SOLUTION TO SGS TRIAL 2004 Mathematics

Question One (12 marks)

(a) $\sqrt{\dfrac{3.4^4}{15.6 \times 12.8}} = 0.818$ ✓ answer ✓ correct rounding

(b) $y = 5x^2 - \cos x$

$y^1 = 10x + \sin x$ ✓✓ one for each

(c) A primitive of $x^3 + 5$

is $\dfrac{x^4}{4} + 5x$ ✓✓ one for each

(d) $\dfrac{40^\circ}{360^\circ} = \dfrac{1}{9}$

Area of a sector $= \frac{1}{9} \pi r^2$ ✓ one for $40^\circ = \frac{\pi}{9}$

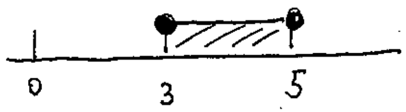
$= \frac{1}{9} \pi \times 9^2$

$= 9\pi$ units ✓

(e) (i) $|x - 4| \leq 1$

$3 \leq x \leq 5$ ✓

(ii)



✓

(f) $\dfrac{3}{3-\sqrt{5}} \times \dfrac{3+\sqrt{5}}{3+\sqrt{5}}$ ✓

$= \dfrac{9 + 3\sqrt{5}}{}$ ✓

Question Two (12 marks)

(a) $P_n = P\left(1 + \frac{r}{100}\right)^n$

$P_4 = 20000 \times (1.06)^4$ ✓

$= \$25249.54$ ✓

(b) $y = e^{2x}$

$y^1 = 2e^{2x}$ ✓

When $x = 0$, gradient $= 2e^0 = 2$ ✓

$y = e^0 = 1$

Equ of tangent is $y - y_1 = m(x - x_1)$

$y - 1 = 2(x - 0)$ ✓

$y = 2x + 1$

(c) (i) midpoint of AC $= \left(\frac{9+6}{2}, \frac{6+-3}{2}\right)$ ✓

$= (7\frac{1}{2}, 1\frac{1}{2})$

(ii) midpoint of BD $= \left(\frac{0+15}{2}, \frac{4+-1}{2}\right)$ ✓

$= (7\frac{1}{2}, 1\frac{1}{2})$

(iii) $m_{BD} = \dfrac{4--1}{0-15} = -\frac{1}{3}$ ✓

(iv) $m_{AC} = \dfrac{6--3}{9-6} = 3$ ✓

$m_{AC} \times m_{BD} = 3 \times -\frac{1}{3} = -1$ ✓

$\therefore AC \perp BD$ ✓

(v) ABCD is rhombus ✓

Question Three (12 marks)

(a) (i) $y = \cos(2x+1)$

$y' = -\sin(2x+1) \times 2$

$= -2\sin(2x+1)$

(ii) $y = \dfrac{x}{\log_e x}$

$y' = \dfrac{\log_e x \times 1 - x \times \frac{1}{x}}{(\log_e x)^2}$

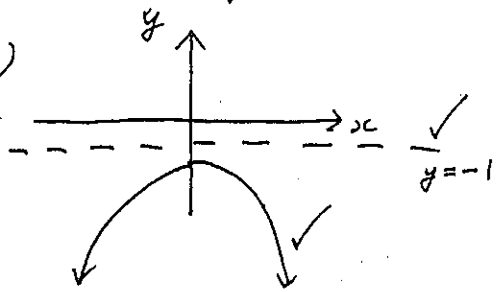
$= \dfrac{\log_e x - 1}{(\log_e x)^2}$

(iii) $y = x^2 \tan x$

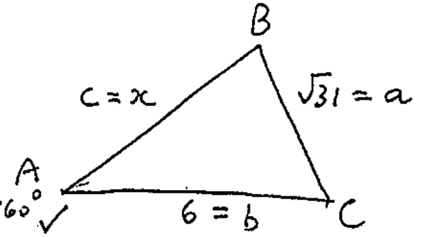
$y' = x^2 \sec^2 x + 2x\tan x$

(b) $x^2 = -4(1)(y+2)$

$V = (0,-2)$



(c)



