4.4 Infinite Series

A sequence is one number after another: a **series** is the sum of a sequence, possibly to infinity.

Let $\{a_k\}$ be an (infinite) sequence.

We define the nth **partial sum**, s_n by

$$s_n = a_0 + a_1 + \cdots + a_n = \sum_{k=0}^{n} a_k.$$

The partial sums form another sequence, and if the sequence $\{s_n\}$ converges to the limit L we say that the (infinite) series $\sum_{k=0}^{\infty} a_k$ converges to L, or is **summable**

If s_n diverges, we say the series $\sum_{k=0}^{\infty} a_k$ diverges.

Example 17 *Let* $a_n = \dfrac{1}{2^n}$ *and* $s_n = \displaystyle\sum_{k=0}^{n} a_k = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k$

Then

$$
\begin{aligned}
s_n &= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} \\
\frac{1}{2} s_n &= \phantom{1 + {}} \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}
\end{aligned}
$$

so $\quad s_n - \dfrac{1}{2} s_n = 1 - \dfrac{1}{2^{n+1}}$

and $\quad s_n = 2 - \dfrac{1}{2^n}$

Hence $\quad \displaystyle\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(2 - \frac{1}{2^n}\right) = 2$

The previous example is a special case of the important **geometric series**:

$$\sum_{k=1}^{\infty} r^k = 1 + r + r^2 + r^3 + \cdots$$

Exactly the same method of proof will lead to the partial sum formula for this case:

$$s_n = 1 + r + r^2 + r^3 + \cdots + r^n$$

$$r\, s_n = \qquad r + r^2 + r^3 + \cdots + r^n + r^{n+1}$$

so $\quad s_n - r\, s_n = 1 - r^{n+1}$

and $\qquad s_n = \dfrac{1 - r^{n+1}}{1 - r}$

With the partial sum formula

$$1 + r + r^2 + r^3 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

it is clear that

a) If $|r| < 1$ then (as $r^{n+1} \to 0$) the series converges and $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$.

b) If $|r| > 1$ then the series diverges since $r^{n+1} \to \infty$.

c) If $r = 1$ then the series diverges, as $s_n = n + 1$.

d) If $r = -1$ then the series is boundedly divergent as $\{s_n\} = \{1, 0, 1, 0, \ldots\}$.

Finding the sum of a series is usually very difficult: proving convergence or divergence is usually easier, and that's what we'll focus on in this course.

One type of series is not too hard to sum though: **telescoping series**:

$$s_n = \sum_{k=0}^{n} (b_k - b_{k+1})$$

The successive terms partially cancel:

$$\begin{aligned} s_n &= \sum_{k=0}^{n} (b_k - b_{k+1}) \\ &= (b_0 - b_1) + (b_1 - b_2) + \cdots + (b_{n-1} - b_n) + (b_n - b_{n+1}) \\ &= b_0 - b_{n+1} \end{aligned}$$

Thus, if $b_k \to \ell$ as $k \to \infty$ then $\displaystyle\sum_{k=0}^{\infty} (b_k - b_{k+1}) = b_0 - \ell$.

Example 18 *Find* $\sum_{k=0}^{\infty} \frac{k}{(k+1)!}$

SOLUTION: This does not look telescoping, but it is:

$$\frac{1}{k!} - \frac{1}{(k+1)!} =$$

So

$$\sum_{k=0}^{n} \frac{k}{(k+1)!} =$$

Hence

$$\sum_{k=0}^{\infty} \frac{k}{(k+1)!} =$$

The Harmonic Series

The **harmonic series** $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Proof.

We first note that we can group terms as follows:

$$
\begin{aligned}
s_n = \sum_{k=1}^{n} \frac{1}{k} \quad = \quad & 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\
& \cdots + \left(\frac{1}{2^{m-1}+1} + \frac{1}{2^{m-1}+2} + \cdots + \frac{1}{2^m}\right) \\
& \quad + \frac{1}{2^m+1} + \cdots + \frac{1}{n}
\end{aligned}
$$

(m is the largest integer so that $2^m \leq n$)

Proof ctd.
Now by comparing grouped terms we have

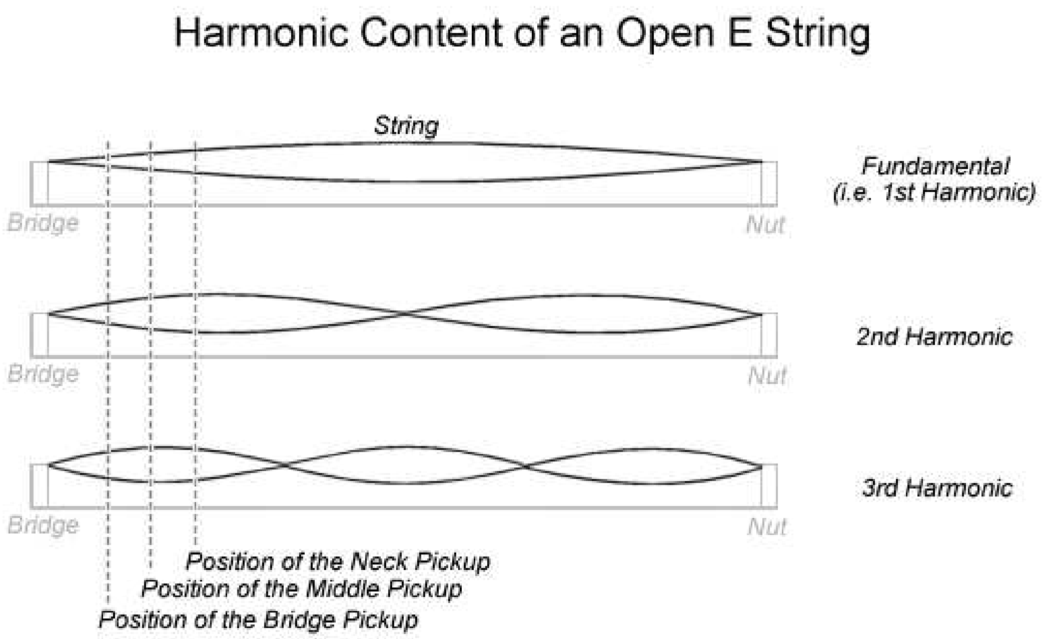
$$\begin{aligned}
s_n \quad &\geq \quad 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\
&\qquad \cdots + \left(\frac{1}{2^m} + \frac{1}{2^m} + \cdots + \frac{1}{2^m}\right) \\
&\qquad\qquad + \frac{1}{n} + \cdots + \frac{1}{n} \\
&\geq \quad 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\
&= \quad 1 + \frac{m}{2} \to \infty \quad \text{as} \quad m \to \infty
\end{aligned}$$

