MATH2019 ENGINEERING MATHEMATICS 2E

Suppose $z = f(x, y)$. Define

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \longrightarrow 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$
$$\frac{\partial f}{\partial y} = \lim_{\Delta y \longrightarrow 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

Notation

$$\frac{\partial f}{\partial x} = f_x = z_x, \qquad \frac{\partial f}{\partial y} = f_y = z_y$$

Hello and welcome to Math2019 Stream B.

You have six Math2019 lectures per week and one Math2019 tutorial per fortnight, as per your timetable. Classes run from Weeks 1-10. The Friday 2-3 lecture is a problem class, where we will go through a selection of examples rather then presenting new content.

• Note that all classes will run from 5 minutes past the hour to 5 minutes to the hour.

• You need to hold a Math1231 pass or better to enrol in Math2019.

• You do not need to purchase a text book.

• The Math2019 Moodle page contains skeleton lecture notes for the entire session, both individually and also as a single pdf. **Please bring printouts to each lecture, as you will need to fill out the notes by hand, in class**.

• On the Math2019 Moodle page you will also find problem class sheets, past exams, extra course notes, tutorial problems for the session and a course outline. You will also find some first year algebra revision lectures which should be read before the eigenvector section of the course. (Lecture 29)

• Read the course outline carefully to ensure that you are completely familiar with the administrative structure of Math2019.

• To each tutorial you should bring your attempted examples for that week together with a printout of tutorial problem set.

• There is no assessable MAPLE in Math2019, though the on-line quizzes and lab tests do use Maple TA.

• Assessment is comprised of two 40 minute lab-based tests valued at 15% each, weekly on-line quizzes making up 8% of your final mark and a writing assignment valued at 2%. The final exam is worth 60%.

Please carefully read the assessment details in the course pack.

LECTURE 1
PARTIAL DIFFERENTIATION

In your previous studies the focus was on functions of a single variable $y = f(x)$ and their rates of change $\dfrac{dy}{dx}$. It is however quite rare for a quantity of interest to depend on only one variable and in complicated physical systems it may be the case that the variable you are concerned with may depend upon dozens of other variables. Partial differentiation is the extension of our usual calculus to functions of several variables.
Given a function of two variables $z = f(x, y)$ we denote the rates of change in the x and y directions as $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ or simply as z_x and z_y. The formal definitions of these derivatives are presented above however in reality we only need to remember a few things to differentiate partially:

The old specific rules of differentiation

y	y'
x^n	nx^{n-1}
e^x	e^x
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\ln(x)$	$\dfrac{1}{x}$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$

The old general rules of differentiation

$$(uv)' = u'v + v'u \qquad \text{Product Rule}$$

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2} \qquad \text{Quotient Rule}$$

The only extra issue that needs to be kept in mind is that when you are differentiating in a particular direction you treat all other variables *exactly* as if they were constant.

Example 1 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $z = x^2 + y^5 + 7$.

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial y} = 5y^4$$

$$\star \quad \frac{\partial z}{\partial x} = 2x, \ \frac{\partial z}{\partial y} = 5y^4 \quad \star$$

Example 2 Suppose that $z = f(x, y) = x^3y^5 + 3x - 8y + 2$. Find the function value and the rate of change of f in the x direction at the point $(1, 2)$.

$$\frac{\partial f}{\partial x} = 3x^2y^5 + 3$$

$$\frac{\partial f}{\partial x}(1, 2) = 3(1)^2(2)^5 + 3 = 99$$

$\bigstar$ $f(1, 2) = 21$, $\dfrac{\partial z}{\partial x}(1, 2) = 99$ $\bigstar$

Example 3 Find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$ if $w = u^3v^4 + \sinh(v^9)$.

$$\frac{\partial w}{\partial u} = 3v^4u^2$$

$$\frac{\partial w}{\partial v} = 4u^3v^3 + 9v^8\cosh(v^9)$$

$\bigstar$ $\dfrac{\partial w}{\partial u} = 3u^2v^4$, $\dfrac{\partial w}{\partial v} = 4u^3v^3 + 9v^8\cosh(v^9)$ $\bigstar$

Example 4 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = \frac{e^{7y}}{x^3+1}$.

