Year 12 Mathematics Extension 2 TRIALS 2006 – Suggested Solutions (LK)

<table>
<tr><th>Solutions to Questions</th><th>Marking Scheme</th><th>Comments</th></tr>
<tr><td>Question 1</td><td></td><td></td></tr>
<tr><td>(a) $\int \frac{2x}{1+2x}\,dx = \int 1-\frac{1}{1+2x}\,dx$

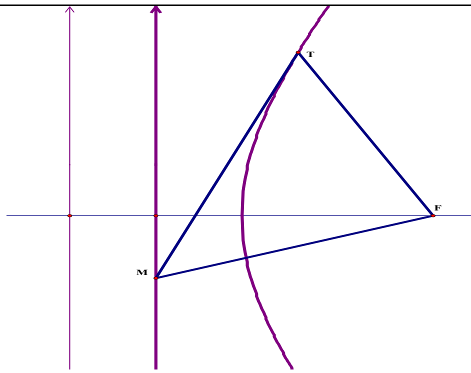
$= x-\frac{1}{2}\ln\left|1+2x\right|+c$</td><td>1 for changing the integral

1 correct integration</td><td></td></tr>
<tr><td>(b) Let the Lee brothers be L_1 and L_2
Let the Abey brothers be A_1, A_2 and A_3.
Then there are 10 others.
$\therefore$ # possibilities is 365 ways.

<table><tr><th>L</th><th>A</th><th>Others</th></tr><tr><td>1</td><td>0</td><td>10</td></tr><tr><td>1</td><td>1</td><td>9</td></tr><tr><td>1</td><td>2</td><td>8</td></tr><tr><td>0</td><td>1</td><td>10</td></tr><tr><td>0</td><td>2</td><td>9</td></tr></table>$\therefore$ 2 + 60 + 270 + 3 + 30 = 365 [using ${}^nC_r$]</td><td>1 for correct answer

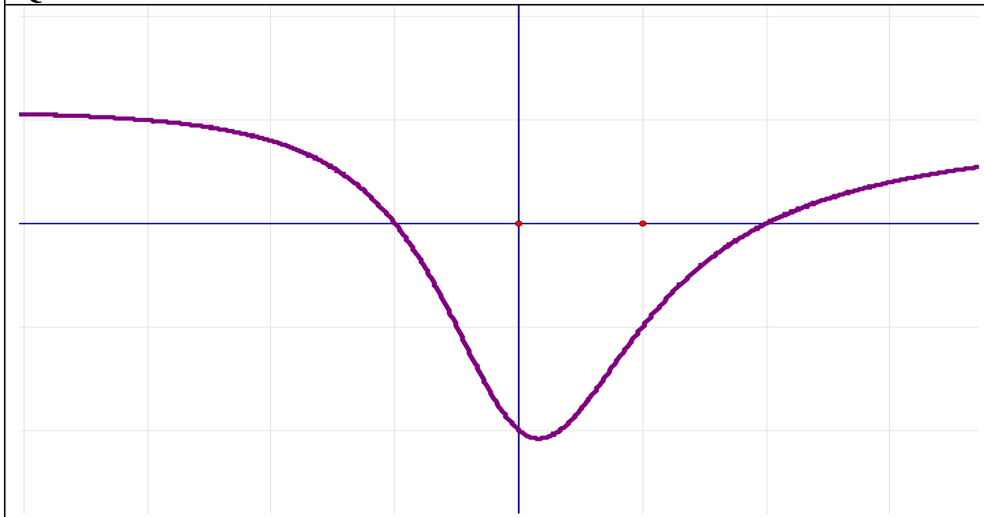
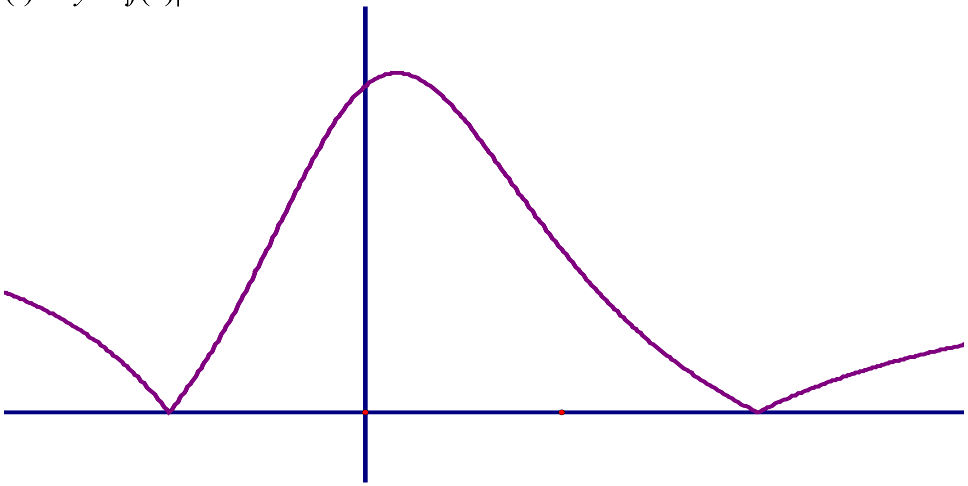
2 for justification</td><td></td></tr>
<tr><td>(c) $\lim_{x\to -5}\frac{\sqrt{20-x}-5}{5+x}\times\frac{\sqrt{20-x}+5}{\sqrt{20-x}+5}$
$=\lim_{x\to -5}\frac{20-x-25}{(5+x)\left(\sqrt{20-x}+5\right)}$
$=\lim_{x\to -5}\frac{-1}{\sqrt{20-x}+5}$
$=-\frac{1}{10}$</td><td>1 for multiplying by conjugate

