MATHEMATICS EXT 1

TRIAL EXAM, 2004

SOLUTIONS

QUESTION 1 : (12 marks)   Calc /6

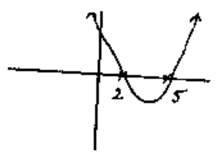
(a) $\frac{3}{x-2} \le 1 \quad , \quad x \neq 2$ ✓

$3(x-2) \le (x-2)^2$ ✓

$3x - 6 \le x^2 - 4x + 4$

$0 \le x^2 - 7x + 10$

$0 \le (x-5)(x-2)$



$\therefore x < 2$ or $x \ge 5$. ✓

(b) $m_1 = 4$ and $m_2 = -\frac{3}{2}$

$\therefore \tan\theta = \left| \frac{4 - (-\frac{3}{2})}{1 + 4(-\frac{3}{2})} \right|$ ✓

$= \frac{11}{10}$

$\therefore \theta = 47°44'$ ✓

(c) $\lim_{x \to 0} \frac{\sin 4x}{8x} = \frac{1}{2} \lim_{x \to 0} \frac{\sin 4x}{4x}$

$= \frac{1}{2} \times 1$

$= \frac{1}{2}$ ✓

(d) $\int_0^{\pi/3} \sin^2 3x \, dx$

$= \frac{1}{2} \int_0^{\pi/3} 1 - \cos 6x \, dx$ ✓

$= \frac{1}{2} \left[x - \frac{1}{6} \sin 6x \right]_0^{\pi/3}$ ✓

$= \frac{1}{2} \left[\left(\frac{\pi}{3} - 0\right) - 0 \right]$

$= \frac{\pi}{6}$ ✓   Calc /3

(e) $\int_0^1 x(1-x)^7 \, dx$

$u = 1 - x$

$\frac{du}{dx} = -1$

when $x = 0$, $u = 1$

$x = 1$, $u = 0$

$= \int_1^0 (1-u) \cdot u^7 \cdot -du$ ✓

$= \int_0^1 u^7 - u^8 \, du$

$= \left[\frac{u^8}{8} - \frac{u^9}{9} \right]_0^1$ ✓

$= \frac{1}{8} - \frac{1}{9}$

$= \frac{1}{72}$ ✓   Calc /3

Comments :

(a) Must state that $x \neq 2$.

(b) Learn formula correctly – complete with absolute value sign!
Be careful with minus sign too.

(c) ✓

(d) Many incorrect substitution for $\sin^2 3x$.

(e) Show all working
Don't forget to change the limits
NB $\int_0^1 f(x)\,dx \neq \int_1^0 f(x)\,dx$.

QUESTION 2 : (12 marks)   Com /1   Reas /3

(a) $y = x^2 \cdot \sin^{-1} 3x$

$u = x^2 \quad v = \sin^{-1} 3x$

$u' = 2x \quad v' = \frac{3}{\sqrt{1 - 9x^2}}$

$\therefore \frac{dy}{dx} = 2x \sin^{-1}(3x) + \frac{3x^2}{\sqrt{1-9x^2}}$ ✓✓

(b) PARABOLA

No. of arrangements $= \frac{8!}{3!}$ ✓✓

$(= 6720)$

(c) $P(x) = ax^3 - 8x^2 - 9$

If divisible by $x - a$, then $P(a) = 0$

$\therefore 0 = a^4 - 8a^2 - 9$ ✓

$0 = (a^2 - 9)(a^2 + 1)$

$\therefore$ Since a is real, $a = \pm 3$. ✓

(d) $y = \cos^{-1} x$ OR $y = 2\tan^{-1}(1-x)$

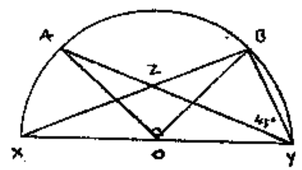
$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$ $\frac{dy}{dx} = \frac{-2}{1+(1-x)^2}$

when $x = 0$, $\frac{dy}{dx} = -1$ when $x = 0$, $\frac{dy}{dx} = -1$ ✓

$\therefore y - \frac{\pi}{2} = -1(x - 0)$

$\therefore x + y - \frac{\pi}{2} = 0$ ✓

(e)



