# Sydney Boys High School

# 4 unit mathematics

## Trial HSC Examination 1994

**1.** **(a)** Find $\int \frac{2x}{1+x^4}\ dx$
**(b)** Evaluate
**(i)** $\int_1^2 \frac{dx}{x^2-x+1}$
**(ii)** $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$
**(iii)** $\int_0^1 e^{\sqrt{x}}\ dx$
**(c)** Prove that $\int_a^b f(mx)\ dx = \frac{1}{m}\int_{ma}^{mb} f(x)\ dx \quad (m \neq 0)$

**2.** **(a)** **(i)** If $z = 3 - 4i$, express $z^2$ and $\frac{1}{z}$ in the form $a + ib$ where $a$ and $b$ are real numbers. Represent them on an Argand diagram.
**(ii)** If $w^2 = z$, express the two values of $w$ in the form $a + ib$.
**(b)** Illustrate on the Argand diagram the region $\{z : 0 \leq \arg(z+4) \leq \frac{2\pi}{3} \wedge |z+4| \leq 4\}$
**(c)** Let $z_1, z_2$ and $z_3$ be three complex numbers represented by $Z_1, Z_2$ and $Z_3$ respectively where $z_1 \times z_3 = (z_2)^2$. Show that $OZ_2$ bisects $\angle Z_1OZ_3$.
**(d)** A sequence $z_1, z_2, z_3, \ldots$ satisfy $z_{m+1} = z_n^2 + z_1$ for all $n \geq 1$. If $z_1 = i$, find the distinct values that occur in the sequence.

**3.** **(a)** A hyperbola has equation $9x^2 - 16y^2 = 144$
**(i)** Prove that the eccentricity is $\frac{5}{4}$.
**(ii)** Find the coordinates of the foci.
**(iii)** Find the equations of the asymptotes.
**(iv)** Find the length of the latus rectum. (A latus rectum is a focal chord parallel to the directrix.)
**(v)** Find the equation of the normal to the hyperbola at the point $(\frac{4\sqrt{10}}{3}, 1)$
**(b)** The tangents at the two points $P$ and $Q$ with parameters $\theta$ and $\phi$ respectively on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intersect at the point $T$. Show that
$T = \left(\frac{a(\sin\theta - \sin\phi)}{\sin(\theta-\phi)}, \frac{b(\cos\phi - \cos\theta)}{\sin(\theta-\phi)}\right)$.
**(c)** Find the condition for the line $px + qy + r = 0$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

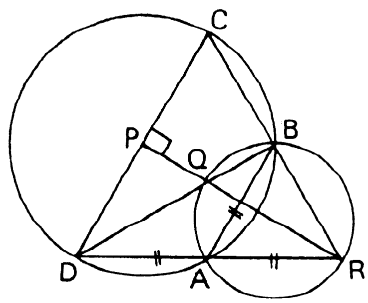
**4.** **(a)** Sketch, on separate diagrams, the curves

**(i)** $y = \frac{x-4}{x}$ **(ii)** $y = \frac{x^2-16}{x^2}$.

(The equation of any asymptotes should be stated, together with the coordinates of any intersections with the axes.) Hence or otherwise sketch the curves

**(iii)** $y = |\frac{x-4}{x}|$ **(iv)** $y^2 = \frac{x^2-16}{x^2}$

**(b)** $AB = AD = AR,\ RP \perp DC.$



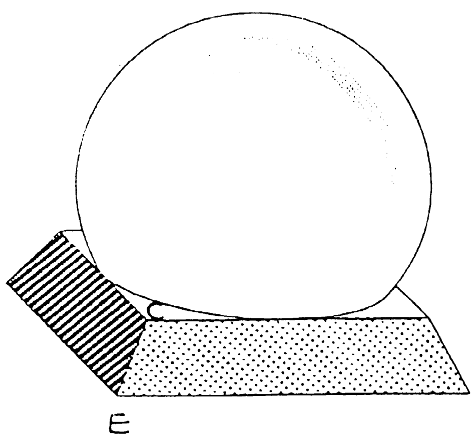

Prove that
**(i)** $BCPQ$ is a cyclic quadrilateral.
**(ii)** $\angle CBD = 90°$
**(iii)** $AB = AP$.

**5.** **(a)** Let $f(x) = (\ln x)^2 + 5$
**(i)** Find the domain of $f$.
**(ii)** Find $f'(x)$
**(iii)** Sketch the graph
**(iv)** Find the maximum domain such that $f(x)$ has an inverse function
**(v)** Find the inverse function $f^{-1}$ of $f$ and state its domain and range
**(vi)** Sketch the graph of $f^{-1}$
**(b)** The sequence of real numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 = 5$ and $u_{n+1} = (u_n + \frac{1}{i_n})^2$ for all $n \geq 1$. Prove by induction that, for every positive integer $n$, $u_n > 2^m$ where $m = 2^n$.

