MATH2019 ENGINEERING MATHEMATICS 2E

Suppose $z = f(x, y)$. Define

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \longrightarrow 0} \frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$
$$\frac{\partial f}{\partial y} = \lim_{\Delta y \longrightarrow 0} \frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

Notation

$$\frac{\partial f}{\partial x} = f_x = z_x, \qquad \frac{\partial f}{\partial y} = f_y = z_y$$

Hello and welcome to Math2019 Stream B.

You have six Math2019 lectures per week and one Math2019 tutorial per fortnight, as per your timetable. Classes run from Weeks 1-10. The Friday 2-3 lecture is a problem class, where we will go through a selection of examples rather then presenting new content.

• Note that all classes will run from 5 minutes past the hour to 5 minutes to the hour.

• You need to hold a Math1231 pass or better to enrol in Math2019.

• You do not need to purchase a text book.

• The Math2019 Moodle page contains skeleton lecture notes for the entire session, both individually and also as a single pdf. **Please bring printouts to each lecture, as you will need to fill out the notes by hand, in class**.

• On the Math2019 Moodle page you will also find problem class sheets, past exams, extra course notes, tutorial problems for the session and a course outline. You will also find some first year algebra revision lectures which should be read before the eigenvector section of the course. (Lecture 29)

• Read the course outline carefully to ensure that you are completely familiar with the administrative structure of Math2019.

• To each tutorial you should bring your attempted examples for that week together with a printout of tutorial problem set.

• There is no assessable MAPLE in Math2019, though the on-line quizzes and lab tests do use Maple TA.

• Assessment is comprised of two 40 minute lab-based tests valued at 15% each, weekly on-line quizzes making up 8% of your final mark and a writing assignment valued at 2%. The final exam is worth 60%.

Please carefully read the assessment details in the course pack.

LECTURE 1
PARTIAL DIFFERENTIATION

In your previous studies the focus was on functions of a single variable $y = f(x)$ and their rates of change $\dfrac{dy}{dx}$. It is however quite rare for a quantity of interest to depend on only one variable and in complicated physical systems it may be the case that the variable you are concerned with may depend upon dozens of other variables. Partial differentiation is the extension of our usual calculus to functions of several variables.
Given a function of two variables $z = f(x, y)$ we denote the rates of change in the x and y directions as $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ or simply as z_x and z_y. The formal definitions of these derivatives are presented above however in reality we only need to remember a few things to differentiate partially:

The old specific rules of differentiation

y	y'
x^n	nx^{n-1}
e^x	e^x
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\ln(x)$	$\dfrac{1}{x}$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$

The old general rules of differentiation

$$(uv)' = u'v + v'u \qquad \text{Product Rule}$$

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2} \qquad \text{Quotient Rule}$$

The only extra issue that needs to be kept in mind is that when you are differentiating in a particular direction you treat all other variables *exactly* as if they were constant.

Example 1 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $z = x^2 + y^5 + 7$.

$$\bigstar \quad \frac{\partial z}{\partial x} = 2x, \ \frac{\partial z}{\partial y} = 5y^4 \quad \bigstar$$

Example 2 Suppose that $z = f(x,y) = x^3y^5 + 3x - 8y + 2$. Find the function value and the rate of change of f in the x direction at the point $(1,2)$.

$$\bigstar \quad f(1,2) = 21, \ \frac{\partial z}{\partial x}(1,2) = 99 \quad \bigstar$$

Example 3 Find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$ if $w = u^3v^4 + \sinh(v^9)$.

$$\bigstar \quad \frac{\partial w}{\partial u} = 3u^2v^4, \ \frac{\partial w}{\partial v} = 4u^3v^3 + 9v^8\cosh(v^9) \quad \bigstar$$

Example 4 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $z = \dfrac{e^{7y}}{x^3+1}$.

★ $\dfrac{\partial z}{\partial x} = \dfrac{-3e^{7y}x^2}{(x^3+1)^2}$, $\dfrac{\partial z}{\partial y} = \dfrac{7e^{7y}}{x^3+1}$ ★

Plotting in Space

Before examining partial derivatives from a geometrical point of view let us consider the issue of sketching in higher dimensions.

Example 5 Plot the points $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ in $\mathbb{R}^3$.

You will observe that plotting in $\mathbb{R}^3$ is somewhat problematic as you are trying to squeeze three dimensions onto a two dimensional page. It gets worse!

Example 6 Plot the point $\begin{pmatrix} 1 \\ 2 \\ 4 \\ 7 \end{pmatrix}$ in $\mathbb{R}^4$.

You will recall that the graph of $y = f(x)$ is generally a curve in $\mathbb{R}^2$......lines, parabolas, hyperbolas etc. The graph of $z = f(x, y)$ is always a *surface* in $\mathbb{R}^3$.

Example 7 Sketch each of the following surfaces in space:

a) $3x + 4y + 6z = 12$

b) $x^2 + y^2 + z^2 = 25$

c) $x^2 + y^2 = 9$

d) $z = 3\sqrt{x^2 + y^2}$

e) $z = x^2 + y^2$

Question How about $x = y^2 + z^2$?

SUMMARY

a) $ax + by + cz = d$ is a plane.

b) $x^2 + y^2 + z^2 = r^2$ is a sphere centre the origin radius r.

c) If a variable is absent it is **!unrestricted!** Extrude the two dimensional curve into the missing direction.

d) $z = \alpha\sqrt{x^2 + y^2}$ is a cone with semi-vertical angle $\tan^{-1}(\frac{1}{\alpha})$.

e) $z = \alpha(x^2 + y^2)$ is a paraboloid of revolution.

There is of course no reason why we must restrict ourselves to two independent variables!!

Example 8 If $f(x_1, x_2, x_3, x_4, x_5, x_6) = x_1^3 x_3^4 + \dfrac{x_5}{\sin(x_6)} + \ln(x_4) - \dfrac{\sinh(x_2)}{e^{x_1}}$
find $\dfrac{\partial f}{\partial x_4}$.

$\bigstar \quad \dfrac{1}{x_4} \quad \bigstar$

As in single variable calculus we make extensive use of second and higher order derivatives. However with partial differentiation we have many more options!

Example 9 If $z = f(x, y) = x^2 \sin(y) + x^3 y + y^5$ find

$$\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial x \partial y}$$

$$\frac{\partial^2 z}{\partial y \partial x}$$

$$\frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial^2 z}{\partial y^2}$$

You will observe in the above example that $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$ This is true for most reasonably well behaved functions. Note however that in general $\dfrac{\partial^2 z}{\partial x^2} \neq \dfrac{\partial^2 z}{\partial y^2}$.

Note also that $\dfrac{\partial^2 z}{\partial x \partial y}$ is most definitely not equal to $(\dfrac{\partial z}{\partial x})(\dfrac{\partial z}{\partial y})$.

[1] You can now do Q's 1 to 4

LECTURE 2
CHAIN RULE

If $z = f(x, y)$ and $x = x(t)$ and $y = y(t)$ then

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

If $z = f(x, y)$ and $x = x(u, v)$ and $y = y(u, v)$ then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

A common situation is that z is a function of x and y with x and y themselves functions of other variables......say u and v. It is then true that z is *ultimately* a function of u and v and thus it makes sense to ask the question "What is $\frac{\partial z}{\partial u}$"? The chain rule enables us to answer this question without actually ever having to produce z as an explicit function of u and v. The chain rule comes in many different flavours, two of which are presented above. However all that needs to be remembered is that you keep on differentiating z with respect to what you can and then always fudge your answer back to what you want.

Note that it is usually more effective to use the chain rule than to explicitly detail the structure of the new function.

Example 1 Suppose that $z = x^2 + 4y$ where $x = u^3 \ln(v)$ and $y = uv^2$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

$$\bigstar \quad \frac{\partial z}{\partial u} = 6u^5(\ln(v))^2 + 4v^2, \ \frac{\partial z}{\partial v} = \frac{2u^6\ln(v)}{v} + 8uv \quad \bigstar$$

Example 2 Given that $z = x^2y^3$ with $x = 2t$ and $y = \sin(4t)$ use the chain rule to find $\frac{dz}{dt}$.

$$\bigstar \quad 8t\sin^3(4t) + 48t^2\sin^2(4t)\cos(4t) \quad \bigstar$$

Example 3 If $z = a^2 + b + c^5 + d^7$ where $a = uv,\ b = 2u + 3v,\ c = u^2$ and $d = v^2$ use the chain rule to find $\dfrac{\partial z}{\partial u}$.

$\bigstar \quad 2uv^2 + 10u^9 + 2 \quad \bigstar$

Example 4 If $w = a^2 - ab^3$ with $a = e^{uv}$ and $b = 3u + 2v$ use the chain rule to find $\dfrac{\partial w}{\partial u}$ when $u = 0$ and $v = 1$.

$\bigstar \quad -42 \quad \bigstar$

Example 5 Suppose that temperature in the plane is given by $T(x, y) = x^2 + y^2$ and that a particle is traveling along a path C defined parametrically by $x = t - 1$ $y = t^3 - 3t^2 + 3t + 1$ where t is time in seconds. Show that $T(t)$ has a stationary point when $t = 1$.

Example 6 If $w = f(u, v)$ with $u = x + y$ and $v = x - y$ show that

$$\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = (\frac{\partial w}{\partial u})^2 - (\frac{\partial w}{\partial v})^2$$

Example 7 Suppose that $z = f(x, y)$ where x and y are expressed in polar coordinates

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

Prove that

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial^2 z}{\partial x^2}\cos^2\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta + \frac{\partial^2 z}{\partial x \partial y}\sin(2\theta)$$

This is going to be tough since f is a random function.

To help with the algebra we will denote $\dfrac{\partial^2 z}{\partial r^2}$ simply by z_{rr}, and $\dfrac{\partial^2 z}{\partial x \partial y}$ by z_{xy} ...etc. Now

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta = z_x\cos\theta + z_y\sin\theta$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(z_x\cos\theta + z_y\sin\theta\right) = \frac{\partial}{\partial r}(z_x\cos\theta) + \frac{\partial}{\partial r}(z_y\sin\theta)$$

$$= \cos\theta\frac{\partial}{\partial r}(z_x) + \sin\theta\frac{\partial}{\partial r}(z_y) \qquad \text{(Why?)}$$

Now the bad news. The partial derivatives z_x and z_y are functions of x and y so we will need to use the chain rule again! Thus

$$\frac{\partial}{\partial r}(z_x) = \frac{\partial}{\partial x}(z_x)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}(z_x)\frac{\partial y}{\partial r} = z_{xx}\frac{\partial x}{\partial r} + z_{yx}\frac{\partial y}{\partial r} = z_{xx}\cos\theta + z_{yx}\sin\theta$$

Similarly

$$\frac{\partial}{\partial r}(z_y) = \qquad\qquad\qquad\qquad = z_{xy}\cos\theta + z_{yy}\sin\theta$$

Noting that $z_{xy} = z_{yx}$ and putting it all together we have

$$\frac{\partial^2 z}{\partial r^2} =$$

★

[2]You can now do Q 5 to 14

LECTURE 3
TAYLOR SERIES AND LINEAR APPROXIMATION

The Taylor Series of $f(x, y)$ about the point (a, b) is

$$f(x,y) = f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b)$$

$$+\frac{1}{2!}\left\{(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b)\right.$$

$$\left. +(y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right\} + \text{higher-order terms.}$$

You have already seen in first year how Taylor and Maclaurin series are used to approximate complicated functions with simple polynomials. Polynomials are very easy to work with and approximations of this sort are often used in engineering and the sciences to simplify complex structures. For example $\sin(x)$ is often approximated by x when x is small.

TAYLOR SERIES FOR A SINGLE VARIABLE

Example 1 Find the Maclaurin series for $g(x, y) = e^{x-2y}$ by considering the series above.

$\bigstar 1 + \frac{(x-2y)}{1!} + \frac{(x-2y)^2}{2!} + \cdots \bigstar$

But this simple approach will not always work!

