Question 1

(a) Differentiate:

    (i) e^{3x-1}

                    1

    (ii) $\log_e (2x - 1)$

                    1

(b) Evaluate $\lim_{x \to 0} \dfrac{\sin 3x}{x}$

                    1

(c) Find the remainder when the polynomial $P(x) = x^3 - 2x$ is divided by $x + 1$

                    2

(d) Evaluate $\displaystyle\int_0^{\sqrt{3}} \dfrac{dx}{\sqrt{4 - x^2}}$

                    2

(e) Find the equation of the tangent to the curve $y = e^{2x} - 3x$ at the point $(0,1)$

                    2

(f) A(-2, -5) and B(1, 4) are two points. Find the acute angle ϑ between the line AB and the line $x + 2y + 1 = 0$, giving the answer correct to the nearest minute.

                    3

Question 2
Start a new page

(a) Given $\log_a x = 0.64$ and $\log_a y = 0.04$ find $\log_a(\dfrac{x}{y})$ 1

(b) $Q(-1, 4)$ and $R(x, y)$ are two points. The point $P(14, -6)$ divides the interval QR externally in the ration 5:3. Find the coordinates of R.   2

(c)

(i) Sketch $y = e^x(-1)$ 2

(ii) Find the exact volume of the solid of revolution formed when the curve $y = e^x - 1$ is rotated about the x-axis from $x = 0$ to $x = 1$ 3

(d)

(i) Show that the function $x^3 - 2x - 5 = 0$ has one root which lies between 2 and 2.2   1

(ii) Using $x = 2.1$ as an approximation to a root of $x^3 - 2x - 5 = 0$, find a better approximation, correct to 2 decimal places, using Newton's method once.   3

Question 3
Start a new page

(a) Find $\int \dfrac{x}{x^2 + 5}\, dx$

(b) Write the expansion of sin (A + B).
Hence, or otherwise, find the exact value of sin $75°$

(c) Solve for x: $\dfrac{4}{x+1} \leq 3$

(d)

    (i)   Show that $(x + 1)$ is a factor of $f(x) = 2x^3 + 7x^2 - 7x - 12$

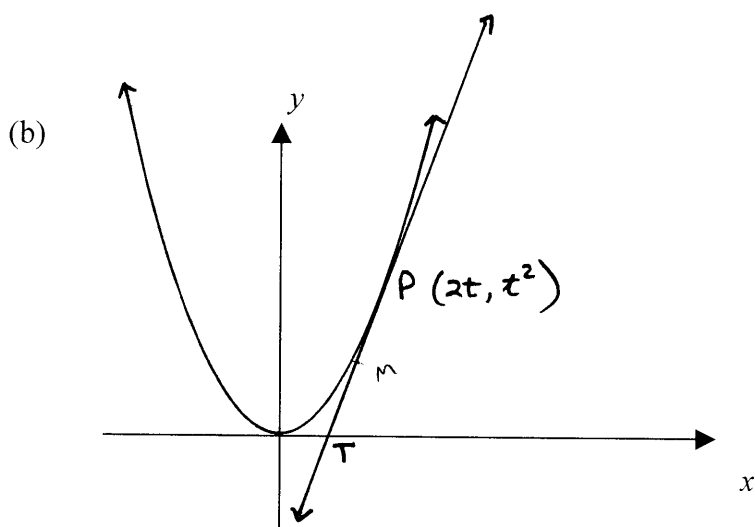   (ii)   Find all roots of $f(x) = 2x^3 + 7x^2 - 7x - 12$ and hence sketch the curve.

Question 4
Start a new page

(a) If α, β, γ are the roots of the equation $3x^3 - 6x^2 + x + 2 = 0$ find the value of

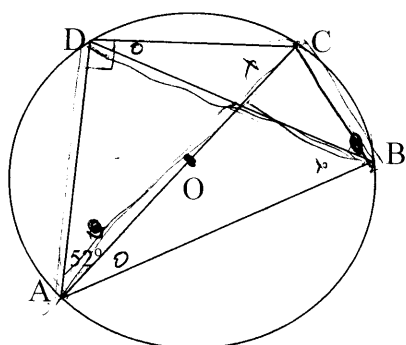$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$$

3

(b)

P $(2t, t^2)$ is a variable point which moves on the parabola $x^2 = 4y$. The tangent to the parabola at P cuts the x-axis at T. M is the midpoint of PT.

