Chapter 4
Taylor Series

4.1 Taylor Polynomials

Many differentiable functions e.g., $\sin, \cosh, \log, \exp, \ldots$ etc, can be well approximated by polynomials:

$$f(x) \approx a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = p_n(x).$$

The higher the degree n the better the approximation, provided we make good choices for $a_0, a_1, \ldots$.

Why would we do this?

- polynomials are easy to manipulate – their derivatives and integrals are also polynomials.
- polynomials are easy to evaluate – a finite number of additions and multiplications.
- polynomial approximations can be used to evaluate functions in computer packages and calculators.

Example 1 *Find approximating polys for $y = e^x$ near $x = 0$.*

SOLUTION: We already know a good approximation by a degree 1 poly: the tangent at $x = 0$. So

$$p_1(x) = 1 + x \approx e^x$$

This p_1 has the same value as e^x at zero, and the same derivative: that's the definition of the tangent.

For p_2 the obvious thing to do is set the second derivatives the same at $x = 0$.

As the Notes point out, this leads us to

$$p_2(x) = 1 + x + \frac{1}{2}x^2 \approx e^x$$

Repeating the idea will give us $p_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$.

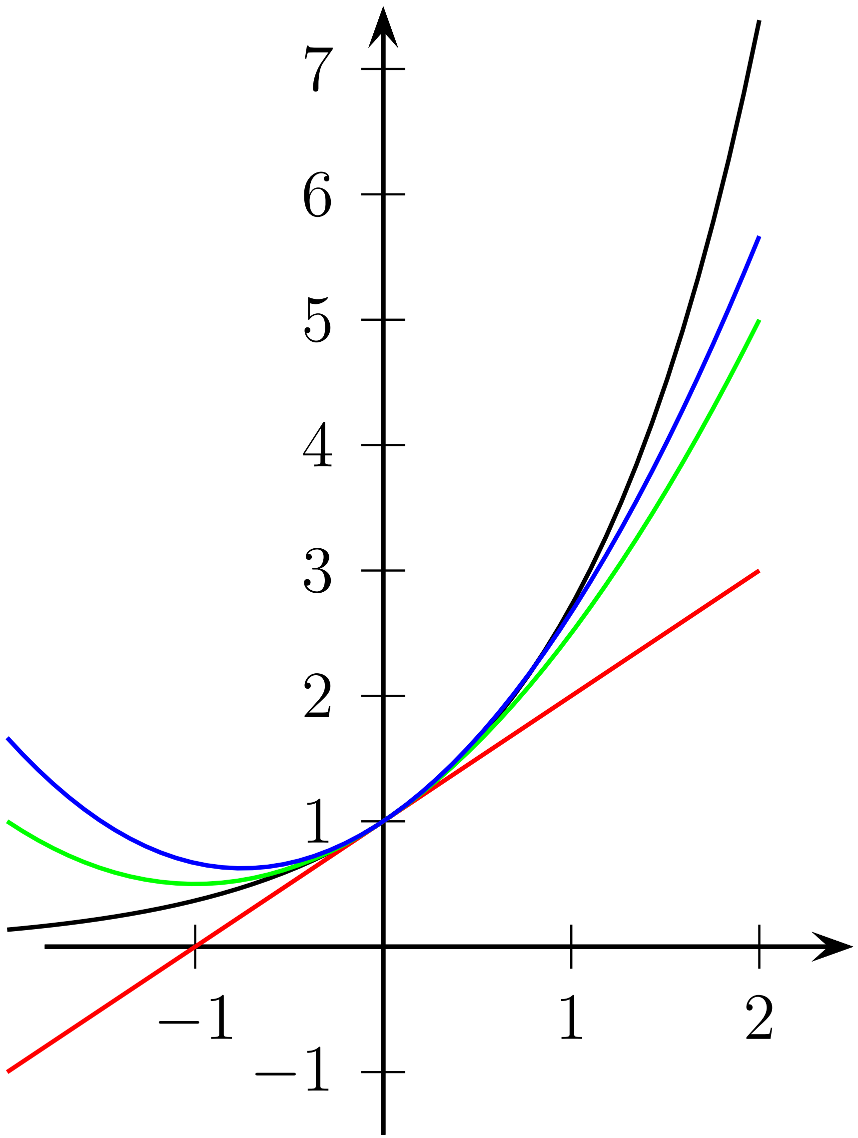
Is this any good?

Well, we can do some calculations, to see if the polys are getting better at finding, say $e^{0.1}$:

n	$p_n(x)$	$p_n(0.1)$	$\lvert f(0.1) - p_n(0.1)\rvert$
1	$1 + x$	1.1	5.17×10^{-3}
2	$1 + x + \frac{x^2}{2}$	1.105	1.71×10^{-4}
3	$1 + x + \frac{x^2}{2} + \frac{x^3}{6}$	1.10516	4.25×10^{-6}

This looks promising.

