NSGHS TRIAL HSC 2010 X1 SOLUTIONS

Question 1

(a) $P(x) = x^4 - 2x^3 + ax + b$

$(x-1)$ factor

$\therefore P(1) = 1 - 2 + a + b = 0$

$\therefore \boxed{a+b=1}$ —①

$(x+1)$ factor

$\therefore P(-1) = 1 + 2 - a + b = 0$

$\boxed{-a+b=-3}$ —②

①+② $2b = -2$

$\therefore b = -1,\ a = 2$

(b) $\dfrac{x+1}{x-1} \geq 2$

$\dfrac{(x+1)(x-1)^2}{(x-1)} \geq 2(x-1)^2,\ x \neq 1$

$2(x-1)^2 \leq (x+1)(x-1)$

$2(x-1)^2 - (x+1)(x-1) \leq 0$

$(x-1)[2x-2-x-1] \leq 0$

$(x-1)(x-3) \leq 0$

$\therefore 1 < x \leq 3$ since $x \neq 1$

(c) B I O L O G I S T

(i) $\dfrac{9!}{2!\,2!} = 90\,720$

(ii) $\boxed{I\ I}$ BOLOGST

$\dfrac{8!}{2!}$

$P(I\text{'s together}) = \dfrac{8!}{2!} \Big/ \dfrac{9!}{2!\,2!}$

$= \dfrac{8! \times 2!}{9!} = \dfrac{2}{9}$

(d) $\tan 45° = \left|\dfrac{m-2}{1+2m}\right|$

$|1+2m| = |m-2|$

$1+2m = m-2$ or $1+2m = -(m-2)$

$m = -3$ $\quad 3m = 1$

$\therefore m = -3, \frac{1}{3}$

(e) $x(x+12) = 8^2$

$x^2 + 12x - 64 = 0$

$(x+16)(x-4) = 0$

$\therefore x = 4\ [x > 0]$

Question 2

(a) $\displaystyle\int \frac{dx}{1+4x^2} = \frac{1}{4}\int \frac{dx}{\left(\frac{1}{4}+x^2\right)}$

$= \frac{1}{4} \cdot \frac{1}{\frac{1}{2}} \tan^{-1}\left(\frac{x}{\frac{1}{2}}\right) + c$

$= \frac{1}{2}\tan^{-1} 2x + c$

(b) let roots $\alpha, \beta, \alpha - \beta$

Sum of roots $\alpha + \beta + \alpha - \beta = -\frac{-4}{4}$

$2\alpha = 1$

$\alpha = \frac{1}{2}$

product of roots $\alpha\beta(\alpha-\beta) = \frac{-15}{4}$

$\frac{1}{2}\beta\left(\frac{1}{2}-\beta\right) = \frac{-15}{4}$

$\beta(1-2\beta) = -15$

$2\beta^2 - \beta - 15 = 0$

$(2\beta+5)(\beta-3) = 0$

$\beta = \frac{-5}{2}, 3$

$\therefore$ roots $\frac{1}{2}, \frac{-5}{2}, \frac{1}{2}-\frac{-5}{2} \rightarrow \frac{1}{2}, \frac{-5}{2}, 3$

$\frac{1}{2}, +3, \frac{1}{2}-3 \rightarrow \frac{1}{2}, 3, \frac{-5}{2}$

NGHS. X1 2010

Question 2 (ctd)

(c) (i) S(0,1) P$(2p, p^2)$ k:l = 1:2

$R\left(\frac{2p+2\times0}{2+1}, \frac{p^2+2}{2+1}\right)$

$\therefore R\left(\frac{2p}{3}, \frac{p^2+2}{3}\right)$

(ii) $\therefore x = \frac{2p}{3}$ $\therefore p = \frac{3x}{2}$

$y = \frac{p^2+2}{3}$

$\therefore 3y = \left(\frac{3x}{2}\right)^2 + 2$

$3y - 2 = \frac{9x^2}{4}$

$9x^2 = 4(3y-2)$

$9x^2 = 12y - 8$

(d) $\sin\theta = \frac{2t}{1+t^2}$ $\cos\theta = \frac{1-t^2}{1+t^2}$

$2\sin\theta + 4\cos\theta = 3$

$2\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right) = 3$

$4t + 4 - 4t^2 = 3 + 3t^2$

$7t^2 - 4t - 1 = 0$

$t = \frac{4 \pm \sqrt{16+28}}{14} = \frac{4 \pm \sqrt{44}}{14}$

$= 0.759\ldots, -0.188\ldots$

$\therefore \frac{\theta}{2} = 0.64\ldots \quad 2.955\ldots$

$\theta = 1.30, \quad 5.91$

Question 3

(a) (ii) exterior angle of cyclic quad ($\angle CDE$)

