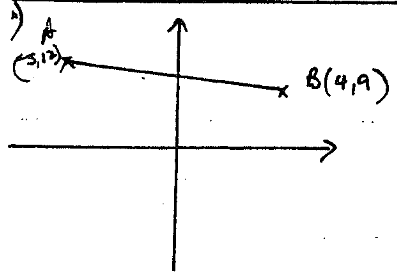
Abbotsleigh Trial Mathematics 3U 1999

1)



(-5,12) (4,9)

(5 : -2)

$$P = \left(\frac{-2\times -5 + 5\times 4}{5-2} , \frac{-2\times 12 + 5\times 9}{5-2} \right)$$

$$P = (10, 7)$$

b) $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$ $m_1 = 2$, $m_2 = 4$

$$= \frac{4-2}{1+2\times 4}$$

$$= \frac{2}{9}$$

$$\theta = 12^\circ 32'$$

c) $f(x) = x^3 + 3x^2 - 10x - 24$

$f(-2) = -8 + 12 + 20 - 24$

$= 0$

$$\begin{array}{r|l}
 & x^2 + x - 12 \\
x+2 & x^3 + 3x^2 - 10x - 24 \\
 & x^3 + 2x^2 \\
 & \quad\; x^2 - 10x \\
 & \quad\; x^2 + 2x \\
 & \qquad -12x - 24 \\
 & \qquad -12x - 24 \\
 & \qquad \cdot\;\cdot
\end{array}$$

$\therefore f(x) = (x+2)(x^2 + x - 12)$

$= (x+2)(x+4)(x-3)$

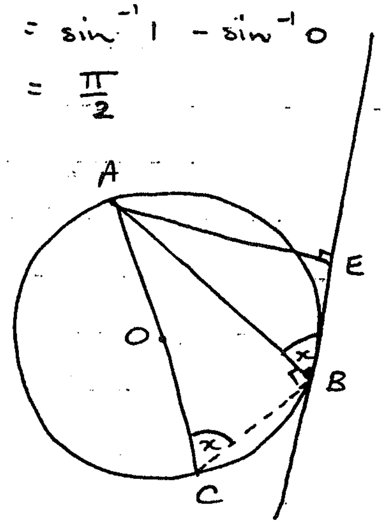
d) $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

$$= \left[\sin^{-1}\frac{x}{3} \right]_0^3$$

$$= \sin^{-1} 1 - \sin^{-1} 0$$

$$= \frac{\pi}{2}$$

e)



$\angle ABE = \angle ACB = x$ (angle in alterna[te] segments equal)

$\angle ABC = 90$ (angle in semicircle = 9[0])

$\therefore \angle CAB = 90 - x$ (angle sum Δ)

$\angle BAE = 90 - x$ (angle sum Δ

$\therefore \angle CAB = \angle BAE$ (both $= 90 - x$)

