MATH1241 Algebra, 2018
Group 2 — Tues 12 pm, Thurs 10 am

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Acknowledgement: Lectures based on Dr. Chi Mak's notes.

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Lecture group 2:

- Tuesday 12:00, *Algebra*, Catherine Greenhill, **Webster A**
- Tuesday 13:00, *Calculus*, John Steele, **Webster A**
- Thursday 9:00, *Calculus*, John Steele, **Webster A**
- Thursday 10:00, *Algebra*, Catherine Greenhill, **Webster A**

MATH1241 Mathematics 1B is a 6UOC course offered in Semester 2.

Excluded courses for MATH1241:

MATH1011, MATH1031, MATH1231, MATH1251,
ECON1202, ECON2291.

You should not be enrolled in any of these courses if you are enrolled in MATH1241.

School's web address:	`http://www.maths.unsw.edu.au`
UNSW Moodle:	`http://moodle.telt.unsw.edu.au`

There is lots of information for current students on the School's webpage. Click on "Current Students $\hookrightarrow$ Student Services".

Visit the MATH1241 Moodle homepage for lecture notes, announcements etc. There is also a help forum.

Check your official UNSW email address regularly!!

`z1234567@student.unsw.edu.au`

Please only send email from this address when contacting academic staff.

We *assume* that we have read *everything* that is sent to this email address!!

Prerequisites

65+ in MATH1141 or MATH1131.

If you do not have this prerequisite then you should unrol yourself from MATH1241 and immediately and enrol in MATH1231. If you need enrolment help, please the **Student Services Centre**, RC-3090, Red Centre (Centre Wing).

Tutorials and tests

Classroom Tutorials

Classroom Tutorials start in week 2 and run up to week 13. Check your timetable on myUNSW at the **end** of Week 1. Make sure you attend the CORRECT tutorial.

Algebra and Calculus tutorials will both be in the *first half of the week*, alternating from week to week.

Class tests

The second tutorial time will only be used for Class Tests.
Algebra and calculus class tests will be conducted together in weeks 6 and 11.

Online Tutorials

Every week you will have an online tutorial. Links for videos and Maple TA exercises for these will be posted on the Ed discussion forum.

Course Packs

All students are strongly advised to buy the MATH1231/MATH1241 Course Pack from the University Bookshop. The coursepack is also available on Moodle.

Please READ the Course Information Booklet, especially the School's policy on assessment. Really. I know it's boring, but it could help you to pass this course.

Do I need a textbook? Wait a week or two: you might find that the lecture notes are enough and that you do not need a textbook.

Dr. Chi Mak has prepared a **summary of MATH1131**, which you can find on the MATH1241 Moodle homepage.
If you have forgotten **Gaussian elimination** (row reduction), revise it NOW!!!

MATH1241 has many abstract concepts which take time to settle into your brain. If you fall behind, we will soon be speaking a different language (basis, kernel, eigenvalue...) Don't fall behind!

Seek help if you need it: your tutor, the Mathematics Drop-in Centre, the staff consultation roster, or me.

Chapter 6 Vector Spaces

You learnt about *vectors* in MATH1231. A vector is used to model a quantity with magnitude and direction. Geometrically a vector is an arrow; algebraically it is an n-tuple of real numbers, that is, an element in $\mathbb{R}^n$.

Vectors can be added, stretched, squished, reflected...

Question: What other objects (quantities) can be modelled by vectors?

Can the following be vectors?

- Matrices?
- Polynomials?
- Functions?

To discuss these questions we need four ingredients:

- A non-empty set of objects, V. Elements of V are called *vectors*.
- A set of scalars (numbers), $\mathbb{F}$. For us, $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.
- An operation, $+$, for adding two elements of V together to produce a new vector. This is called *vector addition*.
- An operation, $*$, for multiplying an element in V by a scalar in $\mathbb{F}$ to produce a new vector called $\lambda * \mathbf{v}$, usually denoted by $\lambda\mathbf{v}$. This is called *multiplication by a scalar* (or just *scalar multiplication*).

