CSSA - NSW

4 Unit Mathematics

Trial HSC Examination 1984

1. (i) Prove that the curve: $y = x^2e^{-x}$ has a minimum turning point at $(0, 0)$ and a maximum turning point at $(2, \frac{4}{e^2})$. Sketch the curve. The roots of the equation $x^2e^{2-x} - 4 = 0$ are 2 and α. In terms of the root(s), determine the values of x for which $x^2e^{2-x} - 4$ is positive. (You are not required to find the value of α.)
(ii) Show that the function $f(x) = \frac{4x^2+9}{x}$ has two stationary points and two asymptotes. Obtain the coordinates of the stationary points and the equations of the two asymptotes. Sketch the function, showing these features.
(iii) A relation is defined implicitly by: $x^2 + xy - 2y^2 = 0$. Show that $\frac{dy}{dx}$ has only two possible values: 1 and $-\frac{1}{2}$. Hence, or otherwise, sketch the relation.

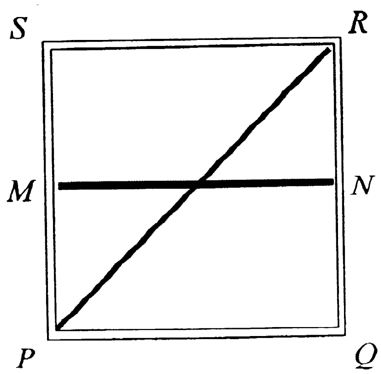
2. (i) Find $\int \operatorname{cosec} x\, dx$ by using the substitution $t = \tan \frac{x}{2}$.
(ii) Find $\int \frac{dx}{x(1+x^2)}$.
(iii) Find **(a)** $\int x\sqrt{x^2 - 1}\, dx$;
(b) $\int_1^2 x\sqrt{3x - 2}\, dx$.
(iv) If $I_n = \int_0^1 x^n e^x\, dx$, where n is a positive integer, show that $I_{n-1} = e - (n+1)I_n$. Hence evaluate $\int_0^{0.2} t^3e^{5t}\, dt$, leaving your answer in terms of e.

3. (i) Indicate on an Argand diagram the region in which z lies given that both $|z - (3 + i)| \leq 3$ and $\frac{\pi}{4} < \arg[z - (1 + i)] \leq \frac{\pi}{2}$ are satisfied.
(ii) Find the locus in the Argand diagram of the point P which represents the complex number z where $z\overline{z} - 4(z + \overline{z}) = 9$.
(iii) Show by geometrical considerations or otherwise that if the complex numbers z_1 and z_2 are such that $|z_1| = z_2$ when $\frac{z_1+z_2}{z_1-z_2}$ is purely imaginary.
(iv) Sketch the curve C with Cartesian equation $x^2 + (y - 1)^2 = 1$. The point P, representing the nonzero complex number z, lies on C.
(a) Express $|z|$ in terms of θ, the argument of z.
(b) Given that $z' = \frac{1}{z}$ find the modulus and argument of z' in terms of θ.
Show that, whatever the position of P on the circle C, the point P' representing z' lies on a certain line and determine the equation of this line.

4. (i) Show that the condition for the line $y = mx + c$ to be tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 + b^2$. Show that the pair of tangents drawn from the point $(3, 4)$ to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ are at right angles to one another.
(ii) Show that the equation of the normal at the point $P(a \sec\theta, b \tan\theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $ax \sin\theta + by = (a^2 + b^2) \tan\theta$. The normal at the point

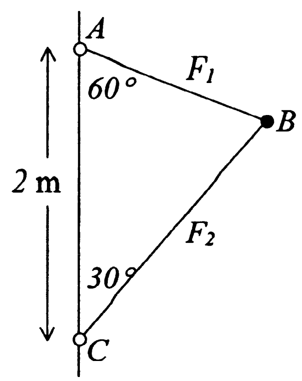
$P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the x axis at G and PN is the perpendicular from P to the x axis. Prove that $OG = e^2.ON$ (where O is the origin).

