S.G.H.S 3U Soln's 2000

Q1a) $\displaystyle\int_0^{0.4} \frac{3\,dx}{4+25x^2} = \frac{3}{2.5}\left[\tan^{-1}\left(\frac{5x}{2}\right)\right]_0^{2/5}$

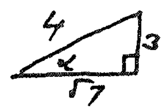
$= \frac{3}{10}\left(\tan^{-1}(1) - \tan^{-1}(0)\right)$

$= \frac{3}{10}\cdot\frac{\pi}{4}$

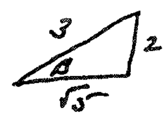
$= \frac{3\pi}{40}$

b) $9\text{ fb},\ 7\text{ n}$, with $9 \downarrow 5$ and $7 \downarrow 3$ $\Rightarrow {}^9C_5 \times {}^7C_3 = 4410$

c) $\sin\alpha = 3/4 \quad 0 < \alpha < \pi/2$



$\sin\beta = 2/3 \quad \pi/2 < \beta < \pi$



i) $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \dfrac{2\times 3}{\sqrt7} \times \dfrac{1}{1-9/7}$

$= \frac{6}{\sqrt7} \times \frac{7}{-2}$

$= -3\sqrt7$

ii) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \frac{\sqrt7}{4}\cdot -\frac{\sqrt5}{3} + \frac{3}{4}\cdot\frac{2}{3}$

$= \frac{1}{2} - \frac{\sqrt{35}}{12}$

d) $2\ln(3x+1) - \ln(x+1) = \ln(7x+4)$

$\therefore (3x+1)^2 = (7x+4)(x+1)$

$9x^2 + 6x + 1 = 7x^2 + 11x + 4$

$\therefore 2x^2 - 5x - 3 = 0$

$(2x+1)(x-3) = 0$

$\therefore x = -1/2,\ 3$

at $x = -\frac{1}{2}$, $\ln(3x+1)$ is undefined

$\therefore x = 3$

Q2 a) $\int_{-1}^{2} x\sqrt{2-x}\,dx$

$u = 2-x$
$du = -dx$
$x = 2-u$

$x=-1,\ u=3$
$x=2,\ u=0$

$= \int_3^0 (2-u)\sqrt{u}\ -du$

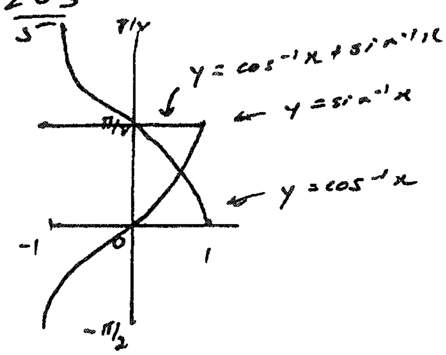
$= \int_0^3 (2u^{1/2} - u^{3/2})\,du$

$= \left[\frac{4}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^3$

$= \frac{4}{3}\cdot 3\sqrt{3} - \frac{2}{5}\cdot 9\sqrt{3}$

$= \frac{2\sqrt{3}}{5}$

b) i)



at $\sin^{-1}x = \cos^{-1}x$

$\therefore x = 1/\sqrt{2}$

c) $\frac{2}{3-x} \geq x$

$\therefore \frac{2(3-x)^2}{3-x} \geq x(3-x)^2 \quad , x \neq 3.$

$\therefore 2(3-x) \geq x(3-x)^2$

$\therefore x(3-x)^2 - 2(3-x) \leq 0$

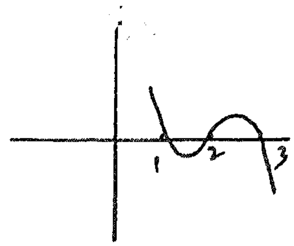
$(3-x)\{x(3-x) - 2\} \leq 0$

$(3-x)(3x - x^2 - 2) \leq 0$

$(3-x)(x^2 - 3x + 2) \cancel{\geq} 0$

$(3-x)(x-1)(x-2) \cancel{\geq} 0 \quad , x \neq 3$

$\therefore$



$\therefore x \leq 1, \quad 2 \leq x < 3$

3 a i) Let $f(x) = \log_e x - \cos x$

$f(1) = \ln 1 - \cos 1 \doteqdot -0.54 < 0$

$f(2) = \ln 2 - \cos 2 \doteqdot 1.11 > 0$

$\therefore$ since $f(x)$ changes sign, there is a root $1 < x < 2$.

