MATHEMATICS EXTENSION 1 - ASSESSMENT 4 TRIAL EXAMINATION AUGUST 2005

Question 1

(a) $\frac{4}{x-1} \geqslant 1$ $\quad \boxed{x \neq 1}$

$4(x-1) \geqslant (x-1)^2$

$(x-1)^2 - 4(x-1) \leqslant 0$

$(x-1)[x-1-4] \leqslant 0$

$(x-1)(x-5) \leqslant 0$

$1 < x \leqslant 5$

(b) $A(-2,-1)$ $\quad B(1,5)$ $\quad 5:-2$

$x = \frac{5\times 1 - 2\times -2}{5-2}$ $\qquad y = \frac{5\times 5 - 2\times -1}{5-2}$

$= 3$ $\qquad\qquad = 9$

$Q(3,9)$

(c) $f(x) = \tan^{-1}(\sin x)$

$f'(x) = \frac{\cos x}{1+\sin^2 x}$

$f'(\pi) = \frac{\cos \pi}{1+\sin^2 \pi} = -1$

(d) $LHS = \frac{1+\sin x - \cos x}{1+\sin x + \cos x}$

$= \frac{1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}$

$= \frac{1+t^2+2t-1+t^2}{1+t^2+2t+1-t^2}$

$= \frac{2t^2+2t}{2+2t}$

$= \frac{2t(t+1)}{2(1+t)} = t = \tan\frac{x}{2}$

(e) $\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{3-4x^2}} = \int_0^{\sqrt{3}/2} \frac{dx}{2\sqrt{\frac{3}{4}-x^2}}$

$= \frac{1}{2}\left[\sin^{-1}\frac{2x}{\sqrt{3}}\right]_0^{\sqrt{3}/2}$

$= \frac{1}{2}\left(\sin^{-1} 1 - \sin^{-1} 0\right)$

$= \frac{\pi}{4}$

Question 2

(a) $2\sin^2\theta = \sin 2\theta \quad , 0 \leqslant \theta \leqslant 2\pi$

$2\sin^2\theta = 2\sin\theta\cos\theta$

$\sin^2\theta - \sin\theta\cos\theta = 0$

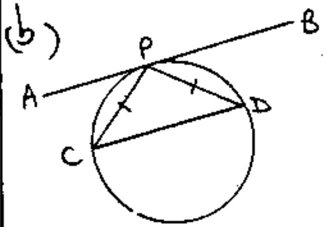
$\sin\theta(\sin\theta - \cos\theta) = 0$

$\sin\theta = 0 \ , \quad \sin\theta = \cos\theta$

$\tan\theta = 1$

$\theta = 0, \pi, 2\pi, \frac{\pi}{4}, \frac{5\pi}{4}$

(b)



$\angle PCD = \angle PDC$ given equal chords ΔPDC isosceles with equal base angle

$\angle BPC = \angle PCD$ angle between tangent + chord = $\angle$ in alternate segment

$\therefore \angle BPC = \angle PDC$

The alternate angles are equal

$\therefore$ AB parallel to CD

(c) (i) $\int \frac{x}{x+9}\,dx = \int \frac{x+9-9}{x+9}\,dx$

$= \int\left(\frac{x+9}{x+9} - \frac{9}{x+9}\right)dx$

$= \int\left(1 - \frac{9}{x+9}\right)dx$

$= x - 9\ln(x+9) + C$

(ii) $\int_0^4 x\sqrt{x^2+9}\,dx$ $\qquad u = x^2+9$, $\frac{du}{dx} = 2x$, $x=0\ u=9$, $x=4\ u=25$

$= \frac{1}{2}\int_0^4 \sqrt{x^2+9} \times 2x\,dx$

$= \frac{1}{2}\int_9^{25} \sqrt{u}\,du$

$= \frac{1}{2}\int_9^{25} u^{\frac{1}{2}}\,du$

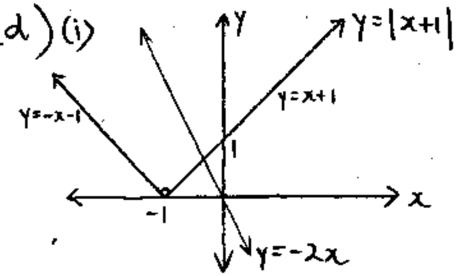
$= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_9^{25}$

