3.8 Second Order Linear ODEs with Constant Coefficients

We now turn to this special (but important) class of second order DEs, which come in two cases:
The **homogeneous** case:

$$\boxed{\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0} \tag{3}$$

and the **non-homogeneous** case:

$$\boxed{\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)} \tag{4}$$

In both cases, a and b are constants.

General Results

Lemma 1. If y_1 and y_2 are solutions of (3) then so is $\alpha y_1 + \beta y_2$.

Lemma 2. (3) has two linearly independent solutions and every solution is a linear combination of these solutions.

Lemma 3. If y_1 and y_2 are solutions of (4) then $(y_1 - y_2)$ is a solution of (3).

Lemma 4. The general solution of (4) is given by $y = y_H + y_P$, where y_H is the general solution of (3) and y_P is a particular solution of (4).

Proof ...

... of Lemma 1. Set $y = \alpha y_1 + \beta y_2$. Then,

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = \alpha\left(\frac{d^2y_1}{dx^2} + a\frac{dy_1}{dx} + by_1\right) + \beta\left(\frac{d^2y_2}{dx^2} + a\frac{dy_2}{dx} + by_2\right) = 0$$

... of Lemma 2. See 2nd year e.g. MATH2601, MATH2221.

... of Lemma 3. Set $y = y_1 - y_2$. Then,

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = \left(\frac{d^2y_1}{dx^2} + a\frac{dy_1}{dx} + by_1\right) - \left(\frac{d^2y_2}{dx^2} + a\frac{dy_2}{dx} + by_2\right)$$

$$= f(x) - f(x) = 0.$$

Proof of Lemma 4.

Let y be the general solution of the inhomogeneous equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x) \tag{4}$$

and y_p be a particular solution of (4).

Then, according to Lemma 3, $y_H = y - y_P$ is a solution of the homogeneous equation (3).

So y is of the form

$$y = y_H + y_P.$$

$\blacksquare$

Remark. It is easy to see that the above lemmas also hold for non-constant coefficients $a(x)$ and $b(x)$.

3.8.1 Homogenous Case

$$\boxed{y''(x) + ay'(x) + by(x) = 0} \tag{3}$$

From Lemma 2 there are two independent solutions: all we have to do is find them.

We already know a solution of the first order linear case $y' + ay = 0$ is e^{-ax}, so it seems worth trying an exponential.

Well, if we set $y = e^{\lambda x}$, then

$$y''(x) + ay'(x) + by(x) = \lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = e^{\lambda x}(\lambda^2 + a\lambda + b)$$

So $e^{\lambda x}$ will be a solution of the DE (3) if and only if λ is a root of the **characteristic equation**

$$\lambda^2 + a\lambda + b = 0. \tag{5}$$

There are three possible cases for the roots of a quadratic:

Case 1: roots real and distinct

If the roots of (5) are $\lambda_1 \neq \lambda_2$, both real, then the general solution to the DE (3) is

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

for arbitrary constants A and B.

Example 16 *Solve* $y'' - y' - 2y = 0$

SOLUTION: The characteristic equation is

so the general solution is

$$y =$$

Case 2: repeated real root

The characteristic equation has one root, λ_1, and is $(\lambda - \lambda_1)^2 = 0$.

In this case the general solution is

$$y = Ae^{\lambda_1 x} + Bx\, e^{\lambda_1 x}$$

for arbitrary constants A and B: note the x.

Example 17 *Solve* $y'' - 6y' + 9y = 0$

SOLUTION: The characteristic equation is

so the general solution is

$$y =$$

Case 3: conjugate roots

The final case is where the roots of the characteristic equation are complex.

Since the characteristic equation is real, the roots must be a conjugate pair $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$.

The general solution is then formally the same as in case 1:

$$y = Ce^{\lambda_1 x} + De^{\lambda_2 x},$$

but for y to be real, C and D must be conjugate.

The Notes prove this, and also show that if $C = \frac{1}{2}(A + iB)$ we can write the general real solution as

$$y = e^{\alpha x}\left(A\cos(\beta x) + B\sin(\beta x)\right).$$

We use this last form as our general solution.

Example 18 *Solve* $y'' - 4y' + 5y = 0$

SOLUTION: The characteristic equation is

and so has roots .

So the general solution of the DE is

$$y =$$

3.8.2 Non-homogeneous Case

$$\boxed{y''(x) + ay'(x) + by(x) = f(x)} \qquad (4)$$

We know the general solution is:

$$y(x) = y_H(x) + y_P(x),$$

where

$$y_H''(x) + ay_H'(x) + by_H(x) = 0.$$

We now know how to find $y_H(x)$ but how do we find a particular solution $y_P(x)$?

