Catholic

SUGGESTED SOLUTIONS TO MATHEMATICS CSSA TRIAL 2003

Question 1

(a) $\dfrac{2.1^2 \times 4.5^2}{2.1^2 + 4.5^2} = \dfrac{89.3}{24.7}$

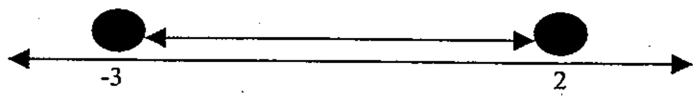
$= 3.6$

(b) $128x - 16x^4 = 16x(8 - x^3)$

$= 16x(2 - x)(4 + 2x + x^2)$

(c) $|2x + 1| \le 5$

$2x + 1 \le 5$	$-2x - 1 \le 5$
$2x \le 5 - 1$	$-2x \le 5 + 1$
$2x \le 4$	$-2x \le 6$
$x \le 2$	$x \ge -3$



(d) $\dfrac{\sqrt{5}}{3\sqrt{2} - 1} = \dfrac{\sqrt{5}}{3\sqrt{2} - 1} \times \dfrac{3\sqrt{2} + 1}{3\sqrt{2} + 1}$

$= \dfrac{3\sqrt{10} + \sqrt{5}}{18 - 1}$

$= \dfrac{3\sqrt{10} + \sqrt{5}}{17}$

(e) $\tan\dfrac{\pi}{3} + \operatorname{cosec}\dfrac{\pi}{4}$

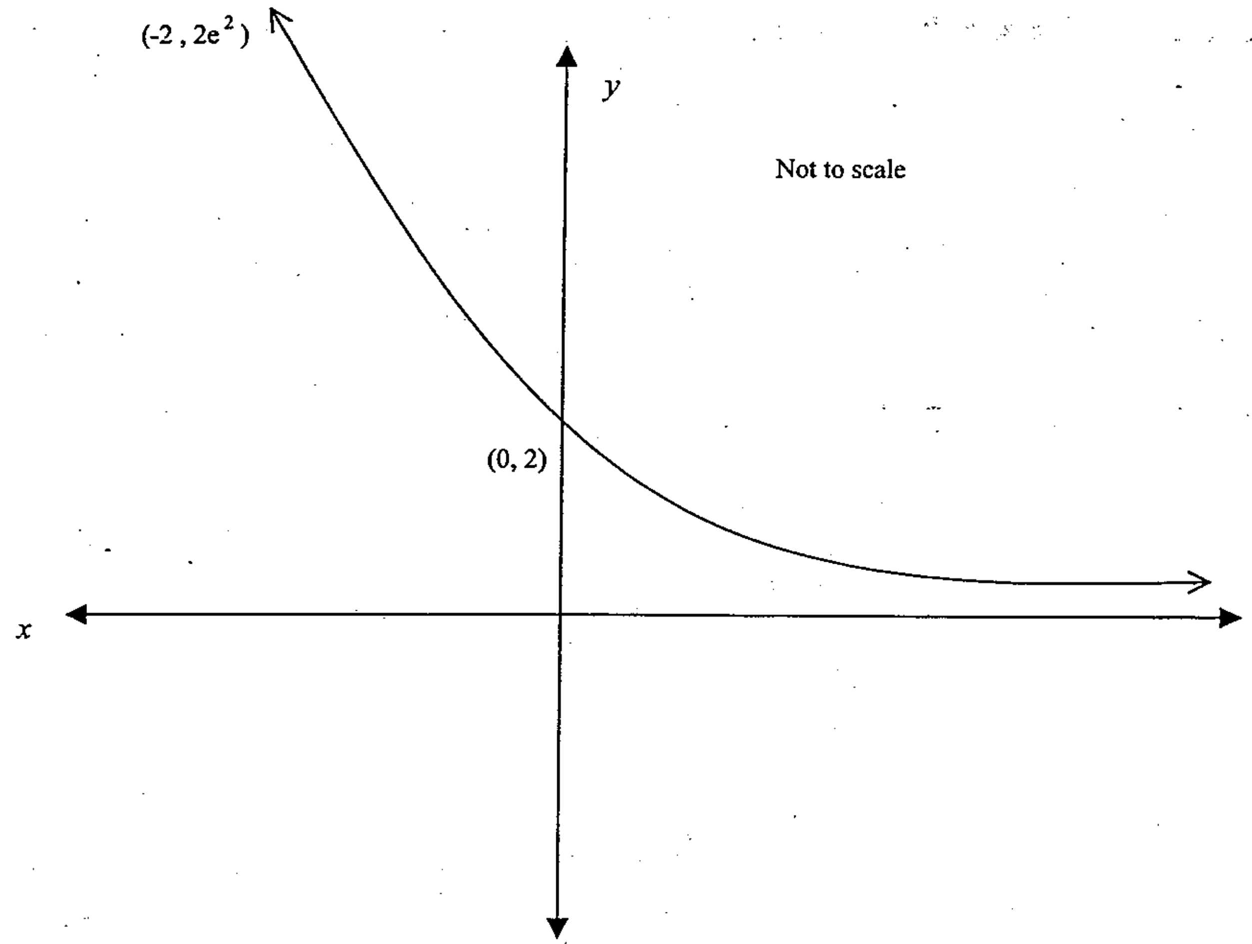
Using the exact triangles $\qquad \tan\dfrac{\pi}{3} = \sqrt{3}$

