LECTURE 22
FIRST ORDER DIFFERENTIAL EQUATIONS

SEPARABLE

These differential equations are of the form

$$\frac{dy}{dx} = f(x)g(y).$$

and are solved by separating and integrating both sides of the equation.

LINEAR

The general first-order linear o.d.e. is

$$\frac{dy}{dx} + P(x)y = Q(x).$$

If we evaluate the integrating factor

$$R(x) = e^{\int P(x)dx}$$

then the solution is given by

$$y = \frac{1}{R(x)} \int R(x)Q(x)\,dx$$

A differential equation (often abbreviated D.E.) is an equation relating a function to its derivatives. For example

$$\frac{dy}{dx} = x^2y^3$$

is a differential equation. It is important to note that a solution to a differential equation is not a number ...... it's a function. What this equation is asking you is 'can you find a function with the property that differentiating the function is the same as cubing the function and then multiplying by x^2?'. We will shortly examine exactly how we would go about finding such a function (and whether or not such a function is unique!). But first some classifying definitions.

The **order** of a differential equation is the biggest derivative that appears in the equation.

A differential equation is said to be **linear** if all terms involving y and its derivatives appear linearly........the x's can do as they please.

Example 1 Find the order of each of the following differential equations and state whether each equation is linear or not.

1. $\dfrac{dy}{dx} = x^2y^3$ non-linear, order : 1

2. $\dfrac{dy}{dx} + y = x^3$ linear, order : 1

3. $y'' + 6y' - 5y = \dfrac{1}{x^2+5}$ linear, order : 2

★ 2 and 3 are linear and the orders are 1,1 and 2 resp ★

How are differential equations solved? It depends completely on the type of equation (very much like the process of integration). First order equations are attacked in a manner totally different from second order equations and unfortunately many differential equations can't be solved at all. There are however some global rules.

• The solution to any D.E. will always involve arbitrary constants. It is crucial to respect the constants and take good care of them.

• The number of constants in solution is always equal to order of the D.E.

• If a D.E. has initial conditions (i.c.'s) attached to it, the i.c.'s are used to knock off the constants.

In this lecture we will look at the two main types of first order problems....separable and linear. In the next lecture we will discuss substitution and applications of first order D.E.'s. Note that much of the material in these two lectures is revision from first year and you are encouraged to check over your first year notes on D.E.'s to refresh your memory.

SEPARABLE

As the name suggests separable first order D.E.'s are solved by separating the variables out and then running two integrals. Not all first order problems are separable!

Example 2 Solve

$$\frac{dy}{dx} = y(2x+3)$$

.

$$\int \frac{1}{y}\, dy = \int 2x + 3 \quad dx$$

$$\ln(y) = x^2 + 3x$$

$$\therefore y = Ae^{(x^2 + 3x)}$$

★ $y = Ae^{x^2+3x}$ ★

Example 3 Solve

$$\frac{dy}{dx} = \frac{\cos(x)}{y} \qquad \text{where} \qquad y(\frac{\pi}{2}) = 5.$$

.

$$y\,dy = \cos x\,dx$$

$$\frac{y^2}{2} = \sin x + C$$

$$y\left(\frac{\pi}{2}\right) = 5: \quad \frac{25}{2} = \sin\frac{\pi}{2} + C$$

$$\therefore C = \frac{23}{2}$$

$$\therefore y^2 = 2\sin x + 23$$

$$\bigstar \quad y^2 = 2\sin(x) + 23 \quad \bigstar$$

LINEAR

The general first-order linear o.d.e. is

$$\frac{dy}{dx} + P(x)y = Q(x).$$

If we evaluate the integrating factor

$$R(x) = e^{\int P(x)dx}$$

then the solution is given by

$$y = \frac{1}{R(x)} \int R(x)Q(x)\,dx$$

- The integrating factor should always be expressed in its simplest possible form.

- Note that the integrating factor $R(x)$ does **NOT** need a $+C$ but the final solution most certainly does.

Example 4 Solve the first order linear D.E.

$$\frac{dy}{dx} + \frac{2}{x}y = 4x$$

Observe that this D.E. is NOT separable and hence cannot be solved using the above techniques

$$h(x) = e^{\int \frac{2}{x}\, dx} = x^2$$

$$x^2 \frac{dy}{dx} + 2xy = 4x^3$$

$$\int \left(x^2 \frac{dy}{dx} + 2xy \right) dx = \int 4x^3\, dx$$

$$x^2 y = x^4 + C$$

$$\therefore y = x^2 + \frac{C}{x^2}$$

$$\star \quad y = x^2 + \frac{C}{x^2} \quad \star$$

Proof of Linear Formulae

$\star$

Example 5 Solve the initial value problem

$$x\frac{dy}{dx} - y = x^3\cos(x) \quad ; \quad y(\pi) = 0.$$

Check that your solution satisfies both the D.E. and the initial condition.

We MUST rewrite the equation: $\dfrac{dy}{dx} - \dfrac{y}{x} = x^2\cos(x)$

$$e^{\int -\frac{1}{x}\,dx} = \frac{1}{x}$$

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = x\cos(x)$$

$$\frac{y}{x} = \left[x\sin x\right] - \int \sin x\,dx$$

$$y = x^2\sin x + x\cos x + Cx$$

$y(\pi) = 0$: $\quad 0 = 0 - \pi + C\pi$

$\therefore\ C = 1$

$$\therefore\ y = x^2\sin x + x\cos x + x$$

★ $y = x^2\sin(x) + x\cos(x) + x$ ★

Example 6 (HOMEWORK) Find the general solution of the differential equation

$$\frac{dy}{dx} + xy = x\,;$$

a) By treating the D.E. as linear.

b) By treating the D.E. as separable.

c) Check that the two solutions are equivalent.

a) $e^{\int x\,dx} = e^{\left(\frac{x^2}{2}\right)}$

$e^{\left(\frac{x^2}{2}\right)} y = \int e^{\left(\frac{x^2}{2}\right)} x\,dx$

$= e^{\left(\frac{x^2}{2}\right)} + A$

$\therefore y = 1 + Ae^{\left(-\frac{x^2}{2}\right)}$

b) $\frac{1}{1-y}\,dy = x\,dx$

$-\ln(1-y) = \frac{x^2}{2} + C$

$1 - y = e^{\left(-\frac{x^2}{2} - C\right)}$

$\therefore y = 1 + Ae^{\left(-\frac{x^2}{2}\right)}$, $A = -e^{-C}$

★ $y = 1 + De^{-\frac{x^2}{2}}$ ★

It is quite rare for a differential equation to be of two completely different types!

[22]You can now do Q 80 and 83