Question 1

(a) $x_P = \dfrac{3.(-3)+1.(5)}{1+3} = \dfrac{-4}{4} = -1$

$y_P = \dfrac{3(4)+1(6)}{1+3} = \dfrac{18}{4} = 4.5$

$P\ (-1, 4.5)$

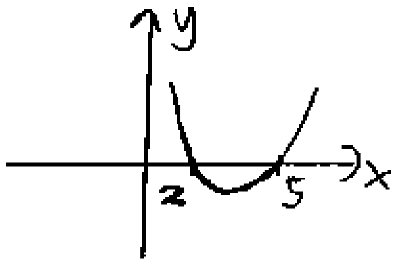
(b) $\dfrac{3}{x-2} \leq 1$

$\times (x-2)^2$: $3(x-2) \leq (x-2)^2$

$3x-6 \leq x^2-4x+4$

$0 \leq x^2-7x+10$

$0 \leq (x-2)(x-5) = y$



$y = x^2-7x+10$, hence concave up (+ve leading term)

$\therefore$ solution:

$x<2$ or $x \geq 5$

since when $x=2$, $\dfrac{3}{x-2}$ is undefined.

(c) $\lim\limits_{x\to 0} \dfrac{3x}{\sin 2x} = \lim\limits_{x\to 0} \dfrac{2x}{\sin 2x} \cdot \dfrac{3}{2}$

$= 1 \cdot \dfrac{3}{2} = \dfrac{3}{2}$

(d) $x = \dfrac{t}{2} \Rightarrow t = 2x$.

$y = 3t^2 = 3(2x)^2$

$\therefore y = 3(4x^2)$

$= 12x^2$

(e) $\displaystyle\int_0^2 \frac{x}{(x^2+1)^3}\,dx$

let $u = x^2+1$

$du = 2x\,dx$

$x=2,\ u=5$

$x=0,\ u=1$

$= \displaystyle\int_1^5 \frac{\frac{1}{2}du}{u^3}$

$= \dfrac{1}{2}\left[\dfrac{u^{-2}}{-2}\right]_1^5 = -\dfrac{1}{4}\left[\dfrac{1}{u^2}\right]_1^5$

$= -\dfrac{1}{4}\left(\dfrac{1}{25}-1\right) = \dfrac{24}{100} = \dfrac{6}{25}$

Question 2 2003 HSC ME1 p2

(a) $y = 3\cos^{-1} 2x$.

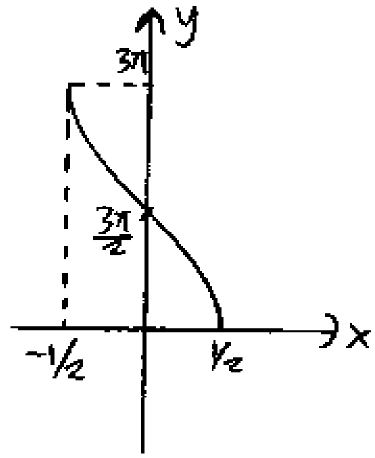
domain: $-1 \leq 2x \leq 1$

$-\frac{1}{2} \leq x \leq \frac{1}{2}$.

range: $0 \leq \cos^{-1} 2x \leq \pi$

$0 \leq 3\cos^{-1} 2x \leq 3\pi$

$0 \leq y \leq 3\pi$.



(b) $\frac{d}{dx}(x\tan^{-1}x)$

$= (\tan^{-1}x)(1) + (x)\left(\frac{1}{1+x^2}\right)$ using product rule

$= \tan x + \frac{x}{1+x^2}$

(c) $\int_0^1 \frac{1}{\sqrt{2-x^2}}\,dx$

$= \left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1$

$= \sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}0$

$= \frac{\pi}{4}$

(d) $(2+x^2)^5$ has general term

$T_k = {}^5C_k\, 2^{5-k}(x^2)^k$, or ${}^5C_k\, 2^k (x^2)^{5-k}$ but we choose the easier form.

we want $2k = 4$.

