THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 2 2017

MATH1231
MATHEMATICS 1B

(1) TIME ALLOWED – TWO (2) HOURS

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS and A STANDARD NORMAL TABLE ARE APPENDED ON THE LAST PAGES

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 1

1. i) Evaluate each of the following integrals.

a) $I_1 = \displaystyle\int \frac{1}{\sqrt{x^2+4x+5}}\,dx$

b) $I_2 = \displaystyle\int \cos^3 x \ \sin^2 x\,dx$

c) $I_3 = \displaystyle\int \frac{x+7}{(x+1)^2(x-2)}\,dx$

ii) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 2x.$$

iii) Let $\mathbf{v}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix}1\\2\\1\end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix}-2\\-1\\1\end{pmatrix}$.

a) Write $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

b) Does there exist a linear map $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ such that

$$T(\mathbf{v}_1) = \begin{pmatrix}1\\6\\3\end{pmatrix}, \quad T(\mathbf{v}_2) = \begin{pmatrix}-2\\16\\2\end{pmatrix}, \quad \text{and} \quad T(\mathbf{v}_3) = \begin{pmatrix}-6\\-3\\-8\end{pmatrix}?$$

c) What is the relationship between $\text{span}(\mathbf{v}_1, \mathbf{v}_2)$ and $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$?

iv) Let $S = \{\mathbf{x} \in \mathbb{R}^3 : x_1^2 - x_2^2 = x_3^2\}$.

a) Prove that S is closed under scalar multiplication.

b) Show that S is not a subspace of $\mathbb{R}^3$.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 2

2. i) The volume V of a tumour can be modelled by the differential equation

$$\frac{dV}{dt} = \alpha\left(1 - \frac{V}{K}\right)V, \qquad (*)$$

where t is time, V is the volume of the tumour at time t and α and K are positive constants. If the initial value $V(0) = V_0$ is imposed, solving (*) as a separable equation gives the non constant solution

$$V(t) = \frac{K}{1 + \left(\frac{K}{V_0} - 1\right)e^{-\alpha t}}.$$

a) Find all constant solutions to equation (*).

b) Find the behaviour of $V(t)$ as $t \to \infty$.

c) Give an interpretation of the constants α and K.

d) Another model for tumour growth is given by the differential equation

$$\frac{dV}{dt} = -\alpha \ln\left(\frac{V}{K}\right)V. \qquad (**)$$

Suppose the same constants α and K are used in the two models. Without solving (**), explain which model predicts faster tumour growth for tumours when V is much smaller than K?

ii) The specific gravity z of a solid heavier than water is given by

$$z = \frac{x}{x-y},$$

where x and y are its weight in air and water respectively. The weights x and y are observed to be 21.3g and 10.2g and each observation is made with an uncertainty whose absolute value is at most 0.1g.

a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

b) Use the total differential approximation for z to estimate the maximum uncertainty in the calculated value of z (to 3 decimal places).

You may find the following Maple session useful.

```
> z := x/(x-y):
> zx := diff(z,x):
> zy := diff(z,y):
> subs(x=21.3, y=10.2, zx);
```

$$-0.08278548821$$

```
> subs(x=21.3, y=10.2, zy);
```

$$0.1728755783$$

iii) The probability density function f of a continuous random variable X is given by

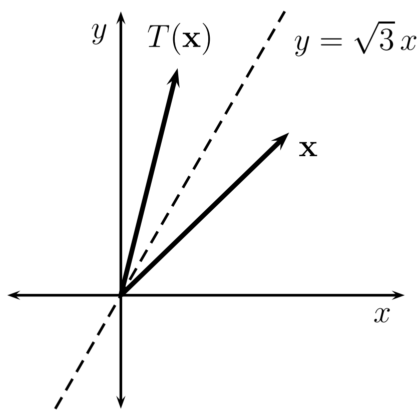
$$f(x) = \begin{cases} kx^2 & \text{for } 0 \le x \le 3 \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant.

a) Find the value of k.

b) Evaluate $E(X)$ and $\mathrm{Var}(X)$.

iv) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which reflects a vector in the line $y = \sqrt{3}\,x$ as show in the diagram.



a) Show that

$$T(\mathbf{e}_1) = \begin{pmatrix} -\dfrac{1}{2} \\ \dfrac{\sqrt{3}}{2} \end{pmatrix}, \quad T(\mathbf{e}_2) = \begin{pmatrix} \dfrac{\sqrt{3}}{2} \\ \dfrac{1}{2} \end{pmatrix}, \quad \text{where } \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

b) Find the matrix A such that

$$A\mathbf{x} = T(\mathbf{x}), \quad \text{for all } \mathbf{x} \in \mathbb{R}^2.$$

c) Find $T\left(\begin{pmatrix} 4 \\ 5 \end{pmatrix}\right)$.

