CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE FINAL TESTS 2000

MATHEMATICS

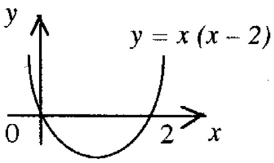
3/4 UNIT COURSE

SUGGESTED SOLUTIONS

Answers to 3 Unit Mathematics CSSA Trial 2000

Question 1.

(a)



$$x^2 \ge 2x$$
$$x^2 - 2x \ge 0$$
$$x(x-2) \ge 0$$
$$x \le 0 \text{ or } x \ge 2$$

(b) The only possible pattern is

G B G B G B G

Arrange G's in 4! ways then arrange B's in 3! ways.

Number of ways is $4! \times 3! = 24 \times 6 = 144$

(c) (i)

$y = \ln x$ $\qquad$ $y = \dfrac{e}{x}$

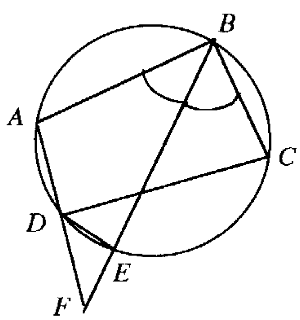
$\dfrac{dy}{dx} = \dfrac{1}{x}$ $\qquad$ $\dfrac{dy}{dx} = \dfrac{-e}{x^2}$

$m_1 = \frac{1}{e}$ $\qquad$ $m_2 = \frac{-1}{e}$

(c) (ii)

$$\tan\theta = \left| \frac{\frac{1}{e} - \left(\frac{-1}{e}\right)}{1 + \frac{1}{e}\left(\frac{-1}{e}\right)} \right|$$
$$= \frac{2e}{e^2 - 1} \qquad (\text{since } e > 1)$$

(d) (i)



(d)(ii)

$C\hat{D}E = C\hat{B}E$ (Angles in the same segment standing on arc CE are equal)

(iii)

$C\hat{B}E = A\hat{B}E$ (given BE bisects $A\hat{B}C$)

$A\hat{B}E = F\hat{D}E$ (In cyclic quad. $ABED$, exterior angle FDE is equal to opposite interior angle ABE)

$\therefore$ $F\hat{D}E = C\hat{D}E$ $\quad (C\hat{D}E = C\hat{B}E = A\hat{B}E = F\hat{D}E)$

and hence DE bisects $C\hat{D}F$.

Question 2

(a)

A $\qquad$ B

(−1, 5) $\qquad$ (5, −4)

2 : 1

$\dfrac{10-1}{2+1}$, $\dfrac{-8+5}{2+1}$ $\qquad$ $P(3,-1)$

(b) $2x^3 - 5x - 1 = 0$ has roots α, β and γ.

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma}$$
$$= \frac{-\left(\frac{5}{2}\right)}{\left(\frac{1}{2}\right)}$$
$$= -5$$

(c) (i) Series is geometric, $a = \tan x$, $r = \tan^2 x$.

$0 \le x < \frac{\pi}{4} \Rightarrow 0 \le \tan x < 1$

Hence $0 \le r < 1$ and since $|r| < 1$, limiting sum S exists.

(c) (ii)

$$S = \frac{\tan x}{1 - \tan^2 x} = \frac{1}{2} \cdot \frac{2\tan x}{1 - \tan^2 x}$$
$$\therefore \quad S = \tfrac{1}{2}\tan 2x$$

Question 2 (cont)

(d) (i)

$$\left.\begin{aligned} y = at^2 &\Rightarrow \frac{dy}{dt} = 2at \\ x = 2at &\Rightarrow \frac{dx}{dt} = 2a \end{aligned}\right\} \quad \therefore \frac{dy}{dx} = \frac{2at}{2a} = t$$

Normal at P has gradient $\frac{-1}{t}$ and equation

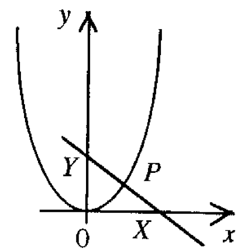
$x + ty = k$, for some constant k.

