2005 ME II TRIAL JRAHS

Q1 a) $\int \frac{dx}{x(\ln x)^2} = \int u^{-2}\,du$ $\quad u = \ln x$, $du = \frac{dx}{x}$ $\quad 1$

$= -\frac{1}{u} + c$

$= \underline{\underline{-\frac{1}{\ln x} + c}}$ $\quad 1$

b) $\frac{x^2 - x - 21}{(2x-1)(x^2+4)} = \frac{A}{2x-1} + \frac{Bx+c}{x^2+4}$

$A(x^2+4) + (Bx+c)(2x-1) = x^2 - x - 21$

when $x = \frac{1}{2}$ $\quad A(\frac{1}{4}+4) = \frac{1}{4} - \frac{1}{2} - 21$

$A = -5$

when $x = 0$

$4A - c = -21$

$-5 \times 4 - c = -21$

$c = 1$

when $x = 1$

$-5(5) + B + 1 = 1 - 1 - 21$

$B = 3$

2

$\therefore \int \frac{-5}{2x-1} + \frac{3x+1}{x^2+4}\,dx = -\frac{5}{2}\ln(2x-1) + \frac{3}{2}\int \frac{2x\,dx}{x^2+4} + \int \frac{dx}{x^2+4} + c$

2 $\quad = \underline{\underline{-\frac{5}{2}\ln(2x-1) + \frac{3}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} + c}}$

c) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x} = \int_0^1 \frac{\frac{2\,dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$ $\quad 1$

$t = \tan\frac{x}{2}$

$dx = \frac{2\,dt}{1+t^2}$ $\quad 1$

$x=0,\ t=0$

$x=\frac{\pi}{2},\ t=1$

$= \int_0^1 \frac{2\,dt}{1+\cancel{t^2}+2t-1+1-\cancel{t^2}}$

$= \int_0^1 \frac{dt}{t+1}$ $\quad 1$

$= \Big[\ln(t+1)\Big]_0^1$ $\quad 1$

$= \underline{\underline{\ln 2}}$

d i) $\int_0^a f(x)\,dx = -\int_a^0 f(a-u)\,du = \int_0^a f(a-u)\,du = \underline{\underline{\int_0^a f(a-x)\,dx}}$

1 $\quad$ 1

$x = a-u$

$dx = -du$

$x=0,\ u=a$

$x=a,\ u=0$

1 d iii) $$\int_0^{\frac{\pi}{4}} \frac{1-\tan x}{1+\tan x}\,dx = \int_0^{\frac{\pi}{4}} \frac{1-\tan\left(\frac{\pi}{4}-x\right)}{1+\tan\left(\frac{\pi}{4}-x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\dfrac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\cdot\tan x}}{1+\dfrac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\cdot\tan x}}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\dfrac{1-\tan x}{1+\tan x}}{1+\dfrac{1-\tan x}{1+\tan x}}\,dx$$ 1

$$= \int_0^{\frac{\pi}{4}} \frac{\cancel{1}+\tan x-\cancel{1}+\tan x}{1+\cancel{\tan x}+1-\cancel{\tan x}}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \tan x\,dx$$ 1

$$= \int_0^{\frac{\pi}{4}} \frac{\sin x\,dx}{\cos x}$$

$$= \Big[-\ln(\cos x)\Big]_0^{\frac{\pi}{4}}$$ $\frac{1}{2}$

$$= -\ln\left(\cos\frac{\pi}{4}\right) + \ln(\cancelto{1}{\cos 0})$$

$$= -\ln\left(\frac{1}{\sqrt{2}}\right) + 0$$

$$= \ln\sqrt{2}$$

$$= \underline{\underline{\tfrac{1}{2}\ln 2}}$$ $\frac{1}{2}$

Q2

(a) $z = \sqrt{3} + i = 2 \operatorname{cis} \frac{\pi}{6}$ 1 $\quad r = \sqrt{3+1} = 2$

$\theta = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$z^7 = \left(2 \operatorname{cis} \frac{\pi}{6}\right)^7 = 2^7 \operatorname{cis} \frac{7\pi}{6}$

$z^7 + 64z = 2^7 \operatorname{cis} \frac{7\pi}{6} + 64\left(2 \operatorname{cis} \frac{\pi}{6}\right)$ 1

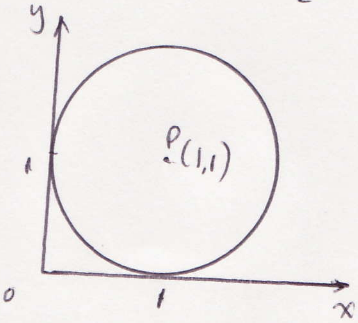
$= 128 \operatorname{cis} \frac{7\pi}{6} + 128 \operatorname{cis} \frac{\pi}{6}$

$= 128\left[\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} + \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right]$

