Chapter 2

Integration Techniques

2.1 Trigonometric Integrals

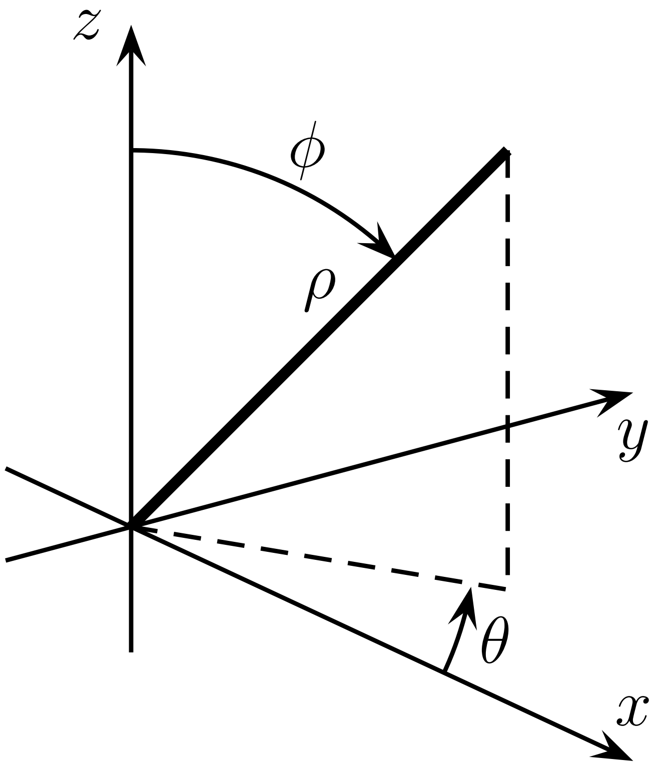
$$\int \sin^3 x \cos^6 x \, dx = -\frac{1}{9} \sin^2 x \cos^7 x - \frac{2}{63} \cos^7 x + C$$

$$\begin{aligned} \int \cos^8 x \, dx = & \frac{1}{8} \cos^7 x \sin x + \frac{7}{48} \cos^5 x \sin x \\ & + \frac{35}{192} \cos^3 x \sin x + \frac{35}{128} \cos x \sin x \\ & + \frac{35}{128} x + C \end{aligned}$$

$$\begin{aligned} \int \tan^2 x \sec^3 x \, dx = & \frac{1}{4} \tan^3 x \sec x + \frac{1}{8} \tan^2 x \sin x + \frac{1}{8} \sin x \\ & - \frac{1}{8} \ln |\sec x + \tan x| + C \end{aligned}$$

Who cares?

Many problems in three dimensional space can be simplified using spherical polar co-ordinates:



$$
\begin{aligned}
x &= \rho \cos\theta \sin\phi \\
y &= \rho \sin\theta \sin\phi \\
z &= \rho \cos\phi
\end{aligned}
$$

Integrals over powers of trig functions then arise, e.g. volume calculations, charge distributions, acoustic vibrations, angular momentum of electrons, trajectories of falling space junk. . .

Images from http://orbitaldebris.jsc.nasa.gov

2.1.1 Powers of Sine and Cosine

Our first type of integral are those of the form

$$\boxed{\int \cos^p x \sin^q x \, dx}$$

There are two cases:

Case (i): p odd

(1) factor out $\cos x$ and use the substitution $u = \sin x$.

(2) Eliminate remaining even powers of $\cos x$, using

$$\cos^{2k} x = (1 - \sin^2 x)^k = (1 - u^2)^k$$

(3) Integrate the polynomial in u

If p is even but q is odd, use an analogous method with roles of $\sin x$ and $\cos x$ interchanged.

