

FORM VI MATHEMATICS EXTENSION 2

Time allowed: 3 hours (plus 5 minutes reading time) **Exam date:** 6th August 2003

Instructions:

- All questions may be attempted.
- All questions are of equal value.
- Part marks are shown in boxes in the right margin.
- All necessary working must be shown.
- Marks may not be awarded for careless or badly arranged work.
- Approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Collection:

- The writing booklets will be collected in one bundle.
- Start each question in a new writing booklet.
- If you use a second booklet for a question, place it inside the first. Don't staple.
- Write your candidate number on each booklet.

Checklist:

- SGS Writing Booklets required — eight booklets per boy.
- Candidature: 54 boys.

QUESTION ONE (Start a new writing booklet)

(a) Find $\int \frac{\sin x}{\cos^5 x} dx$.

Marks

1

(b) Use completion of squares to evaluate $\int_{-2}^{-1} \frac{5}{x^2 + 4x + 5} dx$.

3

(c) (i) Find the real numbers A , B and C such that

3

$$\frac{3x^2 - x + 8}{(1-x)(x^2 + 1)} \equiv \frac{A}{1-x} + \frac{Bx + C}{x^2 + 1}.$$

(ii) Hence find $\int \frac{3x^2 - x + 8}{(1-x)(x^2 + 1)} dx$.

2

(d) Use integration by parts to show that $\int_1^4 \frac{\ln x}{\sqrt{x}} dx = 4(2 \ln 2 - 1)$.

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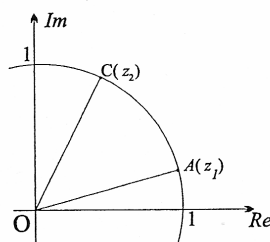
(e) Use the substitution $t = \tan \frac{\theta}{2}$ to find $\int \frac{1}{1 + \cos \theta} d\theta$.

3

QUESTION TWO (Start a new writing booklet)

Marks

- (a) Find the square roots of $9 - 40i$. Give your answers in the form $a + ib$. 3
- (b) Sketch on the Argand diagram the locus $|z - 1| = |z + i|$. 1
- (c) Sketch the region in the Argand diagram that satisfies both the conditions $-\frac{\pi}{2} \leq \arg(z - 2) \leq 0$ and $\operatorname{Im}(z) \leq -1$. 2
- (d) Let $z = 1 - i$ and $w = -1 + i\sqrt{3}$. 1
- (i) Find $\arg z$ and $\arg w$. 1
- (ii) Hence find $\arg(wz)$. 1
- (iii) Hence prove that $\sin \frac{5\pi}{12} = \frac{\sqrt{2}(1 + \sqrt{3})}{4}$. 2
- (e) (i) Let $z = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$. Find z^9 . 1
- (ii) On the Argand diagram, plot and label all complex numbers that satisfy both the conditions $z^9 = -1$ and $\operatorname{Re}(z) \leq 0$. 2
- (f)

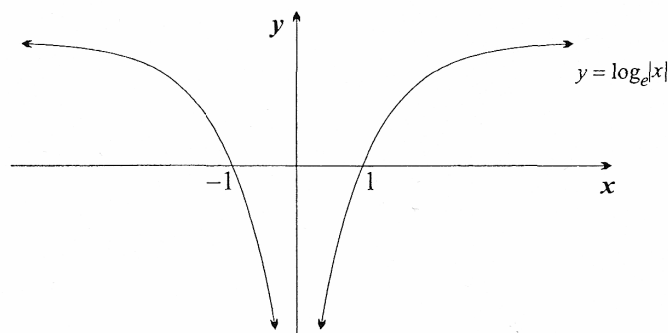


In the Argand diagram above, the two points A and C lie on the circumference of the circle with centre the origin and radius 1. They represent the complex numbers z_1 and z_2 respectively.

