1. (a) $\displaystyle\int \frac{\sin x}{\cos^5 x}\,dx$

$$= \int \sin x \cos^{-5} x\,dx$$
$$= \tfrac{1}{4}\cos^{-4} x + c$$
$$= \tfrac{1}{4}\sec^4 x + c$$
$$\overset{\text{or}}{=} \frac{1}{4\cos^4 x} + c \qquad \boxed{1}$$

(b) $\displaystyle\int_{-2}^{-1} \frac{5}{x^2+4x+5}\,dx$

$$= \int_{-2}^{-1} \frac{5}{(x+2)^2+1}\,dx$$
$$= \Big[5\tan^{-1}(x+2)\Big]_{-2}^{-1}$$
$$= 5(\tan^{-1} 1 - \tan^{-1} 0)$$
$$= \tfrac{5}{4}\pi \qquad \boxed{3}$$

(c) (i) $\displaystyle\frac{3x^2-x+8}{(1-x)(x^2+1)} = \frac{A}{1-x} + \frac{Bx+C}{x^2+1}$

Hence $A = \dfrac{3-1+8}{1^2+1} = 5$

$3x^2 - x + 8 \equiv 5(x^2+1) + (Bx+C)(1-x)$

Hence $5 - B = 3$ and $5 + C = 8$

So $B = 2$ and $C = 3$ $\boxed{3}$

(ii) $\displaystyle\int \frac{3x^2-x+8}{(1-x)(x^2+1)}\,dx = \int \frac{5}{1-x} + \frac{2x+3}{x^2+1}\,dx$

$$= \int \frac{5}{1-x} + \frac{2x}{x^2+1} + \frac{3}{x^2+1}\,dx$$
$$= \ln\left|x^2+1\right| - 5\ln|1-x| + 3\tan^{-1} x + c$$
$$= \ln\left|\frac{x^2+1}{(1-x)^5}\right| + 3\tan^{-1} x + c \qquad \boxed{2}$$

(d) $\displaystyle\int_1^4 \frac{\ln x}{\sqrt{x}}\,dx$

$$= \Big[2\sqrt{x}\ln x\Big]_1^4 - 2\int_1^4 \frac{\sqrt{x}}{x}\,dx$$
$$= 4\ln 4 - 2\int_1^4 \frac{1}{\sqrt{x}}\,dx$$
$$= 4\ln 4 - 4\Big[\sqrt{x}\Big]_1^4$$
$$= 4(2\ln 2 - 1)\text{, as required.} \qquad \boxed{3}$$

(e) $\displaystyle\int \frac{1}{1+\cos\theta}\,d\theta$

$$= \int \frac{2}{(1+t^2)\left(1+\frac{1-t^2}{1+t^2}\right)}\,dt$$

$$= \int \frac{2}{1+t^2+1-t^2}\,dt$$

$$= \int dt$$

$$= t + c$$

$$= \tan\tfrac{\theta}{2} + c \qquad \boxed{3}$$

Let $t = \tan\frac{\theta}{2}$

Hence $\cos\theta = \dfrac{1-t^2}{1+t^2}$

Also $d\theta = \dfrac{2\,dt}{1+t^2}$

2. (a) Let $z = x + iy$, hence $z^2 = 9 - 40i = (x+iy)^2$

So $x^2 - y^2 + 2ixy = 9 - 40i$

Equate real and imaginary parts.

So $x^2 - y^2 = 9$ and $xy = -20$

Hence $x^2 - \frac{400}{x^2} = 9$

So $x^4 - 9x^2 - 400 = 0$

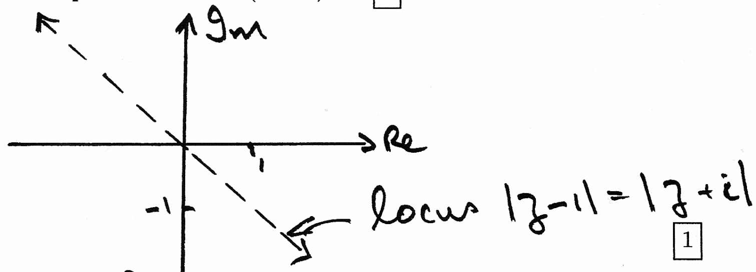
$(x^2 - 25)(x^2 + 16) = 0$

But $x \in \mathbf{R}$, so $x = \pm 5$

$x = \pm 5$ yields $y = \mp 4$

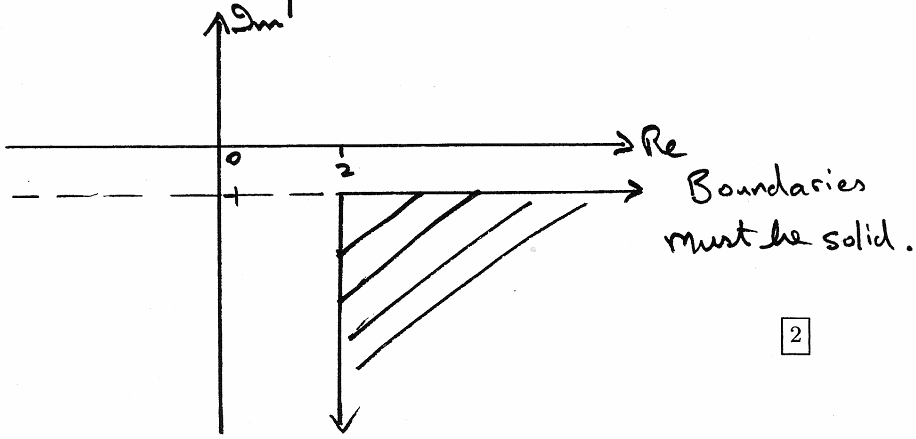
Hence the square roots are $\pm(5 - 4i)$ $\boxed{3}$

