Chapter 5
Applications of Integration

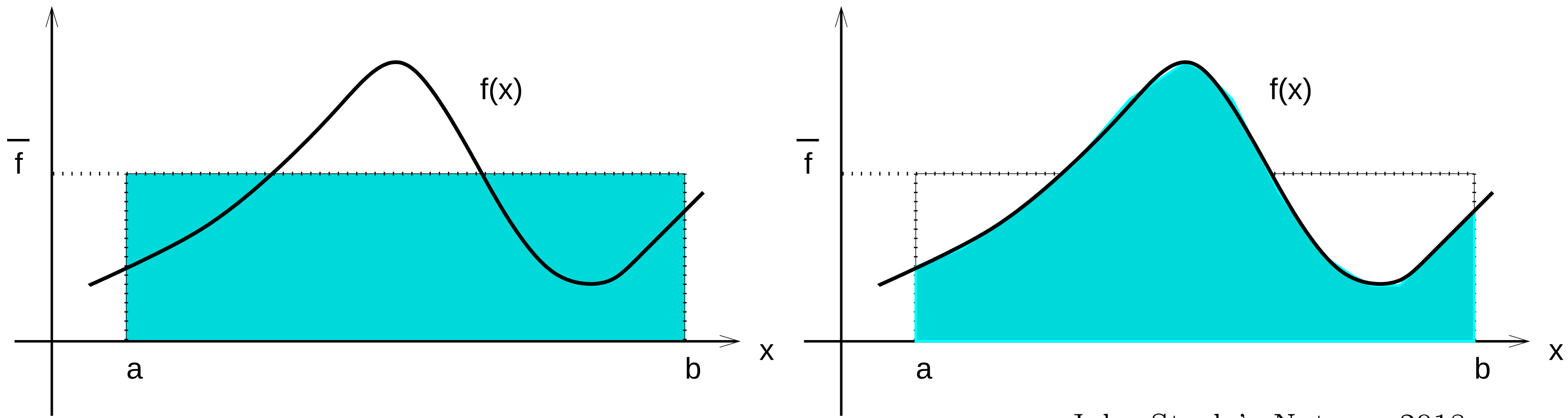
5.1 Average Value of a Function

Suppose a function f is integrable on a closed interval $[a, b]$.

Then the **average value** $\overline{f}$ of $f(x)$ on $[a, b]$ is defined by

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The average value $\overline{f}$ is the unique value such that the area of the rectangle of height $\overline{f}$ and width $b - a$ is equal to the area under the curve $f(x)$ between $x = a$ and $x = b$.



Justification.

Divide the interval into n subintervals of equal width $\Delta = \dfrac{b-a}{n}$ and let x_k denote a point in the kth subinterval. Then,

$$
\begin{aligned}
\frac{\int_a^b f(x)\,dx}{b-a} &= \lim_{n\to\infty} \frac{\overbrace{\sum_{k=1}^{n} f(x_k)\Delta}^{\text{Riemann sum}}}{b-a} \\
&= \lim_{n\to\infty} \frac{\sum_{k=1}^{n} f(x_k)(b-a)}{(b-a)n} \\
&= \lim_{n\to\infty} \underbrace{\frac{1}{n}\left[f(x_1)+f(x_2)+\ldots+f(x_n)\right]}_{\text{arithmetic mean}} = \bar{f}
\end{aligned}
$$

Example 1 *An AC circuit has voltage we can model as $V = V_m \sin \omega t$ for some constants V_m (max voltage) and ω (frequency).*

Over a cycle, say from $t = 0$ to $t = \dfrac{2\pi}{\omega}$ the average voltage is

$$\overline{V} = \int_0^{2\pi/\omega} V_m \sin \omega t \, dt = 0,$$

as is easy to see.

Voltages quoted on appliances are ***root mean square****, V_{rms}, which is $\sqrt{\overline{V^2}}$: square, take the average, then take the square root.*

How do these voltages compare?

Well,

$$V_{\mathsf{rms}}{}^2 = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} V_m^2 \sin^2 \omega t \, dt$$

$$=$$

It's worth pointing out that for periodic functions, that is functions f for which $f(t) = f(t+T)$, then the average value of f over a period

$$\overline{f} = \frac{1}{T}\int_a^{a+T} f(t)\, dt$$

is independent of a i.e. of which period.

