2001 INDEPENDENT TRIALS: MATHEMATICS EXTENSION 1
SAMPLE SOLUTIONS

Question 1:

a.
$$x = \frac{kx_2 + lx_1}{k + l}$$
$$6 = \frac{k \times 3 + l \times -1}{k + l}$$
$$6k + 6l = 3k - l$$
$$3k = -7l$$
$$k:l = -7:3$$

i.e. C divides AB externally in the ratio 7:3

b. Critical points: $x = 1$ and $x - 1 = \dfrac{1}{x - 1}$

Solving: $(x - 1)^2 = 1$
$$x - 1 = \pm 1$$
$$\therefore x = 0,\ 2$$

Testing regions $x < 0$, $0 < x < 1$, $1 < x < 2$ and $x > 2$ gives solutions

$$x < 0 \text{ and } 1 < x \leq 2$$

c. i. $P(1) = 1^3 - 2 \times 1^2 - 1 + 2 = 0$. Hence $x - 1$ is a factor

ii. $P(x) = x^2(x - 2) - (x - 2) = (x - 2)(x^2 - 1) = (x - 2)(x - 1)(x + 1)$

d. i. Book work

ii.
$$\frac{1 - \dfrac{1 - t^2}{1 + t^2}}{\dfrac{2t}{1 + t^2}} = \frac{1 + t^2 - 1 + t^2}{2t}$$
$$= t$$
$$= \tan\frac{\theta}{2}$$

iii. $\tan 15° = \dfrac{1 - \cos 30°}{\sin 30°} = \dfrac{1 - \dfrac{\sqrt{3}}{2}}{\dfrac{1}{2}} = 2 - \sqrt{3}$

Question 2:

a. i. $\dfrac{dy}{dx} = \dfrac{dy/dx}{dx/dt} = \dfrac{4t}{4} = t$; therefore, $m = 3$

ii. Focus $(0, 2)$ and point $(12, 18)$; therefore $m = \dfrac{4}{3}$

Question 2 (continued)

iii. $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\dfrac{3 - \frac{4}{3}}{1 + 3 \times \frac{4}{3}}\right| = \dfrac{1}{3}$

$\therefore\ \theta = 18°26'$

b. $x' = x - \dfrac{f(x)}{f'(x)} = 7 - \dfrac{7\ln 7 - 2 \times 7}{\ln 7 - 1} = 6.5997199..$ so $x = 6.6$

c. i. $(n - 1)! = 5! = 120$

ii. Counting the couple as one, $4! \times 2! = 48$

iii. There are 48 ways they can sit together so there are $120 - 48 = 72$ ways to sit apart

$P(\text{sit apart}) = 72/120 = 3/5$

d. $\angle APC = \angle PDC$ (angles between tangent and chord equals angle in the alt. segment)

$\angle PDC = \angle PCD$ (base angles in isosceles triangle are equal)

$\therefore \angle APC = \angle PCD$ and AB $\parallel$ CD (if alternate angles are equal, lines are parallel)

Question 3

a. Let $p =$ probability of scoring a goal $= .7$

Let $q =$ probability of missing $= .3$

Let $n = 10$ and $r =$ number of goals scored

Then $P(X = r) = \dbinom{n}{r} p^r q^{n-r}$ and

$$P(X \geq 8) = P(X = 8 \text{ or } X = 9 \text{ or } X = 10)$$
$$= \binom{10}{8} 0.7^8 \times 0.3^2 + \binom{10}{9} 0.7^9 \times 0.3 + \binom{10}{10} 0.7^{10}$$
$$= 0.382827864 = 0.38$$

b. Let P(x, y) be a point on the circle. Then $\angle APB = 90°$ (angle in a semicircle is a rt angle)

Hence AP $\perp$ PB and $m_{AP} m_{PB} = -1$

$$\therefore\ \frac{y - y_1}{x - x_1} \times \frac{y - y_2}{x - x_2} = -1$$

whence $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

c. Let $f(n) = 2^{3n} - 3^n$

Then $f(1) = 2^3 - 3 = 5$ which is divisible by 5

Assume that $f(k) = 2^{3k} - 3^k$ is divisible by 5 for k a positive integer, and show that $f(k + 1)$ is therefore also divisible by 5