(b)



$\boxed{1}$

(c)



$\boxed{2}$

(d) (i) $\arg z = -\frac{\pi}{4}$ and $\arg w = \frac{2\pi}{3}$ $\boxed{1}$

(ii) $\arg(wz) = \arg w + \arg z = \frac{5\pi}{12}$ $\boxed{1}$

(iii) Now $wz = \sqrt{3} - 1 + i(\sqrt{3} + 1)$

Hence $\sin\frac{5\pi}{12} = \dfrac{\mathrm{Im}(wz)}{|wz|} = \dfrac{\mathrm{Im}(wz)}{|w||z|}$

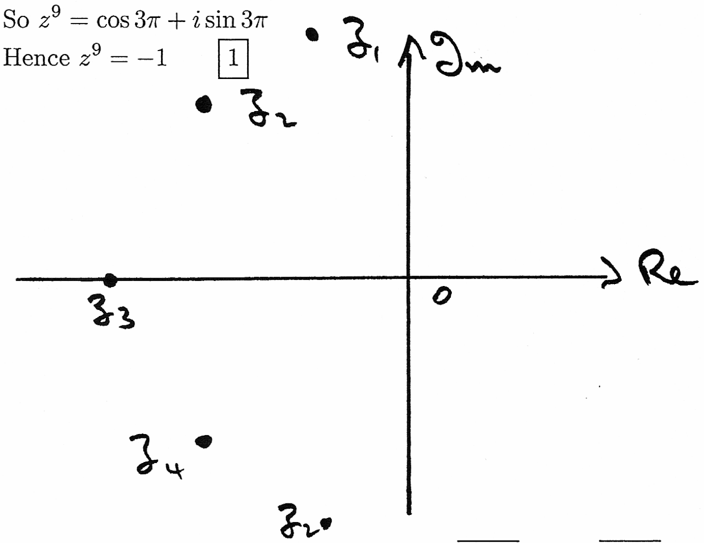
So $\sin\frac{5\pi}{12} = \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \dfrac{\sqrt{2}(1+\sqrt{3})}{4}$ as required. $\boxed{2}$

(e) (i) $z^9 = (\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})^9$

So $z^9 = \cos 3\pi + i\sin 3\pi$

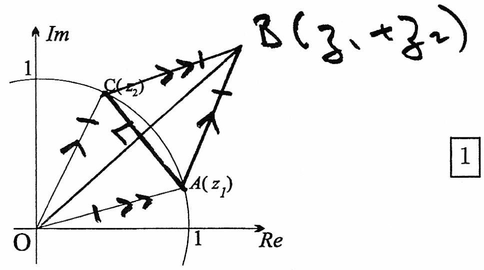
Hence $z^9 = -1$ $\boxed{1}$

(ii)



$z_1 = \text{cis}\,\frac{5\pi}{9}$, $z_2 = \text{cis}\,\frac{7\pi}{9}$, $z_3 = -1$, $z_4 = \overline{\text{cis}\,\frac{7\pi}{9}}$, $z_5 = \overline{\text{cis}\,\frac{5\pi}{9}}$. $\boxed{2}$

(f) (i)

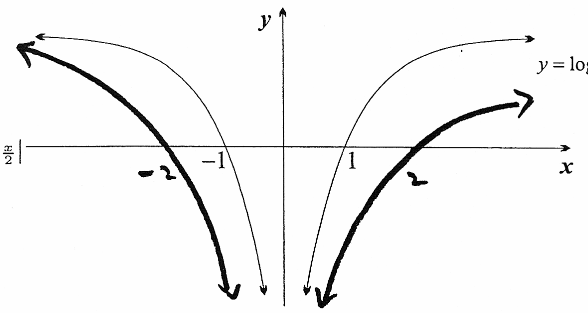


$\boxed{1}$

(ii) $OABC$ is a rhombus and hence the diagonals are perpendicular. $\boxed{1}$

3. (a) (i)

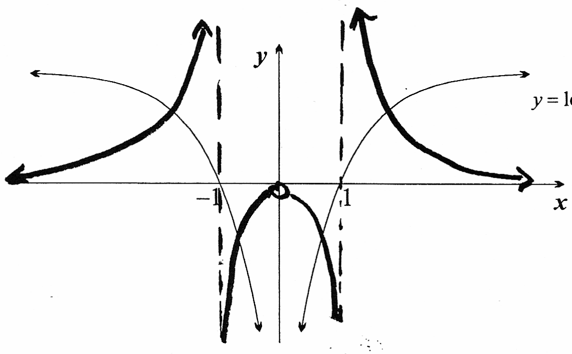
$y = f(\frac{x}{2}) = \log_e|\frac{x}{2}|$



$\boxed{1}$

(ii)

