SET BY: JH

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VIRILE AGITUR

KNOX GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT

2004
TRIAL HSC EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided on the back page of this paper
- All necessary working should be shown in every question

Total marks (84)

- Attempt Questions 1–7
- All questions are of equal value
- Use a **SEPARATE** Writing Booklet for each question
- Please write your **Board of Studies** Student Number and Teachers Initials on the front cover of each of your writing booklets.

NAME:________________________ TEACHER:________________________

Total marks (84)
Attempt questions 1 – 7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Find $\lim_{x \to 0} \frac{\sin 2x}{4x}$. **2**

(b) Find the exact value of $\int_2^3 \left(\frac{x^2}{x^3 - 7}\right) dx$. **3**

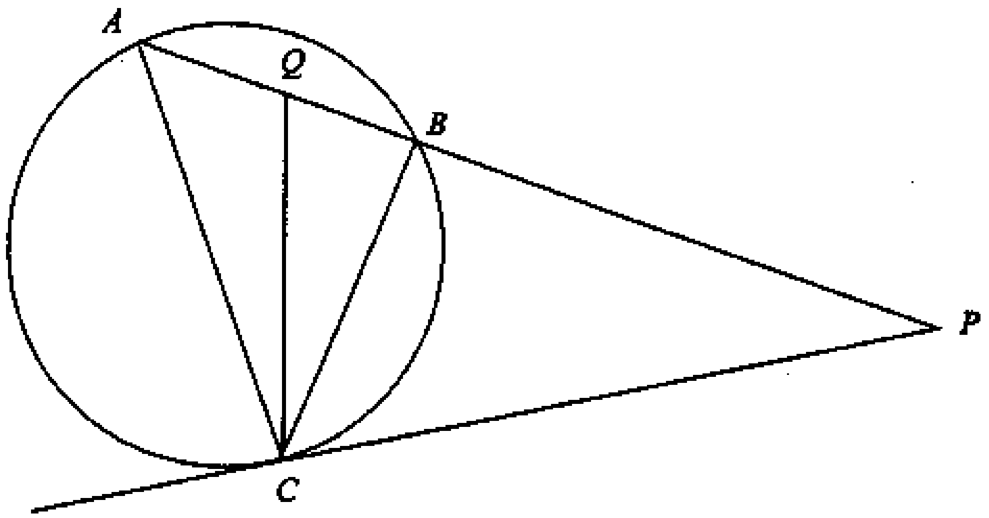
(c) Solve for x: $\frac{2x}{x-1} \leq 1$. **3**

(d) Find $\frac{d}{dx}\left(\tan^{-1} \frac{x}{3}\right)$. **1**

(e) The point $P(19, -15)$ divides an interval AB externally in the ratio 3:2. Find the coordinates of the point $B(x, y)$ given $A(-2, 3)$. **3**

Question 2 (12 marks) Use a SEPARATE writing booklet **Marks**

(a)



In the diagram above, PC is a tangent to the circle at C and QC bisects $\angle ACB$. 3

Copy the diagram into your writing booklet.

Prove, with reasons, that $PC = PQ$.

(b) Use the substitution $u = e^x$ to find: $\displaystyle\int \frac{dx}{e^x + 4e^{-x}}$ 3

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2 2x \; dx$. 3

(d) Find the exact value of $\cos^{-1}\left(\sin\frac{4\pi}{3}\right)$. 3

Question 3 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Find the value of the term independent of x in the expansion of $\left(x-\frac{2}{x^3}\right)^{12}$. **2**

(b) (i) Find the equation of the tangent to the curve $y=x^2-x$ at the point where $x=2$. **2**

(ii) Find the obtuse angle between the line $\frac{x}{3}+\frac{y}{2}=1$ and the tangent found in part (i). Give your answer to the nearest degree. **2**

