NSW Independent Trial HSC 2004

Mathematics Extension 1

Marking Guidelines

1a. Outcomes assessed: PE3, H3

Marking guidelines

Criteria	Marks
• states $\dfrac{5-x}{3-x} > 0$ as basic condition	1
• finds critical points at $x = 3$ and $x = 5$	1
• finds correct domain	1

Answer

$\dfrac{5-x}{3-x} > 0$ whence $x < 3$ and $x > 5$

1b. Outcomes assessed: i. P4 ii. Ask the Board of Studies

Marking guidelines

Criteria	Marks
• shows $x = 0$ gives same y value on each curve	1
• uses a valid method to find the angle	1
• finds the angle	1

Answer

i. (0, 0) lies on both curves

ii. Gradient for $y = x^2 - x$ at $x = 0$ is -1. Therefore angle is 45°

1c. Outcomes assessed: PE3

Marking guidelines

Criteria	Marks
• uses Factor theorem to set up equation	1
• solves to find answer	1

Answer

$P(3)=3^4-3\times3^3+a\times3^2-a\times3-12=0$

$$6a-12=0 \text{ so } a=2$$

1d. Outcomes assessed: PE3

Marking guidelines

Criteria	Marks
• uses Tangent/Secant theorem to set up equation	1
• solves to correct solution	1

Answer

$10^2 = x(x+15)$

$x^2+15x-100=0$

$(x+20)(x-5)=0$

$x=5$

1e. **Outcomes assessed: PE3**

Marking guidelines

Criteria	Marks
• uses appropriate result for circular permutations	1
• allows for the couple and calculates the answer	1

Answer

$4!\times 2! = 48$

2a. **Outcomes assessed: HE6**

Marking guidelines

Criteria	Marks
• finds dx and adjusts limits	1
• correctly substitutes and simplifies to integral of $\cos^2\theta$	1
• finds correct integral and evaluates	1

Answer

$x = 2\sin\theta \Rightarrow dx = 2\cos\theta\, d\theta$

If $x = 1, \theta = \dfrac{\pi}{6}$; $x = -1, \theta = \dfrac{-\pi}{6}$

Therefore, $I = \int_{-\pi/6}^{\pi/6} \sqrt{4 - 4\sin^2\theta}$

$= 4\int_{-\pi/6}^{\pi/6} \cos^2\theta\, d\theta$

$= 8\left[\dfrac{1}{2}(\theta + \dfrac{1}{2}\sin 2\theta\right]_0^{\pi/6} = \dfrac{2\pi}{3} + \sqrt{3}$

2b. **Outcomes assessed: HE3**

Marking guidelines

Criteria	Marks
• correct expansion for binomial expression	1
• calculates correct value of r	1
• calculates answer	1

Answer

$$\left(x - \frac{3}{x}\right)^8 = \sum_{r=0}^{8}\binom{8}{r}x^r\left(\frac{-3}{x}\right)^{8-r} = \sum_{r=0}^{8}\binom{8}{r}(-3)^{8-r}x^{2r-8} \Rightarrow 2r - 8 = 0 \Rightarrow r = 4$$

Therefore, the term is $\binom{8}{4}(-3)^4 = 5670$

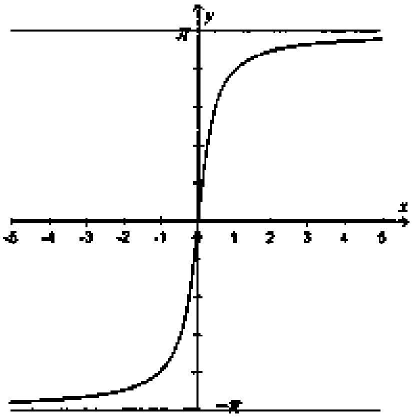
2c. **Outcomes assessed: i. HE4 ii. HE4**

Marking guidelines

Criteria	Marks
i. Correct graph shape with clearly marked axes	1, 1
ii. Correct domain and range	1

Answer

i.



ii. X : x is real
Y : $-\pi \le y \le \pi$

2d. **Outcomes assessed: ask the Board of Studies**

Marking guidelines

Criteria	Marks
• chooses and applies an appropriate method	1
• obtains A and B (method 1) or $t = -\frac{1}{2}$ (method 2)	1
• correct answer	1