- inspired guesswork – **method of undetermined coefficients**
- variation of parameters – see MATH2221, MATH2019 etc

$f(x)$	trial $y_P(x)$
$P(x)$ given poly. of deg n	$Q(x)$ arbitrary poly. of deg n
$P(x)e^{\sigma x}$	$Q(x)e^{\sigma x}$
$P(x)e^{\sigma x}\sin\tau x$ or $P(x)e^{\sigma x}\cos\tau x$	$e^{\sigma x}\left(Q_1(x)\sin\tau x + Q_2(x)\cos\tau x\right)$

If $f(x)$ is a sum of terms in column 1 then $y_P(x)$ is a sum of corresponding terms in column 2 for certain values of the constants in the polynomials $Q(x)$.

If **any** term in the trial solution is a solution of the homogeneous equation then multiply the trial solution by x, and **repeat this step as often as necessary**.

Substitute the trial solution into the equation to determine the unknown constants in the trial solution.

Example 19 *Solve the IVP*

$$y'' - y' - 2y = 1 - 2x^2, \qquad y(0) = 1, \quad y'(0) = 2.$$

SOLUTION: We found the general solution to the corresponding homogeneous equation in example 16:

For our trial solution we would use $\quad y_P =$

So

$$-2y_P - y_P' + y_p'' =$$

$$=$$

This is $1 - 2x^2$ iff

So the general solution of the DE is

$$y =$$

Note: we must find the complete general solution before substituting the initial conditions.

The initial conditions give us

$$y(0) = 1 = \qquad\qquad y'(0) = 2 =$$

This system of linear equation is easily solved to give

So the solution to the IVP is

$$y =$$

Example 20 *Find the general solution of* $y'' - y' - 2y = 3e^{-x}$

SOLUTION: We have the same corresponding homogeneous equation as before.

The natural form to try for y_P is ae^{-x}, but this is a solution of the homogeneous equation.

So we must try axe^{-x}:

$$-2y_P - y'_P + y''_p =$$

This is $3e^{-x}$ iff .

So the general solution is

$$y =$$

Example 21 *Find a particular solution of* $y'' - 6y' + 9y = 6xe^{3x}$

SOLUTION: Although we are not asked for the general solution, we still need the solution of the corresponding homogeneous equation to decide the shape of the answer.

In this case, from example 17, the homogeneous solution is

$y_H =$

The obvious thing to try for y_P is

Thus

$$9y_P - 6y'_P + y''_P =$$

If this is to be $6xe^{3x}$,
Our particular solution is thus

$$y =$$

Example 22 *Find the general solution of*

$$y'' - 4y' + 5y = 8\sin x.$$

SOLUTION: We found the general solution to the corresponding homogeneous equation in example 18:

$$y_H =$$

For a particular solution we try

$$y_P = a\sin x + b\cos x.$$

So

$$5y - 4y' + y''$$

This is $8\sin x$ if and only if

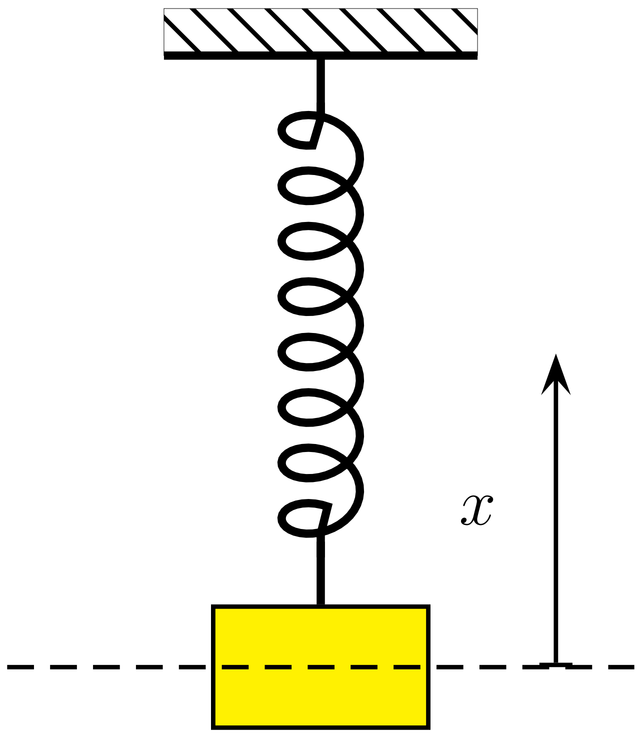
These are easily solved to give .
The general solution is thus

$$y =$$

3.8.3 Vibrations and Resonance

Consider a mass on the end of a spring.

