Ext 1 Trial Solution

(1) (a) (i) $\tan^{-1}x + C$ (ii) $-\frac{1}{(1+x)} + C$ (iii) $\frac{1}{2}\log_e(1+x^2) + C$

(b) $2y = x+1$, $y = \frac{1}{2}x + \frac{1}{2}$, $\therefore m_1 = \frac{1}{2}$

$3x - y - 2 = 0$, $y = 3x - 2$, $\therefore m_2 = 3$

$\therefore \tan\theta = \dfrac{3 - \frac{1}{2}}{1 + 3\cdot\frac{1}{2}} = \dfrac{\frac{5}{2}}{\frac{5}{2}} = 1$

$\therefore \theta = 45°$.

(c) $\cos\frac{5\pi}{12} = \cos 75° = \cos(45° + 30°) = \cos 45°\cdot\cos 30° - \sin 45°\cdot\sin 30°$

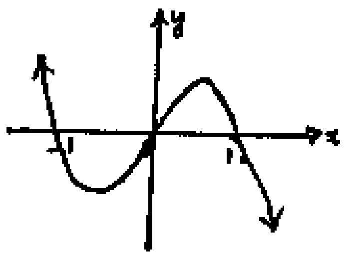
$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

(d) $x - x^3 = 0$

$x(1-x^2) = 0$

$x = 0, \pm 1$



$-1 < x < 0$, $x > 1$

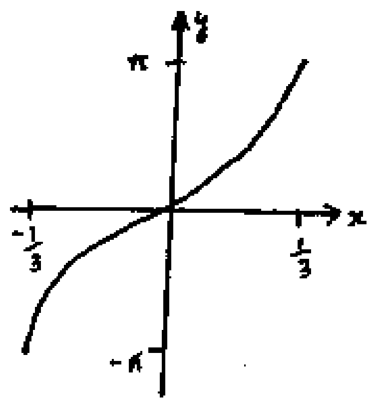
(2) (a) (i) $\frac{y}{2} = \sin^{-1}3x$

D: $-1 \le 3x \le 1$

$-\frac{1}{3} \le x \le \frac{1}{3}$

R: $-\frac{\pi}{2} \le \frac{y}{2} \le \frac{\pi}{2}$

$-\pi \le y \le \pi$



(b) $u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$x = \frac{\pi}{2}$, $u = 0$

$x = 0$, $u = 1$

$\therefore \displaystyle\int_1^0 \frac{-du}{\sqrt{4-u^2}}$

$= -\left[\sin^{-1}\frac{u}{2}\right]_1^0$

$= -\left(\sin^{-1}0 - \sin^{-1}\frac{1}{2}\right) = \frac{\pi}{6}$

(c) $x = \cos t$

$y = \frac{1}{8}(2\cos^2 t - 1)$

$\therefore y = \frac{1}{8}(2x^2 - 1)$

$8y + 1 = 2x^2$

$x^2 = 4y + \frac{1}{2}$

$x^2 = 4\left(y + \frac{1}{8}\right)$

$\therefore$ Vertex is $\left(0, -\frac{1}{8}\right)$

(d) (i) ${}^5C_3 = 10$

(ii) $\dfrac{{}^5C_3}{{}^{11}C_3} = \dfrac{10}{165} = \dfrac{2}{33}$.

(3) (a) $2\sin x \cdot \cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$

$\sin x = 0 \quad \cos x = \frac{1}{2}$

$\therefore x = 0, \pi, 2\pi, \frac{\pi}{3}, \frac{5\pi}{3}.$

(b) $\frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 6x)\,dx$

$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{6}\times 0 - 0 + 0\right)$

$= \frac{\pi}{4}.$

(c) $P(1) = a - 4 + 3b + 2 = 0$

$a + 3b = 2$

$P(-2) = -8a - 16 - 6b + 2 = 15$

$8a + 6b = -29$

$\left.\begin{aligned} 2a + 6b &= 4 \\ 8a + 6b &= -29 \end{aligned}\right\} \quad \begin{aligned} 6a &= -33 \\ a &= -5\tfrac{1}{2} \end{aligned}$

$-44 + 6b = -29$

$6b = 15$

$b = 2\frac{1}{2}$

$\left.\begin{aligned} a &= -5\tfrac{1}{2} \\ b &= 2\tfrac{1}{2} \end{aligned}\right\}$

