STANDARD INTEGRALS

$\int x^n dx = \frac{1}{n+1}x^{n+1} \quad (n \neq -1; x \neq 0 \text{ if } n < 0)$

$\int \frac{1}{x}dx = \log_e x \quad (x > 0)$

$\int e^{ax}dx = \frac{1}{a}e^{ax} \quad (a \neq 0)$

$\int \cos ax\, dx = \frac{1}{a}\sin ax \quad (a \neq 0)$

$\int \sin ax\, dx = -\frac{1}{a}\cos ax \quad (a \neq 0)$

$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax \quad (a \neq 0)$

$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax \quad (a \neq 0)$

$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} \quad (a \neq 0)$

$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} \quad (a > 0, -a < x < a)$

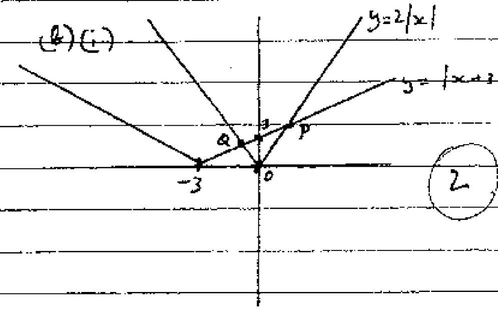
$\int \frac{1}{\sqrt{x^2-a^2}}dx = \log_e\left\{x+\sqrt{x^2-a^2}\right\} \quad (|x| > |a|)$

$\int \frac{1}{\sqrt{x^2+a^2}}dx = \log_e\left\{x+\sqrt{x^2+a^2}\right\}$

1\. (a) $\int_0^{\pi/2} \cos^2 x\, dx = \frac{1}{2}\int_0^{\pi/2}(1+\cos 2x)\,dx$

$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{4}$ 2

(b) (i)



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(ii) $\left.\begin{array}{l} y = x+3 \\ y = 2x \end{array}\right\}$ $0 = x - 3$

$\therefore P = (3, 6)$

$\left.\begin{array}{l} y = x+3 \\ y = -2x \end{array}\right\}$ $0 = 3x + 3$

$\therefore Q = (-1, 2)$

$\therefore 2|x| \leq |x+3|$ if $-1 \leq x \leq 3$ 2

(c) Let a be 1st term, d common difference

T_4, T_6, T_{10} geometric $\therefore (a+3d)^2 = (a+5d)(a+9d)$

$\cancel{a^2} + 6ad + 9d^2 = \cancel{a^2} + 14ad + 45d^2$

$\therefore 8ad + 36d^2 = 0$

$\therefore 4d(2a+9d) = 0$ 2

$\therefore 2a + 9d = 0$ since $d \neq 0$

(i) $S_{10} = \frac{10}{2}(2a+9d)$

$= 5(2a+9d) = 0$ 1

(ii) $S_6 + S_{12} = \frac{6}{2}(2a+5d) + \frac{12}{2}(2a+11d)$

$= 6a + 15d + 12a + 66d$

$= 18a + 81d = 9(2a+9d) = 0$ 2

12

(iii) $S_{12} + S_6 - 2 \times S_{10} = 0$

$\therefore S_{12} - S_{10} = S_{10} - S_6$

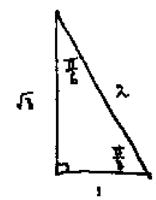
$\therefore T_{12} + T_{11} = T_{10} + T_9 + T_8 + T_7$ 1

Granbrook 3 unit Trial July 2001.

Q2.

