Taylors College, Sydney Campus

4 Unit Mathematics

Trial Higher School Certificate Examinations 1998 - 2002

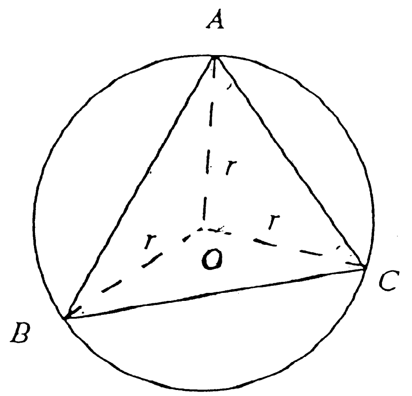
1998

1. **(a)** Find $\int \frac{dx}{x \log_e x}$ **(b)** Find $\int \frac{dx}{\sqrt{3+2x-x^2}}$ **(c)** Find $\int \frac{dx}{(x+1)(x^2+4)}$
(d) Using the substitution $t = \tan\frac{x}{2}$, calculate $\int \frac{15\ dx}{17+8\cos x}$, leaving your answer in terms of t.
(e) **(i)** Differentiate $\frac{x}{\sqrt{x-3}}$ **(ii)** Hence evaluate $\int_4^7 \frac{2x-9}{2(x-3)\sqrt{x-3}}\ dx$

2. **(a)** The complex number z is given by $z = 1 - 2i$. Find in simplest form the values of **(i)** $iz + \overline{z}$ **(ii)** $\frac{1}{z}$
(b) The equation $x^2 - (p + iq)x + 3i = 0$, where p and q are real, has roots α and β. The sum of the squares of the roots is 8.
(i) Write down expressions for the sum of the roots and the product of the roots.
(ii) Hence find the possible values of p and q.
(c) **(i)** Express each of the complex numbers $z = 2i$ and $w = 1 + \sqrt{3}i$ in modulus/argument form. Represent the vectors z, w and $z + w$ on an Argand diagram.
(ii) Find the exact values of $\arg\left(\frac{z}{w}\right)$ and $\arg(z + w)$.
(d) **(i)** On an Argand diagram shade in the region satisfy both the conditions $|z - 2i| \leq 1$ and $0 \leq \arg(z - i) \leq \frac{\pi}{6}$
(ii) Find the exact perimeter and the exact area of the shaded region.

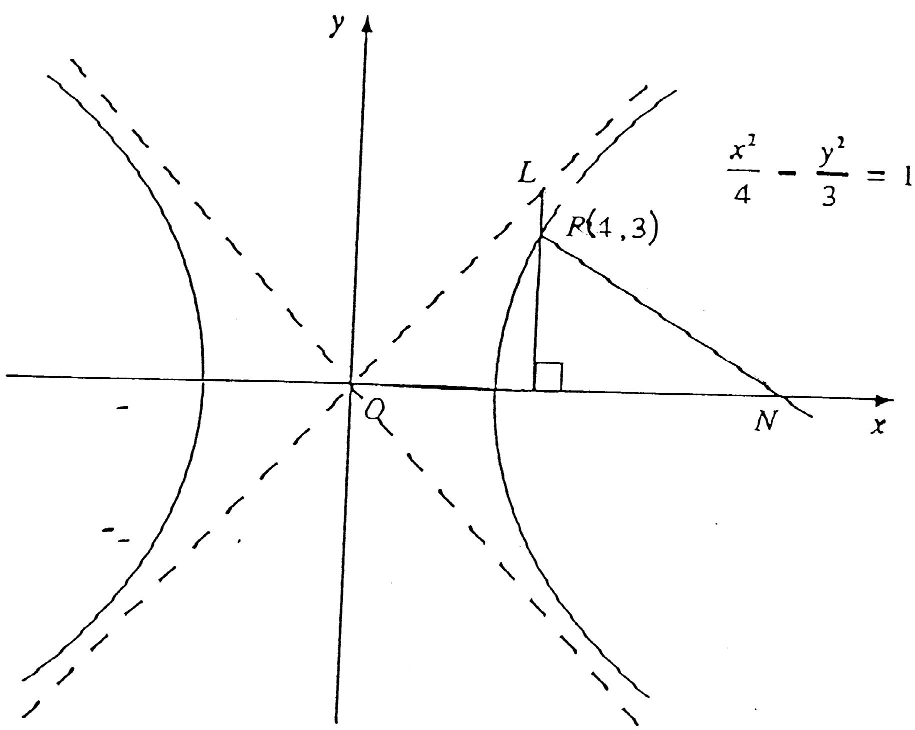
3. **(a)** Show that $\int_0^{\frac{\pi}{4}} x \sin x\ dx = \frac{\sqrt{2}}{8}(4 - \pi)$.
(b) The shape of a particular cake can be represented by rotating the region between the curve $y = \sin x$ and the x-axis, from $x = 0$ to $x = \frac{\pi}{4}$, about the line $x = \frac{\pi}{4}$. Using the method of cylindrical shells, find the volume of the cake.

(c) The circle through the vertices of triangle ABC has centre O and radius r.



(i) Show that $BC = 2r \sin A$.
(ii) Use the fact that Area($\triangle OBC$)+Area($\triangle OCA$)+Area($\triangle OAB$) = Area($\triangle ABC$) to show that $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

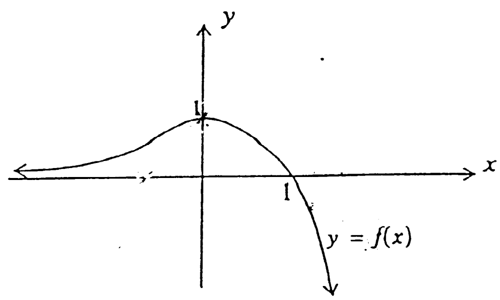
4. **(a)** The diagram shows the graph of the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$.



The point $P(4, 3)$ lies on the hyperbola. The normal at P to the hyperbola meets the x axis at N. The vertical line through P meets the asymptote in the first quadrant at L.
(i) Show that the normal at P to the hyperbola has equation $x + y = 7$.
(ii) Show that LN is perpendicular to OL.
(b) The points $P\left(2p, \frac{2}{p}\right)$ and $Q\left(2q, \frac{2}{q}\right)$ lie on the rectangular hyperbola $xy = 4$. M is the midpoint of the chord PQ. P and Q move on the rectangular hyperbola so

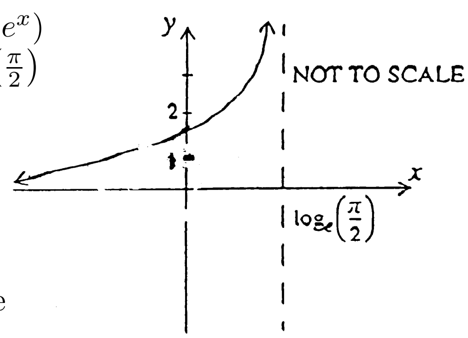
that the chord PQ always passes through the point $R(4,2)$.
(i) Show that the chord PQ has equation $x+pqy=2(p+q)$
(ii) Show that $pq=p+q-2$
(iii) Hence show that the locus of M has equation $y=\frac{x}{x-2}$.
(iv) On the same axes sketch the rectangular hyperbola $xy=4$ and the locus of M, showing clearly the equations of any asymptotes and the point R.

5. (a) The graph of $y=f(x)$ is sketched below. There is a stationary point at $(0,1)$.



Use this graph to sketch the following without using calculus, showing essential features.
(i) $y=f\left(\frac{x}{2}\right)$ **(ii)** $y=x+f(x)$ **(iii)** $y=\frac{1}{f(x)}$ **(iv)** $y=f\left(\frac{1}{x}\right)$.

