BARKER COLLEGE

TRIAL HIGHER SCHOOL CERTIFICATE
2000

MATHEMATICS
3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

BTP
AES
CFR
PJR
MRB
JGD*
JFH*

PM TUESDAY 1 AUGUST

TIME ALLOWED : TWO HOURS
[Plus 5 minutes reading time]

100 copies

DIRECTIONS TO STUDENTS:

- Write your Barker Student Number on EACH AND EVERY page.
- Students are to attempt ALL questions.
  ALL questions are of equal value. [12 marks]
- The questions are not necessarily arranged in order of difficulty.
  Students are advised to read the whole paper carefully at the start of the examination.
- ALL necessary working should be shown in every question.
  Marks may be deducted for careless or badly arranged work.
- Begin your answer to each question on a NEW page. The answers to the questions in this paper are to be returned in SEVEN SEPARATE BUNDLES.
  Write on ONLY ONE SIDE of each page.
- Approved calculators and geometrical instruments may be used.
- A table of Standard Integrals is provided at the end of the paper.

* * * *

QUESTION 1.

(a) Solve for x:

(i) $\dfrac{x+4}{x-2} > 5$ **[3m]**

(ii) $\left(x+\dfrac{1}{x}\right)^2 - 5\left(x+\dfrac{1}{x}\right) + 6 = 0$ **[3m]**

(b) Differentiate with respect to x:

(i) $\cos^3 2x$ **[2m]**

(ii) $e^{x \ln x}$ **[2m]**

(c) AB is a variable interval. M and N divide AB in ratio $-2 : 1$ and $2 : 1$ respectively.

Draw a diagram and decide in what ratio B divides MN. **[2]**

QUESTION 2.

(a) Evaluate: $\displaystyle\lim_{x \to 0} \frac{\sin 5x}{2x}$ **[2m]**

(b) (i) Sketch the curve $y = \sin^{-1}(2x)$

(ii) State the domain and range of this function. **[3m]**

(c) Evaluate: $\displaystyle\int_0^2 \frac{4}{\sqrt{4-x^2}}\,dx$ **[3m]**

(d) Find the obtuse angle, to the nearest minute, between the lines

$3x - 4y + 8 = 0$ and $x + 2y + 1 = 0$ **[4m]**

QUESTION 3.

(a) Prove: $\dfrac{\sin\theta + \sin 2\theta}{1 + \cos\theta + \cos 2\theta} = \tan\theta$ [3m]

(b) By using the substitution $u = \cos x$, or otherwise, evaluate $\int_0^{\frac{\pi}{3}} \tan x \, dx$ [4m]

(c) If ${}^{9}C_4 + {}^{9}C_5 = {}^{10}C_m$, find the value of m. [1m]

(d) Find the derivatives of:

(i) $\ell n(\sec 3x)$

(ii) $\tan^{-1}(2\tan x)$ [4m]

QUESTION 4.

(a) $P(4p, 2p^2)$ is a point on the parabola $x^2 = 8y$ and S is the focus. The tangent to the parabola at P meets the y-axis in M. The perpendicular from the focus S to the tangent PM meets the tangent in N.

(i) Write down the equation of PM and **hence** show that M has coordinates $(0, -2p^2)$. [1m]

(ii) Write down the equation of SN and **hence** find the coordinates of N. [4m]

(iii) Find the coordinates of the midpoint of the interval MN. [1m]

(iv) Find the equation of the locus of the midpoint MN as P varies. [1m]

(b) Use the binomial theorem to find the term in x^5 in the expansion $(1 + 2x)^8$. [2m]

(c) Give the exact value of $\cos^{-1}\left(\sin\dfrac{4\pi}{3}\right)$. [3m]

QUESTION 5.

(a) Prove, by mathematical induction, that $3^{2n} - 1$ is divisible by 8 for all positive integers. [3m]

(b) Rain is falling steadily and is collected in an inverted right cone so that the volume collected increases at a constant rate of $5 \text{ cm}^3/\text{h}$. If the radius r cm of the surface of the water is one third its depth, y cm, find the rate in cm/h at which the depth is increasing when $y = 3.5$. [5m]

(c) Find all angles θ with $0 \leq \theta \leq 2\pi$ for which $\cos 2\theta = \cos\theta$. [4m]

QUESTION 6.

(a) Find the term independent of x in the expansion of $\dfrac{1}{x}\left(3x - \dfrac{1}{2x}\right)^7$. [3m]

(b) A particle moves in a straight line and its position at any time t is given by:

$$x = 2\cos 3t - 5\sin 3t.$$

(i) Find the acceleration in terms of position and **hence** show that the motion is simple harmonic.

