MATH1131 Mathematics 1A
and
MATH1141 Higher Mathematics 1A

PAST EXAM PAPERS
AND
SOLUTIONS

Contents

PAST EXAM PAPERS

Please note that the emphasis of the exam may change from year to year and that it is not possible to test all aspects of the course in a 2-hour examination. Hence, these papers are only a guide as to the style and level of difficulty of our first year examinations.

The solutions to the examination papers contained here have been written by many members of staff of the School of Mathematics and Statistics.

While every care is taken to excluded errors, we cannot guarantee that the solutions are error-free. Please report any serious errors to the Director of First Year Mathematics.

Exam papers from 2016 will be provided on Moodle for practice before the exam. Students are encouraged to produce their own solutions to the 2016 exam papers and discuss these on Moodle.

MATH1131 November 2010

1. i) Let $z = 1 + i$.
   a) Find $|z|$.
   b) Find $\text{Arg}(z)$.
   c) Use the polar form of z to evaluate z^{28} and then express your answer in **Cartesian form**.

   ii) Suppose that $(x + iy)(3 + 2i) = (4 + 7i)$ where $x, y \in \mathbb{R}$.
   Find the value of x and the value of y.

   iii) a) Find all solutions to the equation $z^6 = 1$, where $z \in \mathbb{C}$.
   b) Hence, or otherwise, express $z^6 - 1$ as a product of three quadratic polynomials with real coefficients.

   iv) Let the region S be defined as

   $$S = \{z \in \mathbb{C} : |z + i| = 1\}.$$

   Sketch the region S on a carefully labelled Argand diagram.

   v) Consider the following MAPLE session.

```
> with(LinearAlgebra):
> A:=<<1,0,-sqrt(3)>|<0,2,0>|<sqrt(3),0,1>>;
```

$$A := \begin{bmatrix} 1 & 0 & \sqrt{3} \\ 0 & 2 & 0 \\ -\sqrt{3} & 0 & 1 \end{bmatrix}$$

```
> A^2;
```

$$\begin{bmatrix} -2 & 0 & 2\sqrt{3} \\ 0 & 4 & 0 \\ -2\sqrt{3} & 0 & -2 \end{bmatrix}$$

```
> A^4;
```

$$\begin{bmatrix} -8 & 0 & -8\sqrt{3} \\ 0 & 16 & 0 \\ 8\sqrt{3} & 0 & -8 \end{bmatrix}$$

```
> A^6;
```

$$\begin{bmatrix} 64 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 64 \end{bmatrix}$$

   Use the above Maple session to find the inverse of the matrix A^2.

vi) A car dealer sells three brands of cars: Audis, BMWs and Chevrolets. Her costs consist of registration, GST and the manufacturer's price of the car. She has to pay registration costs of \$200 per Audi, \$600 per BMW and \$400 per Chevrolet. The GST charges are \$1,800 per Audi, \$2,400 per BMW and \$2,000 per Chevrolet. To the manufacturer she pays \$20,000 per Audi, \$30,000 per BMW and \$26,000 per Chevrolet. Last year her total registration bill was \$12,000, the total GST tax bill was \$65,600 and she paid a total of \$784,000 to manufacturers.

Let a, b and c denote the number of Audis, BMWs and Chevrolets she sold last year.

a) Write down a system of linear equations in a, b and c determined by the above information.

b) Convert the system of equations to an augmented matrix form and hence find a, b and c by performing Gaussian elimination on the augmented matrix.

2. i) Let $P = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix}$.

a) Evaluate the matrix product PP^T.

b) State the size of the matrix $P^T P$.

ii) Let A, B and C be points in $\mathbb{R}^3$ with position vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} \quad \text{respectively.}$$

a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b) Find a parametric vector equation of the plane that passes through the points A, B and C.

iii) Determine the coordinates of the point of intersection of the line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{for} \quad t \in \mathbb{R}$$

and the plane $x - 3y + z = 15$.

iv) Let $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 5 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -2 \\ -6 \\ \beta \\ -10 \end{pmatrix}$ be two vectors in $\mathbb{R}^4$.

Find the value of β so that the vectors $\mathbf{v}$ and $\mathbf{w}$ are perpendicular.

v) Let $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}$ be a vector in $\mathbb{R}^3$.

a) Find the magnitude of $\mathbf{u}$.

b) Write down a vector with a magnitude of 10 which is parallel to $\mathbf{u}$.

vi) Find the determinant of the matrix

$$C = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 1 & 7 \\ 1 & 2 & 0 \end{pmatrix}.$$

vii) Suppose that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are distinct non-zero vectors in $\mathbb{R}^n$ with the property that

$$\mathrm{proj}_{\mathbf{w}}(\mathbf{u}) = \mathrm{proj}_{\mathbf{w}}(\mathbf{v}).$$

Prove that $(\mathbf{u} - \mathbf{v})$ is perpendicular to $\mathbf{w}$.

3. i) Evaluate the limit

$$\lim_{x \to 0} \frac{x^2 e^x}{1 - \cos(\pi x)}.$$

ii) Determine all real values of a and b such that the function f given by

$$f(x) = \begin{cases} x^2 & \text{for } x \le 1, \\ -x^2 + ax + b & \text{for } x > 1, \end{cases}$$

is differentiable at $x = 1$.

iii) Let

$$L = \lim_{x \to \infty} \frac{x^2 - 2}{x^2 + 1}.$$

a) Evaluate L.

b) Given any $\varepsilon > 0$, find a constant M (which may depend on ε) such that we have

$$\left| \frac{x^2 - 2}{x^2 + 1} - L \right| < \varepsilon$$

whenever $x > M$.

iv) Let $f(x) = x^3 + \sqrt{3}\, x - 5$ for all real x.

a) Use the Intermediate Value Theorem to prove that f has at least one positive real root.

b) By considering f', or otherwise, show that f has only one real root.

v) Use logarithmic differentiation to calculate $\dfrac{dy}{dx}$ for $y = (\sin x)^x$.

vi) A curve in $\mathbb{R}^2$ is given in polar coordinates as

$$r = 6 \sin \theta, \quad \text{where } 0 \le \theta \le \pi/2.$$

a) Express the equation of the curve using Cartesian coordinates x and y and state the range of x and the range of y.

b) Hence, or otherwise, sketch the curve in the xy-plane.

4. i) Evaluate the following integrals:

a) $I_1 = \displaystyle\int \frac{1-x}{(1+x)^3}\,dx$;

b) $I_2 = \displaystyle\int_0^{\pi} x\cos 2x\,dx$.

ii) Determine whether the improper integral

$$K = \int_1^{\infty} \frac{1+\sin x}{3x^2}\,dx$$

converges or diverges.

iii) a) State the definitions of $\sinh x$ and $\cosh x$ in terms of the exponential function.

b) Hence find an expression for $\dfrac{d}{dx}\cosh(ax)$ in terms of $\sinh(ax)$, where a is a constant.

c) Simplify the expression $\cosh^{-1}\big(\cosh(-4726)\big)$.

iv) A continuous function f satisfies the equation

$$\int_0^x f(t)\,dt = \int_x^1 t^2 f(t)\,dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} - \frac{1}{9}.$$

By differentiating this equation with respect to x, and using the First Fundamental Theorem of Calculus where needed, find $f(x)$.

v) Let g be a function defined by

$$g(x) = \tan^{-1}(x) + \tan^{-1}(1/x).$$

a) What is the (maximal) domain of g? Give reasons for your answer.

b) By examining $g'(x)$, show that g is piecewise constant on its domain.

c) Hence, or otherwise, determine the exact value of

$$\tan^{-1}(-e^{\alpha}) + \tan^{-1}(-e^{-\alpha}), \quad \text{for all } \alpha \in \mathbb{R}.$$

MATH1131 JUNE 2011

1. i) Let $z = -1 - i$.
   a) Find $|z|$.
   b) Find $\mathrm{Arg}(z)$.
   c) Use the polar form of z to evaluate z^{102} and then express your answer in **Cartesian form**.

   ii) a) Simplify $(2+4i)^2$.
   b) Hence, or otherwise, solve the quadratic equation $z^2 - 4z + (7-4i) = 0$.

   iii) Sketch the following region on the Argand diagram
$$S = \{z \in \mathbb{C} : 0 \le \mathrm{Arg}(z-i) \le \frac{\pi}{4}\}.$$

   iv) Evaluate the limit
$$\lim_{x\to\infty} \frac{1}{x - \sqrt{x^2 - 6x - 4}} \ .$$

   v) Evaluate the improper integral
$$\int_1^\infty x^{-5/4}\, dx \ .$$

   vi) A curve in the plane is defined implicitly by the equation
$$x^2 - 3xy^2 + 11 = 0 \ .$$
   a) Show that the curve has slope at the point (x, y) given by
$$\frac{dy}{dx} = \frac{2x - 3y^2}{6xy} \ .$$
   b) Find the equation of the tangent to the curve at the point $(1, 2)$.
   c) Write a Maple command to plot the curve in the region
   $1 \le x \le 4$ and $-5 \le y \le 5$.

