SOLUTIONS : 4U TRIAL, 2001 TOTAL = 120

(1)(a) Let $u = \log_e x$

$\therefore u' = \frac{1}{x}$

Let $v' = x^2$

$\therefore v = \frac{x^3}{3}$

$\therefore \int_1^{e^2} x^2 \log_e x \, dx$

$= \left[\frac{x^3}{3}\log_e x\right]_1^{e^2} - \int_1^{e^2} \frac{x^2}{3}\, dx$

$= \frac{e^6}{3}\cdot 2 - \left(\frac{e^6}{9} - \frac{1}{9}\right)$

$= \frac{5e^6+1}{9}$

(b) $\sin^5 x = \sin x(\sin^2 x)^2$

$= \sin x(1-\cos^2 x)^2$

$= \sin x(1 - 2\cos^2 x + \cos^4 x)$

Let $u = \cos x$

$\therefore du = -\sin x \, dx$

x	0	$\frac{\pi}{2}$
u	1	0

$\therefore \int_0^{\frac{\pi}{2}} \sin^5 x\, dx = \int_0^{\frac{\pi}{2}} \sin x(1 - 2\cos^2 x + \cos^4 x)\, dx$

$= \int_1^0 (1 - 2u^2 + u^4)\cdot -du$

$= \int_0^1 (1 - 2u^2 + u^4)\, du$

(c) $\int \frac{2x+1}{x^2+2x+2}\, dx$

$= \int \frac{(2x+2)-1}{x^2+2x+2}\, dx$

$= \int \frac{2x+2}{x^2+2x+2}\, dx - \int \frac{1}{(x+1)^2+1}\, dx$

$= \ln(x^2+2x+2) - \tan^{-1}(x+1) + c$

[no penalty for omission of c]

(d)(i)

Let $\frac{5-5x^2}{(1+2x)(1+x^2)} = \frac{A}{1+2x} + \frac{Bx+C}{1+x^2}$

$\therefore A(1+x^2) + (Bx+C)(1+2x) = 5 - 5x^2$

Let $x = -\frac{1}{2}$

$\therefore \frac{5A}{4} = 5\cdot\frac{3}{4}$

$\therefore A = 3$

Let $x = 0$

$\therefore 3 + C = 5$

$\therefore C = 2$

Let $x = 1$

$\therefore 6 + (B+2)\cdot 3 = 0$

$\therefore B = -4$

$\therefore \int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)}\, dx$

$= \int_0^1 \frac{3}{1+2x}\, dx + \int_0^1 \frac{-4x+2}{1+x^2}\, dx$

$= \left[\frac{3}{2}\ln(1+2x) - 2\ln(1+x^2) + 2\tan^{-1}x\right]_0^1$

$= \frac{3}{2}\ln 3 - 2\ln 2 + \frac{\pi}{2}$

$= \frac{1}{2}(3\ln 3 - 4\ln 2 + \pi)$

$= \frac{1}{2}\left(\pi + \ln\frac{27}{16}\right)$

(ii) $dx = \frac{2}{1+t^2}\, dt$

x	0	$\frac{\pi}{2}$
t	0	1

$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + 2\sin x + \cos x}\, dx$

$= \int_0^1 \frac{1-t^2}{1+t^2}\cdot\frac{1+t^2}{1+t^2+4t+1-t^2}\cdot\frac{2}{1+t^2}\, dt$

$= \int_0^1 \frac{2(1-t^2)}{(1+t^2)(4t+2)}\, dt$

$= \frac{1}{5}\int_0^1 \frac{5-5t^2}{(1+t^2)(1+2t)}\, dt$

$= \frac{1}{10}\left(\pi + \ln\frac{27}{16}\right)$ [using (i)]

