Problems for Chapter 1

Questions marked with [**R**] are routine, with [**H**] are harder and with [**X**] are for MATH1241 only. You should make sure that you can do the easier questions before you tackle the more difficult questions.

Problems 1.1 : Sketching simple surfaces in $\mathbb{R}^3$

1. [**R**] For each of the following surfaces, sketch some level curves and sketch the yz-profile (which is found by intersecting the surface with the plane $x = 0$). Hence sketch the surface.

   a) $z = x^2 + y^2$ b) $x^2 + y^2 + z^2 = 1$

   c) $z^2 = x^2 + y^2 - 1$ d) $z^2 = x^2 + y^2$

   e) [**H**] $z = x^2 - y^2$

Problems 1.2 : Partial differentiation

2. [**R**] Given that $z = e^{x^2y}$, find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ and $\dfrac{\partial^2 z}{\partial y \partial x}$.

3. [**R**] In each case, find all first and second order partial derivatives and verify that $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$.

   a) $z = x^2y + y^2$ b) $z = \tan^{-1}(y/x)$ c) $z = \sin(x - cy)$

Problems 1.3 : Tangent planes to surfaces

4. [**R**] Find a normal vector $\mathbf{n}$ and the equation of the tangent plane to the surface $\mathcal{S}$ at the point $\mathbf{x}_0$.

   a) $\mathcal{S}: \quad z = x^2 + y^2, \qquad \mathbf{x}_0 = (3, 5, 34).$

   b) $\mathcal{S}: \quad z = 4x^2y, \qquad \mathbf{x}_0 = (2, -1, -16).$

   c) $\mathcal{S}: \quad z = \ln(x^2 + 3y^2), \qquad \mathbf{x}_0 = (2, -1, \ln 7).$

   d) $\mathcal{S}: \quad z^2 + x^2 + y^2 = 1, \qquad \mathbf{x}_0 = \left(\frac{1}{3}, \frac{1}{2}, \frac{\sqrt{23}}{6}\right).$

Problems 1.4 : The total differential approximation

5. [**R**] The volume V of a football in the shape of an ellipsoid of revolution with semi-axes of length a, b and b is given by
$$V = \frac{4}{3}\pi ab^2.$$
   The values of a and b are measured to be 12.0 cm and 7.0 cm respectively, each to the nearest millimetre.

a) Use these measurements to calculate the volume of the football.

b) Use the total differential to estimate the maximum absolute error in the calculated value of V.

c) Hence estimate the percentage error in your answer of (a).

6. [**R**] Suppose that $z = \dfrac{x+1}{y^2+1}$. The measured values of x and y are 3 and 1 respectively and each of the measurements is made with an error whose absolute value is at most 0.02. Use the total differential approximation of z to estimate the maximum error in the calculated value of z.

7. [**R**] Use the total differential approximation of $f(x, y) = \sqrt{x^2 + y^2}$ to estimate $\sqrt{2.98^2 + 4.03^2}$.

8. [**R**] The specific gravity S of a solid is given by

$$S = \frac{A}{A - W},$$

where A and W are its weights in air and water respectively.

a) Find $\dfrac{\partial S}{\partial A}$ and $\dfrac{\partial S}{\partial W}$.

b) If A and W are measured to be 15.1 gm and 5.1 gm respectively, and if each of these measurements is made with an error whose absolute value is at most 0.2 gm, then use the total differential approximation of S to estimate the maximum error in the calculated value of S.

9. [**R**] The specific volume v of a compressible fluid flowing through a section of area A with mean velocity V is given by

$$v = kAV$$

where k is a constant. If v decreases by 5% and A increases by 4%, then estimate the percentage change in V.

10. [**R**] A triangle has two sides of length a and b with an included angle measuring $\pi/3$ radians. Given that a increases by 5%, b decreases by 6% and the included angle increases by 2%, estimate the percentage increase of area of the triangle.

Problems 1.5 : Chain rules

11. [**R**] Use a chain rule to calculate $\dfrac{dw}{dt}$ (as a function of t) when

a) $w = xy, \quad x = e^t, \quad y = t^2;$

b) $w = x^2 + y^2 + z^2, \quad x = \cos t, \quad y = \sin t, \quad z = t.$

12. [**R**] A cylindrical metallic solid is expanding under heat in such a way that its height is increasing at the rate of 0.1 cm/sec and its radius is increasing at the rate of 0.05 cm/sec. Find the rate of increase of its volume at the instant when the height is 10 cm and the radius is 5 cm.

13. [**H**] Suppose that f is a differentiable function of a single variable and $F(x, y)$ is defined by $F(x, y) = f(x^2 - y)$.

 a) Show that F satisfies the partial differential equation

$$\frac{\partial F}{\partial x} + 2x\frac{\partial F}{\partial y} = 0.$$

 b) Given that $F(0, y) = \sin y$ for all y, find a formula for $F(x, y)$.

14. [**H**] Consider the differential equation

$$\frac{\partial^2 u}{\partial t^2} - 16\frac{\partial^2 u}{\partial x^2} = 0.$$

(This is an example of the one dimensional wave equation, which can be used to model, for example, the displacement $u(x, t)$ of a particle at position x along a vibrating guitar string at time t.)

 a) Suppose that g is an arbitrary twice-differentiable function of one variable and that $u(x, t) = g(x + \lambda t)$, where λ is a constant. Calculate u_{xx} and u_{tt}.

 b) Given that $u(x, t) = g(x+\lambda t)$, find all values of λ such that u satisfies the differential equation.

15. [**X**] A function f of two variables is said to be <u>homogeneous of degree</u> n if

$$f(tx, ty) = t^n f(x, y)$$

whenever $t > 0$. Show that such a function f satisfies the equation

$$x\,\frac{\partial f}{\partial x} + y\,\frac{\partial f}{\partial y} = nf.$$

16. [**X**]

 a) Suppose that a and b real numbers. Consider the partial differential equation

$$a\,\frac{\partial u}{\partial x} + b\,\frac{\partial u}{\partial y} = 0. \qquad (1)$$

 Use the chain rule to show that for a suitable choice of constants α and β, every function of the form $u(x, y) = f(\alpha x + \beta y)$, where f is differentiable, satisfies (1).

 b) Generalise this remark to find solutions $u : \mathbb{R}^n \to \mathbb{R}$ to the differential equation

$$a_1\,\frac{\partial u}{\partial x_1} + a_2\,\frac{\partial u}{\partial x_2} + \cdots + a_n\,\frac{\partial u}{\partial x_n} = 0.$$

17. [**X**] A function u of two variables is defined implicitly by

$$u(x, t) = f\big(x - t\,u(x, t)\big),$$

where f is a given bounded, differentiable function of one variable, $f : \mathbb{R} \to \mathbb{R}$.

a) Calculate $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial x}$.

b) Show that $\dfrac{\partial u}{\partial t} + u\,\dfrac{\partial u}{\partial x} = 0$.

c) Given that $f(s) = 1 - \tanh s$, find the smallest positive number t_m such that $\dfrac{\partial u}{\partial x}$ is undefined for precisely one value of x.

d) Sketch $u(x,t)$ as a function of x for several fixed values of t (taking t to be positive), and hence interpret your result in (c). (Maple would be useful for the plots.)

e) For fixed $t > t_m$ is $u(x,t)$ a function of x?

f) Generalise these results and find a way to predict t_m for arbitrary differentiable, bounded, monotonic decreasing functions f.

18. [**H**] A point sits on the hyperboloid

$$F(x,y,z) = x^2 + y^2 - z^2 = 1.$$

The position of the point moves with respect to time as $x(t) = z(t) = t$. The paramerisation of y is unspecified aside from $y(t) \neq 0$.

a) Find the parameterisation of the y co-ordinate and then find $\frac{dy}{dt}$.

b) Calculate the normal to the hyperboloid at the point (x,y,z), this is the vector

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)^T.$$

c) The chain rule states that

$$\frac{dF}{dt} = (\nabla F)\cdot\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)^T.$$

Deduce that $(\nabla F)\cdot\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)^T = 0$ and interpret this equation geometrically. Use this equation to find $\frac{dy}{dt}$ (without first finding y as a function of t as in part a).

d) [**X**] The hyperboloid and the tangent plane at the point (x_0, y_0, z_0) always intersect in two straight lines, find the equation of these lines. Hint: substitute the equation of the tangent plane into the hyperboloid and use the quadratic formula.

