Question 1

(a)(i) let $x = \frac{2}{3}\sin\theta \qquad \frac{3x}{2} = \sin\theta \qquad \theta = \sin^{-1}\left(\frac{3x}{2}\right)$

$dx = \frac{2}{3}\cos\theta \, d\theta$

$$\int \frac{dx}{\sqrt{4-9x^2}} = \int \frac{\frac{2}{3}\cos\theta \, d\theta}{\sqrt{4-9\frac{4}{9}\sin^2\theta}}$$

$$= \frac{2}{3}\int \frac{\cos\theta}{\sqrt{4\cos^2\theta}}\,d\theta$$

$$= \frac{\cancel{2}}{3}\int \frac{\cos\theta}{\cancel{2}\cos\theta}\,d\theta$$

$$= \frac{1}{3}\int d\theta$$

$$= \frac{1}{3}\theta + c$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + c$$

(ii) $u = 1 + \ln x$

$du = \frac{1}{x}dx$

$$\int \frac{1}{x}(1+\ln x)^5 dx = \int u^5 \, du$$

$$= \frac{1}{6}u^6 + c$$

$$= \frac{1}{6}(1+\ln x)^6 + c$$

(b) $\int_0^{\pi/4} x\sec^2 x \, dx$

$$\int_0^{\pi/4} x \frac{d}{dx}(\tan x)\,dx$$

$$= \left[x\tan x\right]_0^{\pi/4} - \int_0^{\pi/4} \frac{d}{dx}(x)\tan x \, dx$$

$$= \frac{\pi}{4}\tan\frac{\pi}{4} - 0 - \int_0^{\pi/4} \tan x \, dx$$

$$= \frac{\pi}{4} - \int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx$$

$$= \frac{\pi}{4} + \left[\ln(\cos x)\right]_0^{\pi/4}$$

$$= \frac{\pi}{4} + \left(\ln\left(\cos\frac{\pi}{4}\right) - \ln(\cos 0)\right)$$

$$= \frac{\pi}{4} + \ln\left(\frac{1}{\sqrt{2}}\right) - \ln 1$$

$$= \frac{\pi}{4} + \ln\left(\frac{1}{\sqrt{2}}\right)$$

(c) let $u = x + 4 \qquad x = u - 4$

$du = dx$

Δ limits:

$x = 0 \quad u = 4$

$x = 5 \quad u = 9$

$$\int_0^5 \frac{x}{\sqrt{x+4}}\,dx$$

$$= \int_4^9 \frac{u-4}{\sqrt{u}}\,du$$

$$= \int_4^9 u^{\frac{1}{2}} - 4u^{-\frac{1}{2}}\,du$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}} - 8u^{\frac{1}{2}}\right]_4^9$$

$$= \left(\frac{2}{3}.9^{\frac{3}{2}} - 8.9^{\frac{1}{2}}\right) - \left(\frac{2}{3}.4^{\frac{3}{2}} - 8.4^{\frac{1}{2}}\right)$$

$$= 18 - 24 - \frac{16}{3} + 16$$

$$= 4\frac{2}{3}$$

(d) $$\frac{8}{(x+2)(x^2+4)} = \frac{a}{x+2} + \frac{bx+c}{x^2+4}$$

$$8 = a(x^2+4) + (bx+c)(x+2)$$

let $x = -2$:

$8 = a(4+4)$

$a = 1$

co-efficient of x^2:

$0 = a + b$

$b = -1$

constant term:

$8 = 4a + 2c$

$c = 2$

$$\therefore \int_0^2 \frac{8}{(x+2)(x^2+4)}\,dx$$

$$= \int_0^2 \frac{1}{x+2} + \frac{-x+2}{x^2+4}\,dx$$

$$= \left[\ln(x+2)\right]_0^2 + -\frac{1}{2}\int_0^2 \frac{2x}{x^2+4}\,dx + 2\int_0^2 \frac{1}{x^2+4}\,dx$$

$$= \ln 4 - \ln 2 - \frac{1}{2}\left[\ln(x^2+4)\right]_0^2 + 2\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

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$= \ln 2 - \frac{1}{2}(\ln 8 - \ln 4) + \quad (\tan^{-1}1 - \tan^{-1}0)$

$= \ln 2 - \frac{1}{2}\ln 2 + \frac{\pi}{4}$

$= \frac{1}{2}\ln 2 + \frac{\pi}{4}$

Question 2

(a)(i) $z = 2+2i$

$r = \sqrt{2^2+2^2}$

$= \sqrt{8}$

$\tan\theta = \frac{2}{2} = 1$

$\theta = \frac{\pi}{4}$ (1st quadrant)

$\therefore \quad z = \sqrt{8}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

(ii) $z^8 = \left[\sqrt{8}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^8$

$= (\sqrt{8})^8\left(\cos\frac{8\pi}{4} + i\sin\frac{8\pi}{4}\right)$ De Moivre's Theorem

$= 4096(\cos 2\pi + i\sin 2\pi)$

$= 4096(1)$

$= 4096$

(b)

ΔOAB equilateral so $OB = OA$
$\angle AOB = \frac{\pi}{3}$

$\therefore$ multiply $1+i\sqrt{2}$ by $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ to get co-ordinates of B

$\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$(1+i\sqrt{2})\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2} + i\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}$

$= \frac{1}{2} - \frac{\sqrt{6}}{2} + i\left(\frac{\sqrt{3}+\sqrt{2}}{2}\right)$

$= \frac{1-\sqrt{6}}{2} + i\left(\frac{\sqrt{3}+\sqrt{2}}{2}\right)$

this is the complex number corresponding to B.

