**PETRUS KY COLLEGE**
**NEW SOUTH WALES**

in partnership with

**VIETNAMESE COMMUNITY IN AUSTRALIA**
**NSW CHAPTER**

**JULY 2006**

# MATHEMATICS EXTENSION 1

## PRE-TRIAL TEST

## SOLUTION

### HIGHER SCHOOL CERTIFICATE (HSC)

Student Number: | | | | |

Student Name: ______________________________

**General Instructions**

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided on Page 2.
- All necessary working should be shown in every question
- Write your Student Number and your Name on all working booklets.

**Total marks – 72**

- Attempt Questions 1–7
- Questions are not of equal value

**Question 1** **10**

(A) By using the substitution method, or otherwise, find the integration of

(i) $\int x.\sqrt{4-x}\ \mathrm{dx}$ 2

Solution : Let $u = 4 - x \quad \therefore x = 4 - u$

$du = -dx \quad \therefore dx = -du$

$$\int x.\sqrt{4-x}.dx = \int (4-u).\sqrt{u}.-du$$

$$= -\int (4u^{1/2} - u^{3/2})\,du$$

$$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + c$$

$$= \frac{2}{5}(4-x)^{5/2} - \frac{2}{3}(4-x)^{3/2} + c$$

$$= \frac{2}{5}(4-x)^2\sqrt{4-x} - \frac{2}{3}(4-x)\sqrt{4-x} + c$$

(ii) $\int \frac{1-\tan x}{1+\tan x}\ \mathrm{dx}$ 2

$$\int \frac{1-\tan x}{1+\tan x}dx = \int \frac{1 - \frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}}dx$$

$$= \int \frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}}dx = \int \frac{\cos x - \sin x}{\cos x + \sin x}dx$$

Let $u = \cos x + \sin x$

$du = (-\sin x + \cos x)\,dx$.

$$\therefore \quad I = \int \frac{du}{u} = \ln u + c = \ln|\cos x + \sin x| + c$$

(iii) $\int \frac{3e^x dx}{4+2e^{2x}}$ 2

$$\int \frac{3e^x}{4+2e^{2x}}dx$$

Let $u = e^x$

$du = e^x.dx$.

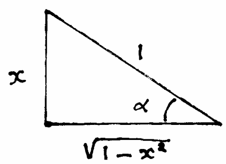
$$\therefore \quad I = \frac{3}{2}\int \frac{du}{2+u^2} = \frac{3}{2\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} + c$$

$$= \frac{3}{2\sqrt{2}}\tan^{-1}\frac{e^x}{\sqrt{2}} + c$$

(B) (i) If $\alpha = \sin^{-1} x$ and $\beta = \tan^{-1} x$ and $\alpha + \beta = \frac{\pi}{9}$, Show that **2**

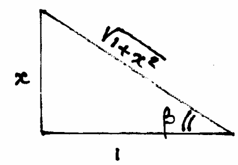
$$\cos(\alpha+\beta) = \frac{\sqrt{1-x^4} - x^2}{\sqrt{1+x^2}}$$

Solution :




$\sin\alpha = x$

$\cos\alpha = \sqrt{1-x^2}$




$\sin\beta = \frac{x}{\sqrt{1+x^2}}$

$\cos\beta = \frac{1}{\sqrt{1+x^2}}$

$$\cos(\alpha+\beta) = \cos\alpha . \cos\beta - \sin\alpha . \sin\beta$$

$$= \sqrt{1-x^2} \times \frac{1}{\sqrt{1+x^2}} - x \times \frac{x}{\sqrt{1+x^2}}$$

$$= \frac{\sqrt{1-x^4} - x^2}{\sqrt{1+x^2}}$$

(ii) Solve the following equation **2**

$$\tan^{-1} 3x - \tan^{-1} x = \tan^{-1} \frac{1}{2}$$

Solve equation : $\tan^{-1} 3x - \tan^{-1} x = \tan^{-1} \frac{1}{2}$

Let $\alpha = \tan^{-1} 3x$ $\therefore$ $\tan\alpha = 3x$

$\beta = \tan^{-1} x$ $\therefore$ $\tan\beta = x$.

