MATH1241 Algebra, 2018
Group 2 — Tues 12 pm, Thurs 10 am

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Chapter 7 Linear transformations

We now study a special type of function between two vector spaces over the same set of scalars. Here V and W are vector spaces over the same set of scalars $\mathbb{F}$, but they may have different operations. Formally, $(V, +, *, \mathbb{F})$ and $(W, \oplus, \otimes, \mathbb{F})$.

For any function $T : V \to W$, we call V the **domain** and W the **codomain** of the function. (Be careful, the codomain of T is usually not the same as the range of T.)

A *linear transformation* is a function $T : V \to W$ satisfying two conditions.

Addition Condition

Addition Condition.
We say T satisfies the **addition condition** if

$$T(\mathbf{v}+\mathbf{v}') = T(\mathbf{v}) \oplus T(\mathbf{v}') \quad \text{for all} \quad \mathbf{v},\mathbf{v}' \in V.$$

That is

$$\begin{array}{ccc} \mathbf{v},\ \mathbf{v}' & \xrightarrow{\quad T \quad} & T(\mathbf{v}),\ T(\mathbf{v}') \\ +\big\downarrow & & \big\downarrow\oplus \\ \mathbf{v}+\mathbf{v}' & \xrightarrow{\quad T \quad} & T(\mathbf{v}+\mathbf{v}') = T(\mathbf{v}) \oplus T(\mathbf{v}') \end{array}$$

Informally, we end up with the same result no matter whether we add the vectors first or we perform the transformation first.

Scalar Multiplication Condition

Scalar Multiplication Condition.
We say T satisfies the **scalar multiplication condition** if

$$T(\lambda * \mathbf{v}) = \lambda \otimes T(\mathbf{v}) \quad \text{for all} \quad \lambda \in \mathbb{F} \text{ and } \mathbf{v} \in V.$$

That is

$$\begin{array}{ccc} \lambda,\ \mathbf{v} & \xrightarrow{\quad T \quad} & \lambda,\ T(\mathbf{v}) \\ *\big\downarrow & & \big\downarrow\otimes \\ \lambda * \mathbf{v} & \xrightarrow{\quad T \quad} & T(\lambda * \mathbf{v}) = \lambda \otimes T(\mathbf{v}) \end{array}$$

Informally, we end up with the same result no matter whether we perform the scalar multiplication first or we perform the transformation first.

Definition of a linear transformation

If there is no confusion about the operations used in V and W, we simply use the $+$ for addition in both V and in W, and we will omit the scalar multiplication sign.

Definition

Let V and W be two vector spaces over the same field $\mathbb{F}$. A function $T : V \to W$ is called a **linear map** or a **linear transformation** if the following two conditions are satisfied:

Addition Condition

$$T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}') \quad \text{for all } \mathbf{v}, \mathbf{v}' \in V,$$

and

Scalar Multiplication Condition

$$T(\lambda \mathbf{v}) = \lambda T(\mathbf{v}) \quad \text{for all} \quad \lambda \in \mathbb{F} \text{ and } \mathbf{v} \in V.$$

To prove that a function is linear.

Example

Show that the function $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = \begin{pmatrix} 4x_2 - 3x_3 \\ x_1 + 2x_2 \end{pmatrix} \quad \text{for} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$$

is a linear map.

Solution

Solution (Continued)

Solution (Continued)

Example

The function $T : \mathbb{R}^3 \longrightarrow \mathbb{P}_1$ is defined by

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a+2b) + (b-2c)x, \quad \text{for all } \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3.$$

Prove that T is linear.

Solution

Solution (Continued)

To prove that a function is not linear.

Proposition.

If $T : V \to W$ is a linear map, then $T(\mathbf{0}) = \mathbf{0}$.

Proof.

$\square$

That is, if $T(\mathbf{0})$ is not the zero vector in W then T is not linear.

Example

Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 - 2 \\ x_1 \end{pmatrix}$ is not linear.

Solution

However, $T(\mathbf{0}) = \mathbf{0}$ does not mean that T is linear.

Example

Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2^2 \end{pmatrix}$$

is not linear.

Solution

Properties of linear maps

Theorem

Suppose that V and W are vector spaces over $\mathbb{F}$. The function $T : V \to W$ is a linear map if and only if for all $\lambda_1, \lambda_2 \in \mathbb{F}$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$,

$$T(\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2) = \lambda_1 T(\mathbf{v}_1) + \lambda_2 T(\mathbf{v}_2).$$

Proof.

Proof (Continued).

