Cranbrook School

Trial HSC Examination

Mathematics Extension 2

2002

1. **(a)** The complex number z is given by $z = \sqrt{3} + \frac{1+i}{1-i}$. Find:
(i) $\Re(z)$ **(ii)** $\Im(z)$ **(iii)** $|z|$ **(iv)** $\arg z$
(b) If $z = \cos\theta + i\sin\theta$, show that $\frac{1}{1+z} = \frac{1}{2}(1 - i\tan\frac{\theta}{2})$
(c) **(i)** On an Argand Diagram, shade in the region for which $0 \le |z| \le$ and $1 \le \Im(z) \le 2$.
(ii) What is the complex number with the largest argument which satisfies the inequalities of **(i)**?
(d) If $z_1 = 1 + \sqrt{3}i$ and $z_2 = \sqrt{3} - i$ show that:
(i) $\frac{z_1^{10}}{z_2^8} = -2 + 2\sqrt{3}i$ **(ii)** $\left(\frac{z_1}{z_2}\right)^{79} = -i$

2. **(a)** Prove that if $P(z) = z^4 + 2z^2 + 1$ has two double roots find these roots and factorise the polynomial $P(z)$ over the complex number field.
(b) It is known that $P(x) = x^4 + 2x^3 + x^2 - 1$ has a zero, $x = \frac{-1+i\sqrt{3}}{2}$. Find all the other zeros of $P(x)$.
(c) Find the sixth-degree polynomial $P(x)$ and the constant A such that $x^4(1-x)^4 \equiv (1+x^2)P(x) + A$.
(d) **(i)** If $P(x) = x^3 - 9x^2 + 24x + c$ for some real number c, find the values of x for which $P'(x) = 0$. Hence find the two values of c for which the equation $P(x) = 0$ has a repeated root.
(ii) Sketch the graphs of $y = P(x)$ for these values of c. hence write down the values of c for which the equation $P(x) = 0$ has three distinct real roots.

3. **(a)** If $P(x_1, y_1)$ is any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $S(ae, 0)$ and $S'(-ae, 0)$ are the foci prove that $|PS'| - |PS| = 2a$.
(b) For the ellipse $x^2 + 4y^2 = 100$:
(i) find the eccentricity and the coordinates of the foci;
(ii) find the equation of the tangent at $P(8, 3)$ in general form.
(iii) If the normal at P meets the major axis at $G(6, 0)$ and the perpendicular from the centre O to the tangent at P meets that tangent at K, prove that $PG.OK$ is equal to the square of the semi-minor axis. [Include a labelled diagram with your answer.]
(c) The condition for the line $L : y = mx + c$ to touch the ellipse $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by $c^2 = b^2 + a^2m^2$.

(i) From a variable point $T(X, Y)$ on L, tangents are drawn to E. Show that $(Y - mX)^2 = b^2 + a^2m^2$.
(ii) If these tangents are at right angles to one another, prove that T lies on the circle $x^2 + y^2 = a^2 + b^2$.

4. (a) Find $\int \frac{dx}{\sqrt{6-x-x^2}}$
(b) Evaluate $\int_1^e x^3 \ln x \; dx$ in exact form.
(c) (i) By using the substitution $t = a - x$, prove that $\int_0^a f(x)\; dx = \int_0^a f(x - x)\; dx$
(ii) Hence prove that $\int_0^3 x(3-x)^{11}\; dx = \frac{3^{12}}{52}$
(d) Find $\int \cos^3\theta\sqrt{1-\sin\theta}\; d\theta$

