TKS TRIAL HSC 2003 SOLUTIONS

Mathematics

Qn 1

(a) $\log_{12} 2003 = \dfrac{\log_{10} 2003}{\log_{10} 12}$ or $\dfrac{\ln 2003}{\ln 12} = \boxed{3.06}$

(b) $\dfrac{d}{dx}(12 - \cos 12x) = - -\sin 12x \times 12 = \boxed{12\sin 12x}$

(c) $120^\circ = \dfrac{2\pi}{3}$ radians , $l = r\theta$

$\therefore \dfrac{2\pi}{3} \times r = 6\pi$

$\Rightarrow r = 6\pi \times \dfrac{3}{2\pi} = 9$

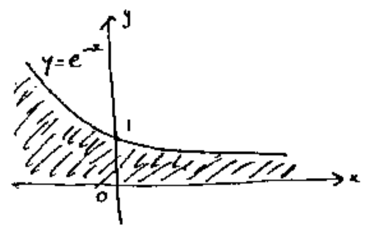
ie radius is $\boxed{9 \text{ cm}}$

or, alternatively, $120^\circ = \frac{1}{3} \times 360^\circ$

$\Rightarrow \frac{1}{3} \times 2\pi r = 6\pi \quad \ldots \quad r = 9$

(d) $x = \dfrac{2\sqrt{3}}{\sqrt{2}} = \sqrt{2}\sqrt{3} = \boxed{\sqrt{6}}$

(e)



(f) $\dfrac{x^2}{x + \dfrac{x}{x-1}} = \dfrac{x^2(x-1)}{x(x-1)+x}$

$= \dfrac{x^2(x-1)}{x^2 - x + x}$

$= \dfrac{x^2(x-1)}{x^2} = \boxed{x-1}$

Qn 2

(a) (i) gradient of $AB = \dfrac{11-3}{7--1} = \dfrac{8}{8} = 1$

$\Rightarrow \tan \angle BPC = 1 \quad \therefore \boxed{\angle BPC = 45^\circ}$

(ii) gradient of $BC = -1 \quad [BC \perp AB]$

$\therefore BC$ is $y - 11 = -(x-7) = -x + 7$

ie $\boxed{y = -x + 18}$

(iii) at C, $y = 0 \Rightarrow -x + 18 = 0$, $x = 18 \quad \therefore \boxed{C = (18, 0)}$

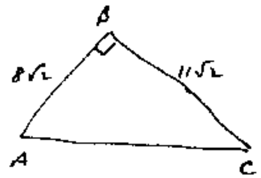
(iv) $AB = \sqrt{(7--1)^2 + (11-3)^2} = \sqrt{8^2 + 8^2} = 8\sqrt{2}$

$BC = \sqrt{(18-7)^2 + (11-0)^2} = \sqrt{11^2 + 11^2} = 11\sqrt{2}$

$\therefore$ Area $= \frac{1}{2} \cdot 8\sqrt{2} \cdot 11\sqrt{2}$ u^2 $= \boxed{88 \ u^2}$

(There are alternatives, of course)

(v)



$\therefore \tan \angle BAC = \dfrac{11\sqrt{2}}{8\sqrt{2}} = \dfrac{11}{8}$

$\Rightarrow \boxed{\angle BAC = 54^\circ}$

(b) AB is $2x - y + 5 = 0$

$\therefore$ perpendicular distance from $(0,0)$ to AB

$= \dfrac{0 - 0 + 5}{\sqrt{2^2 + 1^2}} = \dfrac{5}{\sqrt{5}} = \sqrt{5}$

$\therefore$ Area $= 10 \times \sqrt{5}$ u^2 $= \boxed{10\sqrt{5}}$ units^2

[There are many alternatives]

Qn 3

(a) $\int_0^{0.1} \sec^2(x+1)\,dx = \left[\tan(x+1)\right]_0^{0.1} = \tan 1.1 - \tan 1$
$= \boxed{0.4}$

