

YEAR TWELVE FINAL TESTS 1997

MATHEMATICS
4 UNIT COURSE

Morning session

Wednesday, 6th August 1997

Examiners

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DIRECTIONS TO CANDIDATES :

- * All questions may be attempted.
- * All questions are of equal value.
- * All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- * Standard integrals are printed on a separate page. Approved calculators may be used.

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or the format of the Higher School Certificate Examination. However, the committees responsible for the preparation of these 'Trial Examinations' do hope that they will provide a positive contribution to your preparation for the final examinations.

(a) Consider the curve $y = \cos(\sqrt{x})$, $0 \leq x < 4\pi^2$

(i) Find $\frac{dy}{dx}$ and hence find the limiting tangent as $x \rightarrow 0$.

(ii) Find any stationary points and determine their nature.

(iii) Sketch $y = \cos(\sqrt{x})$, $0 \leq x \leq 4\pi^2$, then complete the sketch to show the

coordinate axes, the limiting tangents at any critical points, and the coordinates of any stationary points.

(iv) If $\frac{dy}{dx} = 0$, then $x = 4\pi^2$

(b) A curve has parametric equations $\left. \begin{array}{l} x = \sin \theta \\ y = \tan \theta \end{array} \right\}, -\pi < \theta \leq \pi, \theta \neq \pm \frac{\pi}{2}$

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Question 2

(b) Use the substitution $t = \tan \frac{\theta}{2}$ to find $\int \frac{1}{1 - \cos \theta - \sin \theta} d\theta$

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(c)

3

(i) Use the substitution $u = \frac{\pi}{2} - x$ to show that $\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x} + e^{\cos x}} dx = \int_0^{\frac{\pi}{2}} \frac{e^{\cos x}}{e^{\sin x} + e^{\cos x}} dx$

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x} + e^{\cos x}} dx$

(d) $I_n = \int_0^1 \frac{x^n}{x^2 + 1} dx$, $n \geq 0$, n integral.

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(i) Show that $I_1 + I_3 = \frac{1}{2}$

(ii) Evaluate I_4 and I_5

Question 3

(a) $\beta = 2 + 3i$

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- (i) On an Argand Diagram show the point B representing β , and vectors representing $\beta - 1$ and $\beta - i$.

- (ii) If θ is the acute angle between vectors $\beta - 1$ and $\beta - i$, show that $\tan \theta = \frac{1}{2}$.

$$|\beta - 1|^2 = 5$$

- (iv) Find $\left(\frac{\beta - 1}{\beta - i}\right)^2$ in the form $a + ib$, a, b real.

(b) $x^2 - 2x + p = 0$, p real and $p > 1$, has roots α and β .

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- (i) Show that α and β are non-real.

- (ii) Show the relative positions of points A and B representing α and β on an Argand Diagram.

(c) z satisfies $|z - 2i| = 1$, and the point P represents z on an Argand Diagram.

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- (i) Sketch the locus of P as z varies.

- (ii) Find the maximum and minimum values of $\arg z$, where $-\pi < \arg z \leq \pi$.

- (iii) Find the value of z when $\arg z$ takes this minimum value, and mark on your sketch the position P_0 of P for this value of z .

Question 4

Marks

- (a) Tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ intersect at T . M is the midpoint of PQ .

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write down the equation of the tangent to the ellipse at Q

- (ii) Show that the line $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{xx_2}{a^2} + \frac{yy_2}{b^2}$ passes through T and M .

