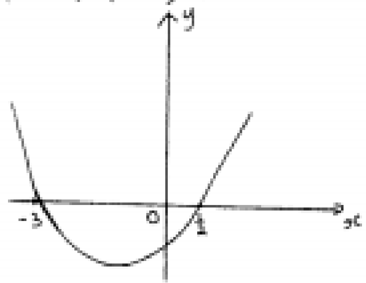
3 Unit Trial [illegible]
Solutions

1(a) $2x^2 + 5x - 3 \geq 0$

$(2x-1)(x+3) \geq 0$



Soln: $x \leq -3$ or $x \geq \frac{1}{2}$ (2)

(b) $\frac{4x-1}{x-2} \leq 3$ $\quad x \neq 2$

let $\frac{4x-1}{x-2} = 3$

$4x - 1 = 3x - 6$

$x = -5$

Critical points:

Sub $x = 0$: $\frac{0-1}{0-2} \leq 3$ T

Soln: $-5 \leq x < 2$

(3)

$\int_0^{\frac{\pi}{4}} \cos x \sin^2 x \, dx$

let $u = \sin x$

$\frac{du}{dx} = \cos x$

When $x = \frac{\pi}{4}$, $u = \frac{1}{\sqrt{2}}$

$x = 0$, $u = 0$

$\int_0^{\frac{\pi}{4}} \cos x \sin^2 x \, dx = \int_0^{\frac{1}{\sqrt{2}}} u^2 \frac{du}{dx} dx$

$= \int^{\frac{1}{\sqrt{2}}} \ldots$

$= \left[\frac{u^3}{3}\right]_0^{\frac{1}{\sqrt{2}}}$

$= \frac{1}{3}\left[\frac{1}{2\sqrt{2}} - 0\right]$

$= \frac{1}{6\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2}}{12}$ (3)

(d) $\int x^2\sqrt{1+3x^3}\, dx$ $\quad u = 1 + 3x^3$, $\frac{du}{dx} = 9x^2$

$= \frac{1}{9}\int u^{\frac{1}{2}} \frac{du}{dx} dx$

$= \frac{1}{9}\int u^{\frac{1}{2}} du$

$= \frac{1}{9} \cdot \frac{2}{3} u^{3/2} + c$

$= \frac{2}{27} u^{3/2} + c$

$= \frac{2}{27}\sqrt{(1+3x^3)^3} + c$ (2)

(e) $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$ $\quad m_1 = 2$, $m_2 = -\frac{3}{2}$

$= \left|\frac{-\frac{3}{2} - 2}{1 + 2 \cdot -\frac{3}{2}}\right|$

$= \frac{-\frac{7}{2}}{-2}$

$\tan\theta = \frac{7}{4}$

$\theta = 60°15'$ to nearest minute

(2)

2(a) $P(6, 9)$ $\quad -3:2$ $\quad A(1, 4)$

$x = \frac{mx_2 + nx_1}{m+n}$ $\quad y = \frac{my_2 + ny_1}{m+n}$

$6 = \frac{-3x_2 + 2.1}{-1}$ $\quad 9 = \frac{-3y_2 + 2.4}{-1}$

$-3x_2 + 2 = -6$ $\quad -9 = -3y_2 + 8$

$-3x_2 = -8$ $\quad 3y_2 = 17$

$x_2 = 2\frac{2}{3}$ $\quad y_2 = 5\frac{2}{3}$

$B: (2\frac{2}{3}, 5\frac{2}{3})$ (2)

(b) Prove $\frac{2\cos A}{\operatorname{cosec} A - 2\sin A} = \tan 2A$

LHS $= \frac{2\cos A}{\frac{1}{\sin A} - 2\sin A}$

$= \frac{2\cos A}{\frac{1 - 2\sin^2 A}{\sin A}}$

$= \frac{2\sin A\cos A}{1 - 2\sin^2 A}$

$= \frac{\sin 2A}{\cos 2A}$

$= \tan 2A$

$=$ RHS

$\therefore \frac{2\cos A}{\operatorname{cosec} A - 2\sin A} = \tan 2A$ (3)

c. Prove: $\sum_{r=1}^{n} \frac{1}{(3r-2)(3r+1)} = \frac{n}{3n+1}$

Step 1. Let $n = 1$, LHS $= \frac{1}{(3-2)(3+1)} = \frac{1}{4}$

RHS $= \frac{1}{3.1+1} = \frac{1}{4}$

$\therefore$ Formula true for $n = 1$

Step 2. Assume true for $n = k$

i.e. $S_k = \frac{1}{(3-2)(3+1)} + \frac{1}{(6-2)(6+1)} + \frac{1}{(9-2)(9+1)} + \ldots$

