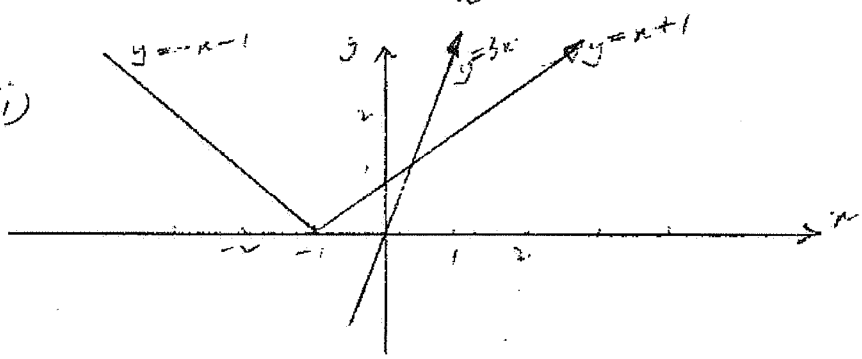
JRAHS 3UNIT TRIAL 2001

QUESTION 1

(a)(i) $y' = 2x\cos x - x^2 \sin x$

(ii) $\int_1^6 \frac{x}{x^2+4}\,dx = \frac{1}{2}\left[\ln(x^2+4)\right]_1^6$
$= \frac{1}{2}(\ln 40 - \ln 5)$
$= \frac{1}{2}\ln 8$

(b)(i)



(ii) $3x = x+1$ (from graph)
$2x = 1$
$x = \frac{1}{2}$

(c)(i) Domain $-\frac{1}{3} \le x \le \frac{1}{3}$
Range $-\pi \le y \le \pi$

(ii) $f(\frac{1}{6}) = 2\sin^{-1}(\frac{3}{6})$
$= \pi/3$

(iii) $f'(x) = 2 \cdot \frac{3}{\sqrt{1-9x^2}}$

$f'(\frac{1}{6}) = \frac{6}{\sqrt{1-9/36}}$

$= \frac{6}{\sqrt{3/4}}$

$= \frac{12}{\sqrt{3}}$ or $4\sqrt{3}$

QUESTION 2.

(a) $P(-7,3)$ $Q(9,15)$ $B(14,0)$

$3:1$

$$A\left(\frac{-7+27}{4}, \frac{3+45}{4}\right) = A(5,12)$$

$$m(PQ) = \frac{15-3}{9+7}$$
$$= 3/4$$

$$m(AB) = \frac{12-0}{5-14}$$
$$= -4/3$$

$$m(PQ) \cdot m(AB) = 3/4 \cdot -4/3$$
$$= -1$$

$\therefore PQ \perp AB$ (prod. of slopes $= -1$)

(b) $x = 0$ $u^2 = 1$
$u = 1$ (take $u > 0$)

$x = 3$, $u^2 = 4$
$u = 2$ (take $u > 0$)

$$u^2 - 1 = x$$
$$\frac{dx}{du} = 2u$$
$$dx = 2u\,du$$

$$\therefore \int_0^3 \frac{x+2}{\sqrt{x+1}}\,dx = \int_1^2 \frac{u^2+1}{\sqrt{u^2}}\,2u\,du$$
$$= 2\int_1^2 \frac{u^2+1}{u}\,du$$
$$= 2\int_1^2 u + \frac{1}{u}\,du$$
$$= 2\left[\frac{1}{2}u^2 + \ln u\right]_1^2$$
$$= 2\left\{\left(\frac{4}{2} + \ln 2\right) - \left(\frac{1}{2} + \ln 1\right)\right\}$$
$$= 3 + 2\ln 2$$

$6\frac{2}{3}$

2(c) $\frac{dt}{dh} = -\frac{1}{k} \cdot h^{-\frac{1}{2}}$

$t = -\frac{1}{k} \cdot 2h^{\frac{1}{2}} + c$

$t = \frac{-2\sqrt{h}}{k} + c$

$t = 0 \quad h = 2500$

$0 = \frac{-100}{k} + c \quad \text{— } (1)$

$t = 5 \quad h = 900$

$5 = \frac{-60}{k} + c \quad \text{— } (2)$

(2) − (1) $\quad 5 = \frac{-60}{k} + \frac{100}{k}$

$5k = 40$

$k = 8$

from (1) $\quad c = \frac{100}{8}$

$\therefore t = \frac{-\sqrt{h}}{4} + 12.5$

when $h = 0 \quad t = 12.5$

$\therefore$ extra time $= 12.5 - 5$

$= 7.5$ min.

