SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE

2003

MATHEMATICS EXTENSION 2

General Instructions:

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Approved calculators may be used
- All necessary working should be shown in every question
- A table of standard integrals is supplied at the back of this paper
- Start each question on a new page
- Attempt all Questions 1 – 8
- All questions are of equal value
- **Total marks 120**

Name: ______________________________

Class: ______________________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Question 7	Question 8	TOTAL

Question 1 **Marks**

a) (i) Find $\int \frac{1}{\cos x + 2}\, dx$ using the substitution $t = \tan \frac{x}{2}$ **3**

Evaluate:

(ii) $\int_2^4 \frac{dx}{x^2 - 4x + 8}$ **3**

(iii) $\int_{-1}^{1} \frac{4 + x^2}{4 - x^2}\, dx$ **4**

b) Let n be a positive integer and let

$$I_n = \int_1^2 (\log_e x)^n \, dx$$

(i) Prove that $I_n = 2(\log_e 2)^n - nI_{n-1}$ **2**

(ii) Hence evaluate $\int_1^2 (\log_e x)^4 \, dx$ as a polynomial in terms of $\log_e 2$ **3**

Question 2

a) The complex number z and its conjugate $\bar{z}$ satisfy the equation $z\bar{z} + 2iz = 12 + 6i$. Find the possible values of z. **4**

b) On an Argand diagram shade the region containing all points representing complex numbers z such that $\text{Re}(z) \leq 1$ *and* $0 \leq \arg(z + i) \leq \frac{\pi}{4}$ **3**

c) Express $z_1 = \frac{7 + 4i}{3 - 2i}$ in the form $a + ib$ where a and b are real. **1**

d) On an Argand diagram sketch the locus of the point representing the complex number z such that $|z - 3 - i| = \sqrt{10}$. Find the greatest value of $|z|$ subject to this condition. **3**

e) (i) Given that w is a complex root of the equation $x^3 = 1$, show that w^2 is also a root of this equation. **2**

(ii) Show that $1 + w + w^2 = 0$, and $1 + w^2 + w^4 = 0$. **2**

Question 3 **Marks**

The ellipse E has Cartesian equation $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$

a) Find

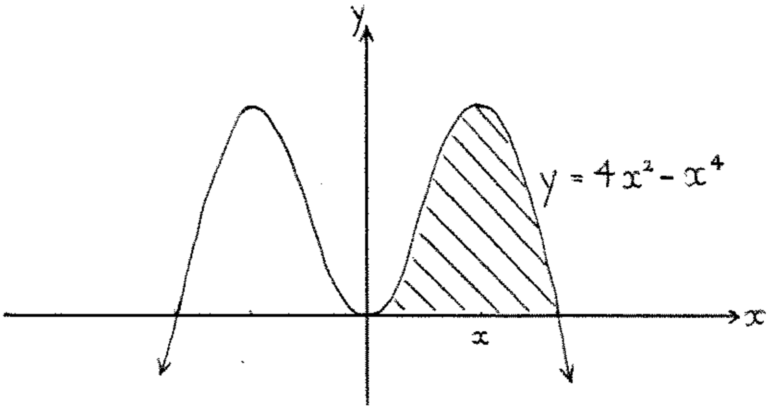
(i) the coordinates of the foci S and S^1 **1**

(ii) Show that any point P on E can be represented by the coordinates $(5\cos\vartheta,\ 4\sin\vartheta)$ and hence or otherwise prove that $PS + PS^1$ is a constant. **3**

(iii) Show that the equation of the normal at the point P on the ellipse is $\dfrac{5x}{\cos\vartheta}-\dfrac{4y}{\sin\vartheta}=9$ **3**

(iv) If this normal meets the x axis at M and the y axis at N, prove that $\dfrac{PM}{PN}=\dfrac{16}{25}$ **4**

b) The region shaded below is rotated about the y-axis to form a solid of revolution.



Using the method of cylindrical shells to calculate the volume of this solid, show that:

(i) The volume δV of a shell at x is given by $\delta V = 2\pi(4x^3 - x^5)\delta x$ **2**

(ii) Hence find the volume of this solid. **2**

Question 4

a) Let $f(x) = -x^2 + 8x - 12$. On separate diagrams, and without calculus, sketch the following graphs. Indicate clearly any asymptotes and intercepts with the axes.

(i) $y = f(x)$ **2**

(ii) $y = |f(x)|$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = \dfrac{1}{f(x)}$ **2**

(v) $y = e^{f(x)}$, giving the coordinates of any turning points by not using calculus. **3**

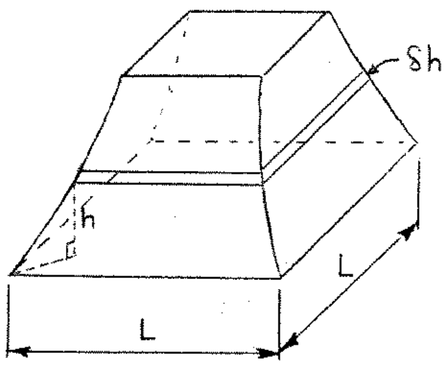
b) Given $p + q \geq 2\sqrt{pq}$ if p and q are positive real numbers

(i) Show that $e^a + e^b \geq 2e^{\frac{a+b}{2}}$ for all real a and b **2**

(ii) Hence find the minimum value of $e^{-2x} + e^{-x} + 1 + e^x + e^{2x}$ for all real x. **2**

Question 5

a)



A stone building of height H metres has the shape of a flat topped square 'pyramid' with curved sides as shown in the figure. The cross-section at height h metres is a square with sides parallel to the sides of the base and of length l, $l = \dfrac{L}{\sqrt{h+1}}$ where L is the side length of the square base in metres.

