SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2000

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Sample Solutions

(Q1) (a) $y = 4\sin^{-1}3x$

D: $-1 \le 3x \le 1$ R: $\frac{-\pi}{2} \le \frac{y}{4} \le \pi/2$

$-\frac{1}{3} \le x \le \frac{1}{3}$ $-2\pi \le y \le 2\pi$

(b) $(x-2)^2 \le 4$

$\therefore$ $-2 \le x-2 \le 2$

$\therefore$ $0 \le x \le 4$

(c) (i) $\frac{d(x\cos^{-1}2x)}{dx} = \cos^{-1}2x - x \times \frac{2}{\sqrt{1-4x^2}}$

$= \cos^{-1}x - \frac{2x}{\sqrt{1-4x^2}}$

(ii) $\frac{d\left(\frac{1}{4+x^2}\right)}{dx} = \frac{d(4+x^2)^{-1}}{dx}$

$= -(4+x^2)^{-2} \times 2x$

$= \frac{-2x}{(4+x^2)^2}$

(d) $x^{3/4} = 10$

$\therefore x = 10^{4/3}$

$\doteqdot 21.544$

e) $P\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$

$A(x_1, y_1)$. $B(6,5)$ $P(11,7)$ $\begin{matrix} m & n \\ 3 & : -1 \end{matrix}$

$\quad\quad\quad\quad x_2 \; y_2$

$\therefore$ $11 = \frac{3\times6 - x_1}{2}$, $7 = \frac{3\times5 - y_1}{2}$

$18 - x_1 = 22$, $15 - y_1 = 14$

$x_1 = -4$ $y_1 = 1$

$A(-4, 1)$

(2) (a) the acute angle is 45°

$$\tan 45 = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \qquad m_2 = \frac{1}{2}$$

$$\therefore \tan 45^\circ = \left|\frac{m_1 - \frac{1}{2}}{1 + m_1 \times \frac{1}{2}}\right|$$

$$\therefore \left|\frac{m_1 - \frac{1}{2}}{1 + \frac{m_1}{2}}\right| = 1 \Rightarrow \left|\frac{m_1 - \frac{1}{2}}{2 + m_1}\right| = 2$$

$$\therefore \frac{m - \frac{1}{2}}{2 + m} = 2 \quad , \quad \frac{m - \frac{1}{2}}{2 + m} = -2$$

$$m - \frac{1}{2} = 4 + 2m \qquad m - \frac{1}{2} = -4 - 2m$$

$$m = 4\tfrac{1}{2} \quad , \quad m = -3\tfrac{1}{2}$$

$$\therefore \boxed{m = -3\tfrac{1}{2},\ 4\tfrac{1}{2}}$$

(b) $u = \sqrt{x} \Rightarrow x = u^2$

$\therefore dx = 2u\,du$

$$\int_1^4 \frac{dx}{x + \sqrt{x}}$$

$x = 1 \Rightarrow u = 1$

$x = 4 \Rightarrow u = 2$

$$= \int_1^2 \frac{2u\,du}{u^2 + u}$$

$$= \int_1^2 \frac{2}{u+2}$$

$$= \ln|u+2| \Big]_1^2$$

$$= \ln 4 - \ln 3$$

$$= \ln\left(\frac{4}{3}\right)$$

(c) $\cos^{-1}\left(\cos \frac{4\pi}{3}\right)$

$= \cos^{-1}\left(\frac{-1}{2}\right)$

$= \frac{2\pi}{3}$

(d) (i) $\displaystyle\int \frac{2\,dx}{\sqrt{1 - 4x^2}} = \sin^{-1}(2x) + C$

(ii) $\displaystyle\int \frac{x}{4 + x^2}\,dx$

$= \displaystyle\frac{1}{2}\int \frac{2x}{4 + x^2}\,dx$

$= \frac{1}{2}\ln(x^2 + 4) + C$

(2) (e)

$$f(x) = e^{-ax}(x-a)$$

$$f'(x) = e^{-ax} + (x-a) \times -ae^{-ax}$$

$$= e^{-ax}\left(1 - a(x-a)\right)$$

$$f'(\tfrac{5}{2}) = 0$$

$e^{-ax} \neq 0 \quad \therefore \; 1 - a\left(\frac{5}{2} - a\right) = 0$

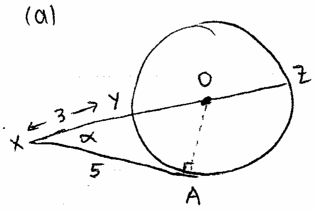
$$\therefore\; 2 - 5a + 2a^2 = 0$$

$$\therefore\; 2a^2 - 5a + 2 = 0$$

$$(2a-1)(a-2) = 0$$

$$a = \tfrac{1}{2}, 2$$

(3) (a)