(i) $\angle AYB = 45°$ because the angle at the centre is twice the angle at the circumference, standing on the same arc, AB. ✓   Com /1

(ii) Also, $\angle XBY = 90°$ ($\angle$ in a semicircle is $90°$) ✓

$\therefore \angle BZY = 45°$ ($\angle$ sum $\triangle = 180°$) ✓

$\therefore BY = BZ$ (sides opposite = angles in an isos. $\triangle$ are =). ✓   Reas /3

Comments:

a) to differentiate $\sin^{-1} f(x)$ it is more successful to use the rule.
$\frac{d}{dx}(\sin^{-1} f(x)) = \frac{-1}{\sqrt{1-(f(x))^2}} \times f'(x)$

b) Well done.

c) MUST BE stated that $P(a) = 0$
The resulting equation is a quadratic. It was solved very badly. You should recognise equations of this form

d) Really only need to find one tangent gradient because it is a common tangent.

e) Word of advice!
Draw a clear/large diagram. Mark on everything you can find, the solution generally reveals itself.

QUESTION 3: (12 marks) Com 1/5

(a) (i) $\sqrt{3}\cos x - \sin x$

$R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$

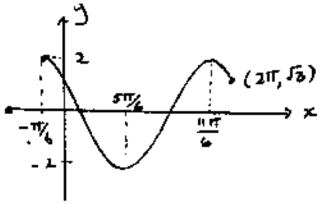
$\therefore R\cos\alpha = \sqrt{3}$
$R\sin\alpha = 1$

$\therefore R = 2$ and $\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \pi/6$

$\therefore \sqrt{3}\cos x - \sin x = 2\cos(x + \pi/6)$

(ii)



(Com 1)

(iii) $2\cos(x+\pi/6) = \sqrt{2}$

$\cos(x+\pi/6) = \frac{1}{\sqrt{2}}$

$\therefore x + \pi/6 = \frac{\pi}{4}, \frac{7\pi}{4}$

$\therefore x = \frac{\pi}{12}, \frac{19\pi}{12}$

(b) (i) $x = a\cos(4t+\alpha)$ (Com 1)

$\dot{x} = -4a\sin(4t+\alpha)$

$\ddot{x} = -16a\cos(4t+\alpha)$

$= -16x$ as required

(ii) $x = 5,\ t=0 \Rightarrow 5 = a\cos\alpha$ ①

$v = -4,\ t=0 \Rightarrow -4 = -4a\sin\alpha$

$1 = a\sin\alpha$ ②

$①^2 + ②^2 \Rightarrow 25 + 1 = a^2$

$\therefore a^2 = 26$

$a = \sqrt{26}$

(iii) Maximum speed is $4\sqrt{26}$ cm/s

(c)

$\underline{n=0}$: $5^0 + 11 = 1 + 11 = 12$
which is divisible by 4

Assume true for $n = k$:

ie $5^k + 11 = 4M$ for some integer M.

Investigate $n = k+1$:

$5^{k+1} + 11 = 5.5^k + 11$

$= 5(4M - 11) + 11$
using assumption

$= 20M - 44$

$= 4(5M - 11)$

$= 4P, \quad (P \in \mathbb{Z})$

$\therefore$ If proposition true for $n=k$, it is also true for $n=k+1$.
Since it is true for $n=0$, it is also true for $n = 1, 2, \ldots$ and hence true for all positive integers by the principle of mathematical induction.

(Com 3)

(a) (ii) mark the endpoints on your curve and make sure it was greater than 1 cycle of the curve.

(iii) Don't forget answer in $\underline{all}$ appropriate quadrants.

(b) (i) careful with derivative. $\frac{d}{dx}(\cos x) = -\sin x$

(ii) Poorly done. Many algebraic errors.