(i) Write an expression for the volume of a slice at height h metres. **2**

(ii) Hence find the volume of the building in terms of L and H. **2**

Question 5

b) The Fibonacci Sequence, F_n, is defined by:

$F_1 = 1$
$F_2 = 1$
$F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$

(i) Write down the first 12 terms of the sequence **1**

(ii) Prove, by mathematical induction, that for all positive integers, n, F_{4n} is divisible by 3. **5**

c) Find $\int \frac{\sin^3 \vartheta}{\cos^4 \vartheta}\, d\vartheta$ **3**

d) Consider the function of $y = \tan^{-1}(\tan x)$

(i) What is its period? **1**

(ii) Hence sketch the function for $-2\pi \leq x \leq 2\pi$ **1**

Question 6

The equation $x^3 + 2x - 1 = 0$ has roots α, β, and γ. In each of the following cases, find an equation with integer coefficients having the roots stated below.

a) (i) $-\alpha, -\beta, -\gamma$ **1**

(ii) $\alpha, -\alpha, \beta, -\beta, \gamma, -\gamma$ **2**

(iii) $\alpha^2, \beta^2, \gamma^2$ **3**

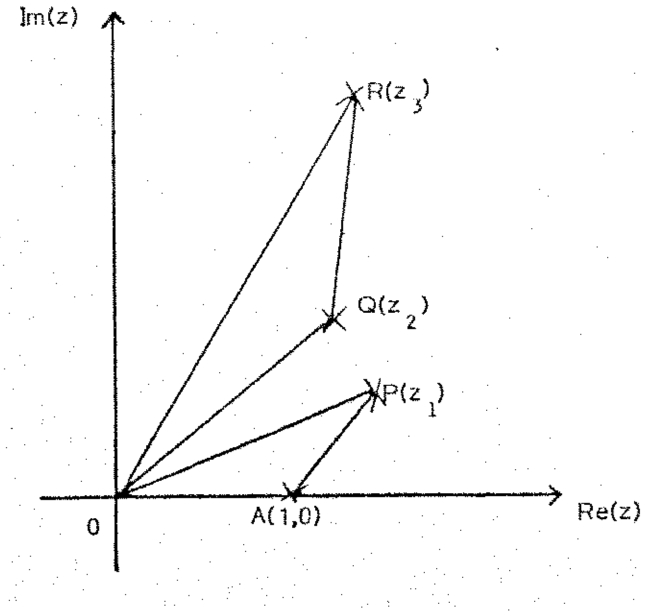
b) (i) Prove that 1 and -1 are both roots of multiplicity 2 of the polynomial $P(x) = x^6 - 3x^2 + 2$ **2**

(ii) Express $P(x)$ as the product of irreducible factors over the field of

(α) rational numbers **1**

(β) complex numbers **1**

c)



In the Argand diagram above, Δ OQR is constructed similar to Δ OAP.

Show that

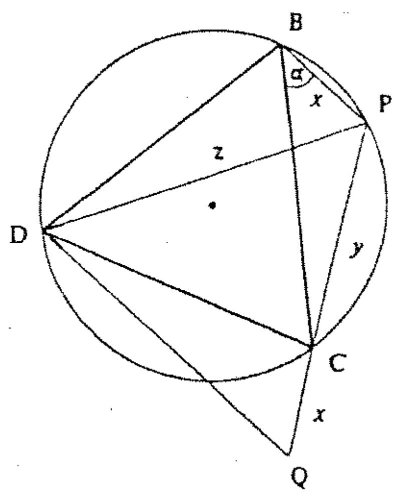
(i) $|z_3| = |z_1|\ |z_2|$ **2**

(ii) $\arg\ z_3 = \arg z_1 + \arg z_2$ **2**

(iii) What is the significance of these results? **1**

Question 7

The figure shows two towns located at B and C. BCD is an equilateral triangle. A road junction is to be placed at P, somewhere on the minor arc BC of the circumscribed circle of the triangle BCD.



Let BP, CP and DP have lengths x, y, z respectively.
The point Q is on the line PC, extended so that BP and CQ have the same length x. Let $< PBC = \alpha$.

Question 7 (cont)

a) (i) Show that <BPD = < CPD =60° **2**

(ii) Find <DCQ in terms of α **1**

(iii) Prove Δ PBD $\equiv \Delta$ QCD. **2**

(iv) Prove ΔDPQ is equilateral **2**

(v) Now show that $z = x + y$ **1**

b) Owing to the tides, the depth of water in an estuary may be assumed to rise and fall with time in simple harmonic motion.
At a certain place there is a danger of flooding when the depth of the water is above 1.25m. One day high tide was 1.5m at 1am and the following low tide was 0.5m at 7:30am.