QUESTION 3.

(a) $T_{r+1} = {}^6C_r \, (3x)^{6-r} \left(\frac{2}{\sqrt{x}}\right)^r$

$= {}^6C_r \, 3^{6-r} \, 2^r \cdot x^{6-r} \, x^{-\frac{1}{2}r}$

$= {}^6C_r \, 3^{6-r} \, 2^r \, x^{6-\frac{3}{2}r}$

for constant term degree of $x = 0$

$\therefore \; 6 - \frac{3}{2}r = 0$

$\frac{1}{2}r = 6$

$r = 4$

$\therefore$ constant $= {}^6C_4\, 3^2 \cdot 2^4$

$= 2160$

(b) (i) 1 1 4 3 2 1

E A

Prob $= \frac{4!}{6!}$ or Prob $= \frac{1}{6}\cdot\frac{1}{5}$

$= \frac{1}{30}$ $= \frac{1}{30}$

(ii)

C E

3 4 3 2 1 2

Prob $= \frac{3.2.4!}{6!}$

$= \frac{1}{5}$

(iii) 2 3 4 2 3 1

G B G B G B

(Place Boys 1st then fill gaps with girls)

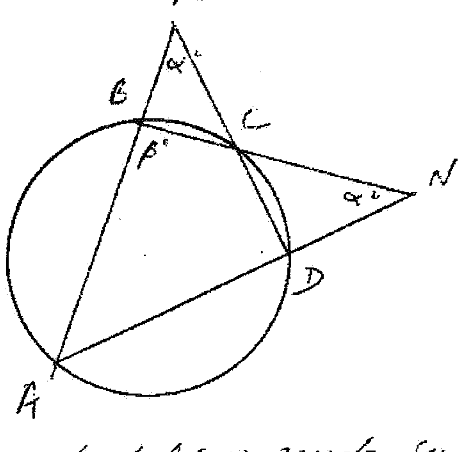
Prob $= \frac{3!\,4.3.2}{6!}$

$= \frac{1}{5}$

(C) (i) let $A\hat{M}D = A\hat{N}B = \alpha^\circ$

& $A\hat{B}C = \beta^\circ$

$B\hat{C}M = (\beta - \alpha)^\circ$ (exterior angle of $\triangle BMC$ equals sum of opposite interior angles)



$D\hat{C}N = (\beta - \alpha)^\circ$ (vertically opposite angles)

$A\hat{D}C = \beta^\circ$ (exterior angle of $\triangle CND$ equals sum of opposite interior angles

$\therefore A\hat{B}C = A\hat{D}C$ (Q if AC)

Q3(c)(ii) $A\hat{B}C + A\hat{D}C = 180°$ (opposite angles of cyclic quad ABCD are supplementary)

$2A\hat{B}C = 180°$ ($A\hat{B}C = A\hat{D}C$, part (i))

$A\hat{B}C = 90°$

$\therefore$ AC is a diameter (angle in semicircle is $90°$)

