MATH1241 – Algebra
Lecture 5 – Linear Dependence/Independence

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Linear dependence/independence

Recall that, for vectors $\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n$ in a vector space V over a field $\mathbb{F}$, we say that $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ if there are scalars $\mu_1, \ldots, \mu_n \in \mathbb{F}$ such that

$$\mathbf{v} = \mu_1 \mathbf{v}_1 + \cdots + \mu_n \mathbf{v}_n.$$

(Informal) Definition for linear dependence/independence

Let S be a subset of vectors in V.

1. If a vector $\mathbf{v} \in S$ is a linear combination of some other vectors in S, we will say then that S is a linearly dependent set.
   Thus, if S is a spanning set of V, then, for any vector $\mathbf{v} \in V - S$, $S \cup \{\mathbf{v}\}$ is linearly dependent. Also, if $\mathbf{0} \in S$, then S is dependent.
2. If no vector in S is a linear combination of other vectors in S, then S is called a linearly independent set.

Testing Linear Dependence/Independence

If S has only two non-zero vectors, the question of independence is easy.

Example

(1) $S_1 = \left\{ \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is dependent because $\begin{pmatrix} 2 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

However $S_2 = \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is independent because the vectors are not scalar multiples of each others.

(2) Consider the set of vectors $S_3 = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 7 \\ 1 \end{pmatrix} \right\}$.

No vector in S_3 is a scalar multiple of another vector in S_3, but

$$\begin{pmatrix} -1 \\ 7 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

The set S_3 is linearly dependent.

An example in $\mathcal{R}(\mathbb{R})$

Recall the vector space $\mathcal{R}(\mathbb{R})$ of all real valued functions.

Example

Is the subset $\{f, g, h\}$ in $\mathcal{R}(\mathbb{R})$, where

$$f(x) = 2, \quad g(x) = \sin^2(x), \quad h(x) = \cos^2(x), \quad \forall x \in \mathbb{R}$$

a linearly dependent set?

Solution

Recall the trigonometric identity

$$1 = \sin^2(x) + \cos^2(x), \quad \forall x \in \mathbb{R}.$$

Hence, $f = 2g + 2h$ *and the set is linearly dependent.*

Definition for Linear Independence/Dependence

The first definition is not very convenient sometimes. For example, to check that none of the vectors in a set can be written as a linear combination of the other vectors in the set, we need to check that every single vector is not a linear combination of the others.

(Formal) Definition for linear dependence/independence

Let S be a subset of a vector space V.

1. If there exist vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in S$ $(n \geqslant 1)$ and **not all zero** scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ such that
$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n = \mathbf{0},$$
then we say that S is a **linearly dependent set** or that the vectors in S are **linearly dependent**.
2. If, for any vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m \in S$ $(m \geqslant 1)$, the **only** solution of
$$x_1 \mathbf{u}_1 + x_2 \mathbf{u}_2 + \cdots + x_m \mathbf{u}_m = \mathbf{0}, \quad (x_i \in \mathbb{F})$$
is $x_1 = x_2 = \cdots = x_m = 0$, then we say that S is a **linearly independent set** or that the vectors in S are **linearly independent**.

Test for Linear Independence in $\mathbb{F}^m$

Example

Show that $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is a linearly independent set.
Can we remove any vector from S and the resulting set still span the same set as span(S)?

Solution

We need to show that the vector equation

$$x_1 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has only the trivial solution $x_1 = x_2 = x_3 = 0$.

Solution (Continued)

Working on the augmented matrix (omitting the last zero column):

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & 3 & -1 \end{pmatrix} \xrightarrow[R_3=R_3-3R_1]{R_2=R_2-R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -4 \end{pmatrix}$$

All columns are leading. So the system $A\mathbf{x} = \mathbf{0}$ has only the zero solution. Hence, S is linearly independent.

Since S is linearly independent, no vector in S can be removed without affecting the span of S.

Example

Suppose that

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix} \text{ and } \mathbf{v}_4 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \text{ in } \mathbb{R}^3.$$

a) Prove that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

b) Find linearly independent subsets of S which have the same span as S.

Solution

Since the vector are in $\mathbb{R}^3$, we may immediately apply row elementary operations to the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & -1 \\ 3 & 1 & 8 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & -3 \\ 0 & -5 & 5 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -5 & 5 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

Column 3 is non-leading. So $A\mathbf{x} = \mathbf{0}$ has a nonzero solution. Hence, S is linearly dependent.

By the echelon form, the non-leading column 3 is a linear combination of columns 1 and 2. (Why? Because $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$ has a solution.) So it can be removed from S without affecting span(S). In fact, by a similar argument, we see that

$$\mathit{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathit{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \mathit{span}\{\mathbf{v}_1, \mathbf{v}_3\} = \mathit{span}\{\mathbf{v}_2, \mathbf{v}_3\}$$

Hence, the possible linearly independent subsets of S with the same span are

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}, \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}, \{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}.$$

How to Test Linear Independence in $\mathbb{F}^m$?