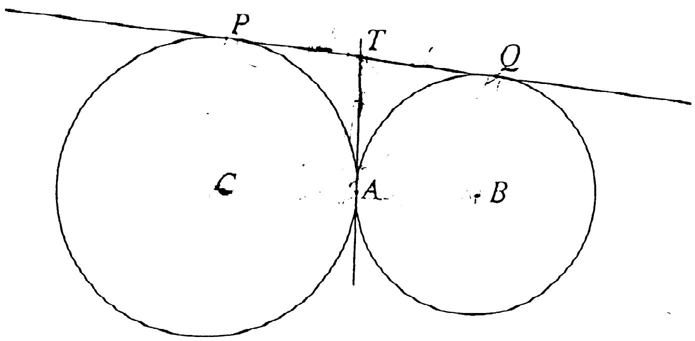
(b) The diagram shows part of the curve $y=\tan(e^x)$ where $x<\log_e\left(\frac{\pi}{2}\right)$. The part to the right of $\log_e\left(\frac{\pi}{2}\right)$ has not yet been drawn.



(i) Find the smallest positive solution to the equation $\tan(e^x)=0$.
(ii) Show that $y=\tan(e^x)$ is an increasing function.
(iii) Copy the diagram and hence sketch the curve $y=\tan(e^x)$ for $x<\log_e\left(\frac{3\pi}{2}\right)$.
(iv) Find the equation of the inverse function of $y=\tan(e^x)$ for the case when
(α) $x<\log_e\left(\frac{\pi}{2}\right)$ **(β)** $\log_e\left(\frac{\pi}{2}\right)<x<\log_e\left(\frac{3\pi}{2}\right)$.

6. (a) (i) Prove that $\tan^{-1}n-\tan^{-1}(n-1)=\tan^{-1}\frac{1}{n^2-n+1}$, where n is a positive integer.
(ii) Hence evaluate $\tan^{-1}1+\tan^{-1}\frac{1}{3}+\cdots+\tan^{-1}\frac{1}{n^2-n+1}$.
(iii) Hence find $\sum_{n=1}^{\infty}\tan^{-1}\frac{1}{n^2-n+1}$.
(b) (i) On the same diagram sketch the graphs of $x^2+y^2=9$ and $x^2-y^2=4$ showing clearly the coordinates of any points of intersection with the x axis or the y axis, and the equations of any asymptotes.
(ii) Shade the region where $(x^2+y^2-9)(x^2-y^2-4)\geq 0$
(c) (i) Use DeMoivre's theorem to show that $(1+i\tan\theta)^n+(1-i\tan\theta)^n=\frac{2\cos n\theta}{\cos^n\theta}$, $(\cos\theta\neq 0)$ where n is a positive integer.

(ii) Hence show that the equation $(1+z)^4+(1-z)^4=0$ has roots $\pm i\tan\frac{\pi}{8}, \pm i\tan\frac{3\pi}{8}$.
(iii) Hence show that $\tan^2\frac{\pi}{8}=3-2\sqrt{2}$.
7. (a)

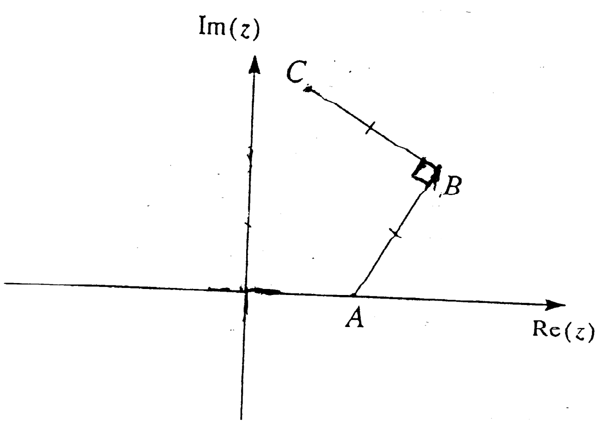


Two circles, centres C and B, touch externally at A. PQ is a direct common tangent touching the circles at P and Q respectively. The common tangent at A meets PQ at T.
(i) Show that the common tangent at A bisects PQ.
(ii) Let M be the midpoint of CB. Prove that $MT||CP$.
(iii) Prove that the circle with BC as diameter touches the line PQ.
(b) A curve is defined by the parametric equations $x=\cos^3\theta, y=\sin^3\theta$ for $0<\theta<\frac{\pi}{4}$.
(i) Show that the equation of the normal to the curve at the point $P(\cos^3\phi, \sin^3\phi)$ is $x\cos\phi - y\sin\phi = \cos 2\phi$
(ii) The normal at P meets the x axis at A and the y axis at B. Show that $AB = 2\cot 2\phi$
(c) (i) If $x\geq 0$ show that $\frac{2x}{1+x^2}\leq 1$.
(ii) Show that $e^a \geq 1+a^2$ for $a\geq 0$.
8. (a) A High School Student Representative Council consists of twelve students, one boy and one girl from each of years 7 to 12. At their meetings the twelve students sit around a circular table. Find how many seating arrangements are possible
(i) without restriction
(ii) if all the boys sit next to each other
(iii) if no two boys sit next to each other
(iv) if the boy and girl from each year group sit opposite each other.
Suppose now that a committee of six students is chosen at random from the members of the Student Representative Council.
(v) Find the probability that the committee contains exactly two students who belong to the same year group.
(b) A sequence is defined by $u_1=2, u_2=8$ and $u_n=4u_{n-1}-4u_{n-2}$, where $n\geq 3$ is a positive integer.
(i) Use the method of Mathematical Induction to show that $u_n=n2^n$ for all positive integers $n\geq 1$.
(ii) Without using the method of Mathematical Induction again, show that $\sum_{r=1}^{n} u_r = 2+(n-1)2^{n+1}$.

1999

1. **(a)** Find $\int \sqrt{e^x}\ dx$. **(b)** Find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(1+\tan^2 x\right) \tan x\ dx$.
(c) Use the substitution $u = \frac{1}{x}$ to show that $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln x}{1+x^2}\ dx = 0$.
(d) Find $\int \frac{x+7}{x^2+16}\ dx$ **(e)** Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x}$.

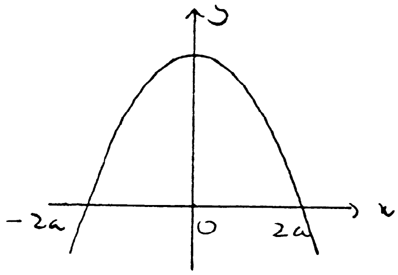
2. **(a)** Given $z = \sqrt{6} - \sqrt{2}i$, find: **(i)** $\Re(z^2)$; **(ii)** $|z|$; **(iii)** $\arg z$; **(iv)** z^4 in the form $x + iy$; **(v)** $\frac{1}{z^3}$ in modulus-argument form.
(b) **(i)** Find that Cartesian equation of the locus represented by $2|z| = 3(z + \overline{z})$.
(ii) Sketch the locus on an Argand diagram.
(c)



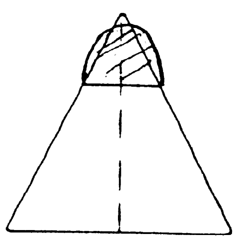
The diagram above shows the fixed points A, B and C in the Argand plane, where $AB = BC$, $\angle ABC = \frac{\pi}{2}$, and A, B and C are in anticlockwise order. The point A represents the complex number $z_1 = 2$ and the point B represents the complex number $z_2 = 3 + \sqrt{5}i$.
(i) Find the complex number z_3 represented by the point C.
(ii) D is the point on the Argand plane such that $ABCD$ is a square. Find the complex number z_4 represented by D.