- (i) Copy the diagram into your answer booklet. Then mark on your diagram the position of the point B that represents the complex number $z_1 + z_2$. 1
- (ii) Explain why AC is perpendicular to OB . 1

QUESTION THREE (Start a new writing booklet)

(a)



The graph above shows the function $y = f(x) = \log_e |x|$.

(i) Use half a page to sketch on a number plane the graph $y = f\left(\frac{x}{2}\right)$.

Mark

1

(ii) Use half a page to sketch on a number plane the graph $y = \frac{1}{f(x)}$.

2

(iii) Use half a page to sketch on a number plane the graph $y^2 = f(x)$.

2

(iv) Use half a page to sketch on a number plane the graph $y = e^{f(x)}$.

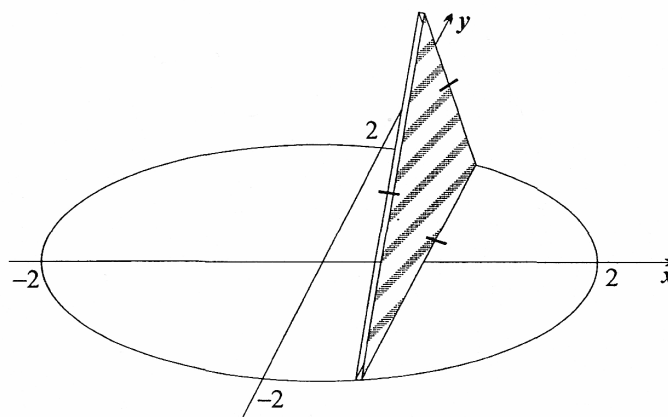
1

(b) Use the method of cylindrical shells to find the volume of the solid formed when the region bounded by the curve $y = x^2 + 3$ and the x -axis between the lines $x = 0$ and $x = 3$ is rotated about the y -axis.

3

(c)

3



The diagram above shows a cross-sectional slice of a solid whose base is the region enclosed by the circle $x^2 + y^2 = 4$. Each such cross-section of the solid is an equilateral triangle. Find the volume of the solid.

(d) The region between the curve $y = \sin x$ and the line $y = 1$, from $x = 0$ to $x = \frac{\pi}{2}$, is rotated around the line $y = 1$. Using a slicing technique find the volume formed.

3

QUESTION FOUR (Start a new writing booklet)

- (a) A mass of 2 kg, on the end of a string 0.6 metres long, is rotating as a conical pendulum, with angular velocity 3π radians per second. Take the acceleration due to gravity to be 10 m/s^2 .

Let θ be the angle that the string makes with the vertical.

- Draw a diagram showing all forces acting on the mass.
- By resolving forces, find the tension in the string.
- Find θ correct to the nearest degree.

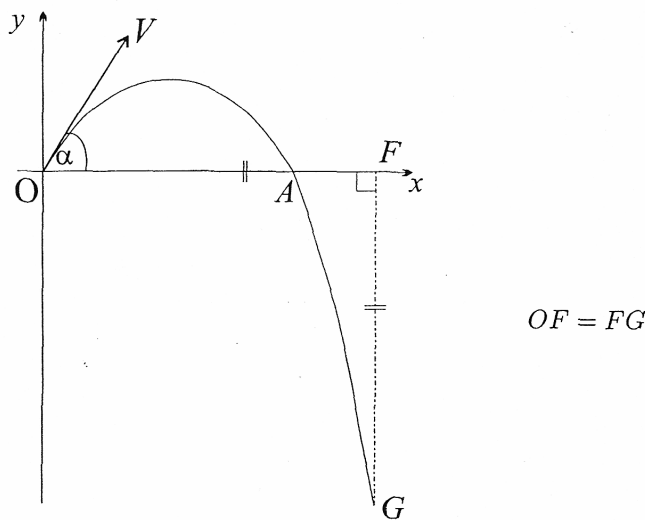
Marks

1

3

1

(b)



In the diagram above, a projectile is fired from a point O at the top of a vertical cliff. Its initial speed is $V \text{ m/s}$ and its angle of elevation is α . Let the acceleration due to gravity be $g \text{ m/s}^2$.