$\square$

Theorem

If $T : V \to W$ is a linear map, $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a subset of V and $\lambda_1, \ldots, \lambda_n$ are scalars, then

$$T(\lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n) = \lambda_1 T(\mathbf{v}_1) + \cdots + \lambda_n T(\mathbf{v}_n).$$

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a function such that

$$T\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}, \quad T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}, \quad T\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}.$$

Show that T is not linear.

Solution

Solution (Continued)

Example

Given that T is a linear map and

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix},\ T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix},\ T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\3\end{pmatrix},$$

find $T\begin{pmatrix}x\\y\\z\end{pmatrix}$.

Solution

The previous example illustrates that we only need the function values of the standard basis of the domain to determine a linear map. More generally:

Theorem

For a linear map, the function values for every vector in the domain are known iff the function values for a basis of the domain are known.

Attempt Problems 7.1.

Matrices define linear maps

Theorem

For each $m \times n$ matrix A, the function $T_A : \mathbb{R}^n \to \mathbb{R}^m$, defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \qquad \textit{for} \quad \mathbf{x} \in \mathbb{R}^n,$$

is a linear map.

Proof.

$\square$

Example

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \end{pmatrix}$. Find the linear map T_A defined in the previous theorem.

Solution

Matrix Representation Theorem

Conversely, given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, we can find an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Example

Given that $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x - y \\ y \end{pmatrix}$ is linear. Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$.

Solution

What if a linear map is not defined by such a simple formula?

Theorem (Matrix Representation Theorem)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let the vectors $\mathbf{e}_j$ for $1 \leqslant j \leqslant n$ be the standard basis vectors for $\mathbb{R}^n$. Then the $m \times n$ matrix A whose columns are given by

$$\mathbf{a}_j = T(\mathbf{e}_j) \quad \text{for} \quad 1 \leqslant j \leqslant n$$

has the property that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

Proof.

□

Example

Use the Matrix Representation Theorem to find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 - 2x_2 + x_3 \\ 4x_2 + 3x_3 \end{pmatrix}.$$

Solution

Attempt Problems 7.2.

Theorem (General Matrix Representation Theorem)

Let $T : V \to W$ be a linear map where $dim(V) = n$ and $dim(W) = m$.

Fix an ordered basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for V and an ordered basis $C = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ for W.

Let A be the $m \times n$ matrix with columns $[T(\mathbf{v}_1)]_C, \ldots, [T(\mathbf{v}_n)]_C$. Then for all $\mathbf{x} \in V$,

$$[T(\mathbf{x})]_C = A\,[\mathbf{x}]_B$$

for $j = 1, \ldots, n$.

Example

The function $T : \mathbb{P}_2 \to M_{22}(\mathbb{R})$ given by

$$T(p) = \begin{pmatrix} p(0) & p(1) \\ p(2) & p(3) \end{pmatrix}$$

is linear. (Exercise: check!) Use the General Matrix Representation Theorem to find the matrix A which represents T with respect to the standard ordered bases

$$B = \{1, x, x^2\}, \qquad C = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

of $\mathbb{P}_2$ and $M_{22}(\mathbb{R})$, respectively.

Solution

Geometric examples

In this section we examine some of the geometric mappings which can be represented by linear maps and matrices.

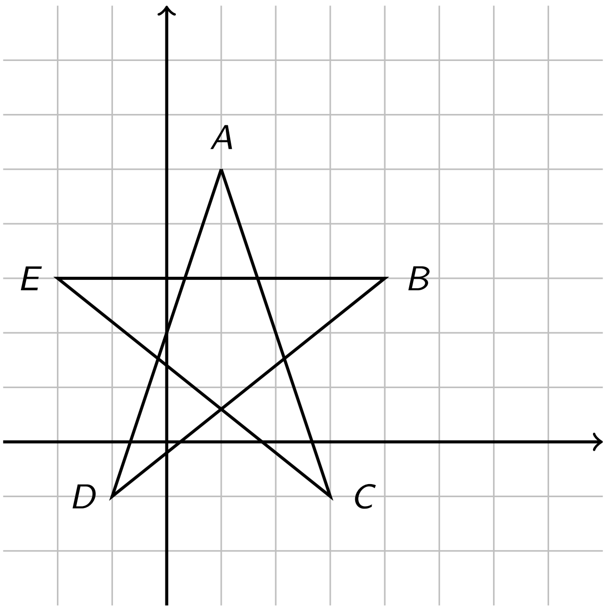
Example (Reflection)

Find the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ which maps $\mathbf{x}$ to $\mathbf{x}'$, where $\mathbf{x}$ (respectively, $\mathbf{x}'$) is the position vector of a point X (respectively, X') such that X' is the reflection of X in the xy-plane.
Prove that there exists a matrix A such that $A\mathbf{x} = T(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$.

