2005 EXT 1
SOLUTIONS

Question 1. (12 marks)

(a) Evaluate $\lim_{x \to 0} \frac{\sin 2x}{3x}$

Solution

$$\lim_{x\to 0}\frac{\sin 2x}{3x} = \lim_{x\to 0}\frac{2}{3}\times\frac{\sin 2x}{2x} = \frac{2}{3}\times\lim_{x\to 0}\frac{\sin 2x}{2x} = \frac{2}{3}$$

Marking Guideline:	1	For correct response

(b) Let A be the point $(8,10)$ and B the point $(-2,4)$.

Find the coordinates of the point P which divides the interval AB externally in the ratio 2:5.

Solution ratio 2:-5

$$x = \frac{nx_1+mx_2}{m+n} = \frac{-5(8)+2(-2)}{2-5} = \frac{-44}{-3} = 14\frac{2}{3}$$

$$y = \frac{ny_1+my_2}{m+n} = \frac{-5(10)+2(4)}{2-5} = \frac{-42}{-3} = 14$$

Marking Guideline:		
	2	For correct response or
	1	One arithmetic mistake or error in ratio

(c) Solve $\frac{1}{x+2} \le 2$

Solution

$\frac{1}{x+2} \le 2$ *Note*: $x \ne -2$

$\frac{1(x+2)^2}{x+2} \le 2(x+2)^2$

$x+2 \le 2\left[x^2+4x+4\right]$

$x+2 \le 2x^2+8x+8$

$2x^2+7x+6 \ge 0$

$(2x+3)(x+2) \ge 0$

$x<-2$ *or* $x \ge -\frac{3}{2}$

Marking Guideline:		
	3	For correct response or
	2	$x \le -2$ *or* $x \ge -\frac{3}{2}$
	1	Correct procedure with errors.

(d) The angle between the line $y=2x$ and the tangent to the curve $y=Ax^2+Ax$ at $x=1$ is $\frac{\pi}{4}$ radians. Find the values of A.

Solution

Finding m_1: $y=2x$, $y'=2 \Rightarrow m_1=2$

Finding m_2: $y=Ax^2+Ax$, $y'=2Ax+A$; at $x=1$ $m_2=3A$

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$\tan\left(\frac{\pi}{4}\right) = \left|\frac{2-3A}{1+6A}\right|$$

$$1 = \left|\frac{2-3A}{1+6A}\right|$$

$|1+6A| = |2-3A|$

$1+6A = 2-3A$ *or* $1+6A = -(2-3A)$

$9A = 1$ $\qquad$ $1+6A = -2+3A$

$A = \frac{1}{9}$ $\qquad$ $A = -1$

Marking Guideline:		
	3	For correct response or
	2	Correct procedure with an error or
	1	Find gradients or stating angle formula

(e) Use the substitution $u = 2x+1$ to evaluate $\int_{-\frac{1}{2}}^{\frac{1}{2}} x\sqrt{2x+1}\, dx$.

Solution

Preparation:

$u = 2x+1$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$x = \frac{1}{2},\ u = 2$

$x = -\frac{1}{2},\ u = 0$

$u = 2x+1$

$2x = u-1$

$x = \frac{u-1}{2}$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} x\sqrt{2x+1}\, dx = \int_0^2 \frac{u-1}{2}\sqrt{u}\ \frac{du}{2}$$

$$= \frac{1}{4}\int_0^2 (u-1)u^{\frac{1}{2}}\ du$$

$$= \frac{1}{4}\int_0^2 u^{\frac{3}{2}} - u^{\frac{1}{2}}\ du$$

$$= \frac{1}{4}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_0^2$$

$$= \frac{1}{4}\left[\left(\frac{2}{5}(2)^{\frac{5}{2}} - \frac{2}{3}(2)^{\frac{3}{2}}\right) - \left((0)^{\frac{3}{2}} - (0)^{\frac{1}{2}}\right)\right]$$

$$= \frac{1}{4}\left[\frac{2}{5}\sqrt{2^5} - \frac{2}{3}\sqrt{2^3}\right]$$

$$= \frac{1}{4}\left[\frac{2}{5}\sqrt{32} - \frac{2}{3}\sqrt{8}\right]$$

$$= \frac{1}{4}\left[\frac{2}{5}\left(4\sqrt{2}\right) - \frac{2}{3}\left(2\sqrt{2}\right)\right]$$

$$= \frac{1}{4}\left[\frac{8\sqrt{2}}{5} - \frac{4\sqrt{2}}{3}\right]$$

$$= \frac{1}{4}\left[\frac{24\sqrt{2}}{15} - \frac{20\sqrt{2}}{15}\right]$$

$$= \frac{1}{4}\left[\frac{4\sqrt{2}}{15}\right]$$

$$= \frac{\sqrt{2}}{15} \quad \text{or} \quad \approx 0{\cdot}09428$$

Marking Guideline:		
	3	For correct response or
	2	Correct procedure with one error or
	1	

Question 2. (12 marks)

(a) Let $f(x) = 4\cos^{-1}\left(\frac{x}{2}\right)$.