At P, $\left.\begin{aligned} x &= 2at \\ y &= at^2 \end{aligned}\right\} \Rightarrow 2at + at^3 = k$

$\therefore$ Equation is $x + ty - 2at - at^3 = 0$

(ii) At X and Y, $x + ty = 2at + at^3$

$X\left(2at + at^3, 0\right) \qquad Y\left(0, 2a + at^2\right)$



(iii) Midpt of XY is $M\left(at + \frac{1}{2}at^3, a + \frac{1}{2}at^2\right)$

Hence if P is the midpoint of XY,

$$at^2 = a + \tfrac{1}{2}at^2$$

$$\tfrac{1}{2}at^2 = a$$

$$t^2 = 2$$

$$t = \sqrt{2} \quad (\text{ since } t > 0)$$

For $t = \sqrt{2}$, $P \equiv M \equiv \left(2\sqrt{2}\,a, 2a\right)$

Question 3.

(a) (i)

$$f(x) = \frac{3x-4}{x-1} = \frac{3(x-1)-1}{x-1} = 3 - \frac{1}{x-1}$$

Throughout domain $\{x : x \neq 1\}$,

$f'(x) = \dfrac{1}{(x-1)^2} > 0$

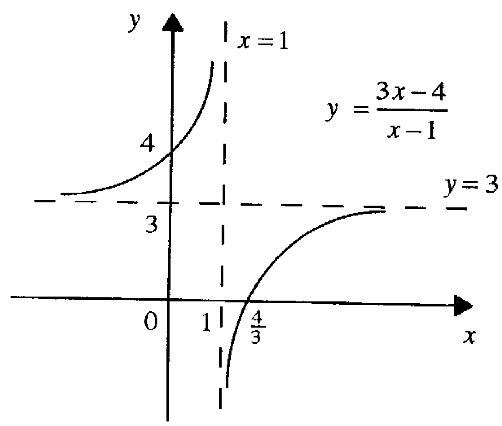
Hence f is increasing.

$$y = 3 - \frac{1}{x-1}$$

As $x \to \infty$, $y \to 3$ $\quad\therefore$ $y = 3$ is an asymptote

As $x \to 1$, $y \to \infty$ $\quad\therefore$ $x = 1$ is an asymptote

(a)(ii)



(iii)

$$mx = \frac{3x-4}{x-1}$$

$$mx^2 - mx = 3x - 4$$

$$mx^2 - (m+3)x + 4 = 0 \qquad *$$

(iv) $y = mx$ is a tangent when $*$ has equal roots and hence discriminant $\Delta = 0$.

$$\Delta = (m+3)^2 - 16m = m^2 - 10m + 9 = (m-9)(m-1)$$

$\Delta = 0 \Rightarrow m = 1$, 9

$\therefore$ tangents are

$y = x$, $y = 9x$

(b)(i) $f(x) = x^3 - kx + 1 \Rightarrow f(0) = 1$, $f(1) = 2 - k$

But if $f(x) = 0$ has exactly one root between 0 and 1, then $f(0)$, $f(1)$ have opposite signs,

$2 - k < 0$. $\quad\therefore k > 2$.

(b)(ii) $f(x) = x^3 - 3x + 1 \Rightarrow f(0\cdot3) = 0\cdot127$

$f'(x) = 3x^2 - 3 \Rightarrow f'(0\cdot3) = -2\cdot73$

$$\alpha \approx 0\cdot3 - \frac{f(0\cdot3)}{f'(0\cdot3)} = 0\cdot3 + \frac{0\cdot127}{2\cdot73} = 0\cdot35$$

Question 4

(a)(i)

Domain	Range
$-1 \le \frac{x}{2} \le 1$	$0 \le \cos^{-1}\frac{x}{2} \le \pi$
$-2 \le x \le 2$	$0 \le 3\cos^{-1}\frac{x}{2} \le 3\pi$
$\{x : -2 \le x \le 2\}$	$\{y : 0 \le y \le 3\pi\}$

