SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

1998

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Sample Solutions

① a) let $x+2=0$

$x=-2$

So $P(-2) = -16+6+2a = 0$

$a = 5$ ②

(b) $T_{n+1} - T_n = 7$

$T_2 - T_1 = 7$ $\quad T_1 = 3$

$T_2 - 3 = 7 \Rightarrow T_2 = 10$

$T_3 - T_2 = 7$

$T_3 - 10 = 7 \Rightarrow T_3 = 17$

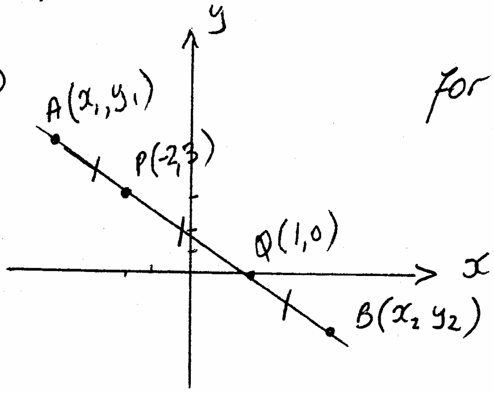
So $\{3+10+17+\dots\}$

$a=3$

$d=7$ $\quad S_{100} = \frac{100}{2}\{6+99\times 7\} = 34950$ ②

$n=100$

(c)



for AQ, midpoint P.

$\frac{x_1+1}{2} = -2 \Rightarrow x_1 = -5$

$\frac{y_1+0}{2} = 3 \Rightarrow y_1 = 6$

$A(-5,6)$.

for PB, midpoint Q

$\frac{-2+x_2}{2} = 1 \Rightarrow x_2 = 4$ ③

$\frac{3+y_2}{2} = 0 \Rightarrow y_2 = -3$

$B(4,-3)$.

(d) $x-y=2$ $\quad m_1 = 1$

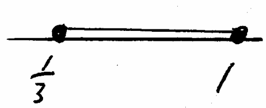
$2x+y=1$ $\quad m_2 = -2$

$\tan\theta = \left|\frac{1+2}{1-2}\right|$ $\quad \theta \doteqdot 72^\circ$ ②

(e) $|2x-1| \le |x|$

$2x-1 \le x$ or $2x-1 \ge -x$

$x \le 1$ or $x \ge \frac{1}{3}$.



③

a) 1 (i) $\int \frac{dx}{4+x^2} = \frac{1}{2}\tan^{-1}\frac{x}{2} + C$

3 (ii) $\int_0^{\pi/2} \cos^2 \frac{t}{2}\, dt = \int_0^{\pi/2} \frac{1}{2}(\cos t + 1)\, dt$

$= \frac{1}{2}\Big[\sin t + t\Big]_0^{\pi/2}$

$= \frac{1}{2}\Big[\left(\sin\frac{\pi}{2} + \frac{\pi}{2}\right) - (0+0)\Big]$

$= \frac{\pi}{4} + \frac{1}{2}$

$\frac{1}{2}(\cos t + 1)$

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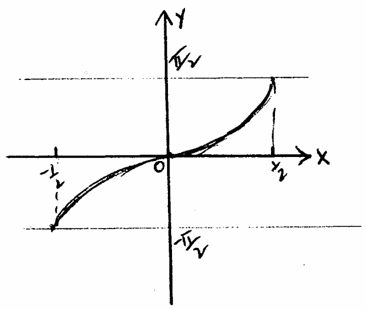
b) 1 (i) $f(x) = \sin^{-1} 2x$

domain: $-1 \le 2x \le 1$

ie $-\frac{1}{2} \le x \le \frac{1}{2}$

range: $-\frac{\pi}{2} \le \sin^{-1} 2x \le \frac{\pi}{2}$

1 (ii)



1 (iii) $f'(x) = \frac{1}{\sqrt{1-4x^2}} \cdot 2$ $\quad \left[\frac{1}{\sqrt{\frac{1}{4} - x^2}}\right]$

$f'\left(\frac{1}{4}\right) = \frac{2}{\frac{\sqrt{3}}{2}} = \frac{4}{\sqrt{3}}$ or $\frac{4\sqrt{3}}{3}$

(c) $-\pi < x < \frac{\pi}{4}$

3 $1 + \cos 2x = \sqrt{3}\sin 2x$

$\cos 2x - \sqrt{3}\sin 2x = -1$ —— (A)

$R\cos(2x + \alpha) = -1$

$\cos\alpha = 1$ and $\sin\alpha = \sqrt{3}$

$\tan\alpha = \sqrt{3} \Rightarrow \boxed{\alpha = \frac{\pi}{3}}$

$R = \sqrt{1^2 + (\sqrt{3})^2} = 2$

∴ (A) becomes

$2\cos\left(2x + \frac{\pi}{3}\right) = -1$ for $-\frac{5\pi}{3} < 2x + \frac{\pi}{3} < \frac{5\pi}{6}$

$\cos\left(2x + \frac{\pi}{3}\right) = -\frac{1}{2}$

$2x + \frac{\pi}{3} = \frac{2\pi}{3}, -\frac{2\pi}{3}, -\frac{4\pi}{3}$

$\Rightarrow x = \frac{\pi}{6}, -\frac{\pi}{2}, -\frac{5\pi}{6}$

(d) 2 $u = \tan x$, $du = \sec^2 x\, dx$

When $x = 0$, $u = 0$

When $x = \pi/4$, $u = \tan \pi/4 = 1$.

