Other Inequalities

3U97-2c)!

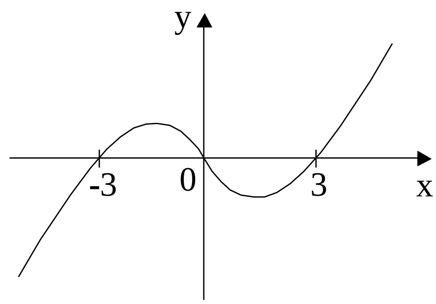
The polynomial $P(x) = x^3 + bx^2 + cx + d$ has roots 0, 3 and -3.

i. Find b, c and d.

ii. Without using calculus, sketch the graph of $y = P(x)$.

iii. Hence, or otherwise, solve the inequality $\frac{x^2-9}{x} > 0$.¤

«→ i) $b = 0$, $c = -9$, $d = 0$ ii)



iii) $-3 < x < 0$ or $x > 3$ »

3U96-1e)!

Solve the inequality $\frac{2}{x-1} \le 1$.¤

«→ $x < 1$ or $x \ge 3$ »

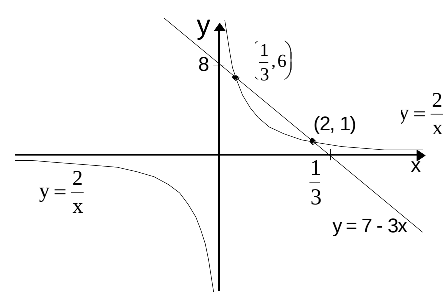
3U94-2c)!

i. Verify that $x = \frac{1}{3}$ and $x = 2$ satisfy the equation $7 - 3x = \frac{2}{x}$.

ii. On the same set of axes, sketch the graphs of $y = 7 - 3x$ and $y = \frac{2}{x}$.

iii. Using part (ii), or otherwise, write down all values of x for which $7 - 3x < \frac{2}{x}$. ¤

«→ i) Proof ii)



iii) $0 < x < \frac{1}{3}$ or $x > 2$ »

3U90-1b)! Solve the inequality $\frac{x^2-4}{x} > 0$. ¤

«→ $-2 < x < 0$ or $x > 2$ »

3U89-1d)! Solve for x: $\frac{4}{5-x} \ge 1$. ¤

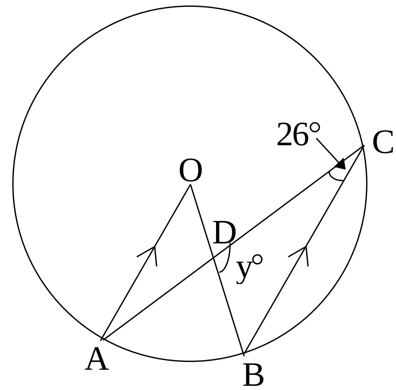
«→ $1 \le x < 5$ »

3U85-1iii)! Find all positive values of x for which $\frac{6}{x} > x - 1$. ¤

«→ $0 < x < 3$ or $x < -2$ »

Circle Geometry

3U97-2a)!



The points A, B and C lie on a circle with centre O. The lines AO and BC are parallel, and OB and AC intersect at D. Also, $\angle ACB = 26°$ and $\angle BDC = y°$, as shown in the diagram. Copy or trace the diagram into your Writing Booklet.

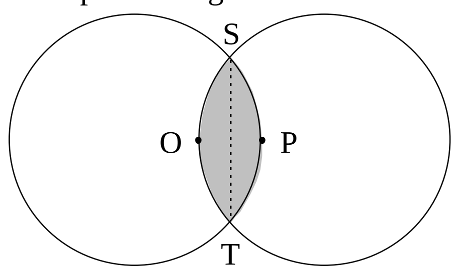
i. State why $\angle AOB = 52°$.

ii. Find y. Justify your answer.¤

«→ i) The angle subtended by an arc at the centre of a circle is twice the angle subtended by the arc at the circumference ii) $y = 102°$ »

3U96-2c)!

The points O and P in the plane are d cm apart. A circle centre O is drawn to pass through P, and another circle centre P is drawn to pass through O. The two circles meet at S and T, as in the diagram.



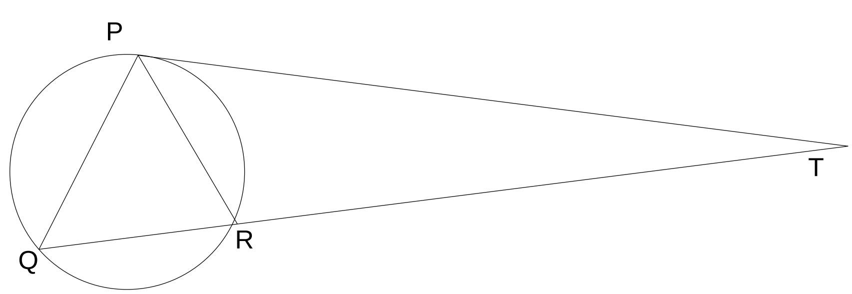
i. Show that triangle SOP is equilateral.

ii. Show that the size of the angle SOT is $\frac{2\pi}{3}$.

iii. Hence find the area of the shaded region in terms of d.¤

«→ i) Proof ii) Proof iii) $\frac{4\pi - 3\sqrt{3}}{6}d^2$ »

3U95-6a)!



PT is a tangent to the circle PRQ, and QR is a secant intersecting the circle in Q and R. The line QR intersects PT at T.

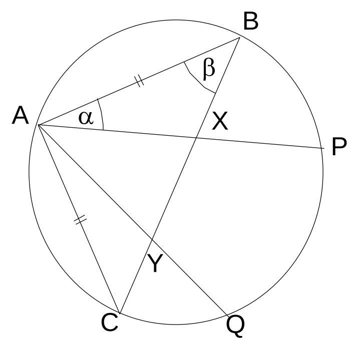
Copy or trace the diagram into your Writing Booklet.

i. Prove that the triangles PRT and QPT are similar.

ii. Hence prove that $PT^2 = QT \times RT$. ¤

«→ Proof »

3U94-2b)!



Let ABPQC be a circle such that AB = AC, AP meets BC at X, and AQ meets BC at Y, as in the diagram. Let $\angle BAP = \alpha$ and $\angle ABC = ß$.

i. Copy the diagram into your Writing Booklet and state why $\angle AXC = \alpha + ß$.

ii. Prove that $\angle BQP = \alpha$.

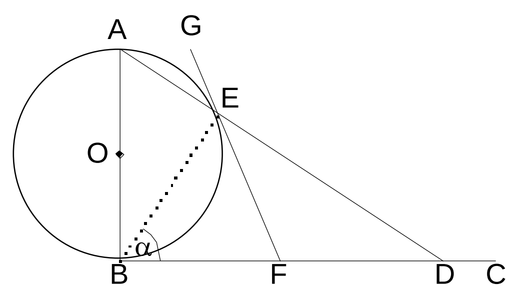
iii. Prove that $\angle BQA = ß$.

iv. Prove that PQYX is a cyclic quadrilateral. ¤

«→ i) $\angle AXC$ is the exterior angle of ΔAXB and as such is equal to the sum of the two interior opposite angles.

ii) iii) iv) Proof »

3U93-4a)!



In the diagram, AB is a diameter of the circle, centre O, and BC is tangential to the circle at B. The line AED intersects the circle at E and BC at D. The tangent to the circle at E intersects BC at F. Let $\angle EBF = \alpha$.

