1 (a) $x^2 = 12y$ —— ①

(i) sub $P(6p, 3p^2)$ into ①

$\therefore LHS = 36p^2$
$RHS = 12(3p^2) = 36p^2 = LHS$

$\therefore P(6p, 3p^2)$ lies on the parabola.

(ii) $M_{PQ} = \left(\frac{6p+6q}{2}, \frac{3p^2+3q^2}{2}\right)$
$= \left(3(p+q), \frac{3(p^2+q^2)}{2}\right)$

(iii) $m_{PQ} = \frac{3q^2-3p^2}{6q-6p} = \frac{3(q-p)(q+p)}{6(q-p)}$

$\therefore m_{PQ} = \frac{q+p}{2}$ $(q \neq p)$

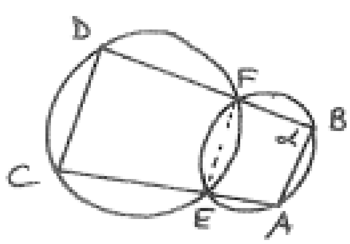
But $m_{PQ} = 1$ $\therefore \frac{q+p}{2} = 1$
$\therefore p+q = 2$

(iv) Now $M_{PQ} = \left(3(p+q), \frac{3(p^2+q^2)}{2}\right)$
$= \left(6, \frac{3(p^2+q^2)}{2}\right)$ using (iii)

when $x=6$, $y=3 \Rightarrow M_{PQ}$ above $y=3$

ie locus of midpoints of chords with a gradient of 1 is $x=6$ $(q \neq p)$, $y > 3$.

(b) (i)




Let $\angle FBA = \alpha$

$\therefore \angle FEC = \alpha$ (ext. $\angle$ of cylic quad. = int. opp. $\angle$)

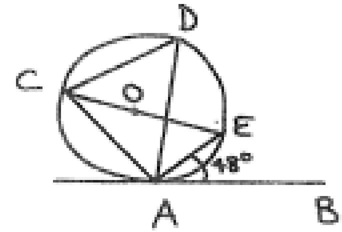
$\therefore \angle CDF = \pi - \alpha$ (opp. $\angle$s in cylic quad are supp.)

But $\angle CDF + \angle ABF = \pi$ $(180°)$

$\therefore$ int. opp. $\angle$s of quad DBAC

$\Rightarrow AB \parallel CD$.

(c) (i)




$\angle ACE = 48°$ ($\angle$ between tangent and chord at pt. of contact = $\angle$ in the alt. segment)

(ii) $\angle CAE = 90°$ ($\angle$ in semi-circle $= 90°$)

$\therefore \angle CEA = 42°$ ($\angle$ sum of $\Delta = 180°$)

$\therefore \angle ADC = 42°$ ($\angle$s at circum. of circle standing on a common arc are equal.)

2 (a) $\cos 2x + \sqrt{3}\sin 2x = 1$

$\therefore 2\left(\frac{1}{2}\cos 2x + \frac{\sqrt{3}}{2}\sin 2x\right) = 1$

$\therefore \cos(2x - \alpha) = \frac{1}{2}$

where $\cos\alpha = \frac{1}{2}$
$\sin\alpha = \frac{\sqrt{3}}{2}$

$\therefore \tan\alpha = \sqrt{3}$ $\therefore \alpha = \frac{\pi}{3}$

$\therefore \cos\left(2x - \frac{\pi}{3}\right) = \frac{1}{2}$

$\therefore \cos\left(2x - \frac{\pi}{3}\right) = \cos\frac{\pi}{3}$

$\therefore 2x - \frac{\pi}{3} = 2\pi n \pm \frac{\pi}{3}$

$\therefore 2x = 2\pi n \pm \frac{\pi}{3} + \frac{\pi}{3}$

$\therefore 2x = 2\pi n + \frac{2\pi}{3}$ or $2\pi n$

$\therefore x = \pi n + \frac{\pi}{3}$ or $\pi n$, where n is any integer

(b) $I = \int \sin^2 6x \, dx$
$= \frac{1}{2}\int 1 - \cos 12x \, dx$
$= \frac{1}{2}\left[x - \frac{1}{12}\sin 12x\right] + c$