$a^2 = b^2 + c^2 - 2bc\cos A$

$31 = 36 + x^2 - 2\times 6 \times x \times \cos 60^\circ$

$31 = 36 + x^2 - 2\times 6x \times \frac{1}{2}$

$31 = 36 + x^2 - 6x$

$x^2 - 6x + 5 = 0$

$(x-5)(x-1) = 0$

$x = 5$ or $x = 1$

Question 4 (12 marks)

(a) (i) $\displaystyle\int_0^{\ln 2} e^{2x}\,dx$

$= \left[\frac{1}{2}e^{2x}\right]_0^{\ln 2}$

$= \frac{1}{2}e^{2\ln 2} - \frac{1}{2}e^0$

$= \frac{1}{2}e^{\ln 4} - \frac{1}{2}$

$= 2 - \frac{1}{2} = 1\frac{1}{2}$

(ii) $\displaystyle\int \frac{2x}{x^2+5}\,dx$

$= \ln(x^2+5) + c$

(d) $A = \displaystyle\int_0^{\pi} 1 + \cos x\,dx$

$= \left[x + \sin x\right]_0^{\pi}$

$= \pi + \sin\pi$

$= \pi$ units^2

(b) $5 + 12 + 19 + \dots 292$

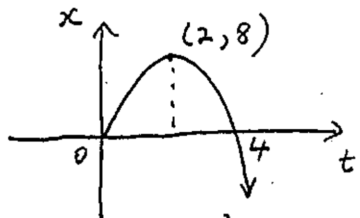
(i) # terms $= \dfrac{292-5}{7} + 1 = 42$

(ii) $S_n = \frac{n}{2}(a+l)$

$S_{42} = 21(5+292)$

$= 6237$

(c) (i)



✓ shape

✓ vertex

$x = 8t - 2t^2$

$= 2t(4-t)$

$V: t = 2,\ x = 8$

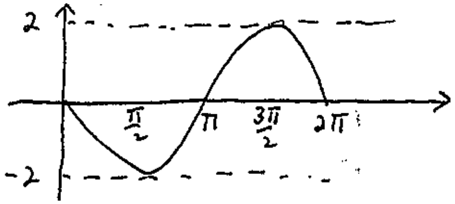
(ii) When $t = 3$, $x = 24 - 18 = 6$ (2 metr from

Distance covered $= 8 + 2$

$= 10$ metres

Question 5 (12 marks)

(a) $y = -2\sin x$ for $0 \le x \le 2\pi$



✓ shape and amplitude

✓ period.

(b) $l = r\theta$

$\frac{7}{12} = \theta$ ✓

$\theta = \left(\frac{7}{12} \times \frac{180}{\pi}\right)$ degrees

$\theta = 33^\circ$ ✓

(c) (i) AC is common

$\angle BAC = \angle ACD$ (alt $\angle$'s, $AB \parallel CD$)

$\angle ABC = \angle ADC$ (given) ✓

$\therefore \triangle ABC \equiv \triangle CDA$ (AAS test) ✓

(ii) $\angle BCA = \angle DAC$ (matching sides of congruent $\triangle$'s) ✓

But these are alternate angles

$\therefore BC \parallel AD$ ✓

$\therefore ABCD$ is a parallelogram since two pairs of opposite sides are parallel ✓

(d) (1) $y = \sqrt{x-1}$

$y^2 = x - 1$

$(y^2+1)^2 = x^2$ ✓

$V = \pi \int_0^3 y^4 + 2y^2 + 1 \, dy$ ✓

$= \pi \left[\frac{y^5}{5} + \frac{2y^3}{3} + y\right]_0^3$ ✓

348π $\pi\left[\frac{3^5}{...} + 2\times 9 + 3\right]$ ✓

Question 6 (12 marks)

(a) $\frac{dy}{dx} = x^{-\frac{1}{2}}$

$y = 2x^{\frac{1}{2}} + C$ ✓

When $x = 4$, $y = -2$: $-2 = 2\sqrt{4} + C$

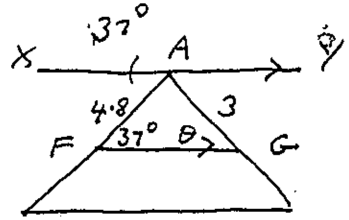
$C = -6$

$y = 2\sqrt{x} - 6$ ✓

(b) (i) $\frac{x}{3} = \frac{8}{5}$ ✓

$x = \frac{24}{5}$

$x = 4.8$ ✓



$\angle AFG = 37^\circ$ (alt $\angle$'s, $XY \parallel FG$)