$= 128\left[\cancel{-\cos \frac{\pi}{6}} \cancel{- i \sin \frac{\pi}{6}} + \cancel{\cos \frac{\pi}{6}} + \cancel{i \sin \frac{\pi}{6}}\right]$

$= \underline{\underline{0}}$ 1

(2b i)



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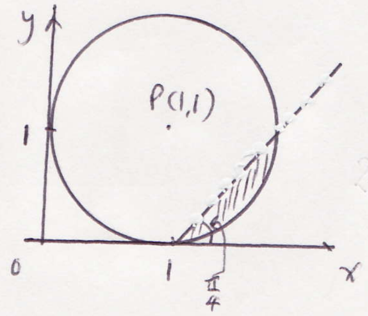
ii) $OP = \sqrt{1^2 + 1^2} = \sqrt{2}$ 1

$\max |z| = \underline{\underline{1 + \sqrt{2}}}$ 1

iii)



1

Shaded Area $= \frac{1}{2} \times \left(\frac{\pi}{2} - \sin \frac{\pi}{2}\right)$

$= \underline{\underline{\left(\frac{\pi}{4} - \frac{1}{2}\right)u^2}}$ 2

2c i) $(w^2)^3 = w^6 = (w^3)^2 = 1$ (since $w^3 = 1$) 1

ii) The roots for $z^3 - 1 = 0$ are $1, w, w^2$

$1 + w + w^2 = \frac{-b}{a} = \underline{\underline{0}}$ 1

Alternatively
$z^3 - 1 = (z-1)(z^2+z+1) = 0$
$z \neq 1$, $z^2 + z + 1 = 0$ i.e. $w^2 + w + 1 = 0$
Since w is a complex root of $z^3 = 1$

iii) $(1-w)(1-w^2)(1-w^4)(1-w^8) = (1-w)(1-w^2)(1-w \cdot w^3)(1-w^2 \cdot w^6)$

$= \left[(1-w)(1-w^2)\right]^2$ (Since $w^3 = w^6 = 1$) 1

$= (1 - w - w^2 + \overset{1}{\cancel{w^3}})^2$

$= \left[2 - (w + w^2)\right]^2$

$= \left[2 - (-1)\right]^2$ 1

$= 3^2$

$= \underline{\underline{9}}$ 1

Q 3

a) Since all coeff of $P(x)$ are real, $(x-i)$ is also of factor of $P(x)$

$(x-i)(x+i) = x^2+1$ 1

$$\begin{array}{r}
x^2 + 3x + 5 \\
x^2+1 \overline{) x^4 + 3x^3 + 6x^2 + 3x + 5} \\
\underline{x^4 \qquad\quad + x^2 \qquad\qquad} \\
3x^3 + 5x^2 + 3x \quad \\
\underline{3x^3 \qquad\quad + 3x \quad} \\
5x^2 \qquad + 5 \\
\underline{5x^2 \qquad + 5}
\end{array}$$ 1

$x^2+3x+5=0$

$x = \dfrac{-3 \pm \sqrt{9-4\times 5}}{2}$

$x = \dfrac{-3 \pm \sqrt{11}\,i}{2}$ 1

$\therefore P(x) = \underline{\underline{(x-i)(x+i)\left(x + \frac{3+\sqrt{11}\,i}{2}\right)\left(x + \frac{3-\sqrt{11}\,i}{2}\right)}}$ 1

b) $P(x) = x^4 - 5x^3 - 9x^2 + 81x - 108$

$P'(x) = 4x^3 - 15x^2 - 18x + 81$

$P''(x) = 12x^2 - 30x - 18 = 6(2x^2 - 5x - 3)$

$P''(x) = 6(2x+1)(x-3) = 0$ when $x = -\frac{1}{2}, 3$ 1

But $P'(-\frac{1}{2}) = 4(-\frac{1}{8}) - 15(\frac{1}{4}) + 9 + 81 = 85\frac{3}{4} \neq 0$

$P'(3) = 108 - 135 - 54 + 81 = 0$

$P(3) = \cancel{81} - 5\times 27 - \cancel{81} + 243 - 108 = 0$

$\therefore$ $x = 3$ is a treble root of $P(x)$ 1

Let γ be the remaining root $3+3+3+\gamma = 5$ $\therefore \gamma = -4$ 1

Alternatively by inspection $\gamma = -4$

$\therefore$ The roots are $\underline{\underline{3, 3, 3, -4}}$

c) $x^3 - 3x^2 + 2x - 1 = 0$

Let $y = \frac{1}{\alpha}$, $\alpha = \frac{1}{y}$

$\therefore \left(\frac{1}{y}\right)^3 - 3\left(\frac{1}{y}\right)^2 + 2\left(\frac{1}{y}\right) - 1 = 0$ 1

