LECTURE 42
FOURIER SERIES PART I

Suppose that a function f has period $T = 2\pi$. Then f may be approximated by the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1}$$

where the Fourier coefficients a_0, a_n, and b_n are given by

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad (n = 1, 2, \dots) \\ b_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad (n = 1, 2, \dots) \end{aligned}\right\} \tag{2}$$

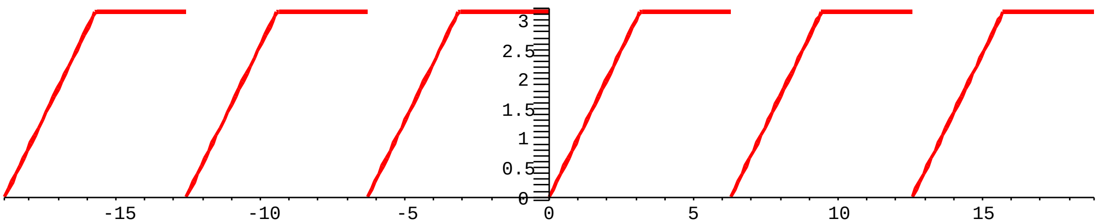
Suppose that we are dealing with a periodic function f of period 2π. Periodic functions are complex in their own special way. We saw earlier that the infinite repetition made the calculation of their Laplace transforms quite a drama. By using Fourier series we can bust such functions up as a sum of much simpler periodic trigonometric components $\sin(nx)$ and $\cos(nx)$. This is of enormous value when dealing with both differential and partial differential equations.

In this lecture we will start off with the simpler case of functions with period $T = 2\pi$. Later on the theory will be generalised to functions of arbitrary period T.

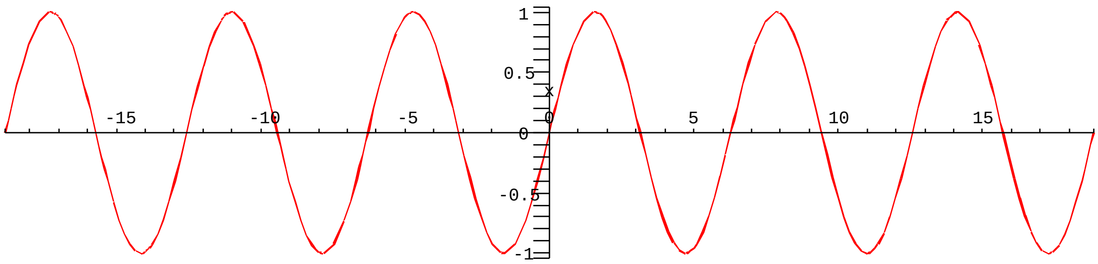
We begin with the periodic function

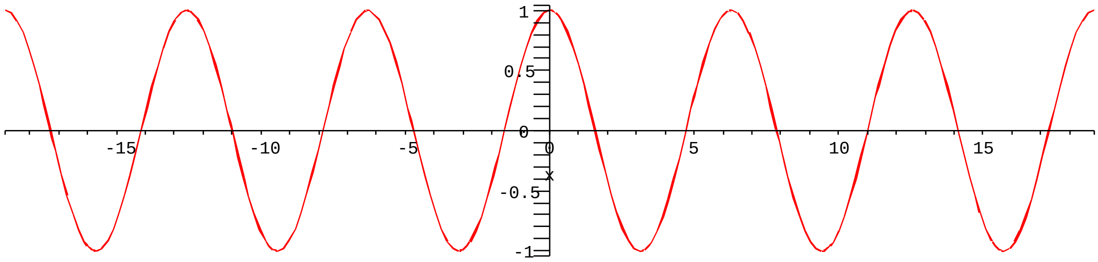
$$f(x) = \begin{cases} x, & 0 \le x < \pi; \\ \pi, & \pi \le x < 2\pi. \\ f(x + 2\pi) & \text{otherwise} \end{cases}$$