$k = 2$

$T_2 = {}^5C_2\, 2^{5-2}\, x^4$

coeff $= 80$.

(e) (i) let $\cos x - \sin x \equiv R\cos(x+\alpha)$, $R > 0$.

$\equiv R\cos x\cos\alpha - R\sin x\sin\alpha$ (compound angle)

equating coefficients of $\cos x$ and $\sin x$,

$1 = R\cos\alpha$ — (1)

$1 = R\sin\alpha$ — (2)

$(1)^2 + (2)^2$: $1 + 1 = R^2\cos^2\alpha + R^2\sin^2\alpha$

$= R^2(\cos^2\alpha + \sin^2\alpha)$

$2 = R^2$

$\therefore R = \sqrt{2}$.

$\left.\begin{array}{l} 1 = \sqrt{2}\cos\alpha \\ 1 = \sqrt{2}\sin\alpha \end{array}\right\}$ α is then in quadrant 1.

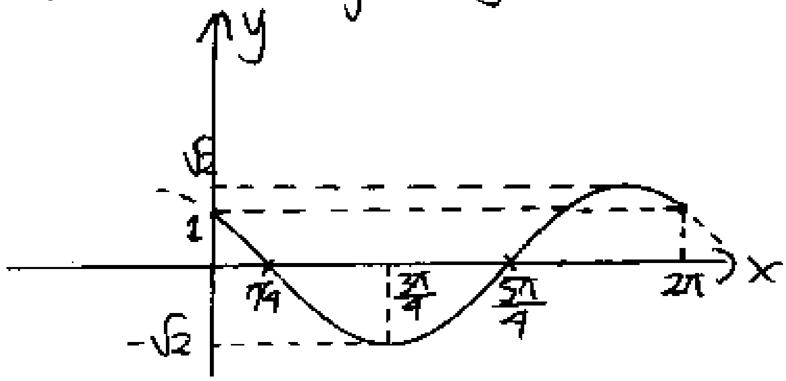
$\alpha = \pi/4$.

$\therefore \cos x - \sin x \equiv \sqrt{2}\cos(x + \pi/4)$

Question 2 2003 HSC ME1 p3

(e)(ii) $y = \sqrt{2}\cos(x + \pi/4)$.

this is $\cos x$ enlarged by $\sqrt{2}$ and shifted $\pi/4$ to left.



put $x=0$

$y = \sqrt{2}\cos\pi/4$

$= 1$

Question 3

(a) ISOSCELES. 9 letters, 3 S's, 2 E's

no of arrangements $= \dfrac{9!}{3! \times 2!} = 30240$

(b) (i) it undergoes SHM if $\ddot{x} = -kx$, $k > 0$ (acceleration is proportional to displacement but acts in opposite direction)

$x = 4\sin(2t + \frac{\pi}{3})$

$\dot{x} = 4.2\cos(2t + \frac{\pi}{3})$

$\ddot{x} = 4.2.2. -\sin(2t + \frac{\pi}{3})$

$= -4x$ as required.

(ii) amplitude $= 4$.

(iii) $\dot{x} = 8\cos(2t + \frac{\pi}{3})$

max speed is when $\cos(2t + \frac{\pi}{3}) = 1$ or -1 (speed, not velocity)

$2t + \frac{\pi}{3} = 0, \pi, 2\pi, \ldots$

$t + \frac{\pi}{6} = 0, \frac{\pi}{2}, \pi, \ldots$

the first $t > 0$ is $t = \frac{\pi}{2} - \frac{\pi}{6}$

$= \frac{\pi}{3}$

Alternatively you can go from $\ddot{x} = 0$

(c) (i) sum of 5 is when the numbers are.

$(1,4), (4,1), (2,3), (3,2) \Rightarrow$ no fav. outcomes $= 4$.

no. possible outcomes $= 6 \times 6 = 36$.

$P = 4/36 = 1/9$.

