MATH1231 Mathematics 1B
MATH1241 Higher Mathematics 1B

ALGEBRA NOTES

Preface

Please read carefully.

These Notes form the basis for the algebra strand of MATH1231 and MATH1241. However, not all of the material in these Notes is included in the MATH1231 or MATH1241 algebra syllabuses. A detailed syllabus is given, commencing on page (ix) of these Notes.

In using these Notes, you should remember the following points:

1. It is essential that you start working right from the beginning of the session and continue to work steadily throughout the session. Make every effort to keep up with the lectures and to do problems relevant to the current lectures.

2. These Notes are **not** intended to be a substitute for attending lectures or tutorials. The lectures will expand on the material in the notes and help you to understand it.

3. These Notes may seem to contain a lot of material but not all of this material is equally important. One aim of the lectures will be to give you a clearer idea of the relative importance of the topics covered in the Notes.

4. Use the tutorials for the purpose for which they are intended, that is, to ask questions about both the theory and the problems being covered in the current lectures.

5. The theory (i.e. the theorems and proofs) is regarded as an essential part of the Algebra course. A list of the theory that you should know is given on page xiii.

6. Some of the material in these Notes is more difficult than the rest. This harder material is marked with the symbol [**H**]. Material marked with an [**X**] is intended for students in MATH1241.

7. It is **essential** for you to do **problems** which are given at the end of each chapter. If you find that you do not have time to attempt all of the problems, you should at least attempt a representative selection of them. The problems set in tests and exams will be similar to the problems given in these notes. Further information on the problems and class tests is on pages (x).

8. You will probably find some of the ideas in Chapters 6 and 7 quite difficult at first because they are expressed in a general and abstract manner. However, as you work through the examples in the chapters and the problems at the ends of the chapters you should find that the ideas become much clearer to you.

9. You will be expected to use the computer algebra package Maple in tests and understand Maple syntax and output for the end of semester examination.

10. You should keep these Notes for use in 2nd year subjects on Linear Algebra.

Note.

These notes have been prepared by a number of members of the University of New South Wales. The main contributors include Peter Blennerhassett, Peter Brown, Shaun Disney, Ian Doust, William Dunsmuir, Peter Donovan, David Hunt, Elvin Moore and Colin Sutherland. Chapter 9 was written by Dr. Thomas Britz based on the notes of Prof. William Dunsmuir. The original problems for this chapter came from MATH1151. They were reorganised and expanded by Dr. Chi Mak. Copyright is vested in The University of New South Wales, ©2018.

Contents

ALGEBRA SYLLABUS AND LECTURE TIMETABLE

The algebra course for both MATH1231 and MATH1241 is based on chapters 6 to 9 of the Algebra Notes. Lecturers will not cover all of the material in these notes in their lectures as some sections of the notes are intended for reference and for background reading.

The following timetable is the basic timetable and syllabus which will be followed by MATH1231 algebra lecturers. MATH1241 lecturers will include extra material in their lectures. Lecturers will try to follow this timetable, but some variations are inevitable.

Chapter 6. Vector Spaces

The aim of this section of the course is to introduce the general theory of vector spaces and to give some basic examples. The majority of examples will be for the real vector space $\mathbb{R}^n$, but occasional examples may be given for the complex vector space $\mathbb{C}^n$, as well as from vector spaces of polynomials.

Lectures 1 and 2. Introduction to vector spaces and examples of vector spaces (6.1).
Properties of vector arithmetic (6.2).
Lecture 3. Subspaces (6.3).
Lectures 4 and 5. Linear combinations and spans (6.4). Linear independence (6.5).
Lectures 6 and 7. Basis and dimension (6.6).

Chapter 7. Linear Transformations

The basic aims of this section are to introduce the general theory of linear transformations, to give some geometric applications of linear transformations and to establish the close relationship between linear functions and matrices.

Lecture 8. Introduction to linear maps (7.1). Linear maps and the matrix equation (7.2).
Lecture 9. Geometrical examples (7.3).
Lecture 10. Subspaces associated with linear maps (7.4).
Lecture 11. Rank, nullity and solutions of $A\mathbf{x} = \mathbf{b}$ (7.4.3). Further applications (7.5).

Chapter 8. Eigenvalues and Eigenvectors

The aims of this section are to introduce the ideas of eigenvalue and eigenvector and to show some applications of these ideas to diagonalization of matrices, evaluation of powers of matrices and solution of simple systems of linear differential equations. Examples will be restricted to 2×2 matrices and very simple 3×3 matrices.

Lecture 12. Definition, examples and geometric interpretation of eigenvalues and eigenvectors (8.1).
Lecture 13. Eigenvectors, bases and diagonalization of matrices (8.2).
Lectures 14 and 15. Applications to powers of matrices and solution of systems of linear differential equations (8.3).

Chapter 9. Probability and Statistics

The main objective of this section is to introduce some of the ideas in mathematical probability and apply these concepts to discrete and continuous valued random variables and their associated probability distributions. The main distributions studied are the binomial and geometric in the

discrete case, and the normal distribution in the continuous case. These are applied to solving a range of problems.
Lecture 16. Revision of set theory (9.1), Mathematical probability (9.2.1, 9.2.2).
Lecture 17. Conditional probability, Bayes' rule, statistical independence (9.2.3, 9.2.4)
Lecture 18. Random variables, discrete random variables, mean of a discrete random variable (9.3.1, 9.3.2)
Lecture 19. Variance of a discrete random variable (9.3.2), special distributions, the binomial distribution (9.4.1)
Lecture 20. Geometric distribution, sign test, (9.4.2, 9.4.3)
Lecture 21. Continuous random variables (9.5)
Lecture 22. The Normal distribution, approximations to the binomial distribution. (9.6)
Lecture 23. Review.

EXTRA ALGEBRA TOPICS FOR MATH1241

The extra topics in the MATH1241 syllabus, marked [**X**] in the notes will be selected from the following:
Vector spaces. Matrices, polynomials and real-valued functions as vector spaces (6.8). Coordinate vectors (6.7). The theoretical treatment of vector spaces in MATH1241 will be at a slightly more sophisticated level than that in MATH1231.
Linear transformations.Linear maps between polynomial and real-valued function vector spaces (7.5). Matrix representations for non-standard bases in domain and codomain (7.6). Matrix arithmetic and linear maps (7.7). Injective, surjective and bijective linear maps (7.8). Proof the rank–nullity theorem (7.9).
Eigenvalues and eigenvectors. Markov Chain Processes (8.3.3). Eigenvalues and eigenvectors for symmetric matrices and applications to conic sections.
Probability and statistics. The Exponential distribution (9.6.2).

PROBLEM SETS

At the end of each chapter there is a set of problems. Some of the problems are very easy, some are less easy but still routine and some are quite hard. To help you decide which problems to try first, each problem is marked with an [**R**], an [**H**] or an [**X**]. The problems marked [**R**] form a basic set of problems which you should try first. Problems marked [**H**] are harder and can be left until you have done the problems marked [**R**]. Problems marked with [**V**] have a video solution available via Moodle. You *do* need to make an attempt at the [**H**] problems because problems of this type will occur on tests and in the exam. If you have difficulty with the [**H**] problems, ask for help in your tutorial.

The problems marked [**X**] are intended for students in MATH1241 – they relate to topics which are only covered in MATH1241.

Extra problem sheets for MATH1241 may be issued in lectures.

There are a number of questions marked [**M**], indicating that MATLAB is required in the solution of the problem. Questions marked with a [V] have a video solution available from the course page for this subject on Moodle.

WEEKLY ALGEBRA SCHEDULES

Solving problems and writing mathematics clearly are two separate skills that need to be developed through practice. We recommend that you keep a workbook to practice *writing* solutions to mathematical problems. The following table gives the range of questions suitable for each week. In addition it suggests specific recommended problems to do before your classroom tutorials.

The Online Tutorials will develop your problem solving skills, and give you examples of mathematical writing. Online Tutorials help build your understanding from lectures towards solving problems on your own. Because this overlaps with the skills developed through homework, there are fewer recommended homework in Online Tutorial weeks.

WEEKLY ALGEBRA HOMEWORK SCHEDULE

Week	**Algebra problems**		**Recommended Homework Problems**
	Chapter	**Problems up to**	
2	6	14	2, 3, 10, 12
3	6	33	16,18,22, 28, 32
4	6	48	34, 38, 41, 43, 44
5	6	61	49, 53, 56, 58
	7	12	2(c), 4, 7, 8, 11(for 2(b))
6	7	23 (Test 1)	13(c), 15, 16, 19
7	7	59	26(b), 31(b), 33(for 25(a)), 37, 38, 47, 56
8	8	15	2, 4, 7(d), 11(for 7(a)), 12
9	8	29	16, 18(for 7(a)), 20, 21(a)
10	9	17	2,4,6,7,11,12
11	9	28 (Test 2)	18, 21,25,26
12	9	40	29, 31, 39
13	9	65	41b, 44, 45, 47,49e, 50d, 56

WEEKLY MATH1231 ALGEBRA TUTORIAL SCHEDULE

The main reason for having tutorials is to give you a chance to tackle and discuss problems which you find difficult or don't fully understand.

There are two kinds of tutorials: Online and Classroom. Algebra Online Tutorials are delivered using MapleTA. These can be completed from home, are available for a two week period, and are due on Sunday night in weeks 2,4,6,8,10 and 12. Algebra Classroom tutorials are delivered in a classroom by an algebra tutor. The topics covered in a classroom tutorial are flexible, and you can (and should) ask your tutor to cover any homework topics you find difficult. You may also be asked to present solutions to homework questions to the rest of the class.

The following table lists the topics covered in each tutorial.

Week	**Location**	**Topics Covered**
2	Online	6.1 : Definitions and examples of vector spaces 6.2 : Vector arithmetic 6.3 : Subspaces upto Pr. 16
3	Classroom	6.3 : Subspaces Pr. 17 onwards 6.4 : Linear combinations and spans
4	Online	6.5 : Linear independence
5	Classroom	6.6 : Basis and dimension
6	Online	7.1 : Introduction to linear maps 7.2 : Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $m \times n$ matrices 7.3 : Geometric examples of linear transformations
7	Classroom	7.4 : Subspaces associated with linear maps 7.5 : Further applications and examples of linear maps
8	Online	8.1 : Definitions and examples 8.2 : Eigenvectors, bases, and diagonalisation
9	Classroom	8.3 : Applications of eigenvalues and eigenvectors
10	Online	9.1 : Some Preliminary Set Theory 9.2 : Probability
11	Classroom	9.3 : Random Variables
12	Online	9.4 : Special Distributions 9.5 : Continuous random variables
13	Classroom	9.6 : Special Continuous Distributions

WEEKLY MATH1241 ALGEBRA TUTORIAL SCHEDULE

MATH1241 Tutorials cover the same material as MATH1231, only in greater detail. The tutorial structure is more flexible, which is designed to allow for classroom discussion. Only a subset of the recommended discussion questions will be discussed in your classroom tutorial, which are held every odd week starting in week 3. Online Tutorial questions for algebra are due at Sunday 23:59 every even week.

weeks	Chapter	Online Tutorial	Recommended Classroom Discussion Questions
2 and 3	6	2/3,16	10,12
		18, 33	22, 28, 32
4 and 5	6	38, 43, 56	34, 41, 44, 49, 53, 58
	7	7/8	2c, 4, 11(for 2b)
6 and 7	7	13, 16	15, 19
		26b, 38	31b, 33(for 25a), 37, 47, 56
8 and 9	8	4, 11(for 7a)	2, 7d, 12
		18, 20	16, 21a
10 and 11	9	4, 6	2, 7, 12
	9	11, 18	21, 25, 26
12 and 13	9	31, 39	29
		45, 49e	41b, 44, 47, 50d, 56

THEORY IN THE ALGEBRA COURSE

The theory is regarded as an essential part of this course and it will be examined both in class tests and in the end of year examination.

You should make sure that you can give DEFINITIONS of the following ideas:

Chapter 6. Subspace of a vector space, linear combination of a set of vectors, span of a set of vectors, linear independence of a set of vectors, spanning set for a vector space, basis for a vector space, dimension of a vector space.
Chapter 7. Linear function, kernel and nullity of a linear function, image and rank of a linear function.
Chapter 8. Eigenvalue and eigenvector, diagonalizable matrix.
Chapter 9. Probability, statistical independence, conditional probability, discrete random variable, expected value (mean) of a random variable, variance of a random variable, binomial distribution, geometric distribution.

You should be able to give STATEMENTS of the following theorems and propositions.

Chapter 6. Theorem 1 of §6.3, Propositions 1 and 3 and Theorem 2 of §6.4, Proposition 1 and Theorems 2, 3, 4, 5 and 6 of §6.5, Theorems 1, 2, 3, 4, 5, 6 and 7 of §6.6.
Chapter 7. Theorems 2, 3 and 4 of §7.1, Theorem 1 and 2 of §7.2, Proposition 7 and Theorems 1, 5, 8, 9 and 10 of §7.4.

Chapter 8. Theorems 1, 2 and 3 of §8.1, Theorem 1 and 2 of §8.2.

You should be able to give PROOFS of the following theorems and propositions.

Chapter 6. Theorem 2 of §6.4, Theorems 2 and 3 of §6.5, Theorem 2 of §6.6.
Chapter 7. Theorem 2 of §7.1, Theorem 1 of §7.2, Theorems 1, 5 and 8 of §7.4.
Chapter 8. Theorem 1 of §8.1.

Chapter 6

VECTOR SPACES

But, keeping still the end in view
To which I hope to come,
I strove to prove the matter true
By putting everything I knew
Into an Axiom
Lewis Carroll, Phantasmagoria.

We have studied geometric vectors and column vectors in Chapter 2nd matrices in Chapter 4 What do the following sets have in common?

- The set of geometric vectors in a three-dimensional space.
- The set of column vectors of n real components, i.e. $\mathbb{R}^n$.
- The set of $m \times n$ matrices with real entries, i.e. $M_{mn}(\mathbb{R})$.

In each of these sets, we can add two elements and we can multiply an element in the set with a scalar (in this case, a scalar is a real number) and remain inside the set we started in. We say that each set is closed under the two operations — addition and multiplication by a scalar. Such a set together with scalars and the two operations satisfy some fundamental properties. As what we have seen in Chapters 2nd 4 addition of vectors and matrices satisfies the associative and commutative laws. There is a special element $\mathbf{0}$ in each set such that $\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$ in the set. For each $\mathbf{v}$ in the set, there is a negative $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0}$. The associative law of scalar multiplication and distributive laws also hold. These are examples of **vector spaces** which we are going to study in this chapter. In a vector space, each element in the set is called a vector. The set of scalars must be a field, generally the set of real or complex numbers.

Besides the above mentioned vector spaces, there are many other examples, such as the set of polynomials with real or complex coefficients, the real functions on a given interval and the differentiable functions defined on an interval, which form a vector space. It is perhaps a remarkable fact that each of these quite different kinds of objects obey similar rules for addition and multiplication by a scalar.

In the present chapter our main objectives are to develop a general theory of vector spaces and to show how this general theory can be applied to the study of particular vector spaces. Although all of the theorems and propositions stated in this chapter are true for **all** vector spaces, we will concentrate on giving examples and applications of the theory for the vector space $\mathbb{R}^n$. The main reason for this is that in $\mathbb{R}^n$ the theoretical results can be more easily understood, as they can often be given a geometric interpretation. Also, $\mathbb{R}^n$ is the most commonly used vector space in practical applications.

As the mathematics developed in this chapter applies to all vector spaces, it is more abstract and theoretical than that of Chapter 2 where there was always an immediate geometric picture available for all results.

Another reason for the difficulty is that an appreciable part of the language of vector spaces will be new to you. Therefore, as with any new language, **it is absolutely essential that you make a special effort to learn the definitions of any new words**. You will find that many of the fundamental vector space ideas discussed in this chapter, such as linear combination, span, linear independence, basis, dimension, and coordinate vector, are generalisations of ideas that we have already discussed in an informal geometric manner in Chapter 2or vectors in $\mathbb{R}^n$.

In addition, you will also find that the solution of most of the problems in this chapter can be obtained by solving systems of linear equations using Gaussian Elimination developed in Chapter 3 In most cases the details are suppressed, but the reader should check them before attempting the exercises.

It is important to keep in mind in this chapter that correct **setting out** is essential, rather than just computation. When you write down a solution to a question, you must make sure it reads correctly, both logically and mathematically.

6.1 Definitions and examples of vector spaces

We start from a mathematical system which consists of the following four things.

1. A non-empty set V of elements called "vectors".

2. A "vector-addition" rule (usually represented by $+$) for combining pairs of vectors from V. For vectors $\mathbf{v},\ \mathbf{w} \in V$, the vector formed by adding $\mathbf{w}$ to $\mathbf{v}$ is denoted by $\mathbf{v}+\mathbf{w}$.

3. A field $\mathbb{F}$ of "scalars". For example, $\mathbb{F}$ could be the rational numbers $\mathbb{Q}$ or the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. There are also other important, but less common, examples of fields which can be used.

4. A "multiplication by a scalar rule" for combining a vector from V and a scalar from $\mathbb{F}$ to form a vector. If λ is a scalar and $\mathbf{v}$ is an element of V, then $\lambda * \mathbf{v}$ means the result of multiplying $\mathbf{v}$ by the scalar λ.

The system is then denoted by $(V, +, *, \mathbb{F})$. However, if we have no problem in distinguishing the product of two scalars $\lambda\mu$ and the multiplication of a vector by a scalar $\lambda * \mathbf{v}$, we shall omit the symbol $*$. Just as we write $2x$ instead of $2 \times x$, we shall write $\lambda\mathbf{v}$ instead of $\lambda * \mathbf{v}$.

We can now give a formal definition of a vector space.

Definition 1. *A* **vector space** V *over the field* $\mathbb{F}$ *is a non-empty set* V *of vectors on which addition of vectors is defined and multiplication by a scalar is defined in such a way that the following ten fundamental properties are satisfied:*

1. ***Closure under Addition***. *If* $\mathbf{u}, \mathbf{v} \in V$, *then* $\mathbf{u} + \mathbf{v} \in V$.
2. ***Associative Law of Addition***. *If* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, *then* $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$.
3. ***Commutative Law of Addition***. *If* $\mathbf{u}, \mathbf{v} \in V$, *then* $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
4. ***Existence of Zero***. *There exists an element* $\mathbf{0} \in V$ *such that, for all* $\mathbf{v} \in V$, $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
5. ***Existence of Negative***. *For each* $\mathbf{v} \in V$ *there exists an element* $\mathbf{w} \in V$ *(usually written as* $-\mathbf{v}$*), such that* $\mathbf{v} + \mathbf{w} = \mathbf{0}$.
6. ***Closure under Multiplication by a Scalar***. *If* $\mathbf{v} \in V$ *and* $\lambda \in \mathbb{F}$, *then* $\lambda\mathbf{v} \in V$.
7. ***Associative Law of Multiplication by a Scalar***. *If* $\lambda, \mu \in \mathbb{F}$ *and* $\mathbf{v} \in V$, *then* $\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$.
8. *If* $\mathbf{v} \in V$ *and* $1 \in \mathbb{F}$ *is the scalar one, then* $1\mathbf{v} = \mathbf{v}$.
9. ***Scalar Distributive Law***. *If* $\lambda, \mu \in \mathbb{F}$ *and* $\mathbf{v} \in V$, *then* $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$.
10. ***Vector Distributive Law***. *If* $\lambda \in \mathbb{F}$ *and* $\mathbf{u}, \mathbf{v} \in V$, *then* $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$.

NOTE.

1. Each of the basic rules is called an **axiom**.
2. In axiom 5, $-\mathbf{v}$ is a symbol for the negative of $\mathbf{v}$. The vector $-\mathbf{v}$ and the vector formed by multiplying $\mathbf{v}$ with the scalar -1 are not the same by definition. We shall prove that they are the same later.
3. Formally, axiom 7 says $\lambda * (\mu * \mathbf{v}) = (\lambda\mu) * \mathbf{v}$.
4. In axiom 9, the addition on the left is the addition of two scalars while the addition on the right is the addition of two vectors. They are different additions.
5. Two systems are the same only when all the four things are the same. However, we seldom discuss different vector spaces with the same set of vectors but we often discuss different sets of vectors with the same set of scalars and the same operations. When there is no confusion, we shall simply call $(V, +, *, \mathbb{F})$, the vector space V.

Example 1 (The Vector Space $\mathbb{R}^n$)**.** The set of vectors is the set of all n-vectors of real numbers,

$$\mathbb{R}^n = \left\{ \mathbf{x} : \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{for} \quad x_1, \dots, x_n \in \mathbb{R} \right\}.$$

The set of scalars is $\mathbb{R}$.
Vector addition is defined by

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}.$$

The multiplication of a vector by a scalar $\lambda \in \mathbb{R}$ is defined by

$$\lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}.$$

To prove that this system is a vector space it is necessary to show that all ten axioms listed in the definition are satisfied by the system.

All the axioms are general statements about arbitrary vectors and scalars. We have to prove the axioms are satisfied by any

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \ \mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \quad \text{and} \quad \lambda, \mu \in \mathbb{R}.$$

1. **Closure under addition.** If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ then $u_1 + v_1, \ \dots, \ u_n + v_n \in \mathbb{R}$ because $\mathbb{R}$ is closed under addition. Hence
$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix} \in \mathbb{R}^n.$$

2. **Associative law of addition**. If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ then
$$(u_1 + v_1) + w_1 = u_1 + (v_1 + w_1), \ \dots, \ (u_n + v_n) + w_n = u_n + (v_n + w_n)$$
because addition in $\mathbb{R}$ is associative. Hence
$$\begin{aligned} (\mathbf{u} + \mathbf{v}) + \mathbf{w} &= \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} (u_1 + v_1) + w_1 \\ \vdots \\ (u_n + v_n) + w_n \end{pmatrix} \\ &= \begin{pmatrix} u_1 + (v_1 + w_1) \\ \vdots \\ u_n + (v_n + w_n) \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix} = \mathbf{u} + (\mathbf{v} + \mathbf{w}). \end{aligned}$$

3. **Commutative law of addition**. If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ then

$$u_1 + v_1 = v_1 + u_1, \ \ldots\ , \ u_n + v_n = v_n + u_n$$

because addition in $\mathbb{R}$ is commutative. Hence,

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix} = \begin{pmatrix} v_1 + u_1 \\ \vdots \\ v_n + u_n \end{pmatrix} = \mathbf{v} + \mathbf{u}.$$

4. **Existence of zero**. There is a special element

$$\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$$

called the **zero vector** which has the property that

$$\mathbf{v} + \mathbf{0} = \begin{pmatrix} v_1 + 0 \\ \vdots \\ v_n + 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \mathbf{v}, \qquad \text{for all } \mathbf{v} \in \mathbb{R}^n.$$

5. **Existence of Negative**. For each $\mathbf{v} \in \mathbb{R}^n$ there exists an element

$$\begin{pmatrix} -v_1 \\ \vdots \\ -v_n \end{pmatrix} \in \mathbb{R}^n,$$

the negative of $\mathbf{v}$, such that

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} -v_1 \\ \vdots \\ -v_n \end{pmatrix} = \begin{pmatrix} v_1 - v_1 \\ \vdots \\ v_n - v_n \end{pmatrix} = \mathbf{0}.$$

6. **Closure under scalar multiplication**. If $\mathbf{v} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ then $\lambda v_1, \ldots, \lambda v_n \in \mathbb{R}$ because $\mathbb{R}$ is closed under multiplication. Hence $\lambda \mathbf{v} \in \mathbb{R}^n$.

7. **Associative law of multiplication by a scalar**. If $\lambda, \mu \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$ then

$$\lambda(\mu v_1) = (\lambda\mu)v_1, \ \ldots\ , \ \lambda(\mu v_n) = (\lambda\mu)v_n$$

because multiplication in $\mathbb{R}$ is associative. Hence,

$$\lambda(\mu\mathbf{v}) = \lambda \begin{pmatrix} \mu v_1 \\ \vdots \\ \mu v_n \end{pmatrix} = \begin{pmatrix} \lambda(\mu v_1) \\ \vdots \\ \lambda(\mu v_n) \end{pmatrix} = \begin{pmatrix} (\lambda\mu)v_1 \\ \vdots \\ (\lambda\mu)v_n \end{pmatrix} = (\lambda\mu) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = (\lambda\mu)\mathbf{v}.$$

8. If $\mathbf{v} \in \mathbb{R}^n$ then $1\mathbf{v} = \begin{pmatrix} 1v_1 \\ \vdots \\ 1v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \mathbf{v}$.

9. **Scalar distributive law**. If $\lambda, \mu \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$ then

$$(\lambda + \mu)v_1 = \lambda v_1 + \mu v_1, \ \ldots \ , \ (\lambda + \mu)v_n = \lambda v_n + \mu v_n,$$

because of the distributive law in $\mathbb{R}$. We then have

$$\begin{aligned} (\lambda+\mu)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} &= \begin{pmatrix} (\lambda+\mu)v_1 \\ \vdots \\ (\lambda+\mu)v_n \end{pmatrix} = \begin{pmatrix} \lambda v_1 + \mu v_1 \\ \vdots \\ \lambda v_n + \mu v_n \end{pmatrix} \\ &= \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix} + \begin{pmatrix} \mu v_1 \\ \vdots \\ \mu v_n \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \mu \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}. \end{aligned}$$

Hence $(\lambda + \mu)\mathbf{v} \ = \ \lambda\mathbf{v} + \mu\mathbf{v}$.

10. **Vector distributive law**. If $\lambda \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ then

$$\lambda(u_1 + v_1) = \lambda u_1 + \lambda v_1, \ \ldots \ , \lambda(u_n + v_n) = \lambda u_n + \lambda v_n,$$

because of the distributive law in $\mathbb{R}$. We then have

$$\lambda \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix} = \begin{pmatrix} \lambda(u_1 + v_1) \\ \vdots \\ \lambda(u_n + v_n) \end{pmatrix} = \begin{pmatrix} \lambda u_1 + \lambda v_1 \\ \vdots \\ \lambda u_n + \lambda v_n \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Hence $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$.

After we have checked that all ten axioms are satisfied, we can conclude that the system is a vector space, or simply $\mathbb{R}^n$ is a vector space over $\mathbb{R}$. $\Diamond$

NOTE. As special cases of $\mathbb{R}^n$, the real number line $\mathbb{R}$, the plane $\mathbb{R}^2$ and three-dimensional space $\mathbb{R}^3$ are all vector spaces over the real numbers.

Example 2 (The Vector Space $\mathbb{C}^n$)**.** The set of vectors is the set of all column vectors with n components of complex numbers

$$\mathbb{C}^n = \left\{ \mathbf{x} : \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{for} \quad x_1, \ldots, x_n \in \mathbb{C} \right\},$$

and the set of scalars is $\mathbb{C}$. Addition is defined by

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}.$$

and multiplication of a vector by a scalar $\lambda \in \mathbb{C}$ is defined by

$$\lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}.$$

To prove that this is a vector space it is necessary to show that the ten vector space axioms are satisfied. This proof is formally identical to that for $\mathbb{R}^n$ over $\mathbb{R}$ since only basic operations are involved. $\Diamond$

Example 3 (The Vector Space $M_{mn} = M_{mn}(\mathbb{R})$ of Real Matrices)**.** M_{mn} is a vector space over $\mathbb{R}$. There is a natural and straightforward generalisation to $M_{mn}(\mathbb{F})$, where the entries come from the field $\mathbb{F}$. In the most important case of $\mathbb{F} = \mathbb{R}$ we often suppress the $\mathbb{R}$. Note that here we are thinking of matrices as vectors!

$$M_{mn} = M_{mn}(\mathbb{R}) = \left\{ A : A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \text{ where } a_{ij} \in \mathbb{R} \text{ for } 1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n \right\}.$$

Using the notation introduced in Chapter 4 the ijth entry (ith row, jth column entry) of A is denoted by $[A]_{ij}$. We define "addition of vectors" to be matrix addition where

$$[A + B]_{ij} = [A]_{ij} + [B]_{ij}, \quad \text{for all} \quad i, j.$$

Similarly we define the "multiplication (of the vector A) by a scalar $\lambda \in \mathbb{R}$" in terms of the multiplication of a matrix by a scalar. That is,

$$[\lambda A]_{ij} = \lambda [A]_{ij} \quad \text{for all} \quad i, j$$

as in Chapter 4

To check that M_{mn} is a vector space is routine. All of the properties are included amongst the properties developed for matrices. For example

$$A + B = B + A \quad \text{for matrices of the same size}$$

hence the commutative law holds for the set M_{mn} regarded as a vector space. The details are left for the reader to check. $\Diamond$

Note. $M_{mn}(\mathbb{R})$ and $M_{mn}(\mathbb{C})$ are widely used in both quantum physics and chemistry.

Example 4 (The Vector Space of Polynomials)**.** One of the most important aspects of vector space theory is that it applies in many quite different situations. The set of all real-valued functions on $\mathbb{R}$ forms a vector space, as does the set of all continuous functions. A simpler example, perhaps, is the set $\mathbb{P}(\mathbb{R})$ of all real polynomials.

Suppose that p is the polynomial given by $p(x) = a_0 + a_1 x + \cdots + a_n x^n = \sum_{k=0}^{n} a_k x^k$ and q is the polynomial given by $q(x) = \sum_{k=0}^{m} b_k x^k$ where a_k and b_k are real. Note that p is a real-valued function while $p(x)$ is the value of the function at x. (You might like to quickly read the brief review of function notation given in Appendix 6.9.)

We all know how to add and subtract these polynomials, and how to multiply a polynomial by a real number. Their sum is just the polynomial $p + q$, where the value of the function at x is

$$(p+q)(x) = p(x) + q(x) = \sum_{k=0}^{\max(n,m)} (a_k + b_k)x^k, \qquad x \in \mathbb{R}.$$

(Of course we just set any missing coefficient equal to zero to do this sum.) The scalar multiple is the polynomial λp where

$$(\lambda p)(x) = \lambda(p(x)) = \sum_{k=0}^{n} \lambda a_k x^k, \qquad x \in \mathbb{R}.$$

The proof that $\mathbb{P}(\mathbb{R})$ is a vector space over $\mathbb{R}$ is straightforward. For example, if p and q are polynomials, then $p + q$ is also a polynomial (Closure under Addition). The zero element of $\mathbb{P}(\mathbb{R})$ is just the polynomial p such that $p(x) = 0$, for all $x \in \mathbb{R}$. It is important when working in $\mathbb{P}(\mathbb{R})$ to remember that saying that two polynomials p and q are equal means that $p(x) = q(x)$ for all $x \in \mathbb{R}$, or equivalently, that the corresponding coefficients for p and q are equal. Please check the details yourself. $\diamondsuit$

Note. The set $\mathbb{P}(\mathbb{F})$ of all polynomials over a field $\mathbb{F}$, with addition and multiplication by a scalar similarly defined, is also a vector space.

For any non-negative integers n, the subset of all polynomials of degrees n or less including the zero polynomial is again a vector space. That is,

$$\mathbb{P}_n(\mathbb{F}) = \{p : p \text{ is a polynomial over } \mathbb{F}, \text{ degree of } p \leqslant n \text{ or } p = 0\},$$

with the same addition and multiplication by a scalar as in $\mathbb{P}(\mathbb{F})$, is a vector space over $\mathbb{F}$.

As a summary, we know that the following are vectors spaces.

- $\mathbb{R}^n$ over $\mathbb{R}$, where n is a positive integer.
- $\mathbb{C}^n$ over $\mathbb{C}$, where n is a positive integer.
- $\mathbb{P}(\mathbb{F})$, $\mathbb{P}_n(\mathbb{F})$ over $\mathbb{F}$, where $\mathbb{F}$ is a field. Usually $\mathbb{F}$ is either $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.
- $M_{mn}(\mathbb{F})$ over $\mathbb{F}$, where m, n are positive integers and $\mathbb{F}$ is a field.

Furthermore, the following set, its subset of all continuous function and its subset of all differentiable functions are vector spaces over $\mathbb{R}$.

- $\mathcal{R}[X]$, the set of all possible real-valued functions with domain X.

When we refer to these vectors spaces, we assume the additions and multiplications by scalars are as defined above. However, there are systems with the same set of vectors but different operations. If we want to emphasise that the operations are the ones defined above, we shall use the terms *usual addition* and *usual multiplication by a scalar*.

We shall concentrate on the vector space $\mathbb{R}^n$. Those who want to see other examples of vector spaces can look at Section 6.8, where other vector spaces are covered in rather more depth.

We are going to end this section by an example of a system which is not a vector space. To prove a system is not a vector space, we need to prove that one of the axioms is not satisfied. To *disprove* a general statement, we only need to give a **counterexample** to illustrate that the statement is false.

Example 5. The system $(\mathbb{R}^2, +, *, \mathbb{R})$ with the usual multiplication by a scalar but the addition defined by — for any $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in $\mathbb{R}^2$,

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ 2u_2 + 2v_2 \end{pmatrix},$$

is not a vector space.

SOLUTION. We are going to give a counterexample to axiom 9. Let $\lambda = \mu = 1$ and $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

We have

$$(\lambda + \mu)\mathbf{v} = 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0+0 \\ 2 \times 1 + 2 \times 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda\mathbf{v} + \mu\mathbf{v}.$$

Hence, axiom 9 is not satisfied by this system and this system is not a vector space. ♢

6.2 Vector arithmetic

The axioms give a minimal set of rules needed to define a vector space. There are several other useful rules which can be proved directly from the axioms and which are therefore true in all vector spaces. In this section we shall discuss some of these properties and give examples of them for $\mathbb{R}^n$.

The first five vector space axioms apply to vector addition, and they are in fact identical to the five basic axioms of addition for integers, real numbers and complex numbers. This means that all the arithmetic properties of vector addition are identical to corresponding properties of addition of numbers. In particular, we have:

Proposition 1. In any vector space V, the following properties hold for addition.

1. **Uniqueness of Zero.** There is one and only one zero vector.
2. **Cancellation Property.** If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ satisfy $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.
3. **Uniqueness of Negatives.** For all $\mathbf{v} \in V$, there exists only one $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$.

Proof. For property 1, Axiom 4 ensures the existence of a zero vector in V.
Now assume that two vectors $\mathbf{0}$ and $\mathbf{0}'$ are both zero vectors in V. Then, for the reasons given in brackets, we have

$$\begin{aligned} \mathbf{0} &= \mathbf{0} + \mathbf{0}' && \text{(axiom 4 applied to the zero vector } \mathbf{0}'\text{)} \\ &= \mathbf{0}' + \mathbf{0} && \text{(axiom 3)} \\ &= \mathbf{0}' && \text{(axiom 4 applied to the zero vector } \mathbf{0}\text{)} \end{aligned}$$

Hence, $\mathbf{0} = \mathbf{0}'$, and there is only one zero vector in V.

For property 2, Axiom 5 ensures the existence of the negative $-\mathbf{u}$. Hence, we have

$$\begin{aligned} (-\mathbf{u}) + (\mathbf{u} + \mathbf{v}) &= (-\mathbf{u}) + (\mathbf{u} + \mathbf{w}) && \text{(axiom 5)} \\ [(-\mathbf{u}) + \mathbf{u}] + \mathbf{v} &= [(-\mathbf{u}) + \mathbf{u}] + \mathbf{w} && \text{(axiom 2)} \\ \mathbf{0} + \mathbf{v} &= \mathbf{0} + \mathbf{w} && \text{(axiom 5)} \\ \mathbf{v} &= \mathbf{w} && \text{(axiom 4)} \end{aligned}$$

For property 3, assume $\mathbf{u}$ and $\mathbf{w}$ are both negatives of $\mathbf{v}$. By axiom 5, we have $\mathbf{v} + \mathbf{u} = \mathbf{0} = \mathbf{v} + \mathbf{w}$. By property 2, we can conclude that $\mathbf{v} = \mathbf{w}$. Hence the inverse is unique. □

Example 1. i) The unique zero vector in $\mathbb{R}^n$ is $\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.

ii) The negative of a vector is used in solving an equation such as

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \mathbf{v} = \begin{pmatrix} -2 \\ 5 \\ 1 \\ 7 \end{pmatrix} \qquad \text{to obtain} \qquad \mathbf{v} = -\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 5 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -2 \\ 3 \end{pmatrix}.$$ $\diamondsuit$

A comparison of the axioms for multiplication of a vector by a scalar with the multiplication properties for fields of numbers (see Chapter 1 also shows strong similarities — the main difference being that in a field, two numbers of the same kind are being multiplied, whereas for vectors the objects being multiplied are of different kinds. As a result, some of the fundamental properties of multiplication of numbers also hold for multiplication of a vector by a scalar. In particular, we have:

Proposition 2. Suppose that V is a vector space over a field $\mathbb{F}$, $\lambda \in \mathbb{F}$, $\mathbf{v} \in V$, 0 is the zero scalar in $\mathbb{F}$ and $\mathbf{0}$ is the zero vector in V. Then the following properties hold for multiplication by a scalar:

1. **Multiplication by the zero scalar.** $0\mathbf{v} = \mathbf{0}$,
2. **Multiplication of the zero vector.** $\lambda\mathbf{0} = \mathbf{0}$.
3. **Multiplication by -1.** $(-1)\mathbf{v} = -\mathbf{v}$ (the additive inverse of $\mathbf{v}$).
4. **Zero products.** If $\lambda\mathbf{v} = \mathbf{0}$, then either $\lambda = 0$ or $\mathbf{v} = \mathbf{0}$.
5. **Cancellation Property.** If $\lambda\mathbf{v} = \mu\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$ then $\lambda = \mu$.

Proof. We shall prove properties 1 and 3. The readers should write out the proofs for the others as exercises. For property 1,

$$\mathbf{v} + \mathbf{0} = \mathbf{v} = 1\mathbf{v} = (1 + 0)\mathbf{v} = 1\mathbf{v} + 0\mathbf{v} = \mathbf{v} + 0\mathbf{v}.$$

You should check carefully which axioms are required. Finally, by Cancellation Property of vector addition, we have $0\mathbf{v} = \mathbf{0}$.

For property 3,

$$\mathbf{v} + (-1)\mathbf{v} = 1\mathbf{v} + (-1)\mathbf{v} = (1 + (-1))\mathbf{v} = 0\mathbf{v} = \mathbf{0}.$$

Hence by Uniqueness of Negatives $(-1)\mathbf{v} = -\mathbf{v}$. $\square$

Example 2. The properties in the previous proposition are true for all vector spaces. In particular, for vectors in $\mathbb{R}^n$, the results can be easily proved by definitions of the operations and properties of scalars. Such as

a) $0\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{0}$,

b) $(-1)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -x_1 \\ \vdots \\ -x_n \end{pmatrix} = -\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$,

c) If $\quad \lambda\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$, then either $\quad \lambda = 0 \quad$ or $\quad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$. ◊

6.3 Subspaces

Before reading this section, you should quickly read the brief review of sets given in Appendix 6.9.

Although all examples in this section are subsets of $\mathbb{R}^n$, the definitions, theorems and corollaries apply to all vector spaces as stated.

In practice, many problems about vectors involve subsets of some vector space. For example, the points on a line in $\mathbb{R}^n$ form a subset of $\mathbb{R}^n$, the points on a plane in $\mathbb{R}^3$ form a subset of $\mathbb{R}^3$, the solutions of a system of m linear equations in n unknowns form a subset of $\mathbb{R}^n$. It is an important problem to determine the conditions under which some subset of a vector space is itself a vector space. It is convenient to begin by looking at some examples.

Example 1. The real-number line is a vector space. The question arises whether some subset of the real-number line is a vector space. For example, we might ask if some interval, for example the interval

$$S = [-5, 5] = \{x \in \mathbb{R} : -5 \leqslant x \leqslant 5\}$$

is a vector space. Geometrically, the set S represents the line segment shown in Figure 1(b).

Solution. The given system is not a vector space, since it is not closed under scalar multiplication. A counterexample — $5 \in S$, but $5 + 5 = 10$ is not an element of S. A picture is given in Figure 1(b). ◊

(a) The real number line is a vector space.

(b) The interval $S = [-5, 5]$ is not a vector space as $5 \in S$ but $5 + 5 = 10 \notin S$.

Figure 1.

Example 2. The plane $\mathbb{R}^2$ is a vector space. Show that the subset S of $\mathbb{R}^2$ given by

$$S = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 \geqslant 0 \right\}$$

is not a vector space.

SOLUTION. There are several ways to solve this problem since there are several axioms which are not satisfied. One method is to note that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S$, whereas $-1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \notin S$. Hence the set S is not closed under scalar multiplication and so S is not a vector space. $\diamondsuit$

NOTE. Geometrically, the subset S contains the position vectors of all the points in the right half-plane as shown.

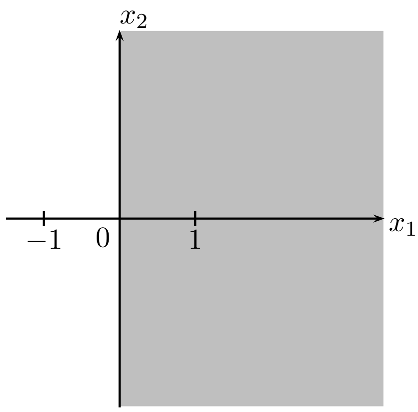


Figure 2.

Example 3. Show that the subset of $\mathbb{R}^2$ given by

$$S_1 = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 + x_2 = 4 \right\}$$

is not a vector space, whereas the subset given by

$$S_2 = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 + x_2 = 0 \right\}$$

is a vector space. (Geometrically, S_1 represents a line in $\mathbb{R}^2$ which does not pass through the origin, whereas S_2 represents a line which does pass through the origin (see Figure 3).)

SOLUTION. The vector $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin S_1$, since $0 + 0 \neq 4$ so S_1 is not a vector space.

It is possible to show that S_2 is a vector space by the usual time-consuming and tedious process of checking that it satisfies all ten of the vector space axioms. $\diamondsuit$

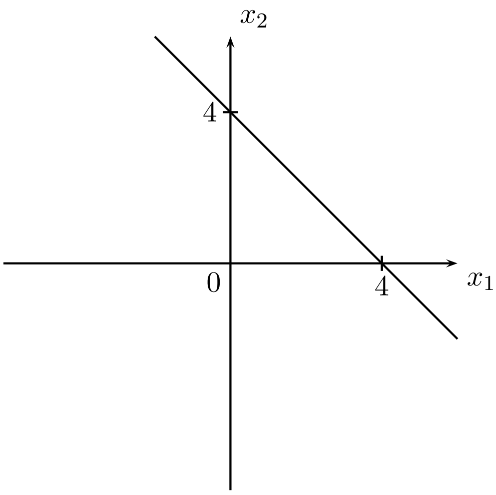


Figure 3(a): The line $x_1 + x_2 = 4$ is not a vector space.

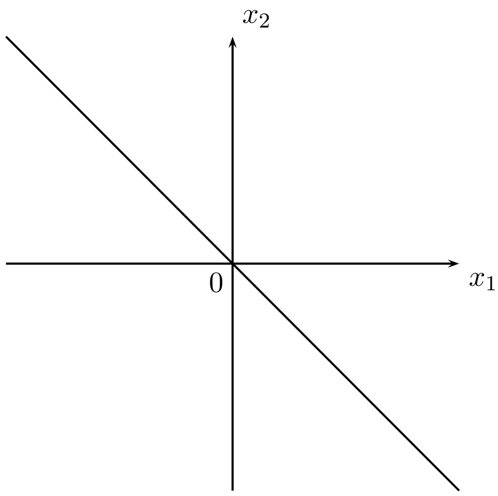


Figure 3(b): The line $x_1 + x_2 = 0$ is a vector space.

NOTE. Points in an n-dimensional space are represented by vectors in $\mathbb{R}^n$. When we say a set of vectors S in $\mathbb{R}^n$ represents a line, or simply S is a line in $\mathbb{R}^n$, we mean S is set of the position vectors of all the points on the line in the n-dimensional space.

In Examples 1, 2 and 3 we have asked whether certain subsets of the vector spaces $\mathbb{R}$ and $\mathbb{R}^2$ are themselves vector spaces. Of course the operations of the systems are the usual addition and the usual multiplication by a scalar. We have seen that it is usually fairly simple to show that a subset is not a vector space, but it will be time-consuming and tedious to show that a given subset is a vector space by checking all ten axioms. We shall now develop a simple general test for determining whether a given subset of a vector space is itself a vector space.

We first make the following definitions.

Definition 1. *A subset S of a vector space V is called a* **subspace** *of V if S is itself a vector space over the same field of scalars as V and under the same rules for addition and multiplication by scalars.*

In addition if there is at least one vector in V which is not contained in S, the subspace S is called a **proper subspace** *of V.*

A simple test for a subspace is given by the following theorem.

Theorem 1 (Subspace Theorem)**.** *A subset S of a vector space V over a field $\mathbb{F}$, under the same rules for addition and multiplication by scalars, is a subspace of V if and only if*

i) The vector $\mathbf{0}$ in V also belongs to S.

ii) S is closed under vector addition, and

iii) S is closed under multiplication by scalars from $\mathbb{F}$.

Proof. We first note that if S is a subspace then it is a vector space.

Conversely, suppose S contains the zero vector and the two closure axioms 1 and 6 are satisfied by the elements of S. Every element of S is an element of V because S is a subset of V. Furthermore, S and V are under the same operations, the vector space axioms 2, 3, 7, 8, 9, 10 are automatically satisfied by all elements of S.

Since S contains the zero vector, if $\mathbf{v} \in S$ then, $\mathbf{0} + \mathbf{v} = \mathbf{0}$ (since this is true in V and hence in S, so axioms 4 follows.

Finally, if $\mathbf{v} \in S$ then $\mathbf{v} \in V$. Hence, from part 3 of Proposition 2 of 6.2, we have $-\mathbf{v} = (-1)\mathbf{v}$. But, as S is closed under multiplication by a scalar, we have $(-1)\mathbf{v} \in S$, and hence $-\mathbf{v} \in S$. Thus, axiom 5 is satisfied for all vectors in S. The proof is complete. $\square$

If we want to check if S is a subspace of V, we should first check if the zero vector of V is in S. If the zero vector is in S we can proceed to verify the two closure axioms. Otherwise, we can draw a conclusion that S is not a subspace of V.

Example 4. Prove that the set

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : 2x_1 - x_2 + 4x_3 = 0 \right\}$$

is a vector subspace of $\mathbb{R}^3$.

Solution. As $2(0) - 0 + 4(0) = 0$, the zero vector of $\mathbb{R}^3$ is in S. We now proceed to show the two closure axioms.

For any vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in S$ and $\lambda \in \mathbb{R}$, we have

$$2u_1 - u_2 + 4u_3 \quad = 0 \qquad (1)$$

$$2v_1 - v_2 + 4v_3 \quad = 0 \qquad (2)$$

If we add (1) and (2), we have

$$(2u_1 - u_2 + 4u_3) + (2v_1 - v_2 + 4v_3) = 0 + 0.$$

Hence, we obtain

$$2(u_1 + v_1) - (u_2 + v_2) + 4(u_3 + v_3) = 0,$$

and so $\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix} \in S$. Thus S is closed under addition.

Now we multiply both sides of (2) by λ, we have

$$\lambda(2v_1 - v_2 + 4v_3) = \lambda 0 \qquad \text{i.e.} \quad 2(\lambda v_1) - \lambda v_2 + 4(\lambda v_3) = 0.$$

Hence $\lambda \mathbf{v} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix} \in S$ and so S is closed under multiplication by a scalar.

By the Subspace Theorem, the set S is a vector subspace of $\mathbb{R}^3$. ◊

Example 5. Prove that a line in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if and only if it passes through the origin.

Solution. Suppose that S represents a line in $\mathbb{R}^n$. If $\mathbf{0} \notin S$, then S is not a subspace. Hence, a line which does not pass through the origin is not a vector subspace.

If S is a line through the origin, we can write

$$S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = t\mathbf{v}, \quad t \in \mathbb{R}\},$$

where $\mathbf{v}$ is a fixed non-zero vector in $\mathbb{R}^n$.

To check if S is a subspace we check the two closure axioms.

Closure under addition. If $\mathbf{x}_1, \mathbf{x}_2 \in S$ then

$$\mathbf{x}_1 = t_1\mathbf{v} \quad \text{and} \quad \mathbf{x}_2 = t_2\mathbf{v} \quad \text{for some } t_1,\, t_2 \in \mathbb{R}.$$

Hence,

$$\mathbf{x}_1 + \mathbf{x}_2 = (t_1 + t_2)\mathbf{v} = t'\mathbf{v},$$

where $t' = t_1 + t_2 \in \mathbb{R}$. Thus, $\mathbf{x}_1 + \mathbf{x}_2 \in S$, and hence S is closed under addition.

Closure under multiplication by a scalar. We have

$$\mathbf{x} = t\mathbf{v} \quad \text{for some} \quad t \in \mathbb{R},$$

and hence, if $\lambda \in \mathbb{R}$,

$$\lambda\mathbf{x} = \lambda(t\mathbf{v}) = (\lambda t)\mathbf{v} = t''\mathbf{v},$$

where $t'' = \lambda t \in \mathbb{R}$. Hence $\lambda\mathbf{x} \in S$, and thus S is closed under multiplication by a scalar.

Therefore, by the Subspace Theorem, the line S is a subspace of $\mathbb{R}^n$ if it passes through the origin. The proof is complete. $\Diamond$

A similar result to that given for lines in Example 5 also holds for planes.

Example 6. Prove that a plane in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if and only if it passes through the origin.

SOLUTION. If the plane does not pass through the origin, then it does not contain the zero vector, and hence is not a subspace.

If the plane passes through the origin, then it is represented by

$$S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2, \quad s_1, s_2 \in \mathbb{R}\},$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are fixed, non-parallel vectors in $\mathbb{R}^n$.

We now check closure under addition and under multiplication by a scalar.

Closure under addition. If $\mathbf{x}_1, \mathbf{x}_2 \in S$, then

$$\mathbf{x}_1 = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \quad \text{and} \quad \mathbf{x}_2 = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \quad \text{for some } s_1,\ s_2,\ t_1,\ t_2 \in \mathbb{R}.$$

Therefore

$$\mathbf{x}_1 + \mathbf{x}_2 = (s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2 = s\mathbf{v}_1 + t\mathbf{v}_2,$$

where $s = s_1 + t_1$ and $t = s_2 + t_2$ are both real numbers. Thus $\mathbf{x}_1 + \mathbf{x}_2 \in S$, and hence S is closed under addition.

The proof that S is closed under multiplication by a scalar is similar and is left as an exercise. $\Diamond$

In practice, some of the most important subspaces of $\mathbb{R}^n$ are connected with systems of linear equations, that is, with the matrix equation $A\mathbf{x} = \mathbf{b}$. Here is an important example of this. follows.

Example 7. Let A be an $m \times n$ matrix with real entries. Show that the subset S of $\mathbb{R}^n$ which consists of all solutions of the matrix equation $A\mathbf{x} = \mathbf{b}$ for given $\mathbf{b} \in \mathbb{R}^m$ is a subspace of $\mathbb{R}^n$ if and only if $\mathbf{b} = \mathbf{0}$.

Formally, the set of all solutions of $A\mathbf{x} = \mathbf{b}$ is given by

$$S = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}.$$

SOLUTION. We first consider the case $\mathbf{b} \neq \mathbf{0}$. Then $\mathbf{0} \in \mathbb{R}^n$ is not a solution of $A\mathbf{x} = \mathbf{b}$ as $A\mathbf{0} = \mathbf{0} \neq \mathbf{b}$, and hence S does not contain the zero vector. Thus S is not a subspace.

We next examine the case $\mathbf{b} = \mathbf{0}$. Then S is the set of solutions of $A\mathbf{x} = \mathbf{0}$. We use the Subspace Theorem to show that S is a subspace.

Closure under addition. If $\mathbf{x} \in S$ and $\mathbf{y} \in S$, then $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$, and hence

$$A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0}+\mathbf{0} = \mathbf{0}.$$

Thus $\mathbf{x}+\mathbf{y} \in S$ and S is closed under addition.
Closure under multiplication by a scalar. If $\mathbf{x} \in S$, we have $A\mathbf{x} = \mathbf{0}$, and hence for all $\lambda \in \mathbb{R}$,

$$A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda\mathbf{0} = \mathbf{0}.$$

Thus $\lambda\mathbf{x} \in S$ and S is closed under scalar multiplication. The result is proved. $\diamondsuit$

NOTE. Example 7 shows that the set of solutions of the matrix equation $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$. This subspace, which is of considerable practical importance, is called the **kernel** of the matrix A (see Section 7.4.1).

An important theoretical and practical problem concerning vector spaces is that of finding all their subspaces. For example, it can be shown that the **only** subspaces of $\mathbb{R}^2$ are (1) the origin, (2) lines through the origin, and (3) $\mathbb{R}^2$ itself. Similarly, for $\mathbb{R}^3$ the only subspaces (see Example 10 of Section 6.6) are (1) the origin, (2) lines through the origin, (3) planes through the origin, and (4) $\mathbb{R}^3$ itself. A listing of subspaces can be given for any vector space. However, before we can investigate this problem satisfactorily, we require further machinery. This machinery will be developed in Sections 6.4 and 6.5. In vector spaces other than $\mathbb{R}^n$ it may be difficult to get a good geometric feel for which subsets are subspaces. Nonetheless, the Subspace Theorem allows one a simple way to check whether a certain set is a subspace or not.

Example 8. Let $V = \mathbb{P}(\mathbb{R})$, the set of all real polynomials. Let $\mathbb{P}_2(\mathbb{R})$ denote the set of all real polynomials of degree less than or equal to 2. That is

$$\mathbb{P}_2(\mathbb{R}) = \{p \in \mathbb{P}(\mathbb{R}) : p(x) = a_0 + a_1x + a_2x^2 \text{ for some } a_0, a_1, a_2 \in \mathbb{R}\}.$$

Show that $\mathbb{P}_2(\mathbb{R})$ is a subspace of $\mathbb{P}(\mathbb{R})$.

SOLUTION. Clearly $\mathbb{P}_2(\mathbb{R})$ contains the zero polynomial. Suppose then that $p, q \in \mathbb{P}_2(\mathbb{R})$. Then there exist coefficients $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{R}$ such that

$$p(x) = a_0 + a_1x + a_2x^2, \qquad q(x) = b_0 + b_1x + b_2x^2.$$

Now $(p+q)(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2$ which is another polynomial of degree less than or equal to 2, i.e. $p+q \in \mathbb{P}_2(\mathbb{R})$.

Suppose that $p \in \mathbb{P}_2(\mathbb{R})$ as above and that $\lambda \in \mathbb{R}$. Then $(\lambda p)(x) = (\lambda a_0) + (\lambda a_1)x + (\lambda a_2)x^2$, and so $\lambda p \in \mathbb{P}_2$.

Thus, by the Subspace Theorem, $\mathbb{P}_2(\mathbb{R})$ is a subspace of $\mathbb{P}(\mathbb{R})$. $\diamondsuit$

Suppose that $\mathbb{P}(\mathbb{F})$ is the set of all polynomials over $\mathbb{F}$. We shall show in Section 6.8 that for any n, the set $\mathbb{P}_n(\mathbb{F})$ consisting of all polynomials over $\mathbb{F}$ of degree less than or equal to n is also a subspace of $\mathbb{P}(\mathbb{F})$.

Example 9. Let S denote the set of all real polynomials of degree exactly 3. Show that S is not a subspace of $\mathbb{P}(\mathbb{R})$.

SOLUTION. This set is not closed under either addition or scalar multiplication! For example, the polynomial p given by $p(x) = x^3$ is in S, but $0p = 0$ which does not lie in S. Also, for example, $(x^3 + x^2 + x) + (-x^3 + x + 3) \notin S$. $\diamondsuit$

6.4 Linear combinations and spans

The two fundamental vector space operations are addition and multiplication by a scalar. By combining these two operations we arrive at the important idea of a sum of scalar multiples of vectors. This leads to the ideas of "linear combination" and "span": a linear combination of a given set of vectors is a sum of scalar multiples of the vectors and the span of a given set of vectors is the set of **all** linear combinations of the vectors. We defined linear combinations and span of two vectors in Chapter 2 These ideas are used to develop the parametric vector forms for planes. In particular, the span of two non-parallel vectors is a plane through the origin. In this section, we generalise the ideas to a finite set of vectors.