2. i) Use De Moivre's Theorem to prove that
$$\cos(4\theta) = 8\cos^4\theta - 8\cos^2\theta + 1.$$

   ii) Consider the line ℓ with parametric vector equation
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} t, \quad \text{for} \quad t \in \mathbb{R}.$$
   a) Give two points on the line.
   b) Give a vector parallel to the line.
   c) Explain why the line ℓ is perpendicular to the plane P with Cartesian equation
$$9x + 6y + 15z = 24.$$

d) Find a point on the line whose y-coordinate is 0.

iii) Consider the following Maple session:

```
> with(LinearAlgebra):
> A:=<<1,1,0>|<-1,1,0>|<0,0,sqrt(2)>>;
```

$$A := \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

```
> A^2;
```

$$\begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

```
> A^4;
```

$$\begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

```
> A^8;
```

$$\begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

Use the above MAPLE session to find the inverse of A^7.

iv) Evaluate the limit

$$\lim_{x \to 1} \frac{(x-1)^2}{1+\cos(\pi x)} .$$

v) Evaluate the indefinite integral

$$\int x \sin(2x)\, dx.$$

vi) The function f has domain $[0,1]$ and is defined by $f(x) = e^x + ax$, where a is a positive constant.

a) Prove that 2 is in the range of f.

b) Prove that f has an inverse function f^{-1}.

c) Find the domain of f^{-1}.

3. i) Let $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 0 & 5 \\ 1 & 0 \end{pmatrix}$.

a) Find AB.

b) What is the size of the matrix BA?

ii) A pet shop has x hamsters, y rabbits and z guinea pigs.
Each hamster eats $50g$ of dry food and $40g$ of fresh vegetables,
and needs $1m^2$ of space.
Each rabbit eats $300g$ dry food and $320g$ of fresh vegetables,
and needs $5m^2$ of space.
Each guinea pig eats $100g$ of dry food and $200g$ of fresh vegetables,
and needs $3m^2$ of space.
Altogether they eat $2900g$ of dry food and $3920g$ of fresh vegetables, and need $63m^2$ of space.

a) Explain why $5x + 30y + 10z = 290$.

b) Write down a system of linear equations that determine x, y and z.

c) Reduce your system to echelon form and solve to find the number of hamsters, rabbits and guinea pigs.

iii) The points A, B and C in $\mathbb{R}^3$ have position vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$$

respectively.
Write down a parametric vector equation of the plane passing through A, B and C.

iv) Consider the following system of linear equations.

$$\begin{aligned} x + y - z &= 2 \\ 2x + 3y + z &= 6 \end{aligned}$$

a) Using Gaussian Elimination find the general solution to the system of equations.

b) Hence, or otherwise, find a solution to the system with the property that the sum of the x, y and z coordinates of the solution is 0.

v) Suppose A, B are two points in $\mathbb{R}^3$ with position vectors $\begin{pmatrix}1\\2\\4\end{pmatrix}$ and $\begin{pmatrix}3\\1\\5\end{pmatrix}$ respectively.

We let O denote the origin.

a) Find $|\overrightarrow{OB}|$.

b) Find the area of triangle AOB.

c) Hence, or otherwise, find the perpendicular distance from A to the line through O and B.

vi) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are non-zero, non-parallel vectors in $\mathbb{R}^3$ of the same magnitude. Prove that $\mathbf{u} - \mathbf{v}$ is perpendicular to $\mathbf{u} + \mathbf{v}$.

4. i) a) Give the definition of $\cosh x$.

b) Use the definition to prove that

$$4\cosh^3 x = \cosh 3x + 3\cosh x \ .$$

ii) Find

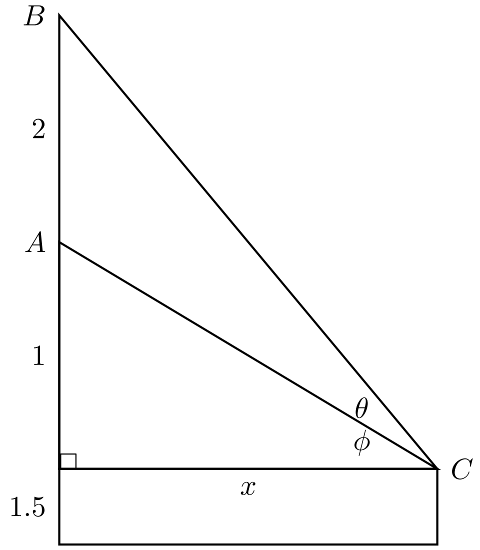
a) $\dfrac{d}{dx}\displaystyle\int_0^x \frac{\cos t}{\sqrt{1+t^2}}\,dt\,;$

b) $\dfrac{d}{dx}\displaystyle\int_0^{\sinh x} \frac{\cos t}{\sqrt{1+t^2}}\,dt\,.$ Give your answer in simplest form.

iii) a) Sketch the curve whose equation in polar coordinates is $r = 6\sin 2\theta$.

b) Find the gradient, $\frac{dy}{dx}$, of this curve at the point where $\theta = \frac{1}{6}\pi$.

iv) A statue 2 metres high stands on a pillar 2.5 metres high. A person, whose eye is 1.5m above the ground, stands at a distance x metres from the base of the pillar.



The diagram shows the above information, with the person's eye being at C.

a) Prove that

$$\frac{d}{dt}\left(\cot^{-1} t\right) = \frac{-1}{1+t^2}.$$

b) Show that

$$\theta = \cot^{-1}\left(\frac{x}{3}\right) - \cot^{-1} x$$

c) Hence find the distance x that maximises the angle θ.

MATH1131 June 2012

1. i) Let $u = 3 + 2i$ and $w = 1 - 5i$.
   a) Find $u - 2w$ in **Cartesian form**.
   b) Find u/w in **Cartesian form**.

   ii) Let $z = \sqrt{3} - i$.
   a) Calculate $|z|$ and $\text{Arg}(z)$.
   b) Express z in polar form.
   c) Hence, or otherwise, express $z^{10} + (\overline{z})^{10}$ in **Cartesian form**.

   iii) Evaluate the determinant
$$\begin{vmatrix} 1 & -1 & 4 \\ 0 & 2 & 7 \\ 0 & 3 & 1 \end{vmatrix}.$$

   iv) a) Evaluate $\displaystyle\lim_{x\to\infty} \frac{3x^2 + \sin(2x^2)}{x^2}$.
   b) Evaluate $\displaystyle\lim_{x\to 0} \frac{3x^2 + \sin(2x^2)}{x^2}$.

   v) Consider the curve in the plane defined by
$$x^2 - 5x \sin y + y^2 = 4.$$
   Find the equation of the tangent line to this curve at the point $(2, 0)$.

   vi) Let $p(x) = x^5 + 5x + 7$.
   a) Explain why p has at least one real root.
   b) Prove that p has exactly one real root.

2. i) Use De Moivre's Theorem to show that
$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta.$$

   ii) Find the intersection of the line $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$, with the plane $5x - 2y + z = 17$.

   iii) Consider the following MAPLE session:

```
> with(LinearAlgebra):
> A:=<<0,0,1>|<0,1,0>|<-1,0,0>>;
```
$$A := \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
```
> A^2;
```
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

```
> A^3;
```