This is (by definition) a function of period 2π and the graph over $-6\pi \le x \le 6\pi$ looks like

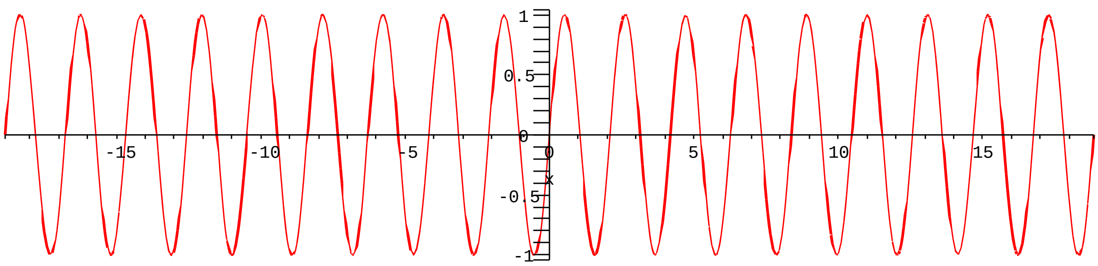


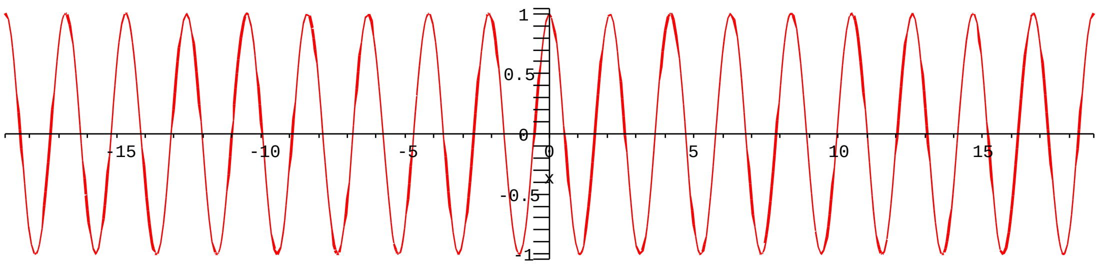
The graphs of $\sin(x)$ and $\cos(x)$ are also of period $T = 2\pi$:





It is fairly clear that there is no way that f could possibly be approximated by taking linear combinations of $\sin(x)$ and $\cos(x)$....... f looks nothing at all like these two functions! But we have a trick up our sleeves! We can make the trig functions busier (by reducing their periods). Consider the graphs of $\sin(3x)$ and $\cos(3x)$:





(Note that in general the period of $\sin(nx)$ and $\cos(nx)$ is $T = \dfrac{2\pi}{n}$).

What we then do to create a Fourier series is to use **all** possible functions $\sin(nx)$ and $\cos(nx)$ together with a constant term a_0 to approximate f as an infinite series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

The coefficients a_0,a_n and b_n are referred to as the Fourier coefficients. The bad news is that their calculation is usually a gruesome process involving the integral formulae

$$\left.\begin{aligned}
a_0 &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
a_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad (n = 1, 2, \ldots) \\
b_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad (n = 1, 2, \ldots)
\end{aligned}\right\}$$