= opposite interior angle ($\angle ABC$)

(iii) $\angle ABC = \angle BAC$ (equal angles isosceles ΔACB)

$\angle BDC = \angle BAC$ (angles in same segment)

$\therefore \angle BDC = \angle ABC$

but $\angle CDE = \angle ABC$ from (ii)

$\therefore \angle BDC = \angle CDE$

$\therefore$ DC bisects $\angle BDE$

(b) $\lim_{x\to0} \frac{\sin 2x}{3x} = \frac{2}{3}\lim_{x\to0}\left(\frac{\sin 2x}{2x}\right) = \frac{2}{3}$

$= \frac{2}{3}\times 1 = \frac{2}{3}$

(c) $I = \int_0^{\ln 2} e^{2x}\sqrt{e^x - 1}\, dx$

let $u = e^x - 1$

$x = 0 \quad u = e^0 - 1 = 0$

$x = \ln 2 \quad u = e^{\ln 2} - 1 = 2 - 1 = 1$

$\frac{du}{dx} = e^x$

$\therefore du = e^x\, dx$

$\therefore I = \int_0^{\ln 2} e^x \sqrt{e^x - 1}\,.\, e^x\, dx$

$= \int_0^1 (u+1)\sqrt{u}\,.\, du$

$= \int_0^1 \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

$= \left[\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right]_0^1$

$= \frac{2}{5} + \frac{2}{3}$

$= \frac{16}{15}$

NSGHS X1 2010

Question 3

(d) (i) $1+\cos 2x = 1$

$\cos 2x = 0$

$2x = \pm \frac{\pi}{2}$

$x = \pm \frac{\pi}{4}$

(ii) $V = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1+\cos 2x)^2\,dx - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1\,dx$

$= 2\pi \int_0^{\frac{\pi}{4}} (1 + 2\cos 2x + \cos^2 2x - 1)\,dx$

$= 2\pi \int_0^{\frac{\pi}{4}} 2\cos 2x + \frac{1}{2}(1+\cos 4x)\,dx$

$= 2\pi\left[\sin 2x + \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right)\right]_0^{\frac{\pi}{4}}$

$= 2\pi\left[\sin\frac{\pi}{2} + \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{4}\sin\pi\right)\right]$

$= 2\pi\left[\frac{\pi}{8} + 1\right]$

$= 2\pi\left[\frac{\pi}{8}+1\right]$

$= \left(\frac{\pi^2}{4} + 2\pi\right)$ units^3

Question 4

(a) $\left(x - \frac{3}{x^2}\right)^{16}$

$T_k = \binom{16}{k}(x)^{16-k}\left(\frac{-3}{x^2}\right)^k$

$= \binom{16}{k} x^{16-k}(-3)^k x^{-2k}$

$= \binom{16}{k}(-3)^k x^{16-3k}$

coeff x: $16-3k=1$ ∴

∴ $3k = 15$

$k = 5$

∴ coefficient $x = \binom{16}{5}(-3)^5$

$= -1\ 061\ 424$

(b) (i) $N = A + Be^{-0.4t}$

$t=0$ $N=500$ ∴ $500 = A+B$

$t\to\infty$ $N\to 100$ but $N\to A$

∴ $A = 100$

∴ $B = 400$

(ii) $110 = 100 + 400e^{-0.4t}$

$\frac{1}{40} = e^{-0.4t}$

$-0.4t = \ln\frac{1}{40}$

$t = \frac{\ln 40}{0.4}$

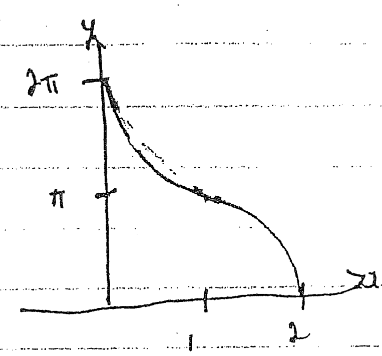
$= 9.22$ yrs

$\doteqdot 111$ months.