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

```
> A^4;
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the Maple session above, find the **inverse** of A^{2001}.

iv) The points C and D have position vectors

$$\mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}.$$

a) Find the cross product $\mathbf{c} \times \mathbf{d}$.

b) Hence, or otherwise, find the area of the parallelogram with adjacent sides OC and OD, where O is the origin.

v) Evaluate the indefinite integral

$$\int x^4 \ln x \, dx.$$

vi) Consider the three functions:

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R}, & f(x) &= \frac{x^2}{1+x^2}, \\ g &: (0,3) \to \mathbb{R}, & g(x) &= (x-1)^2, \\ h &: [1,5] \to \mathbb{R}, & h(x) &= \sqrt{1 + \ln x + \sin x \cos x}. \end{aligned}$$

Only **one** of these functions has a maximum value (on its given domain). Which one is it? Give reasons for your answer.

vii) Sketch the polar curve $r = 2 - 2\cos\theta$. You should show any lines of symmetry, and clearly identify where the curve intersects the x and y axes.

viii) Suppose that $y = x^{\sin x}$. Find $\dfrac{dy}{dx}$.

3. i) The points A and B in $\mathbb{R}^3$ have position vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

a) Find a parametric vector equation of the line l passing through A and B.

b) By evaluating an appropriate dot product, show that the line l from part (a) is perpendicular to the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} t;\ t \in \mathbb{R}$.

ii) Let $P = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 4 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 5 & 0 \end{pmatrix}$.

a) Evaluate PQ^T.

b) What is the size of P^TQ?

c) Does the matrix product PQ exist? Explain your answer.

iii) A system of three equations in three unknowns x, y and z has been reduced to the following echelon form

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & \alpha^2 - 9 & \alpha - 3 \end{array}\right).$$

a) For which value of α will the system have no solution?

b) For which value of α will the system have infinitely many solutions?

c) For the value of α determined in part (b), find the general solution.

iv) The number of \$10, \$20, and \$50 notes in the cash register at Bill's Burger Barn is x, y and z respectively. The total value of all the notes in the register is \$1020. There are 44 notes in total. Also, the number of \$10 notes is equal to the sum of the number of \$20 notes and the number of \$50 notes.

a) Explain why $x + 2y + 5z = 102$.

b) By setting up two further equations and solving the system of three equations in three unknowns x, y and z, determine how many of each type of note is in the cash register.

v) The (non-zero) point Q has position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. The vector $\overrightarrow{OQ}$ makes angles α, β and γ respectively with the X, Y and Z axes.

a) By considering the vector $\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ show that

$$a = \sqrt{a^2 + b^2 + c^2} \cos \alpha.$$

b) Deduce that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

c) If the angles α and β are complementary, that is, their sum is 90°, what can be said about the vector $\overrightarrow{OQ}$?

4. i) Find a quadratic function $q(x) = x^2 + bx + c$ such that the function

$$h(x) = \begin{cases} e^{3x}, & \text{if } x \leq 0 \\ q(x), & \text{if } x > 0 \end{cases}$$

is differentiable at $x = 0$.

ii) Each of the following calculations is expressed in MAPLE. Write each in normal mathematical notation and **evaluate**.

a) `arcsin(sin(7*Pi/3));`

b) `diff(int(exp(t^2),t=0..x^2),x);`

iii) Prove that $\displaystyle\lim_{x\to\infty} \frac{x^2-2}{x^2+3} = 1$ as follows:
Given any real number $\epsilon > 0$, find a real number M (expressed in terms of ϵ), such that if $x > M$ then $\left|\dfrac{x^2-2}{x^2+3} - 1\right| < \epsilon$.

iv) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^3 + \sinh x + 1.$$

a) Explain why f has a differentiable inverse g.

b) What is the domain of g?

c) Evaluate $g'(1)$.

v) A chemical process produces Factor X, which flows into a 50 litre tank which is initially empty.

At time $t \geq 0$, Factor X flows into the tank at the rate of $\dfrac{100}{10+t^2}$ litres per hour.

Will the tank eventually overflow? Explain your answer.

MATH1131 June 2013

1. i) Evaluate the following limits:

   a)
$$\lim_{x\to\infty} \frac{10x^2+3x+\sin x}{5x^2+3x-2},$$

   b)
$$\lim_{x\to 0} \frac{e^{3x}-1}{\sin(7x)}.$$

   ii) A function $f:[0,5]\to\mathbb{R}$ has the following properties:

   - $\lim_{x\to 2^+} f(x)=3,$
   - $\lim_{x\to 2^-} f(x)=1,$
   - $f(2)=4.$

   Draw a possible sketch of the graph of f. (You do not need to give a formula for your function.)

   iii) a) State the definitions of $\cosh x$ and $\sinh x$ in terms of the exponential function.

   b) Prove that $\cosh^2 x - \sinh^2 x = 1$.

   iv) Let $z=5+5i$ and $w=2+i$.

   a) Find $2z+3\overline{w}$.

   b) Find $z(w-1)$.

   c) Find z/w.

   v) Suppose that $(x+iy)(3+4i)=13+9i$, where $x,y\in\mathbb{R}$.
   Find the value of x and the value of y.

   vi) Let the set S in the complex plane be defined by

$$S=\left\{z\in\mathbb{C} \,:\, |z|\le 3 \text{ and } 0\le \mathrm{Im}(z)\le 3\right\}.$$

   a) Sketch the set S on a labelled Argand diagram.

   b) By considering your sketch, or otherwise, find the area of the region defined by S.

vii) Consider the following MAPLE session.

```
> with(LinearAlgebra):
> A:=<<0,1,-1>|<1,0,1>|<-1,1,0>>;
```

$$A := \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

```
> B:=A^2;
```

$$B := \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

```
> C:=A^3;
```

$$C := \begin{bmatrix} -2 & 3 & -3 \\ 3 & -2 & 3 \\ -3 & 3 & -2 \end{bmatrix}$$

```
> F:=3*A-C;
```

$$F := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Without carrying out any row reduction, use the above Maple session to find the inverse of the matrix A.

2. i) A function h is defined by

$$h(x) = \begin{cases} ax^2 + 3x, & \text{if } x \geq 1 \\ 2x + d, & \text{if } x < 1. \end{cases}$$

Given that h is differentiable at $x = 1$, find the values of a and d.

ii) Evaluate

$$\int_0^{\ln 2} 9xe^{3x}\,dx.$$

iii) Find the equation of the tangent at the origin to the curve implicitly defined by

$$e^x + \sin y = xy + 1.$$

iv) Sketch the polar curve whose equation in polar coordinates is given by

$$r = 1 + \cos 2\theta.$$

v) Let $z = \sqrt{2} - \sqrt{2}i$.

a) Find $|z|$.

b) Find $\operatorname{Arg}(z)$.

c) Use the polar form of z to evaluate z^6. Express your answer in **Cartesian form**.

vi) Let $A = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$.

a) Evaluate AB or explain why this product does not exist.

b) Evaluate AB^T or explain why this product does not exist.

vii) a) Find, in polar form, all solutions to the equation $z^5 = -1$, where $z \in \mathbb{C}$.

b) Hence, or otherwise, express $z^5 + 1$ as a product of real linear and real quadratic factors.

3. i) Determine the point of intersection of the line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{for} \quad t \in \mathbb{R}\,.$$

and the plane $4x - 5y + 3z = 0$.

ii) Let $M = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 2 & \alpha \end{pmatrix}$.

a) Evaluate the determinant of M.

b) Determine the value(s) of α for which M does **not** have an inverse.

c) Find the inverse of M when $\alpha = 1$.

iii) Let $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$.

a) Find the cross product $\mathbf{u} \times \mathbf{v}$.

b) Hence find the **Cartesian** equation of the plane parallel to $\mathbf{u}$ and $\mathbf{v}$ and passing through the point $\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$.

iv) Let $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0 \\ 3 \\ -3 \\ \beta \end{pmatrix}$ be two vectors in $\mathbb{R}^4$.

a) Find the value of β so that the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

b) For the value $\beta = 1$, find the projection, $\text{proj}_{\mathbf{u}}(\mathbf{v})$, of $\mathbf{v}$ onto $\mathbf{u}$.

v) Let A, B and D be three points on some circle with centre C in $\mathbb{R}^2$ with position vectors

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

a) Let M be the midpoint of the line joining A and B. Find the position vector $\mathbf{m}$ of the point M.

b) Find a non-zero vector $\mathbf{u}$ that is perpendicular to $\overrightarrow{AB}$.

c) Hence or otherwise, find the parametric vector equation for the line whose points are equidistant from A and B.

d) Given that the parametric vector equation for the line whose points are equidistant from B and D is $\mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mu \in \mathbb{R}$, find the centre C of the circle.

4. i) Find $\displaystyle\int \frac{\cos(\ln(x))}{x}\, dx$.

ii) Use the Pinching theorem to evaluate

$$\lim_{x\to\infty} e^{-x}\sin(x).$$

iii) Show that the improper integral $\displaystyle\int_0^\infty \frac{dx}{x^2+e^x}$ converges.

iv) The following calculation is expressed in MAPLE

