PAST EXAM SOLUTIONS

MATH1131 NOVEMBER 2010 Solutions

1. i) a) $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

b) $\text{Arg}\, z = \frac{\pi}{4}$

c)

$$\begin{aligned} z &= \sqrt{2} e^{i\frac{\pi}{4}} \\ z^{28} &= \left(\sqrt{2} e^{i\frac{\pi}{4}}\right)^{28} \\ &= (2)^{\frac{28}{2}} e^{i\frac{28\pi}{4}} \\ &= 2^{14}\left(\cos 7\pi \ + i\sin 7\pi \ \right) \\ &= -2^{14} (= -16384) \end{aligned}$$

ii)

$$\begin{aligned} (x+iy)(3+2i) &= 4+7i \\ x+iy &= \frac{4+7i}{3+2i} \\ &= \frac{(4+7i)}{(3+2i)}\frac{(3-2i)}{(3-2i)} \\ &= \frac{26+13i}{13} \\ &= 2+i \end{aligned}$$

Equating real and imaginary parts, $x = 2$ and $y = 1$.

iii) a)

$$z^6 = 1 = e^{2k\pi i}, \qquad k \in \mathbb{Z}$$

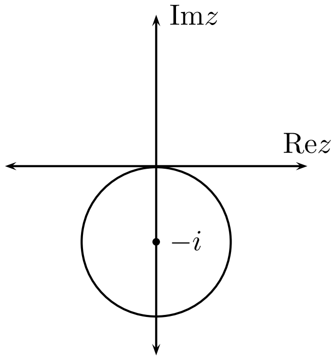
Hence $z = e^{2ki\pi/6}$. Substituting appropriate values for k, the solutions are

$$1, e^{i\pi/3}, e^{-i\pi/3}, e^{2i\pi/3}, e^{-2i\pi/3}, -1.$$

b)

$$\begin{aligned} \left(z^6 - 1\right) &= (z-1)\left(z - e^{\frac{i\pi}{3}}\right)\left(z - e^{-i\frac{\pi}{3}}\right)\left(z - e^{i\frac{2\pi}{3}}\right)\left(z - e^{-i\frac{2\pi}{3}}\right)(z+1) \\ &= (z-1)(z+1)\left(z^2 - 2z\cos(\pi/3) + 1\right)\left(z^2 - 2z\cos(2\pi/3) + 1\right) \\ &= \left(z^2 - z + 1\right)\left(z^2 + z + 1\right). \end{aligned}$$

iv) $|z + i| = 1$ represents a circle, centred at $z = -i$, radius 1.



v) From MAPLE, $A^6 = 64I$. Multiplying by $(A^2)^{-1}$, we have

$$\begin{aligned} A^4 &= 64(A^2)^{-1} \\ \textit{so} \quad \left(A^2\right)^{-1} &= \frac{1}{64}A^4 = \frac{1}{64}\begin{bmatrix} -8 & 0 & -8\sqrt{3} \\ 0 & -16 & 0 \\ 8\sqrt{3} & 0 & -8 \end{bmatrix} \end{aligned}$$

vi) a)

$$\begin{aligned} 200a + 600b + 400c &= 12000 \\ 1800a + 2400b + 2000c &= 65600 \\ 20000a + 30000b + 26000c &= 784000 \end{aligned}$$

b)

$$\begin{aligned} &\left[\begin{array}{ccc|c} 200 & 600 & 400 & 12000 \\ 1800 & 2400 & 2000 & 65600 \\ 20000 & 30000 & 26000 & 784000 \end{array}\right] \\ \longrightarrow &\left[\begin{array}{ccc|c} 2 & 6 & 4 & 120 \\ 18 & 24 & 20 & 656 \\ 20 & 30 & 26 & 784 \end{array}\right] \\ \xrightarrow[R_3 - 10R_1]{R_2 - 9R_1} &\left[\begin{array}{ccc|c} 2 & 6 & 4 & 120 \\ 0 & -30 & -16 & -424 \\ 0 & -30 & -14 & -416 \end{array}\right] \\ \xrightarrow{R_3 + R_2} &\left[\begin{array}{ccc|c} 2 & 6 & 4 & 120 \\ 0 & 30 & 16 & 424 \\ 0 & 0 & 2 & 8 \end{array}\right] \end{aligned}$$

Hence,

$$\begin{aligned} c &= 4 \\ 30b + 16c &= 424 \\ b &= 12 \\ 2a + 6b + 4c &= 120 \\ a &= 16 \end{aligned}$$

i.e. $(a, b, c) = (16, 12, 4)$.

2. i) a)

$$PP^T = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 13 \end{pmatrix}.$$

b) P^TP is a 3×3 matrix.

ii) a) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}, \overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}.$

b) The plane has equation $\mathbf{x} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\-2\\-2\end{pmatrix} + \mu\begin{pmatrix}2\\2\\4\end{pmatrix}, \qquad \lambda, \mu \in \mathbb{R}.$

iii) Substituting the $x = 1+t, y = 2-y, z = 5+t$ into the equation of the plane, we have

$$(1+t) - 3(2-t) + (5+t) = 15 \Rightarrow t = 3.$$

Hence, the point of intersection is $\mathbf{x} = \begin{pmatrix}1\\2\\5\end{pmatrix} + 3\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}4\\-1\\8\end{pmatrix}.$

iv) Since the vectors are perpendicular, their dot product is 0, hence

$$1 \times -2 + 3 \times -6 - \beta + 5 \times -10 = 0 \Rightarrow \beta = -70.$$

v) a) The magnitude of $\mathbf{u}$ is $\sqrt{2^2+1^2+7^2} = \sqrt{54} = 3\sqrt{6}$.
b) The vector $\frac{10}{3\sqrt{6}}\mathbf{u}$ is parallel to $\mathbf{u}$ and has length 10.

vi) One possible method is to row-reduce first, but in this case it is probably easier to expand across the first row.

$$\det(C) = 3\det\begin{pmatrix}1 & 7\\2 & 0\end{pmatrix} - 1 \times \det\begin{pmatrix}4 & 7\\1 & 0\end{pmatrix} + 0 = -35.$$

vii) If $\text{proj}_{\mathbf{w}}(\mathbf{u}) = \text{proj}_{\mathbf{w}}(\mathbf{v})$ then

$$\frac{\mathbf{u}.\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w} = \frac{\mathbf{v}.\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}$$
$$\Rightarrow \mathbf{u}.\mathbf{w} = \mathbf{v}.\mathbf{w}$$

since $\mathbf{w}$ is a non-zero vector. Hence, using the properties of dot product, $(\mathbf{u}-\mathbf{v}).\mathbf{w} = 0$ and so $\mathbf{u}-\mathbf{v}$ is perpendicular to $\mathbf{w}$.

3. i) Since both the numerator and denominator are 0 at $x = 0$, we apply L'Hôpital's rule.

$$L = \lim_{x\to 0} \frac{x^2 e^x}{1-\cos(\pi x)} = \lim_{x\to 0} \frac{(2x+x^2)e^x}{\pi \sin(\pi x)}.$$

Again, both the numerator and denominator are 0 at $x = 0$, so we apply L'Hôpital's rule a second time.

$$L = \lim_{x\to 0} \frac{(2+4x+x^2)e^x}{\pi^2\cos(\pi x)} = \frac{2}{\pi^2}.$$

ii) As a necessary condition, f must be continuous at $x = 1$ so

$$\lim_{x\to 1^+} f(x) = f(1).$$

Hence

$$\lim_{x\to 1^+} (-x^2 + ax + b) = 1 \Rightarrow a + b = 2.$$

For differentiability, we need to show that

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}.$$

Now

$$\text{LHS} = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{(x+h)^2-x^2}{h} = 2$$

and

$$\text{RHS} = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{-(x+h)^2+a(x+h)+b-(-x^2+ax+b)}{h} = -2+a.$$

Hence we have $a=4$ and from $a+b=2$ it follows that $b=-2$.

iii) a)

$$L = \lim_{x\to\infty} \frac{x^2-2}{x^2+1} = \lim_{x\to\infty} \frac{1-\frac{2}{x^2}}{1+\frac{1}{x^2}} = 1.$$

b)

$$\left|\frac{x^2-2}{x^2+1} - 1\right| = \frac{3}{x^2+1} < \frac{3}{x^2}.$$

Hence, if $\frac{3}{x^2} < \varepsilon$ then $\left|\frac{x^2-2}{x^2+1} - 1\right| < \varepsilon$. Now $\frac{3}{x^2} < \varepsilon \Rightarrow x > \sqrt{\frac{3}{\varepsilon}}$ so let $M = \sqrt{\frac{3}{\varepsilon}}$ and then if $x > M$, we have $\left|\frac{x^2-2}{x^2+1} - 1\right| < \varepsilon$.

iv) a) The function f is continuouson the closed interval $[0,2]$ and $f(0) = -5 < 0$, while $f(2) = 3+2\sqrt{3} > 0$. Hence by the intermediate value theorem, f has at least one positive real root in the interval $[0,2]$.

b) Since $f'(x) = 3x^2+\sqrt{3} > 0$, the function f is increasing. Hence f has exactly one real positive root.

v) Using logarithms,

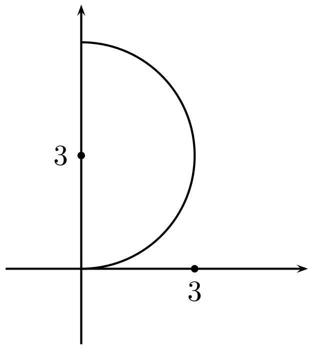
$$\log y = \log(\sin x)^x = x\log(\sin x).$$

Hence

$$\frac{1}{y}\frac{dy}{dx} = \log(\sin x) + x\frac{\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = (\sin x)^x(\log(\sin x) + x\cot x).$$

vi) a) $x = r\cos\theta = 6\sin\theta\cos\theta, y = r\sin\theta = 6\sin^2\theta$,
so $x^2+y^2 = 36\sin^2\theta(\sin^2\theta+\cos^2\theta) = 36\sin^2\theta = 6y$. Completing the square, $x^2+(y-3)^2 = 9$.
From the above, since $0 \le \theta \le \frac{\pi}{2}$, we see that $x \in [0,3], y \in [0,6]$ and so we have a semicircle centre 3 radius 3 in the right half plane.

b)



4. i) a) Using integration by parts with $u = 1 - x, v' = \frac{1}{(1+x)^3}$, we have

$$I_1 = (1-x).\left(\frac{-1}{2(1+x)^2}\right) - \int (-1).\frac{(-1)}{2(1+x)^2}\,dx$$
$$= \frac{x-1}{2(1+x)^2} + \frac{1}{2(1+x)} + C.$$

Alternatively you can make the substitution $x = u - 1$.

b) Using integration by parts with $u = x, v' = \cos 2x$ we have

$$I_2 = \left[x(\frac{1}{2}\sin(2x))\right]_0^{\pi} - \int_0^{\pi} 1.\frac{1}{2}\sin(2x)\,dx = \left[\frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x)\right]_0^{\pi} = \frac{1}{4} - \frac{1}{4} = 0.$$

ii)

$$K = \int_1^{\infty} \frac{1+\sin x}{3x^2}\,dx \le \int_1^{\infty} \frac{2}{3x^2}\,dx$$
$$= \lim_{N\to\infty} \int_1^{N} \frac{2}{3x^2}\,dx = \lim_{N\to\infty} \frac{2}{3} - \frac{2}{3N} = \frac{2}{3}.$$

Hence the improper integral converges by comparison.

iii) a)

$$\sinh x = \frac{e^x - e^{-x}}{2}, \cosh x = \frac{e^x + e^{-x}}{2}.$$

b) Replacing x with ax, we have

$$\frac{d}{dx}\cosh(ax) = \frac{d}{dx}\frac{e^{ax} + e^{-ax}}{2} = \frac{ae^{ax} - ae^{-ax}}{2} = a\sinh(ax).$$

c) Since cosh is an even function,

$$\cosh^{-1}\big(\cosh(-4726)\big) = \cosh^{-1}\big(\cosh(4726)\big) = 4726.$$

iv) Using the First Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_0^x f(t)\,dt = \frac{d}{dx}\int_x^1 t^2 f(t)\,dt + \frac{d}{dx}\left(\frac{x^{16}}{8} + \frac{x^{18}}{9} - \frac{1}{9}\right)$$
$$\Rightarrow f(x) = -\frac{d}{dx}\left(\int_1^x t^2 f(t)\,dt\right) + 2x^{15} + 2x^{17} = -x^2 f(x) + 2x^{15} + 2x^{17}.$$

Solving for $f(x)$, we have

$$f(x) = \frac{2x^{15} + 2x^{17}}{1+x^2} = 2x^{15}.$$

v) a) Since $\tan^{-1}$ is defined for all real x, $g(x) = \tan^{-1}(x) + \tan^{-1}(1/x)$ is defined for all real x except $x = 0$. Hence the maximal domain for g is $(-\infty, 0) \cup (0, \infty)$.

b) For $x > 0$,

$$g'(x) = \frac{1}{1+x^2} + \frac{\frac{-1}{x^2}}{1+(\frac{1}{x})^2} = \frac{1}{x^2+1} - \frac{1}{1+x^2} = 0.$$

Hence on $(0, \infty)$ g is constant. The same calculations hold for $x \in (-\infty, 0)$. Hence g is piecewise constant on its domain. (Note that the constant is not the same in the two intervals. On $(0, \infty)$, $g(x) = \frac{\pi}{2}$, while on $(-\infty, 0)$, $g(x) = -\frac{\pi}{2}$.

c) Since the expression is constant, put $\alpha = 0$ then $\tan^{-1}(-1) + \tan^{-1}(-1) = -\frac{\pi}{2}$.

MATH1131 June 2011 Solutions

1. i) a) $|z| = \sqrt{2}$.

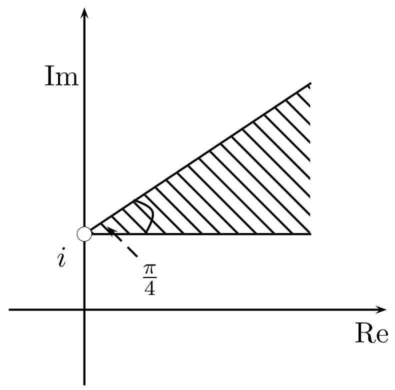
   b) $\mathrm{Arg}(z) = -\frac{3\pi}{4}$.

   c) $z = \sqrt{2}e^{-3\pi i/4}$ and hence $z^{102} = (\sqrt{2})^{102}e^{-153\pi i/2} = 2^{51}e^{-\pi i/2} = -2^{51}i$.

   ii) a) $(2+4i)^2 = -12+16i$.

   b) Applying the quadratic formula, (or by completing the square), we have $z = \frac{4\pm\sqrt{-12+16i}}{2}$. By (a),this simplifies to $z = \frac{4\pm(2+4i)}{2}$, and so the two solutions are $z = 3+2i$ or $z = 1-2i$.

   iii) Diagram as below



   iv) Rationalising the denominator,

$$\lim_{x\to\infty} \frac{1}{x-\sqrt{x^2-6x-4}} = \lim_{x\to\infty} \frac{x+\sqrt{x^2-6x-4}}{(x-\sqrt{x^2-6x-4})(x+\sqrt{x^2-6x-4})}$$

$$= \lim_{x\to\infty} \frac{x+\sqrt{x^2-6x-4}}{(x^2-(x^2-6x-4))}$$

$$= \lim_{x\to\infty} \frac{x+\sqrt{x^2-6x-4}}{6x+4}.$$

We now divide by the highest power of x in the denominator to obtain:

$$\lim_{x\to\infty} \frac{1+\sqrt{1-\frac{6}{x}-\frac{4}{x^2}}}{6+\frac{4}{x}} = \frac{1}{3}.$$

   v) Applying the definition,

$$\int_1^\infty x^{-5/4}\,dx = \lim_{M\to\infty}\int_1^M x^{-5/4}\,dx = \lim_{M\to\infty}\left[-4x^{-\frac{1}{4}}\right]_1^M = \lim_{M\to\infty} 4 - \frac{4}{M^{\frac{1}{4}}} = 4.$$

   vi) a) Differentiating implicitly,

$$2x - 3y^2 - 6xy\frac{dy}{dx} = 0$$

   and the result follows.

   b) At the point $(1,2)$, $\dfrac{dy}{dx} = -\dfrac{5}{6}$, so the equation of the tangent is $y-2 = -\frac{5}{6}(x-1)$ or $5x+6y = 17$.

c) implicitplot($x^2 - 3xy^2 + 11 = 0, x = 1..4, y = -5..5$).

2. i) By De Moivre's Theorem,

$$\cos(4\theta) + i\sin(4\theta) = (\cos\theta + i\sin\theta)^4.$$

Expanding and equating the real parts on both sides, we have

$$\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

Replacing the sine terms,

$$\cos(4\theta) = \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$

and the result follows.

ii) a) Using the values $t = 0, t = 1$ we obtain the two points $\begin{pmatrix}1\\2\\4\end{pmatrix}$ and $\begin{pmatrix}4\\4\\9\end{pmatrix}$.

b) $\begin{pmatrix}3\\2\\5\end{pmatrix}$

c) The vector $\begin{pmatrix}9\\6\\15\end{pmatrix} = 3\begin{pmatrix}3\\2\\5\end{pmatrix}$ is normal to the plane and parallel to the direction of the line. Hence the line is perpendicular to the plane.

d) $t = -1$ gives the required point $\begin{pmatrix}-2\\0\\-1\end{pmatrix}$.

iii) From the MAPLE, we see that $A^8 = 16I$. Multiplying both sides by $(A^7)^{-1}$, we have $(A^7)^{-1} = \frac{1}{16}A = \frac{1}{16}\begin{pmatrix}-1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & \sqrt{2}\end{pmatrix}$.

iv) The expression is indeterminate at $x = 1$ and so we apply L'Hôpital's Rule (twice),

$$\lim_{x\to 1}\frac{(x-1)^2}{1+\cos(\pi x)} = \lim_{x\to 1}\frac{2(x-1)}{-\pi\sin(\pi x)} = \lim_{x\to 1}\frac{2}{-\pi^2\cos(\pi x)} = \frac{2}{\pi^2}.$$

v) Using integration by parts,
$\int x\sin(2x)\,dx = -\frac{x}{2}\cos 2x + \frac{1}{2}\int\cos 2x\,dx = -\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + C.$

vi) a) $f(0) = 1 < 2$ and $f(1) = e + a > 2$ for $a > 0$. Since f is continuous, by the intermediate value theorem, there is a $c \in (0,1)$ such that $f(c) = 2$ and so 2 is in the range of f.

b) $f'(x) = e^x + a > 0$ for $a > 0$ and for all x. Thus the function f is continuous and increasing and so f has an inverse.

c) The domain of f^{-1} is a the range of f and since f is increasing, the range is $[f(0), f(1)] = [1, e + a]$.

