Question 1   2004 Trial Ex 1.

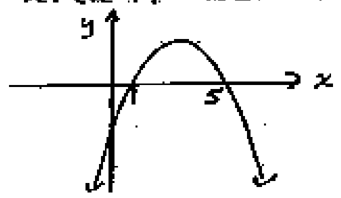
(a) $\frac{4}{5-x} \le 1$

$4(5-x) \le (5-x)^2$

$4(5-x) - (5-x)^2 \le 0$

$(5-x)(4-(5-x)) \le 0$

$(5-x)(x-1) \le 0$



Solution: $x \le 1$ or $x > 5$

(b) Let $P(x) = 4x^3 - x + p$

If $P(x)$ is divisible by $(x+3)$ then $P(-3) = 0$

$P(-3) = -108 + 3 + p$

$-108 + 3 + p = 0$

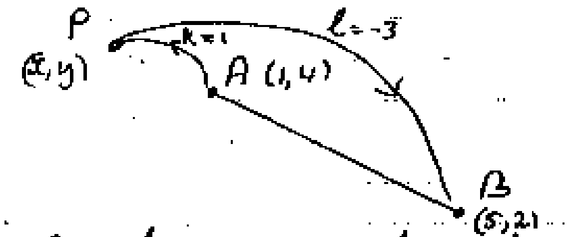
$p = 105$

(c) $(a+\frac{1}{2})^5 = a^5 + 5a^4\frac{1}{2} + 10a^3(\frac{1}{2})^2 + 10a^2(\frac{1}{2})^3$

$+ 5a(\frac{1}{2})^4 + (\frac{1}{2})^5$

$= a^5 + \frac{5}{2}a^4 + \frac{5}{2}a^3 + \frac{5}{4}a^2 + \frac{5}{16}a + \frac{1}{32}$

(d)



$x = \frac{lx_1 + kx_2}{l+k}$, $y = \frac{ly_1 + ky_2}{l+k}$

$= \frac{-3+5}{-2}$, $y = \frac{-12+2}{-2}$

$= -1$, $= 5$

The point is $(-1, 5)$

Give ✓ if they find the pt that divides AB internally $(2, 3\frac{1}{2})$

(e) $\int x(1-x^2)^5\,dx$

$= \int -\frac{1}{2} u^5\,du$

$= -\frac{1}{2}\frac{u^6}{6} + c$

$= -\frac{1}{12}(1-x^2)^6 + c$

$u = 1-x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

Question 2

(a) (i) $x = 4t$ and $y = 2t^2$

$\frac{dx}{dt} = 4$ $\qquad$ $\frac{dy}{dt} = 4t$

so $\frac{dy}{dx} = \frac{4t}{4}$

$= t$

so at $t=4$, the gradient is 4

(ii) When $t=4$, $x=16$, $y=32$

Tangent equation is $y-32 = 4(x-16)$

$y - 32 = 4x - 64$

$y = 4x - 32$

(b)

(i) Tangents to a circle from an external point are equal

$\therefore$ $TX = XA$

So ΔTXA is isosceles and $\angle XTA = \angle XAT = \theta$ (base angles of isosceles triangle)

(ii) Similarly, ΔAYS is isosceles with base angles $\angle YAS$ and $\angle YSA$ equal

But $\angle YAS = \angle TAX = \theta$ (vertically opposite)

So $\angle YSA = \theta$ and $\angle XTA = \theta$

But these are alternate

So $TX \parallel YS$

(c) (i) $(1+mx)^n = 1 + n.mx + \frac{n(n-1)}{2}(mx)^2 + \ldots$

(ii) $1 - 4x + 7x^2 - \ldots = 1 + nmx + \frac{n(n-1)}{2}(mx)^2 + \ldots$

equate coefficents of x: $\quad -4 = nm$ $\quad$ ①

equate coefficients of x^2: $\quad 7 = \frac{n(n-1)}{2}m^2$ $\quad$ ②

from ①, $m = -\frac{4}{n}$, substitute this in ②.

$14 = n(n-1)\frac{16}{n^2}$

$14 = \frac{16(n-1)}{n}$ $\qquad$ for a sensible method

$14n = 16n - 16$

$2n = 16$

$n = 8$

$m = \frac{-4}{8}$

$= -\frac{1}{2}$

$n = 8$ and $m = -\frac{1}{2}$

(d) $\lim_{x \to 0} \frac{5x \cos 2x}{\sin x} = \lim_{x\to 0} \frac{5x}{\sin x} \times \lim_{x \to 0} \cos 2x$

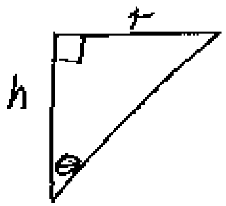
$= 5 \lim_{x \to 0} \frac{x}{\sin x} \times 1$

