Trial HSC Extension 1, 2010 – Solutions

Question 1 (12 marks)

(a) $m_1 = m \qquad m_2 = 2$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$1 = \left|\frac{2-m}{1+2m}\right| \qquad (2)$$

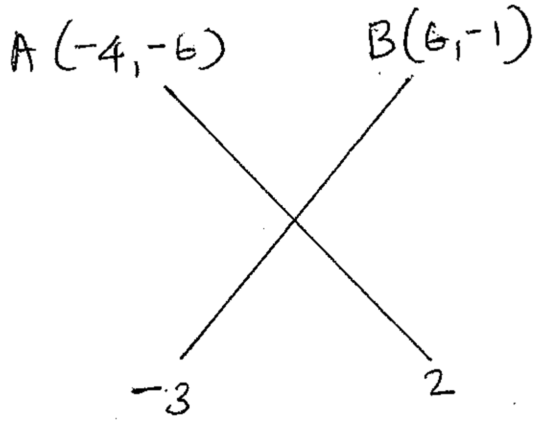
$$\frac{2-m}{1+2m} = 1 \quad \text{OR} \quad \frac{2-m}{1+2m} = -1$$

$$2 - m = 1 + 2m \qquad 2 - m = -1 - 2m$$

$$3m = 1 \qquad -m = 3$$

$$\underline{\underline{m = \frac{1}{3}}} \qquad \underline{\underline{m = -3}}$$

(b)



$$x = \frac{(-3 \times 6) + (2 \times -4)}{-3+2} = 26$$

$$y = \frac{(-3 \times -1) + (2 \times -6)}{-3+2} = 9$$

$$\underline{\underline{P(26, 9)}} \qquad (2)$$

(c) $\dfrac{2x+1}{x-1} \geq 3$

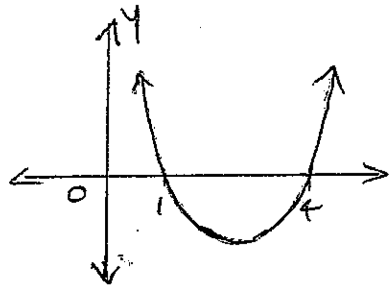
$$(x-1)(2x+1) \geq 3(x-1)^2, \; x \neq 1$$

$$3(x-1)^2 - (x-1)(2x+1) \leq 0$$

$$(x-1)\left[3(x-1) - (2x+1)\right] \leq 0$$

$$(x-1)(3x - 3 - 2x - 1) \leq 0$$

$$(x-1)(x-4) \leq 0$$



(2)

$$\underline{\underline{1 < x \leq 4}}$$

(d) $y = x \tan^{-1}\dfrac{x}{2}$

$$y' = x \times \frac{1}{1 + \frac{x^2}{4}} \times \frac{1}{2} + \tan^{-1}\frac{x}{2} \times 1$$

$$= x \times \frac{1}{\frac{4+x^2}{4}} \times \frac{1}{2} + \tan^{-1}\frac{x}{2}$$

$$= x \times \frac{4}{4+x^2} \times \frac{1}{2} + \tan^{-1}\frac{x}{2}$$

$$= \underline{\underline{\frac{2x}{4+x^2} + \tan^{-1}\frac{x}{2}}} \qquad (3)$$

(c) $\int_1^4 \frac{dx}{x+\sqrt{x}}$, $u=\sqrt{x}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$; $\frac{dx}{du} = 2\sqrt{x}$

$dx = 2\sqrt{x}\,du$
$= 2u\,du$

when $x=1$, $u=\sqrt{1}=1$

when $x=4$, $u=\sqrt{4}=2$

$\int_1^2 \frac{2u\,du}{u^2+u} = \int_1^2 \frac{2u\,du}{u(u+1)}$

$= \int_1^2 \frac{2\,du}{u+1} = 2\int_1^2 \frac{du}{u+1}$

$= 2\left[\log(u+1)\right]_1^2$ (3)