Problems for Chapter 2

Revision problems

1. [**R**] Evaluate each of the following integrals by inspection. Do not use substitution.

a) $\displaystyle\int xe^{2x^2}\,dx$ b) $\displaystyle\int x\sin(x^2)\,dx$ c) $\displaystyle\int x^2\cos(2x^3)\,dx$

d) $\displaystyle\int \frac{x}{5x^2-11}\,dx$ e) $\displaystyle\int \sin x\cos^3 x\,dx$ f) $\displaystyle\int \frac{dx}{x\,\ln x}$

g) $\displaystyle\int \frac{x+2}{\sqrt{x^2+4x+7}}\,dx$ h) $\displaystyle\int x\sqrt{1+x^2}\,dx$ i) $\displaystyle\int x^2\sqrt{9-4x^3}\,dx$

j) $\displaystyle\int \frac{x^2}{\sqrt{9-4x^3}}\,dx$ k) $\displaystyle\int \frac{x^3}{(1+x^4)^3}\,dx$ l) $\displaystyle\int \frac{\sec^2 x}{\tan^4 x}\,dx$

m) $\displaystyle\int \frac{\cos x}{\sin^3 x}\,dx$ n) $\displaystyle\int e^{2x}(4+3e^{2x})^{\frac{1}{3}}\,dx$ o) $\displaystyle\int \frac{1}{x\,(\ln x)^5}\,dx$

2. [**R**] Integrate the following by parts.

a) $\displaystyle\int x^2e^{-x}\,dx$ b) $\displaystyle\int x^3\ln x\,dx$ c) $\displaystyle\int \frac{x}{\cos^2 x}\,dx$ d) $\displaystyle\int \frac{(\ln x)^2}{x^2}\,dx$

e) $\displaystyle\int e^x\cos x\,dx$ f) $\displaystyle\int \ln x\,dx$ g) $\displaystyle\int \tan^{-1} x\,dx$

Problems 2.1 : Trigonometric integrals

3. [**R**] Evaluate the following integrals.

a) $\displaystyle\int_0^{\pi/2} \sin^7 x\cos x\,dx$ b) $\displaystyle\int_0^{\pi} \sin^3 x\cos^2 x\,dx$

c) $\displaystyle\int \sec^3 x\tan x\,dx$ d) $\displaystyle\int \cos^2\theta\,d\theta$

e) $\displaystyle\int \cos x\cos 10x\,dx$ f) $\displaystyle\int \sin 2x\cos 3x\,dx$

Problems 2.2 : Reduction formulae

4. [**R**]

a) By multiplying the integrand by $\dfrac{\tan x+\sec x}{\tan x+\sec x}$, find $\displaystyle\int \sec x\,dx$.

b) The reduction formula

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

is valid whenever $n \geq 2$. Use it to find the following integrals.

i) $\displaystyle\int \sec^4 x\, dx$ ii) $\displaystyle\int \sec^5 x\, dx$

c) [**X**] Prove the reduction formula given above.

5. [**R**] Suppose that

$$I_{m,n} = \int_0^{\pi/2} \cos^m x \sin^n x\, dx$$

whenever m and n are nonnegative integers. Use the reduction formula

$$I_{m,n} = \begin{cases} \left(\frac{m-1}{m+n}\right) I_{m-2,n} & \text{provided that } m \geq 2 \\ \left(\frac{n-1}{m+n}\right) I_{m,n-2} & \text{provided that } n \geq 2 \end{cases}$$

to evaluate the following integrals.

a) $\displaystyle\int_0^{\pi/2} \cos^6 x \sin^4 x\, dx$ b) $\displaystyle\int_0^{\pi/2} \cos^5 x \sin^5 x\, dx$ c) $\displaystyle\int_0^{\pi/2} \cos^3 x \sin^4 x\, dx$

6. [**R**] Suppose that $I_n = \displaystyle\int_0^1 x^n e^{-x}\, dx$. Prove that

$$I_n = n\, I_{n-1} - \frac{1}{e}$$

whenever $n > 0$. Hence evaluate $\displaystyle\int_0^1 x^3 e^{-x}\, dx$.

7. [**R**] It was proven in the notes that if $I_n = \displaystyle\int_0^{\pi/4} \tan^n x\, dx$ then

$$I_n = \frac{1}{n-1} - I_{n-2}$$

whenever $n > 1$. Use this to evaluate I_7 and I_8.

8. [**R**] Suppose that $I_n = \displaystyle\int_1^e x(\ln x)^n\, dx$. Show that

$$I_n = \frac{1}{2}\left(e^2 - nI_{n-1}\right)$$

whenever $n \geq 1$. Hence evaluate I_3.

9. [**R**] By writing $\cos^n x$ as $\cos^{n-1} x \cos x$ and integrating by parts, show that

$$\int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x\, dx$$

whenever $n \geq 2$. Hence find $\displaystyle\int_0^{\pi/2} \cos^8 x\, dx$ and $\displaystyle\int_0^{\pi/2} \cos^7 x\, dx$.

10. **[R]** Suppose that $I_n = \displaystyle\int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx$. Find a reduction formula for I_n.

11. **[H]** Show that $\displaystyle\int_0^1 x^m(1-x)^n\,dx = \frac{m!n!}{(m+n+1)!}$ for all nonnegative integers m and n.

12. **[H]** Suppose that $I_n = \displaystyle\int_0^{\pi/2} \cos^n x\,dx$.

a) By Writing the reduction formula of Question 9 as $I_n = \left(1-\frac{1}{n}\right) I_{n-2}$, show that

$$I_{2m} = \left(1-\frac{1}{2m}\right)\left(1-\frac{1}{2m-2}\right)\cdots\left(1-\frac{1}{2}\right)\frac{\pi}{2} = \frac{\pi}{2}\prod_{k=1}^{m}\left(1-\frac{1}{2k}\right)$$

and

$$I_{2m+1} = \prod_{k=1}^{m}\left(1-\frac{1}{2k+1}\right).$$

b) Deduce that

$$\frac{2}{\pi}\frac{I_{2m}}{I_{2m+1}} = \prod_{k=1}^{m}\left(1-\frac{1}{(2k)^2}\right).$$

c) By considering $\cos x$ on $(0, \frac{\pi}{2})$, show that

$$I_{2m+2} \le I_{2m+1} \le I_{2m}$$

whenever $m \ge 1$.

d) Use the result of (a) and (c) and the pinching theorem to deduce that

$$\lim_{m\to\infty}\frac{I_{2m}}{I_{2m+1}} = 1.$$

e) Conclude that

$$\lim_{m\to\infty}\prod_{k=1}^{m}\left(1-\frac{1}{(2k)^2}\right) = \frac{2}{\pi}.$$

This limit is called *Wallis' product.*

f) Show that

$$\prod_{k=1}^{m}\left(1-\frac{1}{(2k)^2}\right) = \prod_{k=1}^{m}\frac{(2k-1)(2k+1)}{(2k)^2} = \frac{(2m+1)((2m)!)^2}{2^{4m}(m!)^4}$$

and deduce that Wallis' product may be written as

$$\frac{\pi}{2} = \lim_{m\to\infty}\frac{2^{4m}(m!)^4}{(2m+1)((2m)!)^2}.$$

13. [**X**]

a) Show that

$$\int_m^{2m} \ln x\,dx = m\ln m + 2m\ln 2 - m.$$

b) Show that the trapezoidal rule (see SH6 page 457, SH7 page 537 or SH8 page 484) with m subintervals gives the approximate value of this integral as

$$\ln\left(\frac{(2m)!}{m!}\right) - \frac{1}{2}\ln 2.$$

c) It can be shown that if the trapezoidal rule with m intervals of equal width is used to approximate $\int_a^b f(x)\,dx$, where f is a twice differentiable function, then the absolute error in the approximation is no greater than $\dfrac{(b-a)^3 M}{12m^2}$, where M is the maximum of $|f''(x)|$ on $[a,b]$. Use this error bound to show that the error in the approximation in part (b) approaches 0 as $m \to \infty$.

d) Conclude that

$$\frac{(2m)!}{m!} \Big/ \left\{\sqrt{2}\,2^{2m} m^m e^{-m}\right\} \to 1$$

as $m \to \infty$.

14. [**X**] Using the previous two questions (or otherwise), show that

$$\frac{\pi}{2} = \lim_{m\to\infty} \frac{(m!)^2}{4m m^{2m} e^{-2m}},$$

so that

$$\frac{m!}{\sqrt{2\pi}\,m^{m+\frac{1}{2}} e^{-m}} \to 1$$

as $m \to \infty$. This is sometimes called Stirling's formula (or Stirling's approximation for $m!$).

Problems 2.3 : Trigonometric and hyperbolic substitutions

15. [**R**] Evaluate the following integrals.

a) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$ b) $\displaystyle\int \frac{dx}{\sqrt{x^2-6x+13}}$

c) $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx$ d) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+16}}$

e) $\displaystyle\int (1-x^2)^{-\frac{3}{2}}\,dx$ f) $\displaystyle\int_{-1}^1 \frac{dx}{x^2+2x+2}$

16. Evaluate $\displaystyle\int \frac{x}{\sqrt{x^2-4}}\,dx$ by making an appropriate substitution. Are there any other methods or substitutions that could be used? Which one is most efficient?

