1999 NSW INDEPENDENT TRIAL EXAMS

MATHEMATICS
3 UNIT TRIAL 1999

SUGGESTED ANSWERS

1a. Possibilities are

1,5
2,5
3,5
4,5
5,1 5,2 5,3 5,4 5,5 5,6
6,5

∴ Probability of total of 8 $= \frac{2}{11}$

. Let $p =$ prob of supporting A $= 3/10$
$q =$ prob of supporting other $= 7/10$
$n =$ no. of A supporters

Then $P(X=r) = {}^7C_r \left(\frac{3}{10}\right)^r \left(\frac{7}{10}\right)^{7-r}$

& $P(X=4) = {}^7C_4 \left(\frac{3}{10}\right)^4 \left(\frac{7}{10}\right)^3$

$= 0.0972405$
$\doteqdot 0.1$

$x = \dfrac{kx_2 + lx_1}{k+l}$ $\quad$ $y = \dfrac{ky_2 + ly_1}{k+l}$

$-1 = \dfrac{-3 \times x_2 + 1 \times 3}{-3+1}$ $\quad$ $-4 = \dfrac{-3 \times y_2 + 1 \times 2}{-3+1}$

$2 = -3x_2 + 3$ $\quad$ $8 = -3y_2 + 2$

$x_2 = 1/3$ $\quad$ $y_2 = -2$

$\therefore B\left(\frac{1}{3}, -2\right)$

d. $u = \cos x$
$du = -\sin x \, dx$
if $x = \pi/2$, $u = 0$
if $x = \pi/3$, $u = 1/2$

$\therefore I = \int_{.5}^{0} -u^3 \, du$

$= \left[\frac{u^4}{4}\right]_0^{.5}$

$= \frac{.5^4}{4} - 0 = \frac{1}{64}$

e. $\int_0^{\pi/4} \cos^2 \frac{1}{2}x \, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} 1 + \cos x \, dx$

$= \frac{1}{2}\left[x + \sin x\right]_0^{\pi/4}$

$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)$

16

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2a. $\dfrac{1}{x+1} \geq 1-x$

Critical points at $x=-1$ and

$$\frac{1}{x+1} = 1-x$$

$$1 = 1-x^2$$

$$\Rightarrow x=0$$

-1 0

Test $x=-2$ False

Test $x=-\frac{1}{2} \Rightarrow 2 \geq 1\frac{1}{2}$ $\therefore$ True

$x=1 \Rightarrow \frac{1}{2} \geq 0$ $\therefore$ True

Solution: $x > -1$

b. $$\int_0^{2/5} \frac{dx}{\sqrt{16-25x^2}}$$

$$= \int_0^{2/5} \frac{dx}{5\sqrt{\frac{16}{25}-x^2}}$$

$$= \frac{1}{5}\left[\sin^{-1}\frac{x}{4/5}\right]_0^{2/5}$$

$$= \frac{1}{5}\left[\sin^{-1}\frac{5x}{4}\right]_0^{2/5}$$

$$= \frac{1}{5}\left(\sin^{-1}\frac{1}{2} - \sin^{-1}0\right)$$

$$= \frac{1}{5}\cdot\frac{\pi}{6} = \frac{\pi}{30}$$

c. (i) $M\left(a(p+q), \dfrac{a(p^2+q^2)}{2}\right)$

(ii) $M_{PQ} = \dfrac{p+q}{2} = k$, a constant

Then, for the point M,

$$x = a(p+q)$$
$$= a \cdot 2k$$
$$x = 2ak$$

Since a and k are constant, the locus of M is a line parallel to the y-axis

d. $\angle U = \angle V$ (given)

$\angle UZX = \angle VZY$ (vertically opposite)

Now $\angle ZXW = \angle UZX + \angle U$ (exterior angle of triangle)

and $\angle ZYW = \angle VZY + \angle V$ (ditto)

$\therefore \angle ZXW = \angle ZYW$ (equal to sum of equal angles)

In ΔXZW & ΔYZW,

ZW is common

$\angle ZXW = \angle ZYW$ (above)

$\angle XWZ = \angle YWZ$ (given ZW bisects $\angle YWX$)

$\therefore \Delta XZW \equiv \Delta YZW$ (AAS)

and $XW = YW$

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3.(a) $2 - \frac{3}{x+2} = \frac{2(x+2)-3}{x+2}$

