Revision questions

Important Facts from MATH1131

In the next couple of chapters, we shall frequently refer to some subsets of $\mathbb{R}^n$, such as lines and planes, etc. As well as the two operations addition and multiplication by a scalar, we shall also refer some other operations such as dot and cross products, etc. However, for ease of reading some definitions are restated below. When necessary, you should refer to the 1131/41 Algebra Notes.

1. . Suppose that $n \geqslant 1$. A parametric vector equation of a line in $\mathbb{R}^n$ through a point A and parallel to a non-zero vector $\mathbf{v}$ is given by

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}, \quad \lambda \in \mathbb{R},$$

   where $\mathbf{a}$ is the position vector of A (with respect to the origin O) and $\mathbf{x}$ is the position vector of a variable point on the line.

2. Suppose that $n \geqslant 2$. A parametric vector equation of a plane in $\mathbb{R}^n$ through a point A and parallel to two non-zero non-parallel vectors $\mathbf{u}, \mathbf{v}$ is given by

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{u} + \mu\mathbf{v}, \quad \lambda,\, \mu \in \mathbb{R},$$

   where $\mathbf{a}$ is the position vector of A and $\mathbf{x}$ is the position vector of a variable point on the plane.

3. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are two vectors in $\mathbb{R}^n$, $n \geqslant 1$. The dot product is defined by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \cdots + a_n b_n.$$

   The length of a vector $\mathbf{a}$ is defined to be $\sqrt{\mathbf{a} \cdot \mathbf{a}}$. (This definition is equivalent to the one given in the 1131/41 Algebra Notes.)

   The vectors are said to be orthogonal if $\mathbf{a} \cdot \mathbf{b} = 0$. A set of vectors is said to be an orthonormal set if the lengths of each vectors is 1 and the vectors are mutually orthogonal.

   The projection of a vector $\mathbf{a}$ on the vector $\mathbf{b}$ is

$$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\,\mathbf{b}.$$

4. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are two vectors in $\mathbb{R}^3$. An equivalent definition of cross product in determinant form is given by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \qquad \text{where} \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix},\ \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

5. The following is a point-normal form of a plane in $\mathbb{R}^3$ which passes through a point A and has a normal vector $\mathbf{n}$.

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0,$$

where $\mathbf{a}$ is the position vector of A and $\mathbf{x}$ is the position vector of a variable point on the plane.

As revision, you should have a look at the following problems:

1.[**R**] Let A, B, P be points in $\mathbb{R}^3$ with position vectors

$$\mathbf{a} = \begin{pmatrix} 7 \\ -2 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix} \text{ and } \mathbf{p} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Let Q be the point on AB such that $AQ = \frac{2}{3}AB$.

i) Find $\mathbf{q}$, the position vector of Q.

ii) Find the parametric vector equation of the line that passes through P and Q.

2. [**R**] Consider the three points $A(1,1,1)$, $B(2,0,3)$ and $C(3,-1,1)$.

i) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

ii) Find a parametric vector form of the line through A and B.

iii) Find a parametric vector form of the plane through A, B and C.

iv) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

v) Find a point-normal form of the plane through A, B and C.

vi) Find a Cartesian equation of the plane through A, B and C.

3. [**R**] Given the vectors $\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, find $|\mathbf{p}|$, $|\mathbf{q}|$, $\mathbf{p} \cdot \mathbf{q}$, then the cosine of the angle between $\mathbf{p}$ and $\mathbf{q}$.

4. [**R**] Consider the equation

$$\det \begin{pmatrix} x-1 & y-2 & z+1 \\ 1 & 0 & 2 \\ 2 & -1 & 0 \end{pmatrix} = 0$$

i) Show that the equation represents the Cartesian equation of a plane.

ii) Write the equation in point-normal form.

5. [**R**] For the points $P(1,2,0)$, $Q(1,3,-1)$ and $R(2,1,1)$, find $\overrightarrow{PQ} \times \overrightarrow{PR}$ and the area of the triangle with vertices P, Q and R.

6. [**R**] Suppose that A is the point $(2,-1,3)$ and Π is the plane

$$\mathbf{x} = \lambda\begin{pmatrix}1\\0\\1\end{pmatrix} + \mu\begin{pmatrix}1\\-2\\2\end{pmatrix} \qquad \text{for } \lambda,\mu\in\mathbb{R}.$$

for $\lambda,\mu\in\mathbb{R}$.

i) Find a vector $\mathbf{n}$ which is normal to Π.

ii) Find the projection of $\overrightarrow{OA}$ on the direction $\mathbf{n}$.

iii) Hence find the shortest distance of A from Π.

Answers 1i. $\begin{pmatrix}3\\-4\\1\end{pmatrix}$. ii. $\mathbf{x} = \begin{pmatrix}1\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\-1\end{pmatrix}$ for $\lambda\in\mathbb{R}$.

2i. $\begin{pmatrix}1\\-1\\2\end{pmatrix}$, $\begin{pmatrix}2\\-2\\0\end{pmatrix}$. ii. $\mathbf{x} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$ for $\lambda\in\mathbb{R}$.

iii. $\mathbf{x} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix} + \mu\begin{pmatrix}2\\-2\\0\end{pmatrix}$ for $\lambda,\mu\in\mathbb{R}$. iv. $\begin{pmatrix}4\\4\\0\end{pmatrix}$.

v. $\begin{pmatrix}4\\4\\0\end{pmatrix}\cdot\left(\mathbf{x} - \begin{pmatrix}1\\1\\1\end{pmatrix}\right) = 0$.

vi. $x+y-2=0$.

3. $\sqrt{3}$, $\sqrt{6}$, 4, $\dfrac{2\sqrt{2}}{3}$.

4i. $2x+4y-z=11$. ii. $\begin{pmatrix}2\\4\\-1\end{pmatrix}\cdot\left(\begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}0\\0\\-11\end{pmatrix}\right) = 0$

5. $\begin{pmatrix}0\\-1\\-1\end{pmatrix}$, $\dfrac{\sqrt{2}}{2}$.

6i. $\begin{pmatrix}2\\-1\\-2\end{pmatrix}$, ii. $-\dfrac{1}{9}\begin{pmatrix}2\\-1\\-2\end{pmatrix}$, iii. $\dfrac{1}{3}$.

Problems for Chapter 6

Questions marked with [**R**] are routine, [**H**] harder, [**X**] extra for MATH1241, [**M**] Maple. You should try to solve some of the questions in Sections 6.4 to 6.7 with Maple.

Problems 6.1 : Definitions and examples of vector spaces

1. [**R**] Show that the set
$$S = \left\{ \mathbf{x} \in \mathbb{R}^3 : x_1 \leqslant 0, \quad x_2 \geqslant 0 \right\},$$
with the usual rules for addition and multiplication by a scalar in $\mathbb{R}^3$ is not a vector space by showing that at least one of the vector space axioms is not satisfied. Give a geometric interpretation of this result.

2. [**R**] Show that the system S with the usual rules for addition and multiplication by a scalar in $\mathbb{R}^3$, and where
$$S = \left\{ \mathbf{x} \in \mathbb{R}^3 : 2x_1 + 3x_2^3 - 4x_3^2 = 0 \right\},$$
is not a vector space by showing that at least one of the vector space axioms is not satisfied.

3. [**R**] Let $S = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \ : \ (a-b)c = 0 \right\}$.
   a) Write down two non-zero elements of S.
   b) Show that S is not closed under vector addition.

4. [**H**] The set $\mathbb{C}^n$ is a vector space over $\mathbb{C}$ (see Example 2 of Section 6.1). Check that axioms 1, 2, 6, 9 are satisfied by this system.

5. [**X**] Let $M_{mn}(\mathbb{C})$ be the set of all $m \times n$ matrices with complex entries with addition the usual rule for addition of complex matrices, and multiplication by a scalar the usual rule for multiplication of a complex matrix by a complex scalar. Prove that the vector space $M_{mn}(\mathbb{C})$ satisfies axioms 1, 3, 6 and 10.

6. [**H**] Prove that the system $(\mathbb{C}^n, +, *, \mathbb{R})$ with "natural" definitions of $+$ and $*$ is a vector space, whereas the system $(\mathbb{R}^n, +, *, \mathbb{C})$ with "natural" definitions of $+$ and $*$ is not a vector space.

7. [**X**] Consider the system $(\mathbb{R}^2, +', *', \mathbb{R})$ in which the usual operations of "addition" and "multiplication by a scalar" are replaced by the new definitions:
$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} +' \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 - 3b_2 \end{pmatrix}$$
$$\lambda *' \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 4\lambda a_1 \\ \lambda a_2 \end{pmatrix}.$$
Give a list of the vector space axioms satisfied by this system, and a list of any which are not satisfied. Is this system a vector space?

Problems 6.2 : Vector arithmetic

8. **[H]** Prove that the following properties are true for every vector space.

   a) $2\mathbf{v} = \mathbf{v} + \mathbf{v}$.

   b) $n\mathbf{v} = \mathbf{v} + \cdots + \mathbf{v}$, where there are n terms on the right.

9. **[H]** Prove parts 2, 4 and 5 of Proposition 2 of Section 6.2.

Problems 6.3 : Subspaces

10. **[R]** Suppose $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Show that the line segment defined by

$$S = \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \alpha\mathbf{v}, \qquad \text{for} \quad 0 \leqslant \alpha \leqslant 10 \right\}$$

is not a subspace of $\mathbb{R}^3$.

11. **[R]** Show that the set

$$S = \left\{ \mathbf{x} \in \mathbb{R}^3 : 2x_1 + 3x_2 - 4x_3 = 6 \right\},$$

is not a subspace of $\mathbb{R}^3$. Give a geometric interpretation of this result.

12. **[R]** Let S the set

$$S = \left\{ \mathbf{x} \in \mathbb{R}^3 : 2x_1 + 3x_2 - 4x_3 = 0 \right\},$$

   a) Find three distinct members of S.

   b) Show that S is a subspace of $\mathbb{R}^3$.

   c) Give a geometric interpretation of this latter result.

13. **[R]** Show that

$$T = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : -1 \leqslant x + y + z \leqslant 1 \right\}$$

is not a vector subspace of $\mathbb{R}^3$.

14. **[R]** Show that the set

$$S = \left\{ \mathbf{x} \in \mathbb{R}^3 : 2x_1 + 3x_2 - 4x_3 = 4x_1 - 2x_2 + 3x_3 = 0 \right\}$$

is a subspace of $\mathbb{R}^3$.

15. **[R]** Show that the set

$$S = \left\{ \mathbf{x} \in \mathbb{R}^3 : 2x_1 + 3x_2 - 4x_3 = 0 \quad \text{or} \quad 4x_1 - 2x_2 + 3x_3 = 0 \right\}$$

is not a subspace of $\mathbb{R}^3$.

16. **[R]** Show that the set

$$S = \left\{ \mathbf{b} \in \mathbb{R}^2 : \mathbf{b} = A\mathbf{x} \qquad \text{for some} \quad \mathbf{x} \in \mathbb{R}^3 \right\},$$

where

$$A = \begin{pmatrix} 2 & -3 & 1 \\ 4 & 5 & -3 \end{pmatrix},$$

is a subspace of $\mathbb{R}^2$. Explain why each column of the matrix belongs to the set S.

17. **[R]** For each of the following subsets of $\mathbb{R}^3$, either prove that the given subset is a subspace of $\mathbb{R}^3$ or explain why it is not a subspace.

a) $S = \left\{ \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 + 2x_3 \geqslant 0 \right\}.$

b) $T = \left\{ \sum_{i=1}^{4} \lambda_i \mathbf{v}_i : \lambda_i \in \mathbb{R}, 1 \leqslant i \leqslant 4 \right\}$, where $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ are given fixed vectors in $\mathbb{R}^3$.

c) $U = \left\{ A\mathbf{x} : \mathbf{x} \in \mathbb{R}^5 \right\}$, where A is a fixed 3×5 matrix.

18. **[H]** Suppose that $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Show, by the Subspace Theorem that the set

$$S = \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \lambda\mathbf{u} + \mu\mathbf{v}, \qquad \text{for} \quad \lambda,\ \mu \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^3$.

19. **[H]** Prove that the set S in Example 6 on page 15 is closed under multiplication by a scalar.

20. **[H]** Let $\mathbf{a}$ and $\mathbf{b}$ be two fixed non-zero vectors in $\mathbb{R}^5$. Show that

$$W = \left\{ \mathbf{x} \in \mathbb{R}^5 \ : \ \mathbf{x} \cdot \mathbf{a} = \mathbf{x} \cdot \mathbf{b} = 0 \right\}$$

is a subspace of $\mathbb{R}^5$.

If $\mathbf{a} = \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, describe W.

21. **[R]** Show that the set $S = \{p \in \mathbb{P}_2 : p(0) = 1\}$ is NOT a subspace of $\mathbb{P}_2$.