(i) State the domain and range of the function $f(x)$.

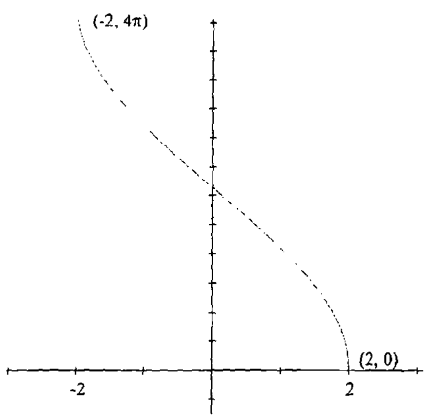
Solution

Domain: $-2 \le x \le 2$ Range: $-4\pi \le y \le 4\pi$

Marking Guideline:	2	For correct response or
	1	One correct answer

(ii) Sketch the graph of $y = f(x)$, indicating clearly the coordinates of the endpoints of the graph.

Solution



Marking Guideline:	1	For correct response

(iii) Find the equation of the tangent to the function $f(x)$ at $x = 1$.

Leave your answer as exact values in gradient intercept form.

Solution

$$f(x) = 4\cos^{-1}\left(\frac{x}{2}\right) \qquad \text{at } x = 1 \qquad f(x) = 4\cos^{-1}\left(\frac{1}{2}\right) = 4\times\frac{\pi}{3} = \frac{4\pi}{3}$$

$$f'(x) = 4\times\frac{-1}{\sqrt{2^2 - x^2}}$$

at $x = 1$

$$f'(1) = \frac{-4}{\sqrt{4-(1)^2}}$$

$$= \frac{-4}{\sqrt{3}}$$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{4\pi}{3} = \frac{-4}{\sqrt{3}}(x-1)$$

$$y = \frac{-4}{\sqrt{3}}x + \frac{4\pi}{3} + \frac{4}{\sqrt{3}}$$

Marking Guideline:	3	For correct response or
	2	Correct procedure with one error or
	1	Finding $f'(x)$

(b) Use the table of standard integrals to evaluate $\int_0^1 \frac{2}{\sqrt{x^2+1}}\,dx$ leaving your answer in exact form.

Solution

$$\int_0^1 \frac{2}{\sqrt{x^2+1}}\,dx = 2\left[\ln\left(x+\sqrt{x^2+1}\right)\right]_0^1$$

$$= 2\left[\left(\ln\left(1+\sqrt{(1)^2+1}\right)\right) - \left(\ln\left(0+\sqrt{(0)^2+1}\right)\right)\right]$$

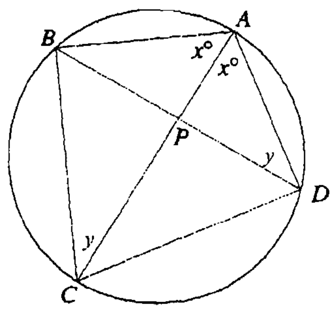
$$= 2\left[\ln\left(1+\sqrt{2}\right) - (\ln 1)\right]$$

$$= \quad 2\ln\left(1+\sqrt{2}\right)$$

Marking Guideline:	2	For correct response or
	1	Correct integration but incorrect evaluation

(c) A, B, C and D are points on the circumference of a circle.
AC and BD intersect at P.

$\angle BAC = \angle DAC = x°$.



(i) State why $\angle ACB = \angle ADB$.

Solution

Angles in the same segment of a circle are equal

Marking Guideline:	1	For correct response

(ii) Prove that $\angle ABC = \angle APD$

Solution

Let $\angle ACB = \angle ADB = y$

$\angle ABC = \quad 180° - (x+y)$ Sum of triangle ABC

$\angle APD = \quad 180° - (x+y)$ Sum of triangle APD

$\therefore \quad \angle ABC = \angle APD$

Marking Guideline:	1	For correct response

(iii) Deduce that $\angle ADC = \angle CPD$.