(i) Find the amplitude in metres and period in minutes of this tidal motion. **2**

(ii) Hence find between what times after 1am was there no danger of flooding. **3**

c) Find $\int \frac{1-x}{1-\sqrt{x}}\,dx$ **2**

Question 8

a) (i) Find the 1st and 2nd derivatives of $P(x) = \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + c$ **1**

(ii) Hence or otherwise show that $P(x) = 0$ has not real roots if $c > \frac{7}{12}$ **3**

b) (i) Write down in mod-arg form, the five roots of $z^5 - 1 = 0$ **3**

(ii) By combining appropriate pairs of these roots, show that for $z \neq 1$, **4**

$$\frac{z^5 - 1}{z - 1} = (z^2 - 2z\cos\frac{2\pi}{5} + 1)\,(z^2 - 2z\cos\frac{4\pi}{5} + 1)$$

(iii) Deduce that $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$ are the roots of the equation **4**

$$4x^2 + 2x - 1 = 0$$

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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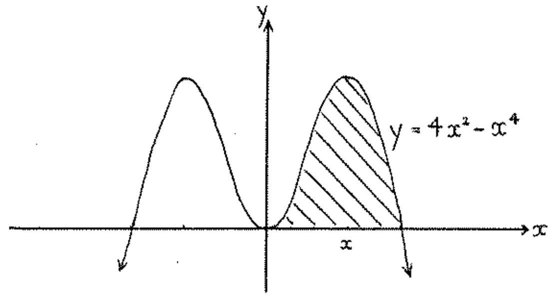
b) The region shaded below is rotated about the y-axis to form a solid of revolution.



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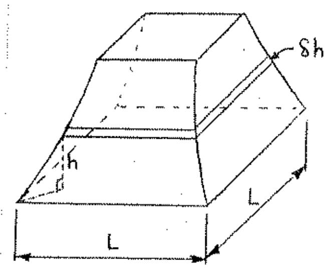

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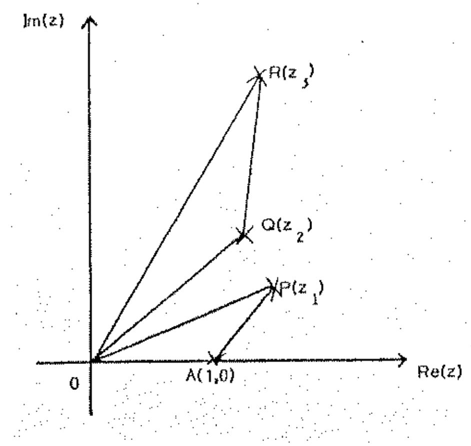

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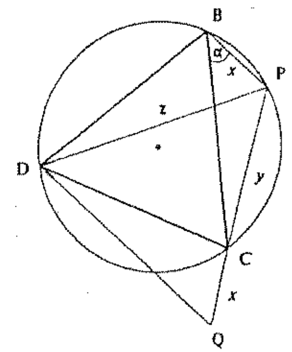

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Question 8

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$$4x^2 + 2x - 1 = 0$$

STANDARD INTEGRALS

$\int x^n\, dx$	$= \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$
$\int \frac{1}{x}\, dx$	$= \ln x, \quad x > 0$
$\int e^{ax}\, dx$	$= \frac{1}{a} e^{ax}, \quad a \neq 0$
$\int \cos ax\, dx$	$= \frac{1}{a} \sin ax, \quad a \neq 0$
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$\int \sec^2 ax\, dx$	$= \frac{1}{a} \tan ax, \quad a \neq 0$
$\int \sec ax \tan ax\, dx$	$= \frac{1}{a} \sec ax, \quad a \neq 0$
$\int \frac{1}{a^2 + x^2}\, dx$	$= \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$
$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx$	$= \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$
$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$	$= \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$
$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx$	$= \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$

Teacher's Name: ______ **Student's Name/N°:** ______

Solutions to 2003 S.T.H.S. 4U Trial

1. a. (i) $\int \frac{1}{\cos x + 2}\, dx$

$= \int \frac{1}{\frac{1-t^2}{1+t^2} + 2} \times \frac{2dt}{1+t^2}$ (1)

$= \int \frac{2dt}{1-t^2+2(1+t^2)}$

$= \int \frac{2dt}{3+t^2}$ (1)

$= \frac{2}{\sqrt{3}} \tan^{-1}\frac{t}{\sqrt{3}} + C$

$= \underline{\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) + C}$ (1)

(ii) $\int_2^4 \frac{dx}{x^2-4x+8}$

$\int_2^4 \frac{dx}{(x-2)^2+4}$ (1)

$\left[\frac{1}{2}\tan^{-1}\frac{x-2}{2}\right]_2^4$ (1)

$= \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0$

$= \underline{\frac{\pi}{8}}$ (1)

(iii) $\frac{4+x^2}{4-x^2} = \frac{8-(4-x^2)}{4-x^2} = \frac{8}{4-x^2} - 1$

Let $\frac{8}{4-x^2} = \frac{A}{2-x} + \frac{B}{2+x}$

Then $8 = A(2+x) + B(2-x)$

Let $x=-2, \quad \therefore B=2$ (1)
$x=2, \quad \therefore A=2$ (1)

$\therefore \int_{-1}^{1} \frac{4+x^2}{4-x^2}\, dx$ becomes

$\int_{-1}^{1} \frac{2}{2-x} + \frac{2}{2+x} - 1 \; dx$

$= \left[-2\log_e(2-x) + 2\log_e(2+x) - x\right]_{-1}^{1}$ (1)

$= -2\log_e 1 + 2\log_e 3 - 1 - (-2\log_e 3 + 2\log_e 1 + 1)$

$= \underline{4\log_e 3 - 2}$ (1)

(ii) (a) $I_n = \int_1^2 (\log_e x)^n\, dx$

$= \int_1^2 (\log_e x)^n \frac{d}{dx} x\, dx$

$= \left[x(\log_e x)^n\right]_1^2 - \int_1^2 x \cdot n(\log_e x)^{n-1} \cdot \frac{1}{x}\, dx$ (1) by parts

$= \underline{2(\log_e 2)^n - nI_{n-1}}$ (1)

