03

SYDNEY GRAMMAR SCHOOL — TRIAL EXAMINATION 2003

MATHEMATICS EXTENSION 1

Time allowed: Two hours (plus 5 minutes reading) **Exam date:** 13th August 2003

Instructions:

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the right margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

Each question will be collected separately.
Start each question in a new answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet:

Checklist:

SGS Examination Booklets required — seven 4-page booklets per boy.
Candidature: 120 boys.

QUESTION ONE (Start a new answer booklet)

Marks

(a) Solve the inequation $\frac{1}{x-3} < 3$. **2**

(b) Evaluate $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$, giving your answer in exact form. **2**

(c) Differentiate with respect to x:

(i) $y = \tan^{-1} 2x$ **1**

(ii) $y = \log_e \cos x$ **2**

(d) Find, correct to the nearest degree, the acute angle between the straight lines $y = 3$ and $y = -\frac{5}{3}x + 2$. **2**

(e) Let α, β and γ be the roots of $2x^3 - x^2 + 3x - 2 = 0$. Find the value of **3**

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}.$$

QUESTION TWO (Start a new answer booklet)

Marks

(a) Use the substitution $u = 1 + \tan x$ to evaluate $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{1+\tan x}}\,dx$. **3**

(b) Find the term independent of x in the expansion of $\left(x^2 - \frac{3}{x^2}\right)^6$. **3**

(c) Using the t-substitutions, or otherwise, prove the identity **3**

$$\frac{\tan 2\theta - \tan\theta}{\tan 2\theta + \cot\theta} = \tan^2\theta.$$

(d) An object, always spherical in shape, is increasing in volume at a constant rate of $8\,\text{m}^3/\text{min}$.

(i) Find the rate at which the radius is increasing when the radius is 4 metres. (Note: You may assume the volume formula $V = \frac{4}{3}\pi r^3$). **2**

(ii) Find the rate at which the surface area is increasing when the radius is 4 metres. (Note: You may assume the surface area formula $S = 4\pi r^2$). **1**

QUESTION THREE (Start a new answer booklet)

Marks

(a) Consider the function $f(x) = 3\sin^{-1}(x+1)$.

(i) Write down the domain and the range of $f(x)$. **2**

(ii) Sketch $y = f(x)$, giving the coordinates of its endpoints and any intercepts with the coordinate axes. **2**

(b) A particle moves according to the equation $v^2 = 2x(6-x)$.

(i) Show that the particle moves in the interval $0 \le x \le 6$. **1**

(ii) Write down the centre of the motion. **1**

(iii) Find the maximum speed of the particle. **1**

(iv) Find the acceleration function. **1**

(c) The expression $\left(2 + \dfrac{x}{3}\right)^n$ is expanded. The ratio of the coefficients of the terms in x^6 and x^7 is $7 : 8$. Find the value of n. **4**

QUESTION FOUR (Start a new answer booklet)

Marks

(a) The polynomial $2x^3 + ax^2 + bx + 6$ has $x - 1$ as a factor and leaves a remainder of -12 when divided by $x + 2$. Find the values of a and b. **4**

(b) Given that the equation $x^3 + px^2 + qx + r = 0$ has a triple root, use the sums and products of roots to show that $pq = 9r$. (Hint: Let the roots be α, α and α). **4**

(c) (i) Show that the coefficient of x^5 in the expansion of $(1+x)^4(1+x)^4$ is given by **3**

$$^4C_0 \times {}^4C_1 + {}^4C_1 \times {}^4C_2 + {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_4.$$

(ii) Hence, by equating the coefficients of x^5 on both sides of the identity **1**

$$(1+x)^4(1+x)^4 = (1+x)^8,$$

prove that $^4C_0 \times {}^4C_1 + {}^4C_1 \times {}^4C_2 + {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_4 = \dfrac{8!}{3! \times 5!}$.

QUESTION FIVE (Start a new answer booklet)

(a) The temperature of a body is changing at the rate $\dfrac{dT}{dt} = -k(T-20)$, where T is the temperature at time t minutes and k is a positive constant.