Problems 2.4 : Integrating rational functions

17. [**R**] Evaluate the following integrals.

a) $\displaystyle\int \frac{1}{x^2+4x+3}\,dx$ b) $\displaystyle\int \frac{5x-7}{x^2-3x+2}\,dx$

c) $\displaystyle\int \frac{(x+1)}{x^2(x-1)}\,dx$ d) $\displaystyle\int \frac{1}{(x^2-1)^2}\,dx$

e) $\displaystyle\int \frac{x^2+1}{x^2-1}\,dx$ f) $\displaystyle\int \frac{18}{(x^2+9)(x-3)}\,dx$

g) $\displaystyle\int \frac{x^2+x+2}{(x+1)(x+2)^2}\,dx$ h) $\displaystyle\int \frac{1-x}{(1+x)^3}\,dx$

Problems 2.5 : Other substitutions and miscellaneous integrals

18. [**R**] Evaluate the following integrals.

a) $\displaystyle\int \frac{x}{x^2+2x+10}\,dx$ b) $\displaystyle\int \frac{x}{\sqrt{x^2+2x+10}}\,dx$

c) $\displaystyle\int \frac{dx}{1+\sqrt{x}}$ d) [**H**] $\displaystyle\int_1^{64} \frac{1}{x^{1/2}+x^{1/3}}\,dx$

19. [**R**] Evaluate $\displaystyle\int_0^1 \frac{x^3}{(4+x^2)^{5/2}}\,dx$ by

a) using the substitution $x = 2\tan\theta$ (since the integrand involves $\sqrt{4+x^2}$);

b) using the substitution $x = 2\sinh\theta$ (since the integrand involves $\sqrt{4+x^2}$);

c) using the substitution $u^2 = 4+x^2$ (aiming for a rational function);

d) using the substitution $u = 4+x^2$.

Which method works best?

20. [**X**] *Caution: for each integral below, it is unlikely to be profitable to seek an indefinite integral.*

a) Use the substitution $x = \pi - u$ to show that $\displaystyle\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$.

b) Show that $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2$.

21. [**X**] Find $\displaystyle\int \frac{x^2-1}{x^2+1}\frac{1}{\sqrt{1+x^4}}\,dx$.

The substitution $u^2 = x^2 + \frac{1}{x^2}$ may be useful, but there are other possible approaches. [Taken from Spivak's *Calculus*.]

22. [**R**] The following integrals were selected from past papers. Evaluate each one.

a) $\displaystyle\int \frac{dx}{x(x^2+x+1)}$

b) $\displaystyle\int 8\sinh x \cosh^4 x\, dx$

c) $\displaystyle\int \frac{3x+5}{x^2+4x+8}\, dx$

d) $\displaystyle\int \sqrt{25-x^2}\, dx$

e) $\displaystyle\int \frac{3x^2-5x+3}{(x-1)\,(x^2-2x+2)}\, dx$

f) $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\, dx$

g) $\displaystyle\int \frac{dx}{(x^2+3)^{3/2}}$

h) $\displaystyle\int \cos(4x)\sin(3x)\, dx$

i) $\displaystyle\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx$. If convergent evaluate the integral.

Problems for Chapter 3

Problems 3.3 : Separable ODEs

1. **[R]** Solve the following differential equations.

   a) $\dfrac{dy}{dt} = t^2(1+y^2)$ b) $\dfrac{dy}{dx} = xy^2$

   c) $\dfrac{dy}{dx} = \dfrac{\sin x}{y^2}$ d) $\dfrac{dy}{dx} = \dfrac{y}{x(x-1)}$

   e) $\dfrac{dy}{dx} = e^{x+y}$ f) $y\cos^2 x\,\dfrac{dy}{dx} = \tan x + 2, \quad y = 2$ when $x = \pi/4$

   g) $x\,\dfrac{dy}{dx} = y \ln x$ h) $\dfrac{dy}{dx} = 3x^2y^2$, given that $y = 1$ when $x = 0$.

2. **[H]** Try to find the general solution for $\dfrac{dy}{dx} = 3y^{2/3}$.
   Your answer will probably be $y = (x+C)^3$.
   Observe $y = 0$ is also a solution and it cannot be expressed as $y = (x+C)^3$ for any value of C.
   How do you account for this? Are there any other solutions?

3. **[R]** Oil is leaking out of a tank in such a way that the depth of oil h in the tank at time t satisfies $\dfrac{dh}{dt} = -\sqrt{2h}$. If the initial height is 4 cm, then find the time taken for the tank to empty.

Problems 3.4 : First order linear ODEs

4. **[R]** Solve the following linear ODEs.

   a) $\dfrac{dy}{dx} - 2y = x^2e^{2x}$ b) $\dfrac{dy}{dx} + 3y = \dfrac{e^{-3x}}{1+x^2}$

   c) $x\,\dfrac{dy}{dx} + (1+x)y = 2$ d) $x\,\dfrac{dy}{dx} - 2y = 6x^5$

   e) $\cos^2 x\,\dfrac{dy}{dx} + y = \tan x$ f) $\dfrac{dy}{dx} = x + 2y\tan 2x$

5. **[R]** Solve $\dfrac{dx}{dt} = t - x$. Sketch the solution curves passing through $(0,1)$, $(0,0)$, $(0,-1)$ and $(0,-2)$.

6. **[R]** An object falling vertically experiences a resistance which is proportional to its velocity.

   a) Explain why its acceleration is given by $\dfrac{dv}{dt} = g - kv$, where g is the acceleration due to gravity and k is a positive constant.

b) Solve this as a linear equation for the initial condition $v = 0$.

c) Solve this as a separable equation.

d) What is the terminal (or limiting) velocity of the object?

Problems 3.5 : Exact ODEs and miscellaneous first order ODEs

7. [**R**] Solve the following exact ODEs.

a) $2xy + (x^2 + y^2)\dfrac{dy}{dx} = 0$ b) $(\sin y - xy^2)\,dx + (x\cos y - x^2y)\,dy = 0$

8. [**R**] Determine which of the following differential equations are exact and solve those which are.

a) $(2xe^y + e^x) + (x^2 + 1)e^y\,\dfrac{dy}{dx} = 0$

b) $y(x^2 + \ln\,y) + x\,\dfrac{dy}{dx} = 0$

c) $\left(\dfrac{y}{1+x^2} + e^y\right)\,dx + \left(\tan^{-1}x + xe^y\right)\,dy = 0$

d) $e^{xy}(y\,\cos\,x - \sin\,x) + xe^{xy}\cos\,x\,\dfrac{dy}{dx} = 0$

9. [**R**] *(Miscellaneous first order ODEs)*
Solve the following ODEs.

a) $(2xy - 3\tan x)\,\dfrac{dy}{dx} = 3y\sec^2 x - y^2$

b) $x - (x^2y + y)\,\dfrac{dy}{dx} = 0$

c) $x^2\,\dfrac{dy}{dx} + xy = 1,\ x > 0$

d) $x^2\,\dfrac{dy}{dx} - xy = y$

e) $x\,\dfrac{dy}{dx} = \dfrac{1+x^2}{y^2}, \qquad y(1) = 3.$

10. [**X**] Solve the differential equation

$$(2x - 10y^3)\frac{dy}{dx} + y = 0$$

by first multiplying it through by some function $\mu(y)$ to make it exact.

Problems 3.6 : Solving ODEs by using a change of variable [X]

11. [**X**] Solve the following ODEs by using the substitution $y(x) = x \cdot v(x)$.

a) $\dfrac{dy}{dx} = \dfrac{xy - y^2}{x^2}$

b) $\dfrac{dy}{dx} = \dfrac{y}{x} + \cos\left(\dfrac{y-x}{x}\right)$

c) $\dfrac{dy}{dx} + \dfrac{2xy}{x^2+y^2} = 0$

d) $\dfrac{dy}{dx} + \dfrac{x^2+y^2}{2xy} = 0$

e) $x\,\dfrac{dy}{dx} = \sqrt{x^2+y^2} + y, \quad y(1) = 0.$

12. [**X**] Use the substitution $y = x \cdot v(x)$ to find the solution to the differential equation

$$\frac{dy}{dx} = \frac{y-x}{y+x}, \qquad y(1) = 0.$$

Use polar coordinates to sketch the corresponding curve.

13. [**X**] Solve $\dfrac{dy}{dx} = (y-x)^2$ using the substitution $u = y - x$.

14. [**X**]

a) Solve $\dfrac{dy}{dt} = y - 2y^2$ by making the substitution $z = \dfrac{1}{y}$ to obtain a linear differential equation in z and t.

b) Use the same trick to solve

$$\frac{dy}{dt} = 5y - ty^2 \qquad \text{given that} \quad y = 1 \quad \text{when} \quad t = 0.$$

Find the maximum value of y when $t \geq 0$.