Solution

$\angle ADC = \quad 180° - \angle ABC$ Opposite angles of a cyclic quadrilateral

$= \quad 180° - \angle APD$ Since $\angle ABC = \angle APD$ from part ii

$\angle CPD = \quad 180 - \angle APD$ Angles of a straight line

$\therefore \quad \angle ADC = \angle CPD$

Marking Guideline:	2	For correct response or
	1	Partial solution

Question 3. (12 marks)

(a) Find $\int_0^{\frac{\pi}{4}} 2\cos^2 x \, dx$

Solution

$$\int_0^{\frac{\pi}{4}} 2\cos^2 x \, dx = 2\int_0^{\frac{\pi}{4}} \frac{1}{2}+\frac{1}{2}\cos 2x \, dx$$

$$= \int_0^{\frac{\pi}{4}} 1+\cos 2x \, dx$$

$$= \left[x+\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{4}+\frac{1}{2}\sin 2\left(\frac{\pi}{4}\right)\right]-\left[0+\frac{1}{2}\sin 2(0)\right]$$

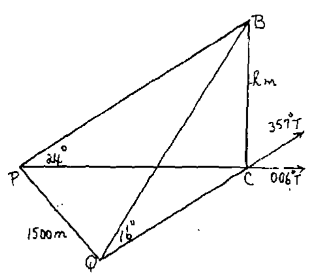
$$= \frac{\pi}{4}+\frac{1}{2}$$

Marking Guideline:	2	For correct response or
	1	Correct procedure with one error

(b) Two observers P and Q are 1500 metres apart.

The bearing of a balloon B from observer P is 006° T while the angle of elevation from P is 24°.

The bearing of balloon B from observer Q is 357° T while the angle of elevation from Q is 16°.



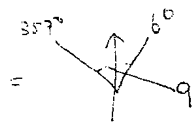
(i) Show that if the height BC is h metres then

$$h = \frac{1500}{\sqrt{\cot^2 24^\circ + \cot^2 16^\circ - 2\cot 24^\circ \cot 16^\circ \cos 9^\circ}}$$

Solution

In triangle BPQ $\quad \frac{PC}{h} = \cot 24^\circ \quad$ In triangle CPQ: $\quad \frac{QC}{h} = \cot 16^\circ$

$PC = h\cot 24^\circ \quad\quad QC = h\cot 16^\circ$



Now in triangle PQC: $\quad \angle PCQ = 9$, since $\angle PCQ =$

$$PQ^2 = QC^2 + PC^2 - 2.PC.QC.\cos(\angle PCQ)$$

$$= (h\cot 16^\circ)^2 + (h\cot 24^\circ)^2 - 2.h\cot 16^\circ .h\cot 24^\circ \cos 9^\circ$$

$$1500^2 = h^2\left[\cot^2 16^\circ + \cot^2 24^\circ - 2\cot 16^\circ \cot 24^\circ \cos 9^\circ\right]$$

$$h = \frac{1500}{\sqrt{\cot^2 16^\circ + \cot^2 24^\circ - 2\cot 16^\circ \cot 24^\circ \cos 9^\circ}}$$

Marking Guideline:	3	For correct response or
	2	Correct procedure with one error or
	1	Finding PC & QC and $\angle PCQ = 9°$

(ii) Hence find h to the nearest metre.

$$h = 1139 \ m$$

Marking Guideline:	1	For correct response

(c) $P(x) = x^3 + 3x^2 + x - 5$.

(i) Show that $x - 1$ is a factor of $P(x)$

If $x - 1$ is a factor then $P(1) = 0$

Test $P(1) = (1)^3 + 3(1)^2 + (1) - 5 = 0$

Marking Guideline:	1	For correct response (must show substitution)

(ii) Hence factorise $P(x)$

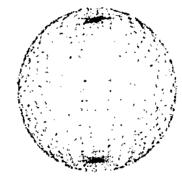
$$\begin{array}{r|l} & x^2 + 4x + 5 \\ x - 1 & x^3 + 3x^2 + x - 5 \\ & x^3 - x^2 \\ \hline & 4x^2 + x \\ & 4x^2 - 4x \\ \hline & 5x - 5 \\ & 5x - 5 \end{array}$$

$\therefore P(x) = (x-1)(x^2 + 4x + 5)$

Note: Irreducible over $\mathbb{R}$

Marking Guideline:	2	For correct response or
	1	Correct procedure with one error or no method

(d) A spherical ball is expanding so that its volume is increasing at the constant rate of $10\,mm^3$ per second.



What is the rate of increase of the radius when the surface area is $400\,mm^2$?

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$ by chain rule

$\frac{dV}{dt} = 10$ from question

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dV} = \frac{1}{4\pi r^2}$

$SA = 4\pi r^2$

$400 = 4\pi r^2$

$\pi r^2 = 100$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$= \frac{1}{4\pi r^2} \times 10$

$= \frac{5}{2\pi r^2}$

$= \frac{5}{200}$

$= 0.025 \quad mm/s$

Marking Guideline:	3	For correct response or
	1	Use of chain rule but incorrect answer
	2	Show link between $\frac{dV}{dt}$ and $\frac{dV}{dr}$

Question 4. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) An aluminium ingot is cooling in a foundry room with a constant temperature of $30^\circ C$.

