THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 2 2015

MATH1241
HIGHER MATHEMATICS 1B

(1) TIME ALLOWED – TWO (2) HOURS

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED “UNSW APPROVED” STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS AND THE NORMAL DISTRIBUTION ARE ATTACHED ON THE LAST PAGE

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 1

1. i) Given that $z = x^2y^4 + 3y + 2$, find $\dfrac{\partial^2 z}{\partial y^2}$.

ii) Evaluate $\displaystyle\int \sin^2(x)\cos^3(x)\,dx$.

iii) Suppose that $z = a^2 + b^3 + c^4$ where

$$\begin{aligned} a &= u - v + w, \\ b &= u + v - w, \\ c &= uvw. \end{aligned}$$

Use the chain rule to find $\dfrac{\partial z}{\partial u}$ at the point $(u, v, w) = (1, 0, 1)$.

iv) A surface S in $\mathbb{R}^3$ has equation $z = 3xy^2$. The point $P(1, 2, 12)$ is a point on the surface S.

a) Find a normal vector to S at P.

b) Find a Cartesian equation for the tangent plane to S at P.

v) Prove that

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 - 2x_2 + 4x_3 = 0 \right\}$$

is a subspace of $\mathbb{R}^3$.

vi) Consider the vectors in $\mathbb{R}^3$,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}.$$

Prove that $\mathbf{b}$ is in $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

vii) Using the following Maple output, answer the questions below. Give reasons.

```
> with(LinearAlgebra):
> A := <<1,1,3,1,2>|<1,2,3,1,3>|<1,-3,3,1,-2>|<2,4,3,-3,5>|
          <2,-8,12,12,-4>>;
```

$$A := \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 1 & 2 & -3 & 4 & -8 \\ 3 & 3 & 3 & 3 & 12 \\ 1 & 1 & 1 & -3 & 12 \\ 2 & 3 & -2 & 5 & -4 \end{bmatrix}$$

```
> LinearSolve(A,<0,0,0,0,0>);
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$$\begin{bmatrix} -5\,\_t_3 - 12\,\_t_5 \\ 4\,\_t_3 + 6\,\_t_5 \\ \_t_3 \\ 2\,\_t_5 \\ \_t_5 \end{bmatrix}$$

```
> B := GaussianElimination(<A|IdentityMatrix(5)>);
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$$B := \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 2 & -10 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 6 & -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & -\frac{5}{3} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\frac{1}{3} & 0 & 1 \end{bmatrix}$$

a) Find a basis for $\ker(A)$.

b) Write down the value of $\operatorname{rank}(A)$.

c) Find a basis for $\mathbb{R}^5$ containing as many columns of A as possible.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 2

2. i) Find

$$\int \frac{2x^2 + 6x + 12}{(x^2 + 4)(x - 6)}\, dx.$$

ii) Solve the initial value problem

$$\frac{dN}{dt} + \frac{1}{1+t} N = \frac{2t}{1+t}, \qquad N(0) = 2.$$

iii) The intensity I of light at a depth of x metres below the surface of a lake is modelled by the differential equation

$$\frac{dI}{dx} = -2xI^2, \qquad \text{for } x \geq 0.$$

a) By referring to the differential equation explain why the light intensity is a decreasing function of depth.

b) Assuming an intensity of $I = 1$ at the surface, solve the differential equation to find a formula for I in terms of x.

c) No plant life is possible when the light intensity falls to 1% of its value at the surface. Find the depth at which this occurs.

iv) Denver attempted an on-line examination in which he answered 50 multiple choice questions. In each question, there were 5 choices with only one correct answer. Denver chose all the answers randomly. Let X be the random variable counting the number of questions he guessed correctly.

a) Find $E(X)$ and $\text{Var}(X)$.

b) Use the normal approximation to the binomial distribution to find the probability that Denver correctly guessed 12 or more questions.

v) Let V be a vector space and $T : V \to V$ be a linear map. Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a subset of V and $R = \{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$.

a) State what it means to say that "R is a linearly independent set".

b) Prove that if R is a linearly independent set then S is a linearly independent set.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 3

3. i) Urn 1 contains 3 red balls and 4 blue balls. Urn 2 contains 4 red balls and 2 blue balls. A ball is drawn at random from Urn 1 and placed in Urn 2, and then a ball is drawn at random from the 7 balls now in Urn 2.

a) What is the probability that the ball drawn from Urn 2 is blue?

b) Given that the ball drawn from Urn 2 is red, what is the probability that the ball transferred was blue?

c) Let A be the event that the ball drawn from Urn 1 is blue and let B be the event that the ball drawn from Urn 2 is blue. Are these events statistically independent? Prove your answer.

ii) Consider the mapping $T : \mathbb{P}_2 \to \mathbb{P}_3$ defined by

$$(Tp)(x) = (x^2 + 1)p'(x) - 2xp(x).$$

a) Prove that T is a linear transformation.

b) Find the nullity of T.

c) Find the matrix of T with respect of the standard bases of $\mathbb{P}_2$ and $\mathbb{P}_3$.

iii) The matrix

$$A = \begin{pmatrix} 5 & 0 & -4 \\ 4 & 9 & 4 \\ -8 & -12 & -5 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = -\lambda^3 + 9\lambda^2 + 9\lambda - 81$.

a) Given that two of the eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 3$, find the remaining eigenvalue λ_3.

b) Find an eigenvector for λ_3.

c) Given that $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ are eigenvectors of A, write down a 3×3 matrix M and a diagonal matrix D such that $D = M^{-1}AM$.

iv) A linear transformation $P : V \to V$ is said to be **idempotent** if $P(P(\mathbf{v})) = P(\mathbf{v})$ for all $\mathbf{v} \in V$.

a) Show that the only possible eigenvalues for an idempotent linear transformation are 0 and 1.

b) Show that if P is idempotent and P is neither the zero nor the identity transformation on V, then both 0 and 1 are eigenvalues.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 4

4. i) Use the substitition $y(x) = 1/v(x)$ to transform

$$\frac{dy}{dx} + 2xy - e^x y^2 = 0$$

into a first order differential equation for $v(x)$. Identify the type of differential equation obtained for $v(x)$. [Do **not** solve the equation.]

ii) The charge, $Q(t)$, in a certain circuit satisfies the differential equation

$$Q'' + Q' - 6Q = 0 \quad \text{with} \quad Q(0) = 3.$$

For what value, if any, of $Q'(0)$, will the charge tend to 0 as t tends to infinity?

iii) Use appropriate tests to determine whether each of the following series converges or diverges.

a) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n \ln n}{n^4 - 2n + 1}$

iv) Given that the power series

$$\sum_{n=1}^{\infty} a_n x^n$$

converges at $x = 4$ and *diverges* at $x = 5$, for which of the points $x = -3$, $x = -4$, $x = -5$ and $x = -6$ can you say for sure whether the series converges or diverges? Justify your answer(s).

v) Consider the curve given parametrically by

$$x(t) = 2t + 3, \quad y(t) = t^{3/2}, \quad 0 \le t \le 1$$

Find the area of the surface obtained when this curve is revolved around the y-axis.

vi) Let $\{a_n\}$ be a sequence of positive terms such that $\sqrt[n]{a_n} \to r < 1$ as $n \to \infty$.

a) Explain why this implies that there is a constant $R < 1$ and an integer N such that $a_n < R^n$ for all $n > N$.

b) Hence or otherwise prove that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Standard normal probabilities $P(Z \leq z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2-x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \qquad |x| < a\\ &= \tfrac{1}{a}\coth^{-1}\tfrac{x}{a} + C, \qquad |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$