$$\therefore \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha . \tan\beta} = \tan\left(\tan^{-1}\frac{1}{2}\right)$$

$$\frac{1}{2} = \frac{3x - x}{1 + 3x^2}$$

$$1 + 3x^2 = 4x$$

$$\therefore 3x^2 - 4x + 1 = 0$$

$$(3x-1)(x-1) = 0$$

Answer : $\boxed{x = 1 \text{ or } \frac{1}{3}}$

**Question 2** **10**

(A) Two points P( 2ap,ap$^2$ ) and Q( 2aq,aq$^2$ ) are on the parabola P : $x^2 = 4ay$

(i) Find the equations of the two tangents to the parabola at P and at Q. Hence find their intersection point T. 2

Equation of tangent at $P(2ap, ap^2)$ 2

$\frac{dy}{dx} = \frac{x}{2a}$, @ $x = 2ap$, gradient of tangent $m_T = p$

Equation of tangent : $y - ap^2 = p(x - 2ap)$

$\therefore \boxed{y = px - ap^2}$ 2

Similar to equation of tangent at Q

$\boxed{y = qx - aq^2}$

Intersection point : $px - ap^2 = qx - aq^2$

$px - qx = ap^2 - aq^2$

$x(p-q) = a(p-q)(p+q)$

$\therefore x = a(p+q)$

Substitute into y

$y = pa(p+q) - ap^2$

$y = apq$

$\therefore$ $T(a(p+q), apq)$

(ii) Find the equation of the two normal at P and Q and their intersection point N.

Equation of normal at $P(2ap, ap^2)$ :

$m_N = -\frac{1}{m_T} = -\frac{1}{p}$

Equation of normal : $y - ap^2 = -\frac{1}{p}(x - 2ap)$

$x + py + 2ap - ap^3 = 0$ (1)

similar to equation of normal at Q

$x + qy + 2aq - aq^3 = 0$ (2)

Intersection point : (1) − (2) :

$y(p-q) = a(p^3 - q^3) - 2a(p-q)$

$y(p-q) = a(p-q)(p^2 + pq + q^2 - 2)$

$\therefore$ $y = a(p^2 + pq + q^2 - 2)$

(1) × q − (2) × p :

$$x(q-p) = aqp^3 - ap.q^3$$
$$= apq(p-q)(p+q)$$
$$\therefore \quad x = -apq(p+q)$$

Intersection point $\quad N\left(-apq(p+q),\ a(p^2+pq+q^2-2)\right)$

(iii) If the two tangents are perpendicular, find the locus of point M, which is the midpoint of T and N .

(B) Show that $\dfrac{1}{n-1}-\dfrac{1}{n+1}=\dfrac{2}{n^2-1}$ 4

Hence find, as a fraction in lowest terms, the sum of the first 100 term of the series $\dfrac{1}{3}+\dfrac{1}{8}+\dfrac{1}{15}+\dfrac{1}{24}+\ldots$

Show that $\dfrac{1}{n-1} - \dfrac{1}{n+1} = \dfrac{2}{n^2-1}$

$$LHS = \frac{n+1-(n-1)}{(n-1)(n+1)} = \frac{2}{n^2-1}$$

Calculate the sum of 100 terms

$$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \cdots\cdots$$

By using the above formulae.

$$\frac{1}{3} = \frac{1}{2^2-1} = \frac{1}{2}\left(\frac{1}{2-1} - \frac{1}{2+1}\right) = \frac{1}{2}\left(1 - \frac{1}{3}\right)$$

$$\frac{1}{8} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$\frac{1}{15} = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) \quad \ldots$$

$$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \cdots + \frac{1}{10200} = \frac{1}{2}\left[1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} + \right.$$
$$\left. + \cancel{\frac{1}{99}} - \frac{1}{101} + \cancel{\frac{1}{100}} - \frac{1}{102}\right]$$
$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{101} - \frac{1}{102}\right)$$
$$\# \ \frac{3}{4} = \left(\frac{7625}{10302}\right)$$

Obtain an expression for $\sum_{r=2}^{n} \frac{1}{r^2-1}$ and hence find the limiting sum of the series.

