N.S.W. DEPARTMENT OF EDUCATION
HIGHER SCHOOL CERTIFICATE EXAMINATION 1973
MATHEMATICS PAPER B (2F) - (EQUIVALENT TO 3U AND 4U - 1ST PAPER)

Instructions: Time allowed 3 hours. All questions may be attempted. (Questions are not of equal value.) In every question, all necessary working should be shown. Marks will be deducted for careless or badly arranged work.
Mathematical tables will be supplied. Approved slide rules or calculators may be used.

QUESTION 1 (12 Marks)

(i) Find the second derivative of $f(x) = \cos(x^2)$.

(ii) Find a primitive of e^{-3x}.

(iii) Find the area under the curve $y = \dfrac{2x}{x^2+1}$ between $x = 0$ and $x = 2\sqrt{2}$.

OMIT (iv) Find the shortest distance from the origin to the plane $x + y + z = 3$.

QUESTION 2 (9 Marks)

(i) Prove that $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$.

(ii) Find the vertex of the parabola $y = x^2 - 8x + 19$.

(iii) An improperly constructed die has the numbers 1, 2, 3, 4, 6 and 6 on its six faces respectively (i.e. the face which should have a 5 on it has a 6 instead). Given that all faces have equal probability of coming up, what is the expected value of the amount shown by the die?

QUESTION 3 (9 Marks)

(i) By using one step of Newton's method, find an approximation to that root of $f(x) = x^5 - 32.01 = 0$ which lies close to $x = 2$.

(ii) The n-th term of a series is $u_n = n + (\frac{1}{2})^n$. Find the sum of the first ten terms.

(iii) Evaluate $\int_0^{\pi/2} \sin 2x \, dx$.

QUESTION 4 (10 Marks)

(i) Evaluate (a) $\int_0^2 \dfrac{1}{x^2+4}\,dx$ (b) $\int_1^{e^3} \dfrac{1}{x}\,dx$

(ii) Write down the formula for the expansion of $\cos(A + B)$ and deduce that $2\sin^2\theta = 1 - \cos 2\theta$

(iii) An alternating electric current i amperes passing through a given radiator produces heat at the instantaneous rate of $12i^2$ calories per second. Find the amount of heat produced in this radiator in one full second by the current $i = 8\sin(100\pi t)$ where t represents time in seconds. (Hint: You may use the result of part (ii) of this question.)

QUESTION 5 (10 Marks)

Consider $f(x) = \dfrac{1-x^2}{x^4}$, $x \neq 0$

(i) Sketch the zeros of this function, i.e. the values of x where $f(x) = 0$.

(ii) Find the turning points of this function, and determine their nature.

(iii) Describe how this function behaves near $x = 0$.

(iv) Describe how this function behaves for large x.

(v) Give a rough sketch of this function (not on graph paper).

QUESTION 6 (10 Marks)

(i) Expand the expression $(u - v)(u^2 + uv + v^2)$

(ii) By letting $u = (x + h)^{1/3}$ and $v = x^{1/3}$, or otherwise, establish the identity: $(x+h)^{1/3} - x^{1/3} = h/\{(x+h)^{2/3} + x^{1/3}(x+h)^{1/3} + x^{2/3}\}$

(iii) Differentiate $f(x) = x^{1/3}$ from first principles (that is, from the definition of a derivative).

QUESTION 7 (10 Marks)

(i) Differentiate $f(x) = \tan^{-1}x + \tan^{-1}(\frac{1}{x})$ for $x \neq 0$.

(ii) By considering a right-angled triangle with two sides equal to 1 and x, respectively, or otherwise find the actual value of $f(x)$ for all $x > 0$.

(iii) By using the fact that $\tan^{-1}(-x) = -\tan^{-1}(x)$ or otherwise, deduce the value of $f(x)$ for all $x < 0$.

(iv) Is it possible to assign a value to $f(0)$ so as to make $f(x)$ a function which is continuous at $x = 0$? Explain your answer with the aid of a sketch.

QUESTION 8 (10 Marks)

(i) If α and β are the roots of the quadratic equation $x^2 + px + q = 0$, write down the values of:

(a) $\alpha + \beta$ (b) $\alpha\beta$ (c) $\alpha^2 + \beta^2$

(ii) Let a, b, c and d be the four roots of the equation:

$x^4 - 2x^3 - 44x^2 + 18x + 314 = 0$

(a) Without attempting to locate the individual roots, state the value of their sum $S = a + b + c + d$

(b) Similarly, state the value of $T = ab + ac + ad + bc + bd + cd$.

(c) From your results for S and T, or otherwise, find the value of $a^2 + b^2 + c^2 + d^2$.

QUESTION 9 (10 Marks)

A man stands on the roof of a tall tower and shoots an arrow in a horizontal direction; the arrow starts 45 metres above ground, with an initial speed of 30 metres per second. You may take the acceleration of gravity g to be 10 metres per second per second as sufficiently accurate for this problem. In parts (i), (ii), and (iii), ignore any effects of air friction.

(i) What time elapses before the arrow strikes the ground?

(ii) How far from the base of the tower does the arrow strike the ground?

(iii) What is the acute angle θ between the path of the arrow and the horizontal at the moment the arrow strikes the ground?

(iv) Now assume that there is an air friction force in a direction opposite to the velocity of the arrow, and that it is an increasing function of the speed of the arrow. Reason qualitatively, without calculating, to decide whether air friction makes the angle θ of part (iii) larger or smaller; give your reasoning. (Hint: Consider the extreme case where air friction is very large, so that it is the dominant force at all but very small speeds.)

QUESTION 10 (10 Marks)

In the 1950's, electronic computers were constructed with valves (rather than transistors). Assume that each particular valve has a probability $p = 0.00002$ of failing during the next hour. (Assume that valve failures are independent events, and are independent of the length of past service of the valve.)

(i) Find the probability q of a particular valve operating correctly during the next hour.

(ii) If the computer contains 10,000 valves, show that the probability P_0 of no valve failing during the next hour is 0.82, to two significant figures. (Hint: In the computation, use the first few terms of a binomial expansion, rather than logarithms.)

(iii) If we are to construct a computer containing 10,000 valves, with a probability of trouble-free operation for one hour at least as high as 0.99, estimate how small must be the probability p of failure of one valve during the next hour.

(iv) With p as in part (i), what is the probability of exactly one valve failure during the next hour?