I
3u

CRANBROOK SCHOOL
3/4u Trial
July 1999.

Q1

Q1.(a) Solve $x^2+x-6<0 \Rightarrow (x+3)(x-2)<0 \Rightarrow -3<x<2$ [2]

Q2.(b) diff $y=\log_e\dfrac{(x-1)}{(x^2+1)} \Rightarrow y=\log_e(x-1)-\log_e(x^2+1)$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{1}{x-1}-\dfrac{2x}{x^2+1}$ [3]

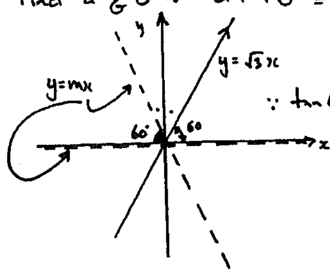
(c) Exact Value of $\int_0^{\frac{\pi}{2}}\cos 2x\,dx = \left[\tfrac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}} = \tfrac{1}{2}\{(\sin\pi)-(\sin 0)\}$ [3]

$= 0$

(d) Find a & b if $a+\sqrt{b}=(2+3\sqrt{2})^2 \Rightarrow a+\sqrt{b}=22+12\sqrt{2}$

$\therefore a=22$ & $b=288.$ [2]

(e)



$\because \tan 60^\circ=\sqrt{3}$

$\therefore y=-\sqrt{3}x$ & $y=0$ are at 60° to $y=\sqrt{3}x$

$\therefore m=-\sqrt{3}$ & 0 [2]

Q2(a) $\log\frac{3}{2}+\log\frac{4}{3}+\log\frac{5}{4}+\dots+\log\frac{n}{n-1}$

LHS $=\log\left(\frac{3}{2}\times\frac{4}{3}\times\frac{5}{4}\times\dots\times\frac{n-1}{n-2}\times\frac{n}{n-1}\right)=\log\frac{n}{2}=$ RHS ✓✓✓ 3.

(b) Y_1 1500×1.055

Y_2 $(1500\times1.055+1500)1.055 = 1500\times1.055^2+1500\times1.055$

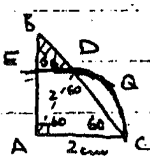
Y_3 $1500\times1.055^3+1500\times1.055^2+1500\times1.055$ ✓

Y_{15} $1500\times1.055^{15}+1500\times1.055^{14}+\dots+1500\times1.055$

$S_{15}=1500\times1.055\,(1.055^{14}+\dots+1)$ ✓

$=1500\times1.055\left(\dfrac{1.055^{15}-1}{0.055}\right)=$ \$35461.71 ✓ ✓ 4

(c)

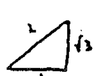


(i) Area $\triangle ABC=\frac{1}{2}.2.2.\tan 60^\circ=2\sqrt{3}$

Area quadrant ACDE: $\dfrac{\pi.2^2}{4}=\pi$ 3

$\tan 60^\circ=\frac{x}{2}$ Area segment DQC $\frac{1}{2}r^2(\theta-\sin\theta)$

$=\frac{1}{2}.2^2\left(\frac{\pi}{3}-\sin\frac{\pi}{3}\right)$



$=2\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)$ ✓✓✓✓

Area BED $=2\sqrt{3}-\pi+\frac{2\pi}{3}-\sqrt{3}$

$=\left(\sqrt{3}-\frac{\pi}{3}\right)\text{ cm}^2$

(ii) Area ACDE: $\pi-2\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)$

$=\pi-\frac{2\pi}{3}+\sqrt{3}$

$=\left(\sqrt{3}+\frac{\pi}{3}\right)\text{ cm}^2$ ✓

16

1230 CRANBROOK TRIAL 1999

QUESTION 3 - MARKED BY [illegible]

(a)

RTP $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3$

Step 1:

Prove true for $n=1$

LHS $= 1$ RHS $= \frac{1}{4}\times 1^2(1+1)^2$

$= \frac{1}{4}\times 1 \times 4$

$= 1$

$=$ LHS

$\therefore$ true for $n=1$

Step 2:

Assuming true for $n=k$ ie

$1^3+2^3+3^3+\ldots+k^3 = \frac{1}{4}k^2(k+1)^2$,

prove true for $n=k+1$

ie. $1^3+2^3+3^3+\ldots+k^3+(k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$

LHS $= 1^3+2^3+3^3+\ldots+k^3+(k+1)^3$

$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$

$= \frac{1}{4}(k+1)^2(k^2+4(k+1))$

$= \frac{1}{4}(k+1)^2(k^2+4k+4)$

$= \frac{1}{4}(k+1)^2(k+2)^2$

$= \frac{1}{4}n^2(n+1)^2$ where $n=k+1$

$\therefore$ true for $n=k+1$

Step 3:

Since true for $n=1$ and true for $n=k+1$, (having assumed true for $n=k$) must be true for $n=1+1=2$, $n=2+1=3$ etc

$\therefore$ true for all +ve integers, n.

