# Chapter 3

# Ordinary Differential Equations

# 3.1 Introduction

In many situations a model will give you an equation relating the rate of change of a quantity to some function of the same quantity: a **differential equation**.

Verhulst's logistic equation (1845). $$\frac{dP}{dt} = k\,P(t)(K - P(t))$$

Black-Scholes equation (1973).

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Nobel Prize for Economics (1997).

Schrödinger equation (1926).

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{x},t) + V(x,t)\psi(\vec{x},t)$$

Nobel Prize for Physics (1933).

A **differential equation** (DE) is an equation which contains the derivatives of one or more **dependent** variables with respect to one or more **independent** variables.

$$m\frac{d^2x}{dt^2} + p(t)\left(\frac{dx}{dt}\right)^3 + k\,x(t)^5 = 0 \qquad \left\{ \begin{array}{ll} x & \text{dependent variable} \\ t & \text{independent variable} \end{array} \right.$$

An **ordinary differential equation** (ODE) contains the ordinary derivatives of one or more dependent variables with respect to just one independent variable.

The above is an example of an ODE.

The **order** of a DE is the order of the highest derivative in the DE.

The above example is of second order

A **partial differential equation** (PDE) contains the partial derivatives of one or more dependent variables with respect to several independent variables.

For example

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

is a second order PDE with 1 dependent variable ($u$) and two independent varables ($t, x$).

We will only look at ODEs in this course: PDEs are studied in such courses as

MATH2121

MATH2221

MATH2019

etc

A **solution** of an $n$th order DE is a function that is $n$ times differentiable and satisfies the DE.

Solutions to DEs can either be **explicit** or **implicit**.

The harmonic oscillator equation $\frac{d^2x}{dt^2} = -\omega^2 x$ has the explicit solution $x = \sin(\omega t)$.

**Proof**: Clearly $\sin(\omega t)$ is twice differentiable, and

$$x' = \omega \cos(\omega t), \qquad x'' = -\omega^2 \sin(\omega t)$$

So

$$x'' = -\omega^2 x$$

$\blacksquare$

EXERCISE: Show $x = A\sin(\omega t) + B\cos(\omega t)$ is also a solution, where $A$ and $B$ are arbitrary constants.

The differential equation $\left(\frac{dy}{dx}\right)^2 = \frac{1-y^2}{1-x^2}$ (for $x$ and $y$ in $(-1,1)$ only) has the implicit solution

$$\sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}.$$

**Proof**: Again, differentiability is not a problem, and using implicit differentiation we get

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0.$$

So

$$\frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

and squaring completes the proof. $\blacksquare$

The simplest method of solving a DE (omitted from Notes) is **direct integration**:

It applies to DEs of the shape $\boxed{\dfrac{dy}{dx} = h(x)}$

If $h$ is continuous in a neighbourhood of some $a$ then, from the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\left(\int_a^x h(s)\,ds\right) = h(x).$$

Hence, $y = \displaystyle\int_a^x h(s)\,ds$ is a solution of $\dfrac{dy}{dx} = h(x)$.

**Note**: $y = \displaystyle\int_a^x h(s)\,ds + C$ is also a solution for any constant $C$.

It is the **general solution** of the DE.

# 3.2 Initial Value Problems

For a DE, the **general solution** is the set of all solutions – if possible – represented as a single equation containing arbitrary constants.

Note that there can be solutions to a DE that are not contained in the set of general solutions.

Consider the DE $\dfrac{d^2y}{dx^2} = \sin x$.

Two applications of the fundamental theorem of calculus gives

$$y = \int \left( \int \sin x \, dx \right) dx = \int (-\cos x + C_1) dx = -\sin x + C_1 x + C_2$$

By construction this is the general solution, containing two constants of integration.

An **Initial Value Problem** (IVP) consists of an $n$th order ODE (for $y$, say) together with specified values for the solution and its first $n-1$ derivatives at a given point, that is

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \cdots, \quad y^{(n-1)}(x_0) = y_{n-1},$$

where the real numbers $x_0, y_0, y_1, \cdots, y_{(n-1)}$ are given.