The temperature of the surrounding environment is 20°C. The initial temperature of the body is 36°C and it falls to 35°C in 5 minutes:

Marks

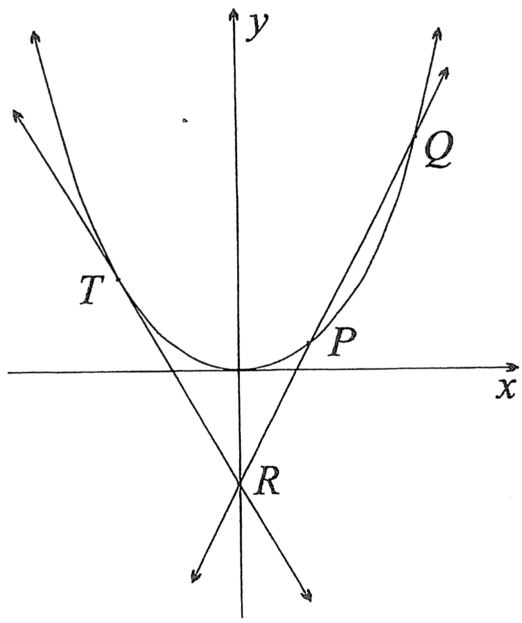
(i) Show that $T = 20 + Ae^{-kt}$ is a solution of $\dfrac{dT}{dt} = -k(T-20)$, where A is a constant. $\boxed{1}$

(ii) Prove that $A = 16$ and $k = -\frac{1}{5}\log_e \frac{15}{16}$. $\boxed{3}$

(iii) Find how long, correct to the nearest minute, it will take the temperature to fall to 27° C. $\boxed{2}$

(iv) Explain why the body will never reach a temperature that is one half of its initial temperature. $\boxed{1}$

(b)



The diagram above shows the parabola $x^2 = 4ay$. The points $T(2at, at^2)$, $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola.

You may assume that the chord PQ has equation $y - \frac{1}{2}(p+q)x + apq = 0$.

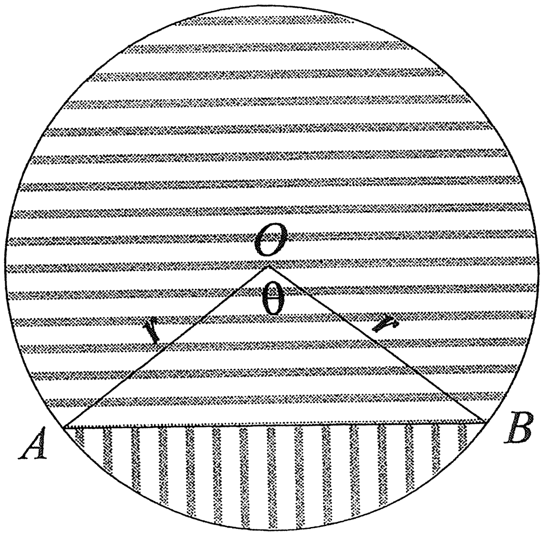
(i) Prove that the equation of the tangent to the parabola at the point $T(2at, at^2)$ is $y - tx + at^2 = 0$. $\boxed{2}$

(ii) Let the tangent at T intersect the axis of the parabola at the point R. Find the coordinates of R. $\boxed{1}$

(iii) Given that the chord PQ also passes through R, show that the parameters p, t and q form a geometric sequence. $\boxed{2}$

QUESTION SIX (Start a new answer booklet)

(a)



In the diagram above, the chord AB subtends an angle of θ radians at the centre O of the circle with radius r.