This follows from

$$\begin{aligned}
\int_a^{a+T} f(t)\, dt &= \int_0^T f(t)\, dt - \int_0^a f(t)\, dt + \int_T^{a+T} f(t)\, dt \\
&= \int_0^T f(t)\, dt - \int_0^a f(t)\, dt + \int_0^a f(s+T)\, ds \\
&= \int_0^T f(t)\, dt.
\end{aligned}$$

Mean Value Theorem for Integrals

Theorem 5.1 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$. Then there is a $c \in [a, b]$ such that*

$$\int_a^b f(t)\,dt = f(c)(b - a),$$

i.e. such that $f(c) = \overline{f}$.

This result is telling us that continuous functions on closed intervals attain their averages.

(We learned last semester that they attain their max and min.)

The result can be strengthened: if g is also cts on $[a, b]$ and positive there then

$$\int_a^b f(t)g(t)\,dt = f(c)\int_a^b g(t)\,dt \quad \text{for some real} \quad c \in [a, b].$$

Proof: Define $F(x) = \int_a^x f(t)\,dt$.

By the Fundamental Theorems of Calculus, F is cts on $[a, b]$, differentiable in (a, b) and $F' = f$.

So by the Mean Value Theorem, there is a $c \in (a, b)$ such that

$$\frac{F(b) - F(a)}{b - a} = F'(c) = f(c)$$

But $F(b) = \int_a^b f(t)\,dt$ and $F(a) = 0$.

The result now follows. $\square$

5.2 Arc Length

We all know that the length of a line is the distance between its end points:

A line from $P = (x_0, y_0)$ to $Q = (x_1, y_1)$ in $\mathbb{R}^2$ has length

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

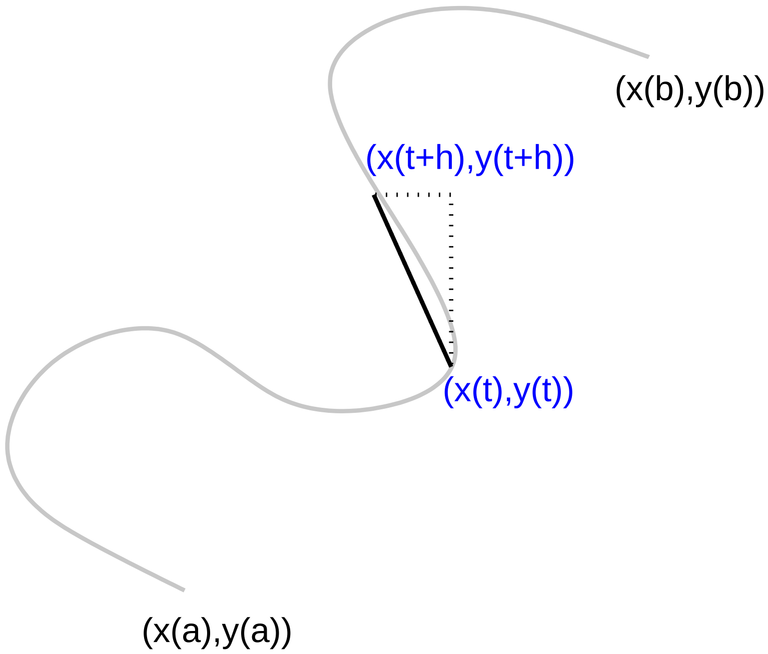
How do we work out the length of a curve, e.g. a circle, a parabola, . . . ?

As in chapter 1, we will look at the 2-dimensional case: for finding **arc lengths**, the higher dimensions follow the same pattern.

The basic idea is to partition the curve and approximate it by straight segments, **secants**, take the limit of finer and finer partitions and only look at those cases where this makes sense.

So suppose a curve is parameterised as $(x(t), y(t))$ for $t \in [a, b]$.

We assume that x and y are differentiable, and that the curve is traced in one consistant direction as t increases.



Let $\ell(s)$ denote the length of the curve from $x(a), y(a)$ to $(x(s), y(s))$.