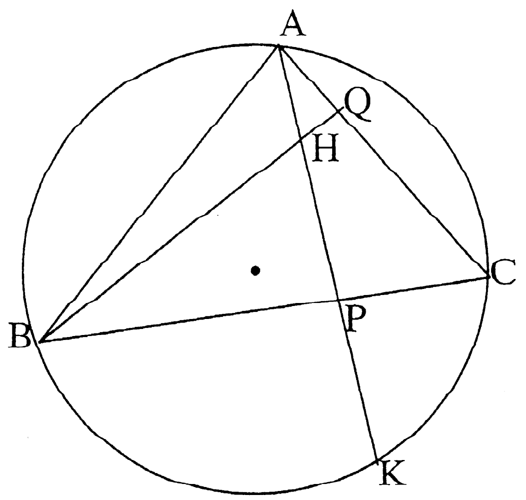
5. (a) (i) Find real numbers a and b such that $\frac{1}{(2t-1)(t+2)} = \frac{a}{2t-1} + \frac{b}{t+2}$
(ii) By using the substitution $t = \tan\frac{\theta}{2}$ evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{3\sin\theta - 4\cos\theta}$
(b) Evaluate $\int_{-1}^{1} \frac{\tan^{-1} x\; dx}{1+\sin^2 x}$
(c) It is given that $I_n = \int \cos^{n-1} x \sin nx\; dx$ and $n \geq 1$ then
$I_n = \frac{1}{2n-1}((n-1)I_{n-1} - \cos^{n-1} x \cos nx)$.
Use this reduction formula to show that: $\int_0^{\frac{\pi}{4}} \cos^2 x \sin 3x\; dx = \frac{1}{60}(28 - \sqrt{2}\,)$

6. (a) A solid is built on the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ such that cross-sectional slices perpendicular to the y-axis are equilateral triangles with one side lying in the base of the solid. Find the exact volume of this solid.
(b) The region bounded by the curve $y = x^2 + 1$ and the line $y = 3 - x$ is rotated about the x-axis. By considering slices perpendicular to the x-axis, prove that the volume generated is $\frac{117\pi}{5}$ unit^3.
(c) (i) The circle $(x-3)^2 + y^2 = 4$ is rotated about the y-axis. By using the method of cylindrical shells show that the volume, V is given by $V = 4\pi\int_1^5 x\sqrt{4-(x-3)^2}\; dx$
(ii) Hence show that the volume is $24\pi^2$ unit^3

7. (a) Solve $\frac{x+1}{(x-1)(x+2)} \geq 3$ for x.
(b) P, Q and R are the vertices of an equilateral triangle. $P = (-3, 2)$ and $Q(3, -2)$ and OR is the perpendicular bisector of PQ where O is the origin.
(i) If m is the gradient of RQ show that the acute angle between PQ and RQ is given by $\tan 60^\circ = \left|\frac{m+\frac{2}{3}}{1-\frac{2m}{3}}\right|$.
(ii) By taking the positive absolute value case show that $m = \frac{3\sqrt{3}-2}{3+2\sqrt{3}}$
(iii) Hence or **otherwise** find the possible coordinates of R.
(c) Consider a pack of 50 playng cards which consist of 5 colours
{Yellow, Green, Blue, Indigo and Violet} containing cards numbered from 1 to 10 inclusive, respectively. A joker is added to the pack. The joker can stand for any card and when there are equal numbers of different cards it takes the value of the higher card. Otherwise, the joker stands for the card which is occurring most often.
e.g., two 4's, two 6's and a joker = two 4's and three 6's;
two 4's, one 6, one 8 and a joker = three 4's, one 6 and one 8.

If five cards are dealt to a player, determine the probability (leaving your answer as a fraction in its simplest form) that the player has received:
(i) four 10's.
(ii) any three of one number and any two of another number, e.g., three 10's and two 8's.

8. (a) (i) On a particular July day with a strong off shore easterly wind blowing and a large sea swell, the motion of a buoy situated off Bondi Beach can be considered to be simple harmonic. If the motion of the buoy is given by $\ddot{x} = -n^2(x - b)$, where $x = b$ is the centre of motion and 'n' is a positive constant show that $x = b + a\cos nt$ (where 'a' is the amplitude) satisfies the buoy's motion.
(ii) The crest height of the buoy is 25m above sea level and the trough height is 20m above sea level. At one instant the buoy is at a crest position and 15 seconds later it is at a trough position. If initially the buoy is a a crest position, how long after this **on the first two occasions**, will the buoy be 24m above sea level? Leave your answers in seconds correct to 2 decimal places.
(b)



ABC is a triangle inscribed in a circle as shown. AP and BQ are altitudes meeting at H. AP produced cuts the circle at K. Prove that $HP = KP$.
(c) Find the general solutions of $\tan 3\theta - \cot 5\theta = 0$.