PETRUS KY COLLEGE

NEW SOUTH WALES

in partnership with

VIETNAMESE COMMUNITY IN AUSTRALIA

NSW CHAPTER

JULY 2006

MATHEMATICS EXTENSION 2

PRE-TRIAL TEST

SOLUTION

HIGHER SCHOOL CERTIFICATE (HSC)

Student Number: | | | | |

Student Name: ______________________

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Write your Student Number and your Name on all working booklets

Total marks – 96

- Attempt Questions 1–8
- Question 8 is optional
- All questions are of equal value

Question 1 **12**

(A) Integrate the following,

(i) $\int \frac{\sin 2x}{\sqrt{1-\cos 2x}}dx$ 1

Let $u = 1-\cos 2x$, $\frac{du}{dx} = 2\sin 2x$.

$$\int \frac{\sin 2x . dx}{\sqrt{1-\cos 2x}} = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} + C$$

$$= \frac{1}{2}\sqrt{1-\cos 2x} + C$$

(ii) $\int \frac{dx}{x\sqrt{x^6-4}}$ (Let $x^3 = 2\sec u$) 2

Let $x^3 = 2\sec u \longrightarrow u = \cos^{-1}\left(\frac{2}{x^3}\right)$

$$x = (2\sec u)^{1/3}$$

$$\frac{dx}{du} = \frac{2\sec u . \tan u . (2\sec u)^{-\frac{2}{3}}}{3}$$

$$= \frac{\tan u . (2\sec u)^{1/3}}{3}$$

$$\therefore \int \frac{dx}{x\sqrt{x^6-4}} = \frac{1}{3}\int \frac{\tan u . (2\sec u)^{1/3} du}{(2\sec u)^{1/3}\sqrt{4\sec^2 u - 4}}$$

$$= \frac{1}{3}\int \frac{\tan u . du}{2\sqrt{\sec^2 u - 1}}$$

$$= \frac{1}{6}\int du = \frac{1}{6}u + C$$

$$\therefore \int \frac{dx}{x\sqrt{x^6-4}} = \frac{1}{6}\cos^{-1}\left(\frac{2}{x^3}\right) + C$$

(iii) $\int_{-1}^{1} \frac{2x}{(x^2+2x+5)^2}\,dx$

$$\int_{-1}^{1} \frac{2x.dx}{(x^2+2x+5)^2} = \int_{-1}^{1} \frac{2x\,dx}{((x+1)^2+4)^2}$$

- Let $x+1 = 2\tan\theta \longrightarrow x = 2\tan\theta - 1$

$$\frac{dx}{d\theta} = 2\sec^2\theta.d\theta$$

- Change the range
  - When $x=-1$, $\tan\theta = 0$, $\theta = 0$
  - when $x = 1$, $\tan\theta = 1$, $\theta = \frac{\pi}{4}$

$$\therefore \int_{-1}^{1} \frac{2x\,dx}{(x^2+2x+5)^2} = \int_0^{\pi/4} \frac{2(2\tan\theta - 1).2\sec^2\theta.d\theta}{(4\tan^2\theta + 4)^2}$$

$$= \frac{4}{16}\int_0^{\pi/4} \frac{(2\tan\theta - 1)\sec^2\theta\,d\theta}{\sec^4\theta\;d\theta}$$

$$= \frac{1}{4}\int_0^{\pi/4} 2\tan\theta \times \cos^2\theta - \cos^2\theta\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/4} 2\sin\theta.\cos\theta - \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/4} \sin 2\theta - \frac{1}{2} - \frac{1}{2}\cos 2\theta\,d\theta$$

$$= -\frac{1}{4}\left[\frac{1}{2}\cos 2\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/4}$$

$$= -\frac{1}{4}\left[\left(0 + \frac{\pi}{8} + \frac{1}{4} - \frac{1}{2}\right)\right]$$

<u>Answer</u> $= \frac{1}{16} - \frac{\pi}{32}$

(iv) $\int \frac{dx}{e^x\sqrt{1-e^{-2x}}}$ 2

$$\int \frac{dx}{e^x\sqrt{1-e^{-2x}}}$$

Let $u = e^{-x}$

$$\frac{du}{dx} = -e^{-x} \quad , -du = \frac{dx}{e^x}$$

$$\therefore \int \frac{dx}{e^x\sqrt{1-e^{-2x}}} = -\int \frac{du}{\sqrt{1-u^2}} = \cos^{-1} u + C$$

$$= \cos^{-1}\left(\frac{1}{e^x}\right) + C$$

(B) By substituting x = a – y , show that 1

$\int_o^a f(x)dx = \int_0^a f(a-x)dx$

Hence use this result to evaluate.

Definite integral

Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Let $x = a - y$ | when $x = 0$, $y = a$

$\frac{dx}{dy} = -1$ | $x = a$, $y = 0$

$$\therefore \int_0^a f(x)\,dx = -\int_a^0 f(a-y)\,dy$$

$$= \int_0^a f(a-y)\,dy$$

$$= \int_0^a f(a-x)\,dx$$

(i) $\int_o^1 x(1-x)^{12}\,dx$ 2

$$\int_0^1 x(1-x)^{12}\,dx = \int_0^1 (1-x)\left(1-(1-x)\right)^{12}dx$$

$$= \int_0^1 (1-x)\,x^{12}\,dx$$

$$= \int_0^1 x^{12} - x^{13}\,dx$$

$$= \left[\frac{x^{13}}{13} - \frac{x^{14}}{14}\right]_0^1$$

$$= \frac{1}{13} - \frac{1}{14} = \frac{1}{182}$$

(ii) $\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin 2x}\,dx$ 2

$$\int_0^{\pi/2} \frac{\cos x - \sin x}{1+\sin 2x}\,dx = \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2}-x\right) - \sin\left(\frac{\pi}{2}-x\right)}{1+\sin 2\left(\frac{\pi}{2}-x\right)}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin(\pi-2x)}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin 2x}\,dx$$

$$\therefore\ 2\int_0^{\pi/2} \frac{\cos x - \sin x}{1+\sin 2x}\,dx = 0$$

Answer = 0

Question 2 **12**

(A) Given $Z_1 = i\sqrt{2}$ and $Z_2 = \dfrac{2}{1-i}$ 4

(i) Express Z_1 and Z_2 in the modulus/argument form.

$$z_1 = \sqrt{2}\ cis\ \frac{\pi}{2}$$

$$z_2 = \frac{2}{1-i} \times \frac{1+i}{1+i} = \frac{2(1+i)}{2} = 1+i$$

$$z_2 = \sqrt{2}\ cis\ \frac{\pi}{4}$$

(ii) If $Z_1 = w.Z_2$ express w in the modulus/argument form.

