Ascham School

Trial Examination

Mathematics Extension 2

2002

1. **(a)** Let $z = -1 - \sqrt{3}i$
(i) Write z in modulus-argument form.
(ii) Show that z^6 is a real number.
(b) **(i)** Simplify $(\sqrt{3} + \sqrt{3}i)^2$
(ii) Solve $z^2 - (1-i)z - 2i = 0$ writing the solutions in the form $x + iy$, where x and y are real.
(c) Sketch the region in the complex number plane in which the following inequalities all hold:

$$|z-4| < |z-4i| \text{ and } |z-4| \leq 4 \text{ and } 0 \leq \arg(z-4) < \tfrac{3\pi}{4}$$

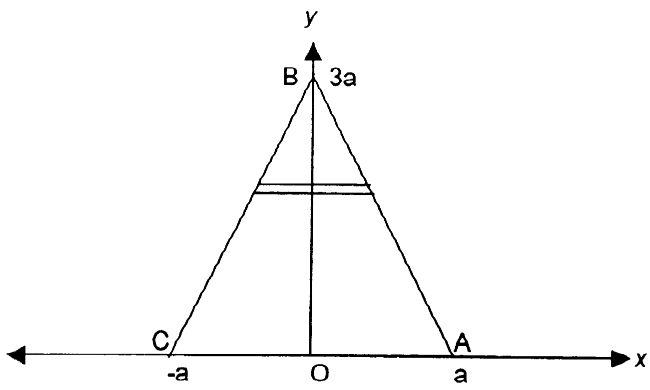
(d) Vertex A of square $ABCD$ is represented by the complex number $5 + 2i$ and its centre X is represented by $2 + i$. Find, in the form $a + ib$ where a and b are real, the complex numbers representing the other three vertices.

2. **(a)** Find
(i) $\int \sec^2 x \tan x \, dx$ by letting $u = \sec x$
(ii) $\int \frac{dx}{\sqrt{x^2-6x+5}}$ by completing the square
(iii) $\int \frac{dx}{5+3\cos x}$ by substituting $t = \tan\frac{x}{2}$
(b) Evaluate $\int_0^{\frac{\pi}{4}} \sin 5x \cos 3x \, dx$
(c) **(i)** If $I_n = \int \tan^n x \, dx$ for integral $n \geq 2$ show that $I_n = \frac{1}{n-1}\tan^{n-1}\theta - I_{n-2}$.
(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} \tan^4 x \, dx$

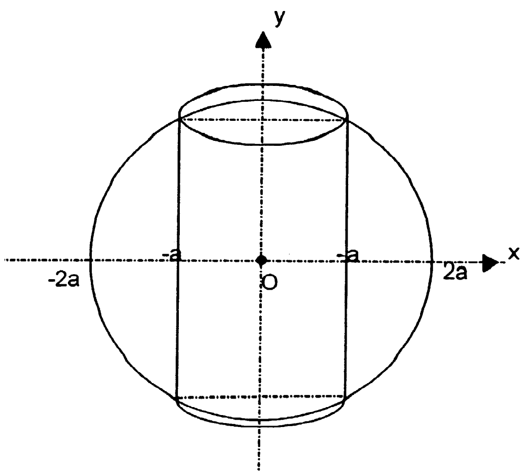
3. **(a)** Show that if α, β and γ are the roots of the equation $x^3 + px^2 + qx + r = 0$ and that $\alpha\beta + 1 = 0$ then $1 + q + pr + r^2 = 0$.
(b) **(i)** Prove that 1 and -1 are the zeroes of multiplicity 2 of the polynomial $x^6 - 3x^2 + 2$. Hence express $x^6 - 3x^2 + 2$ as a product of irreducible factors over the field of:
(α) rational numbers
(β) complex numbers
(c) **(i)** Express $\cos 5\theta$ as a polynomial in terms of $\cos\theta$.
(ii) Hence show that $x = \cos\frac{2k}{5}\pi$ for $k = 0, 1, 2, 3, 4$ are roots of the equation $16x^5 - 20x^3 + 5x - 1 = 0$ and prove that $\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}$.

4. (a) Evaluate $\int_{-2}^{10} x\sqrt{6+x}\ dx$
(b) The foci of an ellipse are $S(4,0)$ and $S'(-4,0)$ and P is any point on the ellipse such that $SP + S'P = 10$. Find the equation of the ellipse.
(c) The hyperbola $xy = 4$ is rotated $45°$ clockwise about its centre. Find the equation of this hyperbola and sketch it labelling the vertices, foci, directrices and asymptotes.
(d) Solve $\cos 4x = \sin 3x$

5. (a)