$$\frac{\partial z}{\partial x} = \frac{0(x^3+1) - e^{7y}(3x^2)}{(x^3+1)^2}$$

$$= -\frac{3e^{7y}x^2}{(x^3+1)^2}$$

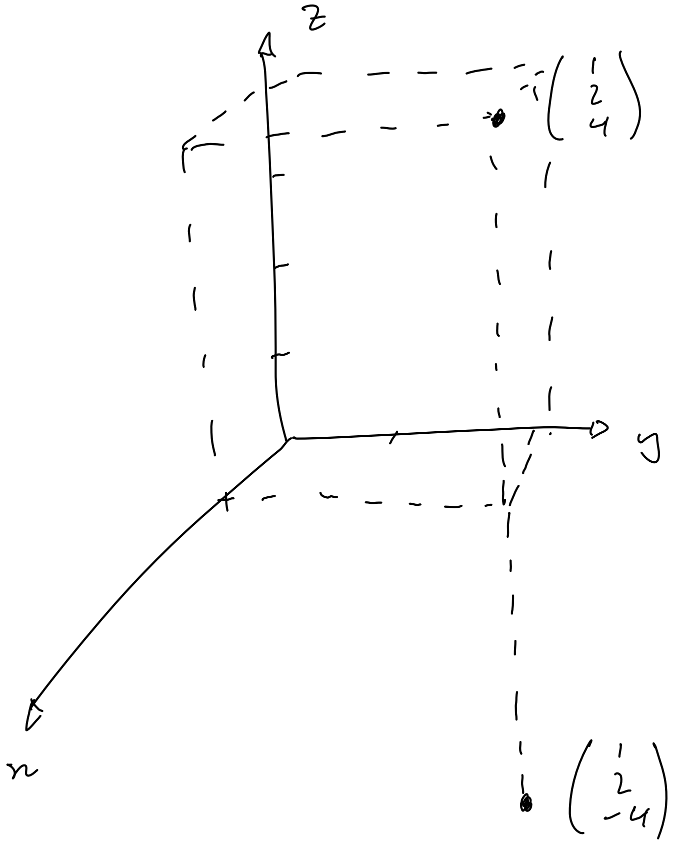
$$\frac{\partial z}{\partial y} = \frac{7e^{7y}}{x^3+1}$$

$$\bigstar \quad \frac{\partial z}{\partial x} = \frac{-3e^{7y}x^2}{(x^3+1)^2},\ \frac{\partial z}{\partial y} = \frac{7e^{7y}}{x^3+1} \quad \bigstar$$

Plotting in Space

Before examining partial derivatives from a geometrical point of view let us consider the issue of sketching in higher dimensions.

Example 5 Plot the points $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ in $\mathbb{R}^3$.



You will observe that plotting in $\mathbb{R}^3$ is somewhat problematic as you are trying to squeeze three dimensions onto a two dimensional page. It gets worse!

Example 6 Plot the point $\begin{pmatrix} 1 \\ 2 \\ 4 \\ 7 \end{pmatrix}$ in $\mathbb{R}^4$.

no

You will recall that the graph of $y = f(x)$ is generally a curve in $\mathbb{R}^2$......lines, parabolas, hyperbolas etc. The graph of $z = f(x, y)$ is always a *surface* in $\mathbb{R}^3$.

Example 7 Sketch each of the following surfaces in space:

a) $3x + 4y + 6z = 12$

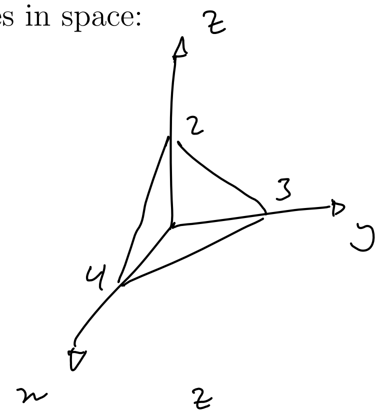
b) $x^2 + y^2 + z^2 = 25$

c) $x^2 + y^2 = 9$

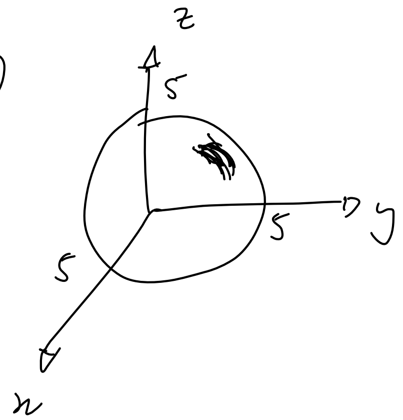
d) $z = 3\sqrt{x^2 + y^2}$

e) $z = x^2 + y^2$

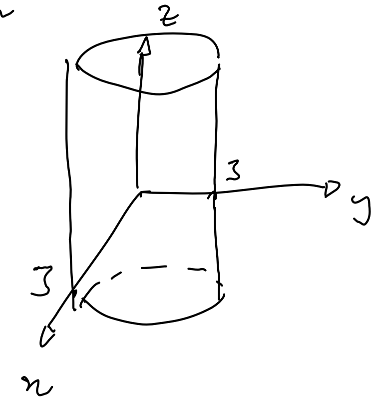
(a)



