Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2006
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics
Extension 1

Afternoon Session
Tuesday 8 August 2006

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided separately
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1-7
- All questions are of equal value

Marks

Question 1 **Begin a new page.**

(a) Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x \; dx$. **2**

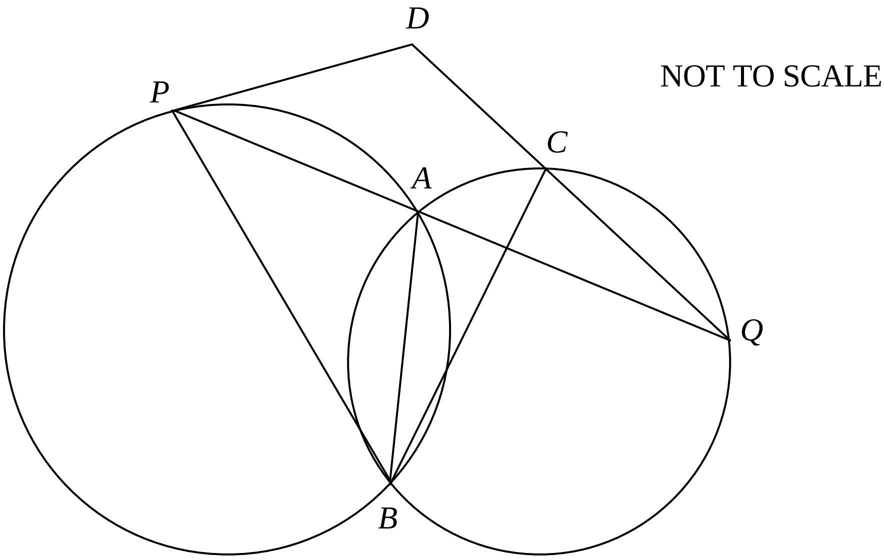
(b) Find the acute angle between the lines $3x - y - 2 = 0$ and $x + 2y - 3 = 0$. **2**
Give the answer correct to the nearest degree.

(c) The polynomial $P(x)$ is given by $P(x) = x^3 + (k-1)x^2 + (1-k)x - 1$ for some real number k.

(i) Show that $x = 1$ is a root of the equation $P(x) = 0$. **1**

(ii) Given that $P(x) = (x-1)(x^2 + kx + 1)$, find the set of values of k such that the equation $P(x) = 0$ has 3 real roots. **3**

(d)



Two circles intersect at A and B. P is a point on the first circle and Q is a point on the second circle such that PAQ is a straight line. C is a point on the second circle. The line QC produced and the tangent to the first circle at P meet at D.

(i) Copy the diagram.

(ii) Give a reason why $\angle DPA = \angle PBA$. **1**

(iii) Give a reason why $\angle CQA = \angle CBA$. **1**

(iv) Hence show that $BCDP$ is a cyclic quadrilateral. **2**

Marks

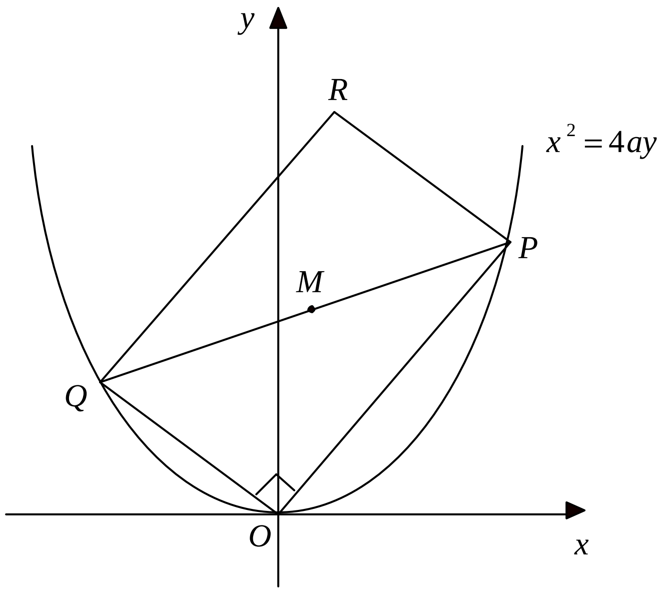
Question 2 **Begin a new page.**

(a) Show that $\dfrac{d}{dx}3^x = 3^x \ln 3$. **2**

(b) $A(-3, 7)$ and $B(4, -2)$ are two points. Find the coordinates of the point P which divides the interval AB internally in the ratio $3 : 2$. **2**

(c) Solve the equation $1 + \cos 2x = \sin 2x$ for $0 \le x \le 2p$. **4**

(d)



$P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points which move on the parabola $x^2 = 4ay$ such that $\angle POQ = 90°$, where $O(0, 0)$ is the origin. $M\left(a(p+q), \tfrac{1}{2}a(p^2+q^2)\right)$ is the midpoint of PQ. R is the point such that $OPRQ$ is a rectangle.

(i) Show that $pq = -4$. **1**

(ii) Show that R has coordinates $\left(2a(p+q), a(p^2+q^2)\right)$. **1**

(iii) Find the equation of the locus of R. **2**

Marks

Question 3 **Begin a new page.**

(a) Consider the function $f(x)=\dfrac{x^2}{x^2-1}$.

