

NSGHS TRIAL HSC 2010 XI SOLUTIONS

Question 1

(a) $P(x) = x^4 - 2x^3 + ax + b$

(x-1) factor

$$\therefore P(1) = 1 - 2 + a + b = 0$$

$$\therefore \boxed{a+b=1} \quad \text{--- (1)}$$

(x+1) factor

$$\therefore P(-1) = 1 + 2 - a + b = 0$$

$$\boxed{-a+b=-3} \quad \text{--- (2)}$$

$$\textcircled{1} + \textcircled{2} \quad 2b = -2$$

$$\therefore b = -1, a = 2$$

$$\frac{x+1}{x-1} \geq 2$$

$$\frac{(x+1)(x+1)^2}{(x-1)} \geq 2(x-1)^2, x \neq 1$$

$$2(x+1)^2 \leq (x+1)(x-1)$$

$$2(x+1)^2 - (x+1)(x-1) \leq 0$$

$$(x-1)[2x-2-x-1] \leq 0$$

$$(x-1)(x-3) \leq 0$$

$$\therefore 1 < x \leq 3 \text{ since } x \neq 1$$

(c) $B \underline{I} \overset{x}{0} \overset{x}{L} \overset{x}{O} \underline{G} \underline{I} \underline{S} \underline{T}$

(i) $\frac{9!}{1!2!} = 90720$

(ii) $\boxed{II} B O L O G S T$

$$\frac{8!}{2!}$$

$$P(\text{I's together}) = \frac{8!}{2!} \div \frac{9!}{2!2!}$$

$$= \frac{8! \times 2!}{9!} = \frac{2}{9}$$

(d) $\tan 45^\circ = \left| \frac{m-2}{1+2m} \right|$

$$|1+2m| = |m-2|$$

$$1+2m = m-2 \text{ or } 1+2m = -(m-2)$$

$$m = -3 \quad 3m = 1$$

$$\therefore m = -3, \frac{1}{3}$$

(e) $x(x+12) = 64$

$$x^2 + 12x - 64 = 0$$

$$(x+16)(x-4) = 0$$

$$\therefore x = 4 [x > 0]$$

Question 2

(a) $\int \frac{dx}{1+4x^2} = \frac{1}{4} \int \frac{dx}{\left(\frac{1}{2} + x^2\right)}$

$$= \frac{1}{4} \cdot \frac{1}{\frac{1}{2}} \tan^{-1}\left(\frac{2x}{\frac{1}{2}}\right) + C$$

$$= \frac{1}{2} \tan^{-1} 2x + C$$

(b) let roots $\alpha, \beta, \alpha - \beta$

Sum of roots $\alpha + \beta + \alpha - \beta = -\frac{-4}{4}$

$$2\alpha = 1$$

$$\alpha = \frac{1}{2}$$

Product of roots $\alpha \beta (\alpha - \beta) = \frac{-15}{4}$

$$\frac{1}{2} \beta \left(\frac{1}{2} - \beta\right) = \frac{-15}{4}$$

$$\beta(1-2\beta) = -15$$

$$2\beta^2 - \beta - 15 = 0$$

$$(2\beta + 5)(\beta - 3) = 0$$

$$\beta = -\frac{5}{2}, 3$$

roots $\frac{1}{2}, -\frac{5}{2}, \frac{1}{2} - \frac{5}{2} \rightarrow \frac{1}{2}, -\frac{5}{2}, 3$

$$\frac{1}{2}, +3, \frac{1}{2} - 3 \rightarrow \frac{1}{2}, 3, -\frac{5}{2}$$

Question 2 (ctd)