22. **[R]** Show that the set

$$S = \left\{ p \in \mathbb{P}_3 \ : \ p''(x) = 0 \quad \text{for all} \quad x \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{P}_3$.

23. **[H]** Is the set

$$S = \{p \in \mathbb{P}_3 : p'(x) + x + 1 = 0 \quad \text{for all} \quad x \in \mathbb{R}\}$$

a subspace of $\mathbb{P}_3$?

24. **[H]** Consider the set

$$S = \{p \in \mathbb{P}_3 \ : \ (x+1)p'(x) - 3p(x) = 0 \quad \text{for all } x \in \mathbb{R}\}.$$

a) Show that S **is** a subspace of $\mathbb{P}_3$, (the set of all real polynomials of degree $\leqslant 3$).

b) Find a polynomial in S where not all the coefficients are zero.

25. **[H]** By constructing a counterexample, show that the union of two subspaces is not, in general, a subspace.

26. **[H]** Let W_1 and W_2 be two subspaces of a vector space V over the field $\mathbb{F}$. Prove that the intersection of W_1 and W_2 (i.e., the set $W_1 \cap W_2$) is a subspace of V.

27. **[X]** Let V be a vector space over the field $\mathbb{F}$.

a) Let $\{W_k : 1 \leqslant k \leqslant m\}$ be m subspaces of V, and let W be the intersection of these m subspaces. Prove that W is a subspace of V.

b) Let S be any set of vectors in V, and let W be the intersection of all subspaces of V which contain S (that is, $\mathbf{x} \in W$ if and only if $\mathbf{x}$ lies in every subspace which contains S). Prove that W is the set of finite linear combinations of vectors from S.

Problems 6.4 : Linear combinations and spans

28. **[R]** Let $\mathbf{a} = \begin{pmatrix} 10 \\ 11 \\ 4 \end{pmatrix}$, $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$.

a) Is $\mathbf{a} \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$? If so, express $\mathbf{a}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$.

b) Do the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$? If not, find condition(s) on $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ for $\mathbf{b}$ to belong to $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and interpret your answer geometrically.

29. **[R]** Repeat the preceding question using

$$\mathbf{a} = \begin{pmatrix} 9 \\ -2 \\ -4 \end{pmatrix}, \ \mathbf{v}_1 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix} \text{ and } \mathbf{v}_3 = \begin{pmatrix} 15 \\ -4 \\ -6 \end{pmatrix}.$$

30. **[R]** Repeat using $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ -9 \\ 1 \end{pmatrix}$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 4 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$.

[Replace $\mathbb{R}^3$ by $\mathbb{R}^4$, of course.]

31. **[R]** Is the vector $\mathbf{b} = \begin{pmatrix} -2 \\ -6 \\ -4 \end{pmatrix} \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}?$$

32. **[R]** Is the set of vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$ a spanning set for $\mathbb{R}^3$?

33. **[R]** Does $\mathbf{v}$ belong to the column space of A, $\operatorname{col}(A)$, where

$$\mathbf{v} = \begin{pmatrix} 2 \\ -5 \\ 19 \\ -13 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 2 & -3 & -5 \\ 1 & 2 & 7 \end{pmatrix}?$$

If so, write $\mathbf{v}$ as a linear combination of the columns of A.

34. **[R]** Is the polynomial $p(x) = 1 + x + x^2$ in $\operatorname{span}\big(1 - x + 2x^2,\ -1 + x^2,\ -2 - x + 5x^2\big)$?

35. **[R]** Is $S = \{1 + x,\ 1 - x^2,\ x + 2x^2\}$ a spanning set for $\mathbb{P}_2$?

36. **[X]** Prove Proposition 1 of Section 6.4.

37. **[X]** Use the vector space axioms to prove that we do not need to use brackets when writing down the linear combination

$$\sum_{k=1}^{n} \lambda_k \mathbf{v}_k = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n.$$

That is, prove that the result of the operations in independent of the order in which the additions are performed.

Problems 6.5 : Linear independence

38. **[R]** Is the set of vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$ linearly independent? Are these three vectors coplanar?

39. **[R]** Is the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}$, a linearly independent set? Are these three vectors coplanar?

40. **[R]** Can a set of linearly independent vectors contain a zero vector? Explain your answer.

41. **[R]** Given the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$, do the following.

    a) Show that S is a linearly dependent set.

    b) Show that at least one of the vectors in S can be written as a linear combination of the others, and find the corresponding linear combination.

    c) Find all possible ways of writing the vector $\begin{pmatrix} 8 \\ 9 \\ 5 \end{pmatrix}$ as a linear combination of the vectors in the set.

    d) Find a linearly independent subset of S with the same span as S, and then show that $\begin{pmatrix} 8 \\ 9 \\ 5 \end{pmatrix}$ can be written as a unique linear combination of this subset.

    e) Give a geometric interpretation of span (S).

42. **[R]** Repeat the previous question for the set of four vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

43. **[R]** Is $\{1 - x + 2x^2,\ -1 + x^2,\ -2 - x + 5x^2\}$ a linearly independent subset of $\mathbb{P}_2$? If the set is not linearly independent express one of the polynomials as a linear combination of the others.

44. **[H]** (For discussion). Let the set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a linearly independent subset of a vector space V. You are standing at the origin of V and set off in the direction of $\mathbf{v}_1$. After a certain length of time, you turn and head in direction $\mathbf{v}_2$ — then in direction $\mathbf{v}_3$ and so on. Is it possible for you to return to the origin? (Note: You may walk any distance that you like along any of the directions, but you are not allowed to retrace your steps).

45. **[H]** What would happen in the previous question if the set S were a linearly dependent set?

46. **[H]** Assume that $m \leqslant n$ and that $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a set of mutually orthogonal, non-zero vectors in $\mathbb{R}^n$, that is, the dot products satisfy (see Section 5.3.1)

$$\begin{aligned} \mathbf{v}_i \cdot \mathbf{v}_j = 0 &\quad \text{for} \quad i \neq j; \quad 1 \leqslant i, j \leqslant m \\ \mathbf{v}_i \cdot \mathbf{v}_i \neq 0 &\quad \text{for} \quad 1 \leqslant i \leqslant m. \end{aligned}$$

Show that S is a linearly independent set.

Problems 6.6 : Basis and dimension

47. **[R]** Is the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}$, a basis for $\mathbb{R}^3$?

48. **[R]** Find a basis for, and the dimension of, $W = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, where
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}.$$

49. **[R]** Without doing any calculation, explain why $\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 4 \\ 0 \end{pmatrix} \right\}$ is not a spanning set for $\mathbb{R}^4$.

    Similarly, without doing any calculation, explain why $\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right\}$ is a linearly dependent set.

50. **[R]** Which of the following statements are true and which are false? Explain your answer.

    a) Any set of 6 vectors in $\mathbb{R}^5$ is linearly dependent.
    b) Some sets of 6 vectors in $\mathbb{R}^5$ are linearly independent.
    c) Any set of 6 vectors in $\mathbb{R}^5$ is a spanning set for $\mathbb{R}^5$.
    d) Some sets of 6 vectors in $\mathbb{R}^5$ span $\mathbb{R}^5$.
    e) Same as in (a) – (d), with 6 replaced by 4.
    f) Any set of 5 vectors in $\mathbb{R}^5$ is a basis for $\mathbb{R}^5$.
    g) Some sets of 5 vectors in $\mathbb{R}^5$ are bases for $\mathbb{R}^5$.
    h) Any set of vectors which spans $\mathbb{R}^5$ is linearly independent.
    i) Any set of 5 vectors which spans $\mathbb{R}^5$ is linearly independent.
    j) Any 5 linearly independent vectors in $\mathbb{R}^5$ form a basis for $\mathbb{R}^5$.

51. **[R]** Let V be a finite dimensional real vector space, and let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a finite set of vectors in V. Suppose also that the dimension of V is ℓ. State, with brief reasons, the relationship, if any, between n and ℓ if

    a) S is linearly independent.
    b) S is linearly dependent.
    c) S spans V.
    d) S is a basis for V.

52. [**H**] Explain why it is impossible to have a set of m mutually orthogonal, non-zero vectors in $\mathbb{R}^n$ with $m > n$.

53. [**R**] Consider the plane P in $\mathbb{R}^3$ whose equation is

$$x + y + z = 0.$$

a) Prove that P is a subspace of $\mathbb{R}^3$.

b) Find a basis for P. Give reasons for your answer.

54. [**R**] Find a basis for, and the dimension of, the column space of the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

55. [**R**] Find a basis for, and the dimension of, col(A), where

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & -1 & -2 & 2 \\ -1 & -1 & 1 & 4 & -1 \\ 1 & 1 & 0 & 4 & 2 \end{pmatrix}.$$

56. [**R**] Show that the columns of the matrix A given below are not a spanning set for $\mathbb{R}^4$. Then find a basis for $\mathbb{R}^4$ which contains as many of the columns of A as possible.

$$A = \begin{pmatrix} 1 & 3 & 3 & -7 & 5 \\ 2 & 6 & 5 & -8 & 1 \\ 3 & 9 & 5 & -3 & -2 \\ 4 & 12 & 5 & 2 & -5 \end{pmatrix}.$$

57. [**R**] Consider the set $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{x}\}$ where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 6 \\ -3 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 4 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} -1 \\ -5 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 6 \\ -3 \\ -1 \\ 20 \end{pmatrix}.$$

a) Find a basis B for span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{x})$.

b) Explain why $\mathbf{x}$ belongs to span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$. Write $\mathbf{x}$ as a linear combination of B.

c) Suppose the matrix A has columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. What is the dimension of the column space of A?

58. [**R**] Show that the set $\left\{1 - x^2 + x^3,\ x + 2x^2,\ 2 + x - x^2 + 2x^3,\ 2x - x^2 + x^3\right\}$ forms a basis for $\mathbb{P}_3$.

59. **[H]** Consider the set of polynomials $S = \{p_1, p_2, p_3, p_4\}$ in $\mathbb{P}_2$, where

$$p_1(z) = 1 + z - z^2, \quad p_2(z) = 2 - z, \quad p_3(z) = 5 - 4z + z^2, \quad p_4(z) = z^2.$$

Show that S is a linearly dependent spanning set for $\mathbb{P}_2$, and then find a subset of S which is a basis for $\mathbb{P}_2$.

60. **[H]** You are given that V is a vector space and that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a subset of V. Suppose that $\mathbf{w} \in \operatorname{span}(S)$. Prove that the set

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}\}$$

is a linearly dependent set.

61. **[X]** Prove that the **only** subspaces of $\mathbb{R}^4$ are (i) $\{\mathbf{0}\}$, (ii) lines through the origin, (iii) planes through the origin, (iv) subspaces of the form span(S), where S is any set of three linearly independent vectors in $\mathbb{R}^4$, and (v) $\mathbb{R}^4$ itself.

Problems 6.7 : [X] Coordinate vectors

62. **[X]** Show that the columns of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 3 & 2 & 0 & -2 \\ 0 & 1 & -1 & 1 \\ 5 & 3 & 0 & -1 \end{pmatrix}$$

are a basis for $\mathbb{R}^4$. Then find the coordinate vector of $\mathbf{v} = \begin{pmatrix} -2 \\ -6 \\ -4 \\ -2 \end{pmatrix}$ with respect to the ordered basis given by the columns of A.

63. **[X]** A vector $\mathbf{v} \in \mathbb{R}^4$ has the coordinate vector $\begin{pmatrix} 1 \\ 6 \\ -1 \\ 4 \end{pmatrix}$ with respect to the ordered basis formed by the columns of the matrix A of the previous question. Find $\mathbf{v}$.

64. **[X]** Find the vector $\mathbf{v}$ that has coordinate vector $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ with respect to the ordered basis $\left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ -5 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 9 \end{pmatrix} \right\}$ of $\mathbb{R}^3$.

65. **[X]** Find the coordinates of the following vectors with respect to the given ordered bases.

a) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ with respect to $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \right\}$.

b) $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ with respect to $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \right\}$.

66. [**X**] With respect to the basis $B = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ -3 \end{pmatrix} \right\}$ of $\mathbb{R}^3$,

a) find the vector $\mathbf{v}$ with coordinate vector $[\mathbf{v}]_B = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}$;

b) find the coordinate vector of $\mathbf{w} = \begin{pmatrix} 7 \\ -3 \\ 11 \end{pmatrix}$.

67. [**X**] Consider the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$.

Without solving systems of linear equations, do the following.

a) Show that S is an orthonormal set of vectors in $\mathbb{R}^3$.

b) Show that S is a basis for $\mathbb{R}^3$.

c) Find the coordinate vector of $\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$ with respect to the ordered basis S.

Hint. See Example 6 of Section 6.6.