Question 3 (continued)

$$\begin{aligned}\text{Then } f(k+1) &= 2^{3(k+1)} - 3^{k+1}\\ &= 2^{3k} \times 2^3 - 3^k \times 3\\ &= 8 \times 2^{3k} - 3 \times 3^k\\ &= 5 \times 2^{3k} + 3 \times 2^{3k} - 3 \times 3^k\\ &= 5 \times 2^{3k} + 3 \times (2^{3k} - 3^k)\end{aligned}$$

The first term is clearly divisible by 5 and $2^{3k} - 3^k$ is also divisible by 5 by our assumption above. Therefore $f(k+1)$ is divisible by 5 if $f(k)$ is divisible by 5

But $f(1)$ is divisible by 5, so $f(2)$ is divisible by 5 and so on for all positive integers n.

d.
$$\begin{aligned}V &= \pi\int_0^{\frac{\pi}{6}} \cos^2 2x\,dx\\ &= \pi \times \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{6}}\\ &= \frac{\pi}{2} \times \left[\left(\frac{\pi}{6} + \frac{1}{4}\sin\frac{2\pi}{3}\right) - (0 - 0)\right]\\ &= \frac{\pi}{2}\left[\frac{\pi}{6} + \frac{\sqrt{3}}{8}\right]\end{aligned}$$

Question 4

a.
$$\begin{aligned}\binom{n}{r} &= \binom{n}{r+1}\\ \therefore \frac{n!}{r!(n-r)!} &= \frac{n!}{(r+1)!(n-r-1)!}\\ \frac{(n-r-1)!}{(n-r)!} &= \frac{r!}{(r+1)!}\\ \frac{1}{n-r} &= \frac{1}{r+1}\\ \therefore r+1 &= n-r\\ n &= 2r+1\end{aligned}$$

and since r is a positive integer, n is odd

b. i. $x^2 + 6x + 13 = x^2 + 6x + 9 + 4 = (x+3)^2 + 4$

ii. $u = x + 3 \Rightarrow du = dx$ so
$$\begin{aligned}\int \frac{dx}{x^2 + 6x + 13} &= \int \frac{dx}{(x+3)^2 + 4}\\ &= \int \frac{du}{u^2 + 4}\\ &= \frac{1}{2}\tan^{-1}\frac{u}{2} + C\\ &= \frac{1}{2}\tan^{-1}\frac{(x+3)}{2} + C\end{aligned}$$

Question 4 (continued)

c. Let $\alpha = \cos^{-1}\left(\frac{4}{5}\right)$ and $\beta = \cos^{-1}\left(\frac{3}{5}\right)$; then $\cos\alpha = \frac{4}{5}$ and $\cos\beta = \frac{3}{5}$

Therefore, $\sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{4}{5}$

Consider
$$\begin{aligned}\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \frac{4}{5} \times \frac{3}{5} - \frac{3}{5} \times \frac{4}{5} \\ &= 0\end{aligned}$$
$$\therefore \cos(\alpha + \beta) = 0$$
$$\alpha + \beta = \frac{\pi}{2}$$
i.e. $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$

d.
$$\begin{aligned}y &= \tfrac{1}{2}(e^x - e^{-x}) \\ 2y &= e^x - \frac{1}{e^x} \\ 2ye^x &= e^{2x} - 1 \\ 0 &= e^{2x} - 2ye^x - 1 \\ \therefore e^x &= \frac{2y \pm \sqrt{4y^2 + 4}}{2} \\ &= \frac{2y \pm 2\sqrt{y^2 + 1}}{2} \\ &= y \pm \sqrt{y^2 + 1}\end{aligned}$$
but $e^x > 0$ and $\sqrt{y^2 + 1} > y$

$\therefore e^x = y + \sqrt{y^2 + 1}$

so $x = \ln(y + \sqrt{y^2 + 1})$

Question 5

a. i. Now $\ddot{x} = \frac{d}{dx}\left[\tfrac{1}{2}v^2\right]$ and $\tfrac{1}{2}v^2 = 6 + 2x - \tfrac{1}{2}x^2$

Therefore $\ddot{x} = 2 - x = -1(x - 2)$ so the motion is Simple Harmonic

ii. Centre of motion is 2 (where $\ddot{x} = 0$) and $n = 1$ so period $T = \frac{2\pi}{n} = 2\pi$