(c) Let $z = x+iy$

$Re(z) = x$

$|z| = \sqrt{x^2+y^2}$

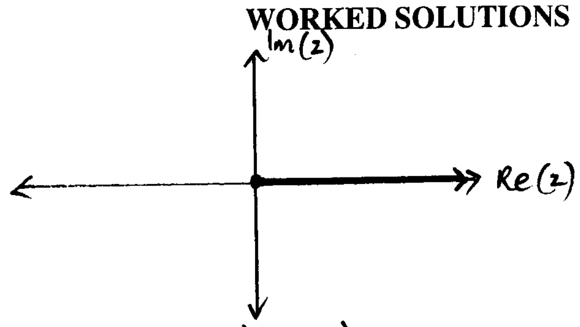
$\therefore \quad x = \sqrt{x^2+y^2}$ so $x \geq 0$

$x^2 = x^2 + y^2$

$y^2 = 0$

$\Rightarrow y = 0$

$\therefore$ locus is the real axis for $x \geq 0$



(d) $\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{c^2+d^2}$

$= \frac{ac-bd}{c^2+d^2} + i\,\frac{bc-ad}{c^2+d^2}$

So $Im\left(\frac{a+ib}{c+id}\right) = \frac{bc-ad}{c^2+d^2}$

Now $bc < ad$

so $bc - ad < 0$

$\frac{bc-ad}{c^2+d^2} < 0$ since c, d real

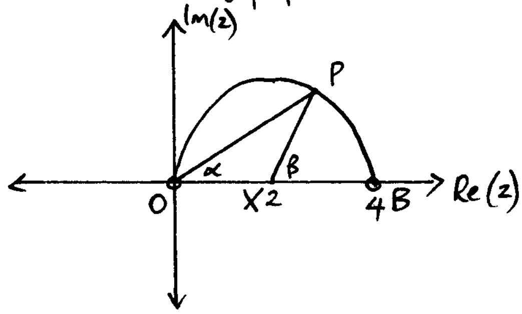
so $Im\left(\frac{a+ib}{c+id}\right) < 0$

(e)(i) $|z^2-2z| = |z(z-2)|$

$= |z||z-2|$

$= 2|z|$

(ii)



$\angle POX = \alpha = \arg z$

$\angle PXB = \beta = \arg(z-2)$

$OX = XP$ (radii)

$\therefore \quad \Delta OXP$ isosceles (2 sides equal)

$\therefore \quad \angle XPO = \alpha$ (base angles of isosceles Δ are equal)

$\therefore \quad \angle PXB = \beta = 2\alpha$ (external angle of Δ)

ie $\arg(z-2) = 2\arg z$

Now $\arg(z-2) = k\arg(z^2-2z)$

$2\arg z = k\arg(z(z-2))$

$2\arg z = k[\arg z + \arg(z-2)]$

$$2\arg z = k(\arg z + 2\arg z)$$
$$2 = 3k$$
$$k = \frac{2}{3}$$

Question 3

(a) $\alpha + \beta + \gamma = -\frac{b}{a} = 0$ sum of roots

α, β, γ are roots so

$2\alpha^3 + 5\alpha + 1 = 0$
$2\beta^3 + 5\beta + 1 = 0$
$2\gamma^3 + 5\gamma + 1 = 0$

Adding:

$2(\alpha^3+\beta^3+\gamma^3) + 5(\alpha+\beta+\gamma) + 3 = 0$
$2(\alpha^3+\beta^3+\gamma^3) + 5(0) + 3 = 0$
$\alpha^3+\beta^3+\gamma^3 = -\frac{3}{2}$

(b)(i) Since the co-efficients of $P(x)$ are real, its complex roots occur in conjugate pairs.
So $1-i$ is a root since $1+i$ is a root.
Let the third root be α.