68. [**X**] Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthonormal set of n vectors in $\mathbb{R}^n$. Prove that S is a basis for $\mathbb{R}^n$, and then show that the coordinate vector for any $\mathbf{v} \in \mathbb{R}^n$ is given by

$$[\mathbf{v}]_S = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad \text{where} \qquad x_j = \mathbf{u}_j \cdot \mathbf{v}.$$

Problems 6.8 : [X] Further important examples of vector spaces

69. [**X**] Let M_{22} be the vector space of all 2×2 matrices with real entries (see Example 3 of Section 6.1). Let S be the set

$$S = \{A \in M_{22} : a_{11} + a_{22} = 5\}.$$

a) Find three matrices in S.

b) Is S a subspace of M_{22}? Give a reason.

70. **[X]** Let T be the set

$$T = \{A \in M_{22} : a_{11} + a_{22} = 0\}.$$

a) Find three matrices in T.

b) Is T a subspace of M_{22}? Give a reason.

71. **[X]** Show that the four matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

form a basis for $M_{22}(\mathbb{R})$. This set is called the **standard basis** for $M_{22}(\mathbb{R})$.

72. **[X]** Show that the four matrices of the previous question also form a basis for the vector space $M_{22}(\mathbb{C})$ of all 2×2 matrices with complex entries.
HINT: Can you see why your proof of the previous question will also be valid for complex numbers?

73. **[X]** Show that the set of four matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

form a basis for $M_{22}(\mathbb{C})$. These matrices are called the Pauli spin matrices, and they are important in quantum physics and chemistry.

74. **[X]** Find the coordinates of the following vectors with respect to the given ordered bases.

a) The matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with respect to the standard basis for M_{22} given in question 71. Note that the results are the same for both real and complex numbers

b) Repeat part (b) for the basis of Pauli spin matrices given in question 73. In this case the entries of A should be regarded as complex numbers.

75. **[X]** Let $\quad R = \left\{ \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 5 & 1 \end{pmatrix} \right\}.$

a) Express $\begin{pmatrix} -4 & 2 \\ -1 & -3 \end{pmatrix}$ as a linear combination of elements of R.

b) Does R span $M_{22}(\mathbb{R})$, the space of all 2×2 matrices? Give a brief reason for your answer.

76. [**X**] Complete the proof of Proposition 2 of Section 6.8 that the system $(\mathcal{R}[X], +, *, \mathbb{R})$ is a vector space.

77. [**X**] Let $\mathcal{C}[X]$ be the set of all complex-valued functions with domain X. Show that the system $(\mathcal{C}[X], +, *, \mathbb{C})$, where $+$ and $*$ are the usual rules for addition and multiplication by a scalar of functions, satisfies vector space axioms 2, 4, 7 and 9. This system is a vector space over the complex field $\mathbb{C}$.

78. [**X**] Show that the set
$$S = \left\{ y \in \mathcal{R}[\mathbb{R}] : \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = 0 \right\}$$
is a subspace of the vector space $\mathcal{R}[\mathbb{R}]$ of all real-valued functions with domain $\mathbb{R}$.

79. [**X**] Is the set
$$S = \left\{ y \in \mathcal{R}[\mathbb{R}] : \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = 5 \right\}$$
a subspace of $\mathcal{R}[\mathbb{R}]$? Prove your answer.

80. [**X**] Let $C^{(k)}[\mathbb{R}]$ be the set of all real-valued functions with domain $\mathbb{R}$ for which the first k derivatives exist and are continuous. Prove that $C^{(k)}[\mathbb{R}]$ is a subspace of the vector space $\mathcal{R}[\mathbb{R}]$ of all real-valued functions with domain $\mathbb{R}$.

81. [**X**] Show that the vector spaces $C^{(k)}[\mathbb{R}]$ defined in the previous question have the property that, if $m \geqslant n$, then $C^{(m)}[\mathbb{R}]$ is a subspace of $C^{(n)}[\mathbb{R}]$.

82. [**X**] Let S be the subset of $\mathcal{R}\,[[-\pi, \pi]]$ defined by
$$S = \left\{ f \in \mathcal{R}\,[[-\pi, \pi]] : \int_{-\pi}^{\pi} \cos(x+t) f(t)dt = 0 \qquad \text{for all} \quad x \in [-\pi, \pi] \right\}.$$
Prove that S is a subspace of the vector space $\mathcal{R}\,[[-\pi, \pi]]$.

83. [**X**] This question generalises the results of question 46 to real-valued functions.

Let $S = \{f_1, \ldots, f_n\}$ be a set of real-valued functions defined on an interval $[a, b]$ with the properties that
$$\int_a^b f_i(x) f_j(x)dx = 0 \qquad \text{for} \quad i \neq j; \quad 1 \leqslant i, j \leqslant n$$
$$\int_a^b f_i^2(x)dx \neq 0 \qquad \text{for} \quad 1 \leqslant i \leqslant n.$$

Prove that S is a linearly independent set.

Note. A set of functions with these properties is said to be mutually orthogonal on the interval $[a, b]$.

84. **[X]** Show that the set

$$S = \left\{ p \in \mathbb{P}_n(\mathbb{R}) : 5p'(6) + 3p(6) = 0 \right\}, \qquad \text{where} \quad p'(x) = \frac{dp}{dx},$$

is a subspace of the vector space $\mathbb{P}_n(\mathbb{R})$ of all real polynomials of degree less than or equal to n.

85. **[X]** Is the set

$$S = \left\{ p \in \mathbb{P}_n(\mathbb{R}) : 5p'(6) + 3p(6) = 8 \right\}, \qquad \text{where} \quad p'(x) = \frac{dp}{dx},$$

a subspace of $\mathbb{P}_n(\mathbb{R})$? Prove your answer.

86. **[X]** Let $\mathbb{P}$ be the set of all polynomials over the complex-number field $\mathbb{C}$. Show that $\mathbb{P}$ is a subspace of the vector space $\mathcal{C}[\mathbf{C}]$ of all complex-valued functions with domain $\mathbb{C}$.

87. **[X]** Is the polynomial $p \in \text{span}\,(p_1, p_2, p_3)$, where the polynomials are defined by

$$p(z) = -6+2z+30z^2, \quad p_1(z) = 1+2z+3z^2, \quad p_2(z) = -4-z+9z^2, \quad p_3(z) = -5-z+12z^2.$$

88. **[X]** Find conditions on the coefficients of the polynomial $p \in \mathbb{P}_2$ for p to be a linear combination of the three polynomials p_1,p_2, p_3, where the polynomials are given by

$$p_1(z) = 2z + 3z^2, \qquad p_2(z) = 5 - 2z - 3z^2, \qquad p_3(z) = 15 - 4z - 6z^2.$$

89. **[X]** Are the polynomials p_1, p_2, p_3 in the previous two questions spanning sets for $\mathbb{P}_2$?

90. **[X]** Is the set of polynomials $S = \{p_1,\, p_2,\, p_3\}$ in $\mathbb{P}_2$, where

$$p_1(z) = 1 + z - z^2, \qquad p_2(z) = 2 - z, \qquad p_3(z) = 5 - 4z + z^2,$$

a linearly independent set? If not, express one of the polynomials as a linear combination of the others.

91. **[X]** Show that

$$\begin{aligned}
p_1(z) &= -2 + 5z - 4z^2 + 15z^3 - 5z^4 + z^5, \\
p_2(z) &= 3z + 4z^2 - 3z^3 + 6z^5, \\
p_3(z) &= 2 + 3z^2 - 4z^3 + 10z^4 - 5z^5, \\
p_4(z) &= 3 + 14z^2 - 5z^3 + 6z^4 - 3z^5, \\
p_5(z) &= 3 + 8z + 17z^2 + 3z^3 + 11z^4 - z^5, \\
p_6(z) &= -3 + 11z - 7z^2 + 10z^3 - z^4 + 11z^5,
\end{aligned}$$

are not a spanning set for $\mathbb{P}_5$, and then construct a basis for $\mathbb{P}_5$ containing as many of the given polynomials as possible.

HINT. Check using Maple.

92. [**X**] Find the coordinate vector for $p(x) = 1 + 2x + x^2$ with respect to the ordered basis $\left\{1 + x,\ 1 - x^2,\ x + 2x^2\right\}$ of $\mathbb{P}_2$.

93. [**X**] Find the coordinate vector of $1 + 2z + 3z^2$ with respect to the ordered basis of $\mathbb{P}_2$ given by

$$\left\{\frac{1}{8}z(z-2), 1 - \frac{1}{4}z^2, \frac{1}{8}z(z+2)\right\}.$$

NOTE. This question and the one that follows do not require Gaussian Elimination.

94. [**X**] Find the coordinate vector of $a_0 + a_1 z + a_2 z^2$ with respect to the ordered basis of $\mathbb{P}_2$ given by

$$\left\{\frac{1}{2}z(z-1), 1 - z^2, \frac{1}{2}z(z+1)\right\}.$$

95. [**X**] Let $S = \{p_1, \dots, p_n\}$ be a set of n polynomials in $\mathbb{P}_{n-1}(\mathbb{R})$ with the property that

$$\int_a^b p_i(x)p_j(x)dx = \begin{cases} 0 & \text{for} \quad i \neq j \\ 1 & \text{for} \quad i = j \end{cases} \qquad \text{for} \quad 1 \leqslant i, j \leqslant n.$$

A set of polynomials with this property is called an orthonormal set of polynomials on the interval $[a, b]$.

Prove that S is a basis for $\mathbb{P}_{n-1}(\mathbb{R})$, and then show that the coordinate vector for any $p \in \mathbb{P}_{n-1}(\mathbb{R})$ is given by

$$[p]_S = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad \text{where} \qquad x_i = \int_a^b p_i(x)p(x)dx.$$

Problems for Chapter 7

Problems 7.1 : Introduction to linear maps

1. [**R**] Explain why the function $S : [-1,1] \to \mathbb{R}$ defined by $S(x) = 5x$ for $x \in [-1,1]$ is not a linear map. Then show that the function $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = 5x$ for $x \in \mathbb{R}$ is a linear map.

2. [**R**] For the following examples, determine whether T is a linear map by using Definition 1.

   a) $T : \mathbb{R}^2 \to \mathbb{R}^4$ defined by

   $$T(\mathbf{x}) = \begin{pmatrix} 3x_1 - x_2 \\ 2x_1 + 4x_2 \\ -3x_1 - 3x_2 \\ x_2 \end{pmatrix} \quad \text{for} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2.$$

   b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

   $$T(\mathbf{x}) = \begin{pmatrix} -2x_1 + 5x_3 \\ 6x_1 - 8x_2 + 2x_4 \\ -2x_1 + 4x_2 - 3x_3 \end{pmatrix} \quad \text{for} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4.$$

   c) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

   $$T(\mathbf{x}) = \begin{pmatrix} 3x_1 + 4 \\ -2x_1 + 3x_2 - x_3 \end{pmatrix} \quad \text{for} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3.$$

   d) $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ by

   $$T(\mathbf{x}) = x_1 \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}.$$

   e) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

   $$T(\mathbf{x}) = \begin{pmatrix} 3x_2^2 - x_3 \\ x_1 - 4x_2 \end{pmatrix} \quad \text{for} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3.$$

3. [**X**] Consider the complex numbers as a real vector space. Specify the "natural" domain and codomain for each of the following functions of a complex number and determine if the function is a linear function with $\mathbb{F} = \mathbb{R}$.

   a) $T(z) = \mathrm{Re}(z)$, b) $T(z) = \mathrm{Im}(z)$, c) $T(z) = |z|$,
   d) $T(z) = \mathrm{Arg}\,(z)$, e) $T(z) = \bar{z}$.

4. [**R**] Show that the sine function $T : \mathbb{R} \to \mathbb{R}$, defined by
$$T(x) = \sin(x) \quad \text{for} \quad x \in \mathbb{R},$$
satisfies parts 1 and 2 of Proposition 1 of Section 7.1 but that it is not a linear map.

5. [**H**] Use proof by induction to prove Theorem 3 of Section 7.1.

6. [**R**] If $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly independent in a real vector space V and $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$, is there a linear map $T : W \to \mathbb{R}^2$ where $W = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2)$ such that
$$T(\mathbf{v}_1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, T(\mathbf{v}_2) = \begin{pmatrix} -3 \\ 2 \end{pmatrix}, T(\mathbf{v}_3) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}?$$

7. [**R**] A linear map $T : \mathbb{R}^3 \to \mathbb{R}^4$ has function values given by
$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 4 \end{pmatrix}, T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -5 \\ 6 \end{pmatrix}.$$ Find $T\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ and $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

8. [**R**] Show that any function $T : \mathbb{R}^3 \to \mathbb{R}^4$ with function values given by $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$,
$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 4 \end{pmatrix}, T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -5 \\ 6 \end{pmatrix}, \text{ and } T\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 1 \end{pmatrix}$$ is not a linear map.