$$\sum_{r=2}^{n} \frac{1}{r^2-1} = \frac{1}{2} \sum_{r=2}^{n} \frac{1}{r-1} - \frac{1}{r+1}$$

$$= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \cdots \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1}\right)$$

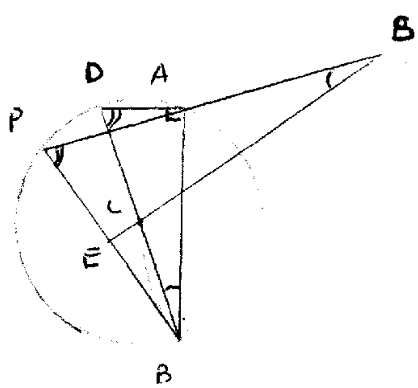
$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \boxed{\frac{1}{4}\left(\frac{3n^2-n-2}{n^2+n}\right)}$$

Limit Sum : $n \to \infty$ $S_\infty = \frac{3}{4}$

**Question 3** **12**

(A) Two different radii circles come across at 2 points A & B. The centre C of smaller circle stays on the circumference of the bigger one. P is a point on the alternate segment (of the smaller circle) outside the common region. From P draw a line through A, that line cuts the bigger circle at S. Show that CS perpendicular to PB. 6




Show that SC perpendicular PB :

Draw diameter BD, form right-angle $\triangle ADB$.

In $\triangle ADB$ and $\triangle SPE$ :

$\angle ABC = \angle ASB$ ( $\angle$s at alt. segment in big circle )

$\angle ADB = \angle APB$ ( $\angle$s at alt. segment in small circle )

$\therefore$ $\triangle ADB \parallel\!| \triangle SPE$ ( equiangular )

$\therefore$ $\angle DAB = \angle SEP = 90^\circ$ ( cor. $\angle$s of similar $\triangle$ )

$\therefore$ $SE \perp PB$.

(B) A research party is held by electing 7 scientists from a department of C.S.I.R.O. There are 7 men and 5 women in that department, and the party will contain 4 men and 3 women.

(i) How many ways the party can be formed ? 2

Number of selections

$$^7C_4 \times {}^5C_3$$ 2

(ii) Find the probability of gaining of party if the oldest man can not be selected together with the youngest woman. 2

Probability of party if oldest man can not be with youngest woman

$$P = \frac{^6C_4 \times {}^5C_3 + {}^7C_4 \times {}^4C_3}{^7C_4 \times {}^5C_3}$$

(iii) Find the probability of gaining a party if both the oldest man and youngest woman present in the party with the condition that no refraction of number of men and women in that 7 people but there are must be at least 3 women present.

Probability if both oldest man and youngest woman and containing at least 3 women.

$$P = \frac{^4C_2 \times {}^6C_3 + {}^4C_3 \times {}^6C_2 + {}^4C_4 \times {}^6C_1}{^{12}C_7}$$

**Question 4** **12**

(A) A ball is thrown with initial velocity 20 m/s at the angle of elevation of $\tan^{-1}\dfrac{4}{3}$

(i) Show that the parabolic path of the ball has the parametric equation 2

$$\begin{cases} x = 12t \\ y = 16t - 5t^2 \end{cases}$$

Find the range of the ball and its greatest height. 2

Angle of projetion : $\tan\alpha = \frac{3}{4}$ , $\therefore \sin\alpha = \frac{3}{5}$ , $\cos\alpha = \frac{4}{5}$

Initial velocity : $V = 20$ m/s. 2

$x = V.t.\cos\alpha = 20 \times \frac{4}{5} t = 12t$

$y = -\frac{1}{2} gt^2 + Vt\sin\alpha = -\frac{10}{2} t^2 + 20 \times \frac{3}{5} \times t = 16t - 5t^2$

3

Total time of flight : Let $y = 0$

$$16t - 5t^2 = 0$$

$$t = \frac{16}{5} = 3.2 \text{ seconds}$$

Range : $R = 12 \times 3.2 = 38.4$ m.