Marks

(i) Show that the ratio of the areas of the two segments is **2**

$$\frac{\text{area of major segment}}{\text{area of minor segment}} = \frac{2\pi - \theta + \sin\theta}{\theta - \sin\theta}.$$

(ii) Now suppose that

$$\frac{\text{area of major segment}}{\text{area of minor segment}} = \frac{\pi - 1}{1}.$$

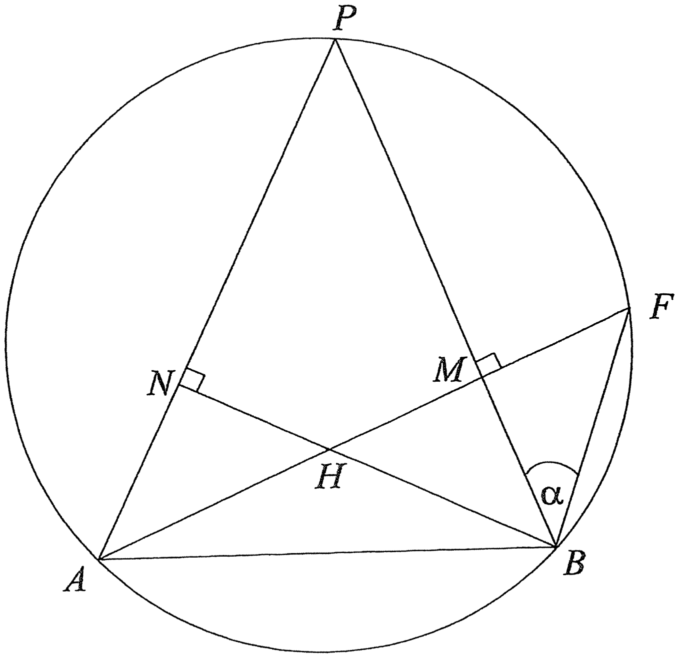
(α) Prove that $\theta - 2 - \sin\theta = 0$. **1**

(β) Show that the equation $\theta - 2 - \sin\theta = 0$ has a root between $\theta = 2$ and $\theta = 3$. **1**

(γ) Taking $\theta = 2{\cdot}5$ as the first approximation, use Newton's method to find a second approximation to the root. Give your answer correct to two decimal places. **1**

(δ) Determine whether the second approximation of θ yields a smaller value of $|\theta - 2 - \sin\theta|$ than the first approximation. **1**

(b)



In the diagram above, ABP is a triangle inscribed in a circle.

The altitudes BN and AM of the triangle intersect at H.

The altitude AM is produced to meet the circumference of the circle at F.

Copy the diagram into your examination booklet.

Let $\angle PBF = \alpha$.

(i) Why is $\angle PAF = \alpha$? $\boxed{1}$

(ii) Why are the points A, N, M, and B concyclic? $\boxed{1}$

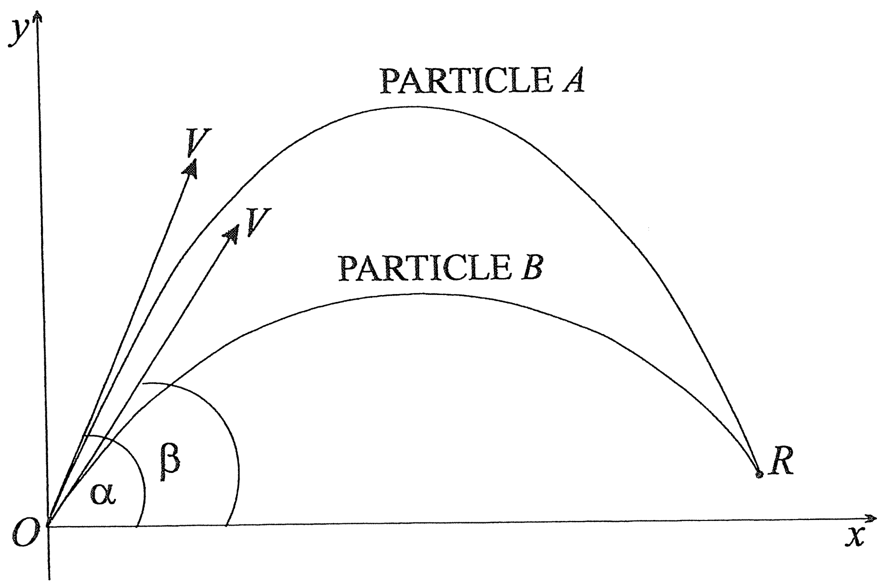
(iii) Why is $\angle NBM = \alpha$? $\boxed{1}$

(iv) Show that M bisects HF. $\boxed{2}$

(v) If AB is a fixed chord of the circle and P moves on the major arc AB, show that α is independent of the position of P. $\boxed{1}$

QUESTION SEVEN (Start a new answer booklet)

(a)



The diagram above shows two particles A and B projected from the origin.

