Question 1

a. **Outcomes Assessed: PE5 , HE4**

Marking Guidelines

Criteria	Marks
• Applies the product rule with correct derivative of $\tan^{-1}x$	1
• Simplifies resulting expression	1

Answer

$$\frac{d}{dx}\left(1+x^2\right)\tan^{-1}x = 2x\tan^{-1}x + \left(1+x^2\right)\frac{1}{1+x^2} = 1 + 2x\tan^{-1}x$$

b. **Outcomes Assessed: PE3**

Marking Guidelines

Criteria	Marks
• uses the remainder theorem to obtain an equation for a	1
• solves the equation to evaluate a.	1

Answer

$P(1) = P(2) \Rightarrow a + 2 = 2a + 9 \quad \therefore a = -7$

c. **Outcomes Assessed: (i) H5 (ii) P4**

Marking Guidelines

Criteria	Marks
i. • writes the expression for $\tan 45°$ in terms of the gradients of the lines	1
• obtains the required equation by putting $\tan 45° = 1$ and rearranging	1
ii. • finds one of the values of m with the corresponding line	1
• finds the second value of m and the equation of the second line	1

Answer

i. $\left|\frac{m-2}{1+2m}\right| = \tan 45° = 1$

$\therefore |m-2| = |1+2m|$

ii. $m - 2 = 1 + 2m \quad or \quad m - 2 = -(1+2m)$

$-3 = m \qquad m - 2 = -1 - 2m$

$3m = 1$

$\therefore m = -3 \ or \ m = \frac{1}{3}$

The required lines are $y = -3x$ and $y = \frac{1}{3}x$

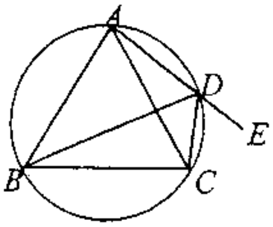
d. **Outcomes Assessed:** **(i) PE3 (ii) PE2 , PE3**

Marking Guidelines

Criteria	Marks
i. •	0
ii. • gives suitable reason referring to appropriate property of cyclic quadrilateral	1
iii. • explains why $\angle BDC = \angle BAC$	1
• explains why $\angle BAC = \angle ABC$	1
• uses these facts to make final deduction about DC	1

Answer

i.



ii. $\angle CDE = \angle ABC$ (*exterior angle of cyclic quadrilateral ABCD is equal to the opposite interior angle*).

iii. $\angle BDC = \angle BAC$ (*$\angle$ s subtended at circumference by same arc BC are equal*)

$\angle BAC = \angle ABC$ (*$\angle$ s opposite equal sides BC and AC in ΔABC are equal*)

$\therefore \angle BDC = \angle ABC$

$\therefore \angle BDC = \angle CDE$ (*both equal to $\angle ABC$*)

$\therefore DC$ bisects $\angle BDE$.

Question 2

a. **Outcomes Assessed:** **P4**

Marking Guidelines

Criteria	Marks
• applies an appropriate formula or pattern for external division	1
• evaluates the coordinates of P.	1

Answer

A $\quad$ B

$(-5,6)$ $\quad$ $(1,3)$

$5 \;:\; -2$

$\left(\frac{5+10}{5-2}, \frac{15-12}{5-2}\right)$ $\quad \therefore P(5,1)$

b. **Outcomes Assessed:** **PE3**

Marking Guidelines

Criteria	Marks
• expresses $\sum \frac{1}{\alpha}$ in terms of $\sum \alpha\beta$ and $\alpha\beta\gamma$.	1
• reads correct values of $\sum \alpha\beta$ and $\alpha\beta\gamma$ from coefficients to evaluate $\sum \frac{1}{\alpha}$	1

Answer

$2x^3+2x^2+4x+1=0$ has roots α,β,γ. $\quad\therefore \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma}=\frac{\left(\frac{4}{2}\right)}{\left(-\frac{1}{2}\right)}=-4$

c. **Outcomes Assessed: (i) H5 (ii) H5**

Marking Guidelines

Criteria	Marks
i. • identifies common ratio as $\cos 2x$	1
• applies condition for existence of limiting sum	1
ii. • writes expression for S in terms of $\sin 2x$ and $\cos 2x$	1
• uses appropriate trig. identities to simplify expression for S.	1