If the system $(V, +, *, \mathbb{F})$ satisfies ten axioms (rules) which we meet next, then we call it a *vector space*.

We often just write V to mean $(V, +, *, \mathbb{F})$, when the field of scalars and the operations are understood.

Definition of a vector space

Definition (Vector space)

A **vector space** V over the set of scalars $\mathbb{F}$ is a non-empty set of objects, called vectors, for which addition of vectors and multiplication of a vector by a scalar are defined and obey the following axioms.

1. **Closure under addition**. If $\mathbf{u}, \mathbf{v} \in V$ then $\mathbf{u} + \mathbf{v} \in V$.
2. **Associative law of addition**.
   If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ then $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
3. **Commutative law of addition**.
   If $\mathbf{u}, \mathbf{v} \in V$ then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
4. **Existence of zero**. There is a special element $\mathbf{0}$ in V called the **zero vector** which has the property that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
5. **Existence of Negative**. For each $\mathbf{v} \in V$ there exists an element $\mathbf{w} \in V$ (the negative of $\mathbf{v}$, usually written as $-\mathbf{v}$) such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$.

- The first five axioms are about the vector addition.

6. **Closure under scalar multiplication**. If $\mathbf{v} \in V$ and $\lambda \in \mathbb{F}$ (that is, λ is a scalar) then $\lambda\mathbf{v} \in V$.
7. **Associative law of multiplication by a scalar**. If $\lambda, \mu \in \mathbb{F}$ and $\mathbf{v} \in V$ then $\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$.
8. If $\mathbf{v} \in V$ then $1\mathbf{v} = \mathbf{v}$.
9. **Scalar distributive law**. If $\lambda, \mu \in \mathbb{F}$ and $\mathbf{v} \in V$ then $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$.
10. **Vector distributive law**. If $\lambda \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in V$ then $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$.

- Axioms 6 to 8 are about scalar multiplication.
- The last two are distributive laws.

Remark

If you know that V is a vector space, then you know that V satisfies these ten axioms. Sometimes that is *all you know*. The axioms form the starting point for proofs of all other facts about vector spaces.

You should know how to

- prove that a system V is a vector space
- prove that V is not a vector space

The addition and scalar multiplication for $\mathbb{R}^n$ defined in MATH1231 are called the *usual addition* and *usual scalar multiplication* for $\mathbb{R}^n$.

For any $\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}, \quad \text{and} \quad \lambda\mathbf{a} = \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{pmatrix}.$$

Example

Prove that $\mathbb{R}^2$ with the **usual** rules for addition and scalar multiplication satisfies the scalar distributive law.

Proof.

- Write the hypothesis at the beginning.
- Write the conclusion at the end.
- Fill in the arguments. Bear in mind what we can assume.

Proof (continued).

$\square$

To prove that $\mathbb{R}^2$ is a vector space, we have to prove that it also satisfies the other nine axioms.

Example

For any positive integer n, the system $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ under the usual addition and multiplication by a scalar.

For a proof, see Section 6.1, Algebra Notes.

Recall that a polynomial over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ of degree k is a function $p : \mathbb{F} \to \mathbb{F}$ such that

$$p(x) = a_0 + a_1 x + \cdots + a_k x^k, \text{ where } a_0, a_1, \ldots, a_k \in \mathbb{F} \text{ and } a_k \neq 0.$$

The set of polynomials of degree at most n is denoted by $\mathbb{P}_n$.

Note that $\mathbb{P}_n$ includes the *zero polynomial*.

For any $p, p_1, p_2 \in \mathbb{P}_n$ and any scalar $\lambda \in \mathbb{F}$, the polynomials $p_1 + p_2$ and λp are defined by

$$(p_1 + p_2)(x) = p_1(x) + p_2(x) \quad \text{and} \quad (\lambda p)(x) = \lambda p(x),$$

for all $x \in \mathbb{F}$.