$\therefore AB$ bisects $\angle CAE$

2a) $x^2 \times x \geq \dfrac{4}{x} \times x^2$ $x \neq 0$

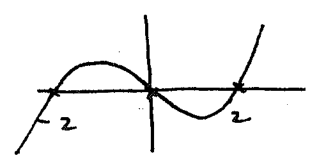
$x^3 \geq 4x$

$x^3 - 4x \geq 0$

$x(x^2 - 4) \geq 0$

$x(x-2)(x+2) \geq 0$

$\therefore x \geq 2$ $-2 \leq x < 0$



b) $y = -3\sin^{-1}\dfrac{x}{2}$

D: $-1 \leq \dfrac{x}{2} \leq 1$

$-2 \leq x \leq 2$

R: $-\dfrac{\pi}{2} \leq \sin^{-1}\dfrac{x}{2} \leq \dfrac{\pi}{2}$

c) $y = x^3 \qquad \frac{dx}{dt} = 2u/s$

$$\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= 3x^2 \times 2$$

$$\frac{dy}{dt} = 6x^2, \text{ when } x = 1$$

$$\frac{dy}{dt} = 6 \ u/s$$

Let m = gradient

$$m = 3x^2$$

$$\frac{dm}{dx} = 6x$$

$$\frac{dm}{dt} = \frac{dm}{dx} \cdot \frac{dx}{dt}$$

$$= 6x \cdot 2$$

$$= 12x$$

when $x = 1 \qquad \frac{dm}{dt} = 12 /s$

rate of change of gradient is 12 per second

4a) $\frac{4^x}{16} = 8^{x+y}$

$$2^{2x-4} = 2^{3x+3y}$$

$$\therefore 2x - 4 = 3x + 3y$$

$$-4 = x + 3y$$

$$2^{2x+y} = 128$$

$$2^{2x+y} = 2^7$$

$$\therefore 2x + y = 7$$

Solve. $\quad x + 3y = -4$

$\quad 2x + y = 7 \qquad \therefore y = 7 - 2x$

$$x + 3(7-2x) = -4$$

$$x + 21 - 6x = -4$$

$$-5x = -25$$

$$x = 5$$

$$y = 7 - 10 = -3$$

b) $x = 2 - \cos t$

$y = 2t + 2\sin t$

$$\frac{dx}{dt} = +\sin t$$

$$\frac{dy}{dt} = 2 + 2\cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{2 + 2\cos t}{\sin t}$$

$$= \frac{2(1 + \cos t)}{\sin t}$$

$$\left[\begin{array}{l} \cos 2\theta = 2\cos^2\theta \\ \cos\theta = 2\cos^2\frac{\theta}{2} \end{array} \right.$$

$$= \frac{\cancel{2}\left(1 + \left(2\cos^2\frac{t}{2} - 1\right)\right)}{\cancel{2}\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= \frac{2\cos^2\frac{t}{2}}{\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= \frac{2\cos\frac{t}{2}}{\sin\frac{t}{2}}$$

$$= 2\cot\frac{t}{2}$$

OR $\quad \frac{dy}{dx} = \frac{2(1+\cos t)}{\sin t}$ $\qquad$ where $t = t$

$$= 2\left(1 + \frac{1-t^2}{1+t^2}\right) \div \frac{2t}{1+t^2}$$

$$= \cancel{2}\left(\frac{1+t^2+1-t^2}{1+t^2}\right) \times \frac{\cancel{1+t^2}}{\cancel{2}t}$$

$$= \frac{2}{t}$$

$$= \frac{2}{\tan\frac{t}{2}}$$

$$= 2\cot\frac{t}{2}$$

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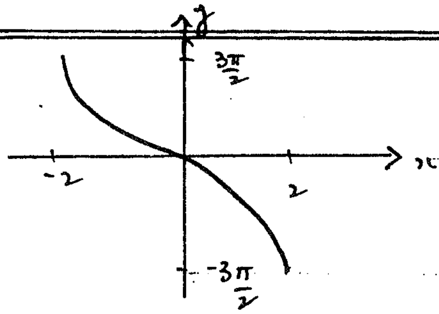


1 $u = 9 - x^2$ when $x = 0$, $u = 9$; $x = 3$, $u = 0$

$\frac{du}{dx} = -2x$

$$\int_0^3 x\sqrt{9-x^2}\,dx = \int_9^0 u^{\frac{1}{2}} x \cdot -\frac{1}{2x}\,du$$

$$= \frac{1}{2}\int_0^9 u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\cdot\frac{2}{3}\Big[u^{3/2}\Big]_0^9$$

$$= \frac{1}{3}\big[27 - 0\big]$$

$$= 9$$

ii) $y = \sin x$

$$V = \pi\int_0^{\pi/2} \sin^2 x\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi/2} 1 - \cos 2x\,dx$$

$$= \frac{\pi}{2}\Big[x - \frac{1}{2}\sin 2x\Big]_0^{\pi/2}$$

$$= \frac{\pi}{2}\Big[\frac{\pi}{2} - \frac{1}{2}\sin\pi - \big(0 - \frac{1}{2}\sin 0\big)\Big]$$

