Doonside Technology High School

Extension 2 Mathematics

Trial HSC Examination 2001

Total marks (120)

Attempt Questions 1-8

All questions are of equal value

Answer each question starting a FRESH SHEET with your name and the question number at the top. Extra writing booklets are available.

Question 1 *(15 marks)* Use a SEPARATE writing booklet **Marks**

(a) Find $\int x\cos(x^2)dx$ 1

(b) Using the substitution $x = 2\sin\theta$ evaluate $\int_0^2 \sqrt{4-x^2}\,dx$ 4

(c) Using the method of partial fractions find $\int \frac{-4dx}{x^2+2x-3}$ 4

(d) Find $\int \frac{x^2+2x-3}{x+1}dx$ 4

(e) Using integration by parts evaluate $\int_1^e \ln x.dx$ 2

Question 2 *(15 marks)* Use a SEPARATE writing booklet **Marks**

(a) If A = 3+4i and B = 2-i

Express the following in the form $x + iy$ where x and y are real numbers :

(i) AB **1**

(ii) $\sqrt{A}$ **2**

(iii) $\dfrac{A}{B}$ **2**

(b) If $z = \sqrt{3} + i$

(i) Find the exact values of mod(z) and arg(z) **2**

(ii) By using your answers to (i) and De Moivre's theorem write z^5 in the form $a + ib$ **2**

(c) On an Argand diagram shade the region containing all the points representing the complex numbers z such that: **2**

$|z-1| \leq 2$ and $\dfrac{\pi}{4} < \arg(z-1) < \dfrac{\pi}{2}$

(d) Explain algebraically or geometrically why the locus described by **2**

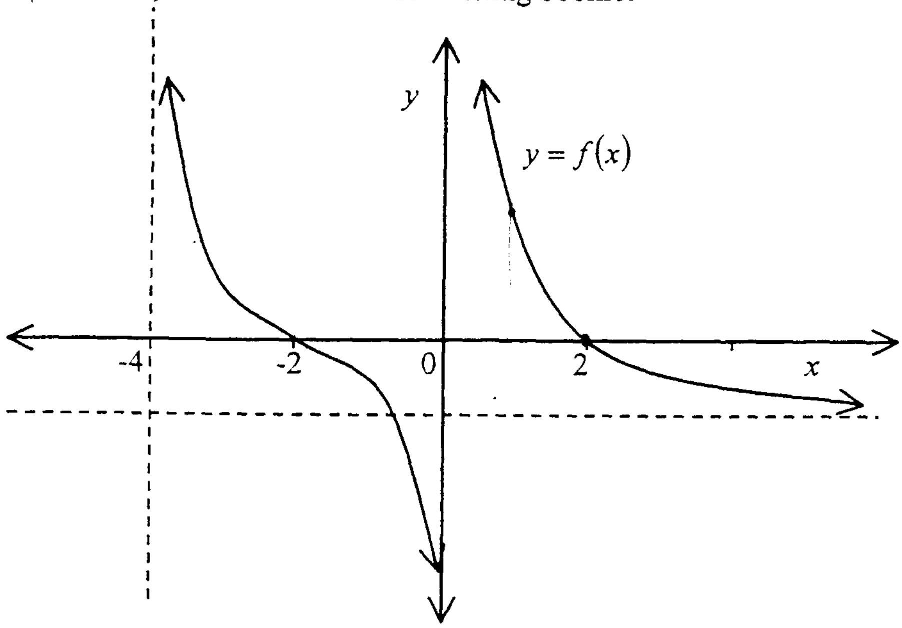
$\arg\left(\dfrac{z}{z-4}\right) = \dfrac{\pi}{2}$ is a circle.

(e) Given that z and w represent two complex numbers, explain why **2**

$|z| + |w| \geq |z - w|$

Question 3 *(15 marks)* Use a SEPARATE writing booklet **Mark**

(a)



The sketch above shows the graph of the function $y = f(x)$. There is a horizontal asymptote at $y = -1$ and vertical asymptotes at $x = 0$ and $x = -4$. Draw separate sketches of the following functions

(i) $y = |f(x)|$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \int f(x)dx$ **2**

(b) Sketch the following curves on separate axes for each part showing all intercepts and turning points.

