MATH1241 CALCULUS
Semester 2 2018

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Overview

Functions of Several Variables

Integration Techniques

Ordinary Differential Equations

Taylor Series

Sequences and Series

Averages, Arc Length, Surface Area

Chapter 1

Functions of Several Variables

a) Sketching in three Dimensions

b) Partial Derivatives

c) Tangent Planes and Surface Normals

d) Total Differential Approximation

e) Chain Rules

f) Functions of Three (or more) Variables

g) Maple Notes

Most functions you meet in applications depend on more than one variable.

Example - Distance from an origin:

1 dimension $d = f(x) = x$

2 dimensions $d = f(x, y) = \sqrt{x^2 + y^2}$

3 dimensions $d = f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

The Notes give other examples.

Geometrically, we have:

$y = f(x)$ – a curve in $\mathbb{R}^2$

$z = f(x, y)$ – a surface in $\mathbb{R}^3$

We will also later define differentiation of functions of several variables: each variable gives a different partial derivative.

1.1 Sketching in 3-dimensions

In general it is difficult to sketch surfaces well.

Mostly it is enough to be able to see what the underlying solid or surface looks like when we sight along the axes, and/or look at the intersections of the surface and the various coordinate planes.

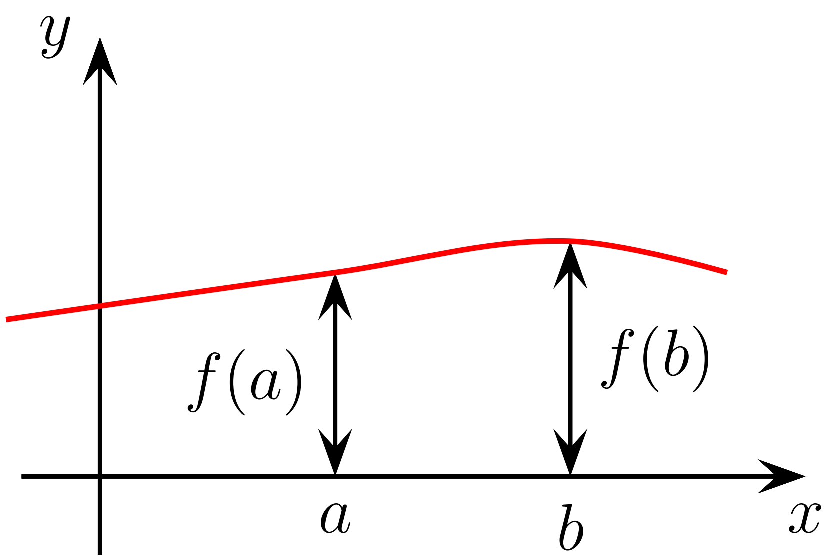
This is where the computer can really help, but we expect you to be able to sketch surfaces without it.

Most of the sketches we will want in this course are for graphs of functions of 2 variables: surfaces of the form

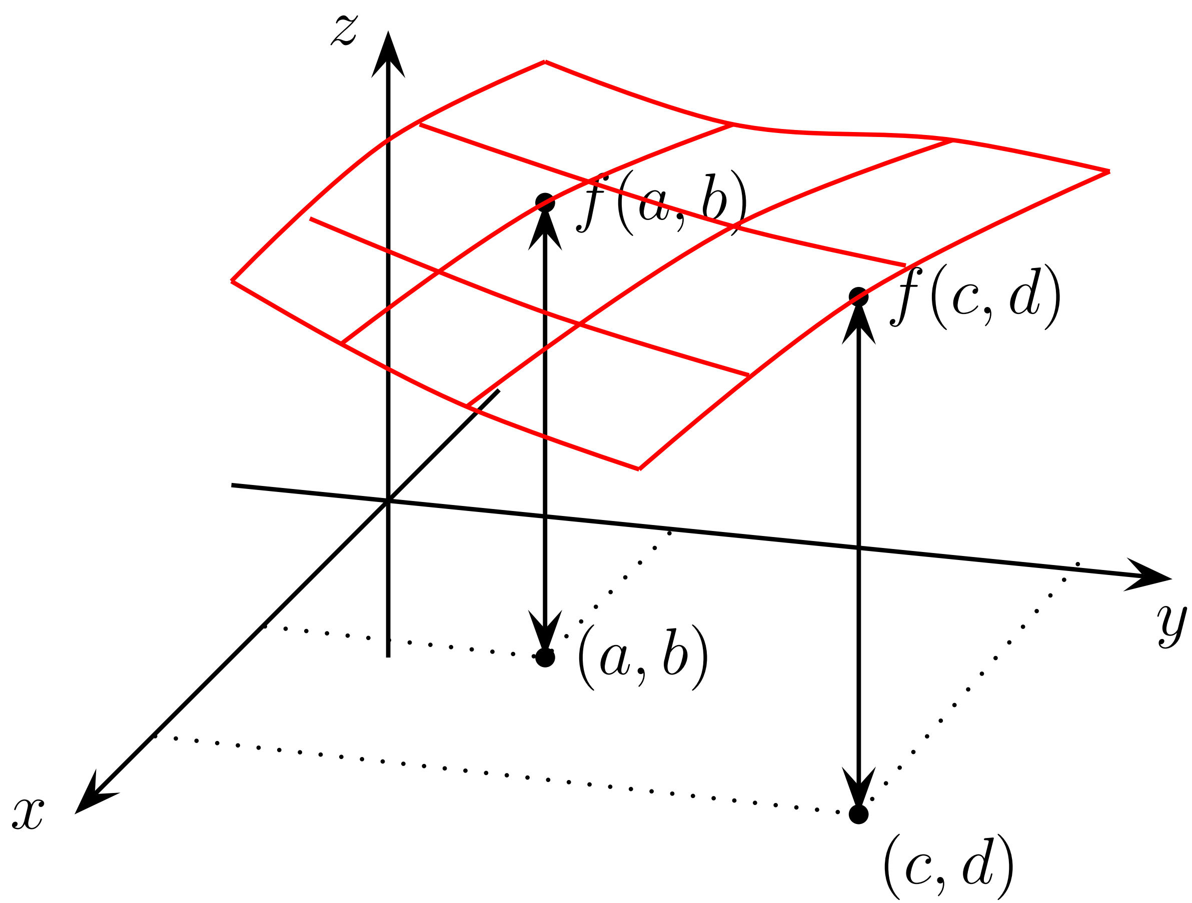
$$z = F(x, y).$$

There are a number of helpful tricks, which I will discuss in lectures, but also see `https://tinyurl.com/yagt9977` two videos I filmed on drawing and sketching, published on YouTube

Recall that for functions of one variable, $f : \mathbb{R} \to \mathbb{R}$, the graph is a subset of $\mathbb{R}^2$, namely the set of points $\{(x, y) : y = f(x)\}$



For functions of two variables, $f : \mathbb{R}^2 \to \mathbb{R}$, the graph is a subset of $\mathbb{R}^3$, namely the set of points $\{(x, y, z) : z = f(x, y)\}$.



