HIGHER SCHOOL CERTIFICATE EXAMINATION 1990

MATHEMATICS - 3/4 UNIT

Directions to Candidates

Time allowed - Two hours (including reading time).

All questions may be attempted. All questions are of equal value.

The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.

All necessary working should be shown in every question. Marks may not be awarded for careless or badly arranged work.

Approved slide-rules or silent calculators may be used.

A table of standard integrals is shown on page 187 .

QUESTION 1.

(a) Evaluate: (i) $\int_0^1 \frac{1}{1 + x^2}dx$ (ii) $\int_0^1 \frac{x}{\sqrt{1 + x}}dx$, using the substitution $u = 1 + x$.

(b) Solve the inequality $\frac{x^2 - 4}{x} > 0$.

(c) The parabolas $y = x^2$ and $y = (x - 2)^2$ intersect at a point P.

(i) Find the co-ordinates of P.

(ii) Find the angle between the tangents to the curves at P. Give your answer to the nearest degree.

QUESTION 2.

(a) (i) Factorise $a^3 + b^3$.

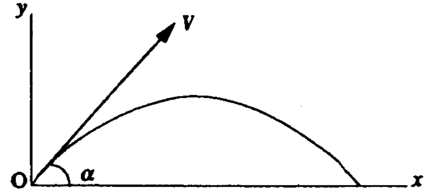
(ii) Hence, or otherwise, show that

$$\frac{2\sin^3 A + 2\cos^3 A}{\sin A + \cos A} = 2 - \sin 2A, \quad \text{if } \sin A + \cos A \neq 0.$$

(b) A polynomial is given by $p(x) = x^3 + ax^2 + bx - 18$.

Find values for a and b if $(x + 2)$ is a factor of $p(x)$ and if -24 is the remainder when $p(x)$ is divided by $(x - 1)$.

(c)



The path of a projectile fired from the origin O is given by

$$x = Vt\cos\alpha$$
$$y = Vt\sin\alpha - 5t^2$$

where V is the initial speed in metres per second and α is the angle of projection as in the diagram and t is the time in seconds.

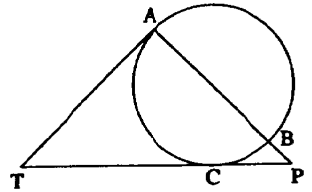
(i) Find the maximum height reached by the projectile in terms of V and α.

(ii) Find the range in terms of V and α.

(iii) Prove that the range is maximum when $\alpha = 45°$.

QUESTION 3.

(a)



(figure not to scale)

AB is a diameter of a circle ABC. The tangents at A and C meet at T. The lines TC and AB are produced to meet at P.

Copy the diagram Join AC and CB.

(i) Prove that $\angle CAT = 90° - \angle BCP$.

(ii) Hence, or otherwise, prove that $\angle ATC = 2\angle BCP$.

(b) The velocity v ms^{-1} of a particle moving in simple harmonic motion along the x axis is given by $v^2 = -5 + 6x - x^2$, where x is in metres.

(i) Between which two points is the particle oscillating?

(ii) Find the centre of motion of the particle.

(iii) Find the maximum speed of the particle.

(iv) Find the acceleration of the particle in terms of x.

(c) You are given that 3.5 is an approximate root of the equation $x^3 - 50 = 0$. Using one application of Newton's method, find a better approximation.

QUESTION 4.

(a) Find (i) $\int \frac{\ln 2x}{x}dx$, using the substitution $u = \ln 2x$

(ii) $\int \cos^2 2x\,dx$.

(b) There are three identical blue marbles and four identical yellow marbles arranged in a row.

(i) How many different arrangements are possible?

(ii) How many different arrangements of just five of these marbles are possible?

(c) (i) State the domain and range of the function given by $y = \cos^{-1}2x$.

(ii) Sketch the graph of the function given by $y = \cos^{-1}2x$.

(iii) Find the slope of the curve at the point where it cuts the y axis.

QUESTION 5

(a) Find all the angles θ with $0 \le \theta \le 2\pi$ for which $\sin\theta + \cos\theta = 1$.

(b) Consider the circle $x^2 + y^2 - 2x - 14y + 25 = 0$.

(i) Show that if the line $y = mx$ intersects the circle in two distinct points then

$(1 + 7m)^2 - 25(1 + m^2) > 0$.

(ii) For what values of m is the line $y = mx$ a tangent to the circle?

(c) The point P $(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$. The focus S is the point $(0, a)$. The tangent at P meets the y axis at Q.

(i) Find the coordinates of Q.

(ii) Prove that SP = SQ.

(iii) Hence show that $\angle PSQ + 2\angle SQP = 180°$.

QUESTION 6.

(a) Sam sits for a multiple choice examination which has 10 questions, each with four possible answers only one of which is correct. What is the probability that Sam answers exactly six questions correctly by chance alone?

(b) Assume that the rate at which a body warms in air is proportional to the difference between its temperature T and the constant temperature A of the surrounding air. This rate can be expressed by the differential equation $\frac{dT}{dt} = k(T - A)$ where t is the time in minutes and k is a constant.

(i) Show that $T = A + Ce^{kt}$, where C is a constant, is a solution of the differential equation.

(ii) A cooled body warms from 5°C to 10°C in 20 minutes. The air temperature around the body is 25°C. Find the temperature of the body after a further 40 minutes have elapsed. Give your answer to the nearest degree.

(iii) By referring to the equation for T, explain the behaviour of T as t becomes large.

(c) (i) Show that $x^n(1 + x)^n(1 + \frac{1}{x})^n = (1 + x)^{2n}$.

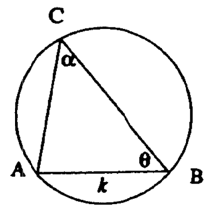
(ii) Hence prove that

$$1 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}.$$

QUESTION 7.

(a) Use mathematical induction to prove that, for every positive integer n, $13 \times 6^n + 2$ is divisible by 5.

(b) Points A, B and C lie on a circle.



The length of the chord AB is a constant k. The radian measures of $\angle ABC$ and $\angle BCA$ are θ and α respectively.

(i) Let ℓ equal the sum of the lengths of chords CA and CB. Show that ℓ is given by

$$\ell = \frac{k}{\sin\alpha}(\sin\theta + \sin(\theta + \alpha)).$$

(ii) Why is α a constant?

(iii) Evaluate $\frac{d\ell}{d\theta}$ when $\theta = \frac{\pi}{2} - \frac{\alpha}{2}$.

(iv) Hence show that ℓ is a maximum when $\theta = \frac{\pi}{2} - \frac{\alpha}{2}$.