9. [**R**] A linear function $T : \mathbb{R}^3 \to \mathbb{R}^2$ has function values given by
$$T\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \qquad T\begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \qquad T\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}.$$
Write $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ as a linear combination of the vectors in the basis $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}$ of $\mathbb{R}^3$. Hence find $T\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$.

   Hint. Use Theorem 3 of Section 7.1.

10. [**H**] Given that $T\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}4\\1\end{pmatrix}$, $T\begin{pmatrix}-2\\1\\4\end{pmatrix}=\begin{pmatrix}-1\\2\end{pmatrix}$ and $T\begin{pmatrix}1\\7\\13\end{pmatrix}=\begin{pmatrix}4\\-2\end{pmatrix}$, show that T is not a linear map.

Problems 7.2 : Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $m \times n$ matrices

11. [**R**] For any function in question 2 which is a linear map, find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x}$ in the domain by using the results of the Matrix Representation Theorem of Section 7.2.

12. [**R**] For any function in question 2 which is a linear map, write a system of linear equations for $T(\mathbf{x})$ for $\mathbf{x}$ in the domain, and hence find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$. Check that the matrices you obtain in this question are the same as the matrices that you obtained in the previous question.

Problems 7.3 : Geometric examples of linear transformations

13. [**R**] For each of the following 2×2 matrices, draw a picture to show $A\mathbf{e}_1$, $A\mathbf{e}_2$, $A\mathbf{b}$, where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the standard basis vectors in $\mathbb{R}^2$ and where $\mathbf{b} = 2\mathbf{e}_1 + 3\mathbf{e}_2$.

    a) $\begin{pmatrix}2 & 0\\0 & 0.7\end{pmatrix}$, b) $\begin{pmatrix}-2 & 0\\0 & 2\end{pmatrix}$, c) $\begin{pmatrix}-2 & 0\\0 & -3\end{pmatrix}$, d) $\begin{pmatrix}6 & -2\\6 & -1\end{pmatrix}$, e) $\begin{pmatrix}4 & -4\\3 & -4\end{pmatrix}$.

14. [**R**] Draw the image of the star in Figure 4(a) on page 90 under each of the transformations defined by the matrices in Quesion 13.

15. [**R**] Let T be the rotation in the plane $\mathbb{R}^2$ through angle $\dfrac{\pi}{3}$ in the anti-clockwise direction. Find the matrix which represents the linear transformation T.

16. [**H**] Let $\mathbf{x}$ be the position vector of a point X in $\mathbb{R}^2$, and let $\mathbf{x}'$ be the position vector of the point X' which is the reflection of X in the x_2-axis. (That is, assume that a mirror is placed along the x_2-axis and that X' is the reflection of X in the mirror.) Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = \mathbf{x}'$ is a linear map. Find a matrix A which transforms $\mathbf{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ into $\mathbf{x}' = \begin{pmatrix}x_1'\\x_2'\end{pmatrix}$.

17. [**H**] Let $\mathbf{x}$ be the position vector of a point X in $\mathbb{R}^3$ and let $\mathbf{x}'$ be the position vector of the point X' which is the reflection of X in the (x_1, x_2)-plane. Show that the function $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = \mathbf{x}'$ is a linear map. Find a matrix A which transforms the position vector $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ of X into the position vector $\begin{pmatrix}x_1'\\x_2'\\x_3'\end{pmatrix}$ of X'.

18. **[X]** Let $\mathbf{p}$ be the position vector of a point P in $\mathbb{R}^n$ and let $\mathbf{q}$ be the position vector of the point Q which is the reflection of P in the line

$$\mathbf{x} = \lambda\mathbf{d}; \quad \lambda \in \mathbb{R}.$$

Show that the function $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\mathbf{p}) = \mathbf{q}$ is a linear map. Find a matrix A which transforms $\mathbf{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$ into $\mathbf{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$.

19. **[R]** Let $\mathbf{b}$ be a fixed vector in $\mathbb{R}^3$. Is the function $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(\mathbf{x}) = \mathbf{b} \times \mathbf{x} \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^3,$$

where $\mathbf{b} \times \mathbf{x}$ is the cross product, a linear map? Prove your answer. Find a matrix A which transforms the vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ into its function value $T(\mathbf{x}) = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$.

20. **[H]** Suppose that $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. Prove that the projection function $T : \mathbb{R}^3 \to \mathbb{R}^3$ of vectors onto $\mathbf{b}$, which is defined by

$$T(\mathbf{a}) = \mathrm{proj}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\,\mathbf{b} \quad \text{for} \quad \mathbf{a} \in \mathbb{R}^3,$$

is a linear map. Find a matrix A which transforms the vector $\mathbf{a}$ into its projection $T(\mathbf{a})$.

21. **[H]** If $\mathbf{a}$ is regarded as a fixed non-zero vector, is the function $S : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$S(\mathbf{b}) = \begin{cases} \mathrm{proj}_{\mathbf{b}}\mathbf{a} & \text{for} \quad \mathbf{b} \in \mathbb{R}^n \backslash \{\mathbf{0}\} \\ \mathbf{0} & \qquad\quad\; \mathbf{b} = \mathbf{0} \end{cases}$$

a linear map? Prove your answer.

22. **[X]** Let A_ϕ, A_θ and $A_{\phi+\theta}$ be the matrices for rotations in the plane by angles ϕ, θ and $\phi + \theta$ respectively (see Example 3 of Section 7.3). Prove that

$$A_\theta A_\phi = A_{\phi+\theta}.$$

What is this saying geometrically?

23. **[X]** Let $B = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be an ordered orthonormal basis (Cartesian coordinate system) for a three-dimensional geometric vector space. Let $\mathbf{a}$ be a three-dimensional geometric vector and let $\mathbf{a}' = R_\alpha(\mathbf{a})$ be the vector obtained by rotating $\mathbf{a}$ anticlockwise by an angle α about an axis parallel to $\mathbf{j}$. If the coordinate vectors of $\mathbf{a}$ and $\mathbf{a}'$ are $[\mathbf{a}]_B = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and

$[\mathbf{a}']_B = \begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix}$, find the rule $R_\alpha \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix}$ and show that it defines a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$. Also find the matrix A such that $\mathbf{a}' = A\mathbf{a}$.

Problems 7.4 : Subspaces associated with linear maps

24. [**R**] Show that the set $\left\{ \begin{pmatrix} \lambda \\ -2\lambda \\ \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\}$ is the kernel of the matrix $\begin{pmatrix} 3 & 1 & -1 \\ 8 & 3 & -2 \end{pmatrix}$.

25. [**R**] Find the kernel and the nullity of each of the following matrices.

    a) $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & -2 & 3 \\ 4 & 1 & -1 \end{pmatrix}$, b) $B = \begin{pmatrix} 0 & 5 & 15 \\ 2 & -2 & -4 \\ 3 & -3 & -6 \end{pmatrix}$,

    c) $C = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 3 & 2 & 0 & -2 \\ 0 & 1 & -1 & 1 \end{pmatrix}$.

    Where possible, give a geometric interpretation of the kernels.

26. [**R**] Find a basis for the kernel, and the nullity, of each of the following matrices.

    a) $D = \begin{pmatrix} 1 & 2 & 5 & 0 \\ 1 & -1 & -4 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}$, b) $E = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 5 & -4 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

27. [**H**] Let $W = \left\{ \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T : x_1 + x_2 + x_3 + x_4 = x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \right\}$. Find a matrix A such that $W = \ker A$.

28. [**R**] Find $\ker(T)$ and $\text{nullity}(T)$ for the linear functions of question 2.

29. [**R**] Find $\ker(T)$ and $\text{nullity}(T)$ for the linear functions of questions 16 through 20. Give a geometric interpretation of the kernels.

30. [**H**] Suppose that $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

    a) Prove that the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$, given by $T(\mathbf{x}) = \mathbf{b} \times \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$, is a linear mapping.

    b) Find the dimension of the kernel of this mapping.

31. [**R**] For each given vector $\mathbf{b}$ and matrix A, determine if $\mathbf{b} \in \text{im}(A)$.

a) $\mathbf{b} = \begin{pmatrix} 11 \\ 10 \\ 4 \end{pmatrix}$, $\quad A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & 3 \\ 4 & 1 & -1 \end{pmatrix}$.

b) $\mathbf{b} = \begin{pmatrix} 9 \\ -2 \\ -4 \end{pmatrix}$, $\quad A = \begin{pmatrix} 0 & 5 & 15 \\ 2 & -2 & -4 \\ 3 & -3 & -6 \end{pmatrix}$.

c) $\mathbf{b} = \begin{pmatrix} -2 \\ -6 \\ -4 \end{pmatrix}$, $\quad A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 3 & 2 & 0 & -2 \\ 0 & 1 & -1 & 1 \end{pmatrix}$.

32. **[R]** Find conditions on b_1, b_2, b_3 for the vector $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3$ for $\mathbf{b}$ to belong to $\operatorname{im}(A)$ for the matrices in the preceding question.

33. **[R]** Find a basis for the image, and the rank, of each of the matrices in questions 25 and 26.

34. **[R]** By comparing the answers to questions 25, 26 and 33, verify the conclusion of the Rank-Nullity Theorem.

35. **[R]** Find a basis for the image, and the rank, of each of the matrices

$$A = \begin{pmatrix} 1 & 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & -1 & -2 & 2 \\ -1 & -1 & 1 & 4 & -1 \\ 1 & 1 & 0 & 4 & 2 \end{pmatrix}.$$

36. **[R]** Find a basis for $\mathbb{R}^3$ which contains a basis of $\operatorname{im}(C)$, where

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -4 & 6 & -2 \\ -1 & 2 & -3 & 1 \end{pmatrix}.$$

37. **[R]** Find a basis for $\mathbb{R}^4$ which contains a basis of $\operatorname{im}(D)$, where

$$D = \begin{pmatrix} 1 & 3 & 3 & -1 & 7 \\ 2 & 6 & 3 & 1 & 8 \\ 3 & 9 & 3 & 4 & 7 \\ 4 & 12 & 0 & 8 & 4 \end{pmatrix}.$$

38. **[R]** A linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ has the property that

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; T\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; T\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \; T\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

a) Write down the matrix representation of T with respect to the standard basis (in both domain and co-domain).

b) Find a basis for the image of T and find the rank of T.

c) State the dimension of the kernel of T.

d) Does the vector $\begin{pmatrix} 8 \\ -3 \\ 1 \\ 2 \end{pmatrix}$ belong to the image of T? Give reasons.

39. **[H]** Let $A \in M_{nn}(\mathbb{R})$. Show that the following statements are equivalent, that is, show that if any statement is true then all are true, whereas if any statement is false then all are false.

a) For all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, $A\mathbf{x} = A\mathbf{y}$ if and only if $\mathbf{x} = \mathbf{y}$.

b) $\ker(A) = \{\mathbf{0}\}$.

c) $\text{nullity}(A) = 0$.

d) $\text{rank}(A) = n$.

e) $\text{im}(A) = \mathbb{R}^n$.

f) The columns of A form a basis for $\mathbb{R}^n$.

40. **[H]** Let $A \in M_{mn}(\mathbb{R})$, and let $\{\mathbf{e}_j \ : \ 1 \leqslant j \leqslant m\}$ be the set of m standard basis vectors of $\mathbb{R}^m$. If A is of rank r, explain why at most r of the m equations $A\mathbf{x}_j = \mathbf{e}_j$ can have solutions.

41. **[H]** Let A and $\mathbf{e}_j$ be as in the previous question. If $\text{nullity}(A) = \nu$, explain why at least $m - n + \nu$ of the m equations $A\mathbf{x}_j = \mathbf{e}_j$ **do not** have solutions.

42. **[H]** Let $A \in M_{nn}(\mathbb{R})$, $\text{rank}(A) = n$, and $\mathbf{e}_j$ be the standard basis vectors for $\mathbb{R}^n$. Prove that each of the n equations $A\mathbf{x}_j = \mathbf{e}_j$, $1 \leqslant j \leqslant n$, has a unique solution.

43. **[X]** Let $T : V \to V$ be a linear map and assume that $\dim(V) = n$. Show that the following statements are equivalent.

a) $T(\mathbf{v}) = T(\mathbf{w})$ if and only if $\mathbf{v} = \mathbf{w}$ for all $\mathbf{v}, \mathbf{w} \in V$.

b) $\ker(T) = \{\mathbf{0}\}$.

c) $\text{nullity}(T) = 0$.

d) $\text{rank}(T) = n$.

e) $\text{im}(T) = V$.

Problems 7.5 : Further applications and examples of linear maps

44. **[X]** Show that the function $T : \mathbb{R}^4 \to M_{22}(\mathbb{R})$ defined by

$$T(\mathbf{a}) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad \text{for} \quad \mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$$

is a linear map.

45. [**X**] Show that the function $T : \mathbb{R}^4 \to M_{22}(\mathbb{R})$ defined by

$$T(a_1, a_2, a_3, a_4) = \begin{pmatrix} 3a_1 - 2a_4 & a_4 + 2a_3 \\ -5a_2 + 3a_3 & a_1 \end{pmatrix}$$

is a linear map.

