Sydney Grammar School

4 unit mathematics

Trial HSC Examination 1994

1. **(a)** Find: **(i)** $\int \frac{e^x \, dx}{(1-e^x)^2}$ **(ii)** $\int \frac{x+3}{x^2-4x+8} \, dx$.

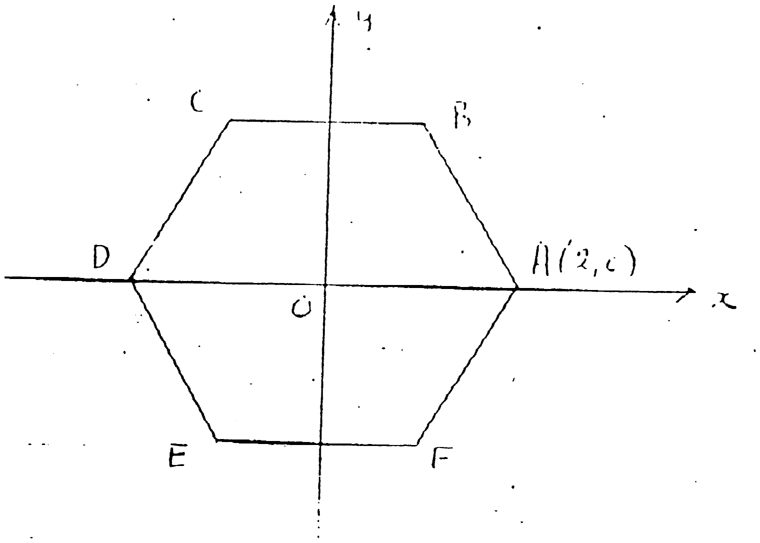
(b) Evaluate $\int_0^{\frac{\pi}{2}} x \cos x \, dx$.

(c) Evaluate $\int_1^3 \frac{dx}{x^2+2x}$

(d) Use the substitution $x = \sin^2 \theta$ to evaluate $\int_0^{\frac{1}{2}} \frac{\sqrt{x} \, dx}{(1-x)^{\frac{3}{2}}}$.

2. **(a)** Find both square roots of $5 - 42i$, expressing them in the form $a + ib$, where a and b are real.

(b)



$ABCDEF$ is a regular hexagon drawn on an Argand diagram with vertex A at the point $(2, 0)$. O is the centre of the hexagon.

(i) Copy the diagram.

(ii) On your diagram show the region within the hexagon in which both the inequalities $|z| \geq 1$ and $-\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{3}$ are satisfied.

(iii) Find, in the form $|z - c| = R$, the equation of the circle through the point O, B and F.

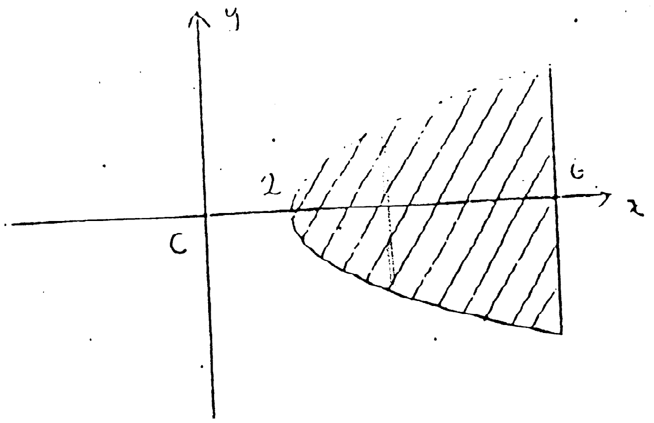
(iv) Find the complex numbers represented by the points C and E.

(v) The hexagon is rotated anticlockwise about the origin through an angle of $\frac{\pi}{4}$. Express in the form $r(\cos \theta + i \sin \theta)$ the complex numbers represented by the new positions of C and E.

