Saint Ignatius' College Riverview

Mathematics Extension 2

Trial HSC Examination 2001

1. **(a)** If $P = 2 - ti$ where t is real, find $\overline{iP}$.
(b) The complex number, u, is given by $u = \frac{\sqrt{3}-i}{1+i}$
Find **(i)** The modulus of u **(ii)** The exact value of $\arg u$ **(iii)** u^6 in the form $a + bi$
(c) On an Argand diagram shade the region satisfied by both of the conditions:
$|z - 2| \geq 1$ and $-\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{6}$.
(d) If the complex number $Q = \frac{z-1}{z-2i}$, is purely imaginary (where $z = x = iy$) determine the Cartesian equation for the locus of z and sketch this locus.

2. **(a)** Find $\int \text{cosec}\theta \, d\theta$ using $t = \tan \frac{\theta}{2}$
(b) Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dy}{1+\cos y}$
(c) Evaluate $\int_0^{\frac{\pi}{4}} e^x \sin^2 x \, dx$
(d) Find $\int \frac{t^2-t-21}{(t^2+4)(2t-1)} \, dt$

3. **(a)** Consider the function $f(x) = 9 - x^2$. On three separate sets of axes, sketch the following, showing all important features.
(i) $y = f(x)$ **(ii)** $y = |f(x)|$ **(iii)** $|y| = f(x)$
(b) Consider the function $y = \sin(\cos^{-1} x)$
(i) Find the domain and range of the function.
(ii) Sketch this function showing the important features.
(c) Consider the function f and g defined by:
$f(x) = \frac{x+1}{x-2}$ for $x \neq 2$ and $g(x) = [f(x)]^2$
(i) Sketch the hyperbola $y = f(x)$, clearly labelling the horizontal and vertical asymptotes and the points of intersection with the x and y axes.
(ii) Sketch the curve $y = g(x)$ on a separate diagram showing all important features including any turning points.
(iii) On a separate diagram sketch the curve given by $y = g(-x)$

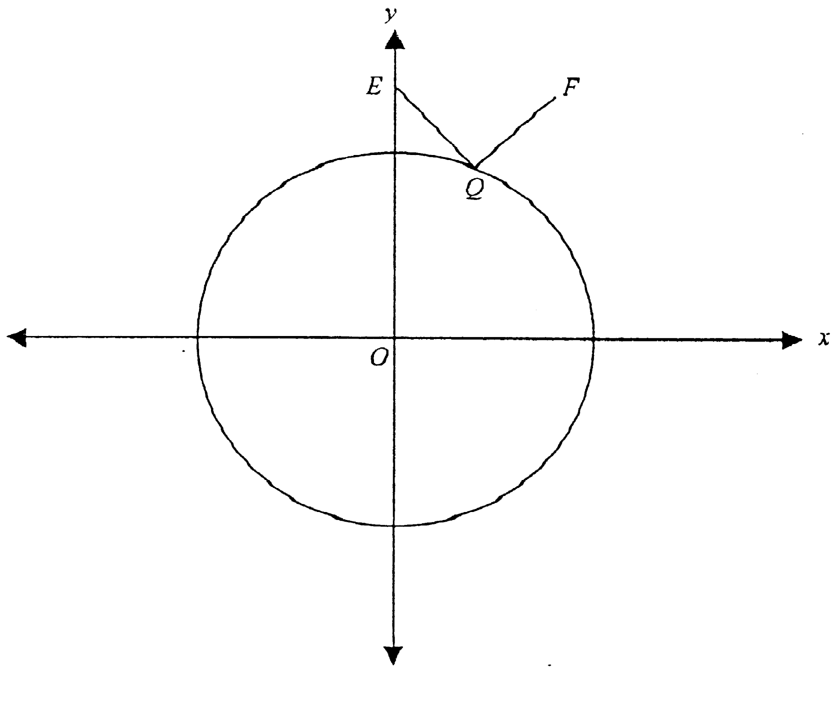
4. **(a)** **(i)** On the same number plane sketch the graphs of $y = |x| - 3$ and $y = 5 + 4x - x^2$
(ii) Hence, or otherwise, solve $\frac{|x|-3}{5+4x-x^2} > 0$
(b) Given the polynomial $Q(x)$, where $Q(x) = kx^{k+1} - (k+1)x^k + 1$ $(k \neq 0)$ prove that $Q(x)$ is divisible by $(x-1)^2$.
(c) The equation $x^3 + 2x - 1 = 0$ has roots p, q and r. Find
(i) the value of $p^2 + q^2 + r^2$
(ii) the equation with roots $-p, -q$ and $-r$

(d) $P(x)$ is a polynomial with the following form:

$$P(x) = Kx^3 + Mx^2 + Lx + N \text{ where } K, M, L \text{ and } N \text{ are real.}$$

$P(x)$ has roots of 5 and i and when divided by $(x-2)$ the remainder is 3. Find $P(x)$.

5. **(a)** **(i)** Show, using differentiation, that the equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\alpha, b\tan\alpha)$ is
$ax\tan\alpha + by\sec\alpha = (a^2+b^2)\sec\alpha\tan\alpha$
(ii) The vertical line through P meets an asymptote of the above hyperbola at M. The normal at P meets the x axis at K. Show that KM is at right angles to the asymptote.
(b) The following diagram shows a circle with centre at the origin O. The point $E(0,a)$ is fixed where $a > 3$. Q lies on the circle such that the angle EQF is a right angle and $EQ = QF$.

