THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 2 2016

MATH1241
HIGHER MATHEMATICS 1B

(1) TIME ALLOWED – TWO (2) HOURS

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS and A STANDARD NORMAL TABLE ARE APPENDED ON THE LAST PAGES

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 1

1. i) Suppose that

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 + 2x_3 \geq 0 \right\}.$$

a) Prove that S is closed under addition.

b) Either prove that S is a subspace of $\mathbb{R}^3$ or explain why it is not a subspace of $\mathbb{R}^3$.

ii) Let $A = \begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 2 & 5 & 1 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$.

a) Is $\mathbf{b}$ in $\ker(A)$? Give reasons.

b) Find a basis for $\text{im}(A)$.

iii) The height of male students in a university is normally distributed with mean 172 cm and standard deviation 5 cm. Calculate the probability that a randomly chosen male student from the university is taller than 180 cm.

iv) Evaluate each of the following integrals.

a) $I_1 = \displaystyle\int \cos^3 x \ \sin^4 x \, dx$

b) $I_2 = \displaystyle\int \frac{3x^2 + 11x + 12}{(x+1)(x+3)^2} \, dx$

v) a) Find the general solution of the equation,

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0.$$

b) What form of solution should you try in order to find a particular solution of the equation,

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 8\sin 2x.$$

[Note that you are NOT asked to find the particular solution.]

vi) Determine the open interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2^n}{(n^2 - n + 1)3^n}(x-3)^n.$$

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 2

2. i) a) Determine whether the sequence

$$\sqrt{n+\sqrt{n}}-\sqrt{n}$$

converges or diverges as $n \to \infty$. If it converges, find its limit.

b) Does the series

$$\sum_{n=2}^{\infty} \sqrt{n+\sqrt{n}}-\sqrt{n}$$

converge? Give a reasons for your answer.

ii) By using an appropriate test, determine whether each of the following series converges or diverges.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

b) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k(\ln k)^2}$

iii) The area A of a rectangle with length x and width y is $A = xy$. Use the total differential approximation for A as a function of x and y to estimate the percentage increase in A when x increases by 5% and y decreases by 6%.

iv) A function $T : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$T(\mathbf{x}) = \begin{pmatrix} x+y \\ 2x \\ -y \end{pmatrix}, \quad \text{for } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

a) Prove that T is linear.

b) Determine the nullity of T. Give reasons.

v) Juan plants three pumpkin seeds. The random variable X denotes the number of seeds that have germinated after three weeks. The distribution of X is given as follows:

k	0	1	2	3
$P(X=k)$	0.01	0.11	c	0.35

a) Find the value of c.

b) Calculate the mean of X.

c) Calculate the variance of X.

vi) Consider the following polynomials in the vector space of polynomials of degree 3 or less, $\mathbb{P}_3$.

$$
\begin{aligned}
p_1(x) &= 1 + 2x + 3x^2 + x^3 \\
p_2(x) &= 1 + x + 3x^2 + x^3 \\
p_3(x) &= 1 + 2x + 4x^2 + x^3 \\
p_4(x) &= 1 - x + 3x^2 + 2x^3 \\
p_5(x) &= 2 + x + x^2 - 4x^3
\end{aligned}
$$

Which of the following statements are true and which are false? Explain your answer.

a) The set $\{p_1, p_2, p_3\}$ is a basis for $\mathbb{P}_3$.

b) The set $\{p_1, p_2, p_3, p_4, p_5\}$ is a linearly independent set in $\mathbb{P}_3$.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 3

3. i) Let

$$A = \begin{pmatrix} 9 & -1 \\ 4 & 5 \end{pmatrix} .$$

a) Find all eigenvalues and eigenvectors of A.

b) Explain carefully why A is not diagonalisable.

c) Find the matrix of the linear mapping

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{where} \quad T(\mathbf{x}) = A\mathbf{x}$$

with respect to the ordered basis $B = \{ \mathbf{v}_1, \mathbf{v}_2 \}$ in both domain and codomain, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} .$$

ii) Let V and W be vector spaces, let $T : V \to W$ be a linear transformation, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in V.

a) Prove, giving detailed reasons, that if $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_m)$ are linearly independent, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent.

b) State, giving reasons, whether the following statement is true or false: if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent, then $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_m)$ are linearly independent.

iii) The field $\mathbb{F} = \mathrm{GF}(4)$ has elements $\{ 0, 1, \alpha, \beta \}$, with addition and multiplication defined by the following tables.