Particle A is projected with initial velocity V m/s at an angle α.

Particle B is projected T seconds later with the same initial velocity V m/s but at an angle of β.

The particles collide at the point R.

Marks

(i) You may assume that the equations of the paths of A and B are: **3**

For A: $y = -\dfrac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha$

For B: $y = -\dfrac{gx^2}{2V^2}\sec^2\beta + x\tan\beta$

Show that the x-coordinate of the point R of collision is

$$x = \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}.$$

(ii) You may assume that the equation of the horizontal displacement of A is

$$x = Vt\cos\alpha.$$

(α) Write down the equation for the horizontal displacement of B. (Remember that B is projected T seconds after A). **1**

(β) Show that the difference T in the times of projection is **2**

$$T = \frac{2V(\cos\beta - \cos\alpha)}{g\sin(\alpha+\beta)}.$$

(b) (i) Prove by mathematical induction that for all positive integers n, **[4]**

$$\sin(n\pi + x) = (-1)^n \sin x.$$

(ii) Let $S = \sin(\pi + x) + \sin(2\pi + x) + \sin(3\pi + x) + \cdots + \sin(n\pi + x)$, for $0 < x < \frac{\pi}{2}$ and for all positive integers n. Show that **[2]**

$$-1 < S \leq 0.$$

GJ

SGS 2003 TRIAL MATHS EXT 1

QUESTION ONE

(a)
$$\frac{1}{x-3} < 3,\ x \neq 3$$
$$\frac{1}{x-3} \times (x-3)^2 < 3(x-3)^2$$
$$x-3 < 3(x-3)^2 \quad \boxed{\checkmark}$$
$$3(x-3)^2 - (x-3) > 0$$
$$(x-3)(3(x-3)-1) > 0$$
$$(x-3)(3x-10) > 0$$
$$x < 3 \text{ or } x > \tfrac{10}{3}. \quad \boxed{\checkmark}$$

(b)
$$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\tfrac{x}{3}\right]_0^3 \quad \boxed{\checkmark}$$
$$= \sin^{-1}1 - \sin^{-1}0$$
$$= \tfrac{\pi}{2}. \quad \boxed{\checkmark}$$

(c) (i)
$$y = \tan^{-1}2x$$
$$\frac{dy}{dx} = \frac{2}{1+4x^2}. \quad \boxed{\checkmark}$$

(ii)
$$y = \log_e \cos x$$
$$\frac{dy}{dx} = -\frac{\sin x}{\cos x} \quad \boxed{\checkmark \text{ for } -\sin x \ \checkmark \text{ for quotient}}$$

(d)
$$\tan\theta = |-\tfrac{5}{3}| \quad \boxed{\checkmark}$$
$$\theta \doteqdot 59^\circ \quad \boxed{\checkmark}$$

(e)
$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma} \quad \boxed{\checkmark}$$
$$= \frac{3}{2} \div 1$$
$$= \frac{3}{2} \quad \boxed{\checkmark\checkmark \text{ any correct method}}$$

QUESTION TWO

(a)
$$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{1+\tan x}}\,dx = \int_1^2 \frac{du}{u^{\frac{1}{2}}} \quad \boxed{\checkmark}$$
$$= \int_1^2 u^{-\frac{1}{2}}\,du$$
$$= \left[2u^{\frac{1}{2}}\right]_1^2 \quad \boxed{\checkmark}$$
$$= 2\sqrt{2}-2 \quad \boxed{\checkmark}$$

Let $u = 1+\tan x$
$du = \sec^2 x\,dx$
When $x=0,\ u=1,$
When $x=\frac{\pi}{4},\ u=2.$

