3/4 UNIT MATHEMATICS FORM VI

Time allowed: 2 hours (plus 5 minutes reading) **Exam date:** 8th August, 2000

Instructions:

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the left margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

Each question will be collected separately.
Start each question in a new 8-leaf answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet.

QUESTION ONE (Start a new answer booklet)

Marks

1 (a) Differentiate $\tan^{-1}(\pi x)$ with respect to x.

2 (b) Find:

(i) $\displaystyle\int \frac{dx}{4+x^2}$,

(ii) $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$.

3 (c) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\, dx$.

2 (d) Find the value of $\tan\alpha$ if α is the acute angle between the lines $y = \frac{1}{2}x$ and $y = -\frac{1}{\sqrt{3}}x + 1$.

2 (e) Prove that $\dfrac{\sin 2\theta}{\sin\theta} - \dfrac{\cos 2\theta}{\cos\theta} = \sec\theta$.

2 (f) Solve the inequation $\dfrac{x-2}{x} \geq 1$.

QUESTION TWO (Start a new answer booklet)

Marks

3 (a) Use the substitution $u = e^x$ to evaluate $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\, dx$.

3 (b) Find the term independent of x in the expansion of $\left(x^2 + \dfrac{2}{x}\right)^6$.

3 (c) The volume V of a spherical balloon is expanding at the rate of $10\,\text{mm}^3/\text{s}$. Find the rate of increase of its radius r when the surface area S is $1000\,\text{mm}^2$. (Note: You may use the formulae $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$).

3 (d) (i) Sketch, without the use of calculus, the polynomial $P(x) = (2x-1)^2(x+1)^3$, showing the x- and y-intercepts.

(ii) Hence solve the inequation $P(x) \geq 0$.

QUESTION THREE (Start a new answer booklet)

Marks

$\boxed{4}$ (a) The velocity v of a particle moving along the x-axis satisfies the equation

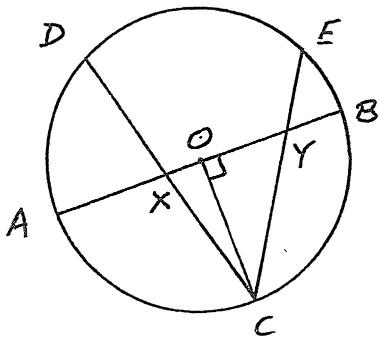
$$v^2 = 5 + 14x - 3x^2.$$

Show that the particle is moving in simple harmonic motion, and find the centre, amplitude and period of the motion.

$\boxed{4}$ (b) (i) Express $\cos 2x - \sin 2x$ in the form $R\cos(2x + \alpha)$, where α is acute and $R > 0$.

(ii) Hence solve the equation $\cos 2x - \sin 2x = 1$, for $0 \leq x \leq \pi$.

$\boxed{4}$ (c)



In the diagram above, AB is the diameter of a circle with centre O. The radius OC is drawn perpendicular to AB. The chords CD and CE intersect the diameter in the points X and Y respectively.

Copy the diagram into your examination booklet.

(i) Prove that $\angle CBA = \angle CAB = 45°$.

(ii) Give a reason why $\angle DCA = \angle DBA$ and $\angle CBD = \angle CED$.

(iii) Prove that $\angle CBD = \angle CXB$. (Hint: Let $\angle DCA = \alpha$).

(iv) Prove that $XYED$ is a cyclic quadrilateral.

QUESTION FOUR (Start a new answer booklet)

Marks

$\boxed{4}$ (a) Prove by mathematical induction that for all positive integers n,

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n+1)! - 1.$$

$\boxed{4}$ (b) The variable point P has coordinates $P(a\cos 2\theta, a\sin\theta)$.

(i) Show that P lies on the curve $y^2 = -\dfrac{a}{2}(x - a)$.

(ii) Sketch the locus of P as θ varies, taking account of any restrictions on x and y.

$\boxed{4}$ (c) The point $P(x, y)$ divides the interval joining the points $A(-1, 3)$ and $B(2, 8)$ internally in the ratio $k : 1$.

(i) Find the coordinates of P in terms of k.

(ii) Hence find the ratio in which the line $5x + 2y - 10 = 0$ divides the interval AB.

QUESTION FIVE (Start a new answer booklet)

Marks

$\boxed{4}$ (a) When the polynomial $P(x)$ is divided by $(x + 1)(x - 2)$, the result can be written as

$$P(x) = (x + 1)(x - 2)Q(x) + R(x), \text{ where } R(x) = ax + b.$$

(i) Given that $P(-1) = 3$, find the value of $R(-1)$.

