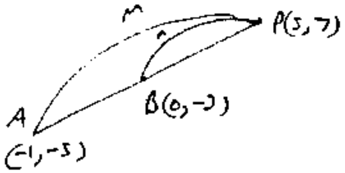
Solutions TKS Trial HSC 2003 Extension 1

Qu 1

(a)



$\Rightarrow \boxed{m : n = 6 : 5}$

(b) $\frac{d}{dx} \tan^{-1}(1+x^2) = \frac{1}{1+(1+x^2)^2} \times 2x = \boxed{\frac{2x}{1+(1+x^2)^2}}$

(c) $\binom{8}{5}$ or, of course, $\binom{8}{3} = \boxed{56}$

(d) gradients of lines are 2 and −3

$\therefore \tan\alpha = \left| \frac{2-(-3)}{1+2(-3)} \right| = \frac{5}{5} = 1$

$\therefore$ acute angle is $\boxed{45^\circ}$

(e) $P(-1) = 0 \Rightarrow (-1)^{2n+1} - (-1)^{2n} + b = 0$

i.e. $-1 - 1 + b = 0 \quad \therefore \boxed{b = 2}$

(f) $\sum_{n=1}^{9} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \dots + \left(\cancel{\tfrac{1}{8}} - \cancel{\tfrac{1}{9}}\right) + \left(\cancel{\tfrac{1}{9}} - \tfrac{1}{10}\right)$

$= 1 - \frac{1}{10}$

$= \boxed{\frac{9}{10}}$

Qu 2

(a) $f'(x) = 5 - 4\cos 4x$

$\geq 5 - 4(1)$ since $-1 \leq \cos 4x \leq 1$

$\Rightarrow f'(x) \geq 1 > 0 \quad \forall x$

$\therefore f(x)$ increases $\forall x$

(b) (i) $R\sin(x-\alpha) = R\cos\alpha \sin x - R\sin\alpha\cos x$

$\equiv \sin x - \sqrt{3}\cos x$

$\Rightarrow R\cos\alpha = 1$, $R\sin\alpha = \sqrt{3}$ $\quad \therefore \tan\alpha = \sqrt{3}$, $\boxed{\alpha = \frac{\pi}{3}}$

and $R = \sqrt{1^2 + \sqrt{3}^2} = \boxed{2}$

(ii) From (i), $2\sin\left(x - \frac{\pi}{3}\right) = \sqrt{2}$

$\therefore \sin\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\therefore x - \frac{\pi}{3} = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

i.e. $x = \boxed{\frac{7\pi}{12} \text{ or } \frac{13\pi}{12}}$

(c) (i) $u = 4 - x^2$ $\qquad x = 0,\ u = 4$

$\frac{du}{dx} = -2x$ or $du = -2x\,dx$ $\qquad x = \sqrt{3},\ u = 1$

$\therefore I = -\frac{1}{2}\int_4^1 \frac{du}{\sqrt{u}} = \frac{1}{2}\int_1^4 u^{-\frac{1}{2}}\,du$

$= \frac{1}{2} \cdot 2\left[u^{\frac{1}{2}}\right]_1^4$

$= 2 - 1 = 1$

(ii) $I = \int_0^{\sqrt{3}} \frac{4}{\sqrt{4-x^2}} - \frac{x}{\sqrt{4-x^2}}\,dx = 4\cdot\left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{3}} - 1$, from (i)

$= 4\frac{\pi}{3} - 1$

Qn 3

(a) $f'(x) = 2x e^{x^2} - 1$

$\therefore x_1 = 1.2 - \dfrac{e^{1.44} - 1.2 - 3}{2.4e^{1.44} - 1} = 1.1977\ldots$

$\therefore$ two decimal approx^n $= \boxed{1.20}$

(b) (i) $\sin 2A = \dfrac{2t}{1+t^2}$

(ii) put $t = \tan A$,

then $\operatorname{cosec} 2A - 3\cot 2A = \dfrac{1+t^2}{2t} - 3 \cdot \dfrac{1-t^2}{2t}$

