2009

Higher School Certificate

Trial Examination

Mathematics

Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A table of standard integrals is provided
- All necessary working should be shown in every question
- Write your student number and/or name at the top of every page

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ....

Student name / number ....

Question 1 **Begin a new booklet** **Marks**

(a) Find $\int \frac{(x+1)^2}{x}\,dx$. **2**

(b)(i) Find constants A, B, C and D such that $\frac{x^3+2x^2+4x+2}{(x^2+1)(x^2+4)} \equiv \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$. **2**

(ii) Hence evaluate $\int_0^2 \frac{x^3+2x^2+4x+2}{(x^2+1)(x^2+4)}\,dx$. **2**

(c) Use the substitution $t=\tan\frac{x}{2}$ to find $\int \frac{1}{5+4\cos x+3\sin x}\,dx$. **3**

(d) Use the substitution $u=\sin x$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx$. **3**

(e) Use the substitution $u=-x$ to evaluate $\int_{-1}^{1} \frac{1}{e^x+1}\,dx$. **3**

Student name / number ....

Marks

Question 2 **Begin a new booklet**

(a) If $z_1 = 3i$ and $z_2 = 1 + i$, find the values of

(i) $|z_1 - z_2|$. **1**

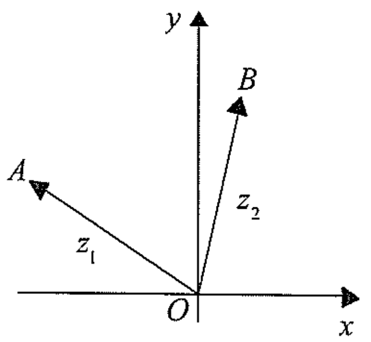
(ii) $z_1 + \overline{z}_2$. **1**

(iii) $\frac{z_1}{z_2}$. **1**

(b)(i) If $z = 1 + i\sqrt{3}$, express z, z^2 and $\frac{1}{z}$ in modulus-argument form. **3**

(ii) If the points A and B represent the complex numbers z^2 and $\frac{1}{z}$ in the Argand diagram, show that A, O and B are collinear, where O is the origin. **1**

(c)



In the Argand diagram, vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the complex numbers $z_1 = 2(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5})$ and $z_2 = 2(\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15})$ respectively.

(i) Show that ΔOAB is equilateral. **2**

(ii) Express $z_2 - z_1$ in modulus-argument form. **2**

(d) z is a complex number such that $\arg z = \frac{\pi}{3}$ and $|z| \leq 2$.

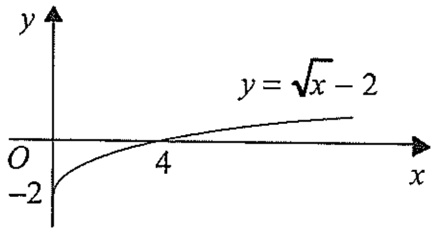
(i) Show the locus of the point P representing z in the Argand diagram. **2**

(ii) Find the possible values of the principal argument of $z - i$ for z on this locus. **2**

Student name / number ....

Question 3 **Begin a new booklet** **Marks**

(a)



This diagram shows the graph of the function $f(x) = \sqrt{x} - 2$.
On separate diagrams sketch the following graphs, showing clearly any intercepts on the coordinate axes and the equations of any asymptotes:

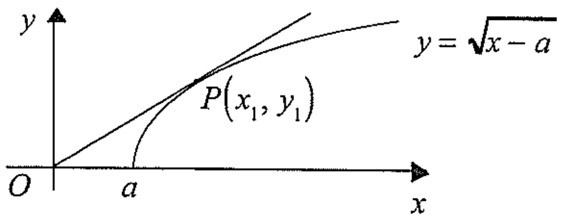
(i) $y = |f(x)|$. **1**

(ii) $y = \{f(x)\}^2$. **1**

(iii) $y = \dfrac{1}{f(x)}$. **2**

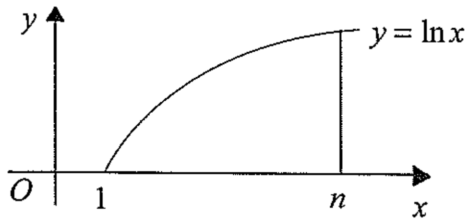
(iv) $y = \log_e f(x)$. **2**

(b)



The tangent to the curve $y = \sqrt{x-a}$, where $a > 0$, at the point $P(x_1, y_1)$ on the curve passes through the origin. Find the coordinates of P. **3**

(c)



(i) Use the trapezoidal rule with n function values to approximate $\int_1^n \ln x \, dx$. **2**

(ii) Show that $\dfrac{d}{dx}(x\ln x - x) = \ln x$ and hence find the exact value of $\int_1^n \ln x \, dx$. **2**

(iii) Deduce that $\ln n! < (n + \frac{1}{2})\ln n - n + 1$. **2**

Student name / number ....

