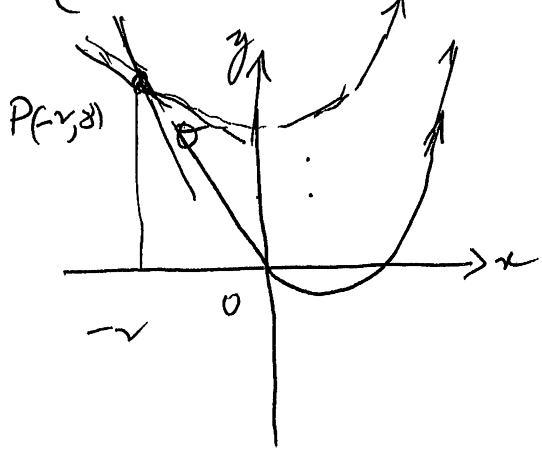
QUESTION 1. (X1)

(a) Find the intersections let $x^2+4 = x^2-2x$.

$$2x = -4$$
$$\underline{x = -2}.$$



now $y = x^2+4$
$y' = 2x$
$\therefore m_1 = -4$

$y = x^2-2x$
$y' = 2x-2$
$m_2 = -6$.

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$
$$= \left|\frac{-4 - -6}{1+ -4\times -6}\right|$$
$$= \left|\frac{-4+6}{1+24}\right|$$
$$= \frac{2}{25}$$
$$\therefore \boxed{\theta = \tan^{-1}\frac{2}{25} = 4^\circ 34'}$$

(b)

$$P = \left(\frac{2\times3 + -1\times-4}{2+-1}, \frac{2\times-1 + -1\times2}{2+-1}\right)$$
$$= \boxed{(10, -4)}$$

(c) $y = \ln(\sin^{-1}x)$

$$y' = \frac{\frac{1}{\sqrt{1-x^2}}}{\sin^{-1}x} = \boxed{\frac{1}{\sin^{-1}x\,\sqrt{1-x^2}}}$$

(d) $\frac{x-1}{x+3} \geqslant -2$

$\frac{x-1}{x+3} + 2 \geqslant 0$

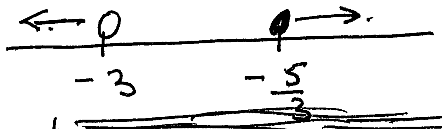
$\frac{x-1+2(x+3)}{x+3} \geqslant 0$

$\frac{x-1+2x+6}{x+3} \geqslant 0$

$\frac{3x+5}{x+3} \geqslant 0$

$\frac{(3x+5)}{(x+3)} \times (x+3)^2 \geqslant 0$

$(3x+5)(x+3) \geqslant 0$



$\therefore \boxed{x < -3, \quad x \geqslant -\frac{5}{3}}$

(e)

$\cos 2B = 1 - 2\sin^2 B$

$= 1 - 2 \times \left(\frac{1}{3}\right)^2$

$= 1 - \frac{2}{9}$

$= \frac{7}{9}$

$\therefore \cos 2B = \frac{7}{9} = \cos A$

$\therefore \boxed{A = 2B}$

[NB doing this question on a calculator is not a proof]

(f) $u = t+1$

$du = dt$

$\therefore \int_0^1 \frac{t}{\sqrt{1+t}}\,dt = \int_1^2 \frac{u-1}{\sqrt{u}}\,du$

$= \int_1^2 \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du$

$= \left[\frac{2}{3}u^{3/2} - 2u^{\frac{1}{2}}\right]_1^2 = \frac{2}{3} \cdot 2^{3/2} - 2 \cdot 2^{\frac{1}{2}} - \left(\frac{2}{3} - 2\right)$

