Question 1

(a) $\int_0^1 \frac{x}{x^2+2}\,dx$ $\qquad u = x^2+2$, $\frac{du}{dx} = 2x$, $du = 2x\,dx$

when $x=0,\ u=2$; $x=1,\ u=3$

$= \frac{1}{2}\int_2^3 \frac{1}{u}\,du$

$= \frac{1}{2}\left[\ln u\right]_2^3$

$= \frac{1}{2}\left(\ln 3 - \ln 2\right)$

$= \frac{1}{2}\ln\left(\frac{3}{2}\right)$

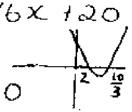
(b) $4(x-2) > 3(x-2)^2$

$4x-8 > 3x^2-12x+12$

$0 > 3x^2-16x+20$

$3x^2-16x+20<0$

$(3x-10)(x-2)<0$



$2 < x < 3\frac{1}{3}$

(c) $\tan\left(2\tan^{-1}\frac{3}{4}\right)$

let $\theta = \tan^{-1}\frac{3}{4}$

$\therefore \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$= \frac{2\times\frac{3}{4}}{1-\frac{9}{16}}$

$= \frac{24}{7}$

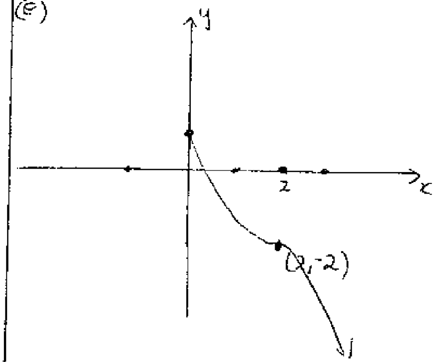
(d) 5R, 4B, 3W

$P(R,B,W)$ or (BRW) or (R,W,B)

$= \left(\frac{5}{12}\times\frac{4}{11}\times\frac{3}{10}\right)\times 6$

$= \frac{3}{11}$

(e)



Question 2

(a) $f(x) = 4\sin^{-1}\left(\frac{x}{3}\right)$

domain $-1 \le \frac{x}{3} \le 1$

$-3 \le x \le 3$

range $-2\pi \le y \le 2\pi$

(b)(i) $F(x) = x^3+x-3$

$F(1.2) = -0.072 < 0$

$F(1.3) = 0.497 > 0$

$F(1.2)<0$ $F(1.3)>0$ $\therefore$ root betwe[en]

(ii) $F(x_1) = 1.2 - \frac{F(1.2)}{F'(1.2)}$

$= 1.2 - \frac{(-0.072)}{3(1.2)^2+1}$

$\doteqdot 1.213533835$

(c) $P(x) = k(x+2)(x+1)(x-1) = 0$

Now $P(2) = 36$

$\therefore 36 = k(4)(3)(1)$

$\therefore k = 3$

$\therefore P(x) = 3(x+2)(x+1)(x-1)$

$= (3x+6)(x^2-1)$

$= 3x^3-3x+6x^2-6$

$P(x) = 3x^3+6x^2-3x-6$

(d) $A+P\ (2ap, ap^2)$ $\quad m = p$

(i) $\therefore y-ap^2 = p(x-2ap)$

$y-ap^2 = px-2ap^2$

$y = px-ap^2$

K is where $y=-a$.

$-a = px-ap^2$.

$px = ap^2-a$.

