Question One

(a) Using the substitution $u = e^x - 1$, find the value of

$$\int_0^{\ln 2} e^x \sqrt{e^x - 1}\, dx$$

(3 marks)

(b) Prove the following trigonometric identity:

$$\frac{\cos 3x}{\cos x} = 1 - 4\sin^2 x$$

(2 marks)

(c) Find the general solution of the trigonometric equation:

$$\sqrt{3}\operatorname{cosec}\theta = -2$$

(2 marks)

(d) Solve:

(i) $2x^2 + 5x - 3 \geq 0$

(ii) $\dfrac{2x^2 + 5x - 3}{x - 1} \geq 0$

(2 marks)

(e) (i) In how many ways can the letters of the word BIOLOGIST be arranged?

(ii) What is the probability that the letters "I" will be next to each other?

(3 marks)

Question Two

(a) Use Mathematical Induction to prove that the expression $2n + n^3$, where n is a positive integer, is always divisible by 3.

(3 marks)

(b) Without using a formula, prove that

$$\frac{a}{p} + a + ap + \ldots + ap^n = \frac{a - ap^{n+2}}{p - p^2}$$

(3 marks)

(c) (i) Use the factor theorem to prove that $x^2 + 2bx - x - 2b$ is a factor of the polynomial $P(x) = x^3 + (2b+1)x^2 + 2(b-1)x - 4b$

(ii) Hence, or otherwise, factorise the polynomial completely.

(3 marks)

(d) The root of the equation $e^x = -x^3$ lies near $x = -1$. Use Newton's method to find a second approximation to the root correct to 3 decimal places.

(3 marks)

Question Three

(a) Solve the cubic equation $4x^3 - 13x + 6 = 0$ given that the product of two of its roots is equal to -1.

(3 marks)

(b) Express $\sin\theta$ and $\cos\theta$ in terms of t, where $t = \tan\frac{\theta}{2}$
Hence solve $2\sin\theta + 4\cos\theta = 3$, $\quad 0 \le \theta \le 360°$

(3 marks)

(c) Determine the coefficient of x^4 in the expansion of

$$(1 - 2x + x^3)(1 - 2x)^7$$

(3 marks)

(d) Find the derivative of $5^{\sqrt{x}}$ and hence evaluate

$$\int_1^4 \frac{5^{\sqrt{x}}}{\sqrt{x}}\,dx$$ correct to 3 significant figures.

(3 marks)

Question Four

(a) Consider the function $f(x) = \dfrac{x^2}{x+2}$

(i) State the equation(s) of the vertical asymptote(s).

(ii) Determine whether there are other asymptotes.

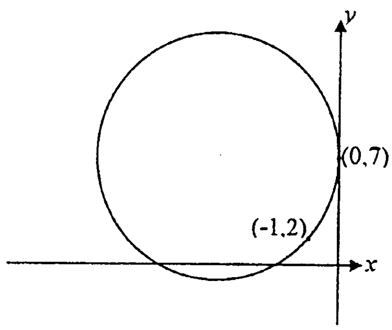
(iii) Find the stationary points (if any) and identify their nature.

(iv) Show that the curve has no points of inflexion.

(v) Draw a neat sketch of the curve.

(6 marks)

(b) A circle touches the y-axis at (0,7) and passes through (-1,2) as shown. Find the coordinates of the centre of the circle.



(3 marks)

(c) (i) Show that the equation of the normal at a point $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ is $x + py = 2ap + ap^3$

(ii) The normal at P meets the y-axis at N and M is the midpoint of PN. Find the coordinates of M.

(iii) Show that the locus of M is another parabola with its vertex equal to the focus of the original parabola. *(3 marks)*

Question Five

(a) A mug of hot coffee at temperature T°C, when placed in a cooler environment, loses heat according to the law:

$$\frac{dT}{dt} = k(T - T_0)$$

when t is the time elapsed in minutes, and T_0 is the temperature of the environment in degrees Celsius.