$+$	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	1	0

$\times$	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	β	1
β	0	β	1	α

For the vectors

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} , \quad \mathbf{b}_2 = \begin{pmatrix} \beta \\ 1 \\ 1 \end{pmatrix} , \quad \mathbf{b}_3 = \begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} ,$$

a) show that $\{ \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \}$ is a basis for $\mathbb{F}^3$;

b) explain without calculation why $\{ \mathbf{b}_1, \mathbf{b}_1 + \mathbf{b}_2, \mathbf{b}_2 + \mathbf{b}_3, \mathbf{b}_3 \}$ is a spanning set but not a basis for $\mathbb{F}^3$;

c) find the coordinate vector of $\mathbf{v}$ with respect to the ordered basis $\{ \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \}$ of $\mathbb{F}^3$.

iv) Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \dfrac{c}{x(1+x^2)} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1, \end{cases}$$

where

$$c = \frac{2}{\ln 2} \ .$$

You are **not** required to prove that f is a probability density function.

a) Find the mean of X.

b) Show that X does not have a finite variance.

c) The median of X is defined to be the real number m such that $P(X \leq m) = \frac{1}{2}$. Show that

$$m > \tan\left(\frac{\pi + \ln 2}{4}\right) \ .$$

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 4

4. i) Find a reduction formula for $\displaystyle\int_{\pi/4}^{\pi/2} \cot^n x\,dx$ and use it to show

$$\int_{\pi/4}^{\pi/2} \cot^4 x = \frac{\pi}{4} - \frac{2}{3}\,.$$

(Note that $1 + \cot^2 x = \operatorname{cosec}^2 x$ and $\dfrac{d}{dx}\cot x = -\operatorname{cosec}^2 x$.)

ii) Using the substitution $v(x) = y^{3/2}$, solve the differential equation

$$xy^{1/2}\frac{dy}{dx} + 2y^{3/2} = e^{x^3}\,.$$

iii) A tank initially contains 100 litres of pure water. Two pipes carry salt water into this tank, one from reservoir A that has a salt concentration of 4 grams per litre, and another from reservoir B that has a salt concentration of 1 gram per litre. The combined flow rate from both reservoirs A and B into the tank is always a constant 1 litre per minute, although a valve can control the proportion coming from each reservoir. An outflow ensures that the tank always holds exactly 100 litres, and the tank is continuously stirred and well mixed. Denote the flow rate coming from reservoir A by p. Let t denote the time measured in minutes, and let $x(t)$ denote the concentration of salt in the tank at time t, measured in grams per litre.

a) By considering the rates of inflow and outflow of salt, we know that $x(t)$ must obey an initial value problem of the form,

$$\frac{dx}{dt} = a + bx, \qquad x(0) = c$$

for some constants a, b, and c. Find the values of these constants.

b) Solve the initial value problem in part (a).

c) What value should p take in order for the outflowing concentration of salt to be 2 grams per litre as $t \to \infty$.

Please see over . . .

iv) An engineer needs to find the approximate value of

$$I = \int_0^1 \tan^{-1}\bigl(\sin x\bigr)\,dx$$

with some estimate of the error. Asking Maple for the exact answer provides something extremely complicated, so she decides to use a series approximation. Maple gives her the result shown below.

```
> series(arctan(sin(x)),x=0,11);
```

$$x - \frac{1}{2}x^3 + \frac{3}{8}x^5 - \frac{83}{240}x^7 + \frac{8375}{24192}x^9 + O\left(x^{11}\right)$$

a) Use the Maple output above to find the first 4 nonzero terms in the Taylor series for $\displaystyle\int_0^1 \tan^{-1}\bigl(\sin x\bigr)\,dx$.

b) Use the first three terms of your answer to (a) to find a rational approximation to I.

c) The series Maple has provided seems to be alternating. Assuming this is the case, give an estimate for the error in your answer to part (b).

v) Write down the integral that gives the area of the surface obtained by rotating the graph of

$$y = \frac{x^3}{2} - \frac{1}{6x}$$

from $x = 1$ to $x = 2$ around the x-axis. [You do NOT need to evaluate this integral.]

vi) Let p_n for $n \geq 2$ denote the number of integers less than or equal to n that are prime. (So for example $p_5 = 3$ as 2, 3 and 5 are prime.)

Do either of the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{p_n}$ and $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{p_n}$ converge? Explain your answers.

Standard normal probabilities $P(Z \leq z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$
$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$
$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$
$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$
$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$
$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$
$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$
$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$
$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$
$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$
$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$
$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$
$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$
$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$
$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$
$$\begin{aligned}\int \frac{dx}{a^2-x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, && |x| < a\\ &= \tfrac{1}{a}\coth^{-1}\tfrac{x}{a} + C, && |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, && x^2 \neq a^2\end{aligned}$$
$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$