Question 1:

a) Evaluate $\displaystyle\int_0^3 \frac{dx}{9+x^2}$ giving your answer in exact form. 2

b) Use the table of standard integrals to find the exact value of 2

$$\int_0^4 \frac{dx}{\sqrt{9+x^2}}$$

c) Show that $\dfrac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$. 4

Hence express $\tan\ 67\frac{1}{2}^{0}$ in simplest form.

d) Solve the inequality $x \geq \dfrac{1}{x}$ 4

Question 2:

Marks:

a) Find the acute angle between the lines $y=\frac{1}{3}x+3$ and $y=\frac{-2}{3}x+3$. 2

Give your answer in radians correct to two decimal places.

b) The polynomial $x^3-3x+1=0$ has roots α,β and γ. Find the exact value of $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ 3

c) i) By using graphs or otherwise show that the curves $y=lnx$ and $y=2-x$ have a point of intersection for which the x co-ordinate is close to 1.5. 1

2

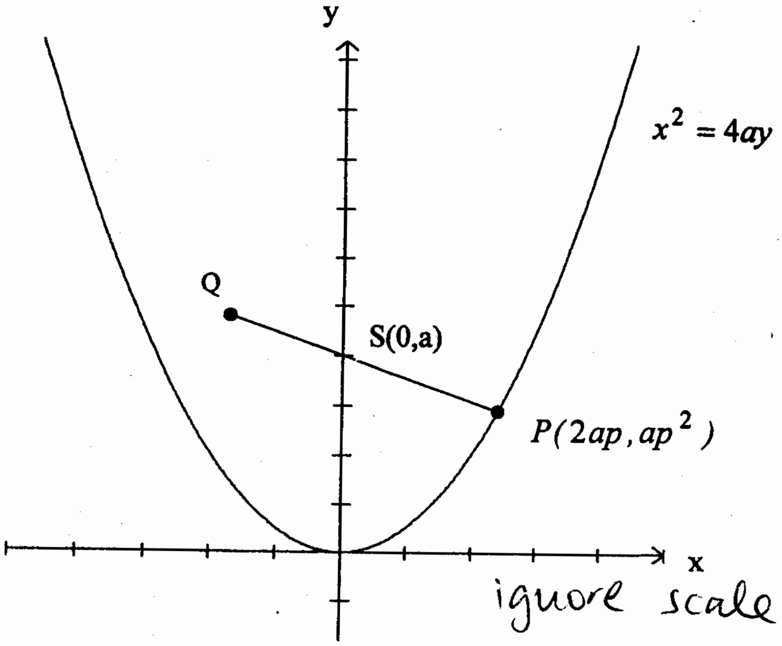
ii) Use $x=1.5$ and one application of Newton's method to find a better approximation of the x co-ordinate of this point of intersection. Give answer correct to two decimal places.

d) Use the substitution $u=x-1$ to evaluate $\int_{2}^{5}\frac{x+1}{\sqrt{x-1}}dx$ 4

Question 3:

Ma

a)



ignore scale.

In the diagram above $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$. The point Q lies on PS produced such that Q divides PS externally in the ratio 3:2.

i) Prove that Q has co-ordinates $(-4ap, a(3-2p^2))$ 2

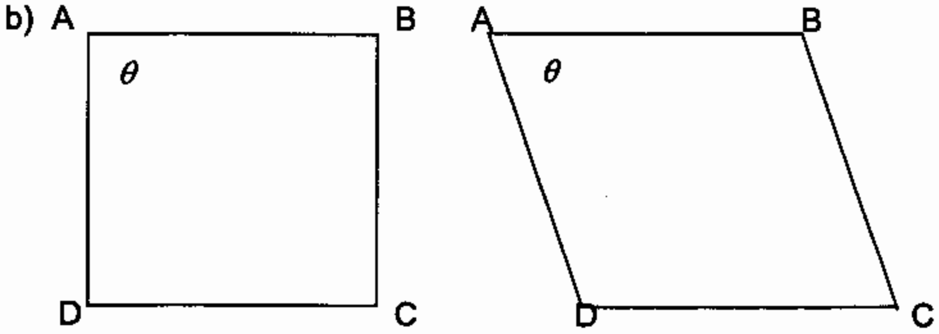
ii) Show that as P varies the locus of Q is another parabola. Find its equation and write down the co-ordinates of its vertex and focus in terms of a. 5

b) Prove by mathematical induction that $\sin(x + n\pi) = (-1)^n \sin x$ where n is a positive integer. 5