The key points when sketching are:

a) use a pencil

b) break a curve that passes behind another curve

c) use dashes for hidden lines

d) draw a surface as if it is below and to the left

e) put the axes in after sketching the surface

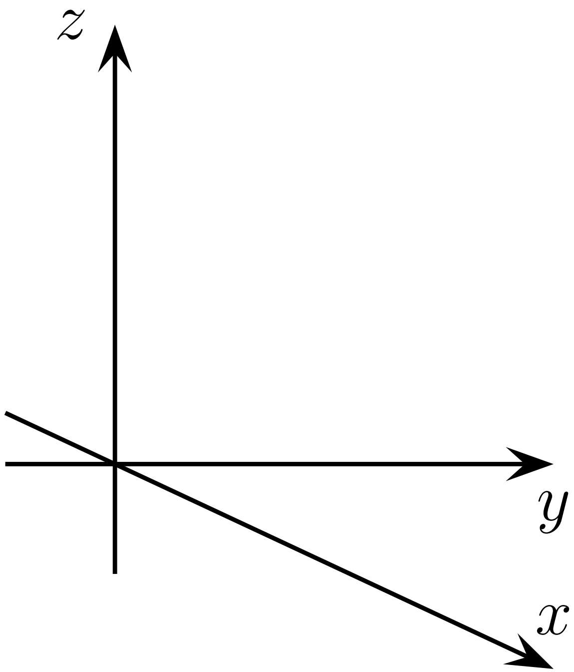
The other problem you face is that drawing in three dimensions needs some awareness of **perspective**: how an object actually looks.

You should all be aware that when you look at parallel lines running away from you they appear to converge.

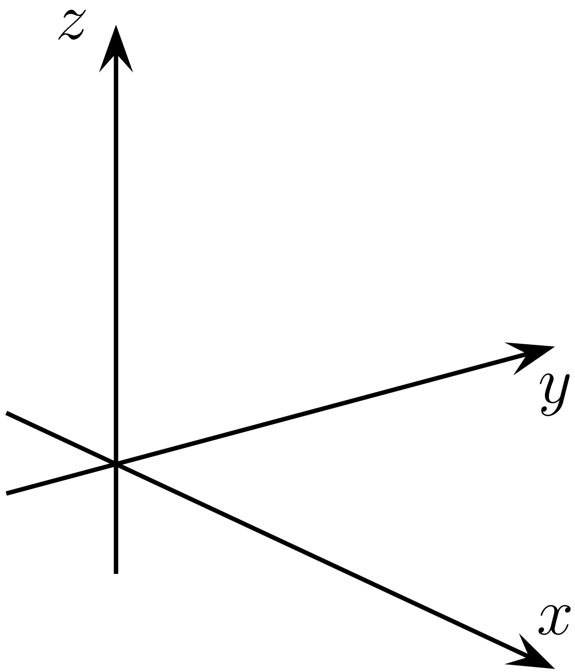
Similarly, when you look at lines that are perpendicular, unless you are looking straight down onto the plane they form, they do not appear at right angles.

The Notes have a number of very badly done diagrams that ignore this last point.

If you actually look at three mutually orthogonal lines in space, then none of the angles they make actually appear to be right angles.



Impossible



Possible

You should also make sure the (apparant) angles between the axes are reasonably large.

In order to get some idea of what a surface looks like, we find level curves and profiles:

A **level curve** of a function $F : \mathbb{R}^2 \to \mathbb{R}$ is a curve in $\mathbb{R}^2$ defined by $F(x, y) = C$ where C is a constant.

A **profile** is obtained from a plot of:

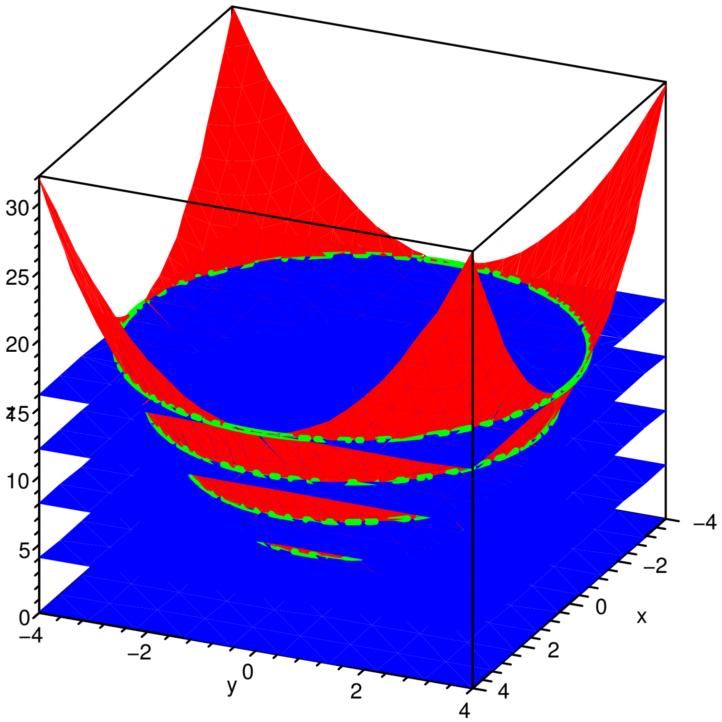
- z versus x (with y constant) or
- z versus y (with x constant).

