2004 Extension 1 Trial Paper Solns

(a) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{3}}$ — $\frac{1}{2}$

$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}0$ — $\frac{1}{2}$

$= \frac{\pi}{3}$ — 1

(b) A $(3,-2)$ B $(1,4)$

$5 : -2$

$x = \frac{-2\times3+5\times1}{5-2}$ $\quad y = \frac{-2\times-2+4\times5}{3}$ — 1

$= \left(\frac{-1}{3}, 8\right)$ — 1

(c) $\frac{X^5Y^3}{Z^2} = \left(\frac{4}{3}\right)^{15}\left(\frac{9}{2}\right)^{12} \div \left(\frac{3}{8}\right)^4$ — 1

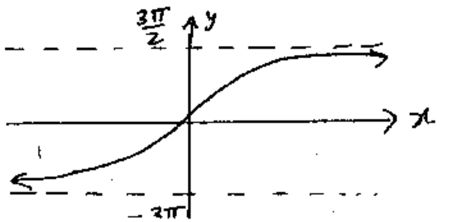
$= \frac{(2^2)^{15}}{3^{15}} \times \frac{(3^2)^{12}}{2^{12}} \times \frac{(2^3)^4}{3^4}$ — 1

$= \frac{2^{30}\times3^{24}\times2^{12}}{3^{15}\times2^{12}\times3^4}$

$= \frac{2^{42}\times3^{24}}{3^{19}\times2^{12}} = 2^{30}\times3^5$ — 1

(d) Domain : all real x

Range : $-\frac{3\pi}{2} < y < \frac{3\pi}{2}$ — 1 (dom + range)



1 (graph shape)

(e) $\int_0^1 x(2-x)^3\,dx$ $\quad$ $u = 2-x$, $du = -dx$, $x=0$ $u=2$, $x=1$ $u=1$ — 1 (RI par.)

$= \int_2^1 (2-u)(u)^3 \times -du$ — $\frac{1}{2}$ (subst)

$= \int_1^2 2u^3 - u^4\,du$ — $\frac{1}{2}$

$= \left[\frac{2u^4}{4} - \frac{u^5}{5}\right]_1^2$ — $\frac{1}{2}$

$= \frac{2\times2^4}{4} - \frac{2^5}{5} - \left(\frac{2\times1^4}{4} - \frac{1}{5}\right)$ — $\frac{1}{2}$

$= 1\frac{3}{10}$ or $\frac{13}{10}$

2) (a) $f(x) = x^3+2x-8$

$f'(x) = 3x^2+2$ — $\frac{1}{2}$

$f(1.6) = 1.6^3 - 2\times1.6 - 8$ — $\frac{1}{2}$

$= -0.704$

$f'(1.6) = 3\times1.6^2+2$ — $\frac{1}{2}$

$= 9.68$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ — $\frac{1}{2}$

$= 1.6 - \frac{(-0.704)}{9.68}$ — $\frac{1}{2}$

$= 1.6729..$

$= 1.7$ to 1 dec place — $\frac{1}{2}$

(b) $\left(x+\frac{5}{x^2}\right)^9$

General term ${}^9C_k\, x^{9-k}\left(\frac{5}{x^2}\right)^k$

$= {}^9C_k\, x^{9-k}\, 5^k \times x^{-2k}$ — 1

$= {}^9C_k\, x^{9-3k} \times 5^k$ — $\frac{1}{2}$

$\therefore 9-3k = 3$ — $\frac{1}{2}$

$3k = 6 \quad k=2$

coeff $= {}^9C_2\, 5^2 = 900$ — 1

(c) $\int \sin^2 6x\,dx = \frac{1}{2}\int(1-\cos 12x)\,dx$ — 1

$= \frac{1}{2}\left[x - \frac{1}{12}\sin 12x\right]$ — 1

$= \frac{x}{2} - \frac{\sin 12x}{24} + c$

(d)(i) $6\cos x + 8\sin x = R\cos(x-\alpha)$

$= R\cos x\cos\alpha + R\sin x\sin\alpha$ — $\frac{1}{2}$

$\therefore R\cos\alpha = 6$

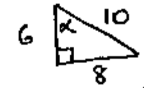
$R\sin\alpha = 8$

$R^2 = 6^2+8^2 \quad \therefore R = 10$ — $\frac{1}{2}$

$\tan\alpha = \frac{8}{4} \quad \alpha = 0.927$ — 1

$\therefore 6\cos x + 8\sin x = 10\cos(x-0.927)$

or $\quad 6\cos x + 8\sin x = 10\left(\frac{6}{10}\cos x + \frac{8}{10}\sin x\right)$