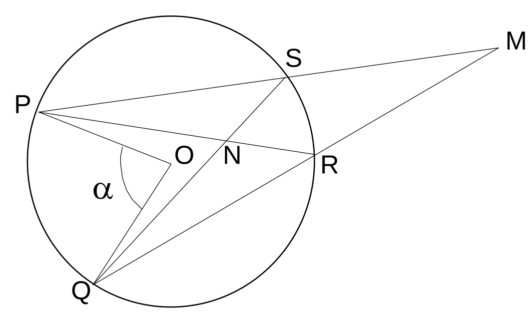
i. Copy the diagram into your Writing Booklet.

ii. Prove that $\angle FED = \dfrac{\pi}{2} - \alpha$.

iii. Prove that BF = FD. ¤

«→ Proof »

3U92-3c)!



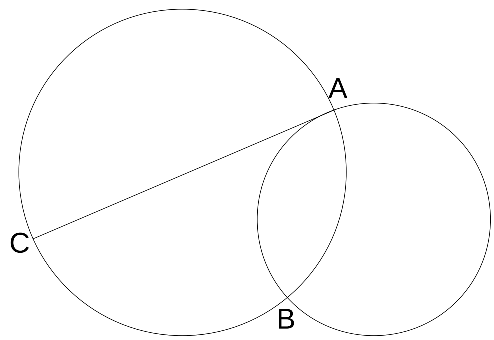
In the diagram P, Q, R, and S are points on a circle centre O, and $\angle POQ = \alpha$. The lines PS and QR intersect at M and the lines QS and PR intersect at N.

i. Explain why $\angle PRM = \pi - \frac{1}{2}\alpha$.

ii. Show that ∠PNQ + ∠PMQ = α. ¤

«→ Proof »

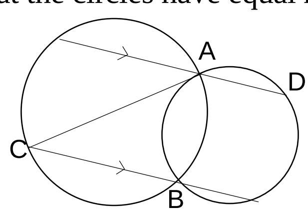
3U91-2c)!



The diagram shows two circles intersecting at A and B. The diameter of one circle is AC. Copy this diagram into your examination booklet.

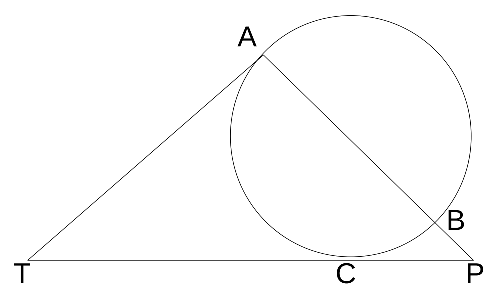
i. On your diagram draw a straight line through A, parallel to CB, to meet the second circle in D.
ii. Prove that BD is a diameter of the second circle.
iii. Suppose that BD is parallel to CA. Prove that the circles have equal radii. ¤

«→ i)



ii) Proof iii) Proof »

3U90-3a)!



(FIGURE NOT TO SCALE)

AB is a diameter of a circle ABC. The tangents at A and C meet at T. The lines TC and AB are produced to meet at P. Copy the diagram into your examination booklet. Join AC and CB.

i. Prove that ∠CAT = 90° - ∠BCP.
ii. Hence, or otherwise, prove that ∠ATC = 2∠BCP. ¤

«→ Proof »

3U89-2a)!

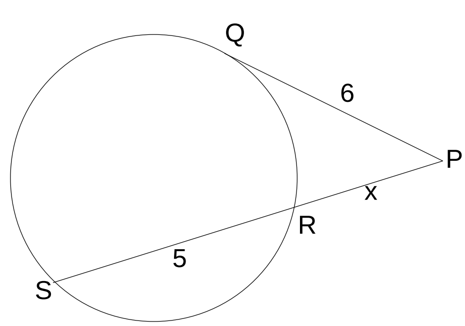


FIGURE NOT TO SCALE

PQ is a tangent to a circle QRS, while PRS is a secant intersecting the circle in R and S, as in the diagram. Given that PQ = 6, RS = 5, PR = x, find x. ¤

«→ x = 4 »

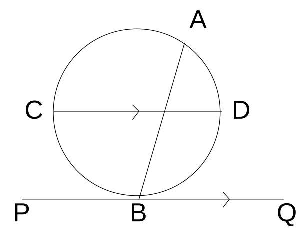
3U89-5a)!

AB and CD are two intersecting chords of a circle and CD is parallel to the tangent to the circle at B.

i. Draw a neat sketch of the above information in your writing booklet.

ii. Prove that AB bisects ∠CAD. ¤

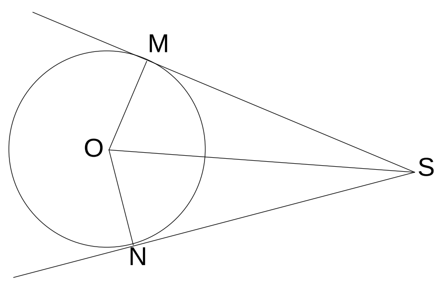
«→ i)



ii) Proof »

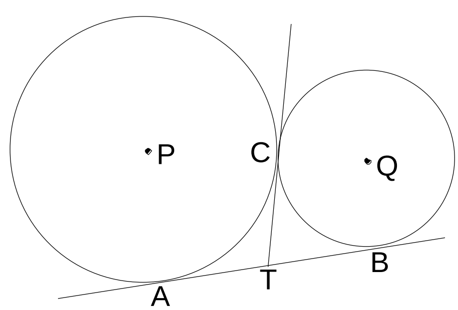
3U88-4a)!

i.



SM and SN are tangents drawn from an external point S to a circle with centre O. The points of contact of these tangents with the circle are M and N. Copy this diagram into your writing booklet. By proving triangle OMS and ONS are congruent show that SM = SN.

ii.



Two circles touch externally at C. The circles, which have centres P and Q, are touched by a common tangent at A and B respectively. The common tangent at C meets AB in T.

α. Copy this diagram in your writing booklet. Using the result from (i) prove that AT = TB.

ß. Show that ACB is a right angle. ¤

«→ Proof »

3U87-2i)!

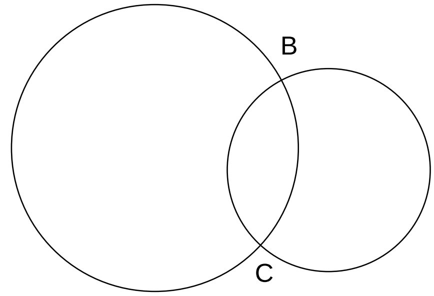
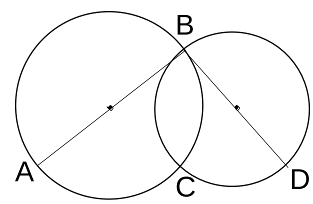


FIGURE NOT TO SCALE

Two circles cut at points B and C as shown in the diagram. A diameter of one circle is AB while BD is a diameter of the other.

a. Draw a neat sketch in your answer book showing the given information.

b. Prove that A, C and D are collinear, giving reasons. ¤



«→ a) b) Proof »

3U86-2ii)!

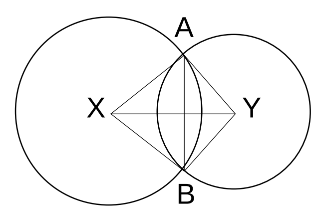
Two circles with centres X and Y intersect at two points A and B.

a. Draw a neat sketch joining XA, XB, YA, YB, XY, AB. Let P be the point where XY meets AB.

b. Prove that the triangles AXY and BXY are congruent.

c. Prove that AP = BP.

d. Given that XA is also a tangent to the circle with centre Y, prove that XAYB is a cyclic quadrilateral. ¤



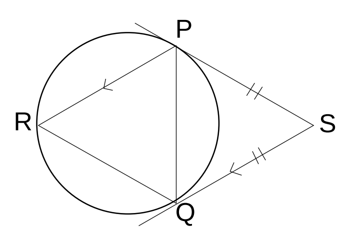
«→ Proof »

3U85-2i)!

