QUESTION ONE

(a) $\dfrac{1}{x-3} < 3,\ x \neq 3$

$\dfrac{1}{x-3} \times (x-3)^2 < 3(x-3)^2$

$x - 3 < 3(x-3)^2$ $\boxed{\checkmark}$

$3(x-3)^2 - (x-3) > 0$

$(x-3)(3(x-3)-1) > 0$

$(x-3)(3x-10) > 0$

$x < 3$ or $x > \frac{10}{3}$. $\boxed{\checkmark}$

(b) $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\tfrac{x}{3}\right]_0^3$ $\boxed{\checkmark}$

$= \sin^{-1} 1 - \sin^{-1} 0$

$= \frac{\pi}{2}$. $\boxed{\checkmark}$

(c) (i) $y = \tan^{-1} 2x$

$\dfrac{dy}{dx} = \dfrac{2}{1+4x^2}$. $\boxed{\checkmark}$

(ii) $y = \log_e \cos x$

$\dfrac{dy}{dx} = -\dfrac{\sin x}{\cos x}$ $\boxed{\checkmark \text{ for } -\sin x\ \checkmark \text{ for quotient}}$

(d) $\tan\theta = |-\frac{5}{3}|$ $\boxed{\checkmark}$

$\theta \doteqdot 59°$ $\boxed{\checkmark}$

(e) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ $\boxed{\checkmark}$

$= \dfrac{3}{2} \div 1$

$= \dfrac{3}{2}$ $\boxed{\checkmark\checkmark \text{ any correct method}}$

QUESTION TWO

(a) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{1+\tan x}}\,dx = \int_1^2 \frac{du}{u^{\frac{1}{2}}}$ $\boxed{\checkmark}$

$\displaystyle = \int_1^2 u^{-\frac{1}{2}}\,du$

$= \left[2u^{\frac{1}{2}}\right]_1^2$ $\boxed{\checkmark}$

$= 2\sqrt{2} - 2$ $\boxed{\checkmark}$

Let $u = 1 + \tan x$

$du = \sec^2 x\,dx$

When $x = 0,\ u = 1$,

When $x = \frac{\pi}{4},\ u = 2$.

(b) General term $= {}^6C_r\,(x^2)^{6-r}\,(-1)^r\,(3x^{-2})^r$

$= {}^6C_r\,(x)^{12-2r}\,(-1)^r\,(3)^r\,(x)^{-2r}$

$= {}^6C_r\,(-1)^r\,(3)^r\,(x)^{12-4r}$ $\boxed{\checkmark}$

Let $12 - 4r = 0$

$r = 3$ $\boxed{\checkmark}$

Term independent of x $= {}^6C_3\,(-1)^3\,(3)^3$

$= -540$. $\boxed{\checkmark}$

(c) $LHS = \dfrac{\tan 2\theta - \tan\theta}{\tan 2\theta + \cot\theta}$

Let $t = \tan\theta$

$LHS = \left(\dfrac{2t}{1-t^2} - t\right) \div \left(\dfrac{2t}{1-t^2} + \dfrac{1}{t}\right)$ $\boxed{\checkmark}$

$= \dfrac{2t - t + t^3}{1-t^2} \times \dfrac{t(1-t^2)}{2t^2 + 1 - t^2}$

$= \dfrac{t(1+t^2)}{1-t^2} \times \dfrac{t(1-t^2)}{t^2+1}$

$\boxed{\checkmark \text{ correct method of simplification of the algebraic fractions}}$

$= t^2$ $\boxed{\checkmark}$

$= \tan^2\theta$

$= RHS$

(d) (i) $V = \frac{4}{3}\pi r^3$

$\dfrac{dv}{dt} = 4\pi r^2 \dfrac{dr}{dt}$ $\boxed{\checkmark}$

$8 = 64\pi \dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{1}{8\pi}$ m/min $\boxed{\checkmark}$

(ii) $S = 4\pi r^2$

$\dfrac{dS}{dt} = 8\pi r \dfrac{dr}{dt}$

$= 8\pi r \times \dfrac{1}{8\pi}$

$= 4\,\text{m}^2/\text{min}$. $\boxed{\checkmark}$

QUESTION THREE

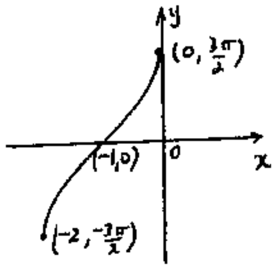
(a) (i) $f(x) = 3\sin^{-1}(x+1)$

Domain: $-1 \le x+1 \le 1$

$-2 \le x \le 0$ $\boxed{\checkmark}$

Range: $-\dfrac{3\pi}{2} \le y \le \dfrac{3\pi}{2}$. $\boxed{\checkmark}$

(ii)



$\boxed{\checkmark \text{ Shape}}$
$\boxed{\checkmark \text{ Labels}}$

(b) (i) $v^2 = 2x(6-x)$

$2x(6-x) \ge 0$

$0 \le x \le 6$ $\boxed{\checkmark}$

(ii) $x = 3$ $\boxed{\checkmark}$

(iii) Maximum speed when $x = 3$.