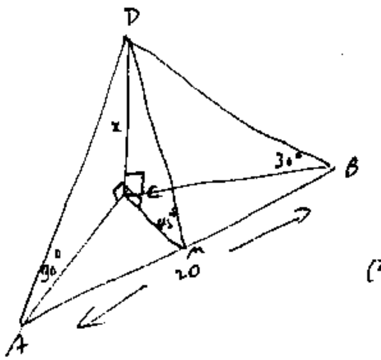
$= \dfrac{1+t^2-3+3t^2}{2t}$

$= \dfrac{4t^2-2}{2t}$

$= 2t - \dfrac{1}{t}$

$= 2\tan A - \cot A$

(c) (i)

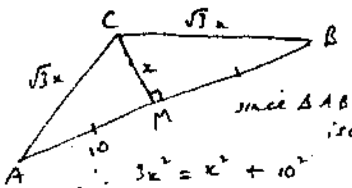


(ii) In ΔACD,

$\tan 30° = \dfrac{x}{AC} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow AC = \sqrt{3}\,x$

(iii) From (ii), we have



since ΔABC is isosceles

$\therefore 3x^2 = x^2 + 10^2$

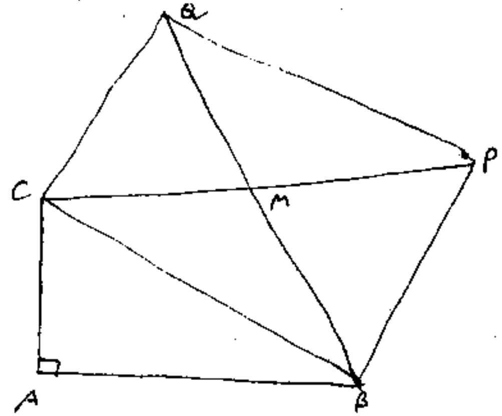
$2x^2 = 100$

$x^2 = 50$

$\therefore x = \sqrt{50}$ or $5\sqrt{2}$

Qn 4

(i)



(ii) $\angle CMB = 90°$, the diagonals of a square meet at right angles

$\therefore$ In $\Diamond ABMC$, $\angle A + \angle M = 180°$

$\Rightarrow \Diamond ABMC$ is cyclic, opposite angles are supplementary

(iii) $MC = MB$, equal diagonals in a square bisect each other.

$\therefore \angle CAM = \angle BAM$, $\angle$s at the circumference of a circle standing on equal arcs.

$\therefore MA$ bisects $\angle BAC$

(b) (i) $t = 0$, $x = 10\cos 0 = 10$

$\dot{x} = -10n \sin nt = -10n \sin 0$ at $t = 0$

$= 0$

$\therefore$ particle is initially at rest at $x = 10$

(ii) $T = \dfrac{2\pi}{n}$ $\therefore n = \dfrac{2\pi}{T}$

$\therefore b = 10\cos\left(\dfrac{2\pi}{T} \cdot \dfrac{T}{3}\right) = 10\cos\left(\dfrac{2\pi}{3}\right)$

i.e. $b = 10\left(-\dfrac{1}{2}\right) = -5$

(iii) $\dot{x} = -10n \sin nt = -10 \cdot \frac{2\pi}{T} \sin\left(\frac{2\pi t}{T}\right)$

$\therefore -20\sqrt{3} = -\frac{20\pi}{T} \sin\left(\frac{2\pi}{3}\right) = -\frac{20\pi}{T} \cdot \frac{\sqrt{3}}{2}$

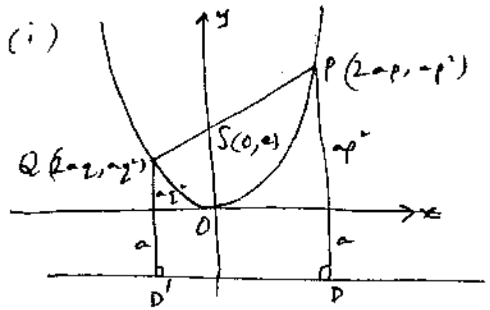
$\therefore T = \frac{\pi}{2}$ (seconds) i.e. period is $\frac{\pi}{2}$ s

(c) $\binom{n}{3} \div \binom{n-1}{2} = \frac{n!}{(n-3)!\,3!} \div \frac{(n-1)!}{(n-3)!\,2!}$