IMPORTANT!!!! Note that p is a *polynomial* (function), while $p(x)$ is a *scalar* which equals the value of p at x. **(NOT THE SAME.)**

Example

Prove that $\mathbb{P}_2$ satisfies the vector distributive law.

Proof.

Proof (continued).

$\square$

Example

The set $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : y \geqslant x \right\}$ with the usual rules for addition and scalar multiplication in $\mathbb{R}^2$ is **not** a vector space.

To prove that a system is not a vector space, we only need to show that it does not satisfy one of the axioms by a **counterexample**.

Proof.

$\square$

Example

Prove that the system $(\mathbb{R}^2, \oplus, *, \mathbb{R})$ is not a vector space. Here, $*$ is the usual scalar multiplication, and the vector addition is defined by

$$\begin{pmatrix} a \\ b \end{pmatrix} \oplus \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + 2c \\ b + 2d \end{pmatrix}.$$

Proof.

$\square$

Some important examples of vector spaces

- The vector spaces $\mathbb{R}^n$ over $\mathbb{R}$, and $\mathbb{C}^n$ over $\mathbb{C}$.
- The set $\mathbb{P}_n(\mathbb{F})$ of all polynomials of degree at most n is a vector space over $\mathbb{F}$.
- The vector space $M_{mn}(\mathbb{F})$ of all $m \times n$ matrices over $\mathbb{F}$ is a vector space over $\mathbb{F}$.
- The set $\mathbb{P}$ of all polynomials over $\mathbb{F}$ (of any degree) is a vector space over $\mathbb{F}$.
- The set $\mathcal{R}[X]$ of all real-valued functions with domain X is a vector space over $\mathbb{R}$.

Attempt Problems 6.1.

Vector arithmetic

The properties stated in this section are easy to prove for the vector spaces mentioned in the last section. However, we can prove that they are true for **all** vector spaces, using our ten axioms and properties of the scalars.

Proposition 1

In any vector space V, the following properties hold.

1. **Uniqueness of Zero.** *There is one and only one zero vector.*
2. **Cancellation Property.** *If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ satisfy*
   $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, *then* $\mathbf{v} = \mathbf{w}$.
3. **Uniqueness of Negatives.** *For all $\mathbf{v} \in V$, there exists only one $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$.*

Proposition 2

Suppose that $\mathbf{v}$ is a vector in a vector space V; λ is a scalar; 0 is the zero scalar; $\mathbf{0}$ is the zero vector in V. Then the following properties hold.

1. $\lambda\mathbf{0} = \mathbf{0}$.
2. $0\mathbf{v} = \mathbf{0}$.
3. $(-1)\mathbf{v} = -\mathbf{v}$. *Here -1 is a scalar and $-\mathbf{v}$ is the additive inverse of $\mathbf{v}$.*
4. *If $\lambda\mathbf{v} = \mathbf{0}$, then either $\lambda = 0$ or $\mathbf{v} = \mathbf{0}$.*
5. *If $\lambda\mathbf{v} = \mu\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$ then $\lambda = \mu$.*

Example

In the vector space $\mathbb{R}^3$, show that $\lambda\mathbf{0} = \mathbf{0}$ for all $\lambda \in \mathbb{R}$.

Proof.

□

Example

Prove the cancellation property for any vector space V.

Proof.

□

Example

Let V be a vector space. Prove that $\lambda\mathbf{0} = \mathbf{0}$ for all scalar λ.

Note: We have to prove the property for any vector space V.

Proof.

$\square$

Attempt Problems 6.2.

Subspaces

We saw that the subset S of $\mathbb{R}^2$ defined by

$$S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : y \geqslant x \right\}$$

is **not** a vector space (example on p. 19). Here we "borrowed" (or "inherited") the field of scalars $\mathbb{R}$, the operations of addition and multiplication by a scalar from $\mathbb{R}^2$.

Now consider the subset $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : y = 2x \right\}$, with the field of scalars, addition and scalar multiplication as in $\mathbb{R}^2$. Is W (or formally $(W, +, *, \mathbb{R})$) a vector space?