(ii) Given also that the remainder is -2 when $P(x)$ is divided by $x - 2$, find the values of a and b.

$\boxed{4}$ (b) Let the expansion of $(2 + 3x)^{12}$ be written in the form $\displaystyle\sum_{r=0}^{12} t_r\, x^r$.

(i) Write down expressions for t_r and t_{r+1}, and show that $\dfrac{t_{r+1}}{t_r} = \dfrac{36 - 3r}{2r + 2}$.

(ii) Hence find the greatest coefficient in the expansion of $(2 + 3x)^{12}$. You need not simplify your answer.

$\boxed{4}$ (c) (i) Show that the coefficient of x^n in the expansion of $(1 + x)^n(1 + x)^n$ is given by

$$\sum_{r=0}^{n} ({}^n\mathrm{C}_r)^2 .$$

(ii) Hence, by equating the coefficients of x^n on both sides of the identity

$$(1 + x)^n(1 + x)^n = (1 + x)^{2n},$$

prove that $\displaystyle\sum_{r=0}^{n} ({}^n\mathrm{C}_r)^2 = \frac{(2n)!}{(n!)^2}$.

QUESTION SIX (Start a new answer booklet)

Marks

$\boxed{5}$ (a) (i) Show that the function $f(x) = x^3 - 3x + 1$ has stationary points at $x = 1$ and at $x = -1$.

(ii) Show that $x^3 - 3x + 1 = 0$ has a root α between $x = 0$ and $x = 0{\cdot}5$.

(iii) Taking $x = 0{\cdot}1$ as a first approximation, use one application of Newton's method to find a closer approximation to α. Give your answer correct to three decimal places.

(iv) Explain, with the aid of a neat sketch of the curve, why $x = 1{\cdot}1$ would not be a suitable first approximation to α.

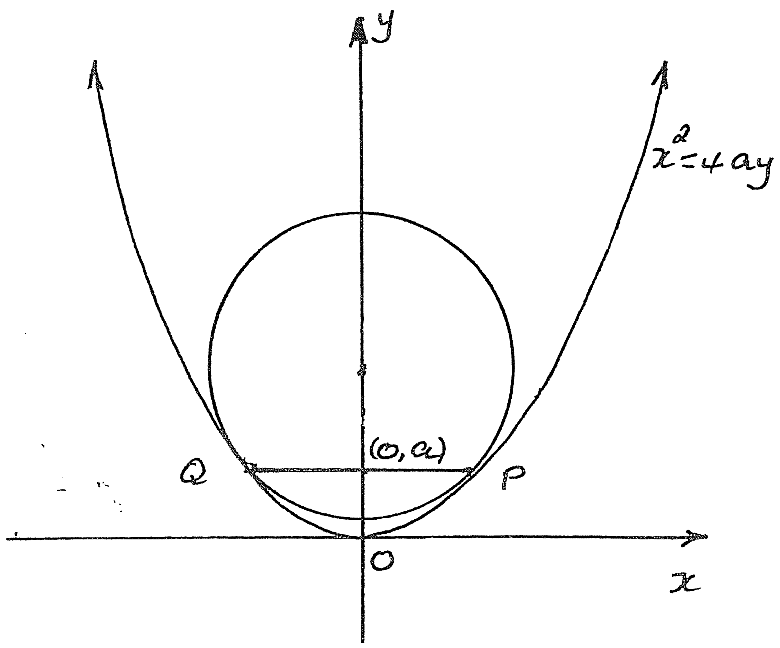
$\boxed{7}$ (b) (i) Show that the equation of the chord PQ joining the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ on the parabola $x^2 = 4ay$ is $y = \frac{1}{2}(p+q)x - apq$.

(ii) Suppose now that PQ is the *latus rectum* of the parabola, that is, the chord parallel to the directrix passing through the focus.

(α) Show that $p + q = 0$ and $pq = -1$, and find the coordinates of P and Q.

(β) Use calculus to find the equations of the normals at P and Q, and show that they intersect at $N(0, 3a)$.