(b) General term
$$= {}^6C_r\,(x^2)^{6-r}\,(-1)^r\,(3x^{-2})^r$$
$$= {}^6C_r\,(x)^{12-2r}\,(-1)^r\,(3)^r\,(x)^{-2r}$$
$$= {}^6C_r\,(-1)^r\,(3)^r\,(x)^{12-4r} \quad \boxed{\checkmark}$$

Let $12-4r=0$
$$r = 3 \quad \boxed{\checkmark}$$

Term independent of x
$$= {}^6C_3\,(-1)^3\,(3)^3$$
$$= -540. \quad \boxed{\checkmark}$$

(c)
$$LHS = \frac{\tan 2\theta - \tan\theta}{\tan 2\theta + \cot\theta}$$

Let $t = \tan\theta$
$$LHS = \left(\frac{2t}{1-t^2} - t\right) \div \left(\frac{2t}{1-t^2} + \frac{1}{t}\right) \quad \boxed{\checkmark}$$
$$= \frac{2t-t+t^3}{1-t^2} \times \frac{t(1-t^2)}{2t^2+1-t^2}$$
$$= \frac{t(1+t^2)}{1-t^2} \times \frac{t(1-t^2)}{t^2+1}$$

$\boxed{\checkmark \text{ correct method of simplification of the algebraic fractions}}$

$$= t^2 \quad \boxed{\checkmark}$$
$$= \tan^2\theta$$
$$= RHS$$

(d) (i)
$$V = \tfrac{4}{3}\pi r^3$$
$$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \boxed{\checkmark}$$
$$8 = 64\pi \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{1}{8\pi}\ \text{m/min} \quad \boxed{\checkmark}$$

(ii)
$$S = 4\pi r^2$$
$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$
$$= 8\pi r \times \frac{1}{8\pi}$$
$$= 4\ \text{m}^2/\text{min}. \quad \boxed{\checkmark}$$

JESTION THREE

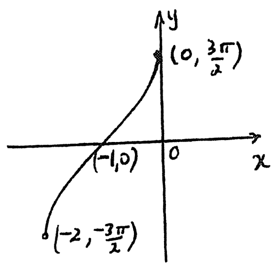
) (i) $f(x) = 3\sin^{-1}(x+1)$

Domain: $-1 \le x+1 \le 1$

$-2 \le x \le 0$ $\boxed{\surd}$

Range: $-\dfrac{3\pi}{2} \le y \le \dfrac{3\pi}{2}$. $\boxed{\surd}$

(ii)



$\boxed{\surd \text{ Shape}}$
$\boxed{\surd \text{ Labels}}$

) (i) $v^2 = 2x(6-x)$

$2x(6-x) \ge 0$

$0 \le x \le 6$ $\boxed{\surd}$

(ii) $x = 3$ $\boxed{\surd}$

(iii) Maximum speed when $x = 3$.

$v^2 = 6 \times 3$

$|v| = 3\sqrt{2}$ $\boxed{\surd}$

(iv) $v^2 = 2x(6-x)$

$\frac{1}{2}v^2 = 6x - x^2$.

$\dfrac{d}{dx}(\frac{1}{2}v^2) = 6 - 2x$

$\ddot{x} = 6 - 2x$ $\boxed{\surd}$

c) Given $\left(2 + \dfrac{x}{3}\right)^n$:

term in $x^6 = {}^nC_6 \times 2^{n-6} \times (\frac{x}{3})^6$

term in $x^7 = {}^nC_7 \times 2^{n-7} \times (\frac{x}{3})^7$ $\boxed{\surd \text{ 1 mark for both answers}}$

$$\text{Ratio of coefficients} = \frac{\dfrac{n!}{6!(n-6)!} \times 2^{n-6} \times (\frac{1}{3})^6}{\dfrac{n!}{7!(n-7)!} \times 2^{n-7} \times (\frac{1}{3})^7}$$

$$= \frac{n!}{6!(n-6)!} \times \frac{7!(n-7)!}{n!} \times 3 \times 2 \quad \boxed{\surd}$$

