Answers to selected problems

Chapter 1

1. a) cup shaped b) unit sphere c) umbrella stand
   d) cone with semi-vertical angle $\pi/4$ e) saddle

2. $2xye^{x^2y},\ x^2e^{x^2y},\ (2x+2x^3y)e^{x^2y}$.

3.

	$\frac{\partial z}{\partial x}$	$\frac{\partial z}{\partial y}$	$\frac{\partial^2 z}{\partial x^2}$	$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$	$\frac{\partial^2 z}{\partial y^2}$
a)	$2xy$	x^2+2y	$2y$	$2x$	2
b)	$\dfrac{-y}{(x^2+y^2)}$	$\dfrac{x}{(x^2+y^2)}$	$\dfrac{2xy}{(x^2+y^2)^2}$	$\dfrac{(y^2-x^2)}{(x^2+y^2)^2}$	$\dfrac{-2xy}{(x^2+y^2)^2}$
c)	$\cos(x-cy)$	$-c\cos(x-cy)$	$-\sin(x-cy)$	$c\sin(x-cy)$	$-c^2\sin(x-cy)$

4. a) $z=6x+10y-34, \quad \mathbf{n}=(6,10,-1)^T$
   b) $z=32-16x+16y, \quad \mathbf{n}=(-16,16,-1)^T$
   c) $4x-6y-7z-14+7\ln 7=0, \quad \mathbf{n}=(4,-6,-7)^T$
   d) $2x+3y+\sqrt{23}z-6=0, \quad \mathbf{n}=(2,3,\sqrt{23})^T$

5. a) $784\pi\,\text{cm}^3$ b) $\frac{217\pi}{15}\,\text{cm}^3$ c) 1.85%

6. 0.05

7. 5.012 (calculator gives 5.012115)

8. $|\Delta S| \leq 0.0404$

9. 9% decrease.

10. 0.21%

11. a) $e^t(t^2+2t)$ b) $2t$

12. 7.5π cubic centimetres per second

13. b) $F(x,y)=\sin(y-x^2)$

14. a) $u_{xx}(x,t)=g''(x+\lambda t), \qquad u_{tt}(x,t)=\lambda^2 g''(x+\lambda t)$
    b) $4,\ -4$

17. a) $u_t(x,t) = \dfrac{-u(x,t)\,f'(x-tu(x,t))}{1+tf'(x-tu(x,t))}$, $\quad u_x(x,t) = \dfrac{f'(x-tu(x,t))}{1+tf'(x-tu(x,t))}$

c) $t_m = 1$

d) Hint: Try the Maple commands

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with (plots):
implicitplot(y-1+tanh(x-t*y),x=-5..7, y=-1..3, gridrefine = 2);
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for a few values of t. Can you animate this in time t?

e) No

18. a) $y = \pm 1$, $\frac{dy}{dt} = 0$.

b) $\nabla F = (2x, 2y, -2z)^T$.

c) The vector $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)^T$ representes the velocity of the raindrop. So the equation states that the velocity is perpendicular to the normal to the curve.

d) If $x_0 = z_0$ then $y_0 = \pm 1$, so the lines are $(0, y_0, 0)^T + \lambda(1, 0, \pm 1)^T$. If $x_0 \neq z_0$ then the lines are

$$\frac{-1}{z_0^2 - x_0^2}\,(x_0 \pm z_0 y_0, 0, z_0 \pm x_0 y_0) + \lambda\left(x_0 y_0 \pm z_0, z_0^2 - x_0^2, z_0 y_0 \pm x_0\right)^T.$$

Chapter 2

1. a) $\dfrac{1}{4}e^{2x^2}$ b) $-\dfrac{1}{2}\cos(x^2)$ c) $\dfrac{1}{6}\sin(2x^3)$

d) $\dfrac{1}{10}\ln|5x^2-11|$ e) $-\dfrac{1}{4}\cos^4 x$ f) $\ln(\ln x)$

g) $\sqrt{x^2+4x+7}$ h) $\dfrac{1}{3}(1+x^2)^{\frac{3}{2}}$ i) $-\dfrac{1}{18}(9-4x^3)^{\frac{3}{2}}$

j) $-\dfrac{1}{6}(9-4x^3)^{\frac{1}{2}}$ k) $-\dfrac{1}{8(1+x^4)^2}$ l) $\dfrac{-1}{3\tan^3 x}$

m) $\dfrac{-1}{2\sin^2 x}$ n) $\dfrac{1}{8}(4+3e^{2x})^{\frac{4}{3}}$ o) $\dfrac{-1}{4(\ln x)^4}$

