The Scots College

Year 12 Mathematics Extension 2

Assessment 4

August 2005

GENERAL INSTRUCTIONS

- Working time - 3 hours
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Standard Integrals Table attached

TOTAL MARKS: **120**

WEIGHTING: **40 %**

- **Start each QUESTION on a new answer booklet**

QUESTION 1 (15 marks)

a) Evaluate $|3+2i|$ **1**

b) i) If $v = \dfrac{1+i\sqrt{3}}{2}$ show that $v^3 = -1$. **2**

ii) Hence calculate v^{10} . **2**

c) If z is a complex number so that $|z| = 2$ and $\arg z = \dfrac{\pi}{6}$, mark clearly on the same Argand diagram the points representing the complex numbers:

i) z ii) iz iii) $\bar{z}$ iv) $\dfrac{1}{z}$ v) $z\,\bar{z}$ vi) z^2 vii) $z^2 + z$ viii) $z^2 - z$ **10**

QUESTION 2 (15 marks)

a) Find $\displaystyle\int \frac{dx}{x^2 - 6x + 13}$ **2**

b) Find $\displaystyle\int \tan x \sec^2 x \, dx$ **2**

c) i) Show that f (x) = $\sin^{-1}x$ is an odd function. **2**

ii) Hence or otherwise find $\displaystyle\int_{-1}^{1} (\sin^{-1} x)^3 dx$ **1**

d) $\displaystyle\int_{0}^{\sqrt{2}} \sqrt{4 - x^2} \, dx$ **4**

e) $\displaystyle\int e^x \cos x \, dx$ **4**

QUESTION 3 (10 marks)

a) Use the method of cylindrical shells to find the volume of the solid (paraboloid) obtained when the region between the curve $y = \frac{1}{2}\sqrt{x-2}$, the x-axis and the line x = 6 is rotated about the x axis. **4**

b) Prove $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$, where x denotes displacement, and v denotes velocity. **2**

c) The acceleration of a particle moving in a straight line is given by $\ddot{x} = xe^x$,

where x is the displacement from 0. The particle is initially at rest.

The particle starts at $x = 0$.

i) Prove that $v^2 = 2e^x(x\text{-}1) + 2$ **3**

ii) Describe the subsequent motion of the particle after it leaves the origin and explain why the particle can only move in one direction **1**

QUESTION 4 (18 marks)

a) The equation $x^3 - x^2 - 3x+2 = 0$ has roots α, β, γ . Find the monic polynomial equation with roots $\alpha^2, \beta^2, \gamma^2$. **4**

b) If $x = \alpha$ is a double root of the equation P(x) = 0, show that $x = \alpha$ is a root of the equation $P'(x) = 0$. **4**

c) i) Show that $1+ \boldsymbol{i}$ is a root of the polynomial Q(x) = $x^3 + x^2 - 4x +6$ **2**

ii) hence resolve Q(x) into irreducible factors over the complex number field. **3**

d) If α , β , γ are the roots of the cubic equation x^3 + qx + r = 0, prove that $(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2 = -6q$. **5**

QUESTION 5 (18 marks)

The ellipse $\mathcal{E}$ has the cartesian equation $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1.$

i) Write down the eccentricity **1**

ii) Write down the coordinates of the foci S and S' **1**

iii) Write down the equations of the directrices. **1**

iv) Sketch the ellipse $\mathcal{E}$. **1**

v) Show that any point P on $\mathcal{E}$ can be represented by the coordinates $(5\cos\theta, 4\sin\theta)$ **1**

vi) Prove that $PS + PS'$ is independent of the position of P on the ellipse $\mathcal{E}$. **3**

vii) Show that the equation of the normal N at the point P on the ellipse $\mathcal{E}$ is **2**
$5\sin\theta\, x - 4\cos\theta\, y = 9\sin\theta\cos\theta$

viii) If this normal meets the major axis of the ellipse in M and the minor axis in N, prove that $\dfrac{PM}{PN}=\dfrac{16}{25}$. **3**

ix) Also show that the line PN bisects the angle $S'PS$. **5**

QUESTION 6 (14 marks)

i) By considering the curve $g(x) = x^6 - 4x^5 + 4x^4$, sketch the graph of $f(x) = x^6 - 4x^5 + 4x^4 - 1$ showing that it has 4 real zeroes. **4**

On different diagrams sketch the curves:

ii) $y = |f(x)|$ **2**

iii) $y = f(|x|)$ **2**

iv) $y^2 = f(x)$ **3**

v) Calculate the slope of the curve $y^2 = f(x)$ at any point x and describe the nature of the curve at a zero of $f(x)$. **3**

QUESTION 7 (15 marks)

a) A parachutist of M kilograms is dropped from a stationary helicopter of height H metres above the ground. The parachutist experiences air resistance during its fall equal to MkV^2, where V is its velocity in metres per second and k is a positive constant. Let x be the distance in metres of the parachutist from the helicopter, measured positively as it falls.

i) Show that the equation of motion of the parachutist is $\ddot{x} = g - kV^2$, where g is the acceleration due to gravity. **1**

ii) Find V^2 as a function of x. **4**

iii) Find the velocity U of the parachutist as he hits the ground in terms of g, k and H. **1**

iv) Find the velocity of the parachutist as he hits the ground if air resistance is neglected. **2**

b)

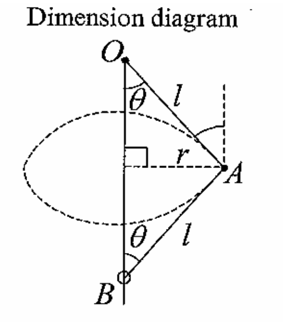
i) Prove the identity $\cos 3A = 4\cos^3 A - 3\cos A$ **2**

ii) Show that $x = 2\sqrt{2}\cos A$ is a root of the equation $x^3 - 6x + 2 = 0$ provided that $\cos 3A = -\frac{1}{2\sqrt{2}}$ **2**

iii) Find the three roots of the equation $x^3 - 6x + 2 = 0$, using the results from part (ii) above. Give your answer to three decimal places. **3**

QUESTION 8 (15 marks)

a) A particle ***A*** of mass ***2m*** is attached by a light inextensible string of length ***l*** to a fixed point ***O*** and is also attached by another light inextensible string of the same length to a small ring ***B*** of mass ***3m*** which can slide on a fixed smooth vertical wire passing through ***O***. The particle ***A*** describes a horizontal circle of radius ***r***, and ***OA*** is inclined at an angle $\theta = \frac{\pi}{3}$ with the downward vertical.



