Question 1:

(a)(i) Find the derivative of $x^2 \cos x$. 2

(ii) Evaluate $\int_1^6 \frac{x}{x^2+4}\,dx$. 2

(b)(i) Sketch $y = |x+1|$. 2

(ii) Hence or otherwise solve $|x+1| = 3x$. 1

(c) If $f(x) = 2\sin^{-1}(3x)$, find

(i) the domain and range of $f(x)$, 2

(ii) $f\left(\frac{1}{6}\right)$, 1

(iii) $f'\left(\frac{1}{6}\right)$. 2

QUESTION 2: **(START A NEW PAGE)**

(a) P(-7,3), Q(9,15) and B(14,0) are three points and A divides the interval PQ in the ratio 3:1. Prove that PQ is perpendicular to AB. 3

(b) By using the substitution $u^2 = x+1$ evaluate $\int_0^3 \frac{x+2}{\sqrt{x+1}}\,dx$. 3

(c) Water flows from a hole in the base of a cylindrical vessel at a rate given by 6

$$\frac{dh}{dt} = -k\sqrt{h}$$

where k is a constant and h mm is the depth of water at time t minutes.
If the depth of water falls from 2500mm to 900mm in 5 minutes, find how much longer it will take to empty the vessel.

QUESTION 3: (START A NEW PAGE)

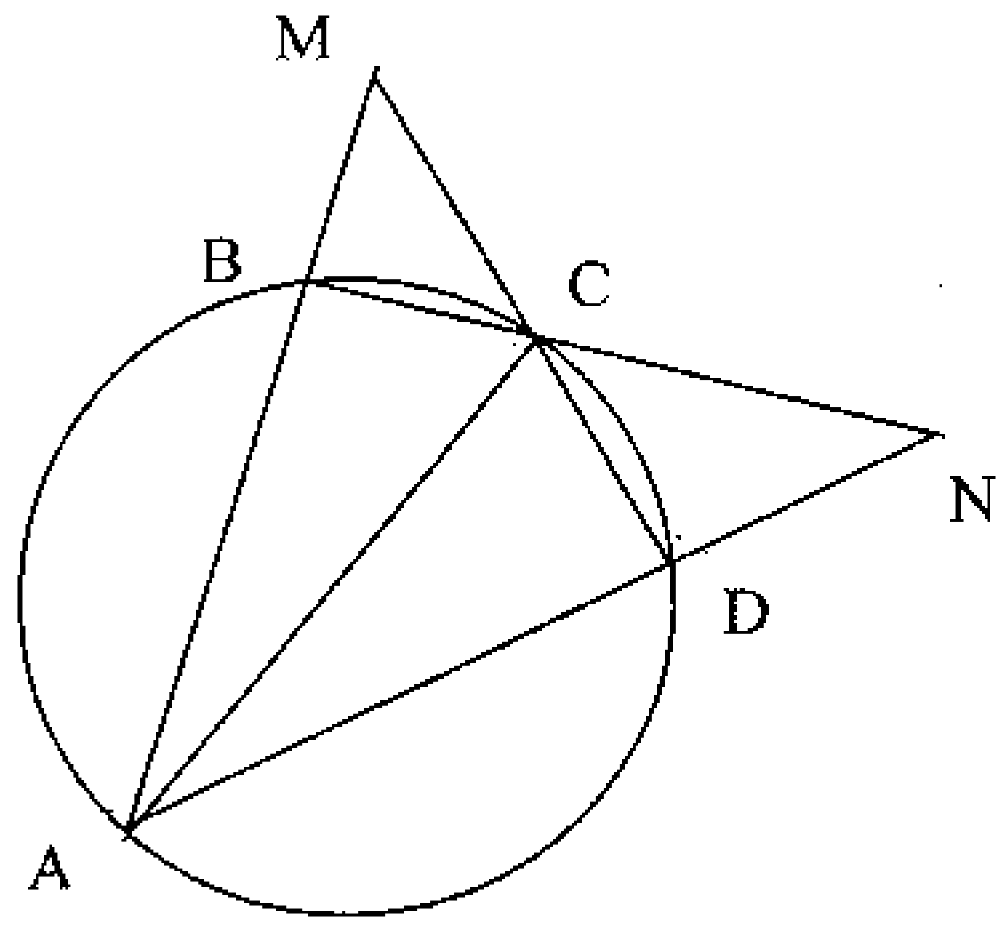
(a) Find the value of the constant term in the expansion of $\left(3x+\frac{2}{\sqrt{x}}\right)^6$. 3

(b) Three boys (Adam, Bruce, Chris) and three girls (Debra, Emma, Fay) form a single queue at random in front of the school canteen window. Find the probability that:

(i) the first two to be served are Emma and Adam in that order, 2

(ii) a boy is at each end of the queue, 1

(iii) no two girls stand next to each other. 1



(c) In the figure ABM, DCM, BCN and ADN are straight lines and $\angle AMD = \angle BNA$.