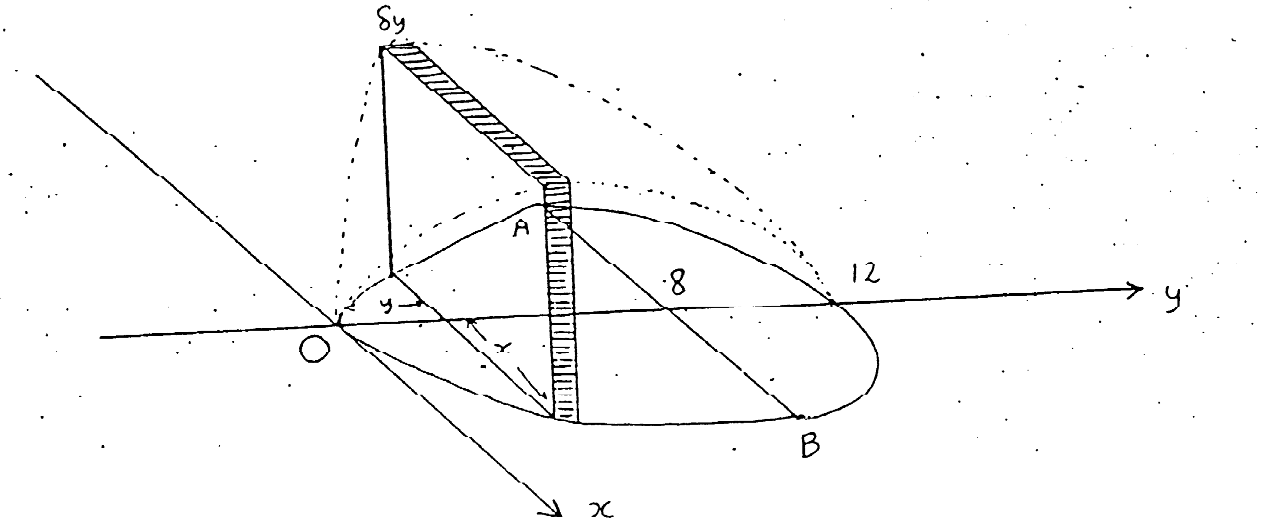
(c) z is a point on the circle $|z - 1| = 1$ and $\arg z = \theta$.

(i) Find $\arg(z-1)$ in terms of θ.
(ii) Hence, or otherwise, find $\arg(z^2-3z+2)$ in terms of θ.

3. (a)



The diagram shows the region bounded by the curve $y^2 = 4(x-2)$ and the line $x = 6$. Use the method of cylindrical shells to find the volume of the solid formed by rotating the given region about the y axis.



The diagram shows a solid with base in the x-y-plane. Every cross-section perpendicular to the y-axis is a square. One part of the base is the segment OAB of the parabola $x^2 = 2y$ cut off by the line $y = 8$. The other part of the base is a semi-circle with diameter AB. Consider a slice S of the solid of width δy and perpendicular to the y-axis as shown.
(i) Find an expression for the volume δV of S in terms of x and δy.
(ii) Find the volume of the solid.

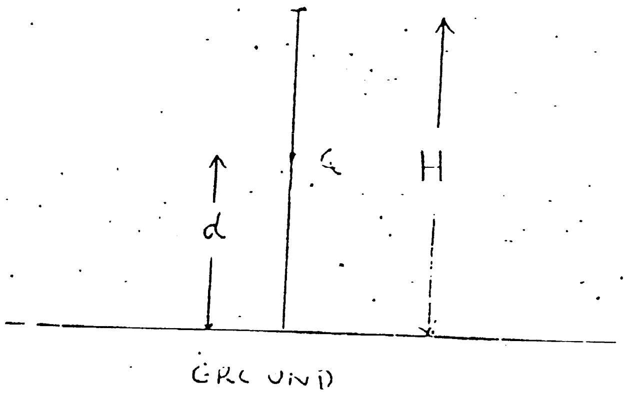
4. (a) (i) If $f(x) = (x+1)(x-2)$, sketch the graphs of the following on separate diagrams.
(α) $y = f(x)$ **(β)** $y = \frac{1}{|f(x)|}$ **(γ)** $y = \ln[f(x)]$.

(ii) If also $g(x) = -x^2$, sketch the graphs of $y^2 = g(x)f(x)$. Use calculus to describe the nature of the curve at $x = -1, x = 0$ and $x = 2$.

(b) Find a general solution (in radians) of the equation $\cos 3x - \cos x + \cos 5x = 0$

(c) If $2\sin 2x + \cos 2x = k$, show that $(1 + k)\tan^2 x - 4\tan x - 1 + k = 0$. Also show that if $\tan x_1$ and $\tan x_2$ are the roots of this quadratic equation in $\tan x$ then $\tan(x_1 + x_2) = 2$.

5. (a)



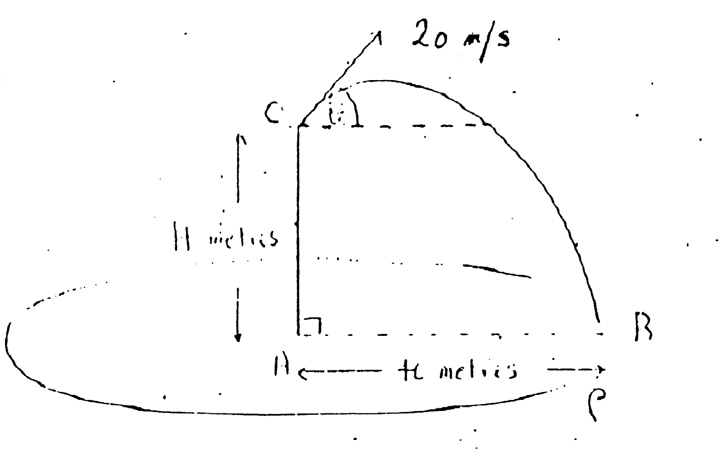
From a point on the ground an object of mass m is projected vertically upwards with an initial speed of u. It reaches a maximum height of H before falling back to the ground. The resistance due to air is equal to mkv^2 and g is the acceleration due to gravity.

