Mathematics Extension 1 Trial HSC 2001

Question 1

a. $\displaystyle\int_0^{2\sqrt{3}} \frac{dx}{4+x^2} = \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^{2\sqrt{3}}$

$= \frac{1}{2}\tan^{-1}\sqrt{3} - 0$

$= \frac{\pi}{6}$ //

[2]

b. $\frac{d}{dx}\left(\cos^3 x\right) = 3\sin x \cdot \cos^2 x$

[2]

c.

$x_1 = 4 \quad x_2 = 13$

$y_1 = 6 \quad y_2 = 5$

$m = 4 \quad n = -1$

$x = \frac{(-1)(4) + (4)(13)}{4-1}$

$= 16$

$y = \frac{(-1)(6) + (4)(5)}{4-1}$

$= \frac{14}{3}$

the point required is $\left(16, \frac{14}{3}\right)$

[2]

d. $y = \frac{2x}{3x-1}$

$x = \frac{1}{3}$ is the equation of the vertical asymptote.

[1]

e

$x \geq -2$ or $x \leq -5$

f. $\displaystyle\int_0^{1/\sqrt{2}} \frac{2x^3}{\sqrt{1-x^4}} \cdot dx$ $\quad u = x^4$

$\frac{du}{dx} = 4x^3$

$= \frac{1}{2}\displaystyle\int_0^{1/4} \frac{1}{\sqrt{1-u}} \cdot du$ $\quad u_1 = \left(\frac{1}{\sqrt{2}}\right)^4$

$= \frac{1}{4}$

$= \frac{1}{2}\displaystyle\int_0^{1/4} (1-u)^{-1/2} \cdot du$ $\quad u_2 = 0^4$

$= 0$

$= \frac{1}{2}\left[-\frac{2}{3}(1-u)^{3/2}\right]_0^{1/4}$

$= \frac{1}{2}\left(-\frac{\sqrt{3}}{4}\right) - \frac{1}{2}\left(\frac{-2}{3}\right)$

$= -\frac{\sqrt{3}}{8} + \frac{1}{3}$ //

[3]

Question 2

a. # arrangements $= \frac{9!}{2!2!2!}$

$= 45360$ //

[2]

b. $\sin\theta + \sqrt{3}\cos\theta = 1 \quad 0 \leq \theta \leq 2\pi$

let $t = \tan\frac{\theta}{2}$

$\Rightarrow \frac{2t}{1+t^2} + \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right) = 1$

$2t + \sqrt{3} - \sqrt{3}.t^2 = 1 + t^2$

$(-\sqrt{3}-1)t^2 + 2t + (\sqrt{3}-1) = 0$

$t = \frac{-2 \pm \sqrt{4 + (\sqrt{3}+1)(\sqrt{3}-1)}}{-2(\sqrt{3}+1)}$

$= \frac{-2 \pm \sqrt{12}}{-2(\sqrt{3}+1)}$

$= 1$ or $\frac{\sqrt{3}-1}{-\sqrt{3}-1}$

$t = 1$: $\tan\frac{\theta}{2} = 1 \quad 0 \leq \frac{\theta}{2} \leq \pi$

$\frac{\theta}{2} = \frac{\pi}{4}$

$\theta = \frac{\pi}{2}$ //

$t = \frac{\sqrt{3}-1}{-\sqrt{3}-1}$:

$\tan\frac{\theta}{2} = \frac{\sqrt{3}-1}{-\sqrt{3}-1} \quad 0 \leq \frac{\theta}{2} \leq \pi$

$\frac{\theta}{2} = \frac{11\pi}{12}$

$\theta = \frac{11\pi}{6}$

$\therefore \theta = \frac{\pi}{2}, \frac{11\pi}{6}$ //

[4]

c.

i $f(x) = 2x^2 + x - 2$

$f(0) = 2(0)^2 + 0 - 2$

$= -2$

$f(1) = 2(1)^2 + 1 - 2$

$= 1$

so $f(0) < 0$ and $f(1) > 0$

$\therefore f(x)$ must cut the x-axis between $x = 0$ and $x = 1$.