(c) (i) Express $\sqrt{12}\sin x+2\cos x$ in the form $A\cos(x-\alpha)$; where $A>0$ and $0<\alpha<\frac{\pi}{2}$. **2**

(ii) Hence, sketch the graph of $y=\sqrt{12}\sin x+2\cos x$, in the domain $0\le x\le 2\pi$. **2**

(iii) State the number of solutions that satisfy the equation $\sqrt{12}\sin x+2\cos x=1$ in the domain $0\le x\le 2\pi$. **1**

(iv) Write down the general solution to $\sqrt{12}\sin x+2\cos x=1$ **1**

Question 4 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Use one application of Newton's method to find a better approximation to the root of the equation $e^{-x} - \log_e x = 0$, given that there is a root near $x = 1.4$. Give your answer to 3 decimal places. **3**

(b) Use the Principle of Mathematical Induction to show that the expression $7^n + 5$ is divisible by 6 for all positive integers n. **4**

(c) (i) Find $\frac{d}{dx}\left(x\sin^{-1}\frac{x}{4} + \sqrt{16 - x^2}\right)$. **3**

(ii) Hence, evaluate $\int_0^4 \sin^{-1}\frac{x}{4}\ dx$. **2**

Question 5 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Newton's Law of Cooling states that when an object at temperature T (°C) is placed in an environment at a temperature R (°C), then the rate of temperature loss is given by the equation $\frac{dT}{dt} = k(T-R)$; where t is the time in seconds and k is a constant.

A packet of peas, initially at $24\,°C$ is placed in a snap-freeze refrigerator in which the internal temperature is maintained at $-40\,°C$. After 5 seconds the temperature of the packet is $19\,°C$. Suppose $T = R + Ae^{kt}$, where A is a constant.

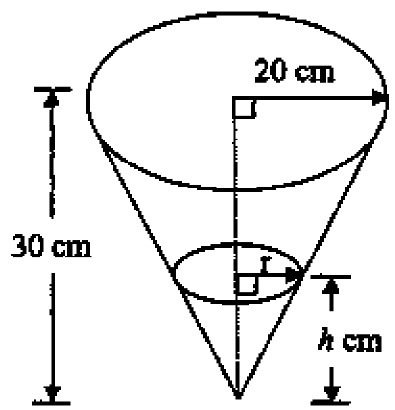
(i) State the value of A. **1**

(ii) Show that $k = \frac{1}{5}\log_e\left(\frac{59}{64}\right)$. **2**

(iii) Hence show that it will take approximately 29 seconds for the packet's temperature to reduce to $0\,°C$. **2**

(b) Prove that: $\tan\left(\frac{\pi}{4}+\theta\right) - \tan\left(\frac{\pi}{4}-\theta\right) = 2\tan 2\theta$ **3**

(c)



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Water is poured into a conical vessel, of base radius 20 cm, and height 30 cm at a constant rate of 24 cm^3 per second. The depth of water is h cm at time t seconds and V is the volume of the water in the vessel at this time.

(i) Explain why $r = \frac{2h}{3}$. **1**

(ii) Hence show that the volume of water in the vessel at any time t is given by $V = \frac{4\pi h^3}{27}$. **1**

(iii) Find the rate of increase of the area (A) of the surface of the water, when the depth is 16cm. **2**

Question 6 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Two points $P(2ap,ap^2)$ and $Q\,(2aq,aq^2)$ lie on the parabola $x^2 = 4ay$ $(a > 0)$.