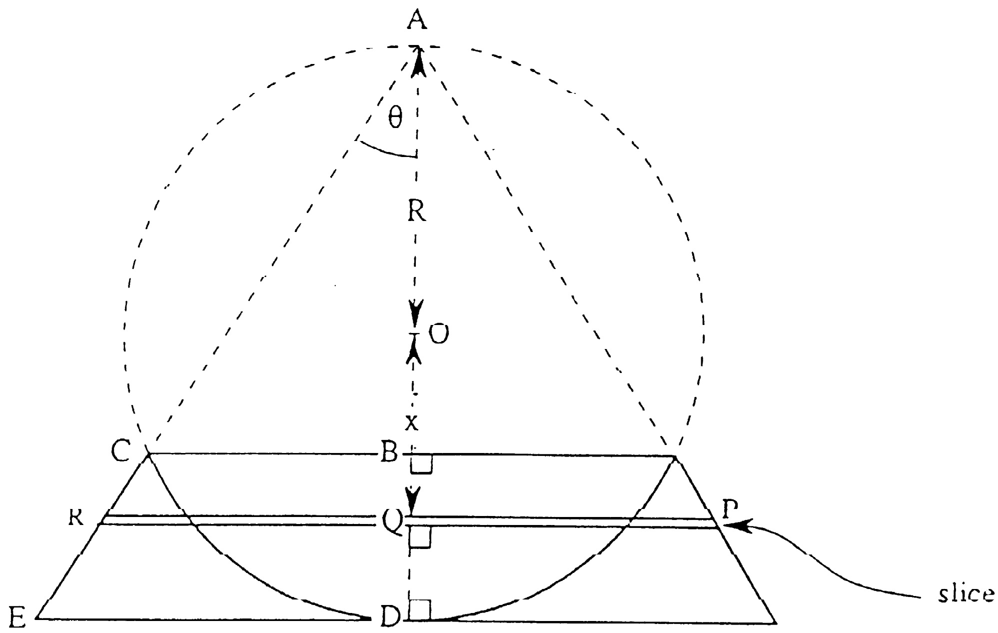
**6.** **(a)** For the equation $x^4 + 2x^3 + 3x^2 + 5x + 1 = 0$
**(i)** Obtain the sum of the squares of the roots of the equation
**(ii)** Show that the equation has two negative roots, $\alpha$ and $\beta$, such that $-2 < \alpha < \beta < 0$
**(iii)** Hence, or otherwise, prove that the equation has no other real roots.
**(b)** The roots of the equation $x^3 + px + q = 0$, $(q \neq 0)$ are $\alpha, \beta$ and $\gamma$
**(i)** Show that $\alpha^{n+3} + p\alpha^{n+1} + q\alpha^n = 0$ where $n$ is a positive integer.
**(ii)** Write down equations involving $\beta$ and $\gamma$ similar to **(i)**.
**(iii)** Deduce that $S_{n+3} + pS_{n+1} + qS_n = 0$, where $S_n = \alpha^n + \beta^n + \gamma^n$
**(iv)** Hence show that $S_3 = -3q$, and find $S_5$ in terms of $p$ and $q$.
**7.** **(a)** Use DeMoivres' theorem to show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. Hence:
**(i)** Solve the equation $16x^4 - 16x^2 + 1 = 0$ and deduce the exact values of $\cos\frac{\pi}{12}$ and $\cos\frac{5\pi}{12}$.

**(b)** **(i)** A particle $P$ is projected, from a point $O$ on the horizontal ground, with speed $V$ at an angle $\theta$ above the horizontal, where $\tan\theta = \frac{1}{3}$. The particle passes through the point with coordinates $(3a, \frac{3a}{4})$ relative to the horizontal and vertical axes at $O$ in the plane of motion. Show that $v^2 = 20ga$.
**(ii)** A particle $Q$ is projected, from a point $O$ at the instant when $P$ is moving horizontally. It strikes the ground at the same place and at the same instant as $P$. Show that the speed of projected of $Q$ is $\sqrt{\frac{145ga}{2}}$ and find the tangent of the angle of projection.

**8.** **(a)**




Mr Keating's crystal ball rests on a solid stand which is in the shape of a square based frustrum as shown.



side view

The stand is constructed such that the crystal ball of radius $R$ fits snugly inside and just touches the centre of the square base. The sides, $EC$, of the base slope so that, if extended, they would pass through the top-most point on the ball at $A$ and make an angle $\theta$ with the vertical $AD$.

**(i)** Show that $OB = R\cos 2\theta$

**(ii)** Consider a slice $PQR$ of thickness $\Delta x$ as shown taken perpendicular to $AD$ such that $OQ = x$ units. Draw a neat sketch of the slice, determine its dimensions and show that it has a volume $\Delta V$ given by $\Delta V \approx [4\tan^2\theta(R+x)^2 - \pi(R^2-x^2)]\Delta x$

**(iii)** Find the volume of such a solid where the angle $\theta = \frac{\pi}{6}$.