2007 THSC Mathematics Extension 1: **Solutions— Question 1**

1. (a) Evaluate $\lim_{x\to 0}\dfrac{\sin 4x}{5x}$. $\boxed{1}$

Solution:
$$\begin{aligned}\lim_{x\to 0}\frac{\sin 4x}{5x} &= \lim_{x\to 0}\frac{\sin 4x}{4x}\times\frac{4}{5},\\ &= \frac{4}{5}\times\lim_{x\to 0}\frac{\sin 4x}{4x},\\ &= \frac{4}{5}.\end{aligned}$$

(b) Calculate the acute angle (to the nearest minute) between the lines $2x+y=4$ and $x-3y=6$. $\boxed{2}$

Solution:
$$\begin{aligned}\tan\alpha &= \frac{|-2-{}^1\!/\!_3|}{1+(-2)\times({}^1\!/\!_3)},\\ &= 7.\\ \therefore\ \alpha &= \tan^{-1}7,\\ &= 81.86989765^\circ \text{ by calculator},\\ &= 81^\circ 52'.\end{aligned}$$

(c) i. Show that $x+1$ is a factor of x^3-4x^2+x+6. $\boxed{1}$

Solution: Putting
$$\begin{aligned}P(x) &= x^3-4x^2+x+6;\\ P(-1) &= -1-4-1+6,\\ &= 0.\end{aligned}$$
$\therefore\ x+1$ is a factor.

ii. Hence or otherwise factorise x^3-4x^2+x+6 fully. $\boxed{2}$

Solution: Possible factors of 6 are 1, 2, 3 or 1, −2, −3.
$$\begin{aligned}P(-2) &= -8-16-2+6\neq 0,\\ P(2) &= 8-16+2+6,\\ &= 0.\end{aligned}$$
$\therefore\ x^3-4x^2+x+6=(x+1)(x-2)(x-3)$.

(d) The point $P(5, 7)$ divides the interval joining the points $A(-1, 1)$ and $B(3, 5)$ externally in the ratio $k : 1$. Find the value of k. $\boxed{2}$

Solution:
$$\begin{aligned} \frac{5 - {}^-1}{5-3} &= \frac{k}{1}, \\ 6 &= 2k, \\ k &= 3. \end{aligned}$$

(e) Find the horizontal asymptote of the function $y = \dfrac{3x^2 - 4x + 1}{2x^2 - 1}$. $\boxed{1}$

Solution: $\displaystyle\lim_{x \to \pm\infty} \frac{3 - {}^4/_x + {}^1/_{x^2}}{2 - {}^1/_{x^2}} = \frac{3}{2}$.
$\therefore\ y = {}^3/_2$ is the horizontal asymptote.

(f) Find a primitive of $\dfrac{1}{\sqrt{4 - x^2}}$. $\boxed{1}$

Solution: From the table of standard integrals,

$$\int \frac{dx}{\sqrt{4 - x^2}} = \sin^{-1}\frac{x}{2} + c.$$

(g) Solve the equation $|x + 1|^2 - 4|x + 1| - 5 = 0$. $\boxed{2}$

Solution:
$$\begin{aligned} \text{Putting } y &= |x+1|; \\ y^2 - 4y - 5 &= 0, \\ (y-5)(y+1) &= 0, \\ \therefore\ y &= 5 \text{ or } -1. \\ \text{But } |x+1| &\geq 0, \\ \text{hence } x + 1 &= 5 \text{ or } x + 1 = -5, \\ \text{so } x &= 4,\ -6. \end{aligned}$$