$\blacksquare$

See the Notes for a useful alternative proof.

The Harmonic series is named for its links to music theory.

A musical note is characterized by a fixed frequency (e.g., C: 261.63Hz) but the same musical note sounds different on different instruments. Why? When a guitar string is played all harmonics are excited. The amplitude of the different harmonics creates the tone.



4.5 Tests for Convergence

As I mentioned earlier, proving summability is usually a lot easier than finding the sum of a series.

There are many tests for convergence (or divergence) of series. Here, we consider the

- k th term test
- integral test
- comparison test
- p series test
- ratio test
- alternating series test
- absolute convergence test

General Notes $(a_k \in \mathbb{R},\ k \in \mathbb{N})$

- $\displaystyle\sum_{k=0}^{\infty} a_k$ converges if and only if $\displaystyle\sum_{k=N}^{\infty} a_k$ converges. (So we can omit the lower limit unless we need the value of the series.)

- If both $\displaystyle\sum^{\infty} a_k$ and $\displaystyle\sum^{\infty} b_k$ converge then

$$\sum^{\infty}(a_k + b_k) = \sum^{\infty} a_k + \sum^{\infty} b_k \quad \text{and} \quad \sum^{\infty}(\alpha a_k) = \alpha \sum^{\infty} a_k$$

- $\displaystyle\sum^{\infty}(a_k + b_k)$ can converge without either of $\displaystyle\sum^{\infty} a_k$ or $\displaystyle\sum^{\infty} b_k$ converging.

  Example: $\quad a_k = k, \quad b_k = -k.$

Theorem 4.7 *Suppose that $\{a_k\}_{k=0}^{\infty}$ is a sequence of positive numbers and let s_n denote the partial sum given by $s_n = \displaystyle\sum_{k=0}^{n} a_k$. If $\{s_n\}_{n=0}^{\infty}$ is a bounded sequence then the infinite series $\sum_{k=0}^{\infty} a_k$ is convergent.*

Proof: For any natural number n,

$$s_{n+1} = s_n + a_{n+1} > s_n,$$

since a_{n+1} is positive. Hence $\{s_n\}_{n=0}^{\infty}$ is a bounded increasing sequence and hence has a limit L. Therefore

$$\sum_{k=0}^{\infty} a_k = L$$

and the series converges. □

Theorem 4.8 *If* $\sum^{\infty} a_k$ *converges then* $a_k \to 0$ *as* $k \to \infty$*.*

Proof: Suppose that $s_n = \sum_{k=0}^{n} a_k \to L$.

Then

$$a_n = s_n - s_{n-1} \to L - L = 0.$$

$\square$

The contrapositive of this theorem gives us **the kth Term Test**

$$\boxed{\text{If } a_k \not\to 0 \text{ as } k \to \infty \text{ then } \sum^{\infty} a_k \text{ cannot converge.}}$$

Example 19 *For which x, if any, does $\displaystyle\sum_{k=1}^{\infty} \frac{k}{x+k}$ converge?*

SOLUTION:
We apply the kth term test as an initial filter.

so that the series converges for

$\blacksquare$

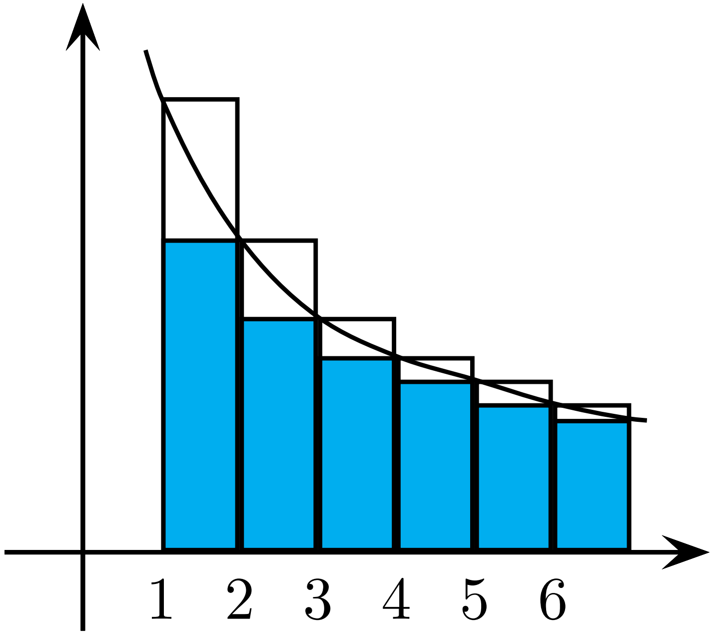
Note: The condition $a_k \to 0$ is necessary but **not** sufficient.