$\frac{\sin\theta}{4.8} = \frac{\sin 37^\circ}{3}$ ✓

$\theta = 74^\circ$ ✓

(c) $\log_a 400$

$= \log_a 25 + \log_a 16$ ✓ 'splitting logs'

$= 2\log_a 5 + 4\log_a 2$

$= 2x + 4y$ ✓ correct answer

(d) (i)

x	0	1	2
y	1.732	2.172	3.064

✓ one correct

✓✓ three correct

(ii) $A \doteqdot \frac{h}{6}\left[f(a) + 4 \times f\left(\frac{a+b}{2}\right) + f(b)\right]$ ✓

$= \frac{2-0}{6}\left[1.732 + 4 \times 2.172 + 3.064\right]$ ✓ (Follow through ...)

Question 7 (12 marks)

(a) (i) $y = x^4 - 4x^3 + 3$

$y' = 4x^3 - 12x^2$

$y'' = 12x^2 - 24x = 12x(x-2)$

Stat pts where $y' = 0$

$4x^3 - 12x^2 = 0$

$4x^2(x-3) = 0$

$x = 0$ or $x = 3$ ✓

When $x = 3$, $y'' = 108 - 72 > 0$ ✓

$\therefore$ Local minimum at $(3, -24)$

When $x = 0$, $y'' = 0$ (test inconclusive) ✓

When $x = -1$, $y' = -16 < 0$

$x = 1$ $y' = -8 < 0$ ✓

$\therefore$ Horizontal point of inflexion at $(0, 0)$

(ii) Possible pts of inflexion where $y'' = 0$

$12x^2 - 24x = 0$

$12x(x-2) = 0$

$x = 0$ or $x = 2$ ✓

At $x = 0$ there is a horizontal pt of inflexion

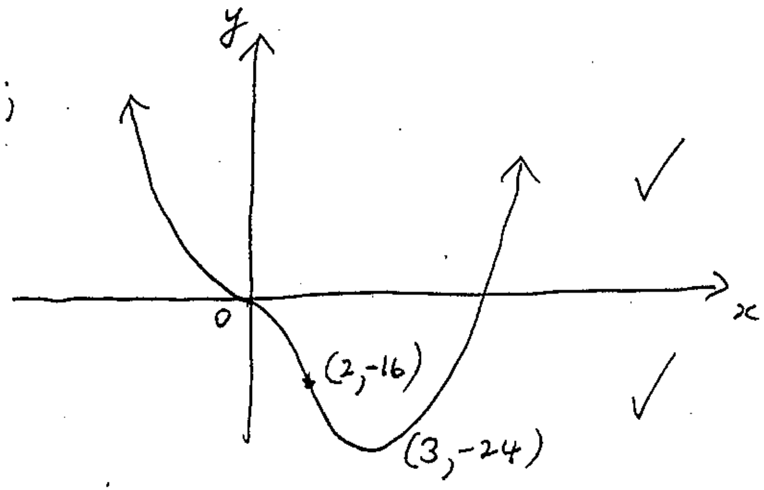
Consider $x = 1{\cdot}9$, $y'' = 12(1{\cdot}9)(-0{\cdot}1) < 0$

$x = 2{\cdot}1$, $y'' = 12(2{\cdot}1)(0{\cdot}1) > 0$ ✓

Change in concavity

There is a point of inflexion at $(2, -16)$

(iii)



✓ ✓

(b) Consider the quadratic

$Kx^2 - 2x\sqrt{6} + (K+1)$

This is positive definite when

$a > 0$ and $b^2 - 4ac < 0$ ✓

ie. $K > 0$ and $(2\sqrt{6})^2 - 4K(K+1) < 0$ ✓

$24 - 4K^2 - 4K < 0$

$K^2 + K - 6 > 0$

$(K+3)(K-2) > 0$

$K > 0$ <u>and</u> $K < -3$ or $K \quad 2$ ✓

The quadratic is positive definite

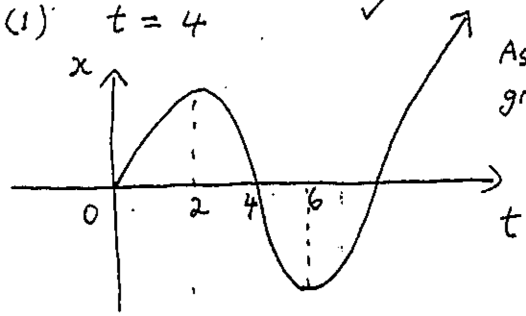
When $K > 2$ ✓

(9)

