CSSA - NSW

4 Unit Mathematics

Trial HSC Examination 1985

1. **(i)** Sketch the curve $y = \frac{x}{x^2-1}$ showing clearly the coordinates of any turning points or points of inflexion and the equations of any horizontal or vertical asymptotes.
(ii) Sketch the curve $9y^2 = x(x-3)^2$ showing clearly the coordinates of any turning points.
(a) Show that the area enclosed between the x axis and that part of the curve which lies in the first quadrant between $x = 0$ and $x = 3$ is $\frac{4\sqrt{3}}{5}$ square units.
(b) Show that the length of that part of the curve which lies in the first quadrant between $x = 0$ and $x = 3$ is $2\sqrt{3}$ units. You may assume that the length required is given by the formula: Length $= \int_0^3 \sqrt{1 + (\frac{dy}{dx})^2}\ dx$.

2. **(i)** Find $\int \frac{dx}{x\sqrt{x^2-1}}$
(ii) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos x}\ dx$.
(iii) Show that $\int_0^1 \frac{\sqrt{x}}{(1+x)}\ dx = 2 - \frac{\pi}{2}$.
(iv) Given that $I_n = \int \sec^n x\, dx$, where $n \geq 2$, show that
$(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$. Hence evaluate $\int_0^{\frac{\pi}{4}} \sec^6 x\, dx$.

3. **(i)** The complex numbers $z_1 = 4i, z_2 = 2\sqrt{3} - 2i, z_3 = -2\sqrt{3} - 2i$ are represented on an Argand diagram by the points A, B, C respectively.
(a) Show that the triangle ABC is equilateral.
(b) Show tht z_1^2 and z_2z_3 are represented by the same point on the Argand diagram.
(ii) Find the exact value of the modulus and argument of the complex number $z = \frac{1+i}{\sqrt{3}-i}$. Find the smallest possible integer n such that z^n is real. For this value of n find the value of z^n
(iii) Given that, in the Argand diagram, the point P represents the complex number z and Q the complex number z^2, show that if P moves on a straight line parallel to (but not coinciding with) the imaginary axis then Q will move on a certain parabola, and that all such parabolas have a common focus. Also state what the locus of Q is when P describes the imaginary axis.

4. **(i)** Show that the point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for all values of θ. If Q is the point $(a\sec\phi, b\tan\phi)$ where $\theta + \phi = \frac{\pi}{2}$ show that the locus of the midpoint of PQ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{y}{b}$.

(ii) Show that the equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a\sec\theta, b\tan\theta)$ is $ax\tan\theta + by\sec\theta = (a^2+b^2)\sec\theta\tan\theta$. The ordinate at P meets an asymptote of the hyperbola at Q. The normal at P meets the x axis at G. Show that GQ is at right angles to the asymptote.

5. **(i)** A solid has a base in the form of an ellipse with major axis 10 units and minor axis 8 units. If every section perpendicular to the major axis is an isosceles triangle with altitude 6 units show that the volume of the solid is 60π cubic units.
(ii) The region R in the first quadrant such that $y \leq 4x^2 - x^4$ is rotated about the y axis to form a solid of revolution. Use the method of decomposition into cylindrical shells to show that the volume of the solid is $\frac{32\pi}{3}$ cubic units.

6. **(i)** A car is travelling round a section of a race track which is banked at an angle of $15°$. The radius of the track is 100 metres. What is the speed at which the car can travel without tending to slip?
(ii) A light inextensible string of length $3L$ is threaded through a smooth vertical ring which is free to turn. The string carries a particle at each end. One particle A of mass m is at rest at a distance L below the ring. The other particle B of mass M is rotating in a horizontal circle whose centre is A. Find the angular velocity of B and find m in terms of M.

7. **(i)** The equation $x^3 + 2x - 1 = 0$ has roots $x = \alpha, \beta, \gamma$. In each of the following cases find an equation with numerical coefficients having the roots stated.
(a) $x = -\alpha, -\beta, -\gamma$
(b) $x = \alpha, -\alpha, \beta, -\beta, \gamma, -\gamma$
(c) $\alpha^2, \beta^2, \gamma^2$.
(ii) Find the general solution of the equation $\cos x + \cos 2x + \cos 3x = 0$.
(iii) Write down, in modulus-argument orm, the five roots of $z^5 = 1$. Show that when these five roots are plotted on an Argand dioagram they form the vertices of a regular pentagon of area $\frac{5}{2}\sin\frac{2\pi}{5}$. By combining appropriate pairs of these roots show that for $z \neq 1$, $\frac{z^5-1}{z-1} = (z^2 - 2z\cos\frac{2\pi}{5} + 1)(z^2 - 2z\cos\frac{4\pi}{5} + 1)$. Deduce that $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$ are the roots of the equation $4x^2 + 2x - 1 = 0$.

8. **(i)** By considering the stationary value of the function $f(x) = x - \ln x$ show that for $x > 0$, $\ln x \leq x - 1$. Deduce that if $a_1, a_2, \ldots, a_n$ are positive numbers and $A = \frac{1}{n}\sum_1^n a_n$ then $\sum_1^a \ln\frac{a_n}{A} \leq \sum_1^a \frac{a_n}{A} - n = 0$. Hence deduce that $\frac{a_1+a_2+\cdots+a_n}{n} \geq (a_1a_2\cdots a_n)^{\frac{1}{4}}$. Hence, or otherwise, prove that if u, v, w are positive numbers and $u + v + w = 1$ then $\frac{1}{u^2} + \frac{1}{v^2} + \frac{1}{w^2} \geq 27$.
(ii) Given that $\sin^{-1} x, \cos^{-1} x$ and $\sin^{-1}(1-x)$ are acute:
(a) show that $\sin(\sin^{-1} x - \cos^{-1} x) = 2x^2 - 1$;
(b) solve the equation $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(1-x)$.