1999 Higher School Certificate Solutions

3 Unit (Additional) and 3/4 Unit (Common) Mathematics

QUESTION 1

(a) $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{3}}$ (standard integral)

$$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}0$$
$$= \frac{\pi}{3} - 0$$
$$= \frac{\pi}{3}.$$

(b) $\displaystyle\frac{d}{dx}\sin^3 x = 3\sin^2 x.\frac{d}{dx}\sin x$

$$= 3\sin^2 x\cos x.$$

(c) $A(-2, 7)$, $B(8, -8)$

Ratio 2 : 3

For P, $\displaystyle x = \frac{2\times 8 + 3\times(-2)}{2+3}$

$$= \frac{10}{5}$$
$$= 2,$$

and $\displaystyle y = \frac{2\times(-8)+3\times 7}{2+3}$

$$= \frac{5}{5}$$
$$= 1.$$

$\therefore$ P is $(2, 1)$.

(d) Asymptote when denominator is zero, that is, $x-3=0$ or $x=3$.

(e) $P(x) = x^3 - 4x$

$P(-3) = -27 + 12$

$= -15$ is the remainder.

(f) If $u = \tan x$

$\displaystyle\frac{du}{dx} = \sec^2 x \Rightarrow du = \sec^2 x\ dx.$

When $x = 0$, $u = 0$.

When $x = \frac{\pi}{3}$, $u = \sqrt{3}$.

$$\therefore \int_0^{\frac{\pi}{3}} \tan^2 x\sec^2 x\ dx = \int_0^{\sqrt{3}} u^2\ du$$
$$= \left[\frac{u^3}{3}\right]_0^{\sqrt{3}}$$
$$= \sqrt{3}.$$

QUESTION 2

(a) Number of ways of choosing 3 females from 7 is ${}^7C_3$. The other two must be male. The number of ways of choosing 2 from 4 is ${}^4C_2$.

$\therefore$ Number of committees $= {}^7C_3 \times {}^4C_2$

$= 210.$

(b) **Method 1:**

$\cos\theta + \sqrt{3}\sin\theta = 1$

Now $R\cos(\theta-\alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$\therefore$ $R\cos\alpha = 1$

$R\sin\alpha = \sqrt{3}.$

$\therefore$ $R^2\left(\sin^2\alpha + \cos^2\alpha\right) = 3 + 1 = 4.$

$\therefore$ $R = 2$

and $\displaystyle\frac{R\sin\alpha}{R\cos\alpha} = \frac{\sqrt{3}}{1},$

$\therefore$ $\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}.$

$\therefore$ $2\cos\left(\theta - \frac{\pi}{3}\right) = 1,\quad -\frac{\pi}{3} \le \left(\theta - \frac{\pi}{3}\right) \le \frac{5\pi}{3}$

$\cos\left(\theta - \frac{\pi}{3}\right) = \frac{1}{2}$

$\theta - \frac{\pi}{3} = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}.$

$\therefore$ $\theta = 0, \frac{2\pi}{3}, 2\pi.$

Method 2:

If $\theta = \pi$, $\cos\pi + \sqrt{3}\sin\pi = -1 + 0 \ne 1$,

$\therefore$ $\theta = \pi$ is not a solution.

If $\theta \ne \pi$, let $t = \tan\frac{\theta}{2}$.

$\therefore$ $\sin\theta = \dfrac{2t}{1+t^2}$ and $\cos\theta = \dfrac{1-t^2}{1+t^2}.$

$$\therefore \frac{1-t^2}{1+t^2} + \sqrt{3}\times\frac{2t}{1+t^2} = 1$$
$$1 - t^2 + 2\sqrt{3}t = 1 + t^2$$
$$2t^2 - 2\sqrt{3}t = 0$$
$$2t\left(t - \sqrt{3}\right) = 0.$$

$\therefore$ $t = 0, \sqrt{3}.$

That is, $\tan\frac{\theta}{2} = 0, \sqrt{3}$.

$\therefore\ \frac{\theta}{2} = 0, \frac{\pi}{3}, \pi \quad (0 \le \theta \le 2\pi)$.

$\therefore\ \theta = 0, \frac{2\pi}{3}, 2\pi$.