[1]

$f\left(\frac{0+1}{2}\right) = f(0.5)$
$= 2(0.5)^2 + 0.5 - 2$
$= -1$

root lies between $x = 0.5$
nd $x = 1$

choose $x = \frac{0.5+1}{2} = 0.75$

$(0.75) = 2(0.75)^2 + 0.75 - 2$
$= -0.125$

root lies between
$= 0.75$ and $x = 1$
$= 0.75$ is our approximation.
[3]

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$) = 2x^2 + x - 2$
$x) = 4x + 1$

osing $x_1 = 0.7$ ...

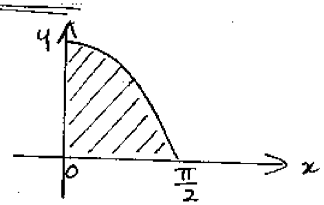
$= 0.7 - \frac{f(0.7)}{f'(0.7)}$

$= 0.7 - \left(\frac{-0.32}{3.8}\right)$

$= 0.784$ // (3 d.p's)
[2]

Question 3:

a.
i



ii $V = \pi \int_0^{\pi/2} \cos^2 x \cdot dx$

$= \frac{\pi}{2} \int_0^{\pi/2} \cos 2x + 1 \cdot dx$

$= \frac{\pi}{2} \left[\frac{1}{2} \sin 2x + x \right]_0^{\pi/2}$

$= \frac{\pi}{2}\left(\frac{\pi}{2}\right) - \frac{\pi}{2}(0)$

$= \frac{\pi}{4}$ ~~sq~~ co. units. //

b. $x^3 + 4x^2 - 6x - 8 = 0$
let the roots be α, β, γ

$\alpha + \beta + \gamma = -4$
$\alpha\beta + \alpha\gamma + \beta\gamma = -6$
$\alpha\beta\gamma = 8$

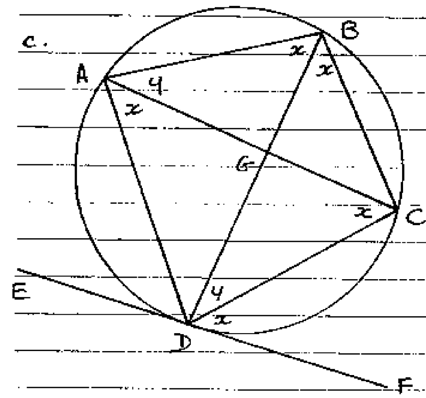
$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{-6}{8}$

$= -\frac{3}{4}$ // [2]

c.



let $\angle DBC = x$
$\angle DAC = x$ ($\angle$'s on same arc)
$\angle ABD = x$ (BD bisects $\angle ABC$)
$\angle ACD = x$ ($\angle$'s on same arc)
let $\angle BAC = y$
$\angle BDC = y$ ($\angle$'s on same arc)
and $\angle BDF = x + y$ ($\angle$ in alt. seg)
$= \angle BAD$
$\therefore \angle CDF = \angle BDF - \angle BDC$
$= (x+y) - y$
$= x$
$= \angle ACD$
$\therefore$ alternate $\angle$'s $\angle CDF$ and $\angle ACD$ are equal
$\therefore AC \parallel EF$ //
[2]

d.
i

$A(2\sin x + \cos x) + B(2\cos x - \sin x)$
$\equiv 7\sin x + 11\cos x$

$2A\sin x + A\cos x + 2B\cos x - B\sin x$
$\equiv 7\sin x + 11\cos x$

$(2A - B)\sin x + (A + 2B)\cos x$
$\equiv 7\sin x + 11\cos x$

$\Rightarrow 2A - B = 7$ — (1)
$A + 2B = 11$ — (2)

from (1)
$B = 2A - 7$
subb'ing into (2) ...