The length of the secant from $(x(t), y(t))$ to $(x(t+h), y(t+h))$ approximates the arc length $\ell(t+h) - \ell(t)$:

$$\ell(t+h) - \ell(t) \approx \sqrt{[x(t+h) - x(t)]^2 + [y(t+h) - y(t)]^2}$$

Thus

$$\frac{\ell(t+h)-\ell(t)}{h} \approx \sqrt{\left(\frac{x(t+h)-x(t)}{h}\right)^2 + \left(\frac{y(t+h)-y(t)}{h}\right)^2}$$

In the limit $h \to 0$, we expect to obtain

$$\frac{d\ell}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

So $$\int_{\ell(a)}^{\ell(b)} d\ell = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The **arc length** between $t = a$ and $t = b$ is given by

$$\ell = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

The Notes fill in some of the gaps in this derivation.

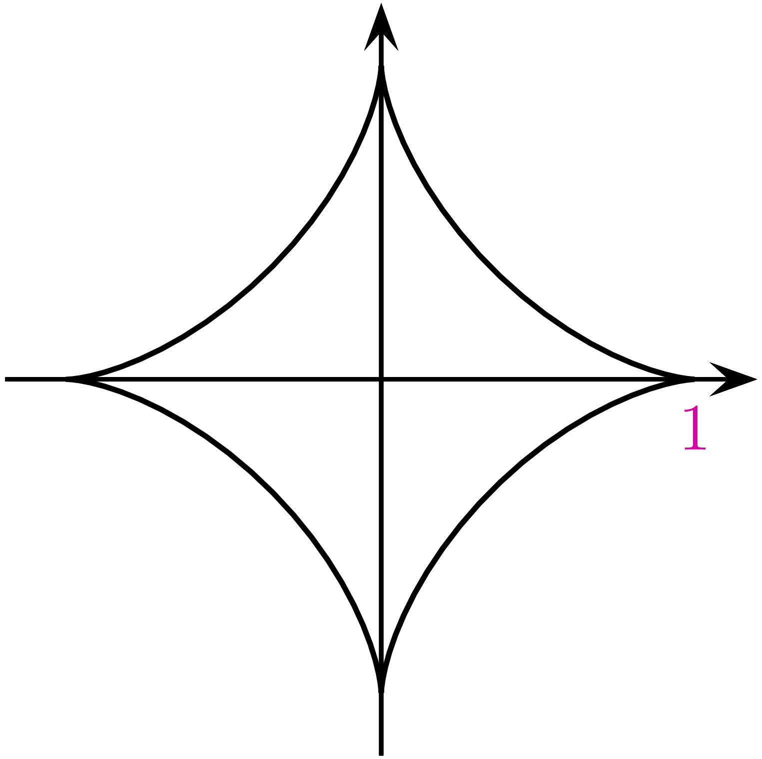
If the curve is the graph of a function, $y = f(x)$, we can simply use x as the parameter: the curve is $(x, f(x))$ and so the arc length from $x = a$ to $x = b$ is

$$\ell = \int_a^b \sqrt{1 + (f'(x))^2)}\, dx$$

Before an example, I should just point out that arc lengths are hardly ever elementary functions: that's the case even for a curve as simple as an ellipse.

The Notes cover the arc length of the **cycloid**, which is the curve you get following the point on the rim of a wheel rolling along a flat surface. . .

Example 2 *If a circle of radius $\frac{1}{4}$ rolls inside the unit circle, a point on its rim traces out the* ***astroid*** $x^{2/3} + y^{2/3} = 1$*, which can be parameterised as* $(x, y) = (\cos^3 t, \sin^3 t)$*,* $-\pi \leq t \leq \pi$*.*



To find the arc length we calculate $(\dot{x})^2 + (\dot{y})^2$ *to get*

By symmetry, the whole length is 4 times the length in the first quadrant.

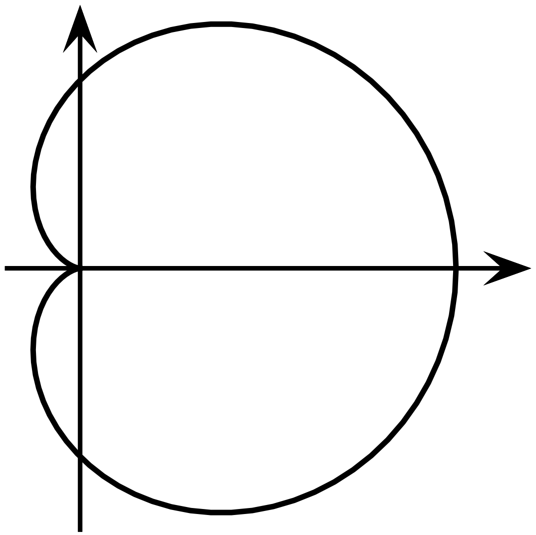
The arc of the astroid in the first quadrant will have length

If the curve is given in polars by $r = r(\theta)$, then we have $(x, y) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$ and the arc length from θ_0 to θ_1 is

$$\ell = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

See the Notes for the proof.