(γ)



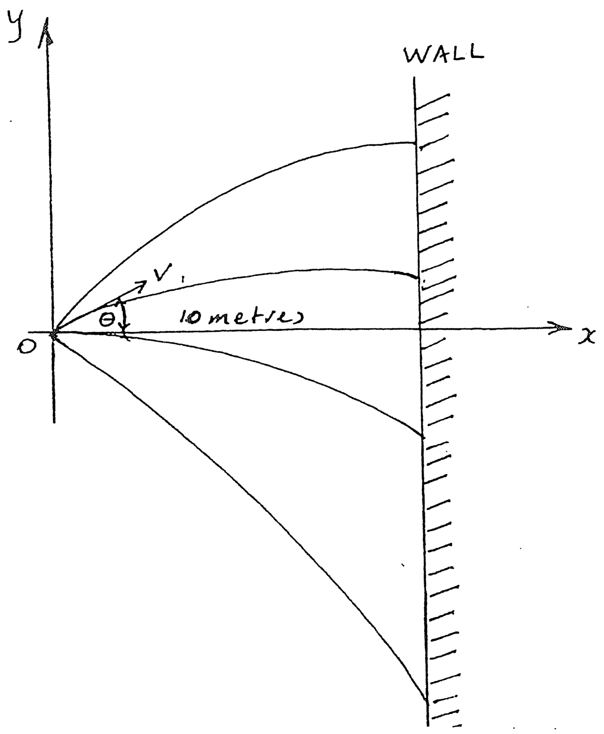
The diagram above shows the circle that touches the parabola $x^2 = 4ay$ at the endpoints of the latus rectum. Use (β) above to prove that the equation of the circle is $x^2 + y^2 - 6ay + a^2 = 0$.

QUESTION SEVEN (Start a new answer booklet)

Marks

3 (a) Find the general solution of the equation $\cos\left(2\pi(1-\frac{1}{3}x)\right) = -\frac{1}{2}$.

9 (b)



In the diagram above, a large number of projectiles are fired simultaneously from O, each with the same velocity V but various angles of elevation θ, at a wall distant 10 metres from O. The projectiles are fired so that their trajectories all lie in the same vertical plane perpendicular to the wall.

You may assume that the equations for the coordinates of a projectile at time t are

$$x = Vt\cos\theta \text{ and } y = -\tfrac{1}{2}gt^2 + Vt\sin\theta.$$

(i) Use the identity $\cos^2\theta + \sin^2\theta = 1$ to eliminate θ from these two equations, and hence prove that the relationship between height y and time t is

$$4y^2 + 4gt^2y + k = 0, \text{ where } k = g^2t^4 + 4x^2 - 4V^2t^2.$$

(ii) Show that the first impact on the wall occurs at time $t = \dfrac{10}{V}$, and that this projectile was fired horizontally. Also find where this projectile hits the wall.

(iii) Show that for $t > \dfrac{10}{V}$, there are two impacts at time t, and that the distance between these impacts is

$$2\sqrt{V^2t^2 - 100}.$$

(iv) Given that $V = 10\,\text{m/s}$, what are the initial angles of elevation of the two projectiles that strike the wall simultaneously $20\sqrt{3}$ metres apart.

GJ

QUESTION ONE

(a) $\dfrac{d}{dx}(\tan^{-1}\pi x) = \dfrac{\pi}{1+\pi^2x^2}$. $\boxed{\surd}$

(b) (i) $\displaystyle\int \frac{dx}{4+x^2} = \tfrac{1}{2}\tan^{-1}\tfrac{x}{2} + C.$ $\boxed{\surd}$

(ii) $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\tfrac{x}{2} + C.$ $\boxed{\surd}$

(c)
$$\begin{aligned}\int_0^{\frac{\pi}{4}} \tan^2 x\,dx &= \int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\,dx, \quad \boxed{\surd}\\ &= \Big[\tan x - x\Big]_0^{\frac{\pi}{4}} \quad \boxed{\surd}\\ &= 1 - \tfrac{\pi}{4}\\ &= \frac{4-\pi}{4}. \quad \boxed{\surd}\end{aligned}$$

(d) Gradients are $\frac{1}{2}$ and $-\frac{1}{\sqrt{3}}$.
$$\begin{aligned}\tan\alpha &= \left|\frac{\frac{1}{2}+\frac{1}{\sqrt{3}}}{1-\frac{1}{2\sqrt{3}}}\right|\\ &= \frac{\sqrt{3}+2}{2\sqrt{3}-1}. \quad \boxed{\surd\surd}\end{aligned}$$

(e)
$$\begin{aligned}LHS &= \frac{\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}{\sin\theta\cos\theta} \quad \boxed{\surd}\\ &= \frac{\sin\theta}{\sin\theta\cos\theta}\\ &= \sec\theta \quad \boxed{\surd}\\ &= RHS.\end{aligned}$$