Answer

$$3\cos\theta - 4\sin\theta = A\cos(\theta + B)$$
$$= A\cos\theta\cos B - A\sin\theta\sin B$$
$$A\cos B = 3$$
$$A\sin B = 4$$
$$\Rightarrow A = 5;\ B = \tan^{-1}\left(\tfrac{4}{3}\right) \Rightarrow B = .927\ rad$$
$$5\cos(\theta + B) = 5,\ -\pi \le \theta \le \pi$$
$$\theta + B = 0 \Rightarrow \theta = -0.93\ rad$$

OR: $$\cos\theta = \frac{1-t^2}{1+t^2};\ \sin\theta = \frac{2t}{1+t^2}$$
$$3\times\frac{1-t^2}{1+t^2} - 4\times\frac{2t}{1+t^2} = 5$$
$$8t^2 + 8t + 2 = 0$$
$$2(4t+1)(4t+1) = 0$$
$$\Rightarrow t = -\tfrac{1}{2} \Rightarrow \tan\tfrac{\theta}{2} = -\tfrac{1}{2}$$
$$\Rightarrow \tfrac{\theta}{2} = -.464\,rad \Rightarrow \theta = -0.93\,rad$$

3a. **Outcomes assessed: PE3**

Marking guidelines

Criteria	Marks
• correct answer	1

Answer

$^{20}C_8 = 125\ 970$

3b. Outcomes assessed: HE6

Marking guidelines

Criteria	Marks
• correctly replaces $\sin^2 2x$ with the appropriate result	1
• finds correct integral	1
• correct answer	1

Answer

$$\int_0^{\pi/12} \sin^2 2x \, dx = \left[\tfrac{1}{2}\left(x - \tfrac{1}{4}\sin 4x\right)\right]_0^{\pi/12}$$

$$= \tfrac{1}{2}\left(\tfrac{\pi}{12} - \tfrac{1}{4}\sin \tfrac{\pi}{3}\right)$$

$$= \tfrac{\pi}{24} - \tfrac{\sqrt{3}}{16}$$

3c. Outcomes assessed: HE2

Marking guidelines

Criteria	Marks
• Verifies the solution when $n = 1$	1
• Attempts to prove that if $S(n)$ is true, then $S(n+1)$ is true	1
• Correctly shows that if $S(n)$ is true, then $S(n+1)$ is true	1

Answer

$S(n): 1 + 5 + 9 + \ldots + 4n - 3 = 2n^2 - n$

$S(1): LHS = 1;\ RHS = 2 \times 1^2 - 1 = 1$

$S(k): 1 + 5 + \ldots 4k - 3 = 2k^2 - k$

$S(k+1): 1 + 5 + 9 + \ldots + 4k - 3 + 4(k+1) - 3$

$= 2k^2 - k + 4k + 1$

$= 2k^2 + 3k + 1$

$= 2(k^2 + 2k + 1) - (k+1)$

$= 2(k+1)^2 - (k+1)$

Therefore, if $S(n)$ is true, then $S(n+1)$ is true.
But $S(1)$ is true, so $S(2)$ is true. Hence $S(3)$ is true and so on for all positive integer values of n

3d. Outcomes assessed: i. HE4 ii. HE1, HE4, HE7

Marking guidelines

	Criteria	Marks
i.	• knows and correctly sets up Newton's Method	1
	• obtains correct answer	1
ii.	• establishes $x_2 = -2x_1$	1
	• concludes $\lvert x_2\rvert > \lvert x_1\rvert$	1
	• gives correct explanation	1

Answer

i. $f(x) = x^{\frac{1}{3}};\ f'(x) = \tfrac{1}{3}x^{-\frac{2}{3}}$

$$x_2 = x_1 - \frac{f(x)}{f'(x)}$$

$$= 1 - \frac{1}{\frac{1}{3} \times 1} = -2$$

ii. $$x_2 = x_1 - \frac{x_1^{\frac{1}{3}}}{\frac{1}{3}x_1^{-\frac{2}{3}}}$$

$$= -2x_1$$

$\therefore |x_2| = |-2x_1| = 2|x_1|$

$\therefore |x_2| > |x_1|$

Method fails because the approximations do not converge

4a. **Outcomes assessed: i. HE3, HE4 ii. P4 iii. H5, P3**

Marking guidelines

		Criteria	Marks
i.	•	finds first and second derivative	1
	•	states $\ddot{x} = -9x$ so motion is SHM	1
ii.	•	answer	1
iii.	•	answer	1

Answer

i.
$$\dot{x} = -6\sin\left(3t + \frac{\pi}{6}\right)$$
$$\ddot{x} = -18\cos\left(3t + \frac{\pi}{6}\right) = -9x$$

