Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE
2004

Mathematics Extension 2

TIME ALLOWED: 3 hours

Instructions:

- Write your name and class at the top of this page, and at the top of each answer sheet.
- At the end of the examination this examination paper must be attached to the front of your answers.
- All questions are of equal value and may be attempted.
- All necessary working must be shown. Marks will be deducted for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.

(FOR MARKERS USE ONLY)

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	TOTAL

QUESTION 1:

2 (a) (i) $\displaystyle\int \frac{dx}{x^2+6x+13}$

2 (ii) $\displaystyle\int_0^4 \frac{dx}{(2x+1)\sqrt{2x+1}}$

3 (b) Show that $\displaystyle\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

Hence find $\displaystyle\int x^2 e^x\,dx$

3 (c) Sketch the locus of the point Z in the Argand plane, which moves so that $\arg(z-1)=\frac{\pi}{2}$

3 (d) (i) Find values of A, B and C so that $\displaystyle\frac{13}{(x^2+4)(x+3)} \equiv \frac{Ax+B}{x^2+4}+\frac{C}{x+3}$

2 (ii) Hence find $\displaystyle\int \frac{13dx}{(x^2+4)(x+3)}$

QUESTION 2:

(a) If $z=1+\sqrt{3}i$ find

4 (i) $\bar{z}$ (ii) $|z|$ (iii) arg z (iv) arg(iz)

(b) Express $z=1+\sqrt{3}i$ in mod-arg form and hence find

3 (i) $\sqrt{z}$ (ii) z^6 (in simplest form)

3 (c) On the same set of axes, sketch $y=|x-1|$ and $y=|x+1|$ and then use this graph, or otherwise, to find the value of k if $|x-1|+|x+1|\geq k$ for all values of x.

3 (d) (i) The point W represents the complex number $w=a+ib$, and $w=\frac{z}{z+1}$ where $z=x+iy$. The point Z representing the complex number z moves along the y-axis only.

Show that $a=\frac{y^2}{1+y^2}$ and $b=\frac{y}{1+y^2}$

2 (ii) Find the locus of W both algebraically and geometrically.

QUESTION 3 :

4 (a) (i) Using the method of Mathematical Induction, prove deMoivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

2 (ii) Prove that the points on the Argand Diagram with co-ordinates representing $(\text{cis}\frac{\pi}{3})^n$ for $n=1,2,3..,6$ are the vertices of a regular hexagon inscribed in a circle of radius 1 unit.

(b) If $f(x) = \cos x + i\sin x$,

1 (i) Find $f(0)$

1 (ii) Show that $\dfrac{f'(x)}{f(x)} = i$

3 (iii) By integrating both sides of part (ii), deduce that $\cos x + i\sin x = e^{ix}$.
This is EULER's THEOREM.

1 (iv) Using Euler's Theorem from part (iii), prove deMoivre's Theorem

1 (c) (i) Describe the locus of the point z, where $|z-a| = r$

2 (ii) If $|z-a| = r$ and $|z-b| = s$ what is the geometric significance when $|a-b| = r+s$

QUESTION 4:

3 (a) By using t-results, or otherwise, find $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta}$ leaving your answer in exact form

(b) Sketch the following on different sets of axes showing all important features

(DO NOT USE CALCULUS)

8 (i) $y = \frac{|x|}{x}$

(ii) $y = \ln(\frac{1}{x^2})$

(iii) $y = |\tan x|$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(iv) $y = \sec x$ for $-2\pi \le x \le 2\pi$

4 (c) Use calculus, or otherwise, to sketch $y = \frac{e^x}{x}$ showing all stationary points and asymptotes, if they exist.

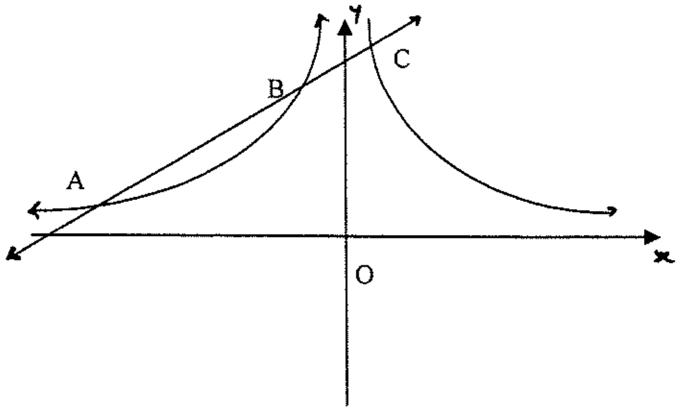
QUESTION 5:

(a) For the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ find

3 (i) the eccentricity
(ii) the co-ordinates of the foci
(iii) the equations of the directrices

3 (b) Given that $P(x)=3x^3-11x^2+8x+4$ has a double root, fully factorise $P(x)$