(i) Copy the diagram onto your answer sheet and prove that $\angle ABC = \angle ADC$. 3

(ii) Hence prove that AC is a diameter. 2

QUESTION 4: (START A NEW PAGE)

(a)(i) Given that $\sin^2 A + \cos^2 A = 1$, prove that $\tan^2 A = \sec^2 A - 1$. 2

(ii) Sketch the curve $y = 4\tan^{-1} x$ clearly showing its range. 2

(iii) Find the volume of the solid formed when the area bounded by the curve $y = 4\tan^{-1} x$, the y-axis and the line $y = \pi$ is rotated one revolution about the y-axis. 2

(b)(i) An object has velocity $v\ ms^{-1}$ and acceleration $\ddot{x}\ ms^{-2}$ at position $x\ m$ from the origin, show that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$. 2

(ii) The acceleration (in ms^{-2}) of an object is given by $\ddot{x} = 2x^3 + 4x$.

(α) If the object is initially 2 m to the right of the origin traveling with velocity 6 ms^{-1}, find an expression for v^2 (the square of its velocity) in terms of x. 2

(β) What is the minimum speed of the object? (Give a reason for your answer) 2

QUESTION 5: **(START A NEW PAGE)**

(a) The curves $y = e^{-2x}$ and $y = 3x + 1$ meet on the y-axis. Find the size of the acute angle between these curves at the point where they meet. 3

(b)(i) Sketch the function $y = f(x)$ where $f(x) = (x-1)^2 - 4$ clearly showing all intercepts with the co-ordinate axes. (Use the same scale on both axes) 2

(ii) What is the largest positive domain of f for which *f(x)* has an inverse $f^{-1}(x)$? 1

(iii) Sketch the graph of $y = f^{-1}(x)$ on the same axes as (i). 1

(c) In tennis a player is allowed a maximum of two serves when attempting to win a point. If the first serve is not legal it is called a fault and the server is allowed a second serve. If the second serve is also illegal then it is called a double fault and the server loses the point. The probability that Pat Smash's first serve will be legal is 0.4. If Pat Smash needs to make a second serve then the probability that it will be legal is 0.7.

(i) Find the probability that Pat Smash will serve a double fault when trying to win a point. 2

(ii) If Pat Smash attempts to win six points, what is the probability that he will serve at least two double faults? (Give answer correct to 2 decimal places) 3

QUESTION 6: **(START A NEW PAGE)**

(a) A spherical bubble is expanding so that its volume is increasing at $10\ cm^3 s^{-1}$. 3
Find the rate of increase of its radius when the surface area is $500\ cm^2$.
(Volume = $\frac{4}{3}\pi r^3$, Surface area = $4\pi r^2$)

(b) Prove by Mathematical Induction that: 4

$$2(1!) + 5(2!) + 10(3!) + \ldots + (n^2 + 1)n! = n(n+1)! \text{ for positive integers } n \geq 1.$$

(c) If $y = \dfrac{\log_e x}{x}$ find $\dfrac{dy}{dx}$ and hence show that $\displaystyle\int_e^{e^2} \frac{1 - \log_e x}{x \log_e x}\, dx = \log_e 2 - 1$. 5

QUESTION 7: **(START A NEW PAGE)**

(i) By considering the expansion of $\sin(X+Y)-\sin(X-Y)$ prove that 3

$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$.

(ii) Also given that $\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$ prove that 2

$\frac{\sin A - \sin B}{\cos A - \cos B} = -\cot\left(\frac{A+B}{2}\right)$.

(iii) Prove that the position of a projectile t seconds after projection from ground level with initial horizontal and vertical velocity components of $V\cos\alpha$ and $V\sin\alpha$ respectively is given by $x = Vt\cos\alpha$ and $y = -\frac{1}{2}gt^2 + Vt\sin\alpha$. 2
(Assume that there is no air resistance)

(iv) Two objects P and Q are projected from the same ground position at the same time with initial speed $V\ ms^{-1}$ at angles α and β respectively (β > α).

(α) If at time t seconds the line joining P and Q makes an acute angle θ with the horizontal prove that $\tan\theta = \left|\frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha}\right|$. 3