Question 4: **Marks:**

a) Solve for $x : \log_{\frac{1}{2}}\left(\frac{1}{x}\right) \geq \log_2(3x-1)$ 3

b)



A square ABCD of side 1 unit is gradually 'pushed over' to become a rhombus. The angle at A (θ) decreases at a constant rate of 0.1^c / second.

i) Show that the area of the rhomubus is equal to $\sin\theta$ 1

ii) At what rate is the area of the rhombus ABCD decreasing when $\theta = \frac{\pi}{6}$? (Give answer correct to 2 decimal places). 2

iii) At what rate is the shorter diagonal of the rhombus ABCD decreasing when $\theta = \frac{\pi}{3}$? (Give answer correct to 2 decimal places). 3

3

c) Prove that $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2$ (for $\sin\theta \neq 0$, $\cos\ \theta \neq\ 0$)

Question 5:

a)

i) Express $\sqrt{3}\ \cos 2t - \sin 2t$ in the form $R\cos(2t+\alpha)$ where **2**

$0 < \alpha < \dfrac{\pi}{2}$

ii) Hence or otherwise find all positive solutions of **3**

$\sqrt{3}\cos 2t - \sin 2t = 0$

b) A particle moves in a straight line and is x metres from a fixed point O after t seconds where:

$x = 5 + \sqrt{3}\cos 2t - \sin 2t$

i) Prove that the acceleration of the particle is $-4(x-5)$. **3**

ii) Between which two points does the particle oscillate. **2**

(You may use your answers from part (a))

iii) At what times does the particle first pass through the point x=5. **2**

Question 6: **Marks:**

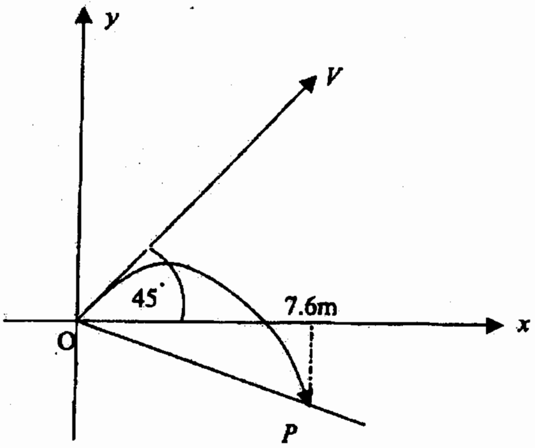
a) The acceleration at any time t of a body moving in a straight line is $-e^{-2x}$.

When t=0, x=0, and v=1.

i) Express its velocity v in terms of x. 3

ii) Express its displacement x in terms of time t. 3

b) A garden hose placed at the top of an incline releases a stream of water with a velocity of 8m/s at an angle of 45^0 with the horizontal. Assuming that $x = vt\cos\alpha$ and $y = vt\sin\alpha - \frac{1}{2}gt^2$ where x and y are the horizontal and vertical displacements of the stream of water from O at any time, $g = 10m/s^2$ and the coordinate axes are taken as shown.



i) Show that the equation of the path of the stream of water is given by $y = x - \frac{5x^2}{32}$ 3

ii) If the stream of water strikes the incline at the point P, 7.6m horizontally from O, find the equation of the incline. 3

Question 7:

a) During the early summer months the rate of increase of the population P of fruit flies is proportional to the excess of the population over 3000.