(2)(a) $\arg((2+i)\overline{w})$

$= \arg((2+i)(-1+3i))$

$= \arg(-5+5i)$

$= \frac{3\pi}{4}$

(b) $x^2 - 12x + 48$

$= (x-6)^2 + 12$

$= (x-6)^2 - (2\sqrt{3}i)^2$

$= (x - 6 - 2\sqrt{3}i)(x - 6 + 2\sqrt{3}i)$

(c) Q represents $-i(a+bi)$

$= b - ai$

(d)(ii) Let $z = x+iy$

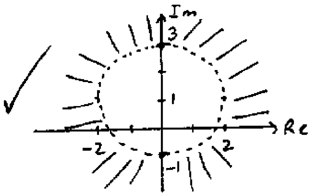
$\therefore |x + i(y+3)| < 2|x+iy|$

$x^2 + (y+3)^2 < 4(x^2+y^2)$

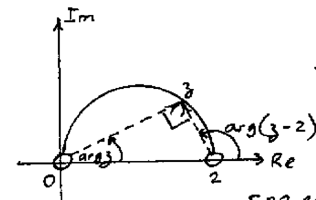
$6y + 9 < 3x^2 + 3y^2$

$x^2 + y^2 - 2y > 3$

$x^2 + (y-1)^2 > 4$



(i) $\arg(z-2) - \arg z = \frac{\pi}{2}$



[no penalty for omission of open circles]

(e)(i) Let $(r\,\text{cis}\,\theta)^3 = -8$

$\therefore r^3\,\text{cis}\,3\theta = 8\,\text{cis}(\pi + 2k\pi)$ where k is a integer

$\therefore r = 2$ and $\theta = (2k+1)\frac{\pi}{3}$

Choose $k = -1, 0, 1$

$\therefore$ the 3 cube roots of -8 are $2\,\text{cis}\left(-\frac{\pi}{3}\right)$, $2\,\text{cis}\frac{\pi}{3}$ and $2\,\text{cis}$

(ii) $2\,\text{cis}\left(-\frac{\pi}{3}\right) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= 1 - \sqrt{3}i$

and $2\,\text{cis}\frac{\pi}{3} = 1 + \sqrt{3}i$

(iii) Let $w_1 = 2\,\text{cis}\frac{\pi}{3}$

and $w_2 = 2\,\text{cis}\left(-\frac{\pi}{3}\right)$.

$\therefore w_1^{6n} + w_2^{6n}$

$= \left(2\,\text{cis}\frac{\pi}{3}\right)^{6n} + \left(2\,\text{cis}\left(-\frac{\pi}{3}\right)\right)^{6n}$

$= 2^{6n}\,\text{cis}(2n\pi) + 2^{6n}\,\text{cis}(-2n\pi)$

$= 2^{6n}\cdot 1 + 2^{6n}\cdot 1$

(since $\cos 2n\pi = 1$ and $\sin 2n\pi = 0$ when n is an integer)

$= 2\cdot 2^{6n}$

$= 2^{6n+1}$

or (Better)

$w_1^{6n} + w_2^{6n} = (w_1^6)^n + (w_2^6)^n$

$= 64^n + 64^n$ (since $w_1^3 = w_2$

$= 2\times 2^{6n}$

$= 2^{6n+1}$

(a)(i) By the quotient rule,

$$y' = \frac{(1+x^2).2 - 2x.2x}{(1+x^2)^2}$$

$$= \frac{2-2x^2}{(1+x^2)^2}$$

$y'=0$ when $x = \pm 1$,
so the turning points are
$A(1,1)$ and $B(-1,-1)$.

(ii) By the quotient rule,

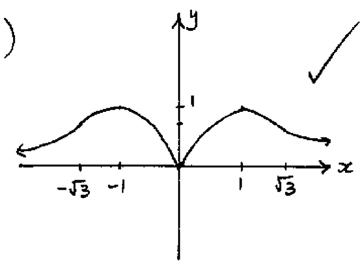
$$y'' = \frac{(1+x^2)^2.(-4x) - 2(1-x^2).4x(1+x^2)}{(1+x^2)^4}$$

$$= \frac{(1+x^2)(-4x-4x^3-8x+8x^3)}{(1+x^2)^4}$$

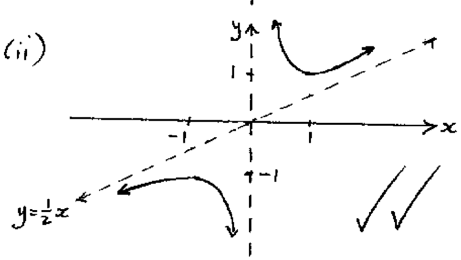
$$= \frac{4x(x^2-3)}{(1+x^2)^3}$$

$y''=0$ when $x = 0$ or $\pm\sqrt{3}$.
It follows that $P = \left(\sqrt{3}, \frac{\sqrt{3}}{2}\right)$
and $Q = \left(-\sqrt{3}, -\frac{\sqrt{3}}{2}\right)$.