The formal definition of 'linear combination' is as follows.

Definition 1. *Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a finite set of vectors in a vector space V over a field $\mathbb{F}$. Then a* ***linear combination*** *of S is a sum of scalar multiples of the form*

$$\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n \quad \textit{with} \quad \lambda_1, \dots, \lambda_n \in \mathbb{F}.$$

Example 1. The vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ is a linear combination of the vectors in the set

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \text{ in } \mathbb{R}^2 \text{ because } \begin{pmatrix} 3 \\ -4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 2 \\ 3 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad \Diamond$$

We know that a vector space (and therefore any subspace of a vector space) is closed under addition and multiplication by scalars, so we would expect that it would also be closed under the operation of forming linear combinations. This is confirmed by the following theorem. The proof of the theorem (which uses induction) is left as an exercise (Problem 36).

Proposition 1 (Closure under Linear Combinations)**.** If S is a finite set of vectors in a vector space V, then every linear combination of S is also a vector in V.

The formal definition of 'span' is as follows.

Definition 2. *Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a finite set of vectors in a vector space V over a field $\mathbb{F}$. Then the* ***span*** *of the set S is the set of all linear combinations of S, that is,*

$$\begin{aligned} \operatorname{span}(S) &= \operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_n) \\ &= \{\mathbf{v} \in V : \mathbf{v} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n \quad \textit{for some} \quad \lambda_1, \dots, \lambda_n \in \mathbb{F}\}. \end{aligned}$$

Example 2. The span of a single non-zero vector $\mathbf{v}$ in $\mathbb{R}^n$ is a line through the origin. In Chapter 2e *defined* "the line in $\mathbb{R}^n$ spanned by $\mathbf{v}$" to mean the set

$$S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \lambda \mathbf{v}, \text{ for some } \lambda \in \mathbb{R}\}.$$

This set is just $\operatorname{span}(\mathbf{v})$. $\Diamond$

Example 3. If $\{\mathbf{v}, \mathbf{w}\}$ is a pair of non-zero, non-parallel vectors in $\mathbb{R}^n$ then $\operatorname{span}(\mathbf{v}, \mathbf{w})$ is a plane containing the origin. $\Diamond$

The following important theorem tells us that the span of a finite non-empty set of vectors in a vector space V is not only a subset of V, it is always a *subspace* of V.

Theorem 2 (A span is a subspace)**.** *If S is a finite, non-empty set of vectors in a vector space V, then* $\operatorname{span}(S)$ *is a subspace of V. Further,* $\operatorname{span}(S)$ *is the smallest subspace containing S (in the sense that* $\operatorname{span}(S)$ *is a subspace of every subspace which contains S).*

Proof. We first note that $\mathbf{0} \in S$, since we may take each scalar to be zero. Proposition 1 tells us that every linear combination of S is a vector in V, so $\operatorname{span}(S)$ is a subset of V.

To prove that $\operatorname{span}(S)$ is a *subspace* we will use the Subspace Theorem, so we set out to prove that $\operatorname{span}(S)$ is closed under addition and under multiplication by scalars. Let S be the set

$$S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$$

where all $\mathbf{v}_j$ belong to V.

To show closure under addition, suppose $\mathbf{u}, \mathbf{w} \in \operatorname{span}(S)$. Then

$$\begin{aligned} \mathbf{u} &= \lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n \quad \text{for some } \lambda_1, \dots, \lambda_n \in \mathbb{F} \quad \text{and} \\ \mathbf{w} &= \mu_1\mathbf{v}_1 + \cdots + \mu_n\mathbf{v}_n \quad \text{for some } \mu_1, \dots, \mu_n \in \mathbb{F}, \\ \text{so} \quad \mathbf{u} + \mathbf{w} &= (\lambda_1 + \mu_1)\mathbf{v}_1 + \cdots + (\lambda_n + \mu_n)\mathbf{v}_n \quad \text{with } \lambda_1 + \mu_1, \dots, \lambda_n + \mu_n \in \mathbb{F}. \end{aligned}$$

This shows that $\mathbf{u} + \mathbf{w}$ belongs to $\operatorname{span}(S)$, so $\operatorname{span}(S)$ is closed under addition. To prove closure under multiplication by a scalar, suppose $\mathbf{u} \in \operatorname{span}(S)$ and $\lambda \in \mathbb{F}$. Then

$$\begin{aligned} \lambda\mathbf{u} &= \lambda(\lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n) \\ &= (\lambda\lambda_1)\mathbf{v}_1 + \cdots + (\lambda\lambda_n)\mathbf{v}_n, \end{aligned}$$

where $\lambda\lambda_1, \dots, \lambda\lambda_n \in \mathbb{F}$. This shows that $\lambda\mathbf{u}$ belongs to $\operatorname{span}(S)$, so $\operatorname{span}(S)$ is closed under multiplication by scalars.

We have now proved that $\operatorname{span}(S)$ is a subspace of V. To show that it is the *smallest* subspace of V containing S, suppose W is any subspace of V containing S. Then W is itself a vector space containing S and, by what we have just proved, $\operatorname{span}(S)$ is a subspace of W. This completes the proof by showing that $\operatorname{span}(S)$ is a subspace of every subspace of V containing S. $\square$

Example 4. If $\mathbf{v}$ is a non-zero vector in $\mathbb{R}^3$, then the line $\operatorname{span}(\mathbf{v})$ is a subspace of every vector space which contains $\mathbf{v}$. In particular, it is a subspace of $\mathbb{R}^3$ and of every plane through the origin parallel to $\mathbf{v}$. Further, there is no subset of the line $\operatorname{span}(\mathbf{v})$ which both contains $\mathbf{v}$ and is a vector space. Thus, for example, a line segment cannot be a vector space. We have already seen a special case of this result in Example 1 of Section 6.3 and Figure 1. $\Diamond$

We often need to find a set S in a vector space V such that $\operatorname{span}(S)$ is the whole of V.

Definition 3. *A finite set S of vectors in a vector space V is called a* ***spanning set*** *for V if* $\operatorname{span}(S) = V$ *or equivalently, if every vector in V can be expressed as a linear combination of vectors in S.*

Note also that we often say that "S spans V" instead of "S is a spanning set for V".

Example 5. As shown in Chapter 2 every geometric vector $\mathbf{a}$ in three dimensions can be written as a linear combination

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad \text{for} \quad a_1, a_2, a_3 \in \mathbb{R},$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors along the directions of the three coordinate axes. Therefore $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a spanning set for the vector space of all geometric vectors in three dimensions. $\Diamond$

Example 6. The set $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \right\}$ is a spanning set of the vector space

$$\left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \lambda \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R} \right\}.$$

The set $S' = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}$ also spans the above vector space. Obviously, the third vector in S' is the sum of the other two, so $\operatorname{span}(S) = \operatorname{span}(S')$. Thus the third vector in S' is somewhat redundant. $\Diamond$

Example 7. Let $\mathbf{v}$ be a fixed non-zero vector in $\mathbb{R}^n$. The spanning set of the vector space $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \lambda\mathbf{v},\ \lambda \in \mathbb{R}\}$ is $\{\mathbf{v}\}$. $\Diamond$

Example 8. Every vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ can be written as $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$. This expresses $\mathbf{x}$ as a linear combination of the set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, where

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ \ldots, \ \mathbf{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Thus $\mathbb{R}^n = \operatorname{span}(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ and the set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ spans $\mathbb{R}^n$. $\Diamond$

Example 9. Let $\mathbb{P}_n$ denote the space of polynomials of degree less than or equal to n. Every polynomial $p \in \mathbb{P}_n$ can be written as a linear combination of the polynomials $\{1, x, x^2, \ldots, x^n\}$, so $\mathbb{P}_n = \operatorname{span}\left(1, x, x^2, \ldots, x^n\right)$. We shall see later that there is no *finite* set of vectors whose span is all of $\mathbb{P}$ (the vector space of *all* polynomials). $\Diamond$

6.4.1 Matrices and spans in $\mathbb{R}^m$

We want to have an effective way to tell whether or not a given vector in $\mathbb{R}^m$ belongs to the span of a set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. From the definition of span, we know that $\mathbf{b}$ belongs to span (S) if and only if there are $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that

$$\mathbf{b} = \lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n.$$

This equivalent to the condition that there is at least one solution to the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{b},$$

where $x_1, \dots, x_n$ are the unknowns. This vector equation represents a set of simultaneous linear equations in n unknowns. Therefore the question of whether $\mathbf{b}$ belongs to span (S) is a question of whether or not a particular set of linear equations has a solution. This is the sort of question which we studied in detail in MATH1131/41.

Furthermore, suppose that $\mathbf{v}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix}$, $\dots$, $\mathbf{v}_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$, and $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.
If that A is the $m \times n$ matrix whose columns are the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ then

$$\begin{aligned} A\mathbf{x} &= \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix} \\ &= x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n. \end{aligned}$$

As a result, we have the following proposition.

Proposition 3 (Matrices, Linear Combinations and Spans)**.** If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a set of vectors in $\mathbb{R}^m$ and A is the $m \times n$ matrix whose columns are the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ then

a) a vector $\mathbf{b}$ in $\mathbb{R}^m$ can be expressed as a linear combination of S if and only if it can be expressed in the form $A\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^n$,

b) a vector $\mathbf{b}$ in $\mathbb{R}^m$ belongs to span (S) if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ in $\mathbb{R}^n$.

Example 10. For the set of three vectors

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 5 \\ 3 \\ 6 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} -2 \\ -3 \\ -5 \\ -6 \end{pmatrix} \in \mathbb{R}^4, \quad \text{we let} \quad A = \begin{pmatrix} 0 & 1 & -2 \\ 5 & 3 & -3 \\ 3 & 4 & -5 \\ 6 & 5 & -6 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

By expanding each side, it can easily be checked that

$$A\mathbf{x} = \begin{pmatrix} & & x_2 & - & 2x_3 \\ 5x_1 & + & 3x_2 & - & 3x_3 \\ 3x_1 & + & 4x_2 & - & 5x_3 \\ 6x_1 & + & 5x_2 & - & 6x_3 \end{pmatrix} = x_1 \begin{pmatrix} 0 \\ 5 \\ 3 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 3 \\ 4 \\ 5 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -3 \\ -5 \\ -6 \end{pmatrix}.$$

In particular,

$$A\begin{pmatrix}1\\1\\3\end{pmatrix} = 1\begin{pmatrix}0\\5\\3\\6\end{pmatrix} + 1\begin{pmatrix}1\\3\\4\\5\end{pmatrix} + 3\begin{pmatrix}-2\\-3\\-5\\-6\end{pmatrix} = \begin{pmatrix}-5\\-1\\-8\\-7\end{pmatrix}.$$

Let us denote the vector $\begin{pmatrix}-5\\-1\\-8\\-7\end{pmatrix}$ by $\mathbf{b}$. Hence $\mathbf{x} = \begin{pmatrix}1\\1\\3\end{pmatrix}$ is a solution of $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{b}$ can be written as the linear combination $\mathbf{v}_1 + \mathbf{v}_2 + 3\mathbf{v}_3$. $\diamond$

Example 11. If A is an $m \times n$ matrix and $\mathbf{e}_j$ is the jth standard basis vector in $\mathbb{R}^n$ then

$$A\mathbf{e}_j = \mathbf{a}_j,$$

where $\mathbf{a}_j$ is the jth column of A. In the case of the matrix A of the last example, we find by direct matrix multiplication that

$$A\mathbf{e}_1 = \begin{pmatrix}0&1&-2\\5&3&-3\\3&4&-5\\6&5&-6\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\5\\3\\6\end{pmatrix} = \mathbf{a}_1;\quad A\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\3\\4\\5\end{pmatrix} = \mathbf{a}_2;\quad A\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\-3\\-5\\-6\end{pmatrix} = \mathbf{a}_3.$$

$\diamond$

When applying the results of Proposition 3, it is convenient to have a special name for the subspace of $\mathbb{R}^m$ spanned by the columns of a given $m \times n$ matrix.

Definition 4. *The subspace of $\mathbb{R}^m$ spanned by the columns of an $m \times n$ matrix A is called the **column space** of A and is denoted by* $\operatorname{col}(A)$.

6.4.2 Solving problems about spans

By Proposition 3, a vector $\mathbf{b}$ in $\mathbb{R}^m$ lies in the span of a set $S = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ in $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution, where A is the matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$. The following examples show how to apply this knowledge to problems about spans in $\mathbb{R}^m$.

Example 12. Is the vector $\mathbf{b} = \begin{pmatrix}1\\4\\1\\2\end{pmatrix}$ in the span of the set $S = \left\{\begin{pmatrix}1\\3\\4\\2\end{pmatrix}, \begin{pmatrix}-4\\-8\\-12\\6\end{pmatrix}\right\}$?

In geometric terms, the question is asking whether the point $(1, 4, 1, 2)$ lies on the plane through the origin parallel to $\begin{pmatrix}1\\3\\4\\2\end{pmatrix}$ and $\begin{pmatrix}-4\\-8\\-12\\6\end{pmatrix}$.

Solution. Let A be the matrix whose columns are the members of S, so

$$A = \begin{pmatrix} 1 & -4 \\ 3 & -8 \\ 4 & -12 \\ 2 & 6 \end{pmatrix}.$$

As a consequence of Proposition 3, we know that $\mathbf{b}$ belongs to span (S) if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution. We form the augmented matrix $(A|\mathbf{b})$ for this system and reduce it to row-echelon form.

$$\left(\begin{array}{cc|c} 1 & -4 & 1 \\ 3 & -8 & 4 \\ 4 & -12 & 1 \\ 2 & 6 & 2 \end{array}\right) \xrightarrow[\substack{R_3 = R_3 - 4R_1 \\ R_4 = R_4 - 2R_1}]{R_2 = R_2 - 3R_1} \left(\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 4 & 1 \\ 0 & 4 & -3 \\ 0 & 14 & 0 \end{array}\right)$$

$$\xrightarrow[R_4 = R_4 - \frac{7}{2}R_2]{R_3 = R_3 - R_2} \left(\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & -4 \\ 0 & 0 & -\frac{7}{2} \end{array}\right) \xrightarrow{R_4 = R_4 - \frac{7}{8}R_3} \left(\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{array}\right).$$

Since the right hand column is a leading column, the system has no solution. Therefore $\mathbf{b}$ does not belong to the span of S. ♢

Example 13. Find conditions which are necessary and sufficient to ensure that a vector $\mathbf{b}$ in $\mathbb{R}^3$ belongs to the span of the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$. Hence, determine if the vector $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \in \text{span}\,(S)$. Then give a geometric interpretation of the span.

Solution. By Proposition 3, the vector $\mathbf{b}$ belongs to span (S) if and only if there is a solution to the system of equations $A\mathbf{x} = \mathbf{b}$, where the three columns of A are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. We reduce the augmented matrix $(A|\mathbf{b})$ to row-echelon form.

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 2 & 1 & 0 & b_2 \\ 3 & -1 & 5 & b_3 \end{array}\right) \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & -1 & 2 & b_2 - 2b_1 \\ 0 & -4 & 8 & b_3 - 3b_1 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 4R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & -1 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 5b_1 - 4b_2 + b_3 \end{array}\right).$$

The system represented by this augmented matrix has a solution if and only if

$$5b_1 - 4b_2 + b_3 = 0.$$

Therefore $\mathbf{b}$ belongs to span (S) if and only if this condition is satisfied.

To check that $\mathbf{v}$ is in the span or not, we substitute the components of $\mathbf{v}$ in the condition. Since $5(2) - 4(1) + 1 = 7 \neq 0$, so $\mathbf{v}$ is not in span (S).

To get a geometric interpretation of this result, note that a vector $\mathbf{b}$ is in the span if and only if its components satisfy the Cartesian equation

$$5x_1 - 4x_2 + x_3 = 0$$

which is a plane through the origin with normal $\begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$. Therefore span (S) is this plane. $\diamondsuit$

Note. As a check, note that each of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ belongs to span $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and should satisfy the above condition. On substituting $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ for $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ we find $5(1) - 4(2) + 3 = 0$, so $\mathbf{v}_1$ does satisfy the above condition. You can check for yourself that $\mathbf{v}_2$ and $\mathbf{v}_3$ also satisfy this condition.

Example 14. Determine whether or not the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, is a spanning set for $\mathbb{R}^3$, where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$ and $\mathbf{v}_4 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$.

Solution. S is a spanning set for $\mathbb{R}^3$ if and only if every vector $\mathbf{b} \in \mathbb{R}^3$ belongs to span (S). By Proposition 3, every vector $\mathbf{b} \in \mathbb{R}^3$ belongs to span (S) if and only if the system $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^3$, where A is the matrix whose columns are the members of S.

By row operations we can reduce the augmented matrix of the system to row-echelon form

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & b_1 \\ 2 & 1 & 0 & 3 & b_2 \\ 3 & -1 & 5 & 5 & b_3 \end{array}\right) \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & b_1 \\ 0 & -1 & 2 & -1 & b_2 - 2b_1 \\ 0 & -4 & 8 & -1 & b_3 - 3b_1 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 4R_2} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & b_1 \\ 0 & -1 & 2 & -1 & b_2 - 2b_1 \\ 0 & 0 & 0 & 3 & b_3 - 4b_2 + 5b_1 \end{array}\right).$$

For every $\mathbf{b} \in \mathbb{R}^3$, the right-hand column is non-leading which means that this system has a solution. This implies that every vector $\mathbf{b} \in \mathbb{R}^3$ belongs to span (S). Hence S *is* a spanning set for $\mathbb{R}^3$. $\diamondsuit$

Note. The equations would still have a solution for all $\mathbf{b} \in \mathbb{R}^3$ if the non-leading column (column 3) were dropped from the row-echelon form matrix. This means that the vector $\mathbf{v}_3$ can be dropped from S and the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ will still span $\mathbb{R}^3$. Thus, in this case,

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4) = \mathbb{R}^3.$$

We shall see that in Section 6.5 Example 6 that in place of $\mathbf{v_3}$ we could drop either $\mathbf{v}_1$ or $\mathbf{v}_2$ from S and obtain the same span. That is

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \text{span}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \text{span}(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4) = \mathbb{R}^3.$$

However, the removal of the vector corresponding to a non-leading column in a row-echelon form matrix gives us a simple criterion to get a subset of S which spans the same subspace span (S). In general, we have the following result.

> *Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a subset of $\mathbb{R}^m$ and A, which is the matrix with the n vectors in S as columns, reduces to a row-echelon from matrix U. If the ith column of U is non-leading, then $\{\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n\}$ spans the same set as S.*

The following example shows that matrix methods can also be used to solve problems about spans in some vector spaces other than $\mathbb{R}^n$.

Example 15. Find conditions on the coefficients of $p \in \mathbb{P}_3(\mathbb{R})$ so that $p \in \operatorname{span}\left(1+x, 1-x^2\right)$.

SOLUTION. Let $p(x) = b_0 + b_1x + b_2x^2 + b_3x^3$ be a polynomial in $\mathbb{P}_3(\mathbb{R})$. From the definition of span, we know that $p \in \operatorname{span}\left(1+x, 1-x^2\right)$ if and only if there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$p(x) = \lambda_1(1+x) + \lambda_2(1-x^2) = (\lambda_1 + \lambda_2) + \lambda_1 x - \lambda_2 x^2.$$

By comparing coefficients, we must have

$$\begin{array}{ccccc} \lambda_1 & + & \lambda_2 & = & b_0 \\ \lambda_1 & & & = & b_1 \\ & & -\lambda_2 & = & b_2 \\ & & 0 & = & b_3. \end{array}$$

This is a system of linear equations in the variables λ_1 and λ_2 and we have to find out which choices of b_0, b_1, b_2, b_3 make it into a system which does have a solution. The augmented matrix for the system is

$$\left(\begin{array}{rr|l} 1 & 1 & b_0 \\ 1 & 0 & b_1 \\ 0 & -1 & b_2 \\ 0 & 0 & b_3 \end{array}\right).$$

This augmented matrix can be reduced to row-echelon form

$$\left(\begin{array}{cc|c} 1 & 1 & b_0 \\ 0 & 1 & b_1 - b_0 \\ 0 & 0 & b_2 - b_1 + b_0 \\ 0 & 0 & b_3 \end{array}\right).$$

This system has a solution if and only if $b_2 - b_1 + b_0 = 0$ and $b_3 = 0$, so these are the conditions under which p belongs to $\operatorname{span}\left(1+x, 1-x^2\right)$. $\diamondsuit$

6.5 Linear independence

Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$ are non-zero vectors. In Chapter 2e saw that span $(\mathbf{v}_1, \mathbf{v}_2)$ represents a plane if $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel to each other, but only a line if they are parallel. Similarly, if $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are given non-zero vectors in $\mathbb{R}^3$ then span $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ represents

i) a line if the three vectors are all parallel to each other,

ii) a plane if they are coplanar, or

iii) the whole of $\mathbb{R}^3$ otherwise.

In this section we shall show how these results can be understood through the ideas of linear independence and linear dependence of a set of vectors.

Definition 1. *Suppose that $S = \{\mathbf{v}_1, \ \dots\ , \ \mathbf{v}_n\}$ is a subset of a vector space. The set S is a* ***linearly independent set*** *if the* ***only*** *values of the scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ for which*

$$\lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n = \mathbf{0} \qquad \text{are} \quad \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.$$

Definition 2. *Suppose that $S = \{\mathbf{v}_1, \ \dots\ , \ \mathbf{v}_n\}$ is a subset of a vector space. The set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a* ***linearly dependent set*** *if it is not a linearly independent set, that is, if there exist scalars $\lambda_1, \dots, \lambda_n$,* ***not all zero****, such that*

$$\lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n = \mathbf{0}.$$

Note. The linear combination

$$\lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n$$

is equal to $\mathbf{0}$ when all the scalars are zero. The essential point of the definition of linear independence is that the **only** way this linear combination is $\mathbf{0}$ is that all the scalars are zero.

Example 1. Show that the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ -6 \\ -9 \\ 5 \end{pmatrix}$ form a linearly independent set.

Solution. Applying the definition, we look for scalars λ_1, λ_2 such that

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ -6 \\ -9 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In order to satisfy this vector equation the scalars must satisfy the four equations

$$\lambda_1 - 3\lambda_2 = 0, \qquad 2\lambda_1 - 6\lambda_2 = 0, \qquad 3\lambda_1 - 9\lambda_2 = 0, \qquad 4\lambda_1 + 5\lambda_2 = 0.$$

Each of the first three equations is satisfied if and only if $\lambda_1 = 3\lambda_2$. By substituting this formula for λ_1 in the fourth equation we get $17\lambda_2 = 0$. Thus the only solution is $\lambda_1 = \lambda_2 = 0$ and this shows that the two given vectors form a linearly independent set. $\diamondsuit$

The vectors in the above example are not parallel because neither is a scalar multiple of the other. This is a special case of the following geometric interpretation of linear dependence for pairs of vectors.

Example 2. Show that two non-zero vectors in $\mathbb{R}^n$ are parallel if and only if they form a linearly dependent set.

SOLUTION. We first show that two parallel vectors form a linearly dependent set. Two non-zero vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ are parallel if one is a (non-zero) scalar multiple of the other, and that is $\mathbf{v}_2 = \lambda \mathbf{v}_1$ for some $\lambda \in \mathbb{R}$. We can rewrite this equation as $\lambda \mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$. The coefficient of $\mathbf{v}_2$ in this expression is -1 (which is non-zero), so this equation proves that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly dependent set.

We next show that two linearly dependent non-zero vectors are parallel. If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly dependent set then there exist λ_1 and λ_2, not both zero, such that

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 = \mathbf{0}.$$

Without loss of generality, we can assume that $\lambda_1 \neq 0$. Dividing by λ_1 and rearranging gives

$$\mathbf{v}_1 = -\frac{\lambda_2}{\lambda_1} \mathbf{v}_2.$$

This shows that $\mathbf{v}_1$ is a scalar multiple of $\mathbf{v}_2$. It also implies that $\lambda_2 \neq 0$ (otherwise $\mathbf{v}_1$ would be $\mathbf{0}$). Hence the two vectors are parallel. $\Diamond$

Example 3. It is easy to verify that

$$3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + (-1) \begin{pmatrix} 5 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

By Definition 2, the set $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 7 \end{pmatrix} \right\}$ is a linearly dependent set. $\Diamond$

Note that no two of the three vectors in the previous example are parallel. The following example gives a geometric interpretation (in terms of coplanarity) for linear dependence of sets of three vectors.

Example 4. Show that three non-zero vectors in $\mathbb{R}^n$ are coplanar if and only if they form a linearly dependent set.

SOLUTION. Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are three non-zero vectors in $\mathbb{R}^n$.

We first show that three coplanar vectors form a linearly dependent set. Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are coplanar. We first consider the case that two of the three vectors are parallel. Without loss of generality, we can assume that $\mathbf{v}_2$ and $\mathbf{v}_3$ are parallel. By Example 2, there exist λ_2, λ_3, not both of them are zero such that

$$\lambda_2 \mathbf{v}_2 + \lambda_3 \mathbf{v}_3 = \mathbf{0}.$$

Hence, we have

$$0\mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \lambda_3 \mathbf{v}_3 = \mathbf{0}.$$

Since, not both λ_2 and λ_3 are zero, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ form a linearly dependent set.

Otherwise, we may assume then that $\mathbf{v}_2$ is not parallel to $\mathbf{v}_3$. Hence, $\mathbf{v}_1$ lies on the plane through the origin parallel to $\mathbf{v}_2$ and $\mathbf{v}_3$. This means that there are scalars λ_2, λ_3 such that

$$\mathbf{v}_1 = \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3.$$

We can rearrange this to get

$$-\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 = \mathbf{0}.$$

At least one coefficient is non-zero (the coefficient of $\mathbf{v}_1$ is -1), so we have shown that the set is linearly dependent.

We now show that if three vectors form a linearly dependent set then they must be coplanar. If the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent then there exist $\lambda_1, \lambda_2, \lambda_3$, not all zero, such that

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 = \mathbf{0}.$$

Without loss of generality we can assume that $\lambda_1 \neq 0$. Then the above equation can be rearranged as

$$\mathbf{v}_1 = -\frac{\lambda_2}{\lambda_1}\mathbf{v}_2 - \frac{\lambda_2}{\lambda_1}\mathbf{v}_3.$$

This shows that $\mathbf{v}_1$ satisfies the parametric vector form

$$\mathbf{x} = \lambda\mathbf{v}_2 + \mu\mathbf{v}_3,$$

which represents either a line or a plane through the origin. In both cases, the three vectors are coplanar. $\Diamond$

6.5.1 Solving problems about linear independence

We have seen that questions about spans in $\mathbb{R}^m$ can be answered by relating them to questions about the existence of solutions for systems of linear equations. The same is true for questions about linear dependence in $\mathbb{R}^m$.

Proposition 1. If $S = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is a set of vectors in $\mathbb{R}^m$ and A is the $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ then the set S is linearly dependent if and only if the system $A\mathbf{x} = \mathbf{0}$ has at least one non-zero solution $\mathbf{x} \in \mathbb{R}^n$.

Proof. As on page 20,

$$A\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n \quad \text{for any vector} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

Therefore $A\mathbf{x} = \mathbf{0}$ has at least one non-zero solution $\mathbf{x} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$ if and only if there are scalars $\lambda_1, \ldots, \lambda_n$, not all zero, such that

$$\lambda_1\mathbf{a}_1 + \cdots + \lambda_n\mathbf{a}_n = \mathbf{0}.$$

In other words, if and only if the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is linearly dependent. $\square$

Example 5. Is the set $S = \left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}$ a linearly independent set?

SOLUTION. Let A be the matrix whose columns are the vectors in S. By Proposition 1, the set S is linearly dependent if and only if the system $A\mathbf{x} = \mathbf{0}$ has at least one non-zero solution. We then reduce the augmented matrix $(A|\mathbf{0})$ to row-echelon form $(U|\mathbf{0})$.

$$\left(\begin{array}{rrr|r} 1 & -2 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 4 & 2 & 2 & 0 \end{array}\right) \xrightarrow[\substack{R_3 = R_3 - 2R_1 \\ R_4 = R_4 - 4R_1}]{R_2 = R_2 - 3R_1} \left(\begin{array}{rrr|r} 1 & -2 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 10 & 2 & 0 \end{array}\right)$$

$$\xrightarrow[R_4 = \frac{1}{2}R_4]{R_2 = \frac{1}{5}R_2} \left(\begin{array}{rrr|r} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 5 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R_4 = R_4 - 5R_2]{R_3 = R_3 - 4R_2} \left(\begin{array}{rrr|r} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_4 = R_4 - R_3} \left(\begin{array}{rrr|r} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

There are no non-leading columns in U, so the system has a unique solution, namely $\mathbf{x} = \mathbf{0}$. Since there are no non-zero solutions, the set S is linearly independent. $\diamondsuit$

Example 6. Suppose that $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$ and $\mathbf{v}_4 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$.

a) Prove that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly independent set.

b) Find all possible ways of writing $\mathbf{0}$ as a linear combination of the vectors in S.

c) Find a linearly independent subset of S with the same span as S.

SOLUTION.

a) Let A be the matrix with $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ as columns. As seen in the previous example, elementary row operations do not affect the zero right-hand column. To see whether or not there is any non-zero solution for the equation $A\mathbf{x} = \mathbf{0}$, we can simply reduce the matrix A

and an equivalent row-echelon form U.

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 0 & 3 \\ 3 & -1 & 5 & 5 \end{pmatrix} \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 2 & -1 \\ 0 & -4 & 8 & 1 \end{pmatrix}$$
$$\xrightarrow{R_3 = R_3 - 4R_2} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

The row-echelon form matrix U has a non-leading column, the homogeneous system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. There must be solutions other than the zero solution. Therefore the given set S is **linearly dependent**.

b) The complete solution of $A\mathbf{x} = \mathbf{0}$ can be found by back substitution, and it is $x_4 = 0$, $x_3 = \lambda$, $x_2 = 2\lambda$, and $x_1 = -\lambda$. By substituting for x_1, x_2, x_3, x_4 in the original linear combination we get

$$\lambda(-\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3) + 0\mathbf{v}_4 = \mathbf{0},$$

which gives all possible ways of writing $\mathbf{0}$ as a linear combination of the vectors in S.

c) Choosing $\lambda = 1$, we have

$$-\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}, \qquad \text{i.e.} \quad \mathbf{v}_3 = -\mathbf{v}_1 + 2\mathbf{v}_2.$$

This shows that $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4) = \operatorname{span}(S)$. On the other hand, if we remove the third column from A, we can reduce the matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & -1 & 5 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 3 \end{pmatrix}.$$

Hence the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a linearly independent subset of S with the same span as S.

Note also that $\mathbf{v}_1 = 2\mathbf{v}_2 + \mathbf{v}_3$ and $\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_3$, so both sets $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$ have the same span as S. It is not difficult to check that these two sets are also linearly independent. $\Diamond$

The next two examples illustrate the fact that matrix methods can be used in vector spaces other than $\mathbb{R}^m$.

Example 7. Show that the set $\{1 + x, 2 - x\}$ is linearly independent in the vector space $\mathbb{P}(\mathbb{R})$ of all polynomials.

Solution. Suppose that $\lambda_1(x + 1) + \lambda_2(2 - x) = 0$ for all $x \in \mathbb{R}$. Expanding the left-hand-side gives

$$(\lambda_1 + 2\lambda_2) + (\lambda_1 - \lambda_2)x = 0.$$

Comparing coefficients shows that we must have

$$\begin{aligned} \lambda_1 + 2\lambda_2 &= 0 \\ \lambda_1 - \lambda_2 &= 0. \end{aligned}$$

The augmented matrix for this system is

$$\left(\begin{array}{rr|r} 1 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right),$$

which can be reduced to the row-echelon form

$$\left(\begin{array}{rr|r} 1 & 2 & 0 \\ 0 & -3 & 0 \end{array}\right).$$

There are no non-leading variables here, so the only solution is $\lambda_1 = \lambda_2 = 0$. Therefore this set of polynomials is linearly independent. $\diamondsuit$

Example 8. Is the set $\{1+x, 2-x, -1+2x\}$ a linearly independent subset of the vector space $\mathbb{P}(\mathbb{R})$?

SOLUTION. Suppose that $\lambda_1(x+1) + \lambda_2(2-x) + \lambda_3(-1+2x) = 0$ for all $x \in \mathbb{R}$. Expanding the left-hand-side gives

$$(\lambda_1 + 2\lambda_2 - \lambda_3) + (\lambda_1 - \lambda_2 + 2\lambda_3)x = 0.$$

Comparing coefficients shows that we must have

$$\begin{aligned} \lambda_1 + 2\lambda_2 - \lambda_3 &= 0 \\ \lambda_1 - \lambda_2 + 2\lambda_3 &= 0. \end{aligned}$$

The augmented matrix for this system is

$$\left(\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right),$$

which can be reduced to the row-echelon form

$$(U|\,\mathbf{0}) = \left(\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right).$$

The third column in U is non-leading so there must be some solutions with λ_1, λ_2 and λ_3 not all zero. Therefore the given set of polynomials is linearly dependent. $\diamondsuit$

NOTE. Although it is not required here, it is a good idea to find a specific nonzero solution and check that the appropriate linear combination is zero. In this example we could use back substitution to get a solution $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 1$. Checking shows that indeed

$$-1(1+x) + 1(2-x) + 1(-1+2x) = 0$$

for all $x \in \mathbb{R}$.

6.5.2 Uniqueness and linear independence

The following theorem gives one reason why the idea of linear independence is important.

Theorem 2 (Uniqueness of Linear Combinations). *Let S be a finite, non-empty set of vectors in a vector space and let $\mathbf{v}$ be a vector which can be written as a linear combination of S. Then the values of the scalars in the linear combination for $\mathbf{v}$ are unique if and only if S is a linearly independent set.*

Proof. It turns out to be easier to prove the theorem by proving the equivalent result that:

> *The values of the scalars in the linear combination are non-unique if and only if S is a linearly dependent set.*

Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. Suppose first that $\mathbf{v}$ has two expressions as a linear combination of S, namely

$$\mathbf{v} = \lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n \qquad \text{and} \qquad \mathbf{v} = \mu_1\mathbf{v}_1 + \cdots + \mu_n\mathbf{v}_n.$$

Subtracting the second equation from the first equation gives

$$(\lambda_1 - \mu_1)\mathbf{v}_1 + \cdots + (\lambda_n - \mu_n)\mathbf{v}_n = \mathbf{0}. \tag{\#}$$

If the two sets of scalars are not identical then there must be at least one value of j (with $1 \leqslant j \leqslant n$) such that $\lambda_j - \mu_j \neq 0$. This means that there is at least one non-zero scalar coefficient in (#) and therefore the set S is a linearly dependent set. Conversely, if S is a linearly dependent set then there are scalars $\alpha_1, \dots, \alpha_n$, not all zero, such that

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}.$$

If $\mathbf{v} = \lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n$ is any expression for $\mathbf{v}$ as a linear combination of S then we can get a second expression by saying

$$\begin{aligned}\mathbf{v} &= \mathbf{v} + \mathbf{0} \\ &= (\lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n) + (\alpha_1\mathbf{v}_1 + \cdots + \alpha\mathbf{v}_n) \\ &= (\lambda_1 + \alpha_1)\mathbf{v}_1 + \cdots + (\lambda_n + \alpha_n)\mathbf{v}_n.\end{aligned}$$

The coefficients in this expression are not all the same as the coefficients in the first expression because $\alpha_1, \dots, \alpha_n$ are not all zero. Therefore the coefficients in an expression for $\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_n$ are not unique. $\square$

The following is a numerical example of the result of Theorem 2.

Example 9. As shown in Example 6, the set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly dependent set, where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$, and $\mathbf{v}_4 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$. Show that $\mathbf{b} = \begin{pmatrix} 7 \\ 7 \\ -4 \end{pmatrix}$ belongs to span (S) and then check that the linear combination for $\mathbf{b}$ in terms of S is not unique.

SOLUTION. We know that $\mathbf{b} \in \text{span}(S)$ if and only if the system $A\mathbf{x} = \mathbf{b}$, where A is the matrix with the vectors in S as columns. We then reduce $(A|\mathbf{b})$ to a row-echelon form $(U|\mathbf{y})$.

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 7 \\ 2 & 1 & 0 & 3 & 7 \\ 3 & -1 & 5 & 5 & -4 \end{array}\right) \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 7 \\ 0 & -1 & 2 & -1 & -7 \\ 0 & -4 & 8 & 1 & -25 \end{array}\right)$$
$$\xrightarrow{R_3 = R_3 - 4R_2} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 7 \\ 0 & -1 & 2 & -1 & -7 \\ 0 & 0 & 0 & 3 & 3 \end{array}\right)$$

The right-hand-side column is not a leading column, so the system does have a solution and $\mathbf{b}$ must be in span (S). The third column in the row-echelon form is non-leading, so the system has infinitely many solutions and hence there are infinitely many expressions for $\mathbf{b}$ as a linear combination of S.
$\diamondsuit$

NOTE. If we want to find these expressions explicitly, we can apply back substitution to the row-echelon form and find the general solution $x_4 = 1$, $x_3 = \lambda$, $x_2 = 6 + 2\lambda$ and $x_1 = -1 - \lambda$, where λ is an arbitrary real parameter. Therefore

$$\begin{pmatrix} 7 \\ 7 \\ -4 \end{pmatrix} = (-1 - \lambda) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (6 + 2\lambda) \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}.$$

We found in Example 6 that if $\mathbf{v}_3$ (which is a non-leading variable in this system) is dropped then the resulting set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is linearly independent.

In this case we find that $\begin{pmatrix} 7 \\ 7 \\ -4 \end{pmatrix}$ is a *unique* linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_4$. In fact, by putting $x_3 = \lambda = 0$ in the above expression we get

$$\begin{pmatrix} 7 \\ 7 \\ -4 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 6 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}.$$

6.5.3 Spans and linear independence

Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are non-zero vectors in $\mathbb{R}^n$. We have seen in Chapter 2hat span $(\mathbf{v}_1, \mathbf{v}_2)$ represents a plane if the vectors are not parallel (in other words, if they are linearly independent), whereas the span represents a line if they are parallel (in other words, linearly dependent). An equivalent way of expressing this result is to say that if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly dependent set then

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2) = \text{span}(\mathbf{v}_1) = \text{span}(\mathbf{v}_2).$$

For three non-zero vectors in $\mathbb{R}^n$, we have also seen that span $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ reduces to either a plane or a line when the three vectors form a linearly dependent set. In either of these cases it is possible to drop at least one vector from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ without changing the span of the set.

In this section we show that similar results are true for the span of any number of vectors in any vector space. To do this, we need two important results. The first, which is a generalisation of the results in Examples 2 and 4, is as follows.

Theorem 3. *A set of vectors S is a linearly independent set if and only if no vector in S can be written as a linear combination of the other vectors in S, that is, if and only if no vector in S is in the span of the other vectors in S.*

NOTE. The theorem is equivalent to:

> *A set of vectors S is a linearly dependent set if and only if at least one vector in S is in the span of the other vectors in S.*

Proof. It is easier to prove the alternative statement because in that case the method of proof can closely follow the solution given for Example 4.

If some vector $\mathbf{v}_i$ in $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is in the span of the other vectors in S then there are scalars $\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_n$ such that

$$\mathbf{v}_i = \mu_1\mathbf{v}_1 + \cdots + \mu_{i-1}\mathbf{v}_{i-1} + \mu_{i+1}\mathbf{v}_{i+1} + \cdots + \mu_n\mathbf{v}_n.$$

An obvious rearrangement then gives

$$\mu_1\mathbf{v}_1 + \cdots - \mathbf{v}_i + \cdots + \mu_n\mathbf{v}_n = \mathbf{0}.$$

At least one coefficient in this expression (the coefficient -1 for $\mathbf{v}_i$) is non-zero, so the set is linearly dependent.

Conversely, if S is a linearly dependent set then there are scalars $\lambda_1, \ldots, \lambda_n$, not all zero, such that

$$\lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n = \mathbf{0}.$$

Let i be such that $\lambda_i \neq 0$. Then we can solve for $\mathbf{v}_i$ in the preceding equation to obtain

$$\mathbf{v}_i = -\frac{1}{\lambda_i}(\lambda_1\mathbf{v}_1 + \cdots + \lambda_{i-1}\mathbf{v}_{i-1} + \lambda_{i+1}\mathbf{v}_{i+1} + \cdots + \lambda_n\mathbf{v}_n),$$

This shows that $\mathbf{v}_i$ is in the span of the other vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n\}$, so the proof is complete. $\square$

Example 10. The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ in Example 6 is a linearly dependent set, and we found that

$$-\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}.$$

We saw that $\mathbf{v}_3 = \mathbf{v}_1 - 2\mathbf{v}_2$. Hence, $\mathbf{v}_3 \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$, which implies $\mathbf{v}_3 \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4)$.

Furthermore, we have $\mathbf{v}_2 \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4)$ and $\mathbf{v}_1 \in \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$. However, $\mathbf{v}_4$ is not in the span of the other three. A geometric interpretation of this result is that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ lie in the same plane, whereas $\mathbf{v}_4$ does not lie in this plane. $\lozenge$

The second key result that we need is as follows.

> *If a vector is added to a set then the span of the new set is equal to the span of the original set if and only if the additional vector is in the span of the original set.*

Formally, we have the following theorem.

Theorem 4. *If S is a finite subset of a vector space V and the vector $\mathbf{v}$ is in V, then*

$$\operatorname{span}(S \cup \{\mathbf{v}\}) = \operatorname{span}(S) \text{ if and only if } \mathbf{v} \in \operatorname{span}(S).$$

[**X**] *Proof.* Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ so that $S \cup \{\mathbf{v}\} = \{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{v}\}$.
Obviously $\mathbf{v} \in \operatorname{span}(S \cup \{\mathbf{v}\})$, so if $\operatorname{span}(S \cup \{\mathbf{v}\}) = \operatorname{span}(S)$ then $\mathbf{v} \in \operatorname{span}(S)$.

To prove the converse, we assume that $\mathbf{v} \in \operatorname{span}(S)$ and prove $\operatorname{span}(S) = \operatorname{span}(S \cup \{\mathbf{v}\})$ by proving firstly that if a vector $\mathbf{u} \in \operatorname{span}(S)$ then $\mathbf{u} \in \operatorname{span}(S \cup \{\mathbf{v}\})$, and secondly that if a vector $\mathbf{u} \in \operatorname{span}(S \cup \{\mathbf{v}\})$ then $\mathbf{u} \in \operatorname{span}(S)$.
For the first proof, suppose that $\mathbf{u} \in \operatorname{span}(S)$. Then

$$\mathbf{u} = \lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n, \qquad \text{for some} \quad \lambda_1, \dots, \lambda_n \in \mathbb{F}.$$

Then $\mathbf{u} = \lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n + 0\mathbf{v}$. Hence $\mathbf{u} \in \operatorname{span}(S \cup \{\mathbf{v}\})$.
For the second proof, suppose that $\mathbf{u} \in \operatorname{span}(S \cup \{\mathbf{v}\})$. Then

$$\mathbf{u} = \lambda\mathbf{v} + \lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n \qquad \text{for some} \quad \lambda, \lambda_1, \dots, \lambda_n \in \mathbb{F}.$$

But $\mathbf{v} \in \operatorname{span}(S)$, and hence

$$\mathbf{v} = \mu_1\mathbf{v}_1 + \cdots + \mu_n\mathbf{v}_n \qquad \text{for some} \quad \mu_1, \dots, \mu_n \in \mathbb{F}.$$

On substituting this linear combination for $\mathbf{v}$ into the previous linear combination for $\mathbf{u}$, we find $\mathbf{u}$ is a linear combination of $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. Hence, $\mathbf{u} \in \operatorname{span}(S)$ and the proof is complete. □

Example 11. For the linearly dependent set of four vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ of Example 6, from Example 10, we have

$$\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4) = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4) = \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$$

which agrees with Theorem 3. ◊

By combining Theorems 3 and 4 we get the following.

> *If S is a linearly dependent set of vectors then it is possible to drop at least one vector from S to obtain a new set with the* **same** *span as S, whereas if S is a linearly independent set then dropping any vector from S results in a new set with a* **smaller** *span than* $\operatorname{span}(()\, S)$.

In formal terms, we have the following theorem.

Theorem 5. *Suppose that S is a finite subset of a vector space. The span of every proper subset of S is a proper subspace of* $\operatorname{span}(S)$ *if and only if S is a linearly independent set.*

Example 12. For the linearly dependent set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ considered in Example 6, we have seen that $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4) = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \mathbb{R}^3$, and the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a linearly independent set.

We have also seen in Example 14 of Section 6.4 that S spans $\mathbb{R}^3$. If we now drop any vector from the linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$, we obtain a set whose span is only a plane in $\mathbb{R}^3$ and not all of $\mathbb{R}^3$. This illustrates Theorem 5. ◊

For use in the next section, we need one more result involving spanning and linear independence.

Theorem 6. *If S is a finite linearly independent subset of a vector space V and $\mathbf{v}$ is in V but not in* span(S) *then $S \cup \{\mathbf{v}\}$ is a linearly independent set.*

Proof. Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $S' = S \cup \{\mathbf{v}\}$. If S' is *not* linearly independent then there exist scalars $\lambda, \lambda_1, \dots, \lambda_n$, not all zero, such that

$$\lambda\mathbf{v} + \lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n = \mathbf{0}. \qquad (*)$$

If $\lambda = 0$ then we must have $\lambda_i \neq 0$ for some $1 \leqslant i \leqslant n$ and this would contradict the assumption that S is linearly independent. Therefore we must have $\lambda \neq 0$ and this means that the equation $(*)$ can be rearranged to express $\mathbf{v}$ as a linear combination of the set S. This contradicts the fact $\mathbf{v}$ is not in span(S), so the supposition that S' was linearly dependent must be false and hence S' is linearly independent. $\square$

6.6 Basis and dimension

In Chapter 2e called the set of vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ the standard "basis" for $\mathbb{R}^n$. We have also talked of a plane as being "two-dimensional" and ordinary space as being "three-dimensional". The main aims of this section are to give precise definitions of the ideas of basis and dimension and to show that these ideas apply to every vector space.

Recall that S is a spanning set for a vector space V when span$(S) = V$, that is, when every vector in V can be written as a linear combination of the vectors in S. We have also shown that if S is a linearly independent set then all linear combinations formed from S are unique. Hence, if S is a **linearly independent spanning set** for V then **every** vector in V can be written as a **unique linear combination** of S.

We have also shown (Theorem 5 of Section 6.5) that a proper subset of a linearly independent set S spans a proper subspace of span(S). This means, in particular, that if S is a **linearly independent spanning set** for a vector space V then **dropping any vector** from S gives a set which is **not a spanning set** for V.

Because of these two properties, linearly independent spanning sets are of fundamental importance in both the theoretical development and the practical applications of vector spaces.

6.6.1 Bases

Definition 1. *A set of vectors B in a vector space V is called a* **basis** *for V if:*

1. *B is a linearly independent set, and*
2. *B is a spanning set for V (that is,* span$(B) = V$*).*

NOTE. We exclude from our discussion the vector space consisting of only the zero vector.

Example 1. The set $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$, where $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, $\dots$, $\mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$, is a linearly independent spanning set for $\mathbb{R}^n$, so this set is a basis for $\mathbb{R}^n$. It is called the **standard basis for** $\mathbb{R}^n$. Each vector $\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ can be written as a **unique** linear combination $a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$. $\Diamond$

Example 2. Show that the set $S = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

SOLUTION. To show that S is a spanning set for $\mathbb{R}^3$, we need to show that every vector $\mathbf{b} \in \mathbb{R}^3$ belongs to span (S). This will be true if and only if the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

has a solution for every right hand side $\mathbf{b} \in \mathbb{R}^3$. The augmented matrix $(A|\mathbf{b})$ can be reduced to the row-echelon form —

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & b_1 \\ 1 & 0 & 1 & b_2 \\ 0 & 1 & -1 & b_3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & b_2 \\ 2 & -1 & 0 & b_1 \\ 0 & 1 & -1 & b_3 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & b_2 \\ 0 & -1 & -2 & b_1 - 2b_2 \\ 0 & 1 & -1 & b_3 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & b_2 \\ 0 & -1 & -2 & b_1 - 2b_2 \\ 0 & 0 & -3 & b_1 - 2b_2 + b_3 \end{array}\right).$$

For all $\mathbf{b} \in \mathbb{R}^3$, the row-echelon matrix has a non-leading right hand column and hence the equation $A\mathbf{x} = \mathbf{b}$ has a solution. Therefore span $(S) = \mathbb{R}^3$.

Moreover, left side of the row-echelon matrix has *no non-leading columns*, so the only solution for a zero right hand side is $x_1 = x_2 = x_3 = 0$. This shows that S is a linearly independent set. We have now proved S is linearly independent spanning set for $\mathbb{R}^3$ and is therefore a basis for $\mathbb{R}^3$. $\Diamond$

Example 3. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, $\mathbf{v}_4 = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$ and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Find a subset of S which is a basis for span (S).

SOLUTION. From the result on page 24, we only need to reduce the matrix with the vectors in S as columns to row-echelon from, then the vectors in S corresponding to the leading columns will span the same set as S.

$$\begin{pmatrix} 1 & 2 & 2 & 1 \\ -1 & 1 & 4 & 5 \\ 2 & 3 & 2 & 0 \end{pmatrix} \xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 + R_1} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 6 & 6 \\ 0 & -1 & -2 & -2 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 + \frac{1}{3}R_2} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 6 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The vectors corresponding to the leading columns are $\mathbf{v}_1$ and $\mathbf{v}_2$. Hence $\{\mathbf{v}_1,\, \mathbf{v}_2\}$ is a spanning set for span (S). Furthermore, if we remove the third and fourth columns in the matrices in the above reduction, we can see that $\{\mathbf{v}_1,\, \mathbf{v}_2\}$ is a linearly independent set. Therefore, this set is a basis for span (S). ◊

Suppose that B is a basis for a finite-dimensional vector space V. Since B is a spanning set, every vector in V can be written as a linear combination of B. Since B is also independent, the linear combination is unique. In formal terms:

Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V over $\mathbb{F}$. Every vector $\mathbf{v} \in V$ can be uniquely written as

$$\mathbf{v} = \lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n, \quad \textit{where } \lambda_1, \ldots, \lambda_n \in \mathbb{F}.$$

Example 4. Write the vector $\mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$ as the unique linear combination of the ordered basis $\left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$ of span $(\mathbf{v}_1, \mathbf{v}_2)$.

Solution. We first write $\mathbf{b}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. From Proposition 3 of Section 6.4 we know that the required scalars are the components of a solution to the system $A\mathbf{x} = \mathbf{b}$, where A is the matrix whose columns are the given vectors $\mathbf{v}_1$, $\mathbf{v}_2$. The augmented matrix $(A|\mathbf{b})$ can be reduced to row-echelon form.

$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 3 & -1 & 5 \end{array}\right) \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & -4 & 8 \end{array}\right) \xrightarrow{R_3 = R_3 - 4R_4} \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

This system has the unique solution $\mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, so $\mathbf{b} = 1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. ◊

Example 5. Show that the set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a basis for $\mathbb{R}^3$

Solution. We have seen in Section 6.4 Example 5 that $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a spanning set. To prove that the set is linearly independent, we use the fact that $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is an orthonormal set of vectors. That is,

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1 \qquad \text{and} \qquad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{i}\cdot\mathbf{k} = 0.$$

If we assume that

$$\mathbf{0} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \tag{\#}$$

then by taking the dot product of (#) with $\mathbf{i}$ we get

$$\begin{aligned} 0 = \mathbf{i}\cdot\mathbf{0} &= \mathbf{i}\cdot(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \\ &= a_1(\mathbf{i}\cdot\mathbf{i}) + a_2(\mathbf{i}\cdot\mathbf{j}) + a_3(\mathbf{i}\cdot\mathbf{k}) \\ &= a_1, \end{aligned}$$

and hence $a_1 = 0$. Similarly, by taking the dot products of (#) with $\mathbf{j}$ and $\mathbf{k}$ in turn, we find that $a_2 = 0$ and $a_3 = 0$. Therefore $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a linearly independent set.

We have now proved that $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a linearly independent spanning set for, and hence a basis for, $\mathbb{R}^3$. $\Diamond$

The set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is an example of a **orthonormal basis**. An orthonormal basis is a basis whose elements are all of length 1 and are mutually orthogonal. The advantage of using an orthonormal basis is that we can write easily any vector as the unique linear combination of the basis by dot product.

Example 6. The set of vectors $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \mathbf{u}_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix},$$

is an orthonormal basis for $\mathbb{R}^3$. Write $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ as the unique linear combination of this basis.

Solution. Suppose that $\mathbf{a} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3$. We could find x_1, x_2, x_3 in the same way as Example 4, but there is a simpler method which uses the orthonormality properties of the basis.

Given that B is orthonormal. That is

$$\mathbf{u}_1 \cdot \mathbf{u}_1 = \mathbf{u}_2 \cdot \mathbf{u}_2 = \mathbf{u}_3 \cdot \mathbf{u}_3 = 1 \qquad \text{and} \qquad \mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0.$$

We have

$$x_1 = \mathbf{u}_1 \cdot (x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3) = \mathbf{u}_1 \cdot \mathbf{a} = \frac{1}{\sqrt{2}}\,a_1 - \frac{1}{\sqrt{2}}\,a_3,$$

and similarly

$$x_2 = \mathbf{u}_2 \cdot \mathbf{a} = \frac{1}{\sqrt{2}}\,a_1 + \frac{1}{\sqrt{2}}\,a_3 \qquad \text{and} \qquad x_3 = \mathbf{u}_3 \cdot \mathbf{a} = -a_2.$$

Therefore, the unique linear combination is given by

$$\mathbf{a} = \frac{1}{\sqrt{2}}\,(a_1 - a_3)\mathbf{u}_1 + \frac{1}{\sqrt{2}}\,(a_1 + a_3)\mathbf{u}_2 - a_2\mathbf{u}_3.$$

$\Diamond$

Example 7. Show that the set $\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathbb{P}_n(\mathbb{R})$. (This is called the **standard basis for** $\mathbb{P}_n(\mathbb{R})$.)

Solution. As we have seen, $\{1, x, x^2, \ldots, x^n\}$ is a spanning set for $\mathbb{P}_n(\mathbb{R})$. This set is also linearly independent because if

$$\lambda_1 1 + \lambda_2 x + \cdots \lambda_{n+1} x^n = 0$$

for all $x \in \mathbb{R}$ then $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1} = 0$. This follows from the theorem that we proved in Chapter 1hich states that two polynomials agree for all values of $x \in \mathbb{R}$ if and only if their corresponding coefficients are equal. $\Diamond$

6.6.2 Dimension

In this section we show that "dimension" can be defined for every vector space which is spanned by a finite set of vectors.

To do this we need two results.

