CSSA - NSW

4 Unit Mathematics

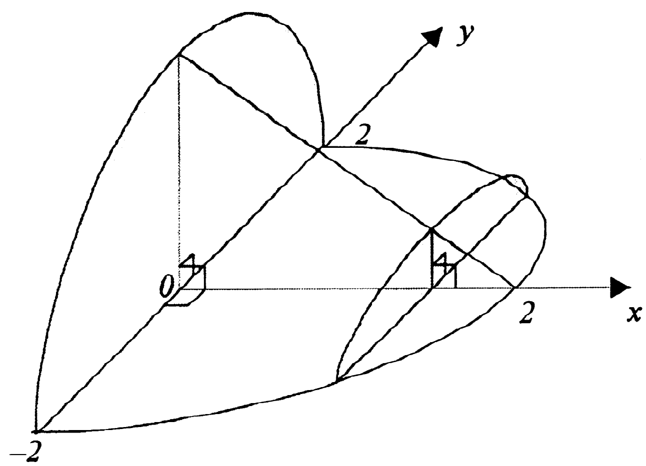
Trial HSC Examination 1987

1. **(i)** **(a)** Obtain the equation of the tangent to the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ at the point $P(x_1, y_1)$ on the curve.
(b) This tangent meets the coordinate axes at Q and R. Show that $OQ + OR = a$ for all positions of P, where O is the origin.
(ii) The function $f(x)$ is given by $f(x) = \frac{4(2x-7)}{(x-3)(x+1)}$.
(a) Express $f(x)$ in partial fractions.
(b) Sketch the graph $y = f(x)$ showing clearly the coordinates of any points of intersection with the x-axis and the y-axis, the coordinates of any turning points and the equations of any asymptotes. (There is no need to investigate points of inflection).
(c) Determine the area of the region bounded by the curve, the x-axis, and the lines $x = 4$ and $x = 6$, expressing your answer as a single logarithm.

2. **(i)** The function $f(x)$ is given by $f(x) = x - \ln(1 + x^2)$.
(a) Show that $f'(x) \geq 0$ for all values of x.
(b) Deduce that $f(x) > 0$ for all positive values of x.
(ii) Find $\int \frac{\sin^3 \theta}{\cos^4 \theta} \, d\theta$.
(iii) **(a)** Show that $\int_{-a}^{0} f(x) \, dx = \int_0^a f(-x) \, dx$.
(b) Deduce that $\int_{-a}^{a} f(x) \, dx = \int_0^a [f(x) + f(-x)] dx$.
(c) Hence evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x} \, dx$.
(iv) Use integration by parts to evaluate $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1+x}} \, dx$.

3. **(i)** The quadratic equation $z^2 + (1 + i)z + k = 0$ has a root $1 - 2i$. Find, in the form $a + ib$, the value of k and the other root of the equation.
(ii) The complex number z satisfies $\arg(z + 3) = \frac{\pi}{3}$.
(a) Sketch the locus of the point P in the argand diagram which represents z.
(b) Find the modulus and argument of z when $|z|$ takes its least value.
(c) Hence find, in the form $a + ib$, z for which $|z|$ is a minimum.

(iii)



A solid figure has a semicircular base of radius 2. Cross sections taken at right angles to the semi-diameter of this base are semi-ellipses. The vertical cross section containing the semi-diameter of the base is a right isosceles triangle as shown.

(a) Given that the area of an ellipse with semi-axes a and b is πab, show that the volume of the solid is given by $V = \frac{\pi}{2}\int_0^2 (2-x)\sqrt{4-x^2}\,dx$.

(b) Hence find the volume of the solid.

4. **(i)** **(a)** Show that the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a^2 > b^2)$ at the point $P(x_1, y_1)$ has equation $a^2y_1x - b^2z_1y = (a^2 - b^2)x_1y_1$.

(b) This normal meets the major axis of the ellipse at G. S is one focus of the ellipse. Show that $GS = e.PS$ where e is the eccentricity of the ellipse.

(ii) **(a)** By using the result in part **(i)** **(a)** above, or otherwise, show that the normal to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at the point $P(5\cos\theta, 3\sin\theta)$ has equation $5x\sin\theta - 3y\sin\theta = 16\sin\theta\cos\theta$.