$\therefore I = \int_0^1 e^u\, du$

$= \left[e^u\right]_0^1$

$= e^1 - e^0$

$= e - 1$

(a)(i) Curves have a simultaneous solution if they meet.

$\therefore y = \dfrac{1}{\sqrt{1-\frac{4y^2}{9}}}$ or $\dfrac{9x^2}{4} = \dfrac{1}{1-x^2}$

$9y^2 - 4y^4 - 9 = 0$ $\qquad$ $9x^2 - 9x^4 - 4 = 0$

$\Delta = 81 - 144$ $\qquad$ $\Delta = 81 - 144$

< 0 $\qquad$ < 0

$\therefore$ No real solutions for y^2 (or x^2), so no real solution for y (or x)

$\therefore$ No point of intersection.

(ii) $\text{Area} = \int_0^{1/2} \left\{ \dfrac{1}{\sqrt{1-x^2}} - \dfrac{3x}{2} \right\} dx$

$= \left[\sin^{-1} x - \dfrac{3x^2}{2} \right]_0^{1/2}$

$= \dfrac{\pi}{6} - \dfrac{3}{16}$

(b) $\dfrac{dV}{dt} = 72\ \text{mm}^3/\text{s}$ $\qquad$ $V = \frac{4}{3}\pi r^3$ $\qquad$ $A = 4\pi r^2$

$\dfrac{dV}{dr} = 4\pi r^2$ $\qquad$ $\dfrac{dA}{dr} = 8\pi r$

$\dfrac{dA}{dt} = \dfrac{dA}{dr} \cdot \dfrac{dr}{dV} \cdot \dfrac{dV}{dt}$

$= \dfrac{8\pi r}{4\pi r^2} \cdot 72$

When $r = 12$, $\dfrac{dA}{dt} = 12\ \text{mm}^2/\text{s}$.

(c) $(3x-2)^{100} = a_{100}x^{100} + a_{99}x^{99} + a_{98}x^{98} + \cdots + a_1 x + a_0$

When $x = 1$

$\text{LHS} = (3-2)^{100}$

$= 1$

$\text{RHS} = a_{100} + a_{99} + a_{98} + \cdots + a_1 + a_0$

$\therefore a_{100} + a_{99} + \cdots a_0 = 1$.

(d) $D_x\left(x^2 \cos^{-1} x\right) = 2x\cos^{-1} x - \dfrac{x^2}{\sqrt{1-x^2}}$

Q 4

(a) $TC^2 = TB \cdot TA$

$= 4.5 \times 2$

$= 9$

$\therefore \quad TC = 3$ ——— 2

(b) To find coeff. of y^{10}, we need:

(i) coefficient of y^9 times 1

(ii) coefficient of y^{10} in $(3y^2-2)^7$ times 1.

Now, $(3y^2-2)^7$

$T_{r+1} = \binom{7}{r}(3y^2)^{7-r}(-2)^r$

$= \binom{7}{r} 3^{7-r} \cdot (-2)^r \cdot y^{14-2r}$.

(a) $14-2r = 9 \Rightarrow r = \frac{5}{2}$ (not possible)

(b) $\therefore 14-2r = 10 \Rightarrow 2r = 4$ i.e $r = 2$.

$\therefore$ coefficient of y^{10}. ——— 3

$\binom{7}{2} 3^5 (-2)^2 = 20412$.

(c). (i) $y = x^2/4$, $\frac{dy}{dx} = x/2$, $\left.\frac{dy}{dx}\right|_{x=-2t} = -t$

$\therefore$ equation of tg t, $y - t^2 = -t(x+2t)$.

$\Rightarrow \quad tx + y + t^2 = 0$. ——— 2

When $y = 0$ $x = -t$. $\Rightarrow$ A $(-t, 0)$.

T $(-2t, t^2)$

$M = \left(\frac{-3t}{2}, \frac{t^2}{2}\right)$. $x = \frac{-3t}{2}$ $\therefore t = -\frac{2x}{3}$.