1 (a) [2] (i) $\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$ ①

(ii) $\sin^{-1}\left(\cos\frac{\pi}{3}\right) = \sin^{-1}\left(\sin\frac{\pi}{6}\right)$ ① $= \frac{\pi}{6}$



(b) [2] $y = \sin^{-1}(1-x^2)$ Domain $|1-x^2| \le 1$

$\Rightarrow -1 \le 1-x^2 \le 1$ ①

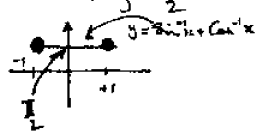
$\Rightarrow -2 \le -x^2 \le 0$ Domain

or $2 \ge x^2 \ge 0$ $\therefore \sqrt{2} \ge x \ge -\sqrt{2}$

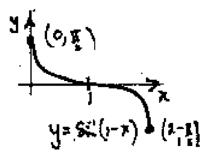
Range ① $|y| \le \frac{\pi}{2}$

(c) [4] (i) $y = \sin^{-1}x + \cos^{-1}x$

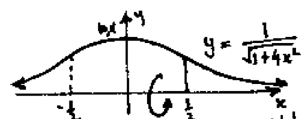
$\Rightarrow y = \frac{\pi}{2}$ ②



(ii) $y = \sin^{-1}(1-x)$ ②



(d) [4] $V = \pi\int_{-\frac{1}{2}}^{\frac{1}{2}} y^2\,dx$ ①



$y = \frac{1}{\sqrt{1+4x^2}} \Rightarrow y^2 = \frac{1}{1+4x^2}$

By Symmetry $V = 2\pi\int_0^{\frac{1}{2}} \frac{1}{1+4x^2}\,dx$ ① ie $V = 2\pi\left[\frac{1}{2}\tan^{-1}2x\right]_0^{\frac{1}{2}}$ $\therefore$ Volume is $\pi \times \frac{\pi}{4}$ units^3 ②

Q3

(a) [4] i let $P(x) = 4x^3 - 8x^2 - 3x + 6$

But $P(2) = 4\times8 - 8\times4 - 3\times2 + 6 = 0$ $\therefore (x-2)$ is a factor of $P(x)$. ①

(ii) $\therefore 4\sin^3\theta = 8\sin^2\theta - 3\sin\theta + 6$

$= (\sin\theta - 2)(4\sin^2\theta - 3)$ ①

$= (\sin\theta - 2)(2\sin\theta - \sqrt{3})(2\sin\theta + \sqrt{3})$

ie $\sin\theta = 2$, $\sin\theta = \pm\frac{\sqrt{3}}{2}$ ①

But $\sin\theta = 2$ generates a range error $\therefore$ No Solution

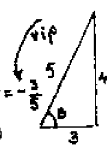
& $\sin\theta = \pm\frac{\sqrt{3}}{2}$ $\therefore \theta = n\pi + (-1)^n\frac{\pi}{3}$ ①

& $\theta = n\pi + (-1)^n\left(-\frac{\pi}{3}\right)$

or not effect. $\theta = (n-1)\pi \pm \pi/3$ where $n = 1, 2, 3 \ldots$

$\frac{\pi}{2} \le \theta \le \pi$

(b) [2] $\sin\theta = \frac{4}{5}$ & $\cos\theta = -\frac{3}{5}$ ①



but $\sin 2\theta = 2\sin\theta\cos\theta$ $\therefore \sin 2\theta = 2\times\frac{4}{5}\times\frac{-3}{5}$ ie $\sin 2\theta = \frac{-24}{25}$ ①

(c) [2] $\frac{\sin 3\phi}{\sin\phi} - \frac{\cos 3\phi}{\cos\phi} = 2$. LHS $= \frac{\sin\phi(3-4\sin^2\phi)}{\sin\phi} - \frac{\cos\phi(4\cos^2\theta-3)}{\cos\phi}$: $\begin{bmatrix}\sin\phi \ne 0\\ \cos\phi \ne 0\end{bmatrix}$

Now $\sin 3\phi = 3\sin\phi - 4\sin^3\phi$ & $\cos 3\phi = 4\cos^3\phi - 3\cos\phi$ ① ①

$= 6 - 4(\sin^2\theta + \cos^2\theta)$ ①

$= 2 = $ RHS. (Various Solutions Accepted)

(d) [4] Consider $\sqrt{3}\sin\theta + \cos\theta$

$= 2\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right)$

$= 2\sin(\theta + \alpha)$ where $\cos\alpha = \frac{\sqrt{3}}{2}$, $\sin\alpha = \frac{1}{2}$ ie $\alpha = \frac{\pi}{6}$ ②

$-\pi \le x \le \pi$

So $\sqrt{3}\sin x + \cos x = 1$ ie $\sin\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$ ①

