# JAMES RUSE AGRICULTURAL HIGH SCHOOL
# YEAR 12 ASSESSMENT TERM 1 1999
# MATHEMATICS 3/4 UNIT

**Time: 85 Minutes**
**All questions to be attempted.**
**Approved calculators may be used.**
**All necessary working must be shown.**
**All questions are of equal value.**
**Each section is to be handed in separately.**

## QUESTION 1 (Start a new page)

(a) Find $\int \sqrt{\sin x}\cos x\,dx$

(b) Differentiate with respect to x:

(i) $y = x\sin 2x$

(ii) $y = \ln\cos x$

(iii) $y = \dfrac{\cos x}{\sin x + 1}$

(c) Find the volume of the solid generated by revolving about the x-axis the region bounded by the co-ordinate axes, the curve $y = \sec x$ and the line $x = \dfrac{\pi}{4}$.

## QUESTION 2 (Start a new page)

(a) Find the exact value of the following definite integrals.

(i) $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-4x^2}}$

(ii) $\displaystyle\int_{-\sqrt{3}}^{0} \frac{4\,dx}{1+x^2}$

(b) (i) For what set of values of x will the infinite geometric series $1 - 2x + 4x^2 - 8x^3 + \ldots$ have a limiting sum?

(ii) If this limiting sum is $\dfrac{3}{5}$, find the value of x.

(c) Find the exact value of $\sec\left[\sin^{-1}\left(-\dfrac{3}{4}\right)\right]$

## QUESTION 3 (Start a new page)

(a) Two concentric circles are expanding. At a certain instant the outer radius (R) is 4m and the inner radius (r) is 95cm. The outer radius is expanding at the rate of 1m/s and the inner radius at 0.25m/s. Find the rate of change of area (A) between the circles. Give your answer correct to 2 significant figures.

(b) Find the sum of the first 1000 terms of the series

$1 - 2 + 3 - 4 + 5 - 6 + \ldots + (-1)^{n+1}n + \ldots$

(c) (i) Differentiate $\sin^{-1}(\cos 2x)$

(ii) Hence, or otherwise sketch the graph of $y = \sin^{-1}(\cos 2x)$ for $-\pi \le x \le \pi$

## QUESTION 4 (Start a new page)

(a) Integrate using the substitution given

$$\int_0^{\frac{\pi}{4}} \frac{dx}{9\cos^2 x + 25\sin^2 x} \qquad \text{where } u = \tan x$$

(b) Express $\sin(2\cos^{-1}x)$ in terms of x only.

(c)