$\operatorname{cosec}\dfrac{\pi}{4} = \dfrac{1}{\sin\frac{\pi}{4}} = \dfrac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$

$\tan\dfrac{\pi}{3} + \operatorname{cosec}\dfrac{\pi}{4} = \sqrt{3} + \sqrt{2}$

Question 1 (continued)

(f)



Notice the curve passes through the y-axis at the point (0, 2)
As $x \to \infty$ $y \to 0$

Question 2

(a) A quadratic function has real roots when $b^2 - 4ac \geq 0$.

$x^2 - (k+2)x + 4 = 0$ a = 1; b = $-k-2$; c = 4

$(-k-2)^2 - 4\times1\times4 \geq 0$

$k^2 + 4k + 4 - 16 \geq 0$

$k^2 + 4k - 12 \geq 0$

$(k+6)(k-2) \geq 0$ $\therefore$ the quadratic has real roots when

$k \leq -6$ *and* $k \geq 2$

(b) (i) Point A is where L_1 intersects the y axis A (0, 2)

Point C is where L_2 intersects the y axis C (0, -4)

(ii) Can solve equations L_1 and L_2 simultaneously or show that the point R(3, -1) satisfies L_1 and L_2 by direct substitution .

Solving simultaneously

L_1 $x + y = 2$

L_2 $x - y = 4$

Adding L_1 and L_2

$2x = 6$

$x = 3$ Substituting $x = 3$ into L_1 or L_2 results in $y = -1$ $\therefore$ R(3, -1)

(iii) Line SR is parallel to the y axis and passes through the point R(3, -1)

$\therefore$ Line SR has equation $x = 3$

(iv) Line L_1 has equation $x + y = 2$ $m = = \frac{-a}{b} = -1$

Using the gradient formula with two points or even $m = \frac{rise}{run}$ with the diagram will also generate the answer to the gradient as -1.

(v) Using $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ with A(0, 2), R(3, -1)

$d = \sqrt{(3-0)^2 + (-1-2)^2}$

$d = \sqrt{9+9}$

$d = \sqrt{18} = 3\sqrt{2}$ units

Question 2 (continued)

(vi) From above (iv) the gradient of L_1 =-1

Line L_2 has equation $x - y = 4$. Using $m = \frac{-a}{b}$ or other methods it can be seen that the gradient of line L_2 is 1.

As $m_{L1} \times m_{L2} = -1$ ΔARC is a right angled triangle.

We know from above (v) the distance of AR = $3\sqrt{2}$ units.
Finding the distance of CR, using the distance formula
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ with C(0, -4) and R(3, -1)

$d = \sqrt{(3-0)^2 + (-1+4)^2}$
$d = \sqrt{9+9}$
$d = \sqrt{18} = 3\sqrt{2}$ units.

The distance of AC along the y axis is 6 units.
As two sides of ΔARC are equal the triangle is isosceles.
$\therefore$ ΔARC is an isosceles, right angled triangle.

(vii) Centre (3, -1) and radius $3\sqrt{2}$ units.

The equation of the circle is $(x-3)^2 + (y+1)^2 = (3\sqrt{2})^2$
$(x-3)^2 + (y+1)^2 = 18$ Or $x^2 - 6x + y^2 + 2y - 8 = 0$

Question 3

(a) (i) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$

$= \frac{1}{2}x^{\frac{-1}{2}}$

$= \frac{1}{2\sqrt{x}}$

(ii) $\frac{d}{dx}\left(x^3 e^{-3x}\right) = \left(x^3\right)\left(-3e^{-3x}\right) + \left(3x^2\right)\left(e^{-3x}\right)$

$= 3x^2 e^{-3x}(1-x)$

(iii) $\frac{d}{dx}\left(\frac{\tan x}{2x+1}\right) = \frac{(2x+1)(\sec^2 x) - (\tan x)(2)}{(2x+1)^2}$

$= \frac{2x\sec^2 x + \sec^2 x - 2\tan x}{(2x+1)^2}$

(b) $\int \frac{e^{2x}}{e^{2x}+4}\, dx = \frac{1}{2}\ln\left(e^{2x}+4\right) + C$