(i) S (0, 1) P (2p, p²) k:l = 1:2

$$R \left(\frac{2p+2 \cdot 0}{2+1}, \frac{p^2+2}{2+1} \right)$$

$$\therefore R \left(\frac{2p}{3}, \frac{p^2+2}{3} \right)$$

(ii) $\therefore x = \frac{2p}{3} \therefore p = \frac{3x}{2}$

$$y = \frac{p^2+2}{3}$$

$$\therefore 3y = \left(\frac{3x}{2} \right)^2 + 2$$

$$3y-2 = \frac{9x^2}{4}$$

$$9x^2 = 4(3y-2)$$

$$9x^2 = 12y - 8$$

(d) $\sin \theta = \frac{2t}{1+t^2} \quad \cos \theta = \frac{1-t^2}{1+t^2}$

$$2 \sin \theta + 4 \cos \theta = 3$$

$$2 \left(\frac{2t}{1+t^2} \right) + 4 \left(\frac{1-t^2}{1+t^2} \right) = 3$$

$$4t + 4 - 4t^2 = 3 + 3t^2$$

$$7t^2 - 4t - 1 = 0$$

$$t = \frac{4 \pm \sqrt{16+28}}{14} = \frac{4 \pm \sqrt{44}}{14}$$

$$\therefore \theta = 0.759 \dots, -0.188 \dots$$

$$2 \theta = 0.64 \dots, 2.955 \dots$$

$$\theta = 1.30, 5.91$$

Question 3

(a) ⁽ⁱⁱ⁾ Exterior angle of cyclic quad ($\angle CDE$)
= opposite interior angle ($\angle ABC$)

(iii) $\angle ABC = \angle BAC$ (equal angles subtend ΔACB)

$\angle BDC = \angle BAC$ (angles in same segment)

$$\therefore \angle BDC = \angle ABC$$

but $\angle CDE = \angle ABC$ from (i)

$$\therefore \angle BDC = \angle CDE$$

$\therefore DC$ bisects $\angle BDE$

(b) $\lim_{x \rightarrow 0} \frac{\sin 2x}{2x} = \frac{2}{3} \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} = 2$
 $= \frac{2}{3} \times 1 = \frac{2}{3}$

(c) $I = \int_0^{\ln 2} e^{2x} \sqrt{e^{2x}-1} dx$

Let $u = e^{2x} - 1$

$x=0 \quad u = e^0 - 1 = 0$

$x=\ln 2 \quad u = e^{\ln 2} - 1 = 2 - 1 = 1$

$$\frac{du}{dx} = e^{2x}$$

$$\therefore du = e^{2x} dx$$

$$\therefore I = \int_0^1 e^{2x} \sqrt{e^{2x}-1} \cdot e^{2x} dx$$

$$= \int_0^1 (u+1) \sqrt{u} \cdot du$$

$$= \int_0^1 (u^{\frac{3}{2}} + u^{\frac{1}{2}}) du$$

$$= \left[\frac{2}{5} u^{\frac{5}{2}} + \frac{2}{3} u^{\frac{3}{2}} \right]_0^1$$

$$= \frac{2}{5} + \frac{2}{3}$$

$$= \frac{16}{15}$$

Question 3

(i) $1 + \cos 2x = 1$

$\cos 2x = 0$

$2x = \pm \frac{\pi}{2}$

$x = \pm \frac{\pi}{4}$

(ii) $V = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + \cos 2x)^2 dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 dx$
 $= 2\pi \int_0^{\frac{\pi}{4}} (1 + 2\cos 2x + \cos^2 2x) dx$

$= 2\pi \int_0^{\frac{\pi}{4}} 2\cos 2x + \frac{1}{2}(1 + \cos 4x) dx$

$= 2\pi \left[\sin 2x + \frac{1}{2} \left(x + \frac{1}{4} \sin 4x \right) \right]_0^{\frac{\pi}{4}}$

$= 2\pi \left[\sin \frac{\pi}{2} + \frac{1}{2} \left(\frac{\pi}{4} + \frac{1}{4} \sin \pi \right) \right]$

$= 2\pi \left[\frac{\pi}{8} + 1 \right]$

$= 2\pi \left[\frac{\pi}{8} + 1 \right]$

$= \left(\frac{\pi^2}{4} + 2\pi \right) \text{ units}^3$

$\therefore \text{coefficient } x = \left(\frac{16}{5} \right) (-3)^5$
 $= -1061424$

(b) (i) $N = A + B e^{-0.4t}$

$t=0 \quad N=500 \quad \therefore 500 = A+B$

$t \rightarrow \infty \quad N \rightarrow 100 \quad \text{but } N \rightarrow 100$