$\frac{dP}{dt} = k(P-3000)$ where k is a constant. At the beginning of summer the population is 4000 and 1 month later it is 10 000.

i) Show that $P = 3000 + Ae^{kt}$ is a solution of the differential equation, A is a constant. 1

ii) Find the value of A and that of k. 2

iii) Find to the nearest 100, the population after 2 ½ months. 1

iv) After how many weeks does the population reach ½ million? (Give your answer to 1 decimal place). 1

b) Consider the function $y = x^3e^{-x}$

i) State the greatest possible domain of the function. 1

ii) Find the maximum value of the function in the domain. 2

iii) Show $\underline{\text{that there are 3 points}}$ of inflexion and that one of them has a horizontal tangent. 3

iv) Sketch the curve for $-1 \le x \le 6$ 1

END OF EXAMINATION

TRIAL HSC 2003 - Extension One SOLUTIONS

	COMMENTS
Question 1.	
a) $\int_0^3 \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\frac{x}{3}\Big]_0^3$ ✓ $= \frac{1}{3}\tan^{-1}1 - \frac{1}{3}\tan^{-1}0$ $= \frac{1}{3} \times \frac{\pi}{4}$ $= \frac{\pi}{12}$ ✓	• some extension II students made this a difficult integral. • some forgot the $\frac{1}{3}$ at the front • answer needs to be in degrees
b) $\int_0^4 \frac{dx}{\sqrt{9+x^2}} = \ln\left(x+\sqrt{x^2+9}\right)\Big]_0^4$ ✓ $= \ln 9 - \ln 3$ $= \ln 3$ ✓	• Some students did not use table of integral • another error $I = \ln(3+\sqrt{9+x^2})$
c) $\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$ ✓ $= \frac{2\sin^2\theta}{2\sin\theta\cos\theta}$ $= \tan\theta$ ✓	• generally well done.
$\tan 67\frac{1}{2}^\circ = \frac{1-\cos 135^\circ}{\sin 135^\circ}$ $= \frac{1+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$ ✓ ✓ $= \sqrt{2}+1$	• some need to revise exact values.
d)	

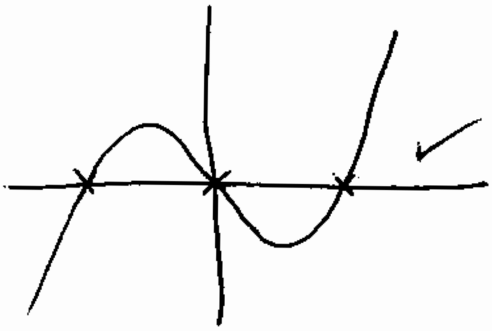
1d) $x \geq \frac{1}{x} \quad x \neq 0.$

$x^3 \geqslant x$

$x^3 - x \geqslant 0$ ✓

$x(x^2-1) \geqslant 0$

$x(x-1)(x+1) \geqslant 0.$ ✓



✓

$-1 \leq x < 0 \quad \text{or} \quad x \geqslant 1$ ✓

OR

If $x > 0$

$x^2 \geqslant 1$ ✓

$\therefore \; x \geqslant 1$ or $x \leq -1$

$\therefore$ Soln $x \geqslant 1.$ ✓

If $x < 0$

$x^2 \leq 1$ ✓

$-1 \leq x \leq 1.$

$\therefore$ Soln $-1 \leq x < 0.$ ✓

COMMENT.

- generally poorly done.

Need to solve by

a) examining critical pts

<u>or</u>

b) mult. b s. by x^2.

<u>or</u>

c) use <u>two</u> cases - give 2 partial soln's and then an <u>o'all</u> soln.