(i)   Show that the tangent PT has equation $tx - y - t^2 = 0$ **2**

(ii)   Show that M has coordinates $(\frac{3t}{2}, \frac{t^2}{2})$ **2**

(iii)   Hence find the Cartesian equation of the locus of M as P moves on the parabola   **2**

(c)

not to scale

Circle, centre O, has diameter AC. $\angle DAC = 52°$

(i)   Explain why $\angle ADC = 90°$ **1**

(ii)   Find $\angle DBA$, giving reasons for your answer.   **2**

5

Question 5
Start a new page

(a) Stephanie invests $150 at the start of each month into a superannuation fund. The interest is compounded monthly at a rate of 3% p.a. The first $150 is invested at the beginning of January 2005 and the last is invested at the beginning of December 2010. Calculate, to the nearest dollar:

    (i)   The amount to which the January 2005 investment will have grown by the end of 2010.   **2**

    (ii)   The amount to which the total will have grown by the end of 2010.   **3**

(b) Use mathematical induction to prove that:

$$\frac{1}{2\times3}+\frac{1}{3\times4}+\frac{1}{4\times5}+\dots\dots+\frac{1}{(n+1)(n+2)}=\frac{n}{2(n+2)}$$
    3

(c) Consider the function $f(x) = 2\sin^{-1}x$

    (i) Sketch the graph of $f(x) = 2\sin^{-1}x$ **1**

    (ii) Find the exact value of $f(\frac{1}{\sqrt{2}})$ **1**

    (iii) Find the equation of the tangent to the curve at the point where $x = \frac{1}{\sqrt{2}}$ **2**

Question 6
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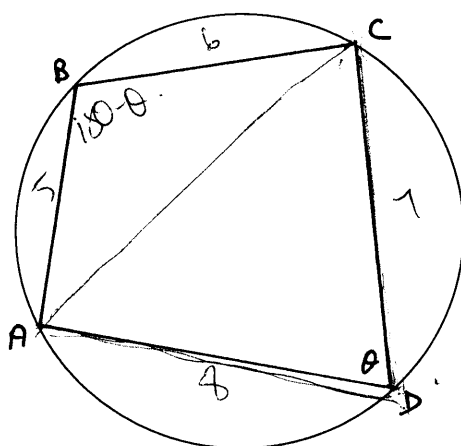
(a) Find $\dfrac{d}{dx}\cos^{-1}2x^3$

2

(b) If $t = \tan\dfrac{1}{2}\theta$, prove $\cos\theta\left(\tan\theta - \tan\dfrac{1}{2}\theta\right) = \tan\dfrac{1}{2}\theta$

3

(c) Show that $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2 x\,dx = \dfrac{\pi}{8} - \dfrac{1}{4}$

3

(d)

not to scale

A, B, C, D are points on the circumference of a circle.

AB = 5cm, BC = 6cm, DC = 7cm, AD = 8cm

(i) If $\angle ADC = \theta$, explain why $\angle ABC = 180° - \theta$

1

(ii) By drawing the diagonal AC, or otherwise

$\cos\angle ADC = \dfrac{13}{43}$

3

Question 7
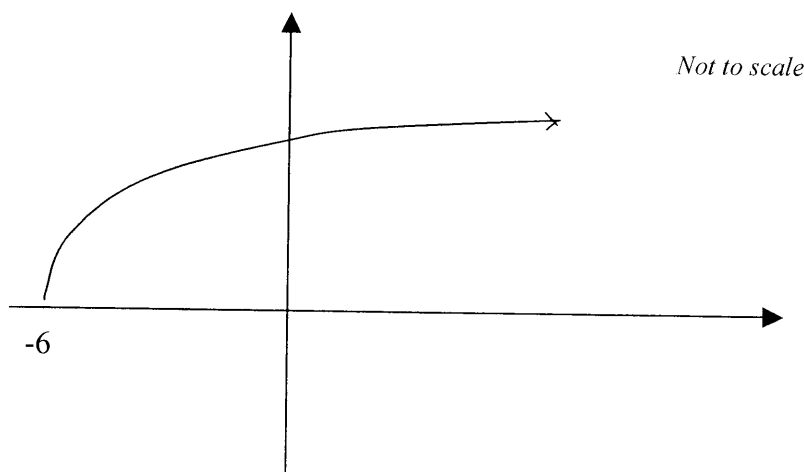**Start a new page**

(a) Using the substitution $u = e^x$, find

$$\int \frac{e^x}{1+e^{2x}}dx$$

2

(b)