Example: $\displaystyle\lim_{k\to\infty} \frac{1}{k} = 0$ but the harmonic series $\displaystyle\sum^{\infty} \frac{1}{k}$ diverges

Theorem 4.9 (The Integral Test) *Suppose $f(x)$ is a **positive, decreasing** function on $[1,\infty)$, and $a_k = f(k)$. Then,*

- *if $\displaystyle\int_1^\infty f(x)\,dx$ is convergent then $\displaystyle\sum^\infty a_k$ converges.*
- *if $\displaystyle\int_1^\infty f(x)\,dx$ is divergent then $\displaystyle\sum^\infty a_k$ diverges.*

Proof:



$$\boxed{\textbf{blue area}} = \sum_{k=2}^{\infty} a_k \le \int_1^\infty f(x)\,dx$$

$$\boxed{\textbf{white area}} = \sum_{k=1}^{\infty} a_k \ge \int_1^\infty f(x)\,dx$$

Now apply theorem 4.7

Example 20 *Test* $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^{3/2}}$ *for convergence*

SOLUTION:

Let $f(x) = \dfrac{1}{x(\ln x)^{3/2}}$, so that the series is $\displaystyle\sum_{n=3}^{\infty} f(n)$.

Now

$$\int_3^{\infty} f(x)\,dx$$

so the integral

Therefore by the integral test, $\displaystyle\sum_{n=3}^{\infty} f(n)$ is

Theorem 4.10 (The Comparison Test) *Suppose that we have two sequences $\{a_k\}$, $\{b_k\}$ and that $0 \leq a_k \leq b_k$ for $k > N$ for some N. Then*

i) if $\sum^{\infty} b_k$ converges, so does $\sum^{\infty} a_k$;

ii) if $\sum^{\infty} a_k$ diverges, so does $\sum^{\infty} b_k$.

Note carefully which way these inequalities go:
if the "larger" series converges, it forces the "smaller" one to converge;
if the "smaller" series diverges, it forces the "larger" one to diverge.

(I've strengthened this result from the one in the Notes: the comparison only has to be true "eventually".)

Proof: (Comparison Test)

For the first part, let $B = \sum^{\infty} b_k$.

Then, $0 \le \sum^{n} a_k \le \sum^{n} b_k \le \sum^{\infty} b_k = B$

Thus, the sequence of partial sums $\left\{ \sum^{n} a_k \right\}$ is bounded and non-decreasing, and therefore has a limit.

The second part is the contrapositive of the first, and so follows.

$\square$

Example 21 *Does* $\sum_{k=1}^{\infty} \frac{1}{k!}$ *converge?*

SOLUTION: As $k!$ grows very quickly, we certainly expect the series to converge.

The comparison test (and the limit comparison test to follow shortly) are most often used in combination with the following result:

Theorem 4.11 (p-series test) *The series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

Proof: Apply the integral test. If $p \neq 1$,

$$\int_1^{\infty} x^{-p}\, dx = \lim_{N \to \infty} \frac{1}{1-p} \left(1 - N^{1-p}\right)$$

which is finite iff $1 - p < 0$, i.e. $p > 1$.

I'll leave the $p = 1$ case as an EXERCISE. $\square$

With the p-series test in hand you can get an intuitive idea whether series involving algebraic terms only (powers and roots) converges or not.

Example 22 *Determine whether either of*

$$(i) \quad \sum^{\infty} \frac{2k-1}{k^3+2k^2+1} \qquad\qquad (ii) \quad \sum^{\infty} \frac{k+1}{(k^4-1)^{1/3}}$$

converge.

SOLUTION: In case (i) the dominant terms (i.e. what happens at large k) give $\dfrac{2k}{k^3} = 2k^{-2}$.

We know that $\sum^{\infty} k^{-2}$ converges (p-series with $p=2$) so we expect convergence.

For (ii) the dominant terms give $\dfrac{k}{k^{4/3}} = k^{-1/3}$ so we expect divergence.

To prove (i) converges we use the first part of the comparison test:

$$\frac{2k-1}{k^3+2k^2+1} < \frac{2k}{k^3} = 2k^{-2}\,.$$

We know that $\sum^{\infty} k^{-2}$ is convergent, the factor 2 does not change this, so by the comparison test series (i) converges.

To prove (ii) diverges we use the second part:

$$\frac{k+1}{(k^4-1)^{1/3}} > \frac{k}{k^{4/3}} = k^{-1/3}\,.$$

We know that $\sum^{\infty} k^{-1/3}$ is divergent, so by the comparison test series (ii) diverges.

$\blacksquare$

These last two examples are pretty easy, but small tweaks to

$$(i) \quad \sum^{\infty} \frac{2k+1}{k^3 - 2k^2 + 1} \qquad\qquad (ii) \quad \sum^{\infty} \frac{k-1}{(k^4+1)^{1/3}}$$

would spoil the proofs.

But the dominant behaviour is still the same, so we would expect the convergence properties to be the same.

