Student Name:____________________ **Teacher Name:**__________________

Saint Mark's Coptic Orthodox College

Mathematics Department

End Semester 2 Examination

2007

Year 11 Mathematics Extension 1

Time Allowed: 2 Hours

DIRECTIONS TO CANDIDATE:

- Attempt all questions.
- Show all necessary working. Marks may be deducted for careless or badly arranged work.
- Only approved calculators may be used.
- This paper contains **6** questions in **4** pages.

Question	1	2	3	4	5	6	Total	Percent
Mark	/13	/13	/12	/11	/11	/10	/70	%

Question 1 (13 Marks) — Marks

a) Find the gradient function of the tangent to the curve $y = \sqrt{x}$ *at* $x = 1$. — 2

b) State the domain and range of

$y = \sqrt{4 - x^2}$ — 2

c) Find the value of : $\lim_{x \to \infty} \frac{3 - 2x^2}{x^2 + x - 3}$ — 1

d) Find the gradient function of

$y = x(5 - x)^6$ — 2

e) i. Expand $\cos(x - y)$.

ii. Hence find the exact value of $\cos 15^\circ$.
Express your answer with a rational denominator. — 3

f) For what values of m will $x^2 + 3x + 6 = m(x + 2)$ have real roots? — 3

Question 2 (13 Marks) — Marks

a) Solve $\frac{1}{x+1} \geq 1 - x$ — 3

b) The point C(-1, -4) divides the interval AB externally in the ratio 3 : 1.
If the coordinates of A are (3, 2), find the coordinates of B. — 3

c) Find the coordinates of the point on the curve $y = x^2 + 3x - 1$ where the tangent is parallel to the line $y = 2x + 6$. — 3

d) i. Find the remainder when $P(x) = x^3 - x^2 - 8x + 12$ is divided by $(x - 2)$.

ii. Factorise completely the polynomial $P(x)$.

iii. Sketch the polynomial. — 4

Question 3 (12 Marks) **Marks**

a) If α, β, λ are roots of the equation $x^3 + 2x^2 - 3x + 5 = 0$ evaluate

i. $\alpha + \beta + \lambda$ 1

ii. $\alpha\beta\lambda$ 1

iii. $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ 2

iv. $(\alpha + 1)(\beta + 1)(\lambda + 1)$ 2

b) If the roots of $x^2 + px + q = 0$ differ by 1, show that $p^2 = 4q + 1$ 3

c) Show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. 3

Question 4 (11 Marks) **Marks**

a) Insert 9 terms in an Arithmetic Progression between 29 and 109, then find their sum. 4

b) A ship (P) is due south of a cliff with a angle of elevation to the top of the cliff (C) is $21°48'$ and from a ship (Q) due east of the cliff is $30°27'$. The distance from P to Q is 100m.

i. Draw the diagram in your answer booklet marking on it all information given. 1

ii. If the height of the cliff is h and A is at the bottom of the cliff, show that AP $\approx$ 2.5h and AQ $\approx$ 1.7h. 3

iii. Hence or otherwise find the height of the cliff to the nearest metre. 3

Question 5 (11 Marks)

Marks

a) Find any values of k, which will make the expression:
$(k+1)x^2 - 2(k-1)x + (2k-5)$ a perfect square. 3

b) The sum of the first seven terms of an arithmetic series is five time the seventh term. Also, the sum of the sixth and seventh term is 40. Find the sum of 15 terms. 4

c) Find the equation of the locus of a point that moves so that it is equidistant from the Line $3x-4y+4=0$ and the line $8x-6y+3=0$. 4

Question 6 (10 Marks)

Marks

a) If $\sec\theta - \tan\theta = x$, show that $x = \dfrac{1-t}{1+t}$ by assuming that $t = \tan\dfrac{\theta}{2}$. 4

b) Show by the principles of Mathematical Induction that:

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$ 6

[[End Of Qns]]

Year 11 - Yearly 2001 (Ext 1).

