THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 1 2017

MATH1141
HIGHER MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS APPENDED TO THE PAPER

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 1

1. i) Evaluate each limit, giving brief reasons for your answer.

a) $\lim_{x \to \infty} \dfrac{2e^x + \cos x}{6e^x - \sin x}$

b) $\lim_{x \to 0} \dfrac{e^{2x} - 1}{x^2 - 3x}$

ii) Evaluate each of the following integrals:

a) $\displaystyle\int \cos x \sin^5 x \, dx$,

b) $\displaystyle\int \frac{1}{\sqrt{25 + 9x^2}} \, dx$.

iii) The floor of x is denoted $\lfloor x \rfloor$ and gives the greatest integer less than or equal to x. For example, $\lfloor 3.1 \rfloor = 3$.
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \lfloor 3x - 1 \rfloor.$$

a) Write down the value of $f(\frac{1}{2})$.

b) By considering left and right limits, state whether or not f is continuous at $x = \frac{1}{3}$.

iv) Let $z = \dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}$.

a) Write z in polar form.

b) Find the smallest positive integer n such that $z^n = 1$.

c) Find all possible positive integers m such that $z^m = 1$.

v) Consider the following Maple output

```
> with(LinearAlgebra):
> A := <<607,207,75>|<-2286,-783,-288>|<1414,483,176>>;
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$$A := \begin{bmatrix} 607 & -2286 & 1414 \\ 207 & -783 & 483 \\ 75 & -288 & 176 \end{bmatrix}$$

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> A^2;
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$$\begin{bmatrix} 1297 & -4896 & 3024 \\ -207 & 783 & -483 \\ -891 & 3366 & -2078 \end{bmatrix}$$

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> A^3;
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$$\begin{bmatrix} 607 & -2286 & 1414 \\ 207 & -783 & 483 \\ 75 & -288 & 176 \end{bmatrix}$$

a) Find A^{100}.

b) Is A invertible? Give reasons for your answer.

vi) a) Express $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$.

b) Find constants a, b and c such that

$$\sin^4\theta = a\cos 4\theta + b\cos 2\theta + c.$$

c) Hence or otherwise evaluate

$$\int \sin^4\theta \, d\theta.$$

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 2

2. i) Find the gradient $\dfrac{dy}{dx}$ of the curve defined by $xy + y^2 = 4$ at the point where $y = 1$.

ii) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x - \cos x$.

a) Show that f has a zero in the interval $\left[0, \dfrac{\pi}{2}\right]$.

b) What is the minimum value of f'?

c) Explain why f has an inverse on $\mathbb{R}$.

d) Find the value of $g'(-1)$, where g is the inverse of f.

iii) Sketch the graph of the polar curve $r = 2 - \cos 2\theta$ for $0 \leq \theta \leq \pi$.

iv) For some values of the real parameters a, b, c and d, the curve

$$ax^2 + by^2 + cx + dy = 1$$

passes through the points $A(1, 1), B(2, 3)$ and $C(0, 1)$.

a) Explain why the following equations can be used to determine the values of a, b, c and d for which the curve passes through the points.

$$\begin{array}{rcrcrcrcl} a & + & b & + & c & + & d & = & 1 \\ 4a & + & 9b & + & 2c & + & 3d & = & 1 \\ & & b & & & + & d & = & 1. \end{array}$$

b) Use Gaussian Elimination to solve the system of linear equations in part (a).

c) Are there zero, one, or infintely many curves of the form $ax^2 + by^2 + cx + dy = 1$ which pass through the points A, B and C?

d) Using your answer from part (b), find the parabola of the form $y = \alpha x^2 + \beta x + \gamma$ which passes through A, B and C.

v) a) Write $\begin{pmatrix} 34 \\ -9 \\ -23 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 4 \\ 1 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

b) What is the volume of the parallelepiped spanned by the vectors $\begin{pmatrix} 34 \\ -9 \\ -23 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 1 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$?

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 3

3. i) a) Let $\mathbf{a}$ and $\mathbf{b}$ be the coordinate vectors of points A and B on a plane. Find a formula, giving reasons, for the coordinate vector of the mid-point of AB in terms of $\mathbf{a}$ and $\mathbf{b}$.

b) Consider a quadrilateral $OABC$ in the plane. Show, using vector geometry, that the diagonals bisect each other if and only if the quadrilateral is a parallelogram.

ii) a) Prove

$$\cos(\theta) + \cos(3\theta) + \ldots + \cos((2n+1)\theta) = \frac{\cos((n+1)\theta)\sin((n+1)\theta)}{\sin\theta}.$$

b) Hence or otherwise show

$$\cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right) = \frac{1}{2}.$$

iii) Consider the line L defined by $\dfrac{x-1}{2} = y+3 = z-1$ and the plane P defined by $x - y + 2z = 9$.

a) Find a parametric form for the line L.

b) Find the point of intersection of the line L with the plane P.

c) Find all values of α such that the line $\dfrac{x-1}{\alpha} = y+3 = z-1$ is parallel to the plane P.

iv) a) Show that

$$\det \begin{pmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{pmatrix} = (t_1 - t_2)(t_1 - t_3)(t_2 - t_3).$$

b) Suppose that t_1, t_2, t_3 are distinct real numbers. Prove that for any $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, there is exactly one polynomial $p(t)$ of degree ≤ 2 with $p(t_i) = \alpha_i, i = 1, 2, 3$.

v) Let P be the plane in $\mathbb{R}^3$ given by the Cartesian equation $x + y + z = 0$. Let L be a line parallel to P and passing through the point $(1, 1, -1)$ and L' be a line parallel to P passing through the point $(1, 2, 2)$. In the case when L and L' are not parallel, find the distance between L and L'.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 4

4. i) a) Clearly state the Mean Value Theorem.

b) Find the derivative of the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \int_1^{e^x} \sin^2(t^2)\, dt.$$

c) Use the Mean Value Theorem to show that

$$f(x) < xe^x$$

for all $x > 0$.

ii) Consider the motion of a point $(x(t), y(t))$ on the plane, where t denotes time.

a) If you are further given that the slope of the curve generated by the point is

$$\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$$

for $y \neq 0$ and $a, b > 0$, use the chain rule to show that the quantity

$$I = b^2x^2 + a^2y^2$$

is constant as a function of time.

b) Whenever $y = b$, the point is moving horizontally. Find the equation of the curve in implicit form.

iii) Consider the function $g : [1, \infty) \to \mathbb{R}$, $g(x) = e^{-x^2} + x^{-2}$. For $b \geq 1$, let $A(b)$ denote the area of the region enclosed by the graph of g, the x-axis and the lines $x = 1$ and $x = b$.

a) Show that $A(b)$ has a limit as $b \to \infty$.

b) Find a number L such that $A(b) \leq L$ for all $b \geq 1$.

c) Show that the graph of A does not have a point of inflection.

iv) a) Show that

$$\lim_{s\to\infty} se^{-s^2} = 0$$

and briefly explain why this implies that

$$\lim_{h\to 0} \frac{e^{-1/h^2}}{h} = 0.$$

b) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is differentiable at 0 with $f'(0) = 0$ and hence determine an expression for $f'(x)$.

c) Given that

$$\lim_{h\to 0} \frac{e^{-1/h^2}}{h^4} = 0,$$

show that $f''(0) = 0$.

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BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2-x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, && |x| < a \\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, && |x| > a > 0 \\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, && x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$