(c)(ii) P (getting it once)

$= {}^7C_1 \times \left(\frac{1}{9}\right)^1 \times \left(\frac{8}{9}\right)^6 = \frac{7.8^6}{9^7}$

P (not getting it at all)

$= {}^7C_0 \times \left(\frac{1}{9}\right)^0 \times \left(\frac{8}{9}\right)^7 = \frac{8^7}{9^7}$

P (getting it at least twice)

= 1 − P (getting it once) − P(not getting it at all)

$= \frac{9^7 - 7.8^6 - 8^7}{9^7}$

$= \frac{850\,809}{9^7}$ or 0.1778

(d) prove for n=1: $\frac{1}{1\times3} = \frac{1}{2+1}$

suppose it's true for n=k, i.e.

$\frac{1}{1\times3} + \frac{1}{3\times5} + \ldots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$ $*$

for n=k+1, $LHS = \frac{1}{1\times3} + \frac{1}{3\times5} + \ldots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$ from $*$

$= \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)}$

$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{(2(k+1)+1)} =$ the expected RHS.

∴ whenever the statement is true for n=k, it's also true for n=k+1.

But it's true for n=1

So it's true for n=1, 2, 3, ...

Question 4

(a) $^{14}C_6 = 3003$ ways

(b) $f(x) = \sin x - \frac{2x}{3}$

$f'(x) = \cos x - \frac{2}{3}$

$x_0 = 1.5$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 1.5 - \dfrac{\sin 1.5 - \frac{2(1.5)}{3}}{\cos 1.5 - \frac{2}{3}}$

$= 1.496$

(c) $2x^3 + x^2 - kx + 6$ has roots $\alpha, \frac{1}{\alpha}, \beta$.

product of roots: $\alpha \cdot \frac{1}{\alpha} \cdot \beta = -\frac{6}{2}$

$\beta = -3$

expression for k: $\alpha\left(\frac{1}{\alpha}\right) + \alpha\beta + \frac{1}{\alpha}\beta = \frac{-k}{2}$

$1 + \beta\left(\alpha + \frac{1}{\alpha}\right) = \frac{-k}{2}$

sum of roots:

$\alpha + \frac{1}{\alpha} + \beta = -\frac{1}{2}$

$\left(\alpha + \frac{1}{\alpha}\right) = 2.5 \longrightarrow 1 + (-3)(2.5) = \frac{-k}{2}$

$k = 13.$

(d) (i) $\angle CPB = \angle CQB$.

$\therefore$ angles on common chord CB are equal.

CPQB is cyclic quad.

(ii) $\angle TPA = 90°$, $\angle TQA = 90°$ } $\angle TPA + \angle TQA = 180°$

$\therefore$ opposite angles are supplementary.

PAQT is cyclic quad.

(iii) try to use parts (i) and (ii).

$\angle QCB = \angle QPB$ (angles on same chord in CPQB)

$\angle QPB = \angle QPT = \angle QAT$ (angles on same chord in PAQT)

$= \angle TAQ$.

$\therefore \angle TAQ = \angle QCB$.

(d)(iv) So far, we haven't used the fact that we have some right angles, so this last part must have something to do with them.

let $\angle TAQ = \angle QCB = \theta$.

using $\triangle AQT$, $\angle ATQ = 90° - \theta$.

$= \angle CTR$ (vertically opposite angles).

Sum of angles in $\triangle CRT$:

$\angle CTR + \angle QCB + \angle CRT = 180°$

$(90° - \theta) + \theta + \angle CRT = 180°$

$\angle CRT = 90°$

$\therefore AR \perp CB$.

