Shore 2004 Maths Ext2 Trial

Question 1 (15 marks) Use a separate page/booklet — **Marks**

(a) Find : $\int x\sqrt{3x-1}\,dx$ — 3

(b) By using the substitution $t = \tan\frac{\theta}{2}$, evaluate

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\sin\theta}$$ — 3

(c) (i) Split into partial fractions: $\frac{8}{(x+2)(x^2+4)}$ — 2

(ii) Hence evaluate: $\int_0^2 \frac{8\,dx}{(x+2)(x^2+4)}$ — 3

(d) If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$, $(n \geq 2)$

(i) Show that $I_n = (n-1)\,I_{n-2} - (n-1)\,I_n$ — 2

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^6 x\,dx$ — 2

Question 2 (15 marks) Use a separate page/booklet — **Marks**

(a) If $z = 3 + 2i$, plot on an Argand diagram

(i) z and $\bar{z}$ — 1

(ii) iz — 1

(iii) $z(1+i)$ — 1

(b) (i) Find all pairs of integers a and b such that $(a+ib)^2 = 8+6i$ — 1

(ii) Hence solve: $z^2 + 2z(1+2i) - (11+2i) = 0$ — 2

(c) (i) If $z = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$, *find* z^6 — 2

(ii) Plot on an argand diagram, all complex numbers that are the solutions of $z^6 = 1$ — 2

(d) Sketch the locus of the following. Draw separate diagrams.

(i) $\arg(z-1-2i) = \frac{\pi}{4}$ — 1

(ii) $z\bar{z} - 3(z+\bar{z}) \leq 0$ — 2

(iii) $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$ — 2

Question 3 (15 marks) Use a separate page/booklet — **Marks**

(a) For the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$

(i) Find the eccentricity. **1**

(ii) Find the coordinates of the foci S and S'. **1**

(iii) Find the equations of the directricies. **1**

(iv) Sketch the curve $\frac{x^2}{25}+\frac{y^2}{16}=1$ **1**

(v) Show that the coordinates of any point P can be represented by $(5\cos\theta, 4\sin\theta)$ **2**

(vi) Show that $PS+PS'$ is independent of the position of P on the curve. **3**

(vii) Show that the equation of the normal at the point P on the ellipse is $5x\sin\theta-4y\cos\theta-9\sin\theta\cos\theta=0$ **3**

(viii) If the normal meets the major axis at L and the minor axis at M, prove that $\frac{PL}{PM}=\frac{16}{25}$ **3**

Question 4 (15 marks) Use a separate page/booklet — **Marks**

(a) The depth of water in a harbour on a particular day is $8\cdot 2\ m$ at low tide and $14\cdot 6\ m$ at high tide. Low tide is at 1:05 pm and high tide is at 7:20 pm.

The captain of a ship drawing $13\cdot 3m$ water wants to leave the harbour on that afternoon. Find between what times he can leave. (Assume that the tide changes in SHM.) **5**

(b) If $a>0$, $b>0$ and $c>0$, show that

(i) $a^2+b^2+c^2-ab-bc-ca\ge 0$ **2**

(ii) $\frac{a+b+c}{3}\ge\sqrt[3]{abc}$ **2**

(iii) $(a+b+d)(b+c+d)(c+a+d)(a+b+c)\ge 81abcd$ **2**

(c) Using mathematical induction prove that $(1+x)^n-nx-1$ is divisible by x^2 for $n\ge 2$, n integer. **4**

Question 5 (15 marks) Use a separate page/booklet **Marks**

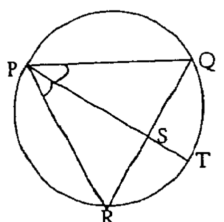
(a) A concrete beam of length $15m$ has plane sides. Cross-sections parallel to the ends are rectangular. The beam measures $4m$ by $3m$ at one end and $8m$ by $6m$ at the other end.

(i) Find an expression for the area of a cross-section at a distance x metres from the smaller end. **3**

(ii) Find the volume of the beam. **2**

(b) Find the volume of the solid generated by rotating the area bounded by the curve $y = \log_e x$, the x - axis and the line $x = 4$. Use the method of *cylindrical shells*. Rotate the area about the y-axis and give your answer correct to 1 decimal place. **4**

(c)



In the diagram, the bisector of the angle RPQ meets RQ in S and the circum-circle of the triangle PQR in T.

(i) Prove that the triangles PSQ and PRT are similar. **2**

(ii) Show that $PQ \times PR = PS \times PT$ **2**

(iii) Prove that $PS^2 = PQ \times PR - RS \times SQ$ **2**

Question 6 (15 marks) Use a separate page/booklet **Marks**

(a) A point is moving in a circular path about O.

(i) Define the angular velocity of the point with respect to O, at any time t. **1**

(ii) Derive expressions for the tangential and normal accelerations of the point at any time t. **4**

(b) A light inextensible string OP is fixed at the end O and is attached at the other end P to a particle of mass m which is moving uniformly in a horizontal circle whose centre is vertically below and distant x from O.

(i) Prove that the period of this motion is $2\pi\sqrt{\dfrac{x}{g}}$, where g is the acceleration due to gravity. **3**

(ii) If the number of revolutions per second is increased from 2 to 3, find the change in x. (Take $g = 10$ m/s^2) Give your answer correct to the nearest millimetre. **3**

(c) The tangent at $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ meets a directrix at Q. S is the corresponding focus.

Given that the equation of the tangent at P is $bx - ay\sin\theta = ab\cos\theta$:

(i) Find the coordinates of Q. **2**

(ii) Show that PQ subtends a right angle at S. **2**

Question 7 (15 marks) Use a separate page/booklet **Marks**

(a) Given $y = \dfrac{x^3}{x^2 - 4}$

(i) Find the coordinates of all stationary points. **2**

(ii) Find the points of intersection with the coordinate axes and the position of all asymptotes. **2**

(iii) Hence sketch the curve $y = \dfrac{x^3}{x^2 - 4}$ **2**

(b) Use the graph $y = \dfrac{x^3}{x^2 - 4}$ to find the number of roots of the equation $x^3 - k(x^2 - 4) = 0$ for varying value of k. **2**

(c) Sketch the following curves:

(i) $y = \log_e(x+1)$ **2**

(ii) $y = \log_e|x+1|$ **1**

(iii) $y = |\log_e(x+1)|$ **1**

(iv) $y = \dfrac{1}{\log_e(x+1)}$ **3**

Question 8 (15 marks) Use a separate page/booklet **Marks**

(a) Find a polynomial $p(x)$ with real coefficients having $3i$ and $1+2i$ as zeros. **3**

(b) A body is projected vertically upwards from the surface of the Earth with initial speed u. The acceleration due to gravity at any point on its path is inversely proportional to the square of its distance from the centre of the Earth.

(i) Prove that the speed v at any position x is given by **3**

$$v^2 = u^2 + 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$$

(ii) Prove that the greatest height H above the Earth's surface is given by $H = \dfrac{u^2 R}{2gR - u^2}$ **3**

(iii) Show that the body will escape from the Earth if $u \geq \sqrt{2gR}$ **1**

(iv) Find the minimum speed in km/s with which the body must be initially projected from the surface of the Earth so as to never return. (Take $R = 6400\,km, g = 10\,m/s^2$) **1**

(v) If $u = \sqrt{2gR}$, prove that the time taken to reach a height $3R$ above the surface of the Earth is $\dfrac{14}{3}\sqrt{\dfrac{R}{2g}}$. **4**