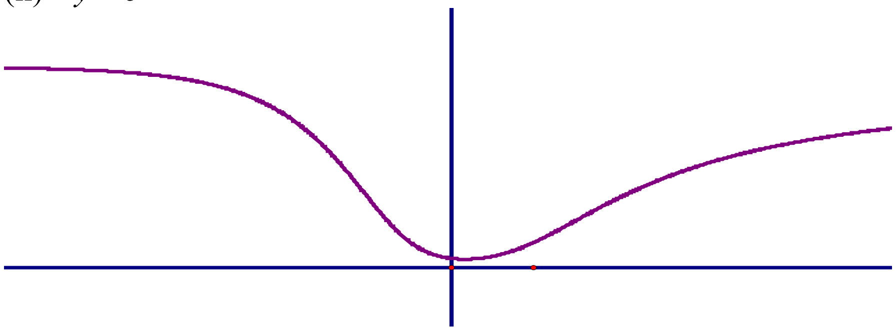
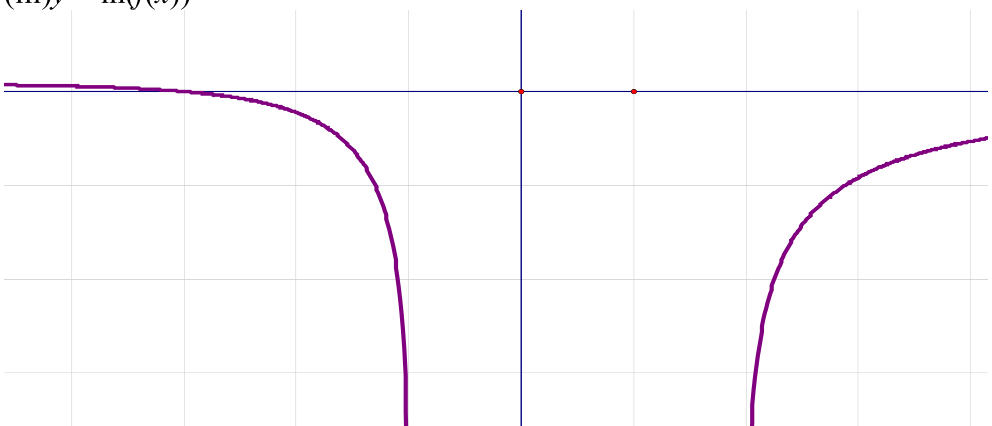
1 correct simplification & answer</td><td></td></tr>
</table>

