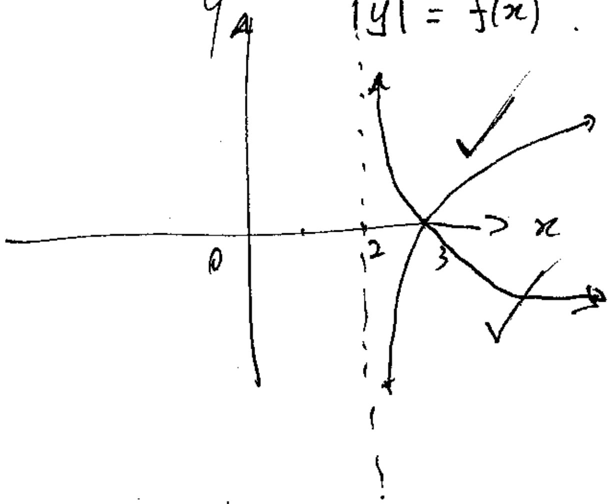
(ii)

$y = f(|x|)$



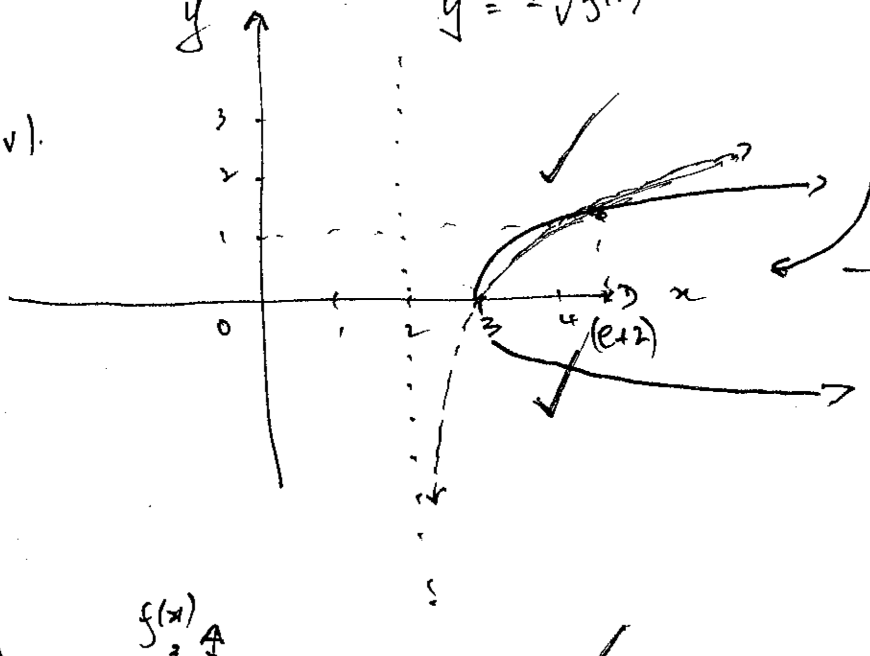
(iii)

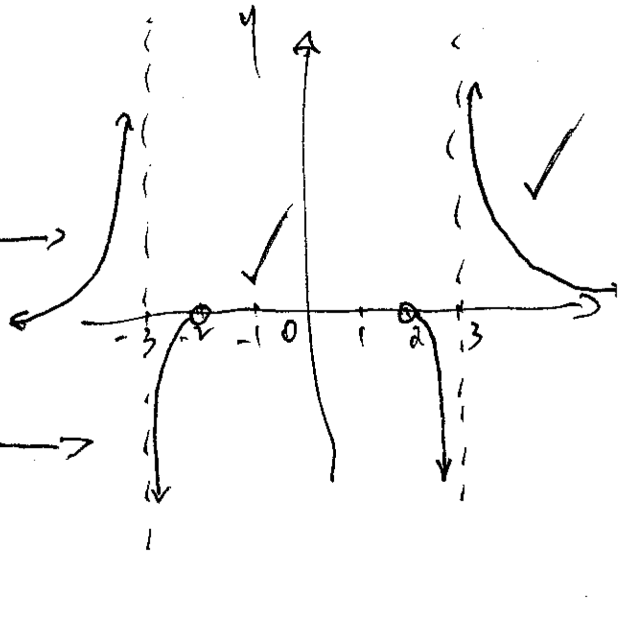
$|y| = f(x)$.