(β) Hence show that $\theta = \frac{1}{2}(\pi - \alpha - \beta)$. 2

THIS IS THE END OF THE EXAMINATION PAPER

JRAHS 3UNIT TRIAL 2001

QUESTION 1

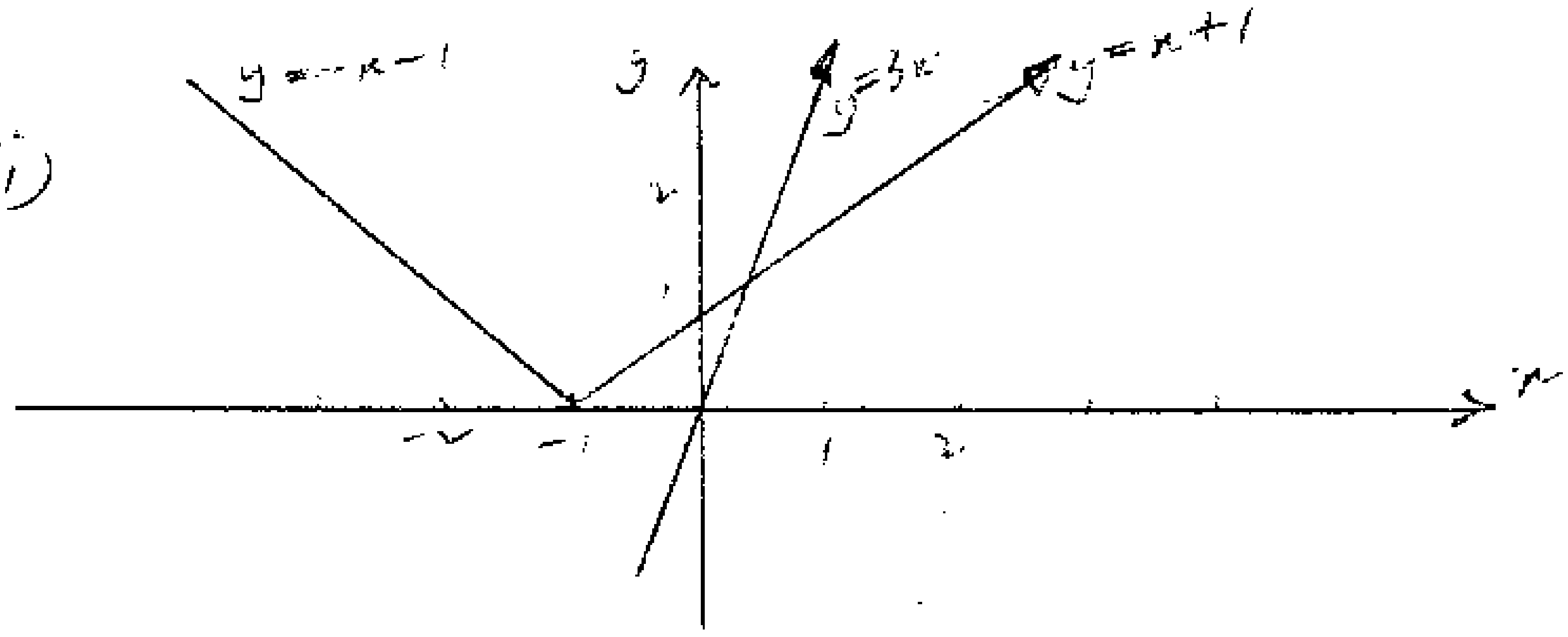
(a)(i) $y' = 2x\cos x - x^2 \sin x$

(ii) $\displaystyle\int_1^6 \frac{x}{x^2+4}\,dx = \frac{1}{2}\left[\ln(x^2+4)\right]_1^6$

$= \frac{1}{2}(\ln 40 - \ln 5)$

$= \frac{1}{2}\ln 8$

(b)(i)



(ii) $3x = x + 1$ (from graph)

$2x = 1$

$x = \frac{1}{2}$

(c)(i) Domain $-\frac{1}{3} \le x \le \frac{1}{3}$

Range $-\pi \le y \le \pi$

(ii) $f(\frac{1}{6}) = 2\sin^{-1}(\frac{3}{6})$

$= \frac{\pi}{3}$

(iii) $f'(x) = 2 \cdot \dfrac{3}{\sqrt{1-9x^2}}$

$f'(\frac{1}{6}) = \dfrac{6}{\sqrt{1-\frac{9}{36}}}$

$= \dfrac{6}{\sqrt{\frac{3}{4}}}$

$= \dfrac{12}{\sqrt{3}}$ or $4\sqrt{3}$

QUESTION 2.

(a) $P(-7,3)$ $\quad$ $Q(9,15)$ $\quad$ $B(14,0)$

$3:1$

$$A\left(\frac{-7+27}{4}, \frac{3+45}{4}\right) = A(5,12)$$

$$m(PQ) = \frac{15-3}{9+7}$$

$$= 3/4$$

$$m(AB) = \frac{12-0}{5-14}$$

$$= -4/3$$

$$m(PQ)\cdot m(AB) = 3/4 \cdot -4/3$$

$$= -1$$

$\therefore PQ \perp AB$ (prod. of slopes $= -1$)

(b) $x=0 \quad u^2 = 1$ $\qquad$ $x = 3, \quad u^2 = 4$

$u = 1$ (take $u > 0$) $\qquad$ $u = 2$ (take $u > 0$)

$$u^2 - 1 = x$$

$$\frac{dx}{du} = 2u$$

$$dx = 2u\,du$$

$$\therefore \int_0^3 \frac{x+2}{\sqrt{x+1}}\,dx = \int_1^2 \frac{u^2+1}{\sqrt{u^2}}\,2u\,du$$

$$= 2\int_1^2 \frac{u^2+1}{u}\,du$$

$$= 2\int_1^2 u + \frac{1}{u}\,du$$

$$= 2\left[\tfrac{1}{2}u^2 + \ln u\right]_1^2$$

$$= 2\left\{\left(\tfrac{4}{2} + \ln 2\right) - \left(\tfrac{1}{2} + \ln 1\right)\right\}$$

$$= 3 + 2\ln 2 \qquad 6\tfrac{2}{3}$$

2(c) $\frac{dt}{dh} = -\frac{1}{k} \cdot h^{-\frac{1}{2}}$

$t = -\frac{1}{k} \cdot 2h^{\frac{1}{2}} + c$

$t = \frac{-2\sqrt{h}}{k} + c$

$t = 0 \quad h = 2500$

$0 = \frac{-100}{k} + c \quad$ — ①

$t = 5 \quad h = 900$

$5 = -\frac{60}{k} + c \quad$ — ②

② − ① $\quad 5 = \frac{-60}{k} + \frac{100}{k}$

$5k = 40$

$k = 8$

from ① $\quad c = \frac{100}{8}$

$\therefore t = \frac{-\sqrt{h}}{4} + 12.5$

when $h = 0 \quad t = 12.5$

$\therefore$ extra time $= 12.5 - 5$

$= 7.5$ min.