$$= \frac{\pi}{2}\Big[\frac{\pi}{2}\Big]$$

$$= \frac{\pi^2}{4}$$

3 a) $3\cos x + 4\sin x$

$= A\cos x\cos\alpha + A\sin x\sin\alpha = A\cos(x\cdot$

$\therefore$ $A\cos\alpha = 3$

$A\sin\alpha = 4$ $\quad$ $\tan\alpha > 0$

$\tan\alpha = \frac{4}{3}$ $\quad$ $\therefore \alpha$ is acute

$\alpha = 53°8'$

$A^2\cos^2\alpha + A^2\sin^2\alpha = 3^2 + 4^2$

$A^2 = 25$

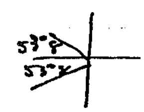
$A = 5$

$\therefore 3\cos x + 4\sin x = 5\cos(x - 53°8')$

$-53°8 \le x - 53°8 \le 36$

$3\cos x + 4\sin x = 5\cos(x - 53°8') = -3$

$\cos(x - 53°8') = -\frac{3}{5}$



$x - 53°8' = 126°52', 233°8'$ $\quad$ 2nd, 3rd quad

$x = 180°, 286°16'$

b) $(3+4x)^{16}$

$T_{k+1} = {}^{16}C_k\, 3^{16-k}\, 4^k$

$T_k = {}^{16}C_{k-1}\, 3^{17-k}\, 4^{k-1}$

$\frac{T_{k+1}}{T_k} \ge 1$ $\quad$ for greatest coeff.

$$\frac{{}^{16}C_k\, 3^{16-k}\, 4^k}{{}^{16}C_{k-1}\, 3^{17-k}\, 4^{k-1}} \ge 1$$

$$\frac{16!\; 3^{16-k}\, 4^k}{(16-k)!\,k!} \times \frac{(17-k)!\,(k-1)!}{16!\; 3^{17-k}\, 4^{k-1}} \ge 1$$

$$\frac{(17-k)4}{3k} \ge 1$$

$68 - 4k \ge 3k$

$7k \le 68$

$k \le 9\frac{5}{7}$ $\quad$ $\therefore k = 9$

i) $\frac{d^2x}{dt^2} = -9x$

) $x = a\cos(3t+\alpha)$

$\frac{dx}{dt} = -3a\sin(3t+\alpha)$

$\frac{d^2x}{dt^2} = -9a\cos(3t+\alpha)$

$= -9x$

$\therefore$ $x = a\cos(3t+\alpha)$ is a solⁿ

$x = a\cos(3t+\alpha)$

i) $v^2 = n^2(a^2 - x^2)$

$3^2 = 3^2(a^2 - 5^2)$

$1 = a^2 - 25$

$a^2 = 26$

$a = \sqrt{26}$

OR

$x = a\cos(3t+\alpha)$ $\qquad$ $\dot{x} = -3a\sin(3t+\alpha)$

$5 = a\cos\alpha$ when $t=0$, $x=5$, $v=3$ $\qquad$ $3 = -3a\sin\alpha$

$\cos\alpha = \frac{5}{a}$ $\qquad$ $\sin\alpha = \frac{-1}{a}$

$\cos^2\alpha + \sin^2\alpha = 1$

$\frac{25}{a^2} + \frac{1}{a^2} = 1$

$a^2 = 26$ $\quad \therefore a = \sqrt{26}$

) max speed when

$\sin(3t+\alpha) = 1$

$\therefore v = -3\sqrt{26}\sin(3t+\alpha)$ is a max

when $\sin(3t+\alpha) = 1$

ie $v = -3\sqrt{26}$

$\therefore$ max speed is $3\sqrt{26}$ m/s

OR $v^2 = n^2(a^2 - x^2)$ for SHM

max velocity occurs when $x = 0$

$v^2 = 9(26 - 0)$

$v = \pm 3\sqrt{26}$

max speed is $3\sqrt{26}$ m/s

5. $2x^3 + 3x^2 - 4 = 0$ $\qquad$ $a=2$, $b=3$, $c=0$, $d=-4$

$\alpha+\beta+\gamma = -\frac{3}{2}$

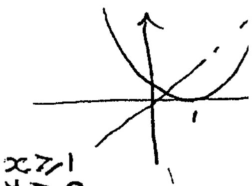
$\alpha\beta\gamma = \frac{4}{2} = 2$

$(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2 - 2(\alpha\beta+\alpha\gamma+$

