XXX 2007 TRIAL

(a)(i) $\frac{a}{x} + \frac{bx+c}{4+x^2} = \frac{4a + ax^2 + bx^2 + cx}{x(4+x^2)}$ 1

$\therefore 4a = 1 \Rightarrow a = \frac{1}{4}$ #

$a + b = 0 \quad b = -\frac{1}{4}$ # 1

$c = 0$ #

(ii) $\int \frac{dx}{x(4+x^2)} = \int \frac{dx}{4x} + \int \frac{-\frac{1}{4}x\,dx}{4+x^2}$

$= \frac{1}{4}\ln|x| - \frac{1}{8}\int \frac{2x}{x^2+4} + c$

$= \frac{1}{4}\ln|x| - \frac{1}{8}\ln|x^2+4| + c$ # 2

(b) $\int_0^2 x\sqrt{2-x}\,dx$ $\quad u = x \quad dv = \sqrt{2-x}$

$du = dx \quad v = -\frac{2}{3}(2-x)^{3/2}$

$= \left[-\frac{2}{3}x(2-x)^{3/2}\right]_0^2 + \frac{2}{3}\int_0^2 (2-x)^{3/2}dx$ 1

$= 0 + \frac{2}{3}\cdot\left[(2-x)^{5/2}\left(-\frac{2}{5}\right)\right]_0^2$ 1

$= \frac{4}{15}2^{5/2}$ # or $\frac{16\sqrt{2}}{15}$ # 1

(c) Since all coeff are real and $2-i$ is a zero, $2+i$ is also a zero 1

$(x-2+i)(x-2-i) = x^2 - 4x + 5$

$$\begin{array}{r} x^2 - x - 2 \\ x^2-4x+5 \overline{\smash{)}\, x^4 - 5x^3 + 7x^2 + 3x - 10} \\ \underline{x^4 - 4x^3 + 5x^2} \\ -x^3 + 2x^2 + 3x \\ \underline{-x^3 + 4x^2 - 5x} \\ -2x^2 + 8x - 10 \\ \underline{-2x^2 + 8x - 10} \end{array}$$ 1

$x^2 - x - 2 = (x-2)(x+1)$

Zeros are $2, -1, 2+i, 2-i$ # 1

d) i) $I_{2n+1} = \int_0^1 x^{2n+1} e^{x^2} dx$ $\quad u = x^{2n} \quad v' = xe^{x^2}$

$\frac{du}{dx} = x^{2n-1}\cdot 2n \quad v = \frac{e^{x^2}}{2}$

$= \int_0^1 x^{2n}\cdot x\cdot e^{x^2} dx$

$= \left[\frac{x^{2n}e^{x^2}}{2}\right]_0^1 - \int_0^1 \frac{1}{2}e^{x^2}2n x^{2n-1} dx$ 1

$= \frac{e}{2} - n\int_0^1 x^{2n-1}e^{x^2}dx$

$= \frac{1}{2}e - nI_{2n-1}$ # 1

ii) $I_5 = \frac{e}{2} - 2I_3 = \frac{e}{2} - 2\left[\frac{e}{2} - I_1\right]$
(n=2)

$I_5 = \frac{e}{2} - e + 2I_1 = -\frac{e}{2} + 2I_1$ 1

$I_1 = \int_0^1 xe^{x^2}dx = \frac{1}{2}\int_0^1 2xe^{x^2}dx = \left[\frac{e^{x^2}}{2}\right]_0^1 = \frac{1}{2}(e-1)$ 1
(n=0)

$\therefore I_5 = -\frac{e}{2} + \frac{2}{2}(e-1) = -\frac{e}{2} + e - 1 = \frac{e}{2} - 1$ # 1

Question 2

a(i) Let $z = x + yi$

$z^2 = x^2 + 2xyi - y^2 = -3 - 4i$ 1

$x^2 - y^2 = -3$ (1)