SOLUTIONS COMMENTS

Question 2.

a) $m_1 = \frac{1}{3} \quad m_2 = -\frac{2}{3}$

$$\tan\theta = \frac{\frac{1}{3} + \frac{2}{3}}{1 + \frac{1}{3} \times -\frac{2}{3}}$$

$$= \frac{1}{1 - 2/9}$$

$$= \frac{9}{7} \quad ✓$$

$\theta = 0.91^c$ to two decimal places. ✓

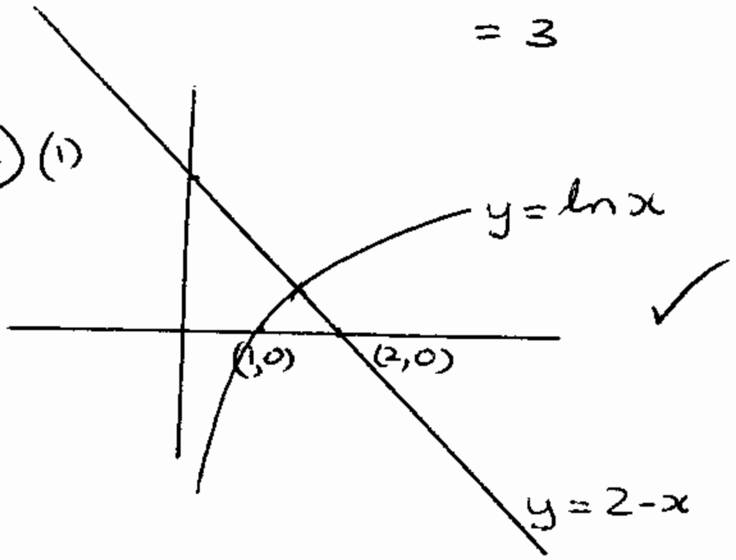
-½ for error finding approximate value of $\tan^{-1} 9/7$. No penalty if in degrees.

b) $$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} \quad ✓$$

$$= \frac{-3 ✓}{-1 ✓}$$

$$= 3$$

c) (i)



✓

Intersection point close to $x = 1.5$

(ii) $P(x) = \ln x - 2 + x$

$P'(x) = \frac{1}{x} + 1$

$$\therefore x_2 = 1.5 - \left(\frac{\ln 1.5 - 0.5}{1^2/3}\right) \quad ✓$$

N.

$= 1.56$ correct to 2 dec. places ✓

d) $\int_2^5 \frac{x+1}{\sqrt{x-1}}\,dx$

$u = x-1$
$du = dx$

$= \int_1^4 \frac{u+2}{\sqrt{u}}\,du$

① correct substition ✓
① correct limits

$= \int_1^4 \left(\sqrt{u} + 2u^{-1/2}\right) du$ ✓

$= \frac{2}{3}u^{3/2} + 4u^{1/2}\Big]_1^4$ ✓

$= \left(\frac{16}{3} + 8\right) - \left(\frac{2}{3} + 4\right)$

✓

$= 8\frac{2}{3}$

No penalty for small arithmetic error to get $8\frac{2}{3}$

SOLUTIONS. COMMENTS

Question 3

a) (i) $P(2ap, ap^2)$ $S(0,a)$

$$\therefore Q = \left(\frac{-2\times 2ap+0}{1}, \frac{-2\times ap^2+3a}{1}\right)$$

$$= (-4ap, \; a(3-2p^2))$$

✓ ✓

(ii) From $x = -4ap$

$$p = \frac{x}{-4a}$$

$$y = a\left(3 - 2\left(\frac{x^2}{16a^2}\right)\right) ✓$$

$$= 3a - \frac{x^2}{8a}$$

$$\frac{x^2}{8a} = -y + 3a$$

$$x^2 = -8a(y-3a) ✓✓$$

Any variation of this quadratic was marked correct.

This is another parabola

Vertex $(0, 3a)$ ✓

Focal length is $2a$. $\therefore$ Focus $(0,a)$ ✓

Many Students did not find focal length $4l = -8a$, $l = -2a$

b) If $n=1$. $\sin(x+\pi) = \sin x\cos\pi + \cos\pi\sin x$

$$= -\sin x + 0$$

$$= (-1)^1 \sin x$$

$\therefore$ True for $n=1$

Assume true for $n = k$ ✓

ie $\sin(x + k\pi) = (-1)^k \sin x$

consider $n = k+1$

$$\sin(x + (k+1)\pi) = \sin(x + k\pi + \pi) \quad ✓$$
$$= \sin(x + k\pi)\cos\pi + \cos(x + k\pi)\sin\pi$$
$$= (-1)^k \sin x \times -1 + 0 \quad ✓$$
$$= (-1)^{k+1} \sin x \quad ✓$$

which is of same form as for $n = k$.