(b) $\int_1^2 (\log_e x)^4 dx = I_4$

$= 2(\log_e 2)^4 - 4I_3$

$= 2(\log_e 2)^4 - 4\left[2(\log_e 2)^3 - 3I_2\right]$

$= 2(\log_e 2)^4 - 8(\log_e 2)^3 +$

$12\left(2(\log_e 2)^2 - 2I_1\right]$ (1)

$\therefore$ Now $I_1 = \int_1^2 \log_e x \, dx$

$= \left[x \log_e x - x\right]_1^2$ by parts (1)

$= 2\log_e 2 - 1$

$\therefore \int_1^2 (\log_e x)^4 \, dx = \underline{2(\log_e 2)^4 - 8(\log_e 2)^3}$ (1)

$\underline{+ 24(\log_e 2)^2 - 48 \log_e 2 + 24}$

/3

2. (a) Let $z = x + iy$

$\therefore z\bar{z} + 2iz = 12 + 6i$ becomes

$(x+iy)(x-iy) + 2i(x+iy) = 12 + 6i$

$x^2 + y^2 + 2ix - 2y = 12 + 6i$

$\therefore x^2 + y^2 - 2y = 12$ and $2x = 6$ (1)

$\therefore 3^2 + y^2 - 2y = 12$ $\quad \therefore x = 3$ (1)

$y^2 - 2y - 3 = 0$

$(y+1)(y-3) = 0$

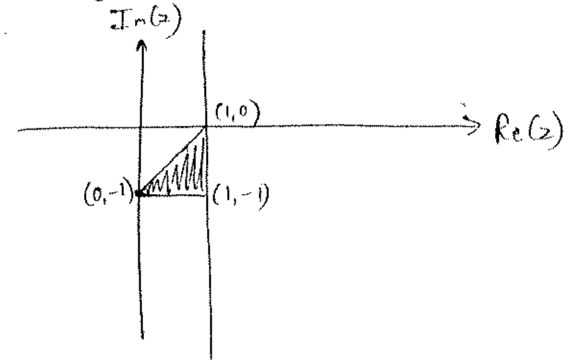
$y = -1$ or 3

$\underline{\therefore z = 3 - i \text{ or } 3 + 3i}$ (2)

(b) $Re(z) \le 1$

$0 \le \arg(z + i) \le \frac{\pi}{4}$

$0 \le \arg(z - (0 - i)) \le \frac{\pi}{4}$ (1)



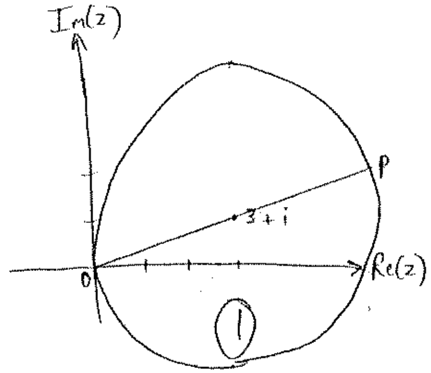
(2)

(c) $z_1 = \frac{7+4i}{3-2i} \times \frac{3+2i}{3+2i}$ (1)

$= \frac{13+26i}{13} = \underline{1 + 2i}$

(d) $|z - 3 - i| = \sqrt{10}$

$\therefore |z - (3 + i)| = \sqrt{10}$ is a circle centre $(3, 1)$ radius $\sqrt{10}$ (1)



(1)

$(0,0)$ lies on the circle. Greatest value of $|z| = OP$

$\underline{= 2\sqrt{10}}$ (1)

/4

(i) If ω is a root of $x^3=1$ then ω satisfies the equation

ie: $\omega^3 = 1$

$\therefore (\omega^3)^2 = \omega^6 = 1$ ie: $\underline{(\omega^2)^3 = 1}$

ie: ω^2 also satisfies $x^3 = 1$.   (2)

(ii) The sum of the roots of $x^3 - 1 = 0$ is $-\frac{b}{a}$ ie: 0

$\therefore \underline{1 + \omega + \omega^2 = 0.}$ (1)

Since $\omega^3 = 1$,

$\omega^4 = \omega$

$\therefore \underline{1 + \omega^4 + \omega^2 = 0.}$ (1)

3. (a) (i) $a = 5$ $b = 4$

$e^2 = 1 - \frac{b^2}{a^2}$

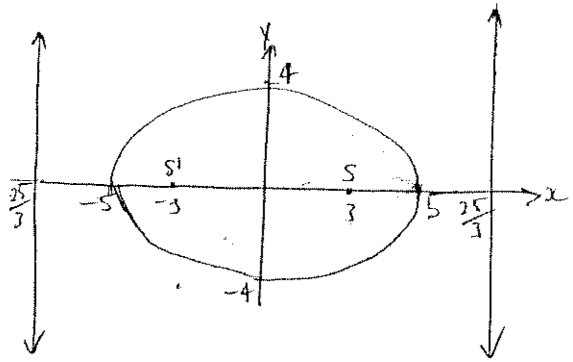
$= 1 - \frac{16}{25}$

$e^2 = \frac{9}{25}$

$e = \frac{3}{5}$

(i) $S(ae, 0)$ $S'(-ae, 0)$

$\underline{S(3, 0)}$ $\underline{S'(-3, 0)}$ (1)



(2)

(ii) Parametric Form of an ellipse

$(a\cos\theta, b\sin\theta)$

$a = 5$, $b = 4$

$\therefore \underline{P(5\cos\theta, 4\sin\theta)}$ (1)

