NORTH SYDNEY
GIRLS HIGH SCHOOL

2009

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK #3

Mathematics Extension 1

SAMPLE SOLUTIONS

Question 1

(a) Evaluate $\int_0^1 \frac{dx}{2x+1}$, leaving your answer in the exact form.

$$\int_0^1 \frac{dx}{2x+1} = \tfrac{1}{2}\int_0^1 \frac{2dx}{2x+1} = \tfrac{1}{2}\Big[\ln|2x+1|\Big]_0^1$$
$$= \tfrac{1}{2}(\ln 3 - \ln 1)$$
$$= \tfrac{1}{2}\ln 3$$

(b) Using the substitution $u = 4 - x^2$, evaluate $\int \frac{x}{\sqrt{4-x^2}}\,dx$

$$u = 4 - x^2 \Rightarrow du = -2x\,dx$$
$$\int \frac{x}{\sqrt{4-x^2}}\,dx = -\tfrac{1}{2}\int \frac{-2x}{\sqrt{4-x^2}}\,dx = -\tfrac{1}{2}\int \frac{du}{\sqrt{u}} = -\tfrac{1}{2}\int u^{-\frac{1}{2}}du$$
$$= -u^{\frac{1}{2}} + c = -\sqrt{4-x^2} + c$$

(c) Let $f(x) = \tfrac{1}{2}\left(e^x + e^{-x}\right)$ and $F(x) = \tfrac{1}{2}\left(e^x - e^{-x}\right)$

Prove that $\left[f(x) + F(x)\right]^n = f(nx) + F(nx)$

$$f(x) + F(x) = \tfrac{1}{2}\left(e^x + e^{-x}\right) + \tfrac{1}{2}\left(e^x - e^{-x}\right)$$
$$= 2\times\left(\tfrac{1}{2}e^x\right) = e^x$$

$$\text{LHS} = \left[f(x) + F(x)\right]^n = \left(e^x\right)^n = e^{nx}$$
$$\text{RHS} = f(nx) + F(nx) = \tfrac{1}{2}\left(e^{nx} + e^{-nx}\right) + \tfrac{1}{2}\left(e^{nx} - e^{-nx}\right)$$
$$= 2\times\tfrac{1}{2}e^{nx} = e^{nx}$$

(d) Evaluate $\int_0^1 \frac{e^x}{e^x+1}\,dx$

$$\int_0^1 \frac{e^x}{e^x+1}\,dx = \Big[\ln\left(e^x+1\right)\Big]_0^1 = \ln(e+1) - \ln(2) = \ln\left(\frac{e+1}{2}\right)$$

Question 2

(a) Solve $e^x = 5$, leaving your answer correct to 3 decimal places

$e^x = 5 \Rightarrow x = \ln 5 \approx 1 \cdot 609437912...$

$x = 1 \cdot 609 \ [3\,\text{dp}]$

(b) Find a primitive of $\dfrac{3x}{1+x^2}$

$$\int \frac{3x}{1+x^2}\,dx = \tfrac{3}{2}\int \frac{2x}{1+x^2}\,dx = \tfrac{3}{2}\ln\left(1+x^2\right)$$

(c) Find $\dfrac{d}{dx}\left(3x\log_e x\right)$

$$\frac{d}{dx}\left(3x\log_e x\right) = 3x\times\frac{1}{x} + 3\times\ln x$$
$$= 3 + 3\ln x$$

(d) Evaluate $\displaystyle\int_0^3 3^x\,dx$

$$\int_0^3 3^x\,dx = \left[\frac{3^x}{\ln 3}\right]_0^3 = \frac{1}{\ln 3}\left(3^3 - 3^0\right) = \frac{26}{\ln 3}$$

(e) Using the substitution $u = \log_e x$, evaluate $\displaystyle\int_1^e \frac{\left(1+\log_e x\right)^2}{x}\,dx$