Solutions to Questions	Marking Scheme	Comments
Question 1 continued		
(d) $(1+z)^8 = \binom{8}{0} + \binom{8}{1}z + \binom{8}{2}z^2 + \cdots + \binom{8}{7}z^7 + \binom{8}{8}z^8$ Let $z = 1$ then $2^8 = \binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \cdots + \binom{8}{8}$ $\therefore$ sum of coefficients = $\mathbf{2^8}$ **or 256**.	1 correct expansion 1 correct substitution of $z = 1$ and the correct answer.	
 (e) (i) Since $3x^2 - y^2 = 12$ then we get $6x - 2y\dfrac{dy}{dx} = 0 \quad \therefore \dfrac{dy}{dx} = \dfrac{3x_1}{y_1}$ at T. $\therefore$ equation of tangent: $y - y_1 = \dfrac{3x_1}{y_1}(x - x_1)$ (ii) since this tangent meets the line $x = 1$ $\therefore y = \dfrac{3x_1 - 12}{y_1}$ & since $F(4, 0)$ $\therefore FM_{\text{grad}} = -\dfrac{(x_1 - 4)}{y_1}$ & $FT_{\text{grad}} = \dfrac{y_1}{x_1 - 4}$ $\therefore FM \times FT = -1 \qquad \therefore FM \perp FT$.	1 correct differential 1 gradient at T. 1 correct tangent equation 1 + 1 for correct gradients 1 correct justification of $FM \perp FT$.	

Solutions to Questions	Marking Scheme	Comments
Question 2		
(a)(i) $z = 2\text{ cis }(-\frac{\pi}{4})$ (ii) $z^{22} = \left[2cis\left(-\frac{\pi}{4}\right)\right]^{22}$ by De Moivre's Theorem $= 2^{22}\left[\cos\left(-\frac{22\pi}{4}\right) + i\sin\left(-\frac{22\pi}{4}\right)\right]$ $= 2^{22}\left[\cos\left(\frac{11\pi}{2}\right) - i\sin\left(\frac{11\pi}{4}\right)\right]$ $= 2^{22}(0 - -i) = \mathbf{2^{22}\, i.}$	1 for modulus 1 for argument 1 correct use of De Moivre's theorem 1 correct simplification 1 correct answer	
(b)(i) Since $a^2 + b^2 > 2ab$ then similiarly $a^2 + c^2 > 2ac$ and $b^2 + c^2 > 2bc$. $\therefore\ 2(a^2 + b^2 + c^2) > 2(ab + bc + ac)$ $\therefore\ a^2 + b^2 + c^2 > ab + bc + ac$ as required (ii) Since $a + b + c = 6$ Then $(a + b + c)^2 = 36$ $\therefore\ a^2 + b^2 + c^2 + 2ab + 2bc + 2ac = 36$ But from part (i) $a^2 + b^2 + c^2 > ab + bc + ac$ $\therefore\ a^2 + b^2 + c^2 + 2ab + 2bc + 2ac > 3(ab + bc + ac)$ $\therefore\ 36 > 3(ab + bc + ac)$ $\therefore\ ab + bc + ac < 12$ as required.	1 correctly writing the other two inequalities. 1 correct inequality 1 correct expansion 1 correct substitution	

