FORM 6 MATHEMATICS EXTENSION I TRIAL EXAM 2003

QUESTION 1

a) $\frac{d}{dx}\sin^{-1}2x = \frac{2}{\sqrt{1-4x^2}}$ (1)

b) $\int \frac{5}{2+3x^2}\,dx = \int \frac{5}{3(\frac{2}{3}+x^2)}\,dx$ ✓

$= \frac{5}{3}\cdot\frac{\sqrt{3}}{\sqrt{2}}\tan^{-1}\frac{\sqrt{3}}{\sqrt{2}}x + c$

$= \frac{5}{\sqrt{6}}\tan^{-1}\sqrt{\frac{3}{2}}\,x + c$ ✓ (2)

$\left(= \frac{5\sqrt{6}}{6}\tan^{-1}\frac{\sqrt{6}}{2}x + c\right)$

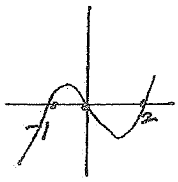
c) $\frac{2}{x} \geq x - 1$

$2x \geq x^3 - x^2$

$x^3 - x^2 - 2x \leq 0$

$x(x^2 - x - 2) \leq 0$

$x(x-2)(x+1) \leq 0$



$x \leq -1$ ✓ or $0 < x \leq 2$ ✓ (2)

d) $y = -x$ $\quad m_1 = -1$

$y = \frac{x}{\sqrt{3}}$ $\quad m_2 = \frac{1}{\sqrt{3}}$

$\tan\alpha = \left|\frac{-1-\frac{1}{\sqrt{3}}}{1+(-1)(\frac{1}{\sqrt{3}})}\right|$ ✓

$= \frac{-\sqrt{3}-1}{\sqrt{3}} \div \frac{\sqrt{3}-1}{\sqrt{3}}$

$= \left|\frac{-\sqrt{3}-1}{\sqrt{3}-1}\right|$ (2)

$\alpha = 75^\circ$ ✓

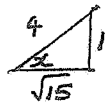
$\therefore$ acute angle between the 2 given lines is 75°.

e) $\cos(\sin^{-1}(-\frac{1}{4})) = \cos(-\sin^{-1}(\frac{1}{4}))$

let $\sin^{-1}(\frac{1}{4}) = x$

$\sin x = \frac{1}{4}$, $\quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$\therefore \cos(\sin^{-1}(-\frac{1}{4}))$ ✓



$= \cos[-x]$

$= \cos x$ (2)

$= \frac{\sqrt{15}}{4}$ ✓

f) $\int_1^2 \frac{1-x}{(1+x)^3}\,dx$

let $u = 1+x$

$du = dx$

$x=1 \quad u=2$

$x=2 \quad u=3$

$x = u-1$

$= \int_2^3 \frac{1-(u-1)}{u^3}\,du$

$= \int_2^3 \frac{2-u}{u^3}\,du$ ✓

$= \int_2^3 \frac{2}{u^3} - \frac{1}{u^2}\,du$

$= \left[\frac{2u^{-2}}{-2} - \frac{u^{-1}}{-1}\right]_2^3$

$= \left[-\frac{1}{u^2} + \frac{1}{u}\right]_2^3$ ✓ (3)

$= -\frac{1}{9} + \frac{1}{3} - \left(-\frac{1}{4} + \frac{1}{2}\right)$

$= -\frac{1}{36}$ ✓

QUESTION 2

a) $\int \sin^2 2x\,dx$

$\cos 4x = 1 - 2\sin^2 2x$

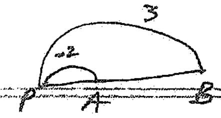
$2\sin^2 2x = 1 - \cos 4x$

$= \frac{1}{2}\int(1-\cos 4x)\,dx$ ✓

$= \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right] + c$

$= \frac{1}{2}x - \frac{1}{8}\sin 4x + c$ (2)

✓



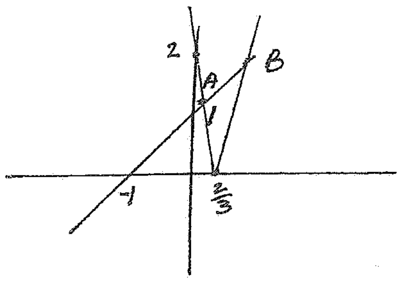
b) $\overset{k}{-2} : \overset{l}{3}$

$$P = \left(\frac{kx_2 + lx_1}{k+l}, \frac{ky_2 + lx_1}{k+l}\right)$$

$$= \left(\frac{-2\times 6 + 3\times 1}{1}, \frac{-2\times 9 + 3\times -4}{1}\right)$$

$$= (-9, -30)$$

c) $|3x - 2| > x + 1$



✓

At A: $-3x + 2 = x + 1$

$-4x = -1$

$x = \frac{1}{4}$

(3)

