CATHOLIC SECONDARY SCHOOLS' ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE FINAL TESTS 1998

MATHEMATICS

3/4 UNIT COMMON PAPER

(i.e. 3 UNIT COURSE – ADDITIONAL PAPER:
4 UNIT COURSE – FIRST PAPER)

Afternoon session

Friday 14th August 1998.

Time Allowed — Two Hours

EXAMINERS

Graham Arnold, Terra Sancta College, Schofield.
Sandra Hayes, All Saints Catholic Senior College, Casula.
Frank Reid, St Ursula's College, Kingsgrove.

DIRECTIONS TO CANDIDATES :

ALL questions may be attempted.

ALL questions are of equal value.

All necessary working should be shown in every question.

Full marks may not be awarded for careless or badly arranged work.

Approved calculators may be used.

Standard integrals are printed at the end of the exam paper.

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or the format of the Higher School Certificate Examination. However, the committees responsible for the preparation of these 'Trial Examinations' do hope that they will provide a positive contribution to your preparation for the final examinations.

Question 1

Marks

(a) (i) Show that $\dfrac{1+\cos 2A}{\sin 2A} = \cot A$ **4**

(ii) Hence find the exact value of $\cot 15°$.

(b) The equation $x^3 - mx + 2 = 0$ has two equal roots. **4**

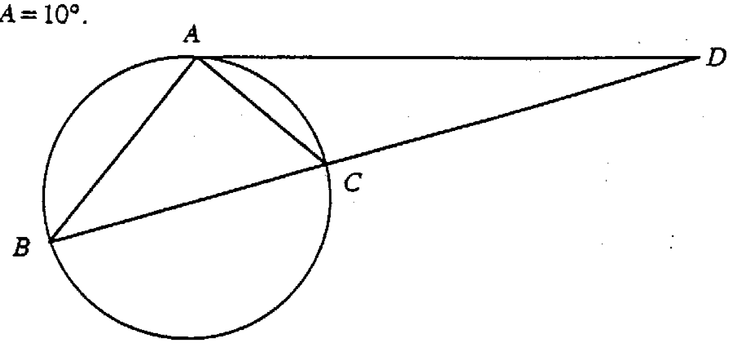
(i) Write down expressions for the sum of the roots and for the product of the roots.
(ii) Hence find the value of m.

(c) The function $y = e^{-kx}$ satisfies $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = 0$. **4**

(i) Show that $k^2 - 4k + 3 = 0$
(ii) Hence find the possible values of k.

Question 2

(a) ABC is a triangle inscribed in a circle. **4.**
The tangent at A meets BC produced at D.
$D\hat{A}C = 40°$, $C\hat{D}A = 10°$.



(i) Copy the diagram showing the above information.
(ii) Show that BC is a diameter of the circle.

(b) Consider the function $f(x) = \dfrac{x}{4-x^2}$. **8**

(i) Find the domain of the function.
(ii) Show that the function is an odd function.
(iii) Show that the function is increasing throughout its domain.
(iv) Sketch the graph of the function showing clearly the coordinates of any points of intersection with the x axis or the y axis and the equations of any asymptotes.
(v) Use the graph of the function to explain whether or not the inverse function exists.

Question 3

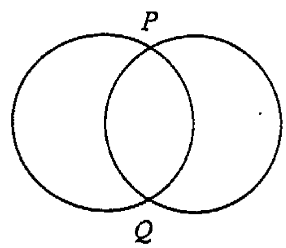
Marks

(a) $T(2t, t^2)$ is a point on the parabola $x^2 = 4y$ with focus F. **4**
P is the point which divides FT internally in the ratio $1 : 2$.

(i) Write down the coordinates of P in terms of t.

(ii) Hence show that as T moves on the parabola $x^2 = 4y$, the locus of P is the parabola $9x^2 = 12y - 8$.

(b) Two circles each with radius 2 cm intersect at P and Q. The common chord PQ subtends an angle θ radians at each centre. **8**



(i) Show that the area A cm^2 of the overlapping part of the circles is given by $A = 4\theta - 4\sin\theta$.

(ii) If the three regions shown in the diagram all have the same area, show that $\theta - \sin\theta - \frac{\pi}{2} = 0$.

Using a starting value of $\theta = 2$, apply Newton's method twice to find the value of θ correct to two decimal paces.

Question 4

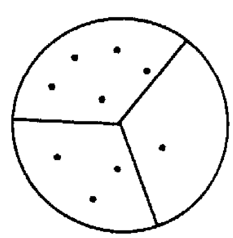
(a) The temperature $T°$ C of a body after t minutes is given by $T = 20 + 60\,e^{-0.1t}$. **3**

(i) Find the initial temperature of the body.

(ii) Find the time taken for the temperature of the body to fall to one half of its original value, giving the answer correct to the nearest minute.

(b) Use the subsitution $x = u^2$, $u > 0$, to find the exact value of $\displaystyle\int_1^3 \frac{1}{\sqrt{x}(1+x)}\,dx$. **4**

(c) Nine points lie inside a circle. No three of the points are collinear. Five of the points lie in sector 1, three lie in sector 2, and the other point lies in sector 3. **5**



(i) Show that 84 triangles can be made using these points as vertices.

One triangle is chosen at random from all the possible triangles.

(i) Find the probability that the vertices of the triangle chosen lie one in each sector.