3. i) a) $AB = \begin{pmatrix}8 & 23\\ 5 & 11\end{pmatrix}$.

b) BA is a 3×3 matrix.

ii) a) Looking at the dry fruit eaten, we have

$$50x + 300y + 100z = 2900.$$

Dividing this equation by 2 gives the desired result.

b)

$$\begin{aligned} 5x + 30y + 10z &= 290 \\ 4x + 32y + 20z &= 392 \\ x + 5y + 3z &= 63 \end{aligned}$$

c)

$$\left(\begin{array}{ccc|c} 5 & 30 & 10 & 290 \\ 4 & 32 & 20 & 392 \\ 1 & 5 & 3 & 63 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 5 & 3 & 63 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 20 \end{array}\right)$$

Back substitution yields, $x = 8, y = 5, z = 10$ so there are 8 hamsters, 5 rabbits, 10 guinea pigs.

iii) $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$, $\lambda, \mu \in \mathbb{R}$.

iv) a)

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & 3 & 1 & 6 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 3 & 2 \end{array}\right)$$

Let $z = \lambda$, then by back substitution, $x = 4\lambda, y = 2 - 3\lambda$.

So $\mathbf{x} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

b) $x + y + z = 2 + 2\lambda = 0$ when $\lambda = -1$ and then $\mathbf{x} = \begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix}$

v) a) $|\overrightarrow{OB}| = \sqrt{35}$.

b) Area of triangle $AOB = \frac{1}{2}|\mathbf{a} \times \mathbf{b}$

$$= \frac{1}{2}\left\| \begin{vmatrix} i & j & k \\ 1 & 2 & 4 \\ 3 & 1 & 5 \end{vmatrix} \right\| = \frac{1}{2}\begin{pmatrix} 6 \\ 7 \\ -5 \end{pmatrix} = \frac{1}{2}\sqrt{110}.$$

c) Since the area of a triangle is half the base time perpendicular height, the perpendicular distance from A to the line through O and B is $\frac{1}{2}\sqrt{110} \div \frac{1}{2}\sqrt{35} = \sqrt{154}/7$.

vi) $(\mathbf{u} - \mathbf{v})(\mathbf{u} + \mathbf{v}) = \mathbf{u}.\mathbf{u} + \mathbf{u}.\mathbf{v} - \mathbf{v}.\mathbf{u} - \mathbf{v}.\mathbf{v} = |\mathbf{u}|^2 - |\mathbf{v}|^2 = 0$ since the vectors have the same magnitude. Hence the vectors $\mathbf{u} - \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ are perpendicular.

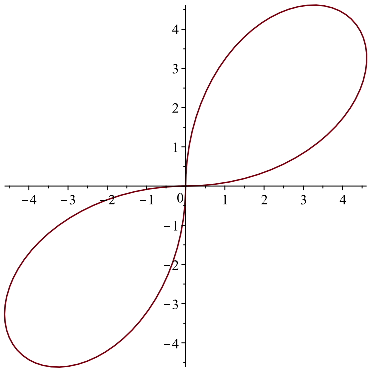
4. i) a) $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

b) $4\cosh^3 x = 4(\frac{1}{2}(e^x + e^{-x}))^3 = \frac{1}{2}(e^{3x} + e^{-3x} + 3e^x + 3e^{-x}) = \cosh 3x + 3\cosh x$.

ii) a) By the Fundamental Theorem of Calculus, $\dfrac{d}{dx}\displaystyle\int_0^x \frac{\cos t}{\sqrt{1+t^2}}\,dt = \frac{\cos x}{\sqrt{1+x^2}}$.

b) By the Fundamental Theorem of Calculus, and the chain rule, $\dfrac{d}{dx}\displaystyle\int_0^{\sinh x} \frac{\cos t}{\sqrt{1+t^2}}\,dt = \cosh x \times \frac{\cos(\sinh x)}{\sqrt{1+\sinh^2 x}} = \cos(\sinh x)$.

iii) a) Diagram as below,



b) $x = r\cos\theta = 6\cos\theta\sin(2\theta)$ and $y = r\sin\theta = 6\sin\theta\sin(2\theta)$.
Hence $\frac{dy}{d\theta} = 12\cos(2\theta)\sin\theta + 6\sin(2\theta)\cos\theta = \frac{15}{2}$ at $\theta = \frac{\pi}{6}$. Similarly, $\frac{dx}{d\theta} = 12\cos(2\theta)\cos\theta - 6\sin(2\theta)\sin\theta = \frac{3}{2}\sqrt{3}$ at $\theta = \frac{\pi}{6}$.
Hence, $\frac{dy}{dx} = \frac{dy}{d\theta}/\frac{dx}{d\theta} = 5\sqrt{3}/3$.

iv) a) Write $y = \cot^{-1} t$, then $t = \cot y$ so $\dfrac{dt}{dy} = -\operatorname{cosec}^2 y = -(1+\cot^2 y) = -(1+t^2)$. The result follows.

b) Write $\theta = (\theta+\phi) - \phi$ and use the two right triangles.

c) $\dfrac{d\theta}{dx} = -\dfrac{3}{9+x^2} + \dfrac{1}{1+x^2} = 0$ for a maximum. Solving this yields $x = \sqrt{3}$ (since $x > 0$). To show it is a maximum, we note that $\frac{d^2\theta}{dx^2}$ is negative at $x = \sqrt{3}$.

MATH1131 June 2012 Solutions

1. i) a) $u - 2w = 1 + 12i$.

   b) $u/w = -\frac{7}{26} + i\frac{17}{26}$.

   ii) a) $|z| = 2$ and $\text{Arg}(z) = -\frac{\pi}{6}$..

   b) $z = 2e^{-i\frac{\pi}{6}}$.

   c) $z^{10} = 2^{10}e^{-\frac{5\pi}{3}}$ and so $\overline{z}^{10} = 2^{10}e^{\frac{5\pi}{3}}$. Hence $z^{10} + (\overline{z})^{10} = 2 \times 2^{10} \cos \dfrac{5\pi}{3} = 2^{10}$.

   iii) Expanding down the first column,

$$\begin{vmatrix} 1 & -1 & 4 \\ 0 & 2 & 7 \\ 0 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 7 \\ 3 & 1 \end{vmatrix} = -19.$$

   iv) a) $\displaystyle\lim_{x\to\infty} \frac{3x^2 + \sin(2x^2)}{x^2} = \lim_{x\to\infty} \frac{3 + \sin(2x^2)/x^2}{1} = 3.$

   b) Applying L'Hopital's Rule,
   $\displaystyle\lim_{x\to 0} \frac{3x^2 + \sin(2x^2)}{x^2} = \lim_{x\to 0} \frac{6x + 4x\cos(2x^2)}{2x} = \lim_{x\to 0} \frac{6 + 4\cos(2x^2)}{2} = 5.$

   v) Differentiating implicitly,

$$2x - 5\sin y - 5x\cos y\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

   and so $\dfrac{dy}{dx} = \dfrac{2}{5}$ at $(2,0)$. Hence the equation of the tangent at this point is $y = \frac{2}{5}(x-2)$ or $2x - 5y = 4$.

   vi) a) p is a polynomial and so is continuous. Since p has degree 5, for large positive x, $p(x) > 0$ and for large negative x, $p(x) < 0$, so by the Intermediate Value Theorem, $p(x)$ has at least one real root.

   b) $p'(x) = 5x^4 + 5 > 0$ for all x and so $p(x)$ is strictly increasing. Hence p has at most one real root.

2. i) We use De Moivre's theorem and then the binomial formula to find

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.\end{aligned}$$

   Equating real parts then gives

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta.\end{aligned}$$

   The result is shown.

   ii) We solve equations simultaneously using $x = 1 + \lambda, y = \lambda, z = 2 + 2\lambda$ to find

$$17 = 5(1+\lambda) - 2\lambda + (2 + 2\lambda) = 5\lambda + 7.$$

Solving for λ we find $\lambda = 2$. The point of intersection can now be obtained from the parametric equation for the line as

$$\mathbf{x} = \begin{pmatrix}1\\0\\2\end{pmatrix} + 2\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}3\\2\\6\end{pmatrix}.$$

iii) Below we use the MAPLE output giving $A^4 = I$ and the output for A^3. The inverse to A^{2001} is

$$A^{-2001} = A^{4\times(-501)+3} = (A^4)^{-501}A^3 = I^{-501}A^3 = A^3 = \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{pmatrix}.$$

iv) a) We compute

$$\mathbf{c} \times \mathbf{d} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ 1 & 2 & 3\\ 4 & 1 & 5\end{pmatrix} = \begin{pmatrix}7\\7\\-7\end{pmatrix}.$$

b) The area of the parallelogram with sides $\mathbf{c}, \mathbf{d}$ is $|\mathbf{c} \times \mathbf{d}| = \sqrt{7^3 + 7^3 + (-7)^3} = 7\sqrt{3}$.

v) We use the integration by parts formula $\int udv = uv - \int vdu$ with $u = \ln x, v = \frac{1}{5}x^5$ so $du = \frac{1}{x}dx, dv = x^4 dx$.

$$\begin{aligned}\int x^4 \ln x dx &= \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^5 \frac{1}{x}dx\\ &= \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^4 dx\\ &= \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 dx + C\end{aligned}$$

where C is an arbitrary constant

vi) We first note that the functions $\ln x, \sin x, \cos x$ are all continuous on their domains. Hence the same is true of $h(x)$. Furthermore, the domain of h is the closed interval $[1, 5]$ which has finite length. It follows by the max-min theorem that $h(x)$ must attain a maximum value.

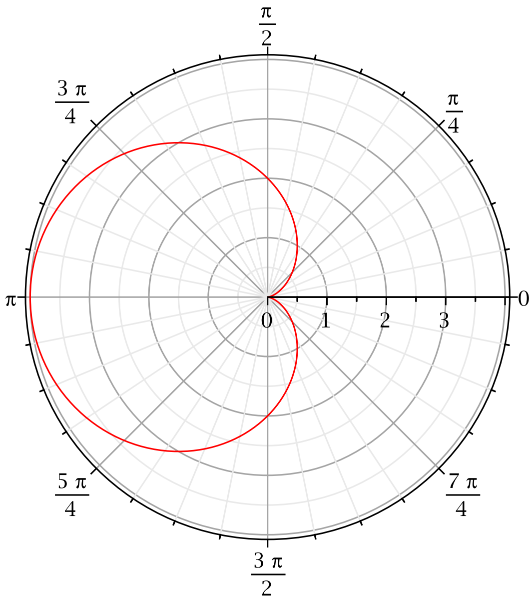
vii) First note that r is an even function of θ so the graph symmetric about the x-axis.

We now compute some values

$$\begin{aligned}r(0) &= 2 - 2\cos 0 = 0\\ r(\pi/2) &= 2 - 2\cos \pi/2 = 2 - 0 = 2\\ r(\pi) &= 2 - 2\cos \pi = 2 - 2(-1) = 4\end{aligned}$$

Note that in general, as θ increases from 0 to π, $r(\theta)$ decreases from 0 to 4. Hence the polar curve spirals out as you rotate anti-clockwise or clockwise from the positive x-axis to the negative x-axis.

Note that the x-intercepts are ± 2 while the y-intercepts are $0, -4$. The only axis of symmetry is the x-axis.



viii) We differentiate implicitly with respect to x, the equation $\ln y = \ln(x^{\sin x}) = \sin x \ln x$.

$$\frac{1}{y}\frac{dy}{dx} = \cos x \ln x + \frac{\sin x}{x}$$

Hence

$$\frac{dy}{dx} = y(\cos x \ln x + \frac{\sin x}{x}) = x^{\sin x}(\cos x \ln x + \frac{\sin x}{x})$$

3. i) a) We have $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$. Therefore l is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}.$$

b) $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = 0 - 6 + 6 = 0$. Therefore the two lines are perpendicular.

ii) a) Note that $Q^T = \begin{pmatrix} 1 & 2 \\ -1 & 5 \\ 1 & 0 \end{pmatrix}$. Thus

$$PQ^T = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 5 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 12 \\ 8 & 1 \end{pmatrix}.$$

b) P^T is a 3×2 and Q is a 2×3 matrix. Therefore, P^TQ is a 3×3 matrix.

c) No as the number of columns of P doesn't equal to the number of rows of Q.

iii) a) Note that $\alpha^2 - 9 = (\alpha - 3)(\alpha + 3)$ and so the matrix becomes:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & (\alpha-3)(\alpha+3) & \alpha - 3 \end{array}\right).$$

If $\alpha = -3$ the last column will be leading, and so this will lead to no solutions.

b) If $\alpha = 3$ both the thirst and last columns will be non-leading, implying infinitely many solutions.

c) If $\alpha = 3$ we get

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Let $z = \lambda$. Back substituting, we get $2y + 4\lambda = 8$ and so $y = 4 - 2\lambda$. Also, $x + 2(4 - 2\lambda) + 3\lambda = 5$ and thus we get $x = -3 + \lambda$. Thus

$$\mathbf{x} = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

iv) a) Since the total value if \$1020 we have

$$10x + 20y + 50z = 1020$$

and after dividing by 10 we get

$$x + 2y + 5x = 102.$$

b) Since there are 44 notes in total, we get

$$x + y + z = 44.$$

The last line in the question implies $x = y + z$ or equivalently

$$x - y - z = 0.$$

Putting this in augmented matrix form and row reducing we get:

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 44 \\ 1 & 2 & 5 & 102 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 2 & 44 \\ 0 & 3 & 6 & 102 \end{array}\right)$$

$$\xrightarrow[R_3 \to \frac{1}{3}R_3]{R_3 \to \frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 22 \\ 0 & 1 & 2 & 34 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 22 \\ 0 & 0 & 1 & 12 \end{array}\right).$$

Back substituting we get $z = 12, y = 10, x = 22$.

v) a) We have

$$\mathbf{a} \cdot \mathbf{e_1} = |\mathbf{a}||\mathbf{e_1}| \cos\alpha$$

and so $a = \sqrt{a^2 + b^2 + c^2} \cos\alpha$.

b) Similarly to (a) we get $b = \sqrt{a^2+b^2+c^2}\cos\beta$ and $c = \sqrt{a^2+b^2+c^2}\cos\gamma$. Squaring and adding the above three equations gives $a^2+b^2+c^2 = (a^2+b^2+c^2)(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$ and so we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

c) If $\alpha+\beta = 90°$ then $\cos\beta = \sin\alpha$. From this we get $\cos^2\alpha+\cos^2\beta = \cos^2\alpha+\sin^2\alpha = 1$ and so from (b) we see that $\cos^2\gamma = 0$. This implies $\gamma = \pi/2$ i.e. $\overrightarrow{OQ}$ is parallel to the XY-plane.

4. i) Let $p(x) = e^{3x}$. For h to be differentiable at $x = 0$, we need

- $p(0) = q(0)$, so that h is continuous at 0;
- $p'(0) = q'(0)$.

As $p'(x) = 3e^{3x}$ and $q'(x) = 2x + b$, these conditions say

- $1 = c$;
- $3 = b$,

and so $q(x) = x^2 + 3x + 1$.

ii) a) In mathematical notation this is $\sin^{-1}(\sin(7\pi/3))$. Evaluating (or just writing this down from thinking about the quadrant)

$$\sin^{-1}(\sin(7\pi/3)) = \sin^{-1}(\sqrt{3}/2) = \frac{\pi}{3}.$$

b) In mathematical notation this is $\displaystyle\frac{d}{dx}\int_0^{x^2} e^{t^2}\,dt$.

To evaluate this, let $F(u) = \int_0^u e^{t^2}\,dt$ and let $u(x) = x^2$. Then

$$\begin{aligned}\frac{d}{dx}\int_0^{x^2} e^{t^2}\,dt &= \frac{d}{dx}F(u(x))\\ &= F'(u(x))\,u'(x) \qquad \text{(by the chain rule)}\\ &= e^{u(x)^2}\,2x \qquad \text{(by the FTC)}\\ &= 2xe^{x^4}.\end{aligned}$$

iii) Suppose that $\epsilon > 0$ is given. Put $M = \sqrt{\frac{5}{\epsilon}}$ then for $x \geq M$,

$$\left|\frac{x^2-2}{x^2+3} - 1\right| = \left|\frac{x^2-2-x^2-3}{x^2+3}\right| = \frac{5}{x^2+3} \leq \frac{5}{x^2} \leq \frac{5}{M^2} = \epsilon.$$

iv) a) f is differentiable and

$$f'(x) = 3x^2 + \cosh x \geq \cosh x \geq 1$$

and so $f'(x)$ is never zero. It follows by the inverse function theorem that f has a differentiable inverse g.

b) As $f(x) \to -\infty$ as $f \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$

$$\mathrm{Dom}(g) = \mathrm{Ran}(f) = \mathbb{R}.$$

c) The inverse function theorem says that

$$g'(x) = \frac{1}{f'(g(x))}.$$

We need to find $y = g(1)$. That is, solve $f(y) = y^3 + \sinh y + 1 = 1$. By inspection, the solution is $y = 0$. Plugging this into $f'(x)$ from (a):

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = \frac{1}{1} = 1.$$

v) Let $V(t)$ denote the volume of Factor X in the tank at time $t \geq 0$.
We are given that $V'(t) = \dfrac{100}{10+t^2}$.
Thus after T hours the amount in the tank is

$$V(T) = \int_0^T \frac{100}{10+t^2}\,dt = \left[10\sqrt{10}\tan^{-1}\frac{t}{\sqrt{10}}\right]_0^T = 10\sqrt{10}\tan^{-1}\frac{T}{\sqrt{10}}.$$

As $T \to \infty$,

$$\tan^{-1}\frac{T}{\sqrt{10}} \to \frac{\pi}{2}$$

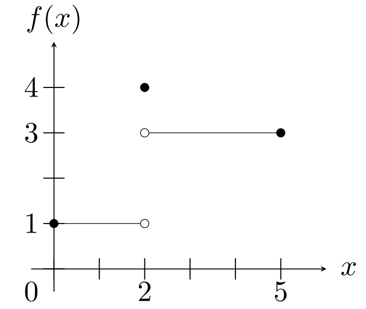
and so $V(T)$ increases towards its limit $5\sqrt{10}\pi \approx 49.6729 < 50$.
In particular, the tank is just big enough to never overflow.

