ASQUITH BOYS HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1996

MATHEMATICS

3 Unit (Additional)
and
3/4 Unit (Common)

Time allowed - TWO hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL question are of equal value.
- ALL necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on the last page. These may be removed for your convenience.
- Board-approved calculators may be used.
- Each question should be started on a new page.

Question 1 (Start a new page) Marks

a. Find the exact value of $\int_0^4 \frac{dx}{x^2+16}$ 3

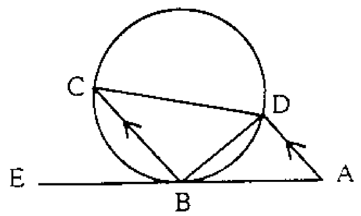
b. Find $\int (1-\cos x)^2 \, dx$ 3

c. Solve the inequality $\frac{3}{x-2} \geq 1 \quad x \neq 2$ 3

d. Find the first derivative of $y = \log_e\left(\frac{1}{\sqrt{\cos x}}\right)$ 3

Question 2 (Start a new page) Marks

a. 4



AB is a tangent at B and $AD \parallel BC$. Prove that $\Delta BCD \,|||\, \Delta DBA$.

b. Find $\int \frac{2x}{(x-1)^2} \, dx$ using the substitution $u = x - 1$ 4

c. Prove by the method of Mathematical Induction that 4

$$\sum_{r=1}^{n} 5^{r-1} = \frac{5^n - 1}{4}$$

Question 3 (Start a new page) **Marks**

a. If ${}^{12}P_r = 120.{}^{12}C_r$ find r. 3

b. The velocity of a particle moving in a straight line is given by 5

$$v^2 = 8x - 2x^2 \quad m/\sec$$

i. Show that the particle is moving in simple harmonic motion.

ii. Find the centre of the motion.

iii. Determine the two end points between which the particle is oscillating.

iv. Find the maximum speed of the particle.

c. A formula for the rate of change in population of a colony of bacteria, is given by $P = 3200 + 400e^{kt}$ 4

If the population doubles after 20 hours, how long would it take to triple the original population.

Question 4 (Start a new page) **Marks**

a. At what points on the curve $y = \cos^{-1} x$, is the gradient equal to $-\frac{2}{\sqrt{3}}$ 3

b. Find the middle term in the expansion of $\left(x^3 - \frac{1}{3x}\right)^8$ 4

c. A capsule is in the shape of a cylinder with hemispherical ends. The radius of the cylindrical section is $r$cm, and the volume of the capsule is 16cm^3. 5

i. If the height of the cylinder is 4cm show that $r^3 + 3r^2 = \frac{12}{\pi}$

ii. Show that one solution of the equation $r^3 + 3r^2 = \frac{12}{\pi}$ lies between 0 and 1.

iii. The equation $r^3 + 3r^2 = \frac{12}{\pi}$ has a root close to $0 \cdot 9$. Use one application of Newton's method to give a better approximation.

Question 5 (Start a new page) **Marks**

a. Solve the equation $3x^3 - 17x^2 - 8x + 12 = 0$ given that the product of two of the roots is 4 **3**

b. The probability that a vaccine succeeds is $\frac{29}{30}$. An experiment is conducted m times with white mice. **4**

i. What is the probability that the experiment will fail at least once?

ii. Show that if the probability that the experiment will fail at least once in m trials, is greater than than $\frac{9}{10}$ then $m > \frac{1}{\log_{10} 30 - \log_{10} 29}$

c. For a particular vessel, the rate of increase of the volume with respect to its depth, is given by $\frac{dV}{dh} = \frac{\pi(h+6)^2}{12}$ $0 \le h \le 10$ **5**

where $V\text{cm}^3$ is the volume and h is the depth of the water.

i. If the container is initially empty, show that the volume as a function of the depth is $V = \frac{\pi h}{36}\left(h^2 + 18h + 108\right)$

ii. Find the volume when the depth is 6cm.

iii. If water is being poured into the vessel at a constant rate of $8\text{cm}^3/\text{s}$ find an expression for the rate of increase in the depth of the water.

iv. At what rate is the depth increasing when the water level is 6 cm, and how long will it take to the nearest second to reach this level.

Question 6 (Start a new page) **Marks**

a. The letters of the word **R E P E T I T I O N** are arranged at random in a row. **3**

i. how many different arrangements are possible?

ii. what is the probability that one particular arrangement will have vowels and consonants alternating?

(Question 6 continued on page 4)

Question 6 Continued Marks

b. i. Write the general expansion of $(1+x)^n$ 3

ii. Hence or otherwise prove that

$${}^nC_0 + \tfrac{1}{2}\,{}^nC_1, \ + \tfrac{1}{3}\,{}^nC_2 + \ldots\ldots + \frac{1}{n+1}\,{}^nC_n = \frac{2^{n+1}-1}{n+1}$$

c. The curve $y = \sin^{-1} x$ intersects the curve $y = \cos^{-1} x$ at P, and the latter intersects the x axis at Q. 6

i. Draw a neat sketch of this information.

ii. Verify that P has co-ordinates $\left(\frac{1}{\sqrt{2}} \ . \ \frac{\pi}{4}\right)$

iii. Prove $\frac{d}{dx}\left(x\sin^{-1}x + \sqrt{1-x^2}\right) = \sin^{-1}x$

iv. If O is the origin, find the area enclosed by the arcs OP and PQ and the x axis using the results in (iii) and the fact that $\frac{d}{dx}\left(x\cos^{-1}x - \sqrt{1-x^2}\right) = \cos^{-1}x$

Question 7 **(Start a new page)** Marks

A projectile fired with velocity V and at an angel $45°$ to the horizontal, just clears the tops of two vertical posts of height $8a^2$, and the posts are $12a^2$ apart. There is no air resistance, and the acceleration due to gravity is g.

a. If the projectile is at the point (x, y) at time t, <u>derive</u> expressions for x and y in terms of t. 3

b. Hence show that the equation of the path of the projectile is $y = x - \frac{gx^2}{V^2}$ 2

c. Using the information in (b) show that the range of the projectile is $\frac{V^2}{g}$ 2

d. If the first post is b units from the origin, show 2

i. $\frac{V^2}{g} = 2b + 12a^2$

ii. $8a^2 = b - \frac{gb^2}{V^2}$

e. Hence or otherwise prove that $V = 6a\sqrt{g}$ 3

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$