Example 3 *The* ***cardioid*** *is given in polars by* $r = 1 + \cos\theta$.



We have $\dfrac{dr}{d\theta} = -\sin\theta$,

so for the arc length we integrate the square root of

By symmetry, the length is twice the upper half,

Example 4 *The space between the grooves of a CD is a constant $s = 1.6 \times 10^{-3}$ mm.*
If the outer track radius is $a = 58$ mm and the inner one $b = 25$ mm, how long is the groove?

SOLUTION: For constant spacing the radius must increase linearly with θ, and after θ increases by 2π, r increases by s. Hence

$$r = b + \frac{s}{2\pi}\theta.$$

This curve is called an **Archimedean spiral**.

The range of θ must be from 0 to $\Theta = \dfrac{2\pi}{s}(a-b)$, so the arc length is

$$L = \int_0^{\Theta} \sqrt{\left(b + \frac{s}{2\pi}\theta\right)^2 + \left(\frac{s}{2\pi}\right)^2}\, d\theta$$

We integrate by a hyperbolic substitution:

$$b + \frac{s}{2\pi}\theta = \frac{s}{2\pi}\sinh u$$

The lower limit will be $u_0 = \sinh^{-1}(2\pi b/s) \approx 12.1$ and the upper limit $u_1 = \sinh^{-1}(2\pi a/s) \approx 13.0$, so the length is

$$L = \int_{u_0}^{u_1} \frac{s}{2\pi} \cosh^2 u \, du$$

5.3 Speed of a Particle

Consider a particle moving along a planar curve with position $(x(t), y(t))$ at time t.

From time $t = 0$ to a later time t it will have gone a distance

$$s(t) = \int_0^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$$

Its speed will be $s'(t)$, that is

$$v(t) = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

Example 5 *A planet orbiting a star obeys Kepler's laws of motion, two of which are:*
(1) the planet moves on an ellipse with the star at a focus;
(2) if r is the distance from the star to the planet and θ the angle of the orbit then $r^2\dot{\theta} = K$ is constant.
Find the speed of the planet in terms of K, θ and the parameters of the ellipse. Where is the speed greatest and least?

SOLUTION: We use polars.

If a path is given in polar coordinates as $(r(t), \theta(t))$ then

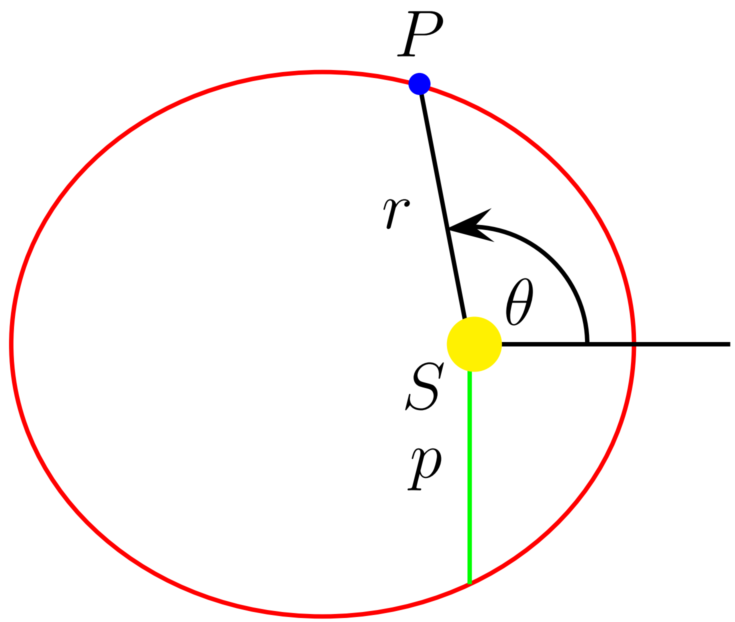
$$\dot{x}(t) = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \quad \dot{y}(t) = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

so that the speed is (EXERCISE)

$$v(t) = \sqrt{(\dot{x})^2 + (\dot{y})^2} = \sqrt{(\dot{r})^2 + r^2\left(\dot{\theta}\right)^2}.$$

The polar form of an ellipse with focus at the origin (see MATH1141) is

$$r = \frac{p}{1 + e\cos\theta},$$



for p the semi-latis rectum and $e \in (0,1)$ the eccentricity (both constant).