$= 2\left[\log 3 - \log 2\right]$

$= 2\log\frac{3}{2}$

Question 2 (12 marks)

(a) $T_{r+1} = {}^{12}C_r (x^2)^{12-r}\left(\frac{2}{x}\right)^r$

$= {}^{12}C_r\, x^{24-2r}\, \frac{2^r}{x^r}$

$= {}^{12}C_r\, x^{24-3r}\, 2^r$

$24-3r = 9$

$3r = 15$

$r = 5$

$T_6 = {}^{12}C_5\, x^{24-15}\, 2^5$ (2)

$= {}^{12}C_5 \times 2^5 x^9$

Coefficient of $x^9 = {}^{12}C_5 \times 2^5$

(b) $\lim_{x\to 0} \frac{\sin 6x}{7x}$

$= \lim_{x\to 0} \frac{\sin 6x}{6x} \times \frac{6x}{7x}$ (2)

$= \frac{6}{7}$ $\left(\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right)$

(c) $f(x) = 4\cos^{-1}\frac{x}{3}$

(i) $D: -1 \le \frac{x}{3} \le 1$

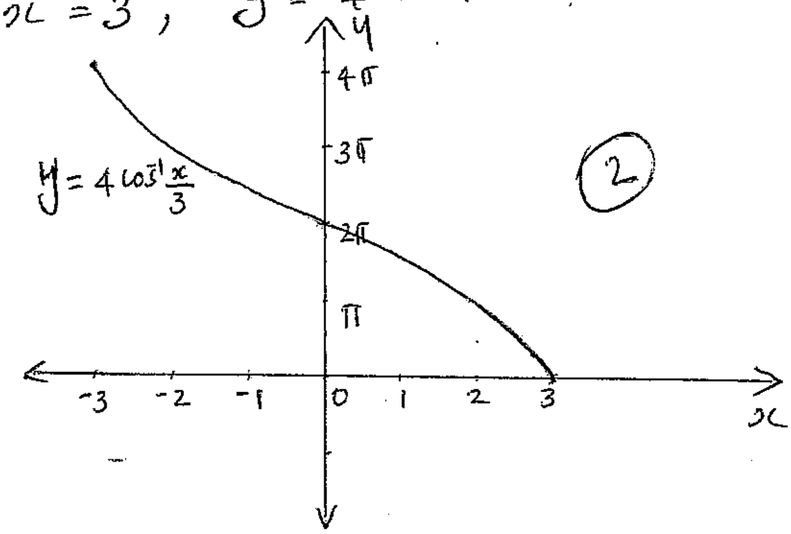
$-3 \le x \le 3$ (2)

$R: 0 \le y \le 4\pi$

$x=-3$, $y = 4\cos^{-1}(-1) = 4\pi$

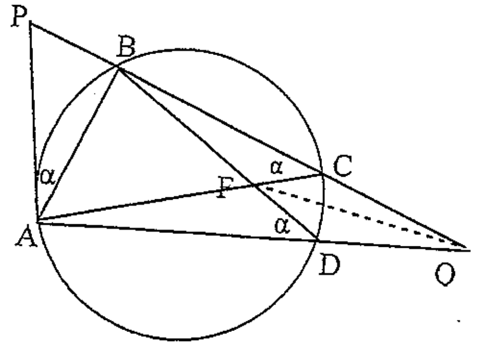
$x=0$, $y = 4\cos^{-1}0 = 4\times\frac{\pi}{2} = 2\pi$

$x=3$, $y = 4\cos^{-1}1 = 0$



(2)

(d)



$\angle PAB = \angle ACB$ (angle between tangent and chord is equal to the angle in the alternate segment)

$\angle ACB = \angle ADB$ (angles at the circumference standing on the same chord)

$\angle BCA = \angle CQF + \angle CFQ$ (exterior angle of ΔFQC)

$\angle ADF = \angle DFQ + \angle DQF$ (exterior angle of ΔFQD)