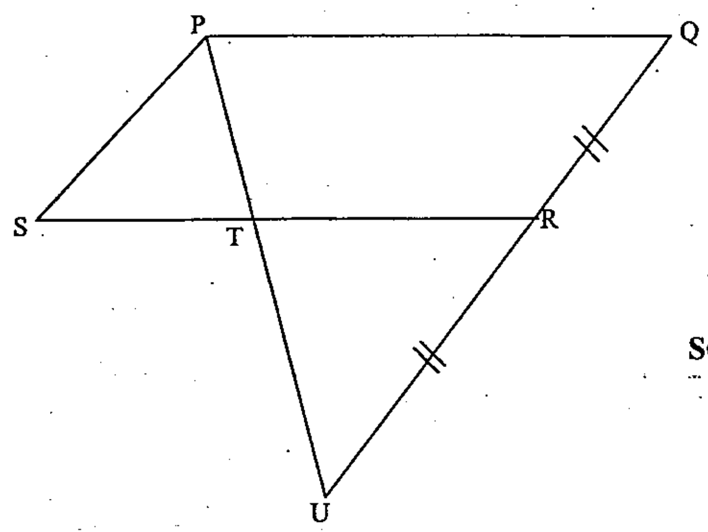
(c) $\int_0^{\frac{\pi}{4}} \frac{1}{2}x + \cos 2x \ dx = \left[\frac{x^2}{4} + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}$

$= \left(\frac{\frac{\pi^2}{16}}{4} + \frac{1}{2}\right) - (0)$

$= \frac{\pi^2 + 32}{64}$

Question 3 (continued)

(d)



NOT TO SCALE

(i) In triangles PST and URT:

PS = RU (RU = QR (given) and PS = QR, opposite sides of parallelogram PQRS.)

$\angle$PST = $\angle$URT(Alternate angles are equal PS || QU)

$\angle$PTS = $\angle$RTU (Vertically opposite angles are equal)

$\therefore \Delta$PTS $\equiv \Delta$URT (two angles and one side)

(ii) Since ΔPST $\equiv \Delta$URT, ST = TR because corresponding sides in congruent triangles are equal.

$\therefore$ T is the midpoint of SR.

Question 4

(a) $\sum_{k=4}^{20} 2k-5 = 3+5+7+\dots\dots\dots+35$

This represents an Arithmetic series with $a=3$, $l=35$ and $n=17$.

Using $S_n = \frac{n}{2}(a+l)$ $\qquad S_{17} = \frac{17}{2}(3+35)$

$= 323$

(b) In a Geometric series, $T_n = ar^{n-1}$ so :

$ar^2 = \frac{3}{4}$

$ar^6 = 12$

Solving simultaneously gives : $\frac{ar^6}{ar^2} \quad \frac{12}{\frac{3}{4}}$

$\therefore r^4 = 16$ $\qquad$ so, $\quad r = \pm 2$

For both $r = 2$ and $r = -2$, $a = \frac{3}{16}$

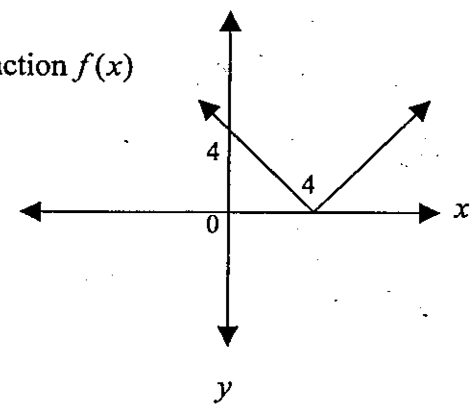
The fourteenth term of the series, $\quad T_{14} = ar^{13}$

$= \frac{3}{16}(\pm 2)^{13}$

$= \pm 1536$

(c) (i)

The required function $f(x)$



(ii) $\int_0^6 f(x)\ dx$ = Area under the curve between $x=0$ and $x=6$

$= (\frac{1}{2}\times 4\times 4) + \left(\frac{1}{2}\times 2\times 2\right) = 10$

(d) $\sqrt[x]{m} = n^3$

$\therefore m = n^{3x}$

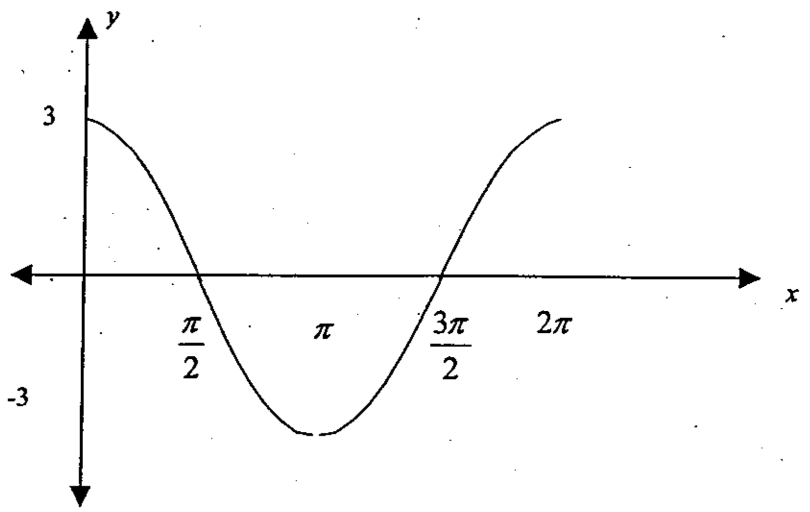
$\log m = \log n^{3x}$

$\log m = 3x \log n$

$x = \frac{\log m}{3\log n}$

Question 4 (continued)