(c) (i) $-1 \le x-1 \le 1$

$0 \le x \le 2$

(ii)



(iii) $y = 2\cos^{-1}(x-1)$

$x = \cos\left(\frac{y}{2}\right)+1$

$A = \int_0^{2\pi}\left(\cos\left(\frac{y}{2}\right)+1\right)dy$

$= \left[2\sin\left(\frac{y}{2}\right)+y\right]_0^{2\pi}$

$= (2\sin\pi + 2\pi) - (2\sin 0 + 0) = 2\pi$ units^2

NSGHS X1 2010

Question 5

a) test $n=1$

$LHS = 2$. $RHS = 2^1 + 1^2 - 1 = 2$

$\therefore$ true for $n=1$

assume true for $n=k$

ie $2+5+9+\dots 2^{k-1}+2k-1 = 2^k + k^2 - 1$

to show true for $n=k+1$

ie $2+5+\dots+[2^{k}+2(k+1)-1] = 2^{k+1}+(k+1)^2-1$

Now $LHS = S_k + T_{k+1}$

$= 2^k + k^2 - 1 \quad + 2^k + 2(k+1) - 1$

$= 2.2^k + k^2 - 1 + 2k + 2 - 1$

$= 2^{k+1} + k^2 + 2k$

$= 2^{k+1} + (k^2+2k+1) - 1$

$= 2^{k+1} + (k+1)^2 - 1$

$\therefore$ True for $n=k+1$

$\therefore$ Since true for $n=1$ and true for $n=k+1$ when true for $n=k$, true for all $n \geq 1$ by induction

b) (i) $y = \dfrac{e^x}{e^x+4}$

$\dfrac{dy}{dx} = \dfrac{e^x(e^x+4) - e^x(e^x)}{(e^x+4)^2}$

$= \dfrac{4e^x}{(e^x+4)^2}$

$\neq 0$ as $4e^x > 0$

$\therefore$ No stationary points

(ii) $y=f(x)$ continuous and increasing.

$\therefore$ always inverse

(iii) $f(x)$ always positive $\therefore y > 0$

denominator > numerator $\therefore y < 1$

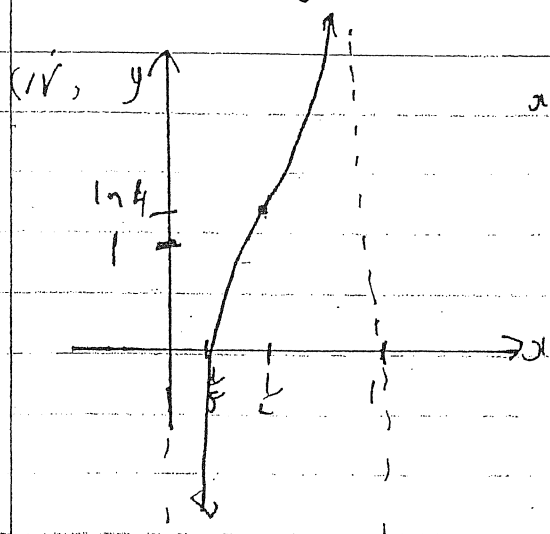
$\therefore 0 < y < 1$

OR $\lim\limits_{x\to\infty} \dfrac{e^x}{e^x+4} = \dfrac{1}{1+\frac{4}{e^x}} = 1$

$\lim\limits_{x\to-\infty} \dfrac{e^x}{e^x+4} = \dfrac{e^{-\infty}}{e^{-\infty}+4} = \dfrac{0}{4} = 0$

$\therefore 0 < y < 1$

(iv)



$x=0 \; y = \dfrac{1}{1+4} = \dfrac{1}{5}$

$\therefore$ x int $= \dfrac{1}{5}$

(v) $y = \dfrac{e^x}{e^x+4}$

Inverse $x = \dfrac{e^y}{e^y+4}$

$xe^y + 4x = e^y$

$e^y(1-x) = 4x$

$e^y = \dfrac{4x}{1-x}$

$y = \ln\left(\dfrac{4x}{1-x}\right)$

X1 TRIAL NGHS 2010

Question 6

(a) (i) $2\cos 5\theta = 1$

$\cos 5\theta = \frac{1}{2}$

$\therefore 5\theta = 2n\pi \pm \frac{\pi}{3}$

$\theta = \frac{2n\pi}{5} \pm \frac{\pi}{15}$

$= (6n \pm 1)\frac{\pi}{15}$

(ii) $(6n+1)\frac{\pi}{15}$

$n=0 \rightarrow \frac{\pi}{15}$ $\quad n=1 \rightarrow \frac{7\pi}{15}$

$(6n-1)\frac{\pi}{15}$

$n=1 \rightarrow \frac{5\pi}{15} = \frac{\pi}{3}$

$\frac{\pi}{15}, \frac{7\pi}{15}, \frac{\pi}{3}$

(b) (i) $\frac{h}{BD} = \tan 45$ $\quad \frac{h}{DC} = \tan 60 = \sqrt{3}$

$\therefore BD = h$ $\quad DC = \frac{h}{\sqrt{3}}$

$\therefore h^2 = x^2 + \frac{h^2}{3} - 2 \times x \cdot \frac{h}{\sqrt{3}} \cdot \cos 30$