Find w if $z_1 = w.z_2$

$$w = \frac{z_1}{z_2}$$

$$\therefore\ |w| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

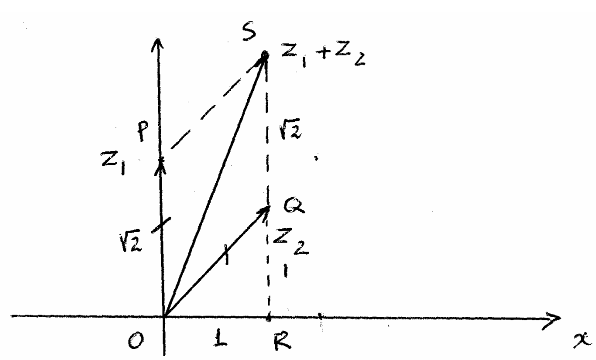
$$Arg(w) = Arg\,z_1 - Arg\,z_2$$
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$\therefore\ w = cis\ \frac{\pi}{4}$$

(iii) Show Z_1, Z_2 and $Z_1 + Z_2$ on an Argand diagram. Hence show that

$$Arg(Z_1 + Z_2) = \frac{3\pi}{8}$$

Use the diagram to find the exact value of $\tan\dfrac{3\pi}{8}$



OPSQ is a rhombus, therefore OS bisects $\angle POQ$

Hence $\angle SOQ = \frac{\pi}{4} \div 2 = \frac{\pi}{8}$

$$\begin{aligned} Arg(z_1 + z_2) &= \angle xOQ + \angle QOS \\ &= \frac{\pi}{4} + \frac{\pi}{8} \\ &= \frac{3\pi}{8} \end{aligned}$$

In ΔOSR, $\tan \angle SOR = \frac{SR}{OR}$

Hence, $\boxed{\tan \frac{3\pi}{8} = \sqrt{2} + 1}$

(B) If Z_1, Z_2 are complex numbers, prove that 4

$$\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$$

Given a complex number $Z = \frac{c+2i}{c-2i}$ where c is real.

Find $|Z|$ and hence describe the exact locus of Z if c varies from -1 to 1.

Prove

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Let $z_1 = |z_1| \text{cis}\,\alpha$

$z_2 = |z_2| \text{cis}\,\beta$

$$\frac{z_1}{z_2} = \frac{|z_1|(\cos\alpha + i\sin\alpha)}{|z_2|(\cos\beta + i\sin\beta)} \times \frac{(\cos\beta - i\sin\beta)}{(\cos\beta - i\sin\beta)}$$

$$= \frac{|z_1|}{|z_2|}\left(\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right)$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} \text{cis}(\alpha - \beta)$$

Therefore, modulus of $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}$

Let $z = \dfrac{c+2i}{c-2i}$

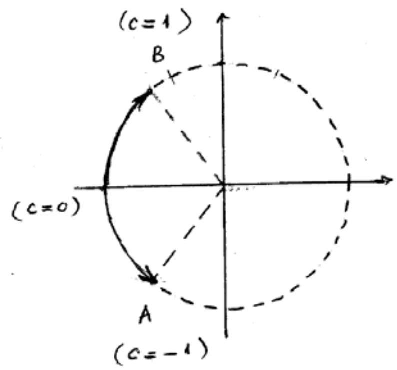
Let $z_1 = c + 2i$ then $|z_1| = \sqrt{c^2+4}$

$z_2 = c - 2i$ then $|z_2| = \sqrt{c^2+4}$

$|z| = \dfrac{|z_1|}{|z_2|} = \dfrac{\sqrt{c^2+4}}{\sqrt{c^2+4}} = 1$

Therefore, in general, the locus of z is the circle, centre at origin and radius = 1, equation $x^2+y^2=1$

When the value of c varies from -1 to 1, Locus of Z becomes only part of that circle, that is it is an arc from point A to B which contains the corner (-1,0) as shown in the following figure:



(C) If $w = 2\sqrt{3}i - 2$, find |w| and arg w, then indicate on an Argand diagram the complex number w, $\overline{w}$, iw, $\dfrac{1}{w}$, -w. 4

Show that $w^2 = 4\overline{w}$

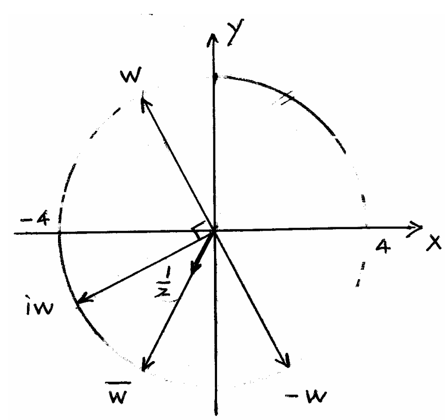
Prove that w is a root of the equation $Z^3 - 64 = 0$. Find other roots.