$A + 2(2A - 7) = 11$
$A + 4A - 14 = 11$
$5A = 25$
$A = 5$
$2(5) - B = 7$
$B = 3$

$\therefore A = 5, B = 3$ // [2]

ii

$$\int_0^{\pi/2} \frac{7\sin x + 11\cos x}{2\sin x + \cos x}\, dx$$

$$\int_0^{\pi/2} \frac{5(2\sin x + \cos x) + 3(2\cos x - \sin x)}{2\sin x + \cos x}\, dx$$

$$\int_0^{\pi/2} 5 + 3\left(\frac{2\cos x - \sin x}{2\sin x + \cos x}\right) dx$$

$$\Big[\, 5x + 3\ln(2\sin x + \cos x) \Big]_0^{\pi/2}$$

$$\left(\frac{5\pi}{2} + 3\ln 2\right) - \left(0 + 3\ln 1\right)$$

$$\frac{5\pi}{2} + 3\ln 2$$

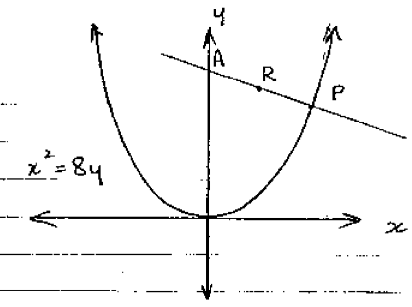
$$\frac{5\pi}{2} + \ln 2^3$$

$$\frac{5\pi}{2} + \ln 8 \;//$$

[3]

Question 4:

a:



i coords of P: $(2ap, ap^2)$
where $a = 2$
$\therefore P$ is $(4p, 2p^2)$.

gradient of normal $= -\dfrac{1}{p}$

$$y - 2p^2 = -\frac{1}{p}(x - 4p)$$

$$py - 2p^3 = -x + 4p$$

$\therefore x + py = 4p + 2p^3$ is the equation of the normal at P.

[2]

ii when $x = 0$, $y = 4 + 2p^2$
$\therefore A$ is $(0, 4 + 2p^2)$

coords of R: $\left(\dfrac{4p}{2}, \dfrac{4 + 2p^2}{2}\right)$

$$= (2p, 2 + p^2)$$

[2]

iii

Coords of R:

$x = 2p$ — (1)
$y = 2 + 2p^2$ — (2)

from (1) $p = \dfrac{x}{2}$

subbing into (2) gives

$$y = 2 + 2\left(\frac{x}{2}\right)^2$$

$$= 2 + \frac{2x^2}{4}$$

$$= 2 + \frac{x^2}{2}$$

$x^2 = 2(y - 2)$ is the locus of R.
This is a parabola with vertex $(0, 2)$.

now focus of $x^2 = 8y$ is $(0, a)$ where $a = 2$ ie $(0, 2)$ which is the same as the vertex of $x^2 = 2(y - 2)$ //.

[3]

i $\int_1^3 \frac{dx}{x}$

$= \left[\ln x \right]_1^3$

$= \ln 3 - \ln 1$

$= \ln 3$ // [1]

$\frac{dx}{x} \doteq \frac{1}{3}\left\{ f(1) + 4f(2) + f(3) \right\}$

$x) = \frac{1}{x}$

$) = 1 \; ; \; f(2) = \frac{1}{2} \; ; \; f(3) = \frac{1}{3}$

$\frac{dx}{x} \doteq \frac{1}{3}\left\{ 1 + 2 + \frac{1}{3} \right\}$

$\doteq \frac{10}{9}$ [2]

from i and ii

$3 \doteq \frac{10}{9}$

$\therefore 3 \doteq e^{\frac{10}{9}}$

$\Rightarrow e \doteq 3^{\frac{9}{10}}$

$\doteq 2.688$ (3 d.p's) [2]

Question 5:

a. Prove...