(i) Show that $f(x)$ is an even function. **1**

(ii) Show that $\lim\limits_{x\to\infty} f(x)=1$. **1**

(iii) Show that the graph $y=f(x)$ has a maximum turning point at the origin $(0\ ,0)$. **2**

(iv) Sketch the graph $y=f(x)$ showing clearly the equations of any asymptotes. **2**

(v) The function $g(x)$ is defined by $g(x)=\dfrac{x^2}{x^2-1}$, $x\geq 0$. Find the equation of the inverse function $g^{-1}(x)$ and state its domain. **2**

(b) Use Mathematical Induction to show that for all positive integers $n\geq 1$ **4**

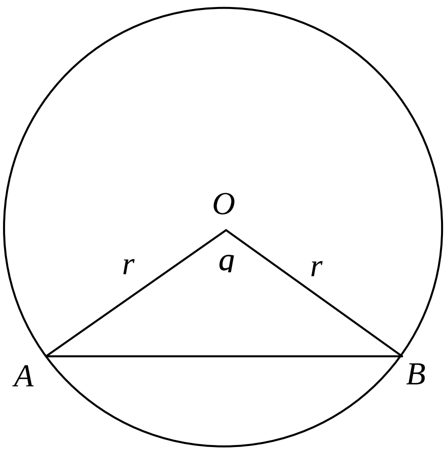
$$\frac{3}{1\times 2\times 2}+\frac{4}{2\times 3\times 2^2}+\ ...\ +\frac{n+2}{n(n+1)2^n}=1-\frac{1}{(n+1)2^n}\ .$$

Marks

Question 4 **Begin a new page.**

(a) The region in the first quadrant bounded by the curve $y = 2\tan^{-1} x$ and the y axis between $y = 0$ and $y = \frac{\pi}{2}$ is rotated through one complete revolution about the y axis. Find the exact volume of the solid of revolution so formed. **4**

(b)



NOT TO SCALE

AB is a chord of a circle of radius r which subtends an angle θ, $0 < \theta < \pi$, at the centre O. The area of the minor segment cut off by chord AB is one half of the area of the sector AOB.

(i) Show that $\theta - 2\sin\theta = 0$. **2**

(ii) Use an initial approximation $\theta_1 = 2$ and one application of Newton's method to find a second approximation to the value of θ. Round your answer to 2 decimal places. **2**

(c) Don guesses at random the answers to each of 6 multiple choice questions. In each question there are 3 alternative answers, only one of which is correct.

(i) Find the probability in simplest exact form that Don answers exactly 2 of the 6 questions correctly. **2**

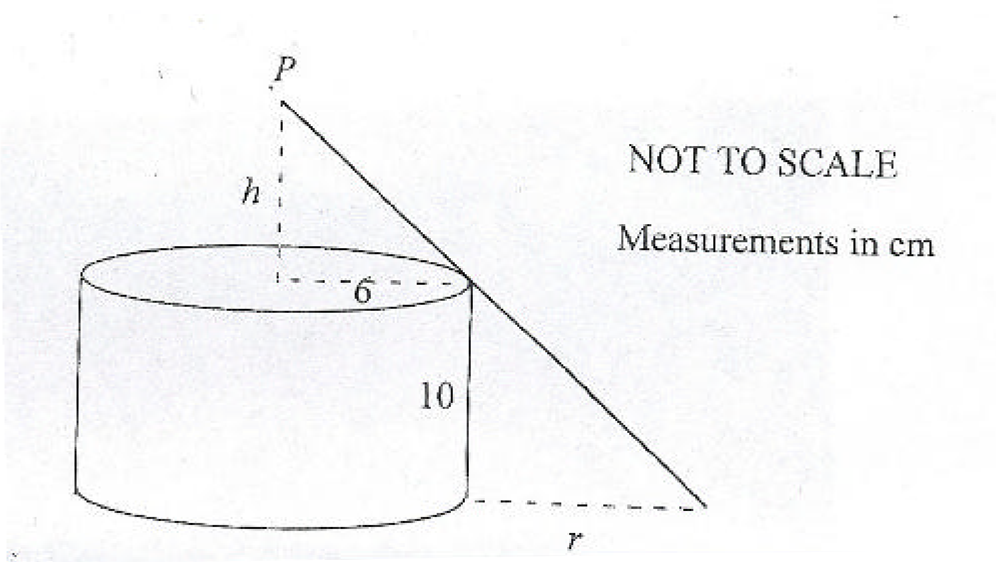
(ii) Find the probability in simplest exact form that the 6th question that Don attempts is only the 2nd question that he answers correctly. **2**

Marks

Question 5 **Begin a new page.**

(a) Use the substitution $u = x - 1$ to evaluate $\int_{0\cdot5}^{1\cdot5} \frac{1}{\sqrt{2x - x^2}}\, dx$. Give the answer in simplest exact form. **4**

(b)



A solid wooden cylinder of height 10 cm and radius 6 cm rests with its base on a horizontal table. A light source P is being lowered vertically downwards from a point above the centre of the top of the cylinder at a constant rate of $0\cdot1 \text{ cm s}^{-1}$. When the light source is h cm above the top of the cylinder the shadow cast on the table extends r cm from the side of the cylinder.