(iii) Poor.

(c) NB Initial value is $\underline{n=0}$!

Also careless e... eg $5.5^k + 11 \neq 5(5^k + 11)$

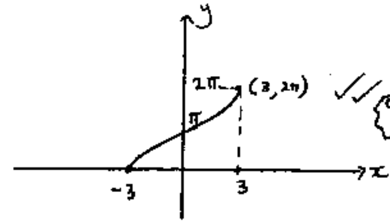
QUESTION 4: (12 marks) Com 1/3, Reas 3, Calc 3/4

(a) $f(x) = \pi + 2\sin^{-1}\left(\frac{x}{3}\right)$

(i) $\underline{\text{Domain}}$: $-3 \leq x \leq 3$

$\underline{\text{Range}}$: $0 \leq f(x) \leq 2\pi$

(ii)



(Com 1/2)

(b) $x^3 + px^2 + q = 0$

(i) Let roots be $\alpha, \frac{1}{\alpha}$ and β.

$\therefore$ $\underline{\text{Product of roots}}$:

$\alpha \cdot \frac{1}{\alpha} \cdot \beta = -q$

$\therefore \beta = -q$

$\therefore$ The third root is $-q$.

(ii) $\underline{\Sigma \text{ of roots}}$:

$\alpha + \frac{1}{\alpha} - q = -p$ ①

$\underline{\Sigma \text{ of pairs of roots}}$:

$\alpha \cdot \frac{1}{\alpha} - \alpha q - \frac{q}{\alpha} = 0$

$1 - q\left(\alpha + \frac{1}{\alpha}\right) = 0$

but from ①: $\alpha + \frac{1}{\alpha} = q - p$

$\therefore 1 - q(q-p) = 0$

$1 - q^2 + pq = 0$

$\therefore pq = q^2 - 1$

$p = q - \frac{1}{q}$

(Reas 3)

(c) $\ddot{x} = -\frac{1}{2}\mu^2 e^{-x}$ where $\mu = 2$

i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{1}{2} \cdot 2^2 \cdot e^{-x}$

$\frac{1}{2}v^2 = \int -2e^{-x}\,dx$

$\frac{1}{2}v^2 = 2e^{-x} + C$

when $x=0, v=2$
$\frac{1}{2} \times 4 = 2e^0 + c$
$2 = 2 + c$
$c = 0$

$\frac{1}{2}v^2 = 2e^{-x}$

$v^2 = 4e^{-x}$

(Calc 1)

ii) $v = \pm\sqrt{4e^{-x}}$

$= \pm 2e^{-x/2}$

Since $e^{-x/2} > 0$ for all x and the initial conditions gives the velocity is 2 m/s. (positive velocity)

$\therefore v > 0$ for all x

$v = 2e^{-x/2}$

(Comm)

iii) $\frac{dx}{dt} = 2e^{-x/2}$

$\frac{dt}{dx} = \frac{1}{2}e^{x/2}$

$t = \int \frac{1}{2}e^{x/2}\,dx$

$t = \frac{1}{2} \times \frac{1}{\frac{1}{2}} e^{x/2} + C$

$t = e^{x/2} + C$

when $t = 0,\ x = 0$

$0 = e^0 + C$
$0 = 1 + C$
$C = -1$

$t = e^{x/2} - 1$

$e^{x/2} = t + 1$

$\frac{x}{2} = \ln(t+1)$

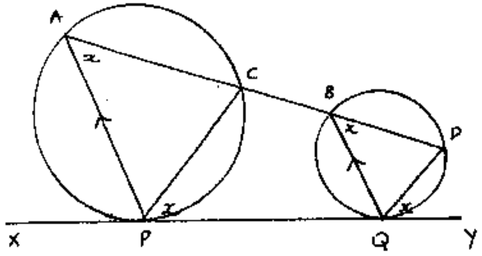
$x = 2\ln(t+1)$

(Calc 3)

iv As $t \to \infty$, $x \to \infty$
$v \to 0$
$a \to 0$

QUESTION 5 : (12 marks) Reas /12

(a)