Example 1 *Find* $\int \cos^3 x \sin^4 x \, dx$

SOLUTION:

$$\int \cos^3 x \sin^4 x \, dx = \int \cos^2 x \sin^4 x \, (\cos x) \, dx$$

Example 2 *Find* $\int_0^{\pi/2} \sin^3 x \cos^2 x \, dx$

SOLUTION:

$$\int_0^{\pi/2} \sin^3 x \cos^2 x \, dx = \int_0^{\pi/2} \sin^2 x \cos^2 x \, (\sin x) \, dx$$

Example 3 *Find* $\int \sin^7 x \, dx$

SOLUTION:

$$\int \sin^7 x \, dx = \int \sin^6 x \, (\sin x) \, dx$$

Case (ii): p, q both even

a) use the identities

$$\cos^2 x = \frac{1+\cos 2x}{2} \quad \text{and} \quad \sin^2 x = \frac{1-\cos 2x}{2}$$

to change the integral into a sum of integrals of the form

$$\int \cos^j 2x \, dx.$$

b) If j is odd, we apply the method of case (i).

c) If j is even, we apply the method of case (ii) again.

Example 4 *Find* $\int \sin^2 x \cos^2 x \, dx$

SOLUTION:

$$\int \sin^2 x \cos^2 x \, dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right) dx$$

2.1.2 Products of multiple angles

Our second type of integral are those of the forms

$$\boxed{\int \sin px \cos qx \, dx} \qquad \boxed{\int \sin px \sin qx \, dx} \qquad \boxed{\int \cos px \cos qx \, dx}$$

The key here is to use the product-to-sum formulas:

$$\begin{aligned}
\sin A \cos B &= \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right] \\
\sin A \sin B &= \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \\
\cos A \cos B &= \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]
\end{aligned}$$

See the Notes about the proofs.

Example 5 *Evaluate* $\int \sin 3x \sin 2x \, dx$

SOLUTION:

$$\int \sin 3x \sin 2x \, dx = \frac{1}{2} \int \left[\cos(3x - 2x) - \cos(3x + 2x)\right] \, dx$$

2.1.3 Power of $\tan$ and $\sec$

We next look at integrals of the form

$$\boxed{\int \tan^p x \sec^q x \, dx,} \qquad q \neq 0$$

The approach depends on the parity of p and q:

a) If q is even, separate off a $\sec^2 x$ and substitute $u = \tan x$.
   Then as $du = \sec^2 x \, dx$, we can use $\sec^2 x = 1 + \tan^2 x$ to eliminate the remaining even powers of $\sec x$.

b) If p is odd, separate off a $\tan x \sec x$ and substitute $u = \sec x$.
   Then as $du = \sec x \tan x \, dx$, we can use $\tan^2 x = \sec^2 x - 1$ to eliminate the remaining even powers of $\tan x$

Before we look at the last case, some examples:

Example 6 *Evaluate* $\int \tan^2 x \sec^4 x \, dx$.

SOLUTION: The power of the secant is even, so

$$\int \tan^2 x \sec^4 x \, dx = \int (\tan^2 x \sec^2 x) \sec^2 x \, dx$$

Example 7 *Evaluate* $\int \tan^3 x \sec^3 x \, dx$.

SOLUTION: The power of the tangent is odd, so

$$\int \tan^3 x \sec^3 x \, dx = \int (\tan^2 x \sec^2 x) \tan x \sec x \, dx$$

$$\boxed{\int \tan^p x \sec^q x \, dx}$$

(c) If $p = 2k$ is **even** and q is **odd** then the integral is

$$\int \tan^p x \sec^q x \, dx = \int (\sec^2 x - 1)^k \sec^q x \, dx$$

which yields a sum of integrals of the form

$$I_j = \int \sec^j x \, dx$$

If j were even we would have case (a), but it is not hard to see that here the powers we get are all odd.

This is case (d):

(d) The two cases $q = 0$ and $p = 0, q$ odd, that is

$$\int \tan^p x \, dx \qquad \text{and} \qquad \int \sec^{(2k+1)} x \, dx$$

both require **reduction formulae**.

Three particular special cases are worth noting:

$$\int \tan^2 x \, dx = \tan x - x + C$$

$$\int \tan x \, dx = \ln |\sec x| + C$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

See Notes or problems.

2.2 Reduction Formulae

A **reduction formula** for an integral is an expression for an integral depending on some integer, n say, in terms of a similar integral involving a smaller integer.

