NEWINGTON

HSC TRIAL 2000 SOLUTIONS.

QUESTION ONE

(a) $-8 + 4a - 4 - 4 = -7$

$4a = 9$

$a = \frac{9}{4}$ ✓

(b) $\tan\theta = \left|\frac{1+3}{1-3}\right| = 2$ ✓ $\theta = 63°26'$ ✓

(c) (i) $\cos 2x = 1 - 2\sin^2 x$ ✓

(ii) Let $\sin^{-1}\frac{\sqrt{3}}{2} = x$

$\therefore \sin x = \frac{\sqrt{3}}{2}, \quad 0 < x < \frac{\pi}{2}$

$\cos\left(2\sin^{-1}\frac{\sqrt{3}}{2}\right) = \cos 2x$

$= 1 - 2\sin^2 x$

$= 1 - 2\left(\frac{3}{4}\right)$ ✓

$= -\frac{1}{2}$ ✓

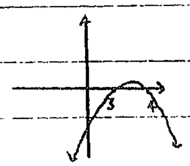
(d) $\frac{1}{x-3} \geq 1$

$x - 3 \geq (x-3)^2, \quad x \neq 3$ ✓

$(x-3)(1-x+3) \geq 0, \quad x \neq 3$

$(x-3)(4-x) \geq 0, \quad x \neq 3$ ✓

$3 < x \leq 4$ ✓



(e) let $u = 1 - x$ when $x = 1$ $u = 0$ when $x = 0$, $u = 1$

$\frac{du}{dx} = -1$

$\int_0^1 x\sqrt{1-x}\,dx = \int_1^0 -(1-u)\sqrt{u}\,du$ ✓

$= \int_0^1 u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$

$= \left[\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^1$ ✓

$= \frac{2}{3} - \frac{2}{5}$

$= \frac{4}{15}$ ✓

QUESTION TWO

(a) LHS $= \frac{\cos x - \cos 2x}{\sin 2x + \sin x}$

$= \frac{\cos x - 2\cos^2 x + 1}{2\sin x\cos x + \sin x}$ ✓

$= \frac{(1+2\cos x)(1-\cos x)}{\sin x(2\cos x + 1)}$ ✓

$= \frac{1-\cos x}{\sin x}$

$= \frac{1}{\sin x} - \frac{\cos x}{\sin x}$ ✓

(b) $f(x) = 2\sin x - 10x + 5$

$f'(x) = 2\cos x - 10$ ✓

If $x_1 = 0.5$, $x_2 = 0.5 - \frac{f(0.5)}{f'(0.5)}$

$= 0.5 - \frac{0.958851}{-8.24483}$ ✓

$= 0.62$ to 2 dec. pls. ✓

(c) let $\sqrt{3}\cos\theta - \sin\theta = r\cos(\theta + \alpha)$

$= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$

$\therefore r\cos\alpha = \sqrt{3}, \quad r\sin\alpha = 1$

$\therefore r^2 = 4$

when $r = 2$ ✓, $\alpha = \frac{\pi}{6}$ ✓

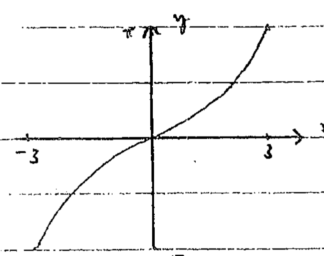
solve $2\cos\left(\theta + \frac{\pi}{6}\right) = 1$

$\cos\left(\theta + \frac{\pi}{6}\right) = \frac{1}{2}$

$\therefore \theta + \frac{\pi}{6} = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ ✓

$\therefore \theta = \frac{\pi}{6}$ or $\frac{3\pi}{2}$ ✓

(d)



✓✓

QUESTION THREE

(a) (i) $y = \frac{x^2}{4a} \quad \frac{dy}{dx} = \frac{x}{2a} = p$ at $P(2ap, ap^2)$

$\therefore$ gradient of normal is $-\frac{1}{p}$ ✓

eqn. of normal: $y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = 2ap + ap^3$ ✓

(ii) When $x = 0$ $y = \frac{2ap + ap^3}{p} = 2a + ap^2$ ✓

$\therefore Q(0, 2a + ap^2)$.

