175

N.S.W. Independent Trial Exams

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1997

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

Time Allowed - Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- Write your student Name / Number on every page of the question paper and your answer sheets.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied.
- Board approved calculators may be used.
- The answers to the seven questions are to be handed in separately clearly marked Question 1, Question 2, etc..
- *The question paper must be handed to the supervisor at the end of the examination.*

STUDENT NUMBER / NAME..

Question 1 (Start a new page) — Marks

a. Find the exact value of $\displaystyle\int_{\frac{3\sqrt{3}}{2}}^{3} \frac{2}{\sqrt{9-x^2}}\,dx$ — 3

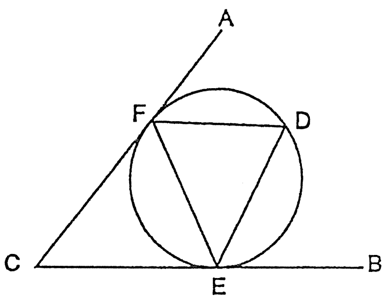
b. Use the substitution $u = 3 - x^2$ to find $\displaystyle\int \frac{x}{\sqrt{3-x^2}}\,dx$ — 3

c. For the expansion of $\left(x - \dfrac{2}{x}\right)^8$, find the term independent of x. — 3

d. Solve the inequality $\dfrac{2x-3}{x} > 1$ — 3

Question 2 (Start a new page)

a. In the diagram, AC and BC are tangents to the circle, touching the circle at F and E respectively. $\angle ACB$ equals 50°. Copy the diagram into your workbook. — 3



Show that $\angle CEF$ is 65° and hence find $\angle EDF$.

b. i. In how many ways can the letters of the word MOUSE be arranged? — 3

ii. How many of these arrangements
1. start with the letter M and end with the letter E?
2. have the *vowels* together?

[A *vowel* is one of the letters A, E, I, O, U]

c. A curve is defined by the parametric equations $x = t - 3,\ y = t^2 - 9$ — 3

i. Find $\dfrac{dy}{dx}$ in terms of t.

ii. Find the equation of the tangent to the curve at the point where $t = -3$

d. The polynomial equation $8x^3 - 36x^2 + 22x + 21 = 0$ has roots which form an arithmetic progression. Find the roots. — 3

Question 3 (Start a new page) — Marks

a. The arc of the curve $y = \cos 3x$ between the lines $x = 0$ and $x = \dfrac{\pi}{6}$ is rotated about the x-axis. — 4

Find the volume of the solid formed.

b. Consider the function $y = x\ln x - 1,\ (x > 0)$ — 6

i. Find the stationary point and determine its nature.

ii. With an initial approximation of $x = 2$, use Newton's Method once to find the x-intercept.

iii. Show that the curve is always concave upwards.

iv. Sketch the curve, showing all of its main features.

c. Sketch the graph of the function $f(x) = 3\sin^{-1}\dfrac{x}{2}$ — 2

Question 4 (Start a new page)

a. Prove by Mathematical Induction that $3^{2n} - 1$ is divisible by 8 when n is an integer greater than 0. — 4

b. From a balloon 500 metres above a road junction, the angle of depression to a point, P, due south of the junction is 42°. To another point, Q, bearing 080° from the junction, the angle of depression is 32°. How far apart are P and Q? — 4

c. It is known that 5% of all gear boxes made in Factory A are faulty whereas 7% of gear boxes made in factory B are faulty. If 10 gear boxes from each factory are bought, find the probability that exactly two are faulty. — 4

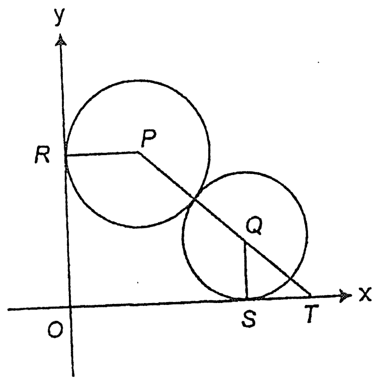
177

Question 7 (Start a new page)