(i) Show that $r = \dfrac{60}{h}$. **1**

(ii) Find the rate at which r is changing when $h = 5$. **3**

(c) A particle is performing Simple Harmonic Motion in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity $v \text{ ms}^{-1}$ given by $v^2 = 32 + 8x - 4x^2$ and acceleration $a \text{ ms}^{-2}$.

(i) Find an expression for a in terms of x. **1**

(ii) Find the centre and amplitude of the motion. **2**

(iii) Find the maximum speed of the particle. **1**

Marks

Question 6 **Begin a new page.**

(a) At time t minutes the volume flow rate R kilolitres per minute of water into a tank is given by $R = 4\sin^2 t$, $0 \le t \le p$.

(i) Find the maximum rate of flow of water into the tank. **1**

(ii) Find the total amount of water which flows into the tank. Give the answer correct to the nearest litre. **3**

(b) At time t years the number N of individuals in a population is given by $N = A + Be^{-t}$ for some real constants A and B. After $\ln 2$ years there are 60 individuals and after $\ln 5$ years there are 36 individuals.

(i) Show that A and B satisfy the equations $2A + B = 120$ and $5A + B = 180$. Hence find the values of A and B . **3**

(ii) Find the limiting population size. **1**

(c) A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line and velocity v ms^{-1} given by $v = \dfrac{x(2-x)}{2}$.
The particle starts 1 metre to the right of O.

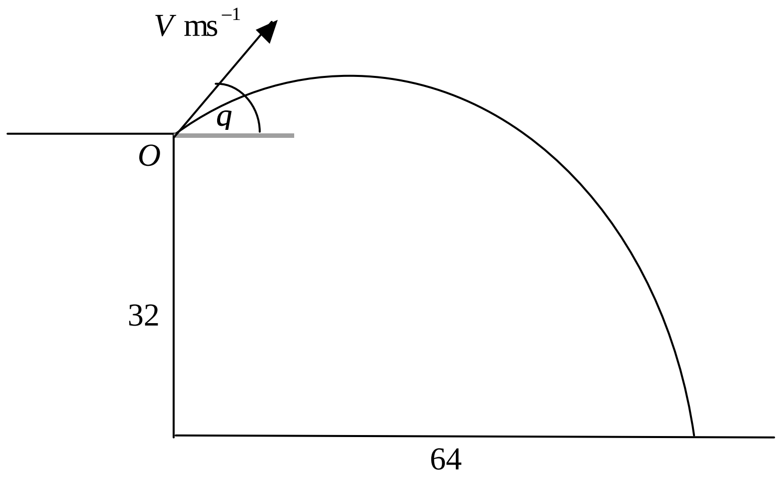
(i) Show that $\dfrac{2}{x(2-x)} = \dfrac{1}{x} + \dfrac{1}{2-x}$. **1**

(ii) Find an expression for x in terms of t. **3**

Marks

Question 7 **Begin a new page.**

(a)



A particle is projected with velocity V ms^{-1} at an angle θ above the horizontal from a point O on the edge of a vertical cliff 32 metres above a horizontal beach. The particle moves in a vertical plane under gravity, and 4 seconds later it hits the beach at a point 64 metres from the foot of the cliff. The acceleration due to gravity is 10 ms^{-2}.

(i) Use integration to show that after t seconds the horizontal displacement x metres and the vertical displacement y metres of the particle from O are given by $x = (V\cos\theta)t$ and $y = (V\sin\theta)t - 5t^2$ respectively. **2**

(ii) Write down two equations in V and θ then solve these equations to find the exact value of V and the value of θ in degrees correct to the nearest minute. **3**

(iii) Find the speed of impact with the beach correct to the nearest whole number and the angle of impact with the beach correct to the nearest minute. **3**

(b)(i) Write down the expansion of $x(1+x)^n$ in ascending powers of x. **1**

(ii) Hence show that $2\,{}^nC_1 + 3\,{}^nC_2 + \ldots + n\,{}^nC_{n-1} = (n+2)(2^{n-1}-1)$. **3**

EXAMINERS

Graham Arnold	Terra Sancta College, Nirimba
Sandra Hayes	Aquinas College, Menai

Question 1

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• writes primitive function	1
• evaluates by substitution of limits	1

Answer

$$\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x \quad dx = \tfrac{1}{2}\left[\sec 2x\right]_0^{\frac{\pi}{6}} = \tfrac{1}{2}\left(\sec\tfrac{\pi}{3} - \sec 0\right) = \tfrac{1}{2}(2-1) = \tfrac{1}{2}$$

b. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
• substitutes gradients into expression for $\tan\theta$	1
• calculates θ to required accuracy	1

Answer

$3x - y - 2 = 0$

$y = 3x - 2$

Gradient is 3

$x + 2y - 3 = 0$

$y = -\tfrac{1}{2}x + \tfrac{3}{2}$

Gradient is $-\tfrac{1}{2}$

Acute angle θ between the lines is given by

$$\tan\theta = \left|\frac{3 - \left(-\frac{1}{2}\right)}{1 + 3\left(-\frac{1}{2}\right)}\right| = 7.$$

$\therefore \theta \approx 82°$ (to the nearest degree)

c. Outcomes assessed : PE3, P4

Marking Guidelines

Criteria	Marks
i • shows $P(1) = 0$ by substitution	1
ii • deduces that equation $P(x) = 0$ has 3 real roots provided $x^2 + kx + 1 = 0$ has real roots.	1
• finds discriminant of this quadratic in terms of k and realizes $\Delta \ge 0$ for real roots	1
• states values of k	1

Answer

i. $P(1) = 1 + (k-1) + (1-k) - 1 = 0$.

ii. Equation $P(x) = 0$ has 3 real roots if equation $x^2 + kx + 1 = 0$ has two real roots.