P, Q are points on a circle and the tangents to the circle at P, Q meet as S. R is a point on the circle so that the chord PR is parallel to QS.

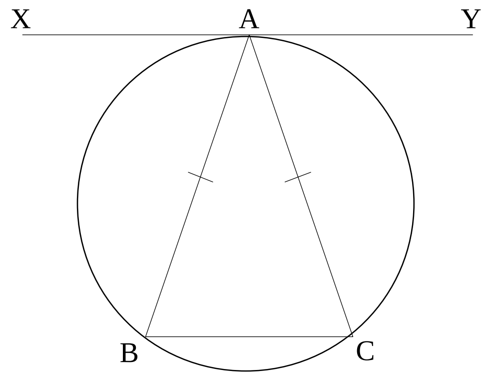
a. Draw a neat sketch in your answer book, showing the given information.

b. Giving reasons, prove carefully that QP = QR. ¤



«→ a) b) Proof »

3U84-2i)!



Given that AB = AC and XY is tangent to circle ABC at A, prove that XY is parallel to BC. ¤

«→ Proof »

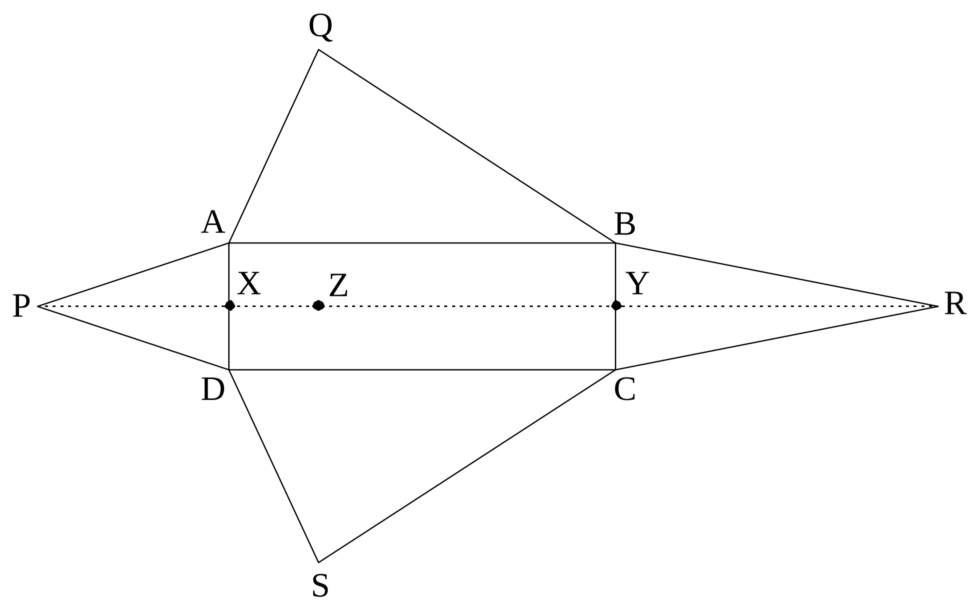
Further Trigonometry
(sums and differences, t formulae, identities and equations)

3U96-4a)!

Prove that $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2$ (for $\sin\theta \neq 0$, $\cos\theta \neq 0$).¤

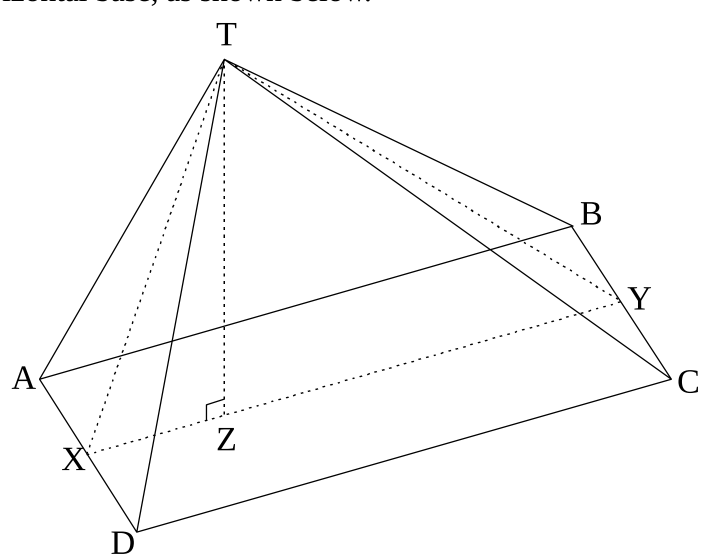
«→ Proof »

3U96-4c)!



NOT TO SCALE

The figure shows the net of an oblique pyramid with a rectangular base. In this figure, PXZYR is a straight line, PX = 15cm, RY = 20cm, AB = 25cm, and BC = 10cm. Further, AP = PD and BR = RC. When the net is folded, points P, Q, R, and S all meet at the apex T, which lies vertically above the point Z in the horizontal base, as shown below.



i. Show that triangle TXY is right-angled.
ii. Hence show that T is 12cm above the base.
iii. Hence find the angle that the face DCT makes with the base.¤

«→ i) Proof ii) Proof iii) 67° 23′(to nearest minute) »

3U94-2a)!

Prove the following identity: $\dfrac{2\tan A}{1+\tan^2 A} = \sin 2A$. ¤

«→ Proof »

3U94-5a)!

Find all angles θ, where $0 \le \theta \le 2\pi$, for which $\sqrt{3}\sin\theta - \cos\theta = 1$. ¤

«→ $\theta = \dfrac{\pi}{3}$ or π »

3U94-7c)!

Uluru is a large rock on flat ground in Central Australia. Three tourists A, B, and C are observing Uluru from the ground. A is due north of Uluru, C is due east of Uluru, and B is on the line-of-sight from A to C and between them. The angles of elevation to the summit of Uluru from A, B, and C are 26°, 28°, and 30°, respectively. Determine the bearing of B from Uluru. ¤

«→ 004°32' (to the nearest minute) »

3U93-2a)!

Prove the following identity: $\dfrac{\sin A}{\cos A + \sin A} + \dfrac{\sin A}{\cos A - \sin A} = \tan 2A$. ¤

«→ Proof »

3U92-2a)!

Solve the equation $2\sin^2\theta = \sin 2\theta$ for $0 \le \theta \le 2\pi$. ¤

«→ $\theta = 0, \dfrac{\pi}{4}, \pi, \dfrac{5\pi}{4}, 2\pi$ »

3U92-3a)!

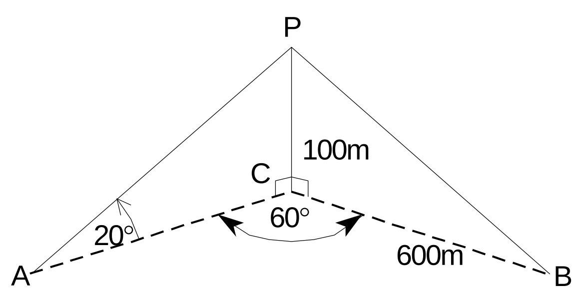


FIGURE NOT TO SCALE

Two yachts A and B subtend an angle of 60° at the base C of a cliff. From yacht A the angle of elevation of the point P, 100 metres vertically above C, is 20°. Yacht B is 600 metres from C.

i. Calculate the length AC.

ii. Calculate the distance between the two yachts. ¤

«→ i) 274.7m (to 1 d.p.) ii) 520m (to the nearest metre) »

3U90-2a)!

i. Factorize $a^3 + b^3$.

ii. Hence, or otherwise, show that $\dfrac{2\sin^3 A + 2\cos^3 A}{\sin A + \cos A} = 2 - \sin 2A$, if $\sin A + \cos A \neq 0$. ¤

«→ i) $(a + b)(a^2 - ab + b^2)$ ii) Proof »

3U90-5a)! Find all the angles θ with $0 \le \theta \le 2\pi$ for which $\sin\theta + \cos\theta = 1$. ¤

«→ $\theta = 0, \dfrac{\pi}{2}, 2\pi$ »

3U89-2b)!