46. [**X**] Is the function $T : M_{23}(\mathbb{R}) \to \mathbb{R}^6$ defined by

$$T(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{21} & a_{22} & a_{23} \end{pmatrix}^T \quad \text{for} \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \in M_{23}(\mathbb{R})$$

a linear map?

47. [**R**] Show that the function $T : \mathbb{P}_2 \to \mathbb{C}^3$ defined by

$$T(a_0 + a_1 z + a_2 z^2) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

is a linear map.

[**X**] Note that T maps a polynomial in $\mathbb{P}_2$ into its coordinate vector with respect to the standard basis $\{1, z, z^2\}$.

48. [**H**] Show that the function $T : \mathbb{C}^4 \to \mathbb{P}_4$ defined by $T(\mathbf{a}) = p$ for $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \in \mathbb{C}^4$, where

$$p(z) = (a_1 - 3a_2) + (2a_3 - 3a_4)z + a_2 z^3 + (3a_1 - a_2 + 2a_3 + 4a_4)z^4 \quad \text{for all} \quad z \in \mathbb{C},$$

is a linear map.

49. [**R**] Show that the function $T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R})$ defined by

$$T(p) = 4p' + 3p, \qquad \text{where} \qquad p'(x) = \frac{dp}{dx},$$

is a linear map.

50. [**R**] Is the function $T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R})$ defined by $T(p) = q$, where

$$q(x) = 4xp'(x) - 8p(x) \qquad \text{for} \quad x \in \mathbb{R},$$

a linear map? Prove your answer.

51. [**H**] Show that the function $T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_4(\mathbb{R})$ defined by $T(p) = q$, where

$$q(x) = \int_0^x p(t)\,dt \qquad \text{for} \quad x \in \mathbb{R},$$

is a linear map.

52. [**X**] Let V be the subset of the vector space $\mathcal{R}[\mathbb{R}]$ of all real-valued functions on $\mathbb{R}$ defined by

$$V = \left\{ f \in \mathcal{R}[\mathbb{R}] : \int_0^x f(t)dt \quad \text{exists for all} \quad x \in \mathbb{R} \right\}.$$

Show that V is a subspace of $\mathcal{R}[\mathbb{R}]$, and then show that the rule $T : V \to \mathcal{R}[\mathbb{R}]$ defined by $T(f) = g$, where

$$g(x) = \int_0^x f(t)dt \quad \text{for} \quad f \in V \quad \text{and} \quad x \in \mathbb{R},$$

is a linear map.

53. [**X**] A function $S : \mathbb{R} \to \mathbb{Z}$ is defined by $S(x) = y$, where y is the integer obtained on rounding x to the nearest integer. Is S a linear map? A function $T : \mathbb{R} \to \mathbb{R}$ is defined by $T(x) = y$, where y is the integer obtained on rounding x to the nearest integer. Is T a linear map?

54. [**X**] Let y be a real-valued function with domain $\mathbb{R}$ such that y and its first two derivatives y' and y'' exist, and such that the Laplace transforms (see Example 7 of Section 7.5) of y, y' and y'' also exist on the interval $(0, \infty)$. Given that $y(0) = 1$ and $y'(0) = 2$ and that y satisfies the differential equation

$$y''(x) + 4y'(x) + 3y = e^{-3x},$$

find an explicit formula for the Laplace transform $y_L(s)$ of y in terms of s.

HINT. Take the Laplace transform of the differential equation and use integration by parts to find formulae for the Laplace transforms of y' and y'' in terms of $y_L(s)$.

55. [**H**] Consider the mapping $T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$T(p(x)) = \begin{pmatrix} a - b \\ c - d \end{pmatrix} \quad \text{where} \quad p(x) = a + bx + cx^2 + dx^3.$$

a) Prove that T is linear.

b) Show that $p(x) = 3x^3 + 3x^2 - 2x - 2$ is in the kernel of T.

56. [**H**] Consider the function $T : \mathbb{R}^4 \to \mathbb{P}_1$ defined by

$$T \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = (a - 2b) + (c + d)x.$$

a) Find $T \begin{pmatrix} 1 \\ -3 \\ 2 \\ -4 \end{pmatrix}$.

b) Show T is a linear transformation.

c) Write down a non-zero vector in $\mathbb{R}^4$ which lies in $\ker(T)$.

57. [**H**] A linear map $T : \mathbb{C}^3 \to \mathbb{P}_3$ has function values given by

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = 1 + (2+i)z - 3z^3, \quad T\begin{pmatrix}0\\1\\0\end{pmatrix} = (4-3i)z + z^2, \quad T\begin{pmatrix}0\\0\\1\end{pmatrix} = -2 \quad \text{for} \quad z \in \mathbb{C}.$$

Find $T\begin{pmatrix}i\\2\\-1\end{pmatrix}$ and $T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$.

58. [**X**] Let $\mathbb{P}_n$ be the real vector space of polynomials of degree less than or equal to n, and take its standard basis to be

$$\{1, x, x^2, \dots, x^n\}.$$

For $p(x) \in \mathbb{P}_3$, let $(T(p))(x) \in \mathbb{P}_4$ be defined by

$$(T(p))(x) = \int_0^x p(t)dt.$$

a) Show that T is a linear transformation from $\mathbb{P}_3$ to $\mathbb{P}_4$.

b) Calculate the matrix A of this linear transformation with respect to the standard bases of $\mathbb{P}_3$ and $\mathbb{P}_4$.

c) Find a basis for the image of T, $\operatorname{im}(T)$.

d) Find a basis for the kernel of T, $\ker(T)$.

59. [**R**] A car manufacturer produces a station wagon, a four-wheel drive, a hatchback and a sedan model. Each model is made from steel, plastics, rubber and glass, and it also requires a number of hours of labour to produce. The requirements per car of these inputs for each model are as shown in the following table.

	steel (tonnes)	plastics (tonnes)	rubber (tonnes)	glass (tonnes)	labour (hours)
station wagon	1	0.5	0.1	0.2	1
4-wheel drive	1.5	0.6	0.2	0.15	1.5
hatchback	0.8	0.7	0.2	0.2	1.1
sedan	0.9	0.6	0.25	0.3	0.9

Construct a matrix which can be used to express the factory *input* as a linear function of the factory *output*.

Problems 7.6 : [X] Representation of linear maps by matrices

60. [**R**] Let $\mathrm{id}_{\mathbb{R}^2}$ be the identity map for $\mathbb{R}^2$. Find a matrix representation of this map with respect to standard bases in domain and codomain.

61. [**X**] Let $\mathrm{id}_{\mathbb{R}^2}$ be the identity map for $\mathbb{R}^2$. Find a matrix representation of this map with respect to the domain basis $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$ and the codomain basis $\left\{\begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$.

62. [**X**] A linear mapping $G : \mathbb{P}_2 \to \mathbb{P}_2$ has matrix representation

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}$$

with respect to the standard basis $\{1, x, x^2\}$ in both domain and co-domain. Find $G(p)$, where $p(x) = -3 + x + 5x^2$.

63. [**X**] For each of the linear maps in questions 48 to 51, find a matrix which represents the linear map for standard bases in the domain and codomain.

64. [**X**] Using your results of the previous question or otherwise, find the kernel, nullity, image and rank of the linear maps in questions 48 to 51.

65. [**X**] For the linear map $T : \mathbb{P}_n(\mathbb{R}) \to \mathbb{P}_n(\mathbb{R})$ defined by $T(p) = q$, where

$$q(x) = x^2 \frac{d^2p}{dx^2} - 3x\frac{dp}{dx} + 3p(x) \qquad \text{for} \quad x \in \mathbb{R},$$

find a matrix which represents T with respect to standard bases in domain and codomain. Hence, or otherwise, find the kernel and nullity of T.

66. [**X**] Let V be a vector space and let $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for V. Let $\mathbf{a} \in V$ be a vector whose coordinate vector with respect to B is $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$. Let $\begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix}$ be the coordinate vector of $\mathbf{a}$ with respect to the basis $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3\}$ given by

$$\begin{aligned} \mathbf{u}'_1 &= \frac{1}{\sqrt{2}}\mathbf{u}_1 + \frac{1}{\sqrt{2}}\mathbf{u}_3, \\ \mathbf{u}'_2 &= -\frac{1}{\sqrt{2}}\mathbf{u}_1 + \frac{1}{\sqrt{2}}\mathbf{u}_3, \\ \mathbf{u}'_3 &= -\mathbf{u}_2 \end{aligned}$$

Show that B' is an orthonormal basis, then show that the rule $T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix}$ is a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$, and find a matrix representation for this function with respect to standard bases for $\mathbb{R}^3$ in domain and codomain. Finally, show that the matrix you have constructed is a matrix representation of the identity map $\mathrm{id}_V : V \to V$ with respect to the basis B in domain and B' in codomain.

Problems 7.7 : [X] Matrix arithmetic and linear maps

67. **[X]** Let $T : V \to W$ and $S : V \to W$ be linear maps. Let B_V be a basis for V and B_W be a basis for W and let A and B be the matrices representing T and S with respect to the bases B_V and B_W. Prove that $A + B$ is the matrix representing the sum function $T + S$ with respect to the bases B_V and B_W.

68. **[X]** Let $T : U \to V$ and $S : V \to W$ be linear maps. Let B_U, B_V and B_W be bases for U, V and W respectively. Let A be the matrix representing T with respect to bases B_U and B_V and let B be the matrix representing S with respect to bases B_V and B_W. Prove that the matrix product BA is the matrix which represents the composition function $S \circ T : U \to W$ with respect to the bases B_U and B_W.

Problems 7.8 : [X] One-to-one, onto and invertible linear maps and matrices

69. **[X]** Let $V = C[\mathbb{R}]$, the vector space of all continuous real-valued functions on $\mathbb{R}$. Let

$$B = \{e^x,\ (x-1)e^x,\ (x-1)(x-2)e^x\} \subseteq V$$

a) Prove that B is linearly independent.

b) Let $W = \operatorname{span}(B)$, and let $D\ :\ W \to W$ denote the linear transformation

$$D(f) = f',$$

where f' is the derivative of f.

i) Find the matrix for D with respect to the ordered basis B of W.

ii) Find the matrix for the linear transformation $T = D \circ D$.

iii) Hence or otherwise, prove that for every $g \in W$, there exists $f \in W$ such that

$$f'' = g.$$

70. **[X]** For a field $\mathbb{F}$, define $T : M_{22}(\mathbb{F}) \to \mathbb{F}^3$ by

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{pmatrix} a_{22} \\ a_{12} - a_{21} \\ 3a_{11} + a_{12} \end{pmatrix}.$$

a) Show that T is a linear transformation.

b) Find the kernel of T and the nullity of T.

c) Find the rank of T.

d) Is T one-to-one (injective)? Give a brief reason for your answer.

e) Find the matrix of T with respect to the standard bases of $M_{22}(\mathbb{F})$ and $\mathbb{F}^3$.

Problems 7.11 : Linear transformations and MAPLE

71. [**M**] Consider the following MAPLE output

```
> with(LinearAlgebra):
> A:=<<2,4,-2,4>|<-3,-6,3,-6>|<1,2,-2,1>|<-1,-3,2,-2>|<1,-1,-1,-1>>;
```

$$A := \begin{bmatrix} 2 & -3 & 1 & -1 & 1 \\ 4 & -6 & 2 & -3 & -1 \\ -2 & 3 & -2 & 2 & -1 \\ 4 & -6 & 1 & -2 & -1 \end{bmatrix}$$

```
> b:=<a1,a2,a3,a4>;
```

$$b := \begin{bmatrix} a1 \\ a2 \\ a3 \\ a4 \end{bmatrix}$$

```
> GaussianElimination(<A|b>);
```

$$\begin{bmatrix} 2 & -3 & 1 & -1 & 1 & a1 \\ 0 & 0 & -1 & 1 & 0 & a3 + a1 \\ 0 & 0 & 0 & -1 & -3 & a2 - 2\,a1 \\ 0 & 0 & 0 & 0 & 0 & a4 - a1 - a3 - a2 \end{bmatrix}$$

a) Find a basis for col(A), the column space of A.

b) What is the dimension of col(A)?

c) Under what conditions does (a_1, a_2, a_3, a_4) belong to col(A).

d) Find a basis for the kernel, or null space, of A.

e) What are the values of the rank and nullity of A?