$= 10\cos(x-\alpha)$ $\quad$ 

$= 10\cos(x-0.927)$ $\quad \alpha = \tan^{-1}\left(\frac{8}{6}\right)$

(ii) $6\cos x + 8\sin x = 5 \quad 0 \le x \le 2\pi$

$10\cos(x-0.927) = 5$

$0 \le x - 0.927 \le 2\pi$

$0.927 \le x \le 7.210$ — $\frac{1}{2}$

$\cos(x-0.927) = \frac{1}{2}$

$x - 0.927 = \frac{\pi}{3}, \; 2\pi - \frac{\pi}{3}$ — $\frac{1}{2}$

$x = \frac{\pi}{3} + 0.927, \; \frac{5\pi}{3} + 0.927$ — 1

or $1.974, \; 6.163$

3(a) (i) $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$ — 1

(ii) Let $x = \tan^{-1}\left(\frac{1}{4}\right), \; y = \tan^{-1}\left(\frac{3}{5}\right)$

Then $\tan x = \frac{1}{4} \quad \tan y = \frac{3}{5}$ — $\frac{1}{2}$

$\tan(x+y) = \frac{1}{4} + \frac{3}{5}$ — $\frac{1}{2}$

3(ii) (cont)

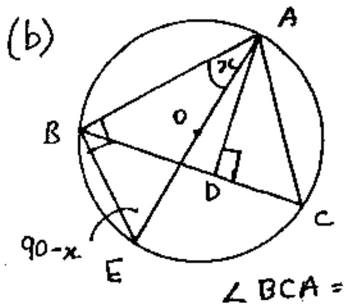
$\tan(x+y) = 1$ ½

$x+y = \tan^{-1}(1)$

$= \frac{\pi}{4}$ ½

$\therefore \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{4}$

(b)



Let $\angle BAE = x$

$\angle ABE = 90^\circ$ (angle in a semicircle) 1

$\therefore \angle BEA = 90 - x$ (angle sum of $\triangle$) 1

$\angle BCA = 90 - x$ ($\angle$'s in same segment) 1

$\therefore \angle DAC = x$ (angle sum of $\triangle$) 1

$\therefore \angle BAE = \angle DAC$

(c) $\frac{dS}{dt} = 0\cdot01$ ½

$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$

$0\cdot01 = 8\pi r \times \frac{dr}{dt}$ ½

$\frac{dr}{dt} = \frac{0\cdot01}{8\pi \times 5} = \frac{0\cdot01}{40\pi}$ ½

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$= 4\pi r^2 \times \frac{0\cdot01}{40\pi}$ ½

$= 4\pi \times 25 \times \frac{0\cdot01}{40\pi}$ ½

$= \underline{0\cdot025 \text{ cm}^3/\text{sec}}$ ½

(d) $\text{LHS} = \frac{2\sin x\cos x}{1 + (2\cos^2 x - 1)}$ ½ ½

$= \frac{\cancel{2}\sin x\cos x}{\cancel{2}\cos^2 x}$

$= \tan x$ 1

4) (i) $f(10) = f(5\times2)$

$= f(5) + f(2)$

$= 12 + 5 = \underline{17}$ 1

(ii) $f(\sqrt{5}) = f(5^{\frac{1}{2}})$

$= \frac{1}{2}f(5)$

$= \frac{1}{2} \times 12 = \underline{6}$ 1

(iii) $f(1) = f(5^0)$

$= 0 \times f(5) = \underline{0}$ 1

(b) $\log_2 x(x-2) = 3$ ½

$2^3 = x^2 - 2x$ 1

$x^2 - 2x - 8 = 0$

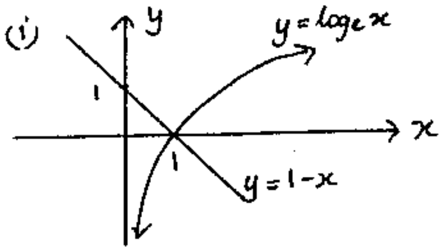
$(x-4)(x+2) = 0$ ½

$x = 4$ or -2 ½

But $x > 2$ (domain of $\log_e(x-2)$) ½

so $\underline{x = 4 \text{ is only soln}}$

(c) (i)



½ (1,0) 1 (ln x) ½ (line)

(ii) $0 < x \leq 1$ ½ (>0) ½ (≤1)

d) $y = x\tan^{-1}x$

$\frac{dy}{dx} = x \times \frac{1}{1+x^2} + \tan^{-1}x$ ½ prod rule

$= \frac{x}{1+x^2} + \tan^{-1}x$ ½

$\int_0^1 \frac{x}{1+x^2} + \tan^{-1}x \, dx = \left[x\tan^{-1}x\right]_0^1$

$\therefore \int_0^1 \tan^{-1}x \, dx = \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2} dx$ 1