$2xy = -4 \therefore xy = -2 \Rightarrow y = -\frac{2}{x}$ (2)

$x^2 - \frac{4}{x^2} = -3$

$x^4 + 3x^2 - 4 = 0$

$(x^2+4)(x^2-1) = 0 \quad \therefore x = \pm 2i$ or ± 1 1

But $x \in R \therefore x = \pm 1$

when $x = 1, \quad y = -2$

$x = -1, \quad y = 2$

$\therefore z = 1 - 2i$ or $-1 + 2i$ # 1

b(i) $x = a\cos\theta \quad \frac{dx}{d\theta} = -a\sin\theta$
$y = b\sin\theta \quad \frac{dy}{d\theta} = b\cos\theta$ $\quad \frac{dy}{dx} = \frac{-b\cos\theta}{a\sin\theta}$ 1

Eq of OQ passing thru (0,0).

$y = \frac{-b\cos\theta}{a\sin\theta}x$

$b\cos\theta x + a\sin\theta y = 0$ is the eq of OQ 1

P is in 1^{st} Quad $\therefore a > 0, \ b > 0$

For point Q

$$\frac{x^2}{a^2} + \frac{\cancel{b^2}\cos^2\theta}{\cancel{b^2}a^2\sin^2\theta}x^2 = 1 \qquad 1$$

$$\frac{x^2}{a^2} + \frac{\cos^2\theta}{a^2\sin^2\theta}x^2 = 1$$

$$x^2(\sin^2\theta + \cos^2\theta) = a^2\sin^2\theta$$

$$x^2 = a^2\sin^2\theta$$

But Q is in 4th Quad ∴ $x_Q > 0$ 1

$$x_Q = a\sin\theta$$

$$y_Q = -\frac{b}{\cancel{a}}\frac{\cos\theta}{\cancel{\sin\theta}}\cancel{(a\sin\theta)}$$

$$y_Q = -b\cos\theta$$

∴ $Q = (a\sin\theta, -b\cos\theta)$ 1

(iii) Perpendicular distance from Q to OP

$$d = \frac{|ab\cos^2\theta + ab\sin^2\theta|}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \qquad 1$$

$$d = \frac{|ab|}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$\text{Area } \triangle OQP = \frac{1}{2} \times \frac{|ab|}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \times OQ$$

$$= \frac{|ab|}{2\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \times \sqrt{a^2\sin^2\theta + b^2\cos^2\theta} \qquad 1$$

$$= \frac{|ab|}{2}$$ independent of position of P 1

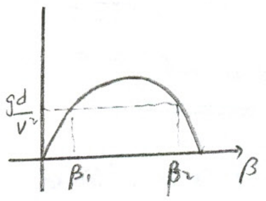
#

Q.3

a) For the particle to hit the wall, the wall must be at a distance d less than the range for angle β

$$d < \frac{V^2\sin 2\beta}{g} \qquad 1$$

$$\sin 2\beta > \frac{gd}{V^2}$$



at β_1, β_2 $\sin 2\beta = \frac{gd}{V^2}$

$$2\beta = n\pi + (-1)^n\sin^{-1}\left(\frac{gd}{V^2}\right) \qquad 1$$

$$\beta = \frac{n\pi}{2} + \frac{1}{2}(-1)^n\sin^{-1}\left(\frac{gd}{V^2}\right)$$

n=0 $\beta_1 = \frac{1}{2}\sin^{-1}\left(\frac{gd}{V^2}\right)$

n=1 $\beta_2 = \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{gd}{V^2}\right) = \frac{\pi}{2} - \beta_1$

∴ the particle will hit wall at an angle α such that

$$\frac{1}{2}\sin^{-1}\left(\frac{gd}{V^2}\right) < \alpha < \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{gd}{V^2}\right) \qquad 1$$

b) $z = \frac{\sqrt{2}}{1-i} \times \frac{1+i}{1+i} = \frac{\sqrt{2}(1+i)}{2} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ 1

