JAMES RUSE AGRICULTURAL HIGH SCHOOL
3/4 Unit Mathematics
Year 12 Term 1 Assessment 2000

TIME ALLOWED : 85 Minutes

- Start each question on a new page
- All questions are of equal value
- Each question is to be handed in separately

QUESTION 1:(9 marks) Start this question on a new page

a) Solve for x : $\log_2 x + 3\log_2 4 = \log_2 128$

b) Differentiate with respect to x: $y = \cos^5 2x$

c) Find the equation of the normal to the curve $y = e^{3x}$ at the point where $x = \frac{1}{3}$.

QUESTION 2:(9 marks) Start this question on a new page

a) Find the following indefinite integral: $\int \frac{x+1}{x^2 + 2x - 9} dx$

b) Prove by mathematical induction that $2^{3n} - 3^n$ is always divisible by 5.

c) Find the area bounded by $y = \sin x$, $y = \tan x$ and the line $x = \frac{\pi}{4}$.

QUESTION 3:(9 marks) Start this question on a new page

a) Evaluate the following definite integral using the substitution given:

$\int_0^{\frac{\pi}{2}} \cos x \sqrt{(\sin x)^3}\, dx$; $u = \sin x$

b) Differentiate with respect to x: $y = \ln\left(\frac{\sqrt{x+1}}{2x-1}\right)$

c) The value of a car when new is $45 000. If it depreciates at the rate of 18% of its value at the beginning of each year, find its value after 8 years (answer to the nearest dollar)

QUESTION 4:(9 marks) Start this question on a new page

(a) Find the volume of the solid of revolution when the area bounded by the curve $y = \cos 3x$, the x-axis and x=0 and $x = \frac{\pi}{6}$ is rotated about the x-axis.

(b) (i) Express $\sqrt{3}\cos x - \sin x$ in the form $A\cos(x + \alpha)$ where A>0 and α is acute.

(ii) Hence, solve the equation $\sqrt{3}\cos x - \sin x = -1$ for $0 \le x \le 2\pi$

(iii) For $y = \sqrt{3}\cos x - \sin x$ find values of x in the domain $0 \le x \le 2\pi$ for which this function is a maximum.

QUESTION 5:(9 marks) Start this question on a new page

(a) Sketch the graph of the function $y = \log_e(x-2)$.

(b) Rotate about the y-axis the region bounded by the curve $y = \log_e(x-2)$, y=0 and y=h to create a bowl. Find the exact volume of the bowl.

(c) The bowl is placed with its axis vertical and water is poured in. If water is poured into the bowl at a rate of 50 cm^3 per second, find the rate at which the water level is rising when the depth of water is 1.5 cm (answer to 3 decimal places).

QUESTION 6:(9 marks) Start this question on a new page

(a) (i) Show that $\frac{d}{dx}\tan^3 x = 3\sec^4 x - 3\sec^2 x$.

(ii) Hence, evaluate $\int_0^{\frac{\pi}{4}} \sec^4 x\, dx$.

(b) (i) Find the difference between the simple interest and compound interest on $5000 invested at 6% p.a. for 4 years (answer to the nearest cent).

(ii) What is the equivalent simple interest rate to earn this compound interest on the same principle over the same time?

QUESTION 7:(9 marks) Start this question on a new page

(a) On January 1st 2000, Sue Bright invested \$1000 in a superannuation scheme. On January 1st of each of the subsequent 14 years, she will make further investments, increasing them by 5% each year to account for inflation. The scheme pays 10% per annum interest, calculated annually, and she will withdraw her funds when the scheme reaches maturity on January 1st, 2015.

Find:

(i) the value of her first investment when it is withdrawn.

(ii) the value of her last investment when it is withdrawn.

(iii) to the nearest dollar, the amount she will withdraw on January 1st, 2015.

