2006 Mathematics Extension I Trial Exam. 1

(a) $\frac{1}{x-2} \geq 2$ $x \neq 2$

$(x-2) \geq 2(x-2)^2$

$(x-2) - 2(x-2)^2 \geq 0$

$(x-2)(1-2(x-2)) \geq 0$

$(x-2)(5-2x) \geq 0$

∴ Solution $2 < x \leq 2\frac{1}{2}$ (3)

b) $\lim_{h\to 0} \left(\frac{\cos 3h - 1}{h}\right)\frac{(\cos 3h+1)}{(\cos 3h+1)}$

$= \lim_{h\to 0} \frac{\cos^2 3h - 1}{h[\cos 3h+1]}$

$= \lim_{h\to 0} \frac{-\sin^2 3h}{h[\cos 3h+1]}$

$= \lim_{h\to 0} \frac{\sin 3h}{3h} \cdot \frac{-3\sin 3h}{\cos 3h+1}$

$= 1 \cdot \frac{0}{2}$

$= 0$ (2)

c) (i) $A(-1,5)$ $r:1$ $B(3,-2)$

$P \equiv \left[\frac{3r-1}{r+1}, \frac{-2r+5}{r+1}\right]$ (2)

ii) ∴ $2\left[\frac{3r-1}{r+1}\right] - 3\left[\frac{-2r+5}{r+1}\right] + 4 = 0$

$6r - 2 + 6r - 15 + 4r + 4 = 0$

$16r = 13$

$r = \frac{13}{16}$ (2)

(d) $\int_0^1 (x^2+1)^3\,dx$

$= \int_0^1 (x^6 + 3x^4 + 3x^2 + 1)\,dx$

$= \left[\frac{x^7}{7} + \frac{3}{5}x^5 + x^3 + x\right]_0^1$

$= 2\frac{26}{35}$ (3)

2 $\frac{dT}{dt} = -k(T - T_0)$

$T = T_0 + Ae^{-kt}$

$T_0 = 22°$ And $t=0$ $T = 55$

∴ $55 = 22 + Ae^0$

$A = 33 \Rightarrow T = 22 + 33e^{-kt}$

And $41 = 22 + 33e^{-10k}$

$e^{-10k} = \frac{19}{33}$

$k = \frac{1}{10}\ln\frac{33}{19}$ (2)

∴ $T = 22 + 33e^{-\frac{t}{10}\ln\left(\frac{33}{19}\right)}$

(i) $t = 25$

$T = 22 + 33e^{-\frac{25}{10}\ln\frac{33}{19}}$

$= 30.3°C$ (1)

(ii) $25 = 22 + 33e^{-\frac{t}{10}\ln\frac{33}{19}}$

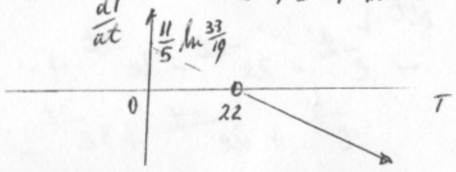
$e^{-\frac{t}{10}\ln\frac{33}{19}} = \frac{3}{33}$

$-\frac{t}{10}\ln\frac{33}{19} = -\ln 11$

$t = \frac{10\ln 11}{\ln\frac{33}{19}}$

$= 43.4$ mins   (2)

iii



(1)

2

2(d) $x = 5\sin 3t - 7\cos 3t$

i) $\dot{x} = 15\cos 3t + 21\sin 3t$

$\ddot{x} = -45\sin 3t + 63\cos 3t$

$= -9[5\sin 3t - 7\cos 3t]$

$\ddot{x} = -9x$ (1)

which is of the form $\ddot{x} = -n^2(x-b)$

$n = 3$ $b = 0$

∴ motion SHM.

ii) Max displacement $= \sqrt{5^2+7^2}$

$= \sqrt{25+49}$

$= \sqrt{74}$ units.   (1)

Max velocity $= \sqrt{15^2+21^2}$

$= 3\sqrt{74}$ m/s   (1)

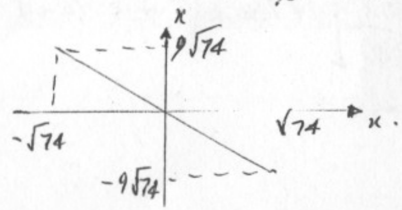
iii) $x = 0$ $5\sin 3t - 7\cos 3t = 0$

$\tan 3t = \frac{7}{5}$

$3t = \tan^{-1}\frac{7}{5}$

$t = 0.32$ s   (2)

iv)



(1)