$\therefore \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 + 2(\alpha\beta+\alpha\gamma+$

$= \left(-\frac{3}{2}\right)^2 + 2\times 0$

$= \frac{9}{4}$



b) $y = x^2 - 2x + 1$

$= (x-1)^2$ $\qquad$ D: $x \geq 1$, R: $y \geq 0$

D: $x \geq 0$, R: $y \geq 1$ for inverse fⁿ.

for inverse fⁿ

$x = (y-1)^2$

$\pm x^{\frac{1}{2}} = y - 1$

$y = 1 \pm \sqrt{x}$

But $y \geq 1$ $\qquad$ $\therefore$ inverse fⁿ is

$y = 1 + \sqrt{x}$

c) $x^2 = 12y$

i) $y = \frac{x^2}{12}$

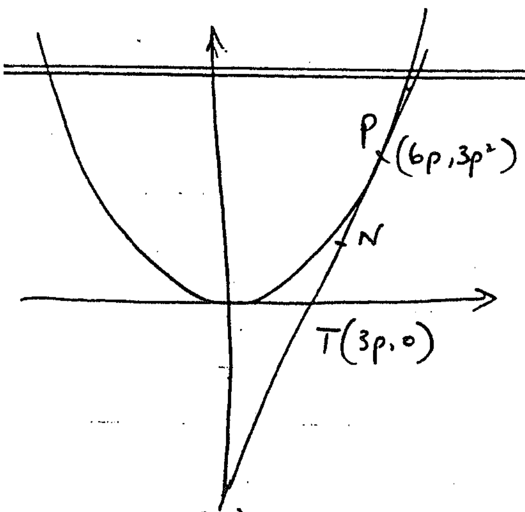
$\frac{dy}{dx} = \frac{2x}{12} = \frac{x}{6}$

when $x = 6p$ $\quad$ $\frac{dy}{dx} = \frac{6p}{6} = p$

eqn tangent $\frac{y - 3p^2}{x - 6p} = p$

$y - 3p^2 = px - 6p^2$

$y = px - 3p^2$



$y = px - 3p^2$

co-ord of T when $y = 0$ $y = px - 3p^2$

$px - 3p^2 = 0$

$p(x - 3p) = 0$ $p = 0$

$x = 3p$

$T\ (3p, 0)$

co-ord of N $P(6p, 3p^2)$ $T(3p, 0)$

$N\left(\frac{6p+3p}{2}, \frac{3p^2}{2}\right)$

$\left(\frac{9p}{2}, \frac{3p^2}{2}\right)$

locus of N $x = \frac{9p}{2}$ $y = \frac{3p^2}{2}$

$\therefore p = \frac{2x}{9}$

$y = \frac{3}{2}\left(\frac{2x}{9}\right)^2$

$= \frac{\cancel{3}}{\cancel{2}}\left(\frac{\cancel{4}^{2}x^2}{\cancel{81}_{27}}\right)$

$\therefore 27y = 2x^2$ is eqn of locus of N.

