a i Solve $\frac{4}{x-1} < 2$

1st critical value (c.v) is $x = 1$

Let $\frac{4}{x-1} = 2$

$4 = 2x - 2$

$6 = 2x$

$x = 3$ is 2nd c.v.

Test $x = 0$ $\quad \frac{4}{-1} < 2$ True

$x = 2$ $\quad \frac{4}{2-1} < 2$ False

$x = 4$ $\quad \frac{4}{3} < 2$ True

ans: $x < 1$, $x > 3$

$P = (2, 5) = x_1\ y_1$

$Q = (6, 3) = x_2\ y_2$

$k_1 : k_2 = 1 : -3$

$x = \frac{k_1 x_2 + k_2 x_1}{k_1 + k_2} = \frac{1 \times 6 - 3 \times 2}{1 - 3} = 0$

$y = \frac{k_1 y_2 + k_2 y_1}{k_1 + k_2} = \frac{1 \times 3 - 3 \times 5}{1 - 3} = \frac{-12}{-2} = 6$

ans $= (0, 6)$

$\int_1^2 \frac{4}{\sqrt{4 - x^2}}\, dx$

$= 4\int_1^2 \frac{1}{\sqrt{2^2 - x^2}}\, dx$

$= 4\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_1^2$

$4\left[\sin^{-1}(1) - \sin^{-1}\left(\frac{1}{2}\right)\right]$

$4\left[\frac{\pi}{2} - \frac{\pi}{6}\right] = \frac{4}{3}\pi$

b $y = \tan^{-1}(4x)$

Let $u = 4x$ $\quad \therefore y = \tan^{-1} u$

$\frac{du}{dx} = 4$ $\quad \frac{dy}{du} = \frac{1}{1+u^2}$

$\frac{dy}{dx} = \frac{dx}{du} \cdot \frac{du}{dx}$

$= \frac{1}{1+u^2} \cdot 4$

$= \frac{4}{1 + 16x^2}$

d $I = \int_{-1}^{0} 2x\sqrt{1+x}\, dx$

Let $u = 1 + x$

$x = -1, u = 0$

$x = 0, u = 1$

$2x\sqrt{1+x}$

$= 2(u-1)u^{1/2}$

$= 2(u^{3/2} - u^{1/2})$

$\frac{du}{dx} = 1$

$\therefore du = dx$

$I = 2\int_0^1 u^{3/2} - u^{1/2}\, du$

$= 2\left[\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3}\right]_0^1$

$= 2\left[\frac{2}{5} - \frac{2}{3}\right]$

$= -\frac{8}{15}$

Question 2

(a) $P(x) = x^3 + mx^2 + nx - 18$

$P(-2) = -8 + 4m - 2n - 18$

$= 4m - 2n - 26 = 0$ (*)

$P(1) = 1 + m + n - 18 = -24$

$m + n + 7 = 0$ (*)

