2007 JRAHS Trial Extension 2

Question 1 **Marks**

(a) (i) Find the real numbers *a, b* and *c* such that $\dfrac{1}{x(4+x^2)} = \dfrac{a}{x} + \dfrac{bx+c}{4+x^2}$. **2**

(ii) Find $\displaystyle\int \frac{1}{x(4+x^2)}dx$. **2**

(b) Evaluate $\displaystyle\int_0^2 x\sqrt{2-x}\,dx$, leaving your answer in exact form. **3**

(c) Find the zeros of $P(x) = x^4 - 5x^3 + 7x^2 + 3x - 10$ over the complex field if $2 - \mathrm{i}$ is a zero. **3**

(d) Given that $I_{2n+1} = \displaystyle\int_0^1 x^{2n+1}e^{x^2}\,dx$ where *n* is a positive integer, show that **2**

$$I_{2n+1} = \frac{1}{2}e - nI_{2n-1}.$$

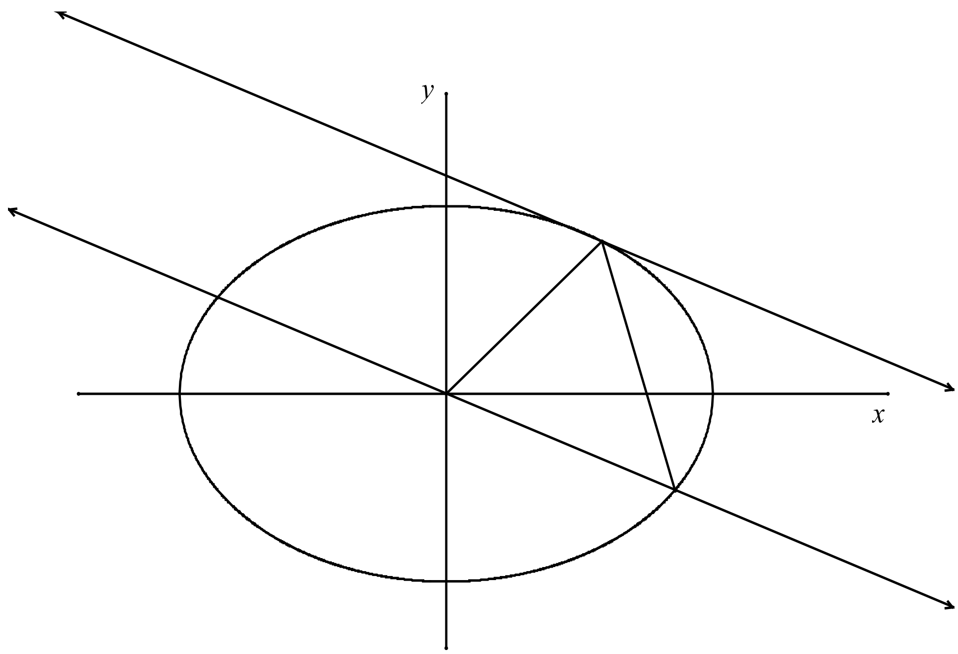
Hence, or otherwise, evaluate $\displaystyle\int_0^1 x^5 e^{x^2}\,dx$. **3**

Question 2 (15 Marks) [START A NEW PAGE] **Marks**

(a)(i)) Given that $z^2 = -3 - 4i$, find z. **4**

(ii) Solve the equation $x^2 - 3x + 3 + i = 0$ over the complex field. **3**

(b)



In the diagram above, $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where P lies in the first quadrant.

A straight line through the origin parallel to the tangent at P meets the ellipse at the point Q, where P and Q both lie on the same side of the y-axis.

(i) Prove that the equation of the line OQ is $xb\cos\theta + ya\sin\theta = 0$. **2**

(ii) Find the coordinates of the point Q given that Q lies in the fourth quadrant. **3**

(iii) Prove that the area of ΔOPQ is independent of the position of P. **3**

Question 3 (15 Marks) [START A NEW PAGE] **Marks**

(a) A particle is projected from the origin with a speed V and an angle of elevation α on level ground. **3**

A vertical wall of "unlimited" height is a distance d from the origin, and the plane of the wall is perpendicular to the plane of the particle's trajectory.

