SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2004

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in 3 bundles.
  Section A (Questions 1 - 3),
  Section B (Questions 4 - 5) and
  Section C (Questions 6 - 7).
- Start each Section in a **NEW** answer booklet.

Total Marks - 84 Marks

- Attempt questions 1- 7
- All questions are of equal value.

Examiner: *R. Boros*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 84
Attempt Questions 1 – 7
All questions are of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

SECTION A (Use a SEPARATE writing booklet)

Question 1 (12 marks) **Marks**

(a) Solve for *x*: $\left(x^2-1\right)\left(x+5\right)>0$ **2**

(b) Differentiate $y=\ln\sqrt{x+1}$ for $x>-1$ **2**

(c) Use the Table of Integrals provided to evaluate **2**

$$\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x \, dx$$

(d) Find the exact value of $\displaystyle\int_0^{\sqrt{3}} \frac{1}{9+x^2}\,dx$ **2**

(e) 8 people including A and B are to be seated around a circle. **2**

How many arrangements are possible if A and B do not wish to sit together?

(f) Show that $\dfrac{1-\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{1+\cos\theta}=2\tan\dfrac{\theta}{2}$ **2**

Question 2 (12 marks) **Marks**

(a) Differentiate $y = \sin^{-1} 2x$ **2**

(b) Find the domain and range of $y = 3\sin^{-1}\sqrt{1-x^2}$ **2**

(c) (i) Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos\left(x+\alpha\right)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$. **2**

(ii) Hence or otherwise, find the general solution for **2**

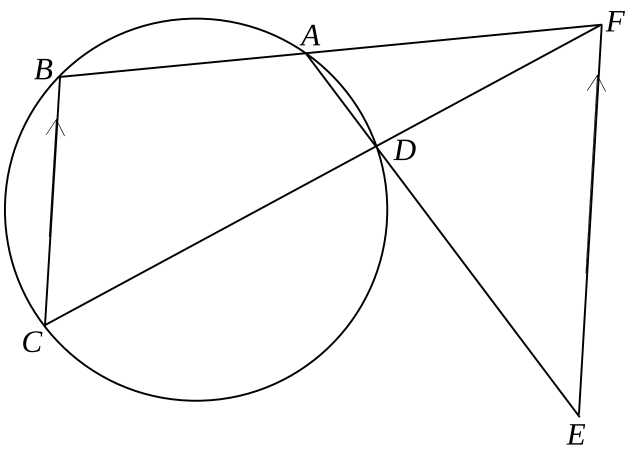
$$\sqrt{3}\cos x - \sin x = 1$$

(d) In the diagram below $ABCD$ is a cyclic quadrilateral.

BA is produced to F.

$BC \parallel FE$

CF and AE meet at D.



Copy or trace the diagram into your answer booklet.

(i) Show that $\Delta DEF \,|||\, \Delta FEA$ **2**

(ii) Hence show that $\left(EF\right)^2 = EA \times ED$ **2**

Section A is continued on page 4

Question 3 (12 marks) **Marks**

(a) Use the Principle of Mathematical Induction to show that $2^{3n}-1$ is divisible by 7 for all integers $n \geq 1$. **3**

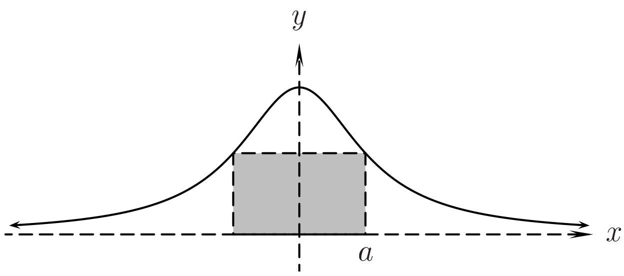
(b) For the curve $y = 1 + 2\cos x - 2\cos^2 x$,

(i) Show that $\dfrac{dy}{dx} = 2\sin x\left(2\cos x - 1\right)$ **1**

(ii) Hence find the stationary point(s) in the interval $-\dfrac{\pi}{6} \leq x \leq \dfrac{\pi}{2}$ **2**

(iii) Sketch the curve and find the greatest and least value of y in $-\dfrac{\pi}{6} \leq x \leq \dfrac{\pi}{2}$ **2**

(c)



A rectangle is inscribed under the curve $y = \dfrac{1}{1+x^2}$, as shown in the diagram above, such that the rectangle is symmetrical about the y axis.

(i) Show that the area of the rectangle is given by $\dfrac{2a}{1+a^2}$. **1**

(ii) Find the value of a that produces the maximum area of the rectangle and what is this maximum area? **3**

END OF SECTION A

SECTION B (Use a SEPARATE writing booklet)

Question 4 (12 marks) **Marks**

(a) (i) Show that the equation of the tangent at $T\left(-2t, t^2\right)$ on the parabola $y = \frac{1}{4}x^2$ is given by $tx + y + t^2 = 0$. **2**

(ii) $M\left(x, y\right)$ is the midpoint of the interval *TA* where *A* is the x intercept of the tangent at *T*. **2**

Find the equation of the locus of *M* as *T* moves on the parabola.