At time t minutes its temperature T decreases according to the equation

$$\frac{dT}{dt} = -k(T-30) \text{ where k is a positive constant.}$$

The temperature of the aluminium ingot is initially $650\ C$ and it cools to $200\ C$ after 10 minutes.

(i) Verify that $T = 30 + Ae^{-kt}$ is a solution of this equation, where A is a constant.

Solution

$T = 30 + Ae^{-kt}$

$\frac{dT}{dt} = -kAe^{-kt}$

$\frac{dT}{dt} = -k(T-30)$

Marking Guideline:	1	For correct response

(ii) Find the values of A and k correct to two decimal places.

Solution

$t = 0,\ T = 650^\circ$

$650 = 30 + Ae^0 \Rightarrow A = 620$

$t = 10,\ T = 200$

$$T = 30 + 620e^{-kt}$$
$$200 = 30 + 620e^{-10k}$$
$$\frac{17}{62} = e^{-10k}$$
$$-10k = \log_e \frac{17}{62}$$
$$k = -\frac{1}{10}\log_e \frac{17}{62} \Rightarrow k = 0\cdot13$$

Marking Guideline:	2	For correct response or
	1	A or k correct **or** correct k for incorrect A

(iii) Most foundry workers can comfortably pick up an ingot by hand when the temperature of the ingot falls to $60^\circ C$ or lower. After how many minutes will most foundry workers first be able to handle the ingot?

Give your answer to the nearest minute.

Solution

$60 = 30 + 620e^{-kt} \Rightarrow 30 = 620e^{-kt}$

$$\frac{3}{62} = e^{-kt}$$
$$t = -\frac{1}{k}\log_e \frac{3}{62}$$
$$= 23.4057...$$
$$Time = 23\,\text{min}\ (\textit{nearest}\ \text{min})$$

or 24 min (*with correct reasoning*)

Marking Guideline:	2	For correct response or
	1	Correct substitution with attempt at $\log_e$

(b) Use mathematical induction to show $5^n > 3^n + 2^n$ for all integers $n \geq 2$.

Solution

<u>Step 1</u> Prove true for $n = 2$

LHS = 5^2 = 25

RHS = $3^2 + 2^2$ = $9 + 4$ = 13

Since LHS > RHS the statement is true for $n = 2$

<u>Step 2</u> Assume true for $n = k$

$$5^k > 3^k + 2^k$$

<u>Step 3</u> Prove true for $n = k+1$ i.e. Show $5^{k+1} > 3^{k+1} + 2^{k+1}$

Now since $5^k > 3^k + 2^k$ is true then

$$5.5^k > 5(3^k + 2^k)$$

$$5^{k+1} > (3+2)(3^k+2^k)$$

$$> 3^{k+1}+2^{k+1} \ + \ 3.2^k+2.3^k \ > 3^{k+1}+2^{k+1}$$

$$\therefore \quad 5^{k+1} > 3^{k+1}+2^{k+1}$$

Step 4 If it is true for $n=k$, it has been proven true for $n = k+1$. And since it has been proven true for $n = 2$ it is therefore true for $n = 3, 4, \ldots$ (i.e. $n \geq 2$)

Marking Guideline:	3	For correct response or
	2	For finding $5^{k+1} > (3+2)(3^k+2^k)$ but not why it is $> 3^{k+1}+2^{k+1}$.
	1	Showing correct working up to step 2.

(c) (i) Show that $\dfrac{\sin\theta}{1+\cos\theta} = \tan\dfrac{\theta}{2}$

Solution

$$\text{LHS} = \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}} \qquad \text{where } t = \tan\frac{\theta}{2}$$

$$= \frac{\frac{2t}{1+t^2}}{\frac{1+t^2}{1+t^2}+\frac{1-t^2}{1+t^2}}$$

$$= \frac{\frac{2t}{1+t^2}}{\frac{2}{1+t^2}}$$

$$= t$$

$$= \text{RHS}$$

Marking Guideline:	2	For correct response or
	1	Correct substitution

(ii) Hence, find the general solutions to the equation $\dfrac{\sin\theta}{1+\cos\theta} = \sqrt{3}$

$$\frac{\sin\theta}{1+\cos\theta} = \sqrt{3}$$

$$\tan\frac{\theta}{2} = \sqrt{3}$$

$$\frac{\theta}{2} = n\pi + \frac{\pi}{3} \qquad \text{where } n \text{ is an integer}$$

$$\theta = 2n\pi + \frac{2\pi}{3} \qquad \text{where } n \text{ is an integer}$$

Marking Guideline:	2	For correct response or
	1	For general solution formula with incorrect evaluation

MARKERS COMMENTS

Trial Ext 1

(a) This question was well-done, some students were not sure of what was required in ~~Ques~~ part (a)(i)

(b) $5^{k+1} = 5.5^k > 5(3^k + 2^k)$
$> 3\cdot 3^k + 2\cdot 2^k$
$= 3^{k+1} + 2^{k+1}$

is a relatively simple proof, but beyond many students. Some students stated
$5^k = 3^k + 2^k$
as part of their proof.
Some simply restated the conclusion
$5^{k+1} > 2^{k+1} + 3^{k+1}$
and called this a proof.

