MATH2019 LECTURES 13 and 13A
SOME PROOFS AND THE THEORY OF LINE INTEGRALS

Line integrals are used to calculate the work done in moving a particle P from A to B along a path $\mathcal{C}$ through a force field $\mathbf{F}$.

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} (F_1 dx + F_2 dy + F_3 dz)$$

In general, this integral depends not only on $\mathbf{F}$ but also on the **path** $\mathcal{C}$ we take from A to B.

$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ denotes the line integral around a **closed** curve $\mathcal{C}$. (That is $A = B$)

Two simple properties

I)
$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = -\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$
where $\mathcal{C}_1$ is the same curve as $\mathcal{C}$ except that we start at B and finish at A. That is, reversing the direction of the path changes the sign of a line integral.

II)
$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$
where $\mathcal{C}_1$ is the curve from A to X, and $\mathcal{C}_2$ the curve from X to B following the path $\mathcal{C}$. In other words breaking the journey does not affect the line integral.

This is quite a long lecture and will be delivered over two hours. We will start with a couple of proofs of claims made in the previous lectures and then move on to the general theory of line integrals. There should be time at the end of the second hour for you to ask any questions you may have regarding the material up to and including line integrals.

Claim: Let $\phi(x, y, z)$ be a scalar field in space and let P be a point on the level surface $\phi(x, y, z) = c$. Then grad(ϕ) at P is perpendicular to the level surface $\phi(x, y, z) = c$ at P.

Proof: Let $\phi(x, y, z)$ be a scalar field in space and let P be a point on a level surface $\phi(x, y, z) = c$. Let $x = x(t)$, $y = y(t)$ and $z = z(t)$ be any parametrically defined path passing throuph P and embedded within the level surface $\phi(x, y, z) = c$. Differentiating with respect to t and implementing the chain rule yields

$$\frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt} = 0$$

and hence

$$\begin{pmatrix} \dfrac{\partial \phi}{\partial x} \\[2ex] \dfrac{\partial \phi}{\partial y} \\[2ex] \dfrac{\partial \phi}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = 0 \rightarrow (\text{grad}\phi) \cdot \mathbf{v} = 0$$

Noting that the velocity vector is tangential to the path, we have shown that gradϕ is perpendicular to all of the tangents to the level surface at P, and thus is perpendicular to the level surface as required. The result and proof in other dimensions is similar.

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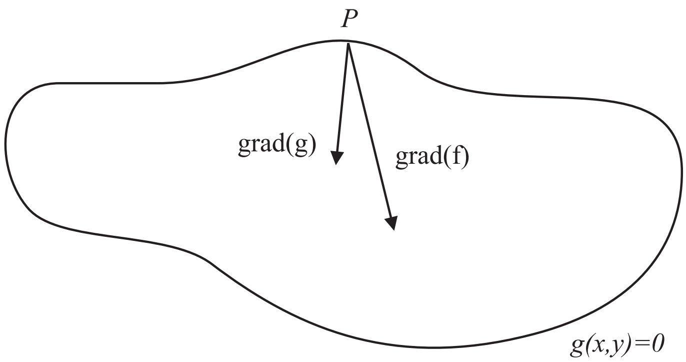
An Intuitive Verification of the Method of Lagrange

Recall that the method of Lagrange Multipliers states that, to find the local minima and maxima of $f(x, y)$ subject to the constraint $g(x, y) = 0$ we find the values of x, y and λ that simultaneously satisfy the equations

$$\frac{\partial f}{\partial x} - \lambda\frac{\partial g}{\partial x} = 0, \quad \frac{\partial f}{\partial y} - \lambda\frac{\partial g}{\partial y} = 0, \quad \text{together with} \quad g(x, y) = 0.$$

We can now justify this result using vector field theory:

The constraint $g(x, y) = 0$ is simply a level curve for the scalar field $g(x, y)$. Pick a point P on the curve $g(x, y) = 0$. At P, grad(g) will point in the direction of maximal increase of $g(x, y)$ and will also point perpendicular to the level curve $g(x, y) = 0$.



However $g(x, y) = 0$ is not a level curve for $f(x, y)$. Thus grad(f) will most likely not point perpendicular to $g(x, y) = 0$. Since grad(f) points in the direction of maximum increase of $f(x, y)$ it will therefore be possible to increase $f(x, y)$ by sliding a little around the level curve from P, in the direction suggested by grad(f).

If the point P is actually sitting at a maximum value of $f(x, y)$ then there will be no component of grad(f) in the tangential direction through which $f(x, y)$ could be increased. In other words if P is positioned so that $f(x, y)$ is maximised (or minimised),then grad(f) will **also** point perpendicular to $g(x, y) = 0$. This means that grad(f) and grad(g) are parallel vectors and hence that grad(f) $= \lambda\,$grad(g). This means that

$$\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \lambda(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j})$$

and so

$$\frac{\partial f}{\partial x} = \lambda\frac{\partial g}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \lambda\frac{\partial g}{\partial y}$$

as required.

