Question One

(a)

$$\begin{aligned}\int_0^2 \frac{dx}{\sqrt{16-x^2}} &= \left[\sin^{-1}\frac{x}{4}\right]_0^2 \\ &= \sin^{-1}\frac{1}{2} - \sin^{-1} 0 \\ &= \frac{p}{6}\end{aligned}$$

(b)

$$\begin{aligned}\frac{d}{dx}\left(x\sin^2 x\right) &= 1.\sin^2 x + x.\frac{d}{dx}\left(\sin^2 x\right) \\ &= \sin^2 x + x.2\sin x\cos x \\ &= \sin^2 x + x\sin 2x\end{aligned}$$

(c)

$$\begin{aligned}\sum_{n=4}^{7}(2n+3) &= 3\times 4 + 2\times(4+5+6+7) \\ &= 12 + 2\times(22) \\ &= 56\end{aligned}$$

(d)

$$\begin{aligned}P \text{ is } &\left(\frac{kx_2+lx_1}{k+l}, \frac{ky_2+ly_1}{k+l}\right) \\ &= \left(\frac{1\times -1 + -2\times 2}{-1+2}, \frac{5\times -1 + 7\times 2}{-1+2}\right) \\ &= \left(\frac{-5}{1}, \frac{9}{1}\right) \\ &= (-5, 9)\end{aligned}$$

(e)

By the factor theorem, $x+3=0$ is a factor of the polynomial if $x=-3$ is a solution of the equation $x^3-5x+12=0$

For $x=-3$:

$$\begin{aligned}(-3)^3 - 5(-3) + 12 &= -27 - 15 + 12 \\ &= 0\end{aligned}$$

Therefore $x+3$ is a factor of $x^3-5x+12$

(f)

Let $u = 1+x \quad \therefore x = u-1 \qquad du = dx$

$$\begin{aligned}15\int_{-1}^{0} \mathrm{x}\sqrt{1+\mathrm{x}}\,dx &= 15\int_{1+-1}^{1+0}(u-1)\sqrt{u}\,du \\ &= 15\int_0^1 u^{\frac{3}{2}} - u^{\frac{1}{2}}\,dx \\ &= 15\left[\frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3}\right]_0^1 \\ &= 15\left[\frac{2}{5} - \frac{2}{3}\right] \\ &= -4\end{aligned}$$

Question Two

(a) $f(x) = 3x^2 + x$

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(h)}{h} \\ &= \lim_{h\to 0} \frac{3(a+h)^2 + a + h - 3a^2 - a}{h} \\ &= \lim_{h\to 0} \frac{3a^2 + 6ah + 3h^2 + a + h - 3a^2 - a}{h} \\ &= \lim_{h\to 0} \frac{6ah + 3h^2 + h}{h} \\ &= \lim_{h\to 0} (6a + 3h + 1) \\ &= 6a + 1 \end{aligned}$$

(b)

(i) $\int \frac{e^x}{1+e^x}\,dx = \ln\left(1+e^x\right) + C$

(ii)
$$\begin{aligned} \int_0^p \cos^2 3x dx &= \int_0^p \frac{1}{2}(1+\cos 6x)\,dx \\ &= \frac{1}{2}\left[x + \frac{1}{6}\sin 6x\right]_0^p \\ &= \frac{1}{2}\left[(p+0) - (0+0)\right] \\ &= \frac{p}{2} \end{aligned}$$

(c)

(i) There are 9! possible permutations of 9 unique letters

But the two As are indistinguishable.

Therefore the number of arrangements as required =

$\frac{9!}{2} = 181440$

(ii)

The 5 unique consonants can be arranged in 5! ways,

the 4 vowels with a repeated A in $\frac{4!}{2}$ ways.

