MATH2019 PROBLEM CLASS

EXAMPLES 2

EXTREMA, METHOD OF LAGRANGE MULTIPLIERS AND DIRECTIONAL DERIVATIVES

2014, S1

1. Find and classify the critical points of

$$f(x,y) = x^3 - 12xy + 8y^3.$$

Solution: First find the critical points.

$$\frac{\partial f}{\partial x} = 3x^2 - 12y = 0 \ \Rightarrow \ y = \frac{1}{4}x^2 \quad (*)$$
$$\frac{\partial f}{\partial y} = -12x + 24y^2 = 0 \ \Rightarrow \ x = 2y^2 \quad (**)\,.$$

Combining $(*)$ and $(**)$ yields

$$y = \frac{1}{4}x^2 = \frac{1}{4}\left(2y^2\right)^2 \ \Rightarrow \ y\left(1 - y^3\right) = 0$$
$$\Rightarrow \ y = 0 (\Rightarrow x = 0) \quad \text{OR} \quad y = 1 (\Rightarrow x = 2)\,.$$

Hence the critical points are $(0,0)$ and $(2,1)$. To classify the critical points we use the 2nd derivative test. The second order partial derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial y\,\partial x} = -12, \quad \frac{\partial^2 f}{\partial y^2} = 48y$$

and therefore

$$\mathcal{D}(x,y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y\,\partial x}\right)^2 = 288xy - 144 = 144(2xy - 1)\,.$$

At each of the critical points we have

$$\mathcal{D}(0,0) = -144 < 0 \ \Rightarrow \ \text{saddle point at } (0,0) \text{ and}$$
$$\mathcal{D}(2,1) = 144 > 0 \ \Rightarrow \ \text{local minimum at } (2,1) \text{ since } \frac{\partial^2 f}{\partial x^2}(2,1) > 0\,.$$

The following is not asked for but included for completeness. The value of f at each of the critical points is

$$f(0,0) = (0)^3 - 12(0)(0) + 8(0)^3 = 0$$
$$f(2,1) = (2)^3 - 12(2)(1) + 8(1)^3 = -8\,.$$

2014, S2

2. Find and classify the critical points of

$$f(x,y) = 2x^3 - 15x^2 + 36x + y^2 + 4y - 16.$$

Also give the function value at the critical points.

Solution: First find the critical points.

$$\frac{\partial f}{\partial x} = 6x^2 - 30x + 36 = 0 \ \Rightarrow \ x^2 - 5x - 6 = (x-2)(x-3) = 0$$
$$\Rightarrow \ x = 2 \quad \text{or} \quad x = 3\,,$$
$$\frac{\partial f}{\partial y} = 2y + 4 = 0 \ \Rightarrow \ y = -2\,.$$

Hence the critical points are $(2, -2)$ and $(3, -2)$. To classify the critical points we use the 2nd derivative test. The second order partial derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 12x - 30, \quad \frac{\partial^2 f}{\partial y\, \partial x} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 2$$

and therefore

$$\mathcal{D}(x, y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y\, \partial x}\right)^2 = 24x - 60 = 12(2x - 5)\,.$$

At each of the critical points we have

$$\begin{aligned}
\mathcal{D}(2, -2) = -12 < 0 \quad &\Rightarrow \quad \text{saddle point at } (2, -2) \text{ and} \\
\mathcal{D}(3, -2) = 12 > 0 \quad &\Rightarrow \quad \text{local minimum at } (3, -2) \text{ since } \frac{\partial^2 f}{\partial x^2}(3, -2) > 0\,.
\end{aligned}$$

The value of f at each of the critical points is

$$\begin{aligned}
f(2, -2) &= 2(2)^3 - 15(2)^2 + 36(2) + (-2)^2 + 4(-2) - 16 = 8\,, \\
f(3, -2) &= 2(3)^3 - 15(3)^2 + 36(3) + (-2)^2 + 4(-2) - 16 = 7\,.
\end{aligned}$$

$\boxed{\text{2015, S2}}$ 3. Find and classify the critical points of

$$h(x, y) = 2x^3 + 3x^2 y + y^2 - y.$$

Also give the function value at the critical points.

Solution: First find the critical points.

$$\begin{aligned}
\frac{\partial f}{\partial x} = 6x^2 + 6xy = 0 \quad &\Rightarrow \quad 6x(x + y) = 0 \\
&\Rightarrow \quad x = 0 \quad \text{OR} \quad y = -x \qquad (*) \\
\frac{\partial f}{\partial y} = 3x^2 + 2y - 1 = 0 \quad &\Rightarrow \quad y = \frac{1}{2}\left(1 - 3x^2\right) \qquad\qquad (**)\,.
\end{aligned}$$

We now investigate the two possibilities of $(*)$ in $(**)$, i.e.,

$$\begin{aligned}
x = 0 \Rightarrow y = \frac{1}{2}\left(1 - 3(0)^2\right) \quad &\Rightarrow \quad y = \frac{1}{2} \quad \text{and} \\
y = -x \Rightarrow -x = \frac{1}{2}\left(1 - 3x^2\right) \quad &\Rightarrow \quad 3x^2 - 2x - 1 = (3x + 1)(x - 1) = 0 \\
&\Rightarrow \quad x = -\frac{1}{3} \quad \text{OR} \quad x = 1\,.
\end{aligned}$$

Hence the critical points are $(0, \frac{1}{2})$, $(-\frac{1}{3}, \frac{1}{3})$ and $(1, -1)$. To classify the critical points we use the 2nd derivative test. The second order partial derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 12x + 6y, \quad \frac{\partial^2 f}{\partial y\, \partial x} = 6x, \quad \frac{\partial^2 f}{\partial y^2} = 2$$

and therefore

$$\mathcal{D}(x, y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y\, \partial x}\right)^2 = 24x + 12y - 36x^2 = -12(3x^2 - 2x - y)\,.$$

At each of the critical points we have

$$\begin{aligned}
\mathcal{D}\left(0,\frac{1}{2}\right) &= +6 > 0 \;\Rightarrow\; \text{local minimum at } \left(0,\frac{1}{2}\right) \text{ since } \frac{\partial^2 f}{\partial x^2}\left(0,\frac{1}{2}\right) > 0,\\
\mathcal{D}\left(-\frac{1}{3},\frac{1}{3}\right) &= -8 < 0 \;\Rightarrow\; \text{saddle point at } \left(-\frac{1}{3},\frac{1}{3}\right) \text{ and}\\
\mathcal{D}(1,-1) &= -24 < 0 \;\Rightarrow\; \text{saddle point at } (1,-1).
\end{aligned}$$

The value of f at each of the critical points is

$$\begin{aligned}
f\left(0,\frac{1}{2}\right) &= 2(0)^3 + 3(0)^2(1/2) + (1/2)^2 - (1/2) = -\frac{1}{4},\\
f\left(-\frac{1}{3},\frac{1}{3}\right) &= 2(-1/3)^3 + 3(-1/3)^2(1/3) + (1/3)^2 - (1/3) = -\frac{5}{27},\\
f(1,-1) &= 2(1)^3 + 3(1)^2(-1) + (-1)^2 - (-1) = 1.
\end{aligned}$$

2017, S1

4. You are given the function $f(x,y) = ax^2 + y^2 - 2y$, where a is a constant not equal to zero. This function has one critical point.

i) Find the critical point of the function.