15. [**X**] Solve the following differential equations by using the given substitution.

a) $y\dfrac{dy}{dx} - 2y^2 = x; \qquad u = y^2.$

b) $x\dfrac{dy}{dx} + y(x^2 + \ln\, y) = 0, \quad x > 0, \quad y > 0; \qquad u = \ln y.$

c) $xyy'' = yy' + x(y')^2; \qquad u = y'/y$ or $u = \ln y$.

16. [**X**] In each case, use the substitution $v = \dfrac{dy}{dt}$ to solve the given ODE.

a) $\dfrac{d^2y}{dt^2} + 2\,\dfrac{dy}{dt} = 6$

b) $t^2\dfrac{d^2y}{dt^2} - \left(\dfrac{dy}{dt}\right)^2 = 0$

17. [**X**] Suppose that $\alpha(x)$ is a solution to the equation

$$\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u = 0.$$

a) Use the substitution $u(x) = \alpha(x)v(x)$ to write down a first order equation for $\dfrac{dv}{dx}$. What method can you see to solve this equation?

b) Verify that $u(x) = x^3$ is one solution to the differential equation

$$x^2\frac{d^2u}{dx^2} - 4x\frac{du}{dx} + 6u = 0.$$

Hence use the technique of (a) to find the general solution.

Problems 3.7 : Modelling with first order ODEs

18. [**R**] The simple population growth model, $\dfrac{dy}{dt} = ky$, is (because of limitations on resources, pollution, ...) unsatisfactory over a 'long' period. We might look at

$$\frac{dy}{dt} = k(y)y \tag{1}$$

(which of course ignores seasonal and other variations with time). Mathematically one of the simplest assumptions we can make (Verhulst 1839) is that $k(y)$ decreases *linearly* as y increases. In this case (1) may be written in the form

$$\frac{dy}{dt} = k\left(1 - \frac{y}{K}\right)y, \tag{2}$$

where k and K are constants.

a) i) Equation (2) has two constant (stationary) solutions. What are they?
   ii) Solve (2), given that $y = y_0$ when $t = 0$ and $0 < y_0 < K$.
   iii) State what happens to y as $t \to \infty$.
   iv) For what value of y is the rate of increase of y a maximum? (Caution — there is a simple, two line, method.)
   v) If $y_0 > K$ then what happens to y as t increases?

b) [**H**] It has been found empirically that for certain bodies $k(y)$ decreases linearly with $\ln y$. (Here y could be the volume, mass or number of cells of the body.) In this case (1) may be written as

$$\frac{dy}{dt} = \alpha \ln\left(\frac{K}{y}\right)y. \tag{3}$$

Solve this differential equation, given that $y = y_0$ when $t = 0$.

19. [**R**] An initially unpolluted lake of 10^9 litres has a river flowing through it at $1,000,000$ litres per day. A factory is built which discharges $10,000$ litres per day of pollutant into the lake. Assume that the total volume of liquid in the lake remains constant at 10^9 litres.

a) What will be the eventual level of pollution in the lake?

b) How long will it take to reach half this level?

c) Is there anything unrealistic about your model?

20. [**R**] A tank can hold 100 litres. Initially it holds 50 litres of pure water. Brine, which contains 2 grams of salt per litre, is run in at the rate of 3 litres per minute. The mixture, which is stirred continuously, is run off at 1 litre per minute. Let $x(t)$ denote the mass of salt (in grams) present in the tank after t minutes.

a) Set up a differential equation in x and t to model the system.

b) Show that when the tank is at the point of overflowing it contains $50(4-\sqrt{2})$ grams of salt.

21. [**R**] A population of size P is subject to seasonal variation. The population has a growth rate given by

$$\frac{dP}{dt} + P = 100 + 50\sin t.$$

a) Solve the differential equation given that $P(0) = 20$.

b) Find the average population size over a long period of time.

22. [**R**] The equation

$$\frac{dy}{dt} = k(1 + a\,\cos(2\pi t))y.$$

is an attempt to model seasonal variation. Solve it.

23. [**R**] It is estimated that the population growth rate of a certain developing country will fall linearly from 2% per year to 1.5% per year over the next decade.

a) Express the population growth rate as a function of time.

b) Given that the present population is 10 million, find the estimated population in 10 years time.

24. [**R**] An object falling in a resisting medium has a constant acceleration due to gravity of 9.8 m/sec^2 and also a drag force, which is approximately proportional to the speed (see Question 6). For a stone falling in water with velocity v m/sec., the acceleration from this drag force is approximately $10v$ m/sec^2.

a) Given that the stone is dropped from rest at the water surface, write a differential equation describing this situation, and solve it to find v.

b) Determine the terminal velocity, $\lim_{t\to\infty} v(t)$.

c) How long does it take before the stone is travelling with 95% of its terminal velocity?

d) What if it starts at a velocity higher than the terminal velocity?

25. [**R**] The differential equation $\dfrac{dy}{dx} = \lambda\dfrac{y}{x}$ is employed as a simple model for comparative growth.

a) Solve this differential equation.

b) The mass y of the large claw of a fiddler crab is compared to the mass x of its body (without claw) over a period of time. The measurements taken (in grams) are recorded in the following table.

x	55	300	536	1080	1449	2233
y	5	72	175	522	773	1498

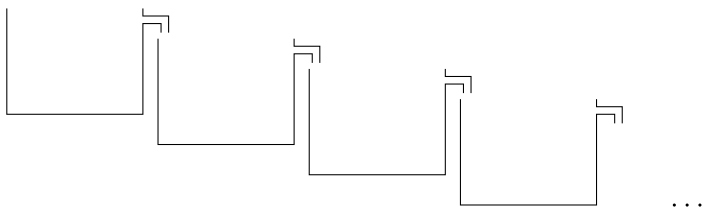
By graphing $f(x)$ against $f(y)$ for a suitable function f, show that the above model appears reasonable in this example. Determine approximately the value of λ and the constant of integration.

26. **[R]** An investor puts \$500,000 into Hitek Bonds, which pay a profit of 20% a year, compounded daily (i.e., if P is the amount of money in the bonds at any time, $\frac{dP}{dt} = 0.2P$ approximately).

    a) How much money would he own at the end of a year?

    b) **[H]** His tax accountant advises that this is overprofitable, and suggests that he use \$200 a week of the interest towards the hire of a Volvo, which would be a tax write–off under the government's Small Business Incentive Scheme, and that at the end of each six months, all the remaining interest should be taken out and invested in the Lake Eyre Ricegrowers' Cooperative, which loses money continuously at a rate of 10% a year $\left(\text{so that } \frac{dP}{dt} = -0.1P\right)$. This would qualify the investor for various advantages under the Rural Rorts Scheme. If the investor takes this advice, what would the total of both his investments be after one year? [Take it that there are 52 weeks in a year.]

27. **[H]** There are $n + 1$ tanks each containing 100 litres and connected as shown.



Throughout, the liquid in every tank is kept well–mixed. The 0-th tank contains 100 litres of pure water with 50 grams of salt dissolved in it. The remainder contain pure water. Pure water is pumped into tank 0 at 3 litres per minute and liquid leaves the system from tank n at 3 litres per minute. Let m_k denote the mass (in grams) of salt in tank k.

    a) Show that

$$\frac{dm_k}{dt} = 0.03(m_{k-1} - m_k), \qquad k = 1, 2, \ldots, n$$

and

$$\frac{dm_0}{dt} = -0.03m_0.$$

    b) Show that

$$m_n = \frac{50(0.03)^n}{n!} t^n e^{-0.03t}.$$

28. **[H]** The decay of one atom of radioactive element A yields one atom of B which is itself radioactive. The decay constants for A and B are 0.25 and 2 (per day) respectively. A pure sample of K atoms of A is placed in a closed container. Let $y_1(t)$ and $y_2(t)$ denote the numbers of atoms of A and B that are present at time t. Write down two differential equations to describe this situation. Solve to obtain a formula for y_2. When is the amount of B in the container a maximum?

29. [**X**] A cyclist freewheeling on a level road experiences a retardation (i.e., a negative acceleration) that is proportional to the square of his speed. His speed is reduced from 20 metres per second to 10 metres per second in a distance of 100 metres. Find his average (with respect to time) speed during this period.