```
> F:=diff(int(sin(sqrt(t)),t=0..x^2),x);
```

a) Write the calculation using standard mathematical notation.

b) Evaluate F.

v) Let $p(x) = x^3 + 4x - 7$.

a) Use the Intermediate Value theorem to show that p has at least one real root in the interval $[1, 2]$.

b) Show that p has **exactly** one real root in the interval $[1, 2]$.

c) Let g be the inverse of p and α be the unique root of p, whose existence is guaranteed in part b). Express $g'(0)$ in terms of α.

vi) Use the Mean Value Theorem to prove that, for $x > 0$,

$$\ln(1+x) > \frac{x}{1+x}.$$

MATH1131 June 2014

1. i) Let $z = 7 + i$ and $w = 4 + 3i$.
   a) Find $2z - \overline{w}$ in $a + ib$ form.
   b) Find $5(w - i)/z$ in $a + ib$ form.
   c) Find $|zw|$.
   d) Find $\text{Arg}(zw)$.
   e) Hence, or otherwise, show that $\text{Arg}(z) + \text{Arg}(w) = \dfrac{\pi}{4}$.
   f) Use the polar form of zw to evaluate $(zw)^{40}$.

   ii) Consider the following system of equations:

$$\begin{array}{rcrcrcl} x & - & y & - & z & = & 1 \\ x & - & 3y & + & z & = & 1 \\ 2x & - & 3y & - & z & = & 2. \end{array}$$

   a) Write the system in augmented matrix form and reduce it to row echelon form.
   b) Solve the system.

   iii) Evaluate the limits
   a)
$$\lim_{x\to\infty} \frac{6x^2 + \sin x}{4x^2 + \cos x};$$
   b)
$$\lim_{x\to 0} \frac{e^{2x} - 2x - 1}{4x^2}.$$

   iv) A function g is defined by

$$g(x) = \begin{cases} \dfrac{|x^2 - 16|}{x - 4} & \text{if } x \neq 4 \\ \alpha & \text{if } x = 4. \end{cases}$$

   By considering the left and right hand limits at $x = 4$, show that no value of α can make g continuous at the point $x = 4$.

   v) Let $f(x) = x^5 + x^3 + x - 2$.
   a) Prove that f has at least one real root in the interval $[0, 2]$, naming any theorems you use.
   b) State, with reasons, the number of real roots of f.

2. i) Use a substitution to find the integral

$$\int \frac{dx}{x(1 + (\log x)^2)}.$$

   ii) a) Give the definitions of $\sinh x$ and $\cosh x$ in terms of the exponential function.
   b) Use your definitions to prove that $\sinh(2x) = 2 \sinh x \cosh x$.

   iii) Evaluate the integral

$$\int_0^{\pi/3} x \sin(2x)\, dx.$$

iv) Simplify the matrix expression $(A^TA)^{-1}(A^TA)^T$, where A is an invertible matrix.

v) Consider the three points A, B, C in $\mathbb{R}^3$ with position vectors $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$, and $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ respectively.

a) Find a parametric vector form for the plane Π that passes through points A, B, and C.

b) Calculate the cross product $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$, showing your working.

c) Hence, or otherwise, find a **Cartesian** equation for the plane Π.

d) Find the area of the triangle ABC.

e) Find the minimal distance from the point $P\begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}$ to the plane Π.

vi) Consider the following MAPLE session.

```
> with(LinearAlgebra):
> A := <<0,1,1>|<1,-1,-1>|<-2,1,0>>;
```

$$A := \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

```
> B := A^2;
```

$$B := \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -3 \\ -1 & 2 & -3 \end{bmatrix}$$

```
> C:=A^3;
```

$$C := \begin{bmatrix} 2 & -3 & 3 \\ -2 & 2 & 1 \\ -1 & 0 & 4 \end{bmatrix}$$

```
> F := 2A + B + C;
```

$$F := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the above MAPLE session, or otherwise, find the 3×3 matrix which is the inverse of the matrix A.

3. i) A block of wood is subject to 3 vector forces:
$\mathbf{F}_1 = 1$, in the direction West, $\mathbf{F}_2 = 1$ in the direction South, $\mathbf{F}_3 = 2$ in the direction North-East, each measured in Newtons.
Let $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ be the resultant force on the block.

a) On a scale diagram draw the 3 forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and the resultant force $\mathbf{F}$.

b) Find the exact value of $|\mathbf{F}|$ and the direction of $\mathbf{F}$.

ii) Let $M = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ -1 & 2 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$.

a) Evaluate the determinant of M.

b) Write down the inverse of N.

iii) Let $p(z) = z^4 + 2z^2 - 3$.

a) Show that $p(1) = p(-1) = 0$.

b) Factor $p(z)$ into two real quadratic polynomials $q(z)$ and $r(z)$.

c) Find the roots of $p(z)$.

d) Factor $p(z)$ into four complex linear polynomials.

iv) Consider the line ℓ and the plane Π given by the following equations:

$$\ell : \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

$$\Pi : 6x + 8y - 9z = 0.$$

Determine the point of intersection of the line ℓ and the plane Π.

v) a) Use De Moivre's Theorem to prove that $4\cos^3\theta = \cos 3\theta + 3\cos\theta$.

b) Deduce that $2\cos\dfrac{\pi}{9}$ is a root of the polynomial $q(z) = z^3 - 3z - 1$.

vi) Let $\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ \beta \end{pmatrix}$ be two vectors in $\mathbb{R}^3$.

a) Find the value of β so that the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

b) For the value $\beta = 0$, find the projection, $\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$, of $\mathbf{u}$ onto $\mathbf{v}$.

c) Find the value of β so that the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\dfrac{\pi}{4}$.

4. i) Use the Fundamental Theorem of Calculus to find

$$\frac{d}{dx}\int_{x^2}^{x^3} \cos\left(\frac{1}{t}\right)\,dt.$$

ii) Let $f(x) = \dfrac{175x^2 - 350x + 10}{x^2 - 2x + 2}$.

Consider the following MAPLE session.

```
> f:=(175*x^2-350*x+10)/(x^2-2*x+2);
```

$$\frac{175\,x^2 - 350\,x + 10}{x^2 - 2\,x + 2}$$

```
> subs(x=0.0,f);
```

$$5.0$$

```
> subs(x=4.0,f);
```

$$141.0$$

```
> solve(f=0,x);
```

$$1+\sqrt{1155}/35,\ 1-\sqrt{1155}/35$$

```
> evalf(%);
```

$$1.971008312,\ 0.0289916875$$

```
> fdash:=diff(f,x);
```

$$\frac{350\,x-350}{x^2-2\,x+2}-\frac{\left(175\,x^2-350\,x+10\right)(2\,x-2)}{\left(x^2-2\,x+2\right)^2}$$