(c) $f(x) = x + \log_e x$

(i) The natural domain is $x > 0$ since $\log_e x$ is defined only for $x > 0$.

(ii) $y = f(x)$ is increasing if $f'(x) > 0$.

$\therefore\ f'(x) = 1 + \frac{1}{x} > 0$, since $x > 0$.

(iii)
$$f(0{\cdot}5) = 0{\cdot}5 + \log_e 0{\cdot}5 \doteqdot -0{\cdot}193 < 0.$$
$$f(1) = 1 + \log_e 1 = 1 > 0.$$

The curve cuts the x axis between $x = 0{\cdot}5$ and $x = 1$, since the sign of $f(x)$ changes and $f(x)$ is continuous.

(iv) Let $f(x) = x + \log_e x$

$f'(x) = 1 + \frac{1}{x}$.

Let x_2 be a second approximation to the root of $x + \log_e x = 0$.

$$\therefore\ x_2 = 0{\cdot}5 - \frac{f(0{\cdot}5)}{f'(0{\cdot}5)}, \text{ by Newton's method,}$$
$$= 0{\cdot}5 - \frac{0{\cdot}5 + \log_e 0{\cdot}5}{1 + \frac{1}{0{\cdot}5}}$$
$$= 0{\cdot}564\ldots$$

N.B. You need to use Newton's method again to see how many of these digits are significant, but this is not required by the question.

QUESTION 3

(a)
$$V = \pi\int_0^{\frac{\pi}{2}} (3\sin x)^2\, dx$$
$$= 9\pi\int_0^{\frac{\pi}{2}} \sin^2 x\, dx$$
$$= \frac{9\pi}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2x)\, dx$$
$$= \frac{9\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$
$$= \frac{9\pi}{2}\left[\left(\frac{\pi}{2} - 0\right) - (0 - 0)\right]$$
$$= \frac{9\pi^2}{4}.$$

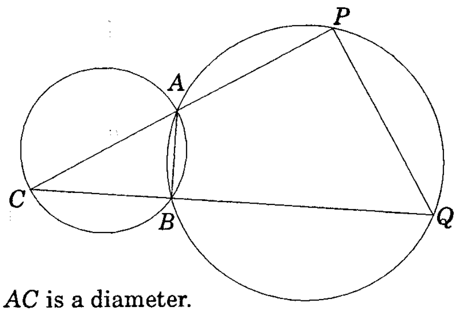
$\therefore$ Volume $= \frac{9\pi^2}{4}$ cubic units.

(b) $P(6) = \frac{1}{6}$, $P(\bar{6}) = \frac{5}{6}$.

Probability of '6' on exactly 2 of 7 throws

$$= {}^7C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^5$$
$$= \frac{7\times 6}{1\times 2}\times\frac{1}{6}\times\frac{5^5}{6^5}$$
$$= \frac{21\,875}{93\,312}$$
$$\doteqdot 0{\cdot}2344.$$

(c)



Data: AC is a diameter.

Construction: Join AB, PQ.

Proof: $\angle ABC = 90°$ (angle in semicircle, given AC is diameter)

$\angle CPQ = \angle ABC$ (exterior angle of cyclic quadrilateral equals interior opposite angle)

$\therefore\ \angle CPQ$ is a right angle.

(d) (i) $A(2\sin x + \cos x) + B(2\cos x - \sin x) \equiv \sin x + 8\cos x$

$\therefore\ (2A - B)\sin x + (A + 2B)\cos x \equiv \sin x + 8\cos x$.

Equating coefficients of $\sin x$ and $\cos x$,

$2A - B = 1$ —①

$A + 2B = 8$ —②

① × 2 → $4A - 2B = 2$ —③

② + ③ → $5A = 10$

$A = 2$.

Substitute $A = 2$ in ②:

$2B = 6$

$B = 3$.

$\therefore\ A = 2,\ B = 3$.

