MATH1241 Algebra, 2018

Group 2 — Tues 12 pm, Thurs 10 am

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Acknowledgement: Lectures based on Dr. Chi Mak's notes.

Chapter 8 Eigenvalues and Eigenvectors

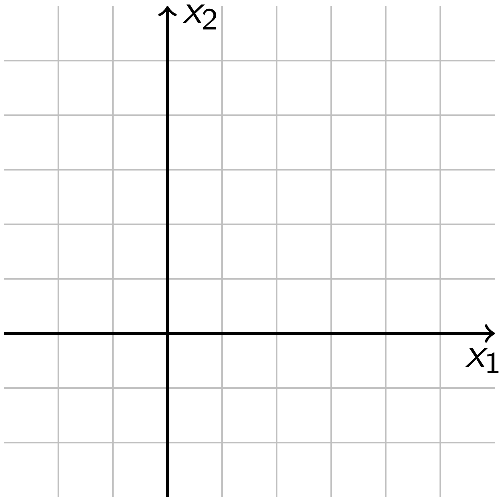
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. If we restrict the domain to a subspace S of $\mathbb{R}^n$, the restricted function is also a linear map. (Why?) We also know that the image of a linear map is a subspace of the codomain. Hence, a linear map will map subspaces to subspaces.

Furthermore, subspaces of $\mathbb{R}^n$ are $\{\mathbf{0}\}$, lines through the origin, planes through the origin, etc. In particular, the image of a line through the origin under a linear transformation is either the origin or a line through the origin. We are interested in a line which has itself or the origin as its image under a given linear transformation.

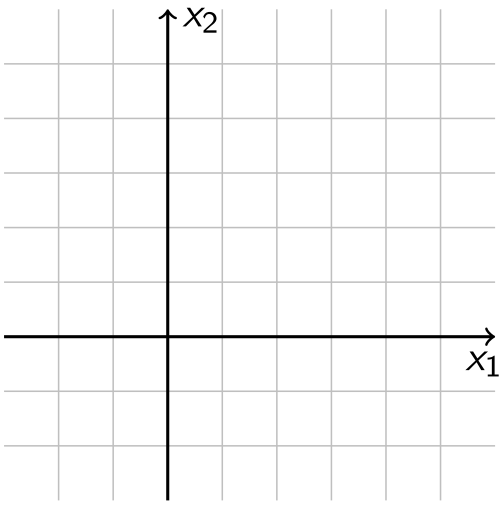
Example

Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$. For each of the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, sketch the span of the vector and the span of the image under A.

Solution



Solution (continued)



Multiplication by A maps the line spanned by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to itself.

Multiplication by A also maps the line spanned by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ to itself.

The vectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are called eigenvectors of A.

Definition of eigenvectors and eigenvalues

Definition (for linear maps)

Suppose that V is a vector space over $\mathbb{F}$. Let $T : V \to V$ be a linear map. If a scalar $\lambda \in \mathbb{F}$ and a non-zero vector $\mathbf{v} \in V$ satisfy

$$T(\mathbf{v}) = \lambda \mathbf{v},$$

then $\lambda \in \mathbb{F}$ is called an **eigenvalue** of T and $\mathbf{v}$ is called an **eigenvector** of T for the eigenvalue λ.

Note: The domain and codomain are the same vector space.

Definition (for matrices)

Let A be an $n \times n$ square matrix. If a scalar $\lambda \in \mathbb{F}$ and non-zero vector $\mathbf{x} \in \mathbb{F}^n$ satisfy

$$A\mathbf{x} = \lambda \mathbf{x},$$

then λ is called an **eigenvalue** of A and $\mathbf{x}$ is called an **eigenvector** of A for the eigenvalue λ.

Example

For the eigenvectors of A in the previous example, find the corresponding eigenvalues.

Solution

Remarks.