Solutions Q2 3 unit Trial HSC 2001 /12

(a)(i) $f(x) = \frac{1}{2}\cos^{-1}\left(\frac{x}{3}\right)$

$y = \cos^{-1}x$ has $-1 \le x \le 1$ D
$0 \le y \le \pi$ R

Domain $-1 \le x \le 1$
$-1 \le \frac{x}{3} \le 1$
$-3 \le x \le 3$ 1

Range: $0 \le \cos^{-1}x \le \pi$
$\frac{1}{2}\times 0 \le \frac{1}{2}\cos^{-1}\left(\frac{x}{3}\right) \le \frac{1}{2}\times\pi$
$0 \le f(x) \le \frac{\pi}{2}$ 1

(ii) $f'(x) = \frac{1}{2} \times \frac{-1}{\sqrt{1-\frac{x^2}{9}}} \times \frac{1}{3}$

$= -\frac{1}{6} \times \frac{1}{\sqrt{\frac{9-x^2}{9}}} = -\frac{1}{6} \times \frac{3}{\sqrt{9-x^2}} = -\frac{1}{2} \cdot \frac{1}{\sqrt{9-x^2}}$

So for $-3 < x < 3$, $f'(x) < 0$ always. 2

(iii) when $x=0$, $f(x) = \frac{1}{2}\cos^{-1}\left(\frac{0}{3}\right)$
$= \frac{1}{2}\cos^{-1}(0)$
$= \frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{4}$

$f'(x) = \frac{-1}{2\sqrt{9-x^2}}$

at $x=0$, $m = \frac{-1}{2\times 3} = -\frac{1}{6}$

$(y-y_1) = m(x-x_1)$
$\left(y-\frac{\pi}{4}\right) = -\frac{1}{6}(x-0)$
$y = -\frac{1}{6}x + \frac{\pi}{4}$
or $\frac{1}{6}x + y - \frac{\pi}{4} = 0$
or $12\times\frac{1}{6}x + 12y - 12\times\frac{\pi}{4} = 0$
$2x + 12y - 3\pi = 0$ 2

(b) $y = \ln(\sin^3 x)$

$y' = \frac{1}{\sin^3 x} \times 3\sin^2 x \times \cos x \times 1$

$= \frac{3\sin^2 x\cos x}{\sin^3 x}$

$= \frac{3\cos x}{\sin x}$

$= 3\cot x$ 2

(c) (i) $1\cos x - \sqrt{3}\sin x$ $\qquad A = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$

$= 2\left(\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x\right)$

$= A(\cos x \cos\alpha - \sin x \sin\alpha)$

$\Rightarrow A = 2.$ (1)

$\left.\begin{array}{l}\cos\alpha = \frac{1}{2} \\ \sin\alpha = \frac{\sqrt{3}}{2}\end{array}\right\} \alpha = 60^\circ = \frac{\pi}{3}$ (1) (1^{st} quad $0 < \alpha < \frac{\pi}{2}$)

So $\cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right).$

(ii) Now $\cos x - \sqrt{3}\sin x + 1 = 0$

$\cos x - \sqrt{3}\sin x = -1$

$2\cos\left(x + \frac{\pi}{3}\right) = -1$

$\cos\left(x + \frac{\pi}{3}\right) = -\frac{1}{2}.$ $\qquad \cos\frac{\pi}{3} = \frac{1}{2}$

$2^{nd}/3^{rd}$ quad.

now $x + \frac{\pi}{3} = \pi - \frac{\pi}{3}$ $\quad 2^{nd}$ quad.

and $x + \frac{\pi}{3} = \pi + \frac{\pi}{3}$ $\quad 3^{rd}$ quad.