At B: $3x - 2 = x + 1$ ✓

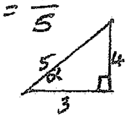
$2x = 3$

$x = \frac{3}{2}$

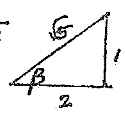
$\therefore |3x - 2| > x + 1$ for

$x < \frac{1}{4}$ or $x > 1\frac{1}{2}$ ✓

d) $\cos\alpha = \frac{3}{5}$



$\sin\beta = \frac{1}{\sqrt{5}}$



$\sin 2\beta = 2\sin\beta\cos\beta$

$= 2 \times \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}}$

$= \frac{4}{5}$ ✓

$\sin\alpha = \frac{4}{5}$ ✓

(2)

$\therefore \sin 2\beta = \sin\alpha$

$\therefore 2\beta = \alpha$

e) $x^3 - 6x^2 + 3x + k = 0$

let roots be $\alpha - d, \alpha, \alpha + d$ ✓

$3\alpha = 6$ ✓

$\alpha = 2$

Sub $\alpha = 2$ in eqn for x: (3)

$8 - 24 + 6 + k = 0$ ✓

$k = 10$

<u>OR</u>

$\alpha(\alpha - d) + \alpha(\alpha + d) + \alpha^2 - d^2 = 3$

$\alpha^2 - \alpha d + \alpha^2 + \alpha d + \alpha^2 - d^2 = 3$

$3\alpha^2 - d^2 = 3$

$12 - d^2 = 3$

$d = \pm 3$

$(\alpha - d)\alpha(\alpha + d) = -k$

$(2 - 3)2(2 + 3) = -k$

$-10 = -k$

$k = 10$.

<u>QUESTION 3</u>

a) $\frac{dV}{dt} = 12 \text{ mm}^3/\text{s}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$ ✓

$4\pi r^2 = 500$

$\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$ ✓

$= \frac{1}{4\pi r^2} \times 12$ ✓

$= \frac{12}{500}$ ✓ (4)

$= \frac{3}{125}$

$\therefore$ radius is increasing at rate of $\frac{3}{125}$ mm/sec.

b) $\sin\theta - \cos\theta = 1$

Let $\tan\frac{\theta}{2} = t$

$\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} = 1$ ✓

$2t - 1 + t^2 = 1 + t^2$

$2t = 2$

$t = 1$

$\therefore \tan\frac{\theta}{2} = 1$

$\frac{\theta}{2} = 45^\circ, 225^\circ \ldots\ldots$

$\theta = 90^\circ, 450 \ldots\ldots$ ✓

Test $\theta = 180^\circ$:

lhs $= 0 - (-1)$ ✓ (4)

$= 1 =$ rhs

$\therefore \theta = 90^\circ + 360n$ or $180^\circ + 360n$

or $\theta = \frac{\pi}{2} + 2\pi n$ or $\pi + 2\pi n$, $n \in J$ ✓

OR Using subsidiary angle method:

$\sin\theta - \cos\theta = \sqrt{2}\sin(\theta - \alpha)$

$\cos\alpha = \frac{1}{\sqrt{2}}$ $\quad \alpha = \frac{\pi}{4}$

$\sqrt{2}\sin(\theta - \frac{\pi}{4}) = 1$

$\sin(\theta - \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$\theta - \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4} \ldots\ldots$

$\theta = \frac{\pi}{2}, \pi \ldots\ldots$

$\theta = \frac{\pi}{2} + 2\pi n$ or $\pi + 2\pi n$

OR square both sides of eqn.

c) Prove

$1\times 2^0 + 2\times 2^1 + 3\times 2^2 + \ldots + n\times 2^{n-1}$

$= 1 + (n-1)2^n$ for $n \geq 1$

Step 1 Prove True for $n=1$

lhs $= 1\times 2^0 = 1$ $\quad$ rhs $= 1 + (1-1)2^1 = 1$ ✓

$\therefore$ True for $n=1$

Step 2 Assume true for $n=k$ ✓

$\therefore 1\times 2^0 + 2\times 2^1 + \ldots k\times 2^{k-1} = 1 + (k-1)2^k$

Step 3 Prove true for $n=k+1$ if true for $n=k$

ie. prove $1\times 2^0 + \ldots k\times 2^{k-1} + (k+1)2^k$

$= 1 + k2^{k+1}$

Proof: lhs $= 1 + (k-1)2^k + (k+1)2^k$

$= 1 + k.2^k - 2^k + k.2^k + 2^k$

$= 1 + 2k.2^k$

$= 1 + k.2^{k+1}$ ✓

$=$ rhs

$\therefore$ True for $n=k+1$ if true for $n=k$

Step 4 Conclusion

Statement true for $n=1$ & true for $n=k+1$ if true for $n=k$. $\therefore$ true for $n = 2, 3, 4 \ldots\ldots$

ie. for all integers ≥ 1

$\therefore$ Proved by induction ✓

(4)

