JAMES RUSE AHS

TRIAL 2001

EXT 1.

QUESTION 1

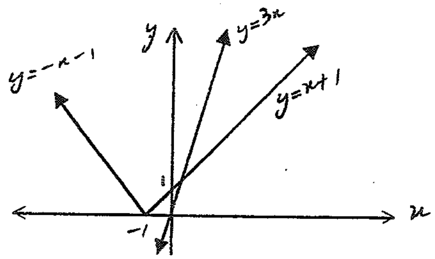
a)(i) $y' = 2x\cos x - x^2\sin x$

(ii) $\int_1^6 \frac{x}{x^2+4}dx = \frac{1}{2}\left[\ln(x^2+4)\right]_1^6$

$= \frac{1}{2}(\ln 40 - \ln 5)$

$= \frac{1}{2}\ln 8.$

b)(i)



(ii)

$3x = x+1$ (*from graph*)

$2x = 1$

$x = \frac{1}{2}$

c)(i) domain: $-\frac{1}{3} \le x \le \frac{1}{3}$

range: $-\pi \le y \le \pi$

(ii) $f\left(\frac{1}{6}\right) = 2\sin^{-1}\left(\frac{3}{6}\right) = \frac{\pi}{3}$

(iii) $f'(x) = 2\cdot\frac{3}{\sqrt{1-9x^2}}$

$f'\left(\frac{1}{6}\right) = \frac{6}{\sqrt{1-\frac{9}{36}}} = \frac{6}{\sqrt{\frac{3}{4}}}$

$= \frac{12}{\sqrt{3}} = 4\sqrt{3}$

QUESTION 2

a) $A = \left(\frac{-7+27}{4}, \frac{3+45}{4}\right) = (5,12)$

$m_{PQ} = \frac{15-3}{9+7} = \frac{3}{4}$

$m_{AB} = \frac{12-0}{5-14} = -\frac{4}{3}$

$m_{PQ}\cdot m_{AB} = \frac{3}{4}\cdot -\frac{4}{3} = -1$

$\therefore PQ \perp AB$ (prod. of slopes is -1)

b) $u^2 = x+1$

$u^2 - 1 = x$

$\frac{dx}{du} = 2u du$

If $x = 0$, $u^2 = 1$, $u = 1$ (take u>0)
If $x = 3$, $u^2 = 4$, $u = 2$ (take u>0)

$\int_0^3 \frac{x+2}{\sqrt{x+1}}dx = \int_1^2 \frac{u^2+1}{\sqrt{u^2}}2u\, du$

$= 2\int_1^2 (u^2+1)\, du$

$= 2\left[\frac{u^3}{3}+u\right]_1^2$

$= 2\left(\frac{8}{3}+2-\frac{1}{3}-1\right)$

$= \frac{20}{3}$

c) $\frac{dt}{dh} = -\frac{1}{k}h^{-\frac{1}{2}}$

$t = -\frac{1}{k}\cdot 2h^{\frac{1}{2}} + c$

$t = \frac{-2\sqrt{h}}{k} + c$

If $t = 0, h = 2500$: $0 = \frac{-100}{k} + c$

If $t = 5, h = 900$: $5 = \frac{-60}{k} + c$

Solving: $5 = \frac{-60}{k} + \frac{100}{k}$

$5k = 40$

$k = 8,\ c = \frac{100}{8}$

$\therefore t = -\frac{\sqrt{h}}{4} + 12.5$

When $h = 0, t = 12.5$:

$\therefore$ *extra time* $= 12.5 - 5 = 7.5$ min

QUESTION 3

a) $T_{r+1} = {}^6C_r (3x)^{6-r}\left(\frac{2}{\sqrt{x}}\right)^r$

$= {}^6C_r 3^{6-r} 2^r x^{6-r} x^{-\frac{1}{2}}$

$= {}^6C_r 3^{6-r} 2^r x^{6-1\frac{1}{2}r}$

for constant term degree of $x = 0$;

$\therefore 6 - \frac{1}{2}r = 0$

$r = 4$

$\therefore \text{constant} = {}^6C_4 3^2 2^4 = 2160$

b)(i) prob $= \frac{4!}{6!} = \frac{1}{30}$

(ii) prob $= \frac{3.2.4!}{6!} = \frac{1}{5}$

(iii) prob $= \frac{3!4.3.2}{6!} = \frac{1}{5}$

c)(i) Let <AMD=<ANB=α^0 and <ABC=β°
<BCM= β−α (ext angle of ΔBMC)
<DCM= β−α (vertically opp angles)
<ADC= β (ext angle of ΔCND)
<ABC=<ADC (both β)