Marks

Question 4 **Begin a new booklet**

(a) The polynomial $P(x)$ leaves a remainder of 9 when divided by $(x-2)$ and a remainder of 4 when divided by $(x-3)$. Find the remainder when $P(x)$ is divided by $(x-2)(x-3)$. **2**

(b) $P\left(ct\,,\ \frac{c}{t}\right)$ and $Q\left(1+\cos\theta\,,\sin\theta\right)$ are points on the hyperbola $xy=c^2$, where $c>0$, and the circle $(x-1)^2+y^2=1$ respectively.

(i) Show by differentiation that the tangent to the hyperbola at P has equation $x+t^2y=2ct$ and the tangent to the circle at Q has equation $x\cos\theta+y\sin\theta=1+\cos\theta$. **3**

(ii) Deduce that PQ is tangent to both the hyperbola and the circle, with points of contact P and Q, if $t^2=\tan\theta$ and $2ct-1=\sec\theta$, where $\left(\frac{t}{c}\right)^3-4\left(\frac{t}{c}\right)+\frac{4}{c^2}=0$. **3**

(iii) By considering the graphs of $y=x^3-4x$ and $y=x^3-4x+\frac{4}{c^2}$, deduce that for every value of $c>0$ there is exactly one point on the third-quadrant branch of the hyperbola where the tangent to the hyperbola is also tangent to the circle. Show that for $c^2>\frac{3\sqrt{3}}{4}$, there are also two such points on the first-quadrant branch of the hyperbola. **3**

(iv) When $c^2=\frac{3\sqrt{3}}{4}$, the hyperbola touches the circle at $P\left(ct\,,\ \frac{c}{t}\right)$ where $\frac{t}{c}$ is a double root of the cubic equation $x^3-4x+\frac{4}{c^2}=0$. Sketch the hyperbola and the circle for $c^2=\frac{3\sqrt{3}}{4}$, showing any common tangents to the curves with their equations. Write numerical values for the coordinates of any points of contact P, Q for these tangents. **4**

Student name / number ....

Marks

Question 5 **Begin a new booklet**

(a) $z = \cos\theta + i\sin\theta$

(i) Show that $z^n + z^{-n} = 2\cos n\theta$ for $n = 1, 2, 3, \ldots$. **1**

(ii) Hence show that $4\cos\theta\cos 2\theta\cos 3\theta = 1 + \cos 2\theta + \cos 4\theta + \cos 6\theta$. **2**

(iii) Hence solve $\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta = 1$, giving general solutions. **3**

(b) A particle is projected vertically downwards under gravity in a medium where resistance is proportional to the speed of the particle. The terminal velocity of the particle is U ms^{-1}, and the speed of projection is equal to half this terminal velocity. At time t seconds, the particle has travelled a distance x metres, has velocity v ms^{-1} and has acceleration $\ddot{x}$ ms^{-2}.

(i) Show $\ddot{x} = \frac{g}{U}(U - v)$, where g ms^{-2} is the acceleration due to gravity. **2**

(ii) Show by integration that $-\frac{g}{U}t = \ln 2\left(1 - \frac{v}{U}\right)$. Hence obtain an expression for $\frac{v}{U}$ in terms of t. **3**

(iii) Show that $x = Ut - \frac{U^2}{g}\left(\frac{v}{U} - \frac{1}{2}\right)$. **2**

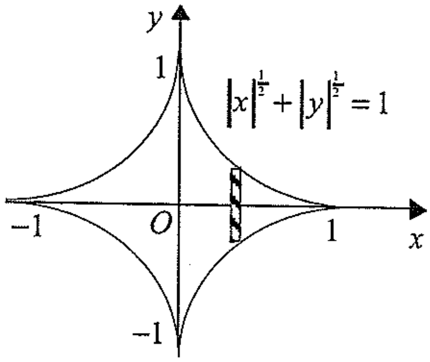
(iv) If $g = 10$ and $U = 100$, find the percentage of the terminal velocity gained during the first second of the motion, and the distance travelled during this time. **2**

Student name / number ....

Question 6 **Begin a new booklet** **Marks**

(a) The roots of the equation $x^3+3x^2+7x+k=0$ are in arithmetic progression. Find the value of the constant k. **2**

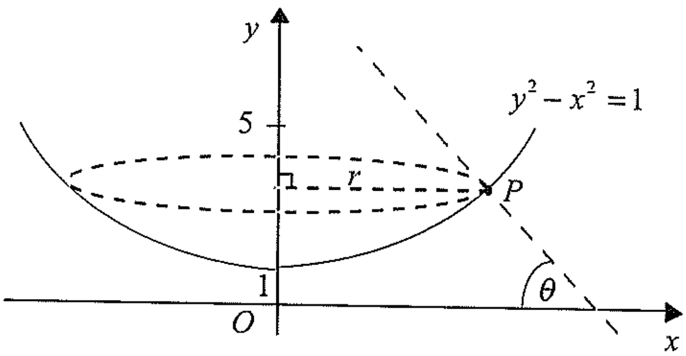
(b) The horizontal base of a solid is the area enclosed by the curve $|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}=1$.
Vertical cross sections taken perpendicular to the x-axis are squares with one side in the base.