(i) A mug of coffee at 96°C is left to stand in a room at a temperature of 18°C. After 3 minutes the coffee cools down to 75°C. Calculate the value of k.

(ii) Kim wishes to drink her coffee at 60°C. How long should she wait before enjoying her coffee?

(b) The vertical velocity V m/s of a buoy moving in simple harmonic motion as waves pass across it is given by

$$v^2 = -12 + 14y - 2y^2 \text{ where } y \text{ is in metres.}$$

(i) Find the acceleration of the buoy in terms of y.

(ii) Calculate the mean position of the buoy.

(iii) Find the period of the oscillation. *(4 marks)*

(c) Six families in a certain street each have 4 children. What is the probability that exactly 2 of these families have two boys and two girls? *(4 marks)*

Question Six

(a) A bank advertises

Fly Now: Pay Later!

An airline ticket and hotel reservations come to \$10 500. The interest is 9% p.a. compound interest on the money owing and is to be paid back monthly over a period of 2 years.

(i) How much is the monthly repayment?

(ii) What is the actual cost of the holiday?

(iii) How much was still owing after one year? *(5 marks)*

(b) If $f(x) = g(x) - \ln[g(x)+1]$

(i) Prove that $f'(x) = \dfrac{g(x)g'(x)}{g(x)+1}$

(ii) Hence evaluate $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{\sin 2x \cos 2x}{\sin 2x + 1}\, dx$ *(5 marks)*

(c) Solve for x: $2^{x+1} - 2^{-x+2} = 7$ *(2 marks)*

Question Seven

(a) Using the Pascal Triangle relationship

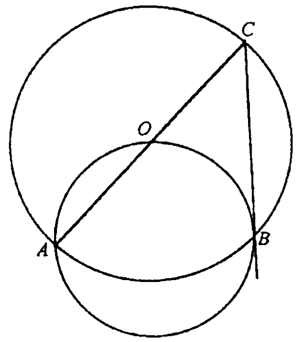
$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$$

prove that

$$\sum_{r=0}^{n} \binom{n+r}{r} = \binom{2n+1}{n}$$

(7 marks)

(b) Two circles intersect at A and B in such a way that the lower circle passes through the centre O of the upper circle. AO produced meets the tangent to the lower circle at B, at C which lies on the upper circle. Prove that the ratio of radius of upper circle to lower circle equals $\sqrt{2} : 1$



(5 marks)

SUGGESTED SOLUTIONS, MATHEMATICS 3 UNIT - 2000

Question 1

(a) $u = e^x - 1$: when $x = 0$, $u = e^0 - 1 = 1 - 1 = 0$

$du = e^x dx$: when $x = \ln 2$, $u = e^{\ln 2} - 1 = 2 - 1 = 1$

$$\therefore \int_0^{\ln 2} e^x\sqrt{e^x - 1}\,dx$$

$$= \int_0^{\ln 2} e^x\sqrt{e^x - 1}\,dx$$

$$= \int_0^1 u^{\frac{1}{2}}\,du$$

$$= \frac{2}{3}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{3}\left[u^{\frac{3}{2}}\right]_0^1$$

$$= \tfrac{2}{3}(1 - 0)$$

$$= \tfrac{2}{3}$$

(b) LHS

$$= \frac{\cos 3x}{\cos x}$$

$$= \frac{\cos(2x + x)}{\cos x}$$

$$= \frac{\cos 2x\cos x - \sin 2x\sin x}{\cos x}$$

$$= \frac{(1 - 2\sin^2 x)\cos x - 2\sin x\cos x.\sin x}{\cos x}$$

$$= 1 - 2\sin^2 x - 2\sin^2 x$$

$$= 1 - 4\sin^2 x$$

$$= \text{RHS}$$

(2 marks)