∴ If true for $n = k$, it is also true for $n = k+1$. Since it is true for $n = 1$, it is true for $n = 2$ and ✓ hence all following positive integers.

One mark for both ! Step 1 and Step 2.
3 marks for Step 3

SOLUTIONS	COMMENTS
Question 4.	
a) $\log_{\frac{1}{2}} \frac{1}{x} \geqslant \log_2 (3x-1)$ $\frac{\log_2 \frac{1}{x}}{\log_2 \frac{1}{2}} \geqslant \log_2 (3x-1)$ ✓ $\frac{\log_2 \frac{1}{x}}{-1} \geqslant \log_2 (3x-1)$ $\log_2 x \geqslant \log_2 (3x-1)$ ✓ $x \geqslant 3x-1$ $2x-1 \leq 0$ $x \leq 1/2$ But $x > 1/3$ since $3x-1>0$. ∴ soln $1/3 \leq x \leq 1/2$ ✓	Many Problems Not Many people realised $3x-1>0$ $\therefore x > \frac{1}{3}$ Straight away ✓ $\boxed{3}$
b)(i) Area $= 2 \times \left(\frac{1}{2} \times 1 \times 1 \times \sin\theta\right)$ $= \sin\theta$ ✓	Good
(ii) $\frac{dA}{dt} = \frac{dA}{d\theta} \frac{d\theta}{dt}$ $= \cos\theta \times -0.1^c$ ✓ when $\theta = \frac{\pi}{6}$ $\frac{dA}{dt} = \frac{\sqrt{3}}{2} \times -0.1$ $= -0.09 \ u^2/s.$ ✓	Ok Substitution caused problems.

(iii) $l^2 = 1^2 + 1^2 - 2\times1\times1\times\cos\theta$

$= 2 - 2\cos\theta$

$l = \sqrt{2}\,(1-\cos\theta)^{1/2}$ ✓

Ok

$\frac{dl}{dt} = \frac{dl}{d\theta}\cdot\frac{d\theta}{dt}$ ✓

$= \left(\sqrt{2}\cdot\times\frac{1}{2}\times(1-\cos\theta)^{-1/2}\times\sin\theta\right)\times -0.1$

Differentiation of l caused problems

A lot of people left off the $\sin\theta$ from chain rule

At $x = \frac{\pi}{3}$ $\frac{dl}{dt} = \sqrt{2}\times\frac{1}{2}\times\sqrt{2}\times\frac{\sqrt{3}}{2}\times -0.1$

$= -0.09$ u/s. ✓

$\boxed{/6}$

e) $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta}$

$= \frac{\sin\theta\cos 2\theta + \cos\theta\sin 2\theta}{\sin\theta} - \frac{\cos\theta\cos 2\theta - \sin\theta\sin 2\theta}{\cos\theta}$ ✓

$= \cos 2\theta + \frac{\cos\theta}{\sin\theta}(\sin 2\theta) - \cos 2\theta + \frac{\sin\theta}{\cos\theta}\cdot\sin 2\theta$ ✓

$= 2\sin\theta\cos\theta\left(\frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}\right)$ ✓

$= 2.$

Well done Overall

$\boxed{/3}$

· SOLUTIONS COMMENTS.