$w = 2\sqrt{3}i - 2$

$|w| = \sqrt{4+12} = 4$

$Arg(w) = \tan^{-1}\left(\dfrac{2\sqrt{3}}{-2}\right) = \dfrac{2\pi}{3}$

Show in Argand Diagram



- show that $w^2 = 4\overline{w}$

$$w^2 = (2\sqrt{3}i - 2)^2 = -12 - 8\sqrt{3}i + 4$$
$$= -(8 + 8\sqrt{3}i)$$
$$4\overline{w} = 4(-2 - 2\sqrt{3}i) = -(8 + 8\sqrt{3}i)$$
$$\therefore \quad w^2 = 4\overline{w}$$

- since $w = 4 \text{ cis } \frac{2\pi}{3}$

Then $w^3 = 4^3 \text{ cis } 2\pi = 64$

Therefore $w^3 - 64 = 0$

Polynomial equation $w^3 - 64 = 0$ has real coefficients and one complex root $\underline{w = -2 + 2\sqrt{3}i}$, then it also has other complex root which is the conjugate $\underline{\overline{w} = -2 - 2\sqrt{3}i}$

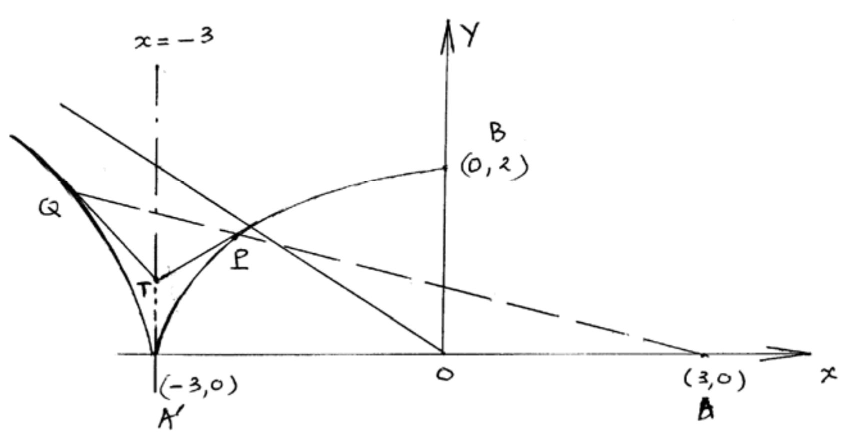
The last root is real, ie $\underline{w = 4}$.

Question 3 **12**

(A) P is the point $(3\cos\theta, 2\sin\theta)$ and Q is the point$(3\sec\theta, 2\tan\theta)$.
Sketch the curves which are the loci, as θ varies, of P and Q, marking on them the range of positions occupied by P and Q respectively as θ varies from $\frac{\pi}{2}$ to π.

(i) Prove that for any value of θ, the line PQ passes through one of the common points of the 2 curves. 2

a) $P(3\cos\theta, 2\sin\theta)$, $Q(3\sec\theta, 2\tan\theta)$, $\frac{\pi}{2} \leqslant \theta \leqslant \pi$



Equation of PQ :

$$\frac{y-2\sin\theta}{x-3\cos\theta} = \frac{2\tan\theta-2\sin\theta}{3\sec\theta-3\cos\theta}$$

$$= \frac{2\sin\theta\,(\sec\theta-1)}{3\cos\theta\,(\sec^2\theta-1)}$$

$$= \frac{2\sin\theta}{3(1+\cos\theta)}$$

Simplify : $\underline{2x\sin\theta - 3y(1+\cos\theta) - 6\sin\theta = 0}$

- Show that PQ passes through common point (3,0)

Substitute (3,0) into equation of PQ

$$6\sin\theta - 3\times 0\,(1+\cos\theta) - 6\sin\theta = 0$$

$$6\sin\theta - 6\sin\theta = 0 \quad (\text{True})$$

Therefore PQ passes through (3,0)

(ii) Show that the tangent at P to the first curve meets the tangent at Q to the second curve in a point which lies on the common tangent to the two curves at their other common point. 2

ii) Equation of tangent to Ellipse at P

$$\frac{x.\cos\theta}{3} + \frac{y\sin\theta}{2} = 1 \qquad (1)$$

Equation of tangent to Hyperbola at Q

$$\frac{x\sec\theta}{3} - \frac{y\tan\theta}{2} = 1 \qquad (2)$$

- Divide equation (1) by $\cos\theta$.

$$\frac{x}{3} + \frac{y\tan\theta}{2} = \sec\theta \qquad (3)$$

Note : the sign of the second term $\frac{y\sin\theta}{2}$ has to be changed to $-$ because θ is in 2^{nd} quadrant, $\sin\theta$ and $\tan\theta$ are opposite sign

- $\underline{\text{point of intersection T}}$: (3) $-$ (2) gives

$$\frac{x}{3} - \frac{x\sec\theta}{3} = \sec\theta - 1$$

$$\frac{x}{3}(1-\sec\theta) = \sec\theta - 1$$

$$\therefore \boxed{x = -3}$$

Therefore, T lies on the line $x = -3$, which is the common tangent of Ellipse and Hyperbola at the common point $(-3, 0)$

(iii) Prove that the two curves have the same length of the latus rectum. 2

iii) Length of Lactus rectum :

- of Ellipse : $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

  $e = \dfrac{\sqrt{5}}{3}$, $S(\sqrt{5}, 0)$.

  Equation of Lactus rectum $x = \sqrt{5}$

  Intersection points : $\dfrac{5}{9} + \dfrac{y^2}{4} = 1$, $y^2 = \dfrac{16}{9}$, $y = \pm\dfrac{4}{3}$

  Therefore, Length of Latus rectum of Ellipse

  $L = \dfrac{8}{3}$

- of Hyperbola : $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$

  $e = \dfrac{\sqrt{13}}{3}$, $S(\sqrt{13}, 0)$, $x = \sqrt{13}$

  Intersection points $\dfrac{13}{9} - \dfrac{y^2}{4} = 1$, $y^2 = \dfrac{16}{9}$, $y = \pm\dfrac{4}{3}$

  Length of lactus rectum $L = 2y = \dfrac{8}{3}$

  Therefore, the 2 curves E & H have the same length of Lactus Rectum.

(B) Show that the condition for a straight line y = mx + c to touch the ellipse E of 3

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{is} \quad c^2 = b^2 + a^2m^2$$

Hence show that the locus of the point P(x, y) from which the 2 tangents to the ellipse E

$\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ are perpendicular together, is a curve with the centre at the origin and a radius of 5.

Condition to be a tangent to an ellipse.

Point of intersection.