Therefore total arrangements $= \frac{4!5!}{2} = 1440$

(d)

$$\begin{aligned} \left(\mathrm{x}^2 - \frac{1}{x}\right)^9 &= \sum_{r=0}^{n}\left[{}^9C_r\left(x^2\right)^{9-r}\left(-\frac{1}{x}\right)^r\right] \\ &= \sum_{r=0}^{n}\left[{}^9C_r\,(-1)^r\,x^{2(9-r)}x^{-r}\right] \\ &= \sum_{r=0}^{n}\left[{}^9C_r\,(-1)^r\,x^{18-3r}\right] \end{aligned}$$

Therefore for constant term, the power to which x is raised = 0

$18 - 3r = 0$

$r = 6$

The term is: ${}^9C_6\,(-1)^6\,x^{18-18}$

$= 84$

Question Three

(a) $f(x) = \sin x + \cos x - x$

$f'(x) = \cos x - \sin x - 1$

Newton's method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $x_0 = 1.2$

Approximate root $x_1 = 1.2 - \dfrac{\sin(1.2) + \cos(1.2) - 1.2}{\cos(1.2) - \sin(1.2) - 1}$

$= 1.26$ (3 significant figures)

(b)

(i)

$\angle AOB = 2\angle APB$ (In circle C_2, the angle at the centre is twice the angle at the circumference subtended by the same arc $\overset{\frown}{AB}$)

$\therefore \angle AOB = 2q$

(ii)

$\angle TAB = \angle AOB$ (In C_1, the angle between the tangent TA and the chord BA equals the angle in the alternate segment $\overset{\frown}{AOB}$)

$\therefore \angle TAB = 2q$

(iii)

$\angle TAB = \angle BPA + \angle APB$ (exterior angle of a triangle theorem)

$2q = q + \angle APB$

$\therefore \angle APB = q$

$\therefore$ **V**BAP is isosceles (base angles $\angle APB$, $\angle ABP$ are equal)

Therefore $PA = BA$ (equal sides of similar triange, **V**BAP)

(c)

(i)

$$
\begin{aligned}
\sin\left(q+2q\right) &= \sin q\cos 2q+\cos q\sin 2q \\
&= \sin q\left(1-2\sin^2 q\right)+\cos q.2\sin q\cos q \\
&= \sin q-2\sin^3 q+2\sin q\cos^2 q \\
&= \sin q-2\sin^3 q+2\sin q\left(1-\sin^2 q\right) \\
&= \sin q-2\sin^3 q+2\sin q-2\sin^3 q \\
&= 3\sin q-4\sin^3 q
\end{aligned}
$$

(ii)

$\sin 3q=2\sin q$

$3\sin q-4\sin^3 q=2\sin q$

$4\sin^3 q-\sin q=0$

For $\sin q=0, q=0,p,2p \qquad 0\le q\le 2p$

For $\sin q\ne 0$:

$4\sin^2 q-1=0$

$\sin q=\pm\frac{1}{2}$

$q=\frac{p}{6},\frac{5p}{6},\frac{7p}{6},\frac{11p}{6} \qquad 0\le q\le 2p$

Therefore for $\sin 3q=2\sin q, 0\le q\le 2p$

$q=0,\frac{p}{6},\frac{5p}{6},p,\frac{7p}{6},\frac{11p}{6},2p$

Question Four

(a) $\dfrac{3\text{x}}{\text{x-2}} \leq 1$

$\dfrac{3x - x + 2}{x - 2} \leq 0$

$\dfrac{x+1}{x-2} \leq 0$

LHS has the same sign as $y = (x+1)(x-2), x \neq 0$. [Draw graph]

Therefore $\dfrac{3\text{x}}{\text{x-2}} \leq 1$ for $-1 \leq x < 2$

(b)

Since $\tan 45^\circ = 1$, the ratio between the magnitudes of the y and x components of velocity $= 1$

But due to the negative direction of motion of the particle vertically, they are in fact opposite:

$\dot{x} = -\dot{y}$ $\quad\quad$ $\dot{x} = V$ $\quad\quad$ $\dot{y} = -10t$ $\quad\quad$ $V = 10t$

But we also know that $x = 4000$ m at this point.