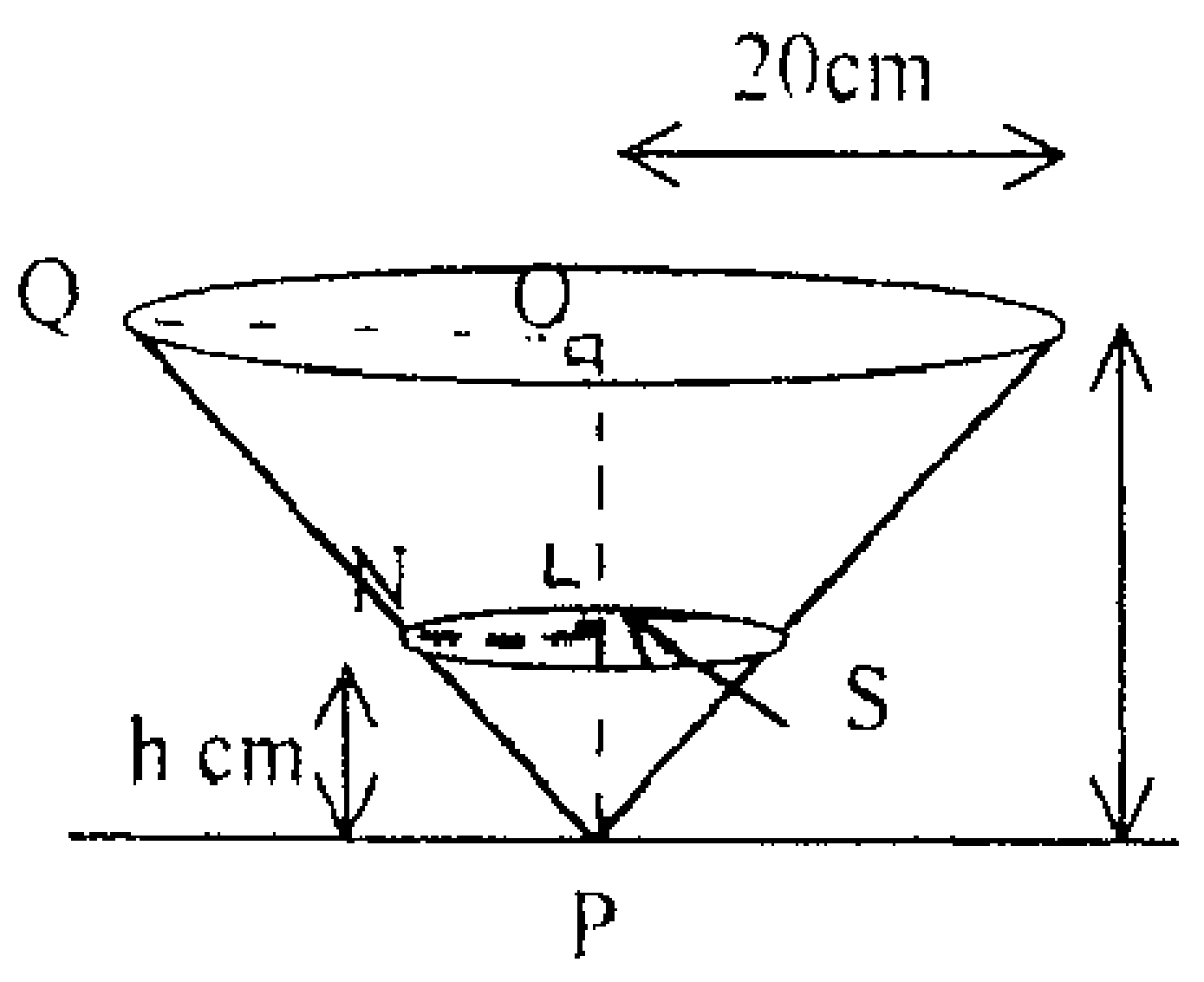

Diagram not to scale

Water is poured into a conical vessel at a constant rate of $10\text{cm}^3$ per second. The depth of water is h cm at any time t seconds. What is the rate of increase of the area of the surface S of the liquid when the depth is 8cm?

## QUESTION 5 (Start a new page)

(a) Katherine borrows \$20 000 at 8% per annum reducible interest, calculated monthly. The loan is to be repaid in 60 equal monthly instalments.

- (i) Show that the monthly repayments should be \$405.53
- (ii) With the 8th repayment, Katherine pays an additional \$2000, so this payment is \$2405.53. After this, repayments continue at \$405.53 per month. How many more repayments will be needed?

(b) Calculate the exact area of the region bounded by the graph of $f(x) = 2\cos^{-1}3x$, the x axis, and the ordinates x = 0 and $x = \frac{1}{6}$.

## QUESTION 6 (Start a new page)

(a) (i) Show that $\frac{d}{dx}(\cos^{-1}(\frac{1}{x}))$ is always positive

(ii) State the domain and range of $y = \cos^{-1}(\frac{1}{x})$

(iii) Hence, sketch the graph of $y = \cos^{-1}(\frac{1}{x})$ for $-2\pi \le x \le 2\pi$

(b) Two circles of unit radius intersect at A and B respectively. If their centres P and Q are 2x units apart, show that the area (A) common to the two circles is given by:

$$A = 2[\cos^{-1}x - x\sqrt{1-x^2}]$$

**END OF PAPER**

## STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

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## QUESTION 1:

(a) $\int \sqrt{\sin x} \cdot \cos x \, dx = \frac{2}{3}\sin^{\frac{3}{2}} x + c$ ②