$= \frac{n!}{(n-3)!\,3!} \times \frac{(n-3)!\,2!}{(n-1)!} = \frac{n}{3}$

Qn 5

(a) (i)



gradient PQ
$= \frac{ap^2 - aq^2}{2ap - 2aq}$
$= \frac{a(p-q)(p+q)}{2a(p-q)}$
$= \frac{p+q}{2}$

$\therefore$ chord PQ is $y - ap^2 = \frac{1}{2}(p+q)(x - 2ap)$

or $y - ap^2 = \frac{1}{2}(p+q)x - ap(p+q)$

$\Rightarrow y - \frac{1}{2}(p+q)x + apq = 0$

(ii) Since $S(0,a)$ is on PQ, then

$a - 0 + apq = 0$

i.e. $pq = -1$ or $q = -\frac{1}{p}$

(iii) $PQ = PS + QS = PD + QD'$, focus-directrix defn (see diagram)

$= a + ap^2 + a + aq^2, \quad q = -\frac{1}{p}$

$= 2a + a\left(p^2 + \frac{1}{p^2}\right)$

(iv) If PQ is a diameter, the radius is $a + \frac{a}{2}\left(p^2 + \frac{1}{p^2}\right)$, from (iii)

& the centre is $\left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$

$= \left(a\left(p - \frac{1}{p}\right), \frac{a}{2}\left(p^2 + \frac{1}{p^2}\right)\right), \quad q = -\frac{1}{p}$

$\therefore$ distance from centre to directrix $y = -a$ is

$\frac{a}{2}\left(p^2 + \frac{1}{p^2}\right) + a$ = radius

$\therefore$ directrix is a tangent to the circle

(b) (i) $A = 6 \times 10^2 + 12 \cdot 6\, t = 600 + 12 \cdot 6\, t$

(ii) If an edge is x, $A = 6x^2$, $V = x^3$

$\therefore$ From (i), $6x^2 = 600 + 12 \cdot 6\, t$

$x^2 = 100 + 2 \cdot 1\, t$

$\therefore V = (100 + 2 \cdot 1\, t)^{3/2}$

& so, $\dfrac{dV}{dt} = \dfrac{3}{2}(100 + 2 \cdot 1\, t)^{\frac{1}{2}}(2 \cdot 1)$

$= 3 \cdot 15 \times \sqrt{121}$ cm^3/s when $t = 10$

$= 34 \cdot 65$ cm^3/s

Alternatively, using $A = 6x^2$, $V = x^3$, $\dfrac{dA}{dt} = 12 \cdot 6$,

we have $\dfrac{dV}{dt} = \dfrac{dV}{dx} \cdot \dfrac{dx}{dt}$

$= \dfrac{dV}{dx} \cdot \dfrac{dx}{dA} \cdot \dfrac{dA}{dt}$

$= 3x^2 \cdot \dfrac{1}{12x} \cdot (12 \cdot 6)$

$= 3 \cdot 15\, x$

But, when $t = 10$, $A = 6 \times 10^2 + 12 \cdot 6 \times 10 = 726$

$\therefore 6x^2 = 726 \Rightarrow x = 11$

$\therefore \dfrac{dV}{dt} = 3 \cdot 15 \times 11$ cm^3/s $= 34 \cdot 65$ cm^3/s

Qu 6

(a) $E(0) = 9^2 - 4^0 = 80$ is a multiple of 5

$\therefore$ assume $E(n) = 9^{n+2} - 4^n = 5q$, q an integer, $n \geq 0$

Then, $E(n+1) = 9^{n+3} - 4^{n+1}$

$= 9(9^{n+2}) - 4^{n+1}$

$= 9(5q + 4^n) - 4^{n+1}$, using the assumption

$= 5(9q) + 4^n(9 - 4)$

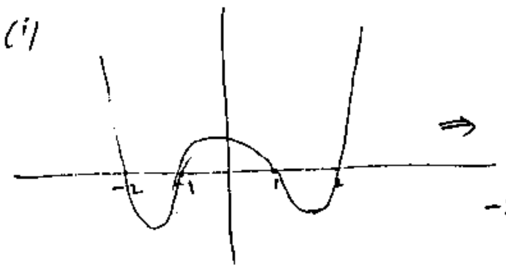
$= 5(9q + 4^n)$ is a multiple of 5 since $9q + 4^n$ is an integer