$\theta = \frac{\pi}{3}$

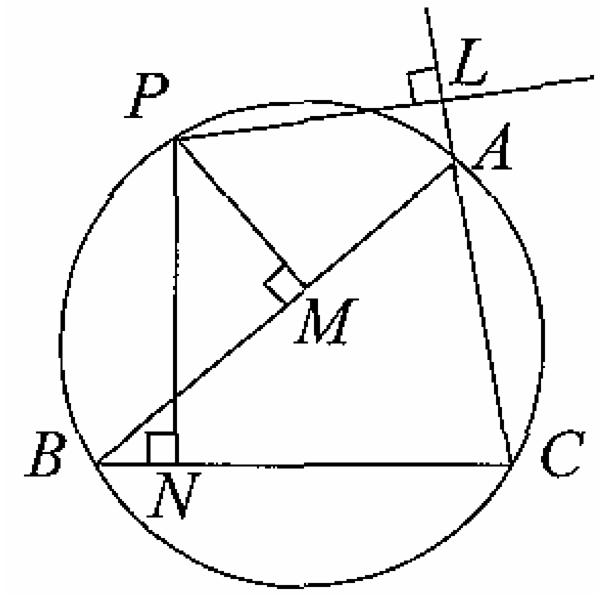
i) Find the tension in the strings ***OA*** and ***AB*** **5**

ii) Find the angular velocity of ***A.*** **3**

iii) Describe what happens to the system as the angular velocity increases. **1**

b) **ABC** is a triangle inscribed in the circle. ***P*** is a point on the minor arc ***AB***. The points ***L***, ***M***, and ***N*** are the feet of the perpendiculars from ***P*** to ***CA*** produced, ***AB***, and ***BC*** respectively.

Copy the diagram into your answer booklet and show that ***L***, ***M*** and ***N*** are collinear.



END OF EXAM

SCOTS

Ext 2: 2005 Trial Solutions

Question 1

a) $\sqrt{3^2 + 2^2} = \sqrt{13}$ ✓ 1

b) i) $v = \dfrac{1 + i\sqrt{3}}{2}$

$v = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$ ✓

$v^3 = \cos \pi + i \sin \pi$ ✓ 2

$v^3 = -1 + 0$

$\boxed{v^3 = -1}$

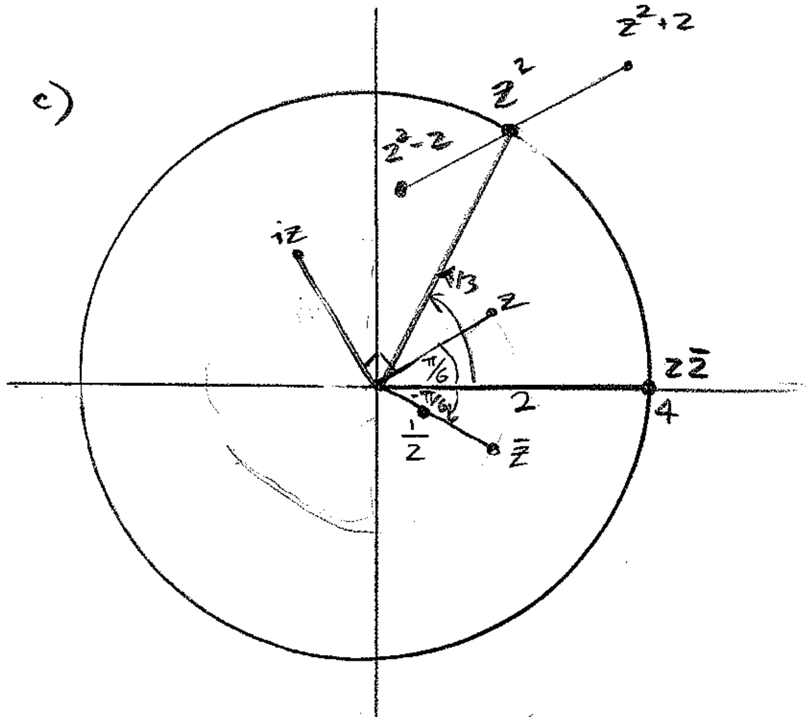
ii) $v^{10} = (v^3)^3 v$ ✓

$= (-1)^3 v$ 2

$= -v$

$= -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ ✓

c)



ii) multiply by 90° or ✓

ii) $z = 2(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})$

$\therefore iz = 2(i \cos \frac{\pi}{6} - \sin \frac{\pi}{6})$

$= 2\left(i\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$ ✓

$= -1 + i\sqrt{3}$

iii) $\bar{z} = 2(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6})$

the conjugate of z is the reflection of pt. z in the x axis. ✓

iv) $\dfrac{1}{z} = \dfrac{1}{2(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})}$

$z^{-1} = \frac{1}{2}\left(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6}\right)$ ✓

$\therefore \frac{1}{z}$ has modulus $= \frac{1}{2}$

argument $= -\frac{\pi}{6}$

v) $z\bar{z} = |r|^2 = 4$ ✓

vi) $(z)^2 = 2^2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$ ✓

by de Moivre Theorem.