(a) (ii) When $x = 0$,

$y = 3\cos^{-1}\frac{x}{2} \Rightarrow y = \frac{3\pi}{2}$

$\frac{dy}{dx} = \frac{-3}{\sqrt{4-x^2}} \Rightarrow \frac{dy}{dx} = -\frac{3}{2}$

Tangent at $\left(0, \frac{3\pi}{2}\right)$ has gradient $-\frac{3}{2}$ and equation

$3x + 2y = k$, k constant

$x = 0$, $y = \frac{3\pi}{2} \Rightarrow 3\pi = k$

$\therefore$ Tangent is $3x + 2y - 3\pi = 0$

(b)

$u = 1 + x$

$du = dx$

$x = 0 \Rightarrow u = 1$

$x = 3 \Rightarrow u = 4$

$\frac{x}{\sqrt{x+1}} = \frac{u-1}{\sqrt{u}}$

$= u^{\frac{1}{2}} - u^{-\frac{1}{2}}$

$\int_0^3 \frac{3x}{\sqrt{1+x}}\,dx$

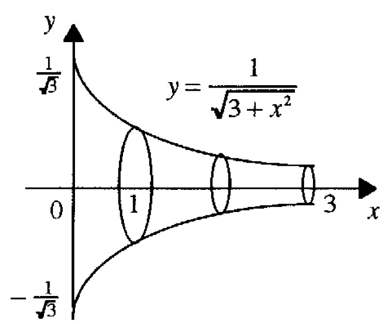
$= 3\int_1^4 u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du$

$= 3\left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^4$

$= 2(8-1) - 6(2-1)$

$= 8$

(c)



$V = \pi \int_1^3 \frac{1}{3+x^2}\,dx$

$= \frac{\pi}{\sqrt{3}}\left[\tan^{-1}\frac{x}{\sqrt{3}}\right]_1^3$

$= \frac{\pi}{\sqrt{3}}\left\{\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right\}$

$= \frac{\pi}{\sqrt{3}}\left\{\frac{\pi}{3} - \frac{\pi}{6}\right\}$

$= \frac{\pi^2\sqrt{3}}{18}$

Volume is $\frac{\pi^2\sqrt{3}}{18}$ cu. units.

Question 5

(a) Let $S(n)$ be the statement $n! > 2^n$, n=4, 5, 6 ...

Consider $S(4)$:

$4! = 24 > 16 = 2^4 \Rightarrow S(4)$ is true

If $S(k)$ is true, then $k! > 2^k$ **

Consider $S(k+1)$, $k \ge 4$:

$(k+1)! = (k+1)\,k!$

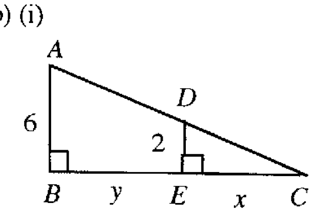
$> (k+1)\,2^k$ if $S(k)$ is true, using **

$> 2 \cdot 2^k$ since $k \ge 4 \Rightarrow k+1 > 2$

$= 2^{k+1}$

Hence if $S(k)$ is true for some $k \ge 4$, then $S(k+1)$ is true. But $S(4)$ is true, hence $S(5)$ is true and then $S(6)$ is true, and so on. Hence by Mathematical induction, $n! > 2^n$ for all integers $n \ge 4$.

(b) (i)



$\Delta ABC \,|||\, \Delta DEC$ (equiangular)

$\therefore \frac{x+y}{x} = \frac{6}{2}$

$\frac{x+y}{6} = \frac{x}{2}$

(b)(ii)

$x + y = 3x$

$2x = y$

$2\frac{dx}{dt} = \frac{dy}{dt}$

$\therefore \frac{dx}{dt} = \frac{2 \cdot 5}{2} = 1 \cdot 25$

Shadow lengthens at rate $1 \cdot 25$ ms^{-1}

Question 5 (cont)

(c) (i)