Find all angles θ with $0 \le \theta \le 2\pi$ for which $\sin 2\theta = \sin\theta$. ¤

«→ $\theta = 0, \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$ or 2π »

3U89-3c)!

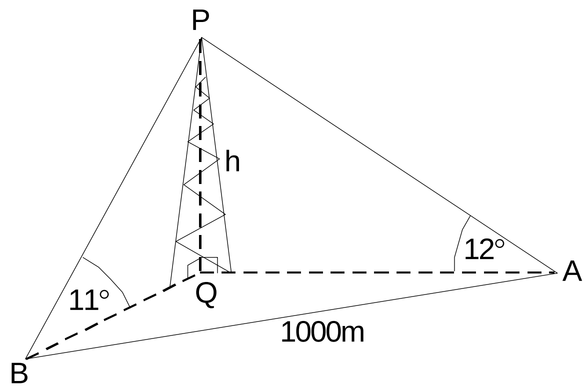


FIGURE NOT TO SCALE

The angle of elevation of a tower PQ of height h metres at a point A due east of it is 12°. From another point B, the bearing of the tower is 051°T and the angle of elevation is 11°. The points A and B are 1000 metres apart and on the same level as the base Q of the tower.

i. Show that $\angle AQB = 141°$.
ii. Consider the triangle APQ and show that AQ = h tan 78°.
iii. Find a similar expression for BQ.
iv. Use the cosine rule in the triangle AQB to calculate h to the nearest metre. ¤

«→ i) Proof ii) Proof iii) BQ = h tan 79 iv) 108m »

3U88-3c)!

Given that $0 < x < \frac{\pi}{4}$, prove that $\tan\left(\frac{\pi}{4}+x\right) = \frac{\cos x + \sin x}{\cos x - \sin x}$. ¤

«→ Proof »

3U87-3i)!

Find, for $0 \le x \le 2\pi$, all solutions of the equation $\sin 2x = \cos x$. ¤

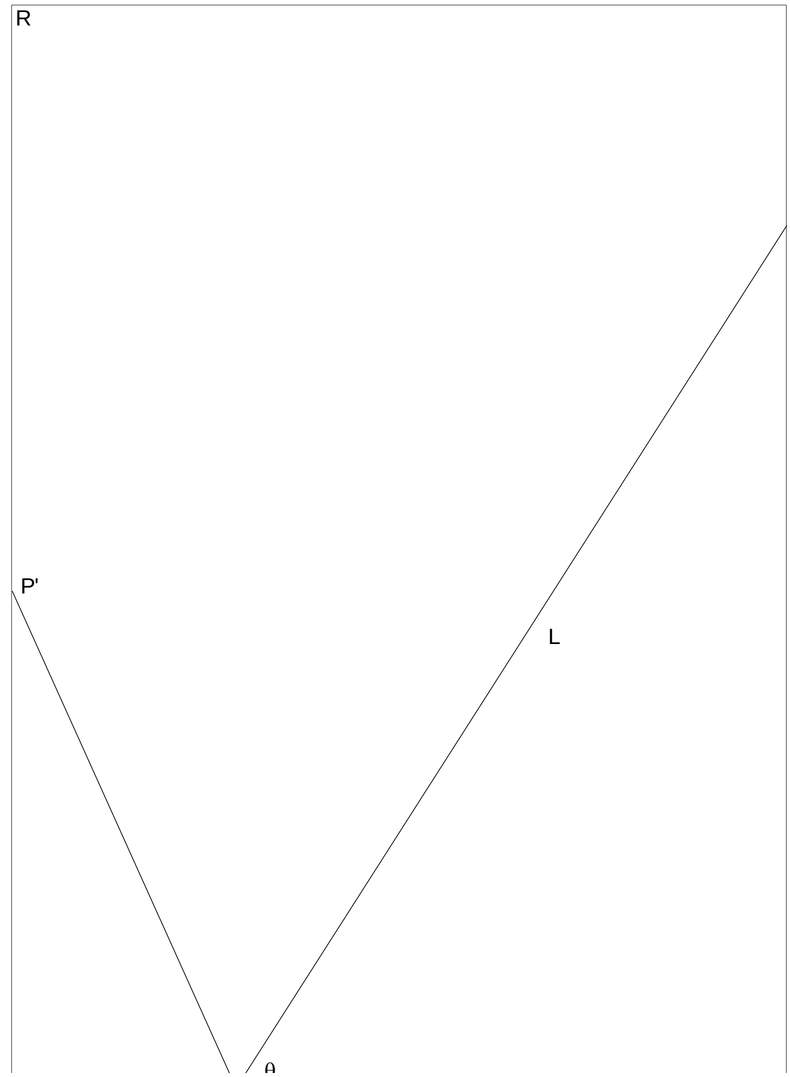
«→ $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$ »

3U87-7)!

The rectangular piece of paper PQRS shown on the next page is folded along a line AB, where A and B lie on edges PQ and PS respectively. This line is so positioned that, after folding, P coincides with a point P' which lies on the edge QR. This fold line AB makes an acute angle θ with the edge PQ. The length of AB is L and that of PQ is w.

a. Show that $\angle P'AQ = (\pi - 2\theta)$.
b. Prove that $L = \frac{w}{\cos\theta(1 - \cos 2\theta)}$.
c. More than one fold line exists such that P coincides with a point on QR after folding. Find the value of θ which corresponds to the fold line of minimum length.
d. Let CD be the fold line of minimum length, where C lies on PQ and D lies on PS. Calculate the length of CP.

(Remove the next page and attach it to your Writing Booklet for this question.)



«→ a) Proof b) Proof c) 54°44' (to the nearest minute) d) $\dfrac{3w}{4}$ »

3U86-1ii)!

a. Write down, in surd form, the values of sin 45°, cos 45°, sin 30°, cos 30°.

b. Hence show that $\sin 75° = \dfrac{1+\sqrt{3}}{2\sqrt{2}}$. ¤

«→ a) $\dfrac{1}{\sqrt{2}}$, $\dfrac{1}{\sqrt{2}}$, $\dfrac{1}{2}$, $\dfrac{\sqrt{3}}{2}$ b) Proof »

3U86-5i)!

Find all values of θ with $0 \le \theta \le \pi$ such that $2\sin\theta + \cos\theta = 1$. ¤

«→ θ = 0, 2.214 (to 4 sig. figs) »

3U84-6ii)!

a. Consider the statement $\cos\left(\dfrac{\pi}{2} + A\right) = \pm\sin A$.

For which sign is this statement true for all A?
For which A is the statement true for both signs?

b. Taking A = 5θ in (a) write down a value of B such that -sin 5θ = cos B. Hence find the least value of θ between 0 and 2π such that sin 5θ + cos 8θ = 0. ¤

«→ a) Negative, A = 0, $n\pi$, where n is an integer b) $\frac{3\pi}{26}$ »

Angles between 2 Lines.
Internal and External Division of Lines into Given Ratios

3U96-1b)!