$= 4\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

c) vii) $z^2 + z$ using vector addition ✓✓

viii) $z^2 - z$ using vector subtraction. ✓✓

Question 2

a) $\int \frac{dx}{x^2 - 6x + 13}$

$= \int \frac{1}{(x-3)^2 + 4} \, dx$

$= \frac{1}{2} \tan^{-1}\left(\frac{x-3}{2}\right) + c$

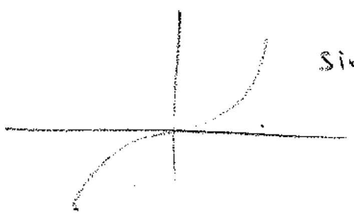
2

b) $\int \tan x \sec^2 x \, dx$

$= \frac{1}{2}(\tan x)^2 + c$ 1

$= \frac{1}{2}\tan^2 x + c$

c) $f(-x) = -f(x)$
for odd functions



$f(x) = \sin^{-1}(x)$

$f(1) = \pi$

$f(-1) = -\pi$

as $f(-x) = -f(x)$
$f(x) = \sin^{-1}(x)$ is an odd function

3

ii) from i) $\sin^{-1}x$ is odd

$\therefore (\sin^{-1}x)^3$ is also odd

$\therefore \int_{-1}^{1} (\sin^{-1}x)^3 \, dx = 0$ due to symmetry.

d) $\int_0^{\sqrt{2}} \sqrt{4 - x^2} \, dx$

$= \int_0^{\pi/4} \sqrt{4 - 4\sin^2\theta} \, . \, 2\cos\theta \, d\theta$

$(x = 2\sin\theta)$

$\frac{dx}{d\theta} = 2\cos\theta$

$= \int_0^{\pi/4} 2\cos\theta \, . \, 2\cos\theta \, d\theta$

$= \int_0^{\pi/4} 4\cos^2\theta \, d\theta$

$= \int_0^{\pi/4} 4\left(\frac{1}{2}(1 + \cos 2\theta)\right) d\theta$

$\int_0^{\pi/4} 2 + 2\cos 2\theta \, d\theta$

$= \left[2\theta + \sin 2\theta\right]_0^{\pi/4}$

$= \frac{\pi}{2} + 1$

4

e) $\int e^x \cos x \, dx$

let
$v' = e^x \quad u = \cos x$
$v = e^x \quad u' = -\sin x$

$= \left[e^x \cos x\right] - \int -\sin x \, (e^x) \, dx$

$= e^x \cos x + \int \sin x (e^x) \, dx$

$= e^x \cos x + e^x \sin x - \int e^x \cos x$

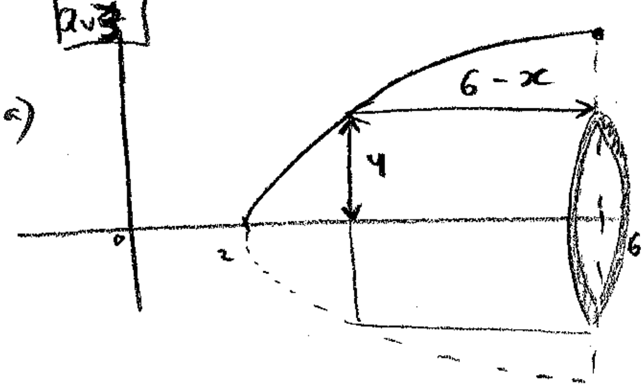
$\therefore 2\int e^x \cos x \, dx = e^x \cos x + e^x \sin x$

$\int e^x \cos x \, dx = \frac{1}{2} e^x (\cos x + \sin x) + c$

4

SECTION B

Q3

a) 

d.

$y = \frac{1}{2}\sqrt{x-2}$, $2y = \sqrt{x-2}$

$4y^2 = x-2$

$r = y$

$\boxed{x = 4y^2+2}$

$h = 6-x$, $6-(4y^2+2) = \boxed{4-4y^2}$

Vol - Shell

$\delta V = 2\pi r h \, \delta y$

$= 2\pi y \times (4-4y^2)\,\delta y$

$= 8\pi y(1-y^2)\,\delta y$ ✓✓

Volume paraboloid

$V = \int_0^1 8\pi y(1-y^2)\,dy$ ✓

$= 8\pi \int_0^1 y - y^3 \, dy$

$= 8\pi \left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1$

$= 2\pi$ ✓ (4)

∴ Volume of paraboloid is 2π cubic units.

b) i) $\frac{d^2x}{dt^2} = \frac{dv}{dt}$

$= \frac{dv}{dx} \cdot \frac{dx}{dt}$

$= v \cdot \frac{dv}{dx}$

and $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\frac{1}{2}v^2 \times \frac{dv}{dx}$

$= v \cdot \frac{dv}{dx}$

$\therefore \boxed{\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)}$ (2)

ii) $\frac{1}{2}v^2 = \int xe^x \, dx$

$= xe^x - \int e^x \, dx$

$= xe^x - e^x + C$

$v^2 = 2xe^x - 2e^x + C$

$0 = -2 + C$ ✓

$\therefore C = 2$

$v^2 = 2xe^x - 2e^x + 2$

$v^2 = 2e^x(x-1) + 2$

ii) $v^2 \geq 0$

$x < 0$, $v^2 < 0$

∴ x must remain +ve.