$PS + PS'$

$\therefore e\,P\,\text{directrix}_1 + e\,P\,\text{directrix}_2$

$e\left(\frac{25}{3} - 5\cos\theta\right) + e\left(5\cos\theta + \frac{25}{3}\right)$ (1) from the defn of an ellipse

$= 2e \times \frac{25}{3}$

$= 2 \times \frac{\cancel{3}}{5} \times \frac{25}{\cancel{3}}$

$= \underline{10}$ (1)

iii) $\frac{x^2}{25} + \frac{y^2}{16} = 1$

Differentiating implicitly,

$$\frac{2x}{25} + \frac{2y\frac{dy}{dx}}{16} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{25} \times \frac{16}{2y}$$

$$= \frac{-16x}{25y}$$

at P, $\frac{dy}{dx} = M_{tangent}$

$$= \frac{-16 \times 5\cos\theta}{25 \times 4\sin\theta}$$

$$= \frac{-4\cos\theta}{5\sin\theta}$$

$M_{\dots} = \frac{5\sin\theta}{}$ (1)

Eq'n of normal:

$$y - 4\sin\theta = \frac{5\sin\theta}{4\cos\theta}(x - 5\cos\theta)$$ (1)

$$4y\cos\theta - 16\sin\theta\cos\theta = 5\sin\theta\, x - 25\sin\theta\cos\theta$$

$$9\sin\theta\cos\theta = 5\sin\theta\, x - 4\cos\theta\, y$$

$$9 = \frac{5\sin\theta\, x - 4\cos\theta\, y}{\sin\theta\cos\theta}$$

$$9 = \frac{5x}{\cos\theta} - \frac{4y}{\sin\theta}$$ (1)

iv) Cuts x axis when $y = 0$

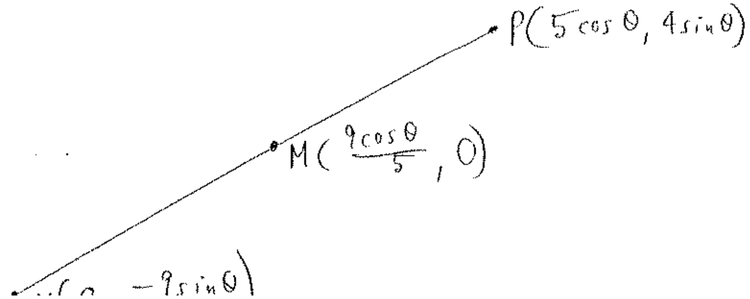
i.e. $9 = \frac{5x}{\cos\theta}$

$x = \frac{9\cos\theta}{5}$ (1)

Cuts y axis when $x = 0$

$9 = -\frac{4y}{\sin\theta}$

$y = \frac{-9\sin\theta}{4}$ (1)



/8

$$\frac{PM}{PN}$$

$$= \frac{\sqrt{(5\cos\theta - \frac{9}{5}\cos\theta)^2 + (4\sin\theta)^2}}{\sqrt{(5\cos\theta)^2 + (4\sin\theta + (\frac{9\sin\theta}{4}))^2}}$$

$$= \frac{\sqrt{(\frac{16}{5}\cos\theta)^2 + 16\sin^2\theta}}{\sqrt{25\cos^2\theta + (\frac{25}{4}\sin\theta)^2}}$$

$$= \frac{\sqrt{\frac{256}{25}\cos^2\theta + 16(1-\cos^2\theta)}}{\sqrt{25\cos^2\theta + \frac{625}{16}(1-\cos^2\theta)}}$$

$$= \sqrt{\frac{-5\frac{19}{25}\cos^2\theta + 16}{-14\frac{1}{16}\cos^2\theta + \frac{625}{16}} \times \frac{400}{400}} \quad 1$$

$$= \sqrt{\frac{-2304\cos^2\theta + 6400}{-5625\cos^2\theta + 15625}}$$

$$= \sqrt{\frac{256\cancel{(25 - 9\cos^2\theta)}}{625\cancel{(25 - 9\cos^2\theta)}}}$$

$$= \frac{16}{25} \text{ as req'd.} \quad 1$$

b. (i) $\delta V = \pi((x+\delta x)^2 - x^2) \times h$ 1

$= \pi(x^2 + 2x\delta x + \delta x^2 - x^2)(4x^2 - x^4)$

$= 2\pi(4x^3 - x^5)\delta x$ since $\delta x^2 \to 0$. 1

(ii) $V = \int_0^a 2\pi(4x^3 - x^5)\,dx$ where a is where the curve cuts the x-axis