We can plot these approximations easily enough too…



$$y = e^x$$

$$y = p_1(x) = 1 + x$$

$$y = p_2(x)$$
$$= 1 + x + \frac{1}{2}x^2$$

$$y = p_3(x)$$
$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

This example has given us a method of determining the coefficients a_k in

$$f(x) \approx a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n$$

$$\begin{aligned}
f(x) &\approx a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots & \Rightarrow \quad f(0) &\approx a_0 \\
f'(x) &\approx a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots & \Rightarrow \quad f'(0) &\approx a_1 \\
f''(x) &\approx 2a_2 + (3)(2)a_3 x + (4)(3)a_4 x^2 + \ldots & \Rightarrow \quad f''(0) &\approx 2a_2 \\
f'''(x) &\approx (3)(2)a_3 + (4)(3)(2)a_4 x + \ldots & \Rightarrow \quad f'''(0) &\approx 3!a_3 \\
f^{(iv)}(x) &\approx (4)(3)(2)a_4 + (5)(4)(3)(2)a_5 x + \ldots & \Rightarrow \quad f^{(iv)}(0) &\approx 4!a_4
\end{aligned}$$

$$\Longrightarrow \quad f^{(k)}(0) = k!a_k \quad \Rightarrow \quad a_k = \frac{f^{(k)}(0)}{k!}$$

Suppose that $f(x)$ is n times differentiable at a then the **nth Taylor polynomial of f about a** is

$$
\begin{aligned}
p_n(x) &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \ldots \\
&\qquad \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^n \\
&= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k
\end{aligned}
$$

The above formula is merely a definition of the polynomials $p_n(x)$.

A priori, it is not known whether or in what sense $p_n(x)$ may be regarded as an approximation of $f(x)$ near $x = a$, or it ever can.

Example 2 *Find Taylor polynomials for $\sin(x)$ about 0.*

SOLUTION: Calculating:

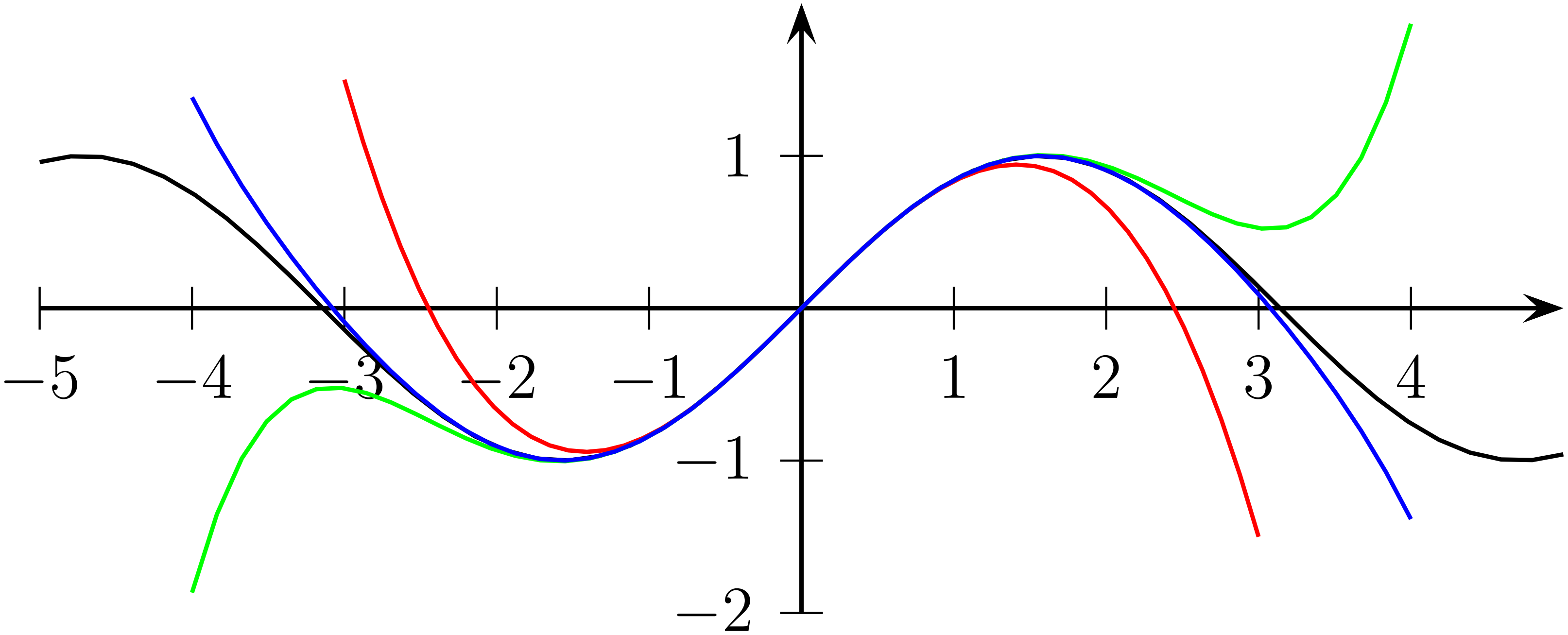
$$
\begin{aligned}
f^{(0)}(x) &= \sin x & \qquad f^{(0)}(0) &= 0 \\
f^{(1)}(x) &= \cos x & \qquad f^{(1)}(0) &= 1 \\
f^{(2)}(x) &= -\sin x & \qquad f^{(2)}(0) &= 0 \\
f^{(3)}(x) &= -\cos x & \qquad f^{(3)}(0) &= -1 \\
f^{(4)}(x) &= \sin x & \qquad f^{(4)}(0) &= 0
\end{aligned}
$$

The pattern repeats from here. So, for example,

$$p_3(x) = x - \frac{x^3}{3!}, \qquad p_4(x) = x - \frac{x^3}{3!}, \qquad p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Note that all the Taylor polynomials around zero are odd – as is the sine.