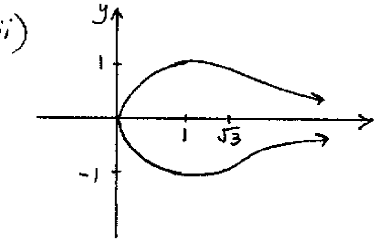
(b)(i)



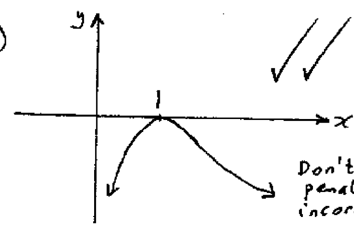
(ii)



(iii)



(iv)



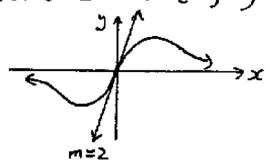
Don't penalize incorrect concavity as $x \to \infty$.

(c)(i) $kx^3 + (k-2)x = 0$

$$2x = kx^3 + kx$$
$$2x = kx(1+x^2)$$
$$\frac{2x}{1+x^2} = kx$$

(ii) Graphical Solution

The curve $y = \frac{2x}{1+x^2}$ has gradient 2 at $(0,0)$.



So $y=kx$ and the curve will intersect exactly once for $k \geq 2$ or $k \leq 0$.

Algebraic Solution (Also worth 2 marks.)

$$x(kx^2 + k - 2) = 0$$

$\therefore x = 0$ or $x^2 = \frac{2-k}{k}$

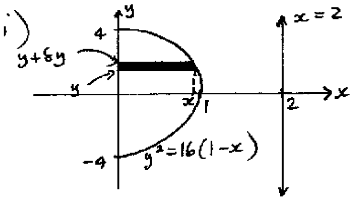
For one real root, $\frac{2-k}{k} \leq 0$

i.e. $\frac{k-2}{k} \geq 0$

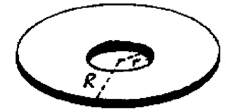
$(\times k^2)$ $k(k-2) \geq 0$ $\therefore k \geq 2$ or $k < 0$

But $k=0$ clearly works so $k \geq 2$ or $k \leq 0$.

(4)(a)(i)



Typical slice :



$R = 2$
$r = 2-x$
$h = \delta y$

$$\delta V \doteqdot (\pi R^2 - \pi r^2)h$$
$$= \pi(R-r)(R+r)h$$
$$= \pi . x . (4-x) . \delta y$$

$$\therefore V = 2\pi \int_0^4 (4x - x^2)\, dy$$

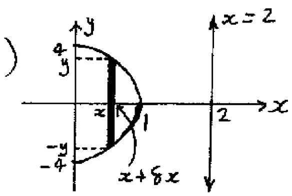
$$= 2\pi \int_0^4 \left[4\left(1-\frac{y^2}{16}\right) - \left(1-\frac{y^2}{16}\right)^2\right] dy$$

$$= 2\pi \int_0^4 \left(3 - \frac{y^2}{8} - \frac{y^4}{256}\right) dy$$

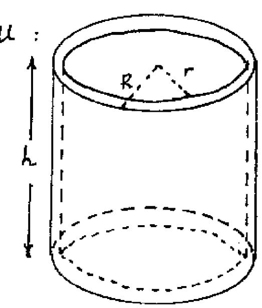
$$= 2\pi \left[3y - \frac{y^3}{24} - \frac{y^5}{1280}\right]_0^4$$

$$= \frac{256\pi}{15} \text{ units}^3$$

(ii)(α)



Typical shell :



$R = 2-x$
$r = 2-(x+\delta x)$
$h = 2y$

$$\delta V \doteqdot \pi(R-r)(R+r)h$$
$$= \pi . \delta x . (4-2x-\delta x) . 2y$$
$$\doteqdot 4\pi y(2-x)\delta x$$

$$\therefore V = \int_0^1 4\pi y(2-x)\, dx$$

$$= \int_0^1 4\pi . 4\sqrt{1-x} . (2-x)\, dx$$

$$= \int_0^1 16\pi(2-x)\sqrt{1-x}\, dx$$

(β) Let $u = 1-x$
$\therefore x = 1-u$
$\therefore dx = -du$

x	0	1
u	1	0

$$V = 16\pi \int_1^0 (u+1).u^{\frac{1}{2}}.-du$$

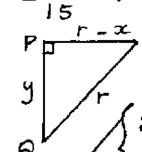
$$= 16\pi \int_0^1 \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$$

$$= 16\pi \left[\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right]_0^1$$

$$= 16\pi\left(\frac{2}{5} + \frac{2}{3}\right)$$

$$= \frac{256\pi}{15} \text{ units}^3$$

(b)(i)