(b) A wooden beam is cut from a solid log so that the cross section of the log is as follows:

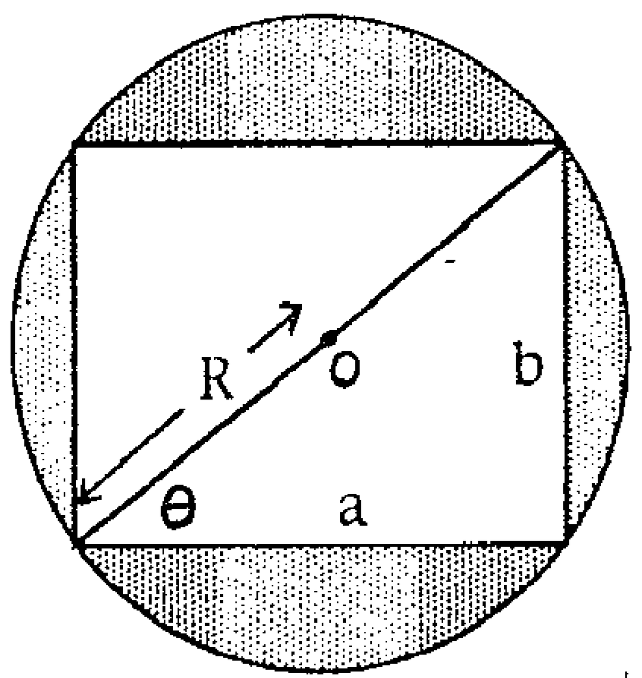


Diagram not to scale

The wooden rectangular beam of ~~length~~ width a cm and height b cm is cut from a circular log of fixed radius R cm. The strength S, of a rectangular beam is given by the formula $S = ka^2b$ where k is a constant and $k > 0$.

(i) Show that the strength of this beam, which can be cut from the circular log has equation $S = 8R^3k\sin\theta\cos^2\theta$.

(ii) Find the value of θ in radians to 3 decimal places, that would maximise the strength of the beam.

THIS IS THE END OF THE PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Qu 1

a) $\log_2 x + \log_2 4^3 = \log_2 128$

$\log_2(64x) = \log_2 128$

$x = 2$ (3)

b) $y = \cos^5 2x$

$\frac{dy}{dx} = 5\cos^4 2x \cdot -2\sin 2x$

$= -10\sin 2x \cdot \cos^4 2x$ (3)

c) $y = e^{3x}$

$\frac{dy}{dx} = 3e^{3x}$

at $x = \frac{1}{3}$: $m = 3e$

$\therefore \perp m = -\frac{1}{3e}$

at $x = \frac{1}{3}$: $y = e$

$y - e = -\frac{1}{3e}\left(x - \frac{1}{3}\right)$

OR

$y = -\frac{1}{3e}x + \frac{1}{9e} + e$ (3)

Qu 2

a) $\int \frac{x+1}{x^2+2x-9}\,dx$ (2)

$= \frac{1}{2}\ln\left(x^2+2x-9\right) + c$

b) Show true for $n = 1$

$2^3 - 3 = 5$ which is ÷ by 5

Assume true for $n = k$

ie $2^{3k} - 3^k = 5M$ $(M \in J)$

Show true for $n = k+1$

ie $2^{3(k+1)} - 3^{k+1}$ is divisible by 5

$2^{3k+3} - 3^{k+1} = 2^{3k} \cdot 2^3 - 3^k \cdot 3$

$= 8(5M + 3^k) - 3 \cdot 3^k$ (by assumption)

$= 40M + 5 \cdot 3^k$

$= 5(8M + 3^k)$

which is divisible by 5 as $8M + 3^k \in J$

As the result is true for $n = 1$ and $n = k+1$ assuming its true for $n = k$ then it is true for $n = 2, 3$, etc and all positive integer values of n (4)

c) $A = \int_0^{\frac{\pi}{4}} (\tan x - \sin x)\,dx$

$= \int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos x} - \sin x\right) dx$

$= \left[-\ln\cos x + \cos x\right]_0^{\frac{\pi}{4}}$

$= \left(-\ln\cos\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - \left(-\ln\cos 0 + \cos 0\right)$

$= \ln\sqrt{2} + \frac{\sqrt{2}}{2} - 1$ (3)