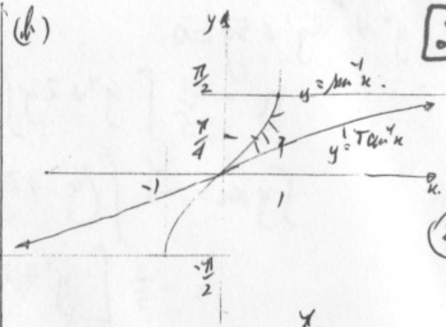
3 (a) $\frac{d}{dx}\cos^{-1}\left(\frac{-1}{x}\right) = \frac{1}{x^2} \cdot \frac{-1}{\sqrt{1 - \frac{1}{x^2}}}$

$= \frac{-\sqrt{x^2}}{x^2\sqrt{x^2-1}}$

$= \frac{-|x|}{x^2\sqrt{x^2-1}}$

$= \frac{-1}{|x|\sqrt{x^2-1}}$ (3)

(b)



(2)

$\int_0^1 \tan^{-1}x\,dx = \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \tan y\,dy$

$= \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \frac{\sin y}{\cos y}\,dy$

$= \frac{\pi}{4} + \left[\ln\cos y\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4} + \ln\left(\frac{\frac{1}{\sqrt{2}}}{1}\right)$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$ (2)

∴ Region $\int_0^1 \sin^{-1}x\,dx - \int_0^1 \tan^{-1}x\,dx$

$= \frac{\pi}{2} - 1 - \left[\frac{\pi}{4} - \frac{1}{2}\ln 2\right]$

$= \frac{\pi}{4} - 1 + \frac{1}{2}\ln 2$ (1)

(c)(i) $y = e^{-x}\sin 2x$

$\frac{dy}{dx} = -e^{-x}\sin 2x + 2e^{-x}\cos 2x$

$= e^{-x}[2\cos 2x - \sin 2x]$ (1)

$\frac{d^2y}{dx^2} = -e^{-x}[2\cos 2x - \sin 2x] + e^{-x}[-4\sin 2x - 2\cos 2x]$ (1)

$= e^{-x}[-3\sin 2x - 4\cos 2x]$

∴ $y'' + 2y' + 5y = e^{-x}[-3\sin 2x - 4\cos 2x + 4\cos 2x - 2\sin 2x + 5\sin 2x]$

$= 0$ (1)

iii) $\boxed{c}$ $y'' + 2y' + 5y = 0$ $\boxed{3}$

$$y = -\frac{1}{5}\left[y'' + 2y'\right]$$

$$\int y\,dx = -\frac{1}{5}\int (y'' + 2y')\,dx$$

$$= -\frac{1}{5}\left[y' + 2y\right] + C$$

$$\int e^{-x}\sin 2x\,dx = -\frac{1}{5}\left[e^{-x}(2\cos 2x - \sin 2x) + 2e^{-x}\sin 2x\right] + C$$

$$= \frac{-e^{-x}}{5}\left[2\cos 2x + \sin 2x\right] + C$$ (1)

4(a)(i) $x = 30t\cos\alpha$ $\quad y = -5t^2 + 30t\sin\alpha$

$$t = \frac{x}{30\cos\alpha} \qquad y = -5\left(\frac{x}{30\cos\alpha}\right)^2 + 30\sin\alpha\,\frac{x}{30\cos\alpha}$$

$$y = \frac{-x^2}{180}\sec^2\alpha + x\tan\alpha$$

or $y = \frac{-x^2}{180}\left[1 + \tan^2\alpha\right] + x\tan\alpha$ (1)

ii) $\alpha = 45^\circ$ $\quad y = 8$ $\quad x = d + 15$

$$\therefore 8 = \frac{-x^2}{180}\cdot(1+1) + x$$

$$x^2 - 90x + 720 = 0$$

$$x = \frac{90 \pm \sqrt{90^2 - 4\times 720}}{2}$$

$d + 15 = 81.12$ or 17.75

$d = 66.12\text{m}$ furthest distance. (2)

iii) $\boxed{1}$ $A(20, 8)$ $\boxed{4}$

$$8 = -\frac{400}{180}\left[1 + \tan^2\alpha\right] + 20\tan\alpha$$

$$72 = -20\left[1 + \tan^2\alpha\right] + 180\tan\alpha$$

$$20\tan^2\alpha - 180\tan\alpha + 92 = 0$$

$$5\tan^2\alpha - 45\tan\alpha + 23 = 0$$

$$\tan\alpha = \frac{45 \pm \sqrt{45^2 - 4\times 5\times 23}}{2\times 5}$$

$= 0.544$ or 8.46

Angle elevation $\alpha = 28^\circ 33'$ as $0 \le \alpha \le 45^\circ$. (2)