$\Rightarrow 2\left(\sin\left(x + \frac{\pi}{6}\right)\right) = 1$ $\therefore \sin\left(x + \frac{\pi}{6}\right) = \sin\frac{\pi}{6}$ ie $x = \frac{\pi}{6} - \frac{\pi}{6}$ etc ①

$\therefore x = 0$, $\frac{2\pi}{3}$

Q4

(a) $y = \frac{1}{2}(p^2 + q^2 + 4)$

$= \frac{1}{2}\left[(p+q)^2 - 2pq + 4\right]$ ✓

$= \frac{1}{2}(x^2 - 4 + 4)$ $\begin{cases} x = p+q \\ pq = 2\end{cases}$

$y = \frac{1}{2}x^2$ or [redacted] ✓

(b) ⑤ $x^2 = 8y$ $\therefore a = 2$

$P = (2ap, ap^2)$ $A = (-4, 8)$

$= (4p, 2p^2)$ ✓

[redacted] ✓

ie $x = -2 + 2p$

$\therefore p = \frac{x+2}{2}$

$y = 4 + p^2$

$= 4 + \frac{(x+2)^2}{4}$ ✓

[redacted]

$\therefore$ focal length = 1 ✓

vertex $(-2, 4)$ ✓

C (i) $P(x) = e^x - 2x - 1$

$P(1\cdot2) = e^{1\cdot2} - 2(1\cdot2) - 1 \doteq -0\cdot08$ ✓

$P(1\cdot3) = e^{1\cdot3} - 2(1\cdot3) - 1 \doteq 0\cdot07$ ✓

[redacted]

Since $P(1\cdot2) < 0$ & $P(1\cdot3) > 0$ ✓

a root exists between $1\cdot2$ & $1\cdot3$

(ii) $x_2 = x_1 - \frac{f(x_1)}{P'(x_1)}$ ✓ $\{P'(x) = e^x - 2$

$= 1\cdot258487\ldots$ ✓

$x_3 = $ [redacted] (3dp) ✓

✓ = 1 MARK

Q5

Reflex $\angle XOZ = 248°$

($\angle$ at centre is twice $\angle$ at circum. on the same arc)

✓ $\angle XOZ = 112°$ ($\angle$s at a point $360°$)

$\angle YXZ = \frac{180-124}{2} = 28°$ ($\angle$ sum 180 $\triangle$)

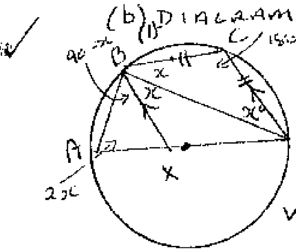
✓ $\angle ZXO = \frac{180-112}{2} = 34°$ ($\angle$ sum 180 $\triangle$, $OX = OZ$ radii)

✓ $b° = \angle YXZ + \angle ZXY$

$= 28° + 34°$

$\therefore a = 112$, $b = 62$

(b) (i) DIAGRAM ✓



(ii) $\angle CBD = x$ (opp = equal sides)

$\angle DBX = x$ (alt $\angle$s, $BX \parallel CD$) ✓

$\angle ABX = 90 - x$ ① ($\angle$ in semi circle is $90°$)

✓ $\angle BCD = 180 - 2x$ ($\angle$ sum $\triangle$)

✓ $\angle BAX = 2x$ (opp $\angle$s of cyclic quad supplement)

$\angle BXA = 180 - (90-x+2x)$

$= 90 - x$ ②

✓ $\therefore$ From (1) & (2) $\angle ABX$ is $= \angle BXA$ ($\triangle$ isosceles) //

(c) (i) Let roots be α, $\frac{1}{\alpha}$, β

✓ $\alpha \times \frac{1}{\alpha} \times \beta = -b$ $\therefore \beta = -b$

(ii) ✓ $\alpha + \frac{1}{\alpha} + \beta = -a$ ie $\alpha + \frac{1}{\alpha} - b = -a$

✓ $\alpha \times \frac{1}{\alpha} + \alpha\beta + \frac{\beta}{\alpha} = 0$ ie $1 + -b\left(\alpha + \frac{1}{\alpha}\right) = 0$