(c)



In the diagram above, the points A, B and C represent the points of intersection of the line $y=4x+8$ and the curve $y=\frac{1}{x^2}$. The x-values of A, B and C are α, β and γ

1 (i) Show that α, β and γ satisfy $4x^3+8x^2-1=0$

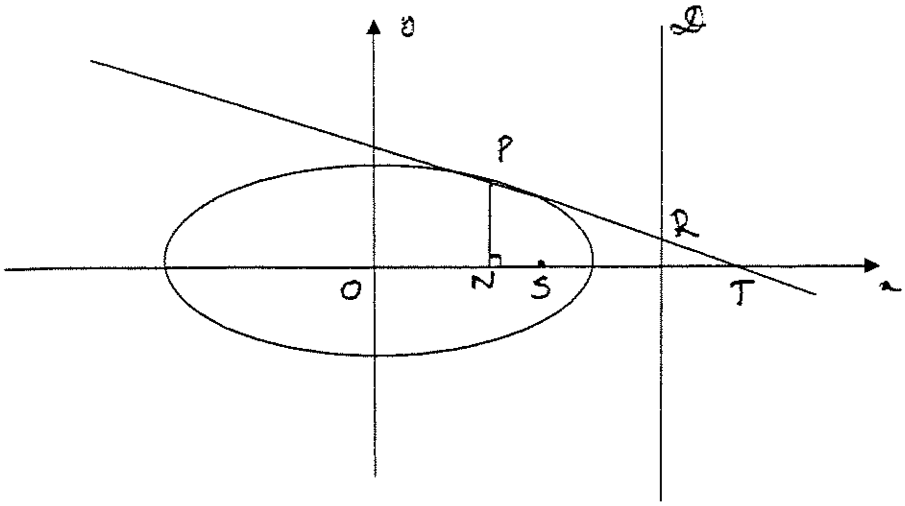
3 (ii) Find a polynomial with roots α^2, β^2 and γ^2

2 (iii) Find $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$

3 (iv) Prove that $OA^2 + OB^2 + OC^2 = 132$ where O is the origin.

QUESTION 6:

(a)



$P(a\cos\theta, b\sin\theta)$ is any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The tangent at P cuts the major axis in T and the Directrix in R. N is the foot of the perpendicular from P to the major axis, O is the centre and S is the focus.

2 (i) Show that the equation of the tangent at P is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

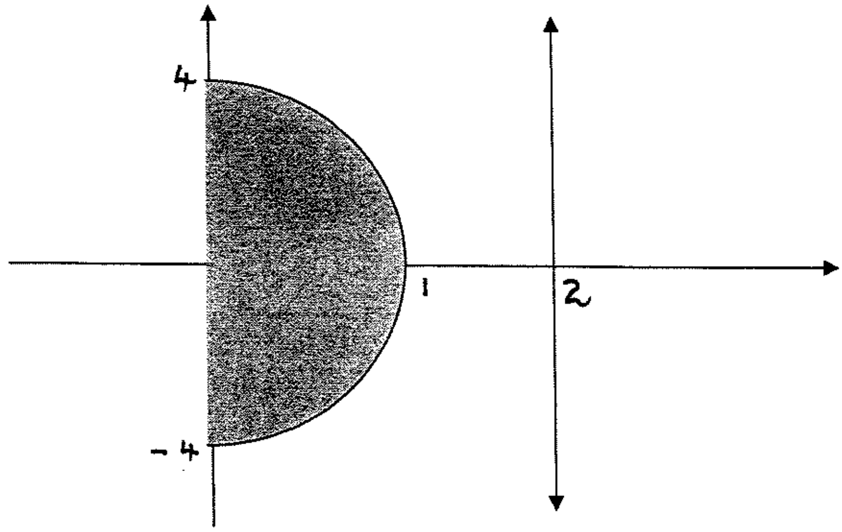
2 (ii) Show that $ON.OT = a^2$

5 (iii) Showing all steps carefully, prove that $\angle PSR = 90°$

QUESTION 6 continues on the next page...)

QUESTION 6 continued...)

(b)



A solid $\boldsymbol{S}$ is formed by rotating the region bounded by the parabola $y^2 = 16(1-x)$ and the y-axis through 2π about the the line x=2.