For this quadratic equation, $\Delta = k^2 - 4 \ge 0$ for $k^2 \ge 4$.

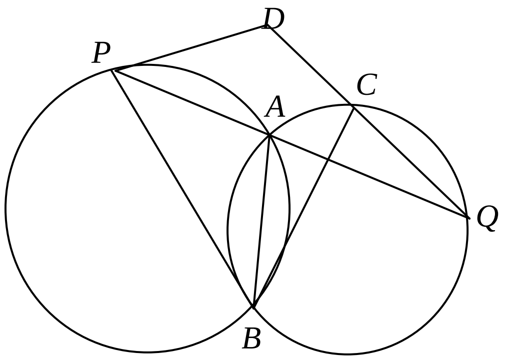
Hence $P(x) = 0$ has 3 real roots for $k \le -2$ or $k \ge 2$.

d. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
ii • quotes the alternate segment theorem in circle *APB*	1
iii • quotes theorem about angles standing on the same arc (or chord) in circle *AQB*	1
iv • writes a sequence of deductions leading to a test for *BCDP* to be cyclic	1
• supports these deductions with reasons	1

Answer



ii. In circle *APB*, angle between tangent *DP* and chord *PA* is equal to the angle subtended by *PA* in the alternate segment at *B*.
Hence $\angle DPA = \angle PBA$.

iii. In circle *AQB*, angles subtended by the same arc *CA* at points *B* and *Q* on the circumference are equal.
Hence $\angle CQA = \angle CBA$.

iv. $\angle QDP + \angle DPQ + \angle DQP = 180°$ (*Angle sum of* ΔQPD *is* $180°$)

But $\angle QDP = \angle CDP$, $\angle DPQ = \angle DPA$, $\angle DQP = \angle CQA$ (*Q, C, D collinear*; *P, A, Q collinear*)

Hence $\angle CDP + \angle DPA + \angle CQA = 180°$.

$\therefore \angle CDP + \angle PBA + \angle CBA = 180°$ ($\angle DPA = \angle PBA$, $\angle CQA = \angle CBA$ *shown above*)

But $\angle PBA + \angle CBA = \angle PBC$ (*by addition of adjacent angles*)

$\therefore \angle CDP + \angle PBC = 180°$

Hence *BCDP* is a cyclic quadrilateral (*one pair of opposite angles supplementary*)

Question 2

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• uses the equivalence of expressions 3^x and $e^{x\ln 3}$	1
• derives the equivalent exponential function with base *e*.	1

Answer

$$3^x = e^{\ln 3^x} = e^{x\ln 3}. \quad \text{Hence} \quad \frac{d}{dx}3^x = \frac{d}{dx}e^{x\ln 3} = \ln 3\, e^{x\ln 3} = 3^x \ln 3$$

b. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
• applies an appropriate process to determine the coordinates	1
• calculates both coordinates correctly	1

Answer

$A(-3\,,7)$ $\qquad\qquad$ $B(4\,,-2)$

$3 \quad : \quad 2$

$$P\left(\frac{3\times 4\ +\ 2\times(-3)}{3+2}\ ,\ \frac{3\times(-2)\ +2\times 7}{3+2}\right)$$

Hence the point of internal division is $P\left(\frac{6}{5}\,,\frac{8}{5}\right)$.

c. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• uses double angle identities for sine and cosine • rearranges and factorises resulting equation • solves $\cos x = 0$ in required domain • solves $\tan x = 1$ in required domain	1 1 1 1

Answer

$$1+\cos 2x = \sin 2x \qquad , \quad 0 \le x \le 2p$$

$$2\cos^2 x = 2\sin x\cos x$$

$$\cos x(\cos x - \sin x) = 0$$

$\therefore \cos x = 0 \quad$ *or* $\quad \cos x = \sin x$

$\qquad\qquad\qquad\qquad 1 = \tan x$

$\therefore\ x = \frac{p}{2}, \frac{3p}{2} \quad$ *or* $\quad x = \frac{p}{4}, \frac{5p}{4}$

$\therefore\ x = \frac{p}{4}\,, \frac{p}{2}\,, \frac{5p}{4}\,, \frac{3p}{2}$

d. Outcomes assessed : P4, PE3

Marking Guidelines

Criteria	Marks
i • finds gradients of OP and OQ and sets product equal to -1 ii • uses appropriate rectangle property to find the coordinates of R. iii • writes y coordinate of R in terms of sum and product of p and q. • substitutes for sum and product of p and q to find Cartesian equation.	1 1 1 1

Answer

i Gradient $OP = \dfrac{ap^2}{2ap} = \frac{1}{2}p$. $\quad$ Similarly gradient $OQ = \frac{1}{2}q$.