$$= \frac{42}{n-6} \quad \boxed{\surd}$$

$$\text{so } \frac{7}{8} = \frac{42}{n-6}$$

$$n - 6 = 48$$

$$n = 54. \quad \boxed{\surd}$$

QUESTION FOUR

(a) Let $P(x) = 2x^3 + ax^2 + bx + 6$

$$P(1) = 2 + a + b + 6$$
$$0 = a + b + 8$$
$$a + b = -8 \qquad \cdots(1) \quad \boxed{\surd \text{ for any correct form}}$$
$$P(-2) = -16 + 4a - 2b + 6$$
$$-12 = 4a - 2b - 10$$
$$4a - 2b = -2$$
$$2a - b = -1 \qquad \cdots(2) \quad \boxed{\surd \text{ for any correct form}}$$

(1) + (2)
$$3a = -9$$
$$a = -3 \quad \boxed{\surd}$$
$$b = -5 \quad \boxed{\surd}$$

(b) $x^3 + px^2 + qx + r = 0$

$$3\alpha = -p \qquad \cdots(1) \quad \boxed{\surd}$$
$$3\alpha^2 = q \qquad \cdots(2) \quad \boxed{\surd}$$
$$\alpha^3 = -r \qquad \cdots(3) \quad \boxed{\surd}$$

(1) × (2)
$$9\alpha^3 = -pq$$
$$-9r = -pq$$
$$pq = 9r \quad \boxed{\surd}$$

(c) (i) $(1+x)^4(1+x)^4 = ({}^4C_0 + {}^4C_1x + {}^4C_2x^2 + {}^4C_3x^3 + {}^4C_4x^4)$
$\times ({}^4C_0 + {}^4C_1x + {}^4C_2x^2 + {}^4C_3x^3 + {}^4C_4x^4)$ $\boxed{\surd}$

Term in $x^5 = {}^4C_1x \times {}^4C_4x^4 + {}^4C_2x^2 \times {}^4C_3x^3 + {}^4C_3x^3 \times {}^4C_2x^2 + {}^4C_4x^4 \times {}^4C_1x$ $\boxed{\surd}$

Coefficient $= {}^4C_1 \times {}^4C_4 + {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_2 + {}^4C_4 \times {}^4C_1$
$= {}^4C_0 \times {}^4C_1 + {}^4C_1 \times {}^4C_2 + {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_4$, by symmetry. $\boxed{\surd}$

(ii) Coefficient of x^5 in $(1+x)^8 = {}^8C_5$
$$= \frac{8!}{3! \times 5!} \quad \boxed{\surd}$$

Now $(1+x)^4(1+x)^4 = (1+x)^8$,

so ${}^4C_0 \times {}^4C_1 + {}^4C_1 \times {}^4C_2 + {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_4 = \dfrac{8!}{3! \times 5!}$.

QUESTION FIVE

(a) (i) Given $T = 20 + Ae^{-kt}$

$$\frac{dT}{dt} = -kAe^{-kt}$$
$$= -k(T - 20). \boxed{\checkmark}$$

So $T = 20 + Ae^{-kt}$ is a solution.

(ii) When $t = 0,\ T = 36$

so $36 = 20 + Ae^0$

$A = 16.$ $\boxed{\checkmark}$

When $t = 5,\ T = 35$

so $35 = 20 + 16e^{-5k}$

$$15 = 16e^{-5k}$$
$$e^{-5k} = \frac{15}{16} \boxed{\checkmark}$$
$$-5k = \log_e \frac{15}{16}$$
$$k = -\frac{1}{5}\log_e \frac{15}{16}. \boxed{\checkmark}$$

(iii) When $T = 27$,

$$27 = 20 + 16e^{-kt}$$
$$e^{-kt} = \frac{7}{16} \boxed{\checkmark}$$
$$t = \frac{\log_e \frac{7}{16}}{-k}$$
$$= 64{\cdot}045\ldots..$$

It will take 64 minutes. $\boxed{\checkmark}$

(iv) As $t \to \infty,\ T \to 20$ from above.
The temperature does not drop below 20°C and so will never reach 18°C. $\boxed{\checkmark}$