2. a) $-e^{-x}(x^2+2x+2)$ b) $\dfrac{1}{4}x^4\ln x - \dfrac{1}{16}x^4$ c) $x\tan x + \ln(\cos x)$

d) $-\dfrac{(\ln x)^2}{x} - 2\dfrac{\ln x}{x} - \dfrac{2}{x}$ e) $\dfrac{1}{2}e^x(\sin x + \cos x)$ f) $x\ln x - x$

g) $x\tan^{-1}x - \dfrac{1}{2}\ln(x^2+1)$

3. a) 1/8 b) 4/15

c) $\frac{1}{3}\sec^3 x + C$ d) $\frac{1}{4}\sin 2\theta + \frac{1}{2}\theta + C$

e) $-\frac{1}{99}(\sin x\cos 10x - 10\cos x\sin 10x) + C$ or $\frac{1}{2}\left(\frac{1}{9}\sin 9x + \frac{1}{11}\sin 11x\right) + C$ f) $-\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C$

4. a) $\ln|\tan x + \sec x| + C$

b) i) $\frac{1}{3}\sec^2 x\tan x + \frac{2}{3}\tan x + C$

ii) $\frac{1}{4}\sec^3 x\tan x + \frac{3}{8}\sec x\tan x + \frac{3}{8}\ln|\tan x + \sec x| + C$

5. a) $3\pi/512$ b) $1/60$ c) $2/35$

6. $6 - 16e^{-1}$

7. $5/12-(\ln 2)/2, \quad \pi/4-76/105$

8. $(e^2+3)/8$

9. $35\pi/256, \quad 16/35$

10. $I_n=(2\sqrt{2}-2nI_{n-1})/(1+2n)$

15. a) $\pi/3-\sqrt{3}/2; \quad x=2\sin\theta$
 b) $\sinh^{-1}\frac{x-3}{2}+C$ or $\ln(x-3+\sqrt{x^2-6x+13})+K$; complete the square.
 c) $9\pi/4$
 d) $-\dfrac{\sqrt{x^2+16}}{16x}+C; \quad x=4\tan\theta$ to obtain $\displaystyle\int\frac{\cos\theta}{\sin^2\theta}\,d\theta.$
 e) $x/\sqrt{1-x^2}+C$
 f) $\tan^{-1}2$

16. $\sqrt{x^2-4}+C$

17. a) $\frac{1}{2}\ln\left|\frac{x+1}{x+3}\right|+C$
 b) $3\log|x-2|+2\log|x-1|+C$
 c) $2\log|x|+\frac{1}{x}+2\log|x-1|+C$
 d) $-\frac{1}{4}\left(\frac{2x}{x^2-1}-\log\left|\frac{x+1}{x-1}\right|\right)+C$
 e) $x+\ln\left|\frac{x-1}{x+1}\right|+C$
 f) $\ln|x-3|-\frac{1}{2}\ln(x^2+9)-\tan^{-1}\frac{x}{3}+C$
 g) $\ln\frac{(x+1)^2}{|x+2|}+\frac{4}{x+2}+C$
 h) $\frac{-1}{(1+x)^2}+\frac{1}{1+x}=\frac{x}{(1+x)^2}+C$

18. a) $\frac{1}{2}\ln(x^2+2x+10)-\frac{1}{3}\tan^{-1}\frac{x+1}{3}+C$
 b) $\sqrt{x^2+2x+10}-\sinh^{-1}\frac{x+1}{3}+C$
 c) $2\sqrt{x}-2\ln(1+\sqrt{x})+C$
 d) $11-6\ln(3/2); \quad x=u^6$ first.