We normally find them using integration by parts.

We can then re-apply the formula to reduce the integer n to get to a case we can integrate directly.

It's probably easiest to see what I mean with an example.

Example 8 *Let* $I_n = \int x^n e^x \, dx$*.*

Then integrating by parts gives

$$I_n = x^n e^x - n \int x^{n-1} e^x \, dx = x^n e^x - n I_{n-1}.$$

Clearly $I_0 = e^x + C$*.*

So

$$\begin{aligned} \int x^3 e^x \, dx = I_3 &= x^3 e^x - 3I_2 \\ &= x^3 e^x - 3(x^2 e^x - 2I_1) \\ &= x^3 e^x - 3x^2 e^x + 6(x e^x - I_0) \\ &= \left(x^3 - 3x^2 + 6x - 6\right) e^x + C \end{aligned}$$

Example 9 *Find a reduction formula for* $I_n = \int \tan^n x\, dx$

SOLUTION:

$$\begin{aligned} I_n &= \int \tan^{n-2} x (\tan^2 x) dx \\ &= \int \tan^{n-2} x (\sec^2 x - 1)\, dx \\ &= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx \end{aligned}$$

Reduction formula: $\boxed{I_n = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2},} \quad n \geq 2$

with

$I_1 = \int \dfrac{\sin x}{\cos x}\, dx = \ln|\sec x| + C$ and $I_0 = \int dx = x + C$.

Applying this to the definite integral

$$\int_0^{\pi/4} \tan^n x \, dx = J_n$$

we get reduction formula $J_n = \dfrac{1}{n-1} - J_{n-2}$, $n \geq 2$. So

$$J_5 = \frac{1}{4} - J_3$$

$$=$$

Example 10 *Find a reduction formula for* $I_n = \int \sec^n x \, dx$

SOLUTION:

$$I_n = \int \sec^{n-2} x \ \overbrace{\sec^2 x}^{\frac{d}{dx}(\tan x)} \ dx$$

(by parts) $= \sec^{n-2} x \tan x - \int (\tan x)(n-2) \sec^{n-3} x \sec x \tan x \, dx$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \ (\sec^2 x - 1) \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}.$$

Thus, if we solve for I_n, we obtain the reduction formula

$$\boxed{I_n = \frac{\sec^{n-2} x \tan x}{(n-1)} + \frac{(n-2)}{(n-1)} I_{n-2}}$$

for $n \geq 2$.

The initial conditions are

$$I_1 = \int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$I_0 = \int dx = x + C \qquad \blacksquare$$

EXERCISE: Show that

$$\int_0^{\pi/4} \sec^5 x \, dx = \frac{7}{8}\sqrt{2} + \frac{3}{8} \ln\left(1 + \sqrt{2}\right)$$

Example 11 *Find a reduction formula for $J_n = \int \sin^n x\, dx$.*

SOLUTION: Integrate by parts with $u = \sin^{n-1} x$ and $\dfrac{dv}{dx} = \sin x$ to obtain

$$J_n = -\sin^{n-1} x \cos x + (n-1)(J_{n-2} - J_n)$$

If we set

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

then

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2} \quad n \geq 2$$

$$I_0 = \frac{\pi}{2}, \quad I_1 = 1.$$

We can also use the same technique to create reduction formulae involving two parameters.

Example 12 *Find a reduction formula for*

$$I_{p,q} = \int x^p (\ln x)^q \, dx, \qquad p, q \geq 0$$

SOLUTION: If $q \geq 1$ then

$$I_{p,q} = \frac{x^{p+1}}{p+1}(\ln x)^q - \frac{1}{p+1}\int x^{p+1} q (\ln x)^{q-1} \frac{1}{x}\, dx$$

$$=$$

$$I_{p,0} = \int x^p \, dx$$

Rather more useful is the following:

$$I_{m,n} = \int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta = \begin{cases} \left(\dfrac{m-1}{m+n}\right) I_{m-2,n} & \text{if} \quad m \geq 2 \\[2ex] \left(\dfrac{n-1}{m+n}\right) I_{m,n-2} & \text{if} \quad n \geq 2 \end{cases}$$