(c) $\sqrt{3}\operatorname{cosec}\theta = -2 \Rightarrow \operatorname{cosec}\theta = -\dfrac{2}{\sqrt{3}}$

$\therefore \quad \sin\theta = -\dfrac{\sqrt{3}}{2}$

$\therefore \quad \sin\theta = \sin\left(-\frac{\pi}{3}\right)$

$\therefore \quad \theta = (-1)^n\left(-\frac{\pi}{3}\right) + n\pi$ (n an integer)

(2 marks)

(d) (i) $2x^2 + 5x - 3 \ge 0$

$(2x-1)(x+3) \ge 0$

-3 0 ½

Test $x = 0$: invalid

$\therefore$ Solution is $x \le -3$ or $x \ge \frac{1}{2}$

(ii) $\dfrac{2x^2 + 5x - 3}{x-1} \ge 0$

On multiplying both sides by $(x-1)^2$:

$(2x^2 + 5x - 3)(x-1) \ge 0$

$(2x-1)(x+3)(x-1) \ge 0$ *(2 marks)*

-3 0 ½ 1

Test $x = 0$: valid

$\therefore$ Solution is $-3 \le x \le \frac{1}{2}$ *or* $x > 1$ (since $x \ne 1$)

(e) (i) Number of ways $= \dfrac{9!}{2!2!} = 90720$

(ii) Number of ways with the letters "I" next to each other: *(3 marks)*

I	I							

$$= 8 \times \frac{7!}{2!} = 20160$$

$\therefore$ The probability of the letters "I" being together

$$= \frac{n(E)}{n(S)}$$

$$= \frac{20160}{90720}$$

$$= \frac{2}{9}$$

Question Two

(a) Put $n = 1$, $2n + n^3 = 2 + 1 = 3$ which is divisible by 3

$\therefore$ The statement is true for $n = 1$

Now assume that it is true for $n = k$ (k a positive integer)

ie. $2k + k^3 = 3M$ (M an integer)............................(1)

Let $n = k + 1$

$\therefore\ 2(k+1) + (k+1)^3$

$= 2k + 2 + k^3 + 3k^2 + 3k + 1$

$= 2k + k^3 + 3k^2 + 3k + 3$

But from (1) $2k + k^3 = 3M$

$\therefore$ Expression $= 3M + 3k^2 + 3k + 3$

$= 3(M + k^2 + k + 1)$ which is divisible by 3.

$\therefore$ It is true for $n = k + 1$ if it is true for $n = k$. But it is true for $n = 1$.

$\therefore$ True for $n = 1 + 1 = 2$, and $2 + 1 = 3$ and so on.

$\therefore$ By Mathematical Induction, it is true for all positive integers n. *(3 marks)*

(b) Let $S_n = \dfrac{a}{p} + a + ap + ... + ap^n$ ....(1)

then $p \times S_n = +a + ap + ... + ap^n + ap^{n+1}$ ....(2)

(1) – (2) gives:

$$S_n - p \times S_n = \frac{a}{p} - ap^{n+1}$$

$$S_n(1-p) = \frac{a - ap^{n+2}}{p}$$

$$\therefore \quad S_n = \frac{a - ap^{n+2}}{p(1-p)}$$

$$= \frac{a - ap^{n+2}}{p - p^2}$$

(3 marks)

(c) (i) $x^2 + 2bx - x - 2b = x(x + 2b) - (x + 2b)$

$$= (x + 2b)(x - 1)$$

$$P(x) = x^3 + (2b + 1)x^2 + 2(b - 1)x - 4b$$

$$P(-2b) = (-2b)^3 + (2b + 1)(-2b)^2 + 2(b - 1)(-2b) - 4b$$

$$= -8b^3 + 8b^3 + 4b^2 - 4b^2 + 4b - 4b$$

$$= 0$$

$\therefore$ $x + 2b$ is a factor of $P(x)$

$$P(1) = 1 + 2b + 1 + 2(b - 1) - 4b$$

$$= 1 + 2b + 1 + 2b - 2 - 4b$$

$$= 0$$

$\therefore$ $x - 1$ is a factor of $P(x)$

$\therefore$ $(x + 2b)(x - 1)$ is a factor of $P(x)$

(ii)