$= 1\tan^{-1}(1) - 0\tan^{-1}0 - \frac{1}{2}\left[\ln(1+x^2)\right]_0^1$ ½

$= \frac{\pi}{4} - \frac{1}{2}(\ln(2) - \ln 1)$ ½

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

5)(a)(i) [illegible]

$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$

$= \frac{-4}{2} = \underline{-2}$

(ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$\alpha+\beta+\gamma = -\frac{b}{a} = \frac{6}{2} = 3$

$\therefore \alpha^2+\beta^2+\gamma^2 = 3^2 - 2\times-2$

$= \underline{13}$

(b)(i) $N = 2000 + Ae^{kt}$ (1)

$\frac{dN}{dt} = kAe^{kt}$

$\frac{dN}{dt} = k(N - 2000)$ from (1)

$\therefore N = 2000 + Ae^{kt}$ is a solution

(ii) $t = 0$ $N = 5000$

$t = 2$ $N = 6000$

$5000 = 2000 + Ae^0$

$\therefore \underline{A = 3000}$ ½

$6000 = 2000 + 3000e^{2k}$ ½

$4000 = 3000e^{2k}$

$\frac{4}{3} = e^{2k}$ ½

$2k = \ln\left(\frac{4}{3}\right)$ $\quad k = \frac{1}{2}\ln\left(\frac{4}{3}\right)$ ½

$\underline{\div 0\cdot143841}$

(iii) $N = 2000 + 3000e^{kt}$

When $t = 5$

$N = 2000 + 3000e^{0.143841\times5}$ ½

$= 2000 + 6158.4029$

$= 8158.4029$

There are $\underline{8158 \text{ sheep}}$ ½

(i) $P(\text{work}) = 0.6$

$P\left(\substack{\text{at least one}\\ \text{works}}\right) = 1 - P\left(\substack{\text{none}\\ \text{work}}\right)$

$= 1 - P(\tilde{w}\tilde{w}\tilde{w})$ — $\frac{1}{2}$

$= 1 - (0.4 \times 0.4 \times 0.4)$ — 1

$= \underline{0.936}$ — $\frac{1}{2}$

(ii) $1 - P\left(\substack{\text{none}\\ \text{work}}\right) = 0.99$

$1 - (0.4)^n = 0.99$ — $\frac{1}{2}$

$(0.4)^n = 0.01$

$n \ln(0.4) = \ln(0.01)$ — $\frac{1}{2}$

$n = \dfrac{\ln(0.01)}{\ln(0.4)}$

$= 5.0258832$ — $\frac{1}{2}$

$\therefore$ You need $\underline{6 \text{ components}}$ — $\frac{1}{2}$

(a) Both are correct — 1

Louise: $\frac{1}{2}\log 2x + k$

$= \frac{1}{2}(\log 2 + \log x) + k$ — $\frac{1}{2}$

$= \frac{1}{2}\log x + \underline{\frac{1}{2}\log 2 + k}$ — $\frac{1}{2}$

This constant is the same as the constant c in Mary's answer.

) $13 \times 6^n + 2$ is divisible by 5

Prove true for $n=1$

$13 \times 6^1 + 2 = 80$ which is $\div$ by 5 — $\frac{1}{2}$

Assume it is true for $n=k$

$13 \times 6^k + 2 = 5m$ (1) m is an integer. — $\frac{1}{2}$

Prove true for $n=k+1$ — $\frac{1}{2}$

$13 \times 6^{k+1} + 2 = 13 \times 6^k \times 6^1 + 2$ — $\frac{1}{2}$

$= (5m-2)6 + 2$ (from (1))

$= 30m - 12 + 2$

$= 30m - 10$ which is $\div$ — $\frac{1}{2}$

true for $n = k+1$. Since it is true for $n=1$ it is true for $n = 1+1 = 2$ and so true for all positive integral n. — $\frac{1}{2}$