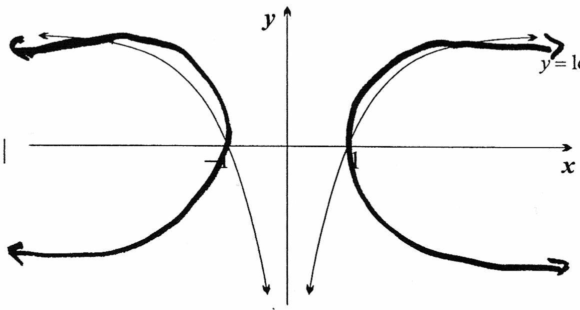
$$y = \frac{1}{f(x)} = \frac{1}{\log_e|x|}$$



$\boxed{2}$

(iii)

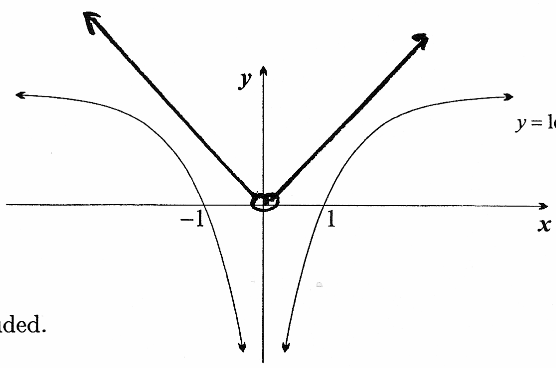
$y^2 = \log_e|x|$



$\boxed{2}$

(iv)

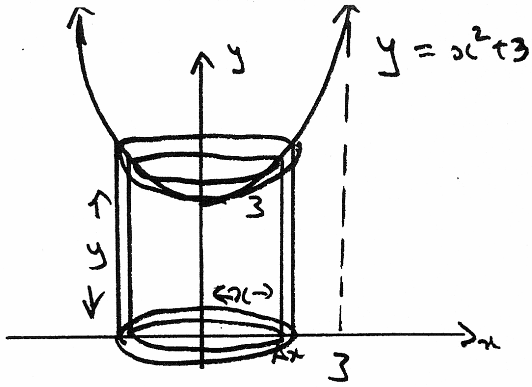
$y = e^{f(x)} = e^{\log_e|x|}$



$\boxed{1}$

Note: $(0, 0)$ is excluded.

(b)



The curved surface of each cylindrical shell is given by $SA = 2\pi xy = 2\pi(x^2+3)$.

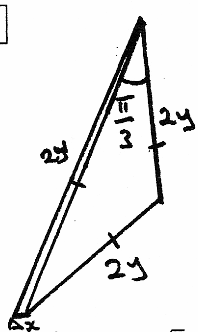
Hence the volume of a shell Δx thick is $\approx 2\pi x(x^2+3)\Delta x$.

So the volume required is $V = 2\pi \int_0^3 x^3 + 3x\,dx$.

So $V = 2\pi \left[\frac{1}{4}x^4 + \frac{3}{2}x^2\right]_0^3$

$V = \frac{135}{2}\pi \overset{\text{or}}{=} 67{\cdot}5\pi$ units^3. $\boxed{3}$

(c)



Area of each cross-sectional slice is $\frac{1}{2}(2y)^2 \sin\frac{\pi}{3} = \sqrt{3}y^2$

Hence the volume of a slice Δx thick is $\approx \sqrt{3}y^2\Delta x = \sqrt{3}(4-x^2)\Delta x$.

So the volume required is $\sqrt{3}\int_{-2}^{2} 4 - x^2\,dx$.

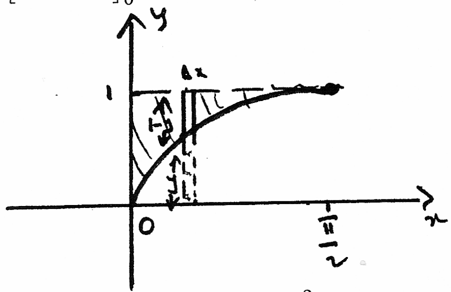
So $V = 2\sqrt{3}\int_0^2 4 - x^2\,dx$

So $V = 2\sqrt{3}\left[4x - \frac{1}{3}x^3\right]_0^2$

$= 2\sqrt{3}(8 - \frac{8}{3})$

So $V = \dfrac{32}{3}\sqrt{3}$ units^3 $\boxed{3}$

(d)



The area of each slice of the solid is $\pi(1-y)^2 = \pi(1-\sin x)^2$.

If the slice is Δx thick then the volume is $\approx \pi(1-\sin x)^2\Delta x$.

$$V = \pi \int_0^{\frac{\pi}{2}} 1 - 2\sin x + \sin^2 x \, dx$$

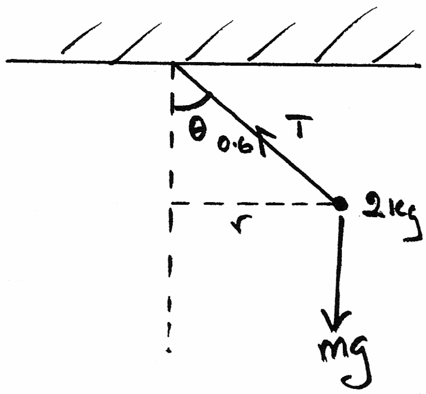
$$V = \int_0^{\frac{\pi}{2}} \tfrac{3}{2} - 2\sin x - \tfrac{1}{2}\cos 2x \, dx$$

$$V = \left[\tfrac{3}{2}x + 2\cos x - \tfrac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$V = \pi\left(\tfrac{3\pi}{4} - 2\right)$$

So $V = \dfrac{(3\pi - 8)\pi}{4}$ units^3. $\boxed{3}$

4. (a) (i)



$\boxed{1}$

(ii) Resolve forces at the mass.