(c) Many errors in going from $\frac{\theta}{2} = n\pi + \frac{\pi}{3}$

to $\theta = 2n\pi + \frac{2\pi}{3}$

Question 5. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The roots α, β and γ of the equation $x^3 - 2x^2 - 4x + 8 = 0$ are in geometric progression.

(i) Show that $\alpha\gamma = \beta^2$

Solution

If the roots are in geometric progression then $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$$\frac{\beta}{\alpha} = \frac{\gamma}{\beta}$$

$$\alpha\gamma = \beta^2$$

Marking Guideline:	1	For correct response

(ii) Show that $\beta = -2$

Solution

$$\alpha\beta\gamma = -\frac{d}{a}$$

$$\alpha\gamma\beta = -8$$

$$\beta^3 = -8$$

$$\beta = -2$$

Marking Guideline:	1	For correct response

(iii) Hence find the values of α and γ. **3**

Solution

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha - 2 + \gamma = 2$$

$$\alpha + \gamma = 4$$

Finding α

Note: $\gamma = \dfrac{\beta^2}{\alpha}$

$$\alpha + \frac{\beta^2}{\alpha} = 4$$

$$\alpha^2 + 4 = 4\alpha$$

$$\alpha^2 - 4\alpha + 4 = 0$$

Note α is a double root so ∴ $\gamma = 2$ as well

$$(\alpha - 2)^2 = 0$$

$$\alpha = 2$$

Finding γ

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

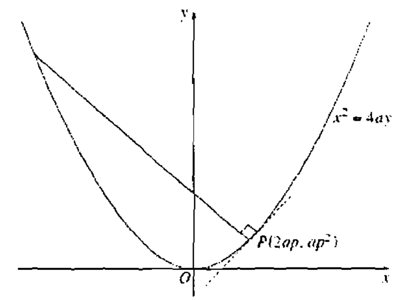
$$2 - 2 + \gamma = 2$$

$$\gamma = 2$$

∴ $\alpha = \gamma = 2$

Marking Guideline:	3	For correct response or
	2	For finding α <u>or</u> γ
	1	For finding showing some correct working e.g. $\alpha^2 + 4 = 4\alpha$

(b)



The diagram shows the normal to the parabola $x^2 = 4ay$ at point $P(2ap, ap^2)$.

(i) Show that the equation of the normal to P is given by

$$x + py = 2ap + ap^3$$

Solution

$$x^2 = 4ay$$

$$y = \frac{1}{4a}x^2$$

$$\frac{dy}{dx} = \frac{1}{2a}x$$

at $x = 2ap$ $\quad \dfrac{dy}{dx} = \dfrac{1}{2a}(2ap) = p$

The gradient of a normal at P

$$m_1 m_2 = -1$$

$$pm_2 = -1$$

$$m_2 = \frac{-1}{p}$$

So equation of the normal at P is

$$y - y_1 = m(x - x_1)$$

$$y - ap^2 = \frac{-1}{p}(x - 2ap)$$

$$x + py = 2ap + ap^3$$

Marking Guideline:	2	For correct response or
	1	For finding the gradient at P

(ii) Find the coordinates of R where the normal at P intersects the y axis. 1

Solution

Intersects the y axis at $x = 0$

$$x + py = 2ap + ap^3$$

$$0 + py = 2ap + ap^3$$

$$y = 2a + ap^2$$

$$y = a\left(p^2 + 2\right)$$

$$\therefore \quad R\left(0, a\left(p^2 + 2\right)\right)$$

Marking Guideline:	1	For correct response

(iii) Hence find the locus of the midpoint of PR.