$= 5 \times 1$

$= 5$

Q3

(a) (i)



$\tan\theta = \frac{r}{h}$

$\frac{1}{2} = \frac{r}{h}$

$r = \frac{1}{2}h$

(ii) $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi (\frac{1}{2}h)^2 h$

$= \frac{1}{12}\pi h^3$

(iii) find $\frac{dh}{dt}$, given $\frac{dV}{dt} = 10$

$\frac{dh}{dt} = \frac{dh}{dV}\,\frac{dV}{dt}$

now $V = \frac{1}{12}\pi h^3$

$\frac{dV}{dh} = \frac{\pi h^2}{4}$ so $\frac{dh}{dV} = \frac{4}{\pi h^2}$

so $\frac{dh}{dt} = \frac{4}{\pi h^2} \times 10$

$= \frac{4}{\pi \times 50^2} \times 10$ when $h = 50$

$= \frac{40}{\pi \times 50 \times 50}$

$= \frac{4}{250\pi}$ cm per minute

(b) The general term is $\binom{11}{r}(2x^2)^{11-r}(-4x)^{-r}$

the index of x is $22 - 2r - r = 0$ if the term is independent of x.

$3r = 22$

$r = \frac{22}{3}$

since r must be an integer, there is no term independent of x.

(c) $\int_{-1}^{0} x\sqrt{1+x}\; dx$

$u = 1+x$

$du = dx$

when $x = 0$, $u = 1$

when $x = -1$, $u = 0$

$= \int_0^1 (u-1)\, u^{\frac{1}{2}}\, du$

$= \int_0^1 (u^{3/2} - u^{\frac{1}{2}})\, du$

$= \left[\frac{2u}{5}^{5/2} - \frac{2u}{3}^{3/2}\right]_0^1$

$= (\frac{2}{5} - \frac{2}{3}) - (0)$

$= -\frac{4}{15}$

(d) $\int \sin x \cos^3 x\, dx$

$= -\frac{1}{4}\cos^4 x + c$

Question 4

a) $y = \frac{1}{200} t e^{-t}$

$\frac{dy}{dt} = \frac{1}{200}\left[t \times (-e^{-t}) + e^{-t}\right]$

$= \frac{1}{200} e^{-t}(1-t)$.

b)(i) $\frac{dA}{dt} = \frac{1}{200}(1-t)e^{-t}$

Now, $\frac{1}{200}(1-t)e^{-t} > 0$ when $1-t>0$ or $t<1$

so, for $0<t<1$, $\frac{dA}{dt}>0$ and A is increasing.

and $\frac{1}{200}(1-t)e^{-t}<0$ when $1-t<0$, $t>1$

so, for $t>1$, $\frac{dA}{dt}<0$ and A is decreasing

(ii) $t=0$, $A = 0.0005$.

$A = \int \frac{1}{200}(1-t)e^{-t}\,dt$

$= \frac{1}{200} t e^{-t} + c$ from (a).

when $t=0$, $0.0005 = 0 + c$ so $c = 0.0005$

$A = \frac{1}{200} t e^{-t} + 0.0005$

(iii) from (i), the maximum A is when $t=1$.

so, if $t=1$, $A = \frac{1}{200} e^{-1} + 0.0005$

$\doteqdot 0.001839 + 0.0005$

$\doteqdot 0.002339$

$\doteqdot 0.0023$

(c) (i) $-\frac{\pi}{2} < x < \frac{3\pi}{2}$

(ii) $x = \frac{\pi}{2} - 2\tan^{-1} y$

$2\tan^{-1} y = \frac{\pi}{2} - x$

$\tan^{-1} y = \frac{1}{2}\left(\frac{\pi}{2} - x\right)$

$y = \tan\left(\frac{\pi}{4} - \frac{1}{2}x\right)$, $-\frac{\pi}{2} < x < \frac{3\pi}{2}$

(iii) $V = \pi \int x^2\,dy$

or, using the inverse function
$V = \pi \int y^2\,dx$

$= \pi \int_0^{\frac{\pi}{2}} \tan^2\left(\frac{\pi}{4} - \frac{1}{2}x\right) dx$

$= \pi \int_0^{\frac{\pi}{2}} \sec^2\left(\frac{\pi}{4} - \frac{1}{2}x\right) - 1 \quad dx$

$= \pi \left[-2\tan\left(\frac{\pi}{4} - \frac{1}{2}x\right) - x\right]_0^{\frac{\pi}{2}}$

$= \pi\left\{\left(-2\tan 0 - \frac{\pi}{2}\right) - \left(-2\tan\frac{\pi}{4} - 0\right)\right\}$

$= \pi\left(2 - \frac{\pi}{2}\right) \quad u^2$

Question 5

a) $\int_0^4 \frac{1}{3+\sqrt{x}}\,dx$

$x = (u-3)^2$
$dx = 2(u-3)\,du$
when $x=4$, $u=5$
when $x=0$, $u=3$.