(iii) ratio of $-2 : 1$

$R\left(\frac{2ap}{-1}, \frac{-2(2a+ap^2) + ap^2}{-1}\right)$ ✓

$= R(-2ap, 4a + ap^2)$ ✓

(v) sub (h, k) into $x + py = 2ap + ap^3$

$h + pk = 2ap + ap^3$ ✓

$ap^3 + (2a-k)p - h = 0$

(vi) 3 (cubic can have at most three distinct solns.) ✓

) (i) $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2 = S$ ✓

(ii) $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$ ✓

$-kS = S \cdot \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = -k$ ✓

i.e. radius is decreasing at a constant rate.

QUESTION FOUR

(a) (i) $-1 \le \cos 3t \le 1$

$\therefore -2 \le 2\cos 3t \le 2$

$\therefore 2 \le 2\cos 3t + 4 \le 6$ ✓

(ii) centre 4, amplitude 2 ✓

(iii) $x = 2\cos 3t + 4$

$\dot{x} = -6\sin 3t$

$\ddot{x} = -18\cos 3t$ ✓

(iv) $\ddot{x} = -18\cos 3t$

but $\cos 3t = \frac{x-4}{2}$

$\therefore \ddot{x} = -18\left(\frac{x-4}{2}\right) = -9(x-4)$ ✓

(v) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9x + 36$

$\frac{1}{2}v^2 = \frac{-9x^2}{2} + 36x + c$ ✓

when $t=0$, $x=6$, $v=0$

$0 = \frac{-9 \times 36}{2} + 36 \times 6 + c$

$0 = -162 + 216 + c$

$\therefore c = -54$ ✓

$\frac{1}{2}v^2 = \frac{-9x^2}{2} + 36x - 54$

$v^2 = -9x^2 + 72x - 108$ ✓

(vi) 6 cm/s ✓

(b) (i) $T = R + Ae^{-kt}$

$\frac{dT}{dt} = -k \cdot Ae^{-kt}$

$\therefore \frac{dT}{dt} = -k(T - R)$ ✓

(ii) $T = 20 + Ae^{-kt}$

$t=0$, $T = 200°c$ $\therefore A = 180$ ✓

$t=1$, $T = 170°c$

$170 = 20 + 180e^{-k}$

$150 = 180e^{-k}$

$\frac{5}{6} = e^{-k}$

$e^{k} = \frac{6}{5}$ $\therefore k = \ln\frac{6}{5}$ ✓

when $T = 50°c$: $50 = 20 + 180e^{-\ln\frac{6}{5} \times t}$

$t \doteqdot 10$ minutes. ✓

QUESTION FIVE

(a) $(3+2x)^{15} = \sum_{k=0}^{15} t_k x^k = t_0x^0 + t_1x^1 + \dots + t_{15}$

(i) $t_k = {}^{15}C_k \, 3^{15-k} \, 2^k$ ✓

(ii) $t_{k+1} = {}^{15}C_{k+1} \, 3^{14-k} \, 2^{k+1}$

$\therefore \frac{t_{k+1}}{t_k} = \frac{15!}{(k+1)!\,(14-k)!} \cdot \frac{k!\,(15-k)!}{15!} \cdot \frac{3^{14-k}}{3^{15-k}} \cdot \frac{2}{1}$ ✓

$= \frac{15-k}{k+1} \cdot \frac{2}{3}$

$= \frac{30-2k}{3k+3}$ ✓

(iii) $t_{k+1} > t_k$ when $30 - 2k > 3k + 3$

$27 > 5k$

$\frac{27}{5} > k$

i.e. when $k = 5, 4, 3, \dots$

i.e. $t_6 > t_5 > t_4 \dots$ ✓

$t_{k+1} < t_k$ when $\frac{27}{5} < k$ i.e. when $k = 6, 7, 8,$

i.e. $t_8 < t_7 < t_6$

$\therefore$ t_6 is the greatest coefficient

i.e. greatest coefficient is ${}^{15}C_6 \, 3^9 \, 2^6$ ✓

(b) (i) $a = 1$, $r = 1+x$, $n+1$ terms

$S = \frac{1\left((1+x)^{n+1} - 1\right)}{(1+x) - 1} = \frac{(1+x)^{n+1} - 1}{x}$ ✓