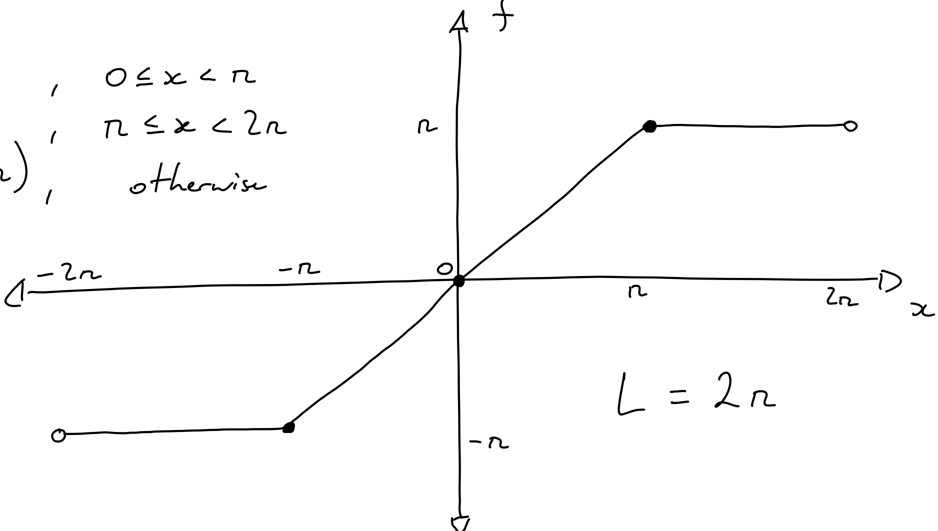
Note that the above equations only apply when the period of the function f is 2π. Minor modifications need to be made for functions of arbitrary period T. We will prove the above formulae in the next lecture but for today lets look at what needs to be done to calculate the Fourier series of our function.

Example 1 Find the Fourier series of the function f above.

Some equations that you must have at your fingertips for Fourier series are:

$$\int \cos(nx)\,dx = \frac{1}{n}\sin(nx)$$
$$\int \sin(nx)\,dx = \frac{-1}{n}\cos(nx)$$
$$\sin(n\pi) = 0$$
$$\cos(n\pi) = (-1)^n$$
$$\sin(-x) = -\sin(x)$$
$$\cos(-x) = \cos(x)$$
$$\sin(0) = 0$$
$$\cos(0) = 1$$

$$f = \begin{cases} x, & 0 \le x < \pi \\ \pi, & \pi \le x < 2\pi \\ f(x+2\pi), & \text{otherwise} \end{cases}$$



$$\bigstar \quad a_0 = \frac{3\pi}{4} \qquad a_n = \frac{(-1)^n - 1}{n^2\pi} \qquad b_n = \frac{-1}{n} \quad \bigstar$$

$$\bigstar \quad f(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty} \{\frac{(-1)^n - 1}{n^2\pi}\} \cos(nx) + \{\frac{-1}{n}\} \sin(nx) \quad \bigstar$$

$$\bigstar \quad f(x) = \frac{3\pi}{4} + \frac{-2}{\pi} \cos(x) - \sin(x) - \frac{1}{2} \sin(2x) - \frac{2}{9\pi} \cos(3x) - \frac{1}{3} \sin(3x) + \dots \quad \bigstar$$

Observe that both a_n and b_n tend to zero as $n \to \infty$. This always happens under normal circumstances and guarantees that the Fourier series will converge for standard functions.