$\angle BCA + \angle ADF$

$= \angle CQF + \angle CFQ + \angle DFQ + \angle DQF$

$= \angle CQF + \angle DQF + \angle CFQ + \angle DFQ$

$= \angle CQD + \angle CFD$

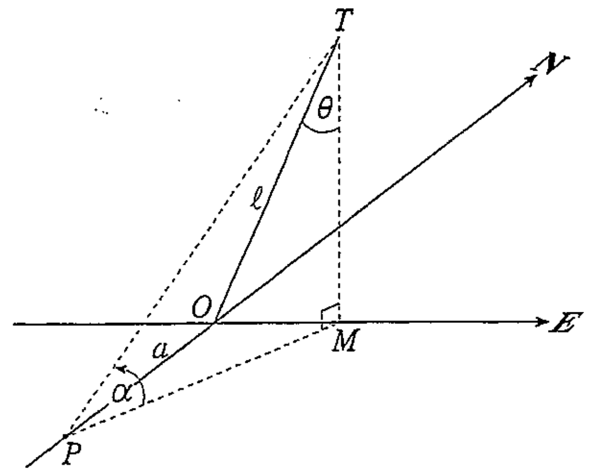
But $\angle BCA + \angle ADF = 2\alpha$

$2\alpha = \angle CQD + \angle CFD$ (4)

$\alpha = \frac{1}{2}(\angle CQD + \angle CFD)$

Question 3 (12 marks)

(a)



(i) $\sin\theta = \frac{OM}{l}$

$OM = l\sin\theta$ (2)

$\cos\theta = \frac{MT}{l}$; $MT = l\cos\theta$

(ii) $\cot\alpha = \frac{PM}{MT}$; $PM = MT\cot\alpha$ (1)

$= l\cos\theta\cot\alpha$

(iii) $PM^2 - OM^2 = a^2$

$l^2\cos^2\theta\cot^2\alpha - l^2\sin^2\theta = a^2$

$l^2(\cos^2\theta\cot^2\alpha - \sin^2\theta) = a^2$

$l^2 = \frac{a^2}{\cos^2\theta\cot^2\alpha - \sin^2\theta}$ (3)

(iv) $l^2 = \frac{25^2}{\cos^2 20^\circ \cot^2 24^\circ - \sin^2 20^\circ}$

$l = 12$ (1)

(b) Total numbers of ways in which 16 people can be seated $= 8P_4 \times 8P_2 \times 10!$ (2)

(c)(i) $P(x=3) = {}^5C_3 (0.4)^3 (0.6)^2$

$= 0.2304$

(ii) $P(x=3) + P(x=4) + P(x=5)$

$= 0.2304 + {}^5C_4 (0.4)^4 (0.6)^1$

$+ {}^5C_5 (0.4)^5 (0.6)^0$

$= 0.31744$ (3)

(iii) $P(x=0) + P(x=1) + P(x=2)$

$= 1 - 0.31744$

$= 0.68256$

Question 4 (12 marks)

(a) when $n = 2$,

$LHS = \log \frac{2}{2-1} = \log 2$

$RHS = \log 2$

$LHS = RHS$

$\therefore$ the result is true for $n = 2$

Assume the result is true for $n = k$

ie $\log 2 + \log(\frac{3}{2}) + \cdots + \log(\frac{k}{k-1})$

$= \log k$ —— (1)

To prove that the result is true for $n = k+1$

ie $\log 2 + \log(\frac{3}{2}) + \log(\frac{4}{3}) + \cdots$

$\cdots + \log(\frac{k}{k-1}) + \log(\frac{k+1}{k}) = \log k+1$

Now

$\log 2 + \log\frac{3}{2} + \cdots + \log(\frac{k}{k-1}) + \log(\frac{k+1}{k})$

$= \log k + \log(\frac{k+1}{k})$

by assumption (1)

$= \log(k \times \frac{k+1}{k})$

$= \log(k+1)$ (3)

$\therefore$ the result is true for $n = k+1$

Hence by the principle of mathematical induction, the result is true for $n \geq 2$