According to Newton's second Law (acceleration proportional to force) and Hooke's Law (spring force proportional to extension), the displacement of the mass will satisfy the equation



$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

We measure x, the displacement from equilibrium

Here (and onwards) we are ignoring gravity, friction, air resistance etc.

The constant ω will depend on the spring and the mass.

It is easy to solve

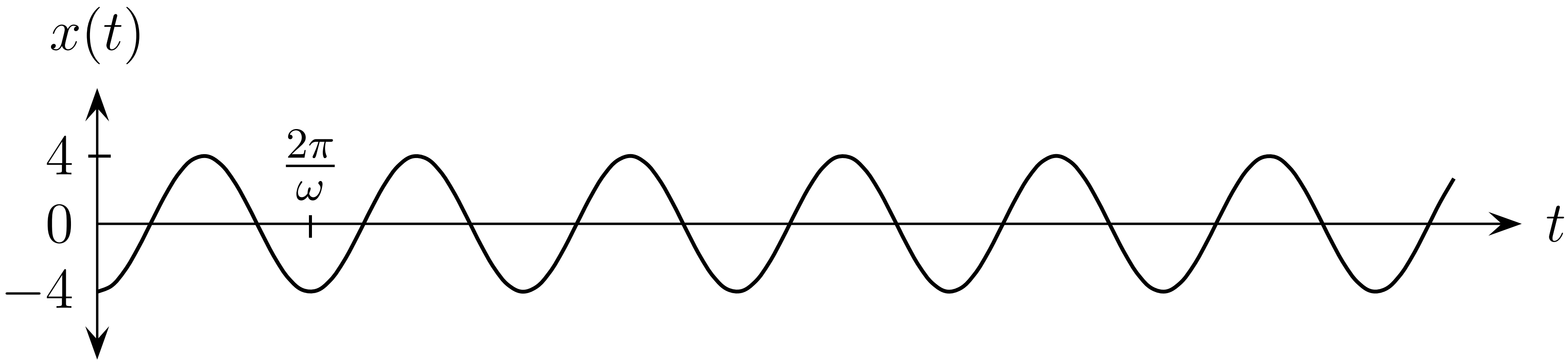
$$\frac{d^2x}{dt^2} + \omega^2 x = 0.$$

The characteristic equation is clearly $\lambda^2 + \omega^2 = 0$.

The general solution is thus

$$x = A\cos\omega t + B\sin\omega t = K\sin(\omega t + \phi_0)$$

The mass undergoes **simple harmonic motion**, with bounded and periodic solution:



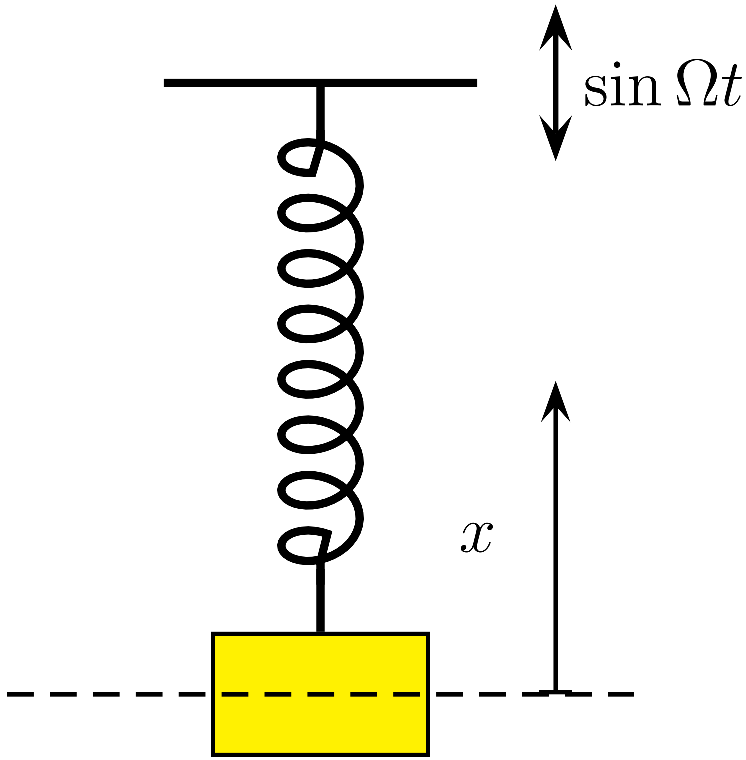
Suppose now the spring's support vibrates with period Ω: we say the spring is **forced**.