(i) By derivation, show that the equation of the chord is: **2**

$$y = \frac{1}{2}(p+q)x - apq\,.$$

(ii) If the chord PQ passes through the focus, S, show that $pq = -1$. **2**

(iii) Using the fact that $PQ = PS + SQ$, or otherwise, show that the chord PQ has length $a\left(p+\dfrac{1}{p}\right)^2$. **3**

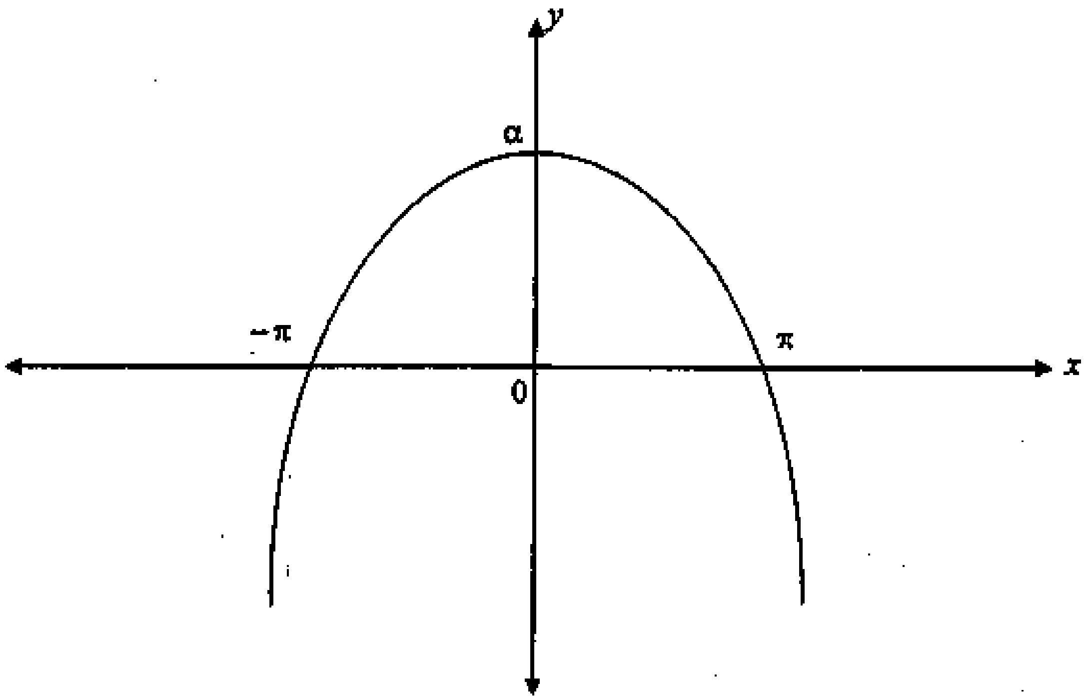
(b) A particle moves along a straight line such that its distance from the origin at time t (s) is x (m) and its velocity is v.

(i) Prove that $\dfrac{d^2x}{dt^2} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$. **2**

(ii) If the acceleration satisfies $\dfrac{d^2x}{dt^2} = -4\left(x+\dfrac{16}{x^3}\right)$ and the particle is initially at rest when $x = 2$, show that $v^2 = 4\left(\dfrac{16-x^4}{x^2}\right)$. **3**

Question 7 (12 marks) Use a SEPARATE writing booklet **Marks**

(a)



The diagram shows a parabola $y = f(x)$, with vertex $(0, \alpha)$ and $\alpha > 0$. The parabola passes through the points $(-\pi, 0)$ and $(\pi, 0)$ as shown.

If a is the focal length of the parabola:

(i) Show that $4a = \dfrac{\pi^2}{\alpha}$. **2**

(ii) Show that $f(x)$ can be expressed in the form $f(x) = \alpha\left(1 - \dfrac{x^2}{\pi^2}\right)$. **2**

(iii) Find the exact value of α given that the area between $y = f(x)$ and the x axis from $x = -\pi$ to $x = \pi$ is 4 square units. **3**

(b) Assume that tides rise and fall in **Simple Harmonic Motion.** A ship needs 11 metres of water to pass down a channel safely. At low tide, the channel is 8m deep and at high tide 12 m deep. Low tide is at 10:00 am and high tide at 4:00 pm. **5**

Find the first time period during which the ship can safely proceed through the channel.

END OF PAPER