22. a) $\ln x-\frac{1}{2}\ln(x^2+x+1)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$
 b) $\frac{8}{5}\cosh^5(x)+C$
 c) $\frac{3}{2}\ln(x^2+4x+8)-\frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+C$
 d) $\frac{1}{2}x\sqrt{25-x^2}+\frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)+C$
 e) $\ln|x-1|+\ln(x^2-2x+2)+\tan^{-1}(x-1)+C$
 f) $-\frac{\sqrt{1+x^2}}{x}+C$
 g) $\frac{1}{3}\ \frac{x}{\sqrt{x^2+3}}+C$
 h) $-\frac{1}{14}\cos(7x)+\frac{1}{2}\cos(x)+C$
 i) $2/e$

Chapter 3

1. a) $y=\tan(t^3/3+C)$
 b) $y=-2/(x^2+A)$
 c) $y=\sqrt[3]{A-3\cos x}$
 d) $y=C(x-1)/x$
 e) $e^{x+y}+Ce^y+1=0$
 f) $y^2=\tan^2 x+4\tan x-1, \quad$ where $y>0$
 g) $(\ln x)^2=2\ln y+C$
 h) $y=1/(1-x^3)$

2. Let a and b be arbitrary real numbers with $a < b$. Then there are solutions of the form

$$y = \begin{cases} (x-a)^3 & \text{if } x < a \\ 0 & \text{if } a \le x \le b \\ (x-b)^3 & \text{if } x > b \end{cases} .$$

3. $2\sqrt{2}$

4. a) $y = (x^3/3 + C)e^{2x}$
   b) $y = e^{-3x}(\tan^{-1} x + C)$
   c) $y = (2 + Ce^{-x})/x$
   d) $y = 2x^5 + Ax^2$
   e) $y = \tan x - 1 + Ae^{-\tan x}$
   f) $y = \frac{x}{2}\tan 2x + \frac{1}{4} + A\sec 2x$

5. $x = t - 1 + Ce^{-t}$

7. a) $3x^2y + y^3 = A$
   b) $x\sin y - \frac{1}{2}x^2y^2 = C$

8. a) $(x^2+1)e^y + e^x = C$
   b) Not exact
   c) $y\tan^{-1} x + xe^y = C$
   d) $e^{xy}\cos x = C$

9. a) $xy^2 - 3y\tan x = C$
   b) $y^2 = \ln(x^2+1) + C$
   c) $y = \dfrac{\ln x + C}{x}$
   d) $y = Axe^{-1/x}$
   e) $y = \left[3\left(\ln|x| + \dfrac{x^2}{2} + \dfrac{17}{2}\right)\right]^{1/3}$

10. $xy^2 = 2y^5 + C$

11. a) $x/y = \ln|x| + C$
   b) $\sec\frac{y-x}{x} + \tan\frac{y-x}{x} = Ax$
   c) $3x^2y + y^3 = A$
   d) $3xy^2 + x^3 = A$
   e) $\sinh^{-1}(y/x) = \ln x$; that is, $y = \frac{1}{2}(x^2 - 1)$

12. $\ln\sqrt{x^2+y^2} = -\tan^{-1}(y/x)$. Logarithmic or equiangular spiral, $r = e^{-\theta}$.

13. $2x + C = \ln\left|\dfrac{y-x-1}{y-x+1}\right|$ or $y = x - 1 + \dfrac{2}{1+De^{2x}}$

14. a) $y = \dfrac{1}{2 + Ae^{-t}}$
   b) $y = 25/(5t - 1 + 26e^{-5t})$; $y_{\max} = \dfrac{25}{\ln 26}$ when $t = \dfrac{\ln 26}{5}$

15. a) $y^2 = Ce^{4x} - \frac{1}{2}x - \frac{1}{8}$ b) $x\ln y + x^3/3 = C$ c) $y = Ae^{Bx^2}$

16. a) $y = Ae^{-2t} + B + 3t$
   b) $y = \dfrac{t}{c} - \dfrac{\ln(1+ct)}{c^2} + d$

17. a) Let $z = \dfrac{dv}{dx}$. We obtain $\alpha z' + (2\alpha' + b\alpha)z = 0,$ which is first order homogeneous linear.
   b) $u = Ax^2 + Bx^3$

18. a) i) $y = 0, \quad y = K$

ii) $y = \dfrac{y_0 K}{y_0 + (K - y_0)e^{-kt}}$

iii) y is strictly increasing and approaches K.

iv) $y = K/2$; work from the differential equation.

v) The solution is the same as for ii); y is strictly decreasing (and concave upwards) with $y = K$ as a horizontal asymptote.

b) $y = K \exp\left\{-\ln(K/y_0)e^{-\alpha t}\right\}$; one way, let $z = \ln y$.