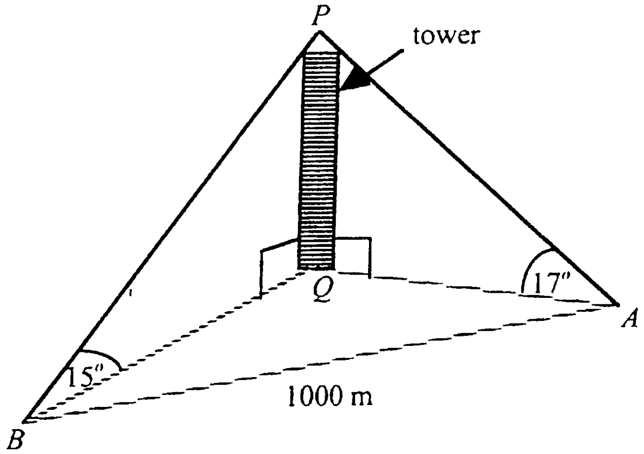
(ii) Find the greatest speed of the particle. [5m]

(c) (i) Show that $\dfrac{d}{dx}\left[e^x(\sin x + \cos x)\right] = 2e^x \cos x$. [4m]

(ii) **Hence**, evaluate: $\displaystyle\int_1^{\frac{\pi}{2}} e^x \cos x \, dx$ (correct to 3 significant figures). [4m]

QUESTION 7.

(a)



The angle of elevation of a tower PQ, of height h metres, at a point A due east of it, is 17°. From another point B, the bearing of the tower is 061°T and the angle of elevation is 15°. The points A and B are 1000 metres apart and on the same level as the base Q of the tower.

(i) Show that $\angle AQB = 151^\circ$.

(ii) Consider the ΔAPQ and show that $AQ = h \tan 73^\circ$.

(iii) Find a similar expression for BQ.

(iv) Calculate h, using the cosine rule, in the ΔAQB. (Answer to nearest metre). **[6m]**

(b) A cricket ball is projected from the ground with an initial velocity of $30\,\text{ms}^{-1}$ at an angle of 40° to the horizontal. The equations of motion taken in the horizontal and vertical directions are $\ddot{x} = 0$, $\ddot{y} = -10$. (Use $g = 10\,\text{ms}^{-2}$).

(i) Calculate the greatest height reached by the ball.

(ii) What is the speed of the ball at the greatest height?

(iii) How high is it after the ball has travelled 40 metres horizontally? **[6m]**

END OF EXAM

ANS (3u) Trial 2000.

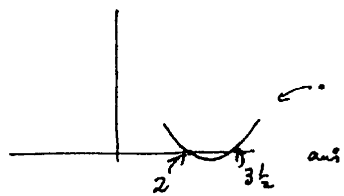
Q1. (a) (i) method 1: $\times (x-2)^2$: $(x+4)(x-2) > 5(x-2)^2$

$x^2+2x-8 > 5(x^2-4x+4)$

$0 > 4x^2-22x+28$

i.e. $2x^2-11x+14 < 0$

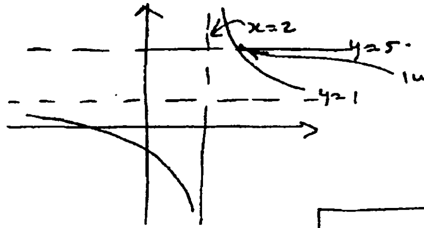
$(2x-7)(x-2) < 0$



ans

$\therefore \boxed{2 < x < 3\tfrac{1}{2}}$

method 2: sketch $y = \dfrac{x-2+6}{x-2} = 1 + \dfrac{6}{x-2}$.



inters: $x+4 = 5(x-2)$.

$= 5x-10$

$4x-14=0$

$x = \frac{14}{4} = \frac{7}{2} = 3\tfrac{1}{2}$.

