SOLUTIONS : Yr 12 TRIAL HSC 2003

1 a) $\lim_{\theta \to 0} \frac{\tan \theta/3}{\theta} = \frac{1}{3} \lim_{\theta \to 0} \frac{\tan \theta/3}{\theta/3}$

$= \frac{1}{3} \lim_{\theta \to 0} \frac{\sin \theta/3}{\theta/3} \cdot \frac{1}{\cos \theta/3}$

$= \frac{1}{3}$ (1)

b) (i) $\frac{9!}{3!\,2!\,2!} = \underline{15{,}120}$ (1)

(ii) with e's together there are $= 3{,}360$ ways of arranging letters

$\therefore$ Prob(e's together) $= \frac{3360}{15120} = \underline{0.22}$ (1)

c) $P(x) = x^3 + 2x^2 - 23x - 60$

if $x = -4$ $P(-4) = -64 + 32 + 92 - 60 = 0$

$\therefore (x+4)$ is a factor of $P(x)$ (1)

$$\begin{array}{r} x^2 - 2x - 15 \\ (x+4)\overline{)x^3 + 2x^2 - 23x - 60} \\ \underline{x^3 + 4x^2} \\ -2x^2 \\ \underline{-2x^2 - 8x} \\ -15x - 60 \\ \underline{-15x - 60} \\ 0 \end{array}$$

$\therefore P(x) = (x+4)(x^2 - 2x - 15)$

$\therefore P(x) = (x+4)(x-5)(x+3)$ (1)

d) $\frac{x+1}{x-2} < 2$ (1)

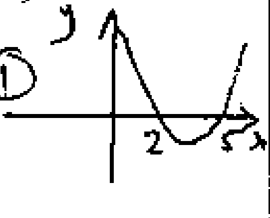
$\therefore (x+1)(x-2) < 2(x-2)^2$

$\therefore x^2 - x - 2 < 2x^2 - 8x + 8$

$\therefore 0 < x^2 - 7x + 10$

$0 < (x-2)(x-5)$ (1)

$\therefore \underline{x < 2,\ x > 5}$ (1)



e) $\frac{d}{dx}(x \sin^2 x)$

$= x \frac{d}{dx}(\sin^2 x) + 1 \cdot \sin^2 x$

$\frac{d}{dx}(\sin^2 x) = \frac{d}{dx}\left\{ \frac{1}{2}(1 - \cos 2x) \right\}$

$= \sin 2x$ (1)

$\therefore \frac{d}{dx}(x \sin^2 x) = x \sin 2x + \sin^2 x$ (1)

f) Let $f(x) = x^2 - 48$ (1)

and $x_1 = 7$

$f'(x) = 2x$

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ (1)

$x_2 = 7 - \frac{1}{14} = \underline{6.93}$ (to 2 d.p.)

Q2) a) Find constant term for $\left(3x + \frac{2}{x^2}\right)^9$,

Expanding gives:- (1)

$\left(3x + \frac{2}{x^2}\right)^9 = {}^9C_0(3x)^9 + {}^9C_1(3x)^8\left(\frac{2}{x^2}\right) + {}^9C_2(3x)^7\left(\frac{2}{x^2}\right)^2$ etc

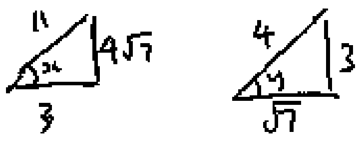
$\therefore$ 4th term is constant term:

$= {}^9C_3 (3x)^6 \cdot \left(\frac{2}{x^2}\right)^3$ (1)

$= 84 \times 729\, x^6 \times \frac{8}{x^6}$

$= \underline{489{,}888}$ (1)

b) Show $\cos^{-1}\frac{3}{11} - \sin^{-1}\frac{3}{4} = \sin^{-1}\frac{19}{44}$



Let $x = \cos^{-1}\frac{3}{11}$ and $y = \sin^{-1}\frac{3}{4}$ (1)

$\therefore$ Prove $x - y = \sin^{-1}\frac{19}{44}$

or $\sin(x - y) = \frac{19}{44}$ (1)

$\therefore \sin(x-y) = \sin x \cos y - \cos x \sin y$

$\therefore \sin(x-y) = \frac{4\sqrt7}{11}\cdot\frac{\sqrt7}{4} - \frac{3}{11}\cdot\frac{3}{4}$