**Example 1** *Solve the IVP* $y' = 2x, \quad y(0) = 5$

SOLUTION: By direct integration the general solution is $y = x^2 + C$

Substituting the initial condition:

$$5 = y(0) = 0^2 + C \quad \text{and so} \quad C = 5.$$

So the solution to the IVP is

$$y = x^2 + 5.$$

**Example 2** *Describe the motion of a point mass under the influence of gravity, given by the IVP* $\dfrac{d^2y}{dt^2} = -g$ *subject to the initial conditions (position and speed)*

$$y(0) = 2, \quad \frac{dy}{dt}(0) = 1.$$

SOLUTION: Integrating once we get $\dfrac{dy}{dt} =$

We might as well put in the derivative condition now: .

Integrating again we get $y =$

And the other initial condition gives us the final solution:

$$y =$$

**Example 3** *Solve the IVP:*

$$\frac{dy}{dx} = 2\sqrt{y}, \qquad y(0) = 0$$

SOLUTION: There is one obvious solution: $y(x) = 0$ for all $x$.

If $y(x)$ never vanishes we can rewrite the ODE as $\dfrac{dx}{dy} = \dfrac{1}{2\sqrt{y}}$.

By direct integration: $x(y) = \sqrt{y} + C$.

Putting in the initial condition we see that $C = 0$, and so

$$y(x) = x^2,$$

which we can check is a solution of the DE. ■

**Lesson**: the solution of a DE (or IVP) need not be unique.

Sometimes when solving a DE for a specific problem we do not get an initial value problem, but a boundary value problem.

An example should suffice:

**Example 4** *Solve* $\dfrac{d^2y}{dx^2} = \sin x, \quad y(0) = 0, \quad y(\pi) = -\pi$

SOLUTION:

General solution: $y = -\sin x + Cx + D$.

First boundary condition: $0 = y(0) \quad \Rightarrow \quad D = 0$

Second boundary condition: $-\pi = y(\pi) \quad \Rightarrow \quad C = -1$

The unique solution is $y = -\sin x - x$.

■

There are 4 important questions we need to ask when faced with a differential equation:

a) Does a solution exist?

b) If a solution exists, does it exist for all values of the independent variables?

c) Is there always a unique solution if a solution exists?

d) How do we find solutions?

The answer to the first 3 questions depends almost completely on what you mean by solution and exist.

In higher year courses you may study these questions.

What we are going to turn to is the last question, at least for certain tractible cases.

# 3.3 Separable Differential Equations

A **first order separable DE** is of the form

$$\boxed{\frac{dy}{dx} = \frac{g(x)}{h(y)}}$$

The method here is to multiply by $h(y)$ and integrate w.r.t. $x$:

$$\int h(y)\frac{dy}{dx}\,dx = \int g(x)\,dx \quad \Rightarrow \quad \boxed{\int h(y)\,dy = \int g(x)\,dx}$$

Two points:

a) Separable DEs usually produce implicit solutions.

b) Since separable DEs are so straightforward, many techniques for solving other cases rely on turning a DE into a separable one (see later).

**Example 5** *Solve* $\dfrac{dy}{dx} = y^2(1+x^2)$

SOLUTION: Separating and integrating we get

$$\int y^{-2}\,dy = \int 1 + x^2\,dx$$

So

Note we only introduce one arbitrary constant $C$

If our separable DE is actually part of an IVP:

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}, \qquad y(x_0) = y_0$$

then we can either solve the DE and substitute the initial condition, or integrate directly:

$$\int_{y_0}^{y} h(s)\, ds = \int_{x_0}^{x} g(t)\, dt \tag{1}$$

- (1) is a solution if $g$ and $h$ are continuous.
- (1) is identically satisfied if $y = y_0,\ x = x_0$.
- If $h'$ is continuous and (1) may be 'solved' for $y$ then (1) is the unique solution of the IVP.

**Example 6** *Solve* $y(x^2-1)\dfrac{dy}{dx} = x(y^2-1)$.

SOLUTION: We can see this is separable by a little manipulation but may as well just separate and integrate:

$$\int \frac{y}{y^2-1}\,dy = \int \frac{x}{x^2-1}\,dx$$

So

**Aside**: It is quite usual to be a little careless about algebraic niceties (e.g. absolute values) when solving DEs.

We often do not bother to take special care about the domain and range of the functions we are dealing with, and just merrily manipulate them as meaningless marks.

This is usually fine, as we can always take our final “solution” and put it back into the DE to check it is actually a solution.

