SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

1999

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Sample Solutions

(1) (a) $\frac{d(\sin^{-1}2x)}{dx} = \frac{2}{\sqrt{1-4x^2}}$

(b) $\tan^{-1}(-1) = -\tan^{-1}1$

$= -\pi/4$

(c) $5x - y - 9 = 0$ | $2x - 3y + 12 = 0$

$y = 5x - 9$ | $3y = 2x + 12$

$m_1 = 5$ | $m_2 = \frac{2}{3}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{5 - 2/3}{1 + 5\times\frac{2}{3}}\right|$

$= 1$ $\quad \therefore \theta = 45°$

(d) $\sin x \doteqdot x$, x small

$\therefore \lim_{x\to 0} \frac{\sin 4x}{7x} = \lim_{x \to 0} \frac{4x}{7x}$

$= \frac{4}{7}$

(e) $x^3 + x^2 - 3 = 0$

(i) $\alpha + \beta + \gamma = -1$

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma = 0$

(iii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 1 - 2\times 0$

$= 1$

(f) $\int_0^{\pi/3} \cos^2 x \, dx = \frac{1}{2}\int_0^{\pi/3} 2\cos^2 x \, dx$

$= \frac{1}{2}\int_0^{\pi/3} (1 + \cos 2x)\, dx$

$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/3}$

$= \frac{1}{2}\left[\pi/3 + \frac{1}{2}\sin\frac{2\pi}{3}\right]$

$= \frac{\pi}{6} + \frac{1}{4}\times\frac{\sqrt{3}}{2}$

$= \pi/6 + \frac{\sqrt{3}}{8}$

(2) (a) $f(x) = \frac{1}{3}\cos^{-1}2x$

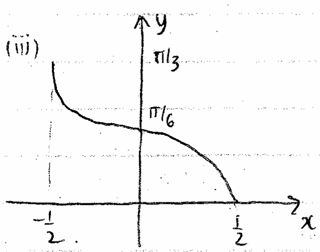
(i) $-1 \le 2x \le 1$

$-\frac{1}{2} \le x \le \frac{1}{2}$

(ii) $0 \le 3y \le \pi$

$0 \le y \le \pi/3$

(iii)



(2) (b) $\quad \begin{matrix} x_1 & y_1 \\ A(3,1) \end{matrix} \quad \begin{matrix} x_2 & y_2 \\ B(-1,4) \end{matrix} \quad \begin{matrix} m : n \\ -2 : 3 \end{matrix}$

$$P\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

$$= \left(\frac{2+9}{1}, \frac{-8+3}{1}\right)$$

$$= (11, -5)$$

(c) (i) $\displaystyle\int \frac{dx}{1+4x^2}$

$$= \frac{1}{4}\int \frac{dx}{\frac{1}{4}+x^2}$$

$$= \frac{1}{4} \times 2\tan^{-1}(2x) + C$$

$$= \frac{1}{2}\tan^{-1}(2x) + C$$

(c) (ii) $\displaystyle\int x\sqrt{2-x}\, dx$

$\left[u = 2-x \Rightarrow x = 2-u \quad dx = -du \right]$

$$= \int -(2-u)\sqrt{u}\, du$$

$$= \int (u^{3/2} - 2u^{1/2})\, du$$

$$= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C$$

$$= \frac{2}{5}(2-x)^{5/2} - \frac{4}{3}(2-x)^{3/2} + C$$

(d) $\log_4 9 \doteqdot 1.585$

$$\log_4 144 = \log_4(9\times 16)$$

$$= \log_4 9 + \log_4 16$$

$$\doteqdot 1.585 + 2\log_4 4$$

$$\doteqdot 1.585 + 2$$

$$= 3.585$$

(3) (a) $\displaystyle\left(x - \frac{2}{x^2}\right)^9 = \sum_{k=0}^{9} \binom{9}{k} x^{9-k}(-2x^{-2})^k$

$$= \sum_{k=0}^{9}\binom{9}{k}(-2)^k x^{9-3k}$$

constant term : $9-3k=0$

$\therefore k = 3$

constant term $= \binom{9}{3}(-2)^3$

$= -672$

(b) $y = x^3 + 3x^2 + 4x$

y is a cubic so cuts at least once.