Solution

Solution (Continued)

Stretching and Compressing



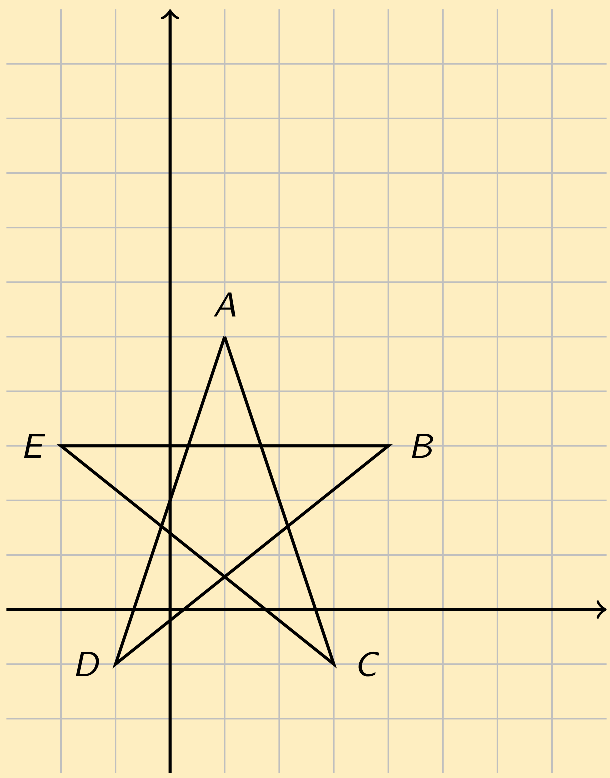
A 5-pointed star with vertices $A(1,5)$, $B(4,3)$, $C(3,-1)$, $D(-1,-1)$ and $E(-2,3)$.

Example

Find and draw the image of the 5-pointed star under the linear map T_M defined by the matrix $M = \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix}$.

Solution

Solution (Continued)



Rotation

Let R_α be the transformation which rotates the $\mathbb{R}^2$ plane through an angle α anticlockwise about the origin. Is R_α linear?

Let **a**, **b** be two vectors, and λ be a scalar. If we rotate **a** and **b** first then add up the results, we get the same vector as we rotate $\mathbf{a} + \mathbf{b}$. So the map satisfies the addition condition.

Similarly, if we rotate **a** first then multiply it by λ, we get the same result as we rotate $\lambda\mathbf{a}$. So the map also satisfies the scalar multiplication condition.

Hence R_α is a linear map.

Example

For the linear map R_α defined above, find the matrix A such that $A\mathbf{x} = R_\alpha(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^2$.

Solution

Example

Let $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the mapping defined by

$$T(\mathbf{x}) = \text{proj}_{\mathbf{b}}\mathbf{x}.$$

Show that T is linear and find the matrix A such that $A\mathbf{x} = T(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$.

Solution

Solution (Continued)

Attempt Problems 7.3.

Subspaces associated with linear maps

There are two very important sets associated with a linear map T.

Definition (Kernel and image of a linear transformation)

Let $T : V \to W$ be a linear map.

The **kernel** of T, written $\ker(T)$, is the set of all *vectors which map to zero under* T. That is,

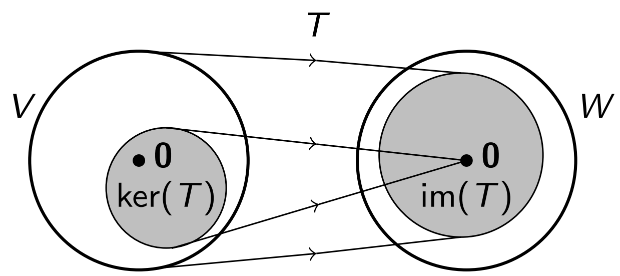
$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}.$$

The **image** of T, written $\mathrm{im}(T)$, is the set of all *function values of* T. That is,

$$\mathrm{im}(T) = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \quad \text{for some} \quad \mathbf{v} \in V\}.$$

The kernel is a subset of the domain V while the image is a subset of the codomain W. Later, we will prove that they are, in fact, subspaces.

Checking membership of the kernel (or image).



Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 - x_3 \\ x_1 + x_2 + x_3 \\ x_2 - 2x_3 \end{pmatrix}.$$

Is $\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$ in $\ker(T)$. How about $\mathrm{im}(T)$? Also try $\begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix}$.