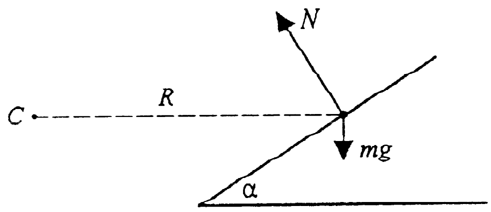


(i) Copy the diagram.
(ii) Show by substitution that $Q(3\cos\alpha, 3\sin\alpha)$ satisfies $x^2+y^2=9$
(iii) Prove, using congruent triangles or otherwise, that F has coordinates
$(3\cos\alpha + a - 3\sin\alpha, 3\cos\alpha + 3\sin\alpha)$
(iv) Find the locus of F as Q moves on the circle.
(v) Prove that the locus of the circle is independent of the value of a.

6. **(a)** The base of a solid is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $b > a$. Sections perpendicular to the y-axis are squares with one side in the base of the solid. Show that the volume of the solid is $\frac{16a^2b}{3}$ cubic units.

(b) The curve $y = \sin x$ is revolved about the straight line $y = 1$. Use a slicing technique to find the volume of the solid of revolution formed by the portion of the curve from $x = 0$ to $x = \frac{\pi}{2}$.
(c) The area enclosed by $y = (x-2)^2$ and the straight line $y = 4$ is rotated about the y-axis. Using the method of cylindrical shells, find the volume of the solid formed.

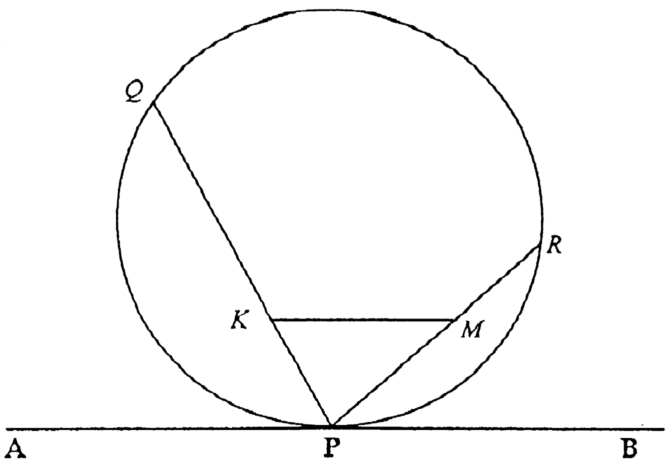
7. **(a)** A certain particle of unit mass moving through air experiences air resistance proportional to the square of its speed, v metres per second.
(i) Explain why the equations of motion with upwards taken as positive are:
$\ddot{x} = -g - kv^2$, when moving upwards and
$\ddot{x} = -g + kv^2$, when moving downwards,
where g is the acceleration due to gravity and k is a positive constant.
(ii) Suppose that the particle is fired vertically upwards from the ground with an initial speed of u metres per second.
(α) Show that the maximum height, H metres, reached by the particle is:
$H = \frac{1}{2k}\ln(1 + \frac{ku^2}{g})$
(β) Show that the time, T seconds, taken to reach maximum height is:
$T = \frac{1}{\sqrt{gk}}\tan^{-1}(\frac{u\sqrt{k}}{\sqrt{g}})$
(b) **(i)** A particle of mass m travels with constant velocity v in a horizontal circle of radius R, centre C, around a track banked at an angle α to the horizontal, as shown in the diagram.



Show that if there is no tendency for the particle to slip sideways then $v = \sqrt{Rg\tan\alpha}$.
(ii) A particle travels in a horizontal circle of radius 1 metre around the lower half of the track where the angle of banking is given by $\tan^{-1}(\frac{5}{18})$. Another particle travels in a horizontal circle of radius 1.2 metres around the upper half of the track where the angle of banking is given by $\tan^{-1}(\frac{16}{27})$. Each particle travels with constant velocity so that it has no tendency to slip sideways. The particles are initially observed to be alongside each other. Taking $g = 10$ metres per second squared, find the time that elapses before the particles are next observed to be alongside each other.

8. **(a)** Prove that:
(i) $p^2 + q^2 \geq 2pq$
(ii) $k^4 + l^4 + m^4 + n^4 \geq 4klmn$, if k, l, m and n are positive.
(b) Given $z = \cos\alpha + i\sin\alpha$, where $\sin\alpha \neq 0$:
(i) Prove that $\frac{1}{1 - z\cos\alpha} = 1 + i\cot\alpha$.
(ii) Hence, by considering $\sum_{k=0}^{\infty}(z\cos\alpha)^k$, deduce the sum of the infinite series $\sin\alpha\cos\alpha + \sin 2\alpha\cos^2\alpha + \cdots + \sin k\alpha\cos^k\alpha + \cdots$.

(c)



AB is a tangent to the circle at P. M and K move on PR and PQ respectively so that KM is parallel to AB. Prove that the point of intersection of the perpendicular bisectors of QK and RM moves on a straight line.