Solution

The midpoint M of PR is

$$x = \frac{2ap + 0}{2} \qquad y = \frac{ap^2 + 2a + ap^2}{2}$$

$$x = ap \quad \Rightarrow \quad p = \frac{x}{a} \qquad y = ap^2 + a$$

$$y = a\left(\frac{x}{a}\right)^2 + a$$

$$y = \frac{x^2}{a} + a$$

$$x^2 = ay - a^2$$

$$x^2 = a(y - a)$$

Marking Guideline:	2	For correct response or
	1	For finding the midpoint

(c) If $y = \tan^{-1}\left(x^2\right)$, find $\frac{d^2y}{dx^2}$

Solution

$$y = \tan^{-1}\left(x^2\right)$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{1 + x^4} \times 2x$$

$$\frac{dy}{dx} = \frac{2x}{1 + x^4}$$

let $u = x^2$

$$\frac{du}{dx} = 2x$$

$$y = \tan^{-1}(u)$$

$$\frac{dy}{du} = \frac{1}{1 + u^2}$$

$$\frac{dy}{du} = \frac{1}{1 + x^4} \quad \text{since } u = x^2$$

$$\frac{d^2y}{dx^2} = \frac{\left(1 + x^4\right)2 - 2x.4x^3}{\left(1 + x^4\right)^2} \quad \text{by the quotient rule}$$

$$\frac{d^2y}{dx^2} = \frac{2 + 2x^4 - 8x^4}{\left(1 + x^4\right)^2}$$

$$\frac{d^2y}{dx^2} = \frac{2 - 6x^4}{\left(1 + x^4\right)^2}$$

Marking Guideline:	2	For correct response or
	1	For finding $\frac{dy}{dx} = \frac{2x}{1 + x^4}$

Question 6. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The velocity v ms^{-1} of a particle moving in simple harmonic motion according to the equation

$$v^2 = -12 + 8x - x^2 \text{, where } x \text{ is in metres.}$$

(i) Between which two values is the particle oscillating?

Solution

Oscillating between the x values when $v = 0$

$0 = -12 + 8x - x^2$

$0 = (x-6)(x-2)$

$x = 2$ *and* 6

Marking Guideline:	1	For correct response

(ii) Find the centre of motion.

Solution

Centre of motion is halfway between 2 and 6 at $x = 4$

Marking Guideline:	1	For correct response

(iii) Find the maximum speed of the particle.

Solution

Max. speed occurs at the centre of the motion i.e. $x = 4$ [from part ii]

$v^2 = -12 + 8(4) - (4)^2$

$v = \pm 2$ max. speed $\Rightarrow |v| = 2$

Marking Guideline:	1	For correct response

(iv) Find the acceleration of the particle in terms of x.

Solution

$v^2 = -12 + 8x - x^2$

$\frac{1}{2}v^2 = -6 + 4x - \frac{1}{2}x^2$

$\ddot{x} = \dfrac{d\frac{1}{2}v^2}{dx}$

$= 4 - x$

$= -(x-4)$

Marking Guideline:	1	For correct response

(v) Find the period of the motion.

Solution

$T = \dfrac{2\pi}{n}$ $\ddot{x} = -(x-4)$ $\Rightarrow$ $n = 1$

$= 2\pi$

Marking Guideline:	1	For correct response

(vi) If initially $x = 4 + \sqrt{3}$, find a function for displacement in terms of time t.

Solution

displacement has the form $x = a\cos(nt + \alpha) + b$

now $a = 2, n = 1$ from previous parts $x = 2\cos(t + \alpha) + 4$

and at $t = 0, x = 4 + \sqrt{3}$ $4 + \sqrt{3} = 2\cos\alpha + 4$

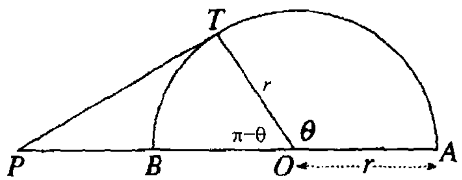
$\cos\alpha = \dfrac{\sqrt{3}}{2}$

$\alpha = \dfrac{\pi}{6}$

$\therefore$ $x = 2\cos\left(t + \dfrac{\pi}{6}\right) + 4$

Marking Guideline:	2	For correct response or
	1	For finding $x = 2\cos(t + \alpha)$

(b) The point T lies on the circumference of a semicircle, radius r and diameter AB, as shown. The point P lies on AB produced and PT is the tangent at T.



The arc AT subtends an angle of θ at the centre, O, and the area of triangle OPT is equal to that of the sector AOT.

(i) Find an expression for $\tan(\pi-\theta)$

Solution

$\tan(\pi-\theta) \quad = \quad -\tan\theta$

Marking Guideline:	1	For correct response

(ii) Hence, or otherwise, show that $\theta+\tan\theta=0$

$$A_{OPT}=A_{AOT}$$

$$\frac{1}{2}\times r\times PT=\frac{1}{2}r^2\theta$$

$$\frac{1}{2}r\left[r\tan(\pi-\theta)\right]=\frac{1}{2}r^2\theta \qquad \text{Note: } \tan(\pi-\theta)=\frac{PT}{r} \Rightarrow PT=r\tan(\pi-\theta)$$

$$r^2(-\tan\theta)=r^2\theta$$

$$0=\theta+\tan\theta$$

Marking Guideline:	2	For correct response or
	1	For not clearly communicating proof

(iii) Taking $\theta=2$ as an initial approximation, use Newton's method once to find a better approximation for a solution to $\theta+\tan\theta=0$, correct to 3 significant figures.