Sometimes you do have to be careful, especially with IVPs, to make sure you pick the right constants, or sign of the square root etc.

One thing you cannot be careless about is introducing arbitrary constants when you integrate.

If you find a purported general solution with no arbitrary constants, it must be wrong.

# 3.4 Linear ODEs

An $n$th order ODE is said to be **linear** if it is a linear combination of the first $n$ derivatives of the dependent function:

$$\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

If $a_0, a_1, \ldots$ are constants (independent of $x$) then the general solution may be found — assuming one can solve a certain $n$th degree polynomial: see section 3.8.

Linear first order ODEs can always be solved by using an **integrating factor**.

To solve a first order linear ODE, begin by writing it in the **standard form**:

$$\boxed{\frac{dy}{dx} + f(x)y(x) = g(x)}$$

Next multiply the standard form by the **integrating factor**

$$\boxed{h(x) = \exp\left(\int f(x)\, dx\right)}$$

We ignore the constant of integration here: that would just multiply the whole DE by a non-zero constant.

The point of doing this is that the right hand side of the new DE is now the derivative of a product, i.e. the DE is

$$\frac{d}{dx}\Big(h(x)y(x)\Big) = g(x)h(x),$$

and we can just integrate (at least in theory!).

**Example 7** *Solve* $x^2 \dfrac{dy}{dx} = -xy + x^2 e^{x^2}$

SOLUTION: In standard form the DE is $\dfrac{dy}{dx} + x^{-1} y = e^{x^2}$.

The integrating factor is then

$$h(x) = \exp\left(\int x^{-1}\, dx\right) = e^{\ln x} = x.$$

Multiplying this through the standard form, the DE becomes

Hence

**Example 8** *Solve the IVP*

$$y' + y\tanh x = e^{\sinh x}, \qquad y(0) = 0$$

SOLUTION: The DE is already in standard form, and as $\int \tanh x\, dx = \ln\cosh x + C$, the integrating factor is

.

Multiplying the DE through we get

and integrating:

The initial condition now gives . So

# 3.5 Exact Differential Equations

The ODE

$$F(x,y) + G(x,y)\frac{dy}{dx} = 0$$

or, in standard form,

$$\boxed{F(x,y)dx + G(x,y)dy = 0}$$

is **exact** if and only if

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$$

**Note**: for checking exactness we compare the $y$ derivative of the function multiplying the "$dx$" to the $x$ derivative of the function multiplying the "$dy$".

**'Theorem'** If the differential equation

$$F(x,y)\,dx + G(x,y)\,dy = 0 \tag{2}$$

is exact then there exists a function $H(x,y)$ such that

$$\frac{\partial H}{\partial x} = F \quad \text{and} \quad \frac{\partial H}{\partial y} = G.$$

The solution of (2) is then given by

$$H(x,y) = C.$$

**'Proof'**

$$0 = \frac{dH}{dx} = \frac{\partial H}{\partial x} + \frac{\partial H}{\partial y}\frac{dy}{dx} = F + G\frac{dy}{dx}$$

**Comments**:

a) Unless the DE is actually given in the standard form

$$F(x, y)\,dx + G(x, y)\,dy = 0 \qquad (2)$$

the idea of an exact DE is not well defined.

b) For any first order DE written as (2), there is in fact always a function $h(x, y)$ (called an **integrating factor**) such that if we multiply (2) through by $h(x, y)$, the DE becomes exact.
Unfortunately, finding the integrating factor is usually harder than solving the original DE.

c) The general solution of an exact DE must be given in the form

$$H(x, y) = C$$

for some constant. Just giving $H(x, y)$ is wrong: there is then no arbitrary constant.

**Example 9** *Show* $(x^2 + y^2)\,dx + 2xy\,dy = 0$ *is exact and solve it.*

SOLUTION: Firstly, to check exactness:

so it is exact.

We have to find one function $H(x, y)$ such that

$$\frac{\partial H}{\partial x} = x^2 + y^2 \qquad \text{and} \qquad \frac{\partial H}{\partial y} = 2xy.$$

The Notes give two ways of finding $H(x, y)$: my advice is use the second way.

$$\frac{\partial H}{\partial x} = x^2 + y^2 \qquad \text{and} \qquad \frac{\partial H}{\partial y} = 2xy.$$

Integrate one of these equations (whichever one looks easiest).