Initially, we assume $\Omega \neq \omega$.

The DE is clearly

$$\frac{d^2x}{dt^2} + \omega^2 x = F \sin \Omega t,$$

for some F.



The trial particular solution is $x_p = C \cos \Omega t + D \sin \Omega t$.

I'll leave it as an EXERCISE to check that $C = 0$, $D = \dfrac{F}{\omega^2 - \Omega^2}$.

So the general solution is

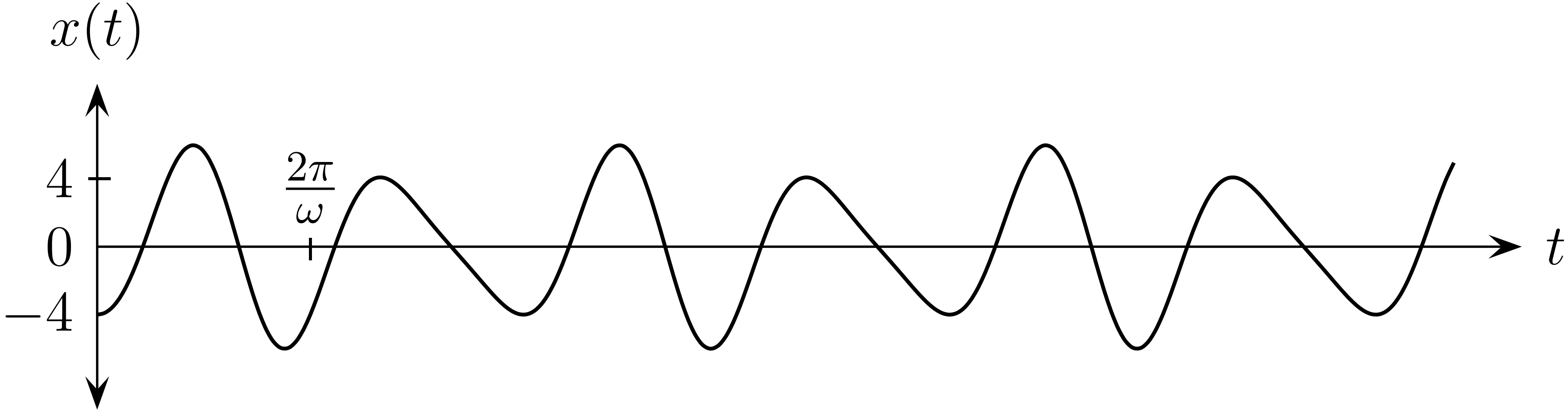
$$x(t) = A \cos \omega t + B \sin \omega t + \frac{F}{\omega^2 - \Omega^2} \sin \Omega t$$

Even without plotting

$$x(t) = A\cos\omega t + B\sin\omega t + \frac{F}{\omega^2 - \Omega^2}\sin\Omega t$$

we can see we still get bounded and oscillating solutions.

In fact we get (here $\Omega = \frac{3}{2}\omega$):



This is a stable solution.

Finally, consider the forced oscillation case but where the forcing is done at the natural frequency ω.