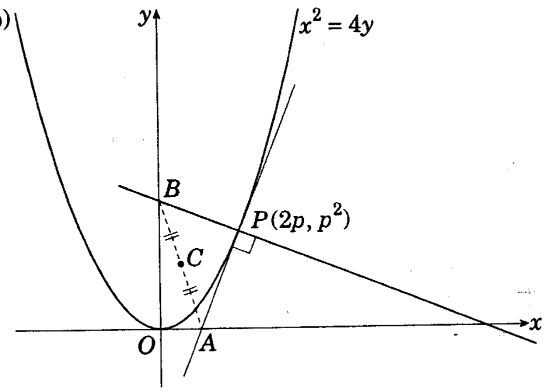
(ii)
$$\int \frac{\sin x + 8\cos x}{2\sin x + \cos x}\, dx$$
$$= \int \frac{2(2\sin x + \cos x) + 3(2\cos x - \sin x)}{2\sin x + \cos x}\, dx \quad \text{from (i)}$$
$$= \int 2\, dx + 3\int \frac{2\cos x - \sin x}{2\sin x + \cos x}\, dx$$
$$= 2x + 3\ln(2\sin x + \cos x) + C.$$

$\left[\textit{Note: } \frac{d}{dx}(2\sin x + \cos x) = 2\cos x - \sin x\right]$

QUESTION 4

(a) $\sum_{k=2}^{5}(-1)^k k = (-1)^2\times 2+(-1)^3\times 3 + (-1)^4\times 4+(-1)^5\times 5$
$= -2.$

(b)



(i) $x^2 = 4y$

$y = \frac{x^2}{4}$

$\frac{dy}{dx} = \frac{x}{2}.$

When $x = 2p$, $\frac{dy}{dx} = \frac{2p}{2} = p$.

Equation of tangent AP is

$y - y_1 = m(x - x_1)$

$y - p^2 = p(x - 2p)$

$y = px - p^2$ —①

(ii) Equation of normal BP is

$y - p^2 = -\frac{1}{p}(x - 2p).$

B lies on BP at $x = 0$.

When $x = 0$, $y = p^2 - \frac{1}{p}(-2p)$

$= p^2 + 2.$

$\therefore$ B is $(0,\ p^2+2)$.

(iii) Substitute $y = 0$ in ①: $0 = px - p^2$

$x = p.$

$\therefore$ A is $(p,\ 0)$.

If $C(x,\ y)$ is the midpoint of $A(p,\ 0)$ and $B(0,\ p^2+2)$, $x = \frac{p+0}{2}$ and $y = \frac{0+(p^2+2)}{2}$.

$x = \frac{p}{2}$ —②

$y = \frac{p^2+2}{2}$ —③

From ②, $p = 2x$.

Substitute in ③: $y = \frac{4x^2+2}{2} = 2x^2 + 1.$

But $p > 0$, $\therefore$ $x > 0$.

$\therefore$ Cartesian equation of locus of C is $y = 2x^2 + 1,\ x > 0$.

(c) (i) $\int_1^2 \frac{dx}{x} = \left[\ln x\right]_1^2$

$= \ln 2 - \ln 1$

$= \ln 2.$

(ii) $\int_1^2 \frac{dx}{x} \doteqdot \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right],$

where $f(x) = \frac{1}{x}$, $a = 1$, $b = 2$.

$= \frac{2-1}{6}\left[\frac{1}{1} + 4\times\frac{1}{1\cdot 5} + \frac{1}{2}\right]$

$= \frac{25}{36}$ $(= 0{\cdot}69\dot{4})$.

(iii) $\ln 2 \doteqdot \frac{25}{36}$

$2 \doteqdot e^{\frac{25}{36}}$

$2^{\frac{36}{25}} \doteqdot e$ $\left(\text{raising both sides to power } \frac{36}{25}\right)$

$\therefore$ $e \doteqdot 2{\cdot}7132\ldots$

$= 2{\cdot}713$ (3 dec. places).

QUESTION 5

(a) Prove $(n+1)(n+2)\cdots(2n-1)2n$
$= 2^n\left[1\times 3\times\cdots\times(2n-1)\right]$

If $n = 1$, LHS $= 1 + 1 = 2$
RHS $= 2^1\times 1 = 2.$

$\therefore$ The statement is true for $n = 1$.