Plotting:



Here we plot:

$$f(x) = \sin(x) \qquad p_3(x) = x - \frac{x^3}{3!}$$

$$p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \qquad p_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

Example 3 *Find the Taylor polynomial* $p_4(x)$ *for* $\ln(x)$ *about* $x = 1$.

SOLUTION: Note that we could not find the polynomials around $x = 0$ as the log is not defined there. But

$$
\begin{array}{llll}
f^{(0)}(x) = \ln x & f^{(0)}(1) = 0 & f^{(1)}(x) = \dfrac{1}{x} & f^{(1)}(1) = 1 \\
f^{(2)}(x) = -\dfrac{1}{x^2} & f^{(2)}(1) = -1 & f^{(3)}(x) = \dfrac{2}{x^3} & f^{(3)}(1) = 2 \\
f^{(4)}(x) = -\dfrac{6}{x^4} & f^{(4)}(1) = -6 & &
\end{array}
$$

So

$$p_4(x) =$$

Taylor Polynomials to Remember

(all around zero note)

$$\begin{aligned}
\frac{1}{1-x} &\to 1 + x + x^2 + x^3 + x^4 + \ldots + x^n \\
\log(1-x) &\to -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots - \frac{x^n}{n} \\
e^x &\to 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} \\
\sin x &\to x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{\frac{1}{2}(n-1)} \frac{x^n}{n!} \qquad n \quad \text{odd} \\
\cos x &\to 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^{\frac{1}{2}n} \frac{x^n}{n!} \qquad n \quad \text{even}
\end{aligned}$$

The sign of the last term in the polynomials for sine and cosine is chosen to fit the pattern of alternating signs.

Example 4 *Prove* $f(x) = \dfrac{1}{1-x}$ *has Taylor polynomial* $\displaystyle\sum_{k=0}^{n} x^k$ *around 0.*

SOLUTION: We need a formula for the derivatives of f.

Calculating the first few:

$$f'(x) = \frac{1}{(1-x)^2}, \quad f''(x) = \frac{2}{(1-x)^3}, \quad f'''(x) = \frac{6}{(1-x)^4} \cdots$$

suggests a pattern:

Clearly we'd prove this by induction.

The base case, $k = 0$, fits this pattern, which is the first stage of the proof.

For the second stage, assume that, for some k,

$f^{(k)}(x) =$

$$f^{(k+1)}(x) =$$

the result follows by induction.

So $f^{(k)}(0) = k!$ for all k, and the Taylor polynomial of degree n is

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{k=0}^{n} x^k$$

as required.

$\blacksquare$

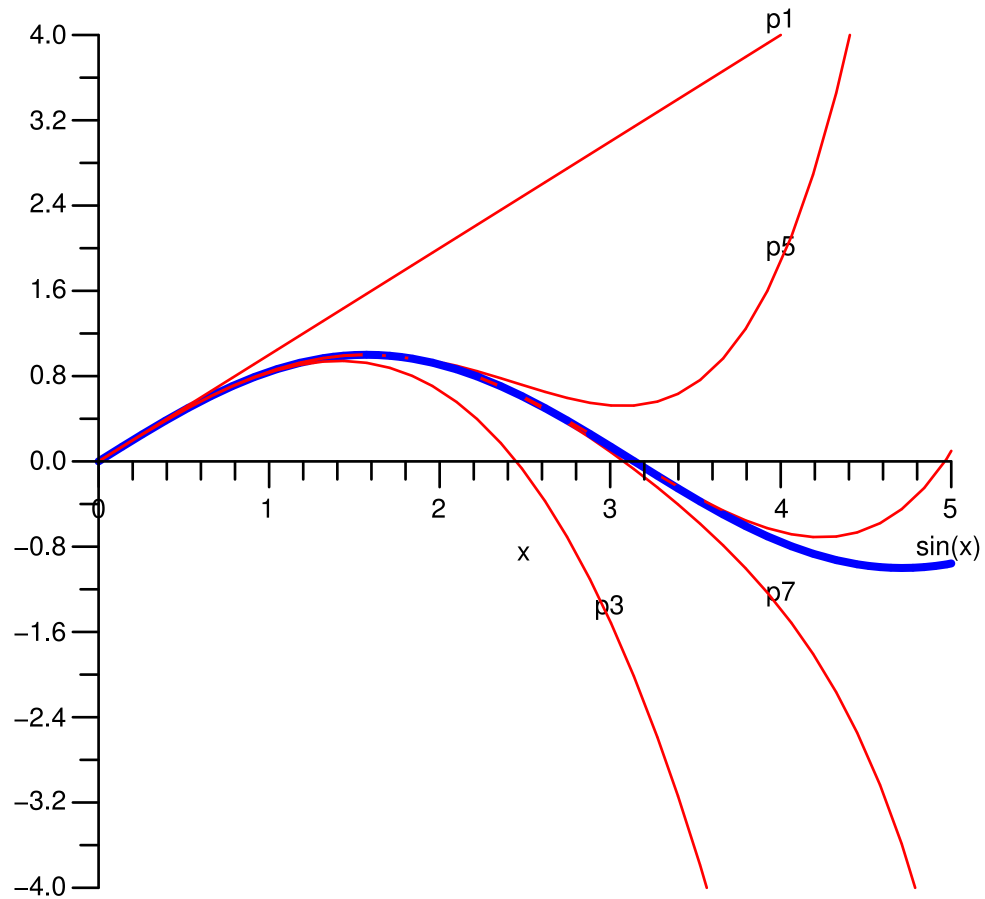
4.2 Taylor's Theorem

Taylor polynomials may provide good approximations to functions.