The DE is then

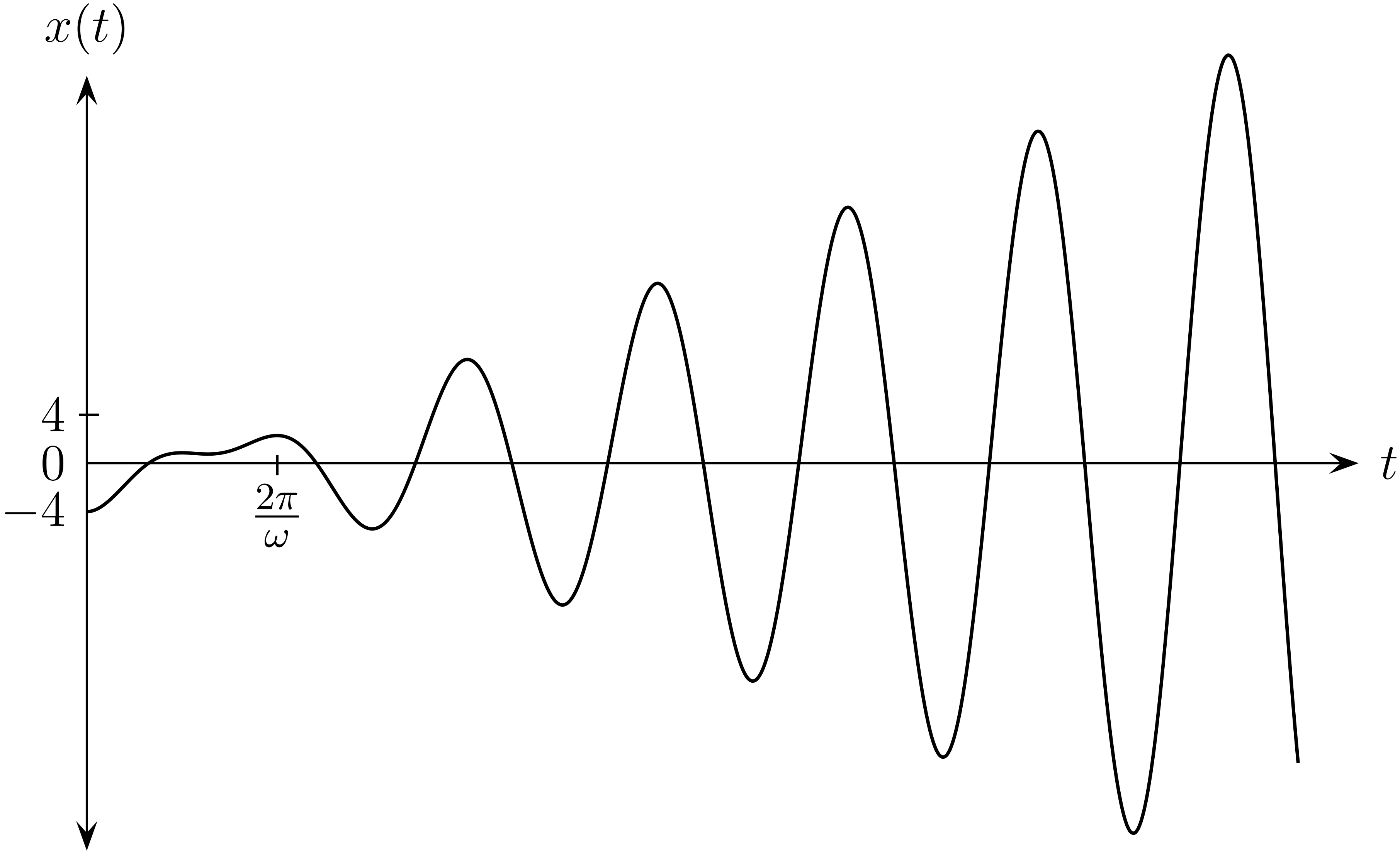
$$\frac{d^2x}{dt^2} + \omega^2 x = F \sin \omega t.$$

Our trial particular solution now has to be $x_p = Ct \cos \omega t + Dt \sin \omega t$.

I'll again leave it as an EXERCISE to check that $D = 0$, $C = -\dfrac{F}{2\omega}$, and so the general solution is

$$x(t) = A \cos \omega t + B \sin \omega t - \left(\frac{F}{2\omega}\right) t \cos \omega t$$

Once again we do not need to plot to see that this solution is oscillatory but unbounded.



We call this effect **resonance**.