With base values

$$I_{1,1} = \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta = \frac{1}{2}, \qquad I_{1,0} = \int_0^{\pi/2} \cos\theta \, d\theta = 1,$$

$$I_{0,1} = \int_0^{\pi/2} \sin\theta \, d\theta = 1, \qquad I_{0,0} = \int_0^{\pi/2} d\theta = \frac{\pi}{2}.$$

Proof: See Notes

Example 13 *Find* $\int_0^{\frac{\pi}{2}} \cos^3 x \sin^4 x \, dx$

$$\int_0^{\frac{\pi}{2}} \cos^3 x \sin^4 x \, dx = I_{3,4}$$

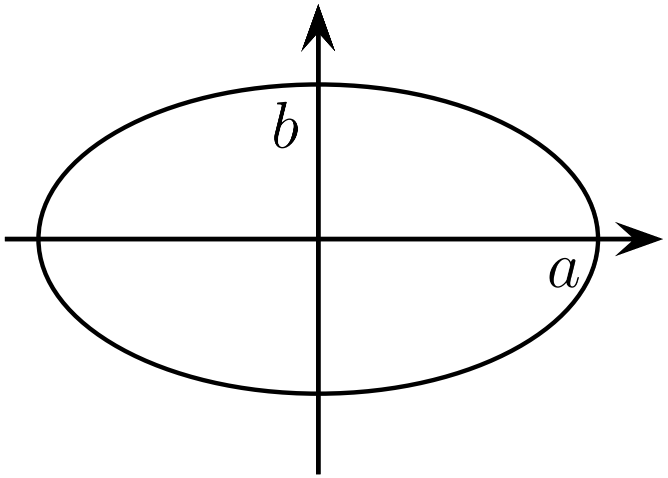
2.3.3 Trigonometric and hyperbolic substitution

Integrals involving square roots of quadratics are quite common and often yield to a substitution by a trigonometric or hyperbolic function.

integral factor	substitutions		identities
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$dx = a\cos\theta\, d\theta$	$\cos^2\theta + \sin^2\theta = 1$
	$x = a\tanh\theta$	$dx = a\,\mathrm{sech}^2\,\theta\, d\theta$	$\mathrm{sech}^2\,\theta + \tanh^2\theta = 1$
$\sqrt{a^2 + x^2}$	$x = a\sinh\theta$	$dx = a\cosh\theta\, d\theta$	
	$x = a\tan\theta$	$dx = a\sec^2\theta\, d\theta$	$\sec^2\theta = \tan^2\theta + 1$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$dx = a\sec\theta\tan\theta\, d\theta$	
	$x = a\cosh\theta$	$dx = a\sinh\theta\, d\theta$	$\cosh^2\theta = \sinh^2\theta + 1$

Example 14 *Find the area inside the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

SOLUTION:



By symmetry, this is 4 times the area under

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

between $x = 0$ and $x = a$. Integrating this will need the substitution $x = a\sin\theta$

So

$$A = 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

Example 15 *Evaluate* $I = \int \frac{dx}{(4+x^2)^{3/2}}$ *using a hyperbolic substitution.*

SOLUTION: We need to set $x = 2\sinh u$. Thus

$$\int \frac{dx}{(4+x^2)^{3/2}} =$$

Example 16 *Find* $I = \int_0^4 x^3 \sqrt{16 - x^2}\, dx$

SOLUTION: Set $x = 4\sin\theta$.

Then $0 \le \theta \le \frac{\pi}{2}$ since $0 \le x \le 4$.

Now in our range $\cos\theta$ is positive, so $\sqrt{16 - x^2} = +4\cos\theta$.

Thus

$$I = \int_0^{\pi/2} (4\sin\theta)^3 (4\cos\theta)\,(4\cos\theta\, d\theta)$$

Example 17 *Find* $I = \int \frac{\ln(x+\sqrt{x^2-1})}{x^2}\,dx$

SOLUTION: Given the radical and the fact that we have a log, it is more natural to use a hyperbolic substitution: put $x = \cosh u$.