$\therefore OP \perp OQ \ \Rightarrow\ \frac{1}{2}p\,.\,\frac{1}{2}q = -1 \qquad \therefore\ pq = -4$

ii The diagonals of a rectangle bisect each other. Hence M is the midpoint of OR.

Hence at R, $\frac{1}{2}(x+0) = a(p+q)$ and $\frac{1}{2}(y+0) = \frac{1}{2}a(p^2+q^2)$.

$$\therefore\ x = 2a(p+q) \quad \text{and} \quad y = a(p^2+q^2)$$

iii At R, $y = a\left\{(p+q)^2 - 2pq\right\} = a\left\{\left(\dfrac{x}{2a}\right)^2 + 8\right\}$

Hence locus of R has equation $x^2 = 4a(y-8a)$.

Question 3

a. Outcomes assessed : P5, P8, H6, HE4

Marking Guidelines

Criteria	Marks
i • shows $f(-x)=f(x)$ to deduce function f is even	1
ii • shows formally that required limit is 1	1
iii • finds the first derivative, showing it is zero at the origin	1
• shows the origin is a maximum turning point by applying first or second derivative test	1
iv • shows the two vertical asymptotes and the central branch of the curve	1
• shows the horizontal asymptote and the remaining branches of the curve	1
v • makes x the subject, interchanges x and y to obtain equation for the inverse g^{-1}	1
• writes the domain of the inverse function	1

Answer

i. $f(-x)=\dfrac{(-x)^2}{(-x)^2-1}=\dfrac{x^2}{x^2-1}=f(x)$,

$x\neq\pm1$. Hence f is an even function.

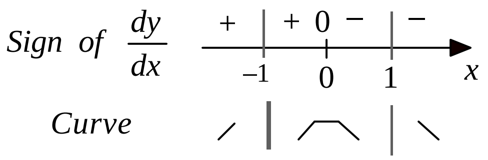
ii. $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\dfrac{1}{1-\frac{1}{x^2}}=\dfrac{1}{1-0}=1$

Curve has horizontal asymptote $y=1$ as $x\to\pm\infty$

iii. $\dfrac{dy}{dx}=\dfrac{2x\left(x^2-1\right)-x^2.2x}{\left(x^2-1\right)^2}$

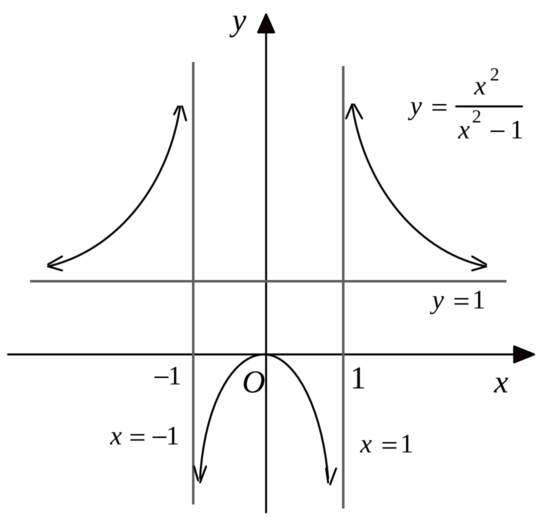
$=\dfrac{-2x}{\left(x^2-1\right)^2}$

$\therefore \dfrac{dy}{dx}=0$ when $x=0$



Hence $(0,0)$ is a maximum turning point.

iv.



v. $y=\dfrac{x^2}{x^2-1}$, $x\geq0$

$y\left(x^2-1\right)=x^2$

$yx^2-y=x^2$

$yx^2-x^2=y$

$x^2(y-1)=y$

$x^2=\dfrac{y}{y-1}$

$\therefore$ for the function g, $x=\sqrt{\dfrac{y}{y-1}}$, since $x\geq0$.

Interchanging x and y, $g^{-1}(x)=\sqrt{\dfrac{x}{x-1}}$.

Inspection of the graph of $y=f(x)$ shows that the range of the function g is $\{y: y\leq0 \text{ or } y>1\}$.

Hence the domain of the inverse function g^{-1} is $\{x: x\leq0 \text{ or } x>1\}$.

b. Outcomes assessed : HE2

Marking Guidelines

Criteria	Marks
• verifies that statement true for $n=1$	1
• writes LHS of $(k+1)^{\text{th}}$ statement in terms of RHS of k^{th} statement (assumed true)	1
• rearranges resulting expression into form of RHS of $(k+1)^{\text{th}}$ statement	1
• deduces the required result, showing understanding of the process of mathematical induction	1

Answer

Let $S(n)$, $n=1, 2, 3, \ldots$ be the sequence of statements $\displaystyle\sum_{r=1}^{n}\frac{r+2}{r\,(r+1)2^{r}} = 1-\frac{1}{(n+1)2^{n}}$, $n=1, 2, 3, \ldots$

Consider $S(1)$: $\quad LHS = \dfrac{3}{1\times 2\times 2} = \dfrac{4-1}{2\times 2^{1}} = 1-\dfrac{1}{2\times 2^{1}} = RHS$. $\quad\therefore S(1)$ is true.