(b) (i) $y = x\sin 2x$

$\frac{dy}{dx} = x \times 2\cos 2x + \sin 2x \times 1$

$= 2x\cos 2x + \sin 2x$ ②

(ii) $y = \ln(\cos x)$

$\frac{dy}{dx} = \frac{1}{\cos x} \times -\sin x$

$= -\tan x$ ②

(iii) $y = \frac{\cos x}{\sin x + 1}$

$\frac{dy}{dx} = \frac{(\sin x+1)\times(-\sin x) - \cos^2 x}{(\sin x+1)^2}$

$= \frac{-1-\sin x}{(\sin x+1)^2}$

$= \frac{-1}{(\sin x+1)}$ ②

(c) $V = \pi\int_0^{\frac{\pi}{4}} \sec^2 x \, dx$

$= \pi\left[\tan x\right]_0^{\frac{\pi}{4}}$

$V = \pi[1-0]$

$V = \pi u^3$ ②

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## QUESTION 2:

(a) (i) $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\left[\sin^{-1} 2x\right]_0^{\frac{1}{2}}$

$= \frac{1}{2}\left[\sin^{-1} 1 - \sin^{-1} 0\right]$

$= \frac{\pi}{4}$ ②

(ii) $\int_{-\sqrt{3}}^{0} \frac{4\,dx}{1+x^2} = 4\left[\tan^{-1} x\right]_{-\sqrt{3}}^{0}$

$= 4\left[\tan^{-1} 0 - \tan^{-1}(-\sqrt{3})\right]$

$= \frac{4\pi}{3}$ ②

(b) (i) $-1 < r < 1$ ②

$-1 < -2x < 1$

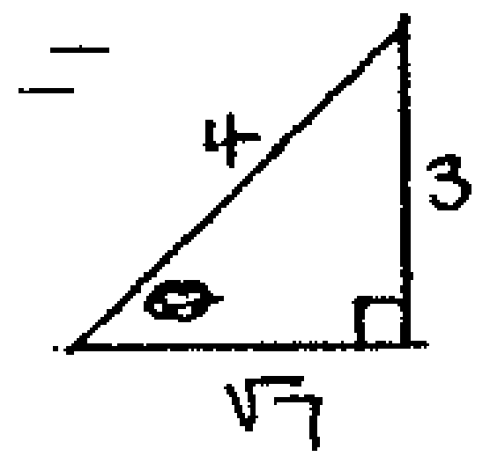
$\therefore -\frac{1}{2} < x < \frac{1}{2}, \; x \neq 0$

(ii) $\frac{3}{5} = \frac{1}{1+2x}$ ②

$5 = 3 + 6x$

$x = \frac{1}{3}$

(c) $\sec\left[\sin^{-1}\left(-\frac{3}{4}\right)\right]$ ②



$\therefore \sec\theta = \frac{4}{\sqrt{7}}$

Let $\theta = \sin^{-1}\left(-\frac{3}{4}\right)$

$\sin\theta = -\frac{3}{4}$

$\therefore \sec\left[\sin^{-1}\left(-\frac{3}{4}\right)\right] = \frac{4}{\sqrt{7}}$

$= \frac{4\sqrt{7}}{7}$

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## QUESTION 3:

(a) $A = \pi(R^2 - r^2)$

$\frac{dA}{dt} = 2\pi\left(R \cdot \frac{dR}{dt} - r \cdot \frac{dr}{dt}\right)$

$= 2\pi(4\times 1 - 0.95 \times 0.25)$

$= 2\pi(3.7625)$

$= 23.64048\ldots$

$= 24 \, m^2/s$ (2 s.f.) ③

(b) $1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n-1} n + \cdots$

$S_{1000} = 500(-1)$

$= -500$ ①

(c) $\frac{d}{dx}\left[\sin^{-1}(\cos 2x)\right]$

let $y = \sin^{-1} u$ where $u = \cos 2x$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot -2\sin 2x$