v) Read the following Maple output and use it to answer the questions below.

```
> with(LinearAlgebra):
> A := <<1,2,7,4,3>|<-1,6,2,8,1>|<2,-4,5,-4,2>|
  <2,3,-1,5,7>|<-1,14,11,20,5>>;
```

$$A := \begin{bmatrix} 1 & -1 & 2 & 2 & -1 \\ 2 & 6 & -4 & 3 & 14 \\ 7 & 2 & 5 & -1 & 11 \\ 4 & 8 & -4 & 5 & 20 \\ 3 & 1 & 2 & 7 & 5 \end{bmatrix}$$

```
> GaussianElimination(A);
```

$$\begin{bmatrix} 1 & -1 & 2 & 2 & -1 \\ 0 & 8 & -8 & -1 & 16 \\ 0 & 0 & 0 & -\frac{111}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> b := <-2,35,16,49,15>:
```

$$b := \begin{bmatrix} -2 \\ 35 \\ 16 \\ 49 \\ 15 \end{bmatrix}$$

```
> LinearSolve(A,b);
```

$$\begin{bmatrix} 1 - \_t_3 - \_t_5 \\ 5 + \_t_3 - 2\,\_t_5 \\ \_t_3 \\ 1 \\ \_t_5 \end{bmatrix}$$

Let A be the matrix A defined in the Maple code above.

a) Give a basis for the kernel of the matrix A.

b) Find one vector in $\mathbf{x} \in \mathbb{R}^5$ such that

$$A\mathbf{x} = \begin{pmatrix} -2 \\ 35 \\ 16 \\ 49 \\ 15 \end{pmatrix}.$$

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 3

3. i) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \subset \mathbb{R}^5$ be a linearly independent set and $V = \text{span}(S)$. Which of the following statements are true and which ones are false. Justify your answer briefly in each case.

a) The set S is a basis for V.

b) The set S spans $\mathbb{R}^5$.

c) There is a set of 5 vectors in V which is linearly independent.

d) The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

ii) Let $A = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$.

a) Show that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ are eigenvectors of A and determine their corresponding eigenvalues.

b) Find an invertible matrix P and a diagonal maxtrix D such that $P^{-1}AP = D$.

c) Hence find $A^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for any positive integer k.

d) Show that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for A^{-1} and find the corresponding eigenvalue.

iii) Mo's mobile phone company produces phones with an average lifetime of 4.2 years. Suppose that the lifetime is an approximately normally distributed random variable with standard deviation 1.3 years. Mo wishes to offer a warranty on his phones, but figures that it is only profitable to do so if he replaces fewer than 2% of the stock he sells, during the warranty period. What warranty length should he offer?

iv) A fair 6 sided die is rolled repeatedly until a 1 is rolled. Let X be the random variable that counts the number of rolls required for this to occur.

a) Find $P(X = 3)$.

b) Find $P(X \geq 2)$.

c) Find the probability that a 5 or 6 is rolled before the 1.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 4

4. i) Consider the differential equation

$$(\tan y - 2x) + (x\sec^2 y - 2y)\frac{dy}{dx} = 0.$$

a) Show that it is exact and find the general solution.

b) Find the solution with initial value $y(1) = \pi/4$.

ii) Let $f(x) = \sin^{-1}(x)$, for $|x| < 1$. The following Maple session may assist you with this question.

```
> f := x -> arcsin(x):
> taylor(f(x), x=0, 7);
```

$$x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + O(x^7)$$

a) Using the Maple output or otherwise, write down the values of $f'(0)$, $f''(0)$ and $f'''(0)$.

b) Using the first two terms of the Maple output above, find a rational number that approximates $\sin^{-1}(0.1)$.

iii) Consider the sequence $\{a_n\}$ given by

$$a_n = \frac{n - \cos^2 n}{n^3}.$$

a) Does the series

$$\sum_{n=1}^{\infty} a_n$$

converge? Give reasons for your answer.

b) Determine the open interval of convergence of the power series

$$\sum_{n=1}^{\infty} a_n(x+2)^n.$$

iv) For a real sequence $\{a_n\}$, the Monotone Convergence Theorem states if the sequence is bounded above and increasing then it is convergent.

Consider the sequence $\{a_n\}$ defined recursively by

$$a_{n+1} = \frac{a_n^2 + \pi^2}{2\pi}, \quad a_0 = 1.$$

a) Show by induction that $a_n \leq \pi$.

b) The AM-GM inequality for $x \geq 0$ and $y \geq 0$ states that $(x+y)/2 \geq \sqrt{xy}$. Using this, or otherwise, show that the sequence is increasing and hence converges.

c) By taking the limit of the left and right hand sides of the defining equation, find the limit of the sequence.

Standard normal probabilities $P(Z \leq z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$
$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$
$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$
$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$
$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$
$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$
$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$
$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$
$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$
$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$
$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$
$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$
$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$
$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$
$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$
$$\begin{aligned}\int \frac{dx}{a^2-x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \qquad |x| < a \\ &= \tfrac{1}{a}\coth^{-1}\tfrac{x}{a} + C, \qquad |x| > a > 0 \\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad x^2 \neq a^2\end{aligned}$$
$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$