$\overset{\text{vert}}{\updownarrow}$ $T\cos\theta = 2g = 20$

$\overset{\text{horoz}}{\leftrightarrow}$ $T\sin\theta = 2r\omega^2$

Hence $T\dfrac{r}{0{\cdot}6} = 2r(3\pi)^2$

So $T = 2 \times 0{\cdot}6 \times 9\pi^2$

i.e. $T = 10{\cdot}8\pi^2 \overset{\text{or}}{\approx} 106{\cdot}6\,\text{N}$ $\boxed{3}$

(iii) $\cos\theta = \dfrac{20}{T}$

So $\cos\theta = \dfrac{20}{10{\cdot}8\pi^2}$

So $\theta = 79^\circ$, to nearest $^\circ$. $\boxed{1}$

(b) (i) $\ddot{x}(t) = 0$

Hence $\dot{x} = C_1$, a constant.

But $\dot{x}(0) = V\cos\alpha = C_1$.

Hence $\dot{x}(t) = V\cos\alpha$.

So $x(t) = V\cos\alpha t + C_2$,

where C_2 is a constant.

But $x(0) = 0 = C_2$.

Hence $x(t) = V\cos\alpha t$.

Also $\ddot{y} = -g$.

So $\dot{y} = -gt + C_3$,

where C_3 is a constant.

But $\dot{y}(0) = V\sin\alpha$,

Hence $C_3 = V\sin\alpha$.

So $\dot{y} = V\sin\alpha - gt$.

So $y = V\sin\alpha t - \frac{1}{2}gt^2 + C_4$,

where C_4 is a constant.

$y(0) = 0 = C_4$.

So $y(t) = V\sin\alpha t - \frac{1}{2}gt^2$.

$\boxed{2}$

(ii) (α) $OF = FG$ hence

$V\sin\alpha t - \frac{1}{2}gt^2 = -V\cos\alpha t$

So $\frac{1}{2}gt = V\sin\alpha + V\cos\alpha$, $(t \neq 0)$

So $t = \dfrac{2V(\sin\alpha + \cos\alpha)}{g}$ seconds. $\boxed{2}$

(β) $OF = V\cos\alpha t$

So $OF = V\cos\alpha\dfrac{2V}{g}(\sin\alpha + \cos\alpha)$

So $OF = \dfrac{V^2}{g}(2\sin\alpha\cos\alpha + 2\cos^2\alpha)$

So $OF = \dfrac{V^2}{g}(\sin 2\alpha + \cos 2\alpha + 1)$ m. $\boxed{2}$

(NOTE: Numerous solutions possible. The most common are below.)

(γ) $OF = \frac{4}{3}OA$, so $\dfrac{V^2}{g}(\sin 2\alpha + \cos 2\alpha + 1) = \dfrac{4}{3}\dfrac{V^2}{g}\sin 2\alpha$

So $3\sin 2\alpha + 3\cos 2\alpha + 3 = 4\sin 2\alpha$

So $\sin 2\alpha - 3\cos 2\alpha = 3$.

$\frac{1}{\sqrt{10}}\sin 2\alpha - \frac{3}{\sqrt{10}}\cos 2\alpha = \frac{3}{\sqrt{10}}$

Hence $\sin(2\alpha - \theta) = \frac{3}{\sqrt{10}}$,

where $\cos\theta = \frac{1}{\sqrt{10}}$

and $\sin\theta = \frac{3}{\sqrt{10}}$.

If $0^o \leq \theta \leq 90^o$ then $\theta = 71^o\ 34'$ to the nearest minute.

So $2\alpha = \sin^{-1}\left(\frac{3}{\sqrt{10}}\right) + \theta = 2\theta$

That is $\alpha = \theta$.

OR Let $t = \tan\alpha$

Hence $\sin 2\alpha = \dfrac{2t}{1+t^2}$

and $\cos 2\alpha = \dfrac{1-t^2}{1+t^2}$

So $\dfrac{2t}{1+t^2} - \dfrac{3(1-t^2)}{1+t^2} = 3$

So $2t - 3 + 3t^2 = 3 + 3t^2$

So $2t = 6$

So $\tan\alpha = 3$

So $\alpha = 72^o$ to the nearest degree. $\boxed{4}$

5. (a) (i) $\tan 4\alpha = 1$

So $4\alpha = n\pi + \frac{\pi}{4}\,,\ n \in Z$

So $\alpha = (4n+1)\frac{\pi}{16}\,,\ n \in Z$ $\boxed{1}$

(ii) $(\cos\alpha + i\sin\alpha)^4 = \cos 4\alpha + i\sin 4\alpha$ (de M. th$^{\underline{m}}$).

But the binomial theorem gives

$(\cos\alpha + i\sin\alpha)^4 = \cos^4\alpha + 4i\cos^3\alpha\sin\alpha - 6\cos^2\alpha\sin^2\alpha$
$-4i\cos\alpha\sin^3\alpha + \sin^4\alpha$

Now equate the real and imaginary parts.

