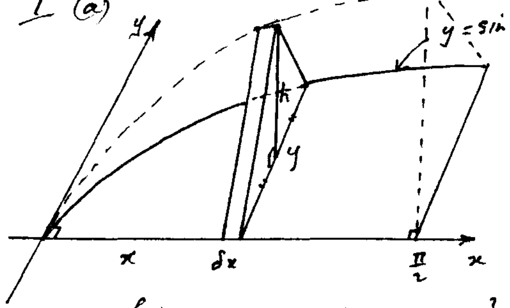
SOLUTIONS TO TERM 2 ASSESSMENT EXTENSION II 2005

1 (a)



Area of triangular section is:

$$A = \frac{1}{2} y^2 \sin 60^\circ$$

$$= \frac{1}{2} \cdot y^2 \cdot \frac{\sqrt{3}}{2}$$

$$\therefore A = \frac{y^2\sqrt{3}}{4}$$

$$\therefore dV = \frac{\sqrt{3}}{4} y^2 \, dx$$

Since $y = \sin x$ $\therefore y^2 = \sin^2 x$

$$\therefore V = \frac{\sqrt{3}}{4}\int_0^{\frac{\pi}{2}} y^2 \, dx = \frac{\sqrt{3}}{4}\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$$

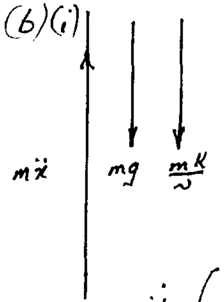
$$= \frac{\sqrt{3}}{8}\int_0^{\frac{\pi}{2}} (1 - \cos 2x) \, dx$$

$$= \frac{\sqrt{3}}{8}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\sqrt{3}}{8}\left(\frac{\pi}{2} - 0\right)$$

$$\therefore V = \frac{\pi\sqrt{3}}{16} \, u^3$$

(b)(i)



$$m\ddot{x} = -mg - \frac{mK}{v}$$

$$\therefore \ddot{x} = -g - \frac{K}{v}$$

$$\therefore \frac{dv}{dt} = -\left(\frac{gv+K}{v}\right)$$

$$\therefore \frac{v\,dv}{gv+K} = -dt$$

$$\therefore \int \frac{\frac{1}{g}(gv+K) - \frac{K}{g}}{gv+K} \, dv = -\int dt \text{ since } \frac{v}{gv+K} = \frac{\frac{1}{g}(gv+K) - \frac{K}{g}}{gv+K}$$

$$\therefore -\frac{1}{g}\int dv + \frac{K}{g}\int \frac{dv}{gv+K} = \int dt$$

$$\therefore -\frac{v}{g} + \frac{K}{g^2}\ln(gv+K) = t + c$$

When $t = 0$, $v = U$

$$\therefore \frac{K}{g^2}\ln(gU+K) - \frac{U}{g} = c$$

$$\therefore -\frac{v}{g} + \frac{K}{g^2}\ln(gv+K) = t + \frac{K}{g^2}\ln(gU+K) - \frac{U}{g}$$

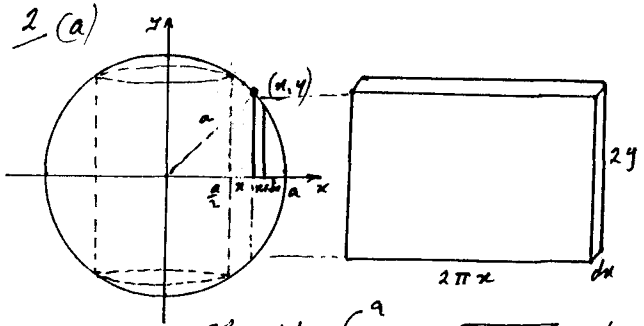
$$\therefore t = \left(\frac{U-v}{g}\right) + \frac{K}{g^2}\ln\left(\frac{gv+K}{gU+K}\right)$$

$$\therefore A = \frac{U-v}{g}, \quad B = \frac{K}{g^2}, \quad C = \frac{gv+K}{gU+K}$$

(ii) Maximum height occurs when $v = 0$

$$\therefore t = \frac{U}{g} + \frac{K}{g^2}\ln\left(\frac{K}{gU+K}\right)$$

2 (a)



