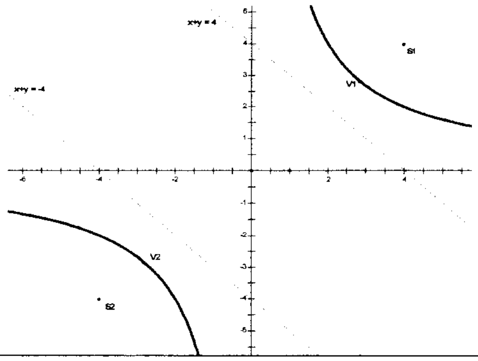
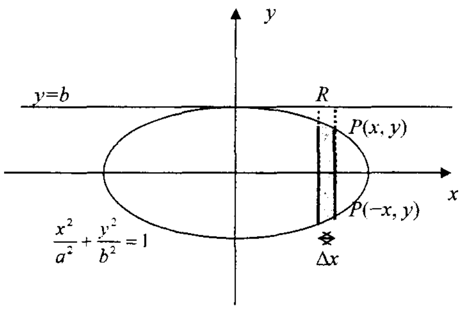
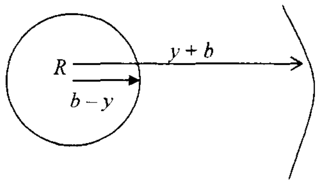
Solutions to Questions	Marking Scheme	Comments
Question 1		
(a) 	• Foci $(\pm 4, \pm 4)$ ➔ **1 mk** • Vertices $\left(\pm 2\sqrt{2}, \pm 2\sqrt{2}\right)$ ➔ **1 mk** • Eqn. of directrices $x + y = \pm 4$ ➔ **1mk** • Shape of graph ➔ **1mk**	Students must **clearly** show the coordinates of the foci and vertices & eqn. of directrices to obtain full marks. **½ mark off is scale is wrong/ no scale given.**
(b) (i) $\frac{dv}{dt} = \frac{1}{3}(3g - 2v)$ $\therefore \int_0^v \frac{dv}{3g - 2v} = \frac{1}{3}\int_0^t dt$ $\therefore -\frac{1}{2}\ln\lvert 3g - 2v\rvert_0^v = \frac{1}{3}[t]_0^t$ $\therefore -\frac{1}{2}\ln\left\lvert\frac{3g - 2v}{3g}\right\rvert = \frac{1}{3}t$ or $\frac{1}{2}\ln\left\lvert\frac{2v - 3g}{3g}\right\rvert = \frac{1}{3}t$ $\therefore \ln\left\lvert\frac{3g - 2v}{3g}\right\rvert = -\frac{2}{3}t$ $\therefore \frac{3g - 2v}{3g} = e^{-\frac{2t}{3}}$ $\therefore v = \frac{3g}{2}\left(1 - e^{-\frac{2t}{3}}\right)$	Correct integral ➔ **1 mk** Correct integration ➔ ½ mk Substitution & simplify➔ **½ mk** Taking e to both sides ➔ **1 mk**	Variation to integral can be: $t = \int \frac{3dv}{3g - 2v}$ When $t = 0$, $v = 0$ ➔ $c = -\frac{3}{2}\ln(3g)$