(4 marks)

(b) $5\sin x - 3\cos x$

(i) $r\sin(x-\alpha) = r\sin x\cos\alpha - r\cos x\sin\alpha$

$\therefore r\cos\alpha = 5$

$r\sin\alpha = 3$

$r = \sqrt{5^2+3^2}$ $\tan\alpha = \frac{3}{5}$

$= \sqrt{34}$ $\therefore \alpha = 30°58'$

$5\sin x - 3\cos x = \sqrt{34}\sin(x - 30°58')$

$\therefore 5\sin x - 3\cos x = 2$

$\sqrt{34}\sin(x-30°58') = 2$

$\sin(x-30°58') = \frac{2}{\sqrt{34}}$

$x - 30°58' = 20°4', 159°56'$ (W.A. $= 20°4'$)

$-200°4', -339°56'$

$\therefore x = 51°2', 190°54', -169°6', -30\cancel{8}°58'$

out of domain   out of domain

(4 marks)

$\therefore x = 51°2', -169°6'$

(c) $v^2 = 9(3+2x-x^2)$

(i) $\frac{d}{dx}(\frac{1}{2}v^2) = \frac{9}{2}(2-2x)$

$= 9(1-x)$

$\therefore \ddot{x} = -9(x-1)$

$= -n^2(x-b)$ $n=3$, $b=1$

$\therefore$ motion is simple harmonic

(2 marks)

(ii) Centre of motion: $x=1$

period $= \frac{2\pi}{3}$

(2 marks)

Question 4 Cranbrook HSC 1999 Trial

a(i) $\frac{dv}{dt} = -2$. Given $r=10$.

To find $\frac{ds}{dt} = \frac{ds}{dr} \times \frac{dr}{dv} \times \frac{dv}{dt}$ (1)

$\therefore S = 4\pi r^2$ $\qquad$ $V = \frac{4}{3} r^3$

$\frac{ds}{dr} = 8\pi r$.

$\frac{dv}{dr} = 4\pi r^2 \qquad \therefore \frac{dr}{dv} = \frac{1}{4\pi r^2}$

From (1) $\therefore \frac{ds}{dt} = 8\pi r \times \frac{1}{4\pi r^2} \times -2$.

$\frac{ds}{dt} = -\frac{4}{r}$

when $r=10$. $\therefore \frac{ds}{dt} = -\frac{4}{10}$

$= -2/5 \ (0.4)$

$\therefore$ Rate of decrease in surface area is $0.4 m^2/s$.

(ii) Find r when $\frac{ds}{dt} = \frac{dv}{dt}$.

$\frac{ds}{dt} = \frac{-4}{r} \qquad \frac{dv}{dt} = -2$.

$\therefore \frac{-4}{r} = -2$

$\therefore r = 2m$.

Radius would be 2m.

b(i) Horizontal Asymptote $y=1$
Vertical Asymptote $x=3$.

b(ii) $x = 1 + \frac{2}{y-3}$ will give inverse function of $f(x)$

$\therefore x-1 = \frac{2}{y-3}$

$y-3 = \frac{2}{x-1}$

(iii) Domain $x \neq 1$

3U Q5

$\frac{\text{IV}}{3u}$

(a) $13 + 21 + \ldots + (n^2+n+1) = \sum_{r=3}^{n} (r^2+r+1)$ (1)

(b) (i) $P(x) = 4x^3 + 3x^2 + 2$

$P'(x) = 12x^2 + 6x$

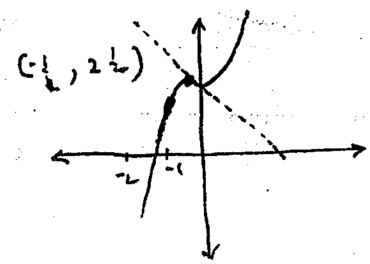
$= 6x(2x+1)$

$= 0$ if $x = 0, -\frac{1}{2}$

$P''(x) = 24x + 6$

>0 if $x = 0$ $\quad \therefore (0,2)$ min T.P.