(b)(i) Normal at P

$x + py = 4p^3 + 8p$ —— (1)

Normal at Q

$x + qy = 4q^3 + 8q$ —— (2)

① − ② gives

$$y(p-q) = 4(p^3-q^3) + 8(p-q)$$

$$y(p-q) = 4(p-q)(p^2+pq+q^2) + 8(p-q)$$

$$y = 4(p^2+pq+q^2) + 8$$
$$= 4(p^2-1+q^2)+8$$
$$= 4p^2-4+4q^2+8$$
$$= 4p^2+4q^2+4$$

Substitute in ①

$$x + p(4p^2+4q^2+4) = 4p^3+8p$$

$$x+4p^3+4pq^2+4p = 4p^3+8p$$

$$x + 4pq^2 = 4p$$

$$x + 4pq \times q = 4p$$

$$x - 4q = 4p$$

$$x = 4p+4q$$
$$= 4(p+q)$$

$$x = 4(p+q) \text{ — ③}$$

$$y = 4p^2+4q^2+4 \text{ — ④}$$

From ③ $p+q = \frac{x}{4}$

From ④ $\frac{y}{4} = p^2+q^2+1$

$$p^2+q^2 = \frac{y}{4} - 1$$

$$p^2+q^2 = (p+q)^2 - 2pq$$
$$= (p+q)^2 - 2\times -1$$
$$= (p+q)^2 + 2$$

$$\frac{y}{4} - 1 = \left(\frac{x}{4}\right)^2 + 2$$

$$\frac{y}{4} - 1 = \frac{x^2}{16} + 2$$

$$4y - 16 = x^2 + 32 \quad ③$$

$$x^2 = 4y - 48$$

$$x^2 = 4(y-12)$$

(c) $y = \frac{2x^2-2}{x^2-9} = \frac{2(x+1)(x-1)}{(x+3)(x-3)}$

Vertical asymptotes

$x = -3$ and $x = 3$

$$\lim_{x\to\infty} \frac{2x^2-2}{x^2-9}$$

$$= \lim_{x\to\infty} \frac{2-\frac{2}{x^2}}{1-\frac{9}{x^2}} \quad ③$$

$$= 2 \left(\text{since } \lim_{x\to\infty} \frac{1}{x} = 0\right)$$