(b) Solve $4x^3 - 12x^2 + 11x - 3 = 0$ if the roots are the terms of an arithmetic series. **3**

(c) (i) Find the points of intersection of the curves $y = 2\cos x$ and $y = \frac{1}{2}\sec x$ in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$. **2**

(ii) The area enclosed between the two curves listed above is rotated $360°$ about the x axis. **3**

Find the volume of the solid of revolution.
(Leave your answer in exact form.)

Section B is continued on page 6

Question 5 (12 marks) **Marks**

(a) A spherical balloon leaks air such that the radius decreases at a rate of 5 cm/second. **2**

Calculate the rate of change of the volume of the balloon when the radius is 100 mm.

[The volume of a sphere is $V = \frac{4}{3}\pi r^3$]

(b) A particle moves in such a way that its displacement x cm from the origin O after a time t seconds is given by

$$x = 2\cos\left(t + \frac{\pi}{6}\right) \text{ cm}$$

(i) Show that the particle moves in Simple Harmonic Motion. **2**

(ii) Evaluate the period of the motion. **1**

(iii) Find the time at which the particle first passes through the origin on its first oscillation. **1**

(iv) Find the velocity when the particle is 1 cm from the origin on its first oscillation. **2**

(c) Find $\int \sqrt{16 - x^2}\, dx$ using the substitution $x = 4\sin\theta$. **4**

END OF SECTION B

SECTION C (Use a SEPARATE writing booklet)

Question 6 (12 marks) **Marks**

(a) Find a primitive function for $\dfrac{3x}{4+x^2}$ **1**

(b) If $P(x) = 8x^3 - 12x^2 + 6x + 13$,

(i) For what values of x is $P(x)$ increasing? **1**

(ii) Show that $P(x)$ has only one zero, x_1 and that $x_1 < 0$. **1**

(iii) Taking $x = -1$ as a first approximation to $P(x) = 0$, find a second approximation for x_1, using Newton's Method. **2**

[Express your answer correct to 2 decimal places.]

(c) At any time *t*, the rate of cooling of the temperature *T* of a body, when the surrounding temperature is *S*, is given by the differential equation

$$\frac{dT}{dt} = -k(T - S)$$

for some constant *k*.

(i) Show that $T = S + Ae^{-kt}$, for some constant *A*, satisfies this differential equation. **2**

(ii) A metal rod has a temperature of $1390°$ C and cools to $1060°$ C in 10 minutes when the surrounding temperature is $30°$ C. **3**

Find how much *longer* it will take the rod to cool to $110°$ C, giving your answer to the nearest minute.

(iii) Sketch the graph of the function $T = S + Ae^{-kt}$, using the values of *S*, *A* and *k* found above. **2**

Section C continues on page 8

SECTION C continued

Question 7 (12 marks) **Marks**

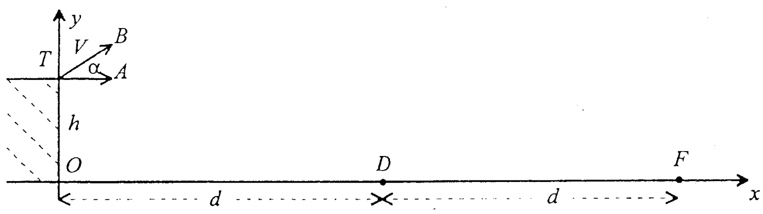
(a) Using the expansion of $(1+x)^n$

(i) Find an expression for $\sum_{r=1}^{n} r\binom{n}{r}$ **2**

(ii) Hence, or otherwise, prove that $\sum_{r=0}^{n}(r+1)\binom{n}{r} = 2^{n-1}(n+2)$ **2**

(b) *T* is the top of a building, *h* metres high. The points *O*, *D* and *F* are in the same line on flat level ground.
O is the base of the building.
D is *d* metres from O, and *F* is a further *d* metres from *D*.
At time $t=0$, two particles *A* and *B* are projected with the same initial velocity *V* m/s from *T*.
Particle *A* is projected horizontally and particle *B* is projected in the same direction, but at an angle α, $\alpha > 0$, to the horizontal.
The equations of motion of both particles are

$$\ddot{x} = 0 \text{ and } \ddot{y} = -g$$



(i) Assuming that the position of particle *A* at time *t* is given by **1**

$$x = Vt,\ y = -\frac{1}{2}gt^2 + h$$

show that the Cartesian equation of the trajectory is given by

$$y = h - \frac{g}{2V^2}x^2$$

(ii) Assuming that the position of particle *B* at time *t* is given by **1**

$$x = Vt\cos\alpha \text{ and } y = -\frac{1}{2}gt^2 + Vt\sin\alpha + h$$

show that the Cartesian equation of the trajectory is given by

$$y = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha + h$$

(iii) If *A* lands at *D* show that $h = \frac{gd^2}{2V^2}$ **1**

(iv) If both *A* and *B* land at *D* show that $\tan\alpha = \frac{d}{h}$ **2**

(v) If *A* lands at *D* and *B* lands at *F* show that $d \geq 2h\sqrt{3}$ **3**

End of paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$