Qu 3

a) $\int_0^{\frac{\pi}{2}} \cos x \cdot \sqrt{(\sin x)^3}\,dx$

$u = \sin x$

$du = \cos x\,dx$

$x = \frac{\pi}{2}$ $u = 1$

$x = 0$ $u = 0$

$\therefore \int_0^1 u^{3/2}\,du$

$= \left[\frac{2}{5}u^{\frac{5}{2}}\right]_0^1$

$= \frac{2}{5}$ (3)

b) $y = \ln\left(\frac{\sqrt{x+1}}{2x-1}\right)$

$= \frac{1}{2}\ln(x+1) - \ln(2x-1)$

$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{x+1} - \frac{2}{2x-1}$

$= \frac{-2x-5}{2(x+1)(2x-1)}$ (3)

c) $I = 45000\left(1 - \frac{18}{100}\right)^8$

$= \$9198.63$

$= \$9199$ (nearest \$) (3)

Qu 4

a) $V = \pi\int_0^{\frac{\pi}{6}} \cos^2 3x\,dx$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{6}} (\cos 6x + 1)\,dx$

$= \frac{\pi}{2}\left[\frac{1}{6}\sin 6x + x\right]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{2}\left[\left(\frac{1}{6}\sin\pi + \frac{\pi}{6}\right) - 0\right]$

$= \frac{\pi^2}{12}\ u^3$ (3)

b) (i) $\sqrt{3}\cos x - \sin x = A\cos(x+\alpha)$

$A = \sqrt{3+1} = 2$

$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\therefore \sqrt{3}\cos x - \sin x = 2\cos\left(x + \frac{\pi}{6}\right)$ (2)

(ii) $\sqrt{3}\cos x - \sin x = -1$

$2\cos\left(x + \frac{\pi}{6}\right) = -1$

$\cos\left(x + \frac{\pi}{6}\right) = -\frac{1}{2}$

For acute $x + \frac{\pi}{6}$, $x + \frac{\pi}{6} = \frac{\pi}{3}$

$\cos\left(x + \frac{\pi}{6}\right)$ is −ve in 2nd & 3rd quad.

$\therefore x + \frac{\pi}{6} = \pi - \frac{\pi}{3},\ \pi + \frac{\pi}{3}$

$x = \pi - \frac{\pi}{3} - \frac{\pi}{6},\ \pi + \frac{\pi}{3} - \frac{\pi}{6}$

$x = \frac{\pi}{2},\ \frac{7\pi}{6}$ (2)

(iii) Max when $2\left(\cos x + \frac{\pi}{6}\right) = 2$

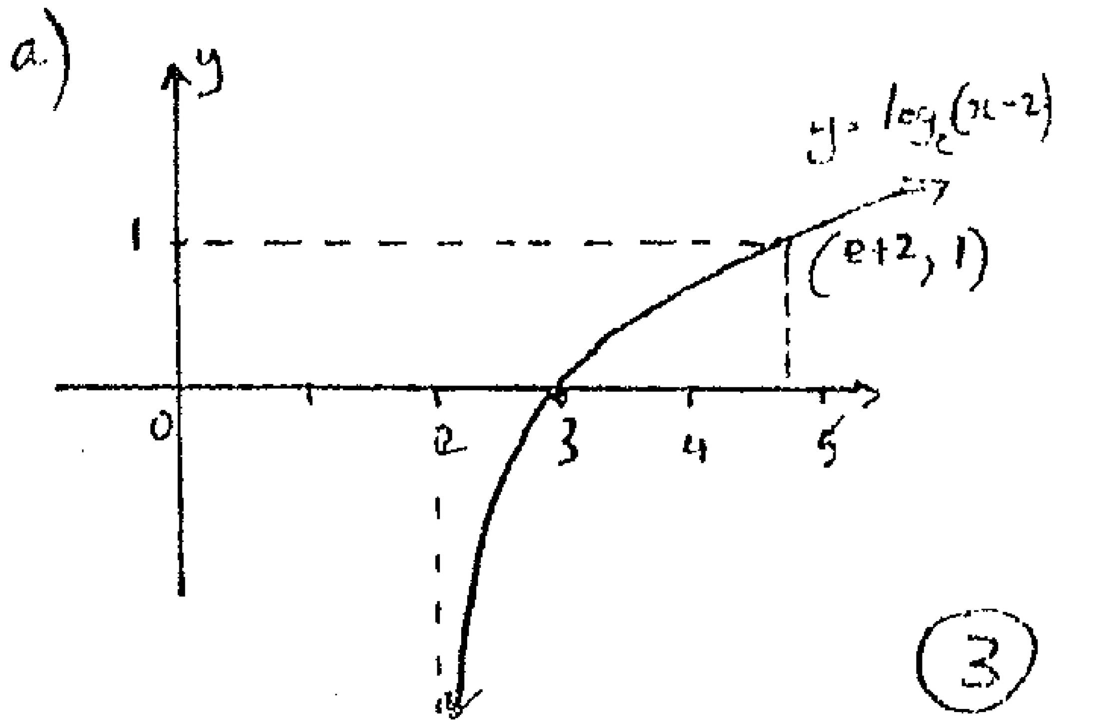
$\cos\left(x + \frac{\pi}{6}\right) = 1$