$= \frac{4\sqrt{2}}{3} - 2\sqrt{2} - \frac{-4}{3}$

$= \frac{4}{3} - \frac{2\sqrt{2}}{3}$

$= \boxed{\frac{4-2\sqrt{2}}{3}}$

Question 2

a) $P(x) = ax^3 + bx^2 \mp 8x + 3$

$P(1) = 0$

$\therefore 0 = a + b - 8 + 3$

$\boxed{a + b = 5 \quad ①}$

$P(-2) = 15$

$15 = -8a + 4b + 16 + 3$

$\boxed{8a - 4b = 4 \quad ②}$

① × 4

$4a + 4b = 20$ ③

② + ③

$12a = 24$

$\boxed{a = 2}$ 1 mark

sub into ①

$2 + b = 5$

$\boxed{b = 3}$ 1 mark

$\therefore P(x) = 2x^3 + 3x^2 - 8x + 3$

$$\begin{array}{r}
2x^2 + 5x - 3 \\
x - 1 \overline{) 2x^3 + 3x^2 - 8x + 3} \\
\underline{2x^3 - 2x^2 } \\
5x^2 - 8x + 3 \\
\underline{5x^2 \mp 5x } \\
-3x + 3 \\
\underline{-3x + 3} \\
\underline{\underline{0}}
\end{array}$$

$$\overset{-6}{2x^2 + 5x - 3}$$

$$= \frac{(2x+6)(2x-1)}{2}$$

$$= \frac{\cancel{2}(x+3)(2x-1)}{\cancel{2}}$$

$\therefore P(x) = (x-1)(x+3)(2x-1)$

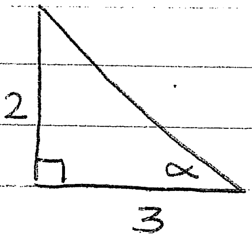
1 mark

b) i $3\sin\theta + 2\cos\theta$

$= R\sin(\theta + \alpha)$

$R = \sqrt{3^2 + 2^2}$

$= \sqrt{13}$ 1 mark



$\tan\alpha = \frac{2}{3}$

$\alpha = \tan^{-1} 2/3$

$= 33°41'$

1 mark

ii $\sqrt{13}\sin(\theta + 33°41') = \frac{5}{2}$

$\sin(\theta + 33°41') = \frac{5}{2\sqrt{13}}$

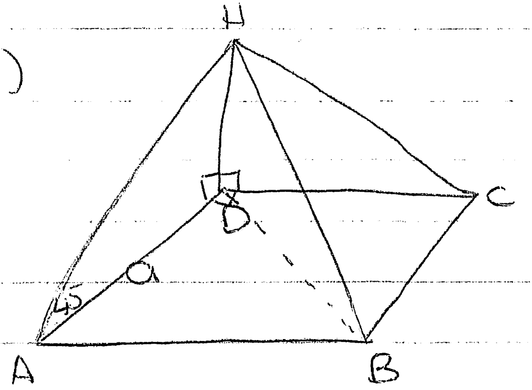
$\theta + 33°41' = \sin^{-1}\frac{5}{2\sqrt{13}}$

$\theta = \sin^{-1}\frac{5}{2\sqrt{3}} - 33°41'$

$\theta = 10°13', 102°25'$

1 mark each

c)



i) $\angle AHD = 45^\circ$

$\therefore \triangle AHD$ is isosceles

$\therefore HD = a$

In $\triangle$'s $HCD + DBA$

$HD = a = DA$

$\angle HDC = 90^\circ = \angle DAB$

(given & properties of a rectangle)

$DC = AB$ (opposite sides of a rectangle)

$\therefore \triangle HCD \equiv \triangle DBA$ (SAS)

