

BY: -----

Trial Higher School Certificate Examination

2005



Mathematics

Extension 1

Total Marks – 84

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Attempt ALL questions.
- Begin each question on a new booklet
- Write your student number on each page
- All necessary working must be shown.
- Diagrams are not to scale.
- Board-approved calculators may be used.
- The mark allocated for each question is listed at the side of the question.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

| Question 1 (12 marks) | Marks |
|--|-------|
| a) Find the coordinates of the point P that divides AB internally in the ratio $2 : 3$ where A is $(-3, 5)$ and B is $(-6, -10)$ | 2 |
| b) Find the possible values of a if the lines $2x + 3y - 5 = 0$ and $ax + 2y + 3 = 0$ are inclined to each other at 45° | 4 |
| c) Solve for x : $\frac{2}{x-1} > 3$ | 3 |
| d) Find $\int \frac{x}{\sqrt{x-1}} dx$ using the substitution $x = u + 1$ | 3 |

Question 2 (12 marks)

- a) (i) Express $\sqrt{3} \sin x + \cos x$ in the form $R \sin(x + \alpha)$ 2
- (ii) Hence, sketch the graph of $y = \sqrt{3} \sin x + \cos x$ for $0 \leq x \leq 2\pi$ 2
- b) (i) Show that $f(x) = 2 \log_e x + 2x$ has a zero between $x = 0.5$ and $x = 1$ 1
- (ii) Starting with $x = 0.5$, use one application of Newton's method to find a better approximation for this zero. Write your answer correct to three significant figures 3
- c) Find $\int \frac{dx}{\sqrt{9 - 4x^2}}$ 2
- d) Find $\int \cos^2 4x \, dx$ 2

Question 3 (12 marks)

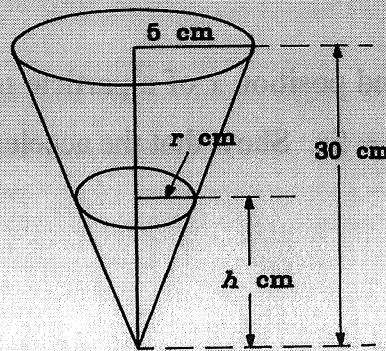
Marks

- a) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $4ay = x^2$ such that the chord PQ subtends a right angle at the vertex O
- (i) Show that $pq = -4$ 2
- (ii) Find the locus of the mid-point of PQ 3
- b) Show that $\int_0^3 \left(\frac{x}{x^2 + 9} + \frac{1}{x^2 + 9} \right) dx = \log_e \sqrt{2} + \frac{\pi}{12}$ 3
- c) If the roots of the equation $x^3 + bx^2 + cx + d = 0$ are in geometric progression show that $c^3 = b^3d$ 4

Question 4 (12 marks)

- a) A container is in the shape of an inverted right circular cone of base radius 5cm and height 30cm. Water is poured into the container at a rate of $2\text{cm}^3/\text{min}$

(i) Show that $r = \frac{h}{6}$



- (ii) Find the rate at which the level of water is rising when the water is 10cm deep

1

3

- b) (i) State the domain and range of $y = 2 \cos^{-1}\left(\frac{x}{3}\right)$

2

(ii) Hence sketch $y = 2 \cos^{-1}\left(\frac{x}{3}\right)$

1

- c) Given $f(x) = \sqrt[3]{x-1}$ for $x > 1$

- (i) Show that the function is monotonic increasing for all x in the given domain

2

- (ii) State the domain and range of $f^{-1}(x)$

1

- (iii) Find $f^{-1}(x)$ and explain why the inverse is a function

2

| Question 5 (12 marks) | Marks |
|--|-------|
| a) By induction show that $7^n - 3^n$ is divisible by 4 for all integers $n \geq 1$ | 3 |
| b) The velocity v and position x of a particle moving in a straight line are connected by the relation $v = 3 + 5x$. Show that the acceleration a of the particle is $5v$ | 2 |
| c) Find the term independent of x in the expansion of $(3 - x)^4 \left(1 + \frac{2}{x}\right)^7$ | 4 |
| d) Evaluate $\cos\left(2 \tan^{-1} \frac{3}{4}\right)$ without the use of a calculator | 3 |

Question 6 (12 marks)

