Mathematics Extension I CSSA HSC Trial Examination 2002
Marking Guidelines

estion 1

Outcomes Assessed: H5, PE5

Marking Guidelines

| Criteria | Marks |
|---|---|
| • finding first derivative | 1 |
| • finding second derivative in form $\frac{e^x}{(e^x+1)^2}$ | 1 |

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$\frac{d}{dx}\ln(e^x+1) = \frac{e^x}{e^x+1}$

$\frac{d^2}{dx^2}\ln(e^x+1) = \frac{e^x.(e^x+1)-e^x.e^x}{(e^x+1)^2} = \frac{e^x}{(e^x+1)^2}$

Outcomes Assessed: H5, PE3, PE6

Marking Guidelines

| Criteria | Marks |
|---|---|
| • interpreting $\Sigma$ notation to write sum of terms in expanded form | 1 |
| • calculating value of sum as $-\frac{5}{8}$ | 1 |

wer

$\sum_{k=1}^{4}\frac{(-1)^k}{k!} = -\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!} = \frac{-24+12-4+1}{24} = -\frac{5}{8}$

Outcomes Assessed: (i) P3 (ii) P3, PE2, HE7

Marking Guidelines

| Criteria | Marks |
|---|---|
| (i) • writing expressions for $1 \pm \cos 2x$ in terms of $\cos^2 x$, $\sin^2 x$ | 1 |
| • simplifying to obtain final result | 1 |
| (ii) • substituting $x = 22\frac{1}{2}°$ and $\cos 45° = \frac{1}{\sqrt{2}}$ to find expression for $\tan^2 22\frac{1}{2}°$ | 1 |
| • using expression for $\tan^2 22\frac{1}{2}°$ to show $\tan 22\frac{1}{2}° = \sqrt{2}-1$ | 1 |

wer

) $\frac{1-\cos 2x}{1+\cos 2x} = \frac{2\sin^2 x}{2\cos^2 x} = \tan^2 x$

(ii) $\tan^2 22\frac{1}{2}° = \frac{1-\cos 45°}{1+\cos 45°} = \frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}-1}{\sqrt{2}+1}$

$\tan^2 22\frac{1}{2}° = \frac{(\sqrt{2}-1)(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} = \frac{(\sqrt{2}-1)^2}{2-1}$

$\therefore \tan 22\frac{1}{2}° = (\sqrt{2}-1)$, since $\tan 22\frac{1}{2}° > 0$

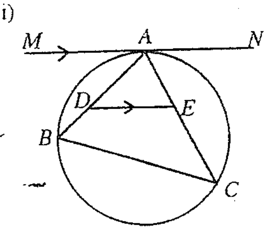
1(d) Outcomes Assessed: (i) (ii) PE3 (iii) H5, PE2, PE3

Marking Guidelines

| Criteria | Mar |
|---|---|
| (i) • copying diagram | 0 |
| (ii) • using alternate segment theorem | 1 |
| (iii) • using equal alternate angles with parallel lines to deduce $A\hat{D}E = M\hat{A}D$ | 1 |
| • deducing $A\hat{D}E = E\hat{C}B$ with explanation | 1 |
| • deducing $BCED$ is cyclic by applying appropriate test | 1 |

Answer

(i)




(ii)
$M\hat{A}B = A\hat{C}B$ (*angle between tangent MAN*
*AB equal to angle in alternate*

(iii)
$A\hat{D}E = M\hat{A}D$ (*Alternate angles equal, DE*
$A\hat{D}E = E\hat{C}B$ (*Both equal to* $M\hat{A}D$)
$\therefore$ $BCED$ is a cyclic quadrilateral
(*Exterior angle* $A\hat{D}E$ = *opposite interior an*

Question 2

2(a) Outcomes Assessed: P4

Marking Guidelines

| Criteria | Ma |
|---|---|
| • finding the $x$ coordinate of $P$ | |
| • finding the $y$ coordinate of $P$ | |

