LECTURE 41
PERIODICITY

- A function f is said to have a period of T if $f(x+T) = f(x)$ for all x.
- $\sin(x)$ and $\cos(x)$ have period 2π.
- $\sin(nx)$ and $\cos(nx)$ have period $\dfrac{2\pi}{n}$.
- A function f is said to be odd if $f(-x) = -f(x)$ for all x.
- A function f is said to be even if $f(-x) = f(x)$ for all x.
- $\sin(-x) = -\sin(x)$ (sin is an odd function).
- $\cos(-x) = \cos(x)$ (cos is an even function).
- Odd× Odd=Even, Odd× Even=Odd, Even× Even=Even.
- Odd±Odd=Odd, Even±Even=Even.
- $\displaystyle\int_{-a}^{a} \text{Odd } dx = 0$ $\qquad$ $\displaystyle\int_{-a}^{a} \text{Even } dx = 2\int_{0}^{a} \text{Even } dx$

This lecture will prepare you for the theory of Fourier series by investigating various issues and definitions surrounding the concept of periodicity. It is quite common for physical systems to exhibit strong repetitive characteristics. For example consider the motion of a piston or the timing on spark plugs. Fourier series provide us with the tools to analyse these systems effectively by making specialised use of the sine and cosine functions which are of course the fundamental periodic objects in mathematics. But first we need to get a feel for how repetition is dealt with at a technical level.

Definition: A function f is said to have a period of T if $f(x+T) = f(x)$ for all x. This means that the function repeats itself every T units.

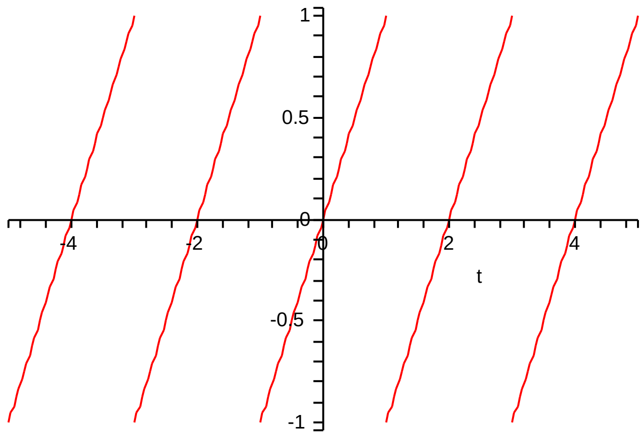
Example 1 Find the period of each of the following functions:

i) $f(x) = \sin(x)$ $\qquad T = 2\pi$

ii) $f(x) = 4\cos(7x)$ $\qquad T = \dfrac{2\pi}{7}$

iii) $f(x) = x^2$ $\qquad T = 0$

iv) f has the following graph:



$T = 2$

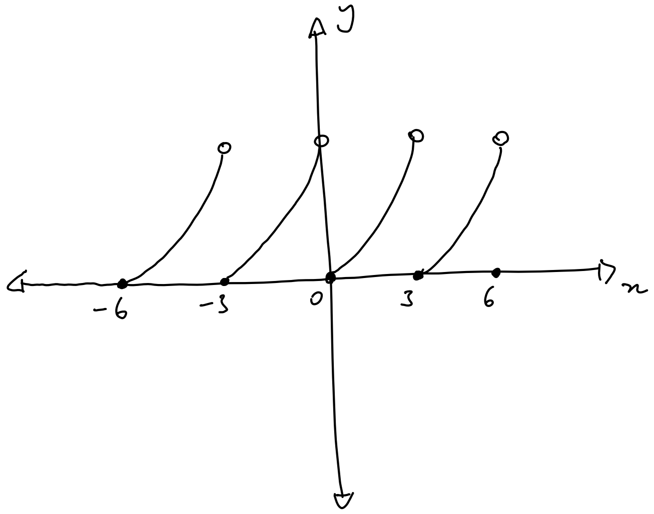
★ 2π, $\dfrac{2\pi}{7}$, no period, 2 ★

Note that $\sin(nx)$ and $\cos(nx)$ both have a period of $\dfrac{2\pi}{n}$.