Horizontal asymptote is

$y = 2$

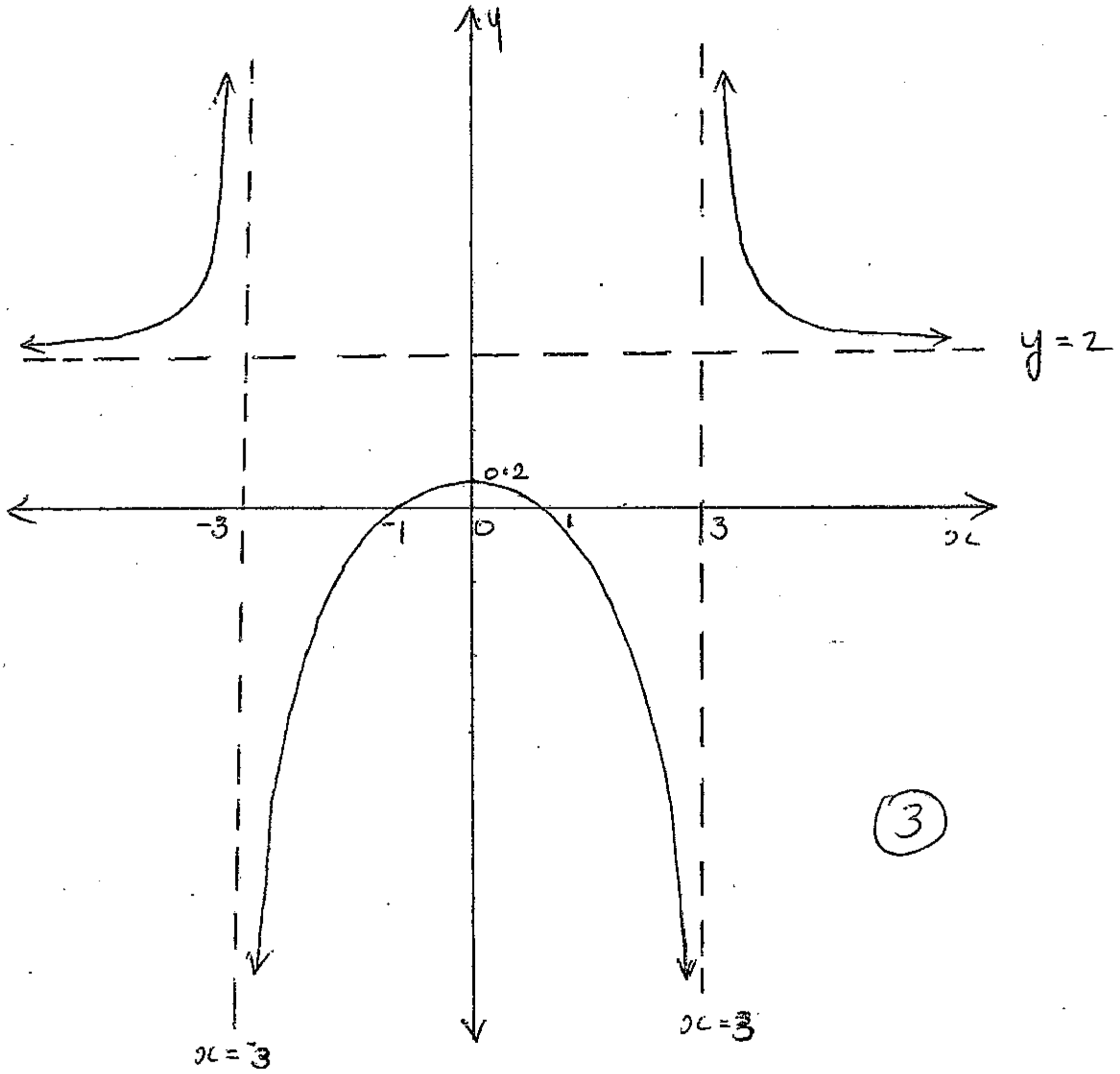
x intercepts

$y=0 \Longrightarrow 2(x+1)(x-1)=0$

$x=-1$ or $x=1$

y intercept

$x=0 \Longrightarrow y=\frac{-2}{-9}=\frac{2}{9}=0{\cdot}2$



(3)

Question 5 (12 marks)

(a) $(1+x)^5\,(1+x)^5=(1+x)^{10}$

$\left[{}^5C_0+{}^5C_1x+{}^5C_2x^2+{}^5C_3x^3+{}^5C_4x^4+{}^5C_5x^5\right]\left[{}^5C_0+{}^5C_1x+{}^5C_2x^2+{}^5C_3x^3+{}^5C_4x^4+{}^5C_5x^5\right]={}^{10}C_0+{}^{10}C_1x+{}^{10}C_2x^2+{}^{10}C_3x^3+{}^{10}C_4x^4$

$+{}^{10}C_5x^5+\cdots+{}^{10}C_{10}x^{10}$

Equating coefficients of x^5 on both sides,

$$^5C_0 \times {}^5C_5 + {}^5C_1 \times {}^5C_4 + {}^5C_2 \times {}^5C_3 + {}^5C_3 \times {}^5C_2 + {}^5C_4 \times {}^5C_1$$

$$+ {}^5C_5 \times {}^5C_0 = {}^{10}C_5$$

$$^5C_0 \times {}^5C_0 + {}^5C_1 \times {}^5C_1 + {}^5C_2 \times {}^5C_2 + {}^5C_3 \times {}^5C_3 + {}^5C_4 \times {}^5C_4$$

$$+ {}^5C_5 \times {}^5C_5 = {}^{10}C_5 \quad (\text{since } {}^nC_r = {}^nC_{n-r})$$

$$({}^5C_0)^2 + ({}^5C_1)^2 + ({}^5C_2)^2 + ({}^5C_3)^2 + ({}^5C_4)^2 + ({}^5C_5)^2 = {}^{10}C_5$$

ie $\displaystyle\sum_{k=0}^{5} ({}^5C_k)^2 = {}^{10}C_5$ (3)