$\frac{1}{y^3} - \frac{3}{y^2} + \frac{2}{y} - 1 = 0$

$1 - 3y + 2y^2 - y^3 = 0$

This is the same as polynomial in x

$\underline{\underline{1 - 3x + 2x^2 - x^3 = 0}}$ 1

3c ii) $y = \alpha^2$, $\alpha = \sqrt{y}$

$(\sqrt{y})^3 - 3(\sqrt{y})^2 + 2\sqrt{y} - 1 = 0$ 1

$y^{3/2} + 2y^{\frac{1}{2}} = 1 + 3y$

$y^{1/2}(y+2) = 1 + 3y$

Squaring both sides

$y(y^2 + 4y + 4) = 1 + 6y + 9y^2$ 1

$y^3 + 4y^2 + 4y = 1 + 6y + 9y^2$

$\underline{\underline{y^3 - 5y^2 - 2y - 1 = 0}}$ 1

d) Let $z_2 = x + iy$

$i\overrightarrow{BC} = \overrightarrow{BA}$

$i[(7-x) + (3-y)i] = (1-x) + i(1-y)$ 1

$-(3-y) + i(7-x) = (1-x) + i(1-y)$

$-(3-y) = 1-x$ and $7-x = 1-y$

$y - 3 = 1 - x$ and $7 - x = 1 - y$

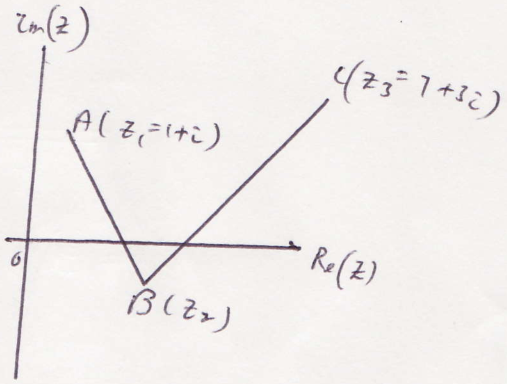
$x + y = 4$ ①

$x - y = 6$ ② 1

① + ② $x = 5$

$\therefore y = -1$

$z_2 = \underline{\underline{5 - i}}$ 1



Q.4

a i) $a = 4$ $b = 3$ $\therefore 9 = 16(1 - e^2)$

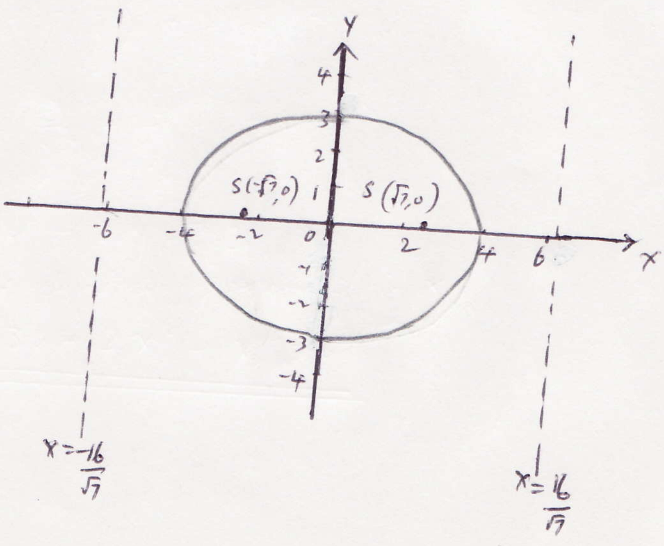
$e^2 = \frac{7}{16}$

$e = \frac{\sqrt{7}}{4}$

$S = (\sqrt{7}, 0)$

$S' = (-\sqrt{7}, 0)$ 1

Directrix : $x = \pm\frac{16}{\sqrt{7}}$ 1



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4a ii) $x = 4\cos\theta \quad y = 3\sin\theta$

$\frac{dx}{d\theta} = -4\sin\theta \quad \frac{dy}{d\theta} = 3\cos\theta$

$\frac{dy}{dx} = -\frac{3\cos\theta}{4\sin\theta}$ 1

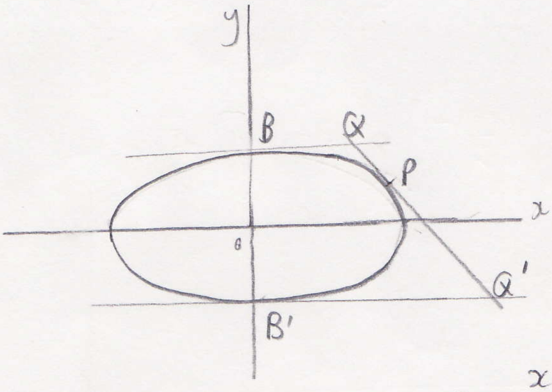
$\therefore y - 3\sin\theta = \frac{-3\cos\theta}{4\sin\theta}(x - 4\cos\theta)$