- By using the equations of motion $\ddot{x} = 0$ and $\ddot{y} = -g$, derive expressions for the horizontal and vertical displacements after t seconds. 2
- Let G be the point on the projectile's path where the distance below the origin equals the distance to the right of the origin. That is, $OF = FG$ on the diagram above.
- (α) Prove that the time taken for the projectile to reach G is 2

$$\frac{2V(\sin \alpha + \cos \alpha)}{g} \text{ seconds.}$$

(β) Show that $OF = \frac{V^2}{g}(\sin 2\alpha + \cos 2\alpha + 1)$ metres. 2

(γ) Let A be the point on the projectile's path where it is level with the point of projection. If $OF = \frac{4}{3}OA$, find α , correct to the nearest degree. 4

You may assume that the distance OA is given by $OA = \frac{V^2 \sin 2\alpha}{g}$ metres.

QUESTION FIVE (Start a new writing booklet)

Marks

(a) (i) Find the general solution of $\tan 4\alpha = 1$. 1

(ii) Use the binomial theorem and de Moivre's theorem to show that 4

$$\tan 4\alpha = \frac{4 \tan \alpha - 4 \tan^3 \alpha}{1 - 6 \tan^2 \alpha + \tan^4 \alpha}.$$

(iii) Hence solve the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$. 3

(iv) Hence show that 3

$$\tan^2 \frac{\pi}{16} + \tan^2 \frac{3\pi}{16} + \tan^2 \frac{5\pi}{16} + \tan^2 \frac{7\pi}{16} = 28.$$

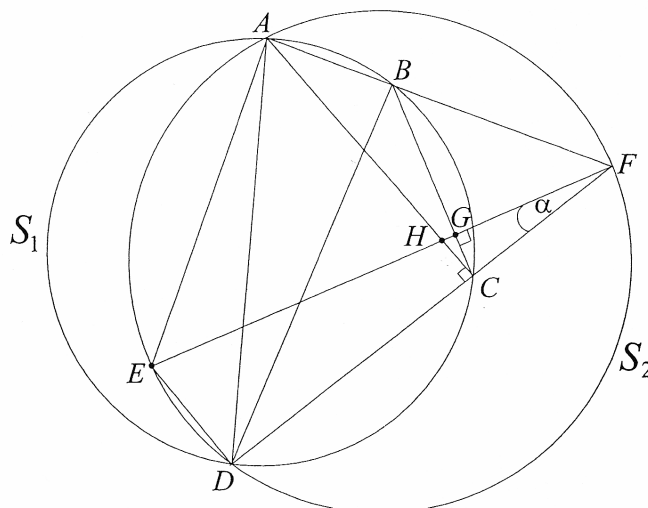
(b) Let α , β and γ be the roots of the equation $x^3 + px^2 + qx + r = 0$.

(i) Show that if the roots form an arithmetic sequence, then $2p^3 - 9pq + 27r = 0$. 2
HINT: If α , β and γ form an arithmetic sequence, then $\alpha + \beta + \gamma = 3\beta$.

(ii) Find a similar identity involving p , q and r that holds if the roots form a geometric sequence. 2

QUESTION SIX (Start a new writing booklet)

(a)



In the diagram above, $ABCD$ is a cyclic quadrilateral inscribed in the circle S_1 , and $AC \perp DC$.

The chords AB and DC produced intersect at F , and S_2 is the circle through A , D and F .

The line through F perpendicular to BC meets BC at G , meets AC at H and meets the circle S_2 at E .

Let $\angle DFE = \alpha$.

- (i) Prove that $\angle HCG = \alpha$.
- (ii) Prove that $AB \perp DB$.
- (iii) Prove that $AE \parallel BD$.
- (iv) Prove that E , A , B and G are concyclic.

Marks

1
1
2
1

(b) Let ω be one of the non-real cube roots of 1.

- (i) Show that $1 + \omega + \omega^2 = 0$.
- (ii) Hence find the value of $(2 - \omega)(2 - \omega^2)(2 - \omega^4)(2 - \omega^5)$.