Assume statement is true for $n = k$, that is,
assume $(k+1)(k+2)\cdots(2k-1)2k$
$= 2^k\left[1\times 3\times\cdots\times(2k-1)\right].$ —①

Hence prove statement is true for $n = k+1$, that is, prove
$(k+2)(k+3)\cdots(2k+1)(2k+2)$
$= 2^{k+1}\left[1\times 3\times\cdots\times(2k+1)\right].$ —②

Now LHS

$= (k+2)(k+3)\cdots(2k+1)(2k+2)$

$= \frac{(k+1)(k+2)(k+3)\cdots(2k-1)2k(2k+1)(2k+2)}{k+1}$

$= \frac{2^k}{k+1}\left[1\times 3\times\cdots\times(2k-1)\right](2k+1)(2k+2)$, from ①

$= \frac{2^k}{\cancel{k+1}}\left[1\times 3\times\cdots\times(2k-1)\right](2k+1)2\cancel{(k+1)}$

$= 2^{k+1}\left[1\times 3\times\cdots\times(2k-1)(2k+1)\right]$

$=$ RHS.

$\therefore$ If the statement is true for $n = k$, it is also true for $n = k+1$. But it is true for $n = 1$.

$\therefore$ It is true for $n = 1 + 1 = 2$ and so on, that is, it is true for all integers $n \geq 1$.

(b) $f(x) = e^x - 1 - x$

(i) $f'(x) = e^x - 1$
$= 0$ only when $x = 0$.

$\therefore$ There is only one stationary point (at $x = 0$).

$f''(x) = e^x > 0$ for all x.

$\therefore$ The graph of $f(x)$ is concave up for all x.

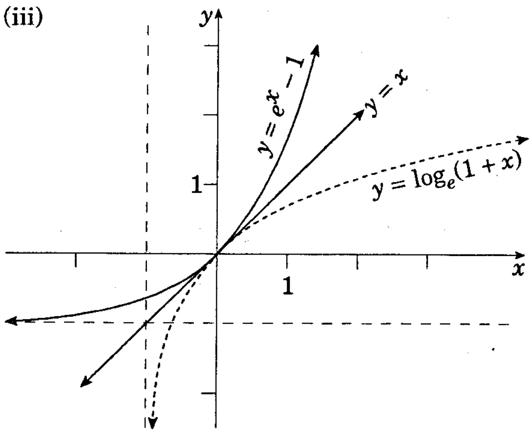
Since $f(x)$ is continuous for all x (being made up of the sum and difference of continuous functions), the stationary point at $x = 0$ is both a local and absolute minimum.

(ii) When $x = 0$, $f(x) = e^0 - 1 - 0 = 0$.

$\therefore$ The least value of $f(x) = 0$.

$\therefore$ $f(x) \geq 0$ for all x.

(iii)



N.B. The gradient of $y = e^x - 1$ at $x = 0$ is 1, so $y = x$ is a tangent at $(0,\ 0)$.
This is also implied by (ii).

(iv) Inverse relation of $y = e^x - 1$ is $x = e^y - 1$.
That is, $e^y = x + 1$
$y = \log_e(x + 1)$
$\therefore$ $g^{-1}(x) = \log_e(x + 1)$.

(v) Domain of $g^{-1}(x)$ is $x + 1 > 0$,
that is, $x > -1$.

(vi) $g(x) = e^x - 1$
$g^{-1}(x) = \log_e(1 + x)$.

The graphs of a pair of inverse functions are symmetrical about the line $y = x$.
The graph of $y = g(x)$ is above the graph of $y = x$ except at $x = 0$ where they coincide.

$\therefore$ The graph of $y = g^{-1}(x)$ is below the graph of $y = x$ except at $x = 0$ where they coincide.

$\therefore$ $\log_e(1 + x) \leq x$ for all $x > -1$.

QUESTION 6

(a) $x = \cos^2 3t, \quad t > 0.$ —①

(i) Substitute $x = \frac{3}{4}$ in ①:

$$\frac{3}{4} = \cos^2 3t$$
$$\cos 3t = \pm\frac{\sqrt{3}}{2}$$
$$3t = \frac{\pi}{6}, \ldots$$
$$t = \frac{\pi}{18}, \ldots$$

Particle is first at $x = \frac{3}{4}$ after $\frac{\pi}{18}$ seconds.

(ii) $v = \dfrac{dx}{dt} = 2\cos 3t . -3\sin 3t$
$= -3\sin 6t.$

When $t = \frac{\pi}{18}$, $\quad v = -3\sin\left(6 \times \frac{\pi}{18}\right)$
$= -3\sin\frac{\pi}{3}$
$= \frac{-3\sqrt{3}}{2} < 0.$

Since $v < 0$, the particle is travelling in the negative direction.