But how good are they?

What is the error?

Clearly it depends on the position x and the degree of the polynomial!



Theorem 4.1 (Taylor's Theorem) *If $f(x)$ has $n+1$ continuous derivatives on an open interval I containing a then for each $x \in I$*

$$f(x) = p_n(x) + R_{n+1}(x)$$

with

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \qquad \textit{(Taylor polynomial)}$$

*where the **remainder** is*

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\, dt\ .$$

The remainder is the error in approximating a function $f(x)$ by a Taylor polynomial.

Proof: (outline)

$$
\begin{aligned}
\int_0^x f'(t)\,dt &= f(x) - f(0) \\
\Rightarrow f(x) &= \underbrace{f(0)}_{p_0(x)} + \underbrace{\int_0^x f'(t)\,dt}_{R_1(x)} \quad \textcolor{red}{\text{integration by parts}} \\
&= f(0) + \int_0^x \underbrace{f'(t)}_{u(t)} \underbrace{1}_{\frac{dv}{dt}}\,dt \quad \textcolor{red}{\text{choose } v \text{ carefully}} \\
&= f(0) + \underbrace{f'(t)}_{u(t)} \underbrace{(t-x)}_{v(t)} \Big|_0^x - \int_0^x \underbrace{(t-x)}_{v(t)} \underbrace{f''(t)}_{\frac{du}{dt}}\,dt \\
&= \underbrace{f(0) + xf'(0)}_{p_1(x)} + \underbrace{\int_0^x f''(t)(x-t)\,dt}_{R_2(x)} \quad \textcolor{red}{\text{integration by parts}}
\end{aligned}
$$

Theorem 4.2 (Lagrange Remainder) *If $f(x)$ has $n+1$ continuous derivatives on an open interval I containing a then for each $x \in I$*

$$f(x) = p_n(x) + R_{n+1}(x)$$

with

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \qquad \textit{(Taylor polynomial)}$$

where the ***Lagrange formula for the remainder*** *is*

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

where c is some real number between a and x.

Proof: The proof of the Lagrange formula is in the tutorial exercises: it's really a quite simple application of one of several mean value theorems for integrals.

$\square$

Note: Taylor's theorem in the form

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

extends the mean value theorem.

Consider the $n = 0$ case. Then,

$$f(x) = f(a) + f'(c)(x-a)$$

$$\Rightarrow f'(c) = \frac{f(x) - f(a)}{x-a} \qquad \text{MVT}$$

The Lagrange form is easier to remember, and more useful.

In particular, if we approximate a function by a Taylor polynomial then the remainder term provides the error in this approximation:

$$|f(x) - p_n(x)| = |R_{n+1}(x)| \equiv |f^{(n+1)}(c)| \left| \frac{(x-a)^{n+1}}{(n+1)!} \right|,$$

where c is between x and a.

It is usually not possible to find c but it is often possible to find an Upper Bound Error.

Suppose we are using $p_n(x)$ to approximate $f(x)$ on some interval $I = [a-b, a+b]$.

If we know that $\left|f^{(n+1)}(x)\right| \leq M$ on I, then

$$\text{error} \leq \frac{M}{(n+1)!} b^{n+1} \qquad \text{for all } x \in I.$$

Example 5 *Consider $f(x) = \exp x$ on $[-0.1, 0.1]$.*

We used the Taylor polynomials to approximate $e^{0.1}$ in example 1 and got $e^{0.1} \approx 1.105$.

We can use this to put a bound on the error in using the Taylor polynomials to approximate e^x on the interval $[-0.1, 0.1]$

The error is given by the remainder, so if $x \in [-0.1, 0.1]$, then the error in using the nth Taylor polynomial to calculate e^x is

$$R_n(x) = \frac{e^c}{(n+1)!} x^{n+1}$$

for some c between x and 0.

We don't know c, but

So looking at an upper bound on the error,

$$|R_n(x)| \leq$$

for $x \in [-0.1, 0.1]$.

For example, we found in example 1 that $p_3(x)$ had an error of 4.25×10^{-6} in $e^{0.1}$.

Our overestimate above gives us error for $p_3(x)$ less than

More generally, we see that in our interval $[-0.1, 0.1]$, each extra term in the Taylor polynomial can be expected to give us at least one extra decimal place of accuracy — again, in line with what we saw in example 1.