Extremes of motion occur when $v = 0$ i.e. when $6 + 2x - \tfrac{1}{2}x^2 = 0 \Rightarrow x = -2, 6$ so the amplitude is 4.

iii. Now $a = 4$, $n = 1$ and the centre of motion, $b = 2$ so $x = 4\sin(t + \theta) + 2$

Further when $t = 0$, $x = 6$ so $6 = 4\sin\theta + 2 \Rightarrow \theta = \frac{\pi}{2}$

$\therefore x = 4\sin(t + \frac{\pi}{2}) + 2$

Question 5 (continued)

b. i. Newton's Law is $\frac{dT}{dt} = k(T - P)$

If $T = P + Ae^{kt}$ then $\frac{dT}{dt} = k \times Ae^{kt} = k(T - P)$

ii. $100 = 23 + Ae^{0} \Rightarrow A = 77$

and $93 = 23 + 77e^{k \times 2} \Rightarrow e^{2k} = \frac{70}{77}$

$\therefore k = \frac{1}{2}\ln\frac{70}{77} = -0.0476550899 = -0.0477$

iii. $80 = 23 + 77 \times e^{-0.0477 \times t} \Rightarrow t = \frac{\ln\frac{57}{77}}{-0.0477} = 6.31106047 \approx 6$ minutes

Question 6

a. i. In the x direction:

$$\ddot{x} = 0 \Rightarrow \dot{x} = \int 0\,dt = C_1$$

When $t = 0,\ \dot{x} = V \Rightarrow C_1 = V$

$$\therefore \dot{x} = V$$

$$x = \int V\,dt = Vt + C_2$$

When $t = 0,\ x = 0 \Rightarrow C_2 = 0$

$$\therefore x = Vt$$

In the y direction:

$$\ddot{y} = -g \Rightarrow \dot{y} = \int -g\,dt = -gt + C_3$$

When $t = 0,\ \dot{y} = 0 \Rightarrow C_3 = 0$

$$\therefore \dot{y} = -gt$$

$$y = \int -gt\,dt = -\tfrac{1}{2}gt^2 + C_4$$

When $t = 0,\ y = h \Rightarrow C_4 = h$

$$\therefore y = -\tfrac{1}{2}gt^2 + h$$

ii. $x = Vt \Rightarrow t = \frac{x}{V}$. Substitute into $y = -\tfrac{1}{2}gt^2 + h$

$$y = -\tfrac{1}{2}g \times \left(\frac{x}{V}\right)^2 + h$$

$$= \frac{-gx^2}{2V^2} + h$$

$$= \frac{-gx^2 + 2V^2h}{2V^2}$$

iii. We require $y = 0$ thus $\frac{-gx^2 + 2V^2h}{2V^2} = 0 \Rightarrow x^2 = \frac{2V^2h}{g} \Rightarrow x = \pm\sqrt{\frac{2V^2h}{g}}$

But the particle is moving in a positive direction so $x = V\sqrt{\frac{2h}{g}}$

Question 6 (continued)

b. i. $\frac{d}{dx}\left[\tfrac{1}{2}v^2\right] = 10x - 2x^3$

$$\therefore \tfrac{1}{2}v^2 = \int 10x - 2x^3 dx = 5x^2 - \frac{x^4}{2} + C$$

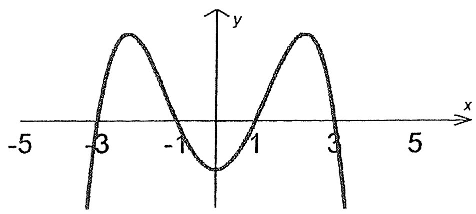
$$v^2 = 10x^2 - x^4 + K \text{ and when } v = 0,\ x = -1 \Rightarrow K = -9$$

$$v^2 = 10x^2 - x^4 - 9$$

$$v = \pm\sqrt{10x^2 - x^4 - 9}$$

ii. $v^2 = -(x^4 - 10x^2 + 9) = -(x^2 - 1)(x^2 - 9) = -(x - 1)(x + 1)(x - 3)(x + 3)$

Hence $v = 0$ when $x = -3, -1, 1, 3$
From graph, between $x = -1$ and $x = 1$, $v^2 < 0$
so the motion cannot exist between $x = -1$ and $x = 1$



iii. If $x = 0$, then acceleration is zero. Since $v = 0$, the particle would remain stationary.