When we jump up a dimension to an arbitrary function $z = f(x, y)$ the theory of Taylor Series is still workable though a little more complicated. We are now approximating complicated surfaces $z = f(x, y)$ with simpler polynomial surfaces. Things to keep in mind are that:

- The production of a Taylor series will involve substantial partial differentiation of the function in question

- A Taylor series about (a, b) will work best near (a, b)

- The more terms you take the better your approximation will be.

Example 2 Calculate the Taylor series of $f(x, y) = e^x \cos(y)$ about $(a, b) = (0, 0)$ up to and including quadratic terms. Compare the value of $f(-0.1, 0.2)$ with its Taylor approximation.

★ $1 + x + \frac{1}{2}(x^2 - y^2)$,0.88680091 vs 0.885 ★

Example 3 Find the Taylor series of $f(x,y) = x^4y^3$ about the point $(1,2)$ up to and including quadratic terms.

$\bigstar$ $8 + 32(x-1) + 12(y-2) + 48(x-1)^2 + 48(x-1)(y-2) + 6(y-2)^2$ $\bigstar$

ERROR ESTIMATES

It is often the case that the independent variables x and y for $z = f(x,y)$ have a degree of uncertainty in their values. That is we need to cope with small errors Δx and Δy in x and y respectively. We may rewrite the linear approximation

$$f(x,y) \approx f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b)$$

as

$$f(x,y) - f(a,b) \approx (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b)$$

and hence

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

This lovely little equation gives you the error in f in terms of the errors in x and y.

Example 4 Suppose that temperature T is given by $T(x, y) = x^2 + y^3$. Find the approximate error in T if $x = 1$, $y = 2$, $\Delta x = 0.1$ and $\Delta y = -0.3$.

$\bigstar - 3.4 \quad \bigstar$

It is rare that we know the exact value and the exact direction of an error! Usually the error could be $\pm$ up to some maximum. We then use

$$|\Delta f| \leq |\frac{\partial f}{\partial x}||\Delta x| + |\frac{\partial f}{\partial y}||\Delta y|.$$

to generate the maximum possible error under given conditions.

Example 5 Suppose that temperature T is given by $T(x, y) = x^2 + y^3$. Find the maximum possible a) absolute error, b) relative absolute error and c) percentage absolute error in T if $x = 1$, $y = 2$, $|\Delta x| \leq 0.1$ and $|\Delta y| \leq 0.3$.

$\bigstar 3.8, \frac{3.8}{9} \approx 0.42, 42\% \bigstar$

Example 6 The volume V of a cone with radius r and perpendicular height h is given by $V = \dfrac{1}{3}\pi r^2 h$. Determine the maximum absolute error and the maximum percentage error in calculating V given that $r = 5$ cm and $h = 3$ cm to the nearest millimetre.

$$\bigstar 2.87, \frac{2.87}{78.54} \times 100 \approx 3.7\% \bigstar$$

[3] You can now do Q 15 to 29

LECTURE 4
LEIBNIZ' THEOREM

Leibniz Rule

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)dt = \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}dt + f(x,v(x))\frac{dv}{dx} - f(x,u(x))\frac{du}{dx}.$$

SOME BASIC INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int \frac{1}{x}dx = \ln|x| + C$$

$$\int e^{ax}dx = \frac{e^{ax}}{a} + C$$

$$\int \sin\, ax\, dx = -\frac{\cos ax}{a} + C$$

$$\int \cos ax\, dx = \frac{\sin ax}{a} + C$$

$$\int \sec^2 ax\, dx = \frac{\tan ax}{a} + C$$

Leibniz' Rule is one of the truly horrible equations in mathematics. It deals with the subtle problem of what happens when we start differentiating integrals of functions of several variables. Differentiation and Integration are of course opposing processes so it would seem reasonable to suspect that differentiating integrals would have some specific consequences! We will motivate the rule with a simple example in Example 2 but first a little revision on integration theory:

Example 1 Evaluate each of the following integrals:

a) $\int e^{7x} dx$

b) $\int \sin(3x) dx$

c) $\int_0^{\frac{\pi}{2}} \cos(\frac{x}{2}) dx$

d) $\int \frac{x}{x^2+1} dx$

★

Example 2 Find

$$\frac{d}{dx}\int_1^{2x} x^6 t^2 dt$$

first directly and then using Leibniz' theorem.

$$\bigstar \quad 24x^8 - 2x^5 \quad \bigstar$$

The problem with the direct approach above is that often the original integral is difficult or even impossible. Leibniz' rule then becomes the only alternative!

It is perhaps best to remember Leibniz in terms of words rather than symbols. It then runs:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)dt =$$

{bring the derivative inside the integral and make it partial}+
{replace variable of integration with the upper limit times the derivative of the upper} −
{replace variable of integration with the lower limit times the derivative of the lower}.

Example 3 Use Leibniz' rule to find

$$\frac{d}{dx}\int_{\sqrt{x}}^{x} \frac{\sin(tx)}{t}dt$$

$$\bigstar \quad 2\frac{\sin(x^2)}{x} - \frac{3}{2}\frac{\sin(x^{\frac{3}{2}})}{x} \quad \bigstar$$

It is important to still be able to implement the rule when the variables are different!!

Example 4

$$\frac{d}{d\alpha}\int_1^{\alpha^2} \ln(1+\beta^8)\, d\beta.$$

$$\bigstar \quad 2\alpha\ln(1+\alpha^{16}) \quad \bigstar$$

Example 5 You are given that $\displaystyle\int_0^{\pi} \frac{1}{\alpha-\cos(\theta)}\, d\theta = \frac{\pi}{\sqrt{\alpha^2-1}}$

Using Leibnitz' rule evaluate $\displaystyle\int_0^{\pi} \frac{1}{(\alpha-\cos(\theta))^2}\, d\theta.$

$$\bigstar \quad \frac{\pi\alpha}{(\alpha^2-1)^{\frac{3}{2}}} \quad \bigstar$$

[4] You can now do Q 30 to 35 and all of Problem Class 1

LECTURE 6
EXTREME VALUES

Second Derivative Test

If f and all its first and second partial derivatives are continuous in the neighbourhood of (a, b) and $f_x(a, b) = f_y(a, b) = 0$ then

(i) f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $\mathcal{D} = f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

(ii) f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $\mathcal{D} = f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

(iii) f has a **saddle point** at (a, b) if $\mathcal{D} = f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).

(iv) If $\mathcal{D} = f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b) the second derivative test is **inconclusive**.

One of the most important applications of calculus of a single variable is the calculation and identification of local maxima and minima. With some minor modifications this theory extends quite naturally into higher dimensions.

Geometrical description of extrema for $z = f(x, y)$

Local maxima and minima for $z = f(x, y)$ are still calculated by setting the first derivative equal to zero. The trouble is that we now have more than one first derivative!! So we have to set both $\dfrac{\partial z}{\partial x} = 0$ and $\dfrac{\partial z}{\partial y} = 0$. This will generate a basket of points which then need to be classified using the tests above. Note that a saddle point is just a 3-D version of a point of inflection. We rarely look specifically for saddle points and they are usually just a nuisance as we search for the crucial extrema. The classification of extrema in higher dimensions is algebraically complicated and you need to take care not to lose solutions along the way. In particular remember to **never divide both sides of an equation by anything that could be zero!**

Example 1 Find and categorise the critical points of the following function:

$$f(x, y) = -x^2 - y^2 - 6x + 4y + 5$$

★ Local maximum at $(-3, 2, 18)$ ★

Example 2 Find and classify the critical points of the following function:

$$f(x,y) = 4xy - x^2y - xy^2$$

★ Saddle points at $(0,0,0),(0,4,0),(4,0,0)$, local max at $(\frac{4}{3},\frac{4}{3},\frac{64}{27})$ ★

Example 3 Ben is an Olympic athlete and a drug cheat. He is soon to run in the 100m sprint final and is taking a mixture of steroids and beta-blockers in order to enhance his performance. He has found that his time T (in seconds) over 100 metres is given by

$$T(s,b) = 0.005s^2 + 0.005b^2 - 0.03s - 0.05b + 10.04$$

where s and b are the number of milligrams of steroids and beta-blockers (respectively) he injects daily.

1. What would his time be if he takes no drugs?
2. What would his time be if he takes 2mg of steroids and 4mg of beta-blockers?
3. How much of each drug should he take in order to minimise his time and what is this minimum time?

★ 10.04 sec, 9.88 sec, 9.87 sec with 3mg of steroids and 5mg of beta-blockers ★

[6] You can now do Q 36 to 38

LECTURE 7
LAGRANGE MULTIPLIERS

The Method of Lagrange Multipliers

To find the local minima and maxima of $z = f(x, y)$ subject to the additional constraint that $g(x, y) = 0$, we find the values of x, y and λ that simultaneously satisfy the three equations

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \quad \text{(I)}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \quad \text{(II)}$$

$$g(x, y) = 0 \quad \text{(III)}$$

In reality we rarely have the opportunity to simply maximise or minimise a quantity. Usually additional constraints must also be met. For example the question 'Find the dimensions of the closed circular can of smallest surface area' is not really a sensible question. Much more interesting is the problem 'Find the dimensions of the closed circular can of smallest surface area whose volume is 16π cm^3. '

Situations where we need to maximise $f(x, y)$ subject to the side condition $g(x, y) = 0$ are best attacked using the method of Lagrange multipliers.

A geometrical view of the problem

We may justify the approach outlined above as follows:

Form the Lagrangian function $L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$. We refer to λ as a Lagrange multiplier, a dummy variable which is introduced to the system to generate solutions. Since $g(x, y) = 0$, L will have max/min exactly when f (constrained by g) has max/min. We now calculate the extrema of $L(x, y, \lambda)$ (a function of 3 variables) in the usual manner:

$$\frac{\partial L}{\partial x} = 0 \to \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0 \to \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial L}{\partial y} = 0 \to \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0 \to \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

$$\frac{\partial L}{\partial \lambda} = 0 \to -g(x, y) = 0 \to g(x, y) = 0$$

We will provide a more intuitive proof in Lecture 11.

These equations, once carefully solved will generate a basket of points for consideration. We *DO NOT* test for max/min as in the previous lecture but rather simply check for max/min by direct evaluation into $f(x, y)$ and or physical considerations.

Example 1

Find the extreme values of $z = f(x, y) = 12 + 3x + 4y$ subject to the constraint

$$x^2 + y^2 - 1 = 0$$

Lets take a careful look at the problem geometrically:

Note that the question 'Find the extreme values of $z = f(x, y) = 12 + 3x + 4y$' makes no sense! A plane takes on neither a max or a min. The side condition $x^2 + y^2 = 1$ is essential here! Let's see how the Method of Lagrange Multipliers works.

★ Maximum of 17 at $(\frac{3}{5}, \frac{4}{5})$. Minimum of 7 at $(-\frac{3}{5}, -\frac{4}{5})$ ★

Observe that we **DO NOT** run any tests here as in the last lecture. We simply see which point yields the biggest value of $f(x, y)$ and which point yields the smallest value of $f(x, y)$.

Example 2 Find the minimum value of $z = f(x, y) = x^4 + y^4$ subject to the condition

$$x + y - 1 = 0$$

Geometrically:

★ Minimum of $\dfrac{1}{8}$ at $(\frac{1}{2}, \frac{1}{2})$ ★

How do we know this is a min and not a max?

Example 3 A closed cylindrical soup can needs to hold 16π cm^3 of soup. If we wish to make the can using a minimal amount of sheet metal, find the radius and the height of the can. Is there a maximal amount of sheet metal for this can?

Surface area $= S = 2\pi r^2 + 2\pi rh$.

Volume$= \pi r^2 h = 16\pi$.