as $x\uparrow$, $v\uparrow$

(

Qu4

a) $\alpha^2, \beta^2, \gamma^2$ each satisfy

$$\left(x^{\frac{1}{2}}\right)^3 - \left(x^{\frac{1}{2}}\right)^2 - 3\left(x^{\frac{1}{2}}\right) + 2 = 0$$

$$x^{3/2} - 3x^{1/2} = x - 2$$

$$x^3 - 6x^2 + 9x = x^2 - 4x + 4$$

$$x^3 - 7x^2 + 13x - 4 = 0$$

is

b) $P(x) = 0$ has a double root $x = \alpha$

$$\therefore P(x) = (x-\alpha)^2 . Q(x)$$

$$\therefore P'(x) = (x-\alpha)^2 . Q'(x) + Q(x) . 2(x-\alpha)$$

$$= (x-\alpha)\left[(x-\alpha) . Q'(x) + 2 . Q(x)\right]$$

$$\therefore P'(\alpha) = 0$$

$\therefore$ $P'(x) = 0$ has a root at $x = \alpha$.

c) i) $(1+i)^2 = 1 + 2i + i^2 = 2i$

$$(1+i)^3 = 2i(1+i) = 2i - 2$$

$$P(1+i) = (1+i)^3 + (1+i)^2 - 4(1+i) + 6$$

$$= 2i + 2i - 2 - 4 - 4i + 6$$

$$= 0$$

hence $(1+i)$ is a root of $P(x)$

ii) as $(1+i)$ is a root of polynomial $P(x)$, the complex roots occur in conjugate pairs.

thus $1-i$ is also a root of $P(x)$.

$P(x)$ has roots $1+i, 1-i, \gamma$

where γ is the 3rd root.

sum of roots $= -1$

$$\therefore 1+i + 1-i + \gamma = -1$$

$$\gamma = -3$$

ii) cont...

factors of $P(x)$ are:

$$\{x - (1+i)\}\{x - (1-i)\}(x+3)$$

---

d) $(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2$

$$= 2(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= 2\{(\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)\}$$

$$- 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

Now

$\sum \alpha = 0$

$\sum \alpha\beta = q$

$\sum \alpha\beta\gamma = -r$

$$= 2\{0 - 2q\} - 2q$$

$$= -6q$$

Qu 5 $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

$\therefore a = 5, \ b = 4$

i) from $b^2 = a^2(1-e^2)$

$16 = 25(1-e^2)$

$\therefore e^2 = 1 - \dfrac{16}{25}$

$e = \dfrac{3}{5}$ ✓

ii) foci $(\pm ae, 0)$

$(\pm 5 \cdot {}^3/_5, 0)$

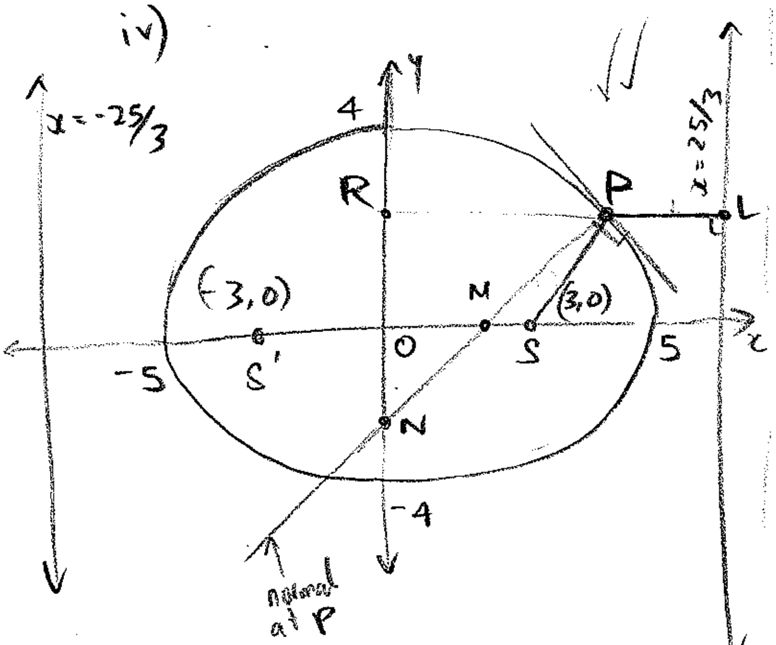
$S(3,0) \quad S'(-3,0)$ ✓

iii) directricies $x = \pm a/e$

$x = \pm 5/(3/5)$

$x = \pm 25/3$ ✓

iv)



v) Sub $P(5\cos\theta, 4\sin\theta)$ into equ. E ✓

$\dfrac{x^2}{25} + \dfrac{y^2}{16} = \dfrac{(5\cos\theta)^2}{25} + \dfrac{(4\sin\theta)^2}{16}$

$= \cos^2\theta + \sin^2\theta = 1$, $\therefore$ P lies on E.

vi) Definition of ellipse

$e = \dfrac{\text{dist. P to S}}{\text{dis. P to directrix}}$

$e = \dfrac{PS}{PL}$

$PS = e \cdot PL$

$= \dfrac{3}{5}\left(\dfrac{25}{3} - 5\cos\theta\right)$

$\boxed{PS = 5 - 3\cos\theta}$ ✓

$\dfrac{PS'}{PL'} = e$

$PS' = ePL'$

$= \dfrac{3}{5}\left(\dfrac{25}{3} + 5\cos\theta\right)$

$\boxed{PS' = 5 + 3\cos\theta}$ ✓

✓

$\therefore PS + PS' = 10$, which is independent of P on curve E

vii) let $x = 5\cos\theta \qquad y = 4\sin\theta$

$\dfrac{dx}{d\theta} = -5\sin\theta \qquad \dfrac{dy}{d\theta} = 4\cos\theta$ ✓

$\dfrac{dy}{dx} = \dfrac{4\cos\theta}{-5\sin\theta}$ equation of tangent

gradient of normal

$= \dfrac{5\sin\theta}{4\cos\theta}$

equ^n normal ✓

$y - 4\sin\theta = \dfrac{5\sin\theta}{4\cos\theta}(x - 5\cos\theta)$

$\therefore 5\sin\theta x - 4\cos\theta y = 9\sin\theta\cos\theta$ is the equ^n of normal.