Show true for S_{k+1}

$S_{k+1} = S_k + \frac{1}{[3(k+1)-2][3(k+1)+1]}$

$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$

$= \frac{3(3k+4)+1}{(3k+1)(3k+4)}$

$= \frac{3k^2+4k+1}{(3k+1)(3k+4)}$

$= \frac{(k+1)(3k+1)}{(3k+1)(3k+4)}$

$= \frac{k+1}{3(k+1)+1}$

Hence, if it is true for $n = k$, it is true for $n = k+1$

Step 3

If it is true for $n = 1$ and it is true for $n = 2$ and so on. Hence it is true for n, a positive integer

(d) $x^2 - x - 2 = (x+1)(x-2)$

$P(x) = x^4 + 3x^3 + ax^2 \ldots$

$P(-1) = 1 - 3 + a + 2$

$= a - b$

Now $a - b = 0$ or $a = \ldots$

$P(2) = 16 + 24 + 4a - 4 \ldots$

$= 4a - b + 36$

Now $4a - b + 36 = 0$ or $4a \ldots$

From ① $4a - a = -36$

$3a = -36$

$\frac{dx}{dt} : 16 \ldots$

a) $V = \frac{4}{3}\pi r^3$ $\quad \frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$

$= \frac{1}{4\pi r^2} \cdot 10$

$\therefore \frac{dr}{dt} = \frac{5}{2\pi r^2}$

$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ $\quad A = 4\pi r^2$, $\frac{dA}{dr} = 8\pi r$

$= 8\pi r \cdot \frac{5}{2\pi r^2}$

$= \frac{20}{r}$

$= 2\cdot5$ when $r = 8$ (3)

Rate of increase $= 2\cdot5\ cm^2/s$.

(b) $P(x) = x^3 - 6x^2 + 3x + 10$

$\alpha + \beta + \gamma = 6$

$\alpha\beta + \beta\gamma + \alpha\gamma = 3$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$= 36 - 2 \times 3$

$= 30$ (3)

3) (i) $\dot{x} = \sqrt{10}\ \sin(2t+\alpha)$

$t = 0$, $x = \sqrt{10}\sin\alpha$

$= \sqrt{10} \times \frac{1}{\sqrt{10}}$ (1)

Initial displacement $= 1$

(ii) Let $\sqrt{10}\sin(2t+\alpha) = 0$

$\sin(2t+\alpha) = 0$

$2t + \alpha = 0, \pi, 2\pi, \ldots$

$2t = -\alpha,\ \pi - \alpha,\ 2\pi - \alpha$

$t = \frac{\pi}{2} - \frac{\alpha}{2},\ \pi - \frac{\alpha}{2}$

$\therefore$ Time lapse $= \frac{\pi}{2}$ s $\quad 1.57$

(iii) $-1 \leq \sin(2t+\alpha) \leq 1$

$-\sqrt{10} \leq \sqrt{10}\sin(2t+\alpha) \leq \sqrt{10}$

Min. displacement $-\sqrt{10}$ cm

Max. $\sqrt{10}$ cm

OR

amplitude $\sqrt{10}$

Min: $\sqrt{10}$ ... Max: ...

(iv) $\frac{dx}{dt} = 2\sqrt{10}\ \cos(2t+\alpha)$

$\frac{d^2x}{dt^2} = -4\sqrt{10}\ \sin(2t+\alpha)$

Let $x = 2$, $\sqrt{10}\sin(2t+\alpha) = 2$

$\sin(2t+\alpha) = \frac{2}{\sqrt{10}}$

$a = -4\sqrt{10}\ \sin(2t+\alpha)$

$= -4\sqrt{10} \cdot \frac{2}{\sqrt{10}}$

$= -8\ cm/s^2$ (6)

OR

$\ddot{x} = -n^2x$

$= -4 \times 2$

$= -8$

4 (a) (i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

$= \frac{2ap}{2a}$ at $P(2ap, ap^2)$

$= p$

$\therefore$ gradient tangent $= p$

gradient normal $= -\frac{1}{p}$

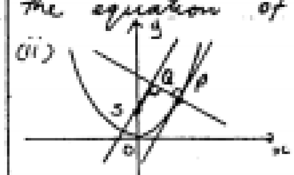
$y - y_1 = m(x - x_1)$

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$\therefore py + x = 2ap + ap^3$ is the equation of the normal.