From the second Law $\quad \dot{\theta} = \dfrac{K}{r^2}$.

Differentiating the expression for r:

$$\dot{r} = \frac{pe\sin\theta}{(1 + e\cos\theta)^2}\dot{\theta} = \frac{Ke}{p}\sin\theta.$$

So the square of the speed is

$$\dot{r}^2 + r^2\dot{\theta}^2 =$$

$$=$$

The maximum speed occurs when

This is at **periastron** – the point of closest approach to the star, when $r =$

The minimum speed occurs when

This is at **apastron** – the point furthest from the star, when $r =$

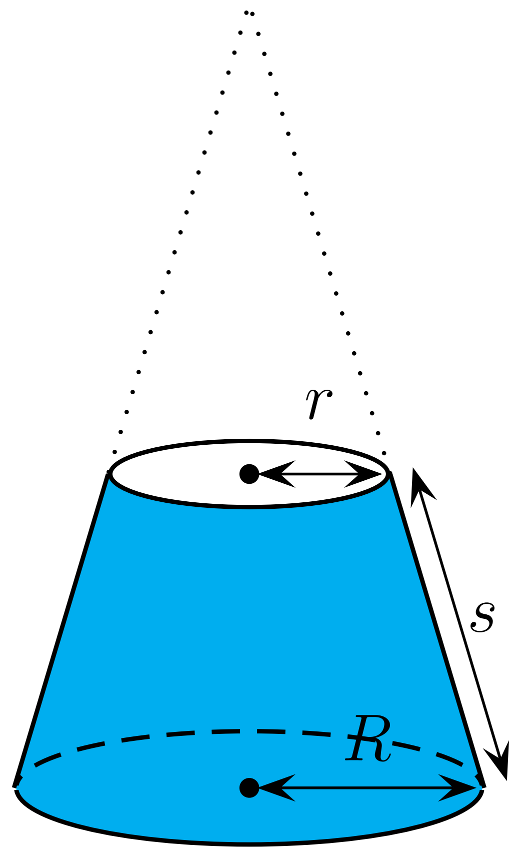
5.4 Surface Area

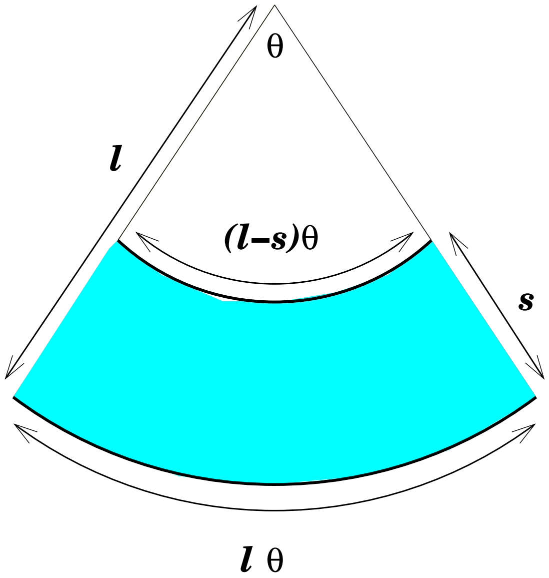
Finding the area of a general surface is covered in our second year courses in several variable calculus (MATH2011, MATH2111, MATH2069).

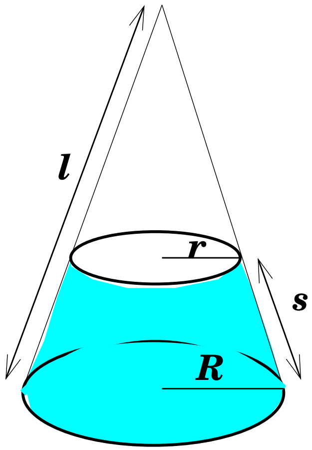
In this course we look at a special case: areas of surfaces given by revolving curves around an axis.