Check the axioms!

Now consider the subset $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : y = 2x \right\}$, with the field of scalars, addition and scalar multiplication as in $\mathbb{R}^2$. Is W (or formally $(W, +, *, \mathbb{R})$) a vector space?

Check the axioms!

- Thanks to the properties of $\mathbb{R}^2$, the subset W satisfies axioms 2, 3, 7, 8, 9 and 10.
- If the zero vector of $\mathbb{R}^2$ belongs to W, then W satisfies axiom 4.
- We need to check that W is closed under addition (axiom 1) and scalar multiplication (axiom 6).
- Since $(-1)\mathbf{v} = -\mathbf{v}$, closure under scalar multiplication implies that W satisfies axiom 5.

Hence W is a vector space.

In general, we have the following definition.

Definition (Subspace)

A subset S of a vector space V is called a **subspace** of V if S is itself a vector space over the same field of scalars as V, under the same rules for addition and multiplication by scalars. In addition, if S is a proper subset of V then S is called a **proper subspace** of V.

Happily, because of the properties of V, we only need to check 3 conditions to conclude that S is a vector space (a subspace of V).

Theorem (Subspace Theorem)

A subset S of a vector space V is a subspace if

i) the zero vector of V belongs to S;

ii) $\mathbf{u} + \mathbf{v} \in S$ for all $\mathbf{u}, \mathbf{v} \in S$; and

iii) $\lambda \mathbf{v} \in S$ for all $\mathbf{v} \in S$, scalars λ.

Prove or disprove that a set is a subspace

Let V be a vector space. To show that $S \subseteq V$ is a subspace of V, we first check that the zero vector of V is in S. If it is **not** then S **cannot** be a subspace. If it is then we proceed to check the closure requirements.

Example

Show that $S = \left\{\mathbf{x} \in \mathbb{R}^3 : x_1 - 2x_2 + 3x_3 = 1\right\}$ is not a subspace.

Solution

Example

Show that $S = \{p \in \mathbb{P}_2 : p(1) = 2\}$ is not a subspace.

Solution

Example

Show that $S = \{\mathbf{x} \in \mathbb{R}^3 : x_1 - 2x_2 + 3x_3 = 0\}$ is a subspace.

Solution

Solution (continued)

Warning: A subset $S \subseteq V$ which contains the zero vector of V may **not** be a subspace.

Example

Prove that $S = \left\{ \mathbf{x} \in \mathbb{R}^3 : x_3 = x_1^2 + x_2^2 \right\}$ is not a subspace of $\mathbb{R}^3$.

Solution

Exercise. Criticise the following proof. Find all the mistakes, and look for parts which are not exactly wrong but could have been expressed better. (The proof runs over two pages.)

Problem. Show that $S = \left\{ \mathbf{x} \in \mathbb{R}^4 : x_1 - 6x_2 + x_4 = 0 \right\}$ a vector space.

Proof. Clearly S contains 0.
Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4$. Then

$$x_1 - 6x_2 + x_4 = 0 \quad \text{and} \quad y_1 - 6y_2 + y_4 = 0,$$

so

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 - 6x_2 + x_4) + (y_1 - 6y_2 + y_4) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

So $\mathbf{x}$ is closed under addition. (PTO)

Proof, continued.

Let $\alpha \in \mathbb{R}$. Then

$$x_1 - 6x_2 + x_3 = 0.$$

Therefore

$$(\alpha x_1) - 6(\alpha x_2) + (\alpha x_3) = \alpha(x_1 - 6x_2 + x_3) = \alpha 0 = 0$$

so it is closed under multiplication.

An example of subspaces of $\mathbb{P}_n$

Example

Show that $S = \{p \in \mathbb{P}_2(\mathbb{R}) : xp'(x) - 2p(x) = 0\}$ is a subspace.

Solution

Solution (continued)

Attempt Problems 6.3.