$\alpha + (1+i) + (1-i) = -\frac{-4}{2} = 2$ sum of roots
$\alpha + 2 = 2$
$\alpha = 0$

$P(\alpha) = 0$
$P(0) = 0$
$P(0) = n$
so $n = 0$

Sum of roots 2 at a time:

$0(1+i) + 0(1-i) + (1-i)(1+i) = \frac{m}{2}$
$1 + 1 = \frac{m}{2}$
$m = 4$

$n = 0, m = 4$

(ii) The zeros of $P(x)$ are $1-i$, $1+i$ and 0

(c) Let the polynomial be $P(x)$

$P(x) = (x^2-9)(x+A) + (x+8)$ ①
and $P(x) = (x)(x^2+Bx+C) + (-4)$ ②

① = ② and expanding

$x^3 + Ax^2 - 9x - 9A + x + 8$
$= x^3 + Bx^2 + Cx - 4$

$x^3 + Ax^2 - 8x + (-9A+8)$
$= x^3 + Bx^2 + Cx - 4$

Equating co-efficients

$A = B$
$-8 = C$
$-9A + 8 = -4$
$-9A = -12$
$A = \frac{4}{3}$
$B = \frac{4}{3}$

So $P(x) = x^3 + \frac{4}{3}x^2 - 8x - 4$

In required form polynomial is

$3x^3 + 4x^2 - 24x - 12$

(d)(i) $\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$ De Moivre
$= c^4 + 4c^3 is - 6c^2s^2 - 4cis^3 + s^4$

Equating real parts

$\cos 4\theta = c^4 - 6c^2s^2 + s^4$
$= c^4 - 6c^2(1-c^2) + (1-c^2)^2$
since $\cos^2\theta + \sin^2\theta = 1$
$= c^4 - 6c^2 + 6c^4 + 1 - 2c^2 + c^4$
$= 8c^4 - 8c^2 + 1$

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$ De Moivre
$= c^3 + 3c^2 is - 3c s^2 - is^3$

Equating real parts

$\cos 3\theta = c^3 - 3cs^2$
$= c^3 - 3c(1-c^2)$ as above
$= c^3 - 3c + 3c^3$
$= 4c^3 - 3c$

$\cos 4\theta = \cos 3\theta$ then becomes

$8c^4 - 8c^2 + 1 = 4c^3 - 3c$
$8c^4 - 4c^3 - 8c^2 + 3c + 1 = 0$

(ii) when $n = 1$, $\theta = \frac{2\pi}{7}$

$\cos 4\theta = \cos\left(\frac{2\pi}{7} \times 4\right) = \cos\left(\frac{8\pi}{7}\right)$
$= \cos\left(2\pi - \frac{8\pi}{7}\right)$
$= \cos\frac{6\pi}{7}$
$= \cos 3\theta$ ✓

when $n = 2$, $\theta = \frac{4\pi}{7}$

$\cos 4\theta = \cos\frac{16\pi}{7} = \cos\frac{2\pi}{7}$
$= \cos\left(-\frac{2\pi}{7}\right)$
$= \cos\left(\frac{12\pi}{7}\right)$
$= \cos 3\theta$ ✓

when $n=3,\ \theta=\frac{6\pi}{7}$

$$\cos 4\theta = \cos\frac{24\pi}{7} = \cos\left(-\frac{3\pi}{7}\right) = \cos\left(\frac{3\pi}{7}\right) = \cos\left(\frac{18\pi}{7}\right) = \cos 3\theta \quad ✓$$

when $n=4,\ \theta=\frac{8\pi}{7}$

$$\cos 4\theta = \cos\frac{32\pi}{7} = \cos\frac{4\pi}{7} = \cos\left(\cancel{4\pi} - \frac{4\pi}{7}\right) = \cos\frac{24\pi}{7} = \cos 3\theta \quad ✓$$

$\cos 4\theta = \cos 3\theta$ can be expressed as a quartic, so these 4 solutions are the only solutions

$\therefore\ \theta = \frac{2n\pi}{7}$ satisfies $\cos 4\theta = \cos 3\theta$.

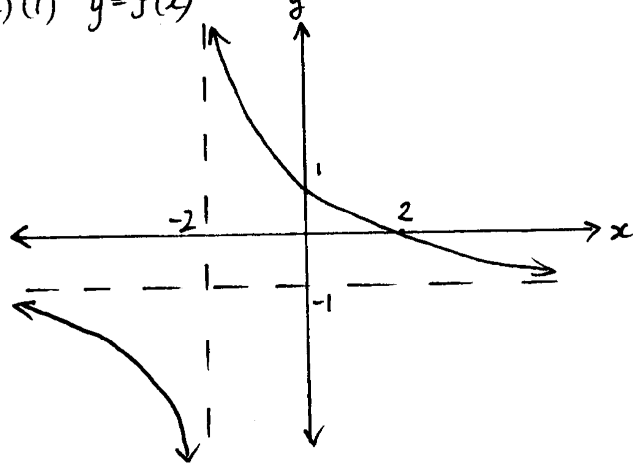
(iii) $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}, \cos\frac{6\pi}{7}$ are 3 roots of $8c^4-4c^3-8c^2+3c+1=0$ since $\theta=\frac{2n\pi}{7}$ satisfies the equivalent $\cos 4\theta = \cos 3\theta$. The 4^{th} root of $8c^4-4c^3-8c^2+3c+1=0$ is $c=1$ by inspection. So to find the equation whose roots are $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}$ and $\cos\frac{6\pi}{7}$, divide by $(c-1)$.