ii $f'(x) = \frac{1}{x} + \sin x$

$f(1.2) \doteqdot -0.18$

$f'(1.2) \doteqdot 1.765$

$$x_0 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\doteqdot 1.2 + \frac{0.18}{1.765}$$

$$\doteqdot 1.30$$

b) Prove $\frac{1}{1\times2} + \frac{1}{2\times3} + \dots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$

at $n=1$, LHS $= \frac{1}{1\times2} = \frac{1}{2}$, RHS $= 1 - \frac{1}{1+1} = \frac{1}{2} =$ LHS

$\therefore$ true for $n=1$

assume true for $n=k$, i.e. assume $\frac{1}{1\times2} + \dots + \frac{1}{k(k+1)} = 1 - \frac{1}{k+1}$

& prove for $n=k+1$, i.e. prove $\frac{1}{1\times2} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = 1 - \frac{1}{k+2}$

Now LHS $= \frac{1}{1\times2} + \frac{1}{2\times3} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

$= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}$ using assumption

$= 1 - \frac{k+2-1}{(k+1)(k+2)}$

$= 1 - \frac{1}{k+2} =$ RHS

$\therefore$ if true for $n=k$ it is true for $n=k+1$.

since true for $n=1$ it is thus true for $n=2$ then $n=3$ & so on for all positive integral n.

c) $(3+2x)^{11}$ $\quad T_{k+1} = {}^nC_k\, a^{n-k} b^k = {}^{11}C_k\, 3^{11-k}(2x)^k$

$T_k = {}^nC_{k-1}\, a^{n-1+k} b^{k-1} = {}^{11}C_{k-1}\, 3^{12-k}(2x)^{k-1}$

$$\therefore \frac{T_{k+1}}{T_k} = \frac{11!}{(11-k)!\,k!}\, 3^{11-k}(2x)^k \, \frac{(12-k)!(k-1)!}{11!} \times \frac{1}{3^{12-k}(2x)^{k-1}}$$