Linear combinations

The definitions of a linear combination and the span of two vectors in $\mathbb{R}^n$ in MATH1231 Algebra Notes (Section 1.5.1) can be generalised to any number of vectors in any vector space V.

Definition (Linear Combination)

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of a vector space V over a field $\mathbb{F}$. Then a **linear combination** of S is a sum of scalar multiples of the form

$$\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n \quad \text{with} \quad \lambda_1, \ldots, \lambda_n \in \mathbb{F}.$$

Proposition

If S is a finite set of vectors in a vector space V, then every linear combination of S is also a vector in V.

Example

Suppose that $S = \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \right\}$. The following vectors are linear combinations of S.

$$2\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} =$$

$$2\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} =$$

Span

Definition (Span)

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of a vector space V. Then the **span** of the set S is the set of all linear combinations of S, and is denoted by $\text{span}(S)$ or $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

If $\text{span}(S) = V$, the set S is called a **spanning set** of V, and S is said to **span** V.

Example

Prove that $\{1,\ x,\ x^2\}$ is a spanning set for $\mathbb{P}_2$.

Proof.

$\square$

How to check if a vector is in a span?

Example

Let $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \in \mathbb{R}^3$, and define the set $S = \left\{ \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$.

Is $\mathbf{u} \in \text{span}(S)$? If so, write $\mathbf{u}$ as a linear combination of S. How about $\mathbf{v}$?

Solution

Solution (continued)

Solution (continued)

Condition(s) for a vector to be in a span

Example (continued from the previous example.)

Is S a spanning set for $\mathbb{R}^3$? If not, find conditions on $\mathbf{b} \in \mathbb{R}^3$ to belong to span(S). Give a geometric interpretation of span(S).

Solution

Solution (continued)

An example in $\mathbb{P}_n$

Example

Is the set

$$S = \left\{1 + 2x + 3x^2,\ 2 + 4x + x^2,\ 1 + 2x + 8x^2,\ 1 - x + 4x^2\right\}$$

a spanning set for $\mathbb{P}_2$?

Solution

Solution (continued)

Solution (continued)

Is there a **proper subset** of S which still spans $\mathbb{P}_2$?

Properties of a span

Theorem

If S is a finite non-empty set of vectors in a vector space V, then span(S) is a subspace of V.

Proof.

Proof (continued)

Proof (continued).

$\square$

Remarks

1. Any subspace of V containing a finite set of vectors S contains $\text{span}(S)$. Hence, $\text{span}(S)$ is the smallest subspace containing S.
2. The only subspaces of $\mathbb{R}^3$ are $\{\mathbf{0}\}$, lines through the origin, planes through the origin, and $\mathbb{R}^3$ itself.

Matrices and spans in $\mathbb{R}^m$

Suppose that $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of vectors in $\mathbb{R}^m$, and $\mathbf{b}$ is a vector in $\mathbb{R}^m$. Let A be the matrix $(\mathbf{v}_1|\cdots|\mathbf{v}_n)$. Then $\mathbf{b} \in \text{span}(S)$ if and only if we can find scalars $x_1, \ldots, x_n$ such that

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{b}.$$

Equivalently, $\mathbf{b}$ is a linear combination of S if and only if it can be written as $A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$.

In other words, $\mathbf{b} \in \text{span}(S)$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} \in \mathbb{R}^n$. [Algebra Notes: Proposition 3 in Section 6.4]

So, the span of the columns of a matrix is a *useful idea*. Let's give it a name...

Column space

Definition

The subspace of $\mathbb{R}^m$ spanned by the columns of an $m \times n$ matrix A is called the **column space** of A, and is denoted by $\text{col}(A)$.

Example

Determine whether the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is in the column space of

$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 1 & 8 \end{pmatrix}$.

Solution

Solution (continued)

Attempt Problems 6.4.