Solution: First find the critical point.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2ax = 0 \;\Rightarrow\; x = 0 \quad \text{since } a \neq 0,\\
\frac{\partial f}{\partial y} &= 2y - 2 = 0 \;\Rightarrow\; y = 1.
\end{aligned}$$

Hence the critical point is $(0,1)$.

ii) Find the value of the function at the critical point.

Solution: $f(0,1) = a(0)^2 + (1)^2 - 2(1) = -1$.

iii) State whether the critical point can be a maximum, a minimum, or a saddle point. Write down the values of a (if they exist) for each case.

Solution: To classify the critical point we use the 2nd derivative test. The second order partial derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 2a, \quad \frac{\partial^2 f}{\partial y\,\partial x} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 2$$

and therefore

$$\mathcal{D}(x,y,a) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y\,\partial x}\right)^2 = 4a.$$

Note in this $\mathcal{D}$ only depends on a.

$$\begin{aligned}
&\text{If } a < 0 \;\Rightarrow\; \mathcal{D} < 0 \;\Rightarrow (0,1) \text{ is a saddle point and}\\
&\text{If } a > 0 \;\Rightarrow\; \mathcal{D} > 0 \;\Rightarrow (0,1) \text{ is a local minimum since } \frac{\partial^2 f}{\partial x^2} = 2a > 0.
\end{aligned}$$

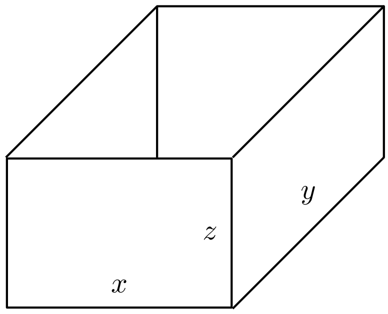
Since all possibilities for a have been exhausted the critical point $(0,1)$ can never be a local maximum.

$\boxed{1995}$ 5. A rectangular box without a lid is to be made from 12 m^2 of sheet metal.

i) If the length, width and height of the box are given by x, y and z metres respectively, show that the constraint function for this problem is given by:

$$g(x, y, z) = 2xz + 2yz + xy - 12 = 0\,.$$

Solution:



The four vertical sides of the box have area $xz + xz + yz + yz$ and the base has area xy. Thus the total surface area of the box is $2xz+2yz+xy$. But the total surface area must be 12m^2. Hence

$$2xz + 2yz + xy = 12 \quad\Rightarrow\quad g(x, y, z) = 2xz + 2yz + xy - 12 = 0\,.$$

ii) Use the method of Lagrange multipliers and the constraint function given in part i) to determine the maximum possible volume of the box.

Solution: The volume V of the box is given by $V(x, y, z) = xyz$.
Let $\mathcal{L}(x, y, z, \lambda) = V(x, y, z) - \lambda g(x, y, z) = xyz - \lambda(2xz + 2yz + xy - 12)$. Thus

$$\frac{\partial \mathcal{L}}{\partial x} = yz - 2\lambda z - \lambda y = 0 \quad\Rightarrow\quad yz = \lambda(2z + y) \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial y} = xz - 2\lambda z - \lambda x = 0 \quad\Rightarrow\quad xz = \lambda(2z + x) \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial z} = xy - 2\lambda(x + y) = 0 \quad\Rightarrow\quad xy = 2\lambda(x + y) \qquad (3)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(2xz + 2yz + xy - 12) = 0 \quad\Rightarrow\quad 2xz + 2yz + xy = 12 \qquad (4)$$

Note if $x = 0$ or $y = 0$ or $z = 0$ the volume V will be zero and we will ignore these cases.
Multiply equation (1) by x and then substitute equation (3), i.e.,

$$\begin{aligned}(1) \times x \quad \Rightarrow \quad xyz = \lambda x(2z + y) \;&\Rightarrow\; 2\lambda(x + y)z = \lambda x(2z + y) \quad \text{substituting (3)}\\ &\Rightarrow\; \lambda y(2z - x) = 0\\ &\Rightarrow\; \lambda = 0 \quad \text{OR} \quad x = 2z\,.\end{aligned}$$

Similarly, multiply equation (2) by y and then substitute equation (3), i.e.,

$$\begin{aligned}(2) \times y \quad \Rightarrow \quad xyz = \lambda y(2z + x) \;&\Rightarrow\; 2\lambda(x + y)z = \lambda y(2z + x) \quad \text{substituting (3)}\\ &\Rightarrow\; \lambda x(2z - y) = 0\\ &\Rightarrow\; \lambda = 0 \quad \text{OR} \quad y = 2z\,.\end{aligned}$$

So overall we have $\lambda = 0$ or $x = y = 2z$. If $\lambda = 0$ then in any of equations (1), (2) or (3) one of the variables x, y or z must be 0. We thus ignore $\lambda = 0$. Substitute $x = y = 2z$ in to equation (4), i.e.,

$$\begin{aligned}2xz + 2yz + xy = 2(2z)z + 2(2z)z + (2z)(2z) = 12 \;&\Rightarrow\; 12z^2 = 12\\ &\Rightarrow\; z = \pm 1\,.\end{aligned}$$

We ignore $z = -1$ since z represents the height of the box so $z \geq 0$. Hence the critical lengths are $x = y = 2\,\mathrm{m}$ and $z = 1\,\mathrm{m}$. The critical volume is $V(2,2,1) = (2)(2)(1) = 4\,\mathrm{m}^3$.

Note that equations (1)–(4) are nonlinear, and there is no standard method to solve them. Moreover, there may be more than one way to solve them. **Here is another solution.**

First note that $x \neq 0$, $y \neq 0$, and $z \neq 0$, because otherwise the volume V will be zero. It follows from (1) that $\lambda \neq 0$. Multiplying (1) by x, (2) by y, and (3) by z, and equating the equations, we deduce

$$xyz = \lambda x(2z+y) = \lambda y(2z+x) = 2\lambda z(x+y),$$

implying (since $\lambda \neq 0$)

$$x(2z+y) = y(2z+x) = 2z(x+y).$$

The first identity gives $x = y$ (since $z \neq 0$) and the second gives $y = 2z$ (since $x \neq 0$). Substituting $x = y = 2z$ into (4) yields

$$\begin{aligned} 2xz + 2yz + xy = 2(2z)z + 2(2z)z + (2z)(2z) = 12 \quad &\Rightarrow \quad 12z^2 = 12 \\ &\Rightarrow \quad z = \pm 1\,. \end{aligned}$$

We ignore $z = -1$ since z represents the height of the box so $z \geq 0$. Hence the critical lengths are $x = y = 2\,\mathrm{m}$ and $z = 1\,\mathrm{m}$. The critical volume is $V(2,2,1) = (2)(2)(1) = 4\,\mathrm{m}^3$.

$\boxed{\text{2014, S1}}$ 6. Use the method of Lagrange multipliers to find the maximum and minimum values of $x + y$ on the circle $x^2 + y^2 - 1 = 0$.