$$XZ \cdot XY = XA^2$$

$$\therefore\; 25 = 3 \times XZ$$

$$XZ = \frac{25}{3} = 8\tfrac{1}{3}$$

$$\therefore\; YZ = 5\tfrac{1}{3}$$

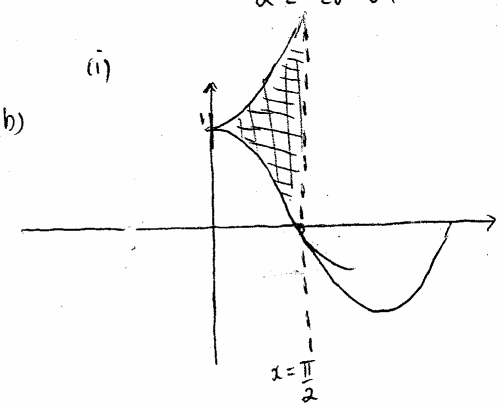
$$\therefore\; OA = 8/3$$

Let $\alpha = \angle AXY$

$$\tan\alpha = \frac{8/3}{5} = 8/15$$

$$\alpha = 28^\circ 04'$$

(b) (i)



$$\text{Area} = \int_0^{\frac{\pi}{2}} (e^x - \cos x)\,dx$$

$$= \left. e^x - \sin x \right]_0^{\pi/2}$$

$$= \left(e^{\pi/2} - \sin\tfrac{\pi}{2}\right) - \left(e^0 - \sin 0\right)$$

$$= e^{\pi/2} - 1 - 1$$

$$= e^{\pi/2} - 2$$

3 (b) (ii)

$$V = \pi \int_0^{\pi/2} (e^{2x} - \cos^2 x)\, dx \qquad \left[\cos^2 x = \frac{1}{2}(1 + \cos 2x)\right]$$

$$= \pi \int_0^{\pi/2} \left(e^{2x} - \frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \pi \left[\frac{1}{2}e^{2x} - \frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi/2}$$

$$= \pi \left[\left(\frac{1}{2}e^{\pi} - \frac{\pi}{4}\right) - \left(\frac{1}{2}\right)\right]$$

$$= \frac{\pi}{2}\left(e^{\pi} - \frac{\pi}{2} - 1\right)$$

(c) (i)

$$\text{RHS} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

$$= \frac{d\left(\frac{1}{2}v^2\right)}{dv} \times \frac{dv}{dx}$$

$$= v\frac{dv}{dx} = \frac{dx}{dt} \cdot \frac{dv}{dx}$$

$$= \frac{dv}{dt}$$

$$= a$$

$$= \ddot{x}$$

$$= \text{LHS}$$

(ii) $v^2 = 36 - 4x^2$

$\frac{1}{2}v^2 = 18 - x^2 \Rightarrow a = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$

(α) $a = -2x$

This is one of the defining equations for SHM, centred at $x=0$

(β) $n^2 = 2 \Rightarrow n = \sqrt{2}$

$T = \frac{2\pi}{n} = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$

$v = 0 \Rightarrow x^2 = 18$

$x = \pm 3\sqrt{2}$

$\therefore$ Amplitude $= 3\sqrt{2}$ m

(4) (a) $P(x) = ax^3 + bx + c$

$P(1) = -4 \quad \Rightarrow \quad a+b+c = -4 \quad -①$

$P(x) = (x^2-4)Q(x) + (-4x+3)$

$\therefore P(2) = -5 \quad \Rightarrow \quad 8a+2b+c = -5 \quad -②$] +

$P(-2) = 11 \quad \Rightarrow \quad -8a-2b+c = 11 \quad -③$

② + ③ : $2c = 6$

$c = 3$

① $\Rightarrow \quad a+b = -7 \quad -④$] −

② $\Rightarrow \quad 8a+2b = -8 \quad -⑤$

$4a+b = -4 \quad -⑥$

⑥ − ④ $\quad 3a = 3$

$a = 1$

sub into ④ $\quad b = -8$

$\therefore \; a = 1, \; b = -8, \; c = 3$

(b) $1 + (1+2) + (1+2+3) + \cdots + (1+2+3+\cdots+n) = \frac{n}{6}(n+1)(n+2), \quad n > 0$

using the sum of an arithmetic series

ie. $1 + (1+2) + \cdots + \frac{n(n+1)}{2} = \frac{n}{6}(n+1)(n+2), \quad n>0 \quad -(*)$