3. **(a)** Consider the function $f(x) = \frac{x-1}{x}$.
(i) Sketch the graph $y = f(x)$ showing clearly the coordinates of any points of intersection with the axes and the equations of any asymptotes.
(ii) Use the graph $y = f(x)$ to sketch on separate axes the graphs
($\boldsymbol{\alpha}$) $y = |f(x)|$ **($\boldsymbol{\beta}$)** $y = f(|x|)$ **($\boldsymbol{\gamma}$)** $y = \{f(x)\}^2$ **($\boldsymbol{\delta}$)** $y = f^{-1}(x)$ **($\boldsymbol{\varepsilon}$)** $y = \sin^{-1} f(x)$
(b) **(i)** Show $\alpha = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ is a root of the equation $z^5 + z - 1 = 0$.
(ii) Hence show $z^2 - z + 1$ is a factor of $z^5 + z - 1$. **(iii)** Simplify $(1-\alpha)^{20}$.

4. (a) (i) Simpson's rule gives the exact answer to the area under a papabola. Use Simpson's rule to find the area enclosed by the x-axis and the parabola shown in the diagram.



(ii) Write down the Cartesian equation of the parabola shown in **(i)**.
(iii) The latus rectum of a parabola is the focal chord perpendicular to the axis of the parabola. Explain why the equation of the latus rectum is $y = 0$ for this parabola.
(iv) The base of a solid is an isosceles triangle with sides 13 cm, 13 cm and 10 cm. The cross-section of the solid is a parabola with its latus rectum lying on this base and perpendicular to the axis of symmetry of the triangle. Find the volume of this solid using the slicing method.



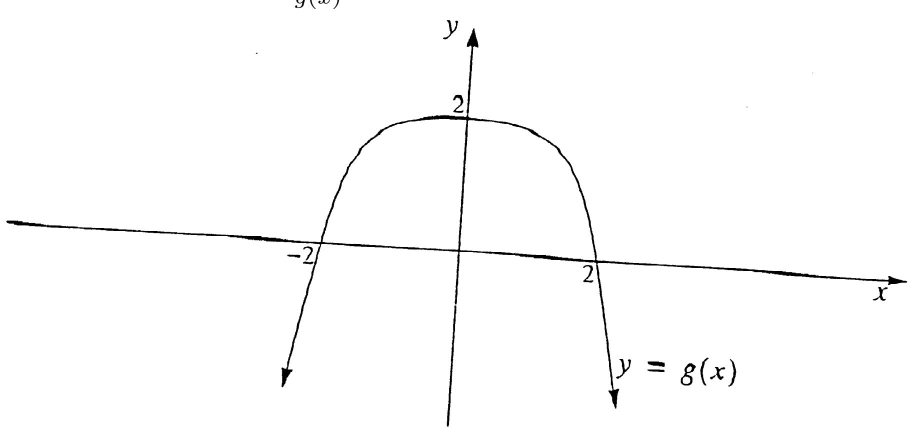
(b) (i) Prove that $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$.
(ii) Hence show that $\int_0^{\frac{\pi}{2}} (a\cos^2 x + b\sin^2 x)\ dx = \int_0^{\frac{\pi}{2}} (a\sin^2 x + b\cos^2 x)\ dx$.
(iii) Deduce that $\int_0^{\frac{\pi}{2}} (a\cos^2 x + b\sin^2 x)\ dx = \frac{\pi(a+b)}{4}$.

5. (a) (i) Show that the tangent to the rectangular hyperbola $xy = 4$ at the point $T\left(2t, \frac{2}{t}\right)$ has equation $x + t^2 y = 4t$.
(ii) This tangent cuts the x-axis at point Q. Show that the line through Q which is perpendicular to the tangent at T has equation $t^2 x - y = 4t^3$.
(iii) This line through Q cuts the rectangular hyperbola at the points R and S. Show that the midpoint M of RS has coordinates $M(2t, -2t^3)$.
(iv) Find the equation of the locus of M as T moves on the rectangular hyperbola.
(b) The hyperbola $xy = c^2$ touches the circle $(x-1)^2 + y^2 = 1$ at the point Q.
(i) Show this information on a diagram.
(ii) Explain why $x^2(x-1)^2 + c^4 = x^2$ has a repeated real root and two complex roots.

(iii) Prove that if k is a repeated real root of the polynomial equation $P(x) = 0$ then k is also a root of $P'(x) = 0$.
(iv) If k is the repeated real root of $x^2(x-1)^2 + c^4 = x^2$, find the value of k and c^2.

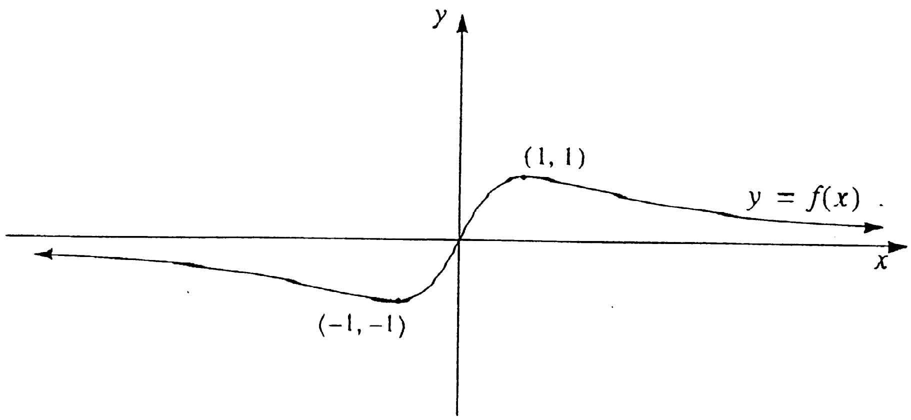
6. (a) (i) Show that $\int x \tan^{-1} x \, dx = \frac{1}{2}(x^2+1)\tan^{-1} x - \frac{1}{2}x + c$, c constant.
(ii) $I_n = \int_0^1 x^n \tan^{-1} x \, dx$, $n = 0, 1, 2, \ldots$. Show that $I_0 = \frac{\pi}{4} - \frac{1}{2}\ln 2$, $I_1 = \frac{\pi}{4} - \frac{1}{2}$ and $I_n = \frac{1}{n+1} \cdot \frac{\pi}{2} - \frac{1}{n(n+1)} - \frac{n-1}{n+1} \cdot I_{n-2}$, $n = 2, 3, 4, \ldots$.
(b) Two tangents are drawn from the external point $T(x_0, y_0)$ to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, meeting it at P and Q.
(i) Write down the equation of the chord PQ.
(ii) If the chord PQ touches the circle $x^2 + y^2 = 9$, then by considering the distance of the chord from the origin, or otherwise, show that the point $T(x_0, y_0)$ statisfies $\frac{9x_0^2}{256} + \frac{y_0^2}{9} = 1$.
(iii) Give a geometrical description of the locus of T.

7. (a) The equation $x^3 - x^2 - 3x + 2 = 0$ has roots α, β and χ.
(i) Use the value of $\alpha + \beta + \chi$ to find the monic polynomial equation with roots $2\alpha + \beta + \chi, \alpha + 2\beta + \chi$ and $\alpha + \beta + 2\chi$.
(ii) Find the monic polynomial equation with roots α^2, β^2 and χ^2.
(b) (i) The diagram shows the graph of $y = g(x)$ where $g(x) = 2 - \frac{x^4}{8}$. Use it to sketch the graph of $y = \frac{1}{g(x)}$.



(ii) Find the area enclosed by $y = \frac{1}{g(x)}$, the x-axis and the lines $x = 1$ and $x = \sqrt{2}$.
(c) (i) Consider the following statements.
(α) If $P(x)$ is an odd function and $Q(x)$ is an even function then $P(Q(x))$ is odd.
(β) If $P(x)$ is an odd function and $Q(x)$ is an even function then $Q(P(x))$ is even.
Indicate whether each of these statements is true or false. Give a reason for your answer.