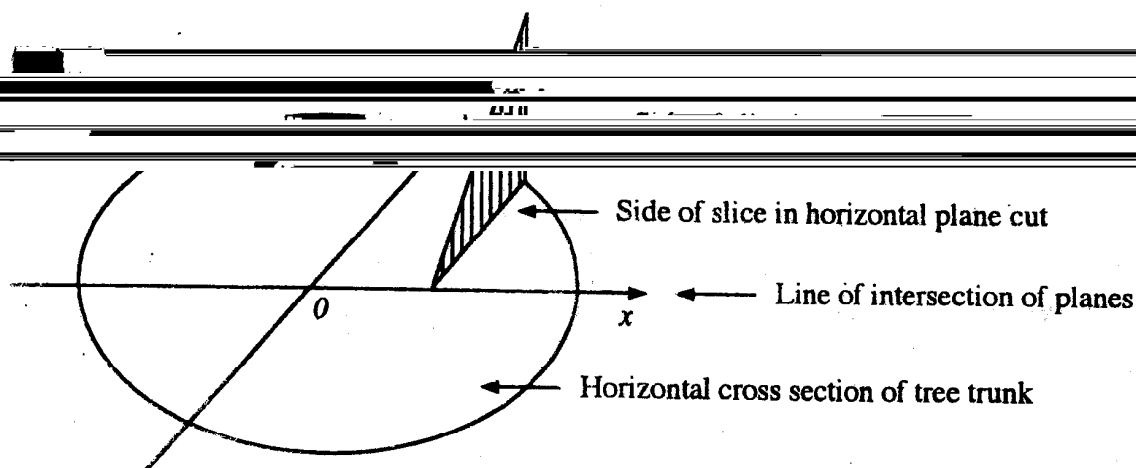
- (iii) Deduce that the points O , T , M are collinear.

- (iv) Show that the product of the gradients of PQ and TM is a constant.

- (b) A vertical tree, circular in cross section with a diameter of 60cm, has a notch cut out by two planes. One plane is horizontal, while the second plane makes an angle of 45° with the first. The two planes meet along a diameter of a circular cross section.

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The volume V of the notch is obtained by taking slices perpendicular to the diameter in



- (i) Show that the volume of the notch is given by $V = \int_{-30}^{30} (900 - x^2) dx$

Question 5

(a)

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(i) Use De Moivre's Theorem to show that $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$.

(ii) Deduce $8x^3 - 6x - 1 = 0$ has solutions $x = \cos \theta$ where $\cos 3\theta = \frac{1}{2}$.

(iii) Find the roots of $8x^3 - 6x - 1 = 0$ in the form $\cos \theta$.

(iv) Hence evaluate $\cos \frac{\pi}{9} \cos \frac{2\pi}{9} \cos \frac{4\pi}{9}$.

(b) $f(x) = x^2(x^2 - 2)$. The tangent to the curve $y = f(x)$ at the point A with x coordinate α meets the curve again at B .

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(i) Show the tangent AB has equation $y = 4\alpha(\alpha^2 - 1)x + \alpha^2(2 - 3\alpha^2)$.

(ii) Deduce that $x^2(x^2 - 2) = 4\alpha(\alpha^2 - 1)x + \alpha^2(2 - 3\alpha^2)$ has real roots $\alpha, \alpha, \beta, \gamma$ for some β, γ .

