1991 Higher School Certificate Examination Paper

3/4 Unit Mathematics

QUESTION ONE

(a) Evaluate:

(i) $\int_0^1 \frac{x}{x^2+1}\,dx$

(ii) $\int_{-1}^1 (1+5x)^4\,dx$, using the substitution $u = 1+5x$.

(b) The polynomial $P(x) = x^3 + ax + 12$ has a factor $(x+3)$. Find the value of a.

(c) The point $P(-3, 8)$ divides the interval AB externally in the ratio $k:1$. If A is the point $(6, -4)$ and B is the point $(0, 4)$, find the value of k.

(d) (i) Sketch the graph of $y = |x-2|$.

(ii) For what values of x is $|x-2| < x$?

QUESTION TWO

(a) Consider $y = e^{kx}$ where k is a constant.

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(ii) Determine the values of k for which $y = e^{kx}$ satisfies the equation

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0.$$

(b) When Mendel crossed a tall strain of pea with a dwarf strain of pea, he found that 3/4 of the offspring were tall and 1/4 were dwarf.

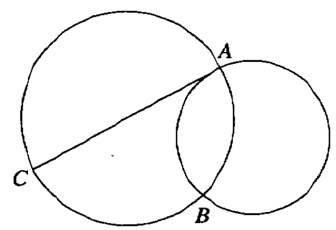
Suppose five such offspring were selected at random. Find the probability that:

(i) all of these offspring were tall;

(ii) at least three of these offspring were tall.

Leave your answers in index form.

(c)



The diagram shows two circles intersecting at A and B. The diameter of one circle is AC.

Copy this diagram into your examination booklet.

(i) On your diagram draw a straight line through A, parallel to CB, to meet the second circle in D.

(ii) Prove that BD is a diameter of the second circle.

(iii) Suppose that BD is parallel to CA. Prove that the circles have equal radii.

QUESTION THREE

(a) Taking $x = 0{\cdot}5$ as a first approximation to the root of $x + \ln x = 0$, use Newton's method to find a second approximation.

(b) Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 3x\,dx$.

(c) If $y = 10^x$, find $\frac{dy}{dx}$ when $x = 1$.

(d) The volume, V, of a sphere of radius r mm is increasing at a constant rate of 200 mm^3 per second.

(i) Find $\frac{dr}{dt}$ in terms of r.

(ii) Determine the rate of increase of the surface area, S, of the sphere when the radius is 50 mm.

$$\left(V = \frac{4}{3}\pi r^3 ; S = 4\pi r^2\right)$$

QUESTION FOUR

(a) Use mathematical induction to prove that, for all positive integers n,

$$1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1.$$

(b) The acceleration of a particle moving in a straight line is given by

$$\frac{d^2x}{dt^2} = 2x - 3.$$

where x is the displacement, in metres, from the origin O and t is the time in seconds. Initially the particle is at rest at $x = 4$.

(i) If the velocity of the particle is v m/s, show that $v^2 = 2(x^2 - 3x - 4)$.

(ii) Show that the particle does not pass through the origin.

(iii) Determine the position of the particle when $v = 10$. Justify your answer.

(c) Containers are coded by different arrangements of coloured dots in a row. The colours used are red, white, and blue.

In an arrangement, at most three of the dots are red, at most two of the dots are white, and at most one is blue.

(i) Find the number of different codes possible if six dots are used.

(ii) On some containers only five dots are used. Find the number of different codes possible in this case. Justify your answer.

QUESTION FIVE

(a) Consider the function $f(x) = 3\sin^{-1}\frac{x}{2}$.

(i) Evaluate $f(2)$.

(ii) Draw the graph of $y = f(x)$.

(iii) State the domain and range of $y = f(x)$.

(b) (i) Sketch the parabola whose parametric equations are $x = t$ and $y = t^2$. On your diagram mark the points P and Q which correspond to $t = -1$ and $t = 2$ respectively.

(ii) Show that the tangents to the parabola at P and Q intersect at $R\left(\frac{1}{2}, -2\right)$.

(iii) Let $T(t, t^2)$ be the point on the parabola between P and Q such that the tangent at T meets QR at the midpoint of QR. Show that the tangent at T is parallel to PQ.

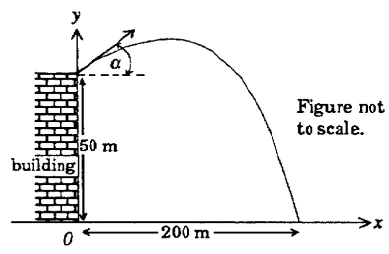
QUESTION SIX

(a) (i) On the same axes, sketch the curves $y = \sin x$, $y = \cos x$ and $y = \sin x + \cos x$ for $0 \le x \le 2\pi$.

(ii) From your graph, determine the number of values of x in the interval $0 \le x \le 2\pi$ for which $\sin x + \cos x = 1$.

(iii) For what values of the constant k does $\sin x + \cos x = k$ have exactly two solutions in the interval $0 \le x \le 2\pi$.

(b)



The diagram shows the path of a projectile launched at an angle of elevation α, with an initial speed of 40 m/s, from the top of a 50 metre high building. The acceleration due to gravity is assumed to be 10 m/s^2.

(i) Given that $\frac{d^2x}{dt^2} = 0$ and $\frac{d^2y}{dt^2} = -10$, show that $x = 40t\cos\alpha$ and

$$y = -5t^2 + 40t\sin\alpha + 50,$$

where x and y are the horizontal and vertical displacements of the projectile in metres from O at time t seconds after launching.

(ii) The projectile lands on the ground 200 metres from the base of the building. Find the two possible values for α. Give your answers to the nearest degree.

QUESTION SEVEN

(a) Let $f(x) = \frac{x}{x^2 - 1}$.

(i) For what values of x is $f(x)$ undefined?

(ii) Show that $y = f(x)$ is an odd function.

(iii) Show that $f'(x) < 0$ at all values of x for which the function is defined.

(iv) Hence sketch $y = f(x)$.

(b)

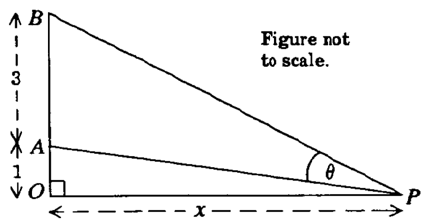


Figure not to scale.

In the diagram, a vertical pole A, 3 metres high, is placed on top of a support 1 metre high. The pole subtends an angle of θ radians at the point P, which is x metres from the base O of the support.

(i) Show that $\theta = \tan^{-1}\frac{4}{x} - \tan^{-1}\frac{1}{x}$.

(ii) Show that θ is a maximum when $x = 2$.

(iii) Deduce that the maximum angle subtended at P is $\theta = \tan^{-1}\frac{3}{4}$.