Hence the Lagrange question is to minimise

$$S = 2\pi r^2 + 2\pi rh$$

subject to the constraint that

$$g(r, h) = \pi r^2 h - 16\pi = 0$$

$$\frac{\partial S}{\partial r} = \lambda \frac{\partial g}{\partial r} \quad \rightarrow \quad 4\pi r + 2\pi h = \lambda 2\pi rh \quad \rightarrow \quad 2r + h = \lambda rh. \qquad (\mathbf{I})$$

$$\frac{\partial S}{\partial h} = \lambda \frac{\partial g}{\partial h} \quad \rightarrow \quad 2\pi r = \lambda \pi r^2 \quad \rightarrow \quad 2 = \lambda r. \qquad (\mathbf{II}) \quad (\text{Note } r \neq 0)$$

We also have:

$$g(r, h) = 0 \quad \rightarrow \quad \pi r^2 h - 16\pi = 0 \quad \rightarrow \quad r^2 h = 16. \qquad (\mathbf{III})$$

We have our three equations in three unknowns! Now:

$(\mathbf{II}) \rightarrow \lambda = \dfrac{2}{r}$. Substituting in $(\mathbf{I})$ yields $2r + h = \dfrac{2}{r} rh \rightarrow 2r + h = 2h \rightarrow 2r = h$.

Finally we sub into $(\mathbf{III})$ to get $r^2(2r) = 16 \rightarrow r^3 = 8 \rightarrow r = 2 \rightarrow h = 4$.

Note finally that we know this yields minimum surface area since there is clearly no maximum surface area. That is, we can make these cans as huge as we wish and still have them hold only 16π cubic centimetres!

★ $r = 2$ cm, $h = 4$ cm ★

The method of Lagrange extends naturally to functions of three variables instead of just 2 variables. The following example is a common quiz question.

Example 4 Use Lagrange multipliers to find the minimum distance between the plane $3x + 4y + 12z = 6$ and the point $(5, 7, 11)$.

Let $P(x, y, z)$ be an arbitrary point in space. The distance from the fixed point $(5, 7, 11)$ to $P(x, y, z)$ is then given by

$$\sqrt{(x-5)^2 + (y-7)^2 + (z-11)^2}$$

Thus we wish to minimise $\sqrt{(x-5)^2 + (y-7)^2 + (z-11)^2}$ subject to the constraint that $g(x, y, z) = 3x + 4y + 12z - 6 = 0$.

That is, we wish to minimize the distance from $(5, 7, 11)$ to $P(x, y, z)$ subject to the condition that the point P lies on the given plane. We have an extra dimension here however the method of Lagrange still works in much the same way except that we will now have 4 equations in 4 unknowns x, y, z and λ !

Note first that we can simplify our calculations by dropping the square root and dealing instead with the equivalent problem:

Minimise $s(x, y, z) = (x-5)^2 + (y-7)^2 + (z-11)^2$ subject to the constraint that $g(x, y, z) = 3x + 4y + 12z - 6 = 0$.

Our four equations in 4 unknowns are:

$$\frac{\partial s}{\partial x} = \lambda \frac{\partial g}{\partial x} \quad \to \quad 2(x-5) = 3\lambda.$$
$$\frac{\partial s}{\partial y} = \lambda \frac{\partial g}{\partial y} \quad \to \quad 2(y-7) = 4\lambda.$$
$$\frac{\partial s}{\partial z} = \lambda \frac{\partial g}{\partial z} \quad \to \quad 2(z-11) = 12\lambda.$$
$$\text{and the side condition} \quad \to \quad 3x + 4y + 12z = 6.$$

For homework please read through the solution on the next page carefully.

$$\frac{\partial s}{\partial x} = \lambda \frac{\partial g}{\partial x} \quad \to 2(x-5) = 3\lambda \quad \to x - 5 = \frac{3}{2}\lambda \quad \to x = 5 + \frac{3}{2}\lambda.$$
$$\frac{\partial s}{\partial y} = \lambda \frac{\partial g}{\partial y} \quad \to 2(y-7) = 4\lambda \quad \to y - 7 = \frac{4}{2}\lambda \quad \to y = 7 + 2\lambda.$$
$$\frac{\partial s}{\partial z} = \lambda \frac{\partial g}{\partial z} \quad \to 2(z-11) = 12\lambda \quad \to z - 11 = \frac{12}{2}\lambda \quad \to z = 11 + 6\lambda.$$

Substituting into $3x+4y+12z = 6$ yields $15+\frac{9}{2}\lambda+28+8\lambda+132+72\lambda = 6$ and hence $84.5\lambda = -169 \to \lambda = -2$. It follows that $x = 5 + \frac{3}{2}(-2) = 2$, $y = 7 + 2(-2) = 3$ and $z = 11 + 6(-2) = -1$. The closest point on the plane is therefore $(2, 3, -1)$ and the shortest distance is $\sqrt{(2-5)^2 + (3-7)^2 + (-1-11)^2} = \sqrt{169} = 13$.

Note that this must be the shortest distance, as the greatest distance is unbounded.

★ shortest distance of 13 at the point on the plane $(2, 3, -1)$ ★

The next lecture (Lecture 8) is a problem class and the lecture after that (Lecture 9) will be a revision of first year linear algebra. In preparation for Lecture 9 please revise all of your first year material on vectors, dot and cross products of vectors, the Cartesian equation of a plane in space and the parametric vector equation of a lines and planes in space. You may wish to look at the revision first year linear algebra notes on Math2019 Moodle pp 49-84.

[7] You can now do Q 39 to 47

LECTURE 9
VECTOR ALGEBRA

Scalars are quantities which have only a magnitude such as temperature, mass, time and speed. Vectors have a magnitude **and a direction**.

Vectors in Physics vs Vectors in Mathematics

Thus from a mathematical perspective we may view vectors as graphical objects $\longrightarrow$ or alternatively as algebraic objects $\mathbf{u} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} = \mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$.

We will now review a host of vector applications covered in first year.

Example 1 Let $\mathbf{u} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$. Find $|\mathbf{u}|$ and hence write down two unit vectors parallel to $\mathbf{u}$.

Recall that two non-zero vectors are parallel iff they are scalar multiples of each other.

★ $\sqrt{42}, \pm\frac{1}{\sqrt{42}}\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$ ★

Example 2 Let $\mathbf{v} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 8 \\ 2 \end{pmatrix} = \mathbf{i} + 8\mathbf{j} + 2\mathbf{k}$. Use the dot product to prove that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

The dot product measures the interaction between vectors. It is zero when the vectors have absolutely nothing to do with each other (perpendicular) and increases in value as the vectors get closer and closer to being parallel. Between these two extremes we can project a vector $\mathbf{u}$ onto a vector $\mathbf{v}$ using the formula $Proj_{\mathbf{v}}(\mathbf{u}) = (\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}})\mathbf{v}$.

Example 3 Let $\mathbf{u} = \begin{pmatrix} 1 \\ -18 \\ 7 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$. Find $Proj_{\mathbf{v}}(\mathbf{u})$ and display this situation in a vector diagram.

$\bigstar \quad \frac{1}{2}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \quad \bigstar$

Example 4 Let $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$. Find the cross product $\mathbf{u} \times \mathbf{v}$.

Check your answer with the dot product.

Some Properties of the Cross Product

$(\mathbf{u} \times \mathbf{v}) \perp \ \mathbf{u}$ and $(\mathbf{u} \times \mathbf{v}) \perp \ \mathbf{v}$
with the direction of the cross product determined by the right hand rule.

$$(\mathbf{u} \times \mathbf{v}) = -(\mathbf{v} \times \mathbf{u})$$

Cross products help us to determine how much of "space" is carved out by a batch of non-zero vectors.

In two dimensions, the magnitude $|\mathbf{u} \times \mathbf{v}|$ of the cross product is the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

It follows that if $\mathbf{u} \times \mathbf{v} = \mathbf{0} \iff \mathbf{u}$ and $\mathbf{v}$ are parallel.

In three dimensions the magnitude of the scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is the volume of the parallelepiped formed by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

It follows that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{0} \iff \mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are coplanar.

Question: What does it mean if $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w} = \mathbf{0}$?

Let's finish off revising the theory of lines and planes in space.

Example 5 Let $\mathcal{P}$ be the plane parallel to the two vectors $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and passing through the point $\begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$.

a) Find a parametric vector equation for $\mathcal{P}$.

b) Find a Cartesian equation for $\mathcal{P}$.

---

a)

b) Recall from Example 4 that $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} -18 \\ 11 \\ -1 \end{pmatrix}$.

This vector is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ and hence perpendicular to the plane $\mathcal{P}$.

Hence the Cartesian equation of $\mathcal{P}$ is

$$-18x + 11y - z = \#$$

★ $18x - 11y + z = 41$ ★

Example 6 Find a parametric vector equation of the line passing through the two points $A = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix}$.

Prove that this line is perpendicular to to the plane $4x+12y+4z = -56$ and determine where the line and the plane meet.

$$\bigstar \quad \begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix} \quad \bigstar$$

[9]You can now do Q 48 to 51

LECTURE 10
VECTOR AND SCALAR FIELDS

$$\nabla\phi = \text{grad}\,\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}.$$

grad : scalar to vector

$$\nabla\cdot\mathbf{F} = \text{div}\,\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

div : vector to scalar

$$\nabla\times\mathbf{F} = \text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

curl : vector to vector

Where the vector differential operator ∇ is given by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

We turn now to the theory of vector and scalar fields, a topic of immense application in Engineering for reasons which will soon become obvious. A scalar field assigns a scalar (think temperature) to each point in space. Imagine measuring the temperature at each point in the room. Each point would have a little tag designating it's temperature and as you moved around the space there would of course be subtle variations in the temperature. Similarly a vector field assigns a vector to each point in space. For example we could measure the wind velocity (speed plus direction) everywhere in the room. There would be big vectors near the AC ducts and small vectors in the corners of the room where there is little turbulence. Vector fields are the ideal tool for studying **flows** whether they be rivers, lava or weather patterns. These concepts may be applied in any dimension, however they are mostly used in $\mathbb{R}^2$ and $\mathbb{R}^3$. In this lecture we will simply get a feeling for how these fields are presented and manipulated. In the next two lectures we will focus on applications.

Example 1 Consider the scalar temperature field $T(x, y, z) = x^2 + y^2 + z^2$. Find the temperature at the point $P(3, 0, 4)$ and describe the surface of all points whose temperature is the same as that of P. (This is called a *level surface*).

★ $T = 25$ and the level surface is all points on a sphere of radius 5 ★

Observe from the above that a scalar field is little more than a function from $\mathbb{R}^3$ or $\mathbb{R}^2$ into $\mathbb{R}$. Vector fields are a little trickier.

Example 2 Consider the vector field $\mathbf{F}(x, y, z) = (x^2 + y^2)\mathbf{i} + xyz\mathbf{j} - e^x z\mathbf{k}$. Find the vector at the point $Q(0, 2, 3)$.

★ $4\mathbf{i} - 3\mathbf{k} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$ ★

Observe how neatly vector fields assign vectors to points!

