

# NEWINGTON COLLEGE



HSC Trial Examination 2000

## 12 MATHEMATICS

### 3 UNIT ADDITIONAL

### 3/4 UNIT COMMON

*Time allowed - 2 hours*

*(plus 5 minutes reading time)*

#### DIRECTIONS TO CANDIDATES:

All questions are of equal value.

All questions may be attempted.

In every question, show all necessary working.

Marks may not be awarded for careless or badly arranged work.

The marks allocated to each part of a question have been indicated on the LHS of each question.

Approved silent calculators may be used.

A table of standard integrals is provided for your convenience.

The answers to the seven questions in this paper are to be returned in separate bundles clearly marked Question 1, Question 2 etc. Start each question on a new page.

Each bundle must show the candidate's computer number.

The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.

Unless otherwise stated candidates should leave their answers in simplest exact form.

QUESTION ONE (12 Marks)

- 1 (a) The polynomial  $P(x) = x^3 + ax^2 + 2x - 4$  has a remainder of  $-7$  on division by  $x + 2$ . Find the value of  $a$ .
- 2 (b) Find, to the nearest degree, the acute angle between the lines  $x - y = 2$  and  $3x + y = 5$ .
- 3 (c) (i) Write  $\cos 2x$  in terms of  $\sin^2 x$ .  
(ii) Without the use of a calculator, evaluate  $\cos\left(2 \sin^{-1} \frac{\sqrt{3}}{2}\right)$ .
- 3 (d) Solve for  $x$ :  $\frac{1}{x-3} \geq 1$
- 3 (e) Evaluate  $\int_0^1 x\sqrt{1-x} \, dx$  using the substitution  $u = 1 - x$ .

QUESTION 2 (12 Marks) Start a new page

- 3 (a) Prove the identity  $\frac{\cos x - \cos 2x}{\sin 2x + \sin x} = \operatorname{cosec} x - \cot x$ .
- 3 (b) The equation  $2 \sin x - 10x + 5 = 0$  has a root near  $x = \frac{1}{2}$ . Use Newton's method once to obtain a better approximation to the root, correct to two decimal places.
- 4 (c) Find all angles  $\theta$ , where  $0 \leq \theta \leq 2\pi$ , for which  $\sqrt{3} \cos \theta - \sin \theta = 1$ .
- 2 (d) Sketch  $y = 2 \sin^{-1} \frac{x}{3}$ .

Question 3 on page 2.....

QUESTION THREE (12 Marks) Start a new page

- 9 (a)  $P(2ap, ap^2)$  is a point on the parabola  $x^2 = 4ay$  ..
- Show that the equation of the normal to the parabola at the point  $P$  is  $x + py = 2ap + ap^3$  .
  - If the normal at  $P$  cuts the  $y$ -axis at  $Q$  show that the coordinates of  $Q$  are  $(0, 2a + ap^2)$ .
  - Show that the coordinates of  $R$  which divided the interval  $PQ$  externally in the ration 2:1 are  $(-2ap, 4a + ap^2)$  .
  - Find the cartesian equation of the locus of  $R$ .
  - Show that if the normal at  $P$  passes through a given point  $(h, k)$  then  $p$  must be a root of the equation  $ap^3 + (2a - k)p - h = 0$  .
  - Hence state the maximum number of normals to the parabola  $x^2 = 4ay$  which can pass through any given point.
- 3 (b) A spherical snowball is melting so that its volume decreases at a rate proportional to its surface area, that is  $\frac{dV}{dt} = -kS$  where  $V$  is the volume at time  $t$ ,  $S$  is the surface area at time  $t$  and  $k$  is a positive constant. Using the results  $V = \frac{4}{3}\pi r^3$  and  $S = 4\pi r^2$  show that:
- $\frac{dV}{dr} = S$  ,
  - the radius of the snowball is decreasing at a constant rate.

Question 4 on page 3....

**QUESTION FOUR (12 Marks) Start a new page**

- 8 (a) A particle is oscillating in simple harmonic motion about a fixed point. Its displacement  $x$  cm from the point  $x = 0$  at time  $t$  seconds is given by  $x = 2 \cos 3t + 4$ .
- (i) Explain why  $2 \leq x \leq 6$ .
  - (ii) Hence write down the amplitude and the centre of the motion.
  - (iii) Find  $\ddot{x}$  as a function of  $t$ .
  - (iv) Hence show that  $\ddot{x} = -9(x - 4)$ .
  - (v) Show that  $v^2 = -9x^2 + 72x - 108$ .
  - (vi) Hence, or otherwise find the greatest speed of the particle.
- 4 (b) (i) Show that the function  $T = R + Ae^{-kt}$  is a solution of the differential equation  $\frac{dT}{dt} = -k(T - R)$ .
- (ii) A metal baking dish is removed from an oven at  $200^\circ\text{C}$ . If the dish takes one minute to cool to  $170^\circ\text{C}$  and the room temperature is  $20^\circ\text{C}$  find the time, correct to the nearest minute, that it takes for the baking dish to cool to  $50^\circ\text{C}$ . (Assume that the baking dish cools at a rate proportional to the difference between the temperature of the baking dish and the temperature of the surrounding air.)

**QUESTION FIVE (12 Marks) Start a new page**

- 5 (a) Suppose that  $(3 + 2x)^{15} = \sum_{k=0}^{15} t_k x^k$ .
- (i) Use the binomial theorem to write down an expression for  $t_k$ .
  - (ii) Show that  $\frac{t_{k+1}}{t_k} = \frac{30 - 2k}{3k + 3}$ .
  - (iii) Hence find the greatest coefficient in the expansion of  $(3 + 2x)^{15}$ .