$\alpha + \frac{1}{\alpha} = \frac{1}{b}$ …… ①

$\therefore \alpha + \frac{1}{\alpha} + \beta = a$ $\{\beta = -b$

✓ $\frac{1}{b} - b = a$

Extension 1 TRIAL 2001

(a) (i) $\frac{dP}{dt} = k(P - 5000)$ — (1)

If $P = 5000 + Ae^{kt}$ — (2)

$\therefore \frac{dP}{dt} = Ake^{kt}$ — (3)

sub (2) and (3) into (1):

$\therefore LHS = Ake^{kt}$

$RHS = k(5000 + Ae^{kt} - 5000)$

$= Ake^{kt}$

$= LHS.$

$\therefore P = 5000 + Ae^{kt}$ is a solution of the differential equation (1).

(ii) When $t = 0$ $P = 5002$

$\therefore 5002 = 5000 + Ae^{0}$

$\therefore A = 2$

$\therefore P = 5000 + 2e^{kt}$

when $t = 6$ $P = 25000$

$\therefore 25000 = 5000 + 2e^{6k}$

$\therefore 10000 = e^{6k}$

$\therefore k = \frac{1}{6}\ln 10000.$

(iii) When $t = 10$ $P = ?$

$\therefore P = 5000 + 2e^{(\frac{1}{6}\ln 10000)10}$

$= 5000 + 2e^{\ln 10000^{10/6}}$

$= 5000 + 2(10000)^{10/6}$

$= 9288177.667\ldots$

$\therefore$ mosquito population after 10 days is 9 288 178 (to nearest mosquito).

(b) (i) $\ddot{x} = -n^2(x - b)$ — (1)

If $x = b - a\cos nt$ — (2)

$\therefore \dot{x} = an\sin nt$

$\ddot{x} = an^2\cos nt$ — (3)

sub (2) and (3) into (1):

$\therefore LHS = an^2\cos nt$

$RHS = -n^2(b - a\cos nt - b)$

$= -n^2(-a\cos nt)$

$= an^2\cos nt$

$= LHS$

$\therefore x = b - a\cos nt$ satisfies the equation for simple harmonic motion.

(ii)

Low Tide	9.25 a.m.	7m
High Tide	3.40 pm	$10\frac{2}{3}$m

Period $= \frac{2\pi}{n}$

$\therefore 2 \times 6\frac{1}{4} = \frac{2\pi}{n}$

$\therefore n = \frac{2\pi}{25/2} = \frac{4\pi}{25}$

$b = \frac{7 + 10\frac{2}{3}}{2} = 8\frac{5}{6}$

$a = \frac{10\frac{2}{3} - 7}{2} = 1\frac{5}{6}$

$\therefore x = 8\frac{5}{6} - 1\frac{5}{6}\cos\frac{4\pi}{25}t$

If $x = 9\frac{1}{2}$ $\therefore 9\frac{1}{2} = 8\frac{5}{6} - 1\frac{5}{6}\cos\frac{4\pi}{25}t$

$\frac{-2/3}{1\frac{5}{6}} = \cos\frac{4\pi}{25}t$

$\therefore -\frac{4}{11} = \cos\frac{4\pi}{25}t$

$\therefore$ basic angle $\frac{4\pi}{25}t = \cos^{-1}\left(\frac{4}{11}\right)$ [require 2nd, 3rd quads]

$\therefore t = \frac{25}{4\pi}\left[\pi - \cos^{-1}\frac{4}{11}\right]$ for earliest time

$\therefore t = 3.865405773\ldots$

ie time after low tide is 3hrs 52min (nearest min.)

$\therefore$ earliest time for $9\frac{1}{2}$m depth is 1.17pm

(7) (a) TO PROVE: $3^n + 7^n$ is always even if $n \in J^+$.

PROOF: Step 1: When $n = 1$ $3^n + 7^n = 3 + 7$

$= 10$, which is even

$\therefore$ it is true for $n = 1$.