Solutions to Questions	Marking Scheme	Comments
Question 2 continued		
(c)(i) $I_n = \frac{1}{n!}\int_0^1 x^n e^{-x}dx$, $n \ge 0$ By integration by parts		
$= \frac{1}{n!}\left[-x^n e^{-x}\right]_0^1 - \frac{1}{n!}\int_0^1\left[-nx^{n-1}e^{-x}\right]dx$	1 + 1 marks	
$= \frac{1}{n!}\left[-e^{-1}\right] + \frac{n}{n!}\int_0^1\left[x^{n-1}e^{-x}\right]dx$	1 mark	
$= -\frac{e^{-1}}{n!} + \frac{1}{(n-1)!}\int_0^1\left[x^{n-1}e^{-x}\right]dx$		
$= I_{n-1} - \frac{e^{-1}}{n!}$ as required.		
(ii) $I_4 = I_3 - \frac{e^{-1}}{4!} \quad = I_2 - \frac{e^{-1}}{3!} - \frac{e^{-1}}{4!}$		
$= I_1 - \frac{e^{-1}}{2!} - \frac{e^{-1}}{3!} - \frac{e^{-1}}{4!}$		
$= I_0 - \frac{e^{-1}}{1!} - \frac{e^{-1}}{2!} - \frac{e^{-1}}{3!} - \frac{e^{-1}}{4!}$	1 up to this line	
$= \frac{1}{0!}\int_0^1 e^{-x}dx - \frac{e^{-1}}{1!} - \frac{e^{-1}}{2!} - \frac{e^{-1}}{3!} - \frac{e^{-1}}{4!}$		
$= -[e^{-x}]_0^1 - e^{-1}\left(\frac{41}{24}\right)$	1 for this line	
$= 1 - \frac{65}{24e}$	1 correct answer	

Solutions to Questions	Marking Scheme	Comments
Question 3		
		
(i) $y = \|f(x)\|$ 	1 shape 1 critical points	

Solutions to Questions	Marking Scheme	Comments
Question 3 continued		
(ii) $y = e^{f(x)}$ 	1 shape 1 asymptote / turning point	
(iii) $y = \ln(f(x))$ 	1 shape 1 x intercept / asymptote	
(b) $(x^2 - 5)(x^2 + 3)$ over rational field $\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)\left(x-3i\right)\left(x+\sqrt{3}i\right)$ over complex field	1 correct factorization 1 correct factorisation	

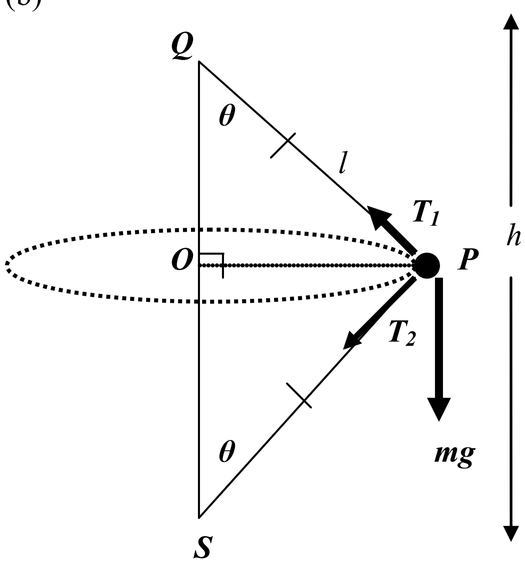
Solutions to Questions	Marking Scheme	Comments
Question 3 continued		
(c)		
(d) $\int_0^{\frac{\pi}{2}} \frac{1-\tan x}{1+\tan x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$ $= \left[\ln\lvert\cos x + \sin x\rvert\right]_0^{\frac{\pi}{2}}$ $= 0$	1 correct conversion to sin & cos 1 correct integration + answer	