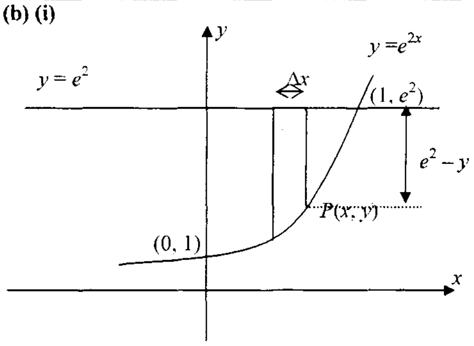
EXT 2 TERM 2 2006

1 (b) (ii) as $t \to \infty$ $v \to \frac{3g}{2}$	Correct answer ➔ **1 mk**	
1 (b) (iii) when $v = \frac{g}{2}$ $\therefore \frac{1}{3} = 1 - e^{-\frac{2t}{3}}$ $\therefore -\frac{2}{3}t = \ln\left(\frac{2}{3}\right)$ $\therefore t = -\frac{3}{2}\ln\left(\frac{2}{3}\right)$ or $t = \frac{3(\ln 3 - \ln 2)}{2}$	Correct substitution & simplification ➔ **1 mk** Taking logs of both sides & simplification ➔ **1 mk**	
1 (c) (i) $P = \frac{QC}{C + e^{-kQt}}$ $\therefore \frac{dP(t)}{dt} = \frac{Q^2Cke^{-kQt}}{\left(C + e^{-kQt}\right)^2}$ $= \frac{kQC}{C + e^{-kQt}} \times \frac{Qe^{-kQt}}{C + e^{-kQt}}$ $= kP\,(Q - P)$ As $(Q - P) = Q - \frac{QC}{C + e^{-kQt}}$ $= \frac{QC + Qe^{-kQt} - QC}{C + e^{-kQt}}$ $= \frac{Qe^{-kQt}}{C + e^{-kQt}}$	Correct differential ➔ **1 mk** Simplification ➔ **1 mk** Showing $(Q - P) = \frac{Qe^{-kQt}}{C + e^{-kQt}}$ ➔ **1 mk**	
1 (c) (ii) as $t \to \infty$, $P \to Q$ as $e^{-kQt} \to 0$	Correct answer with some explaination ➔ **1 mk**	
1 (c) (iii) as $t \to \infty$, $\frac{dP(t)}{dt} \to 0$ as $P \to Q$	Correct answer with some explaination ➔ **1 mk**	

Question 2		
(a) (i) $m\ddot{x} = m(-g) - mkv$ (upwards) $\therefore\ \ddot{x} = -g - kv$ i.e. $\frac{dv}{dt} = -g - kv$ $\therefore\ \frac{1}{k}\int_u^0 \frac{k\,dv}{-(g+kv)} = \int_0^T dt$ $\therefore\ -\frac{1}{k}\left[\ln\lvert g+kv\rvert\right]_u^0 = T$ $\therefore\ -\ln\left\lvert\frac{g}{g+ku}\right\rvert = kT$ $\therefore\ \ln\left\lvert\frac{g+ku}{g}\right\rvert = kT \Rightarrow kT = \ln\left\lvert 1+\frac{ku}{g}\right\rvert$	Correct equation ➔ **1 mk** Correct integral ➔ **1 mk** Correct integration ➔ **1 mk** Correct simplification ➔ **1 mk**	
2 (a) (ii) Highest point reached is when $v = 0$ $\therefore\ \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -g - kv \Rightarrow v\frac{dv}{dx} = -g - kv$ $\therefore -\int_u^0 \frac{v\,dv}{g+kv} = \int_0^h dx$ $\therefore\ -\int_u^0 \frac{1}{k} - \frac{g\,dv}{k(g+kv)} = \int_0^h dx$ $\therefore\ -\int_u^0 1 - \frac{g\,dv}{(g+kv)} = kh$ $\therefore\ -\left[v - \frac{g}{k}\ln(g+kv)\right]_u^0 = kh$ $\therefore\ -\left[-u + \frac{g}{k}\ln\left(\frac{g+ku}{g}\right)\right] = hk$ $\therefore\ hk = u - gT$ as $kT = \ln\left\lvert 1+\frac{ku}{g}\right\rvert$ from part(i)	$v\frac{dv}{dh} = -g - kv$ ➔ **1 mk** Correct integral ➔ **1 mk** $\frac{vdv}{g+kv} = \frac{1}{k} - \frac{gdv}{k(g+kv)}$ ➔ **1 mk** Correct integration ➔ **1 mk** Correct simplification & connecting answer from (i) ➔ **1 mk**	