$\therefore \angle DBA = 30^\circ$

1mark

In $\triangle DAB$

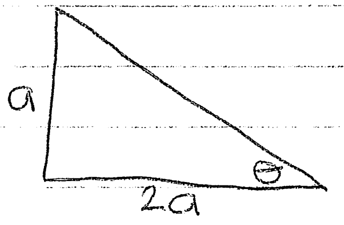
$$\sin 30 = \frac{a}{BD}$$

$$\frac{1}{2} = a/BD$$

$$\therefore BD = 2a$$

1mark

ii) $\triangle HDB$



$$\tan \theta = a/2a$$

$$\tan \theta = \frac{1}{2}$$

$$\theta = 26^\circ 33' 54.18''$$

$$= 26^\circ 34' \text{ (nearest min)}$$

1mark

d) $a_1 = a - \dfrac{f(a)}{f'(a)}$

$f(x) = 2x + \cos x$

$f'(x) = 2 - \sin x$

$a = -\pi/6$

$f(a) = \dfrac{-\pi}{3} + \dfrac{\sqrt{3}}{2}$

$f'(a) = 2 - - \frac{1}{2}$

$= 5/2$

1 mark

$$a_1 = -\pi/6 - \left[\frac{-2\pi + 3\sqrt{3}}{6}\right] \div \frac{5}{2}$$

$$= \frac{-\pi}{6} - \left[\frac{-4\pi + 6\sqrt{3}}{30}\right]$$

$$= \frac{-5\pi}{30} + \frac{4\pi - 6\sqrt{3}}{30}$$

$$= \frac{-\pi - 6\sqrt{3}}{30}$$ 1mark

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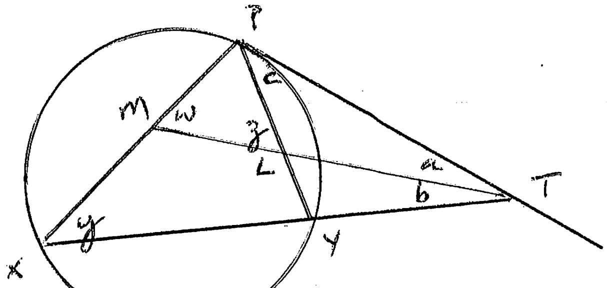
Question 3:



$a = b$ (TM bisects $\angle PTX$, given)

$c = y$ (alternate segment theorem)

$z = a + c$ (exterior $\angle$ of ΔPLT)

$= b + y$

$w = b + y$ (exterior $\angle$ of Δ MTX)

$\therefore z = w$

$\therefore \Delta PLM$ is isosceles (base angles equal) ③

(b) (i) $f^{-1}(x) = \sin^{-1}(3x - 1)$

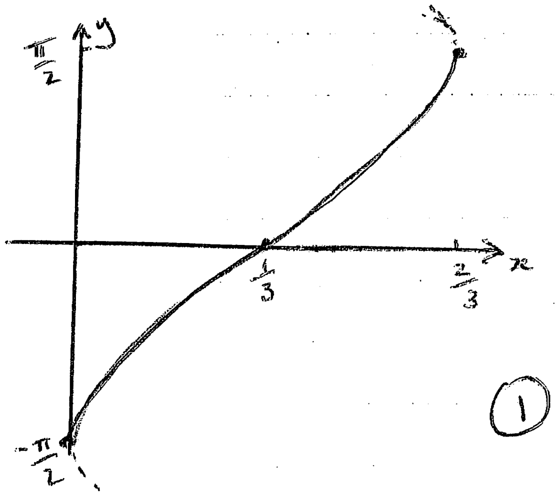
Domain: $-1 \le 3x - 1 \le 1$

$0 \le 3x \le 2$

$0 \le x \le \frac{2}{3}$ ②

Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(ii)



①

(iii) $x = \sin^{-1}(3y - 1)$

$\sin x = 3y - 1$

$3y = \sin x + 1$

$y = \frac{1}{3}(\sin x + 1)$

$f(x) = \frac{1}{3}(\sin x + 1)$

Domain: $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Range: $0 \le y \le \frac{2}{3}$ ③