(i) Show that $H = \frac{1}{2k}\ln(\frac{g+ku^2}{g})$.

(ii) Q is a point of height d above the ground. Let V_1 be the speed of the object at Q on its upward path. Show that $d = \frac{1}{2k}\ln(\frac{g+ku^2}{g+kV_1^2})$

(iii) On the object's downwards path it passes Q with a speed of half that when first at Q. Show that $V_1 = \sqrt{\frac{3g}{k}}$.

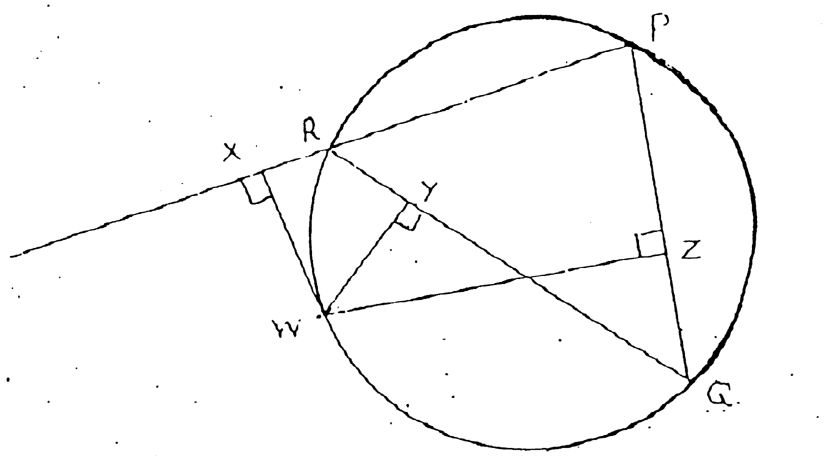
(b)



A particle P is rotating in a circle with a uniform angular velocity of 4 rad/s. The

circle has a centre A and a radius of 40 metres. B is the initial position of the particle P. From a point O, a distance of H metres vertically above A, a stone is projected with a speed of 20m/s at an angle of θ to the horizontal. The stone is projected when P is at its initial position B, and the path of the stone is in the same vertical plane as O, A and B. The stone strikes B at the moment that P has completed exactly 3 revolutions after the stone was projected.
(i) Derive the equations of motion for the stone in flight. (Use $g = 10$ m/s^2 and take O as the origin).
(ii) Show that the time of flight for the stone is $\frac{3\pi}{2}$ seconds.
(iii) Find H and θ (nearest metre and degree respectively).

6. (a) (i) (α) Prove that for any polynomial $P(x)$, if k is a zero of multiplicity 2, then k is also a zero of $P'(x)$.
(β) Show that $x = 1$ is a double root of the equation $x^{2n} - nx^{n+1} + nx^{n-1} - 1 = 0$.
(ii) Find all the roots of $3x^3 - 26x^2 + 52x - 24 = 0$, given that the roots are in geometric progression.
(b)



PQR is a triangle inscribed in a circle with W a point on the arc QR. WX is perpendicular to PR produced, and WZ is perpendicular to PQ.
(i) Copy this diagram.
(ii) Explain why $WXRY$ and $WYZQ$ are cyclic quadrilaterals.
(iii) Show that the points X, Y and Z are collinear.

7. (a) (i) (α) If $z = -1 + i$, express z in mod-arg form.
(β) On an Argand diagram plot the points representing the complex numbers z^4 and $\frac{1}{z^2}$.
(ii) Sketch the locus of those points w such that $|w - z^4| = |w - \frac{1}{z^2}|$. Find the Cartesian equation of this locus.
(iii) (α) Write down, in mod-arg form, the five of the equation $z^5 = 1$.
(β) Show that $z^5 - 1$ can be fully factorized in the form
$z^5 - 1 = (z - 1)(z^2 + 2z\cos\frac{3\pi}{5} + 1)(z^2 + 2z\cos\frac{\pi}{5} + 1)$.

(b) (i) Find the sum of the series $x + x^2 + x^3 + \cdots + x^n$.
(ii) Hence find the sum of the series $x + 2x^2 + 3x^3 + \cdots + nx^n$.

8. (a) (i) If $I_n = \int_0^1 x^n e^{x^2}\, dx$, show that $I_n + (n-1)I_{n-2} = 2e$, $(n \geq 2)$.
(ii) Evaluate I_5.
(b) The function $f(x)$ is given by $f(x) = x - \ln(1 + x^2)$.
(i) Show that $f'(x) \geq 0$ for all values of x.
(ii) Deduce that $e^x > 1 + x^2$ for all positive values of x.
(c) If $u_{n+1} = u_n + u_n^2$ and $u_1 = \frac{1}{3}$, find $\sum_{n=1}^{\infty} \frac{1}{1+u_n}$.