LINE INTEGRALS

We have seen in the previous lectures how partial differentiation may be used to analyse vector fields. Integration also plays a role in the theory through the concept of a line integral. Some of you will have seen the formula for the work done by a force over a distance given by Work=Force$\times$ Distance. We are in a similar but slightly more complicated situation.

We have a force vector field rather that just a force. Also we have a path rather than just a distance. A line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ measures the work done on a particle as it moves through a vector field $\mathbf{F}$ along a path $\mathcal{C}$. The dot product measures the interaction between the path and the field, with the work done being zero if the path sits perpendicular to the field. A positive work integral indicates that the the particle is being worked upon while a negative value indicates that the particle is doing work over the length of the path. It's like swimming with the current $(+)$ or against the current $(-)$.

Line integrals are usually evaluated by parametrising the path and then converting the line integral into a standard integral in terms of the parameter. In certain circumstances (when the vector field is conservative) there is also a dramatic shortcut to their evaluation.

The Meaning of $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$

$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a vector field and $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$ is an increment along the path $\mathcal{C}$. The line integral "sums" all the dot products $\mathbf{F} \cdot d\mathbf{r}$ as we traverse the path. The dot products are large when $\mathbf{F}$ and $d\mathbf{r}$ are close to parallel and small when they are close to perpendicular. As an example consider swimming in a river:

Example 1: Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = (1-2y)\mathbf{i} + 2x\mathbf{j}$ and $\mathcal{C}$ is the path in $\mathbb{R}^2$ along the circle $x^2+y^2=9$ from $(3,0)$ to $(0,3)$ in the first quadrant.

The key to the evaluation of most line integrals is to express the path parametrically with strict limits on the parameter, and then recast the line integral as a simple one dimensional integral in terms of the parameter. The standard parametric representation of a circle of radius r is given by $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

$$\int \tilde{F}\cdot d\tilde{r} = \int_C \begin{pmatrix}1-2y\\ 2x\end{pmatrix}\cdot\begin{pmatrix}dx\\ dy\end{pmatrix}$$

$$= \int_C (1-2y)\,dx + 2x\,dy$$

$$= \int_C (1-6\sin t)(-3\sin t)\,dt + 6\cos t(3\cos t)\,dt$$

$$= \int_C -3\sin t + 18\sin^2 t + 18\cos^2 t\,dt$$

$$= \int_C -3\sin t + 18\,dt$$

$$= \Big[3\cos t + 18t\Big]_0^{\frac{\pi}{2}}$$

$$= 9\pi - 3$$

$x = 3\cos t$ $\qquad y = 3\sin t$

$dx = -3\sin t\,dt$ $\qquad dy = 3\cos t\,dt$

$t \in [0, \frac{\pi}{2}]$

★ $9\pi - 3$ ★

Question: What would the line integral be if we went back the other way along the circle from $(0,3)$ to $(3,0)$?

$3 - 9\pi$

Question: If we went all the way around the circle is it always true that $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$?

Yes

Example 2: Find the work done on a particle traveling through the field $\mathbf{F} = y^2\mathbf{i} + (y^3 + e^z)\mathbf{j} + (x - 2z)\mathbf{k}$ along the straight line from $(1,1,1)$ to $(3,1,4)$ in $\mathbb{R}^3$.

$$\vec{F} = \begin{pmatrix} y^2 \\ y^3 + e^z \\ x - 2z \end{pmatrix}$$

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} t, \quad t \in [0, 1]$$

$$\int \vec{F} \cdot d\vec{r} = \int \begin{pmatrix} y^2 \\ y^3 + e^z \\ x - 2z \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

$$= \int y^2 \, dx + (y^3 + e^z) \, dy + (x - 2z) \, dz$$

$$= \int_0^1 1^2 (2\,dt) + (1^3 + e^{1+3t})(0) + (1 + 2t - 2(1+3t))(3\,dt)$$

$$= \int_0^1 2 + 3 + 6t - 6 - 18t \, dt$$

$$= \int_0^1 -1 - 12t \, dt$$

$$= \Big[-t - 6t^2 \Big]_0^1$$

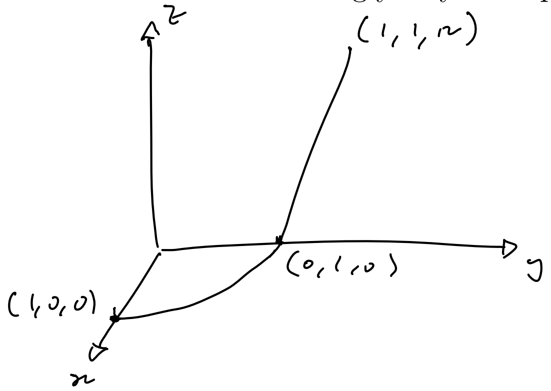
$$= -1 - 6$$

$$= -7$$

★ -7 ★

Example 3 : Find the work done on a particle traveling through the field $\mathbf{F} = (yz+2)\mathbf{i} + xz\mathbf{j} + (xy+\cos(z))\mathbf{k}$ along the circle $x^2+y^2=1$ in the $x-y$ plane from $(1,0,0)$ to $(0,1,0)$ and then along the straight line from $(0,1,0)$ to $(1,1,\pi)$.