Find the acute angle between the lines $2x + y = 4$ and $x - y = 2$, to the nearest degree.¤

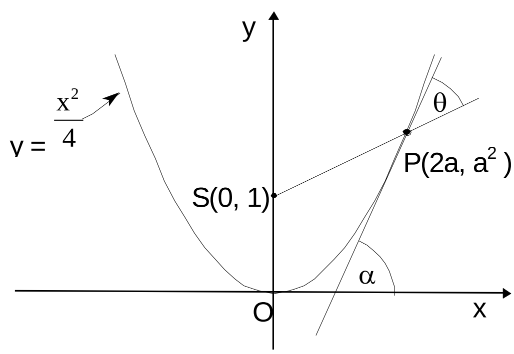
«→ 72° »

3U96-1c)!

Let A(-1, 2) and B(3, 5) be points in the plane. Find the coordinates of the point C which divides the interval AB externally in the ratio 3 : 1.¤

«→ (5, 6.5) »

3U95-3c)!



Let $P(2a, a^2)$ be a point on the parabola $y = \dfrac{x^2}{4}$, and let S be the point (0, 1). The tangent to the parabola at P makes an angle of ß with the x axis. The angle between SP and the tangent is θ. Assume that $a > 0$, as indicated.

i. Show that tan ß = a.

ii. Show that the gradient of SP is $\dfrac{1}{2}\left(a - \dfrac{1}{a}\right)$.

iii. Show that $\tan\theta = \dfrac{1}{a}$.

iv. Hence find the value of θ + ß.

v. Find the coordinates of P if θ = ß. ¤

«→ i) ii) iii) Proof iv) 90° v) P(2, 1) »

3U94-1c)!

The interval AB has end-points A(-2, 3) and B(10, 11). Find the coordinates of the point P which divides the interval AB in the ratio 3 : 1. ¤

«→ P(7, 9) internally or P(11, 9.5) externally »

3U94-1d)!

The graphs of $y = x$ and $y = x^3$ intersect at $x = 1$. Find the size of the acute angle between these curves at $x = 1$. ¤

«→ 26°34' (to the nearest minute) »

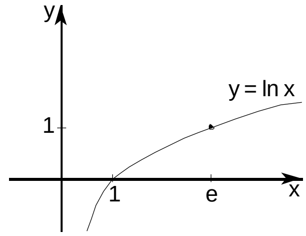
3U93-5b)!

i. Prove that the graph of $y = \ln x$ is concave down for all $x > 0$.

ii. Sketch the graph of $y = \ln x$.

iii. Suppose $0 < a < b$ and consider the points A(a, ln a) and B(b, ln b) on the graph of $y = \ln x$. Find the coordinates of the point P that divides the line segment AB in the ratio 2 : 1.

iv. By using (ii) and (iii) deduce that $\dfrac{1}{3}\ln a + \dfrac{2}{3}\ln b < \ln\left(\dfrac{1}{3}a + \dfrac{2}{3}b\right)$. ¤

«→ i) Proof ii) 

iii) $P\left(\frac{1}{3}a + \frac{2}{3}b, \ \frac{1}{3}\ln a + \frac{2}{3}\ln b\right)$ iv) Proof »

3U92-1e)!

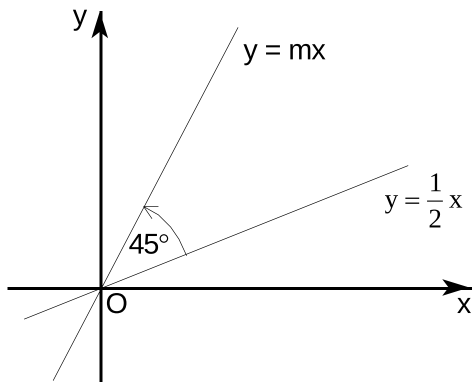


FIGURE NOT TO SCALE

The angle between the lines y = mx and $y = \frac{1}{2}x$ is 45° as shown in the diagram. Find the exact value of m. ¤

«→ 3 »

3U91-1c)!

The point P(-3, 8) divides the interval AB externally in the ratio k : 1. If A is the point (6, -4) and B is the point (0, 4), find the value of k. ¤

«→ -3 »

3U90-1c)!

The parabolas $y = x^2$ and $y = (x - 2)^2$ intersect at a point P.

i. Find the coordinates of P.
ii. Find the angle between the tangents to the curves at P. Give your answer to the nearest degree. ¤

«→ i) P(1, 1) ii) 53° »

3U89-1c)!

Find the coordinates of the point which divides the interval AB with A(1, 4) and B(5, 2) externally in the ratio 1 : 3. ¤

«→ (-1, 5) »

3U88-1b)!

Find the coordinates of the point P which divides the interval AB with end points A(2, 3) and B(5, -7) internally in the ratio 4 : 9. ¤

«→ $P(2\frac{12}{13}, -\frac{1}{13})$ »

3U87-1iii)!

a. Find the equation of the normal n to the curve $y = x^4 + 4x^{\frac{3}{2}}$ at the point A(1, 5).
b. Find, to the nearest degree, the size of the acute angle between the line n and the line L with equation 2x + 3y - 7 = 0. ¤

«→ a) x + 10y - 51 = 0 b) 28° »

3U86-2i)!

Find the coordinates of the point P which divides the interval AB internally in the ratio 2 : 3 where A and B have coordinates (1, -3) and (6, 7) respectively. ¤

«→ P(3, 1) »

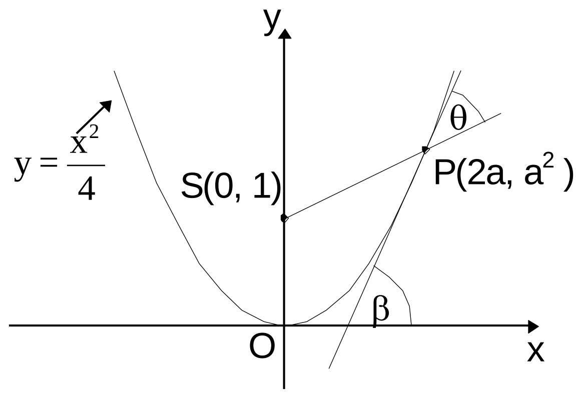
3U85-1iv)!

Find the acute angle between the lines y = -x and $\sqrt{3}\,y = x$. ¤

«→ 75° »

Parametric Representation

3U95-3c)!



Let $P(2a, a^2)$ be a point on the parabola $y = \frac{x^2}{4}$, and let S be the point (0, 1). The tangent to the parabola at P makes an angle of ß with the x axis. The angle between SP and the tangent is θ. Assume that $a > 0$, as indicated.

i. Show that $\tan ß = a$.

ii. Show that the gradient of SP is $\frac{1}{2}\left(a - \frac{1}{a}\right)$.

iii. Show that $\tan\theta = \frac{1}{a}$.

iv. Hence find the value of $\theta + ß$.

v. Find the coordinates of P if $\theta = ß$. ¤

«→ i) ii) iii) Proof iv) 90° v) P(2, 1) »

3U94-3d)!

Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

i. Show that the equation of the tangent to the parabola at P is $y = px - ap^2$.

ii. The tangent at P and the line through Q parallel to the y axis intersect at T. Find the coordinates of T.

iii. Write down the coordinates of M, the midpoint of PT.

iv. Determine the locus of M when $pq = -1$. ¤

«→ i) Proof ii) $T(2ap, 2apq - ap^2)$ iii) $M(a(p + q), apq)$ iv) The directrix of $x^2 = 4ay$, ie $y = -a$ »

3U93-7a)!

Consider the parabola $4ay = x^2$ where $a > 0$, and suppose the tangents at $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ intersect at the point T. Let $S(0, a)$ be the focus of the parabola.