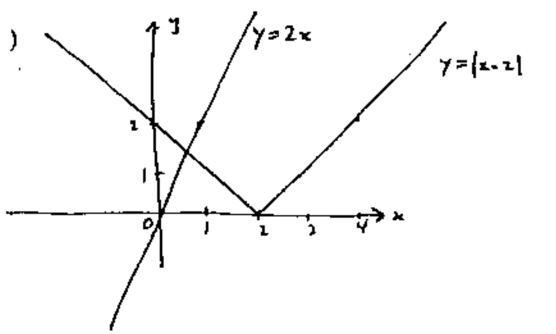
(b) $P(\text{at least 1 wrong}) = 1 - P(\text{none wrong})$
$= 1 - (0.7)^7 = \boxed{0.9}$

(c) In usual notations, $a = -7$, $d = -2-(-7) = 5$

$\Rightarrow -7 + (n-1)5 = 2003$
$5(n-1) = 2010$
$n-1 = 402$
ie. $n = 403$, the number of terms

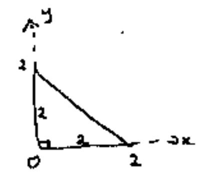
$\therefore -7 + \ldots + 2003 = \frac{403}{2}(-7 + 2003) = \boxed{402194}$

(d) (i)



(ii) The diagram indicates $2x = -(x-2)$
ie $2x = -x + 2$
$\therefore 3x = 2$
or, $\boxed{x = \frac{2}{3}}$

(iii) The diagram indicates $\int_0^4 |x-2|\,dx$ is twice
$= 2 \times \frac{1}{2}\cdot 2 \cdot 2 = \boxed{4}$



Qn 4

(a) $\frac{dy}{dx} = (x+1)e^x + e^x(1) = e^x(x+2)$

at $x=0$, $y=1$, $\frac{dy}{dx} = 2$

$\therefore$ tangent is $y - 1 = 2(x-0)$ ie $\boxed{y = 2x+1}$

(b) $\sec^2 A + \operatorname{cosec}^2 A = \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A}$
$= \frac{\sin^2 A + \cos^2 A}{\cos^2 A \sin^2 A}$
$= \frac{1}{\cos^2 A \sin^2 A} = \sec^2 A \operatorname{cosec}^2 A$

(c) Rewrite as $x^2 + y^2 - 4y = 0$
$\therefore x^2 + (y-2)^2 - 4 = 0$
or $x^2 + (y-2)^2 = 4$
$\therefore$ $\boxed{\text{Centre} = (0,2),\ \text{radius is } 2}$

(d) Solving simultaneously,
$x^2 + x = 2x + c$
ie $x^2 - x - c = 0$
For the line to be a tangent we need $\Delta = 0$
$\Delta = 1 - 4(-c) = 1 + 4c = 0$ if $\boxed{c = -\frac{1}{4}}$

Alternatively, for $y = x^2 + x$ $\quad$ For $y = 2x + c$,
$\frac{dy}{dx} = 2x + 1$ $\quad$ the gradient is 2
$\therefore$ we need $2x + 1 = 2$
$\Rightarrow x = \frac{1}{2}$
& $y = (\frac{1}{2})^2 + \frac{1}{2} = \frac{3}{4}$
$\therefore$ for line to be a tangent,
$\frac{3}{4} = 2 \times \frac{1}{2} + c \Rightarrow c = -\frac{1}{4}$

Qu 5

(a) (i) $\boxed{x = 0,\ 4}$

(ii) $y = x^4 - 4x^3$

$\therefore \frac{dy}{dx} = 4x^3 - 12x^2 = 4x^2(x-3)$

$= 0$ if $x = 0, 3$

i.e. there are stationary points at $x = 0, 3$

(iii) $\frac{d^2y}{dx^2} = 12x^2 - 24x = 12x(x-2)$

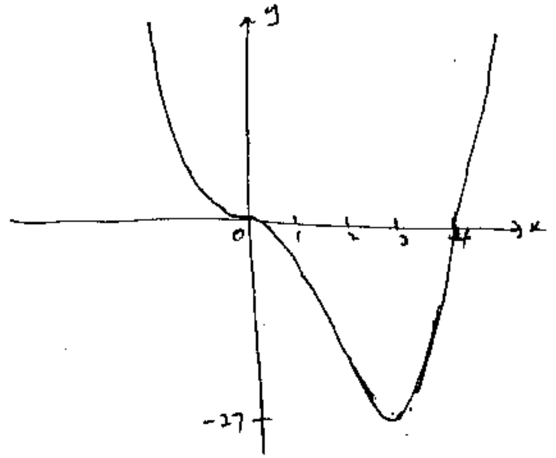
If $x=0$, $\frac{d^2y}{dx^2} = 0$

$\left.\begin{array}{l}\text{and if } x=-1,\ \frac{d^2y}{dx^2} = -12(-3) > 0 \\ \text{if } x = 1,\ \frac{d^2y}{dx^2} = 12(-1) < 0\end{array}\right\} \Rightarrow$ a change in concavity