Damped and Forced Oscillations

Many mathematical models for real world behaviour reduce to

$$y''(t) + \sigma y'(t) + \omega^2 y(t) = f_0 \cos \omega t$$

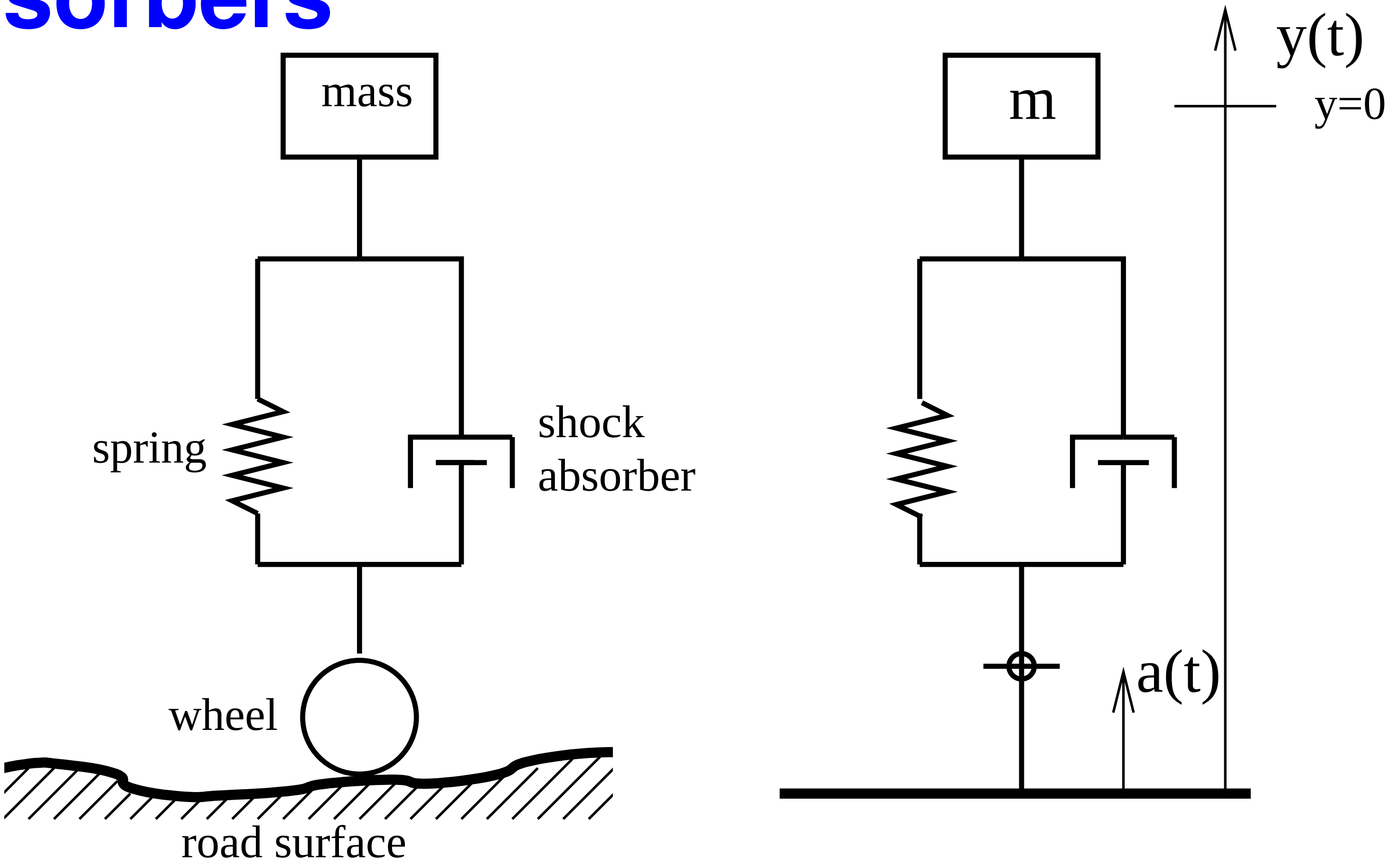
The response $x(t)$ of an accelerometer to the rate of change in velocity in ground shaking $a(t)$ satisfies

$$x''(t) + 2\eta\omega x'(t) + \omega^2 x(t) = -a(t)$$

The charge $q(t)$ on a capacitor in a series RLC circuit with an alternating field $f_0 \cos \omega t$ satisfies

$$Lq''(t) + Rq'(t) + \frac{1}{C} q(t) = f_0 \cos \omega t$$

Application – Car suspension and shock absorbers



$$m\frac{d^2y(t)}{dt^2} = -k\left(y(t)-a(t)\right) - c\frac{d}{dt}\left(y(t)-a(t)\right)$$

$$my'' + cy' + ky = ca' + ka = f(t)$$

The free but damped oscillator

$$\boxed{y''(t) + \sigma y'(t) + \omega^2 y(t) = 0}$$

where σ is 'friction', ω the natural frequency.