I'll pick the second one here:

**Note**: the arbitrary constant is a function of $x$ since we have integrated with respect to $y$.

Differentiate this part-solution with respect to $x$ and compare with what it should be:

$$\frac{\partial H}{\partial x} = \qquad\qquad \text{should be} \quad x^2 + y^2$$

So clearly

Note that we can ignore the constant of integration at this stage – as the Notes explain (remark 3.5.5) it could just be absorbed into the one we're going to introduce on the RHS of the solution.

The general solution is thus

for arbitrary constant $C$, or

if you'd prefer. ■

If the equation were not exact then at the stage where we tried to integrate $f'(x) = \ldots$, we'd find the RHS to be a function of $x$ and $y$, which is a clear contradiction.

**Example 10** *Write* $\dfrac{dy}{dx} = \tan x\,\tan y$ *as an exact DE and solve.*

SOLUTION: The natural thing to do is write:

$$\frac{dy}{dx} = \frac{\sin x}{\cos x}\,\frac{\sin y}{\cos y}$$

and test

$$-\sin x\,\sin y\,dx + \cos x\,\cos y\,dy = 0$$

We get

so our re-write is exact.

We need to find an $H(x, y)$ with

$$\frac{\partial H}{\partial x} = -\sin x \, \sin y \qquad \text{and} \qquad \frac{\partial H}{\partial y} = \cos x \, \cos y.$$

We could go through the integration/differentiation routine, but this example exhibits a typical situation:

It should be obvious that

will do – it's easily checked.

So our solution is

for some arbitrary constant $C$.

# Aside: general advice

When it comes to trying to solve a first order ODE, you should check the type in the order we have covered them:

1) Direct Integration (possibly with the inverse function theorem)
2) Separating
3) Linear
4) Exact

Unless, of course, you are told to use a certain way, or suspect one of the later ones will be easier.

# 3.6 Substitutions

We have seen how to solve separable, linear and exact first order ODEs.

It is sometimes possible to perform a change of variable (either the dependent or independent, or both) to change more general first order ODEs into one of these types.

Finding a substitution to turn one DE into something more tractible is a whole sub-field of the study of DEs.

If we ever ask you to do a substitution in first year, we will supply it.

In higher years you may study other standard substitutions.

The general theory (due to Sophus Lie, 1842-1899) is usually not taught until honours (if then).

**Example 11** *By substituting $y(x) = xv(x)$, solve*

$$2xy\frac{dy}{dx} + x^2 - y^2 = 0.$$

SOLUTION: Differentiating $y = xv$ gives $y' = v + xv'$, and the ODE becomes

$$2x^2v\left(v + xv'\right) + x^2 - x^2v^2 = 0$$

Dividing by $x^2$ and rearranging we get the separable DE

Separating and integrating:

Exponentiating

Resubstituting and multiplying through:

The $y(x) = xv(x)$ substitution works with all ODEs of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

**Example 12** *Show that the substitution* $v(x) = 1/y(x)^4$ *transforms*

$$2xy' - y = 10x^3y^5$$

*into a linear equation, and solve the DE.*

SOLUTION: Firstly, $v = y^{-4}$ implies $v' = -4y^{-5}y'$, so the DE is

$$-\frac{1}{2}xy^5v' - y = 10x^3y^5 \quad \text{or}$$

which is linear and in standard form.

The integrating factor is so the DE is

Integrating:

# 3.7 Modelling with first order ODEs

Derivatives express a rate of change of one variable w.r.t. another and they appear naturally in many modelling equations, e.g.,

bacterial growth $$\frac{dN}{dt} = kN$$

drug decay $$\frac{dy}{dt} = -ky$$

motion $$\frac{d(mv)}{dt} = kF$$

cooling $$\frac{dT}{dt} = -k(T(t) - T_a)$$

mixing $$\frac{dV}{dt} = V_{\text{in}} - V_{\text{out}}$$

MATH3041 – Mathematical Modelling for Real World Systems. Students work in groups to use mathematics to solve real world problems.

# Newton’s Law of Cooling

The rate of decrease of temperature is proportional to the difference between the temperature of an object and its surroundings.