By Pythagoras

$y^2 + (r-x)^2 = r^2$

$y^2 + r^2 - 2xr + x^2 = r^2$

$\therefore y^2 = 2rx - x^2$

(ii) $\delta V \doteqdot \frac{1}{2} . y . 3y . \delta x$

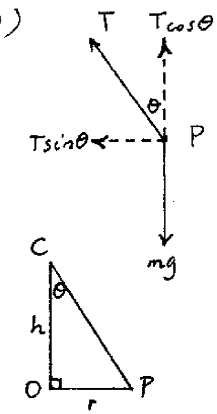
$$\therefore V = 2\int_0^r \frac{3}{2}y^2\, dx$$
$$= 3\int_0^r (2rx - x^2)\, dx$$
$$= 3\left[rx^2 - \frac{1}{3}x^3\right]_0^r$$
$$= 3\left(r^3 - \frac{1}{3}r^3\right)$$
$$= 2r^3 \text{ units}^3$$

(iii) $\frac{2r^3}{\pi . r^2 . 3r} \times 100\,\%$

$$= \frac{200}{3\pi}\,\%$$

$$\doteqdot 21\,\%$$

(5)(a)(i)



Resolving forces vertically at P:

$$T\cos\theta = mg \quad (1)$$

Resolving forces horizontally at P:

$$T\sin\theta = mr\omega^2 \quad (2)$$

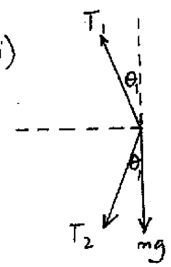
$(2) \div (1)$: $\tan\theta = \dfrac{r\omega^2}{g}$

$$\therefore \frac{r}{h} = \frac{r\omega^2}{g}$$

$$\omega^2 = \frac{g}{h}$$

$$\omega = \sqrt{\frac{g}{h}}$$

(ii)



Resolving forces vertically at P:

$$T_1\cos\theta_1 - T_2\cos\theta_1 = mg \quad (3)$$

Resolving horizontally at P:

$$T_1\sin\theta_1 + T_2\sin\theta_1 = mr\omega^2 \quad (4)$$

$(4) \div (3)$: $\dfrac{T_1 + T_2}{T_1 - T_2}\tan\theta_1 = \dfrac{R\omega^2}{g} \quad (5)$

Now, $\tan\theta_1 = \dfrac{R}{h}$ and $\omega = 3\sqrt{\dfrac{g}{h}}$,

so (5) can be written:

$$\frac{T_1+T_2}{T_1-T_2}\cdot\frac{R}{h} = \frac{R}{g}\cdot\frac{9g}{h}$$

$$\therefore \frac{T_1+T_2}{T_1-T_2} = 9$$

$$T_1 + T_2 = 9T_1 - 9T_2$$

$$10T_2 = 8T_1$$

$$\therefore \frac{T_1}{T_2} = \frac{10}{8} = \frac{5}{4}$$

(b)(i)(α) $\dfrac{d\dot{x}}{dt} = -k\dot{x}$

$$\therefore \frac{dt}{d\dot{x}} = \frac{-1}{k\dot{x}}$$

$$\therefore t = -\frac{1}{k}\ln\dot{x} + c_1$$

when $t=0$, $\dot{x} = V\cos\alpha$

$$\therefore c_1 = \frac{1}{k}\ln(V\cos\alpha)$$

$$\therefore t = \frac{1}{k}\ln\left(\frac{V\cos\alpha}{\dot{x}}\right)$$

$$\therefore e^{kt} = \frac{V\cos\alpha}{\dot{x}}$$

$$\therefore \dot{x} = V\cos\alpha \cdot e^{-kt}$$

(β) $\dfrac{d\dot{y}}{dt} = -g - k\dot{y}$

$$\therefore \frac{dt}{d\dot{y}} = \frac{-1}{g+k\dot{y}}$$

$$\therefore t = -\frac{1}{k}\ln(g+k\dot{y}) + c_2$$

when $t=0$, $\dot{y} = V\sin\alpha$

$$\therefore c_2 = \frac{1}{k}\ln(g + kV\sin\alpha)$$

$$\therefore t = \frac{1}{k}\ln\left(\frac{g+kV\sin\alpha}{g+k\dot{y}}\right)$$

$$\therefore e^{kt} = \frac{g+kV\sin\alpha}{g+k\dot{y}}$$

$$\therefore e^{-kt}(g+kV\sin\alpha) = g+k\dot{y}$$

$$\therefore \dot{y} = \left(\frac{g}{k}+V\sin\alpha\right)e^{-kt} - \frac{g}{k}$$