ie: $4x^2 - x^4 = 0$

$x^2(4 - x^2) = 0$

$x = 2$

$\therefore V = 2\pi\int_0^2 4x^3 - x^5\,dx$ 1

$= 2\pi\left[x^4 - \frac{x^6}{6}\right]_0^2$

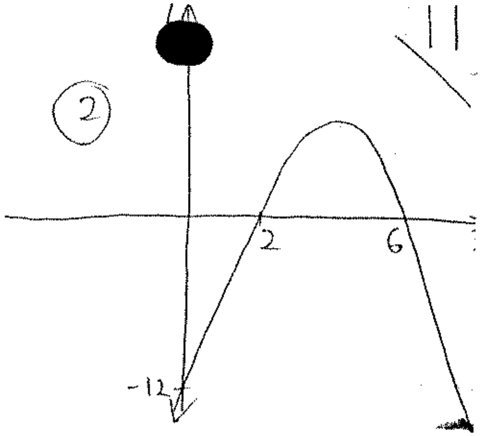
$= 2\pi\left[16 - \frac{64}{6}\right]$

$= \frac{32\pi}{3}$ units^3 1

10 (a) (i) $f(x) = -x^2 + 8x - 12$

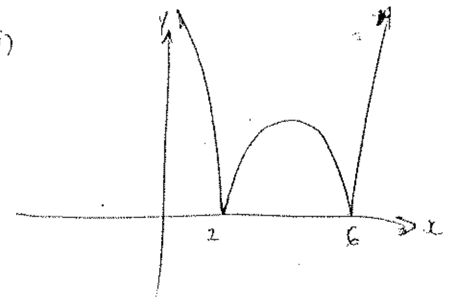
$= -(x^2 - 8x + 12)$

$= -(x-2)(x-6)$ ②



/11

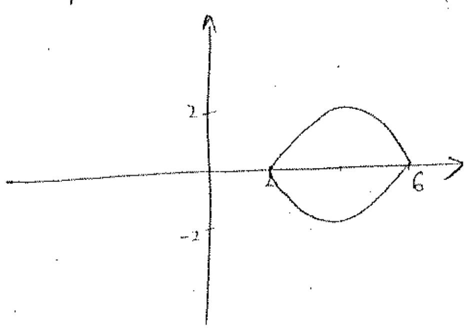
(ii)



②

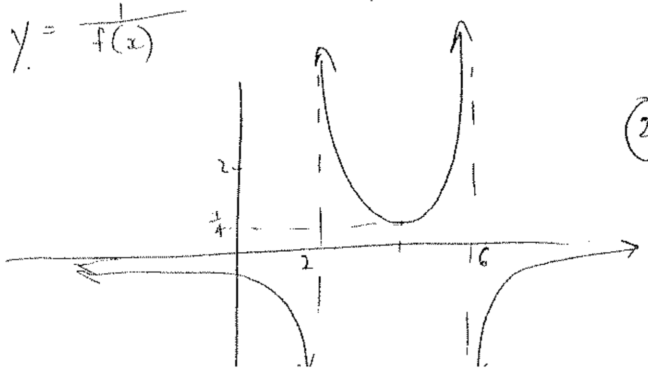
(iii) $y^2 = f(x)$

$\Rightarrow y = \pm\sqrt{-x^2+8x-12}$



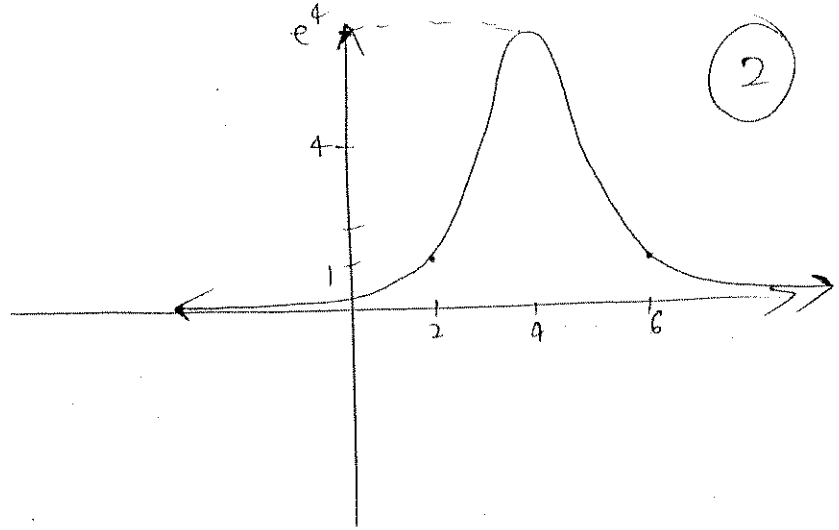
②

(iv) $y = \frac{1}{f(x)}$



②

(v) $y = e^{-x^2+8x-12}$



②

/12

(b) (i) If $p = e^a$, $q = e^b$, both real

$e^a + e^b \geq 2(e^a \times e^b)^{\frac{1}{2}}$ from (i) ①

$e^a + e^b \geq 2e^{\frac{a+b}{2}}$ ①

ii) $e^{-2x} + e^{-x} + 1 + e^{x} + e^{2x} \geq 2e^{\frac{-x+x}{2}} + 1 + 2e^{\frac{-2x+2x}{2}}$ (1)

$\geq 2 + 1 + 2$

≥ 5.

$\therefore$ Min. value is 5 (1)

5. Consider a slice, of thickness δh, at height h.

Area of cross-section $= \ell^2 = \frac{L^2}{h+1}$ m^2 (1)

$\therefore$ Volume of slice $= \frac{L^2}{h+1} \delta h$ m^3 (1)

$\therefore$ Volume of solid $= \int_0^H \frac{L^2}{h+1}\, dh$. (1)

$= \left[L^2 \log_e(h+1) \right]_0^H$

$= \underline{L^2 \log_e(H+1)}$ (1)

(b) (i) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 (1)

(ii) Show true for $n = 1$.

$F_4 = 3$.

$3 = 3$ ✓ (1)

Assume true for $n = k$

ie: $F_{4K} = 3K$ where K is a positive integer (1)

Need now to show result holds for $n = k$

ie: $F_{4(K+1)} = 3L$ where L is a positive integer.