Example 2 Suppose that $f(t) = \begin{cases} t^2, & 0 \le t < 3; \\ f(t+3), & \text{otherwise.} \end{cases}$

Note that the above definition *forces* a periodicity of 3.

Sketch a graph of f for $-6 \le t \le 6$. What is the value of $f(11)$?



$f(11) = f(8)$ (periodicity of 3)

$= f(5)$

$= f(2)$

$= 2^2$

$= 4$

★

$$\text{odd}\times \text{ odd} \to \text{even}$$

$$\text{odd}\times \text{ even} \to \text{odd}$$

$$\text{even}\times \text{ even} \to \text{even}$$

$$\frac{\text{odd}}{\text{even}} \to \text{odd}$$

$$\frac{\text{even}}{\text{odd}} \to \text{odd}$$

$$\frac{\text{even}}{\text{even}} \to \text{even}$$

$$\frac{\text{odd}}{\text{odd}} \to \text{even}$$

$$\text{odd} \pm \text{odd} = \text{odd}$$

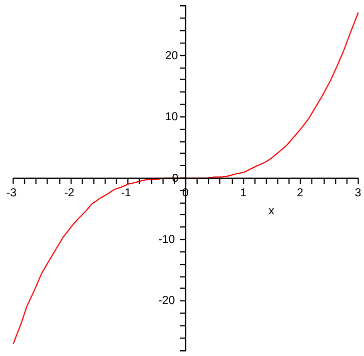
$$\text{even} \pm \text{even} = \text{even}$$

$$\text{even} \pm \text{odd} = \text{nothing}$$

$$\int_{-a}^{a} \text{Odd } dx = 0$$

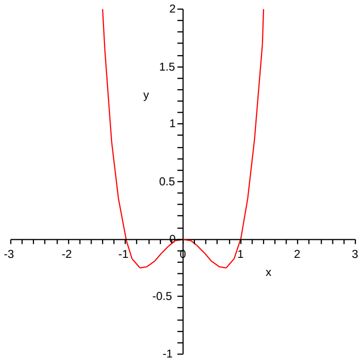
$$\int_{-a}^{a} \text{Even } dx = 2\int_{0}^{a} \text{Even } dx$$

Example 3 Identify each of the following functions as odd, even or neither:



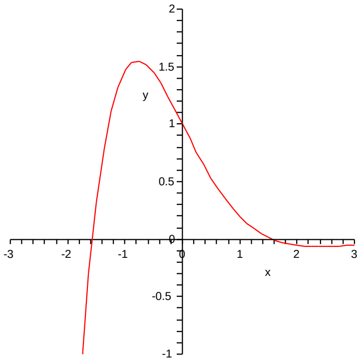
odd

Curve 1



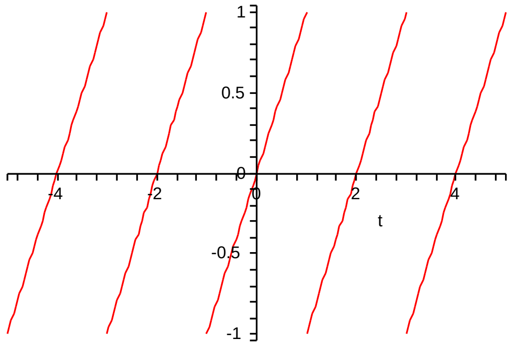
even

Curve 2



neither

Curve 3



odd

Curve 4

★

Example 4 Identify each of the following functions as odd, even or neither:

i) $f(x) = \cos(3x)$ even

ii) $f(x) = x\cos(x)$ odd × even = odd

iii) $f(x) = \sin^2(x)$ odd × odd = even

iv) Prove your answer in iii) from the definition.

$$f(-x) = \sin^2(-x)$$
$$= (-1)^2 \sin^2(x)$$
$$= \sin^2(x)$$
$$= f(x)$$

$\therefore f$ is even

★

Example 5 Evaluate $\displaystyle\int_{-3}^{3} \frac{\overset{\text{even}}{x^2}\,\overset{\text{even}}{\cos(7x)}\,\overset{\text{odd}}{\sin(2x)}}{\underset{\text{even}}{1+x^2+x^4}} = 0$