$\therefore A=100$

$\therefore B=400$

(ii) $110 = 100 + 400 e^{-0.4t}$

$\frac{1}{40} = e^{-0.4t}$

$-0.4t = \ln \frac{1}{40}$

$t = \frac{\ln 40}{0.4}$

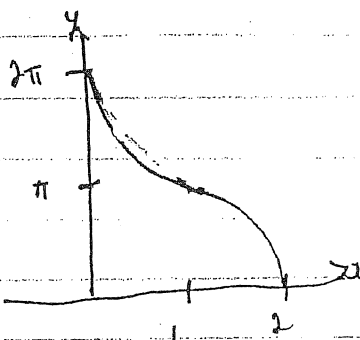
$= 9.22 \text{ yrs.}$

$\approx 111 \text{ months.}$

(c) (i) $-1 \leq x-1 \leq 1$

$0 \leq x \leq 2$

(ii)



(iii) $y = 2 \cos^{-1}(x-1)$

$x = \cos\left(\frac{y}{2}\right) + 1$

$A = \int_0^{2\pi} \left(\cos\left(\frac{y}{2}\right) + 1 \right) dy$

$= \left[2 \sin\left(\frac{y}{2}\right) + y \right]_0^{2\pi}$

$= (2 \sin \pi + 2\pi) - (2 \sin 0 + 0) = 2\pi \text{ units}^2$

Question 4

(a) $\left(x - \frac{3}{x^2} \right)^{16}$

$T_k = \binom{16}{k} (x)^{16-k} \left(\frac{-3}{x^2} \right)^k$

$= \binom{16}{k} x^{16-k} (-3)^k x^{-2k}$

$= \binom{16}{k} (-3)^k x^{16-3k}$

coeff x : $16-3k=1 \quad \therefore$

$\therefore 3k=15$

$k=5$

Question 5

2, test $n=1$

$$LHS = 2. \quad RHS = 2^1 + 1^2 - 1 = 2$$

\therefore true for $n=1$

assume true for $n=k$

$$\therefore 2+5+9+\dots+2k^2+k^2-1 = 2k+k^2-1$$

to show true for $n=k+1$

$$i.e. 2+5+\dots+[2k^2+2(k+1)+1] = 2^{k+1}+(k+1)^2-1$$

$$\text{Now } LHS = S_k + T_{k+1}$$

$$= 2^k + k^2 - 1 + 2 + 2(k+1) - 1$$

$$= 2 \cdot 2^k + k^2 - 1 + 2k + 2 - 1$$

$$= 2^{k+1} + k^2 + 2k$$

$$= 2^{k+1} + (k^2 + 2k + 1) - 1$$

$$= 2^{k+1} + (k+1)^2 - 1$$

\therefore True for $n=k+1$

\therefore Since true for $n=1$ and true for $n=k+1$ when true for $n=k$,
true for all $n \geq 1$ by induction

(i), $y = \frac{e^x}{e^x + 4}$

$$\frac{dy}{dx} = \frac{e^x(e^x + 4) - e^{2x}}{(e^x + 4)^2}$$

$$= \frac{4e^x}{(e^x + 4)^2}$$

$\neq 0$ as $4e^x > 0$

\therefore No stationary points

(ii) $y=f(x)$ continuous and increasing.
 \therefore always inverse

(iii) $f(x)$ always positive $\therefore y > 0$

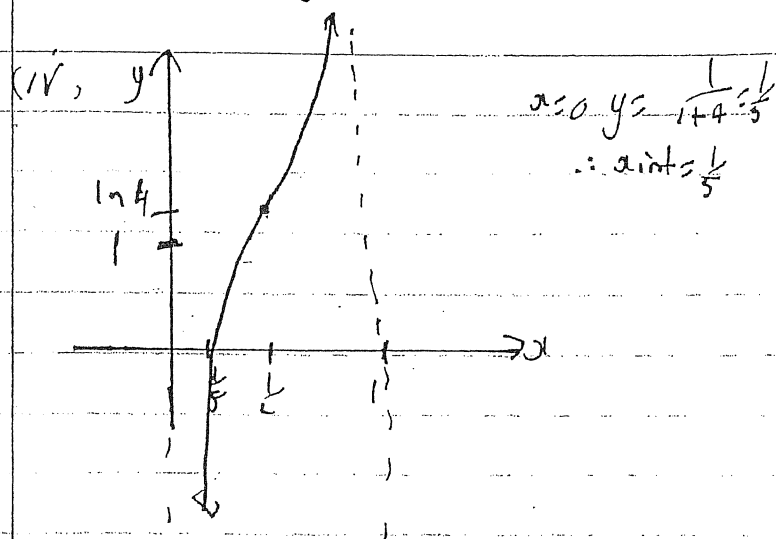
denominator $>$ numerator $\therefore y < 1$