LHS $F_{4K+4} = F_{4K+3} + F_{4K+2}$ from def'n. (

$= F_{4K+2} + F_{4K+1} + F_{4K+1} + F_{4K}$

$= F_{4K+1} + F_{4K} + F_{4K+1} + F_{4K+1} + F$

$= 3F_{4K+1} + 2F_{4K}$ (1)

$= 3F_{4K+1} + 6K$ (from assumptio

$= 3[F_{4K+1} + 2K]$

$= 3L$ as $F_{4K+1} + 2K$ is integr (1)

$\therefore$ Since result is true to $n=1$, $\quad$ /15

$\therefore$ it must also be true for $n = 1+1 = 2$,

$n = 2+1 = 3$ etc. for all positive integral values of n.

3) $\int \frac{\sin^3\theta}{\cos^4\theta}\, d\theta$

$= \int \frac{\sin^2\theta}{\cos^4\theta} \cdot \sin\theta\, d\theta$

$= \int \frac{1-\cos^2\theta}{\cos^4\theta} \sin\theta\, d\theta$ $\quad$ (1)

Let $u = \cos\theta$ $\quad \therefore du = -\sin\theta\, d\theta$

$= \int \frac{1-u^2}{u^4} \cdot -du$ $\quad$ (1)

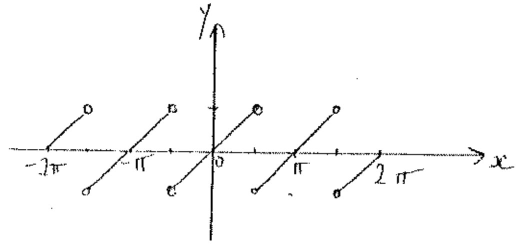
$= \int \frac{1}{u^2} - \frac{1}{u^4}\, du$

$= \frac{-1}{u} + \frac{1}{3u^3} + c$

$= \frac{1}{3}\sec^3\theta - \sec\theta + c$ $\quad$ (1)

i) Period is π (same as for $\tan x$) $\quad$ (1)

ii)



(1)

/16

6. (a) (i) $x^3 + 2x - 1 = 0$ has roots α, β, γ.

$\therefore (-x)^3 + 2(-x) - 1 = 0$

ie: $x^3 + 2x + 1 = 0$ $\quad$ (1)

(ii) $(x^3+2x+1)(x^3+2x-1) = 0$ from given equation and (i) $\quad$ (1)

ie: $x^6 + 4x^4 + 4x^2 - 1 = 0$ $\quad$ (1)

(iii) $(x^{\frac{1}{2}})^3 + 2(x^{\frac{1}{2}}) - 1 = 0$

$x^{\frac{3}{2}} + 2x^{\frac{1}{2}} = 1$

$x^{\frac{1}{2}}(x+2) = 1$ $\quad$ (1)

$x(x+2)^2 = 1$ $\quad$ (1)

$x(x^2+4x+4) = 1$

$x^3 + 4x^2 + 4x - 1 = 0$ $\quad$ (1)

(b) (i) $P(x) = x^6 - 3x^2 + 2$

$P'(x) = 6x^5 - 6x$

$= 6x(x^4 - 1)$

$= 6x(x^2-1)(x^2+1)$

Since $(x-1)$, $(x+1)$ are factors of $P'(x)$ than $x = \pm 1$ are roots of multiplicity 2. 1

$\Rightarrow P(x) = (x-1)^2(x+1)^2(x^2+2)$

(ii)(α) $P(x) = (x-1)(x+1)(x-1)(x+1)(x^2+2)$ 1

(β) $P(x) = (x-1)^2(x+1)^2(x+2i)(x-2i)$ 1

ⓒ (i) By similar Δ's 1

$$\frac{|z_3|}{|z_1|} = \frac{|z_2|}{1}$$

$\therefore |z_3| = |z_1||z_2|$ 1

(ii) $\angle ROQ = \angle POA$ (corr. $\angle$'s in similar Δ's) 1

$\therefore \arg z_3 - \arg z_2 = \arg z_1$

$\therefore \arg z_3 = \arg z_1 + \arg z_2$ 1

(iii) The construction can be used to multiply complex numbers 1

/ (i) $\angle CPD = \angle CBD$ (Angles in same segment)

$= 60°$ (as ΔCBD is equilateral) 1

$\angle BPD = \angle BCD$ (Angles in same segment).

$= 60°$ (as ΔCBD is equilateral) 1

(ii) $\angle BCQ = \angle CBP + \angle BPC$ (exterior angle)

$\angle DCQ + 60 = \alpha + 120$ from (i)

$\therefore \underline{\angle DCQ = \alpha + 60}$ 1

(iii) $BD = DC$ (equilateral triangle)

$BP = CQ = x$ (given) 1

$\angle DBP = \angle DCQ = \alpha + 60$ (from (ii))

$\therefore \Delta PBD \equiv \Delta QCD$ (SAS) 1

(iv) $\angle DPQ = \angle DBC$ (same segment)

$\therefore \angle DPQ = 60^\circ$

$\angle DQC = 180 - \angle DCQ - \angle CDQ$

$= 180 - (\alpha + 60) - \angle BDP$ 1

$= 180 - (\alpha + 60) - (180 - (\alpha + 60) - 60)$

$= 180 - \alpha - 60 - (60 - \alpha)$

$= 180 - 120$

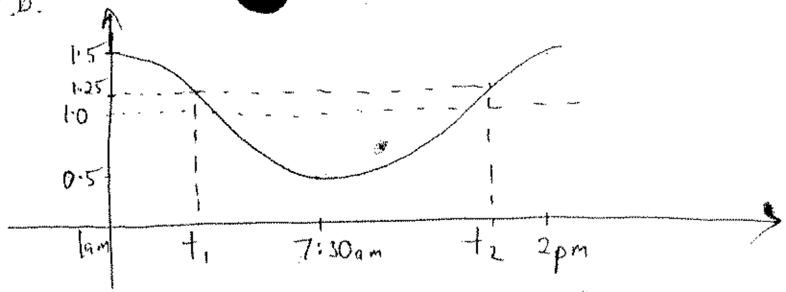
$= 60^\circ$ 1

$\therefore \triangle DPQ$ is equilateral (2 $\angle$'s of 60°).