Solution: Let $f(x,y) = x + y$ and $\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(x^2+y^2-1)$. Thus

$$\frac{\partial \mathcal{L}}{\partial x} = 1 - 2x\lambda = 0 \quad \Rightarrow \quad 2x\lambda = 1 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 1 - 2y\lambda = 0 \quad \Rightarrow \quad 2y\lambda = 1 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x^2+y^2-1) = 0 \quad \Rightarrow \quad x^2+y^2 = 1 \tag{3}$$

Combining (1) and (2) yields

$$\begin{aligned} 1 = 2x\lambda = 2y\lambda \ &\Rightarrow \ 2\lambda\,(x-y) = 0 \\ &\Rightarrow \ \lambda = 0 \quad \text{OR} \quad x = y\,. \end{aligned}$$

Note $\lambda = 0$ is not possible since it doesn't satisfy (1) or (2). Substituting $y = x$ into the constraint $x^2 + y^2 = 1$ yields

$$x^2 + x^2 = 1 \ \Rightarrow \ x = y \pm \frac{1}{\sqrt{2}}\,.$$

Thus

$$\begin{aligned} f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} \qquad \text{MAX} \\ f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) &= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} \quad \text{MIN} \end{aligned}$$

Hence the maximum and minimum values of $x+y$ subject to the constraint $x^2+y^2=1$ are $\sqrt{2}$ and $-\sqrt{2}$ respectively.

Another way to solve (1)–(3)**:** It is clear from (1) and (2) that $x \neq 0$, $y \neq 0$, and $\lambda \neq 0$. Multiplying (1) by y, (2) by x and equating the equations yield $x = y$. Then substituting $x = y$ into (3) gives $2x^2 = 1$. Complete the solution as above.

$\boxed{\text{2014, S2}}$ 7. Use the method of Lagrange multipliers to find the maximum value of the function $f(x,y) = xy$ on the curve $x^2+y^2=1$.

Solution: Let $\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(x^2+y^2-1)$. Thus

$$\frac{\partial \mathcal{L}}{\partial x} = y - 2x\lambda = 0 \quad \Rightarrow \quad y = 2x\lambda \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial y} = x - 2y\lambda = 0 \quad \Rightarrow \quad x = 2y\lambda \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x^2+y^2-1) = 0 \quad \Rightarrow \quad x^2+y^2 = 1 \tag{3}$$

Combining (1) and (2) yields

$$\begin{aligned} y = 2x\lambda = 2\lambda(2\lambda y) = 4\lambda^2 y \ &\Rightarrow \ y\left(1-4\lambda^2\right) = 0 \\ &\Rightarrow \ y = 0 \quad \text{OR} \quad \lambda = \pm\frac{1}{2}. \end{aligned}$$

<u>**Case 1:** $y = 0$**.**</u> This case is not possible since $y = 0$ implies $x = 0$, which doesn't satisfy the constraint $x^2+y^2=1$.

<u>**Case 2:** $\lambda = -1/2$**.**</u> If we substitute $\lambda = -\dfrac{1}{2}$ into either (1) or (2) then $x = -y$. Then substituting this result into the constraint $x^2+y^2=1$ implies $2y^2=1$ which in turn implies $y = \pm\dfrac{1}{\sqrt{2}}$ with $x = \mp\dfrac{1}{\sqrt{2}}$. Thus

$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2}.$$

<u>**Case 3:** $\lambda = 1/2$**.**</u> If we substitute $\lambda = \dfrac{1}{2}$ into either (1) or (2) then $x = y$. Then substitute this result into constraint $x^2+y^2=1 \Rightarrow 2y^2=1$ then $y = \pm\dfrac{1}{\sqrt{2}}$ with $x = \pm\dfrac{1}{\sqrt{2}}$. Thus

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2}.$$

Hence the maximum value of f subject to the constraint $x^2+y^2=1$ is $\dfrac{1}{2}$.

Another way to solve (1)–(3)**:** It follows from (1) and (2) that $x \neq 0$, $y \neq 0$, and $\lambda \neq 0$ because if one of them is zero then all three are zero, and thus the constraint (3) does not hold. Multiplying (1) by x, (2) by y and equating the equations give $x^2 = y^2$, i.e., $x = y$ or $x = -y$. In both cases, (3) gives $2x^2 = 1$. Hence $x = 1/\sqrt{2}$ or $x = -1/\sqrt{2}$. Therefore, the system (1)–(3) has four solutions (by using $x = y$ and $x = -y$)

$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right) \quad \text{and} \quad \left(\pm\frac{1}{2}, \mp\frac{1}{2}\right).$$

The function values are

$$f\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right) = \frac{1}{2} \quad \text{and} \quad f\left(\pm\frac{1}{2}, \mp\frac{1}{2}\right) = -\frac{1}{2}.$$

Hence the maximum value of f subject to the constraint $x^2+y^2=1$ is $\dfrac{1}{2}$.

$\boxed{\text{2015, S1}}$ 8. Use the method of Lagrange multipliers to find the distance from the origin to the curve $5x^2 - 8xy + 5y^2 = 9$.

Solution: The distance $d(x,y)$ from the origin to a point $P(x,y)$ on the curve $5x^2 - 8xy + 5y^2 = 9$ is given by

$$d(x,y) = \sqrt{x^2 + y^2}\,.$$

Rather than optimise the distance d we optimise d^2. Both are optimised at the same location(s). Let $\mathcal{L}(x,y,\lambda) = d^2(x,y) - \lambda(5x^2 - 8xy + 5y^2 - 9) = x^2 + y^2 - \lambda(5x^2 - 8xy + 5y^2 - 9$. Thus

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - 10x\lambda + 8y\lambda = 0 \quad \Rightarrow \quad x(1-5\lambda) = -4\lambda y \qquad (1)$$
$$\frac{\partial \mathcal{L}}{\partial y} = 2y + 8x\lambda - 10y\lambda = 0 \quad \Rightarrow \quad y(1-5\lambda) = -4\lambda x \qquad (2)$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(5x^2 - 8xy + 5y^2 - 9) = 0 \quad \Rightarrow \quad 5x^2 - 8xy + 5y^2 = 9 \qquad (3)$$

Multiplying (1) and (2) yields

$$xy(1-5\lambda)^2 = 16\lambda^2 xy \;\Rightarrow\; xy(9\lambda^2 - 10\lambda + 1) = 0$$
$$\Rightarrow\; xy = 0 \quad \text{OR} \quad 9\lambda^2 - 10\lambda + 1 = 0\,.$$

Note if $xy = 0$ then $x = 0$ or $y = 0$. Then (1) and (2) imply both x and y are zero. But the origin $(0,0)$ is NOT on the curve $5x^2 - 8xy + 5y^2 = 9$ hence the origin is not a solution.

Consider the quadratic in λ, i.e.,

$$9\lambda^2 - 10\lambda + 1 = 0 \;\Rightarrow\; (9\lambda - 1)(\lambda - 1) = 0$$
$$\Rightarrow\; \lambda = \frac{1}{9} \quad \text{OR} \quad \lambda = 1\,.$$

$\underline{\lambda = 1}$ If we substitute $\lambda = 1$ into either (1) or (2) we have

$$x(1-5) = -4x = -4y \;\Rightarrow\; x = y\,.$$

Then substitute this result into constraint $5x^2 - 8xy + 5y^2 = 9 \Rightarrow 2x^2 = 9$ to yield $x = y = \pm\frac{3}{\sqrt{2}}$. Thus

$$d\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = d\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = \sqrt{\frac{9}{2} + \frac{9}{2}} = 3.$$

$\underline{\lambda = \frac{1}{9}}$ If we substitute $\lambda = \frac{1}{9}$ into either (1) or (2) we have

$$x\left(1 - \frac{5}{9}\right) = \frac{4}{9}x = -\frac{4}{9}y \;\Rightarrow\; x = -y\,.$$

Then substitute this result into constraint $5x^2 - 8xy + 5y^2 = 9 \Rightarrow 18x^2 = 9$ to yield $x = -y = \pm\frac{1}{\sqrt{2}}$. Thus

$$d\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = d\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1.$$

Hence the distance from the origin to the curve is 1.

