# The Scots College

## 2003
## Trial HSC Examination

# Mathematics
## Extension 1

### General Instructions

- Reading time - 5 minutes
- Working time - 2 hours
- Write using blue or black pen
- Board approved calculators may be used
- A table of integrals is provided
- All necessary working should be shown

- Start each question on a new booklet
- Attempt Questions 1 - 7
- All questions are of equal value

### Question 1

(a) Find the acute angle between the lines $2x - y = 0$ and $x + 3y = 0$, giving the answer correct to the nearest minute. [2]

(b) Solve the inequality $\frac{x}{x-3} \le 3$ [3]

(c) If $u$, $v$ and $w$ are the roots of $x^3 - 4x + 1 = 0$, find the value of $\frac{1}{u} + \frac{1}{v} + \frac{1}{w}$. [3]

(d) Solve the equation $\sin 2x = \tan x$ for $0 \le x \le \pi$. [4]

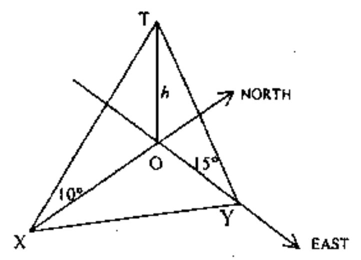
### Question 2 [START A NEW BOOKLET]

(a) A is the point $(-2, 1)$ and B is the point $(x, y)$. The point $P(13, -9)$ divides AB externally in the ratio 5 : 3. Find the values of $x$ and $y$. [3]

(b) (i) Show that the equation of the normal to the parabola $x^2 = 4ay$ at the point $T(2at, at^2)$ is $x + ty = 2at + at^3$. [2]

(ii) Hence show that there is only one normal to the parabola which passes through its focus. [1]

(c) A surveyor at X observes a tower due north. The angle of elevation to the top of the tower is 10°. He then walks 400m to a position Y which is due east of the tower. The angle of elevation from Y to the top of the tower is 15°.




(i) Write an expression for OY in terms of $h$. [1]

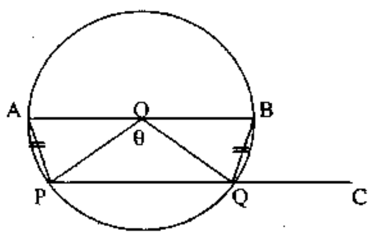
(ii) Calculate $h$ to the nearest metre. [4]

(iii) Find the bearing of Y from X. [1]

**QUESTION 3** **[START A NEW BOOKLET]**

**(a)** Evaluate $\int_0^{2\pi} \cos^2 2x \, dx$. [3]

**(b)**




The points A, B, P and Q lie on the circle with centre at O.

AB is a diameter and PC passes through Q.

AP is equal to BQ and $\angle POQ = \theta$

(i) Express $\angle AOP$ in terms of $\theta$. [1]

(ii) Prove that AB is parallel to PC. [2]

**(c)** By graphing or some other justification, simplify [3]

(i) $\sin^{-1} x + \sin^{-1}(-x)$

(ii) $\tan^{-1} x + \tan^{-1}(-x)$

(iii) $\sin^{-1} x - \cos^{-1}(-x)$

**(d)** Find $\int_0^2 2x\sqrt{1-\frac{x}{2}}\, dx$ using the substitution $u = 1 - \frac{x}{2}$ [3]

**QUESTION 4** **[START A NEW BOOKLET]**

**(a)** The surface area of a cube is increasing at a rate of $10\text{cm}^2$ per second. Find the rate of increase of the volume of the cube when the edge of the cube has length 12cm. [4]

**(b)** $N$ is the number of animals in a certain population at time $t$ years. The population size $N$ satisfies the equation $\frac{dN}{dt} = -k(N-1000)$ for some constant $k$.