$(1+x)^n$ Coeff. of x^4 is ${}^nC_4$

Coeff. of x^2 is ${}^nC_2$

$${}^nC_4 = 6\,{}^nC_2 \Rightarrow \frac{n(n-1)(n-2)(n-3)}{4!} = \frac{6n(n-1)}{2!}$$

$\therefore\ n \geq 4$ and $(n-2)(n-3) = 72$

$n \geq 4$ and $n^2 - 5n - 66 = 0$

(c) (ii)

$n \geq 4$ and $(n-11)(n+6) = 0$

$\therefore\ n = 11$

Question 6

(a) (i)

$N = 1000 - Ae^{-kt}$

$\left.\begin{array}{l} t=0 \\ N=200 \end{array}\right\} \Rightarrow \begin{array}{l} 200 = 1000 - A.1 \\ A = 800 \end{array}$

$\frac{dN}{dt} = kAe^{-kt} = 800ke^{-kt}$

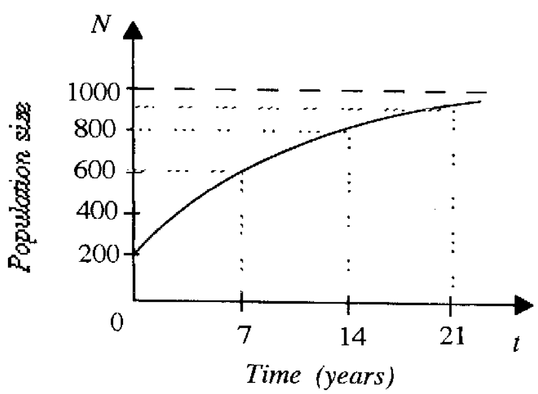
$\left.\begin{array}{l} t=0 \\ \frac{dN}{dt}=80 \end{array}\right\} \Rightarrow \begin{array}{l} 800k = 80 \\ k = 0\cdot 1 \end{array}$

(a) (ii)

$1000 - N = 800e^{-0\cdot 1t} = 800$ when $t = 0$

$1000 - N = 400 \Rightarrow \begin{cases} e^{-0\cdot 1t} = \frac{1}{2} \\ -0\cdot 1\ t = -\ln 2 \\ t = 10\ln 2 \approx 7 \end{cases}$

$\therefore$ $1000 - N$ halves every 7 years and graph has horizontal asymptote at $N = 1000$



(b)(i)

$x = a\cos(2t+\alpha)$

$t=0,\ x=4 \Rightarrow a\cos\alpha = 4$

$t=\frac{\pi}{4},\ x=-3 \Rightarrow a\cos\left(\frac{\pi}{2}+\alpha\right) = -3$

$a\sin\alpha = 3$

(ii)

$a^2\left(\cos^2\alpha + \sin^2\alpha\right) = 4^2 + 3^2$

$\therefore\ a^2 = 25 \qquad \therefore\ a = 5$

$\frac{a\sin\alpha}{a\cos\alpha} = \frac{3}{4} \Rightarrow \tan\alpha = \frac{3}{4} \qquad \therefore \alpha \approx 0\cdot 64$

(c)(i) The number of arrangements of B, B, V, V is ${}^4C_2 = \frac{4!}{2!\,2!} = 6$. Hence

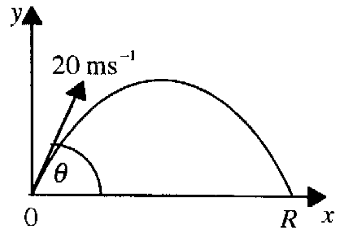
$P(B\ wins\ two\ and\ V\ wins\ two) = 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 = \frac{8}{27}$

(c) (ii) B and V must each win 2 of the first 4 sets, then B must win the 5'th set. Hence

$P(B\ wins\ three\ and\ V\ wins\ two) = \frac{8}{27} \times \frac{2}{3} = \frac{16}{81}$

Question 7

(a)(i)