Therefore, motion is Simple Harmonic

ii. $\frac{2\pi}{3}$

iii. $\dot{x} = -6\sin\left(3t + \frac{\pi}{6}\right) = 0$
$$3t + \frac{\pi}{6} = 0, \pi, 2\pi$$
$$3t = -\frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}$$
$$t = \frac{5\pi}{18}$$

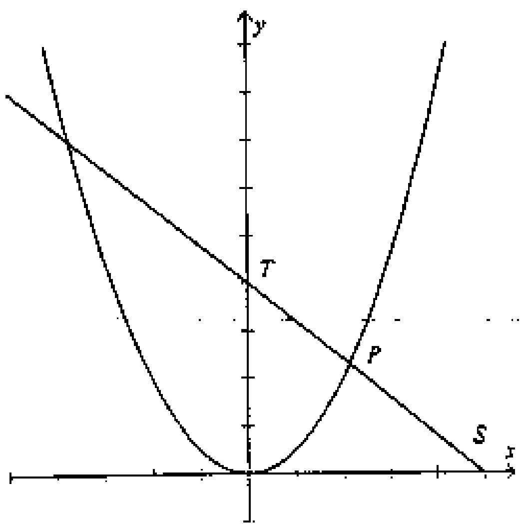
4b **Outcomes assessed: i. P4 ii. PE4 iii. PE4**

Marking guidelines

		Criteria	Marks
i.	•	correct diagram	1
ii.	•	correct formula and coordinates of S and T	1,1,1
iii.	•	correct values	1

Answer

i.



ii. $x + py = 2ap + ap^3$

$T : x = 0 \Rightarrow y = 2a + ap^2$

$S : y = 0 \Rightarrow x = 2ap + ap^3$

iii. If P is the midpoint of ST:

$$2ap = \frac{0 + (2ap + ap^3)}{2}$$
$$4ap = 2ap + ap^3$$
$$ap^3 - 2ap = 0$$
$$ap(p^2 - 2) = 0$$
$$p = 0, p = \pm\sqrt{2}$$
$$\therefore p = \pm\sqrt{2}$$

4c **Outcomes assessed: i. H5 ii. HE3**

Marking guidelines

	Criteria	Marks
i.	• clear and correct explanation	1
ii.	• correctly uses Binomial Probability	1
	• answer	1

Answer

i. There are 2^3 possibilities.
The number of permutations of 2 heads and 1 tail is $3!/2! = 3$
Therefore, the probability is $\frac{3}{8}$

ii. $P(X=r) = {}^{10}C_r\ p^r\ q^{10-r} = {}^{10}C_r\left(\frac{3}{8}\right)^r\left(\frac{5}{8}\right)^{10-r}$

$P(X>1) = 1-\left[{}^{10}C_0\left(\frac{3}{8}\right)^0\left(\frac{5}{8}\right)^{10} + {}^{10}C_1\left(\frac{3}{8}\right)^1\left(\frac{5}{8}\right)^9\right]$

$= 0.936$

5a. **Outcomes assessed: i. HE3 ii. HE3**

Marking guidelines

	Criteria	Marks
i.	• correct demonstration	1
ii.	• finds B	1
	• finds k	1
	• answer	1

Answer

i. $\frac{dT}{dt} = Bke^{kt}$

$= k\ Be^{kt}$

$= k(T-S)$

ii. $100 = 25 + Be^{k\times 0} \Rightarrow B = 75$

$80 = 25 + 75e^{30k} \Rightarrow k = -0.0103$

$t = 60 \Rightarrow T = 25 + 75\,e^{60\times -0.0103}$

$= 65.33^{\circ}$

$= 65$

5b **Outcomes assessed: HE3**

Marking guidelines

Criteria	Marks
• derives results for vertical motion	1
• derives results for horizontal motion and Cartesian form	1
• substitutes parameters and reduces to equation in tanx	1
• uses quadratic formula to obtain angles	1
• states range of values	1