If $d < \dfrac{V^2}{g}$, show that the particle will strike the wall before it hits the ground provided that $\beta < \alpha < \dfrac{\pi}{2} - \beta$ where $\beta = \dfrac{1}{2}\sin^{-1}\left[\dfrac{gd}{V^2}\right]$.

You may assume that the range on the horizontal plane from the point of projection is $\dfrac{V^2 \sin 2\alpha}{g}$.

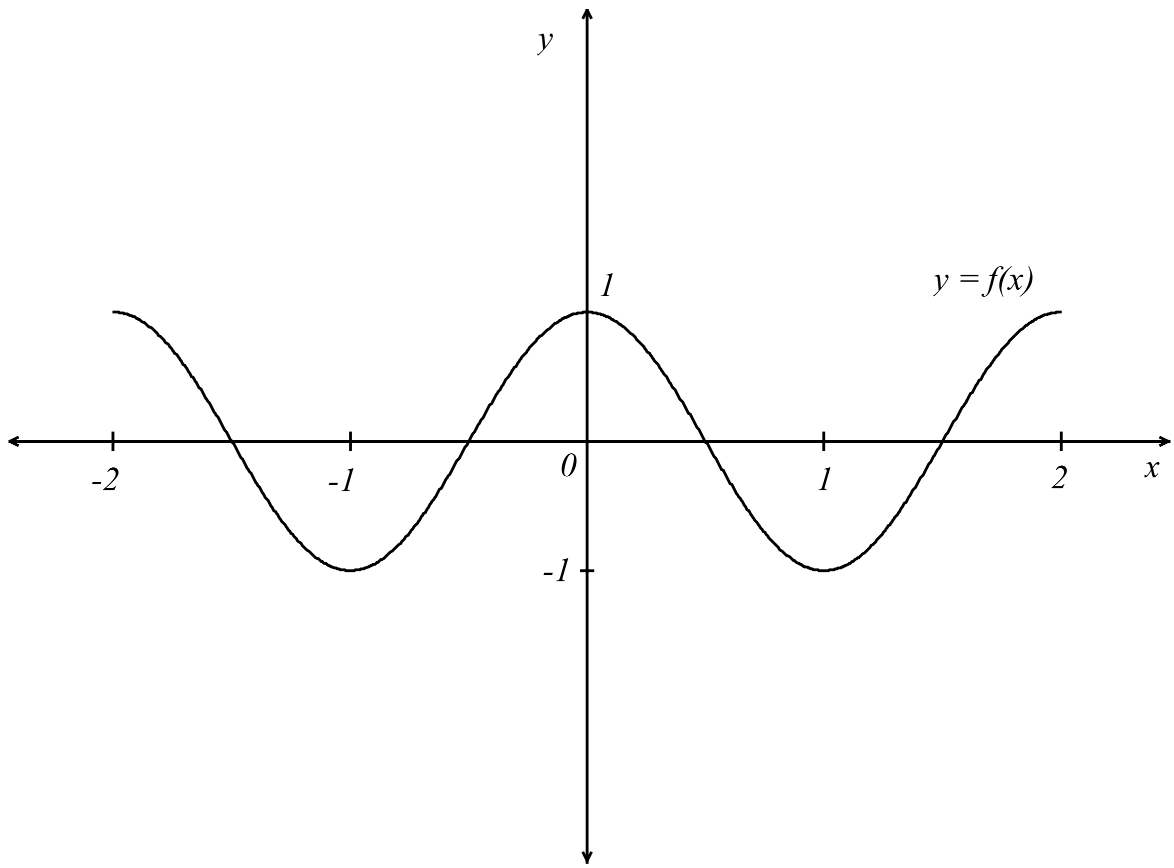
(b) Express $z = \dfrac{\sqrt{2}}{1-i}$ in the modulus-argument form and hence find z^5 in the form of $x + yi$. **4**

Question 3 continues on page 4

Question 3 cont'd

Marks

(c)



The diagram shows the graph of the continuous function $y = f(x)$. Critical points occur at x = -2, -1, 0, 1, 2.