$\cos 2x = 1 - 2s$
$\therefore \sin^2 x = \frac{1}{2}($
$\therefore \sin^2 6x = \frac{1}{2}(1$

$$\int \frac{dx}{x\sqrt{x^2-25}}$$

let $x = 5\sec\theta$

$\therefore dx = 5\sec\theta\tan\theta\, d\theta$

$$\therefore I = \int \frac{\cancel{5\sec\theta}\tan\theta\, d\theta}{\cancel{5\sec\theta}\sqrt{25\sec^2\theta-25}}$$

$$= \int \frac{\tan\theta\, d\theta}{5\sqrt{\sec^2\theta-1}}$$

$$= \frac{1}{5}\int \frac{\tan\theta\, d\theta}{\tan\theta}$$

$$= \frac{1}{5}\int 1\, d\theta$$

$$= \frac{1}{5}\theta + c$$

$$= \frac{1}{5}\sec^{-1}\frac{x}{5} + c$$

(d) $$I = \int_0^{\frac{1}{2}\ln 3} \frac{e^x}{1+e^{2x}}\, dx$$

let $u = e^x$ when $x=0$ $u=1$

$\therefore du = e^x\, dx$ $x = \frac{1}{2}\ln 3$ $u = 3^{\frac{1}{2}}$

$$\therefore I = \int_1^{3^{\frac{1}{2}}} \frac{du}{1+u^2}$$

$$= \left[\tan^{-1} u\right]_1^{\sqrt{3}}$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1} 1$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

3 (a) (i) Step 1: when $n=1$ $(1+x)^n - 1$

$= (1+x)^1 - 1$

$= x$

which is divisible by $x$

$\therefore$ it is true for $n=1$.

Step 2: Assume it is true for $n=k$ and prove it is true for $n=k+1$

$x$ ... any integer)

$\therefore (1+x)^k = Mx + 1$ ——— (1)

If $n=k+1$ $(1+x)^n - 1 = (1+x)^{k+1} - 1$

$= (1+x)^k(1+x) - 1$

$= (Mx+1)(1+x) - 1$

(sub (1))

$= Mx + Mx^2 + 1 + x - 1$

$= Mx + Mx^2 + x$

$= x(M + Mx + 1)$

which is divisible by $x$.

$\therefore$ if it is true for $n=k$ so it is true for $n=k+1$.

Step 3: It is true for $n=1$ and so it is true for $n=1+1=2$. It is true for $n=2$ and so it is true for $n=2+1=3$ and so on for all positive integral value of

(ii) $12^n - 4^n - 3^n + 1$

$= 3^n.4^n - 4^n - 3^n + 1$

$= 4^n(3^n-1) - 1(3^n-1)$

$= (3^n-1)(4^n-1)$

$$= \underbrace{((1+2)^n - 1)}_{\text{divisible by 2}}\underbrace{((1+3)^n - 1)}_{\text{divisible by 3}}$$

divisible by 2 and divisible by 3 using part (i)

$\Rightarrow 12^n - 4^n - 3^n + 1$ is divisible by 2 and 3 is 6, for all positive integers $n \geq 1$.

(b) $$f(x) = \frac{x}{4-x^2}$$

(i) Domain is: all real $x$ except $x = \pm 2$

(ii) $f(x) = \frac{x}{4-x^2}$

$f(-x) = \frac{-x}{4-x^2} = -f(x)$

$\Rightarrow f(x)$ is an odd function.

(iii) $$f'(x) = \frac{(4-x^2).1 - x(-2x)}{(4-x^2)^2}$$

$$= \frac{4+x^2}{(4-x^2)^2} > 0$$ for all $x$ except $x =$

its domain.

(iv) For $x$-intercepts $y=0 \quad \therefore x=0$

For vertical asymptotes $4-x^2=0$

$\therefore x = \pm 2.$

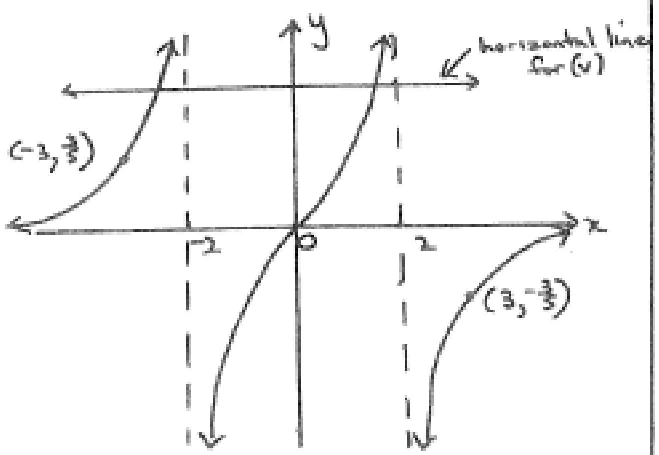
For horizontal asymptotes: $\lim\limits_{x\to\pm\infty} \dfrac{x}{4-x^2}$