$= \frac{1}{3}\left((\sqrt{25})^3 - (\sqrt{9})^3\right)$

$= \frac{1}{3}(125 - 27)$

$= \frac{98}{3}$

(d) (i)



(ii) One pt. of intersection

Solve $x+1 = -2x$

$3x = -1$

$x = -\frac{1}{3}$

$\therefore |x+1| > -2x$ when $x > -\frac{1}{3}$

Question 3

(a) $P(x) = x^3 - kx^2 - x + 2$

(i) $x-1$ is a factor $\therefore P(1) = 0$

$0 = 1 - k - 1 + 2$

$k = 2$

(ii) $P(x) = x^3 - 2x^2 - x + 2$

$= x^2(x-2) - (x-2)$

$= (x-2)(x^2-1)$

$= (x-2)(x-1)(x+1)$

(b) $T_{k+1} = {}^9C_k\left(\frac{2}{x}\right)^k (x^2)^{9-k}$

$= {}^9C_k\, 2^k x^{-k} x^{18-2k}$

$= {}^9C_k\, 2^k x^{18-3k}$

term indept. of x ie x^0

$\therefore 0 = 18 - 3k$

$k = 6$

7^{th} term is independent of x

$T_7 = {}^9C_6\, 2^6$

$= 2^8 \times 3 \times 7$

$= 5376$

c) $f(x) = 4\sin^{-1}(x-2)$

(i) $f(1\tfrac{1}{2}) = 4\sin^{-1}(-\tfrac{1}{2})$

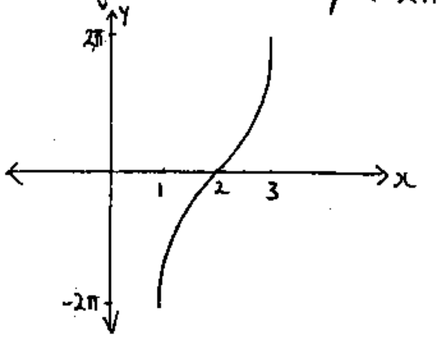
$= 4 \times -\frac{\pi}{6}$

$= -\frac{2\pi}{3}$

(ii) Domain $-1 \leq x-2 \leq 1$

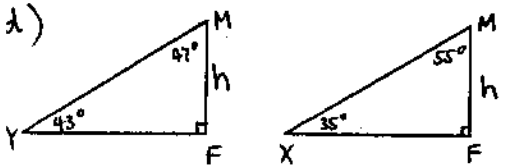
$1 \leq x \leq 3$

Range $-2\pi \leq y \leq 2\pi$



(iii) $\int_1^3 4\sin^{-1}(x-2)\,dx = 0$

d)



In ΔMYF $\quad \tan 47° = \frac{YF}{h}$

$YF = h\tan 47°$

In ΔMXF $\quad \tan 55° = \frac{XF}{h}$

$XF = h\tan 55°$

In ΔYFX

$1200^2 = YF^2 + XF^2$

(Pythagoras)

$1200^2 = h^2\tan^2 47° + h^2\tan^2 55°$

$h^2(\tan^2 47° + \tan^2 55°) = 1200^2$

$h^2 = \frac{1200^2}{(\tan^2 47° + \tan^2 55°)}$

$h = \frac{1200}{\sqrt{\tan^2 47° + \tan^2 55°}}$

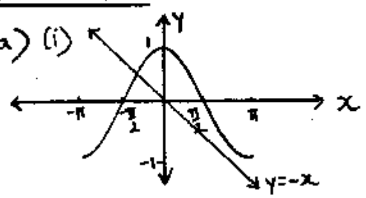
$h = 1200(\tan^2 47° + \tan^2 55°)^{-\frac{1}{2}}$

$h = 671.915\ldots$ m

$h \doteqdot 671.9$ m (to 1 dp)

Question 4

(a) (i)



$\cos x + x = 0 \Rightarrow \cos x = -x$

Consider $y = \cos x$ & $y = -x$

Graphs intersect at one pt. only

$\therefore$ $\cos x + x = 0$ has only one soln.