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$b^2x^2 + a^2m^2x^2 + 2a^2mcx + a^2c^2 - a^2b^2 = 0$$

$$(b^2+a^2m^2)x^2 + (2a^2mc)x + (a^2c^2-a^2b^2) = 0$$

Discriminant : $\Delta = (2a^2mc)^2 - 4(b^2+a^2m^2)(a^2c^2-a^2b^2)$

$$= 4a^4m^2c^2 - 4a^2b^2c^2 + 4a^2b^4 - 4a^4m^2c^2 + 4a^4b^2m^2$$

To be a tangent, the line has only one common point with the ellipse or $\Delta = 0$

$$\Delta = 4a^2b^2(b^2+a^2m^2-c^2) = 0$$

$$\therefore \boxed{c^2 = b^2+a^2m^2}$$

- <u>Locus of $P(x,y)$</u>

Let equation of tangent through $P(x,y)$

$$y = mx + c$$

Using condition above : $c^2 = b^2 + a^2m^2$

$$c^2 = 9 + 16m^2$$

$$\therefore y = mx + \sqrt{(9+16m^2)}.$$

Solving equation in terms of m

$$(y-mx)^2 = 9 + 16m^2$$

$$y^2 - 2xym + m^2x^2 = 9 + 16m^2$$

Quadratic equation :

$$m^2(x^2-16) - 2xym + (y^2-9) = 0$$

Since from external point $P(x,y)$, there are 2 tangents with gradient m_1 and m_2 to the Ellipse

The tangents are perpendicular, then $m_1 \times m_2 = -1$

- m_1 and m_2 are 2 roots of the above quadratic equation, their product $(m_1 \times m_2)$ is equal $\frac{c}{a}$, which is

$$m_1 \times m_2 = \frac{y^2-9}{x^2-16} = 1$$

Therefore $\boxed{x^2+y^2=25}$

So the locus of P is a circle with radius = 5

Question 4 **12**

(A) A function is defined with the polar equation as follows:

$$\begin{cases} x = 8\cos^3\theta \\ y = 8\sin^3\theta \end{cases} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

(i) Find $\frac{dy}{dx}$ in term of θ and show that the graph of this function touches the x and y axis. Sketch the curve. 3

Find $\frac{dy}{dx}$, using chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{24\cos\theta \times \sin^2\theta}{-24\sin\theta \times \cos^2\theta}$$

$$\therefore \frac{dy}{dx} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

- When $\theta = 0$, $\frac{dy}{d\theta} = 0$, $x = 8$, $y = 0$

  $\therefore$ The tangent at the X intercept $(8,0)$ is horizontal, OR the curve touches X axis

- when $\theta = \pm\frac{\pi}{2}$, $\frac{dy}{d\theta} = \infty$, $x = 0$, $y = \pm 8$

  $\therefore$ The tangents at the Y intercepts $(0, \pm 8)$ is vertical, OR the curve touches Y axis at 2 points.

(ii) Show that the Cartesian equation of that function is 3

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$$

Show that the equation of the tangent to the curve at the point P(x_o, y_o) is:

$$y_0^{1/3}x + x_0^{1/3}y = 4x_0^{1/3}.y_0^{1/3}$$

Prove that the segment intercepted on this tangent by the coordinate axis is independent of the position of P on the curve.

, Change to Cartesian form

$$\cos\theta = \frac{x^{1/3}}{2}$$

$$\sin\theta = \frac{y^{1/3}}{2}$$

Then :

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{x^{2/3}}{4} + \frac{y^{2/3}}{4} = 1$$

OR $$\boxed{x^{2/3} + y^{2/3} = 4}$$

Equation of tangent at $P(x_o, y_o)$

Since $$\frac{dy}{dx} = -\frac{\sin\theta}{\cos\theta} = -\frac{y^{1/3}}{x^{1/3}}$$

gradient of tangent $$m_T = -\frac{y_o^{1/3}}{x_o^{1/3}}$$

Equation of tangent

$$y - y_o = -\frac{y_o^{1/3}}{x_o^{1/3}}(x - x_o)$$

$$x_o^{1/3}\cdot y - y_o\cdot x_o^{1/3} = -y_o^{1/3}\cdot x + x_o\cdot y_o^{1/3}$$

$$\therefore\ y_o^{1/3}\cdot x + x_o^{1/3}\cdot y = x_o y_o^{1/3} + y_o x_o^{1/3}$$

$$= x_o^{1/3}\cdot y_o^{1/3}(x_o^{2/3} + y_o^{2/3})$$

$$= 4.x_o^{1/3}\cdot y_o^{1/3}$$

Equation of tangent

$$\boxed{y_o^{1/3}\cdot x + x_o^{1/3}\cdot y = 4x_o^{1/3}\cdot y_o^{1/3}}$$

X and Y intercepts of the tangent.

X intercept, $y=0$, $x = 4x_0^{1/3}$

Y intercept, $x=0$, $y = 4y_0^{1/3}$

Length of XY segment $= \sqrt{(4x_0^{1/3})^2 + (4y_0^{1/3})^2}$

$= \sqrt{16(x_0^{2/3} + y_0^{2/3})}$

$= \sqrt{16 \times 4} = 8$

$\therefore$ This length is independent of (x_0, y_0)

(B) Consider the function $f(x) = 2 - \dfrac{4x}{x^2+1}$

(i) Show that the function is always positive for any value of x. 1

show that $f(x)$ is positive definite

$$f'(x) = \frac{2x^2+2-4x}{x^2+1} = \frac{2(x^2-2x+1)}{x^2+1}$$

$$f(x) = \frac{2(x-1)^2}{x^2+1}$$ always greater or equal zero.

(ii) Find the asymptote (if any) and the stationary point of that curve. 1

Asymptote : Let $x \to \infty$

Limit $f(x) = 2 - \frac{4}{\infty} = 2$

Horizontal asymptote $\boxed{y=2}$.