(ii)(α) $x = V\cos\alpha\int e^{-kt}\,dt$

$$= -\frac{V\cos\alpha}{k}e^{-kt} + c_3$$

when $t=0$, $x=0$

$$\therefore c_3 = \frac{V\cos\alpha}{k}$$

$$\therefore x = \frac{V\cos\alpha}{k}\left(1 - e^{-kt}\right)$$

(β) $y = \left(+\dfrac{g}{k} + V\sin\alpha\right)\int e^{-kt}$

$$- \int \frac{g}{k}\,dt$$

$$= -\frac{1}{k}\left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} -$$

$$+c$$

when $t=0$, $y=0$

$$\therefore c_4 = \frac{1}{k}\left(\frac{g}{k} + V\sin\alpha\right)$$

$$\therefore y = \frac{1}{k}\left(\frac{g}{k}+V\sin\alpha\right) - \frac{1}{k}\left(\frac{g}{k}+V\sin\right.$$

$$-\frac{g}{k}$$

$$\therefore y = \left(\frac{g}{k^2} + \frac{V\sin\alpha}{k}\right)\left(1-e^{-kt}\right) - .$$

(iii) when $\dot{y} = 0$,

$$e^{-kt} = \frac{\frac{g}{k}}{\frac{g}{k} + V\sin\alpha}$$

$$= \frac{g}{g + Vk\sin\alpha}$$

$$\therefore x = \frac{V\cos\alpha}{k}\left(1 - \frac{g}{g+Vk\sin\alpha}\right)$$

$$= \frac{V\cos\alpha}{k}\cdot\frac{g+Vk\sin\alpha - g}{g+Vk\sin\alpha}$$

$$= \frac{V^2\cos\alpha\sin\alpha}{g+Vk\sin\alpha}$$

$$= \frac{V^2\sin 2\alpha}{2(g+Vk\sin\alpha)}$$

)(a)(i) $P'(x) = 3x^2 + c$

Let $P'(x) = 0$

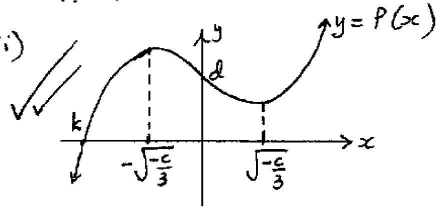
$\therefore\ x^2 = -\frac{c}{3}$

This equation has two real roots if there are two turning points

$\therefore\ -\frac{c}{3} > 0$

$\therefore\ c < 0$

(ii)



(iii) Sum of roots of $P(x)$ is zero.

$\therefore\ 2a + k = 0$

$\therefore\ a = \frac{-k}{2}$

But $k < 0$, so $a > 0$.

(iv) The 3 relationships between the coefficients and zeroes of $P(x)$ are:

$$\begin{cases} 2a + k = 0 & (1) \\ a^2 + b^2 + 2ak = c & (2) \\ k(a^2+b^2) = -d & (3) \end{cases}$$

From (1), $k = -2a$

Substitute into (2) and (3):

$\therefore\ -3a^2 + b^2 = c$ (4)

and $2a(a^2+b^2) = d$ (5)

From (4), $b^2 = 3a^2 + c$

Substitute into (5):

$2a(a^2 + 3a^2 + c) = d$

$\therefore\ d = 8a^3 + 2ac$

(b)(i) In ΔCDE,

$x + y = 90^\circ$ ($\angle$ sum of Δ)

(ii) Since $\angle CPD = \angle CSD = 90^\circ$, quadrilateral CDPS is cyclic (converse of angles in a semicircle).

$\therefore\ \angle CDS = \angle CPS = x$

($\angle$'s at circumference standing on the same arc).

(iii) Similarly, quadrilaterals CERS, CBQP and CAQR are cyclic.