(b) (i)

$$y = \frac{x^2}{4a}$$
$$\frac{dy}{dx} = \frac{x}{2a}$$

At T, $\dfrac{dy}{dx} = \dfrac{2at}{2a}$

$$= t. \boxed{\checkmark}$$

Now $y - at^2 = t(x - 2at)$

$$y - at^2 = tx - 2at^2$$

so $y - tx + at^2 = 0.$ $\boxed{\checkmark}$

(ii) Let $x = 0$

so $y = -at^2$

R is the point $(0, -at^2)$. $\boxed{\checkmark}$

(iii) R lies on PQ.

$$y - \tfrac{1}{2}(p+q)x + apq = 0$$
$$-at^2 + apq = 0 \boxed{\checkmark}$$
$$t^2 = pq,\ a \neq 0$$
$$\frac{t}{p} = \frac{q}{t}$$

So $p, t,$ and q form a geometric sequence. $\boxed{\checkmark}$

QUESTION SIX

(a) (i) Area of minor segment $= \frac{1}{2}r^2(\theta - \sin\theta)$

Area of major segment $= \pi r^2 - \frac{1}{2}r^2(\theta - \sin\theta)$

$$\text{Ratio of areas} = \frac{\pi r^2 - \frac{1}{2}r^2(\theta - \sin\theta)}{\frac{1}{2}r^2(\theta - \sin\theta)} \quad \boxed{\checkmark}$$

$$= \frac{2\pi - \theta + \sin\theta}{\theta - \sin\theta} \quad \boxed{\checkmark}$$

(ii) (α)

$$\frac{2\pi - \theta + \sin\theta}{\theta - \sin\theta} = \frac{\pi - 1}{1}$$

$$\pi\theta - \pi\sin\theta - \theta + \sin\theta = 2\pi - \theta + \sin\theta$$

$$\theta - 2 - \sin\theta = 0 \quad \boxed{\checkmark}$$

(β) Let $f(\theta) = \theta - 2 - \sin\theta$

$$\begin{aligned} f(2) &= -\sin 2 \\ &\doteqdot -0{\cdot}909 \\ &< 0 \\ f(3) &= 1 - \sin 3 \\ &\doteqdot 0{\cdot}859 \\ &> 0. \end{aligned}$$

So the root lies between $\theta = 2$ and $\theta = 3$, $\boxed{\checkmark}$

(γ) $f(\theta) = \theta - 2 - \sin\theta$

$f'(\theta) = 1 - \cos\theta$.

Let θ_0 be the first approximation.

$$\theta_1 = \theta_0 - \frac{\theta_0 - 2 - \sin\theta_0}{1 - \cos\theta_0}$$

$$\theta_1 = 2{\cdot}5 - \frac{2{\cdot}5 - 2 - \sin 2{\cdot}5}{1 - \cos 2{\cdot}5} \quad \boxed{\checkmark}$$

$$\doteqdot 2{\cdot}55$$

(δ) When $\theta = 2{\cdot}5$,
$|\theta - 2 - \sin\theta| \doteqdot 0{\cdot}09847$.

(ε) When $\theta = 2{\cdot}55$,
$|\theta - 2 - \sin\theta| \doteqdot 0{\cdot}00768$. So $\theta = 2{\cdot}55$ yields a smaller value. $\boxed{\checkmark}$

(b) (i) $\angle PAF = \angle PBF$ angles at circumference standing on the same arc $\boxed{\checkmark}$

$\angle PAF = \alpha$.

(ii) $\angle ANB = \angle AMB$ (both given as rightangles).
These lie on the same interval AB and so A,N,M and B are concyclic. $\boxed{\checkmark}$

(iii) $\angle NBM = \angle MAN$ (angles standing on the same arc of circle $ANMB$) $\boxed{\checkmark}$

$\angle NBM = \alpha$.