$$= \frac{12-k}{k} \cdot \frac{2x}{3}$$

For greatest coeff $\frac{T_{k+1}}{T_k} > 1$ $\therefore$

$2(12-k) > 3k$

$24 - 2k > 3k$

$5k < 24$

$\therefore k = 4$

Question 4

a) Given $\frac{dV}{dt} = 2\pi\ mm^3/min$ find $\frac{dA}{dt}$ when $r = 3$

3

$V = \frac{4}{3}\pi r^3$, $A = 4\pi r^2$

$\frac{dV}{dr} = 4\pi r^2$ $\qquad \frac{dA}{dr} = 8\pi r$

$\frac{dA}{dt} = \frac{dV}{dt} \times \frac{dr}{dV} \times \frac{dA}{dr}$

$= 2\pi \times \frac{1}{4\pi r^2} \times 8\pi r$

$= \frac{4\pi}{r}$ when $r = 3$

$\frac{dA}{dt} = \frac{4\pi}{3}\ mm^2/min$

b) i) Coords of Q $\quad x = \frac{2(2)+1(-4)}{3}$, $y = \frac{2(4)+1(1)}{3}$

2

$x = 0 \qquad y = 3$

Coords of R $\quad x = \frac{2(2)-1(-4)}{2-1} \qquad y = \frac{2(4)-1(1)}{2-1}$

$x = 8 \qquad y = 7$

ii) $PA = 2PB$

2

$\sqrt{(x+4)^2+(y-1)^2} = 2\sqrt{(x-2)^2+(y-4)^2}$

$x^2+8x+16+y^2-2y+1 = 4[x^2-4x+4+y^2-8y+16]$

$3x^2+3y^2-24x-30y+63=0$

$x^2-8x+y^2-10y = -21$

$(x-4)^2+(y-5)^2 = -21+16+25$

$(x-4)^2+(y-5)^2 = 20$ ie circle centre $(4,5), r=\sqrt{20}$

1

Midpt QR $\quad x = \frac{0+8}{2} \qquad y = \frac{3+7}{2}$

$= 4 \qquad = 5$ ie centre circle

Radius QR $\quad r = \sqrt{(4-0)^2+(5-3)^2}$

$= \sqrt{20}$

c) i) $T = P + Ae^{-kt} \Rightarrow T-P = Ae^{-kt}$

1

$\frac{dT}{dt} = -k\,Ae^{-kt}$

$= -k(T-P)$

ii) initially $t=0$, $T=1340$, $P=25$

3

$1340 = 25 + A$, $A = 1315$

$T = 25 + 1315e^{-kt}$

when $t = 12 \quad T = 1010$

$1010 = 25 + 1315e^{-12k}$

$e^{-12k} = \frac{197}{263}$

$k = \left[\log_e\left(\frac{197}{263}\right)\right] \div -12 \qquad (k = 0.024\ldots)$

when $T = 60$

$60 = 25 + 1315e^{-kt}$

$t = \log_e\left(\frac{7}{263}\right) \div k$

Question Five

a) i) R.T.P $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

Now $a = \frac{dx}{dt}$

$= \frac{dv}{dx} \cdot \frac{dx}{dt}$ (chain rule)

2 $= v\frac{dv}{dx}$ $\left(v = \frac{dx}{dt}\right)$

~~$= \frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx}$~~

$= \underline{\frac{d}{dx}\left(\frac{1}{2}v^2\right)}$

ii) a) $\ddot{x} = -2e^{-x}$

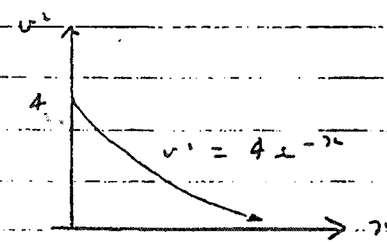
$\therefore \frac{1}{2}v^2 = \int -2e^{-x}\,dx$

2 $\frac{1}{2}v^2 = 2e^{-x} + c$

initially $x = 0$, $v = 2$, $c = 0$

$\therefore \underline{v^2 = 4e^{-x}}$

2 b)



velocity in positive direction and decreasing (2)

b) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots \binom{n}{n}x^n$

1 i) put $x = 2$

$\underline{3^n} = \binom{n}{0} + 2\binom{n}{1} + 4\binom{n}{2} + \ldots 2^n\binom{n}{n}$

ii) d.w.r.t. x

2 $n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + \ldots n\binom{n}{n}x^{n-1}$

put $x = -1$

$\underline{0} = \binom{n}{1} - 2\binom{n}{2} + \ldots (-1)^{n-1}n\binom{n}{n}$

c) $2\cos\theta + \sin\theta = A\cos(\theta - \alpha) = \sqrt{5}$

$2\cos\theta + \sin\theta = A[\cos\theta\cos\alpha - \sin\theta\sin\alpha] = \sqrt{5}$

divide by $A = \sqrt{5}$

$\frac{2}{\sqrt{5}}\cos\theta + \frac{1}{\sqrt{5}}\sin\theta = \cos\theta\cos\alpha - \sin\theta\sin\alpha = 1$

3 ie $\left.\begin{array}{l}\cos\alpha = \frac{2}{\sqrt{5}} \\ \sin\alpha = \frac{1}{\sqrt{5}}\end{array}\right\}$ α acute

$\tan\alpha = \frac{1}{2}$, $\alpha = 0.46\ldots$

Now $\cos(\theta - \alpha) = 1$ $\therefore$ $\theta - \alpha = 0$ 2π

Q6 3U 2000 (Trial HSC)

a) i)

$$\begin{array}{r} x^2 - x - 3 \\ x^2+4\,\overline{)\,x^4 - x^3 + x^2 - x + 1} \\ \underline{x^4 \qquad + 4x^2} \\ -x^3 - 3x^2 - x \\ \underline{-x^3 \qquad - 4x} \\ -3x^2 + 3x + 1 \\ \underline{-3x^2 \qquad -12} \\ 3x + 13 \end{array}$$

or

$$\begin{array}{r} x^2 - x - 3 \\ x^2+4\,\overline{)\,x^4 - x^3 + x^2 + ax + b} \\ \underline{x^4 \qquad + 4x^2} \\ -x^3 - 3x^2 + ax \\ \underline{-x^3 \qquad - 4x} \\ -3x^2 + x(a+4) + \\ \underline{-3x^2 \qquad -12} \\ x(a+4) + b + \end{array}$$