$\therefore$ at $x=0$ there is a (horizontal) point of inflection

(iv) If $x = 3$, $y = 27(-1) = -27$

$\therefore$ combining (i), (ii), (iii) we must have



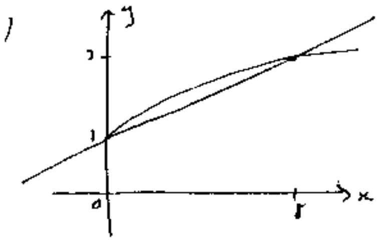
Qu 5

(b) (i) For the line if $x=0, 8$ then $y = 1, 3$

For the curve if $x=0,8$ then $y = 1, \sqrt[3]{8}+1 = 3$

$\therefore$ result

(ii)



(iii) $A = \int_0^8 \sqrt[3]{x} + 1 - (\tfrac{1}{4}x + 1)\, dx$

$= \int_0^8 x^{\frac{1}{3}} - \tfrac{1}{4}x \, dx$

$= \left[\frac{3x^{\frac{4}{3}}}{4} - \frac{1}{4}\cdot\frac{x^2}{2}\right]_0^8$

$= \frac{3}{4}\cdot 16 - \frac{1}{8}\cdot 64 - (0)$

$= \boxed{4\ u^2}$

Qu 6

(a) (i) if $t=0$, $Q = 30e^0 = 30$ $\therefore$ initial condition satisfied

Next, $\frac{dQ}{dt} = 30ke^{kt} = k(30e^{kt}) = kQ$

$\therefore$ equation $\frac{dQ}{dt} = kQ$ is satisfied

(ii) $t=3$, $Q=9 \Rightarrow 9 = 30e^{3k}$

or $e^{3k} = \frac{9}{30} = 0.3$

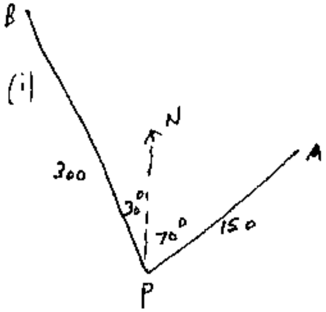
$\therefore 3k = \ln 0.3$

or $k = \frac{1}{3}\ln 0.3 = \boxed{-0.4}$

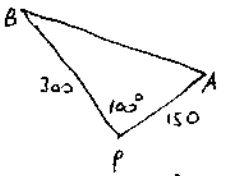
(iii) when $t=4$, $Q = 30e^{-0.4\times4} = 30e^{-1.6}$

$= \boxed{6\,g}$

(b) (i)



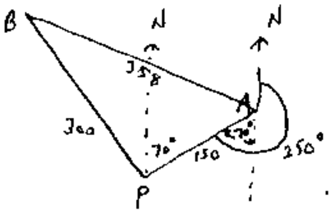
(ii)



$AB^2 = 300^2 + 150^2 - 2\times300\times150\cos100^\circ$

$\Rightarrow \boxed{AB = 358\text{ km}}$

(iii)



In ΔPAB,

$\cos A = \frac{150^2 + 358^2 - 300^2}{2.150.358}$

$\Rightarrow \hat{A} = 56^\circ$, nearest degree

$\therefore$ bearing of B from A is

$250^\circ + 56^\circ = \boxed{306^\circ}$

Q 7

(a) Put $u = (3x-1)^2$

Then, $u^2 - 2u - 8 = 0$

$(u+2)(u-4) = 0$

$\Rightarrow u = -2, 4$

$\therefore (3x-1)^2 = -2$ or 4

& so, $(3x-1)^2 = 4$ only since $(3x-1)^2 \geq 0$

$\therefore 3x-1 = 2$ or -2

Thus, $\boxed{x = 1 \text{ or } -\frac{1}{3}}$

(b) (i) $P(1\ 1\ 1) = \frac{4}{6}\times\frac{3}{5}\times\frac{2}{4} = \boxed{\frac{1}{5}}$