Problems for Chapter 8

Problems 8.1 : Definitions and examples

1. [**R**] Let
$$A = \begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix},$$
and let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the standard basis vectors for $\mathbb{R}^2$.
   a) Write down the eigenvalues and eigenvectors of A, B and C.
   b) Draw a sketch of $\mathbf{e}_1$, $A\mathbf{e}_1$, $\mathbf{e}_2$, $A\mathbf{e}_2$. Then, for some vector $\mathbf{x}$ which is not parallel to either $\mathbf{e}_1$ or $\mathbf{e}_2$ draw a sketch of $\mathbf{x}$ and $A\mathbf{x}$.
   c) Repeat part (b) for the matrix B. Comment on any differences you observe between the results for A and B.
   d) Repeat part (b) for the matrix C. Again comment on any differences you observe between the results for A and C.
   e) For $\mathbf{x} \neq \mathbf{0}$, prove algebraically that $A\mathbf{x}$ is parallel to $\mathbf{x}$ if and only if $\mathbf{x}$ is parallel to either $\mathbf{e}_1$ or $\mathbf{e}_2$, that $B\mathbf{x}$ is parallel to $\mathbf{x}$ for all $\mathbf{x}$ and that $C\mathbf{x}$ is parallel to $\mathbf{e}_1$ for all $\mathbf{x}$.

2. [**R**] Show that the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix $\begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}$ and find the corresponding eigenvalue.

3. [**X**] Let A be a fixed 3×3 matrix and define a linear map $T : M_{33} \to M_{33}$ by $T(X) = AX$. If λ is a real eigenvalue of T corresponding to an invertible eigenvector X, find λ in terms of $\det(A)$.

4. [**H**] Let T be the linear map which reflects vectors in $\mathbb{R}^2$ about the line $y = x$.
   a) Explain why $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of T and give their corresponding eigenvalues.
   b) Find the matrix A such that $T\mathbf{x} = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$.

5. [**R**] Find the eigenvalues and eigenvectors for
   a) $A = \begin{pmatrix} 6 & -2 \\ 6 & -1 \end{pmatrix}$, b) $A = \begin{pmatrix} -5 & 2 \\ -6 & 3 \end{pmatrix}$.

6. [**X**] For each of the matrices in the preceding question find two independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. On one diagram sketch the lines
$$\ell_1 = \{\mathbf{x} : \mathbf{x} = \mu\mathbf{v}_1,\ \mu \in \mathbb{R}\}$$
$$\ell_2 = \{\mathbf{x} : \mathbf{x} = \mu\mathbf{v}_2,\ \mu \in \mathbb{R}\}$$

and the parallelogram

$$P = \{\mathbf{x} : \mathbf{x} = \mu_1 \mathbf{v}_1 + \mu_2 \mathbf{v}_2, \quad \text{for} \quad 0 \leqslant \mu_1 \leqslant 1,\ 0 \leqslant \mu_2 \leqslant 1\}.$$

Then identify and sketch (on a separate diagram)

$$\begin{aligned} &\{\mathbf{y} : \mathbf{y} = A\mathbf{x},\ \mathbf{x} \in \ell_1\} \\ &\{\mathbf{y} : \mathbf{y} = A\mathbf{x},\ \mathbf{x} \in \ell_2\} \\ &\{\mathbf{y} : \mathbf{y} = A\mathbf{x},\ \mathbf{x} \in P\}. \end{aligned}$$

Describe the linear mapping $T\mathbf{x} = A\mathbf{x}$ geometrically.

7. **[R]** Find the eigenvalues and eigenvectors of the following matrices. In each case, note if the eigenvalues are real, occur in complex conjugate pairs, or are general complex numbers. Also note if the eigenvectors form a basis for $\mathbb{C}^2$.

   a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, b) $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, c) $\begin{pmatrix} 3 & 5 \\ 0 & -6 \end{pmatrix}$,

   d) $\begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}$, e) $\begin{pmatrix} 4 & 2i \\ 2i & 6 \end{pmatrix}$, f) $\begin{pmatrix} 4 & -2i \\ 2i & 6 \end{pmatrix}$.

8. **[H]** Show that the eigenvalues of a square row-echelon form matrix U are equal to the diagonal elements of the matrix. (A square row-echelon form matrix is also called an upper triangular matrix).

9. **[R]** Find the eigenvalues and eigenvectors of the row-echelon matrix

$$U = \begin{pmatrix} 2 & -4 & 1 & 3 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}.$$

10. **[R]** Find the eigenvalues and eigenvectors of the following matrices.

   a) $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$. b) $B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -4 & -1 \\ 0 & 6 & 3 \end{pmatrix}$.

Problems 8.2 : Eigenvectors, bases, and diagonalisation

11. **[R]** For each of the matrices in Questions 7, 9 and 10, decide if the matrix is diagonalisable, and if it is find an invertible matrix M and a diagonal matrix D such that $D = M^{-1}AM$.

12. **[H]** Show that if λ is an eigenvalue of A then λ is also an eigenvalue of the matrix $A' = B^{-1}AB$, where B is any invertible matrix. Also show that if $\mathbf{v}$ is an eigenvector of A for eigenvalue λ then $B^{-1}\mathbf{v}$ is an eigenvector of A' for eigenvalue λ.

13. [**H**] Show that if λ is an eigenvalue of A then λ is also an eigenvalue of A^T.

    HINT: Use the characteristic equation and the properties of determinants.

14. [**X**] Let A be an $n \times n$ matrix. Let $T_A : \mathbb{C}^n \to \mathbb{C}^n$ be the linear transformation defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \quad \text{for} \quad \mathbf{x} \in \mathbb{C}^n.$$

    Let the columns of an $n \times n$ matrix B be an ordered basis for $\mathbb{C}^n$. Show that the matrix representing T_A with respect to the basis formed by the columns of B is $B^{-1}AB$.

    HINT. The method used in Example 3 of Section 7.6 might be helpful.

    NOTE. Modern methods of finding eigenvalues search for a change of basis which makes $A' = B^{-1}AB$ into an upper triangular matrix. As shown in question 4 the eigenvalues are then the diagonal elements of the upper triangular matrix. The actual algorithms for finding the change of basis are complicated.

15. [**X**] Let $T : V \to V$ be linear. Show that if B is any basis for V and A is the matrix representing T with respect to the basis B in both domain and codomain of T then the eigenvalues of T and A are the same. What is the relation between the eigenvectors of T and A?

Problems 8.3 : Applications of eigenvalues and eigenvectors

16. [**R**] Let $A = \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}$. Diagonalise A and hence find A^5.

17. [**R**] Let $A = \begin{pmatrix} 0 & 3 \\ 8 & 2 \end{pmatrix}$.

    a) Find the eigenvalues and eigenvectors of A.

    b) Find matrices P and D such that

$$A = PDP^{-1}.$$

    c) Write down an expression for A^n in terms of P and D. Hence evaluate A^nP.

18. [**R**] A first-order linear difference equation (often called a first-order linear recurrence relation) is an equation of the form

$$\mathbf{x}_{k+1} = A\mathbf{x}_k, \qquad \text{where} \quad k = 0, 1, 2, \ldots,$$

    and where A is a fixed matrix.

    The solution of this equation is

$$\mathbf{x}_k = A^k\mathbf{x}_0,$$

    as you can check by direct substitution. For the diagonalisable matrices of questions 7 a,c, find A^k and hence evaluate $\mathbf{x}_k$.

19. **[R]** For each of the diagonalisable matrices of questions 5, 9 and 10 find general solutions of the differential equations
$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}.$$

20. **[R]** a) Find the eigenvalues and eigenvectors of $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$.

b) Hence solve the system of differential equations:
$$\begin{cases} \dfrac{dx_1}{dt} &= 2x_1 + 3x_2, \\ \dfrac{dx_2}{dt} &= x_1 + 4x_2. \end{cases}$$

21. **[R]** Solve the following systems of differential equations, given that $x(0) = y(0) = 100$.

a) $\begin{cases} \dfrac{dx}{dt} = 5x - 8y \\ \dfrac{dy}{dt} = x - y \end{cases}$ b) $\begin{cases} \dfrac{dx}{dt} = 3x - 15y \\ \dfrac{dy}{dt} = x - 5y \end{cases}$

22. **[R]** Solve the following second-order linear differential equations with constant coefficients by the "calculus method" and by the matrix method and compare your answers.

a) $5\dfrac{d^2y}{dt^2} - 6\dfrac{dy}{dt} + y = 0.$

b) $\dfrac{d^2y}{dt^2} - 16y = 0.$

23. **[R]** What happens if you try to solve the second-order equation
$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0$$
by the matrix method?

24. **[X]** Consider the second-order linear differential equation
$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0,$$
where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

a) Assume that the solutions to the characteristic equation $a\lambda^2 + b\lambda + c = 0$ for this second-order differential equation are distinct. By making the substitutions $y_1 = y$ and $y_2 = \dfrac{dy_1}{dt}$, convert the differential equation into a system of first-order linear differential equations
$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}, \qquad \text{where} \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

b) Using matrix methods, show that the general solution of this system is

$$\mathbf{y} = \alpha_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + \alpha_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}.$$

Compare this solution with that obtained using the usual "calculus method" of solving the original second-order linear differential equation.

25. [**X**] A radioactive isotope A decays at the rate of 2% per century into a second radioactive isotope B, which in turn decays at a rate of 1% per century into a stable isotope C.

    a) Find a system of linear differential equations to describe the decay process. If we start with pure A, what are the proportions of A, B, and C after 500 years, after 1000 years, and after 1000000 years?

       First solve this problem using matrix methods, and then try to solve the problem directly by solving the original two differential equations in the right order.

    b) Explain how the problem would be different if the rates of decay of A and B were both 2% per century.

26. [**X**] There are 3 mathematics lecturers A, B and C who are teaching parallel streams in algebra to a total of 900 students. At the first lecture equal numbers go to each lecture group. After each lecture a certain percentage of the students in each group decide to stay with the same lecturer while the remaining percentage divide evenly among the other two lecturers for the next lecture. If 98% of A's students stay with A each time, 96% of B's students stay with B and 94% of C's students stay with C, find the numbers of students in each group in the 12th lecture and in the 24th lecture. Make the assumption that no students stop attending lectures.

    HINT. Set up a model as a difference equation of the type given in Question 18. You may use MAPLE to find all eigenvalues and eigenvectors. Alternatively, if you wish to solve the problem by hand calculations, you will need to know that one of the eigenvalues is 1.

    NOTE. This problem is an example of a Markov chain process. Markov chain processes are important in many areas of mathematics and its applications, such as statistics, psychology, finance, economics, operations research, queueing theory, inventory theory, diffusion processes, theory of epidemics etc.

27. [**X**] Repeat the previous question on the following assumptions. After each lecture, 1% of each group stop attending lectures altogether, and the remaining percentage either stay with the same lecturer or divide equally among the other two lecturers for the next lecture. If 97% of A's students stay with A each time, 95% of B's students stay with B and 93% of C's students stay with C, find the numbers of students in each group in the 12th lecture and in the 24th lecture. Also find the total number of students attending lectures in the 12th lecture and in the 24th lecture.

28. [**X**] Consider a modified version of the population dynamics model of Example 9 of Section 7.5, in which all females are assumed to die at age 74 instead of at age 89, as in the model given. Use eigenvalues and eigenvectors to solve this modified model, given that there are

one million females in each age group at January 1, 1970. What happens to the population for large values of k?

NOTE. You will need to use Maple to find the eigenvalues and eigenvectors of the matrix A, and you may also use Maple, if you wish, to carry out all other matrix manipulations required to solve the problem.

29. [**X**] Let A be a 2×2 matrix with the property that all its entries are non-negative and both its columns sum to 1. Show that $\lambda_1 = 1$ is always an eigenvalue for A, and that if λ_2 is another eigenvalue of A then $-1 \leq \lambda_2 \leqslant 1$.

Problems 8.4 : Eigenvalues and MAPLE

30. [**M**] Show that the matrix of the original population dynamics model of Example 9 of Section 7.5 is not diagonalisable.

HINT. Use Maple to find the eigenvalues, and then show that the eigenvalue $\lambda = 0$ has multiplicity 2 and that $\dim(\ker(A)) = 1$, i.e., a basis for the kernel of $A - 0I$ consists of one vector.

NOTE. The original population dynamics model can be solved by a generalisation of the eigenvalue-eigenvector methods which makes use of "Jordan forms", and is covered in our second year linear algebra subjects.