Stationary point : $f'(x) = \dfrac{4x^2-4}{(x^2+1)^2}$

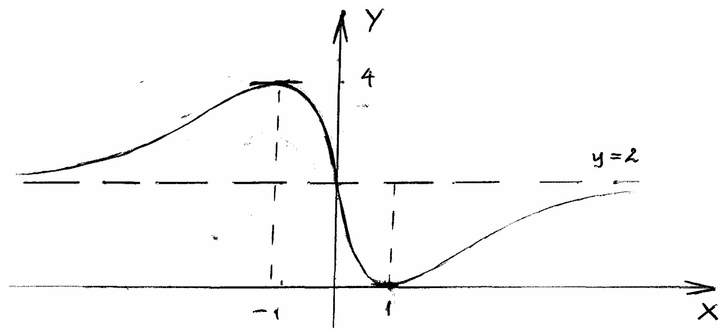
when $x = \pm 1$, $f'(x) = 0$

$y = 0$ OR 4

Maximum point $(-1, 4)$

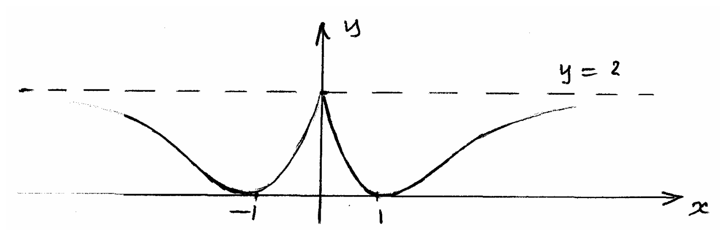
Minimum point $(1, 0)$

(iii) Sketch the curve y = f(x). 1



(iv) On a separate diagram, sketch the relating curves:

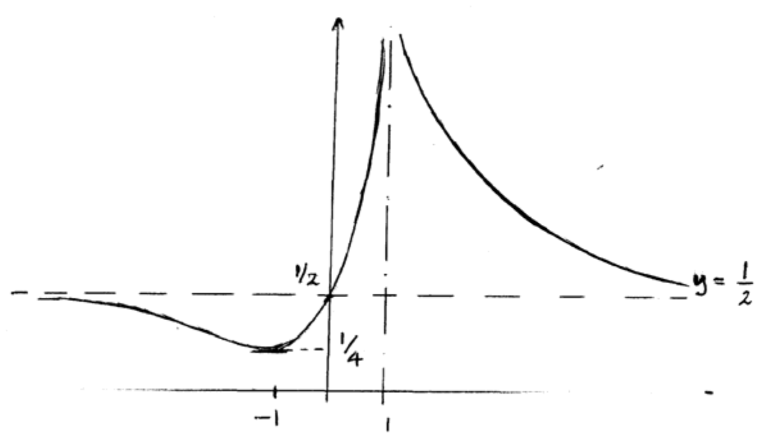
a) y = f(|x|) 1



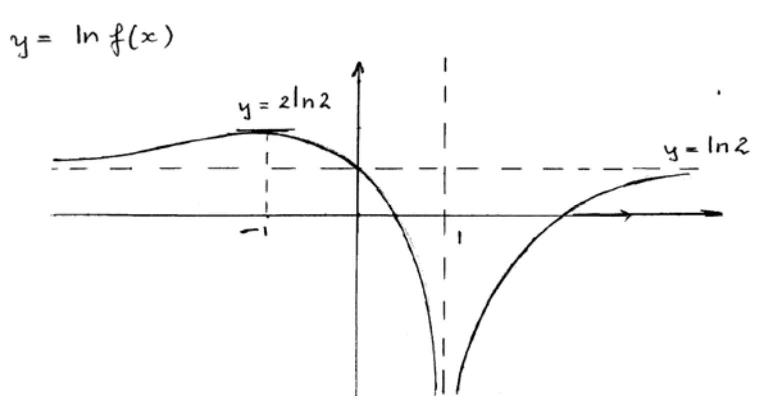
1

1

b) $y = \dfrac{1}{f(x)}$

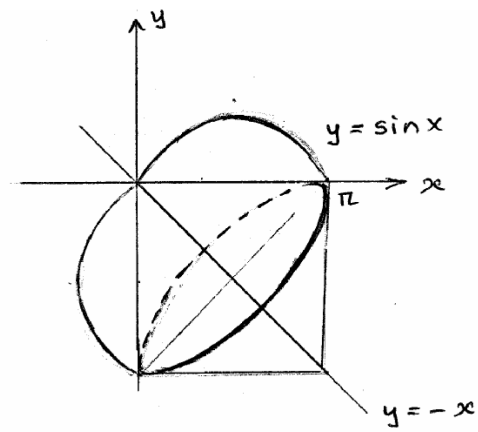


c) y = ln f(x)



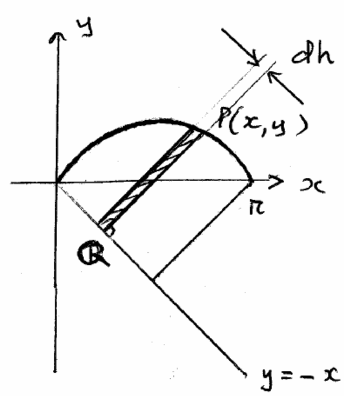
Question 5 **12**

(A) The area bounded by the curve y=sinx, the two lines y=-x and x=π is rotated about the line y=-x. Find the volume of the solid shape of that formation. 6



The solid shape can be divided by 2 separated volumes:

i) The 1st part produced by rotating area bounded by the curve $y = \sin x$, the perpendicular line about the line $y = -x$ as shown in the following figure



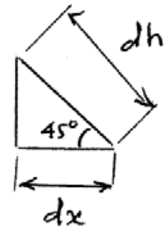
By using the slicing method: The slice is a piece of cylinder, with volume is

$$dV = \pi R^2 dh$$

with R is the perpendicular distance PQ to the

Line $y = -x$ or $x + y = 0$

$$R = PQ = \frac{|x+y|}{\sqrt{2}}$$



Calculate dh by the figure:

$$dh = \sqrt{2}.dx$$

$$\therefore \quad dV = (x+y)^2.\sqrt{2}.dx$$

$$= (x + \sin x)^2.\sqrt{2}\,dx$$

$$= \sqrt{2}\,x^2 + 2\sqrt{2}\,x\sin x + \sqrt{2}\sin^2 x\,dx$$

Therefore:

$$V = \lim_{dx \to 0} \sum dV = \int_0^{\pi} \sqrt{2}\,x^2 + 2\sqrt{2}\,x\sin x + \sqrt{2}\sin^2 x\,dx$$

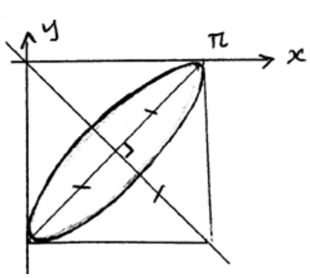
There are 3 separated integrals.