$= \frac{2x+1}{x+2}$

$\therefore \int_0^1 \frac{2x+1}{x+2}\,dx$

$= \int_0^1 2 - \frac{3}{x+2}\,dx$

$= \left[2x - 3\ln(x+2)\right]_0^1$

$= (2 - 3\ln 3) - (0 - 3\ln 2)$

$= 2 + 3\ln(2/3)$

b) Let $\cos x - \sqrt{3}\sin x = A\cos(x+\theta)$

$= A\cos x\cos\theta - A\sin x\sin\theta$

hen $A\cos\theta = 1$

$A\sin\theta = \sqrt{3}$

$\Rightarrow \tan\theta = \sqrt{3}$ and $\theta = \pi/3$

and $A = 2$

$\therefore 2\cos\left(x + \frac{\pi}{3}\right) + 1 = 0$

$\cos\left(x + \frac{\pi}{3}\right) = -1/2$

$x + \frac{\pi}{3} = \ldots\ \frac{4\pi}{3}, \frac{5\pi}{3}, \ldots$

$x = \pi, \frac{4\pi}{3}$ in given domain

)(i) Let $f(x) = x\ln x - 1$

$f(1) = 1.\ln 1 - 1 < 0$

$f(2) = 2\ln 2 - 1 > 0$

$\therefore$ a solution exists between $x = 1$ & $x = 2$ (assuming $f(x)$ is continuous)

(ii) $f'(x) = x.\frac{1}{x} + \ln x = \ln x + 1$

By Newton's method,

$x_1 = x - \frac{f(x)}{f'(x)}$

$= x - \frac{x\ln x - 1}{\ln x + 1}$

if $x = 2$, $x_1 = 2 - \frac{2\ln 2 - 1}{\ln 2 + 1}$

$= +1.77184832$

$= 1.8$

(d)(i) Total no. of possible teams

$= {}^7C_2 \times {}^5C_2 = 210$

Teams with a particular woman

$= {}^6C_1 \times {}^5C_2 = 60$

$\therefore$ Probability of a particular woman

$= \frac{60}{210} = \frac{2}{7}$

(ii) Captain is specified, so the number of possible teams is ${}^4C_1 \times {}^7C_2 = 84$

No. of teams with his brother is ${}^7C_2 = 21$

$\therefore$ Probability of captain and brother

$= \frac{21}{84} = \frac{1}{4}$

OR with the captain as specified, the probability that his brother is chosen from the remaining four men is $\frac{1}{4}$

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4(a) $x^2+y^2-6x+2ky+3k=0$

Completing the squares:

$(x-3)^2+(y+k)^2=k^2-3k+9$

If the centre $(3,-k)$ is on the line $x-3y=0$, then

$3-3x-k=0 \Rightarrow k=-1$

$\therefore C_1: (x-3)^2+(y-1)^2=13$

If C_2 touches the x-axis, the radius is k

$\therefore \sqrt{k^2-3k+9} = k$

$k^2-3k+9 = k^2$

$\Rightarrow k = 3$

$\therefore C_2: (x-3)^2+(y+3)^2=9$

b)(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3+2x$

$\therefore \frac{1}{2}v^2 = \frac{1}{2}x^4+x^2+C$

f $v=2$, $x=1$

$\therefore \frac{1}{2}.2^2 = \frac{1}{2}.1+1+C \Rightarrow C=\frac{1}{2}$

& $\frac{1}{2}v^2 = \frac{1}{2}x^4+x^2+\frac{1}{2}$

$v^2 = x^4+2x^2+1$

$v^2 = (x^2+1)^2$

) so $v = \pm(x^2+1)$

but $v=2\ (>0)$ when $x=1$

$\therefore v = +(x^2+1)$

$\frac{dt}{dx} = \frac{1}{x^2+1}$

so $t = \tan^{-1}x + C$

Now $x=\frac{1}{\sqrt{3}}$ when $t=0$

$\therefore C = -\tan^{-1}\frac{1}{\sqrt{3}} = -\frac{\pi}{6}$

so $t = \tan^{-1}x - \frac{\pi}{6}$

when $x=\sqrt{3}$, $t=\tan^{-1}\sqrt{3}-\frac{\pi}{6}$

$= \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6}$

(c) Let $S(n): 5^{2n}-1=6I$, where I is an integer.

$S(1)$: LHS $= 5^2-1=24=6\times 4$

$\therefore S(1)$ is true

Assume $S(k): 5^{2k}-1=6I$ (I, integer)

Consider $S(k+1)$:

LHS $= 5^{2k+2}-1$

$= 5^{2k}.5^2-1$

$= 25(5^{2k}-1)-1+25$

$= 25.6I+24$ by $S(k)$

$= 6[25I+4]$

Now I is integer, $\therefore 25I+4$ is integer.