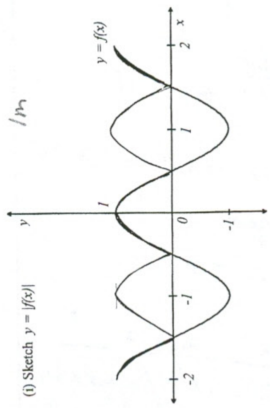
$$z = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\ \text{cis}\frac{\pi}{4}$$

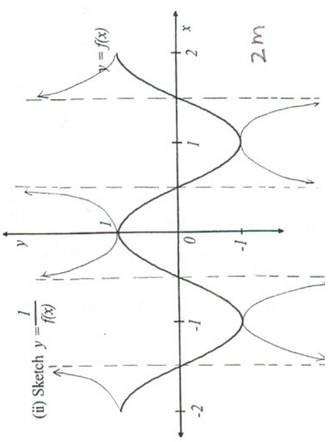
$z = 1.\ \text{cis}\frac{\pi}{4}$ 1

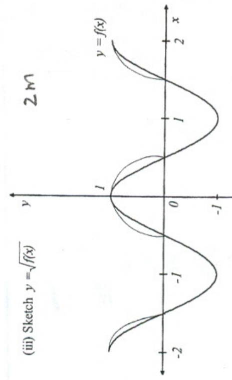
$z^5 = 1^5\ \text{cis}\frac{5\pi}{4} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$ 1

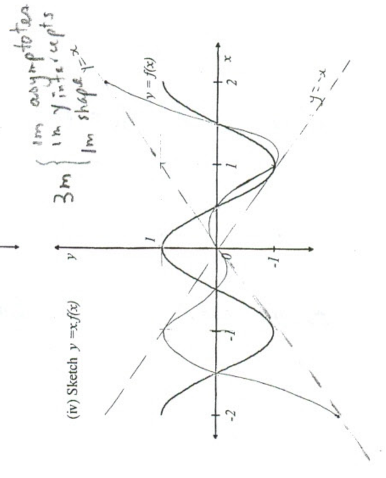
$z^5 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

\# 1









Question 4

(a) $u = \ln x \quad du = \frac{1}{x}dx$

$\int \frac{du}{u^2} = -u^{-1} + c = -\frac{1}{\ln x} + c$ #

(b) No. of ways to choose the 5 letters:

No "i" MOBLTY $^6C_5 = 6$

1 "i" MOBILTY $^6C_4 = 15$

2 "i" MOBILITY $\frac{6C_3}{2!} = 10$

To arrange the 5 letters $= 5!$

$\therefore$ Total No. of different arrangements

$= (6+15+10) \times 5! = 3720$ #

c(i) $\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx} = \frac{-3}{p^2} \times \frac{1}{3} = -\frac{1}{p^2}$

Eq of tangent at P: $y - \frac{3}{p} = -\frac{1}{p^2}(x - 3p)$

$y - \frac{3}{p} = \frac{-x}{p^2} + \frac{3}{p}$

$p^2y + x = 6p$ # —— ①

ii) Similarly eq. of tangent at Q:

$q^2y + x = 6q$ —— ②

solving ① & ② simultaneously

$y(p^2 - q^2) = 6(p-q) \quad \therefore y = \frac{6}{p+q}$

$x = 6p - p^2y = 6p - \frac{6p^2}{p+q} = \frac{\cancel{6p^2} + 6pq - \cancel{6p^2}}{p+q}$