(e)



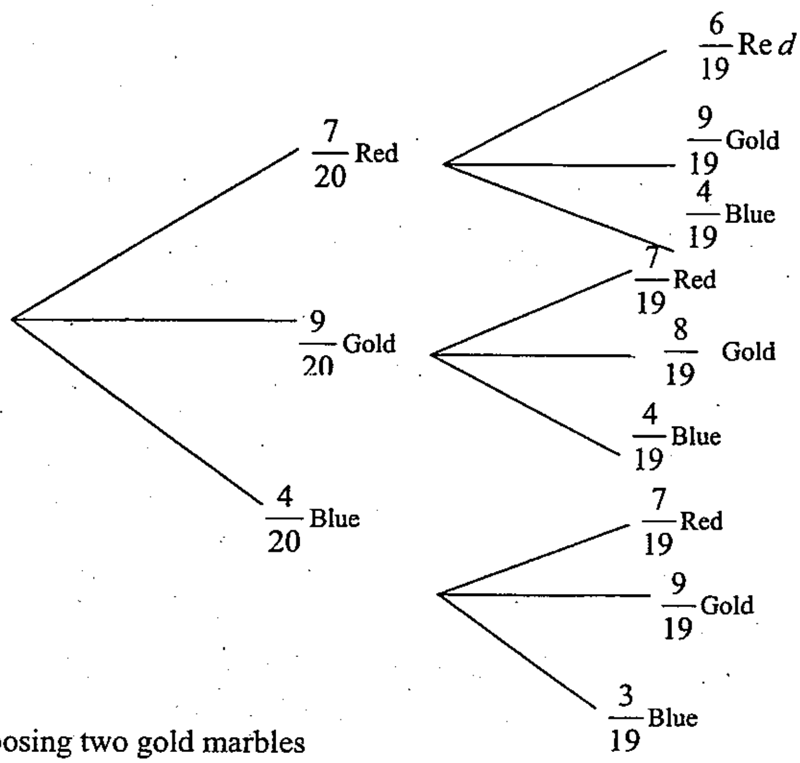
$y = 3cosx$

(f) $\frac{dP}{dt} > 0$ because the function is increasing

$\frac{d^2P}{dt^2} < 0$ because the function is increasing at a *decreasing* rate

Question 5

(a)



(i) Probability of choosing two gold marbles

$$P(\text{Gold, Gold}) = \frac{9}{20}\times\frac{8}{19}=\frac{18}{95}$$

(ii) Probability of choosing marbles of different colour

P(marbles with different colour) = 1 – P(Same colour)

$$= 1-\left[\left(\frac{7}{20}\times\frac{6}{19}\right)+\left(\frac{9}{20}\times\frac{8}{19}\right)+\left(\frac{4}{20}\times\frac{3}{19}\right)\right]$$

$$= \frac{127}{190}$$

(b) $y = 6x^2 - x^3$

(i) Stationary points occur when $\frac{dy}{dx}=0$

$$\frac{dy}{dx}=12x-3x^2=0$$

$$3x(4-x)=0$$

Stationary points occur when $x=0$ and $x=4$

Stationary points are (0, 0) and (4, 32)

Question 5 (continued)

(ii) To determine nature of the stationary points we can use $\frac{d^2y}{dx^2} = 12 - 6x$

When $x = 0$ $\frac{d^2y}{dx^2} = 12$ $\frac{d^2y}{dx^2} > 0$ (Minimum turning point at $x = 0$)

When $x = 4$ $\frac{d^2y}{dx^2} = -12$ $\frac{d^2y}{dx^2} < 0$ (Maximum turning point at $x = 4$)

Minimum turning point (0, 0), maximum turning point (4, 32)