viii) normal at P meets major axis when $y=0$

$$5\sin\theta x = 9\sin\theta\cos\theta$$

$$x = \frac{9\cos\theta}{5}$$

$$M\left(\frac{9\cos\theta}{5}, 0\right)$$

Normal meets minor axis when $x=0$

$$-4\cos\theta y = 9\sin\theta\cos\theta$$

$$y = -\frac{9\sin\theta}{4}$$

$$N\left(0, -\frac{9\sin\theta}{4}\right)$$

Let R be point on y axis from the extension of LP

$\Delta OMN ||| \Delta RPN$ (AAA)

$$\therefore \frac{NM}{NP} = \frac{NO}{NR} = \frac{OM}{RP}$$

$$\frac{NM}{NP} = \frac{9\cos\theta/5}{5\cos\theta} = \frac{9}{25}$$

(taking horizontal distances)

Thus $\frac{PM}{PN} = \frac{25-9}{25} = \frac{16}{25}$

or use distance formula.

$\therefore M\hat{P}S = S'\hat{P}M$

hence normal at P bisects $\angle S'PS$.

ix) Let $\sin\theta = s$
$\cos\theta = c$

$$\text{Grad } PS = \frac{4\sin\theta}{5\cos\theta - 3} = \frac{4s}{5c-3}$$

$$\text{Grad } PN = \frac{9\sin\theta}{4} \times \frac{5}{9\cos\theta} = \frac{5s}{4c}$$

$$\text{Grad } PS' = \frac{4\sin\theta}{5\cos\theta + 3} = \frac{4s}{5c+3}$$

$$\tan M\hat{P}S = \left|\frac{\frac{5s}{4c} - \frac{4s}{5c-3}}{1 + \frac{5s}{4c}\cdot\frac{4s}{5c-3}}\right|$$

$$= \left|\frac{25sc - 15s - 16sc}{20c^2 - 12c + 20s^2}\right|$$

$$= \left|\frac{9sc - 15s}{20 - 12c}\right| = \left|\frac{3s(3c-5)}{4(5-3c)}\right|$$

$$= \frac{3s}{4}$$

$$= \frac{3\sin\theta}{4}$$

$$\tan S'\hat{P}M = \left|\frac{\frac{5s}{4c} - \frac{4s}{5c+3}}{1 + \frac{5s}{4c}\cdot\frac{4s}{5c+3}}\right|$$

$$= \frac{25sc + 15s - 16sc}{20c^2 + 12c + 20s^2}$$

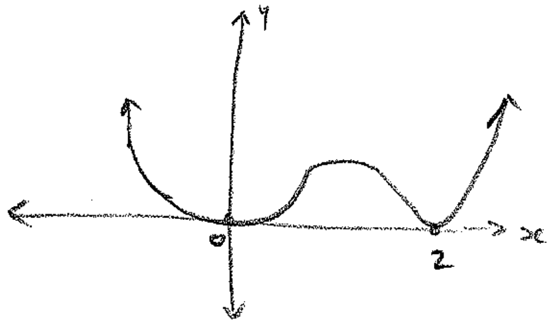
$$= \left|\frac{9sc + 15s}{20 + 12c}\right| = \left|\frac{3s(3c+5)}{4(5+3c)}\right|$$

$$= \frac{3\sin\theta}{4}$$

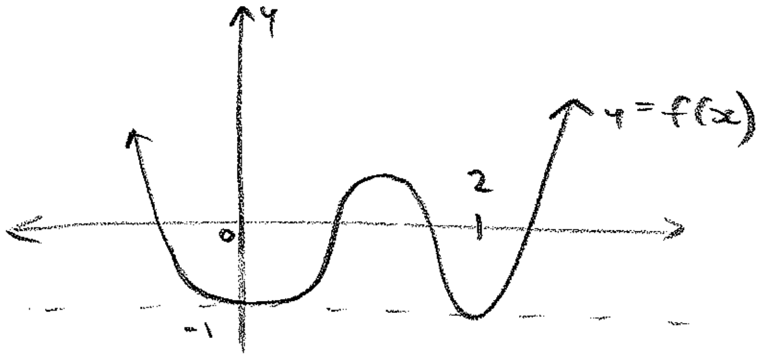
$\boxed{Q6}$ $f(x) = x^6 - 4x^5 + 4x^4 - 1$

a) consider
$y = x^6 - 4x^5 + 4x^4$
$y = x^4(x^2 - 4x + 4)$
$y = x^4(x-2)^2$

$\therefore$ zeroes at $x=0$ and $x=2$



shift curve down 1 unit for $f(x) = x^6 - 4x^5 + 4x^4 - 1$



$\therefore$ two turning points are $(0,-1)$ and $(2,-1)$.
Check for max. t.p b/w $x=0$ + $x=2$. and above x axis

$f'(x) = 6x^5 - 20x^4 + 16x^3$
$= 2x^3(3x^2 - 10x + 8)$
$= 2x^3(x-2)(ax+b)$ as $x=2$ is a solⁿ to $f'(x)$
$= 2x^3(x-2)(3x-4)$ as double root occurs here
by equating coefficients