(b) (i) $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n$ (1)

(ii) $\displaystyle\int_0^1 (1+x)^n\,dx = \int_0^1 ({}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n)\,dx$

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left[{}^nC_0 x + {}^nC_1 \frac{x^2}{2} + {}^nC_2 \frac{x^3}{3} + \dots + {}^nC_n \frac{x^{n+1}}{n+1}\right]_0^1$$

$$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = {}^nC_0 + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + \dots + \frac{{}^nC_n}{n+1}$$

(3)

$$^nC_0 + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + \dots + \frac{{}^nC_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

(c) (i) $T = A + Ce^{kt}$

$$\frac{dT}{dt} = Ce^{kt} \times k$$

$$= k \times Ce^{kt} \quad (2)$$

$$= k(T-A) \quad (\text{since } Ce^{kt} = T-A)$$

$\therefore T = A + Ce^{kt}$ is a solution of the equation

$$\frac{dT}{dt} = k(T-A)$$

(ii) $T = 25 + Ce^{kt}$

when $t=0$, $T = 10°C$

$$10 = 25 + C$$

$$C = 10 - 25 = -15$$

$$\therefore T = 25 - 15e^{kt}$$

when $t = 20$, $T = 15°$

$$15 = 25 - 15e^{20k}$$

$$15e^{20k} = 10$$

$$e^{20k} = \frac{10}{15}$$

$$20k = \log\left(\frac{10}{15}\right)$$

$$k = \frac{1}{20}\log\left(\frac{10}{15}\right)$$

when $t = 50$, $T = ?$

$$T = 25 - 15e^{50k}$$

$$= 25 - 15e^{50 \times \frac{1}{20}\log\left(\frac{10}{15}\right)} \quad (2)$$

$= 20°C$ (to the nearest degree)

(iii) $k = \frac{1}{20}\log\left(\frac{10}{15}\right) = -0.02$

$$T = 25 - 15e^{-0.02t}$$

Since $k < 0$, as $t \to \infty$ $e^{kt} \to 0$

$\therefore$ as t increases indefinitely the object's temperature (1) approaches air temperature.

<u>Question 6</u> (12 marks)

(a) (i) $f(x) = x\sin^{-1}x + \sqrt{1-x^2}$

$$f'(x) = x \times \frac{1}{\sqrt{1-x^2}} + \sin^{-1}x \times 1 + \frac{1}{2\sqrt{1-x^2}} \times -2x$$

$$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x - \frac{x}{\sqrt{1-x^2}} = \sin^{-1}x$$

$$\frac{d}{dx}\left(x\sin^{-1}x + \sqrt{1-x^2}\right) = \sin^{-1}x \quad (3)$$

$$\int_0^{\frac{1}{2}} \sin^{-1}x\,dx = \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^{\frac{1}{2}}$$

$$= \left(\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) + \sqrt{1-\frac{1}{4}}\right) - (0 + \sqrt{1})$$

$$= \frac{1}{2} \times \frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1 = \frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}$$

$$= 0.128 \quad (2)$$

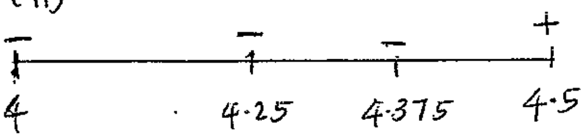
(b) (i) $\tan x - x = 0$

$\tan 4 - 4 = -2.8$

$\tan 4.5 - 4.5 = 0.14$

Since $f(4) < 0$ and $f(4.5) > 0$ there is a root of $f(x) = 0$ between $x = 4$ and $x = 4.5$ (1)