3 (i) Use the method of cylindrical shells to show that the volume of $\boldsymbol{S}$ is given by

$$\int_0^1 16\pi(2-x)\sqrt{1-x}\,dx$$

3 (ii) Calculate this definite integral by using the substitution $u=1-x$ (or otherwise)

QUESTION 7:

(a) If $1, w_1$ and w_2 are the cube roots of unity, prove that

2 (i) $w_1 = \overline{w_2} = w_2^2$

1 (ii) $w_1 + w_2 = -1$

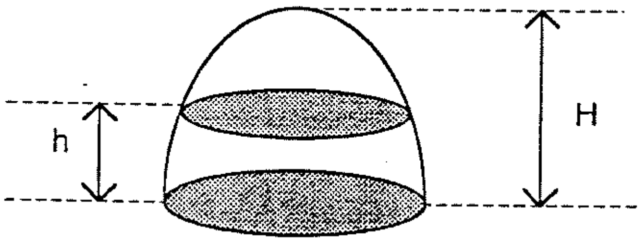
1 (iii) $w_1 w_2 = 1$

(b)

3 (i) By using the substitution $x = a\sin\theta$ or otherwise, verify that $\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{1}{4}\pi a^2$

2 (ii) Deduce that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab

6 (iii)



The diagram above shows a mound of height H. At height h above the horizontal base, the horizontal cross section of the mound is elliptical in shape, with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2 \quad \text{where } \lambda = 1 - \frac{h^2}{H^2}$$

and x and y are co-ordinates in the plane of cross section.

Show that the volume of the mound is $\frac{8\pi abH}{15}$

QUESTION 8:

2 (a) (i) Show that the equation of the tangent to the Hyperbola $xy = c^2$ at the point $P(cp, \frac{c}{p})$ is $x + p^2 y = 2cp$

2 (ii) If the tangents at the points P and $Q(cq, \frac{c}{q})$ meet at the point $R(x_1, y_1)$ prove that

(α) $pq = \frac{x_1}{y_1}$

and that (β) $p + q = \frac{2c}{y_1}$

2 (iii) If the length of the chord PQ is d units, show that

$$d^2 = c^2(p-q)^2\left\{1 + \frac{1}{p^2q^2}\right\}.$$

3 (iv) Further, if d in part (iii) above remains constant, deduce that the locus of R is given by

$$4c^2(x^2+y^2)(c^2-xy) = x^2y^2d^2$$

(b) If n is a positive integer and $f(x) = e^{-x}(1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}), \quad x \geq 0$

3 (i) Show that f(x) is a decreasing function

NOTE: ***n! = 1x2x3x4............(n-1)n***

3 (ii) Deduce that for x>0 and n any positive integer,

$$e^x \geq 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}$$

End of Examination

THSC MATHS EXT 2. 2004 SOLUTIONS.

Question 1:

(a) (i) $\displaystyle\int \frac{dx}{(x+3)^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)+k$ (1) (1)

(ii) $\displaystyle\int_0^4 (2x+1)^{-3/2}\,dx = -(2x+1)^{-1/2}\Big]_0^4$ (1)

$= -9^{-1/2} + 1^{-1/2}$

$= 2/3$ (1)

(b)(i) $\displaystyle\int x^n e^x\,dx = \int x^n \frac{d}{dx} e^x\,dx$

$\displaystyle = e^x x^n - \int e^x . n x^{n-1}\,dx$ (1)

$\underline{n=2}$ $\therefore \displaystyle\int x^2 e^x\,dx = e^x x^2 - 2\int e^x . x^1\,dx$ (1)

$\underline{n=1}$

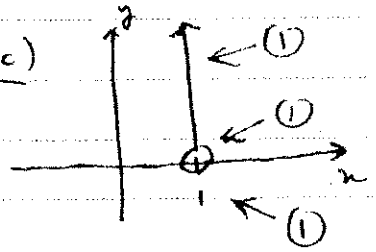
$\displaystyle\int e^x x\,dx = e^x x - \int e^x\,dx$

$= e^x x - e^x$

$\therefore \displaystyle\int x^2 e^x\,dx = e^2 x^2 - 2(e^x x - e^x)$

$= e^2 x^2 - 2e^x x + 2e^x$ (1)

(c)



(d) (i) $\displaystyle\frac{13}{(x^2+4)(x+3)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+3}$

$\therefore (Ax+B)(x+3) + C(x^2+4) = 13$

$\therefore A + C = 0$

$3A + B = 0$

$3B + 4C = 13$

(3) $\left.\begin{array}{l}\underline{\text{Now}}\ C = 1 \\ \therefore\ B = 3 \\ A = -1\end{array}\right\} \Rightarrow$ Exp^n $\displaystyle = \frac{3-x}{x^2+4} + \frac{1}{x+3}$

1 (d) (ii)

$$\int \frac{13\,dx}{(x^2+4)(x+3)} = \int \frac{dx}{x+3} + \int \frac{3-x}{x^2+4}\,dx$$

$$= \underbrace{\ln(x+3)}_{1} + \underbrace{\frac{3}{2}\tan^{-1}\frac{x}{2} - \frac{1}{2}\ln(x^2+4)}_{1}$$

EXAMINER'S COMMENTS:

- Average mark was about 12/15. Generally strongly done
- SPECIFIC COMMENTS:

1 (a) (ii) People forgot about the 2 coming out of the bracket when differentiated:

e.g. $\frac{d}{dx}(2x+1)^{-1/2} = -\frac{1}{2}(2)(2x+1)^{-3/2}$

(b) NOT A BIG ERROR, but the formula could have been used 2 times to evaluate $\int x^2 e^x\,dx$ AND $\int e^x x\,dx$.