19. Let y litres of pollutant be present in the lake after t days.

$$\frac{dy}{dt} = 10^4 - \frac{y(t)}{10^9} \times (10^6 + 10^4), \quad \text{giving} \quad y = \frac{10^9}{101}\left(1 - e^{-1.01t/10^3}\right)$$

a) $y \to 10^9/101$ litres or just under 1%

b) $\dfrac{10^5 \ln 2}{101} \approx 686.3$ days ≈ 1.88 years

c) That there is perfect mixing, that the pollutant does not precipitate or dissolve, that the pollutant does not itself create more pollution, that ....

20. With an inflow of 3 litres per minute and an outflow of 1 litre per minute, the volume of liquid in the tank at time t is $50+2t$ litres. The inflow of salt is 3×2 grams per minute. In running off 1 litre per minute with a concentration of $x/(50+2t)$ grams of salt per litre, the rate of removal of salt is $1 \times x/(50+2t)$ grams per minute. So the net rate of increase of x is given by $\dfrac{dx}{dt} = 6 - x/(50+2t)$ or $\dfrac{dx}{dt} + \dfrac{1}{50+2t}x = 6$. This is a first order linear ODE. (You may find it helpful to consider the outflow over a small time interval $[t,\ t + \Delta t]$.)

21. a) $P(t) = 100 - 25\cos t + 25\sin t - 55e^{-t}$

b) 100

22. $y = A \exp\left\{k(t + \dfrac{a}{2\pi}\sin(2\pi t))\right\}$

23. a) $r(t) = 2\% + (1.5\% - 2\%)/10t = 0.02 - 0.0005t$, where t is measured in years.

b) The differential equation is $\dfrac{dy}{dt} = (0.02 - 0.0005t)y$
$y = 10^7 \exp(0.02t - 0.00025t^2)$. When $t = 10$, $y = 10^7\, e^{0.175} \approx 10^7 \times 1.19$

24. a) $\dfrac{dv}{dt} = g - kv$; with $g = 9.8,\ k = 10$. So $v = Ae^{-kt} + g/k$. With $v = 0$ at $t = 0$,
$v = \frac{g}{k}(1 - e^{-kt}) = \frac{49}{50}(1 - e^{-10t})$

b) $g/k = 49/50$

c) $t \geq \dfrac{\ln 20}{k}$ (≈ 0.3 seconds)

d) $A > 0$ and v decreases (rapidly) towards g/k.

25. a) $y = Ax^\lambda$

b) Graph $\ln y$ against $\ln x$. From the graph, $\lambda \approx 1.54$ and $\ln A \approx -4.55$ (so that $A \approx 0.0106$).

26. a) \$610701.38

b) Treating the car payments as continuous at the rate of \$10400 a year, $\dfrac{dP}{dt} = 0.2P - 10400$. Then $P = 52000 + 448000e^{0.2t}$, for $0 \leq t \leq 1/2$. At $t = 1/2$, $P = 547116.57$. The capital remaining in the cooperative after 6 months is $47116.57\,e^{-0.05} = 44818.67$. At the end of one year, the total is 547116.57 (capital plus new interest in Hitek) + 44818.67 (in the co-op) = 591935.24 dollars.

28. $\dfrac{dy_1}{dt} = -0.25y_1, \quad \dfrac{dy_2}{dt} = 0.25\,y_1 - 2y_2$ with $y_1(0) = K, \quad y_2(0) = 0$. Thus $y_1 = Ke^{-0.25t}$ and hence $y_2 = \frac{K}{7}\,(e^{-0.25t} - e^{-2t})$. The maximum value of y_2 occurs for $t = \frac{12}{7}\ln 2$. That is, after about 1.188 days.