Finding a comparison series can still be done: for example, we can show that

$$\frac{k-1}{(k^4+1)^{1/3}} > \frac{\frac{1}{2}k}{(2k^4)^{1/3}} = 2^{-4/3}k^{-1/3} \quad \text{if} \quad k > 2.$$

But rather that get tied up in this sort of fiddly detail to get a comparison series, we can appeal directly to the idea that the dominant term tells us all we need to know:

Theorem 4.13 *Suppose that* $a_k > 0$, $b_k > 0$ *and*

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L \neq 0,$$

that is, the above limit ***exists and is non-zero****.*

Then, $\sum^{\infty} b_k$ *converges if and only if* $\sum^{\infty} a_k$ *converges.*

Proof: The proof is in the Notes.

It essentially comes down to saying that for some large K

$$(L-1)b_k < a_k < (L+1)b_k \quad \text{if} \quad k > K$$

and then applying the comparison test. $\square$

Example 23 *Analyse the convergence or otherwise of*

$$(i) \quad \sum^{\infty} \frac{2k+1}{k^3-2}, \qquad (ii) \quad \sum^{\infty} \frac{k-1}{(k^4+1)^{1/3}}, \qquad (iii) \quad \sum_{k=0}^{\infty} \frac{3k^2+5k}{2^k(k^2+1)}$$

SOLUTION: (i) Let $a_k = \dfrac{2k+1}{k^3-2}$ and $b_k =$

Then $\displaystyle \lim_{k\to\infty} \frac{a_k}{b_k} =$

(ii) Let $a_k = \dfrac{k-1}{(k^4+1)^{1/3}}$ and $b_k =$

Then $\displaystyle \lim_{k\to\infty} \frac{a_k}{b_k} =$

(iii) Let $a_k = \dfrac{3k^2 + 5k}{2^k(k^2+1)}$.

The limit comparison test is less useful when there are logs involved.

Example 24 *If* $a_k = \dfrac{\ln k}{\sqrt{k^3+1}}$*, does* $\displaystyle\sum_{k=1}^{\infty} a_k$ *converge?*

SOLUTION: Let $b_k = k^{-3/2}$, the dominant part of a_k.

Note that $\sum^\infty b_k$ converges.

Since log grows more slowly that any power, we expect $\sum^\infty a_k$ to behave like $\sum^\infty b_k$ and so converge.

But we cannot use the limit comparison test on a_k and b_k, as

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \left(\lim_{k\to\infty} \ln k\right) \times \lim_{k\to\infty} \frac{k^{3/2}}{\sqrt{k^3+1}}$$

and while the latter limit is 1, the former limit does not exist.

However, we do know that $\lim_{k\to\infty} \dfrac{\ln k}{k^\alpha} = 0$ for all $\alpha > 0$.

So for some suitable K, $\dfrac{\ln k}{k^\alpha} < 1 \quad$ if $\quad k > K$, so we can (eventually) replace the $\ln k$ with k^α for small α.

We need to pick an α that still leaves the denominator suitably dominant.

So choose $\alpha = 1/3$. Then we know that for some K

$$a_k = \frac{\ln k}{\sqrt{k^3+1}} < \frac{k^{1/3}}{\sqrt{k^3+1}} = c_k \quad \text{if} \quad k > K.$$

The dominant behaviour of c_k suggests it behaves as $d_k = k^{-7/6}$, and $\sum^\infty d_k$ converges as it's a p-series with $p = \dfrac{7}{6} > 1$.

Formally,

$$\lim_{k\to\infty} \frac{c_k}{d_k} = \lim_{k\to\infty} \frac{k^{3/2}}{\sqrt{k^3+1}} = 1$$

So by the limit comparison test $\sum^{\infty} c_k$ converges as $\sum^{\infty} d_k$ does.

But we also know that $a_k < c_k$ for $k > K$, and so by the comparison test, $\sum^{\infty} a_k$ converges since $\sum^{\infty} c_k$ does. ■

4.5.6 The Ratio Test

Theorem 4.14 (The Ratio Test) *Suppose that $a_k > 0$ and*

$$\frac{a_{k+1}}{a_k} \to r$$

as $k \to \infty$.

*Then $\sum^{\infty} a_k$ **converges** if $r < 1$ and **diverges** if $r > 1$.*

Proof: See Notes $\square$

Note that the test fails if $r = 1$ — either situation is possible.

Example: with $a_k = \dfrac{1}{k^p}$, the limit ratio of $\dfrac{a_{k+1}}{a_k}$ is always 1, but $\sum^{\infty} a_k$ converges iff $p > 1$.

The ratio test is especially useful for series with exponentials and/or factorials

Example 25 *Does* $\displaystyle\sum_{k=0}^{\infty} \frac{k^2 2^k}{3^k}$ *converge?*

SOLUTION: Applying the ratio test to $a_k = \dfrac{k^2 2^k}{3^k}$ we get

Example 26 *Which, if either, of the following converges?*

$$(i) \quad \sum_{k=0}^{\infty} \frac{(2k)!}{k!k!} \qquad\qquad (ii) \quad \sum_{k=0}^{\infty} \frac{27^k (k!)^3}{(3k)!}$$

SOLUTION: Let $a_k = \dfrac{(2k)!}{k!k!}$, $b_k = \dfrac{27^k (k!)^3}{(3k)!}$. Then

$$\frac{a_{k+1}}{a_k} = \frac{(2k+2)!}{(k+1)!(k+1)!} \frac{k!k!}{(2k)!}$$

$$=$$

So it follows from the ratio test that case (i) .

A similar calculation for case (ii) gives

$$\frac{b_{k+1}}{b_k} =$$

4.5.8 Alternating Series

So far we have only looked at series with positive terms; now we turn to a simple case where there are negatives terms too:

If $\{a_k\}_{k=0}^{\infty}$ is a sequence of positive reals, then the series

$$a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + \cdots = \sum_{k=0}^{\infty} (-1)^k a_k$$

is called an **alternating series**.

