THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 2 2017

MATH1241
HIGHER MATHEMATICS 1B

(1) TIME ALLOWED – TWO (2) HOURS

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS and A STANDARD NORMAL TABLE ARE APPENDED ON THE LAST PAGES

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 1

1. i) Evaluate each of the following integrals.

a) $I_1 = \displaystyle\int \frac{1}{\sqrt{x^2+4x+5}}\,dx$

b) $I_2 = \displaystyle\int \cos^3 x\ \sin^2 x\,dx$

c) $I_3 = \displaystyle\int \frac{x+7}{(x+1)^2(x-2)}\,dx$

ii) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 2x.$$

iii) Let $\mathbf{v}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix}1\\2\\1\end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix}-2\\-1\\1\end{pmatrix}$.

a) Write $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

b) Does there exist a linear map $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ such that

$$T(\mathbf{v}_1) = \begin{pmatrix}1\\6\\3\end{pmatrix}, \quad T(\mathbf{v}_2) = \begin{pmatrix}-2\\16\\2\end{pmatrix}, \quad \text{and} \quad T(\mathbf{v}_3) = \begin{pmatrix}-6\\-3\\-8\end{pmatrix}?$$

c) What is the relationship between $\text{span}(\mathbf{v}_1, \mathbf{v}_2)$ and $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$?

iv) Let $S = \{\mathbf{x} \in \mathbb{R}^3 : x_1^2 - x_2^2 = x_3^2\}$.

a) Prove that S is closed under scalar multiplication.

b) Show that S is not a subspace of $\mathbb{R}^3$.

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 2

2. i) The volume V of a tumour can be modelled by the differential equation

$$\frac{dV}{dt} = \alpha\left(1 - \frac{V}{K}\right)V, \qquad (*)$$

where t is time, V is the volume of the tumour at time t and α and K are positive constants. If the initial value $V(0) = V_0$ is imposed, solving (*) as a separable equation gives the non constant solution

$$V(t) = \frac{K}{1 + \left(\dfrac{K}{V_0} - 1\right)e^{-\alpha t}}.$$

a) Find all constant solutions to equation (*).

b) Find the behaviour of $V(t)$ as $t \to \infty$.

c) Give an interpretation of the constants α and K.

d) Another model for tumour growth is given by the differential equation

$$\frac{dV}{dt} = -\alpha \ln\left(\frac{V}{K}\right)V. \qquad (**)$$

Suppose the same constants α and K are used in the two models. Without solving (**), explain which model predicts faster tumour growth for tumours when V much smaller than K?

ii) The specific gravity z of a solid heavier than water is given by

$$z = \frac{x}{x - y},$$

where x and y are its weight in air and water respectively. The weights x and y are observed to be 21.3g and 10.2g and each observation is made with an uncertainty whose absolute value is at most 0.1g.

a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

b) Use the total differential approximation for z to estimate the maximum uncertainty in the calculated value of z (to 3 decimal places).

You may find the following Maple session useful.

```
> z := x/(x-y):
> zx := diff(z,x):
> zy := diff(z,y):
> subs(x=21.3, y=10.2, zx);
```

$$-0.08278548821$$

```
> subs(x=21.3, y=10.2, zy);
```

$$0.1728755783$$

iii) The probability density function f of a continuous random variable X is given by

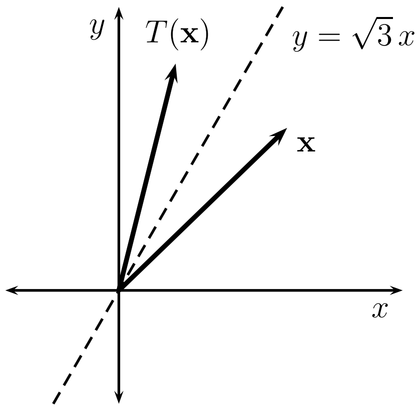
$$f(x) = \begin{cases} kx^2 & \text{for } 0 \le x \le 3 \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant.

a) Find the value of k.

b) Evaluate $E(X)$ and $\mathrm{Var}(X)$.

iv) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which reflects a vector in the line $y = \sqrt{3}\,x$ as show in the diagram.



a) Show that

$$T(\mathbf{e}_1) = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \quad T(\mathbf{e}_2) = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \quad \text{where } \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

b) Find the matrix A such that

$$A\mathbf{x} = T(\mathbf{x}), \quad \text{for all } \mathbf{x} \in \mathbb{R}^2.$$

c) Find $T\left(\begin{pmatrix} 4 \\ 5 \end{pmatrix}\right)$.

v) Read the following Maple output and use it to answer the questions below.

```
> with(LinearAlgebra):
> A := <<1,2,7,4,3>|<-1,6,2,8,1>|<2,-4,5,-4,2>|
  <2,3,-1,5,7>|<-1,14,11,20,5>>;
```

$$A := \begin{bmatrix} 1 & -1 & 2 & 2 & -1 \\ 2 & 6 & -4 & 3 & 14 \\ 7 & 2 & 5 & -1 & 11 \\ 4 & 8 & -4 & 5 & 20 \\ 3 & 1 & 2 & 7 & 5 \end{bmatrix}$$

```
> GaussianElimination(A);
```

$$\begin{bmatrix} 1 & -1 & 2 & 2 & -1 \\ 0 & 8 & -8 & -1 & 16 \\ 0 & 0 & 0 & -\frac{111}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> b := <-2,35,16,49,15>:
```

$$b := \begin{bmatrix} -2 \\ 35 \\ 16 \\ 49 \\ 15 \end{bmatrix}$$