Another solution: First note that if $x = 0$ or $y = 0$ or $\lambda = 0$ then (1) and (2) imply that both x and y equal 0. However, $(0, 0)$ does not satisfy the constraint (3). Hence, $x \neq 0$, $y \neq 0$, and $\lambda \neq 0$.

Multiplying (1) by x, (2) by y and equating the equations give

$$x^2(1 - 5\lambda) = y^2(1 - 5\lambda) \quad \text{or} \quad (x^2 - y^2)(1 - 5\lambda) = 0.$$

This implies $x^2 = y^2$ or $\lambda = 1/5$, i.e., $x = y$ or $x = -y$ or $\lambda = 1/5$.

- ⋆ Due to (1) and (2), $\lambda = 1/5$ implies $x = y = 0$, which does not satisfy (3).
- ⋆ Substituting $x = y$ into (3) gives

$$2x^2 = 9 \quad \text{or} \quad x = \pm\frac{3}{\sqrt{2}}.$$

  Then $y = \pm\frac{3}{\sqrt{2}}$.
- ⋆ Substituting $x = -y$ into (3) gives

$$18x^2 = 9 \quad \text{or} \quad x = \pm\frac{1}{\sqrt{2}}.$$

  Then $y = \mp\frac{1}{\sqrt{2}}$.

The function values are

$$d\left(\pm\frac{3}{\sqrt{2}}, \pm\frac{3}{\sqrt{2}}\right) = \sqrt{\frac{9}{2} + \frac{9}{2}} = 3 \quad \text{and} \quad d\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

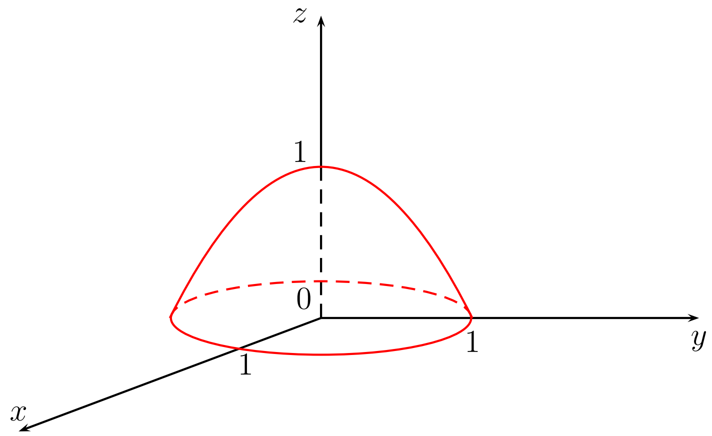
Hence the distance from the origin to the curve is 1.

$\boxed{\text{2016, S1}}$ 9. Consider the function

$$f(x, y) = 1 - x^2 - y^2.$$

i) Sketch the graph of the function f.

Solution:



ii) Using the method of Lagrange multipliers, find the extreme value of $f(x, y)$ subject to the constraint $x + y = 1$.

Solution: Let $\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(x + y - 1)$. Thus

$$\frac{\partial \mathcal{L}}{\partial x} = -2x - \lambda = 0 \quad \Rightarrow \quad \lambda = -2x \tag{1}$$
$$\frac{\partial \mathcal{L}}{\partial y} = -2y - \lambda = 0 \quad \Rightarrow \quad \lambda = -2y \tag{2}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x + y - 1) = 0 \quad \Rightarrow \quad x + y = 1 \tag{3}$$

Combining (1) and (2) yields

$$\lambda = -2x = -2y \quad \Rightarrow \quad x = y\,.$$

Substituting $x = y$ into constraint $x + y = 1$ yields $x = y = \frac{1}{2}$ and thus $f(\frac{1}{2}, \frac{1}{2}) = 1 - \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{1}{2}$. Is this extreme value a maximum or a minimum?

iii) Explain why this extreme value is a maximum and not a minimum.

Solution: The geometrical interpretation of the problem is the "upside down" paraboloid $f(x, y) = 1 - x^2 - y^2$ (see part i)) is cut by the vertical plane $x + y = 1$. The resulting curve of intersection is a "sad face" parabola which has a maximum but no minimum.

$\boxed{\text{2016, S2}}$ 10. i) Use the method of Lagrange multipliers to find the minimum value of $x^2 + y^2$ subject to the constraint $x + y = 6$.

Solution: Let $f(x, y) = x^2 + y^2$ and $\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(x + y - 6)$. Then

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - \lambda = 0 \quad \Rightarrow \quad \lambda = 2x \tag{1}$$
$$\frac{\partial \mathcal{L}}{\partial y} = 2y - \lambda = 0 \quad \Rightarrow \quad \lambda = 2y \tag{2}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x + y - 6) = 0 \quad \Rightarrow \quad x + y = 6 \tag{3}$$

Combining (1) and (2) yields

$$\lambda = 2x = 2y \quad \Rightarrow \quad x = y\,.$$

Substituting $x = y$ into constraint $x + y = 6$ yields $x = y = 3$ and thus $f(3, 3) = 3^2 + 3^2 = 18$. Is this extreme value a maximum or a minimum? The geometrical interpretation of the problem is the paraboloid $f(x, y) = x^2 + y^2$ is cut by the vertical plane $x + y = 6$. The resulting curve of intersection is a "happy face" parabola which has a minimum but no maximum. Hence $f(3, 3)$ corresponds to a minimum.

ii) Using your solution in i) and making no further use of the method of Lagrange multipliers find the maximum value of xy subject to the constraint $x + y = 6$.

Solution: Consider the function $(x + y)^2$ subject to the constraint $x + y = 6$, i.e.,

$$\begin{aligned}(x + y)^2 = 6^2 = x^2 + y^2 + 2xy \quad &\Rightarrow \quad x^2 + y^2 = 36 - 2xy \\ &\Rightarrow \quad xy = 18 - \frac{1}{2}(x^2 + y^2)\,.\end{aligned}$$

From part i) we have $x^2 + y^2$ has minimum value of 18 on $x + y = 6$. So $xy = 18 - \frac{1}{2}(x^2 + y^2)$ will have a maximum value when $x^2 + y^2$ has a minimum value. Thus $xy = 18 - 9 = 9$ is the maximum value of xy subject to the constraint $x + y = 6$.

$\boxed{\text{2017, S2}}$ 11. The temperature in a region of space is given by $T(x,y) = x^2 + y^2$. A sensor measures temperature along a curve given by the equation $xy = 1$.

i) Why does the sensor measure no maximum value of the temperature?

Solution: Along constraint $xy = 1$ there hold $y = \dfrac{1}{x}$ and $y \to \infty$ as $x \to 0^+$ (or $y \to -\infty$ as $x \to 0^-$). Hence as $y \to \pm\infty$ (with $x \to 0^\pm$) then $T(x,y) = x^2 + y^2 \to \infty$. Thus there is NO maximum temperature.

ii) Use the method of Lagrange multipliers to find the minimum temperature measured by the sensor.

