SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2002
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Sample Solutions

Question 1

a) $\displaystyle\int_{-2}^{2} \frac{dx}{\sqrt{16-x^2}} = \left[\sin^{-1}\frac{x}{4}\right]_{-2}^{2}$ (1)

$= \sin^{-1}\left(\tfrac{1}{2}\right) - \sin\left(-\tfrac{1}{2}\right)$

$= \frac{\pi}{6} + \frac{\pi}{6}$ (1)

$= \frac{\pi}{3}$

(b) let $y = e^{x+1}$

inverse $x = e^{y+1}$

$\Rightarrow \log_e x = y+1$

ie $y = \log_e x - 1$ (1)

$\therefore f^{-1}(x) = \log_e x - 1$

Now $f(x) = e^{x+1}$

$\Rightarrow f\left[f^{-1}(x)\right] = e^{f^{-1}(x)+1}$

$= e^{\log_e x - 1 + 1}$

$= e^{\log_e x}$ (1)

$= x$

and

$f^{-1}(x) = \log_e x - 1$

$\Rightarrow f^{-1}(f(x)) = \log_e\left[f(x)\right] - 1$ (1)

$= \log_e\left[e^{x+1}\right] - 1$

$= (x+1)\log_e e - 1$

$= x+1 - 1$

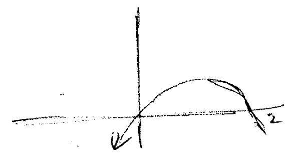
$= x$

(c) $\dfrac{4-x}{x} \le 1$

$x(4-x) \le x^2$

$4x - 2x^2 \le 0$ (2)

$2x - x^2 \le 0$



$\therefore$ ~~x≤0 or x≥2~~

$x \le 0$ or $x \ge 2$

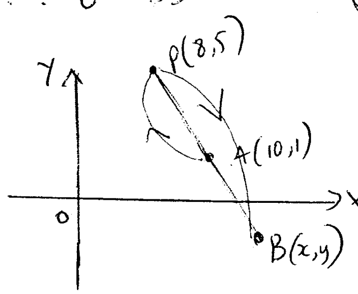
(d) $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\dfrac{\frac{1}{2} + \left(-\frac{1}{\sqrt{3}}\right)}{1 + \frac{1}{2}\left(-\frac{1}{\sqrt{3}}\right)}\right|$ (1)

$= \left|\dfrac{\sqrt{3}+2}{2\sqrt{3}-1}\right|$

$\therefore \theta =$ ~~20°6'~~ 0.99^c (1)

(e)



$AP : PB = 2:3$

$A(x_1, y_1) \quad B(x_2, y_2) \qquad m:n$

ie $A(10,1) \quad B(x_2, y_2)$ (1) $-2:3$

(1) $\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}\right) \equiv (8,5)$ (1)

ie $\left(\dfrac{-2x_2 + 30}{1}, \dfrac{-2y_2 + 3}{1}\right) \equiv (8,5) \quad \therefore B(-11,-1)$

Question 2

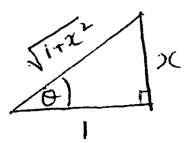
(a) $y = \tan^{-1}(\cot x)$

$$\frac{dy}{dx} = \frac{1}{1+(\cot^2 x)} \cdot \frac{d(\cot x)}{dx} \quad 1$$

$$= \frac{1}{1+\cot^2 x} \cdot \frac{-1}{\sin^2 x} \quad 1$$

$$= \frac{-\operatorname{cosec}^2 x}{1+\cot^2 x} = \frac{-\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x} = -1$$

(b) let $\tan^{-1} x = \theta \Rightarrow \tan\theta = x$



2

$$\sin\theta = \frac{x}{\sqrt{1+x^2}}$$

$$\therefore \theta = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \tan^{-1} x$$

(c) $P(x) = ax^3 + bx^2 - 8x + 3$

$P(1) = a + b - 8 + 3 = 0$ ½

$\therefore a + b = 5$ — (1)