Linear independence

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of a vector space V. The following statements are equivalent:

1. If $\lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n = \mu_1\mathbf{v}_1 + \cdots + \mu_n\mathbf{v}_n$ then $\lambda_1 = \mu_1$, ..., $\lambda_n = \mu_n$. That is, we can "compare coefficients".
2. The zero vector cannot be written as a non-trivial combination of vectors in S. That is, if $\lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n = \mathbf{0}$, then $\lambda_1 = \cdots = \lambda_n = 0$.
3. None of the vectors in S is a linear combination of the other vectors in S.
4. There are no proper subsets $T \subsetneq S$ such that $\text{span}(T) = \text{span}(S)$.

If S satisfies one (equivalently, all) of these conditions then S is **linearly independent**. Otherwise, we say that S is **linearly dependent**.

Example

Is either set linearly independent? $S_1 = \left\{ \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$,

$S_2 = \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$.

It is not quite so easy when there are more than two vectors.

Example

Consider the set of vectors $S_3 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 7 \end{pmatrix} \right\}$.

No vector in S_3 is a scalar multiple of another vector in S_3, but

$$\begin{pmatrix} -1 \\ 7 \end{pmatrix} =$$

The set S_3 is linearly dependent.

Definition (Linear independence / Linear dependence)

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a set of vectors.

1. If we can find scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ **not all zero** such that

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n = \mathbf{0},$$

   then we say that S is a **linearly dependent set**. We say that the vectors in S are **linearly dependent**.
2. If the **only** solution of

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n = \mathbf{0}$$

   is $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$ then we say that S is a **linearly independent set**. We say that the vectors in S are **linearly independent**.

Problems about linear independence

Example

Show that $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is a linearly independent set.

Show that if $\mathbf{b} \in \operatorname{span}(S)$ then there is *only one way* to write $\mathbf{b}$ as a linear combination of vectors in S.

Solution

Solution (continued)

Example

Suppose that

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix} \text{ and } \mathbf{v}_4 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}.$$

a) Prove that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

b) Find a linearly independent subset of S with the same span as S.

Solution

Solution (continued)

An example in $\mathbb{P}_n$

Example

Prove that $\{1 + 2x - x^2,\ -3 - x - 2x^2,\ 2 + 3x + x^2\}$ is a linearly independent subset of $\mathbb{P}_2$.

Proof.

Proof (continued).

$\square$

Example

Recall the set $\mathcal{R}(\mathbb{R})$ of all real-valued functions is a vector space. Let $f, g, h \in \mathcal{R}(\mathbb{R})$ where

$$f(x) = 2, \qquad g(x) = \sin^2(x), \qquad h(x) = \cos^2(x) \quad \text{for all} \quad x \in \mathbb{R}.$$

Is the set of functions $\{f, g, h\}$ linearly independent?

Extracting a maximal linearly independent subset

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^m$. Let $A = (\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_n)$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

We know that $A\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$ is a linear combination of S.

By definition, S is linearly dependent iff $A\mathbf{x} = \mathbf{0}$ has a **non-zero** solution. (The equation will then has infinitely many solutions: why?)

Equivalently, S is linearly independent iff the **only** solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$ (unique solution).

Equivalently, $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly independent iff the **only** solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$ (unique solution).

We omit the right hand zero column of the augmented matrix for $A\mathbf{x} = \mathbf{0}$ and we reduce A to a row-echelon form U. Then

- S is linearly independent iff *all columns* of U are *leading*;
- S is linearly dependent iff *at least one of the columns* of U is *non-leading*;
- the vectors in S corresponding to the *leading columns* of U form a *linearly independent subset* of S with the *same span* as S.

Linear independence and span

Let S be a finite non-empty set of vectors in a vector space V.

1. *Let $\mathbf{v}$ be a vector which can be written as a linear combination of S. The values of the scalars in the linear combination are unique if and only if S is linearly independent.*
2. *S is linearly independent if and only if no vector in S can be written as a linear combination of the other vectors in S.*
3. *For any $\mathbf{v} \in V$, we have $\text{span}(S \cup \{\mathbf{v}\}) = \text{span}(S)$ if and only if $\mathbf{v} \in \text{span}(S)$.*
4. *The span of every proper subset of S is a proper subspace of $\text{span}(S)$ if and only if S is linearly independent.*
5. *If S is linearly independent and $\mathbf{v} \in V$ but not in $\text{span}(S)$, then $S \cup \{\mathbf{v}\}$ is linearly independent.*

Example

Prove the statements (1) and (2).