$\therefore t = \dfrac{4000}{V}$

$V = 10 \times \dfrac{4000}{V}$

$V^2 = 40000$

For $V > 0$, $V = 200\ \text{ms}^{-1}$

(c)

Let $v = \dot{x}$

$$\ddot{x} = \frac{dv}{dt} = \frac{dx}{dt} \times \frac{dv}{dx} = v\frac{dv}{dx} = \frac{d}{dv}\left(\frac{v^2}{2}\right)\frac{dv}{dx} = \frac{d}{dx}\left(\frac{v^2}{2}\right)$$

$$\therefore \frac{d}{dx}\left(\frac{v^2}{2}\right) = -4x$$

$$\frac{v^2}{2} = -2x^2 + \frac{C_1}{2}$$

$$v^2 = -4x^2 + C_1$$

At $x = 3, v = -6\sqrt{3}$ $\quad \therefore 108 = -36 + C_1$

$$C_1 = 144$$

$$v^2 = -4x^2 + 144$$

$$v = \frac{dx}{dt}$$

For $v < 0$:

$$\frac{dx}{dt} = -2\sqrt{36 - x^2}$$

$$\frac{dt}{dx} = -\frac{1}{2\sqrt{36 - x^2}}$$

$$t = \frac{1}{2}\int -\frac{1}{\sqrt{36 - x^2}}\,dx$$

$$= \frac{1}{2}\cos^{-1}\frac{x}{6} + C_2$$

At $t = 0, x = 3$:

$$C_2 = -\cos^{-1}\frac{1}{2}$$

$$= -\frac{\pi}{3}$$

$$\therefore \cos^{-1}\frac{x}{6} = 2t + \frac{2\pi}{3}$$

$$x = 6\cos\left(2t + \frac{2\pi}{3}\right)$$

$$= -6\sin\left(2t + \frac{\pi}{6}\right)$$

Question Five

(a)

(i) $f(0) = 2\cos^{-1} 0 = p$

(ii) $x = 2\cos^{-1}\left(\dfrac{f^{-1}(x)}{3}\right)$

$f^{-1}(x) = 3\cos\left(\dfrac{x}{2}\right)$

(iii)

$$\begin{aligned} A &= \int_0^{f(0)} f^{-1}(x)dx \\ &= 3\int_0^{p} \cos\left(\frac{x}{2}\right) \\ &= 3.2\left[\sin\frac{x}{2}\right]_0^{p} \\ &= 6\,\mathrm{u}^3 \end{aligned}$$

(b)

$$\begin{aligned} LHS &= (q+p)^n - (q-p)^n \\ &= \sum_{r=0}^{n}\binom{n}{r}q^{n-r}p^r - \sum_{r=0}^{n}\binom{n}{r}q^{n-r}(-p)^r \\ &= \sum_{r=0}^{n}\binom{n}{r}q^{n-r}\left[p^r - (-1)^r p^r\right] \\ &= \sum_{r=0}^{n}\binom{n}{r}q^{n-r}p^r\left[1+(-1)^{r+1}\right] \end{aligned}$$

For even r, $(-1)^{r+1} = -1$ and such a term $= \binom{n}{r}q^{n-r}p^r[1-1] = 0$

For odd r, $(-1)^{r+1} = 1$ and such a term $= \binom{n}{r}q^{n-r}p^r[1+1] = 2\binom{n}{r}q^{n-r}p^r$

Therefore the overall sum, $LHS = \sum_{r=0}^{n}\binom{n}{r}q^{n-r}p^r\left[1+(-1)^{r+1}\right]$

$$\begin{aligned} &= 2\binom{n}{1}q^{n-1}p^1 + 2\binom{n}{3}q^{n-3}p^3 + \ldots \\ &= RHS \end{aligned}$$

