1993 HIGHER SCHOOL CERTIFICATE EXAMINATION PAPER

3/4 UNIT MATHEMATICS

QUESTION ONE

(a) Solve the inequality $x^2 - 3x < 4$.

(b) Find the exact value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \, dx$.

(c) Evaluate $\int_{\frac{1}{2}}^{1} 4t(2t-1)^5 \, dt$ by using the substitution $u = 2t - 1$.

(d) If $y = \cos(\ln x)$ find: (i) $\frac{dy}{dx}$ (ii) $\frac{d^2y}{dx^2}$

(e) A class consists of 10 girls and 12 boys.
How many ways are there of selecting a committee of 3 girls and 2 boys from this class?

QUESTION TWO

(a) Prove the following identity:

$$\frac{\sin A}{\cos A + \sin A} + \frac{\sin A}{\cos A - \sin A} = \tan 2A.$$

(b) (i) On the same diagram, sketch the graphs of $y = x$ and $y = |2x - 1|$.

(ii) By using (i) or otherwise, determine for what values of c the equation

$$|2x-1| = x + c$$

has exactly two solutions.

(c) Suppose the cubic $f(x) = x^3 + ax^2 + bx + c$ has a relative maximum at $x = \alpha$ and a relative minimum at $x = \beta$.

(i) Prove that $\alpha + \beta = -\frac{2}{3}a$.

(ii) Deduce that the point of inflexion occurs at $x = \frac{\alpha + \beta}{2}$.

QUESTION THREE

(a) Consider the function $f(x) = 2\cos^{-1}\frac{x}{3}$.

(i) Evaluate $f(0)$.

(ii) Draw the graph of $y = f(x)$.

(iii) State the domain and range of $y = f(x)$.

(b) When the polynomial $P(x)$ is divided by $x^2 - 1$ the remainder is $3x - 1$. What is the remainder when $P(x)$ is divided by $x - 1$?

(c) The velocity v m s^{-1} of a particle moving in simple harmonic motion along the x axis is given by

$$v^2 = 8 + 2x - x^2.$$

(i) Between which two points is the particle oscillating?

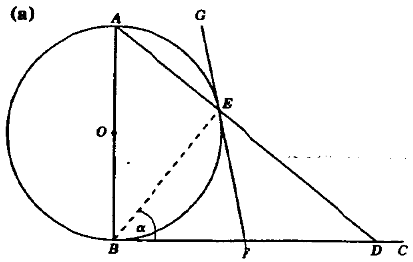
(ii) What is the amplitude of the motion?

(iii) Find the acceleration of the particle in terms of x.

(iv) Find the period of the oscillation.

QUESTION FOUR

(a)



In the diagram, AB is a diameter of the circle, centre O, and BC is tangential to the circle at B. The line AED intersects the circle at E and BC at D. The tangent to the circle at E intersects BC at F. Let $\angle EBF = \alpha$.

(i) Copy the diagram.

(ii) Prove that $\angle FED = \frac{\pi}{2} - \alpha$.

(iii) Prove that $BF = FD$.

(b) Five travellers arrive in a town where there are five hotels.

(i) How many different accommodation arrangements are there if there are no restrictions on where the travellers stay?

(ii) How many different accommodation arrangements are there if each traveller stays at a different hotel?

(iii) Suppose two of the travellers are husband and wife and must go to the same hotel. How many different accommodation arrangements are there if the other three can go to any of the *other* hotels?

(c) (i) Sketch the graph of $y = \cos x$ for $0 \le x \le 2\pi$.

(ii) By using (i), or otherwise, find those values of x satisfying $0 \le x \le 2\pi$ for which the geometrical series

$$1 + 2\cos x + 4\cos^2 x + 8\cos^3 x + \cdots$$

has a limiting sum.

QUESTION FIVE

(a) For $n = 1, 2, 3, \ldots$, let $S_n = 1^2 + 2^2 + \cdots + n^2$.

(i) Use mathematical induction to prove that, for $n = 1, 2, 3, \ldots$,

$$S_n = \frac{1}{6}n(n+1)(2n+1).$$

(ii) By using the result of (i) *estimate* the least n such that $S_n \ge 10^9$.