Marks

a. Nathan, who will soon turn 21, wants to invest some money on his birthday each year so that he will have \$500,000 when he retires on his 65th birthday. He can open an account which will give him 6.2% p.a. compounded yearly over that time period. How much should he invest each year to achieve his goal? **4**

b. The diagram shows two touching circles, with centres P and Q. The circle with centre P has a radius of 4 units and touches the y-axis at R. The circle with centre Q has a radius of 3 units and touches the x-axis at S. PQ produced meets the x-axis at T and $\angle QTS = \theta$. **8**



i. Show that $OR = 3 + 7\sin\theta$ and $OS = 4 + 7\cos\theta$

ii. Show that $RS^2 = 42\sin\theta + 56\cos\theta + 74$

iii. Hence express RS^2 in the form $74 + r\cos(\theta - \alpha)$, clearly stating the values of r and α

iv. Find the maximum length of RS and the value of θ for which this occurs.

QUESTION 1

a) $\int_{\frac{3\sqrt{3}}{2}}^{3} \frac{2}{\sqrt{9-x^2}}\,dx = 2\left[\sin^{-1}\frac{x}{3}\right]_{\frac{3\sqrt{3}}{2}}^{3}$

$= 2\left(\sin^{-1}1 - \sin^{-1}\frac{\sqrt{3}}{2}\right)$

$= 2\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$

$= \frac{\pi}{3}$

b) If $u = 3 - x^2$; $du = -2x\,dx$

$\int \frac{x}{\sqrt{3-x^2}}\,dx = -\frac{1}{2}\int \frac{du}{\sqrt{u}}$

$= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du$

$= -\frac{1}{2}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right] + C$

$= -\sqrt{3-x^2} + C$

c) $\left(x - \frac{2}{x}\right)^8 = \sum_0^8 {}^8C_r\, x^{8-r} \cdot \left(-\frac{2}{x}\right)^r$

Term $= {}^8C_r \cdot x^{8-r} \cdot x^{-r} \cdot (-2)^r$

$= {}^8C_r (-2)^r \cdot x^{8-2r}$

Put $8 - 2r = 0 \Rightarrow r = 4$

$\therefore$ Term $= {}^8C_4 (-2)^4$

$= 1120$

(d) $\frac{2x-3}{x} > 1$

A critical value exists at $x = 0$

Also, at $\frac{2x-3}{x} = 1$

$2x - 3 = x$

$\therefore x = 3$

$x<1$ | $1<x<3$ | $x>3$

0 1 2 3 ... 6

Testing: if $x=6$, $\frac{2\times6-3}{6} > 1$ TRUE

$\therefore$ Solution set is

$x < 1$, $x > 3$

179

QUESTION 4

a) $S(n)$: $3^{2n}-1 = 8I$, where I is an integer

First $S(1)$: $3^2 - 1 = 8 = 8\times 1$

$\therefore S(1)$ is true

Assume $S(k)$: $3^{2k}-1 = 8I$, I an integer

Prove $S(k+1)$.

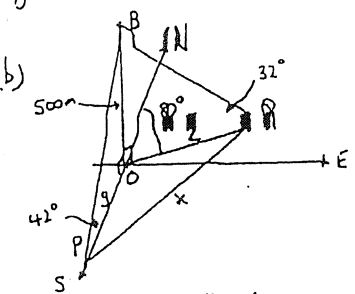
$$\begin{aligned}
LHS &= 3^{2(k+1)} - 1 \\
&= 3^{2k+2} - 1 \\
&= 3^{2k}.3^2 - 1 \\
&= (3^{2k}-1).9 + 8 \\
&= 8I.9 + 8 \\
&= 8(9I+1)
\end{aligned}$$

But I is an integer; so $9I+1$ is an integer

$\therefore$ If $S(k)$ is true, $S(k+1)$ is true.