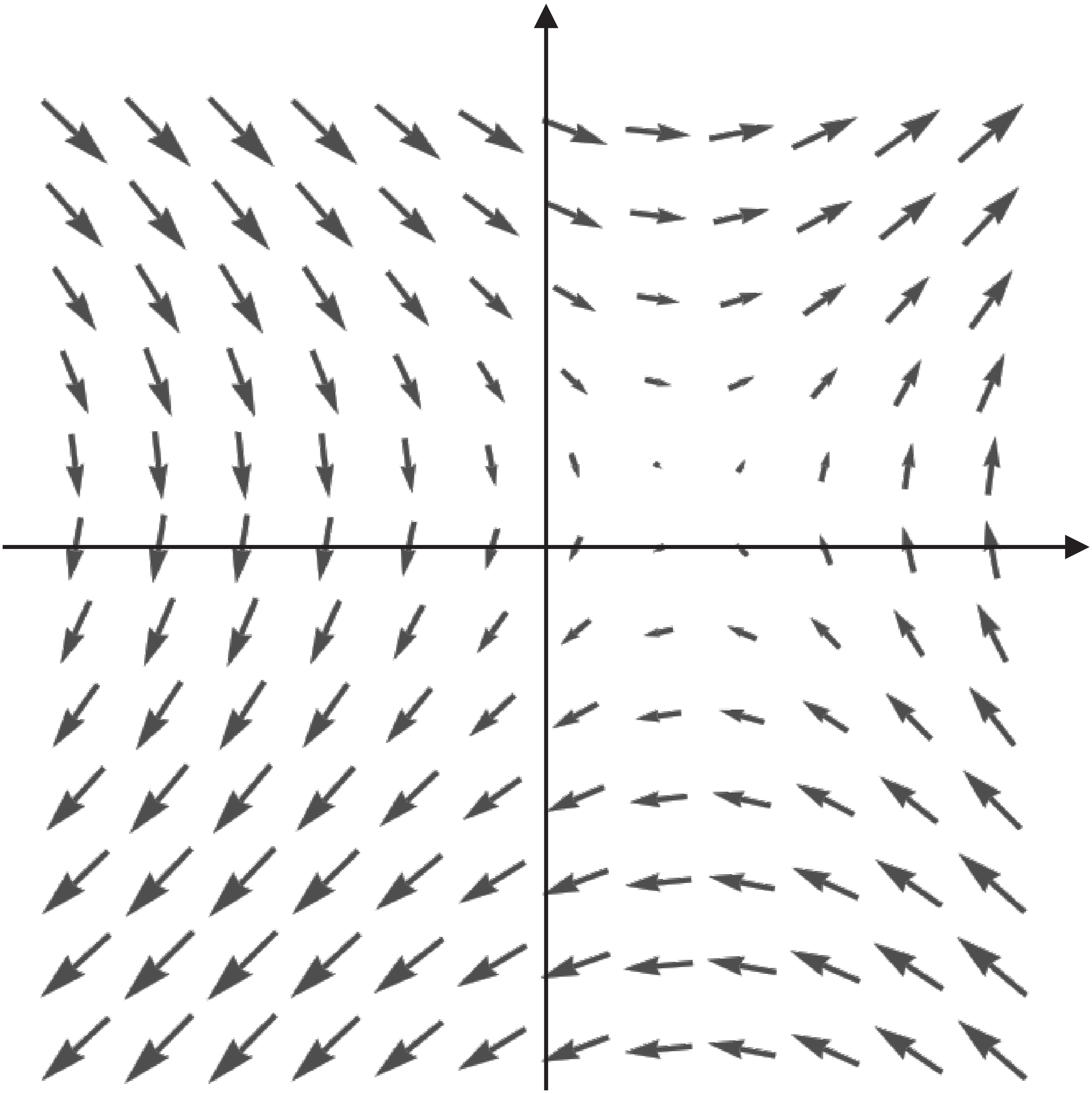
Drawing vector and scalar field is a messy and time consuming process best left to computers. Let's do one in two dimensions just to get a feeling for what they look like:

Example 3 Consider the vector field $\mathbf{T}(x, y) = (x + y)\mathbf{i} + (x^2 + y^2 + 1)\mathbf{j}$ in $\mathbb{R}^2$. Sketch the vectors of the field at the set of points $\{(0, 0), (1, 1), (-1, 1), (1, -1), (-1, -1)\}$.

★

Please observe very carefully in the above analysis that we simply treat vectors as instruments that specify magnitude and direction. They are definitely not stuck at the origin. Indeed they are not fixed anywhere but rather are free to roam across space doing their work wherever it is necessary to point in a particularly direction. The x coordinate tells you how far to go left or right in the x direction and the y coordinate tells you how far to go up or down in the y direction. Where this actually happens is up to you.

Remember that vector fields are an attempt to mathematically analyse flows. An example of a more complicated vector field is:



We now turn to three operators grad, div and curl. These transform vector fields into scalar fields and vice versa. Today we will only look at the technicalities of their calculations.

Example 4 Let $\phi(x, y, z) = x^2y^3z + 2y$ be a scalar field. Calculate grad(ϕ).

$$\nabla\phi = \text{grad}\,\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}. \qquad \textbf{grad : scalar to vector}$$

Note that the differential operator $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ can be used in one way or another to define each of grad, div, and curl. For grad we simply allow ∇ to operate on ϕ.

$$\bigstar \quad 2xy^3z\mathbf{i} + (3x^2y^2z + 2)\mathbf{j} + x^2y^3\mathbf{k} = \begin{pmatrix} 2xy^3z \\ 3x^2y^2z + 2 \\ x^2y^3 \end{pmatrix} \quad \bigstar$$

Example 5 Let $\mathbf{F}(x, y, z) = (x^2 + y^2)\mathbf{i} + xyz\mathbf{j} - e^x z\mathbf{k} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ be a vector field. Find div($\mathbf{F}$). (the divergence of the vector field $\mathbf{F}$)

$$\nabla \cdot \mathbf{F} = \text{div}\,\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \qquad \textbf{div : vector to scalar}$$

Here we are just taking the scalar product of ∇ and $\mathbf{F}$

$$\bigstar \quad 2x + xz - e^x \quad \bigstar$$

Example 6 Given the vector field $\mathbf{F}(x, y, z) = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = 3y\mathbf{i} + xz\mathbf{j} - x^3\mathbf{k}$ find curl($\mathbf{F}$).

$$\nabla \times \mathbf{F} = \text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} \qquad \textbf{curl : vector to vector}$$

For curl we take the cross product of ∇ and $\mathbf{F}$ and thus we need to find a determinant.

$$\bigstar \quad -x\mathbf{i} + 3x^2\mathbf{j} + (z-3)\mathbf{k} = \begin{pmatrix} -x \\ 3x^2 \\ z-3 \end{pmatrix} \quad \bigstar$$

We will fully explore the applications of these processes in the next few lectures . We close however with some special relations between grad, div and curl.

We close with an example of a simple proof:

Theorem $\nabla \times (\nabla \phi) = \mathbf{0}$. (That is curl(grad($\phi$))=$\mathbf{0}$)

Proof: Let $\phi(x, y, z)$ be a scalar field. Then

grad(ϕ) $= \nabla\phi =$

[10] You can now do Q 54,59,60

LECTURE 11
APPLICATIONS OF GRAD

Given a scalar field ϕ, the directional derivative of ϕ in the direction of the vector $\mathbf{b}$ is given by $(\text{grad}\,\phi)\cdot\hat{\mathbf{b}}$ where $\hat{\mathbf{b}}$ is the unit vector in the direction of $\mathbf{b}$.

Given a scalar field ϕ at a point P the direction of maximum increase of ϕ from P is given by $\nabla\phi|_P$ with the magnitude of the increase being the magnitude of $\nabla\phi|_P$.

Given a scalar field ϕ at a point P the direction of maximum decrease of ϕ from P is given by $-\nabla\phi|_P$ with the magnitude of the decrease being the magnitude of $-\nabla\phi|_P$.

$\nabla\phi|_P$ points perpendicular to the level surface (or curve in 2-D) at P.

(**Note that $\nabla\phi|_P$ is just short hand for** $\text{grad}\,\phi$ **at** P)

Directional Derivatives

We have seen earlier that for a scalar field $\phi(x,y,z)$, we can easily find rates of change in the x, y and z directions by using the partial derivatives $\dfrac{\partial\phi}{\partial x}, \dfrac{\partial\phi}{\partial y}$ and $\dfrac{\partial\phi}{\partial z}$. But what if we are immersed in a scalar field and wish to determine the rate of change of the field in some other direction specified by a vector $\mathbf{b}$. This is called a **directional derivative**.

Given a scalar field ϕ, the directional derivative of ϕ in the direction of the vector $\mathbf{b}$ is given by $(\text{grad}\,\phi)\cdot\hat{\mathbf{b}}$ where $\hat{\mathbf{b}}$ is the unit vector in the direction of $\mathbf{b}$.

Proof: After example.

Example 1 Calculate the directional derivative of the scalar field $\phi(x, y, z) = x^2yz$ in the direction $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ at the point $P(-1, 1, 3)$.

$$\star \quad \frac{2}{3} \quad \star$$

This means that if you sit at the point $P(-1, 1, 3)$ in the scalar field $\phi = x^2yz$ and head off in the direction $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ then the instantaneous rate of change of temperature with respect to distance is equal to $\frac{2}{3}$.

Proof of formula: We will prove the result in space. The argument in other dimensions is similar. Let the unit vector $\hat{\mathbf{b}}$ be given by $\hat{\mathbf{b}} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and define a straight-line path in the $\hat{\mathbf{b}}$ direction by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} s$. Then clearly for $s \geq 0$ the fact that $\hat{\mathbf{b}}$ is a unit vector implies that the magnitude of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is just s. Hence s may be interpreted as distance (in the $\hat{\mathbf{b}}$ direction).

Now $\phi(x, y, x)$ depends upon x, y and z which in turn depend upon s. Hence via the chain rule:

$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial s} = \frac{\partial\phi}{\partial x}b_1 + \frac{\partial\phi}{\partial y}b_2 + \frac{\partial\phi}{\partial z}b_3 = \begin{pmatrix} \frac{\partial\phi}{\partial x} \\ \\ \frac{\partial\phi}{\partial y} \\ \\ \frac{\partial\phi}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = (\text{grad}\phi) \cdot \hat{\mathbf{b}}$$

as required. $\star$

Example 2 The pressure in a region of space is given by $P(x,y,z) = \ln(x)y^2e^z + 3xyz$. Calculate the rate of change of the pressure with respect to distance at the point (1,2,0) in the direction $2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

$\bigstar \quad \frac{14}{\sqrt{6}} \quad \bigstar$

We can differentiate a scalar field in any direction. A crucial question we now need to answer is, given a point P in a scalar field ϕ, in which direction should we move in order to increase the scalar field as quickly as possible? In other words which direction $\mathbf{b}$ yields the maximal directional derivative? We have shown above that the directional derivative is given by $(\text{grad}\,\phi) \cdot \hat{\mathbf{b}}$, hence we wish to maximise $|(\text{grad}\,\phi) \cdot \hat{\mathbf{b}}|$.

But from first year we know that $|(\text{grad}\,\phi) \cdot \hat{\mathbf{b}}| = |(\text{grad}\,\phi)||\hat{\mathbf{b}}|\cos(\theta)$, where θ is the angle between grad ϕ and $\hat{\mathbf{b}}$. Now grad ϕ is fixed and $|\hat{\mathbf{b}}| = 1$, hence all we need to do is maximise $\cos(\theta)$ which occurs when $\theta = 0$. That is, when grad ϕ and $\mathbf{b}$ point in the same direction. So the direction $\mathbf{b}$ which maximises the rate of of change is just grad ϕ. Furthermore the magnitude of this maximal directional derivative is just $|\text{grad}\phi| \times 1 \times 1 = |\text{grad}\phi|$. We therefore have:

> Given a scalar field ϕ at a point P the direction of maximum increase of ϕ from P is given by $\nabla\phi|_P$ with the magnitude of the increase being the magnitude of $\nabla\phi|_P$.
>
> Given a scalar field ϕ at a point P the direction of maximum decrease of ϕ from P is given by $-\nabla\phi|_P$ with the magnitude of the decrease being the magnitude of $-\nabla\phi|_P$.
>
> $\nabla\phi|_P$ points perpendicular to the level surface (or curve in 2-D) at P.
>
> (**Note that $\nabla\phi|_P$ is just short hand for** grad ϕ **at** P)

So grad ϕ always points in the direction of max increase of a scalar field. The last point above states that it is also true that grad ϕ is always orientated perpendicular to the level curves and surfaces. In other words the direction of maximal change in a scalar field is always perpendicular to the direction of no change. We will prove this at the end of the next lecture once we have introduced a little vector calculus.

Example 3 Consider the simple temperature field $T(x, y) = x^2 + y^2$ in $\mathbb{R}^2$. Find the direction and magnitude of maximum increase and maximum decrease of T at the point $P(3, 3)$. Describe the level curve at P and verify graphically that the direction of maximum increase of T at P is perpendicular the level curve through P.

★ $6\mathbf{i} + 6\mathbf{j},\ -6\mathbf{i} - 6\mathbf{j},\ \sqrt{72}$ ★

Example 4 Consider the scalar field $\phi(x,y) = \dfrac{5x^2}{y}$ in $\mathbb{R}^2$. Find the direction and magnitude of maximum increase of ϕ at the point $P(2,20)$. Describe the level curve at P and verify graphically that the direction of maximum increase of ϕ at P is perpendicular the level curve through P.