MATH1131 June 2013 Solutions

1. i) a) 2

   b) $\dfrac{3}{7}$

   ii)



   iii) a) $\cosh x = \frac{1}{2}(e^x + e^{-x}) \qquad \sinh x = \frac{1}{2}(e^x - e^{-x})$

   b) **Proof**.

$$\cosh^2 x - \sinh^2 x = \left(\frac{1}{2}(e^x + e^{-x})\right)^2 - \left(\frac{1}{2}(e^x - e^{-x})\right)^2$$
$$= \frac{1}{4}(e^{2x} + e^{-2x} + 2) - \frac{1}{4}(xe^{2x} + e^{-2x} - 2) = \frac{1}{4} \times 4 = 1\,. \qquad \blacksquare$$

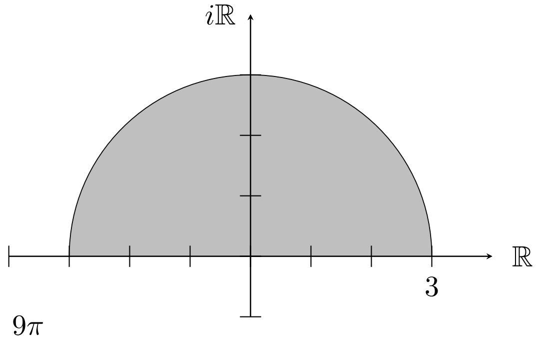
   iv) a) $16 + 7i$

   b) $10i$

   c) $3 + i$

   v) $x = 3$
   $y = -1$

   vi) a)



   b) $\dfrac{9\pi}{2}$

   vii) $A = \dfrac{1}{2}(3I - A^2) = \dfrac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$

2. i)

$$h(x) = \begin{cases} ax^2 + 3x, & \text{if } x \geq 1 \\ 2x + d, & \text{if } x < 1. \end{cases}$$

Both branches are polynomials and are therefore elementary functions. This means they are both continuous and differentiable $x \neq 1$.

Differentiating for $x \neq 1$:

$$h'(x) = \begin{cases} 2ax + 3, & \text{if } x > 1 \\ 2, & \text{if } x < 1. \end{cases}$$

If $h(x)$ is differentiable at $x = 1$ then $h(x)$ is continuous at $x = 1$. i.e.

$$\begin{aligned} h(1) = \lim_{x \to 1^+} h(x) &= \lim_{x \to 1^-} h(x) \\ \lim_{x \to 1^+} ax^2 + 3x &= \lim_{x \to 1^-} 2x + d \\ a + 3 &= 2 + d \\ d &= a + 1. \end{aligned}$$

Also, $h'(x)$ has to be continuous at $x = 1$, i.e. we require

$$\begin{aligned} \lim_{x \to 1^+} h'(x) &= \lim_{x \to 1^-} h'(x) \\ \lim_{x \to 1^+} 2ax + 3 &= \lim_{x \to 1^-} 2 \\ 2a + 3 &= 2 \\ a &= -\frac{1}{2}. \end{aligned}$$

But $d = a + 1$, so

$$a = -\frac{1}{2} \quad \text{and} \quad d = \frac{1}{2}.$$

ii)

$$\begin{aligned} \int_0^{\ln 2} 9xe^{3x} dx &= \left[9x \frac{e^{3x}}{3} \right]_0^{\ln 2} - \int_0^{\ln 2} 9 \frac{e^3 x}{3} dx \\ &= 3(\ln 2) e^{3 \ln 2} - 0 - 3 \left[\frac{e^{3x}}{3} \right]_0^{\ln 2} \\ &= 3(\ln 2) 2^3 - e^{3 \ln 2} + e^0 \\ &= 24 \ln 2 - 7. \end{aligned}$$

iii)

$$e^x + \sin y = xy + 1.$$

Differentiating implicitly with respect to x throughout, using the chain rule and the product rule:

$$e^x + \cos y \frac{dy}{dx} = x \frac{dy}{dx} + 1y + 0$$

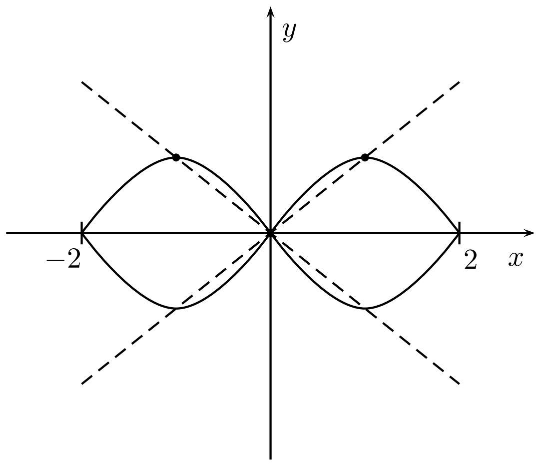
Evaluate this at $(x, y) = (0, 0)$:

$$\begin{aligned} e^0 + \cos 0 \frac{dy}{dx} &= 0 \frac{dy}{dx} + 0 + 0 \\ 1 + \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -1. \end{aligned}$$

The tangent is a line passing through $(x, y) = (0, 0)$ with gradient 1. The equation is therefore

$$\begin{aligned} y - 0 &= -1(x - 0) \\ y &= -x. \end{aligned}$$

iv) Diagram as below,



v) a)

$$|z| = \sqrt{2 + 2} = 2.$$

b)

$$\mathrm{Arg}(z) = \tan^{-1}\left(\frac{-\sqrt{2}}{\sqrt{2}}\right) = -\frac{\pi}{4}.$$

c)

$$\begin{aligned} z &= 2e^{-\frac{\pi}{4}i} \\ z^6 &= 2^6 e^{-\frac{6\pi}{4}i} \\ &= 2^6 e^{-\frac{3\pi}{2}i} \\ &= 2^6 i \\ &= 64i. \end{aligned}$$

vi) a)

$$\begin{aligned} AB &= \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 0 & 2 \\ 2 & -2 & 1 \end{pmatrix}. \end{aligned}$$

b) AB^T does not exist as A is order (2×2) and B^T is order (3×2), and so the number of columns in A is not equal to the number of rows in B^T.

vii) a)

$$z^5 = -1 = e^{i(\pi + 2k\pi)}, \quad k \in \mathbb{Z}.$$

So,

$$\begin{aligned} z &= e^{i\frac{\pi}{5}(2k+1)} \\ &= e^{i\frac{\pi}{5}}, e^{-i\frac{\pi}{5}}, e^{i\frac{3\pi}{5}}, e^{-i\frac{3\pi}{5}}, -1, \quad \text{listing the principal set.} \end{aligned}$$

b) Using these solutions,

$$\begin{aligned} z^5+1 &= \left(z - e^{i\frac{\pi}{5}}\right)\left(z - e^{-i\frac{\pi}{5}}\right)\left(z - e^{i\frac{3\pi}{5}}\right)\left(z - e^{-i\frac{3\pi}{5}}\right)(z+1) \\ &= \left(z^2 - 2\cos\left(\frac{\pi}{5}\right)z+1\right)\left(z^2 - 2\cos\left(\frac{3\pi}{5}\right)z+1\right)(z+1). \end{aligned}$$

3. i) The line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ can be written as $\begin{cases} x = 1+t \\ y = -1+2t \\ z = -1+t \end{cases}$.

For the point of intersection, we substitute the parametric equations for the line into the Cartesian equation of the plane.

$$\begin{aligned} 4(1+t) - 5(-1+2t) + 3(-1+t) &= 0 \\ -3t+6 &= 0 \\ t &= 2 \end{aligned}$$

Hence the point of intersection is $\begin{pmatrix} 1+2 \\ -1+2(2) \\ -1+2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$.

ii) a) Expand $\det(M)$ along the first row.

$$\begin{vmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 2 & \alpha \end{vmatrix} = \begin{vmatrix} 5 & 1 \\ 2 & \alpha \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 0 & \alpha \end{vmatrix} = (5\alpha - 2) - 2(2\alpha) = \alpha - 2.$$

b) M does not have an inverse if and only if $\det(M) = 0$. That is, $\alpha = 2$.

c) When $\alpha = 1$, the matrix M is invertible. We can find the inverse of M by reducing the augmented matrix $(M|I)$ to reduced row-echelon form.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 5 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 4 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 = -R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 2 & -1 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -4 & 2 & -1 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & -2 \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -4 & 2 & -1 \end{array}\right)$$

Hence $M^{-1} = \begin{pmatrix} -3 & 2 & -2 \\ 2 & -1 & 1 \\ -4 & 2 & -1 \end{pmatrix}$.

iii) a) The cross product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{vmatrix} = \mathbf{e}_1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

b) The vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to the plane. Hence, a point-normal form of the plane is

$$\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, the Cartesian equation of the plane is $x - 2y - z = -9$.

iv) a) The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. That is,

$$0 + 3 - 12 + 3\beta = 0, \qquad \text{i.e.} \quad \beta = 3.$$

b) For the value $\beta = 1$,

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}^2|} \right) \mathbf{u} = \frac{0 + 3 - 12 + 3}{2^2 + 1^2 + 4^2 + 3^2} \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \end{pmatrix}.$$

v) a) $\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \begin{pmatrix} 2 \\ \frac{3}{2} \end{pmatrix}$.

b) Since $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we may take $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which is perpendicular to $\overrightarrow{AB}$.

c) The perpendicular bisector of AB is the line whose points are equidistant from A and B. A parametric vector equation for this line is

$$\mathbf{x} = \begin{pmatrix} 2 \\ \frac{3}{2} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

d) Since the centre is equidistant from A, B and D, the centre is the intersection of the two perpendicular bisectors. At the point of intersection, we have

$$\begin{aligned} \begin{pmatrix} 2 \\ \frac{3}{2} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \end{aligned}$$

At the intersection, the value of λ is 1. Hence the position vector of the point of intersection is $\begin{pmatrix} 2 \\ \frac{3}{2} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{3}{2} \end{pmatrix}$.

4. i) Let $u = \ln(x)$, and so $du = \dfrac{dx}{x}$. Substituting:

$$\int \frac{\cos(\ln(x))}{x}\, dx = \int \cos u\, du = \sin u + C = \sin(\ln(x)) + C.$$

ii) Let $f(x) = e^{-x}$ and $g(x) = -e^{-x}$. Then for all x

$$g(x) \le e^{-x} \sin(x) \le f(x).$$

Since

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} f(x) = 0$$

the Pinching theorem implies that $\lim\limits_{x\to\infty} e^{-x}\sin(x)$ exists and also equals 0.

iii) For all $x \ge 0$, we have $0 \le \dfrac{1}{x^2 + e^x} \le e^{-x}$.

Now $\displaystyle\int_0^\infty e^{-x}\, dx = \lim_{R\to\infty} [-e^{-x}]_0^R = \lim_{r\to\infty} \left(-e^{-R} + 1\right) = 1$ converges.

Thus, by the comparison test $\displaystyle\int_0^\infty \frac{dx}{x^2 + e^x}$ converges too.

iv) a) $F = \dfrac{d}{dx} \displaystyle\int_0^{x^2} \sin(\sqrt{t})\, dt.$

b) Let $u(x) = x^2$ and let $g(u) = \int_0^u \sin(\sqrt{t})\, dt$. Then, using the chain rule, $F = \dfrac{d}{dx} g(u(x)) = g'(u(x))\, u'(x)$.

By the Fundamental Theorem of Calculus $g'(u) = \sin(\sqrt{u})$, and so

$$F = \sin(\sqrt{x^2}) \cdot 2x = 2x \sin(|x|).$$

v) a) Since p is a polynomial it is continuous on $[1,2]$. Now $p(1) = -2 < 0$ and $p(2) = 9 > 0$, so by the Intermediate Value theorem there is a value $c \in (1,2)$ so that $p(c) = 0$.

b) $p'(x) = 3x^2 + 4 > 0$ for all $x \in [1,2]$. Therefore p is increasing and hence can only take on any value at most once. Combined with (a) this implies that p has exactly one root in $[1,2]$.

c) The Inverse Function theorem says that g is differentiable with

$$g'(0) = \frac{1}{p'(\alpha)} = \frac{1}{3\alpha^2 + 4}.$$

vi) Let

$$f(x) = \ln(1+x) - \frac{x}{1+x}.$$

We need to show that $f(x) > 0$ for all $x > 0$.

Suppose then that $x > 0$. Since f is continuous on $[0,x]$ and differentiable on $(0,x)$, the Mean Value Theorem says that there exists $c \in (0,x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}.$$

Now, differentiating f at c,

$$f'(c) = \frac{1}{1+c} - \frac{(1+c) - c}{(1+c)^2} = \frac{(1+c) - 1}{(1+c)^2} = \frac{c}{(1+c)^2}$$

and so for any $c \in (0,x)$, $f'(c) > 0$. It follows then that

$$f(x) = xf'(c) > 0.$$

MATH1131 June 2014 Solutions

1. i) a) $2z - \overline{w} = 10 + 5i$

   b) $5(w-i)/z = 3 + i$.

   c) $|zw| = |z||w| = \sqrt{50} \times 5 = 25\sqrt{2}$.

   d) $zw = 25 + 25i$ and so $\text{Arg}(zw) = \frac{\pi}{4}$.

   e) Since both z and w lie in the first quadrant, the sum of their arguments lies between 0 and π. Hence $\text{Arg}(z)+\text{Arg}(w) = \text{Arg}(zw) = \pi/4$.

   f) $zw = 25\sqrt{2}e^{\frac{i\pi}{4}}$ so $(zw)^{40} = (25\sqrt{2})e^{10\pi i} = 5^{80}2^{20}$.

   ii) a)

$$[A|\mathbf{b}] = \left(\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 1 & -3 & 1 & 1 \\ 2 & -3 & -1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

   b) Let $z = \lambda$ then $y = \lambda$ and $x = 1 + 2\lambda$. Hence the general solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}.$$

   iii) a)

$$\lim_{x\to\infty} \frac{6x^2 + \sin x}{4x^2 + \cos x} = \lim_{x\to\infty} \frac{6 + \frac{\sin x}{x^2}}{4 + \frac{\cos x}{x^2}} = \frac{3}{2}$$

   since $\cos x$ and $\sin x$ are bounded.

   b) Using L'Hôpital's Rule twice, and checking the necessary conditions,

$$\lim_{x\to 0} \frac{e^{2x} - 2x - 1}{4x^2} = \lim_{x\to 0} \frac{2e^{2x} - 2}{8x} = \lim_{x\to 0} \frac{4e^{2x}}{8} = \frac{1}{2}.$$

   iv) For $x > 4$, $g(x) = \frac{x^2-16}{x-4} = x + 4$ and so $\lim\limits_{x\to 4^+} g(x) = 8$.

   For $3 < x < 4$, $g(x) = \frac{-(x^2-16)}{x-4} = -(x+4)$ and so $\lim\limits_{x\to 4^-} g(x) = -8$.

   Since these limits have different values, so value of $g(4)$ may be given to make g continuous at $x = 4$.

   v) a) f is a polynomial and so is continuous on $[0, 2]$.
   Now $f(0) = -2 < 0$ and $f(2) = 40 > 0$ and so by the intermediate value theorem, f has at least one zero in the interval $(0, 2)$.

   b) $f'(x) = 5x^4 + 3x^2 + 1 > 1$ for all real x and so f is an increasing function. Hence f has only one real root.

