QUESTION ONE (15 marks) Use a separate writing booklet. **Marks**

(a) Find $\displaystyle\int_0^4 \frac{1}{\sqrt{2x+1}}\,dx$. $\boxed{2}$

(b) Find $\displaystyle\int \tan^3 x \sec^2 x\,dx$. $\boxed{2}$

(c) Find $\displaystyle\int \frac{x}{x^2-4x+8}\,dx$. $\boxed{3}$

(d) (i) Find the values of A and B such that $\dfrac{3x^2-10}{x^2-4x+4} = 3 + \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2}$. $\boxed{2}$

(ii) Find $\displaystyle\int \frac{3x^2-10}{x^2-4x+4}\,dx$. $\boxed{2}$

(e) Use integration by parts twice, to show that $\displaystyle\int_1^e \sin(\ln x)\,dx = \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2}$. $\boxed{4}$

QUESTION TWO (15 marks) Use a separate writing booklet. **Marks**

(a) Simplify $|\cos\theta + i\sin\theta|$. $\boxed{1}$

(b) Express $\dfrac{i^5(1-i)}{2+i}$ in the form $a+ib$ where a and b are rational. $\boxed{2}$

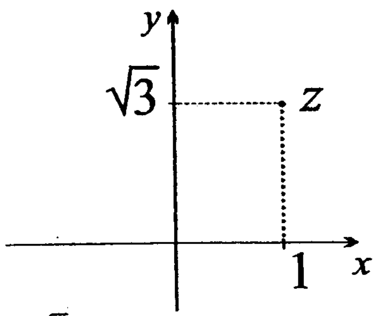
(c) By drawing a diagram, or otherwise, find the solutions of $z^5 = -1$. $\boxed{2}$

(d) Graph the region in the Argand diagram which simultaneously satisfies $\boxed{3}$

$$1 \le |z-i| \le 2 \text{ and } \operatorname{Im} z \ge 0.$$

(e) Find the complex number ϕ if $1+i$ is a root of the equation $z^2 + \phi z - i = 0$. $\boxed{2}$

(f)



Suppose that $z = 1+\sqrt{3}i$ and $\omega = (\operatorname{cis}\alpha)z$ where $-\pi < \alpha \le \pi$.

(i) Find the argument of z. $\boxed{1}$

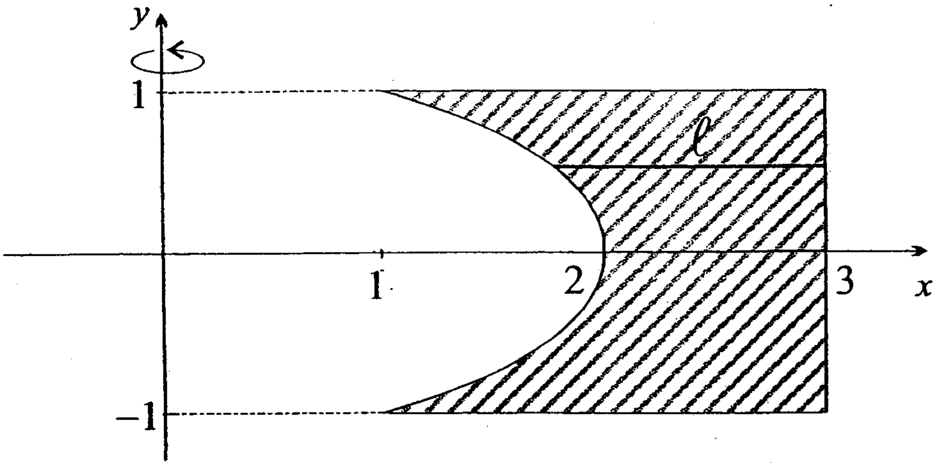
(ii) Find the value of α if ω is purely imaginary and $\operatorname{Im}(\omega) > 0$. $\boxed{2}$

(iii) Find the value of $\arg(z+\omega)$ if ω is purely imaginary and $\operatorname{Im}(\omega) > 0$. $\boxed{2}$

Exam continues next page ...