QUESTION 3.

(a) $T_{r+1} = {}^6C_r (3x)^{6-r} \left(\frac{2}{\sqrt{x}}\right)^r$

$= {}^6C_r \, 3^{6-r} \, 2^r \, x^{6-r} \, x^{-\frac{1}{2}r}$

$= {}^6C_r \, 3^{6-r} \, 2^r \, x^{6-\frac{3}{2}r}$

for constant term degree of $x = 0$

$\therefore \quad 6 - \frac{3}{2}r = 0$

$1\frac{1}{2} r = 6$

$r = 4$

$\therefore$ constant $= {}^{6}C_{4}\, 3^{2} . 2^{4}$

$= 2160$

(b) (i) $\quad 1 \quad 1 \quad 4 \quad 3 \quad 2 \quad 1$

$\quad\;\; E \quad A$

Prob $= \frac{4!}{6!}$ $\quad$ <u>or</u> $\quad$ Prob $= \frac{1}{6} \cdot \frac{1}{5}$

$= \frac{1}{30}$ $\qquad\qquad\qquad\qquad = \frac{1}{30}$

(ii) $\quad \overset{C}{3} \quad 4 \quad 3 \quad 2 \quad 1 \quad \overset{K}{2}$

Prob $= \dfrac{3.2.4!}{6!}$

$= \frac{1}{5}$

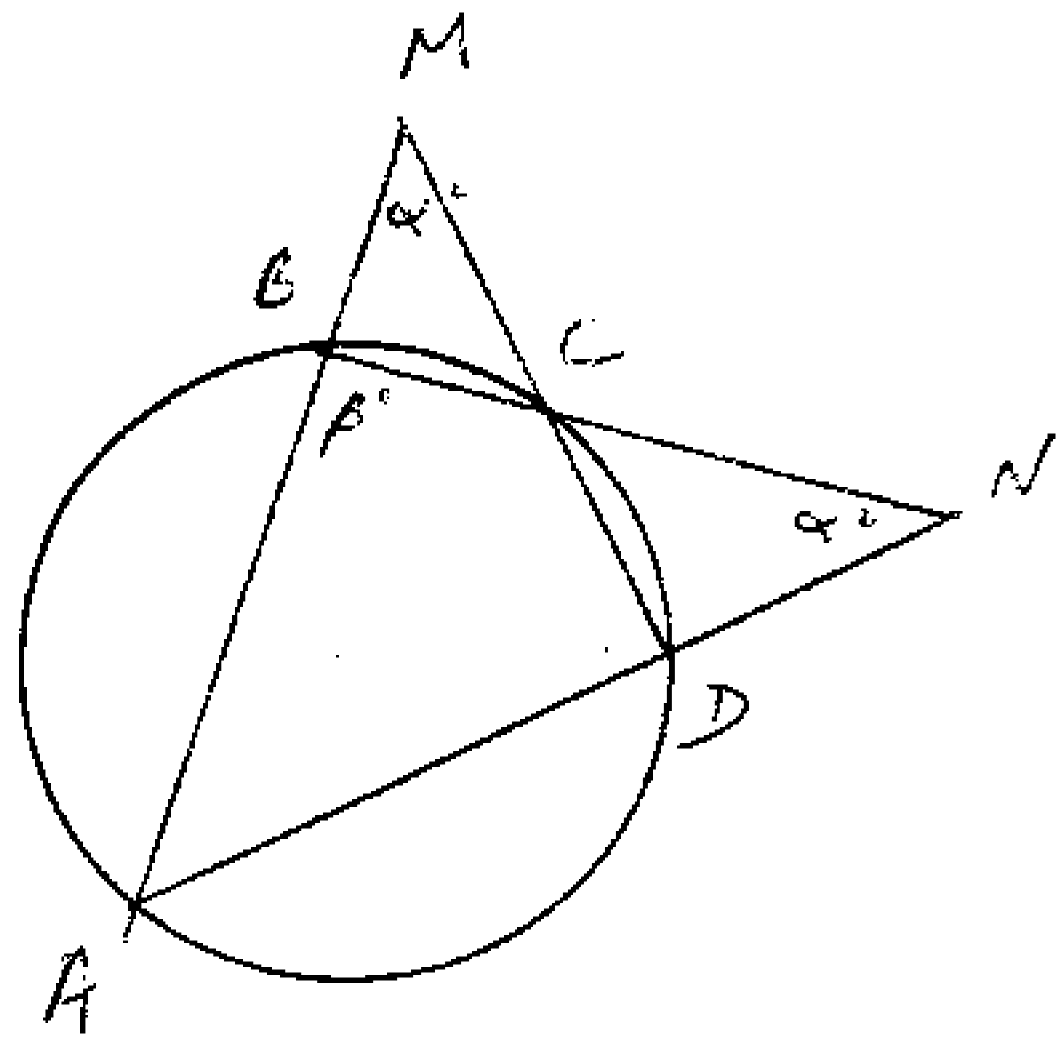
(iii) $\quad 2\ \ 3\ \ 4\ \ 2\ \ 3\ \ 1$