(i) Show that the volume of the solid is given by $V=8\int_0^1\left(1-\sqrt{x}\right)^4 dx$. **2**

(ii) Use the substitution $u=1-\sqrt{x}$ to evaluate this integral. **3**

(c)



A bowl is formed by rotating the hyperbola $y^2-x^2=1$ for $1\le y\le 5$ through $180°$ about the y-axis. Sometime later, a particle P of mass m moves around the inner surface of the bowl in a horizontal circle with constant angular velocity ω.

(i) Show that if the radius of the circle in which P moves is r, then the normal to the surface at P makes an angle θ with the horizontal where $\tan\theta=\dfrac{\sqrt{1+r^2}}{r}$. **2**

(ii) Draw a diagram showing the forces on P. **1**

(iii) Find expressions for the radius r of the circle of motion and the magnitude of the reaction force between the surface and the particle in terms of m, g and ω. **3**

(iv) Find the values of ω for which the described motion of P is possible. **2**

Student name / number ....

Question 7 **Begin a new booklet** **Marks**

(a) $I_n = \int_1^e (1-\ln x)^n \, dx, \quad n = 0, 1, 2, \ldots$

(i) Show $I_n = -1 + nI_{n-1}, \quad n = 1, 2, 3, \ldots$. **2**

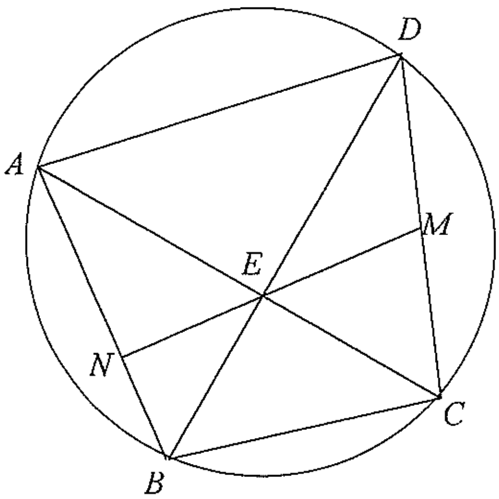
(ii) Hence evaluate $\int_1^e (1-\ln x)^3 \, dx$. **2**

(iii) Show that $\dfrac{I_n}{n!} = e - \displaystyle\sum_{r=0}^{n} \frac{1}{r!}, \quad n = 1, 2, 3, \ldots$. **2**

(iv) Show that $0 \le I_n \le e - 1$. **1**

(v) Deduce that $\displaystyle\lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{r!} = e$. **1**

(b)



$ABCD$ is a cyclic quadrilateral. The diagonals AC and BD intersect at right angles at E. M is the midpoint of CD. ME produced meets AB at N.

(i) Copy the diagram showing the given information. Show that $ME = MC$. **3**

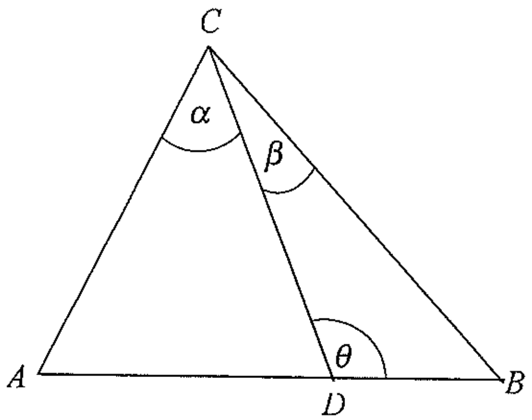
(ii) Hence show that MN is perpendicular to AB. **4**

Student name / number ....

Marks

Question 8 **Begin a new booklet**

(a)



In ΔABC, D is the point on AB that divides AB internally in the ratio $m : n$.
$\angle ACD = \alpha$, $\angle BCD = \beta$ and $\angle CDB = \theta$.

(i) By using the sine rule in each of ΔCAD and ΔCDB, show that **4**

$$\frac{\sin(\theta+\beta)\sin\alpha}{\sin(\theta-\alpha)\sin\beta} = \frac{m}{n}.$$

(ii) Hence show that $\tan\theta = \dfrac{(m+n)\tan\alpha\tan\beta}{m\tan\beta - n\tan\alpha}$. **3**

(b) $f(x)$ and $g(x)$ are continuous and bounded functions.

(i) By considering $\int_0^a \{\lambda f(x) + g(x)\}^2\, dx$, $a > 0$, as a quadratic function of λ, show that **4**

$$\left\{\int_0^a f(x)g(x)\, dx\right\}^2 \le \int_0^a \{f(x)\}^2\, dx \,.\, \int_0^a \{g(x)\}^2\, dx.$$

(ii) Hence show that $\left\{\int_0^1 f(x)\, dx\right\}^2 \le \int_0^1 \{f(x)\}^2\, dx$. **2**

(iii) Deduce that $\left\{\int_0^1 f(x)\, dx\right\}^4 \le \int_0^1 \{f(x)\}^4\, dx$. **2**

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$