Hence $\sin 4\alpha = 4\cos^3\alpha\sin\alpha - 4\cos\alpha\sin^3\alpha$

and $\cos 4\alpha = \cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha$

So $\tan\alpha = \dfrac{4\cos^3\alpha\sin\alpha - 4\cos\alpha\sin^3\alpha}{\cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha}$

Hence $\tan 4\alpha = \dfrac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}$, as required. $\boxed{4}$

(iii) $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

So $4x - 4x^2 = x^4 - 6x^2 + 1$

i.e. $\dfrac{4x - 4x^3}{x^4 - 6x^2 + 1} = 1$

Let $x = \tan\alpha$

So $\dfrac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha} = 1$

Hence $\tan 4\alpha = 1$

So $\alpha = (4n+1)\frac{\pi}{16}\,,\ n \in \mathbf{Z}$

Consider the values when $n = 0\,,\ \pm 1$ and -2.

i.e. $x = \tan\frac{\pi}{16}\,,\ \tan\frac{5\pi}{16}\,,\ -\tan\frac{3\pi}{16}\,(\overset{\text{or}}{=} \tan\frac{13\pi}{16})$ or $-\tan\frac{7\pi}{16}\,(\overset{\text{or}}{=} \tan\frac{9\pi}{16})$ $\boxed{4}$

(iv) $\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16}$

$= \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$

$= (-4)^2 - 2(-6)$

$= 28$, as required. $\boxed{2}$

(b) (i) $\alpha + \beta + \gamma = 3\beta$

So $\beta = -\frac{p}{3}$

$\left(-\frac{p}{3}\right)^3 + p\left(-\frac{p}{3}\right)^2 + q\left(-\frac{p}{3}\right) + r = 0$

So $-p^3 + 3p^2 - 9pq + 27r = 0$

i.e. $2p^3 - 9pq + 27r = 0$ $\boxed{2}$

(ii) $\alpha\beta\gamma = \beta^3$

So $\beta = \sqrt[3]{-r}$

Hence $\left(\sqrt[3]{-r}\right)^3 + p\left(\sqrt[3]{-r}\right)^2$
$+q\left(\sqrt[3]{-r}\right) + r = 0$

So $-r + pr^{\frac{2}{3}} + q(-r)^{\frac{1}{3}} + r = 0$

i.e. $pr^{\frac{2}{3}} = qr^{\frac{1}{3}}$

So $p^3r^2 = q^3r$

Hence $p^3r = q^3$ $\boxed{2}$

6. (a) (i) $\angle GCD = \frac{\pi}{2} + \angle HCG = \frac{\pi}{2} + \alpha$ (Ext. $\angle \triangle CGF =$ sum of the int. opp.$\angle$'s)

Hence $\angle HCG = \alpha$, as required. $\boxed{1}$

(ii) $\angle ABD = \angle ACD = \frac{\pi}{2}$ ($\angle$'s in the same segment)

Hence $AB \perp DB$, as required. $\boxed{1}$

(iii) $\angle EAD = \alpha$ ($\angle$'s in the same segment)

$\angle ADB = \alpha$ ($\angle$'s in the same segment)

So $\angle BAD = \frac{\pi}{2} - \alpha$ ($\angle$ sum $\triangle BAD = \pi$)

Hence $\angle BAE = \alpha + \frac{\pi}{2} - \alpha = \frac{\pi}{2}$.

Hence $AB \perp AE$

So $AE \parallel BD$ (cointerior $\angle$'s are supplementary). $\boxed{2}$

(iv) $\angle BAE + \angle BGE = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ ((iii) and given $FH \perp BC$)

Hence E, A, B and G are concyclic as the opposite $\angle$'s are supplementary. $\boxed{1}$

(b) (i) $1 + \omega + \omega^2$ is a geometric series with common ratio ω.

So $1 + \omega + \omega^2 = \dfrac{\omega^3 - 1}{\omega - 1}$

But $\omega^3 = 1$

Hence $1 + \omega + \omega^2 = 0$, as required. $\boxed{1}$

(ii) $(2-\omega)(2-\omega^2)(2-\omega^4)(2-\omega^5)$

$= (2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2)$, as $\omega^3 = 1$

$= \left((2-\omega)(2-\omega^2)\right)^2$

$= (4 - 2\omega - 2\omega^2 + \omega^3)^2$

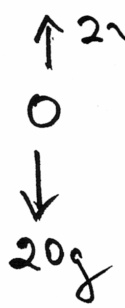
$= \left(5 - 2(\omega + \omega^2)\right)^2$

But $\omega + \omega^2 = -1$ from (i).

Hence $(2-\omega)(2-\omega^2)(2-\omega^4)(2-\omega^5) = (5+2)^2$

i.e. $(2-\omega)(2-\omega^2)(2-\omega^4)(2-\omega^5) = 49$ $\boxed{2}$

(c) (i)



Newton's 2 nd law gives:

$20\ddot{x} = 20g - 2v$

$\ddot{x} = 10 - \dfrac{v}{10}$

$\ddot{x} = \dfrac{100 - v}{10}$ $\boxed{1}$

(ii) $\ddot{x} = \dfrac{dv}{dt} = \dfrac{100 - v}{10}$

So $\displaystyle\int \frac{dv}{100 - v} = \tfrac{1}{10}\int dt$

So $-\ln|100 - v| = \frac{t}{10} + c$, for some constant c.

When $t = 0$, $v = 0$ hence $c = -\ln 100$.