Problems 3.8 : Second order linear ODEs with constant coefficients

30. [**R**] Find the general solutions of the following second order ODEs.

a) $y'' + 3y' + 2y = 0$ b) $y'' + 2y' + 10y = 0$

c) $y' + 3y = 0$ d) $y'' + 4y' + 4y = 0$

31. [**R**] Find the solutions of the following initial value problems.

a) $y'' - 6y' + 5y = 0;$ $y = 1,$ $y' = 0,$ when $x = 0.$
b) $y'' + 2y' + 2y = 0;$ $y = 1,$ $y' = 0,$ when $x = 0.$

32. [**R**] Solve the following differential equations.

a) $y'' + 4y' + 3y = x$ b) $y'' - 6y' + 9y = 5e^{2x}$

c) $y'' + 2y' + 2y = 10 \cos 2x$ d) $y'' - y = e^{-x}$

e) $y'' + 4y = \sin 2x$ f) $y' + 3y = 10e^{2x}$

g) $y'' + 4y = \sin x \quad y = 0,\ y' = 1$ when $x = 0$

h) $y'' - 5y' + 4y = 2e^{2x} \quad y = 1,\ y' = 3$ when $x = 0$

33. [**R**] For each of the following differential equations, find the general solution of the associated homogeneous equation and write down the <u>form</u> of the particular solution you would seek. (Do <u>not</u> evaluate the unknown coefficients.)

a) $2y'' - 3y' - 5y = (x + 4)e^{5x/2}$
b) $y'' + 2y' - 24y = e^{4x} \sin 6x$
c) $y'' + 6y' + 9y = e^{-3x}$

34. [**H**] Find a particular solution for the differential equation $y'' - 4y' + 4y = 6x^2 e^{2x}$.

35. [**X**] By making the substitution $x = e^t$, reduce the linear differential equation

$$x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^5, \qquad x > 0,$$

to one with constant coefficients, and hence solve it.

36. [**X**]

a) Show that the second order differential equation

$$y'' + (m_1 + m_2)y' + m_1 m_2 y = 0,$$

where m_1 and m_2 are constants, can be written in the form

$$\frac{d}{dx}(y' + m_1 y) + m_2(y' + m_1 y) = 0.$$

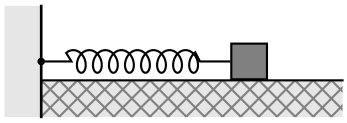
b) Suppose that $m_1 = m_2 = m$. Using the result of (a), solve the first differential equation by solving successively two first order linear equations.

37. [**R**] A cylindrical buoy of 80 cm in diameter floats in water with its axis vertical. When depressed slightly and released, it bobs up and down according to the differential equation

$$m\,\frac{d^2x}{dt^2} = -\pi 40^2 g x,$$

where m is the mass (in grams) of the buoy and x is the displacement (in centimetres) from the equilibrium position. Take the acceleration g due to gravity to be 980 cm/sec^2. The period of oscillation is observed to be 2.5 seconds. What is the mass of the buoy?

38. [**R**] A block of wood is attached to a spring and slides across a surface as shown.



Let $x(t)$ denote the horizontal distance of the block from its 'resting' position at time t. The motion of the block is modelled by the initial value problem

$$\frac{d^2x}{dt^2} + c\,\frac{dx}{dt} + 4x = 0, \qquad x'(0) = 0, \quad x(0) = 1,$$

where c is the 'coefficient of friction' between the block and the surface.

a) By solving the differential equation, describe the motion of the block in the 'ideal' situation when there is no friction between the block and the surface (that is, when $c = 0$).

b) Solve the initial value problem when $c = 2$ and when $c = 5$ and explain why the block oscillates if $c = 2$ but does not if $c = 5$.

c) [**H**] Find the smallest positive c such that the system does not oscillate. (This value of c corresponds to what is known as 'critical damping.')

39. [**R**] A circuit consists of an inductor and capacitor connected in series with a sinusoidal power source. The charge q (in coulombs) stored in a capacitor is given by the differential equation

$$\frac{d^2q}{dt^2} + 10\,000q = 1000 \sin \Omega t,$$

where $\Omega/(2\pi)$ is the frequency of the power source.

a) Find the solution to the corresponding homogeneous equation.

b) Write down the <u>form</u> of particular solution you would seek for the differential equation. (Warning: there are two cases, depending on the value of Ω.)

c) What sinusoidal frequency leads to unbounded oscillatory behaviour (known as 'resonance')?

40. [**R**]

a) Find the general solution of the vibrating system modelled by the equation

$$\frac{d^2y}{dt^2} + 3\,\frac{dy}{dt} + 2y = 20 \sin t.$$

b) The long term behaviour of this system is independent of the initial conditions. What is this 'steady state solution'?

41. [**X**]

a) Solve the previous question by first finding a particular complex-valued function $z(t)$ satisfying

$$\frac{d^2z}{dt^2} + 3\,\frac{dz}{dt} + 2z = 20e^{it}.$$

b) Similarly, find a particular solution of

$$\frac{d^2y}{dt^2} + 4y = \sin 2t$$

by considering

$$\frac{d^2z}{dt^2} + 4z = e^{2it}.$$

42. [**R**] A stationary wave on a guitar string of length L can be (partially) modelled by the boundary value problem

$$y'' = ky; \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0,$$

where k is a constant. Assume that y is not identically zero.

a) Suppose that k is positive and write k as μ^2, where $\mu \geq 0$. Show that the boundary value problem has no nonzero solutions.

b) Are there any nonzero solutions when $k = 0$?

c) Suppose now that k is negative and write k as $-\mu^2$, where $\mu > 0$. Find all possible values of μ such that the boundary value problem has a nonzero solution. Give the corresponding solutions.
(Each such solution corresponds to a natural harmonic of the string.)

43. [**X**]

a) If g is a given function which is continuous and *positive* on the interval $[0, L]$, show that the only solution of the boundary value problem

$$y'' - g(x)y = 0; \qquad y(0) = 0, \quad y(L) = 0, \quad L > 0,$$

is $y = 0$.
(*Hint*: Give a proof by contradiction using the maximum-minimum theorem for continuous functions.)

b) Find all possible values for λ such that there is a solution to the differential equation

$$y'' + 2y' + 2\lambda y = 0$$

satisfying $y(0) = y(L) = 0$ and y not identically zero.

44. **[H]** A simple linear predator-prey model:
Let X denote the number of predators at time t and Y the number of prey. Experience and theoretical considerations suggest that

(α) there is an equilibrium state $(X, Y) = (X_0, Y_0)$ with $X_0, Y_0 \neq 0$;

(β) an increase in the number of prey, from Y_0, results in an increase in the number of predators at a rate (approximately) proportional to the increase in prey, so that

$$\frac{dX}{dt} = a(Y - Y_0), \quad a > 0; \tag{1}$$

and

(γ) an increase in the number of predators, from X_0, results in a decrease in the number of prey at a rate (approximately) proportional to the increase in predators, so that

$$\frac{dY}{dt} = -b(X - X_0), \quad b > 0. \tag{2}$$

[In fact the 'linearisation' of a number of complicated models near an equilibrium state leads to (1) and (2).]

If $x = X - X_0$ and $y = Y - Y_0$ then we obtain

$$\frac{dx}{dt} = ay, \tag{3}$$

and

$$\frac{dy}{dt} = -bx. \tag{4}$$

a) Eliminate dt between (3) and (4) and solve to obtain a relation between x and y (the 'phase trajectories' of the system).

b) Eliminate x by differentiating (4) and substituting from (3). Hence solve for y and then use (4) again to obtain x.

c) Suppose that $a = 0.8$ and $b = 3.2$. If $x = -1.2$ and $y = 3.2$ when $t = 0$, then find x and y in terms of t and also find the phase trajectory.

45. **[X]** *(Note: do not attempt this question until Chapter 7 of MATH1241 Algebra has been completed.)*
Let V denote the vector space of twice differentiable functions on $\mathbb{R}$. Define a linear map L on V by the formula

$$Lu = a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu, \quad \text{where } a, b \text{ and } c \text{ are real numbers.}$$

Suppose that $u_1,\ u_2$ is a basis for the solution space of $L(u) = 0$. Find a basis for the solution space of the fourth order equation $L(L(u)) = 0$. What can you say about the kernels of L and L^2?

Problems for Chapter 4

Problems 4.1 : Taylor polynomials

1. [**R**] For each function f, find the Taylor polynomial of degree 9 for f about 0.

   a) $f(x) = e^x$ b) $f(x) = \sin x$ c) $f(x) = \sinh x$

2. [**R**] Suppose that $f(x) = \sin x$ and $m \geq 0$. Using summation notation, find a formula for the Taylor polynomial p_{2m+1} of degree $2m + 1$ for f about 0.

3. [**R**]

   a) Suppose that $f(x) = \sqrt{x}$. Find the Taylor polynomial of degree 3 for f about 4.
   b) Suppose that $g(x) = \cos x$. Find the Taylor polynomial of degree 4 for g about $\pi/4$.

4. [**R**] Let $f(x) = 1 + x + x^2$. Find the Taylor polynomial $p_n(x)$

   a) of degree $n = 1$ about 1.
   b) of degree $n = 2$ about 1.
   c) of degree $n = 2$ about 2.

Problems 4.2 : Taylor's theorem

5. [**R**] Suppose that $f(x) = \ln(1 + x)$.

   a) Express $f(x)$ in the form $p_1(x) + R_2(x)$, where p_1 is the first Taylor polynomial for f about 0 and R_2 is the Lagrange formula for the remainder.
   b) Suppose that $x \in [-0.1, 0.1]$ and consider the approximation $\ln(1 + x) \approx x$. Use your answer to (a) to show that an upper bound for the absolute error in this approximation is $1/162$.