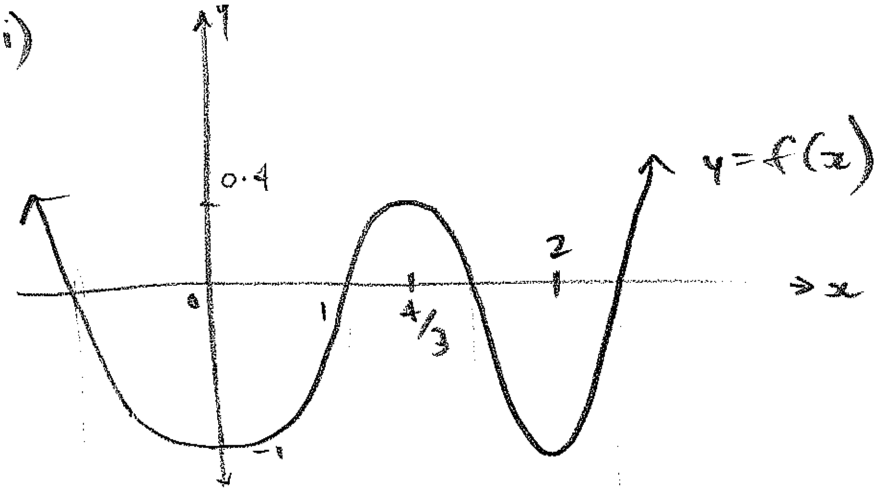
when $f'(x) = 0$
$x = 0,\ x = 2,\ x = 4/3$

Sub. $x = 4/3$ into $y = f(x)$
$y \doteqdot 0.4$ $\therefore$ max t.p above x axis
creating 4 roots.

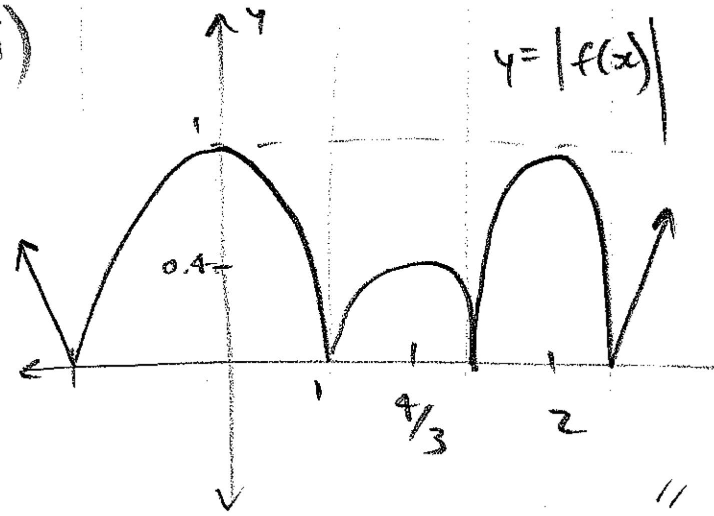
---

or find 4 roots of $y=f(x)$ then sketch curve.

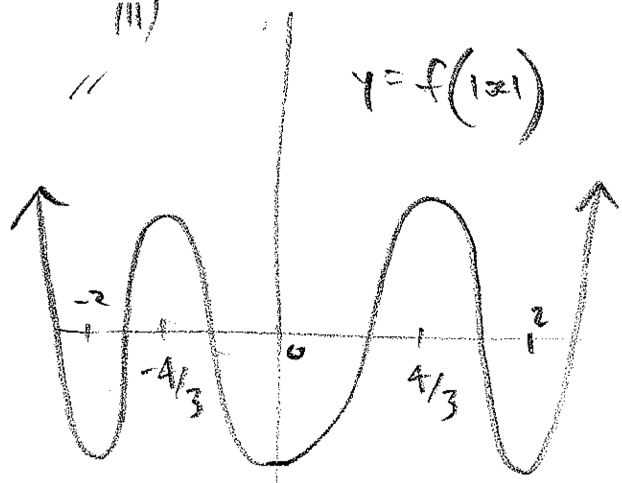
i)



ii)

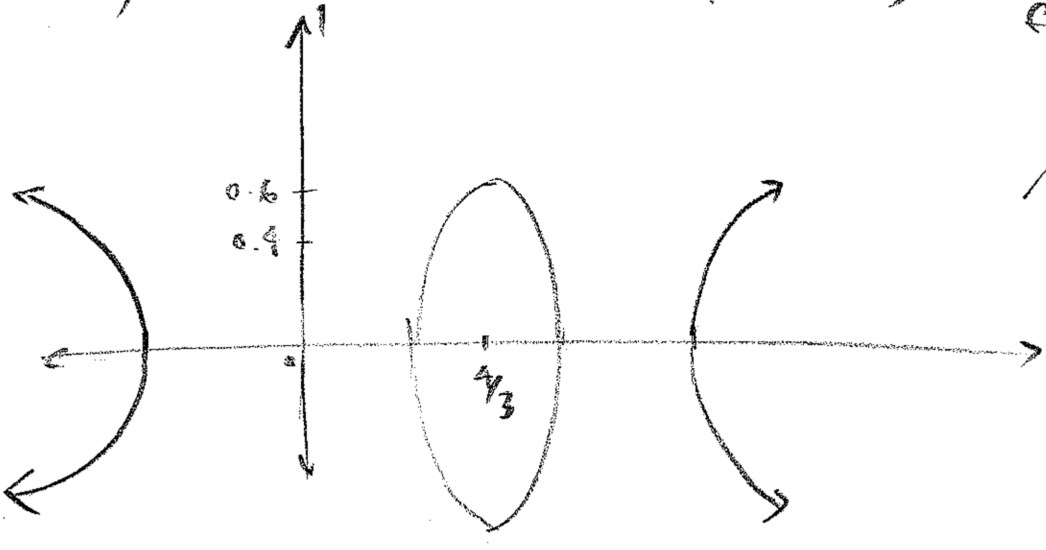


iii)



iii) $y^2 = f(x)$

$\therefore y = \pm\sqrt{f(x)}$, defined for $f(x) \geq 0$



//

This is undefined for $f(x)=0$ $\therefore$ at zeroes of $f(x)$ the curve has vertical tangents.