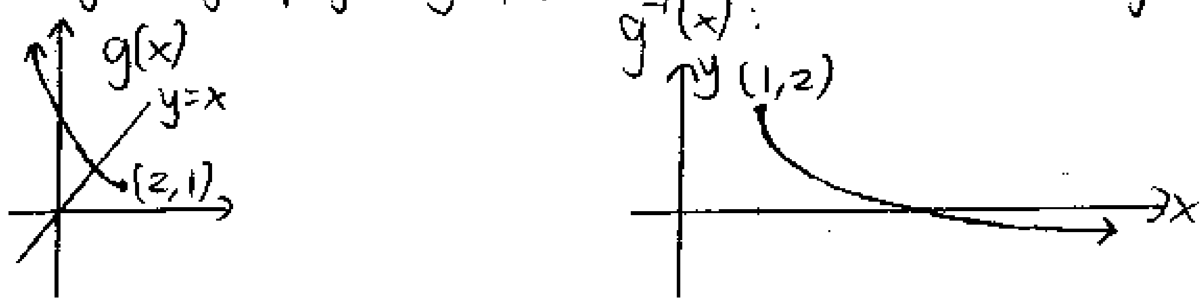
Question 5

(a) $\int \cos^2 3x \, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 6x\right) dx$

$= \frac{1}{2}x + \frac{\sin 6x}{12} + C$

(b) (i) $f(x)$ doesn't pass the horizontal line test.

(ii) inverse function is the original function reflected in $y=x$. (imagine you physically flip the whole curve about line $y=x$)



(iii) domain of $g^{-1}(x) \Rightarrow$ range of $g(x) = y \geq 1$

domain of $g^{-1}(x)$: $x \geq 1$

(iv) $g(x) \Rightarrow y = x^2 - 4x + 5$, $x \leq 2$ (domain of $g(x)$ becomes range of $g^{-1}(x)$)

$g^{-1}(x) \Rightarrow x = y^2 - 4y + 5$, $y \leq 2$

$= y^2 - 4y + 4 + 1$

$= (y-2)^2 + 1$

$(y-2)^2 = x - 1$

$y - 2 = -\sqrt{x-1}$ since $y \leq 2$

$y = 2 - \sqrt{x-1}$

(c) (i) $T = A + Be^{kt}$

$\frac{dT}{dt} = k \cdot Be^{kt}$

$= k(T-A)$ ✓

(ii) $T = 20 + Be^{kt}$ (A is given)

at $t=6$:

$80 = 20 + Be^{6k} \Rightarrow B = \frac{60}{e^{6k}}$

at $t=8$:

$50 = 20 + Be^{8k}$

$= 20 + \left(\frac{60}{e^{6k}}\right)e^{8k}$

$= 20 + 60e^{2k}$

$e^{2k} = \frac{30}{60}$

$2k = \ln \frac{1}{2} \Rightarrow k = \frac{-\ln 2}{2}$ since $\ln \frac{1}{2} = \ln 2^{-1} = -\ln 2$

$B = \frac{60}{e^{6k}} = \frac{60}{e^{-3\ln 2}}$, where $(e^{\ln 2})^{-3} = 2^{-3}$

$B = \frac{60}{2^{-3}} = 60 \cdot 2^3 = 480$

(iii) $T = 20 + 480e^{kt}$.

at $t=0$, $e^{kt} = 1$

$T_0 = 20 + 480 = 500°C$

Question 6

(a)(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 8x(x^2+4)$

$= 8x^3 + 32x$

$\frac{1}{2}v^2 = \int (8x^3 + 32x)\,dx$

$= 2x^4 + 16x^2 + C$

initially, $x=0$ and $v=8$.

$\frac{1}{2}(8)^2 = 2(0)^4 + 16(0)^2 + C \Rightarrow C = 32$

$v^2 = 4x^4 + 32x^2 + 64$.

$v = \sqrt{4x^4 + 32x^2 + 64}$, taking +ve root since initially $v > 0$.

$= \sqrt{4(x^4 + 8x^2 + 16)}$

$= \sqrt{4(x^2+4)^2} = 2(x^2+4)\ ms^{-1}$.