$$\begin{array}{r|l} & x + 2 \\ \hline x^2 + 2bx - x - 2b & x^3 + (2b+1)x^2 + 2(b-1)x - 4b \\ & x^3 + 2bx^2 - x^2 - 2bx \\ \hline & 2x^2 + 4bx - 2x - 4b \\ & 2x^2 + 4bx - 2x - 4b \\ \hline \end{array}$$

$\therefore$ $P(x) = (x + 2b)(x - 1)(x + 2)$ *(3 marks)*

(d) Let $f(x) = e^x + x^3$

$$f'(x) = e^x + 3x^2$$

By Newton's Method:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= -1 - \frac{e^{-1} + (-1)^3}{e^{-1} + 3(-1)^2}$$

$$\approx -0.812$$

(3 marks)

Question Three

(a) Let the roots be $\alpha,\ -\dfrac{1}{\alpha},\ \beta$

then $\quad \alpha-\dfrac{1}{\alpha}+\beta=-\dfrac{b}{a}=0$ ....(1)

$$\alpha\left(\frac{-1}{\alpha}\right)\beta=-\frac{d}{a}=-\frac{3}{2}$$

$$-\beta=-\frac{3}{2}$$

$$\beta=\frac{3}{2}$$

Sub in (1) :
$$\alpha-\frac{1}{\alpha}+\frac{3}{2}=0$$

$$2\alpha^2-2+3\alpha=0$$

$$2\alpha^2+3\alpha-2=0$$

$$(2\alpha-1)(\alpha+2)=0$$

$$\alpha=\frac{1}{2}\ \ or\ \ -2$$

$\therefore$ Roots are $\frac{1}{2}$, -2 and $\frac{3}{2}$

(3 marks)

(b) $\sin\theta=\dfrac{2t}{1+t^2}$, $\quad \cos\theta=\dfrac{1-t^2}{1+t^2}$

$$\therefore\quad \frac{2(2t)}{1+t^2}+\frac{4(1-t^2)}{1+t^2}=3$$

$$4t+4-4t^2=3+3t^2$$

$$7t^2-4t-1=0$$

$$t=\frac{4\pm\sqrt{16-4(7)(-1)}}{14}$$

$$=\frac{4\pm\sqrt{44}}{14}$$

$\therefore$	$\tan\dfrac{\theta}{2}$	$=$	0.7595	or	-0.1881	$0\le\dfrac{\theta}{2}\le180°$	
	$\dfrac{\theta}{2}$	$=$	37°13'	or	180° − 10°39'	$=$	169°21'
	θ	$=$	74°26'	or	338°42'		

(c) $(1-2x+x^3)(1-2x)^7 = (1-2x+x^3)[\ \binom{7}{0}+\binom{7}{1}(-2x)+\binom{7}{2}(-2x)^2+\binom{7}{3}(-2x)^3$
$+\binom{7}{4}(-2x)^4+\ldots+\binom{7}{7}(-2x)^7\ ]$

The terms containing x^4 arise from:

$$1\binom{7}{4}(-2x)^4 - 2x\binom{7}{3}(-2x)^3 + x^3\binom{7}{1}(-2x)$$

$\therefore$ Coefficient of x^4 is $\binom{7}{4}(-2)^4 - 2\binom{7}{3}(-2)^3 + \binom{7}{1}(-2)$

$$= 560 + 560 - 14$$

$$= 1106$$

(d) $\frac{d}{dx}\left[5^{\sqrt{x}}\right] = \dfrac{\ln 5(5^{\sqrt{x}})}{2\sqrt{x}}$