Question 8 (12 marks)

(a) (i) $t = 4$

(ii)



As $t \to \infty$, gradient $\to 2$

(b) (i) $P_0 = 22000$

$P = P_0 e^{kt}$

$t = 5$
$P = 27000$

$27000 = 22000e^{5K}$

$K = \frac{1}{5} \ln \frac{27}{22}$

$\doteqdot 0.04095\ldots$

(ii) $\left.\begin{array}{l} P = 35000 \\ t = ? \end{array}\right\}$ $35000 = 22000 e^{Kt}$

$\frac{35}{22} = e^{Kt}$

$t = \frac{1}{K} \ln \frac{35}{2}$

$= 11 \cdot 3$ years

(c) (i) Pts of intersection where $\sec x = 2$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$

(ii) $V = 2\left\{ \int_0^{\frac{\pi}{3}} 2^2 \, dx - \int_0^{\frac{\pi}{3}} \sec^2 x \, dx \right\}$

$= 2\left\{ \left[4x\right]_0^{\frac{\pi}{3}} - \left[\tan x\right]_0^{\frac{\pi}{3}} \right\}$

$= 2\left\{ \frac{4\pi}{3} - \sqrt{3} \right\} = (8\frac{\pi}{3} - 2\sqrt{3})$ units^3

Question 9 (12 marks

(a) $\frac{dV}{dt} = -2 - \frac{20}{t+1}$

(i) When $t = 0$, $\frac{dV}{dt} = -2 - \frac{20}{1}$

$= -22$ l/min.

(ii) $\frac{dV}{dt} = -2 - \frac{20}{t+1}$

$V = -2t - 20 \ln(t+1) + C$

When $\left.\begin{array}{l} t=0 \\ V = 100 \end{array}\right\}$ $100 = 0 - 20 \ln 0 + C$

$C = 100$

$V = -2t - 20 \ln(t+1) + 100$

When $t = 5$, $V = -10 - 20 \ln 6 + 100$

$\doteqdot 54 \cdot 2$ litres

(b) (i) $A_1 = 250\,000 \times 1.005 - M$

$A_2 = ((250\,000 \times 1.005) - M)\, 1.005 - M$

$= 250000 \times (1.005)^2 - 1.005M - M$

$= 250000 \times (1.005)^2 - M(1 + 1.005)$

$A_3 = 250000 \times (1.005)^3 - M(1 + 1.005 + 1.00$

(ii) Loan is paid of over 240 months = 20y

(iii) $0 = 250\,000 \times (1.005)^{240} - M(1 + 1.005 \cdots + 1.00$

$250000 \times (1.005)^{240} = M\left[\frac{a(r^n - 1)}{r - 1}\right]$

$$250\,000 \times (1.005)^{240} = M\left(\frac{(1.005)^{240} - 1}{1.005 - 1}\right)$$

$$M = \$1791.08$$

(iv) Suppose $M = \$2000$ we need to find n so that

$$250000 \times (1.005)^n = 2000\left(\frac{(1.005)^n - 1}{0.005}\right)$$

$$250000 \times 0.005 \times (1.005)^n = 2000 \times 1.005^n - 2000$$

$$1250 \times (1.005)^n = 2000 \times 1.005^n - 2000$$

$$(1.005)^n (2000 - 1250) = 2000$$

$$(1.005)^n = \frac{8}{3}$$

$$n \ln(1.005) = \ln \frac{8}{3}$$

$$n = \frac{\ln \frac{8}{3}}{\ln 1.005}$$

$$= 196.65 \text{ months}$$

Loan is paid off approximately 43 months earlier.