$\quad\;\; G\ \ B\ \ G\ \ B\ \ G\ \ B$

(Place Boys then fill gaps with girls)

Prob $= \dfrac{3!\,4.3.2}{6!}$

$= \frac{1}{5}$



(c) (i) let $A\hat{M}D = A\hat{N}B = \alpha°$

$\quad$ & $A\hat{B}C = \beta°$

$B\hat{C}M = (\beta - \alpha)°$ (exterior angle of $\triangle BMC$ equals sum of opposite interior angles)

$D\hat{C}N = (\beta - \alpha)°$ (vertically opposite angles)

$A\hat{D}C = \beta°$ (exterior angle of $\triangle CND$ equals sum of opposite interior angles

$\therefore A\hat{B}C = A\hat{D}C$ (0 if ...

Q3(c)(ii) $\hat{ABC} + \hat{ADC} = 180°$ (opposite angles of cyclic quad ABCD are supplementary)

$2\hat{ABC} = 180°$ ($\hat{ABC} = \hat{ADC}$, part (i))

$\hat{ABC} = 90°$

$\therefore$ AC is a diameter (angle in semicircle is $90°$)

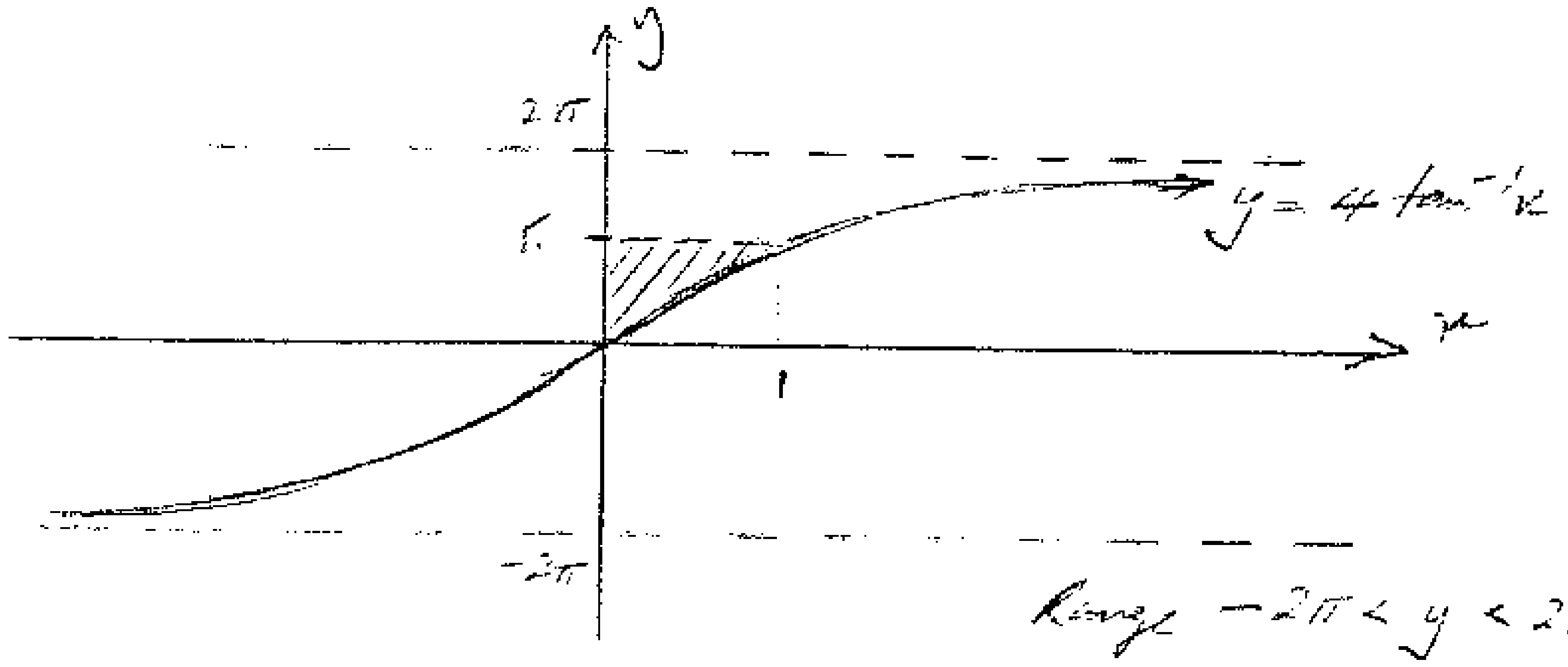
QUESTION 4

(a)(i) $\dfrac{\sin^2 A}{\cos^2 A} + \dfrac{\cos^2 A}{\cos^2 A} = \dfrac{1}{\cos^2 A}$

$\tan^2 A + 1 = \sec^2 A$

$\tan^2 A = \sec^2 A - 1$

(ii)



Range $-2\pi < y < 2\pi$

(iii) $y/4 = \tan^{-1} x$

$x = \tan y/4$.