(ii) The product will be 0 unless all the dice are 1

$\therefore P(\text{zero product}) = 1 - \frac{1}{5} = \boxed{\frac{4}{5}}$

(c) (i) 12% p.a. = 1% p.m = .01

$\therefore A_1 = 10000 + .01\times10000 - M + 10$

$= 10000\times1.01 - (M-10)$

(ii) $A_2 = A_1\times1.01 - M + 10$

$= 10000\times1.01^2 - (M-10)\,1.01 - (M-10)$

$= 10000\times1.01^2 - (M-10)(1+1.01)$

(iii) From (i), (ii),

$A_n = 10000\times1.01^n - (M-10)(1+1.01+\dots+1.01^{n-1})$

& if $n = 5\times12 = 60$, $A_n = 0$

$\therefore (M-10)(1+1.01+\dots+1.01^{59}) = 10000\times1.01^{60}$

or $(M-10)\frac{(1.01^{60}-1)}{.01} = 10000\times1.01^{60}$

i.e. $M - 10 = \frac{10000\times1.01^{60}\times.01}{1.01^{60}-1}$ $\Rightarrow \boxed{\$M = \$232.44}$

Qn 8

(a) (i) $\int \frac{4x^4}{4x^5+1}\,dx = \frac{1}{5}\int \frac{20x^4}{4x^5+1}\,dx = \boxed{\frac{1}{5}\ln(4x^5+1) + c}$

(ii) $\int \frac{4x^5+1}{4x^4}\,dx = \int \frac{4x^5}{4x^4} + \frac{1}{4x^4}\,dx$

$= \int x + \frac{1}{4}x^{-4}\,dx$

$= \frac{x^2}{2} + \frac{1}{4}\cdot\frac{x^{-3}}{-3} + c = \boxed{\frac{x^2}{2} - \frac{1}{12x^3} + c}$

(b) (i) In Δs ABR, CBP

$AB = BC$, sides of square $ABCD$

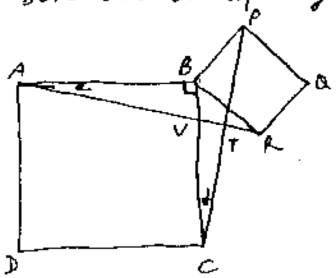
$BR = BP$, sides of square $BPQR$

$\hat{ABR} = \hat{PBC}$, both $90°$ plus (common) angle RBC (angle in a square)

$\therefore \Delta ABR \equiv \Delta CBP$, SAS

(ii) both are corresponding angles in congruent Δs in (i)

(iii)



Let AR meet BC at V & $\angle BAR = \alpha$

Then, $\angle BVR = 90° + \alpha$, exterior $\angle$ theorem in ΔABV ($\angle B = 90°$, angle in a square)

$\therefore \angle VTC = 90° + \alpha - \angle VCT$, ext. $\angle$ thm in ΔVTC

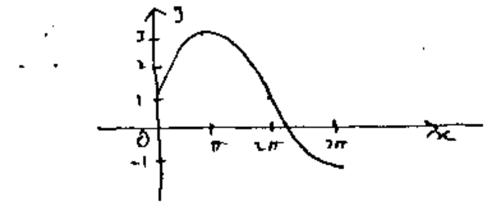
But $\angle VCT = \alpha$ (ii)

$\therefore \angle VTC = 90° \Rightarrow AR \perp PC$

Qn 9

(a) (i)

x	0	π	2π	3π
y	1	3	1	-1



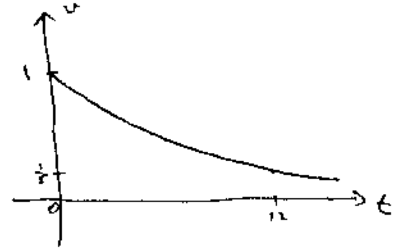
(ii) $V = \pi\int_0^{\pi}\left(2\sin\left(\frac{x}{2}\right)+1\right)^2\,dx$

(iii) $V \simeq \pi\cdot\frac{1}{6}\cdot\pi\left[1^2 + 3^2 + 4\left(2\sin\frac{\pi}{4}+1\right)^2\right]$ u^3

$= \boxed{54.8 \text{ u}^3}$, 1 d.p.