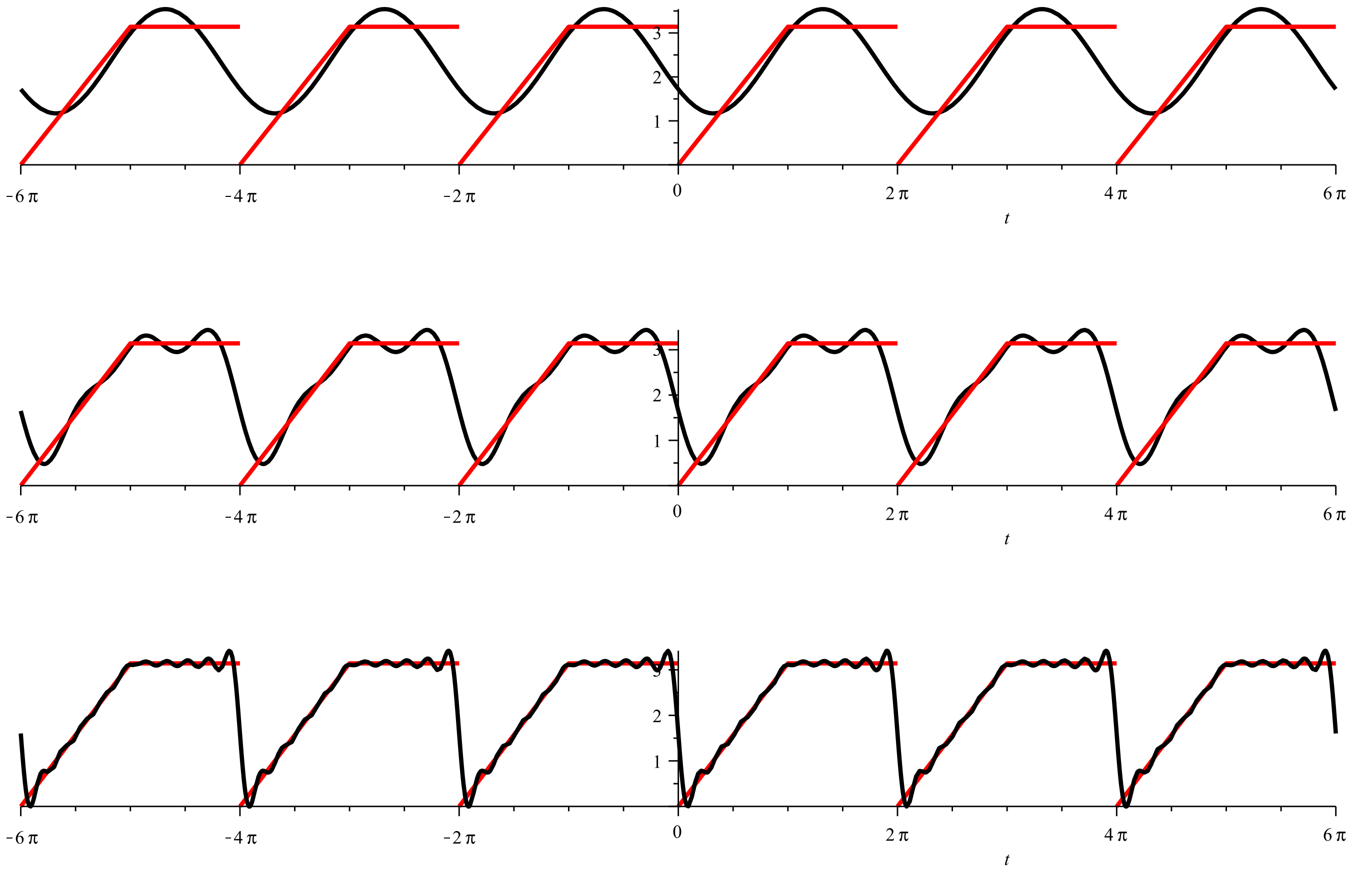
It is fascinating to look at how the Fourier series steps up in accuracy as the number of terms increases. The three graphs below show the first partial sum

$$\frac{3\pi}{4} + \frac{-2}{\pi}\cos(x) - \sin(x)$$

the third partial sum

$$\frac{3\pi}{4} + \frac{-2}{\pi}\cos(x) - \sin(x) - \frac{1}{2}\sin(2x) - \frac{2}{9\pi}\cos(3x) - \frac{1}{3}\sin(3x)$$

and finally the tenth partial sum.



If we were to take infinitely many terms then the Fourier series would sit right over the original function. Observe that the nth partial sums are continuous and yet they do a fine job of approximating discontinuous objects. We will examine carefully at a later stage what actually happens at the points of discontinuity, a situation known as the Gibb's phenomenon.

It is fair to say that the construction of a Fourier series is a major undertaking, but the payoffs are huge. In the next lecture we will still restrict our attention to functions with a period of 2π but will develop some shortcuts which will help us out on occasion.

[42]You can now do Q 108