On the sheets provided draw separate sketches of the graphs of the following :

(i) $y = |f(x)|$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \sqrt{f(x)}$ **2**

(iv) $y = xf(x)$ **3**

Question 4 (15 Marks) [START A NEW PAGE] **Marks**

(a) Find $\int \frac{1}{x(\ln x)^2}\, dx$. **2**

(b) Five letters are chosen from the letters of the word MOBILITY. These five letters are then placed alongside each other to form a five-letter arrangement. **4**

Find the number of different arrangements which are possible.

(c) $P\left(3p, \frac{3}{p}\right)$ and $Q\left(3q, \frac{3}{q}\right)$ are points on different branches of the hyperbola $xy = 9$.

(i) Find the equation of the tangent at P. **2**

(ii) Find the point of intersection T, of the tangents at P and Q. **3**

(iii) If the chord PQ passes through the point (0, 2), find the locus of T. **3**

(iv) Find the restriction on the locus of T. **1**

Question 5 (15 Marks) [START A NEW PAGE]

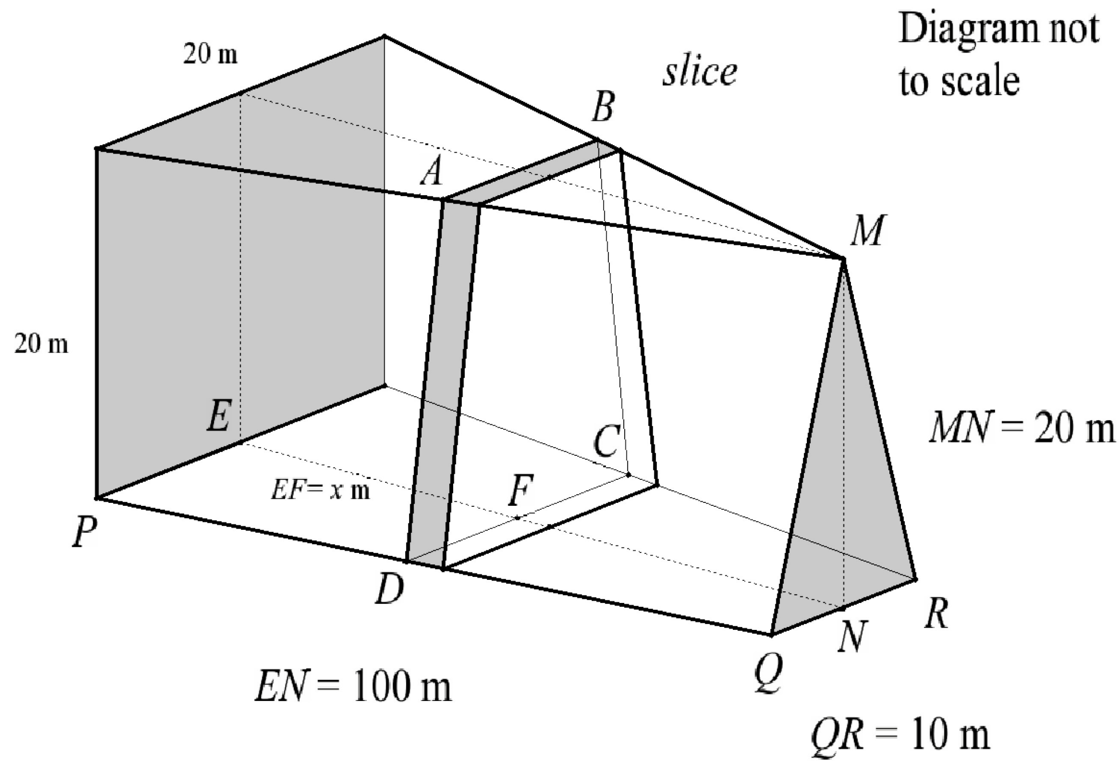
(a) (i) Find the volume generated when the area bounded by $y = \sin x$ and the x-axis, for $0 \le x \le \pi$, is rotated about the x-axis. **3**

(ii) The area described in (i) is now rotated about the line $x = 2\pi$. Find the volume of the solid formed. **4**

Question 5 continues on page 6

Question 5 con'td **Marks**

(b) A boat showroom is built on level ground. The length of the showroom is 100m. At one end of the showroom the shape is a square measuring 20m by 20m and at the other end an isosceles triangle of height 20m and base 10m.