$y = \frac{1}{2}\left(\frac{4x^2}{9}\right) \Rightarrow$ locus $x^2 = \frac{9y}{2}$ ——— 3

(d). $\lim_{x\to 0} \frac{5x(1-2\sin^2 x)}{\sin x} = 5\lim_{x\to 0}\frac{1}{\left(\frac{\sin x}{x}\right)} - 10\lim_{x\to 0} x\sin x$

$= 5$ ——— 2

tion 5

) (i) $x = a\cos(nt+\alpha)$

$a = 8$

$\frac{2\pi}{n} = 6$

$n = \frac{\pi}{3}$ 1

$x = 8\cos(\frac{\pi}{3}t+\alpha)$ cm

(ii) $v^2 = n^2(a^2-x^2)$

Max velocity when $x=0$

$\therefore v^2 = (\frac{\pi}{3})^2 . 64$ 1

$\therefore v = \pm\frac{8\pi}{3}$ cm/s

(iii) $\ddot{x} = -n^2x$

Max acceleration when $x = \pm 8$

$\ddot{x} = -(\frac{\pi}{3})^2 . \pm 8$ 1

$= \mp\frac{8\pi^2}{9}$ cm/s^2

(iv) $v^2 = n^2(a^2-x^2)$

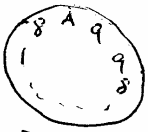
$v^2 = (\frac{\pi}{3})^2(8^2-4^2)$

$v^2 = \frac{\pi^2}{9} . 48$ 1

$\therefore v = \pm\frac{4\pi\sqrt{3}}{3}$ cm/s

(b) (i) $11!$ 1

(ii) $8! \times 4!$ 1

(iii) 

$72 \times 9!$

2

$\therefore$ Probability is $\frac{72\times 9!}{11!}$

$= \frac{72}{110}$

$= \frac{36}{55}$

(c) $\ddot{x} = 0$ $\ddot{y} = -10$

$\dot{x} = V\cos\theta$ $\dot{y} = -10t + V\sin\theta$

$x = V\cos\theta t$ 1 $y = -5t^2 + V\sin\theta t$ 1

$V\cos\theta = \frac{x}{t}$ $V\sin\theta = \frac{y+5t^2}{t}$.

When $t=5$: $V\cos\theta = 12$ $V\sin\theta = \frac{57.5+125}{5}$

$= 36.5$. 1

$\therefore V^2 = 12^2 + 36.5^2$

$= 1476.25$

$\therefore V = 38.42199\ldots$ 1

≈ 38.4 m/s.

(unfortunately this is on the downward flight not upward)

39916800

967680

26127360

6 (a) $P\left(\begin{matrix}\text{multiple of 3 in} \\ \text{a single throw}\end{matrix}\right) = \frac{2}{6} = \frac{1}{3}$

$P(\text{not a multiple of 3}) = \frac{2}{3}$

$P\left(\begin{matrix}\text{no multiples of 3} \\ \text{in } n \text{ tosses}\end{matrix}\right) = \left(\frac{2}{3}\right)^n$

We need $\left(\frac{2}{3}\right)^n < 0\cdot05$

$$n > \frac{\log(0\cdot05)}{\log\left(\frac{2}{3}\right)}$$ 2

$n > 7\cdot38$

$\therefore$ 8 tosses are needed.

(i) $\frac{dT}{dt} = -k(T-S)$ $\qquad \frac{dT}{dt} = -kAe^{-kt}$

$T = S + Ae^{-kt}$ $\qquad \frac{dT}{dt} = -k(T-S)$

(ii) $T = S + Ae^{-kt}$

$S = 30$

When $t = 0$, $T = 1390$

$1390 = 30 + A$

$A = 1360$

$T = 30 + 1360\, e^{-kt}$ 1

When $t = 10$ $\quad 1060 = 30 + 1360\, e^{-10k}$

$T = 1060$

$\frac{1030}{1360} = e^{-10k}$

$k = -\frac{1}{10} \ln\left(\frac{1030}{1360}\right)$ 2

$k = 0.02779\ldots$

When $T = 110$ $\qquad t = ?$

$110 = 30 + 1360\, e^{-kt}$

$\frac{80}{1360} = e^{-kt}$

$t = -\frac{1}{k} \ln\left(\frac{80}{1360}\right)$

$t = 101.9$ $\qquad$ 4

92 mins later.