4.2.1 Classifying Sationary Points

Taylor's theorem can be used to improve the classification of stationary points that you learnt in School:

Example 6 *Examine the behaviour of $f(x) = e^{-x^3}$ near $x = 0$.*

SOLUTION: Now $f'(x) = -3x^2 e^{-x^3}$, so $x = 0$ is a stationary point (SP).

We try the second derivative test:

$$f''(x) = -6xe^{-x^3} + 9x^4 e^{-x^3} = (9x^4 - 6x)e^{-x^3} \quad \text{so} \quad f''(0) = 0$$

and the 2nd derivative test has failed.

Note however that

$$f'''(x) = (36x^3 - 6)e^{-x^3} - 3x^2(9x^4 - 6x)e^{-x^3} \quad \text{so} \quad f'''(0) = -6$$

Now Taylor's theorem tells us that

$$f(x) = f(0) + \frac{1}{3!}(-6)x^3 + \frac{1}{4!}f^{(iv)}(c)x^4 = 1 - x^3 + \frac{1}{24}f^{(iv)}(c)x^4$$

for some $c \in (0, x)$.

We don't need to write the fourth derivative down to see that it will be e^{-c^3} times some polynomial in c.

Thus the remainder term will be x^4 times a polynomial in $c < x$ times e^{-c^3}, and so for small x will be small in comparison to $1 - x^3$.

Thus f ought to behave like $1 - x^3$ near zero, that is it should have a point of inflection.

This is in fact the case, as we could see by noting that the derivative is never positive.

We can generalise the previous example:

Theorem 4.3 *Suppose that f is n times differentiable at a and $f'(a) = 0$. Then, if $k \leq n$ and*

$$f''(a) = f'''(a) = \ldots = f^{(k-1)}(a) = 0 \quad \textit{but} \quad f^{(k)}(a) \neq 0$$

we have

a local minimum at a if k is even and $f^{(k)}(a) > 0$

a local maximum at a if k is even and $f^{(k)}(a) < 0$

an inflection point at a if k is odd

Proof: See Notes $\square$

Example 7 *Show that $f(x) = \sin(x^4)$ has a local minimum at $x = 0$.*

SOLUTION: Clearly $f'(x) = 4x^3 \cos(x^4)$ and so $f'(0) = 0$, hence zero is a SP.

Higher derivatives will need the product rule:

$f''(x) =$ so

Applying our theorem with $a = 0$,

Remark. The previous example shows that we need to address the following two big questions:

- Given the Taylor polynomials p_n of a function f about a, can we make sense of

$$p_\infty(x) := \lim_{n\to\infty} p_n(x)?$$

- If so, under what circumstances is a function represented by its **Taylor series**, that is

$$f(x) = p_\infty(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \ ?$$

4.3 Sequences

A **sequence** is a function

$$f : \mathbb{N} \to \mathbb{R}$$

whose domain is $\mathbb{N}$ and codomain is $\mathbb{R}$.

Notation:

- We usually use a, b etc rather than f and g.
- We usually write a_n instead of $a(n)$ etc.
- We usually denote a sequence by $\{a_n\}_{n=0}^{\infty}$ or $\{a_n\}$.
- A sequence does not have to start at $n = 0$, e.g. $\{a_n\}_{n\geq 1}$

Examples – closed form

a) $a_n = n, \qquad \{0, 1, 2, \ldots\}$

b) $a_n = n^2, \qquad \{0, 1, 4, 9, \ldots\}$

c) $a_n = \dfrac{1}{n+1}, \qquad \left\{1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots\right\}$

d) $a_n = \dfrac{(-1)^n}{n+1}, \qquad \left\{1, -\dfrac{1}{2}, \dfrac{1}{3}, -\dfrac{1}{4}, \ldots\right\}$

Examples – recursion or sum

a) $a_0 = 1, \quad a_n = \dfrac{a_{n-1}}{n}, \qquad n \geq 1$

b) $a_n = a_{n-1}^2 + \dfrac{1}{a_{n-2}}, \qquad a_0 = 1,\ a_1 = 1$

c) $a_n = \begin{cases} 1 & \text{if} \quad n = 1, 2 \\ a_{n-1} + a_{n-2} & \text{if} \quad n \geq 3 \end{cases}$

d) $a_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}$

4.3.1 Limiting behaviour of sequences

We want to know how a sequence $\{a_n\}$ behaves as $n \to \infty$.

Does it settle down; or grow without bound; or oscillate?

Definition: A number L is the **limit** of $\{a_n\}$ if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ whenever $n \geq N$.