Let A be the $m \times n$ matrix whose columns are the vectors in $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset \mathbb{F}^m$, and $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. We have seen that

$A\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$ is a linear combination of vectors in S.

1. S is linearly dependent iff $A\mathbf{x} = \mathbf{0}$ has a **non-zero** solution (i.e., more than 1 solution). So, if $m < n$, then S is linearly dependent.
2. S is linearly independent iff $A\mathbf{x} = \mathbf{0}$ has only zero solution $\mathbf{x} = \mathbf{0}$. (unique solution) So, if S is linearly independent, then $m \geqslant n$.

If A reduces to a row-echelon form U, then

- all columns of U are leading iff S is linearly independent;
- at least one of the columns of U is non-leading iff S is linearly dependent;
- the vectors in S corresponding to the leading columns of U form a linearly independent subset of S with the same span as S.

Interpretation of number of solutions to $A\mathbf{x} = \mathbf{b}$

For $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subset \mathbb{F}^m$ and $\mathbf{b} \in \mathbb{F}^m$, let $A = (\mathbf{v}_1 | \mathbf{v}_2 | \dots | \mathbf{v}_n)$ be the associated $m \times n$ matrix. Then $\text{span}(S) = \mathcal{C}(A)$.

From Math1141, the system of linear equations $A\mathbf{x} = \mathbf{b}$ has either

- no solution (i.e., 0 solution), or
- 1 solution, or
- infinite solutions (i.e., more than 1 solution).

We now have the following interpretation of these numbers:

1. $A\mathbf{x} = \mathbf{b}$ has 0 solution (i.e., no solution) if and only if $\mathbf{b} \notin \text{span}(S)$;
2. $A\mathbf{x} = \mathbf{b}$ has 1 solution if and only if S is linearly independent;
3. $A\mathbf{x} = \mathbf{b}$ has 2 or more solutions if and only if S is linearly dependent.

By the tests above, we may also extract a maximal linearly independent subset from S.

Vector Spaces other than $\mathbb{F}^n$

Example

Prove that $\{1 + 2x - x^2,\ -3 - x - 2x^2,\ 2 + 3x + x^2\}$ is a linearly independent subset of $\mathbb{P}_2$.

Proof.

We need to show that the vector equation

$$x_1(1 + 2x - x^2) + x_2(-3 - x - 2x^2) + x_3(2 + 3x + x^2) = \mathbf{0}$$

has only the solution $x_1 = x_2 = x_3 = 0$. Equating coefficients gives

$$\begin{aligned} x_1 - 3x_2 + 2x_3 &= 0 \\ 2x_1 - x_2 + 3x_3 &= 0 \\ -x_1 - 2x_2 + x_3 &= 0 \end{aligned}$$

Proof (Continued).

Apply row elementary operations to the coefficient matrix:

$$\begin{pmatrix} 1 & -3 & 2 \\ 2 & -1 & 3 \\ -1 & -2 & 1 \end{pmatrix} \xrightarrow[R_3=R_3+R_1]{R_2=R_2-2R_1} \begin{pmatrix} 1 & -3 & 2 \\ 0 & 5 & -1 \\ 0 & -5 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3=R_3+R_2} \begin{pmatrix} 1 & -3 & 2 \\ 0 & 5 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

All columns are leading. So the system has only the solution $x_1 = x_2 = x_3 = 0$. Hence, the set is a linearly independent set.

$\square$

Linear Independence and Span – a Summary

Let S be a finite non-empty set of vectors of a vector space V.

1. *For any $\mathbf{v} \in span(S)$, if S is linearly independent, there is only one way to write $\mathbf{v}$ as a linear combination of vectors in S. On the other hand, if S is dependent, there will be more than one way to write $\mathbf{v}$ as a linear combination of vectors in S.*
2. *S is linearly independent if and only if no vector in S can be written as a linear combination of the other vectors in S.*
3. *For any $\mathbf{v} \in V$, we have $span(S \cup \{\mathbf{v}\}) = span(S)$ if and only if $\mathbf{v} \in span(S)$.*
4. *The span of every proper subset of S is a proper subspace of $span(S)$ if and only if S is linearly independent.*
5. *If S is linearly independent and $\mathbf{v} \in V$ but not in $span(S)$, then $S \cup \{\mathbf{v}\}$ is linearly independent.*

Proof of (1) and (2).