$\therefore \boxed{2 < x < 3\tfrac{1}{2}}$

method 3: cases:

for $x > 2$: $x+4 > 5(x-2) \Rightarrow x+4 > 5x-10$.

i.e. $4x-14 < 0$ $\therefore x < 3\tfrac{1}{2}$

$\therefore 2 < x < 3\tfrac{1}{2}$ is part sol.

for $x < 2$: $x+4 < 5(x-2) \cdots\cdots \Rightarrow x > 3\tfrac{1}{2}$

no part sol. here

$\therefore \boxed{2 < x < 3\tfrac{1}{2}}$

(ii) $y^2-5y+6=0 \Rightarrow (y-2)(y-3)=0$

$\therefore x + \frac{1}{x} = 2, 3$

$x^2-2x+1=0$ or $x^2-3x+1=0$

$(x-1)^2=0$ $\quad x = \dfrac{3 \pm \sqrt{9-4}}{2}$

$\therefore x = 1, \dfrac{3\pm\sqrt{5}}{2}$

Q1. (b) (i) $y = \cos^3 2x$

$y' = 3\cos^2 2x \cdot -\sin 2x \cdot 2$ ← 2 marks, 1 off each mistake.

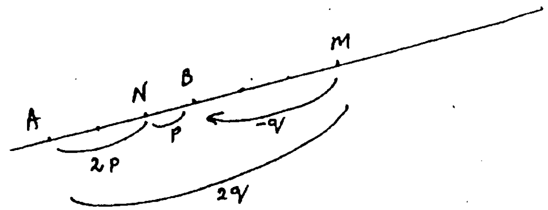
$= -6\sin 2x \cdot \cos^2 2x$

(ii) $y = e^{x \ln x}$

$y' = (1 \cdot \ln x + x \cdot \frac{1}{x}) e^{x\ln x}$ ← 2 marks, 1 off each mistake.

$= (1 + \ln x) e^{x \ln x}$.

(c)

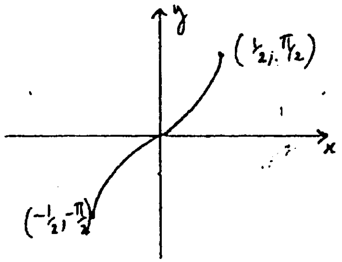


B divides MN in ratio 3:1

Q2. (a) $L = \lim_{x \to 0} \frac{\sin 5x}{2x} = \lim_{5x \to 0} \frac{\sin 5x}{5x} \cdot \frac{5}{2}$

$= 1 \times \frac{5}{2} = 2\frac{1}{2}$

(b) (i) $-1 \le 2x \le 1 \Rightarrow -\frac{1}{2} \le x \le \frac{1}{2}$



$y = \sin^{-1} 2x$

(ii) Domain $-\frac{1}{2} \le x \le \frac{1}{2}$

Range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(c) $I = \int_0^2 \frac{4\,dx}{\sqrt{4-x^2}} = 4\left[\sin^{-1}\frac{x}{2}\right]_0^2$

$= 4\{\sin^{-1} 1 - \sin^{-1} 0\}$

$= 4\{\frac{\pi}{2} - 0\}$

$= 2\pi$

(d) $m_1 = \frac{3}{4}, \quad m_2 = -\frac{1}{2}$

$\tan\theta = \left|\frac{\frac{3}{4} - -\frac{1}{2}}{1 + \frac{3}{4}(-\frac{1}{2})}\right| = \frac{5/4}{5/8} = 2$

acute.

$\therefore \theta = 180° - 63°26' = 116°34'$

Q3.

(a) $LHS = \frac{\sin\theta + 2\sin\theta\cos\theta}{1 + \cos\theta + 2\cos^2\theta - 1}$

$= \frac{\sin\theta(1 + 2\cos\theta)}{\cos\theta \ldots}$

$= \tan\theta$

$= RHS.$

(b) $u = \cos x$, $\quad x = \frac{\pi}{3} \Rightarrow u = \frac{1}{2}$

$du = -\sin x\,dx$, $\quad x = 0 \Rightarrow u = 1$

$I = -\int_0^{\frac{\pi}{3}} \frac{-\sin x\,dx}{\cos x}$

$= -\int_1^{\frac{1}{2}} \frac{du}{u}$

$= \int_{\frac{1}{2}}^1 \frac{du}{u}$

$= [\ln u]_{\frac{1}{2}}^1$

$= \ln 1 - \ln\frac{1}{2}$

$= 0 - -\ln 2$

$= \ln 2$

(c) $LHS = \frac{9!}{5!4!} + \frac{9!}{4!5!} = \frac{2 \times 9!}{5!4!} \times \frac{5}{5} = \frac{10!}{5!5!} = {}^{10}C_5$

$\therefore m = 5$ [note bald answer OK]

(d) (i) $\frac{d}{dx}(\ln(\sec 3x)) = \frac{3 \times \sec 3x \tan 3x}{\sec 3x}$

$= 3\tan 3x$

(ii) $\frac{d}{dx}(\tan^{-1}(2\tan x)) = \frac{2\sec^2 x}{1 + 4\tan^2 x}$.