$= \lim\limits_{x\to\pm\infty} \dfrac{x^2\left(\frac{1}{x}\right)}{x^2\left(\frac{4}{x^2}-1\right)}$

$= \dfrac{0}{0-1} \left(\begin{matrix}\text{as } x\to\pm\infty \\ \frac{1}{x}, \frac{1}{x^2}\to 0\end{matrix}\right)$

$= 0$

$\therefore$ horiz. asymptote at $y=0$ ($x>2$ or $x<-2$)




(v) As a horizontal line can be drawn above, as shown, to intersect the graph at two distinct points $\Rightarrow$ an inverse function will not exist.

4 (a) Let $P(x) = ax^3 + bx^2 + cx + d$

As $P(x)$ is monic $\Rightarrow a=1$

$\therefore P(x) = x^3 + bx^2 + cx + d$

Also $P(0) = -4 \qquad \therefore -4 = d$

$\therefore P(x) = x^3 + bx^2 + cx - 4$

Also when $P(x)$ is divided by $x^2+4$ the remainder is $x+8$.

$x^2+4 \;)\; x^3 + bx^2 + cx - 4$

$\quad -(x^3 \qquad + 4x)$

$\quad bx^2 + x(c-4) - 4$

$\quad -(bx^2 \qquad + 4b)$

$\quad x(c-4) + (-4-4b)$

But the remainder is $x+8$

$\Rightarrow 1 = c-4 \quad \therefore c = 5$

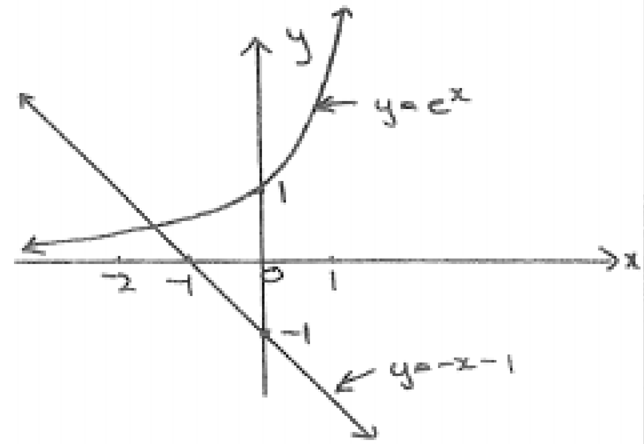
and $8 = -4-4b \quad \therefore 4b = -12 \quad \therefore b = -3$

$\therefore P(x) = x^3 - 3x^2 + 5x - 4.$

(b) (i) If $e^x + x + 1 = 0$

$\therefore e^x = -x-1$

This can be solved graphically by sketching $y=e^x$ against $y=-x-1$.




The sketch indicates that there is only one intersection as shown for $x \doteqdot -1$.

$\Rightarrow e^x + x + 1 = 0$ has only 1 real root and the root is negative.

(ii) By Newton's Method:

$z_2 = z_1 - \dfrac{P(z_1)}{P'(z_1)}$

Let $P(x) = e^x + x + 1$

$\therefore P'(x) = e^x + 1$. Let $z_1 = -1.5$

$\therefore z_2 = -1.5 - \dfrac{P(-1.5)}{P'(-1.5)}$

$= -1.5 - \dfrac{(-0.276869839\ldots)}{1.2231301\ldots}$

$= -1.273638286\ldots$

[As $x \to \infty$, $f(x) \to 1$]

(i) Let $y = 1 + \dfrac{2}{x-3}$ , $x > 3$, $y > 1$

For inverse function interchange $x$ for $y$

$\therefore x = 1 + \dfrac{2}{y-3}$

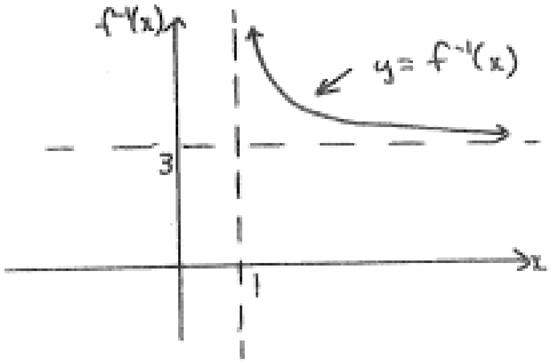
$\therefore x - 1 = \dfrac{2}{y-3}$

$\therefore \dfrac{1}{x-1} = \dfrac{y-3}{2}$

$\therefore y = \dfrac{2}{x-1} + 3$

$\Rightarrow f^{-1}(x) = 3 + \dfrac{2}{x-1}$ , $x > 1$, $y > 3$.