$\therefore 0 < y < 1$

$$\text{or } \lim_{x \rightarrow \infty} \frac{e^x}{e^x + 4} = \frac{1}{1 + \frac{4}{e^x}} = 1$$

$$\lim_{x \rightarrow -\infty} \frac{e^x}{e^x + 4} = \frac{e^{-\infty}}{e^{-\infty} + 4} = \frac{0}{\frac{1}{e^x} + 4} = 0$$

$\therefore 0 < y < 1$



(v) $y = \frac{e^x}{e^x + 4}$

inverse $x = \frac{e^y}{e^y + 4}$

$$x \cdot e^y + 4x = e^y$$

$$e^y(1 - x) = 4x$$

$$e^y = \frac{4x}{1 - x}$$

$$y = \ln\left(\frac{4x}{1 - x}\right)$$

Question 6

(a) (i) $2 \cos 5\theta = 1$
 $\cos 5\theta = \frac{1}{2}$
 $\therefore 5\theta = 2n\pi \pm \frac{\pi}{3}$

$\theta = \frac{2n\pi \pm \pi}{5}$

$= (6n \pm 1) \frac{\pi}{15}$

(ii) $(6n+1) \frac{\pi}{15}$

$n=0 \rightarrow \frac{\pi}{15} \quad n=1 \rightarrow \frac{7\pi}{15}$

$(6n-1) \frac{\pi}{15}$
 $n=1 \rightarrow \frac{5\pi}{15} = \frac{\pi}{3}$

$\frac{\pi}{15}, \frac{7\pi}{15}, \frac{\pi}{3}$

b) (i) $\frac{h}{BD} = \tan 45 \quad \frac{h}{BC} = \tan 60 = \sqrt{3}$
 $\therefore BD \leq h \quad BC = \frac{h}{\sqrt{3}}$

$\therefore h^2 = BD^2 + \frac{h^2}{3} - 2 \times BD \cdot \frac{h}{\sqrt{3}} \cdot \cos 30$

$h^2 = BD^2 + \frac{h^2}{3} - \frac{2ah}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2}$

$h^2 = BD^2 + \frac{h^2}{3} - ah$

$3h^2 = 3BD^2 + h^2 - 3ah$

$2h^2 + 3ah - 3BD^2 = 0$

(ii) $2\left(\frac{h}{a}\right)^2 + 3\left(\frac{h}{a}\right) - 3 = 0$

$\frac{h}{a} = \frac{-3 \pm \sqrt{9 - 4 \times 2 \times (-3)}}{4}$

$= \frac{-3 \pm \sqrt{33}}{4}$

$\therefore \frac{h}{a} = \frac{\sqrt{33} - 3}{4}$ since both > 0

(c)

$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$

(i) $\frac{(1+x)^{n+1}}{n+1} = \binom{n}{0}x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \dots + \frac{1}{n+1}\binom{n}{n}x^{n+1} + c$

let $x < 0 \quad \therefore c < \frac{1}{n+1}$

$\therefore \frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = \binom{n}{0}x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \dots + \frac{1}{n+1}\binom{n}{n}x^{n+1}$

let $x < 1$

$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \dots + \frac{1}{n+1}\binom{n}{n}$

$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \frac{2^{n+1} - 1}{n+1}$

$\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \dots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1} - 1}{n+1}$

(ii)

$\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \dots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1} - 1}{n+1}$

let $n \in \mathbb{N}$ in original

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n \quad \text{--- (2)}$

(2) - (1)

$\binom{n}{1} + \frac{2}{3}\binom{n}{2} + \dots + \left(1 - \frac{1}{n+1}\right)\binom{n}{n} = 2^n - \frac{2^{n+1} - 1}{n+1}$

$\binom{n}{1} + \frac{2}{3}\binom{n}{2} + \dots + \frac{1}{n+1}\binom{n}{n} = \frac{2^n \times n + 2^n - 2^{n+1} + 1}{n+1}$