(b) (i) Prove that the graph of $y = \ln x$ is concave down for all $x > 0$.

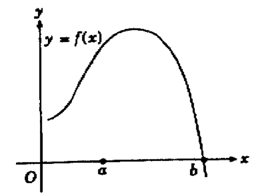
(ii) Sketch the graph of $y = \ln x$.

(iii) Suppose $0 < a < b$ and consider the points $A(a, \ln a)$ and $B(b, \ln b)$ on the graph of $y = \ln x$. Find the coordinates of the point P that divides the line segment AB in the ratio $2:1$.

(iv) By using (ii) and (iii) deduce that

$$\frac{1}{3}\ln a + \frac{2}{3}\ln b < \ln\left(\frac{1}{3}a + \frac{2}{3}b\right).$$

(c)



Consider the above graph of $y = f(x)$. The value a shown on the axis is taken as the first approximation to the solution b of $f(x) = 0$.

Is the second approximation obtained by Newton's method a better approximation to b than a is? Give a reason for your answer.

QUESTION SIX

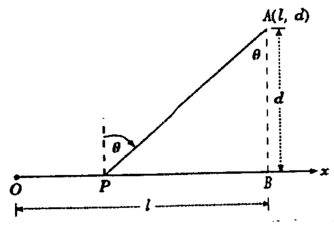
(a) Let T be the temperature inside a room at time t and let A be the constant outside air temperature. Newton's law of cooling states that the rate of change of the temperature T is proportional to $(T - A)$.

(i) Show that $T = A + Ce^{kt}$ (where C and k are constants) satisfies Newton's law of cooling.

(ii) The outside air temperature is 5°C and a heating system breakdown causes the inside room temperature to drop from 20°C to 17°C in half an hour.

After how many hours is the inside room temperature equal to 10°C?

(b)



In the diagram, the x axis represents a major blood vessel, whilst the line PA represents a minor blood vessel that joins the major blood vessel at P. The point A has coordinates (l, d) and PA makes an angle $\theta < \frac{\pi}{2}$ with the normal to the x axis at P, as shown in the diagram.

It is known that the resistance to flow in a blood vessel is proportional to its length, where the constant of proportionality depends upon the particular blood vessel.

Let R be the sum of the resistances to flow in OP and PA.

(i) Show that $R = c_1\left(l - d\tan\theta\right) + c_2 d\sec\theta$, where c_1 and c_2 are constants.

(ii) The blood vessel PA is joined to the blood vessel Ox in such a way that R is minimized.

If $\frac{c_2}{c_1} = 2$, find the angle θ that minimizes R. (You may assume that l is large compared to d.)

QUESTION SEVEN

(a) Consider the parabola $4ay = x^2$ where $a > 0$, and suppose the tangents at $P(2ap,\ ap^2)$ and $Q(2aq,\ aq^2)$ intersect at the point T. Let $S(0, a)$ be the focus of the parabola.

(i) Find the coordinates of T.
(You may assume that the equation of the tangent at P is $y = px - ap^2$.)

(ii) Show that $SP = a(p^2 + 1)$.

(iii) Suppose P and Q move on the parabola in such a way that $SP + SQ = 4a$.
Show that T is constrained to move on a parabola.

(b) A projectile is fired from the origin O with velocity V and with angle of elevation θ, where $\theta \neq \frac{\pi}{2}$. You may assume that

$$x = Vt\cos\theta \text{ and } y = -\frac{1}{2}gt^2 + Vt\sin\theta,$$

where x and y are the horizontal and vertical displacements of the projectile in metres from O at time t seconds after firing.

(i) Show that the equation of flight of the projectile can be written as

$$y = x\tan\theta - \frac{1}{4h}x^2\left(1 + \tan^2\theta\right),$$

where $\frac{V^2}{2g} = h$.

(ii) Show that the point (X, Y), where $X \neq 0$, can be hit by firing at two different angles θ_1 and θ_2 provided

$$X^2 < 4h(h - Y).$$

(iii) Show that no point above the x axis can be hit by firing at two different angles θ_1 and θ_2, satisfying $\theta_1 < \frac{\pi}{4}$ and $\theta_2 < \frac{\pi}{4}$.