St Ives High School

2002 TRIAL EXAMINATION

MATHEMATICS
EXTENSION 1

Time Allowed: Two hours

Directions:

- Answer all questions
- Begin each question on a new page
- All answer pages should be stapled together in the top left hand corner

QUESTION 1

Marks

(a) For what values of x is $|x-2| < x+1$? /2

(b) Evaluate $\int_4^{20} y.dx$ if $xy = 5$ /2

(c) The interval AB lies between $A(-1, 4)$ and $B(5, -3)$. Find the coordinates of the point P which divides the interval AB externally in the ratio 1:3. /2

(d) Evaluate $\lim\limits_{x \to 0} \dfrac{\sin x}{5x}$ /2

(e) Evaluate $\int_1^e \dfrac{\log_e x}{x} \cdot dx$ using the substitution $u = log_e x$. /3

(f) Find the general solution for $\tan\theta = \dfrac{a}{b}$ /1

QUESTION 2

(a) i) Find the gradient of the tangent to the curve $y = x^2 + 3$ at the point $(1, 4)$.

ii) Find the acute angle between the line $y = 3x + 1$ and the curve $y = x^2 + 3$ at the point of intersection $(1, 4)$. Give your answer to the nearest degree. /3

(b) i) Show that $2\sin\left(A + \dfrac{\pi}{3}\right) = \sin A + \sqrt{3}\cos A$

ii) Hence solve $\sin A + \sqrt{3}\cos A = 1, \quad \dfrac{-\pi}{2} \le A \le \pi/2$ /4

(c) State the domain and range of $y = 4\sin^{-1} 2x$ and sketch the curve. / 3

(d) If $y = ae^{bx}$, show $\dfrac{d^2y}{dx^2} = b^2y$ where a, b are constants. /2

QUESTION 3

(a) Use $x = 0.5$ to find an approximation to the root of $\cos x = x$ using one application of Newton's method. (Answer correct to two decimal places). /3

(b) i) Show that $\dfrac{d}{dx}\left(\tan^{-1}(\cos x)\right) = \dfrac{-\sin x}{2 - \sin^2 x}$

ii) Hence evaluate $\displaystyle\int_0^{\pi} \frac{2\sin\theta}{2 - \sin^2\theta}\,.\,d\theta$ /5

(c) A committee of three is to be chosen from a group of six men and four women.

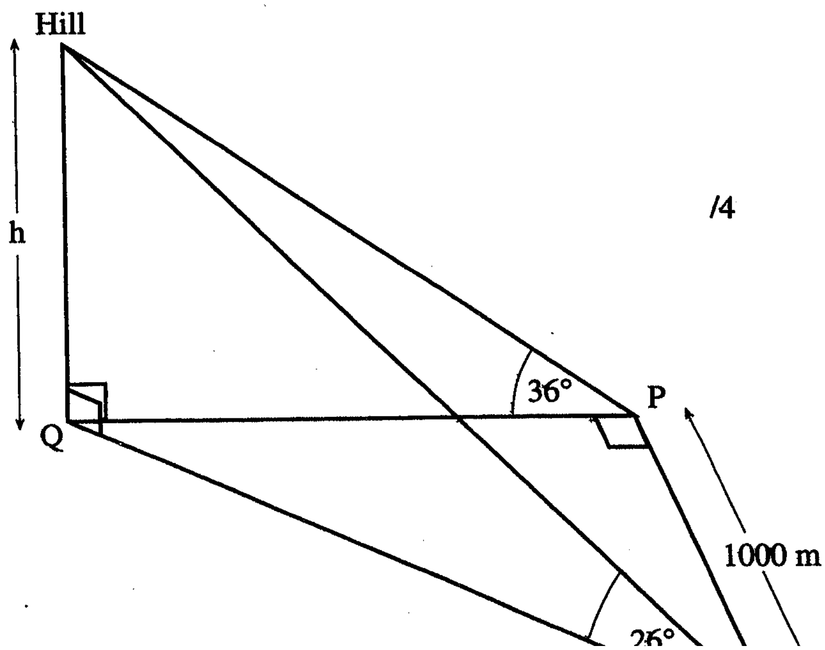
i) How many different committees can be formed if the committee is to contain at least one man and at least one woman?

ii) If the committee is chosen at random, what is the probability that it will contain at least one man and at least one woman? /2

(d) By letting $t = tan\left(\dfrac{\theta}{2}\right)$ prove $\dfrac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = \tan\left(\dfrac{\theta}{2}\right)$ /2

QUESTION 4

(a) The angle of elevation of a hill from a point P (due east of the hill) is 36° and the angle of elevation from a point L (due south of P) is 26°. If the distance between L and P is 1000m



i) Show PQ = hcot 36

ii) Hence, or otherwise, find h, the height of the hill /4

Question 4 cont...