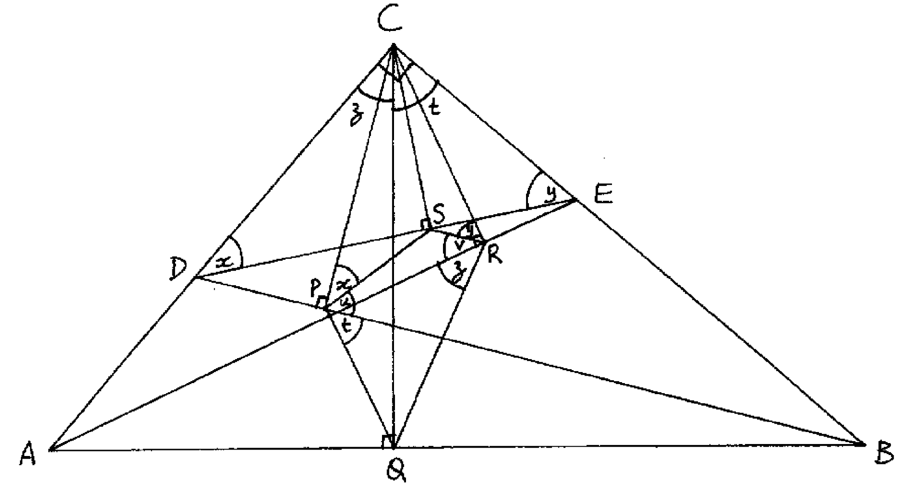
$\therefore\ \angle CES = \angle CRS = y$

and $\angle QCB = \angle QPB = t$

and $\angle ACQ = \angle ARQ = z$

($\angle$'s at the circumference of their respective circles standing on the same arc).

(iv)



Let $\angle SPB = u$ and $\angle SRA = v$.

$x + u = 90^\circ$ and $y + v = 90^\circ$ (adjacent complementary angles)

$\therefore\ x + y + u + v = 180^\circ$

But from part (i), $x + y = 90^\circ$.

$\therefore\ u + v = 90^\circ$

Also, $z + t = 90^\circ$ (since $z + t = \angle ACB$),

so $u + v + z + t = 180^\circ$

i.e. $(u + t) + (v + z) = 180^\circ$

i.e. $\angle SPQ + \angle SRQ = 180^\circ$.

$\therefore$ Quadrilateral PQRS is cyclic (opposite angles supplementary)

(7)(a)(i)

$$I_n = \int_0^1 (1-x^2)^n\,dx$$

$$= \left[x(1-x^2)^n\right]_0^1 - \int_0^1 -2nx^2(1-x^2)^{n-1}\,dx$$

Let $u = (1-x^2)^n$

$\therefore u' = -2nx(1-x^2)^{n-1}$

Let $v' = 1$

$\therefore v = x$

$$= 0 + 2n\int_0^1 x^2(1-x^2)^{n-1}\,dx$$

$$= 2n \cdot J_{n-1}$$

(ii) Continuing from (i), $I_n = 2n\int_0^1 x^2(1-x^2)^{n-1}\,dx$

$$= -2n\int_0^1 \left[(1-x^2)-1\right](1-x^2)^{n-1}\,dx$$

$$= -2n\left(I_n - I_{n-1}\right)$$

$$\therefore I_n(2n+1) = 2n \cdot I_{n-1}$$

$$\therefore I_n = \frac{2n}{2n+1} \cdot I_{n-1}$$

(iii) $J_n = \int_0^1 x^2(1-x^2)^n\,dx$

$$= -\int_0^1 \left[(1-x^2)-1\right](1-x^2)^n\,dx$$

$$= -\left(I_{n+1} - I_n\right)$$

$$= I_n - I_{n+1}$$

$$= I_n - \frac{2n+2}{2n+3} I_n \quad \text{(using (ii))}$$

$$= \frac{2n+3-2n-2}{2n+3} \cdot I_n$$

$$= \frac{1}{2n+3} \cdot I_n$$

(iv) Combining (ii) and (i):

$$J_n = \frac{1}{2n+3} \cdot I_n$$

$$= \frac{2n}{2n+3} \cdot J_{n-1}$$

(7)(b)(i) When $n=1$, $LHS = \frac{1}{2}$

$$RHS = \frac{\sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta} = \frac{1}{2}$$

$\therefore$ the statement is true for $n=1$.