$v^2 = 6 \times 3$

$|v| = 3\sqrt{2}$ $\boxed{\checkmark}$

(iv) $v^2 = 2x(6-x)$

$\frac{1}{2}v^2 = 6x - x^2$

$\dfrac{d}{dx}(\frac{1}{2}v^2) = 6 - 2x$

$\ddot{x} = 6 - 2x$ $\boxed{\checkmark}$

(c) Given $\left(2 + \dfrac{x}{3}\right)^n$:

term in $x^6 = {}^nC_6 \times 2^{n-6} \times (\frac{x}{3})^6$

term in $x^7 = {}^nC_7 \times 2^{n-7} \times (\frac{x}{3})^7$ $\boxed{\checkmark \text{ 1 mark for both answers}}$

$$\text{Ratio of coefficients} = \frac{\dfrac{n!}{6!(n-6)!} \times 2^{n-6} \times (\frac{1}{3})^6}{\dfrac{n!}{7!(n-7)!} \times 2^{n-7} \times (\frac{1}{3})^7}$$

$$= \frac{n!}{6!(n-6)!} \times \frac{7!(n-7)!}{n!} \times 3 \times 2 \quad \boxed{\checkmark}$$

$$= \frac{42}{n-6} \quad \boxed{\checkmark}$$

so $\dfrac{7}{8} = \dfrac{42}{n-6}$

$n - 6 = 48$

$n = 54.$ $\boxed{\checkmark}$

QUESTION FOUR

(a) Let $P(x) = 2x^3 + ax^2 + bx + 6$

$$P(1) = 2 + a + b + 6$$
$$0 = a + b + 8$$
$$a + b = -8 \quad \cdots(1) \quad \boxed{\surd \text{ for any correct form}}$$
$$P(-2) = -16 + 4a - 2b + 6$$
$$-12 = 4a - 2b - 10$$
$$4a - 2b = -2$$
$$2a - b = -1 \quad \cdots(2) \quad \boxed{\surd \text{ for any correct form}}$$

(1) + (2) $\quad 3a = -9$

$$a = -3 \quad \boxed{\surd}$$
$$b = -5 \quad \boxed{\surd}$$

(b) $x^3 + px^2 + qx + r = 0$

$$3\alpha = -p \quad \cdots(1) \quad \boxed{\surd}$$
$$3\alpha^2 = q \quad \cdots(2) \quad \boxed{\surd}$$
$$\alpha^3 = -r \quad \cdots(3) \quad \boxed{\surd}$$

(1) × (2) $\quad 9\alpha^3 = -pq$

$$-9r = -pq$$
$$pq = 9r \quad \boxed{\surd}$$

(c) (i) $(1+x)^4(1+x)^4 = \left({}^4C_0 + {}^4C_1x + {}^4C_2x^2 + {}^4C_3x^3 + {}^4C_4x^4\right)$
$\times \left({}^4C_0 + {}^4C_1x + {}^4C_2x^2 + {}^4C_3x^3 + {}^4C_4x^4\right) \quad \boxed{\surd}$

Term in $x^5 = {}^4C_1x \times {}^4C_4x^4 + {}^4C_2x^2 \times {}^4C_3x^3 + {}^4C_3x^3 \times {}^4C_2x^2 + {}^4C_4x^4 \times {}^4C_1x \quad \boxed{\surd}$

Coefficient $= {}^4C_1 \times {}^4C_4 + {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_2 + {}^4C_4 \times {}^4C_1$

$= {}^4C_0 \times {}^4C_1 + {}^4C_1 \times {}^4C_2 + {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_4$, by symmetry. $\boxed{\surd}$

(ii) Coefficient of x^5 in $(1+x)^8 = {}^8C_5$

$$= \frac{8!}{3! \times 5!} \quad \boxed{\surd}$$

Now $(1+x)^4(1+x)^4 = (1+x)^8$,

so ${}^4C_0 \times {}^4C_1 + {}^4C_1 \times {}^4C_2 + {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_4 = \dfrac{8!}{3! \times 5!}$.

QUESTION FIVE

(a) (i) Given $T = 20 + Ae^{-kt}$

$$\frac{dT}{dt} = -kAe^{-kt}$$
$$= -k(T - 20). \quad \boxed{\checkmark}$$

So $T = 20 + Ae^{-kt}$ is a solution.