$x + \frac{\pi}{6} = 2\pi$

$x = \frac{11\pi}{6}$ only (2)

Qu 5

a)



(3)

b) $y = \log_e(x-2)$ (3)

$e^y = x - 2$

$\therefore x = e^y + 2$

$V = \pi\int_0^h \left(e^y + 2\right)^2 dy$

$= \pi\int_0^h \left(e^{2y} + 4e^y + 4\right) dy$

$= \pi\left[\frac{1}{2}e^{2y} + 4e^y + 4y\right]_0^h$

$= \pi\left(\frac{1}{2}e^{2h} + 4e^h + 4h\right) - \pi\left(\frac{1}{2} + 4\right)$

$= \pi\left(\frac{1}{2}e^{2h} + 4e^h + 4h - \frac{9}{2}\right)$

c) $\frac{dV}{dt} = 50$ cm³/sec (3)

Need $\frac{dh}{dt}$ when $h = 1.5$ cm.

$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$50 = \pi\left(e^{2h} + 4e^h + 4\right) \cdot \frac{dh}{dt}$

at $h = 1.5$: $\frac{dh}{dt} = \frac{50}{\pi\left(e^3 + 4e^{1.5} + 4\right)}$

$= 0.379$ cm/sec (to 3dp)

$\therefore$ Water is rising at a rate of 0.379 cm/sec

Qu6

a) (i) $\frac{d}{dx}\left(\tan^3 x\right)$

$= 3\tan^2 x \cdot \sec^2 x$

$= 3\left(\sec^2 x - 1\right) \cdot \sec^2 x$

$= 3\sec^4 x - 3\sec^2 x$ (2)

(ii) $\frac{d}{dx}\left(\tan^3 x\right) = 3\sec^4 x - 3\sec^2 x$

$\frac{d}{dx}\left(\tan^3 x\right) + 3\sec^2 x = 3\sec^4 x$

$\therefore \int 3\sec^4 x\, dx = \left[\tan^3 x + 3\tan x\right]_0^{\frac{\pi}{4}}$

$\therefore \int \sec^4 x\, dx = \frac{1}{3}\left[\tan^3 x + 3\tan x\right]_0^{\frac{\pi}{4}}$

$= \frac{4}{3}$ (3)

b) (i) $SI = 5000 \times \frac{6}{100} \times 4 = \$1200$

$CI = 5000(1.06)^4 - 5000$

$= \$1312.38$

$\therefore$ Difference $= 1312.38 - 1200$

$= \underline{\$112.38}$ (2)

(ii) $1312.38 = 5000 \times \frac{R}{100} \times 4$

$R = \frac{1312.38 \times 100}{5000 \times 4}$

$R = 6.56$

$\therefore$ Equivalent simple interest rate is $\underline{6.56\%}$ (2)

Qu7 a)

(i) $\frac{1000(1.1)^{15}}{}$ (1)

(ii) $1000(1.05)^{14}(1.1)$ (1)