Solve simultaneously

$2m + 2n + 14 = 0$

$6m - 12 = 0$

$\therefore m = 2$

$n = -9$

(c) $\sin 2\theta = 2\sin^2\theta$, $(0 \le \theta \le 2\pi)$

$2\sin\theta\cos\theta = 2\sin^2\theta$

$2\sin\theta(\sin\theta - \cos\theta) = 0$

$\therefore \sin\theta = 0$ or $\sin\theta = \cos\theta$

$\tan\theta = 1$

$\theta = 0^\circ, \pi, 2\pi, \frac{\pi}{4}, \frac{5\pi}{4}$

(b) $A = \pi$…

$\frac{dA}{dr} = 2\pi$…

$\frac{dA}{dt} = \frac{dA}{dr}$…

$3.2 = 2 \times \pi$…

$\therefore \frac{dr}{dt} = \frac{3.}{(2\times}$…

$= 0.$…

Rate of incr…

is $0.$…

(d) $y = 3\,s$…

i $-1 \le \frac{x}{2}$…

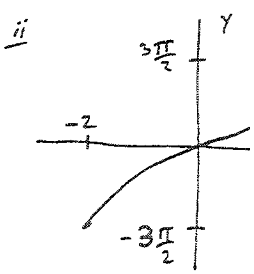
$\therefore$ Domain

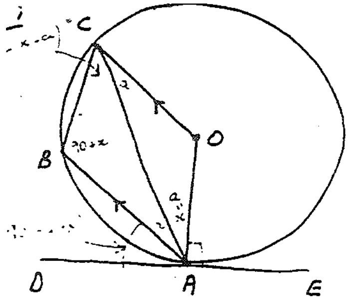
$-\frac{\pi}{2} \le \sin$…

$-\frac{\pi}{2} \times 3 \le 3\sin$…

$\therefore$ Range $-\frac{3\pi}{2} \le y$…

ii





Prove $\angle CAD = \angle BCO$

<u>Proof</u>: Let $\angle CBA = a$

Let $\angle CAO = x$

$\angle OAE = 90°$ ($\angle$ bet. tang & rad.)

$a = x$ (Isos Δ equal rad.)

$\angle CAE = \angle CBA$ ($\angle$ in alt. seg.

$= 90 + x$

$\angle OAD = 90°$ ($\angle$ bet tang & rad.)

$\therefore \angle BAD = 90 - a - x$

$\angle BCA = 90 - x - a$ ($\angle$ sum of Δ)

$\therefore \angle BCO = 90 - x$

also $\angle CAD = 90 - x$

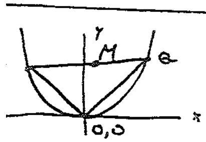
$\therefore \angle CAD = \angle BCO$

ii Aim: Prove $\angle CBA = 90° + \angle CAO$

Proof: $\angle CBA = 90 + x$ (proven above)

$\angle CAO + 90° = x + 90$

$\therefore \angle CBA = 90 + \angle CAO$



$P = 2ap, ap^2 \qquad Q = 2aq, aq^2$

i Grad of $PQ = \dfrac{ap^2 - aq^2}{2ap - 2aq} = \dfrac{a(p-q)(p+q)}{2a(p-q)} = \dfrac{p+q}{2}$

Eqn of PQ is $y - ap^2 = \frac{p+q}{2}(x - 2ap)$

$2y - 2ap^2 = (p+q)x - 2ap^2 - 2apq$

$\therefore (p+q)x - 2y - 2apq = 0$ is chord PQ

Grad OP (m_1) $= \dfrac{ap^2}{2ap} = \dfrac{p}{2}$ Grad OQ (m_2) $= \dfrac{aq^2}{2aq} = \dfrac{q}{2}$

Since $OP \perp OQ$ $\quad p/2 \cdot q/2 = -1 \quad \therefore pq = -4$

Midpoint M (x, y) $= \left(\dfrac{2ap + 2aq}{2}, \dfrac{ap^2 + aq^2}{2}\right) = a(p+q), \dfrac{a(p^2+q^2)}{2}$

$+q = x/a$

$p^2 + q^2 = 2y/a$

$(p+q)^2 - 2pq = 2y/a$

$\dfrac{x^2}{a^2} + 8 = \dfrac{2y}{a}$

$x^2 + 8a^2 = 2ya$

$x^2 = 2ay - 8a^2$

$x^2 = 2a(y - 4a)$ is locus of midpt of PQ

Questions

(c) $f(x) = \tan x - x$ Put $x_1 = 0.6$

$f'(x) = \sec^2 x - 1$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 0.6 - \dfrac{(\tan 0.6 - 0.6)}{(\sec^2 0.6 - 1)}$

$= 0.6 - \dfrac{0.08414}{0.46804}$

$= 0.42$

Question 4.

(a) $y = 10^x$

$\log_e y = \log_e 10^x = x \log_e 10$

$x = \dfrac{1}{\log_e 10} \cdot \log_e y$

$\dfrac{dx}{dy} = \dfrac{1}{\log_e 10} \cdot \dfrac{1}{y}$

$\therefore \dfrac{dy}{dx} = y \cdot \log_e 10$

when $x = 1$, $y = 10$

$\therefore \dfrac{dy}{dx} = 10 \log_e 10$

(b) Prove $\cos 3\theta = 4\cos^3\theta$.