(d) $\left.\begin{aligned} f(x) &= x - 3 + \log_e x \\ f'(x) &= 1 + \frac{1}{x} \end{aligned}\right\}$

$x_1 = x_0 - \dfrac{x_0 - 3 + \log_e x_0}{1 + \frac{1}{x_0}}$

$= \dfrac{x_0 + 1 - x_0 + 3 - \log_e x_0}{\frac{x_0 + 1}{x_0}}$

$= \dfrac{x_0(4 - \log_e x_0)}{x_0 + 1}.$

(4) (a) $x \cdot -\dfrac{1}{\sqrt{1-x^2}} + 1 \cdot \cos^{-1} x - \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot -2x$

$= -\dfrac{x}{\sqrt{1-x^2}} + \cos^{-1} x + \dfrac{x}{\sqrt{1-x^2}}$

$= \cos^{-1} x.$

(b) $2\sin^{-1} x = \frac{\pi}{2} - \sin^{-1} x$

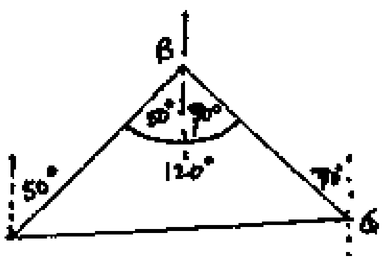
$3\sin^{-1} x = \frac{\pi}{2}$

$\sin^{-1} x = \frac{\pi}{6}$

$\therefore x = \frac{1}{2}$

(c) $4! \times 3 \times 3 \times 2 = 432$ plans

(d) (i)



(ii) $PB = h\cot 30°, \quad BQ = h\cot 45°$

$\therefore 1000^2 = h^2\cot^2 30° + h^2\cot^2 45° - 2h\cot 30° h\cot 45° \cos 120°$

$1000^2 = h^2(\cot^2 30° + \cot^2 45° - 2\cot 30°\cot 45° \cos 120°)$

$\therefore h^2 = \dfrac{1000^2}{\cot^2 30° + \cot^2 45° - 2\cot 30°\cot 45°\cos 120°}$

$\therefore h = \dfrac{10\sqrt{10}}{} = \dfrac{10\sqrt{10}}{}$

(5) (a)(i) $T_{r+1} = {}^7C_r (2x)^{7-r} \cdot \left(-\frac{3}{x}\right)^r$

$= (-1)^r . 3^r . 2^{7-r} . {}^7C_r \; x^{7-r} \; x^{-r}$

$= (-1)^r . 3^r . 2^{7-r} . {}^7C_r \; x^{7-2r}$

let $7-2r = 4$

$\therefore 2r = 3$

$r = \frac{3}{2}$

$\therefore$ As r is not an integer, there is no term in x^4.

(ii) Co-eff of x^3 : $r=2$

$3^2 2^5 \, {}^7C_2$

Co-eff of x^5 : $r=1$

$-3.2^6 \, {}^7C_1$

$\therefore$ Co-eff of x^5 in full expansion

$3^2 2^5 . {}^7C_2 - 3.2^6 \, {}^7C_1$

$= 4704$

(b) $\frac{(n-2)!}{(n-4)!} = 56$

$(n-2)(n-3) = 56$

$n^2 - 5n + 6 = 56$

$n^2 - 5n - 50 = 0$

$(n-10)(n+5) = 0$

$n = 10, -5$

$\therefore n = 10$

(c)(i) $\cos 2x - \sin 2x = R\cos 2x \cos\alpha - R\sin 2x \sin\alpha$

$\therefore \left.\begin{array}{l} R\cos\alpha = 1 \\ R\sin\alpha = 1 \end{array}\right\}$ $R = \sqrt{1^2+1^2} = \sqrt{2}$

$\tan\alpha = 1 \quad \therefore \alpha = \frac{\pi}{4}$

$\therefore \cos 2x - \sin 2x = \sqrt{2}\cos\left(2x + \frac{\pi}{4}\right)$

(ii) $\sqrt{2}\cos\left(2x+\frac{\pi}{4}\right) = 1$

$\cos\left(2x+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$2x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}$.

$2x = 0, \frac{3\pi}{2}, 2\pi$

$x = 0, \frac{3\pi}{4}, \pi$.