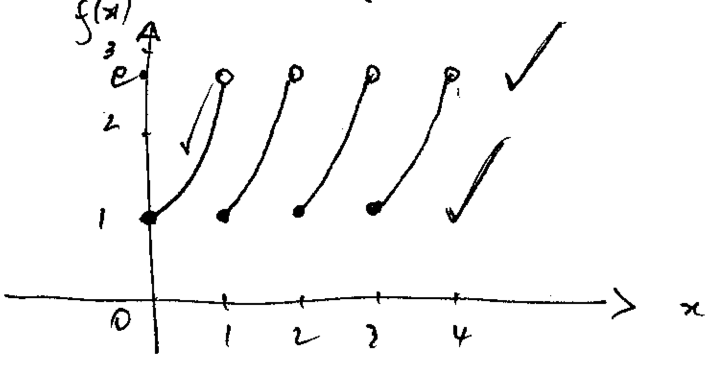
(iv).

$y^2 = f(x)$
$y = \pm\sqrt{f(x)}$





(b).



Domain: x: x All Reals ✓ Range: y: $0 < y < \frac{\pi}{2}$ ✓

Q3.

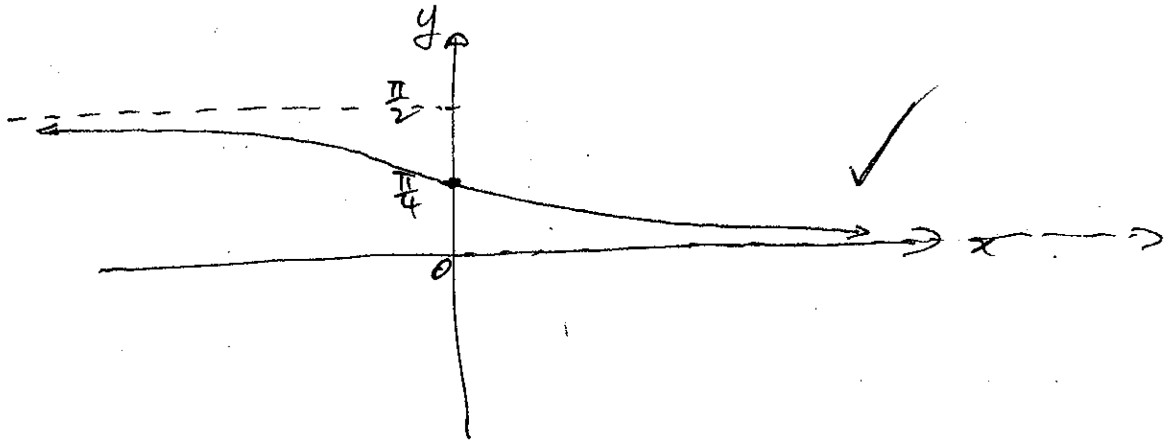
✓ As $x \to -\infty$, $e^{-x} \to \infty$ and $\tan^{-1}(e^{-x}) \to \frac{\pi}{2}$

So $y = \frac{\pi}{2}$ is an asymptote as $x \to -\infty$

For $x = 0$, $e^{-x} = 1$ and $\tan^{-1}(e^{-x}) = \tan^{-1}(1) = \frac{\pi}{4}$

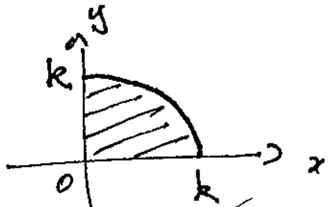
✓ As $x \to \infty$, $e^{-x} \to 0^+$ and $\tan^{-1}(e^{-x}) \to 0^+$

So $y = 0$ is an asymptote as $x \to \infty$



✓

Q.4 (a) $I = \int_0^k \sqrt{k^2 - x^2} = \frac{\pi}{2}$



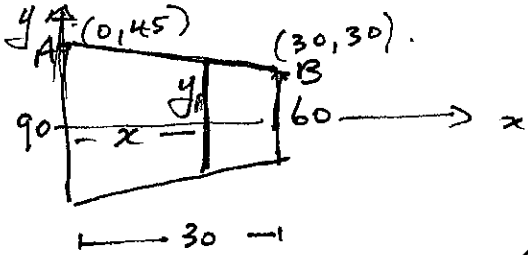
The integral represents $\frac{1}{4}$ of the area of a circle radius k.