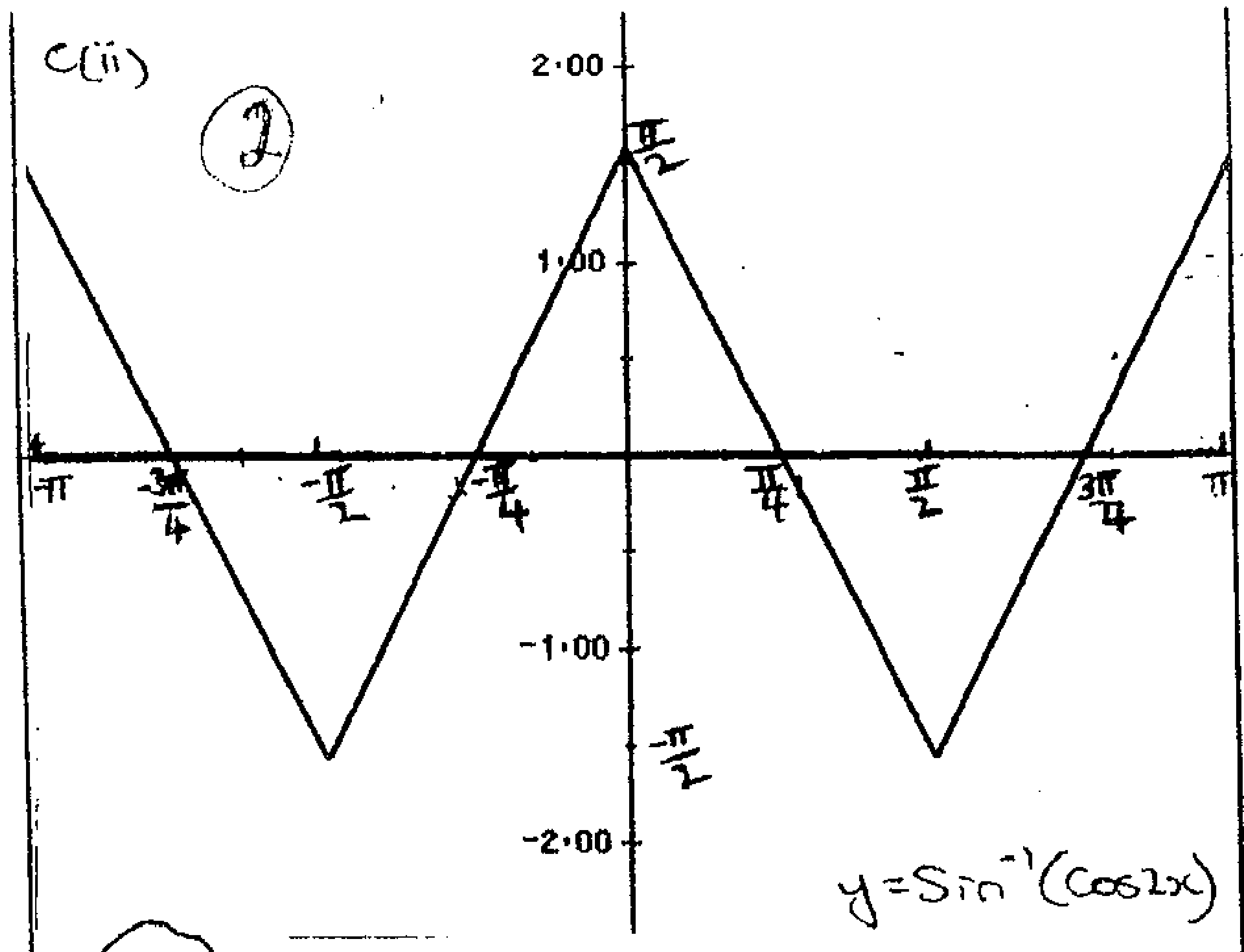
$= \frac{-2\sin 2x}{\sqrt{1-\cos^2 2x}}$ ③

$= \frac{-2\sin 2x}{|\sin 2x|}$

$= -2$ if $\sin 2x > 0$

$2$ if $\sin 2x < 0$

C(ii) ②



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## QUESTION 4:

(a) $\int_0^{\frac{\pi}{4}} \frac{dx}{9\cos^2 x + 25\sin^2 x}$

$u = \tan x$

$du = \sec^2 x \, dx$

when $x = 0$ $u = 0$

$x = \frac{\pi}{4}$ $u = 1$

[1] $= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x \, dx}{9 + 25\tan^2 x}$ ($\div$ by $\cos^2 x$)