$x = 1 \Rightarrow u = \ln 1 = 0$

$x = e \Rightarrow u = \ln e = 1$

$u = \ln x \Rightarrow du = \dfrac{dx}{x}$

$$\int_1^e \frac{\left(1+\log_e x\right)^2}{x}\,dx = \int_1^e \left(1+\log_e x\right)^2 \frac{dx}{x}$$
$$= \int_0^1 \left(1+u\right)^2 du$$
$$= \left[\tfrac{1}{3}\left(1+u\right)^3\right]_0^1$$
$$= \tfrac{1}{3}\left(2^3 - 1^3\right) = \frac{7}{3}$$

Question 3

(a) (i) Show that $\dfrac{5}{\sqrt{5x+3}-\sqrt{5x-2}}=\sqrt{5x+3}+\sqrt{5x-2}$

$$\text{LHS}=\frac{5}{\sqrt{5x+3}-\sqrt{5x-2}}$$

$$=\frac{5}{\sqrt{5x+3}-\sqrt{5x-2}}\times\frac{\sqrt{5x+3}+\sqrt{5x-2}}{\sqrt{5x+3}+\sqrt{5x-2}}$$

$$=\frac{5\left(\sqrt{5x+3}+\sqrt{5x-2}\right)}{\left[(5x+3)-(5x-2)\right]}$$

$$=\frac{5\left(\sqrt{5x+3}+\sqrt{5x-2}\right)}{5}$$

$$=\sqrt{5x+3}+\sqrt{5x-2}$$
$$=\text{RHS}$$

(ii) Hence find $\displaystyle\int\frac{dx}{\sqrt{5x+3}-\sqrt{5x-2}}$

$$\int\frac{dx}{\sqrt{5x+3}-\sqrt{5x-2}}=\int\frac{\left(\sqrt{5x+3}+\sqrt{5x-2}\right)dx}{5}$$

$$=\tfrac{1}{5}\int\left[(5x+3)^{\frac{1}{2}}+(5x-2)^{\frac{1}{2}}\right]dx$$

$$=\tfrac{1}{5}\left[\tfrac{1}{5}\times\tfrac{2}{3}(5x+3)^{\frac{3}{2}}+\tfrac{1}{5}\times\tfrac{2}{3}(5x-2)^{\frac{3}{2}}\right]+C$$

$$=\tfrac{2}{75}\left[(5x+3)^{\frac{3}{2}}+(5x-2)^{\frac{3}{2}}\right]+C$$

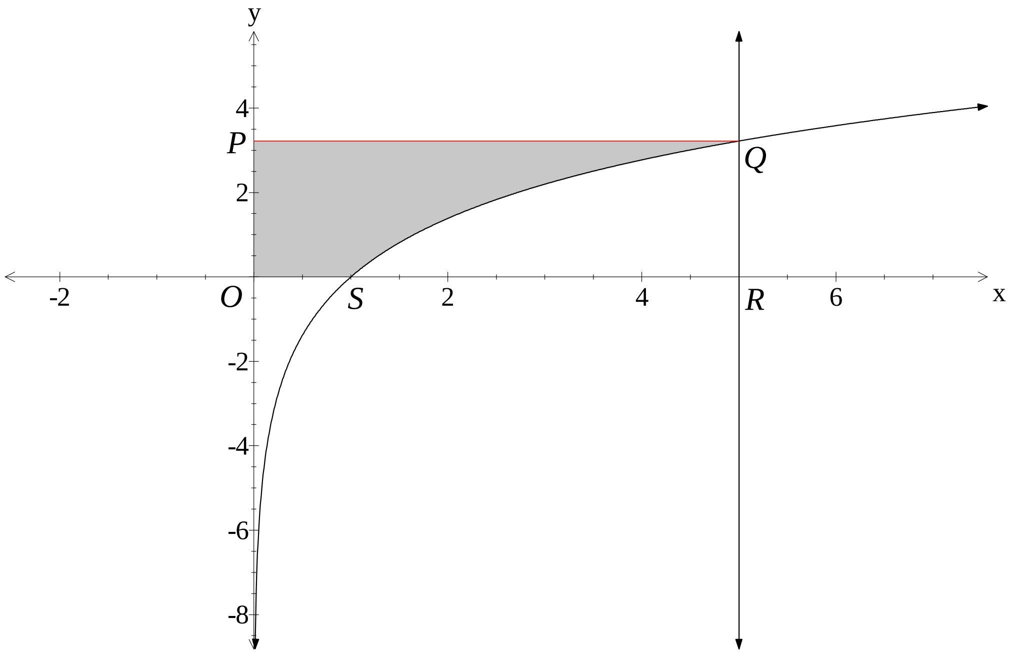
(b) (i) Show that $\dfrac{d}{dx}\left(x\ln x - x\right) = \ln x$.