$h^2 = x^2 + \frac{h^2}{3} - \frac{2xh}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2}$

$h^2 = x^2 + \frac{h^2}{3} - xh$

$3h^2 = 3x^2 + h^2 - 3xh$

$2h^2 + 3xh - 3x^2 = 0$

(ii) $2\left(\frac{h}{x}\right)^2 + 3\left(\frac{h}{x}\right) - 3 = 0$

$\frac{h}{x} = \frac{-3 \pm \sqrt{9 - 4 \times 2 \times -3}}{4}$

$= \frac{-3 \pm \sqrt{33}}{4}$

$\therefore \frac{h}{x} = \frac{\sqrt{33} - 3}{4}$ since both > 0

(c)

$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$

(i) $\frac{(1+x)^{n+1}}{n+1} = \binom{n}{0}x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \dots + \frac{1}{n+1}\binom{n}{n}x^{n+1} + c$

let $x = 0$ $\quad \therefore c = \frac{1}{n+1}$

$\therefore \frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = \binom{n}{0}x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \dots \frac{1}{n+1}\binom{n}{n}x^{n+1}$

let $x = 1$

$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \dots \frac{1}{n+1}\binom{n}{n}$

$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \frac{2^{n+1} - 1}{n+1}$

$\therefore \binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \dots \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1} - 1}{n+1}$

(ii)

$\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \dots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1} - 1}{n+1}$ — ①

let $x = 1$ in original

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$ — ②

② − ①

$\frac{1}{2}\binom{n}{1} + \frac{2}{3}\binom{n}{2} + \dots \left(1 - \frac{1}{n+1}\right)\binom{n}{n} = 2^n - \frac{2^{n+1} - 1}{n+1}$

$\frac{1}{2}\binom{n}{1} + \frac{2}{3}\binom{n}{2} + \dots \frac{n}{n+1}\binom{n}{n} = \frac{2^n \times n + 2^n - 2^{n+1} + 1}{n+1}$

RHS $= \frac{n \cdot 2^n + 2^n - 2 \cdot 2^n + 1}{n+1}$

$= \frac{2^n(n-1) + 1}{n+1}$

NSGHS TRIAL

Question 7

(a) (i) $\angle OBT = 90^\circ$ (tangent ⊥ radius)

$\therefore \cos\theta = \frac{2}{x}$ (right triangle)

(ii) $l = r\theta$

$\angle AOB = \pi - \cos^{-1}\left(\frac{2}{x}\right)$

$\therefore l_L = 2\left[\pi - \cos^{-1}\left(\frac{2}{x}\right)\right]$

$BT^2 = x^2 - 2^2$ (pythagoras)

$\therefore BT = \sqrt{x^2-4}$

$\therefore S = l_L + BT$

$= 2\left[\pi - \cos^{-1}\left(\frac{2}{x}\right)\right] + \sqrt{x^2+4}$

(iii) $S = 2\pi - 2\cos^{-1}\left(\frac{2}{x}\right) + \sqrt{x^2-4}$

$\frac{dS}{dx} = 0 - 2 \cdot \frac{-1}{\sqrt{1-\left(\frac{2}{x}\right)^2}} \cdot -2x^{-2} + \frac{1}{2} \cdot 2x\,(x^2-4)^{-1}$

$= \frac{-4}{x^2\sqrt{1-\frac{4}{x^2}}} + \frac{x}{\sqrt{x^2-4}}$

$= \frac{-4}{x\sqrt{x^2-4}} + \frac{x}{\sqrt{x^2-4}}$

$= \frac{-4+x^2}{x\sqrt{x^2-4}}$

$\frac{dS}{dx} = \frac{\sqrt{x^2-4}}{x}$

(iv) $\frac{dS}{dt} = \frac{dS}{dx} \cdot \frac{dx}{dt}$

$\frac{dx}{dt} = \frac{4 \times \sqrt{8^2-4}}{8}$

$= \frac{\sqrt{60}}{2}$

$= \sqrt{15}$

(b) (i) 2 men + 1 wom $= \binom{n}{2} \cdot \binom{n}{1}$

$= \frac{n(n-1)}{2} \times n$

$= \frac{n^2(n-1)}{2}$

(ii) $n(S) = \binom{2n}{3} = \frac{2n(2n-1)(2n-2)}{3!}$

$= \frac{4n(2n-1)(n-1)}{6}$

$P(\text{2 men 1 woman}) = \frac{n^2(n-1)}{2} \Big/ \frac{4n(2n-1)(n-1)}{6}$

$= \frac{n^2(n-1)}{2} \cdot \frac{6}{4n(2n-1)(n-1)}$

$= \frac{3n}{4(2n-1)} = \frac{3n}{8n-4}$

(iii) least value for $n = 2$

$p = \therefore P(\text{2 men 1 woman}) = \frac{6}{16-4} = \frac{1}{2}$

as $n \to \infty$ $\frac{3n}{8n-4} \to \frac{3}{8}$ $\therefore \frac{3}{8} < p \leq \frac{1}{2}$