Answer

i. $r=\cos 2x,\ 0<x<\frac{\pi}{2}\Rightarrow |r|<1$.
Hence limiting sum S exists.

ii. $S=\dfrac{\sin 2x}{1-\cos 2x}=\dfrac{2\sin x\cos x}{2\sin^2 x}$ $\quad\therefore S=\dfrac{\cos x}{\sin x}=\cot x$

d. **Outcomes Assessed: (i) PE3 , PE4 (ii) PE3**

Marking Guidelines

Criteria	Marks
i. • finds $\frac{dy}{dx}$ to show that the tangent has gradient t	1
• finds the equation of the tangent	1
ii. • finds x and y coordinates of M in terms of t	1
• finds Cartesian equation of locus of M	1

Answer

i. $x=2t\Rightarrow \dfrac{dx}{dt}=2$

$y=t^2\Rightarrow \dfrac{dy}{dt}=2t$

$\therefore \dfrac{dy}{dx}=\dfrac{2t}{2}=t$

Tangent has gradient t and equation

$y-t^2=t(x-2t)$

$y-t^2=tx-2t^2$

$tx-y-t^2=0$

ii. at M, $tx-y-t^2=0$ *and* $y=-tx$

$\therefore 2tx-t^2=0$

$2t\left(x-\frac{1}{2}t\right)=0$

If $t=0$, P and M both lie at the origin.
Otherwise at M $\quad x=\frac{1}{2}t,\ y=-\frac{1}{2}t^2$,
giving $y=-\frac{1}{2}(2x)^2$.

$\therefore$ locus of M has equation $y=-2x^2$.

Question 3

a. **Outcomes Assessed:** **(i) P4 (ii) P4**

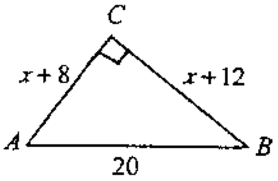
Marking Guidelines

Criteria	Marks
i. • uses Pythagoras to obtain an equation for x	1
• simplifies this equation by expanding squares and collecting like terms	1
ii. • factors this quadratic (or applies an alternative method)	1
• finds the radius of the circle with centre C.	1

Answer

When circles touch, the line joining centres passes through the point of contact, giving the sides of right triangle ABC as shown below

i.



$$(x+8)^2+(x+12)^2=20^2$$
$$2x^2+40x+64+144=400$$
$$2x^2+40x-192=0$$
$$x^2+20x-96=0$$

ii.

$$(x+24)(x-4)=0$$
$$\therefore x>0 \Rightarrow x=4$$

Circle with centre C has radius 4 cm.

b. **Outcomes Assessed:** **(i) P3 (ii) HE6**

Marking Guidelines

Criteria	Marks
i. • rearranges either LHS or RHS to establish result	1
ii. • transforms integral into form $2\int \frac{u}{1+u}\,du$	1
• finds primitive in terms of u	1
• finds primitive in terms of x	1

Answer

i. $$\frac{u}{1+u}=\frac{(1+u)-1}{1+u}$$
$$=1-\frac{1}{1+u}$$

ii. $u \geq 0$

$x=u^2$

$dx=2u\,du$

$$\int \frac{1}{1+\sqrt{x}}\,dx=\int \frac{1}{1+u}2u\,du$$
$$=2\int \frac{u}{1+u}\,du$$
$$=2\int \left(1-\frac{1}{1+u}\right)du$$
$$=2\{u-\ln(1+u)\}+c$$
$$=2\sqrt{x}-2\ln\left(1+\sqrt{x}\right)+c$$

c. **Outcomes Assessed: HE2**

Marking Guidelines

Criteria	Marks
• shows the statement is true for $n=3$	1
• shows that $5^{k+1} > 5(4^k + 3^k)$ if $S(k)$ is true	1
• completes the explanation that $S(k)$ true implies $S(k+1)$ true	1
• makes final statements to complete the Mathematical Induction	1

Answer

Let $S(n)$ be the statement $5^n > 4^n + 3^n$, $n = 3, 4, 5, \ldots$

Consider $S(3)$: $5^3 = 125$, $4^3 + 3^3 = 64 + 27 = 91$. Hence $S(3)$ is true.