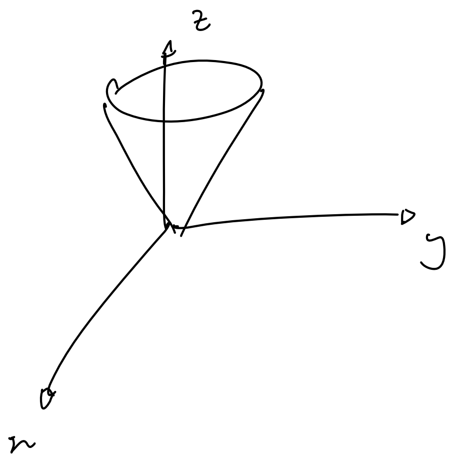
(b)



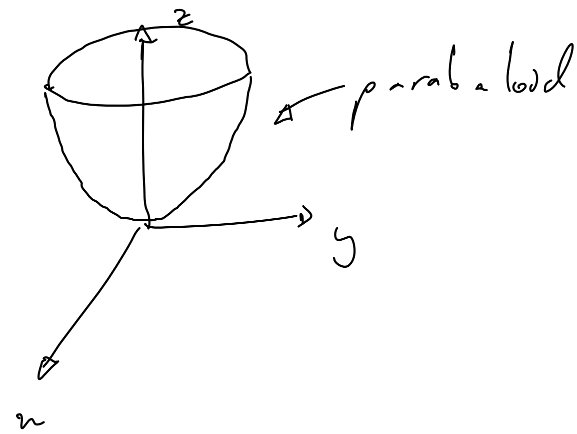
(c)



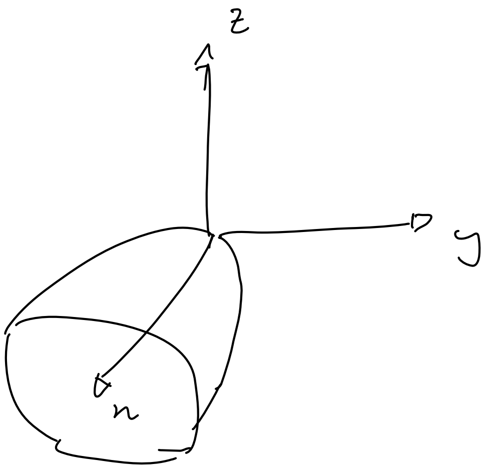
(d)



(e)



Question How about $x = y^2 + z^2$?



SUMMARY

a) $ax + by + cz = d$ is a plane.

b) $x^2 + y^2 + z^2 = r^2$ is a sphere centre the origin radius r.

c) If a variable is absent it is **!unrestricted!** Extrude the two dimensional curve into the missing direction.

d) $z = \alpha\sqrt{x^2 + y^2}$ is a cone with semi-vertical angle $\tan^{-1}(\frac{1}{\alpha})$.

e) $z = \alpha(x^2 + y^2)$ is a paraboloid of revolution.

directional derivative

notes

There is of course no reason why we must restrict ourselves to two independent variables!!

Example 8 If $f(x_1, x_2, x_3, x_4, x_5, x_6) = x_1^3 x_3^4 + \dfrac{x_5}{\sin(x_6)} + \ln(x_4) - \dfrac{\sinh(x_2)}{e^{x_1}}$ find $\dfrac{\partial f}{\partial x_4}$.

$$\frac{\partial f}{\partial x_4} = \frac{1}{x_4}$$

$\bigstar \quad \dfrac{1}{x_4} \quad \bigstar$

As in single variable calculus we make extensive use of second and higher order derivatives. However with partial differentiation we have many more options!

Example 9 If $z = f(x, y) = x^2 \sin(y) + x^3 y + y^5$ find

$$\frac{\partial z}{\partial x} = 2x \sin(y) + 3x^2 y$$

$$\frac{\partial z}{\partial y} = x^2 \cos(y) + x^3 + 5y^4$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2x \cos(y) + 3x^2$$

$$\frac{\partial^2 z}{\partial y \partial x} = 2x \cos(y) + 3x^2$$

$$\frac{\partial^2 z}{\partial x^2} = 2 \sin(y) + 6xy$$

$$\frac{\partial^2 z}{\partial y^2} = -x^2 \sin(y) + 20y^3$$

You will observe in the above example that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ This is true for most reasonably well behaved functions. Note however that in general $\frac{\partial^2 z}{\partial x^2} \neq \frac{\partial^2 z}{\partial y^2}$.

Note also that $\frac{\partial^2 z}{\partial x \partial y}$ is most definitely not equal to $(\frac{\partial z}{\partial x})(\frac{\partial z}{\partial y})$.

[1]You can now do Q's 1 to 4