So $x = \pi - \frac{2\pi}{3} = \frac{3\pi - 2\pi}{3} = \pi/3$ ✓

and $x = \pi + \frac{\pi}{3} - \frac{\pi}{3} = \pi.$ ✓ (2)

Q3. (a) (i) let $f(x) = e^{x} - x - 2.$

now $f(1) = e - 1 - 2 = e - 3 < 0.$ (~ -0.28)

& $f(2) = e^{2} - 2 - 2 = e^{2} - 4 > 0.$ $(\sim 3.38).$

Since $f(x)$ changes sign in $1 < x < 2$ (✓)

$f(x) = 0$ has a solution in $1 < x < 2.$

(ii) now $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ & $f(x) = e^{x} - x - 2.$

$f'(x) = e^{x} - 1.$

$$\therefore x_2 = 1.5 - \frac{f(1.5)}{f'(1.5)}$$

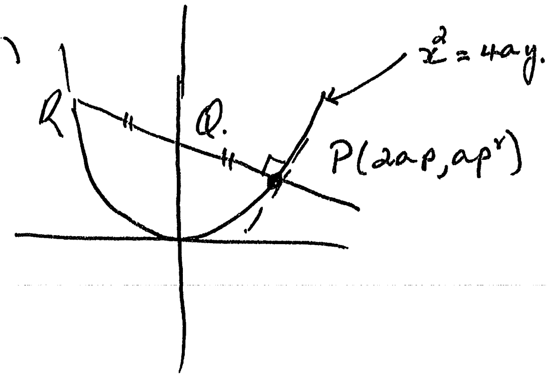
$$= 1.5 - \frac{(e^{1.5} - 1.5 - 2)}{e^{1.5} - 1.}$$

$$\doteqdot 1.5 - \frac{0.98168\ldots}{3.481689\ldots}$$

$$\doteqdot 1.5 - 0.281955\ldots$$ (✓✓)

$\doteqdot \underline{1.218}$

(b)



(i) $y = \frac{1}{4a} x^2$

$y' = \frac{1}{2a} x.$

$\therefore m_T = \frac{2ap}{2a} = p$

$\therefore m_N = \frac{-1}{p}$

N: $\frac{y - ap^2}{x - 2ap} = \frac{-1}{p}$

$py - ap^3 = -x + 2ap$

$\boxed{x + py = 2ap + ap^3.}$ (✓✓)

(ii) Co-ords of Q. $x = 0$ $\therefore py = 2ap + ap^3$

$y = 2a + ap^2$

$\therefore Q\ (0, 2a + ap^2)$ (✓).

(iii) Q is the mid-pt of PR.

let ~~R~~ be (x_1, y_1)

$\therefore \frac{x_1 + 2ap}{2} = 0$

$\therefore x_1 = -2ap.$

$\frac{y_1 + ap^2}{2} = 2a + ap^2$

$y_1 + ap^2 = 4a + 2ap^2$

$y_1 = ap^2 + 4a.$

$\therefore R\ (-2ap, ap^2 + 4a)$ (✓)

(iii) To find the locus of R. $x = -2ap$ $\therefore p = \frac{x}{-2a}$

$\therefore y = a\left(\frac{x}{-2a}\right)^2 + 4a.$

$y = \frac{x^2}{4a} + 4a.$

$$4ay = x^2 + 16a^2$$
$$x^2 = 4ay - 16a^2$$
$$x^2 = 4a(y - 4a)$$

$\boxed{\text{PARABOLA VERTEX } (0, 4a).}$ (✓✓✓)

(c) $$\int_1^5 (2f(x)+1)\,dx = 2\int_1^5 f(x)\,dx + \int_1^5 1.\,dx.$$
$$= 2 \times 3 + [x]_1^5$$
$$= 6 + (5-1).$$
$$= \underline{10}.$$
(✓✓)

Question (4)

(a) Let $u = e^x$.

$du = e^x \, dx$. 1

3

$\int e^{(e^x + x)} dx$

$= \int e^{e^x} \cdot e^x \, dx$ 1

$= \int e^u \, du = e^u + c$

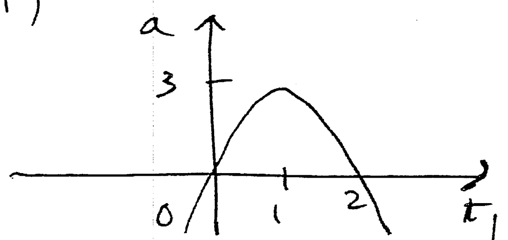
$\Rightarrow \boxed{e^{e^x} + c}$ 1

(b) For $0 \le t \le 2$.

