1999 3U TRIAL (Ascham) (1)

$y = 3x + 2 \quad m_1 = 3 \qquad y = 1 - x \quad m_2 = -1$

$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ ✓

$= \left|\frac{3+1}{1+3(-1)}\right|$

$= 2$ ✓

$\therefore \alpha = 63°26'$

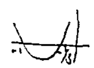
$= 63°$ ✓ (nearest deg.)

$\frac{1}{x+1} < 3 \Rightarrow x + 1 < 3(x+1)^2$ ✓

$= 3(x^2 + 2x + 1)$

$3x^2 + 5x + 2 > 0$

$(3x+2)(x+1) > 0$ ✓



$x < -1$ ½ or $x > -\frac{2}{3}$ ½

2) $2:3 \quad 3, -5$

$P\left(\frac{2\times 3 - 3\times 1}{5}, \frac{2\times -5 + 3\times 2}{5}\right)$ ✓

$P\left(\frac{3}{5}, -\frac{4}{5}\right)$ ✓

$\int_0^1 \frac{t}{\sqrt{t+1}}\,dt \qquad u = t+1 \quad t = u - 1 \quad du = dt \qquad \text{if } t = 0 \;\; u = 1 \quad \text{if } t = 1 \;\; u = 2$

$= \int_1^2 \frac{u-1}{\sqrt{u}}\,du$ ✓

$= \int_1^2 \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du$

$= \left[\frac{2u^{3/2}}{3} - 2u^{1/2}\right]_1^2$ ✓

$= \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2}\right) - \left(\frac{2}{3} - 2\right)$

$= -\frac{2\sqrt{2}}{3} + \frac{4}{3}$ ✓

$= \frac{4 - 2\sqrt{2}}{3}$ (OR)

Q1. (e) $\angle AEL = \angle ACE$ ($\angle$ in alt. segm.) ½

$\angle AEL = \angle MEB$ (vert. opp. $\angle$s) ½

$\angle BDE = \angle MEB$ ($\angle$ in alt. segm) ½

$\therefore \angle ACE = \angle BDE$ ½

But $\angle ACE$ is alternate to $\angle BDE$ $\therefore AC \parallel DB$ ½

Q.2. (a)(i) In ΔADH $\quad AD = DH = a$

In ΔHDC $\quad \tan 30° = \frac{a}{DC}$

$\therefore DC = a\sqrt{3}$ ✓

In ΔBDC $\quad BD^2 = DC^2 + CB^2$

$= (a\sqrt{3})^2 + a^2$

$= 4a^2$

$\therefore BD = 2a$ ✓

(ii) In ΔHDB $\quad \frac{HD}{BD} = \tan H\hat{B}D$

$\frac{a}{2a} =$

$\frac{1}{2} = \tan H\hat{B}D$

$\therefore \angle HBD = 26°34'$ (to nearest minute)

(b) $\frac{d}{dx}(2x + \cos x) = 2 - \sin x$ ✓

$f'\left(\frac{\pi}{6}\right) = 2 + \sin\frac{\pi}{6}$

$= 2 + \frac{1}{2}$

$= \frac{5}{2}$ ½

$f\left(\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{2} - \frac{\pi}{3}$

$= \frac{3\sqrt{3} - 2\pi}{6}$ ½

$x_1 = -\frac{\pi}{6} - \frac{3\sqrt{3} - 2\pi}{6\cdot 3} \times \frac{2}{5}$ ✓

$= -\frac{\pi}{6} - \frac{3\sqrt{3}}{15} + \frac{2\pi}{15}$

$= \frac{-\pi - 6\sqrt{3}}{30}$ ✓

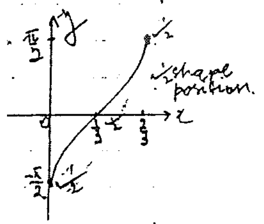
$y = \sin^{-1}(3x-1)$

Domain of $f^{-1}(x)$: $-1 \le 3x-1 \le 1$

$0 \le 3x \le 2$

$0 \le x \le \frac{2}{3}$

Range of $f^{-1}(x)$: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



$y = \sin^{-1}(3x-1)$ is the inverse function of $f(x)$

$\sin y = 3x-1$ is the inverse of $f(x)$

$\sin x = 3y-1$

$y = \frac{1}{3} + \frac{1}{3}\sin x$ is $f(x)$

so $f(x) = \frac{1}{3} + \frac{1}{3}\sin x$. has domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and range $0 \le y \le \frac{2}{3}$