Then

$$I = \int \ln(\cosh u + \sinh u)\frac{\sinh u\,du}{\cosh^2 u} = \int u\frac{\sinh u}{\cosh^2 u}\,du.$$

Note the convenient inversion formula: $u = \ln(x+\sqrt{x^2-1})$.

We can now integrate by parts, as $\int \frac{\sinh u}{\cosh^2 u}\,du = -\operatorname{sech} u$. So

$$I = -u\operatorname{sech} u + \int \operatorname{sech} u\,du.$$

Now $\int \operatorname{sech} u \, du$ is a (slightly obscure) standard integral. Or

$$\int \operatorname{sech} u \, du = \int \frac{2}{e^u + e^{-u}} \, du = 2 \int \frac{e^u}{e^{2u} + 1} \, du$$

$\blacksquare$

ASIDE: Maple will not do this integral, even if you tell it $x > 1$.

Example 18 *An object falling radially into a non-rotating black hole of mass M is forced to the singularity at $r = 0$ once it has passed the event horizon at $r = 2m$, where $m = MG/c^2$ (G is Newton's gravitational constant and c the speed of light).*

It can be shown that the speed of the object is bounded below by $c\left(\frac{2m}{r} - 1\right)^{1/2}$ once it has passed the event horizon.

How long can such an object survive after crossing the event horizon?

SOLUTION: This is asking us for the integral

$$T = \int_{2m}^{0} dt = \int_{2m}^{0} \frac{dt}{dr}\, dr \leq -c^{-1} \int_{2m}^{0} \left(\frac{r}{2m - r}\right)^{1/2} dr$$

Note the minus sign: this is because r is decreasing as the object falls in.

Although there is no quadratic under the radical, we can still use a trig substitution.

We put $r = 2m\sin^2\theta$, so that $dr = 4m\sin\theta\cos\theta\,d\theta$. Then

$$cT \leq -\int_{2m}^{0} \left(\frac{r}{2m - r}\right)^{1/2} dr$$

For a black hole of the mass of the sun, $m = 1444$ metres, this gives a maximum life time after falling in of

2.4 Integrating Rational Functions

The result we are going to prove in this section is:

Every real rational function has an elementary function as an antiderivative.

Some definitions are in order:

A **rational function** has the form $\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions.

The **elementary functions** consist of polynomials, rational functions, trigonometric functions and their inverses, logarithms, exponentials and any composition of such functions.

For example

$\sin^8\left(\log(x^{\sqrt{2}} + e^{\arctan(x+\pi/12)})\right) + \dfrac{\tan^3(x^7 + x)}{\cosh(x^4 - 3/x)}$ is elementary.

A rational function $\dfrac{p(x)}{q(x)}$ is **proper** if the degree of $p(x)$ is strictly less than the degree of $q(x)$, otherwise it is **improper**.

A quadratic is **irreducible** is it has no real linear factors.

We will prove our result by construction: we outline an algorithm that will actually find the antiderivative, modulo being able to factorise polynomials.

Motivation: why do we focus on what looks like a rather special case?

Answers: firstly, we can actually do these integrals.

It is not too hard to show that not all elementary functions have elementary antiderivatives, for example e^{x^2} does not.

Secondly, they do occur in practice quite often: for example a standard population growth model is

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right),$$

solving this is equvalent to integrating $\dfrac{1}{N(K-N)}$

Thirdly, many other integrals can be reduced to rational functions: we will see this later.

2.4.1 Outline

There are three stages:

1. If necessary, reduce the rational function to the sum of a polynomial and a proper rational function.
2. Spilt the proper rational function into a sum of terms of the form

$$\frac{A}{(x-a)^k} \qquad \text{and} \qquad \frac{Bx+C}{(x^2+bx+c)^k}$$

   for constants A, B, C where x^2+bx+c is irreducible.

3. Integrate the resulting decomposition term by term.

This leaves a big open question: can we always do each of these stages?

Stage 1 (where it is needed) is easy: at worst it needs polynomial long division.

