

MATHEMATICS EXTENSION 1

Time allowed: Two hours (plus 5 minutes reading) **Exam date:** 13th August 2003

Instructions:

- All questions may be attempted.
- All questions are of equal value.
- Part marks are shown in boxes in the right margin.
- All necessary working must be shown.
- Marks may not be awarded for careless or badly arranged work.
- Approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Collection:

- Each question will be collected separately.
- Start each question in a new answer booklet.
- If you use a second booklet for a question, place it inside the first. Don't staple.
- Write your candidate number on each answer booklet:

Checklist:

- SGS Examination Booklets required — seven 4-page booklets per boy.
- Candidature: 120 boys.

QUESTION ONE (Start a new answer booklet)

Marks

(a) Solve the inequation $\frac{1}{x-3} < 3$.

2

(b) Evaluate $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$, giving your answer in exact form.

2

(c) Differentiate with respect to x :

(i) $y = \tan^{-1} 2x$

1

(ii) $y = \log_e \cos x$

2

(d) Find, correct to the nearest degree, the acute angle between the straight lines $y = 3$ and $y = -\frac{5}{3}x + 2$.

2

(e) Let α , β and γ be the roots of $2x^3 - x^2 + 3x - 2 = 0$. Find the value of

3

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}.$$

QUESTION TWO (Start a new answer booklet)

Marks

(a) Use the substitution $u = 1 + \tan x$ to evaluate $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{1 + \tan x}} dx$.

3

(b) Find the term independent of x in the expansion of $\left(x^2 - \frac{3}{x^2}\right)^6$.

3

(c) Using the t -substitutions, or otherwise, prove the identity

3

$$\frac{\tan 2\theta - \tan \theta}{\tan 2\theta + \cot \theta} = \tan^2 \theta.$$

(d) An object, always spherical in shape, is increasing in volume at a constant rate of $8 \text{ m}^3/\text{min}$.

(i) Find the rate at which the radius is increasing when the radius is 4 metres.
(Note: You may assume the volume formula $V = \frac{4}{3}\pi r^3$).

2

(ii) Find the rate at which the surface area is increasing when the radius is 4 metres.
(Note: You may assume the surface area formula $S = 4\pi r^2$).

1

QUESTION THREE (Start a new answer booklet)

- (a) Consider the function $f(x) = 3\sin^{-1}(x+1)$. Marks
- (i) Write down the domain and the range of $f(x)$. 2
- (ii) Sketch $y = f(x)$, giving the coordinates of its endpoints and any intercepts with the coordinate axes. 2
- (b) A particle moves according to the equation $v^2 = 2x(6-x)$.
- (i) Show that the particle moves in the interval $0 \leq x \leq 6$. 1
- (ii) Write down the centre of the motion. 1
- (iii) Find the maximum speed of the particle. 1
- (iv) Find the acceleration function. 1
- (c) The expression $\left(2 + \frac{x}{3}\right)^n$ is expanded. The ratio of the coefficients of the terms in x^6 and x^7 is 7 : 8. Find the value of n . 4

QUESTION FOUR (Start a new answer booklet)

- (a) The polynomial $2x^3 + ax^2 + bx + 6$ has $x - 1$ as a factor and leaves a remainder of -12 when divided by $x + 2$. Find the values of a and b . Marks 4
- (b) Given that the equation $x^3 + px^2 + qx + r = 0$ has a triple root, use the sums and products of roots to show that $pq = 9r$. (Hint: Let the roots be α, α and α). 4
- (c) (i) Show that the coefficient of x^5 in the expansion of $(1+x)^4(1+x)^4$ is given by ${}^4C_0 \times {}^4C_1 + {}^4C_1 \times {}^4C_2 + {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_4$. 3
- (ii) Hence, by equating the coefficients of x^5 on both sides of the identity $(1+x)^4(1+x)^4 = (1+x)^8$, 1
- prove that ${}^4C_0 \times {}^4C_1 + {}^4C_1 \times {}^4C_2 + {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_4 = \frac{8!}{3! \times 5!}$.