Solution: Let $\mathcal{L}(x,y,\lambda) = T(x,y) - \lambda(xy - 1)$. Then

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - \lambda y = 0 \quad \Rightarrow \quad x = \frac{\lambda}{2} y\,, \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2y - \lambda x = 0 \quad \Rightarrow \quad y = \frac{\lambda}{2} x\,, \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(xy - 1) = 0 \quad \Rightarrow \quad xy = 1\,. \tag{3}$$

Combining (1) and (2) yields

$$x = \frac{\lambda}{2} y = \frac{\lambda}{2}\frac{\lambda}{2} x \quad \Rightarrow \quad x\left(1 - \frac{\lambda^2}{4}\right) = 0$$
$$\Rightarrow \quad x = 0 \quad \text{OR} \quad \lambda = \pm 2\,.$$

Note $x = 0$ is not possible since it doesn't satisfy the constraint $xy = 1$.

$\underline{\lambda = -2}$ If we substitute $\lambda = -2$ into either (1) or (2) then $x = -y$. But if we substitute this result into constraint $xy = 1$ then $y^2 = -1$. The system (1)–(3) has no real solutions.

$\underline{\lambda = 2}$ If we substitute $\lambda = 2$ into either (1) or (2) then $x = y$. Substitute this result into constraint $xy = 1$ then $y^2 = 1$, i.e., $y = \pm 1$. Hence the locations of the minimum temperature are $(1,1)$ and $(-1,-1)$ with minimum temperature of $T(1,1) = T(-1,-1) = 1^2 + 1^2 = 2$.

Another solution: First we note from (1)–(3) that $x \neq 0$, $y \neq 0$, and $\lambda \neq 0$, because if one of them equals zero then both x and y equal zero, which cannot satisfy (3). Multiplying (1) by x and (2) by y and equating the resulting equations give $x^2 = y^2$. Equation (3) implies that x and y are of the same sign. Hence $x = y$. Substituting this into (3) gives $x^2 = 1$ or $x = \pm 1$. There are two solutions $(x,y) = (1,1)$ and $(x,y) = (-1,-1)$. Hence the locations of the minimum temperature are $(1,1)$ and $(-1,-1)$ with minimum temperature of $T(1,1) = T(-1,-1) = 1^2 + 1^2 = 2$.

$\boxed{\text{2018, S1}}$ 12. A student wants to use the method of Lagrange multipliers to find the point on the surface

$$x^2 - xy + y^2 - z^2 = 1$$

nearest to the origin. Write down the algebraic equations the student needs to solve in order to find this point. You **do not** have to solve these equations.

Solution: The distance d from the origin to a point $P(x,y,z)$ on the surface is given by

$$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}\,.$$

Hence the problem is to minimise d subject to the constraint $x^2 - xy + y^2 - z^2 - 1 = 0$. To make the calculations easier we rather solve the equivalent problem of minimise d^2 subject to the constraint $x^2 - xy + y^2 - z^2 - 1 = 0$ (see lecture notes for explanation). Let

$$\mathcal{L}(x,y,z,\lambda) = d^2 - \lambda(x^2 - xy + y^2 - z^2 - 1) = x^2 + y^2 + z^2 - \lambda(x^2 - xy + y^2 - z^2 - 1)\,.$$

Then the four equations in x, y, z and λ to solve are

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - \lambda(2x - y) = 0\,, \tag{1}$$
$$\frac{\partial \mathcal{L}}{\partial y} = 2y - \lambda(2y - x) = 0\,, \tag{2}$$
$$\frac{\partial \mathcal{L}}{\partial z} = 2z + 2\lambda z = 0\,, \tag{3}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x^2 - xy + y^2 - z^2 - 1) = 0\,. \tag{4}$$

$\boxed{\text{2018, S2}}$ 13. Use the method of Lagrange multipliers to find the extreme values of

$$f(x, y) = 12 + 3x + 4y$$

subject to the constraint

$$g(x, y) = x^2 + y^2 - 1 = 0.$$

Solution: Let $\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda g(x, y)$. Thus

$$\frac{\partial \mathcal{L}}{\partial x} = 3 - 2\lambda x = 0 \quad \Rightarrow \quad 2\lambda x = 3 \tag{1}$$
$$\frac{\partial \mathcal{L}}{\partial y} = 4 - 2\lambda y = 0 \quad \Rightarrow \quad 2\lambda y = 4 \tag{2}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x^2 + y^2 - 1) = 0 \quad \Rightarrow \quad x^2 + y^2 = 1 \tag{3}$$

Multiplying (1) by y and (2) by x and subtracting yields

$$3y = 4x \quad \Rightarrow \quad y = \frac{4}{3}x\,.$$

Substituting $y = \frac{4}{3}x$ into constraint $x^2 + y^2 = 1$ yields $x = \pm\frac{3}{5}$ and therefore $y = \pm\frac{4}{5}$. Thus

$$\begin{aligned} f\left(\frac{3}{5}, \frac{4}{5}\right) &= 12 + 3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = 17 \quad \text{MAX}\,, \\ f\left(-\frac{3}{5}, -\frac{4}{5}\right) &= 12 + 3\left(-\frac{3}{5}\right) + 4\left(-\frac{4}{5}\right) = 7 \quad \text{MIN}\,. \end{aligned}$$

Hence the maximum value of f on the curve (constraint) $x^2 + y^2 = 1$ is 17 and minimum value is 7.

$\boxed{\text{2014, S1}}$ 14. Suppose that the atmospheric pressure P in a certain region of space is given by

$$P(x, y, z) = x^2 + y^2 + z^2.$$

i) Calculate $\boldsymbol{\nabla} P = \operatorname{grad} P$ at the point $T(1, 2, 4)$.
Solution:

$$\begin{aligned} \boldsymbol{\nabla} P = \operatorname{grad} P &= \frac{\partial P}{\partial x}\mathbf{i} + \frac{\partial P}{\partial y}\mathbf{j} + \frac{\partial P}{\partial z}\mathbf{k} \\ &= 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}\,. \end{aligned}$$

Hence the gradient of P at $T(1, 2, 4)$ is given by $\boldsymbol{\nabla} P(1, 2, 4) = 2\,\mathbf{i} + 4\,\mathbf{j} + 8\,\mathbf{k}$.

ii) Find the rate of change of the pressure with respect to distance at the point $T(1,2,4)$ in the direction of the vector $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$.

Solution: The rate of change of pressure with respect to distance at the point $T(1,2,4)$ in the direction $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$ is the directional derivative of P at the point $T(1,2,4)$ in the direction $\mathbf{b}$, i.e.,

$$\boldsymbol{\nabla} P\,(1,2,4)\cdot\widehat{\mathbf{b}} = (2\,\mathbf{i} + 4\,\mathbf{j} + 8\,\mathbf{k})\cdot\frac{1}{13}\,(3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}) = \frac{118}{13}\,.$$

Recall that $\widehat{\mathbf{b}} = \mathbf{b}/\|\mathbf{b}\|$ with $\|\mathbf{b}\|$ being the length (or magnitude or norm) of $\mathbf{b}$.) Hence the rate of change of pressure with respect to distance at the point $T(1,2,4)$ in the direction $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$ is $\dfrac{118}{13}$.

iii) Give a geometrical description of the level surface L of P passing through the point $T(1,2,4)$.

Solution: The level surface L of P sassing through the point $T(1,2,4)$ is given by $x^2 + y^2 + z^2 = P(1,2,4) = 1^2 + 2^2 + 4^2 = 21$, i.e., $x^2 + y^2 + z^2 = 21$. The surface is a sphere of radius $\sqrt{21}$ and centre the origin.

iv) Find a Cartesian equation of the tangent plane to the level surface L of P at the point $T(1,2,4)$.