We derive the formula heuristically from the surface area of the **frustum of a cone**.

The coloured area in the diagram has inner radius r, outer radius R and slant height s.







If we cut and then unwrap the frustum surface we get the diagram on the left here

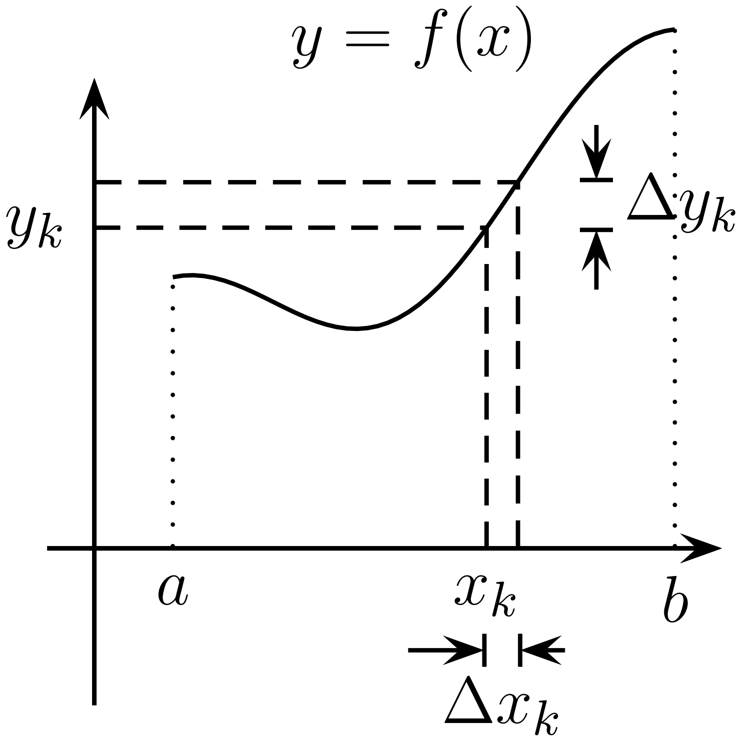
We see that

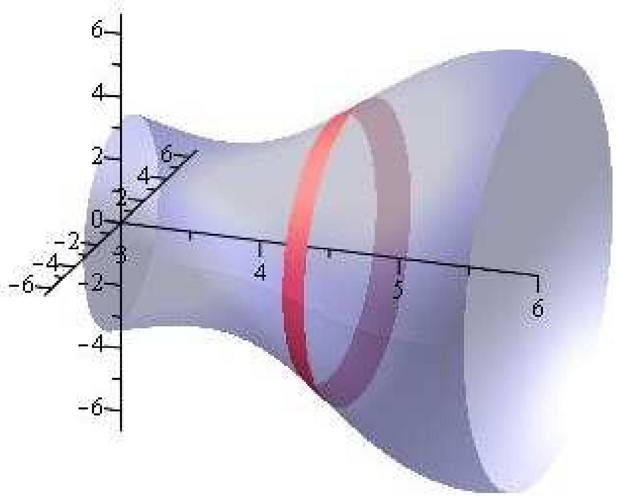
$$2\pi R = \ell\theta, \quad 2\pi r = (\ell - s)\theta \quad \text{so} \quad 2\pi(R - r) = s\theta$$

The area of the frustum is then

$$\left(\frac{\theta}{2\pi}\right)\pi\ell^2 - \left(\frac{\theta}{2\pi}\right)\pi(\ell - s)^2 = \frac{1}{2\theta}\left[\ell\theta - (\ell - s)\theta\right]\left[\ell\theta + (\ell - s)\theta\right]$$

$$= \frac{1}{2\theta}\left(2\pi R - 2\pi r\right)\left(2\pi R + 2\pi r\right) = \pi(r + R)s$$





Let the curve $y = f(x)$ be rotated around the x-axis between $x = a$ and $x = b$, where $f(x) > 0$ on $[a, b]$.