Solutions to Questions	Marking Scheme	Comments
Question 4		
(a) $T \le 20 \times 9.8$ But $T = m\,\omega^2\,r$ $\therefore\ m\,\omega^2\,r \le 20 \times 9.8$ $\therefore\ 4 \times \omega^2 \times ½ \le 20 \times 9.8$ $\therefore\ \omega^2 \le 98$ rads/ sec $\therefore\ \omega \le \sqrt{98} \times 60 \times \frac{1}{2\pi}$ revs / min $\therefore$ greatest no. of revolutions 94 revs/ min	1 $T \le 20 \times 9.8$ 1 1 converting to revs/ min & correct answer	
(b) $\int_a^{a^2} \frac{dx}{x\ln x} = \left[\ln(\ln x)\right]_a^{a^2}$ $= \ln(\ln a^2) - \ln(\ln a)$ $= \ln\left(\frac{\ln a^2}{\ln a}\right)$ $= \ln 2$	1 correct integration 1 correct use of log. Rule 1 correct answer	

(c) (i) $\partial V = 2\pi xy\partial x$ $\therefore$ Total volume = $\int_0^{\frac{\pi}{2}} 2\pi x\left(\cos^2 x - \cos 2x\right)dx$ $= \int_0^{\frac{\pi}{2}} 2\pi\left(x - x\cos^2 x\right)dx$ $= \left[\frac{1}{2}x^2 - \frac{1}{2}x\left(x + \frac{1}{2}\cos 2x\right)\right]_0^{\frac{\pi}{2}}$ $= \frac{\pi^2}{8} - \frac{1}{2}\left(\frac{\pi^2}{8} - \frac{1}{4} - \frac{1}{4}\right)$ $= \left(\frac{\pi^3}{8} + \frac{\pi}{2}\right)$ units^3	1 showing correct volume of slice 1 correct integral + limits 1 correct simplified integral 1 correct use of IBP 1 correct integration by parts 1 correct answer in terms of π.	
(d) $\int_{-4}^{4} \cos x\left(e^x - e^{-x}\right)dx$ Since cos x is an EVEN function & $e^x - e^{-x}$ is an ODD function, EVEN $\times$ ODD ➔ ODD $\therefore \int_{-4}^{4} \cos x\left(e^x - e^{-x}\right)dx = 0$	1 + 1 for correct justification 1 correct answer	**Note: students do not need to physically find the integral!**

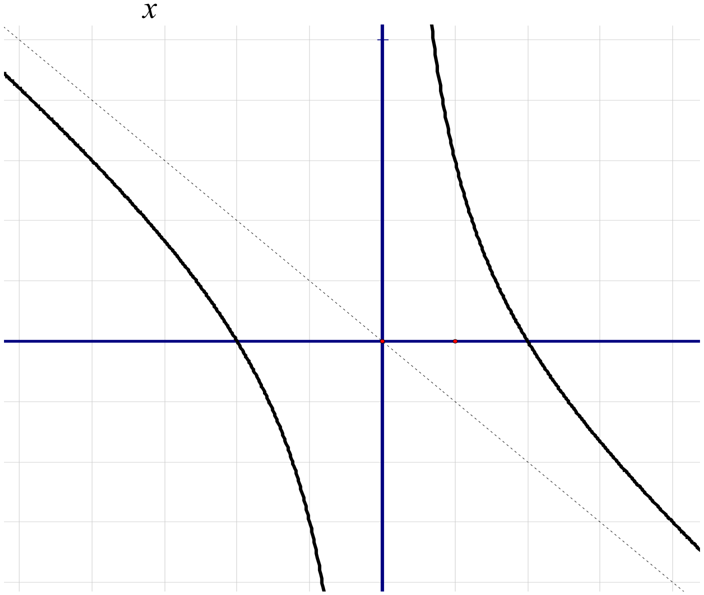
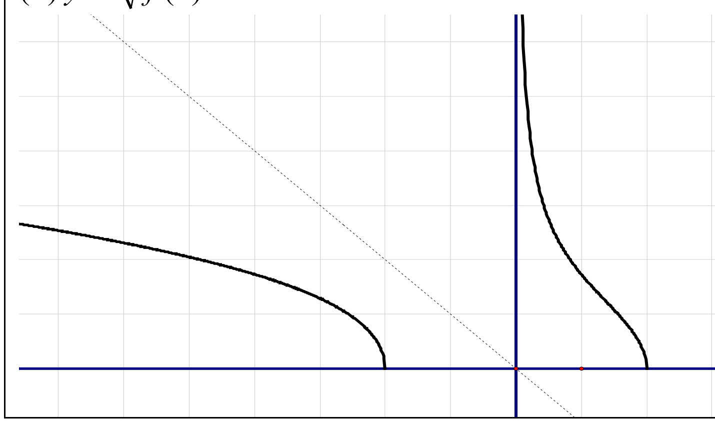
Solutions to Questions	Marking Scheme	Comments
Question 5		
(a) (i) $\frac{x^4}{x^2+1} = A(x^2-1) + \frac{B}{x^2+1}$ Let $x = 0$ ➔ $B - A = 0$ Let $x = 1$ ➔ ½ = ½ × B $\therefore B = 1$ and $A = 1$	1 substituting a reasonable x values 1 correct answers for A and B.	
(ii) $\int_0^1 x^3 \tan^{-1} x \, dx$ $= \left[\frac{x^4}{4}\tan^{-1} x\right]_0^1 - \frac{1}{4}\int_0^1 \frac{x^4}{1+x^2}dx$ $= \frac{\pi}{16} - \frac{1}{4}\left[\frac{x^3}{3} - x + \tan^{-1} x\right]_0^1$ $= \frac{\pi}{16} + \frac{1}{6} - \frac{\pi}{6}$ $= \frac{1}{6}$	1 + 1 for integration by parts 1 for integration 1 substitution 1 correct answer	