Test $n=1$: $\quad LHS = 1$

$RHS = \frac{1}{6}(2)(3) = 1$

$\therefore$ true for $n=1$

Assume $(*)$ is true for some integer $n=k$.

ie. $1 + (1+2) + \cdots + \frac{k(k+1)}{2} = \frac{k}{6}(k+1)(k+2)$

We need to prove $(*)$ is true for the integer $n=k+1$

i.e. $1 + (1+2) + \cdots + \frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)(k+3)}{6}$

4(b)

$$LHS = 1 + (1+2) + \cdots + \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2}$$

$$= \frac{k}{6}(k+1)(k+2) + \frac{(k+1)(k+2)}{2}$$

$$= (k+1)(k+2)\left[\frac{k}{6} + \frac{1}{2}\right]$$

$$= \frac{(k+1)(k+2)(k+3)}{6}$$

$$= \frac{1}{6}(k+1)(k+2)(k+3)$$

$$= RHS$$

$\therefore$ Since the statement is true for $n=k+1$ WHEN the statement is true for $n=k$. By the principle of mathematical induction

$$1 + (1+2) + \cdots + (1+2+\cdots+n) = \frac{n}{2}(n+1)(n+2), \quad n > 0$$

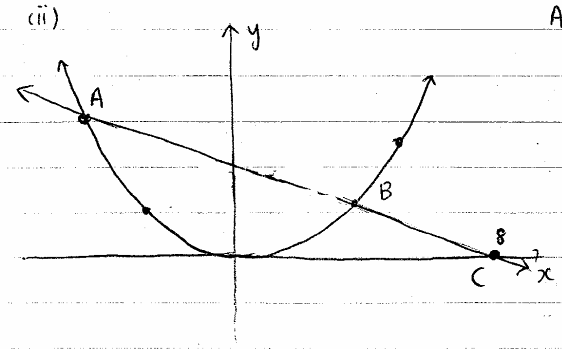
(c) (i) $A(6p, 3p^2)$

$LHS = x^2 = 36p^2$

$RHS = 12y = 12(3p^2) = 36p^2$

$\therefore$ A lies on $x^2 = 12y$

(ii)



$\quad x_1 \quad y_1 \qquad\quad x_2 \quad y_2$

$A(6p, 3p^2) \quad B(6q, 3q^2)$

$$m_{AB} = \frac{3q^2 - 3p^2}{6q - 6p} = \frac{3(q-p)(q+p)}{6(q-p)}$$

$$= \frac{q+p}{2}$$

$$\therefore y - 3q^2 = \frac{q+p}{2}(x - 6q)$$

$$2y - 6q^2 = (q+p)x - 6q(q+p)$$

$$\therefore 2y = (q+p)x - 6qp \quad \text{— (1)}$$

4 (c) (ii) $C(8,0)$ lies on (1)

ie. $0 = (q+p)8 - 6qp$

$\therefore 6qp = 8(q+p) \Rightarrow 3pq = 4(p+q)$ $\quad - (*)$

Midpoint AB $\left(\frac{6p+6q}{2}, \frac{3p^2+3q^2}{2} \right)$

$$X = 3(p+q) \qquad Y = \frac{3}{2}(p^2+q^2)$$

$$= \frac{3}{2}\left[(p+q)^2 - 2pq\right]$$

$$= \frac{3}{2}(p+q)^2 - 3pq$$

$$= \frac{3}{2}(p+q)^2 - 4(p+q) \quad \text{from } *)$$

$$= \frac{3}{2}\left[\frac{x}{3}\right]^2 - 4\left[\frac{x}{3}\right]$$

$$= \frac{x^2}{6} - \frac{4x}{3}$$

$\therefore$ Locus of M is $y = \frac{x^2}{6} - \frac{4x}{3}$

Question 5(a)

$2\tan^{-1}\theta = \tan^{-1}\left(\frac{2\theta}{1-\theta^2}\right)$

$|\theta| < 1$

$\tan(2\tan^{-1}\theta)$

$= \frac{2\tan(\tan^{-1}\theta)}{1 - \tan^2(\tan^{-1}\theta)}$

$= \frac{2\theta}{1-\theta^2}$

$\therefore 2\tan^{-1}\theta = \tan^{-1}\left(\frac{2\theta}{1-\theta^2}\right)$

Now if $|\theta| > 1$

$2\tan^{-1}\theta > \frac{\pi}{2}$ if $\theta > 1$

and $2\tan^{-1}\theta < -\frac{\pi}{2}$ if $\theta < -1$

But $-\frac{\pi}{2} < \tan^{-1}x < \frac{\pi}{2}$

So R.H.S. has

$-\frac{\pi}{2} < \tan^{-1}\left(\frac{2\theta}{1-\theta^2}\right) < \frac{\pi}{2}$

So no valid solution.