Answer on next page

Horizontal Motion

$\ddot{x} = 0$

$\dot{x} = C$ when $t = 0, \dot{x} = 25\cos\alpha$

$\dot{x} = 25\cos\alpha$

$x = 25t\cos\alpha + k,$ when $t = 0, x = 0$

$x = 25t\cos\alpha$

Vertical Motion

$\ddot{y} = -10$

$\dot{y} = -10t + c$ when $t = 0, \dot{y} = 25\sin\alpha$

$\dot{y} = -10t + 25\sin\alpha$

$y = -5t^2 + 25t\sin\alpha + k$ when $t = 0, y = 2$

$y = -5t^2 + 25t\sin\alpha + 2$

Substitute

$t = \dfrac{x}{25\cos\alpha}$

into y $\rightarrow$

$$y = -5\left(\frac{x}{25\cos\alpha}\right)^2 + 25.\frac{x}{25\cos\alpha}.\sin\alpha + 2$$

$$y = -\frac{x^2}{125}\sec^2\alpha + x\tan\alpha + 2$$

And when $x = 20, y = 15$

$15 = -\dfrac{400}{125}\sec^2\alpha + 20\tan\alpha + 2$

$13 = -\dfrac{16}{5}\sec^2\alpha + 20\tan\alpha$

$65 = -16(\tan^2 + 1) + 100\tan\alpha$

$16\tan^2\alpha - 100\tan\alpha + 81$

$\tan\alpha = \dfrac{100 \pm \sqrt{100^2 - 4.16.81}}{32}$

$\tan\alpha = 0.956, 5.29366$

$\alpha = 44°, 79°$

$44° \le \alpha \le 79°$

5c. Outcomes assessed: i. HE3 ii. PE3

Marking guidelines

		Criteria	Marks
i.	•	correctly uses binomial theorem to expand expressions	1
	•	equates coefficients on both sides to obtain answer	1
ii.	•	answer	1

Answer

i. $(1+x)^{n+3} = \binom{n+3}{0} + \binom{n+3}{1}x + \binom{n+3}{2}x^2 + \ldots + \binom{n+3}{n+3}x^{n+3}$

$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n; (1+x)^3 = \binom{3}{0} + \binom{3}{1}x + \binom{3}{2}x^2 + \binom{3}{3}x^3$

Coefficient of x^k on LHS is $\binom{n+3}{k}$; on RHS: $\binom{n}{k}\binom{3}{0} + \binom{n}{k-1}\binom{3}{1} + \binom{n}{k-2}\binom{3}{2} + \binom{n}{k-3}\binom{3}{3}$

Hence: $\binom{n+3}{k} = \binom{n}{k} + 3\binom{n}{k-1} + 3\binom{n}{k-2} + \binom{n}{k-3}$

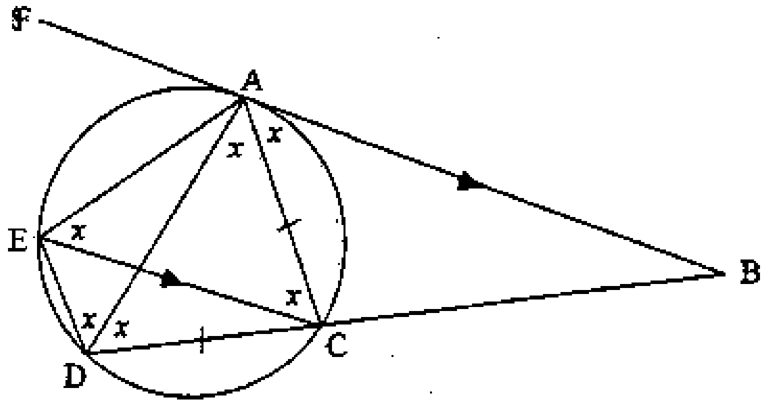
ii. $3 < k < n$

6a. Outcomes assessed: PE3, PE6, HE7

Marking guidelines

Criteria	Marks
• correct use of appropriate geometrical theorems	1
• uses appropriate logical sequence to prove result	1
• uses correct setting out	1
• uses correct notation and terminology	1

Answer



Let $\angle BAC = x$

$\angle AEC = \angle BAC = x$	(The angle between a tangent and a chord is equal to the angle in the alternate segment)
$\angle ADC = \angle AEC = x$	(Angles in the same segment are equal (arc AC))
$\angle ACE = \angle CAB = x$	(Alternate angles AB || EC)
$\angle ADE = \angle ACE = x$	(Angles in the same segment are equal (arc AE))
$\angle DAC = \angle CDA = x$	(ΔACD isosceles given AC = DC. $\therefore$ angles opposite equal sides are equal)

Since $\angle DAC = \angle ADE$ as both are equal to x. AC || ED since alternate angles are equal.