★ $\mathbf{i} - \frac{1}{20}\mathbf{j}$, $\sqrt{\frac{401}{400}}$ ★

Example 5 Consider the scalar field $\phi(x, y, z) = x^2z + 2y^2 - ye^{z^2}$. What is the magnitude and direction of the max rate of change of ϕ at $P(1, 2, 0)$?

$\bigstar$ $7\mathbf{j} + \mathbf{k}$, $\sqrt{50}$ $\bigstar$

[11] You can now do Q 56,57,58

LECTURE 12
VECTOR CALCULUS

$$\nabla\phi = \text{grad}\,\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}.$$

grad : scalar to vector

$$\nabla\cdot\mathbf{F} = \text{div}\,\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

div : vector to scalar

$$\nabla\times\mathbf{F} = \text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

curl : vector to vector

Where the vector differential operator ∇ is given by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Before having a look at how the theories of calculus may be further applied to vectors, we have some final applications of grad, div and curl.

Example 1 Find the tangent plane and the normal line to the surface

$$x^4 + y^4 + 3z^4 = 20$$

at the point $P(2, 1, -1)$.

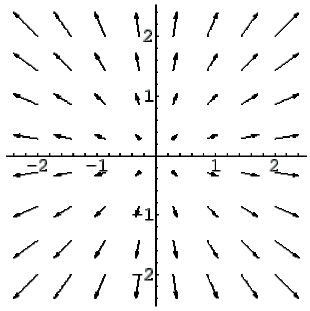
★ $32x + 4y - 12z = 80$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 32 \\ 4 \\ -12 \end{pmatrix} t : t \in \mathbb{R}$. ★

Physical Interpretation of Divergence $\nabla \cdot \mathbf{F} = \text{div}\mathbf{F}$

Imagine that $\mathbf{F}$ is a vector field representing the flow of a fluid and let B be a small wire cube immersed in the field. We wish to measure the rate R per unit volume at which the fluid flows through the cube across its six faces, at any given time. If we place the cube in a constant field (like a strongly flowing river with no turbulence) then $R = 0$, since all fluid that enters the cube also leaves at the same rate. Suppose however that a strong source of fluid is placed in the center of the cube (the end of a hose for example). Then R would be positive as water is leaving the cube on all of its faces. Alternatively if a drain hole is placed in the center of the cube then R would be negative since water would be entering the cube on all of its faces. This is what div measures! If we take the limit of this rate R as the size of the cube goes to zero we obtain

$$\nabla \cdot \mathbf{F} = \text{div}\,\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad ....$$

a measure of the outward flow or expansion of the field from a point. Another way of saying this is that if the divergence is positive at a point P then a tiny cube placed around P would tend to explode whereas if the divergence is negative then the cube would tend to implode.



Positive divergence at the origin

Example 2 Calculate the divergence div($\mathbf{F}$) of the vector field $\mathbf{F} = (x^2)\mathbf{i} + (y^3 z)\mathbf{j} + (4z)\mathbf{k}$ at the origin and also the point $P(-10, 1, 5)$. How is the flow at P different from the flow at the origin?

★ outward flow at the origin, inward flow at P. ★

Physical Interpretion of $\nabla \times \mathbf{F} = \text{curl}\,\mathbf{F}$

Given a vector field $\mathbf{F}$, curl($\mathbf{F}$) measures infinitesimal rotations caused by the vector field. Consider a point P in the field and imagine a tiny sphere whose centre is fixed at P with the sphere still having the freedom of rotation in any direction. Then (the vector!!) curl($\mathbf{F}$) at P is the axis upon which the sphere would rotate under the action of the field (with the orientation of rotation given by the right hand rule). Furthermore the magnitude of the vector curl($\mathbf{F}$) measures the speed of that rotation.

Example 3 Consider the vector field $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j} + 4z\mathbf{k}$. Calculate curl($\mathbf{F}$) at the point $P(3, 1, 2)$ and hence show that this field induces a rotation at P about a vertical axis. Is the rotation clockwise or anticlockwise when viewed from above?

curl($\mathbf{F}$)=

Since curl($\mathbf{F}$) points straight up the positive $\mathbf{k}$ axis at P we have an anticlockwise rotation (when viewed from above) about a vertical axis!

★

Example 4 Find a point Q where the field in the previous example induces no rotation.

★

Vector Calculus

By establishing a time dependent position vector of a particle

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

it is possible to not only specify paths in $\mathbb{R}^3$ but to also to use calculus to analyse the velocity $\mathbf{v}(t) = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} + \dot{z}(t)\mathbf{k}$ and acceleration $\mathbf{a}(t) = \ddot{x}(t)\mathbf{i} + \ddot{y}(t)\mathbf{j} + \ddot{z}(t)\mathbf{k}$ of the particle's motion along the path. Note that the velocity vector is tangential to the path and that by taking the magnitude of the velocity vector we obtain the speed of the particle.

Example 5 Consider the path in $\mathbb{R}^2$ given by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$

Show that the path is a circle and prove that the velocity vector is always tangential to the circle and the acceleration of the particle is always directed to the centre of the circle.

★

Example 6 Consider the path in $\mathbb{R}^2$ given by $\mathbf{r}(t) = \cos(3t)\mathbf{i} + \sin(3t)\mathbf{j}$. How is this motion different from the previous example?

★

Example 7 Suppose that a particle moves through space along the path

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + (t^2)\mathbf{k}.$$

Describe the path and determine the speed and magnitude of the acceleration of the particle after 3 seconds.

★ *speed* $= \sqrt{37}$, *acceleration* $= \sqrt{5}$ ★

Example 8 Suppose that

$$\mathbf{r}_1(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$$

and that

$$\mathbf{r}_2(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$$

are two curves in space.

Prove that $(\mathbf{r}_1 \cdot \mathbf{r}_2)' = \mathbf{r}_1' \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \mathbf{r}_2'$

★

[12]You can now do Q52,53,55

MATH2019 LECTURES 13 and 13A
SOME PROOFS AND THE THEORY OF LINE INTEGRALS

Line integrals are used to calculate the work done in moving a particle P from A to B along a path $\mathcal{C}$ through a force field $\mathbf{F}$.

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} (F_1 dx + F_2 dy + F_3 dz)$$

In general, this integral depends not only on $\mathbf{F}$ but also on the **path** $\mathcal{C}$ we take from A to B.

$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ denotes the line integral around a **closed** curve $\mathcal{C}$. (That is $A = B$)

Two simple properties

I)
$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = -\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

where $\mathcal{C}_1$ is the same curve as $\mathcal{C}$ except that we start at B and finish at A. That is, reversing the direction of the path changes the sign of a line integral.

II)
$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathcal{C}_1$ is the curve from A to X, and $\mathcal{C}_2$ the curve from X to B following the path $\mathcal{C}$. In other words breaking the journey does not affect the line integral.

This is quite a long lecture and will be delivered over two hours. We will start with a couple of proofs of claims made in the previous lectures and then move on to the general theory of line integrals. There should be time at the end of the second hour for you to ask any questions you may have regarding the material up to and including line integrals.

Claim: Let $\phi(x,y,z)$ be a scalar field in space and let P be a point on the level surface $\phi(x,y,z)=c$. Then grad(ϕ) at P is perpendicular to the level surface $\phi(x,y,z)=c$ at P.

Proof: Let $\phi(x,y,z)$ be a scalar field in space and let P be a point on a level surface $\phi(x,y,z)=c$. Let $x=x(t)$, $y=y(t)$ and $z=z(t)$ be any parametrically defined path passing throuph P and embedded within the level surface $\phi(x,y,z)=c$. Differentiating with respect to t and implementing the chain rule yields

$$\frac{\partial\phi}{\partial x}\frac{dx}{dt}+\frac{\partial\phi}{\partial y}\frac{dy}{dt}+\frac{\partial\phi}{\partial z}\frac{dz}{dt}=0$$

and hence

$$\begin{pmatrix} \dfrac{\partial\phi}{\partial x} \\ \\ \dfrac{\partial\phi}{\partial y} \\ \\ \dfrac{\partial\phi}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = 0 \to (\mathrm{grad}\phi)\cdot\mathbf{v}=0$$

Noting that the velocity vector is tangential to the path, we have shown that gradϕ is perpendicular to all of the tangents to the level surface at P, and thus is perpendicular to the level surface as required. The result and proof in other dimensions is similar.

★

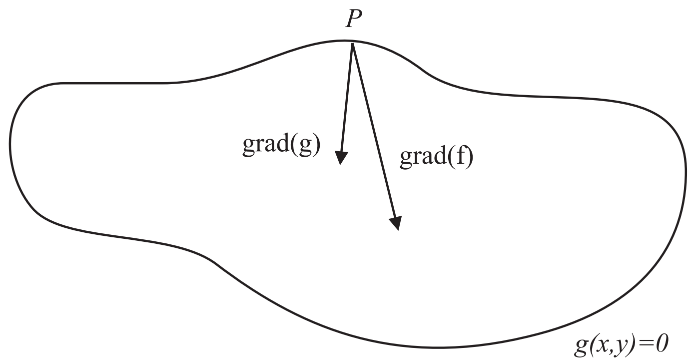
An Intuitive Verification of the Method of Lagrange

Recall that the method of Lagrange Multipliers states that, to find the local minima and maxima of $f(x,y)$ subject to the constraint $g(x,y)=0$ we find the values of x, y and λ that simultaneously satisfy the equations

$$\frac{\partial f}{\partial x} - \lambda\frac{\partial g}{\partial x} = 0, \quad \frac{\partial f}{\partial y} - \lambda\frac{\partial g}{\partial y} = 0, \quad \text{together with} \quad g(x,y) = 0.$$

We can now justify this result using vector field theory:

The constraint $g(x,y)=0$ is simply a level curve for the scalar field $g(x,y)$. Pick a point P on the curve $g(x,y)=0$. At P, grad(g) will point in the direction of maximal increase of $g(x,y)$ and will also point perpendicular to the level curve $g(x,y)=0$.



However $g(x,y)=0$ is not a level curve for $f(x,y)$. Thus grad(f) will most likely not point perpendicular to $g(x,y)=0$. Since grad(f) points in the direction of maximum increase of $f(x,y)$ it will therefore be possible to increase $f(x,y)$ by sliding a little around the level curve from P, in the direction suggested by grad(f).

If the point P is actually sitting at a maximum value of $f(x,y)$ then there will be no component of grad(f) in the tangential direction through which $f(x,y)$ could be increased. In other words if P is positioned so that $f(x,y)$ is maximised (or minimised),then grad(f) will **also** point perpendicular to $g(x,y)=0$. This means that grad(f) and grad(g) are parallel vectors and hence that grad$(f) = \lambda\,$grad(g). This means that

$$\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \lambda(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j})$$

and so

$$\frac{\partial f}{\partial x} = \lambda\frac{\partial g}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \lambda\frac{\partial g}{\partial y}$$

as required.

LINE INTEGRALS

We have seen in the previous lectures how partial differentiation may be used to analyse vector fields. Integration also plays a role in the theory through the concept of a line integral. Some of you will have seen the formula for the work done by a force over a distance given by Work=Force× Distance. We are in a similar but slightly more complicated situation.

We have a force vector field rather that just a force. Also we have a path rather than just a distance. A line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ measures the work done on a particle as it moves through a vector field $\mathbf{F}$ along a path $\mathcal{C}$. The dot product measures the interaction between the path and the field, with the work done being zero if the path sits perpendicular to the field. A positive work integral indicates that the the particle is being worked upon while a negative value indicates that the particle is doing work over the length of the path. It's like swimming with the current (+) or against the current (−).

Line integrals are usually evaluated by parametrising the path and then converting the line integral into a standard integral in terms of the parameter. In certain circumstances (when the vector field is conservative) there is also a dramatic shortcut to their evaluation.