(iii) $a = \dfrac{dv}{dt} = -3 \times 6\cos 6t$
$= -18\cos 6t.$

$\cos 6t = 2\cos^2 3t - 1$
(using $\cos 2x = 2\cos^2 x - 1$)
$= 2x - 1$, from ①.

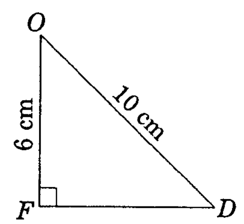
$\therefore \quad a = -18(2x - 1).$

(iv) $a = -18(2x - 1)$
$= -36\left(x - \frac{1}{2}\right)$
$= -6^2\left(x - \frac{1}{2}\right),$

which is of the form $\ddot{x} = -n^2(x - b)$, indicating simple harmonic motion with centre of oscillation at $x = \frac{1}{2}$.

(v) Period $= \frac{2\pi}{n}$ seconds
$= \frac{2\pi}{6}$ seconds
$= \frac{\pi}{3}$ seconds.

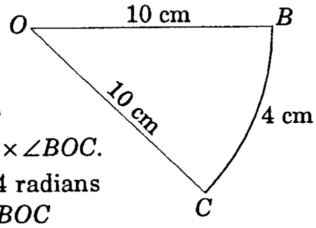
(b) (i)



$OD = 10$ cm (radius)

$\therefore$ $FD = 8$ cm (Pythagoras' theorem).

(ii) OBC is a sector of a circle, centre O, radius 10 cm.



$$l = r\theta$$

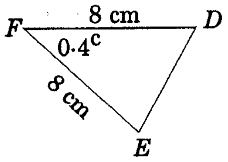
$\therefore \quad 4 = 10 \times \angle BOC.$

$\therefore \ \angle BOC = 0{\cdot}4$ radians

$\angle DFE = \angle BOC$

$\therefore \ \angle DFE = 0{\cdot}4$ radians.

(iii)



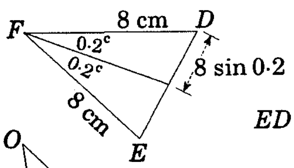
$$ED^2 = FD^2 + FE^2 - 2 \times FD \times FE \cos 0{\cdot}4$$

(by cosine rule)

$$= 8^2 + 8^2 - 2 \times 8 \times 8 \cos 0{\cdot}4$$

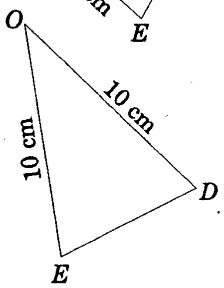
$$= 128(1 - \cos 0{\cdot}4).$$

{ *Alternatively:*



$$ED = 2 \times 8 \sin 0{\cdot}2$$

$$= 16 \sin 0{\cdot}2. \}$$



$$\cos \angle EOD = \frac{OE^2 + OD^2 - ED^2}{2 \times OE \times OD}$$

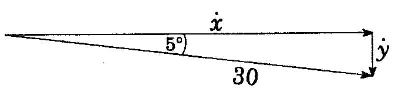
$$= \frac{10^2 + 10^2 - 128(1 - \cos 0{\cdot}4)}{2 \times 10 \times 10}$$

$$\angle DOE = 0{\cdot}3192\ldots$$

$$\approx 0{\cdot}319 \text{ radians (3 dec. places).}$$

QUESTION 7

(a) Initial conditions for velocity:



When $t = 0$,

$\dot{x} = 30\cos(5°), \ \dot{y} = -30\sin(5°).$ —①

(i) $\ddot{x} = 0$

$\therefore \ \dot{x} = C_1$ (constant).

$\therefore \ \dot{x} = 30\cos(5°)$ from ①. —②

$$x = \int 30\cos(5°)\,dt$$

$$= 30t\cos(5°) + C_2.$$

When $t = 0, \ x = 0, \ \therefore \ C_2 = 0.$

$\therefore \quad x = 30t\cos(5°).$

$\ddot{y} = -10$

$\therefore \quad \dot{y} = \int -10\,dt$

$$= -10t + D_1.$$

When $t = 0, \ \dot{y} = -30\cos(5°)$ from ①.