1. When the set of scalars is $\mathbb{C}$, there may be non-real eigenvalues.
2. The vector $\mathbf{0}$ cannot be an eigenvector, but the scalar 0 can be an eigenvalue.
3. Any non-zero vector in $\ker(A)$ is an eigenvector with eigenvalue 0.
4. There are infinitely many eigenvectors with the same eigenvalue, but there is only one eigenvalue for an eigenvector.

Example

Let V be the vector space of all differentiable real functions, and let $T : V \to V$ be given by $T(f) = f'$ (differentiation).
For any $\lambda \in \mathbb{R}$, the function $f(x) = e^{\lambda x}$ satisfies

$$f'(x) = \lambda f(x) \text{ for all } x \in \mathbb{R}.$$

That is, $T(f) = f' = \lambda f$. So f is an eigenvector of T with eigenvalue λ.
Note, T has infinitely many distinct eigenvalues.

Example

Recall the rotation map $R_{\pi/2}$ in $\mathbb{R}^2$. No non-zero vector $\mathbf{v}$ can satisfy $R_{\pi/2}(\mathbf{v}) = \lambda\mathbf{v}$ for any real λ. The rotation map $R_{\pi/2}$ has no real eigenvalues.

Finding eigenvalues and eigenvectors

Because of the Matrix Representation Theorem, we shall focus only on matrices. Only square matrices have eigenvalues and eigenvectors.

Theorem

A scalar λ is an eigenvalue of a square matrix A if and only if $\det(A - \lambda I) = 0$.
Furthermore, $\mathbf{v}$ is an eigenvector of A for the eigenvalue λ if and only if $\mathbf{v}$ is a non-zero solution of the homogeneous equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$, that is, if and only if $\mathbf{v} \in \ker(A - \lambda I)$ and $\mathbf{v} \neq \mathbf{0}$.

Proof.

Proof (continued).

$\square$

Theorem

- *If A is an $n \times n$ matrix over $\mathbb{C}$ and $\lambda \in \mathbb{C}$, then $\det(A - \lambda I)$ is a polynomial of degree n in λ. The polynomial $\det(A - \lambda I)$ is called the* **characteristic polynomial** *for A.*
- *An $n \times n$ matrix always has n eigenvalues in $\mathbb{C}$ (not necessarily distinct).*

2×2 matrices

Example

Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$.

Solution

Solution (continued)

Solution (continued)

In this example, the eigenvalues are distinct. We have two independent eigenvectors, $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Example

Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Solution

Solution (continued)

In this example, there is a repeated eigenvalue. We still have two independent eigenvectors, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Example

Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$.

Solution

Solution (continued)

In this example, there is a repeated eigenvalue. However, the maximal number of independent eigenvectors is 1.

Example

Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Solution

Solution (continued)

Solution (continued)

In this example, the eigenvalues are distinct but they are not real. There are two independent eigenvectors, $\begin{pmatrix} -i \\ 1 \end{pmatrix}$, and $\begin{pmatrix} i \\ 1 \end{pmatrix}$.

Larger square matrices

Example

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} -3 & 4 & 2 & -3 \\ 2 & 12 & -4 & 2 \\ -5 & 12 & 2 & -1 \\ -15 & 4 & 2 & 9 \end{pmatrix}.$$

Solution (Use Maple!)

Solution (continued)

Diagonalisation

Why eigenvectors?

- Suppose that there are n linearly independent eigenvectors for an $n \times n$ matrix A. Then there is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.
- Suppose that there are n linearly independent eigenvectors for a linear transformation T of an n-dimensional vector space V. Then there is a basis for V consisting of eigenvectors of T.
- Using a basis of eigenvectors together with the eigenvalues, it is easy to find the image of a vector under the transformation.

Example

Suppose that the $n \times n$ matrix A has n eigenvectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and the eigenvalues are respectively $\lambda_1, \ldots, \lambda_n$.
Find $A\mathbf{v}$, where $\mathbf{v} = k_1\mathbf{v}_1 + \cdots + k_n\mathbf{v}_n$.

Solution

Independence of eigenvectors

Theorem

If an $n \times n$ matrix has n distinct eigenvalues then it has n linearly independent eigenvectors.