Solution

Solution (Continued)

Solution (Continued)

Example

The function $T : \mathbb{R}^3 \longrightarrow \mathbb{P}_1$ defined by

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a+2b) + (b-2c)x, \quad \text{for all } \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$$

is linear. Find $\ker(T)$ and $\operatorname{Im}(T)$.

Solution

Solution (Continued)

Kernel and image of a matrix

We have seen that, for a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, the matrix A with columns $T(\mathbf{e}_i)$ represents T, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis for the domain. Conversely, for any matrix A, we can associate with it a linear map T_A such that $T_A(\mathbf{x}) = A\mathbf{x}$. This motives the following definition.

Definition (Kernel and image of a matrix)

The **kernel** of an $m \times n$ matrix A is the subset of $\mathbb{R}^n$ defined by

$$\ker(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

The **image** of an $m \times n$ matrix A is the subset of $\mathbb{R}^m$ defined by

$$\operatorname{im}(A) = \{\mathbf{b} \in \mathbb{R}^m : \mathbf{b} = A\mathbf{x} \quad \text{for some} \quad \mathbf{x} \in \mathbb{R}^n\}.$$

Definition (Kernel and image of a matrix)

The **kernel** of an $m \times n$ matrix A is the subset of $\mathbb{R}^n$ defined by

$$\ker(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

The **image** of an $m \times n$ matrix A is the subset of $\mathbb{R}^m$ defined by

$$\operatorname{im}(A) = \{\mathbf{b} \in \mathbb{R}^m : \mathbf{b} = A\mathbf{x} \quad \text{for some} \quad \mathbf{x} \in \mathbb{R}^n\}.$$

Suppose that A is the matrix representing T with respect to standard bases (or, equivalently, $T = T_A$ is the linear map associated with the matrix A, as defined above). Then $\ker(T) = \ker(A)$ and $\operatorname{im}(T) = \operatorname{im}(A)$.

Finding kernels and images of matrices.

Example

Let $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix}$. Find $\ker(A)$ and $\operatorname{im}(A)$.

Solution

Solution (Continued)

Solution (Continued)

Kernel and image are subspaces.

In the previous example, both $\ker(A)$ and $\mathrm{im}(A)$ are subspaces. This is true in general.

Theorem

Let $T : V \to W$ be a linear map. Then $\ker(T)$ is a subspace of the domain V and $\mathrm{im}(T)$ is a subspace of the codomain W.

Let A be an $m \times n$ matrix. Then $\ker(A)$ is a subspace of $\mathbb{R}^n$ and $\mathrm{im}(A)$ is a subspace of $\mathbb{R}^m$.

Proof.

Proof (Continued).

Proof (Continued).

$\square$

Rank and nullity

Since kernels and images are subspaces, we are also interested in their dimensions.

Definition

The **nullity** of a linear map T is the dimension of $\ker(T)$.
The **nullity** of a matrix A is the dimension of $\ker(A)$.

The **rank** of a linear map T is the dimension of $\operatorname{im}(T)$.
The **rank** of a matrix A is the dimension of $\operatorname{im}(A)$.

Example (Continued from the example on p.46)

Let $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix}$ as in the example on p.46.

a) Find a basis for $\ker(A)$, and $\operatorname{nullity}(A)$.

b) Find a basis for $\operatorname{im}(A)$, and $\operatorname{rank}(A)$.

Solution

Rank-Nullity Theorem

Remarks

Let A be an $m \times n$ matrix. Suppose that the columns of A are $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and A reduces to a row-echelon form matrix U.

1. $\ker(A)$ is the solution set of $A\mathbf{x} = \mathbf{0}$.
2. A basis for $\ker(A)$ is a basis for the solution set of $A\mathbf{x} = \mathbf{0}$.
3. The dimension of the solution set is the number of (independent) parameters used in the solution, so

$$\text{nullity}(A) = \text{the number of non-leading columns of } U.$$

4. $\text{im}(A)$ is the set of all vectors of the form

$$A\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n.$$

   Hence $\text{im}(A) = \text{col}(A)$.

5. A maximal set of linearly independent columns of A forms a basis for $\text{im}(A)$. In particular, the set of vectors which are columns of A corresponding to the leading columns of U is a basis for $\text{im}(A)$.
6. $\text{rank}(A) =$ the maximal number of independent columns of A $=$ the number of leading columns of U.

Theorem (Rank-Nullity Theorem for Matrices)

If A is an $m \times n$ matrix, then

$$rank(A) + nullity(A) = n.$$

Proof.

$\square$

In fact, the theorem is true for any linear map between finite dimensional vector spaces.