Answer

$x = \frac{4\times 4+1\times(-2)}{4+1} = 2\cdot 8$, $y = \frac{4\times(-5)+1\times 3}{4+1} = -3\cdot 4$ $\therefore P(2\cdot 8, -3\cdot 4)$

2(b) Outcomes Assessed: PE3

Marking Guidelines

| Criteria | M |
|---|---|
| • using at least one of the factors ${}^7C_2$, $3^5$ | |
| • completing the calculation ${}^7C_2 \times 3^5 = 5103$ | |

Answer

Choose the 2 questions to be answered correctly ${}^7C_2$ ways

**Outcomes Assessed:** (i) PE3 (ii) PE3, PE6

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • partial factorisation $P(x)=(x-1)(x^2+x-2)$ | 1 |
| • completing factorisation $P(x)=(x+2)(x-1)^2$ | 1 |
| (ii) • deducing $x \le -2$ | 1 |
| • including $x=1$ | 1 |

**Answer**

(i)

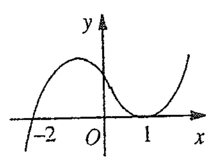
$(x-1)$ is a factor of $P(x)$

$x^3-3x+2=(x-1)(x^2+x-2)$

$=(x-1)(x-1)(x+2)$

$\therefore P(x)=(x+2)(x-1)^2$

(ii)




$y=(x+2)(x-1)^2$

By inspection of the graph,

$x^3-3x+2 \le 0$ when

$x \le -2$ or $x=1$

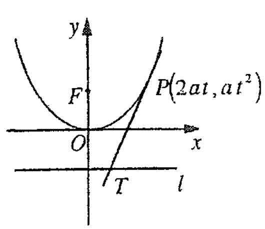
2(d) **Outcomes Assessed:** (i) PE3 (ii) PE3, PE4

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • finding the $x$ coordinate of $T$ | 1 |
| (ii) • finding the gradient of $PF$ | 1 |
| • finding the gradient of $TF$ | 1 |
| • showing the product of the gradients is $-1$ to prove $TF \perp PF$ | 1 |

**Answer**

(i)




At $T$, $\left.\begin{array}{l} y=-a \\ tx-y-at^2=0 \end{array}\right\} \Rightarrow \begin{array}{l} tx=a(t^2-1) \\ x=a\left(t-\frac{1}{t}\right) \end{array}$

$\therefore T\left(a\left(t-\frac{1}{t}\right), -a\right)$

(ii) $F(0,a) \Rightarrow$ *gradient* $PF=\dfrac{a(t^2-1)}{2at}=\dfrac{1}{2}\left(t-\dfrac{1}{t}\right)$ and *gradient* $TF=\dfrac{-2a}{a\left(t-\frac{1}{t}\right)}=-\dfrac{2}{\left(t-\frac{1}{t}\right)}$

$\therefore$ *gradient* $PF$ . *gradient* $TF=-1$ and hence $TF \perp PF$.

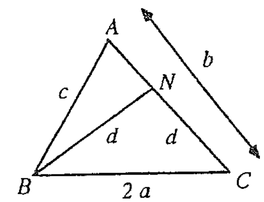
Question 3

(a) **Outcomes Assessed:** (i) H5 (ii) P4

**Marking Guidelines**

| Criteria |
|---|
| (i) • using similarity and sides in proportion to deduce $\frac{d}{2a}=\frac{c}{b}=\frac{b-d}{c}$ |
| • selecting the appropriate relationships to show $bd=2ac$, $c^2=b(b-d)$ |
| • using these simultaneously to show $c^2=b^2-2ac$ |
| (ii) • substitution in expansion of $(a+c)^2$ to show $(a+c)^2=a^2+b^2$ |

**Answer**




(i)

$\Delta ABN ||| \Delta ACB$ (*given*)

$\dfrac{BN}{CB}=\dfrac{AB}{AC}=\dfrac{AN}{AB}$ (*corresponding sides* *$\Delta$'s are in prop*

$\dfrac{d}{2a}=\dfrac{c}{b}=\dfrac{b-d}{c} \Rightarrow \begin{cases} bd=2ac \\ c^2=b(b-d)= \end{cases}$