★

Example 6 Prove that the derivative of an even function is an odd function.

let $f(x) = f(-x)$ (even)

$f'(x) = f'(-x) \times -1$

$\therefore f'(x) = -f'(-x)$

which is odd

★

Example 7 Suppose that $f(t) = \begin{cases} 2t, & 0 \le t < 1; \\ 2, & 1 \le t < 2. \end{cases}$

i) Sketch f over its domain.

ii) Sketch g, the **periodic extension** of f over $-8 \le t \le 8$.
What is the period of g? Evaluate $g(2)$ and $g(8.3)$.

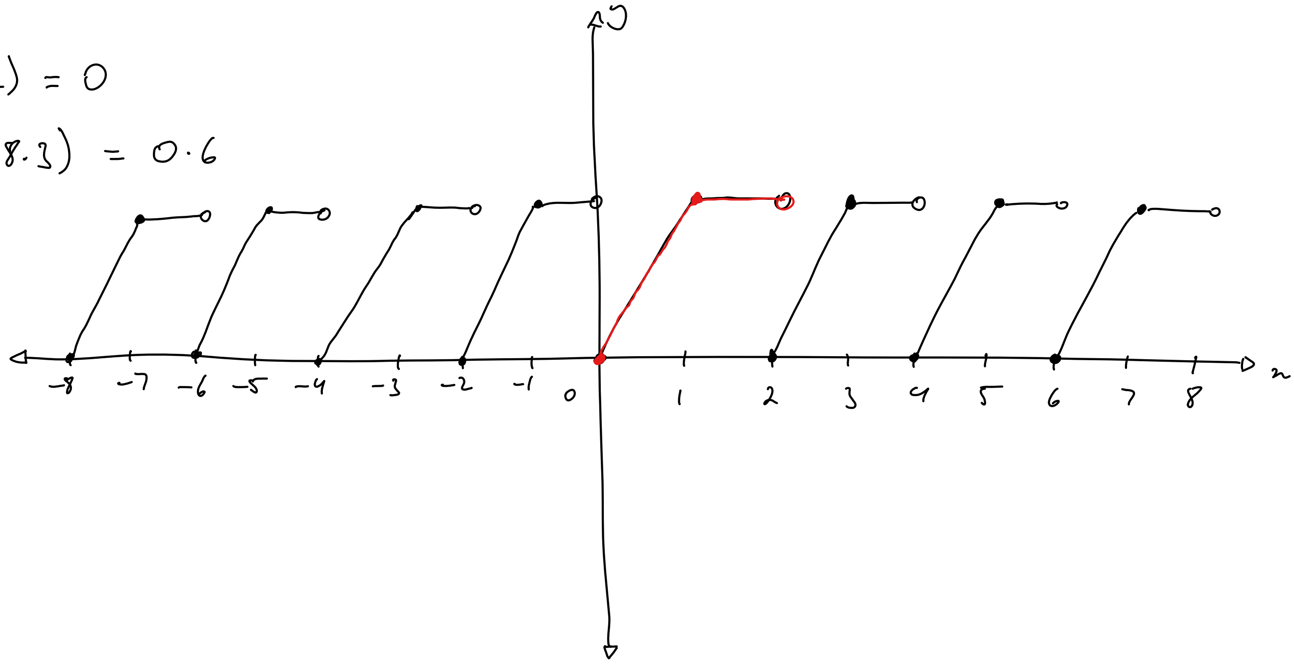
iii) Sketch h, the **even periodic extension** of f over $-8 \le t \le 8$.
What is the period of h? Evaluate $h(15.8)$.

iv) Sketch j, the **odd periodic extension** of f over $-8 \le t \le 8$.
What is the period of j? Evaluate $j(14.7)$.