<0 if $x = -\frac{1}{2}$ $\quad \therefore (-\frac{1}{2}, 2\frac{1}{4})$ max TP



(2)

Because local min^m is >0 as shown on graph, (1) there is only one ZE

(ii) $x_2 = x_1 - \frac{P(x_1)}{P'(x_1)}$

$= -1.2 - \frac{-.592}{10.08}$ (1)

$= -1.1413\ldots$ (1)

(iii) Using $x_1 = -.25$ gives a tangent that slopes away from the zero (eg dotted line on graph) (1)

(c) (i) $8!$ (40320) (1)

(ii) $\frac{4! \times 4! \times 2}{8!}$ (1) $(\frac{1}{35})$ [4! ways of arranging each group ; 2! ways of deciding whether L.H. person is M or F]

(d) (i) $\binom{10}{5}(.6)^5(.4)^5$ (1) $(.2006\ldots)$ [$(.6)^5$ term in expansion of $(.6+.4)^{10}$]

(ii) $.6^5 \times .4^5$ (1) $(.000796\ldots)$

(iii) $1 - \left[\binom{10}{0}.4^{10} + \binom{10}{1}.6 \times .4^9 \ldots).6^2 \times .4^8\right]$ (1) (complement of OH or 1H

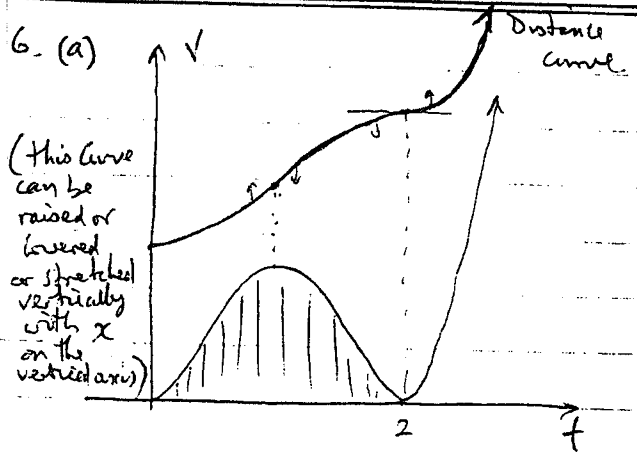
$= 1 - .001677\ldots$

$= \underline{.9983}$

$\frac{V}{3u}$

1999 3U HSC TRIAL – QUESTION 6

6. (a)

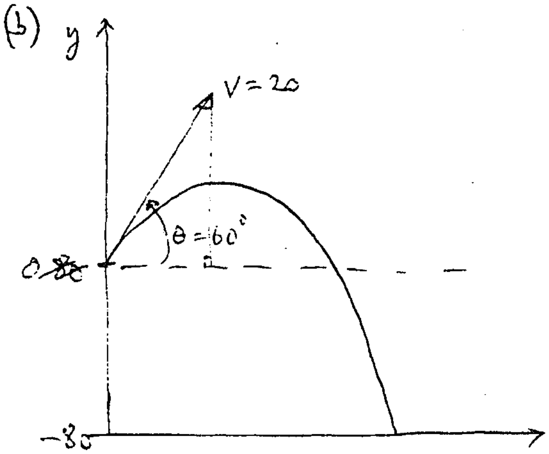


(i) The velocity never becomes negative $\therefore$ the particle does not change direction. $\boxed{1}$

(ii) shaded on diagram. $\boxed{1}$

(iii) drawn on diagram. $\boxed{2}$

(b)



$t=0 \quad V=20 \quad \theta=60^\circ \quad g=10, \; x=0, y=0,$

$\frac{dx}{dt} = 20\cos 60^\circ \qquad \frac{dy}{dt} = 20\sin 60^\circ$

$= 10 \qquad\qquad = 10\sqrt{3}$

(i) $\frac{d^2x}{dt^2} = 0 \qquad \frac{d^2y}{dt^2} = -g = -10$

$\therefore \frac{dx}{dt} = c_1 \qquad \therefore \frac{dy}{dt} = -10t + c_2$

but from boundary conditions above $c_1 = 10$

and $c_2 = 10\sqrt{3}$.

$\therefore \boxed{\frac{dx}{dt} = 10}$ and $\boxed{\frac{dy}{dt} = 10\sqrt{3} - 10t}$

$x = 10t + c_3 \qquad y = 10\sqrt{3}t - \frac{10t^2}{2} + c_4$

but when $t=0$ $x=0, y=0 \therefore c_3 = c_4 = 0$

$\therefore \boxed{x = 10t}$ and $\boxed{y = 10\sqrt{3}t - 5t^2}$ $\boxed{2}$

(ii) Maximum height reached at half time of flight (to position level with projection)

ie when $y=0$ $10\sqrt{3}t - 5t^2 = 0$

$\therefore 5t(2\sqrt{3} - t) = 0$

$\therefore t = 0$ or $2\sqrt{3}$.

so half time of flight is $\sqrt{3}$ seconds

& Maximum height is $y = 10\sqrt{3}\times\sqrt{3} - 5\times\sqrt{3}^2$

$= 30 - 15$

$= 15$ metres

above the ground ie 95 metres $\boxed{2}$

(iii) Projectile hits ground when $y = -80$

ie $10\sqrt{3}t - 5t^2 = -80$.

$\therefore t^2 - 2\sqrt{3}t - 16 = 0$.