Question 10

(a) (i)

$$y = x^{\frac{2}{3}}$$

$$y' = \frac{2}{3} x^{-\frac{1}{3}}$$

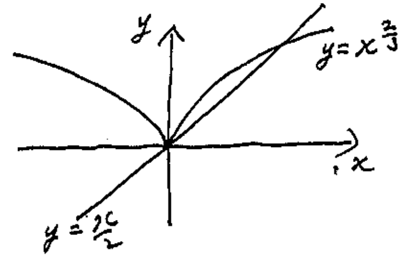
$$y'' = -\frac{2}{9} x^{-\frac{4}{3}}$$

$$y'' = -\frac{2}{9} \sqrt[3]{\frac{1}{x^4}}$$

$x^4 > 0$ for $x \neq 0$

$\therefore y'' < 0$ for $x \neq 0$

(ii) Solve $x^{\frac{2}{3}} = \frac{x}{2}$



Cube both sides

$$x^2 = \frac{x^3}{8}$$

$$x^3 - 8x^2 = 0$$

$$x^2(x - 8) = 0$$

$$x = 0 \text{ or } x = 8$$

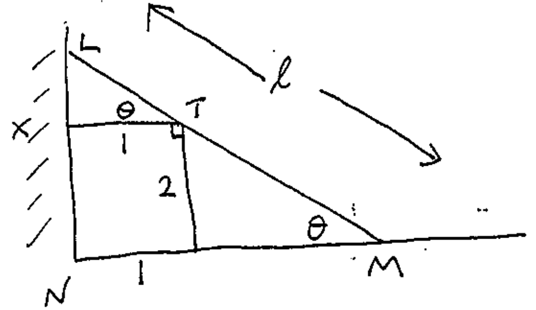
Give one mark for sketch OR

Solution to $x^{\frac{2}{3}} \leq \frac{x}{2}$

is $x \geq 8$ or $x = 0$

Question 10 (12 marks) cont'd

(b)



(i) Draw XT parallel to NM so that $\angle LTX = \theta$.

$\frac{TL}{TX} = \sec\theta$ ✓ OR $\frac{TM}{2} = \operatorname{cosec}\theta$

$TL = \sec\theta$ $\quad TM = 2\operatorname{cosec}\theta$

$\therefore LM = \sec\theta + 2\operatorname{cosec}\theta$

$\ell = \frac{1}{\cos\theta} + \frac{2}{\sin\theta}$ ✓

(ii) $\ell = (\cos\theta)^{-1} + 2(\sin\theta)^{-1}$

$\ell' = -\frac{-\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta}$ ✓

Stat pts where $\ell' = 0$

$\frac{\sin\theta}{\cos^2\theta} = \frac{2\cos\theta}{\sin^2\theta}$ ✓

$\frac{\sin^3\theta}{\cos^3\theta} = 2$

$\tan^3\theta = 2$ ✓

$\tan\theta = \sqrt[3]{2}$

(iii) Minimum ℓ at $\theta = \tan^{-1}\sqrt[3]{2}$

or end points $\theta = 55^\circ$ or $\theta = 70^\circ$

$\theta = \tan^{-1}\sqrt[3]{2}$ ✓
$\doteqdot 51^\circ 34'$ which is outside the safety constraint.

Test ✓ $\theta = 55^\circ$ $\quad \ell = \frac{2}{\sin 55^\circ} + \frac{1}{\cos 55^\circ} \doteqdot 4.18$

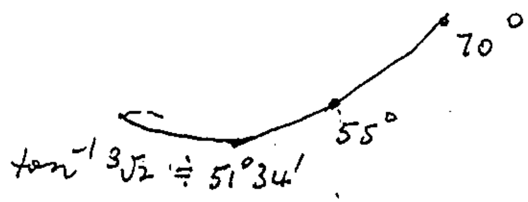
$\theta = 70^\circ$ $\quad \ell = \frac{2}{\sin 70^\circ} + \frac{1}{\cos 70^\circ} \doteqdot 5.05$

So minimum ℓ when $\theta = 55^\circ$ ✓
and the length is 4.18 m

$\boxed{\text{OR}}$ You can show that ℓ is increasing for $\theta > \tan^{-1}\sqrt[3]{2}$

θ	50°	$\tan^{-1}\sqrt[3]{2}$	52°
ℓ'	$-$	0	$+$

So the curve looks like



Clearly minimum at $\theta = 55^\circ$