Proof.

Proof (continued)

Proof (continued).

□

Attempt Problems 6.5.

Basis and dimension

We have seen that a (finite) linearly independent spanning set B of a vector space V has an important property that every vector in V is a *unique* linear combination of B. This will be hugely useful.

Definition (Basis)

A set B of vectors in a vector space V is called a **basis** if B is linearly independent and $V = \text{span}(B)$.

1. The basis for the zero vector space $\{\mathbf{0}\}$ is the empty set: $\emptyset$ or $\{\ \}$.
2. Let $\mathbf{e}_i$ be the vector in $\mathbb{R}^n$ with the i-th entry 1 and all other entries 0. The set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a linearly independent spanning set of $\mathbb{R}^n$. Hence it is a basis, called the **standard basis** of $\mathbb{R}^n$.

Example

In $\mathbb{R}^2$, the standard basis is $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

In $\mathbb{R}^3$, the standard basis is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

The set $\{1, x, \dots, x^n\}$ is the standard basis for $\mathbb{P}_n$.

The basis of a vector space is **not** unique. Besides the standard basis, the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ also form a basis for $\mathbb{R}^2$ and there are many others.

Although standard bases are convenient to use, other bases are essential, like orthonormal bases and bases consisting of eigenvectors. (More later.)

How to prove that a set is a basis?

Example

Do the vectors $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ form a basis for $\mathbb{R}^3$?

Solution

Solution (continued)

Example

Prove that

$$S = \{1 + 2x + x^2,\ 1 + 3x + 2x^2,\ -1 + 2x + 5x^2\}$$

is a basis for $\mathbb{P}_2(\mathbb{R})$.

Solution

Solution (continued)

Orthonormal basis

We learnt in MATH1231 that a set of vectors in $\mathbb{R}^n$ is called *orthonormal* if all its elements have *unit length* and they are mutually orthogonal (perpendicular). Recall two vectors are *orthogonal* if their dot product is zero.

Definition (Orthonormal Basis)

An **orthonormal basis** for $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$ which is an orthonormal set.

The standard basis for $\mathbb{R}^n$ is an orthonormal basis.

Example

Prove that a finite orthonormal set of vectors in $\mathbb{R}^n$ is linearly independent.

Proof.

Proof (continued).

$\square$

Example

Show that the set $\left\{ \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{4}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^3$. Write the vector $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ as a linear combination of the orthonormal basis.

Solution

Solution (continued)

Dimension

In this section, we assume every vector space has a finite spanning set. There are infinitely many bases for a vector spaces V, but all of them have one thing in common. We first state the following, without proof.

Theorem

The number of vectors in any spanning set for a vector space V is always greater than or equal to the number of vectors in any linearly independent set in V.

This theorem leads to the next one, which links different bases for a given vector space by their size.

Theorem

If a vector space V has a finite basis then every basis for V contains the same number of vectors. That is, if $B_1 = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $B_2 = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ are two bases for V then $m = n$.

Proof.

$\square$

Definition (Dimension)

If V is a vector space with a finite basis, then the number of basis vectors is called the **dimension** of V and is denoted by $\dim(V)$.

A vector space with a finite basis is called a **finite dimensional vector space**. We shall only focus on finite dimensional vector spaces.

Example

The standard basis for $\mathbb{R}^n$ consists of n vectors, so $\dim(\mathbb{R}^n) = n$.

Example

What is the dimension of $\mathbb{P}_n$?

Well, the standard basis for $\mathbb{P}_n$ is $\{1, x, \ldots, x^n\}$, so $\dim(\mathbb{P}_n) = n + 1$.