2. i) Let $u = \log x$, $\frac{du}{dx} = \frac{1}{x}$

$$\begin{aligned} \int \frac{dx}{x\left(1 + (\log x)^2\right)} &= \int \frac{du}{1 + u^2} \\ &= \tan^{-1} u + C \\ &= \tan^{-1}(\log x) + C \end{aligned}$$

ii) a)

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

b)

$$\begin{aligned} RHS &= 2\sinh x \cosh x \\ &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\ &= \frac{1}{2}\left(e^{2x} - e^{-2x}\right) \\ &= \sinh(2x) \\ &= RHS \end{aligned}$$

i.e. $\sinh(2x) = 2\sinh x \cosh x$.

iii)

$$\begin{aligned} \int_0^{\frac{\pi}{3}} x\sin(2x)dx &= \left[x\left(-\frac{\cos(2x)}{2}\right)\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} 1\left(-\frac{\cos(2x)}{2}\right)dx \\ &= \frac{\pi}{3}\left(-\frac{\cos\left(\frac{2\pi}{3}\right)}{2}\right) - 0 + \frac{1}{2}\left[\frac{\sin(2x)}{2}\right]_0^{\frac{\pi}{3}} \\ &= \frac{\pi}{12} + \frac{1}{4}\frac{\sqrt{3}}{2} \\ &= \frac{\pi}{12} + \frac{\sqrt{3}}{8}. \end{aligned}$$

iv)

$$\left(A^TA\right)^{-1}\left(A^TA\right)^T = \left(A^TA\right)^{-1}\left(A^TA\right) = I$$

v) a)

$$\begin{aligned} \Pi &= \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \lambda_1\left[\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}\right] + \lambda_2\left[\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}\right], \quad \lambda_1, \lambda_2 \in \mathbb{R} \\ &= \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \lambda_1\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda_2\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}. \end{aligned}$$

b)

$$
\begin{aligned}
\mathbf{n} &= \vec{AB} \times \vec{AC} \\
&= \left[\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}\right] \times \left[\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}\right] \\
&= \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ 3 & 0 & 1 \\ 3 & 1 & 1 \end{vmatrix} \\
&= \mathbf{e_1}(0 \times 1 - 1 \times 1) - \mathbf{e_2}(3 \times 1 - 1 \times 3) + \mathbf{e_3}(3 \times 1 - 0 \times 3) \\
&= \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}.
\end{aligned}
$$

c) Π in cartesian coordinates is

$$
\begin{aligned}
-1(x-0) + 0(y-1) + 3(z-3) &= 0 \\
-x + 3(z-3) &= 0.
\end{aligned}
$$

d)

$$
\begin{aligned}
\text{Area } ABC &= \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right| \\
&= \frac{1}{2}|\mathbf{n}| \\
&= \frac{1}{2}\sqrt{(-1)^2 + (0)^2 + (3)^2} \\
&= \frac{\sqrt{10}}{2}.
\end{aligned}
$$

e)

$$
\vec{PA} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \\ 3 \end{pmatrix}.
$$

$$\begin{aligned}
\text{min distance} &= \frac{\left|\vec{PA}\,\boldsymbol{.}\,\mathbf{n}\right|}{|\mathbf{n}|} \\
&= \frac{\left|\begin{pmatrix}-5\\-2\\3\end{pmatrix}\boldsymbol{.}\begin{pmatrix}-1\\0\\3\end{pmatrix}\right|}{\left|\begin{pmatrix}-1\\0\\3\end{pmatrix}\right|} \\
&= \frac{5+0+9}{\sqrt{(-1)^2+(0)^2+(3)^2}} \\
&= \frac{14}{\sqrt{10}}.
\end{aligned}$$

vi)

$$\begin{aligned}
F &= 2A+B+C \\
I &= 2A+A^2+A^3 \\
A^{-1} &= 2I+A+A^2 \\
&= 2I+A+B \\
&= 2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&1&-2\\1&-1&1\\1&-1&0\end{pmatrix}+\begin{pmatrix}-1&1&1\\0&1&-3\\-1&2&-3\end{pmatrix} \\
&= \begin{pmatrix}1&2&-1\\1&2&-2\\0&1&-1\end{pmatrix}.
\end{aligned}$$

3. i) a) DIAGRAM

b) $|\mathbf{F}| = \sqrt{(\sqrt{2}-1)^2+(\sqrt{2}-1)^2} = \sqrt{6-4\sqrt{2}}$;
Equivalently, $|\mathbf{F}| = 2-\sqrt{2}$ (these two expressions are equal)
and $\mathbf{F}$ has direction North-East.

ii) a)

$$\begin{vmatrix}1&-1&1\\2&1&3\\-1&2&1\end{vmatrix} = \begin{vmatrix}1&-1&1\\0&3&1\\0&1&2\end{vmatrix} = -\begin{vmatrix}1&-1&1\\0&1&2\\0&3&1\end{vmatrix} = -\begin{vmatrix}1&-1&1\\0&1&2\\0&0&-5\end{vmatrix} = 5$$

b) $N^{-1} = \dfrac{1}{3\times 2-1\times 4}\begin{pmatrix}2&-1\\-4&3\end{pmatrix} = \dfrac{1}{2}\begin{pmatrix}2&-1\\-4&3\end{pmatrix}$

iii) a) $p(1) = 1^4+2\cdot 1^2-3 = 0$ and $p(-1) = (-1)^4+2(-1)^2-3 = 0$

b) $p(z) = (z^2-1)(z^2+3)$.

c) $\pm 1, \pm\sqrt{3}\,i$.

d) $p(z) = (z-1)(z+1)(z-\sqrt{3}\,i)(z+\sqrt{3}\,i)$

iv) Substitute $x = 3+\lambda$, $y = 2+2\lambda$, and $z = 1+3\lambda$ into the Cartesian equation:

$$0 = 6(3+\lambda)+8(2+2\lambda)-9(1+3\lambda) = 25-5\lambda\,.$$

We see that $\lambda = 5$,

so the point of intersection is represented by $\vec{x} = \begin{pmatrix}3\\2\\1\end{pmatrix} + 5\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}8\\12\\16\end{pmatrix}$.

v) a) **Proof**. By De Moivre's Theorem,

$$\begin{aligned}\cos 3\theta + i\,\sin 3\theta = e^{i3\theta} &= (e^{i\theta})^3 = (\cos\theta + i\,\sin\theta)^3\\ &= \cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta\,.\end{aligned}$$

By comparing the real parts of the equation, we see that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta\,.$$

Hence, $4\cos^3\theta = 3\cos\theta + \cos 3\theta$. □

b) By part a) with $\theta = \frac{\pi}{9}$,

$$q\Big(2\cos\frac{\pi}{9}\Big) = 8\cos^3\frac{\pi}{9} - 6\cos\frac{\pi}{9} - 1 = 2\Big(3\cos\frac{\pi}{9} + \cos\frac{3\pi}{9}\Big) - 6\cos\frac{\pi}{9} - 1 = 2\frac{1}{2} - 1 = 0\,.$$

Hence, $2\cos\frac{\pi}{9}$ is a root of the polynomial $q(z) = z^3 - 3z - 1$.

vi) a) $\vec{u}\cdot\vec{v} = 1 - \beta$, so the vectors are orthogonal when $\beta = 1$.

b)

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \frac{\begin{pmatrix}0\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\0\end{pmatrix}}{\left|\begin{pmatrix}2\\1\\0\end{pmatrix}\right|^2}\begin{pmatrix}2\\1\\0\end{pmatrix} = \frac{1}{5}\begin{pmatrix}2\\1\\0\end{pmatrix}$$

c)

$$\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} = \frac{\begin{pmatrix}0\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\\beta\end{pmatrix}}{\left|\begin{pmatrix}0\\1\\-1\end{pmatrix}\right|\left|\begin{pmatrix}2\\1\\\beta\end{pmatrix}\right|} = \frac{1-\beta}{\sqrt{2}\sqrt{5+\beta^2}}\,,$$

so $5 + \beta^2 = (1-\beta)^2 = \beta^2 - 2\beta + 1$. Hence, $\beta = -2$.

4. i) Let $F(t) = \int \cos\left(\frac{1}{t}\right)\,dt$ so $F'(t) = \cos\left(\frac{1}{t}\right)$. By the fundamental theorem of calculus

$$\int_{x^2}^{x^3}\cos\left(\frac{1}{t}\right)\,dt = F(x^3) - F(x^2).$$

Then

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{x^3}\cos\left(\frac{1}{t}\right)\,dt &= \frac{d}{dx}\left(F(x^3) - F(x^2)\right)\\ &= F'(x^3)\,\frac{d\,x^3}{dx} - F'(x^2)\,\frac{d\,x^2}{dx}\\ &= \cos\left(\frac{1}{x^3}\right)\,3x^2 - \cos\left(\frac{1}{x^2}\right)\,2x.\end{aligned}$$

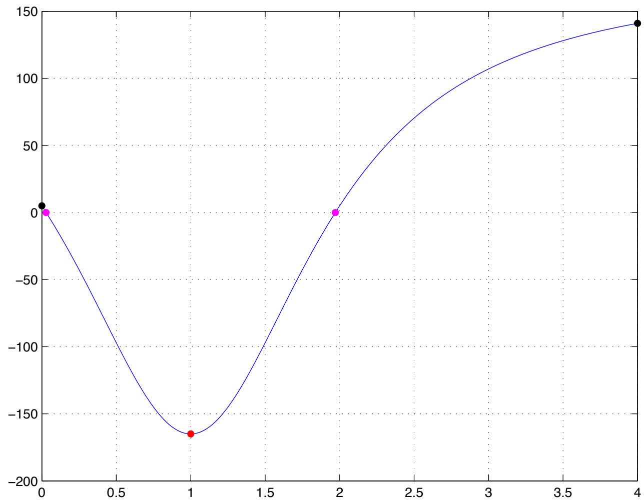
ii) a) The MAPLE session defines the function

$$f(x) = \frac{175x^2 - 350x + 10}{x^2 - 2x + 2}.$$

and you are given that f is differentiable for all real x.
The MAPLE session gives the following information

- $f(0) = 5$ and $f(4) = 141$.
- f has two zeros where $f(x) = 0$, close to $x = 0.029$ and $x = 1.971$.
- f has one stationary point $f'(1) = 0$ where $f(1) = -165$.

Thus a sketch of the function f over the interval $[0, 4]$ is



b)

$$\left|\frac{175x^2 - 350x + 10}{x^2 - 2x + 2}\right| = |f(x)|$$

is the absolute value of the function $f(x)$ from part a). As $|f(x)|$ is continuous on the closed interval $[0, 4]$ the (global) minimum and maximum both exist, and $|f(x)|$ has

- a minimum value of 0 which occurs at the two points $x = 1 \pm \sqrt{1155}/35$
- a maximum value of 165 at $x = 1$.

iii) As $\cos^2 x \geq 0$ for all x,

$$0 \leq \frac{1}{e^{2x} + \cos^2 x} \leq \frac{1}{e^{2x}}.$$

Hence

$$0 \leq K = \int_0^\infty \frac{dx}{e^{2x} + \cos^2 x} \leq \int_0^\infty \frac{dx}{e^{2x}}.$$

Now

$$\int_0^\infty \frac{dx}{e^{2x}} = \lim_{b\to\infty} \int_0^b e^{-2x}\, dx = \lim_{b\to\infty} \left[-\frac{1}{2}e^{-2x}\right]_0^b = \lim_{b\to\infty} -\frac{1}{2}e^{-2b} + \frac{1}{2} = \frac{1}{2}.$$

As $K \leq \int_0^\infty \frac{dx}{e^{2x}}$ which converges, the comparison test implies that

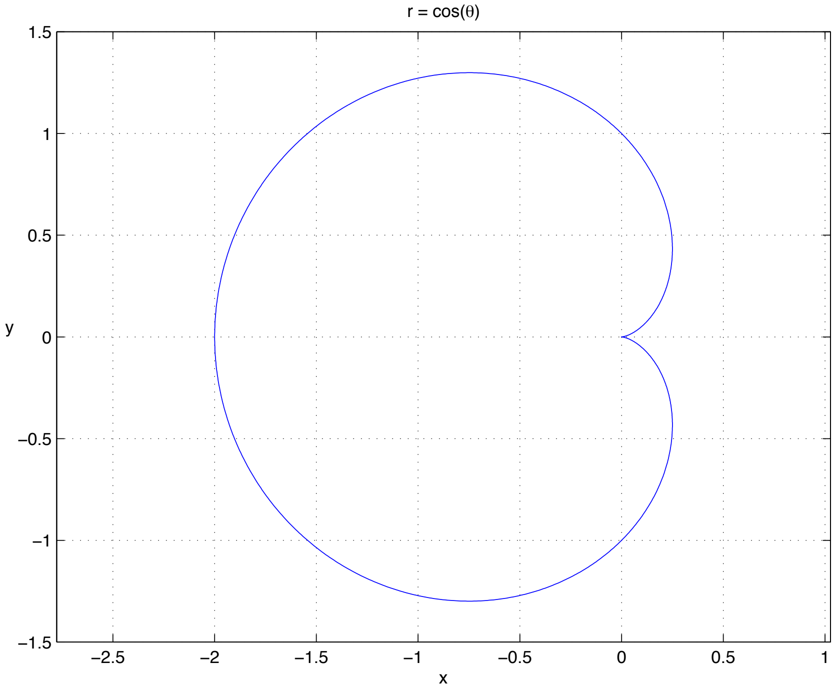
$$K = \int_0^\infty \frac{dx}{e^{2x} + \cos^2 x}$$

also converges.

iv) The polar curve $r = 1 - \cos\theta$ and has the following values

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = 1 - \cos\theta$	0	1	2	1
$x = r\cos\theta$	0	0	-2	0
$y = r\sin\theta$	0	1	0	-1

Noting that the curve is symmetric about $\theta = 0$ ($y = 0$), a plot in the xy-plane is



v) a) Mean Value Theorem: If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), there there exists a $c \in (a, b)$, such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

b) Let $f(t) = \sin^{-1} t$, so

$$f'(t) = \frac{1}{\sqrt{1 - t^2}}.$$

Moreover for any t with $-1 < t < 1$, we have $0 < 1 - t^2 \leq 1$, so $f'(t) \geq 1$. Using the mean value theorem on $[x, y]$ where $-1 < x < y < 1$, there exists a c with $-1 < x < c < y < 1$ such that

$$\frac{\sin^{-1} y - \sin^{-1} x}{y - x} = f'(c) = \frac{1}{\sqrt{1 - c^2}} \geq 1.$$

As multiplying both sides by $y - x > 0$ preserves the inequality, this gives

$$\sin^{-1} y - \sin^{-1} x \geq y - x.$$

vi) a) Let the length of $EP = L_1$ and the length of $FP = L_2$. As the triangles AEP and BFP are right-angled, by Pythagoras $L_1^2 = a^2 + x^2$ and $L_2^2 = b^2 + (c - x)^2$, so

$$L = L_1 + L_2 = \sqrt{a^2 + x^2} + \sqrt{b^2 + (c - x)^2}.$$

b) Differentiating $L = \sqrt{a^2 + x^2} + \sqrt{b^2 + (c-x)^2}$ gives

$$\begin{aligned}\frac{dL}{dx} &= \frac{1}{2}\frac{2x}{\sqrt{a^2+x^2}} + \frac{1}{2}\frac{-2(c-x)}{\sqrt{b^2+(c-x)^2}} \\ &= \frac{x}{L_1} - \frac{(c-x)}{L_2}.\end{aligned}$$

Hence

$$\frac{dL}{dx} = 0 \Longrightarrow \cos\theta = \frac{x}{L_1} = \frac{(c-x)}{L_2} = \cos\phi.$$

c) As the triangles AEP and BFP are similar when $\frac{dL}{dx} = 0$ (right angles, $\theta = \phi$)

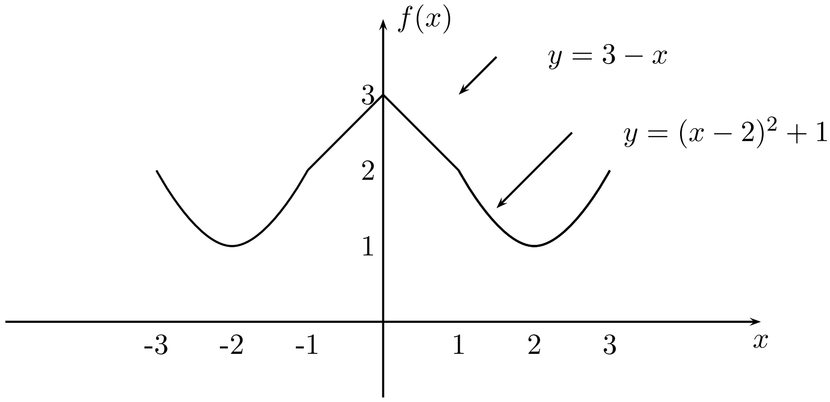
$$\frac{x}{a} = \frac{c-x}{b} \Longrightarrow x = \frac{ac}{a+b}.$$

MATH1131 June 2015 Solutions

1. i) $\lim_{x\to\infty} \dfrac{x+\sin x}{2x} = \lim_{x\to\infty} \dfrac{1+\frac{\sin x}{x}}{2} = \dfrac{1}{2}.$

ii) a) $f(\frac{3}{2}) = \frac{5}{4}$

b)



c)

$$\lim_{x\to 1} f(x) = f(1) = 2.$$

$f'(x) = -1$ for $0 < x < 1$, and $f'(1) = 1$ for $-1 < x < 0$, hence $f'(0)$ does not exist.

iii) Let $x = t^2, t > 0$, then $\frac{dx}{dt} = 2t$ and

$$I_1 = \int \frac{2t}{1+t}\,dt = \int 2 - \frac{2}{1+t}\,dt = 2t - 2\log(1+t) + C$$

$$= 2\sqrt{x} - 2\log(1+\sqrt{x}) + C.$$

iv) Since the limit is of indeterminate form, we apply L'Hopital's rule.

$$\lim_{x\to 0} \frac{1-\cos\left(\frac{x}{2}\right)}{x^2} = \lim_{x\to 0} \frac{\frac{1}{2}\sin\left(\frac{x}{2}\right)}{2x} = \lim_{x\to 0} \frac{\frac{1}{4}\cos\left(\frac{x}{2}\right)}{2} = \frac{1}{8}.$$

v) a) $z - \overline{w} = 2 + 3i$.

b) $10w/(z-2) = 11 + 3i$.

c) Since $|z| = \sqrt{26}$ and $|w| = \sqrt{13}$. Hence $|(z/w)^8| = 2^4 = 16$.

d) $zw = 13(1+i)$ and so $\text{Arg}(zw) = \frac{\pi}{4}$.

e) $|zw|^8 = 13^8 \times 16$ and $\text{Arg}((zw)^8) = 8 \times \frac{\pi}{4} = 2\pi$. Hence $(zw)^8 = 13^8 \times 16$.

vi)

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}, \lambda, \mu \in \mathbb{R}.$$

vii) a) From the MAPLE, $A^3 - 6A^2 - 9A = I$. Hence A^{-1} exists and $A^{-1} = A^2 - 6A - 9I$

$$= \begin{pmatrix} -1 & -2 & 1 \\ -1 & -7 & 2 \\ 1 & 3 & -1 \end{pmatrix}$$

b) $(A^T)^2 = (A^2)^T = B^T$

$$= \begin{pmatrix} 14 & 5 & 25 \\ 4 & 2 & 9 \\ 19 & 8 & 38 \end{pmatrix}$$

2. i)

$$\begin{aligned} 40 + 42i &= z^2 \\ &= (a+ib)^2 \\ &= a^2 - b^2 + i2ab \\ \text{i.e. } a^2 - b^2 &= 40 \\ 2ab &= 42 \\ b &= \frac{21}{a} \\ \text{by inspection, or} & \\ a^2 - \frac{21^2}{a^2} &= 40 \\ a^4 - 441 &= 40a^2 \\ a^2 &= 20 \pm \frac{1}{2}\sqrt{3364} \\ &= -9, 49 \\ a &= \pm 3i, \pm 7 \\ \text{but } a, b \in \mathbb{R} \text{ so } a &= \pm 7, \text{ and} \\ b &= \frac{21}{\pm 7} = \pm 3. \end{aligned}$$