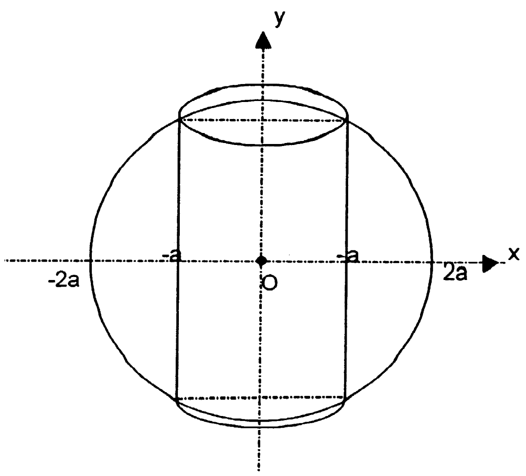
(i) Find the equation of AB.
(ii) Every cross-section perpendicular to OB is the base of a square. Find the volume of the solid formed with ABC as base.
(b) (i) Find the area of an ellipse with semi-major axis of length a units and semi-minor axis of length $\frac{1}{2}a$ units.



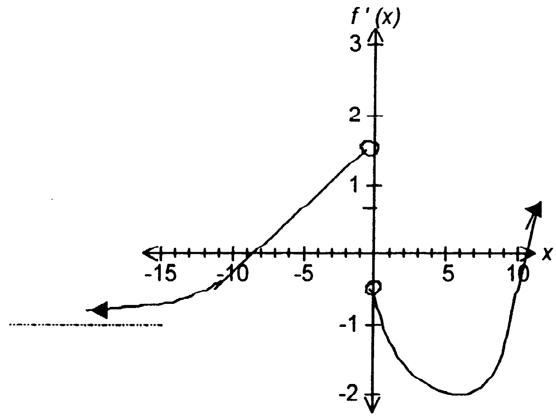
(ii) An elliptical hole with cross-section determined in **(i)** is bored symmetrically through a sphere of radius $2a$ units. Show the total volume remaining is $5\pi a^3\sqrt{3}$ cubic units.
(ii) An elliptical hole with cross-section determined in **(i)** is bored symmetrically through a sphere of radius $2a$ units. Show the total volume remaining is $5\pi a^3\sqrt{3}$

cubic units.

6. **(a)** The diagram shows the graph of $y = f(x)$

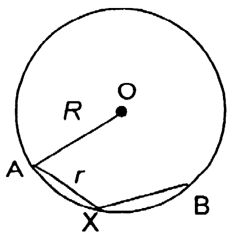


Sketch graphs of
(i) $y = |f(x)|$
(ii)$y = \frac{1}{f(x)}$
(iii) $y^2 = f(x)$
(iv) the inverse function $y = f^{-1}(x)$
(b)



The diagram is a shetch of $y = f'(x)$ with a horizontal asymptote at $y = -1$. Sketch $y = f(x)$ given that it is continuous and $f(-15) = f(5) = 0$, clearly labelling important features.
(c) $ABCD$ is a cyclic quadrilateral and the opposite sides AB and DC are produced to meet at P, and the sides CB and DA meet at Q. If the two circles through the vertices of the triangles PBC and QAB intersect at R:
(i) Draw a diagram showing this information.
(ii) Prove that P, R and Q are collinear.
(iii) Explain why triangle PBQ can never be isosceles.

7. **(a)** Solve $\tan^{-1} x + \tan^{-1}(1-x) = \tan^{-1} \frac{9}{7}$, for x.
(b) **(i)** Prove that if x, y and z are positive $x^2 + y^2 + z^2 \geq xy + yz + xz$
(ii) If x, y and z are positive with constant sum k, show that the least value of $x^2 + y^2 + z^2$ is $\frac{1}{3}k^2$.
(c)



A and B are point on the circumference of a circular pond, centre O of radius R. A toy yacht is tied by means of a string of length r ($r < 2R$) to a point X on the circumference of the pond such that the points A and B are the farthest points of the circumference of the pond that the yacht can reach. If $\angle AOX = \theta$ radians, prove that:
(i) $\angle AXB = (\pi - \theta)$
(ii) $r = 2R \sin \frac{1}{2}\theta$
(iii) the area of the pond in which the yacht can sail is $R^2(\pi - (\pi - \theta)\cos\theta - \sin\theta)$.

8. **(a)** Use mathematical induction to show that
$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \leq \frac{4n+3}{6}\sqrt{n}$ for all integers $n \geq 1$.
(b) A particle moving in a staight line from the origin is subject to a resisting force which produces a retardation of kv^3 where v is the speed at time t and k is a constant. If u is the initial speed, x is the distance moved in time t.
(i) Show that $v = \frac{u}{kux+1}$
(ii) Show that $kx^2 = 2t - \frac{2x}{u}$
(iii) A bullet is fired horizontally at a target 3000m away. The bullet is observed to take 1 second to travel the first 1000m and 1.25 seconds to travel the next 1000m. Assuming that the air resistance is proportional to v^3, and neglecting gravity calculate the time taken to travel the last 1000m.