The Meaning of $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$

$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a vector field and $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$ is an increment along the path $\mathcal{C}$. The line integral "sums" all the dot products $\mathbf{F} \cdot d\mathbf{r}$ as we traverse the path. The dot products are large when $\mathbf{F}$ and $d\mathbf{r}$ are close to parallel and small when they are close to perpendicular. As an example consider swimming in a river:

Example 1: Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (1-2y)\mathbf{i} + 2x\mathbf{j}$ and $\mathcal{C}$ is the path in $\mathbb{R}^2$ along the circle $x^2 + y^2 = 9$ from $(3, 0)$ to $(0, 3)$ in the first quadrant.

The key to the evaluation of most line integrals is to express the path parametrically with strict limits on the parameter, and then recast the line integral as a simple one dimensional integral in terms of the parameter. The standard parametric representation of a circle of radius r is given by $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

★ $9\pi - 3$ ★

Question: What would the line integral be if we went back the other way along the circle from $(0, 3)$ to $(3, 0)$?

Question: If we went all the way around the circle is it always true that $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$?

Example 2: Find the work done on a particle traveling through the field $\mathbf{F} = y^2\mathbf{i} + (y^3 + e^z)\mathbf{j} + (x - 2z)\mathbf{k}$ along the straight line from $(1, 1, 1)$ to $(3, 1, 4)$ in $\mathbb{R}^3$.

★ -7 ★

Example 3: Find the work done on a particle traveling through the field $\mathbf{F} = (yz+2)\mathbf{i} + xz\mathbf{j} + (xy+\cos(z))\mathbf{k}$ along the circle $x^2+y^2=1$ in the $x-y$ plane from $(1,0,0)$ to $(0,1,0)$ and then along the straight line from $(0,1,0)$ to $(1,1,\pi)$.

This particular type of example has a dramatic shortcut which we will discuss in the next example. But first let's do it the ugly way. The picture here is:

The path is in 2 clear pieces so we will need to evaluate 2 independent work integrals. First the little circular bit:

$x = \cos(\theta) \to dx = -\sin(\theta)d\theta$

$y = \sin(\theta) \to dy = \cos(\theta)d\theta$

$z = 0 \to dz = 0d\theta.$

Note finally that $\theta : 0 \to \dfrac{\pi}{2}$. Hence:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \begin{pmatrix} yz+2 \\ xz \\ xy+\cos(z) \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \int (yz+2)dx + xzdy + (xy+\cos(z))dz.$$

Since both z and dz are 0 we obtain:

$$\int_0^{\frac{\pi}{2}} (0+2)(-\sin(\theta))d\theta = \int_0^{\frac{\pi}{2}} -2\sin(\theta)d\theta = [2\cos(\theta)]_0^{\frac{\pi}{2}} = 0-2 = -2.$$

Along the line interval we have:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ \pi \end{pmatrix} t, \quad t : 0 \to 1. \text{ Thus}$$

$x = t \to dx = dt, \quad y = 1 \to dy = 0dt$ and $z = \pi t \to dz = \pi dt$. So

$$\int (yz+2)dx + xzdy + (xy+\cos(z))dz = \int_0^1 (\pi t+2)dt + xz(0dt) + (t+\cos(\pi t))\pi dt$$

$$= \int_0^1 \pi t + 2 + \pi t + \pi\cos(\pi t)dt = \int_0^1 2\pi t + 2 + \pi\cos(\pi t) = [\pi t^2 + 2t + \sin(\pi t)]_0^1$$

$= (\pi+2+0) - (0) = \pi+2$. The total work done is then $-2 + (\pi+2) = \pi$.

$$\bigstar \quad -2 + (\pi+2) = \pi \quad \bigstar$$

Definition: A vector field $\mathbf{F}$ is said to be **conservative** if $F = \text{grad}(\phi)$ for some scalar field ϕ. (Recall that we also write grad(ϕ) as $\nabla\phi$.)

Most vector fields are NOT conservative BUT if $\mathbf{F}$ is conservative then there is a huge shortcut available for the calculation of the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

Fact 1: All line integrals for a conservative vector field given by $\mathbf{F} = \text{grad}(\phi)$ are path independent and depend only upon the starting point P and the finishing point Q of the path $\mathcal{C}$. Furthermore

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \Big[\phi\Big]_P^Q$$

Proof: We have $\mathbf{F} = \nabla\phi = \text{grad}(\phi) = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$.

Thus taking **any** parametrisation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$; $(a \leq t \leq b)$ of $\mathcal{C}$ we have

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} \begin{pmatrix} \dfrac{\partial\phi}{\partial x} \\[2ex] \dfrac{\partial\phi}{\partial y} \\[2ex] \dfrac{\partial\phi}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{dx}{dt} \\[2ex] \dfrac{dy}{dt} \\[2ex] \dfrac{dy}{dt} \end{pmatrix} dt = \int_a^b \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}\ dt$$

and hence via the chain rule

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \frac{d\phi}{dt}\, dt = \Big[\phi\Big]_P^Q$$

$\bigstar$

All the above fact is saying is that if $\mathbf{F} = \text{grad}(\phi)$ for some ϕ then $\mathbf{F}$ is the "derivative" of ϕ and hence when you "integrate" $\mathbf{F}$ you get ϕ.

Please remember however most vector fields are NOT conservative and hence the above fact is useless in general. But we do have a simple test for whether or not a vector field is conservative:

Fact 2: A vector field $\mathbf{F}$ is conservative if and only if curl($\mathbf{F}$)=0.

That is, $\mathbf{F}$ is conservative if there is never a rotational force at a point! Going left is the same as going right so in the end the path doesn't actually matter.

Lets now return to the previous example and do it using our shortcut.

Example 4: For $\mathbf{F} = (yz+2)\mathbf{i} + xz\mathbf{j} + (xy+\cos(z))\mathbf{k}$ of the previous example:

a) Prove $\mathbf{F}$ is conservative.

b) Find a scalar field ϕ such that $\mathbf{F}$=grad(ϕ).

c) Hence check the answer in Example 3 by re-evaluating the work integral making efficient use of ϕ .

---

a)

b) Finding ϕ is a little tricky. We are told that

$$\mathbf{F} = (yz+2)\mathbf{i} + xz\mathbf{j} + (xy+\cos(z))\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} = \text{grad}(\phi)$$

Thus

$$\frac{\partial \phi}{\partial x} = yz + 2 \Longrightarrow \quad \phi =$$

$$\frac{\partial \phi}{\partial y} = xz \Longrightarrow \quad \phi =$$

$$\frac{\partial \phi}{\partial z} = xy + \cos(z) \Longrightarrow \quad \phi =$$

Putting it all together we have $\quad \phi =$

c)

★ π (again) ★

Question: What is the value of $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ if $\mathbf{F}$ is conservative? (Note $\oint_{\mathcal{C}}$ denotes an integral around a closed path $\mathcal{C}$).

★

Example 5: Consider the vector field

$$\mathbf{F} = (2xy^3z^4 + ye^{xy})\mathbf{i} + (3x^2y^2z^4 + xe^{xy} + \frac{z}{y})\mathbf{j} + (4x^2y^3z^3 + \ln y)\mathbf{k}$$

a) (HOMEWORK) Show that $\mathbf{F}$ is conservative by proving that curl($\mathbf{F}$) $= \mathbf{0}$.

b) Find a scalar field ϕ such that $\mathbf{F}$=grad(ϕ).

c) The path $\mathcal{C}$ starts at the point $(3, 4, 7)$ and rotates anticlockwise three complete revolutions in the circle $x^2+y^2 = 25$ within the plane $z = 7$ returning to the point $(3, 4, 7)$.

It then travels along a straight line from $(3, 4, 7)$ to $(6, 8, 12)$ and rotates 3 complete revolutions clockwise in the circle $x^2 + y^2 = 100$ within the plane $z = 12$ returning to the point $(6, 8, 12)$.

The path then drops straight down onto the $x - y$ plane meeting the $x - y$ plane at $(6, 8, 0)$ and finally returns along a straight line to $(3, 4, 7)$.

Sketch the path and evaluate $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

★

The first quiz will cover all of the material up to and including this lecture. We will consolidate this content in the next lecture (a problem class) and then move on to a new topic.

[13]You can now do Q 61-63

LECTURE 15
DOUBLE INTEGRALS

For a region Ω in the $x-y$ plane and a surface $z = f(x,y)$ in $\mathbb{R}^3$ the double integral

$$\iint_\Omega f(x,y)dy\,dx.$$

evaluates the volume of the solid above Ω and below $z = f(x,y)$.

To get a feeling for how double integrals operate, let's compare single integrals $\int_a^b f(x)dx$ with double integrals $\iint_\Omega f(x,y)dy\,dx.$.

$\int_a^b f(x)dx$	$\iint_\Omega f(x,y)dy\,dx.$

When evaluating double integrals keep in mind that the extreme left hand integral must always have constant limits and that we attack all multiple integrals from the middle to the edges.

Example 1 Evaluate $\int_0^1 \int_0^2 12x^2y^3 dy\,dx.$

It is quite rare to have all four limits as constants and a more typical situation has the inner limits as functions.

Example 2 Evaluate $\displaystyle\int_0^1 \int_x^{x^2} 12x^2y^3dy\,dx$.

$\bigstar \quad -\frac{12}{77} \quad \bigstar$

A crucial skill is the ability to create the limits through the consideration of the region Ω over which we are integrating. Interestingly we always have the choice as to whether we use $dA = dxdy$ or $dA = dydx$. When using $dA = dydx$ we are first slicing the cylinder over Ω parallel to the y axis and then accumulating those slices in the x direction, for $dA = dxdy$ it is similar, but the other way around. The result is the same either way, however it is often the case that one approach is significantly quicker than the other!

Example 3 Evaluate $\displaystyle\iint_\Omega 3y\ dA$ where $\Omega = \{(x, y) : 0 \leq x \leq 3 \text{ and } 0 \leq y \leq 2\}$

!!!!! YOU MUST SKETCH Ω IN THE $x - y$ PLANE!!!!!

We will use both $dA = dxdy$ and as a check $dA = dydx$.

★ 18 ★

Question: What is the geometrical interpretation of this answer?

Question: Can $\iint_\Omega f(x, y)\ dA < 0$?

In general the region Ω is much more complicated than a simple rectangle.

Example 4 Evaluate $\iint_\Omega x \, dA$

where Ω is the region in the first quadrant bounded by the parabola $y = 4 - x^2$ and the coordinate axes.

To evaluate general double integrals where the limits still need to be produced:

- Draw Ω.
- Run a tracer parallel to the inner variable to determine the inner limits.
- Consider the degree of freedom in your tracer to determine the outer limits.
- Remember the outer limits must be constants.

We will once again use both $dA = dxdy$ and $dA = dydx$. Note that in practice, only one of these (the easy one!) should be evaluated. We will see in the next lecture that one of the two may well be impossible while the other is quite simple!

★ 4 ★

Example 5 Evaluate $\displaystyle\iint_\Omega 8xy\; dA$

where Ω is the region in the $x-y$ plane bounded by the two curves $y = 3x$ and $y = x^2$.

Recall that to evaluate general double integrals where the limits still need to be produced:

- Draw Ω.
- Run a tracer parallel to the inner variable to determine the inner limits.
- Consider the degree of freedom in your tracer to determine the outer limits.
- Remember the outer limits must be constants.

We will once again use both $dA = dxdy$ and $dA = dydx$. Note that in practice, only one of these (the easy one!) should be evaluated. We will see in the next lecture that one of the two may well be impossible while the other is quite simple!

★ 243 ★

[15] You can now do Q 64 and 66

MATH2019 LECTURE 16
CHANGING THE ORDER OF INTEGRATION AND AREAS

It is important to be able to convert $\displaystyle\iint_\Omega f(x,y)\,dxdy$ into $\displaystyle\iint_\Omega f(x,y)\,dydx$ and vice versa. This should always be done via the production and consideration of the region Ω over which the integration takes place.

$$\iint_\Omega 1\,dA = \text{ area}(\Omega).$$

First a revision example from the last lecture.