(i) Express $\sqrt{3}\sin\vartheta - \cos\vartheta$ in the form $R\sin(\vartheta - \alpha)$, where α is in radians. 2

(ii) Hence, or otherwise, find all angles ϑ, where $0 \le \vartheta \le 2\pi$, for which
$\sqrt{3}\sin\vartheta - \cos\vartheta = 1$ 2

(c) The graph of $f(x) = \sqrt{x+6}$ for $x \ge -6$ is shown in the diagram.

Not to scale

-6

(i) Find the inverse function $f^{-1}(x)$ 1

(ii) On the same diagram sketch the graphs of $y = f(x)$, the line $y = x$, and $y = f^{-1}(x)$. Show clearly the intercepts on the coordinate axes. 2

(iii) What is the domain of $f^{-1}(x)$ 1

(iv) Show that the x coordinate of any points of intersection of the graphs $y = f(x)$ and $y = f^{-1}(x)$ satisfy the equation $x^2 - x - 6 = 0$.
Hence find any points of intersection of the two graphs. 2

END OF EXAMINATION

QUESTION 1

a) i) $y = e^{3x-1}$
$y' = 3e^{3x-1}$

ii) $y = \log_e(2x-1)$
$y' = \frac{2}{2x-1}$

b) $\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{3}{1}$
$= 3$

c) $P(x) = x^3 - 2x$
$P(-1) = (-1)^3 - 2(-1)$
$= -1 + 2$
$= 1$

d) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{3}}$
$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1} 0$
$= \frac{\pi}{3} - 0$
$= \frac{\pi}{3}$

e) $y = e^{2x} - 3x$
$y' = 2e^{2x} - 3$
at $x = 0$, $y' = 2e^0 - 3$
$= 2 - 3$
$= -1$

f) $A(-2,-5)$ $B(1,4)$
ii) $m = \frac{4+5}{1+2}$
$= \frac{9}{3}$
$= 3$

$x + 2y + 1 = 10$
$2y = -x - 1$
$y = -\frac{1}{2}x - \frac{1}{2}$
$\therefore m = -\frac{1}{2}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$
$= \left|\frac{\sqrt{3} + \frac{1}{2}}{1 + \sqrt{3} \cdot -\frac{1}{2}}\right|$
$= 7$
$\theta = 81°52'$

$y - y_1 = m(x - x_1)$
$y - 1 = -1(x - 0)$
$y - 1 = -x$
$x + y - 1 = 0$

c) $(1,-1)$

e) $y = e^{2x} - 3x$
$y' = 2e^{2x} - 3$
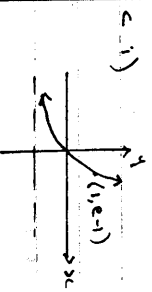$y' = 2e^0 - 3$
$= 2 - 3$
$= -1$

QUESTION 2

a) $\log_a\left(\frac{x^2}{y}\right) = \log_a x^2 - \log_a y$
$= 0.64 - 0.04$
$= 0.6$

b) $Q(-1,4)$ $R(x,y)$ $P(14,-4)$ $5:3$
$14 = \frac{5x - 3(-1)}{2}$ $-6 = \frac{5y - 3(4)}{2}$
$28 = 5x + 3$ $-12 = 5y - 12$
$25 = 5x$ $0 = 5y$
$x = 5$ $y = 0$
$\therefore R(5,0)$

d) i) $f(x) = (x)^3 - 2(x) - 5$
$= -1$
$f(2\cdot2) = (2\cdot2)^3 - 2(2\cdot2) - 5$
$= 1.248$

Since our result is positive then our negative, the root lies between 2 and 2.2.

ii) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$
$f(2\cdot1) = (2\cdot1)^3 - 2(2\cdot1) - 5$
$= 0.061$
$f'(x) = 3x^2 - 2$
$f'(2\cdot1) = 3(2\cdot1)^2 - 2$
$= 11.23$
$x_1 = 2\cdot1 - \frac{0.061}{11.23}$
$= 2.07456891$
$= 2.09$ to 2 dec pl.