(ii) The diagram shows the graph of $y = f(x)$ where $f(x) = \frac{2x}{x^2+1}$.



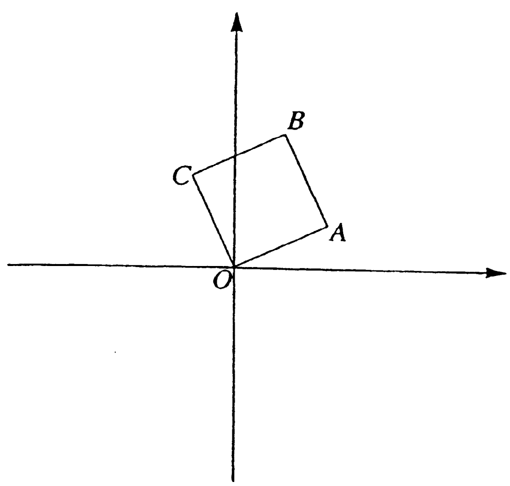
Sketch the graph of $y = g(f(x))$ where $g(x) = 2 - \frac{x^4}{8}$.

8. **(a)** **(i)** Show that for all values of A and B $\sin(A+B) - \sin(A-B) = 2\cos A \sin B$.
(ii) Use the method of mathematical induction to show that for all positive integers n, $\cos x + \cos 2x + \cos 3x + \cdots + \cos nx = \frac{\sin(n+\frac{1}{2})x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$.
(iii) Hence show that $\cos 2x + \cos 4x + \cos 6x + \cdots + \cos 16x = 8\cos 9x \cos 4x \cos 2x \cos x$.
(b) **(i)** If $x > 0, y > 0$ show that $x + y \geq 2\sqrt{xy}$.
(ii) Hence show that if $x > 0, y > 0, z > 0$ then $(x+y)(y+z)(z+x) \geq 8xyz$.
(iii) If a, b, c are the sides of a triangle with semi-perimeter $S = \frac{1}{2}(a+b+c)$ then Heron's formula states that the area Δ of the triangle is given by $\Delta = \sqrt{S(S-a)(S-b)(S-c)}$. By choosing suitable values for x, y, z show that $\Delta^2 \leq \frac{(a+b+c)abc}{16}$.

2000

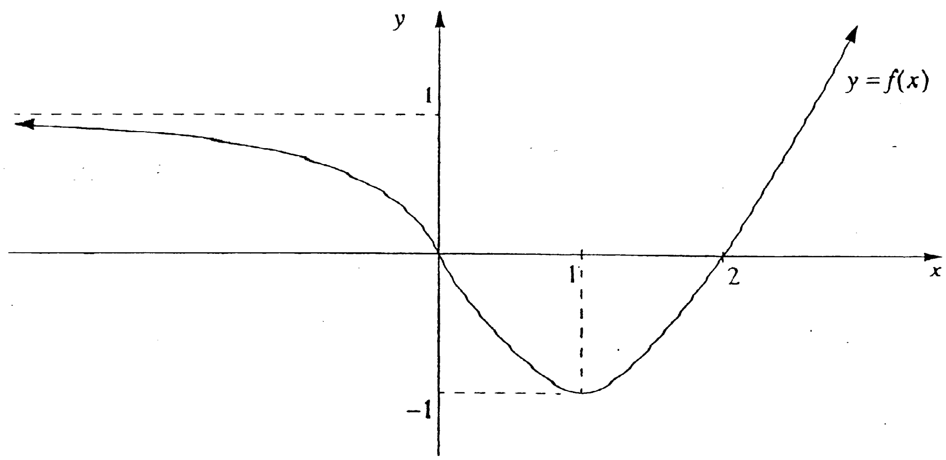
1. **(a)** **(i)** Show that $\frac{1}{1+e^x} = \frac{e^{-x}}{1+e^{-x}}$. **(ii)** hence find $\int \frac{dx}{1+e^x}$.
(b) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{5+13\sin x}\, dx$ using the substitution $t = \tan\frac{x}{2}$.
(c) **(i)** If $I_n = \int_0^1 (x^2-1)^n\, dx$, $n = 0, 1, 2, \ldots$, show that $I_n = \frac{-2n}{2n+1} I_{n-1}$, $n = 1, 2, 3, \ldots$. **(ii)** Evaluate I_1. **(iii)** Hence use the method of mathematical induction to show that $I_n = \frac{(-1)^n 2^{2n} (n!)^2}{(2n+1)!}$ for all positive integers n.

2. **(a)** Let $z = -1 + i\sqrt{3}$.
(i) Write z in mdulus-argumen form.
(ii) Express in the form $a + ib$, where a and b are real
(α) z^5 **(β)** $z\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.
(b)



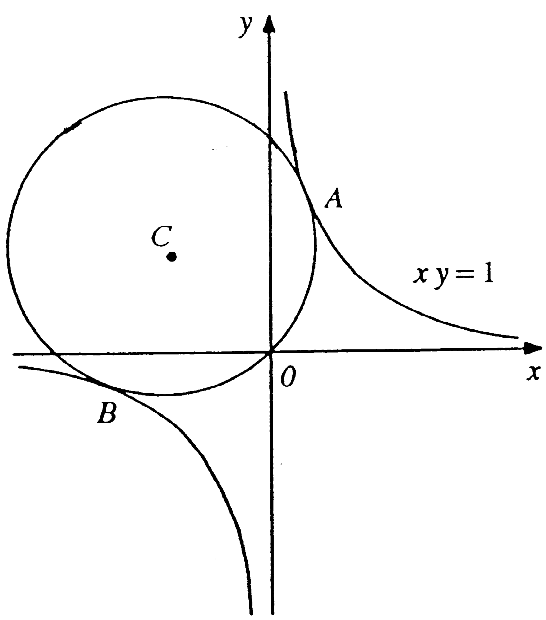
The point A in the Argand diagram sketched above represents the complex number $z = a + ib$, in the first quadrant. The point B represents the complex number $4 + 7i$.
(i) If $OABC$ is a square, find in terms of a and b the complex number represented by the point C. **(ii)** Hence or otherwise evaluate a and b.
(c) **(i)** Sketch and describe the locus of z if $|z - i| = \Im(z)$.
(ii) For what values of m is the line $y = mx$ a tangent to this locus?
(iii) What is the least value of $\arg(z)$ for this locus?
(d) **(i)** Solve $z^3 - 1 = 0$, leaving your answers in modulus-argument form.
(ii) Let ω be one of the non-real roots of $z^3 - 1 = 0$.
(α) Show that $1 + \omega + \omega^2 = 0$. **($\beta$)** Hence simplify $(1+\omega)^8$.

3. (a)



Given the function $y = f(x)$ in the diagram above, sketch on separate diagrams, showing all intercepts, turning points and asymptotes:
(i) $y = f(-x)$, **(ii)** $y = f(|x|)$, **(iii)** $y = f(2x)$, **(iv)** $y = e^{f(x)}$, **(v)** $y = \tan^{-1} f(x)$
(b) **(i)** Graph the function $y = x^3(x-2)$. You need not use calculus, but you must show the behaviour near any x-intercepts.
(ii) Differentiate $y^2 = x^3(x-2)$ implicitly, and hence show that for $y > 0$, $\frac{dy}{dx} = (2x-3)\sqrt{\frac{x}{x-2}}$.
(iii) Sketch $y^2 = x^3(x-2)$, paying particular attention to the behaviour of the curve near its x-intercepts. (You do not need to find the coordinates of any inflections.)

4. (a)



The circle with centre $C(= c, c)$, where $c > 0$, passes through the origin O and touches the curve $xy = 1$ at the points A and B.