$\therefore \quad D_1 = -30\sin(5°)$

$\therefore \quad \dot{y} = -10t - 30\sin(5°).$ —③

$$y = \int -10t - 30\sin(5°)\,dt$$

$$= -5t^2 - 30t\sin(5°) + D_2.$$

When $t = 0, \ y = 0, \ \therefore \ D_2 = 0.$

$\therefore \quad y = -30t\sin(5°) - 5t^2.$ —④

(ii) Ball strikes the ground when $y = -2$.

Substitute $y = -2$ in ④:

$$-2 = -30t\sin 5° - 5t^2$$

$$5t^2 + 30t\sin 5° - 2 = 0$$

$$t = \frac{-30\sin 5° \pm \sqrt{(-30\sin 5°)^2 - 4 \times 5 \times (-2)}}{2 \times 5}$$

$$= \frac{-30\sin 5° + \sqrt{900\sin^2 5° + 40}}{10}$$

(other answer negative and therefore irrelevant)

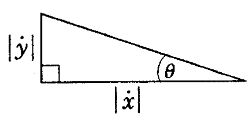
$$= 0{\cdot}4229\ldots$$

$\therefore$ The ball strikes the ground after $0{\cdot}42$ seconds (2 dec. places).

(iii) When $t = 0{\cdot}4229$,

$\dot{x} = 30\cos(5°)$ from ②,

$\dot{y} = -4{\cdot}229 - 30\sin(5°)$, from ③.



$$\tan\theta = \frac{4{\cdot}229 + 30\sin 5°}{30\cos 5°}$$

$$= 0{\cdot}228\,99\ldots$$

$$\theta \doteqdot 12{\cdot}9°.$$

Angle at which the ball strikes the ground is 13° (nearest degree).

(b) $(1-x)^n = \binom{n}{0} - \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n}(-1)^n x^n$

$\left(1+\frac{1}{x}\right)^n = \binom{n}{0}\left(\frac{1}{x}\right)^0 + \binom{n}{1}\left(\frac{1}{x}\right)^1 + \binom{n}{2}\left(\frac{1}{x}\right)^2 + \cdots + \binom{n}{n}(-1)^n\left(\frac{1}{x}\right)^n.$

The term in x^2 in $(1-x)^n\left(1+\frac{1}{x}\right)^n$ is

$\binom{n}{2}\binom{n}{0}x^2\left(\frac{1}{x}\right)^0 - \binom{n}{3}\binom{n}{1}x^3\left(\frac{1}{x}\right)^1 + \binom{n}{5}\binom{n}{3}x^5\left(\frac{1}{x}\right)^3 + \cdots + (-1)^n\binom{n}{n}\binom{n}{n-2}x^n\left(\frac{1}{x}\right)^n.$

$\therefore$ The coefficient of x^2 in $(1-x)^n\left(1+\frac{1}{x}\right)^n$ is

$\binom{n}{2}\binom{n}{0} - \binom{n}{3}\binom{n}{1} + \cdots + (-1)^n\binom{n}{n}\binom{n}{n-2},$

and this is the expression given in the question.

Now $(1-x)^n\left(1+\frac{1}{x}\right)^n = \left[(1-x)\left(1+\frac{1}{x}\right)\right]^n = \left(\frac{1}{x} - x\right)^n.$

The general term of $\left(\frac{1}{x} - x\right)^n$ is

$\binom{n}{r}\left(\frac{1}{x}\right)^{n-r}(-x)^r = \binom{n}{r}(-1)^r x^{2r-n}.$

The term in x^2 has $2r - n = 2$

$r = \frac{n+2}{2}.$

$\therefore$ The coefficient of $x^2 = \binom{n}{\frac{n+2}{2}}(-1)^{\frac{n+2}{2}},$

and only exists if n is even, $\left(\frac{n+2}{2}\right.$ must be an integer$\left.\right)$.

$$\therefore \binom{n}{2}\binom{n}{0} - \binom{n}{3}\binom{n}{1} + \cdots + (-1)^n\binom{n}{n}\binom{n}{n-2} = \begin{cases} \binom{n}{\frac{n+2}{2}}(-1)^{\frac{n+2}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

END OF 3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON) MATHEMATICS SOLUTIONS