Stage 2 requires a technique called **partial fractions**.

It is really just undoing the process of putting rational functions over a common denominator.

You learnt in School how to turn

$$\frac{x}{x^2+2x+2} - \frac{2}{3x-2} \qquad \text{into} \qquad \frac{x^2-6x-4}{3x^3+4x^2+2x-4}$$

Partial fractions is the technique of going from right to left here.

Partial fractions has uses beside integration, though, as most of you will see in future years.

However, it is harder than putting rational functions over a common denominator, which is one reason you do not do that to rational functions without a good reason.

The Notes just say that “[i]t can be shown using algebra” that we can always do this, and then develop the process in several cases depending on the structure of the denominator.

They don’t say that these cases are all you need, but it’s not hard to see that is so — I’ll say more about this later.

For stage 3 (integrating) any polynomial part is easy.

For the rational parts, we use a little simple algebra, completing the square, substitutions etc to leave ourselves something that can be integrated using one of the standard forms

$$\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + C, \qquad \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

and $\int x^k\,dx = \dfrac{1}{k+1}x^{k+1} + C$ ($k \neq -1$).

2.4.2 Partial Fractions

We assume that we have a proper rational function

$f(x) = \dfrac{p(x)}{q(x)}$.

Case 1: $q(x)$ has distinct linear factors

If the factors of $q(x)$ are $x - a_i$ for some distinct a_i, then $f(x)$ can be written as sum of terms $\dfrac{A_i}{x - a_i}$ for constants A_i.

Rather than write out the general form – which obscures what is going on – we will follow the Notes in illustrating with typical examples, both here and in the later cases.

So consider $\dfrac{x+1}{(x-1)(x-2)(x-3)}$. Its partial fraction decomposition is

$$\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$

for some constants A, B and C: our problem is to find A, B and C.

The technique we use is really just to cross-multiply: we get

$$x+1 = A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$$

Now in theory we could expand and compare coefficients: we'd have 3 equations in 3 unknowns.

But we do not do that.

We substitute.

Now

$$x + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

is to be true for all values of x, so we can substitute, in turn, $x = 1$, $x = 2$ and $x = 3$ and it's still true.
Doing this gives us the three equations

$$2 = 2A, \qquad 3 = -B, \qquad 4 = 2C$$

So we have

$$\frac{x+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{3}{x-2} + \frac{2}{x-3}$$

You can easily check this by putting everything over a common denominator.

Case 2: $q(x)$ has repeated linear factors

Two examples should suffice:

$$\frac{x^2+1}{(x+4)^3} = \frac{A}{x+4} + \frac{B}{(x+4)^2} + \frac{C}{(x+4)^3}$$

$$\frac{x^2-2}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

Note carefully how the repeated factors appear on the right-hand side: all lower powers of the repeated factor are required and all have just constants on the numerators.

The calculations are a little more involved than in case 1 but the basic idea is the same: cross multiply and substitute.

Unfortunately, you will not get all the constants this way, and some comparing of coefficients will be needed.

Example 19 *Find the partial fractions decomposition of* $\dfrac{x^2-2}{(x-1)(x-2)^2}$.

SOLUTION: As we saw

$$\frac{x^2-2}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

so cross multiply to get

$$x^2-2 = A(x-2)^2 + B(x-1)(x-2) + C(x-1).$$

Putting $x=1$ and $x=2$ into this in turn gives

We still need B.

There are several thing you could do.

To find B in

$$x^2 - 2 = A(x-2)^2 + B(x-1)(x-2) + C(x-1)$$

the simplest is

This gives

$$\frac{x^2 - 2}{(x-1)(x-2)^2} =$$

Example 20 *Find the p.f.d. of* $\dfrac{x^2 - x - 3}{(x-2)^3}$.