Solution

$f(\theta)=\theta+\tan\theta \qquad f(\theta_0)=2+\tan 2$

$f'(\theta)=1+\sec^2\theta \qquad f'(\theta_0)=1+\sec^2 2$

$$\theta_1=\theta_0-\frac{f(\theta_0)}{f'(\theta_0)}$$

$$=2-\frac{2+\tan 2}{1+\sec^2 2}$$

$$=2\cdot0273\ldots$$

$$=2\cdot03$$

Marking Guideline:	2	For correct response or
	1	For correct procedure with one error

Question 7. (12 marks)

(a) Show that the equation $a\sin x + b\cos x = \sqrt{3}$ has real roots if $a^2 + b^2 \geq 3$

Solution

using t – results:

$$a\sin x + b\cos x = \sqrt{3}$$

$$a\left(\frac{2t}{1+t^2}\right) + b\left(\frac{1-t^2}{1+t^2}\right) = \sqrt{3}$$

$$2at + b - bt^2 = \sqrt{3} + \sqrt{3}t^2$$

$$0 = \sqrt{3}t^2 + bt^2 - 2at + \sqrt{3} - b$$

$$\left(\sqrt{3}+b\right)t^2 - 2at + \sqrt{3} - b = 0$$

$$t = \frac{2a \pm \sqrt{4a^2 - 4\left(\sqrt{3}+b\right)\left(\sqrt{3}-b\right)}}{2\left(\sqrt{3}+b\right)}$$

$$= \frac{2a \pm 2\sqrt{a^2 - \left(3 - b^2\right)}}{2\left(\sqrt{3}+b\right)}$$

$$= \frac{a \pm \sqrt{a^2 + b^2 - 3}}{\left(\sqrt{3}+b\right)}$$

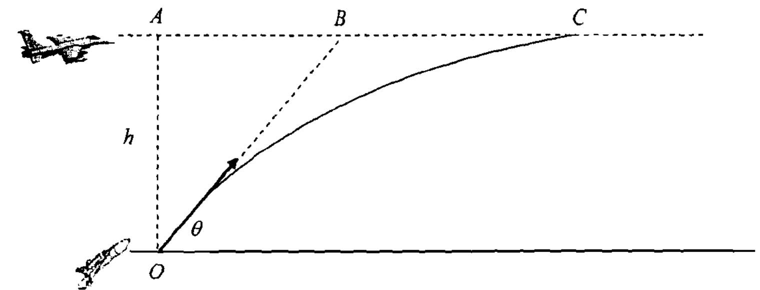
Now $a\sin x + b\cos x = \sqrt{3}$ has real roots if $\Delta \geq 0$

$$a^2 + b^2 - 3 \geq 0$$

$$a^2 + b^2 \geq 3$$

Marking Guideline:	3	For correct response or
	2	For finding $t = \dfrac{a \pm \sqrt{a^2+b^2-3}}{\left(\sqrt{3}+b\right)}$
	1	For using t – results or appropriate method but incorrect

(b) An aircraft is flying with constant velocity $U\ ms^{-1}$ at a constant height h metres above horizontal ground.



When the plane is at A it is directly over a anti-aircraft gun at O.
When the plane is at B a projectile is fired from the gun with velocity $V\ ms^{-1}$ at an angle of elevation θ along OB.
T seconds later the shell hits the aircraft at C. The acceleration due to gravity is $g\ \ ms^{-2}$.

(i) Assume that the equations of motion of the projectile are

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g$$

Show that relative to O the horizontal and vertical displacements of the projectile after time t seconds are given by the equations

$$x = Vt\cos\theta \quad \text{and} \quad y = Vt\sin\theta - \frac{1}{2}gt^2$$

Solution

$$\ddot{x} = 0 \qquad\qquad \ddot{y} = -g$$

$$\dot{x} = 0 + C_1 \qquad\qquad \dot{y} = -gt + D_1$$

At $t = 0$ $\dot{x} = V\cos\theta \Rightarrow C_1 = V\cos\theta$ At $t = 0$ $\dot{y} = V\sin\theta \Rightarrow D_1 = V\sin\theta$

$$\dot{x} = V\cos\theta \qquad\qquad \dot{y} = V\sin\theta - gt$$

$$x = Vt\cos\theta + C_2 \qquad\qquad y = Vt\sin\theta - \frac{1}{2}gt^2 + D_2$$

At $t = 0$ $x = 0 \Rightarrow C_2 = 0$ At $t = 0$ $y = 0 \Rightarrow D_2 = 0$

$$\therefore \quad x = Vt\cos\theta \quad \text{and} \quad y = Vt\sin\theta - \frac{1}{2}gt^2$$