Initial conditions

$x = 0 \qquad \dot{x} = 20\cos\theta$

$y = 0 \qquad \dot{y} = 20\sin\theta$

Horizontal motion

$\ddot{x} = 0$

$\dot{x} = c_1, \quad c_1$ constant

when $t = 0, \ \dot{x} = 20\cos\theta$

$\therefore \ c_1 = 20\cos\theta$

$\therefore \ \dot{x} = 20\cos\theta$

$x = 20t\cos\theta + c_2$

when $t = 0, \ x = 0$

$\therefore \ c_2 = 0$

$\therefore \ x = 20t\cos\theta$

Vertical motion

$\ddot{y} = -10$

$\dot{y} = -10t + c_3, \quad c_3$ constant

when $t = 0, \ \dot{y} = 20\sin\theta$

$\therefore \ c_3 = 20\sin\theta$

$\therefore \ \dot{y} = 20\sin\theta - 10t$

$y = 20t\sin\theta - 5t^2 + c_4$

when $t = 0, \ y = 0$

$\therefore \ c_4 = 0$

$\therefore \ y = 20t\sin\theta - 5t^2$

(a)(ii) $x = R$ when $y = 0$

$20t\sin\theta - 5t^2 = 0$

$5t(4\sin\theta - t) = 0$

$\therefore$ for $y = 0, \ t = 4\sin\theta$

and $x = 20t\cos\theta$

$= 40(2\sin\theta\cos\theta)$

$\therefore R = 40\sin 2\theta$

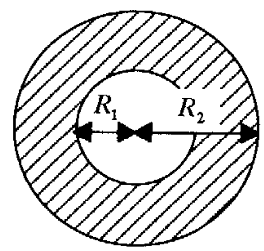
(a) (iii) If the horizontal range R varies so that $R_1 \le R \le R_2$, then the area watered is shaded in the diagram.

$15° \le \theta \le 45°$

$30° \le 2\theta \le 90°$

$0\cdot5 \le \sin 2\theta \le 1$

$20 \le R \le 40$



Shaded area is $\pi(40^2 - 20^2)$

Area watered is 1200π m^2.

(b) (i)

$$v = \tfrac{1}{2}(1 - x^2)$$

$$\frac{dv}{dx} = -x \qquad \therefore \ a = v\frac{dv}{dx} = \frac{x^3 - x}{2}$$

(b) (ii)

$$\frac{1}{1+x} + \frac{1}{1-x} = \frac{(1-x)+(1+x)}{(1+x)(1-x)} = \frac{2}{1-x^2}$$

$$\frac{dx}{dt} = \frac{1-x^2}{2} \ \Rightarrow \ \frac{dt}{dx} = \frac{2}{1-x^2}$$

$$\frac{dt}{dx} = \frac{1}{1+x} + \frac{1}{1-x}$$

$$t = \ln(1+x) - \ln(1-x) + c$$

when $t = 0, \ x = 0 \ \therefore \ c = 0$

$$\therefore \ t = \ln\frac{1+x}{1-x}$$

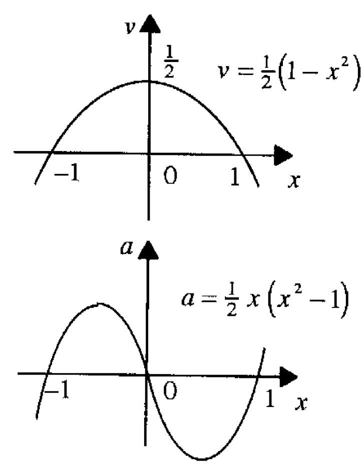
$$\frac{1+x}{1-x} = e^t$$

$$1 + x = e^t - xe^t$$

$$x(e^t + 1) = e^t - 1$$

$$\therefore \quad x = \frac{e^t - 1}{e^t + 1} = \frac{1 - e^{-t}}{1 + e^{-t}}$$

(b) (iii)



Initially the particle is at O, moving right at speed of $0\cdot5$ ms^{-1} and slowing down (since v and a have opposite signs for $0 < x < 1$).

The particle continues to move right while slowing down for $x < 1$. As $t \to \infty$, $x \to \dfrac{1-0}{1+0} = 1$.

Its limiting position is 1 m to the right of O.