

Question 1 (10 marks)**Marks**

- (a) The letters of the word CALCULUS are arranged in a row.
- (i) How many different arrangements are possible ? **1**
- (ii) If one of these arrangements is selected at random, what is the probability that it begins with a “U” and ends with a “U” ? **2**
- (b) Liz heats a mug of milk up to 90°C in a microwave. She takes it out into a room where the temperature is a constant of 26°C . The milk cools to 70°C in 5 minutes. At time t minutes, its temperature T° decreases according to the equation

$$\frac{dT}{dt} = -k(T - 26), \text{ where } k \text{ is a positive constant.}$$

- (i) Verify that $T = 26 + Ae^{-kt}$ is a solution of this equation, where A is a constant. **1**
- (ii) Find the values of A and k . **2**
- (iii) How long will it take for the temperature to cool to 30°C ? **2**
Give your answer to the nearest minute.
- (iv) Sketch the graph of T as a function of t . **2**

Question 2 (10 marks) Start a new page

- (a) A particle P moves along a straight line so that at time t , its displacement from a fixed point O on that line is given by
- $$x(t) = 3t^2(4 + t^3)^{-1}.$$
- (i) Find an expression for the velocity of the particle at time t . **1**
- (ii) Find the time when the particle is momentarily at rest after the motion has started. **1**
- (iii) Show that P is in exactly the same position at both times $t_1 = 1$ and $t_2 = 2 + 2\sqrt{2}$. **2**
- (iv) Graph the displacement – time function. **2**
- (b) Two particles P and Q move along a line, their displacement at time t with respect to a fixed point O being $x(t)$ and $X(t)$ respectively.
- (i) The acceleration of P is given by $\frac{d^2x}{dt^2} = 6 + e^{-t}$. If it begins its motion at $x=0$ with a velocity of -1 , find an expression for $x(t)$. **2**
- (ii) If $X(t) = 2 \sin 5t + 3t^2 + 2$, prove that $X(t) > x(t)$ for all $t \geq 0$. **2**

Question 3 (10 marks) Start a new page**Marks**

- (a) Prove that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d^2x}{dt^2}$, where v is velocity and $\frac{d^2x}{dt^2}$ is acceleration as a function of time. **1**
- (b) The acceleration of a particle moving in a straight line is given by $\frac{d^2x}{dt^2} = (4x - 4)$, where x is the displacement, in metres, from the origin O and t is the time in seconds. Initially the particle is 6 metres to the right of O and its velocity (v m/s) is -8 m/s.
- (i) Show that $v^2 = 4x^2 - 8x - 32$. **2**
- (ii) Find the set of possible values of x where motion can exist and describe the motion of the particle. **3**
- (c) A particle is moving with simple harmonic motion in a straight line with a period of π seconds. Its maximum speed is 12 m/s. Initially the particle has a displacement of 3 metres from the centre of motion and is moving to the right.
- If x is the displacement, in metres, from the centre of motion ($x=0$) and t is the time in seconds, find an expression for x in terms of t . **4**

Question 4 (10 marks) Start a new page

- (a) A particle is moving in a straight line. At time t seconds its displacement x metres from a fixed point O on the line is given by $x = 2 \cos^2 t$.
- (i) Prove that the motion is simple harmonic. **2**
- (ii) Find the amplitude of the motion. **1**
- (b) An employer wishes to choose two people for a job. There are eight applicants, three of whom are women and five of whom are men.
- (i) If each applicant is interviewed separately and all of the women are interviewed before any of the men, find how many ways there are in carrying out the interviews. **1**
- (ii) If the employer chooses two of the applicants at random, find the probability that at least one of those chosen is a woman. **2**

(c) A nine-member Fund Raising Committee consists of four students, three teachers and two parents. The Committee meets around a circular table.

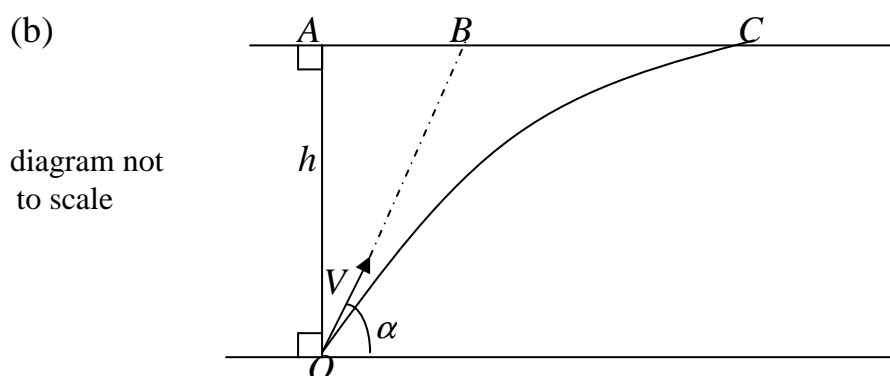
(i) How many different arrangements of the nine members around the table are possible if the students sit together as a group, as do the teachers, but no teachers sit next to a student ? 2

(ii) One student and one parent are related. Given that all arrangements in part (i) are equally likely, what is the probability that these two members sit next to each other ? 2

Question 5 (10 marks)

Start a new page

(a) Find how many groups of one or more digits can be formed from the following digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if repetition is not allowed. 2



In the diagram, an aeroplane is flying with constant velocity U at a constant height h above horizontal ground.

When the plane is at A , it is directly over a gun at O .

When the plane is at B (time $t=0$), a shell is fired from the gun at the plane along the direction OB . The shell is fired with initial velocity V at an angle of elevation α .

The horizontal and vertical components of the displacement of the shell from O at time t are given respectively by

$$x = Vt \cos \alpha \quad \text{and} \quad y = Vt \sin \alpha - \frac{1}{2}gt^2,$$

while g is the acceleration due to gravity.

(i) Show that if the shell hits the plane at point C at time $t=T$, then 2

$$VT \cos \alpha = \frac{h}{\tan \alpha} + UT.$$

(ii) Show that when the shell hits the plane then $2U(V \cos \alpha - U) \tan^2 \alpha = gh$. 3

Marks

- (c) The velocity v of a particle at time t , is given in terms of its displacement, x , by the equation $v = \frac{4}{x}$, where $x \neq 0$. Initially, $x = 8$.
- (i) Find an expression for the acceleration of the particle in terms of x . **1**
- (ii) By expressing v as $\frac{dx}{dt}$, find an expression for x^2 in terms of t . **2**

Question 6 (10 marks)**Start a new page**

- (a) Prove that ${}^{n+1}C_{k+1} = {}^nC_k + {}^nC_{k+1}$, for $1 \leq k < n$ and $n \geq 1$. **2**
(Do not use induction)
- (b) The rate of change of the population of a country is affected by the maximum possible population M of the country. M depends on factors such as the area of land and the amount of raw materials etc.
If P is the population it can be shown that $\frac{dP}{dt} = kP(M - P)$,
where M and k are constants and t is measured in years,
- (i) Verify that $P = \frac{A M e^{Mkt}}{A e^{Mkt} + 1}$ is a solution to the equation where A is a constant. **2**
- (ii) It is known that the maximum possible population (M) of a country is 860 million. In 1790 the population of the country was 4 million people and in 1800 the population was 6 million people. **5**
In what year was the population of the country equal to half of its maximum possible population (i.e. 430 million)? Give your answer to the nearest year.
- (iii) Describe what happens to the population growth rate as P approaches M . **1**

END of PAPER