29. $20\ln 2 \approx 13.8$ m/sec

30. a) $y = Ae^{-2x} + Be^{-x}$ b) $y = e^{-x}(C\cos 3x + D\sin 3x)$
c) $y = Ae^{-3x}$ d) $y = (Ax + B)e^{-2x}$

31. a) $\frac{1}{4}(5e^{x} - e^{5x})$ b) $y = e^{-x}(\cos x + \sin x)$

32. a) $y = Ae^{-3x} + Be^{-x} + \frac{1}{3}x - \frac{4}{9}$ b) $y = (Ax + B)e^{3x} + 5e^{2x}$
c) $y = e^{-x}(A\sin x + B\cos x) + 2\sin 2x - \cos 2x$ d) $y = (A - x/2)e^{-x} + Be^{x}$
e) $y = A\cos 2x + B\sin 2x - \frac{1}{4}x\cos 2x$ f) $y = Ae^{-3x} + 2e^{2x}$
g) $y = \frac{1}{3}(\sin x + \sin 2x)$ h) $y = e^{x} + e^{4x} - e^{2x}$

33. a) $Ae^{-x} + Be^{5x/2}$; seek $y_P = x(ax + b)e^{5x/2}$
b) $Ae^{4x} + Be^{-6x}$; seek $y_P = e^{4x}(a\cos 6x + b\sin 6x)$
c) $(Ax + B)e^{-3x}$; seek $y_P = x^2\,ae^{-3x}$

34. $y = \frac{1}{2}x^4e^{2x}$.

35. You should obtain $\dfrac{d^2y}{dt^2} - 5\dfrac{dy}{dt} + 6y = e^{5t}$. $\quad y = Ax^3 + Bx^2 + \dfrac{1}{6}x^5$

37. $50^2 g/\pi$ grams; about 780 kilograms

38. a) $x(t) = \cos 2t$, so the block oscillates with fixed amplitude.
b) If $c = 2$ then $x(t) = e^{-t}(\cos\sqrt{3}t + \frac{1}{\sqrt{3}}\sin\sqrt{3}t)$, so the system has damped oscillations.
If $c = 5$ then $x(t) = \frac{1}{3}(e^{-t} - e^{-4t})$, so the system does not oscillate.
c) If the characteristic equation has real roots then the solution has no oscillating terms. This happens whenever $c \geq 4$. So the smallest value of c is 4.

39. a) $q(t) = A\cos 100t + B\sin 100t$
b) $x_P = a\cos\Omega t + b\sin\Omega t$ if $\Omega \neq 100$; $\quad x_P = t(a\cos\Omega t + b\sin\Omega t)$ if $\Omega = 100$.
c) $50/\pi$ (which corresponds to when $\Omega = 100$).

40. a) $y = Ae^{-t} + Be^{-2t} + 2\sin t - 6\cos t$
b) $y = -6\cos t + 2\sin t$

41. b) $y_p = -\frac{1}{4}\,t\cos t$

42. b) No
c) $\mu = n\pi/L$, where $n = 1, 2, 3, \ldots$. The corresponding solutions are $y_n(x) = B_n\sin(n\pi x/L)$.

43. b) $2\lambda = 1 + \frac{k^2\pi^2}{L^2}$, $k = 1, 2, 3, \ldots$ with $y_k = B_k e^{-x}\sin(k\pi x/L)$

44. a) $\dfrac{x^2}{a} + \dfrac{y^2}{b} = C$, a family of ellipses.

   b) $\dfrac{d^2y}{dx^2} + aby = 0, \quad y = A\cos\omega t + B\sin\omega t, \quad$ where $\quad \omega^2 = ab$
   and $x = (A\omega/b)\sin\omega t - (B\omega/b)\cos\omega t$.

   c) $\omega = 1.6, \quad A = 3.2, \quad B = 2.4$
   $x = 1.6\sin\omega t - 1.2\cos\omega t, \quad y = 3.2\cos\omega t + 2.4\sin\omega t$ or
   $x = -2\cos(\omega t + \phi) = 2\sin(\omega t + \phi - \pi/2)$ and $y = 4\sin(\omega t + \phi)$
   where $\phi = \sin^{-1}\dfrac{4}{5}$, and $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$. The peaks in the predator population lag behind the peaks in the prey population by a quarter of the period.