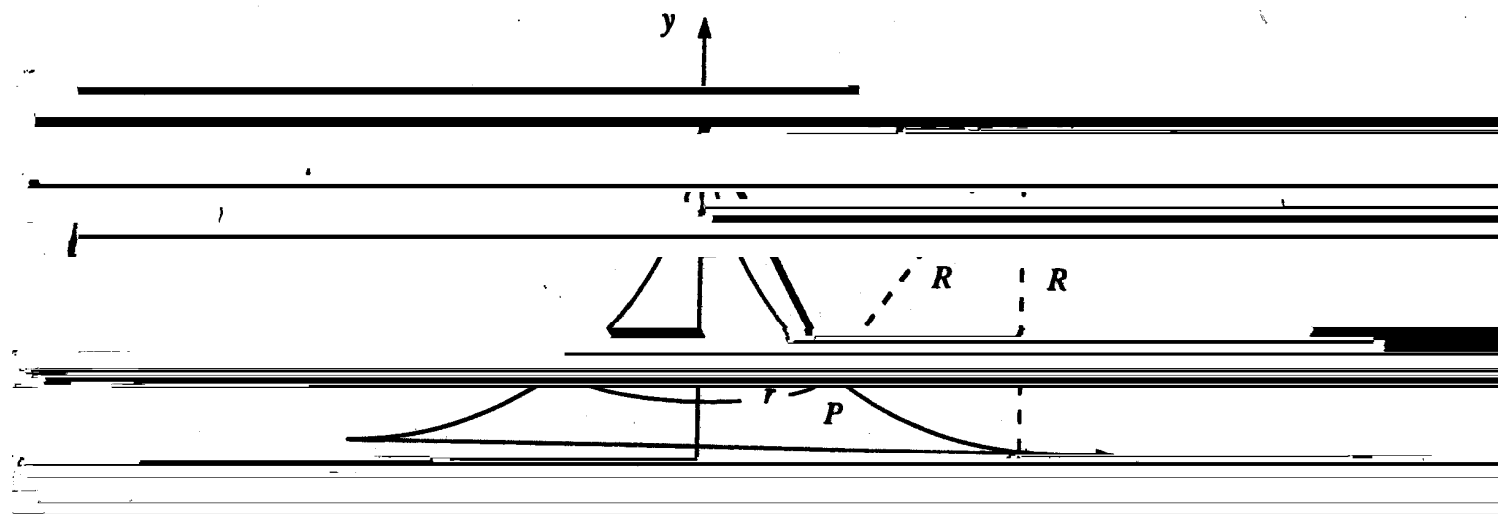
(iii) For $\alpha \neq 0$, find $\beta + \gamma$ and $\beta\gamma$ in terms of α and write down a quadratic equation with roots β, γ .

(iv) Find the possible values of α .

Question 6

Marks

15



AB is an arc of a circle centre C and radius R . A surface is formed by rotating the arc AB through one revolution about the y axis. A light, inextensible string of length l , $l \leq R$, is attached to point A , and a particle of mass m is attached to the other end. The particle is set in motion, tracing out a horizontal circle on the surface with constant angular velocity ω radians per second, while the string stays taut.

(i) When the particle is in the position P shown on the diagram, explain why the direction of the force N exerted by the surface on the particle is towards C .

(ii) If the string makes an angle θ with the vertical, show that $\angle ACP = 2\theta$.

(iii) Show on a diagram the tension force T , the force N and the weight force of magnitude

(iv) Show that

$$T \cos \theta + N \sin 2\theta = mg$$

$$T \sin \theta - N \cos 2\theta = m l \sin \theta \omega^2$$

(v) Show that $N = m l \sin \theta \left(\frac{g}{l} \sec \theta - \omega^2 \right)$

(vi) Deduce that there is a maximum value ω for the motion to occur as described, and write down this maximum value. What happens if ω exceeds this maximum?

