N.S.W. DEPARTMENT OF EDUCATION
HIGHER SCHOOL CERTIFICATE EXAMINATION 1985
MATHEMATICS - 3UNIT/4UNIT COMMON PAPER
(I.E. 3UNIT COURSE - ADDITIONAL PAPER; 4UNIT COURSE - FIRST PAPER)

Instructions. *Time allowed - Two hours*

All questions may be attempted. *All* questions are of equal value.
In every question show all necessary working.
Marks may not be awarded for careless or badly arranged work.
Standard integrals are printed at the end of the paper. Approved slide-rules or silent calculators may be used.

QUESTION 1.

(i) Find the value of the derivative of $\tan^2 x$ at $x = \pi/4$.

(ii) Find $\int_0^1 x(1 + x^2)^2 dx$.

(iii) Find all positive values of x for which $\frac{6}{x} > x - 1$.

(iv) Find the acute angle between the lines $y = -x$ and $\sqrt{3}y = x$.

QUESTION 2.

(i) P, Q are points on a circle and the tangents to the circle at P, Q meet at S. R is a point on the circle so that the chord PR is parallel to QS.

(a) Draw a neat sketch in your answer book, showing the given information.

(b) Giving reasons, prove carefully that QP = QR.

(ii) A circle has equation $x^2 + y^2 - 4x + 2y = 0$.

(a) Find the centre and radius of the circle.

(b) The line $x + 2y = 0$ meets this circle in two points, A, B.

(α) Find the co-ordinates of A and B.

(β) Calculate the distance AB.

(iii) Given that there is a constant c such that

$$(x^4 + y^4) = (x^2 + cxy + y^2)(x^2 - cxy + y^2)$$

identically in x and y, find c.

QUESTION 3.

(i) (a) Derive the equation of the normal to the parabola $x = 2At$, $y = At^2$ at the point where $t = T$.

(b) For the parabola $x = 2t$, $y = t^2$, find the values (if any) of T for which the normal at the point where $t = T$ passes through $(0,6)$.

(ii) The function $x(t)$ is given by $x(t) = 4 - 60 \sin\left[\frac{t}{15}\right]$. Find

(a) M, the maximum value of $x(t)$;

(b) the least positive value of t for which $x(t) = M$;

(c) the values of $x(t)$ for which $|\dot{x}(t)| = 2$.

QUESTION 4.

(i) Using the substitution $u = x^4$, or otherwise, show that

$$\int_0^1 \frac{x^3}{1+x^8}\,dx = \frac{\pi}{16}.$$

(ii) (a) Factorise completely the polynomial

$$p(x) = x^3 - x^2 - 8x + 12,$$

given that the equation $p(x) = 0$ has a repeated root.

(b) The polynomial $q(x)$ has the form $q(x) = p(x)(x + a)$, with $p(x)$ as in (a) and where the constant a is chosen so that $q(x) \geq 0$ for all real values of x. Find all possible values of a.

(iii) Use the Principle of Mathematical Induction to prove that $5^n + 2(11^n)$ is a multiple of 3 for all positive integers n.

QUESTION 5.

(i) Firefighters are forced to stay 60 m away from a dangerous fire burning in a low open tank on horizontal ground. They have two pumps. One, which can eject water in any direction at 30 m s^{-1}, is on the ground, while the other, which can eject water at 40 m s^{-1} but only horizontally, is on a vertical stand 5 m high.

Can both pumps reach the fire? Justify your answer.
(Assume that $g = 10$ m s^{-2}, and that all frictional forces, including air resistance, can be neglected.)

(ii) A scientist found that the amount, $Q(t)$, of a substance present in a mineral at time $t \geq 0$ satisfies the equation

$$4\frac{d^2Q}{dt^2} + 4\frac{dQ}{dt} + Q = 0.$$

(a) Verify that $Q(t) = A(1 + t)e^{-0.5t}$ satisfies this equation for any constant $A > 0$.

(b) If $Q(0) = 10$ mg, find the maximum value of $Q(t)$ and the time at which this occurs.

(c) Describe what happens to $Q(t)$ as t increases indefinitely.

QUESTION 6.

(i) When $(3 + 2x)^n$ is written out as a polynomial in x, the coefficients of x^5 and x^6 have the same value. Find n.

(ii) Prove that

$$1 + \binom{10}{2}3^2 + \binom{10}{4}3^4 + \binom{10}{6}3^6 + \binom{10}{8}3^8 + 3^{10} = 2^9[2^{10} + 1].$$

(iii) David plays a game in which the probability that his score is s is

$$\begin{cases} \frac{1}{8}, & \text{for } s = -4, -3, -2, -1; \\ \frac{1}{2}\binom{4}{s}(0.3)^s(0.7)^{4-s}, & \text{for } s = 0, 1, 2, 3, 4; \\ 0, & \text{for all other values of } s. \end{cases}$$

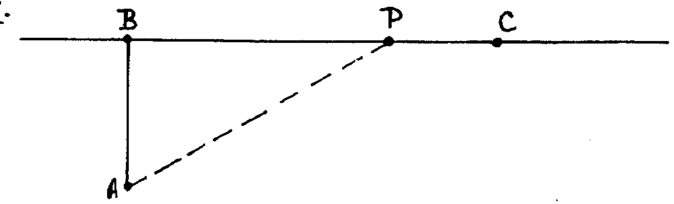
What is

(a) his most likely score?

(b) the probability (expressed as a decimal correct to three places) that his score is positive?

(c) the probability (expressed as a decimal correct to three places) that, after playing the game twice, his total score is -3?

QUESTION 7.



The diagram shows a straight road BC running due East. A four-wheel drive ambulance is in open country at A, 3 km due South of B. It must reach C, 9 km due East of B, as quickly as possible.

The driver knows that she can travel at 80 km per hour in open country and at 100 km per hour along the road. She intends to proceed in a straight line to some point P on the road and then to continue along the road to C. She wishes to choose P so that total time for the journey APC is a minimum.

(a) If the distance BP is x km, derive an expression for $t(x)$, the total journey time from A to C via P, in terms of x.

(b) Show that the minimum time for the total journey APC is $6\frac{3}{4}$ minutes.

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \; n \neq -1; \; x \neq 0, \text{ if } n < 0.$

$\int \frac{1}{x}\,dx = \log_e x, \; x > 0.$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \; a \neq 0.$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \; a \neq 0.$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \; a \neq 0.$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \; a \neq 0.$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \; a \neq 0.$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \; a \neq 0.$

$\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \sin^{-1}\frac{x}{a}, \; a>0, \; -a<x<a.$

$\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx = \log_e\{x + \sqrt{(x^2-a^2)}\}, \; |x| > |a|.$

$\int \frac{1}{\sqrt{(x^2+a^2)}}\,dx = \log_e\{x + \sqrt{(x^2+a^2)}\}.$