Theorem 4.15 (Leibniz Test) *If*

a) $a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq \cdots \geq 0$ *and*

b) $a_k \to 0$ *as* $k \to \infty$

then the alternating series $\sum^{\infty} (-1)^k a_k$ *converges.*

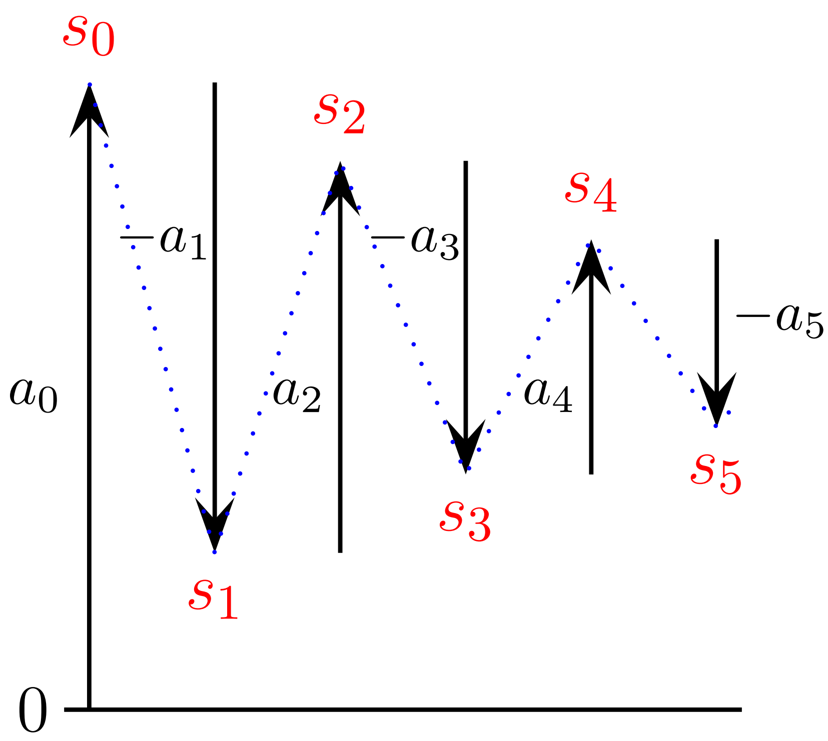
Proof:

Define $s_{2n} = \sum^{2n} (-1)^k a_k$.

$\{s_{2n}\}$ is a decreasing sequence bounded below by 0.

Define $s_{2n+1} = \sum^{2n+1} (-1)^k a_k$.

$\{s_{2n+1}\}$ is an increasing sequence bounded above by a_0.



So $\lim_{n\to\infty} s_{2n} = \ell$, $\lim_{n\to\infty} s_{2n+1} = L$, say. But

$$s_{2n} - s_{2n+1} = a_{2n+1} \to 0$$

hence $\ell = L$, and so $\sum^{\infty} (-1)^k a_k = L$. $\square$

Example 27 *For which sequences a_k is $\sum_{k=1}^{\infty} (-1)^k a_k$ summable?*

$$(i) \quad a_k = \frac{1}{k} \qquad (ii) \quad a_k = \frac{1}{k^2} \qquad (iii) \quad a_k = \frac{(\ln k)^2}{k}$$

SOLUTION: Starting at $k = 1$ is not an issue.

We thus have three things to check:

Firstly, apart from a_1 being zero in case (iii), all the terms are positive (of course this zero term is no issue either).

Secondly it is obvious (or easy to show) that all three sequences tend to zero as $k \to \infty$.

Cases (i) and (ii) are obviously decreasing too, so the Leibniz test tells us they converge.

However, $a_k = \dfrac{(\ln k)^2}{k}$ is not monotonic.

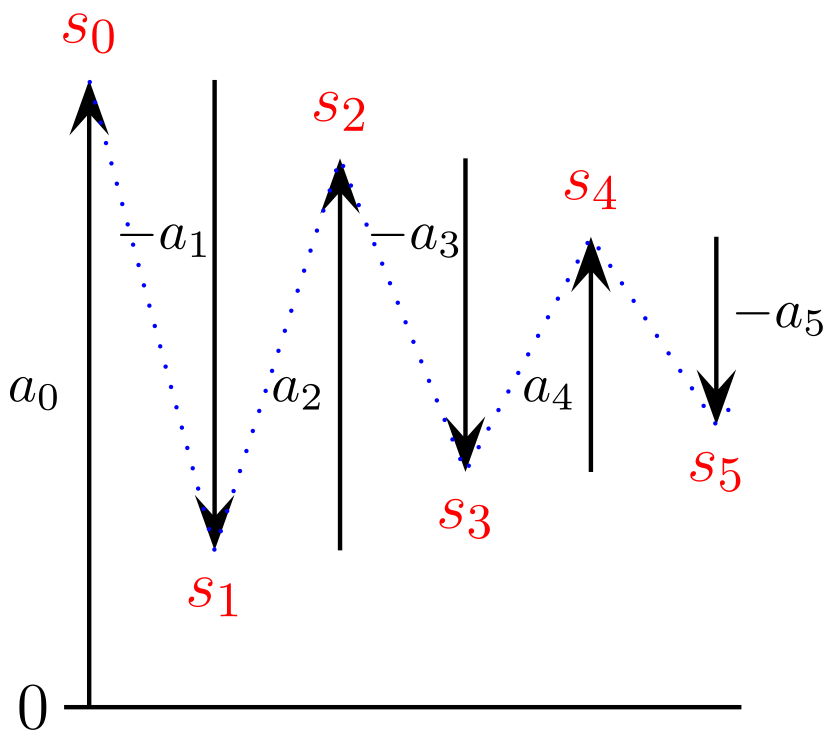
Theorem 4.16 *Suppose that $\{a_k\}_{k=0}^{\infty}$ is a sequence of numbers satisfying the hypotheses of the Leibniz test.*