(i) Show that the x coordinates $x = \alpha$ and $x = \beta$ of the points A and B satisfy the equation $x^4 + 2cx^3 - 2cx + 1 = 0$.

(ii) Explain why the equation $x^4 + 2cx^3 - 2cx + 1 = 0$ has real roots $\alpha, \alpha, \beta, \beta$.

(iii) Use the relationships between the roots and the coefficients of this equation to find the exact values of c, α and β.

(b) **(i)** On the same axes sketch the graphs of $y = \sqrt{1-x^2}$ and $y = \frac{1}{\sqrt{1-x^2}}$.

(ii) The region bounded by the curve $y = \frac{1}{\sqrt{1-x^2}}$, the coordinate axes and the line $x = \frac{1}{2}$ is rotated through one complete revolution about the ine $x = 6$. Use the method of cylindrical shells to show that the volume V unit^3 of the solid of revolution is given by $V = 2\pi \int_0^{\frac{1}{2}} \frac{6-x}{\sqrt{1-x}}\, dx$.

(iii) Hence find the value of V in simplest exact form.

5. **(a)** The point $P(x_0, y_0)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$.

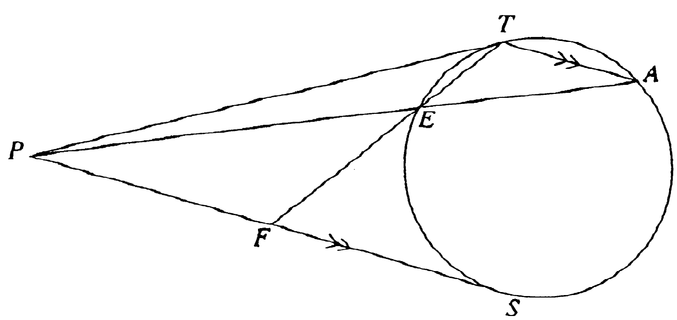
(i) Write down the equations of the two asymptotes of the hyperbola.

(ii) Show that the acute angle α between the two asymptotes satisfies $\tan\alpha = \frac{2ab}{a^2-b^2}$.

(iii) If M and N are the feet of the perpendiculars drawn from P to the asymptotes, show that $MP.NP = \frac{a^2b^2}{a^2+b^2}$.

(iv) Hence show that the area of $\triangle PMN$ is $\frac{a^3b^3}{(a^2+b^2)^2}$ square units.

(b)



The diagram shows two tangents PT and PS drawn to a circle from a point P outside the circle. Through T, a chord TA is drawn parallel to the other tangent PS. The secant PA meets the circle at E, and TE produced meets PS at F.

(i) Prove that $\triangle EFP ||| \triangle PFT$. **(ii)** Hence show that $PF^2 = TF \times EF$.

(iii) Hence or otherwise prove that F is the midpoint of PS.

6. **(a)** Let α, β and χ be the roots of $x^3 - x^2 + 2x - 1 = 0$. Write down an equation with roots

(i) $\alpha + \beta, \beta + \chi$ and $\alpha + \chi$

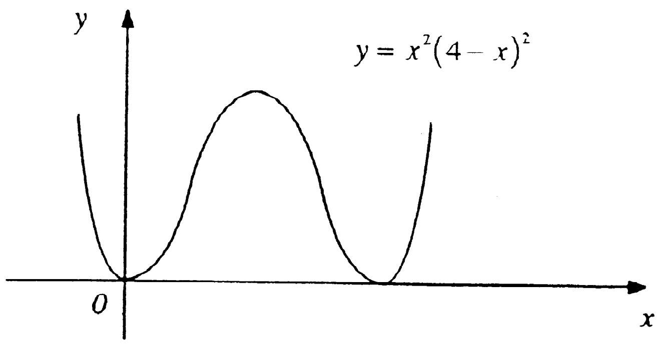
(ii) $\frac{\alpha}{\beta\chi}, \frac{\beta}{\alpha\chi}$ and $\frac{\chi}{\alpha\beta}$.

(b) Conside the hyperbola $xy = c^2$ and the distinct points $P\left(c_1, \frac{c}{t_1}\right)$ and $Q\left(ct_2, \frac{c}{t_2}\right)$ on it.

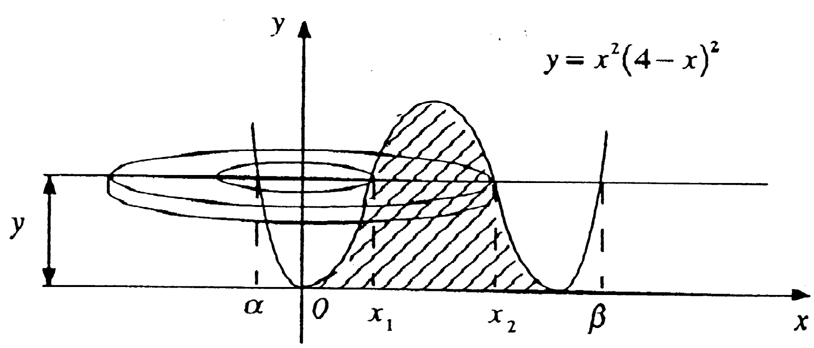
(i) Show that the equation of the tangent at $\left(ct, \frac{c}{t}\right)$, where $t \neq 0$, is $x + t^2y = 2ct$.

(ii) Show that the tangents at P and Q intersect at $M\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$.
(iii) Show that if $t_1t_2 = k$, where k is a non-zero constant, then the locus of M is a line passing through the origin.

7. **(a)** Show that the stationary points of $y = \{f(x)\}^2$ are exactly those points on the curve that have x coordinates which are zeros of either $f(x)$ or $f'(x)$.
(b)



Use the graph of $y = x(4-x)$ to justify the features shown on the graph above. Copy the graph of $y = x^2(4-x)^2$ and mark on the coordinate axes the values of x and y at the stationary points.
(c)



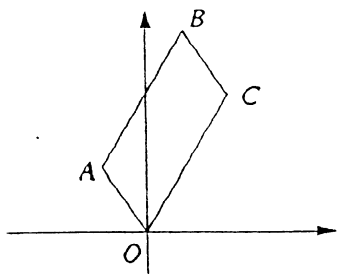
The shaded region is rotated through one revolution about the y axis. The volume of the solid formed is found by taking slices perpendicular to the y axis. The typical slice shown in the diagram is at a height y above the x axis.
(i) Deduce that α, x_1, x_2, β, as shown in the diagram, are roots of $x^4-8x^3+a6x^2-y = 0$.
(ii) Use the symmetry in the graph to explain why $\frac{x_1+x_2}{2} = 2$ and $\frac{\alpha+\beta}{2} = 2$. Hence, by considering the coefficients of the equation in **(i)**, show that $\alpha\beta = -x_1x_2$ and deduce that $x_1x_2 = \sqrt{y}$ and $x_2 - x_1 = 2\sqrt{4-\sqrt{y}}$.
(iii) Show that the volume of the solid of revolution is given by $V = 8\pi\int_0^{16}\sqrt{4-\sqrt{y}}\,dy$. Use the substitution $y = (4-u)^2$ to evaluate this integral and find the exact volume.