But $S(1)$ is true so $S(2)$ is true and so on for all integers, n, greater than 0.

b)



$\tan 42 = 500/y$

$\therefore y = 500/\tan 42$

Similarly, $x = 500/\tan 32$

$\angle POQ = 100°$

By the cosine rule:

$$x^2 = \left(\frac{500}{\tan 42}\right)^2 + \left(\frac{500}{\tan 32}\right)^2 - 2\,\frac{500}{\tan 42}\cdot\frac{500}{\tan 32}\ldots$$

$x = 1050.21$

$\doteqdot 1050$ metres

(c) For A + B: let p = faulty gearbox

q = good gearbox

X = no. of faulty gearboxes

For A: $P(X=r) = {}^{10}C_r(.05)^r(.95)^{10-r}$

B: $P(X=k) = {}^{10}C_k(.07)^k(.93)^{10-k}$

To get 2 faulty gearboxes:

$r=2; k=0$

$\Rightarrow {}^{10}C_2(.05)^2(.95)^8 \times {}^{10}C_0(.07)^0(.93)^{10}$

$r=1; k=1$

$\Rightarrow {}^{10}C_1(.05)^1(.95)^9 \times {}^{10}C_1(.07)^1(.93)^9$

$r=0; k=2$

$\Rightarrow {}^{10}C_0(.05)^0(.95)^{10} \times {}^{10}C_2(.07)^2(.93)^8$

Adding gives

$P(X=2) \approx 0.2248$

QUESTION 5

(a) $\ddot{x} = 0$ $\qquad \ddot{y} = -10$

$\dot{x} = V\cos\alpha$ $\qquad \dot{y} = -10t + V\sin\alpha$

$x = Vt\cos\alpha$ $\qquad y = -5t^2 + Vt\sin\alpha$

(assuming (0,0) at point of impact)

(i) At $t = 1.5$, $x = 60$; $y = 2.25$

$\Rightarrow 60 = V\cos\alpha \times 1.5 \Rightarrow V\cos\alpha = 40$

$\Rightarrow -5\times(1.5)^2 + V\sin\alpha\times 1.5 = 2.25$

$\Rightarrow V\sin\alpha = 9$

whence $V = 41$ m/s

& $\tan\alpha = 9/40$

so $\alpha = 12.68°$

(ii) When $y=0$: $-5t^2 + Vt\sin\alpha = 0$

$t(-5t + V\sin\alpha) = 0$

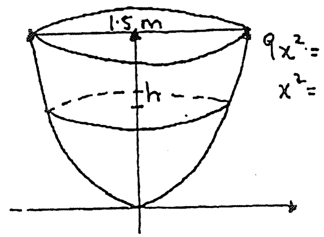
$\Rightarrow t = 0$ or $t = \frac{V\sin\alpha}{5} = \frac{9}{5}$

so $x = Vt\cos\alpha$

$= 40 \times \frac{9}{5}$

$= 72$ metres.

(b)



$9x^2 = 4y$

$x^2 = \frac{4}{9}y$

$V = \pi\int x^2\,dy$

$$\begin{aligned}
V &= \pi\int_0^h \frac{4}{9}y\,dy \\
&= \frac{4\pi}{9}\left[\frac{y^2}{2}\right]_0^h \\
&= \frac{2\pi h^2}{9}
\end{aligned}$$

(ii) If $x = 0.75$, $y = 1.2656$ so the volume when full is

$V = \frac{2\pi}{9}\times(1.2656)^2 = 1.1183\ m^3$

$\therefore$ when half full, $V = 0.5591$

i.e. $\frac{2\pi}{9}.h^2 = 0.5591$

$h^2 = 0.8009$

$h = 0.8949$

$\doteqdot 0.9$ m.

(iii) $\frac{dV}{dt} = \frac{dV}{dh}\times\frac{dh}{dt}$

$= \frac{4\pi}{9}.h\times\frac{dh}{dt}$

$\frac{dV}{dt} = 0.1\ m^3/s$; $h = 0.2$ m

so $0.1 = \frac{4\pi}{9}.0.2\times\frac{dh}{dt}$

$\frac{dh}{dt} = 0.0143$ m/s.

i.e about 1.4 cm/s.

/8