Question 6

(a)(ii) $\frac{dx}{dt} = 2(x^2+4)$

$\frac{dt}{dx} = \frac{1}{2} \cdot \frac{1}{4+x^2}$

$t = \frac{1}{2} \cdot \frac{1}{2} \tan^{-1}\frac{x}{2} + C$

initially $x=0 \Rightarrow C=0$.

put $x=2$: $t = \frac{1}{4}\tan^{-1}\frac{2}{2}$

$= \frac{\pi}{16}$ seconds.

(b)(i) using ΔABE: $\tan\alpha = \frac{1}{p}$

$\alpha = \tan^{-1}\frac{1}{p}$ or $\cot^{-1}p$.

Now, let $\angle EAG = \theta$

$= \angle HCF$ (isosceles ΔADC)

hence $\angle HFC = \beta$ (angle sum in Δ).

using ΔBFC, $\tan\beta = \frac{1}{q}$

$\beta = \tan^{-1}\frac{1}{q}$ or $\cot^{-1}q$

(ii) notice that $\angle DAC = \pi/4$ (45°)

so $\alpha + \beta = 3\pi/4$.

$\tan^{-1}1/p + \tan^{-1}1/q = 3\pi/4$

taking tan: $\tan(\tan^{-1}1/p + \tan^{-1}1/q) = -1$

compound angle: $\frac{1/p + 1/q}{1 - 1/p \, 1/q} = -1$

$\times pq$: $\frac{q+p}{pq-1} = -1 \Rightarrow p+q = 1-pq$.

(iii) area = area of rectangle − area of triangles

area $\Delta ABE = \frac{1}{2} \times 1 \times p$

area $\Delta CBF = \frac{1}{2} \times 1 \times q$

$\Big\} = \frac{1}{2}(p+q) = \frac{1}{2}(1-pq)$ doesn't seem to work...

from (ii), $q + pq = 1-p \Rightarrow q(1+p) = 1-p \Rightarrow q = \frac{1-p}{1+p}$.

area $= 1 - \frac{1}{2}p - \frac{1}{2}q = 1 - \frac{p}{2} - \frac{1-p}{2(1+p)} = 1 - \frac{p}{2} + \frac{(p-1)}{2(1+p)}$

Question 6.

(b) (iv) let $A = 1 - \frac{p}{2} + \frac{p-1}{2(1+p)}$

$$\frac{dA}{dp} = -\frac{1}{2} + \frac{1}{2}\left[\frac{(1+p)(1)-(p-1)(1)}{(1+p)^2}\right]$$
$$= -\frac{1}{2} + \frac{1}{2}\frac{2}{(1+p)^2}$$
$$= -\frac{1}{2} + \frac{1}{(1+p)^2}$$

put $\frac{dA}{dp} = 0$: $0 = -\frac{1}{2} + \frac{1}{(1+p)^2}$

$\times 2(1+p)^2$: $0 = -(1+p)^2 + 2$

$(1+p) = \sqrt{2}$ since $(1+p) > 0$

$p = \sqrt{2} - 1$

p	0.4	$\sqrt{2}-1$	0.5
$\frac{dA}{dp}$	+ve	0	−ve

⇐ don't calculate at all. you know it's going to be +ve, 0, −ve.

∴ maximum.

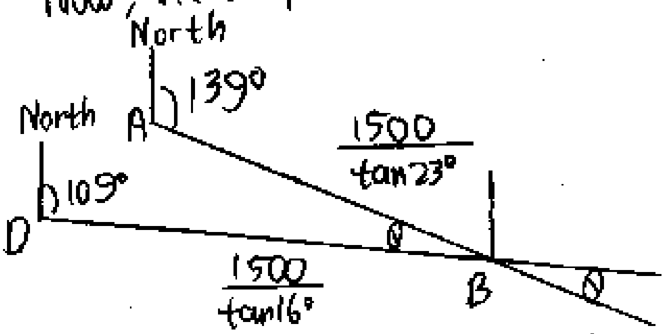
$$A = 1 - \frac{\sqrt{2}-1}{2} + \frac{\sqrt{2}-2}{2(\sqrt{2})}$$
$$= 1 - \frac{1}{2}\left(\sqrt{2}-1 - \frac{\sqrt{2}-2}{\sqrt{2}}\right)$$
$$= 1 - \frac{1}{2}\left(\sqrt{2}-1-\frac{1}{2}(2-2\sqrt{2})\right)$$
$$= 2-\sqrt{2} \text{ units}^2.$$

