Name : ____________________
Class : 12 MTZ___

Pymble Ladies' College
Mathematics Department

KW

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2001 AP4

YEAR 12 TRIAL HSC

MATHEMATICS EXTENSION II

[4 UNIT]

Time allowed - 3 hours
(plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES:

- Attempt ALL questions.
- All questions are of equal value.
- Standard Integrals are provided.
- Approved calculators may be used.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Each question attempted is to be returned on a new page clearly marked Question 1, Question 2, etc on the top of the page.
- Each page must show your class and your name.

Students are advised that this is a school based Trial Examination *only* and cannot in any way guarantee the complete content nor format of the Higher School Certificate Examination.

QUESTION 1. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\displaystyle\int_0^3 \frac{x\,dx}{\sqrt{16+x^2}}$. **3**

(b) Find $\displaystyle\int \frac{dx}{x^2+6x+13}$. **2**

(c) Find $\displaystyle\int xe^{-x}\,dx$. **2**

(d) Find $\displaystyle\int \cos^3\theta\,d\theta$. **3**

(e) (i) Find constants A, B and C such that **3**

$$\frac{x^2-4x-1}{(1+2x)(1+x^2)} \equiv \frac{A}{1+2x} + \frac{Bx+C}{1+x^2}.$$

(ii) Hence find $\displaystyle\int \frac{x^2-4x-1}{(1+2x)(1+x^2)}\,dx$. **2**

QUESTION 2. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given that $z = 1 + i$ and $w = -3$, find, in the form $x + iy$:

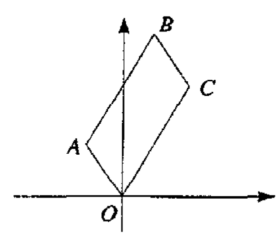
(i) wz^2. **1**

(ii) $\dfrac{z}{z+w}$. **2**

(b) Using de Moivre's theorem, simplify $(-1 - i\sqrt{3})^{-10}$, expressing the answer in the form $x + iy$. **3**

(c) Find the values of real numbers a and b such that $(a+ib)^2 = 5-12i$. **2**

(d) **3**



In the diagram above, $OABC$ is a parallelogram with $OA = \frac{1}{2}OC$.

The point A represents the complex number $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

If $\angle AOC = 60°$, what complex number does C represent?

(e) z_1 and z_2 are complex numbers.

(i) Show that $|z_1||z_2| = |z_1 z_2|$. **1**

(ii) By taking $z_1 = 2+3i$ and $z_2 = 4+5i$, express 533 (the product of 13 and 41) as a sum of squares of two positive integers. **1**

(iii) By taking other values for z_1 and z_2, express 533 as a sum of squares of two other positive integers. **2**

QUESTION 3. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) *On separate number planes, draw graphs of the following functions, showing essential features.*

(i) $y = \dfrac{x+1}{x-1}$ **2**

(ii) $y = \sqrt{\dfrac{x+1}{x-1}}$ **2**

(iii) $y = \ln\left(\dfrac{x+1}{x-1}\right)$ **2**

(b) z is a variable complex number which is represented by the point P. Find the locus of P if $|z - 2i| = \text{Im}(z)$ **2**

(c) The fixed complex number α is such that $0 < \arg\alpha < \frac{\pi}{2}$. In an Argand diagram α is represented by the point A while $i\alpha$ is represented by the point B. z is a variable complex number which is represented by the point P.

(i) Draw a diagram showing A, B and the locus of P if $|z-\alpha| = |z - i\alpha|$. **1**

(ii) Draw a diagram showing A, B and the locus of P if $\arg(z-\alpha) = \arg(i\alpha)$. **1**

(iii) Find in terms of α the complex number represented by the point of intersection of the two loci in (i) and (ii). **1**

(d) Consider the function $y = \sin^{-1}(e^x)$.

(i) Find the domain and range of the function. **2**

(ii) Sketch the graph of the function showing clearly the coordinates of any endpoints and the equations of any asymptotes. **2**

QUESTION 4. (15 marks) Use a SEPARATE writing booklet. **Marks**

Consider the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$.

(a) (i) Find the eccentricity of the ellipse. **1**

(ii) Find the coordinates of the foci and the equations of the directrices of the ellipse. **2**

(iii) Sketch the graph of the ellipse showing clearly all of the above features and the intercepts on the coordinate axes. **2**

(b) (i) Use differentiation to derive the equations of the tangent and the normal to the ellipse at the point $P(2,3)$. **3**

(ii) The tangent and normal to the ellipse at P cut the y axis at A and B respectively. Find the coordinates of A and B. **1**

(c) (i) Show that AB subtends a right angle at the focus S of the ellipse. **2**

(ii) Show that the points A, P, S and B are concyclic. **1**

(iii) Find the centre and radius of the circle which passes through the points A, P, S and B. **3**

QUESTION 5. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Let $P(x)$ be a degree 4 polynomial with a zero of multiplicity 3. Show that $P'(x)$ has a zero of multiplicity 2. **2**

(ii) Hence or otherwise find all zeros of $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, given that it has a zero of multiplicity 3. **2**

(iii) Sketch $y = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, clearly showing the intercepts on the coordinate axes. You do not need to give the coordinates of turning points or inflections. **1**

(b) (i) Show that the general solution of the equation $\cos 5\theta = -1$ is given by $\theta = (2n+1)\frac{\pi}{5}$, $n = 0, \pm 1, \pm 2, \dots$. **2**

Hence solve the equation $\cos 5\theta = -1$ for $0 \le \theta \le 2\pi$.

