MATH2019 PROBLEM CLASS

EXAMPLES 3

DIV, GRAD, CURL AND LINE INTEGRALS

$\boxed{1996}$ 1. A moving particle has position vector

$$\mathbf{r}(t) = \cos(\omega t)\,\mathbf{i} + \sin(\omega t)\,\mathbf{j} + t\,\mathbf{k}$$

where ω is a positive constant and t is time.

i) Find the acceleration of the particle and show that it has constant magnitude.

ii) Describe the path of the particle.

iii) Evaluate $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathcal{C}$ is the portion of the path of the particle between $t = 0$ and $t = 2\pi/\omega$ and

$$\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}\,.$$

[Hint: Show that $\mathbf{F} = \nabla(xyz)$.]

$\boxed{2014, S1}$ 2. Given the vector field $\mathbf{G} = yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}$ calculate:

i) div $\mathbf{G}$.

ii) curl $\mathbf{G}$.

$\boxed{2014, S1}$ 3. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be a path in space embedded within the surface $\phi(x, y, z) = 1$. Assuming that all relevant derivatives exist use the chain rule to show that grad ϕ is perpendicular to the velocity vector $\mathbf{v}(t)$ for all t.

$\boxed{2014, S2}$ 4. Given the vector field $\mathbf{F} = \sin x\ \mathbf{i} + \cos x\ \mathbf{j} + xyz\ \mathbf{k}$ calculate:

i) div $\mathbf{F}$.

ii) curl $\mathbf{F}$.

$\boxed{2015, S1}$ 5. Given the vector field $\mathbf{F} = xz\ \mathbf{i} + y^2\ \mathbf{j} + yz\ \mathbf{k}$ calculate:

i) div $\mathbf{F} = \nabla \cdot \mathbf{F}$,

ii) curl $\mathbf{F} = \nabla \times \mathbf{F}$ and

iii) div (curl $\mathbf{F}$) $= \nabla \cdot (\nabla \times \mathbf{F})$.

$\boxed{2016, S2}$ 6. i) Suppose that

$$\mathbf{r}_1(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$$

and

$$\mathbf{r}_2(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$$

are two curves in $\mathbb{R}^3$. Prove that

$$[\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)]' = \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t) + \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t).$$

ii) Suppose that a particle P moves along a curve $\mathcal{C}$ in $\mathbb{R}^3$ in such a manner that its velocity vector is always perpendicular to its position vector. Using part i) prove that the path $\mathcal{C}$ lies on the surface of a sphere whose centre is the origin.

2016, S2 7. Consider the vector field $\mathbf{F} = (27y - y^3)\mathbf{i} + (x^3)\mathbf{j} + (x - xz)\mathbf{k}$.

i) Calculate $\operatorname{curl} \mathbf{F}$.

ii) Sketch the curve $\mathcal{C}$ in $\mathbb{R}^3$ for which $\operatorname{curl} \mathbf{F}=\mathbf{0}$.

2017, S2 8. A vector field is given by

$$\mathbf{F}(x, y, z) = \sin x \sin y \, \mathbf{k}.$$

i) Calculate $\nabla \times \mathbf{F}$.

ii) Calculate $\nabla \times (\nabla \times \mathbf{F})$.

iii) Hence, or otherwise, evaluate $\nabla \times (\nabla \times (\nabla \times (\nabla \times \mathbf{F})))$.

2014, S1 9. By evaluating an appropriate line integral calculate the work done on a particle traveling in $\mathbb{R}^3$ through the vector field $\mathbf{F} = -y\mathbf{i} + xyz\mathbf{j} + x^2\mathbf{k}$ along the straight line from $(1, 2, 3)$ to $(2, 2, 5)$.

2014, S2 10. Let $\mathcal{C}$ denote the path taken by a particle travelling in a straight line from point $P(-2, 3, 0)$ to point $Q(-2, 0, 3)$.

i) Write down a vector function $\mathbf{r}(t)$ that describes the path $\mathcal{C}$ and give the value of t at the start and the end of the path.

ii) If $\mathbf{F} = y^2 \, \mathbf{i} + xyz \, \mathbf{j} - z^2 \, \mathbf{k}$ evaluate the line integral $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

2015, S1 11. Let $\mathcal{C}$ denote the path taken by a particle travelling anticlockwise around the unit circle, starting *and* ending at the point $(1, 0)$ [i.e., the particle travels completely around the circle].

i) Write down a vector function $\mathbf{r}(t)$ that describes the path $\mathcal{C}$ and give the value of t at the start and the end of the path.

ii) If $\mathbf{F} = -3y \, \mathbf{i} + 3x \, \mathbf{j}$ evaluate the line integral $\displaystyle\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

2015, S2 12. Given a vector field

$$\mathbf{F} = 8e^{-x}\mathbf{i} + \cosh z \, \mathbf{j} - y^2\mathbf{k}$$

i) Compute $\nabla \cdot \mathbf{F}$ (i.e., div $\mathbf{F}$) and $\nabla \times \mathbf{F}$ (i.e., curl $\mathbf{F}$).

ii) Calculate the line integral $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathcal{C}$ is the straight line path from $A(0, 1, 0)$ to $B(\ln(2)\ 1,\ 2)$.

2017, S1 13. A charged particle moves in an electric field given by

$$\mathbf{F}(x, y, z) = 3y\mathbf{i} - 3x\mathbf{j}.$$

Let $\mathcal{C}$ denote the path taken by the particle travelling anticlockwise around the unit circle, starting at $(1, 0)$ and ending at $(0, 1)$.

i) Write down a vector function $\mathbf{r}(\theta)$ that describes the path $\mathcal{C}$ and give the values of θ at the start and the end of the path.

ii) Calculate the work done on the particle as it moves along the path $\mathcal{C}$ by evaluating the line integral $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

$\boxed{\text{2018, S1}}$ 14. Consider the scalar field

$$\phi(x, y, z) = xe^{z-1} + \cos y$$

and let $\mathbf{F} = \nabla\phi$.

i) Calculate $\mathbf{F}$.

ii) What is $\nabla \times \mathbf{F}$?

iii) Hence, or otherwise, calculate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ along the straight line path $\mathcal{C}$ from $(1, 0, 1)$ to $(5, \pi, 1)$.

$\boxed{\text{2018, S2}}$ 15. Consider the vector field

$$\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + (2xyz + 3)\mathbf{k}.$$

i) Calculate div $\mathbf{F}$.

ii) Show that $\mathbf{F}$ is conservative by evaluating curl $\mathbf{F}$.

iii) The path $\mathcal{C}$ in $\mathbb{R}^3$ starts at the point $(3, 4, 7)$ and subsequently travels anticlockwise four complete revolutions around the circle $x^2 + y^2 = 25$ within the plane $z = 7$, returning to the starting point $(3, 4, 7)$. Using part ii) or otherwise, evaluate the work integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.