$$\text{So } I = \frac{1}{4}\pi k^2$$

$$\text{So } \frac{1}{4}\pi k^2 = \frac{\pi}{2}$$

$$k^2 = 2 \text{ and } k = \sqrt{2}.$$

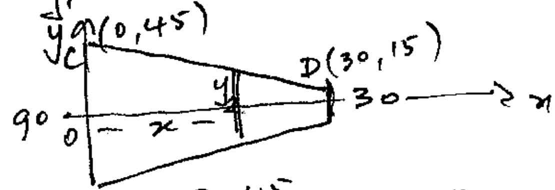
(b). From the front.



(i) Equation AB is $y_1 = \frac{30-45}{30-0}x + 45$

$$y_1 = -\frac{1}{2}x + 45 \quad \text{So } \boxed{2y_1 = 90 - x}$$

From the side



Equat. CD is $y_2 = \frac{15-45}{30-0}x + 45$

$$y_2 = -x + 45 \quad \text{So } \boxed{2y_2 = 90 - 2x}$$

Now $\Delta V = (2y_1)(2y_2).\Delta x$

$$= 4(90-x)(90-2x)\Delta x$$

$$= 82(90-x)(45-x)\Delta x.$$

(ii) Now $V = 2\int_0^{30}(90-x)(45-x)\,dx$

$$= 2\int_0^{30} 4050\phi - 135x + x^2\,dx$$

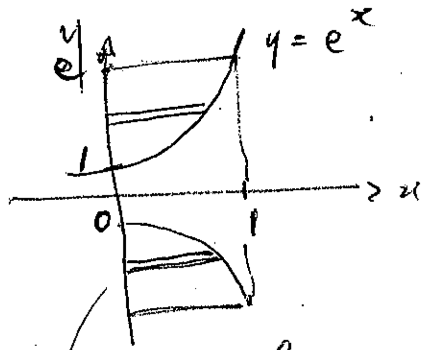
$$= 2\left[4050\phi x - \frac{135}{2}x^2 + \frac{1}{3}x^3\right]_0^{30}$$

$$= 2\left\{4050 \times 30 - \frac{135}{2} \times 900 + \frac{1}{3} \times 27000\right\}$$

$$= 2\left\{121500 - 60750 + 9000\right\}$$

$$= 139500 \text{ m}^3$$

4(c) $y = e^x$



$x = \ln y$

By shells $V = \pi (e)^2 \times 1 - 2\pi \int_1^e (\ln y)\, y \, dy$ ✓

(i) $= \pi e^2 - 2\pi \int_1^e x \ln x \, dx$ ✓

(ii) $V = \pi e^2 - 2\pi \left[\int_1^e \frac{d(\frac{1}{2}x^2)}{dx} \ln x \, dx \right]$

$= \pi e^2 - 2\pi \left[\left[\tfrac{1}{2}x^2 \ln x\right]_1^e - \int_1^e \tfrac{1}{2}x^2 \frac{d(\ln x)}{dx} dx \right]$ ✓

$= \pi e^2 - 2\pi \left[\left[(\tfrac{1}{2}e^2 \ln e) - (\tfrac{1}{2}\ln 1)\right] - \int_1^e \tfrac{1}{2}x \, dx \right]$ ✓

$= \pi e^2 - 2\pi \left[\tfrac{1}{2}e^2 - \left[\tfrac{1}{4}x^2\right]_1^e \right]$

$= \pi e^2 - 2\pi \left[\tfrac{1}{2}e^2 + \left[\tfrac{1}{4}e^2 - \tfrac{1}{4}\right] \right]$