(i) If EF is x m in length, show that the length of DC is $\left[20 - \frac{x}{10}\right]$ m. **2**

(ii) By considering trapezoidal slices parallel to the ends of the showroom, find the volume enclosed by the showroom in m^3 . **6**

Question 6 (15 Marks) [START A NEW PAGE] **Marks**

(a) Find $\int \frac{dx}{x^2 - 6x + 13}$. **2**

(b) A food parcel of 1 kg is dropped from a helicopter which is hovering 800 metres above a group of stranded bushwalkers. After 10 seconds a parachute opens automatically. Air resistance is neglected for the first 10 seconds but the effect of the open parachute is to cause a retardation of $2v$ newtons where v ms^{-1} is the velocity of the parcel after t seconds ($t \geq 10$).

Take the position of the helicopter as the origin, the downwards direction as positive and the value of g, the acceleration due to gravity as 10m s^{-2}.

(i) Write down the equation of motion of the parcel before the parachute opens. **1**

(ii) Determine the velocity and the distance fallen by the parcel at the end of the 10 seconds. **4**

(iii) Write down the equation of motion for $t \geq 10$. **1**

(iv) What is the terminal velocity of the parcel? **1**

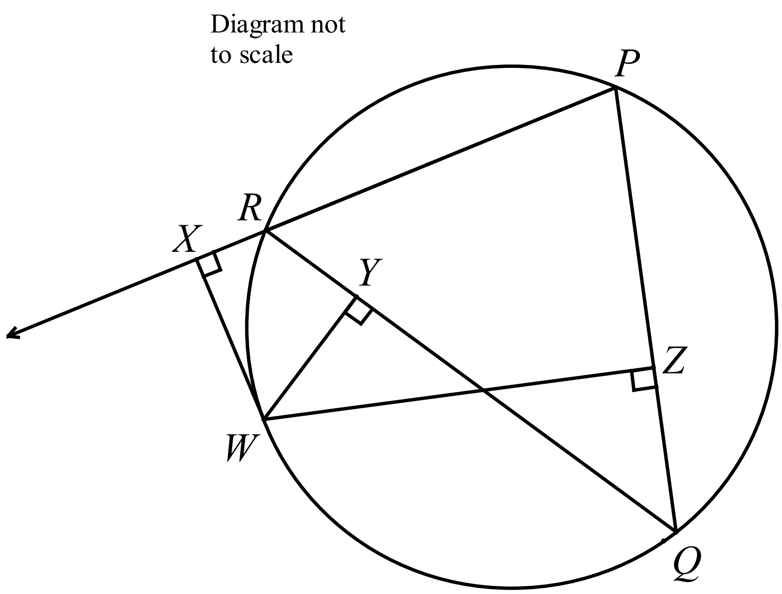
(v) Show that the velocity of the parcel after the parachute has opened is given by : **3**

$$v = 5 + 95e^{-2(t-10)}, \quad t \geq 10.$$

(vi) Find the distance fallen, x, as a function of t and hence find the distance the parcel has fallen 1 <u>minute</u> after it leaves the helicopter. **3**

Question 7 (15 Marks) [**START A NEW PAGE**] **Marks**

(a)



PQR is a triangle inscribed in a circle. W is a point on the arc QR.
From W, perpendiculars are drawn to PR (produced), QR and PQ, meeting them at X, Y and Z respectively.

Copy the diagram.