8. (a) (i) If $p > 0$ and $q > 0$ are positive real numbers, show that $p + q \geq 2\sqrt{pq}$.
(ii) Hence show that
($\boldsymbol{\alpha}$) $\left(\sqrt[3]{p} + \sqrt[3]{p^2}\right)\left(\sqrt[3]{q^4} + \sqrt[3]{p^5}\right) \geq 4pq$
($\boldsymbol{\beta}$) $\sqrt[3]{\frac{p}{q}} + \sqrt[3]{\frac{q^2}{p^2}} + \sqrt[3]{\frac{q^4}{p^4}} + \sqrt[3]{\frac{p^5}{q^5}} \geq 4$
(b) Amy and Zoe both applied for Olympic Games tickets to the finals in Athletics, Basketball, Hockey, Swimming and Tennis. When Amy applied, the probability of getting a ticket to any one of these finals was one in five. When Zoe applied, the probability of getting a ticket to any one of the Athletics, Hockey or Tennis finals was still one in five, but the probability of getting a ticket to either one of the Basketball or Swimming finals was one in ten. Find the probability, correct to 4 decimal places, that
(i) Amy gets tickets to exactly four finals.
(ii) Zoe gets a ticket to exactly one final.
(iii) Amy gets tickets to four more finals than Zoe.
(iv) Only one of Amy and Zoe gets tickets to both the Basketball and Swimming finals.

2001

1. **(a)** Find $\int \frac{e^x}{e^x+1}\,dx$ **(b)** Find $\int \frac{e^x}{(e^x+1)^2}\,dx$ **(c)** Find $\int \frac{4x\ dx}{\sqrt{x^4+4}}$ **(d)** Find $\int \frac{dx}{x^2+6x+13}$
(e) Evaluate $\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x}\,dx$ using the substitution $t = \tan\frac{x}{2}$.
(f) **(i)** Use the substitution $x = u^2, u > 0$, to show that $\int_4^{16} \frac{\sqrt{x}}{x-1}\,dx = 4+2\ln 3-\ln 5$.
(ii) Hence use integration by parts to evaluate $\int_4^{16} \frac{\ln(x-1)}{\sqrt{x}}\,dx$ in simplest exact form.

2. **(a)** Find x and y if $x + iy = \frac{1}{(-1-i\sqrt{3})^{10}}$.
(b)

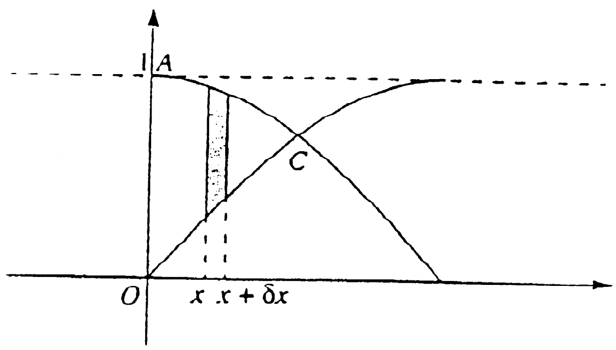


In the diagram above, $OABC$ is a parallelogram with $OA = \frac{1}{2}OC$. The point A represents the complex number $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$. If $\angle AOC = 60°$, what complex number does C represent?
(c) In an Argand diagram, the complex number α and $i\alpha$ are represented by the points A and B. z is a variable complex number represented by the point P. $0 < \arg\alpha < \frac{\pi}{2}$.
(i) Draw a diagram showing A, B and the locus of P if $|z-\alpha| = |z-i\alpha|$.
(ii) Draw a diagram showing A, B and the locus of P if $\arg(z-\alpha) = \arg(i\alpha)$.
(iii) Find, in terms of α, the complex number represented by the point of intersection of the two loci in **(i)** and **(ii)**.
(d) It is given that $z = \cos\theta + i\sin\theta$ where $0 < \arg z < \frac{\pi}{2}$.
(i) Show that $z+1 = 2\cos\frac{\theta}{2}\big(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\big)$ and express $z-1$ in modulus/argument form.
(ii) Hence show that $\Re\big(\frac{z-1}{z+1}\big) = 0$.

3. **(a)** Consider the functions $f(x) = |x|+1$ and $g(x) = \frac{6}{|x|}$.
(i) Solve the equation $f(x) = g(x)$.
(ii) Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same set of axes.
(iii) Solve the inequality $g(x) > f(x)$.
(b) $f(x) = 1 - 2\cos x$ for $-\pi \le x \le \pi$. Sketch the graphs of
(i) $y = f(x)$ **(ii)** $y = f(x)^2$ **(iii)** $yO^2 = f(x)$
(c) Consider the function $y = \sin^{-1}(e^x)$
(i) Find the domain and range of the function.

(ii) Sketch the graph of the function showing clearly the coordinates of any endpoints and the equations of any asymptotes.

4. **(a)** **(i)** Write down the cartesian equation of the curve with the parametric equations $x = 3\cos\theta$ and $y = 4\sin\theta$.
(ii) Sketch the graph of the curve.
(iii) Find the coordinates of the foci, S_1 and S_2.
(iv) Find the equations of the directrices.
(v) Prove $PS_1 + PS_2 = 8$ where P is any point on the curve.
(b) The diagram below shows part of the graphs of $y = \cos x$ and $y = \sin x$. The graph of $y = \cos x$ meets the y-axis at A, and the C is the first point of intersection of the two graphs to the right of the y-axis.

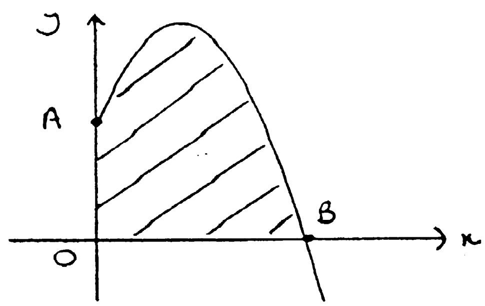


The region OAC is to be rotated about the line $y = 1$.
(i) Write down the coordinates of the point C.
(ii) The shaded stip of width δx shown in the diagram is rotated about the line $y = 1$. Show that the volume δV of the resulting slice is given by $\delta V = \pi(2\cos x - 2\sin x + \sin^2 x - \cos^2 x)\delta x$.
(iii) Hence evaluate the total volume when the region OAC is rotated about the line $y = 1$.

5. **(a)** **(i)** Let $P(x)$ be a degree 4 polynomial with a zero of multiplicity 3. Show that $P'(x)$ has a zero of multiplicity 2.
(ii) Hence or otherwise find all zeros of $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, given that it has a zero of multiplicity 3.
(iii) Sketch $y = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, clearly showing the intercepts on the coordinate axes. You do not need to give the coordinates of turning points or inflections.
(b) **(i)** Solve the $\cos 5\theta = -1$ for $0 \leq \theta \leq 2\pi$.
(ii) Use De Moivre's Theorem to show that $\cos 5\theta = 16\cos^2\theta - 20\cos^3\theta + 5\cos\theta$.
(iii) Find the exact trigonometric roots of the equation $16x^5 - 20x^3 + 5x + 1 = 0$.
(iv) Hence find the exact values of $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ and $\cos\frac{\pi}{5}.\cos\frac{3\pi}{5}$ and factorise $16x^5 - 20x^3 + 5x + 1$ into irreducible factors over the rational numbers.

6. **(a)** **(i)** Show that $\frac{t^n}{1+t^2} = t^{n-2} - \frac{t^{n-2}}{1+t^2}$.

(ii) Let $I_n = \int \frac{t^n}{1+t^n}\,dt$. Show that $I_n = \frac{t^{n-1}}{n-1} - I_{n-2},\ n > 2$.
(iii) Show that $\int_0^1 \frac{t^6}{1+t^2}\,dt = \frac{13}{15} - \frac{\pi}{4}$.
(b)



The graph shows part of the curve whose parametric equations are $x = 2t + 1$, $y = (5 - 2t)(3 + 2t)$.
(i) Find the values of t corresponding to the points A and B on the curve.
(ii) The volume V of the solid formed by rotating the shaded area about the y axis is to be calculated using cylindrical shells. Express V in the form $V = 2\pi \int_a^b f(t)\,dt$. Specify the limits of integration a and b and the function $f(t)$. You may leave $f(t)$ in unexpanded form. Do NOT evaluate this integral.
(c) $P\big(cp, \frac{c}{p}\big)$ and $Q\big(cq, \frac{c}{q}\big)$ are two points on the hyperbola $xy = c^2$ where $p, q > 0$.
(i) Show the point of intersection, T of the tangents at P and Q is $\big(\frac{2cpq}{p+q}, \frac{2c}{p+q}\big)$. [You may assume the equation of the tangent at P is $x + p^2y = 2cp$.]
(iii) The chord PQ produced, passes through $(0, c)$. Find the equation of the locus of T **precisely**. [You may assume the equation of the chord PQ is $x+pqy = c(p+q)$.]

