SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2008
TRIAL
HIGHER SCHOOL CERTIFICATE

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 2 Hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Start each question in a new booklet
- The questions are of equal value
- Marks may NOT be awarded for messy or badly arranged work.
- All necessary work should be shown in every question.
- Full marks will NOT be given unless the method of the solution is shown.

Total Marks – 84

- Attempt all questions

Examiner: *R. Boros*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Start each question in a new answer booklet.

Question 1 (12 marks). **Marks**

a) Find the acute angle between the intersection of the curves $y = x^2 + 4$ and $y = x^2 - 2x$, correct to the nearest minute. 2

b) A is the point (-4, 2) and B is the point (3, -1). Find the coordinates of the point P which divides the interval AB externally in the ratio 2:1 2

c) Differentiate $y = \log_e\left(\sin^{-1} x\right)$ 2

d) Solve the inequality $\dfrac{x-1}{x+3} \geq -2$ 2

e) If $\cos A = \dfrac{7}{9}$ and $\sin B = \dfrac{1}{3}$ where A and B are acute angles,

Prove that $A = 2B$. 2

f) Use the substitution $u = t + 1$ to evaluate $\displaystyle\int_0^1 \frac{t}{\sqrt{t+1}}\,dt$ 2

End of Question 1.

Start a new booklet.

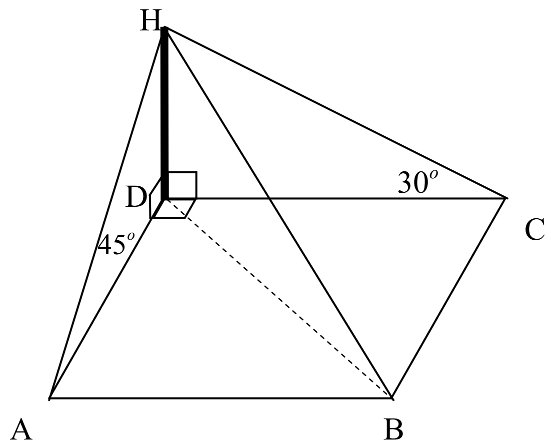
Question 2 (12 Marks). **Marks**

a) The polynomial $P(x) = ax^3 + bx^2 - 8x + 3$ has a factor of $(x-1)$ and leaves a remainder of 15 when divided by $(x+2)$. Find the values of a and b and hence fully factorise $P(x)$. 3

b) (i) Express $3\sin\theta + 2\cos\theta$ in the form $R\sin(\theta+\alpha)$ where α is an acute angle.

(ii) Hence, or otherwise solve the equation $3\sin\theta + 2\cos\theta = 2.5$ for $0^\circ \le \theta \le 360^\circ$. Answer correct to the nearest minute. 4

c) A post HD stands vertically at one corner of a rectangular field $ABCD$ The angle of elevation of the top of the post H from the nearest corners A and C are 45^o and 30^o respectively.



(i) If $AD = a$ units, find the length of BD in terms of a 2

(ii) Hence, find the angle of elevation of H from the corner B to the nearest minute. 1

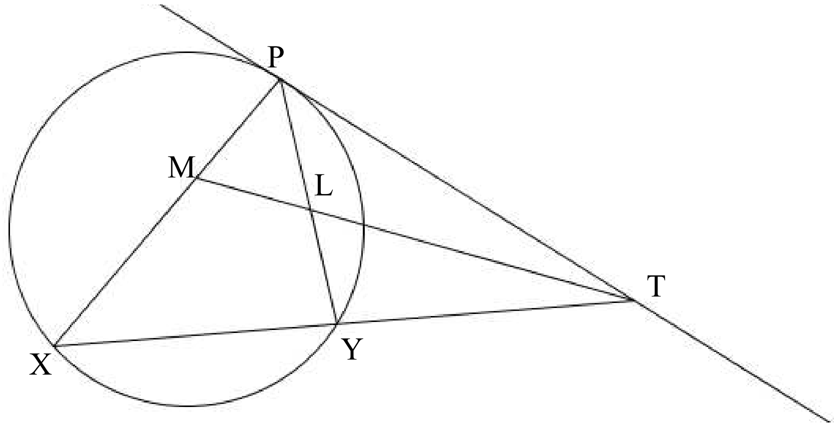
d) Taking $x = \dfrac{-\pi}{6}$ as a first approximation to the root of the equation $2x + \cos x = 0$, use Newton's method once to show that a second approximation to the root of the equation is $\dfrac{-\pi - 6\sqrt{3}}{30}$. 2