(ii)




(b) $E = \sin\left[\tan^{-1}\left(\frac{3}{2}\right) + \cos^{-1}\left(\frac{2}{3}\right)\right]$

let $\alpha = \tan^{-1}\frac{3}{2}$ , let $\beta = \cos^{-1}\frac{2}{3}$

$\therefore \tan\alpha = \frac{3}{2}$ $\quad\therefore \cos\beta = \frac{2}{3}$

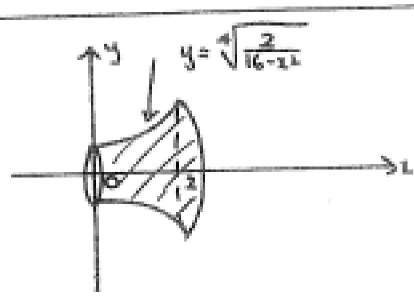

$\therefore E = \sin[\alpha + \beta]$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \dfrac{3}{\sqrt{13}} \cdot \dfrac{2}{3} + \dfrac{2}{\sqrt{13}} \cdot \dfrac{\sqrt{5}}{3}$

$= \dfrac{6 + 2\sqrt{5}}{3\sqrt{13}}$

(c)




$= \pi \displaystyle\int_0^2 \frac{\sqrt{2}}{\sqrt{16-x^2}}\, dx$

$= \sqrt{2}\pi \left[\sin^{-1}\frac{x}{4}\right]_0^2$

$= \sqrt{2}\pi\left[\sin^{-1}\frac{1}{2} - 0\right]$

$= \sqrt{2}\pi\left(\frac{\pi}{6}\right)$

$= \dfrac{\sqrt{2}\pi^2}{6}$ units$^3$

6(a) $\dfrac{x^2 - 5x}{4 - x} \le -3$ $\quad (x \ne 4)$

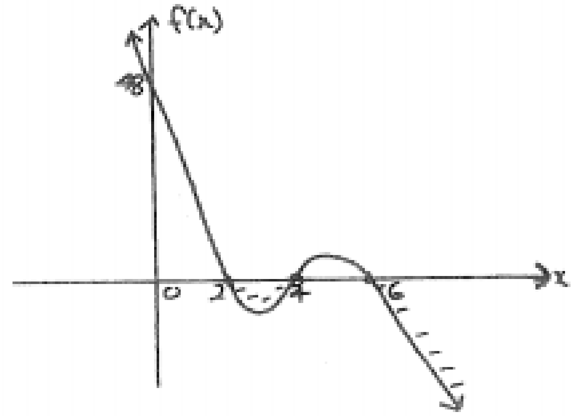
b.s. $\times (4-x)^2$ $\quad \therefore (4-x)(x^2-5x) \le -3(4-x)^2$

$\therefore 3(4-x)^2 + (4-x)(x^2-5x) \le 0$

$\therefore (4-x)[3(4-x) + x^2 - 5x] \le 0$

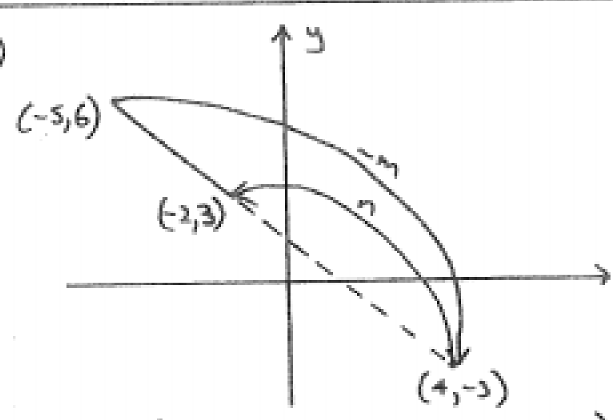
$\therefore (4-x)(x^2 - 8x + 12) \le 0$

$\therefore (4-x)(x-2)(x-6) \le 0$




$\Rightarrow 2 \le x < 4$ or $x \ge 6$

(b)




$(4,-3) = \left(\dfrac{-m \times -2 + n \times -5}{-m+n}, \dfrac{-m \times 3 + n \times 6}{-m+n}\right)$

$\therefore -4m+4n = 2m-5n$

$\therefore -6m = -9n$

$\therefore \frac{m}{n} = \frac{9}{6} = \frac{3}{2}$

ie $m:n = 3:2$

(c) $v^2 = 15+2x-x^2$

(i) At end points of motion $v=0$

$\therefore 15+2x-x^2=0$

$\therefore x^2-2x-15=0$

$(x-5)(x+3)=0$

$\therefore x=5$ or $-3$.

ie end points of motion occur at $x=-3$ and $x=5$.