RHS = $\frac{n \cdot 2^n + 2^n - 2^{n+1} + 1}{n+1}$

$= \frac{2^n(n-1) + 1}{n+1}$

Question 7

(a) (i) $\angle OBT = 90^\circ$ (tangent \perp radius)

$$\therefore \cos \theta = \frac{2}{5} \text{ (right triangle)}$$

(ii) $L = r\theta$

$$\angle AOB = \pi - \cos^{-1}\left(\frac{2}{5}\right)$$

$$\therefore L = 2\left[\pi - \cos^{-1}\left(\frac{2}{5}\right)\right]$$

$$BT^2 = 5^2 - 2^2 \text{ (pythagoras)}$$

$$\therefore BT = \sqrt{21}$$

$$\therefore S = L + BT$$

$$= 2\left[\pi - \cos^{-1}\left(\frac{2}{5}\right)\right] + \sqrt{21}$$

(iii) $S = 2\pi - 2\cos^{-1}\left(\frac{2}{5}\right) + \sqrt{21}$

$$\frac{dS}{dx} = 0 - 2 \cdot \frac{-1}{\sqrt{1 - \left(\frac{2}{5}\right)^2}} \cdot -2x^{-2} + \frac{1}{2} \cdot 2x (x^2 - 4)^{-1/2}$$

$$= \frac{-4}{5^2 \sqrt{1 - \frac{4}{25}}} + \frac{x}{\sqrt{x^2 - 4}}$$

$$= \frac{-4}{25 \sqrt{\frac{21}{25}}} + \frac{x}{\sqrt{x^2 - 4}}$$

$$= \frac{-4 + 5x}{5 \sqrt{x^2 - 4}}$$

$$\frac{dS}{dx} = \frac{\sqrt{x^2 - 4}}{5}$$

(iv) $\frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt}$

$$\frac{dS}{dt} = 4 \times \frac{\sqrt{8^2 - 4}}{2}$$

$$= \frac{\sqrt{60}}{2}$$

$$= \sqrt{15}$$

(b) (i) $2n+1 \text{ women} = \binom{n}{2} \cdot \binom{n}{1}$

$$= \frac{n(n-1)}{2} \times n$$

$$= \frac{n^2(n-1)}{2}$$

(ii) $n(5) = \binom{2n}{3} = \frac{2n(2n-1)(2n-2)}{3!}$

$$= \frac{4n(2n-1)(n-1)}{6}$$

$$P(2n \text{ women}) = \frac{n^2(n-1)}{2} \cdot \frac{4n(2n-1)(n-1)}{6}$$

$$= \frac{n^2(n-1)}{2} \cdot \frac{8}{4n(2n-1)(n-1)}$$

$$= \frac{3n}{4(2n-1)} = \frac{3n}{8n-4}$$

(iii) least value for $n \geq 2$

$$p = \therefore P(2n \text{ women}) = \frac{6}{16-4} = \frac{1}{2}$$

$$\text{as } n \rightarrow \infty \quad \frac{3n}{8n-4} \rightarrow \frac{3}{8} \therefore \frac{3}{8} < p < \frac{1}{2}$$

Question 1

$$a^2 + b^2 = a^4 - 2a^3 + a^2 + b$$

(a) Factor

$$\therefore P(D) = 1 - 2 + a + b = 0$$

$$\therefore \boxed{a+b=1} \quad -①$$

(b) Factor

$$\therefore P(-1) = 1 + 2 - a + b = 0$$

$$\therefore \boxed{-a+b=-3} \quad -②$$

$$① + ② \quad 2b = -2$$

$$\therefore b = -1, a = 2$$

$$\frac{a+1}{a-1} \geq 2$$

$$\frac{(a+1)(a-1)^2}{(a-1)^2} \geq 2(a-1)^2, a \neq 1$$

$$2(a+1)^2 \leq (a+1)(a-1)$$

$$2(a+1)^2 - (a+1)(a-1) \leq 0$$

$$(a-1)[2a+2 - a-1] \leq 0$$

$$(a-1)(a+3) \leq 0$$

$$\therefore 1 < a \leq 3 \text{ since } a \neq 1$$

$$8 \text{ I } 0 \text{ L } 0 \text{ G I } 5 \text{ T}$$

$$\frac{9!}{2!2!} = 90720$$

$$(ii) \text{ I I } 0 \text{ L } 0 \text{ G I } 5 \text{ T}$$

$$\frac{8!}{2!}$$

$$(I' \text{ I } 5 \text{ I } 0 \text{ G I } 5 \text{ T}) = \frac{8!}{2!} \cdot \frac{1}{2!2!}$$