Question 5 continued on page 4....

- 4 (b) Consider the geometric series  $S = 1 + (1+x) + (1+x)^2 + (1+x)^3 + \dots + (1+x)^n$ .

(i) Show that  $S = \frac{(1+x)^{n+1} - 1}{x}$ .

(ii) Hence show that:

$$S = {}^{n+1}C_1 + {}^{n+1}C_2 x + {}^{n+1}C_3 x^2 + \dots + {}^{n+1}C_{r+1} x^r + \dots + {}^{n+1}C_{n+1} x^n$$

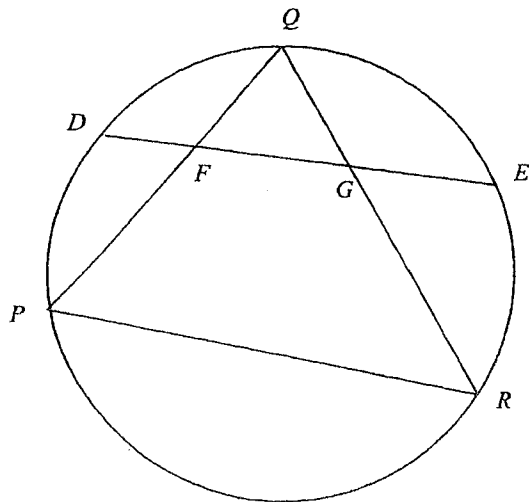
(iii) By considering terms in  $x^r$  from each expression for  $S$ , prove that

$${}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + \dots + {}^rC_r = {}^{n+1}C_{r+1}.$$

- 3 (c) If one root of the equation  $x^3 - px^2 + qx - r = 0$  is equal to the product of the other two, show that  $(q+r)^2 = r(p+1)^2$ .

**QUESTION SIX (12 Marks) Start a new page**

- 4 (a)  $\triangle PQR$  is inscribed in a circle.  $D$  and  $E$  are the midpoints of the arcs  $PQ$  and  $QR$  respectively. The line  $DE$  intersects  $PQ$  and  $QR$  at  $F$  and  $G$  respectively.



- (i) Let angle  $QDE = \alpha$ . Explain why angle  $RQE = \alpha$ .  
 (ii) Prove that  $\triangle QFG$  is isosceles.

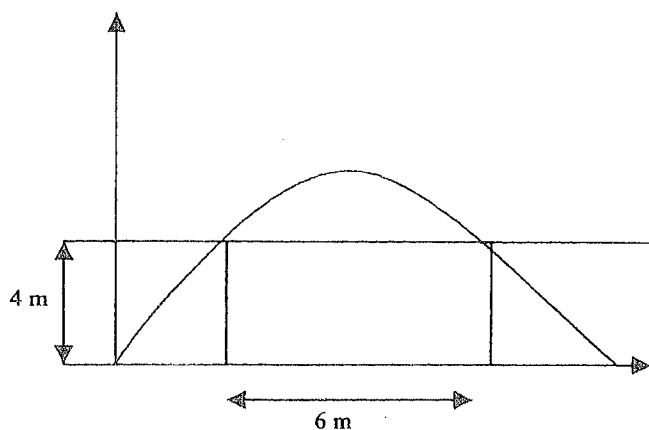
- 4 (b) If  $x = \tan \theta + \sec \theta$ , use the  $t$ -formulae to show that  $\frac{x^2 - 1}{x^2 + 1} = \sin \theta$ .

Question 6 continued on page 5....

- 4 (c) (i) Prove that  $\frac{d}{dx} \left( \tan^{-1} x + \tan^{-1} \frac{1}{x} \right) = 0$ .
- (ii) Hence sketch the curve  $y = \tan^{-1} x + \tan^{-1} \frac{1}{x}$ .

**QUESTION SEVEN (12 Marks) Start a new page**

- 5 (a) A particle is projected from a point  $O$  with an initial velocity of  $v$  m/s and with an angle of projection  $\alpha$ , where  $0^\circ \leq \alpha \leq 90^\circ$  and where  $g \text{ m/s}^2$  is the acceleration due to gravity. Under these circumstances the equations for the horizontal and vertical displacements at time  $t$  are given by  $x = v \cos \alpha t$  and  $y = -\frac{1}{2}gt^2 + v \sin \alpha t$  respectively.
- (i) Prove that  $y = x \tan \alpha - \frac{gx^2}{2v^2} \sec^2 \alpha$ .
- (ii) If  $R$  is the range of the projectile on the horizontal plane, prove that  $y = x \left( 1 - \frac{x}{R} \right) \tan \alpha$ .
- (iii) If  $\alpha = 45^\circ$  and the particle just clears two walls 6 m apart, both at a height of 4 m, find  $R$ .



Question 7 continued on page 6 ...

7 (b) (i) Show that  $\sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$ .

(ii) Prove that  $\frac{d}{dx}(e^x \sin x) = \sqrt{2}e^x \sin\left(x + \frac{\pi}{4}\right)$ .

(iii) Prove by mathematical induction that if  $y = e^x \sin x$ , then

$$\frac{d^n y}{dx^n} = (\sqrt{2})^n e^x \sin\left(x + \frac{n\pi}{4}\right) \text{ where } n \text{ is a positive integer.}$$

END OF PAPER