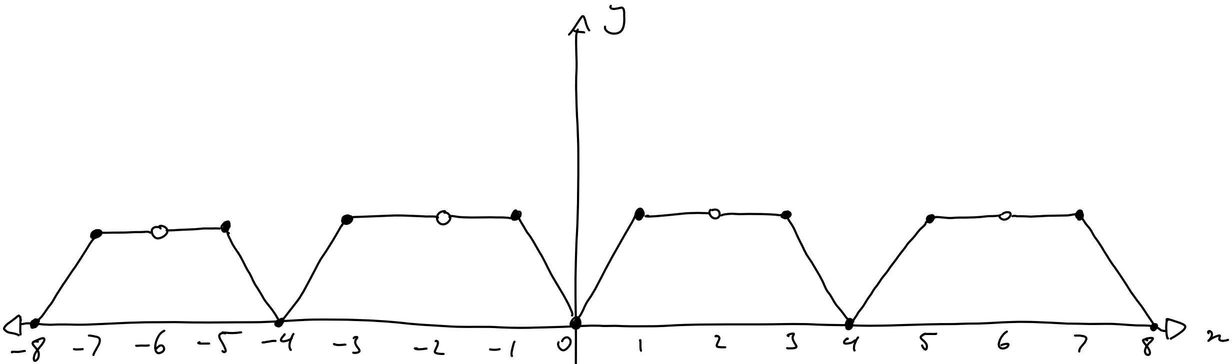
v) Sketch the derivative of j over $-8 \le t \le 8$.

$g(2) = 0$

$g(8.3) = 0.6$

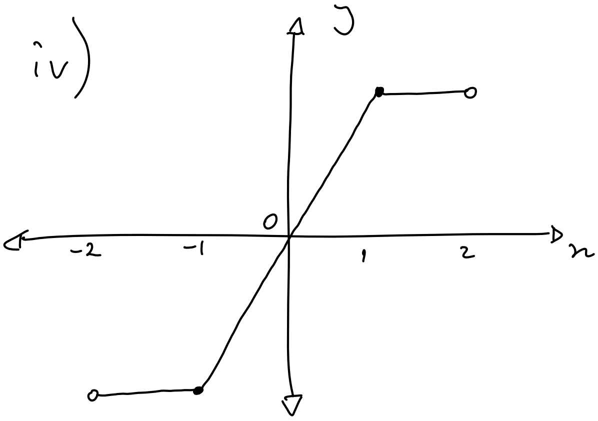


iii)

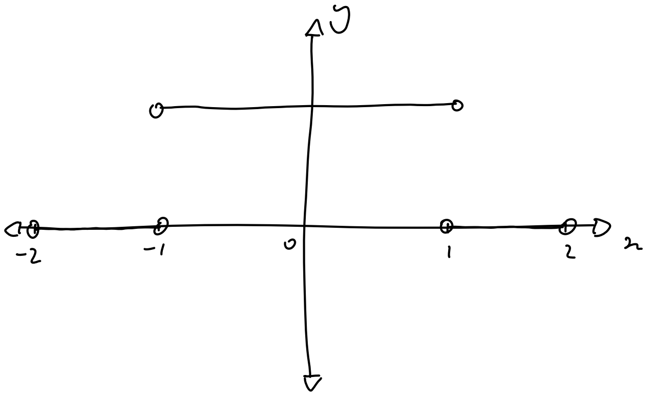


$$h(15.8) = h(3.8)$$
$$= h(-0.2)$$
$$= h(0.2)$$
$$= 0.4$$

iv)



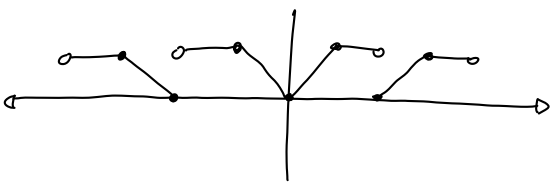
v)



$$j(14.7) = j(-1.3)$$
$$= -j(1.3)$$
$$= -2$$

★

One last comment regarding a very common error when sketching the even periodic extension h. Consider the following (incorrect) sketch of h:



This is certainly **even** but it is **not** periodic! For a periodic function we need to have a single fundamental component graph repeated endlessly.

[41]You can now do Q 106 107