Example 1: Evaluate $\displaystyle\iint_\Omega \frac{x}{y}\,dA$ where Ω is the region in the first quadrant bounded by the four lines:

$$y = x \quad y = 2x \quad x = 1 \text{ and } x = 2.$$

Observe that using $dxdy$ would be a bit of a disaster here as we would no longer have clear curves of entry and exit. We will have a go at the $dxdy$ version at the start of the next lecture.

★ $\frac{3}{2}\ln(2)$ ★

Example 2: Evaluate $\displaystyle\int_0^{\pi}\int_x^{\pi} \frac{\sin(y)}{y}\, dydx$ by first changing the order of integration.

Note firstly that the integral is impossible to evaluate directly! When changing the order of integration it is absolutely essential to sketch the region of integration first.

★ 2 ★

Example 3: Evaluate $\int_{-1}^{1} \int_{y^2}^{1} 2\sqrt{x} e^{x^2} \; dxdy$ by first changing the order of integration.

★ $2e - 2$ ★

Example 4: Evaluate $\displaystyle\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\ dxdy$ by first changing the order of integration.

$$= \int_1^2 4x - x^3 + 8 - 4x^2 + \frac{1}{2}x^4\, dx = \left[2x^2 - \frac{1}{4}x^4 + 8x - \frac{4}{3}x^3 + \frac{1}{10}x^5\right]_1^2$$

$$= (8 - 4 + 16 - \frac{32}{3} + \frac{32}{10}) - (2 - \frac{1}{4} + 8 - \frac{4}{3} + \frac{1}{10}) = \frac{188}{15} - \frac{511}{60} = \frac{241}{60}.$$

★ $\frac{241}{60}$ ★

Areas Via Double Integrals

Although designed to evaluate volumes the double integral may be tricked into the evaluation of areas by simply replacing $f(x,y)$ with 1. That is

$$\boxed{\iint_\Omega 1 \; dA = \text{ area}(\Omega).}$$

This works since $\displaystyle\iint_\Omega 1 \; dA$ is the volume above Ω below the horizontal plane $z = 1$, which is in turn equal to the (area of Ω) $\times$ 1

Example 5: Use double integration to find the area bounded by $x = y^2$ and $x = 2y - y^2$.

★ $\frac{1}{3}$ square unit ★

Observe that use of $dydx$ would be a disaster here!

Example 6: Prove that standard formula for the area of a parallelogram

$$A = bh$$

using double integrals.

Let's construct a parallelogram with vertices at $(0,0)$, $(0,b)$, (h,a) and $(h,a+b)$ where $a,b,h > 0$.

Sketch:

The two interesting lines are $y = \frac{a}{h}x$ and $y = \frac{a}{h}x + b$. Thus

Area $= A =$

---

[16] You can now do Q 65, 67, 70 and 71

LECTURE 18
DOUBLE INTEGRALS IN POLAR COORDINATES

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

$$r = \sqrt{x^2+y^2}$$

$$\tan(\theta) = \frac{y}{x}$$

$$dA = dxdy = dydx = rdrd\theta$$

First a revision example from the previous lecture.

Example 1 Evaluate $\displaystyle\int_1^2\int_x^{2x} \frac{x}{y}\,dydx$ by first changing the order of integration.

This is an example we did at the start of the previous lecture using $dydx$ and easily got an answer of $\frac{3}{2}\ln(2)$. It is much harder using $dxdy$. First a sketch:

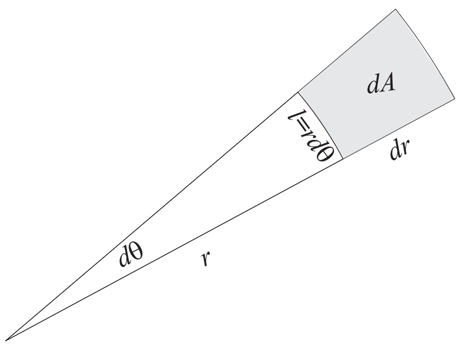
For homework evaluate the two integrals above and convince yourself that you still get an answer of $\frac{3}{2}\ln(2)$. The option $dxdy$ is poor. Always question your choice if you need to split regions.

★ $\dfrac{3}{2}\ln(2)$ ★

DOUBLE INTEGRALS IN POLAR COORDINATES

It is sometimes advantageous to abandon the rectangular coordinate system and replace it with polars. This is particularly true when dealing with circular objects or sums of squares.

Let us begin by defining polar coordinates and proving the transformation equations above



The shaded area dA is approximately a rectangle and hence its area is given by

$$dA \approx (rd\theta)dr = rdrd\theta$$

.

Example 2 Find the polar coordinates of the point P whose Cartesian coordinates are $P(1, \sqrt{3})$.

★ $(2, \frac{\pi}{3})$ ★

Example 3 Find the Cartesian coordinates of the point Q whose polar coordinates are $Q(\sqrt{2}, \frac{3\pi}{4})$.

★ $(-1, 1)$ ★

Example 4 Express the equation $r = 2\cos(\theta)$ in Cartesian form

★ $(x-1)^2 + y^2 = 1$ ★

When converting or presenting a double integral in polar form it is crucial to carefully consider a sketch of the underlying region Ω and to always remember that dA or $dydx$ or $dxdy$ are NOT replaced by $drd\theta$ bit rather by $rdrd\theta$. Unlike the Cartesian situation we almost never reverse the order and use $rd\theta dr$. It's *always* $rdrd\theta$.

Example 5 Evaluate $\displaystyle\iint_\Omega 3x\ dydx$ where Ω is the region in the first quadrant given by $\Omega = \{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$. That is Ω is the interior of a quarter circle centre the origin. First evaluate the integral in Cartesian coordinates and then in polars.

★ 1 ★

Example 6 Convert the following Cartesian integral into an equivalent integral in polar coordinates and evaluate:

$$\int_0^{\frac{1}{\sqrt{2}}} \int_y^{\sqrt{1-y^2}} x \ dxdy$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{3} \cos(\theta) d\theta = \frac{1}{3}[\sin(\theta)]_0^{\frac{\pi}{4}} = \frac{1}{3\sqrt{2}}$$

$$\bigstar \quad \frac{1}{3\sqrt{2}} \quad \bigstar$$

Example 7 Evaluate $\displaystyle\iint_\Omega 2xy \ dydx$ where Ω is the region in the first quadrant between the circles of radius 2 and radius 5 centered at the origin.

$= \frac{609}{4} \int_0^{\frac{\pi}{2}} \sin(2\theta) d\theta = \frac{609}{4}[-\frac{1}{2}\cos(2\theta)]_0^{\frac{\pi}{2}} = \frac{609}{4}[-\frac{1}{2}(-1) + \frac{1}{2}] = \frac{609}{4}$

$$\bigstar \quad \frac{609}{4} \quad \bigstar$$

Example 8 Suppose that Ω is the finite region bounded by the curve $r = 2\cos(\theta)$.

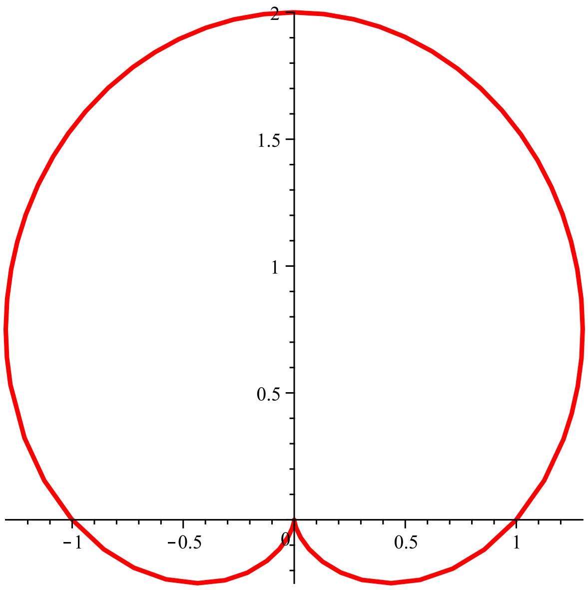
a) Sketch the region Ω in the plane.

b) Use polar coordinates to evaluate $\displaystyle\iint_\Omega \frac{3}{\sqrt{x^2+y^2}}\, dA$.

We saw in Example 3 that $r = 2\cos(\theta)$ is a circle centre at $(1,0)$ with a radius of 1.

$\bigstar$ 12 $\bigstar$

Example 9 Find the area of the region in the first two quadrants bounded by the cardioid $r = 1+\sin(\theta)$.



$= \frac{1}{2}\int_0^\pi (1+\sin(\theta))^2 d\theta = \frac{1}{2}\int_0^\pi 1+2\sin(\theta)+\sin^2(\theta) d\theta = \frac{1}{2}\int_0^\pi 1+2\sin(\theta)+\frac{1}{2}(1-\cos(2\theta))d\theta$

$= \frac{1}{2}\int_0^\pi \frac{3}{2} + 2\sin(\theta) - \frac{1}{2}\cos(2\theta)d\theta = \frac{1}{2}[\frac{3}{2}\theta - 2\cos(\theta) - \frac{1}{4}\sin(2\theta)]_0^\pi$

$= \frac{1}{2}\{(\frac{3\pi}{2} - 2(-1) - 0) - (0-2-0)\} = \frac{1}{2}(\frac{3\pi}{2} + 4)$.

$\bigstar$ $\dfrac{3\pi}{4} + 2$ $\bigstar$

Jacobian Transformation (Optional)

The equation $dA = dxdy = rdrd\theta$ is a special case of a Jacobian transformation. If $x = x(u, v)$, $y = y(u, v)$ is a change of variable in a double integral then

$$\iint_{\Omega} f(x, y) dx\, dy = \iint_{\Omega^*} f\left(x(u, v), y(u, v)\right) |J| du\, dv$$

where Ω^* is the region in the (u, v) plane corresponding to Ω in the (x, y) plane and J is the Jacobian Determinant

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

If $x = x(r, \theta) = r\cos(\theta)$ and $y = y(r, \theta) = r\sin(\theta)$ then:

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} = \cos(\theta) r\cos(\theta) + r\sin(\theta)\sin(\theta) = r(\cos^2(\theta) + \sin^2(\theta)) = r.$$

Hence

$$\iint_{\Omega} f(x, y) dx\, dy = \iint_{\Omega^*} f\left(x(r, \theta), y(r, \theta)\right) r dr\, d\theta$$

[18] You can now do Q 68

LECTURE 19
DENSITY, MASS AND CENTRE OF MASS

Consider a lamina Ω of varying composition (for example a thin sheet of metal) in the $x-y$ plane with density $\delta(x,y)$ at the point (x,y). Then

$$\text{Mass}(\Omega) = M = \iint_\Omega \delta(x,y)dA.$$

If the centre of mass of Ω is $(\bar{x},\bar{y})$ then

$$\bar{x} = \frac{\iint_\Omega x\ \delta(x,y)dA}{M} = \frac{M_y}{M}$$

and

$$\bar{y} = \frac{\iint_\Omega y\ \delta(x,y)dA}{M} = \frac{M_x}{M}$$

$M_y = \iint_\Omega x\ \delta(x,y)dA$ is called the first moment of the lamina about the y axis.

$M_x = \iint_\Omega y\ \delta(x,y)dA$ is called the first moment of the lamina about the x axis.

If the density is uniform with $\delta(x,y) = 1$ then the centre of mass is referred to as a *centroid*. Note that the mass is then

$$\text{Mass}(\Omega) = M = \iint_\Omega 1dA = \text{Area}(\Omega).$$

When Isaac Newton was considering the gravitational forces between bodies a fundamental problem was how to measure the distance between two planets. Do we use the closest distance, the furthest distance or some average? Newton eventually understood that all of the mass of the planet could be treated as if it was concentrated at the centre of mass and thus the planets could be treated as abstract mathematical points. This simple idea greatly assisted the development of his theories.