$y' = 3x^2 + 6x + 4$

$\Delta = 36 - 4\times 3\times 4$

$= -12$

< 0

$\therefore y$ is strictly increasing

$\therefore y$ has <u>only</u> one x-intercept.

(3) (c) $LHS = \cos^4 x + \sin^2 x$

$= (1-\sin^2 x)^2 + \sin^2 x$ $\quad [\because \cos^2 x = 1-\sin^2 x]$

$= \sin^4 x - 2\sin^2 x + 1 + \sin^2 x$

$= \sin^4 x + 1 - \sin^2 x$

$= \sin^4 x + \cos^2 x$

$= RHS$

(d) $4 \times 6^n + 1 = 5M$, $n > 0$ (where M is an integer)

Test $n=1$: $LHS = 4 \times 6 + 1 = 25$

$= 5 \times 5$

$\therefore$ The statement is true for $n=1$

Assume the statement is true for some integer $n=k$

i.e. $4 \times 6^k + 1 = 5P$ (P an integer) $\quad (*)$

We need to prove the statement true when $n=k+1$

ie. $4 \times 6^{k+1} + 1 = 5Q$ (Q an integer)

$LHS = 4 \times 6^{k+1} + 1$

$= 4 \times 6 . 6^k + 1$

$= 4 \times 6 . 6^k + 1$

$= 6(4 \times 6^k + 1) - 6 + 1$

$= 6(5P) - 5$

$= 5(6P - 1)$

$= 5Q$ $\quad$ [Q is an integer, since P is an integer]

$= RHS$

So when the statement is true for $n=k$, it is true for $n=k+1$

So by the principle of mathematical induction $4 \times 6^n + 1 = 5M$, for $n > 0$

④ (a) (i) $\sqrt{3}\sin 3t - \cos 3t \equiv R\sin(3t-\alpha)$

$= R\sin 3t\cos\alpha - R\sin\alpha\cos 3t$

$R\cos\alpha = \sqrt{3}$ — ①

$R\sin\alpha = 1$ — ②

$\Rightarrow \boxed{R = 2}$

$\Downarrow$

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\boxed{\alpha = \pi/6}$

(ii) "otherwise" is better approach as

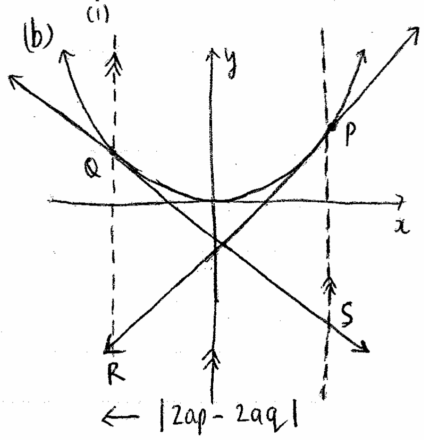
$\sqrt{3}\sin 3t = \cos 3t$

$\tan 3t = \frac{1}{\sqrt{3}}$

$3t = n\pi + \pi/6 \quad (n \in \mathbb{Z})$

$\boxed{t = \frac{n\pi}{3} + \pi/18}$

(b) (i)



(ii) $P(2ap, ap^2)$

$x^2 = 4ay$

$\therefore 2x = 4a\frac{dy}{dx}$

$P: 2(2ap) = 4a\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = p$

$y - ap^2 = p(x - 2ap)$

$y = px - 2ap^2 + ap^2$

$\boxed{y = px - ap^2}$

(iii) $R: x = 2aq$

$y = px - ap^2$

$\therefore y = 2apq - ap^2$

$\therefore QR = |aq^2 - 2apq + ap^2| = |a|(p-q)^2$

$\therefore x = 2ap$

$y = qx - aq^2 \Rightarrow y = 2apq - aq^2$

$PS = |ap^2 - 2apq + aq^2| = |a|(p-q)^2$

(4) (b) (iii)

$RQ \parallel SP$ and $RQ = SP \Rightarrow PQRS$ is a parallelogram.