- a) The cooling rate of a body is proportional to the difference between the temperature of the body and that of a surrounding medium ie. $\frac{dT}{dt} = -k(T - T_1)$ where T is the temperature of the cooling body and T_1 is the temperature of the surrounding medium
- (i) Show that $T - T_1 = Ae^{-kt}$ satisfies this equation 2
- (ii) A cup of coffee cools from 80° to 40° in 10 minutes when placed in a room with temperature 18° . How long will it take for the coffee's temperature to fall to 20° ? 4
- b) A particle is moving in a straight line such that its acceleration at time t seconds is $\ddot{x} = -4x$, where x is the displacement in metres from the origin. The particle is initially 6m to the right of the origin.
- (i) Find its displacement in terms of time 3
- (ii) Find the position and time when the particle first obtains a velocity of 6m/s 3

Question 7 (12 marks)

Marks

- a) (i) Differentiate $x(1+x)^n$

1

- (ii) Write the binomial expansion for $x(1+x)^n$

1

(iii) Hence show that $\sum_{r=0}^n (r+1) {}^n C_r = (n+2) 2^{n-1}$

3

- b) A particle is projected from a point O with an initial velocity of 60m/s at an angle of 30° to the horizontal. At the same instant a second particle is projected in the opposite direction with an initial velocity of 50m/s from a point level with O and 100m from O.

- (i) Show that the horizontal and vertical displacement equations of the first particle are given by:

$$x = 60\cos 30^\circ t \text{ and } y = 60\sin 30^\circ t - \frac{1}{2} gt^2$$

where g is acceleration due to gravity

- (ii) Find the angle of projection of the second particle if they collide

3

- (iii) Find the time at which the two particles collide

2

Extension 1 - Trial HSC Solutions 2005

(1) a) A(3, 5) B(-6, -10) Ratio 2:3

$$x = \frac{3x-3+2x-6}{2+3} \quad y = \frac{3+5+2x-10}{2+3}$$

$$= -\frac{21}{5} \quad = -1$$

$$\therefore P \text{ is } (-4\frac{1}{5}, -1)$$

b) $2x+3y-5=0 \quad ax+2y+3=0$

$$m_1 = -\frac{2}{3} \quad m_2 = -\frac{a}{2}$$

$$\therefore \tan 45^\circ = 1 = \left| \frac{-\frac{2}{3} + \frac{a}{2}}{1 + -\frac{2}{3} \times -\frac{a}{2}} \right|$$

$$1 = \left| \frac{\frac{-4+3a}{6}}{\frac{6+2a}{6}} \right|$$

$$\therefore |6+2a| = |3a-4|$$

$$\therefore 6+2a = 3a-4 \quad \text{or} \quad 6+2a = 4-3a$$

$$10 = a \quad 5a = -2 \quad a = -\frac{2}{5}$$

c) $\frac{2}{x-1} > 3 \quad x \neq 1$

$$2(x-1) > 3(x-1)^2$$

$$2(x-1) - 3(x-1)^2 > 0$$

$$(x-1)[2-3(x-1)] > 0$$

$$(x-1)(5-3x) > 0.$$

$$\therefore 1 < x < \frac{5}{3}$$

d) $\int \frac{x}{\sqrt{x-1}} dx \quad x = u+1 \quad du = dx$

$$= \int \frac{u+1}{\sqrt{u}} du$$

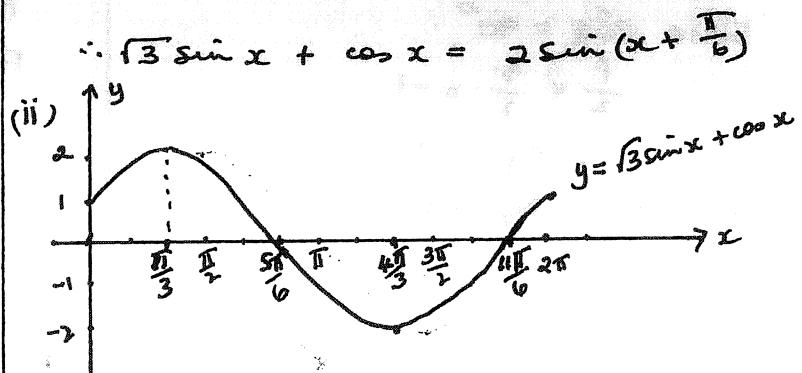
$$= \int u^{\frac{1}{2}} + u^{-\frac{1}{2}} du$$