$= \int_0^1 \frac{du}{9 + 25u^2}$

$= \frac{1}{25}\int_0^1 \frac{du}{u^2 + \frac{9}{25}}$

[1] $= \frac{1}{25} \times \frac{5}{3}\left[\tan^{-1}\frac{5u}{3}\right]_0^1$

[1] $= \frac{1}{15}\tan^{-1}\frac{5}{3}$ OR $\doteqdot 3.94$

(b) $\sin(2\cos^{-1} x)$

Let $\cos^{-1} x = \alpha$

$\therefore \cos\alpha = x$

$\sin(2\alpha) = 2\sin\alpha\cos\alpha$

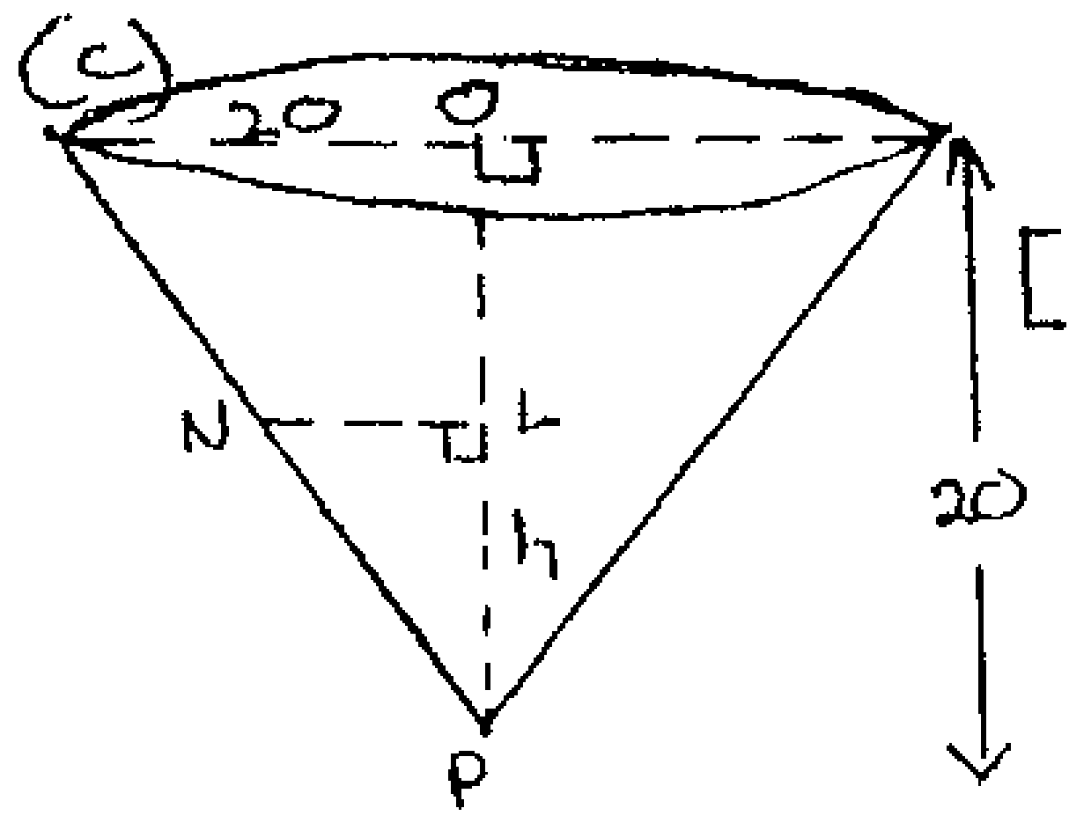
$\sin\alpha = \sqrt{(1-\cos^2\alpha)}$

$= \sqrt{1-x^2}$ ③

$\therefore 2\sin\alpha\cos\alpha = 2\left[\sqrt{1-x^2}\right] \cdot x$

$\therefore \sin(2\cos^{-1} x) = 2x\sqrt{1-x^2}$

(c)