$\therefore \sin(x-y) = \frac{4\times 7 - 9}{44} = \frac{19}{44}$

$\therefore$ LHS = RHS $\therefore$ Proved. (1)

c) (i) $v = \frac{dx}{dt}$

$\therefore v\frac{dv}{dx} = \frac{dx}{dt}\cdot\frac{dv}{dx} = \frac{dv}{dt} = \ddot{x}$

$\therefore \ddot{x} = v\frac{dv}{dx}$. (1)

(ii) $v = 6-2x$

$\ddot{x} = v\frac{dv}{dx} = (6-2x).(-2)$

$= \underline{4x - 12}$

$\therefore$ when $x = 0$, $\underline{\ddot{x} = -12\,ms^{-2}}$ (1)

(iii) $v = 6-2x = \frac{dx}{dt}$

$\therefore \int dt = \int \frac{dx}{6-2x}$

$\therefore t = -\frac12 \int \frac{-2}{6-2x}\,dx$ (1)

$= -\frac12 \log(6-2x) + C$ (1)

$= -\frac12 \log 6 - \frac12 \log\left(1 - \frac{x}{3}\right) + C$

when $t = 0$ $x = 0$

$\therefore C = +\frac12 \log 6.$

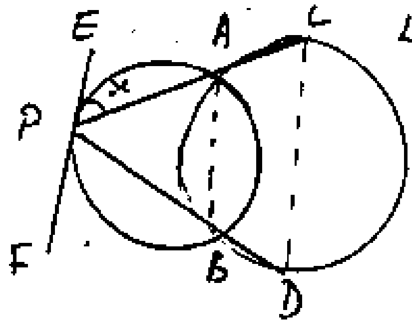
$\therefore \underline{t = -\frac12 \log\left(1-\frac{x}{3}\right)}$ (1)

$\therefore 2t = -\log\left(1-\frac{x}{3}\right)$

$\therefore e^{-2t} = 1 - \frac{x}{3}$

$\therefore x = 3\left(1 - e^{-2t}\right)$ (1)

Q3) Let $\angle EPA = x$



$\therefore x = \angle ABP$ ($\angle$ subtended by a chord is equal to the angle between the chord and the tangent.)

$\therefore \angle ABD = 180 - x$ (supp. $\angle$'s)

$\therefore \angle ACD = x$ (opp. $\angle$'s in a cyclic quad)

$\therefore \angle EPA = \angle ACD$

$\therefore EF \parallel CD$ (equal alternate $\angle$'s)

(3 marks, deduct 1 for missing reasons)

b) Let $S_n = 1.2^0 + 2.2^1 + 3\times 2^2 \ldots n.2^{n-1}$

$\therefore$ Prove: $S_n = 1 + (n-1)2^n$

for $n = 1$, $S_n = 1$

and $1 + (1-1).2^1 = 1 + 0 = 1$

$\therefore$ True for $n = 1$ (1)

Assume True for $n = k$ (1)

$\therefore S_k = 1 + (k-1)2^k$

$\therefore T_{k+1} = k+1^{\text{th}}$ term of S_n

$= (k+1)2^k$ (1)

$S_{k+1} = S_k + T_{k+1}$

$= 1 + (k-1)2^k + (k+1)2^k$

$= 1 + (k-1+k+1)2^k$

$= 1 + 2k.2^k$

$= 1 + k.2^{k+1}$ (1)

$\therefore$ Statement is true for $n = k+1$

$\therefore$ By principle of mathematical [illegible]

c) There are $^8C_3$ ways of selecting 3 women from 8 and $^7C_4$ ways for selecting 4 men from 7 (1)

$\therefore$ No of Committees is $^8C_3 \cdot {}^7C_4 = \underline{1960 \text{ ways}}$

If both women A and woman B are on the committee there are $^6C_1$ ways of selecting the remaining women.

$\therefore$ $^6C_1 \cdot {}^7C_4 = 210$ ways of having both women serve. (1)

(1)

$\therefore$ There are $1960 - 210 = \underline{1750}$ ways of selecting the committee <u>without</u> both serving.