$x = \frac{6pq}{p+q}$

$\therefore T = \left(\frac{6pq}{p+q}, \frac{6}{p+q}\right)$

(iii) slope of chord PQ $= \frac{3(\frac{1}{p} - \frac{1}{q})}{3(p-q)}$

$= \frac{\frac{q-p}{pq}}{p-q} = -\frac{1}{pq}$

Eq of PQ: $y - \frac{3}{p} = -\frac{1}{pq}(x - 3p)$

$y - \frac{3}{p} = -\frac{x}{pq} + \frac{3}{q}$

$pqy + x = 3(p+q)$

PQ passes throu' $(0,2)$

$pq(2) + 0 = 3(p+q)$

$2pq = 3(p+q)$

$\therefore \frac{6pq}{p+q} = 6 \times \frac{2}{3} = 9$

$\frac{6}{p+q} = 6 \times \frac{3}{2pq} = \frac{9}{pq}$

$\therefore T = (9, \frac{9}{pq})$

$\therefore$ Locus of T is $x = 9$

iv) Since P, Q are on different branches of the rectangular hyperbola

$p \cdot q < 0$

Restriction on the locus of T: $x = 9$ and $y < 0$

$\therefore$ T must be in 4th Quad. #

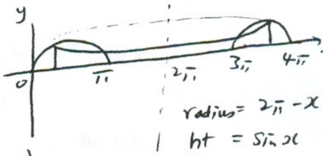
Question 5:

a) i) $V = \pi \int_0^{\pi} (\sin x)^2 dx$

$= \pi \int_0^{\pi} \left(\frac{1 - \cos 2x}{2}\right) dx$

$= \frac{\pi}{2} \left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{\pi}{2}(\pi - 0) = \frac{\pi^2}{2}$ unit^3 #

ii) $V = \int_0^{\pi} 2\pi(2\pi - x)\sin x \, dx$



$= \int_0^{\pi} (4\pi^2 \sin x - 2\pi x \sin x) dx$

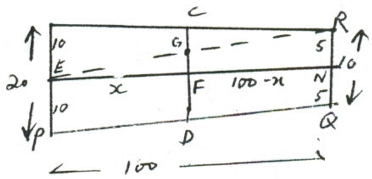
$= 4\pi^2 (\cos x)\Big]_0^{\pi} - 2\pi \int_0^{\pi} x \sin x \, dx$
$\qquad\qquad\qquad\qquad\qquad u \quad dv$

$= -4\pi^2(-1-1) - 2\pi \left[-x\cos x\right]_0^{\pi} - \int_0^{\pi} \cos x \, dx$

$= 8\pi^2 - 2\pi(\pi) - [\sin x]_0^{\pi}$

$= 6\pi^2$ #

b) Join ER cutting CF at G



Base Plan

$\frac{FG}{5} = \frac{x}{100}$

$FG = \frac{x}{20}$

$\frac{CG}{10} = \frac{100-x}{100}$

$CG = \frac{100-x}{10}$

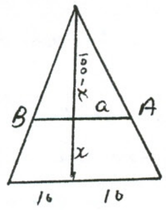
$CF = CG + FG = \frac{x}{20} + \frac{100-x}{10} = \frac{x + 200 - 2x}{20}$

$= \frac{200-x}{20} =$

$CD = 2 \times CF = \frac{200-x}{20} \times 2 = \frac{200-x}{10} = 20 - \frac{x}{10}$ m #

Q5 Roof plan

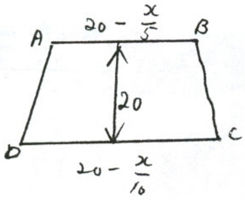
$$\frac{a}{10} = \frac{100-x}{100}$$



$$a = \frac{100-x}{10} = 10 - \frac{x}{10}$$

$$AB = 2a = 20 - \frac{x}{5}$$

Each trapezoidal slice :



$$\text{Area} = \frac{20}{2}\left[20 - \frac{x}{5} + 20 - \frac{x}{10}\right]$$

$$= 10\left(40 - \frac{3x}{10}\right)$$

$$= 400 - 3x$$

ΔV {volume slice} $= (400-3x)\,\Delta x$

Volume of the showroom $= \lim_{\Delta x \to 0} \sum \Delta V$

$$= \int_0^{100} (400-3x)\,dx = \left[400x - \frac{3x^2}{2}\right]_0^{100}$$

$$= 25000 \text{ m}^3 \ \#$$

Question 6

(a) $\displaystyle\int \frac{dx}{(x^2-6x+9)+4} = \int \frac{dx}{(x-3)^2+4}$ Let $u = x-3$, $du = dx$

$$= \int \frac{du}{u^2+2^2} = \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + c = \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right) + c \ \#$$