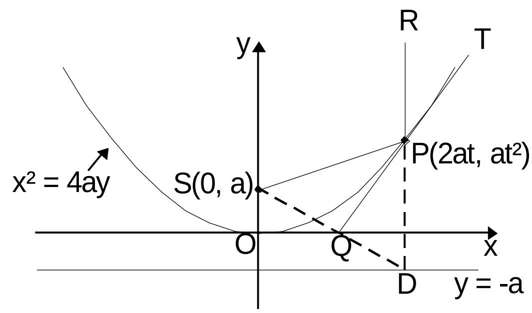
i. Find the coordinates of T. (You may assume that the equation of the tangent at P is $y = px - ap^2$.)

ii. Show that $SP = a(p^2 + 1)$.

iii. Suppose P and Q move on the parabola in such a way that $SP + SQ = 4a$. Show that T is constrained to move on a parabola. ¤

«→ i) $T(a(p + q), apq)$ ii) iii) Proof: the parabola $x^2 = 2a^2 + 2ay$ »

3U92-5a)!



The diagram shows the parabola $x^2 = 4ay$ with focus S(0, a) and directrix y = -a. The point P($2at$, at^2) is an arbitrary point on the parabola and the line RP is drawn parallel to the y axis, meeting the directrix at D. The tangent QPT to the parabola at P intersects SD at Q.

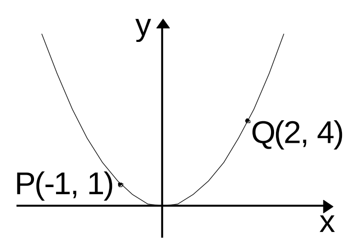
i. Explain why SP = PD.
ii. Find the gradient m_1 of the tangent at P.
iii. Find the gradient m_2 of the line SD.
iv. Prove that PQ is perpendicular to SD.
v. Prove that $\angle RPT = \angle SPQ$. ¤

«→ i) A parabola is the locus of a point P which moves such that its distance from a fixed point S (the focus) is equal to its distance from a fixed line (the directrix, y = -a). PD is perpendicular to the directrix. ∴ SP = PD ii) t iii) $-\frac{1}{t}$ iv) Proof v) Proof »

3U91-5b)!

i. Sketch the parabola whose parametric equations are x = t and $y = t^2$. On your diagram mark the points P and Q which correspond to t = -1 and t = 2 respectively.
ii. Show that the tangents to the parabola at P and Q intersect at $R(\frac{1}{2}, -2)$.
iii. Let T(t, t^2) be the point on the parabola between P and Q such that the tangent at T meets QR at the mid-point of QR. Show that the tangent at T is parallel to PQ. ¤

«→ i)



ii) Proof iii) Proof »

3U90-5c)!

The point P($2ap$, ap^2) lies on the parabola $x^2 = 4ay$. The focus S is the point (0, a). The tangent at P meets the y axis at Q.

i. Find the coordinates of Q.
ii. Prove that SP = SQ.
iii. Hence show that $\angle PSQ + 2\angle SQP = 180°$. ¤

«→ i) Q(0, $-ap^2$) ii) Proof iii) Proof »

3U89-6b)!

Two points P($2ap$, ap^2) and Q($2aq$, aq^2) lie on the parabola $x^2 = 4ay$.

i. Derive the equation of the tangent to the parabola at P.
ii. Find the coordinates of the point of intersection T of the tangents to the parabola at P and Q.
iii. You are given that the tangents at P and Q in (ii) intersect at an angle of 45°. Show that p - q = 1 + pq.

iv. By evaluating the expression $x^2 - 4ay$ at T, or otherwise, find the locus of T when the tangents at P and Q intersect as given in (iii). ¤

«→ i) $y = px - ap^2$ ii) $T(a(p + q), apq)$ iii) Proof iv) $x^2 - 4ay = (a + y)^2$, an hyperbola »

3U88-6a)!

i. Show that the equation of the tangent to the parabola $x^2 = 4Ay$ at any point $P(2Ap, Ap^2)$ is given by $px - y - Ap^2 = 0$.

ii. S is the focus of the parabola and T the point of intersection of the tangent and the y-axis. Prove that SP = ST.

iii. Hence show that $\angle SPT$ is equal to the acute angle between the tangent and the line through P parallel to the axis of the parabola. ¤

«→ Proof »

3U87-6ii)!

Two points $P(2Ap, Ap^2)$ and $Q(2Aq, Aq^2)$ lie on the parabola $x^2 = 4Ay$, where $A > 0$. The chord PQ passes through the focus.

a. Show that $pq = -1$.

b. Show that the point of intersection T of the tangents to the parabola at P and Q lies on the line $y = -A$.

c. Show that the chord PQ has length $A\left(p+\frac{1}{p}\right)^2$. ¤

«→ Proof »

3U85-3i)!

a. Derive the equation of the normal to the parabola $x = 2At$, $y = At^2$ at the point where $t = T$.

b. For the parabola $x = 2t$, $y = t^2$, find the values (if any) of T for which the normal at the point where $t = T$ passes through (0, 6). ¤

«→ a) $x + Ty = AT^3 + 2AT$ b) $T = -2, 0, 2$ »

3U84-5)!

The straight line $y = mx + b$ meets the parabola $x = 2At$, $y = At^2$ in real distinct points P, Q which correspond respectively to the values $t = p$, $t = q$.

a. Prove that $pq = -\frac{b}{A}$.

b. Prove that $p^2 + q^2 = 4m^2 + \frac{2b}{A}$.

c. Show that the equation of the normal to the parabola at P is $x + py = 2Ap + Ap^3$.

d. The point N is the point of intersection of the normals to the parabola at P and Q. Prove that the coordinates of N are $(-Apq\{p + q\}, A\{2 + p^2 + pq + q^2\})$, and express these coordinates in terms of A, m, and b.

e. Suppose that the chord PQ is free to move while maintaining a fixed gradient. Show that the locus of N is a straight line and verify that this straight line is a normal to the parabola. ¤

«→ a) Proof b) Proof c) Proof d) Proof, $N(2mb, 2A + 4Am^2 + b)$ e) Proof »

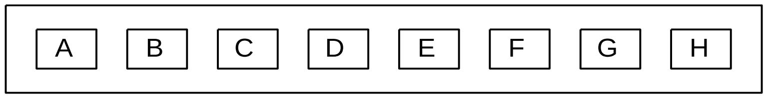
Permutations and Combinations

3U96-1d)!

A committee of 3 men and 4 women is to be formed from a group of 8 men and 6 women. Write an expression for the number of ways this can be done.¤

«→ $^8C_3 \times {^6C_4}$ »

3U95-3a)!



A security lock has 8 buttons labelled as shown. Each person using the lock is given a 3-letter code.

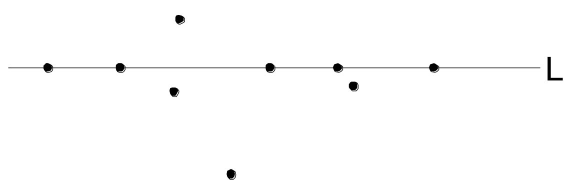
I. How many different codes are possible if letters can be repeated and their order is important?

ii. How many different codes are possible if letters cannot be repeated and their order is important?

iii. Now suppose that the lock operates by holding 3 buttons down together, so that order is NOT important. How many different codes are possible? ¤

«→ i) 512 ii) 336 iii) 56 »

3U94-7b)!



The figure shows 9 points lying in the plane, 5 of which lie on the line L. No other set of 3 of these points is collinear.

i. How many sets of 3 points can be chosen from the 5 points lying on L?

ii. How many different triangles can be formed using the 9 points as vertices?¤

«→ i) 10 ii) 74 »

3U93-1e)!

A class consists of 10 girls and 12 boys. How many ways are there of selecting a committee of 3 girls and 2 boys from this class? ¤

«→ 7920 »

3U93-4b)!