[Let $y = 5^{\sqrt{x}}$, put $\sqrt{x} = u$

then $y = 5^u$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \ln 5\left(5^u\right)\cdot\frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{\ln 5\left(5^{\sqrt{x}}\right)}{2\sqrt{x}}]$$

$$\therefore \int_1^4 \frac{5^{\sqrt{x}}}{\sqrt{x}}\,dx = \frac{2}{\ln 5}\left[5^{\sqrt{x}}\right]_1^4$$

$$= \frac{2}{\ln 5}(25-5)$$

$$= \frac{40}{\ln 5}$$

$$= 24.9 \quad (3 \text{ s.f.})$$

(3 marks)

Question Four

(a) (i) One vertical asymptote: $x = -2$

(ii) $\dfrac{x^2}{x+2} = x - 2 + \dfrac{4}{x+2}$

As $x \to \infty$, $y \to x-2$ from above

As $x \to -\infty$, $y \to x-2$ from below

$\therefore$ The line $y = x-2$ is an oblique asymptote.

(iii)

$$f(x)=\frac{x^2}{x+2}$$

$$f'(x)=\frac{(x+2)2x-x^2}{(x+2)^2}$$

$$=\frac{x^2+4x}{(x+2)^2}$$

$$=0 \text{ for stationary points}$$

$$x(x+4)=0$$

$$x=0,\quad x=-4$$

$$y=0,\quad y=\tfrac{16}{-2}=-8$$

$$f''(x)=\frac{(x+2)^2(2x+4)-2(x^2+4x)(x+2)}{(x+2)^4}$$

$$=\frac{(x+2)[(x+2)(2x+4)-2(x^2+4x)]}{(x+2)^4}$$

$$=\frac{2x^2+8x+8-2x^2-8x}{(x+2)^3}$$

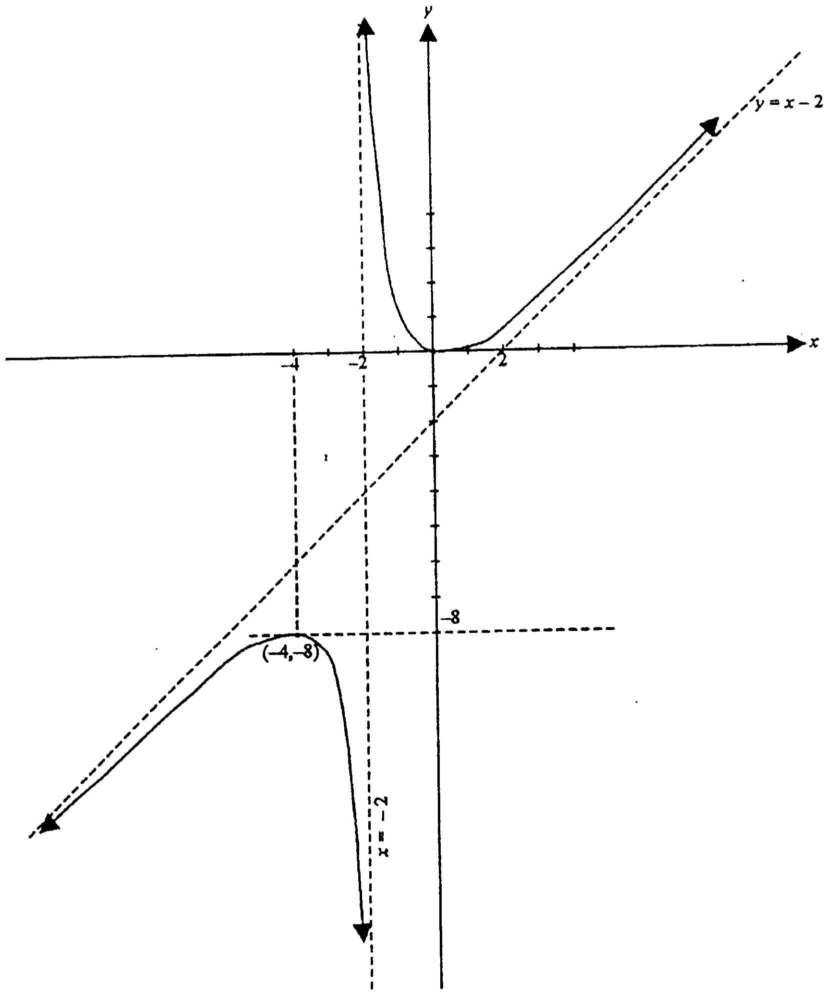
$$=\frac{8}{(x+2)^3}$$

$f''(0)=1>0$ $\quad\therefore$ minimum turning point at (0,0)

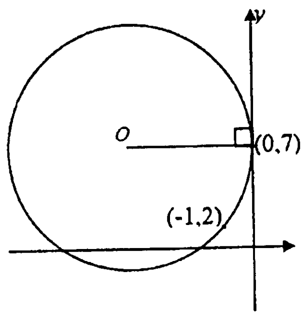
$f''(-4)=\dfrac{8}{(-2)^3}=-1<0$ $\quad\therefore$ maximum turning point at (–4,–8)

(iv) Since $\dfrac{8}{(x+2)^3}\neq 0$ there are no points of inflexion.

(v)