Question 1 (13 Marks)

a) $y = \sqrt{x}$
$= x^{1/2}$
$y' = \frac{1}{2}x^{-1/2}$

at $x = 1$ $\quad y' = \frac{1}{2\sqrt{1}}$
$= \frac{1}{2}$

/2

b) $y = \sqrt{4 - x^2}$

$4 - x^2 \geq 0$
$(2-x)(2+x) \geq 0$ $\quad -2 \quad 2$
D: $-2 \leq x \leq 2$
R: $0 \; y \leq 2$

/2

c) $\lim_{x \to \infty} \frac{3 - 2x^2}{x^2 + x - 3}$
$= \lim_{x \to \infty} \frac{3/x^2 - 2}{1 + 1/x - 3/x^2}$
$= -2$

/1

d) $y = x(5-x)^6$
$y' = (5-x)^6 + 6x(5-x)^5 \times -1$
$= (5-x)^5(5-x-6x)$
$= (5-7x)(5-x)^5$

/2

e) i) $\cos(x-y)$
$= \cos x \cos y + \sin x \sin y$

/1

ii) $\cos 15^\circ = \cos(60 - 45^\circ)$
$= \cos 60 \cos 45 + \sin 60 \sin 45$
$= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$
$= \frac{1+\sqrt{3}}{2\sqrt{2}}$
$= \frac{\sqrt{2}+\sqrt{6}}{4}$

/2

f) $x^2 + 3x + 6 = m(x+2)$
Real roots $\Delta \geq 0$

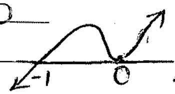
$x^2 + (3-m)x + 6 - 2m = 0$
$\Delta = b^2 - 4ac$
$(3-m)^2 - 4(6-2m) \geq 0$
$9 - 6m + m^2 - 24 + 8m \geq 0$
$m^2 + 2m - 15 \geq 0$
$(m+5)(m-3) \geq 0$ $\quad -5 \quad 3$
$m \leq -5,\ m \geq 3.$

/3

Question 2 (13 Marks)

a) $\frac{1}{x+1} \geq 1 - x \quad \times (x+1)^2$

$x + 1 \geq (1-x)(x+1)^2$
$(x+1)[(1-x)(x+1) - 1] \leq 0$
$(x+1)[x + 1 - x^2 - x - 1] \leq 0$
$\therefore -x^2(x+1) \leq 0$
$x^2(x+1) \geq 0$
$x > -1$



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b) $A(3,2) \quad B(x,y) \quad C(-1,-4)$
exernal ie $-3:1$.

$-1 = \frac{-3x+3}{-2} \qquad -4 = \frac{-3y+2}{-2}$

$-3x = -1 \qquad -3y = 6$
$x = \frac{1}{3} \qquad y = -2.$
$\therefore (1/3, -2)$

/3

c) $y = x^2 + 3x - 1$
$y' = 2x + 3$

grad of $y = 2x + 6$ is 2.
$\therefore 2x + 3 = 2$
$2x = -1$
$x = -\frac{1}{2}$
$y = (-\frac{1}{2})^2 - 3 \times \frac{1}{2} - 1$
$= -2\frac{1}{4}$
$\therefore (-\frac{1}{2}, -2\frac{1}{4})$

/3

Q2 d) (i) $P(x) = x^3 - x^2 - 8x + 12$
$P(2) = 2^3 - 2^2 - 16 + 12$
$= 8 - 4 - 16 + 12$
$= 0$
$\therefore (x-2)$ is a factor