(ii) $f(x) = \cos x + x$

$f'(x) = -\sin x + 1$

$x_1 = -1$ (in radians)

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= -1 - \frac{\cos(-1) - 1}{1 - \sin(-1)}$

$= -0.75036\ldots$

2nd approximation is

$x \doteqdot -0.75$ (to 2 dp)

b) (i) $V = \pi\int_0^h \frac{9y}{8}\,dy$

$= \frac{9\pi}{8}\left[\frac{1}{2}y^2\right]_0^h$

$= \frac{9\pi}{8} \times \frac{1}{2}h^2 - 0$

$= \frac{9\pi h^2}{16}$

(ii) $\frac{dV}{dt} = 20 \qquad \frac{dV}{dh} = \frac{18\pi h}{16} = \frac{9\pi h}{8}$

Find h when $\frac{1}{2}$ full.

Full $\Rightarrow h = 8$ cm $\quad V = 36\pi$

$\frac{1}{2}$ full $\Rightarrow V = 18\pi = \frac{9\pi h^2}{16}$

$h^2 = 32$

$h = 4\sqrt{2}$.

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$20 = \frac{9\pi h}{8} \times \frac{dh}{dt} \qquad h = 4\sqrt{2}$

$\frac{20 \times 8}{9\pi \times 4\sqrt{2}} = \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{40}{9\pi\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{20\sqrt{2}}{9\pi}$ cm/sec

(c) $2^{3n} - 3^n$ divisible by 5.

Show true for $n=1$

$2^3 - 3^1 = 8 - 3$

$= 5$ which is $\div$ by 5

Assume true for $n=k$

ie assume $2^{3k} - 3^k = 5M$ where

M is a positive integer

Show true for $n = k+1$

$2^{3(k+1)} - 3^{k+1} = 2^{3k+3} - 3^{k+1}$

$= 2^3 \times 2^{3k} - 3 \times 3^k$

$= 2^3(5M + 3^k) - 3 \times 3^k$

using assumption

$= 40M + 8 \times 3^k - 3 \times 3^k$

$= 40M + 5 \times 3^k$

$= 5(8M + 3^k)$ which

is divisible by 5

$\therefore$ true for $n=k+1$ if true for $n=k$

Since true for $n=1$ it is also true for $n = 1+1 = 2$ & thus true for $n = 2+1 = 3$ and so on for all positive integers n.

Question 5

(a) (i) $y = \frac{x^2}{4a} \Rightarrow y' = \frac{x}{2a}$

at $P(2ap, ap^2)$ $\quad m_T = \frac{2ap}{2a} = p$

eq. tangent $\quad y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y - px + ap^2 = 0$

(ii) at Q tangent eq. $y - qx + aq^2 = 0$

$y = px - ap^2 \quad , \quad y = qx - aq^2$

$px - ap^2 = qx - aq^2$

$px - qx = ap^2 - aq^2$

$x(p-q) = a(p-q)(p+q)$

$x = a(p+q) \quad p \neq q$

$y = ap(p+q) - ap^2$

$y = apq$

$R\,(a(p+q),\ apq)$

(iii) $m_{OP} = \dfrac{ap^2 - 0}{2ap - 0} = \dfrac{p}{2}$

Similarly $m_{OQ} = \dfrac{q}{2}$

$OP \perp OQ \quad \therefore m_{OP} \times m_{OQ} = -1$

ie $\dfrac{pq}{4} = -1 \Rightarrow pq = -4$

(iv) at R $\quad x = a(p+q)$

$y = apq$

from (iii) $pq = -4$

$y = -4a$ indept. of $p+q$

$\therefore$ locus of R is $y = -4a$

(b) $f(x) = (1+x)^n$

$$\int_0^1 (1+x)^n\,dx = \left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1$$

