ST IGNATIUS COLLEGE RIVERVIEW

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2000

MATHEMATICS

3/4 UNIT COMMON

Time allowed: Two hours
(plus 5 minutes reading time)

Instructions to Candidates

- Attempt all questions
- All questions are of equal value.
- Show all necessary working. Marks may be deducted for missing or poorly arranged work.
- Standard integrals are provided
- Board approved calculators may be used.
- Each question attempted must be returned in a *separate* writing booklet clearly marked Question 1, Question 2 etc, on the cover
- **Each booklet must have your student number and the name of your Class Teacher.**

Students are advised that this is a trial examination only and cannot in any way guarantee the content or format of the 2000 3/4 unit Mathematics Higher School Certificate Examination

Question 1 **(12 marks)** **Start a new booklet**

(a) Solve $|x-3| > 5$ *2 marks*

(b) Find the exact value of $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ *1 mark*

(c) Differentiate with respect to x: $e^{-\ln x}$ *2 marks*

(d) Show that $\int_{-2}^{2\sqrt{3}} \frac{dx}{\sqrt{16-x^2}} = \frac{\pi}{2}$ *2marks*

(e) Find the coefficient of x^5 in the expansion of $\left(x+\frac{1}{x}\right)^{13}$ *2 marks*

(f) (i) Sketch $y=\frac{1}{x}$. *1 mark*

(ii) Hence or otherwise find the values of x for which $\frac{1}{x} > x$ *2 marks*

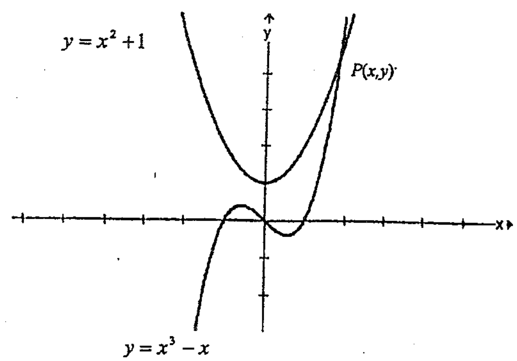
Question 2 **(12 marks)** **Start a new booklet**

(a) Find $\int_e^{e^3} \frac{\ln x}{x}dx$ using the substitution $u = \ln x$ *3marks*

(b) (i) Prove that $\frac{1-\cos x}{\sin x} = \tan\frac{x}{2}$ *2 marks*

(ii) Hence sketch $y=\frac{1-\cos x}{\sin x}$ for $-\pi < x < \pi$ *2 mark*

(c) The graphs of $y=x^3-x$ and $y=x^2+1$ intersect at $P(x,y)$ as shown in the diagram.



(i) Show that $1<x<2$. *2marks*

(ii) Taking $x=1.8$ as a first approximation to the x-value of P, use one application of Newton's method to find a closer value for x. *3marks*

Question 3 (12 marks) **Start a new booklet**

(a)

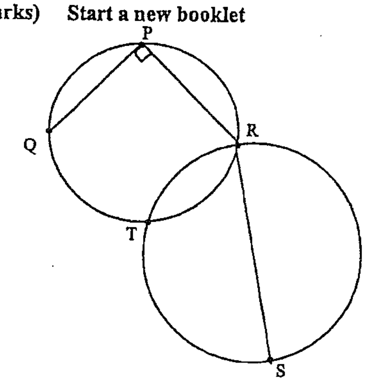


Diagram not to scale

RS is a diameter. PQ is perpendicular to PR.
Prove that Q, T and S are collinear.

3marks

(b) (i) State the domain and range of

$$y=2\sin^{-1}3x.$$ *2marks*

(ii) Sketch $y=2\sin^{-1}3x$. *1mark*

(iii) The graph of $y=2\sin^{-1}3x$ is rotated about the y-axis. Show that the volume generated is $\frac{\pi^2}{9}$ $units^3$. *4marks*

(c) Julian has 10 different pairs of socks where the left sock and right sock of each pair are indistinguishable.

Find the number of odd pairs of socks (ie a pair which do not match) that Julian can wear. Explain your reasoning.

2marks

Question 4 (12 marks) **Start a new booklet**

(a) Find all solutions to $\cos x=\frac{\sqrt{3}}{2}$. *2marks*

(b) (i) Show that $\frac{1}{(x+1)(x+2)}=\frac{1}{x+1}-\frac{1}{x+2}$ *1mark*

(ii) Hence evaluate $\int_0^1 \frac{dx}{(x+1)(x+2)}$ *2marks*

(c) Tangents from the point $T(x_0,y_0)$ touch the parabola $x^2=4y$ at $P(x_1,y_1)$ *and* $Q(x_2,y_2)$.

(i) State the equation of the chord of contact. *1 mark*

(ii) Show that the x-values of P and Q are given by the roots of the equation $x^2-2x_0x+4y_0=0$ *2marks*

(iii) Hence or otherwise prove that the midpoint M of QP is $\left(x_0, \frac{1}{2}x_0^{\ 2} - y_0\right)$. *2marks*

(iv) If T moves on the line $y=x-1$ find the equation of the locus of M. *2marks*

Question 5 **(12 marks)** **Start a new booklet**

(a) Prove by Mathematical Induction that the expression 5^n-1 is divisible by 4 for all positive integers n. *4marks*

(b) Metal Fatigue is a phenomenon where a piece of steel will fail when repeatedly subjected to a force F. The endurance limit is the force below which the steel will not break even if subjected to an infinite number of applications of that force. Let the number of applications be n.

