N.S.W. DEPARTMENT OF EDUCATION

HIGHER SCHOOL CERTIFICATE EXAMINATION 1971

MATHEMATICS - PAPER B (2F) - EQUIVALENT TO 3U AND 4U - 1st PAPER

Instructions: Time allowed 3 hours. All questions may be attempted. In every question, all necessary working should be shown. Marks will be deducted for careless or badly arranged work. Mathematical tables will be supplied.

QUESTION 1 (12 Marks)

(i) Find the second derivative of e^{3x}.

(ii) Find the area above the x-axis and under the curve $y = e^{-x}$ between $x = 0$ and $x = 1$.

(iii) Find a primitive (indefinite integral) of $1/(3x)$.

(iv) Differentiate $\tan^{-1}(3x)$.

QUESTION 2 (9 Marks)

(i) Use one step of Newton's method to find an improved value of that root of $f(x) = 9/(9x + 10) - e^{1.19x}$ which lies close to $x = 0$.

(ii) Find the stationary point of the curve $y = (1 + x^4)^{-1}$ and determine its nature.

(iii) Find, for $-\pi/4 < x < \pi/4$, the sum to infinity $1 - \tan^2 x + \tan^4 x - \tan^6 x + \ldots$ and express it in the simplest form.

QUESTION 3 (9 Marks)

(i) Find the derivative of $\sqrt{(1 + x^3)}$.

(ii) Find the equation of the curve $y = x(x-4)/(x^2 - 4x + 6)$ in the new X-Y coordinate system with origin at $x = 2$, $y = 1$, and the axes unchanged in direction.

(iii) If $f(x) = \log_e\{x + \sqrt{(x^2 - 1)}\}$ what is the largest possible domain of the function? What is the derivative of this function?

QUESTION 4 (10 Marks)

(i) Determine the equation of the tangent to the curve $C: y = 2x^2$ at the point $P: (t, 2t^2)$.

(ii) The point Q lies on the curve $C': y = x^2 + 1$, on the same vertical line (i.e., with the same abscissa) as the point P of part (i). Show that the equation of the tangent to C' at Q is: $y = 2tx + (1 - t^2)$.

(iii) Find the precise locus of the points of intersection of these two tangents, as the common abscissa t of the points P and Q assumes all positive values. Indicate this locus on a sketch.

QUESTION 5 (10 Marks)

(i) A monic polynomial of degree three has three zeros located at 2, 0, and -2, respectively. Find this polynomial. Is there any other correct answer?

(ii) A monic polynomial $P(x)$ of degree four is known to have exactly two zeros, at 2 and at -2, and is known to be an even function of x, that is, $P(-x) = P(x)$. Show that there are many such polynomials, and find their general form.

(iii) The polynomial of part (ii) is known to have the value 55 when $x = 3$. Show that this determines the polynomial uniquely to be $P(x) = x^4 - 2x^2 - 8$.

QUESTION 6 (10 Marks)

A biassed coin has a probability of coming up heads equal to $p = 0.6$.

(i) Find the probability of getting four heads in four tosses of the coin.

(ii) Find the probability of getting at least three heads in four tosses of this coin.

(iii) Find the expected value of the number of heads in four tosses of this coin.

QUESTION 7 (10 Marks)

(i) Differentiate $\cos^m t$ and hence, or otherwise, evaluate

$$\int_0^{\pi/2} (\cos t)^{2k} \sin t \, dt.$$

(ii) Show that, for any positive integer n,

$$\int_0^{\pi/2} (\sin t)^{2n+1} dt = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \cdot {}^nC_k.$$

{Hint: Write the integrand as $(1 - \cos^2 t)^n \sin t$}.

(iii) Hence, or otherwise, evaluate $\int_0^{\pi/2} \sin^5 t \, dt$ (leave your result as a fraction).

QUESTION 8 (10 Marks)

(i) A function f, whose domain is all real numbers, is (strictly) increasing. What is meant by its inverse function f^{-1}?

(ii) The graph $y = f(x)$ of the function in (i) is reflected in the straight line $y = x$ (that is, every point P on the curve is associated with a point Q on the opposite side of the line, at the same distance from the line). Establish a relationship between the equation of the reflected curve (of the locus of the points Q) and the inverse function f^{-1}.

(iii) The curve $y = e^{x+1}$ is reflected in the straight line $y = x$. Determine the equation of the reflected curve.

(OMIT)

QUESTION 9 (10 Marks)

A line L in three-dimensional space is defined by the equations $y = x$, $z = 3 + 4x$.

(i) What is the equation of the plane containing the line L and the z-axis? Describe this plane in geometrical terms.

(ii) Find the cosine of the acute angle between the line L and the z-axis.

(iii) By substitution of the equations of L into the equation of the surface of the sphere of unit radius with centre at the origin, or otherwise, show that there is exactly one point in common between the line L and this sphere. State the geometrical interpretation of this fact.

QUESTION 10 (10 Marks)

A pendulum is constructed by attaching a weight to a light rigid rod (not to a flexible string) of length L; the weight can swing in a vertical plane around a fixed point P to which the other end of the rod is attached. Let θ be the angle between the pendulum rod and the (downward) vertical direction, and let g be the acceleration of gravity. Then the equation of motion of the pendulum can be shown to be $d^2\theta/dt^2 = -(g/L)\sin\theta$.

(i) Show that the quantity $E = \frac{1}{2}(\frac{d\theta}{dt})^2 - (g/L)\cos\theta$ does not change with time.

(ii) The pendulum is started from its equilibrium position (from $\theta = 0$) with an initial angular velocity equal to ω. What is the value of the quantity E of part (i)?

(iii) Assume that the pendulum is started moving rapidly enough so that it swings all the way around the fixed point P in a full circle. By considering the instant when it is at the very top of the circle, or otherwise, show that the value of the quantity E for such a motion must be greater than g/L.

(iv) By combining the results of parts (ii) and (iii), or otherwise, show that the pendulum will swing all the way around in a full circle if and only if the absolute value of its initial angular velocity ω exceeds $2\sqrt{(g/L)}$.