$$= \frac{8! \cdot 2!}{9!} = \frac{2}{9}$$

$$(d) \tan 45^\circ = \left| \frac{m-2}{1+m} \right|$$

$$|1+2m| = |m-2|$$

$$1+2m = m-2 \text{ or } 1+2m = -(m-2)$$

$$m = -3 \quad 3m = 1$$

$$\therefore m = -3, \frac{1}{3}$$

$$(e) x(x+12) = 8^2$$

$$x^2 + 12x - 64 = 0$$

$$(x+16)(x-4) = 0$$

$$\therefore x = 4 \text{ [} x > 0 \text{]}$$

Question 2

$$(a) \int \frac{dx}{1+4x} = \frac{1}{4} \int \frac{dx}{\frac{1}{4} + x} = \frac{1}{4} \cdot \frac{1}{\frac{1}{4}} \tan^{-1} \left(\frac{1}{4} + x \right) + c$$

$$= \frac{1}{4} \cdot \frac{1}{\frac{1}{4}} \tan^{-1} \left(\frac{1}{4} + x \right) + c$$

$$= \frac{1}{4} \tan^{-1} 2x + c$$

(b) Let roots $\alpha, \beta, \alpha-\beta$

$$\text{Sum of roots } \alpha + \beta + \alpha - \beta = -\frac{4}{4}$$

$$2\alpha = 1$$

$$\alpha = \frac{1}{2}$$

$$\text{Product of roots } \alpha\beta(\alpha-\beta) = -\frac{15}{4}$$

$$\frac{1}{2}\beta(1-\beta) = -\frac{15}{4}$$

$$\beta(1-2\beta) = -15$$

$$2\beta^2 - \beta - 15 = 0$$

$$(2\beta+5)(\beta-3) = 0$$

$$\beta = -\frac{5}{2}, 3$$

$$\therefore \text{roots } \frac{1}{2}, -\frac{5}{2}, 1-\frac{5}{2} \Rightarrow \frac{1}{2}, -\frac{5}{2}, 3$$

$$\frac{1}{2}, +3, \frac{1}{2}-3 \rightarrow \frac{1}{2}, 3, -\frac{5}{2}$$

Question 2 (ctd)

$$(c) (i) 5(a,1) \quad P(2p, p^2) \quad k:l = 1:2$$

$$R\left(\frac{p^2+2p}{2+1}, \frac{p^2+2}{2+1}\right)$$

$$\therefore R\left(\frac{2p}{3}, \frac{p^2+2}{3}\right)$$

$$(ii) \therefore a = \frac{2p}{3} \therefore p = \frac{3a}{2}$$

$$y < \frac{p^2+2}{3}$$

$$\therefore 3y = \left(\frac{3x}{2}\right)^2 + 2$$

$$3y-2 = \frac{9x^2}{4}$$

$$9x^2 = 4(3y-2)$$

$$9x^2 = 12y-8$$

$$(d) \sin \theta = \frac{2x}{1+x} \quad \cos \theta = \frac{1-x}{1+x}$$

$$2 \sin \theta + 4 \cos \theta = 3$$

$$2\left(\frac{2x}{1+x}\right) + 4\left(\frac{1-x}{1+x}\right) = 3$$

$$4x + 4 - 4x = 3 + 3x$$

$$7x = 4x - 1 = 0$$

$$x = \frac{4 \pm \sqrt{16+8}}{7} = \frac{4 \pm \sqrt{24}}{7}$$

$$\therefore \theta = 0.789 \dots, -0.188 \dots$$

$$\therefore \theta = 1.30, 5.91$$

Question 3

(a) Exterior angle of cyclic quad ($< CDE$) = opposite interior angle ($< ABC$)