(b) (i) $t=0$, $v = \boxed{1}$ m/s ; $t=12$, $v = \frac{1}{\sqrt{25}}$ m/s $= \boxed{\frac{1}{5}}$ m/s

(ii)



(iii) $v = (2t+1)^{-\frac{1}{2}}$

$\therefore \ddot{x} = -\frac{1}{2}(2t+1)^{-\frac{3}{2}}\cdot 2 = -\frac{1}{(2t+1)^{3/2}}$

$t=12$, $\ddot{x} = -\frac{1}{25^{3/2}}$ m/s^2 $= \boxed{-\frac{1}{125}\text{ m/s}^2}$

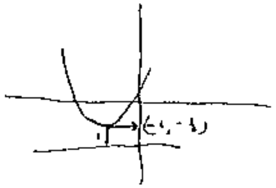
(iv) $x = \int (2t+1)^{-\frac{1}{2}}\,dt = \frac{2(2t+1)^{\frac{1}{2}}}{2} + c = \sqrt{2t+1} + c$

$t=0, x=0 \Rightarrow 0 = 1 + c$, $c = -1$ $\boxed{\therefore x = \sqrt{2t+1} - 1}$

Qn 10

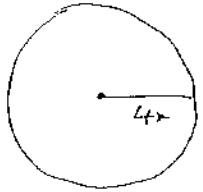
(a) Rewrite as $(x+1)^2 = 4(y+\frac{1}{2})$

$\therefore$ Vertex $= (-1, -\frac{1}{2})$, focal length $= 1$



$\Rightarrow$ directrix is $\boxed{y = -1\frac{1}{2}}$

(b)



(i) Circumference of circle $= 2\pi(4x)$
$= 8\pi x$

$\therefore$ each side of the squares

$= \dfrac{200 - 8\pi x}{8} = 25 - \pi x$

(ii) We must have $4x \geq 0$ and $25 - \pi x \geq 0$

$\Rightarrow x \geq 0$ and $\pi x \leq 25$
ie $x \leq \frac{25}{\pi}$

$\therefore \quad 0 \leq x \leq \frac{25}{\pi}$

(iii) $A = \pi(4x)^2 + 2(25 - \pi x)^2$

$= 16\pi x^2 + 2(625 - 50\pi x + \pi^2 x^2)$

$= 16\pi x^2 + 1250 - 100\pi x + 2\pi^2 x^2$

$= 2\pi(8+\pi)x^2 - 100\pi x + 1250$

(iv) $\dfrac{dA}{dx} = 4\pi(8+\pi)x - 100\pi$

$= 0$ if $x = \dfrac{100\pi}{4\pi(8+\pi)} = \dfrac{25}{8+\pi}$

$\dfrac{d^2A}{dx^2} = 4\pi(8+\pi) > 0$ for all x

$\therefore$ curve for A (a parabola) is concave upward

$\therefore$ least A occurs when $x = \dfrac{25}{8+\pi}$

(v) minimum $A = 2\pi(8+\pi)\dfrac{625}{(8+\pi)^2} - 100\pi \cdot \dfrac{25}{8+\pi} + 1250$

$= \dfrac{1250\pi}{8+\pi} - \dfrac{2500\pi}{8+\pi} + 1250$

$= 1250 - \dfrac{1250\pi}{8+\pi}$

$= \dfrac{1250(8+\pi) - 1250\pi}{8+\pi} = \dfrac{10000}{8+\pi}$

(vi) From (ii) and (iv), maximum area occurs when $x = 0$ or $x = \frac{25}{\pi}$

The axis of symmetry of the parabola $A = 2\pi(8+\pi)x^2 - 100\pi x + 1250$

is $x = \dfrac{25}{8+\pi} \approx 2.24$

and $\dfrac{25}{\pi} \approx 7.96$, further from 2.24 than 0

$\Rightarrow$ maximum A occurs when $x = \dfrac{25}{\pi}$