L.H.S $= \cos(2\theta + \theta)$

$= \cos 2\theta \cos\theta - \sin 2\theta$.

$= (2\cos^2\theta - 1)\cos\theta - 2\sin^2$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta(1-$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta +$

$= 4\cos^3\theta - 3\cos\theta$

$=$ R.H.S

(c) $x^3 + ax^2 + 15x - 7 = 0$

Let roots $= \alpha, \alpha, \beta$

$2\alpha + \beta = -a$

$\alpha^2 + \alpha\beta + \alpha\beta = 15$

$\alpha^2\beta = 7$

$\alpha^2 + 2\alpha\beta = 15$

$\beta = 7/\alpha^2$

$\alpha^2 + 2\alpha\left(\frac{7}{\alpha^2}\right) = 15$

$\alpha^2 + \frac{14}{\alpha} = 15$

$\alpha^3 + 14 = 15\alpha$

Thus $2 + 7 = -a$

$\therefore a = -9$

(d) $\dfrac{dv}{dt} = -k(v - A)$

i $v = A + Ce^{-kt}$

$\dfrac{dv}{dt} = 0 - Cke^{-kt}$

$-k(v - A) = -k(A + Ce^{-kt} - A)$

$= -Cke^{-kt}$

Thus $v = A + Ce^{-kt}$ is a solution

Question 4

d ii $v = A + Ce^{-kt}$

$0 = 500 + Ce^{0}$

$\therefore \underline{C = -500}$

$21 = 500 - 500e^{-5k}$

$500e^{-5k} = 479$

$e^{-5k} = \frac{479}{500}$

$-5k \log_e e = \log_e\left(\frac{479}{500}\right)$

$\therefore k = 0.0085815$

iii $v = 500 - 500e^{-0.0085815 \times 20}$

$= 78.9$ m/s

iv $v = 500 - \frac{500}{e^{.0085815 \times t}}$

as $t \to \infty$, $v \to 500$ m/s

$\therefore$ max Velocity $= 500$ m/s

Questions 5

(a) $\left(\frac{2}{x^3} - \frac{x}{3}\right)^8$

$T_{k+1} = {}^nC_k a^{n-k} b^k$

$= {}^8C_k \left(\frac{2}{x^3}\right)^{8-k}\left(\frac{-x}{3}\right)^k$

$= {}^8C_k \frac{2^{8-k}}{x^{24-3k}} \cdot (-1)^k \frac{x^k}{3^k}$

$= {}^8C_k \frac{2^{8-k}}{3^k}(-1)^k x^{4k-24}$

For term independent of x

$4k - 24 = 0$

$k = 6$

$\therefore T_7 = (-1)^6 \, {}^8C_6 \frac{2^2}{3^6}$

$= \frac{112}{729}$ = Term indep. of x.

(b) a $\ddot{x} = -4x = -n^2x$

$\therefore n = 2$

Period: $T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$.

b $v^2 = n^2(a^2 - x^2)$

$3^2 = 2^2(a^2 - 0^2)$

$\therefore a = 3/2 = 1.5$

= amplitude.

c $v^2 = n^2(a^2 - x^2)$

$= 2^2\left(\frac{3}{2}^2 - 1^2\right)$

$= 4\left(\frac{9}{4} - 1\right)$

$= 9 - 4$

$= 5$

$\therefore v = \pm\sqrt{5}$ m/s

(c) Put $n = 1$

$13 \times 6^n + 2 = 13 \times 6^1 + 2 = 80$

This is divisible by 5

$\therefore$ True for $n = 1$

Assume true for $n = k$

$13 \times 6^k + 2 = 5m$, for integer m

Prove true for $n = k+1$

$13 \times 6^{k+1} + 2 = 6(13 \times 6^k + 2) - 10$

$= 6(5m) - 5 \times 2$

$= 5(6m - 2)$

This is divisible by 5.