Question 7

(a) using ΔABH : $1500 = AB . \tan 23°$

using ΔDBH : $1500 = DB \tan 16°$

Now, view from above (bearing):



$\theta = 139° - 109°$
$= 30°$

using cosine rule in ΔDBA:

$AD^2 = \frac{1500^2}{\tan^2 23°} + \frac{1500^2}{\tan^2 16°} - 2(AB)(DB)\cos\theta$

$AD = 2798.960894$
$= 2799$ m.

using sine rule : $\frac{\sin\angle DAB}{DB} = \frac{\sin\theta}{AD}$

$\angle DAB = 69° 9'$ or $110° 51'$
$= ?$

(a) Suppose $\angle DAB = 69°9'$

we know $DB > AB > AD$

$\angle A > \angle D > \angle B$

if $\angle A = 69°9'$, sum of angles can't be $180°$

$\therefore \angle DAB = 110°51'$

bearing of D from A $= 139° + 110°51'$

$= 249°51'$

(b) (i) $\frac{dy}{dt} = v\sin\alpha - gt = 0$ at max height.

$t = \frac{v\sin\alpha}{g}$ —①

put ① to y: $y = v\left(\frac{v\sin\alpha}{g}\right)\sin\alpha - \frac{1}{2}g\left(\frac{v\sin\alpha}{g}\right)^2$

$= \frac{v^2\sin^2\alpha}{2g}$

(ii) put $y=0$: $0 = t\left(v\sin\alpha - \frac{1}{2}gt\right)$

$t = \frac{2v\sin\alpha}{g}$

$x = vt\cos\alpha$

$= \frac{2v^2\sin\alpha\cos\alpha}{g} = \frac{v^2\sin 2\alpha}{g}$

(iii) max range d is when $\alpha = \pi/4$. But there is a ceiling which can restrict it.

using (i) we put $h = (H-S)$ as the max height.

$(H-S) \geq \frac{v^2\sin^2\alpha}{2g}$ and put $\alpha = \pi/4$ then make v^2 the subject.

$v^2 \leq 4g(H-S)$

so if $v^2 \leq 4g(H-S)$ then it can be thrown at $\alpha = \pi/4$.

Otherwise, also from (i) using $h = (H-S)$,

$\sin^2\alpha = \frac{2g(H-S)}{v^2} \Rightarrow \sin\alpha = \frac{\sqrt{2g(H-S)}}{v}$

$1-\cos^2\alpha = \frac{2g(H-S)}{v^2}$

$\cos^2\alpha = 1 - \frac{2g(H-S)}{v^2} \Rightarrow \cos\alpha = \frac{\sqrt{v^2-2g(H-S)}}{v}$

from (ii), $x = \frac{v^2}{g}\sin 2\alpha = \frac{v^2}{g}.2\sin\alpha\cos\alpha$

Question 7

(iii) (continued)

when $v^2 \geqslant 4g(H-S)$,

$$d = \frac{v^2}{g} 2 \cdot \frac{\sqrt{2g(H-S)}}{v} \cdot \frac{\sqrt{v^2-2g(H-S)}}{v}$$

$$= \frac{2}{g} \sqrt{2gv^2(H-S) - 4g^2(H-S)^2}$$

$$= 2\sqrt{\frac{2v^2(H-S)}{g} - 4(H-S)^2}$$

$$= 4\sqrt{\frac{v^2(H-S)}{2g} - (H-S)^2}$$, as required.

when $v^2 \leqslant 4g(H-S)$, $\alpha = \pi/4$.

$$d = \frac{v^2}{g} \sin 2\alpha$$

$$= \frac{v^2}{g}$$, as required.