Suppose it's true for $n=k$, where k is a positive integer

i.e. suppose $\frac{1}{2} + \cos\theta + \ldots + \cos(k-1)\theta = \frac{\sin\frac{1}{2}(2k-1)\theta}{2\sin\frac{1}{2}\theta}$

Prove it's true for $n=k+1$.

i.e. prove that $\frac{1}{2} + \cos\theta + \ldots + \cos(k-1)\theta + \cos k\theta = \frac{\sin\frac{1}{2}(2k+1)\theta}{2\sin\frac{1}{2}\theta}$

$$LHS = \frac{\sin\frac{1}{2}(2k-1)\theta}{2\sin\frac{1}{2}\theta} + \cos k\theta \quad \text{(by the assumption)}$$

$$= \frac{\sin\frac{1}{2}(2k-1)\theta + 2\sin\frac{1}{2}\theta\cos k\theta}{2\sin\frac{1}{2}\theta}$$

$$= \frac{\sin\frac{1}{2}(2k-1)\theta + \sin\left(k\theta+\frac{1}{2}\theta\right) - \sin\left(k\theta-\frac{1}{2}\theta\right)}{2\sin\frac{1}{2}\theta}$$

using the identity $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$$= \frac{\cancel{\sin\frac{1}{2}(2k-1)\theta} - \cancel{\sin\frac{1}{2}(2k-1)\theta} + \sin\frac{1}{2}(2k+1)\theta}{2\sin\frac{1}{2}\theta}$$

$$= RHS$$

So the statement is true for $n=k+1$ if it's true for $n=k$. But it's true for $n=1$, so by mathematical induction it's true for all positive integer values of n.

1)(b)(ii)(α) The width of each rectangle is $\frac{\pi}{6n}$.

$$\therefore\ S_n = \frac{\pi}{6n}\left[\cos\frac{\pi}{6n} + \cos\frac{2\pi}{6n} + \cos\frac{3\pi}{6n} + \dots + \cos\frac{(n-1)\pi}{6n} + \cos\frac{\pi}{6}\right]$$

$$= \frac{\pi}{6n}\left[\frac{\sqrt{3}}{2} + \left(\frac{1}{2} + \cos\frac{\pi}{6n} + \cos\frac{2\pi}{6n} + \dots + \cos\frac{(n-1)\pi}{6n}\right) - \frac{1}{2}\right]$$

$$= \frac{\pi}{6n}\left[\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) + \frac{\sin\frac{1}{2}(2n-1)\frac{\pi}{6n}}{2\sin\frac{1}{2}\cdot\frac{\pi}{6n}}\right]$$, using part (i) with $\theta = \frac{\pi}{6n}$,

$$= \frac{\pi}{12n}\left[(\sqrt{3}-1) + \frac{\sin(2n-1)\frac{\pi}{12n}}{\sin\frac{\pi}{12n}}\right]$$, as required.

(β) $\lim\limits_{n\to\infty} S_n = \lim\limits_{h\to 0} S_{\frac{1}{h}}$

$$= \lim_{h\to 0}\left[\frac{h\pi(\sqrt{3}-1)}{12} + \frac{\frac{\pi h}{12}}{\sin\frac{\pi h}{12}}\cdot\sin\left(\frac{2}{h}-1\right)\frac{\pi h}{12}\right]$$

$$= \lim_{h\to 0}\left[\frac{h\pi(\sqrt{3}-1)}{12} + \frac{\frac{\pi h}{12}}{\sin\frac{\pi h}{12}}\cdot\sin\left(\frac{\pi}{6} - \frac{\pi h}{12}\right)\right]$$

$$= 0 + 1.\sin\left(\frac{\pi}{6} - 0\right)$$

$$= \frac{1}{2}$$

(8)(a)(i) $m_{AB} = m_{AC}$

$$\therefore\ \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1}$$

$$\therefore\ x_2y_3 - x_2y_1 - x_1y_3 + \cancel{x_1y_1} = x_3y_2 - x_1y_2 - x_3y_1 + \cancel{x_1y_1}$$

$$\therefore\ x_1y_2 + x_2y_3 + x_3y_1 = x_1y_3 + x_2y_1 + x_3y_2,$$

as required.