Question 1

(a) $P(x) = x^4 - 2x^3 + ax + b$

$(x-1)$ factor

$\therefore P(1) = 1 - 2 + a + b = 0$

$\therefore \boxed{a + b = 1}$ — ①

$(x+1)$ factor

$\therefore P(-1) = 1 + 2 - a + b = 0$

$\boxed{-a + b = -3}$ — ②

①+② $2b = -2$

$\therefore b = -1,\ a = 2$

$\frac{x+1}{x-1} \geq 2$

$\frac{(x+1)(x-1)^2}{(x-1)} \geq 2(x-1)^2,\ x \neq 1$

$2(x-1)^2 \leq (x+1)(x-1)$

$2(x-1)^2 - (x+1)(x-1) \leq 0$

$(x-1)[2x-2-x-1] \leq 0$

$(x-1)(x-3) \leq 0$

$\therefore 1 < x \leq 3$ since $x \neq 1$

(c) B I O L O G I S T

(i) $\frac{9!}{2!\,2!} = 90720$

(ii) $\boxed{I\ I}$ B O L O G S T

$\frac{8!}{2!}$

P(I's together) $= \frac{8!}{2!} \Big/ \frac{9!}{2!\,2!}$

$= \frac{8! \times 2!}{9!} = \frac{2}{9}$

(d) $\tan 45° = \left|\frac{m-2}{1+2m}\right|$

$|1+2m| = |m-2|$

$1+2m = m-2$ or $1+2m = -(m-2)$

$m = -3$ $\qquad 3m = 1$

$\therefore m = -3, \frac{1}{3}$

(e) $x(x+12) = 8^2$

$x^2 + 12x - 64 = 0$

$(x+16)(x-4) = 0$

$\therefore x = 4\ [x > 0]$

Question 2

(a) $\int \frac{dx}{1+4x^2} = \frac{1}{4}\int \frac{dx}{\frac{1}{4}+x^2}$

$= \frac{1}{4} \cdot \frac{1}{\frac{1}{2}} \tan^{-1}\left(\frac{x}{\frac{1}{2}}\right) + c$

$= \frac{1}{2}\tan^{-1} 2x + c$

(b) let roots $\alpha, \beta, \alpha-\beta$

Sum of roots $\alpha + \beta + \alpha - \beta = -\frac{-4}{4}$

$2\alpha = 1$

$\alpha = \frac{1}{2}$

product of roots $\alpha\beta(\alpha-\beta) = \frac{-15}{4}$

$\frac{1}{2}\beta\left(\frac{1}{2}-\beta\right) = \frac{-15}{4}$

$\beta(1-2\beta) = -15$

$2\beta^2 - \beta - 15 = 0$

$(2\beta+5)(\beta-3) = 0$

$\beta = \frac{-5}{2}, 3$

$\therefore$ roots $\frac{1}{2}, \frac{-5}{2}, \frac{1}{2}-\frac{-5}{2} \rightarrow \frac{1}{2}, \frac{-5}{2}, 3$

$\frac{1}{2}, +3, \frac{1}{2}-3 \rightarrow \frac{1}{2}, 3, \frac{-5}{2}$

Question 2 (ctd)

(c) (i) $S(0,1)$ $P(2p, p^2)$ $k:l = 1:2$

$R\left(\frac{2p+2\times 0}{2+1}, \frac{p^2+2}{2+1}\right)$

$\therefore R\left(\frac{2p}{3}, \frac{p^2+2}{3}\right)$

(ii) $\therefore x = \frac{2p}{3}$ $\therefore p = \frac{3x}{2}$

$y = \frac{p^2+2}{3}$

$\therefore 3y = \left(\frac{3x}{2}\right)^2 + 2$

$3y - 2 = \frac{9x^2}{4}$

$9x^2 = 4(3y-2)$

$9x^2 = 12y - 8$

(d) $\sin\theta = \frac{2t}{1+t^2}$ $\cos\theta = \frac{1-t^2}{1+t^2}$

$2\sin\theta + 4\cos\theta = 3$

$2\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right) = 3$

$4t + 4 - 4t^2 = 3 + 3t^2$

$7t^2 - 4t - 1 = 0$

$t = \frac{4 \pm \sqrt{16+28}}{14} = \frac{4 \pm \sqrt{44}}{14}$

$= 0.759\ldots, -0.188\ldots$

$\therefore \frac{\theta}{2} \doteqdot 0.64\ldots \quad 2.955\ldots$

$\theta = 1.30, \quad 5.91$

Question 3

(a) (i) exterior angle of cyclic quad ($\angle CDE$)