So $z = 7 + 3i, -7 - 3i$.

ii) Converting to augmented form,

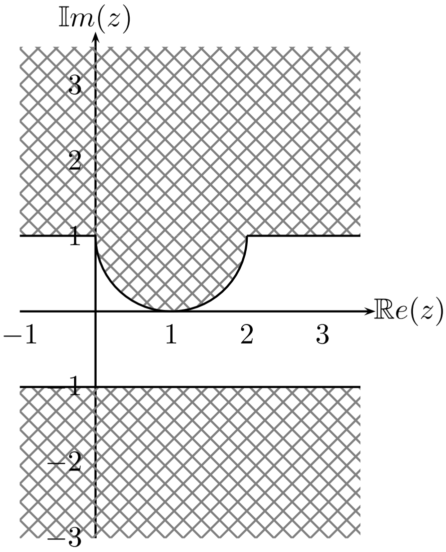
$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 1 & 1 & -1 & b_2 \\ 2 & 1 & -3 & b_3 \end{array}\right)$$

Row reducing,

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -3 & -3 & b_3 - 2b_1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - 2b_1 - 3(b_2 - b_1) \end{array}\right)$$

In order for a solution to exist, $b_3 - 2b_1 - 3(b_2 - b_1) = 0$, i.e. $b_3 + b_1 - 3b_2 = 0$.

iii) 

iv) a)

$$\begin{aligned}\mathbf{u}.\mathbf{v} &= |\mathbf{u}||\mathbf{v}|\cos\theta \\ 2-1+2 &= \sqrt{4+1+1}\sqrt{1+1+4}\cos\theta \\ 3 &= 6\cos\theta \\ \cos\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{3}.\end{aligned}$$

b)

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -1 & -2 \end{vmatrix} \\ &= \mathbf{i}(-2-1) - \mathbf{j}(-4+1) + \mathbf{k}(-2-1) \\ &= \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix} = 3\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.\end{aligned}$$

v) a)

$$f(x) = \begin{cases} 1+ax^2, & x \le 1, \\ bx + 2x^3, & x > 1, \end{cases}$$

a)

$$\left.\begin{array}{lclcl} \lim_{x\to1^-} f(x) & = & \lim_{x\to1^-} bx+2x^3 & = & b+2 \\ \lim_{x\to1^+} f(x) & = & \lim_{x\to1^+} 1+ax^2 & = & 1+a \end{array}\right]$$

$$\text{i.e. } b+2 = 1+a \text{ for } \lim_{x\to1} f(x) \text{ to exist.}$$

$$a = b+1.$$

b)

$$f'(x) = \begin{cases} 2ax, & x < 1, \\ b + 6x^2, & x > 1, \end{cases}$$

$$\left.\begin{array}{lclcl} \lim_{x\to 1^-} f'(x) & = & \lim_{x\to 1^-} b + 6x^2 & = & b + 6 \\ \lim_{x\to 1^+} f'(x) & = & \lim_{x\to 1^+} 2ax & = & 2a \end{array}\right]$$

$$\text{i.e. } b + 6 = 2a \text{ for } \lim_{x\to 1} f'(x) \text{ to exist.}$$

Also require f to be continuous at $x = 1$, so additionally $a = b + 1$ from part (a).

$$\begin{aligned} b + 6 &= 2a \\ b + 6 &= 2(b + 1) \\ b &= 4 \\ a &= b + 1 = 5. \end{aligned}$$

So $a = 5$ and $b = 4$ for f to be differentiable at $x = 1$.

vi)

$$\begin{aligned} I_2 &= \int x^3 \ln x dx \\ &= \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \frac{1}{x} dx \\ &= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 dx \\ &= \frac{x^4}{4} \ln x - \frac{1}{4} \frac{x^4}{4} + C \\ &= \frac{x^4}{4} \left(\ln x - \frac{1}{4} \right) + C. \end{aligned}$$

vii) a) $(x, y) = (0, 1) \implies (r, \theta) = \left(1, \frac{\pi}{2}\right)$.

$$\begin{aligned} x &= r\cos\theta = (1 + \cos\theta)\cos\theta \text{ and} \\ y &= r\sin\theta = (1 + \cos\theta)\sin\theta. \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{\frac{d}{d\theta}\left((1 + \cos\theta)\sin\theta\right)}{\frac{d}{d\theta}\left((1 + \cos\theta)\cos\theta\right)} \\ &= \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\cos\theta\sin\theta} \\ &= \frac{2\cos^2\theta + \cos\theta - 1}{-\sin\theta - 2\cos\theta\sin\theta} \end{aligned}$$

When $\theta = \frac{\pi}{2}$,

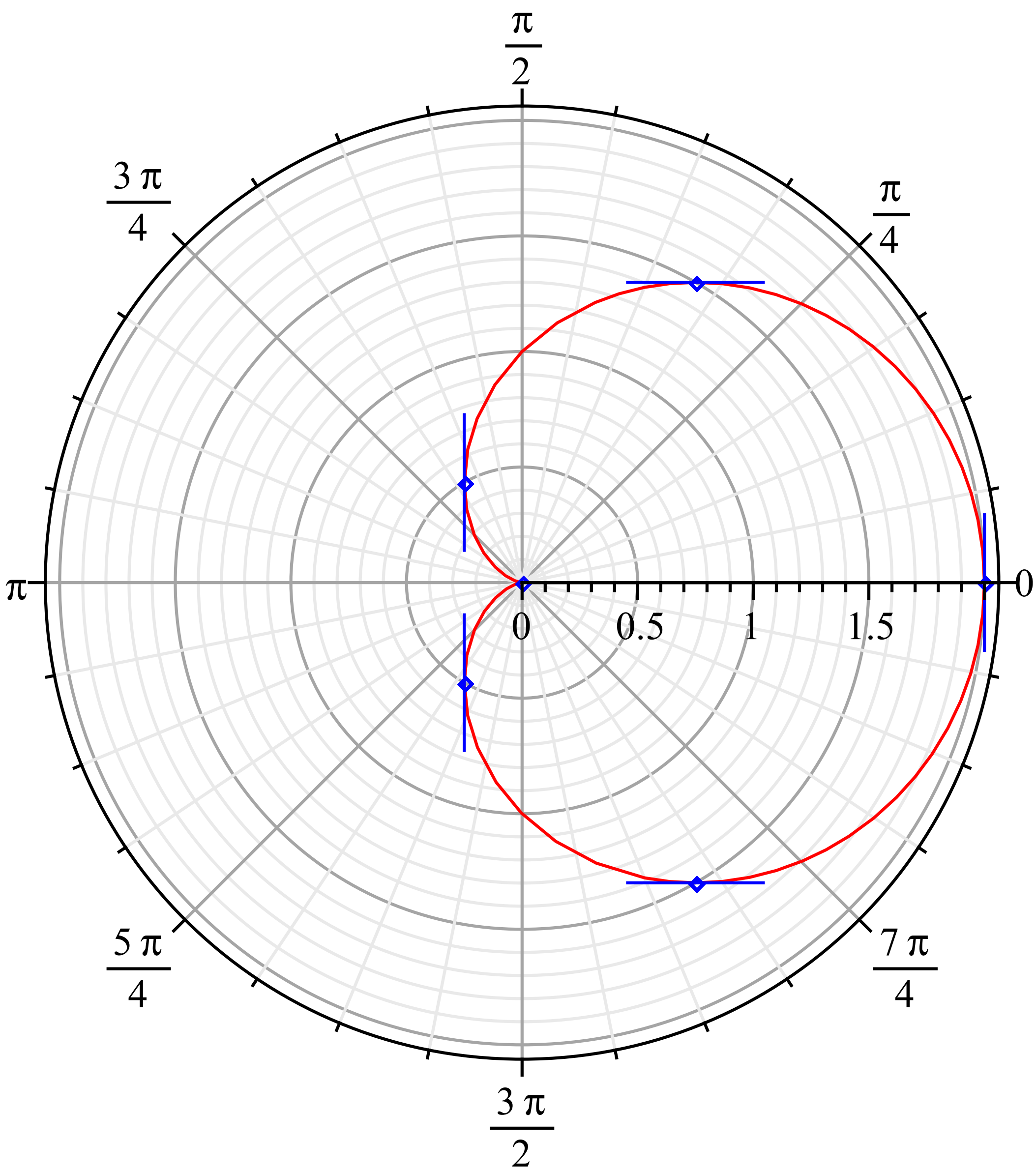
$$\frac{dy}{dx} = \frac{-1}{-1} = 1.$$

i.e. the slope of the tangent at $(x, y) = (0, 1)$ is 1.

b)

$$\frac{dy}{dx} = 0 \implies \frac{dy}{d\theta} = 2\cos^2\theta + \cos\theta - 1 = 0$$
$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$
$$\cos\theta = \frac{1}{2}, -1$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \pi.$$

However, at $\theta = \pi$, both $\frac{dy}{d\theta} = 0$, and $\frac{dx}{d\theta} = 0$.
If we calculate the limiting value of $\frac{dy}{dx}$ at $\theta = \pi$ we get 0, indicating a cusp point.
So the points where the tangent is horizontal are $(r, \theta) = \left(\frac{3}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$.



c)

3. i) Consider the equation $2|z-3i| = |z+3i|$. Squaring both sides and using the identity $|z|^2 = z\bar{z}$ yields

$$\begin{aligned} 4\left(z\bar{z}+3i(z-\bar{z})+9\right) &= z\bar{z}+3i(z-\bar{z})+9 \\ \Rightarrow \quad z\bar{z}+5i(z-\bar{z})+9 &= 0 \\ \Rightarrow \quad z\bar{z}+5i(z-\bar{z})+25 &= 16 \\ \Rightarrow \quad |z-5i|^2 &= 16 \\ \Rightarrow \quad |z-5i| &= 4\,. \end{aligned}$$

This is the equation of a circle, centre $(0,5)$ (on an Argand diagram) with radius 4.

ii) a) Since $z = 2i$ is a root of a polynomial with real coefficients then another root of the polynomial is the complex conjugate, i.e., $z = -2i$. Hence a quadratic factor of the polynomial is z^2+4. Using long division yields

$$\begin{aligned} z^4-3z^3+6z^2-12z+8 &= (z^2+4)(z^2-3z+2) \\ &= (z^2+4)(z-1)(z-2)\,. \end{aligned}$$

b) Thus the four roots of $p(z)$ are $\pm 2i, 1, 2$.

iii) a) To find a point-normal form for the equation of the plane Π we need a point Q on the plane (to then create the position vector $\overrightarrow{OQ}$ where O is the origin) and a normal vector $\mathbf{n}$ to the plane. A point on the plane Π is $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and hence a position vector to Q on the plane Π is $\overrightarrow{OQ} = \begin{pmatrix}1\\0\\0\end{pmatrix}$. The coefficients of the cartesian form of the plane Π yield the normal vector $\mathbf{n} = \begin{pmatrix}1\\-3\\2\end{pmatrix}$. Hence a point-normal form for the plane Π is given by

$$\left(\mathbf{x}-\overrightarrow{OQ}\right)\cdot\mathbf{n} = 0 \quad \Rightarrow \quad \begin{pmatrix}x-1\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\-3\\2\end{pmatrix} = 0\,.$$

b) To show that the line ℓ is parallel to the plane Π we can show that the normal $\mathbf{n}$ to the plane Π and the direction vector $\mathbf{v}$ for the line ℓ are perpendicular, i.e., $\mathbf{n}\cdot\mathbf{v} = 0$. The direction vector $\mathbf{v}$ for the line ℓ is $\mathbf{v} = \begin{pmatrix}2\\0\\-1\end{pmatrix}$. Hence

$$\begin{pmatrix}1\\-3\\2\end{pmatrix}\cdot\begin{pmatrix}2\\0\\-1\end{pmatrix} = 2+0-2 = 0\,.$$

Thus the plane Π and line ℓ are parallel.

c) The shortest distance between a point P and a plane is given by the length of the projection of a vector from a point on the plane, say Q, and P, i.e., $\overrightarrow{QP}$ and a normal vector $\mathbf{n}$ to the plane. From part a) we have $\mathbf{n} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $\overrightarrow{QP} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$.

Thus

$$|\operatorname{proj}_{\mathbf{n}} \overrightarrow{QP}| = \frac{\left| \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \right|}{\sqrt{\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}}} = \frac{1}{\sqrt{14}}\,.$$

Hence the shortest distance between the point $P(4, 2, 2)$ and the plane Π is $\dfrac{1}{\sqrt{14}}$.

iv) a) Using elementary row operations to calculate the determinant yields

$$\begin{aligned}
\begin{vmatrix} 1 & -1 & 3 \\ 3 & a & 2 \\ 2 & 1 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & -1 & 3 \\ 0 & a+3 & -7 \\ 0 & 3 & -5 \end{vmatrix} \qquad R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1 \\
&= \begin{vmatrix} a+3 & -7 \\ 3 & -5 \end{vmatrix} \\
&= 6 - 5a\,.
\end{aligned}$$

b) The matrix A will not have an inverse if $\det(A) = 0$, i.e., $a = \dfrac{6}{5}$.

c) If matrix B is an inverse for matrix A then $AB = I$ where I is the 3×3 identity matrix, i.e.,

$$\begin{pmatrix} 1 & -1 & 3 \\ 3 & a & 2 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 4 & -5 \\ 1 & b & 7 \\ 1 & -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If we multiply row 2 of the first matrix by the 1st column of the second matrix yields $-3 + a + 2 = 0$ with $a = 1$. Also if we multiply the 1st row of the first matrix by the 2nd column of the second matrix yields $4 - b - 9 = 0$ with $b = -5$. (A quick check is to multiply the 2nd row of the first matrix by the 2nd column of the second matrix to verify it is equal to 1). Thus

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad A^{-1} = B = \begin{pmatrix} -1 & 4 & -5 \\ 1 & -5 & 7 \\ 1 & -3 & 4 \end{pmatrix}.$$

v) By definition, $XX^{-1} = I$, where I is the $n \times n$ identity matrix. Hence

$$\begin{aligned}
\det(XX^{-1}) = \det(I) &= 1 \\
\Rightarrow \quad \det(X)\det(X^{-1}) &= 1 \qquad \text{since} \det(AB) = \det(A)\det(B) \\
\Rightarrow \quad 1 \times \det(X^{-1}) &= 1 \qquad \text{since} \det(X) = 1 \\
\Rightarrow \quad \det(X^{-1}) &= 1 \qquad \text{as required.}
\end{aligned}$$

4. i) a)

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

b)

$$\frac{d(\cosh(6x))}{dx} = \frac{d}{dx}(\frac{e^{6x} + e^{-6x}}{2}) = 6(\frac{e^{6x} - e^{-6x}}{2}) = 6\sinh(6x).$$

c) Consider the definition in a) and note that

$$\sinh x = \frac{e^x - e^{-x}}{2} < e^x/2.$$

Taking logs of both sides yields

$$\ln(\sinh(x)) < \ln\left(\frac{e^x}{2}\right) = x - \ln 2.$$

ii) Using the fact that for $x > 0$,

$$\ln x < x$$

it implies that

$$\frac{\ln x}{x^3} < \frac{1}{x^2}.$$

Integrating both sides of the inequality from 1 ti ∞ yields

$$\int_1^\infty \frac{\ln x}{x^3}\, dx < \int_1^\infty \frac{dx}{x^2}.$$

To determine the integral, we write it terms of a proper integral, that is

$$\lim_{R\to\infty} \int_1^R \frac{dx}{x^2} = \lim_{R\to\infty} \left[1 - \frac{1}{R}\right] = 1.$$

Thus, the improper integral is bounded by 1,

$$\int_1^\infty \frac{\ln x}{x^3}\, dx < 1.$$

Alternatively,
we can tackle this question directly by parts:

$$\int_1^\infty \frac{\ln x}{x^3}\, dx = \lim_{R\to\infty} \left(\left[-\frac{\ln x}{2x^2} \right]_1^R + \int_1^R \frac{-1}{2x^3}\, dx \right) = 1/4.$$

iii) (a) Using the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\left(\int_0^{x^3} \cos(t^2)\, dt \right) = 3x^2 \cos(x^6).$$

iv) a) Let $p(x) := x^3 + 3x + 1$.
Note that on the interval $[-1, 0]$ that $p(-1) = -3$ and $p(0) = 1$. The function p is a continuous function and thus by the Intermediate Value Theorem there is a point $c \in (-1, 0)$ such that $p(c) = 0$.

b) Notice that the function is always increasing since

$$p'(x) = 3x^2 + 3 > 0.$$

This means that the function is $1-1$ and has an inverse function $g(x)$.

c) Using the chain rule,

$$p'(x)g'(p(x)) = 1.$$

That is,

$$g'(p(x)) = \frac{1}{p'(x)}.$$

Now, if we solve the equation

$$p(x) = 1$$

then $x = 0$. Thus,

$$g'(1) = g'(p(0)) = \frac{1}{3}.$$

v) a) If f is differentiable on the open interval (a, b) and continuous on the $[a, b]$, then there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

b) Let $f(t) = \cos t$ and consider the interval $[x, y]$ then by the MVT, we have

$$\frac{\cos(y) - \cos(x)}{y - x} = \sin(c).$$

Now, the $|\sin c| \leq 1$ for any $c \in (x, y)$. If we take the absolute value of both sides, we obtain

$$\left|\frac{\cos(y) - \cos(x)}{y - x}\right| = |\sin(c)| \leq 1.$$

Rearranging, proves the inequality

$$|\cos(y) - \cos(x)| \leq |y - x|.$$

PAST HIGHER EXAM SOLUTIONS

MATH1141 June 2011 Solutions

2. i) a) No solutions requires both $6 - a^2 - a = 0$ and $2 - a \neq 0$. i.e.

$$\begin{aligned} -a^2 - a + 6 &= 0 \\ -(a-2)(a+3) &= 0 \\ a &= 2, -3 \end{aligned}$$

