Chapter 1: Introduction to Vectors (based on Ian Doust's notes)

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A typical problem

Chewie points his crossbow south-east. If bolts fly at 5ms^{-1}, it's easy to determine where it is at any point in time.

Question

But what if he fires it from the Millenium Falcon which is moving north at 10ms^{-1}?

Pictorial Ans

Q What if the Millenium Falcon is in the Death Star which is moving ...?

Goals of this chapter

Note that to answer the question above, you need to know both the magnitudes $5\text{ms}^{-1}, 10\text{ms}^{-1}$ and the directions of motion.

In this chapter we'll

- intro new mathematical objects called *vectors* which encode info about magnitudes & directions.
- Note real numbers only encode magnitudes and sign (i.e. $+$ or $-$).
- study how vectors combine to answer questions such as that above.
- intro coordinates to reduce vector calculations to calculations involving numbers.

What's a (geometric) vector?

A (geometric) *vector* is often represented by an arrow or a directed line segment. It can represent a velocity like 5ms^{-1} south east.

The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector. Typical notation for a vector: $\mathbf{v}$, $\vec{v}$, or $\underset{\sim}{v}$.

We denote the magnitude of $\mathbf{v}$ by $|\mathbf{v}|$. (No notation for the direction!)

Equality of vectors

We say vectors $\mathbf{u}$ & $\mathbf{v}$ are *equal*, and write $\mathbf{u} = \mathbf{v}$, if $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude and direction. **E.g.**

The *zero vector*, denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.

Vector addition

Given two vectors $\mathbf{u}$ and $\mathbf{v}$ we can 'add' them "head-to-tail" to produce a new vector $\mathbf{u} + \mathbf{v}$ as follows:

Adding the zero vector $\mathbf{0}$ does nothing as $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v}$.

There are many physical interpretations of this addition. E.g. 3-way tug-o-war

We let $-\mathbf{v}$ denote the vector with the same magnitude but the opposite direction so $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

We define subtraction by $\mathbf{u} - \mathbf{v} := \mathbf{u} + (-\mathbf{v})$.

Reminder on real numbers

Primary and high school arithmetic uses numbers that we call *real*. For example, the numbers

$$0,\ 73,\ -2\frac{1}{5},\ \sqrt{2},\ \pi - e^2, \ldots .$$

and so on are real numbers.

We will denote the real number system by $\mathbb{R}$. We visualize $\mathbb{R}$ as an infinitely long line:

Within the set $\mathbb{R}$ we have

- Natural numbers, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
- Integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- Rational numbers, $\mathbb{Q} = \left\{ \frac{p}{q} \, : \, p, q \in \mathbb{Z},\ q \neq 0 \right\}$.
- Positive numbers, $\mathbb{R}^+ = \{x \in \mathbb{R} \, : \, x > 0\}$.
- Irrational numbers, i.e. those that aren't in $\mathbb{Q}$.

and many other subsets of numbers. Sometimes, we refer to numbers as *scalars* (as opposed to vectors).

Scalar multiplication

There is another operation we can perform on vectors, *scalar multiplication*. If $\lambda \in \mathbb{R}$ is a non-negative scalar and $\mathbf{v}$ a vector, then $\lambda\mathbf{v}$ denotes the vector with the same direction, but magnitude is scaled by λ, i.e. $|\lambda\mathbf{v}| = \lambda|\mathbf{v}|$.

If $\lambda < 0$ then we define $\lambda\mathbf{v} := |\lambda|(-\mathbf{v})$.
E.g. $(-1)\mathbf{v} = ??$

If $\mathbf{u} = \lambda\mathbf{v}$, then we say that $\mathbf{u}$ and $\mathbf{v}$ are *parallel*.

Properties of vector addition

Question

In what sense is vector addition a type of "addition"?

A You can manipulate vector sums much as you can numbers because of the

Commutative law $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ (Why?)

Associative law $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (Why?)

Since these sums equal, we'll write it simply as $\mathbf{u} + \mathbf{v} + \mathbf{w}$. Similarly, we may omit brackets when adding 4 or more vectors together. **Challenge Q** Why?

Properties of scalar multiplication

Question

In what sense is scalar multiplication a type of “multiplication”?

Associative law $\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$ (Why?)