If $S(k)$ *is true* : $5^k > 4^k + 3^k$ **

Consider $S(k+1)$: $5^{k+1} = 5.5^k$

$$\begin{aligned} &> 5(4^k + 3^k) \quad \textit{if } S(k) \textit{ is true, using } ** \\ &= 5.4^k + 5.3^k \\ &> 4.4^k + 3.3^k \\ &= 4^{k+1} + 3^{k+1} \end{aligned}$$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(3)$ is true, hence $S(4)$ is true and then $S(5)$ is true and so on. Hence by Mathematical Induction $5^n > 4^n + 3^n$ for all integers $n \geq 3$.

Question 4

a. **Outcomes Assessed: PE3**

Marking Guidelines

Criteria	Marks
• writes an expression for the general term in the expansion	1
• identifies the term independent of x	1
• calculates the term independent of x	1

Answer

General term is ${}^{15}C_r\left(-\frac{2}{x^2}\right)^r x^{15-r} = {}^{15}C_r(-2)^r x^{15-3r}$, $r = 0, 1, 2, \ldots, 15$

Constant term has $15 - 3r = 0 \Rightarrow r = 5$

$\therefore$ term independent of x is ${}^{15}C_5\,(-2)^5 = -96\,096$.

b. **Outcomes Assessed:** (i) **HE3** **(ii) H3**

Marking Guidelines

Criteria	Marks
i. • uses given information to show one of $A = 100$ or $A + B = 500$	1
• shows the second result about A, B and deduces the values of A and B	1
ii. • obtains $t \geq 2\ln 40$	1
• calculates the time to nearest month	1

Answer

i. $N = A + Be^{-0.5t}$

$t = 0, \quad N = 500 \Rightarrow A + B = 500$

$t \to \infty,, N = 100 \Rightarrow A + 0 = 100$

$\therefore A = 100, \quad B = 400$

ii. $N \leq 110 \Rightarrow 100 + 400\, e^{-0.5t} \leq 110$

$$400\, e^{-0.5t} \leq 10$$

$$e^{-0.5t} \leq \tfrac{1}{40}$$

$$e^{0.5t} \geq 40$$

$$\tfrac{1}{2}t \geq \ln 40$$

$$t \geq 2\ln 40$$

Population falls within 10 of limiting size after $7 \cdot 38$ yrs ≈ 7 yrs 5 months.

c. **Outcomes Assessed:** (i) **PE3** **(ii) PE3**

Marking Guidelines

Criteria	Marks
i. • shows $f(0)$, $f(1)$ have opposite signs	1
• notes continuity of f to justify deduction.	1
ii. • obtains expression for α by substitution into Newton's formula	1
• calculates at least one of $f(0 \cdot 7)$, $f'(0 \cdot 7)$ correctly	1
• approximates α to 2 decimal places	1

Answer

i. $f(x) = x - \cos x$

f is a continuous function and

$f(0) = 0 - 1 < 0$

$f(1) = 1 - \cos 1 > 0$

$\therefore f(\alpha) = 0$ for some α such that $0 < \alpha < 1$.

ii. $f'(x) = 1 + \sin x$

$$\alpha \approx 0 \cdot 7 - \frac{0 \cdot 7 - \cos 0 \cdot 7}{1 + \sin 0 \cdot 7}$$

$$\approx 0 \cdot 7 - \frac{-0 \cdot 065}{1 \cdot 644}$$

$$\approx 0 \cdot 74$$

Question 5

a. **Outcomes Assessed** (i) **HE3** (ii) **HE3**:

Marking Guidelines

Criteria	Marks
i. • writes appropriate expression for binomial probability	1
ii. • interprets *at most* as either sum or complement of appropriate binomial probabilities	1
• calculates the probability in fraction or decimal form	1

Answer

Binomial distribution: $n = 4,\ p = \frac{2}{5},\ q = \frac{3}{5}$

i. ${}^4C_3\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right) = \frac{96}{625}$ ii. $1 - {}^4C_4\left(\frac{2}{5}\right)^4 = 1 - \frac{16}{625} = \frac{609}{625}$

b. **Outcomes Assessed**: (i) **P3** (ii) **HE 5**

Marking Guidelines

Criteria	Marks
i. • obtains required expression for S in terms of h	1
ii. • writes expression for $\frac{dS}{dt}$ in terms of $\frac{dh}{dt}$	1
• evaluates $\frac{dS}{dt}$ when $h=2$	1
• interprets negative value and provides appropriate units	1

Answer

i. The surface of the water is a circle with radius x when the depth is y, where $x^2 = 4 - y$.

