Hornsby Girls High School 2007 X1
Solutions

Question 1 (12 marks)

$-1 \qquad 3$

a) $A(3,4)$ and $B(-2,5)$

$x = \frac{2+9}{2}$ $\qquad y = \frac{-5+12}{2}$

$x = 5\frac{1}{2}$ $\qquad y = 3\frac{1}{2}$

$C\left(5\frac{1}{2}, 3\frac{1}{2}\right)$ ✓

b) 8 boys 6 girls

↓

4 boys 4 girls

i) $n(S) = {}^8C_4 \times {}^6C_4$

$= 70 \times 15$

$= 1050$ ✓

$n(E) = {}^7C_3 \times {}^5C_4$

$= 35 \times 5$

$= 175$

$P(E) = \frac{175}{1050}$ ✓

$= \frac{1}{6}$

$3! \times 4!$

$= 144$ ✓

$f(x) = \frac{1}{2}\cos^{-1}\frac{1}{3}x$

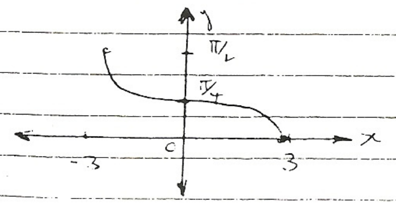
$2y = \cos^{-1}\frac{x}{3}$

Range $0 \le 2y \le \pi$

$0 \le y \le \frac{\pi}{2}$ ✓

Domain $-1 \le \frac{x}{3} \le 1$

$-3 \le x \le 3$ ✓



✓

(d)

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$

$\sin\frac{\pi}{12} = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$ ✓

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$ ✓

(e)

$f(x) = \ln x$	$f(x) = -x^2+1$
$f'(x) = \frac{1}{x}$	$f'(x) = -2x$
$f'(1) = 1$	$f'(1) = -2$
$m_1 = 1$	$m_2 = -2$

✓

$\tan\theta = \frac{m_1 - m_2}{1+m_1m_2}$

For acute angle θ

$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$

✓

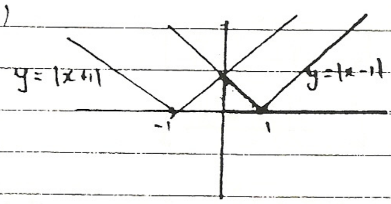
$\tan\theta = \left|\frac{1+2}{1-2}\right|$

$\tan\theta = 3$ ✓

$\theta = 72^\circ$ (Nearest degree)

Question 2 12 marks.

(a)



✓

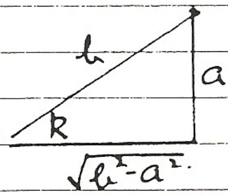
$|x-1| \le |x+1|$ ✓

$x \ge 0$

) $\cos\left(2\sin^{-1}\frac{a}{b}\right)$

Let $\sin^{-1}\frac{a}{b} = k$

$\sin k = \frac{a}{b}$



$\cos 2k = 2\cos^2 k - 1$

$= 2\left(\frac{b^2-a^2}{b^2}\right) - 1$

$= \frac{2b^2-2a^2-b^2}{b^2}$

$= \frac{b^2-2a^2}{b^2}$

$V = \pi\int y^2\,dx$

$= \pi\int_{\pi/12}^{\pi} \sin^2 x\,dx$

$= \frac{\pi}{2}\int_{\pi/12}^{\pi} 1-\cos 2x\,dx$

$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\pi/12}^{\pi}$

$= \frac{\pi}{2}\left[[\pi - 0] - [\frac{\pi}{12} - \frac{1}{4}]\right]$

$= \frac{\pi}{2}\left[\frac{11\pi}{12} + \frac{1}{4}\right]$

$= \frac{\pi}{2}\left[\frac{11\pi+3}{12}\right]$

$= \frac{\pi}{24}\left[11\pi+3\right]$ cubic units

(d)