$= \pi e^2 - \pi e^2 + 2\pi (\tfrac{1}{4}e^2 - \tfrac{1}{4})$

$= \frac{\pi e^2}{2} - \frac{\pi}{2}$ ✓

$= \frac{\pi}{2}(e^2 - 1)$

(iii) By slices $V = \pi \int_0^1 (e^x)^2 \, dx$

$= \pi \int_0^1 e^{2x} \, dx$

So $\pi \int_0^1 e^{2x} \, dx = \frac{\pi}{2}(e^2 - 1)$ ✓

and $\int_0^1 e^{2x} \, dx = \frac{1}{2}(e^2 - 1)$

Q5 (a) $\dfrac{2}{(1-x)(1+x^2)} = \dfrac{A}{1-x} + \dfrac{Bx+C}{1+x^2}$

So $2 \equiv A + Ax^2 + Bx + C - Bx^2 - Cx$

$2 = (A-B)x^2 + (B-C)x + A + C$

$A - B = 0 \quad —(1)$

$B - C = 0 \quad —(2)$

(1)+(2) $A - C = 0 \quad —(3)$

and $A + C = 2$

so $2A = 2 \longrightarrow \boxed{A=1}, \boxed{B=1}, \boxed{C=1}$

So $\dfrac{2}{(1-x)(1+x^2)} = \dfrac{1}{1-x} + \dfrac{x+1}{1+x^2}$

(b) Since $x=\alpha$ satisfies $\alpha^3 - a\alpha + b = 0$

and $8\left(\frac{1}{2}\alpha\right)^3 - 2a\left(\frac{1}{2}\alpha\right) + b = 0$

So $x = \frac{1}{2}\alpha$ is a root of $8x^3 - 2ax + b = 0$

(c) Let α be a root of $x^{10} - 5x^3 + x - 4 = 0$

so $\alpha^{10} - 5\alpha^3 + \alpha - 4 = 0$

and $1 - 5\left(\frac{1}{\alpha}\right)^7 + \left(\frac{1}{\alpha}\right)^9 - 4\left(\frac{1}{\alpha}\right)^{10} = 0$

So $x = \frac{1}{\alpha}$ is a root of $1 - 5x^7 + x^9 - 4x^{10} = 0$

or $4x^{10} - x^9 + 5x^7 - 1 = 0$

(d) So $5x^4 - 3ax^2 = 0$ has a root which is the multiple of $x^5 - ax^3 + b = 0$

Now $5x^4 - 3ax^2 = 0$

or $x^2(5x^2 - 3a) = 0$

$x = 0$ or $x = \sqrt{\frac{3a}{5}}$. Since $x=0$ is not a root of $x^5 - ax^3 + b = 0$, $x = \sqrt{\frac{3a}{5}}$ is the multiple root.

Substituting $\left(\sqrt{\frac{3a}{5}}\right)^5 - a\left(\sqrt{\frac{3a}{5}}\right)^3 + b = 0$

$\dfrac{9\sqrt{3}\,a^{\frac{5}{2}}}{25\sqrt{5}} - \dfrac{3\sqrt{3}\,a^{\frac{5}{2}}}{5\sqrt{5}} + b = 0$

$\dfrac{-6\sqrt{3}\,a^{\frac{5}{2}}}{25\sqrt{5}} = -b$

So $\dfrac{108a^5}{3125} = b^2$

$108a^5 - 3125b^2 = 0$

Q5 (e) If $x = -1+2i$ is a zero then
$x = -1-2i$ is a zero.

So $(x-(-1+2i))(x-(-1-2i))$ is a factor
$= (x+(1-2i))(x+(1+2i))$
$= x^2+2x+5$ a quadratic factor. ✓

(i) $P(\overline{2i-1}) = 0$

(ii)

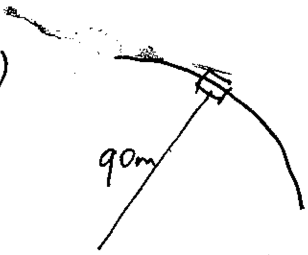
$$\begin{array}{r l}
 & x^2+4 \\
x^2+2x+5 \,\big) & x^4+2x^3+9x^2+8x+20 \\
 & x^4+2x^3+5x^2 \\
\hline
 & 4x^2+8x+20 \\
 & 4x^2+8x+20 \\
\hline
 & 0
\end{array}$$
✓ ✓