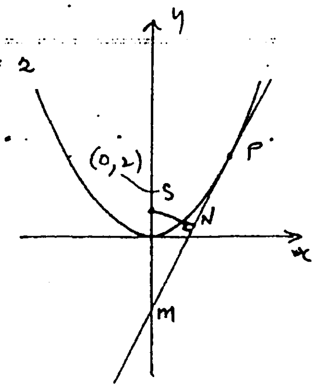
Q4.

(a) (i) PM is $x \cdot 4p = 4(y+2p^2)$ $\quad a=2$

i.e. $px = y + 2p^2$

cuts y axis: $x=0 \therefore y = -2p^2$

i.e. M is $\underline{(0,-2p^2)}$



(ii) gr. PM $= p$

$\therefore$ SN is $y - 2 = -\frac{1}{p}(x-0)$

i.e. $\underline{y = 2 - \frac{x}{p}}$.

N: $px = (2 - \frac{x}{p}) + 2p^2$

i.e. $p^2 x = 2p - x + 2p^3$

$x(p^2+1) = 2p(p^2+1)$

$\therefore x = 2p$, since $p^2+1 > 0$

& $y = 2 - \frac{2p}{p} = 0$

$\therefore$ $\underline{\text{N is } (2p,0)}$

(iii) mid pt of MN : $\left(\frac{0+2p}{2}, \frac{-2p^2+0}{2}\right)$

i.e. $\underline{(p,-p^2)}$

(iv) locus : $y = -p^2 = -x^2$

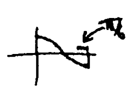
i.e. $\underline{y = -x^2}$

(b) $(1+2x)^8 = \binom{8}{0} + \binom{8}{1}(2x) + \dots + \binom{8}{5}(2x)^5 + \dots + (2x)^8$

$\therefore$ coeff of x^5 is $\binom{8}{5} \times 2^5 = \frac{8\times7\times6}{1\times2\times3} \times 32 = 7 \times 256$

$= 1792.$

(c) $\sin\frac{4\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$, $\cos^{-1}(-\frac{\sqrt{3}}{2})$



$\therefore$ Ans $= \pi - \frac{\pi}{6} = \underline{5\pi}$

Q5. (a) $n=1 \Rightarrow 3^{2\times1} - 1 = 9-1 = 8 \therefore$ div by 8 when $n=1$

Say $(n=k)$, $3^{2k} - 1 = 8P$ for some pos. int. k, P

then $3^{2(k+1)} - 1 = 3^{2k+2} - 1$

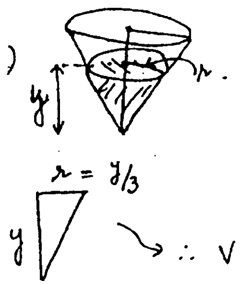
$= 9 \times (3^{2k} - 1) + 8$

$= 9 \times 8P + 8$

$= 8(9P+1)$ & $(9P+1)$ is an int.

$\therefore$ If div by 8 for some value of n then div by 8 for next value of n, and shown true for $n=1$ $\therefore$ true for all pos. int. n.

(b)



$V = \frac{1}{3}\pi r^2 y$

$\frac{dV}{dt} = 5$ cm³/h

want $\frac{dy}{dt}$ when $y = 3.5$

$\therefore V = \frac{\pi}{3} \cdot \frac{y^3}{3} = \frac{\pi y^3}{9}$

$\frac{dV}{dy} = \frac{\pi y^2}{3}$

And. $\frac{dV}{dt} = \frac{dV}{dy} \cdot \frac{dy}{dt}$

$5 = \frac{\pi y^2}{3} \cdot \frac{dy}{dt}$

$\therefore \frac{dy}{dt} = 5 \times \frac{3}{\pi \times (3.5)^2}$ at $y = 3.5$

$\underline{\doteq 1.2 \text{ cm/h}}.$

(c) $2\cos^2\theta - \cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

$\cos\theta = -\frac{1}{2}, 1$

$\theta = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}, 0, 2\pi$

i.e. $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, 0, 2\pi$.