$$\frac{d}{dx}\left(x\ln x - x\right) = x\times\frac{1}{x} + 1\times\ln x - 1$$
$$= 1 + \ln x - 1$$
$$= \ln x$$

(ii) Hence, or otherwise, find $\displaystyle\int \ln x^2 \, dx$.

$$\int \ln x^2 \, dx = 2\int \ln x \, dx = 2\left(x\ln x - x\right) + C$$

(iii) The graph below shows the curve $y = \ln x^2$ ($x > 0$) which meets the line $x = 5$ at Q.
Using your answers above, or otherwise, find the area of the shaded region.



P has coordinates $\left(0, \ln 25\right)$

The required area = area rectangle $OPQR - \displaystyle\int_1^5 \ln x^2 dx$

$$= 5\times\ln 25 - \left[2\left(x\ln x - x\right)\right]_1^5$$
$$= 5\ln 25 - 2\left[\left(5\ln 5 - 5\right) - \left(\ln 1 - 1\right)\right]$$
$$= 5\ln 25 - 10\ln 5 + 10 - 2$$
$$= 5\ln 25 - 5\ln 25 + 8$$
$$= 8 \text{ u}^2$$

Question 4

(a) Find $\int \frac{x+1}{x^2}\,dx$ **2**

$$\int \frac{x+1}{x^2}\,dx = \int \left(\frac{1}{x}+\frac{1}{x^2}\right)dx = \int \left(\frac{1}{x}+x^{-2}\right)dx$$