Solution: Note $\boldsymbol{\nabla} P(1,2,4) = 2\,\mathbf{i} + 4\,\mathbf{j} + 8\,\mathbf{k}$ is normal (perpendicular) to the level surface $x^2 + y^2 + z^2 = 21$ at $T(1,2,4)$. Using the point-normal form for the tangent plane we can determine a cartesian equation to the level surface $x^2 + y^2 + z^2 = 21$, i.e.,

$$\begin{aligned}
\boldsymbol{\nabla} P(1,2,4)\cdot\left(\begin{pmatrix} x\\ y\\ z\end{pmatrix} - \begin{pmatrix} 1\\ 2\\ 4\end{pmatrix}\right) = 0 \;\Rightarrow\; & \begin{pmatrix} 2\\ 4\\ 8\end{pmatrix}\cdot\begin{pmatrix} x-1\\ y-2\\ z-4\end{pmatrix} = 0\\
\Rightarrow\; & 2(x-1) + 4(y-2) + 8(z-4) = 0\\
\Rightarrow\; & x + 2y + 4z = 21\,.
\end{aligned}$$

$\boxed{\text{2014, S2}}$ 15. Suppose the atmospheric pressure P in a certain region of space is given by

$$P(x,y,z) = \mathrm{e}^{z}(x^3 + y).$$

i) Calculate $\operatorname{grad} P$ at the point $(1,-2,0)$.

Solution:

$$\begin{aligned}
\boldsymbol{\nabla} P = \operatorname{grad} P &= \frac{\partial P}{\partial x}\mathbf{i} + \frac{\partial P}{\partial y}\mathbf{j} + \frac{\partial P}{\partial z}\mathbf{k}\\
&= 3x^2e^z\,\mathbf{i} + e^z\,\mathbf{j} + e^z(x^3 + y)\,\mathbf{k}\,.
\end{aligned}$$

Hence the gradient of P at $(1,-2,0)$ is given by $\boldsymbol{\nabla} P(1,-2,0) = 3\,\mathbf{i} + \mathbf{j} - \mathbf{k}$.

ii) Find the rate of change of pressure with respect to distance at the point $(1,-2,0)$ in the direction $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

Solution: The rate of change of pressure with respect to distance at the point $(1,-2,0)$ in the direction $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is the directional derivative of P at the point $(1,-2,0)$ in the direction $\mathbf{b}$, i.e.,

$$\boldsymbol{\nabla} P\,(1,-2,0)\cdot\widehat{\mathbf{b}} = (3\,\mathbf{i} + \mathbf{j} - \mathbf{k})\cdot\frac{1}{3}\,(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = \frac{5}{3}\,.$$

Hence the rate of change of pressure with respect to distance at the point $(1,-2,0)$ in the direction $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is $\dfrac{5}{3}$.

$\boxed{\text{2015, S1}}$ 16. The temperature T in a certain region of space is given by

$$T(x, y, z) = \sin(xyz).$$

i) Calculate grad T at the point $(\frac{1}{2}, \frac{1}{2}, \pi)$.

Solution:

$$\begin{aligned} \boldsymbol{\nabla} T = \operatorname{grad} T &= \frac{\partial T}{\partial x}\,\mathbf{i} + \frac{\partial T}{\partial y}\,\mathbf{j} + \frac{\partial T}{\partial z}\,\mathbf{k} \\ &= yz\cos(xyz)\,\mathbf{i} + xz\cos(xyz)\,\mathbf{j} + xy\cos(xyz)\,\mathbf{k}\,. \end{aligned}$$

Hence the gradient of T at $(\frac{1}{2}, \frac{1}{2}, \pi)$ is given by $\boldsymbol{\nabla} T(\frac{1}{2}, \frac{1}{2}, \pi) = \dfrac{\pi}{2\sqrt{2}}\,\mathbf{i} + \dfrac{\pi}{2\sqrt{2}}\,\mathbf{j} + \dfrac{1}{4\sqrt{2}}\,\mathbf{k}$.

ii) Find the rate of change of temperature with respect to distance at the point $(\frac{1}{2}, \frac{1}{2}, \pi)$ in the direction $\mathbf{b} = \mathbf{i} + \mathbf{j}$.

Solution: The rate of change of temperature with respect to distance at the point $(\frac{1}{2}, \frac{1}{2}, \pi)$ in the direction $\mathbf{b} = \mathbf{i}+\mathbf{j}$ is the directional derivative of T at the point $(\frac{1}{2}, \frac{1}{2}, \pi)$ in the direction $\mathbf{b}$, i.e.,

$$\boldsymbol{\nabla} T\left(\frac{1}{2}, \frac{1}{2}, \pi\right) \cdot \widehat{\mathbf{b}} = \left(\frac{\pi}{2\sqrt{2}}\,\mathbf{i} + \frac{\pi}{2\sqrt{2}}\,\mathbf{j} + \frac{1}{4\sqrt{2}}\,\mathbf{k}\right) \cdot \frac{1}{\sqrt{2}}\,(\mathbf{i} + \mathbf{j} + 0\,\mathbf{k}) = \frac{\pi}{2}\,.$$

Hence the rate of change of temperature with respect to distance at the point $(\frac{1}{2}, \frac{1}{2}, \pi)$ in the direction $\mathbf{b} = \mathbf{i} + \mathbf{j}$ is $\dfrac{\pi}{2}$.

$\boxed{\text{2015, S2}}$ 17. For the scalar field

$$\phi(x, y, z) = x^2 + 3y^2 + 4z^2$$

find:

i) grad ϕ at the point $P(1, 0, 1)$,

Solution:

$$\begin{aligned} \boldsymbol{\nabla}\phi = \operatorname{grad}\phi &= \frac{\partial \phi}{\partial x}\,\mathbf{i} + \frac{\partial \phi}{\partial y}\,\mathbf{j} + \frac{\partial \phi}{\partial z}\,\mathbf{k} \\ &= 2x\,\mathbf{i} + 6y\,\mathbf{j} + 8z\,\mathbf{k}\,. \end{aligned}$$

Hence the gradient of ϕ at $P(1, 0, 1)$ is given by $\boldsymbol{\nabla}\phi(1, 0, 1) = 2\,\mathbf{i} + 8\,\mathbf{k}$.

ii) the directional derivative of ϕ at the point $P(1, 0, 1)$ in the direction of the vector $\mathbf{u} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$ and

Solution: The directional derivative of ϕ at the point $P(1, 0, 1)$ in the direction $\mathbf{u} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$ is given by

$$\boldsymbol{\nabla}\phi(1, 0, 1) \cdot \widehat{\mathbf{u}} = (2\,\mathbf{i} + 0\,\mathbf{j} + 8\,\mathbf{k}) \cdot \frac{1}{\sqrt{3}}\,(-\mathbf{i} - \mathbf{j} + \mathbf{k}) = 2\sqrt{3}\,.$$

The directional derivative of ϕ at the point $P(1, 0, 1)$ in the direction $\mathbf{u} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$ is $2\sqrt{3}$.

iii) the maximum rate of change of ϕ at the point $P(1, 0, 1)$.

Solution: The maximum rate of change of ϕ at the point $P(1, 0, 1)$ is given by

$$\|\boldsymbol{\nabla}\phi(1, 0, 1)\| = \sqrt{\boldsymbol{\nabla}\phi(1, 0, 1) \cdot \boldsymbol{\nabla}\phi(1, 0, 1)} = \sqrt{2^2 + 8^2} = \sqrt{68} = 2\sqrt{17}\,.$$

$\boxed{\text{2016, S2}}$ 18. Suppose that the pressure ϕ in a region of space is given by the scalar field

$$\phi(x, y, z) = xy^2z^3.$$

i) Calculate grad ϕ at the point $A(1,2,1)$.