On 1/1/00 \$1000 invested amounts to $1000(1.1)^{15}$ after 15 yrs

On 1/1/01 $1000(1.05)$ " " " $1000(1.1)^{14}\cdot(1.05)$ " 14 yrs

On 1/1/02 $1000(1.05)^2$ " " " $1000(1.05)^2\cdot(1.1)^{13}$ " 13 yrs

On 1/1/03 $1000(1.05)^3$ " " " $1000(1.05)^3\cdot(1.1)^{12}$ " 12 yrs

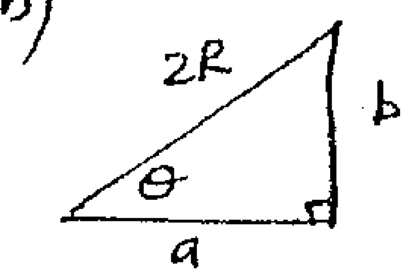
⋮

On 1/1/14 $1000(1.05)^{14}$ " " " $1000(1.05)^{14}(1.1)^1$ " 1 yr

(iii) $1000\Big[\underbrace{(1.1)^{15} + (1.1)^{14}(1.05) + (1.1)^{13}(1.05)^2 + \cdots + (1.1)^1(1.05)^{14}}_{\text{GP where } a = 1.1^{15} \quad r = \frac{1.05}{1.1} \quad n = 15}\Big]$

$= 1000\left[\frac{1.1^{15}\left(\left(\frac{1.05}{1.1}\right)^{15} - 1\right)}{\frac{1.05}{1.1} - 1}\right] = \underline{\$46\,163}$ (2)

b)



$\cos\theta = \frac{a}{2R}$ $\qquad a = 2R\cos\theta$

$\sin\theta = \frac{b}{2P}$ $\qquad b = 2R\sin\theta$

(i) $S = ka^2b$

$= k\left(2R\cos\theta\right)^2\left(2R\sin\theta\right)$

$S = 8kR^3\cos^2\theta\sin\theta \qquad 0 < \theta < \frac{\pi}{2}$ (2)

(ii) $S = 8R^3k \sin\theta \cos^2\theta$ (3)

$\frac{dS}{d\theta} = 8R^3k\left[\sin\theta \cdot 2\cos\theta \cdot -\sin\theta + \cos^2\theta \cdot \cos\theta\right]$

$= 8R^3k\left[-2\sin^2\theta\cos\theta + \cos^3\theta\right]$

For max/min: $\frac{dS}{d\theta} = 0$

$8R^3k\left[\cos^3\theta - 2\sin^2\theta\cos\theta\right] = 0$

$\underset{0}{\cos\theta}\left(\underset{0}{\cos^2\theta - 2\sin^2\theta}\right) = 0$

$\cos\theta = 0$ $\qquad \cos^2\theta - 2\sin^2\theta = 0$

$\theta = 90$ $\qquad 1 - \sin^2\theta - 2\sin^2\theta = 0$

but $\theta \neq 90$ $\qquad 1 - 3\sin^2\theta = 0$

$\sin^2\theta = \frac{1}{3}$

$\sin\theta = \pm\frac{1}{\sqrt{3}}$.

θ can only be acute

$\therefore \theta = 0.615$

Check $\theta = 0.615$ is a max:

$\frac{d^2S}{d\theta^2} = 8R^3k\left[-2\left(\sin^2\theta \cdot -\sin\theta + \cos\theta \cdot 2\sin\theta\cos\theta\right) + 3\cos^2\theta \cdot -\sin\theta\right]$

$= 8R^3k\left[2\sin^3\theta - 4\sin\theta\cos^2\theta - 3\sin\theta\cos^2\theta\right]$

$= 8R^3k\left[2\sin^3\theta - 7\sin\theta\cos^2\theta\right]$

at $\theta = 0.615$, $\frac{d^2S}{d\theta^2} \doteq 8R^3k\left(-2.310\right)$

< 0 as $R^3 > 0$ and $k > 0$

$\Rightarrow$ Concave down

$\Rightarrow$ local max at $\theta = 0.615$.

Since this is a continuous function and there is only 1 maximum value, then $\theta = 0.615$ is the absolute maximum.