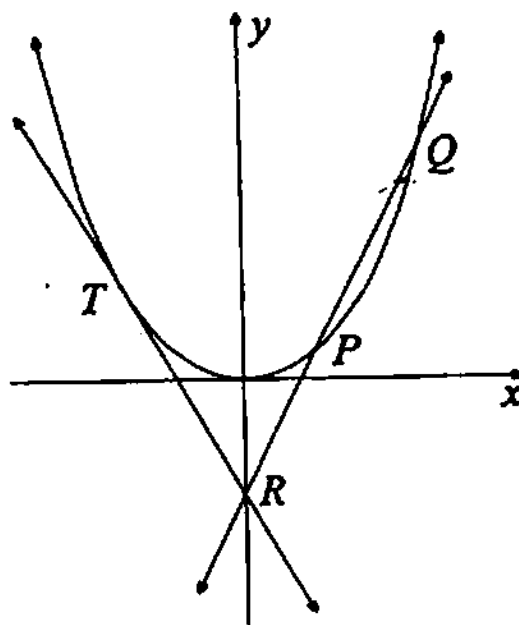
QUESTION FIVE (Start a new answer booklet)

- (a) The temperature of a body is changing at the rate $\frac{dT}{dt} = -k(T - 20)$, where T is the temperature at time t minutes and k is a positive constant.

The temperature of the surrounding environment is 20°C . The initial temperature of the body is 36°C and it falls to 35°C in 5 minutes:

- (i) Show that $T = 20 + Ae^{-kt}$ is a solution of $\frac{dT}{dt} = -k(T - 20)$, where A is a constant. Marks 1
- (ii) Prove that $A = 16$ and $k = -\frac{1}{5}\log_e \frac{15}{16}$. 3
- (iii) Find how long, correct to the nearest minute, it will take the temperature to fall to 27°C . 2
- (iv) Explain why the body will never reach a temperature that is one half of its initial temperature. 1

(b)



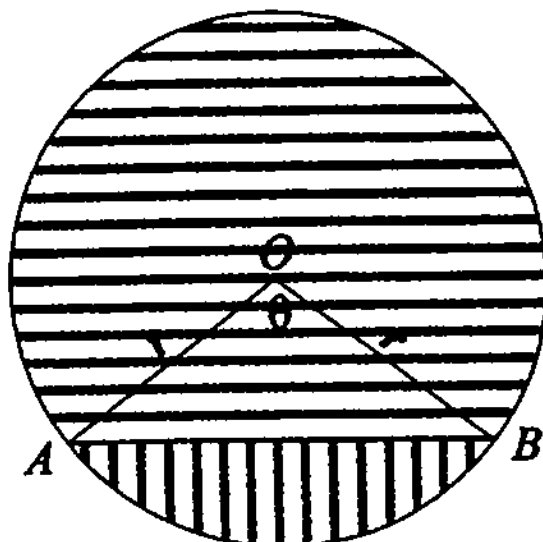
The diagram above shows the parabola $x^2 = 4ay$. The points $T(2at, at^2)$, $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola.

You may assume that the chord PQ has equation $y - \frac{1}{2}(p + q)x + apq = 0$.

- (i) Prove that the equation of the tangent to the parabola at the point $T(2at, at^2)$ is $y - tx + at^2 = 0$. 2
- (ii) Let the tangent at T intersect the axis of the parabola at the point R . Find the coordinates of R . 1
- (iii) Given that the chord PQ also passes through R , show that the parameters p , t and q form a geometric sequence. 2

QUESTION SIX (Start a new answer booklet)

(a)



In the diagram above, the chord AB subtends an angle of θ radians at the centre O of the circle with radius r .

Marks

- (i) Show that the ratio of the areas of the two segments is

2

$$\frac{\text{area of major segment}}{\text{area of minor segment}} = \frac{2\pi - \theta + \sin \theta}{\theta - \sin \theta}.$$

- (ii) Now suppose that

$$\frac{\text{area of major segment}}{\text{area of minor segment}} = \frac{\pi - 1}{1}.$$

- (α) Prove that $\theta - 2 - \sin \theta = 0$.

1

- (β) Show that the equation $\theta - 2 - \sin \theta = 0$ has a root between $\theta = 2$ and $\theta = 3$.

1

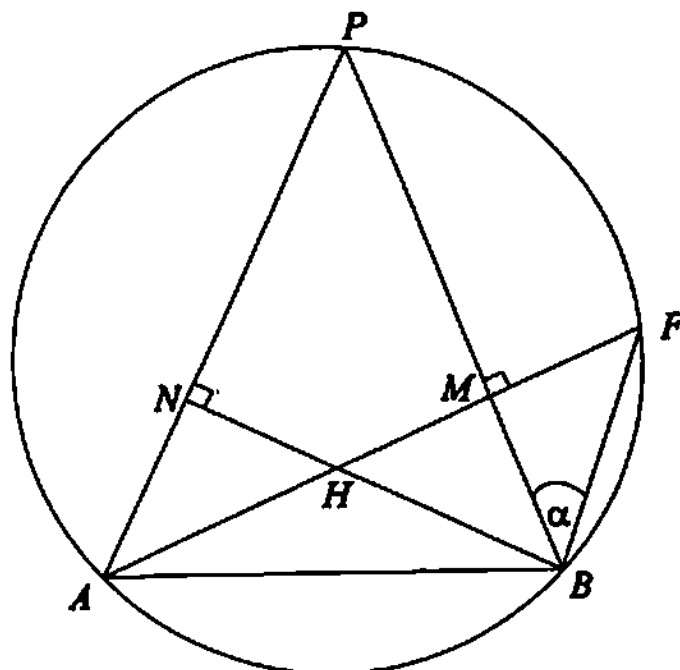
- (γ) Taking $\theta = 2.5$ as the first approximation, use Newton's method to find a second approximation to the root. Give your answer correct to two decimal places.