(iv) $\text{Area} = |ap^2 - 2apq + aq^2| \times |2a(p-q)|$

$= 2a^2 |p^2 - 2pq + q^2| \times |p-q|$

$= 2a^2 |(p-q)^2| \times |p-q|$

$= 2a^2 |p-q|^3$

(5) $x = 5\sin 2t$

(a) (i) $\dot{x} = 10\cos 2t$

$\ddot{x} = -20\sin 2t$

$= -4(5\sin 2t)$

$= -4x$

(ii) $\dot{x} = 10\cos 2t$

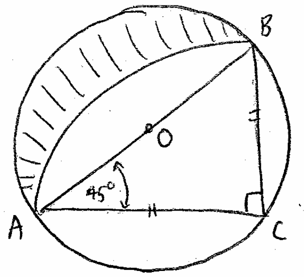
max $|\dot{x}| = 10$ m/s

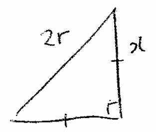
(iii) $\ddot{x} = -20\sin 2t$

max acc $= 20$ m/s^2

(iv) $x = 0 \Rightarrow a = 0$ m/s^2

(b)





$\therefore 2x^2 = 4r^2$

$x = \sqrt{2}r$

$\Delta ABC = \frac{1}{2}(\sqrt{2}r)^2$

$= r^2$

segment $AOB = \frac{1}{4}\pi(\sqrt{2}r)^2 - r^2$

$= \frac{\pi r^2}{2} - r^2$

$= r^2(\pi/2 - 1)$

$= r^2\left(\frac{\pi - 2}{2}\right)$

semi-circle $= \frac{1}{2}\pi r^2$

$\therefore$ shaded area $= \frac{1}{2}\pi r^2 - \left(r^2(\frac{\pi}{2} - 1)\right)$

$= r^2$

Q.E.D.

⑤ (c) (i) $T = D + Ce^{-kt}$

LHS $\dfrac{dT}{dt} = -k \times Ce^{-kt}$

$= -k(T-D)$

$= RHS$

Q.E.D.

(ii) $D = -10$ $\quad t=0,\ T=25$

$t=12,\ T=15$

$T = -10 + Ce^{-kt}$

$\therefore T = -10 + 35e^{-kt} = 35e^{-kt} - 10$

$\therefore 15 = 35e^{-12k} - 10$

$35e^{-12k} = 25$

$e^{-12k} = 5/7$

$\therefore -12k = \ln(5/7)$

$k = \frac{1}{12}\ln(7/5)$

$T = 0$

$35e^{-kt} - 10 = 0$

$e^{-kt} = \frac{10}{35} = 2/7$

$-kt = \ln(2/7)$

$t = \frac{1}{k}\ln(7/2) \doteqdot 44.7$

$\therefore$ An additional 32.7 minutes

(6) (a) 6W , 2R

Total arrangements $= \frac{8!}{6!2!} = \binom{8}{2} = 28$

(i) R X X X X X X R

$P(\text{Reds at end}) = \frac{1}{28}$

(ii) RR X X X X X X $\frac{7!}{6!} = 7$ ways

RXR X X X X X $\frac{6 \cdot 6!}{6!} = 6$ ways

RXXR X X X X $\frac{6 \cdot 5 \cdot 5!}{6!} = 5$ ways

Total = 18 ways

$P(\text{At least 3 separated}) = 1 - \frac{18}{28} = \frac{10}{28} = \frac{5}{14}$

(b) $f(x) = x^4 - 110$, $f'(x) = 4x^3$, $x_0 = 3.2$

$f(3.2) \doteqdot -5.1424$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3.2 - \frac{(3.2^4 - 110)}{4(3.2)^3}$