Time to reach greatest height $= \frac{1}{2} \times 3.2 = 1.6$ seconds

Greatest height $H = 16 \times 1.6 - 5 \times 1.6^2 = 12.8$ m.

(ii) Show that in order to reach ¾ of the greatest hight (on the way up), the ball just spends ¼ of the total time.

To reach $3/4$ greatest height $= \frac{3}{4} \times 12.8 = 9.6$ m

Time to reach 9.6m : Let $y = 9.6$m

$$-5t^2 + 16t = 9.6$$

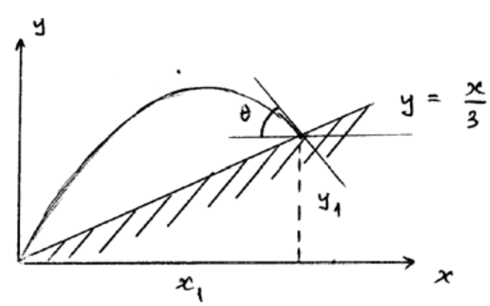
$$5t^2 - 16t + 9.6 = 0$$

$$t = \frac{16 \pm \sqrt{16^2 - 4 \times 5 \times 9.6}}{10} = 0.8 \text{ or } 2.4 \text{ s}$$

on the way up, time to reach $3/4$ H is 0.8 s, ie

$\frac{0.8}{3.2} = \frac{1}{4}$ total time of flight.

(iii) Suppose that the ball is thrown up a road inclined at angle $\alpha = \tan^{-1}\frac{1}{3}$ to the horizontal. Find the time, distance along the road and the angle when the ball hit the road.




- with $\alpha = \tan^{-1}\frac{1}{3}$, ie $\tan\alpha = \frac{1}{3}$, gradient of the road $m = \frac{1}{3}$, Equation of the road $y = \frac{1}{3}x$.
- Equation of motion $y = 16\left(\frac{x}{12}\right) - 5\left(\frac{x}{12}\right)^2$

$$= \frac{4x}{3} - \frac{5x^2}{144}$$

- Intersection point = where the ball hits the road

$$\frac{4x}{3} - \frac{5x^2}{144} = \frac{x}{3} \qquad \therefore \frac{5x^2}{144} - x = 0$$

$$x = \frac{144}{5} = 28.8 \text{ m}$$

$$y = \frac{28.8}{3} = 9.6 \text{ m}$$

- Total time $\quad t = \frac{x}{12} = \frac{28.8}{12} = 2.4$ second.

- Angle when it hits the road

Let $\theta$ be the angle of the ball with the horizontal gound. $\quad \tan\theta = \left|\frac{\dot{y}}{\dot{x}}\right| = \left|\frac{-10 \times 2.4 + 16}{12}\right|$

$$\theta = 34^\circ$$

Angle make with the road $= \theta + \alpha = 34 + 18 = 52^\circ$

(B) Using the mathematic induction method of proving to show that $7^n + 11^n$ is divisible by 9 for odd $n \geq 1$. 3

: By using mathematic induction method, prove that $7^n + 11^n$ is divisible by 9, $n$ is odd positive integer.

Prove true for $n = 1$ : $7^1 + 11^1 = 18$ divisible by 9

$\therefore$ The statement is true for $n = 1$

Assume that the statement is true for $n = k$,

ie $7^k + 11^k = 9m$ : ($m$ is integer)

Prove true for $n = k+2$, ie

$7^{k+2} + 11^{k+2} = 9n$ : ($n$ is integer)

Proof:

$$\begin{aligned}
7^{k+2} + 11^{k+2} &= 49 \times 7^k + 11^{k+2} \\
&= 49(9m - 11^k) + 11^{k+2} \\
&= 49 \times 9m - 49 \times 11^k + 121 \times 11^k \\
&= 49 \times 9m - 72 \times 11^k = 9(49m - 8 \times 11^k)
\end{aligned}$$

$$7^{k+2} + 11^{k+2} = 9n$$

$\therefore$ The statement is true for $n = k+2$

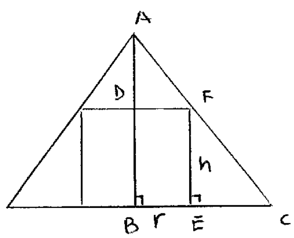
since the statement is true for $n = 1$, it is also true for $n = 1 + 2 = 3$, and so on it is true for all values of $n$ as odd integer.