Solutions to Questions	Marking Scheme	Comments
Question 5 continued		
(b)(i) particle moving upwards & acceleration is acting downwards $\therefore\ a = -\dfrac{k}{x^2}$ When $x = R$, $a = g$ $\therefore\ k = gR^2$.	1 for finding k in terms of g and R.	
$\therefore\ \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = \dfrac{-gR^2}{x^2}$ since $a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$	1 for finding a in terms of g and R	
$\therefore\ \dfrac{1}{2}v^2 = \dfrac{gR^2}{x} + c$	1 integration	
Now when $x = R$, $v = u$ initially $\therefore\ c = \dfrac{1}{2}u^2 - gR$ $\therefore\ v^2 = u^2 - 2gR + \dfrac{2gR^2}{x}$	1 for c	
(ii) when $u = \sqrt{2gR} \Rightarrow v^2 = \dfrac{2gR^2}{x}$ $\therefore\ v = \dfrac{dx}{dt} = \sqrt{\dfrac{2gR^2}{x}}$, $v > 0$	1 v^2 equation	
$\therefore\ \dfrac{dt}{dx} = \dfrac{\sqrt{x}}{\sqrt{2gR^2}}$	1 for $\dfrac{dt}{dx}$	
$\therefore\ t = \displaystyle\int_R^{4R} \frac{x^{\frac{1}{2}}}{\sqrt{2gR^2}}dx = \frac{1}{\sqrt{2gR^2}}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_R^{4R}$	1 correct integration for t	
$\therefore\ t = \dfrac{14}{3}\sqrt{\dfrac{R}{2g}}$	1 correct answer	

Solutions to Questions	Marking Scheme	Comments
Question 6		
(a) **4 players** There are ${}^{4}C_1$ ways of choosing the team Then we need four players of the 10: ${}^{10}C_4$ ways The 5th player is chosen from the remaining 30 in ${}^{30}C_1$ way $\therefore$ total number of ways = ${}^{4}C_1 \times {}^{10}C_4 \times {}^{30}C_1$ = **25200** **5 players ➔** ${}^{4}C_1 \times {}^{10}C_5 = 1008$ ways $\therefore$ p(at least 4 players from same team) $= \dfrac{25200+1008}{\binom{40}{5}} = \dfrac{1}{25}$	1 correct answer + 1 justification 1 correct answer 1 correct probability	
(b) 		