In $\Delta OQP$ and $\Delta LNP$

$\angle OPQ$ is common

[1] $\angle QOP = \angle LNP$ (corresp. angles $OQ \parallel LN$)

$\therefore \Delta PLN \parallel\!\mid \Delta POQ$ (equiangular)

$\therefore \Delta PLN$ is isosceles

[1] $\therefore LN = LP = h$

Now $V = \frac{1}{3}\pi h^3$

$S = \pi h^2$

$\frac{dV}{dt} = \pi h^2 \cdot \frac{dh}{dt}$ $\qquad \frac{dV}{dt} = 10$

$\therefore \frac{dh}{dt} = \frac{10}{\pi h^2}$

at $h = 8$

$\frac{dh}{dt} = \frac{5}{32\pi}$ cm/s [1]

$\frac{dS}{dt} = 2\pi h \cdot \frac{dh}{dt}$

$\left[\frac{dS}{dt}\right]_{h=8} = 2\pi \times 8 \times \frac{5}{32\pi}$

$\therefore \frac{dS}{dt} = \frac{5}{2}$ cm$^2$/s [1]

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## QUESTION 5:

(a) (i) $M = \frac{20000 \times 1.006^{60} \times 0.006}{1.006^{60} - 1}$

[2] $= \$405.53$ (nearest cent)

(ii) after 8th normal payment, amt. owing is:

$A = 20000 \times \left(1\frac{1}{150}\right)^8 - \$405.53\frac{\left(\left(1\frac{1}{150}\right)^8 - 1\right)}{\left(1\frac{1}{150} - 1\right)}$

$\therefore A = \$17770.93$ [1]

Repays \$2000

$\therefore$ Amt. owing $= \$15770.93$

$0 = 15770.93 \times \left(1\frac{1}{150}\right)^n - 405.53\frac{\left(\left(1\frac{1}{150}\right)^n - 1\right)}{\frac{1}{150}}$

[1] $= 15770.93 \times \left(1\frac{1}{150}\right)^n - 60829.50\left(\left(1\frac{1}{150}\right)^n - 1\right)$

$= \left(1\frac{1}{150}\right)^n\left[15770.93 - 60829.5\right] + 60829.5$

$= \left(1\frac{1}{150}\right)^n\left[-45058.57\right] + 60829.5$

$\therefore \frac{60829.5}{45058.57} = \left(1\frac{1}{150}\right)^n$ [1]