(ii)



gradient of $SQ = p$

$y - y_1 = m(x - x_1)$

$y - a = p(x - 0)$

$\therefore$ Equation of SQ: $px - y = -a$

(iii) $py + x = 2ap + ap^3$ — ①

$px - y = -a$ — ②

From ②: $y = px + a$ ... sub into ①

$p(px + a) + x = 2ap + ap^3$

7(a) $f(x) = 2x^3 + x - 4$

$f(1) = 2 + 1 - 4 < 0$

$f(2) = 16 + 2 - 4 > 0$

$\therefore$ root bet. $x = 1, x = 2$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f'(x) = 6x^2 + 1$

$f'(1\tfrac{1}{2}) = \ldots$

$f(1\tfrac{1}{2}) = \ldots$

$x_1 = \frac{3}{2} - \ldots$

$= \ldots = 1.18$ (2)

(iv) G: $x = 0$, $py + x = 2ap + ap^3$

$py = 2ap + ap^3$

$y = 2a + ap^2$

G: $(0, a(2 + p^2))$

Midpoint of PG:

$x = \frac{2ap + 0}{2}$, $y = \frac{ap^2 + a(2+p^2)}{2}$

$\therefore x = ap$ $\quad = \frac{a(p^2 + 2 + p^2)}{2}$

$y = a(p^2 + 1)$

$\therefore$ the coords. of Q.

$\therefore$ Q is the midpoint of PG (9)

5 (a) $\int \frac{1}{9 + 4x^2}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{9}{4} + x^2\right)}\,dx$

$= \frac{1}{4} \cdot \frac{2}{3} \tan^{-1}\frac{2x}{3} + C$

$= \frac{1}{6}\tan^{-1}\frac{2x}{3} + C$ (2)

(b) (i) Let $y = x\sin^{-1}x + \sqrt{1 - x^2}$

$= x\sin^{-1}x + (1 - x^2)^{\frac{1}{2}}$

$\frac{dy}{dx} = \sin^{-1}x + x\frac{1}{\sqrt{1-x^2}} + \frac{1}{2}(1 - x^2)^{-\frac{1}{2}} \cdot -2x$

$= \sin^{-1}x$

The derivative of $\cos^{-1}x$ is $\frac{-1}{\sqrt{1-x^2}}$. Using the result above, $\frac{d}{dx}(x\cos^{-1}x - \sqrt{1-x^2})$ must be $\cos^{-1}x$

Hence the primitive of $\cos^{-1}x$ is $x\cos^{-1}x - \sqrt{1-x^2}$

(ii) (α) If $y = \sin^{-1}x$, $\sin y = x$

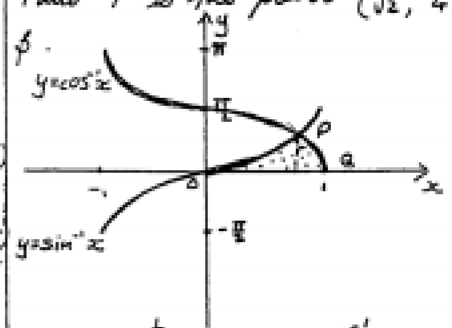
If $y = \cos^{-1}x$, $\cos y = x$

At P: $\sin y = \cos y$

$x = \sin y = \cos y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Thus P is the point $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$

β.



Area $= \int_0^{\frac{1}{\sqrt{2}}} \sin^{-1}x\,dx + \int_{\frac{1}{\sqrt{2}}}^{1} \cos^{-1}x\,dx$

$= \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^{\frac{1}{\sqrt{2}}} + \left[x\cos^{-1}x - \sqrt{1-x^2}\right]$

$= \left[\frac{1}{\sqrt{2}} \cdot \frac{\pi}{4} + \sqrt{1 - \frac{1}{2}} - \sqrt{1}\right] + \left[1.0 - \sqrt{0} - \frac{1}{\sqrt{2}}\frac{\pi}{4} + \ldots\right]$

$= 2 \cdot \frac{1}{\sqrt{2}} - 1$

$= \sqrt{2} - 1$ units^2

6 (a) (i) $t = 2x^2 - 5x + 3$

$\frac{dt}{dx} = 4x - 5$

$\therefore \frac{dx}{dt} = \frac{1}{4x - 5}$

i.e., $v = \frac{1}{4x - 5}$

(ii) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{1}{2}v^2 = \frac{1}{2}\left(\frac{1}{4x-5}\right)^2$

$= \frac{1}{2}(4x - 5)^{-2}$

$a = -2 \cdot \frac{1}{2}(4x - 5)^{-3}$

$= \frac{-4}{(4x - 5)^3}$

(iii) $t = 6$: $2x^2 - 5x + 3 = 6$

$2x^2 - 5x - 3 = 0$

(iii) cont.