(iv) $\triangle BHM \equiv \triangle BFM$ (AAS test) $\boxed{\checkmark}$

$HM = MF$ (matching sides of congruent triangles) $\boxed{\checkmark}$

(v) $\angle APB$ stands on fixed chord AB and its size is independent of the position of P (angles at circumference standing on the same chord). So α is independent of the position of P. $\boxed{\checkmark}$

QUESTION SEVEN

(a) (i) For A: $y = -\frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha \cdots (1)$

For B: $y = -\frac{gx^2}{2V^2}\sec^2\beta + x\tan\beta \cdots (2)$

At R the coordinates are identical, so substitute (1) in (2).

$$-\frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha = -\frac{gx^2}{2V^2}\sec^2\beta + x\tan\beta \quad \boxed{\surd}$$

$$\frac{gx^2}{2V^2}(\sec^2\alpha - \sec^2\beta) = x(\tan\alpha - \tan\beta)$$

$$\frac{gx}{2V^2}(\tan^2\alpha - \tan^2\beta) = (\tan\alpha - \tan\beta),\ x \neq 0 \quad \boxed{\surd}$$

$$\frac{gx}{2V^2} = \frac{(\tan\alpha - \tan\beta)}{(\tan^2\alpha - \tan^2\beta)}$$

$$x = \frac{2V^2}{g} \times \frac{1}{\tan\alpha + \tan\beta},\ \tan\alpha \neq \tan\beta$$

$$= \frac{2V^2}{g} \times \frac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

$$= \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)} \quad \boxed{\surd}$$

(ii) (α) $x = V(t - T)\cos\beta$. $\boxed{\surd}$

(β) When A is at R:

$$Vt\cos\alpha = \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}$$

$$t = \frac{2V\cos\beta}{g\sin(\alpha+\beta)} \quad \cdots (3) \quad \boxed{\surd}$$

When B is at R:

$$V(t - T)\cos\beta = \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}$$

$$t - T = \frac{2V\cos\alpha}{g\sin(\alpha+\beta)}$$

$$T = t - \frac{2V\cos\alpha}{g\sin(\alpha+\beta)}$$

$$= \frac{2V\cos\beta}{g\sin(\alpha+\beta)} - \frac{2V\cos\alpha}{g\sin(\alpha+\beta)},\ \text{from (3)}$$

$$= \frac{2V(\cos\beta - \cos\alpha)}{g\sin(\alpha+\beta)} \quad \boxed{\surd}$$

(b) (i) Prove by mathematical induction the proposition that for all positive integers

$$\sin(n\pi + x) = (-1)^n\sin x,\ \text{for } 0 < x < \tfrac{\pi}{2}.$$

A. When $n = 1$,

$$LHS = \sin(\pi + x)$$
$$= -\sin x$$
$$= RHS.$$

The proposition is true for $n = 1$. $\boxed{\surd}$

B. Assume the proposition is true for some positive integer k so that $\sin(k\pi + x) = (-1)^k\sin x \cdots (*)$

We are required to prove the proposition true for $n = k + 1$.

That is, $\sin[(k+1)\pi + x] = (-1)^{k+1}\sin x$. $\boxed{\surd}$

Now

$$LHS = \sin[(k+1)\pi + x]$$
$$= \sin[\pi + (k\pi + x)] \quad \boxed{\surd}$$
$$= \sin\pi\cos(k\pi + x) + \cos\pi\sin(k\pi + x)$$
$$= -1 \times \sin(k\pi + x)$$
$$= -1 \times (-1)^k\sin x,\ \text{from } (*) \quad \boxed{\surd}$$
$$= (-1)^{k+1}\sin x$$
$$= RHS$$

It follows from A and B by mathematical induction that for all positive integ n, $\sin(n\pi + x) = (-1)^n\sin x$, for $0 < x < \frac{\pi}{2}$.

(ii)

$$S = \sin(\pi + x) + \sin(2\pi + x) + \sin(3\pi + x) + \cdots + \sin(n\pi +$$
$$= -\sin x + \sin x - \sin x + \cdots + \sin(n\pi + x)$$

When n is odd $\quad S = -\sin x$

so $\quad -1 < S < 0$, for $0 < x < \frac{\pi}{2}$. $\boxed{\surd}$

When n is even $\quad S = 0$.

So $\quad -1 < S \leq 0$. $\boxed{\surd}$