End of Question 2.

Start a new booklet.

Question 3 (12 marks). **Marks**

a) Diagram not to scale.



XY is any chord of a circle. XY is produced to T and TP is a tangent to the circle. The bisector of $\angle PTX$ meets XP in M and cuts PY at L. Prove that $\triangle MPL$ is isosceles. 3

b) (i) Find the domain and range of $f^{-1}(x) = \sin^{-1}(3x-1)$. 2

(ii) Sketch the graph of $y = f^{-1}(x)$. 1

(iii) Find the equation representing the inverse function $f(x)$ and state the domain and range. 3

c) Newton's Law of Cooling states that the rate of cooling of a body is proportional to the excess of the temperature of a body above the surrounding temperature. This rate can be represented by the differential equation $\frac{dT}{dt} = -k(T - T_0)$, where T is the temperature of the body, T_0 is the temperature of the surroundings, t is the time in minutes and k is a constant.

(i) Show that $T = T_0 + Ae^{-kt}$, where A is a constant, is a solution to the differential equation $\frac{dT}{dt} = -k(T - T_0)$. 1

(ii) A cup of coffee cools from 85^oC to 80^oC in one minute in a room temperature of 25^oC. Find the temperature of the cup of coffee after <u>a further</u> 4 minutes have elapsed. Answer to the nearest degree. 2

End of Question 3.

Start a new booklet.

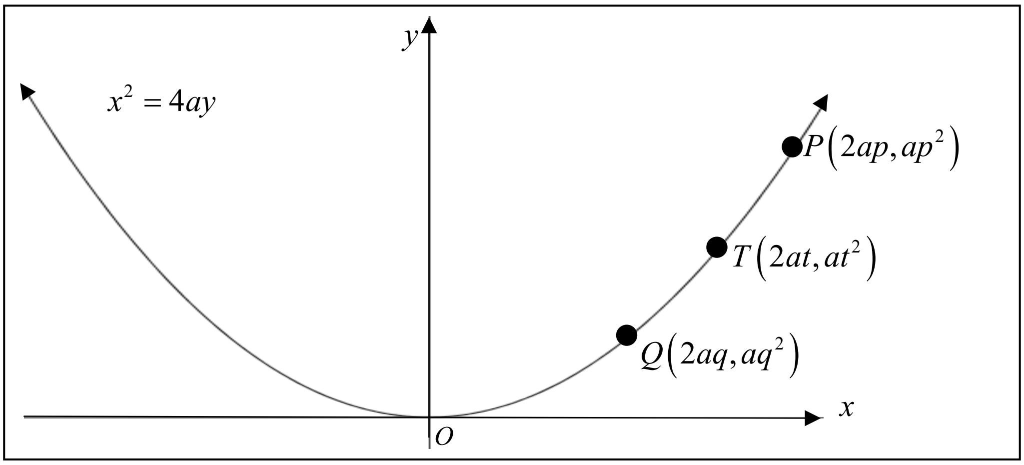
Question 4 (12 marks). **Marks**

a) Find the number of ways of seating 5 boys and 5 girls at a round table if:

(i) A particular girl wishes to sit between two particular boys. 1

(ii) Two particular persons do not wish to sit together. 1

b) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are the points on the parabola $x^2 = 4ay$



It is given that the chord PQ has the equation $y - \frac{1}{2}(p+q)x + apq = 0$

(i) Derive the equation of the tangent to the parabola $x^2 = 4ay$ at the point $T(2at, at^2)$. 2

(ii) The tangent at T cuts the y-axis at the point R. Find the coordinates of the point R. 1

(iii) If the chord PQ passes through the point R show that p, t and q are terms of a geometric series. 2

c) A particle moves so that its distance x cm from a fixed point O at time t seconds is $x = 2\cos 3t$.