Solution:

$$\begin{aligned}\boldsymbol{\nabla}\phi = \operatorname{grad}\phi &= \frac{\partial \phi}{\partial x}\,\mathbf{i} + \frac{\partial \phi}{\partial y}\,\mathbf{j} + \frac{\partial \phi}{\partial z}\,\mathbf{k} \\ &= y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}\,.\end{aligned}$$

Hence the gradient of ϕ at $A(1,2,1)$ is given by $\boldsymbol{\nabla}\phi(1,2,1) = 4\,\mathbf{i} + 4\,\mathbf{j} + 12\,\mathbf{k}$.

ii) Find the rate of change of the pressure with respect to distance at the point $A(1,2,1)$ in the direction $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Solution: Let $\mathbf{b} = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$. The rate of change of pressure with respect to distance at the point $A(1,2,1)$ in the direction $\mathbf{b} = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ is the directional derivative of ϕ in the direction $\mathbf{b}$, i.e.,

$$\boldsymbol{\nabla}\phi(1,2,1)\cdot\widehat{\mathbf{b}} = (4\,\mathbf{i} + 4\,\mathbf{j} + 12\,\mathbf{k})\cdot\frac{1}{3}\,(2\mathbf{i}+\mathbf{j}-2\mathbf{k}) = -4\,.$$

Hence the rate of change of pressure with respect to distance at the point $A(1,2,1)$ in the direction $2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ is -4.

iii) Write down a unit normal to the level surface $\phi(x,y,z) = 4$ at the point $A(1,2,1)$.

Solution: Note $\boldsymbol{\nabla}\phi(1,2,1) = 4\,\mathbf{i}+4\,\mathbf{j}+12\,\mathbf{k}$ is normal (perpendicular) to the level surface $\phi(1,2,1) = 4$ at $P(1,2,1)$. Hence a unit normal $\widehat{\mathbf{n}}$ to the level surface $\phi(x,y,z) = 4$ at the point $A(1,2,1)$ is given by

$$\widehat{\mathbf{n}} = \frac{\boldsymbol{\nabla}\phi(1,2,1)}{\|\boldsymbol{\nabla}\phi(1,2,1)\|} = \frac{1}{\sqrt{11}}\,(\mathbf{i}+\mathbf{j}+3\,\mathbf{k})\;.$$

2017, S1

19. Suppose the temperature in a region of space is given by the scalar field

$$T(x,y,z) = x^4 + y^4 + z^4.$$

i) Calculate the gradient of T at the point $P(1,1,1)$.

Solution:

$$\begin{aligned}\boldsymbol{\nabla}T = \operatorname{grad}T &= \frac{\partial T}{\partial x}\,\mathbf{i} + \frac{\partial T}{\partial y}\,\mathbf{j} + \frac{\partial T}{\partial z}\,\mathbf{k} \\ &= 4x^3\,\mathbf{i} + 4y^3\,\mathbf{j} + 4z^3\,\mathbf{k}\,.\end{aligned}$$

Hence the gradient of T at $P(1,1,1)$ is given by $\boldsymbol{\nabla}T(1,1,1) = 4\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}$.

ii) Find the rate of change of temperature with respect to distance at the point $P(1,1,1)$ in the direction $\mathbf{i}+\mathbf{j}$.

Solution: Let $\mathbf{b} = \mathbf{i}+\mathbf{j}$. The rate of change of temperature with respect to distance at the point $P(1,1,1)$ in the direction $\mathbf{b} = \mathbf{i}+\mathbf{j}$ is the directional derivative of T at the point $P(1,1,1)$ in the direction $\mathbf{b}$, i.e.,

$$\boldsymbol{\nabla}\phi(1,1,1)\cdot\widehat{\mathbf{b}} = (4\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k})\cdot\frac{1}{\sqrt{2}}\,(\mathbf{i}+\mathbf{j}+0\,\mathbf{k}) = 4\sqrt{2}\,.$$

Hence the rate of change of temperature with respect to distance at the point $P(1,1,1)$ in the direction $\mathbf{i}+\mathbf{j}$ is $4\sqrt{2}$.

iii) Write down the equation of the tangent plane to the surface $T(x,y,z) = 3$ at the point $P(1,1,1)$.

Solution: Note $\boldsymbol{\nabla} T(1,1,1) = 4\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}$ is normal (perpendicular) to the level surface $T(1,1,1) = 3$ at $P(1,1,1)$. Using the point-normal form for the tangent plane we can determine a cartesian equation to the level surface $T(1,1,1)$, i.e.,

$$\boldsymbol{\nabla} T(1,1,1) \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) = 0 \;\Rightarrow\; \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} = 0$$
$$\Rightarrow\; 4(x-1) + 4(y-1) + 4(z-1) = 0$$
$$\Rightarrow\; x + y + z = 3\,.$$

$\boxed{\text{2017, S2}}$ 20. Consider the scalar field

$$\phi(x,y,z) = x^2 - y^2 + z^2.$$

i) Calculate the gradient of ϕ at the point $P(1,1,0)$.
Solution:

$$\begin{aligned} \boldsymbol{\nabla}\phi = \operatorname{grad}\phi &= \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} \\ &= 2x\,\mathbf{i} - 2y\,\mathbf{j} + 2z\,\mathbf{k}\,. \end{aligned}$$

Hence the gradient of ϕ at $P(1,1,0)$ is given by $\boldsymbol{\nabla}\phi(1,1,0) = 2\,\mathbf{i} - 2\,\mathbf{j}$.

ii) Find the direction and magnitude of the maximum rate of increase of ϕ at $P(1,1,0)$.
Solution: The direction of maximum rate of increase of ϕ at $P(1,1,0)$ is $\boldsymbol{\nabla}\phi(1,1,0) = 2\,\mathbf{i} - 2\,\mathbf{j}$.
The magnitude of maximum rate of increase of ϕ at $P(1,1,0)$ is

$$\|\boldsymbol{\nabla}\phi(1,1,0)\| = \sqrt{\boldsymbol{\nabla}\phi(1,1,0) \cdot \boldsymbol{\nabla}\phi(1,1,0)} = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}\,.$$

iii) Write down any non-zero vector $\mathbf{b}$ that is perpendicular to the gradient of ϕ at the point $P(1,1,0)$.
Solution: Let $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$. For $\mathbf{b}$ to be perpendicular to $\boldsymbol{\nabla}\phi$ at $P(1,1,0)$ the scalar (dot) product of $\mathbf{b}$ and $\boldsymbol{\nabla}\phi(1,1,0)$ will be zero, i.e.,

$$\mathbf{b} \cdot \boldsymbol{\nabla}\phi(1,1,0) = (b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}) \cdot (2\,\mathbf{i} - 2\,\mathbf{j} + 0\,\mathbf{k}) = 2b_1 - 2b_2 = 0,$$

implying $b_1 = b_2$. Examples of such a non-zero vector $\mathbf{b}$ are

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

iv) What is the rate of change of ϕ at the point $P(1,1,0)$ in the direction $\mathbf{b}$ found in part iii)?
Solution: The rate of change of ϕ at $P(1,1,0)$ in the direction $\widehat{\mathbf{b}}$ is ZERO, i.e.,

$$\widehat{\mathbf{b}} \cdot \boldsymbol{\nabla}\phi(1,1,0) = 0\,.$$

$\boxed{\text{2018, S1}}$ 21. Consider the function

$$f(x,y) = 2e^{y-1}\sin x\,.$$

i) Calculate the Taylor series expansion of f about the point $\left(\frac{\pi}{6}, 1\right)$ up to and including **linear** terms.