(ii) <ABC + <ADC=180
(opp angles of cyclic quad. are supp)
2<ABC=180 (<ABC=<ADC; above)
<ABC=90°
AC is a diameter (angle in semicircle is 90°)

QUESTION 4

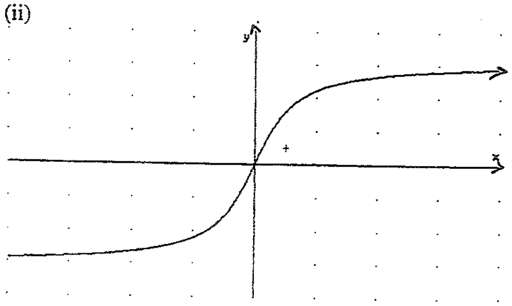
a)

(i) $\frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\cos^2 A} = \frac{1}{\cos^2 A}$

$\tan^2 A + 1 = \sec^2 A$

$\tan^2 A = \sec^2 A - 1$

(ii)



(iii) $\frac{y}{4} = \tan^{-1} x$

$x = \tan\frac{y}{4}$

$V = \pi\int_0^\pi x^2\,dy$

$= \pi\int_0^\pi \tan^2\frac{y}{4}\,dy$

$= \pi\int_0^\pi (\sec^2\frac{y}{4} - 1)\,dy$

$= \pi\left[4\tan\frac{y}{4} - y\right]$

$= \pi\left\{\left(4\tan\frac{\pi}{4} - \pi\right) - (4\tan 0 - 0)\right\}$

$= \pi(4-\pi)u^3$

b)(i)

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right)\cdot\frac{dv}{dx}$

$= v\frac{dv}{dx}$

$= \frac{dx}{dt}\cdot\frac{dv}{dx}$

$= \frac{dv}{dt}$

$= \dot{x}$

(ii)(α) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 4x$

$\frac{1}{2}v^2 = \frac{x^4}{2} + 2x^2 + c$

If t=0, x=2, v=6:

$18 = 8 + 8 + c$

$c = 2$

$v^2 = x^4 + 4x^4 + 4$ or

$v^2 = (x^2+2)^2$

(β) $v^2 = (x^2+2)^2$

$\therefore v^2 \geq 4 \quad \therefore v \neq 0$

Here the object never changes direction. Thus, it always moves to the right with increasing speed since the initial velocity >0 and acceleration>0 for x>0. Therefore the min. speed is the initial speed. Therefore min. speed=6m/s.

QUESTION 5

a) $y' = -2e^{-2x}$

when $x = 0,\ y' = -2e^{0}$

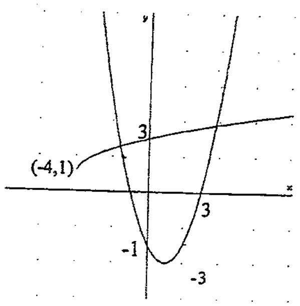
$\therefore m_1 = -2$

$m_2 = 3$

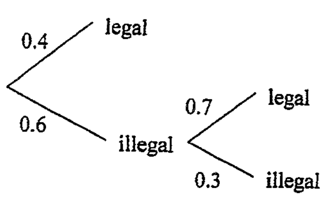
$\tan\theta = \left|\dfrac{3+2}{1+(3)(2)}\right| = 1$

$\theta = \dfrac{\pi}{4}$ *or* 45^0

b)(i) and (iii)



(ii) $x \geq 1$
(iii)see graph
c) (i)



P(double fault) $= 0.6 \times 0.3 = 0.18$

(ii) $(0.82 + 0.18)^6$

$P(\textit{at least 2 double faults})$

$= 1 - \{P(0\ \textit{double faults}) + P(1\ \textit{double faults})\}$

$= 1 - \{{}^6C_0(0.82)^6(0.18)^0 + {}^6C_1(0.82)^5(0.18)^1\}$

$\cong 0.30$

QUESTION 6

a)

Given $V = \dfrac{4}{3}\pi r^3$

and $\dfrac{dV}{dr} = 4\pi r^r$

By chain rule:

$\dfrac{dr}{dt} = \dfrac{dr}{dV}\cdot\dfrac{dV}{dt}$

$= \dfrac{1}{4\pi r^2}\cdot 10$

$= \dfrac{10}{4\pi r^2}$

When $SA = 500 \quad (= 4\pi r^2)$

$\dfrac{dr}{dt} = \dfrac{10}{500}$

$= \dfrac{1}{50}\, cm/s$

b) Step 1: Show true for n=1.
LHS= 2(1!)=2 RHS= 1(2!)=2
$\therefore$ true for n=1
Step 2: Assume true for n=k
ie. $2(1!) + 5(2!) + \ldots + (k^2+1)k! = k(k+1)!$
Show true for n=k+1
ie. $2(1!) + 5(2!) + \ldots + (k^2+1)k! + [(k+1)^2+1](k+1)! = (k+1)(k+2)!$