If $S(k)$ is true : $\quad \displaystyle\sum_{r=1}^{k}\frac{r+2}{r\,(r+1)2^{r}} = 1-\frac{1}{(k+1)2^{k}} \quad **$

Consider $S(k+1)$:

$$
\begin{aligned}
LHS &= \sum_{r=1}^{k+1}\frac{r+2}{r\,(r+1)2^{r}} \\
&= \sum_{r=1}^{k}\frac{r+2}{r\,(r+1)2^{r}} + \frac{(k+1)+2}{(k+1)(k+2)2^{k+1}} \\
&= 1 - \frac{1}{(k+1)2^{k}} + \frac{(k+1)+2}{(k+1)(k+2)2^{k+1}} \qquad \textit{if } S(k) \textit{ is true, using } ** \\
&= 1 - \frac{2(k+2)-(k+3)}{(k+1)(k+2)2^{k+1}} \\
&= 1 - \frac{k+1}{(k+1)(k+2)2^{k+1}} \\
&= 1 - \frac{1}{(k+2)2^{k+1}} \\
&= RHS
\end{aligned}
$$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true, and then $S(3)$ is true and so on. Hence by mathematical induction $S(n)$ is true for all positive integers $n\ge 1$.

Question 4

a. Outcomes assessed : HE4

Marking Guidelines

Criteria	Marks
• makes x the subject of the equation of the curve	1
• expresses the volume as a definite integral with respect to y with integrand $\tan^{2}\left(\frac{1}{2}y\right)$	1
• uses an appropriate trig. identity to find the primitive function	1
• substitutes the limits to evaluate the exact volume	1

Answer

$$y = 2\tan^{-1} x$$
$$\tfrac{1}{2} y = \tan^{-1} x$$
$$\tan\left(\tfrac{1}{2} y\right) = x$$

Hence volume is V cubic units where

$$V = p\int_0^{\frac{p}{2}} \tan^2\left(\tfrac{1}{2} y\right)\ dy$$

$$V = p\int_0^{\frac{p}{2}} \left\{\sec^2\left(\tfrac{1}{2} y\right) - 1\right\}\ dy$$
$$= p\ \left[2\tan\left(\tfrac{1}{2} y\right) - y\right]_0^{\frac{p}{2}}$$
$$= p\ \left\{2\left(\tan\tfrac{p}{4} - \tan 0\right) - \left(\tfrac{p}{2} - 0\right)\right\}$$
$$= p\ \left\{2 - \tfrac{p}{2}\right\}$$

Hence volume is $\tfrac{1}{2} p\ (4 - p)$ cubic units.

b. Outcomes assessed : H5, PE3

Marking Guidelines

Criteria	Marks
i • writes equation using expressions for areas of segment and sector	1
• simplifies to obtain required equation	1
ii • writes second approximation in terms of $f(2)$, $f'(2)$ where $f(q) = q - 2\sin q$	1
• evaluates expression for second approximation correct to 2 decimal places	1

Answer

i. *area segment* $= \tfrac{1}{2}$ *area sector*

$$\tfrac{1}{2} r^2 q - \tfrac{1}{2} r^2 \sin q = \tfrac{1}{4} r^2 q$$
$$\tfrac{1}{4} r^2 q - \tfrac{1}{2} r^2 \sin q = 0$$
$$r^2\left(q - 2\sin q\right) = 0$$
$$\therefore r \neq 0 \ \Rightarrow\ q - 2\sin q = 0$$

ii. Let $f(q) = q - 2\sin q$

Then $f'(q) = 1 - 2\cos q$

Using Newton's method with $q_1 = 2$,

$$q_2 = 2 - \frac{f(2)}{f'(2)} \approx 2 - \frac{0\cdot 1814}{1\cdot 8323}$$

Hence second approximation is $1\cdot 90$ (to 2 dec. pl.)

c. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • writes numerical expression for required probability	1
• evaluates probability as a fraction	1
ii • writes numerical expression for required probability	1
• evaluates probability as a fraction	1

Answer

Probability distribution is Binomial with $n = 6$, $p = \tfrac{1}{3}$, $q = \tfrac{2}{3}$.

i. $P\left(exactly\ 2\ correct\right) = {}^6C_2 \left(\tfrac{1}{3}\right)^2 \left(\tfrac{2}{3}\right)^4 = 15 \times \tfrac{16}{729} = \tfrac{80}{243}$

ii. $P\left(exactly\ 1\ correct\ out\ of\ \ first\ 5,\ then\ 6^{th}\ correct\right) = {}^5C_1 \left(\tfrac{1}{3}\right)^1 \left(\tfrac{2}{3}\right)^4 \times \tfrac{1}{3} = 5 \times \tfrac{16}{243} \times \tfrac{1}{3} = \tfrac{80}{729}$

Question 5

a. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• writes dx in terms of du and converts x limits to u limits	1
• writes integrand in terms of u	1
• finds primitive function in terms of u	1
• evaluates integral in simplest exact form by substitution of limits	1

