Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Kings 2004 Maths Ext 2 trial

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Find a, b, c if $\dfrac{1}{x(x+1)^2} \equiv \dfrac{a}{x} + \dfrac{b}{x+1} + \dfrac{c}{(x+1)^2}$ **2**

(ii) Find $\displaystyle\int \frac{dx}{x(x+1)^2}$ **2**

(b) Evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{x}{\sqrt{4-x^2}}\,dx$ **3**

(c) (i) Simplify $\dfrac{1}{\sqrt{2}}\left(\dfrac{1}{\sqrt{2}+x} + \dfrac{1}{\sqrt{2}-x}\right)$ **1**

(ii) Show that $\displaystyle\frac{1}{\sqrt{2}}\int_{-1}^{0}\left(\frac{1}{\sqrt{2}+x} + \frac{1}{\sqrt{2}-x}\right)dx = \sqrt{2}\ln\left(\sqrt{2}+1\right)$ **2**

(iii) Use completion of square to evaluate $\displaystyle\int_0^1 \frac{2}{1-t^2+2t}\,dt$ **3**

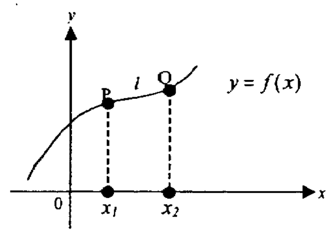
(d) Use the substitution $t = \tan\dfrac{x}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{\cos x + \sin x}$ **2**

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) If $x^2 + y^2 = 1$, show that $\dfrac{dy}{dx} = \dfrac{-x}{y}$, $y \neq 0$ **1**

(ii)



In the diagram, the length, l, of the arc PQ is given by $l = \displaystyle\int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

Use this result to prove that the length of the arc of the circle $x^2 + y^2 = 1$ between the points $(0, 1)$ and $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$ is $\dfrac{\pi}{6}$ **3**

(b) (i) $f(x)$, $f'(x)$ and $f''(x)$ exist for $a \le x \le b$

Show that

(i) $\displaystyle\int_a^b f'(x)\,dx = \int_a^b f'(a+b-x)\,dx$ **2**

(ii) $\displaystyle\int_a^b x f''(x)\,dx = bf'(b) - f(b) - (af'(a) - f(a))$ **3**

(c) Let $z = 1 - i$ and $w = -3 + 3i$

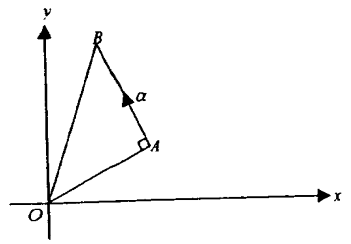
(i) Find $\dfrac{1}{z}$ in the form $x + iy$ **1**

(ii) Find $\arg(z+w)$ **2**

Question 2 continues next page

Question 2 (continued) **Marks**

(d)



In the Argand diagram, ΔOAB is isosceles and right angled at A. $\vec{AB}$ represents the complex number α

(i) What complex number corresponds to the vertex A? **1**

(ii) What complex number corresponds to the vertex B? **1**

(iii) Show that the area of ΔOAB is $\frac{1}{2}\alpha\overline{\alpha}$ **1**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider $(x+iy)^3 = i$, x, y real

(i) Show that $|x+iy| = 1$ **1**

(ii) Solve the equation $(x+iy)^3 = i$ **3**

(b) z is any complex number such that $|z-1| = 1$

(i) Sketch the locus of z in the Argand diagram. **1**

(ii) Hence, or otherwise, show that $|z| + |z-2| \geq 2$ **2**

(iii) If z were not on the locus in (i) would the result in (ii) still be true? Give a reason for your answer. **1**

(iv) If $0 < \arg z < \frac{\pi}{2}$, find the value of $\arg\left(\frac{z-2}{z}\right)$ **2**

(c) α, β, γ are the roots of $x^3 + x - 1 = 0$.