$\therefore c^2=b^2-2ac$

(ii)

$\left.\begin{array}{l} b^2=c^2+2ac \quad (\textit{from (i)}) \\ (a+c)^2=a^2+c^2+2ac \end{array}\right\} \Rightarrow (a+c)^2$

(b) **Outcomes Assessed:** (i) PE2, PE3 (ii) PE3

**Marking Guidelines**

| Criteria |
|---|
| (i) • establishing that $P(0)$, $P(1)$ have opposite signs |
| • noting that $P(x)$ is continuous to deduce existence of root $\alpha$, $0<\alpha<1$ |
| (ii) • quoting correct expression for approximate value of $\alpha$ using Newton's method |
| • calculating approximate value of $\alpha$ correct to 2 decimal places |

**Answer**

(i) $P(x)=x^3+3x^2+6x-5 \Rightarrow \begin{cases} P(0)=-5<0 \\ P(1)=5>0 \end{cases}$ and $P(x)$ is continuous

$\therefore P(x)=0$ has a root $\alpha$, $0<\alpha<1$.

(ii) $P'(x)=3x^2+6x+6$ $\alpha \approx 0\cdot5-\dfrac{P(0\cdot5)}{P'(0\cdot5)}=0\cdot5-\dfrac{(-1\cdot125)}{9\cdot75} \approx 0\cdot62$ (to 2 dec

(c) **Outcomes Assessed:** HE6

**Marking Guidelines**

| Criteria |
|---|
| • using substitution process correctly to obtain new integrand in terms of $u$ |
| • finding the new limits for the integral in terms of $u$ |
| • obtaining the primitive function $2\sin^{-1}u$ |
| • evaluating the definite integral by substitution of the limits |

$u > 0$

$du$

$u = \frac{1}{2}$

$u = \frac{1}{\sqrt{2}}$

$$I = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx = \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{1}{u\sqrt{1-u^2}}\,2u\,du$$

$$I = 2\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-u^2}}\,du = 2\left[\sin^{-1}u\right]_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}}$$

$$I = 2\left(\sin^{-1}\tfrac{1}{\sqrt{2}} - \sin^{-1}\tfrac{1}{2}\right) = 2\left(\tfrac{\pi}{4}-\tfrac{\pi}{6}\right) = \tfrac{\pi}{6}$$

n 4

utcomes Assessed: (i) H5 (ii) H8

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • obtaining the primitive function $\frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)$ | 1 |
| • evaluation of the definite integral by substitution of the limits | 1 |
| • using the pattern for Simpson's rule with correct $x$ values, $h$ value and multipliers. | 1 |
| • calculation of 3 function values and final approximation for definite integral | 1 |

(ii)

$$= \int_0^{\frac{\pi}{2}} \sin^2 x\,dx$$

$$= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x)\,dx$$

$$= \tfrac{1}{2}\left[x - \tfrac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{2}\left(\tfrac{\pi}{2} - 0\right) = \tfrac{\pi}{4}$$

$f(x) = \sin^2 x$, $h = \frac{\pi}{4}$

| $x$ | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ |
|---|---|---|---|
| $f$ | 0 | $\frac{1}{2}$ | 1 |
| | ×1 | ×4 | ×1 |

$$I \approx \tfrac{h}{3}\{f_0 + 4f_1 + f_2\}$$
$$= \tfrac{\pi}{12}\{0+2+1\}$$
$$= \tfrac{\pi}{4}$$

utcomes Assessed: (i) PE3 (ii) PE3

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • determining that there are 3 appropriate sets of three cards for a sum of 9 | 1 |
| • calculating $\frac{3}{{}^9C_3} = \frac{1}{28}$ as the required probability | 1 |
| ) • realising that there are now ${}^8C_2$ possible sets of three cards given 2 is selected | 1 |
| • calculating $\frac{2}{{}^8C_2} = \frac{1}{14}$ as the required probability | 1 |

r

Exactly 3 sets of cards have a sum of 9: 1+2+6, 1+3+5, 2+3+4

$P(sum\ is\ 9) = \frac{3}{{}^9C_3} = \frac{3.3.2.1}{9.8.7} = \frac{1}{28}$

If the set of cards contains the number 2, exactly two such sets have a sum of 9.
The two cards chosen to complete the set of 3 are selected from the remaining 8 cards.