Q4) a(i) If $T = A + (I - A)e^{-kt}$

$$\frac{dT}{dt} = -kIe^{-kt} + kAe^{-kt}$$

$$\frac{dT}{dt} = -k(I - A)e^{-kt}$$ (1)

$\therefore$ Yes, $T = A + (I - A)e^{-kt}$ is a solution

ii) $1200 = 20 + (1500 - 20)e^{-k.5}$

$\therefore 1180 = 1480e^{-5k}$

$\therefore -5k = \log\frac{1180}{1480}$

$\therefore k = -\frac{1}{5}\log\frac{1180}{1480}$ (1)

$\therefore k = 0.04531$

$\therefore$ After 1 hr. (60 mins)

$T = 20 + 1480e^{-(0.04531 \times 60)}$

$\therefore \underline{T = 117.6^\circ}$ (1)

b) $y = \frac{x-1}{x^2}$

This has an asymptote at $x = 0$

$$\frac{dy}{dx} = \frac{x^2 \cdot 1 - (x-1)2x}{x^4} = \frac{2-x}{x^3}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{x^3(-1) - (2-x) \cdot 3x^2}{x^6}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{2x-6}{x^4}$$ (1)

Turning points exist at $\frac{dy}{dx} = 0$

$\therefore 2 - x = 0, \ x = 2$

$\therefore$ pt. is $(2, \frac{1}{4})$

At $x = 2$, $\frac{d^2y}{dx^2} = \frac{-2}{16}$ $\therefore$ pt is a <u>max</u> (1)

Points of inflexion where

$\frac{d^2y}{dx^2} = 0 \quad 2x - 6 = 0$

$\therefore x = 3$ (1)

$\therefore$ At $(3, \frac{2}{9})$

checking for curvature change:

x	3^-	3	3^+
$\frac{d^2y}{dx^2}$	$-$	0	$+$

(1)

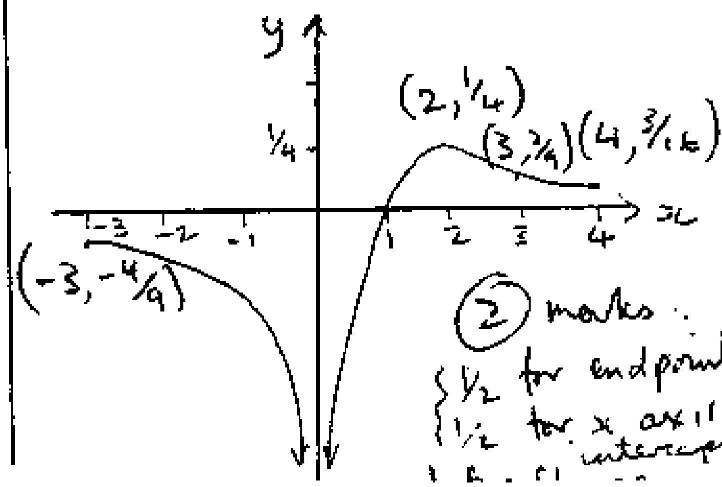
As $x \to 0^+$ $y \to -\infty$

As $x \to 0^-$ $y \to -\infty$

At $x = -3$ $y = -\frac{4}{9}$

At $x = +4$ $y = \frac{3}{16}$

When $y = 0$ $0 = \frac{x-1}{x^2}$ $\therefore x = 1$



(2) marks.

{½ for endpoints

½ for x axis intercept

9) $I = \int_1^9 \frac{dx}{x+\sqrt{x}}$

Let $x = u^2 \quad \therefore \frac{dx}{du} = 2u$ (1)

$\therefore dx = 2u\,du$.

When $x = 1 \quad u = \sqrt{x}$, gives $u = 1$

when $x = 9 \quad u = 3$

$\therefore I = \int_1^3 \frac{2u\,du}{u^2+u} \quad (u \neq 0)$

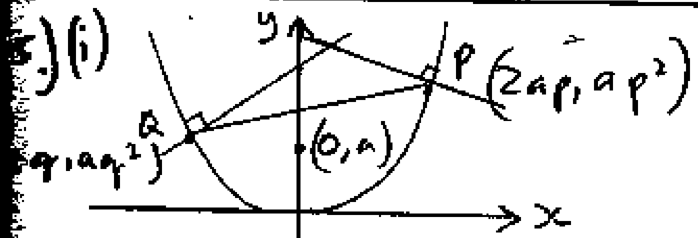
$\therefore I = \int_1^3 \frac{2}{u+1}\,du$. (1)

$I = 2\left[\log_e(1+u)\right]_1^3$

$= 2\left[\log_e 4 - \log_e 2\right]$

$\therefore \underline{I = 2\log_e 2}$ (or $\log_e 4$) $= 1.386$ (1)

5.) (i)



Equation of Chord:

$\frac{y - ap^2}{aq^2 - ap^2} = \frac{x - 2ap}{2aq - 2ap}$ (1)

$\frac{y - ap^2}{a(q^2-p^2)} = \frac{x - 2ap}{2a(q-p)}$

$\therefore 2(y - ap^2) = (p+q)(x - 2ap)$

$\therefore 2y - \cancel{2ap^2} = (p+q)x - \cancel{2ap^2} - 2apq$

$\therefore y = \left(\frac{p+q}{2}\right)x - apq$.