(b) Let O be the centre of the circle. Since radius of a circle is perpendicular to a tangent drawn to the circle, O has y-coordinate equal to 7 and $\therefore$ $O\ (h,7)$.



Equation of circle:

$(x-h)^2+(y-7)^2=h^2$

Sub $(-1,2)$:

$(-1-h)^2+(2-7)^2=h^2$

$1+2h+h^2+25=h^2$

$2h=-26$

$h=-13$

$\therefore$ Centre of circle is $O(-13,7)$.

(c) (i) At P: $x = 2ap$, $y = ap^2$

$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$

$= 2ap \times \frac{1}{2a}$

$= p$ which is the gradient of the tangent at P

$\therefore$ the gradient of the normal at P is $-\frac{1}{p}$

$\therefore$ the equation of the normal using $y - y_1 = m(x - x_1)$ is:

$$y - ap^2 = -\frac{1}{p}(x - 2ap)$$

$$py - ap^3 = -(x - 2ap)$$

$$\therefore \quad x + py = 2ap + ap^3$$

(ii) The normal meets the y-axis when x = 0

$$\therefore \quad py = 2ap + ap^3$$

$$y = 2a + ap^2$$

$\therefore$ N is $(0, 2a + ap^2)$

The midpoint of PN, M is:

$$\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}$$

$$\frac{0 + 2ap}{2}, \frac{ap^2 + 2a + ap^2}{2}$$

$$\therefore \quad M(ap, ap^2 + a)$$

(iii) Eliminating p from $\left.\begin{array}{l} x = ap \\ y = ap^2 + a \end{array}\right\}$ $\qquad p = \frac{x}{a}$

$$\therefore \quad y = a\left(\frac{x}{a}\right)^2 + a$$

$$y = \frac{x^2}{a} + a$$

$$ay = x^2 + a^2$$

$x^2 = a(y - a)$ which is another parabola with vertex $(0, a)$, the focus of the original parabola *(3 marks)*

Question Five

(a) (i) Given $\frac{dT}{dt} = k(T - T_0)$

$T = T_0 + Ae^{kt}$ where A is a constant.