$$\begin{array}{r|l} & 8c^3+4c^2-4c-1 \\ \hline c-1 & 8c^4-4c^3-8c^2+3c+1 \\ & 8c^4-8c^3 \\ \hline & \quad 4c^3-8c^2 \\ & \quad 4c^3-4c^2 \\ \hline & \quad\quad -4c^2+3c \\ & \quad\quad -4c^2+4c \\ \hline & \quad\quad\quad -c+1 \\ & \quad\quad\quad -c+1 \\ \hline & \quad\quad\quad\quad 0 \end{array}$$

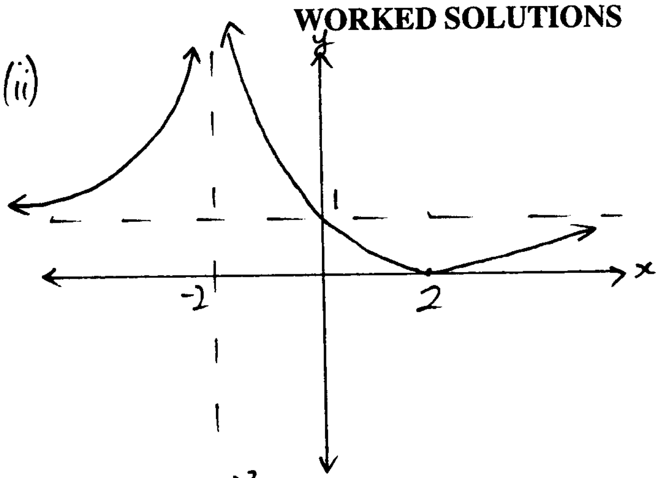
The equation is $8c^3+4c^2-4c-1=0$

Question 4

(a) (i) $y=f(x)$



(ii)



$$y = \frac{(2-x)^2}{(2+x)^2}$$

$$y' = \frac{-2(2-x)(2+x)^2 - 2(2+x)(2-x)^2}{(2+x)^4}$$

$$= \frac{-2(2-x)(2+x)(2+x+2-x)}{(2+x)^4}$$

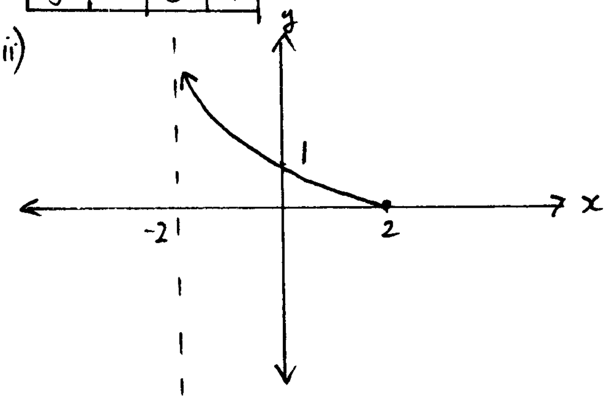
$$= \frac{-8(2-x)}{(2+x)^3}$$

turning point at $x=2$

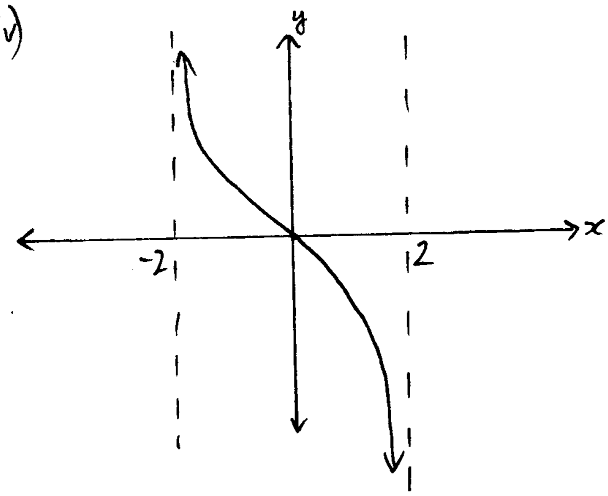
x	2^-	2	2^+
y'	$-$	0	$+$

$\therefore$ min at $x=2$

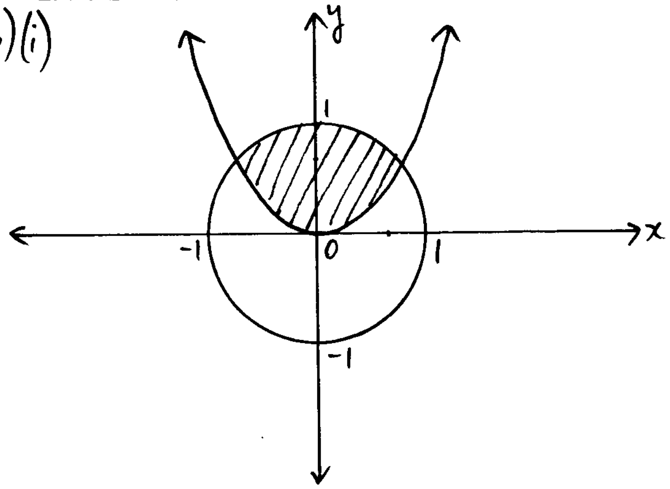
(iii)