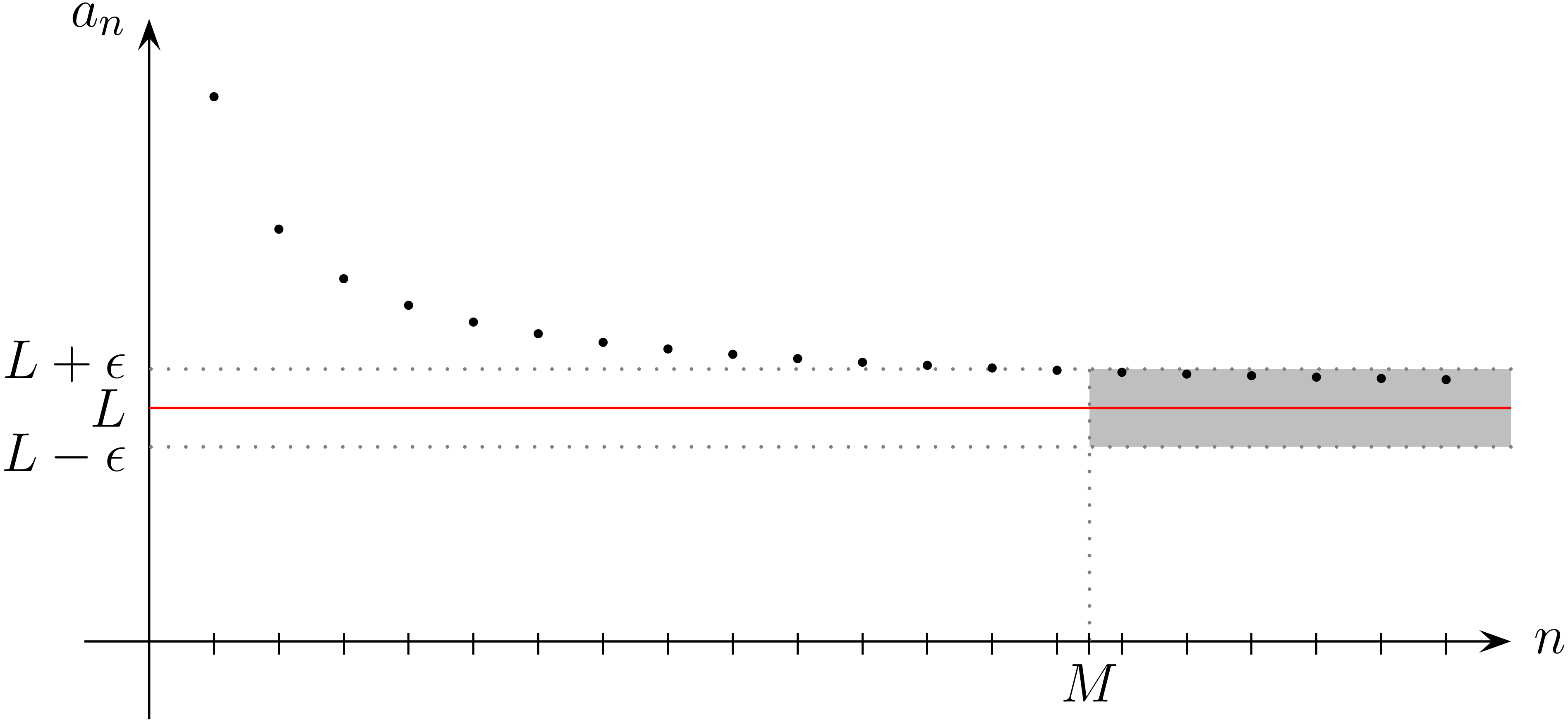
In other words, the members of the sequence get close to L and stay close to L as n gets large.

Notation:

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty$$

A sequence that has a limit is said to **converge** or be **convergent**.

Geometrically, the definition looks like this:



Example 8 *Prove that* $\lim_{n\to\infty} \frac{n-1}{n+1} = 1$.

SOLUTION: Let $a_n = \frac{n-1}{n+1}$.

As with functions, we begin with

$$|a_n - L| = \left|\frac{n-1}{n+1} - 1\right| =$$

Given any $\epsilon > 0$, we want $|a_n - L| < \epsilon$, and we can achieve this by making

Or in other words,

So take $N =$, and if $n > N$ then $|a_n - 1| < \epsilon$.

A sequence that does not converge **diverges**, or is **divergent**.

There are several ways a sequence can diverge.

(i) $\boxed{a_n \text{ **diverges to** } \infty}$ if $a_n \to \infty$ as $n \to \infty$.

(For every $M \in \mathbb{R}$ there exists an $N \in \mathbb{N}$ such that $a_n > M$ whenever $n \geq N$.)

(ii) $\boxed{a_n \text{ **diverges to** } -\infty}$ if $a_n \to -\infty$ as $n \to \infty$.

(iii) $\boxed{a_n \text{ is **boundedly divergent**}}$ if $\{a_n\}$ remains bounded ($|a_n| < K$ for every n) but does not approach a limit.

(iv) $\boxed{\{a_n\} \text{ is **unboundedly divergent**}}$ if diverges but not as in i), ii) or iii).

Examples

a) $a_n = n^2$

b) $a_n = -n$

c) $a_n = (-1)^n$

d) $a_n = (-1)^n n$

e) $\sin(n)\left(1 + \frac{1}{n}\right)^{3n}$

4.3.2 Calculating limits of sequences

Suppose $a_n \to L$ and $b_n \to \ell$ as $n \to \infty$. Then

Rule 1 $a_n + b_n \to L + \ell$ unless $L + \ell$ has the form $\infty - \infty$

Rule 2 $a_n b_n \to L\ell$ unless $L\ell$ has the form $0 \cdot (\pm\infty)$

Rule 3 $\dfrac{a_n}{b_n} \to \dfrac{L}{\ell}$ unless $\dfrac{L}{\ell}$ has the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$

Rule 4 $\alpha a_n \to \alpha L$ for all real α.

Summary: limits of sequences behave nicely under the usual arithmetic operations.