$1.3500096 = \left(1\frac{1}{150}\right)^n$

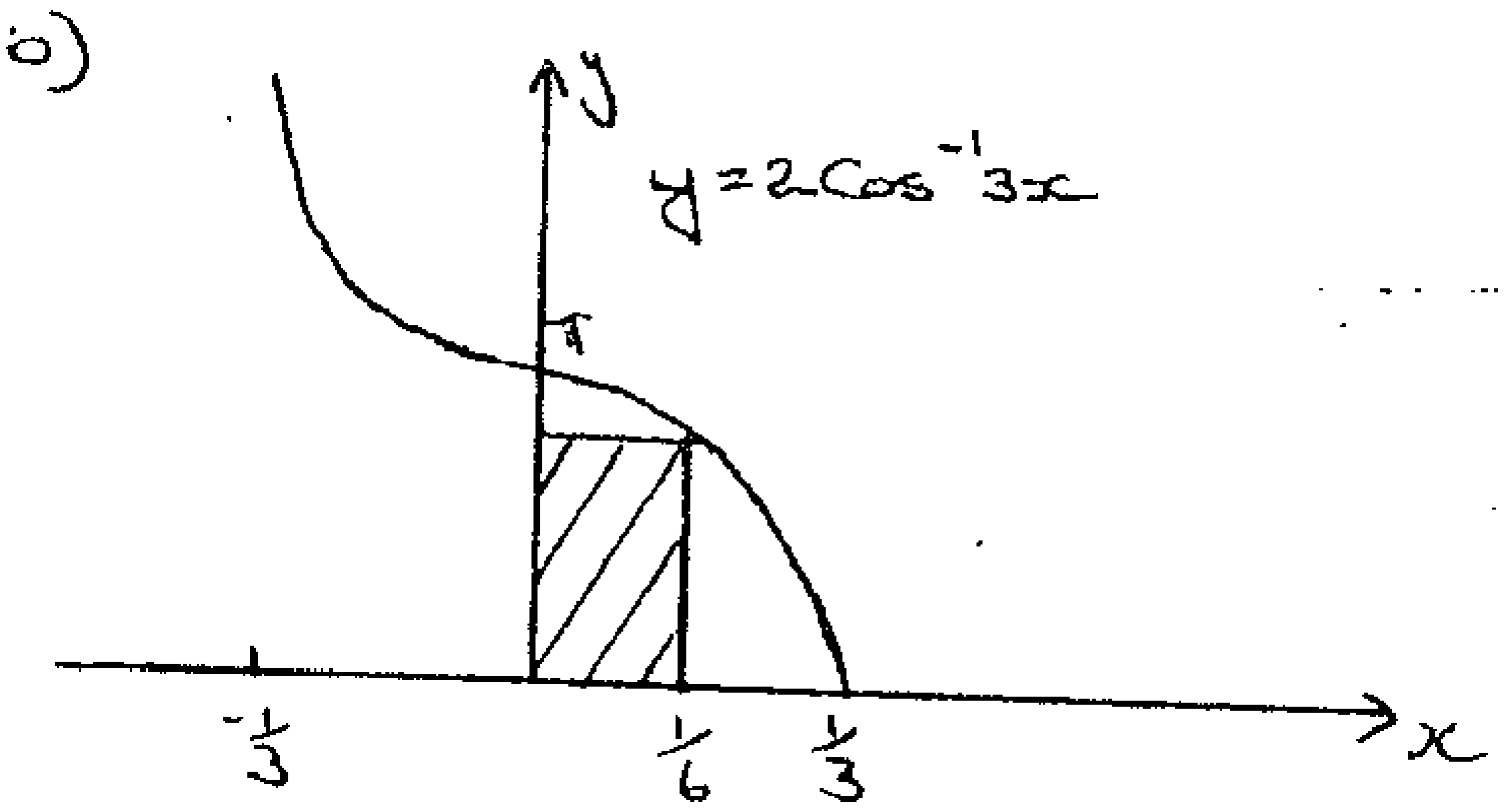
$\therefore n = 45.17$

[1] $n = 46$ repayments

④

QUESTION 5 (cont.)

(a)



$A = \int_0^{\frac{1}{6}} (2\cos^{-1}3x)\,dx$

Let $y = 2\cos^{-1}3x$

$\frac{y}{2} = \cos^{-1}3x$

$3x = \cos\frac{y}{2}$

$x = \frac{1}{3}\cos\frac{y}{2}$ [1]

$A = \text{Area of Rectangle} + \int_{\frac{2\pi}{3}}^{\pi} \frac{1}{3}\cos\frac{y}{2}\,dy$ [1]

$= \frac{\pi}{9} + \left[\frac{2}{3}\sin\frac{y}{2}\right]_{\frac{2\pi}{3}}^{\pi}$ [1]