Distributive law $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$ (Vector)
$(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$ (Scalar)

Vector arithmetic

E.g. Let $\mathbf{w} = \mathbf{u} + 2\mathbf{v}$. Are $2\mathbf{w} - 4\mathbf{v}$ & $\mathbf{u}$ parallel?
A We simplify $2\mathbf{w} - 4\mathbf{v} = 2(\mathbf{u} + 2\mathbf{v}) - 4\mathbf{v}$

Note To perform this arithmetic, we only needed the basic properties of vector addition and scalar multiplication above. In general, we will meet lots of contexts where we have a vector addition and scalar multiplication satisfying these *axioms*. These sets will be called vector spaces. In these cases, we can perform arithmetic as above.

Co-ordinates

To put co-ordinates on the plane we need to specify:

- an origin point O where (co-ord axes cross), and
- a pair of vectors $\mathbf{i}$ and $\mathbf{j}$ which have length 1 and which are at right angles to one another. (The convention is to choose $\mathbf{j}$ at $\pi/2$ anticlockwise from $\mathbf{i}$.)

So $\mathbf{i}, \mathbf{j}$ give direction of coord axes & scale.

Fact-Definition

Every geometric vector $\mathbf{a}$ in the plane can be written as $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ for some unique pair of numbers $a_1, a_2 \in \mathbb{R}$.

The *co-ordinates or coordinate vector of* $\mathbf{a}$ (with respect to $\mathbf{i}, \mathbf{j}$) is $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

Co-ordinate arithmetic reflects vector arithmetic

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ so coords are $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

The coords of

$$\mathbf{a} \pm \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j}) \pm (b_1\mathbf{i} + b_2\mathbf{j}) = (a_1 \pm b_1)\mathbf{i} + (a_2 \pm b_2)\mathbf{j}$$

are $\begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \end{pmatrix}$.

Sim, the coords of

$$\lambda\mathbf{a} = \lambda(a_1\mathbf{i} + a_2\mathbf{j}) = \lambda a_1\mathbf{i} + \lambda a_2\mathbf{j}$$

are $\begin{pmatrix} \lambda a_1 \\ \lambda a_2 \end{pmatrix}$.

Upshot This says to sum, subtract or multiply vectors, we need only sum, subtract or multiply coords.

Challenge Q How do coords change if you replace $\mathbf{i} \mapsto \mathbf{j}, \mathbf{j} \mapsto -\mathbf{i}$?

Example

To find coords recall given a right angle triangle,

Question

I walk 1km due west, then 4km on a bearing 30° east of north. Where do I end up?

Solution. Take $\mathbf{i}$ pointing east and $\mathbf{j}$ pointing north & units are km.

3-dimensional version

You can do all this with 3-dimensional geometric vectors too.
Here you need basis vectors **i**, **j** and **k**.
You can write vectors in form

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

which have coords $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Again

$$\text{coords of } \mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}, \qquad \text{coords of } \lambda\mathbf{a} = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{pmatrix}.$$

The space $\mathbb{R}^n$

We can generalise coordinate vectors to any number of components!
Let n be a positive integer. An *n-tuple* or *n-vector* is an ordered list of n numbers $a_1, a_2, \ldots, a_n$, written as either a column vector or (less often in this course) a row vector:

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad \text{or} \quad \mathbf{a} = (a_1, a_2, \ldots, a_n).$$

The set of all n-tuples is denoted $\mathbb{R}^n$. Thus

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2, \qquad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^3, \qquad \begin{pmatrix} 0 \\ 1 \\ -1 \\ \pi \end{pmatrix} \in \mathbb{R}^4.$$

Hence coord vectors of 3-dim vectors lie in $\mathbb{R}^3$ whilst those of 2-dim vectors lie in $\mathbb{R}^2$.

Arithmetic on $\mathbb{R}^n$

We can define vector addition and scalar multiplication on $\mathbb{R}^n$ *coordinatewise* as we saw for coordinate vectors:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} := \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + a_n \end{pmatrix}, \qquad \lambda \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} := \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{pmatrix}.$$

The 'zero element' is $\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ and the negative is given by $-\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} -a_1 \\ \vdots \\ -a_n \end{pmatrix}$ (so we can define subtraction!).

Note: each $\mathbb{R}^n$ is a separate system. You can't add a 3-tuple to a 7-tuple!

Q Compute $3\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} =$

Properties of arithmetic on $\mathbb{R}^n$

This 'vector addition and scalar multiplication inherit good properties from addition and multiplication on $\mathbb{R}$. That is, if $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$ then

- Commutative: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$,
- Associative: $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$,
- Distributive: $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$,
- Distributive: $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$.
- Cancellation: $\lambda\mathbf{a} = \mathbf{0}$ if and only if $\lambda = 0$ or $\mathbf{a} = \mathbf{0}$,
- etc

Moral: You can do algebra in $\mathbb{R}^n$ without running into any problems!