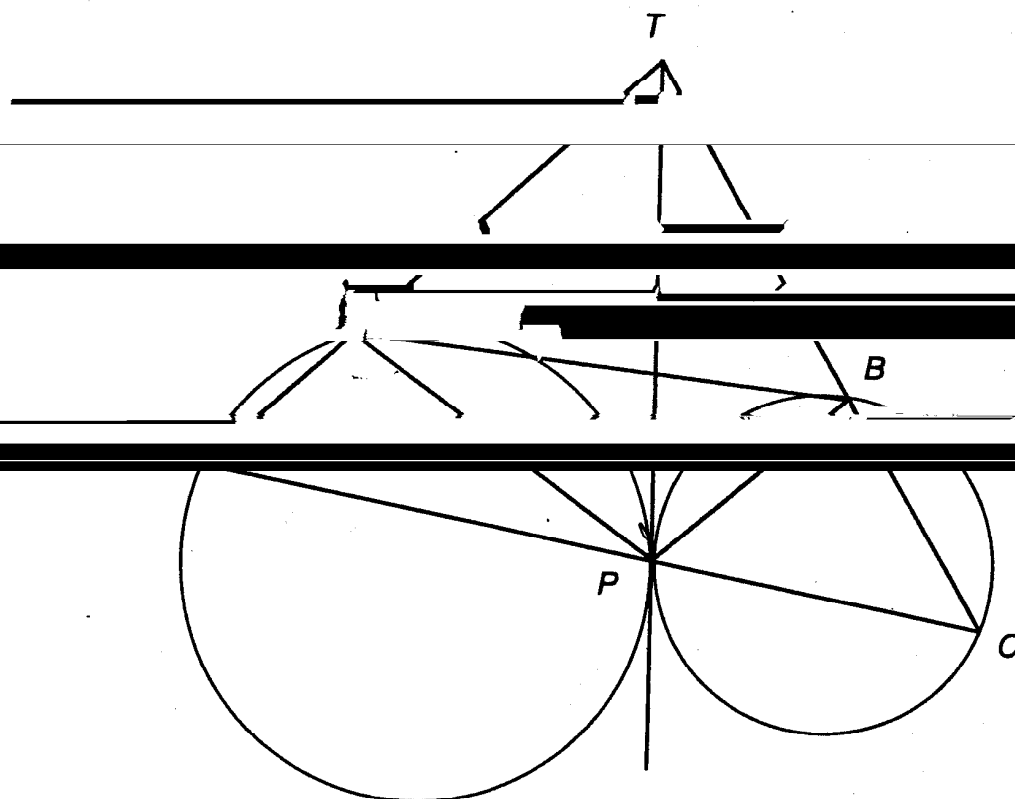
(vii) If $l = R$, find T in terms of l , m and ω^2 . Describe what happens to the tension in the string and the force the particle exerts on the surface as ω increases.

Question 7

Marks

(a)

7



Circles APD and BPC touch at P . D , P and C are collinear. TP is the common tangent at P . TC cuts circle BPC in B while TD cuts circle APD in A .

(i) Copy the diagram.

(ii) Show that $ATBP$ is a cyclic quadrilateral.

(iii) Show that $ABCD$ is a cyclic quadrilateral.

(b)

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(ii) Use the method of mathematical induction to show that for all positive integers $n \geq 2$, if $x_j > 1$, $j = 1, 2, 3, \dots, n$ then

$$\ln(x_1 + x_2 + \dots + x_n) > \frac{1}{2^{n-1}} (\ln x_1 + \ln x_2 + \dots + \ln x_n)$$

Question 8

(a)

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(i) Show that for $a > 0$ and $n \neq 0$, $\log_{a^n} x = \frac{1}{n} \log_a x$.

(ii) If $\log_2 2 = 1$, $\log_2 4 = 2$, $\log_2 8 = 3$, $\log_2 16 = 4$, $\log_2 32 = 5$, $\log_2 64 = 6$, $\log_2 128 = 7$, $\log_2 256 = 8$, $\log_2 512 = 9$, $\log_2 1024 = 10$, $\log_2 2048 = 11$, $\log_2 4096 = 12$, $\log_2 8192 = 13$, $\log_2 16384 = 14$, $\log_2 32768 = 15$, $\log_2 65536 = 16$, $\log_2 131072 = 17$, $\log_2 262144 = 18$, $\log_2 524288 = 19$, $\log_2 1048576 = 20$, $\log_2 2097152 = 21$, $\log_2 4194304 = 22$, $\log_2 8388608 = 23$, $\log_2 16777216 = 24$, $\log_2 33554432 = 25$, $\log_2 67108864 = 26$, $\log_2 134217728 = 27$, $\log_2 268435456 = 28$, $\log_2 536870912 = 29$, $\log_2 1073741824 = 30$, $\log_2 2147483648 = 31$, $\log_2 4294967296 = 32$, $\log_2 8589934592 = 33$, $\log_2 17179869184 = 34$, $\log_2 34359738368 = 35$, $\log_2 68719476736 = 36$, $\log_2 137438953472 = 37$, $\log_2 274877906944 = 38$, $\log_2 549755813888 = 39$, $\log_2 1099511627776 = 40$, $\log_2 2199023255552 = 41$, $\log_2 4398046511104 = 42$, $\log_2 8796093022208 = 43$, 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