Since subspaces are vector spaces, we can find bases and dimension of a subspace. In particular, the span of a finite set is a subspace spanned by this set, so a **maximal independent subset of the spanning set** will be a basis for the subspace.

Example

Find a basis for the subspace spanned by

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \right\}.$$

What is the dimension of span(S)?

Solution

Solution (continued)

Size of an independent set, size of a spanning set, and dimension

Theorem

For any finite dimensional vector space V,

1. *the number of vectors in any spanning set for V is greater than or equal to the dimension of V,*
2. *the number of vectors in any linearly independent set in V is less than or equal to the dimension of V,*
3. *if the number of vectors in a spanning set for V is equal to the dimension of V, then that set is linearly independent and so forms a basis for V,*
4. *if the number of vectors in a linearly independent subset of V equals the dimension of V, then those vectors also span V and thus form a basis for V.*

For proofs see Algebra Notes Section 6.6, Theorem 3.

Example

Can 5 vectors in $\mathbb{R}^4$ be linearly independent?

Solution

Example

Does there exist a spanning set of $\mathbb{P}_3$ which contains only 3 polynomials?

Solution

Example

Construct a set S of 4 vectors in $\mathbb{R}^3$ such that $\text{span}(S) \neq \mathbb{R}^3$.

Solution

Construction of bases

The following theorems guarantee the existence of a basis, for any vector space with a finite spanning set.

Theorem

Let S be a finite spanning set for a vector space V. Then S contains a subset which is a basis for V.

This subset is a maximal independent subset of S.

Theorem

Every linearly independent subset of a vector space V which has a finite spanning set can be extended to a basis for V.

We keep adding vectors which do not belong to the span, one by one, until we reach a basis.

When S is dependent, we can construct a basis of V which contains as many elements in S as possible.

Example

Let $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 3 \\ 6 \end{pmatrix} \right\}$.

i) Find a basis for span(S).

ii) Find a basis for $\mathbb{R}^4$ containing as many as possible of the vectors in S.

Solution

Solution (continued)

Solution (continued)

The methods used in the previous example are summarised in the following theorems.

Theorem

Let S be a finite subset of $\mathbb{R}^m$ and let A be a matrix whose columns are the vectors in S. If U is a row-echelon form for A then the columns of A corresponding to leading columns in U form a basis for span(S).

Theorem

Let S be a finite subset of $\mathbb{R}^m$ and let A be a matrix whose columns are the vectors in S, followed by the standard basis. If U is a row-echelon form for A then the columns of A corresponding to leading columns in U form a basis for $\mathbb{R}^m$ and this basis contains as many vectors in S as possible.

In particular, if S is independent then the resulting basis will contain S as a subset.

Attempt Problems 6.6.

Coordinate vectors

A basis $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for a vector space V defines a **coordinate system** for V.

The coordinate vector of an element $\mathbf{x} \in V$ is the unique vector

$$[\mathbf{x}]_B = \boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} \in \mathbb{R}^n$$

such that $\mathbf{x} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n$.

The map $\varphi : V \to \mathbb{R}^n$ with $\varphi(\mathbf{x}) = [\mathbf{x}]_B$ has very nice properties.

- φ is a bijection (one-to-one and onto)
- $\varphi^{-1} : \mathbb{R}^n \to V$ is easy: $\varphi^{-1}(\boldsymbol{\lambda}) = \lambda_1 \mathbf{v}_1 + \cdots \lambda_n \mathbf{v}_n$.
- φ respects the vector space structure of V.

φ is called a (vector space) **isomorphism** and we say that V and $\mathbb{R}^n$ are **isomorphic vector spaces**.

Example

The set $B = \{1 + 3x - 2x^2, 2 - 5x + 7x^2, 1 + x + x^2\}$ is a basis for $\mathbb{P}_2$. Let $p(x) = 2 - x + x^2$. Find $[p]_B$, the coordinate vector of p with respect to this basis.

Solution

Solution (continued)

End of Chapter 6