2 2.2.1 1

4(c) Outcomes Assessed: PE2, HE5

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • finding the relationship between $\frac{dV}{dt}$ and $\frac{dr}{dt}$ | 1 |
| • finding the relationship between $\frac{dL}{dt}$ and $\frac{dV}{dt}$, where the equator has length $L$ cm | 1 |
| • using the numerical values of $\frac{dV}{dt}$ and $r$ to show $\frac{dL}{dt} = 0\cdot 125$ | 1 |
| • interpreting this to deduce that length of equator is increasing at a rate of $0\cdot 125\text{cm s}^{-1}$ | 1 |

**Answer**

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi . 3r^2 \frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$L = 2\pi r$

$\frac{dL}{dt} = 2\pi \frac{dr}{dt} = 2\pi . \frac{1}{4\pi r^2}\frac{dV}{dt}$

$\therefore \frac{dL}{dt} = \frac{1}{2r^2}\frac{dV}{dt} = \frac{25}{2\times 10^2} = 0\cdot 125$ when $r = 10$

Length of equator is increasing at a rate of $0\cdot 125\text{cm s}^{-1}$ when the radius is 10 cm

**Question 5**

(a) Outcomes Assessed: (i) HE4 (ii) P5, HE4

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • finding the equation of the inverse function $f^{-1}(x)$ | 1 |
| (ii) • showing intercepts on the coordinate axes and asymptotes for both curves | 1 |
| • showing intersection point (1, 1) | 1 |
| • correct shapes with curves as reflections in $y = x$ | 1 |

**Answer**

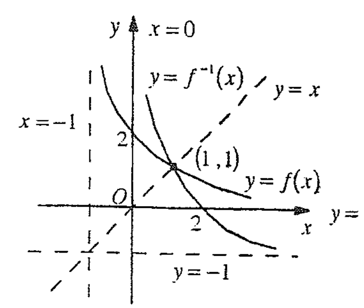
(i)

$y = \frac{2}{x+1} \Rightarrow (x+1) = \frac{2}{y} \Rightarrow x = \frac{2}{y} - 1$

$f(x) = \frac{2}{x+1},\ x > -1 \Rightarrow f^{-1}(x) = \frac{2}{x} - 1,\ y > -1$

$\therefore f$ has inverse $f^{-1}(x) = \frac{2}{x} - 1,\ x > 0$

Curves are reflections in $y = x$ and hence intersect on $y = x$ where $\frac{2}{x+1} = x \Rightarrow x = 1$



) **Outcomes Assessed:** (i) HE5 (ii) H5, PE2

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • obtaining expression for $a$ in terms of $x$ | 1 |
| (ii) • integrating expression for $\frac{dt}{dx}$ to obtain primitive function (even if $+c$ omitted) | 1 |
| • including and evaluating the constant of integration to find $t$ in terms of $x$ | 1 |
| • finding $x$ in terms of $t$ by rearrangement. | 1 |

**Answer**

(i) $v=-x^2 \Rightarrow a=v\frac{dv}{dx}=-x^2.(-2x)=2x^3$

(ii)

$$\frac{dx}{dt}=-x^2 \Rightarrow \frac{dt}{dx}=-\frac{1}{x^2} \Rightarrow t=\frac{1}{x}+c,\quad c \text{ constant}$$

$$\left.\begin{array}{l} t=0 \\ x=1 \end{array}\right\} \Rightarrow 0=1+c \Rightarrow c=-1 \Rightarrow t=\frac{1}{x}-1 \quad \therefore x=\frac{1}{t+1}$$

5(c) **Outcomes Assessed:** PE3

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • writing general term with appropriate binomial coefficient and powers of $x^2$ and $\frac{a}{x}$ | 1 |
| • showing term independent of $x$ is ${}^6C_4 a^4$ or ${}^6C_2 a^4$ | 1 |
| • deducing ${}^6C_4 a^4=240$ or ${}^6C_2 a^4=240$ and hence $a^4=16$ | 1 |
| • stating both solutions $a=\pm 2$ | 1 |