(b) Solve $\cos^2 x - \cos 2x = 0$ for $0 \le x \le 2\pi$ /3

(c) A cup of coffee is poured at 85°. It cools according to Newton's Law of Cooling so that $\frac{dT}{dt} = -K(T - T_0)$. After 10 minutes it has cooled to 60° where the air temperature is 25°.

i) Show that $T = T_0 + Ae^{-Kt}$ is a solution to the differential equation.

ii) How long does it take for the coffee to cool to 40°?

iii) As $t \to \infty$, what happens to T? /5

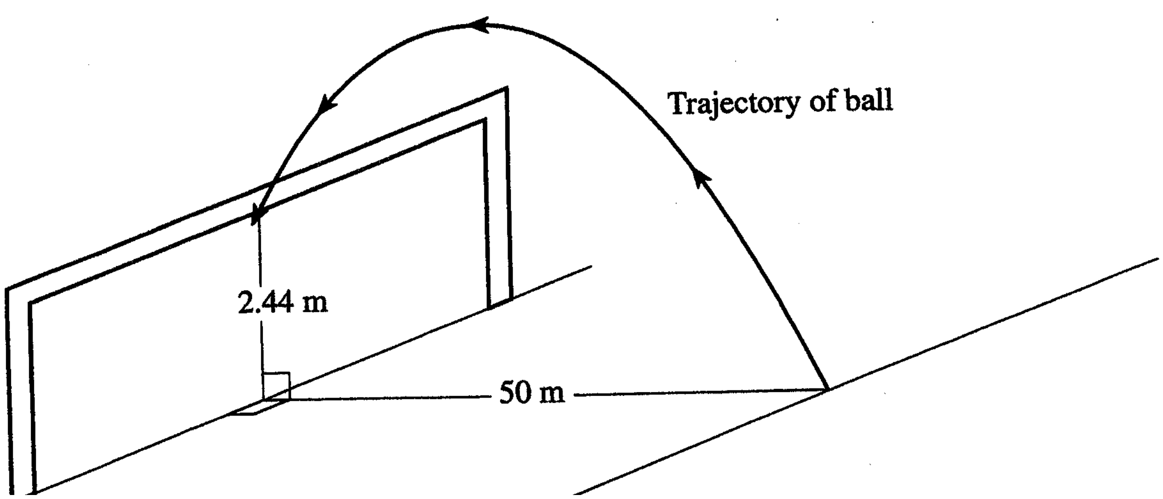
QUESTION 5

(a) A collection of "random digits" is a long sequence of digits, each of which was selected at random from the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Among five such digits taken at random, what is the probability that:

i) 0 does not appear?
ii) 0 appears exactly twice?
iii) 0 appears at least twice? /3

(b) In the World Cup Soccer series a player struck a ball on the half way line, 50m from the goal. You may assume that the angle of projection was 45°, there is no air resistance, and that acceleration due to gravity is $10 m/s^2$.

i) Write down the equations of motion for x and y.
ii) The height of the goal under which the ball must pass is 2.44m. Find the maximum velocity that the ball may be struck in order to score a goal (in the plane of the central axis as shown in the diagram below).
iii) What is the maximum height that the ball reaches in its path if it is hit with this velocity from part (ii)? /6



Question 5 cont...

(c) Find the value of the term that is independent of x in the expansion

$$\left(2x^2 + \frac{1}{x^3}\right)^{10}$$ /3

QUESTION 6

(a) A spacecraft has 10 on-board computers, each of which has a 95% reliability during a flight. If 3 or more computers fail during the flight, the spacecraft will **not** return safely to Earth. What is the probability that the spacecraft will return safely to Earth, correct to three significant figures. /3

(b) Use mathematical induction to prove that $3^{2n+4} - 2^{2n}$ is divisible by 5, for $n \geq 1$. /3

(c) i) Sketch the graph of $x = a\cos(nt)$ for $0 \leq t \leq \frac{2\pi}{n}$, where a and n are constants.

ii) On a particular day, the high tide mark at a wharf was 0.80 metres below the top of a wharf, and the low tide mark was 3.40 metres below the top of the wharf. If the motion on this day is represented by $x = a\cos(nt)$, determine the values of a and n.

iii) Tied up at the wharf is a boat whose deck is 1.70 metres above water level. Find the first time after 10 am at which the deck of the boat is level with the top of the wharf. Give answer to nearest minute. /6

QUESTION 7

(a) A student received a grant of \$50 000 to pay for her university course. The money was paid into an account at the beginning of her first year at university. At the beginning of each year, a fixed amount, M dollars was withdrawn to pay the fees for that year. The balance in the account earned interest at the rate of 8% per annum paid into the account at the end of each year, just before the next amount of M dollars is withdrawn.

i) Show that at the end of the second year, just before the third withdrawal, the amount of money in the account in dollars is $58320 - 2.2464M$.

ii) At the end of the sixth year, \$2000 remains in the account. Find the value of M to the nearest dollar. /5

(b) $P(4p, 2p^2)$ and $Q(4q, 2q^2)$ are two variable points on the parabola $x^2 = 8y$. R is the point of intersection of the tangents at P and Q.

i) Show that the coordinates of R are $(2(p + q), 2pq)$.

ii) Show that the cartesian equation of the locus of R, if $p^2 + q^2 = 8$ is $x^2 = 4(y + 8)$. /5

(c) Let s and t be positive consecutive integers, with $t = s + 1$.
Show that $s^{2n} + 2nt - 1$ is divisible by t^2. /2

End of paper