$3\sin x - \sqrt{3}\cos x = A\sin(x-\alpha)$

$= A\sin x\cos\alpha - A\cos x\sin\alpha$

$\frac{A\sin\alpha}{A\cos\alpha} = \frac{+\sqrt{3}}{3}$ $\therefore \tan\alpha = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$ $\therefore \alpha = \frac{\pi}{6}$

$A^2\sin^2\alpha + A^2\cos^2\alpha = 9+3$ $\therefore A^2 = 12$, $A = 2\sqrt{3}$

$\therefore 3\sin x - \sqrt{3}\cos x = 2\sqrt{3}\sin(x - \frac{\pi}{6})$

min. value $= -2\sqrt{3}$

$2\sqrt{3}\sin(x-\frac{\pi}{6}) = \sqrt{3}$

$\sin(x-\frac{\pi}{6}) = \frac{1}{2}$

$x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$ $\therefore x = \frac{\pi}{3}$ or $x = \pi$ for $0 \le x \le \pi$

Q3 (b)(i) $T = T_0 + Ae^{-kt}$ i.e. $T - T_0 = Ae^{-kt}$

$\frac{dT}{dt} = -kAe^{-kt}$

$= -k(T - T_0)$

(ii) $85 = 25 + Ae^0$ $\therefore A = 60$.

$T = 25 + 60e^{-kt}$

$80 = 25 + 60e^{-k}$

$e^{-k} = \frac{55}{60}$

$-k = \ln\frac{11}{12}$ (or $k = \ln\frac{12}{11}$)

$\therefore T = 25 + 60e^{(\ln\frac{11}{12})t}$

when $t = 5$ $T = 25 + 60e^{-5\ln\frac{12}{11}}$

$\doteqdot 64$

$T \doteqdot 64°$ after a further 4 minutes.

Q.4 (a) $x^2 = 4ay$ $\frac{dy}{dx} = \frac{x}{2a}$ at $x = 2ap$

$\frac{dy}{dx} = p$

$\therefore$ gradient of normal $= -\frac{1}{p}$

Eqn: $y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = 2ap + ap^3$

(3)

Normal at P $\quad x + py = 2ap + ap^3 \quad$ (1)

" " Q $\quad x + qy = 2aq + aq^3 \quad$ (2)

$-$ (2)

$$y(p-q) = 2a(p-q) + a(p^3-q^3)$$
$$y = 2a + a(p^2+q^2+pq)$$
$$y = a(p^2+q^2+pq+2)$$

$q \quad xq + pqy = 2apq + aqp^3$

$p \quad -xp - pqy = -2apq - apq^3$

$$x(q-p) = apq(p-q)(p+q)$$
$$x = -apq(p+q)$$

$\therefore [-apq(p+q), a(p^2+q^2+pq+2)] = N(X, Y)$

$m_{PQ} = \frac{ap^2-aq^2}{2ap-2aq} \Rightarrow m_{PQ} = \frac{a(p+q)\cancel{(p-q)}}{2a\cancel{(p-q)}}$

$= \frac{p+q}{2}$

Chord PQ: $\quad y - ap^2 = \frac{p+q}{2}(x - 2ap)$

sub. $S(0, a)$: $\quad a - ap^2 = \frac{p+q}{2}(-2ap)$

$a - \cancel{ap^2} = -\cancel{ap^2} - apq$

$\therefore pq = -1$ if PQ is through S.

$X = -apq(p+q) \Rightarrow X = a(p+q) \Rightarrow p + q = \frac{X}{a}$ (3)

$Y = a(p^2+q^2+pq+2) \Rightarrow Y = a(p^2+q^2+1) \Rightarrow p^2+q^2 = \frac{Y}{a} - 1$ (4)

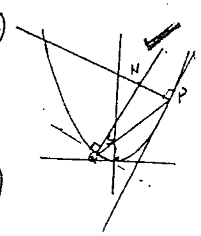
$p^2+q^2 = (p+q)^2 + 2$

(5) $\quad \frac{Y}{a} - 1 = \frac{X^2}{a^2} + 2$

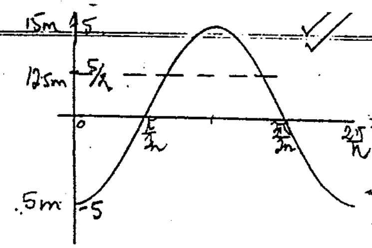
$$\frac{Y}{a} = \frac{X^2}{a^2} + 3$$

$\therefore Y = \frac{X^2}{a} + 3a$ or $X^2 = a(Y - 3a)$

(e)