$= \int_3^5 \frac{2(u-3)}{u}\,du$

$= 2\int_3^5 \left(1 - \frac{3}{u}\right) du$

$= 2\left[u - 3\log_e u\right]_3^5$

$= 2\left[(5-3\log 5) - (3 - 3\log 3)\right]$

$= 2\left(2 - 3\log 5 + 3\log 3\right)$

$= 2\left(2 + 3\log \frac{3}{5}\right)$

(b)(i) $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots \binom{n}{n}x^n$

$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots n\binom{n}{n}x^{n-1}$

(ii) let $x=1$

$LHS = n(2)^{n-1}$

$RHS = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots n\binom{n}{n}$

(iii) let $x=1$ in expansion of $(1+x)^n$

then $2^n = 1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots \binom{n}{n}$

and $n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots n\binom{n}{n}$

adding $2^n + n2^{n-1} = 1 + 2\binom{n}{1} + 3\binom{n}{2} + 4\binom{n}{3} + \cdots (n+1)\binom{n}{n}$

so $2^n + n2^{n-1} - 1 = 2\binom{n}{1} + 3\binom{n}{2} + 4\binom{n}{3} + \cdots (n+1)\binom{n}{n}$

(c) $\sqrt{3}\sin\theta - \cos\theta = R\cos(\theta+\alpha)$
$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

so $-1 = R\cos\alpha$ and $\sqrt{3} = -R\sin\alpha$

x

$R = \sqrt{3+1}$
$= 2$

and $\alpha = \frac{4\pi}{3}$

(d) If $n=1$

$LHS = \frac{1}{2!} = \frac{1}{2}$

$RHS = \frac{2!-1}{2!}$

$= \frac{1}{2}$

so the statement is true when $n=1$.

Now suppose the statement is true for some value of n, k.

ie $\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k}{(k+1)!} = \frac{(k+1)!-1}{(k+1)!}$

We now prove the result for $n = k+1$

That is prove that $\frac{1}{2!} + \frac{2}{3!} + \cdots \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+2)!-}{(k+2)!}$

Now, $LHS = \frac{(k+1)!-1}{(k+1)!} + \frac{k+1}{(k+2)!}$ using induction hypothesis

$$LHS = \frac{(k+1)!-1}{(k+1)!} + \frac{k+1}{(k+2)!}$$

$$= \frac{(k+2)(k+1)! - (k+2) + (k+1)}{(k+2)!}$$

$$= \frac{(k+2)!-1}{(k+2)!}$$

$$= RHS.$$

So, the statement if true for $k+1$ as long as it is true for k. Hence, by the principle of mathematical induction, it is true for all positive integers n.

Question 6

(a) i) At A, $y = 2\sin x$ and $y = \frac{1}{3}x$

so we want $2\sin x = \frac{1}{3}x$

or $2\sin x - \frac{1}{3}x = 0$

ii) Let $f(x) = 2\sin x - \frac{1}{3}x$

$f'(x) = 2\cos x - \frac{1}{3}$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, $x_0 = 3$

$= 3 - \frac{2\sin 3 - 1}{2\cos 3 - \frac{1}{3}}$

$\doteqdot 3 - 0 \cdot 3102$

$\doteqdot 2 \cdot 7$

(b) (i) $T = S + Ae^{-kt}$

$\frac{dT}{dt} = -kAe^{-kt}$, $Ae^{-kt} = T - S$

$= -k(T-S)$

(ii) (α) $T = 30° + Ae^{-kt}$

when $t=0$, $470° = 30° + Ae^{0}$

$A = 440°$

when $t = 10$, $250 = 30 + 440e^{-10k}$

$440e^{-10k} = 220$

$e^{-10k} = \frac{1}{2}$

$-10k = \log_e \frac{1}{2}$

$k = -\frac{1}{10}\log\frac{1}{2}$

$= \frac{1}{10}\log 2$

(b) Find t when $T = 70°$

$$70 = 30 + 440e^{-kt}, \quad k = \tfrac{1}{10}\ln 2.$$

$$e^{-kt} = \frac{40}{440} = \frac{1}{11}$$

$$-kt = \log_e \frac{1}{11}$$

$$t = \frac{\log_e \frac{1}{11}}{-\frac{1}{10}\log_e 2} \doteqdot 35 \text{ min}$$

(c) (i) $\angle AFB = \frac{\pi}{2}$ (the angle in a semicircle is a right angle).

So $\angle ABF = \pi - (\frac{\pi}{2} + \beta)$ (the angle sum of ΔAFB is π)
$= \frac{\pi}{2} - \beta$

(ii)
$\angle BAD = \frac{\pi}{2}$ (angle between tangent and radius is $\frac{\pi}{2}$).