$\therefore$ if $E(n)$ is a multiple of 5, so is $E(n+1)$

But, $E(0)$ is a multiple of 5

$\therefore E(n)$ is a multiple of 5 by induction

(b) (i)



$\Rightarrow (x^2 - 1)(x^2 - 4) \leq 0$ has solutions $-2 \leq x \leq 1$ or $1 \leq x \leq 2$

(ii) (a) $\dfrac{d(\frac{1}{2}v^2)}{dx} = 10x - 4x^3$

$\therefore \frac{1}{2}v^2 = 5x^2 - x^4 + c$, c a constant

ie. $\frac{1}{2}v^2 + x^4 - 5x^2 = c$

When $x = \sqrt{2}$, $v = 2$

$\therefore 2 + 4 - 10 = c = -4$

(β) we have $\frac{1}{2}v^2 = 5x^2 - x^4 - 4$

or $\frac{1}{2}v^2 = -(x^4 - 5x^2 + 4)$

$= -(x^2-1)(x^2-4)$

Now, $\frac{1}{2}v^2 \geq 0 \Rightarrow (x^2-1)(x^2-4) \leq 0$

$\therefore$ Using (b)(i) and when $x = \sqrt{2}$, $v = 2$, we have the particle oscillates between $x = 1$ and $x = 2$

(c) (i) $P(5 \text{ males}, 5 \text{ females}) = \binom{10}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5$

$= 0.246$, 3 d.p.

(ii) $P(\text{more females}) = P(\text{more males})$, since $P(M) = P(F) = \frac{1}{2}$

$\therefore$ using (i), $P(\text{more females}) = \dfrac{1 - 0.246}{2} = 0.377$

Qn 7

(a) if $x+1>0$, then $-2x>0$

i.e. $x>-1$ i.e. $x<0$

$\therefore$ solution is $-1<x<0$

if $x<-1$, we'd have $x>0$ $\Rightarrow$ no further solutions

$\therefore$ $-1<x<0$

(b)(i) We need $\dfrac{-2x}{x+1}>0$ and $-2x>0$ and $x+1>0$.

All 3 inequalities are "true" from (a)

(ii) From (i), $y = \ln(-2x) - \ln(x+1)$

$\therefore \dfrac{dy}{dx} = \dfrac{-2}{-2x} - \dfrac{1}{x+1}$

$= \dfrac{1}{x} - \dfrac{1}{x+1} = \dfrac{x+1-x}{x(x+1)}$

$= \dfrac{1}{x(x+1)} \neq 0$ for any x

$\therefore$ there are no stationary points

(iii) Since $-1<x<0$, then $x(x+1)<0$

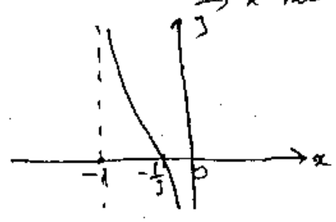
$\therefore$ From (ii) $\dfrac{dy}{dx}<0$ for $-1<x<0$

i.e. curve is decreasing for $-1<x<0$

When $y=0$, $\dfrac{-2x}{x+1} = 1$

$\therefore -2x = x+1$

$\Rightarrow$ x intercept is $-\frac{1}{3}$



(iv) since curve is decreasing, the inverse function

is $x = \ln\left(\frac{-2y}{y+1}\right)$

$\therefore \; -\frac{2y}{y+1} = e^{x}$

$-2y = e^{x}y + e^{x}$

or $y(e^{x}+2) = -e^{x}$

ie inverse function is $y = -\frac{e^{x}}{e^{x}+2}$

(v) $A = \left| \int_0^2 x \, dy \right| = \int_0^2 \frac{e^{y}}{e^{y}+2} \, dy$, from (iv)

$= \left[\ln(e^{y}+2)\right]_0^2$

$= \ln(e^{2}+2) - \ln 3 \quad u^2$