(c) (i) $\frac{dT}{dt} = \frac{d}{dt}(T_0 + Ae^{-kt})$

$= Ae^{-kt} \times -k$

$= (T - T_0) \times -k$

$= -k(T - T_0)$ ①

(ii) When $t = 0$: $T = 85$

$\therefore 85 = 25 + A$

$\therefore A = 60$

$\therefore T = 25 + 60e^{-kt}$

When $t = 1$: $80 = 25 + 60e^{-k}$

$\therefore 55 = 60e^{-k}$

$\therefore e^{-k} = \frac{55}{60}$

$k = -\ln\left(\frac{55}{60}\right)$

When $t = 5$: $T = 25 + 60e^{-5k}$

$= 63.8336\ldots$

$\approx 64°$ ②

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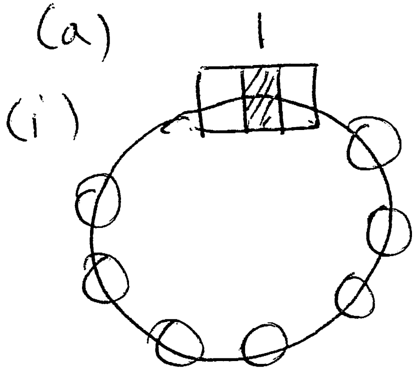
Solution to Q(4)

(a)

(i)



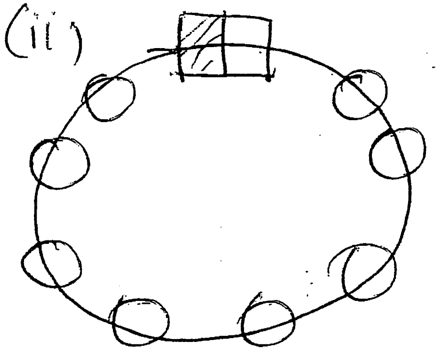
$\boxed{1}$

$7! \times 1 \times 2!$

the rest → $7!$; particular girl → 1 ; 2 boys → $2!$

$= \boxed{2 \times 7!}$
$= \boxed{10080}$

(ii)



Sitting together:

$= 8! \times 2!$

∴ Separate to $= \boxed{9! - 8! \times 2!}$

$= \boxed{282240}$ $\boxed{1}$

(b) $y = x^2/4a$

(i) $\frac{dy}{dx} = \frac{x}{2a}$ $\boxed{2}$

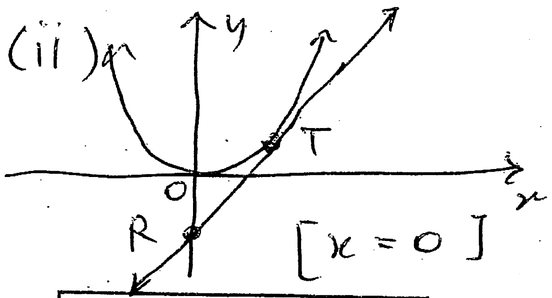
$\left.\frac{dy}{dx}\right|_{x=2at} = \frac{2at}{2a} = t$ ✓

∴ Equ. of tgt

$y - at^2 = t(x - 2at)$

$\boxed{y = tx - at^2}$ ✓

(ii)



$[x = 0]$

$\boxed{R(0, -at^2)}$ $\boxed{1}$

If PQ passes through R. then Coordinates of R satisfy equation of PQ

i.e

$-at^2 + apq = 0$ ✓

$t^2 = pq$.

i.e $\boxed{\frac{t}{p} = \frac{q}{t}}$ ✓ $\boxed{2.}$

∴ p, t, q are terms of a geometric series

(c) $x = 2\cos 3t$, $\dot{x} = -6\sin 3t$ ✓

$\ddot{x} = -9(2\cos 3t) = -9x$ ✓

(i) ∴ $n^2 = 9$.