Q5 (a)



(b) $a = 5$ $\frac{1}{2}$

$T = \frac{2\pi}{n}$ $\frac{1}{2}$

$\frac{25}{2} = \frac{2\pi}{n}$ $\frac{1}{2}$

$n = \frac{4\pi}{25}$ $\frac{1}{2}$

$\therefore x = -5\cos\frac{4\pi}{25}t$

when $x = \frac{5}{2}$

$$\frac{5}{2} = -5\cos\frac{4\pi}{25}t$$
$$-\frac{1}{2} = \cos\frac{4\pi}{25}t$$
$$\frac{4\pi}{25}t = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \dots$$
$$t = \frac{25}{6}, \frac{25}{3}, \dots$$

$\therefore$ the times between which the ship may enter the harbour are 8:40 am and 12:50 pm.

(b) (i) $\quad LHS = \sin(\sin^{-1}x - \cos^{-1}x)$

$= \sin(\alpha - \beta)$

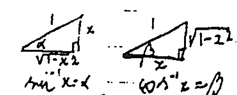
$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= x \times x - \sqrt{1-x^2}\sqrt{1-x^2}$

$= 2x^2 - 1$

$= RHS$

where



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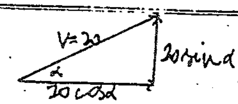
a) 

at $t=0$

$g=-10$, $\dot{x}=20\cos\theta$, $\dot{y}=20\sin\theta$, $x=0$, $y=0$

$\ddot{x}=0$

$\dot{x}=20\cos\alpha$

$x=20t\cos\alpha$ ①

$\ddot{y}=-10$

$\dot{y}=-10t+20\sin\alpha$

$y=-5t^2+20t\sin\alpha$ ②

) Sub $t=\frac{x}{20\cos\alpha}\rightarrow$ ②

$y=\frac{x\,20\sin\alpha}{20\cos\alpha}-5\frac{x^2}{400\cos^2\alpha}$

$y=x\tan\alpha-\frac{x^2}{80}\sec^2\alpha$ ✓

) when $x=20$, $y=h$ ✓

$\therefore h=20\tan\alpha-\frac{400}{80}\sec^2\alpha$

$h=20\tan\alpha-5\sec^2\alpha$ ✓

$\frac{dh}{d\alpha}=20\sec^2\alpha-10\sec^2\alpha\tan\alpha$ ✓

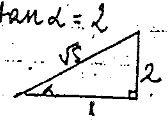
$\therefore 10\sec^2\alpha(2-\tan\alpha)=0$ for max

$\sec\alpha=0$ or $\tan\alpha=2$ ✓

$\alpha=63°$ (nearest deg)

60°	63°	70°
+	0	−

$\therefore h$ is max. when $\tan\alpha=2$ ✓



$h_{max}=20\times2-5\times\sqrt{5}^2$
$=15$ metres ✓

(ii) $V=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$ ✓ (6)

$=\sqrt{(20\cos\alpha)^2+(20\sin\alpha-10t)^2}$ ✓

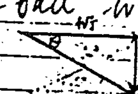
$=\sqrt{\left(\frac{20}{\sqrt5}\right)^2+\left(\frac{20}{1}\cdot\frac{2}{\sqrt5}-10\times\sqrt5\right)^2}$

$=\sqrt{80+(8\sqrt5-10\sqrt5)^2}$

$=\sqrt{80+20}$

$=10$ m/s is the speed of ball when hits the wall on its way ↓. ✓

To find t:
$20=Vt\cos\alpha$
$20=20t\times\frac{1}{\sqrt5}$
$t=\sqrt5$ ✓



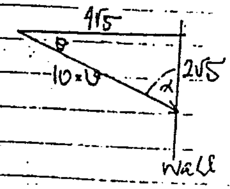
$\frac{dy}{dt}=20\sin\alpha-10t$
$=-2\sqrt5$

$\frac{dx}{dt}=20\cos\alpha$
$=\frac{20}{\sqrt5}$
$=4\sqrt5$

$\tan\theta=\frac{2\sqrt5}{4\sqrt5}$ ✓

$\theta=\tan^{-1}\left(\frac{1}{2}\right)$ ✓

hits wall at



$\alpha=\tan^{-1}\left(\frac{4\sqrt5}{2\sqrt5}\right)$

$=\tan^{-1}2$ ✓