(iv)



(b)(i)



(ii) Typical shell

$$\Delta V = \pi h(R^2 - r^2)$$

$x^2 + y^2 = 1$ for circle

so $y = \sqrt{1-x^2}$ for this region

$y = \frac{3x^2}{8}$ for parabola

$\therefore\ h = \sqrt{1-x^2} - \frac{3x^2}{8}$

So
$$\Delta V = \pi\left(\sqrt{1-x^2} - \frac{3x^2}{8}\right)\left((x+\Delta x)^2 - x^2\right)$$
$$= \pi\left(\sqrt{1-x^2} - \frac{3x^2}{8}\right)\left(x^2 + 2x\Delta x + (\Delta x)^2 - x^2\right)$$
$$= \pi\left(\sqrt{1-x^2} - \frac{3x^2}{8}\right)(2x\,\Delta x)$$
$$\Delta x^2 \approx 0$$
$$= 2\pi x\left(\sqrt{1-x^2} - \frac{3x^2}{8}\right)\Delta x$$

Now $V \doteqdot \sum_{x=0}^{1} \Delta V$

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{1} \Delta V$$
$$= \lim_{\Delta x \to 0} \sum_{x=0}^{1} \left(2\pi x\sqrt{1-x^2} - 2\pi \frac{3x^3}{8}\right)\Delta x$$
$$= 2\pi \int_0^1 x\sqrt{1-x^2} - \frac{3x^3}{8}\, dx$$
$$= 2\pi\left[-\frac{1}{3}(1-x^2)^{\frac{3}{2}} - \frac{3x^4}{32}\right]_0^1$$
$$= 2\pi\left[\left(-\frac{3}{32}\right) - \left(-\frac{1}{3}\right)\right]$$
$$= 2\pi \times \frac{23}{96}$$
$$= \frac{23\pi}{48} \text{ units}^3$$

Question 5

(a) (i) $a^2 = 9 \quad b^2 = 4$

$$b^2 = a^2(1-e^2)$$
$$4 = 9(1-e^2)$$
$$\frac{4}{9} = 1 - e^2$$
$$e^2 = \frac{5}{9}$$
$$e = \frac{\sqrt{5}}{3}$$

foci at $(\pm ae, 0)$. $a = 3$

$S(\sqrt{5}, 0) \qquad S'(-\sqrt{5}, 0)$

directrices at $x = \pm \frac{a}{e}$

$$x = \pm \frac{9}{\sqrt{5}}$$

(ii) $\frac{x^2}{9} + \frac{y^2}{4} = 1$

$$\frac{2x}{9} + \frac{2y}{4}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{4x}{9y}$$

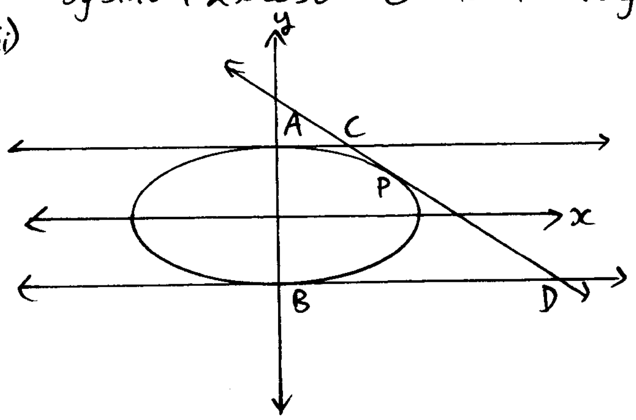
at P, $\frac{dy}{dx} = \frac{-\overset{2}{\cancel{4}}.\cancel{3}\cos\theta}{3\cancel{9}.\cancel{2}\sin\theta}$

$$= -\frac{2}{3}\frac{\cos\theta}{\sin\theta}$$

$$y - y_1 = m(x - x_1)$$
$$y - 2\sin\theta = -\frac{2}{3}\frac{\cos\theta}{\sin\theta}(x - 3\cos\theta)$$
$$3y\sin\theta - 6\sin^2\theta = -2x\cos\theta + 6\cos^2\theta$$

$3y\sin\theta + 2x\cos\theta = 6$ is the tangent

(iii)



A is the y-intercept of the ellipse ie $(0,2)$
so tangent at A is $y=2$
Similarly, tangent at B is $y=-2$
Finding C: C has y-coord 2
Put into eqn. of tangent at P

$$6\sin\theta + 2x\cos\theta = 6$$
$$2x\cos\theta = 6 - 6\sin\theta$$
$$x = \frac{3-3\sin\theta}{\cos\theta}$$