- $\int_0^{\pi} \sqrt{2}\,x^2\,dx = \frac{\sqrt{2}}{3}\pi\sqrt{\pi}$
- $\int_0^{\pi} 2\sqrt{2}\,x.\sin x\,dx = \Big[-x\cos x + \sin x\Big]_0^{\pi}$

  $= 2\sqrt{2}\pi$
- $\int_0^{\pi} \sqrt{2}\sin^2 x\,dx = \frac{\sqrt{2}}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\sqrt{2}\pi}{2}$

Therfore $\quad V = \frac{\sqrt{2}}{3}\pi\sqrt{\pi} + 2\sqrt{2}\pi + \frac{\sqrt{2}}{2}\pi . = 11.34\ u^3$

ii) The 2^{nd} part of that volume is the right-angular cone with the radius and the height equal to $\frac{\pi}{\sqrt{2}}$



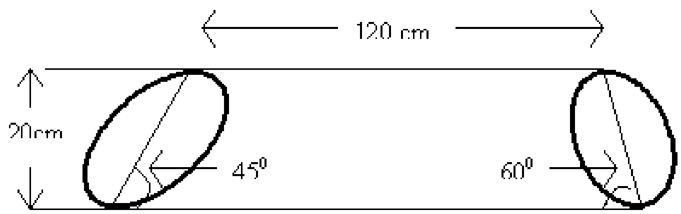
The volume of a cone : $V = \frac{1}{3}\pi R^2 h$

$$= \frac{1}{3}\pi\left(\frac{\pi}{\sqrt{2}}\right)^2\left(\frac{\pi}{\sqrt{2}}\right)$$

$$= \frac{\pi^4}{6\sqrt{2}} = 11.49\ u^3$$

Total volume of the solid shape = 22.8 unit cube

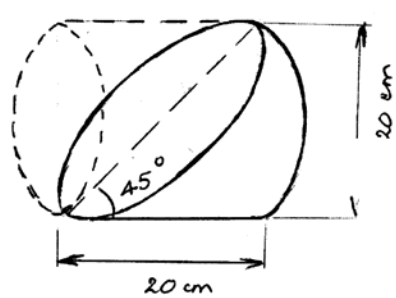
(B) A cylindrical timber is chopped at two ends by 2 planes which are inclined 60^0 and 45^0 respectively. If the two ends are tilt toward each other and the shortest length of 2 ends is 120cm. Find the volume of that timber. Give the radius of timer is 10cm. 6



Volume of a chopped cylindrical timber :

The total volume can be divided by 3 parts, The 2 ends are the pieces of timber which are chopped into half, and the body is the full cylinder.

- One end :



$$V = \frac{1}{2}\pi 10^2 \times 20 = 1000\pi$$

- Other end : is similar, except the length of that piece is $\frac{20}{\sqrt{3}}$ cm

$$V = \frac{1}{2}\pi 10^2 \times \frac{20}{\sqrt{3}} = \frac{1000\pi}{\sqrt{3}}$$

- The body is the full cylinder

$$V = \pi \times 10^2 \times 120 = 12000\pi$$

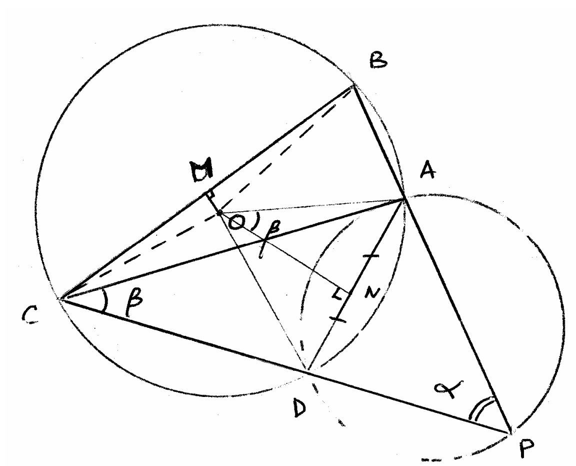
Therefore total volume

$$V = 1000\pi + \frac{1000\pi}{\sqrt{3}} + 12000\pi = \underline{42654 \text{ cm}^3}$$

$$\boxed{V = 42.65 \text{ Litre}}$$

Question 6 **12**

(A)



Two circles intersect at A and D. P is the point on the major arc of one circle. The other circle has the radius r. PA produced and PD produced meet the other circle at B and C respectively. Let $\angle APD = \alpha$ and $\angle ACD = \beta$.

(i) Show that $BC = 2r\sin(\alpha + \beta)$. 2

From centre O, draw OM ⊥ bisects BC

$\angle BOC = 2\angle BAC$ (angle at the centre)

$\angle BAC = \alpha + \beta$ (ext. ∠ of △ ACP)

$\therefore \angle BCC = 2(\alpha + \beta)$

$\therefore \angle BOM = \alpha + \beta$ (in isoceles △ BOC)

In Right angle △ OBM, $\sin(\alpha+\beta) = \frac{BM}{OB}$

$\therefore BM = r.\sin(\alpha+\beta)$

Then $BC = 2BM = 2r\sin(\alpha+\beta)$

(ii) As P moves along the major arc AD on its circle, show that the length of the chord BC is independent of the position of P. 2

Prove BC is independent of the position of P.