In mathematical form:

$$\frac{dT}{dt} = -k\left(T(t) - T_A\right) \qquad \begin{cases} T_A & \text{ambient temp. in } {}^\circ C \\ T(t) & \text{temp. of object after time } t \\ k & \text{unknown positive constant} \end{cases}$$

This law is widely used in forensics.

The DE that has resulted is both separable and first order linear, so we can solve it in one of two ways.

**Example 13** *A dead body is discovered by a cleaner in a hotel room at* $11$ *am. A forensic scientist arrives at the scene at* $11.30$ *am and records the temperature of the body at* $24.5°C$*.*
*The body temperature is recorded again an hour later at* $24.0°C$*.*
*Estimate the time of death given that the hotel room is at constant* $20°C$ *and normal body temp. is* $36.5°C$*.*

SOLUTION: Treating Newton's Law as a separable DE:

$$\int_{T(t_1)}^{T(t_2)} \frac{dT}{T - T_A} = \int_{t_1}^{t_2} -k\,dt \quad \Rightarrow \quad \ln\left|\frac{T(t_2) - T_A}{T(t_1) - T_A}\right| = -k(t_2 - t_1)$$

So

$$T(t_2) = T_A + (T(t_1) - T_A)e^{-k(t_2 - t_1)}.$$

(Do not forget: $k$ here is positive.)

The next step is to estimate $k$:

Let $t_1 = 0$ at $11.30$ am and $t_2 = 60$ at $12.30$ pm with $T(t_1) = 24.5$, $T(t_2) = 24.0$ and $T_A = 20$.

So

$$k = \frac{1}{t_2 - t_1} \ln \left| \frac{T(t_1) - T_A}{T(t_2) - T_A} \right|$$

$$=$$

($k$ positive, as it needs to be.)

To find the time of death $t_0 < t_1$, we now set $T(t_0) = 36.5$ and substitute into

$$T(t_1) = T_A + (T(t_0) - T_A)e^{-k(t_1 - t_0)}$$

We solve this for $t_0$:

$$t_0 = t_1 - \frac{1}{k} \ln \left| \frac{T(t_0) - T_A}{T(t_1) - T_A} \right|$$

This is about before the scientist's initial measurement.

So the time of death was approximately

Mixing problems are common questions, and they typically come down to the same equation:

$$\text{rate of change of substance} = \text{rate coming in} - \text{rate going out}$$

Usually these rates on the right are either constant or of the form

$$\text{concentration} \times \text{flow rate},$$

and we assume perfect mixing.

It is usually best to set up an equation for an amount of a substance rather than its concentration, even if the problem is worded for concentration.

For a solid substance disolved in a liquid (for example brine, which is salt in water), the concentration is

$$\frac{\text{mass of solid}}{\text{volume of solution}}$$

For one liquid disolved in another (e.g. ethanol in water) we usually use volume percent, $\%$v$/$v, that is

$$\frac{\text{volume of solute}}{\text{volume of solution}}$$

because volumes of liquids are not usually additive.

**Example 14** *A $50$ litre tank is half full of a $10\%$v$/$v ethanol solution. A $5\%$v$/$v ethanol solution is run in at $3$ litres per minute, and the resulting mixture runs out at $2$ litres per minute.*
*What is the concentration of the solution at the point where the tank overflows?*

We work with the total amount of ethanol: let $x(t)$ be the amount of ethanol in the tank at time $t$, measured in minutes.

Our initial condition is then that

The DE is the usual one for mixing:

$$\frac{dx}{dt} = \text{rate coming in} - \text{rate going out}$$

The rate going in is easy: $3\,\ell$ of a $5\%$ solution means the inflow rate is

For the rate going out, we have $x\,\ell$ of ethanol in a total solution volume of litres, flowing out at $2\,\ell$ per minute, so the outflow rate is

Our IVP is then

Now is a linear DE, and its integrating factor is

Multiplying this through, the DE is

So

Putting in the initial condition gives

The solution to our IVP is thus

The tank overflows when the total volume is $50\,\ell$, that is after 25 minutes, and

$$x(25) =$$

The concentration at overflow is thus

Other common modelling problems involve populations.

The Notes discuss three common models:

1) Exponential growth: $\dfrac{dP}{dt} = kP$ for $k > 0$ (Malthus).

2) Bounded growth: $\dfrac{dP}{dt} = k(P_c - P)$.