Example 9 *Find* $\lim_{n\to\infty} \sqrt{n^2+3n} - n$

SOLUTION:

$$\lim_{n\to\infty} \sqrt{n^2+3n} - n = \lim_{n\to\infty} \frac{(\sqrt{n^2+3n} - n)(\sqrt{n^2+3n} + n)}{(\sqrt{n^2+3n} + n)}$$

$$=$$

Theorem 4.4 *Let f be a function which is continuous at a and $\{a_n\}$ be a sequence with the following properties:*

i) a_n lies in the domain of f for all n,

ii) $a_n \to a$ as $n \to \infty$

Then $f(a_n) \to f(a)$ as $n \to \infty$.

This is a very useful result: it can best be remembered as saying that if f is continuous then

$$f\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} f(a_n)$$

i.e. we can swap the limit and the function.

The commonest use of this result is with the exponential and log.

Example 10 *Find* $\lim_{n\to\infty} \alpha^{\frac{1}{n}}$ *for* $\alpha > 0$*.*

SOLUTION: We apply a standard trick: $a_n = \exp(\ln(a_n))$.
As the exponential is continuous everywhere:

$$\lim_{n\to\infty} \alpha^{\frac{1}{n}} = \exp\left(\lim_{n\to\infty} \frac{1}{n} \ln \alpha\right)$$

$$=$$

$$=$$

■

Theorem 4.5 *Given a sequence $f(n)$ for $n \in \mathbb{N}$, if one can extend the sequence to a function $f : \mathbb{R} \to \mathbb{R}$ with*

$$\lim_{x\to\infty} f(x) = L$$

then

$$\lim_{n\to\infty} f(n) = L.$$

Notes:

a) This theorem enables us to calculate limits of sequences in a similar fashion to limits of functions.

b) In particular, it allows us to call on L'Hôpital's Rule.

c) Note that $\lim_{n\to\infty} f(n)$ might converge but $\lim_{x\to\infty} f(x)$ diverge, for example $f(x) = x\sin(\pi x)$.

Example 11 *Find* $\lim\limits_{n\to\infty} \dfrac{\ln n}{n}$*.*

SOLUTION: Let $f(x) = \dfrac{\ln x}{x}$, so that $a_n = \dfrac{\ln n}{n} = f(n)$.

It is a routine application of L'Hôpital's rule to find $\lim\limits_{x\to\infty} f(x)$:

Let $g(x) = \ln x$ and $h(x) = x$, so that $f(x) = \dfrac{g(x)}{h(x)}$.

Both g and h are differentiable and

$$\lim_{x\to\infty} \frac{g'(x)}{h'(x)} =$$

so by L'Hôpital's rule, $\lim\limits_{x\to\infty} f(x) =$

Hence $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} =$

Example 12 *Find* $\lim_{n\to\infty} n^{\frac{1}{n}}$*.*

SOLUTION:

The Notes prove the important result

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

which you must know.

Pay careful attention to the method of proof.

They also quote the useful:

Theorem 4.6 (Pinching Theorem) *Suppose that* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are sequences and that for some positive integer* N *the inequality*

$$a_n \le b_n \le c_n$$

is satisfied whenever $n > N$.

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ *then* $\lim_{n\to\infty} b_n = L$.

Example 13 *Let* $a_n = \dfrac{n!}{n^n}$*. Find* $\lim\limits_{n\to\infty} a_n$*, if it exists.*

SOLUTION: We have

$$a_n = \frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n}{n}$$
$$\leq \frac{1}{n}\cdot\frac{n}{n}\cdot\frac{n}{n}\cdots\frac{n}{n} = \frac{1}{n}$$

whenever $n \geq 1$.

On the other hand, a_n is always positive.

Thus

$$0 \leq a_n \leq \frac{1}{n}.$$

As $n \to \infty$ we conclude that $a_n \to 0$ by the pinching theorem. ■

Which is the biggest of them all?

a_n	growth as $n \to \infty$
1	constant
$\ln n$	slow growth
$n^k \quad (k > 0)$	faster for larger k
$c^n \quad (c > 1)$	faster for larger c
$n!$	rapid
n^n	very rapid

Theorem 4.7 *i) Let a_n be an increasing sequence that is bounded above, i.e.,*

$$a_0 \le a_1 \le \ldots \le a_{n-1} \le a_n \le \ldots \le K.$$

Then, a_n converges to some real number $L \le K$.

ii) Let a_n be a decreasing sequence that is bounded below, i.e.,

$$a_0 \ge a_1 \ge \ldots \ge a_{n-1} \ge a_n \ge \ldots \ge K$$

Then, a_n converges to some real number $L \ge K$.

Sequences in this theorem are called **monotonic**.

The proof of this result depends crucially on a deep property of the real numbers.

If the sequence is rational then the limit may not be rational.

Example 14 *Define the sequence a_n by $a_0 = 3$, and*

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right) \quad \textit{for } n \geq 1.$$

Show that $a_n \geq \sqrt{3}$ for all n, that $\{a_n\}$ is decreasing and find its limit.

SOLUTION: The am/gm inequality says that for positive numbers x and y

$$\frac{1}{2}(x+y) \geq \sqrt{xy}$$

(proof: EXERCISE). Applying this to the formula for a_{n+1} gives

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right) \geq \sqrt{3}$$

proving the first part.

