# Sydney Girls High School
4 unit mathematics
Trial HSC Examination 1991

1. (a) (i) If $Z = -2 - i$ show on an Argand diagram the points A, B, C representing the complex numbers $Z, iZ, Z + iZ$ respectively.
(ii) Show that the geometric figure $OABC$, where O is the origin is a square.
(b) (i) Express $(5 + 4i)(3 + 2i)$ in the form $a + ib$, where a and b are real. Hence deduce a pair of factors of $7 - 22i$.
(ii) Using your results in **(i)** express $7^2 + 22^2$ as the product of two positive integers.
(c) If Z is a complex number and $\Re\left[\frac{z-2i}{2z-1}\right] = 0$ show that the locus of the point P, representing Z in the complex plane, is a circle with centre at $\frac{1}{4} + i$ and radius $\frac{\sqrt{17}}{4}$ units.
(d) The point P represents the complex number Z. The reflection of P in the line OR, making an angle α with the positive direction of the real axis, is the point Q, representing the complex number z_1.
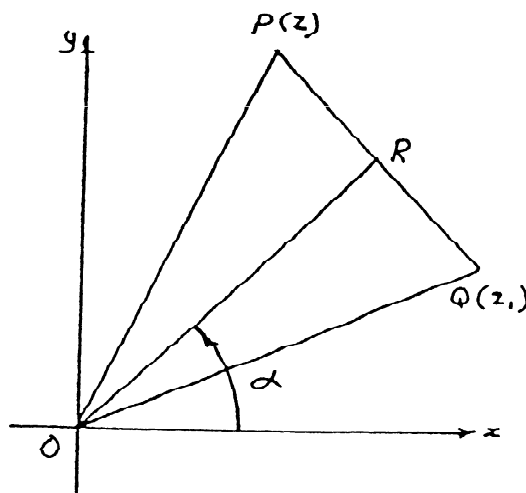
(i) Show that $|z_1| = |z|$
(ii) Also show that $\arg z_1 + \arg z = 2\alpha$
(iii) Hence prove that $zz_1 = |z|^2(\cos 2\alpha + i \sin 2\alpha)$

2. (a) Show that $\int x(x + 1)^{10} \, dx = \frac{(x+1)^{11}(11x-1)}{132} + c$
(b) Assuming the identity $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$, evaluate $\int_0^{\pi/6} \sin 2x \cos 4x \, dx$ expressing your answer in rational form.
(c) Use the substitution $x = a \sin \theta$ to show that

$\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}a^2 \sin^{-1}\frac{x}{a} + \frac{1}{2}x\sqrt{a^2 - x^2} + c$

(d) (i) Use the product rule of differentiation to find $\frac{d}{dx}[xf(x)]$. Hence show that $\int xf'(x)\, dx = xf(x) - \int f(x)\, dx$.

(ii) Using your result in **(i)** at some stage, show that
$\int_0^1 (x - \frac{1}{2})f'(x)\, dx = \frac{1}{2}[f(0) + f(1)] - \int_0^1 f(x)\, dx$.

3. (a) The ellipse $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a, b are the lengths of the semi-major and semi-minor axes of E, passes through the foci of the hyperbola $H : \frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$, where A, B are the lengths of the semi-major and semi-minor axes of H. H also passes through the foci of E. Show that $2A^2 + B^2 = 2a^2 - b^2$

(b) (i) Show that the equation of the tangent to the ellipse $\frac{x^2}{36} + \frac{y^2}{25} = 1$ at the point $(6\cos\theta, 5\sin\theta)$ is $y = -\frac{5}{6}\cot\theta x + 5\operatorname{cosec}\theta$.

(ii) Deduce that the equation of the tangent derived in **(i)** can be expressed in the form $y = mx \pm \sqrt{36m^2 + 25}$, where m is the gradient of the tangent.

(iii) Hence, or otherwise, find the equation of the tangents to the ellipse in **(i)** which are perpendicular to the line $x + y = 0$.

4. (a) Find the values of the real numbers a and b, such that $(1 - i)$ is a root of the equation $z^3 + az + b = 0$.