(ii) $T = \frac{2\pi}{n} \boxed{= \frac{2\pi}{3}}$ $\boxed{1}$

(iii) $\cos 3t = 1/2$ ✓, $3t = \frac{\pi}{3}, \frac{5\pi}{3}$

$\Rightarrow t = \frac{\pi}{9}, \frac{5\pi}{9}$ ✓

∴ $\dot{x} = -6\sin\frac{\pi}{3}$, ✓ $6\sin\frac{5\pi}{3}$

$= \boxed{\pm 3\sqrt{3} \text{ cm/sec}}$.

$\doteqdot \pm 5.196$ cm/se $\boxed{2}$

QUESTION 5.

(a) $\frac{\sin\theta}{\cos\theta} = 2\sin\theta\cos\theta$

$\sin\theta = 2\sin\theta\cos^2\theta.$

$2\sin\theta(1-\sin^2\theta) - \sin\theta = 0.$

$2\sin^3\theta - \sin\theta = 0.$

$\sin\theta(2\sin^2\theta - 1) = 0.$

$\sin\theta = 0$ $\qquad$ $\sin\theta = \pm\frac{1}{\sqrt{2}}$

$\theta = \pi n \quad n \in \mathbb{Z}$ $\qquad$ $\theta = \pi n \pm \frac{\pi}{4} \quad n \in \mathbb{Z}$

(b) (i) $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{1}{2}.$

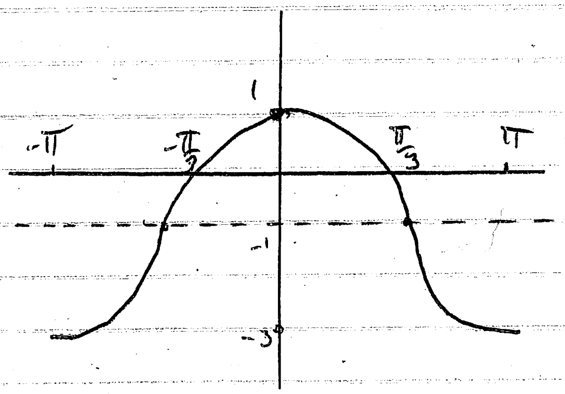
(ii) $\alpha\beta\gamma = \frac{1}{2}.$

$\frac{1}{\cos a} + \frac{1}{\cos b} + \frac{1}{\cos c} = \frac{\cos a \cos b + \cos a \cos c + \cos b \cos c}{\cos a \cos b \cos c.}$

$= \frac{\frac{1}{2}}{\frac{1}{2}}$

$= 1$

(c) (i)



$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$x = \pm\frac{\pi}{3}.$

(iii)

$$V = \pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 4\cos^2 x - 4\cos x + 1 \, dx.$$

$$= \pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2\cos 2x + 2 - 4\cos x + 1 \, dx.$$

$$= \pi \left[\sin 2x - 4\sin x + 3x \right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}.$$

$$= 2\pi \left(\sin\frac{2\pi}{3} - 4\sin\frac{\pi}{3} + \pi \right).$$

$$= 2\pi^2 - 3\pi\sqrt{3}. \text{ units}^3.$$

Question 6

a) i) $\frac{d}{dy}(\ln \cos y) = \frac{-\sin y}{\cos y}$

$= \underline{\underline{-\tan y}}$

ii) $A = \int_0^{\frac{\pi}{4}} x \, dy$

$= \int_0^{\frac{\pi}{4}} \tan y \, dy$

$= -\int_0^{\frac{\pi}{4}} -\tan y \, dy$

$= -\left[\ln(\cos y)\right]_0^{\frac{\pi}{4}}$ using (i)

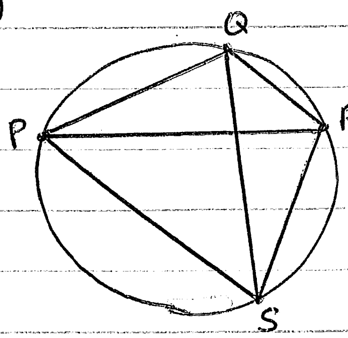
$= -\left(\ln\left(\cos\frac{\pi}{4}\right) - \ln(\cos 0)\right)$