QUESTION THREE (15 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows the region bounded by the curve $x = 2 - y^2$ and the lines $x = 3, y = 1$ and $y = -1$. This region is rotated about the y-axis to form a solid. The interval ℓ at height y sweeps out an annulus.

(i) Show that the annulus at height y has area equal to **2**

$$\pi(5 + 4y^2 - y^4).$$

(ii) Find the volume of the solid. **2**

(b) Consider the function $f(x) = \dfrac{1}{1 + x^3}$.

(i) Show that there is a horizontal point of inflexion at $x = 0$. **2**

(ii) Find the vertical asymptote and the horizotal asymptote. **2**

(iii) Sketch $y = f(x)$ showing the features from parts (a) and (b) and the y-intercept. **2**

(iv) On a separate diagram sketch $y = |f(x)|$. **1**

(v) On a separate diagram sketch $y^2 = f(x)$. **2**

(vi) On a separate diagram sketch $y = e^{f(x)}$. **2**

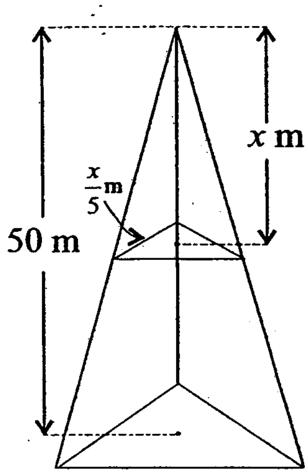
QUESTION FOUR (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the polynomial equation $x^3 - 3x^2 + x - 5 = 0$ which has roots α, β and γ.

(i) Show that $\alpha + \beta = 3 - \gamma$. **1**

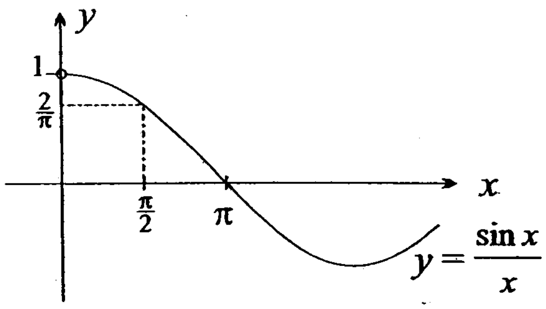
(ii) Write down similar expressions for $\alpha + \gamma$ and $\beta + \gamma$ and hence find a polynomial equation which has the roots $\alpha + \beta, \alpha + \gamma$ and $\beta + \gamma$. **2**

(b)



The diagram above shows a monument 50 metres high. A horizontal cross section x metres from the top is an equilateral triangle with sides $\frac{x}{5}$ metres. Use integration to find the volume of the monument. **3**

(c)



Use the method of cylindrical shells to find the volume of the solid formed when the region bounded by the curve $y = \dfrac{\sin x}{x}$ and the lines $y = 0$ and $x = \dfrac{\pi}{2}$ is rotated about the y-axis.

(d) An hyperbola is defined parametrically by $x = 3\sec\theta$ and $y = 4\tan\theta$.