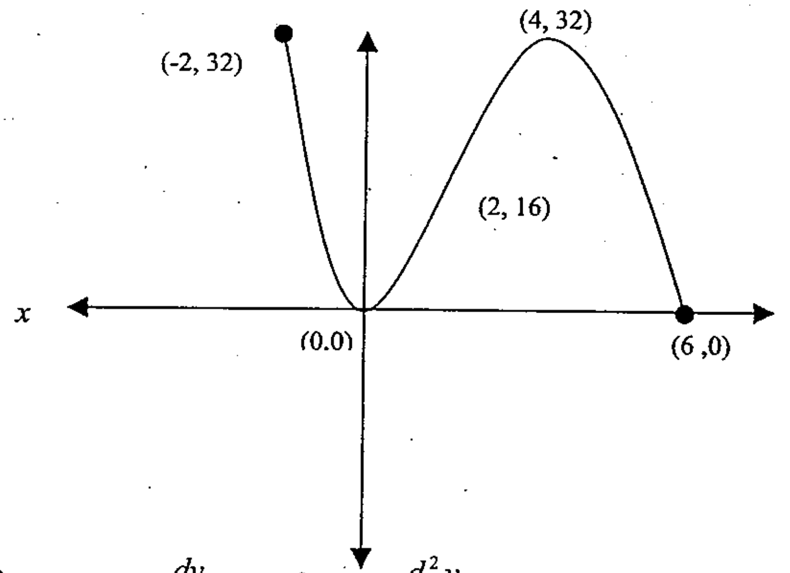
(iii) Point of inflexion when $\frac{d^2y}{dx^2} = 0$ and concavity changes.

$$\frac{d^2y}{dx^2} = 12 - 6x = 0$$
$$6x = 12$$
$$x = 2$$

Change in concavity and point of inflexion when $x = 2$
Point of inflexion is (2 , 16)

x	<2	2	>2
$\frac{d^2y}{dx^2}$	+	0	-

(iv)



(c) If $y = 3e^{-2x}$ then $\frac{dy}{dx} = -6e^{-2x}$ and $\frac{d^2y}{dx^2} = 12e^{-2x}$

So, $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 2(12e^{-2x}) + 3(-6e^{-2x}) - 2(3e^{-2x})$
$= 24e^{-2x} - 18e^{-2x} - 6e^{-2x}$
$= 0$ as required.

Question 6

(a) $y = x \sin x$ $\qquad \frac{dy}{dx} = x \cos x + \sin x$

When $x = \frac{\pi}{2}$ $\qquad m_T = 1 \qquad \therefore m_N = -1$

$y = \frac{\pi}{2}$

$\therefore$ The equation of the normal at $(\frac{\pi}{2}, \frac{\pi}{2})$ is

$$y - \frac{\pi}{2} = -1\,(x - \frac{\pi}{2})$$
$$y - \frac{\pi}{2} = -x + \frac{\pi}{2}$$
$$x + y - \pi = 0$$

(b) Using Simpson's rule with h = 1

$$\int_0^4 f(x)\ dx \approx \frac{h}{3}[f(0) + 4\{f(1) + f(3)\} + 2\{f(2)\} + f(4)]$$
$$\approx \frac{1}{3}[2 + 4\{3 + 35\} + 2\{12\} + 80]$$
$$\approx 86$$

(c) (i) Solving simultaneously to find the points of intersection between $y = x^2$ and $y = 3x + 4$.

$$x^2 = 3x + 4$$
$$x^2 - 3x - 4 = 0$$
$$(x+1)(x-4) = 0$$
$$x = -1,\ x = 4$$

$\therefore$ At A, x = -1 and at B, x = 4

(ii) Area = $\int_{-1}^{4} 3x + 4\ dx - \int_{-1}^{4} x^2\ dx$

$$= \left[\frac{3x^2}{2} + 4x - \frac{x^3}{3}\right]_{-1}^{4}$$

$$= \left(\frac{3\times 4^2}{2} + 4\times 4 - \frac{4^3}{3}\right) - \left(\frac{3\times -1^2}{2} + 4\times -1 - \frac{-1^3}{3}\right)$$

$$= 18\frac{2}{3} + 2\frac{1}{6}$$

$$= 20\frac{5}{6} \text{ units}^2 \qquad (20.8\dot{3} \text{ units}^2)$$

Question 6 (continued)

(d) $\text{Volume} = V = \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} y^2 \, dx \quad = \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cot x \, dx$

$$= \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos x}{\sin x} \, dx$$

$$= \pi [\ln(\sin x)]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \pi \left[\ln(\sin \frac{\pi}{3}) - \ln(\sin \frac{\pi}{4}) \right]$$

$$= \pi \left[\ln(\frac{\sqrt{3}}{2}) - \ln(\frac{1}{\sqrt{2}}) \right]$$

$$= \pi \left[\ln \frac{\sqrt{6}}{2} \right] \text{ units}^3.$$

Question 7

(a) $\log_e\left(\dfrac{2x+1}{3x-7}\right) = \log_e(2x+1) - \log_e(3x-7)$

$$\frac{dy}{dx} = \frac{2}{2x+1} - \frac{3}{3x-7}$$

(b) (i) $x = 3t - 2\ln(1+t) + c$ *where c is a constant*

Since the particle is initially 1 metre to the right of the origin, when $t = 0$, $x = 1$

$\therefore \quad 1 = 3t - 2\ln(1+t) + c$

$\therefore \quad c = 1$

$\therefore \quad x = 3t - 2\ln(1+t) + 1$

(ii) Since $\dfrac{2}{1+t}$ can never be 0, v will never be 3.