Question 5.

a) (i) $\sqrt{3}\cos 2t - \sin 2t = \overset{\checkmark}{2}\cos(2t+\alpha)$

$$\left[\begin{aligned} \tan\alpha &= \tfrac{1}{\sqrt{3}} \\ \alpha &= \tfrac{\pi}{6} \end{aligned}\right] \checkmark$$

$$= 2\cos\left(2t + \frac{\pi}{6}\right)$$

(ii) $\cos\left(2t + \frac{\pi}{6}\right) = 0.$

$$2t + \frac{\pi}{6} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$2t = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \ldots$$

$$t = \frac{\pi}{6}, \frac{4\pi}{6}, \frac{7\pi}{6}, \ldots$$

$$= \frac{(3n+1)\pi}{6} \text{ for } n = 0, 1, 2, \ldots$$

Many did not know. general solution Either this way OR

$$2t + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{2}$$

$$\therefore 2t = 2n\pi - \frac{\pi}{6} \pm$$

$$t = n\pi - \frac{\pi}{12} \pm \frac{\pi}{4}$$

b) (i) $x = 5 + \sqrt{3}\cos 2t - \sin 2t$

$$\dot{x} = -2\sqrt{3}\sin 2t - 2\cos 2t \checkmark$$

$$\ddot{x} = -4\sqrt{3}\cos 2t + 4\sin 2t \checkmark$$

$$= -4(\sqrt{3}\cos 2t - \sin 2t)$$

$$= -4(x-5). \checkmark$$

Well done.

(ii) Motion is Simple Harmonic Motion centre of motion is 5, ✓ amplitude is 2 from part (a). ✓

$\therefore$ Particle oscillates between 3 and 7

Many did not use part (a) hence made extra work for themsel

(iii) At $x = 5$: $5 = 5 + \sqrt{3}\cos 2t - \sin 2t$ ✓

$$0 = \sqrt{3}\cos 2t - \sin 2t \checkmark$$

$$\therefore t = \frac{\pi}{6} \text{ (from (ii))} \checkmark$$

SOLUTIONS

COMMENT

Question 6.

a) (i) $\ddot{x} = -e^{-2x}$

$$\frac{d}{dx}\frac{1}{2}v^2 = -e^{-2x}$$

$$\frac{1}{2}v^2 = \frac{-e^{-2x}}{-2} + C \checkmark$$

$$\frac{v^2}{2} = \frac{e^{-2x}}{2} + C$$

when $v = 1$, $x = 0$

$$\frac{1}{2} = \frac{1}{2} + C \quad \therefore \quad C = 0. \checkmark$$

$$\therefore v^2 = e^{-2x}$$

$$v = \pm\sqrt{e^{-2x}}$$

Since $v = 1$ when $x = 0$

$$v = \sqrt{e^{-2x}} \checkmark = e^{-x}$$

(ii) $v = e^{-x}$

$$\frac{dx}{dt} = e^{-x}$$

$$\frac{dt}{dx} = e^{x} \checkmark$$

$$t = e^{x} + C$$

when $t = 0$, $x = 0$ $\therefore$ $C = -1$

$$t = e^{x} - 1 \checkmark$$

$$e^{x} = t + 1$$

$$x = \ln(t+1) \checkmark$$

Most students wh used $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2$ performed well on this questio However, many did not explai why v is the positive squar root.

Some students used $\frac{dx}{dt}$ correctl but did not make x the subject. Some used an incorrect formu from (i) and could not achie a final result.

SOLUTIONS

c) (i) $x = Vt\cos\alpha$, $y = Vt\sin\alpha - \frac{1}{2}gt^2$

$g = 10$, $V = 8$, $\alpha = 45^\circ$.

$\therefore x = 4\sqrt{2}t$, $y = 4\sqrt{2}t - 5t^2$ ✓

$\therefore y = 4\sqrt{2}\left(\frac{x}{4\sqrt{2}}\right) - 5 \cdot \frac{x^2}{32}$ ✓

$= x - \frac{5x^2}{32}$. ✓

(ii) Equation of line of incline is

$y = mx$. ✓

$\therefore mx = x - \frac{5x^2}{32}$

when $x = 7.6$

$7.6m = 7.6 - \frac{5 \times 7.6^2}{32}$ ✓

$m = 1 - \frac{5 \times 7.6}{32}$

$= -\frac{3}{16}$

$\therefore y = -\frac{3x}{16}$ is required equation. ✓

$= 0{\cdot}1875x$

COMMENTS

Generally performed well

Some students made this question much more complicated than it was meant to be

QUESTION 7 : SOLUTIONS

COMMEN

a)(i) $P = 3000 + Ae^{kt}$ so $Ae^{kt} = P - 3000$

$$\frac{dP}{dt} = kAe^{kt}$$

$$= k(P - 3000) \quad ✓$$

many students were unclear in their starting point, and what substitutions they were making.