(b) The cubic equation $3x^3 + 2x^2 - x + 1 = 0$ has roots α, β, γ. Write down the cubic equation whose roots are

(i) $2\alpha, 2\beta, 2\gamma$ **(ii)** $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ (NOTE: You need not deduce the results)

(c) Express $z^4 + 1$, where z is a complex number, as the product of two real quadratic factors. Hence, or otherwise, find the fourth roots of -1, expressing them in the form $a + ib$, where a, b are real.

(d) When a monic cubic polynomial $A(x)$ is divided by $x^2 + x + 1$, the remainder is $2x + 3$. When $A(x)$ is diveded by $x(x + 3)$ the remainder is $5(x + 1)$. Find $A(x)$.

5. (a) The region bounded by the parabola $y = x^2$, the ordinates $x = 1$, $x = 3$ and the line $y = 1$ is rotated about the x-axis.
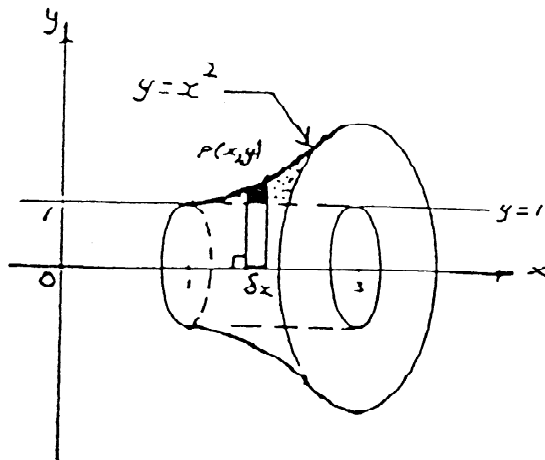

(i) A slice of thickness δx and volume δv is drawn through the rotated region and perpendicular to the x-axis. Show that $\delta v \doteqdot \pi(y^2 - 1)\delta x$.

(ii) Hence find the volume of the solid so formed, expressing your answer in exact form.

(b) The semicircle $y = \sqrt{1 - x^2}$ is rotated about the y-axis.
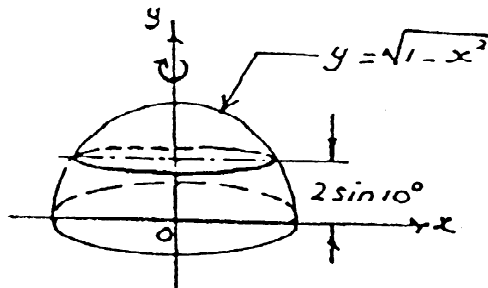
(i) Find the exact volume of the hemisphere so formed.

(ii) A plane is drawn parallel to the base of the hemisphere at a distance $2\sin 10°$ from the base. Prove that this plane bisects the volume of the hemisphere. [Hint: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$]

(c)
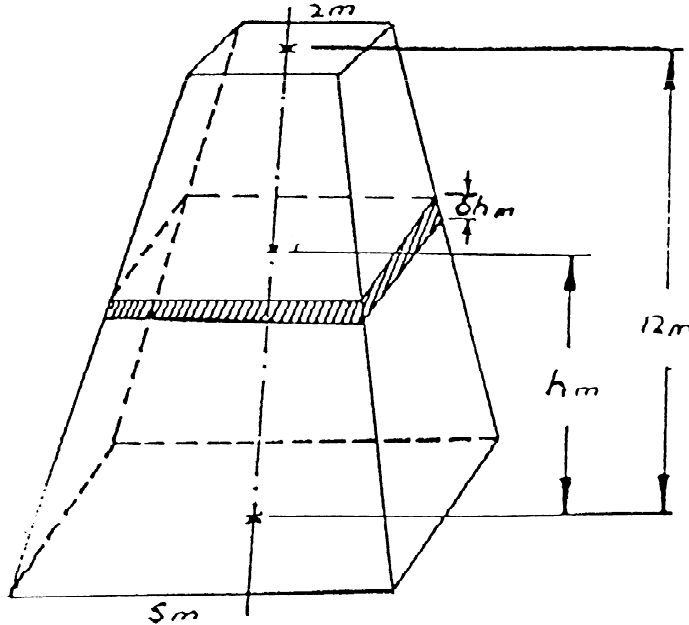
A solid with a square base has square cross sections whose sides vary in length from 5m at the base to 2m at the top. The height of the solid is 12m. Consider an element of thickness δh_m and h_m from the base.