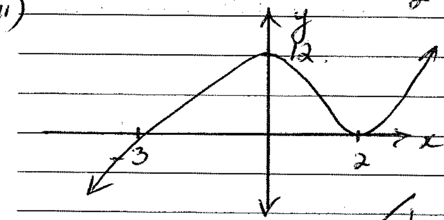
/1

(ii)
$$\begin{array}{r|l} & x^2 + x - 6 \\ x - 2 & x^3 - x^2 - 8x + 12 \\ & x^3 - 2x^2 \\ & \quad x^2 - 8x \\ & \quad x^2 - 2x \\ & \qquad -6x + 12 \\ & \qquad -6x + 12 \end{array}$$

$\therefore P(x) = (x-2)(x^2 + x - 6)$
$= (x-2)(x+3)(x-2)$
$= (x-2)^2(x+3)$

/2

(iii)



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Now $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$
$= -3$

$\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{+3}{5}$

/2

(iv) $(\alpha+1)(\beta+1)(\gamma+1)$
$= (\alpha\beta + \alpha + \beta + 1)(\gamma + 1)$
$= \alpha\beta\gamma + \alpha\gamma + \beta\gamma + \gamma + \alpha\beta + \alpha + \beta + 1$
$= \alpha\beta\gamma + \alpha\beta + \alpha\gamma + \beta\gamma + \alpha + \beta + \gamma + 1$
$= -5 - 3 + -2 + 1$
$= -9$

/2.

b) $x^2 + px + q = 0$
Let roots be $\alpha, \alpha+1$
$2\alpha + 1 = -p$ --- ①
$\alpha^2 + \alpha = q$ --- ②
from ①
$\alpha = \frac{-p-1}{2}$
subst in ②
$\frac{(-p-1)^2}{4} + \frac{-p-1}{2} = q$
$p^2 + 2p + 1 - 2p - 2 = 4q.$
$p^2 = 4q + 1$
$\therefore$ Shown.

/3

Question 3 (12 Marks)

a) $x^3 + 2x^2 - 3x + 5 = 0$
(i) $\alpha + \beta + \gamma = \frac{-b}{a}$
$= -2$

/1

(ii) $\alpha\beta\gamma = \frac{-d}{a} = -5$

/1

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$
$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

c) Show $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
LHS $= \sin(\theta + 2\theta)$
$= \sin\theta\cos 2\theta + \cos\theta\sin 2\theta$
$= \sin\theta(1 - 2\sin^2\theta) + 2\sin\theta\cos^2\theta$
$= \sin\theta - 2\sin^3\theta + 2\sin\theta(1 - \sin^2\theta)$
$= \sin\theta - 2\sin^3\theta + 2\sin\theta - 2\sin^3\theta$
$= 3\sin\theta - 4\sin^3\theta$
$=$ RHS.

/3

Question 4 (11 Marks)

a) 29, 9 terms, 109. AP

$a = 29$, $T_{11} = 109$

$29 + 10d = 109$

$d = 8$

29, 37, 45, 53, 61, 69, 77, 85, 93, 101, 109

$S_n = \frac{n}{2}[2a + (n-1)d]$

or $= \frac{n}{2}[a + L]$

$S_{11} = \frac{11}{2}[29 + 109]$

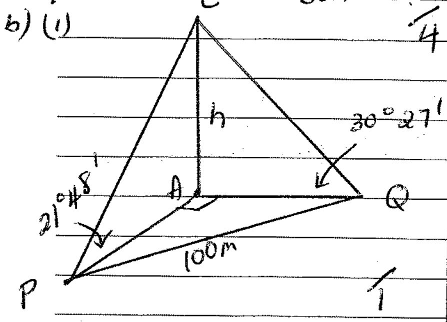
$= 759$

or, Sum of the 9 = $759 - (29+109)$

$= 621$

4

b) (i)



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(ii) Show that $AP \approx 2.5h$, $AQ \approx 1.7h$.

$\tan 21°48' = \frac{h}{AP}$

$AP = \frac{h}{\tan 21°48'}$

$\approx 2.5\,h$.

$\tan 30°27' = \frac{h}{AQ}$

$AQ \approx \frac{h}{\tan 30°27'}$

$\approx 1.7h$.

