James Ruse Agricultural High School

4 unit mathematics

Trial HSC Examination 1990

1. Find the exact value of:
(a) $\int_1^2 x(x^2+1)^3\,dx$ **(b)** $\int_0^{0.5} \cos^{-1} x\,dx$ **(c)** $\int_2^3 \frac{dx}{x(x^2+4)}$ **(d)** $\int_0^2 \sqrt{16-x^2}\,dx$
(e) $\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x}$

2. **(a)** Sketch $f(x)=\frac{(x+1)(x-2)}{(x+2)}$ showing the points of intersection with the coordinate axes, the equations and positions of all asymptotes and the coordinates of the turning points.
(b) Without using calculus, draw separate graphs, with the main features clearly labelled, of:
(i) $g(x)=\cos 2x$ for $0 \le x \le 4\pi$ **(ii)** $b(x)=|g(x)|$ **(iii)** $k(x)=\frac{1}{g(x)}$

3. **(a)** The complex number $z=\sqrt{3}+i$ is represented on an Argand diagram by the point A.
(i) Write z in modulus-argument form.
(ii) Write down the modulus and the principal argument of z^5.
(iii) B, C, D and E are the points representing $-z, iz, 1-z$ and $\overline{z}$ respectively. Mark clearly on an Argand diagram the points A, B, C, D and E. Clearly indicate all important geometrical relationships between these points.
(iv) F is a point in the second quadrant such that triangle ABF is equilateral. Find the coordinates of F.
(b) Sketch the locus of z if $z=x+iy$ and:
(i) $\Im(z)<2$ **(ii)** $|z-2|=2$.

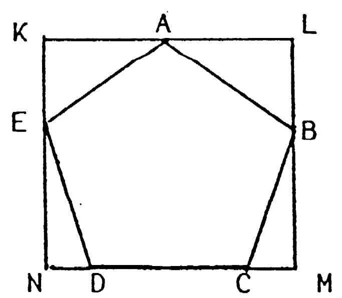
4. The hyperbola h has equation $\frac{x^2}{16}-\frac{y^2}{9}=1$.
(a) Write down its eccentricity, the coordinates of its foci S and S', the equation of each directrix and the equation of each asymptote. Sketch the curve and indicate on your diagram the foci, directrices and asymptotes.
(b) P is a point $(4\sec\theta, 3\tan\theta)$ on h. Prove that the equation of the tangent to h at P is $\frac{x\sec\theta}{4}-\frac{y\tan\theta}{3}=1$.
(c) Write down the coordinates of C and D, the points where this tangent cuts the X-axis and Y-axis respectively.
(d) If $OCBD$ is a rectangle, write down the coordinates of B.
(e) Find the equation of the locus of B as P moves along the hyperbola h.

5. **(a)** A body of mass 1 kilogram is fired vertically upwards with an initial speed of 50ms^{-1}. At any instant the body is acted on by gravity and a resistance of

magnitude $\frac{1}{5}v$ where $v\text{ms}^{-1}$ is the speed of the body at that instant. Taking the acceleration due to gravity as 10ms^{-2}, prove that:
(i) the time taken for the body to reach its maximum height is $5\ln 2$ seconds.
(ii) the maximum height reached is $(250 - 250\ln 2)$ metres.
(b) Investigate the maximum and minimum values of $\frac{\sin x}{\sqrt{2+\sin x}}$.

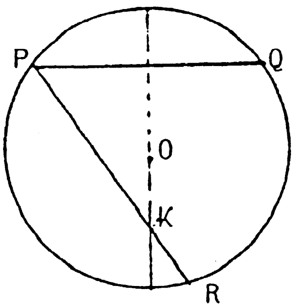
6. (a) (i) On a number plane draw a large neat sketch of the functions $y = 1 - x^2$ and $y = (1 - x^2)^{\frac{1}{3}}$ for $0 \le x \le 1$.
(ii) Show that the volume of a right circular cylindrical shell of height h with inner and outer radii x and $x + \delta x$ respectively is $2\pi \times h\delta x$ when δx is sufficiently small for terms involving $(\delta x)^2$ to be neglected.
(iii) The region bounded by the coordinate axes and $y = (1 - x^2)^{\frac{1}{3}}$ for $0 \le x \le 1$ is rotated about the Y-axis. By summing volumes of cylindrical shells find the volume V of the solid.
(b) An orchestra has $2n$ cellists, n being female and n male. From the $2n$ cellists a committee of 3 members is formed which contains more females than males.
(i) How many possible committees are there consisting of 2 females and 1 male?
(ii) How many committees are there consisting of 3 females?
(iii) Using the results above, or otherwise, prove that $n\binom{n}{2} + \binom{n}{3} = \frac{1}{2}\binom{2n}{3}$
(iv) If in fact the orchestra has 6 cellists, including Mary and Peter, find the probability that the committee chosen has Mary in it if it is known that Peter has been chosen.

7. (a)



The diagram shows a regular pentagon $ABCDE$ with all sides 1 unit in length. The pentagon is inscribed in a rectangle $KLMN$.
(i) Deduce from this diagram that $2\cos 36° = 1 + 2\cos 72°$
(ii) Use this relation to prove that $\cos 36° = \frac{\sqrt{5}+1}{4}$
(iii) Hence calculate $\cos 72°$.
(iv) Prove that the exact length of BD is $\frac{1}{2}(\sqrt{5} + 1)$ units.
(b) (i) Using the binomial theorem write down the expansion of $(1 + i)^{2m}$, where $i = \sqrt{-1}$, m is a positive integer.
(ii) Hence show that $1 - \binom{2m}{2} + \binom{2m}{4} - \binom{2m}{6} + \cdots + (-1)^m\binom{2m}{2m} = 2^m \cos(\frac{1}{2}m\pi)$ where m is a positive integer.

8. (a)



PQ is a chord of a circle. The diameter of the circle perpendicular to PQ meets another chord PR at K such that $OK = KR$.

(i) Prove that quadrilateral $OKRQ$ is cyclic.

(ii) Hence deduce that KQ bisects $O\hat{Q}R$.

(b) Two sequences $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ of positive integers, are defined by $x_1 = 2$, $y_1 = 1$ and by equating rational and irrational parts in the equation $x_{n+1} + y_{n+1}\sqrt{3} = (x_n + y_n\sqrt{3})^2$, $(n = 1, 2, 3, \ldots)$

(i) Prove that an equivalent definition is $x_1 = 2$, $y_1 = 1$ and by equating rational and irrational parts in the equation $x_{n+1} - y_{n+1}\sqrt{3} = (x_n - y_n\sqrt{3})^2$, $(n = 1, 2, 3, \ldots)$

(ii) Prove by induction that $x_n^2 - 3y_n^2 = 1$, for all n a positive integer.

(iii) Prove that $\frac{x_n}{y_n}$ and $\frac{3y_n}{x_n}$ tend to the limit $\sqrt{3}$, from above and below respectively.

(iv) Hence obtain two rational numbers (in the form $\frac{p}{q}$ where p and q are integers) which enclose $\sqrt{3}$ and differ from one another by less than 5×10^{-9}.