**Answer**

General term in expansion of $\left(x^2+\frac{a}{x}\right)^6$ is ${}^6C_r\left(\frac{a}{x}\right)^r\left(x^2\right)^{6-r}={}^6C_r\, a^r x^{12-3r}$, $r=0,1,2,\ldots,6$

Then term independent of $x$ is ${}^6C_4\, a^4 x^0=15a^4 \Rightarrow 15a^4=240 \Rightarrow a^4=16 \quad \therefore a=\pm 2$

## Question 6

6(a) **Outcomes Assessed:** (i) H5, HE4 (ii) P4, HE7

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • showing $\tan\theta=\frac{A+B}{1-AB}$ | 1 |
| (ii) • showing $6x^2+5x-1=0$ | 1 |
| • solving this quadratic equation | 1 |
| • rejecting the solution $x=-1$ with explanation | 1 |

**Answer**

(i) Let $x=\tan^{-1}A$ and $y=\tan^{-1}B$. Then $\theta=x+y$, $\tan x=A$, $\tan y=B$ and hence

$$\tan\theta=\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x.\tan y}=\frac{A+B}{1-AB}$$

(ii) $\tan^{-1}3x+\tan^{-1}2x=\frac{\pi}{4} \Rightarrow \frac{3x+2x}{1-3x.2x}=\tan\frac{\pi}{4}$, using $A=3x$, $B=2x$ in (i)

$$\frac{5x}{1-6x^2}=1 \Rightarrow \begin{array}{c} 6x^2+5x-1=0 \\ (6x-1)(x+1)=0 \\ x=\frac{1}{6} \text{ or } x=-1 \end{array}$$

But $x=-1 \Rightarrow \left\{\begin{array}{l} \tan^{-1}3x<0 \text{ and } \tan \\ \therefore \tan^{-1}3x+\tan^{-1}2 \end{array}\right.$

Hence $x\neq -1$. $\therefore x=\frac{1}{6}$

6(b) **Outcomes Assessed:** (i) H3, HE3 (ii) H3, HE3

**Marking Guidelines**

| Criteria |
|---|
| (i) • finding value of $A$ |
| • finding exact value of $k$ |
| (ii) • showing $t=\frac{\ln 2}{k}$ |
| • finding the further time 5 min 38 s |

**Answer**

(i) $T=20+Ae^{-kt}$

$$\left.\begin{array}{l} t=0 \\ T=100 \end{array}\right\} \Rightarrow \begin{array}{l} 100=20+Ae^0 \\ 100=20+A \end{array} \quad \therefore A=80 \text{ and } T=20+80e^{-kt}$$

T

$$\left.\begin{array}{l} t=4 \\ T=80 \end{array}\right\} \Rightarrow \begin{array}{l} 80=20+80e^{-4k} \\ e^{-4k}=\frac{60}{80}=\frac{3}{4} \end{array} \quad \therefore \begin{array}{l} -4k=\ln\frac{3}{4} \\ k=-\frac{1}{4}\ln\frac{3}{4}=\frac{1}{4}\ln\frac{4}{3} \end{array}$$

(ii)

$$\left.\begin{array}{l} T=20+80e^{-kt} \\ T=60 \end{array}\right\} \Rightarrow \begin{array}{l} e^{-kt}=\frac{40}{80}=\frac{1}{2} \\ -kt=\ln\frac{1}{2}=-\ln 2 \end{array} \quad \therefore t=\frac{\ln 2}{\left(\frac{1}{4}\ln\frac{4}{3}\right)}\approx 9\cdot 6377$$

Hence it falls to 60°C after 9 min 38 sec, that is after a further 5 min 38 sec.