(b) $$\int \frac{4x-7}{2x^2+1}\,dx = \int\left(\frac{4x}{2x^2+1} - \frac{7}{2x^2+1}\right)dx$$

$$= \int\left(\frac{4x}{2x^2+1} - \frac{7}{2}\cdot\frac{1}{x^2+\frac{1}{2}}\right)dx$$

$$= \ln(2x^2+1) - \frac{7}{2}\cdot\frac{1}{\sqrt{\frac{1}{2}}}\tan^{-1}\frac{x}{\sqrt{\frac{1}{2}}} + c$$

$$= \ln(2x^2+1) - \frac{7}{\sqrt{2}}\tan^{-1}(x\sqrt{2}) + c.$$ (2)

(c)(i) $e^{-t} + e^{-2t} + e^{-3t} \ldots = \frac{e^{-t}}{1 - e^{-t}}$ as $|e^{-t}| < 1$ for $t > 0$. (1)

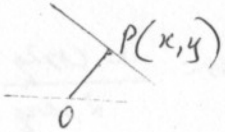
$$= \frac{1}{e^t - 1}$$

(ii) Now $\frac{d}{dt}\left[e^{-t} + e^{-2t} + e^{-3t} + \ldots\right] = \frac{d}{dt}(e^t - 1)^{-1}$

$$-e^{-t} - 2e^{-2t} - 3e^{-3t} + \ldots = -(e^t - 1)^{-2}\cdot e^t$$

$$\therefore e^{-t} + 2e^{-2t} + 3e^{-3t} \ldots = \frac{e^t}{(e^t-1)^2}$$ (1)

(d) $y = \sqrt{r^2 - x^2}$



Gradient Tangent $m_1 = \frac{dy}{dx} = \frac{-x}{\sqrt{r^2-x^2}}$ (1)

Gradient $OP = \frac{y-0}{x-0}$

$= \frac{y}{x}$

$m_2 = \frac{\sqrt{r^2-x^2}}{x}$

$\therefore m_1 \times m_2 = \frac{-x}{\sqrt{r^2-x^2}} \cdot \frac{\sqrt{r^2-x^2}}{x}$

$= -1$ (1)

$\therefore$ Tangent $\perp$ radius.

5. $(a+b)^n \equiv \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$

$\therefore T_{r+1} = \binom{11}{r} (4x)^{11-r} 5^r$

$T_r = \binom{11}{r-1} (4x)^{12-r} 5^{r-1}$

$\frac{T_{r+1}}{T_r} = \frac{11!}{r!(11-r)!} \cdot \frac{(r-1)!(12-r)!}{11!} \cdot \frac{4^{11-r}}{4^{12-r}} \cdot \frac{5^r}{5^{r-1}} \cdot \frac{x^{11-r}}{x^{12-r}}$

$= \frac{12-r}{r} \cdot \frac{5}{4} \cdot \frac{1}{x}$ (1)

For largest coefficient $\frac{5(12-r)}{4r} \geq 1$ (1)

$60 - 5r \geq 4r$

$9r \geq 60$

$r \geq 7$ $T_8 < T_7$

$\therefore$ Largest coefficient $\binom{11}{6} 4^5 5^6$ (1)

$\boxed{6}$

(b) (i) No motion does not oscillate. 1

(ii) $\frac{d}{dx}\left(\frac{v^2}{2}\right) = \frac{d}{dv}\left(\frac{v^2}{2}\right) \cdot \frac{dv}{dx}$

$= \frac{2v}{2} \cdot \frac{dv}{dx}$

$= v \frac{dv}{dx}$

$= \frac{dx}{dt} \cdot \frac{dv}{dx}$

$= \frac{dv}{dt}$

$= \ddot{x}$ 2

(iii)

$\ddot{x} = -625x$

$\frac{d}{dx}\left(\frac{v^2}{2}\right) = -625x$

$\frac{v^2}{2} = -625\frac{x^2}{2} + C$ But $v=0$ $x=-1$

$\therefore 0 = -\frac{625}{2} + C$

$C = +\frac{625}{2}$

$\therefore \frac{v^2}{2} = \frac{625}{2} - 625\frac{x^2}{2}$

$v^2 = 625(1-x^2)$

$v = 25\sqrt{1-x^2}$ $v>0$. 2

(iv) At Surface $x=0$ $\therefore v = 25$ m/s 1

(v) $\frac{d^2x}{dt^2} = -g$

$\frac{d}{dx}\left(\frac{v^2}{2}\right) = -10$

7

(v) $\frac{v^2}{2} = -10x + C$ $x=0$ $v=25$

$\therefore C = \frac{25^2}{2}$

$\frac{v^2}{2} = -10x + \frac{625}{2}$

$v^2 = -20x + 625$

$x = \frac{625 - v^2}{20}$ (2)