$t = \frac{2\sqrt{3} \pm \sqrt{(2\sqrt{3})^2 - 4\times1\times-16}}{2}$

$= \frac{2\sqrt{3} \pm \sqrt{12+64}}{2}$

$= \frac{2\sqrt{3} \pm \sqrt{76}}{2}$

Since $t > 0$

$t = \frac{2\sqrt{3} + 2\sqrt{19}}{2}$ $\boxed{2}$

$= 6.0909\ldots$ seconds.

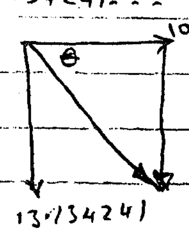
(iv) When $t = 3.04545\ldots$

$\frac{dx}{dt} = 10$ and $\frac{dy}{dt} = 10\sqrt{3} - 10\times3.045\ldots$

$= -13.134241\ldots$

$\tan\theta = \frac{13.134241\ldots}{10}$

$\therefore \theta = 52.7^\circ$



$\therefore$ the direction is 52.7° downwards from the horizontal.

7. (a) $P(x) = x^3 - (h+1)x^2 + hx + 12$

(i) $P(3) = 27 - 9(h+1) + 3h + 12$

$= 27 - 9h - 9 + 3h + 12 = \underline{30 - 6h}$ 2

(ii) $P(x)$ divisible by $(x-3)$ if $30 - 6h = 0$

i.e. $\underline{h = 5}$ 1

(b) (i) $\displaystyle\int_0^3 x\sqrt{(x^2+1)^3}\,dx$ Set $u = x^2+1$

$\dfrac{du}{dx} = 2x$

$\dfrac{dx}{du} = \dfrac{1}{2x}$ 1 If $x = 0,\ u = 1$; $x = 3,\ u = 10$

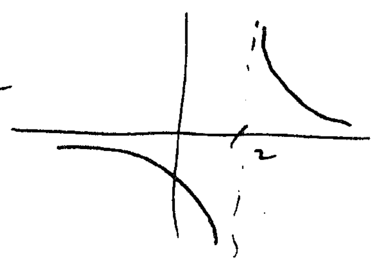
$= \displaystyle\int_1^{10} \cancel{x}\, u^{3/2} \cdot \frac{1}{2\cancel{x}}\,du$

$= \dfrac{1}{2}\left[\dfrac{2}{5}u^{5/2}\right]_1^{10}$ 1 $= \dfrac{1}{5}\left[10^{5/2} - 1^{5/2}\right] = \underline{\dfrac{100\sqrt{10} - 1}{5}}$ 1

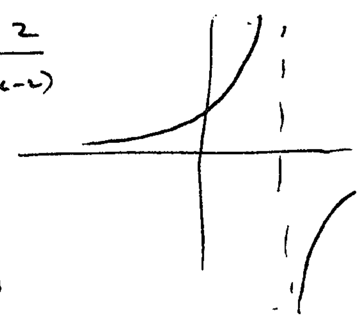
(ii) $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int \frac{2x}{(x^2+1)^{1/2}}\,dx$

$= \dfrac{1}{2} \cdot 2(x^2+1)^{1/2} + c = \underline{\sqrt{x^2+1} + c}$ 2

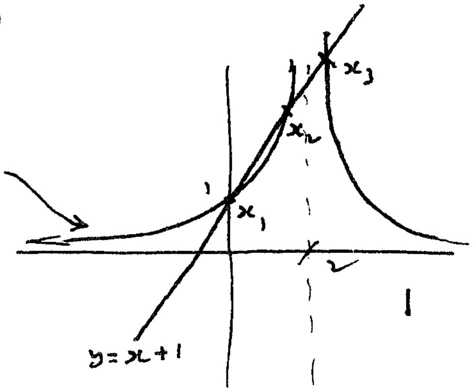
(c) Graphs $y = \dfrac{2}{x-2}$



$y = \dfrac{2}{-(x-2)}$



$\therefore\ y = \dfrac{2}{|x-2|}$



x coordinates of pts of intersection are x_1, x_2, x_3

1

x_1, x_2: $\dfrac{2}{2-x} = x + 1$

$\therefore\ 2 = 2x + 2 - x^2 - x$

$x^2 - x = 0$

$x(x-1) = 0$

$\underline{x = 0, 1}$ 1

$x_1 = 0\quad x_2 = 1$

x_3: $\dfrac{2}{x-2} = x + 1$

$2 = x^2 - x - 2$

$x^2 - x - 4 = 0$ 1

$x_3 = \dfrac{1 + \sqrt{17}}{2}$

$\therefore$ From graph

$\dfrac{2}{|x-2|} > x + 1$

if $\underline{x < 0,\ 1 < x < \dfrac{1+\sqrt{17}}{2}}$ 1