- **characteristic equation**:

$$\lambda^2 + \sigma\lambda + \omega^2 = 0 \qquad \Rightarrow \qquad \lambda_\pm = \frac{-\sigma \pm \sqrt{\sigma^2 - 4\omega^2}}{2}$$

- **overdamped case**: $\sigma > 2\omega$

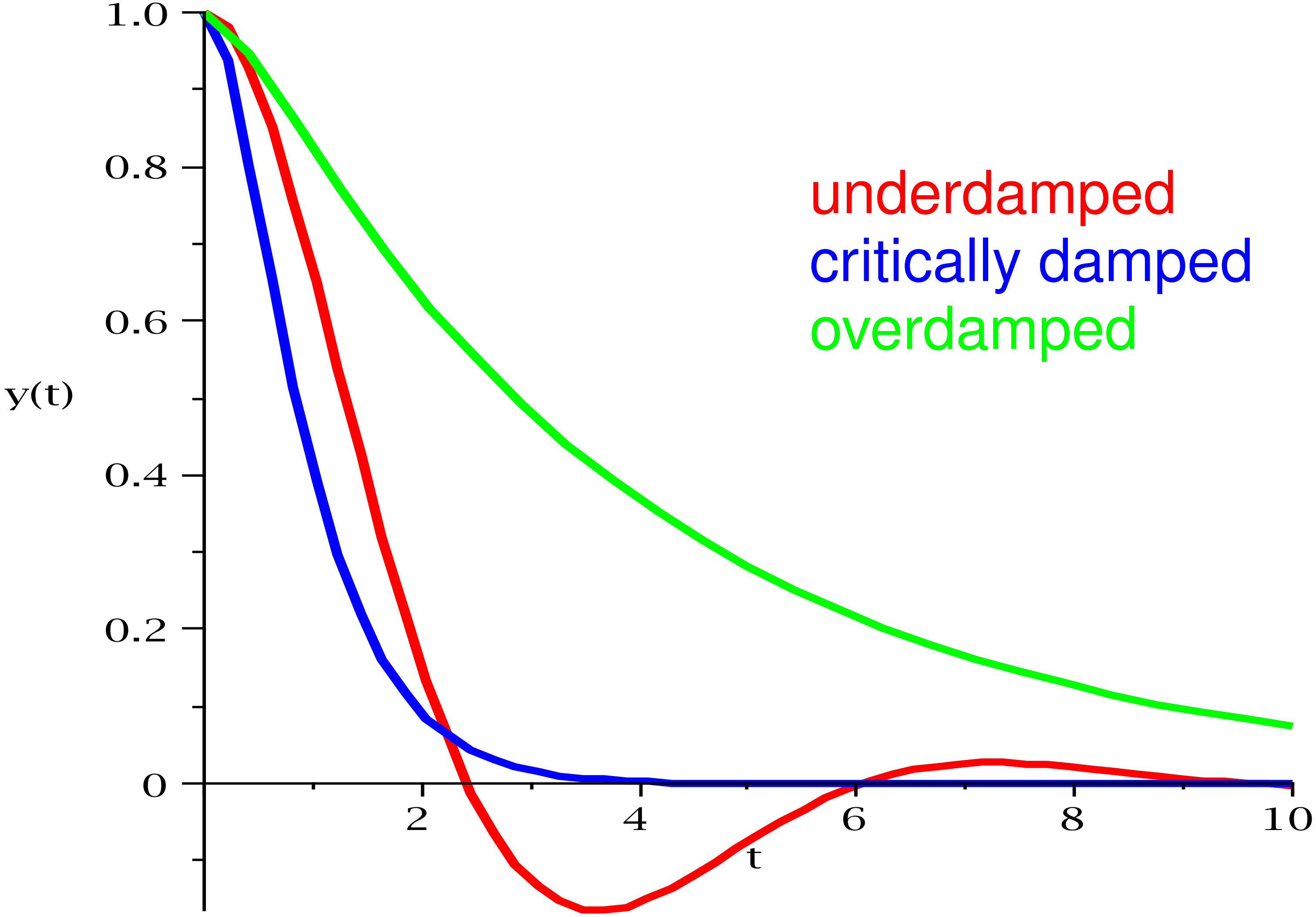
$$y(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t}, \qquad \lambda_\pm < 0$$

- **underdamped case**: $\sigma < 2\omega$

$$y(t) = e^{-\frac{\sigma}{2}t}\left(A\cos\Omega t + B\sin\Omega t\right), \qquad \Omega = \sqrt{\omega^2 - \frac{\sigma^2}{4}}$$

- **critically damped case**: $\sigma = 2\omega$

$$y(t) = Ae^{-\frac{\sigma}{2}t} + Bte^{-\frac{\sigma}{2}t}$$



Critical damping is the fastest way to return to equilibrium without oscillations (overshooting). Car shock absorbers are designed to achieve critical damping.

The forced and damped oscillator

$$\boxed{y''(t) + \sigma y'(t) + \omega^2 y(t) = \cos \alpha t}$$

The preceding analysis shows that $y = y_H + y_P \to y_P$ as $t \to \infty$!