6(c) **Outcomes Assessed:** (i) PE2, HE3 (ii) H5, HE3

**Marking Guidelines**

| Criteria |
|---|
| (i) • finding values of $v$ and $a$ when $t=0$ |
| • interpreting these values to deduce particle is moving right and slowing down |
| (ii) • showing if particle is at $O$ at time $t$, then $\tan 2t=-3$ |
| • solving this equation to find the first such time. |

**Answer**

(i)

$x=3\cos 2t+\sin 2t$

$v=-6\sin 2t+2\cos 2t$

$a=-12\cos 2t-4\sin 2t$

$\therefore t=0 \Rightarrow x=3,\ v=2,\ a=-12$

Hence particle is initially 3 m to the right of $O$, moving to the right (since $v>0$) and

$\tan^{-1}3x + \tan^{-1}2x = \frac{\pi}{4} \Rightarrow \dfrac{3x+2x}{1-3x.2x} = \tan\frac{\pi}{4}$, using $A = 3x$, $B = 2x$ in (i)

$\dfrac{5x}{1-6x^2} = 1 \Rightarrow$ $6x^2+5x-1=0$, $(6x-1)(x+1)=0$, $x=\frac{1}{6}$ or $x=-1$

But $x=-1 \Rightarrow \begin{cases} \tan^{-1}3x<0 \text{ and } \tan^{-1}2x<0 \\ \therefore \tan^{-1}3x + \tan^{-1}2x \neq \frac{\pi}{4} \end{cases}$

Hence $x \neq -1$. $\therefore x = \frac{1}{6}$

**Outcomes Assessed:** (i) **H3, HE3** (ii) **H3, HE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • finding value of $A$ | 1 |
| • finding exact value of $k$ | 1 |
| (ii) • showing $t = \dfrac{\ln 2}{k}$ | 1 |
| • finding the further time 5 min 38 s | 1 |

er

$T = 20 + A e^{-kt}$

$\left.\begin{matrix} t=0 \\ T=100 \end{matrix}\right\} \Rightarrow$ $100 = 20 + A e^0$, $100 = 20 + A$ $\quad \therefore A = 80$ and $T = 20 + 80 e^{-kt}$ Then

$\left.\begin{matrix} t=4 \\ T=80 \end{matrix}\right\} \Rightarrow$ $80 = 20 + 80 e^{-4k}$, $e^{-4k} = \frac{60}{80} = \frac{3}{4}$ $\quad \therefore$ $-4k = \ln\frac{3}{4}$, $k = -\frac{1}{4}\ln\frac{3}{4} = \frac{1}{4}\ln\frac{4}{3}$

$\left.\begin{matrix} T = 20 + 80e^{-kt} \\ T = 60 \end{matrix}\right\} \Rightarrow$ $e^{-kt} = \frac{40}{80} = \frac{1}{2}$, $-kt = \ln\frac{1}{2} = -\ln 2$ $\quad \therefore t = \dfrac{\ln 2}{\left(\frac{1}{4}\ln\frac{4}{3}\right)} \approx 9 \cdot 6377$

Hence it falls to 60°C after 9 min 38 sec, that is after a further 5 min 38 sec.

**utcomes Assessed:** (i) **PE2, HE3** (ii) **H5, HE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| ) • finding values of $v$ and $a$ when $t = 0$ | 1 |
| • interpreting these values to deduce particle is moving right and slowing down | 1 |
| i) • showing if particle is at $O$ at time $t$, then $\tan 2t = -3$ | 1 |
| • solving this equation to find the first such time. | 1 |

r

$= 3\cos 2t + \sin 2t$

$= -6\sin 2t + 2\cos 2t$

$= -12\cos 2t - 4\sin 2t$

$\therefore t = 0 \Rightarrow x = 3,\ v = 2,\ a = -12$

Hence particle is initially 3 m to the right of $O$, moving to the right (since $v > 0$) and

(ii) At $O$, $x = 0$

$3\cos 2t + \sin 2t = 0$

$\sin 2t = -3\cos 2t$

$\tan 2t = -3$

smallest positive such $t$ is given by

$2t = \pi - \tan^{-1}3 \Rightarrow t = \frac{1}{2}(\pi - \tan^{-1}3) \approx 0 \cdot 95$