$P(-2) = -8a + 4b + 16 + 3 = 15$ ½

$\therefore -8a + 4b = -4$ — (2)

$\Rightarrow a = 2 \quad b = 3$ 1

(d) $$\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2}$$

$$\therefore \frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1-\ln x}{x^2} \quad 1$$

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right) + \frac{1}{x^2} = \frac{2-\ln x}{x^2}$$

$$\text{ie } \frac{d}{dx}\left[\frac{\ln x}{x} - \frac{1}{x}\right] = \frac{2-\ln x}{x^2}$$

$\therefore$ primitive of $\frac{2-\ln x}{x^2}$ is 1

$$\frac{\ln x}{x} - \frac{1}{x} \text{ or } \frac{\ln x - 1}{x}$$

(e) $^5C_1 \times {}^3C_1 \times {}^6C_1 \times 3! = 540$ 2

(f) ½ $S\hat{R}P = S\hat{Q}P$ (angles on same arc SP at cir.

½ $B\hat{R}P = B\hat{A}P$ (angles on sam arc BP at circ

$\Rightarrow S\hat{Q}P = B\hat{A}P$ 1 which are corresp. angles formed by line SQ and BA, transversal QA

Since corresp. angles equal, lines SQ and BA must be parallel

Question 3

$$p(n): \sum_{r=1}^{n} a^{-r} = \frac{a^n - 1}{(a-1)a^n}$$

$p(1)$: Test for $n=1$

$$\text{LHS} = \frac{1}{a} \qquad \text{RHS} = \frac{a-1}{(a-1)a}$$

$$= \frac{1}{a}$$

$$= \text{LHS}$$

$\therefore p(1)$ is true

$p(k)$: Assume $p(n)$ is true when $n=k$ $(k \in J^+)$.

i.e. $$\sum_{r=1}^{k} a^{-r} = \frac{a^k - 1}{(a-1)a^k}$$

$p(k+1)$: Required to prove that $p(k) \rightarrow p(k+1)$.

i.e. $$\sum_{r=1}^{k+1} a^{-r} = \frac{a^{k+1} - 1}{(a-1)a^{k+1}}$$

$$\text{LHS} = \sum_{r=1}^{k} a^{-r} + a^{-(k+1)}$$

$$= \frac{a^k - 1}{(a-1)a^k} + a^{-(k+1)}$$

(by assumption)

$$= \frac{a(a^k - 1)}{(a-1)a^{k+1}} + \frac{1}{a^{k+1}}$$

$$= \frac{a^{k+1} - a + (a-1)}{(a-1)a^{k+1}}$$

$$= \frac{a^{k+1} - 1}{(a-1)a^{k+1}}$$

$$= \text{RHS}$$

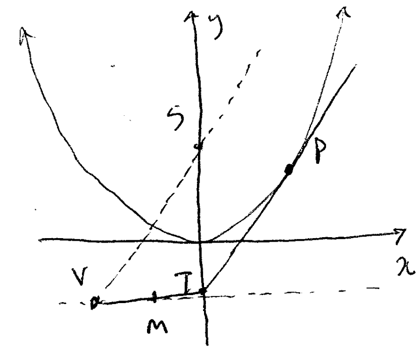
$\therefore p(k) \rightarrow p(k+1)$.

Since $p(1)$ is true,

$p(1) \rightarrow p(2) \rightarrow p(3) \rightarrow$

By Principle of Mathematical induction, $p(n)$ is true for positive integral n.