$\sqrt{3}\cos 2x - 1\sin 2x = 2$

use Subsidiary angle

$R = \sqrt{4}$ $\quad \tan\theta = \frac{1}{\sqrt{3}}$

$R = 2$ $\quad \theta = \frac{\pi}{6}$

$\therefore\ 2\cos\left(2x + \frac{\pi}{6}\right) = 2$

$\cos\left(2x + \frac{\pi}{6}\right) = 1$

$\cos\left(2x + \frac{\pi}{6}\right) = \cos 0$

$\therefore\ 2x + \frac{\pi}{6} = 2n\pi \pm 0$

$2x = 2n\pi - \frac{\pi}{6}$

$x = n\pi - \frac{\pi}{12}$

(e) G E O L I S T
G O

$\boxed{E/G}$ E O L I S T
O

$= \frac{8!}{2!}$

$= 20160$

Question 3

(a) $\displaystyle\int_0^{\sqrt{\frac{27}{2}}} \frac{1}{9+2x^2}\,dx$

$= \dfrac{1}{2}\displaystyle\int_0^{\sqrt{\frac{27}{2}}} \frac{1}{\frac{9}{2}+x^2}\,dx$

$= \dfrac{1}{2}\displaystyle\int_0^{\sqrt{\frac{27}{2}}} \frac{1}{\left(\frac{3}{\sqrt{2}}\right)^2+x^2}\,dx$

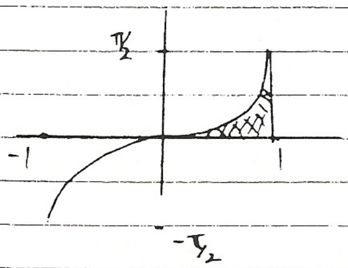
$= \dfrac{1}{2}\times\dfrac{\sqrt{2}}{3}\left[\tan^{-1}\dfrac{\sqrt{2}x}{3}\right]_0^{\sqrt{\frac{27}{2}}}$

$= \dfrac{\sqrt{2}}{6}\left(\left[\tan^{-1}\dfrac{\sqrt{2}\times 3\sqrt{3}}{3\sqrt{2}} - \tan^{-1}0\right]\right)$

$= \dfrac{\sqrt{2}}{6}\left(\tan^{-1}\sqrt{3}\right]$

$= \dfrac{\sqrt{2}}{6}\times\dfrac{\pi}{3}$

$= \dfrac{\sqrt{2}\,\pi}{18}$



$A = \dfrac{\pi}{2} - \displaystyle\int_0^{\frac{\pi}{2}} \sin x\,dx$

$\dfrac{\pi}{2} - \left(\left[-\cos x\right]_0^{\frac{\pi}{2}}\right)$

$= \dfrac{\pi}{2} - \left(0+1\right)$

$= \dfrac{\pi}{2} - 1$ sq units.

(c)

LHS $= \dfrac{2\sin\theta\cos\theta + \sin\theta}{1+2\cos^2\theta - 1 + \cos\theta}$

$= \dfrac{\sin\theta\left(2\cos\theta+1\right)}{\cos\theta\left(2\cos\theta+1\right)}$

$= \tan\theta$

$=$ RHS

(d) Let $g = \frac{5}{8}$

$h = \frac{3}{8}$

$(g+h)^{100} =$

max Coef : $\dfrac{n-r+1}{r}\cdot\dfrac{\frac{3}{8}}{\frac{5}{8}} \geq 1$

$\dfrac{101-r}{r}\cdot\dfrac{3}{5} \geq 1$

$303-3r \geq 5r$

$8r \leq 303$

$r \leq 37.875$

$r = 37$

$\therefore$

MAX Coef Term $\ {}^{100}C_{37}\left(\frac{5}{8}\right)^{63}\left(\frac{3}{8}\right)^{37}$