31. [**M**] Using the following Maple session,

```
> with(LinearAlgebra):
> M:=<<6|2|2>,<-2|8|4>,<0|1|7>>;
```

$$M := \begin{bmatrix} 6 & 2 & 2 \\ -2 & 8 & 4 \\ 0 & 1 & 7 \end{bmatrix}$$

```
> I3:=IdentityMatrix(3);
```

$$I3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
> p:=Determinant(M-t*I3);
```

$$p := 336 - 146\,t + 21\,t^2 - t^3$$

```
> solve(p,t);
```

$$6, 7, 8$$

```
> NullSpace(M-6*I3);
```

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

```
> NullSpace(M-7*I3);
```

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

```
> NullSpace(M-8*I3);
```

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

a) state the eigenvalues and corresponding eigenvectors for the matrix M,

b) find a matrix A such that $A^{-1}MA$ is a diagonal matrix D and write down D,

c) calculate A^{-1} and hence find an explicit formula for M^k where k is a positive integer.

Problems for Chapter 9

Problems 9.1 : Some Preliminary Set Theory

1. [**R**] Let $A = \{a, c, d, e\}$ and $B = \{d, e, f\}$.
   Suppose that the universal set is $S = \{a, b, c, d, e, f\}$.
   Write down the following sets.

   a) $A - B$, b) $B - A$, c) $A \cup A^c$, d) $B \cap B^c$,
   e) $A \cup B^c$, f) $A^c \cap B$, g) $(A \cup B)^c$, h) $A^c \cap B^c$.

2. [**R**] A survey was carried out in a new development area to gain data on home-delivered newspapers. 110 homes were selected at random and the occupants asked whether they had the daily paper or the weekend paper home delivered. 74 received the daily paper, 58 received the weekend paper and 10 received no paper at all. How many homes visited in this survey received both the daily and weekend papers.

3. [**R**] Of the students taking PHYS1121 and MATH1131 in a hypothetical year, 90% passed MATH1131, 85% passed PHSY1121 and 6% passed neither. What percentage passed both? What percentage of those who passed PHYS1121 also passed MATH1131?

4. [**R**] A brewery brews one type of beer which is marketed under three different brands. In a survey of 150 first year students, 58 drink at least brand A, 49 drink at least brand B and 57 drink at least brand C. 14 drink brand A and brand C, 13 drink brand A and brand B and 17 drink both brand B and brand C. 4 students drink all three brands. How many students drink none of these three brands?

5. [**R**] Suppose A, B and C represent three events. Using unions, intersections and complements, find expressions representing the events

   a) only A occurs,
   b) at least one event occurs,
   c) at least two events occur,
   d) exactly one event occurs,
   e) exactly two events occur.

Problems 9.2 : Probability

6. [**R**] Two fair dice are thrown.

   a) What is the probability that the sum of the two numbers obtained is 6?
   b) What is the probability that both dice show the same number?
   c) What is the probability that at least one of the dice shows an even number?

7. [**R**] Suppose that 30% of computer users use a Macintosh, 50% use a Microsoft Windows PC and that 20% use Linux. Also suppose that 60% of the Macintosh users have succumbed to a computer virus, 80% of the Windows PC users get the virus and 10% of the Linux users get the virus. A computer user is selected at random and it is found that her computer was infected with the virus. What is the probability that she is a Windows PC user?

8. [**R**] Employment data at a large company reveal that 72% of the workers are married, that 44% are university graduates and that half of the university graduates are married. What is the probability that a randomly chosen worker

   a) is neither married nor a university graduate?
   b) is married but not a university graduate?
   c) is married or is a university graduate?

9. [**R**] On the basis of the health records of a particular group of people, an insurance company accepted 60% of the group for a 10 year life policy. Ten years later it examined the survival rates for the whole group and found that 80% of those accepted for the policy had survived the 10 years, while 50% of those rejected had survived the 10 years. What percentage of the group did not survive 10 years? If a person did survive 10 years, what is the probability that they had been refused cover?

10. [**R**] Urn 1 contains 2 red balls and 3 black balls. Urn 2 contains 4 red balls and 5 black balls.

    a) If an urn is randomly selected and a ball drawn at random from it, what is the probability that the ball is red?
    b) Suppose a ball is drawn at random from Urn 1 and placed into Urn 2. If a ball is then drawn at random from the 10 balls in Urn 2, what is the probability that it is red?
    c) In the previous part, given that the ball drawn from Urn 2 is red, what is the probability that the ball transferred from Urn 1 was black?

11. [**R**] Down's syndrome is a disorder that affects 1 in 270 babies born to mothers aged 35 or over. A new blood test for the condition has a sensitivity (*i.e.* the probability of a positive test result given the Down's syndrome is present) of 89%. The specificity (*i.e.* the probability of a negative test result given that Down's syndrome is absent) of the new test is 75%.

    a) What proportion of women over age 35 would test positive on this new blood test?
    b) A mother over age 35 receives a positive test result. What is the chance that Down's syndrome is actually present?
    c) A mother over age 35 receives a negative test result. What is the chance that Down's syndrome is actually present?

12. [**R**] The following is a table of the annual promotion probabilities at a particular workplace, broken down by gender.

	Promoted	Not promoted	Total
Male	0.17	0.68	0.85
Female	0.03	0.12	0.15

Is there gender bias in promotion?

13. [**R**] A system has n independent components and each fail with probability p. Calculate the probability that the system will fail when

    a) the components are in parallel, so the system fails only when all of the components fail;

    b) the components are in series, so the system fails if any one of the components fail;

14. [**X**] Tom and Bob play a game by each tossing a fair coin. The game consists of tossing the two coins together, until for the first time either two heads appear when Tom wins, or two tails appear when Bob wins.

    a) Show that the probability that Tom wins at or before the n^{th} toss is $\dfrac{1}{2} - \dfrac{1}{2^{n+1}}$.

    b) Show that the probability that the game is decided at or before the n^{th} toss is $1 - \dfrac{1}{2^n}$.

15. [**X**] Extend the Multiplication Rule of section 9.2.3 to 3 events A_1, A_2, A_3 and show that

$$P(A_1 \cap A_2 \cap A_3) \quad = \quad P(A_3|A_1 \cap A_2)P(A_2|A_1)P(A_1).$$

    The same pattern applies to higher numbers of events. Write this down.

    This law is particularly useful when we have a sequence of dependent trials. To gain entry to a selective high school students must pass 3 tests. 20% fail the first test and are excluded. Of the 80% who pass the first, 30% fail the second and are excluded. Of those who pass the second, 60% pass the third.

    What proportion of students pass the first two tests? Use the multiplicative law to answer this question.

    What proportion of students gain entry to the selective high school?

    What proportion pass the first two tests, but fail the third?

16. [**X**] Use the additive law of probability to establish, using mathematical induction, Boole's Law:

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) \leq P(A_1) + P(A_2) + \cdots + P(A_n)$$

17. [**X**] Establish, using mathematical induction, Bonferoni's inequality:

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) \geq 1 - [P(A_1^c) + P(A_2^c) + \cdots + P(A_n^c)]$$

Problems 9.3 : Random Variables

18. [**R**] Show that each of the following sequences p_k satisfies $p_k \geq 0$ and $\sum_{k=0}^{\infty} p_k = 1$. Note that in the distributions below $p_k = P(X = k)$ where X is the random variable under consideration.

    a) **Uniform Distribution**.

    $$p_k = \frac{1}{n} \qquad \text{for} \quad 1 \leq k \leq n$$

    and 0 otherwise. Here, n is a fixed positive integer.

    b) **Binomial Distribution.**

    $$p_k = B(n, p, k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \text{for} \quad 0 \leq k \leq n,$$

    and 0 otherwise. Here, p is a constant with $0 < p < 1$.

    c) **Geometric Distribution.**

    $$p_k = G(p, k) = (1-p)^{k-1} p \qquad \text{for} \quad 1 \leq k < \infty,$$

    where p is a constant with $0 < p < 1$.

    d) **Poisson Distribution.**

    $$p_k = e^{-\lambda} \frac{\lambda^k}{k!} \qquad \text{for} \quad 0 \leq k < \infty,$$

    where $\lambda > 0$ is a constant. To solve this question you will need the Maclaurin series for e^{λ}.

19. [**R**] A box contains four red and two black balls. Two balls are drawn from the box. Let X be the number of red balls obtained. Find the probability distribution for X.

20. [**R**] A busy switchboard receives 150 calls an hour on average. Assume that the probability, p_k, of getting k calls in a given minute is

    $$p_k = e^{-\lambda} \frac{\lambda^k}{k!}$$

    where λ the average number of calls per minute. (This is called a *Poisson* distribution.)

    a) Find the probability of getting exactly 3 calls in a given minute.
    b) Find the probability of getting at least 2 calls in a given minute.

21. [**R**] In a biased lottery with tickets numbered 1 to 50, the probability that ticket number n wins is

    $$p_n = \frac{n}{1275} \quad \text{for } n = 1, 2, 3, \ldots, 50.$$

    What is the probability that the winning ticket bears a number less than or equal to 25?

22. [**H**] Let X be a random variable with probability distribution

$$P(X = k) = \frac{c}{k!}, \quad \text{for } k = 0, 1, 2, \ldots.$$

a) Determine the value of c.

b) Calculate $P(X = 2)$.

c) Calculate $P(X < 2)$.

d) Calculate $P(X \geqslant 4)$.

23. [**X**] In a biochemical experiment, n organisms are placed in a nutrient medium, and the number of organisms X which survive for a given period is recorded. The probability distribution of X is assumed to be given by

$$P(X = k) = \frac{2(k+1)}{(n+1)(n+2)} \qquad \text{for} \qquad 0 \le k \le n,$$

and 0 otherwise.

a) Check that $\sum_{k=0}^{n} P(X = k) = 1$.

b) Calculate the probability that at most a proportion α of the organisms survive, and deduce that for large n this is approximately α^2.

c) Find the smallest value of n for which the probability of there being at least one survivor among the n organisms is at least 0.95.

24. [**H**] A genetic experiment on cell division can give rise to at most $2n$ cells. The probability distribution of the number of cells X recorded is

$$P(X = k) = \frac{\theta^k (1-\theta)}{1 - \theta^{2n+1}} \qquad \text{for} \qquad 0 \le k \le 2n,$$

where θ is a constant with $0 < \theta < 1$.

What are the probabilities that

a) an odd number of cells is recorded,

b) at most n cells are recorded?

25. [**R**] Let X be a discrete random variable with the following probability distribution

k:	0	1	2	3	4
$P(X = k) = p_k$:	0.1	$2c$	0.2	0.1	$4c$

a) Find the value of c.

b) Calculate $E(X)$ and $\text{Var}(X)$.

c) Let $Y = 1 - 4X$. Calculate $E(Y)$ and $\text{Var}(Y)$.

26. **[R]** Find the mean and variance for the uniform distribution in Question 18(a).

27. **[X]** Let X be a random variable with the Poisson probability distribution given in Question 18(d). Find $E\big((1+X)^{-1}\big)$.

28. **[X]** Assuming that one can differentiate a power series term by term, one obtains from the formula

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x| < 1$$

the formulas

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}; \quad \sum_{k=2}^{\infty} k(k-1)x^{k-2} = \frac{2}{(1-x)^3}, \quad |x| < 1.$$

(You will see that this is justified in your MATH1231/41 Calculus lectures).
From these formulas, show that

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}; \quad \sum_{k=0}^{\infty} k^2x^k = \frac{x(x+1)}{(1-x)^3}, \quad |x| < 1.$$

and hence calculate the mean and the variance for geometric distribution in Question 18(c).

Problems 9.4 : Special Distributions

29. **[R]** A coin is tossed 50 times. What is the probability of it coming down heads exactly 25 times?

30. **[R]** A test paper contains 8 multiple choice questions, each with 4 potential answers to choose from. A correct answers gains 1 mark, a wrong answer 0 marks and 4 is the pass mark. If a student simply guesses, what is probability that she will pass?

31. **[R]** The probability of dying from a particular disease is 0.3. 10 people in a hospital are suffering from the disease. Find the probability that at least 8 survive.

32. **[R]** How many times must a coin be tossed until the probability of getting 2 or more heads exceeds 0.99? (You need to try different n values after an initial guess.)

33. **[X]** For the $B(n,p)$ distribution, by considering $\dfrac{p_k}{p_{k-1}}$, show that p_k is largest when $k = \lfloor (n+1)p \rfloor$. This k is called the "mode" of the distribution.

34. **[R]** Consider the game of "rock, scissors, paper" in which two players instantaneously choose one of "rock", "scissors" or "paper". If both players pick the same item, they play again; if the two players make different choices one of them wins (rock beats scissors, scissors beats paper and paper beats rock).

    Let X be the number of times the game is played until someone wins. Find the probability distribution for X when each player chooses randomly from rock, scissors or paper.

35. [**H**] A die is rolled repeatedly.

    a) Find the probability that the third time a '6' shows is on the 20th roll.

    b) [**X**]Generalise to the kth time on the nth roll. This is an example of the "negative binomial" distribution, which generalises the geometric distribution.

36. [**X**] a) Show that for non-negative integers k, m, n,

$$\binom{k}{m}\binom{n}{k} = \binom{n}{m}\binom{n-m}{k-m}$$

    b) Show that

$$\sum_{k=m}^{n}\binom{k}{m}\binom{n}{k}\, p^k\,(1-p)^{n-k} = \binom{n}{m}p^m.$$