(i) Verify where A is constant, that $N = 1000 + Ae^{-kt}$ is a solution of the equation. [2]

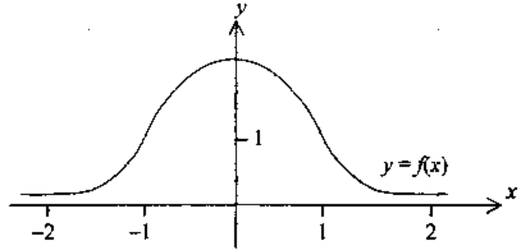
(ii) Initially there are 2500 animals but after 2 years there are only 2200 left. Find the values of $A$ and $k$, to 2 decimal places. [2]

(iii) Find when the number of animals has fallen to 1300. [2]

(iv) Sketch the graph of the population size against time. [2]

**QUESTION 5** **[START A NEW BOOKLET]**

**(a)** The graph below shows the derivative of $y = 2\tan^{-1} x$.




(i) Where does $y = 2\tan^{-1} x$ have its greatest slope and what is this slope? [2]

(ii) Calculate the $x$ values correspond with $\frac{dy}{dx} = \frac{1}{3}$? [1]

(iii) Write an integral that represents the area in the first quadrant bounded by this curve, the $x$ axis and $x = k$, where $k > 0$. [1]

(iv) By considering the limit as $k \to \infty$ determine the total area bounded by this curve and the $x$ axis. [1]

**(b)** (i) Sketch the graph of function $f(x) = e^x - 4$. [1]

(ii) On the same diagram sketch the graph of the inverse function $f^{-1}$. [2]

(iii) Explain why the $x$ coordinate of any point of intersection of the graphs $y = f(x)$ and $y = f^{-1}(x)$ satisfies the equation $e^x - x - 4 = 0$. [1]

(iv) Show that the equation $e^x - x - 4 = 0$ has a root between $x = 1$ and $x = 2$. Use one application of Newton's method to approximate the root, to 2 decimal places. [3]

**QUESTION 6** **[START A NEW BOOKLET]**

**(a)** Prove by Mathematical Induction that $1 + 4 + 7 + \ldots + (3n-2) = \frac{n(3n-1)}{2}$ for all positive integers $n$. [5]

**(b)** A particle moves in a straight line so that its displacement $x$ from a fixed point 0 at time $t$ is given by $x = 3\sin 2t + 4\cos 2t$.

(i) If the motion is expressed in the form of $x = R\sin(2t + \alpha)$ where $\alpha$ is in radians, evaluate the constants $R$ and $\alpha$, to 2 decimal places. [3]

(ii) Show that the motion is Simple Harmonic. [1]

(iii) What is the period of oscillation? [1]

(iv) Determine the maximum displacement from the centre of motion. [2]

**QUESTION 7** [START A NEW BOOKLET]

**(a)** A projectile has an initial velocity $V$ and an angle of projection $\theta$.

**(i)** Assuming $\frac{d^2y}{dt^2} = -10$, $\frac{d^2x}{dt^2} = 0$ and initially $x = 0, y = 10$, find expressions for $x$ and $y$. [3]

**(ii)** If $V = 13ms^{-1}$ and $\theta = \tan^{-1}\left(\frac{5}{12}\right)$ find the range of the projectile. [2]

**(b)** **(i)** Use the Chain Rule to show that

$$\frac{dv}{dt} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$ [1]

**(ii)** The acceleration due to gravity is inversely proportional to the square of the distance $x$ from the centre of the earth.

This can be written as $\frac{dv}{dt} = \frac{-k}{x^2}$. Find $k$ if $\frac{dv}{dt} = -g$ when $x = R$. [1]

**(iii)** Hence show that $v^2 = \frac{2R^2g}{x} + u^2 - 2gR$ where the initial velocity of a rocket is $u$ ms$^{-1}$, $g$ is the acceleration due to gravity and $R$ is the radius of the earth. [2]

**(iv)** Find the maximum distance that the rocket will travel from the centre of the earth. (Answer in terms of $g$, $R$ and $u$). [2]

**(v)** Taking $g = 9.8$ms$^{-2}$, $R = 6400$km find the value of $u$ in ms$^{-1}$ for which the rocket will escape the gravity of the earth. [1]