(b)



At $P(2ap, ap^2)$ $\quad \frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$

$$= 2ap \cdot \frac{1}{2a}$$

$$= p$$

$\therefore$ Tgt at P: $y - ap^2 = p(x - 2ap)$

$\therefore px - y - ap^2 = 0$

This line cuts y-axis when $x=0$

$\therefore y = -ap^2$

$\therefore T$ is $(0, -ap^2)$

The line thro' S $\parallel$ PT is

$$y - a = p(x - 0)$$

$$px - y + a = 0$$

This line cuts $y = -a$

$$px + a + a = 0$$

$$x = -\frac{2a}{p}$$

$\therefore V$ is $\left(\frac{-2a}{p}, -a\right)$

Q3/2)

$\therefore$ For M: $x = \dfrac{\frac{-2a}{p} + 0}{2}$

$\therefore x = -\frac{a}{p}$ —①

$y = \dfrac{-a + -ap^2}{2}$

$= \dfrac{-a(1+p^2)}{2}$ —②

For locus, eliminate p.

From equation ① $p = -\frac{a}{x}$

$\therefore y = \dfrac{-a\left(1+\left(\frac{-a}{x}\right)^2\right)}{2}$

$2y = -a - a \times \frac{a^2}{x^2}$

$2y = -a\left(1+\frac{a^2}{x^2}\right)$

$y = -\frac{a}{2}\left(1+\frac{a^2}{x^2}\right)$

(c) $f(x) = x - 3 + \ln x$

$f(1) = 1 - 3 + 0$
$= -2$

$f(3) = 3 - 3 + \ln 3$
$= \ln 3$

Since the sign of $f(x)$ changes between 1 and 3 and it is continuous in the domain, there must be at least one root.

Newton's method state

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

Here $f'(x) = 1 + \frac{1}{x}$

$\therefore x_2 = x_1 - \dfrac{f(x_1)}{1+\frac{1}{x_1}}$

$= x_1 - \dfrac{x_1 - 3 + \ln x_1}{1 + \frac{1}{x_1}}$

$= x_1 - \dfrac{x_1^2 - 3x_1 + x_1 \ln x_1}{1 + x_1}$

$= \dfrac{x_1(1+x_1) - x_1(x_1 - 3 + \ln x_1)}{1+x_1}$

$= \dfrac{x_1(4 - \ln x_1)}{1 + x_1}$

Now if $x_1 = 2$

$x_2 = \dfrac{2(4 - \ln 2)}{1+2}$

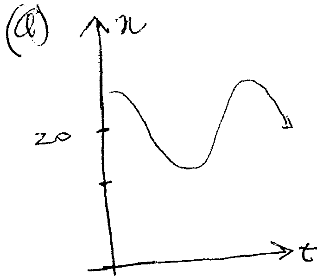
$= \dfrac{8 - 2\ln 2}{3}$

$\doteqdot 2.20$

IKIAH HSC 3HS Ext 1 2002

Question 4

(a)



$x = 20 + A\cos(nt+\alpha)$

Trough to crest $= 28 - 12$

$= 16$

$\therefore A = 8$

Period $= 2 \times$ (Trough to crest)

$= 2 \times 7$

$= 14$

$\therefore 14 = \frac{2\pi}{n}$

$\therefore n = \frac{\pi}{7}$

Let $t=0$ at 2:00pm.

We seek.

$22 = 20 + 8\cos\left(\frac{\pi t}{7} + \alpha\right)$

Now $12 = 20 + 8\cos(0+\alpha)$

$-8 = 8\cos\alpha$

$\cos\alpha = -1$

$\alpha = \pi$

$\therefore 22 = 20 + 8\cos\left(\frac{\pi t}{7} + \pi\right)$

$\frac{1}{4} = \cos\left(\frac{\pi}{7}t + \pi\right)$

$\frac{\pi}{7}t + \pi = \cos^{-1}\left(\frac{1}{4}\right) + k\pi$

$\frac{\pi}{7}t = \cos^{-1}\frac{1}{4} - \pi + k\pi$

$t = \frac{7}{\pi}\left(\cos^{-1}\frac{1}{4} + (k-1)\pi\right)$

$= 9.936$ when $k=$

$\therefore$ Time 2:00p + 9hr 56' 13"

= 11:56:13 pm until

$= 16.936$ when $k=3$

$\therefore$ Tide remains above 22m from 11:56pm until 4:56am.

