N.S.W. DEPARTMENT OF EDUCATION

HIGHER SCHOOL CERTIFICATE EXAMINATION 1977

MATHEMATICS 3 UNIT (AND 4 UNIT - FIRST PAPER)

Instructions: Time allowed 3 hours. All questions may be attempted. All questions are of equal value. In every question, all necessary working should be shown. Marks will be deducted for careless or badly arranged work.

Mathematical tables will be supplied. Approved slide rules or calculators may be used.

QUESTION 1

(i) Differentiate with respect to x: (a) $\frac{1}{1+x^2}$ (b) $x \tan x$

(ii) Write down primitives (indefinite integrals) of

(a) e^{2x} (b) $\frac{2x}{1+x^2}$ (c) $\sin x + \sec^2 x$

(iii) Express $\sin 50^0 \sin 15^0$ in the form $\frac{1}{2}(\cos A - \cos B)$

(iv) If $x = 2 \sin \theta$ and $y = \cos \theta$, find a relation between x and y which does not involve θ.

QUESTION 2

(i) For what values of the constant B are the graphs of the linear equations $2x + 3y = 6$, $6x + By = 9$

(a) parallel (b) perpendicular?

(ii) Indicate, using a sketch (not on graph paper) the region of the Cartesian plane for which $2x + 3y < 6$ and $6x + 9y > 9$

(iii) Evaluate (a) $\int_0^{\pi/2} \sin 2x \, dx$ (b) $\int_2^3 \frac{1}{x+1} \, dx$

QUESTION 3

(i) Sketch (not on graph paper) the graph of $y = \cos^{-1} x$

(ii) (a) Show, by completing the square or otherwise, that the expression $10 + 2x - x^2$ can never be greater than 11.

(b) Find the set of values of x for which this expression exceeds 7.

(iii) Use the method of mathematical induction to show that the sum of the squares of the first n positive integers is $\frac{1}{6} n (n+1)(2n+1)$.

QUESTION 4

(i) Show that the tangent to the parabola $x^2 = 8y$ at the point $(4, 2)$ intersects the y- axis at $(0, -2)$.

(ii) Transform the parabola $y = x^2 - 4x + 3$ to the form $X^2 = 4aY$ and hence find its focal length.

(iii) Sketch the region in the Cartesian (x, y) plane bounded by the parabola $y = x^2 - 4x + 3$, its tangent at $(3, 0)$ and the line $x = 2$. Find the equation of this tangent and hence calculate the area of this region.

QUESTION 5

(i) Use Simpson's Rule with three function values to find an approximate value of $\int_0^1 e^{-x^2} dx$.

(ii) Find the acute angle between the two planes:
$2x + y - 2z = 5$, $3x - 6y - 2z = 9$

(iii) For the curve $y = f(x)$ it is given that $f'(x) = \sin^2 x$ and the curve passes through the point $(\pi/4, \pi/8)$. Find the equation of the curve.

QUESTION 6

(i) (a) Define $|x|$,(i.e. the absolute value of x) for a real number x

(b) Sketch (not on graph paper) the curve $|x| + |y| = 1$

(ii) (a) Explain the terms 'function', 'domain', and 'range'.

(b) Discuss whether the circle $x^2 + y^2 = a^2$, in the Cartesian plane, is the graph of a function.

(iii) Use one application of Newton's method to estimate that root of $x^3 - 6x^2 + 24 = 0$ which lies near $x = 3$.

QUESTION 7

(i) Find the maximum and minimum values of $-x \log_e x$ for $0 < x \leq 1$.

(ii) Express $\cos\theta - \sin\theta$ in the form $A\cos(\theta + \alpha)$ in which A is a positive number.
Hence or otherwise solve the equation $\cos\theta - \sin\theta = 1$, $0 \leq \theta \leq \pi$.

QUESTION 8

(i) Write down the binomial expansion of $(1 + x)^n$ and use it to show that $2^n = \sum_{k=0}^{n} {}^nC_k$

(ii) (a) For a particle moving in the x- axis with velocity v at time t show that the acceleration is given by $\frac{d}{dx}(\frac{1}{2}v^2)$

(b) An object falling directly to the earth from space moves according to the equation $\frac{d^2x}{dt^2} = \frac{-k}{x^2}$, where x is the distance of the object from the centre of the earth at time t. The constant k is related to g, the value of gravity at the earth's surface, and the radius of the earth, R, by the formula $k = gR^2$.
Show that, if the object started from rest at a distance of 10^9 metres from the earth's centre, then it will reach the earth with a velocity of approximately 11200 m/s. (Assume $R = 6.4 \times 10^6$m, $g = 9.8$ m/s^2)

QUESTION 9

(i) The first two terms of a geometric sequence are $\frac{1}{2}$, $\frac{1}{8}$. Find

(a) the n-th term

(b) the sum of the first n terms

(c) the 'sum to infinity'

(ii) A and B, in order, continue to toss a coin until a head is thrown, and the first to throw a head wins. Using the results of (i), or otherwise, find the probabilities

(a) the game ends at the ninth toss

(b) the game ends in less than $2n$ tosses and A wins

(c) A wins.

QUESTION 10

(i) Calculate the volume of the solid generated when the region in the first quadrant bounded by the curve $y = x^2$, the axis $x = 0$ and the line $y = 4$ is rotated about

(a) the y- axis (b) the x- axis

(ii) Almost always in the application of mathematics to the real world, one has available information which, though interesting, is not really essential for the solution of the problem in hand. The

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following question contains some of this, so-called redundant, information.

As aspirin tablets deteriorate, a quantity of salicylic acid is formed; to be acceptable, the tablets must not contain more free salicylic acid than 0.15 per cent of the weight of aspirin stated to be in the tablet. From a large supply of tablets, ten are taken at random and tested. If nine or more meet the requirement, the lot is passed as satisfactory; otherwise it is rejected. Find the probability of acceptance if, in the supply as a whole, there are 10% unsatisfactory tablets.