2 (b) $xy = 4$ Eqn. of normal is $y = p^2x - 2p^3 + \frac{2}{p}$ **(i)** $\therefore$ coordinates of Q are $\left(2p - \frac{2}{p^3}, 0\right)$	Correct coordinates ➔ **1 mk**	
(ii) $M = \left[\left(\frac{2p + 2p - \frac{2}{p^3}}{2}\right), \left(\frac{\frac{2}{p}}{2}\right)\right]$ $M = \left[\left(2p - \frac{1}{p^3}\right), \frac{1}{p}\right]; p \neq 0$	Correct midpoint formula ➔ **1 mk** Correct simplification & restriction for p ➔ **1 mk**	Alternatively can award 1 mk each for x and y coordinate of midpoint.
(iii) $x = 2p - \frac{1}{p^3}; y = \frac{1}{p}$ $\therefore p = \frac{1}{y}\ ; y \neq 0$ $\therefore x = \frac{2}{y} - y^3$ **is the locus of** M; $y \neq 0$	➔ **1 mk** ➔ **1 mk** equation ➔ **1 mk** restriction $y \neq 0$	
Question 3		
(a) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = \frac{b}{a}\sqrt{a^2 - x^2}$ $\therefore$ Area $= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$ $= \frac{4b}{a} \times \frac{1}{4}\pi a^2$ since $\int_0^a \sqrt{a^2 - x^2}\,dx$ is a quadrant of a circle, centre O radius a units. $\therefore$ Area $= \pi ab$ sq. units.	➔ **1 mk** For $\frac{1}{4}\pi a^2$ ➔ **1 mk** ➔ **1 mk** explanation of using $\frac{1}{4}\pi a^2$	

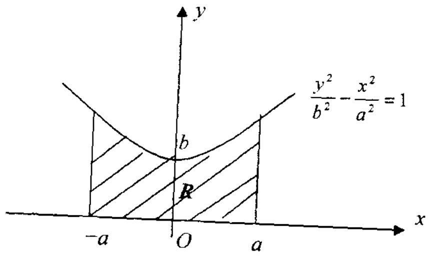
3 (a) (ii) 		
A slice taken through the ellipse perpendicular to the x – axis is the annulus with inner radius $(b - y)$ and outer radius $(b + y)$.		
$\therefore$ Area of cross- section of slice $= \pi[(b+y)^2 - (b-y)^2] = 4\pi by$	Showing area of cross section is $4\pi ay$ ➔ **1 mk**	
$\therefore$ Volume of slice $\Delta V = 4\pi by\ \Delta x$ $\therefore$ Volume of solid $V = 4\pi b\int_{-a}^{a} y\,dx$		
$= 8\pi b\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$	Correct integral➔ **1 mk**	
$= \frac{8\pi b^2}{a}\times\frac{1}{4}\pi a^2$ from part (i) $= 2\pi^2 ab^2$ cubic units.	Correct Answer ➔ **1 mk**	

3 (b) (i) 		
Area of rectangular slice; $A(x) = 2\pi x(e^2 - e^{2x})$	➔ **1 mk**	
$\therefore \Delta V = 2\pi x(e^2 - e^{2x})\Delta x$		
$\therefore$ Vol $= \lim_{\Delta x\to 0}\sum_0^1 2\pi x(e^2 - e^{2x})\Delta x$	➔ **1 mk**	
3 (b) (ii) $\therefore$ Vol $= \int_0^1 2\pi x\left(e^2 - e^{2x}\right)dx$		
$= \left[2\pi e^2\cdot\frac{1}{2}x^2\right]_0^1 - \pi\int_0^1 2xe^{2x}\,dx$	➔ **1 mk**	
$= \pi e^2 - \pi\left[xe^{2x}\right]_0^1 + \pi\int_0^1 e^{2x}\,dx$	➔ **1 mk**	
$= \pi e^2 - \pi e^2 + \frac{1}{2}\pi\left[e^{2x}\right]_0^1 = \frac{1}{2}\pi\left(e^2 - 1\right)$	➔ **1 mk**	