Therefore, if n is odd, the last term is $2\binom{n}{n}q^{n-n}p^n = 2p^n$

If n is even, the last term cancels to 0 and so $r = n-1$ becomes the final term in the expansion:

$$\begin{aligned} &= 2\binom{n}{n-1}q^{n-n+1}p^{n-1} \\ &= 2nqp^{n-1} \end{aligned}$$

(c)

(i) Probability of rolling r 6s is:

$$P_r = \binom{n}{r}\left(\frac{1}{6}\right)^r\left(\frac{5}{6}\right)^{n-r}$$

(ii)

Let $p = \frac{1}{6}$, $q = \frac{5}{6}$

The probability an odd number of 6s are rolled is the probability that 1 six is rolled or 3 sixes are rolled or 5 sixes are rolled and so on...

$$\begin{aligned}
\therefore P_{odd} &= P_1 + P_3 + P_5 + \ldots \\
&= \binom{n}{1}(p)^1(q)^{n-1} + \binom{n}{3}(p)^3(q)^{n-3} + \ldots \\
&= \frac{1}{2}\left\{(q+p)^n - (q-p)^n\right\} \quad \text{from part (b)} \\
&= \frac{1}{2}\left\{\left(\frac{5}{6}+\frac{1}{6}\right)^n + \left(\frac{5}{6}-\frac{1}{6}\right)^n\right\} \\
&= \frac{1}{2}\left\{1^n + \left(\frac{4}{6}\right)^n\right\} \\
&= \frac{1}{2}\left\{1 + \left(\frac{2}{3}\right)^n\right\} \quad \text{as required}
\end{aligned}$$

Question six

(a)

For $n=1$:

$$1^3+(1+1)^3+(1+2)^3 = 1^3+2^3+3^3$$
$$= 1+8+27$$
$$= 36 \text{ which is divisible by 9.}$$

Therefore the proposition is true for $n=1$

Assume the proposition true for $n=k,\ k\in\mathbb{N}\geq 1$

Ie, assume $k^3+(k+1)^3+(k+2)^3=9N,\quad N\in\mathbb{Z}$

We need to prove the proposition true for $n=k+1$

Ie, prove that $(\text{k+1})^3+(k+2)^3+(k+3)^3$ is divisible by 9

$$(\text{k+1})^3+(k+2)^3+(k+3)^3 = (\text{k+1})^3+(k+2)^3+k^3+3k^2 3+3k3^2+3^3$$
$$= k^3+(\text{k+1})^3+(k+2)^3+9k^2+27k+27$$
$$= 9N+9k^2+27k+27 \qquad \text{by assumption}$$
$$= 9\left(N+k^2+3k+3\right) \qquad \text{which is divisible by 9}$$

Therefore the proposition is true for $n=k+1$ if it is true for $n=k\in\mathbb{N}\geq 1$

But it is also true for $n=1$.

Therefore by mathematical induction it is true for all $n\in\mathbb{N}\geq 1$, ie $n=1,2,3,\ldots$

(b)

(i)

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$$
$$= 2at\times\frac{1}{2a}$$
$$= t$$

Therefore the gradient of the normal at P is $-\frac{1}{t}$

Hence the equation of the normal is:

$$y-at^2 = -\frac{1}{t}(x-2at)$$
$$yt-at^3 = 2at-x$$
$$x+ty = 2at+at^3$$

(ii)

Let Q be $\left(2aq, aq^2\right)$

The gradient of the normal $m_{PR} = -\dfrac{1}{t}$

For $PR \perp QR$ the tangent at Q must be P to the normal at P

Ie, at Q, $\dfrac{dy}{dx} = -\dfrac{1}{t}$

$\therefore q = -\dfrac{1}{t}$

Q is therefore $\left(-\dfrac{2a}{t}, \dfrac{a}{t^2}\right)$

(iii)

The equation of PR is $x + ty = at^3 + 2at$ (1)