5. **(i)** The base of a solid is the circle $x^2 + y^2 = 16x$ and every plane section perpendicular to the x axis is a rectangle whose height is twice the distance of the plane of the section from the origin. Show that the volume of the solid os 1024π cubic units.
(ii) The Anti-Bureaucracy Organisation wants to install a rectangular notice board $PQRS$ of fixed area A square metres for each of its offices. The notice board is to be subdivided by two thin strips of red tape PR and MN (where MN is parallel to PQ) as shown below.



Find, in terms of A, the dimensions of the notice board so that the length of red tape used is a minimum.

6. **(i)** A mass of 10kg, centre B is connected by light rods to sleeves A and C which revolve freely about the vertical axis AC but do not move vertically.

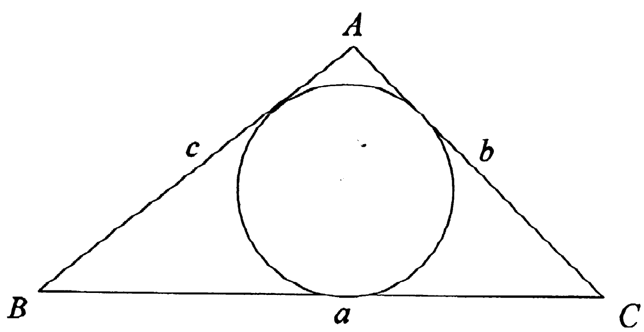


(a) Given $AC = 2$ metres, show that the radius of the circular path of rotation of B ia $\sqrt{\frac{3}{2}}$ metres.
(b) Find the tensions in the rods AB, BC when the mass makes 90 revolutions per minute about the vertical axis.
(ii) A small satellite of mass m revolves uniformly in a circular orbit of radius a about a fixed spherical planet of mass M and radius c. Given that the gravitational attraction between two spherical bodies of masses m_1, m_2 whose centres are r apart

is $\frac{Gm_1m_2}{r^2}$ (where G is a universal constant), show that the period T of revolution of the satellite is $2\pi\sqrt{\frac{a^3}{gc^2}}$ where g is the force per unit mass at the planet's surface due to its own gravity.

7. **(i)** Show that if a polynomial equation $P(x) = 0$ has a double root $x = \alpha$ then the polynomial equation $P'(x) = 0$ has a root $x = \alpha$. Show also that the condition on a, b, c such that the cubic equation $ax^3 + bx^2 + c = 0$ has a double root is $27a^2c + 4b^3 = 0$ where a, b, c are all non-zero.
(ii) Given that
$L + m + n = -3$
$L^2 + m^2 + n^2 = 29$
$Lmn = -6$
form the cubic equation whose roots are $x = L, m, n$. Hence find the values of L, m, n.
(iii) Prove that if $x = \alpha, \beta$ are the roots of the equation $t^2 - 2t + 2 = 0$ then $\frac{(x+\alpha)^n - (x+\beta)^n}{(\alpha-\beta)} = \frac{\sin n\theta}{\sin^n \theta}$ where $\cot\theta = x + 1$.

8. **(i)** If $p + q = 1$ and $p^2 + q^2 = 2$ determine the values of $p^3 + q^3$ and $p^4 + q^4$ without finding the values of p and q.
(ii) The positive integers are bracketed as follows: (1), (2,3), (4,5,6), ... where there are r integers in the rth bracket. Prove that the sum of the integers in the rth bracket is $\frac{1}{2}r(r^2 + 1)$.
(iii) The triangle ABC has sides a, b, c in length.
(a) Given $s = \frac{1}{2}(a + b + c)$ show that $\cos\frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$ and $\sin\frac{1}{2}A = \sqrt{\frac{(s-c)(s-b)}{bc}}$.
(b) Deduce that the area of the triangle is $\sqrt{s(s-a)(s-b)(s-c)}$.



Show that the radius of a circle inscribed in the triangle is $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.