**Question 5** **10**

(A) A cylinder is inscribed in a cone whose base diameter is 10 cm and whose hight is 12 cm. If the highest of the cylinder is h cm and the radius of its base is r cm , Show that:

(i) $5h + 12r = 60$

2




Show that : $5h + 12r = 60$

By proving 2 similar triangles

2

$\Delta CEF \parallel \Delta CAB$ ( equiangular ).

$\therefore \frac{EF}{AB} = \frac{CF}{CB}$ $\qquad \frac{h}{12} = \frac{5-r}{5}$

2

$5h = 60 - 12r$

$\therefore 5h + 12r = 60$

Show that the volume of the cylinder is: $V = \dfrac{\pi r^2(60-12r)}{5}$

Hence find the dimension r and h of which the volume of that cylinder is maximum. Find the maximum volume.

Volume of the cylinder : $V = \pi r^2 h$

$V = \pi r^2 \left( \frac{60-12r}{5} \right)$

Maximum of volume : $\frac{dV}{dr} = 0$

$\frac{dV}{dr} = 12\pi \times 2r - \frac{12}{5}\pi \times 3r^2 = 0$

$r = 0$ OR $r = \frac{2}{3} \times 5 = \frac{10}{3}$ cm

$h = 4$ cm

Maximum volume $V = \pi \left(\frac{10}{3}\right)^2 . 4 = \frac{400\pi}{9}$ cm$^3$

(B) What is the domain and range of the function 2

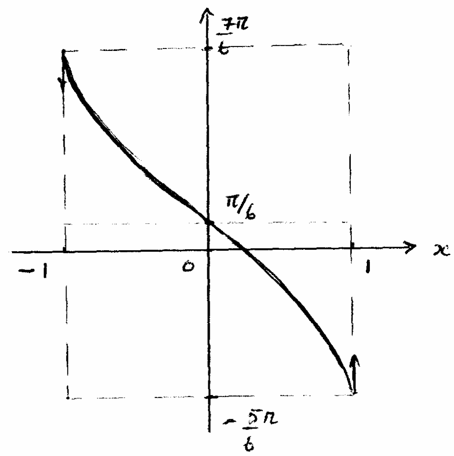
$y = \dfrac{\pi}{6} - 2\sin^{-1} x^2$ Sketch that curve.

2

Inverse function : $y = \frac{\pi}{6} - 2\sin^{-1} x^2$

Domain : $-1 \leq x \leq 1$.

Range : $-\frac{5\pi}{6} \leq y \leq \frac{7\pi}{6}$




**Question 6** **8**

(A) If the equation $6x^4 - 13x^3 - 90x^2 + 208x - 96 = 0$ have 4 distinct roots of $\alpha$, $-\alpha$, $\beta$ and $\frac{1}{\beta}$, then solve the equation. 3

Equation : $6x^4 - 13x^3 - 90x^2 + 208x - 96 = 0$

have 4 roots : $\alpha, -\alpha, \beta$ and $\frac{1}{\beta}$

Product of 4 roots : $(\alpha)(-\alpha)(\beta)\left(\frac{1}{\beta}\right) = -\frac{96}{6}$

$-\alpha^2 = -16$

$\alpha = 4, \ -\alpha = -4$.

$\therefore \quad P(x) = (x-4)(x+4) \cdot Q(x)$

Divide $P(x)$ for $x^2 - 16$ : $Q(x) = 6x^2 - 13x + 6 = 0$

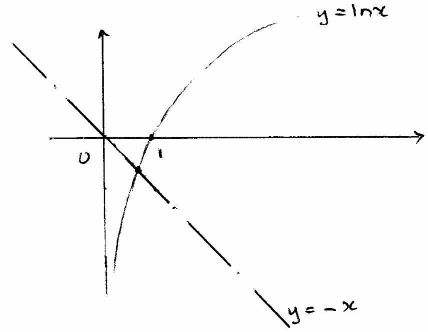
$(3x-2)(2x-3) = 0$

$\therefore \quad \beta = \frac{2}{3}$ or $\frac{1}{\beta} = \frac{3}{2}$

Four roots $4, -4, \frac{2}{3}, \frac{3}{2}$

(B) By sketching the 2 separate functions, show that the equation $x + \ln x = 0$ has only one root from [0, 1]