SOLUTION: The form required is

$$\frac{x^2 - x - 3}{(x-2)^3} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}.$$

Cross-multiplying:

$$x^2 - x - 3 = A(x-2)^2 + B(x-2) + C.$$

So

$$\frac{x^2 - x - 3}{(x-2)^3} =$$

Example 21 *Find* $\int \frac{8}{(x^2-2)x^2}\,dx$

SOLUTION: In this case the p.f.d. is

$$\frac{8}{(x^2-2)x^2} = \frac{A}{x+\sqrt{2}} + \frac{B}{x-\sqrt{2}} + \frac{C}{x} + \frac{D}{x^2}.$$

The appearance of surds is to be expected: in general polynomials have surds (often very complicated ones) in their roots.

Cross-multiplying as usual:

$$8 = A(x-\sqrt{2})x^2 + B(x+\sqrt{2})x^2 + Cx(x^2-2) + D(x^2-2)$$

Our usual method of substitution puts x, in turn, So

Then

Thus

$$\int \frac{8}{(x^2-2)x^2}\,dx =$$

$$=$$

Note: You could do this example using a standard integral:

$$\int \frac{1}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C.$$

The method is similar to the next case . . .

Case 3: $q(x)$ has distinct irreducible quadratic factors

Again, two examples:

$$\frac{x^2+x}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$
$$\frac{x^3-2x+4}{(x^2+5)(x^2+x+1)} = \frac{Ax+B}{x^2+5} + \frac{Cx+D}{x^2+x+1}$$

As before, note carefully how the irreducible quadratic appears on the right-hand side.

The constants A, B, C and D in each case can be determined by algebra.

Example 22 *Integrate* $\dfrac{2x^2-2x+5}{(x-1)(x^2+2x+2)}$.

SOLUTION: We begin with the p.f.d. We have

$$\frac{2x^2-2x+5}{(x-1)(x^2+2x+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+2}$$

so

$$2x^2-2x+5 = A(x^2+2x+2) + (Bx+C)(x-1).$$

Put $x=1$ to immediately get

What next?

Well, putting $x=0$ will now give us C:

So

$$\frac{2x^2-2x+5}{(x-1)(x^2+2x+2)}$$

Integrating the linear term is easy, so we focus on the quadratic one.

We re-write the numerator as a multiple of $2x+2$, which is the derivative of x^2+2x+2, plus a constant term:

where we have completed the square in the last term.

This can now be integrated with standard integrals:

$$\int \frac{2x^2-2x+5}{(x-1)(x^2+2x+2)}$$

Case 4: $q(x)$ has repeated irreducible quadratic factors

You will only rarely meet this type in first year, but here are two typical examples:

$$\frac{x^2+x}{(x^2+9)^3} = \frac{Ax+B}{x^2+9} + \frac{Cx+D}{(x^2+9)^2} + \frac{Ex+F}{(x^2+9)^3}$$

$$\frac{x^3-2x+4}{(x-2)(x^2+x+1)^2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}.$$

Again, note carefully how the repeated factors appear on the right-hand side: all lower powers of the repeated factor are required and all have just linear polynomials on the numerators.

The integration of the repeated quadratic terms is done by a similar process to the non-repeated one (see examples).

Example 23 *Integrate* $\dfrac{2x+5}{(x^2+4x+5)^2}$.

SOLUTION: There is no need to do a partial fractions here, we just split up the numerator into the appropriate parts:

$$\frac{2x+5}{(x^2+4x+5)^2} = \frac{2x+4}{(x^2+4x+5)^2} + \frac{1}{((x+2)^2+1)^2},$$

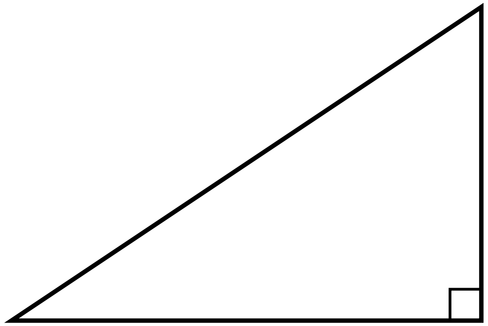
completing the square again.