7. **(a)** A, B and C are three distinct points on a horizontal straight line that is on the same level as the foot P of a vertical tower PQ of height h. The distances AB and BC are both equal to d and the angles of elevation of the top Q of the tower from the points A, B, C are equal to α, β, γ respectively.
(i) If the line ABC passes through the foot P of the tower so that A, B, C are all on the same side of P, show that $2\cot\beta = \cot\alpha + \cot\gamma$.
(ii) If the line ABC does not pass through the foot P of the tower, use the cosine rule in each of $\triangle ABP, \triangle CBP$ to show that $h^2(\cot^{\alpha} - 2\cot^2\beta + \cot^2\gamma) = 2d^2$.
(b) After t minutes the number N of bacteria in a culture is given by $N = \frac{a}{1+be^{-ct}}$ for some constants $a > 0, b > 0$ and $c > 0$. Initially there are 300 bacteria in the culture and the number of bacteria is initially increasing at a rate of 20 per minute. As t increases indefinitely the number of bacteria in the culture approaches a limiting value of 900.
(i) Show that $\frac{dN}{dt} = \frac{c}{a}(a - N)N$.
(ii) Find the values of a, b and c.
(iii) Show that the maximum rate of increase in the number of bacteria occurs when $N = 450$. Sketch the graph of N against t.

8. (a) Two students were asked to find $\frac{dy}{dx}$ for the curve $\frac{x^2}{y} + y = 3$. Student A used the quotient rule and found $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$. Student B first multiplied the equation by y and then differentiated to get a different expression for $\frac{dy}{dx}$. Has one of the students made a mistake or can the two students be reconciled? Justify your answer.

(b) (i) ($\boldsymbol{\alpha}$) Differentiate $y = \log_e(1+x)$, and hence draw $y = x$ and $y = \log_e(1+x)$ on one set of axes.

($\boldsymbol{\beta}$) Using this graph, explain why $\log - e(1+x) < x$, for all $x > 0$.

(ii) ($\boldsymbol{\alpha}$) Differentiate $y = \frac{x}{1+x}$, and hence draw $y = \frac{x}{1+x}$ and $y = \log_e(1+x)$ on one set of axes.

($\boldsymbol{\beta}$) Using this graph, explain why $\frac{x}{1+x} < \log_e(1+x)$, for all $x > 0$.

(iii) Use the inequalities of parts **(i)** and **(ii)** to show that

$$\frac{\pi}{8} - \frac{1}{4}\log_e 2 < \int_0^1 \frac{\log_e(1+x)}{1+x^2}\, dx < \frac{1}{2}\log_e 2.$$

(c) You may assume that, for all positive real numbers a and b, $\sqrt{ab} \leq \frac{a+b}{2}$.

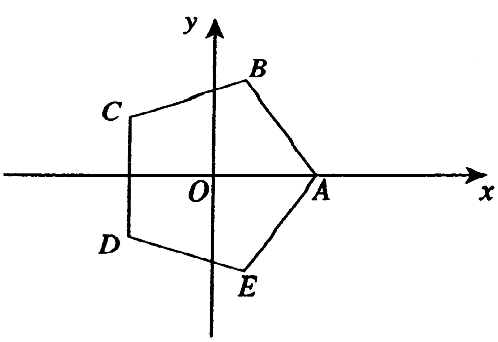
(i) Show that for all positive integers n, ${}^nC_0 + {}^nC_1 + \cdots + {}^nC_n = 2^n$.

(ii) Prove that for all positive integers n, $\left(\sqrt{{}^nC_1} + \sqrt{{}^nC_2} + \cdots + \sqrt{{}^nC_n}\right)^2 \leq n(2^n - 1)$. You may use the identity $(x_1 + x_2 + \cdots + x_n)^2 = (x_1^2 + x_2^2 + \cdots + x_n^2) + \sum_{i<j} 2x_ix_j$.

2002

1. **(a)** Find $\int \frac{dx}{\sqrt{9+4x^2}}$.
(b) **(i)** Find real constants A, B and C such that $\frac{x^2+5x+2}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$.
(ii) Hence find $\int \frac{x^2+5x+2}{(x^2+1)(x+1)}\ dx$.
(c) Evaluate $\int_1^5 x\sqrt{2x-1}\ dx$.
(d) Evaluate $\int_0^1 x^5 e^{x^3}\ dx$.
(e) **(i)** Simplify $\sin(A-B)+\sin(A+B)$.
(ii) Hence find $\int \sin 5x \cos 3x\ dx$.

2. **(a)** Let $z = \frac{2-4i}{1+i}$.
(i) Find $\overline{z}$, giving your answer in the form $a+bi$, where a and b are real.
(ii) Find iz.
(b) Find a and b if $(a+ib)^2 = 3-4i$, where a and b are real and $a > 0$.
(c) Consider the region defined by $|z-4i| \leq 3$.
(i) Sketch the region.
(ii) Determine the maximum value of $|z|$.
(iii) Determine the maximum value of $\arg z$, where $-\pi < \arg z \leq \pi$.
(d)

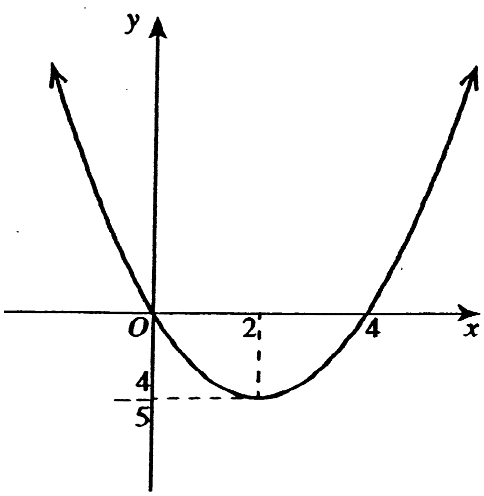


In the diagram above, the complex numbers z_0, z_1, z_2, z_3, z_4 are represented by the vertices of a regular polygon with centre O and vertices A, B, C, D, E respectively. Given that $z_0 = 2$:
(i) Express z_2 in modulus-argument form.
(ii) Find the value of z_2^5.
(iii) Show that the perimeter of the pentagon is $20 \sin \frac{\pi}{5}$.

3. **(a)** Let α, β and γ be the roots of $x^3 - 7x^2 = 18x - 7 = 0$.
(i) Find a cubic equation that has roots $1+\alpha^2, 1+\beta^2$ and $1+\gamma^2$.
(ii) Hence, or otherwise, find the value of $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)$.
(b) a lifebelt mould is made by rotating the circle $x^2+y^2 = 64$ through one complete revolution about the line $x = 28$.
(i) Use the method of slicing to show that the volume, V, of the lifebelt is given by $V = 112\pi \int_{-8}^{8} \sqrt{64-y^2}\ dy$.