```
> solve(fdash=0,x);
```

$$1$$

```
> subs(x=1.0,f);
```

$$-165.0$$

a) Use the information in the MAPLE output to give a rough sketch of $f(x) = \dfrac{175x^2-350x+10}{x^2-2x+2}$, for $0 \le x \le 4$.

b) Hence, or otherwise, find the maximum and minimum values of $f(x) = \left|\dfrac{175x^2-350x+10}{x^2-2x+2}\right|$ over the closed interval $[0,4]$.

iii) Determine, with reasons, whether the improper integral

$$K = \int_0^\infty \frac{dx}{e^{2x}+\cos^2 x}$$

converges or diverges.

iv) Sketch in the xy-plane, the graph of the polar curve given by $r = 1-\cos\theta$. (You are NOT required to find the slope of the curve.)

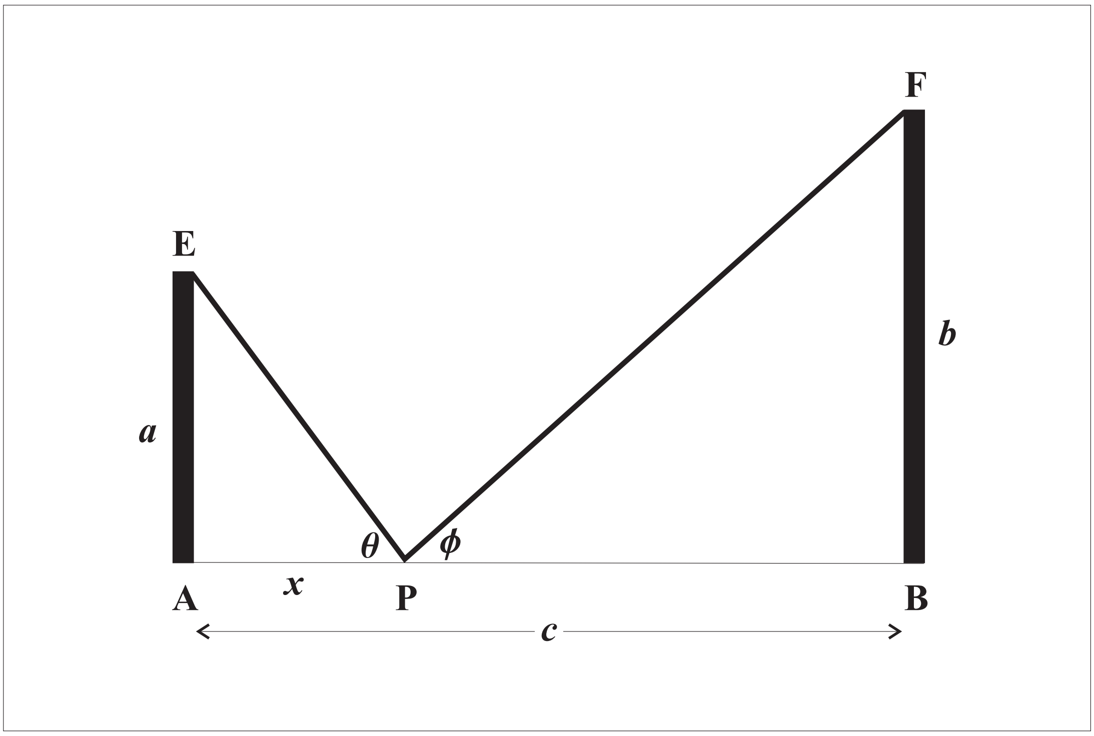
v) a) State carefully the Mean Value Theorem.

b) Suppose $-1 < x < y < 1$. By applying the Mean Value Theorem to the function $f(t) = \sin^{-1} t$ on the interval $[x, y]$, prove that

$$\sin^{-1} y - \sin^{-1} x \ge y - x.$$

vi) Two poles A and B, of heights a metres and b metres respectively, are c metres apart on the horizontal ground. A single tight rope runs from the top of pole A to the point P on the ground between A and B and then to the top of pole B.

Assume that the distance from A to P is x, and that the angles θ and ϕ are as shown in the diagram.



a) Explain why the length L of the rope is given by

$$L = \sqrt{a^2 + x^2} + \sqrt{b^2 + (c - x)^2}.$$

b) Prove that $\cos\theta = \cos\phi$ when $\dfrac{dL}{dx} = 0$.

c) Assuming that $\cos\theta = \cos\phi$ minimizes L, using similar triangles, or otherwise, find the value of x that minimizes L.

MATH1131 June 2015

1. i) Find $\lim\limits_{x\to\infty} \dfrac{x+\sin x}{2x}$. (Give brief reasons for your answer.)

   ii) The function f is defined by

$$f(x) = \begin{cases} 3-x, & 0 \le x < 1, \\ (x-2)^2+1, & 1 \le x \le 3, \end{cases}$$
$$\text{and} \quad f(-x) = f(x) \quad \text{for all } x.$$

   a) Find $f(\frac{3}{2})$.

   b) Sketch the graph of $f(x)$ over the interval $-3 \le x \le 3$.

   c) Which, if any, of the following exists? If it exists, state its value. (Give brief reasons for your answer.)

$$\lim_{x\to 1} f(x), \quad f'(0).$$

   iii) Find, for $x > 0$,

$$I_1 = \int \frac{dx}{1+\sqrt{x}}.$$

   iv) Evaluate

$$\lim_{x\to 0} \frac{1-\cos\left(\frac{x}{2}\right)}{x^2}.$$

   v) Let $z = 5 + i$ and $w = 3 + 2i$.

   a) Find $z - \overline{w}$ in $a + ib$ form.

   b) Find $10w/(z-2)$ in $a + ib$ form.

   c) Find $|(z/w)^8|$.

   d) Find $\text{Arg}(zw)$.

   e) Use the polar form of zw to evaluate $(zw)^8$.

   vi) Consider the three points A, B, C in $\mathbb{R}^3$ with position vectors

$$\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix},$$

   respectively.

   Find a parametric vector form for the plane Π that passes through points A, B, and C.

vii) Consider the following MAPLE session.

```
> with(LinearAlgebra):
> A := <<1,1,3>|<1,0,1>|<4,1,5>>;
```

$$A := \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 4 & 1 & 5 \end{bmatrix}$$

```
> B := A^2;
```

$$B := \begin{bmatrix} 14 & 4 & 19 \\ 5 & 2 & 8 \\ 25 & 9 & 38 \end{bmatrix}$$

```
> C := A^3;
```

$$C := \begin{bmatrix} 94 & 33 & 141 \\ 39 & 13 & 57 \\ 186 & 63 & 274 \end{bmatrix}$$

```
> F := C - 6B - 9A;
```

$$F := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) Using the above MAPLE session, or otherwise, find the 3×3 matrix which is the inverse of the matrix A.

b) State $(A^T)^2$.

2. i) By writing $z = a + ib$, or otherwise, solve $z^2 = 40 + 42i$, giving your answers in Cartesian form.

ii) Find condition(s) on b_1, b_2, b_3 to ensure that the following system has a solution.

$$\begin{aligned} x + 2y \qquad &= b_1 \\ x + y - z &= b_2 \\ 2x + y - 3z &= b_3 \end{aligned}$$

iii) Sketch the following region on the Argand diagram:

$$S = \Big\{ z \in \mathbb{C} \;:\; |z - i - 1| \leq 1 \quad \text{or} \quad |\text{Im}(z)| \geq 1 \Big\}.$$

iv) Let

$$\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}.$$

a) Calculate the acute angle θ between the vectors $\mathbf{u}$ and $\mathbf{v}$.

b) Find $\mathbf{u} \times \mathbf{v}$.

v) Consider the function

$$f(x) = \begin{cases} 1 + ax^2, & x \leq 1, \\ bx + 2x^3, & x > 1, \end{cases}$$

where $a, b \in \mathbb{R}$.

a) Find all values of a and b such that f is continuous at $x = 1$.

b) Find all values of a and b such that f is differentiable at $x = 1$.

vi) Evaluate

$$I_2 = \int x^3 \ln x \, dx.$$

vii) Consider the polar curve $r = 1 + \cos\theta$, for $0 \leq \theta \leq 2\pi$.

a) Find the slope of the tangent to the curve at the point with Cartesian coordinates $(0, 1)$.

b) Find the polar coordinates of the points at which the tangent to the curve is horizontal.

c) Sketch the polar curve.

3\. i) Show that the set of points z in the complex plane that satisfy the equation

$$2|z-3i| = |z+3i|$$

lie on a circle. State the radius and centre of the circle.

ii) Consider the complex polynomial $p(z) = z^4 - 3z^3 + 6z^2 - 12z + 8$.

a) Given that $p(2i) = 0$, factorise p into linear and quadratic factors with real coefficients.

b) Find all roots of p.

iii) Consider the plane Π in $\mathbb{R}^3$ with Cartesian equation

$$x - 3y + 2z = 1\,.$$

a) Find a point-normal form for the plane Π.

b) Show that the line ℓ with parametric vector form

$$\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \qquad \lambda \in \mathbb{R}$$

is parallel to the plane Π.

c) Find the shortest distance between the point $P\,(4, 2, 2)$ and the plane Π.

iv) a) Find the determinant of

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & a & 2 \\ 2 & 1 & 1 \end{pmatrix}.$$

b) For what value of a does the matrix A **not** have an inverse.

c) Determine the value of a and the value of b for which $A^{-1} = B$, where

$$B = \begin{pmatrix} -1 & 4 & -5 \\ 1 & b & 7 \\ 1 & -3 & 4 \end{pmatrix}.$$

v) Let X be an $n \times n$ matrix with determinant $|X| = 1$.
Prove that the inverse of X also has determinant equal to 1.

4. i) a) Define the functions $\sinh x$ and $\cosh x$ in terms of the exponential function.

   b) From the definitions in (a), show that

$$\frac{d}{dx}\left(\cosh 6x\right) = 6 \sinh 6x.$$

   c) Show that, for $x > 0$,

$$\ln\left(\sinh x\right) < x - \ln 2.$$

ii) Prove carefully that the improper integral $\displaystyle\int_1^{\infty} \frac{\ln x}{x^3}\, dx$ converges.

iii) Find $\displaystyle\frac{d}{dx}\left(\int_0^{x^3} \cos\left(t^2\right)\, dt\right)$.

iv) Consider the polynomial $p(x) = x^3 + 3x + 1$ defined on $\mathbb{R}$.

   a) Use the Intermediate Value Theorem to show that the equation $p(x) = 0$ has at least one real root.

   b) Show that the function p has an inverse function g.

   c) Find the value of $g'(1)$.

v) a) State the Mean Value Theorem.

   b) Apply the Mean Value Theorem to $f(t) = \cos t$, on a suitably chosen interval, to show that $|\cos y - \cos x| \leq |y - x|$, for all $x, y \in \mathbb{R}$.

PAST HIGHER EXAM PAPERS

MATH1141 JUNE 2011

2. i) You may use the following Maple session to assist you in answering the question below.