(f)
$$\begin{aligned}\frac{x-2}{x} &\geq 1\\ (x-2)x &\geq x^2,\ x \neq 0 \quad \boxed{\surd}\\ x^2 - 2x &\geq x^2\\ 2x &\leq 0\\ \text{so} \qquad x &< 0 \quad \boxed{\surd}\end{aligned}$$

QUESTION TWO

(a) Let $u = e^x$

$du = e^x\,dx.$

When $x = 0$

$u = 1.$

When $x = ln\sqrt{3}$

$u = \sqrt{3}.$

$$\int_0^{ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx = \int_1^{\sqrt{3}} \frac{du}{1+u^2} \quad \boxed{\surd}$$

$$= \Big[\tan^{-1} u\Big]_1^{ln\sqrt{3}} \quad \boxed{\surd}$$

$$= \tfrac{\pi}{3} - \tfrac{\pi}{4}$$

$$= \tfrac{\pi}{12}. \quad \boxed{\surd}$$

(b) General term $= {}^6\mathrm{C}_r(x^2)^{6-r}(2x^{-1})^r$

$= {}^6\mathrm{C}_r \times 2^r \times x^{12-3r}$ $\boxed{\surd}$

$12 - 3r = 0$

$r = 4.$ $\boxed{\surd}$

Required term $= {}^6\mathrm{C}_4 \times 2^4$

$= 240.$ $\boxed{\surd}$

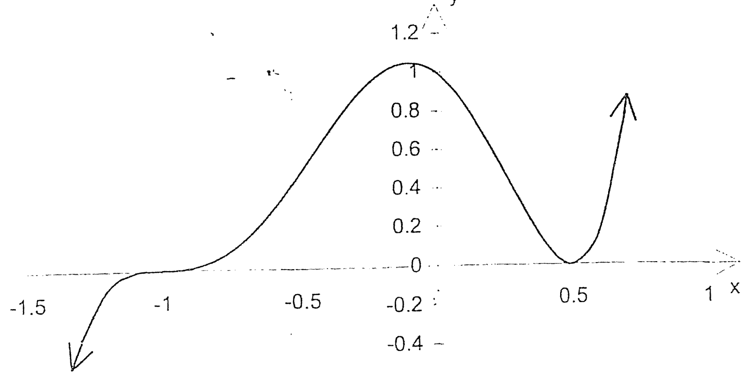
(c) $\dfrac{dV}{dt} = \dfrac{dV}{dr} \times \dfrac{dr}{dt}$

$10 = 4\pi r^2 \times \dfrac{dr}{dt}$ $\boxed{\surd}$

$10 = 1000 \times \dfrac{dr}{dt}$ $\boxed{\surd}$

$\dfrac{dr}{dt} = 0{\cdot}01\,\mathrm{mm/s}.$ $\boxed{\surd}$

(d) (i)



$\boxed{\surd\surd}$

(ii) $x \geq -1.$ $\boxed{\surd}$

QUESTION THREE

(a)
$$\begin{aligned} v^2 &= 5 + 14x - 3x^2 \\ \frac{d}{dx}\left(\tfrac{1}{2}v^2\right) &= \tfrac{1}{2}(14 - 6x) \\ &= 7 - 3x \\ &= -3\left(x - \tfrac{7}{3}\right), \quad \boxed{\surd} \end{aligned}$$

which is of the required form.

The centre of motion is $x = \frac{7}{3}$. $\boxed{\surd}$

Let
$$\begin{aligned} v^2 &= 0 \\ 3x^2 - 14x - 5 &= 0 \\ (3x + 1)(x - 5) &= 0 \\ x &= -\tfrac{1}{3} \text{ or } 5. \end{aligned}$$

$$\begin{aligned} \text{Amplitude } &= 5 - \tfrac{7}{3} \\ &= \tfrac{8}{3}. \quad \boxed{\surd} \end{aligned}$$

$$n = \sqrt{3}$$

$$\text{Period } = \frac{2\pi}{\sqrt{3}}. \quad \boxed{\surd}$$

(b) (i) Let
$$\begin{aligned} \cos 2x - \sin 2x &= R\cos(2x + \alpha) \\ &= R\cos 2x \cos\alpha - R\sin 2x \sin\alpha \end{aligned}$$

So $R\cos\alpha = 1$

and $R\sin\alpha = 1$.