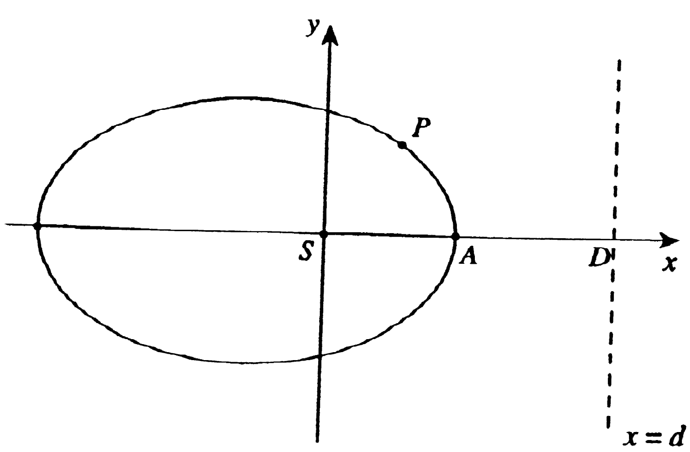
(ii) Find the exact volume of the lifebelt.
(c) Let $P(x) = x^4 + ax^3 + 36x^2 - 35x + b$, where a and b are real numbers. It is known that $x = 5$ and $x = \frac{1-i\sqrt{5}}{2}$ are zeroes of $P(x)$.
(i) Explain why $x^2 - x + 1$ must be a factor of $P(x)$.
(ii) Find a and b.

4. (a)



The sketch above shows the parabolic curve $y = f(x)$ where $f(x) = \frac{x^2-4x}{5}$. Without the use of calculus, draw sketches of the following, showing interceps, asymptotes and turning points:
(i) $y = |f(x)|$, **(ii)** $y = \frac{1}{f(x)}$, **(iii)** $y = \frac{x}{5}|x-4|$, **(iv)** $y = \tan^{-1}(f(x))$.
(b) Find the set of values of x for which the limiting sum exists for the series $1 + (\frac{2x-3}{x+1}) + (\frac{2x-3}{x+1})^2 + (\frac{2x-3}{x+1})^3 + \cdots$

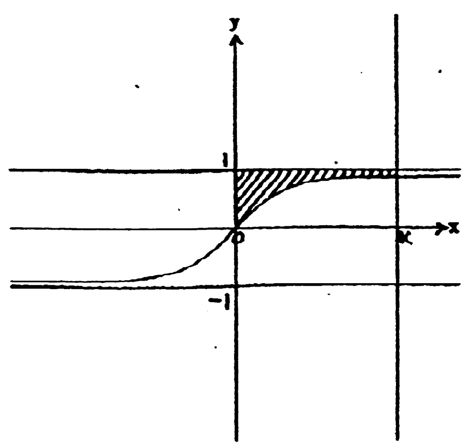
5. (a)



Consider the ellipse sketch obove of eccentricity e with one focus S at the origin and its corresponding directrix at $x = d$.
(i) If P corresponds to the complex number z, where $z = r(\cos\theta + i\sin\theta)$, use the

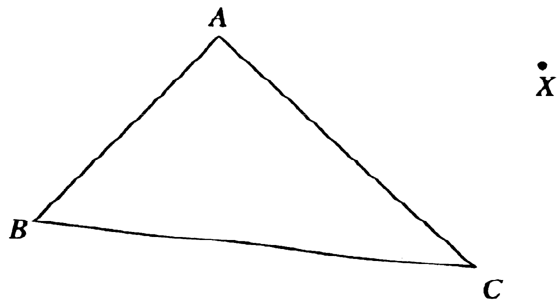
focus-directrix definition of an ellipse to show that $r = \frac{ed}{1+e\cos\theta}$.
(ii) Hence draw the ellipse represented by $r = \frac{33}{5+3\cos\theta}$ showing the coordinates of the points A and D. [There is no need to find the coordinates of any other point, or to write the Cartesian equation of the ellipse.]
(b) Consider the curve defined by the equation $3x^2 + y^2 - 2xy - 8x + 2 = 0$.
(i) Show that $\frac{dy}{dx} = \frac{3x-y-4}{x-y}$.
(ii) Find the coordinates of the points on the curve where the tangent to the curve is parallel to the line $y = 2x$.
(c) Consider the complex number $z = \cos\theta + i\sin\theta$.
(i) Using de Moivre's theorem, show that $z^n + \frac{1}{z^2} = 2\cos n\theta$, for any integer n.
(ii) Hence or otherwise express $(z+\frac{1}{z})^6$ in the form $A\cos 6\theta + B\cos 4\theta + C\cos 2\theta + D$, where A, B, C and D are real constants.
(iii) Hence evaluate $\int_0^{\frac{\pi}{4}} \cos^6\theta \, d\theta$.

6. (a)



The sketch above shows the curve with equation $y = \frac{e^{2x}-1}{e^{2x}+1}$ which has asymptotes at $y = \pm 1$. The line $x = K$ (where $K > 0$) is also shown.
(i) Using the substitution $u = e^{2x}$ or otherwise, show that the shaded area is given by $A = \ln 2 + 2K - \ln(e^{2K} + 1)$
(ii) Explain why this area is always less than $\ln 2$ no matter how large K is.
(b) (i) Use the substitution $u = 1 + x$ to evaluate $\int_0^1 x(1+x)^n \, dx$
(ii) Use the binomial theorem to write an expansion of $x(1+x)^n$
(iii) Prove that $\sum_{r=0}^{n} \frac{1}{r+2} {}^nC_r = \frac{n.2^{n+1}+1}{(n+1)(n+2)}$.
(iv) Find the largest integer value of n such that $\sum_{r=0}^{n} \frac{1}{r+2} {}^nC_r < 50$

7. (a)



The diagram above shows a point X outside a triangle ABC. Show that $AX + BX + CX > \frac{AB+BC+CA}{2}$.

(b) (i) Show that the normal at the point $P(cp, \frac{c}{p})$ to the rectangular hyperbola $xy = c^2$ is given by $p^3x - py = c(p^4 - 1)$.

(ii) If this normal meets the hyperbola again at $Q(cq, \frac{c}{q})$, show that $p^3q = -1$.

(iii) Hence find the area of the triangle PQR, where R is the point of intersection of the tangent at P with the y-axis. You may assume that the equation of the tangent is given by $x + p^2y = 2cp$.

(iv) What is the value of p that produces a triangle of minimum area?

(c) (i) Using t results, or otherwise, find $\int \frac{1}{1+\sin x}\ dx$.

(ii) Hence find $\int_0^{\frac{\pi}{3}} \frac{2}{2+\sin x+\sqrt{3}\cos x}\ dx$.

8. (a) P, Q represent complex numbers α, β respectively in an Argand diagram, where O is the origin and O, P, Q are not collinear. In $\triangle OPQ$, the line from O to the midpoint M of PQ meets the line from Q to the midpoint N of OP in the point R, where R represents the complex number z.

(i) Show this information on a sketch.

(ii) Explain why there are positive real numbers k, l such that $kz = \frac{1}{2}(\alpha + \beta)$ and $l(z - \beta) = \frac{1}{2}\alpha - \beta$

(iii) Show that $z = \frac{1}{3}(\alpha + \beta)$

(iv) If S is the midpoint of OQ show that R lies on PS.

(b) Let $J_n = \int_0^1 x^n e^{-x}\ dx$, where $n \geq 0$.

(i) Show that $J_0 = 1 - \frac{1}{e}$.

(ii) Show that $J_n = nJ_{n-1} - \frac{1}{e}$, for $n \geq 1$.

(iii) Show that $J_n \to 0$ as $n \to \infty$.

(iv) Deduce by the principle of mathematical induction that for all $n \geq 0$, $J_n = n! - \frac{n!}{e}\sum_{r=0}^{n} \frac{1}{r}$.

(v) Conclude the $e = \lim\limits_{n\to\infty} (\sum_{r=0}^{n} \frac{1}{r!})$.