Is the converse of this theorem true?

Example

Suppose that A is a 3×3 matrix with eigenvalues 1, 3, -2 and corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Explain why the eigenvectors are linearly independent.

Find $A(2\mathbf{v}_1 - \mathbf{v}_2 - 2\mathbf{v}_3)$, $A^2(2\mathbf{v}_1 - \mathbf{v}_2 - 2\mathbf{v}_3)$ and also $A^n(2\mathbf{v}_1 - \mathbf{v}_2 - 2\mathbf{v}_3)$.

Solution

Continuing from the previous example, let $M = (\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3)$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. Then

$$\begin{aligned} AM &= A(\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3) = (A\mathbf{v}_1|A\mathbf{v}_2|A\mathbf{v}_3) = (\mathbf{v}_1|3\mathbf{v}_2|-2\mathbf{v}_3) \\ &= (\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} = MD. \end{aligned}$$

Since the columns of M are linearly independent, the matrix M is invertible. Hence, we have

$$M^{-1}AM = D \text{ or } A = MDM^{-1}.$$

This diagonal matrix D is closely related to the linear transformation T_A. (In fact, D is a matrix for T_A with respective to the basis of the eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.)
The argument holds in general for any $n \times n$ matrix A with independent eigenvectors.

Theorem (Diagonalisation)

If an $n \times n$ matrix A has n linearly independent eigenvectors, then there exists an invertible matrix M and a diagonal matrix D such that

$$M^{-1}AM = D.$$

Further, the diagonal elements of D are the eigenvalues of A, and the jth column of M is an eigenvector of A with the jth element of the diagonal of D as eigenvalue.

Conversely, if $M^{-1}AM = D$ with D diagonal then the columns of M form a set of n linearly independent eigenvectors of A.

Definition

A square matrix A is said to be a **diagonalisable matrix** if there exists an invertible matrix M and diagonal matrix D such that $M^{-1}AM = D$.

Example

Are the matrices $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ diagonalisable?

For each of the matrix which is diagonalisable, find the diagonal matrix D and the corresponding invertible matrix M which satisfy the condition in the previous theorem.

Solution

Solution (continued)

Solution (continued)

Powers of a matrix

When a matrix A is diagonalisable, we can find invertible matrix M and diagonal matrix D such that

$$M^{-1}AM = D \text{ or } A = MDM^{-1}.$$

Hence, we can find A^n by

$$A^n = \left(MDM^{-1}\right)^n = \left(MDM^{-1}\right)\cdots\left(MDM^{-1}\right) = MD^nM^{-1}.$$

Example

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$. Find A^5 and A^n.

Solution

Solution (continued)

Solution (continued)

System of first order differential equations

Another application of the eigenvalues and eigenvectors is finding the solutions to a system of first order linear differential equations.

Example

A species of bird and a species of tree enjoy a symbiotic relationship. That is, the tree provides shelter for the bird and food in the form of berries, while the bird supplies the means for the tree to distribute it seeds over a large territory. Hence the rate of increase of birds depends not only on how many birds there are, but also on the number of trees and the number of tree depends not only on the number of seeds produced but also on the number of birds.

Suppose that the rate of increase of birds is numerically the sum of the number of birds and twice the number of trees, while the rate of increase of trees is numerically the sum of three times the number of birds and twice the number of trees.

Solution

If we let $x(t)$ be the number of birds at time t and $y(t)$ be the number of trees at time t, then the rates of change are governed by the system of differential equations.

$$\begin{cases} \dfrac{dx}{dt} = \\ \dfrac{dy}{dt} = \end{cases}$$

To solve the system by matrix method, we begin by writing the system in matrix form. Let $\mathbf{y} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A =$, so

$$\mathbf{y}' = \begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} \quad & \quad \end{pmatrix} \mathbf{y} \Rightarrow \mathbf{y}' = A\mathbf{y}.$$