For the second part

$$a_{n+1} - a_n = \frac{1}{2}\left(\frac{3}{a_n} - a_n\right) = \frac{1}{2a_n}(3 - a_n^2) \leq 0$$

by what we have just proved.

So $\{a_n\}$ is a decreasing bounded sequence and hence converges.

Now

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{2}\left(\lim_{n\to\infty} a_n + \frac{3}{\lim_{n\to\infty} a_n}\right).$$

If the limit is L, we have

$$L = \frac{1}{2}\left(L + \frac{3}{L}\right) \quad \text{i.e.} \quad L^2 = 3$$

and as $L \geq \sqrt{3}$ we must have $L = \sqrt{3}$. ■

4.3.3 Suprema and Infima

Suppose that $\{a_n\}_{n=0}^{\infty}$ is a sequence of real numbers.

- M is an **upper bound** if $a_n \le M$ for all $n \in \mathbb{N}$.
- M is a **lower bound** if $a_n \ge M$ for all $n \in \mathbb{N}$.
- K is **the least upper bound** or **supremum** if K is an upper bound and $K \le M$ for any other upper bound M. (**supremum** is latin for the greatest)
- K is **the greatest lower bound** or **infimum** if K is a lower bound and $K \ge M$ for any other lower bound M. (**infimum** is latin for lowest part)

Notation

Suppose that $\{a_n\}_{n=0}^{\infty}$ is a sequence of real numbers.

- The least upper bound or **supremum** is written as

$$\sup_{n \geq 0} a_n \qquad \text{or} \qquad \sup \{a_n : n \geq 0\}$$

- The greatest lower bound or **infimum** is written as

$$\inf_{n \geq 0} a_n \qquad \text{or} \qquad \inf \{a_n : n \geq 0\}$$

Notes

- Every non-empty set of reals bounded from above has a unique supremum.
- Every non-empty set of reals bounded from below has a unique infimum.
- The supremum is like the largest element of a set.
  The infimum is like the smallest element of a set.
- The supremum and infimum do not need to be elements of the set.
- If the set has a maximum then the maximum is the supremum.
- If the set has a minimum then the minimum is the infimum.

Example 15 *Let* $A = (-1, 1)$ *and* $B = \left\{ \frac{1}{n}, n = 1, 2, \ldots \right\}$*.*

Find $\sup A$*,* $\sup B$*,* $\inf A$ *and* $\inf B$*.*

SOLUTION: It is easy to see that $\sup A = 1$: 1 is an upper bound for A and no number less that 1 is an upper bound.

Similarly, $\inf A = -1$.

Note that neither $\sup A$ nor $\inf A$ are in A.

Since $B = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\}$, 1 is the maximum element of B and so $\sup B = 1$.

All elements of B are positive, so 0 is a lower bound for B.

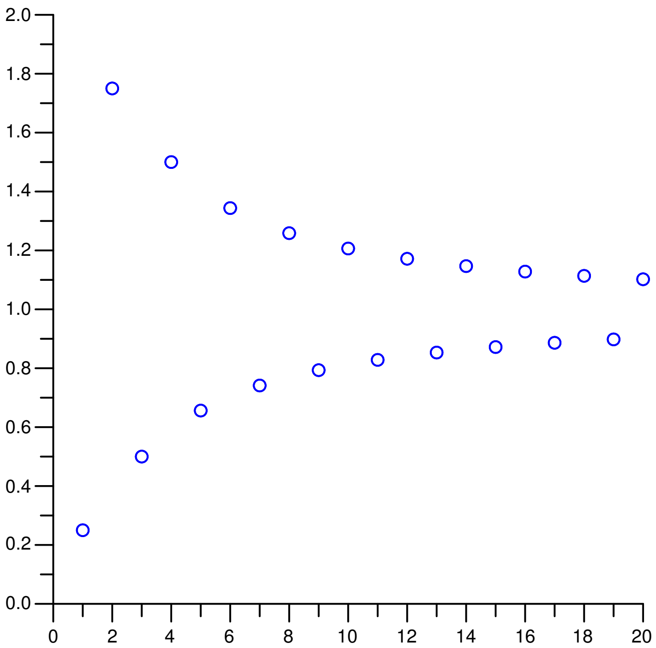
In fact, $0 = \inf B$: for any number $\epsilon > 0$, let $N = \lceil \epsilon^{-1} \rceil + 1$.

Then $\epsilon > \frac{1}{N} \in B$, so ϵ is not a lower bound for B.

Hence $0 = \inf B$. ■

Example 16 *Find* $\sup\{a_n : n \geq 1\}$ *and* $\inf\{a_n : n \geq 1\}$ *given that*

$$a_n = \frac{n^2 + (-1)^n n + 1}{n^2 + (-1)^{n+1} n + 2}$$



$$\sup\{a_n\} = a_2 = \frac{7}{4}$$

$$\inf\{a_n\} = a_1 = \frac{1}{4}$$

To prove this show that $\{a_{2k}\}$ is a decreasing sequence bounded below by 1 and $\{a_{2k+1}\}$ is an increasing sequence bounded above by 1.