HMX1

CRANBROOK 1997 3U TRIAL

SOLUTIONS

1. (a) $\dfrac{2x-5}{x+3} \geq 1 \qquad x \neq -3$

$(x+3)^2 \times \dfrac{2x-5}{x+3} \geq (x+3)^2$

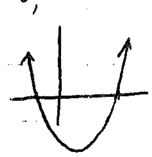
$(x+3)(2x-5) \geq (x+3)^2$

$2x^2+x-15 \geq x^2+6x+9$

$x^2-5x-24 \geq 0 \qquad -8, 3$

$(x-8)(x+3) \geq 0$

$x < -3$ or $x \geq 8$



3

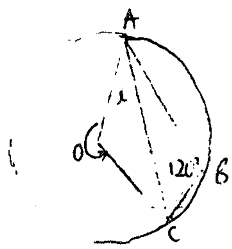
(b) $(5,3) \quad -3$
$(1,-3) \quad 2$

$P = \left(\dfrac{2\times 5 - 3\times 1}{2-3}, \dfrac{2\times 3 - 3\times(-3)}{2-3}\right)$

$= (-7, -15)$

2

(c)



reflex $\angle AOC = 2\times 120°$ (angle at centre = twice angle
$= 240°$ at circumference standing on same arc)

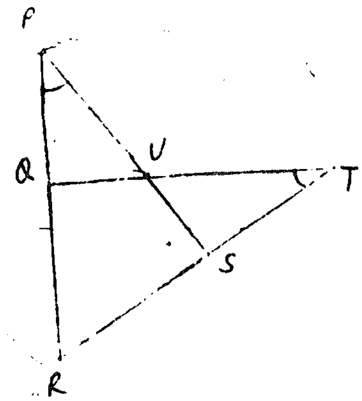
$\therefore \angle AOC = 120°$ (angles at a point = 360°)

$2x+120° = 180°$ (angle sum in isos ΔAOC equal radii)

$\therefore x = 30°$

3

(d)



Data: $\angle RPS = \angle QTR$

(i) RTP: $\angle UQR = \angle USR$

Proof:

Considering Δs PRS, QRT

$\angle RPS = \angle QTR$ (given)

$\angle R$ common

$\therefore \angle PSR = \angle TQR$ (remaining angle in Δ)

QED

2

(ii) $\angle UQR + \angle USR = 180°$ (opp angles cyclic quad)

$\therefore \angle UQR = \angle USR = 90°$ since $\angle UQR = \angle USR$

Since angle in a semi-circle $= 90°$,

UR is a diameter.

2

→ alternative soln:

Produce AO to D on circumference

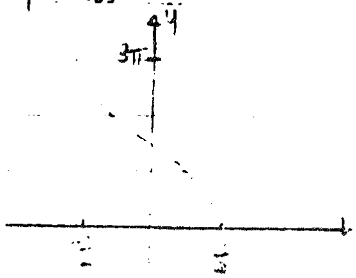
$\therefore$ AD diameter.

$\angle ABD = 90°$ (angle in a semi-circle)

$\therefore \angle CBD = 30°$ $(120° - 90°)$

Now, $\angle OAC\ (x°) = 30°$ (angles at circumference standing on same arc)

2 (a)(i) $y = 3\cos^{-1} 2x$



①

(ii) domain: $-1 \le 2x \le 1$

$\{x : -\frac{1}{2} \le x \le \frac{1}{2}\}$ ②

range: $0 \le y \le 3\pi$

(iii) $y = 3\cos^{-1} 2x$

$\frac{dy}{dx} = 3 \times \frac{-2}{\sqrt{1-4x^2}}$

$= \frac{-6}{\sqrt{1-4x^2}}$

$= \frac{-6}{\sqrt{1-4(\frac{1}{4})^2}}$ when $x = \frac{1}{4}$

$= \frac{-6}{\frac{\sqrt{3}}{2}}$

$= \frac{-12}{\sqrt{3}}$

$= -4\sqrt{3}$

eqn. $y - y_1 = m(x - x_1)$

$y = 3\cos^{-1}\frac{1}{2}$

$= \pi$ ③

$y - \pi = -4\sqrt{3}(x - \frac{1}{4})$

$4\sqrt{3}x + y = \pi + \sqrt{3}$

(b) $P(\text{hits target}) = \frac{1}{3}$

(i) $P(3 \text{ successes}) = P(X=3)$

$= {}^5C_3 \left(\frac{1}{3}\right)^3 \times \left(\frac{2}{3}\right)^2$

$= 10 \times \frac{1}{27} \times \frac{4}{9}$

$= \frac{40}{243}$ ①

(ii) $P(X \geq 1) = 1 - P(X=0)$

> 0.9

$1 - \left(\frac{2}{3}\right)^n > 0.9$ where n = no. of torpedoes fired

$0.1 > \left(\frac{2}{3}\right)^n$

$\ln 0.1 > n \ln \frac{2}{3}$

$n \; \frac{\ln 0.1}{\ln \frac{2}{3}}$ since $\ln \frac{2}{3} < 0$

$n > 5.67\ldots$ ③

∴ 6 or more must be fired.