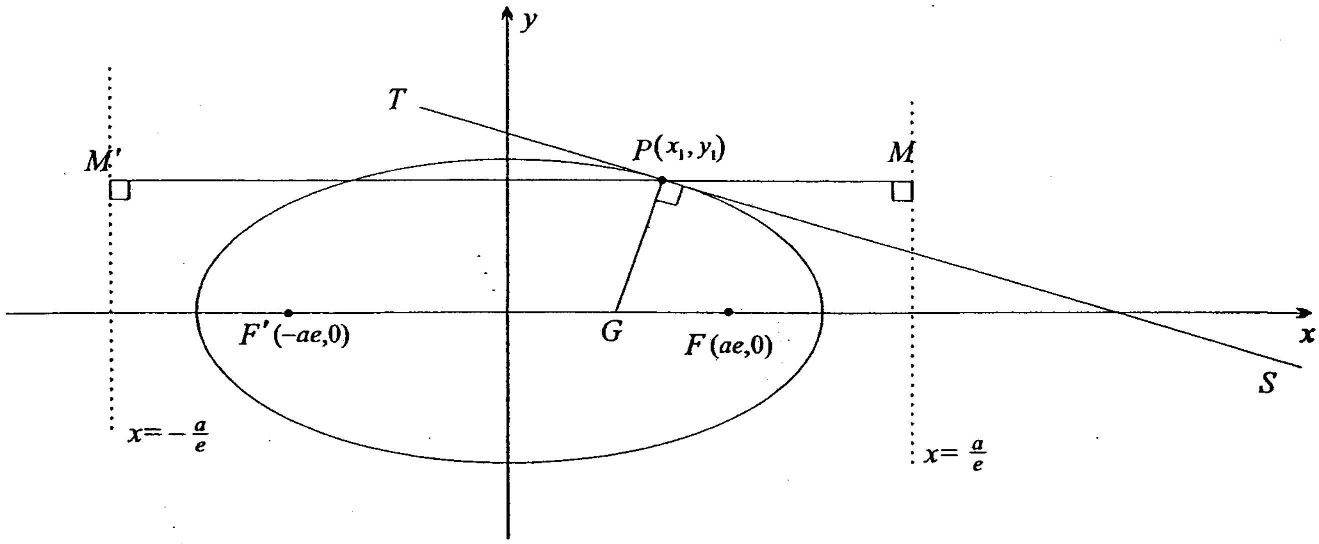
(i) Write the equation of the curve in Cartesian form and show that the eccentricity is $\frac{5}{3}$. **2**

(ii) Sketch the curve showing its x-intercepts, foci, directrices and asymptotes. **4**

Exam continues next page ...

QUESTION FIVE (15 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with foci $F(ae,0)$ and $F'(-ae,0)$.
$P(x_1,y_1)$ is any point on the ellipse.
Let M and M' be the feet of the perpendiculars from P to the directrices $x=\dfrac{a}{e}$ and $x=-\dfrac{a}{e}$.
Line TS is a tangent to the ellipse at P and G is the point where the normal at P meets the x-axis.

(i) Show that the equation of the normal at P is $\dfrac{a^2x}{x_1}-\dfrac{b^2y}{y_1}=a^2-b^2$. **3**

(ii) Show that the point G has co-ordinates $(e^2x_1,0)$. **3**

(iii) Show that the distance PF is $a-ex_1$. **2**

(iv) Show that $\dfrac{PF}{FG}=\dfrac{PF'}{F'G}$. **2**

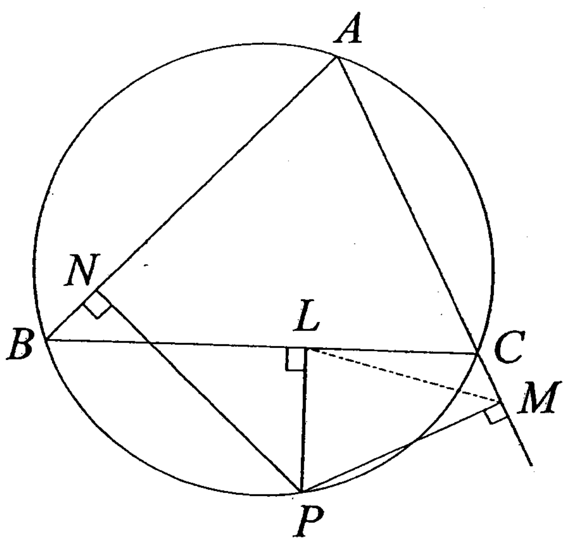
(b) (i) Show that $1-\cos 2\theta - i\sin 2\theta = 2\sin\theta(\sin\theta - i\cos\theta)$. **2**

(ii) Given that $\dfrac{z-1}{z}=\operatorname{cis}\frac{2\pi}{5}$, show that $z=\dfrac{1}{2}(1+i\cot\frac{\pi}{5})$. **3**

Exam continues overleaf ...