Theorem 1. *The number of vectors in any **spanning** set for a vector space V is always greater than or equal to the number of vectors in any **linearly independent** set in V.*

[**X**] *Sketch Proof.* Suppose $I = \{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ is a linearly independent set and $S = \{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ is a spanning set. Then we can write

$$\begin{array}{rcl} \mathbf{v}_1 & = & a_{11}\,\mathbf{w}_1 \;+ \cdots +\; a_{1n}\mathbf{w}_n \\ & \vdots & \\ \mathbf{v}_m & = & a_{m1}\,\mathbf{w}_1 \;+ \cdots +\; a_{mn}\,\mathbf{w}_n. \end{array} \tag{1}$$

Suppose $m > n$ and consider the matrix $A = (a_{ij})$. A has more rows than columns, so if we reduce A to row-echelon form by the Gaussian elimination algorithm then we must end up with at least one row of zeros. Apart from a row swap to get it into the right position, this row of zeros will have been obtained from some row of our original matrix A, say row R_k, by subtracting from it multiples of other rows or linear combinations of other rows of A. Therefore there are scalars $\alpha_1, \dots, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_m$ such that

$$R_k - (\alpha_1\, R_1 + \cdots + \alpha_{k-1}\, R_{k-1} + \alpha_{k+1}\, R_{k+1} + \cdots + \alpha_m\, R_m) \tag{2}$$

is an all-zero row.

If we use the equations in (1) to express the vector

$$\mathbf{v}_k - (\alpha_1\, \mathbf{v}_1 + \cdots + \alpha_{k-1}\, \mathbf{v}_{k-1} + \alpha_{k+1}\, \mathbf{v}_{k+1} + \cdots + \alpha_m\, \mathbf{v}_m) \tag{3}$$

as a linear combination of $\mathbf{w}_1, \dots, \mathbf{w}_n$ then the coefficient of each $\mathbf{w}_i$ will be the same as the ith entry in the row which is defined by (2). But we know that this row is all zeros, so the vector defined by (3) must be the zero vector. We now have a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_m$ which equals zero and at least one coefficient in this combination (the coefficient of $\mathbf{v}_k$) is non-zero. This is not compatible with the fact that the set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is linearly independent. By assuming that m is greater than n we have reached a contradiction, therefore m must be less than or equal to n. □

The second important theorem which guarantees the existence of a dimension for a vector space is as follows:

Theorem 2. *If a vector space V has a finite basis then every set of basis vectors for V contains the same number of vectors, that is, if $B_1 = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ and $B_2 = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ are two bases for the same vector space V then $m = n$.*

Proof. Using the results of Theorem 1 and the fact that a basis is a linearly independent spanning set, we have

$m \geqslant n$, since B_1 spans V and B_2 is linearly independent, and

$n \geqslant m$, since B_2 spans V and B_1 is linearly independent.

Therefore $m = n$ and the proof is complete. □

Since every basis for a vector space with a finite basis contains exactly the same number of vectors, the following definition makes sense for every vector space with a finite basis.

Definition 2. *If V is a vector space with a finite basis, the* ***dimension*** *of V, denoted by $\dim(V)$, is the number of vectors in any basis for V. V is called a* ***finite dimensional vector space****.*

Example 8. a) $\mathbb{R}^n$ has a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of n vectors, and hence $\dim(\mathbb{R}^n) = n$.

b) The space of geometric vectors in ordinary physical space has a basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ of three vectors and therefore its dimension is 3.

c) The subspace span (S) in Example 3 has a basis of two vectors. The dimension of span (S) is 2.

d) We define the dimension of the vector space consisting only of the zero vector to be 0.

e) The space $\mathbb{P}_n$ of polynomials of degree less than or equal to n has a basis $\{1, x, x^2, \dots, x^n\}$, so $\dim(\mathbb{P}_n) = n + 1$.

The following theorem summarises some useful results connecting spanning sets, linearly independent sets and dimension.

Theorem 3. *Suppose that V is a finite dimensional vector space.*

1. *the number of vectors in any spanning set for V is greater than or equal to the dimension of V;*
2. *the number of vectors in any linearly independent set in V is less than or equal to the dimension of V;*
3. *if the number of vectors in a spanning set is equal to the dimension then the set is also a linearly independent set and hence a basis for V;*
4. *if the number of vectors in a linearly independent set is equal to the dimension then the set is also a spanning set and hence a basis for V.*

Proof. Assume that V is a vector space of dimension n.

The dimension of a vector space is equal to the number of vectors in a basis and a basis is a linearly independent spanning set. Therefore there is a linearly independent set in V which contains n vectors and there is also a spanning set for V which contains n vectors.

(1) and (2) follow from Theorem 1.

3. Assume that a spanning set S contains n vectors. Then, as no spanning set for V can contain fewer than n vectors, there is no proper subset of S which is a spanning set for V. Hence no proper subset of S has the same span as S. Thus, by Theorem 5 of Section 6.5, S is a linearly independent set, and is a basis of V.

4. Assume that $I = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a linearly independent set of n vectors in V and let $\mathbf{v}$ be any vector in V. If $\mathbf{v}$ does not belong to span (I) then, by Theorem 6 of Section 6.5, the set $I \cup \{\mathbf{v}\}$ is linearly independent. This implies that V contains a linearly independent set with $n+1 > \dim(V)$ vectors. This would contradict the result of (2), so we must have $\mathbf{v} \in$ span (I) for all $\mathbf{v} \in V$. Therefore I is a spanning set for V and hence a basis for V. □

Some of the uses of Theorem 3 are illustrated in the next example.

Example 9. a) Obviously, the two vectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ are non-parallel and so they are linearly independent. Hence, the set of two vectors $\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \end{pmatrix} \right\}$ is a basis for the two-dimensional space $\mathbb{R}^2$.

b) The set of 3 linearly independent vectors $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ in Example 2 is a basis for the three-dimensional space $\mathbb{R}^3$.

c) A set of three vectors is not a spanning set for $\mathbb{R}^4$ as $\dim(\mathbb{R}^4) = 4 > 3$.

d) Any set of 10 vectors which spans $\mathbb{R}^{10}$ is a basis for $\mathbb{R}^{10}$ since $\dim(\mathbb{R}^{10}) = 10$.

e) Any linearly independent set of 325 vectors in $\mathbb{R}^{325}$ is a basis for $\mathbb{R}^{325}$ since $\dim(\mathbb{R}^{325}) = 325$.

f) A set of 1200 vectors in $\mathbb{R}^{1209}$ is not a spanning set as $\dim(\mathbb{R}^{1209}) = 1209 > 1200$. $\Diamond$

Example 10. Show that the only subspaces in $\mathbb{R}^3$ are (1) the origin, (2) lines through the origin, (3) planes through the origin, and (4) $\mathbb{R}^3$ itself.

SOLUTION. By part 2 of Theorem 3, no subspace of $\mathbb{R}^3$ can have dimension greater than 3, otherwise a basis for the subspace would be a linearly independent set with more than three members. It follows that the only possible dimensions for subspaces (other than the subspace $\{\mathbf{0}\}$) are 1, 2 and 3.

A subspace of dimension 1 must be of the form $\operatorname{span}(\mathbf{v})$, where $\mathbf{v}$ is non-zero. We know that this represents a line through the origin.

A subspace of dimension 2 must be of the form $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$, where the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. We know that this represents a plane through the origin.

If a subspace is of dimension 3, it must be the whole of $\mathbb{R}^3$ because a basis for it will be a set of 3 linearly independent vectors in $\mathbb{R}^3$ and hence a basis for $\mathbb{R}^3$ itself. $\Diamond$

6.6.3 Existence and construction of bases

In this section we examine the following two problems. Firstly, is it always possible to find a basis for a given vector space V? Secondly, if we know that a basis does exist for a given vector space V, how can we find a basis for V?

For the existence of a basis, we have seen in Example 3 that a finite set of vectors in $\mathbb{R}^3$ contains a subset which is a basis for $\operatorname{span}(S)$. This is generally true for any vector space spanned by a finite set of vectors.

Theorem 4. *If S is a finite non-empty subset of a vector space then S contains a subset which is a **basis** for* $\operatorname{span}(S)$.

*In particular, if V is any non-zero vector space which can be spanned by a **finite** set of vectors then V has a basis.*

Proof. Let S be a finite non-empty set of vectors. If S is linearly independent then it is a basis for span (S) and there is nothing more to be done. If not, then (by Theorem 5 of Section 6.5) there must be a vector which can be dropped from S without changing the span. This gives a new set with fewer vectors which still spans span (S). If this new set is linearly independent, we have a basis. If not, we can again remove a vector and get a smaller set which still spans the same subspace. If we continue in this way we must eventually get a set which is linearly independent and spans span (S) (the process cannot continue indefinitely because the original set S had only a finite number, say n, of members and after n steps we would have no vectors left in the set).

A non-zero vector space V contains the zero vector, so it cannot be spanned by an empty set. Suppose that S is a non-empty finite spanning set for V. By the above result, there exists a subset of S which is a basis for span $(S) = V$. $\square$

This theorem shows that a spanning set can always be converted into a basis by removing some vectors from it. The next theorem proves a result about going in the opposite direction.

Theorem 5. *Suppose that V is a vector space which can be spanned by a finite set of vectors. If S is a linearly independent subset of V then there exists a basis for V which contains S as a subset. In other words, every linearly independent subset of V can be extended to a basis for V.*

Proof. Suppose S is a linearly independent set in V and that V can be spanned by a set of n vectors. If S spans V then there is nothing more to be done. If not, then there is a vector $\mathbf{v} \in V$ which is not in span (S). If we add $\mathbf{v}$ to S then we get a new set $S \cup \{\mathbf{v}\}$ which is still linearly independent (by Theorem 6 of Section 6.5). If this new set spans V then we can stop. Otherwise, we can repeat the previous step and add another vector to get a larger linearly independent set. This process cannot continue beyond n steps, otherwise we would have a linearly independent set with more than n members (which is more than the number of members in a spanning set) and this would contradict Theorem 1. But the process only ends when we get a set which does span V. So eventually we must have a linearly independent spanning set or, in other words, a basis, $\square$

Note carefully that the last two theorems only apply to vector spaces which can be spanned by a finite set of vectors. Any vector space which cannot be spanned by any finite set of vectors is said to be an **infinite-dimensional** vector space. The vector space $\mathbb{P}$ of *all* polynomials is an example of an infinite-dimensional vector space (see Example 22 of Section 6.8).

In the proofs of the last two theorems we used step-by-step procedures to reduce a spanning set to a basis and to extend a linearly independent set to a basis. These procedures could be translated into algorithms for finding a basis but they would not be efficient because they involve re-testing (for linear independence or spanning) each time a vector is added or deleted. In practice, at least in $\mathbb{R}^m$, we can do all the adding or all the deleting at the same time. We form a suitable matrix, reduce it to echelon form and examine the echelon form to see which vectors we should add or delete. The details of the procedures are given in the next two theorems.

Theorem 6 (Reducing a spanning set to a basis in $\mathbb{R}^m$)**.** *Suppose that $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is any subset of $\mathbb{R}^m$ and A is the matrix whose columns are the members of S. If U is a row-echelon form for A and S' is created from S by deleting those vectors which correspond to non-leading columns in U then S' is a basis for* span (S).

Proof. Let U' be the matrix created by deleting any non-leading columns from U and let A' be created by deleting the same-numbered columns from A (so that the columns of A' are the members of S'). The matrix U' has no non-leading columns, so the homogeneous system $A'\mathbf{y} = \mathbf{0}$ has no solutions other than the zero solution. This implies (by Proposition 1 of Section 6.5) that the set S' is linearly independent. In removing non-leading columns from U, we cannot create any new all-zero rows, so the system $A'\mathbf{y} = \mathbf{b}$ has a solution whenever $A\mathbf{x} = \mathbf{b}$ has a solution. This implies (by Proposition 3 of Section 6.4) that S' spans the same subspace as S. This completes the proof that S' is a basis for span$\,(S)$. $\square$

Example 11. Find a basis and the dimension for the subspace of $\mathbb{R}^4$ spanned by the set

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ -6 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ -4 \\ -3 \end{pmatrix} \right\}.$$

Solution. As an exercise, show that the matrix A with the members of S as its columns can be reduced by elementary row operations to echelon form

$$U = \begin{pmatrix} 1 & 2 & -3 & 1 & -2 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The third and fifth columns of U are non-leading, so we remove the third and fifth members from S and get

$$S' = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 3 \\ 6 \end{pmatrix} \right\}$$

as a basis for span$\,(S)$, which is 3 dimensional. $\Diamond$

Note. Do not confuse the dimension of a subspace with the with the dimension of the vector space it lies in. In the above example, span$\,(S)$ is a 3 dimension subspace of $\mathbb{R}^4$. This has nothing to do with $\mathbb{R}^3$.

Example 12. Show that the vectors $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$, $\mathbf{v}_4 = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$ span $\mathbb{R}^3$ and find a basis for $\mathbb{R}^3$ which is a subset of S.

Solution. Suppose that A is the matrix whose columns are the four given vectors. One way to show that S spans $\mathbb{R}^3$ is to reduce the augmented matrix $(A|\mathbf{b})$ to the echelon form as in Section 6.4.1. The augmented matrix involves a column of indeterminate $\mathbf{b}$.

By Theorem 6, we can first find a basis for span$\,(S)$ simply by reducing A to a row-echelon form. If the dimension of the span is 3 then the result follows.

$$\begin{pmatrix} 0 & 2 & 3 & 5 \\ 1 & -1 & 2 & 4 \\ 2 & -2 & 4 & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -1 & 2 & 4 \\ 0 & 2 & 3 & 5 \\ 2 & -2 & 4 & 2 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 - 2R_1} \begin{pmatrix} 1 & -1 & 2 & 4 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & -6 \end{pmatrix}$$

The matrix U has one non-leading column, the third, so we delete the third member from the given set and find that the subset $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis for span (S). However B is a linearly independent set of 3 vectors in $\mathbb{R}^3$, so it is also a basis for $\mathbb{R}^3$. ♢

Theorem 7 (Extending a linearly independent set to a basis in $\mathbb{R}^m$).
Suppose that $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a linearly independent subset of $\mathbb{R}^m$ and A is the matrix whose columns are the members of S followed by the members of the standard basis for $\mathbb{R}^m$. If U is a row-echelon form for A and S' is created by choosing those columns of A which correspond to leading columns in U then S' is a basis for $\mathbb{R}^m$ containing S as a subset.

Proof. Let S'' be the set of $n+m$ vectors from the columns of A. Since S'' includes all the standard basis vectors for $\mathbb{R}^m$, so $\mathbb{R}^m = \text{span}\,(S'')$. By Theorem 6, the set S' is a basis for $\mathbb{R}^m$.

To see that S' contains S we need to prove that the first n columns of U (which correspond to the members of S in A) are leading columns. Let B be the submatrix formed from the first n columns of A and P be the submatrix formed from the first n columns of U. Since A reduces to U, we also have B reduces to P. By Proposition 1 of Section 6.5, the linear independence of S implies $B\mathbf{x} = \mathbf{0}$ has unique solution $\mathbf{x} = \mathbf{0}$. Hence all columns of P, i.e. the first n columns of U, are leading. Hence the result follows. □

Example 13. Find a basis for $\mathbb{R}^4$ containing the members of the linearly independent set

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 10 \\ -5 \end{pmatrix} \right\}.$$

Solution. We form the matrix A whose columns are the members of S followed by the standard basis vectors for $\mathbb{R}^4$ and reduce it to row-echelon form

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 1 & 0 & 0 \\ 4 & 10 & 0 & 0 & 1 & 0 \\ -2 & -5 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{R_3 = R_3 - 4R_1 \\ R_4 = R_4 + 2R_1}]{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 2 & -4 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_4 = R_4 + R_2]{R_3 = R_3 - 2R_2} \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_4 = R_4 + \frac{1}{2}R_3} \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{pmatrix} = U.$$

The first, second, fourth and fifth columns are the leading columns in U, so we take the corresponding columns in A and get a basis

$$S' = \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 10 \\ -5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

for $\mathbb{R}^4$. ♢

NOTE. The procedure stated in the last theorem can also be used in situations where we have a set S which is neither linearly independent nor spanning and we want to find a basis which contains as many members of S as possible. We still form a matrix A from the members of S followed by the standard basis vectors, reduce to echelon form U and pick out from A the columns which correspond to leading columns in U. The only difference is that the first n columns are not necessarily all leading columns (because S is not necessarily linearly independent), so the new set S' does not necessarily include all the members of S.

Example 14. Find a basis for $\mathbb{R}^4$ containing as many as possible of the members of the set $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -3 \\ 6 \end{pmatrix} \right\}$.

SOLUTION. We form the matrix A whose columns are the members of S followed by the standard basis vectors for $\mathbb{R}^4$ and reduce it to row-echelon form.

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 & 0 \\ 4 & 1 & -3 & 0 & 0 & 1 & 0 \\ -2 & 4 & 6 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{R_3 = R_3 - 4R_1 \\ R_4 = R_4 + 2R_1}]{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & -7 & -7 & -4 & 0 & 1 & 0 \\ 0 & 8 & 8 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_4 = R_4 - 8R_2]{R_3 = R_3 + 7R_2} \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -18 & 7 & 1 & 0 \\ 0 & 0 & 0 & 18 & -8 & 0 & 1 \end{pmatrix} \xrightarrow{R_4 = R_4 + R_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -18 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

The first, second, fourth and fifth columns are the leading columns in the row-echelon form matrix, so we take the corresponding columns in A and get a basis

$$S' = \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

for $\mathbb{R}^4$.

Note that this basis only contains two of the three vectors in the original set because S was not linearly independent (Question, what is $\dim(\operatorname{span}(S))$?) $\diamondsuit$

In Chapter 7 we shall need the following proposition which follows from Theorems 3 and 4.

Proposition 8. If V is a finite-dimensional space and W is a subspace of V and $\dim(W) = \dim(V)$ then $W = V$.

Proof. By Theorem 4, there exists B a basis for W. So B is a linearly independent set in V. By Theorem 3 part 4, B is also a basis for V and $V = \operatorname{span}(B) = W$. $\square$

6.7 [X] Coordinate vectors

In Chapter 2 we have seen that the position vector **a** of a point in an n-dimensional space could be represented by a (column) coordinate vector $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$, where the coordinates are the scalars in the linear combination which expresses **a** in terms of the standard basis vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$.

As remarked in the last section, any basis B for a finite-dimensional vector space V over $\mathbb{F}$ has the property that — every vector $\mathbf{v} \in V$ can be written as a unique linear combination of B. If we specify a fixed ordering of the basis $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ then we can associate with every vector **v** the unique n-vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$, where $\mathbf{v} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$.

Note that the order of vectors in B is important because changing the order of the vectors also changes the order of the coefficients and therefore changes the n-vector corresponding to **v**.

This generalise the notion of coordinates of a point in an n-dimensional real space to coordinates of a vector in any finite dimensional vector space. In this case, we can represent a vector in any vector space by a coordinate vector which is an n-vector in $\mathbb{F}^n$. Consequently, we can use all the techniques which we have learnt in the previous sections to study vector spaces by matrices over $\mathbb{F}$.

Now, we introduce the definition of coordinate vectors.

Definition 1. *Let V be an n-dimensional vector space and let the ordered set of vectors $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V. If*

$$\mathbf{v} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$$

then the vector

$$[\mathbf{v}]_B = \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

*is called the **coordinate vector of v with respect to the ordered basis** B.*

Example 1. With respect to the ordered basis $B = \left\{ \begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -6 \\ 2 \\ 1 \\ 4 \end{pmatrix} \right\}$ of $\mathbb{R}^4$, a vector $\mathbf{v} \in \mathbb{R}^4$ has the coordinate vector $[\mathbf{v}]_B = \begin{pmatrix} 1 \\ -3 \\ 4 \\ 2 \end{pmatrix}$. Find **v**.

SOLUTION.

$$\mathbf{v} = 1\begin{pmatrix} 0 \\ 1 \\ 3 \\ -1 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 5 \\ -3 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 4 \\ -1 \\ 0 \\ 2 \end{pmatrix} + 2\begin{pmatrix} -6 \\ 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ -14 \\ 14 \\ 12 \end{pmatrix}.$$

$\diamondsuit$

Example 2. Find the coordinate vector for the vector $\mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$ with respect to the ordered basis $\left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$ of $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$.

SOLUTION. From the result of Example 4 in Section 6.6, the coordinate vector for $\mathbf{b}$ with respect to the ordered basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. $\diamondsuit$

Example 3. The set of vectors $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \mathbf{u}_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix},$$

is an orthonormal basis for $\mathbb{R}^3$. Find the coordinate vector of $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ with respect to this basis.

SOLUTION. From the result of Example 6 in Section 6.6, the required coordinate vector is therefore

$$[\mathbf{a}]_B = \begin{pmatrix} \frac{1}{\sqrt{2}}(a_1 - a_3) \\ \frac{1}{\sqrt{2}}(a_1 + a_3) \\ -a_2 \end{pmatrix}.$$

$\diamondsuit$

One of the most important aspects of coordinate vectors is that by using them we can reduce problems in any finite-dimensional real vector space to problems in $\mathbb{R}^n$, where of course we have powerful matrix techniques available.

Example 4. Find the coordinate vector of $p(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{P}_n(\mathbb{R})$ with respect to the ordered basis $B = \{1, x, x^2, \ldots, x^n\}$.

SOLUTION. Of course there is no need to do any calculation in this case — we already know how to write p as a linear combination of elements of B. The coordinate vector of p with respect to B is $[p]_B = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^{n+1}$. $\diamondsuit$

Things are more difficult if we are given a nonstandard basis for $\mathbb{P}_n(\mathbb{R})$.

Example 5. As a special case of Example 17 in 6.8.3, the set $\mathbb{P}_2(\mathbb{R})$ of all real polynomials which have degree two or less is a vector space. You are given an ordered basis $B = \{1+x, x+x^2, 1+x^2\}$ for $\mathbb{P}_2(\mathbb{R})$. Find the coordinate vector for $p(x) = 1 - x^2$ with respect to the ordered basis B.

SOLUTION. We need to find scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$p(x) = 1 - x^2 = \alpha_1(1+x) + \alpha_2(x+x^2) + \alpha_3(1+x^2).$$

Expanding the right-hand-side and comparing coefficients shows that we must have

$$\begin{aligned} \alpha_1 + \alpha_3 &= 1 \\ \alpha_1 + \alpha_2 &= 0 \\ \alpha_2 + \alpha_3 &= -1. \end{aligned}$$

This reduces the problem to that of solving a system of linear equations in the unknowns α_1, α_2 and α_3. Solving these equations by Gaussian elimination gives the unique solution $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = 0$. Therefore $[p]_B = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. ◊

Because coordinate vectors can be obtained in any finite-dimensional vector space, we can turn problems involving vectors in the vector space into problems involving coordinate vectors in $\mathbb{F}^n$. There are three important results which are fundamental to this approach.

Theorem 1. *If B is an ordered basis for a vector space V over a field $\mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in V$ and $\lambda \in \mathbb{F}$, then*

(*a*) $\mathbf{u} = \mathbf{v}$ *if and only if* $[\mathbf{u}]_B = [\mathbf{v}]_B$, *that is, two vectors are equal if and only if the corresponding coordinate vectors are equal.*

(*b*) $[\mathbf{u}+\mathbf{v}]_B = [\mathbf{u}]_B + [\mathbf{v}]_B$, *that is, the coordinate vector of the sum of two vectors is equal to the sum of the two corresponding coordinate vectors.*

(*c*) $[\lambda\mathbf{v}]_B = \lambda[\mathbf{v}]_B$, *that is, the coordinate vector of a scalar multiple of a vector is equal to the same scalar multiple of the corresponding coordinate vector.*

Proof. **Equality.** If $\mathbf{u}$ and $\mathbf{v}$ have the same coordinates then they are equal to the same linear combination of B and must therefore be equal to each other. Conversely, if $\mathbf{u} = \mathbf{v}$ then, because B is a basis and (by Theorem 2 of Section 6.5) no vector can have two different expressions as a linear combination of a given basis, any expressions for $\mathbf{u}$ and $\mathbf{v}$ as linear combinations of B must be the same. The coefficients in these linear combinations are the coordinates of $\mathbf{u}$ and $\mathbf{v}$, so $[\mathbf{u}]_B = [\mathbf{v}]_B$.

Addition. Let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $[\mathbf{u}]_B = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$ and $[\mathbf{v}]_B = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$. By the definition of coordinate vectors, we have

$$\mathbf{u} = \lambda_1\mathbf{b}_1 + \cdots + \lambda_n\mathbf{b}_n$$

and

$$\mathbf{v} = \mu_1\mathbf{b}_1 + \cdots + \mu_n\mathbf{b}_n.$$

By adding these two equations we get

$$\mathbf{u} + \mathbf{v} = (\lambda_1 + \mu_1)\mathbf{b}_1 + \cdots + (\lambda_n + \mu_n)\mathbf{b}_n$$

and this implies (by the definition of coordinate vectors) that

$$[\mathbf{u}+\mathbf{v}]_B = \begin{pmatrix} \lambda_1+\mu_1 \\ \vdots \\ \lambda_n+\mu_n \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = [\mathbf{u}]_B + [\mathbf{v}]_B.$$

Multiplication by a scalar. The proof is similar to that for addition, and is omitted. ☐

6.8 [X] Further important examples of vector spaces

All of the material in this section is regarded as being more difficult than the material in previous sections of this chapter.

In the preceding sections we have developed a general theory of vector spaces. However, for simplicity, we have concentrated on examples which illustrate the applications of the theory to the particular vector space $\mathbb{R}^n$. In this section we shall examine three other important examples of vector spaces, namely the vector spaces of matrices, of real-valued functions, and of polynomials. We shall show how the vector space ideas of subspace, linear combination and span, linear independence, basis, dimension, and coordinate vector apply to these spaces.

Before the discussion of these vector spaces, we introduce an alternative way of proving a subset to be a subspace.

Theorem 1 (Alternative Subspace Theorem)**.** *A subset S of a vector space V over a field $\mathbb{F}$ is a subspace of V if and only if S contains the zero vector and it satisfies the closure condition:*

$$\text{If} \;\; \mathbf{v}_1, \mathbf{v}_2 \in S, \;\; \text{then} \;\; \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 \in S \;\; \text{for all} \;\; \lambda_1, \lambda_2 \in \mathbb{F}. \qquad (\#)$$

Proof. We prove that the closure condition (#) is satisfied if and only if both closure under addition and closure under scalar multiplication are also satisfied.

We first assume that (#) is satisfied. Then, for the special case of $\lambda_1 = \lambda_2 = 1$, condition (#) becomes

$$\mathbf{v}_1 + \mathbf{v}_2 \in S \quad \text{for all} \quad \mathbf{v}_1, \mathbf{v}_2 \in S,$$

and hence S is closed under addition. Further, for the special case of $\lambda_2 = 0$, condition (#) becomes

$$\lambda_1\mathbf{v}_1 \in S \quad \text{for all} \quad \mathbf{v}_1 \in S \quad \text{and for all} \quad \lambda_1 \in \mathbb{F},$$

and hence S is closed under multiplication by a scalar. Thus, if (#) is satisfied then closure under addition and closure under scalar multiplication are also satisfied.

We now assume that both closure conditions are satisfied. Then, from closure under multiplication by a scalar, we have, for all $\mathbf{v}_1, \mathbf{v}_2 \in S$ and for all $\lambda_1, \lambda_2 \in F$, that

$$\lambda_1\mathbf{v}_1 \in S \qquad \text{and} \qquad \lambda_2\mathbf{v}_2 \in S.$$

Then, on adding these scalar multiples and using closure under addition, we have that

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 \in S,$$

and hence (#) is satisfied.

Thus, (#) is satisfied if and only if closure under addition and under scalar multiplication are both satisfied. We then use the Subspace Theorem of Section 6.3 to complete the proof. ☐

6.8.1 Vector spaces of matrices

We have seen in Example 3 in Section 6.1 that $M_{mn}(\mathbb{R})$ the set of all $m \times n$ real matrices is a vector space over $\mathbb{R}$ and $M_{mn}(\mathbb{C})$ the set of $m \times n$ complex matrices is a vector space over $\mathbb{C}$. In this section, we are going to see some examples of their subspaces, their bases and coordinate vectors with respect to these bases.

Example 1. Show that the set

$$S = \{A \in M_{22}(\mathbb{R}) : [A]_{11} = [A]_{22} = 1\}$$

is not a vector subspace of $M_{22}(\mathbb{R})$.

SOLUTION. Since the zero matrix is not in S, then vector space axiom 4 **Existence of Zero** is not satisfied. Therefore the set S is not a vector subspace. ♢

Example 2. Prove that the set of $n \times n$ real symmetric matrices is a subspace of $M_{nn}(\mathbb{R})$.

SOLUTION. Recall that a square matrix A is called symmetric if $A = A^T$. Let S be the set of $n \times n$ symmetric matrices. Obvious, the zero matrix is symmetric, and so belongs to S.

Suppose that A, $B \in S$ and λ, $\mu \in \mathbb{R}$. By a property of transpose in Chapter 4nd the fact that A, B are symmetric, we have

$$(\lambda A + \mu B)^T = \lambda A^T + \mu B^T = \lambda A + \mu B.$$

Therefore, $\lambda A + \mu B$ is symmetric and so it is in S. Hence by the Alternative Subspace Theorem, S is a subspace. ♢

Example 3. For $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, let E_{ij} be the $m \times n$ matrix with all entries 0 except that the ijth entry is 1. Show that the set $B = \{E_{ij} : 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$ is a basis for $M_{mn}(\mathbb{C})$.

SOLUTION. For any $A = (a_{ij}) \in M_{mn}(\mathbb{C})$, we have

$$A = a_{11}E_{11} + \cdots + a_{1n}E_{1n} + a_{21}E_{21} + \cdots + a_{mn}E_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E_{ij}.$$

Thus, S is a spanning set for $M_{mn}(\mathbb{C})$.

Suppose that $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij}E_{ij} = \mathbf{0}$, where $\mathbf{0}$ is the $m \times n$ zero matrix. Note that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij}E_{ij} = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_{m1} & \cdots & \lambda_{mn} \end{pmatrix}$$

is the zero matrix. Hence $\lambda_{11} = \cdots = \lambda_{mn} = 0$ and so B is an independent set. Therefore B is a basis for $M_{mn}(\mathbb{C})$. ♢

NOTE. In general, the set B is a basis called the **standard basis** for $M_{mn} = M_{mn}(\mathbb{F})$ for $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. As a result the dimension of M_{mn} is mn.

Example 4. The coordinate vector of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with respect to the standard basis of M_{22} is $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$.

We can also solve problems of independent sets, spanning sets and bases by Gaussian Elimination. Be careful not to mix up the augmented matrix formed from the linear combinations and the elements in M_{mn}.

Example 5. Show that set

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \right\}$$

is a basis for M_{22}.

Solution. Note that the dimension of M_{22} is 4 and the number of vectors in the given set is also 4. To prove this set is a basis, by Theorem 3 part 4 in Section 6.6.2 we only need to show that this set is independent.

Suppose that

$$\lambda_1 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \lambda_3 \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

By equating the corresponding entries in both sides, we have

$$\begin{array}{ccccccccc} \lambda_1 & + & \lambda_2 & + & \lambda_3 & + & 0 & = & 0 \\ \lambda_1 & + & \lambda_2 & + & 0 & + & \lambda_4 & = & 0 \\ \lambda_1 & + & 0 & + & \lambda_3 & + & \lambda_4 & = & 0 \\ 0 & + & 2\lambda_2 & + & 2\lambda_3 & + & 2\lambda_4 & = & 0 \end{array}$$

Since the right hand sides are all zeros, we can simply reduce the coefficient matrix to row-echelon form.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 2 \end{pmatrix} \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 2 & 2 & 2 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 2 & 2 & 2 \end{pmatrix} \xrightarrow{R_4 = R_4 + 2R_2 + 2R_3} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

Thus the system of equations has unique solution $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. Hence the set of matrices is a basis. $\Diamond$

Note. The four columns of the coefficient matrix are the coordinate vectors of the four matrices with respect to the standard basis.

6.8.2 Vector spaces associated with real-valued functions

Before reading this section, you might like to quickly read the brief review of function notation given in Appendix 6.9.

We know how to add two functions and how to multiply a function by a real number, so it is natural to ask whether or not these operations satisfy the axioms of a vector space. In this section we shall see that they do.

The Vector Space of Real-Valued Functions. Let X be a non-empty set. Consider the set (which we call $\mathcal{R}[X]$) of all possible real-valued functions with domain X, that is,

$$\mathcal{R}[X] = \{f : X \to \mathbb{R}\}.$$

We also let $+$ be the usual rule for adding real functions, i.e., $f + g$ is the function given by

$$(f + g)(x) = f(x) + g(x) \quad \text{for all} \quad x \in X,$$

and we let $*$ represent the usual rule for multiplying a real function by a real number, i.e., $\lambda * f$ is the function given by

$$(\lambda * f)(x) = (\lambda f)(x) = \lambda f(x) \text{ for all } x \in X.$$

We then have the following result:

Proposition 2. The system $(\mathcal{R}[X], +, *, \mathbb{R})$ is a vector space over the real-number field $\mathbb{R}$.

Proof. The proof of this proposition follows the usual procedure of proving that all ten of the vector space axioms are satisfied. We give proofs of two of the axioms and leave the proofs of the other ones as exercises.

Closure under Addition. If $f, g \in \mathcal{R}[X]$, then $f(x)$ and $g(x)$ are defined and are real numbers for all $x \in X$. Then, using the usual rule for function addition, we have $(f + g)(x) = f(x) + g(x)$ for all $x \in X$.

Thus, $(f + g)(x)$ is also defined and is a real number for all $x \in X$. Hence $f + g : X \to \mathbb{R}$ and therefore $f + g \in \mathcal{R}[X]$.

Closure under Multiplication by Scalars. If $f \in \mathcal{R}[X]$ and $\lambda \in \mathbb{R}$, then, from the usual rule for multiplication of a function by a real number, we have

$$(\lambda f)(x) = \lambda f(x) \text{ for all } x \in X.$$

Thus $(\lambda f)(x)$ is defined and is a real number for all $x \in X$. Hence, $\lambda f : X \to \mathbb{R}$ and therefore $\lambda f \in \mathcal{R}[X]$. □

In the next examples we briefly mention some subspaces of the vector space of real-valued functions. These subspaces are of importance in many areas of modern mathematics, science and engineering. The most important one, the subspace of polynomials will be discussed in the next subsection.

Example 6. Let (a, b) be an interval of $\mathbb{R}$, and let $C[(a, b)]$ be the set of all continuous real-valued functions on (a, b). Show that $C[(a, b)]$ is a subspace of the vector space $\mathcal{R}[(a, b)]$ of all real-valued functions with domain (a, b).

SOLUTION. We use the Alternative Subspace Theorem.

The set $C[(a,b)]$ contains the zero function, as the zero function is continuous and so is in this set.

From calculus, we know that if f and g are continuous on an interval, then $\lambda_1 f + \lambda_2 g$ is also continuous on the same interval for all real λ_1 and λ_2. Also, $\lambda_1 f + \lambda_2 g$ is a real function, and hence $C[(a,b)]$ is a subset of $\mathcal{R}[(a,b)]$ which satisfies the condition (#) of the Alternative Subspace Theorem. Thus $C[(a,b)]$ is a subspace of $\mathcal{R}[(a,b)]$. $\diamondsuit$

Calculus provides a very rich source of subspaces of the vector space of real-valued functions.

Example 7. Let $C^{(1)}[(a,b)]$ be the set of all real-valued functions whose first derivative exists and is continuous on an interval (a,b) of $\mathbb{R}$. Show that $C^{(1)}[(a,b)]$ is a subspace of the vector space $\mathcal{R}[(a,b)]$.

SOLUTION. We use the Alternative Subspace Theorem.

The set $C^{(1)}[(a,b)]$ contains the zero function, as the zero function has a continuous first derivative and so is in this set.

From calculus, we know that if the first derivatives of the real functions f and g exist and are continuous on an interval, then, for all $\lambda_1, \lambda_2 \in \mathbb{R}$, the function $\lambda_1 f + \lambda_2 g$ is also a real-valued function whose first derivative exists and is continuous on the same interval. Thus, $C^{(1)}[(a,b)]$ is a subset of $\mathcal{R}[(a,b)]$ which satisfies condition (#) of the Alternative Subspace Theorem, and hence it is a subspace of $\mathcal{R}[(a,b)]$.

Note that $C^{(1)}[(a,b)]$ is also a subspace of the vector space $C[(a,b)]$ of real-valued, continuous functions on (a,b) given in Example 6. Can you see why? $\diamondsuit$

An important class of function subspace is defined by the solutions of homogeneous, linear differential equations.

Example 8. Let S be the subset of the vector space $\mathcal{R}[\mathbb{R}]$ of real-valued functions on $\mathbb{R}$ defined by

$$S = \left\{ f \in \mathcal{R}[\mathbb{R}] : \frac{d^2 f}{dx^2} - 6\frac{df}{dx} + 5f = 0 \right\}.$$

Show that S is a subspace of $\mathcal{R}[\mathbb{R}]$.

SOLUTION. As the zero function satisfies the equation, so it belongs to S. For $f_1, f_2 \in S$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, we have, on using the properties of derivatives, that

$$\begin{aligned} &\frac{d^2}{dx^2}(\lambda_1 f_1 + \lambda_2 f_2) - 6\frac{d}{dx}(\lambda_1 f_1 + \lambda_2 f_2) + 5(\lambda_1 f_1 + \lambda_2 f_2) \\ &\quad = \lambda_1 \left(\frac{d^2 f_1}{dx^2} - 6\frac{df_1}{dx} + 5f_1 \right) + \lambda_2 \left(\frac{d^2 f_2}{dx^2} - 6\frac{df_2}{dx} + 5f_2 \right) \\ &\quad = \lambda_1 0 + \lambda_2 0 = 0. \end{aligned}$$

Hence, $\lambda_1 f_1 + \lambda_2 f_2 \in S$, and therefore S is a subspace. $\diamondsuit$

Subspaces can also be defined by integration.

Example 9. Let S be the subset of the vector space $\mathcal{C}\,[[-\pi,\pi]]$ of all real-valued continuous functions on the interval $[-\pi,\pi]$ defined by

$$S = \left\{ f \in \mathcal{C}\,[[-\pi,\pi]] : \int_{-\pi}^{\pi} f(x)g(x)\,dx = 0 \right\},$$

where g is a fixed continuous function. Clearly the product $f(x)g(x)$ is integrable on $[-\pi,\pi]$, since f and g are continuous. Prove that S is a subspace of $\mathcal{C}\,[[-\pi,\pi]]$.

Solution. The zero function is in S. For all $f_1,\ f_2 \in S$ and for all $\lambda_1,\ \lambda_2 \in \mathbb{R}$, we have $\lambda_1 f_1 + \lambda_2 f_2 \in \mathcal{C}\,[[-\pi,\pi]]$ and

$$\begin{aligned}\int_{-\pi}^{\pi} \big(\lambda_1 f_1(x) + \lambda_2 f_2(x)\big) g(x)\,dx &= \lambda_1 \int_{-\pi}^{\pi} f_1(x)g(x)\,dx + \lambda_2 \int_{-\pi}^{\pi} f_2(x)\,g(x)dx \\ &= \lambda_1 0 + \lambda_2 0 = 0.\end{aligned}$$

Hence $\lambda_1 f_1 + \lambda_2 f_2 \in S$, and thus S is a subspace of $\mathcal{C}\,[[-\pi,\pi]]$. $\Diamond$

Example 10. As shown in courses on differential equations, the homogeneous, linear differential equation in Example 8 has the solution

$$f(x) = \lambda_1 e^{5x} + \lambda_2 e^{x} \quad \text{for} \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Hence the set S of solutions is given by $S = \operatorname{span}\left(e^{5x}, e^{x}\right)$, and thus $\{e^{5x}, e^{x}\}$ is a spanning set for S. $\Diamond$

Example 11. Show that the set $S = \{\sin(x), \cos(x)\}$ is a linearly independent subset of the vector space $\mathcal{R}\,[[-\pi,\pi]]$ of all real-valued functions on the interval $[-\pi,\pi]$.

Solution. We have to show that if

$$f(x) = \lambda_1 \sin(x) + \lambda_2 \cos(x) = 0 \qquad \text{for all} \quad x \in [-\pi,\pi]$$

then λ_1 and λ_2 are zero.

We first note that if the linear combination $f(x)$ is zero for all $x \in [-\pi,\pi]$ then $f(x)$ must also be zero for the values $x = 0$ and $x = \frac{\pi}{2}$. We then obtain

$$0 = f(0) = \lambda_2 \qquad \text{and} \qquad 0 = f\left(\frac{\pi}{2}\right) = \lambda_1.$$

Thus the scalars are zero, and hence the set is linearly independent. $\Diamond$

Example 12. Show that the subset S_n of the vector space $\mathcal{R}[\mathbb{R}]$ defined by

$$S_n = \{\sin(kx) \ : \ k = 1, \dots, n \quad \text{and} \quad x \in \mathbb{R}\}$$

is a linearly independent set.

Solution. We will use a trick which is based on an extension to the vector space of real functions of the idea of orthogonality of geometric vectors and vectors in $\mathbb{R}^n$ (see Example 5 and 6 of Section 6.6).

We first prove that if k and m are integers then

$$\int_0^{\pi} \sin(kx)\sin(mx)dx = \begin{cases} 0 & \text{for} \quad k \neq m \\ \frac{\pi}{2} & \text{for} \quad k = m. \end{cases}$$

From trigonometry, we have

$$\sin(kx)\sin(mx) = \frac{1}{2}\left(\cos(k-m)x - \cos(k+m)x\right).$$

Then, for $k \neq m$,

$$\begin{aligned}\int_0^{\pi} \sin(kx)\sin(mx)dx &= \frac{1}{2}\int_0^{\pi} \cos(k-m)x\,dx - \frac{1}{2}\int_0^{\pi} \cos(k+m)x\,dx \\ &= \left[\frac{1}{2(k-m)}\sin(k-m)x\right]_0^{\pi} - \left[\frac{1}{2(k+m)}\sin(k+m)x\right]_0^{\pi} \\ &= 0\end{aligned}$$

as $\sin(0) = 0$, $\sin(k-m)\pi = 0$, and $\sin(k+m)\pi = 0$ for integers k and m. We leave the proof of the result for $k = m$ as a simple exercise.

Now suppose that

$$\sum_{k=1}^{n} \lambda_k \sin(kx) = 0.$$

On multiplying this expression by $\sin(mx)$ and integrating from 0 to π, we have

$$0 = \sum_{k=1}^{n} \lambda_k \int_0^{\pi} \sin(kx)\sin(mx)\,dx = \lambda_m \frac{\pi}{2}$$

Hence, $\lambda_m = 0$ for $1 \leqslant m \leqslant n$, and thus the set S_n is linearly independent. ♢

Example 13. Use the previous example, we can show that the vector space $\mathcal{R}[\mathbb{R}]$ cannot be spanned by a finite set and hence it is an infinite dimensional vector space. If it can be spanned by a finite set of m elements, by Theorem 1 in Section 6.6, all independent set has at most m elements. We can choose an integer n such that $n > m$, by the previous example, S_n will be a linearly independent set of more than m elements. Hence $\mathcal{R}[\mathbb{R}]$ cannot be spanned by a finite set.♢

6.8.3 Vector spaces associated with polynomials

Polynomials can be added, subtracted and multiplied. From the point of view of vector spaces, only addition (and subtraction) and multiplication of a polynomial by a scalar are relevant.

We will be concerned with polynomials defined over either the real or complex fields. Although it is possible to generalise all of the results in this section to the rational field $\mathbb{Q}$ and to also generalise many of the results to the finite fields $\mathbb{Z}_p$, (p a prime), we will not do so here. In the following, therefore, the field $\mathbb{F}$ should be taken as either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$.

We begin by quickly reviewing the definitions of polynomial function, polynomial addition, multiplication of a polynomial by a scalar, and equality of polynomials.

Definition 1. *A function $p : \mathbb{F} \to \mathbb{F}$ is called a* **polynomial function** *over $\mathbb{F}$ if there is a natural number $n \in \mathbb{N}$ and numbers $a_0, a_1, \ldots, a_n \in \mathbb{F}$ such that*

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n = \sum_{k=0}^{n} a_k z^k \text{ for all } z \in \mathbb{F}.$$

For brevity we will usually refer to a polynomial function as a polynomial even though it is important in advanced mathematics courses to distinguish between the two.

Polynomials may be added and multiplied by scalars in such a way as to produce other polynomials. The formal definitions follow the usual definitions of addition and multiplication by a scalar for functions (see Appendix 6.9).

Definition 2. *If p and q are polynomials over the same field $\mathbb{F}$ given by*

$$p(z) = \sum_{k=0}^{n} a_k z^k \qquad \text{and} \qquad q(z) = \sum_{k=0}^{m} b_k z^k \qquad \text{for all } z \in \mathbb{F},$$

then the **sum** *polynomial is the polynomial $p + q$ given by*

$$(p+q)(z) = p(z) + q(z) = \sum_{k=0}^{\max(n,m)} (a_k + b_k) z^k \qquad \text{for all } z \in \mathbb{F}.$$

That is, the rule is to add corresponding coefficients. The rule for subtraction of polynomials follows immediately from the addition rule, and it is to subtract corresponding coefficients.

Definition 3. *If $\lambda \in \mathbb{F}$ and p is a polynomial over $\mathbb{F}$ given by*

$$p(z) = \sum_{k=0}^{n} a_k z^k \text{ for all } z \in \mathbb{F}$$

then the **scalar multiple** *λp of p is the polynomial given by*

$$(\lambda p)(z) = \lambda(p(z)) = \sum_{k=0}^{n} (\lambda a_k) z^k \text{ for all } z \in \mathbb{F}.$$

That is, the rule is to multiply each coefficient by the scalar.

The last main property of polynomials that we need is given by the following Uniqueness Proposition (see Chapter 1or a proof for complex polynomials).

Proposition 3 (Uniqueness Proposition for Real and Complex Polynomials)**.** Let p and q be polynomials over $\mathbb{F}$ given by

$$p(z) = \sum_{k=0}^{n} a_k z^k \qquad \text{and} \qquad q(z) = \sum_{k=0}^{n} b_k z^k \qquad \text{for all } z \in \mathbb{F}.$$

Then, if the field $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, we have that $p(z) = q(z)$ for all $z \in \mathbb{F}$ if and only if $a_k = b_k$ for all $k = 0, 1, 2, \ldots, n$.

An immediate consequence of Proposition 3 is the following important result.

Corollary 4. *If $\mathbb{F}$ is the field $\mathbb{R}$ or $\mathbb{C}$, then a polynomial p over $\mathbb{F}$ has the property that $p(z) = 0$ for all $z \in \mathbb{F}$ if and only if all of its coefficients are zero.*

NOTE. that this is not true for other fields such as $\mathbb{Z}_p$, p a prime.

The unique polynomial, whose function values are all zero and which has all of its coefficients equal to zero, is called the **zero polynomial**.

The fundamental vector space associated with polynomials is defined in the following example.

Example 14 (The Vector Space of Polynomials over $\mathbb{F}$)**.** The vector space of polynomials over a field $\mathbb{F}$ is the system $(\mathbb{P}(\mathbb{F}), +, *, \mathbb{F})$ defined as follows. The set of "vectors" is the set $\mathbb{P}(\mathbb{F})$ of all possible polynomials over the field $\mathbb{F}$, i.e.,

$$\mathbb{P}(\mathbb{F}) = \{p : p(z) = a_0 + a_1 z + \cdots + a_n z^n \quad \text{for} \quad n \in \mathbb{N}, \quad a_j \in \mathbb{F}, \quad 1 \leqslant j \leqslant n, \quad z \in \mathbb{F}\}.$$

The rule of "vector addition" is the polynomial addition rule given in Definition 2 and the rule of multiplication is the scalar multiplication rule for polynomials given in Definition 3.

To prove that this system is a vector space it is necessary to check that all ten of the vector space axioms are satisfied. We leave the checking of the axioms as an exercise. $\diamondsuit$

Example 15. As special cases of Example 14, we have the vector space of real polynomials $(\mathbb{P}(\mathbb{R}), +, *, \mathbb{R})$ (we have seen this in Example 4 in Section 6.1) and the vector space of complex polynomials $(\mathbb{P}(\mathbb{C}), +, *, \mathbb{C})$. $\diamondsuit$

Notation. We will usually talk of the polynomial vector space $\mathbb{P}$ instead of the more formal $(\mathbb{P}(\mathbb{F}), +, *, \mathbb{F})$ when there can be no possibility of confusion over the field ($\mathbb{R}$ or $\mathbb{C}$) being used. When necessary, the vector space of real polynomials will be referred to as $\mathbb{P}(\mathbb{R})$ and the vector space of complex polynomials as $\mathbb{P}(\mathbb{C})$.

As shown in the following example, the field for the polynomials and the field for the scalars must be, in some sense, compatible.

Example 16. Show that the system $(\mathbb{P}(\mathbb{C}), +, *, \mathbb{R})$ of complex polynomials and real scalars is a vector space, whereas the system $(\mathbb{P}(\mathbb{R}), +, *, \mathbb{C})$ of real polynomials and complex scalars is not a vector space.

SOLUTION. It can be checked without too much difficulty that both systems satisfy nine of the ten vector space axioms. The axioms satisfied by both are the five axioms of addition, the three scalar-multiplication axioms of associativity, commutativity, multiplication by 1, and the scalar and vector distributive axioms.

The remaining axiom to check is closure under scalar multiplication.

For the system of complex polynomials and real scalars, we have that if $p \in \mathbb{P}(\mathbb{C})$ is a complex polynomial and $\lambda \in \mathbb{R}$, then λp is also a complex polynomial. The closure under scalar multiplication axiom is therefore satisfied, and hence the system of complex polynomials and real scalars is a vector space.

For the system of real polynomials and complex scalars, we note that $x \in \mathbb{P}(\mathbb{R})$, $i \in \mathbb{C}$, $ix \notin \mathbb{P}(\mathbb{R})$. The closure under scalar multiplication axiom is therefore not satisfied, and hence the system of real polynomials with complex scalars is not a vector space. $\diamondsuit$

We now consider some of the subspaces of the polynomial vector space $\mathbb{P} = \mathbb{P}(\mathbb{F})$. The most important subspaces of $\mathbb{P}$ are the vector spaces of polynomials of degree less than or equal to n, for some n.

Example 17. Let $\mathbb{P}$ be the vector space of polynomials over $\mathbb{F}$, and let $\mathbb{P}_n$ be the subset of $\mathbb{P}$ consisting of all polynomials of degree less than or equal to some fixed integer $n > 0$, that is,

$$\mathbb{P}_n = \{p \in \mathbb{P} : \deg(p) \leqslant n\}.$$

Show that $\mathbb{P}_n$ is a subspace of $\mathbb{P}$.

Solution. If $p, q \in \mathbb{P}_n$, then there exist coefficients $\{a_0, \dots, a_n\}$ and $\{b_0, \dots, b_n\}$ such that

$$\begin{aligned} p(z) &= a_0 + a_1 z + \cdots + a_n z^n \\ q(z) &= b_0 + b_1 z + \cdots + b_n z^n \end{aligned}$$

for all $z \in \mathbb{F}$. Then, for $\lambda p + \mu q$ with $\lambda, \mu \in \mathbb{F}$, we have

$$(\lambda p + \mu q)(z) = \lambda p(z) + \mu q(z) = (\lambda a_0 + \mu b_0) + \cdots + (\lambda a_n + \mu b_n) z^n \quad \text{for all} \quad z \in \mathbb{F}.$$

But the coefficients $\lambda a_j + \mu b_j$, $1 \leqslant j \leqslant n$, are also scalars in $\mathbb{F}$, and hence $\lambda p + \mu q \in \mathbb{P}_n$. Thus the condition in the Alternative Subspace Theorem is satisfied, and $\mathbb{P}_n$ is a subspace of $\mathbb{P}$. $\diamondsuit$

Note that the subset of all polynomials of degree **exactly** n is not a subspace, since this subset does not contain the zero polynomial.

Subspaces of the space of all polynomials can also be formed by selecting all polynomials which have their roots at given points.

Example 18. Let $\mathbb{P}_n$ be the vector space of polynomials of degree less than or equal to n over $\mathbb{F}$. Show that the subset S of $\mathbb{P}_n$ given by

$$S = \{p \in \mathbb{P}_n : p(5) = \alpha\}$$

is a subspace of $\mathbb{P}_n$ if and only if $\alpha = 0$.

Solution. We use the Alternative Subspace Theorem.

If $\alpha \neq 0$, then the zero polynomial is not in S and so S is not a subspace.

If $\alpha = 0$, then the zero polynomial is in S and so S is not empty. If $p, q \in S$ and $\lambda_1, \lambda_2 \in \mathbb{F}$, then $p(5) = 0$ and $q(5) = 0$ and

$$(\lambda_1 p + \lambda_2 q)(5) = \lambda_1 p(5) + \lambda_2 q(5) = 0.$$

Hence $\lambda_1 p + \lambda_2 q \in S$, and S is a subspace.

Therefore, S is a subspace of $\mathbb{P}_n$ if and only if $\alpha = 0$.

Note that this subspace is the set of all polynomials of degree less than or equal to n over $\mathbb{F}$ which have a root at $z = 5$. $\diamondsuit$

As the ideas of linear combination, span and linear independence apply to all vector spaces, they apply to spaces of polynomials. The methods used in Sections 6.4 and 6.5 for $\mathbb{R}^n$ can be used to solve problems of spanning sets and independent sets.

Example 19. Does the complex polynomial p belong to span (p_1, p_2), where p, p_1, p_2 are complex polynomials defined by

$$p(z) = 4 + z + 2z^2, \qquad p_1(z) = 1 + z - z^2 \qquad \text{and} \qquad p_2(z) = 2 - z \qquad \text{for all} \quad z \in \mathbb{C}?$$

Solution. $p \in \text{span}\,(p_1, p_2)$ if and only if p is a linear combination of p_1 and p_2, that is, if and only if there are scalars x_1, x_2 such that

$$p(z) = x_1 p_1(z) + x_2 p_2(z) \qquad \text{for all} \quad z \in \mathbb{C}.$$

That is,

$$\begin{aligned} 4 + z + 2z^2 &= x_1(1 + z - z^2) + x_2(2 - z) \\ &= (x_1 + 2x_2) + (x_1 - x_2)z + (-x_1)z^2 \qquad \text{for all} \quad z \in \mathbb{C}. \end{aligned}$$

From the Uniqueness Proposition for Polynomials (Section 6.8.3), we know that polynomials are equal if and only if coefficients of corresponding powers of z are equal. Equating coefficients of equal powers then gives the system of linear equations

$$x_1 + 2x_2 = 4, \qquad x_1 - x_2 = 1, \qquad -x_1 = 2,$$

with augmented matrix

$$(A|\mathbf{b}) = \left(\begin{array}{rr|r} 1 & 2 & 4 \\ 1 & -1 & 1 \\ -1 & 0 & 2 \end{array} \right).$$

Then, p is in the span of p_1 and p_2 if and only if these equations have a solution. It is easy to see that these equations have no solution, and hence p is not in span$(p_1,\ p_2)$. $\diamondsuit$

Example 20. Is the set $\{p_1, p_2, p_3\}$ of polynomials, where

$$p_1(z) = 1 + 2z - z^2; \qquad p_2(z) = -3 - z + 2z^2; \qquad p_3(z) = 2 + 3z + z^2,$$

a linearly independent subset of $\mathbb{P}_2$?

Solution. Following the usual test for linear independence, we look for scalars x_1, x_2, x_3 such that $x_1p_1 + x_2p_2 + x_3p_3 = 0$, that is, such that

$$x_1(1 + 2z - z^2) + x_2(-3 - z + 2z^2) + x_3(2 + 3z + z^2) = 0 \quad \text{for all} \quad z \in \mathbb{C}.$$

Thus, the polynomial on the left is the zero polynomial, and hence the coefficient of each power of z is zero, that is,

$$x_1 - 3x_2 + 2x_3 = 0; \qquad 2x_1 - x_2 + 3x_3 = 0; \qquad -x_1 + 2x_2 + x_3 = 0.$$

This system of equations corresponds to the homogeneous system $A\mathbf{x} = \mathbf{0}$, where the matrix A and an equivalent row-echelon form U are

$$A = \begin{pmatrix} 1 & -3 & 2 \\ 2 & -1 & 3 \\ -1 & 2 & 1 \end{pmatrix}; \qquad U = \begin{pmatrix} 1 & -3 & 2 \\ 0 & 5 & -1 \\ 0 & 0 & \frac{14}{5} \end{pmatrix}.$$

Then, as all columns of U are leading columns, the only solution for the scalars is $x_1 = x_2 = x_3 = 0$, and hence the set is linearly independent. $\Diamond$

As we can discuss spans and independence of sets of polynomials, we also have the notion of dimensions of subspaces of polynomials. In the next example, we construct a standard basis for $\mathbb{P}_n$.