6. [**R**] Suppose that $f(x) = \sqrt{1 + x}$ and let p_2 denote the second Taylor polynomial for f about 0. If $x \in [0, 1]$ then show that the absolute error in the approximation $f(x) \approx p_2(x)$ does not exceed $\frac{1}{16}$.

7. [**R**] Suppose that $f(x) = \cos x$ and that n is a positive even integer.

   a) Find the nth Taylor polynomial p_n for f about 0 and the Lagrange formula for the remainder R_{n+1}.
   b) Use the mean value theorem to prove that
   $$\sin x < x \quad \text{whenever } x > 0.$$
   c) Use parts (a) and (b) to find an upper bound for the absolute error in the approximation $f(1/10) \approx p_n(1/10)$.

d) Hence find a value for n such that the absolute error in the approximation of (c) is less than 10^{-6}.

e) The value of $\cos 2$ is estimated using $p_n(2)$ for some n. Explain why it is better *not* to use the inequality of (b) to find an upper bound for $|R_{n+1}(2)|$.

f) Find a value for n such that the absolute error in the approximation $f(2) \approx p_n(2)$ is less than 10^{-6}.

g) The Taylor polynomial p_{10} is used to approximate f on the interval $[-a, a]$, where a is a positive real number. Find a value for a such that the absolute error in the approximation $f(x) \approx p_{10}(x)$ is less than 10^{-6} whenever $x \in [-a, a]$.

8. [**X**] Below is a statement of the **mean value theorem for integrals**:

If f and g are continuous on $[a, b]$ and g is nonnegative on $[a, b]$, then

$$\int_a^b f(t)g(t)\,dt = f(c)\int_a^b g(t)\,dt$$

for some real number c in $[a, b]$.

This theorem is used in several of the problems below. We give a proof of it in this question.

a) Suppose that f is continuous on $[a, b]$ and let m and M denote the minimum and maximum values of f on $[a, b]$ respectively. If $m \le z \le M$, then explain why there exists a real number c in $[a, b]$ such that $f(c) = z$.

b) By considering a lower and upper bound for the integral $\int_a^b f(t)g(t)\,dt$ in terms of m and M, prove the mean value theorem for integrals.

9. [**X**] In this question, we will prove the Lagrange formula for the remainder under some simple assumptions. Suppose that f is $(n+1)$-times differentiable on some open interval I containing 0 and that $f^{(n+1)}$ is continuous on I. Suppose that $x \in I$ and recall that the remainder term in Taylor's theorem is given by

$$R_{n+1}(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt.$$

Use the mean value theorem for integrals (see Question 8) to deduce that

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1},$$

where c is some real number between 0 and x.

10. [**R**]

a) Suppose that

$$f(x) = x^7 + 5x^6 + 3x^5 - 17x^4 - 16x^3 + 24x^2 + 16x - 11.$$

Verify that 1 and -2 are stationary points for f and use the corollary to Taylor's theorem to classify each of these stationary points.

b) Suppose that

$$f(x) = x^7 - 7x^6 + 10x^5 + 22x^4 - 43x^3 - 35x^2 + 48x + 40.$$

Verify that -1, 2 and 3 are stationary points for f and classify each of these stationary points.

11. [**H**] Use the following outline to show that e is irrational.

a) If e were rational, it would be of the form $e = \dfrac{p}{q}$, where p and q are positive integers. Select an integer k such that $k \geq 3$ and $k \geq q$. Use Taylor's Theorem to show that

$$\frac{p}{q} = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{k!} + \frac{e^z}{(k+1)!}$$

for some z in $[0, 1]$.

b) Suppose that

$$s_k = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{k!}.$$

Show that $k!(e - s_k)$ is an integer.

c) Show that $0 < k!(e - s_k) < 1$.

d) Conclude that e is irrational.

Problems 4.3 : Sequences

12. [**R**] Describe the limiting behaviour of the following sequences. If the sequence converges, then state its limit.

a) $\dfrac{n^2 - 2n + 1}{2n^2 + 4n - 1}$

b) $\dfrac{\ln n}{n^a}$, where $a > 0$.

c) $\dfrac{n \sin(n\pi/4)}{\sqrt{n^2 + 1}}$

d) $\dfrac{n + \cos n\pi}{n - \cos n\pi}$

e) $\dfrac{n!}{n^n}$

f) $\dfrac{(2n)!}{(n!)^2}$

g) $(a^n + b^n)^{1/n}$, where $a \geq b > 0$.

13. [**H**] Suppose that $a_n = \dfrac{n^2 - 2n + 1}{2n^2 + 4n - 1}$.

a) Find L, where $L = \displaystyle\lim_{n\to\infty} a_n$.

b) For each positive number ϵ, find a number N such that

$$|a_n - L| < \epsilon \qquad \text{whenever } n > N.$$

14. [**R**] Use the result $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$ and standard results about limits to evaluate the following limits.

a) $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{4n}$

b) $\displaystyle\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$

15. [**X**] *The limit of a recursively defined sequence.*
Suppose that $a_1 = 1$ and $a_{n+1} = \sqrt{1 + a_n}$ whenever $n \geq 1$.

   a) Show that $\sqrt{1+x} \in [1,2]$ whenever $x \in [1,2]$.
   b) Use induction to show that the sequence $\{a_n\}$ is bounded.
   c) Use induction to show that $\{a_n\}$ is an increasing sequence.
   d) Explain why $\lim\limits_{n\to\infty} a_n$ exists.
   e) Find $\lim\limits_{n\to\infty} a_n$. (That is, find $\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$.)

16. [**X**] Find the supremum and infimum of each of the following sets.

   a) $\left\{ \dfrac{n}{1+n^2} \,:\, n = 1, 2, \ldots \right\}$
   b) $\left\{ \dfrac{n}{1+n^2} \,:\, n \in \mathbb{Z} \right\}$
   c) $\{x \in \mathbb{Q} \,:\, x^2 < 2\}$
   d) $\left\{ \dfrac{(-1)^n}{n} + \sin n \,:\, n = 1, 2, \ldots \right\}$
   e) $\{x \in (0, \infty) \,:\, \sin x < 0\}$
   f) $\{1 + \tan^{-1} x \,:\, x \in \mathbb{R}\}$

17. [**X**] Suppose that A and B are nonempty subsets of $\mathbb{R}^2$. Define the distance $d(A,B)$ between A and B by the formula

$$d(A,B) = \inf\{|\mathbf{a} - \mathbf{b}| \,:\, \mathbf{a} \in A,\ \mathbf{b} \in B\}.$$

   a) Explain why this infimum always exists.
   b) Suppose that $A = \{(x,y) \,:\, x^2 + y^2 < 1\}$ and $B = \{(x,y) \,:\, x^2 - y^2 > 9\}$. Find $d(A,B)$.
   c) Find two disjoint sets A and B such that $d(A,B) = 0$.

18. [**X**] Suppose that $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence. Given a positive integer m, define K_m by the formula

$$K_m = \sup\{a_m, a_{m+1}, a_{m+2}, \ldots\}.$$

   a) Explain why $K_m \geq \inf\limits_{n\geq 1} a_n$ for all m.
   b) Explain why $K_{m+1} \leq K_m$ for all m.
   c) Explain why the limit $\lim\limits_{m\to\infty} K_m$ exists.

   Let K denote the limit of part (c). We call K the **limit superior** of the sequence and write $K = \limsup\limits_n a_n$. It is the largest number to which any subsequence of $\{a_n\}$ converges. The smallest number to which any subsequence converges is called the **limit inferior**, $\liminf\limits_n a_n = \lim\limits_{m\to\infty} \left(\inf\limits_{n\geq m} a_n \right)$.

   d) Prove that $\liminf\limits_n a_n = \limsup\limits_n a_n$ if and only if $\{a_n\}$ converges.

Problems 4.4 : Infinite series

19. **[R]** *Examples of telescoping series.*

    a) Consider the series $\displaystyle\sum_{k=1}^{\infty} a_k$, where $a_k = \dfrac{1}{k(k+1)}$.

       i) Find the partial fractions decomposition of a_k.

       ii) Find a simple formula for the partial sum s_n, where $s_n = \displaystyle\sum_{k=1}^{n} a_k$.

       iii) Hence find $\displaystyle\sum_{k=1}^{\infty} a_k$.

    b) Repeat the question for the series $\displaystyle\sum_{k=2}^{\infty} a_k$, where $a_k = \dfrac{1}{k^2-1}$.

20. **[R]** Let s_n denote the nth partial sum of the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots.$$

    a) Show that $s_n > \sqrt{n}$ whenever $n > 1$.

    b) Hence explain why the series diverges.