So $-\dfrac{t}{10} = \ln\left|\dfrac{100 - v}{100}\right|$

So $100e^{-\frac{t}{10}} = 100 - v$

$v = 100\left(1 - e^{-\frac{t}{10}}\right)$ $\boxed{2}$

(iii) Terminal velocity attained when either $t \to \infty$ or $\ddot{x} = 0$

Hence the terminal velocity is $100\,\text{m/s}$ $\boxed{1}$

(iv) Now $\ddot{x} = \dfrac{100 - v}{10}$

So $v\dfrac{dv}{dx} = \dfrac{100 - v}{100}$

So $\dfrac{dv}{dx} = \dfrac{100 - v}{10v}$

So $\dfrac{dx}{dv} = \dfrac{10v}{100 - v} = \dfrac{1000 - 10(100 - v)}{100 - v}$

So $\displaystyle\int dx = \int \frac{1000}{100 - v} - 10\,dv$

So $x = -1000\ln|100 - v| - 10v + c$, for some constant c

But $x = 0$ when $v = 0$

So $c = 1000\ln 100$ and from (iii) $v < 100$.

So $x = 1000\left(\ln 100 - \ln(100 - v)\right) - 10v$.

So $x = 1000\ln\left(\dfrac{100}{100 - v}\right) - 10v\,\text{m}$, as required. $\boxed{2}$

(v) Let $v = 50$

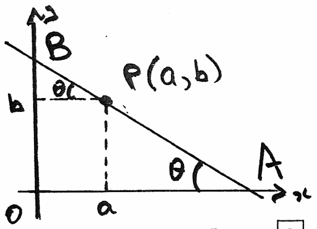
So $x = 1000\ln 2 - 500$

So $x = 500(\ln 4 - 1)$

So $x = 193{\cdot}15$

Hence the object has fallen approximately $193{\cdot}15$ metres. $\boxed{1}$

7. (a) (i)



$\boxed{2}$

$AP = b \operatorname{cosec}\theta$
and $PB = a\sec\theta$.
$AB = a\sec\theta + b\operatorname{cosec}\theta$

(ii) $\dfrac{d}{d\theta}(AB) = a\sec\theta\tan\theta - b\operatorname{cosec}\theta\cot\theta$

If $\dfrac{d}{d\theta}AB = 0$

then $a\sec\theta\tan\theta = b\operatorname{cosec}\theta\cot\theta$

So $\dfrac{\operatorname{cosec}\theta\cot\theta}{\sec\theta\tan\theta} = \dfrac{a}{b}$

So $\dfrac{\cot\theta}{\sec\theta\sin\theta\tan\theta} = \dfrac{a}{b}$

So $\dfrac{\cot\theta}{\tan^2\theta} = \dfrac{a}{b}$

So $\cot^3\theta = \dfrac{a}{b}$

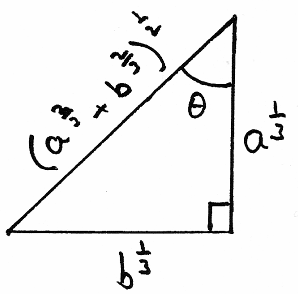
Hence $\cot\theta = \left(\dfrac{a}{b}\right)^{\frac{1}{3}}$

Now $\dfrac{d^2}{d\theta^2}AB = a\sec\theta\tan^2\theta + a\sec^3\theta + b\operatorname{cosec}\theta\cot^2\theta + b\operatorname{cosec}^3\theta$

But $0 \le \theta \le \frac{\pi}{2}$ and hence all the trigonometric functions are positive so $\dfrac{d^2}{d\theta^2}AB > 0$.

So $\cot\theta = \left(\dfrac{a}{b}\right)^{\frac{1}{3}}$ minimises AB. $\boxed{3}$

(iii) $\cot\theta = \dfrac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$ and as θ is acute we can represent θ as shown in the right triangle.



Hence $r^2 = a^{\frac{2}{3}} + b^{\frac{2}{3}}$

So $r = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}$

Hence $\sec\theta = \dfrac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})}{a^{\frac{1}{3}}}$ and $\operatorname{cosec}\theta = \dfrac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})}{b^{\frac{1}{3}}}$

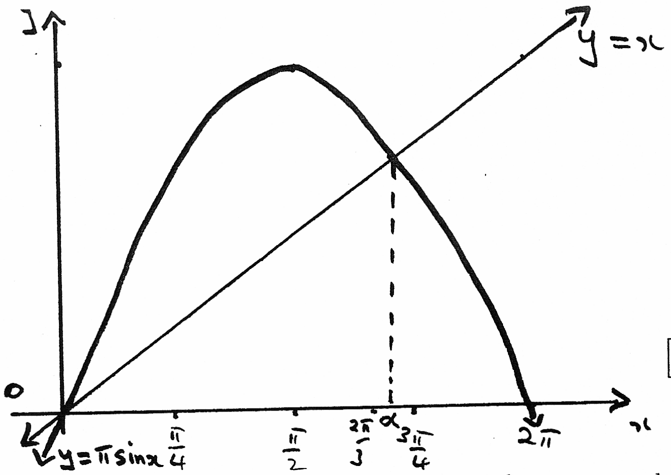
So the minimum length of AB is:

$a\dfrac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})}{a^{\frac{1}{3}}} + b\dfrac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})}{b^{\frac{1}{3}}}$

$= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}}\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)$.

Hence the minimum length of $AB = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$, as required. $\boxed{2}$

(b) (i)



$\boxed{1}$

(ii) As $y = x$ and $y = \pi \sin x$ intersect there exists some value, α say such that $\pi \sin \alpha = \alpha$.

Consider the function $g(x) = \pi \sin x - x$.