(c) $2\sin 2x + \sqrt{3} = 0 \qquad 0 \le x \le 360°$

$\sin 2x = -\frac{\sqrt{3}}{2} \qquad Q3, 4 \quad \alpha = 60°$

$2x = 240°, 300°, 600°, 660°$

$x = 120°, 150°, 300°, 330°$ ②

3. $V = \pi \int_0^{3\sqrt{3}} \left(\frac{1}{\sqrt{x^2+9}}\right)^2 dx$

$= \pi \int_0^{3\sqrt{3}} \frac{dx}{x^2+9}$

$= \frac{\pi}{3} \int_0^{3\sqrt{3}} \frac{3}{x^2+9} dx$

$= \frac{\pi}{3} \left[\tan^{-1} \frac{x}{3}\right]_0^{3\sqrt{3}}$

$= \frac{\pi}{3} \left(\tan^{-1}\sqrt{3} - \tan^{-1} 0\right)$

$= \frac{\pi}{3}\left(\frac{\pi}{3} - 0\right)$

$= \frac{\pi^2}{9}$ units^3 (3)

(c) $a = -\frac{1}{2} e^{-x}$

when $x = 0$, $v = 1$ $t = 0$

(i) $\frac{1}{2}v^2 = \int -\frac{1}{2} e^{-x} dx$

$\frac{1}{2}v^2 = \frac{1}{2} e^{-x} + c_1$

$\frac{1}{2} \times 1^2 = \frac{1}{2} e^0 + c_1$

$\therefore c = 0$ (3)

$\therefore v^2 = e^{-x}$

$v = \sqrt{e^{-x}}$ (-ve does not apply) since when $x=0$, $v > 0$

(ii) $\frac{dx}{dt} = \sqrt{e^{-x}} = e^{-\frac{x}{2}}$

$\frac{dt}{dx} = e^{\frac{x}{2}}$

$\therefore t = \int e^{\frac{x}{2}} dx$

$= 2e^{\frac{x}{2}} + c_2$

$0 = 2e^0 + c_2$

$c_2 = -2$

$\therefore t = 2e^{\frac{x}{2}} - 2$

$e^{\frac{x}{2}} = \frac{t+2}{2}$

$\frac{x}{2} = \ln \frac{t+2}{2}$ (3)

$x = 2 \ln \frac{t+2}{2}$

(d) $\cos\left(\tan^{-1}\sqrt{3}\right) = \cos \frac{\pi}{3}$

$= \frac{1}{2}$ (1)

(b) $\sec^2 x - 3\tan x - 3 = 0$ $\quad -\pi \leq x \leq \pi$.

$1 + \tan^2 x - 3\tan x - 3 = 0$

$\tan^2 x - 3\tan x - 2 = 0$

$\tan x = \frac{3 \pm \sqrt{9 - 4(-2)1}}{2}$

$= \frac{3 \pm \sqrt{17}}{2}$

$= 3.525..,\ -0.512...$

Q1, 3 $\qquad$ Q2, 4

$x = 1.297, -\pi + 1.297$ OR $x = \pi - 0.512, -0.512$

$= 1.297, -1.844$ $\qquad = 2.630, -0.512$

$\therefore x = -1.844, -0.512, 1.297, 2.630$ (all to 3d.p)

(3)

4. (a) $\frac{dV}{dt} = 50$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ $\qquad V = \frac{4}{3}\pi r^3$

$50 = 4\pi r^2 \times \frac{dr}{dt}$ $\qquad \therefore \frac{dV}{dr} = 4\pi r^2$

$\frac{50}{4\pi \times 20^2} = \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{1}{32\pi}$

Now,

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$ $\qquad A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$

$= 8\pi r \times \frac{1}{32\pi}$

$= 8\pi \times 20 \times \frac{1}{32\pi}$

$= 5$ $\qquad$ (4)

surface area increasing at rate of $5\ mm^2/s$

(b) RTS $\tan^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4} = \frac{\pi}{4}$

Let $x = \tan^{-1}\frac{3}{5}$, $y = \tan^{-1}\frac{1}{4}$

$\tan x = \frac{3}{5}$ $\qquad \tan y = \frac{1}{4}$

$\sqrt{34}$, 3, 5, x $\qquad$ 1, $\sqrt{17}$, 4, y

Taking tan of both sides

LHS $= \tan(x+y)$

$= \frac{\tan x + \tan y}{1 - \tan x \tan y}$

$= \frac{\frac{3}{5} + \frac{1}{4}}{1 - \frac{3}{5}\times\frac{1}{4}}$ $\qquad$ (3)