$(4\sin\theta)y - 12\sin^2\theta = -3\cos\theta\, x + 12\cos^2\theta$ 1

$(3\cos\theta)x + (4\sin\theta)y = 12(\sin^2\theta + \cos^2\theta)$
$= 12$

$\frac{x\cos\theta}{4} + \frac{y\sin\theta}{3} = 1$ 1

iii)



$B(0, 3) \quad B'(0, -3)$ when $y = 3$, $\frac{x\cos\theta}{4} = 1 - \sin\theta$

$x = \frac{4(1-\sin\theta)}{\cos\theta}$

when $y = -3$, $\frac{x\cos\theta}{4} = 1 + \sin\theta$

$x = \frac{4}{\cos\theta}(1 + \sin\theta)$,

$\therefore Q\left[\frac{4}{\cos\theta}(1 - \sin\theta), 3\right]$ and $Q' = \left[\frac{4}{\cos\theta}(1 + \sin\theta, -3\right]$

$BQ = \frac{4}{\cos\theta}(1 - \sin\theta)$ and $B'Q' = \frac{4}{\cos\theta}(1 + \sin\theta)$ 1

$BQ \times B'Q' = \frac{16}{\cos^2\theta}(1 - \sin^2\theta) = \frac{16}{\cos^2\theta}\cos^2\theta = 16$

$\therefore BQ \times B'Q' = \underline{\underline{16}}$ 1

Q 4b

bi) Take strips of thickness Δx parallel to the y-axis
$\therefore$ volume of resulting shell is given by

$$\Delta V = 2\pi(10-x)y\,\Delta x$$

$$V = \lim_{\Delta x \to 0} \sum \Delta V$$

$$= \lim_{\Delta x \to 0} \sum 2\pi(10-x)y\Delta x \qquad 1$$

$$V = \int_0^3 2\pi(10-x)\frac{5}{x^2+1}\,dx$$

$$= \int_0^3 \frac{100\pi - 10\pi x}{x^2+1}\,dx \qquad 1$$

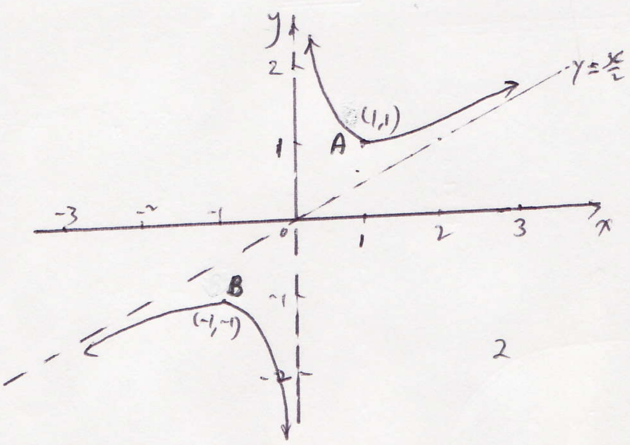
ii) $$V = 100\pi\int_0^3 \frac{dx}{x^2+1} - 5\pi\int_0^3 \frac{2x\,dx}{x^2+1}$$

$$= 100\pi\left[\tan^{-1}x\right]_0^3 - 5\pi\left[\ln(x^2+1)\right]_0^3 \qquad 1$$

$$= 100\pi\tan^{-1}(3) - 5\pi \times \ln(10) \qquad 1$$

$$= 356 \text{ cm}^3 \text{ (nearest cm}^3\text{)} \qquad 1$$

Q 5a)
i α) $y = \dfrac{1+x^2}{2x}$



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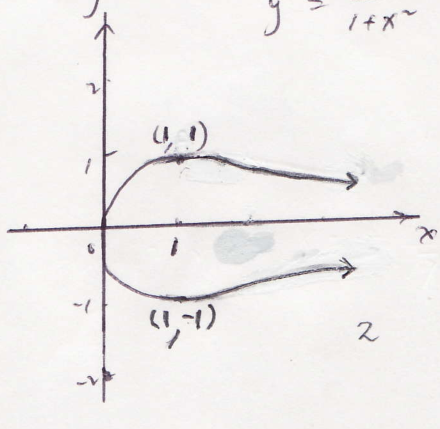
Note $\dfrac{1+x^2}{2x} = \dfrac{1}{2}\left[x + \dfrac{1}{x}\right]$

i β) $y^2 = \dfrac{2x}{1+x^2}$



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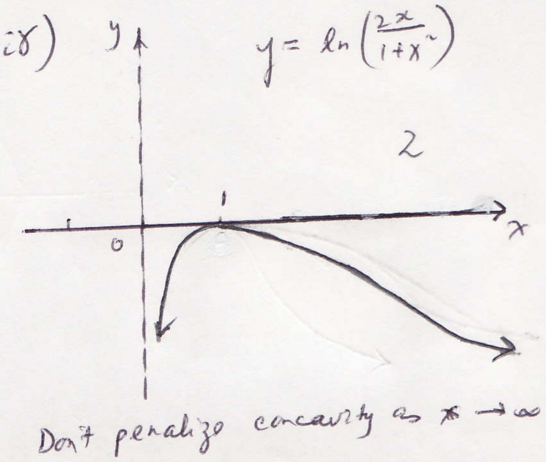
i γ) $y = \ln\left(\dfrac{2x}{1+x^2}\right)$