When $t = 0$, $T = 96$ and $T_0 = 18$, $\therefore\ A = 96 - 18 = 78$

$\therefore\quad T = 18 + 78e^{kt}$

When $t = 3$, $T = 75$

$\therefore\quad 75 = 18 + 78e^{3k}$

$57 = 78e^{3k}$

$e^{3k} = \frac{57}{78}$

$3k = \ln\left(\frac{57}{78}\right)$

$k \approx -0.10455$

(ii) Again $T = 18 + 78e^{kt}$

Sub $T = 60$, $k = -0.10455\ldots$

$60 = 18 + 78e^{-0.10455t}$

$\frac{42}{78} = e^{-0.10455t}$

$t = \frac{\ln\frac{42}{78}}{-1.0455}$

$\approx 5.9\,\text{min}$

(b) (i) $v^2 = -12 + 14y - 2y^2$

$\frac{1}{2}v^2 = -6 + 7y - y^2$

$\therefore\quad \ddot{y} = \frac{d}{dy}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dy}(-6 + 7y - y^2)$

$= 7 - 2y$

ie. acceleration of buoy $= 7 - 2y$

(ii) $7-2y=-2\left(y-\frac{7}{2}\right)$

$\therefore$ Mean position of buoy $=\frac{7}{2}$

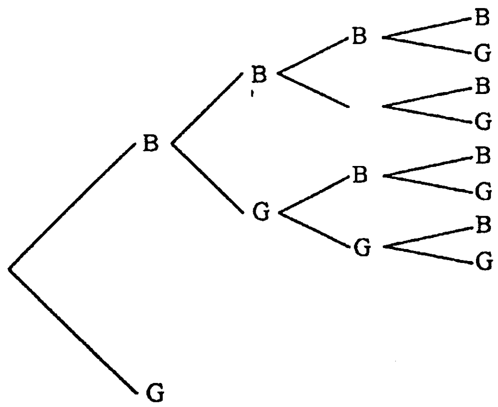
(iii) $n^2=2$

$n=\sqrt{2}$

$\therefore\ T=\frac{2\pi}{n}=\frac{2\pi}{\sqrt{2}}=\sqrt{2}\pi$ seconds

4 marks

(c)



4 marks

P(2 boys and 2 girls) = $\frac{6}{16}=\frac{3}{8}$

Let

$p=$ success (2 boys and 2girls) $=\frac{3}{8}$

$q=$ failure (any other combination) $=\frac{5}{8}$

For 6 families: $(q+p)^6=\sum_{k=0}^{6}\binom{6}{k}q^{6-k}\cdot p^k$

$\therefore\ P$(exactly 2 families having 2 boys and 2 girls) $=\binom{6}{2}\left(\frac{5}{8}\right)^4\left(\frac{3}{8}\right)^2$

$=\frac{15\times 625\times 9}{262144}$

$=\frac{84375}{262144}$

≈ 0.322(3 d.p.)

Question Six

(a) Let A_n = amount owing after n months and let M = monthly repayment
9% p.a. = 0.75% per month

(i) $A_1 = 10500(1.0075) - M$

$A_2 = A_1(1.0075) - M$

$= \{10500(1.0075) - M\}(1.0075) - M$

$= 10500(1.0075)^2 - M(1 + 1.0075)$

Similarly $A_3 = 10500(1.0075)^3 - M(1 + 1.0075 + 1.0075^2)$

and so

$A_{24} = 10500(1.0075)^{24} - M(1 + 1.0075 + 1.0075^2 ... + 1.0075^{23})$

But $A_{24} = 0$

$$\therefore\ 10500(1.0075)^{24} = \frac{M[1.0075^{24} - 1]}{1.0075 - 1}$$

$$M = \frac{10500(1.0075)^{24} \times 0.0075}{1.0075^{24} - 1}$$

$= \$479.69$

(ii) Total cost of holiday $= 479.69 \times 24$

$= \$11\ 512.56$

(iii)
$$A_{12} = 10500(1.0075)^{12} - 479.69\left[\frac{1.0075^{12} - 1}{1.0075 - 1}\right]$$