QUESTION 4

a) i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx}$

$$= v\frac{dv}{dx}$$

$$= \frac{dx}{dt}\cdot\frac{dv}{dx}$$

$$= \frac{dv}{dt}$$

$$= \frac{d^2x}{dt^2}$$

(2)

ii) $\frac{d^2x}{dt^2} = n^2(3-x)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = n^2(3-x)$

$\frac{1}{2}v^2 = n^2\left(3x - \frac{x^2}{2}\right) + c$ ✓

$\left.\begin{matrix}x=0\\ v=0\end{matrix}\right]$ $0 = n^2(0-0) + c$

$c = 0$ ✓

$\therefore \frac{1}{2}v^2 = n^2\left(3x - \frac{1}{2}x^2\right)$

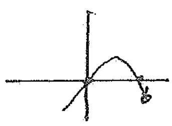
$\therefore \frac{1}{2}v^2 - n^2\left(3x - \frac{1}{2}x^2\right) = 0$ (2)

iii) $\frac{1}{2}v^2 \geq 0$ for all x.

$\therefore 3x - \frac{1}{2}x^2 \geq 0$ ✓

$6x - x^2 \geq 0$

$x(6-x) \geq 0$



$0 \leq x \leq 6$

$\therefore$ particle never moves ✓
outside the interval
$0 \leq x \leq 6$. (2)

b) i) $y = \frac{x+4}{x(x+8)}$ ✓

Vert asymptotes: $x=0$, $x=-8$

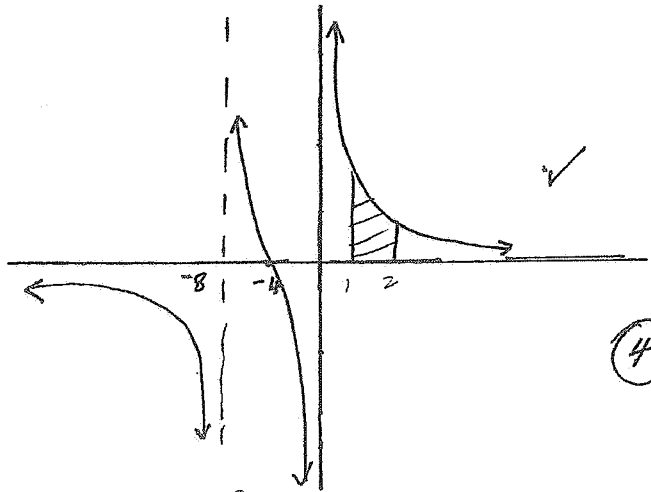
x intercept : $x=-4$ ✓

Horiz asymptote: $y: \lim_{x\to\infty} \frac{x+4}{x^2+8x}$

$= \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{4}{x^2}}{1+8/x}$

$= 0.$ ✓

Sign Diag: $-$ | $+$ | $-$ | $+$ at -8, -4, 0



✓

(4)

ii) Area $= \int_1^2 \frac{x+4}{x^2+8x}\,dx$

$= \frac{1}{2}\left[\log(x^2+8x)\right]_1^2$ ✓

$= \frac{1}{2}(\log 20 - \log 9)$

$= \frac{1}{2}\log\frac{20}{9}$ u^2 ✓ (2)

QUESTION 5

a) $1 - \tan^2 x + \tan^4 x - \ldots$

lim sum exists if $|r| < 1$

ie. $-1 < -\tan^2 x < 1$ ✓

$1 > \tan^2 x > -1$

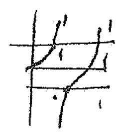
$-1 < \tan^2 x < 1$

But $\tan^2 x$ always +ve

ie $-\tan^2 x$ always > -1

So solve $\tan^2 x < 1$ ✓

ie. $-1 < \tan x < 1$



$0 < x < \frac{\pi}{4}$ or $\frac{3\pi}{4} < x < \frac{5\pi}{4}$

or $\frac{7\pi}{4} < x < 2\pi$ ✓✓ (4)

b) $\frac{d^2x}{dt^2} = -4x$

i) $x = a\cos(2t+\beta)$

$v = -2a\sin(2t+\beta)$

$\ddot{x} = -4a\cos(2t+\beta)$

$= -4x$ (1)

ii) $t=0$ $v=2$ $x=4$.