6a) $\lim_{x \to 0} \frac{\sin 3x}{5x}$

$= \frac{1}{5} \lim_{x \to 0} \frac{\sin 3x}{3x} \times 3$

$= \frac{3}{5}$

b) $\frac{dN}{dt} = 0.1(N - 1.2 \times 10^6)$

$N = P + Ae^{kt}$

$t = 0$ $N = 2.7 \times 10^6$ $P = 1.2 \times 10^6$

$2.7 \times 10^6 = 1.2 \times 10^6 + Ae^0$

$1.5 \times 10^6 = A$

$\therefore N = 1.2 \times 10^6 + 1.5 \times 10^6 e^{0.1t}$

i) when $t = 3.5$

$N = 1.2 \times 10^6 + 1.5 \times 10^6 \times e^{0.1 \times 3.5}$

$= 3.32 \times 10^6$

ii) $3N = 8.1 \times 10^6$

$8.1 \times 10^6 = 1.2 \times 10^6 + 1.5 \times 10^6 \times e^{0.1t}$

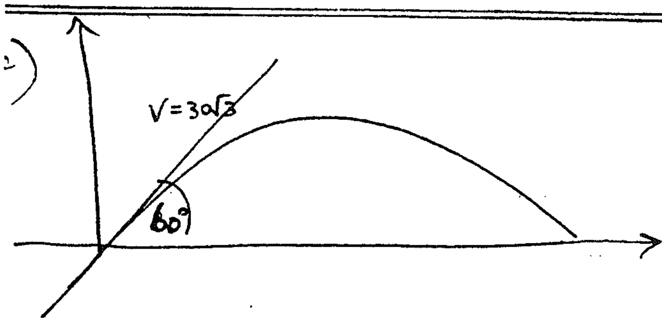
$4.6 = e^{0.1t}$

$0.1t = \ln 4.6$

$t = \frac{\ln 4.6}{0.1}$

$= 15.2$

$\therefore$ on 16th day the pop has tripled



$\ddot{x} = 0$

$\dot{x} = C_1$

when $t=0$ $\quad \dot{x} = 30\sqrt{3}\cos 60$

$= 15\sqrt{3}$

$\therefore \dot{x} = 15\sqrt{3}$

$x = 15\sqrt{3}t + C_2$

when $t=0$ $x=0$ $\therefore C_2 = 0$

$x = 15\sqrt{3}t$

$\ddot{y} = -10$

$\dot{y} = -10t + C_3$

when $t=0$ $\dot{y} = 30\sqrt{3}\sin 60$

$= 45$

$\therefore \dot{y} = -10t + 45$

$y = -5t^2 + 45t + C_4$

when $t=0$ $y=0$ $\therefore C_4 = 0$

$\therefore y = -5t^2 + 45t$

for time of flight $\quad y=0$

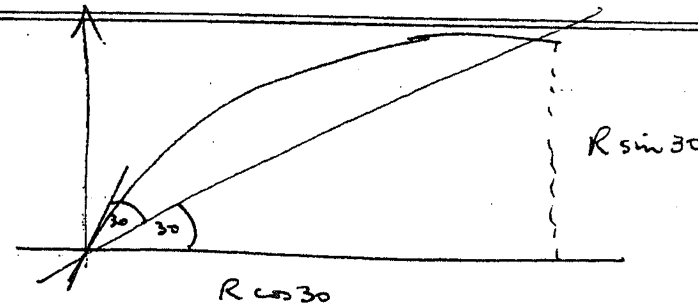
$-5t^2 + 45t = 0$

$5t(-t+9) = 0$

$t=0 \quad t=9 \quad \therefore$ time of flight $= 9s$

for range $\quad x = 15\sqrt{3}t$ & $t=9$

$x = 135\sqrt{3}$ m.



$x = R\cos 30 = \frac{R\sqrt{3}}{2} = 15\sqrt{3}t$

$y = R\sin 30 = \frac{R}{2} = -5t^2 + 45t$

$\therefore R = 30t$

$\frac{R}{2} = -5t^2 + 45t$

$15t = -5t^2 + 45t$

$5t^2 - 30t = 0$

$5t(t-6) = 0$

$t=0 \quad t=6$

$\therefore$ time of flight is 6 secs.

OR

$$\tan 30 = \frac{R\sin\theta}{R\cos 30} = \frac{y}{x}$$

$\frac{1}{\sqrt{3}} = \frac{-5t^2+45t}{15\sqrt{3}t}$

$15t = -5t^2 + 45t$

$5t^2 - 30t = 0$

$5t(t-6) = 0$

$\therefore t=0 \quad t=6$

$\therefore$ time of flight is 6 secs.

2) $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}$ for all n.