$(0, a)$ ... $= 0$ (1)

(ii) If chord is a focal chord it passes through $(0, a)$

$\therefore -a - apq = 0$

$\therefore apq = -a$

$\therefore pq = -1$

$\therefore p = -\frac{1}{q}$ (1)

But $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{4a}\right) = \frac{2x}{4a} = \frac{x}{2a}$

$\therefore$ At P gradient $= \frac{2ap}{2a} = p$

and At Q gradient $= \frac{2aq}{2a} = q$

$\therefore$ Gradients of normals are $-\frac{1}{p}$ and $-\frac{1}{q}$. (1)

If chord is focal $p = -\frac{1}{q}$

$\therefore$ Gradients of normals are $\frac{-1}{-1/q} = q$ and $-1/q$ $\therefore$ Product of gradients of normals $= -1$ $\therefore$ Normals to curve at P and Q are perpendicular to each other if PQ is a focal chord. (1)

b) $2\sin 2x\cos x = \sqrt{3}\sin 2x$

$\therefore 2\sin 2x\cos x - \sqrt{3}\sin 2x = 0$

$\therefore (2\cos x - \sqrt{3}).\sin 2x = 0$

$\therefore$ Either $2\cos x - \sqrt{3} = 0$ (1)

$\therefore \cos x = \frac{\sqrt{3}}{2} \quad x = \pi/6, \; 11\pi/6$ etc

or $\sin 2x = 0 \quad \therefore 2x = 0, \pi, 2\pi \ldots$

$\therefore x = 0, \pi/2, \pi, 3\pi/2 \ldots$ (1)

$\therefore$ General Solution is

$x = \frac{n\pi}{2}$ or $2n\pi \pm \pi/6$ for $n \in R$. (1)

c)(i) Initially $x=0$, $y=0$, $v=30\,m/s$

$\dot{x} = V\cos 0 = 30 \qquad \dot{y} = V\sin 0 = 0$

$\therefore$ Equations of motion are.

$\ddot{x} = 0 \qquad \ddot{y} = -g = -10$.

Integrating :-

$\dot{x} = c_1$ and $\dot{y} = -10t + c_2$

Initial conditions give $\left.\begin{array}{l}\underline{\dot{x} = 30}\\ \underline{\dot{y} = -10t}\end{array}\right\}$ 1

Integrating again:

$x = 30t + c_3 \qquad y = -5t^2 + c_4$

using $t=0$, $x = y = 0$, c_3 and $c_4 = 0$

$\therefore \underline{x = 30t}$, $\underline{y = -5t^2}$ 1

when $y = -80$ $\quad -5t^2 = -80$

$\therefore \underline{t = 4\text{ secs}}$ 1/2

$\therefore$ when $t = 4$ $\quad x = 30 \times 4 = 120$

$\therefore$ Distance to boat $= \underline{120\,m}$ 1/2

(ii) At $t = 4$, $\dot{x} = 30$, $\dot{y} = -40$

$\therefore v^2 = \dot{x}^2 + \dot{y}^2 = 30^2 + (-40)^2$

$\therefore$ Speed, $\therefore \underline{v = 50\,m/s}$ 1

Q6(i)(a) Let r be radius at time t,

$A = 4\pi r^2$ = Surface Area.

$\frac{dA}{dt} = 24$ (given). 1/2

$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ 1/2 $\therefore 24 = 8\pi r \cdot \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{24}{8\pi r} = \frac{3}{\pi r}$

$\therefore$ when $r = 12$, $\frac{dr}{dt} = \frac{1}{4\pi}$ cms/sec 1

(ii) Volume $= \frac{4\pi}{3}r^3 \quad \therefore \frac{dV}{dr} = 4\pi r^2$

$\therefore \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$ 1/2

$\therefore \frac{dV}{dt} = \frac{3}{\pi r} \cdot 4\pi r^2 = 12r$ 1/2

$\therefore$ when $r = 12$ cm $\frac{dV}{dr} = 144\,cm^3/sec$ 1

b) Find $I = \int_{-1/3}^{1/3} \frac{dx}{\sqrt{4 - 9x^3}}$ 1

$I = \int_{-1/3}^{1/3} \frac{dx}{3\sqrt{4/9 - x^2}} = \frac{1}{3}\int_{-1/3}^{1/3} \frac{dx}{\sqrt{4/9 - x^2}}$