$4 = a\cos\beta$ — (1) ✓

$2 = -2a\sin\beta$ ✓

$1 = -a\sin\beta$ — (2)

$4^2+1^2 = a^2\cos^2\beta + a^2\sin^2\beta$

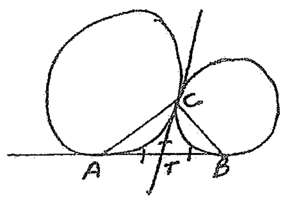
$17 = a^2$

$a = \sqrt{17}$ ✓ (3)

iii) $v = -2\sqrt{17}\sin(2t+\beta)$

$\therefore$ max $v = 2\sqrt{17}$ m/sec. (1)

b)



✓

AT = TC (tangents from ext pt
CT = TB (to circle are equal)

Let $\angle TAC = x$

$\therefore \angle TCA = x$ (base $\angle$s isos Δ TAC)

Let $\angle TCB = y$ ✓

$\therefore \angle TBC = y$ (base $\angle$s isos Δ TCB)

$2x + 2y = 180°$ ($\angle$sum Δ ABC) ✓

$\therefore x + y = 90°$

$\therefore \angle ACB = 90°$

(3)

(−2 marks if use 45°)

QUESTION 6

a) $x = 30t\cos\theta$ (i) $\quad y = 30t\sin\theta - 5t^2$ (ii)

$\dot{y} = 30\sin\theta - 10t$

Max range when $y = 0$

$30t\sin\theta - 5t^2 = 0$

$t(30\sin\theta - 5t) = 0$

$t = 6\sin\theta$ — (iii) ✓

$y = 15$ when $\dot{y} = 0$

$30\sin\theta - 10t = 0$

$t = 3\sin\theta$ ✓

Sub in (ii)

$15 = 30(3\sin\theta)\sin\theta - 5(9\sin^2\theta)$

$15 = 90\sin^2\theta - 45\sin^2\theta$

$15 = 45\sin^2\theta$

$\sin^2\theta = \frac{1}{3}$

$\sin\theta = \pm\frac{1}{\sqrt{3}}$ ✓

But $0 < \theta < \frac{\pi}{2}$, $\therefore \sin\theta = \frac{1}{\sqrt{3}}$ (IV)

Sub iii) & IV) in i)

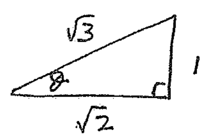
$x = 30(6\sin\theta)\cos\theta$ ✓

$= 180 \times \frac{1}{\sqrt{3}} \times \frac{\sqrt{2}}{\sqrt{3}}$

$= 60\sqrt{2}$

✓

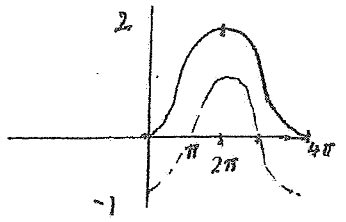
$\therefore$ max horiz range is $60\sqrt{2}$ metres.



(5)

b) $f(x) = 1 - \cos\frac{x}{2}, \; 0 \le x \le a$

i)



Period $= \frac{2\pi}{\frac{1}{2}}$

$= 4\pi$ ✓

Largest a is 2π ✓ (2)

ie. $f^{-1}(x)$ exists for $0 \le x \le 2\pi$

ii) $y = 1 - \cos\frac{x}{2}$

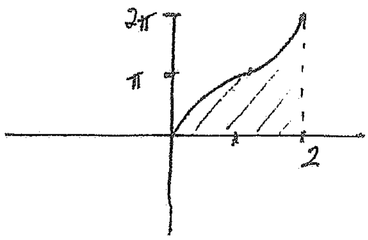
For inv: $x = 1 - \cos\frac{y}{2}$

$\cos\frac{y}{2} = 1 - x$

$\frac{y}{2} = \cos^{-1}(1-x)$

$y = 2\cos^{-1}(1-x)$ ✓ (1)

iii)



(2)

R: $0 \le y \le 2\pi$

D: $-1 \le 1 - x \le 1$

$-2 \le -x \le 0$

$2 \ge x \ge 0$

$x = 2 \quad y = 2\cos^{-1}(-1)$

$= 2\pi$

iv) Reqd area $= \frac{1}{2}$ Area of rect

$= \frac{1}{2}\, 2 \times 2\pi$

$= 2\pi \; u^2$

(2)

OR →

Area $= 4\pi - \int_0^{2\pi} (1 - \cos y/2)\, dy$

$= 4\pi - \left[y - 2\sin y/2\right]_0^{2\pi}$

$= 4\pi - [2\pi - 0 - (0 - 0)]$

$= 2\pi u^2$.