(ii) When $t = 0,\ T = 36$

so $36 = 20 + Ae^0$

$A = 16. \quad \boxed{\checkmark}$

When $t = 5,\ T = 35$

so $35 = 20 + 16e^{-5k}$

$$15 = 16e^{-5k}$$
$$e^{-5k} = \frac{15}{16} \quad \boxed{\checkmark}$$
$$-5k = \log_e \frac{15}{16}$$
$$k = -\frac{1}{5}\log_e \frac{15}{16}. \quad \boxed{\checkmark}$$

(iii) When $T = 27$,

$$27 = 20 + 16e^{-kt}$$
$$e^{-kt} = \frac{7}{16} \quad \boxed{\checkmark}$$
$$t = \frac{\log_e \frac{7}{16}}{-k}$$
$$= 64{\cdot}045\ldots\ldots$$

It will take 64 minutes. $\boxed{\checkmark}$

(iv) As $t \to \infty,\ T \to 20$ from above.
The temperature does not drop below 20°C and so will never reach 18°C. $\boxed{\checkmark}$

(b) (i)

$$y = \frac{x^2}{4a}$$
$$\frac{dy}{dx} = \frac{x}{2a}$$

At T,
$$\frac{dy}{dx} = \frac{2at}{2a}$$
$$= t. \quad \boxed{\checkmark}$$

Now
$$y - at^2 = t(x - 2at)$$
$$y - at^2 = tx - 2at^2$$

so $y - tx + at^2 = 0. \quad \boxed{\checkmark}$

(ii) Let $x = 0$

so $y = -at^2$

R is the point $(0, -at^2)$. $\boxed{\checkmark}$

(iii) R lies on PQ.

$$y - \tfrac{1}{2}(p+q)x + apq = 0$$
$$-at^2 + apq = 0 \quad \boxed{\checkmark}$$
$$t^2 = pq,\ a \neq 0$$
$$\frac{t}{p} = \frac{q}{t}$$

So p, t, and q form a geometric sequence. $\boxed{\checkmark}$

QUESTION SIX

(a) (i) Area of minor segment $= \frac{1}{2}r^2(\theta - \sin\theta)$

Area of major segment $= \pi r^2 - \frac{1}{2}r^2(\theta - \sin\theta)$

$$\text{Ratio of areas} = \frac{\pi r^2 - \frac{1}{2}r^2(\theta - \sin\theta)}{\frac{1}{2}r^2(\theta - \sin\theta)} \quad \boxed{\checkmark}$$

$$= \frac{2\pi - \theta + \sin\theta}{\theta - \sin\theta} \quad \boxed{\checkmark}$$

(ii) (α)

$$\frac{2\pi - \theta + \sin\theta}{\theta - \sin\theta} = \frac{\pi - 1}{1}$$

$$\pi\theta - \pi\sin\theta - \theta + \sin\theta = 2\pi - \theta + \sin\theta$$

$$\theta - 2 - \sin\theta = 0 \quad \boxed{\checkmark}$$

(β) Let $f(\theta) = \theta - 2 - \sin\theta$

$$\begin{aligned} f(2) &= -\sin 2 \\ &\doteqdot -0{\cdot}909 \\ &< 0 \\ f(3) &= 1 - \sin 3 \\ &\doteqdot 0{\cdot}859 \\ &> 0. \end{aligned}$$

So the root lies between $\theta = 2$ and $\theta = 3$. $\boxed{\checkmark}$

(γ) $f(\theta) = \theta - 2 - \sin\theta$

$f'(\theta) = 1 - \cos\theta$.

Let θ_0 be the first approximation.

$$\theta_1 = \theta_0 - \frac{\theta_0 - 2 - \sin\theta_0}{1 - \cos\theta_0}$$

$$\theta_1 = 2{\cdot}5 - \frac{2{\cdot}5 - 2 - \sin 2{\cdot}5}{1 - \cos 2{\cdot}5} \quad \boxed{\checkmark}$$

$$\doteqdot 2{\cdot}55$$

(δ) When $\theta = 2{\cdot}5$,
$|\theta - 2 - \sin\theta| \doteqdot 0{\cdot}09847$.

(ε) When $\theta = 2{\cdot}55$,
$|\theta - 2 - \sin\theta| \doteqdot 0{\cdot}00768$. So $\theta = 2{\cdot}55$ yields a smaller value. $\boxed{\checkmark}$

(b) (i) $\angle PAF = \angle PBF$ angles at circumference standing on the same arc $\boxed{\checkmark}$

$\angle PAF = \alpha$.

(ii) $\angle ANB = \angle AMB$ (both given as rightangles).
These lie on the same interval AB and so A,N,M and B are concyclic. $\boxed{\checkmark}$

(iii) $\angle NBM = \angle MAN$ (angles standing on the same arc of circle $ANMB$) $\boxed{\checkmark}$

$\angle NBM = \alpha$.