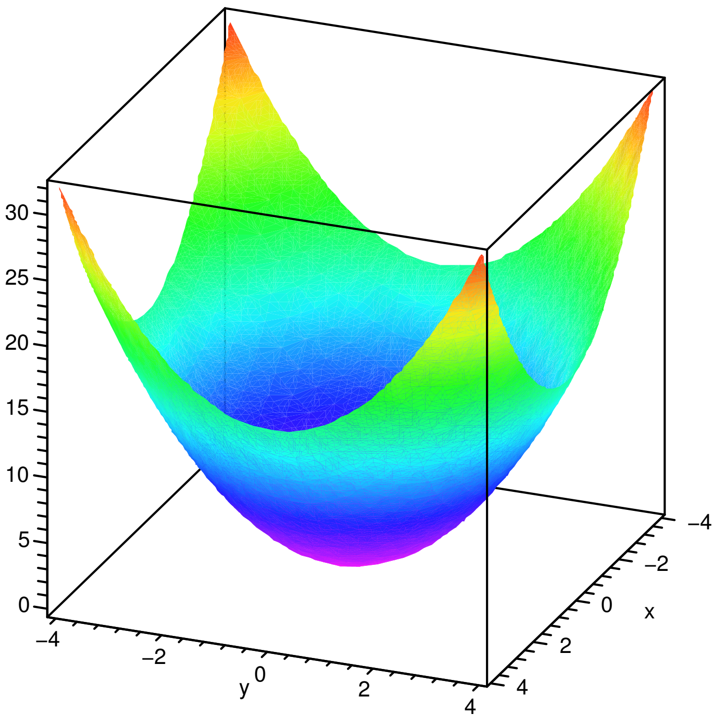
You then combine level curves and profiles to sketch the surface.

Note: this is particularly easy if the function F is a function of $x^2 + y^2$, as the graph then arises from spinning a profile curve around the z-axis.

Example 1 *Find the profiles and sketch the level curves of* $F(x, y) = x^2 + y^2$.

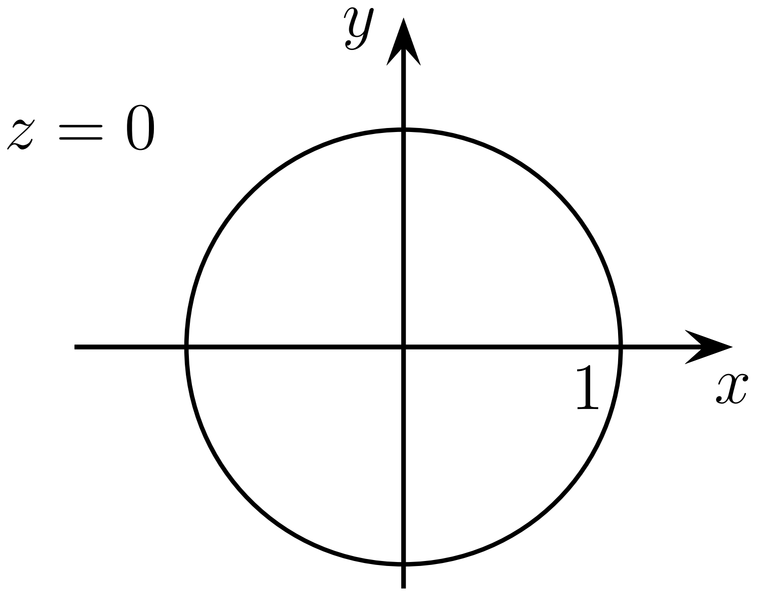
Same example: $F(x, y) = x^2 + y^2$:

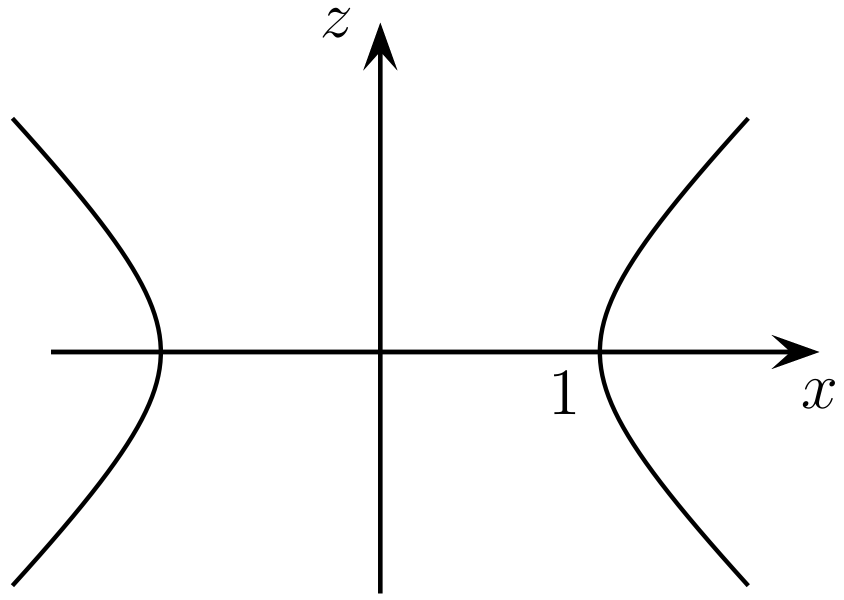




Example 2 *What sort of shape is the surface given by $x^2 + y^2 - z^2 = 1$?*

We note that the intersection of the surface with $z = 0$ is the curve $x^2 + y^2 = 1$, which is of course the unit circle.