(ii) $m_{AB} = \dfrac{y_2 - y_1}{x_2 - x_1}$, $m_{AT} = \dfrac{(u-1)y_1 + vy_2 + wy_3}{(u-1)x_1 + vx_2 + wx_3}$

So the condition for T to lie on the line through A, B and C is:

$$m_{AB} = m_{AT}$$

i.e.

$$(u-1)x_1y_2 + \cancel{vx_2y_2} + wx_3y_2 - (u-1)x_1y_1 - vx_2y_1 - wx_3$$

$$= (u-1)x_2y_1 + \cancel{vx_2y_2} + wx_2y_3 - (u-1)x_1y_1 - vx_1y_2 - wx_1$$

i.e. $(1-u-v)x_1y_2 + wx_2y_3 + wx_3y_1 = wx_1y_3 + (1-u-v)x_2y_1 + wx_3$

i.e. $w(x_1y_2 + x_2y_3 + x_3y_1) = w(x_1y_3 + x_2y_1 + x_3y_2)$ [since $u+v+w=1$

i.e. $x_1y_2 + x_2y_3 + x_3y_1 = x_1y_3 + x_2y_1 + x_3y_2$ [since $w \neq$

which is the condition established in part (i),

so T lies on the line through A, B and C.

(b)(i)(α) $\overrightarrow{AB}$ represents $z_2 - z_1$, so $\overrightarrow{AM}$ represents $\frac{1}{2}(z_2 - z_1)$

(β) $MP = AM\cot\frac{\alpha}{2}$

$\therefore$ $\overrightarrow{MP}$ represents $\frac{1}{2}(z_2 - z_1).\,i\cot\frac{\alpha}{2}$

(ii) $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AM} + \overrightarrow{MP}$,

so P represents $z_1 + \frac{1}{2}(z_2 - z_1) + \frac{1}{2}(z_2 - z_1).\,i\cot\frac{\alpha}{2}$

$$= z_1\left(1 - \frac{1}{2} - \frac{1}{2}i\cot\frac{\alpha}{2}\right) + z_2\left(\frac{1}{2} + \frac{1}{2}i\cot\frac{\alpha}{2}\right)$$

$$= \frac{1}{2}\left(1 - i\cot\frac{\alpha}{2}\right)z_1 + \frac{1}{2}\left(1 + i\cot\frac{\alpha}{2}\right)z_2$$

(c) Suppose we consider the diagram to be drawn in the Argand diagram, and let A, B and C represent the complex numbers z_1, z_2 and z_3 respectively.
Then from (b),

P represents the complex number $\frac{1}{2}\left(1 - i\cot\frac{\alpha}{2}\right)z_1 + \frac{1}{2}\left(1 + i\cot\frac{\alpha}{2}\right)z_2$

and Q " " " " $\frac{1}{2}\left(1 - i\cot\frac{\alpha}{2}\right)z_2 + \frac{1}{2}\left(1 + i\cot\frac{\alpha}{2}\right)z_3$.

Now apply the same result to ΔPRQ, noting that Q – R – P is the clockwise cyclic orientation corresponding to A – P – B and B – Q – C.

$\therefore$ The point R represents the complex number

$$\frac{1}{2}(1-ic)\left[\frac{1}{2}(1-ic)z_2 + \frac{1}{2}(1+ic)z_3\right] + \frac{1}{2}(1+ic)\left[\frac{1}{2}(1-ic)z_1 + \frac{1}{2}(1+ic)z_2\right]$$

(where $c = \cot\frac{\alpha}{2}$) ✓✓

$$= \frac{1}{4}(1-ic)^2 z_2 + \frac{1}{4}(1-ic)(1+ic)z_3 + \frac{1}{4}(1+ic)(1-ic)z_1 + \frac{1}{4}(1+ic)^2 z_2$$

$$= \frac{1}{4}\left(1-c^2-\cancel{2ci}\right)z_2 + \frac{1}{4}(1+c^2)z_3 + \frac{1}{4}(1+c^2)z_1 + \frac{1}{4}\left(1-c^2+\cancel{2ci}\right)z_2$$

$$= \frac{1}{2}(1-c^2)z_2 + \frac{1}{4}(1+c^2)z_3 + \frac{1}{4}(1+c^2)z_1$$ ✓✓

which is of the form $uz_1 + vz_2 + wz_3$, where

$$u+v+w = \frac{1}{4}(1+c^2) + \frac{1}{2}(1-c^2) + \frac{1}{4}(1+c^2)$$
$$= 1.$$

So by part (a), R lies on the line through A, B and C. ✓