Remainder $= 0$

$\therefore a + 4 = 0 \quad a = -4$

$b + 12 = 0 \quad b = -$

$f(x) = (x^2+4)(x^2-x-3) + 3x + 13$

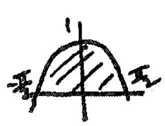
so,

$f(x) - 3x - 13 = (x^2+4)(x^2-x-3)$

$x^4 - x^3 + x^2 - x + 1 - 3x - 13$

$= x^4 - x^3 - x^2 - 4x - 12 \qquad \therefore \; a = -4, \; b = -12$

b)



$V = \pi \int y^2 \, dx$

$y = \cos x$

$y = \cos^2 x$

$= \frac{1}{2}(\cos 2x + 1)$

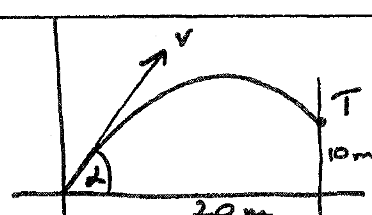
$= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(\cos 2x + 1)\, dx$

$= \frac{\pi}{2} \times 2 \int_0^{\frac{\pi}{2}} \cos 2x + 1 \quad dx$

$= \pi \left[\frac{\sin 2x}{2} + x \right]_0^{\frac{\pi}{2}} = \pi \left[0 + \frac{\pi}{2} - \right.$

$\therefore \text{Vol} = \frac{\pi^2}{2} \, u^3$

c)



$t = 0, \; x = 0, \; y = 0$

$V = 20$

$\dot{x} = 20\cos\alpha$

$\dot{y} = 20\sin\alpha$

i) $\ddot{x} = 0$

$\dot{x} = c_1 \quad (t = 0, \dot{x} = 20\cos\alpha)$

$\dot{x} = 20\cos\alpha$

$x = 20t\cos\alpha + c_2$

$x = 0 \; t = 0 \; \therefore c_2 = 0$

$\therefore x = 20t\cos\alpha$

$\ddot{y} = -g$

$\dot{y} = -gt + c_3$

$20\sin\alpha = c_3$

$\therefore \dot{y} = -gt + 20\sin\alpha$

$y = -\frac{1}{2}gt^2 + 20t\sin\alpha$

$c_4 = 0 \quad (\text{when } t = 0, y$

$\therefore y = -\frac{1}{2}gt^2 + 20t\sin$

$= -5t^2 + 20t\sin$

ii) when $x = 20, \; y = 10$

ie $\quad 20 = 20t\cos\alpha \quad \Rightarrow t = \frac{1}{\cos\alpha}$

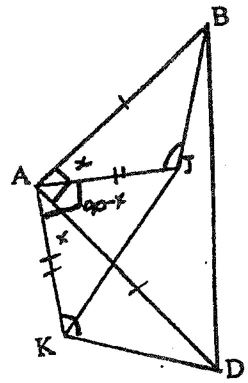
and $\quad 10 = -5t^2 + 20t\sin\alpha$

$\tan^2\alpha - 4\tan\alpha + 3 = 0$

$(\tan\alpha - 3)(\tan\alpha - 1) = 0$

Q7. 3U 2000 (Trial HSC)

i)



$AB = AD$ let $\angle BAJ = x$

$AJ = AK$ $\therefore \angle JAD = 90 - x$ (adj. compl. $\angle$s $\angle BAD = 90°$)

also $\angle KAD = x$ (adj comp $\angle$s $\angle JAK = 90°$)