(v) $z = x + y$ (equal sides of equilateral $\triangle$). 1

19

7b.



(i) amplitude $= 0.5$ m 1

Period $= 13$ hrs $\times 60$

$= 780$ minutes.

(ii) Let equation of motion be

$x = A\cos(nt) + C$

$T = \frac{2\pi}{n}$

$780 = \frac{2\pi}{n}$ $\therefore n = \frac{2\pi}{780} = \frac{\pi}{390}$

$\therefore x = 0.5\cos\left(\frac{\pi t}{390}\right) + C$

When $t = 0$, $x = 1.5$

$\therefore 1.5 = 0.5\cos 0 + C$

$1.5 = 0.5 + C$

$\therefore C = 1$

$\therefore x = 0.5\cos\left(\frac{\pi t}{390}\right) + 1$ is the equation 1

20

Need to find t when [illegible]

$$1.25 = 0.5\cos\left(\frac{\pi t}{390}\right) + 1$$

$\therefore \cos\left(\frac{\pi t}{390}\right) = 0.5$ (1)

$\frac{\pi t}{390} = \frac{\pi}{3}, \frac{5\pi}{3}$

$\therefore t = 130$ minutes, 650 minutes after 1am

ie: no danger of flooding between

3:10 am and 11:50 am (1)

8 (a)(i) $P'(x) = x^3 + x^2 + x + 1$

$P''(x) = 3x^2 + 2x + 1$ (1)

(ii) $P'(x) = 0$

$x^3 + x^2 + x + 1 = 0$

$x^2(x+1) + 1(x+1)$

$(x^2+1)(x+1) = 0$

when $x = -1$. (1)

$P''(x) = 3x^2 + 2x + 1$

$a = 3, \ \Delta < 0$

$\therefore$ positive definite

$\therefore$ curve is always concave up. (1)

So curve must have a minimum turning point at $x = -1$.

$P(-1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} - 1 + C$

$= \frac{-7}{12} + C$ (1)

$\therefore$ if $c > \frac{7}{12}$, curve will always be above the x-axis and have no real roots.

(b)(i) $z^5 = 1 = \cos 0 + i\sin 0$

$z = r(\cos\theta + i\sin\theta)$

$\therefore z^5 = r^5(\cos 5\theta + i\sin 5\theta)$ (1)

23

$\therefore r^5 = 1 \quad \therefore r = 1$

$\therefore 5\theta = 0 + 2k\pi$

$\theta = \frac{2k\pi}{5}$

$\therefore z^5 = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$ for $k = 0, 1, 2, 3, 4$ 1

When

$k=0 \quad z_1 = \cos 0 + i\sin 0$

$k=1, \quad z_2 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$k=2, \quad z_3 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$k=3, \quad z_4 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = \overline{z_3}$

$k=4, \quad z_5 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \overline{z_2}$ 1

ii) $z^5 - 1 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)$

$z_1 = 1, \quad z_4 = \overline{z_3}, \quad z_5 = \overline{z_2}$ 1

$\therefore z^5 - 1 = (z-1)(z-z_2)(z-\overline{z_2})(z-z_3)(z-\overline{z_3})$ 1

$\therefore \frac{z^5-1}{z-1} = \left[z^2 - z(\overline{z_2}+z_2) + z_2\overline{z_2}\right]\left[z^2 - z(z_3+\overline{z_3}) + z_3\overline{z_3}\right]$ 1

$= \left[z^2 - z(2\cos\frac{2\pi}{5}) + 1\right]\left[z^2 - z(2\cos\frac{4\pi}{5}) + 1\right]$

$= (z^2 - 2z\cos\frac{2\pi}{5} + 1)(z^2 - 2z\cos\frac{4\pi}{5} + 1)$ 1

24

(iii) For $z^5 = 1$,

Sum of roots is

$z_1 + z_2 + z_3 + z_4 + z_5 = 0$

ie: $z_1 + z_2 + \overline{z_2} + \overline{z_3} + z_3 = 0$

$1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0$

$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = \frac{-1}{2}$. 1

Sum of pairs of roots for $z^5 - 1 = 0$

$z_1z_2 + z_1\overline{z_2} + z_3z_1 + z_1\overline{z_3} + z_2\overline{z_2} + z_2z_3$
$+ z_2\overline{z_3} + \overline{z_2}z_3 + \overline{z_2}\overline{z_3} + z_3\overline{z_3} = 0$

$2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} + 1 + z_2(2\cos\frac{4\pi}{5}) + \overline{z_2}(2\cos\frac{4\pi}{5})$
$+ 1 = 0$ 1

$-1 + 1 + (2\cos\frac{4\pi}{5})(2\cos\frac{2\pi}{5}) + 1 = 0$ (from above).

$\therefore \cos\frac{4\pi}{5}\cos\frac{2\pi}{5} = \frac{-1}{4}$ 1

$\therefore$ The quadratic equation whose roots are $\cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}$ is

$x^2 - (\frac{-1}{2})x + (\frac{-1}{4}) = 0$