$P(x) = (x^2+2x+5)(x^2+4)$ ✓
Remaining zeros are $\pm 2i$ ✓

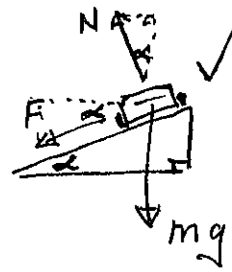
(iii) $P(x) = (x^2+2x+5)(x^2+4)$ ✓

(iv) $P(x) = (x+2i)(x-2i)(x+1-2i)(x+1+2i)$. ✓

Q6. (a)



(i)



N - Normal reaction
F - Frictional force
mg - Weight

(ii) Resolving vertically

$$N\cos\alpha = F\sin\alpha + mg$$

$$N\cos\alpha - F\sin\alpha = mg \quad -(1)$$

horizontally $\quad N\sin\alpha + F\cos\alpha = m\frac{v^2}{r} \quad -(2)$

From (1) and (2)

$$N\cos\alpha\sin\alpha - F\sin^2\alpha = mg\sin\alpha \quad -(3)$$

$$N\sin\alpha\cos\alpha + F\cos^2\alpha = m\frac{v^2}{r}\cos\alpha \quad -(4)$$

Subtract (4) - (3)

$$F(\cos^2\alpha + \sin^2\alpha) = m\frac{v^2}{r}\cos\alpha - mg\sin\alpha$$

$$F = m\frac{v^2}{r}\cos\alpha - mg\sin\alpha \quad *$$

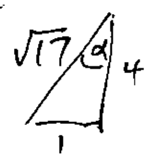
No sideways slip means $F = 0$

and so $\quad mg\sin\alpha = m\frac{v^2}{r}\cos\alpha$

$$\tan\alpha = \frac{v^2}{rg}$$

For $v = \frac{54 \times 1000}{60 \times 60}$ ms^{-1} and $r = 90$

$$\tan\alpha = \frac{15 \times 15}{90 \times 10} = \frac{225}{900} = \frac{1}{4}$$



$$\alpha = \tan^{-1}\left(\frac{1}{4}\right).$$

(iii). Now $F = m\cos\alpha\left(\frac{v^2}{r} - g\tan\alpha\right) \quad *$

$\cos\alpha = \frac{4}{\sqrt{17}}$

$$F = 1200 \cdot \frac{4}{\sqrt{17}}\left(\left(\frac{72 \times 1000}{60 \times 60}\right)^2 \cdot \frac{1}{90} - \frac{10}{4}\right)$$

$$= \frac{4800}{\sqrt{17}}\left(\frac{400}{90} - \frac{10}{4}\right)$$

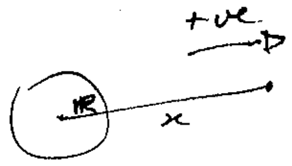
$$= \frac{4800}{\sqrt{17}}\left(\frac{800 - 450}{180}\right)$$

$$= \frac{4800}{\sqrt{17}} \times \frac{350}{180}$$

$$\doteqdot 2263.665\ldots$$

$$= 2263.7 \text{ N}$$

Q6 (b) (i) $\ddot{x} \propto \frac{1}{x^2}$



So $\ddot{x} = -\frac{k}{x^2}$

now at the earth's surface

$g = \frac{k}{R^2}$

$k = gR^2$

so $\ddot{x} = -\frac{gR^2}{x^2}$

Assume that when $t = 0$, $v = u$, $x = R$.