(i) Explain why $WXRY$ and $WYZQ$ are cyclic quadrilaterals. **2**

(ii) Prove that the points X, Y and Z are collinear. **4**

(b)(i) By considering the expansion of $(\cos\theta + i\,sin\,\theta)^5$ and by using De Moivre's Theorem show that **2**

$$\cos 5\theta = 16\,cos^5\,\theta - 20\,cos^3\,\theta + 5\,cos\,\theta.$$

(ii) Hence finds all the four roots of the equation **2**

$$16x^4 - 20x^2 + 5 = 0.$$

(iii) Hence or otherwise, show that $cos\frac{\pi}{10}cos\frac{3\pi}{10} = \frac{\sqrt{5}}{4}$. **3**

(iv) Find the exact value of $\sin\frac{3\pi}{5}\sin\frac{6\pi}{5}$. **2**

Question 8 (15 Marks) [START A NEW PAGE] **Marks**

(a) The region R in the Argand diagram is defined by:

$$|z-1| \le |z-i| \text{ and } |z-2-2i| \le 1.$$

(i) Sketch the region R. **3**

(ii) If z describes the boundary of the region R, find **2**

(α) the value of $|z|$ when $\arg(z)$ has the smallest value.

(β) z in the form of $a+ib$ when $\arg(z-1) = \dfrac{\pi}{4}$. **3**

(b) A certain type of merry go-round consists of seats hung from pivots attached to the rim of a horizontal circular disc. The disc is rotated by a motor attached to the vertical axle. As the angular velocity increases, the seats swing out and move up. The seat is represented by a point with mass m kg suspended by a rod of length h metres below the pivot, which is a metres from the axle of rotation.

Neglecting the mass of the rod, assume that when the disc rotates with constant angular velocity w radians per second, there is an equilibrium position such that the rod makes an angle θ with the vertical as shown in the diagram on the following page.

Question 8 continues on page 10

Question 8 cont'd **Marks**

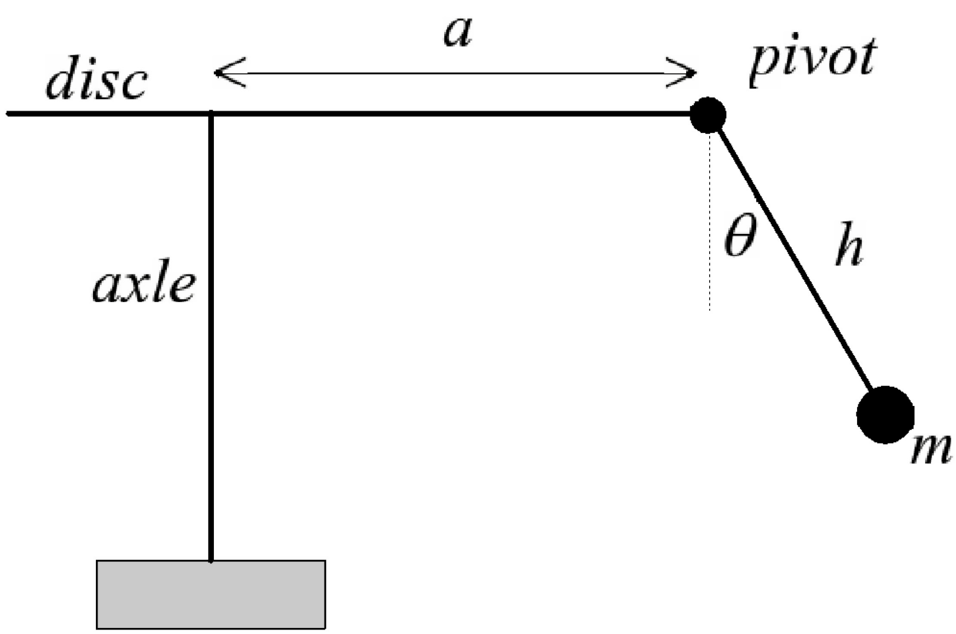


Diagram not to scale

(i) Show that w and θ satisfy the equation **3**

$$(a + h\sin\theta)w^2 = g\tan\theta$$

where g is the acceleration due to gravity.

(ii) Use graphical methods to show that for a given w, there is only one value of θ in the domain $0 \le \theta \le \frac{\pi}{2}$, which satisfies the above equation. **3**

(iii) Given $a = 4$, $h = 2.5$, $\theta = 30^0$ and using $g = 10\text{ms}^{-2}$, find the angular velocity w correct to 3 significant figures when the merry-go-round is in equilibrium. **1**

END