We know that $y = \alpha e^{kt}$ is a solution of the equation $y' = ky$. We guess that the solution to $\mathbf{y}' = A\mathbf{y}$ might look like

$$\mathbf{y} = \mathbf{v}e^{\lambda t} = \begin{pmatrix} ue^{\lambda t} \\ ve^{\lambda t} \end{pmatrix}, \quad \text{where } \mathbf{v} = \begin{pmatrix} u \\ v \end{pmatrix},$$

and u, v, λ are constants, i.e., a product of $e^{\lambda t}$ and a constant vector. Hence, by differentiating the entries we obtain

$$\mathbf{y}' = \begin{pmatrix} \lambda ue^{\lambda t} \\ \lambda ve^{\lambda t} \end{pmatrix} = \lambda \mathbf{v}e^{\lambda t}.$$

Substituting into $\mathbf{y}' = A\mathbf{y}$, we have

$$\lambda \mathbf{v}e^{\lambda t} = A\mathbf{v}e^{\lambda t}.$$

Simplifying this we obtain $A\mathbf{v} = \lambda\mathbf{v}$, so our guess will be correct provided that we choose $\mathbf{v}$ to be an eigenvector and λ to be an eigenvalue.

Conversely, if $\mathbf{v}$ is an eigenvector of A with eigenvalue λ then $\mathbf{y} = \alpha\mathbf{v}e^{\lambda t}$ is a solution of the equation $\mathbf{y}' = A\mathbf{y}$ because $\mathbf{y}' = \alpha\lambda\mathbf{v}e^{\lambda t} = A\mathbf{y}$.

Theorem

Let A be an $n \times n$ matrix. Then $\mathbf{y}(t) = \mathbf{v}e^{\lambda t}$ is a solution of $\mathbf{y}' = A\mathbf{y}$ if and only if λ is an eigenvalue of A and $\mathbf{v}$ is an eigenvector for the eigenvalue λ.

In particular, if $\lambda_1, \ldots, \lambda_n$ are n distinct eigenvalues and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the n corresponding eigenvectors, then the general solution to $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y}(t) = \alpha_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + \alpha_n e^{\lambda_n t}\mathbf{v}_n.$$

Solution (of the birds/trees example)

Solution (continued)

Example

Find the general solution to

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y} \quad \text{for} \quad A = \begin{pmatrix} -3 & 4 & 2 & -3 \\ 2 & 12 & -4 & 2 \\ -5 & 12 & 2 & -1 \\ -15 & 4 & 2 & 9 \end{pmatrix}.$$

Solution

Example

Find the population of the birds and trees for the birds/trees example, provided that initially there are 100 birds and 1000 trees.

Solution

We can turn a second order linear differential equation with constant coefficients into a system of first order equations with constant coefficients.

Example

Solve $y'' + 4y' - 5y = 0$ by letting

$$y_1 = y \quad \text{and} \quad y_2 = y_1' = y',$$

then write the differential equation as the system of first order differential equations. Hence find the general solution to the second order differential equation.

Solution

Solution (continued)

Solution (continued)

Markov chains

A Markov chain is a time-dependent process on a finite set S of states, such that the probability of going from state j to state i (in one step) is given by the (i, j)th entry of a matrix P.

Example

Consider Sydney, Melbourne and Everywhere Else. Assume 0% population growth from births, deaths, and that each year

- 5% of Sydney people move to Melbourne and 3% move elsewhere.
- 4% of Melbourne people move to Sydney and 2% move elsewhere.
- Of everyone else, 7% move to Sydney and 6% move to Melboune.

Represent this population model using a Markov chain with matrix

$$P = \begin{pmatrix} 0.92 & 0.04 & 0.07 \\ 0.05 & 0.94 & 0.06 \\ 0.03 & 0.02 & 0.87 \end{pmatrix}.$$

What is the equilibrium population ratio?

Solution

Solution (continued)

Fact: P has eigenvalue 1.

Proof.

$\square$

In fact, the largest eigenvalue is $\lambda = 1$ and the corresponding eigenvector is a stable equilibrium.

End of Chapter 8