$\frac{d(\frac{1}{2}v^2)}{dx} = -gR^2x^{-2}$

Integrate w.r.to x $\quad \frac{1}{2}v^2 = +gR^2x^{-1} + c$

$v^2 = +\frac{2gR^2}{x} + c$

For $x = R$, $v = u$, $\quad u^2 = +\frac{2gR^2}{R} + c$

$c = u^2 - 2gR$

and $v^2 = u^2 - 2gR + \frac{2gR^2}{x}$

$v^2 = u^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right)$

(ii) Now at greatest height H, $v = 0$

so $2gR^2\left(\frac{1}{R} - \frac{1}{x}\right) = u^2$

$\frac{1}{R} - \frac{1}{x} = \frac{u^2}{2gR^2}$

$\frac{1}{x} = \frac{1}{R} - \frac{u^2}{2gR^2}$

$\frac{1}{x} = \frac{2gR - u^2}{2gR^2}$

$x = \frac{2gR^2}{2gR - u^2}$

so $H = \frac{2gR^2}{2gR - u^2} - R$

$H = \frac{\cancel{2gR^2} - \cancel{2gR^2} + u^2R}{2gR - u^2}$

$H = \frac{u^2R}{2gR - u^2}$

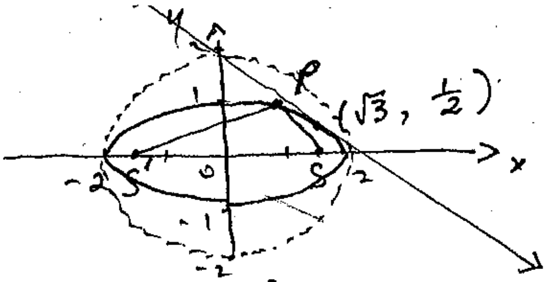
(iii) As $x \to \infty$, $v^2 \to u^2 - 2gR$ $\quad$ Note: $\frac{1}{x} \to 0^+$

so $u^2 - 2gR > 0$

$u^2 > 2gR$

$u > \sqrt{2gR}$

Q7. (a) (i)



(ii) Since $\frac{x^2}{4} + y^2 = 1$, $a = 2$ and $b = 1$

$a^2e^2 = a^2 - b^2$, So $4e^2 = 4 - 1$

$e = \frac{\sqrt{3}}{2}$

Focii are: $S'(-\sqrt{3}, 0)$, $S(\sqrt{3}, 0)$.

(iii) At $x = \sqrt{3}$, $y = \frac{1}{2}$ and

equation Tangent is $y - \frac{1}{2} = m(x - \sqrt{3})$

Since $\frac{x^2}{4} + y^2 = 1$

differentiating $\frac{2x}{4} + 2y \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2x}{4} \cdot \frac{1}{2y} = -\frac{x}{4y}$

At $x = \sqrt{3}$, $m = \frac{-\sqrt{3}}{2}$

Equat. tangent is $y - \frac{1}{2} = -\frac{\sqrt{3}}{2}(x - \sqrt{3})$

$2y - 1 = -\sqrt{3}x + 3$

So $\sqrt{3}x + 2y - 4 = 0$.

(iv) Perimeter $\Delta S'PS = PS' + PS + S'S$

$= 2a + 2ae$

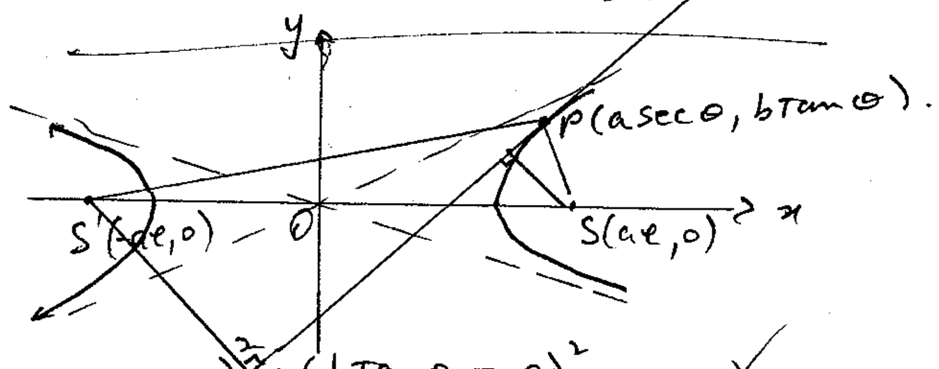
$= 2a(1 + e)$

$= 4(1 + \frac{\sqrt{3}}{2})$

$= 2(2 + \sqrt{3})$.