MOST COMMON ERROR: The question required you to show the result, not just quote the result because you knew it was true. This was the most common:

" $u = x^n$ $u' = nx^{n-1}$ $v = e^x$ $v' = e^x$ "

then quoting the question! This leaves me to work out the in-between part - which was your job!!

(d) TROUBLE was with the $\int \frac{3-x}{x^2+4}$ part.

And the negatives involved!

QUESTION 2

(a) $z = 1+\sqrt{3}i$

Deal.

(i) $\bar{z} = 1-\sqrt{3}i$

(ii) $|z| = \sqrt{1+3}$
$= 2$

(iii) $\arg z = \tan^{-1}\frac{\sqrt{3}}{1}$
$= \pi/3$

(iv) $\arg iz = \arg(-\sqrt{3}+i)$
$= \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)$
$= 5\pi/6$

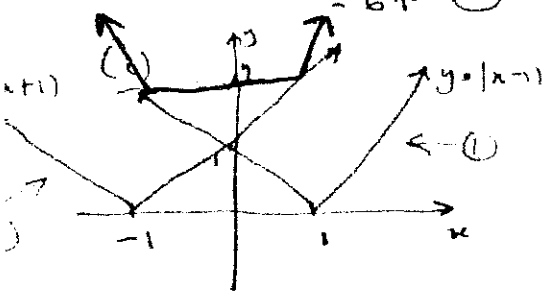
OR just $\theta = \pi/3 + \pi/2$
$= 5\pi/6$

(b) $z = 2\operatorname{cis}\pi/3$ (1) (1)

$\therefore$ (i) $\sqrt{z} = \sqrt{2}\operatorname{cis}\pi/6 \left(\frac{\sqrt{6}}{2} + \frac{\sqrt{2}i}{2}\right)$

(ii) $z^6 = 2^6 \operatorname{cis} 2\pi$
$= 64$ (1)

(c)



$k = 2$ (1)

(d) $w = \frac{z}{z+1} = a+ib$

$\therefore a+ib = \frac{x+iy}{x+iy+1}$

Now since z moves along the y-axis

$x = 0$ (1)

$\therefore a+ib = \frac{iy}{iy+1} \times \frac{1-iy}{1-iy}$

$= \frac{iy(1-iy)}{1+y^2}$

$= \frac{iy+y^2}{1+y^2}$ (1)

$\therefore a = y^2/(1+y^2)$ and $b = \frac{y}{1+y^2}$ (1)

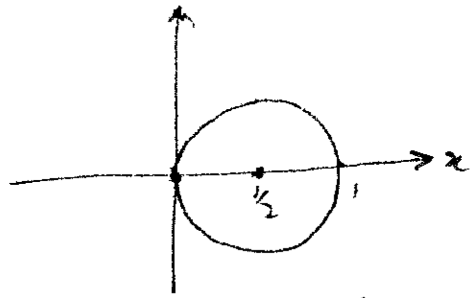
(ii) $y=0 \quad a=b=0.$

$y \to \pm\infty \quad a \to 1$

$y \to \pm\infty \quad b \to 0.$

$y = 1 \quad a = 1/2 \quad b = 1/2$

$y = -1 \quad a = 1/2 \quad b = -1/2.$



circle centre $(1/2, 0)$, $r = 1/2$

OR

$a = \frac{y^2}{1+y^2} \qquad b = \frac{y}{1+y^2}$

$a^2 = \frac{y^4}{(1+y^2)^2} \qquad b^2 = \frac{y^2}{(1+y^2)^2}$

$= \frac{y^2(1+y^2)}{(1+y^2)^2} - \frac{y^2}{(1+y^2)^2}$

$= \frac{y^2}{(1+y^2)} - b^2$

$= a - b^2.$

$\therefore a^2 - a + b^2 = 0$

$\therefore (a-\tfrac{1}{2})^2 + b^2 = 0 + 1/4$

circle, centre $(1/2, 0)$ $r = 1/2.$

QUESTION 3

(a) (i) For $n=1$ $(\cos\theta + i\sin\theta)^1 = \cos 1\theta + i\sin 1\theta$

$\therefore$ True for $n=1$ (1)

Assume the formula is true for $n=k$,

i.e. $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

For $n=k+1$

$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$ (2)

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= \cos\theta\cos k\theta - \sin\theta\sin k\theta + i(\cos\theta\sin k\theta + \sin\theta\cos k\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

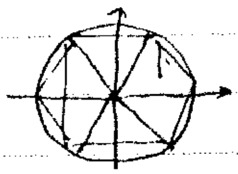
$\therefore$ If the formula is true for $n=k$, it is true for $n=k+1$

But it is true for $n=1$, $\therefore$ it is true for $n=2$ and so on (1)

i.e. true $\forall n$.