(ii) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}[15+2x-x^2]\right)$

$= \frac{1}{2}[2-2x]$

when $x=-2$ $\quad \ddot{x} = \frac{1}{2}[2+4] = 3$

ie acc'n of particle is $3\ ms^{-2}$ in $\rightarrow$.

7 (a) $\frac{dP}{dt} = k(P-10000)$ —— (1)

(i) $P = 10000 + P_0e^{kt}$ —— (2)

Sub (2) into (1):

LHS of (1) $= \frac{dP}{dt}$

$= \frac{d}{dt}\left(10000+P_0e^{kt}\right)$

$= kP_0e^{kt}$

$= k(P-10000)$ (from (2))

$=$ RHS of (1)

$\Rightarrow P = 10000+P_0e^{kt}$ is a solution of the differential equation.

$\therefore 15000 = 10000 + P_0e^0$

$\therefore \underline{P_0 = 5000}$

$\therefore P = 10000+5000e^{kt}$

When $t=6$, $P=25000$

$\therefore 25000 = 10000+5000e^{6k}$

$\therefore \underline{k = \frac{1}{6}\ln 3}$.

(iii) Now $P = 10000+5000e^{(\frac{1}{6}\ln 3)t}$

when $t=12$, $P=?$

$\therefore P = 10000+5000e^{(\frac{1}{6}\ln 3).12}$

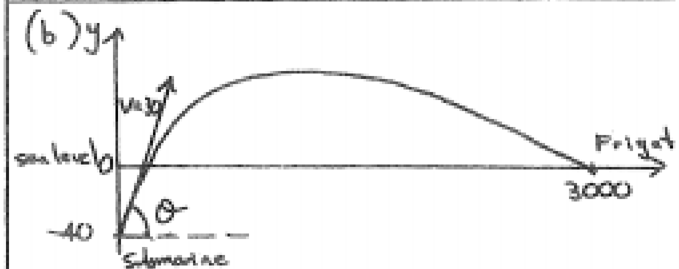
$= 10000+5000e^{\ln 9}$

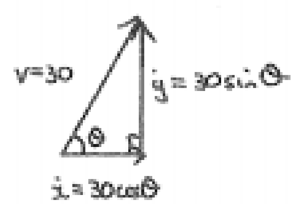
$= 10000+45000$

$= 55000$

$\therefore$ After 1 year there are 55000 tsetse flies.

(b)




(i) Initially $\ddot{x}=0$, $\ddot{y}=-10$

$\therefore \dot{x}=c_1$, $\dot{y}=-10t+c_2$

when $t=0$ $\dot{x}=30\cos\theta$, $\dot{y}=30\sin\theta$

$\therefore 30\cos\theta=c_1$, $30\sin\theta=c_2$

$\therefore \dot{x}=30\cos\theta$, $\dot{y}=-10t+30\sin\theta$

$\therefore x=30t\cos\theta+c_3$, $y=-5t^2+30t\sin\theta+$

when $t=0$, $x=0$, $y=-40$

$\therefore c_3=0$, $c_4=-40$

$\therefore x=30t\cos\theta$, $y=30t\sin\theta-5t^2-40$ are the parametric equations of motion after $\frac{3\sqrt{3}}{2}$ s.

Now as $\dot{y} = -10t + 30\sin\theta$

$\therefore 0 = \frac{-30\sqrt{3}}{2} + 30\sin\theta$

$\therefore \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore \theta = \frac{\pi}{3}$

ie angle of projection, $\theta = \frac{\pi}{3}$ or $60°$.

(iii) The missile strikes the frigate when $x = 3000$.

Now as $x = 30t\cos\theta$

$\therefore 3000 = 30 \times t \times \cos\frac{\pi}{3}$

$\therefore t = \frac{3000}{30 \times \frac{1}{2}}$

$\therefore t = 200$

$\therefore$ missile strikes the frigate after 200 seconds.