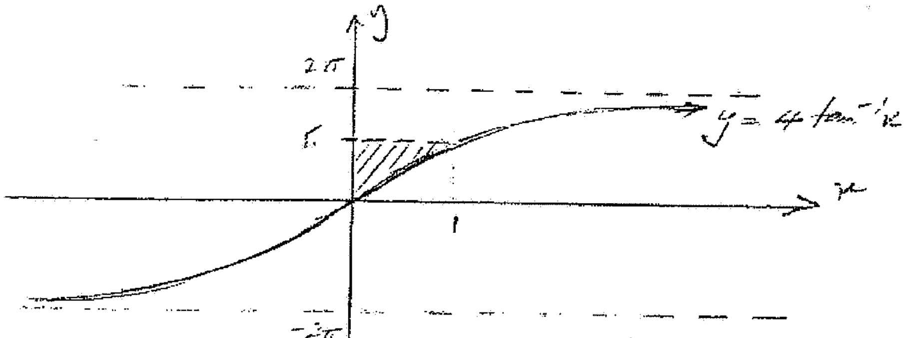
QUESTION 4

(a)(i) $\dfrac{\sin^2 A}{\cos^2 A} + \dfrac{\cos^2 A}{\cos^2 A} = \dfrac{1}{\cos^2 A}$

$\tan^2 A + 1 = \sec^2 A$

$\tan^2 A = \sec^2 A - 1$

(ii)



Range $-2\pi < y < 2\pi$

(iii) $y/4 = \tan^{-1} x$

$x = \tan y/4$

$$V = \pi \int_0^{\pi} x^2 \, dy$$

$$= \pi \int_0^{\pi} \tan^2 y/4 \, dy$$

$$= \pi \int_0^{\pi} \sec^2 y/4 - 1 \, dy$$

$$= \pi \Big[4 \tan y/4 - y \Big]_0^{\pi}$$

$$= \pi \Big\{ (4 \tan \pi/4 - \pi) - (4 \tan 0 - 0) \Big\}$$

$$= \pi(4 - \pi) \ u^3.$$

(b)(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right) \cdot \frac{dv}{dx}$