Example 21. Show that the set $\{1, z, \dots, z^n\}$ is a basis for $\mathbb{P}_n(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

SOLUTION. A polynomial p of degree less than or equal to n over a field $\mathbb{F}$ is a function of the form

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n \qquad \text{with} \quad a_j \in \mathbb{F} \quad \text{for} \quad 0 \leqslant j \leqslant n \qquad \text{and for} \quad z \in \mathbb{F},$$

where any or all of the coefficients may be zero. Hence $\operatorname{span}(1, z, \dots, z^n) = \mathbb{P}_n(\mathbb{F})$.

Furthermore, if

$$a_0 + a_1 z + \cdots + a_n z^n = 0 \qquad \text{for all} \quad z \in \mathbb{C}.$$

This linear combination is the zero polynomial, and hence, from the Uniqueness Proposition for Polynomials of Section 6.8.3, all of the coefficients are zero. Hence $\{1, z, \dots, z^n\}$ is independent.

Therefore, this set is a basis for $\mathbb{P}_n(\mathbb{F})$. $\Diamond$

An important result which follows immediately from Example 21 is the following.

Proposition 5. The vector space $\mathbb{P}_n$ of polynomials of degree less than or equal to n has dimension $n + 1$.

As a consequence, $\mathbb{P}$ is not a finite dimensional space.

Example 22. The vector space $\mathbb{P}$ of all polynomials cannot be spanned by a finite set.

SOLUTION. We can use the same argument used for $\mathcal{R}[\mathbb{R}]$ in Example 13. However, we now use another approach.

Assume the contrary that some set S containing a finite number of polynomials is a spanning set for $\mathbb{P}$. Then, since the number of polynomials in S is finite, there must be a highest-degree polynomial in S. Let N be the degree of this polynomial. Then, no polynomial p with $\deg(p) > N$ is in $\operatorname{span}(S)$. Hence, $\mathbb{P}$ is not spanned by any finite set of polynomials. $\Diamond$

Example 23. Show that the set $S = \{2 + z, -1 + z^2, z - z^2\}$ is a basis for $\mathbb{P}_2$.

SOLUTION. If $p \in \mathbb{P}_2$, then p is given by

$$p(z) = a_0 + a_1 z + a_2 z^2 \qquad \text{for all} \quad z \in \mathbb{C}.$$

Then $p \in \operatorname{span}(S)$ if there exist scalars x_1, x_2, x_3 such that

$$a_0 + a_1 z + a_2 z^2 = x_1(2 + z) + x_2(-1 + z^2) + x_3(z - z^2) \qquad \text{for all} \quad z \in \mathbb{C}.$$

On equating coefficients of powers of z, we obtain the system of linear equations with augmented matrix

$$(A|\mathbf{b}) = \left(\begin{array}{ccc|c} 2 & -1 & 0 & a_0 \\ 1 & 0 & 1 & a_1 \\ 0 & 1 & -1 & a_2 \end{array}\right)$$

On using Gaussian elimination, we obtain the augmented matrix

$$(U|\mathbf{y}) = \left(\begin{array}{ccc|c} 2 & -1 & 0 & a_0 \\ 0 & \frac{1}{2} & 1 & -\frac{1}{2}a_0 + a_1 \\ 0 & 0 & -3 & a_0 - 2a_1 + a_2 \end{array}\right).$$

Then, as there are no zero rows in U, there is a solution for all right hand sides. Thus, for every polynomial $p \in \mathbb{P}_2$, we have $p \in \text{span}\,(S)$, and hence S is a spanning set for $\mathbb{P}_2$. Further, as U has no non-leading columns, the only solution for a zero right hand side is $x_1 = x_2 = x_3 = 0$, and hence S is linearly independent. S is therefore a basis for $\mathbb{P}_2$. $\diamondsuit$

The coordinate-vector idea applies immediately to the finite-dimensional vector space $\mathbb{P}_n$.

Example 24. For the standard basis of $\mathbb{P}_n$ consisting of powers of z, that is, $\{1, z, \ldots, z^n\}$, the coordinate vector consists of the coefficients of the polynomial. For example, the polynomial $p \in \mathbb{P}_n$ defined by

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

has the coordinate vector $\begin{pmatrix} a_0 & a_1 & \cdots & a_n \end{pmatrix}^T$ with respect to the standard basis.

Note. The order is important. For example, the coordinate vector of p with respect to the ordered basis $\{z^2, z, 1, z^3, \ldots, z^n\}$ would be $\begin{pmatrix} a_2 & a_1 & a_0 & a_3 & \cdots & a_n \end{pmatrix}^T$. $\diamondsuit$

Example 25. Find the coordinate vector for the polynomial $p_3(z) = -1 + 5z^2$ with respect to the ordered basis $\{p_1, p_2\}$ of span (p_1, p_2), where $p_1(z) = 1 + 2z + 3z^2$ and $p_2 = 1 + z - z^2$.

Solution. We must find the scalars in the expression for p_3 as a linear combination of p_1 and p_2. On writing $p_3 = x_1 p_1 + x_2 p_2$ and equating coefficients of equal powers of z, we get the system of equations with augmented matrix

$$(A|\mathbf{b}) = \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 3 & -1 & 5 \end{array}\right).$$

The solution is $x_1 = 1$, $x_2 = -2$. Hence the coordinate vector of p_3 with respect to the ordered basis $\{p_1, p_2\}$ is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, i.e.,

$$p_3(z) = -1 + 5z^2 = 1p_1(z) - 2p_2(z) = 1(1 + 2z + 3z^2) - 2(1 + z - z^2).$$

$\diamondsuit$

Example 26. A polynomial p has a coordinate vector $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ with respect to the ordered basis $\{p_1 = 1, p_2 = 1 + z, p_3 = 2 - z + z^2\}$ of $\mathbb{P}_2$. Find p.

Solution.

$$p = 2p_1 - p_2 + 4p_3 = 9 - 5z + 4z^2.$$

$\diamondsuit$

6.9 A brief review of set and function notation

6.9.1 Set notation.

A set is any collection of elements. Sets are usually defined either by listing their elements or by giving a rule for selection of the elements. The elements of a set are usually enclosed in braces $\{\}$.

Example 1. $S = \{1, 4, -7\}$ is the set whose elements are 1,4, and -7. ◊

Common notation for a set defined by a rule is shown in the following example.

Example 2. The notation

$$S = \{\mathbf{x} \in \mathbb{R}^n : x_1 \geqslant 0,\ x_3 \leqslant 4\}$$

is read as: the set S of vectors $\mathbf{x}$ in $\mathbb{R}^n$ such that x_1 is greater than or equal to zero and x_3 is less than or equal to 4. Note that the colon (:) is read as "such that" and the comma (,) is read as "and". ◊

Definition 1. *Two sets A and B are* **equal** *(notation $A = B$) if every element of A is an element of B, and if every element of B is an element of A.*

To prove that $A = B$ it is necessary to prove that the two conditions:

1. if $x \in A$ then $x \in B$, and
2. if $x \in B$ then $x \in A$

are both satisfied.

Definition 2. *A set A is a* **subset** *of another set B (notation $A \subseteq B$) if every element of A is also an element of B.*

To prove that $A \subseteq B$ it is necessary to prove that the condition:
if $x \in A$ then $x \in B$
is satisfied.

Definition 3. *A is said to be a* **proper subset** *of B if A is a subset of B and at least one element of B is not an element of A.*

To prove that A is a proper subset of B it is necessary to prove that the two conditions:

1. if $x \in A$ then $x \in B$, and
2. for some $x \in B$, x is not an element of A

are both satisfied.

Definition 4. *The **intersection** of two sets A and B (notation: $A \cap B$) is the set of elements which are common to both sets.*

That is,

$$A \cap B = \{x : x \in A \quad \text{and} \quad x \in B\}.$$

Definition 5. *The **union** of two sets A and B (notation: $A \cup B$) is the set of all elements which are in either or both sets.*

That is,

$$A \cup B = \{x : x \in A \quad \text{or} \quad x \in B\}.$$

6.9.2 Function notation

The notation $f : X \to Y$ (which is read as "f is a function (or map) from the set X to the set Y") means that f is a rule which associates exactly one element $y \in Y$ to each element $x \in X$. The y associated with x is written as $y = f(x)$ and is called the "value of the function f at x" or "the image of x under f". The set X is often called the **domain** of the function f and the set Y is often called the **codomain** of the function f.

Equality of Functions. Two functions $f : X \to Y$ and $g : X \to Y$ are defined to be equal if and only if $f(x) = g(x)$ for all $x \in X$.

Addition of Functions. If $f : X \to Y$ and $g : X \to Y$, and if elements of Y can be added, then the sum function $f + g$ is defined by

$$(f + g)(x) = f(x) + g(x) \quad \text{for all} \quad x \in X.$$

Multiplication by a Scalar. If $f : X \to Y$ and $\lambda \in \mathbb{F}$, where $\mathbb{F}$ is a field, and if elements of Y can be multiplied by elements of $\mathbb{F}$, then the function λf is defined by

$$(\lambda f)(x) = \lambda\big(f(x)\big) \quad \text{for all} \quad x \in X.$$

Multiplication of Functions. If $f : X \to Y$ and $g : X \to Y$, and if elements of Y can be multiplied, then the product function fg is defined by

$$(fg)(x) = f(x)g(x) \quad \text{for all} \quad x \in X.$$

Composition of Functions. If $g : X \to W$ and $f : W \to Y$, then the composition function $f \circ g : X \to Y$ is defined by

$$(f \circ g)(x) = f\big(g(x)\big) \quad \text{for all} \quad x \in X.$$

6.10 Vector spaces and MAPLE

Most of the problems in this chapter can be solved using matrix methods, which of course means that Maple can be very helpful. You might look at your session 1 notes to refresh your memory as to how Maple handles vectors and matrices. As usual, you should type

```
with(LinearAlgebra);
```

in order to load the linear algebra package.

The following commands, for example, can be used to put the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ as the columns in a 3×3 matrix.

```
v1:=<1,2,3>;
v2:=<0,-1,2>;
v3:=<3,-1,3>;
A:=<v1|v2|v3>;
```

We could now test whether these three vectors are linearly independent by performing Gaussian elimination on A.

```
GaussianElimination(A);
```

In this particular example, there are no non-leading columns so the vectors are linearly independent (and hence form a basis for $\mathbb{R}^3$).

Most of the other problems concerning problems in $\mathbb{R}^n$ can be solved using similar methods. Actually, the `LinearAlgebra` package contains many ready-made commands for performing the standard calculations. For example

```
Basis({v1,v2,v3});
```

returns a subset of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ which forms a basis for span $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

Problems in other vector spaces can often be solved by using coordinate vectors to convert the problem to one in $\mathbb{R}^n$. Finding the coordinate vector for a vector $\mathbf{b} \in \mathbb{R}^n$ with respect to a new basis is quite simple. For example, with the ordered basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, the commands

```
b:=<1,1,1>;
coordvect:=LinearSolve(A,b);
```

find the coordinate vector of $\mathbf{b}$. The `firstyear` package contains commands to find the coordinate vector of a polynomial in $\mathbb{P}_n$ with respect to the standard basis $\{1, x, x^2, \ldots, x^n\}$ (be careful with the capital letters).

```
with(firstyear);
p:=3*x^2-x+4;
v:=polytoVect(p,2);
Vecttopoly(v,x);
```

Chapter 7

LINEAR TRANSFORMATIONS

"But I don't need a Sillygism, you know,
to prove that mathematical axiom you mentioned."
"Nor to prove that 'all angles are equal', I suppose?"
"Why, of course not! One takes such a simple truth as that for granted!"
Lewis Carroll, Sylvie and Bruno.

The purpose of this chapter is to give an introduction to an extremely important class of functions called "linear transformations" or "linear maps". Mathematical examples of linear transformations include geometric transformations such as stretching, reflection and rotation, algebraic operations such as matrix multiplication, and calculus operations such as differentiation and integration. Linear transformations are also widely used in many applications of mathematics, and objects which are often modelled (either exactly or approximately) by linear transformations are related to radio and TV sets, amplifiers and hi-fi equipment, atomic spectra, molecular vibrations, sound waves, ocean waves, oil refineries, chemical plants, profit of a company, inventory of a factory or shop, and the state of an economy.

7.1 Introduction to linear maps

Before reading this chapter you should quickly read the brief review of function notation given in Appendix 6.9.

As stated in Appendix 6.9, a function f with domain X and codomain Y (notation $f : X \to Y$) is a rule which associates exactly one element $y = f(x)$ of Y to each element $x \in X$. Note that an element x in the domain X is called an "argument" of the function and the corresponding element $y = f(x)$ in the codomain Y is usually called either "the function value of x" or the "image of x under f".

Linear maps are an important special class of functions, in which both the domain and the codomain are vector spaces (that is, all arguments and values of the functions are vectors), and in which the two vector-space operations of addition and scalar multiplication are "preserved" by the function in the sense that:

Addition Condition. The function value of a sum of the two vectors is equal to the sum of the function values of the vectors.

Scalar Multiplication Condition. The function value of a scalar multiple of a vector is equal to the scalar multiple of the function value of the vector.

A more formal mathematical definition of a linear map is as follows.

Definition 1. *Let V and W be two vector spaces over the same field $\mathbb{F}$. A function $T : V \to W$ is called a **linear map** or **linear transformation** if the following two conditions are satisfied.*

Addition Condition. $T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}')$ *for all* $\mathbf{v}, \mathbf{v}' \in V$, *and*

Scalar Multiplication Condition. $T(\lambda \mathbf{v}) = \lambda T(\mathbf{v})$ *for all* $\lambda \in \mathbb{F}$ *and* $\mathbf{v} \in V$.

The domain and codomain of a linear map can be any vector spaces. In Section 7.5, we shall be concentrate specifically on linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Unless otherwise stated, the following propositions and theorems are true for all linear maps.

The adjective "linear" in "linear map" suggests that the idea of a linear map arose from the geometric idea of a line. The connection between the equation of a line and a linear map is shown in Figure 1 (a) and (b) and in Example 1 below.

Example 1. Show that the function $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = a_0 + a_1 x \quad \text{for} \quad x \in \mathbb{R},$$

where $a_0, a_1 \in \mathbb{R}$ are constants, is a linear map if and only if $a_0 = 0$.

Solution. We check the conditions of the definition of a linear map. Firstly, the domain $\mathbb{R}$ and codomain $\mathbb{R}$ are both vector spaces as $\mathbb{R}$ is a vector space. Further, we have,
for $x, x' \in \mathbb{R}$,

$$T(x + x') = a_0 + a_1(x + x'),$$

whereas,

$$T(x) + T(x') = (a_0 + a_1 x) + (a_0 + a_1 x') = 2a_0 + a_1(x + x').$$

Thus, the addition condition is satisfied if and only if $a_0 = 2a_0$, that is, if and only if $a_0 = 0$.
For $a_0 = 0$, we check the scalar multiplication condition, and obtain

$$T(\lambda x) = a_1(\lambda x) = \lambda(a_1 x) = \lambda T(x),$$

as required.

Thus, the conditions for T to be a linear map are satisfied if and only if $a_0 = 0$. ♢

Note.

1. The equation $y = T(x) = a_0 + a_1 x$ is the equation of a line in $\mathbb{R}^2$. Example 1 shows that the equation of a line defines a linear map if and only if the line goes through the origin.

2. The function $T(x) = a_0 + a_1 x$ is a polynomial of degree 1 and is often called a linear polynomial. Example 1 shows that a "linear polynomial" is a "linear map" if and only if the constant term in the polynomial is zero.

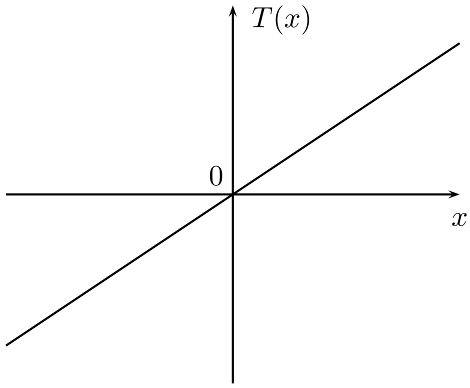


Figure 1(a). A linear map.

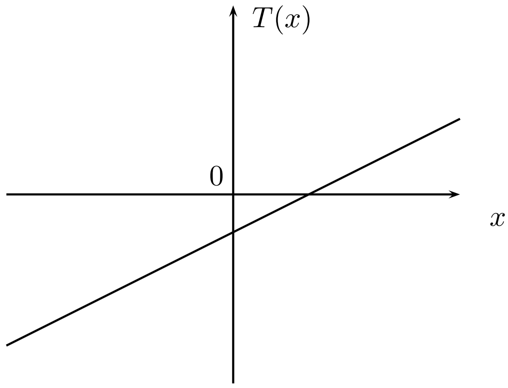


Figure 1(b). A linear polynomial which is NOT a linear map.

Example 2. Show that the function $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = \begin{pmatrix} -5x_2 + 4x_3 \\ x_1 + 2x_3 \end{pmatrix} \quad \text{for} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3,$$

is a linear map.

Solution. The domain $\mathbb{R}^3$ and codomain $\mathbb{R}^2$ are both vector spaces. We next check the addition and scalar multiplication conditions.

Addition condition. For $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^3$, we have $\mathbf{x} + \mathbf{x}' = \begin{pmatrix} x_1 + x_1' \\ x_2 + x_2' \\ x_3 + x_3' \end{pmatrix} \in \mathbb{R}^3$, and hence

$$\begin{aligned} T(\mathbf{x} + \mathbf{x}') &= \begin{pmatrix} -5(x_2 + x_2') + 4(x_3 + x_3') \\ (x_1 + x_1') + 2(x_3 + x_3') \end{pmatrix} \\ &= \begin{pmatrix} -5x_2 + 4x_3 \\ x_1 + 2x_3 \end{pmatrix} + \begin{pmatrix} -5x_2' + 4x_3' \\ x_1' + 2x_3' \end{pmatrix} = T(\mathbf{x}) + T(\mathbf{x}'). \end{aligned}$$

Thus the addition condition is satisfied.

Scalar multiplication condition. For $\mathbf{x} \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$, we have $\lambda\mathbf{x} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix} \in \mathbb{R}^3$, and hence

$$T(\lambda\mathbf{x}) = \begin{pmatrix} -5(\lambda x_2) + 4(\lambda x_3) \\ \lambda x_1 + 2(\lambda x_3) \end{pmatrix} = \lambda \begin{pmatrix} -5x_2 + 4x_3 \\ x_1 + 2x_3 \end{pmatrix} = \lambda T(\mathbf{x}).$$

Thus, the scalar multiplication condition is also satisfied, and therefore T is a linear map. $\diamondsuit$

We shall now summarise some useful properties that are true for all linear maps. In all of the following propositions and theorems the domain V and the codomain W are assumed to be vector spaces over the same field $\mathbb{F}$.

Here is a useful proposition regarding linear maps.

Proposition 1. If $T : V \to W$ is a linear map, then

1. $T(\mathbf{0}) = \mathbf{0}$ and
2. $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v} \in V$.

An informal way of stating these results is that a linear map always:

1. transforms the zero vector in the domain into the zero vector in the codomain, and
2. transforms the negative of a vector $\mathbf{v}$ in the domain into the negative of the corresponding function value $T(\mathbf{v})$ in the codomain.

Proof. (1). Since V is a vector space, we have from Proposition 2 of Section 6.2 that $0\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in V$. Thus,

$$T(\mathbf{0}) = T(0\mathbf{v}) = 0T(\mathbf{v}) = \mathbf{0},$$

where we have first used the scalar multiplication condition of a linear map and then applied Proposition 2 of Section 6.2 to the vector $T(\mathbf{v}) \in W$.
(2). Since V is a vector space, we have from Proposition 2 of Section 6.2 that $-\mathbf{v} = (-1)\mathbf{v}$ for all $\mathbf{v} \in V$. Hence,

$$T(-\mathbf{v}) = T\big((-1)\mathbf{v}\big) = (-1)T(\mathbf{v}) = -T(\mathbf{v}),$$

where we have again used the scalar multiplication condition of a linear map, and then Proposition 2 of Section 6.2 applied to the vector $T(\mathbf{v}) \in W$. $\square$

Proposition 1 may often be used to provide a quick proof that some given function is not linear.

Example 3. Show that the function $T : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 4x_1 + 3(x_2 - 6)$$

is not linear.

Solution. $T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = -18 \neq 0$, and hence T is not linear. $\Diamond$

Example 4. Show that the function $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = x^2$$

is not linear.

Solution. $T(3) = 9$, but $T(6) = 36 \neq 2 \times 9$. Hence T is not linear. $\Diamond$

To prove that a given map is not linear, it is easiest to provide a specific example that contravenes one of the conditions. One should also check first that the given map, takes the zero vector to the zero vector.

Warning. The converses of the two results in Proposition 1 are **not** true in general as shown in the following example.

Example 5. The function $T(x) = x^3$ satisfies both $T(-x) = -T(x)$ and $T(0) = 0$. However, it is not a linear map by the following counterexample. For $x = 1$,

$$T(2 \times 1) = 8 \neq 2(1)^3 = 2T(1).$$

$\Diamond$

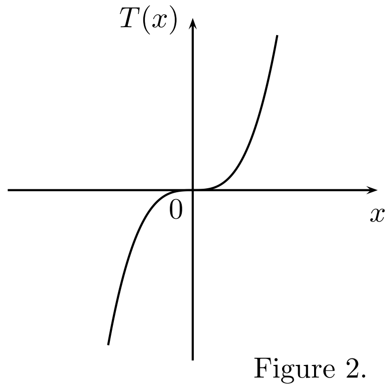


Figure 2.

The two conditions in the definition of a linear map are closely related to the two fundamental vector operations of addition and scalar multiplication. We therefore expect there to be a close relationship between linear combinations and linear maps. This relationship is given in Theorems 2 and 3 below.

Theorem 2. *A function $T : V \to W$ is a linear map if and only if for all $\lambda_1, \lambda_2 \in \mathbb{F}$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$*

$$T(\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2) = \lambda_1 T(\mathbf{v}_1) + \lambda_2 T(\mathbf{v}_2). \qquad (\#)$$

Proof. Let T be a linear function. Then,

$$\begin{aligned} T(\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2) &= T(\lambda_1\mathbf{v}_1) + T(\lambda_2\mathbf{v}_2) && \text{(from the addition condition)} \\ &= \lambda_1 T(\mathbf{v}_1) + \lambda_2 T(\mathbf{v}_2) && \text{(using the scalar multiplication condition twice),} \end{aligned}$$

and hence (#) is satisfied.

Conversely, suppose (#) is satisfied. Then, for $\lambda_1 = \lambda_2 = 1$, the condition the (#) becomes the addition condition, while for $\lambda_2 = 0$ the condition reduces to the scalar multiplication condition. The proof is complete. $\square$

Theorem 2 can be used to simplify the test for linearity, since it means that only one condition must be checked instead of the two separate conditions of the original definition.

Example 6. Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^3$, defined by

$$T(\mathbf{x}) = \begin{pmatrix} 3x_1 - x_2 \\ 4x_2 \\ 5x_1 + 6x_2 \end{pmatrix} \quad \text{for} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2,$$

is a linear map.

SOLUTION. For $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$ and $\lambda, \lambda' \in \mathbb{R}$, we have

$$\lambda\mathbf{x} + \lambda'\mathbf{x}' = \begin{pmatrix} \lambda x_1 + \lambda' x_1' \\ \lambda x_2 + \lambda' x_2' \end{pmatrix} \in \mathbb{R}^2,$$

and hence

$$\begin{aligned} T(\lambda\mathbf{x}+\lambda'\mathbf{x}') &= \begin{pmatrix} 3(\lambda x_1+\lambda' x_1')-(\lambda x_2+\lambda' x_2') \\ 4(\lambda x_2+\lambda' x_2') \\ 5(\lambda x_1+\lambda' x_1')+6(\lambda x_2+\lambda' x_2') \end{pmatrix} \\ &= \lambda \begin{pmatrix} 3x_1-x_2 \\ 4x_2 \\ 5x_1+6x_2 \end{pmatrix} + \lambda' \begin{pmatrix} 3x_1'-x_2' \\ 4x_2' \\ 5x_1'+6x_2' \end{pmatrix} = \lambda T(\mathbf{x})+\lambda' T(\mathbf{x}'). \end{aligned}$$

Thus, from Theorem 2, T is a linear map. ♢

A generalisation of Theorem 2 is also of considerable importance in the theory and applications of linear maps.

Theorem 3. *If T is a linear map with domain V and S is a set of vectors in V, then the function value of a linear combination of S is equal to the corresponding linear combination of the function values of S, that is, if $S=\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ and $\lambda_1,\ldots,\lambda_n$ are scalars, then*

$$T(\lambda_1\mathbf{v}_1+\cdots+\lambda_n\mathbf{v}_n)=\lambda_1 T(\mathbf{v}_1)+\cdots+\lambda_n T(\mathbf{v}_n).$$

Proof. This, left as an exercise (see question 5), is based on an easy inductive argument.

Theorem 3 has many uses. Some examples of its use are as follows.

Example 7. Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be a linear map with values

$$T\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}3\\7\end{pmatrix} \quad T\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-5\\6\end{pmatrix}, \quad T\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}-2\\8\end{pmatrix}.$$

Find the function value at $\mathbf{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$.

Solution. We have

$$\mathbf{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=x_1\begin{pmatrix}1\\0\\0\end{pmatrix}+x_2\begin{pmatrix}0\\1\\0\end{pmatrix}+x_3\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

From Theorem 3, the function value at $\mathbf{x}$ is

$$\begin{aligned} T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} &= x_1T\begin{pmatrix}1\\0\\0\end{pmatrix}+x_2T\begin{pmatrix}0\\1\\0\end{pmatrix}+x_3T\begin{pmatrix}0\\0\\1\end{pmatrix} \\ &= x_1\begin{pmatrix}3\\7\end{pmatrix}+x_2\begin{pmatrix}-5\\6\end{pmatrix}+x_3\begin{pmatrix}-2\\8\end{pmatrix} \\ &= \begin{pmatrix}3x_1-5x_2-2x_3\\7x_1+6x_2+8x_3\end{pmatrix}. \end{aligned}$$

♢

Example 8. Show that the function $T : \mathbb{R}^3 \to \mathbb{R}^2$ with function values, $T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}3\\7\end{pmatrix}$, $T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-5\\6\end{pmatrix}$, $T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-2\\8\end{pmatrix}$, and $T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-4\\20\end{pmatrix}$, is not a linear map.

SOLUTION. We have

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Hence, if T is a linear map, we have from Theorem 3 that

$$\begin{aligned} T\begin{pmatrix}1\\1\\1\end{pmatrix} &= T\begin{pmatrix}1\\0\\0\end{pmatrix} + T\begin{pmatrix}0\\1\\0\end{pmatrix} + T\begin{pmatrix}0\\0\\1\end{pmatrix} \\ &= \begin{pmatrix}3\\7\end{pmatrix} + \begin{pmatrix}-5\\6\end{pmatrix} + \begin{pmatrix}-2\\8\end{pmatrix} = \begin{pmatrix}-4\\21\end{pmatrix}. \end{aligned}$$

But, $T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-4\\20\end{pmatrix} \neq \begin{pmatrix}-4\\21\end{pmatrix}$, and hence T is not a linear map. ◊

Examples 7 and 8 are actually special cases of the following extremely important result.

Theorem 4. *For a linear map $T : V \to W$, the function values for every vector in the domain are known if and only if the function values for a basis of the domain are known.*

Further, if $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for the domain V then for all $\mathbf{v} \in V$ we have

$$T(\mathbf{v}) = x_1 T(\mathbf{v}_1) + \cdots + x_n T(\mathbf{v}_n),$$

where $x_1, \dots, x_n$ are the scalars in the unique linear combination $\mathbf{v} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$ of the basis B.

Proof. It follows from Theorem 3 that

$$T(\mathbf{v}) = x_1 T(\mathbf{v}_1) + \cdots + x_n T(\mathbf{v}_n).$$

The theorem follows immediately. □

7.2 Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $m \times n$ matrices

If you look at the examples of functions with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ given in the previous section, you will see that if $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $\mathbf{x} = \begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} \in \mathbb{R}^n$, then we can write $T(\mathbf{x})$ as $A\mathbf{x}$ where A is an $m \times n$ matrix. In this section we are going to show that every matrix A represents a linear map and conversely that every linear map with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ can be represented by a matrix. Because of the close relation between T and the corresponding A and we prefer to write $A\mathbf{x}$ for $T(\mathbf{x})$ instead of $\mathbf{x}A$, the vector $\mathbf{x}$ **must be a column vector**.

We begin with the following theorem.

Theorem 1. *For each $m \times n$ matrix A, the function $T_A : \mathbb{R}^n \to \mathbb{R}^m$, defined by*

$$T_A(\mathbf{x}) = A\mathbf{x} \qquad \text{for} \quad \mathbf{x} \in \mathbb{R}^n,$$

is a linear map.

Proof. We check the addition and scalar multiplication conditions, using the properties $A(\mathbf{x}+\mathbf{x}') = A\mathbf{x} + A\mathbf{x}'$ and $A(\lambda\mathbf{x}) = \lambda A\mathbf{x}$ from Chapter 4

Addition Condition. For all $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$, we have

$$T_A(\mathbf{x}+\mathbf{x}') = A(\mathbf{x}+\mathbf{x}') = A\mathbf{x} + A\mathbf{x}' = T_A(\mathbf{x}) + T_A(\mathbf{x}').$$

Scalar Multiplication Condition. For all $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, we have

$$T_A(\lambda\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda T_A(\mathbf{x}).$$

Thus, since both the addition and scalar multiplication conditions are satisfied, $T_A(\mathbf{x}) = A\mathbf{x}$ is a linear map. $\square$

The matrix equation $A\mathbf{x} = \mathbf{y}$ therefore has the interpretation that $\mathbf{y} = T_A(\mathbf{x}) = A\mathbf{x}$ is the function value of T_A at the point $\mathbf{x}$, or, for linear equations, the vector $\mathbf{y}$ may be regarded as the function value of the vector $\mathbf{x}$.

Example 1. Find a linear map T_A such that $T_A(\mathbf{x}) = A\mathbf{x}$ for the matrix

$$A = \begin{pmatrix} 3 & 4 \\ -1 & 0 \\ -5 & 6 \end{pmatrix}.$$

Solution. Since A has 3 rows and 2 columns, the domain is $\mathbb{R}^2$, the codomain is $\mathbb{R}^3$, and the map $T_A : \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$T_A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\mathbf{x} = \begin{pmatrix} 3x_1 + 4x_2 \\ -x_1 \\ -5x_1 + 6x_2 \end{pmatrix}.$$

$\diamondsuit$

Theorem 1 and Example 1 show that a matrix can be used to define a linear map. We shall now show that every linear map with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ can be represented by an $m \times n$ matrix with real entries. The basic theorem which establishes this result is the following.

Theorem 2 (Matrix Representation Theorem)**.** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let the vectors $\mathbf{e}_j$ for $1 \leqslant j \leqslant n$ be the standard basis vectors for $\mathbb{R}^n$. Then the $m \times n$ matrix A whose columns are given by*

$$\mathbf{a}_j = T(\mathbf{e}_j) \quad \text{for} \quad 1 \leqslant j \leqslant n$$

has the property that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

Proof. Every vector $\mathbf{x} \in \mathbb{R}^n$ can be written as a unique linear combination of the standard basis vectors, that is,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n.$$

Then, from Theorem 3 of Section 7.1, we have

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) \\ &= x_1T(\mathbf{e}_1) + \cdots + x_nT(\mathbf{e}_n) \\ &= x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n, \end{aligned}$$

where $\mathbf{a}_j = T(\mathbf{e}_j)$. Now $\mathbf{a}_j \in \mathbb{R}^m$ for $1 \leqslant j \leqslant n$, and hence from Proposition 3 of Section 6.4 the linear combination can be rewritten in the matrix form $A\mathbf{x}$, where A is the matrix with the $\mathbf{a}_j$ as its columns. Thus, $T(\mathbf{x}) = A\mathbf{x}$ and the proof is complete. ☐

The Representation Theorem can be used to construct a matrix for any given linear map with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$, or more generally from $\mathbb{F}^n$ to $\mathbb{F}^m$ for any field $\mathbb{F}$.

Example 2. Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 - 5x_2 + 6x_3 \\ 5x_2 + 31x_3 \end{pmatrix}.$$

[Notice that we are using columns.]

SOLUTION. The first column of the matrix A is the vector given by

$$T(\mathbf{e}_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix},$$

the second column is given by

$$T(\mathbf{e}_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix},$$

and the third column is given by

$$T(\mathbf{e}_3) = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 31 \end{pmatrix}.$$

Thus, the matrix A is

$$A = \begin{pmatrix} 3 & -5 & 6 \\ 0 & 5 & 31 \end{pmatrix}.$$

◊

An alternative method, which is often simpler, of writing down the matrix for a given linear function is shown in the following example.

Example 3. Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for the linear map $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$T_A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 - 3x_2 + 4x_3 - 5x_4 \\ -2x_1 + 3x_4 \\ x_1 - 5x_2 + 6x_3 - 8x_4 \end{pmatrix}.$$

Solution. As usual for a linear map with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ (here $n = 4$ and $m = 3$), the components of the function value look like the left hand side of a system of linear equations. This system of equations is given by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 & - & 3x_2 & + & 4x_3 & - & 5x_4 \\ -2x_1 & & & & & + & 3x_4 \\ x_1 & - & 5x_2 & + & 6x_3 & - & 8x_4 \end{pmatrix} = \begin{pmatrix} 2 & -3 & 4 & -5 \\ -2 & 0 & 0 & 3 \\ 1 & -5 & 6 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Then the coefficient matrix, namely

$$A = \begin{pmatrix} 2 & -3 & 4 & -5 \\ -2 & 0 & 0 & 3 \\ 1 & -5 & 6 & -8 \end{pmatrix},$$

has the required property that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^4$. $\Diamond$

In this section, we have shown that a matrix always defines a linear map and that a linear map between the vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ can always be represented by a matrix.

This result can easily be generalised to linear maps between any two finite–dimensional vector spaces. This theorem is of fundamental importance in both the mathematical theory of linear maps and in applying the ideas of linear maps to practical problems. See Section 7.6.

7.3 Geometric examples of linear transformations

In this section we shall examine some of the geometric mappings which can be represented by linear maps and matrices. These mappings include stretching and compression, reflections, rotations, projections, and the dot and cross products with a fixed vector.

We shall begin by looking at geometric interpretations which can be given to simple types of matrices.

Example 1 (Reflection in $\mathbb{R}^2$)**.** The simplest examples of reflections in $\mathbb{R}^2$ are reflections in one of the coordinate axes. An example of a reflection in the x_1-axis is shown in Figure 3. Note that the reflection of the point with the position vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is the point represented by the position vector $\mathbf{x}' = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$. This reflection can be represented by the 2×2 diagonal matrix with a negative diagonal entry given by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

since

$$A\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} = \mathbf{x}'.$$

Note that we know that this reflection is a linear map since we have found a matrix that describes the effect of the reflection. $\Diamond$

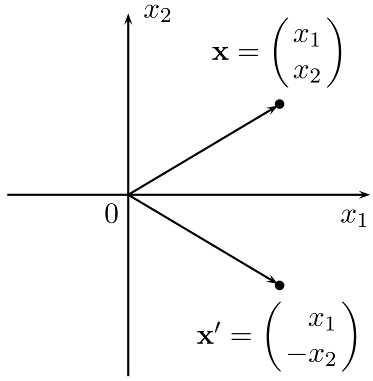


Figure 3: A reflection in the x_1-axis.

Note that a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ will map the position vector of a point in an n-dimensional space to the position vector of a point in an m-dimensional space. The following proposition tells us a linear transformation will map a line to a line or a point.

Proposition 1. Suppose that $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map. It maps a line in $\mathbb{R}^n$ to either a line or a point in $\mathbb{R}^m$.

Proof. In Chapter 2 a line in $\mathbb{R}^n$ through a point represented by $\mathbf{a}$ parallel to $\mathbf{v} \neq \mathbf{0}$ is defined to be the set

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{a} + \lambda\mathbf{v} \text{ for some } \lambda \in \mathbb{R}\}.$$

By Theorem 2 in Section 7.1, $T(\mathbf{a} + \lambda\mathbf{v}) = T(\mathbf{a}) + \lambda T(\mathbf{v})$. Hence T maps the line to the following subset of $\mathbb{R}^m$.

$$\{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = T(\mathbf{a}) + \lambda T(\mathbf{v}) \text{ for some } \lambda \in \mathbb{R}\}.$$

This set is a line when $T(\mathbf{v}) \neq \mathbf{0}$ and it contain a single vector $T(\mathbf{a})$ otherwise. $\square$

Remark. Using similar argument, we can show that T maps a line segment with end points of position vectors $\mathbf{a}$ and $\mathbf{b}$ to a line segment with end points of position vectors $T(\mathbf{a})$ and $T(\mathbf{b})$.

Example 2 (Stretching and compression in $\mathbb{R}^2$)**.** Let A be a 2×2 diagonal matrix with positive diagonal entries, that is, a matrix of the form

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{with } \lambda_1 > 0, \ \lambda_2 > 0.$$

Then the function value of $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is $\mathbf{y} = T_A(\mathbf{x}) = A\mathbf{x}$, where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \end{pmatrix}.$$

Thus, $b_1 = \lambda_1 x_1$ and $b_2 = \lambda_2 x_2$, and hence the effect of the matrix is simply to multiply the first component x_1 by the scalar λ_1 and the second component x_2 by the scalar λ_2. Note that the first standard basis vector $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is transformed into $\begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} = \lambda_1 \mathbf{e}_1$, that is, its direction remains the same but it is either stretched (if $\lambda_1 > 1$) or compressed (if $\lambda_1 < 1$). Similarly, the second standard basis vector $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is transformed into $\begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix} = \lambda_2 \mathbf{e}_2$ with a resulting stretching if $\lambda_2 > 1$ or compression if $\lambda_2 < 1$.

Figure 4(a) shows a picture of a 5-point star with vertices $A(1,5)$, $B(4,3)$, $C(3,-1)$, $D(-1,-1)$ and $E(-2,3)$. Suppose X is the point $(1,3)$ on the line segment BE.

Hence the position vectors of these points are respectively

$$\mathbf{a} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix},$$
$$\mathbf{c} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \qquad \mathbf{d} = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$
$$\mathbf{e} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

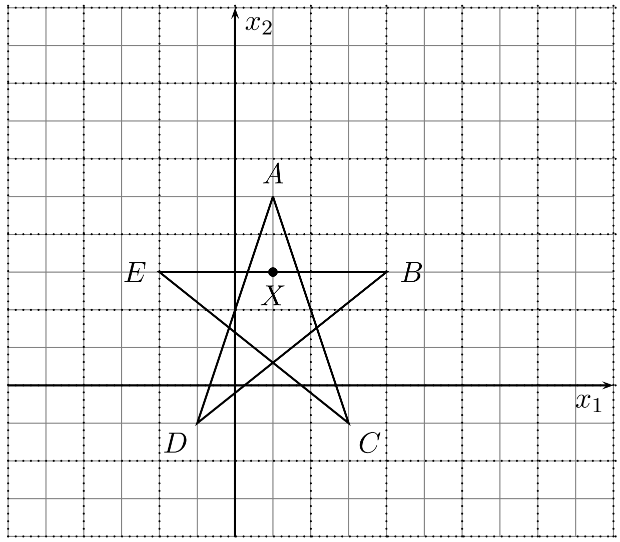


Figure 4(a): A 5-point star.

When $\lambda_1 = \lambda_2 = 2$, the matrix A is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. The points in Figure 4(a) will be "transformed" to A', B', C', D', E' and X' according to

$$A\mathbf{a} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix},$$

and similarly,

$$A\mathbf{b} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}, \quad A\mathbf{c} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}, \quad A\mathbf{d} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \quad A\mathbf{e} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} \text{ and } A\mathbf{x} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}.$$

By Theorem 1 and the remark after it, the line segment AB will be transformed to $A'B'$ and so on. The star will be transformed to one shown in Figure 4(b).

When $\lambda_1 = \lambda_2 = 0.5$, the matrix A is $\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$. The points in Figure 4(a) will be "transformed" according to

$$A\mathbf{a} = \begin{pmatrix} 0.5 \\ 2.5 \end{pmatrix}, \; A\mathbf{b} = \begin{pmatrix} 2 \\ 1.5 \end{pmatrix}, \; A\mathbf{c} = \begin{pmatrix} 1.5 \\ -0.5 \end{pmatrix}, \; A\mathbf{d} = \begin{pmatrix} -0.5 \\ -0.5 \end{pmatrix}, \; A\mathbf{e} = \begin{pmatrix} -1 \\ 1.5 \end{pmatrix} \quad \text{and} \quad A\mathbf{x} = \begin{pmatrix} 0.5 \\ 1.5 \end{pmatrix}.$$

The star will be transformed to one shown in Figure 4(c).

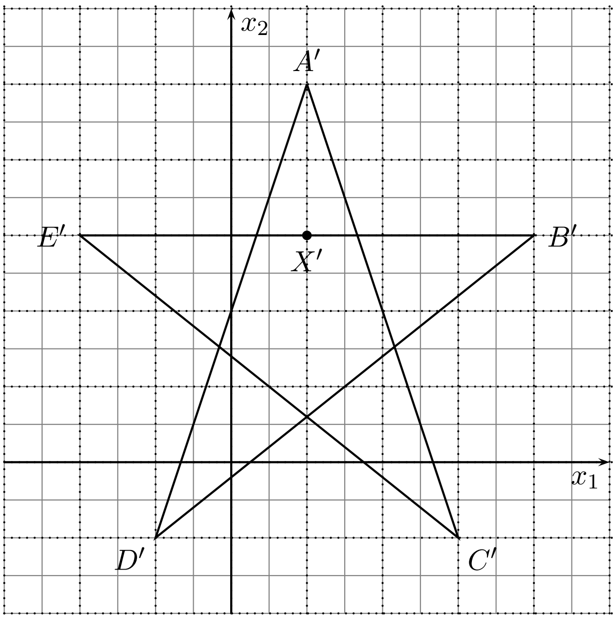


Figure 4(b): Image under $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

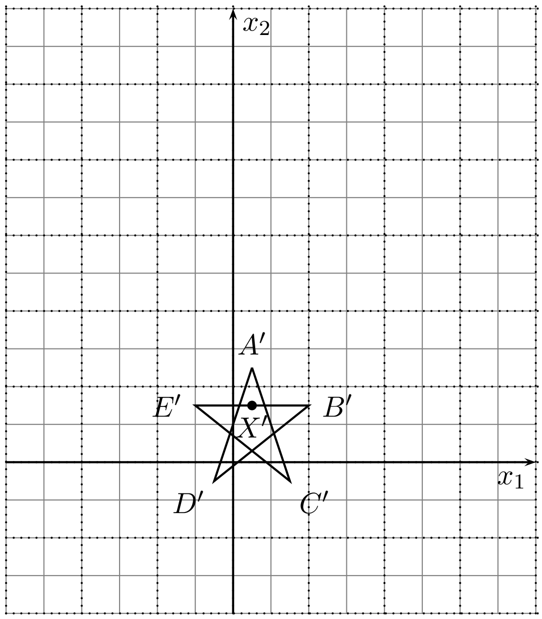


Figure 4(c): Image under $A = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$.

Figure 4(d) shows the image of the 5-point star when $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

The star is stretched to twice the width horizontally.

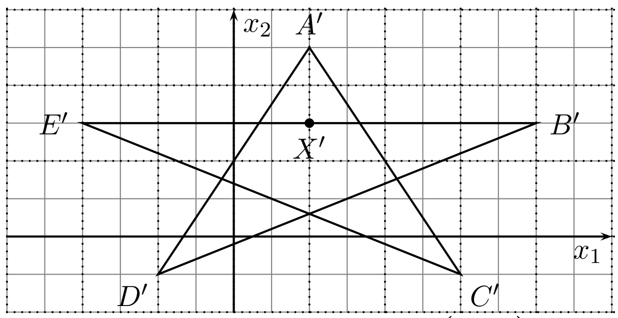


Figure 4(d): Image under $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

$\diamond$

Example 3 (Rotation in a plane)**.** Suppose that X is an arbitrary point in a plane and then X is rotated about the origin O anticlockwise by an angle α to a new position X'. Let $\mathbf{x}$ and $\mathbf{x}'$ be the position vectors of the points X and X' respectively. Show that the function $R_\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $R_\alpha(\mathbf{x}) = \mathbf{x}'$ is a linear transformation. Find the matrix A_α such that $A_\alpha \mathbf{x} = \mathbf{x}'$.

SOLUTION. We know that R_α will be linear if, for all vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ and scalar $\lambda \in \mathbb{R}$, the following two conditions are satisfied.

Addition condition	$R_\alpha(\mathbf{a}+\mathbf{b}) = R_\alpha(\mathbf{a}) + R_\alpha(\mathbf{b})$
Scalar multiplication condition	$R_\alpha(\lambda\mathbf{a}) = \lambda R_\alpha(\mathbf{a})$.

We can see the addition condition from Figure 5. The vector formed adding $\mathbf{a}$ and $\mathbf{b}$ first then rotating the sum $\mathbf{a}+\mathbf{b}$ is the same as the one formed by first rotating $\mathbf{a}$ and $\mathbf{b}$ then adding these rotated vectors.

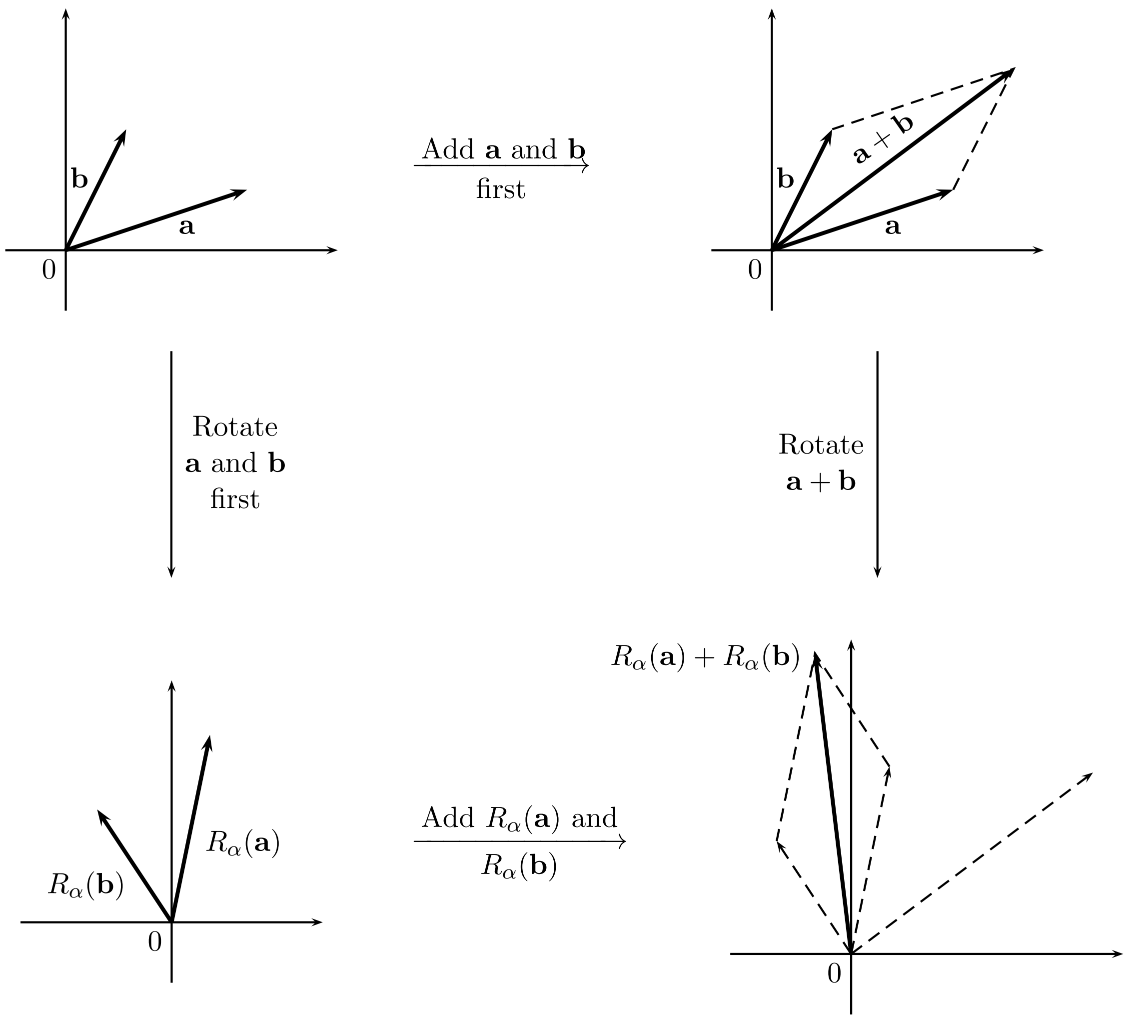


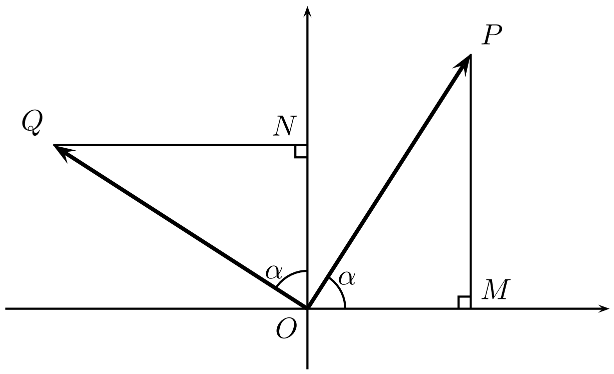
Figure 5: The geometry of the addition condition for rotations.

You should attempt to draw a picture to illustrate the scalar multiplication condition. In any case, since both the addition condition and the scalar multiplication condition hold, R_α is a linear transformation.

By Theorem 2 in Section 7.2, the columns of the matrix A_α are $R_\alpha(\mathbf{e}_1)$ and $R_\alpha(\mathbf{e}_2)$. From

Figure 6 and the fact that the length of both $R_\alpha(\mathbf{e}_1)$ and $R_\alpha(\mathbf{e}_2)$ are 1, we have

$$R_\alpha(\mathbf{e}_1) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \quad \text{and} \quad R_\alpha(\mathbf{e}_2) = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}.$$



$$\begin{aligned} R_\alpha(\mathbf{e_1}) &= \overrightarrow{OP} \\ R_\alpha(\mathbf{e_2}) &= \overrightarrow{OQ} \\ OM &= ON = \cos\alpha \\ PM &= QN = \sin\alpha \end{aligned}$$

Figure 6.

The matrix

$$A_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

is called the **rotation matrix** for angle α. $\Diamond$

Example 4 (Projections)**.** The projection of a vector $\mathbf{x} \in \mathbb{R}^n$ on a fixed, non-zero vector $\mathbf{b} \in \mathbb{R}^n$ is given by

$$\text{proj}_{\mathbf{b}}\mathbf{x} = \frac{\mathbf{x}\cdot\mathbf{b}}{|\mathbf{b}|^2}\,\mathbf{b}.$$

Show that the function $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$T(\mathbf{x}) = \text{proj}_{\mathbf{b}}\mathbf{x} \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^n.$$

is a linear map.

Solution. Clearly, the domain and codomain are vector spaces. Instead of proving that T is linear by geometric properties, we use the algebraic properties of the dot product. For all $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$

$$T(\mathbf{x}+\mathbf{x}') = \text{proj}_{\mathbf{b}}(\mathbf{x}+\mathbf{x}') = \frac{(\mathbf{x}+\mathbf{x}')\cdot\mathbf{b}}{|\mathbf{b}|^2}\,\mathbf{b} = \frac{\mathbf{x}\cdot\mathbf{b}+\mathbf{x}'\cdot\mathbf{b}}{|\mathbf{b}|^2}\,\mathbf{b} = T(\mathbf{x}) + T(\mathbf{x}'),$$

and hence the addition condition is satisfied.

Finally, for all $\mathbf{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,

$$T(\lambda\mathbf{x}) = \frac{(\lambda\mathbf{x})\cdot\mathbf{b}}{|\mathbf{b}|^2}\,\mathbf{b} = \lambda\left(\frac{\mathbf{x}\cdot\mathbf{b}}{|\mathbf{b}|^2}\,\mathbf{b}\right) = \lambda T(\mathbf{x}).$$

Thus, the scalar multiplication condition is also satisfied, and therefore T is a linear map. $\Diamond$

Example 5 (Dot Product)**.** Let $\mathbf{b}$ be a fixed vector in $\mathbb{R}^n$. Show that the function $T : \mathbb{R}^n \to \mathbb{R}$, defined by

$$T(\mathbf{x}) = \mathbf{b} \cdot \mathbf{x} \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^n,$$

is a linear map.

The proof that T is a linear map is similar to the previous example and is left as an exercise. $\Diamond$

The following examples show the importance of the linear maps defined by dot product.

Example 6. From Example 5, for each $1 \leqslant i \leqslant n$ the function $P_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined by

$$P_i(\mathbf{x}) = \mathbf{e}_i \cdot \mathbf{x} \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^n,$$

where $\mathbf{e}_i$ is the ith standard basis element. It is not difficult to see that if $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, the value $P_i(\mathbf{x})$ is simply x_i, the ith component of $\mathbf{x}$. $\Diamond$

This example can be generalised to any basis of $\mathbb{R}^n$.

Example 7. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$ and $1 \leqslant i \leqslant n$. The function $P_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined by

$$P_i(\mathbf{x}) = \mathbf{v}_i \cdot \mathbf{x} \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^n.$$

By the argument used in Example 6 in Section 6.6 we can prove that if $\mathbf{x} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n$, the value $P_i(\mathbf{x})$ is simply λ_i, the coefficient of $\mathbf{v}_i$ in the unique way of writing $\mathbf{x}$ as a linear combination of the basis vectors. $\Diamond$

7.4 Subspaces associated with linear maps

There are two important subspaces associated with a linear map. These subspaces are called the **kernel** (or **null space**, which is the name Maple uses) of the linear map and the **image** (or range) of the map. Informally, the kernel is the set of zeroes of the function, and the image is the set of all function values. This is shown diagrammatically in Figure 7. You should of course not take this picture too literally — all the sets drawn as discs are vector spaces!

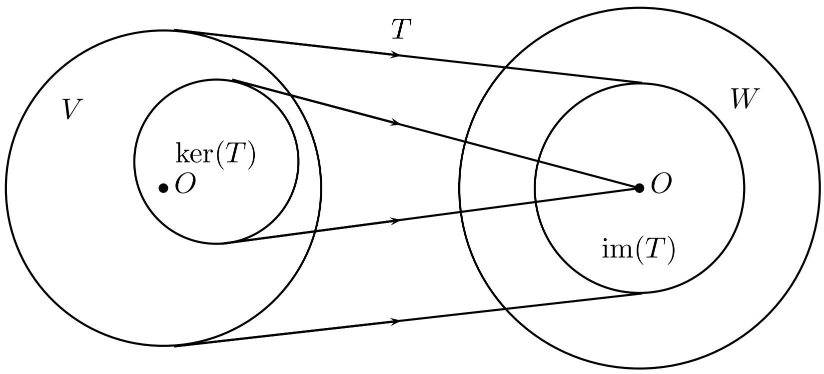


Figure 7: Kernel and image.