(i) Show that $\delta v \doteqdot (5 - \frac{h}{4})^2 \delta h$.

(ii) Hence find the volume of the solid.

6. (a) (i) Express $\frac{(x-2)(x+1)}{(x-3)(x+2)}$ in the form $a + \frac{b}{x-3} + \frac{c}{x+2}$ where a, b, c are numerical constants. Hence show that the gradient of the function $y = \frac{(x-2)(x+1)}{(x-3)(x+2)}$ is positive when $x < \frac{1}{2}$ and negative when $x > 1$.

(ii) Find the coordinates of the points of intersection of the function in **(i)** with the coordinate axes.
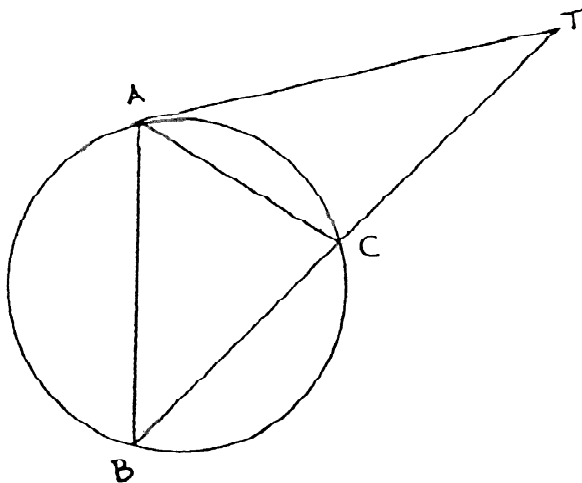
(iii) Find the equation(s) of the asymptote (s) of the function in **(i)**.

(iv) Use the data in **(i)**, **(ii)** and **(iii)** to sketch the graph of the function and use the graph to show that the equation $x^2(x-3)(x+2) - (x-2)(x+1) = 0$ has four real roots.

(b) Sketch the curve $y = (x+1)(x-3)$. Hence, or otherwise, on different diagrams sketch the following curves:

(i) $y = |(x+1)(x-3)|$ **(ii)** $y = \sqrt{(x+1)(x-3)}$

7. (a) Give the general solution to the equation $\tan 2x = 2\sin x \cos x$

(b)

In the figure, TA is the tangent to the circle at A, and TCB is a secant cutting the circle in C, B.

(i) Draw the figure neatly on your paper.

(ii) Prove that $T\hat{C}A = T\hat{A}B$

(iii) Point D is taken on CB so that $TA = TD$. Prove that AD bisects angle BAC.

(c) a and b are the heights of two vertical towers subtending the same angle γ at a point P on the line joining their feet A, B. Point Q is taken in the horizontal plane such that angle AQB is a right angle. The towers subtend angles α, β at Q.

(i) Draw a neat diagram showing all the given data.

(ii) Prove that $(a + b)^2 \cot^2 \gamma = a^2 \cot^2 \alpha + b^2 \cot^2 \beta$.

8. (a) A sequence $\{u_n\}$ is defined by the formula $u_{n+1} = \sqrt{3u_n}$.

(i) If $u_1 = 1$, show that $u_2 = 3^{\frac{1}{2}}$, $u_3 = 3^{\frac{1}{2}+\frac{1}{4}}$, $u_4 = 3^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}$.

(ii) Hence prove by mathematical induction that $u_n = 3^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}}} = 3^{1-2^{1-n}}$ where $n \geq 2$.

(iii) Find the limiting value of u_n as $n \to \infty$.

(b) (i) If x, y, z are real and distinct show that $z^2 + y^2 > 2xy$ and hence deduce that $x^2 + y^2 + z^2 > xy + xz + yz$

(ii) Using the result in **(i)**, or otherwise, show that $(x + y + z)^2 > 3(xy + xz + yz)$ and $x^3 + y^3 + z^3 > 3xyz$

(iii) (α) If $x + y + z = 12$, show that $xy + xz + yz \leq 48$

(β) If $ax = by = cz = d$, where a, b, c are positive integers and $d = abc = 6$, show that $x^3 + y^3 + z^3 > 108$.