LECTURE 9
VECTOR ALGEBRA

Scalars are quantities which have only a magnitude such as temperature, mass, time and speed. Vectors have a magnitude **and a direction**.

Vectors in Physics vs Vectors in Mathematics

Thus from a mathematical perspective we may view vectors as graphical objects $\longrightarrow$ or alternatively as algebraic objects $\mathbf{u} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} = \mathbf{i}+5\mathbf{j}+4\mathbf{k}$.

We will now review a host of vector applications covered in first year.

Example 1 Let $\mathbf{u} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$. Find $|\mathbf{u}|$ and hence write down two unit vectors parallel to $\mathbf{u}$.

Recall that two non-zero vectors are parallel iff they are scalar multiples of each other.

$$|\underset{\sim}{u}| = \sqrt{1^2 + 5^2 + 4^2} = \sqrt{42}$$

$$\pm \frac{1}{\sqrt{42}} \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} \parallel \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$$

$$\bigstar \quad \sqrt{42}, \pm\frac{1}{\sqrt{42}} \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} \quad \bigstar$$

Example 2 Let $\mathbf{v} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 8 \\ 2 \end{pmatrix} = \mathbf{i} + 8\mathbf{j} + 2\mathbf{k}$. Use the dot product to prove that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

$$\underset{\sim}{v} \cdot \underset{\sim}{w} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 8 \\ 2 \end{pmatrix} = 2 - 8 + 6 = 0$$

The dot product measures the interaction between vectors. It is zero when the vectors have absolutely nothing to do with each other (perpendicular) and increases in value as the vectors get closer and closer to being parallel. Between these two extremes we can project a vector $\mathbf{u}$ onto a vector $\mathbf{v}$ using the formula $Proj_{\mathbf{v}}(\mathbf{u}) = (\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}})\mathbf{v}$.

Example 3 Let $\mathbf{u} = \begin{pmatrix} 1 \\ -18 \\ 7 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$. Find $Proj_{\mathbf{v}}(\mathbf{u})$ and display this situation in a vector diagram.

$$proj_{v}(\vec{u}) = \frac{\begin{pmatrix} 1 \\ -18 \\ 7 \end{pmatrix}\cdot\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}}{\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}\cdot\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

$$= \frac{3-18+28}{9+1+16}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

$$\star \quad \frac{1}{2}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \quad \star$$

Example 4 Let $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$. Find the cross product $\mathbf{u}\times\mathbf{v}$.

Check your answer with the dot product.

$$\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ 1 & 2 & 4 \\ 3 & 5 & 1 \end{vmatrix}$$

$$= \begin{pmatrix} -18 \\ 11 \\ -1 \end{pmatrix}$$

Check:

$$\begin{pmatrix} -18 \\ 11 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = -54 + 55 - 1 = 0$$

$$\begin{pmatrix} -18 \\ 11 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = -18 + 22 - 4 = 0$$

Some Properties of the Cross Product

$(\mathbf{u}\times\mathbf{v})\perp\ \mathbf{u}$ and $(\mathbf{u}\times\mathbf{v})\perp\ \mathbf{v}$
with the direction of the cross product determined by the right hand rule.

$$(\mathbf{u}\times\mathbf{v}) = -(\mathbf{v}\times\mathbf{u})$$

Cross products help us to determine how much of "space" is carved out by a batch of non-zero vectors.

In two dimensions, the magnitude $|\mathbf{u}\times\mathbf{v}|$ of the cross product is the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

It follows that if $\mathbf{u}\times\mathbf{v}=\mathbf{0} \iff \mathbf{u}$ and $\mathbf{v}$ are parallel.

In three dimensions the magnitude of the scalar triple product $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ is the volume of the parallelepiped formed by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

It follows that $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\mathbf{0} \iff \mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are coplanar.

Question: What does it mean if $(\mathbf{u}\cdot\mathbf{v})\times\mathbf{w}=\mathbf{0}$?

$\vec{w} = \vec{0}$

Let's finish off revising the theory of lines and planes in space.

Example 5 Let $\mathcal{P}$ be the plane parallel to the two vectors $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and passing through the point $\begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$.

a) Find a parametric vector equation for $\mathcal{P}$.

b) Find a Cartesian equation for $\mathcal{P}$.

---

a)

$$\mathcal{P} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}$$

b) Recall from Example 4 that $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} -18 \\ 11 \\ -1 \end{pmatrix}$.

This vector is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ and hence perpendicular to the plane $\mathcal{P}$.

Hence the Cartesian equation of $\mathcal{P}$ is

$$-18x + 11y - z = \#$$

$(2,0,5)$: $\quad -18(2) + 11(0) - 5 = -41$

$\therefore \; \mathcal{P}: \; 18x - 11y + z = 41$

$\bigstar \quad 18x - 11y + z = 41 \quad \bigstar$

Example 6 Find a parametric vector equation of the line passing through the two points $A = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix}$.

Prove that this line is perpendicular to to the plane $4x+12y+4z = -56$ and determine where the line and the plane meet.

$$l = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$$P: 4x + 12y + 4z = -56$$

$$\tilde{n} = \begin{pmatrix} 4 \\ 12 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

$$\therefore\ l \parallel \tilde{n} \quad \text{so} \quad l \perp P$$

$$l: \quad x = 3 + \lambda$$
$$y = 1 + 3\lambda$$
$$z = 2 + \lambda$$

$$4(3+\lambda) + 12(1+3\lambda) + 4(2+\lambda) = -56$$
$$\therefore\ \lambda = -2$$

$$\therefore\ \text{P.O.I}: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + (-2)\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix}$$

$\star \begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix} \star$

[9]You can now do Q 48 to 51