$$= \frac{2u^{3/2}}{3} + 2u^{1/2} + C$$

$$= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C$$

(2) a) i) $\sqrt{3}\sin x + \cos x \equiv R \sin(x+\alpha)$
 $\sqrt{3}\sin x + \cos x \equiv R \sin x \cos \alpha + R \cos x \sin \alpha$
 $\therefore R \cos \alpha = \sqrt{3} \quad R \sin \alpha = 1.$
 $\therefore R^2 (\cos^2 \alpha + \sin^2 \alpha) = 3+1$
 $\therefore R = 2 \quad R > 0.$

and $\frac{R \sin \alpha}{R \cos \alpha} = \frac{1}{\sqrt{3}}$
 $\therefore \tan \alpha = \frac{\pi}{6}.$



b) i) $f(x) = 2 \log_e x + 2x.$

$$f(0.5) \doteq -0.386$$

$$f(1) = 2.$$

∴ Since sign change a zero lies between $\frac{1}{2}$ and 1.

ii) $f'(x) = \frac{2}{x} + 2.$

$$\text{If } z_1 = 0.5$$

$$\text{then } z_2 = 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$$= 0.5 - \frac{(2 \ln 0.5 + 1)}{6}$$

$$\doteq 0.56438 \dots$$

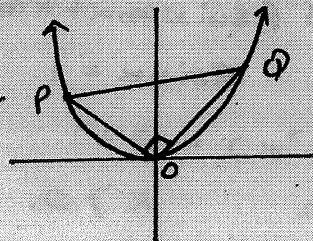
$$\doteq 0.564 \text{ (to 3 sig. fig.)}$$

c) $\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2} \sin^{-1} \frac{2x}{3} + C$

d) $\int \cos^2 4x dx = \frac{1}{2} \int 1 + \cos 8x dx$
 $= \frac{1}{2} \left(x + \frac{\sin 8x}{8} \right) + C$

$$= \frac{x}{2} + \frac{\sin 8x}{16} + C$$

$$(3) a) P(2ap, ap^2), Q(2aq, aq^2)$$



$$i) \text{ m of } OP = \frac{ap^2 - 0}{2ap - 0} = \frac{p}{2}$$

$$\text{m of } OQ = \frac{aq^2 - 0}{2aq - 0} = \frac{q}{2}$$

Since $\hat{POQ} = 90^\circ$

$$\frac{p}{2} \times \frac{q}{2} = -1$$

$$\therefore pq = -4$$

$$ii) \text{ midpt } PQ = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2} \right)$$

$$= \left(a(p+q), a\left(\frac{p^2+q^2}{2}\right) \right)$$

$$x = a(p+q)$$

$$\therefore p+q = \frac{x}{a}$$

$$y = \frac{a(p^2+q^2)}{2}$$

$$\frac{2y}{a} = (p+q)^2 - 2pq$$

$$= \left(\frac{x}{a}\right)^2 - 2x - 4$$

$$\frac{2y}{a} = \frac{x^2}{a^2} + 8$$

$$2ay = x^2 + 8a^2$$

$$x^2 = 2a(y - 4a)$$

$$(b) \int_0^3 \frac{x}{x^2+9} + \frac{1}{x^2+9} dx$$

$$= \left[\frac{1}{2} \ln(x^2+9) + \frac{1}{3} \tan^{-1} \frac{x}{3} \right]_0^3$$

$$= \left(\frac{1}{2} \ln 18 + \frac{1}{3} \tan^{-1} 1 \right) - \left(\frac{1}{2} \ln 9 + 0 \right)$$