3

(iii) $PQ^2 = AP^2 + AQ^2$

$100^2 = (2.5h)^2 + (1.7h)^2$

$h^2 = \frac{100^2}{2.5^2 + 1.7^2}$

$h^2 = 1094.09\ldots$

$h \approx 33$ m.

3

Question 5 (11 Marks)

a) $(k+1)x^2 - 2(k-1)x + (2k-5)$ is a perfect square when $\Delta = 0$.

$4(k-1)^2 - 4(k+1)(2k-5) = 0$

$4k^2 - 8k + 4 - 4(2k^2 - 3k - 5) = 0$

$-4k^2 + 4k + 24 = 0$

$k^2 - k + 6 = 0$

$(k-3)(k+2) = 0$

$k = 3$ or -2.

3

b) $5 \times T_7 = S_7$ of AP.

$5(a + 6d) = \frac{7}{2}[2a + 6d]$

$10a + 60d = 14a + 42d$.

$4a = 18d$

$a = \frac{9}{2}d$ — (1)

$T_6 + T_7 = 40$

$a + 5d + a + 6d = 40$

$2a + 11d = 40$ — (2).

subst (1) in (2)

$9d + 11d = 40$

$d = 2$.

$\therefore a = 9$

$S_{15} = \frac{15}{2}[18 + 14 \times 2]$

$= 345$

4

Q5 c) $P(x, y)$ is equidistant from $3x - 4y + 4 = 0$ — L_1 and $8x - 6y + 3 = 0$ — L_2

$\therefore PL_1 = PL_2$.

$\frac{|3x - 4y + 4|}{\sqrt{9+16}} = \frac{|8x - 6y + 3|}{\sqrt{64+36}}$

$\frac{|3x - 4y + 4|}{5} = \frac{|8x - 6y + 3|}{10}$

$2|3x - 4y + 4| = |8x - 6y + 3|$

2 solutions.

$6x - 8y + 8 = 8x - 6y + 3$

$2x + 2y - 5 = 0$

or $6x - 8y + 8 = -8x + 6y - 3$

$14x - 14y + 11 = 0$.

$\therefore$ P lies on either lines

$2x + 2y - 5 = 0$

or $14x - 14y + 11 = 0$.

4

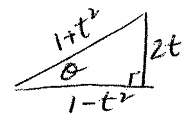
Question 6 (10 Marks)

a) If $\sec\theta - \tan\theta = x$

show $x = \frac{1-t}{1+t}$

assuming $t = \tan\frac{\theta}{2}$.

$\therefore$



$LHS = \frac{1+t^2}{1-t^2} - \frac{2t}{1-t^2}$

$= \frac{t^2 - 2t + 1}{1-t^2}$

$= \frac{(t-1)^2}{(1-t)(1+t)}$

$= \frac{(t-1)^2}{-(t-1)(1+t)}$

$= \frac{-(t-1)}{1+t}$

$= \frac{1-t}{1+t}$

$= x$.

$\therefore$ Shown.

4

Question 6 (con't)

b) Show $\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}---+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

Step 1: When $n=1$

$LHS=\frac{1}{1\times3}=\frac{1}{3}$

$RHS=\frac{1}{2+1}=\frac{1}{3}$. $\therefore$ True Statement.

Step 2: Assume that the statement is true for $n=k$

$\therefore S_k=\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+---\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$

When $n=k+1$, aim to show

$S_{k+1}=\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+---\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$.

Now $S_{k+1}=S_k+T_{k+1}$

$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$

$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$

$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$

$=\frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}$

$=\frac{k+1}{2k+3}$

$=RHS$.

$\therefore$ Statement is true for $n=k+1$.

Step 3:

Since the statement is true for $n=1$, its true for $n=2$,

$\therefore$ its true for $n=3$ and so on.

$\therefore$ It's true for all n.

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