$= v\frac{dv}{dx}$

$= \frac{dx}{dt} \cdot \frac{dv}{dx}$

$= \frac{dv}{dt}$

$= \ddot{x}$

(ii)(α) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 4x$

$\frac{1}{2}v^2 = \frac{x^4}{2} + 2x^2 + c$

$t = 0,\ x = 2,\ v = 6$

$18 = 8 + 8 + c$

$c = 2$

$v^2 = x^4 + 4x^2 + 4$

(β) $v^2 = (x^2+2)^2 \quad \therefore v^2 \geq 4 \quad \therefore v \neq 0$

$\therefore$ object never changes direction

$\therefore$ always moves to right with increasing speed

since initial vel > 0 & accel > 0 for $x > 0$

$\therefore$ min speed is initial speed

$\therefore$ min. speed $= 6\ ms^{-1}$

QUESTION 5

(a) $y' = -2e^{-2x}$

when $x = 0$, $y' = -2e^0$

$\therefore m_1 = -2$

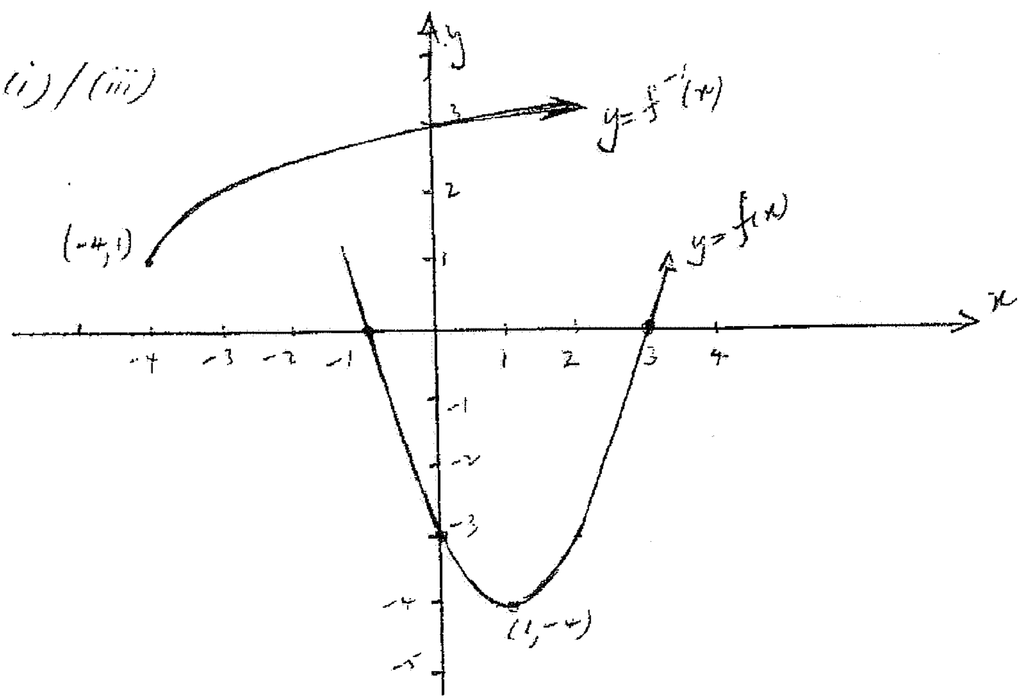
$m_2 = 3$

$\tan\theta = \left|\frac{3+2}{1+(3)(-2)}\right|$

$= 1$

$\theta = \pi/4$, or 45°

(b) (i)/(iii)

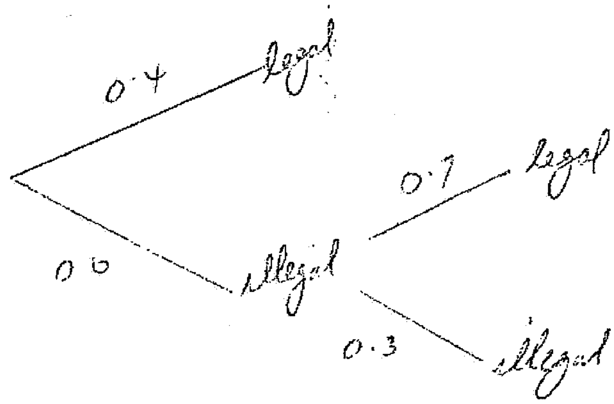


(ii) $x \geq 1$

(iii) see graph.

(c)

(i)



$$P(\text{double fault}) = 0.6 \times 0.3$$
$$= 0.18$$

(ii) $(0.82 + 0.18)^6$

$P(\text{at least 2 double faults}) = 1 - \{P(\text{0 double faults}) + P(\text{1 double fault})$

$= 1 - \{ {}^6C_0 (0.82)^6 (0.18)^0 + {}^6C_1 (0.82)^5 (0.18)^1 \}$

$\doteqdot 0.30.$

QUESTION 6.

(a) $\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$= \frac{1}{4\pi r^2} \cdot 10$

$= \frac{10}{4\pi r^2}$

when $SA = 500$ $(= 4\pi r^2)$

$\frac{dr}{dt} = \frac{10}{500}$

$= \frac{1}{50}$ cm/s.

(b) when $n=1$, $LHS = 2(1!)$ $= 2$ $\quad RHS = 1(2!)$ $= 2$.

$\therefore$ true for $n=1$

assume true for $n=k$

ie. $2(1!) + 5(2!) + \cdots + (k^2+1)k! = k(k+1)!$

to prove true for $n=k+1$

ie $2(1!) + 5(2!) + \cdots + (k^2+1)k! + [(k+1)^2+1](k+1)!$
$= (k+1)(k+2)!$

Now $LHS = 2(1!) + 5(2!) + \cdots + (k^2+1)k! + (k^2+2k+2)(k+1)!$

$= k(k+1)! + (k^2+2k+2)(k+1)!$ (by assumption) Q??

$= (k+1)!\{k + k^2 + 2k + 2\}$

$= (k+1)!(k^2+3k+2)$

$= (k+1)!(k+2)(k+1)$

$= (k+2)!(k+1)$

$= RHS$

$\therefore$ if true for $n=k$ then true for $n=k+1$ &
since true for $n=1$ then true for all
$n \geq 1$.