QUESTION 7

a) $x_1 = x_0 - \frac{P(x)}{P'(x)}$ ✓ (1)

i) $x_0 = 1\cdot 8$

ii) $P(x) = f(x) + 5$

$= a\sin x + bx + 5$ ✓

$P(1\cdot 8) = -0\cdot 10$

$\therefore a\sin 1\cdot 8 + 1\cdot 8b + 5 = -0\cdot 10$

or ✓

$a\sin 1\cdot 8 + 1\cdot 8b + 5\cdot 10 = 0$

$P'(1\cdot 8) = -5\cdot 91$

$P'(x) = a\cos x + b$

$\therefore a\cos 1\cdot 8 + b = -5\cdot 91$

✓

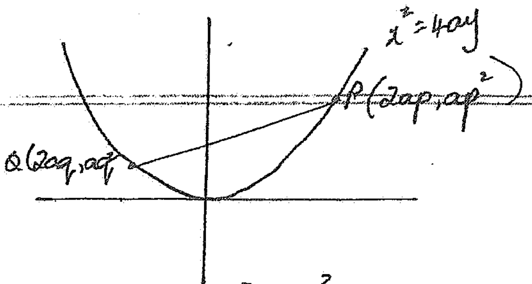
(3)

-1 mark if

$a\sin 1\cdot 8 + b1\cdot 8 = -0\cdot 1$

$a\cos 1\cdot 8 + b = -5\cdot 91$

b)



i) For PQ: $m = \dfrac{ap^2-aq^2}{2ap-2aq}$

$= \dfrac{a(p-q)(p+q)}{2a(p-q)}$

$= \dfrac{p+q}{2}$

$y - ap^2 = \frac{p+q}{2}(x-2ap)$

$2y - 2ap^2 = (p+q)x - 2ap^2 - 2apq$

$2y = (p+q)x - 2apq$

$y = \left(\frac{p+q}{2}\right)x - apq$ ✓

This can be written as

$y = mx + b$

$\therefore -apq = b$

$pq = -b/a$ —— (1) ✓ (2)

ii) $\frac{p+q}{2} = m$ —— (ii) ✓

$\frac{p^2+2pq+q^2}{4} = m^2$

$p^2+q^2+2pq = 4m^2$ (2)

$p^2+q^2+2(-b/a) = 4m^2$ ✓

$p^2+q^2 = 4m^2 + \frac{2b}{a}$

iii) $N\left[-apq(p+q), a(2+p^2+pq+q^2)\right]$

$x = -apq(p+q)$

$x = -a \times \frac{-b}{a}(p+q)$ ✓

$x = b(p+q)$ $\therefore x = b \times 2m$

$= 2bm$

from (ii)

$y = a(2+p^2+pq+q^2)$

$= a\left(2+4m^2+\frac{2b}{a}-\frac{b}{a}\right)$

$= a\left(2+4m^2+\frac{b}{a}\right)$

$= 2a + 4am^2 + b$. ✓ (2)

v) Locus of N:

$x = 2bm$ $\quad y = 2a + 4am^2 + b$

$b = \frac{x}{2m}$ → $y = 2a + 4am^2 + \frac{x}{2m}$

$2my = 4am + 8am^3 + x$

$x - 2my = -4am - 8am^3$

$x + (-2m)y = 2a(-2m) + a(-2m)^3$ ✓

Compare this with equation of normal to parabola at P

$x + py = 2ap + ap^3$

p has been replaced by $-2m$. ✓

$\therefore$ locus of N is a straight line which is a normal to the parabola at point $(2a \times -2m, a(-2m)^2)$

ie. at $(-4am, 4am^2)$.

(2)

$7x - 3y = -41$

$5x - 4y = -33 \quad -②$

$-3y = -41 - 7x$

$3y = 41 + 7x$

$y = \frac{41 + 7x}{3} \quad -①$

sub ① → ②

$5x - 4\left(\frac{41 + 7x}{3}\right) = -33$

$5x - \frac{(164 + 28x)}{3} = -33$

$15x - 164 - 28x = -99$

$-13x = 65$

$x = -5$

$y = \frac{41 + 7(-5)}{3}$

$= 2.$