(i) $v = 3t^2 - t^3$ 1

$a = \frac{dv}{dt} = \boxed{6t - 3t^2} = \boxed{3t(2-t)}$

(ii)



1

From the graph of a versus t.

2

The max. acceleration occurs when $t = 1$

$\therefore a_{max} = 3 \times 1 = \boxed{3 \text{ m/s}^2}$

(iii) Let $d(t)$ be the total distance travelled (which is 41m).

$\therefore d(t) = 2\int_0^2 v(t)\,dt + \int_2^{T-2} 4 \, dt$

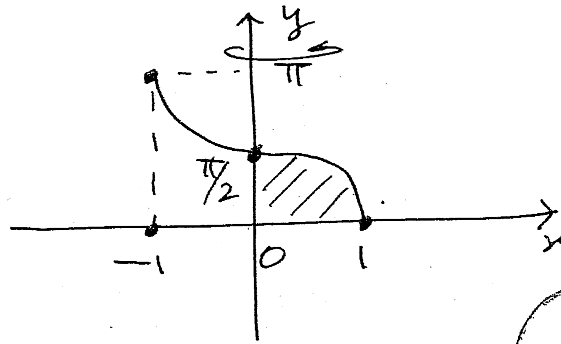
$\therefore 41 = 2\int_0^2 (3t^2 - t^3)\,dt + \left[4t\right]_2^{T-2}$. 2

$\therefore \boxed{41 = 2\left[t^3 - \frac{t^4}{4}\right]_0^2 + 4(T-2) - 8}$

$\therefore 41 = 2(8-4) + 4T - 16$.

$41 = 4T - 8 \Rightarrow \boxed{T = 12\frac{1}{4}}$.

(c)



1

$y = \cos^{-1} x$.

$\Rightarrow x = \cos y, \begin{cases} -1 \le x \le 1 \\ 0 \le y \le \pi \end{cases}$

$\therefore V = \pi \int_0^{\pi/2} x^2 \, dy$ ✓

$\therefore x = \cos y$, $V = \pi \int_0^{\pi/2} \cos^2 y \, dy$

(ii) $V = \pi \int_0^{\pi/2} \cos^2 y \, dy$.

3

but $\cos^2 y = \left(\frac{1 + \cos 2y}{2}\right)$ 1

$= \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2y)\, dy$

$= \frac{\pi}{2}\left[y + \frac{\sin 2y}{2}\right]_0^{\pi/2}$

$= \frac{\pi}{2}\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$ 1

EXI QUESTION 5

(a) if $n=1$, $1\times 1! = (1+1)! - 1$

$\therefore P(1)$ is true

Assume $P(k)$ is true $1\times 1! + 2\times 2! \ldots + k\times k! = (k+1)! - 1$

if $P(k+1)$ is $1\times 1! + 2\times 2! \ldots + k\times k! + (k+1)(k+1)! = (k+2)! - 1$

LHS is $(k+1)! - 1 + (k+1)(k+1)!$ using assumption

$= (k+1)!(1 + k+1) - 1$

$= (k+1)!(k+2) - 1 = (k+2)! - 1 = RHS$

$\therefore P(k+1)$ is true if $P(k)$ is true. $P(1)$ is true + by Mathematical induction $\sum_{r=1}^{n} r\times r! = (n+1)! - 1$

(b)