(c)(i) $y - \frac{1}{2}(p+q)x + 3pq = 0$

sub $(4, -3)$

$-3 - \frac{1}{2}(p+q) \times 4 + 3pq = 0$ — $\frac{1}{2}$

$-3 - 2p - 2q + 3pq = 0$ — $\frac{1}{2}$

$3pq = 3 + 2(p+q)$

(ii) $y = \dfrac{x^2}{12}$

$\dfrac{dy}{dx} = \dfrac{2x}{12}$ — $\frac{1}{2}$

At $x = 6p$ $\quad \dfrac{dy}{dx} = \dfrac{12p}{12} = p$ — $\frac{1}{2}$

$y - 3p^2 = p(x - 6p)$ — 1

$y = px - 3p^2$

(iii) tangent at Q has eqn $y = qx - 3q^2$

$px - 3p^2 = qx - 3q^2$ — $\frac{1}{2}$

$px - qx = 3p^2 - 3q^2$ — $\frac{1}{2}$

$x(p-q) = 3(p-q)(p+q)$ — $\frac{1}{2}$

$x = 3(p+q)$ $\quad$ since $p - q \neq 0$

When $x = 3(p+q)$ $\quad y = p \times 3(p+q) - 3p^2 = 3pq$ — $\frac{1}{2}$

$T\,(3(p+q), 3pq)$

(iv) From (i) $3pq = 3 + 2(p+q)$ — $\frac{1}{2}$

From (iii) $x = 3(p+q)$ $\quad y = 3pq$ — $\frac{1}{2}$

$\therefore \underline{y = 3 + 2\left(\frac{x}{3}\right)}$ by substitution

$\underline{3y = 9 + 2x}$ or $\underline{y = \frac{2x}{3} + 3}$

This is a straight line with gradient $\frac{2}{3}$, y intercept 3 — 1

7 (a) $\lim\limits_{x \to 0} \dfrac{\sin 3x}{7x} = \dfrac{3}{7} \lim\limits_{x \to 0} \dfrac{\sin 3x}{3x}$

$= \underline{\underline{\dfrac{3}{7}}}$ — 1

$\dfrac{d}{dx}\left(\frac{1}{2}v^2\right) = 4x + 2$ — 1

$\frac{1}{2}v^2 = \int 4x + 2 \, dx$ — $\frac{1}{2}$

$\frac{1}{2}v^2 = 2x^2 + 2x + c$ — $\frac{1}{2}$

When $x = 0$ $v = -1$ $\quad \frac{1}{2} = c$

$\frac{1}{2}v^2 = 2x^2 + 2x + \frac{1}{2}$ — $\frac{1}{2}$

$\underline{\underline{v^2 = 4x^2 + 4x + 1}}$

(ii) $v^2 = (2x+1)^2$ — $\frac{1}{2}$

$v = \pm(2x+1)$

but initially $x=0$ $v=-1$ and motion stops at $x = -\frac{1}{2}$ (as $v = 0$, $\ddot{x} = 0$) so

$v = -(2x+1)$ — $\frac{1}{2}$

$\dfrac{dx}{dt} = -(2x+1)$ $\quad \therefore \dfrac{dt}{dx} = \dfrac{-1}{2x+1}$ — $\frac{1}{2}$

$t = -\frac{1}{2}\ln(2x+1) + k$ — $\frac{1}{2}$

$t = 0$ $\; x = 0$ $\quad \therefore 0 = -\frac{1}{2}\ln 1 + k$ $\; k = 0$ — $\frac{1}{2}$

$t = -\frac{1}{2}\ln(2x+1)$

$-2t = \ln(2x+1)$

$2x + 1 = e^{-2t}$ — $\frac{1}{2}$

$2x = e^{-2t} - 1$ $\qquad x = \frac{1}{2}\left(e^{-2t} - 1\right)$

(iii) As $t \to \infty$ $\; e^{-2t} \to 0$ so $\underline{x \to -\frac{1}{2} \text{ from above}}$ — 1

(c) (i) $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$ — 1

(ii) $S = 1 + (1+x) + \ldots (1+x)^n$ — $\frac{1}{2}$

Geometric series $a = 1$ $r = (1+x)$ $n = n+1$ — $\frac{1}{2}$

so $S = \dfrac{a(r^n - 1)}{r-1} = \dfrac{1\left((1+x)^{n+1} - 1\right)}{(1+x) - 1}$ — $\frac{1}{2}$ — $\frac{1}{2}$

(iii) $(1+x)^{n+1} = 1 + \binom{n+1}{1}x^1 + \binom{n+1}{2}x^2 + \ldots + \binom{n+1}{n+1}x^{n+1}$ — $\frac{1}{2}$

$S = \dfrac{(1+x)^{n+1} - 1}{x}$

$= \dfrac{\left[1 + \binom{n+1}{1}x + \binom{n+2}{2}x^2 + \ldots + \binom{n+1}{n+1}x^{n+1}\right] - 1}{x}$ — 1

$= \dfrac{\cancel{x}\left[\binom{n+1}{1} + \binom{n+2}{2}x + \ldots + \binom{n+1}{n+1}x^n\right]}{\cancel{x}}$ — $\frac{1}{2}$

$S = \binom{n+1}{1} + \binom{n+1}{2}x + \ldots \binom{n+1}{r+1}x^r + \ldots + \binom{n+1}{n+1}x^n$ — 1