$I = \left[\frac{1}{3} \cdot \sin^{-1}\frac{3x}{2}\right]_{-1/3}^{1/3}$ 1

$\left(\text{for } -\frac{2}{3} < x < \frac{2}{3}\right)$

$\therefore I = \frac{1}{3}\left\{\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right\}$

$= \frac{1}{3}\left\{\pi/6 - -\pi/6\right\}$

$\therefore I = \frac{1}{3} \cdot \frac{\pi}{3} = \underline{\underline{\frac{\pi}{9}}}$. 1

c) $y = \cos^{-1}(\sin x) \qquad$ Let $\sin x = u$

$\therefore \frac{du}{dx} = \cos x$

$\therefore \frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}} \cdot \cos x$ 1

$= \frac{-1}{\sqrt{1 - \sin^2 x}} \cdot \cos x$

$\therefore \frac{dy}{dx} = -1$ 1

36 d) $2x+3y=4$

$P(2,2)$, $Q(-1,-2)$

Let ratio $\frac{PC}{CQ} = \frac{m}{n}$

$\therefore C(x_3, y_3)$ is given by

$\therefore x_3 = \frac{-m+2n}{m+n}$

$\therefore y_3 = \frac{-2m+2n}{m+n}$ (1)

(x_3, y_3) lies on $2x+3y=4$ (1)

$\therefore 2\left(\frac{-m+2n}{m+n}\right) + 3\left(\frac{-2m+2n}{m+n}\right) = 4$

$\therefore -2m+4n-6m+6n = 4m+4n$

$\therefore 10n-8m = 4m+4n$

$\therefore 6n = 12m$ (1)

$\therefore$ Ratio is $2:1$ $\left(\text{or } \frac{PC}{CQ} = \frac{1}{2}\right)$

Q7 a)(i) $v^2 = 36-6x-2x^2$

$\ddot{x} = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = \frac{d}{dx}\left(18-3x-x^2\right)$

$\therefore \ddot{x} = -3-2x$ (1)

$\ddot{x} = -2(1.5+x)$ (1)

$\therefore$ This has the form $-n^2(x-x_0) = \ddot{x}$

$\therefore$ Motion is SHM with $n=\sqrt{2}$ and motion centred round the point $x=-1.5$. (1)

(ii) Period $= \frac{2\pi}{n} = \sqrt{2}\,\pi$ secs (1)

Max. amplitude occurs when $v=0$

$\therefore 0 = 18-3x-x^2$

$0 = (6+x)(3-x)$

$\therefore x = -6$ or $x = +3$

Motion is centred round -1.5

$\therefore$ Amplitude $= 4.5$ m (1)

(iii) Max. speed occurs as particle passes $x=-1.5$

$\therefore v^2 = 36+9-2(-1.5)^2$

$\therefore v^2 = 40.5$ (1)

$\therefore$ Max. speed $= 6.4$ m/s (1)

b) Equating coefficients of x^5 and x^6 from $(3+2x)^n$ we have.

${}^nC_5 \cdot 3^{n-5}\, 2^5 = {}^nC_6\, 3^{n-6} \cdot 2^6$ (1)

$\therefore {}^nC_5 \cdot 3 = {}^nC_6 \cdot 2$

$\therefore \frac{n!}{5!(n-5)!} \cdot 3 = \frac{n!}{6!(n-6)!} \cdot 2$ (1)

$\therefore \frac{3}{n-5} = \frac{2}{6}$

$\therefore 18 = 2(n-5)$ $\therefore n=14$ (1)

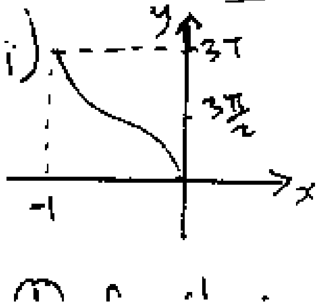
c)(i) $y-3 = e^x$

swapping x and y

$\therefore x-3 = e^y$

$\therefore y = \log_e(x-3)$ (1)

(ii)



$-1 \le 2x+1 \le 1$

Domain: $-1 \le x \le 0$

Range: $0 \le y \le 3\pi$