This particular type of example has a dramatic shortcut which we will discuss in the next example. But first let's do it the ugly way. The picture here is:



The path is in 2 clear pieces so we will need to evaluate 2 independent work integrals. First the little circular bit:

$x = \cos(\theta) \to dx = -\sin(\theta)d\theta$

$y = \sin(\theta) \to dy = \cos(\theta)d\theta$

$z = 0 \to dz = 0d\theta.$

Note finally that $\theta : 0 \to \dfrac{\pi}{2}$. Hence:

$$\textstyle\int_C \mathbf{F}\cdot d\mathbf{r} = \displaystyle\int_C \begin{pmatrix} yz+2 \\ xz \\ xy+\cos(z) \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \int (yz+2)dx + xzdy + (xy+\cos(z))dz.$$

Since both z and dz are 0 we obtain:

$\int_0^{\frac{\pi}{2}} (0+2)(-\sin(\theta))d\theta = \int_0^{\frac{\pi}{2}} -2\sin(\theta)d\theta = [2\cos(\theta)]_0^{\frac{\pi}{2}} = 0-2 = -2.$

Along the line interval we have:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ \pi \end{pmatrix} t, \quad t : 0 \to 1. \text{ Thus}$$

$x = t \to dx = dt, \quad y = 1 \to dy = 0dt$ and $z = \pi t \to dz = \pi dt$. So

$\int (yz+2)dx + xzdy + (xy+\cos(z))dz = \int_0^1 (\pi t+2)dt + xz(0dt) + (t+\cos(\pi t))\pi dt$

$= \int_0^1 \pi t + 2 + \pi t + \pi\cos(\pi t)dt = \int_0^1 2\pi t + 2 + \pi\cos(\pi t) = [\pi t^2 + 2t + \sin(\pi t)]_0^1$

$= (\pi+2+0) - (0) = \pi+2$. The total work done is then $-2 + (\pi+2) = \pi$.

$\bigstar \quad -2 + (\pi+2) = \pi \quad \bigstar$

Definition: A vector field $\mathbf{F}$ is said to be **conservative** if $F = \text{grad}(\phi)$ for some scalar field ϕ. (Recall that we also write $\text{grad}(\phi)$ as $\nabla\phi$.)

Most vector fields are NOT conservative BUT if $\mathbf{F}$ is conservative then there is a huge shortcut available for the calculation of the line integral $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

Fact 1: All line integrals for a conservative vector field given by $\mathbf{F} = \text{grad}(\phi)$ are path independent and depend only upon the starting point P and the finishing point Q of the path $\mathcal{C}$. Furthermore

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \Big[\phi\Big]_P^Q$$

Proof: We have $\mathbf{F} = \nabla\phi = \text{grad}(\phi) = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$.

Thus taking **any** parametrisation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$; $(a \leq t \leq b)$ of $\mathcal{C}$ we have

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} \begin{pmatrix} \dfrac{\partial\phi}{\partial x} \\[2ex] \dfrac{\partial\phi}{\partial y} \\[2ex] \dfrac{\partial\phi}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{dx}{dt} \\[2ex] \dfrac{dy}{dt} \\[2ex] \dfrac{dy}{dt} \end{pmatrix} dt = \int_a^b \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}\, dt$$

and hence via the chain rule

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \frac{d\phi}{dt}\, dt = \Big[\phi\Big]_P^Q$$

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All the above fact is saying is that if $\mathbf{F} = \text{grad}(\phi)$ for some ϕ then $\mathbf{F}$ is the "derivative" of ϕ and hence when you "integrate" $\mathbf{F}$ you get ϕ.

Please remember however most vector fields are NOT conservative and hence the above fact is useless in general. But we do have a simple test for whether or not a vector field is conservative:

Fact 2: A vector field $\mathbf{F}$ is conservative if and only if curl($\mathbf{F}$)=0.

That is, $\mathbf{F}$ is conservative if there is never a rotational force at a point! Going left is the same as going right so in the end the path doesn't actually matter.

Lets now return to the previous example and do it using our shortcut.