$= 1$

RHS $= \tan\frac{\pi}{4}$

$= 1$

$\therefore \tan^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4} = \frac{\pi}{4}$

(c) (i) $\ddot{x} = 0$ $\qquad \ddot{y} = -10$

$\dot{x} = V\cos\theta$ $\qquad \dot{y} = -10t + V\sin\theta$

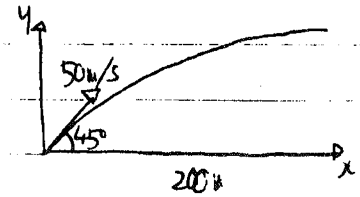
$= 50\cos 45°$ $\qquad = 50\sin 45° - 10t$

$= 25\sqrt{2}$ $\qquad = 25\sqrt{2} - 10t$

$x = Vt\cos\theta$ $\qquad y = 25t\sqrt{2} - 5t^2$

$= 25t\sqrt{2}$ $\qquad$ (3)

(ii)



$200 = 25t\sqrt{2}$

$\therefore t = \frac{8}{\sqrt{2}}$

$= 4\sqrt{2}\ s$ $\qquad$ (2)

(iii) height, $y = 25 \times 4\sqrt{2} \times \sqrt{2} - 5 \times (4\sqrt{2})^2$

$= 40\ m$ $\qquad$ (1)

Alternative solution to (a)

(finding $\frac{dV}{dt}$ without first finding $\frac{dr}{dt}$)

since r is not a constant

$V = \frac{4}{3}\pi r^3$ $\qquad S = 4\pi r^2$

$\therefore \frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt}$ $\qquad \frac{dS}{dt} = 8\pi r\frac{dr}{dt}$

$\frac{\frac{dV}{dt}}{\frac{dS}{dt}} = \frac{r}{2}$

ie. $\frac{dV}{dt} = \frac{r}{2} \times \frac{dS}{dt}$

$50 = \frac{20}{2} \times \frac{dS}{dt}$ $\qquad$ when $r = 20$

$\therefore \frac{dS}{dt} = 5$

ie. surface area is increasing at rate of $5\ mm^2/s$.

or $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$

$= 8\pi r \times \frac{1}{4\pi r^2} \times 50$

Cranbrook 3unit Trial 1997. (1)

Q5 (a) i: let $f(x) = 3\sin 2x - x$. Consider $f(1\cdot3) \doteqdot 0\cdot247$ & $f(1\cdot4) = -0\cdot395$ ∴ root lies between $1\cdot3$ & $1\cdot4$

ii) Applying Newton's Method for a better root (x_2) & using $x_1 = 1\cdot3$ since $f(1\cdot3) \doteqdot 0\cdot247$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ where $f'(x) = 6\cos 2x - 1$ gives $x_2 = 1\cdot3 - \dfrac{f(1\cdot3)}{f'(1\cdot3)} = 1\cdot340138539$.

∴ $x_3 = 1\cdot34 - \dfrac{f(1\cdot34)}{f'(1\cdot34)} = 1\cdot339669957$ ∴ a better root is $1\cdot34$ (2)

Q5(b)

If $x=1$ is a double root for $f(x) = ax^3 + bx^2 + cx + d$ then $f(1) = 0$ (1) & $a+b+c+d = 0$

$f'(1) = 0$ (1) $\Rightarrow 3a + 2b + c = 0$ & since min value exists at -1 $f'(-1) = 0$ (1) $\Rightarrow 3a - 2b + c = 0$

& $f(-1) = -4$ (1) $\Rightarrow -a + b - c + d = -4$. ∴ $3a + c = 0$ & $4b = 0$ together with $2b + 2d = -4$ or $b + d = \ldots$

∴ $b = 0$, $a = -1$, $c = 3$ & $d = -2$ (1)

Q5(c)

Let $P(n)$ be the proposition that $n^3 + 2n = 3t_n$ (where $t_n \in \mathbb{N}$) & let S be the truth set for $P(n)$.

Consider $P(1)$ LHS $= 3 =$ RHS if $t_1 = 1$ ∴ $1 \in S$.