Question 7

a. Now $PT^2 = AP \times BP$ (On a circle, the square of the length of the tangent from an external point equals the product of the intercepts of the secant through the point)

Therefore $x^2 = a \times (a + 6) \Rightarrow x = \sqrt{a(a + 6)}$

b. i. Now $\tan\alpha = \frac{x}{a + 6}$, $\tan\beta = \frac{x}{a}$, $\theta = \beta - \alpha$

$$\therefore \tan\theta = \tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$$

$$= \frac{\frac{x}{a} - \frac{x}{a+6}}{1 + \frac{x}{a} \times \frac{x}{a+6}} = \frac{(a + 6)x - ax}{a(a + 6) + x^2} = \frac{6x}{a^2 + 6a + x^2}$$

ii. $\frac{dT}{dx} = \frac{(a^2 + 6a + x^2) \times 6 - 6x(2x)}{(a^2 + 6a + x^2)^2} = \frac{6a^2 + 36a - 6x^2}{(a^2 + 6a + x^2)^2} = 0$ when $x = \sqrt{a(a + 6)}$

When $x < \sqrt{a(a + 6)}$, $\frac{dT}{dx} > 0$; when $x > \sqrt{a(a + 6)}$, $\frac{dT}{dx} < 0$; therefore this is a max.

iii. $T = \tan\theta = \frac{6\sqrt{a(a + 6)}}{a^2 + 6a + (a^2 + 6a)} = \frac{3}{\sqrt{a^2 + 6a}} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{\sqrt{a^2 + 6a}}\right)$

iv. $x = 12.64911064 \approx 12.65$ m and $\theta = 13°20'33'' = 13°21'$

v. The maximum value of θ occurs when $x = \sqrt{a(a + 6)}$. Using the result from part a., we see that, because the square of the tangent equals the product of the intercepts of the secant, the goal posts and the point P from which the kick is taken lie on a circle, with PT a tangent.

NSW INDEPENDENT TRIAL EXAMS –2001 MAPPING GRID

for Mathematics Extension 1

Q'n	Marks	Syllabus Area	Outcome	Draft Perf. Band
1a	3	Linear Functions and Lines	PE3	E2-E3
1b	3	Basic Arithmetic and Algebra	PE3	E2-E4
1c	2	Polynomials	PE3	E2-E3
1d	4	Further Trigonometry	PE2	E2-E3
2a.i,ii	2	Parametric Representation	PE4	E2-E3
2a.iii	2	Linear Functions and Lines	PE3	E2-E3
2b	3	Iterative methods	HE3	E2-E4
2c.i,ii	2	Permutations and Combinations	PE3	E2-E3
2c.iii	1	Probability	H5	E2-E3
2d	2	Circle Geometry	PE3	E2-E3
3a	2	Further Probability	HE3	E2-E3
3b	2	Linear Functions and Lines:Harder applications	H5	E3-E4
3c	4	Induction	HE2	E2-E4
3d	4	Primitive of $\cos^2 x$; Harder applications	H8; HE6	E2-E4
4a	3	Binomial Theorem	HE7!?	E3-E4
4b.i	1	Basic Arithmetic and Algebra	H3	E2-E3
4b.ii	2	Methods of Integration and Inverse Functions	HE4; HE6	E2-E3
4c	3	Inverse Trigonometric Functions	HE4	E2-E4
4d	3	Harder applications	H3	E3-E4
5a	6	Simple Harmonic Motion	HE3	E2-E4
5b	6	Equation $\frac{dN}{dt} = k(N - P)$	HE3	E2-E4
6a	6	Projectile Motion	HE3	E2-E4
6b	6	Velocity and acceleration as a function of x	HE5; HE7	E3-E4
7a	2	Circle Geometry	PE3	E2-E3
7b.i	3	Trigonometry and Further Trigonometry	HE1; HE7	E2-E4
7b.ii,iii,iv	6	Harder applications	H5; HE1; HE4	E2-E4
7b.v	1	Circle Geometry	HE1	E3-E4