$$= \frac{1}{2} \ln 2 + \frac{1}{3} \times \frac{\pi}{4}$$

$$= \ln \sqrt{2} + \frac{\pi}{12} \quad \text{as req.}$$

$$x^3 + bx^2 + cx + d = 0$$

Let roots be $\frac{\alpha}{r}, \alpha, \alpha r$

$$\therefore \frac{\alpha}{r} + \alpha + \alpha r = -b \quad (1)$$

$$\frac{\alpha^2}{r} + \alpha^2 + \alpha^2 r = c \quad (2)$$

$$\frac{\alpha}{r} \times \alpha \times \alpha r = -d \quad (3)$$

$$\therefore \alpha^3 = -d$$

$$\text{From (1)} : \alpha \left(\frac{1}{r} + 1 + r \right) = -b$$

$$\text{From (2)} : \alpha^2 \left(\frac{1}{r} + 1 + r \right) = c$$

$$\therefore \frac{1}{r} = -\frac{b}{c}$$

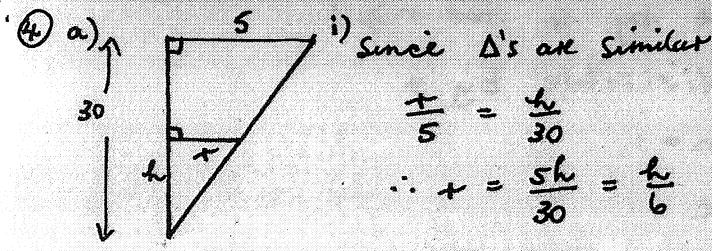
$$\alpha = \frac{c}{-b}$$

$$\therefore \left(\frac{c}{-b} \right)^3 = -d$$

$$\frac{c^3}{-b^3} = -d$$

$$\therefore c^3 = b^3 d$$

as req.



ii) Vol of cone = $\frac{1}{3}\pi r^2 h$

$$\therefore V = \frac{1}{3}\pi \times \left(\frac{h}{6}\right)^2 \times h$$

$$= \frac{\pi h^3}{108}$$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$2 = \frac{3\pi h^2}{108} \times \frac{dh}{dt}$$

∴ when $h = 10$: $2 = \frac{300\pi}{108} \times \frac{dh}{dt}$

$$\therefore \frac{dh}{dt} = \frac{216}{300\pi}$$

$$= \frac{18}{25\pi}$$

∴ water is rising at $\frac{18}{25\pi}$ cm/min

(b) i) $y = 2 \cos^{-1} \frac{x}{3}$

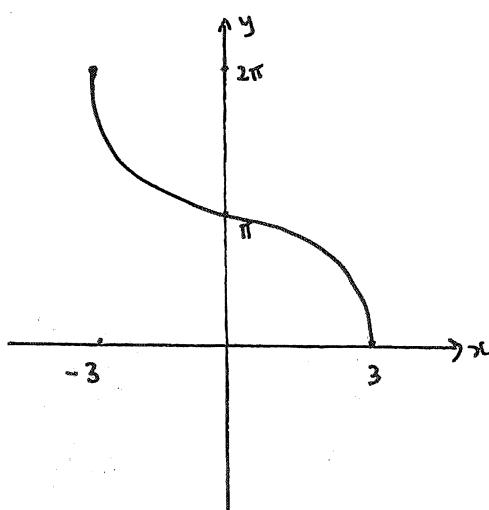
D: $-1 \leq \frac{x}{3} \leq 1$

$-3 \leq x \leq 3$

R: $0 \leq \frac{y}{2} \leq \pi$

$0 \leq y \leq 2\pi$

ii)



c) $f(x) = \sqrt[3]{x-1} \quad x > 1$

i) $f(x) = (x-1)^{1/3}$

$$f'(x) = \frac{1}{3}(x-1)^{-2/3}$$

$$= \frac{1}{3\sqrt[3]{(x-1)^2}}$$

Since $(x-1)^2$ is positive for all x

$$\sqrt[3]{(x-1)^2} > 0$$

$$\therefore \frac{1}{3\sqrt[3]{(x-1)^2}} > 0 \quad \text{for all } x$$

∴ $f(x)$ is monotonic increasing

(ii) For $f(x)$: D: $x > 1$

R: $y > 0$

∴ For $f^{-1}(x)$: D: $x > 0$

R: $y > 1$

(iii) $y = (x-1)^{1/3}$

For inverse: $x = (y-1)^{1/3}$

$$x^3 = y-1$$

$$\therefore y = x^3 + 1$$

Since $f(x)$ is monotonic increasing and it passes horizontal line test inverse will also be a function

5. a) Assertion: that $7^n - 3^n$ is divisible by 4 for $n \geq 1$

For $n=1$: $7^1 - 3^1 = 4$ which is divisible by 4

\therefore Assertion is true for $n=1$.

Assume assertion is true for $n=k$

i.e. that $7^k - 3^k$ is divisible by 4

i.e. $7^k - 3^k = 4M$ (where M is a positive integer)

We need to prove that:

$7^{k+1} - 3^{k+1}$ is also divisible by 4.

$$7^{k+1} - 3^{k+1} = 7 \cdot 7^k - 3 \cdot 3^k$$

$$= (8-1) \cdot 7^k - (4-1) \cdot 3^k$$

$$= 8 \cdot 7^k - 7^k - 4 \cdot 3^k + 3^k$$

$$= 8 \cdot 7^k - 4 \cdot 3^k - (7^k - 3^k)$$

$$= 8 \cdot 7^k - 4 \cdot 3^k - 4M \text{ using assumption}$$

$$= 4(2 \cdot 7^k - 3^k - M)$$

$$= 4J \text{ where } J \text{ is a positive integer}$$

$\therefore 7^{k+1} - 3^{k+1}$ is divisible by 4.