```
> with(LinearAlgebra):
> A:=<<1,2,1>|<1,3,a>|<-1,a,3>>;
```

$$A := \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & a \\ 1 & a & 3 \end{bmatrix}$$

```
> t:=<1,3,2>;
```

$$t := \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

```
> B:=<A|t>;
```

$$B := \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & a & 3 \\ 1 & a & 3 & 2 \end{bmatrix}$$

```
> G:=GaussianElimination(B);
```

$$G := \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & a+2 & 1 \\ 0 & 0 & 6-a^2-a & 2-a \end{bmatrix}$$

For which values of a will the system $A\mathbf{x} = \mathbf{t}$ have:

a) no solutions,

b) unique solution,

c) infinitely many solutions?

ii) Evaluate the determinant

$$\begin{vmatrix} 2 & 0 & -1 \\ 1 & 3 & 0 \\ 5 & 7 & 3 \end{vmatrix}.$$

iii) Find the point of intersection, if any, of the line $\boldsymbol{x} = \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\lambda \in \mathbb{R}$, with the plane

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mu, \nu \in \mathbb{R}.$$

iv) Consider the function f defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

a) Given that $\lim_{x \to \infty} xe^{-x} = 0$, evaluate the limit

$$\lim_{h \to 0} \frac{e^{-1/h^2}}{h}.$$

b) Using the definition of a derivative, determine whether f is differentiable at $x = 0$.

v) Consider the function f defined by

$$f(x) = \begin{cases} \dfrac{2}{x} & \text{for } x < -1, \\ x^2 - 1 & \text{for } -1 \leq x \leq \dfrac{6}{\pi}, \\ \left(\dfrac{72}{\pi^2} - 2\right) \sin \dfrac{1}{x} & \text{for } x > \dfrac{6}{\pi}. \end{cases}$$

a) Find all critical points of f and determine their nature. In particular, distinguish between local and global extrema.

b) Find all horizontal asymptotes for f.

c) Given that f does not have a point of inflexion, sketch the graph of f, clearly indicating all of the above information.

3. i) Let

$$S = e^{i\theta} + \frac{e^{3i\theta}}{3} + \frac{e^{5i\theta}}{3^2} + \frac{e^{7i\theta}}{3^3} + \cdots.$$

a) Prove that

$$S = \frac{3(3e^{i\theta} - e^{-i\theta})}{10 - 6\cos(2\theta)}.$$

b) Hence, or otherwise, find the sum

$$T = \sin(\theta) + \frac{\sin(3\theta)}{3} + \frac{\sin(5\theta)}{3^2} + \frac{\sin(7\theta)}{3^3} + \cdots.$$

ii) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are non-zero, non-parallel vectors in $\mathbb{R}^3$ of the same magnitude. Prove that $\mathbf{u} - \mathbf{v}$ is perpendicular to $\mathbf{u} + \mathbf{v}$.

iii) Let $A = (a_{ij})$ be a real $n \times n$ matrix and let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be the standard basis vectors for $\mathbb{R}^n$.

a) Prove that $\boldsymbol{e}_i^T A \boldsymbol{e}_j = a_{ij}$ for all $1 \le i, j \le n$.

b) Prove that if A is symmetric then $\boldsymbol{x}^T A \boldsymbol{y} = (A\boldsymbol{x})^T \boldsymbol{y}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

c) Conversely, prove that if $\boldsymbol{x}^T A \boldsymbol{y} = (A\boldsymbol{x})^T \boldsymbol{y}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, then A is symmetric.

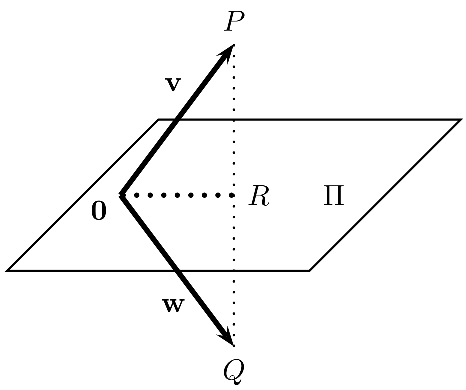
iv) The matrix

$$M = \frac{1}{9}\begin{pmatrix} 1 & 4 & -8 \\ 4 & 7 & 4 \\ -8 & 4 & 1 \end{pmatrix}$$

has the following property.

If $\boldsymbol{v}$ is the position vector of any point P in $\mathbb{R}^3$, then $\boldsymbol{w} = M\boldsymbol{v}$ is the vector obtained by reflecting $\boldsymbol{v}$ in a fixed plane Π, which passes through the origin.

Hence, in the diagram, $|\boldsymbol{v}| = |\boldsymbol{w}|$ and $|RP| = |RQ|$, where R is the foot of the perpendicular from P (or Q) to the plane.



a) Show that M reflects the vector $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ to the vector $-\boldsymbol{a}$.

b) Hence write down the Cartesian equation of Π.

c) Find a **non-zero** vector $\boldsymbol{u}$ such that $M\boldsymbol{u} = \boldsymbol{u}$.

d) Find the shortest distance from the point B with position vector $\boldsymbol{b} = \begin{pmatrix} 6 \\ -6 \\ 0 \end{pmatrix}$ to the plane.

4. i) a) Using L'Hôpital's rule or otherwise, indicate why

$$\lim_{x\to\infty} e^{-x}x^n = 0$$

for any $n \in \mathbb{N}$.

b) Show that for all $x \geq 1$,

$$e^{-x}x^n < Cx^{-2},$$

where C is a positive constant.

c) Hence, or otherwise, show that the improper integral

$$\int_1^\infty e^{-x}x^n\,dx$$

converges for any $n \in \mathbb{N}$.

ii) Suppose that $f : [0,2] \to [0,12]$ is continuous on its domain and twice differentiable on $(0,2)$. Further suppose that $f(0) = 0, f(2) = 12$.

a) Explain why $f'(c) = 6$ for some real number $c \in (0,2)$.

b) Suppose further that $f'(0) = 0$, prove that $f''(d) > 3$ for some real number $d \in (0,c)$.

iii) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = x - a\tanh x$$

for some constant $a \in \mathbb{R}$.

a) Explain why $\operatorname{sech} x < 1$ for $x > 0$.

b) By considering the derivative, or otherwise, find the values of a for which $f(x) > 0$ for all $x > 0$? Give reasons for your answer

c) For which values of a does the equation

$$x = a\tanh x$$

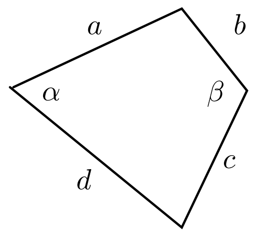
have a positive solution?

iv) The area $A(t)$ of an arbitrary convex quadrilateral with given side lengths a, b, c, d depends on the sum $t = \alpha + \beta$ of either pair of opposite angles, and is given by

$$A(t) = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd(1+\cos t)},$$

where

$$s = \frac{1}{2}(a+b+c+d).$$



a) Explain why the area A of a convex quadrilateral, with fixed side lengths a, b, c, d, is maximal if the sum of either pair of opposite angles is π.

b) Show that the area function $A : [0, \pi] \to \mathbb{R}$ as defined above is invertible, and that the inverse function B is differentiable on $(A(0), A(\pi))$.

c) Show that

$$B'(A_0) = \frac{4A_0}{abcd},$$

where

$$A_0 = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd}.$$

MATH1141 JUNE 2012

2. i) Consider the function $f : (0, 2\sqrt{\pi}\,] \to \mathbb{R}$ defined by

$$f(x) = x^2 + \cos(x^2).$$

a) Find all critical points of f and determine their nature.

b) Explain why f is invertible, state the domain of f^{-1} and find $f^{-1}(5\pi/2)$.

c) Where is f^{-1} differentiable?

ii) Let

$$f(x) = \int_0^{x^2-9x} e^{-t^2}\, dt.$$

a) Use the Mean Value Theorem to show that f has a stationary point x_0 in the interval $[0, 9]$.

b) Find the value of x_0 and determine the nature of the stationary point.

iii) Suppose that z lies on the unit circle in the complex plane.

a) Show that $z + \dfrac{1}{z}$ is real.

b) Find the maximum value of $z + \dfrac{1}{z}$.

iv) Use De Moivre's theorem to prove that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

v) Consider the plane P with parametric vector form

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}, \qquad \lambda_1, \lambda_2 \in \mathbb{R}.$$

a) Does the point $\mathbf{a} = \begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix}$ lie on P?

b) Is the vector $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ parallel to P?

c) Is the vector $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ orthogonal to P?

3. i) Suppose that A is a 4×4 matrix with $\det(A) = 5$. The matrix B is the result of performing the following three elementary row operations on A:

1. first multiply the 3rd row by 7;
2. then replace the second row with twice the first row plus the second row;
3. then swap the first and last rows.

What is the value of $\det(B)$?

ii) Consider the line in $\mathbb{R}^3$,

$$x - 4 = -y = z - 5.$$

a) Write this line in parametric vector form.

b) Find the point on the line closest to the origin $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

iii) Consider the following Maple session, which defines a matrix A and a vector $\mathbf{b} \in \mathbb{R}^3$:

```
> with(LinearAlgebra):
> A:=<<1,3,2>|<1,2,a>|<-2,2*a,4>>;
```

$$A := \begin{bmatrix} 1 & 1 & -2 \\ 3 & 2 & 2a \\ 2 & a & 4 \end{bmatrix}$$

```
> b:=<1,2,-2>;
```

$$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

```
> GaussianElimination(<A|b>);
```

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & -1 & 2a+6 & -1 \\ 0 & 0 & -4+2a^2+2a & -2-a \end{bmatrix}$$

For which values of a will the system $A\mathbf{x} = \mathbf{b}$ have

a) a unique solution,

b) no solutions,

c) infinitely many solutions?

iv) A matrix $Q \in M_{nn}(\mathbb{R})$ is said to be *nilpotent* (of degree 2) if $Q^2 = \mathbf{0}$, the zero matrix.

a) Give an example of a non-zero 2×2 nilpotent matrix.

b) Explain why a nilpotent matrix cannot be invertible.

Suppose now that $S, Q \in M_{nn}(\mathbb{R})$ commute, that S is invertible and that Q is nilpotent (of degree 2).

c) Prove that $S^{-1}Q = QS^{-1}$.

d) Show that $S + Q$ is invertible by finding an integer k such that

$$(S + Q)(S^{-1} - S^{-k}Q) = I.$$

v) Consider the non-zero vector $\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in $\mathbb{R}^3$ which makes angles α, β, γ with the three coordinate axes respectively.

By considering dot products with the standard basis vectors $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$, (or otherwise), prove that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

vi) Suppose that A is an $n \times n$ matrix with the property that every vector $\mathbf{b} \in \mathbb{R}^n$ can be written uniquely as a linear combination of the **columns** of A. Prove that every vector $\mathbf{b} \in \mathbb{R}^n$ can also be written uniquely as a linear combination of the **rows** of A.

4. i) Consider the polar curve $r = 1 + \cos 2\theta$.

a) Prove that the curve is symmetric about the x axis and also about the y axis.

b) Sketch the curve.
(You are NOT required to find the derivative.)

ii) Let $f(x) = \tanh x$ (the hyperbolic tangent).

a) Express $\tanh x$ in terms of exponentials.

b) Sketch the graph $y = f(x)$.

c) Show that

$$\lim_{x \to \infty} \frac{1 - \tanh x}{e^{-2x}} = 2\,.$$

d) Explain why the improper integral

$$\int_0^\infty (1 - \tanh x)\, dx$$

converges.

e) Compute

$$\int_0^\infty (1 - \tanh x)\, dx,$$

justifying your calculations.

iii) Suppose that f is a function whose derivative is continuous and hence bounded on $[a, b]$, with $|f'(x)| \le L$ for all $x \in [a, b]$.

a) Show that for any $n > 0$,

$$\int_a^b f(x) \sin nx\, dx = \frac{K(n)}{n} + \frac{1}{n} \int_a^b f'(x) \cos nx\, dx,$$

where $K(n) = f(a)\cos(na) - f(b)\cos(nb)$.

b) Explain why

$$\left| \int_a^b f'(x) \cos nx\, dx \right| \le (b - a)L.$$

c) Find, with reasons,

$$\lim_{n \to \infty} \int_a^b f(x) \sin nx\, dx.$$

MATH1141 JUNE 2013

2. i) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^3 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0. \end{cases}$$

a) Explain why f is differentiable everywhere and determine $f'(x)$.

b) Explain why the function g defined by $g(x) = f'(x)$ is continuous at $x = 0$.

c) Use the definition of the derivative to determine whether g is differentiable at $x = 0$.

ii) Consider the function $f(x) = \dfrac{1}{1+x}$ defined on $[0, 1]$ and let P be the partition $\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\}$.

a) Show that the lower Riemann sum $L_P(f)$ is given by

$$L_P(f) = \sum_{k=1}^{n} \frac{1}{n+k}.$$

b) Assuming that the limits of the upper and lower Riemann sums are equal, evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k}.$$

iii) Let A, B and C be three points in the plane with corresponding position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

a) Let M be the midpoint of the line joining A and B. What is the position vector $\mathbf{m}$ of M?

b) Write a parametric vector equation for the line through C and M.

c) Suppose that

$$(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a}) = \frac{1}{2}|\mathbf{b} - \mathbf{a}|\,|\mathbf{c} - \mathbf{a}| \quad \text{and} \quad (\mathbf{c} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \frac{1}{2}|\mathbf{c} - \mathbf{b}|\,|\mathbf{a} - \mathbf{b}|.$$

Explain why the triangle ABC is equilateral.

iv) Consider the system of equations

$$\begin{aligned} x + y - z &= 2 && (1) \\ x - y + 3z &= 6 && (2) \\ x^2 + y^2 + z^2 &= 10 && (3) \end{aligned}$$

[Note that equation (3) is NOT linear.]

a) Give, in parametric vector form, the set of points which satisfy the first two equations (that is, (1) and (2)).

b) Describe this solution set geometrically.

c) Using the answer to (a), or otherwise, find all the points which satisfy all three equations.

3. i) Find the shortest distance from the plane

$$\mathbf{x} = \begin{pmatrix}1\\2\\4\end{pmatrix} + \lambda_1 \begin{pmatrix}0\\2\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}-1\\1\\3\end{pmatrix}, \qquad \lambda_1, \lambda_2 \in \mathbb{R}$$

to the point $\mathbf{p} = \begin{pmatrix}3\\2\\3\end{pmatrix}$.

ii) Let $p(z) = z^4 - z^3 - z^2 - z + 2$. Denote the roots of p by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, where α_1 is an **integer**.

a) Find the value of α_1.

b) Given that at least one of the roots of p is not real, deduce how many of the roots are real.

c) By considering the sum of the roots, or otherwise, prove that at least one of the roots has negative real part.

d) Prove that $|\alpha_j| > \frac{1}{2}$ for $j = 1, 2, 3, 4$.

iii) Which of the following statements are true **for all** non-zero 2×2 matrices $A, B, C \in M_{2,2}(\mathbb{R})$? For those statements which are not always true, give a counterexample.

a) $AB = BA$.

b) $\det(AB) = \det(BA)$.

c) If $\det(AB) = \det(AC)$ then $\det(B) = \det(C)$.

d) If $AB = AC$ then $B = C$.

iv) a) Define what it means for a set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ to be an **orthonormal set** in $\mathbb{R}^n$.

b) Let M be the matrix whose columns consist of the n orthonormal vectors, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$. By considering $M^T M$ or otherwise, find, with reasons, all possible values for $\det(M)$.

4. i) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \int_0^{x^3} e^{-t^2}\,dt.$$

a) Determine, with reasons,

$$\lim_{t\to\infty} t^2 e^{-t^2}.$$

b) Does the improper integral

$$I = \int_0^{\infty} e^{-t^2}\,dt$$

converge? Give reasons for your answer.

c) Find all critical points and asymptotes of f.

d) Carefully sketch the graph of f, clearly indicating the above information and any other relevant features.

ii) Let f be a differentiable function on (a, b), and take $c \in (a, b)$. Define

$$q(x) = \frac{f(x) - f(c) - f'(c)(x - c)}{(x - c)^2},$$

where $a < x < b$ and $x \neq c$.

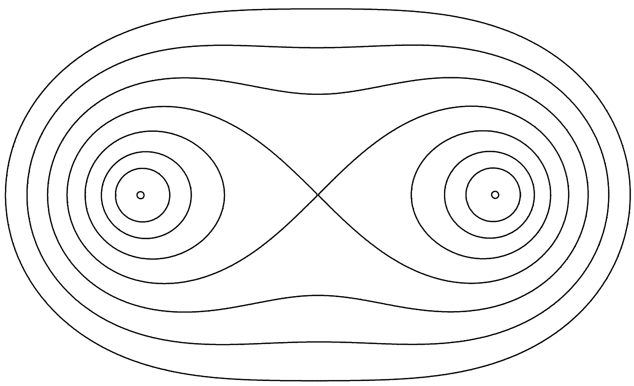
Show that if $f''(c)$ exists, then

$$\lim_{x \to c} q(x) = \frac{f''(c)}{2}.$$

iii) An oval of Cassini is a curve on the (x, y)-plane defined implicitly by

$$\left(x^2 + y^2\right)^2 - 2\left(x^2 - y^2\right) + 1 = b.$$