AND

$$\begin{aligned} 2 - a &\neq 0 \\ a &\neq 0. \end{aligned}$$

Putting these together we have, for no solutions, $a = -3$.

b) For a unique solution, we require $6 - a^2 - a \neq 0$, which (using part a) gives

$$a \neq 2 \quad \text{and} \quad a \neq -3.$$

c) For infinite solutions we require both $6 - a^2 - a = 0$ and $2 - a = 0$.
i.e. to satisfy both $a = 2$.

ii)

$$\begin{aligned} \begin{vmatrix} 2 & 0 & -1 \\ -1 & 3 & 0 \\ 5 & 7 & 3 \end{vmatrix} &= 2\begin{vmatrix} 3 & 0 \\ 7 & 3 \end{vmatrix} - 0\begin{vmatrix} -1 & 0 \\ 5 & 3 \end{vmatrix} - 1\begin{vmatrix} -1 & 3 \\ 5 & 7 \end{vmatrix} \\ &= 2(9-0) - 0(3-0) + (-1)(7-15) \\ &= 18 + 8 \\ &= 26. \end{aligned}$$

Alternatively an upper triangular form could be used, multiplying the pivots and adjusting for the number of swaps of rows performed to get to the reduced form.

iii)

$$\lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

i.e.

$$\begin{aligned} \lambda &= 1 + \mu + \nu \\ 2\lambda &= \mu \\ 3\lambda &= -\nu \end{aligned}$$

Substituting the last 2 equations into the first we have

$$\begin{aligned} \lambda &= 1 + 2\lambda - 3\lambda \\ 2\lambda &= 1 \\ \lambda &= \frac{1}{2}. \end{aligned}$$

i.e.

$$\boldsymbol{x} = \frac{1}{2}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\1\\\frac{3}{2}\end{pmatrix},$$

Alternatively you could rearrange

$$\lambda\begin{pmatrix}1\\2\\3\end{pmatrix} - \mu\begin{pmatrix}1\\1\\0\end{pmatrix} - \nu\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & -1 & -1\\2 & -1 & 0\\3 & 0 & 1\end{pmatrix}\begin{pmatrix}\lambda\\\mu\\\nu\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

and put the problem in an augmented matrix:

$$\left(\begin{array}{ccc|c}1 & -1 & -1 & 1\\2 & -1 & 0 & 0\\3 & 0 & 1 & 0\end{array}\right)$$

This can be row-reduced and back-substituted to find (λ, μ, ν) which can be substituted into the line (or plane) to find $\boldsymbol{x}$.
Another alternative is to convert the plane to cartesian form and then substitute the line and solve for λ.

iv) a)

$$\begin{aligned}\lim_{h\to 0}\frac{e^{-1/h^2}}{h} &= \lim_{h\to 0}\frac{e^{-1/h^2}}{h^2}h\\&= \lim_{h\to 0}\frac{e^{-1/h^2}}{h^2}\lim_{h\to 0}h.\end{aligned}$$

Letting $x = \frac{1}{h^2}$, when $h \to 0$ then $x \to \infty$. i.e.

$$\begin{aligned}\lim_{h\to 0}\frac{e^{-1/h^2}}{h} &= \lim_{h\to 0}\frac{e^{-1/h^2}}{h^2}\lim_{h\to 0}h\\&= \lim_{x\to\infty}xe^{-x}\lim_{h\to 0}h\\&= 0\cdot 0\\&= 0.\end{aligned}$$

Alternatively, you could use

$$\begin{aligned}\lim_{h\to 0}\frac{e^{-1/h^2}}{h} &= \lim_{h\to 0}\frac{h^{-1}}{e^{1/h^2}}\\&= \text{"}\frac{\infty}{\infty}\text{"} \qquad \Longrightarrow \text{ use L'Hopital's rule}\\&= \lim_{h\to 0}\frac{-h^{-2}}{-2h^{-3}e^{1/h^2}}\\&= \lim_{h\to 0}\frac{h}{2e^{1/h^2}}\\&= \frac{0}{\infty}\\&= 0.\end{aligned}$$

b) Firstly, you should check $f(x)$ is continuous at $x = 0$.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{-1/x^2} = 0$$
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} e^{-1/x^2} = 0$$
$$f(0) = 0.$$

i.e. $f(x)$ is continuous at $x = 0$.
The definition of the derivative at $x = 0$ is

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{e^{-1/h^2} - 0}{h} \\ &= \lim_{h\to 0} \frac{e^{-1/h^2}}{h} \quad \text{which from part a)} \\ &= 0. \end{aligned}$$

Thus $f(x)$ is differentiable at $x = 0$ (with $f'(0) = 0$).

v) a) The derivative at points other than the split points is

$$f(x) = \begin{cases} -\frac{2}{x^2} & \text{for } x < -1, \\ 2x & \text{for } -1 < x < \frac{6}{\pi}, \\ -\frac{1}{x^2}\left(\frac{72}{\pi^2} - 2\right)\cos\frac{1}{x} & \text{for } x > \frac{6}{\pi}. \end{cases}$$

This is a split funciton, so the positions of the splits are possible critical points.

$x = -1$:

$$\lim_{x\to -1^-} f(x) = \frac{2}{-1} = -2$$
$$\lim_{x\to -1^+} f(x) = (-1)^2 - 1 = 0$$

i.e. $\lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x)$ so $x = -1$ **is a critical point** due to the function being discontinuous (derivative undefined) at the is point.

$x = \frac{6}{\pi}$:

$$\lim_{x\to \frac{6}{\pi}^-} f(x) = \left(\frac{6}{\pi}\right)^2 - 1 = \frac{36}{\pi^2} - 1$$
$$\lim_{x\to \frac{6}{\pi}^+} f(x) = \left(\frac{72}{\pi^2} - 2\right)\sin\frac{\pi}{6} = \left(\frac{72}{\pi^2} - 2\right)\frac{1}{2} = \frac{36}{\pi^2} - 1$$

i.e. $\lim_{x\to \frac{6}{\pi}^-} f(x) = \lim_{x\to \frac{6}{\pi}^+} f(x)$ so the function is continuous at $x = \frac{6}{\pi}$. This point is, however the global maximum of the function.

Checking the derivative:

$$\lim_{x\to\frac{6}{\pi}^-} f'(x) = 2\frac{6}{\pi} = \frac{12}{\pi}$$
$$\lim_{x\to\frac{6}{\pi}^+} f'(x) = -\frac{\pi^2}{36}\left(\frac{72}{\pi^2}-2\right)\cos\frac{\pi}{6} = -\frac{\pi^2}{36}\left(\frac{72}{\pi^2}-2\right)\frac{\sqrt{3}}{2}$$

i.e. $\lim_{x\to\frac{6}{\pi}^-} f'(x) \neq \lim_{x\to\frac{6}{\pi}^+} f'(x)$, i.e. the derivative is not continuous at $x=\frac{6}{\pi}$, so $x=\frac{6}{\pi}$ **is critical point**.

Now look for any stationary points in the intervals:

$x<-1$:
$f'(x) = -\frac{2}{x^2} \neq 0$ for any $x \in \mathbb{R}$. i.e. no stationary points.
$-1<x<\frac{6}{\pi}$:
$f'(x) = 2x = 0$ for $x=0$, so there is a stationary point at $x=0$. As $f''(0)=2>0$, this is a local minimum. Thus $x=0$ **is also a critical point**.

$x>\frac{6}{\pi}$:
$f'(x) = -\dfrac{1}{x^2}\left(\dfrac{72}{\pi^2}-2\right)\cos\dfrac{1}{x} = 0$ for $x\to\pm\infty$ or $\cos\dfrac{1}{x}=0$. For the latter case this means $x=\dfrac{2}{(2k+1)\pi}$, for $k\in\mathbb{Z}$. However, we are restricted to $x>\frac{6}{\pi}$, i.e.

$$\begin{aligned}\frac{2}{(2k+1)\pi} &> \frac{6}{\pi}\\ \frac{1}{2k+1} &> 3\\ 2k+1 &> 3(2k+1)^2\\ 0 &> 12k^2+10k+2\\ 0 &> (2k+1)(3k+1)\end{aligned}$$

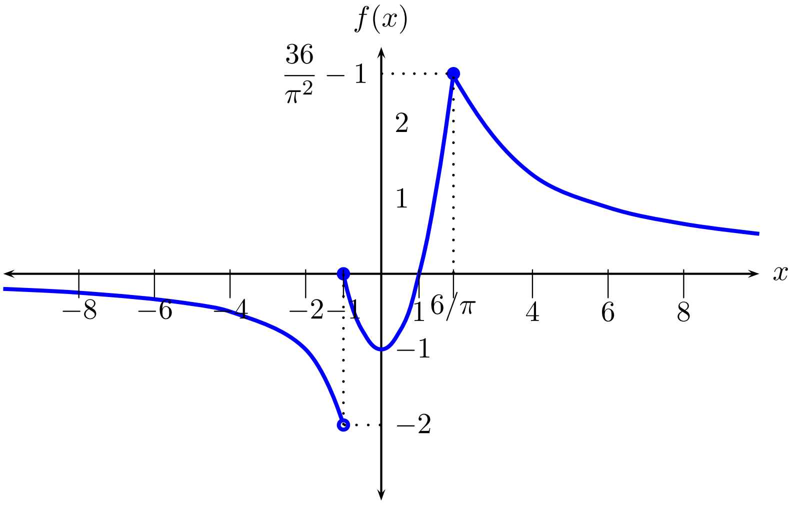
i.e. $k\in(-1/2,-1/3)$, but $k\in\mathbb{Z}$, so there is no solution. i.e. there are no stationary points for $x>\frac{6}{\pi}$.

Putting it altogether, we have 3 critical points:
$x=-1$, where the function is discontinuous, and the derivative is undefined,
$x=0$, where we have a local minimum, and
$x=\dfrac{6}{\pi}$, where the derivative is undefined.

b)

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{2}{x} = 0$$
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{72}{\pi^2}-2\right)\sin\frac{1}{x} = 0$$

Thus there is a horizontal asymptote at $f(x)=0$.



c)

3. i) a) We recognise that S is the sum of a geometric progression with common ratio $e^{2i\theta}/3$. Therefore

$$\begin{aligned} S &= e^{i\theta}\sum_{k=0}^{\infty}\left(\frac{e^{2i\theta}}{3}\right)^k \\ &= \frac{e^{i\theta}}{1-e^{2i\theta}/3} \\ &= \frac{3e^{i\theta}}{3-e^{2i\theta}}\cdot\frac{3-e^{-2i\theta}}{3-e^{-2i\theta}} \\ &= \frac{3\left(3e^{i\theta}-e^{-i\theta}\right)}{(3-\cos(2\theta))^2+\sin^2(2\theta)} \\ &= \frac{3\left(3e^{i\theta}-e^{-i\theta}\right)}{10-6\cos(2\theta)} \end{aligned}$$

as required.

b) We observe that $T = \mathrm{Im}(S)$, and hence using part (a) we calculate

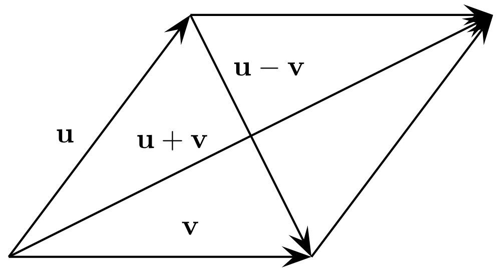
$$\begin{aligned} T &= \frac{3(3\sin(\theta)-\sin(-\theta))}{10-6\cos(2\theta)} \\ &= \frac{3\times 4\sin(\theta)}{10-6\cos(2\theta)} \\ &= \frac{6\sin(\theta)}{5-3\cos(2\theta)}. \end{aligned}$$

ii) To show that $\mathbf{u}-\mathbf{v}$ is perpendicular to $\mathbf{u}+\mathbf{v}$, we show that the dot product of the two vectors is zero. Now, using properties of the dot product and the fact that $\mathbf{u}$ and $\mathbf{v}$ have the same length, we have

$$\begin{aligned}(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) &= \mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}\\ &= \mathbf{u}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}\\ &= \|\mathbf{u}\|^2-\|\mathbf{v}\|^2\\ &= 0,\end{aligned}$$

as required.

Alternatively, note that $\mathbf{u}-\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are the diagonals of the rhombus spanned by the vectors $\mathbf{u}$ and $\mathbf{v}$, as shown in the figure.



We know that the diagonals of a rhombus are perpendicular, completing the proof.

iii) a) Recall that $\mathbf{e}_j$ is the $n\times 1$ vector with a 1 in the jth position and zeros everywhere else. Therefore

$$A\mathbf{e}_j = \begin{pmatrix} a_{1j}\\ a_{2j}\\ \vdots\\ a_{nj}\end{pmatrix},$$

which is the jth column of A. Similarly, $\mathbf{e}_i^T$ is a $1\times n$ matrix with a 1 in the ith column and zeros everywhere else. Hence

$$\mathbf{e}_i^T A\mathbf{e_j} = \mathbf{e}_i^T \begin{pmatrix} a_{1j}\\ a_{2j}\\ \vdots\\ a_{nj}\end{pmatrix} = a_{ij},$$

since a_{ij} is the ith entry of the jth column of A. This completes the proof.

b) If A is symmetric then $A^T = A$, and hence

$$(A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T A^T\mathbf{y} = \mathbf{x}^T A\mathbf{y},$$

as required.

c) Suppose that $\mathbf{x}^T A\mathbf{y} = (A\mathbf{x})^T\mathbf{y}$ for all vectors $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$. Rewriting this gives

$$\mathbf{x}^T A\mathbf{y} = \mathbf{x}^T A^T\mathbf{y} \tag{$*$}$$

for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$. Now let $\mathbf{x}=\mathbf{e}_i$ and $\mathbf{y}=\mathbf{e}_j$, where i,j are arbitrary integers in $\{1,\ldots,n\}$. Applying part (a) to condition $(*)$ gives

$$a_{ij} = (A^T)_{ij} = a_{ji}.$$

This proves that A is symmetric, since i and j were arbitrary.

iv) a) To show that M reflects the vector $\mathbf{a}$ to the vector $-\mathbf{a}$, we must show that $M\mathbf{a} = -\mathbf{a}$. We calculate

$$M\mathbf{a} = \frac{1}{9}\begin{pmatrix} 1 & 4 & -8 \\ 4 & 7 & 4 \\ -8 & 4 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 2-4-16 \\ 8-7+8 \\ -16-4+2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = -\mathbf{a}$$

as required.

b) Every vector in the plane is perpendicular to the vector $\mathbf{a}$. Therefore the Cartesian equation of the plane Π is

$$2x - y + 2z = 0.$$

c) Any nonzero vector $\mathbf{u}$ which is the position vector of a point in the plane Π satisfies $M\mathbf{u} = \mathbf{u}$. One such vector is $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

d) The shortest distance from B to the plane is equal to the length of the projection of $\mathbf{b}$ onto $\mathbf{a}$. We calculate

$$|\operatorname{proj}_{\mathbf{a}}(\mathbf{b})| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{\|\mathbf{a}\|} = \frac{|12+6+0|}{\sqrt{4+1+4}} = \frac{18}{3} = 6.$$

The shortest distance from B to the plane Π is 6 units.

4. i) a) We apply L'Hôpital's rule repeatedly (checking each time that he required conditions are satisfied)

$$\lim_{x\to\infty}\frac{x^n}{e^x} = \lim_{x\to\infty}\frac{nx^{n-1}}{e^x} = \ldots = \lim_{x\to\infty}\frac{n!}{e^x} = 0.$$

b) By (a), $e^{-x}x^{n+2} \to 0$ as $n \to \infty$ and so $e^{-x}x^{n+2} < 1$ for sufficiently large n, say $n > M$. Also by continuity, $e^{-x}x^{n+2} < K$ on the interval $[1, M]$. Thus if $C = \max\{K, 1\}, e^{-x}x^{n+2} < C$ for all $x \geq 1$ and the result follows.

c)

$$\int_1^\infty e^{-x}x^n\,dx < \int_1^\infty \frac{C}{x^2}\,dx.$$

The latter integral converges by the p-test and so the given integral converges for any $n \in \mathbb{N}$.

ii) a) Applying the Mean Value Theorem to f on $[0,2]$, we have, for some $c \in (0,2)$,

$$f'(c) = \frac{f(2)-f(0)}{2-0} = 6.$$

b) Applying the Mean Value Theorem to f' on $[0,c]$, we have, for some $d \in (0,c)$,

$$f''(d) = \frac{f(c)-f(0)}{c-0} = \frac{6}{c} > 3.$$

iii) a) Since $\cosh x > 1$ for all $x > 0$,the result follows.

b)

$$f'(x) = 1 - a\,\operatorname{sech}^2 x > 0$$

if $a \leq 1$. Thus f is strictly increasing on $(0,\infty)$. Also $f(0) = 0$, hence $f(x) > 0$ for all $x > 0$ when $a \leq 1$.

c) For f to have a positive zero, $f'(x) = 1 - a\ \mathrm{sech}^2 x = 0$ for some $x > 0$ so $a = \frac{1}{\mathrm{sech}^2 x} > 1$.

iv) a)

$$A(t) = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd(1+\cos t)}$$

has a maximum when $1 + \cos t = 0$, that is, when $\alpha + \beta = \pi$.

b) Now $1 + \cos t$ is decreasing on $[0, \pi]$ and so $-(1 + \cos t)$ is increasing. Thus $A(t)$ is increasing and continuous on $[0, \pi]$ and so has an inverse.