$\therefore$ C is $\left(\frac{3-3\sin\theta}{\cos\theta}, 2\right)$

$\therefore$ $AC = \left|\frac{3-3\sin\theta}{\cos\theta}\right|$

Similarly for D:

$$-6\sin\theta + 2x\cos\theta = 6$$
$$x = \frac{3+3\sin\theta}{\cos\theta}$$
$$BD = \left|\frac{3+3\sin\theta}{\cos\theta}\right|$$

$$\therefore AC.BD = \left|\frac{3-3\sin\theta}{\cos\theta}\right|\left|\frac{3+3\sin\theta}{\cos\theta}\right|$$
$$= \left|\frac{(3-3\sin\theta)(3+3\sin\theta)}{\cos^2\theta}\right|$$
$$= \left|\frac{9-9\sin^2\theta}{\cos^2\theta}\right|$$
$$= \left|\frac{9\cos^2\theta}{\cos^2\theta}\right|$$
$$= 9$$

(b)(i) $(x-(\omega+\omega^4))(x-(\omega^2+\omega^3)) = 0$

$$x^2 - x(\omega+\omega^2+\omega^3+\omega^4) + (\omega+\omega^4)(\omega^2+\omega^3) = 0$$
$$x^2 - x(\omega+\omega^2+\omega^3+\omega^4) + (\omega^3+\omega^4+\omega^6+\omega^7) = 0$$

Now $\omega^5 = 1$
so $(\omega^2)^5 = (\omega^3)^5 = (\omega^4)^5 = 1$
Also $1^5 = 1$
so $1, \omega, \omega^2, \omega^3, \omega^4$ are 5 roots of $z^5 - 1 = 0$

so $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$
$\omega + \omega^2 + \omega^3 + \omega^4 = -1$

Also, $\omega^6 = (\omega^5)(\omega) = 1 \times \omega = \omega$
$\omega^7 = \omega^5 . \omega^2 = 1 . \omega^2 = \omega^2$

so $\omega^3 + \omega^4 + \omega^6 + \omega^7$
$= \omega + \omega^2 + \omega^3 + \omega^4$
$= -1$

so equation is
$$x^2 - x(-1) + (-1) = 0$$
$$x^2 + x - 1 = 0$$

(ii) $X^2 + Y^2 = (X+iY)(X-iY)$
$$= z\bar{z}$$
$$= w^n\,\overline{(w^n)}$$
$$= w^n(\bar{w})^n$$
$$= (w\bar{w})^n$$
$$= [(x+iy)(x-iy)]^n$$
$$= (x^2+y^2)^n$$

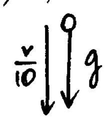
Question 6

(a) $\frac{\sqrt{x}}{\sqrt{u}} + \frac{\sqrt{y}}{\sqrt{v}} = 1$

$$\frac{\frac{1}{2}x^{-\frac{1}{2}}}{\sqrt{u}} + \frac{\frac{1}{2}y^{-\frac{1}{2}}}{\sqrt{v}}\frac{dy}{dx} = 0$$
$$\frac{1}{\sqrt{vy}}\frac{dy}{dx} = \frac{-1}{\sqrt{xu}}$$
$$\frac{dy}{dx} = -\sqrt{\frac{vy}{xu}}$$

this is undefined at $x=0$
$\therefore$ there is a vertical tangent at the y-axis.
Now $x \geq 0$ and $y \geq 0$
so the curve cannot cross the y-axis
$\therefore$ it touches the y-axis.

(b)(i)



$$F_{down} = -g - \frac{v}{10}$$
$$F = ma$$
$$m = 1$$
so $a = -g - \frac{v}{10}$
$$a = \frac{dv}{dt}$$
$$\therefore \frac{dv}{dt} = -\frac{v}{10} - g$$

(ii) $\frac{dv}{dt} = \frac{-v-10g}{10}$

$$\frac{dt}{dv} = \frac{10}{-v-10g} = \frac{-10}{v+10g}$$
$$t + c = -10\ln|(v+10g)|$$

When $t=0$, $v=10(20-g)$

$$c = -10\ln(200-10g+10g)$$
$$= -10\ln 200$$

When at greatest height, $v=0$

$$T - 10\ln 200 = -10\ln(10g)$$
$$T = -10\ln(10g) + 10\ln(200)$$
$$= 10\ln\left(\frac{200}{10g}\right)$$
$$= 10\ln\left(\frac{20}{g}\right)$$

(iii) $a = v\frac{dv}{dx}$

so $v\frac{dv}{dx} = -\frac{v}{10} - g$

$$\frac{dv}{dx} = -\frac{1}{10} - \frac{g}{v}$$
$$= \frac{-v-10g}{10v}$$
$$\frac{dx}{dv} = \frac{-10v}{v+10g}$$
$$x + c = -10\int \frac{v}{v+10g}\,dv$$
$$= -10\int \frac{v+10g-10g}{v+10g}\,dv$$
$$= -10\int 1 - 10g\left(\frac{1}{v+10g}\right)dv$$
$$= -10\left(v - 10g\ln|v+10g|\right)$$

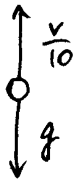
When $x=0$, $v=10(20-g)$

$$\therefore\ c = -10\left(10(20-g) - 10g\ln 200\right)$$
$$= -10(200-10g-10g\ln 200)$$

