N.S.W. DEPARTMENT OF EDUCATION

HIGHER SCHOOL CERTIFICATE EXAMINATION 1972

MATHEMATICS PAPER B (2F) - (EQUIVALENT TO 3 UNIT AND 4 UNIT - 1ST PAPER)

Instructions Time allowed 3 hours. All questions may be attempted. In every question, all necessary working should be shown. Marks will be deducted for careless or badly arranged work.

Mathematical tables will be supplied. Slide rules (or calculators) may be used.

QUESTION 1 (12 Marks)

(i) Find the second derivative of $1/\sqrt{x}$

(ii) Find $\int_0^1 \dfrac{dx}{\sqrt{4 - x^2}}$

(iii) Write down the equation satisfied by all points in the x-y plane at a distance of two units from the point $(3, 4)$.

(iv) The curve $y = 8x^3$ is reflected in the line $y = x$. Find the equation of the resulting curve.

QUESTION 2 (9 Marks)

(i) Find the area under the curve $y = 1/x$ between $x = e$ and $x = e^2$

(ii) It is known that the equation $e^{-x^2} - 5x^2 - 0.99 = 0$ has a positive root close to the origin. Attempt to find the root using Newton's method, starting with the trial value $x_0 = 0$. Explain why this method fails.

(iii) Find the derivative of $e^{\sin x}$

QUESTION 3 (9 Marks)

(i) A die has the numbers 1, 2, 3, 4, 5, and 6, respectively, marked on its six faces. Find the expected value of the number which is obtained by casting the die at random.

(ii) Does the curve $y = \tan^{-1}x$ have a maximum? Explain your answer with the aid of a sketch.

(iii) Write down the sum S of the geometric series $n + n^2 + n^3 + \ldots + n^k$ where k is a positive integer. Does S approach a limit as n tends to infinity, k remaining constant?

QUESTION 4 (10 Marks)

Consider the parabola $y = x^2$.

(i) Find the equation of the tangent ℓ to the parabola at the point P: (t, t^2).

(ii) Show that the line passing through the focus of the parabola, and perpendicular to ℓ, has the equation $y = \frac{t - 2x}{4t}$

(iii) Hence, or otherwise, find the locus of the foot of the perpendicular from the focus to the tangent to the parabola at any point.

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QUESTION 5 (10 Marks)

(i) Write down the equation describing the surface S of a sphere centred at the origin and of radius five units.

(ii) Use geometrical terms to describe the set of points T satisfying the equation $x^2 + z^2 = 16$.

(iii) Let $U = S \cap T$ be the intersection of the above two sets S and T. Describe the set U in precise geometrical terms (including shape, size, and location).

QUESTION 6 (10 Marks)

A function f is called odd if $f(-x) = -f(x)$ for all x.

(i) Prove that every odd function is zero at $x = 0$.

(ii) Prove that every odd polynomial $P(x)$ is divisible by x.

(iii) The polynomial $P(x)$ is known to be monic, to be an odd function, and to have a root at $x = -5$. Show that $P(x)$ has degree no less than 3.

(iv) Find a polynomial $Q(x)$ of degree 3 with the properties given in part (iii). State, with reasons, whether there are any other polynomials of degree 3 with these properties.

(v) State the form of the most general polynomial with the properties given in part (iii) and with degree d in the range $4 \leq d \leq 6$.

QUESTION 7 (10 Marks)

A particle moving along the x- axis starts at the origin with an initial velocity v_0. Its acceleration is given by $\frac{d^2x}{dt^2} = 4x^3 - 16x$.

(i) Show that the quantity $E = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - x^4 + 8x^2$ does not change with time.

(ii) Given that the initial velocity is $v_0 = \sqrt{(31/8)}$, find the value of E.

(iii) Hence, or otherwise, establish that the particle with this initial velocity remains at all times in the region $-\frac{1}{2} \leq x \leq \frac{1}{2}$. State, giving your reasons, whether the motion is simple harmonic.

(iv) Find the value of E when the initial velocity is $v_0 = 6$. Use this result to discuss the nature of the subsequent motion.

QUESTION 8 (10 Marks)

A deck of cards consists of four groups of seven cards, each group having a distinctive colour. The cards within any one group are labelled with the integers "1" through "7".

(i) Determine the total number of different (i.e. distinct and unordered) five-card subsets that can be chosen from this deck. Leave your answer unsimplified.

(ii) Find the probability that five cards selected at random consist of exactly four cards labelled with the same integer, plus one other card. Express your answer as an unsimplified fraction.

(iii) The five cards are replaced in the deck, and the deck is shuffled. Five cards are selected at random. Find the probability that these five cards consist of three cards each of which is labelled with the integer "2", plus two cards each of which is labelled with the integer "6". Express your answer as an unsimplified fraction.

QUESTION 9 (10 Marks)

(i) Write down an expression for $\cos (a + b)$ and hence prove that $\cos (2q) = 1 - 2 \sin^2 q$.

(ii) Prove the identity $\frac{\cos y - \cos (y + 2q)}{2 \sin q} = \sin (y + q)$.

(iii) Use mathematical induction and the result of part (ii) to prove the identity $\sin q + \sin 3q + \sin 5q + \ldots + \sin (2N - 1)q = \frac{1 - \cos 2Nq}{2 \sin q}$

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QUESTION 10 (10 Marks)

Consider the integral $J = \int_0^{\pi} \sin x \, dx$.

(i) Determine the value of J.

(ii) By dividing the interval from $x = 0$ to $x = \pi$ into N equal sub-intervals and using the mid-ordinate approximation, show that an approximate value for J is given by

$K_N = 2q_N \left[\sin q_N + \sin 3q_N + \sin 5q_N + \ldots + \sin (2N - 1)q_N\right]$ where

$q_N = \pi/(2N)$.

(iii) Evaluate K_2 to two significant figures; compare this with the exact value of J and state the percentage error of the approximation, to the nearest percent.

(iv) Starting with the formula of part (ii), and using the result stated in question 9 (iii), deduce the exact value of J.

$\left[\text{It may be assumed that } \lim\limits_{h \to 0} \dfrac{h}{\sin h} = 1\right]$.