Let $\sum_{k=0}^{\infty}(-1)^k a_k = L$ and let the nth partial sum of the same series be s_n. Then

$$|s_n - L| \le a_{n+1} \quad \textit{for all } n.$$

In effect, this results says that if you chop the series off after the nth term, the error in approximation will be less than the $(n+1)$st term.

The proof is given in the Notes, or consider the diagram.



Example 28 *Estimate* $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2}$ *to within* 0.01.

SOLUTION: It is easy to see that the series satisfies the hypotheses of the Leibniz test.

So to be sure of an error less than 0.01 we need to find n such that

4.5.8 Absolute and Conditional Convergence

Consider the series

$$1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \frac{1}{49} + \frac{1}{64} - \frac{1}{81} + \cdots . \qquad (*)$$

It does not consist of positive terms, and is not alternating, so none of our previous results will help us decide if it converges.

However, since $\displaystyle\sum^{\infty} \frac{1}{k^2}$ converges, and the series $(*)$ cannot be any bigger than this, we'd expect $(*)$ to converge too.
To prove it does, we need a definition.

A series $\sum^{\infty} a_k$ **converges absolutely** if $\sum^{\infty} |a_k|$ converges.

Theorem 4.17 *If a series converges absolutely, then it converges.*

Proof: See Notes $\square$

Example 29 *Explain why* $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ *converges.*

SOLUTION:

A series is **conditionally convergent** if it is convergent but not absolutely convergent.

The basic example here is $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, which we saw earlier is convergent.

But we also know that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (a p-series with $p = 1$).

So $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ is conditionally convergent.

A Paradox

According to MAPLE, $\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{1}{k+1} = L = \ln 2$ but

$$
\begin{aligned}
L &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots \\
&= (1 - \frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + (\frac{1}{5} - \frac{1}{10}) - \frac{1}{12} + (\frac{1}{7} - \frac{1}{14}) + \cdots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \cdots \\
&= \frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots) \\
&= \frac{L}{2}
\end{aligned}
$$

This seems to suggest that $L = 0$!

What's going on?

A **rearrangement** of a series $\sum a_k$ is a series that has exactly the same terms but that is summed in a different order.

Our previous example leads us to suspect that rearranging a series is a problem. In fact:

Theorem 4.18 *Suppose that* $\sum_{k=0}^{\infty} a_k$ *is an infinite series.*

(i) If $\sum a_k$ converges absolutely, then every rearrangement of the series converges absolutely and all rearrangements have the same limit as $\sum a_k$.

(ii) If $\sum a_k$ converges conditionally, then given any real number L, the series has a rearrangement that converges to L.
Moreover, every conditionally convergent series has a rearrangement that diverges to ∞, and another rearrangement that diverges to $-\infty$.

MAPLE notes

`sum(f(k), k=m..n);` computes the sum of `f(k)` as `k` runs from `m` to `n`.

```
> sum(k^2, k=1..4);
```

$$30$$

```
> sum(k^2, k=1..n);
```

$$\frac{1}{3}(n+1)^3 - \frac{1}{2}(n+1)^2 + \frac{1}{6}n + \frac{1}{6}$$

```
> sum(1/k^2, k=1..infinity);
```

$$\frac{1}{6}\pi^2$$

4.6 Taylor Series

- Suppose that $f(x)$ is infinitely differentiable at a then

$$P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

  is the **Taylor series** of $f(x)$ about a.

- The Taylor series of $f(x)$ about $a = 0$ is called a **Maclaurin series**.

Key question: Under what circumstances and for which x is

$$f(x) = P(x)?$$

Let I be an interval and f be infinitely differentiable at some point a. We say that

- the Taylor series for f about a **converges on I** if the series converges for each $x \in I$.
- the Taylor series for f about a **converges to f on I** if

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

  for each $x \in I$ and x is in the domain of f.
  We say f is **analytic on I** in this case.
- the Taylor series for f about a **diverges on I** if the series diverges for each point $x \in I$ (except a of course).

Theorem 4.19 (Corollary to Taylor) *Suppose that f is infinitely differentiable at a and x is in the domain of f. If*

$$\lim_{n\to\infty} R_{n+1}(x) = 0,$$

where

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

is Lagrange's remainder, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \,.$$

Note that c may depend on n in this result.

Example 30 *Find the Taylor series for e^x about $a = 1$ and show that it converges to e^x for all x.*

SOLUTION: Firstly, the Tayor series coefficients are

$$a_n = \frac{f^{(n)}(a)}{n!} = \qquad\qquad .$$

So the Taylor series is

$$P(x) =$$

The remainder term is then

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)} =$$

Now, c lies between x and 1 and hence

$$e^c \leq M(x) = \max\{e^1, e^x\}$$

Accordingly (note that x is fixed),

$$0 \leq \left| e^c \frac{(x-1)^{(n+1)}}{(n+1)!} \right| \leq M \left| \frac{(x-1)}{(n+1)} \frac{(x-1)}{n} \cdots \frac{(x-1)}{1} \right| \to 0$$

as $n \to \infty$.