(iii) $v = 3 - 2(1+t)^{-1}$

$$a = \frac{dv}{dt}$$

$$a = 2(1+t)^{-2}$$

$$a = \frac{2}{(1+t)^2}$$

When $t = 2$ seconds, $a = \dfrac{2}{(1+2)^2} = \dfrac{2}{9}$ m/s^2

(c) (i)
$$\begin{aligned}
\text{LHS} &= (\text{cosec}^2 A - 1)\sin^2 A \\
&= \left(\frac{1}{\sin^2 A} - 1\right)\sin^2 A \\
&= 1 - \sin^2 A \\
&= \cos^2 A \\
&= \text{RHS}
\end{aligned}$$

(ii)
$$\begin{aligned}
(\text{cosec}^2 A - 1)\sin^2 A &= \tfrac{3}{4} \\
\cos^2 A &= \tfrac{3}{4} \\
\cos A &= \pm\frac{\sqrt{3}}{2} \qquad -\pi \le A \le \pi \\
A &= \pm\frac{\pi}{6},\ \pm\frac{5\pi}{6}.
\end{aligned}$$

Question 8

(a) (i) 6 % p.a = 1.5 % per quarter.

After 1 quarter $A_1 = 480000(1 + \frac{1.5}{100})^1 - \$P.$

$= \$487\,200 - \$P$

(ii) $A_1 = \$487\,200 - \$P$

$A_2 = A_1 \times (1 + \frac{1.5}{100}) - \$P.$

$A_2 = \$480000(1.015)^2 - \$P(1 + 1.015)$

$A_3 = A_2(1 + \frac{1.5}{100}) - \$P.$

$A_3 = \$480000(1.015)^3 - \$P(1 + 1.015 + 1.015^2)$

(iii) 20 years = 80 repayments.

Pattern continues.....

$A_{80} = \$480000(1.015)^{80} - \$P(1 + 1.015 + 1.015^2 + 1.015^3 + \ldots\ldots 1.015^{79})$

$A_{80} = 0$ (Loan repaid)

$\$480000(1.015)^{80} - \$P(1 + 1.015 + 1.015^2 + 1.015^3 \ldots\ldots\ldots + 1.015^{79}) = 0$

$\$480000(1.015)^{80} = \$P(1 + 1.015 + 1.015^2 + 1.015^3 \ldots\ldots\ldots + 1.015^{79})$

$$\$P = \frac{\$480000(1.015)^{80}}{(1 + 1.015 + 1.015^2 + 1.015^3 + \ldots\ldots 1.015^{79})}$$

The denominator is the sum of a geometric series where

a = 1 r = 1.015 n = 80 $\therefore$ $S = \frac{1(1.015^{80} - 1)}{1.015 - 1}$

$$\$P = \frac{\$480000(1.015)^{80}}{\frac{1(1.015^{80} - 1)}{1.015 - 1}}$$

$\$P = \$\,10\,343.20$ (nearest cent)

(b) (i) Since the volume is changing at a rate proportional to the present volume, $\frac{dV}{dt} = kV$

and $V = V_0 e^{-kt}$ can be used.

Since initial volume is 1 000 L, $V_0 = 1\,000$.

$\therefore V = 1\,000\, e^{-kt}$

When $t = 40$ minutes, $V = 800$ L, so :

$800 = 1\,000\, e^{-40k}$

$0.8 = e^{-40k}$

$\ln 0.8 = \ln e^{-40k}$

$k = \frac{\ln 0.8}{-40} = 5.5786 \times 10^{-3}$

When $t = 60$, $V = 1\,000\, e^{-60k}$

$(k = 5.5786 \times 10^{-3})$

$V = 715.54175\ldots$..litres

$V = 716$ L (to the nearest litre)

Question 8 (continued)

(ii) When $V = 1$ then $1\,000e^{-kt} = 1 \quad (k = 5.5786 \times 10^{-3})$

$$e^{-kt} = 0.001$$

$$\ln e^{-kt} = \ln 0.001$$

$$kt = \ln 0.001$$

$$t = 1238.2621\ldots \text{ minutes}$$

The storage tank will reach the last litre after 20 hours and 38 minutes.

(c) (i) A limiting sum exists as $|r| < 1$

$$r = \sin^2 x \qquad 0 < x < \frac{\pi}{2}$$

Note $|\sin^2 x| < 1$ does hold and a limiting sum exists.