When the depth is h, $S = \pi x^2 = \pi(4-h)$

ii. $$\therefore \frac{dS}{dt} = -\pi\frac{dh}{dt} = -\pi\frac{10}{\pi(4-h)} = -5 \text{ when } h = 2$$

When depth is 2 cm, surface area of the water is decreasing at a rate of $5\,\text{cm}^2\text{s}^{-1}$.

c. **Outcomes Assessed**: (i) **H5** (ii) **H5** (iii) **PE3**

Marking Guidelines

Criteria	Marks
i. • finds $f''(x)$ and notes $f''(x) > 0$ for all x	1
ii. • finds coordinates of stationary point	1
• states nature of stationary point	1
iii. • deduces that $f(x) \ge 1$ for all x	1
• uses this result to deduce $e^x \ge x+1$ for all x	1

Answer

i. $f(x) = e^x - x$

$f'(x) = e^x - 1$

$f''(x) = e^x$

$f''(x) > 0$ for all x, hence curve is concave up for all x.

ii. $f'(x) = 0 \Rightarrow e^x = 1$ $\therefore$ stationary point is $(0,1)$

Since curve is concave up, $(0,1)$ is a minimum turning point

iii. $f(x) \ge 1$ for all $x \Rightarrow e^x - x \ge 1$ for all x

$\therefore e^x \ge x + 1$ for all x.

Question 6

a. **Outcomes Assessed:** **(i) HE4 (ii) HE4 (iii) H8**

Marking Guidelines

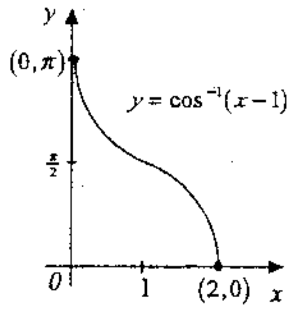
Criteria	Marks
i. • states domain of function	1
ii. • sketches curve with correct shape and position	1
• shows endpoints with correct coordinates	1
iii. • writes integral for V in terms of y	1
• finds primitive function	1
• evaluates V by substitution of correct limits	1

Answer

i. $f(x)=\cos^{-1}(x-1) \quad \Rightarrow -1\le x-1\le 1$

$\therefore$ Domain is $\{x: 0\le x\le 2\}$

ii.



iii. $V=\pi\int_0^{\pi} x^2 \; dy,$

where $\cos y = x-1 \Rightarrow x=1+\cos y.$

$$\therefore\; V=\pi\int_0^{\pi}(1+\cos y)^2 \; dy$$

$$=\pi\int_0^{\pi}\left(1+2\cos y+\cos^2 y\right)\; dy$$

$$=\pi\int_0^{\pi}\left(1+2\cos y+\tfrac{1}{2}(1+\cos 2y)\right)\; dy$$

$$=\pi\int_0^{\pi}\left(\tfrac{3}{2}+2\cos y+\tfrac{1}{2}\cos 2y\right)\; dy$$

$$=\pi\left[\tfrac{3}{2}y+2\sin y+\tfrac{1}{4}\sin 2y\right]_0^{\pi}$$

$$=\pi\left(\tfrac{3}{2}\pi+0+0\right)$$

Volume is $\frac{3}{2}\pi^2$ cubic units.

b. **Outcomes Assessed:** **(i) HE3 (ii) HE3 (iii) HE3**

Marking Guidelines

Criteria	Marks
i. • expresses x in terms of $\cos 2t$	1
• expresses $\ddot{x}$ in required form	1
ii. • finds possible values for x	1
• states period of motion	1
iii. • finds smallest t for which $x=0$	1
• finds initial x and deduces distance travelled	1

Answer

i. $x = 4\cos^2 t - 2\sin^2 t$

$= 2(1+\cos 2t) - (1-\cos 2t)$

$= 1 + 3\cos 2t$

$\dot{x} = -6\sin 2t$

$\ddot{x} = -12\cos 2t$

$= -4(x-1)$

$\ddot{x} = -2^2(x-1)$

ii.