(i) Let $\angle CPQ = x$

$\therefore \angle PAC = x$ ($\angle$ in the alt seg = $\angle$ between a tangent + chord). ✓

$\therefore \angle DBQ = x$ (corresponding $\angle$ = as $AP \parallel BQ$) ✓

$\therefore \angle DQY = x$ ($\angle$ in alt seg = $\angle$ between tangent + chord). ✓

$\therefore \angle CPQ = \angle DQY$

$\therefore CP \parallel DQ$ (corresponding $\angle$ =) ✓

(ii) $\therefore \angle CBQ = 180 - x$ ($\angle$ Σ st. line $= 180°$) ✓

$\therefore$ PQBC is a cyclic quadrilateral since opposite angles are supplementary. ✓

(b) $y = x^2$, $P(t, t^2)$

Tangent $y = 2tx - t^2$.

(i) $m = -\frac{1}{2t}$ ✓

Focus $(0, \frac{1}{4})$

$\therefore y - \frac{1}{4} = -\frac{1}{2t}(x - 0)$ ✓

$\therefore y = -\frac{1}{2t}x + \frac{1}{4}$

$\therefore y = \frac{t - 2x}{4t}$.

(ii) Solving simultaneously,

$y = 2tx - t^2$ ①

$y = \frac{t - 2x}{4t}$ ②

$\therefore 2tx - t^2 = \frac{t - 2x}{4t}$

$8t^2x - 4t^3 = t - 2x$

$x(8t^2 + 2) = t + 4t^3$

$x = \frac{t(1 + 4t^2)}{2(4t^2 + 1)}$ ✓

$= t/2$

$\therefore y = \cancel{2}t \cdot \frac{t}{\cancel{2}} - t^2$

$= 0$ ✓

$\therefore F(t/2, 0)$

(iii) $P(t, t^2)$ $F(t/2, 0)$

$M = (3t/4, \frac{t^2}{2})$ ✓

$\therefore x = \frac{3t}{4}$ and $y = \frac{t^2}{2}$

$\therefore t = \frac{4x}{3}$ ✓

$\therefore y = \frac{1}{2} \cdot \left(\frac{4x}{3}\right)^2$

$= \frac{8x^2}{9}$ ✓

Comments:

(a) many no. attempts.

(b) (i) Line passes thru S, ~~not~~ P.

(ii) Need to solve simult ~~as~~ sub both lines

(iii) still some careless, but improving.

Reas /12

QUESTION 6 : (12 marks) Calc /2 Reas /6

(a) (i) ${}^{11}C_4 = 330$ ✓

(ii) ${}^{9}C_2 = 36$ ✓

(b) (i) $A \doteq \frac{\frac{1}{2}}{3}\left(1 + 4 \cdot \frac{4}{5} + \frac{1}{2}\right)$ ✓

$\doteq \frac{47}{60}$

$\doteq 0.7833$ ✓

(ii) $A = \int_0^1 \frac{1}{1 + x^2}\, dx$ ✓

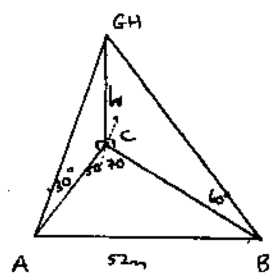
$= \left[\tan^{-1} x\right]_0^1$

$= \pi/4$ ✓ Calc /2

(iii) $\therefore \pi/4 \doteq 0.7833$

$\therefore \pi \doteq 3.13$ ✓ Reas /1

(c)



$\tan 60 = \frac{h}{BC}$

$\therefore BC = \frac{h}{\sqrt{3}}$ ✓

$\tan 30 = \frac{h}{AC}$

$\therefore AC = h\sqrt{3}$ ✓

$\therefore \cos 120 = \frac{AC^2 + BC^2 - 52^2}{2 \cdot AC \cdot BC}$ ✓

$-\frac{1}{2} = \frac{3h^2 + \frac{h^2}{3} - 52^2}{2 \cdot h\sqrt{3} \cdot \frac{h}{\sqrt{3}}}$