Let the volume of a cylindrical shell be:

$$dV = 2\pi x \cdot 2y \, dx$$

Since $y = \sqrt{a^2 - x^2}$

$$\therefore dV = 4\pi x\sqrt{a^2-x^2} \, dx$$

$$\text{Now } V = \int_{\frac{a}{2}}^{a} 4\pi x\sqrt{a^2-x^2} \, dx$$

Let $u = a^2 - x^2$

$\therefore du = -2x \, dx$

When $x = \frac{a}{2}$ $u = \frac{3a^2}{4}$

when $x = a$ $u = 0$

and $x = \sqrt{a^2 - u^2}$

$$= -2\pi\int_{\frac{3a^2}{4}}^{0} u^{\frac{1}{2}} \, du$$

$$= 2\pi\int_0^{\frac{3a^2}{4}} u^{\frac{1}{2}} \, du$$

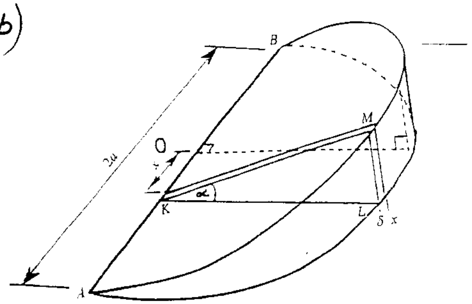
$$= 2\pi\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^{\frac{3a^2}{4}}$$

$$= \frac{4\pi}{3}\left[\left(\frac{3}{4}a^2\right)^{\frac{3}{2}} - 0\right]$$

$$= \frac{4\pi}{3} \cdot \frac{3\sqrt{3}}{8} a^3$$

$$\therefore V = \frac{\sqrt{3}}{2}\pi a^3 \, u^3$$

2(b)



(i) From the diagram $KL^2 = OL^2 - OK^2$ by Pythagoras.

$\therefore KL^2 = a^2 - x^2$

$\therefore KL = \sqrt{a^2 - x^2}$

Now $\tan\alpha = \frac{ML}{KL}$

$\therefore ML = KL\tan\alpha$

$\therefore ML = \sqrt{a^2-x^2}\,\tan\alpha$

(ii) Area of ΔKLM is $\frac{1}{2}\cdot KL\cdot LM$

$\therefore A = \frac{1}{2}\cdot\sqrt{a^2-x^2}\cdot\sqrt{a^2-x^2}\,\tan\alpha$

$\therefore A = \frac{1}{2}(a^2-x^2)\tan\alpha$

$\therefore \delta V = \frac{1}{2}(a^2-x^2)\tan\alpha\,\delta x$

$\therefore V = \frac{\tan\alpha}{2}\times 2\int_0^a (a^2-x^2)\,dx$

$= \tan\alpha\left[a^2x - \frac{x^3}{3}\right]_0^a$

$= \tan\alpha\left(a^3 - \frac{a^3}{3}\right)$

$\therefore V = \frac{2}{3}a^3\tan\alpha$

(iii) As $\tan\alpha$ decreases,

$\therefore \tan\alpha \to \alpha$

$\therefore V \to \frac{2a^3}{3}\alpha$

For n identical wedges we have:

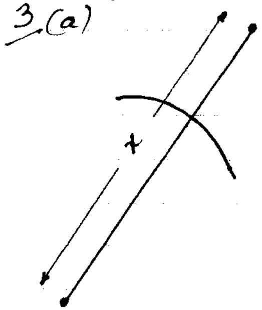
$V_n = n\cdot\frac{2a^3}{3}\alpha$

For $\alpha = \frac{2\pi}{n}$

$\therefore V_n = \cancel{n}\cdot\frac{2a^3}{3}\cdot\frac{2\pi}{\cancel{n}}$

$\therefore V_n = \frac{4}{3}\pi a^3$

3(a)