2

Don't penalize concavity as $x \to \infty$

5 a ii)

$kx^3+(k-2)x=0$

$2x = kx^3+kx$

$2x = kx(1+x^2)$

$\therefore kx = \dfrac{2x}{1+x^2}$ #   1

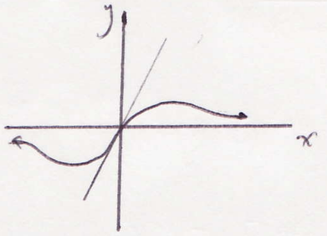
iii) $y = \dfrac{2x}{1+x^2}$

$y' = \dfrac{2(1+x^2)-2x(2x)}{(1+x^2)^2}$

$y' = \dfrac{-2x^2+2}{(1+x^2)^2}$

At $x=0$, $y'=2$

$\therefore$ y has gradient 2 at $(0,0)$ 1



Alternatively

$x(kx^2+k-2)=0$

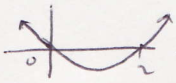
$x=0$ or $x^2 = \dfrac{2-k}{k}$ $(k \neq 0)$

For one real root $\dfrac{2-k}{k} \leq 0$ 1

$\dfrac{k-2}{k} \geq 0$

$k(k-2) \geq 0$

$k < 0$ or $k \geq 2$

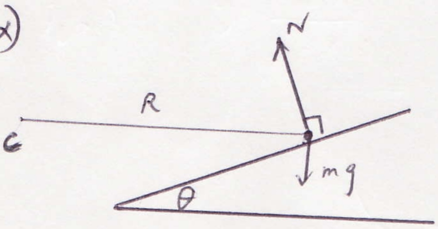


$\frac{1}{2}$

$\therefore$ $y=kx$ and $y=\dfrac{2x}{1+x^2}$ will intersect exactly once for $k \geq 2$ or $k<0$ $\frac{1}{2}$

When $k=0$, $y=0.x=0$ $\therefore$ $k=0$ is also a solution $\therefore$ $k \geq 2$ or $k \leq 0$ $\frac{1}{2}$

5 b i)



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p) Considering the forces acting on the particle

Vertically $N\cos\theta = mg$ ①

Horizontally $N\sin\theta = \dfrac{mv^2}{R}$ ②

② ÷ ① $\dfrac{N\sin\theta}{N\cos\theta} = \dfrac{\frac{mv^2}{R}}{mg}$ 1

$\therefore \tan\theta = \dfrac{v^2}{Rg}$

$v = \sqrt{Rg\tan\theta}$ 1

ii) For the first particle

speed $= \sqrt{1 \times 10 \times \frac{5}{18}} = 5/3$ m/s $\therefore$ first particle completes its circuit in $2\pi \div 5/3 = 1.2\pi$ sec   1

For the second particle

speed $= \sqrt{1.2 \times 10 \times 16/27} = 8/3$ m/s $\therefore$ 2nd particle completes its circuit in $1.2 \times 2\pi \div 8/3$

1 $= 0.9\pi$ sec

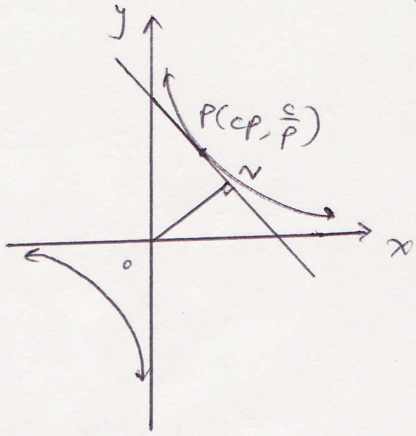
$\therefore$ the particles are next observed to be alongside each other after 3.6π sec

1

Q 6 a i)

$y = \frac{c^2}{x}$

$y' = -\frac{c^2}{x^2}$



m = slope at $P(cp, \frac{c}{p}) = -\frac{c^2}{(cp)^2} = -\frac{1}{p^2}$

Eq of tangent at P:

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$yp^2 - cp = -x + cp$

$x + yp^2 = 2cp$ ① /

Eq of No is:

$y - 0 = p^2(x - 0)$ /

$y = p^2 x$ ②

Solving ① ② simultaneously to find coord. of N:

$x = 2cp - p^2 y$ and $x = \frac{y}{p^2}$

$2cp - p^2 y = \frac{y}{p^2}$

$2cp = y\left[\frac{1}{p^2} + p^2\right]$

$y = \frac{2cp^3}{1+p^4}$ /

$x = \frac{2cp^3}{1+p^4} \times \frac{1}{p^2} = \frac{2cp}{1+p^4}$

$\therefore$ coordinates of N is $\left(\frac{2cp}{1+p^4}, \frac{2cp^3}{1+p^4}\right)$ /

a ii) Since $x(1+p^4) = 2cp$ and $\frac{y}{x} = p^2$

$x^2(1+p^4)^2 = (2cp)^2$ /

$x^2\left(1 + \frac{y^2}{x^2}\right)^2 = 4c^2\left(\frac{y}{x}\right)$

$x^2\left(1 + \frac{2y^2}{x^2} + \frac{y^4}{x^4}\right) = 4c^2\left(\frac{y}{x}\right)$ /

$x^2 + 2y^2 + \frac{y^4}{x^2} = 4c^2\frac{y}{x}$

$x^4 + 2y^2x^2 + y^4 = 4c^2xy$

$(x^2+y^2)^2 = 4c^2xy$ /

6 bi) $\Delta COD \parallel\!\!| \Delta FHD$

$\therefore \frac{FH}{CO} = \frac{HD}{OD}$

$\frac{FH}{a} = \frac{h-x}{h}$

$FH = \frac{a}{h}(h-x)$ 1

ii) $c'F = a - FH$

$= a - \frac{a}{h}(h-x)$

$= \frac{ax}{h}$ 1

$\cos \angle HC'G = \frac{C'F}{a} = \frac{x}{h}$

$\therefore \angle HC'G = \cos^{-1}\left(\frac{x}{h}\right)$ 1

iii) Shaded area = area sector $C'EHG - 2 \times$ area $\Delta C'FG$

$= \frac{1}{2}a^2 \cdot 2\cos^{-1}\left(\frac{x}{h}\right) - 2 \cdot \frac{1}{2} C'F.FG$ 1

$= a^2 \cos^{-1}\left(\frac{x}{h}\right) - \frac{ax}{h}\sqrt{a^2 - \frac{a^2x^2}{h^2}}$

$= a^2 \cos^{-1}\left(\frac{x}{h}\right) - \frac{a^2x}{h}\sqrt{1-\left(\frac{x}{h}\right)^2}$

$= a^2\left[\cos^{-1}\frac{x}{h} - \frac{x}{h}\sqrt{1-\left(\frac{x}{h}\right)^2}\right]$ 1

iv) Volume $= \int_0^h a^2\left[\cos^{-1}\left(\frac{x}{h}\right) - \frac{x}{h}\sqrt{1-\left(\frac{x}{h}\right)^2}\right]dx$

$= a^2 h \int_0^1 \cos^{-1}\theta - \theta\sqrt{1-\theta^2}\, d\theta \qquad \left(\begin{array}{r} \theta = \frac{x}{h} \\ h\, d\theta = dx \end{array}\right)$ 1

$= a^2 h \int_0^1 \cos^{-1}\theta\, d\theta - \frac{a^2h}{2}\int_0^1 2\theta\sqrt{1-\theta^2}\, d\theta$

$= a^2 h\left[\theta\cos^{-1}\theta - \sqrt{1-\theta^2}\right]_0^1 + \left[\frac{a^2h\cancel{2}}{\cancel{2}\times 3}(1-\theta^2)^{3/2}\right]_0^1$ 1

$= a^2h - \frac{a^2h}{3}$

$= \frac{2a^2h}{3}$ u^3 # 1

Q. 7

(i)

$\Downarrow +ve$ $\quad \uparrow \frac{1}{100}v^2$ $\quad \downarrow mg$

$m\ddot{x} = mg - \frac{1}{100}v^2$

$m = 1,$

$\ddot{x} = g - \frac{v^2}{100}$ 1

* without indicating positive direction of motion will lose $\frac{1}{2}$ mark

ii) $\ddot{x} = 0$ when $g = \frac{v^2}{100}$

$\therefore$ terminal velocity $V^2 = 100g$ 1

$V = \sqrt{100g}$ (motion going down $V > 0$)

iii) $v\frac{dv}{dx} = g - \frac{v^2}{100}$

$v\frac{dv}{dx} = \frac{100g - v^2}{100}$

1 $-\frac{1}{2}\int \frac{-2v\,dv}{100g - v^2} = \int \frac{dx}{100}$

1 $-\frac{1}{2}\ln(100g - v^2) + c = \frac{x}{100}$

Put $c = \frac{\ln A}{2}$, $-\frac{1}{2}\left[\ln(A(100g - v^2))\right] = \frac{x}{100}$