\therefore If statement is true for $n=k$, it is true for $n=k+1$.

\therefore Since statement is true for $n=1$, it is true for $n=2$ and by induction it is true for all $n \geq 1$.

b) $v = 3 + 5x$

Since $\frac{d}{dx}(\frac{1}{2}v^2) = \ddot{x}$

$$\begin{aligned} \frac{d}{dx} \left(\frac{1}{2}(3+5x)^2 \right) &= 2x \cdot \frac{1}{2} (3+5x) \times 5 \\ &= 5(3+5x) \\ &= 5v \end{aligned}$$

$$\therefore \text{acceleration} = 5v.$$

c) $(3-x)^4 \left(1 + \frac{2}{x}\right)^7$

$$(3-x)^4 = {}^4C_0 3^4 - {}^4C_1 3^3 x + {}^4C_2 3^2 x^2 - {}^4C_3 3 x^3 + {}^4C_4 x^4$$

$$\left(1 + \frac{2}{x}\right)^7 = {}^7C_0 + {}^7C_1 \frac{2}{x} + {}^7C_2 \frac{4}{x^2} + {}^7C_3 \frac{8}{x^3} + {}^7C_4 \frac{16}{x^4} \dots$$

$$\therefore \text{Term independent of } x = {}^4C_0 {}^7C_0 - {}^4C_1 {}^7C_1 x^2 + {}^4C_2 {}^7C_2 x^4 - {}^4C_3 {}^7C_3 x^8 + {}^4C_4 {}^7C_4 x^{16}$$

$$= 81 - 1512 + 4536 - 3360 + 560$$

$$= 305$$

d) next page

$$(6) \text{ a) i) } \frac{dT}{dt} = -k(T - T_1)$$

$$T - T_1 = Ae^{-kt}$$

$$\therefore T = T_1 + Ae^{-kt}$$

$$\text{LHS} = \frac{dT}{dt}$$

$$= -k \cdot Ae^{-kt}$$

$$\text{RHS} = -k(T - T_1)$$

$$= -k(T_1 + Ae^{-kt} - T_1)$$

$$= -k \cdot Ae^{-kt}$$

$$= \text{LHS}$$

$\therefore T - T_1 = Ae^{-kt}$ satisfies eqn. (2)

$$\text{ii) } T_1 = 18$$

$$t=0 : T = 80$$

$$\therefore 80 = 18 + A \times 1$$

$$\therefore A = 62$$

$$\therefore T = 18 + 62e^{-kt}$$

$$t=10, T = 40$$

$$40 = 18 + 62e^{-10k}$$

$$\frac{22}{62} = e^{-10}$$

$$\therefore k = \frac{\ln \frac{11}{31}}{-10}$$

$$T = 20 :$$

$$20 = 18 + 62e^{-kt}$$

$$\frac{2}{62} = e^{-kt}$$

$$t = \frac{\ln \frac{1}{31}}{-\left(\frac{\ln \frac{1}{31}}{-10}\right)}$$

$$= 33.14 \text{ min (2 d.p.)}$$

b) i) Since $\ddot{x} = -4x$ particle is moving in SHM about origin

$$\therefore x = a \cos(nt + \alpha)$$

$$= b \cos(2t + \alpha)$$

$$t=0, x=b :$$

$$\therefore b = b \cos \alpha$$

$$\cos \alpha = 1$$

$$\therefore \alpha = 0$$

$$\therefore x = b \cos 2t$$

$$\text{ii) } v = -12 \sin 2t$$

$$v=6 : 6 = -12 \sin 2t$$

$$\therefore \sin 2t = -\frac{1}{2}$$

$$2t = \frac{7\pi}{6}$$

$$t = \frac{7\pi}{12}$$

1.83

$$t = \frac{7\pi}{12} : x = b \cos 2t \times \frac{7\pi}{12}$$

$$= b \cos \frac{7\pi}{6}$$

$$= 6 \times -\frac{\sqrt{3}}{2}$$

$$= -3\sqrt{3}$$

5.196

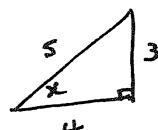
\therefore Particle first reaches 6 m/sec after $\frac{7\pi}{12}$ secs., $3\sqrt{3}$ metres to left of origin.