Slice the surface into strips, approximate each strip by a frustum with radii y_k and $y_k + \Delta y_k$, and slant height

$$s_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

Area of the kth frustum is $A_k = \pi[y_k + (y_k + \Delta y_k)]s_k$.
So

$$A_k = \pi(2y_k + \Delta y_k)\sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\,\Delta x_k.$$

The **surface area** is the sum of the frustum areas in the limit $\Delta x \to 0$.
That is

$$\begin{aligned} A &= \lim_{\Delta x \to 0} \sum_k \pi(2y + \Delta y)\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x \\ &= \int_a^b 2\pi y(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \end{aligned}$$

If the curve were given by $(x(t), y(t))$, with $y(t) > 0$, the kth slant height would be

$$s_k \approx \sqrt{(x'(t)\Delta t)^2 + (y'(t)\Delta t)^2} = \sqrt{(x'(t))^2 + (y'(t))^2}\,\Delta t$$

leading to

$$A = \int_a^b 2\pi\, y(t)\, \boldsymbol{y(t)} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

For rotating around the y-axis, swap x and y, so the parametric case would be

$$A = \int_a^b 2\pi \boldsymbol{x(t)} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

(We need $x(t) > 0$ for this to work.)

If the curve is given in polars by $r(\theta)$, $\theta_0 \leq \theta \leq \theta_1$, then the kth slant height is

$$s_k \approx \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,\Delta\theta$$

So we get surface areas

$$\int_{\theta_0}^{\theta_1} 2\pi r(\theta)\,\boldsymbol{\sin\theta}\sqrt{r^2(\theta) + \left(\frac{dr}{d\theta}\right)^2}\,d\theta \quad \text{around } x\text{-axis}$$

and

$$\int_{\theta_0}^{\theta_1} 2\pi r(\theta)\,\boldsymbol{\cos\theta}\sqrt{r^2(\theta) + \left(\frac{dr}{d\theta}\right)^2}\,d\theta \quad \text{around } y\text{-axis}$$

Example 6 *Find the area swept out when the upper half of the cardioid $r = 1 + \cos\theta$, $0 \leq \theta \leq \pi$ (see example 3) is rotated around the x-axis.*

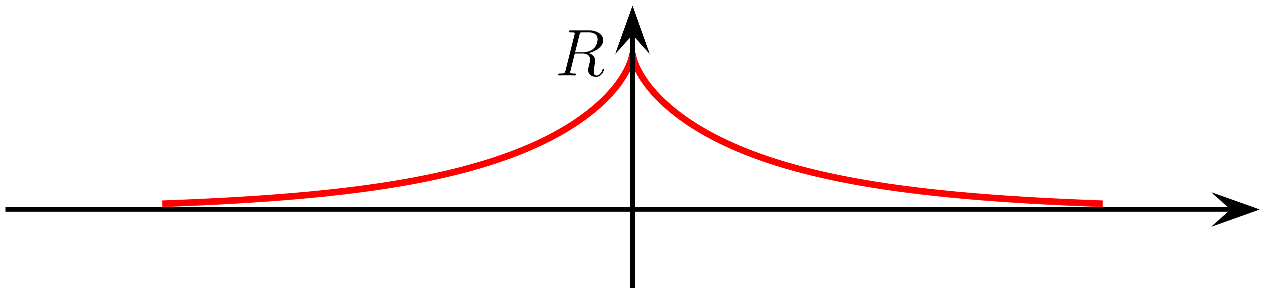
SOLUTION: We know that $\left(r^2 + (r_\theta)^2\right)^{1/2}$ from example 3. So the area is

Example 7 *Find the area of the surface formed by rotating the curve*

$$(R(t - \tanh t), R\operatorname{sech} t), \quad t \in \mathbb{R}$$

around the x-axis.

SOLUTION: This curve is called the **tractrix**:



We have

$$\begin{aligned}(x')^2 + (y')^2 &= R^2(1 - \operatorname{sech}^2 t)^2 + (-R\operatorname{sech} t \tanh t)^2 \\ &= R^2 \left(\tanh^4 + \operatorname{sech}^2 t \tanh^2 t\right) \\ &= R^2 \tanh^2 t\end{aligned}$$

We appeal to symmetry, and for the area of the whole surface find the area for $t \geq 0$ and double:

$$A = 2\int_0^\infty 2\pi R \,\mathrm{sech}\, t \, R \tanh t \, dt =$$

Note that the length of the tractrix is $\displaystyle\int_0^\infty R \tanh t \, dt = \infty$,

giving us another paradox similar to the Gabriel's trumpet one in the Notes: an infinite curve is rotated to give a finite area. (And a finite volume in this case.)