iv)

$$y^2 = f(x)$$

$$\frac{d}{dy} y^2 \cdot \frac{dy}{dx} = \frac{d}{dx} f(x)$$

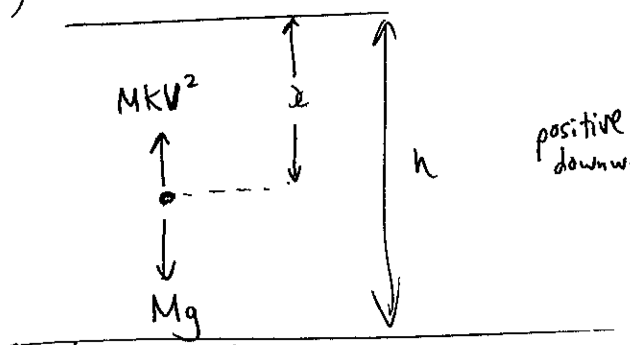
$$2y \cdot \frac{dy}{dx} = f'(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{2y}$$

$$= \pm \frac{f'(x)}{2\sqrt{f(x)}}$$

Qu7

i)



positive downwards

Equation of Motion

$$M\ddot{x} = Mg - MkV^2$$

$$\ddot{x} = g - kV^2$$

ii) $v\dfrac{dv}{dx} = g - kv^2$

$$\frac{v}{g-kv^2}\,dv = dx$$

$$\int \frac{v\,dv}{g-kv^2} = \int dx$$

$$-\frac{1}{2k}\log_e\left(g-kv^2\right) = x + C$$

when $x=0$, $v=0$

so $C = -\frac{1}{2k}\log_e g$

$$-\frac{1}{2k}\log_e\left(g-kv^2\right) = x - \frac{1}{2k}\log_e g$$

$$x = \frac{1}{2k}\log_e\left(\frac{g}{g-kv^2}\right)$$

$$2kx = \log_e\left(\frac{g}{g-kv^2}\right)$$

$$e^{2kx} = \frac{g}{g-kv^2}$$

$$\frac{g-kv^2}{g} = e^{-2kx}$$

$$g-kv^2 = ge^{-2kx}$$

$$kv^2 = g\left(1-e^{-2kx}\right)$$

$$v^2 = \frac{g}{k}\left(1-e^{-2kx}\right)$$

iii) when $x=h$, $v=V$

$$V^2 = \frac{g}{k}\left(1-e^{-2kh}\right)$$

$$V = \sqrt{\frac{g}{k}\left(1-e^{-2kh}\right)}$$

iv) Without air resistance, the equation of motion is

$$\ddot{x} = g$$

$$v\frac{dv}{dx} = g$$

$$\frac{1}{2}v^2 = gx + C$$

when $x=0$, $v=0$, $c=0$

$$\therefore v^2 = 2gx$$

when $x=h$, $v=V$

so $V^2 = 2gh$

$$V = \sqrt{2gh}$$

$\boxed{7b}$

a) i) $\cos 3A = \cos(2A + A)$

$$= \cos 2A \cos A - \sin 2A \sin A$$

$$= (2\cos^2 A - 1)\cos A - 2\cos A \sin A \sin A$$

$$= 2\cos^3 A - \cos A - 2\cos A \sin^2 A$$

$$= 2\cos^3 A - \cos A - 2\cos A(1 - \cos^2 A)$$

$$\boxed{\cos 3A = 4\cos^3 A - 3\cos A}$$

ii) Sub $x = 2\sqrt{2}\cos A$ into $x^3 - 6x + 2 = 0$

$$16\sqrt{2}\cos^3 A - 12\sqrt{2}\cos A + 2 = 0$$

$$16\cos^3 A - 12\cos A = \frac{-2}{\sqrt{2}}$$

$$4\cos^3 A - 3\cos A = \frac{-1}{2\sqrt{2}}$$

$$\cos 3A = \frac{-1}{2\sqrt{2}} \quad \text{(from i)}$$

$\dfrac{-1}{2\sqrt{2}} = \dfrac{-1}{2\sqrt{2}}$ (from qu) $\quad \therefore\ x = 2\sqrt{2}\cos A$ is a root of the equation $x^3 - 6x + 2 = 0$

iii) $\cos 3A = -\dfrac{1}{2\sqrt{2}}$

$\therefore 3A = \pm \cos^{-1}\left(-\dfrac{1}{2\sqrt{2}}\right) + 2n\pi$, where n is a +ve integer

Using $n = 1, 2, 3$ & +ve values of $\cos^{-1}$.

$3A = 1.93216, \quad 8.21535, \quad 14.49853$

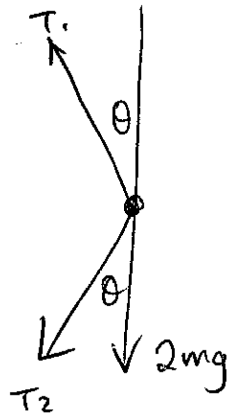
$A = 0.64405, \quad 2.73845, \quad 4.83284$

$\therefore x = 2\sqrt{2}\cos A$

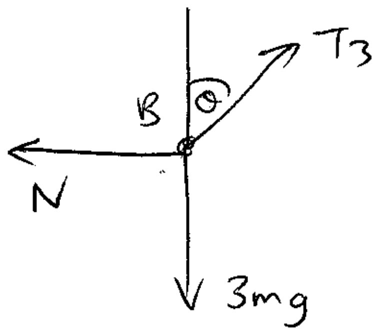
$x = 2.262, \ -2.602, \ 0.340$ to 3 dec place

8 a)

Forces on A



Forces on B



i)