```
> LinearSolve(A,b);
```

$$\begin{bmatrix} 1 - \_t_3 - \_t_5 \\ 5 + \_t_3 - 2\,\_t_5 \\ \_t_3 \\ 1 \\ \_t_5 \end{bmatrix}$$

Let A be the matrix A defined in the Maple code above.

a) Give a basis for the kernel of the matrix A.

b) Find one vector in $\mathbf{x} \in \mathbb{R}^5$ such that

$$A\mathbf{x} = \begin{pmatrix} -2 \\ 35 \\ 16 \\ 49 \\ 15 \end{pmatrix}.$$

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 3

3. i) Let $\mathbb{P}_2$ be the vector space of all real polynomials of degree at most 2.

a) Find three polynomials f_1, f_2, f_3 in $\mathbb{P}_2$ such that $f_i(0) = 1$ for $i = 1, 2, 3$ and $\{f_1, f_2, f_3\}$ is linearly independent.

b) Suppose that $P = \{p_1, p_2, p_3\}$ is a subset of $\mathbb{P}_2$ such that $p_i(1) = 0$ for $i = 1, 2, 3$. Show that P is a linearly dependent set.

ii) Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

a) Given that the eigenvalues of A are $1, 2, 3$, explain why A is diagonalisable.

b) Find an eigenvector of A for the eigenvalue $\lambda = 3$.

c) Let $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and

$$f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6.$$

Show that $f(D) = D^3 - 6D^2 + 11D - 6I$ is the zero matrix $\mathbf{0}$.

d) Hence, prove that $f(A) = \mathbf{0}$.

e) Compute A^{-1} as a linear combination of A^2, A, I.

iii) Let $\mathbb{F} = \{0, 1\}$ be the field of 2 elements, where $1 + 1 = 0$, and let $V = \mathbb{F}^3$ be the set of all 3-tuples with elements in $\mathbb{F}$. For example,

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

You are **given** that V forms a vector space over the field $\mathbb{F}$. Two (not necessarily distinct) vectors $\mathbf{v}_1$, $\mathbf{v}_2$ are chosen at random from V. Let S denote the sample space. Define E to be the event that $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent.

a) Find $|S|$.

b) Show that $P(E) = \dfrac{21}{32}$.

c) Consider the discrete random variable $X = \dim\big(\text{span}(\mathbf{v}_1, \mathbf{v}_2)\big)$. Copy and complete the following table for the probability distribution $p_k = P(X = k)$.

k	0	1	2
p_k			

iv) Mo's mobile phone company produces phones with an average lifetime of 4.2 years. Suppose that the lifetime is an approximately normally distributed random variable with standard deviation 1.3 years. Mo wishes to offer a warranty on his phones, but figures that it is only profitable to do so if he replaces fewer than 2% of the stock he sells, during the warranty period. What warranty length should he offer?

USE A SEPARATE BOOK CLEARLY MARKED QUESTION 4

4. i) Consider the differential equation

$$x^2y' = xy - \frac{1}{2}e^x y^3.$$

a) Show that the substitution $y(x) = xu(x)$ leads to a separable equation for $u(x)$.

b) Solve the inital value problem $y(1) = 1/\sqrt{e}$.

ii) Consider the power series expansion

$$\frac{1}{\sqrt{1-2cx+x^2}} = \sum_{n=0}^{\infty} a_n x^n,$$

where c is a constant.

a) Find the first two coefficients a_0 and a_1.

b) Show that

$$(c-x)\sum_{n=0}^{\infty} a_n x^n = (1-2cx+x^2)\sum_{n=1}^{\infty} n a_n x^{n-1}$$

and, hence, or otherwise, determine a_2.

iii) Consider the sequence $\{a_n\}$ defined recursively by

$$a_{n+1} = \frac{a_n^2+\pi^2}{a_n+\pi}, \quad a_0 = 1.$$

The following Maple session may be useful.

```
> factor(Pi - (a[n]^2 + Pi^2)/(a[n] + Pi));
```

$$\frac{a_n(\pi - a_n)}{a_n + \pi}$$

a) Show by induction that $a_n < \pi$.

b) By considering $a_{n+1} - a_n$, show that the sequence is monotonically increasing.

c) State clearly why the sequence converges and determine its limit.

iv) Consider the sequence $\{a_n\}$ given by

$$a_n = \frac{\cos n + n}{n^3 - e^{-n}}.$$

a) Does the series

$$\sum_{n=1}^{\infty} a_n$$

converge? Give reasons for your answer.

b) Determine the interval of convergence of the power series

$$\sum_{n=1}^{\infty} a_n (x+2)^n.$$

You may use the fact that the sequence $\{a_n\}$ is monotonically decreasing.

Standard normal probabilities $P(Z \leq z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2 - x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \qquad |x| < a \\ &= \tfrac{1}{a}\coth^{-1}\tfrac{x}{a} + C, \qquad |x| > a > 0 \\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$