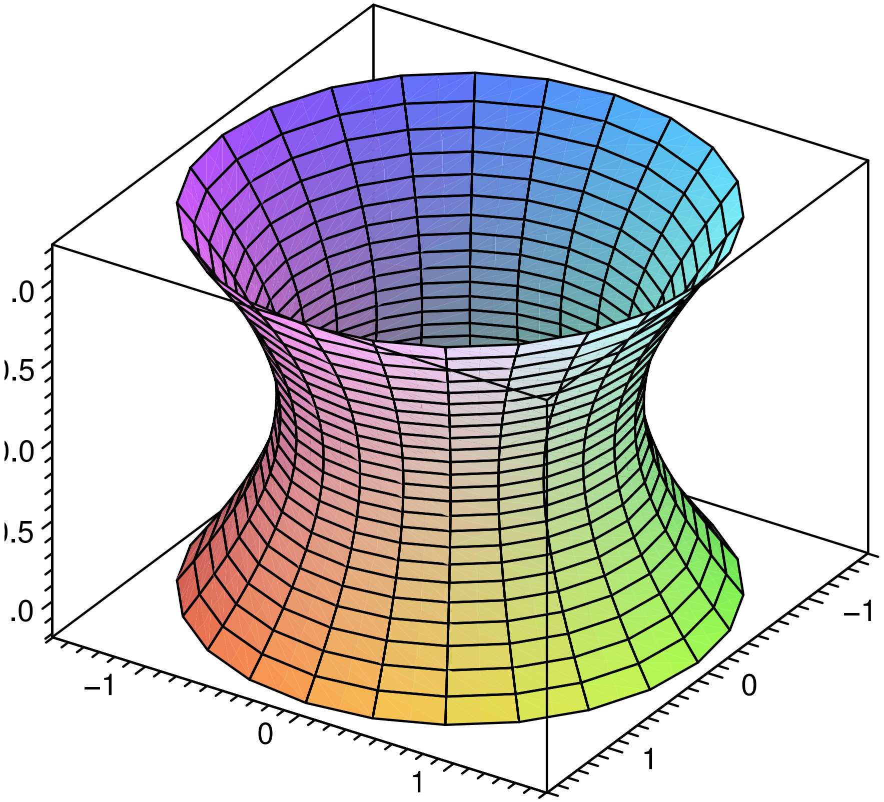


$x^2 + y^2 = 1$ $\qquad\qquad$ $x^2 - z^2 = 1$

The intersection with $y = 0$ is $x^2 - z^2 = 1$, which you ought to recognise as a hyperbola. In fact, the intersection of the surface and any plane of the form $y = ax$ (which contains the z-axis) is a congruent hyperbola.

The level curves are the intersections of the surface and the plane z constant and are also a circles, but of radius $\sqrt{1+z^2}$.

This tells us the surface is formed from revolving the hyperbola $x^2 - z^2 = 1$ around the z-axis to form a surface known as a **hyperboloid of one sheet**.



1.2 Partial Derivatives

Now we wish to investigate the rate of change in each direction x or y of the function F depending on x and y.

The (first) **partial derivative**, F_x, with respect to x, of a function F of two variables is the function of two variables obtained by holding y constant and differentiating with respect to x.

$$F_x(x,y) = \lim_{h\to 0} \frac{F(x+h,y) - F(x,y)}{h}$$

or

$$F_x(a,b) = \lim_{x\to a} \frac{F(x,b) - F(a,b)}{x-a}\,.$$

Similarly for F_y we have

$$F_y(x,y) = \lim_{h\to 0} \frac{F(x,y+h) - F(x,y)}{h}$$

or

$$F_y(a,b) = \lim_{y\to b} \frac{F(a,y) - F(a,b)}{y-b} \,.$$

There are several different notations for partial derivatives, for example, if $z = F(x,y)$ we can write

$$F_x(x,y) = F_x = \frac{\partial F}{\partial x} = \frac{\partial}{\partial x} F(x,y) = \frac{\partial z}{\partial x} = D_x F = F_{,1} = D_1(F)$$

My personal preference is for F_x, $F_{,1}$ or $D_1(F)$, but you will meet all of these in different texts.

Partial differentiation satisfies the same rules as ordinary differentiation (with the same proofs), to wit:

If $F = F(x,y), \quad G = G(x,y), \quad H = H(x), \quad C =$ const.

$$\begin{aligned}
\frac{\partial}{\partial x}(F \pm G) &= \frac{\partial F}{\partial x} \pm \frac{\partial G}{\partial x} \\
\frac{\partial}{\partial x}(F\,G) &= \frac{\partial F}{\partial x} G + F \frac{\partial G}{\partial x} \\
\frac{\partial}{\partial x}\left(\frac{F}{G}\right) &= \frac{\frac{\partial F}{\partial x} G - F \frac{\partial G}{\partial x}}{G^2} \qquad \text{if} \quad G(x,y) \neq 0 \\
F(x,y) = C \quad \forall x, y &\Rightarrow \frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y} \\
\frac{\partial}{\partial x}\left[H(F(x,y))\right] &= \frac{dH}{dF}\frac{\partial F}{\partial x}
\end{aligned}$$

And note: $$\frac{\partial H}{\partial y} = 0$$

Example 3 *Find* F_x *and* F_y *for* $F(x,y) = x^3y + y^2 + 2x$

SOLUTION:

■

Example 4 *Find* G_x *and* G_y *for* $G(x,y) = e^{xy^3}\sin(y)$

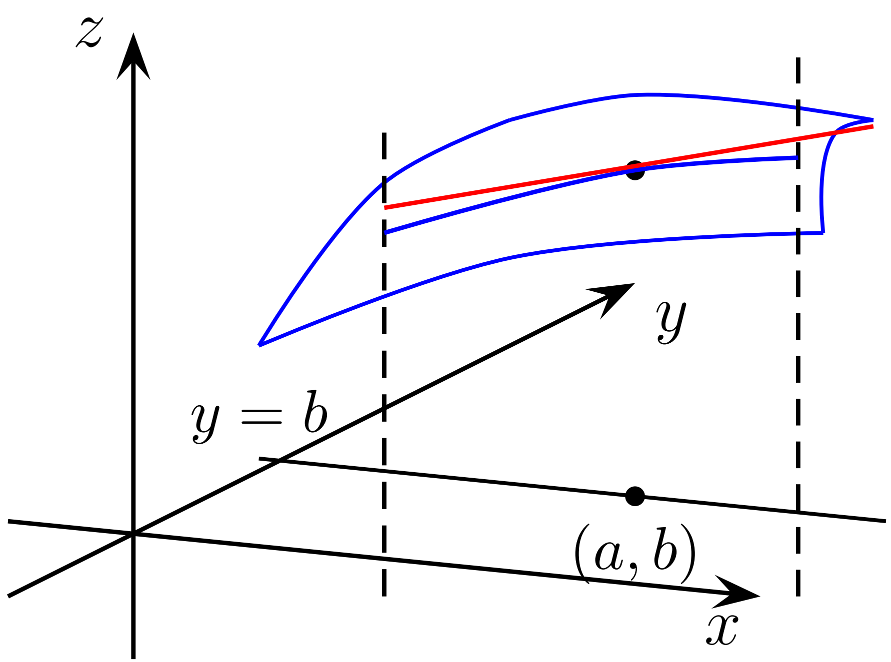
SOLUTION:

■

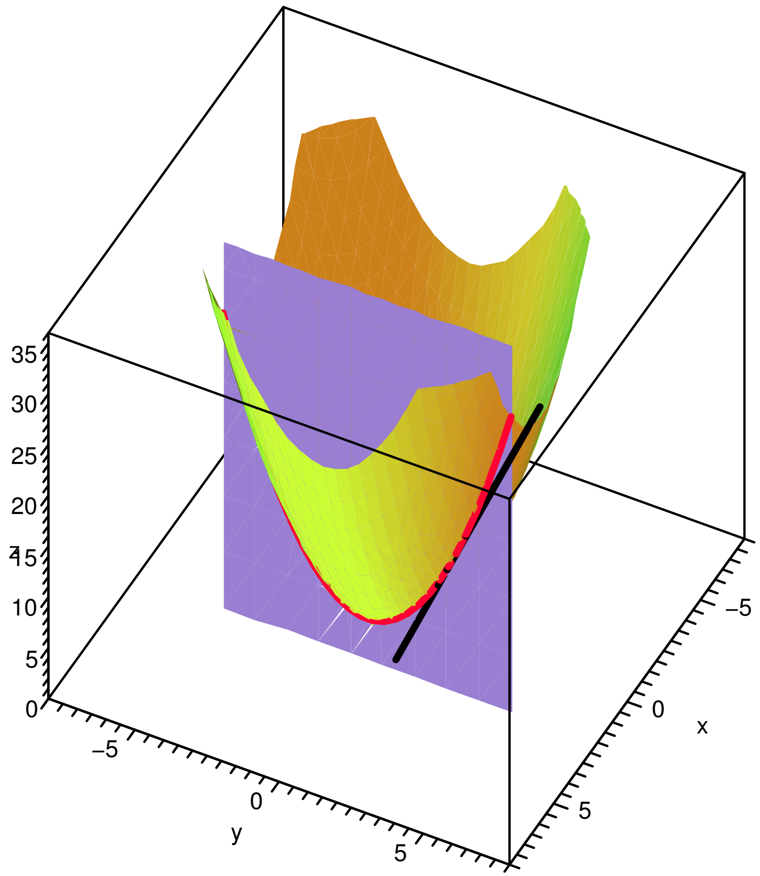
Geometrical Interpretation

$\dfrac{\partial F}{\partial x}$ is the slope of the surface $z = F(x, y)$ in the x direction.

$F_x(a, b)$ is the gradient of the tangent to the cross section at (a, b) when the surface $z = F(x, y)$ is intersected with the plane $y = b$.



Example 5 *Consider* $z = F(x, y) = x^2 + y^2$



$F_y(2,3)$ is the slope of the tangent to the cross section at $(2,3)$ when the surface $z = x^2 + y^2$ intersects with the plane $x = 2$.

The cross-section of the surface $z = x^2 + y^2$ with the plane $x = 2$ is the line $z = 4 + y^2$ and this has slope $\dfrac{dz}{dy} = 2y$ so that at $y = 3$ we have slope 6.

$$F_y(2,3) = \left.\frac{\partial F}{\partial y}\right|_{y=3} = 2y\,|_{y=3} = 6$$

Unlike ordinary differentiation, the existence of partial derivatives at a point does not imply continuity at a point:

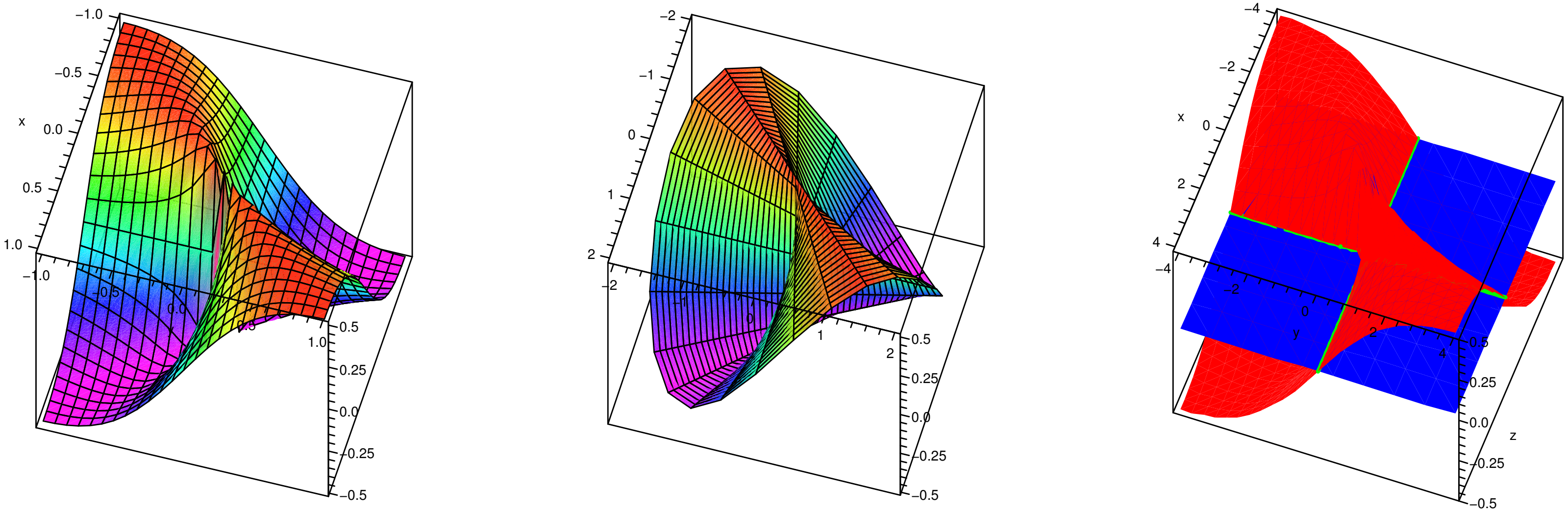
Example 6 *Consider the function given in Cartesian coordinates by:*

$$z = F(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

In polar coordinates this is

$$z = \begin{cases} \dfrac{1}{2}\sin 2\theta & \text{for } r \neq 0 \\ 0 & \text{for } r = 0 \end{cases}$$

$$z = F(x,y) = \frac{xy}{x^2+y^2}, \qquad z = \frac{1}{2}\sin 2\theta$$



$F_x(0,0) = F_y(0,0) = 0$ but $F(x,y)$ is not continuous at $x = 0$ since

$$\lim_{r\to 0} z = \lim_{r\to 0} \frac{1}{2}\sin 2\theta$$

does not exist.

If $z = f(x, y)$ we can write the **second partial derivatives** of f

$$(f_x)_x = f_{xx} = f_{,11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{,12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{,21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{,22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Example 7 *Calculate all first and second partial derivatives for the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$f(x, y) = x^4 + 2x^2y^3 + \sin(xy).$$

I’ve been very careful with the order in the **mixed second derivatives** f_{xy} and f_{yx}, but the following very useful result means we do not need to be (usually):

Theorem 1.1 (Mixed Derivatives) *If $F = F(x, y)$ is a continuous function of two variables (x, y) and all of its first and second order partial derivatives are continuous then*

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

In particular, this theorem tells us that for a function whose components are built up from polynomials, sine, cosine, the exponential etc, all derivatives of all orders are continuous and so we get equality of the mixed derivatives.