Proof Easy from definitions but take up space e.g.

Displacement vector

Put coords on the plane by specifying $O, \mathbf{i}, \mathbf{j}$ as usual.
Given any 2 points A, B on the plane, we define the *displacement vector* $\overrightarrow{AB}$ to be the geometric vector with tail A & head B i.e.

From the picture we see $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

Important Remark

In high school you would have considered coords of the point A (as opposed to a vector).

- coords $A =$ coords $\overrightarrow{OA}$.
- coords $\overrightarrow{AB} =$ coords B - coords A.

$\overrightarrow{OA}$ is called the *position vector* of A (with respect to O). These observations also hold if the points are in space with coords specified.

Simple geometric applications of vectors

E.g. Are $A = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, B = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, C = \begin{pmatrix} 6 \\ -2 \\ 5 \end{pmatrix}$ collinear?

E.g. Are $A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, C = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, D = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ the vertices of a parallelogram?

Length and distance in $\mathbb{R}^n$

Pythagoras' thm $\Longrightarrow$ the length of a geometric vector with coords $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is $\sqrt{a_1^2 + a_2^2}$. This suggests the following generalisation of the length concept to $\mathbb{R}^n$.

Definition

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$.

- The *length* of $\mathbf{a}$ is defined to be $|\mathbf{a}| = \sqrt{a_1^2 + \cdots + a_n^2}$.
- the *distance* between $\mathbf{a}$ and $\mathbf{b}$ is defined to be $\mathrm{dist}(\mathbf{a}, \mathbf{b}) = |\mathbf{b} - \mathbf{a}|$.

Example. a) What is $\left| \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right|$?

b) Suppose that the point A has coordinates $(1, 2, 3)$ and the point B has coordinates $(-1, 2, 5)$. What is the distance between A and B?

Lines in $\mathbb{R}^n$

Using our 2 and 3-dim intuition, a line L in $\mathbb{R}^n$ should be determined by

- a point A on the line, say with $\mathbf{a} = \overrightarrow{OA}$ and,
- a direction, say given by a non-zero vector $\mathbf{v}$

Let's determine what the general point of L ought to be:

$$\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}, \ \ \lambda \in \mathbb{R}$$

This is called the *parametric vector form* of the line L. We call λ the parameter, and as it varies over $\mathbb{R}$, the variable $\mathbf{x}$ varies over all the points of the line L.

Definition

A *line* in $\mathbb{R}^n$ is any set of the form

$$\{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{x} = \mathbf{a} + \lambda \mathbf{v}, \ \ \lambda \in \mathbb{R}\}$$

for some fixed vectors $\mathbf{0} \neq \mathbf{v}, \mathbf{a} \in \mathbb{R}^n$. Note $\mathbf{a}$ gives a point on the line and $\mathbf{v}$ its direction.

Vectors and MAPLE

See MAPLE file

Midpoints

Midpoints

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ be the position vectors for the points A, B. The midpoint of AB has coordinates

$$\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{a} + \mathbf{b}).$$

Why?

Q Show that the diagonals of a parallelogram bisect each other.

Challenge Q What's the 3-dim version of this result?

Finding parametric forms for lines from Cartesian form

In high school, you express a line in the plane in Cartesian form $ax + by = c$.

Question

Find a parametric vector form for the line $y = 3x + 2$ in $\mathbb{R}^2$.

A1 Consider 2 points on the line

A2 We introduce the parameter $\lambda =$

N.B. There are many other solutions! (What are they?)

Parametric to Cartesian form

Question

Write the line $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$ in Cartesian form.

The secret is to eliminate the extra variable λ!

Solution. Write $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda \\ -1 + \lambda \end{pmatrix}$. Then

$$\lambda =$$

What about $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$?

Cartesian form for lines and planes in $\mathbb{R}^3$

Recall that a plane in *xyz*-space can be described by an equation

$$ax + by + cz = d$$

where not all a, b, c are 0. This is called the *Cartesian form* for the plane. The terms in this equation can of course be re-arranged many ways (see below).

To obtain the cartesian form for a line L, we need 2 such equations. Each defines a plane P_1, P_2 and solving simultaneously gives the solution $P_1 \cap P_2$. This will be a line unless . . . .