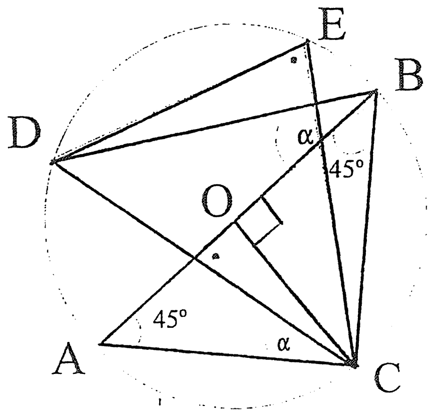
So $R = \sqrt{2}$

and $\alpha = \dfrac{\pi}{4}$.

So $\cos 2x - \sin 2x = \sqrt{2}\cos(2x + \frac{\pi}{4})$. $\boxed{\surd}$

(ii) Now $\sqrt{2}\cos(2x + \frac{\pi}{4}) = 1$
$$\begin{aligned} \cos(2x + \tfrac{\pi}{4}) &= \tfrac{1}{\sqrt{2}}, \text{ for } \tfrac{\pi}{4} \le 2x + \tfrac{\pi}{4} \le \tfrac{9\pi}{4} \quad \boxed{\surd} \\ 2x + \tfrac{\pi}{4} &= \tfrac{\pi}{4}, \tfrac{7\pi}{4}, \text{ or } \tfrac{9\pi}{4} \\ x &= 0, \tfrac{3\pi}{4} \text{ or } \pi. \quad \boxed{\surd\surd} \end{aligned}$$

(c)



(i) $\angle CBA = \angle CAB = 45°$ (angle at centre is twice angle at circumference). $\boxed{\checkmark}$

(ii) $\angle DCA = \angle DBA$ and $\angle CBD = \angle CED$,
(angles at the circumference standing on the same arc). $\boxed{\checkmark}$

(iii) Let $\angle DCA = \angle DBA = \alpha$.
$\angle CBD = \alpha + 45°$ (from (i) and (ii)).
But $\angle CXB = \alpha + 45°$ (exterior angle of $\triangle AXC$ equals sum of interior opposite angles)
so $\angle CBD = \angle CXB$. $\boxed{\checkmark}$

(iv) $\angle DEY = \alpha + 45°$ from (ii) above,
$\angle CXB = \alpha + 45°$ from (iii) above,
so $\angle DEY = \angle CXB$,
so $XYED$ is a cyclic quadrilateral as the exterior angle equals interior opposite angle. $\boxed{\checkmark}$

QUESTION FOUR

(a) Prove that for positive integers n,

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n+1)! - 1.$$

STEP 1.
When $n = 1$:

$$\text{LHS} = 1$$
$$\text{RHS} = 2! - 1 = 1.$$

So proposition is true for $n = 1$. $\boxed{\checkmark}$

STEP 2.
Assume the proposition true for some positive integer k so that,
$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + k \times k! = (k+1)! - 1$. $\boxed{\checkmark}$
We are required to prove the proposition true for $k+1$. That is,
$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + k \times k! + (k+1) \times (k+1)! = (k+2)! - 1$.

Now LHS $= 1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + k \times k! + (k+1) \times (k+1)!$.

$$= (k+1)! - 1 + (k+1) \times (k+1)!, \text{ from the assumption } \boxed{\checkmark}$$
$$= (k+1)! \times (1 + k + 1) - 1$$
$$= (k+2)(k+1)! - 1$$
$$= (k+2)! - 1$$
$$= \text{RHS. } \boxed{\checkmark}$$

It follows from steps one and two above by mathematical induction that for all positive integers n,

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n+1)! - 1.$$

(b) (i) $y^2 = -\dfrac{a}{2}(x - a)$.

$$\text{LHS} = y^2$$
$$= a^2 \sin^2 \theta$$
$$\text{RHS} = -\frac{a}{2}(a \cos 2\theta - 1)$$
$$= -\frac{a^2}{2}(1 - 2\sin^2\theta - 1) \quad \boxed{\checkmark}$$
$$= a^2 \sin^2 \theta$$
$$= \text{LHS. } \boxed{\checkmark}$$

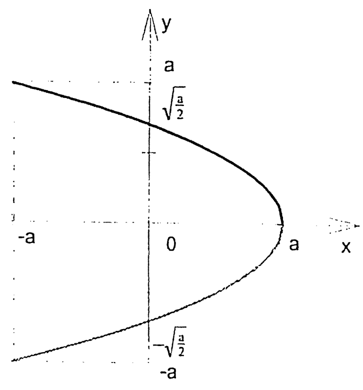
So P lies on the curve.