(ii) Using $S = \dfrac{a}{1-r}$ where $|r| < 1$ $\quad r = \sin^2 x\,, \ a = \sin^2 x$

$$\therefore \quad S = \frac{\sin^2 x}{1 - \sin^2 x}$$

$$\therefore \quad S = \frac{\sin^2 x}{\cos^2 x}$$

$$\therefore \quad S = \tan^2 x$$

Question 9

(a) (i) The area of a sector is given by $Area = A = \frac{1}{2}r^2\theta$

The perimeter of the sector is given by $r + r + r\theta$ (Where length of arc $= r\theta$)
The perimeter is given to be 375 metres.

$$\therefore 2r + r\theta = 375$$

$$\theta = \frac{375 - 2r}{r}$$

Substituting $\theta = \frac{375-2r}{r}$ into $Area = A = \frac{1}{2}r^2\theta$

gives $A = \frac{1}{2}r^2(\frac{375-2r}{r})$

$$A = \frac{r}{2}(375 - 2r)$$

(ii) Greatest Area occurs when $\frac{dA}{dr} = 0$ and $\frac{d^2A}{dr^2} < 0$

$$\frac{dA}{dr} = \frac{375}{2} - 2r = 0 \qquad r = 93.75 \text{ metres}$$

$$\frac{d^2A}{dr^2} = -2$$

As $\frac{d^2A}{dr^2} < 0 \therefore$ maximum area occurs when $r = 93.75$ metres.

Maximum area is $A = \frac{93.75}{2}(375 - 2 \times 93.75)$

$= 8789.06 \; m^2$ (2 decimal places)

(iii) The maximum area is 8789.06 m^2

Using $Area = \frac{1}{2}r^2\theta$

$8789.06 = \frac{1}{2} \times 93.75^2 \times \theta$

$\theta =$ 2 radians.

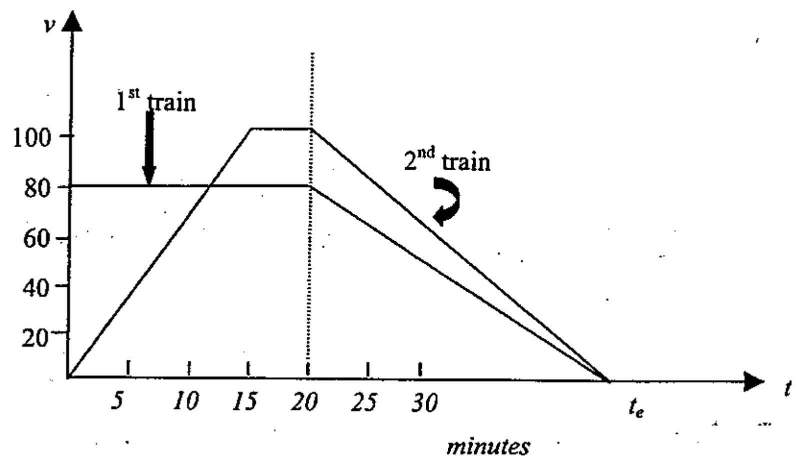
$\theta = 115^\circ$ (nearest degree)

This is the angle required to produce max area from (ii).

The maximum area found in part (ii) created with a radius of 93.75 m would not be possible with an angle less than 110°. θ is required to be 2 radians. (115° to the nearest degree).

Question 9 (continued)

(b) (i)



Let t_e be the time at which the trains stop at the next station.
Since both trains cover the same distance, the areas between each velocity-time graph and the time axis are equal. We use this to find the time taken for the journey, t_e.

$$1\,600 + 40\,t_e - 800 = \tfrac{1}{2}(20 + 5) \times 100 + \tfrac{1}{2} \times 100(t_e - 20)$$
$$10\,t_e = 550$$
$$t_e = 55 \text{ minutes}$$

(ii) Converting time to hours because velocity is measured in km/h we can calculate the distance as the area under either velocity graph.

$$\text{Distance between the two stations} = 80 \times \frac{20}{60} + \tfrac{1}{2} \times 80 \times \left(\frac{55-20}{60}\right)$$

$$= 50 \text{ kilometres.}$$

The stations are 50 kilometres apart.

Question 10

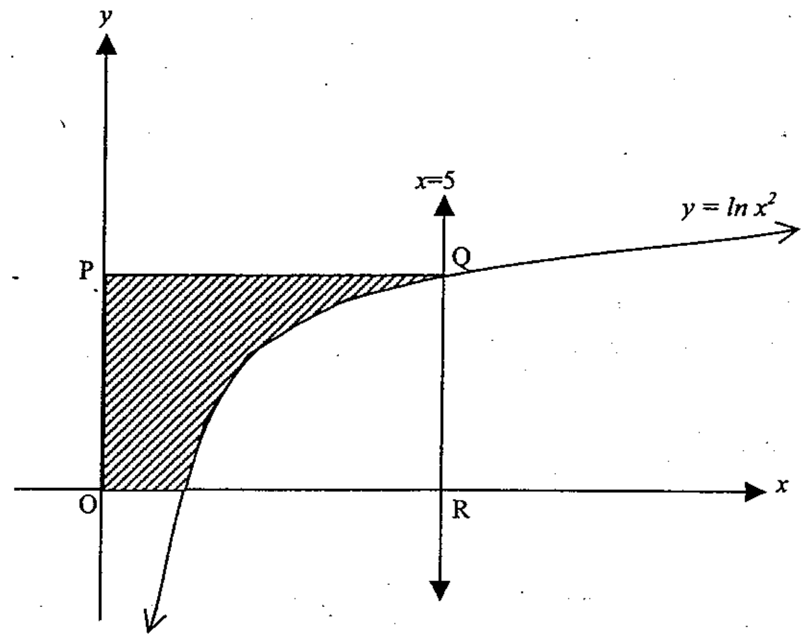
(a) (i) $\frac{d}{dx}(x\ln x - x) = x.\frac{1}{x} + 1.\ln x - 1$