7.4.1 The kernel of a map

You will be familiar with the fact that one of the important properties of functions (for example of quadratics or polynomials) is the values of their zeroes. The set of zeroes of a linear map is also of importance.

Definition 1. *Let $T : V \to W$ be a linear map. Then the* **kernel** *of T (written* $\ker(T)$*) is the set of all zeroes of T, that is, it is the subset of the domain V defined by*

$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}.$$

Example 1. Showing that a vector $\mathbf{v}$ is in the kernel of a linear map T is simply a verification that $T(\mathbf{v}) = \mathbf{0}$. In particular, $\mathbf{0} \in \ker(T)$ for any linear map T, since $T(\mathbf{0}) = \mathbf{0}$. ♢

Example 2 (Dot Product)**.** In Example 5 of Section 7.3, we showed that the function $T : \mathbb{R}^n \to \mathbb{R}$ defined by $T(\mathbf{x}) = \mathbf{b} \cdot \mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$ is a linear map. The kernel of T is

$$\ker(T) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{b} \cdot \mathbf{x} = 0\},$$

that is, $\ker(T)$ is the set of vectors which are orthogonal to the given fixed vector $\mathbf{b}$.

For the special case that $\mathbf{x} \in \mathbb{R}^3$, the equation $\mathbf{b} \cdot \mathbf{x} = 0$ is the point-normal form of the equation of a plane in $\mathbb{R}^3$, and hence $\ker(T)$ corresponds to the points on a plane with normal $\mathbf{b}$ which passes through the origin. ♢

For the important special case of a linear map $T_A : \mathbb{R}^n \to \mathbb{R}^m$ associated with an $m \times n$ matrix A, the kernel has a simple interpretation. For matrices, the definition of kernel becomes:

Definition 2. *For an $m \times n$ matrix A, the* **kernel** *of A is the subset of $\mathbb{R}^n$ defined by*

$$\ker(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\},$$

that is, it is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Example 3. Suppose that $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Since

$$A\mathbf{x} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \times 2 + 2 \times (-1) \\ 3 \times 2 + 6 \times (-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which means that $\mathbf{x} \in \ker(A)$ by definition.

To find the kernel of a matrix A, we need to find the solution set of the equation $A\mathbf{x} = \mathbf{0}$.

Example 4. Find the kernel of the matrix A, where

$$A = \begin{pmatrix} 1 & 4 & 2 & 7 \\ 3 & 6 & 0 & 15 \\ 2 & -4 & -8 & 2 \end{pmatrix}.$$

Solution. The kernel is the set of all solutions of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$. An equivalent row-echelon form U for A is

$$U = \begin{pmatrix} 1 & 4 & 2 & 7 \\ 0 & -6 & -6 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We then set parameters to the variables of the non-leading columns — $x_3 = \lambda_1$ and $x_4 = \lambda_2$. By back substitution, we obtain the solution of $A\mathbf{x} = \mathbf{0}$ as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ -1 \\ 0 \\ 1 \end{pmatrix},$$

and hence,

$$\ker(A) = \left\{ \mathbf{x} \in \mathbb{R}^4 : \mathbf{x} = \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{for} \quad \lambda_1, \lambda_2 \in \mathbb{R} \right\}.$$

In this example, the kernel can be interpreted geometrically as a plane in $\mathbb{R}^4$ through the origin parallel to $\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ -1 \\ 0 \\ 1 \end{pmatrix}$. ◊

A very important property of the kernel of a linear map is given in the following theorem.

Theorem 1. *If $T : V \to W$ is a linear map, then* $\ker(T)$ *is a subspace of the domain V.*

Proof. We use the Subspace Theorem (Theorem 1) of Section 6.3 and prove that $\ker(T)$ is a non-empty subset of V which is closed under addition and scalar multiplication.

It is not the empty set, since $T(\mathbf{0}) = \mathbf{0}$ and so $\mathbf{0} \in \ker(T)$.

Suppose that $\mathbf{v}, \mathbf{v}' \in \ker(T)$ and $\lambda \in \mathbb{F}$. Since T is linear, it satisfies the addition and scalar multiplication conditions, so we have

$$T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}') = \mathbf{0} \qquad \text{and} \qquad T(\lambda\mathbf{v}) = \lambda T(\mathbf{v}) = \mathbf{0},$$

and hence both $\mathbf{v} + \mathbf{v}'$ and $\lambda\mathbf{v}$ are in $\ker(T)$. Thus, $\ker(T)$ is closed under addition and scalar multiplication, and the proof is complete. ☐

The dimension of the kernel is important and is given a special name.

Definition 3. *The* ***nullity*** *of a linear map T is the dimension of* $\ker(T)$. *The* ***nullity*** *of a matrix A is the dimension of* $\ker(A)$.

Proposition 2. Let A be an $m \times n$ matrix with real entries and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ the associated linear transformation. Then

$$\ker(T_A) = \ker(A)$$

Proof. $T_A(\mathbf{x}) = \mathbf{0} \Leftrightarrow A\mathbf{x} = \mathbf{0}$. ☐

The nullity of a matrix A can be easily obtained from the properties of row-echelon forms by using the following result.

Proposition 3. For a matrix A:

$$\begin{aligned} \text{nullity}(A) &= \text{maximum number of independent vectors in the solution space of } A\mathbf{x} = \mathbf{0} \\ &= \text{number of parameters in the solution of } A\mathbf{x} = \mathbf{0} \text{ obtained by Gaussian} \\ &\quad\ \text{elimination and back substitution} \\ &= \text{number of non-leading columns in an equivalent row-echelon form } U \text{ for } A. \end{aligned}$$

Although a general proof of this proposition is not difficult to construct, we shall restrict ourselves to looking at an example.

Example 4 (continued)**.** Find nullity(A), and a basis for $\ker(A)$, for

$$A = \begin{pmatrix} 1 & 4 & 2 & 7 \\ 3 & 6 & 0 & 15 \\ 2 & -4 & -8 & 2 \end{pmatrix}.$$

SOLUTION. We have found that any vector $\mathbf{x} \in \ker(A)$ can be written as

$$\mathbf{x} = \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

This is a linear combination of the two vectors in the parametric vector form for the solution of $A\mathbf{x} = \mathbf{0}$. These two vectors are linearly independent, since if $\mathbf{x} = \mathbf{0}$ then the parameters λ_1 and λ_2 are both zero (look at the third and fourth rows of the linear combination). Thus, we obtain a basis for $\ker(A)$,

$$\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

and hence $\text{nullity}(A) = \dim(\ker(A)) = 2$. This illustrates Proposition 3. $\Diamond$

For matrices, there is a close relationship between linear independence of the columns and the nullity of the matrix.

Proposition 4. The columns of a matrix A are linearly independent if and only if $\text{nullity}(A) = 0$.

Proof. From Proposition 1 of Section 6.5, the columns of A are linearly independent if and only if $\mathbf{x} = \mathbf{0}$ is the only solution of $A\mathbf{x} = \mathbf{0}$. That is, if and only if $\mathbf{0}$ is the only element of $\ker(A)$, in which case, $\text{nullity}(A) = \dim(\ker(A)) = 0$. $\square$

7.4.2 Image

The range or image of a function is the set of all function values (see, for example, Appendix 7.10). In this course we will usually use the term *image* rather than *range*. A formal definition of the image of a linear map is as follows.

Definition 4. *Let $T : V \to W$ be a linear map. Then the* ***image*** *of T is the set of all function values of T, that is, it is the subset of the codomain W defined by*

$$\text{im}(T) = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \quad \textit{for some} \quad \mathbf{v} \in V\}.$$

For the special case of a linear map associated with a real $m \times n$ matrix, the definition becomes:

Definition 5. *The* ***image*** *of an $m \times n$ matrix A is the subset of $\mathbb{R}^m$ defined by*

$$\text{im}(A) = \{\mathbf{b} \in \mathbb{R}^m : \mathbf{b} = A\mathbf{x} \quad \textit{for some} \quad \mathbf{x} \in \mathbb{R}^n\}.$$

We have met this set several times before. In the language of linear equations, this set $\text{im}(A)$ is just the set of all right-hand-side vectors $\mathbf{b}$ for which the equation $A\mathbf{x} = \mathbf{b}$ has a solution, and in vector-space language it is just the span of the columns of the matrix A, that is, the column space of A. Thus, we have

$$\begin{aligned}\operatorname{range}(A) = \operatorname{im}(A) = \operatorname{col}(A) &= \operatorname{span}(\text{columns of } A) \\ &= \{\mathbf{b} \in \mathbb{R}^m : A\mathbf{x} = \mathbf{b} \text{ has a solution}\}.\end{aligned}$$

These connections mean that any questions about the image of a matrix can be solved by the methods previously given for linear equations and spans in Chapter 6. It is useful to give an example, as it will serve to review some of the previous results on vector spaces and linear equations.

Example 4 (continued)**.** Find conditions on a vector $\mathbf{b}$ for $\mathbf{b}$ to be in $\operatorname{im}(A)$ where

$$A = \begin{pmatrix} 1 & 4 & 2 & 7 \\ 3 & 6 & 0 & 15 \\ 2 & -4 & -8 & 2 \end{pmatrix}.$$

SOLUTION. We look for conditions on $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3$ for $A\mathbf{x} = \mathbf{b}$ to have a solution.

For hand calculations on a small system of equations, the simplest method of solution is as follows. Instead of put b_1, b_2, b_3 on the right hand side, the three right-hand columns of the following augmented matrix are coefficients of b_1, b_2, b_3.

$$(A|\mathbf{b}) = \left(\begin{array}{cccc|ccc} 1 & 4 & 2 & 7 & 1 & 0 & 0 \\ 3 & 6 & 0 & 15 & 0 & 1 & 0 \\ 2 & -4 & -8 & 2 & 0 & 0 & 1 \end{array}\right).$$

On reduction to row-echelon form using Gaussian elimination, we find

$$(U|\mathbf{y}) = \left(\begin{array}{cccc|ccc} 1 & 4 & 2 & 7 & 1 & 0 & 0 \\ 0 & -6 & -6 & -6 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 & -2 & 1 \end{array}\right).$$

This system of equations has a solution if and only if the components of $\mathbf{b}$ satisfy

$$4b_1 - 2b_2 + b_3 = 0.$$

In this case, $\operatorname{im}(A)$ has a geometric interpretation as a plane through the origin in $\mathbb{R}^3$ with normal $\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$. In vector-space language, $\operatorname{im}(A)$ is a two-dimensional subspace of $\mathbb{R}^3$. $\diamondsuit$

Note that for larger matrices it is preferable to use computer packages such as Maple to solve the equations.

An extremely important property of the image of a linear map is as follows.

Theorem 5. *Let $T : V \to W$ be a linear map between vector spaces V and W. Then $\operatorname{im}(T)$ is a subspace of the codomain W of T.*

Proof. We use the Subspace Theorem (Theorem 1 of Section 6.3) and show that $\mathrm{im}(T)$ is a non-empty subset of W which is closed under vector addition and scalar multiplication.

We note first that $\mathrm{im}(T)$ is a subset of W. Since, from Proposition 1 of Section 7.1, $T(\mathbf{0}) = \mathbf{0}$, we see that $\mathbf{0} \in \mathrm{im}(T)$.

Closure under addition. If $\mathbf{w}, \mathbf{w}' \in \mathrm{im}(T)$, then

$$\mathbf{w} = T(\mathbf{v}) \quad \text{for some} \quad \mathbf{v} \in V \qquad \text{and} \qquad \mathbf{w}' = T(\mathbf{v}') \quad \text{for some} \quad \mathbf{v}' \in V,$$

and hence,

$$\mathbf{w} + \mathbf{w}' = T(\mathbf{v}) + T(\mathbf{v}') = T(\mathbf{v} + \mathbf{v}').$$

But, since V is a vector space, $\mathbf{v} + \mathbf{v}' \in V$, and therefore $\mathbf{w} + \mathbf{w}' \in \mathrm{im}(T)$ and $\mathrm{im}(T)$ is closed under addition.

Closure under scalar multiplication. If $\mathbf{w} \in \mathrm{im}(T)$ and $\lambda \in F$, then

$$\mathbf{w} = T(\mathbf{v}) \quad \text{for some} \quad \mathbf{v} \in V, \qquad \text{and hence} \quad \lambda\mathbf{w} = \lambda T(\mathbf{v}) = T(\lambda\mathbf{v}).$$

But, since V is a vector space, $\lambda\mathbf{v} \in V$, and therefore $\lambda\mathbf{w} \in \mathrm{im}(T)$ and $\mathrm{im}(T)$ is closed under scalar multiplication.

The proof is complete. □

The next result is obvious.

Proposition 6. Let A be an $m \times n$ matrix with real entries and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ the associated linear transformation. Then

$$\mathrm{im}(A) = \mathrm{im}(T_A)$$

We have shown in Theorem 5 that $\mathrm{im}(T)$ is always a subspace of the codomain of T. Thus, the fundamental vector-space properties of basis and dimension must apply to $\mathrm{im}(T)$. The dimension of the image is very important and it has therefore been given a special name.

Definition 6. *The **rank** of a linear map T is the dimension of* $\mathrm{im}(T)$. *The **rank** of a matrix A is the dimension of* $\mathrm{im}(A)$.

The rank is usually regarded as one of the most important properties of a matrix, since it is the maximum number of linearly independent right-hand-side vectors for which a solution to $A\mathbf{x} = \mathbf{b}$ can be found. Some important properties of the rank of a matrix are summarised in the following proposition.

Proposition 7. For a matrix A:

$$\begin{aligned} \mathrm{rank}(A) &= \textit{maximal number of linearly independent columns of } A \\ &= \textit{number of leading columns in a row-echelon form } U \textit{ for } A \end{aligned}$$

Proof. From before, $\mathrm{im}(A) = \mathrm{col}(A) = \mathrm{span}\,(\text{columns of } A)$. A basis for span(columns of A) is a maximal set of linearly independent columns of A. One maximal set of linearly independent columns of A are the columns which reduce to leading columns in a row-echelon form U. Hence,

$$\begin{aligned}\text{number of leading columns} &= \text{number of linearly independent columns of } A\\ &= \text{number of vectors in basis for } \mathrm{col}(A)\\ &= \dim(\mathrm{col}(A)) = \dim(\mathrm{im}(A)) = \mathrm{rank}(A).\end{aligned}$$

$\square$

Example 4 (continued)**.** Find rank(A), and a basis for im(A), for

$$A = \begin{pmatrix} 1 & 4 & 2 & 7 \\ 3 & 6 & 0 & 15 \\ 2 & -4 & -8 & 2 \end{pmatrix}.$$

SOLUTION. Since there are two leading columns (1 and 2) in the row-echelon form

$$U = \begin{pmatrix} 1 & 4 & 2 & 7 \\ 0 & -6 & -6 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

hence $\mathrm{rank}(A) = 2$.

A basis for im(A) therefore contains two vectors. One maximal set of linearly independent columns of A is columns 1 and 2 of A, and hence a basis for im(A) is $\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ -4 \end{pmatrix} \right\}$. $\diamond$

7.4.3 Rank, nullity and solutions of $A\mathbf{x} = \mathbf{b}$

Example 4 illustrates the following important fact about the rank and nullity of a matrix.

Theorem 8 (Rank-Nullity Theorem for Matrices)**.** *For any matrix A,*

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = \textit{number of columns of } A.$$

Proof. Let U be an equivalent row-echelon form for A obtained by the Gaussian elimination algorithm. Then, from Proposition 7, $\mathrm{rank}(A) = \text{number of leading columns in } U$. Also, from Proposition 3, $\mathrm{nullity}(A) = \text{number of non-leading columns in } U$. But, of course, number of leading columns + number of non-leading columns = total number of columns in U = total number of columns in A, and the result is proved. $\square$

The above theorem is equivalent to the following result for linear maps between finite dimensional vector spaces.

Theorem 9 (Rank-Nullity Theorem)**.** *Suppose V and W are finite dimensional vector spaces and $T : V \to W$ is linear. Then*

$$\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(V).$$

A proof of Theorem 9, using a suitably constructed basis of V, is given in Section 7.9.

For matrices, a very common use of rank and nullity is to classify the types of solution of a system of linear equations $A\mathbf{x} = \mathbf{b}$. The basic results are summarised in the following proposition.

Theorem 10. *The equation* $A\mathbf{x} = \mathbf{b}$ *has:*

1. *no solution if* $\operatorname{rank}(A) \neq \operatorname{rank}([A|\mathbf{b}])$, *and*

2. *at least one solution if* $\operatorname{rank}(A) = \operatorname{rank}([A|\mathbf{b}])$. *Further,*

   i) if $\operatorname{nullity}(A) = 0$ *the solution is unique, whereas,*

   ii) if $\operatorname{nullity}(A) = \nu > 0$, *then the general solution is of the form*

$$\mathbf{x} = \mathbf{x}_p + \lambda_1 \mathbf{k}_1 + \cdots + \lambda_\nu \mathbf{k}_\nu \qquad \textit{for} \quad \lambda_1, \ldots, \lambda_\nu \in \mathbb{R},$$

where $\mathbf{x}_p$ *is any solution of* $A\mathbf{x} = \mathbf{b}$, *and where* $\{\mathbf{k}_1, \ldots, \mathbf{k}_\nu\}$ *is a basis for* $\ker(A)$.

Proof. Let U and $(U|\mathbf{y})$ be equivalent row-echelon forms for A and $(A|\mathbf{b})$ obtained by the Gaussian elimination algorithm.

Now, from Chapter 3 we know that $A\mathbf{x} = \mathbf{b}$ has a solution if and only if the right-hand-side column $\mathbf{y}$ is a non-leading column, and hence if and only if the numbers of leading columns in U and $[U|\mathbf{y}]$ are equal. But, from Proposition 7, $\operatorname{rank}(A) =$ number of leading columns in U, and $\operatorname{rank}([A|\mathbf{b}]) =$ number of leading columns in $(U|\mathbf{y})$, and thus $A\mathbf{x} = \mathbf{b}$ has a solution if and only if the ranks of A and $(A|\mathbf{b})$ are equal.

The proof of parts 2(i) and 2(ii) follows immediately from the relation between solutions of a non-homogeneous system and the corresponding homogeneous system (see Chapter 3 and the fact that $\operatorname{nullity}(A)$ is equal to the maximum number of linearly independent solutions of the homogeneous equation $A\mathbf{v} = \mathbf{0}$. $\square$

NOTE. A similar type of solution to the general solution in 2(ii) above is also obtained as the solution of a **linear** differential equation. In this differential equation case, $\mathbf{x}_p$ is called a "particular solution" and the parametric terms are called the "complementary function". The similarity between the two types of solution is due to the fact that both the matrix and differential equation problems involve linear functions.

Example 5. Illustrate the above rules with the system of equations given by the augmented matrix

$$(A|\mathbf{b}) = \left(\begin{array}{ccccc|c} 0 & 0 & 2 & -1 & 3 & 1 \\ 1 & -2 & \frac{1}{2} & 3 & 4 & -1 \\ 3 & 2 & 1 & 4 & 6 & 0 \end{array}\right).$$

SOLUTION. Gaussian elimination gives

$$(U|\mathbf{y}) = \left(\begin{array}{ccccc|c} 1 & -2 & \frac{1}{2} & 3 & 4 & -1 \\ 0 & 8 & -\frac{1}{2} & -5 & -6 & 3 \\ 0 & 0 & 2 & -1 & 3 & 1 \end{array}\right).$$

The system has a solution as $\mathbf{y}$ is a non-leading column. The number of leading columns in U is 3, and hence $\operatorname{rank}(A) = 3$. Similarly, $\operatorname{rank}(A|\mathbf{b}) = 3$.

On back substitution, the parametric vector form of the solution is found to be

$$\mathbf{x} = \begin{pmatrix} -\frac{7}{16} \\ \frac{13}{32} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -\frac{31}{16} \\ \frac{21}{32} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -\frac{31}{16} \\ \frac{21}{32} \\ \frac{3}{2} \\ 0 \\ 1 \end{pmatrix} = \mathbf{x}_p + \lambda_1 \mathbf{k}_1 + \lambda_2 \mathbf{k}_2.$$

Note that

$$A \begin{pmatrix} -\frac{7}{16} \\ \frac{13}{32} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}; \qquad A \begin{pmatrix} -\frac{31}{16} \\ \frac{21}{32} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \qquad A \begin{pmatrix} -\frac{31}{16} \\ \frac{21}{32} \\ \frac{3}{2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and that the number of non-leading columns of $U = 2 = \text{nullity}(A) =$ the number of parameters in solution. This parametric vector form is the general solution of $A\mathbf{x} = \mathbf{b}$, and as expected it is the sum of a "particular solution" of $A\mathbf{x} = \mathbf{b}$ and a "complementary function" which is a linear combination of two linearly independent solutions of $A\mathbf{x} = \mathbf{0}$ (a basis for $\ker(A)$). $\diamondsuit$

7.5 Further applications and examples of linear maps

Although the theory that we have developed so far in this chapter applies to all linear maps, most examples have been restricted to maps for which the domain is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$. In this section we will give some examples of linear maps in which the domain and codomain are other kinds of vector spaces.

A simple, but useful, map in any vector space is the map which takes a vector to itself.

Example 1. The **identity map** $\text{id}_V : V \to V$ on a vector space V is defined by

$$\text{id}_V(\mathbf{v}) = \mathbf{v} \qquad \text{for all} \quad \mathbf{v} \in V.$$

This map is linear, since for all $\mathbf{v}, \mathbf{v}' \in V$ and all scalar λ,

$$\begin{aligned} &\text{id}_V(\mathbf{v} + \mathbf{v}') = \mathbf{v} + \mathbf{v}' = \text{id}_V(\mathbf{v}) + \text{id}_V(\mathbf{v}') \\ \text{and} \qquad &\text{id}_V(\lambda \mathbf{v}) = \lambda \mathbf{v} = \lambda\, \text{id}_V(\mathbf{v}). \end{aligned}$$

$\diamondsuit$

In Theorems 2 and 3 of Section 7.1 we showed that linear maps had the important property that they preserved linear combinations. As the next example shows, every linear combination can also be regarded as the image of a linear map.

[**H**] **Example 2.** Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a subset of a vector space V and let $x_1, \dots, x_n \in \mathbb{R}$. Show that the map T given by

$$T(\mathbf{x}) = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n \qquad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

is a linear map.

Solution. The rule obviously defines a function, since $T(\mathbf{x})$ is uniquely determined for each $\mathbf{x} \in \mathbb{R}^n$.

To prove that T is linear we use Theorem 2 of Section 7.1.

Suppose $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$ and $\lambda, \lambda' \in \mathbb{R}$. Then

$$\lambda\mathbf{x} + \lambda'\mathbf{x}' = (\lambda x_1 + \lambda' x'_1, \dots, \lambda x_n + \lambda' x'_n),$$

and hence

$$\begin{aligned} T(\lambda\mathbf{x} + \lambda'\mathbf{x}') &= (\lambda x_1 + \lambda' x'_1)\mathbf{v}_1 + \cdots + (\lambda x_n + \lambda' x'_n)\mathbf{v}_n \\ &= \lambda(x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n) + \lambda'(x'_1\mathbf{v}_1 + \cdots + x'_n\mathbf{v}_n) \\ &= \lambda T(\mathbf{x}) + \lambda' T(\mathbf{x}'). \end{aligned}$$

Thus T is a linear map. $\Diamond$

This example shows that all properties of linear combinations discussed in Chapter 6 can in fact be restated in the language of linear maps.

Example 3. Let V be a vector space over the real numbers, and let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an ordered basis for V. For any $\mathbf{v} \in V$, we can write the vector uniquely as a linear combination of B,

$$\mathbf{v} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n.$$

Show that the rule $T : V \to \mathbb{R}^n$ defined by

$$T(\mathbf{v}) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{for} \quad \mathbf{v} \in V$$

is a linear map.

Solution. Obviously the function T has domain V and codomain $\mathbb{R}^n$ which are vector spaces.

To prove that this function is a linear map, we check the addition and scalar multiplication conditions. For all $\lambda \in \mathbb{R}$ and $\mathbf{v}, \mathbf{v}' \in V$, we can write in a unique way that

$$\mathbf{v} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n \quad \text{and} \quad \mathbf{v}' = x'_1\mathbf{v}_1 + \cdots + x'_n\mathbf{v}_n.$$

Since

$$\mathbf{v} + \mathbf{v}' = (x_1 + x'_1)\mathbf{v}_1 + \cdots + (x_n + x'_n)\mathbf{v}_n \quad \text{and} \quad \lambda\mathbf{v} = (\lambda x_1)\mathbf{v}_1 + \cdots + (\lambda x_n)\mathbf{v}_n$$

we have

$$T(\mathbf{v}+\mathbf{v}') = \begin{pmatrix} x_1 + x_1' \\ \vdots \\ x_n + x_n' \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = T(\mathbf{v}) + T(\mathbf{v}')$$

$$\text{and} \qquad T(\lambda\mathbf{v}) = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \lambda T(\mathbf{v}),$$

and hence T is a linear map. $\Diamond$

[**X**] REMARK: The above example simply says that the function which maps a vector to its coordinate vector with respect to a basis is linear.

We shall now give some examples of linear maps associated with the vector spaces of polynomials and real-valued functions.

Example 4. Show that the function $T : \mathbb{C}^3 \to \mathbb{P}_2(\mathbb{C})$ defined by $T\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = p$, where $a_0, a_1, a_2 \in \mathbb{C}$ and

$$p(z) = a_0 + a_1 + (a_2 + 3a_0)z + a_1 z^2 \qquad \text{for} \quad z \in \mathbb{C},$$

is a linear map.

Before we solve this problem, note that an argument of T is a complex vector $\mathbf{a} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \in \mathbb{C}^3$, while the corresponding function value $p = T(\mathbf{a})$ is a complex polynomial of degree less than or equal to 2. Some function values are the polynomials given by

$$\left(T\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right)(z) = 1 + 2 + (3+3)z + 2z^2 = 3 + 6z + 2z^2,$$

$$\left(T\begin{pmatrix} -2 \\ 0 \\ i \end{pmatrix}\right)(z) = -2 + (i-6)z, \qquad \left(T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)(z) = 0.$$

SOLUTION. We use 7.1.2. Let $\lambda, \lambda' \in \mathbb{C}$, $\mathbf{a} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \in \mathbb{C}^3$, $\mathbf{a}' = \begin{pmatrix} a_0' \\ a_1' \\ a_2' \end{pmatrix} \in \mathbb{C}^3$ and let $s = T(\lambda\mathbf{a} + \lambda'\mathbf{a}')$, $p = T(\mathbf{a})$, and $q = T(\mathbf{a}')$. Then

$$s = T\begin{pmatrix} \lambda a_0 + \lambda' a_0' \\ \lambda a_1 + \lambda' a_1' \\ \lambda a_2 + \lambda' a_2' \end{pmatrix},$$

and hence

$$\begin{aligned} s(z) &= \lambda a_0 + \lambda' a_0' + \lambda a_1 + \lambda' a_1' + \big(\lambda a_2 + \lambda' a_2' + 3(\lambda a_0 + \lambda' a_0')\big) z + (\lambda a_1 + \lambda' a_1') z^2 \\ &= \lambda\big(a_0 + a_1 + (a_2 + 3a_0)z + a_1 z^2\big) + \lambda'\big(a_0' + a_1' + (a_2' + 3a_0')z + a_1' z^2\big) \\ &= \lambda p(z) + \lambda' q(z). \end{aligned}$$

Thus, $T(\lambda \mathbf{a} + \lambda' \mathbf{a}') = s = \lambda p + \lambda' q = \lambda T(\mathbf{a}) + \lambda' T(\mathbf{a}')$, and hence $\diamondsuit$

As the next examples show, calculus provides many important applications of linear maps as differentiation and integration are both associated with linear maps.

Example 5 (Differentiation of polynomials)**.** Let $\mathbb{P}_n(\mathbb{R})$ be the vector space of real polynomials of degree less than or equal to n. Show that the function $D : \mathbb{P}_n(\mathbb{R}) \to \mathbb{P}_{n-1}(\mathbb{R})$, defined by

$$D(p) = p', \qquad \text{where} \qquad p'(x) = \frac{dp}{dx} \qquad \text{for} \quad p \in \mathbb{P}_n(\mathbb{R}) \quad \text{and} \quad x \in \mathbb{R},$$

is a linear map.

Solution. Firstly, we note that if p is a polynomial of degree k then the derivative p' exists and is a polynomial of degree $k-1$. Hence, if $p \in \mathbb{P}_n(\mathbb{R})$ then $p' \in \mathbb{P}_{n-1}(\mathbb{R})$, and thus $D : \mathbb{P}_n(\mathbb{R}) \to \mathbb{P}_{n-1}(\mathbb{R})$ is a function.

We now prove D is linear by checking the addition and scalar multiplication conditions of the definition of a linear map.

For all $p, q \in \mathbb{P}_n(\mathbb{R})$, we have from the properties of derivatives that

$$(p+q)'(x) = \frac{d}{dx}\big(p(x) + q(x)\big) = \frac{d}{dx}p(x) + \frac{d}{dx}q(x) = p'(x) + q'(x).$$

Hence, $D(p+q) = (p+q)' = p' + q' = D(p) + D(q)$, and the addition condition is satisfied.

Further, for all $p \in \mathbb{P}_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$, we have that

$$\frac{d}{dx}(\lambda p(x)) = \lambda \frac{d}{dx} p(x),$$

and hence $D(\lambda p) = \lambda D(p)$ and the scalar multiplication condition is also satisfied. Thus, D is a linear map. (Note that nullity $(D) = 1$ and $\text{rank}(D) = n$.) $\diamondsuit$

Example 6 (Integration of Polynomials)**.** Show that the function $I : \mathbb{P}_n(\mathbb{R}) \to \mathbb{P}_{n+1}(\mathbb{R})$, defined by $I(p) = q$, where

$$q(x) = \int_0^x p(t)\,dt \qquad \text{for} \quad p \in \mathbb{P}_n(\mathbb{R}) \quad \text{and} \quad x \in \mathbb{R},$$

is a linear map.

Before solving this problem, we give some examples of function values of I. If p is the zero polynomial we have

$$I(p) = \int_0^x 0\,dx = 0,$$

whereas if p is the polynomial of degree 2 defined by

$$p(x) = 1 - 3x + 4x^2 \qquad \text{then} \qquad I(p) = \int_0^x (1 - 3t + 4t^2)dt = x - \frac{3}{2}x^2 + \frac{4}{3}x^3$$

is a polynomial of degree 3.

SOLUTION. We will prove that I is a linear map by using Theorem 2 of Section 7.1.

For ease of writing, we let $q_1 = I(p_1)$, $q_2 = I(p_2)$ and $q = I(\lambda_1 p_1 + \lambda_2 p_2)$, where $p_1, p_2 \in \mathbb{P}_n(\mathbb{R})$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. Then, from the properties of integration,

$$\begin{aligned} q(x) &= \int_0^x \big(\lambda_1 p_1(t) + \lambda_2 p_2(t)\big)\,dt \\ &= \lambda_1 \int_0^x p_1(t)dt + \lambda_2 \int_0^x p_2(t)dt \\ &= \lambda_1 q_1(x) + \lambda_2 q_2(x). \end{aligned}$$

Thus, $I(\lambda_1 p_1 + \lambda_2 p_2) = q = \lambda_1 q_1 + \lambda_2 q_2 = \lambda_1 I(p_1) + \lambda_2 I(p_2)$, and hence I is a linear map. (Note that nullity $(I) = 0$ and rank$(I) = n + 1$.) $\Diamond$

The next example is one of a class of so-called **integral transforms** which have many uses in mathematics, science, engineering and economics.

[X] **Example 7. The Laplace transform**. Let s and a be real numbers, and let V_a be the set of real-valued functions on the interval $(0, \infty)$ defined by

$$V_a = \left\{ f \in \mathcal{R}[(0,\infty)] : \int_0^\infty e^{-st} f(t)dt \quad \text{exists for} \quad a < s < \infty \right\}.$$

Now, from the theory of integration, if $f, g \in V_a$ and $\lambda, \mu \in \mathbb{R}$ then

$$\int_0^\infty e^{-st}(\lambda f(t) + \mu g(t))dt \qquad \text{exists for} \quad a < s < \infty,$$

and thus $\lambda f + \mu g \in V_a$. Hence, from the Alternative Subspace Theorem of Section 6.8, V_a is a subspace of the vector space $\mathcal{R}[(0,\infty)]$ of all real-valued functions with domain $(0,\infty)$.

We now define a function $L : V_a \to \mathcal{R}[(a,\infty)]$ with function values $L(f) = f_L$, where f_L is the function from the domain (a,∞) to the codomain $\mathbb{R}$ defined by

$$f_L(s) = \int_0^\infty e^{-st} f(t)dt \qquad \text{for} \quad a < s < \infty.$$

f_L is called the **Laplace transform** of the function f. $\Diamond$

We shall now prove that L is a linear map.

Proof. Let $f, g \in V_a$ and $\lambda, \mu \in \mathbb{R}$. Then, as noted above, the function $h = \lambda f + \mu g$ is an element of V_a, and its Laplace transform $h_L = L(\lambda f + \mu g)$ satisfies

$$\begin{aligned} h_L(s) &= \int_0^\infty e^{-st} h(t)dt \\ &= \int_0^\infty e^{-st}\big(\lambda f(t) + \mu g(t)\big)dt \\ &= \lambda \int_0^\infty e^{-st} f(t)dt + \mu \int_0^\infty e^{-st} g(t)dt \\ &= \lambda f_L(s) + \mu g_L(s). \end{aligned}$$

Thus,

$$L(\lambda f + \mu g) = h_L = \lambda f_L + \mu g_L = \lambda L(f) + \mu L(g),$$

and hence L is a linear map. □

The Laplace transform is widely used in, for example, the solution of linear differential equations and the theory of dynamical systems. To understand the technique, work through question 54. It has extensive applications in electrical engineering, computer science, physics, applied and pure mathematics, and so on.

To finish this section, we shall describe some applications of linear maps in the areas of optics, chemical engineering, electrical engineering, and population dynamics.

[**H**] **Example 8** (Optics)**.** White light is made up of the seven colours: red, orange, yellow, green, blue, indigo, and violet. Assume that a green filter transmits 0% of red and violet, 5% of orange and indigo, 20% of yellow and blue, and 90% of the green light that falls on it. This green filter can be represented by a linear map, and the kernel and image of the map have a simple interpretation.

We let a_1, a_2, a_3, a_4, a_5, a_6 and a_7 be the intensities of the red, orange, yellow, green, blue, indigo and violet light respectively in the incoming light. Then the filter can be represented by the very simple linear map $T : \mathbb{R}^7 \to \mathbb{R}^7$ given by

$$T(\mathbf{a}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix}.$$

The kernel of the filter is the set of all possible incident light such that there is no transmitted light. The filter will transmit no light if the incoming light contains only red and violet light. Thus, a basis for the kernel of the map T which models the filter is $[1\,0\,0\,0\,0\,0\,0]^T$ (red light only) and $[0\,0\,0\,0\,0\,0\,1]^T$ (violet light only), and the nullity is 2.

The image of the filter is the transmitted light. The transmitted light can only contain orange, yellow, green, blue, and indigo, and these colours may be taken as a basis for the image. As five basic colours are transmitted, the rank is 5. Mathematically, a basis for the image of the map T is the set of 5 vectors $[0\,1\,0\,0\,0\,0\,0]^T$, $[0\,0\,1\,0\,0\,0\,0]^T$, $[0\,0\,0\,1\,0\,0\,0]^T$, $[0\,0\,0\,0\,1\,0\,0]^T$ and $[0\,0\,0\,0\,0\,1\,0]^T$. ♢

[X] **Example 9** (Population Dynamics). As a simple model of the growth of a human population in a given country, we neglect males, and divide females into the six age groups of 0–14, 15–29, 30–44, 45–59, 60–74 and 75–89. It is found that, on average, 5% of the 0–14 group, 3% of the 15–29 group, 5% of the 30–44 group, 10% of the 45–59 group, 40% of the 60–74 group and 100% of the 75–89 group die in a fifteen-year period, whereas, on average, 0% of the 0–14 group, 50% of the 15–29 group, 45% of the 30–44 group, 6% of the 45–59 group and 0% of the 60–74 and 75–89 groups give birth to a female baby in a fifteen-year period.

The population at any time can be represented by a linear-transformation model as follows.

Starting at some convenient time, say January 1 1970, at which the population of the country is known, we divide time into intervals of length 15 years. Let k represent the kth of these 15-year periods, starting with $k = 0$ in the period 1970–1984. Thus, $k = 1$ represents 1985–1999, k=2 represents 2000–2014, etc. We then let

x_1(k)	number of females of age 0–14 in interval k
x_2(k)	number of females of age 15–29 in interval k
x_3(k)	number of females of age 30–44 in interval k
x_4(k)	number of females of age 45–59 in interval k
x_5(k)	number of females of age 60–74 in interval k
x_6(k)	number of females of age 75–89 in interval k

Now, if we know the values of $x_j(k), 1 \leqslant j \leqslant 6$ in a given interval k, we can calculate the values of $x_j(k+1), 1 \leqslant j \leqslant 6$ in the $k+1$th interval from the given data. For example, the females in the age group 15–29 in interval $k+1$ are the survivors of those in the age group 0–14 in interval k. Thus, $x_2(k+1) = 0.95x_1(k)$. The numbers in all groups other than the 0–15 group can be obtained in a similar fashion. In our model, the only way that females can enter the 0–15 group is to be born from mothers in the 15–29, 30–44 and 45–59 groups. Thus,

$$x_1(k+1) = 0.50x_2(k) + 0.45x_3(k) + 0.06x_4(k).$$

We therefore obtain the model

$$\mathbf{x}(k+1) = \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \\ x_5(k+1) \\ x_6(k+1) \end{pmatrix} = \begin{pmatrix} 0 & 0.5 & 0.45 & 0.06 & 0 & 0 \\ 0.95 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.97 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.95 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.90 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.60 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \\ x_6(k) \end{pmatrix} = A\mathbf{x}(k).$$

Thus, the population vector in time interval $k+1$ is the image under the linear map whose matrix is given above of the population vector in interval k. $\diamondsuit$

7.6 [X] Representation of linear maps by matrices

We have seen in Section 7.2 that every linear map between the vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ can be represented by a matrix. The next theorem shows that this result can be generalised to any linear map between any finite-dimensional vector spaces.

Theorem 1 (General Matrix Representation Theorem). *Let $T : V \to W$ be a linear map from an n-dimensional vector space V to an m-dimensional vector space W, and let $B_V = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an ordered basis for V and $B_W = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ be an ordered basis for W. Then, there is a unique $m \times n$ matrix A such that*

$$[T(\mathbf{v})]_{B_W} = A[\mathbf{v}]_{B_V}.$$

Further, A is the matrix whose columns are

$$\mathbf{a}_j = [T(\mathbf{v}_j)]_{B_W} \qquad \text{for} \quad 1 \leqslant j \leqslant n.$$

Proof. Let $\mathbf{w} = T(\mathbf{v}) \in W$ and

$$[\mathbf{v}]_{B_V} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad \text{i.e.} \quad \mathbf{v} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n.$$

By Theorem 3 of Section 7.1, we have

$$\mathbf{w} = T(\mathbf{v}) = T(x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n) = x_1T(\mathbf{v}_1) + \cdots + x_nT(\mathbf{v}_n).$$

Now, we have shown (Example 3 of Section 7.5) that taking coordinate vectors is a linear map, and hence, on using Theorem 3 of Section 7.1, we have

$$[\mathbf{w}]_{B_W} = [T(\mathbf{v})]_{B_W} = x_1[T(\mathbf{v}_1)]_{B_W} + \cdots + x_n[T(\mathbf{v}_n)]_{B_W} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n,$$

where $\mathbf{a}_j = [T(\mathbf{v}_j)]_{B_W}$. Finally, we note that $\mathbf{a}_j \in \mathbb{R}^m$ for $1 \leqslant j \leqslant n$, and hence, from Proposition 3 of Section 6.4, the linear combination can be rewritten in the matrix form $A\mathbf{x}$, and the theorem is proved. $\square$

NOTE.

1. This theorem says that if T maps $\mathbf{v}$ to $T(\mathbf{v})$ then the matrix A transforms the coordinate vector $[\mathbf{v}]_{B_V}$ for $\mathbf{v}$ with respect to an ordered basis B_V of the domain into the coordinate vector $[T(\mathbf{v})]_{B_W}$ for $T(\mathbf{v})$ with respect to an ordered basis B_W of the codomain.

2. It is important to note that the matrix A depends **only** on the map T, the ordered basis B_V and the ordered basis B_W. It does **not** depend on the particular vector $\mathbf{v}$ of V whose image is being found.

Theorem 1 provides a straightforward algorithm for finding a matrix representation.

Algorithm 1. Constructing a matrix representation for a linear map.

1. Find a basis $B_V = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for the domain V and a basis $B_W = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ for the codomain W.

2. Find the function values $T(\mathbf{v}_j)$, $1 \leqslant j \leqslant n$, of the domain basis vectors.

3. Find the coordinate vectors $[T(\mathbf{v}_j)]_{B_W}$ of the function values $T(\mathbf{v}_j)$ with respect to the codomain basis B_W.

4. Construct the $m \times n$ matrix A with the coordinate vectors $[T(\mathbf{v}_j)]_{B_W}$, $1 \leqslant j \leqslant n$, as its **columns**.

Example 1. Construct the matrix representation of the derivative map $D : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R})$, defined by

$$D(p) = p', \qquad \text{where} \qquad p'(x) = \frac{dp}{dx} \quad \text{for} \quad x \in \mathbb{R},$$

with respect to the standard bases of $\mathbb{P}_3(\mathbb{R})$ and $\mathbb{P}_2(\mathbb{R})$.

Solution. As shown in Example 5 of Section 7.5, D is a linear map, and hence it can be represented by a matrix. We again follow algorithm 1.

For the domain $\mathbb{P}_3(\mathbb{R})$, the standard basis is $\{1, x, x^2, x^3\}$. The function values of the basis vectors are

$$D(1) = 0, \qquad D(x) = 1, \qquad D(x^2) = 2x, \qquad D(x^3) = 3x^2.$$

The coordinate vectors of these function values with respect to the standard basis $\{1, x, x^2\}$ of the codomain are $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$ respectively. Hence the matrix is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

As an example of the use of this matrix, we find $D(p)$ for

$$p(x) = 1 - 3x + 4x^2 + 7x^3.$$

The coordinate vector for p with respect to the standard basis $\{1, x, x^2, x^3\}$ of the domain $\mathbb{P}_3(\mathbb{R})$ is $\begin{pmatrix} 1 \\ -3 \\ 4 \\ 7 \end{pmatrix}$. Then the coordinate vector for $D(p)$ with respect to the standard basis $\{1, x, x^2\}$ of the codomain $\mathbb{P}_2(\mathbb{R})$ is

$$A \begin{pmatrix} 1 \\ -3 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ 21 \end{pmatrix},$$

and hence $D(p)$ is given by

$$\big(D(p)\big)(x) = -3 + 8x + 21x^2.$$

This is clearly the derivative of the polynomial p. $\diamondsuit$

A similar procedure to that given in Example 1 can be used to find a matrix representation for definite integration of polynomials.

In the simple case described in Example 1, it is obviously a waste of time to go through the matrix formalism. However, in more complicated examples, it is often useful to be able to use the powerful and efficient algorithms of matrix algebra to solve problems involving differentiation of polynomials.

Another example involving polynomials is as follows.

Example 2. Find the matrix with respect to standard bases in domain and codomain for the linear transformation $T : \mathbb{P}_2 \to \mathbb{C}^3$ defined by

$$T(a_0 + a_1 z + a_2 z^2) = \begin{pmatrix} 2a_0 + 3a_2 \\ -a_2 \\ 4a_1 + 6a_2 \end{pmatrix}.$$

Solution. For the domain, the standard basis is $\{1, z, z^2\}$ and hence the images of the domain basis vectors are

$$T(1) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \qquad T(z) = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}, \qquad T(z^2) = \begin{pmatrix} 3 \\ -1 \\ 6 \end{pmatrix}.$$

For the codomain, the standard basis for $\mathbb{C}^3$ is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. The coordinate vectors for this basis are just the three image vectors given above, and hence the matrix is

$$A = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 0 & -1 \\ 0 & 4 & 6 \end{pmatrix}.$$

$\diamond$

In the above examples, we have restricted the choice of bases to standard bases in domain and codomain. However, it is frequently possible to achieve great simplifications in calculations involving linear maps by using special choices of bases. Two examples of the simplification which can be achieved in this way are given below.

Example 3. A linear map $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T_A(\mathbf{x}) = A\mathbf{x}, \qquad \text{where} \qquad A = \begin{pmatrix} 3.12 & 0.16 & -0.32 \\ 4.76 & -1.32 & -7.36 \\ 2.8 & -1.6 & -1.8 \end{pmatrix}.$$

Find the matrix which represents T_A with respect to the bases in both domain and codomain given by the columns of the matrix

$$B = \begin{pmatrix} 1 & 0 & 4 \\ -3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

Solution. The images of the basis vectors for the domain basis are

$$T_A \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = A \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 4 \end{pmatrix}, \qquad T_A \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -10 \\ -5 \end{pmatrix}, \qquad T_A \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 3 \\ 6 \end{pmatrix}.$$

From algorithm 1, the columns of the matrix representing T_A with respect to the given codomain basis are just the coordinate vectors for the above images with respect to the codomain basis.

These are $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$ respectively. The required matrix has these three coordinate vectors as its columns, and it is therefore

$$X = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

◊

Note that, in this example, we have found a diagonal matrix to represent a transformation T_A. A general theory which shows the conditions under which a diagonal matrix can be found to represent a linear map will be developed in Chapter 8.

Example 4. A linear map $T_A : \mathbb{R}^3 \to \mathbb{R}^4$ is defined by

$$T_A(\mathbf{x}) = A\mathbf{x}, \qquad \text{where} \qquad A = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 4 & -1 \\ -2 & 0 & 5 \\ 3 & -4 & 4 \end{pmatrix}.$$

Find the matrix U which represents T_A for the standard basis in the domain $\mathbb{R}^3$ and for the basis in the codomain $\mathbb{R}^4$ which consists of the column vectors of the matrix

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 3 & 2 & 6 & 1 \end{pmatrix}.$$

SOLUTION. The standard basis for $\mathbb{R}^3$ is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, and hence the images of the domain basis vectors are

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \\ 3 \end{pmatrix}, \qquad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \\ -4 \end{pmatrix}, \qquad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 5 \\ 4 \end{pmatrix}.$$

These images are just the columns of the matrix A.

Now, following Algorithm 1, we must find the coordinate vectors for $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ with respect to the codomain basis given by the columns of L, using the standard Gaussian elimination algorithm.

They turn out to be $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 4 \\ -8 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -7 \\ 2 \\ 0 \end{pmatrix}$. Thus

$$U = \begin{pmatrix} 1 & 4 & 2 \\ 0 & -8 & -7 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that L is "lower triangular", U is "upper triangular", and that $LU = A$. $\Diamond$

Both of the above examples illustrate the importance of suitable choices of bases in solving problems about linear maps. If the bases are suitably chosen, a matrix representation for a map will usually take on a very simple form from which the properties of the map can be immediately read off. We have, of course, used a special case of this approach repeatedly throughout these notes, where we have solved most problems about linear equations, vector spaces, and linear maps by constructing row-echelon form matrices from which the solutions can be immediately read off.

7.7 [X] Matrix arithmetic and linear maps

We have seen in Sections 7.2 and 7.6 that every matrix defines a linear map and that every linear map between finite-dimensional vector spaces can be represented by a matrix. We would therefore expect that properties of matrices — such as matrix addition, multiplication of a matrix by a scalar, matrix multiplication, inverse of a matrix — should also have an interpretation in terms of properties of linear maps. In this section, we shall show that matrix addition corresponds to addition of linear maps, that multiplication of a matrix by a scalar corresponds to multiplication of a linear map by a scalar, and that matrix multiplication corresponds to composition of linear maps. The relationship between inverses of linear maps and matrix inverses is discussed in Section 7.8.

Proposition 1 (Addition)**.** Let A and B be real $m \times n$ matrices and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ and $T_B : \mathbb{R}^n \to \mathbb{R}^m$ be the linear maps given by $T_A(\mathbf{x}) = A\mathbf{x}$ and $T_B(\mathbf{x}) = B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then the sum $T = T_A + T_B$ is the linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\mathbf{x}) = (A + B)\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Proof. By definition of function addition, the sum $T = T_A + T_B$ is the linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ given by

$$T(\mathbf{x}) = (T_A + T_B)(\mathbf{x}) = T_A(\mathbf{x}) + T_B(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

Hence, on using the definition of T_A and T_B and the distributive law for matrix addition and multiplication, we have

$$T(\mathbf{x}) = A\mathbf{x} + B\mathbf{x} = (A + B)\mathbf{x} \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

$\square$

Proposition 2 (Multiplication by a Scalar)**.** Let A be a real $m \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear map defined by $T_A(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then the scalar multiple $T = \lambda T_A$ is the linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\mathbf{x}) = (\lambda A)\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Proof. By definition of multiplication of a function by a scalar, the function $T = \lambda T_A$ is given by

$$T(\mathbf{x}) = (\lambda T_A)(\mathbf{x}) = \lambda T_A(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

Hence, on using the definition of T_A, we have

$$T(\mathbf{x}) = \lambda(A\mathbf{x}) = (\lambda A)\mathbf{x} \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

$\square$

Proposition 3 (Multiplication and Composition)**.** Let A be a real $m \times n$ matrix and B be a real $n \times p$ matrix and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear map defined by $T_A(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$ and let $T_B : \mathbb{R}^p \to \mathbb{R}^n$ be the linear map defined by $T_B(\mathbf{x}) = B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^p$. Then the composite $T_A \circ T_B$ is the linear map $T : \mathbb{R}^p \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = (AB)\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^p$.

Proof. The composite $T_A \circ T_B$ is the linear map $T : \mathbb{R}^p \to \mathbb{R}^m$ defined by

$$T(\mathbf{x}) = (T_A \circ T_B)(\mathbf{x}) = T_A(T_B(\mathbf{x})) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^p.$$

Hence, on substituting the function values for T_A and T_B and using the associative law of matrix multiplication, we have

$$T(\mathbf{x}) = T_A(B\mathbf{x}) = A(B\mathbf{x}) = (AB)\mathbf{x} \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^p.$$

$\square$

NOTE. B is acting first, then A.

Similar relationships hold between linear transformations on general finite-dimensional vector spaces and their corresponding matrices. We shall give just one example of these more general theorems, whose proof we shall leave to the exercises.

Proposition 4. Let U, V and W be finite-dimensional vector spaces with bases B_U, B_V and B_W respectively. Let $T : U \to V$ and $S : V \to W$. Then

a) $S \circ T$ is a linear transformation from $U \to W$.

b) If the matrix for T with respect to B_U and B_V is A_T, the matrix for S with respect to B_V and B_W is A_S and the matrix for $S \circ T$ with respect to B_U and B_W is A_{ST}, then $A_{ST} = A_S A_T$.

7.8 [X] One-to-one, onto and invertible linear maps and matrices

In this section we shall discuss the main results involving the ideas of one-to-one, onto and inverses for linear maps and matrices and we shall show how these results for linear maps compare with the results for general functions summarised in Appendix 7.10.

7.8.1 Linear maps

The definitions of one-to-one, onto and inverse for linear maps are virtually identical to the definitions for general functions. However, for ease of reading it is convenient to restate them here. On applying the definitions of one-to-one, onto and inverse given in Appendix 7.10 to linear maps, we obtain the following definitions.

Definition 1. *A linear map $T : V \to W$ is said to be:*

a) ***one-to-one*** *if for all $\mathbf{v}_1, \mathbf{v}_2 \in V$, $T(\mathbf{v}_1) = T(\mathbf{v}_2)$ only if $\mathbf{v}_1 = \mathbf{v}_2$.*

b) ***onto*** *if for all $\mathbf{w} \in W$ there exists $\mathbf{v} \in V$ such that $\mathbf{w} = T(\mathbf{v})$, that is, if $\mathrm{im}(T) = W$.*

Definition 2. *Let $T : V \to W$ be a linear map. Then a function $S : W \to V$ is called an* ***inverse*** *of T if it satisfies the two conditions:*

a) *$S \circ T = \mathrm{id}_V$, where id_V is the identity map on V defined by $\mathrm{id}_V(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$.*

b) *$T \circ S = \mathrm{id}_W$, where id_W is the identity map on W defined by $\mathrm{id}_W(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w} \in W$.*

For linear maps, there is a simple connection between a map being one-to-one and the kernel of the map. This connection, which is not in general true for all functions, is as follows.

Proposition 1. A linear map $T : V \to W$ is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$, that is, if and only if $\mathrm{nullity}(T) = 0$.

Proof. We first prove that if T is one-to-one then $\ker(T) = \{\mathbf{0}\}$. Now, as T is a linear map $T(\mathbf{0}) = \mathbf{0}$, and as T is one-to-one $T(\mathbf{v}) = T(\mathbf{0})$ only if $\mathbf{v} = \mathbf{0}$. Thus, $T(\mathbf{v}) = \mathbf{0}$ only if $\mathbf{v} = \mathbf{0}$, and hence $\ker(T) = \{\mathbf{0}\}$. We next prove that if $\ker(T) = \{\mathbf{0}\}$ then T is one-to-one. We let $\mathbf{v}_1, \mathbf{v}_2$ satisfy $T(\mathbf{v}_1) = T(\mathbf{v}_2)$, that is, $T(\mathbf{v}_1) - T(\mathbf{v}_2) = \mathbf{0}$. But, as T is a linear map, $T(\mathbf{v}_1) - T(\mathbf{v}_2) = T(\mathbf{v}_1 - \mathbf{v}_2)$. Then as $\ker(T) = \{\mathbf{0}\}$, we have $T(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}$ only if $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$. Thus, $T(\mathbf{v}_1) = T(\mathbf{v}_2)$ only if $\mathbf{v}_1 = \mathbf{v}_2$, and hence T is one-to-one. $\square$

The result stated in Proposition 1 is not true in general for non-linear functions — for example, it is not true for the function of Example 1 of Appendix 7.10 which is neither one-to-one nor onto even though $f(x) = x^2 = 0$ only for $x = 0$.

Proposition 2. If the codomain W of a linear function $T : V \to W$ is finite dimensional then T is onto if and only if $\mathrm{rank}(T) = \dim(W)$.

Proof. We know that $\mathrm{im}(T)$ is a subspace of W and $\mathrm{rank}(T) = \ dim(\mathrm{im}(T))$ so, by Theorem 8 of Section 6.6, $\mathrm{rank}(T) = \dim(W)$ if and only if $\mathrm{im}(T) = W$, that is, if and only if T is onto. $\square$

The following result is also of importance for linear maps.

Proposition 3. If the domain and codomain of a linear map $T : V \to W$ are finite-dimensional then:

a) If T is one-to-one and onto then $\dim(V) = \dim(W)$.

b) If $\dim(V) = \dim(W)$ and T is one-to-one then T is onto.

c) If $\dim(V) = \dim(W)$ and T is onto then T is one-to-one.

Proof. The proofs of the three parts are based on the Rank-Nullity Theorem and Propositions 1 and 2 above.

a) If T is one-to-one then $\text{nullity}(T) = 0$, and if T is onto then $\text{rank}(T) = \dim(W)$. Therefore, if T is one-to-one and onto, we have from the Rank-Nullity Theorem that

$$\dim(V) = \text{rank}(T) + \text{nullity}(T) = \dim(W) + 0 = \dim(W).$$

b) If $\dim(V) = \dim(W)$ and T is one-to-one then $\text{nullity}(T) = 0$, and so

$$\text{rank}(T) = \dim(V) - \text{nullity}(T) = \dim(W),$$

and therefore T is onto.

c) If $\dim(V) = \dim(W)$ and T is onto then $\text{rank}(T) = \dim(W)$, and so

$$\text{nullity}(T) = \dim(V) - \text{rank}(T) = \dim(V) - \dim(W) = 0,$$

and therefore T is one-to-one.