The shape of the curve depends on the value of the positive constant b. The plot below shows ovals of Cassini for several different values of b.



a) Show that the points on an oval of Cassini where the tangent is horizontal either lie on the unit circle $x^2 + y^2 = 1$ or lie on the y-axis.

b) Determine all such points (x, y) for which the corresponding tangent is horizontal and state carefully for which values of $b > 0$ these exist.

iv) Suppose $f : [0, 2] \to [0, 8]$ is continuous and differentiable on its domain.

a) By considering the function $g(x) = f(x) - x^3$, prove that there is a real number $\xi \in [0, 2]$ such that $f(\xi) = \xi^3$, stating any theorems you use.

b) Now suppose that $f(0) = 0$ and $f(2) = 8$. Explain why $f'(\eta) = 4$ for some real $\eta \in (0, 2)$, stating any theorems you use.

MATH1141 JUNE 2014

3. i) Let $g(x) = 3x - \cos 2x - 1$, $x \in \mathbb{R}$. Explain why g has a differentiable inverse function $h = g^{-1}$ and calculate $h'(-2)$.

ii) a) State carefully the Mean Value Theorem.

b) Use the Mean Value Theorem to prove that if $a < b$ then

$$0 < \tan^{-1} b - \tan^{-1} a \leq b - a.$$

c) Using (b) or otherwise, prove that the improper integral

$$I = \int_1^{\infty} \tan^{-1}\left(t + \frac{1}{t^2}\right) - \tan^{-1} t \, dt$$

converges.

iii) Use the ϵ-M definition of the limit to prove that

$$\lim_{x\to\infty} \frac{e^x}{\cosh x} = 2.$$

iv) Consider the polar curve $r = 1 + \cos 4\theta$.

a) Determine the values of $\theta \in [0, 2\pi]$ for which r has the smallest and largest values.

b) Hence, or otherwise, sketch this polar curve. (You are not required to find the slope.)

v) For $x > 0$, let $f(x) = x^{x \ln x}$.

a) Evaluate $f'(x)$.

b) Determine the values of x for which $f'(x) > 0$ and the values of x for which $f'(x) < 0$.

c) Given that $\lim_{x\to 0^+} f(x) = 1$, sketch the graph $y = f(x)$ for $0 \leq x \leq 2$.

4. i) Find the conditions on b_1, b_2, b_3 which ensure that the following system has a solution.

$$\begin{array}{rcrcrcl} 2x & & & - & 4z & = & b_1 \\ 3x & + & y & - & 2z & = & b_2 \\ -2x & - & y & & & = & b_3 \end{array}$$

ii) Let I, J and K be the points in $\mathbb{R}^3$ whose position vectors are the three standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ respectively.

By considering vectors of the form $\begin{pmatrix} x \\ x \\ x \end{pmatrix}$, find the position vector of a point A, not the origin, such that the distances from A to I, J and K are all 1.

iii) Consider the complex matrix $A = \begin{pmatrix} 2 & i \\ 1+i & \alpha \end{pmatrix}$.

a) Find A^{-1} in the case when $\alpha \in \mathbb{R}$.

b) Find all values of $\alpha \in \mathbb{C}$ for which $\det(A^2) = -1$.

iv) You may assume that $(z^9 - 1) = (z^3 - 1)(z^6 + z^3 + 1)$.

a) Explain why the roots of $z^6 + z^3 + 1 = 0$ are $e^{\pm\frac{2\pi i}{9}}, e^{\pm\frac{4\pi i}{9}}, e^{\pm\frac{8\pi i}{9}}$.

b) Divide $z^6 + z^3 + 1$ by z^3 and let $x = z + \dfrac{1}{z}$.
Find a cubic equation satisfied by x.

c) Deduce that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0$.

v) The norm $\|M\|$ of an $n \times n$ matrix M is the maximum value that $|M\mathbf{u}|$ takes for all unit vectors $\mathbf{u} \in \mathbb{R}^n$.

a) Show that for any vector $\mathbf{x} \in \mathbb{R}^n$,

$$|M\mathbf{x}| \leq \|M\||\mathbf{x}|.$$

b) Suppose that M and N are any two $n \times n$ matrices. By considering $MN\mathbf{u}$, or otherwise, show that

$$\|MN\| \leq \|M\|\|N\|.$$

c) What is the norm of the matrix $\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$

MATH1141 JUNE 2015

3. i) a) Carefully state the Mean Value Theorem.

   b) Assume that a differentiable function f on $\mathbb{R}$ is such that $f'(x) \le 1$ for all $x \in \mathbb{R}$. Given that $f(2) = 2$, show that $f(x) \ge x$ for all $x \le 2$.

ii) Let f be a continuous function on $\mathbb{R}$ and

$$g(x) = \frac{\int_0^x f(t)\,dt - xf(0)}{x^2}.$$

Use L'Hôpital's rule to show that if $f'(0)$ exists then

$$\lim_{x \to 0} g(x) = \frac{f'(0)}{2}.$$

iii) Let

$$f(x) = \int_0^{x^3} (t^2 - 1)e^{t^2}\,dt.$$

a) Show that f is an odd function, that is, $f(-x) = -f(x)$.

b) Find the stationary points of f.

c) By examining the sign of $f'(x)$ in a neighbourhood of each stationary point, or otherwise, determine the nature of each stationary point.

iv) Consider the curve in the (x, y)-plane defined by the relation

$$xy(x^2 + y^2) = 1. \tag{1}$$

a) Wherever y is an implicit function of x, determine $\frac{dy}{dx}$ in terms of x and y.

b) For any point (x, y) on the curve with $x, y \ge 0$, consider the area $A = xy$ of a rectangle of edge lengths x and y. Find the point (x, y) on the curve for which A is stationary.

c) Find the polar form of the curve (1) and express A in terms of the radial polar coordinate r.

d) Explain why the stationary point of A corresponds to the rectangle which has the largest area.

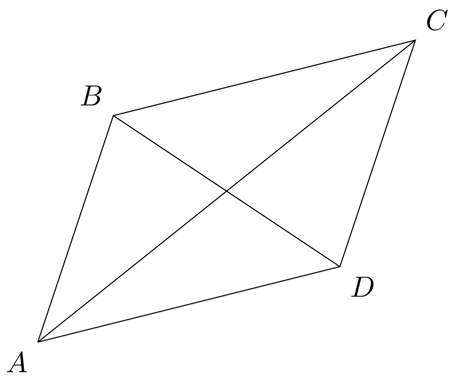
4. i) Find the shortest distance from the point $P(1, 2, 0)$ to the line with parametric vector equation

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

ii) a) By using the dot product, or otherwise, show that for any vectors $\mathbf{a}$ and $\mathbf{b}$, the following identity holds:

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2.$$

b) Consider a parallelogram $ABCD$, as pictured below.



If $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$, describe clearly in words the geometric interpretation of the result in (a).

iii) a) Use De Moivre's Theorem to express sin 5θ as a polynomial in $x = \sin\theta$.

b) Consider the polynomial $p(x) = 16x^5 - 20x^3 + 5x - 1$.
Show that $\sin\frac{\pi}{10}$ is a root of $p(x)$.

c) Using the fact that

$$16x^5 - 20x^3 + 5x - 1 = (x-1)(4x^2 + 2x - 1)^2$$

find the distinct roots of $p(x)$.

d) Evaluate $\sin\frac{\pi}{10}$ in surd form.

iv) Consider the matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. Suppose that $\det(A) = 7$.

Find the value of the determinant of each of the following matrices

$$B = \begin{pmatrix} d & e & f \\ a & b & c \\ d-3a & e-3b & f-3c \end{pmatrix} \text{ and } C = \begin{pmatrix} g-3a & 2a & a-d \\ h-3b & 2b & b-e \\ i-3c & 2c & c-f \end{pmatrix}.$$

v) Recall that the dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ can be written as $\mathbf{a}^T\mathbf{b}$ and that a square matrix Q is said to be *orthogonal* if $Q^TQ = I$.

Let Q be a real orthogonal $n \times n$ matrix and suppose $\mathbf{v}$ is a non-zero vector in $\mathbb{R}^n$.

a) Prove that the vectors $Q\mathbf{v}$ and $\mathbf{v}$ have the same length.

b) If $\mathbf{v}$ is a non-zero vector in $\mathbb{R}^n$, such that $Q\mathbf{v} = \lambda\mathbf{v}$, for some $\lambda \in \mathbb{R}$, show that $\lambda = 1$ or $\lambda = -1$.

THE UNIVERSITY OF NEW SOUTH WALES
BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2-x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, && |x| < a\\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, && |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, && x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$