$= -\ln\left(\frac{1}{\sqrt{2}}\right) + \cancel{\ln 1}$

$= -\ln(2)^{-\frac{1}{2}}$

$= \frac{1}{2}\ln 2$ units^2

b)



In $\triangle PQR$

$\frac{\sin P\hat{Q}R}{PR} = \frac{\sin Q\hat{R}P}{PQ}$

In $\triangle QPS$

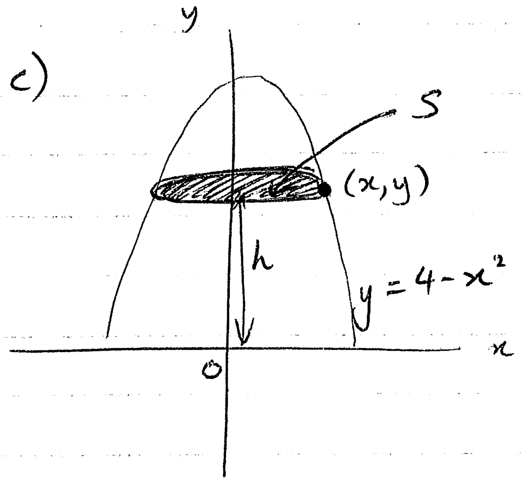
$\frac{\sin Q\hat{P}S}{QS} = \frac{\sin P\hat{S}Q}{PQ}$

$Q\hat{R}P = P\hat{S}Q$ (angles in same segment (arc PQ))

$\therefore \frac{\sin P\hat{Q}R}{PR} = \frac{\sin Q\hat{P}S}{QS}$

$\frac{\sin P\hat{Q}R}{\sin Q\hat{P}S} = \frac{PR}{QS}$

c)



i) $S = \pi r^2$

$S = \pi x^2$

when $y = h$

$h = 4 - x^2$

$x^2 = 4 - h$

$\therefore \; S = \pi(4-h)$

ii) $S = 4\pi - \pi h$

$\frac{dS}{dh} = -\pi$

$\frac{dS}{dt} = \frac{dS}{dh} \times \frac{dh}{dt}$

$= -\pi \times \frac{10}{\pi(4-h)}$

$= \frac{-10}{4-h}$

when $h = 2$

$\frac{dS}{dt} = \frac{-10}{(4-2)}$

$= \underline{\underline{-5}}$ cm^2/s

QUESTION 7

(a) $x = 20t\cos\alpha$

$y = -5t^2 + 20t\sin\alpha$

(i)

$$\Rightarrow y = -5\left(\frac{x}{20\cos\alpha}\right)^2 + 20\left(\frac{x}{20\cos\alpha}\right)\sin\alpha$$

$$y = -\frac{1}{80}x^2\sec^2\alpha + x\tan\alpha$$

ie $y = -\frac{1}{80}(\tan^2\alpha + 1)x^2 + (\tan\alpha)x$

(ii) When $x = 20$, $y = h$

$$\Rightarrow h = -\frac{1}{80}(\tan^2\alpha + 1)400 + 20\tan\alpha$$

ie $h = -5\tan^2\alpha + 20\tan\alpha - 5$

$$\boxed{h = 20\tan\alpha - 5(1 + \tan^2\alpha)}$$

(iii)

$$h = -5\tan^2\alpha + 20\tan\alpha - 5$$

Max. value of h occurs when $\tan\alpha = \frac{-20}{2(-5)} = 2$

ie $\boxed{\tan\alpha = 2}$

Max height is

$$-5(2)^2 + 20(2) - 5 = \boxed{15 \text{ metres}}$$

(b) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9x + 5(x-2)^{-2}$

$$\frac{1}{2}v^2 = \frac{-9x^2}{2} + \frac{5}{2-x} + C$$

$\left.\begin{matrix} x=0 \\ v=0 \end{matrix}\right\} \Rightarrow C = -\frac{5}{2}$