(ii) $x = a \cos 2x$

so $-a \leq x \leq a$,

and $y = a \sin \theta$

so $-a \leq y \leq a$. $\boxed{\checkmark}$



$y^2 = -\frac{a}{2}(x - a)$. $\boxed{\checkmark}$

(c) (i) Given points are $A(-1, 3)$ and $(2, 8)$.

For P $x = \dfrac{k \times 2 + 1 \times -1}{k+1}$

$$= \frac{2k - 1}{k + 1}$$

and $y = \dfrac{k \times 8 + 1 \times 3}{k + 1}$

$$= \frac{8k + 3}{k + 1}.$$

P is the point $\left(\dfrac{2k-1}{k+1}, \dfrac{8k+3}{k+1}\right)$. $\boxed{\checkmark\checkmark}$

(ii) Let P lie on $5x + 2y - 10 = 0$,

$$\text{so } 5 \times \left(\frac{2k-1}{k+1}\right) + 2 \times \left(\frac{8k+3}{k+1}\right) - 10 = 0 \quad \boxed{\surd}$$

$$10k - 5 + 16k + 6 - 10k - 10 = 0$$

$$16k - 9 = 0$$

$$k = \tfrac{9}{16}. \quad \boxed{\surd}$$

The line $5x + 2y - 10 = 0$ divides the interval in the ratio $9 : 16$.

QUESTION FIVE

(a) (i)

$$P(x) = (x+1)(x-2)Q(x) + R(x)$$

$$P(-1) = R(-1)$$

$$\text{so } R(-1) = 3. \quad \boxed{\surd}$$

(ii)

$$P(2) = 0 + R(2)$$

$$\text{so} \quad R(2) = -2. \quad \boxed{\surd}$$

$$\text{Now } -a + b = 3$$

$$\text{and } \quad 2a + b = -2$$

$$\text{so} \quad 3a = -5$$

$$a = -\tfrac{5}{3} \quad \boxed{\surd}$$

$$\text{and} \quad b + \tfrac{5}{3} = 3$$

$$b = \tfrac{4}{3}. \quad \boxed{\surd}$$

(b) (i)

$$t_r = {}^{12}\mathrm{C}_r \times 2^{12-r} \times 3^3 \quad \boxed{\surd}$$

$$t_{r+1} = {}^{12}\mathrm{C}_{r+1} \times 2^{11-r} \times 3^{r+1}$$

$$\frac{t_{r+1}}{t_r} = \frac{{}^{12}\mathrm{C}_{r+1} \times 2^{11-r} \times 3^{r+1}}{{}^{12}\mathrm{C}_r \times 2^{12-r} \times 3^3}$$

$$= \frac{12!}{(r+1)!(11-r)!} \times \frac{r!(12-r)!)}{12!} \times \frac{3}{2} \quad \boxed{\surd}$$

$$= \frac{12-r}{r+1} \times \frac{3}{2}$$

$$= \frac{36-3r}{2r+2}. \quad \boxed{\surd}$$

(ii) For increasing coefficients,

$$\frac{t_{r+1}}{t_r} > 1$$

$$\frac{36-3r}{2r+2} > 1$$

$$36 - 3r > 2r + 2$$

$$r < 6\tfrac{4}{5}.$$

Since $t_r < t_{r+1}$, $t_1 < t_2 < t_3 < \cdots < t_7$.

So the greatest coefficient is $t_7 = {}^{12}\mathrm{C}_7 \times 2^5 \times 3^7$. $\boxed{\surd}$

(c) (i) $(1+x)^n \times (1+x)^n = \left(\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{n-1}x^{n-1}+\binom{n}{n}x^n\right)^2.$

Term in x^n

$= \binom{n}{0}\binom{n}{n}x^n+\binom{n}{1}\binom{n}{n-1}x^n+\cdots+\binom{n}{n}\binom{n}{0}x^n$

$= \left(\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\cdots+\binom{n}{n}\binom{n}{0}\right)x^n$

Coefficient of term in x^n

$= \binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\cdots+\binom{n}{n}\binom{n}{0}$ ☑

$= \binom{n}{0}^2+\binom{n}{1}^2+\cdots+\binom{n}{n}^2$, since $\binom{n}{r}=\binom{n}{n-r}$