(vi) Max height $v = 0$

$x = \frac{625}{20}$

$= 31.25$ m. (1)

6(a) Ways $= \binom{6}{2}\binom{8}{2} = 420$ (2)

(b) (i) Number words $\frac{6!}{3!} = 120$ (2)

(ii) C Δ C Δ C Δ Δ C Δ C Δ C

C $\Delta\Delta$ C Δ C

C Δ C $\Delta\Delta$ C (2)

Total $= 4 \times 3!$

$= 24$

(iii) CCC Y L I or I CCC YL

Ways I $= 2!$ Ways I $= 2!$

Ways C $= 1$ Ways [CCCYL] $= \frac{5!}{3!} = 20$

Total $= 2 \times 3!$ Total $= 2 \times 20$

I end $= 12$ $= 40$

Probability $\left(\begin{array}{c}\text{If end is I} \\ \text{Cs together}\end{array}\right) = \frac{12}{40}$

$= \frac{3}{10}$ (2)

8

(b) Step 1 $n=1$

LHS $= \sin\theta$ RHS $= \frac{1-\cos 2\theta}{2\sin\theta}$

$= \frac{1-(1-2\sin^2\theta)}{2\sin\theta}$

$= \frac{2\sin^2\theta}{2\sin\theta}$

$= \sin\theta$

$=$ LHS (1)

$\therefore$ True $n=1$.

Step 2

Assume true $n=k$ $\sin\theta + \sin 3\theta + \ldots \sin(2k-1)\theta = \frac{1-\cos 2k\theta}{2\sin\theta}$

To prove true $n=k+1$ $\sin\theta + \sin 3\theta + \ldots \sin(2k+1)\theta = \frac{1-\cos 2(k+1)\theta}{2\sin\theta}$

LHS $= \sin\theta + \sin 3\theta + \ldots + \sin(2k-1)\theta + \sin(2k+1)\theta$

$= \frac{1-\cos 2k\theta}{2\sin\theta} + \sin(2k+1)\theta$ (By assumption)

$= \frac{1-\cos 2k\theta + 2\sin\theta\sin(2k+1)\theta}{2\sin\theta}$ (1)

$= \frac{1-\cos[(2k+1)\theta - \theta] + 2\sin\theta\sin(2k+1)\theta}{2\sin\theta}$

$= \frac{1-\{\cos(2k+1)\theta\cos\theta + \sin(2k+1)\theta\sin\theta\} + 2\sin\theta\sin(2k+1)\theta}{2\sin\theta}$ (1)

$= \frac{1-\{\cos(2k+1)\theta\cos\theta - \sin\theta\sin(2k+1)\theta\}}{2\sin\theta}$

$= \frac{1-\cos[(2k+1)\theta+\theta]}{2\sin\theta}$ (1)

$= \frac{1-\cos 2(k+1)\theta}{2\sin\theta}$

$\therefore$ If statement true $n=k$ it is also true $n=k+1$.

Since true for $n=1$ it is also true for $n=1+1=2$, $n=2+1=3$ and so on

i) (c) 11

$$\frac{dP}{d\theta} = r\left[2 - 2\cos\left(\frac{\pi}{3}-\theta\right)\right]$$

$$\frac{d^2P}{d\theta^2} = 2r\sin\left(\frac{\pi}{3}-\theta\right)$$

For maximum Perimeter $\frac{dP}{d\theta} = 0$

$$r\left[2 - 2\cos\left(\frac{\pi}{3}-\theta\right)\right] = 0$$

$r \neq 0$ $\quad \cos\left(\frac{\pi}{3}-\theta\right) = 1$

$\frac{\pi}{3} - \theta = 0$ for $0<\theta<\frac{\pi}{2}$ 1

$\therefore \theta = \frac{\pi}{3}$

For nature test $\frac{d^2P}{d\theta^2}$ for concavity

at $\theta = \frac{\pi}{3}$ $\quad \frac{d^2P}{d\theta^2} = r \times 0$

$= 0$

Test gradients:

θ	1	$\frac{\pi}{3}$	1.1
$\frac{dP}{d\theta}$	$2\times10^{-3}r$	0	$2\times10^{-3}r$
	/	—	/

1

gradients same sign

$\therefore$ Inflexion point at $\theta = \frac{\pi}{3}$ and monotonic increasing, continuous for $0<\theta<\frac{\pi}{2}$

$\therefore$ Maximum perimeter at end points of domain ie $\theta = \frac{\pi}{2}$. 1