QUESTION 3

a) $\int \frac{x}{x^2+5}\,dx = \frac{1}{2}\int \frac{2x}{x^2+5}\,dx$
$= \frac{1}{2}\ln(x^2+5) + c$

b) $\sin(A+B) = \sin A\cos B + \cos A\sin B$
$\sin 75 = \sin(45+30) = \sin 45\cos 30 + \cos 45\sin 30$
$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$
$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

c) $\frac{4}{x+1} \le 3(x+1)$
$4(x+1) \le 3(x+1)^2$
$0 \le 3(x+1)^2 - 4(x+1)$
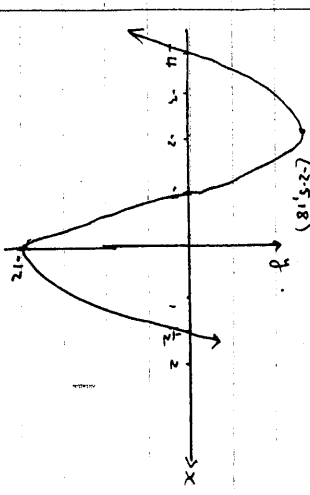$0 \le (x+1)[3(x+1) - 4]$
$0 \le (x+1)(3x-1)$
$(x+1)(3x-1) \ge 0$

$x \le -1$ and $x > \frac{1}{3}$
but $x \ne -1$
$x < -1$, $x > \frac{1}{3}$

d) i) $f(x) = 2x^3 + 7x^2 - 7x - 12$
$f(-1) = 2(-1)^3 + 7(-1)^2 - 7(-1) - 12$
$= -2 + 7 + 7 - 12$
$= 0$

Since the remainder is 0, $(x+1)$ is a fact[or]

ii)
$$\begin{array}{r} 2x^2 + 5x - 12 \\ x+1 \overline{)\,2x^3 + 7x^2 - 7x - 12} \\ 2x^3 + 2x^2 \\ \hline 5x^2 - 7x \\ 5x^2 + 5x \\ \hline -12x - 12 \\ -12x - 12 \\ \hline \end{array}$$

$\therefore f(x) = (x+1)(2x^2 + 5x - 12)$
$= (x+1)(2x-3)(x+4)$, $x = -1, \frac{3}{2}, -4$

f) ii) $V_{x-axis} = \pi\int_a^b y^2\,dx$
$= \pi\int_0^1 (e^x - 1)^2\,dx$
$= \pi\int_0^1 e^{2x} - 2e^x + 1\,dx$
$= \pi\left[\frac{e^{2x}}{2} - 2e^x + x\right]_0^1$
$= \pi\left[\left(\frac{e^2}{2} - 2e^1 + 1\right) - \left(\frac{e^0}{2} - 2e^0 + 0\right)\right]$
$= \pi\left(\frac{e^2}{2} - 2e + 2\frac{1}{2}\right)$ units³

$\alpha+\beta+\gamma = -\frac{b}{a} = \frac{6}{3} = 2$

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a} = \frac{3}{3} = 1$

$\alpha\beta\gamma = -\frac{d}{a} = -\frac{2}{3}$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$