$=$ opposite interior angle ($\angle ABC$)

(ii) $\angle ABC = \angle BAC$ (equal angles isosceles ΔACB)

$\angle BDC = \angle BAC$ (angles in same segment)

$\therefore \angle BDC = \angle ABC$

but $\angle CDE = \angle ABC$ from (i)

$\therefore \angle BDC = \angle CDE$

$\therefore DC$ bisects $\angle BDE$

(b) $\lim_{x\to 0} \frac{\sin 2x}{3x} = \frac{2}{3}\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right) = \frac{2}{3}$

$= \frac{2}{3} \times 1 = \frac{2}{3}$

(c) $I = \int_0^{\ln 2} e^{2x}\sqrt{e^x - 1}\,dx$

let $u = e^x - 1$

$x = 0$ $\quad u = e^0 - 1 = 0$

$x = \ln 2$ $\quad u = e^{\ln 2} - 1 = 2 - 1 = 1$

$\frac{du}{dx} = e^x$

$\therefore du = e^x\,dx$

$\therefore I = \int_0^{\ln 2} e^x\sqrt{e^x - 1}\,.\,e^x\,dx$

$= \int_0^1 (u+1)\sqrt{u}\,.\,du$

$= \int_0^1 \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$

$= \left[\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right]_0^1$

$= \frac{2}{5} + \frac{2}{3}$

$= \frac{16}{15}$

Question 5

test $n=1$

$LHS = 2.\quad RHS = 2^1 + 1^2 - 1 = 2$

$\therefore$ true for $n=1$

assume true for $n=k$

ie $2+5+9+\ldots 2^{k-1}+2k-1 = 2^k + k^2 - 1$

to show true for $n=k+1$

ie $2+5+\ldots+[2^{k}+2(k+1)-1] = 2^{k+1}+(k+1)^2-1$

$LHS = S_k + T_{k+1}$

$= 2^k + k^2 - 1 + 2^k + 2(k+1) - 1$

$= 2.2^k + k^2 - 1 + 2k + 2 - 1$

$= 2^{k+1} + k^2 + 2k$

$= 2^{k+1} + (k^2 + 2k + 1) - 1$

$= 2^{k+1} + (k+1)^2 - 1$

$\therefore$ True for $n=k+1$

$\therefore$ Since true for $n=1$ and true for $n=k+1$ when true for $n=k$, true for all $n \geq 1$ by induction

(i) $y = \dfrac{e^x}{e^x+4}$

$\dfrac{dy}{dx} = \dfrac{e^x(e^x+4) - e^x(e^x)}{(e^x+4)^2}$

$= \dfrac{4e^x}{(e^x+4)^2}$

$\neq 0$ as $4e^x > 0$

$\therefore$ No stationary points

(ii) $y=f(x)$ continuous and increasing.

$\therefore$ always inverse

$f(x)$ always positive $\therefore y > 0$

denominator $>$ numerator $\therefore y < 1$

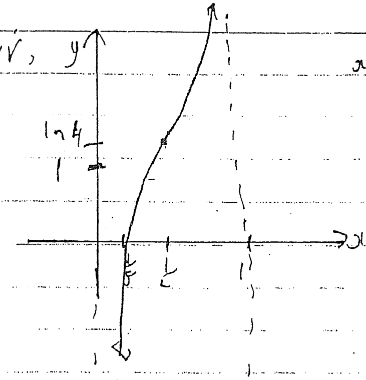
$\therefore 0 < y < 1$

OR $\displaystyle\lim_{x\to\infty} \frac{e^x}{e^x+4} = \frac{1}{1+\frac{4}{e^x}} = 1$

$\displaystyle\lim_{x\to-\infty} \frac{e^x}{e^x+4} = \frac{e^{-\infty}}{e^{-\infty}+4} = \frac{0}{4} = 0$

$\therefore 0 < y < 1$

(iv)