Proof: See comments in notes. A proper proof (not given) requires a formal treatment of continuity.

Example 8 *Calculate* $f_{xy}(0,0)$ *and* $f_{yx}(0,0)$ *if*

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & x = y = 0 \end{cases}.$$

SOLUTION:
We cannot just use the quotient rule here. We use the definitions:

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}, f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

Now

$$f_x(0,0) =$$

For the other parts of the derivatives:

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h}$$

So

$$f_{xy}(0,0) =$$

and

$$f_{yx}(0,0) =$$

1.3 Tangent Planes and Surface Normals

The **tangent plane** to a surface generalises the idea of a tangent line to a curve.

The tangent line to a curve at a point P on the curve is a line that just touches the curve at P: the line and curve share a direction at P.

The tangent plane to surface at a point P on the surface is a plane that just touches the surface at P.

We can build up the tangent plane at P by taking the tangent lines to all curves in the surface that go through P.

This is the idea behind how the Notes introduce the tangent plane.

I'm going to take a slightly different approach that I find more useful in practice: I'll come back to the "all tangents to all curves" idea later.

Suppose that $\mathbf{p} = (x_0, y_0)$ is a point that lies on the curve $y = f(x)$.

The tangent line (if it exists) has equation

$$y - y_0 = f'(x_0)(x - x_0).$$

Now write the tangent line in vector form as

$$\begin{pmatrix} f'(x_0) \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = 0.$$

Or more usefully as $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \mathbf{n} = \begin{pmatrix} f'(x_0) \\ -1 \end{pmatrix}.$$

Here the vector $\mathbf{n}$ is a **normal** to the tangent line.

The expression

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$$

ought to remind you of something from last session: it looks like the point-normal form of a plane — in fact it is the point-normal form of a line.

The normal to the curve is by definition orthogonal to the direction of the curve, and hence to the tangent line.

If the tangent plane to a surface at P consists of the tangent lines to all curves in the surface through P, then the normal to the tangent plane will be orthogonal to all these tangent lines.

So if we can define a **normal to a surface**, we can then just use the point-normal form to get the tangent plane.

Writing $y = f(x)$ in vector form as $\begin{pmatrix} x \\ f(x) \end{pmatrix}$, tangent is $\begin{pmatrix} 1 \\ f'(x) \end{pmatrix}$.

The normal to this tangent (if it exists) is $\begin{pmatrix} f'(x_0) \\ -1 \end{pmatrix}$.

A surface $z = F(x, y)$ in vector form is $\begin{pmatrix} x \\ y \\ F(x, y) \end{pmatrix}$.

Following the above pattern, the normal vector must be orthogonal to both

$$\begin{pmatrix} 1 \\ 0 \\ F_x \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ F_y \end{pmatrix}.$$

To get such a vector we use the cross product.

So if the surface has a normal at (x_0, y_0), then

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \\ F_x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ F_y \end{pmatrix} = - \begin{pmatrix} F_x(x_0, y_0) \\ F_y(x_0, y_0) \\ -1 \end{pmatrix}.$$

is normal to the surface.

The point-normal form of the tangent plane at $\mathbf{p} = (x_0, y_0, z_0)$ is then $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$.

Or on expanding out

$$z = z_0 + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0)$$

Example 9 *Find the tangent plane and normal vector to* $z = F(x, y) = -\frac{x^2}{4} - y^2$ *at* $P = (2, -1, -2)$*.*

SOLUTION:

$$\frac{\partial F}{\partial x} = \qquad\qquad F_x(2, -1) =$$

$$\frac{\partial F}{\partial y} = \qquad\qquad F_y(2, -1) =$$

So a normal vector at P is $\mathbf{n} =$.

And the tangent plane

For a picture in MAPLE use:

```
with(plots):

Sf:=plot3d(-x∧2/4-y∧2,x=-3..3,y=-3..3,
color=red,axes=BOXED):

TP:=plot3d(-x+2*y+2,x=-3..3,y=-3..3,color=blue):

display(Sf,TP);
```

As the Notes point out, for a surface given in the form $g(x, y, z) =$ constant, where it exists the normal is parallel to

$$\begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix}$$

and then the same technique applies.

I have continually said things like where the normal exists, because there are cases where it does not.

If there is a tangent plane at a point P, there is a normal at P and the partial derivatives exist at P.

This means that at points where the partial derivatives do not exist we cannot define the tangent plane.

But as the Notes point out there are cases where the partials exist but there is still no tangent plane (example 6 is a case).

1.4 Total Differential Approximation

If $f = f(x)$ then near a given point x_0, the tangent line to $y = f(x)$ gives

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Define

$$\Delta f = f(x) - f(x_0).$$

Then

$$\Delta f \approx f'(x_0)(x - x_0)$$

is the **differential approximation** to f.

Geometrically this approximates points on the curve $y = f(x)$ near x_0 by points on the tangent line near this point.

We seek to generalise to higher dimensions.

If $F = F(x, y)$ then near a given point x_0, y_0, the tangent plane to $z = F(x, y)$ gives

$$F(x, y) \approx F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0)$$

Define

$$\Delta F = F(x, y) - F(x_0, y_0).$$

Then

$$\Delta F \approx F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0)$$

is the **total differential approximation** to f.

Geometrically this approximates points on the surface $z = F(x, y)$ near (x_0, y_0) by points on the tangent plane near this point.

Uses of the total differential approximation

- To estimate changes in output given changes in input
- To estimate upper bounds on errors:

$$
\begin{aligned}
\text{Error} &= |\Delta F| \\
&\approx \left| \frac{\partial F}{\partial x} \Delta x + \frac{\partial F}{\partial y} \Delta y \right| \\
&\leq \left| \frac{\partial F}{\partial x} \right| |\Delta x| + \left| \frac{\partial F}{\partial y} \right| |\Delta y|
\end{aligned}
$$

Example 10 *A right circular cone is measured to have a height of 10.0cm and a base radius of 30.0cm. Estimate the maximum error in the volume if each measurement is to the nearest mm.*

SOLUTION:

$$V = \frac{\pi r^2 h}{3} \quad \Longrightarrow \quad \Delta V \approx \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial h}\Delta h$$

$$=$$

Measurement to the nearest mm means that $|\Delta r| \leq 0.05$ cm, similarly $|\Delta h| \leq 0.05$

So

$$\text{Error} = |\Delta V| \leq$$

1.5 Chain Rules

We calculate the derivative of $f \circ g$ in one dimension using the chain rule: $(f \circ g)'(x) = f'(g(x)).g'(x)$.