(b) i) $\ddot{y} = 10$

ii) $\dot{y} = 10t + C_1$ $\quad t=0, \dot{y}=0, \therefore C_1 = 0$

$\dot{y} = 10t$

$y = 5t^2 + C_2$ $\quad t=0, y=0, \therefore C_2 = 0$

$y = 5t^2$

$t=10,\quad \dot{y} = 10 \times 10 = 100$ m/s #

$\quad y = 5 \times 10^2 = 500$ m #

iii) For $t \geq 10$ $\downarrow 10 \quad \uparrow 2V \quad \ddot{y} = \dfrac{dV}{dt} = 1(10-2V) = 10-2V$

$\ddot{y} = 10 - 2V$ #

iv) terminal velocity when $\ddot{y} = 0$

$10 - 2V = 0 \qquad V = 5$ m/s #

v) $\dfrac{dV}{dt} = 10 - 2V$

$$\int \frac{dV}{10-2V} = \int dt$$

$$\int \frac{dV}{2V-10} = -\int dt$$

$$\ln|V-5| = -2t + K$$

$t=10,\ V=100$

$$\ln|100-5| = -20 + K$$

$$K = 20 + \ln 95$$

$$\ln|V-5| = -2t + 20 + \ln 95$$

$$\frac{V-5}{95} = e^{-2(t-10)}$$

$$V = 5 + 95e^{-2(t-10)}, \quad t \geq 10$$

vi) $\dfrac{dx}{dt} = 5 + 95e^{-2(t-10)}$

$$x = 5t + 95\frac{e^{-2(t-10)}}{-2} + c_2$$

$t=10,\ x = 500$

$$500 = 50 - \frac{95}{2}e^0 + c_2$$

$$497.5 = c_2$$

$$\therefore x = 5t - \frac{95}{2}e^{-2(t-10)} + 497.5$$

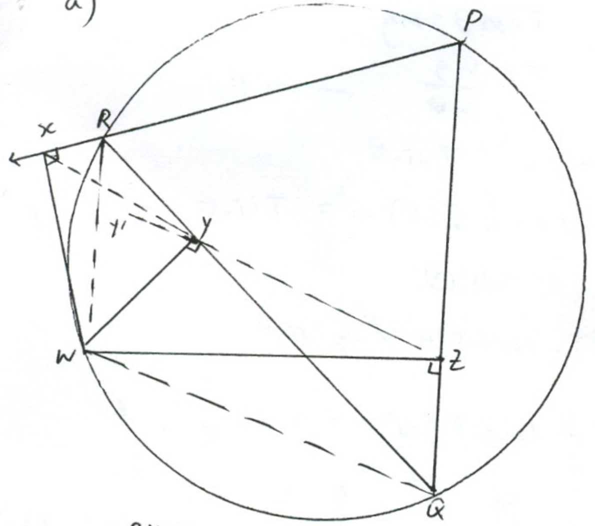
After 1 minutes ($t = 60$)

$$x = 5 \times 60 - \frac{95}{2}e^{-2(50)} + 497.5$$

$$x = 797.5 \text{ m} \ \#$$

$\therefore$ The particle has fallen 797.5 m. after 1 min.

a)



$\angle RXW = \angle QYW = 90°$ (given)