(ii)



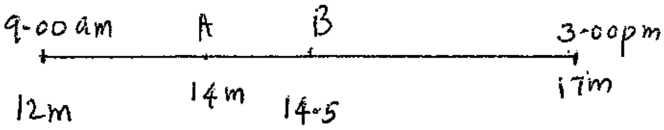
$f(4.25) = \tan 4.25 - 4.25$
$= -2.24$

$f(4.375) = \tan 4.375 - 4.375$
$= -1.52$

Approximate value of the root (2)

$= \frac{4.375 + 4.5}{2} = 4.4$

(c) (i)



$T = 2 \times (3.00\text{pm} - 9.00\text{am})$
$= 2 \times 6h$
$= 12h$

$T = \frac{2\pi}{n}$ $\quad n = \frac{2\pi}{T} = \frac{2\pi}{12} = \frac{\pi}{6}$

Let 9.00 am denote $t = 0$

$x = a\cos(nt + \pi)$
$= 2.5\cos\left(\frac{\pi}{6}t + \pi\right)$

when the harbour is 14m deep

$x = -0.5$

$-0.5 = 2.5\cos\left(\frac{\pi}{6}t + \pi\right)$

$\cos\left(\frac{\pi}{6}t + \pi\right) = \frac{-1}{5}$

$\frac{\pi}{6}t + \pi = \pi - \cos^{-1}\left(\frac{1}{5}\right), \pi + \cos^{-1}\left(\frac{1}{5}\right)$

$\frac{\pi t}{6} = -1.3694, 1.369$

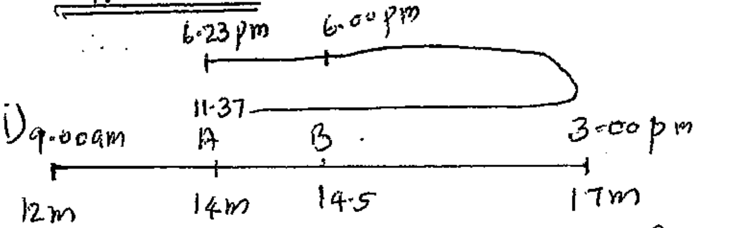
$\frac{\pi t}{6} = 1.3694$ (t can't be negative)

$t = 1.3694 \times \frac{6}{\pi}$
$= 2.6154$
$= 2$ hr 37 minutes. (3)

Time taken from A to B
$= 3$ hr $- 2$ hr 37 min
$= 23$ min

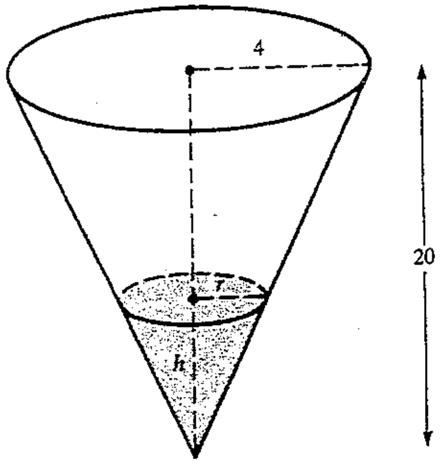
The ship can go into the harbour at 11.37 am

(ii)



The ship must depart before 5.53 pm. (1)

Question 7 (12 marks)



$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = 12\ cm^3/s$

By similar triangles

$\frac{r}{4} = \frac{h}{20}$

$r = \frac{4h}{20} = \frac{h}{5}$

$V = \frac{1}{3}\pi \times \frac{h^2}{25} \times h$

$= \frac{\pi h^3}{75}$ (3)

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$\frac{dV}{dh} = \frac{\pi \times 3h^2}{75} = \frac{\pi h^2}{25}$

$12 = \frac{\pi h^2}{25} \times \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{12 \times 25}{\pi h^2}$

when $h = 5$,

$\frac{dh}{dt} = \frac{12 \times 25}{\pi \times 25} = 3.82$

The fluid level is dropping at the rate of 3.82 cm/s.