Answer

$$u = x - 1$$
$$du = dx$$

$$x = 0\cdot 5 \Rightarrow u = -0\cdot 5$$
$$x = 1\cdot 5 \Rightarrow u = 0\cdot 5$$

$$2x - x^2 = 2(u+1) - (u^2 + 2u + 1)$$
$$= 1 - u^2$$

$$\int_{0\cdot 5}^{1\cdot 5} \frac{1}{\sqrt{2x - x^2}}\, dx = \int_{-0\cdot 5}^{0\cdot 5} \frac{1}{\sqrt{1-u^2}}\, du$$
$$= \left[\sin^{-1} u\right]_{-0\cdot 5}^{0\cdot 5}$$
$$= \tfrac{\pi}{6} - \left(-\tfrac{\pi}{6}\right)$$
$$= \tfrac{\pi}{3}$$

b. Outcomes assessed : P4, HE5, HE7

Marking Guidelines

Criteria	Marks
i • uses similar triangles or tangent ratio to write r in terms of h	1
ii • writes $\frac{dr}{dt}$ in terms of $\frac{dh}{dt}$	1
• substitutes values of h and $\frac{dh}{dt}$	1
• finds required rate	1

Answer

i. The ray of light from P makes equal angles with the horizontal in both right triangles. Corresponding sides in these similar triangles are in proportion.

$$\therefore \frac{r}{6} = \frac{10}{h} \text{ and hence } r = \frac{60}{h}$$

ii. $$\frac{dr}{dt} = \frac{dr}{dh} \cdot \frac{dh}{dt} = -\frac{60}{h^2} \times \frac{dh}{dt}$$

But $\frac{dh}{dt} = -0\cdot 1$. Hence when $h = 5$,

$$\frac{dr}{dt} = \frac{60}{25} \times 0\cdot 1 = 0\cdot 24$$

Hence r is increasing at a rate of $0\cdot 24 \text{ cm s}^{-1}$.

c. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • differentiates $\frac{1}{2}v^2$ to find a in terms of x	1
ii • states the centre of the motion	1
• states the amplitude of the motion	1
iii • finds the maximum speed	1

Answer

i. $a = \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = \dfrac{d}{dx}\left(16 + 4x - 2x^2\right) \qquad \therefore a = 4 - 4x$

ii. $v^2 = 4\left(-x^2 + 2x + 8\right)$

$\therefore v^2 = 4\left(x+2\right)\left(4-x\right)$

$v^2 \geq 0 \;\Rightarrow\; -2 \leq x \leq 4$

The midpoint of this interval is $x = 1$.
Hence centre of motion is 1 m to the right of O
and the amplitude is 3 m.

iii. Maximum speed occurs at the centre of the motion.
$x = 1 \;\Rightarrow\; v^2 = 36.$ Hence maximum speed is $6\ \text{ms}^{-1}$

Question 6

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • states maximum rate of flow	1
ii • expresses total amount of water as a definite integral	1
• uses an appropriate trig. identity to find the primitive	1
• evaluates by substitution of limits, giving answer to nearest litre	1

Answer

i.
$0 \leq \sin^2 t \leq 1$

$\therefore\ 0 \leq R \leq 4$

Maximum rate of flow is 4 kL/min ,
since $R = 4$ when $t = \frac{p}{2}$.

ii. $\displaystyle\int_0^p 4\sin^2 t \ \ dt = 2\int_0^p \left(1 - \cos 2t\right)\ dt$

$= \left[2t - \sin 2t\right]_0^p$

$= 2\left(p - 0\right) - \left(\sin 2p - \sin 0\right)$

$= 2p$

$\therefore 2p$ kL $\approx 6\cdot 283$ kL (to the nearest L) flows into the tank.

b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • substitutes one pair of N, t values to obtain one equation in A and B	1
• similarly obtains a second equation in A and B	1
• solves simultaneously to evaluate A and B	1
ii • states limiting value of N.	1

Answer

i. $N = A + Be^{-t}$

$60 = A + B\,e^{-\ln 2} \qquad\qquad 36 = A + B\,e^{-\ln 5}$

$= A + B\,e^{\ln\frac{1}{2}} \qquad\qquad = A + B\,e^{\ln\frac{1}{5}}$

$= A + \frac{1}{2}B \qquad\qquad = A + \frac{1}{5}B$

$\therefore\ 120 = 2A + B$ and $180 = 5A + B$

By subtraction, $3A = 60$
$\therefore A = 20$ and $B = 80$

ii. As $t \to \infty$, $N \to A + B \times 0 = 20$
Hence limiting population size is 20.

c. Outcomes assessed : P4, HE5

Marking Guidelines

Criteria	Marks
i • establishes result algebraically	1
ii • writes $\frac{dt}{dx}$ as sum of two algebraic fractions using (i)	1
• integrates and evaluates constant to find t as a function of x	1
• rearranges to find x as a function of t	1

Answer

i. $\frac{1}{x}+\frac{1}{2-x}=\frac{(2-x)+x}{x(2-x)} = \frac{2}{x(2-x)}$

ii. Initially particle is at $x=1$ moving right with $v=\frac{1}{2}$.