$\ddot{x} = -\frac{k}{x^2}$ $\quad\therefore v\frac{dv}{dx} = -\frac{k}{x^2}$

$\therefore v\,dv = -\frac{k}{x^2}\,dx$

$\therefore \int v\,dv = -k\int\frac{dx}{x^2}$

$\therefore \frac{1}{2}v^2 = \frac{k}{x} + c$

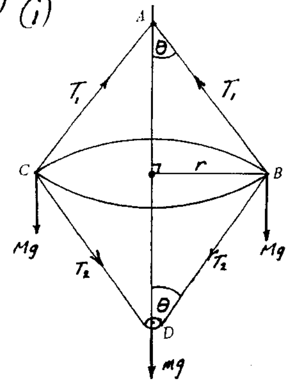
when $x = a$, $v = 0$ $\quad\therefore c = -\frac{k}{a}$

$\therefore \frac{1}{2}v^2 = \frac{k}{x} - \frac{k}{a}$

$\therefore v^2 = 2k\left(\frac{1}{x} - \frac{1}{a}\right) = 2k\left(\frac{a-x}{ax}\right)$

$\therefore v = \sqrt{\frac{2k(a-x)}{ax}}$ for $v > 0$

(b) (i)



(ii) Resolving forces at B:

Vertically: $(T_1 - T_2)\cos\theta = Mg$ — (1)

Horizontally: $(T_1 + T_2)\sin\theta = Mr\omega^2$ — (2)

Forces at D: $2T_2\cos\theta = mg$ — (3)

Now $T_2 = \frac{mg}{2}\sec\theta$ from (3)

Sub. into (1): $\left(T_1 - \frac{mg}{2}\sec\theta\right)\cos\theta = Mg$

$\therefore T_1 = Mg\sec\theta + \frac{mg}{2}\sec\theta$

$= \left(Mg + \frac{mg}{2}\right)\sec\theta$ — (4)

Sub. (4) into (2): $\left[\left(Mg + \frac{mg}{2}\right)\sec\theta + \left(\frac{mg}{2}\sec\theta\right)\right]\sin\theta = Mr\omega^2$

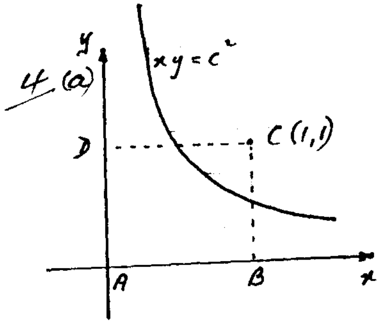
Since $\sin\theta = \frac{L}{r}$ $\quad\therefore \left[\left(Mg + \frac{mg}{2}\right)\sec\theta + \frac{mg}{2}\sec\theta\right]\cancel{\sin\theta} = M\omega^2 L\cancel{\sin\theta}$

$\therefore r = L\sin\theta$

$\therefore (Mg + mg)\sec\theta = M\omega^2 L$

$\therefore \sec\theta = \frac{M\omega^2 L}{(M+m)g}$

4 (a)



Now $C(1,1) \equiv C(c\sqrt{2}, c\sqrt{2})$

$\therefore x = y = c\sqrt{2} \qquad \therefore c\sqrt{2} = 1 \qquad \therefore c^2 = \frac{1}{2}$

Since $xy = c^2$

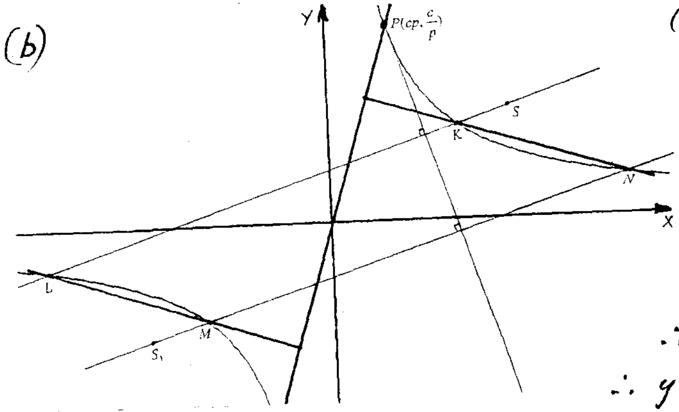
$\therefore xy = \frac{1}{2}$

When $x = 1$, $y = \frac{1}{2}$

When $y = 1$, $x = \frac{1}{2}$

$\therefore (1, \frac{1}{2})$ is mid-point of BC, and $(\frac{1}{2}, 1)$ mid-point of CD.