Step 1. Test for $n=1$

$$LHS = \frac{1}{2!} = \frac{1}{2} \qquad RHS = \frac{2!-1}{2!} = \frac{1}{2}$$

$\therefore$ true for $n=1$

Step 2 Assume true for $n=k$

ie $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots \frac{k}{(k+1)!} = \frac{(k+1)! - 1}{(k+1)!}$

& show true for $n=k+1$

ie show $\frac{1}{2!} + \frac{2}{3!} + \cdots \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+2)! - 1}{(k+2)!}$

$$LHS = \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+2)!}$$

$$= \frac{(k+2)((k+1)! - 1) + k+1}{(k+2)!}$$

$$= \frac{(k+2)! - (k+2) + k+1}{(k+2)!}$$

$$= \frac{(k+2)! - k - 2 + k + 1}{(k+2)!}$$

$$= \frac{(k+2)! - 1}{(k+2)!}$$

$= RHS$

$\therefore$ if it is true for $n=k$, then it is true for $n=k+1$.

Step 3. Since it is true for $n=1$, by step 2, it must be true for $n=1+1=2$ & since it is true for $n=2+1=3$ & so on $\therefore$ it is true for all n.

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!} \quad \text{for all } n$$

b) i) $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$$y' = \frac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x+e^{-x})^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - (e^{2x} + e^{-2x} - 2)}{(e^x+e^{-x})^2}$$

$$= \frac{4}{(e^x+e^{-x})^2}$$

$\neq 0$ for all x

$\therefore$ there are no stationary pts

ii) as $x \to \infty$ $\quad e^{-x} \to 0$ $\quad \therefore y = \dfrac{e^x - e^{-x}}{e^x+e^{-x}} \to \dfrac{e^x}{e^x} = 1$

as $x \to -\infty$ $\quad e^{x} \to 0$ $\quad \therefore y = \dfrac{e^x - e^{-x}}{e^x+e^{-x}} \to \dfrac{-e^{-x}}{e^{-x}} = -1$

$\therefore y = \pm 1$ are the asymptotes.

OR $\quad$ for $y = 1$ $\quad 1 = \dfrac{e^x - e^{-x}}{e^x+e^{-x}}$

$e^x + e^{-x} = e^x - e^{-x}$

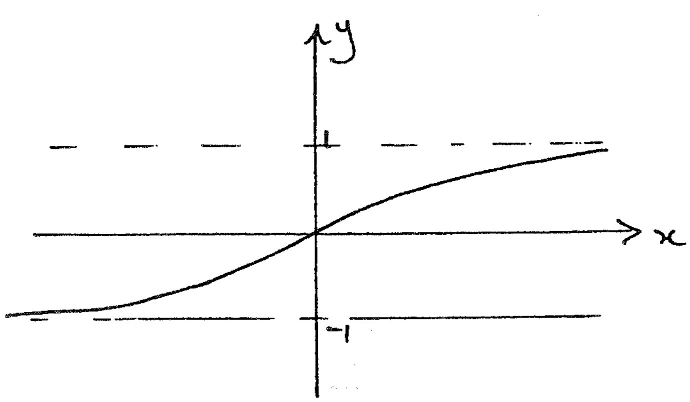
$2e^{-x} = 0$ $\quad$ No solⁿ $\quad \therefore y = 1$ is an asymp

for $y = -1$ $\quad -1 = \dfrac{e^x - e^{-x}}{e^x+e^{-x}}$

$-e^x - e^{-x} = e^x - e^{-x}$

$2e^x = 0$ $\quad$ No solⁿ $\quad \therefore y = -1$ is an asympt

iii)



iv) $$\text{Area} = k - \int_0^k \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$$

$$= k - \left[\log_e\left(e^x + e^{-x}\right)\right]_0^k$$

$$= k - \left[\log_e\left(e^k + e^{-k}\right) - \log_e\left(e^0 + e^0\right)\right]$$

$$= k - \left[\log_e\left(e^k + e^{-k}\right) - \log_e 2\right]$$

v) $$k - \left[\log_e\left(e^k + e^{-k}\right) - \log_e 2\right]$$

$$= \log_e e^k - \log_e\left(e^k + e^{-k}\right) + \log_e 2$$

$$= \log_e \frac{e^k}{e^k + e^{-k}} + \log_e 2$$

Now $\log_e \frac{e^k}{e^k + e^{-k}} < 0$ because $\frac{e^k}{e^k + e^{-k}} < 1$

$\therefore$ max value is $\log_e 2$