$LHS = 2(1!) + 5(2!) + \ldots\ldots + (k^2+1)k! + (k^2+2k+2)(k+1)!$

$= k(k+1)! + (k^2+2k+2)(k+1)!$ *(by assumption)*

$= (k+1)!\{k + k^2 + 2k + 2\}$

$= (k+1)!\{k^2 + 3k + 2\}$

$= (k+1)!(k+2)(k+1)$

$= (k+2)!(k+1)$

$= RHS$

Step 3: If true for n=k then true for n=k+1
and since true for n=1 then true for n≥1.

c)

$$\frac{dy}{dx}=\frac{x.\left(\frac{1}{x}\right)-1.(\ln x)}{x^2}$$

$$=\frac{1-\ln x}{x^2}$$

$$\int_e^{e^2}\frac{1-\ln x}{x\ln x}dx=\int_e^{e^2}\frac{\frac{1-\ln x}{x^2}}{\frac{\ln x}{x}}dx$$

$$=\left[\ln\left(\frac{\ln x}{x}\right)\right]_e^{e^2}$$

$$=\ln\left(\frac{\ln e^2}{e^2}\right)-\ln\left(\frac{\ln e}{e}\right)$$

$$=\ln\left(\frac{2}{e^2}\right)-\ln\left(\frac{1}{e}\right)$$

$$=\ln\left(\frac{2}{e}\right)$$

$$=\ln 2-\ln e$$

$$=\ln 2-1$$

QUESTION 7

(i) $\sin(X+Y)-\sin(X-Y)$
$=(\sin X\cos Y+\cos X\sin Y)-(\sin X\cos Y-\cos X\sin Y)$
$=2\cos X\sin Y$

let X+Y=A and X-Y=B

$2X=A+B$ $\qquad$ $2y=A-B$

$X=\frac{A+B}{2}$ $\qquad$ $Y=\frac{A-B}{2}$

Thus, $\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

(ii)

$$\frac{\sin A-\sin B}{\cos A-\cos B}=\frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)}$$

$$=\frac{\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{-\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$$

$$=-\cot\left(\frac{A+B}{2}\right)$$

(iii)

$\ddot{x}=0$ $\qquad$ $\ddot{y}=-g$

$\dot{x}=c_1$ $\qquad$ $\dot{y}=-gt+c_3$

When $t=0,\dot{x}=v\cos\alpha$ $\qquad$ *When* $t=0,\dot{y}=v\sin\alpha$

$\therefore v\cos\alpha=c_1$ $\qquad$ $\therefore c_3=v\sin\alpha$

$\dot{x}=v\cos\alpha$ $\qquad$ $\dot{y}=-gt+v\sin\alpha$

$x=vt\cos\alpha+c_2$ $\qquad$ $y=-\frac{1}{2}gt^2+vt\sin\alpha+c_4$

When $t=0,x=0$ $\qquad$ *When* $t=0,y=0$

$\therefore c_2=0$ $\qquad$ $\therefore c_4=0$

$x=vt\cos\alpha$ $\qquad$ $y=-\frac{1}{2}gt^2+vt\sin\alpha$

(iv)(α) Particle P

$x_p=vt\cos\alpha\quad y_p=-\frac{1}{2}gt^2+vt\sin\alpha$

Particle Q

$x_p=vt\cos\beta\ y_p=-\frac{1}{2}gt^2+vt\sin\beta$

$\tan\theta$
$=$ slopePQ

$$=\left|\frac{\left(-\frac{1}{2}gt^2+v\sin\beta\right)-\left(-\frac{1}{2}gt^2+v\sin\alpha\right)}{vt\cos\beta-vt\cos\alpha}\right|$$

$$=\left|\frac{vt(\sin\beta-\sin\alpha)}{vt(\cos\beta-\cos\alpha)}\right|$$

$$=\left|\frac{(\sin\beta-\sin\alpha)}{(\cos\beta-\cos\alpha)}\right|$$

(β)

$$\tan\theta=\left|-\cot\left(\frac{\alpha+\beta}{2}\right)\right| \textit{from(ii)}$$

$$=\tan\left(\frac{\pi}{2}-\left(\frac{\alpha+\beta}{2}\right)\right) \alpha,\beta,\theta \textit{ acute}$$

$$\theta=\frac{\pi}{2}-\left(\frac{\alpha+\beta}{2}\right)$$

$$\theta=\frac{1}{2}(\pi-\alpha-\beta)$$