The Scots College 4 Unit trial 1991.

Question 1. (15 marks)

a) Evaluate

i) $\displaystyle\int_0^1 \frac{dx}{(x+1)(x+3)}$

ii) $\displaystyle\int_0^1 \sqrt{4-x^2}\,dx$

iii) $\displaystyle\int_{-1}^2 x\sqrt{2-x}\,dx$

b) Find $\displaystyle\int x^2 e^{-x}dx$

c) If $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x\,dx$, show that $I_n = \dfrac{n-1}{n} I_{n-2}$.

Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x\,dx$

Question 2. (15 marks)

a) Reduce the polynomial $P(x) = x^4 - 2x^2 - 15$ into irreducible factors over
i) the rational field Q;
ii) the real field R;
iii) the complex field C.

b) Divide the polynomial $x^3 + 5ix^2 - 7ix - 3$ by $(x - 2i)$ using long division.

c) Show that $2 - \sqrt{3}$ is a zero of the polynomial $a(x) = x^3 - 15x + 4$.
Hence reduce $a(x)$ to irreducible factors over the real field.

d) Given that the polynomial $P(x) = x^4 + x^2 + 6x + 4$ has a rational zero of multiplicity 2, find all the zeros of $P(x)$ over the complex field.

e) If α, β, γ are the roots of the equation
$x^3 + qx + r = 0$,
where $r \neq 0$, obtain as functions of q and r, in their simplest forms, the coefficients of the cubic equation whose roots are $\dfrac{1}{\alpha^2}, \dfrac{1}{\beta^2}, \dfrac{1}{\gamma^2}$.

Question 3. (15 marks)

a) i) Define the modulus $|z|$ of a complex number z.

ii) Given two complex numbers z_1, z_2 prove that $|z_1 z_2| = |z_1||z_2|$;

b) Given

$$w = \frac{2-3i}{1-i}$$

determine

i) $|w|$ (the modulus of w);
ii) $\bar{w}$ (the conjugate of w);
iii) $w + \bar{w}$.

c) Describe, in geometric terms, the locus (in the Argand plane) represented by
$2|z| = z + \bar{z} + 4$.

Question 4. (15 marks)

a) Determine the real values of k for which the equation
$$\frac{x^2}{19-k} + \frac{y^2}{7-k} = 1$$
defines respectively an ellipse and a hyperbola.

Sketch the curve corresponding to the value $k = 3$.

Describe how the shape of this curve changes as k increases from 3 towards 7. What is the limiting position of the curve as 7 is approached?

b) P is a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with centre O. A line drawn through O, parallel to the tangent to the ellipse at P, meets the ellipse at Q and R.

Prove that the area of triangle PQR is independent of the position of P.

Question 5. (15 marks)

a) Sketch the curve $y^2 = x^2(x-2)(x-3)$.

b) In the Cartesian plane sketch the curve

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

and prove that the lines $y = \pm 1$ are asymptotes.

Also, if k is a positive constant, find the area in the positive quadrant enclosed by the above curve and the three lines
$y = 1,\ x = 0,\ x = k$
and prove that this area is always less than $ln2$, however large k may be.

Question 6. (15 marks)

a) The area bounded by the curve $y = \frac{1}{x+1}$, the x-axis, the line $x = 2$ and the line $x = 8$, is rotated about the y-axis. Find the volume of the solid generated using the method of cylindrical shells.

b) i) Using the substitution $x = a\sin\theta$, or otherwise, verify that

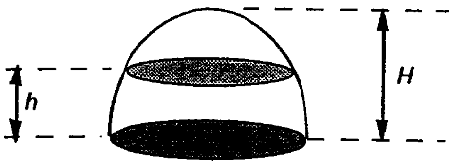
$$\int_0^a \sqrt{a^2 - x^2}\,dx = \tfrac{1}{4}\pi a^2.$$

ii) Deduce that the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is πab.

iii)



The diagram shows a mound of height H. At height h above the horizontal base, the horizontal cross-section of the mound is elliptical in shape, with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2,$$

where

$$\lambda = 1 - \frac{h^2}{H^2},$$

and x, y are appropriate coordinates in the plane of cross-section.

Show that the volume of the mound is $\frac{8\pi abH}{15}$.

Question 7. (15 marks)

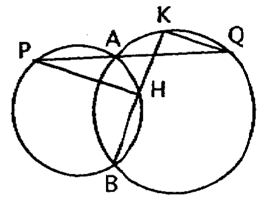
a) Six letters are chosen from the letters of the word AUSTRALIA. These six letters are then placed alongside one another to form a six letter arrangement. Find the number of distinct six letter arrangements which are possible, considering all the choices.

b) Solve for x the following inequation

$$\frac{x^2 - 5x}{4 - x} \le -3$$

and show the solutions on a number line.

c)



In the figure PAQ and BHK are straight lines. Prove that PH is parallel to KQ.

d) Two circles, centres B and C, touch externally at A. PQ is a direct common tangent touching the circles at P and Q respectively.
i) Draw a neat diagram depicting the given information;
ii) Prove that the circle with BC as diameter touches the line PQ.

Question 8. (15 marks)

a) An aeroplane flies horizontally due East at a constant speed of 240 km/h. From a point P on the ground the bearing of the plane at one instant is 311°T and 3 minutes later the bearing of the plane is 073°T whilst its elevation then is 21°. If h metres is the altitude of the plane, show that

$$h = 12000\sin 41^\circ \tan 21^\circ \operatorname{cosec} 58^\circ$$

and calculate h correct to the nearest metre.

b) The magnitude and direction of the acceleration due to gravity at a point outside the Earth at a distance x from the Earth's centre is equal to $-\frac{k}{x^2}$, where k is a constant.

i) Neglecting atmospheric resistance, prove that if an object is projected upwards from the Earth's surface with speed u, its speed v in any position is given by

$$v^2 = u^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right)$$

where R is the Earth's radius and g is the magnitude of the acceleration due to gravity at the Earth's surface.

ii) Show that the greatest height, H, above the Earth's surface reached by the particle is given by

$$H = \frac{u^2R}{2gR - u^2}.$$

iii) Hence, if the radius of the Earth is approximately 6400km, and the acceleration due to gravity at the Earth's surface is 9.8 m/s^2, find the speed required by the particle to escape the Earth's gravitational influence.

End of Paper.