Hence, $\lim_{n\to\infty} R_{n+1}(x) = 0$ for all $x \in \mathbb{R}$.

We conclude that

$$e^x = \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n \quad \forall\, x \in \mathbb{R}.$$

■

Functions which cannot be represented by a Taylor series are not difficult to construct and are also important in applications.

Example 31 *Consider $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} e^{-1/x} & \textit{if } x > 0 \\ 0 & \textit{if } x \le 0 \end{cases}$$

Taylor series to know

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad x \in (-1,1)$$

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^k}{k}, \quad x \in (-1,1]$$

$$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}, \quad x \in [-1,1]$$

And for all x we have

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \qquad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!},$$

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \qquad \cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

4.7 Power Series

A **power series** (around $x = 0$) is a (formal) series of the form

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{k=0}^{\infty} a_k x^k,$$

where $\{a_k\}$ is a given sequence of real constants and x is a real variable.

The series $\displaystyle\sum_{k=0}^{\infty} a_k (x-a)^k$ is also a power series but this time written in powers of $(x - a)$.

Both Taylor and Maclaurin series are examples of power series.

We use the same terminology for convergene and divergence at points and on intervals for general power series as we do for Taylor series (see Notes).

The interesting problem is: given a power series, where does it converge?

Theorem 4.20 *Let $\{a_k\}$ be a sequence. Then, there exists an $R \in [0, \infty) \cup \{\infty\}$ such that*

- $\displaystyle\sum_{k=0}^{\infty} a_k(x-a)^k$ *converges absolutely for* $|x-a| < R$
- $\displaystyle\sum_{k=0}^{\infty} a_k(x-a)^k$ *diverges for* $|x-a| > R$

A proof will have to wait for 2nd year courses on complex analysis.

Remarks

- The power series may or may not converge at the endpoints $x = a \pm R$.
  It may converge at both endpoints, only one or neither.

- The number R is called the **radius of convergence** of this power series.
  If $R = \infty$ then the series converges for all x.
  If $R = 0$ then the series only converges at $x = a$.

- The interval $(a - R, a + R)$ together with those endpoints at which the series converges is called the **interval of convergence** of the power series.

Big Question: How do we find R?

Big Answer: Use the ratio test.

Example 32 *Find the interval of convergence for*
$$\sum_{k=1}^{\infty} \frac{k+1}{2^k k^2}(x-1)^k.$$

SOLUTION: We look for absolute convergence: let

$$c_k = \left| \frac{k+1}{2^k k^2}(x-1)^k \right| = \frac{k+1}{2^k k^2}|x-1|^k.$$

Then we apply the ratio test to $\displaystyle\sum_{k=1}^{\infty} c_k$:

$$\frac{c_{k+1}}{c_k} = \frac{(k+2)|x-1|^{k+1}}{2^{k+1}(k+1)^2} \times \frac{2^k k^2}{(k+1)|x-1|^k}$$

Now the ratio test says that the series $\sum_{k=1}^{\infty} c_k$ converges if

So the radius of convergence is

For the upper end point,

For the lower end point,

So the interval of convergence of our power series is

Theorem 4.21 *Suppose that $\{a_k\}_{k=0}^{\infty}$ is a sequence of reals and that the power series $\sum_{k=0}^{\infty} a_k (x-a)^k$ has radius of convergence R.*
Then, if $\lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right|$ exists, this limit has value R.

Proof: Apply the ratio test exactly as we did in the previous example. $\square$

I prefer not to use this result, but to rely on the ratio test: in the end you are less likely to get things wrong by going back to the ratio test.

There are also examples where the ratio in this theorem does not exist: the Maclaurin series for $\sin(x)$ is one example.

Example 33 *Show that the Maclaurin series for* $\tan^{-1}(x)$ *converges on* $[-1, 1]$*.*

SOLUTION: The series is $\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$.

Applying the ratio test to the absolute series:

$$\left|(-1)^{k+1}\frac{x^{2(k+1)+1}}{2(k+1)+1}\right| \times \left|(-1)^k \frac{2k+1}{x^{2k+1}}\right| = \frac{2k+3}{2k+1}|x|^2 \to |x|^2$$

as $k \to \infty$, so the radius of convergence is 1.

At $x = \pm 1$, $x^{2k+1} = \pm 1$ and at these points the series is

$\displaystyle\pm\sum_{k=1}^{\infty}\frac{(-1)^k}{2k+1}$, which converges using Leibniz (EXERCISE).

So the series converges for $-1 \le x \le 1$. ■

Note that our results now allow us to write series expansions for certain familiar constants:

$$\begin{aligned} e = e^1 &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \\ \ln 2 = \ln(1+1) &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \\ \frac{\pi}{4} = \tan^{-1} 1 &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \end{aligned}$$

Although the first of these converges very quickly, we already know the second one converges only conditionally.

In fact both the second and third converge very slowly — the third one (**Gregory's Series**) is a very poor way of finding π.

With a bit of trickery the series for $\tan^{-1}(x)$ can be used to get better approximations to π though.

Maple exploration

a) Show that the first 6 terms of Gregory's series give π with an error of about 5%.

b) Show that $(5+2i)(7+3i) = 29+29i$ and deduce that

$$\pi = 4\left(\tan^{-1}\frac{2}{5} + \tan^{-1}\frac{3}{7}\right)$$

c) Use the first 3 terms in the series for $\tan^{-1}(x)$ to approximate $\tan^{-1}\dfrac{2}{5}$ and $\tan^{-1}\dfrac{3}{7}$ and show that they give π to 2 decimal places (as does $22/7$).

d) Using the above ideas, see if you can cook up more expressions for π (sums of three angles maybe) that can give better approximations.