$$(ii) < ABC < < AOC \text{ (equal angles subtend } \Delta ACB)$$

$$< BDC < < BAC \text{ (angles in same segment)}$$

$$\therefore < BDC < < ABC$$

$$\text{but } < CDE < < AOC \text{ from (i),}$$

$$\therefore < BDC < < CDE$$

$$\therefore DC \text{ bisects } < BDE$$

$$(b) \lim_{x \rightarrow 0} \frac{\sin 2x}{3x} = \frac{2}{3} \lim_{x \rightarrow 0} \frac{\cos 2x}{3} = \frac{2}{3} \times 1 = \frac{2}{3}$$

$$(c) \int_0^{\ln 2} e^{2x} \sqrt{e^{2x}-1} dx$$

$$\text{Let } u = e^{2x} - 1$$

$$u = 0 \quad u \leq e^{2x} - 1 < 0$$

$$u = \ln 2 \quad u = e^{2 \ln 2} - 1 = 2 - 1 = 1$$

$$\frac{du}{dx} = e^{2x}$$

$$\therefore du = e^{2x} dx$$

$$\therefore \int_0^{\ln 2} e^{2x} \sqrt{e^{2x}-1} \cdot e^{2x} dx$$

$$= \int_0^1 (u+1) \sqrt{u} \cdot du$$

$$= \int_0^1 (u^{\frac{3}{2}} + u^{\frac{1}{2}}) du$$

$$= \left[\frac{2}{5} u^{\frac{5}{2}} + \frac{2}{3} u^{\frac{3}{2}} \right]_0^1$$

$$= \frac{2}{5} + \frac{2}{3}$$

$$= \frac{16}{15}$$

Question 6

(a) (i) $2 \cos 5\theta = 1$

$\cos 5\theta = \frac{1}{2}$

$\therefore 5\theta = 2n\pi \pm \frac{\pi}{3}$

$\theta = \frac{2n\pi \pm \frac{\pi}{3}}{5}$

$= (6n \pm 1) \frac{\pi}{15}$

(ii) $(6n+1) \frac{\pi}{15}$

$n: 0 \Rightarrow \frac{\pi}{15} \quad n: 1 \Rightarrow \frac{7\pi}{15}$

$(6n-1) \frac{\pi}{15}$

$n: 1 \Rightarrow \frac{5\pi}{15} = \frac{\pi}{3}$

$\frac{\pi}{15}, \frac{7\pi}{15}, \frac{\pi}{3}$

(b) (i) $\frac{h}{BD} = \tan 45^\circ \quad \frac{h}{BC} = \tan 60^\circ = \sqrt{3}$

$\therefore BD < h \quad BC < \frac{h}{\sqrt{3}}$

$\therefore h < BD + \frac{h}{\sqrt{3}} - 2 \times 0.1 \cdot \frac{h}{\sqrt{3}} \cdot \cos 30^\circ$

$h < 0.1 + \frac{h}{\sqrt{3}} - \frac{2 \times 0.1 \cdot h}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2}$

$h < 0.1 + \frac{h}{\sqrt{3}} - 0.1h$

$3h < 3 \times 0.1 + h - 3 \times 0.1h$

$2h + 3 \times 0.1h - 3 \times 0.1 = 0$

(ii) $2 \left(\frac{h}{\sqrt{3}} \right)^2 + 3 \left(\frac{h}{\sqrt{3}} \right) - 3 = 0$

$h = \frac{-3 \pm \sqrt{9 - 4 \times 2 \times 3}}{4}$

$= \frac{-3 \pm \sqrt{3}}{4}$

$\therefore h = \frac{\sqrt{3}-3}{4} \quad \text{Since } h > 0$

(c)

$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$

(i) $\frac{(1+x)^{n+1}}{n+1} = \binom{n+1}{0}x + \binom{n+1}{1}x^2 + \binom{n+1}{2}x^3 + \dots + \frac{1}{n+1} \binom{n+1}{n}x^n + c$

Let $x < 0 \quad \therefore c < \frac{1}{n+1}$

$\therefore \frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = \binom{n+1}{0}x + \binom{n+1}{1}x^2 + \binom{n+1}{2}x^3 + \dots + \frac{1}{n+1} \binom{n+1}{n}x^n$