(b) $\frac{dV}{dt} = 30 \quad \left(V = \frac{4}{3}\pi r^3\right)$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ —— (1)

$\therefore 30 = 4\pi r^2 \times \frac{dr}{dt}$.

$\Rightarrow \frac{dr}{dt} = \frac{15}{2\pi r^2}$ —— (2)

$S = 4\pi r^2$

$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$ —— (3)

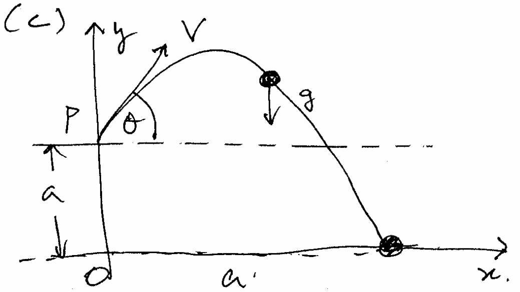
Subst (2) into (3)

$= \overset{4}{\cancel{8}}\cancel{\pi}\cancel{r} \times \frac{15}{\cancel{2}\cancel{\pi}r^{\cancel{2}}}$

$\frac{dS}{dt} = \frac{60}{r}$

When $r = 20$, $\frac{dS}{dt} = 3$

(c)



$\ddot{x} = 0$, $\dot{x} = V\cos\theta$

$x = (V\cos\theta)t$ —— (1)

$\ddot{y} = -g$, $\dot{y} = (V\sin\theta) - gt$

$y = (V\sin\theta)t - \frac{gt^2}{2} + a$. —— (2)

When $x = a$, $y = 0$

When $x = a$

$t = \frac{a}{V\cos\theta}$ —— (3)

and When $t = \frac{a}{V\cos\theta}$, $y = 0$.

Subst. (3) into (2)

We have

$0 = V\sin\theta\left(\frac{a}{V\cos\theta}\right) - \frac{g}{2}\left(\frac{a}{V\cos\theta}\right)t + a$

divide each term by a and rearrange.

$0 = \tan\theta - \frac{gt}{2V\cos\theta} + 1$.

$\frac{gt}{2V\cancel{\cos\theta}} = \frac{\sin\theta + \cos\theta}{\cancel{\cos\theta}}$

$\therefore t = \frac{2V(\sin\theta + \cos\theta)}{g}$. —— (4)

$a = (V\cos\theta)t$ —— (5)

Subst (4) into (5) we have

$$a = \frac{(V\cos\theta)(2V)(\sin\theta + \cos\theta)}{g}$$

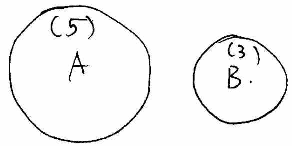
$$= \frac{V^2(2\sin\theta\cos\theta + 2\cos^2\theta)}{g}$$

$$= \frac{V^2(2\sin\theta\cos\theta + (2\cos^2\theta - 1) + 1)}{g}$$

$$= \frac{V^2(\sin 2\theta + \cos 2\theta + 1)}{g}.$$

Question (6)

(a)



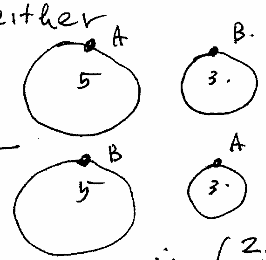
(a) (i) The 1st person has 8 choices, the 2nd person has 7 choices . . . .

$\therefore \dfrac{8!}{5!\,3!}$.

(ii)

either

(α) or (β)



$$\therefore P(E) = \frac{\left(\frac{2\times 6!}{5!\,3!}\right)}{\frac{8!}{5!\,3!}}$$

$$= \frac{2}{8\times 7} = \frac{1}{28}.$$

(b) (i) $f(x) = u(x) - \ln[u(x)+1]$.

$$f'(x) = u'(x) - \frac{u'(x)}{u(x)+1}.$$

$$= u'(x)\left[1 - \frac{1}{u(x)+1}\right]$$

$$= u'(x)\left[\frac{u(x) + \cancel{1} - \cancel{1}}{u(x)+1}\right].$$

(ii)

$$\int_0^{\pi/2} \frac{\sin x \cos x}{1 + \sin x}\,dx$$

$$= \left[\sin x - \ln(\sin x + 1)\right]_0^{\pi/2}$$

$$= (1 - \ln 2) - (0)$$

$$= 1 - \ln 2.$$

(c).