Usually, (but not always) we can write the 2 equations in the form

$$\frac{x - a_1}{v_1} = \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3}$$

for some constants a_i, v_i.

Cartesian to parametric form for lines in $\mathbb{R}^3$

Question

Find the parametric form for $\dfrac{x-2}{3} = \dfrac{y+1}{6} = \dfrac{z-3}{-2}$.

A We can find a point on the line and a direction vector or just introduce the parameter

Parametric to cartesian form for lines in $\mathbb{R}^3$

Question

Find a cartesian form for

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \qquad \lambda \in \mathbb{R}$$

What about $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}$?

Parametric vs Cartesian form

- The Cartesian form expresses a line or plane as the solutions to some equations. (Top down)
- The parametric form expresses the line or plane by a sophisticated way of listing elements, where running through the list is by letting the parameter range over a set.(Bottom up)
- In mathematics, these are the two usual general ways to describe any set.
- Both have their uses.
- For lines, the parametric form is closest to our geometric picture of the line.

Why bother defining lines in $\mathbb{R}^n$?

- Suppose we are solving equations in n unknowns $x_1, \ldots, x_n$. If $n = 3$, it is often good to visualise the solution set in $x_1x_2x_3$-space.
- For example, solving simultaneously

$$a_1x_1 + a_2x_2 + a_3x_3 = a, b_1x_1 + b_2x_2 + b_3x_3 = b$$

  should on geometric grounds, give either a line, plane or the empty set.
- In particular, you can't get a point or two points etc.
- If $n > 3$, we can use our geometric intuition to understand solutions to many equations provided we generalise our notions of things like lines in $\mathbb{R}^3$ to lines in higher dimensions.

"Directions" of a plane

Thought experiment

What are the "directions" of a plane $P \subset \mathbb{R}^3$?

Since we are interested in directions only, let's suppose P passes through O and that $\mathbf{v}, \mathbf{w} \in P$ are not parallel

Hence

Fact

Any vector parallel to P has the form $\lambda \mathbf{v} + \mu \mathbf{w}$ for some $\lambda, \mu \in \mathbb{R}$.

i.e. all other "directions" of P can be obtained from $\mathbf{v}, \mathbf{w}$ by combining them using vector operations.

Linear combination

More generally, we consider

Definition

Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. A *linear combination* of these vectors is a vector of the form

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_k \mathbf{v}_k$$

with $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$.

Q Is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$?

Span

Definition

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. The *span* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, written $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. i.e.

$$\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_k \mathbf{v}_k | \lambda_1, \ldots, \lambda_k \in \mathbb{R}\}.$$

Ex. Describe $\mathrm{span} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

More gen, $\mathrm{span}(\mathbf{v})$ is the

Ex. Describe $\mathrm{span} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$.

Planes in $\mathbb{R}^n$

Definition

A *plane in* $\mathbb{R}^n$ is defined to be a set of the form

$$S = \{\mathbf{a} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 | \lambda_1, \lambda_2 \in \mathbb{R}\},$$

where $\mathbf{a}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are fixed vectors in $\mathbb{R}^n$, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel.

The expression $\mathbf{x} = \mathbf{a} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2,\ \lambda_1, \lambda_2 \in \mathbb{R}$ is a *parametric vector form* for *the plane through* $\mathbf{a}$ *parallel to the vectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$.

The above picture shows that when $n = 3$, our definition agrees with our old one.

Q What if $\mathbf{v}_1, \mathbf{v}_2$ above are parallel?

Parametric form for plane determined by 3 points

Question

Find a parametric vector equation for the plane through the points $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

Finding Cartesian form for planes from parametric form

Question

Find the Cartesian equation of the plane in $\mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Meanwhile back at the Death Star

Question

The Millenium Falcon, at coords $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is flying in direction $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Will it hit the Death Star wall, a plane with Cartesian eqn $x - y - z = 1$?

A Without the Death Star, the flight trajectory would be the *ray* with parametric equation

Standard basis vectors for $\mathbb{R}^n$

In $\mathbb{R}^n$, the vector $\mathbf{e}_j$ is the n-tuple with 1 in the jth position and zeros elsewhere.

$\mathbb{R}^2$: $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$\mathbb{R}^3$: $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Obviously, every vector in $\mathbb{R}^n$ can be written uniquely as a linear combination of $\mathbf{e}_1, \ldots, \mathbf{e}_n$, eg

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3.$$

The vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are called the *standard basis vectors* for $\mathbb{R}^n$.