QUESTION SIX (15 marks) Use a separate writing booklet. **Marks**

(a)



In the diagram above, ABC is a triangle with the circumcircle through points A, B and C drawn. P is another point on the minor arc BC. Points L, M and N are the feet of the perpendiculars from P to the sides BC, CA and AB respectively.

(i) Copy the diagram and explain why P, L, N and B are concyclic. **1**

(ii) Explain why P, L, C and M are concyclic. **1**

(iii) Let $\angle PLM = \alpha$.

(α) Show that $\angle ABP = \alpha$. **2**

(β) Hence show that M, L and N are collinear. **2**

(b) A particle of unit mass is thrown vertically downwards with an initial velocity of v_0. It experiences a resistive force of magnitude kv^2 where v is its velocity. Taking downwards as the positive direction, the equation of motion of the particle is given by

$$\ddot{x} = g - kv^2 .$$

Let V be the terminal velocity of the particle.

(i) Explain why $V = \sqrt{\dfrac{g}{k}}$. **1**

(ii) Show that $v^2 = V^2 + (v_0^2 - V^2)e^{-2kx}$. **4**

(c) Let $z = x + iy$ be any non-zero complex number such that $z + \dfrac{1}{z} = k$, where k is a real number.

(i) Prove that either $y = 0$ or $x^2 + y^2 = 1$. **2**

(ii) Show that if $y = 0$ then $|k| \geq 2$. **2**

Exam continues next page ...

QUESTION SEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Write down $\cos 2\theta$ in terms of $\tan\theta$. **1**

(ii) Show that $\cos 4\theta = \dfrac{1 - 6\tan^2\theta + \tan^4\theta}{1 + 2\tan^2\theta + \tan^4\theta}$. **3**

(iii) Deduce that $\tan^2 \frac{\pi}{8} + \tan^2 \frac{3\pi}{8} = 6$. **3**

(b) Consider the equation $z^7 = 1$.

This equation has seven roots $1, \rho, \rho^2, \ldots, \rho^6$, where $\rho = \operatorname{cis} \frac{2\pi}{7}$.

Let $\alpha = \rho + \rho^2 + \rho^4$ and $\theta = \rho^3 + \rho^5 + \rho^6$.

(i) Express ρ^9 as a lower positive power of ρ. **1**

(ii) Simplify $\alpha + \theta$. **2**

(iii) Simplify $\alpha\theta$. **2**

(iv) Form a quadratic equation with α and θ as roots. **1**

(v) Deduce that $\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{8\pi}{7} = -\frac{1}{2}$. **2**

Exam continues overleaf ...

QUESTION EIGHT (15 marks) Use a separate writing booklet. **Marks**

(a) The sequence $a_1, a_2, \ldots, a_n$ is defined by $a_n = \dfrac{(2n)!}{2^n n!}$. **4**

Show by induction on n that a_n is an odd positive integer.

(b) Suppose that $y = f(x)$ is an increasing function for $x \geq 1$.
Suppose also that $f(x) \geq 0$ for $x \geq 1$.

(i) Explain, with the aid of a diagram, why **2**

$$f(1) + f(2) + \cdots + f(n-1) < \int_1^n f(x)\,dx < f(2) + f(3) + \cdots + f(n).$$

(ii) Show that $\displaystyle\int_1^n \ln x\,dx = n\ln n - n + 1$. **2**

(iii) Use parts (i) and (ii) to deduce that, for $n > 1$:

(α) $n! > \dfrac{n^n}{e^{n-1}}$ **3**

(β) $n! < \dfrac{n^{n+1}}{e^{n-1}}$ **2**

(iv) Find $\displaystyle\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$. (You may assume that $\displaystyle\lim_{n\to\infty} \sqrt[n]{n} = 1$.) **2**

END OF EXAMINATION