(5)

Q6(c) $\dfrac{dy}{dx} = \dfrac{(x)(1/x) - (1)(\ln x)}{x^2}$

$= \dfrac{1 - \ln x}{x^2}$

$$\int_e^{e^2} \frac{1-\ln x}{x\ln x}\,dx = \int_e^{e^2} \frac{\frac{1-\ln x}{x^2}}{\frac{\ln x}{x}}\,dx$$

$$= \left[\ln\left(\frac{\ln x}{x}\right)\right]_e^{e^2}$$

$$= \ln\left(\frac{\ln e^2}{e^2}\right) - \ln\left(\frac{\ln e}{e}\right)$$

$$= \ln(2/e^2) - \ln(1/e)$$

$$= \ln\left(\frac{2}{e^2} \times \frac{e}{1}\right)$$

$$= \ln(2/e)$$

$$= \ln 2 - \ln e$$

$$= \ln 2 - 1$$

QUESTION 7

(i) $\sin(x+y) - \sin(x-y) = (\sin x\cos y + \cos x\sin y) - (\sin x\cos y - \cos x\sin y)$

$= 2\cos x\sin y$ (1)

let $x+y = A$ & $x-y = B$

$\therefore 2x = A+B$ $\quad\quad 2y = A-B$

$x = \dfrac{A+B}{2}$ (1) $\quad\quad y = \dfrac{A-B}{2}$ (1)

$\therefore \sin A - \sin B = 2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$

(ii) $\dfrac{\sin A - \sin B}{\cos A - \cos B} = \dfrac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)}$ (1)

$= \dfrac{\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{-\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$ (1)

$$= -\cot\left(\frac{A+B}{2}\right)$$

(iii) $\ddot{x} = 0$ $\qquad$ $\ddot{y} = -g$

$\dot{x} = c_1$ $\qquad$ $\dot{y} = -gt + c_3$

$t=0,\ \dot{x} = V\cos\alpha$ $\qquad$ $t=0,\ \dot{y} = V\sin\alpha$

$\therefore V\cos\alpha = c_1$ $\qquad$ $V\sin\alpha = c_3$

$\dot{x} = V\cos\alpha$ $\qquad$ $\dot{y} = -gt + V\sin\alpha$

$x = Vt\cos\alpha + c_2$ $\qquad$ $y = -\frac{1}{2}gt^2 + Vt\sin\alpha + c_4$

$t=0,\ x=0 \ \therefore c_2 = 0$ $\qquad$ $t=0,\ y=0 \ \therefore c_4 = 0$

$x = Vt\cos\alpha$ ① $\qquad$ $y = -\frac{1}{2}gt^2 + Vt\sin\alpha$ ①

(iv)(α) Particle P

$x_P = Vt\cos\alpha \qquad y_P = -\frac{1}{2}gt^2 + Vt\sin\alpha$

Particle Q

$x_Q = Vt\cos\beta \qquad y_Q = -\frac{1}{2}gt^2 + Vt\sin\beta$

$\tan\theta$ = slope PQ ①

$$= \left|\frac{(-\frac{1}{2}gt^2 + Vt\sin\beta) - (-\frac{1}{2}gt^2 + Vt\sin\alpha)}{Vt\cos\beta - Vt\cos\alpha}\right|$$

$$= \left|\frac{Vt(\sin\beta - \sin\alpha)}{Vt(\cos\beta - \cos\alpha)}\right| \quad ①$$

$$= \left|\frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha}\right| \quad ①$$

(β) $\tan\theta = \left|-\cot\left(\frac{\alpha+\beta}{2}\right)\right|$ ECF from (ii) ECF

$$= \tan\left(\frac{\pi}{2} - \left(\frac{\alpha+\beta}{2}\right)\right)$$ ① α, β, θ acute

$$\therefore \theta = \frac{\pi}{2} - \left(\frac{\alpha+\beta}{2}\right) \quad ①$$

$$\theta = \frac{1}{2}(\pi - \alpha - \beta)$$