Solution: We first calculate all the partial derivatives of f up to including first order terms at $\left(\frac{\pi}{6}, 1\right)$.

$$\begin{aligned}
f(x,y) &= 2e^{y-1}\sin x, & f\left(\frac{\pi}{6},1\right) &= 2e^{1-1}\sin\frac{\pi}{6} = 2e^0\frac{1}{2} = 1\,,\\
\frac{\partial f}{\partial x} &= 2e^{y-1}\cos x, & f_x\left(\frac{\pi}{6},1\right) &= 2e^{1-1}\cos\frac{\pi}{6} = 2e^0\frac{\sqrt{3}}{2} = \sqrt{3}\,,\\
\frac{\partial f}{\partial y} &= 2e^{y-1}\sin x = f(x,y), & f_y(\left(\frac{\pi}{6},1\right) &= f\left(\frac{\pi}{6},1\right) = 1\,,
\end{aligned}$$

Thus

$$\begin{aligned}
f(x,y) &\approx f\left(\frac{\pi}{6},1\right) + \left(x-\frac{\pi}{6}\right)f_x\left(\frac{\pi}{6},1\right) + (y-1)f_y\left(\frac{\pi}{6},1\right)\\
&= 1+\sqrt{3}\left(x-\frac{\pi}{6}\right)+(y-1)\,.
\end{aligned}$$

A check for the answer is to determine the expansion using the Maclaurin series for e^t, $\sin t$ and $\cos t$ (since f is a product of functions), i.e.,

$$e^t = 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\,,\quad \sin t = t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots\,,\quad \cos t = 1-\frac{t^2}{2!}+\frac{t^4}{4!}-\cdots\,.$$

Thus

$$\begin{aligned}
f(x,y) &= 2e^{y-1}\sin x\\
&= 2e^{y-1}\sin\left(\frac{\pi}{6}+\left(x-\frac{\pi}{6}\right)\right) \quad \text{since we wish to expand about } \left(\frac{\pi}{6},1\right)\\
&= 2e^{y-1}\left[\sin\left(\frac{\pi}{6}\right)\cos\left(x-\frac{\pi}{6}\right)+\cos\left(\frac{\pi}{6}\right)\sin\left(x-\frac{\pi}{6}\right)\right]\\
&\qquad \text{using the identity } \sin(A+B) = \sin A\cos B+\cos A\sin B\\
&= 2e^{y-1}\left[\frac{1}{2}\cos\left(x-\frac{\pi}{6}\right)+\frac{\sqrt{3}}{2}\sin\left(x-\frac{\pi}{6}\right)\right]\\
&= e^{y-1}\left[\cos\left(x-\frac{\pi}{6}\right)+\sqrt{3}\sin\left(x-\frac{\pi}{6}\right)\right]\\
&= \left(1+(y-1)+\frac{(y-1)^2}{2!}+\frac{(y-1)^3}{3!}+\cdots\right)\\
&\quad\times\left[\left(1-\frac{\left(x-\frac{\pi}{6}\right)^2}{2!}+\cdots\right)+\sqrt{3}\left(\left(x-\frac{\pi}{6}\right)-\frac{\left(x-\frac{\pi}{6}\right)^3}{3!}+\cdots\right)\right]\\
&\approx 1+\sqrt{3}\left(x-\frac{\pi}{6}\right)+(y-1) \quad \text{up to and including linear terms}
\end{aligned}$$

ii) Determine the **direction** from the point $\left(\frac{\pi}{6},1\right)$ for which the change in f with distance

α) is a minimum;

β) is zero.

Solution: The answer to part i) can be written in the form

$$f(x,y)-f\left(\frac{\pi}{6},1\right) \approx \begin{pmatrix}\sqrt{3}\\1\end{pmatrix}\cdot\begin{pmatrix}x-\frac{\pi}{6}\\y-1\end{pmatrix} \;\Rightarrow\; \Delta f \approx \begin{pmatrix}\sqrt{3}\\1\end{pmatrix}\cdot\begin{pmatrix}x-\frac{\pi}{6}\\y-1\end{pmatrix}.$$

Thus the **direction** from the point $\left(\frac{\pi}{6},1\right)$ for which the change in f, Δf, with distance

α) is a minimum when $\begin{pmatrix} x-\frac{\pi}{6} \\ y-1 \end{pmatrix}$ is in the direction $-\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$, i.e., when the scalar (dot) is the most negative and

β) is zero when $\begin{pmatrix} x-\frac{\pi}{6} \\ y-1 \end{pmatrix}$ is in the direction $-\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$, i.e., when the scalar (dot) is zero.

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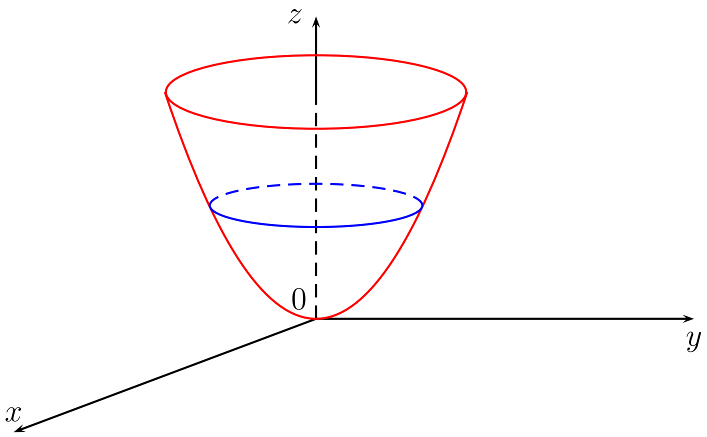
22. Suppose that the temperature T, at a point (x, y, z) in space is given by

$$T(x, y, z) = z - x^2 - y^2.$$

i) Sketch the level surface of all points with a temperature of zero.

Solution: The level surface when the temperature is zero, i.e., $T(x, y, z) = 0$, is given by the equation

$$0 = z - x^2 - y^2 \quad \Rightarrow \quad z = x^2 + y^2, \quad \text{(paraboloid)}.$$



ii) Find $\operatorname{grad} T$.

Solution:

$$\begin{aligned} \operatorname{grad} T = \boldsymbol{\nabla} T &= \frac{\partial T}{\partial x}\,\mathbf{i} + \frac{\partial T}{\partial y}\,\mathbf{j} + \frac{\partial T}{\partial z}\,\mathbf{k} \\ &= -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}\,. \end{aligned}$$

iii) Calculate the rate of change of the temperature T at the point $P(1, 1, 0)$ in the direction of the vector $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$.

Solution: The rate of change of the temperature T at the point $P(1, 1, 0)$ in the direction of the vector $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$ is the directional derivative of T at the point $P(1, 1, 0)$ in the direction $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$, i.e.,

$$\boldsymbol{\nabla} T(1,1,0) \cdot \widehat{\mathbf{b}} = (-2\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}) \cdot \frac{1}{13}\,(3\,\mathbf{i} + 4\,\mathbf{j} + 12\,\mathbf{k}) = \frac{1}{13}\,(-6 - 8 + 12) = -\frac{2}{13}\,.$$

Hence the rate of change of the temperature T at the point $P(1, 1, 0)$ in the direction of the vector $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$ is $-\dfrac{2}{13}$.