$\square$

Proposition 4. If a linear map has an inverse then the inverse is also a linear map.

Proof. Let $T : V \to W$ be a linear map with inverse $S : W \to V$. Then, for all $\mathbf{v} \in V$ and for all $\mathbf{w} \in W$, we have $\mathbf{v} = S(\mathbf{w})$ if and only if $\mathbf{w} = T(\mathbf{v})$.

To prove S is linear, we must show that for all $\mathbf{w}_1, \mathbf{w}_2 \in W$ and all scalars $\lambda_1, \lambda_2 \in F$

$$S(\lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2) = \lambda_1 S(\mathbf{w}_1) + \lambda_2 S(\mathbf{w}_2).$$

Now if $S(\mathbf{w}_1) = \mathbf{v}_1$ and $S(\mathbf{w}_2) = \mathbf{v}_2$, we have $\mathbf{w}_1 = T(\mathbf{v}_1)$ and $\mathbf{w}_2 = T(\mathbf{v}_2)$, and hence on using the fact that T is linear we obtain

$$\lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 = \lambda_1 T(\mathbf{v}_1) + \lambda_2 T(\mathbf{v}_2) = T(\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2).$$

Thus, by definition of S,

$$\begin{aligned} S(\lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2) &= \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 \\ &= \lambda_1 S(\mathbf{w}_1) + \lambda_2 S(\mathbf{w}_2). \end{aligned}$$

The proof is complete. $\square$

A fundamental result which summarises the main properties of inverses of linear maps is as follows.

Theorem 5. *If V and W are finite-dimensional vector spaces and if $T : V \to W$ is a linear map then the following statements are equivalent:*

1. *T is invertible, that is, there exists a linear map $S : W \to V$ such that $(S \circ T)(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$ and such that $(T \circ S)(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w} \in W$.*
2. *T is one-to-one and onto.*
3. *$\dim(V) = \dim(W)$ and there exists $S : W \to V$ such that $(T \circ S)(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w} \in W$.*
4. *$\dim(V) = \dim(W)$ and there exists $S : W \to V$ such that $(S \circ T)(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$.*

Further, the map S in statements 1, 3 and 4 is the inverse of T.

Note. The phrase "The statements are equivalent" means that if **one** statement is true then **all** statements are true, and also that if **one** statement is false then **all** statements are false.

Proof. The equivalence of statements 1 and 2 follows immediately from Theorem 1 of Section 7.10. We shall now show that statements 1 and 3 are equivalent by first showing that statement 1 implies statement 3 and then by showing that statement 3 implies statement 1.

1 implies 3. If T is invertible, then there exists $S : W \to V$ such that $(T \circ S)(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w} \in W$. Further, as T is invertible it is also one-to-one and onto, and hence, from part (a) of Proposition 3, $\dim(V) = \dim(W)$.

3 implies 1. We first prove that T is onto. Let $\mathbf{w} \in W$. Then, as S is a function from W to V, there exists $\mathbf{v} \in V$ such that $S(\mathbf{w}) = \mathbf{v}$. Then, on first taking the function value of $\mathbf{v}$ under T and next using the property of $T \circ S$ given in statement 3, we have

$$T(\mathbf{v}) = T(S(\mathbf{w})) = (T \circ S)(\mathbf{w}) = \mathbf{w}.$$

Thus, we have shown that for all $\mathbf{w} \in W$, there exists $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$, and hence T is onto. Then, from part (c) of Proposition 3, T is also one-to-one, and thus T is one-to-one and onto and therefore invertible.

To complete the proof we prove that 1 and 4 are equivalent by proving first that 1 implies 4 and then that 4 implies 1.

1 implies 4. The proof of this is virtually identical to the proof given above that 1 implies 3 and hence we omit it.

4 implies 1. We first prove T is one-to-one. Let $T(\mathbf{v}_1) = T(\mathbf{v}_2)$ for $\mathbf{v}_1, \mathbf{v}_2 \in V$. On taking the composite of both sides with S, we have

$$(S \circ T)(\mathbf{v}_1) = (S \circ T)(\mathbf{v}_2).$$

Then, using the property of $S \circ T$ given in 4, we have

$$\mathbf{v}_1 = (S \circ T)(\mathbf{v}_1) = (S \circ T)(\mathbf{v}_2) = \mathbf{v}_2.$$

Thus, $T(\mathbf{v}_1) = T(\mathbf{v}_2)$ only if $\mathbf{v}_1 = \mathbf{v}_2$, and hence T is one-to-one. Then, from part (b) of Proposition 3, T is also onto, and thus T is one-to-one and onto and therefore invertible.

The proof is complete. $\square$

7.9 [X] Proof of the Rank-Nullity Theorem

In this section we give a proof of the general Rank-Nullity Theorem stated in Section 7.4.3.

Theorem 1 (Rank-Nullity Theorem)**.** *If V is a finite dimensional vector space and $T : V \to W$ is linear then*

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

Proof. Recall that $\text{rank}(T) = \dim(\text{im}(T))$ = number of vectors in a basis for $\text{im}(T)$, and that $\text{nullity}(T) = \dim(\ker(T))$ = number of vectors in a basis for $\ker(T)$.

Let $\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$, where $r = \text{rank}(T)$, be a basis for $\text{im}(T)$. Then, since $\mathbf{w}_j \in \text{im}(T)$, there exists an element $\mathbf{v}_j \in V$ such that $\text{T}(\mathbf{v}_j) = \mathbf{w}_j$.

Let $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_{r+\nu}\}$, where $\nu = \text{nullity}(T)$, be a basis for $\ker(T)$.

We shall now prove that the combined set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_{r+\nu}\}$ is a basis for the domain V, and hence that $r + \nu = \dim(V)$.

We first prove that S is linearly independent.

Suppose,

$$\lambda_1\mathbf{v}_1 + \cdots + \lambda_{r+\nu}\mathbf{v}_{r+\nu} = \mathbf{0}. \qquad (\#)$$

Taking the image of this linear combination and using the fact that linear maps preserve linear combinations, we have

$$\lambda_1 T(\mathbf{v}_1) + \cdots + \lambda_{r+\nu} T(\mathbf{v}_{r+\nu}) = T(\mathbf{0}) = \mathbf{0}.$$

Now, for $j > r$, $\mathbf{v}_j \in \ker(T)$, and hence $T(\mathbf{v}_j) = \mathbf{0}$ for $j > r$. Further, for $j \leqslant r$, $T(\mathbf{v}_j) = \mathbf{w}_j$. On substituting these results in the previous equation, we have

$$\lambda_1\mathbf{w}_1 + \cdots + \lambda_r\mathbf{w}_r = \mathbf{0}.$$

But the set $\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$ is linearly independent (since it is a basis for $\text{im}(T)$), and thus $\lambda_j = 0$ for $j \leqslant r$. On substituting these values of the scalars in (#) we have

$$\lambda_{r+1}\mathbf{v}_{r+1} + \cdots + \lambda_{r+\nu}\mathbf{v}_{r+\nu} = \mathbf{0}.$$

But the set $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_{r+\nu}\}$ is linearly independent (it is a basis for $\ker(T)$), and thus $\lambda_j = 0$ for $j > r$. We have therefore proved that (#) is satisfied only if $\lambda_j = 0$ for $1 \leqslant j \leqslant r + \nu$, and hence S is linearly independent.

Now we show that S is a spanning set for V.

Since $S \subseteq V$, $\text{span}(S)$ is a subset of V. To complete the proof that $\text{span}(S) = V$ we must prove that $V \subseteq \text{span}\,(S)$, that is, if $\mathbf{v} \in V$ then $\mathbf{v} \in \text{span}\,(S)$.

Suppose $\mathbf{v} \in V$. Then $\mathbf{w} = T(\mathbf{v})$ exists and is an element of $\text{im}(T)$.

Now the set $\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$ is a basis for $\text{im}(T)$, and hence there exist scalars such that

$$\mathbf{w} = \lambda_1\mathbf{w}_1 + \cdots + \lambda_r\mathbf{w}_r.$$

Using these scalars we now form the linear combination

$$\mathbf{v}_I = \lambda_1\mathbf{v}_1 + \cdots + \lambda_r\mathbf{v}_r.$$

Now, by definition of $\mathbf{v}_I$ and the $\mathbf{v}_j$ for $j \leqslant r$, and on using the fact that linear maps preserve linear combinations, we have $T(\mathbf{v}_I) = \mathbf{w}$. We now define

$$\mathbf{v}_R = \mathbf{v} - \mathbf{v}_I.$$

Now $\mathbf{v}_R$ satisfies

$$T(\mathbf{v}_R) = T(\mathbf{v} - \mathbf{v}_I) = T(\mathbf{v}) - T(\mathbf{v}_I) = \mathbf{w} - \mathbf{w} = \mathbf{0},$$

and hence $\mathbf{v}_R \in \ker(T)$. Then, since $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_{r+\nu}\}$ is a basis for $\ker(T)$, there exist scalars such that

$$\mathbf{v}_R = \lambda_{r+1}\mathbf{v}_{r+1} + \cdots + \lambda_{r+\nu}\mathbf{v}_{r+\nu}.$$

Then, on adding the linear combinations for $\mathbf{v}_I$ and $\mathbf{v}_R$, we have

$$\mathbf{v} = \mathbf{v}_I + \mathbf{v}_R = \lambda_1\mathbf{v}_1 + \cdots + \lambda_r\mathbf{v}_r + \lambda_{r+1}\mathbf{v}_{r+1} + \cdots + \lambda_{r+\nu}\mathbf{v}_{r+\nu},$$

and hence $\mathbf{v} \in \operatorname{span}(S)$.

We have therefore established that S is a linearly independent spanning set for V, and hence $\dim(V) = r + \nu = \operatorname{rank}(T) + \operatorname{nullity}(T)$ as asserted in the theorem. $\square$

7.10 One-to-one, onto and inverses for functions

A fundamental problem about functions is the relationship between the points in the codomain and the points in the domain. Now, we know that by the definition of function each point in the domain of a function has exactly one point in the codomain as its function value. However, for a given point in the codomain it is possible in general that either (i) it is not the function value of any point in the domain, (ii) it is the function value of exactly one point in the domain, or (iii) it is the function value of more than one point in the domain. To cover these possibilities the following definitions are introduced.

Definition 1. *The **range** or **image** of a function is the set of all function values, that is, for a function $f : X \to Y$,*

$$\mathit{im}(f) = \{y \in Y : y = f(x) \ \text{ for some } \ x \in X\}.$$

Definition 2. *A function is said to be **onto** (or surjective) if the codomain is equal to the image of the function, that is, a function $f : X \to Y$ is onto if for all $y \in Y$ there exists an $x \in X$ such that $y = f(x)$.*

Definition 3. *A function is said to be **one-to-one** (or injective) if no point in the codomain is the function value of more than one point in the domain, that is, a function $f : X \to Y$ is one-to-one if $f(x_1) = f(x_2)$ if and only if $x_1 = x_2$.*

Note that a function is onto when every point in the codomain is a function value and that it is one-to-one when each function value corresponds to exactly one point in the domain. Further, a function is one-to-one and onto if and only if every point in the codomain is the function value of exactly one point in the domain.

Example 1. The function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = x^2 \qquad \text{for} \quad x \in \mathbb{R},$$

is neither one-to-one nor onto. It is not one-to-one, since, for example, $x_1^2 = x_2^2$ is true for $x_1 = 3$ and $x_2 = -3 \neq x_1$. It is not onto, since there are some $y \in \mathbb{R}$ $(y < 0)$ for which no $x \in \mathbb{R}$ exists such that $y = x^2$. ◊

Example 2. The function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(x) = x^2 \qquad \text{for} \quad x \geqslant 0$$

is one-to-one but not onto. It is one-to-one, since if $x_1^2 = x_2^2$ (and $x_1 \geqslant 0$ and $x_2 \geqslant 0$) then $x_1 = x_2$. However, it is not onto for the reason given in Example 1. ◊

Example 3. The function $f : [0, \infty) \to [0, \infty)$ defined by

$$f(x) = x^2 \qquad \text{for} \quad x \geqslant 0$$

is both one-to-one and onto. It is one-to-one for the reason given in Example 2. The function is also onto, since for all $y \geqslant 0$ there is an $x \geqslant 0$ such that $y = f(x) = x^2$. ◊

The definition of inverse of a function is as follows.

Definition 4. *Let $f : X \to Y$ be a function. Then a function $g : Y \to X$ is called an* ***inverse*** *of f if it satisfies the two conditions:*

a) $g \circ f = \mathrm{id}_X$, where id_X is the identity function in X with the property that $\mathrm{id}_X(x) = x$ for all $x \in X$.

b) $f \circ g = \mathrm{id}_Y$, where id_Y is the identity function in Y with the property that $\mathrm{id}_Y(y) = y$ for all $y \in Y$.

An alternative way of stating (a) is that $(g \circ f)(x) = g(f(x)) = x$ for all $x \in X$, and an alternative way of stating (b) is that $(f \circ g)(y) = f(g(y)) = y$ for all $y \in Y$.

An important connection between the existence of an inverse and the properties of one-to-one and onto is given in the following proposition.

Theorem 1. *A function has an inverse if and only if the function is both one-to-one and onto.*

[**X**] *Proof.* Let $f : X \to Y$ be a function with domain X and codomain Y.

We first prove that if f has an inverse $g : Y \to X$ then f is one-to-one and onto.

To prove one-to-one, we note that if $f(x_1) = f(x_2)$ then on taking the composition with the inverse g we obtain $g(f(x_1)) = g(f(x_2))$, and hence from condition (a) of the definition of inverse

$$x_1 = g(f(x_1)) = g(f(x_2)) = x_2.$$

Thus, for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ only if $x_1 = x_2$ and f is one-to-one.

We now prove that f is onto. From condition (b) of the definition of inverse, $y = f(g(y))$ for all $y \in Y$. But g is a function with codomain X, and hence $g(y) = x$ for $x \in X$. Thus, for all $y \in Y$ there exists an $x = g(y) \in X$ such that $y = f(x)$, and hence f is onto.

To complete the proof of the theorem, we must prove that if f is one-to-one and onto then f has an inverse function $g : Y \to X$.

Now, as f is an onto function, for each $y \in Y$ there exists $x \in X$ such that $y = f(x)$. Further, as f is one-to-one, $y_1 = f(x_1) = f(x_2) = y_2$ only if $x_1 = x_2$, and hence for each $y \in Y$ there exists a unique $x \in X$ such that $y = f(x)$. We can therefore define a function $g : Y \to X$ by the rule:

For each $y \in Y$ define $g(y) = x$ where x is the unique $x \in X$ such that $y = f(x)$.

This function g then has the property that $y = f(x) = f(g(y)) = (f \circ g)(y)$ for all $y \in Y$, and hence g satisfies condition (b) of the definition of inverse.

To complete the proof we must prove that the function g also satisfies condition (a). Now, as f is a function with domain X and codomain Y, for each $x \in X$ there exists a unique $y \in Y$ such that $y = f(x)$. But, as $y \in Y$, we have from the definition of g that $x = g(y)$. Therefore $x = g(y) = g(f(x)) = (g \circ f)(x)$ for all $x \in X$. The proof is complete. $\square$

7.11 Linear transformations and MAPLE

The main result of this chapter is the Matrix Representation Theorem. Accordingly, we can do calculations with linear transformations by getting Maple to manipulate the corresponding matrices. Usually you will have to calculate the appropriate matrix by hand, but Maple can handle some linear transformations directly. [Be aware that Maple displays vectors as rows but treats them as columns.] Consider for example the linear transformation of projecting onto a fixed vector $\mathbf{b} \in \mathbb{R}^n$ (see Section 7.3). In the following, the `LinearAlgebra` command `Norm(b,2)` calculates $||\mathbf{b}||$.

```
with(LinearAlgebra):
b:=<1,2,3>;
T:=a->(b.a/norm(b,2)^2).b);
a:=<1,0,-1>;
T(a);
```

We could then get Maple to find the matrix for T (with respect to the standard basis) by using

```
A:=< T(e1) | T(e2) | T(e3) >;
```

where `e1`, `e2`, `e3` have been previously defined as the standard basis vectors in $\mathbb{R}^3$. Indeed, you can even get Maple to check that the operator T defined above is linear.

```
x:=Vector(3,i->X[i]);
y:=Vector(3,i->Y[i]);
T(x+y)-T(x)-T(y);
simplify(%);
```

This last calculation will give zero, showing that $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$. You should have a go at showing that T preserves scalar multiplication as well (this turns out to be just a little more complicated for Maple!). You can also check that the matrix defined above actually does give the linear transformation T by using:

```
simplify(A.x - T(x));
```

Many of the standard calculations concerning linear transformations have special commands in the `LinearAlgebra` package. In particular you should look at the procedures `NullSpace`, `ColumnSpace`, and `Rank`. As usual, the details of these commands are available using the on-line help facility. For example to find out about `NullSpace` (which calculates the kernel), enter the command `?NullSpace`.

Chapter 8

EIGENVALUES AND EIGENVECTORS

...she set to work very carefully, nibbling first at one and then at the other, and growing sometimes taller, and sometimes shorter,...
Lewis Carroll, Alice in Wonderland.

Eigenvalues and eigenvectors are of great theoretical and practical importance. Some practical applications of eigenvalues and eigenvectors include the following.

1. Oscillations. For example, vibrating strings, organ pipes, wing flutter on an aircraft, vibrations of buildings and bridges, etc.

2. Quantum Physics and Chemistry. Structure of atoms, molecules, nuclei, solids etc.

3. Electronics and Electrical Engineering. Microwave oscillators, amplifiers, signal transmission, communications networks, etc.

4. Economics. Stability of economic systems, dynamic econometric models, Leontief input-output models, inventory models, stock market models, etc.

5. Biological and Ecological Systems. Solution of population models, stability of ecological systems etc.

In this chapter we shall only be able to give a brief introduction to this extremely important topic. A general theory of eigenvalues and eigenvectors and some applications of them is given in the second year mathematics courses.

8.1 Definitions and examples

We are concerned with linear maps in which the domain and the codomain are the same vector space, that is, with linear maps of the form $T : V \to V$. The fundamental questions asked are:

1. Given a map T, are there vectors $\mathbf{v} \in V$ which are related in a very simple way to their images $T(\mathbf{v}) \in V$?

2. [**X**] Is there a choice of basis for V such that the matrix representing T for this basis takes on a very simple form?

The answer to both of these questions is yes.

For question 1, we look for vectors for which $T(\mathbf{v})$ is a multiple of $\mathbf{v}$. Formally, we have

> **Definition 1.** *Let $T : V \to V$ be a linear map. Then if a scalar λ and* **non-zero** *vector $\mathbf{v} \in V$ satisfy*
> $$T(\mathbf{v}) = \lambda \mathbf{v},$$
> *then λ is called an* ***eigenvalue*** *of T and $\mathbf{v}$ is called an* ***eigenvector*** *of T for the eigenvalue λ.*

NOTE. An eigenvector is non-zero, but zero can be an eigenvalue.

Example 1. For infinitely differentiable real-valued functions f, the derivative

$$D(f) = f', \quad \text{where} \quad f'(x) = \frac{df}{dx} \quad \text{for} \quad x \in \mathbb{R}$$

defines a linear map D. The exponential function satisfies

$$D(e^{\lambda x}) = \lambda e^{\lambda x},$$

and hence $e^{\lambda x}$ is an eigenvector of D with eigenvalue λ. It should be noted that the great importance of exponential functions in calculus is due to the fact that they are the only functions f where f' is a multiple of f. $\diamondsuit$

Calculus and its applications provides a very rich source of eigenvalue and eigenvector problems. However, we are mainly concerned in this course with algebraic problems involving linear maps between finite-dimensional vector spaces. These linear maps can always be represented by matrices, and hence we will be concerned in the remainder of this chapter with eigenvalues and eigenvectors of matrices.

When dealing with eigenvalues and eigenvectors of matrices we will be forced to use complex numbers for our scalar field. The fundamental reason for this is that the eigenvalues of a matrix are actually zeroes of some polynomial and, as we have seen in Chapter 1 we can only be certain of finding zeroes when the polynomials are complex polynomials. Thus, the "natural" field of scalars for eigenvalues and eigenvectors is the set of complex numbers $\mathbb{C}$, and the "natural" vector spaces are the complex vector spaces $\mathbb{C}^n$ (see Example 2 of Section 6.1).

For the special case of a matrix, Definition 1 becomes:

> **Definition 2.** *Let $A \in M_{nn}(\mathbb{C})$ be a square matrix. Then if a scalar $\lambda \in \mathbb{C}$ and non-zero vector $\mathbf{v} \in \mathbb{C}^n$ satisfy*
> $$A\mathbf{v} = \lambda \mathbf{v},$$
> *then λ is called an* ***eigenvalue*** *of A and $\mathbf{v}$ is called an* ***eigenvector*** *of A for the eigenvalue λ.*

Example 2. For the diagonal 2×2 matrix,

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

the standard basis vectors $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ satisfy

$$\begin{aligned} A\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \\ A\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Thus, $\mathbf{e}_1$ is an eigenvector of A with eigenvalue λ_1 and $\mathbf{e}_2$ is an eigenvector of A with eigenvalue λ_2. A picture of this result is shown in Figure 1 for the special case of $\lambda_1 = 3$ and $\lambda_2 = -2$. ♢

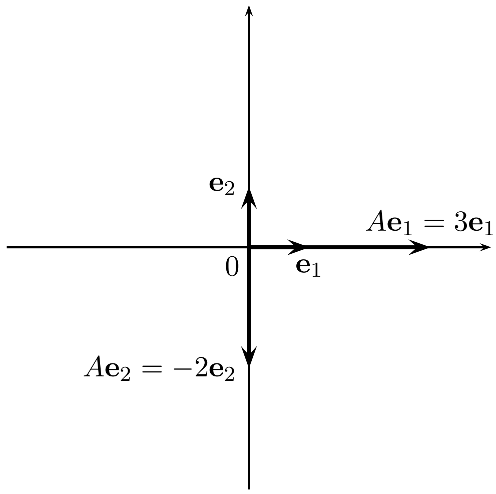


Figure 1: The eigenvectors of the diagonal matrix $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$.

Example 3. For the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 20 & -24 & 9 \end{pmatrix} \qquad \text{and the vector} \qquad \mathbf{v} = \begin{pmatrix} 1 \\ 5 \\ 25 \end{pmatrix}$$

by matrix multiplication, we have $A\mathbf{v} = 5\mathbf{v}$, and hence $\mathbf{v}$ is an eigenvector of A for the eigenvalue $\lambda = 5$. ♢

8.1.1 Some fundamental results

The fundamental theoretical results for eigenvalues and eigenvectors draw on results given in previous chapters on linear equations, polynomials, vector spaces, linear maps, and determinants.

The following theorem is extremely important.

Theorem 1. *A scalar λ is an eigenvalue of a square matrix A if and only if $\det(A - \lambda I) = 0$, and then $\mathbf{v}$ is an eigenvector of A for the eigenvalue λ if and only if $\mathbf{v}$ is a non-zero solution of the homogeneous equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$, i.e., if and only if $\mathbf{v} \in \ker(A - \lambda I)$ and $\mathbf{v} \neq \mathbf{0}$.*

Proof. From Definition 2, A is a square matrix, and an eigenvalue λ and corresponding eigenvector $\mathbf{v}$ of A satisfy the equation

$$A\mathbf{v} = \lambda\mathbf{v}, \qquad \text{where} \quad \mathbf{v} \neq \mathbf{0}.$$

This equation can be rearranged in the form

$$\mathbf{0} = A\mathbf{v} - \lambda\mathbf{v} = A\mathbf{v} - \lambda I\mathbf{v} = (A - \lambda I)\mathbf{v},$$

where I is an identity matrix of the same size as A.

Now, $A - \lambda I$ is a square matrix, and hence (by a proposition in Chapter 4 the equation

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

can have a non-zero solution if and only if $\det(A - \lambda I) = 0$. Thus, λ is an eigenvalue if and only if $\det(A - \lambda I) = 0$ and the first part of the theorem is proved.

Then, if λ is an eigenvalue, $\mathbf{v}$ is an eigenvector if and only if it is a non-zero solution of the above homogeneous equation, that is, if and only if $\mathbf{v} \in \ker(A - \lambda I)$ and $\mathbf{v} \neq \mathbf{0}$. The proof is complete. $\square$

NOTE. The set of all eigenvectors of A for eigenvalue λ is therefore equal to $\ker(A - \lambda I)$ with $\mathbf{0}$ removed. Also, there are infinitely many eigenvectors corresponding to a single eigenvalue.

A second fundamental result for the theory of eigenvalues is the following.

Theorem 2. *If A is an $n \times n$ matrix and $\lambda \in \mathbb{C}$, then $\det(A - \lambda I)$ is a complex polynomial of degree n in λ.*

This theorem can be proved in a straightforward, but tedious, fashion by direct expansion of the determinant $\det(A - \lambda I)$. For example, for $n = 3$, we have

$$A - \lambda I = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{pmatrix}.$$

Then, by direct evaluation of $\det(A - \lambda I)$ by expansion along the first column, we obtain

$$\begin{aligned} \det(A - \lambda I) &= (a_{11} - \lambda)\big((a_{22} - \lambda)(a_{33} - \lambda) - a_{32}a_{23}\big) \\ &\quad - a_{21}\big(a_{12}(a_{33} - \lambda) - a_{32}a_{13}\big) + a_{31}\big(a_{12}a_{23} - (a_{22} - \lambda)a_{13}\big) \\ &= -\lambda^3 + \text{terms containing } \lambda^2,\ \lambda \text{ and constants.} \end{aligned}$$

Hence, for $n = 3$, $\det(A - \lambda I)$ is a complex polynomial of degree 3 as stated in Proposition 2.

Definition 3. *For a square matrix A, the polynomial $p(\lambda) = \det(A - \lambda I)$ is called the **characteristic polynomial** for the matrix A.*

Example 4. For the 3×3 matrix,

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & -4 & -1 \\ 5 & 1 & 2 \end{pmatrix},$$

the characteristic polynomial is the cubic

$$p(\lambda) = \begin{vmatrix} 1 - \lambda & -1 & 2 \\ 3 & -4 - \lambda & -1 \\ 5 & 1 & 2 - \lambda \end{vmatrix} = -\lambda^3 - \lambda^2 + 16\lambda + 50.$$

$\lozenge$

We can now apply the theory of roots of complex polynomials developed in Chapter 1o obtain the following fundamental result.

Theorem 3. *An $n \times n$ matrix A has exactly n eigenvalues in $\mathbb{C}$ (counted according to their multiplicities). These eigenvalues are the zeroes of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.*

Proof. From Proposition 2, the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree n over the complex field. Thus, from the Factorisation Theorem of Chapter 1 p has exactly n zeroes (counted according to their multiplicities) which from Theorem 1 are the eigenvalues of A. $\square$

Example 5. For the matrix A of Example 4, the roots of the cubic characteristic polynomial are (to 4-figure accuracy) 4.688, $-2.844 + 1.605i$, $-2.844 - 1.605i$, and these are the three eigenvalues of A. $\diamondsuit$

NOTE.

1. The equation $p(\lambda) = 0$ is called the **characteristic equation** for A.
2. Theorem 3 is of fundamental theoretical importance, as it proves the existence of eigenvalues of a matrix. However, with the exception of 2×2 and specially constructed larger matrices, modern methods of finding eigenvalues of matrices do not make use of this theorem. These efficient modern methods are currently available in standard matrix software packages such as MAPLE, MATLAB.

8.1.2 Calculation of eigenvalues and eigenvectors

As stated above, Theorem 3 provides a practical method for finding eigenvalues of 2×2 or specially constructed larger matrices. The corresponding eigenvectors can then be obtained from Theorem 1. Some examples of the calculation of eigenvalues and eigenvectors for simple matrices are as follows:

Example 6. For a diagonal matrix the diagonal entries are eigenvalues and the standard basis vectors are eigenvectors. For example, if

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

then

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \begin{vmatrix} \lambda_1 - \lambda & 0 \\ 0 & \lambda_2 - \lambda \end{vmatrix} = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = 0$$

has the solutions $\lambda = \lambda_1$ and $\lambda = \lambda_2$.

Then for the eigenvector corresponding to $\lambda = \lambda_1$, we solve

$$(A - \lambda_1 I)\mathbf{v} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 - \lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}.$$

It is obvious that one of the solutions is $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{e}_1$.

Similarly, $\mathbf{e}_2$ is a solution of the homogeneous equation $(A - \lambda_2 I)\mathbf{v} = \mathbf{0}$, and hence $\mathbf{e}_2$ is one of the eigenvectors corresponding to the eigenvalue λ_2. $\diamondsuit$

Example 7. Find eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

SOLUTION. The first step is to find the eigenvalues from the characteristic polynomial.

We have

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda.$$

Note that $A - \lambda I$ is obtained from A by subtracting λ from each diagonal element of A, and that the characteristic polynomial is a quadratic.

The roots of the characteristic equation are

$$\lambda_1 = 0 \qquad \text{and} \qquad \lambda_2 = 4.$$

Note that, as asserted in Theorem 3, there are two eigenvalues for the 2×2 matrix A.

The next step is to find an eigenvector for each eigenvalue by finding $\ker(A - \lambda I)$, first for $\lambda = 0$, and then for $\lambda = 4$.

For eigenvalue $\lambda_1 = 0$, the eigenvectors are the non-zero vectors in $\ker(A)$. By row reduction,

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right) \xrightarrow{R_2 = R_2 - R_1} \left(\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

and then, back substitution gives $\ker(A) = \operatorname{span}(\mathbf{v}_1)$ where $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The set of eigenvectors corresponding to the eigenvalue 0 is then

$$\left\{ t \begin{pmatrix} -1 \\ 1 \end{pmatrix} : t \neq 0 \right\}.$$

For $\lambda_2 = 4$, the required eigenvectors are $\ker(A - 4I)$ (with $\mathbf{0}$ deleted) where

$$A - 4I = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}.$$

Solving $(A - 4I)\mathbf{v} = \mathbf{0}$ in the same way, we find that $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a basis for $\ker(A - 4I)$ and the set of eigenvectors corresponding to the eigenvalue 4 is then

$$\left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} : t \neq 0 \right\}.$$

[Note that the scalar field is assumed to be $\mathbb{C}$, so $t \in \mathbb{C}$.] $\Diamond$

Example 8. Find eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}.$$

SOLUTION. The eigenvalues are solutions of the characteristic equation

$$\begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda)+1 = (\lambda-3)^2 = 0.$$

Hence, there is one eigenvalue $\lambda = 3$ with multiplicity 2.
Eigenvectors. The eigenvectors are vectors $\mathbf{v} \neq \mathbf{0} \in \ker(A-3I)$, where

$$A - 3I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

On solving $(A-3I)\mathbf{v} = \mathbf{0}$, we find that the only solution is $\mathbf{v} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $t \in \mathbb{C}$. A matrix with fewer linearly independent eigenvectors than columns, as in this example, is called a **defective matrix** (poor thing). ♢

As the next example shows, it is also possible to have a 2×2 matrix A with one eigenvalue (with multiplicity 2) and two linearly independent eigenvectors.

Example 9. The matrix

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

has eigenvalue $\lambda = 3$ (with multiplicity 2) and $\ker(A-3I)$ is span $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. ♢

Example 10. Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

SOLUTION. As usual the eigenvalues are solutions of the characteristic equation $\det(A-\lambda I) = 0$, that is, of

$$\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5 = 0.$$

In this case the roots of the quadratic are the complex numbers

$$\lambda_1 = 1 + 2i \qquad \text{and} \qquad \lambda_2 = 1 - 2i.$$

The eigenvectors for $\lambda_1 = 1 + 2i$,

$$A - (1+2i)I = \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix}.$$

An equivalent row-echelon form is

$$U = \begin{pmatrix} -2i & 2 \\ 0 & 0 \end{pmatrix},$$

and the eigenvectors are $\mathbf{v} = t\begin{pmatrix} -i \\ 1 \end{pmatrix}$ with $t \in \mathbb{C},\ t \neq 0$.

For $\lambda_2 = 1 - 2i$ the eigenvectors are $\mathbf{v} = t\begin{pmatrix} i \\ 1 \end{pmatrix}$ with $t \in \mathbb{C},\ t \neq 0$. $\Diamond$

The characteristic polynomial of a real 2×2 matrix has real coefficients, so has two real roots, one real root with multiplicity 2, or a pair of distinct conjugate complex roots, so the matrix has two real eigenvalues, one real eigenvalue with multiplicity 2, or two distinct conjugate complex eigenvalues. Examples 7 – 10 above show all these possibilities.

If A is a complex 2×2 matrix, its characteristic polynomial has complex coefficients, and either two distinct complex roots or one complex root with multiplicity 2, and the matrix has two eigenvalues or one eigenvalue with multiplicity 2.

For each eigenvalue the space spanned by its corresponding eigenvector(s) is called the **eigenspace** for that eigenvalue. Thus, when we write down *the* eigenvectors for a given eigenvalue, we are really recording the basis vectors for the corresponding eigenspace.

8.2 Eigenvectors, bases, and diagonalisation

In the examples of the preceding section, we have seen that, with one exception (Example 8), we obtain two linearly independent eigenvectors for a 2×2 matrix. Since a 2×2 matrix A represents a linear map whose domain is $\mathbb{C}^2$, these two eigenvectors form a basis for the domain. The matrix of Example 8 has one independent eigenvector, and it does not form a basis for the domain.

These results can be generalised to matrices of arbitrary size.

Theorem 1. *If an $n \times n$ matrix has n distinct eigenvalues then it has n linearly independent eigenvectors.*

[**X**] *Proof.* Let the set of n distinct eigenvalues of the $n \times n$ matrix A be $\{\lambda_1, \ldots, \lambda_n\}$ and let $\mathbf{v}_i$ be a corresponding eigenvectors for λ_i, $1 \leqslant i \leqslant n$. We shall now prove that

$$S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}.$$

is linearly independent.

Suppose

$$\mu_1\mathbf{v}_1 + \cdots + \mu_n\mathbf{v}_n = \mathbf{0}. \qquad (\#)$$

We show that $\mu_1 = 0$. In similar fashion, $\mu_2 = \cdots = \mu_n = 0$, so $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are linearly independent. Apply the matrix $(A - \lambda_2 I)\ (A - \lambda_3 I) \cdots (A - \lambda_n I)$ to both sides of $\#$, then we have If $j \neq 1$,

$$\begin{aligned}
(A - \lambda_2 I) \cdots (A - \lambda_n I)\mathbf{v}_j &= (\lambda_j - \lambda_2)\ (\lambda_j - \lambda_3) \cdots (\lambda_j - \lambda_n)\mathbf{v}_j = \mathbf{0} \quad \text{if } j \neq 1, \\
(A - \lambda_2 I) \cdots (A - \lambda_n I)\mathbf{v}_1 &= (\lambda_1 - \lambda_2)\ (\lambda_1 - \lambda_3) \cdots (\lambda_1 - \lambda_n)\mathbf{v}_1 \neq \mathbf{0}.
\end{aligned}$$

So $\mu_1(\lambda_1 - \lambda_2) \cdots (\lambda_1 - \lambda_n)\mathbf{v}_1 = \mathbf{0}$, and that is $\mu_1 = 0$. $\square$

NOTE. Even if the $n \times n$ matrix does not have n distinct eigenvalues, it may have n linearly independent eigenvectors.

In Examples 2 and 6 of Section 8.1, we have seen that it is very easy to write down eigenvalues and eigenvectors of diagonal matrices. The next theorem shows that it is sometimes possible to find an equivalent diagonal matrix for a given matrix.

Theorem 2. *If an $n \times n$ matrix A has n linearly independent eigenvectors, then there exists an invertible matrix M and a diagonal matrix D such that*

$$M^{-1}AM = D.$$

Further, the diagonal elements of D are the eigenvalues of A and the columns of M are the eigenvectors of A, with the jth column of M being the eigenvector corresponding to the jth element of the diagonal of D.
Conversely if $M^{-1}AM = D$ with D diagonal then the columns of M are n linearly independent eigenvectors of A.

[**X**] *Proof.* Let the n linearly independent eigenvectors of A be $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. We now form the matrix M with these vectors as its columns, i.e.,

$$M = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}.$$

Then, from the usual rules of matrix multiplication, we have

$$AM = \begin{pmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{pmatrix},$$

and from $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ we have

$$AM = \begin{pmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \end{pmatrix}.$$

Following the usual rules of matrix multiplication, we can rewrite this equation in the matrix form

$$AM = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = MD,$$

where

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

is the diagonal matrix of eigenvalues. Thus, $AM = MD$. Further, since the columns of M are a basis for $\mathbb{C}^n$, the equation $M\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{C}^n$, and hence M is invertible. Then, on multiplying the equation $AM = MD$ on the left by M^{-1}, we have $M^{-1}AM = D$.
Conversely if $M^{-1}AM = D$ then $AM = MD$ and M is invertible. Let

$$M = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

then from the first columns of the matrix products on the two sides of $AM = MD$, we have $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$. Similarly $A\mathbf{v}_i = \lambda_i\mathbf{v}_i,\ 1 \leqslant i \leqslant n$.

Finally the columns of an invertible matrix are linearly independent. $\square$

> **Definition 1.** *A square matrix A is said to be a* ***diagonalisable matrix*** *if there exists an invertible matrix M and diagonal matrix D such that $M^{-1}AM = D$.*

Example 1. Show that the matrix

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

is diagonalisable and find an invertible matrix M and diagonal matrix D such that $M^{-1}AM = D$.

SOLUTION. We first find the eigenvalues and eigenvectors of A in the usual way. The eigenvalues are $\lambda_1 = 5, \lambda_2 = 1$ and corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Clearly, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. (Theorem 1 guarantees this, since $\lambda_1 \neq \lambda_2$.) Thus we may apply Theorem 2, letting D be a diagonal matrix with the eigenvalues as its diagonal elements, and M be the matrix with corresponding eigenvectors as its columns. For example,

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

are the required diagonal matrix D and a suitable invertible matrix M. ♢

NOTE.

1. The results can be checked by direct multiplication of $M^{-1}AM$. In the above example, we readily obtain
$$M^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$
and then
$$M^{-1}AM = M^{-1}\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} = D.$$

2. The choice of D and M is not unique. For example, we could reverse the order of the eigenvalues and set
$$D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$
Also non-zero multiples of eigenvectors are eigenvectors, so multiplying any column of M by a non-zero scalar would produce another valid diagonalising matrix.

8.3 Applications of eigenvalues and eigenvectors

Some important practical applications have already been noted at the beginning of this chapter. Many of these applications arise from the study of dynamical systems. A dynamical system is essentially any system which changes in time. Some examples of such systems include an electrical power network, a bridge oscillating in a wind, the population of a city or country, an ant colony, a forest, the Australian economy, an atom, an atomic nucleus.

8.3.1 Powers of A

A typical problem in, for example, the study of dynamical systems is to find A^k for positive integers k. There are two results which enable us to easily solve this problem.

Proposition 1. Let D be the diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

Then, for $k \geqslant 1$,

$$D^k = \begin{pmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{pmatrix}.$$

Proof. The proof is by induction.
The result is obviously true for $k = 1$.
Now assume that the result is true for $k = m$. Then, on multiplying out,

$$D^{m+1} = DD^m = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \begin{pmatrix} \lambda_1^m & 0 & \dots & 0 \\ 0 & \lambda_2^m & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^m \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1^{m+1} & 0 & \dots & 0 \\ 0 & \lambda_2^{m+1} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^{m+1} \end{pmatrix}.$$

Hence, if the result is true for m it is also true for $m+1$. But, we have already seen that it is true for $m = 1$, and hence it is true for all positive integers k. ☐

The second result that we need is as follows:

Proposition 2. If A is diagonalisable, that is, if there exists an invertible matrix M and diagonal matrix D such that $M^{-1}AM = D$, then

$$A^k = MD^kM^{-1} \qquad \text{for integer} \quad k \geqslant 1.$$

Proof. The proof is by induction.
On multiplying $M^{-1}AM = D$ on the left by M and on the right by M^{-1}, we obtain

$$A = MDM^{-1},$$

and hence the statement of the proposition is true for $k = 1$.

Now suppose the statement of the proposition is true for $k = m$. Then

$$A^{m+1} = AA^m = MDM^{-1}MD^mM^{-1} = MDD^mM^{-1} = MD^{m+1}M^{-1},$$

and hence the statement of the proposition is also true for $m + 1$. Thus, the statement of the proposition is true for all positive integers k. ☐

Example 1. Find A^k for

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

SOLUTION. The first step is to check that A is diagonalisable, and, if it is, to find the matrix M of eigenvectors and diagonal matrix D of eigenvalues such that $A = MDM^{-1}$. From Example 1 of Section 8.2, suitable matrices are:

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}; \qquad M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; \qquad M^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Then,

$$\begin{aligned} A^k = MD^kM^{-1} &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5^k & 0 \\ 0 & 1^k \end{pmatrix} M^{-1} \\ &= \begin{pmatrix} 5^k & 1 \\ 5^k & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(5^k+1) & \frac{1}{2}(5^k-1) \\ \frac{1}{2}(5^k-1) & \frac{1}{2}(5^k+1) \end{pmatrix}. \end{aligned}$$

As a check on this solution, note that we obtain I if we substitute $k = 0$, and A if we substitute $k = 1$. ♢

NOTE. [X] Given a diagonalisable matrix A, we can give meaning to its exponential, using the power series expansion of e^x. We substitute A into the expansion

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...$$

replacing 1 by I.

Since A is diagonalisable, we can write $A = PDP^{-1}$, with D diagonal, and a simple calculation shows that

$$I + PDP^{-1} + \frac{1}{2!}(PDP^{-1})^2 + \frac{1}{3!}(PDP^{-1})^3 + ... = P(I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + ...)P^{-1}.$$

We define this to be the exponential of the matrix.

In the case of a 2×2 matrix with distinct eigenvalues λ_1, λ_2, by adding the entries in the matrix, we have

$$e^A = P \begin{pmatrix} 1 + \lambda_1 + \frac{1}{2!}\lambda_1^2 + \frac{1}{3!}\lambda_1^3 + ... & 0 \\ 0 & 1 + \lambda_2 + \frac{1}{2!}\lambda_2^2 + \frac{1}{3!}\lambda_2^3 + ... \end{pmatrix} P^{-1}$$

$$= P \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix} P^{-1}.$$

In a similar way, one can define the sine and cosine (etc) of a matrix.

8.3.2 Solution of first-order linear differential equations

A typical problem in many applications is to find the solution of a pair of first-order linear differential equations with constant coefficients of the form

$$\begin{aligned} \frac{dy_1}{dt} &= a_{11}y_1 + a_{12}y_2 \\ \frac{dy_2}{dt} &= a_{21}y_1 + a_{22}y_2, \end{aligned}$$

with initial conditions $y_1(0)$ and $y_2(0)$ given.

If t represents time, this pair of equations represents a simple "continuous-time dynamical system." For example, in a model of a population, $y_1(t)$ might be the number of females at time t and $y_2(t)$ the number of males at time t. The system of equations then describes how the numbers of females and males change with time.

One method of solution of this system is as follows. We first write the equations in matrix form, with

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

and obtain

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}, \qquad \text{with} \qquad \mathbf{y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}.$$

In this matrix form it is clear that there is no real restriction on the number of components of the vector $\mathbf{y}$. Equations of this type are important in the study of dynamical systems, where they are given the special name of **state-space equations**. The vector $\mathbf{y}$ in the equation is then called the **state vector** and t represents time.

This type of equation is a generalisation of the one-dimensional, first-order, linear differential equation with constant coefficients that you have met in calculus, and which is of the form

$$\frac{dy}{dt} = ay; \qquad y(0) = y_0 = \text{constant}.$$

This equation has a solution

$$y(t) = y_0 e^{at}.$$

It is therefore plausible to guess that the n- dimensional equation will have a similar exponential type of solution. We therefore guess an exponential solution of the form:

$$\mathbf{y} = \mathbf{u}(t) = \mathbf{v}e^{\lambda t},$$

where λ is a constant scalar and $\mathbf{v}$ is a constant vector. On substituting this guess or "trial solution" into the matrix equation we obtain

$$\frac{d\mathbf{y}}{dt} = \lambda\mathbf{v}e^{\lambda t} = A\mathbf{y} = A\mathbf{v}e^{\lambda t},$$

which can be rearranged to give

$$e^{\lambda t}(A\mathbf{v} - \lambda\mathbf{v}) = \mathbf{0}.$$

Now, $e^{\lambda t} \neq 0$ for all t, and hence our guess is actually a solution only if $(A - \lambda I)\mathbf{v} = \mathbf{0}$. We therefore arrive at the result:

Proposition 3. $\mathbf{y}(t) = \mathbf{v}e^{\lambda t}$ is a solution of

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}$$

if and only if λ is an eigenvalue of A and $\mathbf{v}$ is an eigenvector for the eigenvalue λ.

Example 2. Find solutions of

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y} \qquad \text{where} \qquad A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

SOLUTION. We first find the eigenvalues and eigenvectors of A. We have obtained these previously, and they are:

$$\lambda_1 = 5, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\lambda_2 = 1, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence, two solutions of the equation are

$$\mathbf{u}_1(t) = e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{u}_2(t) = e^{t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

◊

The next point to notice is that the linearity of the differential equation leads to the following proposition.

Proposition 4. If $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ are two solutions of the equation

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y},$$

then any linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ is also a solution.

Proof. Let

$$\mathbf{y}(t) = \alpha_1\mathbf{u}_1(t) + \alpha_2\mathbf{u}_2(t),$$

where α_1 and α_2 are scalars. Then

$$\begin{aligned} \frac{d}{dt}\big(\alpha_1\mathbf{u}_1(t) + \alpha_2\mathbf{u}_2(t)\big) &= \alpha_1\frac{d\mathbf{u}_1}{dt} + \alpha_2\frac{d\mathbf{u}_2}{dt} \\ &= \alpha_1 A\mathbf{u}_1 + \alpha_2 A\mathbf{u}_2 \\ &= A(\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2), \end{aligned}$$

and the result is proved. □

Example 2 (continued)**.** In our example, we therefore have that

$$\mathbf{y}(t) = \alpha_1 e^{5t}\begin{pmatrix}1\\1\end{pmatrix} + \alpha_2 e^t \begin{pmatrix}1\\-1\end{pmatrix}.$$

is a solution of the linear differential equation.

Although we have not proved it, the above solution is the general solution of the original differential equation, that is, every solution of the differential equation is of the above form.

Now, since there are two unknown scalars in the general solution, two extra conditions must be specified in order to completely determine the solution. Typical conditions are that the value of $\mathbf{y}(t) = \begin{pmatrix}y_1(t)\\y_2(t)\end{pmatrix}$ is given at some t, for example, at $t = 0$. ♢

Example 2 (continued)**.** Find the solution of

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y} \qquad \text{for} \qquad A = \begin{pmatrix}3 & 2\\2 & 3\end{pmatrix}, \qquad \text{given that} \qquad \mathbf{y}(0) = \begin{pmatrix}1\\-2\end{pmatrix}.$$

Solution. On substituting $t = 0$ into our general solution of the differential equation, and equating $\mathbf{y}(0)$ to the given vector, we obtain

$$\mathbf{y}(0) = \alpha_1\begin{pmatrix}1\\1\end{pmatrix} + \alpha_2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}.$$

We can now obtain α_1 and α_2 by solving this pair of linear equations in the usual way. We find $\alpha_1 = -\frac{1}{2}$ and $\alpha_2 = \frac{3}{2}$, and hence the solution of the differential equation is

$$\mathbf{y}(t) = -\frac{1}{2}e^{5t}\begin{pmatrix}1\\1\end{pmatrix} + \frac{3}{2}e^t\begin{pmatrix}1\\-1\end{pmatrix}.$$

♢

One reason for considering these linear first-order systems of differential equations is that every linear differential equation can be written as a system of linear first-order differential equations. We will illustrate the method with an example.

Example 3. Convert the second-order differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 5y = 0$$

to a system of first-order differential equations.

Solution. First define new variables by

$$y_1 = y \qquad \text{and} \qquad y_2 = \frac{dy_1}{dt} = \frac{dy}{dt}.$$

Then, on differentiating y_2 and using the differential equation, we find

$$\frac{dy_2}{dt} = \frac{d^2y}{dt^2} = 5y - 4\frac{dy}{dt} = 5y_1 - 4y_2.$$

The original second-order equation is therefore equivalent to the pair of first-order equations

$$\begin{aligned}\frac{dy_1}{dt} &= y_2\\ \frac{dy_2}{dt} &= 5y_1 - 4y_2.\end{aligned}$$

This pair of equations can then be rewritten in matrix form as

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}, \qquad \text{where} \qquad A = \begin{pmatrix} 0 & 1 \\ 5 & -4 \end{pmatrix}.$$

$\Diamond$

It is useful to compare the matrix method of solution of a second-order linear differential equation with the method of solution usually taught in calculus courses. The final results obtained by the two methods are, of course, the same.

Example 4 (Matrix method)**.** Guess a solution of form $\mathbf{y}(t) = \mathbf{u}(t) = e^{\lambda t}\mathbf{v}$ and substitute in the differential equation. Then, $\mathbf{u}$ is a solution if λ and $\mathbf{v}$ satisfy the eigenvector equation $A\mathbf{v} = \lambda\mathbf{v}$.

The eigenvalues are solutions of the characteristic equation $\det(A - \lambda I) = 0$, that is, of

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 5 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda - 5 = 0.$$

The roots of the quadratic give the eigenvalues $\lambda_1 = -5$ and $\lambda_2 = 1$. A solution to $(A+5I)\mathbf{v}_1 = 0$ is the eigenvector $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, and a solution to $(A - I)\mathbf{v}_2 = 0$ is the eigenvector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The general solution is therefore

$$\mathbf{y}(t) = \alpha_1 e^{-5t}\begin{pmatrix} -1 \\ 5 \end{pmatrix} + \alpha_2 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since $y(t) = y_1(t)$, the solution for $y(t)$ in the original second-order equation is

$$y(t) = y_1(t) = -\alpha_1 e^{-5t} + \alpha_2 e^{t}.$$

$\Diamond$

Example 5 (Calculus method)**.** We first guess a solution $y(t) = e^{\lambda t}$, and substitute in the original second-order differential equation to obtain the so-called characteristic equation

$$\lambda^2 + 4\lambda - 5 = 0.$$

Note that this characteristic equation is identical to the characteristic equation $\det(A - \lambda I) = 0$ of the matrix method. See question 24 of the problems for a generalisation of this result.

The roots of the quadratic are $\lambda_1 = -5$ and $\lambda_2 = 1$, and hence the general solution is

$$y(t) = \alpha_1 e^{-5t} + \alpha_2 e^{t},$$

which is identical to the solution from the matrix method. $\Diamond$

In the above example, it is clear that the calculus method gives a much quicker solution than the matrix method. However, the matrix method has the great advantage that it works for a much larger class of differential equations than does the calculus method. One reason that the matrix method works for a larger class of differential equations is that any single higher-order differential equation can be easily converted into a system of first-order equations, but it is extremely difficult to convert a given system of first-order equations into a single higher-order differential equation. It should also be pointed out that the matrix method described above will not work if the matrix A is not diagonalisable. However, an extension of the matrix method which uses “Jordan forms” can be developed to handle this case.

Example 6. The atoms in a laser can exist in two states, an “excited state” and a “ground state”. The laser is initially pumped so that it has 80% of its atoms in the excited state and the remaining 20% in the ground state. When the laser is operating, 70% of the excited atoms decay to the ground state per second, whereas 40% of the ground state atoms are pumped up to the excited state per second.

Find the percentage of atoms in each state at a time t seconds after the laser starts to operate.

Solution. Let

$$\begin{aligned} x_1(t) &= \% \text{ of atoms in excited state at time } t \\ x_2(t) &= \% \text{ of atoms in ground state at time } t. \end{aligned}$$

During operation the laser is described by the pair of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= -70x_1(t) + 40x_2(t) \\ \frac{dx_2}{dt} &= 70x_1(t) - 40x_2(t) \end{aligned}$$

That is, in matrix form,

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t), \qquad \text{where} \qquad A = \begin{pmatrix} -70 & 40 \\ 70 & -40 \end{pmatrix}.$$

The eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = -1.1$, and corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The general solution is therefore

$$\mathbf{x}(t) = \alpha_1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} + \alpha_2 e^{-1.1t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The initial condition of the laser is given as $\mathbf{x}(0) = \begin{pmatrix} 80 \\ 20 \end{pmatrix}$, and hence the values of α_1 and α_2 can be determined from the equations

$$\mathbf{x}(0) = \begin{pmatrix} 80 \\ 20 \end{pmatrix} = \alpha_1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

for which the solution is $\alpha_1 = 9\frac{1}{11}$ and $\alpha_2 = -43\frac{7}{11}$.

Thus, the complete solution is

$$x_1(t) = \frac{1}{11}\left(400 + 440e^{-1.1t}\right)$$
$$x_2(t) = \frac{1}{11}\left(700 - 440e^{-1.1t}\right)$$

Note that, as $t \to \infty$, $e^{-1.1t} \to 0$, and hence the laser settles into a "steady-state" operation in which there are $400/11 = 36\frac{4}{11}\%$ of the atoms in the excited state and $700/11 = 63\frac{7}{11}\%$ of the atoms in the ground state. The "steady-state" solution for large t is a scalar multiple of the eigenvector $\mathbf{v}_1 = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$ corresponding to the eigenvalue $\lambda_1 = 0$. $\Diamond$

8.3.3 [X] Markov chains

Matrices are very useful in studying many discrete-time dynamical systems. Dynamical systems are ones where the state of the system at stage $k+1$ depends solely on the state at stage k.

Example 7. In a certain experiment, a psychologist was testing the learning abilities of rats by getting them to run a maze. The experimenter started with 100 rats, none of which had previously run the maze. She then set each of the 100 rats in turn at the maze and noted whether it successfully ran the maze. She then repeated the process several more times.

She found that, on average, 10% of the rats which failed at one attempt were successful on their next attempt, whereas 95% of the rats which were successful at one attempt were also successful at their next attempt. (These numbers are meant for illustration only. They are not taken from actual experimental data).

For this experiment, calculate the approximate number of rats which successfully run the maze on the 3rd run, the 20th run and the 50th run.

Solution. Let

$x_1(k)$ = number of rats successfully completing the maze at the kth run
$x_2(k)$ = number of rats failing the maze at the kth run.

Then, in the $(k+1)$th run, we have

$$\begin{array}{rcrcr} x_1(k+1) & = & 0.95x_1(k) & + & 0.10x_2(k) \\ x_2(k+1) & = & 0.05x_1(k) & + & 0.90x_2(k), \end{array}$$

which can be written in matrix form as

$$\mathbf{x}(k+1) = A\mathbf{x}(k), \qquad \text{where} \qquad A = \begin{pmatrix} 0.95 & 0.10 \\ 0.05 & 0.90 \end{pmatrix}.$$

We note that the unique solution of the equation is

$$\mathbf{x}(k) = A^k\mathbf{x}(0),$$

as can easily be checked by direct substitution in $\mathbf{x}(k+1) = A\mathbf{x}(k)$.

In our problem, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 100 \end{pmatrix}$, since at the beginning there were 100 rats, none of which had successfully run the maze.