Let $x < 1$

$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \binom{n+1}{0}x + \binom{n+1}{1}x^2 + \binom{n+1}{2}x^3 + \dots + \frac{1}{n+1} \binom{n+1}{n}x^n$

$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \frac{2^{n+1}-1}{n+1}$

$\binom{n+1}{0} + \binom{n+1}{1}x + \binom{n+1}{2}x^2 + \dots + \frac{1}{n+1} \binom{n+1}{n}x^n = \frac{2^{n+1}-1}{n+1}$

(ii)

$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \frac{1}{n+1} \binom{n}{n}x^n = \frac{2^{n+1}-1}{n+1}$

Let $x=1$ in original

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n - 1$

(i)

$\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n} = 2^n - 2$

$\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n} = \frac{2^{n+1}-2}{n+1}$

$RHS = \frac{n \cdot 2^n + 2^n - 2 \cdot 2^n + 1}{n+1}$

$= \frac{2^n(n-1)+1}{n+1}$

Question 7

(a) (i) $\angle OBT = 90^\circ$ (Tangent is radius)

$\therefore \cos \theta = \frac{2}{\sqrt{13}}$ (right triangle)

(ii) $L = r\theta$

$\leq AB \leq \pi - \cos^{-1} \left(\frac{2}{\sqrt{13}} \right)$

$\therefore L = 2 \left[\pi - \cos^{-1} \left(\frac{2}{\sqrt{13}} \right) \right]$

$\theta \neq 90^\circ \quad x^2 - 2^2 = (y^2 + 4 \cos^2 \theta)$

$\therefore BT = \sqrt{13-4}$

$\therefore S = L + BT$

$= 2 \left[\pi - \cos^{-1} \left(\frac{2}{\sqrt{13}} \right) \right] + \sqrt{13-4}$

(iii) $S = 2\pi - 2 \cos^{-1} \left(\frac{2}{\sqrt{13}} \right) + \sqrt{13-4}$

$\frac{dS}{dx} = 0 - 2 \cdot \frac{-1}{\sqrt{1-(\frac{2}{\sqrt{13}})^2}} \cdot -2x^{-2} + \frac{1}{2} \cdot 2x \cdot (2x-4)^{-1/2}$

$= \frac{-4}{2x^2 \sqrt{1-\frac{4}{13}}} + \frac{x}{\sqrt{2x-4}}$

$= \frac{-4}{2x^2 \sqrt{\frac{9}{13}}} + \frac{x}{\sqrt{2x-4}}$

$= \frac{-4}{2x^2 \sqrt{\frac{9}{13}}} + \frac{x}{\sqrt{2x-4}}$

$= \frac{-4}{2x^2 \sqrt{\frac{9}{13}}} + \frac{x}{\sqrt{2x-4}}$

$= \frac{-4}{2x^2 \sqrt{\frac{9}{13}}} + \frac{x}{\sqrt{2x-4}}$

$\frac{dS}{dx} = \frac{\sqrt{2x-4}}{2x}$

(iv) $\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$\frac{dy}{dx} = \frac{4 \times \sqrt{8x-4}}{8}$

$= \frac{\sqrt{8x-4}}{2}$

$= \sqrt{5}$

(b) (i) $2n(n+1) \text{ ways} = \binom{n}{2} \cdot \binom{n}{1}$

$= \frac{n(n-1)}{2} \times n$

$= \frac{n^2(n-1)}{2}$

(ii) $n(5) = \binom{2n}{3} = \frac{2n(2n-1)(2n-2)}{3!}$

$= \frac{4n(2n-1)(n-1)}{6}$

$P(2nm | nm) = \frac{n!(n-1)}{2} \cdot \frac{1}{4n(2n-1)}$

$= \frac{n!(n-1)}{2} \cdot \frac{1}{4n(2n-1)}$

$= \frac{3n}{4(2n-1)} = \frac{3n}{8n-4}$

$= \frac{3n}{4(2n-1)} = \frac{3n}{8n-4}$

(iii) Least value for $n=2$

$P = P(2nm | nm) = \frac{6}{16-4} = \frac{1}{2}$

$\text{As } n \rightarrow \infty \quad \frac{3n}{8n-4} \rightarrow \frac{3}{8} < \frac{1}{2}$