(b)

(i)



$SP^2 = (a\sec\theta - ae)^2 + (b\tan\theta - 0)^2$

$= a^2\sec^2\theta - 2a^2e\sec\theta + a^2e^2 + b^2\tan^2\theta$

$= a^2\sec^2\theta - 2a^2e\sec\theta + a^2e^2 + b^2\sec^2\theta - b^2$

$= (a^2 + b^2)\sec^2\theta - 2a^2e\sec\theta + a^2$ NOTE: $a^2e^2 = a^2 + b^2$

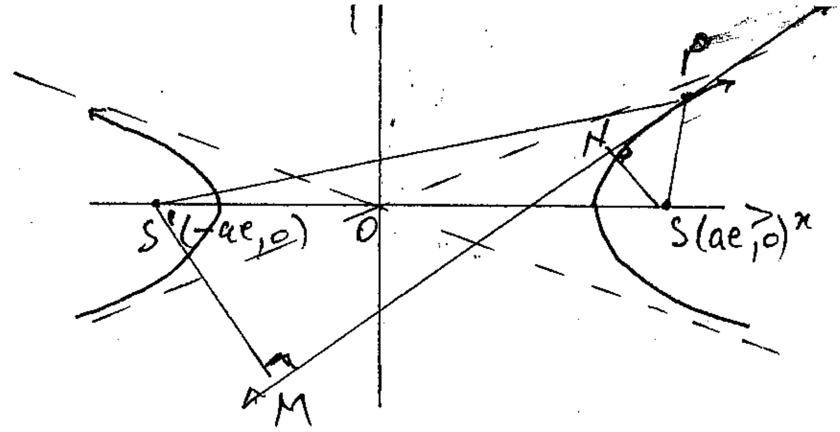
$= a^2e^2\sec^2\theta - 2a^2e\sec\theta + a^2$

$= a^2(e\sec\theta - 1)^2$

$SP = a(e\sec\theta - 1)$ and similarly $S'P = a(e\sec\theta + 1)$

OVER

Q 7(b) ii



Since equation tangent is $\frac{(\sec\theta)x}{a} - \frac{(\tan\theta)y}{b} - 1 = 0$

Perp. distance $S'M = \left| \dfrac{-\frac{ae\sec\theta}{a} - 0 - 1}{\sqrt{\frac{\sec^2\theta}{a^2} + \frac{\tan^2\theta}{b^2}}} \right|$

$= \left| \dfrac{-e\sec\theta - 1}{\frac{1}{ab}\sqrt{\sec^2\theta + \tan^2\theta}} \right|$

$= + \dfrac{(e\sec\theta + 1)ab}{\sqrt{\sec^2\theta + \tan^2\theta}}$

Similarly for distance $SM = \left| \dfrac{\frac{ae\sec\theta}{a} - 0 - 1}{\sqrt{\frac{\sec^2\theta}{a^2} + \frac{\tan^2\theta}{b^2}}} \right|$

$= \dfrac{(e\sec\theta - 1)ab}{\sqrt{\sec^2\theta + \tan^2\theta}}$

So $\dfrac{S'M}{SM} = \dfrac{e\sec\theta + 1}{e\sec\theta - 1} = \dfrac{S'P}{SP}$

and $\dfrac{S'M}{S'P} = \dfrac{SM}{SP}$

and so $\sin\angle S'PM = \sin\angle SPN$

$\angle S'PM = \angle SPN$

Q8. (a) (i) RHS $= \int_0^a f(a-x)\,dx$ Let $u = a - x$ For $x = 0$, $u = a$

$\frac{du}{dx} = -1$ For $x = a$, $u = 0$

$-du = dx$

$$= -\int_a^0 f(u)\,du$$

$$= \int_0^a f(u)\,du$$

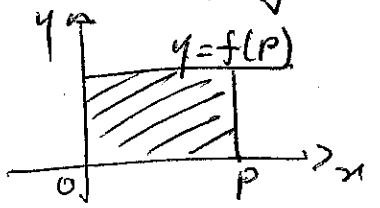
$$= \int_0^a f(x)\,dx.$$

(ii) Given $f(x) + f(p-x) = f(p)$ — a constant.

$$\int_0^p f(x)\,dx + \int_0^p f(p-x)\,dx = \int_0^p f(p)\,dx$$

and by part (i) $\quad 2\int_0^p f(x)\,dx = \int_0^p f(p)\,dx$

The definite integral $\int_0^p f(p)\,dx$ can be evaluated by determining the area under the curve $y = f(p)$ between $x = 0$ and $x = p$.