(ii) The points are $\text{cis}\,\pi/3$, $\text{cis}\,2\pi/3$, $\text{cis}\,\pi$, $\text{cis}\,4\pi/3$, $\text{cis}\,5\pi/3$, $\text{cis}\,2\pi$ (1)

which are 6 points equally-spaced about the unit circle.



So cut off equal chords (1)

$\therefore$ form a regular hexagon

(b) $f(x) = \cos x + i\sin x$.

(i) $f(0) = 1$ (1)

(ii) $f'(x) = -\sin x + i\cos x$

$= i(\cos x + i\sin x)$

$= i f(x)$

$\therefore \dfrac{f'(x)}{f(x)} = i$ (1)

(iii) $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \int i\,dx.$

$\therefore \log_e f(x) = ix + c$ (1)

At $x=0$ $\log_e(1) = c \Rightarrow c = 0$ (1)

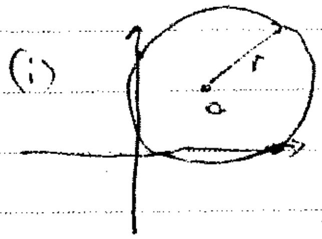
$\therefore \log_e f(x) = ix \Rightarrow f(x) = e^{ix}$ (1)

Q 3 (b) (iv) $\cos x + i\sin x = e^{ix}$

$\therefore (\cos x + i\sin x)^n = e^{inx} = e^{i(nx)}$ (1)

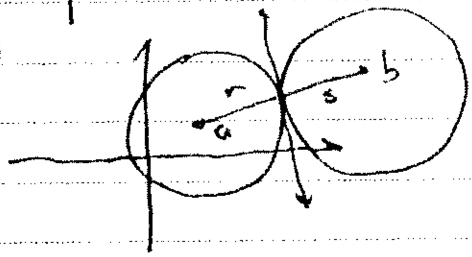
$= \cos nx + i\sin nx$ 1

(c) (i)



a circle of radius r, centre the point a. (1)

(ii)

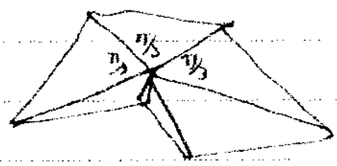


(1) Two circles, centres a and b which share a common tangent (ie just touch). (1)

EXAMINERS COMMENTS:

(a) Unless you don't know how to progress from one step to another, don't shortcut – it looks like a fudge! And, after all, the answer is already there! – Induction is a very formal method. In general reasonably done!

(a)(ii) The pertinent points here are that the angles are equal and that the moduli are all 1. ie. evenly spaced about a unit circle. ↑ this was frequently omitted, and without it the diagram could have been



(b) (iii) 75% of the candidates left off the + k when integrating. To evaluate the k you needed your information from part (i).

(c) (i) Well done!

(ii) Awfully done!

QUESTION 4

a(a) $\displaystyle\int_{\pi/3}^{\pi/2} \frac{d\theta}{\sin\theta} = \int_{1/\sqrt{3}}^{1} \frac{2\,dt}{1+t^2}\cdot\frac{1+t^2}{2t}$ by t-results

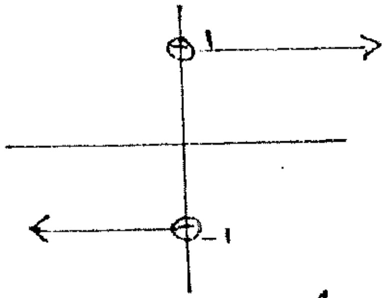
$\displaystyle = \int_{1/\sqrt{3}}^{1} \frac{dt}{t}$

$\displaystyle = \Big[\ln t\Big]_{1/\sqrt{3}}^{1}$

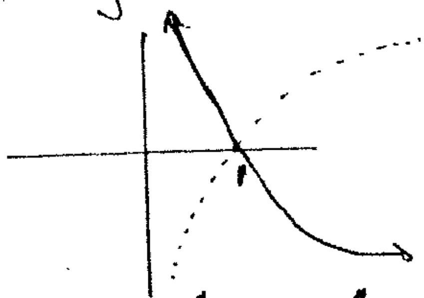
$= \ln\sqrt{3}$ or $\tfrac{1}{2}\ln 3$

b)

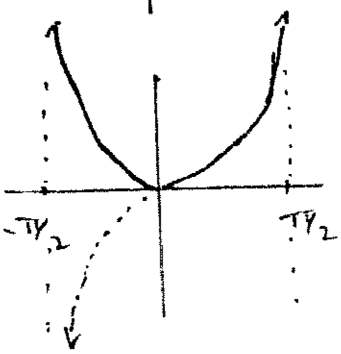
i)