$x=0 \quad y = \frac{1}{1+4} = \frac{1}{5}$

$\therefore$ a int $= \frac{1}{5}$

(v) $y = \dfrac{e^x}{e^x+4}$

Inverse $x = \dfrac{e^y}{e^y+4}$

$xe^y + 4x = e^y$

$e^y(1-x) = 4x$

$e^y = \dfrac{4x}{1-x}$

$y = \ln\left(\dfrac{4x}{1-x}\right)$

Question 3

(a)(i) $1+\cos 2x = 1$

$\cos 2x = 0$

$2x = \pm\frac{\pi}{2}$

$x = \pm\frac{\pi}{4}$

(ii) $V = \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1+\cos 2x)^2\,dx - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1\,dx$

$= 2\pi\int_0^{\frac{\pi}{4}} (1 + 2\cos 2x + \cos^2 2x - 1)\,dx$

$= 2\pi\int_0^{\frac{\pi}{4}} 2\cos 2x + \frac{1}{2}(1+\cos 4x)\,dx$

$= 2\pi\left[\sin 2x + \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right)\right]_0^{\frac{\pi}{4}}$

$= 2\pi\left[\sin\frac{\pi}{2} + \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{4}\sin\pi\right)\right]$

$= 2\pi\left[\frac{\pi}{8} + 1\right]$

$= 2\pi\left[\frac{\pi}{8} + 1\right]$

$= \left(\frac{\pi^2}{4} + 2\pi\right)$ units^3

Question 4

(a) $\left(x - \frac{3}{x^2}\right)^{16}$

$T_k = \binom{16}{k}(x)^{16-k}\left(\frac{-3}{x^2}\right)^k$

$= \binom{16}{k} x^{16-k}(-3)^k x^{-2k}$

$= \binom{16}{k}(-3)^k x^{16-3k}$

coeff x: $16-3k = 1$ $\therefore$

$\therefore 3k = 15$

$k = 5$

$\therefore$ coefficient $x = \binom{16}{5}(-3)^5$

$= -1\ 061\ 424$

(b)(i) $N = A + Be^{-0.4t}$

$t=0\quad N=500 \quad\therefore 500 = A+B$

$t\to\infty \quad N\to 100$ but $N \to A$

$\therefore A = 100$

$\therefore B = 400$

(ii) $110 = 100 + 400e^{-0.4t}$

$\frac{1}{40} = e^{-0.4t}$

$-0.4t = \ln\frac{1}{40}$

$t = \frac{\ln 40}{0.4}$

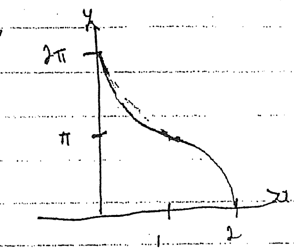
$= 9.22$ yrs

$\doteqdot 111$ months.

(c)(i) $-1 \leq x - 1 \leq 1$

$0 \leq x \leq 2$

(ii)



(iii) $y = 2\cos^{-1}(x-1)$

$x = \cos\left(\frac{y}{2}\right)+1$

$A = \int_0^{2\pi}\left(\cos\left(\frac{y}{2}\right)+1\right)dy$

$= \left[2\sin\left(\frac{y}{2}\right) + y\right]_0^{2\pi}$

$= (2\sin\pi + 2\pi) - (2\sin 0 + 0) = 2\pi$ units^2

Question 6

(a) (i) $2\cos 5\theta = 1$

$\cos 5\theta = \frac{1}{2}$

$\therefore 5\theta = 2n\pi \pm \frac{\pi}{3}$

$\theta = \frac{2n\pi}{5} \pm \frac{\pi}{15}$

$= (6n \pm 1)\frac{\pi}{15}$

(ii) $(6n+1)\frac{\pi}{15}$

$n=0 \to \frac{\pi}{15}$ $\quad n=1 \to \frac{7\pi}{15}$

$(6n-1)\frac{\pi}{15}$

$n=1 \to \frac{5\pi}{15} = \frac{\pi}{3}$

$\frac{\pi}{15}, \frac{7\pi}{15}, \frac{\pi}{3}$

(b) (i) $\frac{h}{BD} = \tan 45$ $\quad \frac{h}{BC} = \tan 60 = \sqrt{3}$

$\therefore BD = h$ $\quad BC = \frac{h}{\sqrt{3}}$

$\therefore h^2 = x^2 + \frac{h^2}{3} - 2 \times x \cdot \frac{h}{\sqrt{3}} \cdot \cos 30$

$h^2 = x^2 + \frac{h^2}{3} - \frac{2xh}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2}$