(b) $\frac{d^2x}{dt^2} = -\frac{24}{x^2}$

When $t=0$, $x=3$, $v=4$

(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{24}{x^2}$

$\int \frac{d}{dx}\left(\frac{1}{2}v^2\right)dx = -24\int \frac{dx}{x^2} + C$

$\frac{1}{2}v^2 = \frac{24}{x} + C$

$v^2 = \frac{48}{x} + C'$

When $x=3$, $v=4$

$16 = 16 + C'$

$C' = 0$

Q4/(2)

$\therefore V^2 = \dfrac{48}{x}$

Now $v = \pm \dfrac{\sqrt{48}}{\sqrt{x}}$

Since $v > 0$ initially, we choose the positive root.

$\therefore V = \dfrac{4\sqrt{3}}{\sqrt{x}}$

(ii) $\dfrac{dx}{dt} = \dfrac{4\sqrt{3}}{\sqrt{x}}$

$\therefore \dfrac{dt}{dx} = \dfrac{\sqrt{x}}{4\sqrt{3}}$

So $\int \dfrac{dt}{dx}\,dx = \int \dfrac{\sqrt{x}}{4\sqrt{3}}\,dx + D$

$t = \dfrac{x^{3/2}}{3/2 \times 4\sqrt{3}} + D$

$t = \dfrac{x\sqrt{x}}{6\sqrt{3}} + D$

When $t=0$, $x=3$

$0 = \dfrac{3\sqrt{3}}{6\sqrt{3}} + D$

$0 = \frac{1}{2} + D$

$\therefore D = -\frac{1}{2}$

$t = \dfrac{x\sqrt{x}}{6\sqrt{3}} - \dfrac{1}{2}$

(iii) When $x = 10$

$t = \dfrac{10\sqrt{10}}{6\sqrt{3}} - \frac{1}{2}$

$= 2.543$ sec

(5)

(i)

$\ddot{x} = 0$

$\dot{x} = c_1 \quad t=0, \dot{x} = 5\cos\theta.$

$\dot{x} = 5\cos\theta$

$x = 5t\cos\theta + c_2$

$\boxed{x = 5t\cos\theta}$

$c_2 = 0$ as $x = 0$ when $t=0$

$\ddot{y} = -10$

$\dot{y} = -10t + c_3$ clearly $t=0, \dot{y} = 5\sin\theta$

$\dot{y} = -10t + 5\sin\theta$

$y = -5t^2 + 5t\sin\theta + c_4$

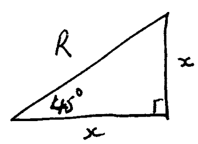
clearly when $t=0, y=0$

$\therefore \boxed{y = -5t^2 + 5t\sin\theta}$

3

($2\frac{1}{2}$ marks if not evaluating c_2 etc.)

(ii)



clearly $\frac{y}{x} = \tan 45° = 1$

$\therefore y = x.$

and $\frac{x}{R} = \sin 45°$

$x = \frac{R}{\sqrt{2}}.$

$\therefore \boxed{y = x = \frac{R}{\sqrt{2}}}$

2

(iii) If $x = y$: $\quad 5t\cos\theta = -5t^2 + 5t\sin\theta$

$5t^2 + 5t\cos\theta - 5t\sin\theta = 0$

$5t(t + \cos\theta - \sin\theta) = 0$

$t = 0, \sin\theta - \cos\theta.$

$\therefore \quad x = 5(\sin\theta - \cos\theta)\cos\theta$

$\therefore \boxed{R = 5\sqrt{2}(\sin\theta\cos\theta - \cos^2\theta).}$

3

(iv) $R' = 5\sqrt{2}\left[\sin\theta \times -\sin\theta + \cos\theta . \cos\theta - 2\cos\theta . -\sin\theta\right]$