Example 4: For $\mathbf{F} = (yz+2)\mathbf{i} + xz\mathbf{j} + (xy+\cos(z))\mathbf{k}$ of the previous example:

a) Prove $\mathbf{F}$ is conservative.

b) Find a scalar field ϕ such that $\mathbf{F}$=grad(ϕ).

c) Hence check the answer in Example 3 by re-evaluating the work integral making efficient use of ϕ .

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a)

$$\mathrm{Curl}(\vec{F}) = \begin{vmatrix} \frac{\partial \vec{F}}{\partial x} & \frac{\partial \vec{F}}{\partial y} & \frac{\partial \vec{F}}{\partial z} \\ yz+2 & xz & xy+\cos z \end{vmatrix} = \begin{pmatrix} x-x \\ y-y \\ z-z \end{pmatrix} = \vec{0}$$

b) Finding ϕ is a little tricky. We are told that

$$\mathbf{F} = (yz+2)\mathbf{i} + xz\mathbf{j} + (xy+\cos(z))\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} = \mathrm{grad}(\phi)$$

Thus

$$\frac{\partial \phi}{\partial x} = yz+2 \Longrightarrow \quad \phi = xyz + 2x + g_1(y,z)$$

$$\frac{\partial \phi}{\partial y} = xz \Longrightarrow \quad \phi = xyz + g_2(x,z)$$

$$\frac{\partial \phi}{\partial z} = xy+\cos(z) \Longrightarrow \quad \phi = xyz + \sin z + g_3(x,y)$$

Putting it all together we have $\phi = xyz + 2x + \sin z$

c) $$\int_C \vec{F}\cdot d\vec{r} = \Big[\phi\Big]_{\text{start}}^{\text{finish}} = \Big[xyz + 2x + \sin z\Big]_{(1,0,0)}^{(1,1,\pi)}$$

$$= \pi + 2 + 0 - (0 + 2 + 0)$$

$$= \pi$$

★ π (again) ★

Question: What is the value of $\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$ if $\mathbf{F}$ is conservative? (Note $\oint_{\mathcal{C}}$ denotes an integral around a closed path $\mathcal{C}$).

$$\oint_C \vec{F}\cdot d\vec{r} = 0$$

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Example 5: Consider the vector field

$$\mathbf{F} = (2xy^3z^4 + ye^{xy})\mathbf{i} + (3x^2y^2z^4 + xe^{xy} + \frac{z}{y})\mathbf{j} + (4x^2y^3z^3 + \ln y)\mathbf{k}$$

a) (HOMEWORK) Show that $\mathbf{F}$ is conservative by proving that $\text{curl}(\mathbf{F}) = \mathbf{0}$.

b) Find a scalar field ϕ such that $\mathbf{F}$=grad(ϕ).

c) The path $\mathcal{C}$ starts at the point $(3, 4, 7)$ and rotates anticlockwise three complete revolutions in the circle $x^2+y^2 = 25$ within the plane $z = 7$ returning to the point $(3, 4, 7)$.

It then travels along a straight line from $(3, 4, 7)$ to $(6, 8, 12)$ and rotates 3 complete revolutions clockwise in the circle $x^2 + y^2 = 100$ within the plane $z = 12$ returning to the point $(6, 8, 12)$.

The path then drops straight down onto the $x - y$ plane meeting the $x - y$ plane at $(6, 8, 0)$ and finally returns along a straight line to $(3, 4, 7)$.

Sketch the path and evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

b) $\frac{\partial \phi}{\partial x} = 2xy^3z^4 + ye^{xy}$

$\phi = x^2y^3z^4 + e^{xy} + g_1(y,z)$

$\frac{\partial \phi}{\partial y} = 3x^2y^2z^4 + xe^{xy} + \frac{z}{y}$

$\phi = x^2y^3z^4 + e^{xy} + z\ln(y) + g_2(x,z)$

$\frac{\partial \phi}{\partial z} = 4x^2y^3z^3 + \ln(y)$

$\phi = x^2y^3z^4 + z\ln(y) + g_3(x,y)$

$\therefore \phi = x^2y^3z^4 + e^{xy} + z\ln(y)$

c) $\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = 0$

a)

$$\begin{vmatrix} \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ 2xy^3z^4 + ye^{xy} & 3x^2y^2z^4 + xe^{xy} + \frac{z}{y} & 4x^2y^3z^3 + \ln y \end{vmatrix}$$

$$= \begin{pmatrix} 12x^2y^2z^3 + \frac{1}{y} - (12x^2y^2z^3 + \frac{1}{y}) \\ -8xy^3z^3 + 8xy^3z^3 \\ 6xy^2z^4 + e^{xy} + xye^{xy} - (6xy^2z^4 + e^{xy} + xye^{xy}) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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The first quiz will cover all of the material up to and including this lecture. We will consolidate this content in the next lecture (a problem class) and then move on to a new topic.

[13] You can now do Q 61-63