$$= \ln|x| - x^{-1} + C$$

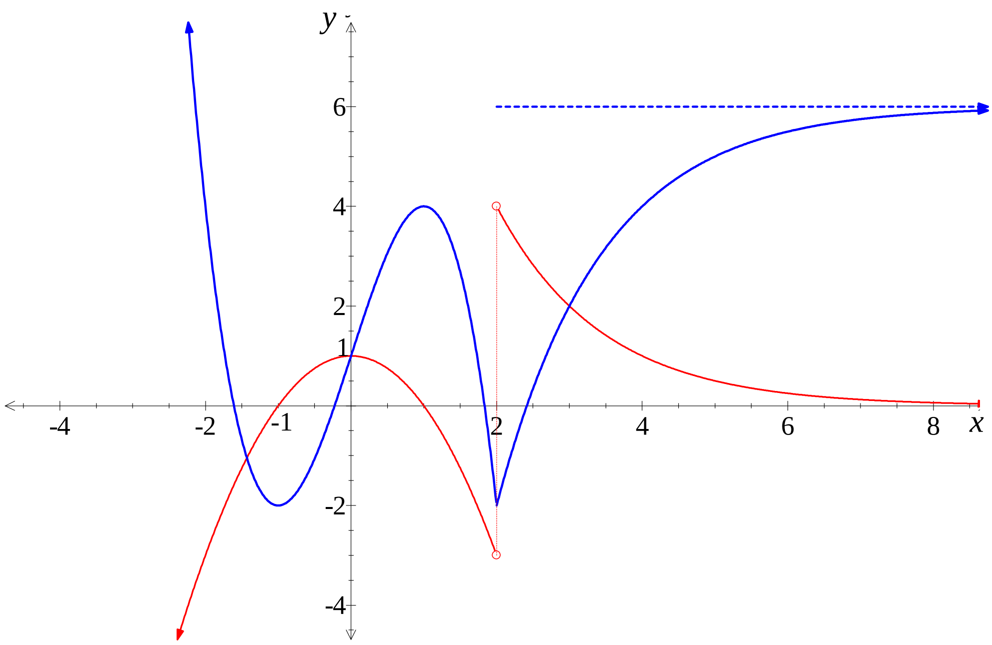
$$= \ln|x| - \frac{1}{x} + C$$

(b) The following graph shows the gradient function $y = f'(x)$. **3**

The graph shows that $f'(1) = f'(-1) = 0$.

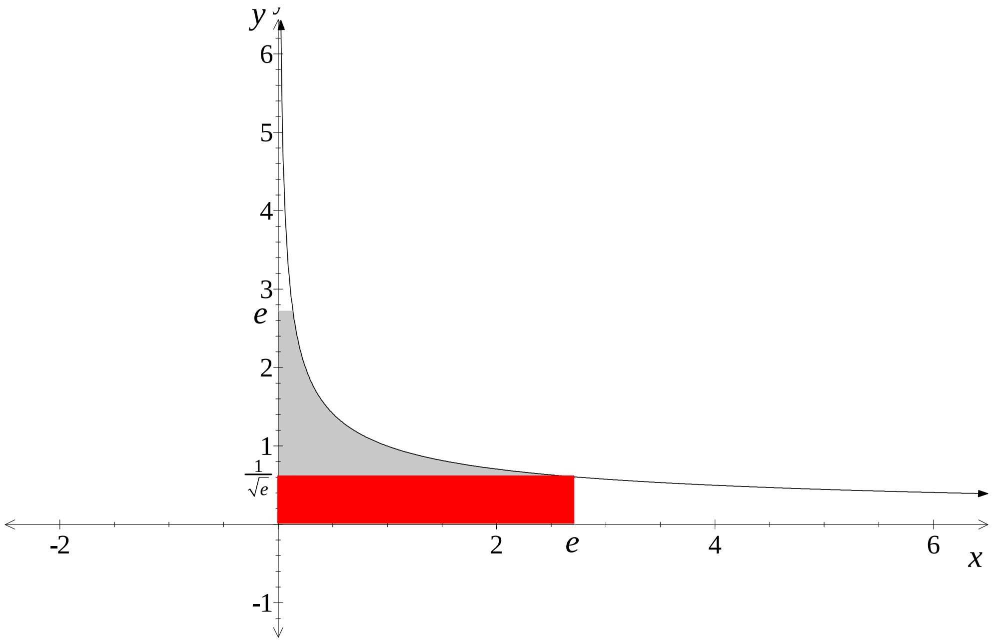
Sketch the graph of $y = f(x)$, given that $f'(x)$ is continuous everywhere except at $x = 2$ and that $f(0) = 1$ and $f(-1) = -2$

A possible solution:



(c) The shaded region below is that bounded by $y = \dfrac{1}{\sqrt{x}}$, the coordinate axes and the lines $x = e$ and $y = e$. **4**

Find the volume when the shaded region is rotated about the *y*-axis, correct to 2 significant figures.



The volume V is the sum of two volumes V_1 and V_2.

V_1 is the volume formed by rotating the curve $y = \dfrac{1}{\sqrt{x}}$ from $y = \dfrac{1}{\sqrt{e}}$ to $y = e$ about the *y*-axis. $y = \dfrac{1}{\sqrt{x}} \Rightarrow x = \dfrac{1}{y^2} \Rightarrow x^2 = \dfrac{1}{y^4} = y^{-4}$

V_2 is the cylinder formed by rotating the line $x = e$ about the *y*-axis.