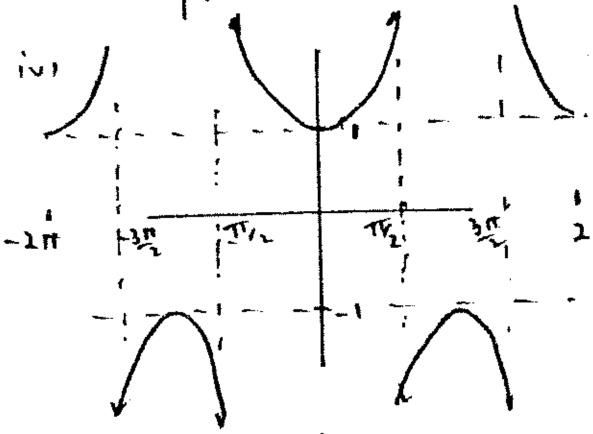
ii) $y = -2\ln x$



iii



iv)



c) $y = \dfrac{e^x}{x}$

$\dfrac{dy}{dx} = \dfrac{x.e^x - e^x}{x^2}$

$\dfrac{dy}{dx} = 0 \quad x = 1$

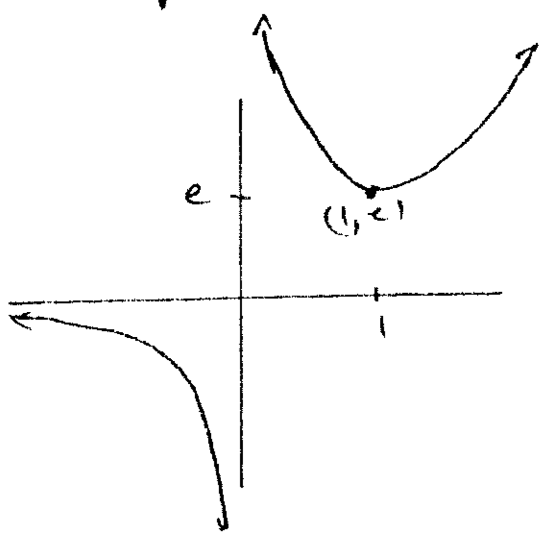
$x = 1,\ y = e$

undefined at $x = 0$

$x \to 0^+ \quad y \to \infty$

$x \to 0^- \quad y \to -\infty$

$x \to -\infty \quad y \to 0$



QUESTION 5:

(a) $x^2/9 + y^2/4 = 1 \Rightarrow a = 3,\ b = 2$

(i) $e = \pm \sqrt{5}/3$ (1)

(ii) $S: (\pm\sqrt{5}, 0)$ (1)

(iii) $\mathcal{D}: \ x = \pm 9/\sqrt{5}$ (1)

(b) $P'(x) = 9x^2 - 22x + 8$

FOR DOUBLE ROOT, $P'(x) = 0$

$\therefore (9x - 4)(x - 2) = 0$

$\therefore x = 4/9$ or $x = 2$ (1)

$P(2) = 0 \Rightarrow x = 2$ is the double root (1)

By INSPECTION $P(x) = (x-2)^2(3x+1)$

(c)(i) $y = 4x + 8 = 1/x^2$

$\therefore 4x^3 + 8x^2 - 1 = 0$ solves both equations (1)

(ii) $P(\sqrt{x}) = 4(\sqrt{x})^3 + 8(\sqrt{x})^2 - 1 = 0$ (1)

$\therefore 4x^{3/2} + 8x - 1 = 0$

$4x^{3/2} = -8x + 1$ (1)

$\therefore 16x^3 = 64x^2 - 16x + 1$ *

$\therefore 16x^3 - 64x^2 + 16x - 1 = 0$ (1)

(iii) $1/\alpha^2 + 1/\beta^2 + 1/\gamma^2 = \dfrac{\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2}{\alpha^2\beta^2\gamma^2}$ (1)

$= \dfrac{\text{Sum of roots} \times 2}{\text{Product of roots}}$ from equation *

$= \dfrac{1}{1/16} = 16.$ (1)

(iv) $OA^2 = \alpha^2 + \left(\frac{1}{\alpha^2}\right)^2 = \alpha^2 + 1/\alpha^4$

$OB^2 = \beta^2 + 1/\beta^4$

$OC^2 = \gamma^2 + 1/\gamma^4$

$\therefore OA^2 + OB^2 + OC^2 = \alpha^2 + \beta^2 + \gamma^2 + \frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4}$