The equation of QR is $x + qy = aq^3 + 2aq$

Substituting $q = -\dfrac{1}{t}$:

$$x - \frac{y}{t} = -\frac{a}{t^3} - \frac{2a}{t}$$

$$\frac{y}{t} - x = \frac{a}{t^3} + \frac{2a}{t} \qquad (2)$$

Adding (1) and (2) to eliminate x we get:

$$\frac{y}{t} + ty = \frac{a}{t^3} + \frac{2a}{t} + at^3 + 2at$$

$$y\left(t + \frac{1}{t}\right) = a\left(t^3 + 2t + \frac{2}{t} + \frac{1}{t^3}\right)$$

$$y = a\left(\frac{t^3 + t}{t + \frac{1}{t}} + \frac{t + \frac{1}{t}}{t + \frac{1}{t}} + \frac{\frac{1}{t} + \frac{1}{t^3}}{t + \frac{1}{t}}\right)$$

$$= a\left(t^2 + 1 + \frac{1}{t^2}\right) \qquad (3)$$

Sub (3) into (1):

$$x = at^3 + 2at - at\left(t^2 + 1 + \frac{1}{t^2}\right)$$

$$= at^3 + 2at - at^3 - at - \frac{a}{t}$$

$$= a\left(t - \frac{1}{t}\right)$$

(iv)

$$\begin{aligned} x^2 &= a^2\left(t-\frac{1}{t}\right)^2 \\ &= a^2\left(t^2-2+\frac{1}{t^2}\right) \\ &= a^2\left(t^2+1+\frac{1}{t^2}-3\right) \\ &= a^2\left(\frac{y}{a}-3\right) \\ &= ay-3a^2 \end{aligned}$$

$$\therefore y=\frac{x^2}{a}+3a$$

Question Seven

(a)

(i)

$\frac{dv}{dt} = \frac{d}{dx}\left(\frac{v^2}{2}\right)$

$\therefore \frac{d}{dx}\left(\frac{v^2}{2}\right) = x-1$

$\frac{v^2}{2} = \frac{x^2}{2} - x + \frac{C_1}{2}$

$v^2 = x^2 - 2x + C_1$

For $x = 0,\ v = 1,\ \therefore C_1 = 1$

$v^2 = x^2 - 2x + 1$

$\quad = (x-1)^2$

(ii)

$v = 1 - x$ Taking $v > 0$ at $x = 0$

$\frac{dx}{dt} = 1 - x$

$\frac{dt}{dx} = \frac{1}{1-x}$

$t = \ln|1-x| + C_2$

At $t = 0,\ x = 0$:

$0 = \ln 1 + C_2$

$C_2 = 0$

$|1-x| = e^t$

For $v = 1$ initially, $x - 1 = e^t$ holds true

But under this motion, the particle always has a positive velocity

$\therefore x = e^t + 1$

(b)

(i) By the cosine rule:

$$\begin{aligned} AP^2 &= AO^2 + PO^2 - 2AO.PO\cos\frac{p}{3} \\ &= AO^2 + PO^2 - \frac{2}{2}AO.PO \\ &= AO^2 + PO^2 - AO.PO \end{aligned}$$

Now $AO = OT\cot 45^\circ = h$

And $PO = OT\cot a = h\cot a$

Therefore:

$$AP^2 = h^2 + h^2\cot^2 a - h^2\cot a \qquad (1)$$

(ii)

$$\begin{aligned} AP^2 &= AT^2 + PT^2 - 2AT.PT\cos q \\ \cos q &= \frac{AT^2 + PT^2 - AP^2}{2AT.PT} \qquad (2) \end{aligned}$$

$$\begin{aligned} AT^2 &= AO^2 + TO^2 \\ &= h^2 + h^2 \\ &= 2h^2 \\ AT &= \sqrt{2}h \qquad (3) \\ PT^2 &= PO^2 + TO^2 \\ &= h^2\cot^2 a + h^2 \\ PT &= h\sqrt{\cot^2 a + 1} \end{aligned}$$