$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{n(n+1)}$

$= \frac{n}{n+1}$ for $n \geq 1$

let $n = 1$:

$LHS = \frac{1}{1\times 2} = \frac{1}{2}$

$RHS = \frac{1}{1+1} = \frac{1}{2} = LHS$

$\therefore$ true for $n = 1$ //

Assume true for $n = k$

ie $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

Prove true for $n = k+1$

$HS = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$

$= \frac{k(k+2)+1}{(k+1)(k+2)}$

$= \frac{k^2 + 2k + 1}{(k+1)(k+2)}$

$= \frac{(k+1)^2}{(k+1)(k+2)}$

$= \frac{k+1}{k+2}$

$RHS = \frac{k+1}{(k+1)+1}$

$= \frac{k+1}{k+2}$

$= LHS$

$\therefore$ true for $n = k+1$

$\therefore$ since true for $n = 1$

then true for $n = 2, n = 3, \ldots$

$\therefore$ true for all $n \geq 1$ // [3]

b.

i $y = x^2 + 6x$

$\frac{dy}{dx} = 2x + 6$

for monotonic increasing ... $\frac{dy}{dx} > 0$

$2x + 6 > 0$

$2x > -6$

$x > -3$

$\therefore$ the function is monotonic increasing when $x > -3$ [2]

ii let $x = y^2 + 6y$

$x + 9 = y^2 + 6y + 9$

$= (y+3)^2$

$y + 3 = \pm\sqrt{x+9}$

$y = -3 \pm \sqrt{x+9}$

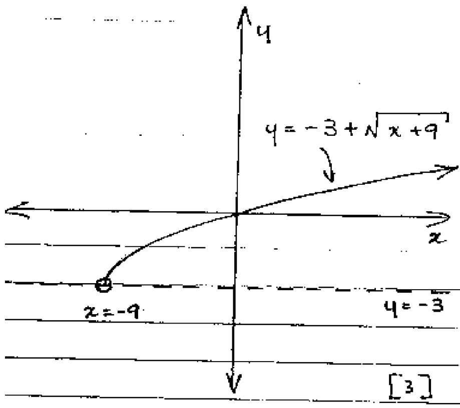
but the range will be $y > -3$

$\therefore y = -3 + \sqrt{x+9}$ //

is the inverse function.

domain: $x \geq -9$

range: $y > -3$



[3]

i $\cos\left[\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right]$

$= \cos\left(-\frac{\pi}{6}\right)$

$= \cos\left(\frac{\pi}{6}\right)$

$= \frac{\sqrt{3}}{2}$ // [1]

$\therefore\ y = 3\sin^{-1}\left(\frac{x}{2} - 1\right)$

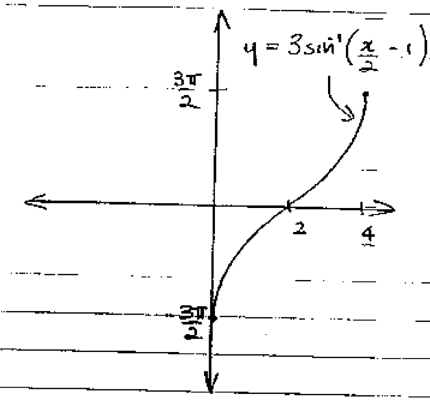
$-1 \leq \frac{x}{2} - 1 \leq 1$

$2 \leq x - 2 \leq 2$

$0 \leq x \leq 4$

domain : $0 \leq x \leq 4$

range : $-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}$



[3]

Question 6:

i $\frac{dT}{dt} = -k(T - S)$

let $T = S + Ae^{-kt}$

$\frac{dT}{dt} = -Ake^{-kt}$

$-k(T-S) = -k(S + Ae^{-kt} - S)$

$= -Ake^{-kt}$

$= \frac{dT}{dt}$

$\therefore T = S + Ae^{-kt}$ is a soln. of the differential equation

[2]

ii Initial conditions:

$t = 0$, $S = 15$, $\frac{dT}{dt} = -1$, $T = 65$

$T = S + Ae^{-kt}$

$65 = 15 + A.1$

$\therefore A = 50$

$\frac{dT}{dt} = -k(T - S)$

$-1 = -k(65 - 15)$

$\therefore k = \frac{1}{50}$ // [2]

iii $T = 15 + 50e^{-\frac{t}{50}}$

$= 15 + 50e^{-\frac{20}{50}}$

$= 49°$ (to nearest degree) [2]

iv $35 = 15 + 50e^{-\frac{t}{50}}$

$20 = 50e^{-t/50}$

$e^{-t/50} = 0.4$

$-\frac{t}{50} = \ln(0.4)$

$t = -50\ln(0.4)$

$= 45.8$ minutes // [2]

b.

i

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4\left(x + 16x^{-3}\right)$

$\therefore \frac{1}{2}v^2 = \int -4x - 64x^{-3} \cdot dx$

$= -2x^2 + 32x^{-2} + C$

$\therefore v^2 = -4x^2 + \frac{64}{x^2} + D$

now $v = 0$ when $x = 2$

$\therefore 0 = -16 + 16 + D$

$\therefore D = 0$

so $v^2 = \frac{64}{x^2} - 4x^2$

$= 4\left(\frac{16}{x^2} - x^2\right)$

$= 4\left(\frac{16 - x^4}{x^2}\right)$ // [3]

ii when P is halfway to the origin $x = 1$

$v^2 = 4\left(\frac{16-1}{1}\right)$

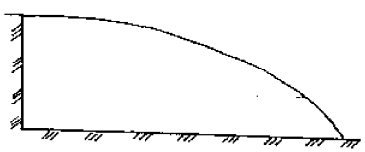
$= 4 \times 15$

$\therefore v = \pm 2 \times \sqrt{15}$

hence speed is $2\sqrt{15}$ m/s. [1]

Question 7:

a.



horizontal motion:

$\ddot{x} = 0$

$\dot{x} = \int 0 \cdot dt$

$= C$

since $\dot{x} = 60$ when $t = 0$

$C = 60$

$\therefore \dot{x} = 60$ and is constant.

$x = \int 60 \cdot dt$

$= 60t + D$

$x = 0$ when $t = 0$

$\therefore D = 0$

$\therefore x = 60t$ //

vertical motion:

$\ddot{y} = -10$

$\dot{y} = \int -10 \cdot dt$

$= -10t + E$

$= 0$ when $t = 0$

$\therefore E = 0$

$y = \int -10t \cdot dt$

$= -5t^2 + F$

$= 20$ when $t = 0$

$\therefore F = 20$

$\therefore y = -5t^2 + 20$ // [3]

ii $0 = -5t^2 + 20$

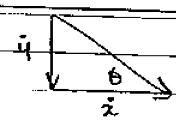
$5t^2 = 20$

$t^2 = 4$

$t = 2$ s. // [2]

iii when $t = 2$, $x = 120$ m. [1]

iv.



$\tan\theta = \dot{y} / \dot{x}$

when $t = 2$, $\dot{y} = -20$

$\dot{x} = 60$

$\therefore \tan\theta = -1/3$

$\theta = \tan^{-1}(1/3)$

$= 18.4°$ with the horizontal.

[2]

b.

i # ways $= {}^{10}C_5 \times {}^{10}C_2 \times {}^{10}C_1$

$= 113400$ // [2]

ii # ways of choosing Joe and Fred $= {}^{9}C_4 \times {}^{9}C_1 \times {}^{10}C_1$

$\therefore P(\text{Joe, Fred}) = 11340 / 113400$ [2]

$= 0.1$ //