(i) $WXRY$ is a cyclic quad.
(exterior angle equals interior opposite angle) #

$\angle WYQ = \angle WZQ = 90°$ (given)
$WYZQ$ is a cyclic quad.
(line interval WQ subtends equal angles at Y, Z, on the same side of the line interval, the 4 end points $WQZY$ are then concyclic) #

b) $(\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta + 10i^3\cos^2\theta\sin^3\theta + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta(1-\cos^2\theta) - 10i\cos^2\theta\sin^3\theta + 5\cos\theta(1-\cos^2\theta)^2 + i\sin^5\theta$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta + 10\cos^5\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta(1-2\cos^2\theta+\cos^4\theta) + i\sin^5\theta$

$= c$

By De Moivre's Theorem,

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$\cos 5\theta = \underline{16\cos^5\theta - 20\cos^3\theta + 5\cos\theta}$

ii) Let $x = \cos\theta$. Then $\cos 5\theta = 0$ means $x(16x^4 - 20x^2 + 5) =$

Hence the roots of $16x^4 - 20x^2 + 5 = 0$ are the non-zero values of $\cos\theta$, where θ is a solution of $\cos 5\theta$

$\cos 5\theta = 0$ when $5\theta = 2n\pi \pm \frac{\pi}{2}$, $n \in \mathbb{Z}$

There are 4 distinct non-zero values of $\cos\theta$, namely $\frac{\pi}{10}, \frac{3\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$

$\therefore$ the four roots are $\cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{7\pi}{10}, \cos\frac{9\pi}{10}$ #

iii) Product of all roots $= \left(\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\cos\frac{7\pi}{10}\cos\frac{9\pi}{10}\right)$

$= \frac{e}{a} = \frac{5}{16}$

But $\cos\frac{9\pi}{10} = -\cos\frac{\pi}{10}$, $\cos\frac{7\pi}{10} = -\cos\frac{3\pi}{10}$

$\therefore \left(\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\right)^2 = \frac{5}{16}$

$\cos\frac{\pi}{10}\cos\frac{3\pi}{10} = +\frac{\sqrt{5}}{4}$ # (Since $\cos\frac{\pi}{10} > 0$, $\cos\frac{3\pi}{10} > 0$ $\therefore \cos\frac{\pi}{10}\cos\frac{3\pi}{10} > 0$)

iv) $\sin\frac{3\pi}{5} = \cos\left(\frac{\pi}{2} - \frac{3\pi}{5}\right) = \cos\left(-\frac{\pi}{10}\right) = \cos\frac{\pi}{10}$

$\sin\frac{6\pi}{5} = \cos\left(\frac{\pi}{2} - \frac{6\pi}{5}\right) = \cos\left(-\frac{7\pi}{10}\right) = \cos\frac{7\pi}{10}$
$= -\cos\frac{3\pi}{10}$

$\therefore \sin\frac{3\pi}{5} \cdot \sin\frac{6\pi}{5} = \cos\frac{\pi}{10}\left(-\cos\frac{3\pi}{10}\right) = -\frac{\sqrt{5}}{4}$ # (from part iii)

Q7a ii)

Construction: Join XY, WQ, RW
Join ZY and extend it to Y'

Proof: $\angle XRW = \angle XYW$ (angles at circumference in same segment)

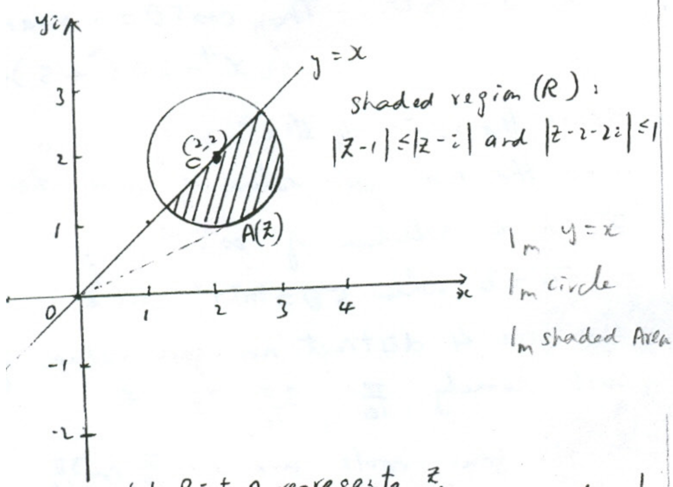
$\angle XRW = \angle ZQW$
(exterior angle equals to interior opposite angle in cyclic quadrilateral $RPQW$)