$-\ln\left[A(100g - v^2)\right] = \frac{x}{50}$

$x = 0,\ v = 0 \quad \therefore \quad -\ln\left[A(100g - 0)\right] = 0$

$\ln\left[A(100g)\right] = 0$

$A(100g) = 1$

$A = \frac{1}{100g}$

1 $\quad \therefore A = \frac{1}{V^2}$

$\therefore \frac{x}{50} = -\ln\left[\frac{100g - v^2}{V^2}\right]$

$e^{-\frac{x}{50}} = \frac{100g - v^2}{V^2} = 1 - \frac{v^2}{V^2}$ because $V^2 = 100g$

$\frac{v^2}{V^2} = 1 - e^{-\frac{x}{50}}$

1 $\quad v^2 = V^2\left[1 - e^{-\frac{x}{50}}\right]$

Alternatively

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g - \frac{v^2}{100}$

$\frac{2\,dx}{d(v^2)} = \frac{100}{100g - v^2}$ 1

$\int \frac{dx}{50\,d(v^2)} = \int \frac{1}{V^2 - v^2}$

$\frac{x}{50} = -\ln\left(A(V^2 - v^2)\right)$ A is a constant 1

$x = 0,\ v = 0,\quad A = \frac{1}{V^2}$ 1

$0 = -\ln A(V^2 - 0)$

$AV^2 = 1$

$-\frac{x}{50} = \ln\left(1 - \frac{v^2}{V^2}\right)$

$1 - \left(\frac{v}{V}\right)^2 = e^{-\frac{x}{50}}$

$v^2 = V^2\left(1 - e^{-\frac{x}{50}}\right)$ 1

1

iv) $\left(\frac{v}{V}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ $\quad \frac{-x}{50} = \ln\left(1-\frac{1}{4}\right) = \ln\frac{3}{4}$

$x = 50\ln\frac{4}{3} \doteqdot 14.4\,m\ (1dp)$

distance fallen is 14.4 m (1dp) 1

v) $x = 50$ $\quad \left(\frac{v}{V}\right)^2 = 1 - e^{-\frac{50}{50}} = 1 - e^{-1}$ 1

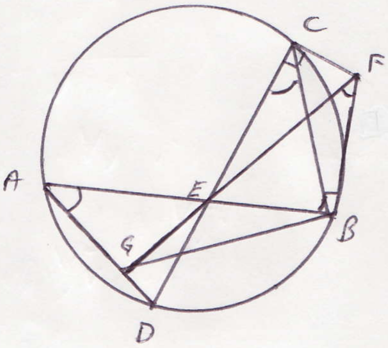
$\frac{v}{V} = \sqrt{1-e^{-1}} = \sqrt{\frac{e-1}{e}}$ $\quad (v>0, V>0)$

$v = \sqrt{\frac{e-1}{e}} \times 100\%$ of V

$v \doteqdot \underline{\underline{79.5\% \text{ of } V}}$ 1

Q 8a)

To prove FE produced is perpendicular to AD



Proof: Join AD, Extend FE to meet AD at G.

Join BG, BC

$\angle ABF = \angle DCF = 90°$ (given)

$\therefore \angle ECF + \angle EBF = 180°$

1 $\therefore$ ECFB is a cyclic quadrilateral (opposite angles supplementary)

$\angle BAD = \angle ECB$ (angles in same segment of the given circle)

$\angle ECB = \angle EFB$ (angles in same segment of circle ECFB

1 $\therefore \angle BAD = \angle EFB$

1 $\therefore$ AFBG is a cyclic quadrilateral

(line interval BG subtends equal angles at 2 points on the same side of it, the end points of the intervals and the 2 points are concyclic)

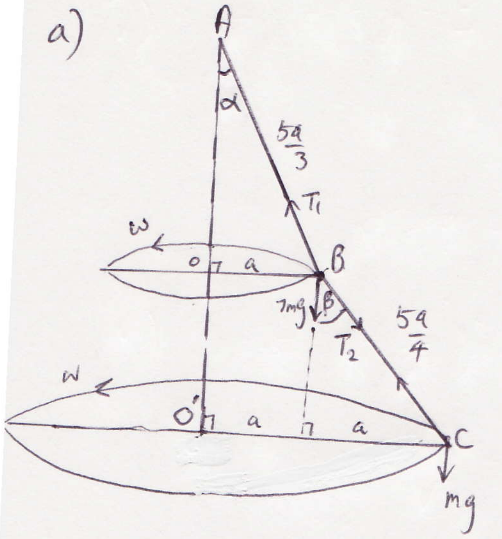
1 $\angle AGF = \angle ABF = 90°$ (angles in same segments of circle AFBG)

ie. $FG \perp AD$

1 ie FE produced is perpendicular to AD

Q 8

a)



The particle at B, C rotate about the vertical AO' with the same angular velocity w, in horizontal circles of radii a and 2a respectively. The strings AB, BC are inclined at fixed angles α, β to the vertical.