(ii) $S = \frac{1}{x}\left[\cancel{{}^{n+1}C_0} + {}^{n+1}C_1 x + {}^{n+1}C_2 x^2 + \dots + {}^{n+1}C_{n+1} x^{n+1} \cancel{-1}\right]$

$= {}^{n+1}C_1 + {}^{n+1}C_2 x + \dots + {}^{n+1}C_{n+1} x^n$. ✓

(iii) From (ii) coefficient of x^r is ${}^{n+1}C_{r+1}$ ✓

From (5) $= 1+(1+x)+(1+x)^2+\cdots+\underbrace{(1+x)^r+\cdots+(1+x)^n}_{\text{contain terms in } x^r}$

coefficient of x^r is ${}^{r}C_r + {}^{r+1}C_r + \cdots + {}^{n-1}C_r + {}^{n}C_r$

$\therefore\ {}^{n+1}C_{r+1} = {}^{n}C_r + {}^{n-1}C_r + {}^{n-2}C_r + \cdots + {}^{r}C_r.$

c) $x^3 - px^2 + qx - r = 0$

let roots be α, β and $\alpha\beta$

then
$$\left.\begin{aligned} \alpha+\beta+\alpha\beta &= p \\ \alpha\beta+\alpha^2\beta+\alpha\beta^2 &= q \\ \alpha^2\beta^2 &= r \end{aligned}\right\}$$

$$\begin{aligned} \text{LHS} &= (q+r)^2 \\ &= \left[\alpha\beta(\alpha+\beta+\alpha\beta+1)\right]^2 \\ &= (\alpha\beta)^2(\alpha+\beta+\alpha\beta+1)^2 \\ &= r(p+1)^2. \end{aligned}$$

QUESTION SIX

a) (i) $\angle QDE = \alpha$

$\therefore \angle RQE = \alpha$ (angle subtended by equal arc)

(ii) let $\angle QED = \beta$

$\therefore \angle DQF = \beta$ (angle subtended by equal arc)

$\therefore \angle QFG = \angle QGF = \alpha+\beta$ (exterior $\angle$ of Δ)

$\therefore \Delta QFG$ is isosceles (base angles equal)

(b)
$$\begin{aligned} x &= \tan\theta + \sec\theta \\ &= \frac{2t}{1-t^2} + \frac{1+t^2}{1-t^2} \\ &= \frac{(1+t)^2}{(1-t)(1+t)} \\ &= \frac{1+t}{1-t} \end{aligned}$$

$\therefore\ t = \dfrac{x-1}{x+1}$

$$\sin\theta = \frac{2t}{1+t^2} = \frac{\frac{2(x-1)}{x+1}}{1+\frac{(x-1)^2}{(x+1)^2}}$$

$$= \frac{2(x^2-1)}{2(x^2+1)}$$

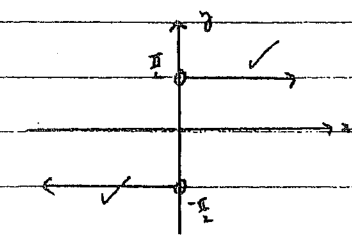
$$= \frac{x^2-1}{x^2+1}$$

(c) (i)
$$\begin{aligned} \frac{d}{dx}\left(\tan^{-1}x + \tan^{-1}\frac{1}{x}\right) &= \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}}\cdot -\frac{1}{x^2} \\ &= \frac{1}{1+x^2} - \frac{1}{x^2+1} \\ &= 0 \end{aligned}$$

(ii) $\tan^{-1}x + \tan^{-1}\frac{1}{x} = \text{constant}$

when $x>0$, $\tan^{-1}x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}$

$x<0$, $\tan^{-1}x + \tan^{-1}\frac{1}{x} = -\frac{\pi}{2}$



QUESTION SEVEN

(a) $x = V\cos\alpha . t \qquad y = -\frac{1}{2}gt^2 + V\sin\alpha . t$

$t = \dfrac{x}{V\cos\alpha}$

$$\begin{aligned} y &= -\tfrac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2 + V\sin\alpha\left(\frac{x}{V\cos\alpha}\right) \\ &= \frac{-gx^2}{2V^2}\sec^2\alpha + x\tan\alpha \end{aligned}$$