Step 2: Assume it is true for $n = k$ and prove it is true for $n = k + 1$ ie $\frac{3^k + 7^k}{2} = M$ (where $M \in J$)

$\therefore 3^k = 2M - 7^k$ — (1)

If $n = k + 1$ $3^n + 7^n = 3^{k+1} + 7^{k+1}$

$= 3.3^k + 7.7^k$

$= 3(2M - 7^k) + 7.7^k$ (substituting (1))

$= 6M + 4.7^k$

$= 2(3M + 2.7^k)$, which is even.

$\therefore$ if it is true for $n = k$ so it is true for $n = k + 1$.

Step 3: It is true for $n = 1$ and so it is true for $n = 1 + 1 = 2$. It is true for $n = 2$ and so it is true for $n = 2 + 1 = 3$ and so on for all positive integral values of n.

(b) (i) After 1 instalment the debt remaining $D_1 = P\left(1 + \frac{r}{100}\right) - F$

after 2 instalments the debt remaining $D_2 = D_1\left(1 + \frac{r}{100}\right) - F$

$= \left(P\left(1 + \frac{r}{100}\right) - F\right)\left(1 + \frac{r}{100}\right) - F$

$= P\left(1 + \frac{r}{100}\right)^2 - F\left(1 + \left(1 + \frac{r}{100}\right)\right)$

after 3 instalments the debt remaining $D_3 = D_2\left(1 + \frac{r}{100}\right) - F$

$= \left[P\left(1 + \frac{r}{100}\right)^2 - F\left(1 + \left(1 + \frac{r}{100}\right)\right)\right]\left(1 + \frac{r}{100}\right) - F$

$= P\left(1 + \frac{r}{100}\right)^3 - F\left(1 + \left(1 + \frac{r}{100}\right) + \left(1 + \frac{r}{100}\right)^2\right)$

$\therefore$ continuing this pattern after n instalments the debt remaining

$D_n = P\left(1 + \frac{r}{100}\right)^n - F\underbrace{\left[1 + \left(1 + \frac{r}{100}\right) + \left(1 + \frac{r}{100}\right)^2 + \ldots + \left(1 + \frac{r}{100}\right)^{n-1}\right]}_{GP \quad a = 1;\ r = 1 + \frac{r}{100},\ n = n}$

$$\therefore D_n = P\left(1+\frac{r}{100}\right)^n - F\left[\frac{1\left[\left(1+\frac{r}{100}\right)^n - 1\right]}{1+\frac{r}{100} - 1}\right]$$

$$\therefore D_n = P\left(1+\frac{r}{100}\right)^n - F\left[\frac{\left(1+\frac{r}{100}\right)^n - 1}{\frac{r}{100}}\right].$$

(ii) Now if $D_n = 0$ $\therefore P\left(1+\frac{r}{100}\right)^n = F\left[\frac{\left(1+\frac{r}{100}\right)^n - 1}{\frac{r}{100}}\right]$

$$\therefore \frac{rP}{100}\left(1+\frac{r}{100}\right)^n = F\left(1+\frac{r}{100}\right)^n - F$$

$$\therefore F = \left(1+\frac{r}{100}\right)^n\left[F - \frac{rP}{100}\right]$$

$$\therefore \left(1+\frac{r}{100}\right)^n = \frac{F}{F - \frac{rP}{100}}$$

$$\therefore n \log_e\left(1+\frac{r}{100}\right) = \log_e\left[\frac{F}{F - \frac{rP}{100}}\right]$$

$$\therefore n = \frac{\log_e\left[\frac{F}{F-\frac{rP}{100}}\right]}{\log_e\left(1+\frac{r}{100}\right)}.$$

(iii) $P = 47000$, $\frac{r}{100} = \frac{7.8}{100 \times 26} = 0.003$, $F = 500$

$$\therefore n = \frac{\log_e\left[\frac{500}{500 - 47000 \times 0.003}\right]}{\log_e\left[1 + 0.003\right]}$$

$$= 110.5941301 \ldots$$

ie the debt would be repaid in $\frac{110.5941301\ldots}{26}$ years

ie 4 years and 6.59413... fortnights

$\Rightarrow$ 4 years and 7 fortnights (to pay off loan).

$\therefore$ The loan will be repaid in October, 2005.