Since AD is common chord of both circles, AD is constant, therefore angle α and β

subtend AD on both circles will be constant, no matter of P moves along the arc. Therefore length of BC will not change.

(iii) If the 2 circles have equal radii, show that 2

BC = 2 cosα .AD

If the 2 circles are the same, then

$\alpha = \beta$ (angles subtend equal arcs)

Hence $BC = 2r\sin 2\cancel{\beta}\alpha = 2r.\,2\sin\alpha.\cos\alpha$

Draw ON $\perp$ bisects AD,

$\angle AOD = 2\angle ACD = 2\beta$

$\angle AON = \beta$

Hence in $\triangle AON$, $\sin\beta = \dfrac{AN}{OA}$

$AN = r.\sin\beta$

$AD = 2r\sin\beta$

Since $\alpha = \beta$, then $AD = 2r\sin\alpha$.

Substitute into BC

$$\boxed{BC = 2\,AD\cos\alpha}$$

(B) P is any point (ct, c/t) on the Hyperbola $xy = c^2$, whose centre is O.

(i) M and N are perpendicular roots of P to the 2 asymptotes. Prove that PM.PN is constant. 1

(ii) Find the equation of tangent at P, and show that OP and this tangent are equally inclined to the asymptotes. 2

(iii) If the tangent at P meet the asymptotes at A and B, and Q is the fourth vertex of the rectangle OAQB, find the locus of Q. 1

(iv) Show that PA=PB and hence conclude that the area of ΔOAB is independent of position of P. 2

Question 7 **12**

(A) If the polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ has two zeros $(a + ib)$ and $(a - 2ib)$ where a and b are real, then find the values of a and b. 4

Hence find the zeros of P(x) over the complex field C, and express P(x) as the product of 2 quadratic factors with rational coefficients.

$P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ with 2 zeros $(a+ib)$ and $(a-2ib)$, then find a, b.

Apply the conjugate zeros, P(x) has other 2 zeros which are conjugate to the above. They are $a-ib$ and $a+2ib$. Let $P(x)=0$

$$x^4 - 4x^3 + 11x^2 - 14x + 10 = 0$$

The sum of 4 roots $= -\frac{b}{a}$

$$a+ib + a-ib + a+2ib + a-2ib = 4$$

$$4a = 4$$

$$a = 1$$

The products of 4 roots $= \frac{e}{a}$

$$(1+ib)(1-ib)(1+2ib)(1-2ib) = 10$$

$$(1+b^2)(1+4b^2) = 10$$

$$4b^4 + 5b^2 - 9 = 0$$

$$(4b^2+9)(b^2-1) = 0$$

$$b^2 = 1$$

$$b = 1.$$

- Therefore, 4 zeros of P(x) are $1+i$, $1-i$, $1+2i$, $1-2i$

- Factorise P(x) in real set.

$$P(x) = (x-1+i)(x-1-i)(x-1+2i)(x-1-2i)$$

$$= \left((x-1)^2+1\right)\left((x-1)^2+4\right)$$

$$\boxed{P(x) = (x^2-2x+2)(x^2-2x+5)}$$

(B) Show that if the polynomial P(x) = 0 has a root a of multiplicity m, then 4

P'(x) 0 has a root α of multiplicity (m - 1).

Given that $P(x) = x^4 + x^3 - 3x^2 - 5x — 2 = 0$ has a 3-fold root, find all the roots of P(x).

b) If $P(x)=0$ has $x=\alpha$ as a multiple root., then

$P(x) = (x-\alpha)^m . Q(x)$ with m is the multiplicity

$P'(x) = m(x-\alpha)^{m-1} . Q(x) + (x-\alpha)^m . Q'(x)$

$P'(x) = (x-\alpha)^{m-1}\left[m . Q(x) + (x-\alpha) . Q'(x)\right]$

Let $x=\alpha$, $P'(\alpha) = 0$, so α is also the root of $P'(x)$.

Let $P(x) = x^4 + x^3 - 3x^2 - 5x - 2 = 0$

$P'(x) = 4x^3 + 3x^2 - 6x - 5$

$P''(x) = 12x^2 + 6x - 6 = 0$

$6(2x-1)(x+1) = 0$

$\therefore \quad x = -1$ or $\frac{1}{2}$

Substitute $x=-1$ into $P(x)$ and $P'(x)$ we get

$P(-1) = P'(-1) = P''(-1) = 0$

$\therefore \quad x = -1$ is a multiple root multiplicity = 3

Let the 4^{th} root be α, using the sum of 4 roots $= -\frac{b}{a}$

$-1 \times 3 + \alpha = -1$

$\alpha = 2$

All the roots are, $-1, -1, -1$ and 2

(C) Find the cubic roots of unity and express them in the form $r(\cos\theta + i \sin\theta)$.

Show these roots on an Argand diagram.

If w is one of the complex roots, prove that the other root is w^2 and show that $1 + w + w^2 = 0$.

(i) Prove that if n is a positive integer, then $1 + w^n + w^{2n} =$ **3 or 0** depending on whether n **is or is not** a multiply of 3. 2

c) Find the cube roots of unity $z^3 = 1$

$$|z^3| = |z|^3 = 1 \quad \therefore \quad |z| = 1$$

Let $z = \cos\theta + i\sin\theta$

$$z^3 = \cos 3\theta + i\sin 3\theta = 1$$

$$\cos 3\theta = 1$$

$$3\theta = 0,\ 2\pi,\ 4\pi$$

$$\theta = 0,\ \frac{2\pi}{3},\ \frac{4\pi}{3}$$

The 3 roots are: $z = \cos 0 + i\sin 0 = 1$

$$z = w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$

$$z = w^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

Since $z^3 - 1 = 0$

Sum of 3 roots $= -\frac{b}{a} = 0$

Then $1 + w + w^2 = 0$

- Show that $\boxed{1 + w^n + w^{2n} = 3 \text{ or } 0}$
  - i) if n is a multiple of 3, let $n = 3p$.

    then $1 + w^n + w^{2n} = 1 + (w^3)^p + (w^3)^{2p}$

    $= \underline{1 + 1 + 1 = 3}$

    $(\text{since } w^3 = 1)$
  - ii) if n is NOT a multiple of 3, let $n = 3p + 1$ or $3p + 2$