3 (c) Substituting $y^2 = 4ax$ into $xy = c^2$ we get $y^3 = 4ac^2 = 2a^3$ as $2c^2 = a^2$.		
Let P be the point of intersection where $\therefore\ y = a\sqrt[3]{2}$ and $x = \dfrac{a\left(\sqrt[3]{4}\right)}{4}$	→ **1mk**	
By differentiating $xy = c^2$ we get $\dfrac{dy}{dx} = -\dfrac{y}{x}$	→ **½ mk**	
$\therefore$ the gradient of the hyperbola $xy = c^2$ at P is $m = \dfrac{-a\left(\sqrt[3]{2}\right)}{\left(\dfrac{a}{4}\right)\left(\sqrt[3]{4}\right)} = \dfrac{-4}{\sqrt[3]{2}} \rightarrow \boxed{A}$	→ **1 mk**	
By differentiating $y^2 = 4ax$ we get $\dfrac{dy}{dx} = \dfrac{2a}{y}$	→ **½ mk**	
$\therefore$ gradient of $y^2 = 4ax$ is given by $M = \dfrac{2a}{a\left(\sqrt[3]{2}\right)} = \dfrac{2}{\sqrt[3]{2}} \rightarrow \boxed{B}$	→ **1 mk**	
$\therefore$ A ÷ B → $m = -2M$ as required.		

4(a) (i) $\ddot{x} = -\dfrac{k}{x^2}$ $\therefore\ \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = -\dfrac{k}{x^2}$		
$\therefore \dfrac{1}{2}v^2 = -\int \dfrac{k}{x^2}dx = \dfrac{k}{x} + c$	→ **½ mk**	
Now $v =$ o when $x = a\ \therefore\ c = -\dfrac{k}{x}$ $\therefore\ v^2 = 2k\left(\dfrac{1}{x} - \dfrac{1}{a}\right)$	→ **½ mk**	
Now $0 < x < a$ but motion is moving towards the origin for $t > 0$. $\therefore v = -\sqrt{2k\left(\dfrac{1}{x} - \dfrac{1}{a}\right)}$	→ **1 mk**	
For $x = ½\,a$, $v = -\sqrt{\dfrac{2k}{a}}$	Correct answer→ **1 mk**	
4 (a) (ii)$\therefore\ \dfrac{1}{v} = \dfrac{dt}{dx} = -\dfrac{1}{\sqrt{2k\left(\dfrac{a-x}{ax}\right)}} = -\dfrac{1}{\sqrt{\dfrac{2k}{a}\left(\dfrac{a-x}{x}\right)}}$	Correct $\dfrac{1}{v}$ equation→ **1 mk**	
$= -\sqrt{\dfrac{a}{2k}}\cdot\sqrt{\dfrac{x}{a-x}} \quad = \sqrt{\dfrac{a}{2k}}\cdot\int_{\frac{a}{2}}^{a}\sqrt{\dfrac{x}{a-x}}$	$\sqrt{\dfrac{a}{2k}}\cdot\int_{\frac{a}{2}}^{a}\sqrt{\dfrac{x}{a-x}}$ → **1 mk**	
$= -\sqrt{\dfrac{a}{2k}}\left[\sqrt{x(a-x)} + \dfrac{1}{2}a\sin^{-1}\left(\dfrac{a-2x}{a}\right)\right]_{\frac{a}{2}}^{a}$ $= -\sqrt{\dfrac{a}{2k}}\left[\dfrac{1}{2}a\sin^{-1}(-1) - \dfrac{a}{2} - \dfrac{1}{2}a\sin(0)\right]$ $\therefore t = \dfrac{(\pi+2)a^{\frac{3}{2}}}{4\sqrt{2k}}$	Correct substitution → **1mk**	

4 (b) (i)