$= \$5485.21$

5 marks

(b) (i) $f(x) = g(x) - \ln[g(x) + 1]$

$$\therefore\quad f'(x) = g'(x) - \frac{g'(x)}{g(x)+1}$$

$$= \frac{g'(x)[g(x)+1] - g'(x)}{g(x)+1}$$

$$= \frac{g'(x)g(x) + g'(x) - g'(x)}{g(x)+1}$$

$$= \frac{g(x)g'(x)}{g(x)+1}$$

(ii) $$\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{\sin 2x\cos 2x}{\sin 2x+1}\,dx = \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{\sin 2x.(2\cos 2x)}{\sin 2x+1}\,dx$$

$$= \frac{1}{2}\left[\sin 2x + \ln(\sin 2x+1)\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[1+\ln 2 - \left\{\frac{1}{2} + \ln\left(\frac{1}{2}+1\right)\right\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} + \ln 2 - \ln\frac{3}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} + \ln\frac{4}{3}\right]$$

(5 marks)

(c) $$2^{x+1} - 2^{-x+2} = 7$$

$$2.(2^x) - 2^2.(2^{-x}) = 7$$

$$2(2^x) - \frac{4}{2^x} = 7$$

$$2(2^{2x}) - 7(2^x) - 4 = 0$$

Let $2^x = a$

$2a^2 - 7a - 4 = 0$

$(2a+1)(a-4) = 0$

$2a+1=0$, $a-4=0$

$a = -\frac{1}{2}$, $a = 4$

$2^x = -\frac{1}{2}$, $2^x = 4$

NO SOLUTION , $x = 2$

(2 marks)

Question Seven

(a) $\binom{2n+1}{n}=\binom{2n}{n}+\binom{2n}{n-1}$

$=\binom{2n}{n}+\binom{2n-1}{n-1}+\binom{2n-1}{n-2}$

$=\binom{2n}{n}+\binom{2n-1}{n-1}+\binom{2n-2}{n-2}+\binom{2n-2}{n-3}$

and so on until

$\binom{2n+1}{n}=\binom{2n}{n}+\binom{2n-1}{n-1}+\binom{2n-2}{n-2}+\binom{2n-3}{n-3}+\ldots+\binom{n+4}{3}$

$=\binom{2n}{n}+\binom{2n-1}{n-1}+\binom{2n-2}{n-2}+\binom{2n-2}{n-3}+\ldots+\binom{n+3}{3}+\binom{n+3}{2}$

$=\binom{2n}{n}+\binom{2n-1}{n-1}+\binom{2n-2}{n-2}+\binom{2n-2}{n-3}+\ldots+\binom{n+3}{3}+\binom{n+2}{2}+\binom{n+2}{1}$

$=\binom{2n}{n}+\binom{2n-1}{n-1}+\binom{2n-2}{n-2}+\binom{2n-2}{n-3}+\ldots+\binom{n+3}{3}+\binom{n+2}{2}+\binom{n+1}{1}+\binom{n+1}{0}$

$=\binom{2n}{n}+\binom{2n-1}{n-1}+\binom{2n-2}{n-2}+\binom{2n-2}{n-3}+\ldots+\binom{n+3}{3}+\binom{n+2}{2}+\binom{n+1}{1}+\binom{n}{0}$

since $\binom{n+1}{0}=\binom{n}{0}=1$

ie. $\sum_{r=0}^{n}\binom{n+r}{r}=\binom{2n+1}{n}$

(b) Join AB and OB: $\angle ABC = 90°$ (angle in semi-circle in upper circle)

$\therefore$ AB passes through the centre H of the lower circle (angle between tangent and diameter = 90°) ie. AB is diameter of lower circle.

$\therefore$ $\angle AOB = 90°$ (angle in semi-circle in lower circle)

Let the radius of the upper circle = 1 unit

i.e. AO = 1 = OC = OB

$\therefore$ AB = $\sqrt{2}$ (Pythagoras' Theorem in ΔAOB)

$\therefore$ Ratio of AC : AB = $2:\sqrt{2}$ ie. $\sqrt{2}:1$

i.e. Ratio of radius of upper circle to lower circle equals $\sqrt{2}:1$.