$-h^2 = 3h^2 + \frac{h^2}{3} - 52^2$ ✓

$\frac{13h^2}{3} = 52^2$

$\therefore h^2 = 624$

$\therefore h = \sqrt{624}$

$= 4\sqrt{39}$ ✓

Reas /5

Comments:

a) well done.

b) i) Learn Simpson's rule properly.

ii) Very easy! Use the standard integral page.

iii) Hence means you must use your answers from parts i) and ii)

c) Draw a clear diagram. It is easier to solve this problem using the simplified expressions for BC and AC. Watch your rearranging of algebra!

QUESTION 7 : (12 marks) Reas /4 Calc /5

(a) $\cos^2\theta + \frac{\sqrt{3}}{2}\sin 2\theta = 0$

$\cos^2\theta + \sqrt{3}\sin\theta\cos\theta = 0$

$\therefore \cos\theta\left(\cos\theta + \sqrt{3}\sin\theta\right) = 0$

$\therefore \cos\theta = 0$ or $\cos\theta + \sqrt{3}\sin\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ $\quad \tan\theta = -\frac{1}{\sqrt{3}}$

$\theta = \frac{5\pi}{6}, 2\pi - \frac{\pi}{6}$

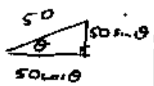
$3\pi - \pi/6, 4\pi - \pi/6, \ldots$

$\therefore \theta = (2n+1)\frac{\pi}{2} \quad n \in \mathbb{Z}$

$\theta = n\pi - \frac{\pi}{6}$

Reas 4

(b) (i) $\boxed{\ddot{x} = 0}$



$\dot{x} = C$

when $t=0$, $\dot{x} = 50\cos\theta \therefore C = 50\cos\theta$

$\therefore \boxed{\dot{x} = 50\cos\theta}$

$\therefore x = 50t\cos\theta + C$

when $t=0$, $x = 0 \therefore C = 0$

$\therefore \boxed{x = 50t\cos\theta}$

$\boxed{\ddot{y} = -10}$

$\dot{y} = -10t + C$

when $t=0$, $\dot{y} = 50\sin\theta \therefore C = 50\sin\theta$

$\therefore \boxed{\dot{y} = -10t + 50\sin\theta}$

$\therefore y = -5t^2 + 50t\sin\theta + C$

when $t=0$, $y = 80 \therefore C = 80$

$\therefore \boxed{y = -5t^2 + 50t\sin\theta + 80}$

(ii) when $x = 300$, $y = 2$.

$300 = 50t\cos\theta$

$\therefore t = \frac{6}{\cos\theta}$

$\therefore 2 = -5 \cdot \frac{36}{\cos^2\theta} + 50 \cdot \frac{6}{\cos\theta} \cdot \sin\theta + 80$

$0 = -180\sec^2\theta + 300\tan\theta + 78$

$= -180\left(\tan^2\theta + 1\right) + 300\tan\theta + 78$

$= -180\tan^2\theta + 300\tan\theta - 102$

$0 = 180\tan^2\theta - 300\tan\theta + 102$

$\therefore \tan\theta = \frac{300 \pm \sqrt{300^2 - 4 \cdot 180 \cdot 102}}{2 \times 180}$

$\doteqdot 1.19$ or 0.475

$\therefore \theta = 49°58'$ or $25°27'$

$\therefore$ The initial angle of projection would be $50°$ or $25°$ to the nearest degree.

Calc 8

Comments:

(a) · Factorise!!

· "All solutions" means find the general solution

· really should indicate that n is an integer

(b) (i) 'derive' means you must show all steps, NOT just quote a formula.

(ii) Finding a t value first wastes far too much time. Eliminate t and find the angles straight away.