$\therefore$ particle first reaches $O$ after $0 \cdot 95$ s (to 2 de

**Question 7**

7(a) **Outcomes Assessed:** (i) **P5, PE6** (ii) **P5, PE6, HE2** (iii) **P5, PE2, PE6**

**Marking Guidelines**

| Criteria | Mar |
|---|---|
| (i) • showing $f(0) = 1$ | 1 |
| • showing $f(-x) = \dfrac{1}{f(x)}$ | 1 |
| (ii) • noting that $S(1)$ is true | 1 |
| • showing that if $S(k)$ is true, then $S(k+1)$ is true | 1 |
| • deducing the truth of $S(n)$ for all positive integers | 1 |
| (iii) • using (i) and (ii) to deduce that $f(-nx) = [f(x)]^{-n}$ | 1 |

**Answer**

(i)

$f(0+0) = f(0).f(0)$

$f(0) - f(0).f(0) = 0$

$f(0)\,[1 - f(0)] = 0$

$\therefore f(0) > 0 \Rightarrow f(0) = 1$

$f(x + [-x]) = f(x).f(-x)$

$\therefore f(x).f(-x) = f(0) = 1$

$\therefore \quad f(-x) = \dfrac{1}{f(x)}$

(ii) Let $S(n)$ be the statement $f(nx) = [f(x)]^n$, $n = 1, 2, 3, \ldots$

Clearly $S(1)$ is true, since $f(1.x) = [f(x)]^1$.

If $S(k)$ is true for some positive integer $k$, then $f(kx) = [f(x)]^k$ **

Consider $S(k+1)$ : $f([k+1]x) = f(kx + x) = f(kx).f(x)$

$= [f(x)]^k.f(x)$

$= [f(x)]^{k+1}$ if $S(k)$ is true, using

Hence if $S(k)$ is true for some positive integer $k$, then $S(k+1)$ is true. But $S(1)$ is true. Hence $S(2$ true, and then $S(3)$ is true and so on. Hence $S(n)$ is true for all positive integers $n$.

(iii) If $n$ is a positive integer, $f(-nx) = \dfrac{1}{f(nx)} = \dfrac{1}{[f(x)]^n}$, using (i) and (ii), and hence

$f(-nx) = [f(x)]^{-n}$.

**Outcomes Assessed:** (i) **HE3** (ii) **PE2, HE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| (i) • writing expressions for horizontal displacements of both particles | 1 |
| • writing expressions for vertical displacements of both particles | 1 |
| (ii) • showing $U\cos\alpha = V\cos\beta$ | 1 |
| • showing $UT\sin\alpha = h + VT\sin\beta$ | 1 |
| • eliminating $V$ from this relationship | 1 |
| • rearrangement to obtain $T$ in required form | 1 |

**swer**

(i) For particle projected from $O$

$$x_O = Ut\cos\alpha$$
$$y_O = Ut\sin\alpha - \tfrac{1}{2}gt^2$$

For particle projected from $A$

$$x_A = Vt\cos\beta$$
$$y_A = h + Vt\sin\beta - \tfrac{1}{2}gt^2$$

(ii) Particles collide at time $T$, having equal horizontal displacements and equal vertical displacements.

$$UT\cos\alpha = VT\cos\beta \quad \Rightarrow \quad U\cos\alpha = V\cos\beta \qquad (1)$$
$$UT\sin\alpha - \tfrac{1}{2}gT^2 = h + VT\sin\beta - \tfrac{1}{2}gT^2 \quad \Rightarrow \quad UT\sin\alpha = h + VT\sin\beta \qquad (2)$$

From (2):
$$T(U\sin\alpha - V\sin\beta) = h$$
$$T(U\sin\alpha\cos\beta - V\cos\beta\sin\beta) = h\cos\beta$$

Using (1):
$$T(U\sin\alpha\cos\beta - U\cos\alpha\sin\beta) = h\cos\beta$$
$$UT\sin(\alpha-\beta) = h\cos\beta$$
$$\therefore \quad T = \frac{h\cos\beta}{U\sin(\alpha-\beta)}$$