Theorem (Rank-Nullity Theorem for linear maps)

Suppose V and W are finite dimensional vector spaces. If $T : V \to W$ is a linear map then

$$rank(T) + nullity(T) = dim(V).$$

Example

Let $T : \mathbb{R}^3 \to \mathbb{R}$ be a linear map defined by

$$T(\mathbf{x}) = \mathbf{x} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

Find $\ker(T)$, and hence find $\text{nullity}(T)$ and $\text{rank}(T)$.

Solution

Solution (Continued)

Example

Prove that if $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear then the following are equivalent.

a) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\quad T(\mathbf{x}) = T(\mathbf{y}) \;\Leftrightarrow\; \mathbf{x} = \mathbf{y}$.

b) $\text{rank}(T) = n$.

Solution

Solution (Continued)

Rank, nullity and solutions of $A\mathbf{x} = \mathbf{b}$

Theorem

The equation $A\mathbf{x} = \mathbf{b}$ has:

1. *no solution if rank$(A) \neq$ rank$([A|\mathbf{b}])$, and*
2. *at least one solution if rank$(A) =$ rank$([A|\mathbf{b}])$. Further,*
   i) *If nullity$(A) = 0$ then the solution is unique.*
   ii) *If nullity$(A) = \nu > 0$ then the general solution is of the form*

$$\mathbf{x} = \mathbf{x}_p + \lambda_1\mathbf{k}_1 + \cdots + \lambda_\nu\mathbf{k}_\nu \quad \text{for } \lambda_1, \ldots, \lambda_\nu \in \mathbb{R},$$

where $\{\mathbf{k}_1, \ldots, \mathbf{k}_\nu\}$ is a basis for ker(A), and $\mathbf{x}_p$ is any solution of $A\mathbf{x} = \mathbf{b}$.

Example

Let A be a 5×4 matrix with real entries, and let $\mathbf{b}$ be the second column of A. It is given that

$$\ker(A) = \left\{ \mathbf{x} : \mathbf{x} = \mu \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \text{ for some } \mu \in \mathbb{R} \right\}.$$

a) Find a solution $\mathbf{x}_p$ to $A\mathbf{x} = \mathbf{b}$.

b) Find the general solution to $A\mathbf{x} = \mathbf{b}$. Give a geometric interpretation of the general solution.

c) Find $\text{rank}(A)$. Give reasons.

Solution

Solution (Continued)

Attempt Problems 7.4.

Matrix arithmetic and linear maps

Suppose that $S, T : \mathbb{R}^n \to \mathbb{R}^m$, such that

$$S(\mathbf{x}) = A\mathbf{x} \qquad \text{and} \qquad T(\mathbf{x}) = B\mathbf{x}$$

for all $x \in \mathbb{R}^n$. That is, A is the matrix which represents S and B is the matrix which represents T (with respect to the standard bases of $\mathbb{R}^n$, $\mathbb{R}^m$).

Facts:

With respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$,

- The matrix which represents $S + T$ is $A + B$.
- The matrix which represents λS is λA, for any $\lambda \in \mathbb{F}$.

Theorem (Composition of linear maps)

(i) *Suppose that V, W and U are vector spaces over a field $\mathbb{F}$. If $T : V \to W$ and $S : W \to U$ are linear maps then the composition*

$$S \circ T : V \to U$$

is a linear map.

(ii) *Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ are linear maps. Let A, B be the matrices which represent S and T with respect to the standard bases. Then the matrix which represents $S \circ T$ is AB.*

Proof.

Proof, continued.

$\square$

Vector space isomorphism

Recall that a function $f : X \to Y$ is **one-to-one** if
for all $y \in Y$ there is at most one $x \in X$ such that $f(x) = y$.

Also recall that f is **onto** if
for all $y \in Y$ there is at least one $x \in X$ such that $f(x) = y$.

If f is both one-to-one and onto then it is a **bijection**.

Definition

Suppose that V, W are vector spaces over a field $\mathbb{F}$ and $T : V \to W$ is a linear map. If T is also a bijection then we say that T is a **vector space isomorphism**, and we say that V and W are **isomorphic**.

Example

Let V be a vector space over a field $\mathbb{F}$ with ordered basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Let $T : V \to \mathbb{F}^n$ be defined by

$$T(\mathbf{x}) = [\mathbf{x}]_B,$$

which maps $\mathbf{x} \in V$ to its coordinate vector $[\mathbf{x}]_B$.
Then T is a vector space isomorphism.

Hence any n-dimensional vector space over $\mathbb{F}$ is isomorphic to $\mathbb{R}^n$ (!!!).

Proof.

Proof, continued.

□

End of Chapter 7