Similarly, $\angle Y'YW = \angle ZQW$ since in part (i) we have proved $WYZQ$ is a cyclic quadrilateral

$\therefore \angle XRW = \angle XYW = \angle Y'YW$

Since $\angle XYW = \angle Y'YW$, X, Y, Z must be collinear.

Question 6



Shaded region (R): $|z-1| \le |z-i|$ and $|z-2-2i| \le 1$

1m $y = x$

1m circle

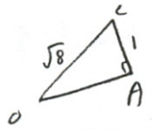
1m shaded Area

ii) α) Let Point A represents z.

When arg(z) has the smallest value, 1

OA is tangent to the circle

$OC = \sqrt{2^2+2^2} = \sqrt{8}$



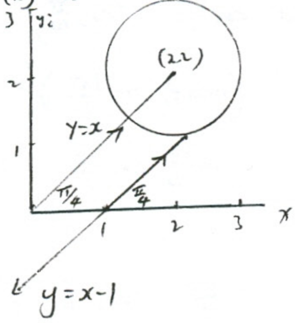
$\therefore OA = \sqrt{8-1} = \sqrt{7}$ 1

$\therefore |z| = \sqrt{7}$ when arg(z) has the smallest value.

β)

$Arg(z-1) = \frac{\pi}{4}$ is

the line $y = x-1$ ①



Pt of intersection with the circle

$(x-2)^2+(y-2)^2=1$

$(x-2)^2+(x-3)^2=1$

$x^2-4x+4+x^2-6x+9=1$

$2x^2-10x+13=1$

$x^2-5x+6=0$

$(x-3)(x-2)=0$

$\therefore x = 2$ or 3 1

When $x=2$, $y=1$

$x=3$, $y=2$

$z = 2+i$ or $3+2i$ 1

b) i) $T\cos\theta = mg$ 1

$T = \frac{mg}{\cos\theta}$ ——— ①

$mr\omega^2 = T\sin\theta$

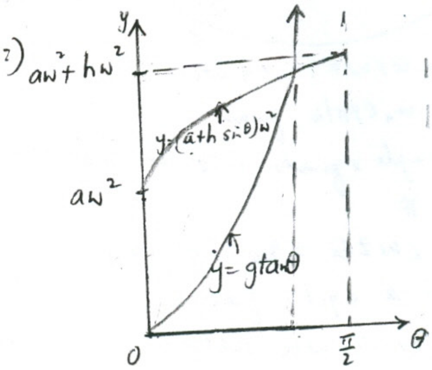
$m(a+h\sin\theta)\omega^2 = T\sin\theta$ ——— ② 1

Sub ① into ②

$\cancel{m}(a+h\sin\theta)\omega^2 = \frac{\cancel{m}g}{\cos\theta}\sin\theta$

$(a+h\sin\theta)\omega^2 = g\tan\theta$ # 1

ii)



1m $y=(a+h\sin\theta)\omega^2$

1m $y = g\tan\theta$

1m for pt of intersection

As there is only 1 pt of intersection there is only one value of θ that satisfies $(a+h\sin\theta)\omega^2 = g\tan\theta$

iii) $(a+h\sin\theta)\omega^2 = g\tan\theta$

$(4+2.5\sin 30^\circ)\omega^2 = 10\tan 30^\circ$

$5.25\omega^2 = \frac{10\sqrt{3}}{3}$

$\omega^2 = 1.0997$

$\omega = 1.04867$

$\omega = 1.05$ radian/second 1