It has radius e and height $\dfrac{1}{\sqrt{e}}$.

$$V_1 = \pi \int_{\frac{1}{\sqrt{e}}}^{e} x^2\, dy = \pi \int_{\frac{1}{\sqrt{e}}}^{e} y^{-4}\, dy$$

$$= \pi \left[-\tfrac{1}{3} y^{-3} \right]_{\frac{1}{\sqrt{e}}}^{e} = \frac{\pi}{3} \left[-\frac{1}{y^3} \right]_{\frac{1}{\sqrt{e}}}^{e}$$

$$\left[\text{NB} \left(\sqrt{e} \right)^3 = e\sqrt{e} \right]$$

$$= \frac{\pi}{3} \left[-\frac{1}{e^3} + \frac{1}{\frac{1}{e\sqrt{e}}} \right] = \frac{\pi}{3} \left[\frac{\sqrt{e}}{e^2} - \frac{1}{e^3} \right]$$

$$= \frac{\pi}{3} \left(e\sqrt{e} - \frac{1}{e^3} \right)$$

$$V_2 = \pi \times e^2 \times \frac{1}{\sqrt{e}}$$

$$= \pi e^{\frac{3}{2}}$$

$$V = \frac{\pi}{3} \left(e\sqrt{e} - \frac{1}{e^3} \right) + \pi e^{\frac{3}{2}} \approx 19 \ \text{u}^3$$

Question 5

Consider the function $y = \dfrac{\ln x}{x}$

(a) What is the domain of this function? $x > 0$

(b) Show that $\dfrac{d}{dx}\left(\dfrac{\ln x}{x}\right) = -\left(\dfrac{\ln x - 1}{x^2}\right)$

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{x \times \frac{1}{x} - \ln x \times 1}{x^2} = \frac{1 - \ln x}{x^2} = -\left(\frac{\ln x - 1}{x^2}\right)$$

(c) Describe the behaviour of the function as x

(i) approaches zero. $y \to -\infty$

(ii) increases indefinitely $y \to 0$

(d) Find any stationary points and determine their nature.

$y' = 0 \Rightarrow \ln x - 1 = 0 \Rightarrow \ln x = 1$

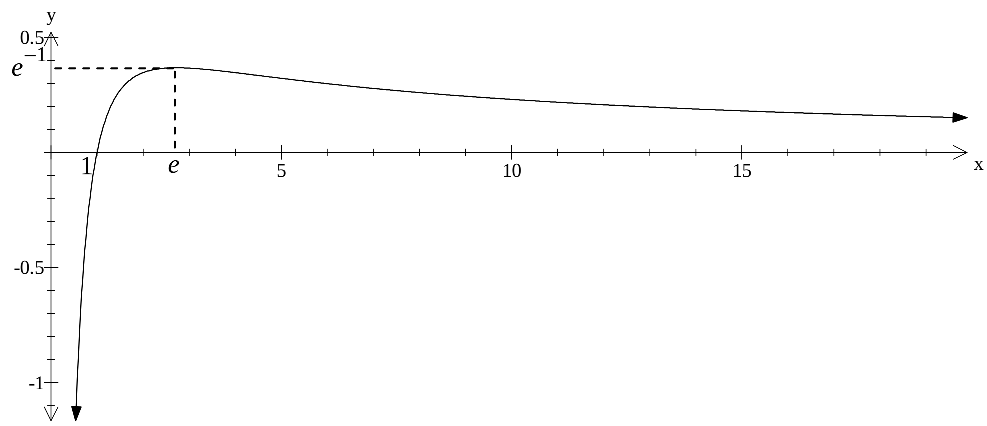
$\therefore x = e \Rightarrow \left(e, \dfrac{1}{e}\right)$ is the stationary point

x	2	e	3
y'	$0 \cdot 3$	0	$-0 \cdot 1$

Only need to check $(1 - \ln x)$ as $x^2 > 0$.

So (e, e^{-1}) is a maximum turning point.

(e) Sketch the curve of this function.



(f) Hence find the value(s) of k for which $e^{kx} = x$ has no solutions.