$= 5\sqrt{2}\left[\cos^2\theta - \sin^2\theta + \sin 2\theta\right]$

$= 5\sqrt{2}\left[\cos 2\theta + \sin 2\theta\right]$

If $R' = 0$

$\cos 2\theta + \sin 2\theta = 0$

$\sin 2\theta = -\cos 2\theta$

$\tan 2\theta = -1.$

$2\theta = \frac{3\pi}{4}$

$R'' = 5\sqrt{2}\left[-2\sin 2\theta + 2\cos 2\theta\right]$

$= +10\sqrt{2}\left[\cos 2\theta - \sin 2\theta\right]$

2

($1\frac{1}{2}$ if not verifying max)

$\boxed{\theta = \frac{3\pi}{8}}$, $R'' = +10\sqrt{2}\left[-\frac{1}{\sqrt{2}} - \dots\right]$

$= -20 < 0$

$\therefore$ MAX

(v) $R = 5\sqrt{2}\left(\cos\frac{3\pi}{8}\sin\frac{3\pi}{8} - \cos^2\frac{3\pi}{8}\right)$

$= 5\sqrt{2}\left(\frac{1}{2\sqrt{2}} - \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)\right)$

$= 5\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)$

$= 5\sqrt{2}\left(\frac{2-\sqrt{2}}{2\sqrt{2}}\right)$

$= \boxed{\frac{5}{2}\left(2-\sqrt{2}\right).}$ (2 for exact answer 1½ for approximation)

2

NB $\cos\frac{3\pi}{4} = 2\cos^2\frac{3\pi}{8} - 1$

$2\cos^2\frac{3\pi}{8} = 1 + \cos\frac{3\pi}{4}$

$\cos^2\frac{3\pi}{8} = \frac{1}{2} + \frac{1}{2}\cos\frac{3\pi}{4}$

$= \frac{1}{2} + \frac{1}{2} \times -\frac{1}{\sqrt{2}}.$

$= \frac{1}{2} - \frac{1}{2\sqrt{2}}$

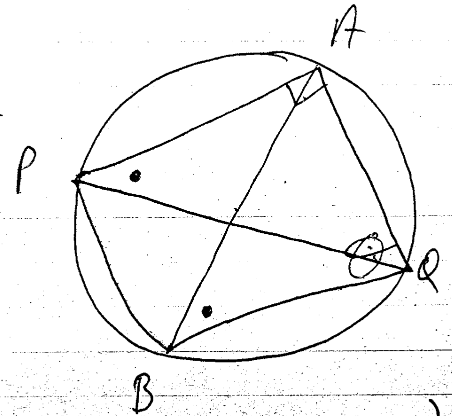
also $\cos\frac{3\pi}{8}\sin\frac{3\pi}{8} = \frac{1}{2}\sin\frac{3\pi}{4}$

$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$

$= \frac{1}{2\sqrt{2}}.$

NB $\frac{5}{2}\left(2-\sqrt{2}\right) \doteqdot 1.464.$

Q6



i) $\frac{AQ}{\sin A\hat{B}Q} = \frac{AB}{\sin\theta}$ (2)

So $\sin A\hat{B}Q = \frac{AQ}{AB}\sin\theta$

ii) Now $\sin A\hat{B}Q = \sin A\hat{P}Q$ (equal angles on same arc)

$\therefore \frac{AQ}{AB}\sin\theta = \frac{AQ}{PQ}$ (3)

$\therefore \frac{\sin\theta}{AB} = \frac{1}{PQ}$

$\therefore PQ = \frac{AB}{\sin\theta}$

(iii) Now $APB = 180 - \cos\theta$, ($\angle$'s of cyclic quad)

In $\triangle APB$: $AB^2 = BP^2 + PA^2 - 2BP\,PA\cos(180-\theta)$

$AB^2 = AP^2 + BP^2 + 2AP\times BP\cos\theta$ (2)

$AB = \sqrt{AP^2+BP^2+2AP\,BP\cos\theta}$

but $AB = PQ\sin\theta$ $\therefore PQ = \frac{\sqrt{AP^2+BP^2+2AP\,BP\cos\theta}}{\sin\theta}$