$T_{k+1} = {}^{15}C_k (2x)^{n-k} (x^{-2})^k$

for term independent of x $n-k-2k=0$, $k=5$

$Cf\ {}^{15}C_5 \times 2^{10} = 3075072$

(c)(i) $\frac{d(\frac{1}{2}v^2)}{dx} = 8x(x^2+1) = 8x^3+8x$

$\frac{1}{2}v^2 = 2x^4 + 4x^2 + C$

$v = -2 \quad x = 0 \quad C = 2$

$v^2 = 4x^4 + 8x^2 + 4 = 4(x^4 + 2x^2 + 1)$

$v = \pm 2(x^2+1)^2$

(ii) if $\frac{dx}{dt} = 2(x^2+1)$

$\frac{dt}{dx} = \frac{1}{2}\left(\frac{1}{x^2+1}\right)$

$t = \frac{1}{2}\tan^{-1}x + C$

$t = 0 \quad x = 0, \quad C = 0$

$2t = \tan^{-1}x$

$x = \tan 2t$

(iii)

$t = \pi/8, \quad x = \tan \pi/4$

$= 1$

$v = 2(1+1)$ from (i)

$v = 4\ m/s$

Question 6

(a)

$\angle CBD = 60^\circ$ (alternate segment theorem)

$\angle BCD = 90^\circ$ (angle in semicircle)

$\therefore \angle CDB = 30^\circ$ (angle sum of triangle)

$\therefore \angle CAB = 30^\circ$ (angles at circumference on same arc)

(b) (i) $(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_{n-1}x^{n-1} + x^n$

Differentiating with respect to x:

$n(1+x)^{n-1} = {}^nC_1 + 2\,{}^nC_2x + 3\,{}^nC_3x^2 + \ldots + n\,{}^nC_nx^{n-1}$

Let $x = 1$:

$n2^{n-1} = {}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \ldots + n\,{}^nC_n$

QED

(ii) Multiplying $(1+x)^n$ by x:

$x(1+x)^n = {}^nC_0x + {}^nC_1x^2 + {}^nC_2x^3 + \ldots + {}^nC_nx^{n+1}$

Differentiating with respect to x:

$xn(1+x)^{n-1} + (1+x)^n = {}^nC_0 + 2\,{}^nC_1x + 3\,{}^nC_2x^2 + \ldots + (n+1)\,{}^nC_nx^n$

Let $x = 1$:

$n(2)^{n-1} + (2)^n = 1 + 2\,{}^nC_1 + 3\,{}^nC_2 + \ldots + (n+1)\,{}^nC_n$

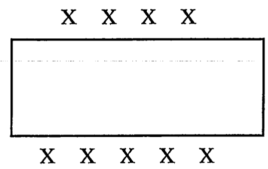
Thus

$$\begin{aligned} 2\,{}^nC_1 + 3\,{}^nC_2 + \ldots + (n+1)\,{}^nC_n &= n(2)^{n-1} + (2)^n - 1 \\ &= (n+2)2^{n-1} - 1 \end{aligned}$$

(c)

$$\begin{aligned} f(x+2) &= x^2 + 2 \\ f(x) &= (x-2)^2 + 2 \\ &= x^2 - 4x + 4 + 2 \\ &= x^2 - 4x + 6 \end{aligned}$$

(d)



(i) If J&M sit on the short side, they can be arranged in 12 ways, and the other guests in 7! ways. Thus $12 \times 7!$ ways.
If J&M sit on the long side they can be arranged in 20 ways, and the other guests in 7! Ways. Thus $20 \times 7!$

Hence there are $32 \times 7! = 161\ 280$ ways.

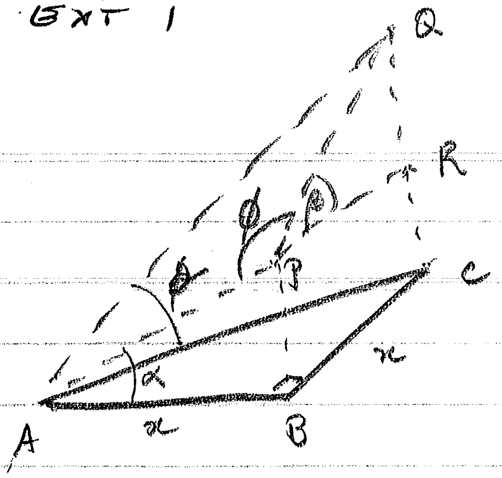
(ii) If John sits on the short side he has four seats available, and Mary (on the long side) has 5, thus $20 \times 7!$

But Mary may be the one on the short side.