(i) $y = \cos 3x$ and hence $y = \cos^2 3x$ (in the domain $-\pi \le x \le \pi$) 4

(ii) $y = \dfrac{(x-1)(x+3)}{(x+2)(x-2)}$ (in the domain $-5 \le x \le 4$) 5

Question 4 *(15 marks)*

(a) An ellipse has equation $\frac{x^2}{25}+\frac{y^2}{9}=1$

(i) Show that this is the equation of the locus of a point P*(x,y)* moving such that the sum of its distances from A (4, 0) and B (- 4, 0) is 10 units. **4**

(ii) Calculate the eccentricity of this ellipse. **1**

(iii) State the equations of the directrices of this ellipse. **1**

(iv) Find the equation of the tangent to the curve at a point Q (a, b) which lies on the ellipse. **2**

) $P\left(5p,\frac{5}{p}\right), p>0$ and $Q\left(5q,\frac{5}{q}\right), q>0$ are two points on the hyperbola, *H*, $xy=25$.

i) Derive the equation of the chord *PQ*, 2

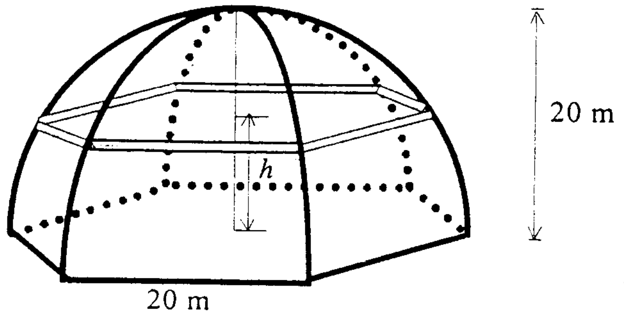
ii) State the equations of the tangents at *P* and *Q*, 1

iii) If the tangents at *P* and *Q* intersect at *R*, find the co-ordinates of *R*. 2

iv) If the secant *PQ* passes through the point $S(15,0)$, find the locus of *R*. 2

Question5 *(15 marks)*

a A dome is sitting on a regular hexagonal base of side 20 metres. The height of the dome is also 20 metres. Each strut of the dome is a quarter of a circle with its centre at the centre of the hexagonal base.