(b) This normal cuts the major and minor axes of the ellipse at G and H respectively. Show that as P moves on the ellipse the mid point of GH describes another ellipse with the same eccentricity as the first.

(c) On the same axes sketch the two ellipses showing clearly the coordinates of the intercepts on the coordinate axes.

5. **(i)** The transformation $w = (z+1)^2 + 3$ maps the complex number $z = x + iy$ to the complex number $w = u + iv$.

(a) Show that as z moves along the y-axis from the origin to the point $(0, 2)$ in the z-plane, w moves from the point $(4, 0)$ to the point $(0, 4)$ along a curve in the w plane.

(b) Find the Cartesian equation of this curve.

(ii) **(a)** Write down the general solution of $\tan 4\theta = 1$.

(b) Use De Moivre's theorem to express $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$. Hence show that $\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$.

(c) Find the roots of the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ in trigonometric form. Hence show that $\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16} = 28$.

6. A particle of mass m is projected vertically upwards under gravity with initial speed v_0. Air resistance to its motion has magnitude mkv^2 where v is the speed of the particle and k is a constant.
(i) (a) Draw a diagram to show the forces acting on the particle during this upward motion. Hence write down the equation of motion of the particle during its upward motion.
(b) Show that the greatest height attained is $\frac{1}{2k}\ln\left(\frac{g+kv_0^2}{g}\right)$
(ii) The speed of the particle is v when it has fallen a distance y from its maximum height.
(a) Draw a diagram to show the forces acting on the particle during this downward motion. Hence write down the equation of motion of the particle during its downward motion.
(b) Show that $y = \frac{1}{2k}\ln\left(\frac{g}{g-kv^2}\right)$.
(c) Deduce that these is an upper bound for v. Briefly discuss the significance of this.
(iii) The speed of the particle is v_1 when it returns to its point of projection. Show that $(g+kv_0^2)(g-kv_1^2) = g^2$.

7. **(i)** By considering the stationary values of $f(x) = x^3 - 3px^2 + 4q$, where p and q are positive real constants, show that the equation $f(x) = 0$ has three distinct real roots if and only if $p^3 > q$.
(ii) (a) Show that $\frac{x^4+x^2+1}{x^2} \geq 3$ for all real values of x.
(b) State the range of the functions $y = \tan^{-1}(\frac{1}{1+x^2})$ and $y = \tan^{-1}(\frac{x^2}{1+x^2})$.
(c) Show that $\tan^{-1}(\frac{1}{1+x^2}) + \tan^{-1}(\frac{x^2}{1+x^2}) = \tan^{-1}(1 + \frac{x^2}{1+x^2+x^4})$. Hence determine the range of the function $y = \tan^{-1}(\frac{1}{1+x^2}) + \tan^{-1}(\frac{x^2}{1+x^2})$.
(d) Hence sketch the graph of $y = \tan^{-1}(\frac{1}{1+x^2})+\tan^{-1}(\frac{x^2}{1+x^2})$ without using calculus, by deducing from the above that the curve has one minimum and two maximum turning point, and one horizontal asymptote.
(iii) (a) By considering the points of intersection of the curve $y = x^3 - x + 2$ and the line $y = mx$, show that there is just one tangent to the curve $y = x^3 - x + 2$ which passes through the origin.
(b) Find the equation of this tangent and its point of contact with the curve.

8. **(i)** A vertical mast stands on the north bank of a river with straight parallel banks running from east to west. The angle of elevation of the top of the mast is α when measured from from a point A on the south bank distant $3a$ to the east of the mast, and β when measured from another point B on the south bank distant $5a$ to the west of the mast.
(a) Show that the height of the mast is $\frac{4a}{(\cot^2\beta - \cot^2\alpha)^{\frac{1}{2}}}$.
(b) Show that the angle of elevation θ of the top of mast measured from a point midway between A and B is given by the equation $2\cot^2\theta = 3\cot^2\alpha - \cot^2\beta$.
(ii) A closed container in the shape of a square pyramid is to have given surface area A.

(a) If the side of the base is of length x, show that the volume V is given by $36V^2 = A^2x^2 - 2Ax^4$.
(b) Hence find the value of x for which V is a maximum, and show that this maxiumum value of V is $\sqrt{\frac{A^3}{288}}$.