Particular solution:

$$y_P = a \sin \alpha t + b \cos \alpha t$$

$$\Rightarrow \quad a = \frac{\sigma\alpha}{\Delta^2 + \sigma^2\alpha^2}, \quad b = \frac{\Delta}{\Delta^2 + \sigma^2\alpha^2}, \quad \Delta = \omega^2 - \alpha^2$$

Amplitude:

$$\mathrm{AMP} = \sqrt{a^2 + b^2} = \frac{1}{\sqrt{\Delta^2 + \sigma^2\alpha^2}}$$

A natural question to ask is: for what forcing frequency is the steady state amplitude at a maximum?

$$
\begin{aligned}
\frac{d\mathrm{AMP}}{d\alpha} = 0 \quad &\Leftrightarrow \quad \frac{d}{d\alpha}(\Delta^2 + \sigma^2\alpha^2) = 0 \\
&\Rightarrow \quad \alpha^2 = \omega_{\mathrm{max}}^2 = \omega^2 - \frac{\sigma^2}{2} \\
&\Rightarrow \quad \mathrm{AMP}_{\mathrm{max}} = \frac{1}{\sigma}\frac{1}{\sqrt{\omega^2 - \frac{\sigma^2}{4}}}
\end{aligned}
$$

Conclusion:

If σ is ‘small’ then $\omega_{\mathrm{max}} \approx \omega$ and $\mathrm{AMP}_{\mathrm{max}}$ is ‘large’.

This is resonance.

To see these phenomena, I'll leave you to experiment using Maple:

```
DE:=diff(x(t),t,t) + sigma*diff(x(t),t) +
omega**2*x(t)=cos(alpha*t);
```

This define the general DE, and then use `subs` and `dsolve`:

```
SOL:=dsolve(subs(sigma=.2,omega=1,alpha=2,DE),
x(0)=0,D(x)(0)=1,x(t));
```

And then plot:

```
plot(rhs(SOL),t=0..100);
```

What you should find is that as you repeat the last two commands, changing the parameters σ and α, you should be able to arrange over and under damping to the steady solution as well as beats (when $\alpha \approx \omega$ and σ small) etc.

Note: for the damping phenomena, it is the ratio σ/ω that matters, so you may as well stick to $\omega = 1$ and just vary α and σ.

3.8.4 Linear Algebra and DEs

The Linear Algebra you are covering in the algebra lectures can usefully be applied to the study of linear ODEs.

In particular, we need it to prove our Lemma 2 from earlier: recall that is the result that essentially tells us that we only need 2 solutions for the second order homogeneous equation.

I'm not going to prove that (it's a piece of pure algebra that you will see later), but it is useful to look at our linear ODEs using the terminology from the algebra lectures.

To deal with second order DEs, we need to look at functions that can be differentiated at least twice: this gives us the stage to play on:

Let V denote the vector space of all twice differentiable functions $y : \mathbb{R} \to \mathbb{R}$.

Define the map $T : V \to V$ by

$$T(y) = y'' + ay' + by \quad \text{for} \quad a, b \in \mathbb{R}$$

It is a routine exercise to prove that T is linear on V.

If y_H is a solution of $y'' + ay' + by = 0$ then $T(y_H) = 0$ so that y_H is in the kernel of T, $\ker(T)$.

If y is a solution of

$$y'' + ay' + by = f \tag{6}$$

then $T(y) = f$ so that f is in the image of T.

If $y_H \in \ker(T)$ and y_P is any solution to (6), then

$$T(y_P + y_H) = T(y_P) + T(y_H) = T(y_P) = f,$$

so $y_P + y_H$ is a solution to (6).

Finally, if y and y_P are both solutions of

$$y'' + ay' + by = f \tag{6}$$

then

$$T(y - y_P) = T(y) - T(y_P) = f - f = 0$$

so that $y - y_P \in \ker(T)$.

It follows that $y - y_P = y_H$ for some $y_H \in \ker(T)$, and so all solutions of (6) are of the form $y_H + y_P$.

The point here is that just as the general solution of the set of linear equations $A\mathbf{x} = \mathbf{b}$ is the sum of an arbitrary vector in $\ker(A)$ plus any one solution to $A\mathbf{x} = \mathbf{b}$,
the general solution to $T(y) = f$ is the sum of an arbitrary vector (function) in $\ker(T)$, plus any one solution to $T(y) = f$.