$g\left(\frac{2\pi}{3}\right) = \pi.\dfrac{\sqrt{3}}{2} - \frac{2\pi}{3}$

$= \dfrac{3\sqrt{3} - 4}{6}\pi \approx 0{\cdot}626 > 0$.

$g\left(\frac{3\pi}{4}\right) = \dfrac{\pi}{\sqrt{2}} - \frac{3\pi}{4}$

$= \frac{1}{4}(2\sqrt{2} - 3)\pi \approx -0{\cdot}135 < 0$.

So $g(x)$ being continuous between $\frac{2\pi}{3}$ and $\frac{3\pi}{4}$ and as $g\left(\frac{2\pi}{3}\right).g\left(\frac{3\pi}{4}\right) < 0$ there exists a zero α such that $\frac{2\pi}{3} < \alpha < \frac{3\pi}{4}$. $\boxed{1}$

(iii) (α) $f(-x) = \sqrt{\pi^2 - (-x)^2}\cos(-x) - (-x)\sin(-x)$

$= \sqrt{\pi^2 - x^2}\cos x - x \sin x$

Hence $f(-x) = f(x)$

That is $f(x)$ is even. $\boxed{1}$

(β) $f(0) = \pi$.

$f\left(\frac{\pi}{3}\right) = \sqrt{\pi^2 - \frac{\pi^2}{9}}.\frac{1}{2} - \frac{\pi}{3}.\frac{\sqrt{3}}{2}$

$f\left(\frac{\pi}{3}\right) = \dfrac{2\sqrt{2} - \sqrt{3}}{6}\pi \approx 0{\cdot}574$.

$f\left(\frac{\pi}{2}\right) = \sqrt{\pi^2 - \frac{\pi^2}{4}}.0 - \frac{\pi}{2}.1$

$f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$.

$f(\pi) = \sqrt{\pi^2 - \pi^2}. - 1 - \pi \sin \pi$

$f(\pi) = 0$. $\boxed{1}$

(γ) $f(\alpha) = \sqrt{\pi^2 - \alpha^2}\cos \alpha - \alpha \sin \alpha$

$= \pi\sqrt{\cos^2 \alpha}\cos \alpha - \pi \sin^2 \alpha$

But $\frac{2\pi}{3} < \alpha < \frac{3\pi}{4}$ so $\cos \alpha < 0$ hence $\sqrt{\cos^2 \alpha} = -\cos \alpha$

$= \pi(-\cos^2 \alpha - \sin^2 \alpha)$

So $f(\alpha) = -\pi$. $\boxed{1}$

(δ) $f'(x) = \frac{1}{2}\dfrac{1}{\sqrt{\pi^2 - x^2}} . -2x\cos x . - \sin x\sqrt{\pi^2 - x^2} - x\cos x - \sin x$

So $f'(x) = -\left(\dfrac{x\cos x}{\sqrt{\pi^2 - x^2}} + \sin x\sqrt{\pi^2 - x^2} + x\cos x + \sin x\,.\right)$

So $f'(\alpha) = \dfrac{-\alpha\cos\alpha}{\sqrt{\pi^2 - \alpha^2}} - \dfrac{\sqrt{\pi^2 - \alpha^2}\sin\alpha}{1} - \alpha\cos\alpha - \sin\alpha$

That is $f'(\alpha) = \sin\alpha + \pi\cos\alpha\sin\alpha - \pi\cos\alpha\sin\alpha - \sin\alpha$

So $f'(\alpha) = 0$.

Hence $x = \alpha$ is a stationary point.

Now $\frac{\pi}{2} < \frac{2\pi}{3} < \alpha < \frac{3\pi}{4}$.

So $f'\left(\frac{\pi}{2}\right) = -\left(\dfrac{\sqrt{3}}{2}\pi + 1\right) < 0$

and $f'\left(\frac{3\pi}{4}\right) = -\left(-\dfrac{3}{\sqrt{14}} + \dfrac{\sqrt{7}}{4\sqrt{2}}\pi - \dfrac{3}{4\sqrt{2}}\pi + \dfrac{1}{\sqrt{2}}\right) \approx 0{\cdot}29 > 0$

Hence $(\alpha, -\pi)$ is a minimum.

But $f(x)$ is even so $(-\alpha, -\pi)$ is a minimum.

As $f(x)$ is continuous there must be a maximum between the two minimums above. As $f(x)$ is even the only possible maximum must occur at $x = 0$. That is there is a maximum at $(0, \pi)$.

So the turning points and their nature are:

$$\begin{cases} (-\alpha, -\pi) & \text{minimum,} \\ (0, \pi) & \text{maximum,} \\ (\alpha, -\pi) & \text{minimum.} \end{cases} \qquad \boxed{3}$$

[As a matter of interest $\alpha \approx 2{\cdot}31373413208$.]

8. (a) (i) $\sin n\theta + \sin(n-2)\theta$

$= 2\sin(n-1)\theta\cos\theta$

Hence $k = n-1$. $\boxed{1}$

(ii) $I_n + I_{n-2}$

$$= \int (\sin n\theta + \sin(n-2)\theta)\sec\theta\, d\theta$$

$$= 2\int \sin(n-1)\theta\cos\theta\sec\theta\, d\theta$$

$$= 2\int \sin(n-1)\theta\, d\theta$$

$$= -\frac{2}{n-1}\cos(n-1)\theta + C \text{, for some constant } C.$$

So $I_n + I_{n-2} = \dfrac{2\cos(n-1)\theta}{1-n} + C$ as required. $\boxed{2}$

(iii) $\dfrac{\cos 5\theta \sin\theta}{\cos\theta} = \sec\theta(\tfrac{1}{2}\sin 6\theta - \tfrac{1}{2}\sin 4\theta)$.