(ii) Find the probability that the vertices of the triangle chosen lie all in the same sector.

Question 5

Marks

(a) (i) Find the gradients of the tangents to the curve $y=\sin^{-1}x$ when $x=0$ and $x=\frac{\sqrt{3}}{2}$. **3**

(ii) Find the acute angle between these two tangent lines, giving the answer correct to the nearest degree.

(b) (i) Expand $\left(x-\frac{1}{x}\right)^3$ in ascending powers of x. **4**

(ii) If $x-\frac{1}{x}=1$ find the value of $x^3-\frac{1}{x^3}$.

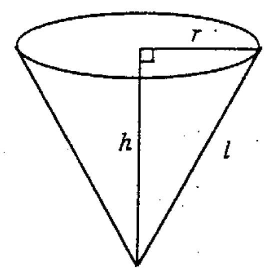
(c) The area between the curve $y=\sin^2 x$ and the x axis between $x=0$ and $x=\frac{\pi}{2}$ is rotated through one complete revolution about the x axis. **5**

(i) Find the exact value of the area of the region.

(ii) Use Simpson's rule with three function values to find an approximation to the volume of the solid of revolution, leaving the answer in terms of π.

Question 6

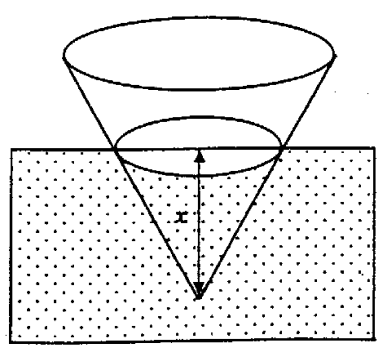
(a) A cone with base radius r, vertical height h and slant edge length l has curved surface area $S=\pi rl$ and volume $V=\frac{1}{3}\pi r^2 h$. **6**



(i) If a cone has vertical height x and its base radius is $\frac{3}{4}$ of its vertical height, show that

$$S=\frac{15\pi x^2}{16} \quad \text{and} \quad V=\frac{3\pi x^3}{16}.$$

A container is filled to overflowing with water. A solid cone with its axis vertical and its vertex pointing downwards is being lowered into the water at a constant rate. The base radius of the cone is $\frac{3}{4}$ of its vertical height. When the vertex of the cone is 2 m under water (with the cone not fully submerged), the surface area of the cone is being covered with water at a rate of $0\cdot5\,\text{m}^2\text{s}^{-1}$.



(ii) Show that the cone is being lowered into the water at a rate of $\frac{2}{15\pi}\ \text{ms}^{-1}$.

(iii) Find the rate at which the water is overflowing the container when the vertex of the cone is 2 m under water.

Marks

(b) A particle P is performing Simple Harmonic Motion. At a time t seconds its acceleration is given by $\frac{d^2x}{dt^2} = -4(x-3)$, where x metres is the displacement from the origin O. Initially the particle is at O and its velocity is 8 ms^{-1}. 6

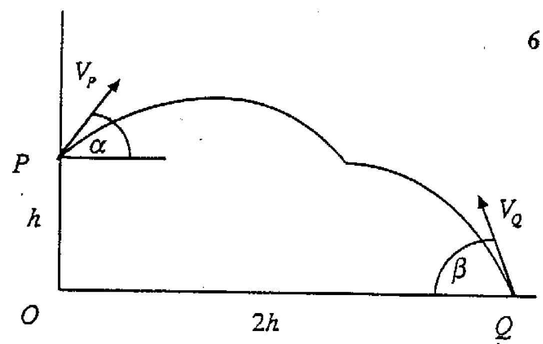
(i) Find the centre and period of the motion.

(ii) Show that $v^2 = 64 + 24x - 4x^2$, where $v \text{ ms}^{-1}$ is the velocity of P.
Hence find the amplitude of the motion.

(iii) Find the maximum speed of the particle.

Question 7

(a) O and Q are two points $2h$ metres apart on horizontal ground. P is a point h metres directly above O. A particle is projected from P towards Q with speed V_p ms^{-1} at an angle α above the horizontal. At the same time another particle is projected from Q towards P with speed V_Q ms^{-1} at an angle β above the horizontal. The two particles collide T seconds after projection. 6



(i) For the particle going from P towards Q, the equations of motion are $\ddot{x} = 0$ and $\ddot{y} = -g$.
Use calculus to show that at time t seconds, its horizontal distance x_P from O and its vertical distance y_P from O are given by

$$x_P = (V_P \cos\alpha)\,t \qquad \text{and} \qquad y_P = (V_P \sin\alpha)\,t - \tfrac{1}{2}gt^2 + h$$

(ii) For the particle going from Q towards P, write down expressions for its horizontal distance x_Q from Q and its vertical distance y_Q from Q at time t seconds.

(iii) Hence show that $\dfrac{V_P}{V_Q} = \dfrac{2\sin\beta - \cos\beta}{2\sin\alpha + \cos\alpha}$.

(b) Consider the statement 6
$S(n)$: $2^n - (-1)^n$ *is divisible by* 3, where n is a positive integer.

(i) Show that $S(1)$ and $S(2)$ are true.

(ii) Show that if $S(k)$ is true for some positive integer k then $S(k+2)$ is also true.

(iii) Deduce that $S(n)$ is true for all positive integers n.