Find the values of

(i) $\alpha^3 + \beta^3 + \gamma^3$ **2**

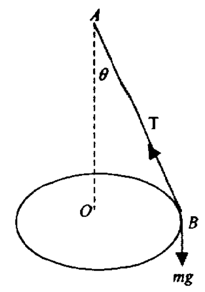
(ii) $\alpha^5 + \beta^5 + \gamma^5$ **2**

and (iii) write down an equation with roots $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$ **1**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



A particle of mass m kg is attached to one end of a light string at B. The other end of the string is fixed at a point A. The particle rotates in a horizontal circle of radius r metres at g *rad/s*, the centre O of the circle being directly below A.

The forces acting on the particle are the tension in the string T and the gravitational force mg.

Let $\angle BAO = \theta$

(i) Show that $T \sin\theta = mg^2r$ **2**

(ii) Prove that $\theta = \tan^{-1}(gr)$ **2**

(iii) Prove that $T = mg\sqrt{1+g^2r^2}$ **2**

Question 4 continues next page

Question 4 (continued) **Marks**

(b) A is the series $x + x^2 + x^3 + \ldots + x^n = \dfrac{x(1-x^n)}{1-x}$, $|x| < 1$ and

B is the series $1 + 2x + 3x^2 + \ldots + nx^{n-1}$, $|x| < 1$

(i) Deduce that the sum of series B is $\dfrac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$ **2**

[DO NOT USE INDUCTION]

(ii) Prove the result in (i) by induction, $n \geq 1$ **4**

(iii) The limiting sum of series A is 1. Find the limiting sum of series B. **3**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the function $f(x)=\frac{1}{2}\left(x\sqrt{x^2-1}-\ln\left(x+\sqrt{x^2-1}\right)\right)$

(i) Find the domain of f. **2**

(ii) Show that $f'(x)=\sqrt{x^2-1}$ **3**

(iii) Sketch the function. **1**

(b) (i) Sketch the hyperbola $x^2-y^2=1$, showing its foci, directrices and asymptotes. **3**

(ii) A particular solid has as its base the region bounded by the hyperbola $x^2-y^2=1$ and the line $x=2$. Cross-sections perpendicular to this base and the x axis are semi-circles whose diameters are in the base.

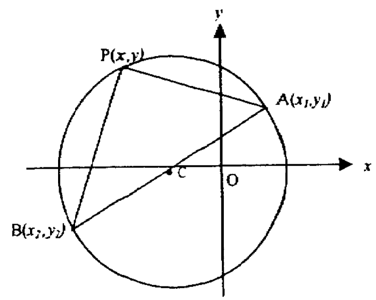
Find the volume of this solid. **4**

(iii) Show that the area of the base of the solid in (ii) is $2\sqrt{3}-\ln\left(2+\sqrt{3}\right)$ **2**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



$P(x,y)$ is any point on the circle, centre C. $A(x_1,y_1)$ and $B(x_2,y_2)$ are the end points of a diameter of the circle.

Show that the equation of the circle is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$ **2**

(b) $A(a,o)$ and $A'(-a,o)$ are the vertices of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $a>0$

$P(a\sec\theta, b\tan\theta)$ is any point on the hyperbola, $P\neq A$ or A'.

The tangent at P meets the tangents at A, A' at Q, R respectively.

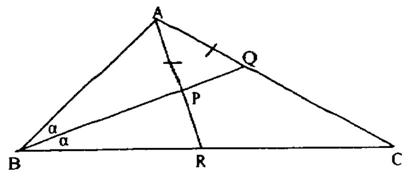
(i) Prove that the equation of the tangent at P is $\frac{\sec\theta}{a}x-\frac{\tan\theta}{b}y=1$ **3**

(ii) Find the coordinates of Q and R. **2**

(iii) Prove that the circle with QR as a diameter passes through the two foci of the hyperbola. **4**

Question 6 continues next page

(c)

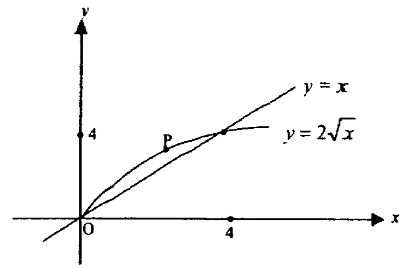


In the diagram, BQ bisects $\angle ABC$ and P is the point on BQ so that $AP = AQ$.