(6)(a)(i) $y = \frac{x^2}{4}$

$\frac{dy}{dx} = \frac{x}{2}$

$m_T = p$

$m_N = -\frac{1}{p}$

$\therefore$ normal is

$y - p^2 = -\frac{1}{p}(x - 2p)$

$py - p^3 = -x + 2p$

$x + py = p(p^2+2)$

(ii) when $x = 0$

$y = p^2 + 2$

ie $Q(0, p^2+2)$

Mid pt of PQ is

(p, p^2+1)

Parametrics of locus are

$\left.\begin{array}{l} x = p \\ y = p^2+1 \end{array}\right\}$

$\therefore$ Eqn is

$y = x^2 + 1$.

(b) $6\frac{\cos^2\theta}{\sin^2\theta} - 4\cos^2\theta = 1$

$6\frac{\cos^2\theta}{1-\cos^2\theta} - 4\cos^2\theta = 1$

$6\cos^2\theta - 4\cos^2\theta + 4\cos^4\theta = 1 - \cos^2\theta$

$4\cos^4\theta + 3\cos^2\theta - 1 = 0$

$(4\cos^2\theta - 1)(\cos^2\theta + 1) = 0$

$\cos^2\theta = \frac{1}{4}$ $\quad$ $\cos^2\theta = -1$

$\cos\theta = \pm\frac{1}{2}$ $\quad$ no sol

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

(c) (i) $t=0$: $\left.\begin{array}{l} x=0 \\ y=h \end{array}\right\}$ $\left.\begin{array}{l} \dot{x}=V \\ \dot{y}=0 \end{array}\right\}$

$\ddot{x} = 0$

$\dot{x} = c$

$\therefore \dot{x} = V$

$x = Vt + c$

$c = 0$

$\therefore x = Vt$

$\ddot{y} = -10$

$\dot{y} = -10t + c$

$c = 0$

$\therefore \dot{y} = -10t$

$y = -5t^2 + c$

$c = h$

$\therefore y = -5t^2 + h$

(ii) when $x = d, y = 0$

$d = Vt \quad \therefore t = \frac{d}{V}$

$\therefore 0 = -5 . \frac{d^2}{V^2} + h$

$\frac{5d^2}{V^2} = h$

$\therefore 5d^2 = V^2 h$.

(7) (a) (i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(8x - x^2 - 7\right)$

$= 8 - 2x$

$\therefore \ddot{x} = -2(x-4)$

hence SHM

(ii) when $v = 0$

$8x - x^2 - 7 = 0$

$x^2 - 8x + 7 = 0$

$(x-7)(x-1) = 0$

$x = 7, 1$

$\therefore$ Stations are 6 km apart

(b) (i) Co-ords of all points on $y = \frac{1}{4}x^2$ are $(x, \frac{1}{4}x^2)$

$\therefore$ Perp distance

$= \left| \frac{3x - 4 \times \frac{1}{4}x^2 + 4}{\sqrt{3^2 + (-4)^2}} \right|$

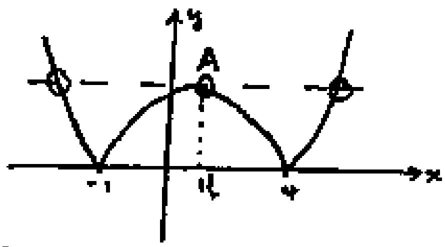
$= \frac{1}{5}\left|3x - x^2 + 4\right|$

$= \frac{1}{5}\left|x^2 - 3x - 4\right|$

(ii) $x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

$x = 4, -1$



(iii) Co-ords of A are $(1\frac{1}{2}, 1\frac{1}{4})$

$\therefore y = 1\frac{1}{4}$ meets $y = \frac{1}{5}\left|x^2 - 3x - 4\right|$ in exactly 3 places

(c) $\frac{dV}{dt} = 5$, $\frac{dS}{dt} = ?$, $r = 20$

$\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dV} \cdot \frac{dV}{dt}$

$= 8\pi r \cdot \frac{1}{4\pi r^2} \cdot 5$

$= \frac{2}{r} \cdot 5$

$\therefore \frac{dS}{dt} = \frac{10}{20} = 0.5 \text{ cm}^2/\text{sec}$