$V = \pi \int_0^{\pi} x^2 \, dy$

$= \pi \int_0^{\pi} \tan^2 y/4 \, dy$

$= \pi \int_0^{\pi} \sec^2 y/4 - 1 \, dy$

$= \pi \left[4 \tan y/4 - y \right]_0^{\pi}$

$= \pi \left\{ (4 \tan \pi/4 - \pi) - (4 \tan 0 - 0) \right\}$

$= \pi(4 - \pi) \; u^3$.

(b)(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right)\cdot\frac{dv}{dx}$

$= v\frac{dv}{dx}$

$= \frac{dx}{dt}\cdot\frac{dv}{dx}$

$= \frac{dv}{dt}$

$= \ddot{x}$

(ii)(α) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 4x$

$\frac{1}{2}v^2 = \frac{x^4}{2} + 2x^2 + c$

$t=0,\ x=2,\ v=6$

$18 = 8 + 8 + c$

$c = 2$

$v^2 = x^4 + 4x^2 + 4$

(β) $v^2 = (x^2+2)^2 \quad \therefore v^2 \geq 4 \quad \therefore v \neq 0$

$\therefore$ object never changes direction

$\therefore$ always moves to right with increasing speed

since initial vel > 0 & accel > 0 for x > 0

$\therefore$ min speed is initial speed

$\therefore$ min. speed = 6 ms^{-1}

QUESTION 5

(a) $y' = -2e^{-2x}$

when $x=0$, $y' = -2e^0$

$\therefore m_1 = -2$

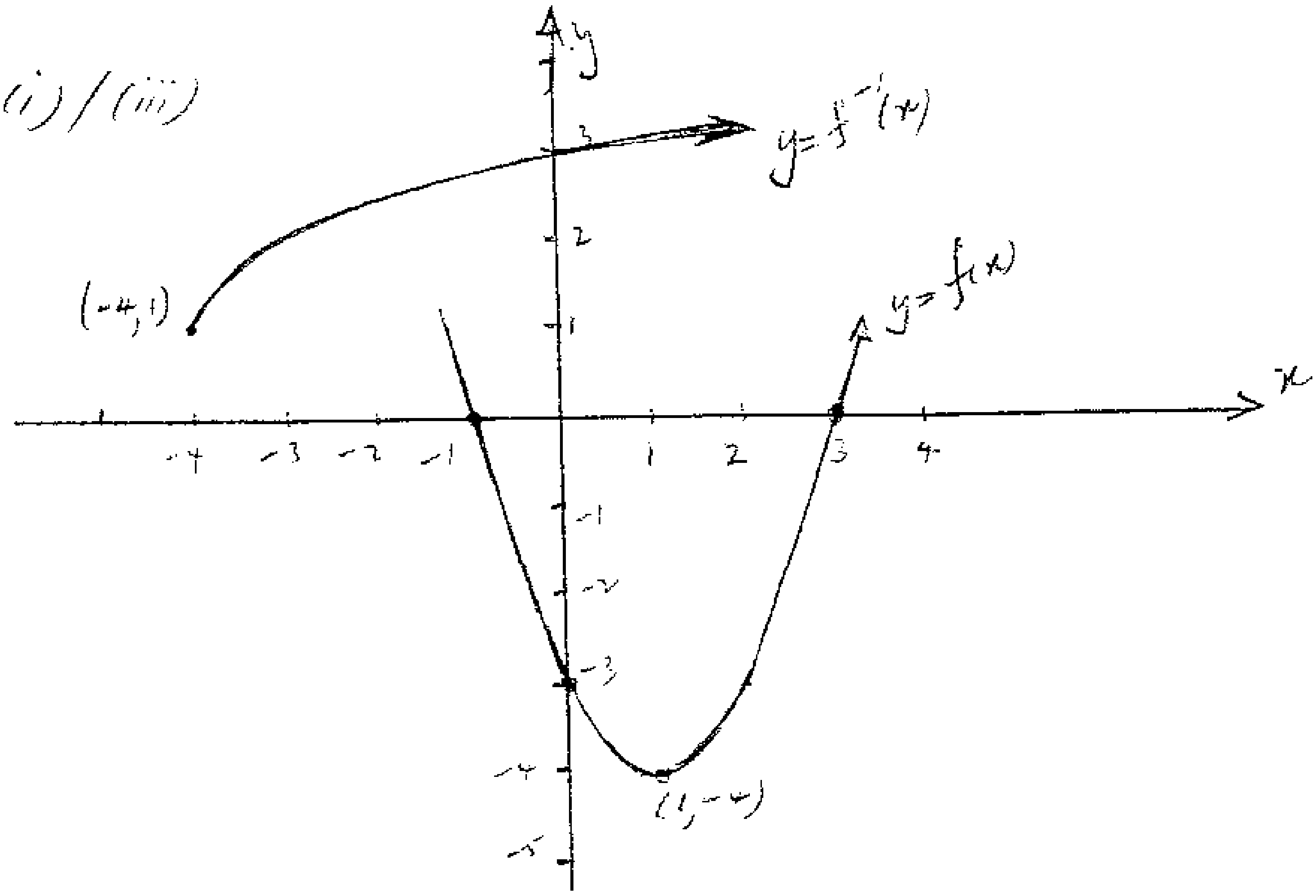
$m_2 = 3$

$\tan\theta = \left|\frac{3+2}{1+(3)(-2)}\right|$

$= 1$

$\theta = \pi/4$, or $45°$

(b) (i)/(iii)