The force and the number of applications are related by the differential equation

$\frac{dF}{dn} = -k(F - F_0)$ where k and F_0 are constants.

(i) Show that $F = 275e^{-k(n-1)} + F_0$ is a solution to $\frac{dF}{dn} = -k(F - F_0)$ *1 mark*

(ii) If F=350 when n=1, find the value of F_0 *1mark*

(iii) Find the endurance limit. *1 mark*

(iv) Find the value of k if F=80 when n=200. *2marks*

(c) In today's society, statistics show that 28% of Australian women will never have children. Three women are selected at random. Find the probability that

(i) they will all have children *1marks*

(ii) at least one of them will have children *2marks*

Question 6 **(12 marks)** **Start a new booklet**

(a) Consider the function $f(x) = e^{-x^2}$

(i) Show that the function is even. *1mark*

(ii) Find the stationary point of $y=f(x)$. *1mark*

(iii) Show that $\frac{d^2y}{dx^2} = -2e^{-x^2}(1 - 2x^2)$ and hence find any points of inflexion. *2marks*

(iv) Sketch the curve of $y=f(x)$, $x \ge 0$. *1 mark*

(v) Sketch the inverse function $f^{-1}(x)$ of $f(x) = e^{-x^2}$ $x \ge 0$. *1mark*

(vi) Find the equation of $f^{-1}(x)$ and state its domain. *3marks*

(b) Malaysia has invented a minirail system which is completely automated, running to precision timing (ignoring passenger boarding and alightment). The journey between two stations A and B, where A is to the west of B, can be modelled by the equation $v^2 = 2(8x - x^2 - 7)$ where v is velocity in km/h and x is displacement in km from the central automated control office.

(i) Show that the motion is simple harmonic. *1mark*

(ii) Find the distance between the two stations. *1mark*

(iii) Where is the control office in relation to A and B? *1mark*

Question 7 (12 marks) Start a new booklet

(a)

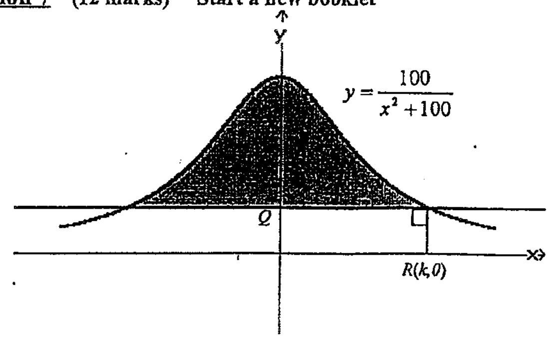


Diagram not to scale

The cross-section of a light fabric structure for a stadium roof is described by the equation $y=\frac{100}{x^2+100}$.
Dimensions are in metres.

(i) If Q is the point $\left(0,\frac{1}{4}\right)$ find the value of k. *1 mark*

(ii) Show that the shaded area is $\frac{5\left(4\pi-3\sqrt{3}\right)}{3}$ square metres. *2marks*

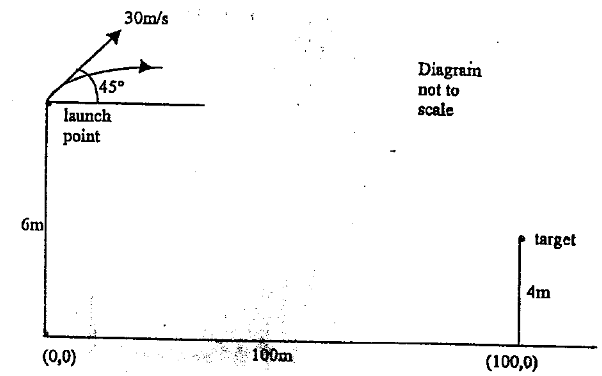
(iii) By considering the integral $\int_{-k}^{k}\frac{100}{x^2+100}\,dx$ or otherwise show that the area of the cross-section will never exceed 10π square metres. *2marks*

(b)

By considering the coefficient of x^{n+1} on both sides of the identity $(x+1)^n(x+1)^n=(x+1)^{2n}$ prove that

$${}^nC_0\,{}^nC_1+{}^nC_1\,{}^nC_2+{}^nC_2\,{}^nC_3+\ldots\ldots+{}^nC_{n-1}\,{}^nC_n=\frac{(2n)!}{(n-1)!(n+1)!}$$ *3marks*

(c)



One of the great historic problems which prompted the development of calculus was whether a cannonball would reach a target. Using the origin as shown and assuming $\ddot{x}=0$ and $\ddot{y}=-10$, if a cannonball is fired at an angle of 45 degrees at a velocity of 30m/s,

(i) show that

$x=15t\sqrt{2}$

$y=-5t^2+15t\sqrt{2}+6$ *2marks*

(ii) Hence determine whether or not the ball will reach its target. *2marks*