Q6 (a) $\frac{1}{x}\cdot\binom{7}{k}(3x)^{7-k}\left(-\frac{1}{2x}\right)^k$

we want $x^{-1} \times x^{7-k} \times x^{-k} = 1 = x^0$

i.e. $6-2k=0$

$\therefore \underline{k=3}$

$\therefore \binom{7}{\cancel{k}3} 3^{7-\cancel{k}3}\left(-\frac{1}{2}\right)^{\cancel{k}3}$ is the req. term

i.e. $-\frac{7\times 6\times 5}{1\times 2\times 3}\times\frac{3^4}{2^3} = -\frac{35\times 81}{8}$

$\therefore \underline{-354\,3/8}$

(b) $x = 2\cos 3t - 5\sin 3t$

$\dot{x} = -6\sin 3t - 15\cos 3t$

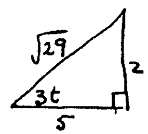
$\ddot{x} = -18\cos 3t + 45\sin 3t$

(i) $\therefore \underline{\ddot{x} = -9x}$ which is SHM

(ii) max speed is when $\ddot{x} = 0$

i.e. $x = 0 \therefore 2\cos 3t = 5\sin 3t$

$\therefore \frac{2}{5} = \tan 3t$



& max speed $= \left|-6\times\frac{2}{\sqrt{29}} - 15\times\frac{5}{\sqrt{29}}\right|$

$= \frac{87}{\sqrt{29}} \doteqdot 16.155 \doteqdot$ 16 speed units

(c)(i) $\frac{d}{dx}\left[e^x(\sin x + \cos x)\right] = e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$

$= 2e^x\cos x$. as req.

(ii) $I = \int_1^{\pi/2} e^x \cos x\,dx$

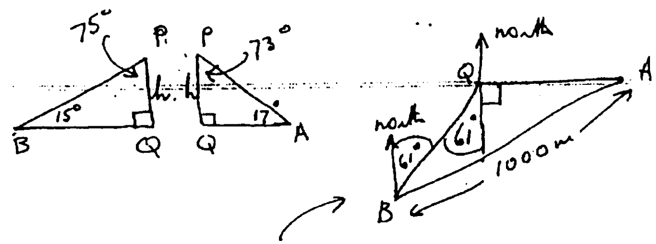
$= \frac{1}{2}\int_1^{\pi/2} 2e^x\cos x\,dx$

$= \frac{1}{2}\left[e^x(\sin x + \cos x)\right]_1^{\pi/2}$

$= \frac{1}{2}\left\{e^{\pi/2}(1+0) - e(\sin 1 + \cos 1)\right\}$ $\frac{1}{2}\{4.81047... - 3.75608...\}$

$\doteqdot \underline{0.527}$

Q7. (a)



(i) $\angle AQB = 61° + 90° = 151°$

(ii) in ΔAPQ: $\tan 73° = \frac{AQ}{h}$ $\therefore AQ = h\tan 73°$

(iii) in ΔBPQ: $\tan 75° = \frac{BQ}{h}$ $\therefore BQ = h\tan 75°$

(iv) in ΔABQ:

$1000^2 = (h\tan 73°)^2 + (h\tan 75°)^2 - 2(h\tan 73°)(h\tan 75°)\cos 151°$

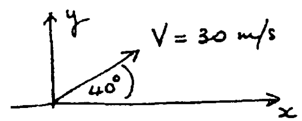
$= h^2\left[\tan^2 73° + \tan^2 75° - 2\tan 73°\tan 75°\cos 151°\right]$

$= h^2 \times 45.9796....$

$\therefore h = \frac{1000}{\sqrt{45.97....}} = 147.47...$

$\underline{= 147\ m.}$

(b)



$\ddot{x} = 0$ | $\ddot{y} = -10$

$\dot{x} = 30\cos 40°$ | $\dot{y} = -10t + 30\sin 40°$

$x = 30t\cos 40°$ | $y = -5t^2 + 30t\sin 40°$

(i) max ht: when $\dot{y} = 0$: $t = 3\sin 40°$

then ht $= -5(3\sin 40°)^2 + 90\sin^2 40°$

$= 45\sin^2 40°$

$\doteqdot \underline{18.6\ m.}$

(ii) Speed at top pt ($\dot{y}=0$) $= \dot{x} = 30\cos 40° \doteqdot 23$ m/s

(iii) $x = 40 \Rightarrow t = \frac{4}{3\cos 40°} \Rightarrow y = -5\times\left(\frac{4}{3\cos 40°}\right)^2 + \frac{4}{3\cos 40°}\times 30\sin 40°$

$= -15.14743... + 33.56....$

$\doteqdot \underline{18.4\ m.}$