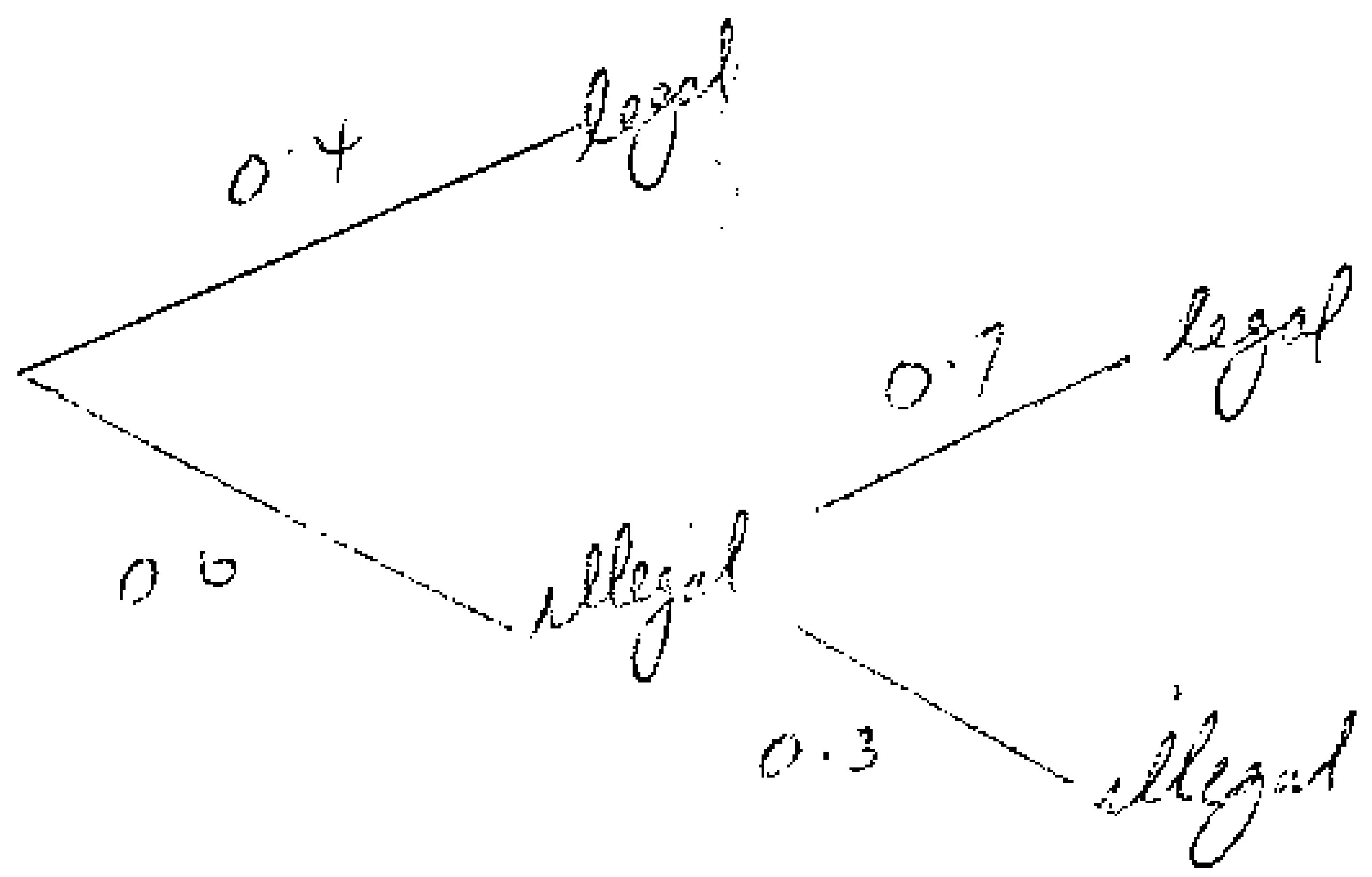


(ii) $x \geq 1$

(iii) see graph

(c)

(i)



$$P(\text{double fault}) = 0.6 \times 0.3$$
$$= 0.18$$

(ii) $(0.82 + 0.18)^6$

$P(\text{at least 2 double faults}) = 1 - \{P(\text{0 double faults}) + P(\text{1 double fault})\}$

$$= 1 - \{ {}^6C_0 (0.82)^6 (0.18)^0 + {}^6C_1 (0.82)^5 (0.18)^1 \}$$

$$\doteqdot 0.30.$$

QUESTION 6.