1

- (δ) Determine whether the second approximation of θ yields a smaller value of $|\theta - 2 - \sin \theta|$ than the first approximation.

1

(b)



In the diagram above, ABP is a triangle inscribed in a circle.

The altitudes BN and AM of the triangle intersect at H .

The altitude AM is produced to meet the circumference of the circle at F .

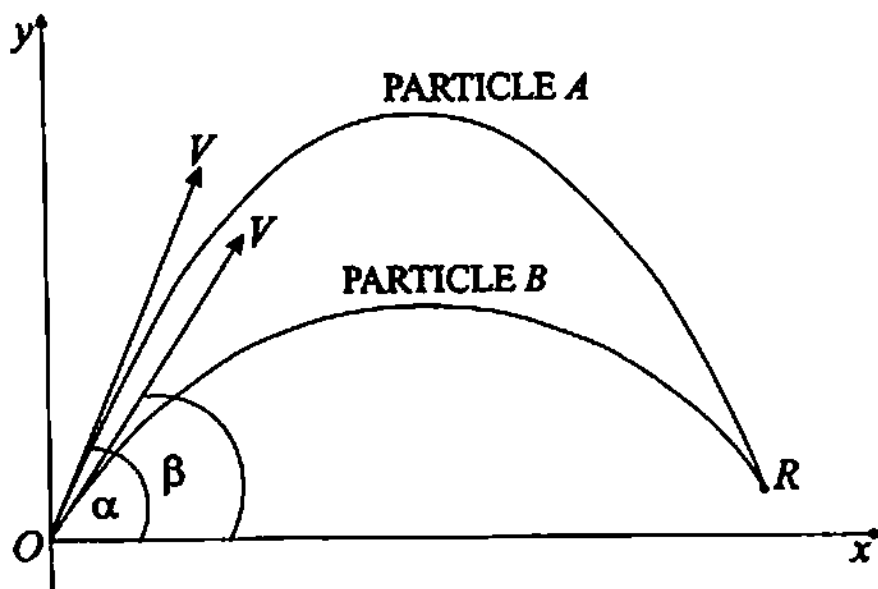
Copy the diagram into your examination booklet.

Let $\angle PBF = \alpha$.

- (i) Why is $\angle PAF = \alpha$? 1
- (ii) Why are the points A, N, M , and B concyclic? 1
- (iii) Why is $\angle NBM = \alpha$? 1
- (iv) Show that M bisects HF . 2
- (v) If AB is a fixed chord of the circle and P moves on the major arc AB , show that α is independent of the position of P . 1

QUESTION SEVEN (Start a new answer booklet)

(a)



The diagram above shows two particles A and B projected from the origin.

Particle A is projected with initial velocity V m/s at an angle α .

Particle B is projected T seconds later with the same initial velocity V m/s but at an angle of β .

The particles collide at the point R .

(i) You may assume that the equations of the paths of A and B are:

Marks

3

$$\text{For A: } y = -\frac{gx^2}{2V^2} \sec^2 \alpha + x \tan \alpha$$

$$\text{For B: } y = -\frac{gx^2}{2V^2} \sec^2 \beta + x \tan \beta$$

Show that the x -coordinate of the point R of collision is

$$x = \frac{2V^2 \cos \alpha \cos \beta}{g \sin(\alpha + \beta)}.$$

(ii) You may assume that the equation of the horizontal displacement of A is

$$x = Vt \cos \alpha.$$

(α) Write down the equation for the horizontal displacement of B. (Remember that B is projected T seconds after A). 1

(β) Show that the difference T in the times of projection is 2

$$T = \frac{2V(\cos \beta - \cos \alpha)}{g \sin(\alpha + \beta)}.$$

- (b) (i) Prove by mathematical induction that for all positive integers n ,

4

$$\sin(n\pi + x) = (-1)^n \sin x.$$

- (ii) Let $S = \sin(\pi + x) + \sin(2\pi + x) + \sin(3\pi + x) + \cdots + \sin(n\pi + x)$, for $0 < x < \frac{\pi}{2}$ and for all positive integers n . Show that

2

$$-1 < S \leq 0.$$

GJ