$= 4 + \frac{256}{2}$

$= 132.$

$\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} = \frac{\alpha^4\beta^4 + \alpha^4\gamma^4 + \beta^4\gamma^4}{\alpha^4\beta^4\gamma^4}$

and

$\alpha^4\beta^4 + \alpha^4\gamma^4 + \beta^4\gamma^4$

$= (\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2)^2 - 2\alpha^2\beta^2\gamma^2(\alpha^2 + \beta^2 + \gamma^2)$

$= 1^2 - 2 \times \frac{1}{16} \times 4$

$= 1/2$

$\alpha^4\beta^4\gamma^4 = \frac{1}{256}$

QUESTION 6

a) i) $x = a\cos\theta$

$\frac{dx}{d\theta} = -a\sin\theta$

$y = b\sin\theta$

$\frac{dy}{d\theta} = b\cos\theta$

$\therefore \frac{dy}{dx} = \frac{b\cos\theta}{-a\sin\theta}$

Eqn of tangent:

$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$\frac{y\sin\theta}{b} - \sin^2\theta = -\frac{x\cos\theta}{a} + \cos^2\theta$

$\frac{x\cos\theta}{a} + y\frac{\sin\theta}{b} = \sin^2\theta + \cos^2\theta$

$\frac{x\cos\theta}{a} + y\frac{\sin\theta}{b} = 1$

ii $N = (a\cos\theta, 0)$

when $y = 0$

$x = \frac{a}{\cos\theta}$

$\therefore T = (\frac{a}{\cos\theta}, 0)$

$ON . OT = a\cos\theta . \frac{a}{\cos\theta}$

$= a^2$

iii

$P\ (a\cos\theta, b\sin\theta)$

$S\ (ae, 0)$

$R\ (\frac{a}{e}, \frac{b(e-\cos\theta)}{e\sin\theta}$

$M_{PS} = \frac{b\sin\theta}{a\cos\theta - ae}$

$$\frac{\overline{\quad e\sin\theta \quad}}{\frac{a}{e} - ae}$$

$= \frac{be - b\cos\theta}{e\sin\theta} \times \frac{e}{a - ae^2}$

$= \frac{b(e-\cos\theta)}{a(1-e^2)\sin\theta}$

$M_{PS} \times M_{SR}$

$= \frac{b\sin\theta}{a\cos\theta - ae} \times \frac{b(e-\cos\theta)}{a(1-e^2)\sin\theta}$

$= \frac{-b^2}{a^2(1-e^2)}$

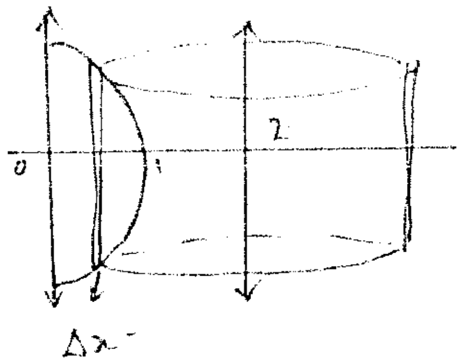
but $b^2 = a^2(1-e^2)$ for ellipse

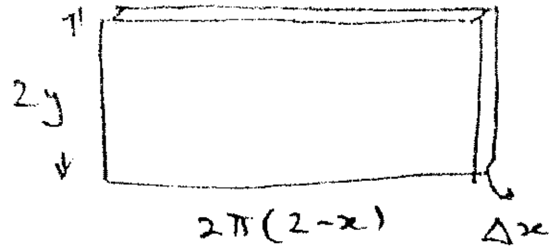
$= -1$

$\therefore \angle PSR = 90^\circ$

QUESTION 6. (cont)

b ii





$$\Delta V = 2\pi(2-x)\cdot 2y \cdot \Delta x$$

$$\text{Volume} = \lim_{\Delta x \to 0} \sum_{0}^{1} 2\pi(2-x)\, 8\sqrt{1-x}\, \Delta x$$

$$= 16\pi \int_0^1 (2-x)\sqrt{1-x}\, dx$$

ii) Let $u = 1-x$

$\therefore du = -dx$

$$\text{Volume} = 16\pi \int_1^0 (1+u)\sqrt{u} \cdot -du$$

$$= 16\pi \int_0^1 \sqrt{u} + u\sqrt{u}\, du.$$

$$= 16\pi \left[\tfrac{2}{3} u^{3/2} + \tfrac{2}{5} u^{5/2} \right]_0^1$$

$$= \frac{256\pi}{15}$$

QUESTION 7

a. i) roots of $z^3 = 1$ are

$$1$$
$$w_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$
$$w_2 = \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$$

$$\therefore \quad \overline{w_2} = \overline{\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}}$$
$$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$
$$= w_1$$

$$w_2^2 = \left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)^2$$
$$= \cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3}$$
$$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$
$$= w_1$$

ii) as $1, w_1, w_2$ are the roots of $z^3 - 1 = 0$

$\therefore$ Sum of roots $\left(\frac{-b}{a}\right)$

$$1 + w_1 + w_2 = 0$$
$$w_1 + w_2 = -1$$

iii) product of roots $\left(\frac{c}{a}\right)$

$$\therefore \quad 1 \times w_1 \times w_2 = 1$$
$$w_1 w_2 = 1$$

b. i) integral represents the area of $\frac{1}{4}$ of a circle radius a.