(i) Show that the particle satisfies the equation of motion $\ddot{x} = -n^2 x$ where n is a constant. 2

(ii) What is the period of the motion? 1

(iii) What is the velocity when the particle is first 1cm from O. 2

End of Question 4.

Start a new booklet.

Question 5 (12 marks). **Marks**

a) Find the general solution of the equation $\tan\theta = \sin 2\theta$ 3

b) The cubic equation $2x^3 - x^2 + x - 1 = 0$ has roots α, β and γ. Evaluate

(i) $\alpha\beta + \beta\gamma + \alpha\gamma$ 1

(ii) $\alpha\beta\gamma$ 1

The equation $2\cos^3\theta - \cos^2\theta + \cos\theta - 1 = 0$ has roots $\cos a$, $\cos b$ and $\cos c$.

Using appropriate information from parts (i) and (ii), prove that 2

$\sec a + \sec b + \sec c = 1$.

c) (i) Sketch the curve $y = 2\cos x - 1$ for $-\pi \le x \le \pi$. Mark clearly where the graph crosses each axis. 2

(ii) Find the volume generated by the rotation through a complete revolution about the x axis of the region between the x-axis and that part of the curve $y = 2\cos x - 1$ for which $|x| \le \pi$ and $y \ge 0$ 3

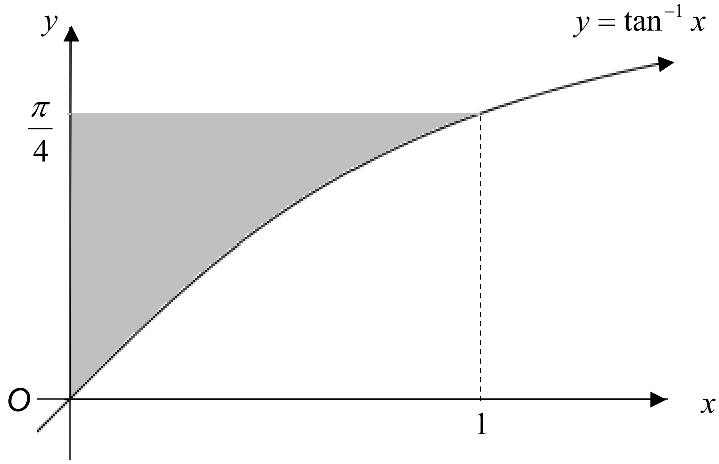
End of Question 5

Start a new booklet.

Question 6 (12 marks). **Marks**

a) (i) Find $\frac{d}{dy}\left(\ln \cos y\right)$. 1

(ii)



Show that the shaded area is given by $A = \frac{1}{2}\ln 2$ units^2 3

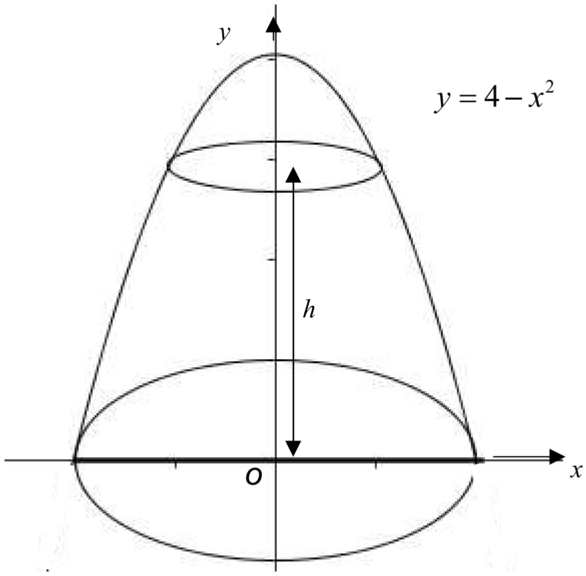
b) *P, Q, R* and *S* are four points taken in order on a circle. Prove that:

$$\frac{PR}{QS} = \frac{\sin P\hat{Q}R}{\sin Q\hat{P}S}$$ 3

Question 6 continued next page.

Question 6 continued

c)



A mould for a container is made by rotating the part of the curve $y = 4 - x^2$ which lies in the first quadrant through one complete revolution about the y-axis. After sealing the base of the container, water is poured through a hole in the top. When the depth of water in the container is h cm, the depth is changing at a rate of $\dfrac{10}{\pi(4-h)}$ cms^{-1}.

(i) Show that when the depth is h cm, the surface area S cm^2 of the top of the water is given by $S = \pi(4-h)$. **2**

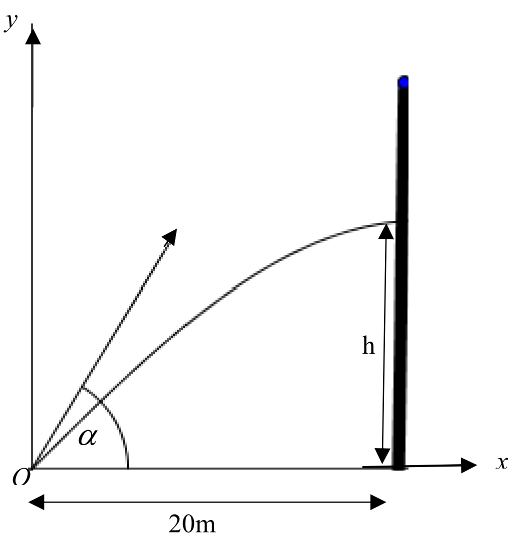
(ii) Find the rate at which the surface area of the water is changing when the depth of the water is 2cm. **3**

End of Question 6.

Start a new booklet.

Question 7 (12 marks). **Marks**

a) A softball player hits the ball from ground level with a speed of 20 m/s and an angle of elevation α . It flies toward a high wall 20m away on level ground. Taking the origin at the point where the ball is hit, the derived expressions for the horizontal and vertical components of x and y of displacement at the time t seconds, taking $g = 10\,\text{m/s}^2$, are $x = 20t\cos\alpha$ and $y = -5t^2 + 20t\sin\alpha$



(i) Hence find the equation of the path of the ball in flight in terms of x, y and α . 1

(ii) Show that the height h at which the ball hits the wall is given by $h = 20\tan\alpha - 5\left(1 + \tan^2\alpha\right)$ 1

(iii) Using part (ii) above, show that the maximum value of h occurs when $\tan\alpha = 2$ and find this maximum height 2

Question 7 continued next page.

Question 7 continued

b) A particle of unit mass moves in a straight line. It is placed at the origin on the x-axis and is then released from rest. When at position x, its acceleration is given by:

$$-9x+\frac{5}{(2-x)^2}\ .$$

Prove that the particle ultimately moves between two points on the x-axis and find these points. 3

c) (i) For any angles α and β show that

$$\tan\alpha+\tan\beta=\tan(\alpha+\beta)[1-\tan\alpha\tan\beta]$$ 1

(ii) Prove, by mathematical induction, that

$$\tan\theta\tan 2\theta+\tan 2\theta\tan 3\theta+...+\tan n\theta\tan(n+1)\theta=\tan(n+1)\theta\cot\theta-(n+1)$$

for all positive integers n 4

End of Question 7.

End of Examination.

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