21. **[R]** Let s_n denote the nth partial sum of the harmonic series series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$. When $n = 2^{k-1}$, the terms of s_n may be bracketed as shown:

$$s_n = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \ldots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-2}+1} + \frac{1}{2^{k-2}+2} + \ldots + \frac{1}{2^{k-1}}\right).$$

Hence use an argument similar to that in Question 20 to show that the harmonic series diverges.

22. **[R]** The following three questions are similar to Example 4.4.3.

    a) By drawing a diagram and interpreting each term in sum as the area of rectangle, show that

$$\sum_{k=2}^{n} \frac{1}{k^2} \le \int_1^n \frac{dx}{x^2}.$$

    b) Deduce that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges.

23. **[R]**

    a) By drawing a diagram and interpreting each term in sum as the area of rectangle, show that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \ge \int_1^{n+1} \frac{dx}{\sqrt{x}}.$$

b) Deduce that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{x}}$ diverges.

24. [**R**] By using the technique of the previous two questions, determine whether or not the sum $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ converges.

Problems 4.5 : Tests for series convergence

25. [**R**] Use the integral test to examine the convergence of

a) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^2+4}$ b) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k(\ln k)}$ c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+9)^3}$

26. [**R**] Use a comparison test to determine whether or not each series converges.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k^3+3)^{1/2}}$ b) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(k^2-1)^{1/3}}$ c) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^2-6}$

d) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\ln k}$ e) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(\ln k)^9}$ f) $\displaystyle\sum_{k=1}^{\infty} \sin^2 \frac{1}{k}$

27. [**R**] Use the ratio test to examine the convergence of each series.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{2^k}$ b) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k!}$ c) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$ d) $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{2^k+4^k}$

28. [**R**] Determine which of the following alternating series converge. Which are absolutely convergent?

a) $\displaystyle\sum_{k=3}^{\infty} \frac{(-1)^k}{\ln k}$ b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k^k}{(k+1)^k}$

c) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k(k^2-2)}}$ d) [**X**] $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\sqrt{k}+(-1)^k}$

29. [**R**] Consider the alternating series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k^3+1}$. Let L denote the value of the series and let s_n denote the nth partial sum of the series whenever $n \geq 0$.

a) Verify that the series is convergent.

b) Calculate s_4 and give an upper bound for the absolute error in the approximation $L \approx s_4$.

c) Find a value for n such that the absolute error in the approximation $L \approx s_n$ is less than 10^{-6}.

30. [**R**] By using an appropriate test, determine whether or not each series converges.

a) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ b) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k^3}$ c) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{1+7^k}$

d) $\displaystyle\sum_{k=2}^{\infty} \frac{2^k k!}{k^k}$ e) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^{k+1}k^2}{\sqrt{4k^4+1}}$ f) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin\big((2k-1)\pi/4\big)}{2^k}$

31. [**X**] Determine the convergence or divergence of $\sum a_k$, for each a_k given below.

a) $\dfrac{\sin k}{k^2}$ b) $\dfrac{\sqrt{k}}{k^2+1}$ c) $\dfrac{k}{(\ln k)^k}$ d) $\dfrac{(\ln k)^3}{\sqrt{k^3-3k^2+1}}$

e) $\dfrac{(-1)^k 2^k}{k^3}$ f) $\dfrac{1}{k^{1+\frac{1}{k}}}$ g) $\dfrac{\ln(k!)}{k^3}$

32. [**H**] Discuss the convergence of the series

$$\sum_{k=1}^{\infty} \frac{1}{a_1 a_2 \dots a_k} = \frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \cdots ,$$

where $\{a_k\}$ is a strictly increasing sequence and $a_1 > 0$.

33. [**X**]

a) Prove that if $\displaystyle\sum_k a_k^2$ and $\displaystyle\sum_k b_k^2$ converge, then $\displaystyle\sum_k a_k b_k$ converges.

b) Prove that if $\displaystyle\sum_k a_k^2$ converges, then $\displaystyle\sum_k \frac{a_k}{k}$ converges.

Problems 4.6 : Taylor series

34. [**R**] Find, in each case, the Maclaurin series for f. Express your answer using summation notation. (You may find your answers to some questions of Problems 4.1 : Taylor polynomials helpful.)

a) $f(x) = e^x$ b) $f(x) = \sin x$ c) $f(x) = \sinh x$

d) $f(x) = \ln(1+x)$ e) $f(x) = \dfrac{1}{x-1}$

35. [**R**] Suppose that $f(x) = e^x$.

a) Express f in the form $f(x) = p_n(x) + R_{n+1}(x)$, where p_n is the nth Taylor polynomial for f about 0 and R_{n+1} is the Lagrange formula for the remainder.

b) Fix x in $\mathbb{R}$ and show that $R_{n+1}(x) \to 0$ as $n \to \infty$.

c) Hence write down the Taylor series expansion for $f(x)$, stating clearly where the expansion is valid.

36. [**H**] Repeat the previous question in the case when $f(x) = \sinh x$.

37. [**H**] Suppose that $f(x) = \ln(1+x)$.

a) Express f in the form $f(x) = p_n(x) + R_{n+1}(x)$, where p_n is the nth Taylor polynomial for f about 0 and R_{n+1} is the Lagrange formula for the remainder.

b) Suppose that $0 \le x \le 1$ and show that $R_{n+1}(x) \to 0$ as $n \to \infty$.

c) Hence write down the Taylor series expansion for $f(x)$, when $0 \le x \le 1$.

d) [**X**] We will show that this Taylor series expansion is also valid on $(-1, 0)$. Suppose that $-1 < x < 0$.

i) By using the integral form for the remainder, show that

$$R_{n+1}(x) = \int_0^x \left(\frac{t-x}{1+t}\right)^n \frac{1}{1+t}\, dt.$$

ii) Use the mean value theorem for integrals (when $g(t) = 1$; see Question 8) to show that

$$|R_{n+1}(x)| < \left(\frac{c_n + |x|}{1 + c_n}\right)^n \left(\frac{|x|}{1+x}\right)$$

for some number c_n between x and 0.

iii) Deduce that

$$|R_{n+1}(x)| < |x|^n \left(\frac{|x|}{1+x}\right)$$

and hence that $R_{n+1}(x) \to 0$ as $n \to \infty$.

38. [**H**] Let I denote the interval $(x_0 - R, x_0 + R)$, where x_0 and R are real numbers. Suppose that a function f has derivatives of all orders on I and that all these derivatives have a common bound (that is, $|f^{(n)}(x)| \le M$ for all x in I and for all positive integers n). Show that f is represented by its Taylor series about x_0 on I.

39. [**X**] Suppose that

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0. \end{cases}$$

a) Show that f is differentiable everywhere and find a formula for f'.

b) Show that f' is differentiable everywhere and find a formula for f''.

c) Suppose that k is a positive integer. Prove that $\dfrac{d}{dx}\left(\dfrac{e^{-1/x^2}}{x^k}\right)$ is a linear combination of functions of the form $\dfrac{e^{-1/x^2}}{x^m}$, where m is a positive integer.

d) Hence deduce that $f^{(n)}(0) = 0$ for every natural number n.

e) Write down the Maclaurin series for f. Where does the Maclaurin series converge? Where does it converge to f?

40. [**H**] *(This exercise illustrates that a conditionally convergent series, when rearranged, can have a different sum.)*
Consider the series s and t, given by

$$s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

and

$$t = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots .$$

Note that second series is a rearrangement of the first.

a) Explain why s is conditionally convergent.

b) By considering an appropriate Maclaurin series (see Theorem 4.6.5), find the exact value of s.

c) Denote the sum of the first n terms of s by s_n and of t by t_n. By using induction, show that $t_{3n} = \frac{1}{2}\, s_{2n}$ whenever $n \geq 1$.

d) Hence find the value of $\lim\limits_{n\to\infty} t_{3n}$.

e) Explain why $\lim\limits_{n\to\infty} t_{3n+1} = \lim\limits_{n\to\infty} t_{3n+2} = \lim\limits_{n\to\infty} t_{3n}$.

f) Hence write down the value of t.