$= \frac{1}{-\frac{2}{3}}$

$= -\frac{3}{2}$

b)i) $y = \frac{x^2}{4}$

$y' = \frac{2x}{4} = \frac{x}{2}$

ii) $x = 2t, \quad y' = \frac{2t}{2} = t$

$y-y_1 = m(x-x_1)$

$y-t^2 = t(x-2t)$

$y-t^2 = tx - 2t^2$

$0 = tx - y - t^2$

ii) If $y=0$, $\quad tx-0-t^2=0$

$x = \frac{t^2}{t} = t$

$\therefore T(t, 0)$

midpoint $= \left(\frac{2t+t}{2}, \frac{t^2+0}{2}\right)$

$= \left(\frac{3t}{2}, \frac{t^2}{2}\right)$

ii) $x = \frac{3t}{2}$

$t = \frac{2x}{3}$

$y = \frac{t^2}{2} = \frac{\left(\frac{2x}{3}\right)^2}{2}$

$= \frac{4x^2}{9} \times \frac{1}{2}$

$18y = 4x^2$

$9y = 2x^2$

c)i) $\angle ADC = 90°$ angle in a semi-circle

ii) $\angle DCA = 180-90-52$ (angle sum of Δ)

$= 38°$

$\angle DBA = 38°$ (angles in the same segment are equal)

a) $3\% \, pa = \frac{1}{6}\% \, pm = 0.0025$

i) $A = P(1+r)^n$

$= 150(1.0025)^{72}$

$= \$179.54$

ii) $A_1 = 150(1.0025)^{72}$

$A_2 = 150(1.0025)^{71}$

$A_{72} = 150(1.0025)^1$

Total $A = 150(1.0025^{72} + 1.0025^{71} + \cdots + 1.0025)$

$= 150\left(\frac{1.0025(1.0025^{72}-1)}{1.0025-1}\right)$

$= \$11846.45$

b) If $n=1$, LHS $= \frac{1}{2\times3} = \frac{1}{6}$

RHS $= \frac{1}{2(3)} = \frac{1}{6} =$ LHS

Assume true for $n=k$

$\frac{1}{2\times3} + \frac{1}{3\times4} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k}{2(k+2)}$

To prove true for $n=k+1$

ie $\frac{1}{2\times3} + \cdots + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)} = \frac{k+1}{2(k+3)}$

LHS $= \frac{k}{2(k+2)} + \frac{1}{(k+2)(k+3)}$

$= \frac{k(k+3) + 1(2)}{2(k+2)(k+3)}$

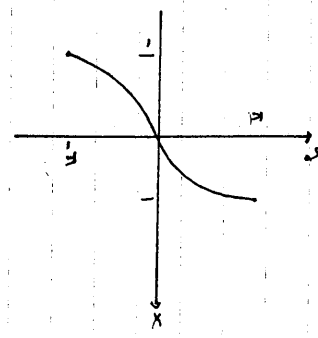$= \frac{k^2+3k+2}{2(k+2)(k+3)}$

$= \frac{(k+2)(k+1)}{2(k+2)(k+3)}$

$= \frac{k+1}{2(k+3)}$

$\therefore$ true for $n=k+1$.

Since statement is true for $n=1$.

it is true for $n=1+1=2$, and $n=2+1=3$

etc.

c)i)

a) $\frac{d}{dx} \cos^{-1}(2x^3) = \frac{-1}{\sqrt{1-(2x^3)^2}} \cdot 6x^2$

$= \frac{-6x^2}{\sqrt{1-4x^6}}$

b) $\cos\theta(\tan\theta - \tan\tfrac{1}{2}\theta)$

$= \frac{1-t^2}{1+t^2}\left(\frac{2t}{1-t^2} - t\right)$

$= \frac{1-t^2}{1+t^2}\left(\frac{2t - t + t^3}{1-t^2}\right)$

$= \frac{t+t^3}{1+t^2}$

$= \frac{t(1+t^2)}{1+t^2}$

$= t = \tan\tfrac{1}{2}\theta$.

ii) $f\left(\frac{1}{\sqrt{2}}\right) = 2\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= 2 \cdot \frac{\pi}{4}$

$= \frac{\pi}{2}$

iii) $f(x) = 2\sin^{-1}x$

$f'(x) = 2 \cdot \frac{1}{\sqrt{1-x^2}}$

$f'\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{1-\frac{1}{2}}}$

$= \frac{2}{\sqrt{\frac{1}{2}}}$

$= 2\sqrt{2}$

$y-y_1 = m(x-x_1)$

$y - \frac{\pi}{2} = 2\sqrt{2}\left(x - \frac{1}{\sqrt{2}}\right)$

$y - \frac{\pi}{2} = 2\sqrt{2}x - 2$

$y = 2\sqrt{2}x - 2 + \frac{\pi}{2}$

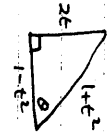c) $\int_{\pi/4}^{\pi/2} \cos^2 x \, dx =$