$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ can be written as $y=\pm\frac{b}{a}\sqrt{a^2+x^2}$

$\therefore$ Area of region R is given by

$$\text{Area}=\frac{2b}{a}\int_0^a\sqrt{a^2+x^2}$$

Let $x=a\tan\theta$. $\therefore$ at $x=a$, $\tan\theta=1$ $\therefore$ $\theta=\frac{\pi}{4}$

At $x=0$, $\tan\theta=0$ so $\theta=0$. Also $dx=a\sec^2\theta\,d\theta$

$$\therefore\ \text{Area}=A=\frac{2b}{a}\int_0^{\frac{\pi}{4}}\sqrt{a^2+a^2\tan^2\theta}\cdot\sec^2\theta\,d\theta$$ → **1 mk**

$$=2ab\int_0^{\frac{\pi}{4}}\sqrt{1+\tan^2\theta}\cdot\sec^2\theta\,d\theta$$

$$=2ab\int_0^{\frac{\pi}{4}}\sec^3\theta\,d\theta \text{ where } I=\int\sec^3\theta\,d\theta$$ → **1 mk**

4 (b) (i) Continued

Now $I=\int\sec\theta(\sec^2\theta)d\theta=\int\sec\theta(1+\tan^2\theta)d\theta$ → **1 mk**

$$\therefore\quad I=\int\sec\theta\,d\theta+\int(\sec\theta\tan\theta)\tan\theta\,d\theta$$

$$=\int\frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}d\theta+\int\tan\theta\cdot\frac{d}{d\theta}(\sec\theta)d\theta$$ → **1mk**

$$=\ln(\sec\theta+\tan\theta)+\tan\theta\sec\theta-I$$

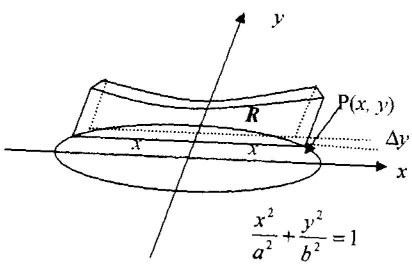
$$\therefore\ I=\tfrac{1}{2}\ln(\sec\theta+\tan\theta)+\tfrac{1}{2}\tan\theta\sec\theta$$

So Area $=ab[\ln(\sec\theta+\tan\theta)+\tan\theta\sec\theta]_0^{\frac{\pi}{4}}$ → **1mk**

$$=ab\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]$$

Since $L=2a$ area generated is $A=\frac{Lb}{2}\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]$ unit^2

4 (b) (ii)



Consider the cross section at $P(x, y)$ on the ellipse of thickness Δy (See diagram). The area of this cross section from (i) is A square units.

Note $L = 2a = 2x$ so $a = x$. $\therefore$ Area $= A = xb\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$ But $x = \frac{a}{b}\sqrt{b^2 - y^2}$ and let $K = \left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$ $\therefore A(y) = \frac{LK}{2}\sqrt{b^2 - y^2}$ Now volume of slice, $\Delta V = A(y)\,\Delta y$ $\therefore \Delta V = \frac{LK}{2}\sqrt{b^2 - y^2}\,\Delta y$ $\therefore$ volume of sum of slices, $V = \frac{LK}{2}\lim_{\Delta y \to 0}\sum_{-b}^{b}\sqrt{b^2 - y^2}\,\Delta y$ $= \frac{LK}{2}\int_{-b}^{b}\sqrt{b^2 - y^2}\,dy$ $= \frac{LK}{2}\cdot\frac{1}{2}\pi b^2$ (Note: this integral gives area of semi circle radius b) $\therefore$ Volume of $S = \frac{\pi L b^2}{4}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$ in terms of L and b or $= \frac{\pi a b^2}{2}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$ in terms of a and b.	$A(y)$ **➔ 1 mk** **➔ 1 mk** **➔ 1 mk** **➔ 1mk**	

~ End of Test ~