Furthermore, $A'(t) = \frac{\frac{1}{2}abcd \sin t}{2A(t)}$ so that A is differentiable. By the inverse function theorem, $B = A^{-1}$ is differentiable on $(A(0), A(\pi))$.

c) If $t = \frac{\pi}{2}$, then $A(\frac{\pi}{2}) = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd} = A_0$. By the inverse function theorem,

$$B'(A_0) = \frac{1}{A'(\frac{\pi}{2})} = \frac{1}{\frac{abcd}{4A(\pi)}} = \frac{4A_0}{abcd}.$$

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2. i) a)

$$\begin{aligned}
f(x) &= x^2 + \cos\left(x^2\right), \quad x \in \left(0, 2\sqrt{\pi}\right) \\
f'(x) &= 2x\left(1 - \sin\left(x^2\right)\right) \\
\text{Note} \;\; f'(x) &= 0, \quad x \in \left(0, 2\sqrt{\pi}\right] \\
f''(x) &= 2\left(1 - \sin\left(x^2\right)\ \right) + 4x^2\cos\left(x^2\right) \\
f'(x) &= 0 \quad \Rightarrow \\
x &= 0, \quad \text{which is not in the domain, or} \\
x^2 &= \frac{\pi}{2}, \frac{5\pi}{2} \\
x &= \sqrt{\frac{\pi}{2}}, \sqrt{\frac{5\pi}{2}}. \qquad\qquad \text{Note } 3\sqrt{\frac{\pi}{2}} \text{ is not in the domain.}
\end{aligned}$$

$$f''\left(\sqrt{\frac{\pi}{2}}\right) = 0, f''\left(\sqrt{\frac{5\pi}{2}}\right) = 0.$$

So the critical points are $x = \sqrt{\frac{\pi}{2}}, \sqrt{\frac{5\pi}{2}}$, the stationary points (both points of inflection) and $x = 2\sqrt{\pi}$, the endpoint, and global maximum of the function.

b)

$$\begin{aligned}
f\left(2\sqrt{\pi}\right) &= 4\pi + 1 \\
f(0) &= 1 \\
f'(x) &= 2x\left(1 - \sin\left(x^2\right)\right) \geq 0 \quad \text{for } x \in \left(0, 2\sqrt{\pi}\right]
\end{aligned}$$

So f is an increasing function on its domain.
It is also continuous (combination of elementary functions) and thus is invertible.
$Range\,(f) = (1, 4\pi + 1]$, so $Dom\left(f^{-1}\right) = (1, 4\pi + 1]$.
From above we can see $f\left(\sqrt{\frac{5\pi}{2}}\right) = \frac{5\pi}{2}$. Thus$f^{-1}\left(\frac{5\pi}{2}\right) = \sqrt{\frac{5\pi}{2}}$.

c) From the inverse function theorem, $\frac{d}{dx}f^{-1}(x) = -\frac{1}{f'\left(f^{-1}(x)\right)}$. So $f^{-1}(x)$ is not differentiable where $f'\left(f^{-1}(x)\right) = 0$. i.e. $f^{-1}(x) = \sqrt{\frac{\pi}{2}}, \sqrt{\frac{5\pi}{2}}$, or $x = \frac{\pi}{2}, \frac{5\pi}{2}$. i.e. $f^{-1}(x)$ is differentiable in $(1, 4\pi + 1)$, $x \neq \frac{\pi}{2}, \frac{5\pi}{2}$.

ii) a)

$$\begin{aligned}
f(x) &= \int_0^{x^2 - 9x} e^{-t^2}\, dt \\
f'(x) &= e^{-\left(x^2 - 9x\right)^2}(2x - 9)
\end{aligned}$$

Stationary point: $f'(x) = 0$.
Mean value theorem: $f(x)$ is continuous on $[0, 9]$ and differentiable on $(0, 9)$. There-

fore

$$\frac{f(9)-f(0)}{9-0} = f'(x_0), \quad x_0 \in (0,9).$$
$$f(9) = \int_0^0 e^{-t^2}\, dt = 0$$
$$f(0) = \int_0^0 e^{-t^2}\, dt = 0$$

Therefore,

$$f^{'}(x_0) = 0, \quad x_0 \in (0,9).$$

b)

$$f^{'}(x) = e^{-\left(x^2-9x\right)^2}(2x-9) = 0$$

i.e. the stationary point is $x = \frac{9}{2}$.

$$f^{''}(x) = \left(2 - 2\left(x^2-9x\right)(2x-9)\right)e^{-\left(x^2-9x\right)^2}$$
$$f^{''}\left(\frac{9}{2}\right) = \left(2-2\left(\left(\frac{9}{2}\right)^2 - 9\frac{9}{2}\right)\left(2\frac{9}{2}-9\right)\right)e^{-\left(\left(\frac{9}{2}\right)^2-9\frac{9}{2}\right)^2} = 2e^{-\left(\frac{81}{4}\right)^2} > 0$$

iii) a) Since z lies on the unit circle, $z = e^{i\theta}$ for some $\theta \in [0, 2\pi]$. Hence $z + \dfrac{1}{z} = e^{i\theta} + e^{-i\theta} = 2\cos\theta \in \mathbb{R}$.

b) By (a), $z + \dfrac{1}{z}$ has a maximum value of 2 (when $\theta = 0$).

iv)

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$
$$e^{i4\theta} = (\cos\theta + i\sin\theta)^4 \quad \text{by de Moivre's theorem}$$
$$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta \quad \text{(binomial expansion)}$$

$\cos 4\theta$

$$= \mathbb{R}e\left(e^{i4\theta}\right)$$
$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$= \cos^4\theta - 6\cos^2\theta\left(1-\cos^2\theta\right) + \left(1-\cos^2\theta\right)^2$$
$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$
$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

v) a)

$$x = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}$$
$$\begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 2 \\ -7 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 3 & 3 \\ 1 & -1 & 2 \\ -3 & -1 & -7 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -4 & -1 \\ 0 & 8 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -4 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

This system has unique solution, so the point lies on the plane.

b) If $\mathbf{b}$ is parallel to P then $\mathbf{b}= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 3 & 1 \\ 1 & -1 & 3 \\ -3 & -1 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -4 & 2 \\ 0 & 8 & 5 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 9 \end{array}\right)$$

i.e. no solution, and $\mathbf{b}$ is not parallel to P.

c)

$$\mathbf{c} \cdot \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} = 0, \quad \text{and} \mathbf{c} \cdot \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = 0.$$

Therefore $\mathbf{c}$ is orthogonal to P.

3. i) $\det(B) = -35$.

ii) a) $\mathbf{x} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ where $\lambda \in \mathbb{R}$.

b) The quadratic distance from $\mathbf{x}$ to $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is

$$(4+\lambda)^2 + (-\lambda)^2 + (5+\lambda)^2 = 3\lambda^2 + 18\lambda + 41$$

which is minimal when $\lambda = -\frac{18}{2 \cdot 3} = -3$.

The closest point is thus given by $\mathbf{x} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

iii) a) All $a \neq 1, -2$.
b) $a = 1$.
c) $a = -2$.

iv) a) For example, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

b) If Q is nilpotent (of degree 2), then $|\det Q| = \sqrt{|\det Q^2|} = \sqrt{|\mathbf{0}|} = 0$, so Q is not invertible.

c) Proof. $S^{-1}Q = S^{-1}QI = S^{-1}QSS^{-1} = S^{-1}SQS^{-1} = IQS^{-1} = QS^{-1}$. □

d) By c),

$$(S+Q)(S^{-1}-S^{-2}Q) = I+QS^{-1}-S^{-1}Q-QS^{-2}Q = I-S^{-1}QQS^{-1} = I-S^{-1}\mathbf{0}S^{-1} = I\,.$$

Hence, $S+Q$ is invertible, and $k = 2$.

v) PROOF. We see that

$$a = \mathbf{x} \cdot \mathbf{e}_1 = |\mathbf{x}||\mathbf{e}_1| \cos\alpha = |\mathbf{x}| \cos\alpha\,;$$
$$b = \mathbf{x} \cdot \mathbf{e}_2 = |\mathbf{x}||\mathbf{e}_2| \cos\beta = |\mathbf{x}| \cos\beta\,;$$
$$c = \mathbf{x} \cdot \mathbf{e}_3 = |\mathbf{x}||\mathbf{e}_3| \cos\gamma = |\mathbf{x}| \cos\gamma\,.$$

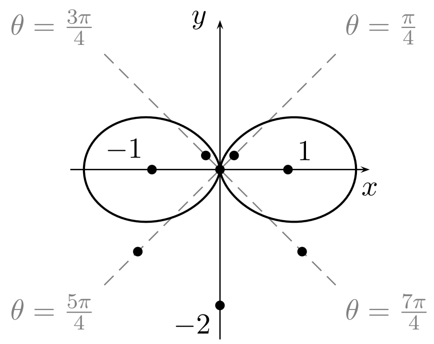
Hence,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a^2}{|\mathbf{x}|^2} + \frac{b^2}{|\mathbf{x}|^2} + \frac{c^2}{|\mathbf{x}|^2} = \frac{|\mathbf{x}|^2}{|\mathbf{x}|^2} = 1\,. \quad \square$$

vi) PROOF. Since $\det(A^T) = \det(A) \neq 0$, every vector $\mathbf{b} \in \mathbb{R}$ can be written uniquely as a linear combination of columns of A^T, or in other words, rows of A. $\square$

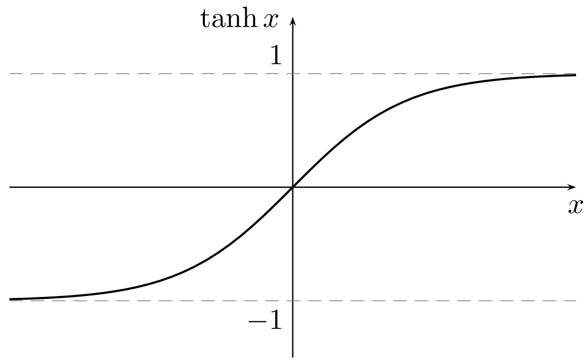
4. i) a) Since $\cos(-2\theta) = \cos\theta$ the curve is symmetric about the x axis and since $\cos 2(\pi - \theta) = \cos 2\theta$ the curve is symmetric about the y axis.

b) Diagram as below



ii) a) $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

b) Diagram as below



c) Since $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = 1 - \dfrac{2e^{-x}}{e^x + e^{-x}}$, $(1 - \tanh x)/e^{-2x} = \dfrac{2e^x}{e^x + e^{-x}}$ which tends to 2 as $x \to \infty$.

d) Since $\displaystyle\int_0^\infty e^{-2x}\,dx$ converges, it follows from (c) by the limit comparison test that $\displaystyle\int_0^\infty (1 - \tanh x)\,dx$ converges also.

e)

$$\int_0^\infty (1 - \tanh x)\,dx = \lim_{R\to\infty} \int_0^R (1 - \tanh x)\,dx = \lim_{R\to\infty} \left[x - \log(\cosh x)\right]_0^R$$

$$= \lim_{R\to\infty} R - \log(\cosh R) = \lim_{R\to\infty} \log \frac{2e^R}{e^R + e^{-R}} = \log 2$$

iii) a) Integrating by parts with $u = f(x)$, we have

$$\int_a^b f(x) \sin nx \, dx = \left[-\frac{\cos(nx)}{n} \right]_a^b + \frac{1}{n} \int_a^b f'(x) \cos(nx) \, dx$$

$$= \frac{f(a)\cos(nx) - f(b)\cos(nb)}{n} + \frac{1}{n} \int_a^b f'(x) \cos(nx) \, dx.$$

b)

$$\left| \int_a^b f'(x) \cos nx \, dx \right| \le \int_a^b \left| f'(x) \cos nx \right| \, dx \le \int_a^b |f'(x)| \, dx \le \int_a^b L \, dx = L(b-a).$$

c)

$$\lim_{n\to\infty} \int_a^b f(x) \sin nx \, dx = \lim_{n\to\infty} \frac{f(a)\cos(nx) - f(b)\cos(nb)}{n} + \frac{1}{n} \int_a^b f'(x) \cos(nx) \, dx$$

$$\le \lim_{n\to\infty} K(n)/n + L(b-a)/n = 0$$

since $K(n)$ is bounded.

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2. i) a) Away from 0 f is differentiable. Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 0$, f is continuous at $x = 0$ and since the derivatives of the two constituent functions have the same derivative at $x = 0$, by the 'split function theorem', f is differentiable at $x = 0$ and hence everywhere.

$$f'(x) = \begin{cases} 3x^2 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0. \end{cases}$$

b) Since $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x) = 0$, so f' is continuous at $x = 0$

c) For $x < 0$, $\lim_{h\to 0^-} \dfrac{f'(0+h) - f'(0)}{h} = 0$, while for $x > 0$, $\lim_{h\to 0^+} \dfrac{f'(0+h) - f'(0)}{h} = 2$. Hence f' is not differentiable at $x = 0$.

ii) a)

$$L_P(f) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} = \sum_{k=1}^{n} \frac{1}{n+k}.$$

b) Under the given assumption,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \int_0^1 \frac{1}{1+x} = \log 2.$$

iii) a) $\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

b) $\mathbf{x} = \mathbf{c} + \lambda(\frac{1}{2}(\mathbf{a}+\mathbf{b}) - \mathbf{c}), \lambda \in \mathbb{R}$.

c) In triangle ABC, usin the dot product formula, $\cos A = \frac{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}|\,|\mathbf{c}-\mathbf{a}|} = \frac{1}{2}$ and so $A = \pi/3$. Similarly $\cos B = \frac{1}{2}$ so $B = \pi/3$. Hence ABC is equilateral.

iv) a)

$$\mathbf{x} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}.$$

b) This is the line of intersection of two planes.

c) Substituting the line into the thrid equation (which is a sphere), gives $3\lambda^2 - 8\lambda + 5 = 0$ and so $\lambda = 1, 5/3$. The value $\lambda = 1$ gives the point $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and $\lambda = 5/3$ gives the point $\frac{1}{3}\begin{pmatrix} 7 \\ 4 \\ 5 \end{pmatrix}$.

3. i) A vector normal to the plane

$$\boldsymbol{n} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}.$$

A vector from the point $\begin{pmatrix}1\\2\\4\end{pmatrix}$ on the plane to $\mathbf{p}$:

$$\begin{pmatrix}1\\2\\4\end{pmatrix} - \begin{pmatrix}3\\2\\3\end{pmatrix} = \begin{pmatrix}-2\\0\\1\end{pmatrix}.$$

The length of the projection of this vector onto $\boldsymbol{n}$ is the shortest distance from $\mathbf{p}$ to the plane

$$\frac{\left|\begin{pmatrix}-2\\0\\1\end{pmatrix} \cdot \begin{pmatrix}7\\1\\2\end{pmatrix}\right|}{\left|\begin{pmatrix}7\\1\\2\end{pmatrix}\right|} = \frac{4}{\sqrt{6}}.$$

ii) a) $\alpha_1 = 1$.

b) Complex roots come in conjugate pair because p has real coefficients. Hence there are two complex roots and two real roots.

c) Assume that α_2 is the second real root, and that α_3 and α_4 are the conjugate complex roots ($\alpha_4 = \bar{\alpha}_3$). Since $\alpha_1 + \cdots + \alpha_4 = 1$ and $\alpha_1 = 1$, there holds

$$\alpha_2 + \alpha_3 + \alpha_4 = \alpha_2 + 2a = 0, \tag{2}$$

where we denote by a the real part of α_3. Noting that 0 is not a root, we deduce that either $\alpha_2 < 0$ or $a < 0$.

d) If α is a root satisfying $|\alpha| \le 1/2$ then from $p(\alpha) = 0$ we deduce $\alpha^4 - \alpha^3 - \alpha^2 - \alpha = -2$, so that

$$2 = |\alpha^4 - \alpha^3 - \alpha^2 - \alpha| \le |\alpha|^4 + |\alpha|^3 + |\alpha|^2 + |\alpha| \le \frac{1}{2^4} + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2} = \frac{15}{16},$$

contradiction! Hence $|\alpha_j| > 1/2$ for $j = 1, 2, 3, 4$.

iii) a) False. Take for example

$$A = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}$$

b) True. (Note: $\det(AB) = \det(A)\det(B)$.)

c) False. Since $\det(AB) = \det(A)\det(B)$ and $\det(AC) = \det(A)\det(C)$ we need to choose an example such that $\det(A) = 0$. For example

$$A = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \quad B = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

d) False. Choose an example such that A is not invertible, i.e., $\det(A) = 0$ as in c). (The statement is true if A is invertible.)

iv) a) $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal set if

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j. \end{cases}$$

b) Since the jth row of M^T is $\mathbf{v}_j$, we have

$$M^T M = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 & \cdots & \mathbf{v}_1 \cdot \mathbf{v}_n \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 & \cdots & \mathbf{v}_2 \cdot \mathbf{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_n \cdot \mathbf{v}_1 & \mathbf{v}_n \cdot \mathbf{v}_2 & \cdots & \mathbf{v}_n \cdot \mathbf{v}_n \end{pmatrix} = I$$

i.e., M is an orthogonal matrix. Hence

$$\det(M)^2 = \det(M^T)\det(M) = \det(M^T M) = \det(I) = 1,$$

implying $\det(M) = \pm 1$.

4. i) a)

$$\lim_{t\to\infty} t^2 e^{-t^2} = \lim_{t\to\infty} \frac{t^2}{e^{t^2}} = \lim_{t\to\infty} \frac{2t}{2te^{t^2}} = \lim_{t\to\infty} \frac{1}{e^{t^2}} = 0.$$

b) Since $e^{t^2} \geq t^2$ for $t \geq 1$, we have that $\int_1^\infty e^{-t^2} dt \leq \int_1^\infty \frac{1}{t^2} dt$, which converges by the p-test. Also $\int_0^1 e^{-t^2} dt < \int_0^1 1\, dt = 1$ and so the original integral converges.

c) $f'(x) = 3x^2 e^{-x^6}$ and so the only critical point is $(0,0)$ which is an inflection point. As $x \to \infty$ $f(x) \to I$.

d) DIAGRAM

ii)

$$\lim_{x\to c} \frac{f(x) - f(c) - f'(c)(x-c)}{(x-c)^2} = \lim_{x\to c} \frac{f'(x) - f'(c)}{2(x-c)} = \lim_{x\to c} \frac{f''(x)}{2} = \frac{f''(c)}{2}.$$

iii) a) Differentiating implicitly, $2(x^2+y^2)(2x+2yy') - 2(2x-2yy') = 0$, so $y' = \dfrac{x(1-(x^2+y^2))}{y(1+x^2+y^2)}$. The tangents correspond to when the numerator is 0, and this occurs when $x = 0$, i.e. on the y axis or when $x^2 + y^2 = 1$.

b) The condition $x = 0$ yields the point, $(x, y) = (0, \pm\sqrt{-1+\sqrt{b}})$, provided $b \geq 1$. The condition $x^2 + y^2 = 1$ yields $4y^2 = b$; $0 < b \leq 4$ and hence the points $(x, y) = (\frac{\sqrt{4-b}}{2}, \pm\frac{\sqrt{b}}{2})$.

iv) a) $g(0) = f(0) \geq 0$ and $g(2) = f(2) - 8 \leq 0$ and so by the IVT, g has at least one root ξ in the interval $[0, 2]$. Thus $f(\xi) = \xi^3$.

b) Applying the MVT to f on $[0, 2]$ we have $\frac{f(2)-f(0)}{2-0} = f'(\eta)$ for some $\eta \in (0, 2)$ and the result follows.