1
2

(c) An object of mass 20 kg is dropped in a medium where the resistance at speed v m/s has a magnitude of $2v$ newtons. The acceleration due to gravity is 10 m/s^2 .

- (i) Draw a diagram to show the forces on the object and show that the equation of

1

motion is $\ddot{x} = \frac{100 - v}{10}$.

- (ii) Find an expression for the velocity at time t seconds after the object is dropped. 2
- (iii) Find the terminal velocity of the object. 1
- (iv) Show that the distance x metres travelled when the speed is v m/s is given by 2

$$x = 1000 \log_e \left(\frac{100}{100 - v} \right) - 10v.$$

- (v) Hence find the distance the object has fallen before reaching half its terminal velocity. 1

QUESTION SEVEN (Start a new writing booklet)

- (a) A straight line is drawn through a fixed point $P(a, b)$ in the first quadrant on a number plane. The line cuts the positive part of the x -axis at A and the positive part of y -axis at B . Let $\angle OAB = \theta$.

Marks

- (i) Prove that the length of AB is given by 2

$$AB = a \sec \theta + b \operatorname{cosec} \theta.$$

- (ii) Show that the length of AB will be a minimum if 3

$$\cot \theta = \left(\frac{a}{b} \right)^{\frac{1}{3}}.$$

- (iii) Show that the minimum length of AB is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}} \right)^{\frac{3}{2}}$. 2

- (b) (i) On the same number plane, sketch the graphs $y = \pi \sin x$ and $y = x$, for $0 \leq x \leq \pi$. 1

- (ii) Explain why there is a number α between 0 and π such that $\pi \sin \alpha = \alpha$. Furthermore, show that $\frac{2\pi}{3} < \alpha < \frac{3\pi}{4}$. Do **NOT** try to evaluate α . 1

- (iii) Let $f(x) = \sqrt{\pi^2 - x^2} \cos x - x \sin x$, for $-\pi \leq x \leq \pi$.

- (α) Prove that $f(x)$ is an even function. 1

- (β) Evaluate $f(x)$ at $x = 0$, $\frac{\pi}{3}$, $\frac{\pi}{2}$ and π . 1

- (γ) If α is the number defined in part (ii), show that $f(\alpha) = -\pi$. 1

- (δ) Show that $f'(\alpha) = 0$, and hence find 3 stationary points of $f(x)$ and determine their nature. 3

QUESTION EIGHT (Start a new writing booklet)

Marks

- (a) (i) Find k in terms of n if $\sin n\theta + \sin(n-2)\theta = 2 \sin k\theta \cos \theta$. **1**

- (ii) If n is an integer greater than 1 and $I_n = \int \sin n\theta \sec \theta d\theta$, prove that **2**

$$I_n + I_{n-2} = \frac{2 \cos(n-1)\theta}{1-n} + C, \text{ where } C \text{ is a constant of integration.}$$

- (iii) Hence prove that $\int_0^{\frac{\pi}{2}} \frac{\cos 5\theta \sin \theta}{\cos \theta} d\theta = \frac{23}{15}$. **4**

- (b) (i) Let a_1, a_2, \dots, a_{k+1} be positive real numbers. Define the function $\psi(x)$ by **3**

$$\psi(x) = a_1 + a_2 + \dots + a_k + x - (k+1)(a_1 a_2 \dots a_k x)^{\frac{1}{k+1}}, \text{ for } x > 0.$$

Show that the minimum value of $\psi(x)$ occurs at $x = x_0$, where

$$x_0 = (a_1 a_2 \dots a_k)^{\frac{1}{k}}.$$

- (ii) Let $A_n = \frac{a_1 + a_2 + \dots + a_n}{n}$ and $G_n = \sqrt[n]{a_1 a_2 \dots a_n}$. By considering $\psi(a_{k+1})$ **5**
from part (i) and using mathematical induction, prove that $A_n \geq G_n$.

REP