$L(0) = 30$

$\therefore 30 = p + q$.

$L'(0) = -14$.

Now, $L'(x)$

$$= \frac{p}{3}e^{\frac{x}{3}} - \frac{2q}{3}e^{-\frac{2x}{3}}.$$

$\therefore -14 = \dfrac{p}{3} - \dfrac{2q}{3}$.

$\therefore p - 2q = -42$ —— (1)

$p + q = 30$ —— (2)

$\Longrightarrow p = 6$, $6 + q = 30$

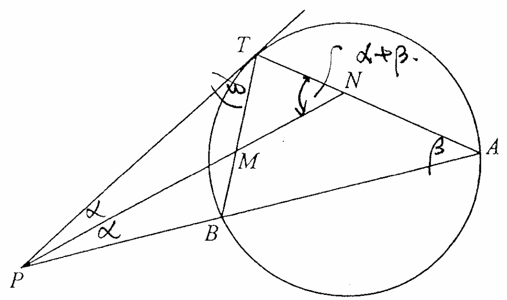
$\therefore q = 24$.

$\because L'(0) = -14 < 0$

and $L'(3) = 2e - 16e^{-2} > 0$.

$\therefore L(x_1)$ must be minimum for $0 < x_1 < 3$.

Question 7



Let $\angle PAT = \beta$.

$\therefore \angle PTB = \beta$ (alternate segment theorem.)

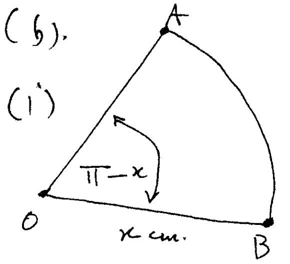
Now $\angle TNP = \alpha + \beta$ (ext. $\angle$ = sum of int. opp. $\angle$'s).

Similarly In $\triangle TPM$, $\angle TMN = \alpha + \beta$.

$\therefore \triangle TMN$ is Isosceles.

(b).

(i)



$P = 2x + x(\pi - x)$

$\therefore P = (\pi+2)x - x^2$.

$\frac{dP}{dx} = \pi + 2 - 2x$.

$\frac{dP}{dx} = 0$, $2x = \pi + 2$

$\therefore x = \frac{\pi+2}{2}$

$\frac{d^2P}{dx^2} = -2 < 0$.

$\therefore P$ is max when $x = \frac{\pi+2}{2}$

$P_{max} = \pi + 2 + \frac{\pi+2}{2}\left(\frac{\pi}{2} - 1\right)$

$= \pi + 2 + \left(\frac{\pi}{2}+1\right)\left(\frac{\pi}{2}-1\right)$

$= \pi + 2 + \frac{\pi^2}{4} - 1$

$= \frac{\pi^2}{4} + \pi + 1$

$= \frac{\pi^2 + 4\pi + 4}{4}$

$t(x) = \frac{x^2}{2}\sin(\pi - x)$.

$\sin(\pi - x) = \sin x$

$\therefore t(x) = \frac{x^2 \sin x}{2}$.

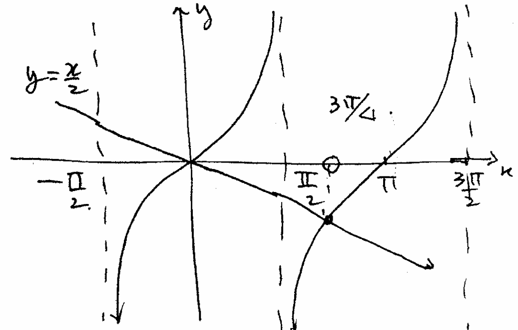
$\frac{dt(x)}{dx} = x\sin x + \frac{x^2}{2}\cos x$.

$\frac{dt(x)}{dx} = 0$, $x\left(\sin x + \frac{x\cos x}{2}\right) = 0$

$\therefore \sin x = \frac{-x\cos x}{2}$

$\Longrightarrow \tan x = \frac{-x}{2}$.

$\therefore 2\tan x = -x$.



$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= \frac{3\pi}{4} - \frac{-2 + \frac{3\pi}{4}}{1 + 3\pi/4}$.

$=$