$$\therefore \int_0^a \sqrt{a^2 - x^2}\, dx = \frac{1}{4}\pi a^2$$

or $\int_0^a \sqrt{a^2 - x^2}\, dx \qquad x = a\sin\theta$

$$dx = a\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2\theta}\,.\, a\cos\theta\, d\theta$$

$$= a^2 \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} 1 + \cos 2\theta\, d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\pi a^2$$

ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\therefore \quad y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\therefore \quad y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\therefore \text{area} = 4 \times \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

$$= \frac{4b}{a} \times \frac{1}{4}\pi a^2$$

$$= \pi ab$$

iii) area of cross-section

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2$$

or $\frac{x^2}{a^2\lambda^2} + \frac{y^2}{b^2\lambda^2} = 1$

is given by $A = \pi ab\lambda^2$

Q7 (cont)

but $\lambda^2 = 1 - \frac{h^2}{H^2}$

$\therefore \quad A = \pi ab\left(1 - \frac{h^2}{H^2}\right)^2$

$\therefore \quad V = \pi ab \int_0^H \left(1 - \frac{h^2}{H^2}\right)^2 dh$

$= \pi ab \int_0^H 1 - \frac{2h^2}{H^2} + \frac{h^4}{H^4} \; dh$

$= \pi ab \left[h - \frac{2h^3}{3H^2} + \frac{h^5}{5H^4} \right]_0^H$

$= \pi ab \times \frac{8H}{15}$

$= \frac{8\pi ab H}{15}$

QUESTION 8

a i) $y = c^2 x^{-1}$

$\frac{dy}{dx} = -c^2 x^{-2}$

when $x = cp$

$m_T = \frac{-c^2}{c^2p^2}$

$= \frac{-1}{p^2}$

$\therefore \quad y - \frac{c}{p} = \frac{-1}{p^2}(x - cp)$

$p^2 y - cp = -x + cp$

$x + p^2 y = 2cp$

ii) Solve simultaneously to find (x_1, y_1)

$x + p^2 y = 2cp$

$x + q^2 y = 2cq$ — subtract

$p^2 y - q^2 y = 2cp - 2cq$

$y = \frac{2c(p-q)}{p^2 - q^2}$

$= \frac{2c}{p+q}$

$\therefore \quad p + q = \frac{2c}{y_1} \qquad (\beta)$

$\therefore \quad x + p^2\left(\frac{2c}{p+q}\right) = 2cp$

$x = 2cp - \frac{2cp}{p+q}$

$= \frac{2cp(p+q) - 2cp}{p+q}$

$= \frac{2cpq}{p+q}$

$\therefore \quad \frac{x_1}{y_1} = \frac{\frac{2cpq}{p+q}}{\frac{2c}{p+q}}$

$= pq \qquad (\alpha)$

iii) $d^2 = (cp - cq)^2 + \left(\frac{c}{p} - \frac{c}{q}\right)^2$

by distance formula

$= c^2\left[(p-q)^2 + \frac{(q-p)^2}{p^2q^2}\right]$

$= c^2\left[(p-q)^2\left(1 + \frac{1}{p^2q^2}\right)\right]$

iv) $d^2 = c^2(p-q)^2\left(1 + \frac{1}{p^2q^2}\right)$

sub: $\frac{x}{y} = pq$

$\frac{2c}{y} = p + q$

$(p-q)^2 = (p+q)^2 - 4pq$

$$\therefore d^2 = c^2\left(\frac{4c^2}{y^2} - \frac{4x}{y}\right)\left(1 + \frac{y^2}{x^2}\right)$$

$$d^2 = 4c^2\left(\frac{c^2 - xy}{y^2}\right)\left(\frac{x^2+y^2}{x^2}\right)$$

$$x^2y^2d^2 = 4c^2(c^2 - xy)(x^2+y^2)$$

b. i) $f(x) = e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right) \qquad x \geq 0$

$$f'(x) = -e^{-x}\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right) + e^{-x}\left(1 + \frac{x}{1} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!}\right)$$

$$= -e^{-x}\left(\frac{x^n}{n!}\right)$$

< 0 for all $x \geq 0$

as $e^{-x} > 0$

$\frac{x^n}{n!} > 0$

$\therefore$ curve is decreasing for all $x \geq 0$

as $f'(x) < 0$

ii) when $x = 0$ $f(0) = 1$

also curve is always decreasing

$\therefore f(x) \leq 0$

$$\therefore e^{-x}\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right) \leq 1$$

$$\therefore e^{x} \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$