$= \frac{1}{2}\int_{\pi/4}^{\pi/2} (\cos 2x+1)\,dx$

$= \frac{1}{2}\left[\frac{1}{2}\sin 2x + x\right]_{\pi/4}^{\pi/2}$

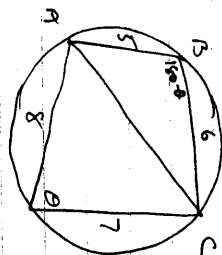$= \frac{1}{2}\left[\left(\frac{1}{2}\sin\pi + \frac{\pi}{2}\right) - \left(\frac{1}{2}\sin\frac{\pi}{2} + \frac{\pi}{4}\right)\right]$

$= \frac{1}{2}\left[\left(0+\frac{\pi}{2}\right) - \left(\frac{1}{2}+\frac{\pi}{4}\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right]$

$= \frac{\pi}{8} - \frac{1}{4}$

$\cos 2x = \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

a) ∠ABC = 180 - θ   opposite ∠s of cyclic quadrilaterals are supplementary.

cos θ = $\dfrac{8^2 + 7^2 - AC^2}{2.8.7}$

= $\dfrac{113 - AC^2}{112}$

AC² = 5² + 6² - 2.5.6 cos(180-θ)
= 61 + 60 cos θ

∴ cos θ = $\dfrac{113 - (61+60\cos\theta)}{112}$

112 cos θ = 113 - 61 - 60 cos θ

172 cos θ = 52

cos θ = $\dfrac{52}{172}$

= $\dfrac{13}{43}$

QUESTION 7

e) u = eˣ

$\dfrac{du}{dx}$ = eˣ

dx = $\dfrac{du}{e^x}$

∫ $\dfrac{e^x}{1+e^{2x}}$ dx = ∫ $\dfrac{u}{1+u^2}$ · $\dfrac{du}{u}$

= ∫ $\dfrac{du}{1+u^2}$

= tan⁻¹ u + c

= tan⁻¹ eˣ + c

---

b) i) R = √(a²+b²)   a=√3, b=1

R = $\sqrt{(\sqrt3)^2 + 1^2}$

= 2

tan α = $\dfrac{b}{a}$

tan α = $\dfrac{1}{\sqrt3}$

α = 30° = $\dfrac{\pi}{6}$ 0 < α < $\dfrac{\pi}{2}$

R sin(θ - α) = 2 sin(θ - $\dfrac{\pi}{6}$)

ii) 2 sin(θ - $\dfrac{\pi}{6}$) = 1

sin(θ - $\dfrac{\pi}{6}$) = $\dfrac{1}{2}$

θ - $\dfrac{\pi}{6}$ = $\dfrac{\pi}{6}$, θ - $\dfrac{\pi}{6}$ = $\dfrac{5\pi}{6}$

θ = $\dfrac{\pi}{3}$ θ = π

c) i) f(x) = √(x+6)

y = √(x+6)

x = √(y+6)

x² = y+6

y = x² - 6

f⁻¹(x) = x² - 6

ii)
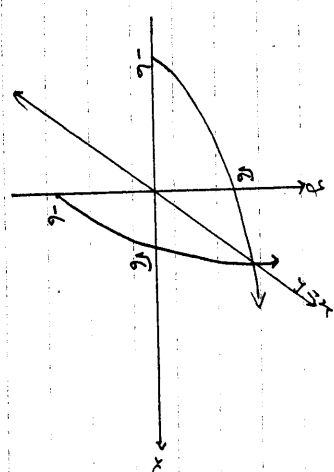
iii) D: x ≥ 0

iv)

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y = x² - 6 intersects with y = x.

x = x² - 6

x² - x - 6 = 0.

y = √(x+6) intersects with y=x

x = √(x+6)

x² = x+6

x² - x - 6 = 0.

x² - x - 6 = 0

(x-3)(x+2) = 0

x = 3, -2

but x ≠ -2

∴ x = 3

Point of intersection is (3,3)