$$\therefore v^2 = -9x^2 + \frac{5}{2-x} - 5$$

(b)(ctd) $v^2 = -9x^2 + \frac{10}{2-x} - 5$

For motion to exist then

$v^2 \geq 0$

ie $-9x^2 + \frac{10}{2-x} - 5 \geq 0$

$$-9x^2(2-x)^2 + 10(2-x) - 5(2-x)^2 \geq 0$$

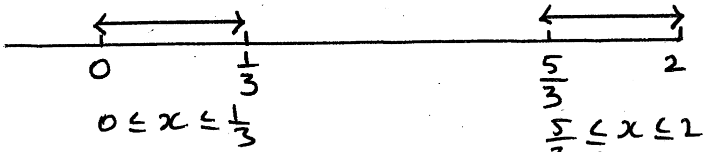
$$(2-x)\left[-9x^2(2-x) + 10 - 5(2-x)\right] \geq 0$$

ie $(2-x)(-18x^2 + 9x^3 + 5x) \geq 0$

ie $(2-x).x(9x^2 - 18x + 5) \geq 0$

$$x(2-x)(3x-5)(3x-1) \geq 0$$

SOLUTION



However since particle starts at zero and changes direction at $x = \frac{1}{3}$ it can never be outside the interval $0 \leq x \leq \frac{1}{3}$.

Note For $\frac{1}{3} < x < \frac{5}{3}$ $v^2 < 0$

∴ impossible to move in this interval and therefore cannot move in $\frac{5}{3} \leq x \leq 2$.

Ultimately moves in interval

$$0 \leq x \leq \frac{1}{3}$$

⑦ (c) (i)

$$\text{RHS} = \tan(\alpha+\beta)\left[1-\tan\alpha\tan\beta\right]$$

$$= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\left[1-\tan\alpha\tan\beta\right]$$

$$= \tan\alpha+\tan\beta$$

$$= \text{LHS}$$

(ii) When $n=1$ $\quad \tan\theta.\tan 2\theta = \tan 2\theta\cot\theta - 2$

$$\text{RHS} = \frac{2\tan\theta}{1-\tan^2\theta}\cdot\frac{1}{\tan\theta} - 2 = \frac{2\tan^2\theta}{1-\tan^2\theta} = \tan\theta.\frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \tan\theta.\tan 2\theta = \text{LHS}$$

- Assume $\quad \tan\theta\tan2\theta + \dots + \tan k\theta\tan(k+1)\theta = \tan(k+1)\theta\cot\theta - (k+1)$

RTP $\quad \tan\theta\tan2\theta + \dots + \tan k\theta + \tan(k+1)\theta + \tan(k+1)\theta\tan(k+2)\theta = \tan(k+2)\theta\cot\theta - (k+2)$

Now $\quad \text{LHS} = \tan(k+1)\theta.\cot\theta - (k+1) \quad + \tan(k+1)\theta\tan(k+2)\theta$

$$= \cot\theta\left[\tan(k+1)\theta + \tan(k+1)\theta.\tan(k+2)\theta.\tan\theta\right] - (k+1)$$

$$= \cot\theta\left[\tan(k+1)\theta + \tan(k+2)\theta\left(1 - \frac{\tan(k+1)\theta+\tan\theta}{\tan(k+2)\theta}\right] - (k+1) \xleftarrow{\text{using(i)}}$$

$$= \cot\theta\left[\tan(k+1)\theta + \tan(k+2)\theta - \tan(k+1)\theta - \tan\theta\right] - (k+1)$$

$$= \cot\theta\left[\tan(k+2)\theta - \tan\theta\right] - (k+1)$$

$$= \cot\theta\left[\tan(k+2)\theta\right] - 1 - (k+1)$$

$$= \cot\theta\left[\tan(k+2)\theta\right] - (k+2) \quad = \text{RHS}$$