$$AO = \sqrt{\left(\frac{5a}{3}\right)^2 - a^2} = \frac{4a}{3}$$
$$BD = \sqrt{\left(\frac{5a}{4}\right)^2 - a^2} = \frac{3a}{4}$$
(Pythagoras' Th.)

$$\cos\beta = \frac{BD}{BC} = \frac{3a/4}{5a/4} = \frac{3}{5}$$

1 $$\tan\beta = \frac{a}{\frac{3a}{4}} = \frac{4}{3}$$

$$\cos\alpha = \frac{4a/3}{5a/3} = \frac{4}{5}$$

i) Forces acting on particle C are its weight and tension T_2

1 vertically : $mg = T_2 \cos\beta$ (1)

$$mg = T_2 \times \frac{3}{5}$$

1 Tension in BC $= T_2 = \frac{5mg}{3}$

(ii) Horizonally along CO'

$$m.\ 2a \cdot w^2 = T_2 \sin\beta \quad (2)$$

1 (2) ÷ (1) $$\frac{2aw^2 \cancel{m}}{\cancel{m}g} = \tan\beta$$

$$w^2 = \frac{\tan\beta}{2a} \cdot g$$

$$w^2 = \frac{4}{3} \times \frac{g}{2a} = \frac{2g}{3a}$$

1 $$w = \sqrt{\frac{2g}{3a}}.$$

∴ the angular velocity of the particle at B and C is given by $w = \sqrt{\frac{2g}{3a}}$

The forces acting on particle B are its weight and tension T_1 in string AB

Vertically : $7mg + T_2\cos\beta = T_1\cos\alpha$ 1

$$7mg + \frac{5mg}{\cancel{3}} \times \frac{\cancel{3}}{5} = T_1 \cdot \frac{4}{5}$$

$$8mg = T_1 \frac{4}{5}$$

Tension in AB $= T_1 = 10\,mg$ 1

iii) Since particle B moves with angular velocity w in a circle with radius a, its speed $= |V| = a\sqrt{\frac{2g}{3a}} = \sqrt{\frac{2ag}{3}}$ 1

Particle C moves with angular velocity w in a circle with radius 2a its speed $= |V| = 2a\sqrt{\frac{2ag}{3}} = \sqrt{\frac{8ag}{3}}$ 1

Alternative way to find T_1:

Horizontally along BO :

$$7m\,a \cdot w^2 = T_1\sin\alpha - T_2\sin\beta$$

$$7m\cancel{a} \cdot \frac{2g}{3\cancel{a}} = T_1\frac{3}{5} - \frac{5mg}{3} \cdot \frac{4}{5}$$

$$\frac{14mg + 4mg}{3} = T_1 \times \frac{3}{5}$$

$$T_1 = 10mg$$

8bi) $I_n = \int \frac{dx}{(x^2+a^2)^n}$

Let $u = \frac{1}{(x^2+a^2)^n}$ $\quad du = \frac{-2nx}{(x^2+a^2)^{n+1}}$

$V = x \qquad dv = dx$

$$I_n = u.v - \int u'v = \frac{x}{(x^2+a^2)^n} + \int \frac{2nx^2\,dx}{(x^2+a^2)^{n+1}}$$ /

But $\int \frac{x^2 dx}{(x^2+a^2)^{n+1}} = \int \frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}}dx = \int \frac{dx}{(x^2+a^2)^n} - a^2\int\frac{dx}{(x^2+a^2)^{n+1}}$

$$= I_n - a^2 I_{n+1}$$

$$\therefore \quad I_n = \frac{x}{(x^2+a^2)^n} + 2n\, I_n - 2a^2 n\, I_{n+1}$$ /

We have a higher order integral on the RHS, we solve this difficulty by changing the subject of the equations adjusting the order in the last step by replacing n by n-1

$$2a^2 n\, I_{n+1} = \frac{x}{(x^2+a^2)^n} + (2n-1) I_n$$ /

$$I_{n+1} = \frac{1}{2a^2(n)}\left(\frac{x}{(x^2+a^2)^n} + (2n-1)I_n\right)$$

$$I_n = \frac{1}{2a^2(n-1)}\left[\frac{x}{(x^2+a^2)^{n-1}} + (2n-3) I_{n-1}\right]$$ /

bii) Put $n=2$

$$I_2 = \frac{1}{2a^2}\left[\frac{x}{(x^2+a^2)} + I_1\right]_0^1 = \frac{1}{2a^2}\left[\frac{x}{x^2+a^2} + \int_0^1 \frac{dx}{a^2+x^2}\right]$$ /

$$I_2 = \frac{\left[\frac{x}{x^2+a^2}\right]_0^1 + \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^1}{2a^2} = \underline{\underline{\left[\frac{1}{1+a^2} + \frac{1}{a}\tan^{-1}\frac{1}{a}\right]\frac{1}{2a^2}}}$$ /