45. Care is required if the characteristic equation of $Lu = 0$ has a double root. Otherwise a basis for $\ker(L^2)$ is $\{u_1,\ xu_1,\ u_2,\ xu_2\}$.
   The two dimensional kernel of L is a subspace of the four dimensional kernel of L^2.

Chapter 4

1. a) $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^9}{9!}$

   b) $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!}$

   c) $x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \dfrac{x^9}{9!}$

2. $p_{2m+1}(x) = \displaystyle\sum_{k=0}^{m} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$

3. a) $2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$

   b) $\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}\left(x - \dfrac{\pi}{4}\right) - \dfrac{1}{2\sqrt{2}}\left(x - \dfrac{\pi}{4}\right)^2 + \dfrac{1}{6\sqrt{2}}\left(x - \dfrac{\pi}{4}\right)^3 + \dfrac{1}{24\sqrt{2}}\left(x - \dfrac{\pi}{4}\right)^4$

4. a) $3 + 3(x-1)$

   b) $3 + 3(x-1) + (x-1)^2$

   c) $7 + 5(x-2) + (x-2)^2$

5. a) $p_1(x) = x$ and $R_2(x) = -\dfrac{x^2}{2(1+c)^2}$ for some c between 0 and x.

7. a) $p_n(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots + (-1)^k\dfrac{x^n}{n!}$ where $n = 2k$;

   $R_{n+1}(x) = \dfrac{(-1)^{k+1}\sin c}{(n+1)!}x^{n+1}$ for some c between 0 and x.

   c) $|R_{n+1}(x)| < \dfrac{1}{10^{n+2}(n+1)!}$

   d) $n = 4$

   e) The estimate $\sin x \le 1$ (whenever $x \in \mathbb{R}$) is less crude in the case when $x \ge 1$.

   f) $n = 14$

g) Rearranging the inequality $|\text{error}| \leq \dfrac{a^{11}}{11!} < 10^{-6}$ gives $a < \sqrt[11]{11!10^{-6}}$. So a less than 1.398 will do.

10. a) Horizontal point of inflexion at 1; local maximum at -2.
    b) Horizontal point of inflexion at -1; local minima at 2 and 3.

12. a) 1/2 b) 0 c) boundedly divergent d) 1
    e) 0 f) diverges to ∞ g) a

13. a) 1/2 b) $N = 2/\epsilon$ works

14. a) e^4 b) $1/e$

15. e) $(1+\sqrt{5})/2$

16. a) $1/2,\ 0$ b) $1/2,\ -1/2$ c) $\sqrt{2},\ -\sqrt{2}$
    d) $1/2+\sin 2,\ -1/5+\sin 5$ e) $\infty,\ \pi$ f) $1+\pi/2,\ 1-\pi/2$

17. b) 2

19. a) iii) 1 b) iii) 3/4

24. diverges

25. a) divergent b) divergent c) convergent

26. a) convergent b) divergent c) divergent
    d) divergent e) divergent f) convergent

27. a) convergent b) convergent c) convergent d) divergent

28. a) conditionally convergent
    b) divergent (by the kth term test)
    c) absolutely convergent
    d) diverges to ∞; $\dfrac{(-1)^k}{\sqrt{k}+(-1)^k} = \dfrac{(-1)^k\sqrt{k}}{k-1} - \dfrac{1}{k-1}$

29. b) $s_4 = 9677/16380 \approx 0.59078$; $|\text{error}| \leq 1/126$
    c) n equal to 99 will do.

30. a) convergent b) divergent c) convergent
    d) convergent e) divergent f) convergent

31. Only e) and f) are divergent.

32. Converges if $\lim\limits_{k\to\infty} a_k > 1$ and diverges otherwise.

34. a) $\displaystyle\sum_{k=0}^{\infty}\frac{x^k}{k!}$ b) $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$ c) $\displaystyle\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}$

d) $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$ e) $\displaystyle -\sum_{k=0}^{\infty}x^k$

35. a) $e^x = 1 + x + \dfrac{x^2}{2!}\cdots + \dfrac{x^n}{n!} + \dfrac{e^{c_n}x^{n+1}}{(n+1)!}$, for some c_n between 0 and x.