We now calculate A^k to complete the solution. The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 0.85$, and corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Thus, A is diagonalisable and suitable choices for M, D and M^{-1} are

$$M = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 0.85 \end{pmatrix}, \qquad \text{and} \quad M^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Thus,

$$\begin{aligned} \mathbf{x}(k) = MD^kM^{-1}\mathbf{x}(0) &= \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1^k & 0 \\ 0 & (0.85)^k \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 100 \end{pmatrix} \\ &= \frac{100}{3} \begin{pmatrix} 2\left(1-(0.85)^k\right) \\ \left(1+2(0.85)^k\right) \end{pmatrix}. \end{aligned}$$

Note: As a check, if we substitute $k = 0$ in this expression, we obtain $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 100 \end{pmatrix}$. Further, for $k = 1$, we have

$$\mathbf{x}(1) = \frac{100}{3} \begin{pmatrix} 2(1-0.85) \\ (1+2(0.85)) \end{pmatrix} = \begin{pmatrix} 10 \\ 90 \end{pmatrix},$$

which equals

$$A\mathbf{x}(0) = \begin{pmatrix} 0.95 & 0.10 \\ 0.05 & 0.90 \end{pmatrix} \begin{pmatrix} 0 \\ 100 \end{pmatrix} = \begin{pmatrix} 10 \\ 90 \end{pmatrix}.$$

Then, for $k = 3$ the solution is $\mathbf{x}(3) = (25.72, 74.28)$, and hence approximately 26 rats will successfully complete the maze on the third run. For $k = 20$, the solution is $\mathbf{x}(0) = \begin{pmatrix} 64.08 \\ 35.92 \end{pmatrix}$, and hence approximately 64 rats will successfully complete the maze on the 20th run. For $k = 50$, the solution is $\mathbf{x}(50) = \begin{pmatrix} 66.65 \\ 33.35 \end{pmatrix}$, and hence approximately 67 rats will successfully complete the maze on the 50th run. Note that, for large values of k, $\mathbf{x}(k)$ is approximately equal to $\begin{pmatrix} 66\frac{2}{3} \\ 33\frac{1}{3} \end{pmatrix}$, which corresponds to approximately 67 rats successfully completing the maze on a given run. Note that this solution for large values of k is a scalar multiple of the eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ corresponding to the eigenvalue $\lambda_1 = 1$, which is the eigenvalue of A with largest magnitude. $\Diamond$

Systems such as the one in Example 7 are called Markov chains. In these systems the objects or individuals can be in one of a certain number of states and the system is modelled by the matrix equation

$$\mathbf{x}(k+1) = A\mathbf{x}(k)$$

where $\mathbf{x}(k) = \begin{pmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{pmatrix}$ gives the number of individuals in each of the n states at time k. The $n \times n$ matrix A has the property that all its entries are non-negative, and, for $j = 1, \ldots, n$, $\sum_{i=1}^{n} a_{ij} = 1$. In other words, each column sums to 1. The number a_{ij} is the probability that an individual changes from state j to state i. Usually what we are interested in is finding the long term behaviour of such a system. That is, how A^k behaves as $k \to \infty$. It turns out that the behaviour exhibited in Example 7 is typical of these systems. For any such matrix, $\lambda = 1$ is an eigenvalue, and indeed is the eigenvalue of largest magnitude. In almost all cases, $A^k\mathbf{x}(0)$ converges to a multiple of the eigenvector corresponding to $\lambda = 1$. The limit vector in these cases depends only on the number of individuals involved, and not on the initial distribution of the individuals into the particular states. We conclude this section by proving that $\lambda = 1$ is always an eigenvalue of such matrix, i.e. the columns sums are all one.

Lemma 5. *If λ is an eigenvalue of A, then λ is also a eigenvalue of A^T.*

Proof. Question 13 in the problems for this chapter. □

Theorem 6. *Suppose that A is $n \times n$ matrix and that the sum of each of the columns of A is 1. Then A has 1 as an eigenvalue.*

Proof. The hypothesis on $A = (a_{ij})$ is that

$$\begin{aligned} a_{11} + a_{21} + \cdots + a_{n1} &= 1 \\ a_{12} + a_{22} + \cdots + a_{n2} &= 1 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{1n} + a_{2n} + \cdots + a_{nn} &= 1 \end{aligned}$$

or equivalently,

$$A^T \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Thus, 1 is an eigenvalue of A^T, and by the preceding lemma, is thus an eigenvalue of A. (Note that, in general, $(1 \ \ 1 \ \ \cdots)^T$ will not be its eigenvector.) □

8.4 Eigenvalues and MAPLE

The `LinearAlgebra` package in Maple has procedures for doing all the calculations described in this section. The command

```
with(LinearAlgebra):
```

loads the `LinearAlgebra` commands. If A is a square matrix then

```
Determinant(A-t);
```

produces the characteristic polynomial for A. Actually, Maple has a command which will also do this directly. A slight complication here is that

```
CharacteristicPolynomial(A,t);
```

gives the polynomial $\det(tI - A)$. This of course has the same roots as the polynomial we use. You can (and should at least once!) use solve or fsolve to find the roots of this equation, or you can use the eigenvalues command directly. You can then use kernel to find the eigenvectors.

```
evals:=Eigenvalues(A);
NullSpace(A-evals[1]);
```

This will give you a set containing a basis for the eigenspace. Use `op` to strip off the braces if necessary.

You can also get Maple to do all this for you at once, and more. The command

```
EV:=Eigenvectors(A);
```

returns a sequence with two elements. The first is a Vector with the eigenvalues as entries and the second is a Matrix whose columns are the eigenvectors in the same order. This matrix is thus a diagonalising matrix for A, if one exists. Thus if you then do

```
 EV[2]^(-1).A.EV[2]
```

you will get a diagonal matrix — the same matrix that

```
 DiagonalMatrix(EV[1]);
```

would give.

Chapter 9

INTRODUCTION TO PROBABILITY AND STATISTICS

"What IS the use of repeating all that stuff?"
the Mock Turtle interrupted . . .
Lewis Carroll, Alice in Wonderland.

This chapter introduces mathematical probability, random variables, and probability distributions. The concepts, methods, and applications are required in statistics courses that include MATH2801/MATH2901 – (Higher) Theory of Statistics, a core subject for the mathematics and statistics majors, and MATH2089/MATH2099, which are compulsory courses for many second year engineering students.

Statistics is the science of turning raw data into reliable information on which decisions can be made, given randomness or variation in the original data. As a science, it aims to uncover patterns in observations that can be described by mathematical or heuristic models. It is also concerned with formulating and testing various hypotheses about the context from which the data are drawn.

For instance in order to predict voting patterns in an election, opinions are sought from voters by carrying out opinion polls. It might not be possible to obtain the views of all voters, so the preferences of a relatively small sample of voters are obtained. As a result, the opinion poll can only provide an *estimate* of the true proportion of voters who favour a particular political party. It is important to quantify how accurate this estimate can be expected to be.

How many voters must be polled in order for this estimate to be reasonably accurate? How big is the measurement error? Statistical science has the answers to these frequently asked and important questions. These answers have had immediate utility. It traditionally cost time and resources to poll voters, so it mattered whether 1,000 voters formed an adequate sample or whether 10,000 voters were required.

This simple but typical example illustrates nicely the three essential aspects of statistical science: data production, data analysis, and statistical inference.

Data production

How many sample units should be taken, how should they be selected, and what data should be measured on each unit? An important part of data production is controlling the measurement error that invariably arises.

Data Analysis

In order to be presented transparently, data must be organised into easily understood forms, often as graphics, tables, and summary "statistics", such as sample means or sample proportions. For opinion polling, a simple summary of the sample proportions provide the information sought.

Statistical Inference

This is the process of drawing valid conclusions about a whole population based on information obtained from a part of the population. An essential ingredient here is a **random sample** from the population. Given the data obtained from a random sample of voters, what can one infer about the general voting patterns?

The transition from population to random sample is one instance in which the notion of probability becomes important. To create a random sample, we must know the probability that a given member of the population will be selected in the sample. Conversely, probability allows us to model and forecast real-world behaviour in terms of random processes. In these ways, probability theory and statistics play important roles in countless contexts, such as clinical trials, weather forecasting, finance, or traffic control, to name just a few.

First, let us recollect some background set theory and notation.

9.1 Some Preliminary Set Theory

A probability model consists of two components:

1. A set of possible outcomes;

2. The probability of each outcome or set of outcomes.

In this section, we present basic set theory as background material for the first of these components. The second component will be addressed in Section 9.2.

Definition 1. *A* ***set*** *is a collection of objects. These objects are called* ***elements****.*

We write $x \in A$ to express that x is an element of a set A. If x is not an element of A, then we write $x \notin A$.

Example 1. The set $A = \{1, 2, 3\}$ has elements 1, 2, and 3. Thus, $1 \in A$ but $4 \notin A$, say. $\Diamond$

The above definition is circular and imprecise. For instance, it is vulnerable to Russell's Paradox (briefly discussed in MATH1081 Discrete Mathematics). One could improve the definition by insisting that each set must have the property that each conceivable element is either completely in the set or completely outside of the set, but not both. However, one must improve the definition further in order to guard it from contradiction, and this is in fact difficult. Fortunately for our purposes, the above naive definition suffices.

Definition 2.

- *A set A is a **subset** of a set B (written $A \subseteq B$) if and only if each element of A is also an element of B; that is, if $x \in A$, then $x \in B$.*
- *The **power set** $\mathcal{P}(A)$ of A is set of all subsets of A.*
- *The **universal set** S is the set that denotes all objects of given interest.*
- *The **empty set** $\emptyset$ (or $\{\}$) is the set with no elements.*

Example 2. The set $A = \{1, 2, 3\}$ has eight subsets. For instance, $\{2, 3\} \subseteq A$. The power set of A is the set of these eight subsets, namely

$$\mathcal{P}(A) = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\big\}\,. \qquad \Diamond$$

Example 3. For problems in 3-dimensional vector geometry, the universal set is usually $S = \mathbb{R}^3$. Points, lines, and planes are then subsets of S. $\Diamond$

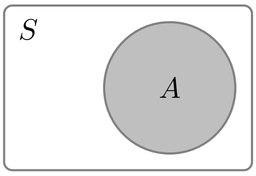
Definition 3. *A set S is **countable** if its elements can be listed as a sequence.*

More formally, S is countable if and only if there is a one-to-one function from S to $\mathbb{N}$.

Example 4.

- Every finite set is countable.
- The integers are countable since we can list them as follows: $0, 1, -1, 2, -2, \ldots$.
- The rationals are countable. (**Challenge**: can you list them as a sequence?)
- The reals are not countable; this can be shown by a simple and elegant proof known as Cantor's Diagonal Argument.

Sets are often visualised by a **Venn diagram** as regions in the plane. For instance, here is a Venn diagram of a universal set S containing a set A:

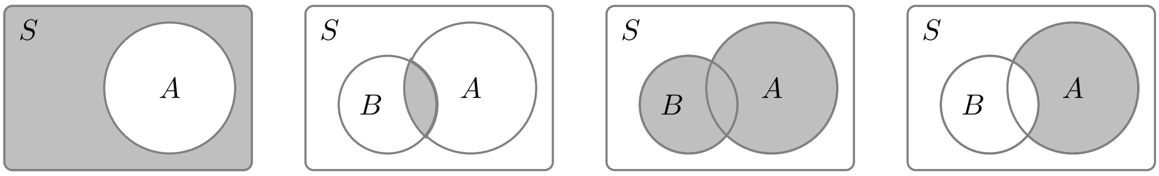


Definition 4. *For all subsets $A, B \subseteq S$, define the following **set operations**:*

- ***complement** of A:* $A^c = \{x \in S : x \notin A\}$
- ***intersection** of A and B:* $A \cap B = \{x \in S : x \in A \text{ and } x \in B\}$
- ***union** of A and B:* $A \cup B = \{x \in S : x \in A \text{ or } x \in B\}$
- ***difference**:* $A - B = \{x \in S : x \in A \text{ but } x \notin B\} = A \cap B^c$

Following mathematical convention, "or" in the union definition means "one or the other or both".

Venn diagrams describing the above set operations are given below.



A^c $\quad$ $A \cap B$ $\quad$ $A \cup B$ $\quad$ $A - B$

Example 5. Let S be all students enrolled in MATH1231, let A be those who are 20 years or older, and let B be those who are engineering students. Then

A^c are MATH1231 students at most 19 years old

$A \cap B$ are MATH1231 students studying engineering and who are 20 years or older

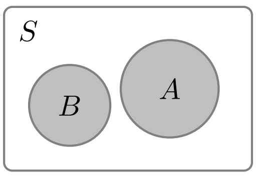
$A \cup B$ are MATH1231 students who study engineering or are 20 years or older

$A - B$ are MATH1231 students who do not study engineering but who are 20 years or older.$\Diamond$

Definition 5. *Sets A and B are* ***disjoint*** *(or* ***mutually exclusive****) if and only if*

$$A \cap B = \emptyset$$

A Venn diagram showing two disjoint sets A and B is given below:



$A \cap B = \emptyset$

Example 6. Let S be the set of people in Australia, let A be the set of people enrolled in MATH1231, and let B be the set of people aged under 10. Then A and B are disjoint. $\Diamond$

Definition 6. *Disjoint subsets $A_1, \dots, A_k$* ***partition*** *a set B if and only if*

$$A_1 \cup \cdots \cup A_k = B$$

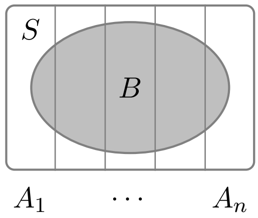
Note that A and A^c partition the universal set S for each subset A of S.

Example 7. The sets $A_1 = \{1, 3\}$, $A_2 = \{2\}$, $A_3 = \{4, 5\}$ partition the set $B = \{1, 2, 3, 4, 5\}$. $\Diamond$

The following simple result will often be used in the rest of the chapter, sometimes implicitly.

Lemma 1. *If $A_1, \ldots, A_n$ partition S and B is a subset of S, then $A_1 \cap B, \ldots, A_n \cap B$ partition B.*

This result is illustrated below.



There are many laws governing set operations. Here are just as few:

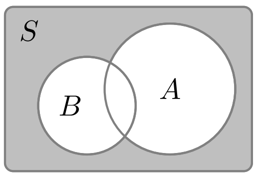
Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

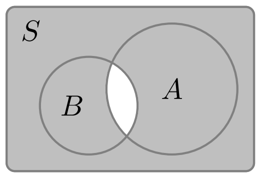
De Morgan's Laws

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

These laws can be proved by logical arguments or by sketching the Venn diagrams for the left-hand and right-hand sides of the identities. Venn diagrams for De Morgan's Laws are given here:



$(A \cup B)^c = A^c \cap B^c$



$(A \cap B)^c = A^c \cup B^c$

Definition 7. *If A is a set, then $|A|$ is the* ***number of elements*** *in A.*

Note that if A and B are disjoint, then

$$|A \cup B| = |A| + |B|$$

The Inclusion-Exclusion Principle. $|A \cup B| = |A| + |B| - |A \cap B|$

This result is clear once a Venn diagram is drawn.
The Inclusion-Exclusion Principle may be extended to any finite number of sets. For instance,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| .$$

Note that for any subset A of S, we have $S = A \cup A^c$ and so $|A^c| = |S| - |A|$. Hence, for example, $|(A \cup B)^c| = |S| - |A \cup B|$. The following example makes use of this idea.

Example 8. Of 20 music students, 7 play guitar, 8 play piano, and 3 play both guitar and piano. How many play neither guitar nor piano?

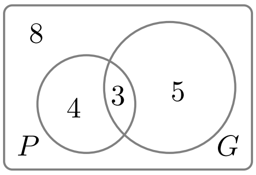
SOLUTION. Let S be the set of all the music students; let G be the set of students who play guitar; and let P be the set of students who play piano. By the information given,

$$|S| = 20 \qquad |G| = 7 \qquad |P| = 8 \qquad |P \cap G| = 3\,.$$

By the Inclusion-Exclusion Principle, the number of students who play neither piano nor guitar is

$$\begin{aligned} |(G \cup P)^c| &= |S| - |G \cup P| \\ &= |S| - (|G| + |P| - |G \cap P|) = 20 - (7 + 8 - 3) = 8\,. \end{aligned}$$

Alternatively, we can draw a Venn diagram of the problem and deduce the answer by filling in the number of elements in each region:



◊

9.2 Probability

The notion of luck is ancient and has often been seen as an inherent quality that individuals or objects might possess and whose nature is determined by Fate, whims of the Gods, karmic justice, mana-like association with other instances of luck, and many other mechanisms. By associating with lucky individuals or objects, by acting righteously, or by appealing to the Gods, one might improve one's luck during one's present life. Gambling is the competitive realisation of this belief in influencing one's luck, and it too is ancient. Good gamblers have appeared throughout history, and many prominent and talented mathematicians have focused much of their work on gambling problems and strategies, particularly in the 16-18th centuries. However, most of this work addressed specific problems and was stunted by incorrect intuitions and by an unfortunate focus on ratios and odds. This focus is still present in gambling today, where odds are given, rather than percentages.

Apart from a few important exceptions, it was only relatively recently, in the first half of the 20th century, that the notion of luck was treated rigorously and systematically by mathematicians. Of note, A. Kolmogorov put forth a set of axioms in 1933 that provided a solid framework for dealing mathematically with the notion of luck, or in mathematical terms: **probability**.

9.2.1 Sample Space and Probability Axioms

In order to develop a framework for probability, we will first think of any given situation that leads randomly to a set of outcomes as an **experiment**. Thus, the roll of a die is seen as an experiment, as is the Melbourne Cup; countless other such experiments abound, including financial markets, the weather, election outcomes, or what grade you might get for this course.

Definition 1. *A* ***sample space*** *of an experiment is a set of all possible outcomes.*

Outcomes are also called **sample points**.

Example 1. Tossing a coin may be seen as an experiment. An appropriate sample space is the set $S = \{H, T\}$ where H ("head") and T ("tail") are the two possible outcomes.

Example 2. Tossing a coin 3 times can be seen as another experiment. If the object of the experiment is to determine the resulting coin-flip sequence, then an appropriate sample space is

$$S_1 = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

On the other hand, if the object of the experiment were to determine the number of resulting heads, then an appropriate sample space is

$$S_2 = \{0, 1, 2, 3\}.$$

Thus, the experiment and its sample space depends on the type of data that we wish to observe.

It is often useful to consider sets of outcomes, particularly if the number of outcomes is large. This leads to the next definition.

Definition 2. *An **event** is a subset of a sample space.*

Note that the set of all events in a sample space S is exactly the power set $\mathcal{P}(S)$.
Note also that the empty set $\emptyset$ and the whole space S are events.

Example 3. Toss a coin 3 times and consider the event A that we toss 2 heads. This is the subset of the sample space S_1 of Example 2 given by

$$A = \{HHT, HTH, THH\}.$$

Note that each of the outcomes in A forms an event by itself: $\{HHT\}$, $\{HTH\}$, $\{THH\}$.

In each of the above examples, each possible outcome has equal probability. This is not generally true, so we must define probability in full generality.

Definition 3. *A **probability** P on a sample space S is any real function on $\mathcal{P}(S)$ that satisfies the following conditions:*

(a) $0 \leqslant P(A) \leqslant 1$ *for all* $A \subseteq S$;

(b) $P(\emptyset) = 0$;

(c) $P(S) = 1$;

(d) If A and B are disjoint, then $P(A \cup B) = P(A) + P(B)$.

Example 4. Toss a coin and observe whether H or T is tossed. The appropriate sample space is the set $S = \{H, T\}$. Define the probability P on S as follows, for each event $A \subseteq S$:

$$P(A) = \frac{|A|}{2}\,.$$

Then $P(\{H\})$ is the probability of tossing H, namely $P(\{H\}) = \frac{1}{2}|\{H\}| = \frac{1}{2}$. Similarly, $P(\{T\}) = \frac{1}{2}$. Note that the probability of tossing neither H nor T is $P(\emptyset) = 0$, and that the probability of tossing either H or T is $P(S) = 1$.

This probability is exactly the probability that one would usually think of when tossing a coin. However, there are many other possible probabilities. For instance, let p be some real number between 0 and 1, and define the function Q on $\mathcal{S}$ by

$$Q(\emptyset) = 0 \qquad Q(\{H\}) = p \qquad Q(\{T\}) = 1 - p \qquad Q(S) = 1\,.$$

It is easy to verify that Q is a probability on S. To find a physical interpretation of this probability, one could think of a coin that is twisted or bent, so that the probability of tossing H is not necessarily the same as that of tossing T. Bear in mind, however, that a probability is a mathematical object that need not always model a real-world phenomenon.

Example 5. Toss a coin 3 times and observe the resulting (ordered) sequence of H and T, as in Example 2 above. Let S be the natural sample space consisting of all 8 such sequences. The appropriate probability P is then given as follows, for each event $A \subseteq S$:

$$P(A) = \frac{|A|}{8}\,.$$

For instance, consider the event A that we toss 2 heads. The probability of this happening is

$$P(A) = \frac{|A|}{8} = \frac{|\{HHT, HTH, THH\}|}{8} = \frac{3}{8}\,.$$

Now consider the event $B = \{HHH\}$ that we toss 3 heads. Since $|B| = 1$, we see that $P(B) = \frac{1}{8}$.

The probability of tossing at least 2 heads is $P(A \cup B)$ which, since A and B are disjoint, equals

$$P(A \cup B) = P(A) + P(B) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}\,.$$

Example 6. Roll a die and observe the resulting number. An appropriate sample space is then $S = \{1, \ldots, 6\}$. The appropriate probability P is given as follows, for each event $A \subseteq S$:

$$P(A) = \frac{|A|}{6}\,.$$

For example, consider the event A that we roll an even number. The probability of this occurring is

$$P(A) = \frac{|A|}{6} = \frac{|\{2, 4, 6\}|}{6} = \frac{3}{6} = \frac{1}{2}\,.$$

Theorem 1. *Let P be a probability on a sample space S, and let A be an event in S.*

1. *If S is finite (or countable), then* $P(A) = \sum_{a \in A} P(\{a\})\,.$

2. *If S is finite and $P(\{a\})$ is constant for all outcomes $a \in S$, then* $P(A) = \dfrac{|A|}{|S|}$.
3. *If S is finite (or countable), then* $\displaystyle\sum_{a \in S} P(\{a\}) = 1$.

Note that if S is finite, then $P(A)$ may be seen as *size*, or *ratio*, of A compared to S. In general, $P(A)$ may be seen as a *measure* of how large A is compared to S. Outcomes whose probabilities are all equal are often referred to as "equally likely".

Proof. The finite case of statement 1 follows by induction using the additive condition (d) in the definition of a probability. We will ignore the general case of statement 1 but note that it is often given as an axiom for probabilities. Statement 3 follows immediately from statement 1 and by noting that $P(S) = 1$. Let us now prove the statement 2, so suppose that S is finite and that $P(\{a\})$ is equal to the constant p for all outcomes $a \in S$. By statement 3,

$$1 = \sum_{a \in S} P(\{a\}) = \sum_{a \in S} p = |S|p\,,$$

so $p = \dfrac{1}{|S|}$. By statement 1,

$$P(A) = \sum_{a \in A} P(\{a\}) = \sum_{a \in A} \frac{1}{|S|} = \frac{|A|}{|S|}\,. \qquad \Diamond$$

Example 7. The natural probabilities P in Examples 4–6 may each be expressed as

$$P(A) = \frac{|A|}{|S|}$$

where A is any event in S. This reflects the fact that each outcome is equally likely.

Example 8. Pick a ball at random from a bag containing 3 red balls and 7 blue balls. If each ball has the same chance as being picked as any other ball, then the chance of picking a red ball is $\frac{3}{10}$.

Let us express this in mathematical terms. Let S be the sample space consisting of all 10 balls. Next, let A be the event that a red ball is chosen; A is then the set containing the three red balls. Since the probability of each outcome is the same, the probability of picking a red ball is

$$P(A) = \frac{|A|}{|S|} = \frac{3}{10}\,,$$

as expected. $\Diamond$

The definition of probability only states what is required of a probability; it does not help us decide upon an appropriate probability for a given experiment. This sort of decision is called "allocating the probabilities" and is generally based on one of the following three methods:

Method 1. Allocate the probabilities on the basis of any inherent symmetry in the situation. This is what is applied in games of chance, as illustrated by the die-rolling or coin-tossing examples that we have seen. It is how you calculated with probability at school by using counting, permutations, and combinations with equally likely outcomes.

It is also used to allocate probabilities in the following sort of experiment. A "wheel of fortune" wheel is spun. The probability that it points to some region which subtends an angle θ is $\frac{\theta}{360}$. This is an example of an experiment with a sample space that is not countable, let alone finite.

Method 2. Allocate the probabilities on the basis of experience or large amounts of data.
This is what actuaries regularly do when creating life tables. The probability of a typical Australian male aged 60 living to age 65 is 95.4%, based on past history of lots of males aged 60 years.

Method 3. Guess or use intuition.
This is common in society but should be avoided by non-experts when dealing with serious issues. Indeed, the gambling industry, insurances, supermarket prices, and so on, and even much of politics and lawmaking, all rely on the common person's inability to properly understand probabilities, particularly when it comes to odds of winning or risk of injury. Actuaries and mathematicians educate themselves to avoid falling for common misconceptions; however, even these experts should not rely heavily on their intuition of probability.

9.2.2 Rules for Probabilities

Theorem 2. *Let A and B be events of a sample space S.*

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (**Addition Rule**)
2. $P(A^c) = 1 - P(A)$
3. *If* $A \subseteq B$, *then* $P(A) \leqslant P(B)$.

Proof.

1. This result is connected to the Inclusion-Exclusion Principle.
2. The sets A and A^c partition S, so $P(A) + P(A^c) = P(A \cup A^c) = P(S) = 1$.
3. The sets A and $B - A$ partition B, so $P(A) \leqslant P(A) + P(B - A) = P(B)$. ☐

Example 9. What is the probability that at least two of n people share the same birthday?

Solution. Ignoring leap years, let Y be the set of the 365 days of the year. An experiment could here be to discover the n birthdays, and an associated sample space is

$$S_n = \{(b_1, \ldots, b_n) \,:\, b_1, \ldots, b_n \in Y\}\,.$$

We wish to calculate $P(A_n)$ for the event $A_n \subseteq S_n$ that at least two of the n people share the same birthday. Assume that the probability of each person being born on a given date does not depend on the person or on the date. The outcomes of the sample space then have constant probability, namely $\frac{1}{|S_n|} = \frac{1}{365^n}$. Therefore, $P(A_n) = 1 - P(A_n^c) = 1 - \frac{|A_n^c|}{|S_n|}$. Now, A_n^c is the event that none of the n birthdays are the same. Therefore,

$$P(A_n) = 1 - P(A_n^c) = 1 - \frac{|A_n^c|}{|S_n|} = 1 - \frac{365 \times 364 \times \cdots (365 - n + 1)}{365^n}\,.$$

It is thus slightly more likely than not that at least two of 23 people have the same birthday ($P(A_{23}) = 50.7\%$), and it is highly likely that at least two of 57 people share the same birthday ($P(A_{57}) = 99.01\%$). Of course, there will always be at least two people with the same birthday whenever there are more people than days in the year, and this is expressed by the probability $P(A_n) = 1$ for $n > 365$. ◊

Example 10. In some town, 80% of the population has comprehensive car policies, 60% has house cover, and 10% has neither. What percent has both covers?

SOLUTION. Let A be the event "a person has comprehensive car cover" and let B be the event "a person has house cover". For any random person,

$$P(A) = 0.8\,, \qquad P(B) = 0.6\,, \qquad \text{and} \qquad P(A^c \cap B^c) = 0.1\,.$$

Hence, $P(A \cup B) = 1 - P((A \cup B)^c) = 1 - P(A^c \cap B^c) = 1 - 0.1 = 0.9$. Therefore,

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.8 + 0.6 - 0.9 = 0.5\,.$$

In other words, 50% of the population has both covers. ◊

9.2.3 Conditional Probabilities

We now consider what happens if we restrict the sample space from S to some event in S.

Example 11. In Example 10 above, 80% of people have comprehensive car cover.
However, of those people who have house cover, the percentage who also have comprehensive car cover is

$$\frac{50}{60} = 0.833 \quad \text{or} \quad 83.3\%\,.$$

Thus when we restrict our sample space to those having house cover, the percentage of those having comprehensive cover changes. We say that the conditional probability of a person having comprehensive cover **given** that they have house cover is 0.833. ◊

Definition 4. *The* ***conditional probability*** *of A* ***given*** *B is denoted and defined by*

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \textit{provided that} \quad P(B) \neq 0\,.$$

Lemma 3. *For any fixed event B, the function $P(A|B)$ is a probability on S.*

Proof. Check that the probability conditions are satisfied for $P(A|B)$. ☐

Since $P(S) = 1$, we can write, for each event A of S,

$$P(A) = \frac{P(A \cap S)}{P(S)} = P(A|S)\,.$$

Just as $P(A)$ can be seen as a measure of A compared to S, $P(A|B) = \frac{P(A\cap B)}{P(B)}$ can be seen as a measure of A (or the part of A that is contained in B) compared to B. This is illustrated by the following Venn diagrams:

Example 12. We roll a die and let A and B be the events that we roll a six and that we roll an even number, respectively. Then $P(A) = \frac{1}{6}$ and $P(B) = \frac{3}{6} = \frac{1}{2}$. Since in this case $A \cap B = A$,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{\left(\frac{1}{6}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{3}\,.$$

In other words, given that we rolled an even number, the probability of having rolled a six is $\frac{1}{3}$. This is as one would expect since there are 3 even rolls (2,4,6) of which one is six.

In contrast, the probability of rolling an even number, given that we rolled a six, is

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1\,. \qquad \Diamond$$

Example 13. Consider a bag containing 3 red balls and 3 blue balls. First draw one ball from the bag and then draw another. Let R_i be the events that a red ball is chosen on the ith draw, where $i = 1, 2$, and define $B_1 (= R_1^c)$ and $B_2 (= R_2^c)$ similarly for the blue balls. The probability of first drawing a red ball or a blue is the same, namely $P(R_1) = P(B_1) = \frac{3}{6} = \frac{1}{2}$.

Now, suppose that we first draw a red ball. The bag then contains 2 red balls and 3 blue balls, so the probability of drawing a red ball on the second draw is $P(R_2|R_1) = \frac{2}{5}$. We can also calculate this probability from the definition of conditional probability. Of the $\frac{6\times 5}{2} = 15$ ways of choosing two of the six balls without order, three of ways give us two red balls. Therefore, we see that $P(R_1 \cap R_2) = \frac{3}{15} = \frac{1}{5}$, so

$$P(R_2|R_1) = \frac{P(R_1 \cap R_2)}{P(R_1)} = \frac{\left(\frac{1}{5}\right)}{\left(\frac{1}{2}\right)} = \frac{2}{5}\,.$$

In contrast, $P(R_2) = P(B_2) = \frac{1}{2}$ since there are equally many red and blue balls to begin with.

Rearranging the terms in the definition of conditional probability yields the following identities.

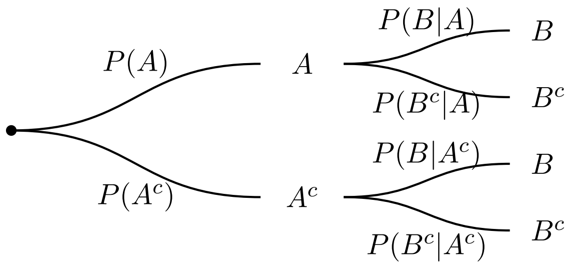
Multiplication Rule $\quad P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$

Example 14. Consider the bag of red and blue balls in Example 13. We saw that the probability of drawing a red ball on the second draw, given that we first drew a red ball, is $P(R_2|R_1) = \frac{2}{5}$. Similarly, it is easy to see that $P(R_2|B_1) = \frac{3}{5}$. To calculate $P(R_2)$ without using the symmetry argument in Example 13, first note that R_2 is partitioned by $R_2 \cap R_1$ and $R_2 \cap B_1$: either a red ball is first drawn, followed by another red ball, or a blue ball and then a red ball are drawn. Hence by the Multiplication Rule,

$$\begin{aligned} P(R_2) &= P((R_2 \cap R_1) \cup (R_2 \cap B_1)) = P(R_2 \cap R_1) + P(R_2 \cap B_1) \\ &= P(R_2|R_1)P(R_1) + P(R_2|B_1)P(B_1) = \frac{2}{5} \times \frac{1}{2} + \frac{3}{5} \times \frac{1}{2} = \frac{1}{2}\,, \end{aligned}$$

as expected. $\Diamond$

Conditional probabilities are implicitly used whenever a tree diagram is drawn. Thus in a typical two stage experiment we have the following tree diagram



Example 15. Consider 3 urns containing red and blue balls:

- Urn 1 contains 10 balls, of which 3 are red and 7 are blue;
- Urn 2 contains 20 balls, of which 4 are red and 16 are blue; and
- Urn 3 contains 10 balls, of which 0 are red and 10 are blue.

First, an urn is chosen at random; then a ball is chosen from it at random.

(a) What is the probability that a red ball is chosen from urn 2?

(b) What is the probability of choosing a red ball?

(c) If a red ball were chosen, what is then the probability that it came from Urn 2?

Solution. Assume that we are equally likely to choose any urn and, given an urn, are equally likely to choose any ball in it. Let U_1, U_2, and U_3 denote the event of choosing Urn 1, 2, and 3, respectively, and let R and B denote the event of then choosing a red or blue ball, respectively.

Then $P(U_1) = P(U_2) = P(U_3) = \frac{1}{3}$ and

$$\begin{aligned} P(R|U_1) &= \frac{3}{10} & P(R|U_2) &= \frac{4}{20} = \frac{1}{5} & P(R|U_3) &= 0 \\ P(B|U_1) &= \frac{7}{10} & P(B|U_2) &= \frac{16}{20} = \frac{4}{5} & P(B|U_3) &= 1\,. \end{aligned}$$

(a) Therefore, $P(R \cap U_2) = P(R|U_2)P(U_2) = \dfrac{1}{5} \times \dfrac{1}{3} = \dfrac{1}{15}$.

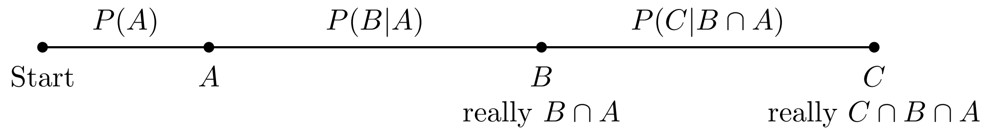
(b) Similarly, $P(R \cap U_1) = \dfrac{3}{10} \times \dfrac{1}{3} = \dfrac{1}{10}$ and $P(R \cap U_3) = \dfrac{0}{10} \times \dfrac{1}{3} = 0$.
Now, $R = (R \cap U_1) \cup (R \cap U_2) \cup (R \cap U_3)$ and the three terms are disjoint, so

$$P(R) = P(R \cap U_1) + P(R \cap U_2) + P(R \cap U_3) = \frac{1}{10} + \frac{1}{15} + 0 = \frac{1}{6}\,.$$

(c) Finally, $P(U_2|R) = \dfrac{P(U_2 \cap R)}{P(R)} = \dfrac{\left(\frac{1}{15}\right)}{\left(\frac{1}{6}\right)} = \dfrac{2}{5}$.

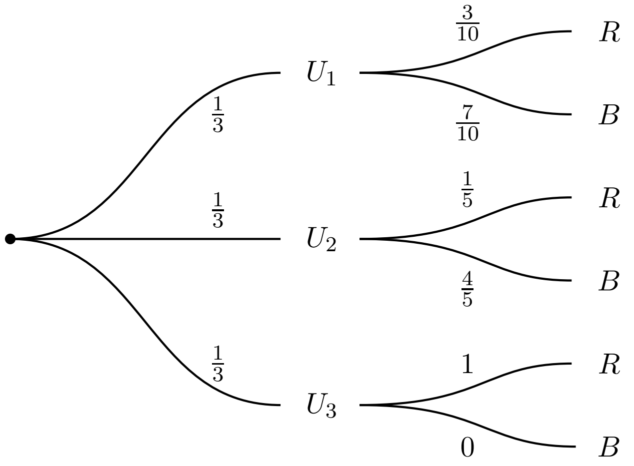
$\diamondsuit$

The above example shows one instance of a multi-stage experiment. The conditional probabilities and the multiplication rule for these sort of experiments can be illustrated and applied using **tree diagrams**. Such diagrams illustrate all possible outcomes of the multi-stage experiment as well as the way in which one is able to arrive at those outcomes via the various stages. The branches carry the conditional probability of the right hand node given that you are at the left node. The probability of getting from one node to a node to its right is obtained by multiplying the probability on the connecting branches. A typical sequence of branches is of the form



Here,
$$P(C \cap B \cap A) = P(C|B \cap A)P(B \cap A) = P(C|B \cap A)P(B|A)P(A)\,.$$

Example 16. The tree diagram for the multi-stage experiment given in Example 15 is as follows:



$\leftarrow P(R \cap U_1) = \frac{1}{3} \times \frac{3}{10} = \frac{1}{10}$

$\leftarrow P(B \cap U_1) = \frac{1}{3} \times \frac{7}{10} = \frac{7}{30}$

$\leftarrow P(R \cap U_2) = \frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$

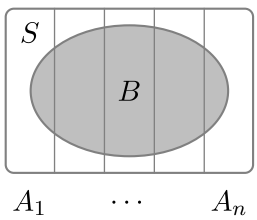
$\leftarrow P(B \cap U_2) = \frac{1}{3} \times \frac{4}{5} = \frac{4}{15}$

$\leftarrow P(R \cap U_3) = \frac{1}{3} \times 0 = 0$

$\leftarrow P(B \cap U_3) = \frac{1}{3} \times 1 = \frac{1}{3}$

Since sequences of branches represent disjoint events,
$$P(R) = P(R \cap U_1) + P(R \cap U_2) + P(R \cap U_3) = \frac{1}{10} + \frac{1}{15} + 0 = \frac{1}{6}\,. \qquad \Diamond$$

Tree diagrams are very useful for visualising and calculating problems involving small numbers of conditional probabilities. However, tree diagrams are infeasible when these numbers are large or only implicitly given. We now derive a mathematical rule that enables us to deal with such cases. In particular, suppose that the n events $A_1, \ldots, A_n$ partition the sample space S:



Since the sets $A_1 \cap B, \ldots, A_n \cap B$ partition the event B, we see that
$$P(B) = P(A_1 \cap B) + \cdots + P(A_n \cap B)\,.$$

Applying the Multiplication Rule to each of these n terms yields the following rule.

Total Probability Rule
If $A_1, \ldots, A_n$ partition S and B is an event, then $\displaystyle P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)\,.$

Note that we have already used this rule implicitly, for instance to calculate $P(R_2)$ in Example 13, and to calculate $P(R)$ in part (b) of Example 15.

Although the Total Probability Rule is a very simple and almost obvious result, it is very useful. Furthermore, it implies Bayes' Rule, which is non-trivial and which often offers surprising results.

Bayes' Rule
If $A_1, \ldots, A_n$ partition S and B is an event, then $P(A_j|B) = \dfrac{P(B|A_j)P(A_j)}{\displaystyle\sum_{i=1}^{n} P(B|A_i)P(A_i)}$.

Proof. Apply the Total Probability Rule to the identity $P(A_j|B) = \dfrac{P(A_j \cap B)}{P(B)}$. $\square$

Example 17. Consider the urns and the ball of Example 15 above, and suppose that we drew a blue ball from one of the urns. What is then the probability $P(U_1|B)$ that we drew it from Urn 1?

Solution. Now, U_1, U_2, U_3 partition S since we must choose exactly one urn. Thus by Bayes' Rule,

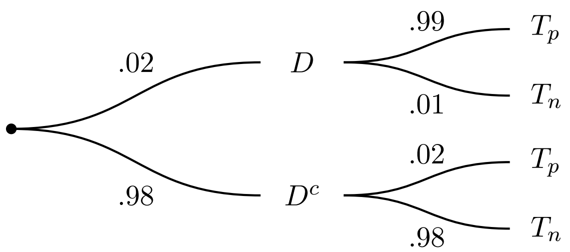
$$P(U_1|B) = \frac{P(B|U_1)P(U_1)}{P(B|U_1)P(U_1) + P(B|U_2)P(U_2) + P(B|U_3)P(U_3)} = \frac{\frac{7}{10} \times \frac{1}{3}}{\frac{7}{10} \times \frac{1}{3} + \frac{4}{5} \times \frac{1}{3} + 1 \times \frac{4}{5}} = \frac{7}{25} \quad \diamondsuit$$

This example illustrates how Bayes' Rule allows reverse-inference. To some, this can seem counter-intuitive and has even caused contention and controversy. Nevertheless, Bayes' Rule remains a very useful statistical tool that is widely used in medical trials, court cases, and elsewhere.

Example 18. A certain diagnostic test for a disease X indicates with 99% accuracy that a person has X when that person actually has it. Similarly, the test indicates with 98% accuracy that someone does not have X when they do not in fact have it. In medical terms, the test is "positive" if it indicates that a person has the disease and is "negative" otherwise. Suppose that 2% of the population has the disease.

Find the probability of a false positive, namely that a person without X still tests positive.

Solution. One might guess that this probability would be very small since the test seems so accurate. Let us calculate whether or not this is true. Thus, let D be the event that the person has disease X, let T_p be the event that the test shows positive, and set $T_n = T_p^c$. The tree diagram is



so by Bayes' Rule,

$$P(D^c|T_p) = \frac{P(T_p|D^c)P(D^c)}{P(T_p|D^c)P(D^c) + P(T_p|D)P(D)} = \frac{.02 \times .98}{.02 \times .98 + .99 \times .02} = 0.497\,.$$

Thus, almost 50% positives are false, which might seem outrageously inaccurate. Fortunately, the probability of a false negative, that is, someone with X testing negative, is almost negligible:

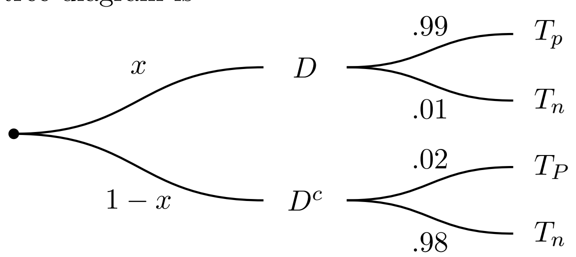
$$P(D|T_n) = \frac{P(T_n|D)P(D)}{P(T_n|D)P(D) + P(T_n|D^c)P(D^c)} = \frac{.01 \times .02}{.01 \times .02 + .98 \times .98} = .000208\,.$$

Thus, if someone is tested for disease X and the test shows that they do not have X, then they almost certainly do not have it. On the other hand, if the test is positive, then they might possibly have X, and more accurate (and presumably more expensive or time-consuming) tests can then made made to determine whether they do in fact have X. ♢

Example 19. We modify Example 18 slightly by supposing that fraction x of population has disease X and that 5% test positive when a large random sample is tested.

What percentage of the population has disease X?

Solution. Here, the tree diagram is



By the Total Probability Rule,

$$0.05 = P(T_p) = P(T_p|D)P(D) + P(T_p|D^c)P(D^c) = .99x + .02(1-x) = .97x + .02\,.$$

Therefore, $x = \frac{0.03}{0.97} \approx .031 = 3.1\%$ of the population has the disease. ♢

9.2.4 Statistical Independence

Intuitively, two events A and B are mutually independent if one does not influence the probability of the other. This can be expressed as $P(A|B) = P(A)$ and $P(B|A) = P(B)$. That is, the probability of A does not depend on whether or not B is given, and the same is true for the probability of B. Since $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$, we can express this independence quite elegantly:

Definition 5. *Events A and B are* ***(statistically) independent*** *if and only if*

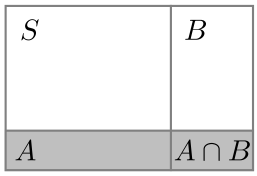
$$P(A \cap B) = P(A)P(B)$$

Note that in contrast to conditional probability, this definition allows all probabilities, including 0.

To visualise statistical independence, let A and B be independent events with $P(B) \neq 0$. Then, since $P(S) = 1$,

$$\frac{P(A)}{P(S)} = \frac{P(A \cap B)}{P(B)}\,.$$

Thus, the probability measure of A is just as great in comparison to the whole sample space S as is $A \cap B$, the part of A in B, when compared to B. The following Venn diagram illustrates this:



Note that independence and disjointness are not the same concept, as might be supposed. Indeed, these two concepts are almost opposite in nature: non-empty events A and A^c are disjoint but are strongly dependent, for if A occurs, then A^c cannot (independently) occur, and vice versa.

Example 20. We roll a die and let A and B be the events that we roll a six and that we roll an even number, respectively. Then $P(A) = \frac{1}{6}$ and $P(B) = \frac{3}{6} = \frac{1}{2}$. Since in this case $A \cap B = A$,

$$P(A \cap B) = P(A) = \frac{1}{6} \qquad \text{and} \qquad P(A)P(B) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12},$$

so $P(A \cap B) \neq P(A)P(B)$. Therefore, A and B are not independent; that is, they are dependent. This is as one would expect since if A occurs, then B must necessarily occur.

Now, define A' to be the event that we roll either a five or a six. Then since $A' \cap B = A$,

$$P(A') = \frac{2}{6} = \frac{1}{3} \qquad \text{and} \qquad P(A' \cap B) = P(A) = \frac{1}{6},$$

so $P(A')P(B) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6} = P(A' \cap B)$. Therefore, A' and B are independent. In other words, the probability of rolling an even number is the same, namely $\frac{1}{2}$, whether or not one of the numbers five and six is rolled, and the converse is equally true. $\Diamond$

Example 21. Roll a dice twice and, for each $i = 1, \ldots, 6$, let X_i and Y_i denote the events that we get i on the first roll and second roll, respectively. Under usual conditions, we may assume that the first and second throws have no influence on each other. Therefore, X_i and Y_j are independent for all i, j, so

$$P(X_i \cap Y_j) = P(X_i)P(Y_j) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}.$$

This identity allows us to calculate more complicated events, such as the event S_4 that the sum of the two rolls is 4. Since S_4 is partitioned by $X_1 \cap Y_3$, $X_2 \cap Y_2$, $X_3 \cap Y_1$, we see that

$$\begin{aligned} P(S_4) &= P((X_1 \cap Y_3) \cup (X_2 \cap Y_2) \cup (X_3 \cap Y_1)) \\ &= P(X_1 \cap Y_3) + P(X_2 \cap Y_2) + P(X_3 \cap Y_1) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{1}{12}. \end{aligned}$$

This result could also have been obtained by viewing this experiment as having a sample space of the 36 equally likely outcomes

$$S = \{(1,1), (1,2), \ldots, (6,6)\}$$

where, for instance, $(3,5)$ denotes that we first rolled three and then five. Then

$$P(S_4) = \frac{|\{(1,3), (2,2), (3,1)\}|}{|S|} = \frac{3}{36} = \frac{1}{12},$$

as before. $\Diamond$

We now consider the statistical independence of any finite number of events.

Definition 6. *Events $A_1, \ldots, A_n$ are* ***mutually independent*** *if and only if, for any $A_{i_1}, \ldots, A_{i_k}$ of these,*

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \times \cdots \times P(A_{i_k}) .$$

Example 22. Events A, B, C are mutually independent if and only if these four identities all hold:

$$\begin{aligned} P(A \cap B) &= P(A)P(B) \\ P(A \cap C) &= P(A)P(C) \\ P(B \cap C) &= P(B)P(C) \\ P(A \cap B \cap B) &= P(A)P(B)P(C) \end{aligned}$$

$\Diamond$

In general for n events to be mutually independent, they must satisfy non-trivial $2^n - n - 1$ identities such as the ones above, one for each subset of $\{1, \ldots, n\}$ with at least two elements. None of these many identities imply any of the others in general, so we cannot make do with a smaller set of identities. This is illustrated by the following example.

Example 23. Draw a ball from a bag containing four balls marked 0 to 3. For $i = 0, \ldots, 3$, let A_i be the event that ball i is drawn, and let $B_i = A_0 \cup A_i$ be the event that ball 0 or ball i is drawn. Then for all distinct $i, j = 1, 2, 3$,

$$P(B_i) = P(A_0 \cup A_i) = \frac{2}{4} = \frac{1}{2} \qquad \text{and} \qquad P(B_i \cap B_j) = P(A_0) = \frac{1}{4} .$$

Hence, $P(B_i \cap B_j) = P(B_i)P(B_j)$, so B_i and B_j are independent. In contrast,

$$P(B_1 \cap B_2 \cap B_3) = P(A_0) = \frac{1}{4} \neq \frac{1}{8} = P(B_1)P(B_2)P(B_3) ,$$

so B_1, B_2, B_3 are not mutually independent. $\Diamond$

If events A and B are independent, then A and B^c are also independent:

$$P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c) .$$

By modifying these calculations slightly and using induction, we can proof the more general result:

Theorem 4. *If events $A_1, \ldots, A_n$ are mutually independent and B_i is either A_i or A_i^c for each $i = 1, \ldots, n$, then $B_1, \ldots, B_n$ are also mutually independent.*

Suppose that events A, B, and C are mutually independent. Then by Theorem 4,

$$\begin{aligned} P(A \cap (B \cup C)) &= P(A \cap ((B - C) \cup (B \cap C) \cup (C - B))) \\ &= P(A \cap B \cap C^c) + P(A \cap B \cap C) + P(A \cap C \cap B^c) \\ &= P(A)P(B \cap C^c) + P(A)P(B \cap C) + P(A)P(C \cap B^c) \\ &= P(A)(P(B - C) + P(B \cap C) + P(B - C)) \\ &= P(A)P(B \cup C) . \end{aligned}$$

We see that A and $B \cup C$ are also independent. By generalising the above calculations, one may prove the following result.

Theorem 5. *If events $A_{1,1}, \ldots, A_{1,n_1}, A_{2,1}, \ldots, A_{m,n_m}$ are mutually independent and for each $i = 1, \ldots, m$, the event B_i is obtained from $A_{i,1}, \ldots, A_{i,n_i}$ by taking unions, intersections, and complements, then $B_1, \ldots, B_n$ are also mutually independent.*

Example 24 (A reliability example)**.**
A 3-engine plane has a central engine and two wing engines. The plane will crash if the central engine and at least one of the wing engines fail. On any given flight, the central engine fails with probability 0.005, and each wing engine fails with probability 0.008. Assuming that the three engines fail mutually independently, find the probability that the plane will crash during a flight.

Solution. Let A be the event that the port engine fails, let B be the event that the starboard engine fails, and C be the event that the central engine fails. Then $P(A) = P(B) = 0.008$ and $P(C) = 0.005$. Let D denote the event that the plane crashes and note that $D = C \cap (A \cup B)$. Since A, B, and C are mutually independent, C and $A \cup B$ are independent by Theorem 5. Therefore,

$$\begin{aligned} P(D) = P(C \cap (A \cup B) = P(C)P(A \cup B) &= P(C)[P(A) + P(B) - P(A \cap B)] \\ &= P(C)[P(A) + P(B) - P(A)P(B)] \\ &= 0.005[0.008 + 0.008 - 0.008 \times 0.008] = 0.00007968\,. \end{aligned}$$

(Note: There are other ways to do this problem.)

Under our assumptions, the plane will crash on a given flight with probability slightly less than eighty in one million. These are dangerous assumptions, however, since it is highly optimistic to hope that the engines will fail independently of each other. For instance, if engine failure is caused by volcanic ash, then all three engines will be at risk of failure, and there are countless other factors, like shared electric wiring, that might similarly introduce dependence. In view of this, the real probabilities might be considerably higher than stated. ♢

9.3 Random Variables

It can often be useful to label the outcomes of an experiment by numbers. This often makes event notation more flexible, and it allows us to perform arithmetic on the outcomes.

Definition 1. *A* ***random variable*** *is a real function defined on a sample space.*

Example 1. Toss a coin and let $S = \{H, T\}$ be the associated sample space.
Two random variables X and Y on S are given as follows, for each outcome $s \in S$:

$$X(s) = \begin{cases} 1 & \text{if } s = H \\ 0 & \text{if } s = T \end{cases} \qquad Y(s) = \begin{cases} 1 & \text{if } s = H \\ -1 & \text{if } s = T \end{cases}$$

Example 2. Roll a die and let $S = \{1, \ldots, 6\}$ be the associated sample space.
Two random variables X and Y on S are given as follows, for each outcome $s \in S$:

$$X(s) = s \qquad Y(s) = \begin{cases} -1 & \text{if } s \text{ is odd} \\ 1 & \text{if } s \text{ is even} \end{cases}$$

Definition 2. *For a random variable X on some sample space S, define for all subsets $A \subseteq S$ and real numbers $r \in \mathbb{R}$,*

- $\{X \in A\} = \{s \in S \,:\, X(s) \in A\}$
- $\{X = r\} = \{s \in S \,:\, X(s) = r\}$
- $\{X \leqslant r\} = \{s \in S \,:\, X(s) \leqslant r\}$
- *... and so on.*

We suppress the curly brackets when expressing the probability of these events. For instance, $P(\{X = r\})$ is written as $P(X = r)$.

Example 3. Roll a die and let X and Y be the random variables defined in Example 2. Then

$$P(X > 4) = P(\{5, 6\}) = \frac{1}{3} \qquad P(Y = 1) = P(\{2, 4, 6\}) = \frac{1}{2} \qquad P(Y = \pi) = P(\emptyset) = 0\,.$$

Example 4. Toss a coin three times and let the random variable X count the number of heads tossed. Then $X(S) = \{0, 1, 2, 3\}$ and

$$\begin{aligned} P(X = 0) &= P(\{TTT\}) = \frac{1}{8} \\ P(X = 1) &= P(\{HTT, THT, TTH\}) = \frac{3}{8} \\ P(X = 2) &= P(\{HHT, HTH, THH\}) = \frac{3}{8} \\ P(X = 3) &= P(\{HHH\}) = \frac{1}{8}\,. \end{aligned}$$

Definition 3. *The* ***cumulative distribution function*** *of a random variable X is given by*

$$F_X(x) = P(X \leqslant x) \qquad \text{for} \quad x \in \mathbb{R}\,.$$

We often refer to $F_X(x)$ as just $F(x)$. Note that F is non-decreasing and that if $a \leqslant b$, then $P(a < X \leqslant b) = F(b) - F(a)$ and

$$0 = \lim_{x \to -\infty} F(x) \leqslant F(a) \leqslant F(b) \leqslant \lim_{x \to \infty} F(x) = 1\,.$$

Example 5. Toss a coin three times and let random variable X be the number of heads tossed, as in Example 4. Then

$$F\Big(\frac{3}{2}\Big) = P\Big(X \leqslant \frac{3}{2}\Big) = P(X = 0) + P(X = 1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}\,.$$

9.3.1 Discrete Random Variables

The *image* of a function is the set of its function values.

> **Definition 4.** *A random variable X is* **discrete** *if its image is countable.*

The random variables in Examples 1–5 are each discrete since their images are finite and thus countable. We shall for now only consider discrete random variables but will consider certain non-discrete random variables in Section 9.5, namely those that are *continuous*.

> **Definition 5.** *The* **probability distribution** *of a discrete random variable X is some description of all the probabilities of all events associated with X.*

We sometimes write the probabilities as $p_k = P(X = x_k)$.

Note that for a discrete random variable X, the cumulative distribution function $F(x)$ is

$$F(x) = \sum_{k \leqslant x} p_k\,.$$

Thus, in practice to show that $\{p_k\}_{k\geqslant 0}$ is a probability distribution, we need to show that:

(i) $p_k \geqslant 0$ and

(ii) $\displaystyle\sum_k p_k = 1$.