(e) most likely green faces is 63

Probability of 63 green faces $= \binom{100}{37}\left(\frac{5}{8}\right)^{63}\left(\frac{3}{8}\right)^{37}$

$= 0.082$

Term in x^5

${}^nC_5\, 3^{n-5}\, 2^5$

Term in x^6

${}^nC_6\, 3^{n-6}\, 2^6$

${}^nC_5\, 3^{n-5}\, 2^5 = {}^nC_6\, 3^{n-6}\, 2^6$

${}^nC_5 \div {}^nC_6 = \frac{2}{3}$

$\frac{n!}{(n-5)!\cdot 5!} \times \frac{(n-6)!\, 6!}{n!} = \frac{2}{3}$

$\frac{6}{(n-5)} = \frac{2}{3}$

$2n - 10 = 18$

$2n = 28$

$n = 14$

Question 4

$v^2 = 15 + 2x - x^2$

$acel = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{15}{2} + x - \frac{1}{2}x^2\right)$

$= 1 - x$

$= -1(x-1)$

$= -n^2(x-h)$ where $n = 1$, $h = 1$

Simple harmonic motion

Centre is at $x = 1$

(α) Centre is $x = 1$

(β) Particle stops $x^2 - 2x - 15 = 0$

$(x-5)(x+3) = 0$

$x = 5$ and $x = -3$

$\therefore$ Amplitude $= 4$ units.

(γ) Period $T = \frac{2\pi}{n}$

$= 2\pi$ Seconds

(iii) Max velocity is at $x = 1$

(ie) $v^2 = 15 + 2 - 1$

$v^2 = 16$

$v = \pm 4$

Max speed is 4 m/s.

Max acc is at $x = 5$ or $x = -3$

(ie) acc $= -1(5-1)$ or $-1(-3-1)$

Max acc $= \pm 4$ m/s^2

(iv) $x = 4 \sin 1t$

$x = 4 \sin t$

Vel $= \dot{x} = 4 \cos t$

When $t = \frac{\pi}{4}$

vel $= 4 \times \frac{1}{\sqrt{2}}$

$= 2\sqrt{2}$ m/s

(b) When $n = 1$

LHS $= 2 \times 1!$

$= 2$

RHS $= 1(2)!$

$= 2$

$\therefore$ True for $n = 1$

Assume true for $n = k$

(ie) Assume

$\boxed{2(1!) + 5\times(2!) + \dots + (k^2+1)k! = k(k+1)!}$

RQTP

$2(1!) + 5\times(2!) + \dots + (k^2+1)k! + ((k+1)^2+1)(k+1)!$

$= (k+1)(k+2)!$

LHS $= k(k+1)! + ((k+1)^2+1)(k+1)!$

$= k(k+1)! + (k^2+2k+2)(k+1)!$

$= (k+1)\,[\,k\,k! + (k^2+2k+2)k!$

$= (k+1)\, k!\, [k + k^2 + 2k + 2]$

$= (k+1)\, k!\, [k^2 + 3k + 2]$

$= (k+1)\, k!\, (k+2)(k+1)$

$= (k+1)\, (k+2)!$

$=$ RHS

True for $n = k+1$

ince True for $n = 1$ Then

True for $n = 2, 3, 4$ all $n \geq 1$

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uestion 5

a) Let $P(x) = 2x^3 + ax^2 + bx + 6$

$P(1) = 0$

(ie) $2 + a + b + 6 = 0$

$a + b + 8 = 0$ — (1)

$P(2) = -12$

(ie) $16 + 4a + 2b + 6 = -12$

$4a + 2b + 34 = 0$

$2a + b + 17 = 0$ — (2)

(2) − (1) $a + 9 = 0$

$a = -9$

$b = 1$

$x^3 + 2x^2 - 5x - 6 = 0$

Let Roots be $\alpha, \beta, (\alpha+\beta)$

um: $2\alpha + 2\beta = -2$

$\alpha + \beta = -1$ — (1)

oduct: $\alpha\beta(\alpha+\beta) = 6$ — (2)

rom (1)