6b. Outcomes assessed: i. HE5 ii. HE5 iii. HE5

Marking guidelines

	Criteria	Marks
i.	• answer	1
ii.	• inverts integrand	1
	• integrates and finds c	1
	• rearranges to obtain expression for t	1
iii.	• answer	1

Answer

i. $$a = v\frac{dv}{dx} = (2-x)^2 \times -2(2-x)$$
$$= -2(2-x)^3$$

ii. $$\frac{dx}{dt} = (2-x)^2 \Rightarrow \frac{dt}{dx} = (2-x)^{-2}$$
$$\Rightarrow t = (2-x)^{-1} + c$$
$$t = 0, x = 0 \Rightarrow c = -\tfrac{1}{2}$$
$$t = \frac{1}{2-x} - \frac{1}{2} \Rightarrow x = 2 - \frac{2}{2t+1} = \frac{4t}{2t+1}$$

iii. $$(2-x)^2 = 1$$
$$2-x = 1 \text{ or } 2-x = -1$$
$$x = 1, 3$$

But when $x = 3$, $t < 0$

So $x = 1$

6c. Outcomes assessed: i. HE4 ii. HE4

Marking guidelines

	Criteria	Marks
i.	• makes statement about symmetry	1
	• specifies the line of symmetry is $y = x$	1
ii.	• gives a correct example	1

Answer

i. The function must be symmetrical about the line $y = x$

ii. Examples

7a. Outcomes assessed: i. PE5 ii. PE5, HE4

Marking guidelines

	Criteria	Marks
i.	• uses appropriate procedures to find derivative of each term	1
	• answer	1
ii.	• adjusts functions to y-axis and sets up integral to find area	1
	• uses part i. result to obtain integral	1
	• evaluates integral	1

Answer

i. $\dfrac{d}{dx}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)$

$=\cos^{-1}x+x\times\dfrac{-1}{\sqrt{1-x^2}}-\dfrac{1}{2\sqrt{1-x^2}}\times-2x$

$=\cos^{-1}x$

ii. $A=\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\cos^{-1}y\,dy$

$=\left[y\cos^{-1}y-\sqrt{1-y^2}\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$

$=\left[\dfrac{\sqrt{3}}{2}\times\dfrac{\pi}{6}-\sqrt{1-\dfrac{3}{4}}\right]-\left[\dfrac{1}{2}\times\dfrac{\pi}{3}-\sqrt{1-\dfrac{1}{4}}\right]$

$=\dfrac{\pi}{12}\left(\sqrt{3}-2\right)+\dfrac{\sqrt{3}-1}{2}$

7b. Outcomes assessed: i. HE1 ii. PE2, PE6 iii. HE1, HE7

Marking guidelines

	Criteria	Marks
i.	• answer	1
ii.	• finds both expressions for area in terms of m	1
	• finds values of m	1, 1
iii.	• finds both expressions for area in terms of n	1
	• constructs expression for ratio and simplifies	1
	• finds answers and justifies conclusion	1

Answer on next page

Answer

i. $0 \le m \le \frac{1}{2}$

ii. $P(1, m)$, $Q(2, 2m)$

Area of trapezium, $APQD$, is $\frac{3m}{2}$

Area of $PBCQ$ is $1-\frac{3m}{2}$

Ratio: either $\dfrac{\frac{3m}{2}}{1-\frac{3m}{2}} = \dfrac{2}{1} \Rightarrow m = \dfrac{4}{9}$

or $\dfrac{\frac{3m}{2}}{1-\frac{3m}{2}} = \dfrac{1}{2} \Rightarrow m = \dfrac{2}{9}$

iii. Now $\frac{1}{2} \le n \le 1$

$S(1, n)$, $T(\frac{1}{n}, 1)$

Area of triangle SBT is $(1-n)(\dfrac{1}{n}-1)$

Area of remainder is $1 - (1-n)(\dfrac{1}{n}-1)$

Ratio: $\dfrac{\frac{1}{n}-2+n}{1-\left(\frac{1}{n}-2+n\right)} = \dfrac{1-2n+n^2}{-1+3n-n^2}$

$\therefore \dfrac{1-2n+n^2}{-1+3n-n^2} = \dfrac{1}{2}$

$3n^2-7n+3=0$

$n = \dfrac{7 \pm \sqrt{49-4\times3\times3}}{2\times3} = \dfrac{7\pm\sqrt{13}}{6}$

$n = 1.7676, 0.5657$

Both are outside the limit above so k cannot divide the square in the ratio 2:1