Hence, if $S(k)$ is true, $S(k+1)$ is true

But $S(1)$ is true, so $S(2)$ is true, and then $S(3)$ is true and so on for all integer values of n.

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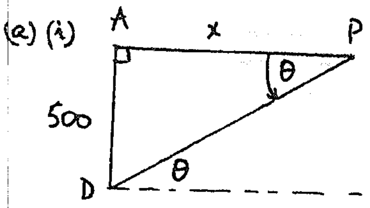
(a) (i)



$\tan\theta = \frac{500}{x}$

$\therefore x = 500\cot\theta$

$$\frac{dx}{d\theta} = -500\,\text{cosec}^2\theta$$

(ii) $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$

$= \frac{1}{-500\,\text{cosec}^2\theta} \times 20$

$= -\frac{1}{25}\sin^2\theta$

iii) At 9:01, $t=60$, $x=1200$

Then $PD = 1300$ (Pythagoras' Theorem)

so $\sin\theta = \frac{500}{1300} = \frac{5}{13}$

$\therefore \frac{d\theta}{dt} = -\frac{1}{25} \times \left(\frac{5}{13}\right)^2$

$= -\frac{1}{169}$ degrees/sec.

(i) $\ddot{x} = 0$ $\qquad \ddot{y} = -10$

$\dot{x} = c_1$ $\qquad \dot{y} = -10t + c_2$

Initially $\dot{x} = 50\cos\alpha$ $\qquad \therefore \dot{x} = 50\cos\alpha$

and $\dot{y} = 50\sin\alpha$ $\qquad \therefore \dot{y} = -10t + 50\sin\alpha$

$x = 50t\cos\alpha + C_3$ $\qquad y = -5t^2 + 50t\sin\alpha + C_4$

Since $x=0$ when $t=0$, and $y=0$ when $t=0$

$C_3 = 0$ $\qquad C_4 = 0$

$x = 50t\cos\alpha$ $\qquad \therefore y = -5t^2 + 50t\sin\alpha$

when $x = 150$, $150 = 50t\cos\alpha$

so $3 = t\cos\alpha$ ... (1)

when $y=0$, $0 = -5t^2 + 50t\sin\alpha$

$= -5t(t - 10\sin\alpha)$

$\Rightarrow t = 10\sin\alpha$ (2)

Solving (1) + (2):

$3 = 10\sin\alpha\cos\alpha$

$= 5\sin 2\alpha$

$\therefore \sin 2\alpha = 3/5$

$2\alpha = 36°52'$, $143°08'$

$\therefore \alpha = 18°26'$ or $71°29'$

(ii) $\ddot{x} = 0$ $\qquad \ddot{y} = -10$

$\dot{x} = C_1$ $\qquad \dot{y} = -10t + C_2$

Initially, $\dot{x} = 55\cos\alpha$, $\dot{y} = 55\sin\alpha$

$\therefore \dot{x} = 55\cos\alpha$ $\qquad \dot{y} = -10t + 55\sin\alpha$

$\dot{x} = 55$ $\qquad \dot{y} = -10t$ since $\alpha = 0$

Then $x = 55t + C_3$ $\qquad y = -5t^2 + C_4$

when $t=0$, $x=0$ and $y=30$

$\Rightarrow C_3 = 0$ $\qquad C_4 = 30$

$\therefore x = 55t$ $\qquad y = -5t^2 + 30$

Now when $y=0$, $-5t^2 + 30 = 0$

$\therefore t^2 = 6$

$\therefore t = \sqrt{6}$

At $t = \sqrt{6}$, $x = 55\sqrt{6}$

≈ 135 m

$\therefore$ Group B cannot reach the target

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) (i) $x = 2\sin t - 3\cos t$

$\dot{x} = 2\cos t + 3\sin t$

$\ddot{x} = -2\sin t + 3\cos t$

$= -(2\sin t + 3\cos t)$

$= -x$

∴ motion is simple harmonic.