$\beta = -1 - \alpha$

into (2) $\alpha(-1-\alpha)(-1) = 6$

$\alpha + \alpha^2 = 6$

$\alpha^2 + \alpha - 6 = 0$

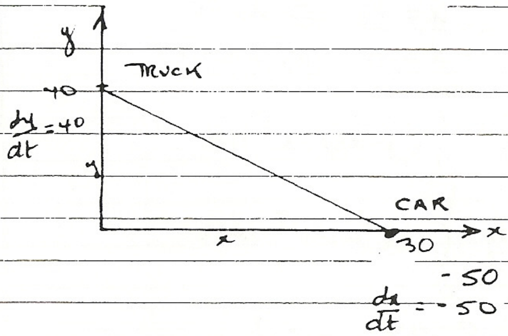
$(\alpha + 3)(\alpha - 2) = 0$

$\alpha = 3$ $\alpha = 2$ and $\alpha + \beta = -1$

$\beta = 2$ $\beta = -3$

oots $-3, 2, -1$

(c)



$z^2 = x^2 + y^2$

$\cancel{2}z \frac{dz}{dt} = \cancel{2}x \frac{dx}{dt} + \cancel{2}y \frac{dy}{dt}$

$\frac{dz}{dt} = \frac{30 \times -50 + 40 \times 40}{50}$

$= \frac{-1500 + 1600}{50}$

$= 2$ Km/h.

(d)

$\frac{d}{dx} \sin^{-1}(\tfrac{1}{2}\sin x)$

$= \frac{1}{\sqrt{1 - \frac{1}{4}\sin^2 x}} \times \frac{1}{2}\cos x$

$= \frac{\cos x}{\sqrt{4 - \sin^2 x}}$

$\therefore \int_0^{\pi/2} \frac{\cos x \, dx}{\sqrt{4 - \sin^2 x}} = \left[\sin^{-1}(\tfrac{1}{2}\sin x)\right]_0^{\pi/2}$

$= [\sin^{-1}(\tfrac{1}{2})] - [\sin^{-1} 0]$

$= \frac{\pi}{6}$

$I = \int_0^{1/3} 3x(1-3x)^4\,dx$

Let $u = 1-3x$

$du = -3dx$

$dx = -\frac{1}{3}du$

Terminal Values

$x = \frac{1}{3} \Rightarrow u = 0$

$x = 0 \Rightarrow u = 1$

$I = -\frac{1}{3}\int_1^0 (1-u)u^4\,du$

$= \frac{1}{3}\int_0^1 u^4 - u^5\,du$

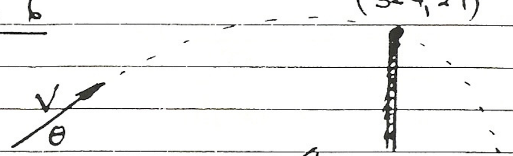
$= \frac{1}{3}\left[\frac{u^5}{5} - \frac{u^6}{6}\right]_0^1$

$= \frac{1}{3}\left[\frac{1}{5} - \frac{1}{6}\right] - [0]$

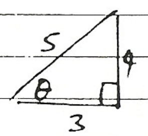
$= \frac{1}{90}$

Question 6

(324, 27)



$\tan\theta = \frac{4}{3}$



Vertical	Horizontal
$\ddot{y} = -10$	$\ddot{x} = 0$
$\dot{y} = -10t + \frac{4V}{5}$	$\dot{x} = \frac{3V}{5}$
$y = -5t^2 + \frac{4Vt}{5}$ ✓	$x = \frac{3Vt}{5}$ ✓

(2)