When $x=H$, $v=0$

$$\therefore\ H - 10(200-10g-10g\ln 200) = -10(-10g\ln 10g)$$
$$H - 2000 + 100g + 100g\ln 200 = 100g\ln 10g$$
$$H = 2000 - 100g(1 + \ln 200 - \ln 10g)$$
$$H = 2000 - 100g\left(1 + \ln\left(\frac{20}{g}\right)\right)$$

(iv)(α)



$$\frac{dv}{dt} = \frac{v}{10} - g$$
$$= \frac{v-10g}{10}$$
$$\frac{dt}{dv} = \frac{10}{v-10g} = \frac{-10}{10g-v}$$

$$t + c = -10\ln|10g - v|$$

When $t=0$, $v=0$

$$\therefore\ c = -10\ln(10g)$$
$$\therefore\ t - 10\ln(10g) = -10\ln|10g-v|$$
$$t = 10\ln\left|\frac{10g}{10g-v}\right|$$
$$e^{\frac{t}{10}} = \frac{10g}{10g-v}$$
$$10g - v = 10g e^{-\frac{t}{10}}$$
$$v = 10g\left(1 - e^{-\frac{t}{10}}\right)$$

as $t \to \infty$, $v \to 10g$

so terminal velocity is $10g$

(β) Time taken to reach maximum height will be greater than time taken to fall from the maximum height. In the upward motion, the magnitude of the acceleration against the direction of motion is $\frac{v}{10} + g$. This is greater than the magnitude of acceleration going down, $\frac{v}{10} - g$, so the upward journey takes longer.

Question 7

(a)(i) 9 lines can be drawn through the first point

8 other lines can be drawn through the second point

$\vdots$

$$\text{Total} = 9+8+7+\ldots+1$$
$$= 45$$

(ii) No. of diagonals $= 45 -$ no. of sides
$$= 45 - 10$$
$$= 35$$

(b) The roots of $(1-x)^n - 1 = 0$ are $0, 1-\alpha_1, 1-\alpha_2, \ldots, 1-\alpha_n$.

Now
$$(1-x)^n - 1 = (-1)^n x^n + \binom{n}{1}(-1)^{n-1}x^{n-1} + \ldots + \binom{n}{n-1}(-x) + 1 - 1$$
$$= (-1)^n x^n + \binom{n}{1}(-1)^{n-1}x^{n-1} + \ldots - nx$$

So the sum of the product of the roots $(n-1)$ at a time is $\dfrac{(-n)(-1)^{n-1}}{(-1)^n} = n$

Since 0 is a root, the sum of the product of the roots (n-1) at a time is just the product of all non-zero roots. Thus

$$(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_{n-1}) = n$$

(c) Since $x>0$, $t>0$ so

$$\frac{t^{n-1}}{1+t} < t^{n-1}$$

so
$$\int_0^x \frac{t^{n-1}}{1+t}\,dt < \int_0^x t^{n-1}\,dt$$
$$= \left[\frac{t^n}{n}\right]_0^x$$
$$= \frac{x^n}{n}$$

(d)(i) Let $\angle YZ'Z = \alpha$

Then $\angle ACZ' = \alpha$ (alternate angles, $AB \parallel ZZ'$)

$\therefore$ $\angle YC'X = \alpha$ (vertically opposite)

Also $\angle YDZ = \angle YZ'Z$ (angles in same segment)
$= \alpha$

ie $\angle YDX = \angle YC'X$

$\therefore$ C', X, Y and D are concyclic since XY subtends equal angles at C' and D

(ii) In Δs $C'XY$, CXY

XY common

$\angle C'XY = \angle CXY = \frac{\pi}{2}$ (given $- \; OX \perp AB$)

$AX = XB$ ($\perp$ from centre to chord bisects the chord)

$AC' = BC$ (given)

$\therefore$ $AX - AC' = XB - BC$ (subtracting)

$\therefore$ $C'X = CX$

$\therefore$ $\Delta C'XY \equiv \Delta CXY$ (side, angle, side)

$\therefore$ $C'Y = CY$ (corresponding sides)

$C'XYD$ is a cyclic quadrilateral (above)

$\angle C'XY = \frac{\pi}{2}$ (given)

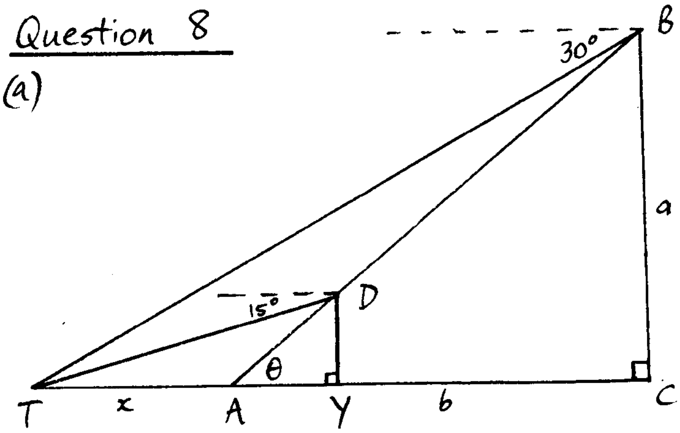
$\therefore$ $C'Y$ is a diameter of the circle with C', X, Y, D on its circumference (diameter subtends right angle at circumference)