$x = \frac{ap^2-a}{p}$

$\therefore$ K is $\left(\frac{ap^2-a}{p},\ -a\right)$

(ii) $m_{PS} = \frac{ap^2-a}{2ap} = \frac{p^2-1}{2p}$

$m_{SK} = \frac{a+a}{\frac{-ap^2+a}{p}}$ $\qquad \frac{-2ap}{ap^2-a}$

$= \frac{2ap}{a-ap^2}$

$= \frac{2p}{1-p^2}$

Since $m_{PS}\times m_{SK} = -1$

$\angle PSK$ is $90°$

Question 3

(a) $\ddot{x} = 4(1+x)$

$\frac{d(\frac{1}{2}v^2)}{dx} = 4(1+x)$

(i) $\frac{1}{2}v^2 = 4x+2x^2+c$

when $v=2,\ x=0$

$2 = c$

$\therefore \frac{1}{2}v^2 = 4x+2x^2+2$

$v^2 = 8x+4x^2+4$

$v = \sqrt{4x^2+8x+4}$

$v = 2\sqrt{x^2+2x+1}$

$v = 2\sqrt{(x+1)^2}$

$v = 2(x+1)$.

(ii) $x = e^{2t}-1$

Now $\frac{dx}{dt} = 2(x+1)$

$\frac{dt}{dx} = \frac{1}{2(x+1)}$

$t = \frac{1}{2}\int\frac{1}{x+1}\,dx$

$t = \frac{1}{2}\ln(x+1)+k$

when $t=0,\ x=0,\ \therefore k=0$

$2t = \ln(x+1)$

$e^{2t} = x+1$

$\therefore x = e^{2t}-1$

(ii) $\dot{x} = 2e^{2t}$

$\ddot{x} = 4e^{2t}$

when $t=1$

$\ddot{x} = 4e^2\ m/s^2$

b) $\sin 2\theta = \sin\theta$

$2\sin\theta\cos\theta = \sin\theta$

$2\sin\theta\cos\theta - \sin\theta = 0$

$\sin\theta(2\cos\theta - 1) = 0$

$\sin\theta = 0, \quad \cos\theta = \frac{1}{2}$

$\therefore \theta = \ +(-1)^n\sin^{-1}0, \quad 2\pi n \pm \cos^{-1}\frac{1}{2}$

$= n\pi + (-1)^n.0, \quad 2\pi n \pm \frac{\pi}{3}$

$\theta = n\pi, \ 2\pi n \pm \frac{\pi}{3}$

c)(i) $\int \frac{dx}{\sqrt{9-4x^2}}$

$= \frac{1}{2}\sin^{-1}\frac{2x}{3} + c$

(ii) $\int \sin^2 x\,dx$

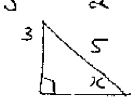
$= \int \frac{1}{2} - \frac{1}{2}\cos 2x\,dx$

$= \frac{x}{2} - \frac{1}{4}\sin 2x + c$

Question ④

a) $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$

let $x = \cos^{-1}\frac{4}{5}$



$y = \cos^{-1}\frac{3}{5}$

$\cos(x+y) = \cos x\cos y - \sin x\sin y$

$= \frac{4}{5}\times\frac{3}{5} - \frac{3}{5}\times\frac{4}{5}$

$= 0$

$\therefore \ x + y = \cos^{-1}(0)$

$x + y = \frac{\pi}{2}$

$\therefore \cos^{-1}\frac{4}{5} + \cos^{-1}\frac{3}{5} = \frac{\pi}{2}$

(b)(i) $P(1+r)^n = P(1\cdot12)^1$

(ii) $B_2 = P(1\cdot12)^2 + P(1\cdot12)$

$B_3 = P(1\cdot12)^3 + P(1\cdot12)^2 + P(1\cdot12)$

$= P(1\cdot12)(1\cdot12^2 + 1\cdot12 + 1)$

(iii)

$B_{20} = P(1\cdot12)(1\cdot12^{19} + 1\cdot12^{18} + \ldots + 1)$

$1000\,000 = P(1\cdot12)\underbrace{(1\cdot12^{19} + \ldots + 1)}_{a=1,\,n=20,\,r=1}$