But $v=\frac{x(2-x)}{2}$ and $a=v\frac{dv}{dx}$. Hence if particle reaches $x=2$, $v=a=0$ and particle will remain at rest at this point. Hence $1\le x\le 2$.

$$\frac{dx}{dt}=\frac{x(2-x)}{2}$$
$$\frac{dt}{dx}=\frac{2}{x(2-x)}$$
$$=\frac{1}{x}+\frac{1}{2-x}$$
$$t=\ln x-\ln(2-x)+c$$
$$=\ln\left(\frac{x}{2-x}\right)+c \qquad (c \text{ constant})$$
$$\left.\begin{array}{l} t=0\\ x=1\end{array}\right\}\Rightarrow \begin{array}{l}\ln 1+c=0\\ \therefore\ c=0\end{array}$$

$$\therefore\quad t=\ln\left(\frac{x}{2-x}\right)$$
$$-t=\ln\left(\frac{2-x}{x}\right)$$
$$e^{-t}=\frac{2-x}{x}$$
$$e^{-t}=\frac{2}{x}-1$$
$$1+e^{-t}=\frac{2}{x}$$
$$\therefore\quad x=\frac{2}{1+e^{-t}}$$

Question 7

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • uses integration to find expression for x	1
• uses integration to find expression for y	1
ii • substitutes given values to write two equations in V and α	1
• finds exact value of V	1
• finds required approximate value of α to required accuracy	1
iii • finds horizontal and vertical components of impact velocity	1
• finds speed of impact to required accuracy	1
• finds angle of impact to required accuracy	1

Answer

i. *Horizontal component* | *Vertical component*

$\ddot{x} = 0$

$\dot{x} = c_1$, c_1 *const*.

$\left.\begin{array}{l} t = 0 \\ \dot{x} = V\cos q \end{array}\right\} \Rightarrow c_1 = V\cos q \quad \therefore\ \dot{x} = V\cos q$

$x = \left(V\cos q\right)t + c_2$, c_2 *const*

$\left.\begin{array}{l} t = 0 \\ x = 0 \end{array}\right\} \Rightarrow c_2 = 0 \quad \therefore\ x = \left(V\cos q\right)t$

$\ddot{y} = -10$

$\dot{y} = -10\,t + c_3$, c_3 *const*.

$\left.\begin{array}{l} t = 0 \\ \dot{y} = V\sin q \end{array}\right\} \Rightarrow c_3 = V\sin q \quad \therefore\ \dot{y} = -10\,t + V\sin q$

$y = -5\,t^2 + \left(V\sin q\right)t + c_4$, c_4 *const*

$\left.\begin{array}{l} t = 0 \\ y = 0 \end{array}\right\} \Rightarrow c_4 = 0 \quad \therefore\ y = \left(V\sin q\right)t - 5t^2$

ii. When $t = 4$, $x = 64$ and $y = -32$

$$\left.\begin{array}{r} 4\,V\cos q = 64 \\ 4V\sin q - 80 = -32 \end{array}\right\} \quad \therefore \left.\begin{array}{l} V\cos q = 16 \\ V\sin q = 12 \end{array}\right\}$$

$$\therefore V^2\left(\cos^2 q + \sin^2 q\right) = 16^2 + 12^2$$

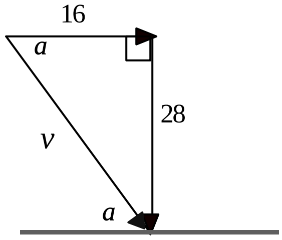
$$\therefore V^2 = 4^2\left(4^2 + 3^2\right)$$

Also $\cos q = \frac{4}{5}$ and $\sin q = \frac{3}{5}$

$\therefore V = 20, \quad q \approx 36°52'$

iii. When $t = 4$,

$\dot{x} = V\cos q = 16$ and $\dot{y} = -40 + V\sin q = -28$



$v^2 = 16^2 + 28^2 \Rightarrow v \approx 32 \cdot 2$

$\tan a = \frac{7}{4} \Rightarrow a \approx 60°15'$

Speed of impact is $32\ \mathrm{ms}^{-1}$ (to nearest integer)
Angle of impact with beach is $60°15'$ (nearest minute).

b. Outcomes assessed : H9, HE3

Marking Guidelines

Criteria	**Marks**
i • writes expansion as required	1
ii • differentiates both sides with respect to x	1
• substitutes $x = 1$	1
• rearranges to obtain required identity	1

Answer

i. $x\left(1 + x\right)^n \equiv x + {}^nC_1 x^2 + {}^nC_2 x^3 + \ldots + {}^nC_{n-1} x^n + x^{n+1}$

ii. Differentiation with respect to x gives

$$\left(1 + x\right)^n + nx\left(1 + x\right)^{n-1} \equiv 1 + 2\ {}^nC_1 x + 3\ {}^nC_2 x^2 + \ldots + n\ {}^nC_{n-1} x^{n-1} + \left(n + 1\right)x^n$$

Substituting $x = 1$, $\quad 2^n + n \,.\, 2^{n-1} = 1 + 2\ {}^nC_1 + 3\ {}^nC_2 + \ldots + n\ {}^nC_{n-1} + \left(n + 1\right)$

$\therefore\ 2\ {}^nC_1 + 3\ {}^nC_2 + \ldots + n\ {}^nC_{n-1} = \left(n + 2\right)2^{n-1} - \left(n + 2\right) = \left(n + 2\right)\left(2^{n-1} - 1\right)$

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$