$= \sum_{r=0}^{n}\binom{n}{r}^2.$ ☑

(ii) For coefficient of term in x^n in the expansion of $(1+x)^{2n}$,

$$\begin{aligned} \text{coefficient} &= \binom{2n}{n} \\ &= \frac{(2n)!}{(n!)^2} \quad ☑ \end{aligned}$$

$$\text{so} \quad \sum_{r=0}^{n}\binom{n}{r}^2 = \frac{(2n)!}{(n!)^2} \quad ☑$$

QUESTION SIX

(a) (i)

$$\begin{aligned} f(x) &= x^3-3x+1 \\ f'(x) &= 3x^2-3. \\ \text{Let} \quad f'(x) &= 0 \\ 3x^2-3 &= 0 \\ 3(x-1)(x+1) &= 0 \\ x &= 1 \text{ or } -1. \quad ☑ \end{aligned}$$

(ii)

$$\begin{aligned} f(0) &= 1 \\ &> 0 \\ \text{and } f(0{\cdot}5) &= -0{\cdot}375 \\ &< 0, \end{aligned}$$

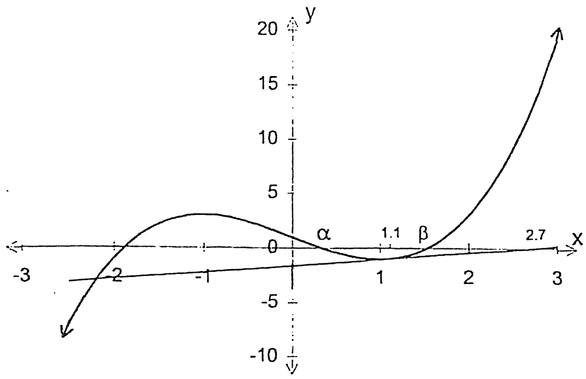
so α is a root between $x=0$ and $x=0{\cdot}5$. ☑

(iii) $x_2 = 0{\cdot}1 - \dfrac{f(0{\cdot}1)}{f'(0{\cdot}1)}$

$= 0{\cdot}1 + \dfrac{0{\cdot}701}{2{\cdot}97}$ ☑

$= 0{\cdot}336$ (correct to three decimal places). ☑

(iv)



The tangent at $x = 1{\cdot}1$ crosses the x-axis to the right of the root β shown, where $\alpha < \beta$. Further applications will approximate to the root at β. $\boxed{\surd}$

(b) (i) Gradient $\displaystyle = \frac{ap^2 - aq^2}{2ap - 2aq}$

$\displaystyle = \frac{p+q}{2}$

Equation of the chord is,

$$y - ap^2 = \frac{p+q}{2}(x - 2ap)$$

$$y = \frac{p+q}{2}x - apq. \quad \boxed{\surd}$$

(ii) (α) $S(0, a)$ lies on the chord,

so $a = 0 - apq$

$pq = -1.$

PQ is parallel to x-axis so gradient is zero.

$$\frac{p+q}{2} = 0$$

so $\quad p + q = 0 \quad \boxed{\surd \text{ for both results}}$

and so $\quad p = 1$ and $q = -1$.

It follows that the coordinates are $P(2a, a)$ and $Q(-2a, a)$. $\boxed{\surd}$

(β) $\displaystyle y = \frac{1}{4a}x^2$

$$\frac{dy}{dx} = \frac{x}{2a}.$$

At P, $\dfrac{dy}{dx} = 1$, so gradient of normal is -1.

Equation of normal:

$y - a = -1(x - 2a)$

$y = -x + 3a \ldots (1)$

At Q, $\dfrac{dy}{dx} = -1$, so gradient of normal is 1. $\boxed{\surd \text{ for either gradient}}$

Equation of normal:

$y - a = 1(x + 2a)$

$y = x + 3a \ldots (2) \quad \boxed{\surd \text{ for normals}}$

For N, $(1)+(2)$:

$2y = 6a$

so $\quad y = 3a$

and $3a = x + 3a$

so $\quad x = 0.$

N is the point $(0, 3a)$.

(γ) Since the tangents to the parabola at P and Q are also tangents to the circle, the normals at these points are radii of the circle. These intersect at the centre of the circle.