5d

$$\cos(2 \tan^{-1} \frac{3}{4}) = \cos 2x$$

$$\text{where } x = \tan^{-1} \frac{3}{4}$$

$$\therefore \tan 2x = \frac{3}{4}$$



$$\therefore \cos 2x = 2 \cos^2 x - 1$$

$$= 2 \left(\frac{4}{5}\right)^2 - 1$$

$$= \frac{32}{25} - 1$$

$$= \frac{7}{25}$$

(4)

$$\textcircled{7} \quad \text{i)} \quad \frac{d}{dx} \left(x(1+x)^n \right) = (1+x)^n \times 1 + x \times n (1+x)^{n-1}$$

$$= (1+x)^n + nx (1+x)^{n-1}.$$

$$\text{ii)} \quad x(1+x)^n = x^0 C_0 + x^1 C_1 x + x^2 C_2 x^2 + \dots + x^n C_n x^n$$

$$= {}^n C_0 x + {}^n C_1 x^2 + {}^n C_2 x^3 + \dots + {}^n C_n x^{n+1}$$

$$\text{iii)} \quad \sum_{r=0}^n (r+1) {}^n C_r = {}^n C_0 + 2 {}^n C_1 + 3 {}^n C_2 + \dots + (n+1) {}^n C_n$$

from (ii): $\frac{d}{dx} x(1+x)^n = {}^n C_0 + 2 {}^n C_1 x + 3 {}^n C_2 x^2 + \dots + (n+1) {}^n C_n x^n$

\therefore from (i): $(1+x)^n + nx(1+x)^{n-1} = {}^n C_0 + 2 {}^n C_1 x + 3 {}^n C_2 x^2 + \dots + (n+1) {}^n C_n x^n$

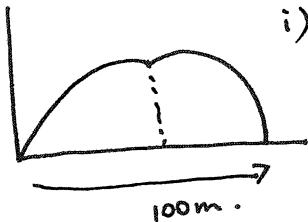
Let $x=1$:

$$2^n + n(2)^{n-1} = {}^n C_0 + 2 {}^n C_1 + 3 {}^n C_2 + \dots + (n+1) {}^n C_n$$

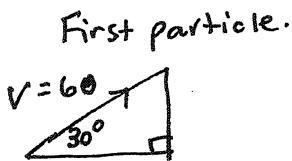
$$\therefore 2^{n-1}(2+n) = {}^n C_0 + 2 {}^n C_1 + 3 {}^n C_2 + \dots + (n+1) {}^n C_n.$$

$$\therefore \sum_{r=0}^n (r+1) {}^n C_r = 2^{n-1} (n+2)$$

b)



i)



First particle.

$$\begin{aligned} x &= 0 \\ \dot{x} &= c = 60 \cos 30 \\ x &= \int 60 \cos 30 dt \\ &= 60 \cos 30 t + k \end{aligned}$$

$$t=0 \quad x=0 \quad \therefore k=0$$

$$\therefore x = 60 \cos 30 t$$

$$\begin{aligned} \ddot{x} &= -g \\ \ddot{y} &= -gt + N \\ t=0 \quad \ddot{y} &= 60 \sin 30 \\ \therefore \ddot{y} &= -gt + 60 \sin 30 \\ y &= \int \ddot{y} dt \\ &= -\frac{gt^2}{2} + 60 \sin 30 t \\ t=0 \quad y=0 \quad \therefore N &= 0 \end{aligned}$$

$$\therefore y = -\frac{gt^2}{2} + 60 \sin 30 t$$

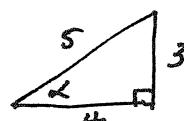
ii) When particles collide height above ground is same

$$\therefore 60 \sin 30 t - \frac{1}{2} gt^2 = 50 \sin 36^\circ 52' t - \frac{1}{2} gt^2$$

$$30t = 50 \sin 36^\circ 52'$$

$$\therefore \sin 36^\circ 52' = \frac{3}{5}$$

$$\therefore \alpha = 36^\circ 52'$$



iii) x values add to 100 m.

$$\therefore 60 \cos 30 t + 50 \sin 36^\circ 52' t = 100$$

$$30\sqrt{3} t + 50 \times \frac{4}{5} t = 100$$

$$\frac{100}{100} = 1.087 \text{ secs (to 3dp)}$$