    c) By considering the cases $m = 1$ and $m = 2$ in the preceding formula, prove the variance formula for the Binomial distribution, as stated in Theorem 2 of Section 9.4.

37. [**R**] A certain type of car is known to sustain damage 25% of the time in 15 km/hr crash tests. A modified bumper design has been proposed in an effort to decrease this percentage. In a trial, cars with the new type of bumper were damaged in one of 15 test crashed.

    a) What distribution could be used to model the number cars damaged in trials, if the new bumpers perform no better than the old ones?

    b) Calculate a tail probability that measures how unusual it would be to observe as few as one damaged car, under the assumptions of part (a).

    c) Does the trial indicate that the new bumpers protect cars from damage better than the old bumpers?

38. [**R**] An extensive study was conducted in the northern hemisphere of butterfly distributions, comparing where butterfly species are presently found with where they were found a century ago (Parmesan *et al* 1999, Nature 399, 579-583). It was hypothesised that due to climate change, more species would have shifted northwards in distribution than shifted southwards.

    It was found that of the 23 butterfly species whose distribution had shifted, 22 had shifted northwards in distribution, and 1 had shifted southwards in distribution.

    a) What distribution could be used to model the number of butterfly species moving northward, if they are just as likely to move north as south (that is, if there is no influence of climate change on butterfly species)?

    b) Calculate a tail probability that measures how unusual it would be to observe as many as 22 butterfly species moving northwards, under the assumptions of part (a).

    c) Do these data provide evidence that climate change has affected the distribution of butterfly species?

39. [**R**] Sydney's dam levels recently reached historic lows, which may in part be due to lower than average rainfall over recent years. The following data are total annual rainfall in Sydney for eight recent calendar years.

Year	2000	2001	2002	2003	2004	2005	2006	2007
Annual rainfall	812.6	1359.0	860.0	1207.6	995.2	816.0	994.0	1499.2

Historic rainfall levels prior to the year 2000 was 1302.2mm per year. We will use a sign test approach to see how much evidence there is that the recent Sydney rainfall is less than historic levels (that is, less than 1302.2mm).

It is reasonable to assume that annual rainfall is independent across years.

a) In how many years was the annual rainfall less than 1302.2mm?
b) What distribution could be used to model the number of years in which rainfall was less than 1302.2mm rather than being greater than this value, if both outcomes were equally likely?
c) Calculate a tail probability that measures how unusual it would be to observe as many years with a total rainfall less than 1302.2mm as was observed, if annual rainfall was just as likely to be greater than 1302.2 as less than 1302.2.
d) Do you think there is evidence that Sydney rainfall has decreased in recent years? (That is, that the values are systematically smaller than 1302.2mm?)

40. [**R**] Do ravens intentionally fly towards gunshot sounds (to scavenge on the carcass they expect to find)? Crow White addressed this question by counting raven numbers at a location, firing a gun, then counting raven numbers 10 minutes later (Ecology 2005, 86:1057-1060). He did this in 12 locations. Results:

location	1	2	3	4	5	6	7	8	9	10	11	12
before	1	0	1	0	0	0	0	5	1	1	2	0
after	2	3	2	2	1	1	2	2	4	2	0	3

We would like to find out if there is evidence that ravens fly towards the location of gunshots.

a) In how many locations was there an increase in number of ravens, after the gunshot?
b) In how many locations was there a change in number of ravens after the gunshot?
c) What distribution could be used to model the number of locations in which there was an increase in the number of ravens rather than a decrease, if both outcomes were equally likely?
d) Calculate a tail probability that measures how unusual it would be to observe as many locations with an increase in number of ravens as was observed, if increases and decreases were equally likely.
e) Do you think there is evidence that the ravens fly towards gunshot sounds? (That is, was there a systematic increase in the number of ravens present after the gunshot sound?)

Problems 9.5

41. [**R**] Verify that the following functions f are probability densities, that is, that

$$f(x) \geqslant 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Also sketch the graph of each function.

a) **Uniform Distribution (Assume $a < b$).**

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for} \quad a \leqslant x \leqslant b \\ 0 & \text{otherwise.} \end{cases}$$

b) **Pareto Distribution**. For $k > 0$,

$$f(x) = \begin{cases} \dfrac{k}{x^{k+1}} & \text{for} \quad x \geqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

c) **Gamma Distribution** f_n. For $n > 0$,

$$f_n(x) = \begin{cases} \dfrac{1}{n!}x^n e^{-x} & \text{for} \quad x \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

NOTE. The formula $\displaystyle\int_0^{\infty} x^n e^{-x}\,dx = n!$ will be useful. You can prove it using integration by parts.

d) **Laplace Distribution.**

$$f(x) = \frac{1}{2}e^{-|x|} \qquad \text{for} \quad -\infty < x < \infty.$$

42. [**R**] Calculate the means and variances for the distributions in the preceding question. (Note that for the Pareto distribution the mean is only defined if $k > 1$ and the variance is only defined if $k > 2$.)

43. [**R**] The probability density of the **Cauchy Distribution** is given by

$$f(x) = \frac{\alpha}{1+x^2} \qquad \text{for} \quad -\infty < x < \infty.$$

a) Find the value of α.

b) If X has a Cauchy distribution, find a number c such that

$$P(X \leqslant c) = 0.25.$$

c) [**H**]What can be said about $E(X)$ and $\mathrm{Var}(X)$ for the Cauchy distribution?

44. [**R**] X has the probability density function

$$f(x) = \begin{cases} \frac{1}{8}x & \text{for} \quad 3 \leqslant x \leqslant 5 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate

a) $P(X \geqslant 4)$, b) $P(X \leqslant 4)$, c) $P(3.2 \leqslant X \leqslant 4.1)$, d) $E(X)$.

45. [**R**] Y has the probability density function

$$f(y) = \begin{cases} \dfrac{c}{y} & \text{for} \quad 10 \leqslant y \leqslant 100 \\ 0 & \text{otherwise.} \end{cases}$$

a) Determine the value of the constant c.

b) Obtain the cumulative distribution function of Y.

c) Find b such that $P(Y \leqslant b) = 0.50$.

d) Find the expected value of $E(Y)$.

46. [**R**] Let F be the function defined by

$$F(x) = \begin{cases} 0 & \text{for} \quad x < 2 \\ \frac{1}{4}(x-2) & \text{for} \quad 2 \leqslant x < 3 \\ \frac{1}{4} + \frac{3}{8}(x-3) & \text{for} \quad 3 \leqslant x < 5 \\ 1 & \text{for} \quad x \geqslant 5. \end{cases}$$

a) Sketch the graph of F.

b) Find a probability density function f which would have F as its cumulative distribution function. Sketch the graph of f.

c) Find $E(X)$ for this probability density function.

47. [**H**] Suppose X has a probability density given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & \text{for} \quad x \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the mean and variance of $Y = 2X + 3$.

Problems 9.6

48. [**R**] Suppose Z is the standard normal random variable. Use the standard normal tables to write down

    a) $P(Z \leqslant 1.23)$
    b) $P(Z \leqslant -2.3)$
    c) $P(Z \geqslant 0.36)$
    d) $P(Z \geqslant -1.24)$
    e) $P(1 \leqslant Z \leqslant 2)$
    f) $P(-0.5 \leqslant Z \leqslant 0.5)$

49. [**R**] Suppose X is normally distributed with mean μ and standard deviation σ. Use the standard normal tables to find:

    a) $P(X \leqslant 12)$ with $\mu = 10, \sigma = 2$
    b) $P(X \geqslant 53)$ with $\mu = 50, \sigma = 4.5$
    c) $P(X \leqslant 47)$ with $\mu = 50, \sigma = 4.5$
    d) $P(X \geqslant 32)$ with $\mu = 36, \sigma = 8$
    e) $P(21 \leqslant X \leqslant 24)$ with $\mu = 20, \sigma = 3$
    f) $P(15 \leqslant X \leqslant 20)$ with $\mu = 18, \sigma = 1.5$

50. [**R**] Suppose X is normally distributed with mean μ and standard deviation σ. Use the standard normal tables to find the value of c such that:

    a) $P(X \leqslant c) = 0.8238$ with $\mu = 0, \sigma = 1$
    b) $P(X \geqslant c) = 0.0495$ with $\mu = 0, \sigma = 1$
    c) $P(X \leqslant c) = 0.2514$ with $\mu = 50, \sigma = 4.5$
    d) $P(X \geqslant c) = 0.6915$ with $\mu = 36, \sigma = 8$

51. [**R**] The mean heights of men in a certain country is estimated to be 1.69m with a standard deviation of 0.06m. We assume that the heights are normally distributed.

    a) Find the approximate probability that a man chosen at random from this country has a height between 1.60m and 1.74m.
    b) If 400 men are chosen from this country, how many would you expect on average to have heights greater than 1.74?

52. [**R**] The length of life of a particular make of T.V. is approximately normally distributed, with a mean of 3.1 years and a standard deviation of 1.2 years. If this type of T.V. is guaranteed for one year, what fraction of the T.V.s sold will require replacement under the guarantee?

53. [**R**] Experience has shown that the I.Q. scores of University students are normally distributed with a mean of 112 and a standard deviation of 8. Calculate the percentage of students who will have an I.Q. score

    a) higher than 130 b) lower than 100 c) between 105 and 125.

54. [**R**] The lengths of studs turned out by a certain automatic machine are normally distributed with a mean of 3.220 cm and a standard deviation of 0.003cm. If the acceptable length of a stud is between 3.226 and 3.212 cm, determine to one decimal place the percentage rejected as under size and over size respectively.

55. [**R**] An unbiased die is tossed 600 times. Use the normal approximation to the binomial to find the approximate probability that a 6 appears more than 120 times.

56. [**R**] Olof Jonsson was a controversial psychic whose psychic abilities were tested in an experiment. In the experiment a computer showed 4 cards to the subject and (randomly) picked one of them, and the subject was to guess which card they thought the computer had picked. This process was repeated 288 times, and Olof managed to pick the right card 88 times.

    a) What distribution could be used to model the number of cards Olof correctly selects, if he is not psychic (and so can do no better than random guessing)?
    b) Write down an expression for a tail probability that measures how unusual it is to correctly pick as many cards as Olof did.
    c) Use the normal approximation to the binomial to find this probability, under the assumption that Olof is not psychic.
    d) Is this evidence that Olof was psychic?

57. [**X**] If X is a normal random variable with mean μ and variance σ^2, show that

$$E(|X-\mu|) = \sqrt{\frac{2}{\pi}}\,\sigma.$$

58. [**X**] Find $E(X)$ and $\mathrm{Var}(X)$ for the random variable X with probability density function proportional to

$$e^{-x^2+x}.$$

59. [**X**] Let T be a continuous random variable and $T \sim \mathrm{Exp}(\lambda)$; that is T has an exponential distribution with parameter λ. Prove that

$$\mathrm{Var}(T) = \frac{1}{\lambda^2}.$$

60. [**X**] Suppose a continuous random variable T has an exponential distribution with parameter λ and

$$P(T \leqslant t) = p, \qquad 0 < p < 1.$$

a) Find t in terms of p.

b) Hence find the median m. In other words, find m such that $P(T \leqslant m) = 0.5$.

61. [**X**] (Memoryless property) A continuous random variable T has an exponential distribution and $T \sim \text{Exp}(\lambda)$. Prove that

$$P(T \geqslant t + t_0 | T \geqslant t_0),$$

is independent of t_0 for positive t and t_0.

62. [**X**] Suppose the time, in minutes, required by any particular student to complete a certain two-hour examination has the exponential distribution for which the mean is 90 minutes. The examination starts at 10:00 am.

a) What is the probability that a student completes the examination before 11:00 am?

b) What is the probability that a student completes the examination between 11:00 am and 11:30 am?

c) What is the probability that a student could not complete the examination within 2 hours?

d) What is the median time for completing the examination?

63. [**X**] During a whale watch season in Sydney, the time, measured in hours from the moment that a cruise enters the whale watch area, to spot the first whale can be modelled by the exponential distribution with parameter $\lambda = 0.4$. A person joins a Sydney whale watch tour during the season. After entering the area,

a) what is the probability that no whale can be spotted in the first hour?

b) what is the probability that the time to spot the first whale exceeds the mean by one standard deviation?

c) The organiser claims a 90 % success rate of finding whales in a trip. How long should the cruise stay in the whale watching area to achieve that?

64. [**X**] A system consists of 3 independent components connected in series. Hence the system fails when at least one of the components fails. Suppose the lengths of life of the components, measured in hours, have the exponential distribution Exp(0.001), Exp(0.003), Exp(0.004). Find the probability that the system can last at least 100 hours.

65. [**X**] A system consists of n independent components connected in series. The lengths of life of the components, in hours, have the exponential distributions $\text{Exp}(\lambda_i)$, $1 \leqslant i \leqslant n$. Let T be the random variable that gives the time until the system fails.

a) Find $P(T \leqslant t)$ and hence write down the cumulative distribution function of the random variable T.

b) Find the probability density function of T.

c) Name the probability distribution of T, and find the expected value and variance of T.