When it comes to functions of several variables, there several possible chain rules:

We illustrate by looking at derivatives like

$$\frac{d}{dt}F(x(t), y(t)) \qquad \text{and} \qquad \frac{\partial}{\partial x}F(u(x,y), v(x,y))$$

The methods I give can easily be generalised.

It's a general rule that the hard step in multivariable calculus is going from 1 dimension to 2: going any higher is just a matter of notation.

Now $F(x(t), y(t))$ is really only a function of t and to differentiate we start with the total differential approximation:

$$\Delta F \approx \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y$$

Divide by Δt:

$$\frac{\Delta F}{\Delta t} \approx \frac{\partial F}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial F}{\partial y}\frac{\Delta y}{\Delta t}$$

As $\Delta t \to 0$, the ratios of Δs become derivatives:

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$$

To remember the rule, consider the following **chain diagram**

$$
\begin{tikzcd}
& x \arrow[dr, "\frac{dx}{dt}"] & \\
F \arrow[ur, "\frac{\partial F}{\partial x}"] \arrow[dr, "\frac{\partial F}{\partial y}"'] & & t \\
& y \arrow[ur, "\frac{dy}{dt}"'] &
\end{tikzcd}
$$

To construct the diagram, draw an arrow from each function to each of its variables.

Then $\dfrac{dF}{dt}$ is the sum of all paths (left to right) from F to t, where the derivatives are multiplied along each path.

The chain rule as given is technically imprecise: the x in $\frac{\partial F}{\partial x}$ is just a place holder meaning “the first variable in F”, but the x in $\frac{dx}{dt}$ is a function of t.

Until you get used to it, this can cause confusion about where you evaluate each term in the chain rule.

A more precise statement is in the Notes:

Theorem 1.2 (Chain Rule) *Suppose that F is a function of two variables and that x and y are both functions of one variable. Define the function ϕ by $\phi(t) = F(x(t), y(t))$ and the point (x_0, y_0) by $(x_0, y_0) = (x(t_0), y(t_0))$. If x and y are both differentiable at t_0 and the partial derivatives of F exist and are continuous at (x_0, y_0), then ϕ is differentiable at t_0 and*

$$\phi'(t_0) = D_1F(x_0, y_0)x'(t_0) + D_2F(x_0, y_0)y'(t_0).$$

Example 11 *A wire has a (figure eight) shape defined by the parametric equations $x = \sin 2t$ and $y = \cos t$. The temperature at a point (x, y) on the wire is given by $T = x^2y + 2xy^3$. Find the rate of change of temperature w.r.t. t when $t = 0$.*

SOLUTION:

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

$$=$$

We don't bother subbing in x and y as functions of t since we only want the value.

At $t = 0$ we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$. So

$$\frac{dT}{dt}(0) =$$

Let us revisit the tangent plane.

Suppose our surface is $z = F(x, y)$ and P is the point (x_0, y_0, z_0) where $z_0 = F(x_0, y_0)$.

Any curve on the surface through P can be given by defining x and y as functions of t, such that $x(0) = x_0$ and $y(0) = y_0$.

So the curve will be given by $\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$, where

$z(t) = F(x(t), y(t))$.

The tangent line at P will be

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + s \begin{pmatrix} x'(0) \\ y'(0) \\ z'(0) \end{pmatrix}, \qquad s \in \mathbb{R}$$

By the chain rule $\quad z'(0) = F_x(x_0, y_0)x'(0) + F_y(x_0, y_0)y'(0)$.
So we can expand the expression for the tangent line to

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + sx'(0) \begin{pmatrix} 1 \\ 0 \\ F_x(P) \end{pmatrix} + sy'(0) \begin{pmatrix} 0 \\ 1 \\ F_y(P) \end{pmatrix}$$

If we take all such curves through P, $x'(0)$ and $y'(0)$ can independently take on any real values, and so the tangent plane has parametric vector form

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ F_x(P) \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ F_y(P) \end{pmatrix}$$

where λ and μ run over all real numbers.

The converse is true too: given any vector $\mathbf{v}$ in the plane

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ F_x(P) \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ F_y(P) \end{pmatrix}$$

we can find a curve in the surface with tangent $\mathbf{v}$ at P.
But the plane above has normal

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \\ F_x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ F_y \end{pmatrix} = \begin{pmatrix} F_x(x_0, y_0) \\ F_y(x_0, y_0) \\ -1 \end{pmatrix}.$$

So we are back to where we were earlier, but have now proved that the tangent plane at P is the collection of all tangents to all possible curves in the surface going through P.

For our other main example of a chain rule, suppose $F = F(x, y)$ where $x = x(s, t)$ and $y = y(s, t)$.

Then F is a function of s and t, and so we would look for its derivatives $\frac{\partial F}{\partial t}$ and $\frac{\partial F}{\partial s}$.

We find

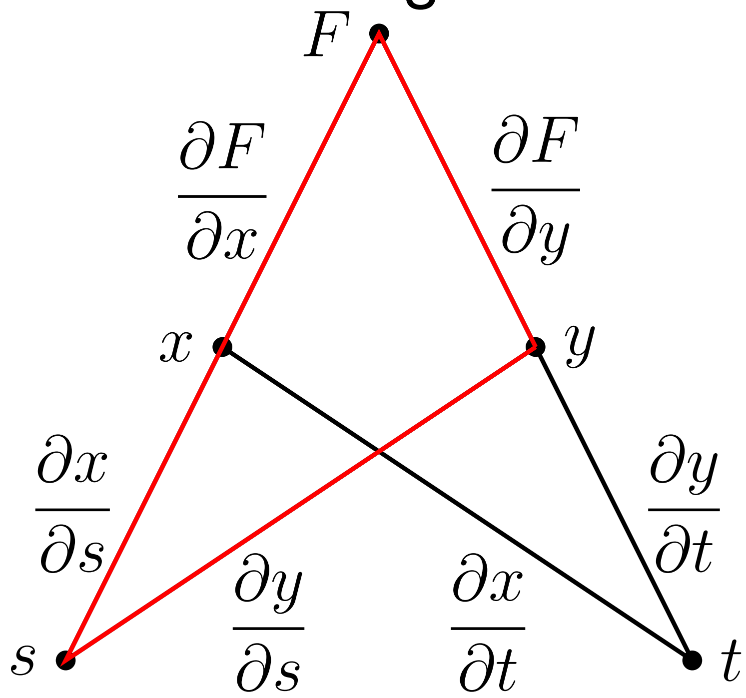
$$\boxed{\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t}} \qquad \boxed{\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s}}$$

To prove the first of these, apply our previous chain rule, holding s fixed.