So $\angle ADB = \pi - (\frac{\pi}{2} + (\frac{\pi}{2} - \beta))$ (angle sum of ΔADB is π).
$= \beta$

Now, $\angle BAF = \angle BEF$ (both subtended at the circumference by arc BF)
$= \beta$

So, $\angle BEF = \angle CDF$
So, $CDFE$ is cyclic (exterior angle equals interior opposite angle)

Question 2.

(a) (i) For car A

$\ddot{x} = -k$, since the car is decelerating

$\frac{1}{2}v^2 = -kx + c$

When $x = 0$, $v = V_A$.

so $\frac{1}{2}V_A^2 = 0 + c$ making $c = \frac{1}{2}V_A^2$

$v^2 = -2kx + V_A^2$

and speed $= \sqrt{V_A^2 - 2kx}$

(ii) For car A:

$\ddot{x} = -k$

Integrating, $\dot{x} = -kt + c_1$

When $t = 0$, $\dot{x} = V_A$ making $c_1 = V_A$

So $\dot{x} = -kt + V_A$

Integrating, $x = -\frac{1}{2}kt^2 + tV_A + C_2$

When $t = 0$, $x = 0$, taking the origin of displacement at car A, so $C_2 = 0$

We have $x = tV_A - \frac{1}{2}kt^2$

For car B:

$\ddot{x} = 0$

Integrating, $\dot{x} = C_3$

When $t = 0$, $\dot{x} = V_B$, making $C_3 = V_B$

So $\dot{x} = V_B$

Integrating, $x = tV_B + c_4$

When $t = 0$, $x = D$, car B is D metres in front of car A, making $C_4 = D$

So $x = tV_B + D$.

When the cars collide, their displacement are equal, so we have

$$tV_B + D = tV_A - \frac{1}{2}kt^2$$

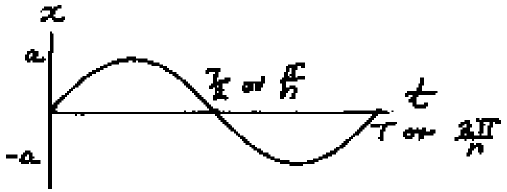
This is a quadratic in t.
For t to have a real value, the discriminant must be positive

$$\tfrac{1}{2}kt^2 - tV_A + tV_B + D = 0$$
$$kt^2 - 2t(V_A - V_B) + 2D = 0$$

$$\Delta = 4(V_A - V_B)^2 - 8kD$$
$$4(V_A - V_B)^2 - 8kD > 0$$
$$(V_A - V_B)^2 > 2kD$$
$$V_A - V_B > \sqrt{2kD}$$

since $V_A > V_B$ and so $V_A - V_B$ is positive.

(b) (i)

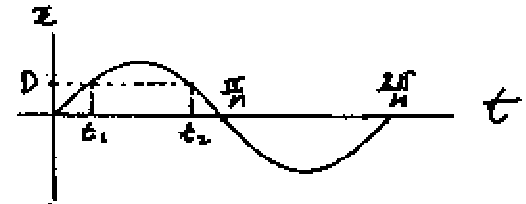


$T = \frac{2\pi}{n}$ so $n = \frac{2\pi}{T}$

(ii)

$$x = a\sin nt$$
$$\dot{x} = na\cos nt$$
$$= \frac{2\pi a}{T}\cos\frac{2\pi}{T}t$$

(iii)



At P we have $D = a\sin\frac{2\pi}{T}t$ ①

And $V = \frac{2\pi a}{T}\cos\frac{2\pi}{T}t$ ②

① ÷ ② $\quad \frac{D}{V} = \frac{a\,T}{2\pi a}\tan\frac{2\pi t}{T}$

$$\frac{D\,2\pi}{VT} = \tan\frac{2\pi t}{T}$$

Let t_1 and t_2 be the first two times when the particle is at P

Then $\frac{2\pi t_1}{T} = \tan^{-1}\frac{2\pi D}{VT}$

$$t_1 = \frac{T}{2\pi}\tan^{-1}\frac{2\pi D}{VT}$$

And $\quad t_2 = \frac{T}{2} - \frac{T}{2\pi}\tan^{-1}\frac{2\pi D}{VT}$

So the difference in times is

$$t_2 - t_1 = \frac{T}{2} - 2\frac{T}{2\pi}\tan^{-1}\frac{2\pi D}{VT}$$
$$= \frac{T}{\pi}\left(\frac{\pi}{2} - \tan^{-1}\frac{2\pi D}{VT}\right)$$
$$= \frac{T}{\pi}\tan^{-1}\frac{VT}{2\pi D}, \text{ using}$$

complementary angles.