$e^{kx} = x \Rightarrow kx = \ln x$

$\therefore k = \dfrac{\ln x}{x}$

So the solutions to $e^{kx} = x$ are found by intersecting the line $y = k$ with $y = \dfrac{\ln x}{x}$.

So there will be no solutions when $k > \dfrac{1}{e}$.

Question 6

(a) Use mathematical induction to show that the following statement is true

$$n^3 + 2n \text{ is a multiple of } 12$$

where n is an <u>even</u> positive integer

Test $n = 2$

$2^3 + 2\times 2 = 12$

Clearly $n = 2$ is true.

Assume true for $n = 2k$ i.e. $(2k)^3 + 2(2k) = 12N,\ N \in \mathbb{Z}$

$\therefore 8k^3 + 4k = 12N$.

NTP true for $n = 2k + 2$ i.e. $(2k+2)^3 + 2(2k+2) = 12M,\ M \in \mathbb{Z}$

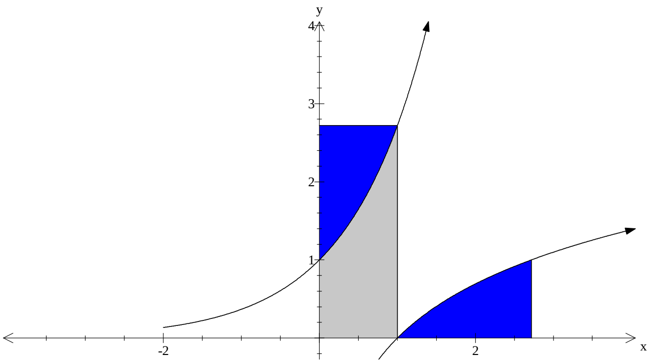
$$\begin{aligned}(2k+2)^3 + 2(2k+2) &= 8(k+1)^3 + 4(k+1)\\ &= 8\left(k^3 + 3k^2 + 3k + 1\right) + 4k + 4\\ &= \left(8k^3 + 4k\right) + 24k^2 + 24k + 12\\ &= 12\left(N + 2k^2 + 2k + 1\right)\\ &= 12M \qquad \left[\because N + 2k^2 + 2k + 1 \in \mathbb{Z}\right]\end{aligned}$$

So the statement is true for $n = 2k + 2$ provided it is true for $n = 2k$.

So by the principle of mathematical induction it is true for all positive even integers.

(b) By use of an appropriate diagram and reasons, evaluate the following sum.
Do NOT evaluate any primitive functions.

$$\int_0^1 e^x dx + \int_1^e \ln x\, dx$$



By symmetry the integral $\int_1^e \ln x\, dx$ produces the same area as that of e^x next to the y-axis for $1 \le y \le e$.

So $\int_0^1 e^x dx + \int_1^e \ln x\, dx$ is the area of the rectangle with dimensions $1 \times e$

$\therefore \int_0^1 e^x dx + \int_1^e \ln x\, dx = e$

(c) (i) Show $\frac{1}{u}-\frac{1}{u+1}=\frac{1}{u(u+1)}$

$$\frac{1}{u}-\frac{1}{u+1}=\frac{u+1-u}{u(u+1)}=\frac{1}{u(u+1)}$$

(ii) Using the substitution $x=\ln u$, find $\int \frac{dx}{1+e^x}$

$x=\ln u \Rightarrow dx=\frac{du}{u}$

$x=\ln u \Rightarrow u=e^x$

$$\int \frac{dx}{1+e^x}=\int \frac{1}{1+e^x}\times dx=\int \left(\frac{1}{1+u}\right)\frac{du}{u}$$

$$=\int \left[\frac{du}{u(u+1)}\right]=\int \left(\frac{1}{u}-\frac{1}{u+1}\right)du$$

$$=\ln u-\ln(u+1)+C$$

$$=\ln\left(\frac{e^x}{1+e^x}\right)+C \qquad \left[=x-\ln\left(1+e^x\right)+C\right]$$

End of Solutions