Assume $K \in S$ ie $K^3 + 2K = 3t_K$. Consider $P(K+1)$. LHS $= (K+1)^3 + 2(K+1)$

(1) or LHS $= (K^3 + 3K^2 + 3K + 1) + 2K + 2$

$= K^3 + 2K + 3K^2 + 3K + 3$

(1) ∴ $P(K+1)$ is true & if $K \in S$ then $K+1 \in S$, $1 \in S$ $= 3t_K + 3(K^2 + K + 1)$

∴ $P(n)$ is true for $\forall n \in \mathbb{N}$. $= 3t_{K+1}$ where $t_{K+1} = t_K + (K^2 + K + 1)$

Q6(a)

For $\left(x^2 + \frac{2}{x}\right)^{10}$ $T_n = {}^{10}C_{n-1} (x^2)^{11-n} \left(\frac{2}{x}\right)^{n-1}$ or ${}^{10}C_{n-1}\, x^{23-3n} . 2^{n-1}$ (1)

For term in x^2 (1) $23 - 3n = 2 \Rightarrow n = 7$. ∴ we need to evaluate T_7. (1)

so $T_7 = {}^{10}C_6 . 2^6 . x^2$ ∴ Coefficient of x^2 is 210×64 or 13440. (1)

Q6 (b) (i) $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$ let $x = -1$ then $(1-1)^n = \sum_{r=0}^{n} (-1)^n \binom{n}{r}$ ie $\sum_{r=0}^{n} (-1)^n \binom{n}{r} = 0$ (1)

(ii) let $x = 2$ (1) $\sum_{r=0}^{n} \binom{n}{r} 2^r = 3^n$. Consider $\dfrac{d(1+x)^n}{dx}$

$\Rightarrow n(1+x)^{n-1} = \sum_{r=0}^{n} \binom{n}{r} r (x)^{r-1}$ let $x = 1$

then $\sum_{r=0}^{n} \binom{n}{r} r = n.2^{n-1}$ (2)

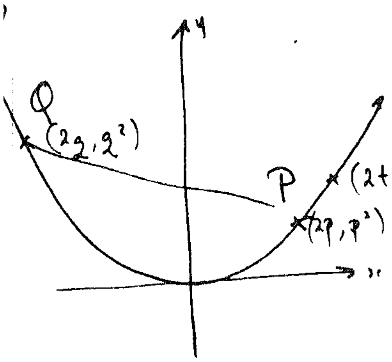
Q6(c)

$\displaystyle\int_0^1 \frac{x^3}{1 + x^8}\,dx$ let $x^4 = u$ $\Rightarrow 4x^3 dx = du$ & when $x = 1$, $u = 1$; $x = 0$, $u = 0$ (1)

$= \dfrac{1}{4}\displaystyle\int_0^1 \frac{du}{1 + u^2}$ (1)

$= \dfrac{1}{4}\left[\tan^{-1}(u)\right]_0^1$ (1) or $\dfrac{1}{4}\left\{(\tan^{-1}(1)) - (\tan^{-1}(0))\right\}$ or $\dfrac{1}{4} \times \dfrac{\pi}{4}$

ranbrook 3unit Trial 1997.



(i) For Equation of Tangent at P: $y = px - p^2$ (2)

ii For Chord $\dfrac{y - p^2}{x - 2p} = \dfrac{p^2 - q^2}{2(p - q)}$

$\Rightarrow \dfrac{y - p^2}{x - 2p} = \dfrac{p+q}{2}$ or $y = \left(\dfrac{p+q}{2}\right)x - pq$.

∴ Eq of Chord is

$2y - (p+q)x + 2pq = 0$ (2)

(iii) Eq of tangent at P $y = px - p^2$ $\Rightarrow$ $(p-q)x = p^2 - q^2$ or $x = p+q$.

.. . . . Q $y = qx - q^2$

if $x = p+q$ then $y = p(p+q) - p^2$ ie $y = pq$ ∴ $M \equiv (p+q, pq)$ (2)

(iv) For M to lie on $x^2 + 4y = 0$ then $(p+q)^2 + 4pq = 0 \Rightarrow q = (-3 \pm 2\sqrt{2})p$, $p \neq$

Now M has coords $((-2 \pm 2\sqrt{2})p, (-3 \pm 2\sqrt{2})p^2)$ (4)

(v) Chord PQ is $2y = (p+q)x - 2pq$. & intersects $x^2 = -4y$

when $x^2 = -2((p+q)x - 2pq)$ or $x^2 = -2(p+q)x + 4pq$

ie $x^2 + 2(p+q)x - 4pq = 0$ & for PQ to be a tangent to $x^2 = -4ay$ we must have one & one only root to quadratic (ie $\Delta = 0$). (2)

∴ $4(p+q)^2 + 16pq = 0$ or $(p+q)^2 + 4pq = 0$

which was one equation in part (iv) & gave M coords $((-2 \pm 2\sqrt{2})p, (-3 \pm 2\sqrt{2})$

∴ Chord PQ is a tangent to $x^2 = -4y$.