Solutions to Questions	Marking Scheme	Comments
Question 6 continued		
(b) (i) By trig. $R = l\sin\theta$ and $h = 2l\cos\theta$. **Resolving forces:** **Vertically**: $0 = mg + T_2\cos\theta - T_1\cos\theta$. $\therefore\ mg = (T_1 - T_2)\cos\theta$. ➔ ① **Horizontally:** $m\omega^2 r = T_1\sin\theta + T_2\sin\theta$ $\therefore\ m(l\sin\theta)\ \omega^2 = (T_1 + T_2)\sin\theta$. ➔ ② From ① ➔ $T_1 - T_2 = \dfrac{mg}{\cos\theta} = \dfrac{2mgl}{h}$ From ② ➔ $T_1 + T_2 = ml\omega^2$ ① + ② ➔ $T_1 = ml(\frac{1}{2}\omega^2 + \dfrac{g}{h})$ → tension in PQ.	1 $\sin\theta$ and $\cos\theta$ relationships 1 vertical force 1 horizontal force 1 rearrangements of equation 1 correct addition & simplification	
(ii) ② − ① ➔ $T_2 = ml\left(\dfrac{1}{2}\omega^2 - \dfrac{g}{h}\right)$	1 for answer	
(iii) String PS (hence PQ) will only remain stretched is $T_2 > 0\ \therefore \left(\dfrac{1}{2}\omega^2 - \dfrac{g}{h}\right) > 0$ $\therefore\ \omega > \sqrt{\dfrac{2g}{h}}$	1 knowing $T_2 > 0$ 1 showing answer is true	
(iv) If $T_1 : T_2 = 2 : 1$ then $\omega^2 = \dfrac{6g}{h}$ $\therefore\ \omega = \sqrt{\dfrac{6g}{h}}\ \therefore$ period of motion $= \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\sqrt{\dfrac{h}{6g}}}$	1 getting ω^2 1 for ω + 1 period	

Solutions to Questions	Marking Scheme	Comments
Question 7		
(a)(i) $\alpha^3+\beta^3+\gamma^3 = (\alpha+\beta+\gamma)^3 + 3\alpha\beta\gamma$ $\therefore (\alpha+\beta+\gamma)^3 + 3\alpha\beta\gamma$ $= (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\alpha\gamma-\beta\gamma) + 3\alpha\beta\gamma$ $= 0 + 3(-q) = -3q$ (ii) Let $X = \dfrac{\alpha}{\beta\gamma} = \dfrac{\alpha^2}{\alpha\beta\gamma} = -\dfrac{\alpha^2}{q}$ $\therefore \alpha^2 = -qX$ But $\alpha^3 + p\alpha + q = 0$ as α is a root $\therefore -qX\alpha + p\alpha + q = 0$ $(qX-p)\alpha = q$ $(qX-p)^2a^2 = q^2$ $\therefore (qX-p)^2(-qX) = q^2$ $\therefore X(qX-p)^2 = -q$	(1) correct relationship (1) correct expansion (1) correct answer (1) a^2 equation (1) (1) correct solution	
(b) (i) $\therefore$ gradient of normal at P is t^2 $\therefore$ equation is given by $\left(y-\dfrac{c}{t}\right) = t^2(x-ct)$ Which leads to the required equation. (ii) $N\left(\dfrac{c-tc^3}{t(t^2-1)}, \dfrac{c-tc^3}{t(t^2-1)}\right)$ (iii) tan $\angle NOP = \left\|\dfrac{1+\frac{1}{t^2}}{1-\frac{1}{t^2}}\right\| = \left\|\dfrac{1+t^2}{t^2-1}\right\| = \left\|\dfrac{1+t^2}{1-t^2}\right\|$ tan $\angle ONP = \left\|\dfrac{1+t^2}{1-t^2}\right\|$ $\therefore \angle NOP = \angle ONP$	(1) correct gradient & use of point – gradient formula. (1) correct coordinates. (1) Finding angles + (1) conclusion $\therefore \Delta ONP$ is isosceles (equal base angles)	**Can also show *PN* = *OP***