(c) (i) RTP $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$$RHS = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= v \times \frac{dv}{dx}$$ since v is a function of x.

$$= \frac{dx}{dt} \times \frac{dv}{dx}$$

2

$$= \frac{dv}{dt} = \ddot{x} = LHS$$

(ii) $$v^2 = 24 - 6x - 3x^2$$

$$\frac{1}{2}v^2 = 12 - 3x - \frac{3}{2}x^2$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -3 - 3x$$ /

Greatest displacement when $v = 0$

$$24 - 6x - 3x^2 = 0$$

$$x^2 + 2x - 8 = 0$$ /

$$(x+4)(x-2) = 0$$

$$x = -4, \ 2$$

Acceleration at maximum displacement

is $\ddot{x} = -3 - 3(-4) = 9 \text{ m/s}^2$ /

or $\ddot{x} = -3 - 3(2) = -9 \text{ m/s}^2$

7 (a) (i) $P(x) = 8x^3 - 12x^2 + 6x + 13$

$P'(x) = 24x^2 - 24x + 6$

$= 6(4x^2 - 4x + 1)$

$= 6(2x-1)^2$

Stat pt where $P'(x) = 0$

$(2x-1)^2 = 0$

$x = \frac{1}{2}$

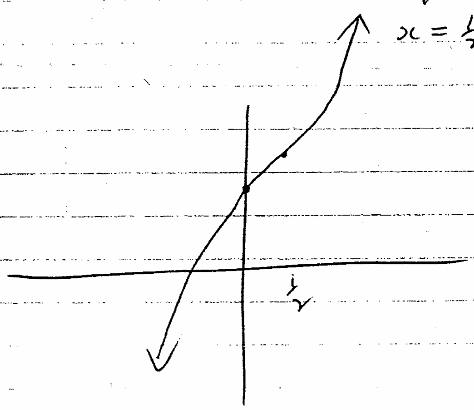
$P'(0.4) > 0$ and $P'(0.6) > 0$

$\therefore$ Horizontal pt of inflexion where

$x = \frac{1}{2}$, $y = 1 - 3 + 3 + 13$ 2

$= 14$

When $x = 0$, $y = 13$



From sketch it is clear that there is only one root which is negative

(ii) $P(-1) = -8 - 12 - 6 + 13 < 0$

$\therefore$ Req'd value of c is $c = -1$

(iii) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_1 = -\frac{1}{2} - \frac{f(-\frac{1}{2})}{f'(-\frac{1}{2})}$

$= -0.75$

(b) (i) $\tan^{-1} y = 2\tan^{-1} x$

Take tan of both sides

$$y = \tan(2\tan^{-1} x)$$

$$y = \tan 2\alpha \qquad \text{where } \alpha = \tan^{-1} x$$

$$y = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$y = \frac{2x}{1-x^2}$$

(ii) $$\frac{dy}{dx} = \frac{(1-x^2)2 - 2x(-2x)}{(1-x^2)^2}$$

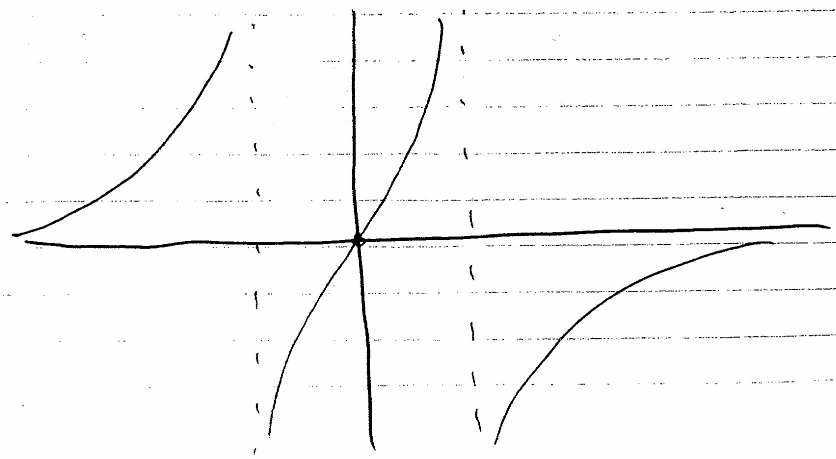
$$= \frac{2-2x^2+4x^2}{(1-x^2)^2}$$

$$= \frac{2x^2+2}{(1-x^2)^2} > 0$$

$\therefore$ No turning pts

(iii) D : all real x except $x=1$ or $x=-1$

(iv)



(c) $a \tan^2\theta + b\tan\theta + c = 0$

(i) $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$ 1

$= \dfrac{\frac{-b}{a}}{1-\frac{c}{a}} = \dfrac{-b}{a-c}$ 1 2

(ii) $\tan^2(\alpha-\beta) = \left(\dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}\right)^2$

$= \dfrac{\tan^2\alpha + \tan^2\beta - 2\tan\alpha\tan\beta}{(1+\tan\alpha\tan\beta)^2}$ 1

$= \dfrac{(\tan\alpha+\tan\beta)^2 - 4\tan\alpha\tan\beta}{(1+\tan\alpha\tan\beta)^2}$

$= \dfrac{\left(\frac{-b}{a}\right)^2 - 4\frac{c}{a}}{\left(1+\frac{c}{a}\right)^2}$

$= \dfrac{b^2-4ac}{(a+c)^2}$ 2