(ii) when $y=0$, $x=R$.

$$0 = \frac{-gR^2}{2V^2}\sec^2\alpha + R\tan\alpha$$

$$\frac{gR^{\cancel{2}}}{2V^2}\sec^2\alpha = \cancel{R}\tan\alpha$$

$$\sec^2\alpha = \frac{2V^2\tan\alpha}{gR}$$

$$\begin{aligned} \therefore\ y &= \frac{-\cancel{g}x^2}{\cancel{2V^2}}\left(\frac{\cancel{2V^2}\tan\alpha}{\cancel{g}R}\right) + x\tan\alpha \\ &= \frac{-x^2\tan\alpha}{R} + x\tan\alpha \\ &= x\tan\alpha\left(1-\frac{x}{R}\right). \end{aligned}$$

(iii) when $y=4$, $x = \dfrac{R-6}{2}$, $\tan\alpha = 1$

$$\therefore\ 4 = \left(\frac{R-6}{2}\right)\left(1-\frac{R-6}{2R}\right)$$

$$\frac{8}{R-6} = \frac{R+6}{2R}$$

$$16R = R^2-36$$

$$R^2-16R-36 = 0$$

$$(R-18)(R+2) = 0.$$

) (i) RHS $= \sqrt{2}\sin(x+\frac{\pi}{4})$

$= \sqrt{2}\left(\sin x\cos\frac{\pi}{4} + \cos x\sin\frac{\pi}{4}\right)$

$= \sqrt{2}\left(\sin x.\frac{1}{\sqrt{2}} + \cos x.\frac{1}{\sqrt{2}}\right)$

$= \sin x + \cos x$

$=$ LHS ✓

(ii) $\frac{d}{dx}(e^x\sin x) = e^x\cos x + e^x\sin x$

$= e^x(\sin x + \cos x)$

$= \sqrt{2}e^x\sin(x+\frac{\pi}{4})$ from (i) ✓

(iii) $y = e^x\sin x$, $\frac{d^ny}{dx^n} = (\sqrt{2})^n e^x\sin(x+\frac{n\pi}{4})$

). when $n=1$, $\frac{dy}{dx} = \sqrt{2}e^x\sin(x+\frac{\pi}{4})$ which is true from (ii)

i.e. statement is true when $n=1$. ✓

let statement be true when $n=k$

i.e. $\frac{d^ky}{dx^k} = (\sqrt{2})^k e^x\sin(x+\frac{k\pi}{4})$

prove the statement is true when $n=k+1$

i.e. prove that $\frac{d^{k+1}y}{dx^{k+1}} = (\sqrt{2})^{k+1}e^x\sin(x+\frac{(k+1)\pi}{4})$

LHS $= \frac{d^{k+1}y}{dx^{k+1}}$

$= \frac{d}{dx}\left(\frac{d^ky}{dx^k}\right)$

$= \frac{d}{dx}\left((\sqrt{2})^k e^x\sin(x+\frac{k\pi}{4})\right)$ by the induction hypothesis

$= (\sqrt{2})^k e^x\sin(x+\frac{k\pi}{4}) + (\sqrt{2})^k e^x\cos(x+\frac{k\pi}{4})$

✓✓ $= (\sqrt{2})^k e^x\left[\sin(x+\frac{k\pi}{4}) + \cos(x+\frac{k\pi}{4})\right]$

$= (\sqrt{2})^k e^x\left[\sqrt{2}\sin\left(x+\frac{k\pi}{4}+\frac{\pi}{4}\right)\right]$ from (i)

$= (\sqrt{2})^k e^x\sqrt{2}\sin\left(x+\frac{(k+1)\pi}{4}\right)$

$= (\sqrt{2})^{k+1}e^x\sin\left(x+\frac{(k+1)\pi}{4}\right)$

$=$ RHS.

) steps A and B and the axiom of mathematical duction, the statement is true for all ositive integers n. ✓