$-1 \le \cos 2t \le 1$

$-3 \le 3\cos 2t \le 3$

$-2 \le 1 + 3\cos 2t \le 4$

$\therefore -2 \le x \le 4$

Period if the motion is $\dfrac{2\pi}{n} = \pi$ s

iii. $x = 0 \Rightarrow \cos 2t = -\frac{1}{3}$

Smallest such t is $\frac{1}{2}\cos^{-1}\left(-\frac{1}{3}\right) \approx 1 \cdot 0$

Initially particle is at $x = 4$.

Hence particle first passes through O after $1 \cdot 0$s when particle has travelled a distance of 4m.

Question 7

a. **Outcomes Assessed: (i) HE5 (ii) HE5 (iii) HE5, HE7**

Marking Guidelines

Criteria	Marks
i. • uses chain rule then simplifies using trig. identities	1
ii. • writes expression for $\dfrac{dt}{dx}$	1
• finds expression for t in terms of x, evaluating the constant of integration	1
• finds expression for x in terms of t	1
iii. • states limiting position	1
• sketches graph of x against t with correct shape, endpoint and asymptote	1

Answer

i.

$$\frac{d}{dx}\ln(\tan x) = \frac{\sec^2 x}{\tan x}$$

$$= \frac{1}{\cos^2 x}\,\frac{\cos x}{\sin x}$$

$$= \frac{1}{\sin x \cos x}$$

ii. $v = \sin x \cos x$

$$\frac{dx}{dt} = \sin x \cos x$$

$$\frac{dt}{dx} = \frac{1}{\sin x \cos x}$$

$$t = \ln(\tan x) + c$$

$t = 0,\ x = \frac{\pi}{4} \Rightarrow c = 0$

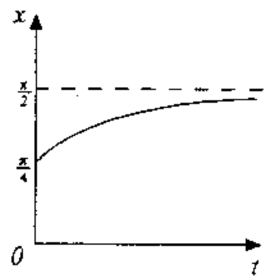
$\therefore t = \ln(\tan x)$

$e^t = \tan x$

$x = \tan^{-1}(e^t)$

iii. as $t \to \infty,\ x \to \frac{\pi}{2}$

$\therefore$ limiting position is $\frac{\pi}{2}$ metres to the right of O.



b. **Outcomes Assessed:** **(i) HE3 (ii) HE3 (iii) HE3**

Marking Guidelines

Criteria	Marks
i. • writes horizontal and vertical displacements for particle projected from A	1
• writes horizontal and vertical displacements for particle projected from O	1
ii. • equates expressions for x and y to obtain equations (1) and (2) if particles collide	1
• solves simultaneously to find $\cos\theta$, $\sin\theta$ and t if collision occurs	1
iii. • obtains values for $\dot{x}$, $\dot{y}$ for each particle for $t=1$ and $\theta=\tan^{-1}2$	1
• deduces that if particles collide, their velocities are perpendicular at that time	1

Answer

i. Partical projected from A :

horizontal displacement $x=10t$

vertical displacement $y=20-5t^2$

Particle projected from O :

horizontal displacement $x=10\sqrt{5}\,t\cos\theta$

vertical displacement $y=10\sqrt{5}\,t\sin\theta-5t^2$

ii. If the particles collide at some time t

$$10t=10\sqrt{5}\,t\cos\theta \qquad (1) \text{ and}$$

$$20-5t^2=10\sqrt{5}\,t\sin\theta-5t^2$$

$$20=10\sqrt{5}\,t\sin\theta \qquad (2)$$

From (1), $\cos\theta=\frac{1}{\sqrt{5}}$. $\therefore \sin\theta=\frac{2}{\sqrt{5}}$

Substituting in (2) gives $t=1$

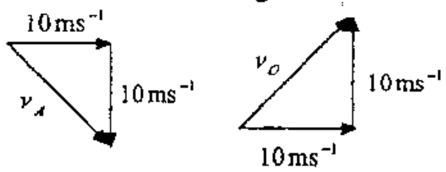
Hence the particles collide if $\theta=\tan^{-1}2$, and in this case they collide after 1 s.

iii. If $\theta=\tan^{-1}2$, when $t=1$

the particle from A has $\dot{x}=10$ and $\dot{y}=-10$

the particle from O has $\dot{x}=10$ and $\dot{y}=20-10=10$

Hence the particles have velocities v_A and v_O as shown in the diagrams below :



Hence if the particles collide, when they do so the particle from A is travelling in a direction 45° below the horizontal while the particle from O is travelling in a direction 45° above the horizontal, and their paths of motion are perpendicular to each other.