Area is $p\,f(p)$

and so $\quad 2\int_0^p f(x)\,dx = p\,f(p)$

$$\int_0^p f(x)\,dx = \tfrac{1}{2}\,p\,f(p).$$

(iii) $$I = \int_0^\pi \frac{x\,dx}{4+\sin^2 x} = \int_0^\pi \frac{(\pi - x)\,dx}{4+\sin^2(\pi - x)}$$

$$= \int_0^\pi \frac{\pi\,dx}{4+\sin^2 x} - \int_0^\pi \frac{x\,dx}{4+\sin^2 x}$$

and so $$2I = \pi\int_0^\pi \frac{dx}{4+\sin^2 x}$$

$$= \pi\int_0^\pi \frac{\sec^2 x\,dx}{4\sec^2 x + \tan^2 x}$$ (DIVIDING BY $\cos^2 x$ TOP + BOTTOM)

$$= \pi\int_0^\pi \frac{\sec^2 x\,dx}{4 + 5\tan^2 x}$$

Let $u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x\,dx$

For $x = 0$, $u = 0$

For $x = \frac{\pi}{2}$, $u = \infty$

$$= \frac{2\pi}{5}\int_0^{\frac{\pi}{2}} \frac{\sec^2 x\,dx}{\frac{4}{5} + \tan^2 x}$$ NOTE: EVEN FUNCTION

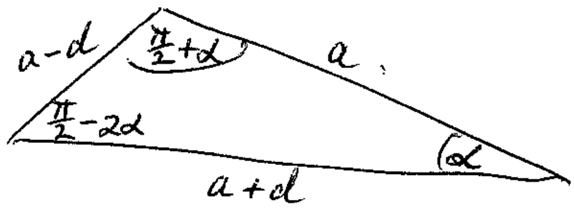
$$= \frac{2\pi}{5}\int_0^\infty \frac{du}{\left(\frac{2}{\sqrt{5}}\right)^2 + u^2}$$

$$= \frac{2\pi}{5}\cdot\frac{\sqrt{5}}{2}\left[\tan^{-1}\frac{\sqrt{5}\,u}{2}\right]_0^\infty$$

$$= \frac{\pi}{\sqrt{5}}\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$2I = \frac{\pi}{\sqrt{5}}\times\frac{\pi}{2} \quad\text{and}\quad I = \frac{\pi^2}{4\sqrt{5}}$$

Q8. (b)



BY THE SIN RULE :

$$\frac{a}{\sin(\frac{\pi}{2}-2\alpha)} = \frac{a+d}{\sin(\frac{\pi}{2}+\alpha)} \quad ✓$$

$$\frac{a}{\cos 2\alpha} = \frac{a+d}{\cos\alpha}.$$

$$\frac{a}{a+d} = \frac{\cos 2\alpha}{\cos\alpha} = \frac{2\cos^2\alpha - 1}{\cos\alpha} \quad ✓$$

$$\frac{a}{a+d} = 2\cos\alpha - \frac{1}{\cos\alpha}. \quad \text{——} (1).$$

BY THE COSINE RULE :

$$\cos\alpha = \frac{a^2+(a+d)^2-(a-d)^2}{2a(a+d)}$$

$$= \frac{a^2+\left[\left((a+d)-(a-d)\right)\left((a+d)+(a-d)\right)\right]}{2a(a+d)} \quad ✓$$

$$= \frac{a^2+\left[(2d)(2a)\right]}{2a(a+d)}$$

$$\cos\alpha = \frac{a+4d}{2(a+d)} \quad ✓$$

SUBSTITUTE IN (1)

$$\frac{a}{a+d} = \frac{a+4d}{a+d} - \frac{2(a+d)}{a+4d} \quad ✓$$

$$a(a+4d) = (a+4d)^2 - 2(a+d)^2$$

$$\cancel{a^2}+4ad = \cancel{a^2}+8ad+16d^2-2a^2-4ad-2d^2$$

$$2a^2 = 14d^2$$

$$d^2 = \frac{1}{7}a^2$$

$$d = \frac{1}{\sqrt{7}}a$$

SIDES IN RATIO $\quad a-\frac{1}{\sqrt{7}}a \; : \; a \; : \; a+\frac{1}{\sqrt{7}}a$

$$1-\frac{1}{\sqrt{7}} \; : \; 1 \; : \; 1+\frac{1}{\sqrt{7}}$$

$$\sqrt{7}-1 \; : \; \sqrt{7} \; : \; \sqrt{7}+1 \quad ✓$$