Then $\frac{dF}{dt}$ will really be $\frac{\partial F}{\partial t}$, and similarly for the derivatives of x and y.

The other is proved similarly, keeping t fixed instead.

As before, we can use chain diagrams:



We link every function to all those variables that directly determine it.
Each line is a partial derivative.
Each path starting from the top most function (F here) gives you a product of partial derivatives.
You add up all the paths from the top variable to the appropriate bottom (independent) variable to get the chain rule

Example 12 *Suppose $f(x,y) = x^2y + y^3$, $x = r\cos\theta$ and $y = r\sin\theta$. Find $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial \theta}$ using the chain rule.*

SOLUTION: We have

$$\begin{aligned}\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\\ &= \\ &= \end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}\\ &= \\ &= \end{aligned}$$

Example 13 *Find $\frac{\partial^2 F}{\partial \theta^2}$ when $F = F(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$ in terms of x and y derivatives of F.*

SOLUTION: As before

$$\frac{\partial F}{\partial \theta} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \theta}$$

So using the product rule

$$\frac{\partial^2 F}{\partial \theta^2}$$

For $\frac{\partial F_x}{\partial \theta}$ we use the same chain rule that we used for $\frac{\partial F}{\partial \theta}$:

$$\frac{\partial F_x}{\partial \theta} = \frac{\partial F_x}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial F_x}{\partial y}\frac{\partial y}{\partial \theta}$$

And again for $\dfrac{\partial F_y}{\partial \theta}$:

$$\frac{\partial F_y}{\partial \theta} = \qquad\qquad\qquad\qquad\qquad\qquad .$$

Assuming that $F_{xy} = F_{yx}$ we can collect the terms together:

$$\frac{\partial^2 F}{\partial \theta^2} =$$

■

EXERCISE: Find $\dfrac{\partial^2 F}{\partial \theta\, \partial r}$ and $\dfrac{\partial^2 F}{\partial r^2}$.

For one last useful set of chain rules:
If $F = F(u)$ and $u = u(x, y)$ then

$$\boxed{\frac{\partial F}{\partial x} = \frac{dF}{du}\frac{\partial u}{\partial x}} \qquad \boxed{\frac{\partial F}{\partial y} = \frac{dF}{du}\frac{\partial u}{\partial y}}$$

Prove these by applying the usual 1-variable chain rule.

The chain diagram corresponding to this situation is

$$\begin{tikzcd}
 & & x \\
F \arrow[r, "\frac{dF}{du}"] & u \arrow[ur, "\frac{\partial u}{\partial x}"] \arrow[dr, "\frac{\partial u}{\partial y}"'] & \\
 & & y
\end{tikzcd}$$

1.6 Functions of Three (or more) Variables

If $F = F(x, y, z)$ then

$$
\begin{aligned}
D_1F \equiv F_x(x,y,z) \equiv \frac{\partial F}{\partial x} &= \lim_{h\to 0} \frac{F(x+h,y,z) - F(x,y,z)}{h} \\
D_2F \equiv F_y(x,y,z) \equiv \frac{\partial F}{\partial y} &= \lim_{h\to 0} \frac{F(x,y+h,z) - F(x,y,z)}{h} \\
D_3F \equiv F_z(x,y,z) \equiv \frac{\partial F}{\partial z} &= \lim_{h\to 0} \frac{F(x,y,z+h) - F(x,y,z)}{h}
\end{aligned}
$$

wherever these limits exists.

This is no different to what we did before: differentiate w.r.t. one of the variables by holding the other variables fixed.

There are chain rules as before, and similar (but larger) chain diagrams can keep track of the formulas.

As a simple case, if $F = F(u)$ and $u = u(x, y, z)$ then

$$\boxed{\frac{\partial F}{\partial x} = \frac{dF}{du}\frac{\partial u}{\partial x}} \qquad \boxed{\frac{\partial F}{\partial y} = \frac{dF}{du}\frac{\partial u}{\partial y}} \qquad \boxed{\frac{\partial F}{\partial z} = \frac{dF}{du}\frac{\partial u}{\partial z}}$$

For a more complicated example, suppose we have $f(x, y, z)$ with x, y and z dependent on r, s and t.

Then, for example

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r},$$

The chain diagram is in the Notes.

Example 14 *The potential energy of a particle is given by* $\phi = F(r)$ *where* $r = \sqrt{x^2 + y^2 + z^2}$*.*
Show that the force acting on the partical is radial:

$$\begin{pmatrix} \phi_x \\ \phi_y \\ \phi_z \end{pmatrix} = f(r) \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

SOLUTION: Firstly $\dfrac{\partial r}{\partial x} =$

Similarly, ϕ_y So

$$\text{force} \sim \begin{pmatrix} \phi_x \\ \phi_y \\ \phi_z \end{pmatrix} =$$

If (x_0, y_0, z_0, w_0) is a point on the **hypersurface** $w = F(x, y, z)$ in $\mathbb{R}^4$ then by the same argument as before the **tangent hyperplane** to the hypersurface at this point has equation

$$\begin{aligned} w &= w_0 + F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) \\ &\quad + F_z(x_0, y_0, z_0)(z - z_0) \end{aligned}$$

And similarly, the total differential approximation to $F(x, y, z)$ is

$$\Delta F \approx \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y + \frac{\partial F}{\partial z}\Delta z$$

The generalisations are easy.
As I said before, the hard part is going from 1 to 2 dimensions: going further just requries more notation.

1.7 Maple Notes

The MAPLE `plot3d` command is useful for visualization.

Try the following:

```
addcoords(zcylinder,[z,r,theta],
  [2*cos(theta)+r*cos(theta/2),
  2*sin(theta)+r*cos(theta/2),z]);
plot3d(r*sin(theta/2),r=-1/2..1/2,theta=0..2*Pi,
coords=zcylinder, title=`Mobius Strip`);
```

Note that the MAPLE `diff` command carries out partial differentiation automatically.

For example `diff(f(x,y),x)` computes $\dfrac{\partial f}{\partial x}$