(b) $f(x) = x^3 + x^2 - 10$

$f'(x) = 3x^2 + 2x$

$a = 2$

$a_1 = 2 - \frac{f(2)}{f'(2)}$

$= 2 - \left[\frac{8+4-10}{3\times 4+4}\right] = 2 - \frac{1}{8} = 1.875$

$= 1.9$

$a_2 = 1.875 - \frac{f(1.875)}{f'(1.875)}$

$= 1.875 - \left[\frac{(1.875)^3 + (1.875)^2 - 10}{3\times(1.875)^2 + 2\times 1.875}\right]$

$= 1.867 = 1.9$ (2)

$a_1 = a_2 = 1.9$

$\therefore$ the root of $f(x) = 0$ correct to one decimal place is $\underline{\underline{1.9}}$

(c) (i) When the particle strikes the ground $y = 0$

$Vt\sin\alpha - \frac{1}{2}gt^2 = 0$

$t(V\sin\alpha - \frac{1}{2}gt) = 0$

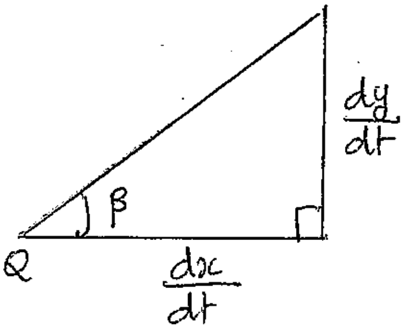
$t = 0$ or $V\sin\alpha = \frac{1}{2}gt$

$t = 0$ or $2V\sin\alpha = gt$

$t = 0$ or $t = \dfrac{2V\sin\alpha}{g}$

Now $t=0$ refers to the instant of projection (2) and hence $t = \dfrac{2V\sin\alpha}{g}$ is the required time, the time of flight.

(ii)



(1)

$$\tan\beta = \frac{dy/dt}{dx/dt}$$

$$= \frac{V\sin\alpha - gt}{V\cos\alpha}$$

(iii) $$\frac{\sin\beta}{\cos\beta} = \frac{V\sin\alpha - gt}{V\cos\alpha}$$

$$V\sin\beta\cos\alpha = V\sin\alpha\cos\beta - gt\cos\beta$$

$$gt\cos\beta = V\sin\alpha\cos\beta - V\sin\beta\cos\alpha$$

$$gt\cos\beta = V(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$

$$gt\cos\beta = V\sin(\alpha - \beta)$$ (2)

$$t = \frac{V\sin(\alpha-\beta)}{g\cos\beta}$$

(iv) when $\beta = \frac{\alpha}{2}$ we have

$$t = \frac{V\sin(\alpha - \frac{\alpha}{2})}{g\cos\frac{\alpha}{2}} = \frac{V\sin\frac{\alpha}{2}}{g\cos\frac{\alpha}{2}}$$

$$= \frac{V}{g}\tan\frac{\alpha}{2}$$

Given that $\dfrac{V}{g}\tan\dfrac{\alpha}{2} = \dfrac{1}{3}\,\dfrac{2V\sin\alpha}{g}$

$$\tan\frac{\alpha}{2} = \frac{1}{3} \times 2\sin\alpha$$

$$3\tan\frac{\alpha}{2} = 2\sin\alpha$$

$$= 2 \times \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

$$3 = \frac{4}{1+\tan^2\frac{\alpha}{2}}$$

$$4 = 3 + 3\tan^2\frac{\alpha}{2}$$ (2)

$$3\tan^2\frac{\alpha}{2} = 1$$

$$\tan^2\frac{\alpha}{2} = \frac{1}{3}$$

$$\tan\frac{\alpha}{2} = \frac{1}{\sqrt{3}} \quad \left(\frac{\alpha}{2} \text{ is acute}\right)$$

$$\frac{\alpha}{2} = \frac{\pi}{6}$$

$$\alpha > \frac{\pi}{3}$$