THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 1 2016

MATH1141
HIGHER MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS APPENDED TO THE PAPER

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book clearly marked Question 1

1. i) For each of the following, either evaluate the limit or explain why it does not exist.

a) $\displaystyle\lim_{x\to 2} \frac{|x^2-4|}{x-2}$

b) $\displaystyle\lim_{x\to\infty} \frac{2\sin x + x}{3x-1}$

ii) Let

$$f(x) = \begin{cases} x^2 & x \geqslant 0 \\ 0 & x < 0. \end{cases}$$

a) Show that $f(x)$ is differentiable at $x = 0$ and find $f'(0)$.

b) Determine $f'(x)$ for all x.

iii) Let $g(x) = x^7 + 4x + 2$, defined for all real x.

a) Use the Intermediate Value Theorem to show that $g(x) = 0$ has at least one real solution.

b) Show that g has an inverse function with domain $\mathbb{R}$.

iv) Let $z = 1 + 3i$ and $w = 2 - 4i$. Find, in $a + ib$ form:

a) $\overline{w} + 2z$.

b) z/w.

v) Let $u = -\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$.

a) Calculate $|u|$ and $\operatorname{Arg}(u)$.

b) Hence, or otherwise, find u^{30} in its simplest form.

vi) Let $P = \begin{pmatrix} 1 & 4 \\ 3 & 5 \\ 0 & 7 \end{pmatrix}$ and $Q = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$.

a) Evaluate PQ^T.

b) What is the size of PQP^T?

Use a separate book clearly marked Question 2

2. i) a) State the Mean Value Theorem.

b) By using the Mean Value Theorem show that

$$\sin x \le x \quad \text{for all} \quad x \ge 0.$$

ii) Sketch the polar curve

$$r = 1 - \cos\theta \quad \text{for} \quad 0 \leqslant \theta < 2\pi.$$

iii) Use logarithmic differentiation to find $\dfrac{dy}{dx}$ if $y = 2^{\sin x}$.

iv) The volume of a spherical balloon is increasing at a constant rate of 4 cm^3/min. How fast is the radius increasing when the radius is 12 cm? You are given that a sphere of radius r has volume $V = \dfrac{4}{3}\pi r^3$.

v) Let the set S in the complex plane be defined by

$$S = \left\{ z \in \mathbb{C} : -\frac{\pi}{4} \le \operatorname{Arg}(z) \le \frac{\pi}{4} \text{ and } \operatorname{Re}(z) \le 3 \right\}.$$

a) Sketch the set S on a labelled Argand diagram.

b) Let w be the complex number in S with greatest imaginary part. By considering your sketch or otherwise find w in $a + ib$ form.

vi) The points A and B in $\mathbb{R}^3$ have position vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}.$$

a) Find a parametric vector equation of the line l passing through A and B.

b) Hence find a point P on the line such that the sum of the x, y and z coordinates of P is equal to 62.

vii) Let $A = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 3 & 8 \\ 0 & 2 & 6 \end{pmatrix}$.

a) Calculate the determinant of A.

b) Does A have an inverse?

c) Write down the determinant of $5A$.

Please see over ...

Use a separate book clearly marked Question 3

3. i) Let f and g be differentiable functions. Find y', where

$$y = f(\sin^2[g(x)]).$$

ii) a) Sketch the graph of a function $y = F(x)$ which has the following properties:
F has domain $(0, \infty)$;
F has a vertical asymptote at $x = 0$;
F is increasing on the domain $(0, \infty)$.

b) Give a formula for a rational function F which has all the properties listed in part (a). Justify your answer.

iii) A curve is defined by the parametric equations

$$x = t^2, \quad y = t^3 - 3t.$$

a) Find the points on the curve corresponding to $t = 1, -1$.

b) Find all the points on the curve where $y = 0$.

c) Show that the curve passes twice through the point $(3, 0)$ and find the equations of the corresponding tangents at this point.

d) Find the points on the curve where the tangent is horizontal or vertical.

e) Sketch the curve, displaying the above features and the behaviour as $t \to \pm\infty$.

iv) a) Carefully state the first fundamental theorem of calculus.

b) For $\alpha > 0$ and $n > 0$, determine whether the improper integral

$$\int_0^{\cancel{x}\ \infty} u e^{\alpha u^n}\, du$$

converges or diverges. Give reasons for your answer.

c) Using L'Hôpital's rule, find, without integration, $\lim_{x \to \infty} f(x)$, where

$$f(x) = \frac{\left(\displaystyle\int_0^x u e^{3u^2}\, du\right)^2}{\displaystyle\int_0^x u e^{6u^2}\, du}.$$

d) Show that the function f defined in (c) is an even function, that is, $f(-x) = f(x)$.

Use a separate book clearly marked Question 4

4. i) Consider the plane $\overset{A}{\cancel{\Pi}}$ given by the equation $3x - y + 2z = 5$.

a) Find the distance from the point $P(1, 2, -1)$ to the plane A.

b) Find all points of intersection of the plane A and the line with parametric equation

$$\mathbf{x} = \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} \quad \text{for } \lambda \in \mathbb{R}.$$

ii) Suppose z and w are two nonzero complex numbers z and w with $\text{Arg}(z) < \text{Arg}(w)$ satisfying

$$z^2 + w^2 = zw.$$

a) Find a formula for z in terms of w.

b) Hence, or otherwise, show that the points in the Argand plane corresponding to z and w are the vertices of an equilateral triangle whose third vertex is the origin.

iii) Consider the polynomial $p(z) = z^6 - 4z^5 + 5z^4 + z^2 - 4z + 5$.

a) Given that $2 - i$ is a root of $p(z)$, find all of the roots of $p(z)$.

b) Express $p(z)$ as a product of linear and quadratic terms with real coefficients.

iv) Are the two planes in $\mathbb{R}^4$

$$\mathbf{x} = \begin{pmatrix} 4 \\ -7 \\ 0 \\ 11 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 0 \\ 1 \\ -2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ -2 \\ -1 \end{pmatrix} \quad \text{for } \lambda_1, \lambda_2 \in \mathbb{R}$$

and

$$\mathbf{x} = \begin{pmatrix} -3 \\ 6 \\ 4 \\ -3 \end{pmatrix} + \mu_1 \begin{pmatrix} 6 \\ 0 \\ 2 \\ -4 \end{pmatrix} + \mu_2 \begin{pmatrix} 7 \\ 6 \\ -3 \\ -4 \end{pmatrix} \quad \text{for } \mu_1, \mu_2 \in \mathbb{R}$$

parallel? Give reasons for your answer.

v) A square matrix Q is said to be a unitary matrix if it has the property that $\overline{Q}^T Q = I$, where $\overline{Q}$ is the matrix obtained from Q by taking complex conjugates of each entry of Q.

a) Give an example of a 2×2 unitary matrix with non-real entries.

b) Show that the determinant of a unitary matrix has the form $e^{i\theta}$ for some real number θ.

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BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2-x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \qquad |x| < a\\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, \qquad |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$