Now in $\triangle$s BAJ and DAK

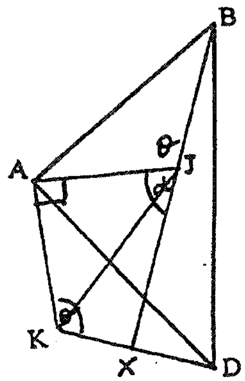
$AB = AD$ (equal sides of isosc $\triangle$ BAD)

$AJ = AK$ (" " " isosc $\triangle$AJK)

$\angle BAJ = \angle KAD$ (proven above)

$\therefore \triangle BAJ \equiv \triangle DAK$ (SAS)

Hence, $\angle BJA = \angle DKA$ (corresp $\angle$s of congr. $\triangle$s)



ii) $\angle AJB + \angle AJX = 180°$ (adj suppl. $\angle$s)

$\therefore \angle AJX = 180 - \angle BJA$

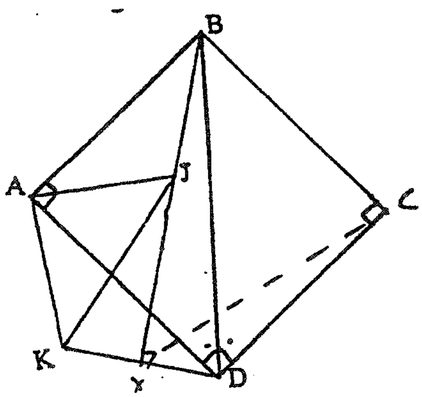
Now, $\angle JAK + \angle AKX + \angle KXJ + \angle AJX = 360°$ ($\angle$ sum of quad)

$90° + \angle AKX + \angle KXJ + 180 - \angle BJA = 360$

($\angle$KX = $\angle$DKA = $\angle$BJA) ie $\angle BJA + \angle KXJ - \angle BJA = 90°$

$\therefore \angle KXJ = 90°$ $\therefore JX \perp KD$

iii)



since $\angle BCD = 90°$ and $\angle BXD = 90°$

then BCDX are concyclic

with BD a diameter.

Now, BD bisects $\angle ADC$

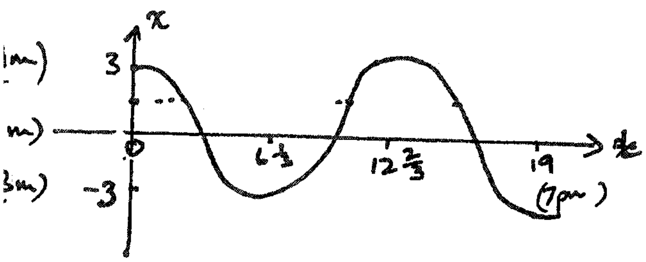
(diagonal of square)

$\therefore \angle BDC = 45°$

and $\angle BDC = \angle BXC$ ($\angle$s on same arc)

$\therefore \angle BXC = 45°$

7(b) high tide $= 9m$ at 4am
low tide $= 3m$ at 10.20 am



let 6m be equilibrium (ie $x=0$)

$\therefore$ High tide $x = 3$
low tide $x = -3$

let $t = 0$ be at 4am

$\therefore$ $t = 6\frac{1}{3}$ is at 10.20am

$\therefore$ period $= 12\frac{2}{3} \Rightarrow n = \frac{3\pi}{19}$

amplitude $= 3$

$$x = 3\cos nt$$
$$= 3\cos \frac{3\pi}{19} t$$

$x = 1\cdot 5$ (ie 7.5m deep)

$$1\cdot 5 = 3\cos \frac{3\pi t}{19}$$
$$\frac{1}{2} = \cos \frac{3\pi t}{19}$$
$$\therefore \frac{3\pi t}{19} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3},$$
$$t = \frac{19}{9}, \frac{95}{9}, \frac{133}{9}, \ldots$$

ie after $\frac{19}{9}$ hours

$\Rightarrow$ 4am + 2h 6min

6.06 am

$x \geq 1\cdot 5$ (from graph)

$$0 \leq t \leq \frac{19}{9}, \quad \frac{95}{9} \leq t \leq \frac{133}{9}$$

ie between 4am and 6.06am

and 4am + 10h 33min and 4am + 14h 46min

between 2.33pm to 6.46 pm