(a) $\frac{dr}{dt} = \frac{dr}{dv} \cdot \frac{dv}{dt}$

$V = \frac{4}{3}\pi r^3$

$\frac{dv}{dr} = 4\pi r^2$

$= \frac{1}{4\pi r^2} \cdot 10$

$= \frac{10}{4\pi r^2}$

when $SA = 500 \quad (= 4\pi r^2)$

$\frac{dr}{dt} = \frac{10}{500}$

$= \frac{1}{50}$ cm/s.

(b) when $n=1$, $LHS = 2(1!)$ $= 2$ $\qquad RHS = 1(2!)$ $= 2$.

$\therefore$ true for $n=1$

assume true for $n=k$

ie. $2(1!) + 5(2!) + \cdots + (k^2+1)k! = k(k+1)!$

to prove true for $n=k+1$

ie $2(1!) + 5(2!) + \cdots + (k^2+1)k! + [(k+1)^2+1](k+1)! = (k+1)(k+2)!$

Now $LHS = 2(1!) + 5(2!) + \cdots + (k^2+1)k! + (k^2+2k+2)(k+1)!$

$= k(k+1)! + (k^2+2k+2)(k+1)!$ (by assumption) Q??

$= (k+1)!\{k + k^2+2k+2\}$

$= (k+1)!(k^2+3k+2)$

$= (k+1)!(k+2)(k+1)$

$= (k+2)!(k+1)$

$= RHS$

$\therefore$ if true for $n=k$ then true for $n=k+1$ & since true for $n=1$ then true for all $n \geq 1$.

Q6(c) $\frac{dy}{dx} = \frac{(x)(1/x) - (1)(\ln x)}{x^2}$

$= \frac{1 - \ln x}{x^2}$

$$\int_e^{e^2} \frac{1 - \ln x}{x \ln x}\, dx = \int_e^{e^2} \frac{\frac{1-\ln x}{x^2}}{\frac{\ln x}{x}}\, dx$$

$$= \left[\ln\left(\frac{\ln x}{x}\right)\right]_e^{e^2}$$

$$= \ln\left(\frac{\ln e^2}{e^2}\right) - \ln\left(\frac{\ln e}{e}\right)$$

$$= \ln(2/e^2) - \ln(1/e)$$

$$= \ln\left(\frac{2}{e^2} \times \frac{e}{1}\right)$$

$$= \ln(2/e)$$

$$= \ln 2 - \ln e$$

$$= \ln 2 - 1$$

QUESTION 7

(i) $\sin(x+y) - \sin(x-y) = (\sin x\cos y + \cos x \sin y) - (\sin x \cos y - \cos x \sin y)$ 1

$= 2\cos x \sin y$

let $x + y = A$ & $x - y = B$

$\therefore 2x = A + B$ $\qquad 2y = A - B$

$x = \frac{A+B}{2}$ 1 $\qquad y = \frac{A-B}{2}$ 1

$\therefore \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

(ii) $\frac{\sin A - \sin B}{\cos A - \cos B} = \frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)}$ 1

$= \frac{\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{-\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$ 1

$$= -\cot\left(\frac{A+B}{2}\right) \ldots$$

(iii) $\ddot{x} = 0$

$\dot{x} = c_1$

$t=0,\ \dot{x} = V\cos\alpha$

$\therefore V\cos\alpha = c_1$

$\dot{x} = V\cos\alpha$

$x = Vt\cos\alpha + c_2$

$t=0,\ x=0\ \therefore c_2 = 0$

$x = Vt\cos\alpha$ (1)

$\ddot{y} = -g$

$\dot{y} = -gt + c_3$

$t=0,\ \dot{y} = V\sin\alpha$

$V\sin\alpha = c_3$

$\dot{y} = -gt + V\sin\alpha$

$y = -\frac{1}{2}gt^2 + Vt\sin\alpha + c_4$

$t=0,\ y=0\ \therefore c_4 = 0$

$y = -\frac{1}{2}gt^2 + Vt\sin\alpha$ (1)

(iv) (α) Particle P

$x_P = Vt\cos\alpha \qquad y_P = -\frac{1}{2}gt^2 + Vt\sin\alpha$

Particle Q

$x_Q = Vt\cos\beta \qquad y_Q = -\frac{1}{2}gt^2 + Vt\sin\beta$

$\tan\theta$ = slope PQ (1)

$$= \left| \frac{(-\frac{1}{2}gt^2 + Vt\sin\beta) - (-\frac{1}{2}gt^2 + Vt\sin\alpha)}{Vt\cos\beta - Vt\cos\alpha} \right|$$

$$= \left| \frac{Vt(\sin\beta - \sin\alpha)}{Vt(\cos\beta - \cos\alpha)} \right| \quad (1)$$

$$= \left| \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} \right| \quad (1)$$

(β) $\tan\theta = \left| -\cot\left(\frac{\alpha+\beta}{2}\right) \right|$ from (ii)

$$= \tan\left(\frac{\pi}{2} - \left(\frac{\alpha+\beta}{2}\right)\right) \quad (1)\ \alpha, \beta, \theta \text{ acute}$$

$$\therefore \theta = \pi/2 - \left(\frac{\alpha+\beta}{2}\right) \quad (1)$$

$$\theta = \frac{1}{2}(\pi - \alpha - \beta)$$