(b)



(i) Let equation be

$y - y_1 = m(x - x_1)$

$\therefore y - c\sqrt{2} = m(x - c\sqrt{2})$ at P

$\frac{dy}{dx} = -\frac{c^2}{x^2}$

$= -\frac{1}{p^2}$ at $x = cp$

$\therefore m = p^2$

$\therefore y - c\sqrt{2} = p^2(x - c\sqrt{2})$

$\therefore y - c\sqrt{2} = p^2x - p^2c\sqrt{2}$

$\therefore y - p^2x = c\sqrt{2}\,(1 - p^2)$ —(1)

(ii) Let $K(x,y) \equiv K(ck, \frac{c}{k})$

Since K lies on (1)

$\therefore \frac{c}{k} - p^2ck = c\sqrt{2}(1 - p^2)$

$\therefore 1 - p^2k^2 = \sqrt{2}k - \sqrt{2}p^2k$

$\therefore p^2k^2 + \sqrt{2}k - \sqrt{2}p^2k - 1 = 0$

$\therefore p^2k^2 + \sqrt{2}(1-p^2)k - 1 = 0$

$\therefore k = \dfrac{\sqrt{2}(p^2-1) \pm \sqrt{[\sqrt{2}(1-p^2)]^2 + 4p^2}}{2p^2}$

$= \dfrac{\sqrt{2}(p^2-1) \pm \sqrt{2(1+p^4)}}{2p^2}$

$\therefore k = \dfrac{\sqrt{2}\left[(p^2-1) \pm \sqrt{1+p^4}\right]}{2p^2}$

$\therefore k = \dfrac{\sqrt{2}\left[(p^2-1) + \sqrt{1+p^4}\right]}{2p^2}$ for k in first quadrant.

(iii) Since L lies on SKL, then "l" $= \dfrac{\sqrt{2}\left[(p^2-1) - \sqrt{1+p^4}\right]}{2p^2}$

Similarly, for M and N which lie on the line through $S_1(-c\sqrt{2}, -c\sqrt{2})$, the parameters m and n are $m = \dfrac{-\sqrt{2}\left[(p^2-1) + \sqrt{1+p^4}\right]}{2p^2}$ and $n = \dfrac{-\sqrt{2}\left[(p^2-1) - \sqrt{1+p^4}\right]}{2p^2}$

$\therefore m = -k$ and $l = -n$ ie; diagonals of KLMN bisect each other at origin. $\therefore$ KLMN is a parallelogram.

(iv) The gradient of diameter OP is $m_{OP} = \dfrac{(\frac{c}{p})}{cp} = \dfrac{1}{p^2}$

The gradient of KN is:

$m_{KN} = \dfrac{(\frac{c}{k} - \frac{c}{n})}{(ck - cn)}$ for $K(ck, \frac{c}{k})$ and $N(cn, \frac{c}{n})$

$= \dfrac{\cancel{c}(n-k)}{\cancel{c}(k-n)kn}$

$= -\dfrac{1}{kn}$

$= -1 \div \left\{\dfrac{[\sqrt{2}(p^2-1) + \sqrt{1+p^4}]}{2p^2} \times \dfrac{[-\sqrt{2}(p^2+1) - \sqrt{1+p^4}]}{2p^2}\right\}$

$= -1 \div \left\{\dfrac{2}{4p^4}\left[\sqrt{1+p^4} + (p^2-1)\right]\left[\sqrt{1+p^4} - (p^2-1)\right]\right\}$

$= -1 \div \left\{\dfrac{1}{2p^4}\left[(1+p^4) - (p^4 - 2p^2 + 1)\right]\right\}$

$= -1 \div \left[\dfrac{1}{2p^4}\left(\cancel{1} + \cancel{p^4} - \cancel{p^4} + 2p^2 - \cancel{1}\right)\right]$

$= -1 \div \dfrac{\cancel{2}p^2}{\cancel{2}p^4}$

$= -p^2$

Now $m_{OP} \times m_{KN} = \dfrac{1}{p^2} \times -p^2$

$= -1$

$\therefore KN \perp OP$

Since $KN \parallel LM$ (opposite sides of parm KLMN parallel)

$\therefore LM \perp OP$