4.8 Manipulation of Power Series

If $\displaystyle\sum_{k=0}^{\infty} c_k(x-a)^k$ converges for $|x-a|<R$ then we can define a function $f:(a-R,a+R)\to\mathbb{R}$ by

$$f(x)=\sum_{k=0}^{\infty} c_k(x-a)^k$$

Sometimes we can find a closed form for these series (see Notes for an example), but often the power series is all we have — and it's usually good enough.

Theorem 4.22 *Suppose that both power series* $f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$ *and* $g(x) = \sum_{k=0}^{\infty} b_k (x-a)^k$ *are convergent on an interval* I*. Then:*

i) $f(x) \pm g(x) = \sum_{k=0}^{\infty} (a_k \pm b_k)(x-a)^k$

ii) $f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) (x-a)^k$

The second of these is just a generalisation of the rule for multiplying polynomials.

In practice, that's how you use it.

Example 34 *Find the first four non-zero terms of the Maclaurin series for* $f(x) = \dfrac{e^x}{1-x}$*. Where does it converge?*

SOLUTION: We have

$$f(x) = \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) \times \left(1 + x + x^2 + x^3 + \cdots\right)$$

Theorem 4.23 *Suppose that* $f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$ *is a convergent power series for* $|x-a| < R$*. Then,*

i) $f(x)$ *is continuous and differentiable for* $|x-a| < R$ *with*

$$\frac{df}{dx} = \sum_{k=0}^{\infty} \frac{d}{dx}\left(a_k (x-a)^k\right) = \sum_{k=1}^{\infty} k a_k (x-a)^{k-1}$$

ii) $f(x)$ *is integrable on* $|x-a| < R$ *with*

$$\int f(x)\,dx = \sum_{k=0}^{\infty} \int \left(a_k (x-a)^k\right)\,dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-a)^{k+1} + C$$

Convergent power series can be differentiated and integrated term by term (as if they were infinite polynomials).

Remarks on above result

a) This result might not surprise you, but we are swapping two infinite processes here (sum and limits), and we've already seen problem with infinite processes and conditional convergence.
Key point: you cannot always swap infinite processes (see Notes).

b) This result also tells us that for a function f defined by a power series about a, that series is the Taylor series of f.

c) This result is the basis of many applications of power series, see Notes for some.

d) Among others, we can also see how we could use the previous result to solve differential equations: if we assume a DE has a power series as solution, we can feed a formal series into the DE and get a formula (maybe) for the coefficients.

Example 35 *Find a power series solution for the IVP* $f'(x) = f(x)$*,* $f(0) = 1$*.*

SOLUTION: Assume that $f(x) = \sum_{k=0}^{\infty} a_k x^k$. Then

$$f'(x) - f(x) = \sum_{k=0}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} a_k x^k$$

Example 36 *Find a closed form for* $f(x) = \sum_{k=0}^{\infty} k^2 x^k$*.*

SOLUTION: The closest series we know is $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.

We can introduce k as a coefficient by differentiating:

$$\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} k x^{k-1}$$

To match up the power and the coefficient, multiply by x:

$$\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} k x^k.$$

From

$$\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} kx^k$$

repeat this whole process to get the series we want:

Application to limits (cf. L'Hôpital's Rule).

Suppose that f and g are represented by their Taylor series about x_0 and

$$\begin{aligned} f(x_0) = & \cdots = f^{(k-1)}(x_0) = 0 \\ g(x_0) = & \cdots = g^{(k-1)}(x_0) = 0, \quad g^{(k)}(x_0) \neq 0. \end{aligned}$$

Then,

$$\begin{aligned} \frac{f(x)}{g(x)} &= \frac{\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(k+1)}(x_0)}{(k+1)!}(x-x_0)^{k+1} + \cdots}{\frac{g^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{g^{(k+1)}(x_0)}{(k+1)!}(x-x_0)^{k+1} + \cdots} \\ &= \frac{f^{(k)}(x_0) + \frac{f^{(k+1)}(x_0)}{k+1}(x-x_0) + \cdots}{g^{(k)}(x_0) + \frac{g^{(k+1)}(x_0)}{k+1}(x-x_0) + \cdots} \end{aligned}$$

$$\Rightarrow \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f^{(k)}(x_0)}{g^{(k)}(x_0)}$$

Example 37 *Find* $\lim_{x\to 0} \dfrac{\sin x - x}{[\ln(x+1)]^3}$

SOLUTION: We could use L'Hôpital (3 times!), but we instead use Maclaurin series:

$$\sin x - x = -\frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\ln(x+1) = \int \frac{dx}{x+1} = \int \left(1 - x + x^2 - x^3 + \cdots\right)\, dx$$

$$=$$

So $[\ln(x+1)]^3$

Hence

$$\lim_{x\to 0} \frac{\sin x - x}{[\ln(x+1)]^3}$$

4.9 MAPLE Notes

`taylor(expr, x=a, k);`
computes Taylor series of `expr` about `x=a` up to the term of order `k`.

`convert(taylor(expr,x=a,k),polynom);`
computes the Taylor polynomial of order `k-1` for `expr` about `x=a`.

`readlib(coeftayl);`
`coeftayl(expr,x=a,k);`
computes the `k`th coefficient in the Taylor series expansion `expr` about `x=a`.