Prove that BA is a tangent to the circle through the points A, R and C. **4**

End of Question 6

(a)



The region bounded by the curve $y = 2\sqrt{x}$ and the line $y = x$ is revolved about the line $y = x$.

Let $P\left(x, 2\sqrt{x}\right)$ be a point on $y = 2\sqrt{x}$, $0 \le x \le 4$.

By considering slices through P perpendicular to the line $y = x$, find the volume of the solid of revolution. **5**

(b) A particle of mass M moves in a straight line with velocity v under the action of two propelling forces $\dfrac{Mu^2}{v}$ and Mk^2v, u, k positive constants.

(i) Show that the acceleration equation of motion is $\dfrac{u^2 + k^2v^2}{v}$ **1**

(ii) Show that the distance travelled by the particle in increasing its velocity from $\dfrac{u}{k}$ to $\dfrac{2u}{k}$ is $\dfrac{u}{k^3}\left(1 - \tan^{-1}\dfrac{1}{3}\right)$ **5**

Question 7 continues next page

(c) (i) $x^2 + Ax + B = 0$ has integer coefficients. If $\alpha + \sqrt{\beta}$ is a root, α, β, rational, $\beta \geq 0$, show that $\alpha - \sqrt{\beta}$ is also a root. **1**

(ii) $f(x) = x^4 - 4x^3 - 4x^2 + 16x + 16 = 0$ is known to have only real roots. Further, it is also known that there is at least one double root.

Express $f(x)$ as a product of factors with integer coefficients. **3**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Given that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ ... when expressed as an infinite series, show that $\lim_{x \to \infty} \left(\frac{x^n}{e^x} \right) = 0, \quad n = 0, 1, 2, 3, \ldots$ **1**

(ii) Let $u_n = \int_0^x t^n e^{-t}\, dt, \; n = 0, 1, 2, 3, \ldots$

Show that $u_n = n u_{n-1} - x^n e^{-x}, \quad n \geq 1$ **2**

(iii) Let $f(n) = \lim_{x \to \infty} \int_0^x t^n e^{-t}\, dt = \int_0^\infty t^n e^{-t}\, dt, \quad n = 0, 1, 2, 3, \ldots$

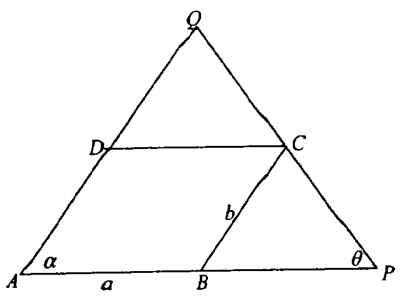
Deduce that $f(n) = n!$ **2**

(iv) Evaluate $\int_0^\infty t^7 e^{-t^2}\, dt$ **2**

Question 8 continues next page

(b) Show that $\frac{d}{d\theta}\left(\frac{\sin\theta}{\sin(\theta+\alpha)}\right)=\frac{\sin\alpha}{\sin^2(\theta+\alpha)}$, α constant **1**

(c)



In the diagram, $ABCD$ is a fixed parallelogram where $AB=a$ and $BC=b$. $\angle DAB=\alpha$.

A variable line through C meets AB produced at P and AD produced at Q.

Let $\angle BPC=\theta$, $0<\theta<\pi-\alpha$

(i) Show that the area of ΔAPQ is given by

$$A(\theta)=\frac{1}{2}\sin\alpha\left(\frac{a^2\sin\theta}{\sin(\theta+\alpha)}+\frac{b^2\sin(\theta+\alpha)}{\sin\theta}+2ab\right)$$ **2**

(ii) Show that as $\theta\to 0$, $A(\theta)\to\infty$ **1**

(iii) Prove that the minimum area of ΔAPQ occurs when $\cot\theta=\frac{a}{b}\operatorname{cosec}\alpha-\cot\alpha$ **3**

(iv) Draw a diagram to show clearly the position of the side PQ for which the area of ΔAPQ is a minimum. Include on your diagram the parallelogram $ABCD$. **1**

End of Examination