) Amplitude $= \sqrt{2^2 + 3^2}$

$= \sqrt{13}$ cm

) $\dot{x} = 2\cos t + 3\sin t$

$\ddot{x} = -2\sin t + 3\cos t$

Max velocity when $\ddot{x} = 0$

$-2\sin t + 3\cos t = 0$

$3\cos t = 2\sin t$

$3/2 = \tan t$

$t = 0{\cdot}983, 4{\cdot}1243 \ldots$ etc

∴ reaches maximum velocity when $t = 0{\cdot}983$

(i) $T = T_0 + Ae^{kt}$

$\frac{dT}{dt} = k.Ae^{kt}$

$= k(T - T_0)$

(ii) When $t = 0$, $T = 95$, $T_0 = -10$

$\Rightarrow A = 105$

when $t = 5$, $T = 65$

∴ $65 = -10 + 105e^{5k}$

$e^{5k} = 75/105 = 5/7$

$\ln e^{5k} = \ln 5/7$

∴ $k = \frac{1}{5}\ln\frac{5}{7}$

(iii) When $t = 0$, $T_0 = 26$ + $T = 65$

∴ $65 = 26 + Be^{k.0}$

∴ $B = 39$

Therefore, at $t = 5$,

$T = 26 + 39e^{5k}$ with $k = \frac{1}{5}\ln\frac{5}{7}$

so $T = 53{\cdot}86°$

$= 54°$ (to the nearest degree)

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7(a) $\left(3x - \frac{1}{x^2}\right)^6 = \sum_{r=0}^{6} {}^6C_r \,(3x)^{6-r}\left(-\frac{1}{x^2}\right)^r$

Typical term, T_r, is

$T_r = {}^6C_r\, 3^{6-r} \cdot x^{6-r} \cdot (-1)^r \cdot (x^{-2})^r$

$= {}^6C_r\, 3^{6-r} (-1)^r\, x^{6-3r}$

Constant term when $6-3r=0$

$r = 2$

then $T_2 = {}^6C_2\, 3^4\, (-1)^2$

$= 1215$

(i) $x^4 + x^2 - 1 = 0$

$x^2 = \frac{-1 \pm \sqrt{1-4\times1\times-1}}{2}$

$= \frac{-1 \pm \sqrt{5}}{2}$

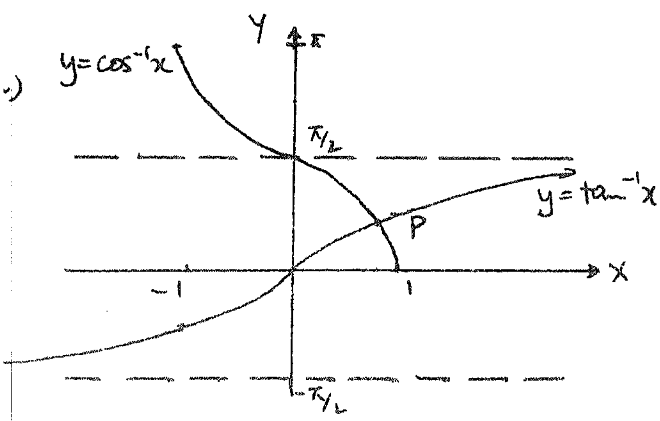
$\therefore x^2 = \cancel{\frac{-1-\sqrt{5}}{2}}$ or $\frac{-1+\sqrt{5}}{2}$

$x^2 = 0.618033988$

$x = \pm 0.786151377$

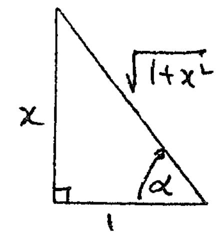
$= \pm 0.79$

(ii)



(iii) let $\tan^{-1}x = \alpha$

$\therefore x = \tan\alpha$



At P, $\cos^{-1}x = \tan^{-1}x = \alpha$

$\therefore$ at P $\cos^{-1}x = \alpha$ + $x = \cos\alpha$

But $\cos\alpha = \frac{1}{\sqrt{1+x^2}}$ (from diagram)

$\therefore \quad x = \frac{1}{\sqrt{1+x^2}}$

Squaring, $x^2 = \frac{1}{1+x^2}$

+ $x^4 + x^2 = 1$

$x^4 + x^2 - 1 = 0$

$\therefore x = 0.79$ (from (i))

and $y = \tan^{-1} 0.79 = 0.6686$

so $P(0.79,\ 0.67)$

(iv) $A = \int_0^{0.67} \tan y\, dy + \int_{0.67}^{\pi/2} \cos y\, dy$

$= \left[-\ln|\cos y|\right]_0^{0.67} + \left[\sin y\right]_{0.67}^{\pi/2}$

$= -\ln|\cos 0.67| + \sin\frac{\pi}{2} - \sin 0.67$

$= 0.62258$

$= 0.62$ (to 2-decimal places)