Five travellers arrive in a town where there are five hotels.

i. How many different accommodation arrangements are there if there are no restrictions on where the travellers stay?

ii. How many different accommodation arrangements are there if each traveller stays at a different hotel?

iii. Suppose two of the travellers are husband and wife and must go to the same hotel. How many different accommodation arrangements are there if the other three can go to any of the other hotels? ¤

«→ i) 3125 ii) 120 iii) 320 »

3U91-4c)!

Containers are coded by different arrangements of coloured dots in a row. The colours used are red, white, and blue. In an arrangement, at most three of the dots are red, at most two of the dots are white, and at most one is blue.

i. Find the number of different codes possible if six dots are used.

ii. On some containers only five dots are used. Find the number of different codes possible in this case. Justify your answer. ¤

«→ i) 60 ii) 60 »

3U90-4b)!

There are three identical blue marbles and four identical yellow marbles arranged in a row.

i. How many different arrangements are possible ?

ii. How many different arrangements of just five of these marbles are possible ? ¤

«→ i) 35 ii) 25 »

3U89-4c)!

Let each different arrangement of all the letters of DELETED be called a word.

i. How many words are possible?

ii. In how many of these words will the D's be separated? ¤

«→ i) 420 ii) 300 »

3U88-1d)!

Find the number of six-letter arrangements that can be made from the letters in the word SYDNEY. ¤

«→ 360 »

Polynomials

3U97-2c)!

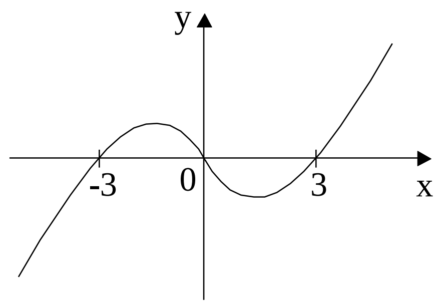
The polynomial $P(x) = x^3 + bx^2 + cx + d$ has roots 0, 3 and -3.

i. Find b, c and d.

ii. Without using calculus, sketch the graph of $y = P(x)$.

iii. Hence, or otherwise, solve the inequality $\dfrac{x^2 - 9}{x} > 0$.¤

«→ i) $b = 0, c = -9, d = 0$ ii)



iii) $-3 < x < 0$ or $x > 3$ »

3U96-1a)!

$(x - 2)$ is a factor of the polynomial $P(x) = 2x^3 + x + a$. Find the value of a.¤

«→ $a = -18$ »

3U95-2c)!

Consider the equation

$$x^3 + 6x^2 - x - 30 = 0.$$

One of the roots of this equation is equal to the sum of the other two roots.
Find the values of the three roots. ¤

«→ -5, -3 and 2 »

3U94-4a)!

When the polynomial P(x) is divided by $(x + 1)(x - 4)$, the quotient is Q(x) and the remainder is R(x).

i. Why is the most general form of R(x) given by $R(x) = ax + b$?

ii. Given that $P(4) = -5$, show that $R(4) = -5$.

iii. Further, when P(x) is divided by $(x + 1)$, the remainder is 5. Find R(x). ¤

«→ i) The degree of R(x) must be less than the degree of $(x + 1)(x - 4)$ ii) Proof iii) $R(x) = -2x + 3$ »

3U93-3b)!

When the polynomial P(x) is divided by $x^2 - 1$ the remainder is $3x - 1$. What is the remainder when P(x) is divided by $x - 1$? ¤

«→ 2 »

3U92-6a)!

Show that $(x - 1)(x - 2)$ is a factor of $P(x) = x^n(2^m - 1) + x^m(1 - 2^n) + (2^n - 2^m)$ where m and n are positive integers. ¤

«→ Proof »

3U91-1b)!

The polynomial $P(x) = x^3 + ax + 12$ has a factor $(x + 3)$. Find the value of a. ¤

«→ -5 »

3U90-2b)!

A polynomial is given by $p(x) = x^3 + ax^2 + bx - 18$. Find values for a and b if $(x + 2)$ is a factor of p(x) and if -24 is the remainder when p(x) is divided by $(x - 1)$. ¤

«→ $a = 2, b = -9$ »

3U89-4b)!

i. The polynomial equation $P(x) = 0$ has a double root at $x = a$. By writing $P(x) = (x - a)^2Q(x)$, where Q(x) is a polynomial, show that $P'(a) = 0$.

ii. Hence or otherwise find the values of a and b if x = 1 is a double root of $x^4 + ax^3 + bx^2 - 5x + 1 = 0$. ¤

«→ i) Proof ii) a = -5, b = 8 »

3U88-2c)!

If α, ß and γ are the roots of $x^3 - 3x + 1 = 0$ find:

i. $\alpha + \beta + \gamma$

ii. $\alpha\beta\gamma$

iii. $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}$. ¤

«→ i) 0 ii) -1 iii) 3 »

3U88-4b)!

i. Divide the polynomial $f(x) = 2x^4 - 10x^3 + 12x^2 + 2x - 3$ by $g(x) = x^2 - 3x + 1$.

ii. Hence write f(x) = g(x)q(x) + r(x) where q(x) and r(x) are polynomials and r(x) has degree less than 2.

iii. Hence show that f(x) and g(x) have no zeros in common. ¤

«→ ii) $2x^4 - 10x^3 + 12x^2 + 2x - 3 = (x^2 - 3x + 1)(2x^2 - 4x - 2) - 1$ iii) Proof »

3U87-5)!

The polynomial equation $f(x) = 8x^3 + 12x^2 - 18x - 20 = 0$ has a root at x = -2.

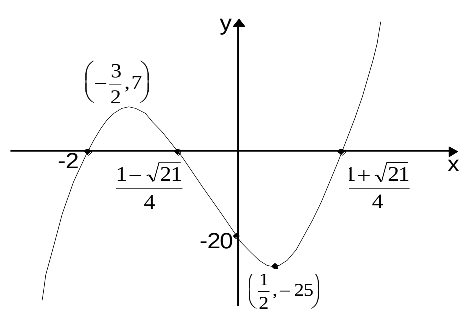
a. Find all roots of f(x) = 0.

b. Draw a sketch of the graph of y = f(x) showing the coordinates of its points of intersection with the axes and all stationary points.

c. Apply Newton's method once to approximate a root of f(x) = 0 beginning with an initial approximation $x_1 = 1$.

d. Willy chose an initial approximation of $x_1 = 0.49$ and used Newton's method a number of times in order to approximate a root of f(x) = 0. State, giving reasons, the root of f(x) = 0 to which Willy's approximations are getting closer. (It is not necessary to do additional calculations.) ¤

«→ a) x = -2, $\dfrac{1\pm\sqrt{21}}{4}$ b)



c) 1.6 d) x = -2 »

3U86-3ii)!

Consider the curve $y = x^4 + 4x^3 - 16x + 1$.

a. Verify that the curve has a minimum at x = 1.

b. Factorise $\dfrac{dy}{dx}$ completely, and hence determine the location and nature of any other stationary points of the curve. ¤

«→ a) Proof b) $\dfrac{dy}{dx} = 4(x - 1)(x + 2)^2$, (-2, 17) is a point of inflexion »

3U85-2iii)!

Given that there is a constant c such that $(x^4 + y^4) = (x^2 + cxy + y^2)(x^2 - cxy + y^2)$ identically in x and y, find c. ¤

«→ $\pm\sqrt{2}$ »

3U85-4ii)!

a. Factorise completely the polynomial $p(x) = x^3 - x^2 - 8x + 12$, given that the equation $p(x) = 0$ has a repeated root.

b. The polynomial $q(x)$ has the form $q(x) = p(x)(x + a)$, with $p(x)$ as in (a) and where the constant a is chosen so that $q(x) \geq 0$ for all real values of x. Find all possible values of a. ¤

«→ a) $p(x) = (x - 2)^2(x + 3)$ b) $a = 3$ »

3U84-1ii)!