(ii) Use De Moivre's Theorem to show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **3**

(iii) Find the exact trigonometric roots of the equation $16x^5 - 20x^3 + 5x + 1 = 0$. **2**

(iv) Hence find the exact values of $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ and $\cos\frac{\pi}{5} . \cos\frac{3\pi}{5}$ and factorise $16x^5 - 20x^3 + 5x + 1$ into irreducible factors over the rational numbers. **3**

QUESTION 6. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A lifebelt mould is made by rotating the circle $x^2 + y^2 = 64$ through one complete revolution about the line $x = 28$, where all the measurements are in centimetres.

(i) Use the method of slicing to show that the volume $V\text{cm}^3$ of the lifebelt is given by **5**

$$V = 112\pi \int_{-8}^{8} \sqrt{64 - y^2}\ dy.$$

(ii) Find the exact volume of the lifebelt. **2**

(b) (i) Show that $\dfrac{t^n}{1+t^2} = t^{n-2} - \dfrac{t^{n-2}}{1+t^2}$. **1**

(ii) Let $I_n = \displaystyle\int \frac{t^n}{1+t^2}dt$. **1**

Show that $I_n = \dfrac{t^{n-1}}{n-1} - I_{n-2},\ n \geq 2$.

(iii) Show that $\displaystyle\int_0^1 \frac{t^6}{1+t^2}dt = \frac{13}{15} - \frac{\pi}{4}$. **3**

(c) In a series of five games played by two equally matched teams, team A and team B, the team that wins three games first is the champion.

(i) If team B wins the first two games, what is the probability that team A is the champion? **1**

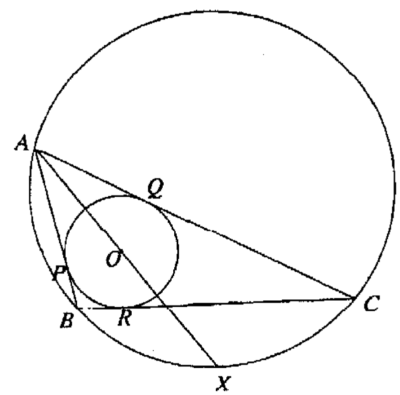
(ii) If team A has won the first game, what is the probability that team A is the champion? **2**

QUESTION 7. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) In the diagram below, ABC is a triangle.

The incircle tangent to all three sides has centre O, and touches the sides AB, AC and BC at P, Q and R respectively.

The circumcircle through A, B and C meets the line AO produced at X.



(i) Show that $\angle CBX = \angle CAX$. **1**

(ii) Use congruence to prove that $\angle OBA = \angle OBC$. **2**

(iii) Prove that ΔXBO is an isosceles triangle. **3**

(iv) Prove that $BX = XC$. **1**

(b) (i) α. Differentiate $y = \log_e(1+x)$, and hence draw $y = x$ and $y = \log_e(1+x)$ on one set of axis. **1**

β. Using this graph, explain why **1**

$$\log_e(1+x) < x, \text{ for all } x > 0.$$

(ii) α. Differentiate $y = \dfrac{x}{1+x}$, and hence draw $y = \dfrac{x}{1+x}$ and $y = \log_e(1+x)$ on one set of axis. **1**

β. Using this graph, explain why **1**

$$\frac{x}{1+x} < \log_e(1+x), \text{ for all } x > 0.$$

(iii) Use the inequalities of parts (i) and (ii) to show that **4**

$$\frac{\pi}{8} - \frac{1}{4}\log_e 2 < \int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx < \frac{1}{2}\log_e 2.$$

QUESTION 8. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) At a dinner party there are twelve people, consisting of the six State Premiers and their partners. Each couple was representing one of the six States: New South Wales, Victoria, Western Australia, South Australia, Tasmania and Queensland.

(i) The dinner took place at a circular table. Find how many seating arrangements are possible if:

α. there are no restrictions, **1**

β. the males and females are in alternate positions. **1**

(ii) A committee of six is to be formed from the Premiers and their partners, where not more than one State can have two representatives. How many such committees are possible? **2**

(b) It is given that if a, b, c are any three positive real numbers, then $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$.

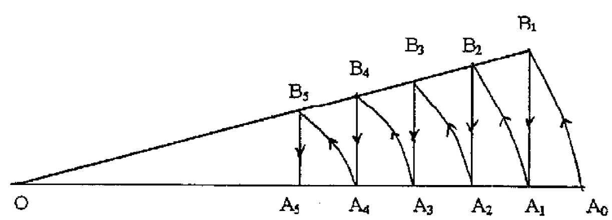
If $a>0$, $b>0$ and $c>0$ are real numbers such that $a+b+c=1$, use the given result to show that

(i) $\frac{1}{abc} \geq 27$ **1**

(ii) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9$ **2**

(iii) $(1-a)(1-b)(1-c) \geq 8abc$ **2**

(c)



An ant walks along the circular arc from A_0 to B_1, then down the straight line to A_1, along the circular arc to B_2, then down to A_2, and so on, until it reaches O.

The length of OA_0 is 1, while angle A_0OB_1 is x radians, $0 < x \leq \frac{\pi}{2}$.

(i) Show that the total distance the ant walks by the time it reaches O is given by $y = \frac{x+\sin x}{1-\cos x}$ **2**

(ii) Find the derivative of y with respect to x and explain why the derivative of y is always negative for all $0 < x \leq \frac{\pi}{2}$ **2**

(iii) Hence find the shortest possible distance the ant needs to walk from A_0 to O. **2**

End of Paper