$\therefore$ True for $n = k+1$

If the result is true for $n = k$

Then it is true for $n = k+1$

Since it is true for $n = 1$, then

it is true for $n = 2$, $n = 3$ etc.

d $\sqrt{3}\sin\theta - \cos\theta = 1$ F

0≤

Using $t = \tan\frac{\theta}{2}$

$\sin\theta = \frac{2t}{1+t^2}$ $\cos\theta = \frac{1-}{1}$

$\frac{2\sqrt{3}t}{1+t^2} - \frac{1-t^2}{1+t^2} = 1$

$2\sqrt{3}t - 1 + t^2 = 1 + t^2$

$t = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\tan\frac{\theta}{2} = \frac{1}{\sqrt{3}}$

$\frac{\theta}{2} = \frac{\pi}{6}$

$\therefore \theta = \frac{\pi}{3}$

Also test $\theta = \pi$ since

t method won't prove this

$\sqrt{3}\sin\pi - \cos\pi = -(-1)$

$\therefore \theta = \pi$ is a soluti

Ans : $\theta = \frac{\pi}{3}$ and π

$$y = -x \qquad x - \sqrt{3}y = 0$$

$$\therefore m_1 = -1 \qquad y = \frac{1}{\sqrt{3}}x$$

$$m_2 = \frac{1}{\sqrt{3}}$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}}\right| \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{1 + \sqrt{3}}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1}$$

$$= \frac{1 + 3 + 2\sqrt{3}}{2}$$

$$= 2 + \sqrt{3}$$

$$\therefore \theta = 75^\circ$$

(b) $$\frac{2(\sin^3 x + \cos^3 x)}{\sin x + \cos x}$$

$$= \frac{2(\sin x + \cos x)(\sin^2 x - \sin x\cos x + \cos^2 x)}{(\sin x + \cos x)}$$

$$= \frac{2(1 - \sin x\cos x)}{1}$$

$$= 2 - 2\sin x\cos x$$

$$= 2 - \sin 2x$$ (if $\sin x + \cos x \neq$ ...

c i Consider vertical motion up being +

$$\ddot{y} = -g$$

$$\dot{y} = -gt + c$$

$$V\sin\theta = c$$

$$\dot{y} = -gt + V\sin\theta$$

$$y = -\frac{gt^2}{2} + Vt\sin\theta + k$$

$$h = k$$

$$\therefore y = Vt\sin\theta - \frac{1}{2}gt^2 + h$$

Consider horizontal motion

$$\ddot{x} = 0$$

$$\dot{x} = c$$

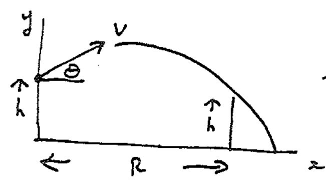
$$V\cos\theta = c$$

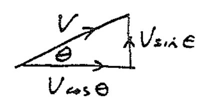
$$\dot{x} = V\cos\theta$$

$$x = Vt\cos\theta + c$$

$$0 = c$$

$$x = Vt\cos\theta$$





$$\left\{\begin{array}{l} t = 0 \\ \dot{y} = V\sin\theta \\ y = h \\ \dot{x} = V\cos\theta \\ x = 0 \end{array}\right.$$

ii Eliminate t

$$t = \frac{x}{V\cos\theta}$$

$$y = h + \frac{V\sin\theta\, x}{V\cos\theta} - \frac{g x^2}{2V^2\cos^2\theta}$$

$$y = h + x\tan\theta - \frac{x^2 g}{2V^2\cos^2\theta}$$

iii For ball to clear the fence

$x = R \qquad y > h$

$$h + R\tan\theta - \frac{R^2 g}{2V^2\cos^2\theta} > h$$

$$R\tan\theta > \frac{R^2 g}{2V^2\cos^2\theta}$$

$$2V^2\cos^2\theta > \frac{Rg}{\tan\theta}$$

$$V^2 > \frac{gR}{2\frac{\sin\theta}{\cos\theta}\cdot\cos^2\theta}$$

$$\therefore V^2 > \frac{gR}{2\sin\theta\cos\theta}$$

Question 7

a $(1+x)^n = 1 + {}^nC_1 x + \dots + {}^nC_r x^r + \dots + x^n$ ①

$(1+x)(1+x)^{n-1} = (1+x)(1 + {}^{n-1}C_1 x + \dots + {}^{n-1}C_{r-1} x^{r-1} + {}^{n-1}C_r x^r + \dots x^{n}$