But $\cos^2 a + \sin^2 a = 1$

$\therefore \cot^2 a + 1 = \operatorname{cosec}^2 a$

$$PT = h\operatorname{cosec} a \qquad (4)$$

Subbing (1), (3), (4) into (2):

$$\cos q = \frac{2h^2 + h^2\left(\cot^2 a + 1\right) - \left(h^2 + h^2\cot^2 a - h^2\cot a\right)}{2\sqrt{2}h.h\operatorname{cosec} a}$$

$$\begin{aligned} \cos q &= \frac{2h^2 + h^2\cot a}{2\sqrt{2}h^2\operatorname{cosec} a} \\ &= \frac{1}{2\sqrt{2}}\frac{2 + \cot a}{\operatorname{cosec} a} \\ &= \frac{1}{2\sqrt{2}}\frac{2 + \frac{\cos a}{\sin a}}{\frac{1}{\sin a}} \\ &= \frac{1}{2\sqrt{2}}\left(2\sin a + \cos a\right) \end{aligned}$$

1 1

(iii)

$$\frac{1}{\sqrt{2}}\sin a + \frac{1}{2\sqrt{2}}\cos a = R\cos\left(a - f\right)$$

Feel free to derive this - I won't!

$$\begin{aligned} R &= \sqrt{\frac{1}{\sqrt{2}^2} + \frac{1}{\left(2\sqrt{2}\right)^2}} \\ &= \sqrt{\frac{1}{2} + \frac{1}{8}} \\ &= \sqrt{\frac{5}{8}} \end{aligned}$$

$$\begin{aligned} f &= \tan^{-1}\frac{\frac{1}{\sqrt{2}}}{\frac{1}{2\sqrt{2}}} \\ &= \tan^{-1} 2 \end{aligned}$$

$$\therefore \frac{1}{\sqrt{2}}\sin a + \frac{1}{2\sqrt{2}}\cos a = \sqrt{\frac{5}{8}}\cos\left(a - \tan^{-1} 2\right)$$

$$q = \cos^{-1}\left(\sqrt{\frac{5}{8}}\cos\left(a - \tan^{-1} 2\right)\right)$$

$$q' = \frac{-1}{\sqrt{1 - \frac{5}{8}\cos^2\left(a - \tan^{-1} 2\right)}} \times -\sqrt{\frac{5}{8}}\sin\left(a - \tan^{-1} 2\right)$$

Stationary points where $q' = 0$

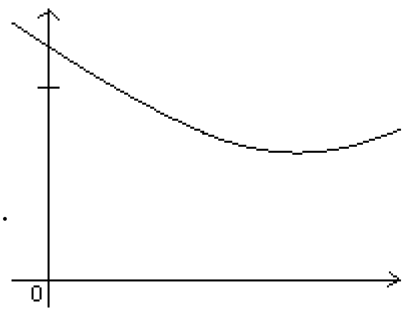
$\sin\left(a - \tan^{-1} 2\right) = 0$

$a = \tan^{-1} 2 \quad$ for $0 \le a \le \frac{p}{2}$

If $a = \tan^{-1} 2$, $q = \cos^{-1}\sqrt{\frac{5}{8}} = 0.659...$

If $a = \tan^{-1} 2 - 0.1$, $q = 0.665...$ If $a = \tan^{-1} 2 + 0.1$, $q = 0.665...$

Therefore there is a local minimum at $\left(\tan^{-1} 2, \cos^{-1}\sqrt{\frac{5}{8}}\right)$



As $a^{+} \to 0$, q increases towards $\cos^{-1}\frac{1}{2\sqrt{2}}$

As $a^{-} \to \frac{p}{2}$, q increases towards $\cos^{-1}\frac{1}{\sqrt{2}}$