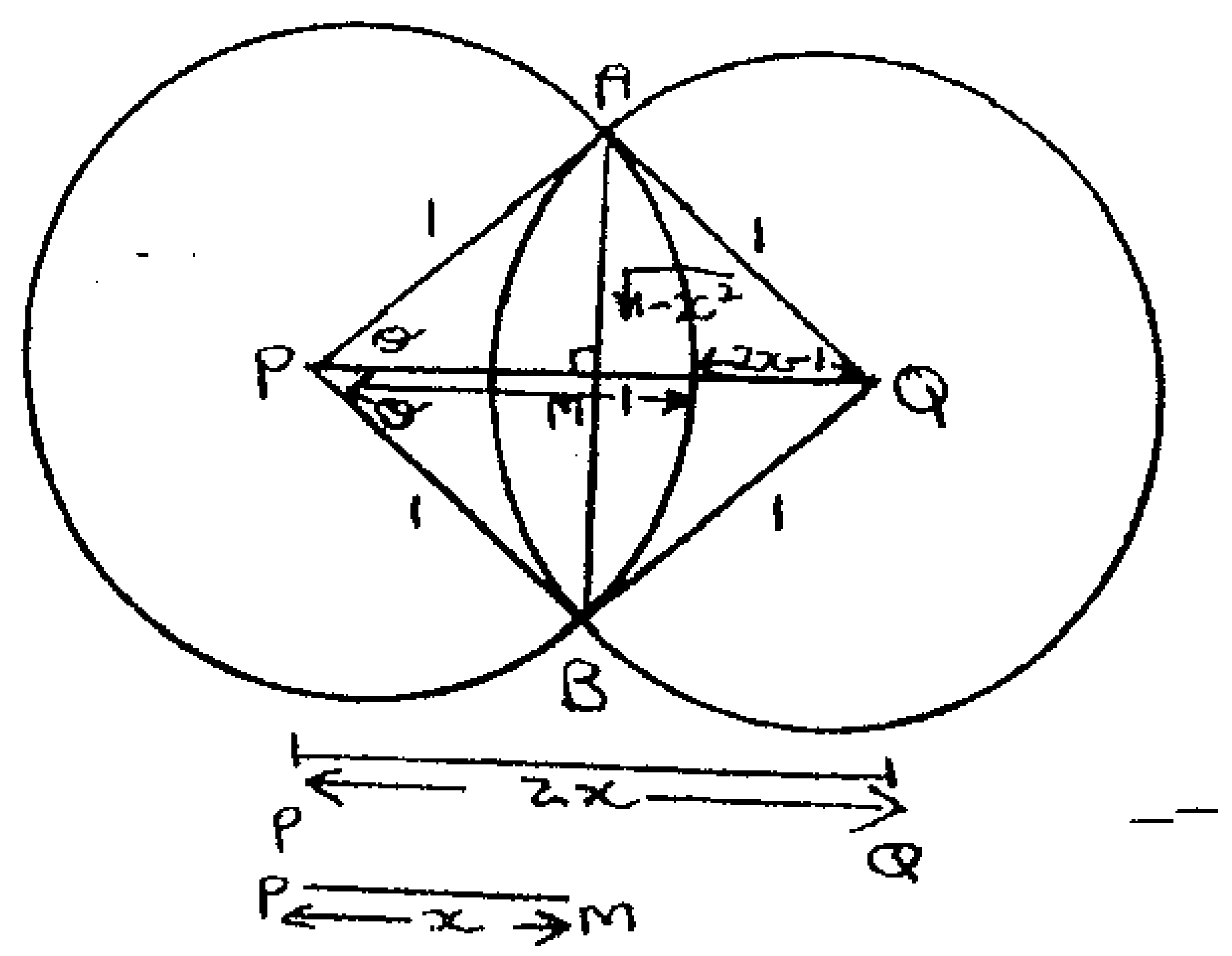
$= \frac{\pi}{9} + \frac{2}{3}\left(1 - \frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{9} + \frac{2-\sqrt{3}}{3}$ [1]

④

⑩

(b)



③

$A = 2 \times \frac{1}{2} r^2 (2\theta - \sin 2\theta)$

$A = r^2(2\theta - \sin 2\theta)$ $\qquad \cos\theta = r$

$A = r^2(2\theta - (2\sin\theta\cos\theta))$

$A = r^2(2\theta - 2\sqrt{1-x^2} \times x)$ $\qquad r = 1$

$A = 2(\cos^{-1}x - x\sqrt{1-x^2})$

⑩

QUESTION 6:

(a)(i) $\frac{d}{dx}\left(\cos^{-1}\left(\frac{1}{x}\right)\right) = \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \cdot \frac{-1}{x^2}$

$= \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}$

$= \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$

$= \frac{1}{\sqrt{x^4-x^2}} > 0$

②

(ii) Domain: $-1 \leq \frac{1}{x} \leq 1$

$\therefore x \leq -1$ and $x \geq 1$ ①

Range: $0 \leq y \leq \pi$ but $y \neq \frac{\pi}{2}$ ①

(iii)

③