(iv) $\triangle BHM \equiv \triangle BFM$ (AAS test) $\boxed{\checkmark}$

$HM = MF$ (matching sides of congruent triangles) $\boxed{\checkmark}$

(v) $\angle APB$ stands on fixed chord AB and its size is independent of the position of P (angles at circumference standing on the same chord). So α is independent of the position of P. $\boxed{\checkmark}$

QUESTION SEVEN

(a) (i) For A: $y = -\dfrac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha \cdots (1)$

For B: $y = -\dfrac{gx^2}{2V^2}\sec^2\beta + x\tan\beta \cdots (2)$

At R the coordinates are identical, so substitute (1) in (2).

$$-\frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha = -\frac{gx^2}{2V^2}\sec^2\beta + x\tan\beta \quad \boxed{\checkmark}$$

$$\frac{gx^2}{2V^2}\left(\sec^2\alpha - \sec^2\beta\right) = x\left(\tan\alpha - \tan\beta\right)$$

$$\frac{gx}{2V^2}\left(\tan^2\alpha - \tan^2\beta\right) = \left(\tan\alpha - \tan\beta\right),\ x \neq 0 \quad \boxed{\checkmark}$$

$$\begin{aligned}\frac{gx}{2V^2} &= \frac{(\tan\alpha - \tan\beta)}{(\tan^2\alpha - \tan^2\beta)}\\ x &= \frac{2V^2}{g} \times \frac{1}{\tan\alpha + \tan\beta},\ \tan\alpha \neq \tan\beta\\ &= \frac{2V^2}{g} \times \frac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta}\\ &= \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)} \quad \boxed{\checkmark}\end{aligned}$$

(ii) (α) $x = V(t-T)\cos\beta$. $\boxed{\checkmark}$

(β) When A is at R:

$$\begin{aligned}Vt\cos\alpha &= \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}\\ t &= \frac{2V\cos\beta}{g\sin(\alpha+\beta)} \quad \cdots (3) \quad \boxed{\checkmark}\end{aligned}$$

When B is at R:

$$\begin{aligned}V(t-T)\cos\beta &= \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}\\ t - T &= \frac{2V\cos\alpha}{g\sin(\alpha+\beta)}\\ T &= t - \frac{2V\cos\alpha}{g\sin(\alpha+\beta)}\\ &= \frac{2V\cos\beta}{g\sin(\alpha+\beta)} - \frac{2V\cos\alpha}{g\sin(\alpha+\beta)},\ \text{from (3)}\\ &= \frac{2V(\cos\beta - \cos\alpha)}{g\sin(\alpha+\beta)} \quad \boxed{\checkmark}\end{aligned}$$

(b) (i) Prove by mathematical induction the proposition that for all positive integers

$$\sin(n\pi + x) = (-1)^n\sin x,\ \text{for } 0 < x < \tfrac{\pi}{2}.$$

A. When $n = 1$,

$$\begin{aligned}LHS &= \sin(\pi + x)\\ &= -\sin x\\ &= RHS.\end{aligned}$$

The proposition is true for $n = 1$. $\boxed{\checkmark}$

B. Assume the proposition is true for some positive integer k so that $\sin(k\pi + x) = (-1)^k\sin x \cdots (*)$

We are required to prove the proposition true for $n = k+1$.

That is, $\sin[(k+1)\pi + x] = (-1)^{k+1}\sin x$. $\boxed{\checkmark}$

Now

$$\begin{aligned}LHS &= \sin[(k+1)\pi + x]\\ &= \sin[\pi + (k\pi + x)] \quad \boxed{\checkmark}\\ &= \sin\pi\cos(k\pi + x) + \cos\pi\sin(k\pi + x)\\ &= -1 \times \sin(k\pi + x)\\ &= -1 \times (-1)^k\sin x,\ \text{from } (*) \quad \boxed{\checkmark}\\ &= (-1)^{k+1}\sin x\\ &= RHS\end{aligned}$$

It follows from A and B by mathematical induction that for all positive integ n, $\sin(n\pi + x) = (-1)^n\sin x$, for $0 < x < \frac{\pi}{2}$.

(ii)

$$\begin{aligned}S &= \sin(\pi + x) + \sin(2\pi + x) + \sin(3\pi + x) + \cdots + \sin(n\pi +\\ &= -\sin x + \sin x - \sin x + \cdots + \sin(n\pi + x)\end{aligned}$$

When n is odd $S = -\sin x$

so $-1 < S < 0$, for $0 < x < \frac{\pi}{2}$. $\boxed{\checkmark}$

When n is even $S = 0$.

So $-1 < S \leq 0$. $\boxed{\checkmark}$