The first term integrates to

For the second, we use the trig substitution:

The integral is then

To turn this back to x we use a triangle:



Thus

So

$$\int \frac{2x+5}{(x^2+4x+5)^2} =$$

A case where we put it all together:

Example 24

$$\frac{x^2-4x+4}{x^9-7x^8+23x^7-45x^6+56x^5-44x^4+20x^3-4x^2}$$

$$=\frac{x^2-4x+4}{x^2(x-1)^3(x^2-2x+2)^2}$$

$$=\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{(x-1)}+\frac{a_4}{(x-1)^2}+\frac{a_5}{(x-1)^3}+\frac{a_6+a_7x}{(x^2-2x+2)}+\frac{a_8+a_9x}{(x^2-2x+2)^2}$$

$$=$$

Courtesy of Maple: `convert(...,parfrac,x);`

The method of partial fractions is computationally very expensive, and not always practical:

a) You have to factorise $q(x)$ into linear and quadratic factors. In practice, for general polynomials this is impossible.

b) Even if you can factorise $q(x)$ it will, in general, lead to non-rational numbers: Consider

$$\int \frac{3x^2+1}{(x^3+x+1)^3}\,dx = -\frac{1}{2(x^3+x+1)^2} + C$$

The p.f.d. of this integrand will involve numbers like $\left(108+12\sqrt{93}\right)^{1/3}$, all of which cancel out.

There are methods that allow you to find the integral and avoid all needlessly complicated non-rationals: these methods are programmed into Maple etc.

2.5 Other Substitutions

Now we have a method that is “guaranteed” to work for rational functions, we have a new general strategy for other integrals:

$\boxed{\text{Substitute to make the integrand rational}}$

Example 25 *Find* $I = \displaystyle\int \frac{x^{1/3}}{(x^{2/3} + x^{4/5})^2}\,dx$

SOLUTION: Use a polynomial substitution: let $x = u^{15}$.
Then $dx = 15u^{14}\,du$ and

$$I = 15\int \frac{u^{19}}{(u^{10} + u^{12})^2}\,du = 15\int \frac{1}{u(1+u^2)^2}\,du$$

$$=$$

So we need to find constants with

Putting

Then putting

Hence

$$I =$$

Example 26 *Find* $I = \int_{\ln 3}^{\ln 8} (e^x + 1)^{-3/2}\, dx$

SOLUTION: We get rid of the square root and the exponential simultaneously (a standard strategy).

So set $u^2 = e^x + 1$, and then

Thus

$$I =$$

$$=$$

Example 27 *Find* $I = \int \frac{\sin\theta}{1+\cos\theta} d\theta$*.*

SOLUTION: There is a standard substitution for integrals of the form $\frac{1}{a + b\sin\theta + c\cos\theta}$ where a, b and c are constant: $t = \tan\frac{\theta}{2}$.

Double-angle and derivative formulas give

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}, \quad d\theta = \frac{2dt}{1+t^2}.$$

I'll leave checking these as an EXERCISE in trig identities.

In our case

$$
\begin{aligned}
I &= \int \frac{\sin\theta}{1+\cos\theta} d\theta \\
&= \int \frac{\dfrac{2t}{1+t^2}\,\dfrac{2dt}{1+t^2}}{\left(1+\dfrac{1-t^2}{1+t^2}\right)} \\
&=
\end{aligned}
$$

In fact, this particular examples yields rather more quickly to a trick that often helps with $1 + \cos\theta$ examples: half-angle formulas.

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1, \qquad \sin\theta = 2\sin\frac{\theta}{2}\,\cos\frac{\theta}{2}$$

So,

$$\int \frac{\sin\theta}{1+\cos\theta}d\theta = \int \frac{2\sin\frac{\theta}{2}\,\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}d\theta$$

2.6 Maple

You should all know Maple's standard integrating command
`int`

To convert to partial fractions, you use a command like:

```
> convert(1/(x*(x-1)),parfrac,x);
```

Note that you need to tell Maple the name of the variable.

This means you can find p.f.ds with parameters:

```
> convert(1/((x-a)*(x+b)),parfrac,x);
```

Be aware that Maple is clever but not infallible: we've already seen that it fails to find

$$\int \frac{\ln(x+\sqrt{x^2-1})}{x^2}\,dx.$$