Marking Guideline:	2	For correct response or
	1	For one correct answer

(ii) Show that the projectiles path is given by the Cartesian equation

$$y = x\tan\theta - \frac{g\sec^2\theta}{2v^2}x^2$$

Solution

$$x = Vt\cos\theta \qquad \Rightarrow \qquad t = \frac{x}{V\cos\theta}$$

Substituting $t = \dfrac{x}{V\cos\theta}$ into y:

$$y = V\left(\frac{x}{V\cos\theta}\right)\sin\theta - \frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2$$

$$y = x\tan\theta - \frac{1}{2}g \times \frac{x^2}{V^2\cos^2\theta}$$

$$y = x\tan\theta - \frac{g\sec^2\theta}{2V^2}x^2$$

Marking Guideline:	2	For correct response or
	1	For omitting some working

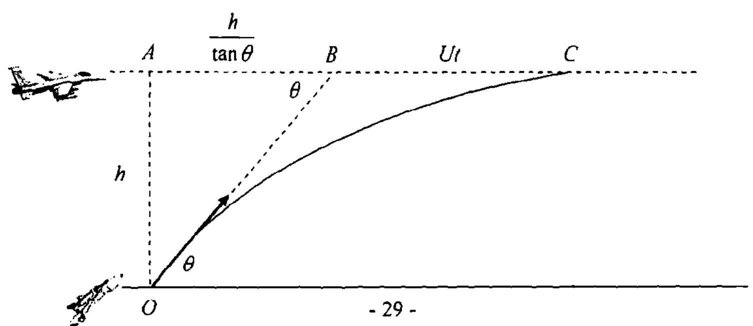
(iii) Show that the time until the projectile hits the aircraft is given by

$$T = \frac{h}{(V\cos\theta - U)\tan\theta}$$

Solution

From the diagram: $\tan\theta = \dfrac{h}{AB} \quad \Rightarrow \quad AB = \dfrac{h}{\tan\theta}$

and BC = speed of plane × time = Ut



Now comparing horizontal distances travelled by the plane and the projectile when they collide noting they collide at time T

Plane		Projectile
$\dfrac{h}{\tan\theta} + UT$	=	$VT\cos\theta$
$VT\cos\theta - UT$	=	$\dfrac{h}{\tan\theta}$
$T(V\cos\theta - U)$	=	$\dfrac{h}{\tan\theta}$
T	=	$\dfrac{h}{(V\cos\theta - U)\tan\theta}$

Marking Guideline:	2	For correct response or
	1	For distance AB

(iv) Hence show that $gh = 2U(V\cos\theta - U)\tan^2\theta$

Solution

Substituting $T = \dfrac{h}{(V\cos\theta - U)\tan\theta}$ and $y = h$ into $y = Vt\sin\theta - \dfrac{1}{2}gt^2$

$$h = V\left(\frac{h}{(V\cos\theta - U)\tan\theta}\right)\sin\theta - \frac{1}{2}g\left(\frac{h}{(V\cos\theta - U)\tan\theta}\right)^2$$

$$2h = \frac{2hV\sin\theta}{(V\cos\theta - U)\tan\theta} - \frac{gh^2}{(V\cos\theta - U)^2\tan^2\theta}$$

$$\frac{gh^2}{(V\cos\theta - U)^2\tan^2\theta} = \frac{2hV\sin\theta}{(V\cos\theta - U)\tan\theta} - 2h$$

$$gh = 2V\sin\theta(V\cos\theta - U)\tan\theta - 2(V\cos\theta - U)^2\tan^2\theta$$

$$gh = 2(V\cos\theta - U)\tan\theta\left[V\sin\theta - (V\cos\theta - U)\tan\theta\right]$$

$$gh = 2(V\cos\theta - U)\tan\theta[V\sin\theta - V\tan\theta\cos\theta + U\tan\theta]$$

$$gh = 2(V\cos\theta - U)\tan\theta[V\sin\theta - V\sin\theta + U\tan\theta]$$

$$gh = 2U(V\cos\theta - U)\tan^2\theta$$

Marking Guideline:	3	For correct response or
	2	For correct procedure but with one error
	1	For using substituting $T = \dfrac{h}{(V\cos\theta - U)\tan\theta}$ and $y = h$ into y