$h^2 = x^2 + \frac{h^2}{3} - xh$

$3h^2 = 3x^2 + h^2 - 3xh$

$2h^2 + 3xh - 3x^2 = 0$

(ii) $2\left(\frac{h}{x}\right)^2 + 3\left(\frac{h}{x}\right) - 3 = 0$

$\frac{h}{x} = \frac{-3 \pm \sqrt{9 - 4 \times 2 \times -3}}{4}$

$= \frac{-3 \pm \sqrt{33}}{4}$

$\therefore \frac{h}{x} = \frac{\sqrt{33} - 3}{4}$ since both > 0

(c) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 \ldots + \binom{n}{n}x^n$

(i) $\frac{(1+x)^{n+1}}{n+1} = \binom{n}{0}x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \ldots + \frac{1}{n+1}\binom{n}{n}x^{n+1} + c$

let $x=0$ $\quad \therefore c = \frac{1}{n+1}$

$\therefore \frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = \binom{n}{0}x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \ldots \frac{1}{n+1}\binom{n}{n}x^{n+1}$

Let $x=1$

$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \ldots \frac{1}{n+1}\binom{n}{n}$

$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \frac{2^{n+1} - 1}{n+1}$

$\therefore \binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \ldots \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$

(ii)

$\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \ldots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$ — ①

let $x=1$ in original

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$ — ②

② − ①

$\frac{1}{2}\binom{n}{1} + \frac{2}{3}\binom{n}{2} + \ldots \left(1 - \frac{1}{n+1}\right)\binom{n}{n} = 2^n - \frac{2^{n+1}-1}{n+1}$

$\frac{1}{2}\binom{n}{1} + \frac{2}{3}\binom{n}{2} + \ldots \frac{n}{n+1}\binom{n}{n} = \frac{2^n \times n + 2^n - 2^{n+1} + 1}{n+1}$

$RHS = \frac{n \cdot 2^n + 2^n - 2 \cdot 2^n + 1}{n+1}$

$= \frac{2^n(n-1)+1}{n+1}$

Question 7

(a) (i) $\angle OBT = 90°$ (tangent ⊥ radius)

$\therefore \cos\theta = \frac{2}{x}$ (right triangle)

(ii) $l = r\theta$

$\angle AOB = \pi - \cos^{-1}\left(\frac{2}{x}\right)$

$\therefore l = 2\left[\pi - \cos^{-1}\left(\frac{2}{x}\right)\right]$

$BT^2 = x^2 - 2^2$ (pythagoras)

$\therefore BT = \sqrt{x^2 - 4}$

$\therefore S = l + BT$

$= 2\left[\pi - \cos^{-1}\left(\frac{2}{x}\right)\right] + \sqrt{x^2+4}$

(iii) $S = 2\pi - 2\cos^{-1}\left(\frac{2}{x}\right) + \sqrt{x^2-4}$

$\frac{ds}{dx} = 0 - 2 \cdot \frac{-1}{\sqrt{1 - \left(\frac{2}{x}\right)^2}} \cdot -2x^{-2} + \frac{1}{2} \cdot 2x\,(x^2-4)^{-1}$

$= \frac{-4}{x^2\sqrt{1 - \frac{4}{x^2}}} + \frac{x}{\sqrt{x^2-4}}$

$= \frac{-4}{x\sqrt{x^2-4}} + \frac{x}{\sqrt{x^2-4}}$

$= \frac{-4 + x^2}{x\sqrt{x^2-4}}$

$\frac{ds}{dx} = \frac{\sqrt{x^2-4}}{x}$

(iv) $\frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt}$

$\frac{dx}{dt} = \frac{4 \times \sqrt{8^2-4}}{8}$

$= \frac{\sqrt{60}}{2}$

$= \sqrt{15}$

(b) (i) 2 men + 1 wom $= \binom{n}{2} \cdot \binom{n}{1}$

$= \frac{n(n-1)}{2} \times n$

$= \frac{n^2(n-1)}{2}$

(ii) $n(S) = \binom{2n}{3} = \frac{2n(2n-1)(2n-2)}{3!}$

$= \frac{4n(2n-1)(n-1)}{6}$

$P(\text{2 men} \mid \text{woman}) = \frac{n^2(n-1)}{2} \Big/ \frac{4n(2n-1)(n-1)}{6}$

$= \frac{n^2(n-1)}{2} \cdot \frac{6}{4n(2n-1)(n-1)}$

$= \frac{3n}{4(2n-1)} = \frac{3n}{8n-4}$

(iii) least value for $n = 2$

$p = \therefore P(\text{2 men} \mid \text{woman}) = \frac{6}{16-4} = \frac{1}{2}$

as $n \to \infty$ $\quad \frac{3n}{8n-4} \to \frac{3}{8}$ $\quad \therefore \frac{3}{8} < p \leq \frac{1}{2}$