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3. i)

$$g'(x) = 3 + 2\sin 2x = 2(1 + \sin 2x) + 1 > 0.$$

Therefore, $g(x)$ is an increasing function. It is also continuous (because it is a composition of elementary functions) so it is invertible. By the inverse function theorem, the inverse is also continuous and differentiable.

By inspection, $g(0) = -2$ so $h(-2) = 0$. Thus

$$h'(-2) = \frac{1}{g'(h(-2))} = \frac{1}{g'(0)} = \frac{1}{3}.$$

ii) a) Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists at least one real number c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

b) Let $f(x) = \tan^{-1} x$ *(1 mark)*. Then $f'(x) = 1/(1+x^2)$ and the Mean Value Theorem states that there exists a c in (a, b) such that

$$f'(c) = \frac{\tan^{-1} b - \tan^{-1} a}{b - a}. \qquad \textit{(1 mark)}$$

But $0 < 1/(1 + c^2) \leq 1$ so

$$0 < \frac{\tan^{-1} b - \tan^{-1} a}{b - a} \leq 1$$

or

$$0 < \tan^{-1} b - \tan^{-1} a \leq b - a.$$

c) Let $I = \int_1^\infty g(t)\, dt$ where $g(t) = \tan -1(t+t^{-2}) - \tan^{-1} t$. From (b), $0 \leq g(t) \leq h(t)$ where $h(t) = t + t^{-2} - t = t^{-2}$.

Since $\int_1^\infty h(t)dt = \int_1^\infty t^{-2} dt$ converges (by the p-test) then so does $\int_1^\infty g(t)dt$ (by the comparison test).

iii) Let $L = 2$. Then

$$\begin{aligned}
|f(x) - L| &= \left|\frac{e^x}{\cosh x} - 2\right| \\
&= \frac{e^x}{\cosh x}\left|1 - 2\frac{\cosh x}{e^x}\right| \\
&= \frac{e^x}{\cosh x}\left|1 - 1 - e^{-2x}\right| \\
&= \frac{e^x}{\cosh x}\left|-e^{-2x}\right| \\
&= \frac{e^{-x}}{\cosh x} \leq e^{-x},
\end{aligned}$$

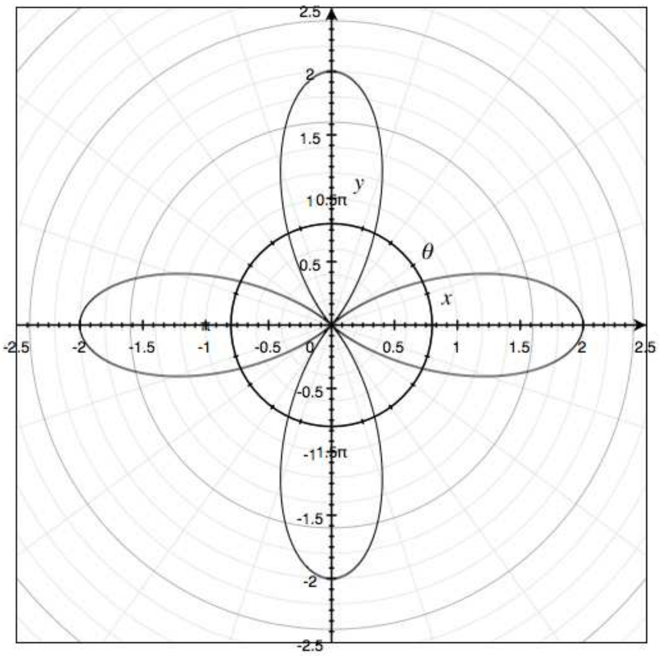
since $\cosh x \geq 1$.

Let $\epsilon > 0$. Then $e^{-x} < \epsilon$ if and only if $-x < \ln \epsilon$ or $x > -\ln \epsilon = \ln \epsilon^{-1}$.

Let $M = \ln \epsilon^{-1}$. Then we have shown that if $x > M$ then there exists an $\epsilon > 0$ such that $|f(x) - L| < \epsilon$, as required.

iv) a) $r' = -4 \sin 4\theta = 0$ when $\theta = n\pi/4$ for $n = 0, 1, 2, \cdots$. Thus:
r takes a maximum value of 2 at $\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi$.
r takes a minimum value of 0 at $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

b) Diagram as below

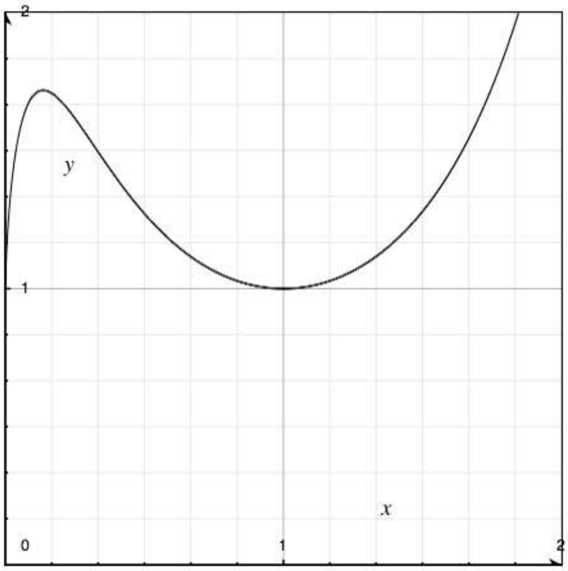


v) a) $f(x) = x^{x \ln x} = (e^{\ln x})^{x \ln x} = e^{x(\ln x)^2}$. So

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x(\ln x)^2\right) e^{x(\ln x)^2} \\ &= \left((\ln x)^2 + x\frac{2\ln x}{x}\right) e^{x(\ln x^2} \\ &= \left((\ln x)^2 + 2\ln x\right) x^{x \ln x}. \end{aligned}$$

b) Since $x^{x \ln x} = e^{x(\ln x)^2} > 0$ for $x > 0$, the sign of $f'(x)$ will be determined by the sign of $g(x) = (\ln x)^2 + 2\ln x$.
Let $z = \ln x$. Then $g = z^2 + 2z = z(z + 2)$. This negative for $-2 < z < 0$. Thus, $f'(x)$ is negative for $e^{-2} < x < 1$ and positive for $x < e^{-2}$ or $x > 1$.

c) Diagram as below



4. i) We first write the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 0 & -4 & b_1 \\ 3 & 1 & -2 & b_2 \\ -2 & -1 & 0 & b_3 \end{array}\right).$$

and row reduce to get:

$$\left(\begin{array}{ccc|c} 2 & 0 & -4 & b_1 \\ 0 & 2 & 8 & 2b_2 - 3b_1 \\ 0 & 0 & 0 & 2b_3 + 2b_2 - b_1 \end{array}\right).$$

The system has a solution if the final column in a row reduced form of the augmented matrix is not a leading column; this condition corresponds to the equation

$$-b_1 + 2b_2 + 2b_3 = 0.$$

ii) The distance from the point represented by the vector $\begin{pmatrix} x \\ x \\ x \end{pmatrix}$ to I (which is represented by the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$) is given by the distance formula:

$$d = \sqrt{(x-1)^2 + (x-0)^2 + (x-0)^2} = \sqrt{3x^2 - 2x + 1}.$$

The distance will be 1 if and only if the square of the distance is 1, so we get the equation

$$3x^2 - 2x + 1 = 1,$$

or

$$3x^2 - 2x = 0,$$

which has the solution

$$x = 0 \text{ or } x = \frac{2}{3}.$$

To get a point distinct from the origin, we take $x = \frac{2}{3}$ and the corresponding vector is $\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$.

iii) a) The inverse is given by

$$\frac{1}{\det(A)} \begin{pmatrix} \alpha & -i \\ -1-i & 2 \end{pmatrix} = \frac{1}{2\alpha + 1 - i} \begin{pmatrix} \alpha & -i \\ -1-i & 2 \end{pmatrix}.$$

b) Suppose that $\det(A^2) = -1$. Then we have

$$-1 = \det(A^2) = (\det(A))^2,$$

so $\det(A) = \pm i$. On the other hand, we have

$$\det(A) = 2\alpha + 1 - i = \pm i,$$

so we get the solutions

$$\alpha = -\frac{1}{2} \text{ or } \alpha = \frac{2i-1}{2}.$$

iv) a) The roots of $z^9 - 1$ are the ninth roots of unity: 1 and $e^{\pm\frac{2\pi k}{9}}, k = 1, 2, 3, 4$. From the factorization

$$(z^9 - 1) = (z^3 - 1)(z^6 + z^3 + 1)$$

we see that the roots $z^6 + z^3 + 1$ are those ninth roots of unity which are not also cube roots of unity, which leaves the six roots $e^{\pm\frac{2\pi k}{9}}, k = 1, 2, 4$.

b) Dividing $z^6 + z^3 + 1 = 0$ by z^3 gives the equation

$$z^3 + 1 + \frac{1}{z^3} = 0,$$

or

$$z^3 + \frac{1}{z^3} = -1.$$

We have

$$x^3 = (z + \frac{1}{z})^3 = z^3 + \frac{1}{z^3} + 3z + \frac{3}{z} = -1 + 3x,$$

so

$$x^3 - 3x + 1 = 0.$$

c) Let $z_1 = e^{\frac{2\pi i}{9}}$, $z_2 = e^{\frac{4\pi i}{9}}$, $z_3 = e^{\frac{8\pi i}{9}}$. Then by parts (a) and (b), the three numbers $z_1 + \frac{1}{z_1}, z_2 + \frac{1}{z_2}, z_3 + \frac{1}{z_3}$ are the roots of the cubic polynomial $x^3 - 3x + 1$. Since this polynomial has no quadratic term, the sum of these roots must be 0, and we have

$$0 = z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_3 + \frac{1}{z_3} = 2(\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9}).$$

v) a) We can write any vector as $\mathbf{x} = |\mathbf{x}|\mathbf{u}$, with $\mathbf{u}$ a unit vector. Then

$$|M\mathbf{x}| = |M(|\mathbf{x}|\mathbf{u})| = |\mathbf{x}| \cdot |M\mathbf{u}| \leq |\mathbf{x}|||M||$$

by the definition of $||M||$.

b) For any unit vector $\mathbf{u}$, we have

$$|(MN)\mathbf{u}| = |M(N\mathbf{u})| \leq ||M|| \cdot |N\mathbf{u}| \leq ||M|| \cdot ||N||,$$

where we have used part (a) for the first inequality and the definition of norm for the second.
Therefore $||M|| \cdot ||N||$ is at least as big as the maximum of $|(MN)\mathbf{u}|$ for all unit vectors $\mathbf{u}$, which is the norm of MN.

c) The matrix given takes an arbitrary unit vector $\begin{pmatrix} u \\ v \end{pmatrix}$ to the vector $\begin{pmatrix} v \\ -2u \end{pmatrix}$.
The second vector is clearly not more than twice as long as the first vector. On the other hand, if $v = 0$ then is it exactly twice as long. Therefore the norm is 2.

MATH1141 June 2015 Solutions

2. i) a) If f is differentiable on the open interval (a, b) and continuous on the $[a, b]$, then there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

b) Let x be a real number, $x \leq 2$. The function f satisfies the requirements of the Mean Value Theorem on $[x, 2]$ so

$$\frac{f(2) - f(x)}{2 - x} = f'(c)$$

for some $c \in (2, x)$. Hence

$$\frac{f(2) - f(x)}{2 - x} \leq 1 \Rightarrow f(x) \geq x.$$

ii) The limit satisfies the conditions for L'Hopital's rule, so $\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{f(x)-f(0)}{2x} = \lim_{x\to 0} \frac{f'(x)}{2} = \frac{f'(0)}{2}$.

iii) a) Now

$$f(-x) = \int_0^{(-x)^3} (t^2 - 1)e^{t^2} dt.$$

Replace $-t$ with t in the integral, so replace dt with $-dt$, then

$$f(-x) = -\int_0^{x^3} (t^2 - 1)e^{t^2} dt = -f(x).$$

Hence f is odd.

b) Set $f'(x) = 3x^2[(x^6 - 1)e^{x^6}] = 0$ for a stationary point, giving $x = 0, 1, -1$.

c) Around $x = 0$, $f'(x)$ is positive so we have an inflection point at $x = 0$.
Around $x = 1$, $f'(x)$ moves from negative to positive so we have a minimum at $x = 1$.
Around $x = -1$, $f'(x)$ moves from positive to negative so we have a maximum at $x = -1$.

iv) a) Differentiating implicitly and solving for y' we have

$$y' = -\frac{y}{x}\frac{3x^2 + y^2}{x^2 + 3y^2}.$$

b)

$$\frac{dA}{dx} = y + xy' = 0$$

for a stationary point. Substituting and noting that $x, y > 0$ we have $y = x$. Substituting this back into the equation fo curve we obtain $x = y = \alpha$, where $\alpha = \frac{1}{\sqrt{4}2}$.

c) Passing to polar coordinates, we have $r^4 \sin\theta \cos\theta = 1$ or $r^4 \sin(2\theta) = 2$.
Also $A = xy = r^2 \sin\theta \cos\theta = \frac{1}{r^2}$.

d) We can write $A = \frac{\sqrt{\sin(2\theta)}}{\sqrt{2}}$ which has a maximum when $\theta = \frac{\pi}{4}$. Thus, $y = x$ and this is the line on which A has the maximum point $x = y = \alpha$, where $\alpha = \frac{1}{\sqrt{42}}$.

3. i) Let A be the point $(1, 0, 1)$ and let $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; then $\overrightarrow{AP} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$,

$\overrightarrow{AP} \cdot \mathbf{v} = \mathbf{1}$ and $\mathbf{v} \cdot \mathbf{v} = \mathbf{2}$.
The distance is then

$$|\overrightarrow{AP} - \mathrm{proj}_{\mathbf{v}}\overrightarrow{AP}| = \left|\overrightarrow{AP} - \frac{\overrightarrow{AP} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}\right| = \left|\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right| = \frac{3}{2}\left|\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right| = \frac{3}{\sqrt{2}}.$$

ii) a) $|\mathbf{a}+\mathbf{b}|^{\mathbf{2}} + |\mathbf{a}-\mathbf{b}|^{\mathbf{2}} = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) + (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{2a}\cdot\mathbf{a} + \mathbf{2a}\cdot\mathbf{b} = \mathbf{2}|\mathbf{a}|^{\mathbf{2}} + \mathbf{2}|\mathbf{b}|^{\mathbf{2}}$

b) Now, $|\mathbf{a}|$ and $|\mathbf{b}|$ are the side lengths of the parallellogram
whereas $|\mathbf{a} + \mathbf{b}| = |\overrightarrow{AB} + \overrightarrow{AD}| = |\overrightarrow{AB} + \overrightarrow{BC}| = |\overrightarrow{AC}|$
and $|\mathbf{a} - \mathbf{b}| = |\overrightarrow{AB} - \overrightarrow{AD}| = |\overrightarrow{AB} + \overrightarrow{DA}| = |\overrightarrow{DB}|$ are the diagonal lengths.
Therefore, a) states that the sum of the squared diagonal lengths is twice the sum of the squared side lengths.

b) Now, $|\mathbf{a}|$ and $|\mathbf{b}|$ are the side lengths of the parallellogram
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Therefore, a) states that the sum of the squared diagonal lengths is twice the sum of the squared side lengths.

iii) a)

$$\begin{aligned} \cos 5\theta + i\,\sin 5\theta &= e^{i5\theta} = (e^{i\theta})^5 = (\cos\theta + i\,\sin\theta)^5 \\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta \\ &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta) \end{aligned}$$

By comparing the imaginary parts, we see that

$$\begin{aligned} \sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\ &= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta \\ &= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta \\ &= 16x^5 - 20x^3 + 5x\,. \end{aligned}$$

b) Set $\theta = \frac{\pi}{10}$. By a), $p(x) = 16x^5 - 20x^3 + 5x - 1 = \sin 5\theta - 1 = \sin\frac{5\pi}{10} - 1 = 1 - 1 = 0$.

c) 1, $\frac{-1\pm\sqrt{5}}{4}$.

d) By parts b) and c), $\sin\frac{\pi}{10}$ must be one of the roots 1, $\frac{-1\pm\sqrt{5}}{4}$.
Since it is not 1 and is not negative, we see that

$$\sin\frac{\pi}{10} = \frac{-1+\sqrt{5}}{4}\,.$$

iv) $\det(B) = 0$ and $\det(C) = -2(-1)^2\det(A) = -14$
since the third row of B is a linear combination of the first two rows
and since swapping rows twice on the transpose C^T gives A with rows scaled by $-1, 2$.

v) a) **Proof**. $|Q\mathbf{v}|^2 = (Q\mathbf{v}) \cdot (Q\mathbf{v}) = \mathbf{v}^T Q^T Q\mathbf{v} = \mathbf{v}^T I\mathbf{v} = \mathbf{v}^T\mathbf{v} = \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$, so $|Q\mathbf{v}| = |\mathbf{v}|$.

b) **Proof**. By a), $|\lambda||\mathbf{v}| = |\lambda\mathbf{v}| = |\mathbf{Qv}| = |\mathbf{v}|$.
Since $|\mathbf{v}| \neq \mathbf{0}$, we see that $|\lambda| = 1$, so $\lambda = \pm 1$.