<table>
<tr><th>Solutions to Questions</th><th>Marking Scheme</th><th>Comments</th></tr>
<tr><td>Question 7 continued</td><td></td><td></td></tr>
<tr><td>(c) (i) $y = \frac{4}{x} - x$

(ii) $y = \sqrt{f(x)}$
</td><td>(1) correct shape

(1) correct roots & Asymptotes

(1) + (1) for each arm</td><td></td></tr>
</table>

Solutions to Questions	Marking Scheme	Comments
Question 8		
(a)(i) Since $y = x\tan\theta - \dfrac{gx^2\sec^2\theta}{2V^2}$ (÷ by $gsec^2\theta / 2V^2$) $\therefore x^2 - \dfrac{2V^2\tan\theta}{g\sec^2\theta}x = \dfrac{-2V^2}{g\sec^2\theta}y$ By completing the square: $\therefore x^2 - \dfrac{2V^2}{g}\sin\theta\cos\theta x + \left(\dfrac{V^2}{g}\sin\theta\cos\theta\right)^2$ $= \dfrac{-2V^2\cos^2\theta}{g}y + \dfrac{V^4}{g^2}\sin^2\theta\cos^2\theta$ $\therefore \left[x - \dfrac{V^2}{g}\sin\theta\cos\theta\right]^2 = -\dfrac{2V^2\cos^2 x}{g}\left[y - \dfrac{V^2}{2g}\sin^2\theta\right]$ Which is of the form $(x-h)^2 = 4a(y-k)$.	(1) (1) completing the square (1) putting in the form $(x-h)^2 = 4a(y-k)$.	
(ii) The **focal length** is $\dfrac{1}{4} \times -\dfrac{2V^2\cos^2\theta}{g} = \dfrac{V^2\cos\theta}{2g}$ **Horizontal range** when $y = 0$ $\therefore x\tan\theta - \dfrac{gx^2\sec^2\theta}{2V^2} = 0 \Rightarrow x\left(\tan\theta - \dfrac{gx\sec^2\theta}{2V^2}\right) = 0$ $\therefore$ ignore $x = 0$, then $x = \dfrac{V^2\sin 2\theta}{g}$ $\therefore$ since $\dfrac{2V^2\cos\theta}{2g} = \dfrac{V^2\sin 2\theta}{g}$ we get $\cos\theta = \sin 2\theta = 2\sin\theta\cos\theta$; $\mathbf{\cos\theta \neq 0}$ $\therefore \sin\theta = ½ \Rightarrow \theta = \dfrac{\pi}{3}$	(1) focal length (1) for RANGE (1) answer	

Solutions to Questions	Marking Scheme	Comments
Question 8 continued		
(b)(i) $\frac{1}{2}(p+q) \geq \sqrt{pq}$. RTP $\frac{1}{4}(p+q)^2 - pq \geq 0$ LHS = $\frac{1}{4}(p^2 + 2pq + q^2 - 4pq)$ $= \frac{1}{4}(p-q)^2 \geq 0$ (ii) $\frac{1}{2}(p+q) \geq \sqrt{pq}$. now ÷ by $\sqrt{q} > 0$ $\Rightarrow \frac{1}{2}\left(\frac{p}{\sqrt{q}} + \frac{q}{\sqrt{q}}\right) \geq \sqrt{p}$ $\Rightarrow \frac{1}{2}\left(\frac{p}{\sqrt{q}} + \sqrt{q}\right) \geq \sqrt{p}$ as required	(1) mark (1) mark	
(c)(i) see attached		