Example 6. Roll a die and define random variables X and Y as in Example 2. The probability distributions of X and Y can for instance be represented as

$$P(X = x) = \begin{cases} \frac{1}{6} & \text{if } x \in \{1, \dots, 6\} \\ 0 & \text{otherwise} \end{cases}$$

and

y_k	-1	1
$p_k = P(Y = y_k)$	$\frac{1}{2}$	$\frac{1}{2}$

Clearly in each case $p_k \geqslant 0$ and for the random variable X, we have $\displaystyle\sum_k p_k = 6 \times \frac{1}{6} = 1$, while for the random variable Y, $\displaystyle\sum_k p_k = \frac{1}{2} + \frac{1}{2} = 1$. Hence, these are probability distributions for the random variables X and Y respectively. $\diamondsuit$

Example 7. Roll a die twice and let $S = \{(i, j) : i, j = 1, \dots, 6\}$ be the associated sample space. Let X be the random variable defined by $X(i, j) = i + j$, that is, X is the sum of the numbers showing. The probability distribution of X is

x_k	2	3	4	5	6	7	8	9	10	11	12
p_k	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Clearly, $p_k \geqslant 0$. Also,

$$\sum_k p_k = \frac{1}{36}(1 + 2 + 3 + 4 + 5 + 6 + 5 + 4 + 3 + 2 + 1) = 1.$$

Hence p_k is a probability distribution. $\Diamond$

Note that, in the above example, we did not actually need to specify the sample space; it would suffice to define X to be the sum of rolls. Indeed, we often specify the probability distribution without defining the sample space or even, at times, a random variable with that distribution.

Example 8. The probability distribution of discrete random variable X is given as follows:

x_k	0	1	2	4	7
$p_k = P(X = x_k)$	0.2	$0.3c$	0.2	c^2	0.5

(a) Find the value of c.

(b) Find $P(X \geq 4)$.

SOLUTION.

(a) $\sum p_k = c^2 + 0.3c + 0.9 = 1$, or $c^2 + 0.3c - 0.1 = 0$.
Solving this gives $c = -0.5$ or $c = 0.2$. Since $0.3c = p_1 \geq 0$, we conclude that $c = 0.2$.

(b) By (a), the probability distribution is

x_k	0	1	2	4	7
p_k	0.2	0.06	0.2	0.04	0.5

Hence, $P(X \geq 4) = P(X = 4) + P(X = 7) = 0.04 + 0.5 = 0.54\,.$ $\Diamond$

9.3.2 The Mean and Variance of a Discrete Random Variable

As we have seen in the above examples, the use of random variables can simplify the description of events and their probabilities. We will now see how they also enable us to perform arithmetic on outcomes. In particular, we can calculate the weighted averages of outcome values; this is the expected value, or mean. We can also measure the average of the squares of the distances from the mean to the outcome values; this is called the variance.

Definition 6. *The* ***expected value*** *(or* ***mean****) of a discrete random variable X with probability distribution p_k*

$$E(X) = \sum_{\text{all } k} x_k p_k\,.$$

The expected value $E(X)$ is often denoted by μ or μ_X.

Example 9. Toss a coin three times and let X count the number of heads tossed as in Example 4. The probability distribution of X is

x_k	0	1	2	3
p_k	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

so the expected value of X is

$$E(X) = \sum_k p_k\, x_k = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = \frac{3}{2}\,.$$

This agrees with our intuition: on average, half of the throws will be heads.

Example 10. Roll a die twice and let the random variable X be the sum of the rolls. Since a die roll i is as likely as the roll $7-i$, we see that X and $14-X$ have the same probability distribution, so the expected value of X is $E(X) = \frac{14}{2} = 7$. Let us check this using the definition of $E(X)$ and the probabilities $P(X=x)$ given in Example 10:

$$E(X) = 2 \times \frac{1}{36} + \cdots + 7 \times \frac{6}{36} + \cdots + 12 \times \frac{1}{36} = 7\,.$$

◊

Theorem 1. *Let X be a discrete random variable with probability distribution $p_k = P(X = x_k)$. Then for any real function $g(x)$, the expected value of $Y = g(X)$ is*

$$E(Y) = E(g(X)) = \sum_k g(x_k)p_k\,.$$

[**X**] *Proof.* Let $\{y_j\} = \{g(x_k)\}$ be the set of function values of Y, and note that

$$P(Y = y_j) = P(\{s \in S\,:\, g(X(s)) = y_j\}) = P\Big(\bigcup_{k\,:\,g(x_k)=y_j} \{s \in S\,:\, X(s) = x_k\}\Big) = \sum_{k\,:\,g(x_k)=y_j} P(X = x_k)\,.$$

By changing order of summation, we therefore see that

$$\begin{aligned} E(Y) = \sum_j y_j P(Y = y_j) = \sum_j y_j \sum_{k\,:\,g(x_k)=y_j} P(X = x_k) &= \sum_j \sum_{k\,:\,g(x_k)=y_j} g(x_k)p_k \\ &= \sum_k \sum_{j\,:\,y_j=g(x_k)} g(x_k)p_k \\ &= \sum_k g(x_k)p_k\,. \end{aligned}$$

The final equality is valid because the second sum only sums over a single element j. □

Example 11. Toss a coin three times and let X be the number of heads tossed, as in Examples 4 and 9. By Theorem 1, the expected value of X^2 is

$$E(X^2) = \sum_k p_k\, x_k^2 = 0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8} = 3\,.$$

◊

The expected value of a random variable X describes where the values of X are centred. We can also measure how widely the values of X spread, namely by the average distance (squared) between the values and the mean.

Definition 7. *The* ***variance*** *of a discrete random variable* X *is*

$$Var(X) = E\left((X - E(X))^2\right) .$$

The ***standard deviation*** *of* X *is* $SD(X) = \sqrt{\text{Var}(X)}$.

The standard deviation is often denoted by σ or σ_X, and the variance is often written as σ^2 or σ_X^2.

Theorem 2. $\text{Var}(X) = E(X^2) - (E(X))^2$.

This formula is often useful in hand calculations.

Proof. Write μ for $E(X)$. By the definition of the variance we have,

$$\begin{aligned}
\text{Var}(X) &= \sum_k (x_k - \mu)^2 p_k \\
&= \sum_k (x_k^2 - 2x_k\mu + \mu^2) p_k \\
&= \sum_k x_k^2 p_k - 2\mu \sum_k x_k p_k + \mu^2 \sum_k p_k = E(X^2) - 2\mu^2 + \mu^2 = E(X^2) - (E(X))^2 .
\end{aligned}$$ □

Example 12. Toss a coin three times and let X be the number of heads tossed. We saw in Examples 9 and 11 that $E(X) = \frac{3}{2}$ and $E(X^2) = 3$. Thus by Theorem 2, the variance of X is

$$\text{Var}(X) = E(X^2) - (E(X))^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4} .$$

◊

Example 13. Consider a random variable X with probability distribution given below:

x_k	0	1	2	4	7
p_k	0.2	0.06	0.2	0.04	0.5

The expected values of X and X^2 are

$$\begin{aligned}
E(X) &= \sum x_k\, p_k = 0 \times 0.2 + 1 \times 0.06 + 2 \times 0.2 + 4 \times 0.04 + 7 \times 0.5 = 4.12 \\
E(X^2) &= \sum x_k^2\, p_k = 0^2 \times 0.2 + 1^2 \times 0.06 + 2^2 \times 0.2 + 4^2 \times 0.04 + 7^2 \times 0.5 = 26.0
\end{aligned}$$

and the variance of X is

$$\text{Var}(X) = E(X^2) - (E(X))^2 = 26 - (4.12)^2 = 9.256 .$$

Thus, the average root mean square distance of the values of X to the mean $E(X)$ is roughly $\sqrt{9.256} \approx 3$. ◊

There is generally no easily-described relationship between $\text{Var}(Y)$ and $\text{Var}(X)$ when $Y = g(X)$. However, if $Y = aX + B$ is a linear function of X, then we have the following simple identities:

Theorem 3. *If a and b are constants, then*

$$\begin{aligned} E(aX+b) &= aE(X)+b \\ \mathrm{Var}(aX+b) &= a^2\,\mathrm{Var}(X) \\ \mathrm{SD}(aX+b) &= |a|\,\mathrm{SD}(X)\,. \end{aligned}$$

Proof. Write μ for $E(X)$, we have

$$E(aX+b) = \sum(ax_k+b)p_k = a\sum x_k p_k + b\sum p_k = aE(X)+b\,.$$

By Theorem 2 and the above identity,

$$\begin{aligned} \mathrm{Var}(aX+b) = E(aX+b-E(aX+b))^2 &= E(aX-aE(X))^2 \\ &= a^2E(X-E(X))^2 = a^2\,\mathrm{Var}(X)\,. \end{aligned}$$

The third statement follows by definition from the second. □

The variance $\sigma^2 = \mathrm{Var}(X)$ of a random variable X gives a measure of the average square distance from the expected value $E(X)$ to the values of X.

9.4 Special Distributions

Often, statistical models incorporate specific classes of probability distributions whose expected value and variance are known. In this section, we shall consider two such classes, namely the Binomial distributions and the Geometric distributions. These both involve **Bernoulli trials** and **Bernoulli processes**. A Bernoulli trial is an experiment with two outcomes, often "success" and "failure", or Y(es) and N(o), or $\{1,0\}$, where $P(Y)$ and $P(N)$ are denoted by p and $q = 1-p$, respectively. A Bernoulli process is an experiment composed of a sequence of identical and mutually independent Bernoulli trials. More particularly, the events A_i, denoting the success of the ith trial, are mutually independent. We have already seen examples of Bernoulli processes in previous sections, such as tossing a coin repeatedly and considering head-outcomes ($p = \frac{1}{2}$); rolling a die multiple times to obtain sixes ($p = \frac{1}{6}$); or asking each of several people whether it is their birthday ($p = \frac{1}{365}$).

Example 1. Tossing a coin three times is a Bernoulli process with $n = 3$ identical trials that each result in either H or T, with probabilities $p = q = \frac{1}{2}$. The trials are mutually independent since the coin tosses do not influence each other. Let us formally verify this claim. Let A_1, A_2, and A_3 to be the events that H is tossed on the 1st, 2nd, and 3rd toss, respectively. Then

$$P(A_1) = P(\{HTT, HTH, HHT, HHH\}) = \frac{4}{8} = \frac{1}{2}\,.$$

Similarly, $P(A_2) = P(A_3) = \frac{1}{2}$. Therefore,

$$P(A_1\cap A_2) = P(\{HHT, HHH\}) = \frac{2}{8} = \frac{1}{4} = P(A_1)P(A_2)\,,$$

and, similarly, $P(A_2\cap A_3) = P(A_2)P(A_3)$ and $P(A_1\cap A_3) = P(A_1)P(A_3)$. Finally,

$$P(A_1\cap A_2\cap A_3) = P(\{HHH\}) = \frac{1}{8} = P(A_1)P(A_2)P(A_3)\,.$$

We see that the events A_1, A_2, A_2 are indeed mutually independent. ◊

Throughout the remainder of this section, let p be a real number with $0 < p < 1$ and let $q = 1 - p$.

9.4.1 The Binomial Distribution

Recall that the expression $\binom{n}{k}$ denotes the number of ways to select k objects from n distinct objects, with order unimportant, and is given by

$$\frac{n!}{k!(n-k)!}.$$

Definition 1. *The* ***Binomial distribution*** $B(n,p)$ *for* $n \in \mathbb{N}$ *is the function*

$$B(n,p,k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{where } k = 0, 1, \ldots, n\,.$$

Note that $B(n,p,k)$ is a probability distribution. To see this, we can use the Binomial Theorem:

$$\sum_k B(n,p,k) = \sum_{k=0}^{n} B(n,p,k) = \sum_{k=0}^{n} \binom{n}{k} p^k\, q^{n-k} = (p+q)^n = 1^n = 1\,.$$

Since $B(n,p,k)$ is nonnegative, we conclude that $0 \leqslant B(n,p,k) \leqslant 1$ for all k.

Theorem 1. *If* X *is the random variable that counts the successes of some Bernoulli process with* n *trials having success probability* p*, then* X *has the binomial distribution* $B(n,p)$*.*

We write $X \sim B(n,p)$ to denote that X is a random variable with this distribution.

Proof. The variable X can assume values $k = 0, 1, \ldots, n$ so we must calculate $p_k = P(X = k)$ for these values. Suppose that the first k trials each results in Y(es) and the rest each results in N(o):

$$\underbrace{Y \cdots Y}_{k}\underbrace{N \cdots N}_{n-k}$$

The trials are independent, so this outcome has probability $p^k(1-p)^{n-k}$. In general, there are

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

ways for precisely n trials with k Y's to occur. Therefore,

$$p_k = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} = B(n,p,k)\,.$$

$\square$

Example 2. Toss a coin $n = 3$ times and let X be the random variable counting the number of resulting heads (H). The tosses are identical and mutually independent with probability $p = \frac{1}{2}$ of resulting in H. Thus, Theorem 1 implies that $X \sim B(3, \frac{1}{2})$. This tells us everything about the probabilities of X; for instance, $P(X = 2) = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{3-2} = \frac{3}{8}$. $\Diamond$

Probabilities such as $P(X \geqslant t), P(X > t)$, and $P(|X - E(X)| > t)$ are each referred to as a **tail probability**.

Example 3. Roll a die $n = 12$ times and let X be the number of resulting sixes. The rolls are identical and mutually independent with probability $p = \frac{1}{6}$ of resulting in a six, so by Theorem 1, $X \sim B(12, \frac{1}{6})$. Thus for instance, we can calculate the following tail probability:

$$\begin{aligned} P(X > 9) &= P(X = 10) + P(X = 11) + P(X = 12) \\ &= \binom{12}{10}\left(\frac{1}{6}\right)^{10}\left(\frac{5}{6}\right)^{2} + \binom{12}{11}\left(\frac{1}{6}\right)^{11}\left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)^{12} \approx 7.86 \times 10^{-7}\,. \end{aligned}$$

We see that the likelihood of rolling more than nine sixes is less than one in a million. ♢

Example 4. Ask $n = 40$ people whether today is their birthday and let X count the Yes-answers. Assume that these questions form identical and mutually independent trials with probability $p = \frac{1}{365}$ of resulting in a Yes-answer. By Theorem 1, $X \sim B(40, \frac{1}{365})$. The likelihood of today being the birthday of at least one of these people is

$$P(X \geqslant 1) = 1 - P(X = 0) = 1 - \left(\frac{364}{365}\right)^{40} \approx 10.4\%\,.$$

♢

Theorem 2. *If X is a random variable and $X \sim B(n, p)$, then*

- $E(X) = np$ *;*
- $\mathrm{Var}(X) = npq = np(1 - p)$*.*

Proof. First note that for $k \geq 1$,

$$k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = n\frac{(n-1)!}{(k-1)!(n-1-(k-1))!} = n\binom{n-1}{k-1}\,.$$

Hence,

$$\begin{aligned} E(X) = \sum_{k=0}^{n} kp_k = \sum_{k=0}^{n} k\binom{n}{k}p^k q^{n-k} &= \sum_{k=1}^{n} k\binom{n}{k}p^k q^{n-k} \\ &= n\sum_{k=1}^{n}\binom{n-1}{k-1}p^k q^{n-k} \\ &= n\sum_{j=0}^{n-1}\binom{n-1}{j}p^{j+1}\, q^{n-1-j} \\ &= np\sum_{j=0}^{n-1}\binom{n-1}{j}p^{j}\, q^{n-1-j} = np\sum_{j=0}^{n-1} B(n-1, p, j) = np\,. \end{aligned}$$

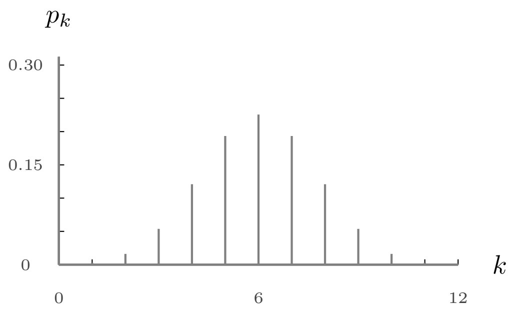
See Problem 36 for the second half of the proof. □

Example 5. Toss a coin $n = 3$ times and let X count the number of ensuing heads. By Example 1, $X \sim B(3, \frac{1}{2})$, so by Theorem 2, the expected number of resulting heads is $E(X) = np = \frac{3}{2}$ and the average square distance between the number of heads and $E(X)$ is $\mathrm{Var}(X) = npq = \frac{3}{4}$.

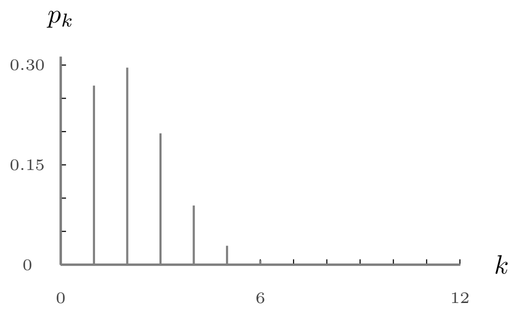
If we now toss the coin $n = 12$ times, then $X \sim B(12, \frac{1}{2})$, so the expected number of resulting heads is $E(X) = np = \frac{12}{2} = 6$, and $\mathrm{Var}(X) = npq = 3$. ♢

Example 6. If we roll a die $n = 12$ times, then one might intuitively expect to roll $12 \times \frac{1}{6} = 2$ sixes on average. This is also what we find by the following calculations. If X is the random variable counting the number of sixes rolled, then by Example 3, $X \sim B(12, \frac{1}{6})$. Thus by Theorem 2, the expected number of resulting sixes is $E(X) = np = 12 \times \frac{1}{6} = 2$, as we expected. $\diamondsuit$

The distributions $B(12, \frac{1}{2})$ and $B(12, \frac{1}{6})$ appearing in Example 2 and 6 are illustrated below. Note that the function values $B(12, \frac{1}{2}, k)$ are centered symmetrically around $E(X) = 6$ and spread out gradually to the extremities $k = 0, 12$. In contrast, the function values $B(12, \frac{1}{6}, k)$ are clustered asymmetrically about the expected value $E(X) = 2$ and taper rapidly off, so that $B(12, \frac{1}{6}, k)$ is nearly zero for $k \geqslant 7$. Thus, it is extremely highly unlikely that we would roll at least seven sixes when rolling a die twelve times.



$B(12, \frac{1}{2})$



$B(12, \frac{1}{6})$

9.4.2 Geometric Distribution

Definition 2. *The* ***Geometric distribution*** $G(p)$ *is the function*
$$G(p,k) = (1-p)^{k-1}p = q^{k-1}p \quad \text{where } k = 1, 2, \ldots .$$

Note that $G(p, k)$ is a probability distribution since $0 \leqslant G(p, k) \leqslant 1$ for all k and since

$$\sum_k G(p,k) = \sum_{k=1}^{\infty} G(p,k) = \sum_{k=1}^{n} q^{k-1}\, p = p \sum_{k=0}^{n} q^k = p\frac{1}{1-q} = p\frac{1}{p} = 1\,.$$

Theorem 3. *Consider an infinite Bernoulli process of trials each of which has success probability p. If the random variable X is the number of trials conducted until success occurs for the first time, then X has the geometric distribution $G(p)$.*

We write $X \sim G(p)$ to denote that X has this distribution. Note that it is theoretically possible for a success never to happen ($X = \infty$); however, this has zero probability. We therefore omit the all-failure outcome from the sample space so that X is a well-defined finite number.

Proof. The variable X can assume values $k = 1, 2, \ldots$, so we must find $p_k = P(X = k)$ for these values. The event $\{X = k\}$ consists of the outcome in which the first $k - 1$ trials each result in N(o) and the kth trial results in Y(es):

$$\underbrace{N \cdots N}_{k-1}\, Y$$

The trials are independent, so this outcome has probability

$$p_k = P(X = k) = (1 - p)^{k-1} p = G(p, k)\,. \qquad \square$$

Example 7. Toss a coin until H(ead) is tossed and let X count the number of these tosses. The tosses are identical and mutually independent with probability $p = \frac{1}{2}$ of resulting in H. Theorem 3 then implies that $X \sim G(\frac{1}{2})$. Thus, the likelihood of having to toss the coin seven times before tossing H is

$$P(X = 7) = \left(1 - \frac{1}{2}\right)^{7-1} \frac{1}{2} = \frac{1}{2^7} \approx 0.8\%\,. \qquad \Diamond$$

Tail probabilities are very easily expressed for geometrically distributed random variables:

Theorem 4. *If $X \sim G(p)$ and n is a positive integer, then $P(X > n) = (1 - p)^n = q^n$.*

Proof. $P(X > n) = \displaystyle\sum_{k=n+1}^{\infty} P(X = k) = \sum_{k=n+1}^{\infty} q^{k-1} p = q^n \sum_{k=1}^{\infty} q^{k-1} p = q^n \sum_{k=1}^{\infty} G(p, k) = q^n\,.$ $\square$

Theorem 4 gives us a simple expression for the cumulative distribution function $F(x)$ of X:

Corollary 5. *If $X \sim G(p)$, then the cumulative distribution function F is given by $F(x) = P(X \leqslant x) = 1 - (1 - p)^{\lfloor x \rfloor} = 1 - q^{\lfloor x \rfloor}$ for $x \in \mathbb{R}$.*

Note that $\lfloor x \rfloor$ denotes the largest integer less or equal to x.

Example 8. Roll a die until six is rolled, and let X count the number of these rolls. The rolls are identical and mutually independent with probability $p = \frac{1}{6}$ of resulting in six. Theorem 3 implies that $X \sim G(\frac{1}{6})$. By Theorem 4, the likelihood of rolling a six within at most four rolls is

$$F(4) = P(X \leqslant 4) = 1 - P(X > 4) = 1 - \left(1 - \frac{1}{6}\right)^4 \approx 52\%\,,$$

or close to half. Similarly, $F(6) = P(X \leqslant 6) \approx \frac{2}{3}$ and $P(X > 7) \approx 28\%$. $\Diamond$

Theorem 6. *If X is a random variable and $X \sim G(p)$, then*

- $E(X) = \frac{1}{p}$;
- $\mathrm{Var}(X) = \frac{1-p}{p^2}$.

Proof. First note that for $x \neq 0$, using Power Series results from Calculus,

$$\sum_{k=0}^{\infty} k x^{k-1} = \frac{d}{dx} \sum_{k=0}^{\infty} x^k = \frac{d}{dx} \frac{1}{1 - x} = \frac{1}{(1 - x)^2}\,.$$

Hence,

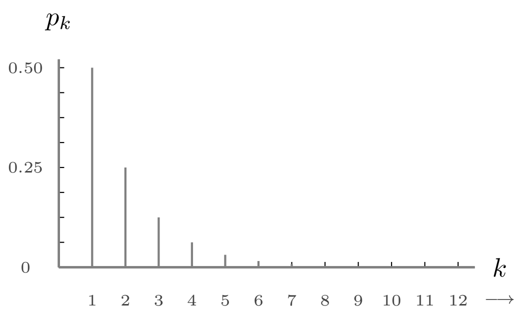
$$E(X) = \sum_{k=1}^{\infty} k q^{k-1} p = \frac{p}{(1 - q)^2} = \frac{p}{p^2} = \frac{1}{p}\,.$$

The second part of the proof is left as an exercise. $\square$

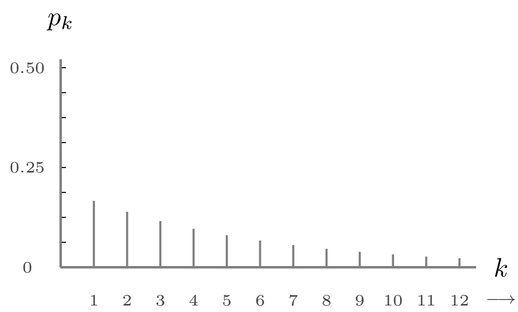
Example 9. Toss a coin until H(ead) is tossed and let X count the number of these tosses. As seen in Example 7, $X \sim G(\frac{1}{2})$. Thus, we must expect to on average have to toss the die $E(X) = \frac{1}{p} = (\frac{1}{2})^{-1} = 2$ times in order to toss a head; this is presumably what most would expect intuitively. Note that this is a relatively accurate estimate of the average since the average squared distance from $E(X) = 2$ to the infinitely many values of X is only $\text{Var}(X) = \frac{1-p}{p^2} = (1-\frac{1}{2})/(\frac{1}{2})^2 = 2$. Indeed by Theorem 4, the likelihood that we must toss the coin more than three times to get a head is $P(X > 3) = (\frac{1}{2})^3 = \frac{1}{8} = 12.5\%$, which is relatively small. ♢

Example 10. Roll a die until six is rolled, and let X count the number of these rolls as in Example 7. In that example, we saw that $X \sim G(\frac{1}{6})$, so we must expect on average to have to toss the die $E(X) = \frac{1}{p} = (\frac{1}{6})^{-1} = 6$ times in order to toss a head. As in the coin-tossing example above, this expected value is what one might guess intuitively. However, in contrast to that example, the present expected value $E(X) = 6$ is not a particularly precise estimate of an average tossing count. In particular, the average squared distance from $E(X) = 2$ to the infinitely many values of X is $\text{Var}(X) = \frac{1-p}{p^2} = (1-\frac{1}{6})/(\frac{1}{6})^2 = 30$. Indeed by Example 7, the likelihood of requiring at most four rolls in order to roll a six is (slightly) more than half, and the likelihood of requiring at least eight rolls is more than a quarter. ♢

The distributions $G(\frac{1}{2})$ and $G(\frac{1}{6})$ appearing in Example 8 and 10 are illustrated below. Note that the function values $G(\frac{1}{2}, k)$ are large to begin with but very quickly decrease, which is reflected by the small expected value and variance that both equal $E(X) = \text{Var}(X) = 2$. In contrast, the function values $G(\frac{1}{6}, k)$ are small to begin with but only gradually decrease. This is indicated by the expected value $E(X) = 6$ and the large variance $\text{Var}(X) = 30$. Thus, it is very likely that we would toss a head after only a few tosses of a coin; whereas it would not be possible to form such a good estimate about the number of die-rolls required to roll a six.



$G(\frac{1}{2})$



$G(\frac{1}{6})$

9.4.3 Sign Tests

Often, we have a sample of data consisting of independent observations of some quantity of interest, and it might be of interest to see whether the observed values differ systematically from some fixed and pre-determined value.

Example 11. Crop research shows that a new variety of corn yields, in bushels per acre, for 15 plots of land:

138.0	139.1	113.0	132.5	140.7
109.7	118.0	134.8	109.6	127.3
115.6	130.4	130.2	117.7	105.5

A variety of corn currently used yields 110 bushels per acre. We want to know whether the new variety improves on the existing one – that is, are the above values centred around a true value for yield of 110, or are they systematically different from the value 110?

To answer this question, one may use a "sign test" approach as follows:

1. Count the number of observations that are strictly greater than the target value ("+").
2. Count the total number of observations that are either strictly greater ("+") or strictly smaller ("–") than the target value.
3. Calculate the tail probability that measures how often one would expect to observe as many increases ("+") as were observed, if there were equal probability of "+" and "–".

Using this approach, we can now determine whether the new variety of corn has a higher yield than the current variety.

1. The yield was strictly greater ("+") than 110 bushels/acre in 12 plots.
2. The yield was either strictly greater ("+") than or strictly smaller ("-") than 110 bushels/acre in all 15 plots.
3. Assuming that probabilities of greater-than ("+") yield probabilities for each plot are identical and mutually independent, we can model these yields binomially. In particular, let X be the random variable that counts the yields exceeding the average yield of 110 bushels per acre; then $X \sim B(15, \frac{1}{2})$. The probability $p = \frac{1}{2}$ is set under the assumption that smaller-than ("-") yields are as likely as greater-than ("+") yields and that no equal-to yields occur; this assumes that the new crop has the same yield as the old one. The tail probability that 12 or more plots have a greater-than ("+") yield is then

$$P(X \geqslant 12) = \sum_{k=12}^{15} \binom{15}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{15-k} = 1.76\% \,.$$

Under the assumption that the average yield for the new variety has not improved, it is quite unlikely (less than 2%) that we would have observed 12 of the 15 yields above the old average yield. We therefore conclude that the new variety has improved yield.

In this course, we will say that if the tail probability is less than 5% then we will regard this as significant.

9.5 Continuous random variables

In the previous sections, we considered discrete random variables. These can assume countably many values assigned to the outcomes of an experiment. Although there may be infinitely many of these values, this is not sufficient to model many real-life experiments in which outcomes may be assigned any real values from some interval, such as the height or weight of individuals, or the half-life of a radioactive isotope. We will therefore now consider a type of non-discrete random variable, called a **continuous** variable X. In contrast to the discrete random variables, these cannot be defined by probabilities $P(X = x)$ of single values x, since these probabilities each equal 0. Instead, we will define continuous random variables in terms of the cumulative distribution function

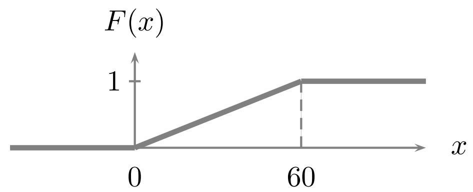
$$F(x) = F_X(x) = P(X \leqslant x) \qquad \text{for} \quad x \in \mathbb{R}\,.$$

Definition 1. *Random variable X is* ***continuous*** *if and only if $F_X(x)$ is continuous.*

Strictly speaking, $F_X(x)$ must actually be *piecewise differentiable*, which means that $F_X(x)$ is differentiable except for at most countably many points. However, the above definition is good enough for our present purposes.

Example 1. At a random point during the day, we take note of the time, ignoring the date and number of hours. This gives us a real number (of minutes) that we can represent by a random variable X with function values lying in the interval $[0, 60)$. If x is one of these values, then, assuming that any time is as likely as another, our intuition tells us that $F(x) = P(X \leqslant x) = P(0 \leqslant X \leqslant x)$ is the size of the interval $[0, x]$ compared the size of $[0, 60)$; measured in lengths, this is $\frac{x}{60}$.
We therefore find that $F(x)$ is the function

$$F(x) = P(X \leqslant x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ \frac{x}{60} & \text{if } 0 < x \leqslant 60 \\ 1 & \text{if } x > 60 \end{cases}$$



This is a continuous function, so X is a continuous random variable. $\Diamond$

For discrete random variables X, the cumulative distribution function $F(x)$ is a sum over probability distribution values $p_k = P(X = x_k)$. For continuous random variables, $F(x)$ is an integral over continuous function analogues of the discrete probability distributions. These analogues are given in the following definition.

Definition 2. *The* ***probability density function*** *$f(x)$ of a continuous random variable X is defined by*

$$f(x) = f_X(x) = \frac{d}{dx}F(x)\,,\ x \in \mathbb{R}$$

if $F(x)$ is differentiable, and $\lim\limits_{x\to a^-} \frac{d}{dx}F(x)$ if $F(x)$ is not differentiable at $x = a$.

Since $F(x)$ is non-decreasing and $\lim\limits_{x\to\infty} F(x) = 1$, the probability density function satisfies

$$f(x) \geqslant 0 \quad \text{for all } x \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x)dx = 1\,.$$

Theorem 1. $F(x) = \displaystyle\int_{-\infty}^{x} f(t)dt$.

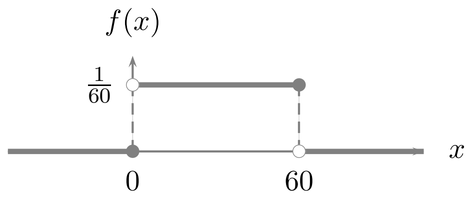
Proof. This follows from the Fundamental Theorem of Calculus since $\lim\limits_{x\to-\infty} F(x) = 0$. $\square$

Note that if $a \leqslant b$, then

$$P(a \leqslant X \leqslant b) = P(a < X \leqslant b) = F(b) - F(a) = \int_a^b f(x)dx\,.$$

Example 2. At a random point during the day, we take note of the time as in Example 1, and let X again be the continuous random variable X that tell us how far past the hour the time is. The density function $f(x)$ is calculated by differentiating the cumulative distribution function $F(x)$ found in Example 1:

$$f(x) = \frac{d}{dx}F(x) = \begin{cases} \frac{1}{60} & , 0 < x \leqslant 60 \\ 0 & , \text{otherwise}\,. \end{cases}$$



or, more compactly written,

$$f(x) = \frac{1}{60} \quad \text{for} \quad x \in (0, 60]\,.$$

Here, we differentiated $F(x)$ from the left at the points $x = 0, 60$.

The probability that we noted the time between a quarter past and half past the hour is

$$P(15 \leqslant X \leqslant 30) = \int_{15}^{30} f(x)dx = \int_{15}^{30} \frac{1}{60}\,dx = \frac{1}{4}\,,$$

as we would intuitively expect. $\Diamond$

9.5.1 The mean and variance of a continuous random variable

The mean of continuous random variable is obtained (using the notion of Riemann Sums) by replacing sums by integrals, and probability distributions by probability density functions:

Definition 3. *The* ***expected value*** *(or* ***mean****) of a continuous random variable X with probability density function $f(x)$ is defined to be*

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)dx\,.$$

Here, and in the following, we assume that all improper integrals converge.

The following theorem is the continuous analogue of Theorem 1 in Section 9.3.2.

Theorem 2. *If X is a continuous random variable with density function $f(x)$, and $g(x)$ is a real function, then the expected value of $Y = g(X)$ is*

$$E(Y) = E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx\,.$$

The variance of a continuous random variable is defined exactly as for discrete random variables:

Definition 4. *The* ***variance*** *of a continuous random variable X is*

$$\mathrm{Var}\,(X) = E((X - E(X))^2) = E(X^2) - (E(X))^2\,.$$

The standard deviation of X is $\sigma = \mathrm{SD}(X) = \sqrt{\mathrm{Var}\,(X)}$.

Note that by Theorem 2,

$$E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\, dx\,.$$

Example 3. At a random point during the day, we take note of the time as in Examples 1 and 2, and let X again be the continuous random variable giving the number of minutes past the hour. In Example 2, we found the density function $f(x)$ to be $f(x) = \frac{1}{60}$ for $x \in (0, 60]$, so

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{0} 0\, dx + \int_{0}^{60} \frac{x}{60}\, dx + \int_{0}^{\infty} 0\, dx = 0 + \frac{1}{60}\left[\frac{1}{2}x^2\right]_0^{60} = 30\,,$$

as we would expect. Similarly by Theorem 2,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{0}^{60} x^2 \frac{1}{60}\, dx = \frac{1}{60}\left[\frac{1}{3}x^3\right]_0^{60} = 1200\,,$$

so $\mathrm{Var}(X) = 1200 - 30^2 = 300$ and $\mathrm{SD}(X) = \sqrt{300} \approx 17.3$. $\Diamond$

The mean and variance have the same properties under linear scaling as in the discrete case.

Theorem 3. *If a and b are constants, then*

$$\begin{aligned} E(aX + b) &= aE(X) + b \\ \mathrm{Var}(aX + b) &= a^2\mathrm{Var}(X) \\ \mathrm{SD}(aX + b) &= |a|\,\mathrm{SD}(X)\,. \end{aligned}$$

An immediate consequence of these properties is

Theorem 4. *If $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$, and $Z = \dfrac{X - \mu}{\sigma}$, then $E(Z) = 0$ and $\mathrm{Var}(Z) = 1$.*

The random variable $Z = \frac{X-\mu}{\sigma}$ is referred to as the **standardised** random variable obtained from X. Note that this theorem holds for discrete and continuous random variables alike.

Proof. By Theorem 3,

$$E(Z) = E\left(\frac{X - \mu}{\sigma}\right) = \frac{1}{\sigma}E(X) - \frac{\mu}{\sigma} = \frac{\mu}{\sigma} - \frac{\mu}{\sigma} = 0\,;$$

$$\mathrm{Var}(Z) = \mathrm{Var}\left(\frac{X - \mu}{\sigma}\right) = \frac{1}{\sigma^2}\mathrm{Var}(X) = \frac{\sigma^2}{\sigma^2} = 1\,.$$

$\square$

9.6 Special Continuous Distributions

In this section, we consider two well-known continuous probability distributions, namely the normal and exponential distributions. It turns out that these are limiting cases of the discrete probability distributions that we have already seen, namely the binomial and geometric distributions, respectively, but we will not prove this here.

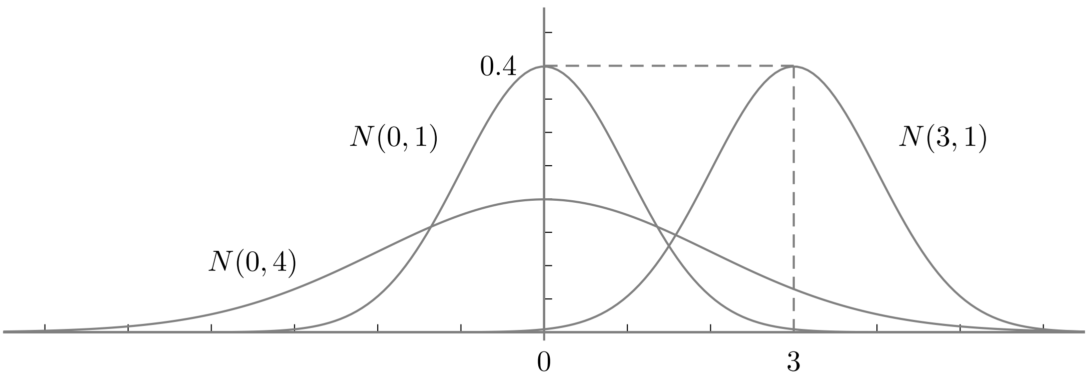
9.6.1 The Normal Distribution

A widely used probability distribution in statistics is the **normal** or **Gaussian** distribution.

Definition 1. *A continuous random variable X has **normal distribution** $N(\mu, \sigma^2)$ if it has probability density*

$$\phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \textit{where} \quad -\infty < x < \infty\,.$$

We write $X \sim N(\mu, \sigma^2)$ to denote that X has the normal distribution $N(\mu, \sigma^2)$. The normal probability density is bell-shaped, symmetric about the value $x = \mu$, and narrower for smaller σ. The probability densities for $N(0,1)$, $N(3,1)$, and $N(0,4)$ are illustrated below.



The distribution $N(0,1)$ is called the **standard normal distribution**.
The mean and variance of a random variable $X \sim N(\mu, \sigma^2)$ are simply μ and σ^2:

Theorem 1. *If X is a continuous random variable and $X \sim N(\mu, \sigma^2)$, then*

- $E(X) = \mu$
- $\mathrm{Var}(X) = \sigma^2$.

Proof. These are left as an exercise. □

Theorem 2. *If $X \sim N(\mu, \sigma^2)$, then $\dfrac{X-\mu}{\sigma} \sim N(0,1)$.*

Proof. This (almost) follows from Theorem 1 above and Theorem 4 from the previous section, but a proof is required that the new random variable is actually normal.

□

Note that when we standardise a normal random variable, the resulting distribution is also normal.

To find a probability involves evaluating the integral of the density function which is very hard, since this function does not have an elementary primitive. Thus, if X is normally distributed with mean μ and standard deviation σ,

$$P(X \leqslant x) = F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} \, dt \, .$$

To evaluate this integral, we convert to the standard normal distribution $Z \sim N(0,1)$, using the change of variable $Z = \dfrac{X-\mu}{\sigma}$ outlined above. This gives

$$P(Z \leqslant z) = F_Z(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \, dt \, .$$

The value of this integral for various z has been tabulated numerically and is available either via a calculator or the table given on the following page. This table gives the values of this integral for z in the range -3 to 3. For z less than -3, the value is essentially zero, while for z greater than 3, the value is essentially 1.

Example 1. Suppose X is normally distributed with mean $\mu = 20$ and standard deviation $\sigma = 3$. Find $P(X \leqslant 24)$.
Solution. We change to the standard normal distribution, so

$$P(X \leqslant 24) = P\left(\frac{X-\mu}{\sigma} \leqslant \frac{24-20}{3}\right) \approx P(Z \leqslant 1.33) \approx 0.9082 \, ,$$

from the tables. ♢

Example 2. Suppose that the weekly wages of secretaries are normally distributed with mean \$800 and standard deviation \$50. What is the probability of a secretary having a weekly wage higher than \$900 and how many secretaries out of a group of 2000 randomly selected secretaries would you expect to have a weekly wage greater than \$900?

Solution. Let X denote the weekly wage of a secretary. Then the mean of X is $\mu = 800$ and the standard deviation of X is $\sigma = 50$. Since $X \sim N(800, 50)$,

$$Z = \frac{X - 800}{50} \sim N(0,1) \qquad \text{and} \qquad X = 900 \quad \text{when} \quad Z = \frac{900-800}{50} = 2 \, .$$

Thus,

$$P(X > 900) = P(Z > 2) = 1 - P(Z \leqslant 2) = 1 - 0.9772 = 0.0228 \, .$$

Therefore, in a group of 2000 secretaries we would expect $.0228 \times 2000 = 45.6$ (i.e., about 46) of them to have a weekly wage in excess of \$900. ♢

In the above example, we used the fact that $P(Z \geqslant a) = 1 - P(Z \leqslant a)$; this is true because $P(Z = a) = 0$ since Z is continuous. Note also that for $a \leqslant b$, $P(a \leqslant Z \leqslant b) = P(Z \leqslant b) - P(Z \leqslant a)$.

Standard normal probabilities $P(Z \leqslant z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

The normal distribution is used, among other things, to approximate the binomial distribution $B(n,p)$ when n grows large. Before the advent of powerful computers, calculations involving many $B(n,p)$ terms was very laborious or even impossible, so it was easier to approximate such calculations by ones involving integration of the probability density of the normal distribution $N(\mu,\sigma^2)$ with the same mean ($\mu = np$) and variance ($\sigma^2 = np(1-p)$) as the binomial distribution. These integrals can be evaluated by transforming to the standard normal distribution as we did above and using tables.

These days, computers can calculate most probabilities involving binomial sums; however, the normal distribution is occasionally still used instead.

Example 3. Toss a coin $n = 10$ times and let X be the random variable counting the number of resulting heads. As we have seen previously, $X \sim B(10, \frac{1}{2})$. The probability of tossing at most four heads is thus

$$P(X \le 4) = F_X(4) = \sum_{k=0}^{4} \binom{10}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{10-k} = \frac{193}{512} \approx 37.7\%\,.$$

We could also have approximated this probability by calculating $F_Y(4)$ of a continuous random variable Y with $Y \sim N(\mu,\sigma^2)$ where $\mu = E(X) = 5$ and $\sigma^2 = \mathrm{Var}(X) = \frac{5}{2}$. To get an even better approximation, we can calculate $F_Y(4.5)$ since $P(X \leqslant 4) = P(X < 5)$ and 4.5 lies in the middle of the interval from 4 to 5:

$$P(X \le 4) = F_X(4) \approx P(Y \leqslant 4.5) \approx P(Z \leqslant -0.32) \approx 37.45\%\,,$$

which differs from the true value by only 0.25%.

Now toss a coin $n = 50$ times and let X again be the random variable counting the number of resulting heads. Then $X \sim B(50, \frac{1}{2})$. The probability of tossing at most 23 heads is thus

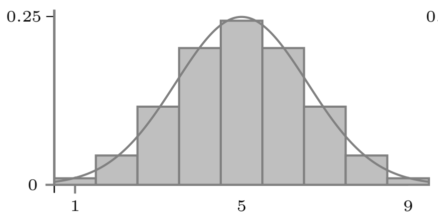
$$P(X \le 23) = F_X(23) = \sum_{k=0}^{23} \binom{50}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{50-k} \approx 33.59\%\,.$$

The above calculation is obtained at once when using Maple but would be very time-consuming to calculate by hand (however, can you find a simple short-cut to perform this particular calculation with far less effort?). We could also have approximated this probability by calculating $F_Y(23.5)$ of a continuous random variable Y with $Y \sim N(\mu,\sigma^2)$ where $\mu = E(X) = 25$ and $\sigma^2 = \mathrm{Var}(X) = \frac{25}{2}$:

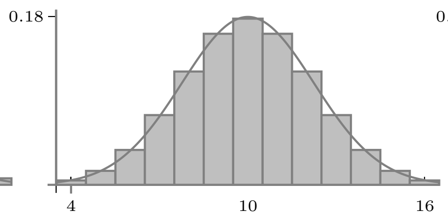
$$P(X \le 23) \approx P(Y \leqslant 23.5) \approx P(Z \leqslant -0.42) \approx 33.72\%\,,$$

which only differs from the true value by 0.13%. $\Diamond$

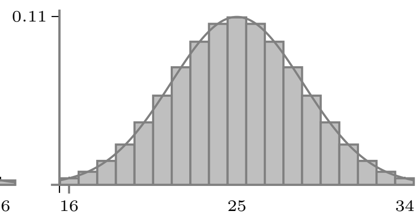
The three figures below show how binomial distributions may be approximated by normal distributions. The first and third figures illustrate $B(10, \frac{1}{2})$ and $N(5, \frac{5}{2})$, and $B(50, \frac{1}{2})$ and $N(25, \frac{25}{2})$, respectively, that appeared in the above example. Note that the first and second coordinates of the three figures are differently scaled and truncated.



$B(10, \frac{1}{2})$ and $N(5, \frac{5}{2})$



$B(20, \frac{1}{2})$ and $N(10, 5)$



$B(50, \frac{1}{2})$ and $N(25, \frac{25}{2})$

More generally, normal distributions are used to model experiments involving large numbers of identical and independent trials that have several possible outcomes. Typical examples include the final-grade distributions of the high-school graduates in a particular country in a given year; or distributions of height, or of weight, or of IQ-test results, of the citizens of a country, and so on.

Example 4. A six-sided die, which is believed to be biased, is rolled 720 times and shows a '6' 100 times.

a. Write down the formula for the tail probability of getting 100 or less 6's in 720 rolls of a fair die.

b. Using the normal approximation to the binomial distribution, calculate the probability in (i), giving your answer to 3 decimal places.

Solution: a. Let X be the number of sixes in 720 rolls of a fair die.
Then X is binomial with $n = 720, p = \frac{1}{6}$.
Hence

$$P(X \leqslant 100) = \sum_{k=0}^{100} \binom{720}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{720-k}.$$

b. Now X can be approximated by the continuous random variable $Y \sim N(\mu, \sigma^2)$, with

$$\mu = E(X) = np = 720 \times \frac{1}{6} = 120,$$

$$\sigma^2 = Var(X) = np(1-p) = 720 \times \frac{1}{6} \times \frac{5}{6} = 100.$$

Then

$$P(X \leqslant 100) \approx P(Y \leqslant 100.5) = P\left(Z \leqslant \frac{100.5 - 120}{10}\right) = P(Z \leqslant -1.95) = 0.026,$$

where $Z = \frac{Y-\mu}{\sigma} \sim N(0, 1)$.

Since $0.026 = 2.6\% < 5\%$, the tail probability is significantly low and so there is good evidence that the die is biased.

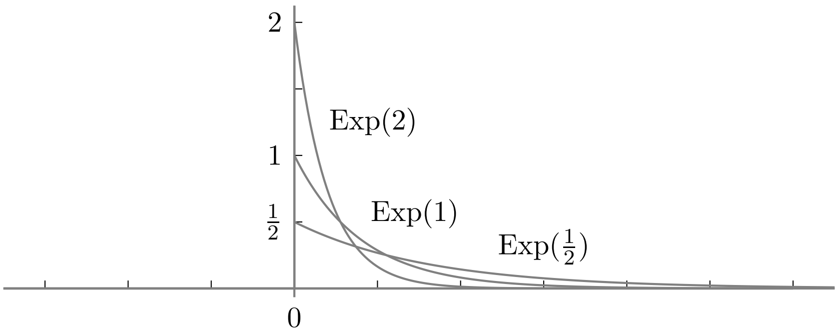
9.6.2 [X] The Exponential Distribution

In the previous subsection, we saw that the binomial distribution can be approximated by the normal distribution. Similarly, the geometric distribution has an analogous continuous probability distribution, namely the **exponential distribution**.

Definition 2. *A continuous random variable T has* ***exponential distribution*** $\mathrm{Exp}(\lambda)$ *if it has probability density*

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geqslant 0 \\ 0 & \text{if } t < 0\,. \end{cases}$$

We write $T \sim \mathrm{Exp}(\lambda)$ to denote that T has the exponential distribution $\mathrm{Exp}(\lambda)$. The probability densities for $\mathrm{Exp}(\frac{1}{2})$, $\mathrm{Exp}(1)$, and $\mathrm{Exp}(2)$ are illustrated below.



The mean and variance associated with the exponential distribution are given as follows:

Theorem 3. *If T is a continuous random variable and $T \sim \mathrm{Exp}(\lambda)$, then*

- $E(T) = \dfrac{1}{\lambda}$
- $\mathrm{Var}(T) = \dfrac{1}{\lambda^2}$.

Proof. Using integration by parts,

$$\begin{aligned} E(T) = \int_{-\infty}^{\infty} t f(t)dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{\infty} t\lambda e^{-\lambda t}dt \\ = 0 + \left[-te^{-\lambda t}\right]_0^\infty + \int_0^\infty e^{-\lambda t}dt = 0 - 0 + \frac{1}{\lambda}\left[-e^{-\lambda t}\right]_0^\infty = \frac{1}{\lambda}(0-(-e^0)) = \frac{1}{\lambda}\,. \end{aligned}$$

The variance $\mathrm{Var}(T)$ is calculated similarly. □

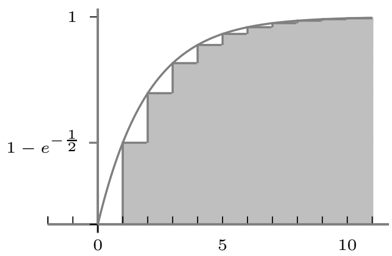
The cumulative distribution function $F_T(t)$ of an exponentially distributed random variable $T \sim \mathrm{Exp}(\lambda)$ is easily expressed:

$$F_T(t) = P(T \leqslant t) = \int_{-\infty}^{t} f(x)\,dx = \begin{cases} 1 - e^{-\lambda t} & , t \geqslant 0 \\ 0 & , t < 0\,. \end{cases}$$

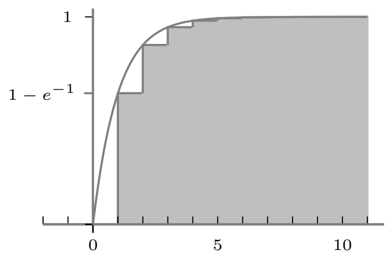
If we set $p = 1 - e^{-\lambda}$ and let $n \in \mathbb{Z}$ be an integer, then

$$F_T(n) = \begin{cases} 1 - (1-p)^n & , n \geqslant 0 \\ 0 & , n < 0\,. \end{cases}$$

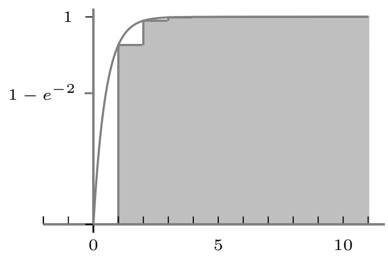
By Corollary 5 of Theorem 9.4.2, this is the value $F_X(n)$ of the cumulative distribution function of a discrete random variable X that is geometrically distributed with parameter $p = 1 - e^{-\lambda}$. In other words, the exponential distribution $\mathrm{Exp}(\lambda)$ is approximated by the geometric distribution $G(p)$:



$\mathrm{Exp}(\frac{1}{2})$ and $G\big(1 - e^{-\frac{1}{2}}\big)$



$\mathrm{Exp}(1)$ and $G\big(1 - e^{-1}\big)$



$\mathrm{Exp}(2)$ and $G\big(1 - e^{-2}\big)$

Conversely, the geometric distribution $G(p)$ is interpolated by the exponential distribution $\mathrm{Exp}(\lambda)$ where $\lambda = \ln \frac{1}{1-p}$.

Example 5. An insurance company has collected data on one of its insurance policies, and it turns out that, on average, $p = 0.0502$ of these policies are claimed each year. For one of these policies, find the

(a) probability that the first claim occurs within the first six years;

(b) probability that the first claim occurs within the first 6.5 years;

(c) probability that the first claim occurs during the first half of the sixth year;

(d) expected number of years until the first claim occurs.

Solution. We assume that claims occur independently of each other and with equal probability $p = 0.0502$. If claims only occurred at the end of each year, then we could model the behaviour of the first occurring claim by a discrete random variable $X \sim G(p)$ that counted the number of years until that first claim occurred. However, claims might occur at any positive time from the policy's inception, so the discrete model does not suffice; instead, let T be continuous random variable that gives the time until the first claim. Then $T \sim \mathrm{Exp}(\lambda)$ where $\lambda = \ln \frac{1}{1-0.0502} \approx 0.0515$.

(a) The probability that the first claim occurs within the first six years is

$$P(X \leqslant 6) = F_X(6) = 1 - (1-p)^6 = 1 - (1 - 0.0502)^6 \approx 26.58\% .$$

We could also have calculated this probability as follows:

$$P(T \leqslant 6) = F_T(6) = 1 - e^{-\lambda \times 6} = 1 - e^{-0.0515 \times 6} \approx 26.58\% .$$

(b) The probability that the first claim occurs within the first 6.5 years is

$$P(T \leqslant 6.5) = F_T(6.5) = 1 - e^{-\lambda \times 6.5} = 1 - e^{-0.0515 \times 6.5} \approx 28.45\% .$$

(c) The probability that the first claim occurs during the first half of the sixth year is

$$\begin{aligned} P(5 < T \leqslant 5.5) = P(T \leqslant 5.5) - P(T \leqslant 5) &= F_T(5.5) - F_T(5) \\ &= (1 - e^{-\lambda \times 5.5}) - (1 - e^{-\lambda \times 5}) \\ &= e^{-0.0515 \times 5} - e^{-0.0515 \times 5.5} \\ &\approx 1.965\% \, . \end{aligned}$$

A second way to calculate this probability is as follows:

$$P(5 < T \leqslant 5.5) = \int_5^{5.5} \lambda e^{-\lambda t} dt = \left[-e^{-\lambda t}\right]_5^{5.5} = e^{-0.0515 \times 5} - e^{-0.0515 \times 5.5} \approx 1.965\% \, .$$

(d) The expected number of years until the first claim occurs is $E(T) = \frac{1}{\lambda} = \frac{1}{0.0515} \approx 19.42$.
This is roughly approximated by $E(X) = \frac{1}{p} = \frac{1}{0.0502} \approx 19.53$.
These values are what we might intuitively estimate: "just under $\frac{1}{0.05} = 20$". ◊

9.6.3 Useful Web Applets to Illustrate Probability Reasoning

Web applets for long-run frequency illustrations:

http://www.shodor.org/interactivate/activities/AdjustableSpinner/

http://socr.stat.ucla.edu/Applets.dir/DiceApplet.html

An applet to illustrate conditional probability and independence:

http://www.stat.berkeley.edu/users/stark/Java/Html/Venn.htm

An applet for Bayes' Rule:

http://www.stat.berkeley.edu/users/stark/Java/Html/Venn.htm

An applet for the Birthday Problem:

http://www.mste.uiuc.edu/reese/birthday/

9.7 Probability and MAPLE

Most of the problems in this chapter reduce to sums of various probability expressions. Maple's ability to do summation symbolically is especially useful here, as is its ability to sum infinite series. Even for finite sums, Maple can simplify the calculations significantly. For example, we can implement the binomial distribution function $B(n, p, k) = \binom{n}{k} p^k (1-p)^{n-k}$ as follows

```
B := proc(n,p,k) binomial(n,k)*p^k*(1-p)^(n-k) end proc;
```

Maple allows us to check that the function values of $B(n, p, k)$ sum to 1 (even though we have not specified what n and p are!):

```
simplify(sum(B(n,p,k), k = 0..n));
```

Indeed, Maple allows us to perform many exact calculations on discrete distributions. For example, it is somewhat time-consuming and awkward to calculate by hand and calculator the probability that 1000 coin tosses result in between 510 and 530 heads. However, this calculation is very quick and easy to conduct with Maple:

```
n := 1000;
p := 1/2;
sum(B(n,p,k), k = 510..530);
evalf(%);
```

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