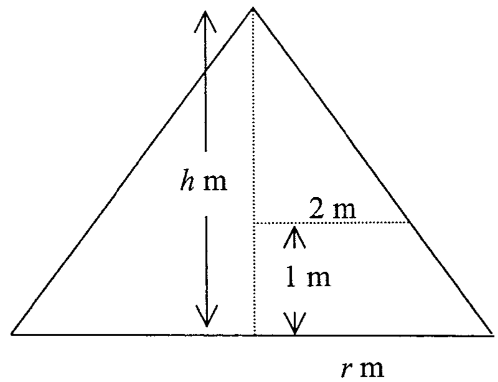
b i) $\frac{\sin 2x}{1-\cos 2x} = \cot x \Rightarrow$ LHS $= \frac{2\sin x\cos x}{1-(1-2\sin^2 x)} = \frac{2\sin x\cos x}{2\sin^2 x}$

$= \frac{\cos x}{\sin x} = \cot x.$ (3)

ii) $\cot 67\frac{1}{2}^\circ = \frac{\sin 135^\circ}{1-\cos 135^\circ} = \frac{\sqrt{2}}{2} \div \left(1+\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}\times\frac{2}{2+\sqrt{2}}$ (2)

$= \frac{\sqrt{2}}{2+\sqrt{2}} = \underline{\underline{\sqrt{2}-1}}$ or $\frac{1}{\sqrt{2}+1}$

(7)



(i)

$$S = \pi r^2$$

$$\frac{r}{h} = \frac{2}{h-1} \text{(similar triangles)}$$

$$\therefore r = \frac{2h}{h-1}$$

$$\therefore S = \frac{4\pi h^2}{(h-1)^2}$$

(ii)

$$\frac{dS}{dt} = \frac{dS}{dh} \times \frac{dh}{dt}$$

$$\boxed{\frac{dh}{dt} = -\frac{1}{8}}$$

$$S = \frac{4\pi h^2}{(h-1)^2}$$

$$\frac{dS}{dh} = \frac{(8\pi h) \times (h-1)^2 - 4\pi h^2 \times 2(h-1)}{(h-1)^4}$$

$$= \frac{8\pi h(h-1)\left[(h-1)-h\right]}{(h-1)^4}$$

$$= -\frac{8\pi h}{(h-1)^3}$$

$$\frac{dS}{dt} = -\frac{8\pi h}{(h-1)^3} \times -\frac{1}{8} = \frac{\pi h}{(h-1)^2}$$

$$= 2\pi \text{ m}^2/\text{s when } h = 2$$

(iii)

$$V = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi \times \frac{4h^2}{(h-1)^2} \times h = \frac{4\pi h^3}{3(h-1)^2}$$

(iv)

$$V=\frac{4\pi h^3}{3(h-1)^2}$$

$$\frac{dV}{dh}=\frac{3(h-1)^2\times 12\pi h^2-4\pi h^3\times 6(h-1)}{9(h-1)^4}$$

$$=\frac{12\pi h^2(h-1)\left[3(h-1)-2h\right]}{9(h-1)^4}$$

$$=\frac{4\pi h^2(h-3)}{9(h-1)^3}$$

Minimum when $\dfrac{dV}{dh}=0$

$\dfrac{dV}{dh}=\dfrac{4\pi h^2(h-3)}{9(h-1)^3}=0\Rightarrow h=0,3$

$\because 1<h\le 5\Rightarrow h=3$

h	2	3	4
$\dfrac{dV}{dh}$	- 1	0	$\frac{1}{8}$

NB We only need to test $\dfrac{(h-3)}{(h-1)^3}$ $\because \dfrac{4\pi h^2}{9}>0$

So there is a *relative* minimum at $h=3$

$V=9\pi$

Testing end points $h=5$, $V=\frac{125}{12}\pi$

So the minimum value of V is 9π, when $h=3$

Note that $\dfrac{dS}{dh}\neq 0$ for $1<h\le 5$

So the minimum value of S will occur when $h=5$, so the two minimums don't coincide for the same value of h.