$$= \frac{2^{n+1}}{n+1} - \frac{1^{n+1}}{n+1}$$

$$= \frac{2^{n+1}-1}{n+1}$$

Also

$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{r}x^r + \dots$

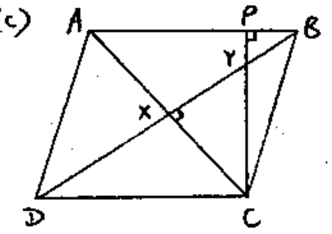
$$\int_0^1 (1+x)^n\,dx = \left[\binom{n}{0}x + \tfrac{1}{2}\binom{n}{1}x^2 + \tfrac{1}{3}\binom{n}{2}x^3 + \tfrac{1}{r+1}\binom{n}{r}x^{r+1} + \dots \tfrac{1}{n+1}\binom{n}{n}x^{n+1}\right]_0^1$$

$$= \binom{n}{0} + \tfrac{1}{2}\binom{n}{1} + \tfrac{1}{3}\binom{n}{2} + \dots + \tfrac{1}{r+1}\binom{n}{r} + \dots + \tfrac{1}{n+1}\binom{n}{n} - 0$$

$$= \sum_{r=0}^{n} \frac{1}{r+1}\binom{n}{r}$$

$$\therefore \sum_{r=0}^{n} \frac{1}{r+1}\binom{n}{r} = \frac{2^{n+1}-1}{n+1}$$

(c)



$\angle BPC = 90°$ given

$\angle BXC = 90°$ given rhombus; diagonals bisect each other at right angles.

Interval BC subtends equal angles on the same side of it at points P and X

$\therefore$ B, C, X, P are concyclic

Question 6

(a) $\int \sin^2 x \cos^2 x \; dx$

$= \int (\sin x \cos x)^2\,dx$

$= \int \left(\tfrac{1}{2}\sin 2x\right)^2 dx$

$\int \sin^2 x \cos^2 x\,dx = \frac{1}{4}\int \sin^2 2x\,dx$

$= \frac{1}{4}\int \frac{1}{2}(1 - \cos 4x)\,dx$

$= \frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + C$

$= \frac{x}{8} - \frac{1}{32}\sin 4x + C$

(b) (i) $\dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 2x^3 - 10x$

$\frac{1}{2}v^2 = \frac{x^4}{2} - 5x^2 + C$

$v = 0$ when $x = -1$

$0 = \frac{1}{2} - 5 + C$

$C = \frac{9}{2}$

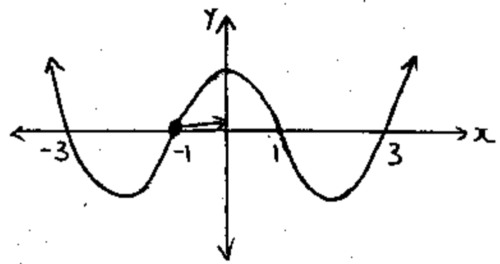
$\frac{1}{2}v^2 = \frac{x^4}{2} - 5x^2 + \frac{9}{2}$

$v^2 = x^4 - 10x^2 + 9$

$v = \pm\sqrt{x^4 - 10x^2 + 9}$

(ii) $v^2 = (x^2 - 9)(x^2 - 1)$

$v^2 = (x-3)(x+3)(x-1)(x+1)$



Starts at $x = -1$ and $v^2 \geq 0$

$\therefore$ only movement from $x = -1$ is to $x = 1$ + back. Exists between

(iii) If $x = 0$ then $a = 0$

With $v = 0$ and $a = 0$ there is no acceleration + no velocity

$\therefore$ particle would not move.