    Then $1 + w^n + w^{2n} = 1 + w^{3p+1} + w^{6p+2}$

    $= \underline{1 + w^3 . w + w^{6p} . w^2}$

    $= 1 + w + w^2$

    $= 0$

(ii) If $x = a + b$, $y = aw + bw^2$ and $z = aw^2 + bw$, show that 2

$z^2 + y^2 + x^2 = 6ab$

If $x = a+b$, $y = aw + bw^2$ and $z = aw^2 + bw$

Show that $z^2 + y^2 + x^2 = 6ab$

- $z^2 = (aw^2 + bw) = a^2w^4 + 2abw^3 + b^2w^2$
$$= \underline{a^2w + b^2w^2 + 2ab} \quad (1)$$
- $y^2 = (aw + bw^2) = a^2w^2 + 2abw^3 + b^2w^4$
$$= \underline{a^2w^2 + b^2w + 2ab} \quad (2)$$
- $x^2 = (a+b)^2 = \underline{a^2 + b^2 + 2ab} \quad (3)$

$$\therefore \quad z^2 + y^2 + x^2 = a^2(1 + w + w^2) + b^2(1 + w + w^2) + 6ab$$
$$= 6ab$$

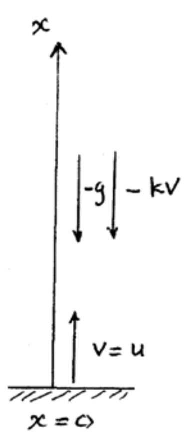
Question 8 - Optional **12**

(A) A particle is projected vertically upward with initial speed u. The air resistance is proportional to the speed of the particle.

(a) If $\ddot{x} = -(g + kw)$ with k is the constant, then find the maximum height reached by the particle and the time to do so. 2

<u>Vertical projectile motion</u> :

a) <u>Project upward</u> :



Equation of motion

$$a = \frac{v\,dv}{dx} = -g - kv$$

$$\int \frac{v\,dv}{g + kv} = \int -\,dx$$

$$\frac{1}{k}\int \frac{kv + g}{kv + g}\,dv - \frac{g}{k}\int \frac{dv}{kv + g} = -x + c$$

$$\therefore \quad x + c = \frac{g}{k^2}\ln(kv + g) - \frac{v}{k}$$

when $x = 0$, $v = u$,

$$c = \frac{g}{k^2}\ln(ku + g) - \frac{u}{k}$$

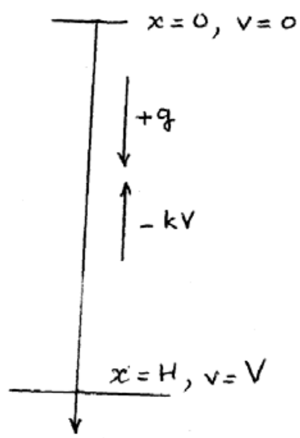
Maximum height, when $v = 0$, $x = H$

$$H = \frac{u}{k} + \frac{g}{k^2}\ln g - \frac{g}{k^2}\ln(ku+g)$$

$$\boxed{H = \frac{u}{k} - \frac{g}{k^2}\ln\left(\frac{ku+g}{g}\right)}$$

(b) Set up the differential equation for the downward motion. 2

b) <u>Downward motion</u>



Equation of motion

$$a = g - kv$$

$$\int \frac{v\,dv}{g-kv} = \int dx$$

$$-\frac{1}{k}\int \frac{-kv+g}{g-kv}\,dv + \int \frac{g\,dv}{g-kv} = x + c$$

$$\therefore\ x + c = -\frac{v}{k} - \frac{g}{k^2}\ln(g-kv)$$

when $x = 0$, $v = 0$

$$c = -\frac{g}{k^2}\ln g$$

when $x = H$, $v = V$

$$\frac{u}{k} + \cancel{\frac{g}{k^2}\ln g} - \frac{g}{k^2}\ln(ku+g) - \cancel{\frac{g}{k^2}\ln g} = -\frac{V}{k} - \frac{g}{k^2}\ln(g-kV)$$

Simplify both sides by k^2:

$$ku + kV = g\ln(ku+g) - g\ln(g-kV)$$

$$\therefore\ \boxed{k(u+V) = g\ln\left[\frac{ku+g}{g-kV}\right]}$$

(c) Show that the particle returns to its point of projection with speed v given by 2

$$k(u+v) = g\log_e\left[\frac{g+ku}{-g-kV}\right]$$

(B) Show that for $n \geq 1$ 3

$$1.\ln\frac{2}{1} + 2\ln\frac{3}{2} + 3\ln\frac{4}{3} + \ldots + n\ln\left(\frac{n+1}{n}\right) = \ln\left(\frac{(n+1)^n}{n!}\right.$$

Show that for $n \geq 1$

$$1\ln\frac{2}{1} + 2\ln\frac{3}{2} + 3\ln\frac{4}{3} + \ldots + n\ln\frac{(n+1)}{n} = \ln\frac{(n+1)^n}{n!}.$$

$$LHS = \ln\left(\frac{2}{1}\right)^1 + \ln\left(\frac{3}{2}\right)^2 + \ln\left(\frac{4}{3}\right)^3 + \cdots + \ln\left(\frac{n+1}{n}\right)^n$$

$$= \ln\left[\frac{2^1 . 3^2 . 4^3 . 5^4 . \ldots . n^{n-1} . (n+1)^n}{1 . 2^2 . 3^3 . 4^4 . 5^5 \ldots (n-1)^{n-1} . n^n}\right]$$

Simplify the indices of the top and bottom :

$$LHS = \ln\left[\frac{(n+1)^n}{1.2.3\ldots n}\right] = \ln\left(\frac{(n+1)^n}{n!}\right) = RHS$$

(C) By using the induction method, prove that 3

$(35)^n + 3\times 7^n + 3\times 5^n + 6$ is divisible by 12 for $n \geq 1$

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Blank Page