Thus the total is $40 \times 7! = 201\ 600$

2007 EXT 1

Q7



(i) $\tan\alpha = \frac{BP}{x} = \frac{BP}{AB}$

$\therefore BP = x\tan\alpha$

$\tan\beta = \frac{QR}{x} = \frac{QR}{PR}$

$\therefore QR = x\tan\beta$

$QC = x\tan\alpha + x\tan\beta$

$= x(\tan\alpha + \tan\beta)$

$\tan\theta = \frac{QC}{AC}$

$= \frac{x(\tan\alpha + \tan\beta)}{\sqrt{2}\,x}$

$= \frac{\tan\alpha + \tan\beta}{\sqrt{2}}$ ← GIVEN

2

(ii) $\cos\phi = \frac{AP^2 + PQ^2 - AQ^2}{2AP.PQ}$

$= \frac{\frac{x^2}{\cos^2\alpha} + \frac{x^2}{\cos^2\beta} - \frac{2x^2}{\cos^2\theta}}{2 \cdot \frac{x}{\cos\alpha} \cdot \frac{x}{\cos\beta}}$

$= \frac{\sec^2\alpha + \sec^2\beta - 2\left(1 + \frac{(\tan\alpha + \tan\beta)^2}{2}\right)}{2\sec\alpha\sec\beta}$

$= \frac{\sec^2\alpha + \sec^2\beta - (2 + \tan^2\alpha + 2\tan\alpha\tan\beta + \tan^2\beta)}{2\sec\alpha\sec\beta}$

$= \frac{\cancel{\sec^2\alpha} + \cancel{\sec^2\beta} - \cancel{\sec^2\alpha} - 2\tan\alpha\tan\beta - \cancel{\sec^2\beta}}{2\sec\alpha\sec\beta}$

$= -\sin\alpha\sin\beta$ ← GIVEN

2

$\ddot{x} = 0$ $\qquad$ $\ddot{y} = -10$

$\dot{x} = 12\cos 60^\circ$ $\qquad$ $\dot{y} = -10t + 12\sin 60^\circ$

$= 6$ $\qquad$ $= -10t + 6\sqrt{3}$

$x = 6t$ $\qquad$ $y = -5t^2 + 6\sqrt{3}t$.

$\therefore y = -5.\left(\frac{x}{6}\right)^2 + 6\sqrt{3} \times \frac{x}{6}$

$= \frac{-5x^2}{36} + \sqrt{3}x$. $\qquad$ 3

If $x = -y$, $-x = \frac{-5x^2}{36} + \sqrt{3}x$.

$\therefore \frac{5x^2}{36} - (\sqrt{3}+1)x = 0$

$\therefore x(5x - 36(\sqrt{3}+1)) = 0$

$\therefore x = 0$ or $x = \frac{36(\sqrt{3}+1)}{5}$

GIVEN

$\therefore T = \frac{6(\sqrt{3}+1)}{5}$ $\qquad$ 3

$\dot{x} = 6$ $\qquad$ $\dot{y} = -10 \times \frac{6(\sqrt{3}+1)}{5} + 6\sqrt{3}$

$= -12(\sqrt{3}+1) + 6\sqrt{3}$ $\qquad$ $= -6\sqrt{3} - 12$

Speed $= \sqrt{36 + (-6\sqrt{3} - 12)^2}$

$= \sqrt{36 + 108 + 144 + 144\sqrt{3}}$

$= \sqrt{288 + 144\sqrt{3}}$

$= 12\sqrt{2 + \sqrt{3}}$

$= 23.2$ $\qquad$ 2