Now $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos 5\theta\sin\theta}{\cos\theta}\, dh = \tfrac{1}{2}\Big[I_6 - I_4\Big]_0^{\frac{\pi}{2}}$

$$= \tfrac{1}{2}\Big[I_6 + I_4 - 2I_4\Big]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{2}\Big[I_6 + I_4\Big]_0^{\frac{\pi}{2}} - \Big[I_4\Big]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{2}\left[\frac{\cos 5\theta}{-5}\right]_0^{\frac{\pi}{2}} - \Big[I_4 + I_2 - I_2\Big]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{5} - \left[\frac{2\cos 3\theta}{-3}\right]_0^{\frac{\pi}{2}} + \Big[I_2\Big]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{5} - \tfrac{2}{3} + \Big[I_2 + I_0 - I_0\Big]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{5} - \tfrac{2}{3} + \left[\frac{2\cos\theta}{-1}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 0\, d\theta$$

$$= \tfrac{1}{5} - \tfrac{2}{3} + 2 - 0$$

$= \tfrac{23}{15}$ as required. $\boxed{4}$

(b) (i) $\psi(x) = a_1 + a_2 + \cdots + a_k + x - (k+1)\left(a_1 a_2 \cdots a_k x\right)^{\frac{1}{k+1}}$.

So $\psi'(x) = 1 - \left(a_1 a_2 \cdots a_k x\right)^{\frac{1}{(k+1)}-1}\left(a_1 a_2 \cdots a_k\right)$

$\psi'(x) = 1 - \left(a_1 a_2 \cdots a_k\right)^{\frac{1}{(k+1)}} x^{\frac{1}{k+1}-1}$

$\psi'(x) = 1 - \left(a_1 a_2 \cdots a_k\right)^{\frac{1}{(k+1)}} x^{-\frac{k}{k+1}}$

When $\psi'(x) = 0$ then

$\left(a_1 a_2 \cdots a_k\right)^{\frac{1}{(k+1)}} x^{-\frac{k}{k+1}} = 1$

So $x^{-\frac{k}{k+1}} = \left(a_1 a_2 \cdots a_k\right)^{-\frac{1}{k+1}}$

So $x^k = \left(a_1 a_2 \cdots a_k\right)$

Hence $\psi'(x) = 0$, when $x = \left(a_1 a_2 \cdots a_k\right)^{\frac{1}{k}} = x_0$.

Now $\psi''(x) = \left(\dfrac{k}{k+1}\right)\left(a_1 \cdots a_k\right)^{\frac{1}{k+1}} x^{-\frac{2k+1}{k+1}}$

So $\psi''(x_0) = \dfrac{k}{k+1}\left(a_1 \cdots a_k\right)^{\frac{1}{k+1}}\left(\left(a_1 \cdots a_k\right)^{\frac{1}{k}}\right)^{-\frac{2k+1}{k+1}}$

So $\psi''(x_0) = \dfrac{k}{k+1}\left(a_1 \cdots a_k\right)^{\frac{1}{k+1}-\frac{2k+1}{k(k+1)}}$

That is $\psi''(x_0) = \dfrac{k}{k+1}\left(a_1 \cdots a_k\right)^{-\frac{1}{k}} \overset{\text{or}}{=} \dfrac{k}{(k+1)G_k} > 0$, as $k, G_k > 0$.

Hence the minimum value of $\psi(x)$ occurs at $x = x_0$. $\boxed{3}$

(ii) Consider the proposition that

"if $A_n = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$ and $G_n = \sqrt[n]{a_1 a_2 \cdots a_n}$ then $A_n \geq G_n$".

Now $A_1 = a_1$ and $G_1 = \sqrt[1]{a_1} = a_1$ hence $A_1 \geq G_1$.

Hence the proposition is true for $n = 1$.

Let k be some positive integer such that the proposition is true.

That is $A_k \geq G_k$.

From (i) $\psi(a_{k+1}) \geq \psi(x_0)$.

That is $a_1 + a_2 + \cdots + a_k + a_{k+1} - (k+1)(a_1 a_2 \cdots a_{k+1})^{\frac{1}{k+1}}$

$\geq a_1 + a_2 + \cdots + a_k + G_k - (k+1)\left(a_1 a_2 \cdots a_k G_k\right)^{\frac{1}{k+1}}$.

$\left(\left(a_1 a_2 \cdots a_k G_k\right)^{\frac{1}{k+1}} = \left(\left(a_1 \cdots a_k\right)^{1+\frac{1}{k}}\right)^{\frac{1}{k+1}} = \left(\left(a_1 \cdots a_k\right)^{\frac{k+1}{k}}\right)^{\frac{1}{k+1}} = G_k\right)$

That is $(k+1)\left(A_{k+1} - G_{k+1}\right) \geq kA_k + G_k - (k+1)G_k$

So $(k+1)(A_{k+1} - G_{k+1}) \geq k(A_k - G_k) \geq 0$

Hence $A_{k+1} \geq G_{k+1}$.

As $A_1 \geq G_1$ and $A_k \geq G_k$ implies $A_{k+1} \geq G_{k+1}$ for some positive integer k then by the principle of mathematical induction $A_n \geq G_n$ for all positive integers n. $\boxed{5}$