(c) (i) $t = 0 \quad N = 1$

$1 = \dfrac{400}{1 + ke^0}$

$1 + k = 400$

$k = 399$

find t when $N = 200$

$200 = \dfrac{400}{1 + 399e^{-400t}}$

$1 + 399e^{-400t} = 2$

$399e^{-400t} = 1$

$e^{-400t} = \frac{1}{399}$

$-400t = \ln\frac{1}{399}$

$t = -\frac{1}{400}\ln\frac{1}{399}$

$= 0.0149\ldots$ years

$\doteqdot 5.5$ days

(ii) as $t \to \infty$

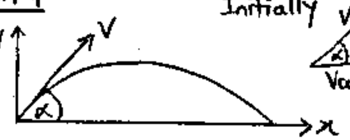
$e^{-400t} \to 0$

$\therefore N \to \dfrac{400}{1+0} = 400$

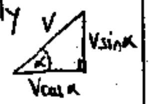
ie all the ants will be infected.

Question 7

(i)



Initially



Horizontal	Vertical
$\ddot{x} = 0$	$\ddot{y} = -g$
$\dot{x} = c_1$	$\dot{y} = -gt + c_3$
when $t=0$ $\dot{x} = V\cos\alpha$	when $t=0$ $\dot{y} = V\sin\alpha$
$\therefore c_1 = V\cos\alpha$	$\therefore c_3 = V\sin\alpha$
$\dot{x} = V\cos\alpha$	$\dot{y} = V\sin\alpha - gt$
$x = Vt\cos\alpha + c_2$	$y = Vt\sin\alpha - \frac{1}{2}gt^2 + c_4$
when $t=0$ $x=0$	when $t=0$ $y=0$
$\therefore c_2 = 0$	$\therefore c_4 = 0$
$x = Vt\cos\alpha$	$y = Vt\sin\alpha - \frac{1}{2}gt^2$

(ii) greatest ht when $\dot{y} = 0$

$$V\sin\alpha = gt$$
$$t = \frac{V\sin\alpha}{g}$$

$$y = V \times \frac{V\sin\alpha}{g}\sin\alpha - \frac{1}{2}g\,\frac{V^2\sin^2\alpha}{g^2}$$
$$= \frac{V^2\sin^2\alpha}{2g}$$

(iii) range $\Rightarrow y = 0$

$$0 = t\left(V\sin\alpha - \frac{1}{2}gt\right)$$
$$t = 0,\ t = \frac{2V\sin\alpha}{g}$$

$$x = V \times \frac{2V\sin\alpha}{g}\cos\alpha$$
$$= \frac{V^2 \times 2\sin\alpha\cos\alpha}{g} = \frac{V^2\sin 2\alpha}{g}$$

(iv) max. range when $\sin 2\alpha = 1$

$$\therefore R = \frac{V^2}{g}$$

$$h_1 = \frac{V^2\sin^2\alpha}{2g} \qquad h_2 = \frac{V^2\sin^2(90-\alpha)}{2g}$$

$$h_1 + h_2 = \frac{V^2}{2g}\left(\sin^2\alpha + \cos^2\alpha\right)$$
$$= \frac{V^2}{2g}$$
$$= \frac{1}{2}R$$

(b) (i) $B_n = 3^2 + 7^2 + \ldots + (4n-1)^2$

n-th term $(4n-1)^2$

(ii) $S_{2n} = A_n - B_n$

$$= 1^2 + 5^2 + \ldots + (4n-3)^2$$
$$- 3^2 - 7^2 - \ldots - (4n-1)^2$$

$$S_{2n} = (1^2-3^2) + (5^2-7^2) + (9^2-11^2) + \ldots + \left[(4n-3)^2 - (4n-1)^2\right]$$

$$= (1-3)(1+3) + (5-7)(5+7) + \ldots + (9-11)(9+11) + \ldots + (4n-3-4n+1)(4n-3+4n-1)$$
$$= -2(4) - 2(12) - 2(20) - \ldots - 2(8n-4)$$
$$= -2\left(4 + 12 + 20 + \ldots + (8n-4)\right)$$
$$= -2\left(\frac{n}{2}\left(8 + (n-1)8\right)\right)$$
$$= -n(8 + 8n - 8)$$
$$= -8n^2$$

(iii) $101^2 - 103^2 + 105^2 - \ldots + 2001^2 - 2003^2$

starts from $n=26$ to $n=501$

$$S_{501} - S_{25} = -8(501)^2 + 8(25)^2$$
$$= -2003008$$