$= (1 + \dots + {}^{n-1}C_r x^r + \dots + x^{n-1}) + (x + \dots + {}^{n-1}C_{r-1} x^r + \dots + x^n)$ ②

Equating coefficient of x^r in Line ① with Coeff of x^r in Line ②

$\therefore {}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$

---

b i Income I = Number of cars rented × Rate per car per day

Let \$$x$ = additional amount over \$30

$I = (200 - 5x).(30 + x)$

$= 6000 + 200x - 150x - 5x^2$

$I = 6000 + 50x - 5x^2$

$\frac{dI}{dx} = 50 - 10x$

$\frac{d^2I}{dx^2} = -10 < 0 \quad \therefore \max I$

Now for maximum I, $\frac{dI}{dx} = 0$

$50 - 10x = 0$

$\therefore x = 5$

Thus the rate which produces maximum daily Income

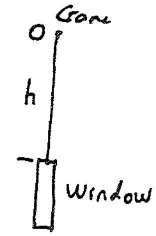
$=$ \$30 + \$5 = \$35 per car per day.

ii Maximum Income $= 6000 + (50 \times 5) - 5 \times 5^2$

$=$ \$6125

Question 7

(c) Consider the downward direction as positive ↓



Let h = height of crane above top o

Take vertical motion $t = 0,\ y = 0,$

$\ddot{y} = +g$

$\dot{y} = gt + c$

$0 = 0 + c$

$\dot{y} = gt$

$y = \frac{gt^2}{2} + c$

$0 = 0 + c$

$\therefore y = \frac{gt^2}{2}$

Let $t = T$ secs to reach top of window

Velocity at top of window $\dot{y} = gT$

Displacement at top of window $= h = \frac{g}{2} T^2$

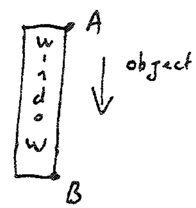
$\therefore \frac{2h}{g} = T^2$

Time to reach top of window $= T = \sqrt{\frac{2}{}}$

$\therefore$ Vel at top of window $= g\sqrt{\frac{2h}{g}} =$

Now consider Motion from top to bottom of window

Let $t = 0$, $y = 0$, $\dot{y} = \sqrt{2gh}$ at A



$\ddot{y} = g$

$\dot{y} = gt + c$

$\sqrt{2gh} = 0 + c$

$\dot{y} = gt + \sqrt{2gh}$

$y = \frac{gt^2}{2} + \sqrt{2gh}\,.\,t + c$

$0 = 0 + 0 + c$

$y = \frac{gt^2}{2} + t\sqrt{2gh}$

at B, $y = 2$, $t = 1/10$

$2 = \frac{9.8}{2} \times \frac{1}{100} + \frac{1}{1}$

$2 = 0.049 + \frac{1}{1}$

$1.951 \times 10 = \sqrt{}$

$19.51 = \sqrt{19.}$

$380.6401 = 19$

$\therefore h = 19.42$

Thus the crane

Q 7c

2m $\Big]$ $t = \frac{1}{10}$ sec

$\ddot{x} = 9.8$

$\dot{x} = 9.8t + c$

at $t = 0$, $\dot{x} = 0 = c$

$\therefore \dot{x} = 9.8t$

$x = 4.9t^2 + c_2$

at $t = 0$, $x = 0 = c_2$

$\therefore x = 4.9t^2$

at $t = t + 0.1$, $x = x + 2$

$\therefore x + 2 = 4.9(t + 0.1)^2$

& sub $x = 4.9t^2$, $4.9t^2 + 2 = 4.9(t^2 + 0.2t + 0.01)$

$\therefore 4.9t^2 + 2 = 4.9t^2 + 0.98t + 0.049$

$\therefore 2 - 0.049 = 0.98t$

$\therefore t = \dfrac{1.951}{0.98} = \dfrac{195.1}{98}$

$\therefore x = 4.9\left(\dfrac{195.1}{98}\right)^2 \doteqdot 19.4$

$\therefore$ height above window is 19.4 m.