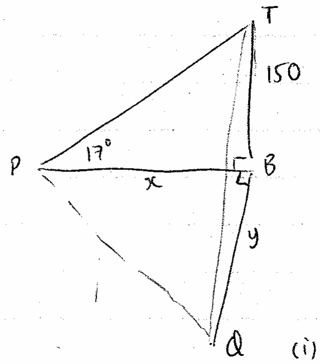
$\doteqdot 3.\underline{239}233398 \doteqdot 3.24$

$f(x_1) \doteqdot 0.953474$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \doteqdot 3.\underline{238}532068 \doteqdot 3.24$

$f(x_2) \doteqdot 0.0000309$ No. change to 2 decimal place.

6 (c)



$\tan 17^\circ = \frac{150}{x}$

$x = \frac{150}{\tan 17^\circ} = 150 \cot 17^\circ = 150 \tan 73^\circ$

$\therefore y = 150 \tan 75^\circ$

(i) $PQ^2 = x^2 + y^2$

$= 150^2 (\tan^2 73^\circ + \tan^2 75^\circ)$

$PQ = 150 \sqrt{\tan^2 73^\circ + \tan^2 75}$

(ii) $PT = \frac{150}{\sin 17^\circ}$, $TQ = \frac{150}{\sin 15^\circ}$

$\cos \angle PTQ = \frac{PT^2 + TQ^2 - PQ^2}{2 . PT . TQ}$

$\therefore \angle TPQ = 85^\circ 40'$

7 (a) (i) $PQ = PT - QT = ST - RT = SR$ (tangents from a point)

(ii) ΔTQR & ΔTPS are isosceles

$\because \angle T$ is common $\Rightarrow \angle QRT = \angle PST$

(base angles of isosceles Δ's with common vertex are equal)

$\therefore PS \parallel QR$

$\therefore PSRQ$ is a trapezium

(iii) $\angle QRT = \angle PSR$ (parallel)

$= \angle SPR$ (isos. Δ)

$\therefore$ exterior angle = opposite interior angle

$\therefore PSRQ$ is a cyclic quad (converse of exterior angle theorem of cyclic quads)

(iv) let $\angle PAS = x$ $\Rightarrow \angle PSR = x$ (alternate seg. theorem)

$\Rightarrow \angle PQR = 180 - x$ (opp. angles of cyclic quad)

$\Rightarrow \angle QBR = 180 - x$ (alternate seg. theorem

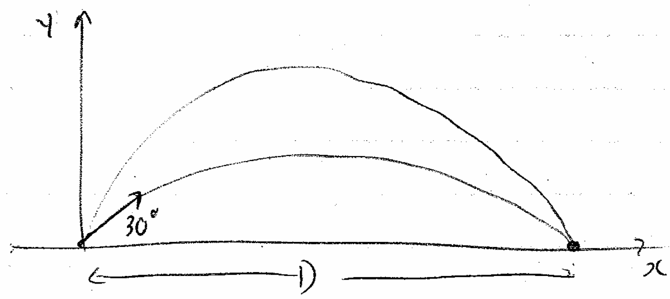
$\therefore \angle PAS + \angle QBR = 180^\circ$

7(b) Assume the target is on the ground (i.e. $y=0$) at same horizontal height as cannon.

[Too many variables otherwise]

Formulae quoted without proof:

(i)



$$D = \frac{v^2 \sin 2\theta}{g} = \frac{150^2 \sin 60}{10} = \frac{150^2 \sin 120^\circ}{10}$$

$$= 1125\sqrt{3}$$

(ii) $\theta = 60$ is the other angle.

(iii) $T = \dfrac{2v\sin\theta}{g}$

$\therefore$ time elapse $= \dfrac{2v}{g}(\sin 60 - \sin 30)$

$$= \frac{2\times 150}{10}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$$

$$= 15(\sqrt{3}-1)$$

$\doteqdot$ 11.0 secs difference