St Catherine's School

Year: 12
Subject: 3 Unit Mathematics
Time Allowed: 2 hours (plus 5 mins reading time)
Date: August 2000

Exam number:_______________

Directions to candidates:

- All questions are to be attempted.
- All questions are of equal value.
- All necessary **working** must be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Each question attempted should be started on a **new page.**
- Approved calculators and geometrical instruments are required.
- Attach the question paper to the front of **Section A.**
- Write a cover page for **Section B and C** and include your number.
- Hand in your work in **3 bundles:**
  Section A Questions 1, 2 and 3.
  Section B Questions. 4 and 5
  Section C Questions. 6 and 7.

TEACHER'S USE ONLY Total Marks
A
B
C
TOTAL

Section A

Question 1

a) Differentiate $e^{2x}\sin x$

b) Find the acute angle between the lines $2x+y=4$ and $x-y=2$

c) A committee of 3 men and 4 women is to be formed from a group of 8 men and 6 women. Write an expression for the number of ways this can be done.

d) Evaluate $\int_0^2 \frac{dx}{4+x^2}$

e) Using the substitution $u=2x+1$ or otherwise, find $\int_0^1 \frac{4x}{2x+1}\,dx$

Question 2

a) A particle is moving in simple harmonic motion. It's displacement, x, at time, t, is given by $x=3\sin(4t+\frac{\pi}{4})$.

i) find the period and amplitude of the motion

ii) find the velocity of the particle when t = 0.

iii) find the maximum acceleration of the particle.

iv) find the speed of the particle when $x = 2$

b) The polynomial $P(x)=x^3+bx^2+cx+d$ has roots at 0, 3 and -3.

i) find b, c and d

ii) without using calculus, sketch the graph of $y=P(x)$

iii) Hence or otherwise solve the inequality $\frac{x^2-9}{x}\geq 0$.

Question 3

a) i) Find $\frac{dy}{dx}$ if $y = \tan^{-1}(\sin x)$ **2**

ii) Evaluate $\int_0^1 \frac{dx}{\sqrt{2-x^2}}$ **2**

b) A cup of hot coffee at temperature T degrees Celsius loses heat when placed in a cooler environment. It cools according to the law $\frac{dT}{dt} = k(T - T_0)$ where time, t is the time elapsed in minutes and T_0 is the temperature of the environment in degrees Celsius.

i) A cup of coffee at 100 ^0C is placed in an environment at -20 ^0C for 3 minutes and then cools to 70 ^0C. Find k. **2**

ii) The same cup of coffee at 70^0C is then placed in an environment at 20 ^0C assuming k stays the same, find the temperature of the coffee after a further 15 minutes. **3**

c) The function *h(x)* is given by $h(x) = \sin^{-1} x + \cos^{-1} x$ for $-1 \le x \le 1$.

i) show that $h'(x) = 0$ **1**

ii) sketch the graph of $y = h(x)$ **2**

SECTION B (Start a new page)

Question 4

a) A spherical balloon is expanding so that its volume is increasing at the constant rate of $10\,mm^3$ per second. What is the rate of increase of the radius when the surface area is $500\,mm^2$.
$(V = \frac{4}{3}\pi r^3 \quad SA = 4\pi r^2)$ **4**

b) Find the constant term in the expansion of $(3x^2 - \frac{1}{2x})^9$. **4**

c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The equation of chord PQ is given by $y - ap^2 = \frac{p+q}{2}(x - 2ap)$.

i) If PQ is a focal chord show that $pq = -1$ **1**

ii) Find M, the midpoint of PQ. **1**

iii) Find the equation of the locus of M **2**

Question 5

A dangerous fire is burning in a low open tank on horizontal ground. Fire fighters are forced to stay 60m away from the fire. They are using a pump which is on the ground and can eject water at *30m/s* at any angle to the horizontal, α.
(Assume that g = *10m/s/s* and that all frictional forces, including air resistance, can be neglected.)

a) Show that the expression for the vertical motion is $y = -5t^2 + 30t\sin\alpha$ **$1\frac{1}{2}$**

b) Show that the expression for horizontal motion is $x = 30t\cos\alpha$ **$1\frac{1}{2}$**

c) Show that the range of the projectile is given by $x = 90\sin 2\alpha$ **2**

d) Find the maximum horizontal distance the pump can reach. **1**

e) Find the two ~~angle~~ angles of projection needed for the pumped water to reach the fire. **3**

f) Another other pump is on a vertical stand 5m high and can eject water at *40m/s* but only horizontally. Can this pump reach the fire? Justify your answer. **3**
(You may use the formulas for the horizontal distance; $x = Vt\cos\alpha$ and vertical distance $y = -\frac{1}{2}gt^2 + Vt\sin\alpha$, where V is the initial velocity and α is the angle of projection and g=10m/s/s).

SECTION C (Start a new page)

Question 6

a) i) Show that $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 x \, dx = \frac{\pi}{8} - \frac{1}{4}$ **2**

ii) The function $g(x)$ is given by $g(x) = 2 + \cos x$. The graph $y = g(x)$ for $\frac{\pi}{4} \le x \le \frac{\pi}{2}$ is rotated about the x axis. Find the volume of the solid generated. (You may use the result of a(i)). Give your answer in exact form. **3**

b) The velocity of a point moving along the x axis is given by $v^2 = 16x - 4x^2 + 20$.

i) Show that $\ddot{x} = -4(x-2)$ **2**

ii) State the centre of motion **1**

iii) What is the amplitude of the motion **2**

iv) What is the period of the motion **1**

v) Find the maximum speed of the particle **1**

Question 7

a) If $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$ show that $\sum_{r=1}^{n} r \binom{n}{r} = n\,2^{n-1}$ **3**

b) Consider the function $f(x) = (x-2)^2 + 1$

i) Sketch the parabola $y = f(x)$, showing clearly any intercepts with the axes, and the coordinates of its vertex. Use the same scale on both axes. **1.5**

ii) What is the largest domain containing the value $x = 3$, for which the function has an inverse function $f^{-1}(x)$? **1**

iii) Sketch the function $y = f^{-1}(x)$ on the same set of axes as your graph in part(i). Label the two graphs clearly. **1.5**

(iv) What is the domain of the inverse function? **1**

(v) Let a be a real number not in the domain found in part (ii). Find $f^{-1}[f(a)]$. **2**

(vi) Find the x coordinate of any points of intersection of the two curves $y = f(x)$ and $y = f^{-1}(x)$. **2**

END OF EXAMINATION

Table of Standard Integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \ne -1; \quad x \ne 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \ne 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \ne 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \ne 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \ne 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \ne 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \ne 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$