Problems 4.7 : Power series

41. [**R**] Determine the open interval of convergence for each of the following power series. (Students studying MATH1241 should also examine the behaviour of the power series at the endpoints.)

a) $\displaystyle\sum_{k=0}^{\infty} \left(\frac{x}{6}\right)^k$ b) $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k^2+1}$ c) $\displaystyle\sum_{k=0}^{\infty} \frac{kx^k}{2^k}$

d) $\displaystyle\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^3}$ e) $\displaystyle\sum_{k=2}^{\infty} \frac{(3x-2)^k}{k\,\ln k}$ f) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{(k+1)3^k}$

g) [**X**] $\displaystyle\sum_{k=1}^{\infty} \frac{(\ln k)^k x^k}{k^k}$

42. [**R**] Consider the power series

$$\text{(i)} \quad \sum \frac{n!}{n^n}\, x^n \qquad \text{and} \qquad \text{(ii)} \quad \sum \frac{n^n}{n!}\, x^n.$$

a) Show that their radii of convergence are e and e^{-1} respectively.

b) [**X**] Show, by any method, that $a(n) = \left(1 + \dfrac{1}{n}\right)^n$ is a strictly increasing sequence (whose limit is e).

c) [**X**] Deduce from your working in (a) and (b) that (i) diverges when $x = \pm e$.

d) [**X**] Here you may assume Stirling's formula:

$$n! \Big/ \left\{\sqrt{2\pi}\, n^{n+1/2}\, e^{-n}\right\} \to 1 \qquad \text{as} \quad n \to \infty.$$

Show that series (ii) diverges when $x = e^{-1}$. Using Stirling's formula and (b), show that (ii) is conditionally convergent when $x = -e^{-1}$.

Problems 4.8 : Manipulation of power series

43. [**R**] Use your answers to Question 34 to deduce the Maclaurin series for each function g.

a) $g(x) = (x+1)e^x$ b) $g(x) = \sin(x^2)$ c) $g(x) = (x-1)^{-2}$

44. [**R**] Suppose that $f(x) = (1-x)^{-1}$. Write down the Maclaurin series for f and hence find the Maclaurin series for each function g given below. On what open interval is each function represented by its Maclaurin series?

a) $g(x) = (1+x)^{-1}$ b) $g(x) = (1+x^2)^{-1}$ c) $g(x) = \tan^{-1} x$

45. [**R**] Suppose that $f(x) = x^2 \sin(x^3)$.

a) By using the Maclaurin series for sine, find the Maclaurin series for f.
b) Hence show that 0 is a stationary point for f.
c) Is 0 a local maximum point, local minimum point or a horizontal point of inflexion? Explain.

46. [**R**] Consider the Maclaurin series representation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots ,$$

which is valid whenever $|x| < 1$.

a) By first integrating the above Maclaurin series, deduce that

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right)$$

whenever $|x| < R$, for some real number R.

b) What is the largest possible value of R?
c) Use the first two terms of the series of (a) to find a rational number that approximates $\ln 2$.

47. [**R**] The function $\mathrm{Si} : \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathrm{Si}(x) = \int_0^x f(t)\, dt,$$

where

$$f(t) = \begin{cases} \frac{\sin t}{t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$

The Si function is used in signal processing and by surveyors for GPS.

a) Show that Si has a stationary point at π and classify this stationary point.
b) By first writing down the Maclaurin series for the sine function, find the Maclaurin series for Si.
c) Hence find an estimate for the value of $\mathrm{Si}(\pi)$ such that the absolute error is less than $1/100$.

48. **[H]** A function f is defined by the rule

$$f(x) = \sum_{k=1}^{\infty} kx^k.$$

a) What is the largest open interval I on which the function well-defined?

b) By manipulating the power series expansion

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \qquad |x| < 1,$$

find a closed formula for $f(x)$.

49. **[H]** Consider the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = e^x - 1.$$

Suppose that a solution y has a power series representation given by

$$y = \sum_{k=0}^{\infty} a_k x^k,$$

where the coefficients a_k are to be determined.

a) Write down, in summation notation, the Maclaurin series representation of $e^x - 1$.

b) Write down the power series representations of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

c) By expressing both the left- and right-hand sides of the differential equation as a power series, determine the value of each coefficient a_k.

d) Hence write down a solution to the differential equation, stating the values of x for which the solution is valid.

Note: This method for solving differential equations will be developed in some second year courses.

Problems for Chapter 5

Problems 5.1 : The average value of a function

1. [**R**] It can be shown that when a cable is hanging between two poles the curve that it forms is always the graph of a hyperbolic cosine function. Suppose that the height h (in metres) of a cable above the ground is given by
$$h(x) = 4\cosh\left(\frac{x-10}{20}\right),$$
where $0 \le x \le 20$. Find the average height of the cable above the ground.

2. [**R**] Suppose that the air temperature $T(t)$, measured in degrees Celsius t hours after noon, is given by
$$T(t) = 25 + 2t - \frac{t^2}{3}.$$
Find the average temperature between noon and 5 p.m.

Problems 5.2 : The arc length of a curve

3. [**R**] Calculate the lengths of the given arcs given by
   a) $y = x^{3/2}$, where $0 \le x \le 1$;
   b) $x = t - \sin t$, $y = 1 - \cos t$, where $0 \le t \le 2\pi$; and
   c) $x = t^3$, $y = t^2$ from $(0,0)$ to $(8,4)$.

4. [**R**] The astroid
$$x^{2/3} + y^{2/3} = a^{2/3} \qquad (1)$$
has a parametrisation given by
$$x(\theta) = a\cos^3\theta, \qquad y(\theta) = a\sin^3\theta.$$
Its graph was sketched, in the case when $a = 1$, in one of the problems from Chapter 7 of MATH1131.
   a) Use the parametric form to calculate the arc length of the astroid.
   b) [**H**]
      i) Show that the improper integral $\displaystyle\int_0^1 x^{-1/3}\,dx$ converges to $3/2$ by calculating the limit
      $$\lim_{h\to 0^+}\int_h^1 x^{-1/3}\,dx.$$
      ii) Hence calculate the arc length of the astroid by using the implicit equation (1).

5. [**R**] Find the length of the curve $r = e^\theta$, where $0 \le \theta \le 2\pi$.

6. [**R**] Find the length of the cardioid $r = 1 + \cos\theta$.

Problems 5.3 : The speed of a moving particle

7. [**R**] A projectile is fired from an elevated cannon. Its horizontal distance x (in metres) from the cannon and height y (in metres) above the ground, exactly t seconds after it is fired, is given by

$$x(t) = 40t, \qquad y(t) = -5t^2 + 40t + 45, \qquad 0 \le t \le t_1,$$

where t_1 is the time the projectile hits the ground.

a) Find t_1.
b) Find the speed of the projectile immediately prior to impact.
c) What was the average height of the projectile above the ground during the period after it was fired and before impact?
d) [**X**] What distance did the projectile travel during this period?

8. [**H**] The position $(x(t), y(t))$ of a particle P at time t is given by

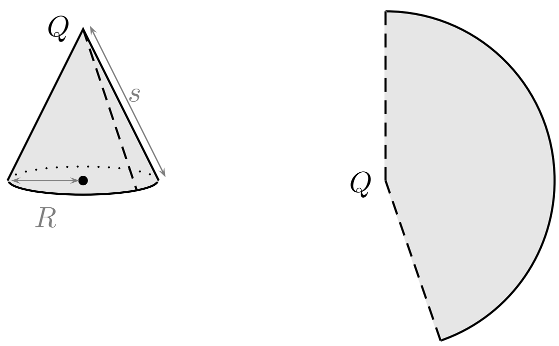
$$x(t) = \cos\left(\frac{\pi}{2}(\cos \pi t - 1)\right), \qquad y(t) = -\sin\left(\frac{\pi}{2}(\cos \pi t - 1)\right),$$

where $t \ge 0$.

a) Find a formula for the speed $v(t)$ of the particle at time t.
b) For what values of t is the speed of the particle (i) a maximum and (ii) a minimum?
c) What curve does the trajectory of the particle trace out?
d) What is the length of the curve of (c)?
e) Find the distance that the particle travels during the time interval $[0, 3]$.

Problems 5.4 : Surface area

9. [**R**] In this question we will show that the surface area of the frustum of a right circular cone is given by $\pi(R + r)s$. (For a diagram, see the beginning of Section 5.4.)
Consider a truncated right circular cone with slant height s and base radius R. Cut a line from the vertex Q to the base and flatten the cone as shown below.



a) Explain why the flattened surface is the sector of a circle.

b) Find, in terms of R and s, the area of the sector and hence the surface area of the cone.

c) Hence show that the surface area of a frustum of radii r and R and of slant height s is given by $\pi(R+r)s$.

10. [**R**] Find the area of the surface of revolution formed when the given curve is rotated about the x-axis.

    a) $y = x^3$, where $0 \le x \le 2$. b) $x = t - \sin t$, $y = 1 - \cos t$, where $0 \le t \le 2\pi$.

11. [**R**] Show that if $-a \le b < c \le a$ then the surface area of the sphere $x^2 + y^2 + z^2 = a^2$ between the planes $x = b$ and $x = c$ is $2\pi a(c - b)$.

12. [**R**] Suppose that $0 < r < R$. A surface (doughnut) is formed by rotating a circle of radius r and centre $(0, R)$ about the x-axis. By finding a suitable parametrisation for the circle, show that the area of the surface is $(2\pi R)(2\pi r)$.

13. [**R**] Find the area of the surface formed when the polar curve $r = 1 + \cos\theta$, where $0 \le \theta \le \pi$, is rotated about the x-axis.