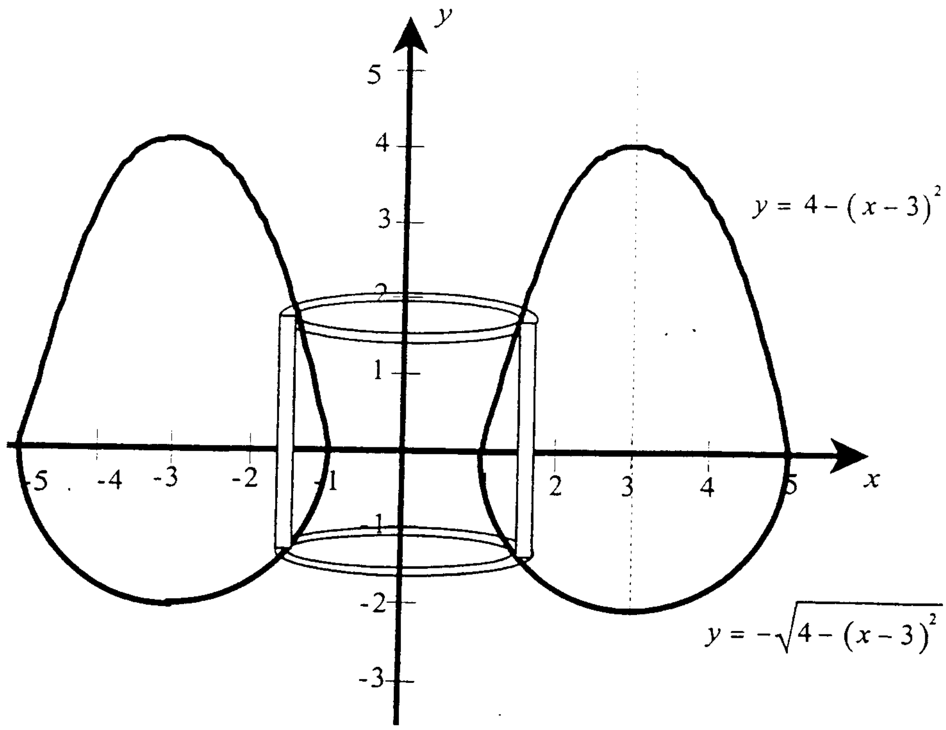


i) If the slice is h metres above the base, show that the length of each side is $\sqrt{400-h^2}$ [2]

ii) Show that the area of the cross-section is $A = \frac{3\sqrt{3}}{2}\left(400-h^2\right)$ 2

iii) Hence, or otherwise calculate the volume of the solid. 3

b)



The area in the diagram is composed of a parabola $y = 4-(x-3)^2$ surmounted on a semi-circle $y = -\sqrt{4-(x-3)^2}$, as shown. This area is rotated about the y-axis.

Use the method of cylindrical shells to calculate the volume generated. 8

[Hint: Explain and show that $V = \int_1^5 2\pi x[4-(x-3)^2+\sqrt{4-(x-3)^2}\,]dx$. For the second part of the integral, use the substitution $x-3 = 2\sin\theta$]

Question 6 *(15 marks)*

a) When $x^3 - kx^2 - 10kx + 25$ is divided by $x - 2$ the remainder is 9. Find the value of k. 2

b) A polynomial function is $P(x) = x^5 + x^4 + 13x^3 + 13x^2 - 48x - 48$. Factorise $P(x)$ over the field of

i) real numbers, 2

ii) complex numbers. 1

(c) Factorise $x^4 - 16$ fully over the complex field. 2

(d) Solve the equation $4x^3 - 8x^2 + 5x - 1 = 0$ given that it has a double root. 3

(e) The equation $x^3 - 6x^2 + 7x - 3 = 0$ has roots $\alpha, \beta,$ and γ

Write an equation which has roots $\alpha^2, \beta^2,$ and γ^2. 2

ii) Write an equation which has roots $\frac{1}{\alpha}, \frac{1}{\beta},$ and $\frac{1}{\gamma}$. 2

(iii) It is known that the solution to given a problem is the average of the roots of the equation $x^3 - 6x^2 + 7x - 3 = 0$ 1

Without finding the roots determine the solution to the problem.

Question 7 *(15 marks)*

(a) Prove the identity $\dfrac{\cos y - \cos(y+2q)}{2\sin q} = \sin(y+q)$ 2

(b) Use mathematical induction and the result in part (a) to prove the identity

$$\sin q + \sin 3q + \sin 5q + \ldots + \sin(2n-1)q = \frac{1-\cos 2nq}{2\sin q}$$ 3

(c) (i) Find the domain of $f(x) = \sin^{-1}(2x - 1)$ 1

(ii) Sketch the graph of $y = \sin^{-1}(2x - 1)$ 1

(iii) Solve $\sin^{-1}(2x - 1) = \cos^{-1} x$. 4

(c) A box contains 6 cards, two of which are identical. From this box 3 cards are drawn without replacement.

(i) How many different selections could be made. 2

(ii) What is the probability that a selection will include the two identical cards. 2

Question 8 *(15 marks)*

(a) Solve for z if $z^5 = 1$

(b) By noting that $z^n + z^{-n} = 2\cos n\theta$ and that z is the complex number $\cos\theta + i\sin\theta$, show that

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

i) Show that, if $I_n = \int_0^i \left(1 + x^2\right)^n dx$, then $I_n = \frac{2n}{2n+1} I_{n-1}$ 4

ii) Hence find $\int_0^i \left(1 + x^2\right)^3 dx$ 2

‹) By expanding $(1 + x)^{n+2}$ in two different ways, show that 3

$$\binom{n+2}{r} = \binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$$