(ii) Trajectory

$y = x\tan\theta - \frac{gx^2}{2V^2}(1+\tan^2\theta)$ ← given

where $\tan\theta = \frac{4}{3}$

$x = \frac{3vt}{5}$ $\quad x = 324$

$y = 27$

$t = \frac{5x}{3v}$ $\quad g = 10$

$y = -5\left(\frac{5x}{3v}\right)^2 + \frac{4v}{5}\left(\frac{5x}{3v}\right)$

$27 = 324 \times \frac{4}{3} - \frac{5 \times (324)^2}{V^2}\left(\frac{25}{9}\right)$ ✓

$9 \times 27 = 12 \times 324 - \frac{125 \times 324^2}{V^2}$

$243 = 3888 - \frac{13122000}{V^2}$

$3645V^2 = 125 \times 324^2$

$V^2 = \frac{125 \times 324^2}{3645}$

$V = 60$ ✓ 3

(b)

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n$

Integrate Both sides with respect to x

$\frac{(1+x)^{n+1}}{n+1} = {}^nC_0 x + \frac{1}{2}{}^nC_1 x^2 + \frac{1}{3}{}^nC_2 x^3 + \cdots + \frac{{}^nC_n x^{n+1}}{(n+1)} + k$ ✓

Let $x=0$ To Find k

$\frac{1}{n+1} = k$ ✓

$\therefore \frac{(1+x)^{n+1} - 1}{n+1} = {}^nC_0 x + \frac{1}{2}{}^nC_1 x^2 + \frac{1}{3}{}^nC_2 x^3 + \cdots + \frac{{}^nC_n x^{n+1}}{(n+1)}$

Let $x = -1$

✓

$\frac{-1}{n+1} = -{}^nC_0 + \frac{1}{2}{}^nC_1 - \frac{1}{3}{}^nC_2 + \cdots \frac{(-1)^{n+1}\,{}^nC_n}{(n+1)}$

3

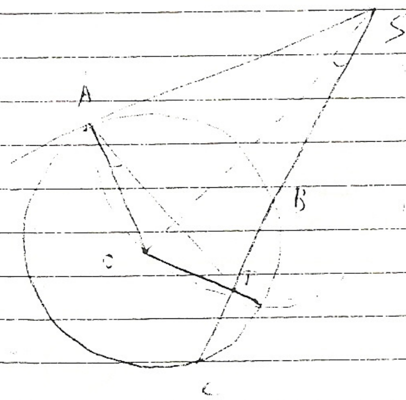
(i) $\angle SAO = 90^\circ$ (Tangent $\perp$ Radius)

$\angle OTB = 90^\circ$ ($\because$ Line joining centre of circle to mid point of chord is 90° to chord)

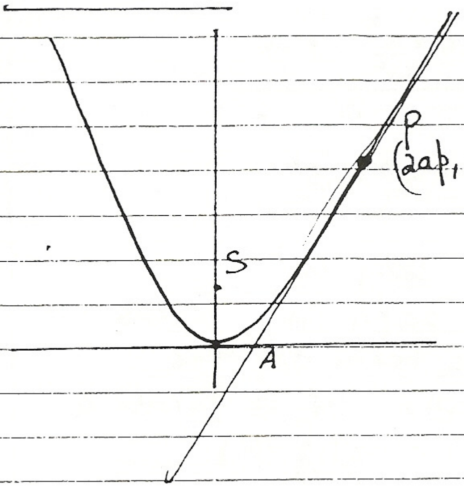
$\therefore$ TOAS is a cyclic quad because opposite angles are supplementary (2)

$\angle OAT = \angle OST$ (angles at the circumference, standing on same arc $\underline{\underline{OT}}$) ($\because$ TOAS cyclic quad)

(2)



Question 7



(i)

Tangent at $(2ap, ap^2)$

$y = px - ap^2$.