(1) We want to prove that S is an independent set if and only if the following condition holds for all $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in S$: for all $\mathbf{v} \in V$,

$$(*) \qquad \mathbf{v} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i = \sum_{i=1}^{n} \mu_i \mathbf{v}_i \ (\mathbf{v}_i \in S) \implies \lambda_i = \mu_i \, \forall i.$$

Clearly, $\mathbf{v} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i = \sum_{i=1}^{n} \mu_i \mathbf{v}_i$ is equivalent to $\sum_{i=1}^{n} (\lambda_i - \mu_i) \mathbf{v}_i = \mathbf{0}$. So S independent implies $\lambda_i = \mu_i$ for all i. Conversely, if (*) is true, then $\sum_{i=1}^{n} \mu_i \mathbf{v}_i = \mathbf{0} = \sum_{i=1}^{n} 0\mathbf{v}_i$ ($\mathbf{v}_i \in S$) implies $\mu_i = 0$. So the $\mathbf{v}_i$'s are linearly independent.

(2) If one of the vectors in S can be written as a linear combination of the others, then the zero vector is a linear combination of S with some coefficients non-zero. Conversely, if

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n = \mathbf{0},$$

with, say, $\lambda_1 \neq 0$, then $\mathbf{v}_1 = -\frac{1}{\lambda_1}(\lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n)$. □

Example

Prove the statement (5).

Proof.

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a linearly independent set. We want to show that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{v}\}$ is linearly independent.

Suppose there are scalars $\mu_1, \ldots, \mu_n, \mu \in \mathbb{F}$ such that

$$\mu_1\mathbf{v}_1 + \mu_2\mathbf{v}_2 + \cdots + \mu_n\mathbf{v}_n + \mu\mathbf{v} = \mathbf{0}. \qquad (*)$$

Since $\mathbf{v} \notin \operatorname{span}(S)$, we must have $\mu = 0$.
(Otherwise, $\mathbf{v} = -\mu^{-1}(\mu_1\mathbf{v}_1 + \mu_2\mathbf{v}_2 + \cdots + \mu_n\mathbf{v}_n) \in \operatorname{span}(S)$.)
Thus,

$$\mu_1\mathbf{v}_1 + \mu_2\mathbf{v}_2 + \cdots + \mu_n\mathbf{v}_n = \mathbf{0}.$$

But S is linearly independent forces all $\mu_i = 0$. This proves that the only solution to (*) is the zero solution. Hence, $S \cup \{\mathbf{v}\}$ is linearly independent.

Example

For $n \geqslant 0$, let $p_n(x) = 1 + x + \cdots + x^n$. Prove that

$$S = \{p_0, p_1, p_2, \ldots, p_n, \ldots\}$$

is a linearly independent subset of the real polynomial space $\mathbb{P}(\mathbb{R})$.

Proof.

We want to prove that every finite subset F of S is linearly independent. Since F is finite, there exists a positive integer n such that $F \subseteq S_n := \{p_0, p_1, p_2, \ldots, p_n\}$. Thus, it suffices to show that S_n is linearly independent as any subset of a set of linearly independent vectors is linearly independent.

Suppose $\sum_{i=0}^{n} \lambda_i p_i = \mathbf{0}$. Equating the coefficients of x^n gives $\lambda_n = 0$. Thus, $\sum_{i=0}^{n-1} \lambda_i p_i = \mathbf{0}$. So $\lambda_{n-1} = 0$. Since from $\lambda_n = \lambda_{n-1} = ... = \ldots = \lambda_{n-i} = 0$ we see that $\lambda_{n-i-1} = 0$, by induction, $\lambda_n = \lambda_{n-1} = \ldots = \lambda_1 = \lambda_0 = 0$.Hence, S_n is linearly independent and consequently, S is linearly independent.

Recall the dot product: $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Example

(1) Orthogonal sets are linearly independent. Specifically, if $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subset \mathbb{R}^n$ satisfied $\mathbf{v}_i \neq 0$ and $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$, then S is linearly independent.
(2) The subset $T = \{f_1, f_2, \ldots, f_n, \ldots\}$ of $\mathcal{R}(\mathbb{R})$, where $f_m = \sin(mx)$ $(m \in \mathbb{Z}_{>0})$, is linearly independent.

Proof.

(1) Suppose (*) $\sum_{i=1}^n x_i \mathbf{v}_i = \mathbf{0}$ for some $x_i \in \mathbb{R}$. Then

$$\left(\sum_{i=1}^n x_i \mathbf{v}_i \right) \cdot \mathbf{v}_j = \mathbf{0} \cdot \mathbf{v}_j = 0$$

But, the LHS$= \sum_{i=1}^n x_i (\mathbf{v}_i \cdot \mathbf{v}_j) = x_j (\mathbf{v}_j \cdot \mathbf{v}_j)$. Hence, $x_j = \frac{0}{\mathbf{v}_j \cdot \mathbf{v}_j} = 0$, where $1 \leqslant j \leqslant n$. So (*) has only zero solution. Hence, S is independent.
(2) Use the fact that $\int_{-\pi}^{\pi} \sin(nx) \sin(mx) \neq 0$ if and only if $m = n$.
Recall: $\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y))$.