$= 1 + \ln x - 1$

$= \ln x$ as required.

(ii) $\ln x^2 = 2 \ln x$

$\therefore$ a primitive of $2 \ln x$ is $2\ (x \ln x - x)$ [+ a constant]



(iii) The shaded area in the diagram is found by taking the area under the curve $y = \ln x^2$ between the line $x = 5$ and the x-axis from the area of the rectangle OPQR.

Area OPQR : P has co-ordinates (0, ln25)

$\therefore$ the rectangle has dimensions $5 \times \ln 25$

Area OPQR $= 5 \ln 25$

$= 5 \ln 5^2$

$= 10 \ln 5$ units^2

The curve crosses the x-axis at (0, 1)

Area under the curve is found by evaluating $\int_1^5 \ln x^2 dx$

$\int_1^5 \ln x^2 dx = 2[x \ln x - x]_1^5$

$= 2\ \{5 \ln 5 - 5 - (\ln 1 - 1)\}$

$= 10 \ln 5 - 8$

Shaded area $= 10 \ln 5 - (10 \ln 5 - 8) = 8\ units^2$

Question 10 (continued)

(b) (i) $f(x) = e^{-x}\cos x \quad 0 \le x \le 2\pi$

Stationary points occur when $f'(x) = 0$

$$f'(x) = (e^{-x})(-\sin x) + (-e^{-x})(\cos x) = 0$$
$$= -e^{-x}(\sin x + \cos x) = 0$$

Stationary points occur when $\sin x = -\cos x$

$\therefore$ when $\tan x = -1$

$$x = \frac{3\pi}{4} \ , \ \frac{7\pi}{4}$$

(ii) To determine nature of the stationary points we can use a before and after test with the first derivative.

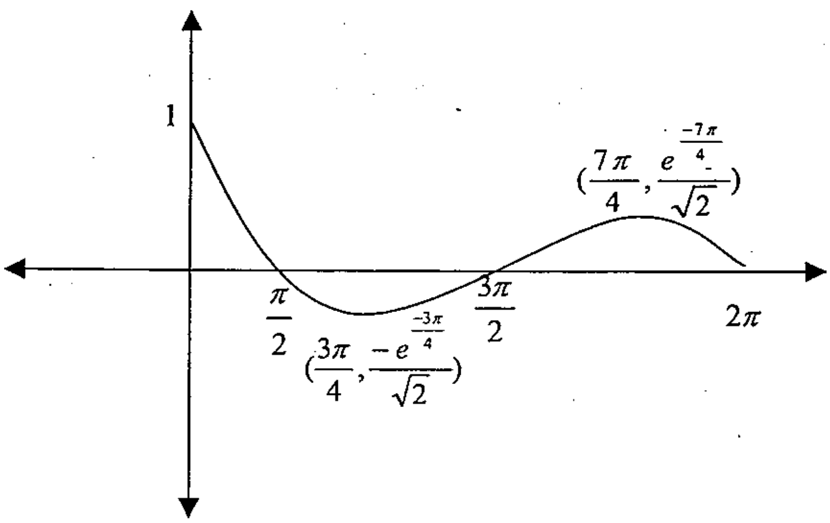
x	$\frac{3\pi}{4} - \varepsilon$	$\frac{3\pi}{4}$	$\frac{3\pi}{4} + \varepsilon$
$f'(x)$	-	0	+

Minimum turning point at $x = \frac{3\pi}{4}$

x	$\frac{7\pi}{4} - \varepsilon$	$\frac{7\pi}{4}$	$\frac{7\pi}{4} + \varepsilon$
$f'(x)$	+	0	-

Maximum turning point at $x = \frac{7\pi}{4}$

(iii)



Question 10 (continued)

(iv) The equation $e^{-x}\cos x - \frac{1}{2}x = 0$ can be solved graphically

Sketching $f(x) = e^{-x}\cos x$ and $f(x) = \frac{1}{2}x$

As $e^{-x}\cos x = \frac{1}{2}x$. The curve and the line intersect at only one point.

$\therefore$ One solution exists for the equation $e^{-x}\cos x - \frac{1}{2}x = 0$ $(0 \le x \le 2\pi)$