Sketch the 2 curves $y = \ln x$ and $y = -x$ 2




From the graph, there is only one intersection point between 2 curves 2

$\therefore$ equation $\ln x = -x$

or $\ln x + x = 0$

has only one root 1

between $(0, 1)$

(i) By using the half-interval method three times, find the approximate value of the root.

Solve equation $\ln x + x = 0$ by using half-interval method.

Let $x_1 = 0.5$, $\ln 0.5 + 0.5 = -0.193$

Let $x_2 = \frac{0.5+1}{2} = 0.75$, $\ln 0.75 + 0.75 = 0.4623$

Let $x_3 = \frac{0.5+0.75}{2} = 0.625$, $\ln 0.625 + 0.625 = 0.155$.

Approximation answer $x = 0.625$

(ii) By using the approximation Newton's method 2 times, find the closest root of this equation.

By using Newton's method; 2 times : $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Let $x_0 = 0.5$

$$x_1 = 0.5 - \frac{(\ln 0.5 + 0.5)}{\frac{1}{0.5} + 1} = 0.5643$$

$$x_2 = 0.5643 - \frac{(\ln 0.5643 + 0.5643)}{\frac{1}{0.5643} + 1}$$

$$x_2 = 0.5672$$

(iii) By comparison the two answers of i) and ii) above, which method is more appropriate?

comparing 2 solutions, show that $x_2 = 0.5672$ is the better answer : $\ln 0.5676 + 0.5672 = 0.00014$

**Question 7** **10**

(A) Using the binomial expansion or else show that 2

$(3+\sqrt{5})^6+(3-\sqrt{5})^6 = 20608$

show that $(3+\sqrt{5})^6 + (3-\sqrt{5})^6 = 20608$

$$(3+\sqrt{5})^6 = {}^6C_0 3^6 + {}^6C_1 3^5.\sqrt{5} + {}^6C_2 3^4.(\sqrt{5})^2 + {}^6C_3 3^3.(\sqrt{5})^3 + \dots \; {}^6C_6(\sqrt{5})^6$$

$$+\ (3-\sqrt{5})^6 = {}^6C_0 3^6 - {}^6C_1 3^5\sqrt{5} + {}^6C_2 3^4(\sqrt{5})^2 - {}^6C_3 3^3(\sqrt{5})^3 + \quad {}^6C_6(\sqrt{5})^6$$

$$(3+\sqrt{5})^6 + (3-\sqrt{5})^6 = 2\left({}^6C_0 3^6 + {}^6C_2 3^4.5 + {}^6C_4 3^2.5^2 + {}^6C_6.5^3\right)$$

$$= 20608.$$

(B) The position x cm of a particle relative to a fixed point 0 at any time t is:

$x = 5 - 2\cos^2 t$

(i) Show, by finding its acceleration in term of x that the motion is simple harmonic. 2

show this motion is S.H.M

$$x = 5-(1+\cos 2t) = 4-\cos 2t$$

$$\dot{x} = 2\sin 2t$$

$$\ddot{x} = 4\cos 2t = 4(4-x)$$

$$\boxed{\ddot{x} = -4(x-4)} \;\therefore\; \text{It's S.H.M}$$

(ii) Find the centre of the motion, the period and the amplitude. 2

Centre of motion, period, amplitude.

centre $c = 4$

period : $T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$

Amplitude : $A = 1$

(iii) Find the initial velocity and acceleration. 2

Initial velocity : $t=0$ ; $\dot{x} = 0$

Initial acceleration $t=0$ $\ddot{x} = 4\ m/s^2$

(iv) Find the velocity when the particle passing the centre of motion. 2

when $x$ = centre of motion $= 4$

$\dot{x}$ = maximum value

$\dot{x} = 2\ m/s$