$1000000 = P(1\cdot12)\left(\frac{1\cdot12^{20}-1}{0\cdot12}\right)$

$120000 = P(1\cdot12)(1\cdot12^{20}-1)$

$P = \frac{120\,000}{(1\cdot12)(1\cdot12^{20}-1)}$

$= \$1239 1\cdot77$

(c)(i) $P = N + Ae^{0\cdot1t}$

at $t=0$, $P = 2\cdot7\times10^6$

$2\cdot7\times10^6 = 1\cdot2\times10^6 + A$

$A = 1\cdot5\times10^6$

$\therefore P = 1\cdot2\times10^6 + 1\cdot5\times10^6e^{0\cdot1t}$

when $t = 3\cdot5$

$P = 1\cdot2\times10^6 + 1\cdot5\times10^6e^{0\cdot35}$

$= 3328601\cdot323$

$\doteqdot 3\cdot3\times10^6$

when $P = 8\cdot1\times10^6$

$8\cdot1\times10^6 = 1\cdot2\times10^6 + 1\cdot5\times10^6e^{0\cdot1t}$

$6\cdot9\times10^6 = 1\cdot5\times10^6e^{0\cdot1t}$

$4\cdot6 = e^{0\cdot1t}$

$\ln(4\cdot6) = 0\cdot1t$

$t = 15\cdot26$

$\therefore$ On 16th day

Question ⑤

(a) $\frac{dA}{dt} = 15$

$\frac{dV}{dt} = \frac{dV}{dr}\times\boxed{\frac{dr}{dt}}$ $\qquad V = \frac{4}{3}\pi r^3$

$= 4\pi r^2\times\boxed{\frac{dr}{dt}}$ $\qquad \frac{dV}{dr} = 4\pi r^2$

Now $\frac{dr}{dt} = \frac{dr}{dA}\times\frac{dA}{dt}$

$= \boxed{\frac{1}{8\pi r}\times15}$ $\qquad A = 4\pi r^2$, $\ \frac{dA}{dr} = 8\pi r$

$\therefore \frac{dV}{dt} = 4\pi r^2\times\frac{15}{2\,8\pi r}$

$= \frac{15r}{2}$

When $r=5$

$\frac{dV}{dt} = 37\tfrac{1}{2}\ mm^3/s$

(b) $y = e^{mx}$

$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$

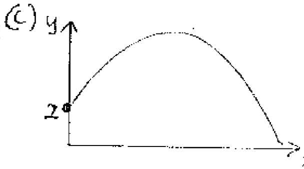
$m^2e^{mx} - me^{mx} - 6e^{mx} = 0$

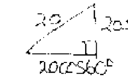
$e^{mx}(m^2 - m - 6) = 0$

$e^{mx}(m-3)(m+2) = 0$

$m = -2, 3$

(c)





$\ddot{x} = 0 \qquad \ddot{y} = -10$

$\dot{x} = 10 \qquad \dot{y} = -10t + 10\sqrt{3}$

$x = 10t \qquad y = -5t^2 + 10\sqrt{3}t + 2$

(i) height reached $\to \dot{y} = 0$

$-10t + 10\sqrt{3} = 0$

$t = \sqrt{3}\ s$

$\therefore \ y = -5(3) + 10(3) + 2$

$= -15 + 30 + 2$

$y = 17\ m$

(ii) time of flight

$y = 0$

$-5t^2 + 10\sqrt{3}t + 2 = 0$

$5t^2 - 10\sqrt{3}t - 2 = 0$

$t = \frac{10\sqrt{3} \ \sqrt{300+40}}{10}$

$t = \sqrt{3}\cdot\sqrt{340}$

$t \doteqdot 31\cdot937s$

(iii) when $t = \sqrt{1020}$

$x = 10\sqrt{1020}\ m \quad (319\cdot37$

Question ⑥

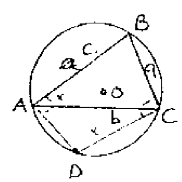
(a) $\frac{1}{2}V^2 = 18 - 2x^2$

(i) $\frac{d\frac{1}{2}v^2}{dx} = -4x \quad$ S.H.M.