(ii) $t = 0, \quad P = 4000$

$$4000 = 3000 + Ae^{0}$$

$$\therefore A = 1000 \quad ✓$$

$t = 1, \quad P = 10,000$

$$\therefore 7000 = 1000e^{k}$$

$$e^{k} = 7$$

$$k = \ln 7 \quad ✓$$

$$\doteqdot 1.946 \text{ to } 3 \text{ dec places.}$$

(iii) $t = 2.5$

$$P = 3000 + 1000e^{2.5k} \quad ✓$$

$$= 132600 \text{ to nearest } 100$$

(iv) $500,000 = 3000 + 1000e^{kt}$

$$497000 = 1000e^{kt}$$

$$e^{kt} = 497$$

$$kt = \ln 497$$

$$t = 3.19\ldots \text{ months}$$

$$= 12.8 \text{ weeks} \quad ✓$$

correct to 1 dec. place.

• many could not conver months to weeks! (and thus lost the mar

· SOLUTIONS

b) (I) $D = \{x : x \in R\}$. ✓

(II) $y' = -x^3e^{-x} + 3x^2e^{-x}$

$= x^2e^{-x}(-x+3)$

$= 0$ if $x = 0$ or $x = 3$.

$y'' = x^3e^{-x} + -e^{-x} \cdot -3x^2 + -3x^2e^{-x}$
$+ 6xe^{-x}$

$= xe^{x}(x^2 - 6x + 6)$ ✓

At $x = 0$ $y'' = 0$ ∴ horizontal inflection

$x = 3$ $y'' < 0$ ∴ max at $x = 3$

Maximum value is $3^3e^{-3} = \frac{27}{e^3}$ ✓

(III) Possible pts of inflection at

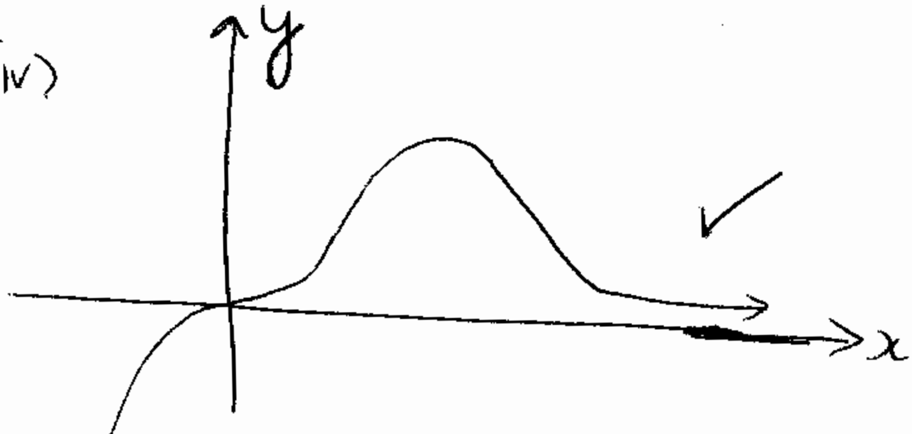
$y'' = 0$ ie $x = 0$ or $x = \frac{6 \pm \sqrt{12}}{2}$ ✓

$= 3 \pm \sqrt{3}$. ✓

At $x = 0$ horizontal inflection

Since changes concavity at each point - all are points of inflection ✓

(IV)



✓

COMMENTS

① showing $x = 3$ is a max (any method) - this was often left out.

① max value - many did not read the question and left this out.

① 3 points from $\frac{d^2y}{dx^2} = 0$

① 3 points are correct

① show $x = 0$ is horizontal ⇒ indicate $y' = 0$.

Many lost the graph mark because horizontal inflection at $x = 0$ was not clear!