$\frac{1}{4x-5} \neq 0$

particle does not change direction. As particle is initially at 1.5cm to the right and moves away from the origin $x > 3$ only

At $x = 3$, $v = \frac{1}{4.3 - 5}$

$\therefore$ velocity $= \frac{1}{7}$ cm s^{-1} when $t = 6$ s.

(5)

(i) $\ddot{x} = 0$ $\qquad \ddot{y} = -g$

$\dot{x} = V\cos\alpha$ $\qquad \dot{y} = -gt + V\sin\alpha$

$x = V\cos\alpha\, t$ $\qquad y = -\frac{1}{2}gt^2 + V\sin\alpha\, t$

$t = \frac{x}{V\cos\alpha}$, sub for y

$\therefore y = -\frac{1}{2}\frac{gx^2}{V^2\cos^2\alpha} + \frac{Vx\sin\alpha}{V\cos\alpha} + 2$

$= x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha} + 2$

$= x\tan\alpha - \frac{gx^2}{2V^2}(1+\tan^2\alpha) + 2$

$V = 21$, $g = 9.8$

$y = x\tan\alpha - \frac{9.8}{2\times 21^2}x^2(1+\tan^2\alpha) + 2$

$= x\tan\alpha - \frac{x^2}{90}(1+\tan^2\alpha) + 2$

(ii) Sub $x = 15$, $y = 12$ into equation of path

$12 = -\frac{15^2}{90}(1+\tan^2\alpha) + 15\tan\alpha + 2$

$900 = -225 - 225\tan^2\alpha + 1350\tan\alpha$

$1125 = -225\tan^2\alpha + 1350\tan\alpha$

$225(\tan^2\alpha - 6\tan\alpha + 5) = 0$

$\tan\alpha = 1$ or $\tan\alpha = 5$

$\alpha = 45°$ $\qquad \alpha = 78°41'$

But $\alpha < 50°$ (condition)

$\therefore \alpha = 45°$

(iii) For $\alpha = 45°$, $y = \frac{-x^2}{90}(1+1) + x + 2$

$= \frac{-x^2}{45} + x + 2$

For $x = 36$, $y = \frac{-36^2}{45} + 36 + 2 = 9.2$ m

$\therefore$ C is 9.2m above the ground.

(iv) $\frac{dy}{dx} = 0$ for max. height

i.e. $\frac{-2x}{45} + 1 = 0$, $x = 22\frac{1}{2}$

$\frac{d^2y}{dx^2} < 0$ for all x

$\therefore$ Max when $x = 22\frac{1}{2}$

i.e. Max. height $= 13.25$ m. (7)

7.(a)

$t(x) = \frac{\sqrt{x^2+9}}{80} + \frac{9-x}{100}$ (6)

(b) $t(x) = \frac{(x^2+9)^{\frac{1}{2}}}{80} + \frac{9-x}{100}$

$t'(x) = \frac{\frac{1}{2}(x^2+9)^{-\frac{1}{2}} \cdot 2x}{80} - \frac{1}{100}$

$= \frac{x}{80\sqrt{x^2+9}} - \frac{1}{100}$

Let $t'(x) = 0$, $\frac{x}{80\sqrt{x^2+9}} = \frac{1}{100}$

$10x = 8\sqrt{x^2+9}$

$5x = 4\sqrt{x^2+9}$

$25x^2 = 16x^2 + 144$

$9x^2 = 144$

$x^2 = \frac{144}{9}$

$x = 4$ $\quad (x > 0)$

(6)

$t(x) = \frac{\sqrt{16+9}}{80} + \frac{9-4}{100}$

$= \frac{5}{80} + \frac{5}{100}$

$= \frac{9}{80}$ h

$= \frac{9}{80} \times 60$

x	3	4	5
$t'(x)$	-	0	+
Min	when $x = 4$		