(ii) $T = \frac{2\pi}{2} = \pi$

When $v=0 \qquad 18 - 2x^2 = 0$

$\therefore a = 3 \qquad x = \pm3$



$\angle BCD = 90^\circ$ ($\angle$ in Semi-circle)
$\angle BDC = \angle BAC$ ($\angle$'s standing on same arc)

$\therefore \sin\angle BDC = \dfrac{a}{BD}$

$\sin\angle BDC = \dfrac{a}{2R}$

$\therefore \sin\angle BAC = \dfrac{a}{2R}$

(ii) Area $\triangle ABC = \dfrac{1}{2}\cdot c\cdot b\cdot \sin\angle BAC$

$= \dfrac{1}{2}. c. b. \dfrac{a}{2R}$

$= \dfrac{abc}{4R}$

(c) (i) $\sin x + \sqrt{3}\cos x \equiv A\sin(x+\alpha)$

$A = \sqrt{1+3} = 2$

$\alpha = \tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$

$\therefore \sin x + \sqrt{3}\cos x = 2\sin(x + \frac{\pi}{3})$

(ii) $\sin x + \sqrt{3}\cos x = \sqrt{3}$

$\therefore 2\sin(x + \frac{\pi}{3}) = \sqrt{3}$

$\sin(x + \frac{\pi}{3}) = \sqrt{3}$

$x + \dfrac{\pi}{3} = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{7\pi}{3}$

$x = 0, \pi/3, 2\pi$

Question (7)

(a) Induction.

Step ① Assume true $n=k$

$9^{k+2} - 4^k = 5M$ M tve int

Step ② Prove true for $n=k+1$

R.T.P.

$9^{k+3} - 4^{k+1}$ is divisible by 5

Step ③ Proof

$9^{k+3} - 4^{k+1} = 9.9^{k+2} - 4^{k+1}$

$= 9(5M + 4^k) - 4^{k+1}$

$= 45M + 9.4^k - 4.4^k$

$= 45M + 5.4^k$

$= 5(9M + 4^k)$

which is divisible by 5

Step ④

Hence statement is true for $n=k+1$ when it is true for $n=k$.

Step ⑤

For $n=1$

$9^2 - 4 = 80$

$\therefore$ True for $n=1$

Step ⑥

Since true for $n=1$ by step ④ it will be true for $n=2$ and then $n=3$ and so on for all integers.

(b) $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{1+4x^2} = \int_0^{\frac{1}{2}} \frac{dx}{4(\frac{1}{4}+x^2)}$

$= \dfrac{1}{2}\Big[\tan^{-1} 2x\Big]_0^{\frac{1}{2}}$

$= \dfrac{1}{2}\Big[\tan^{-1}(1) - \tan^{-1}(0)\Big]$

$= \dfrac{1}{2}\Big[\dfrac{\pi}{4}\Big]$

$= \dfrac{\pi}{8}$

(c) $A(-2,3)$. $B(4,3)$

$m_{AB} = 0$.

Equation $y = 3$.

sub into $y = 2x+2$

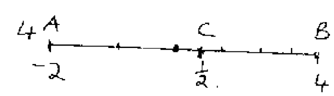
$3 = 2x+2$

$2x = 1$

$x = \dfrac{1}{2}$.

$\therefore C$ is $(\frac{1}{2}, 3) \to (m,n)$

$\therefore$



$\therefore AC : CB$

$2\frac{1}{2} : 3\frac{1}{2}$

$5 : 7$

(a) $y = \frac{1}{2}(e^x - e^{-x})$

$2y = e^x - e^{-x}$

$0 = e^{2x} - 1 - 2y.e^x$

$0 = e^{2x} - 2ye^x - 1$

$e^x = \dfrac{2y \pm \sqrt{4y^2+4}}{2}$

$e^x = \dfrac{\cancel{2}y \pm \cancel{2}\sqrt{y^2+1}}{\cancel{2}}$

$e^x = y \pm \sqrt{y^2+1}$

$e^x > 0 \quad \sqrt{y^2+1} > 0$

$e^x = y + \sqrt{y^2+1}$

$x = \ln(y + \sqrt{y^2+1})$