36. a) $R_{n+1}(x) = \dfrac{\sinh c_n}{(n+1)!}x^{n+1}$ if n is odd and $R_{n+1}(x) = \dfrac{\cosh c_n}{(n+1)!}x^{n+1}$ if n is even. In each case, c_n lies between 0 and x.

37. a) $R_{n+1}(x) = \dfrac{(-1)^n x^{n+1}}{(n+1)(1+c_n)^{n+1}}$ for some c_n between 0 and x.

39. a) $f'(x) = 2e^{-1/x^2}/x^3$ if $x \neq 0$; $f'(0) = 0$.

b) $f''(x) = 4e^{-1/x^2}/x^6 - 6e^{-1/x^2}/x^4$ if $x \neq 0$; $f''(0) = 0$.

e) The Maclaurin series is $0 + 0x + 0x^2 + 0x^3 + \cdots$, and hence converges everywhere. It only converges to f at 0.

40. b) $\ln 2$

f) $\frac{1}{2}\ln 2$

41. Students studying MATH1231 only need to state the corresponding open interval in each case.

a) $(-6, 6)$ b) $[-1, 1]$ c) $(-2, 2)$

d) $[1, 3]$ e) $[1/3, 1)$ f) $(-3, 3]$

g) $(-\infty, \infty)$

43. a) $\displaystyle\sum_{k=0}^{\infty}\frac{(k+1)x^k}{k!}$ b) $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k x^{4k+2}}{(2k+1)!}$

c) Differentiate 34(e): $\displaystyle\sum_{k=1}^{\infty}kx^{k-1}$

44. All series are valid on $(-1, 1)$.

a) $\displaystyle\sum_{k=0}^{\infty}(-1)^k x^k$ b) $\displaystyle\sum_{k=0}^{\infty}(-1)^k x^{2k}$ c) $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$

45. a) $f(x) = x^5 - \dfrac{x^{11}}{3!} + \dfrac{x^{17}}{5!} - \dfrac{x^{23}}{7!} + \cdots = \displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{6k-1}}{(2k-1)!}$

c) A horizontal point of inflexion.

46. $R = 1$, $\ln 2 \approx \frac{56}{81}$.

47. a) A local maximum

b) $\mathrm{Si}(x) = x - \dfrac{x^3}{3!3} + \dfrac{x^5}{5!5} - \dfrac{x^7}{7!7} + \cdots$

c) $\mathrm{Si}(\pi) \approx \pi - \frac{\pi^3}{3!3} + \frac{\pi^5}{5!5} - \frac{\pi^7}{7!7} \approx 1.84$

48. a) $(-1, 1)$ b) $f(x) = \dfrac{x}{(1-x)^2}$

49. a) $\sum_{k=1}^{\infty} \frac{x^k}{k!}$ b) $y' = \sum_{k=1}^{\infty} k a_k x^{k-1}$, $y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$

c) $a_k = \frac{1}{k^2 k!}$ whenever $k \geq 1$ d) $y = \sum_{k=1}^{\infty} \frac{x^k}{k^2 k!}$ whenever $x \in \mathbb{R}$.

Chapter 5

1. $8 \sinh(1/2) \approx 4.17$ metres

2. $27\frac{2}{9}\,{}^\circ C$

3. a) $(13^{3/2} - 8)/27$ b) 8 c) $\frac{8}{27}(10\sqrt{10} - 1)$

4. $6a$

5. $\sqrt{2}(e^{2\pi} - 1)$

6. 8

7. a) 9 seconds
   b) $10\sqrt{41} \approx 64.03$ metres per second
   c) 90 metres
   d) $80\sqrt{2} - 80\ln(\sqrt{2} - 1) + 25\sqrt{41} + 160\ln 2 - 80\ln(\sqrt{41} - 5) \approx 427.53$ metres

8. a) $v(t) = \frac{\pi^2}{2}|\sin(\pi t)|$
   b) (i) $t = \frac{1}{2} + k$, where k is a positive integer. (ii) $t = n$, where n is a positive integer.
   c) A semicircle of centre $(0, 0)$ and radius 1 in the upper half-plane.
   d) π
   e) 3π

10. a) $\frac{\pi}{27}(145^{3/2} - 1)$ b) $64\pi/3$

13. $32\pi/5$