The centre of the circle is $N(0, 3a)$.

$$\begin{aligned} \text{radius } &= \sqrt{(2a-0)^2 + (3a-a)^2} \\ &= \sqrt{4a^2 + 4a^2} \\ &= 2a\sqrt{2}. \quad \boxed{\checkmark} \end{aligned}$$

The equation of the circle is:

$$(x-0)^2 + (y-3a)^2 = (2a\sqrt{2})^2$$

$$x^2 + y^2 - 6ay + 9a^2 = 8a^2$$

$$x^2 + y^2 - 6ay + a^2 = 0. \quad \boxed{\checkmark}$$

QUESTION SEVEN

(a) $\cos\left(2\pi(1 - \tfrac{1}{3}\,x)\right) = -\dfrac{1}{2}$

$$2\pi(1 - \tfrac{1}{3}\,x) = 2n\pi + \tfrac{2\pi}{3} \text{ or } 2n\pi - \tfrac{2\pi}{3}, \text{ where } n \text{ is an integer.} \quad \boxed{\checkmark}$$

$$1 - \tfrac{1}{3}x = n + \tfrac{1}{3} \text{ or } n - \tfrac{1}{3}$$

$$-\tfrac{1}{3}x = n - \tfrac{2}{3} \text{ or } n - \tfrac{4}{3}$$

$$x = 2 - 3n \text{ or } 4 - 3n \quad \boxed{\checkmark\checkmark}$$

(Note: Check variations.)

(b) (i)

$$\begin{aligned} V\cos\theta &= \tfrac{x}{t} \\ V\sin\theta &= \tfrac{y}{t} + \tfrac{1}{2}gt \\ \left(\tfrac{x}{t}\right)^2 + \left(\tfrac{y}{t} + \tfrac{1}{2}gt\right)^2 &= V^2 \\ x^2 + \left(y + \tfrac{1}{2}gt^2\right)^2 &= V^2t^2 \\ 4x^2 + (2y + gt^2)^2 &= 4V^2t^2 \end{aligned}$$

$$\text{so } 4y^2 + 4ygt^2 + (4x^2 + g^2t^4 - 4V^2t^2) = 0$$

$$4y^2 + 4gt^2y + k = 0 \quad \boxed{\checkmark}$$

(ii) Now $t = \dfrac{x}{V\cos\theta}$ which, for fixed x and V, is a minimum when $\cos\theta$ is a maximum.

$$\text{So minimum } t = \frac{x}{V} \text{ when } \theta = 0 \quad \boxed{\checkmark}$$
$$= \frac{10}{V}. \quad \boxed{\checkmark}$$

So
$$y = -\frac{1}{2}g \times \left(\frac{10}{V}\right)^2$$
$$= -\frac{50g}{V^2}. \quad \boxed{\checkmark}$$

(iii) For $t > \dfrac{10}{V}$ there is a solution for θ, which means that the projectile hits the wall and hence there will be a solution for y.

Now for the quadratic in y :
$$\Delta = (4gt^2)^2 - 16(g^2t^4 + 4 \times 10^2 - 4V^2t^2)$$
$$= 64(V^2t^2 - 100).$$
$$\text{So } \Delta > 64\left(V^2 \times \frac{10^2}{V^2} - 100\right) \text{ since } t > \frac{10}{V}$$
$$> 0. \quad \boxed{\checkmark}$$

Hence there are two real and distinct roots and so there are two impacts at time t. $\boxed{\checkmark}$

Now distance between impacts equals difference between roots.
$$\text{Difference} = \frac{-b + \sqrt{\Delta}}{2a} - \frac{-b - \sqrt{\Delta}}{2a}$$
$$= \frac{2\sqrt{\Delta}}{2a}$$
$$= \frac{\sqrt{\Delta}}{a}.$$
$$\text{So Distance} = \frac{8\sqrt{V^2t^2 - 100}}{4}$$
$$= 2\sqrt{V^2t^2 - 100}. \quad \boxed{\checkmark}$$

(iv)

$$
\begin{aligned}
\text{Distance} &= 2\sqrt{V^2t^2 - 100.} \\
&= 2\sqrt{100t^2 - 100} \\
&= 20\sqrt{t^2 - 1}
\end{aligned}
$$

so

$$
\begin{aligned}
20\sqrt{t^2 - 1} &= 20\sqrt{3} \\
\sqrt{t^2 - 1} &= \sqrt{3} \\
t^2 - 1 &= 3 \\
t^2 &= 4 \\
t &= 2. \quad \boxed{\surd}
\end{aligned}
$$

Now

$$
\begin{aligned}
\cos\theta &= \frac{x}{Vt} \\
&= \frac{10}{10 \times 2} \\
&= \frac{1}{2}
\end{aligned}
$$

and so $\theta = 60^\circ$ and -60° are the angles of elevation. $\boxed{\surd}$

GJ