Sub $y = 0$

$px = ap^2$

$x = ap$

$\therefore$ A is $(ap, 0)$ (2)

(ii) $S(0, a)$ $A(ap, 0)$

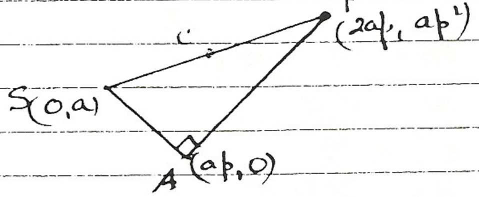
Gradient SA $= \frac{a}{-ap}$

$= -\frac{1}{p}$

Gradient AP $= p$

$\therefore$ AP $\perp$ SA (1)

(iii)



Centre of Circle $\left(ap, \frac{ap^2+a}{2}\right)$

$x = ap$ $y = \frac{ap^2+a}{2}$.

$p = \frac{x}{a}$

$y = \dfrac{a \times \frac{x^2}{a^2} + a}{2}$

$2ay = x^2 + a^2$

$x^2 = 2ay - a^2$

$x^2 = 2a(y - \frac{a}{2})$ Locus

is is The Form of The
neral equation of a Parabola

$(x-h)^2 = 4a(y-k)$

The Locus is The parabola

$(x-0)^2 = 2a(y - \frac{a}{2})$

hich has vertex $(0, \frac{a}{2})$

(3)

(i) $T = S + Ae^{kt}$

$\frac{dT}{dt} = kAe^{kt}$

$= k(T-S)$

u may assume This in This question

(ii) Boiling water.

$T = 25 + Ae^{kt}$

$100 = 25 + A$

$A = 75$

$\therefore T = 25 + 75e^{k_1 t}$

$55 = 25 + 75e^{5k_1}$

$30 = 75e^{5k_1}$

$e^{5k} = \frac{2}{5}$

$k_1 = \frac{1}{5}\ln\frac{2}{5}$

$T = 25 + 75e^{\frac{t}{5}\ln\frac{2}{5}}$

Iced water.

$T = 25 + Ae^{k_2 t}$

$\left.\begin{array}{l} t=0 \\ T=0 \end{array}\right]$ $0 = 25 + A$

$A = -25$

$T = 25 - 25e^{k_2 t}$

$\left.\begin{array}{l} T=15 \\ t=5 \end{array}\right]$ $15 = 25 - 25e^{5k_2}$

$-10 = -25e^{5k_2}$

$e^{5k_2} = \frac{3}{5}$

$k_2 = \frac{1}{5}\ln\frac{3}{5}$

$T = 25 - 25e^{\frac{t}{5}\ln\frac{3}{5}}$

Difference in Temps = 10°C

$10 = \left(25 + 75e^{\frac{t}{5}\ln\frac{2}{5}}\right) - \left(25 - 25e^{\frac{t}{5}\ln\frac{2}{5}}\right)$

$10 = 100e^{\frac{t}{5}\ln\frac{2}{5}}$

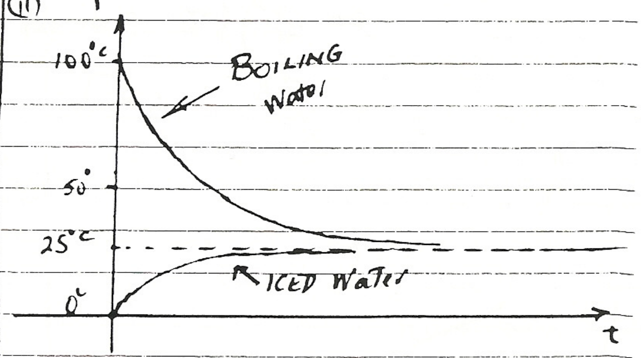
$.1 = e^{\frac{t}{5}\ln\frac{2}{5}}$

$\frac{t}{5}\ln\frac{2}{5} = \ln .1$

$t = 5 \times \frac{\ln .1}{\ln \frac{2}{5}}$

$t = 12.56$ mins (4)

(ii)



(2)