3) Logistic growth: $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{K}\right)$ (Verhulst).

Malthusian exponential growth – which typically applies to micro-organisms – is straightforward.

The same equation (with $k < 0$) applies to radioactive decay.

Bounded growth also applies to e.g. diffusion of chemicals and uptake of technology.

Logistic growth also applies to e.g. the spread of diseases, rumours.

**Example 15** *Buggoil plc grows bacteria and harvests them to make oil. If the bacteria are not harvested the population $P$ follows the logistic equation:*

$$\frac{dP}{dt} = kP\left(P_c - P\right).$$

*where the company's scientists can modify $k$ and $P_c$ by genetic engineering, changing the feedstock etc. Buggoil want to harvest at a constant rate $H$ and need to know a maximum value of $H$ as a function of $k$ and $P_c$, and what that implies for the long term population.*

ANALYSIS: With harvesting, our DE will be

$$\frac{dP}{dt} = kP\left(P_c - P\right) - H.$$

The DE is separable, so not too hard to solve.

We begin by tidying up with a change of variables:

Let $P = (X + \frac{1}{2})P_c$, $t = T/(kP_c)$.

The DE then becomes (EXERCISE)

$$\frac{dX}{dT} = -X^2 + L, \quad \text{where} \quad L = \frac{1}{4} - \frac{H}{kP_c^2}$$

The shape of the solution will now depend on the sign of $L$.

One thing we can see without solving the DE is that if $L < 0$ (i.e. $H > kP_c^2/4$) then $dX/dT < 0$, so $dP/dt < 0$ and the bacteria will die out, so there is a limit to the harvest rate.

Suppose the initial population is $P_0$: since $t = 0$ when $T = 0$, the initial conditions for our new variables are

$$X = X_0 = \frac{P_0}{P_c} - \frac{1}{2} \quad \text{at} \quad T = 0.$$

The simple case is $L = 0$, i.e. $H = kP_c^2/4$, when the DE is
$\dfrac{dX}{dT} = -X^2$.
Separating and integrating:

$$\int -\frac{dX}{X^2} = \int dT \quad \Rightarrow \quad \frac{1}{X} = T + C,$$

and the initial conditions imply $C = 1/X_0$. So $X =$ .

Now $t \to \infty$ iff $T \to \infty$, and so in the long term if Buggoil harvest at $H = kP_c^2/4$ then $X \to 0$. So

QUESTION: is is sensible to harvest at this rate?

Now suppose that $L > 0$: let $L = \lambda^2$ (more tidying up).
Separating the DE we get

$$\int \frac{dX}{X^2 - \lambda^2} = -\int dT \;\Rightarrow\; \int \left( \frac{1}{X - \lambda} - \frac{1}{X + \lambda} \right) dX = \int -2\lambda \, dT$$

Integrating:

We could exponentiate and solve for $X$ (or $P$).
But we can answer the important questions without doing that.

We have

We see that as $T \to \infty$, the term inside the log must tend to

But

The maximum limiting population is clearly

If there is harvesting, as long as the rate is less than $kP_c^2/4$, the population settles down to

I will leave you to solve the $L < 0$ case explicitly and confirm that over-harvesting will kill off the bacteria.

Before we finish, we look again at the maximum sustainable harvesting rate:

$$H_{\text{max}} = \frac{kP_c^2}{4}$$

This maximum rate depends linearly on $k$ but quadratically on $P_c$.

So Buggoil's scientists get more for increasing the maximum sustainable (non-harvested) population $P_c$ than the growth rate $k$.

I'm not sure I expected that! ■

Our analysis will apply to similar situations where a population that naturally grows logistically is exploited at a constant rate: fishing and logging are obvious cases.

One last thing. Our new variables $X$ and $T$ (and $L$) have a happy property that makes what we have done especially useful: they are dimensionless.

What I mean by this is that if the populations are measured in numbers of bacteria (or thousands, or millions, or billions. . . ) then $X = P/P_c$ is a pure number.

Similarly, if $t$ is in seconds (say), $dP/dt$ must have units of numbers per second.

It follows that $k$ has dimensions of inverse (numbers $\times$ seconds), and so $T = kP_ct$ is also a pure number.

$H$ should have units numbers per second, and I'll leave you to check $L$ is a pure number.