Define the function f by $f(x) = x^3 + 3x^2 - 9x - 27$.

a. Show that $(x - 3)$ is a factor of $x^3 + 3x^2 - 9x - 27$, and factorise this expression completely.

b. Find where the graph of $y = f(x)$ meets the axes.

c. Find the stationary points of f and determine their nature.

d. Find the point(s) of inflexion (if any) of f. ¤

«→ a) Proof, $(x - 3)(x + 3)^2$ b) (-3, 0), (3, 0) and (0, -27) c) (-3, 0) is a relative maximum turning point and (1, -32) is a relative minimum turning point. d) (-1, -16) »

Harder Applications of the Preliminary 2 Unit Course

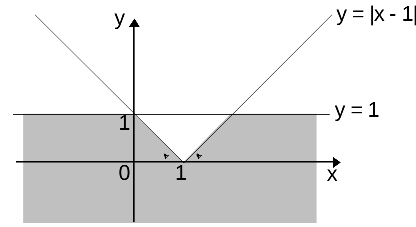
3U97-1b)!

Find the perpendicular distance from the point (1, 2) to the line $y = 3x + 4$.¤

«→ $\frac{\sqrt{10}}{2}$ »

3U95-1a)!

On a number plane, indicate the region specified by $y \le |x - 1|$ and $y \le 1$. ¤

«→ 

»

3U95-1d)!

Factorize $2^{n+1} + 2^n$, and hence write $\frac{2^{1001} + 2^{1000}}{3}$ as a power of 2. ¤

«→ $3(2^n)$, 2^{1000} »

3U93-1a)!

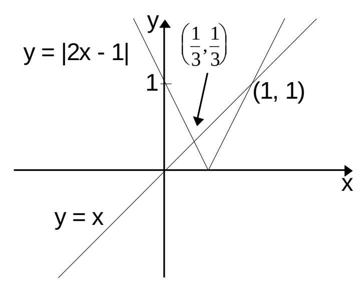
Solve the inequality $x^2 - 3x < 4$. ¤

«→ $-1 < x < 4$ »

3U93-2b)!

i. On the same diagram, sketch the graphs of $y = x$ and $y = |2x - 1|$.

ii. By using (i) or otherwise, determine for what values of c the equation $|2x - 1| = x + c$ has exactly two solutions. ¤

«→ i) 

ii) $c > -\frac{1}{2}$ »

3U92-1a)!

Solve $x^2 - x - 2 > 0$.¤

«→ $x > 2, x < -1$ »

3U92-1b)!

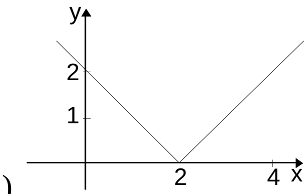
Differentiate $\frac{1}{\sqrt{1+x^2}}$. ¤

«→ $\frac{-x}{\sqrt{(1+x^2)^3}}$ »

3U91-1d)!

i. Sketch the graph of $y = |x - 2|$.

ii. For what values of x is $|x - 2| < x$?¤

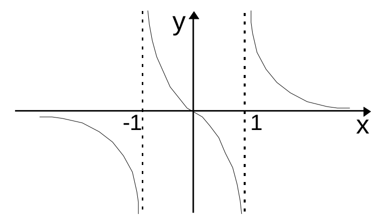
«→ i) 

ii) $x > 1$ »

3U91-7a)!

Let $f(x) = \frac{x}{x^2 - 1}$.

i. For what values of x is f(x) undefined?
ii. Show that y = f(x) is an odd function.
iii. Show that $f'(x) < 0$ at all values of x for which the function is defined.
iv. Hence sketch y = f(x). ¤

«→ i) ±1 ii) Proof iii) Proof iv) 

»

3U90-5b)!

Consider the circle $x^2 + y^2 - 2x - 14y + 25 = 0$.

i. Show that if the line y = mx intersects the circle in two distinct points, then $(1 + 7m)^2 - 25(1 + m^2) > 0$.
ii. For what values of m is the line y = mx a tangent to the circle? ¤

«→ i) Proof ii) $m = -\frac{4}{3}, \frac{3}{4}$ »

3U89-1a)!

Factorize $2m^3 - 128$. ¤

«→ $2(m - 4)(m^2 + 4m + 16)$ »

3U87-1i)!

Differentiate:

a. $\frac{1}{3+x^2}$
b. $e^x \log_e(2x)$. ¤

«→ a) $\frac{-2x}{(3+x^2)^2}$ b) $e^x\{\frac{1}{x} + \ln(2x)\}$ »

3U87-1iii)!

a. Find the equation of the normal n to the curve $y = x^4 + 4x^{\frac{3}{2}}$ at the point A(1, 5).
b. Find, to the nearest degree, the size of the acute angle between the line n and the line L with equation $2x + 3y - 7 = 0$. ¤

«→ a) $x + 10y - 51 = 0$ b) 28° »

3U87-2ii)!

Find the values of x for which $(x - 2)^2 \geq 4$. ¤

«→ $x \geq 4$ or $x \leq 0$ »

3U86-1i)!

Write $\frac{1+\sqrt{7}}{3-\sqrt{7}}$ in the form $a + b\sqrt{7}$ where a and b are rational. ¤

«→ $5 + 2\sqrt{7}$ »

3U86-1iv)!

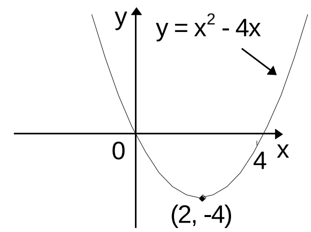
Find all real numbers x such that $x^2 + 4x > 5$. ¤

«→ $x < -5$ or $x > 1$ »

3U86-4i)!

a. Find the coordinates of the vertex and the focus and the equation of the directrix of the parabola $y = x^2 - 4x$. Draw a sketch of the curve.

b. A line whose equation is $y = mx - 4$ passes through the point (0, -4) and is a tangent to the parabola $y = x^2 - 4x$. Find the two possible values of m. ¤



«→ a) V(2, -4), F$(2, -3\frac{3}{4})$ and $y = -4\frac{1}{4}$, b) 0, -8 »

3U85-2ii)!

A circle has equation $x^2 + y^2 - 4x + 2y = 0$.

a. Find the centre and radius of the circle.

b. The line $x + 2y = 0$ meets this circle in two points, A, B.

  α. Find the co-ordinates of A and B.

  ß. Calculate the distance AB. ¤

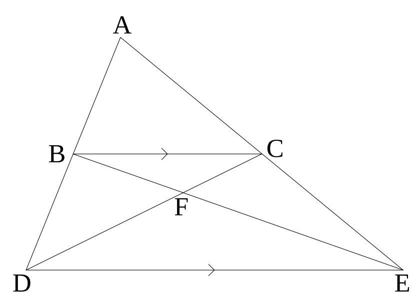
«→ a) centre (2, -1), radius = $\sqrt{5}$ units b) α) A(0, 0), B(4, -2) ß) $2\sqrt{5}$ units »

3U85-2iii)!

Given that there is a constant c such that $(x^4 + y^4) = (x^2 + cxy + y^2)(x^2 - cxy + y^2)$ identically in x and y, find c. ¤

«→ $\pm\sqrt{2}$ »

3U84-2ii)!



In the figure, BC $\parallel$ DE and AB:BD = 3:5. Show that

a. ΔABC is similar to ΔADE,

b. ΔBFC is similar to ΔEFD,

c. DF : FC = 8 : 3. ¤

«→ Proof »