Sum Vertical forces

$$2mg + T_2 \cos\theta = T_1 \cos\theta \quad - 1$$

Sum Horizontal forces

$$T_2 \sin\theta + T_1 \sin\theta = 2mr\omega^2 \quad - 2$$

Sum vert. forces

$$T_3 \cos\theta = 3mg \quad - 3$$

Sum horizontal forces

$$T_3 \sin\theta = N \quad - 4$$

$$T_2 = T_3$$

$$\therefore T_1 = T_2 + \frac{2mg}{\cos\theta} \quad (\text{from } 1)$$

$$T_1 = \frac{3mg}{\cos\theta} + \frac{2mg}{\cos\theta} \quad (\text{from } 3)$$

$$\boxed{T_1 = 10mg}$$

$$\boxed{T_2 = 6mg} \quad (\text{from } 3)$$

5

ii) $T_1 \sin\theta + T_2 \sin\theta = 2mr\omega^2$

sub $r = l\sin\theta$

$$T_1 \cancel{\sin\theta} + T_2 \cancel{\sin\theta} = 2ml\omega^2$$

3

$$\omega^2 = \frac{T_1 + T_2}{2ml}$$

$$\omega^2 = \frac{16mg}{2ml} \qquad \therefore \omega = \sqrt{\frac{8g}{l}}$$

iii) height above centre of circle decreases, or radius of circle increases.

1

Qu 8 b)

In order to prove L, M and N are collinear, we can show $\angle LMA = \angle NMB$

Step 1

In Δ's PKM + BKN

$\angle BKN = \angle PKM$ (vert opp.)

$\angle BNK = \angle PMK$ (90° given)

$\therefore \Delta PKM ||| \Delta BKN$ (AAA)

$\therefore \frac{BK}{PK} = \frac{NK}{MK}$

In Δ's PKB + MKN

$\angle PKB = \angle MKN$

$\frac{BK}{NK} = \frac{MK}{PK}$

$\therefore \Delta PKB ||| \Delta MKN$ (2 sides ratio + included $\angle$)

$\therefore \angle NMB = \angle BPN$ (corr. $\angle$'s $|||$ Δ's)

Step 2 PACB is a cyclic quad

$\angle PAC + \angle PBC = 180°$ (opp. $\angle$'s cyclic quad supp.)

and $\angle PBC = \angle PAL$ (ext. $\angle$ cyclic quad = opp. interior $\angle$)

$\therefore \Delta PNB ||| \Delta PLA$ (AAA)

$\therefore \angle BPN = \angle APL$ (corr. $\angle$'s $|||$ Δ's)

Step 3

$\Delta ALS ||| \Delta PMS$ (AAA)

$\therefore \frac{PS}{AS} = \frac{MS}{LS}$

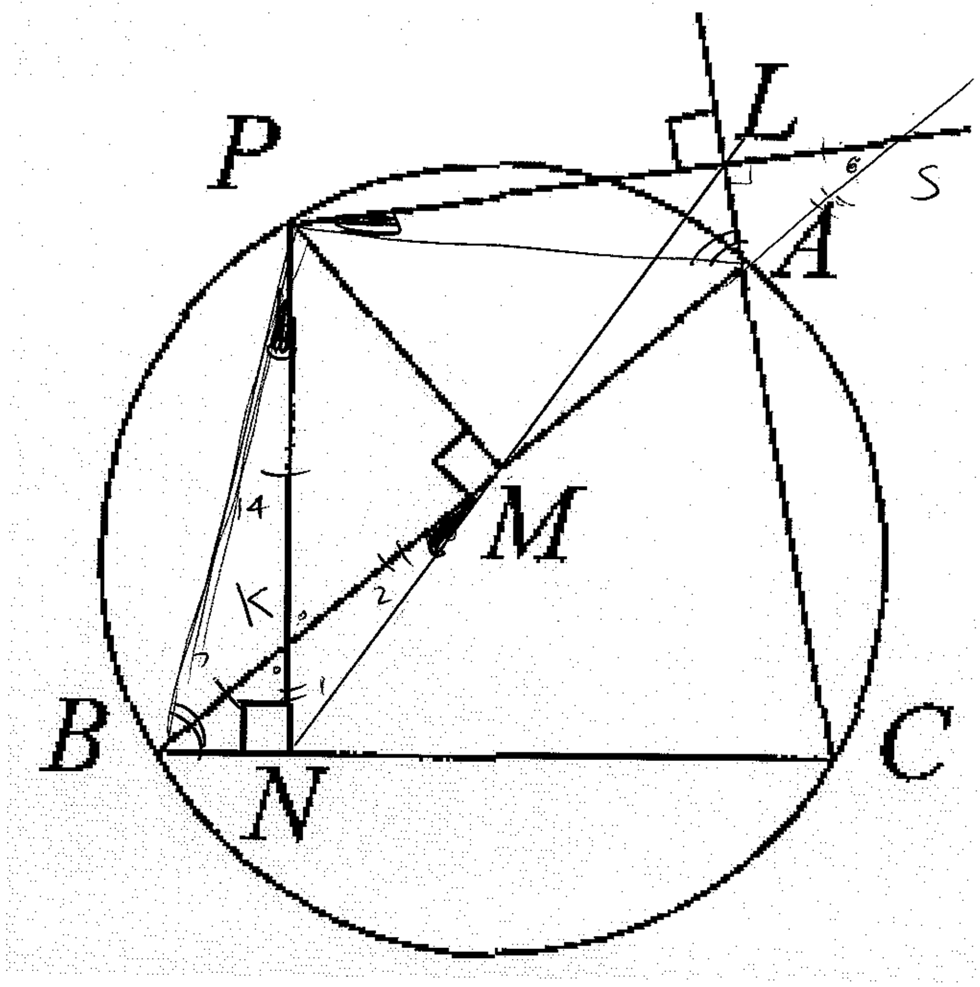
$\therefore \Delta MLS ||| \Delta PAS$ (2 sides ratio + included $\angle$)

$\therefore \angle SPA = \angle LMA$ (corr. $\angle$'s $|||$ Δ's)

$\therefore \angle LMA = \angle NMB$

and $\therefore$ LMN are collinear.

6



P
L
S
A
M
B
C
N
K