$\therefore$ $C'Y \geq XD$ (diameter is longest chord in the circle)

$\therefore$ $CY \geq XD$ $\quad (CY = C'Y)$

Question 8

(a)



Let $AT = x$
$AC = b$
$BC = a$

so $\cot\theta = \dfrac{b}{a}$

$\angle BTC = 30°$ (alternate angle to angle of depression at B)

so $\cot 30° = \dfrac{b+x}{a}$

$\sqrt{3} = \dfrac{b}{a} + \dfrac{x}{a}$

$\sqrt{3} = \cot\theta + \dfrac{x}{a}$ ①

$\angle DTA = 15°$ (alternate angle to angle of depression at D)

Construct $DY \perp AC$

In Δs ADY, ABC

$\angle A$ common

$\angle AYD = \angle ACB = 90°$ (by construction)

$\therefore \Delta ADY \,|||\, \Delta ABC$ (equiangular)

Now $\dfrac{AD}{AB} = \dfrac{1}{4}$ (given)

so $\dfrac{DY}{BC} = \dfrac{1}{4}$ (sides of similar Δs in same ratio)

$\dfrac{DY}{a} = \dfrac{1}{4}$

$DY = \dfrac{a}{4}$

Similarly $AY = \dfrac{b}{4}$

So in ΔTDY

$\cot 15° = \dfrac{x + \frac{b}{4}}{a/4}$

$\cot 15° = \dfrac{4x}{a} + \dfrac{b}{a}$

$\cot 15° = \dfrac{4x}{a} + \cot\theta$

Now $\cot 15° = \cot(45°-30°)$

$$= \frac{1}{\tan(45°-30°)}$$

$$= \frac{1+\tan 45°\tan 30°}{\tan 45° - \tan 30°}$$

$$= \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$= \frac{(\sqrt{3}+1)^2}{2}$$

$$= \frac{4+2\sqrt{3}}{2}$$

$$= 2+\sqrt{3}$$

so $\quad 2+\sqrt{3} = \frac{4x}{a} + \cot\theta \quad (2)$

From (1) $\quad \frac{x}{a} = \sqrt{3} - \cot\theta$

Sub into (2)

$$2+\sqrt{3} = 4\sqrt{3} - 4\cot\theta + \cot\theta$$

$$3\cot\theta = -2 + 3\sqrt{3}$$

$$\cot\theta = \sqrt{3} - \frac{2}{3}$$

(b) (i) $\quad (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

(ii) $\quad \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!}\,\frac{1}{n^k}$$

$$= \sum_{k=0}^{n} \frac{n(n-1)(n-2)\dots(n-k+1)}{n^k}\cdot\frac{1}{k!}$$

Now $\frac{n-c}{n} \to 1$ as $n \to \infty$, and there are k terms in the numerator

so $\quad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!}$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}$$

$$= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

as required

(iii) Using induction

1. Prove true for first term $n=3$

LHS $= \frac{1}{n!} = \frac{1}{3!} = \frac{1}{6}$

RHS $= \frac{1}{2^{n-1}} = \frac{1}{2^2} = \frac{1}{4}$

$\frac{1}{6} < \frac{1}{4} \quad \therefore$ true for $n=3$

2. Assume true for $n=k \quad k \in \mathbb{Z}^+, k \geq 3$

ie $\quad \frac{1}{k!} < \frac{1}{2^{k-1}}$

3. Prove true for $n=k+1$

LHS $= \frac{1}{(k+1)!}$

$$= \frac{1}{(k+1)\,k!}$$

$$< \frac{1}{(k+1)2^{k-1}} \quad \text{by assumption}$$

$$< \frac{1}{2.2^{k-1}} \quad \text{since } k+1 > 2$$

$$= \frac{1}{2^k}$$

$\therefore$ true for $n=k+1$ when assumed true for $n=k$

$\therefore \frac{1}{n!} < \frac{1}{2^{n-1}}$ for $n \geq 3$ by the principle of mathematical induction

(iv) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$ (above)

$$> 2$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$$

$$= 1 + 1 + \frac{1}{2} + \sum_{k=3}^{\infty}\frac{1}{k!}$$

$$< 2\tfrac{1}{2} + \sum_{k=3}^{\infty}\frac{1}{2^{k-1}} \quad \text{(from (iii))}$$

$$= 2\tfrac{1}{2} + S_\infty$$

where $\quad S_\infty = \frac{a}{1-r} \quad$ (limit exists since $|r| = \frac{1}{2} < 1$)

$$= \frac{\frac{1}{4}}{1-\frac{1}{2}}$$

$$= \frac{1}{2}$$

So $\quad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n < 2\tfrac{1}{2} + \frac{1}{2}$

$$= 3$$

$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = N$ where $2 < N < 3$