So the centre of mass of an object is the unique point where we can pretend that all of the mass lies for our calculations. In this course will will only examine the centre of mass of laminas, that is flat sheets of material with negligible thickness. Think of a thin tile or plate. The centre of mass $(\bar{x},\bar{y})$ of such an object is the unique point where you could balance the plate on the tip of your finger.

Generally density $= \dfrac{\text{mass}}{\text{volume}}$, however since the lamina is assumed to have negligible thickness we can say that density $= \dfrac{\text{mass}}{\text{area}}$. Thus $\delta(x,y) = \dfrac{dm}{dA}$ implying that

$dm = \delta(x,y)dA$. Integrating to obtain the total mass we have

$\text{Mass}(\Omega) = M = \displaystyle\iint_\Omega \delta(x,y)dA$. That is, **mass is the integral of density**.

When calculating the x coordinate of the centre of mass via $\bar{x} = \dfrac{\iint_\Omega x\ \delta(x,y)dA}{M}$ we are simply taking a weighted average of all the $x's$ in Ω with the values of x of higher density making the greater contribution. Similarly for $\bar{y}$.

Please note that I will set up all the relevant double integrals in this lecture but will leave the actual evaluation to you. Answers are at the bottom of each example.

Note also that you should invoke symmetry wherever possible.

Other formulae of interest are:

Moments of Inertia

The moments of inertia of the above lamina about the x and y axes are

$$\begin{aligned} I_x &= \iint_\Omega y^2 \delta(x,y)dA \\ I_y &= \iint_\Omega x^2 \delta(x,y)dA. \end{aligned}$$

The polar moment of inertia about the origin is defined by

$$I_0 = I_x + I_y = \iint_\Omega (x^2+y^2)\delta(x,y)dA.$$

Moments of inertia measure an objects resistance to spinning about a particular axis. A spinning ice skater with her arms out has a larger moment of inertia about the vertical axis than one with her arms tucked into her body.

Example 1 A thin plate Ω is the region in the first quadrant bounded by the coordinate axes and $x + y = 1$. Find the mass M and centre of mass $(\bar{x}, \bar{y})$ of the plate assuming

a) uniform density $\delta(x, y) = 1$.

b) non uniform density given by $\delta(x, y) = xy$.

---

a) (For uniform density the centre of mass is called the centroid)

$$\bigstar \quad a) \quad M = \tfrac{1}{2},\ M_x = \tfrac{1}{6},\ M_y = \tfrac{1}{6},\ (\bar{x}, \bar{y}) = (\frac{M_y}{M}, \frac{M_x}{M}) = (\frac{1}{3}, \frac{1}{3}) \quad \bigstar$$

$$\bigstar \quad b) \quad M = \tfrac{1}{24},\ M_x = \tfrac{1}{60},\ M_y = \tfrac{1}{60},\ (\bar{x}, \bar{y}) = (\frac{M_y}{M}, \frac{M_x}{M}) = (\frac{2}{5}, \frac{2}{5}) \quad \bigstar$$

Example 2 Find the mass and the centre of mass of a thin plate bounded by the curves $y = x^2$ and $y = 6x - x^2$ with variable density given by $\delta(x, y) = 10x + 10$.
Also find the moment of inertia I_y about the $y-$ axis. Recall that $I_y = \iint_\Omega x^2\delta(x, y)dA$.

★ $M = 225,\ M_x = 1134,\ M_y = 378,\ (\bar{x}, \bar{y}) = (\frac{M_y}{M}, \frac{M_x}{M}) = (\frac{42}{25}, \frac{126}{25}),\ I_y = 729$ ★

Example 3 Find the mass and the centre of mass of a lamina bounded by $y = \sqrt{1 - x^2}$ and the x axis assuming:

a) uniform density $\delta(x, y) = 1$.

b) variable density given by $\delta(x, y) = y$.

c) variable density given by $\delta(x, y) = r$.

$$\bigstar \quad a) \quad M = \tfrac{\pi}{2},\ M_x = \tfrac{2}{3},\ M_y = 0,\ (\bar{x}, \bar{y}) = (\frac{M_y}{M}, \frac{M_x}{M}) = (0, \frac{4}{3\pi}) \quad \bigstar$$

$$\bigstar \quad b) \quad M = \tfrac{2}{3},\ M_x = \tfrac{\pi}{8},\ M_y = 0,\ (\bar{x}, \bar{y}) = (\frac{M_y}{M}, \frac{M_x}{M}) = (0, \frac{3\pi}{16}) \quad \bigstar$$

$$\bigstar \quad c) \quad M = \tfrac{\pi}{3},\ M_x = \tfrac{1}{2},\ M_y = 0,\ (\bar{x}, \bar{y}) = (\frac{M_y}{M}, \frac{M_x}{M}) = (0, \frac{3}{2\pi}) \quad \bigstar$$

[19] You can now do Q 72 to 76

LECTURE 20
FURTHER VOLUMES

Recall that for a region Ω in the $x-y$ plane and a surface $z = f(x, y)$ in $\mathbb{R}^3$ the double integral

$$\iint_\Omega f(x,y)dy\,dx.$$

evaluates the volume of the solid above Ω and below $z = f(x, y)$.

If the solid is pinned between two surfaces $z = f(x, y)$ and $z = g(x, y)$ with $f \geq g$ then its volume will be

$$\iint_\Omega [f(x,y) - g(x,y)]\; dy\,dx = \iint_\Omega [\text{top surface} - \text{bottom surface}]\; dydx.$$

where Ω is the projection of the solid back onto the $x-y$ plane.

Revision of the meaning of the double integral:

Example 1 Find the volume of the tetrahedron bounded by the plane $3x+2y+z=6$ and the coordinate planes.

$$= \int_0^2 [6y-3xy-y^2]_0^{3-\frac{3}{2}x}dx = \int_0^2 6(3-\frac{3}{2}x)-3x(3-\frac{3}{2}x)-(3-\frac{3}{2}x)^2dx$$

$$= \int_0^2 18-9x-9x+\frac{9}{2}x^2-(3-\frac{3}{2}x)^2dx = \int_0^2 18-18x+\frac{9}{2}x^2-(9-9x+\frac{9}{4}x^2)dx$$

$$= \int_0^2 9-9x+\frac{9}{4}x^2dx = \left[9x-\frac{9}{2}x^2+\frac{9}{12}x^3\right]_0^2 = (18-18+6)-(0)=6.$$

★ 6 ★

Example 2 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

★ $\frac{\pi}{2}$ ★

Example 3 Find the volume of the solid containing the origin, bounded by the 5 planes $x = 0,\ y = 0,\ z = 0,\ 3x + 2y + z = 6$ and $x + y = 1$.

$$\int_0^1 6(1-x) - 3x(1-x) - (1-x)^2\,dx = \int_0^1 6 - 6x - 3x + 3x^2 - (1 - 2x + x^2)\,dx$$

$$= \int_0^1 5 - 7x + 2x^2\,dx = \left[5x - \frac{7}{2}x^2 + \frac{2}{3}x^3\right]_0^1 = 5 - \frac{7}{2} + \frac{2}{3} = \frac{13}{6}.$$

★ $\dfrac{13}{6}$ ★

Example 4 Find the volume of the solid lying between the paraboloids $z = x^2 + y^2$ and $3z = 4 - x^2 - y^2$.

★ $\frac{2\pi}{3}$ ★

Example 5 Find the volume of the solid bounded above by the paraboloid $z = x^2 + y^2$, below by the $x - y$ plane and lying inside the cylinder $x^2 + y^2 = 2x$.

$$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(\theta) d\theta.$$

Noting that cos is an even function and that $\cos^2(x) = \dfrac{1}{2}(1 + \cos(2x))$ we have:

$$= 8\int_0^{\frac{\pi}{2}} \cos^4(\theta) d\theta = 8\int_0^{\frac{\pi}{2}} \cos^2(\theta)\cos^2(\theta) d\theta = 8\int_0^{\frac{\pi}{2}} \{\frac{1}{2}(1 + \cos(2\theta))\}^2 d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (1 + \cos(2\theta))^2 d\theta = 2\int_0^{\frac{\pi}{2}} 1 + 2\cos(2\theta) + \cos^2(2\theta) d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 1 + 2\cos(2\theta) + \frac{1}{2}(1 + \cos(4\theta)) d\theta = 2\int_0^{\frac{\pi}{2}} \frac{3}{2} + 2\cos(2\theta) + \frac{1}{2}\cos(4\theta) d\theta$$

$$= 2\left[\tfrac{3}{2}\theta + \sin(2\theta) + \tfrac{1}{8}\sin(4\theta)\right]_0^{\frac{\pi}{2}} = 2\left[\frac{3}{2}\frac{\pi}{2}\right] = \frac{3\pi}{2}.$$

★ $\frac{3\pi}{2}$ ★

HOMEWORK: Attempt the polar integrals in this lecture using Cartesian coordinates instead. Some may be almost impossible using $dydx$.

IMPORTANT NOTE: In 5 lectures time, we will start on the theory of eigenvalues and eigenvectors. This will need a firm understanding of the theory of determinants and also Gaussian elimination of systems of linear equations. This material is covered in Math1131 linear algebra here at UNSW. If you are a rusty UNSW student or an overseas student who has not had seen this content before please read the revision first year algebra notes (available on Moodle) before Lecture 28.

[20] You can now do Q 69,77,78,79

LECTURE 22
FIRST ORDER DIFFERENTIAL EQUATIONS

SEPARABLE

These differential equations are of the form

$$\frac{dy}{dx} = f(x)g(y).$$

and are solved by separating and integrating both sides of the equation.

LINEAR

The general first-order linear o.d.e. is

$$\frac{dy}{dx} + P(x)y = Q(x).$$

If we evaluate the integrating factor

$$R(x) = e^{\int P(x)dx}$$

then the solution is given by

$$y = \frac{1}{R(x)} \int R(x)Q(x)\, dx$$

A differential equation (often abbreviated D.E.) is an equation relating a function to its derivatives. For example

$$\frac{dy}{dx} = x^2y^3$$

is a differential equation. It is important to note that a solution to a differential equation is not a number ...... it's a function. What this equation is asking you is 'can you find a function with the property that differentiating the function is the same as cubing the function and then multiplying by x^2?'. We will shortly examine exactly how we would go about finding such a function (and whether or not such a function is unique!). But first some classifying definitions.

The **order** of a differential equation is the biggest derivative that appears in the equation.

A differential equation is said to be **linear** if all terms involving y and its derivatives appear linearly........the x's can do as they please.

Example 1 Find the order of each of the following differential equations and state whether each equation is linear or not.

1. $\dfrac{dy}{dx} = x^2y^3$
2. $\dfrac{dy}{dx} + y = x^3$
3. $y'' + 6y' - 5y = \dfrac{1}{x^2+5}$

★ 2 and 3 are linear and the orders are 1,1 and 2 resp ★

How are differential equations solved? It depends completely on the type of equation (very much like the process of integration). First order equations are attacked in a manner totally different from second order equations and unfortunately many differential equations can't be solved at all. There are however some global rules.

• The solution to any D.E. will always involve arbitrary constants. It is crucial to respect the constants and take good care of them.

• The number of constants in solution is always equal to order of the D.E.

• If a D.E. has initial conditions (i.c.'s) attached to it, the i.c.'s are used to knock off the constants.

In this lecture we will look at the two main types of first order problems....separable and linear. In the next lecture we will discuss substitution and applications of first order D.E.'s. Note that much of the material in these two lectures is revision from first year and you are encouraged to check over your first year notes on D.E.'s to refresh your memory.

SEPARABLE

As the name suggests separable first order D.E.'s are solved by separating the variables out and then running two integrals. Not all first order problems are separable!

Example 2 Solve

$$\frac{dy}{dx} = y(2x+3)$$

.

★ $y = Ae^{x^2+3x}$ ★

Example 3 Solve

$$\frac{dy}{dx} = \frac{\cos(x)}{y} \qquad \text{where} \qquad y(\frac{\pi}{2}) = 5.$$

.

$\bigstar \quad y^2 = 2\sin(x) + 23 \quad \bigstar$

LINEAR

The general first-order linear o.d.e. is

$$\frac{dy}{dx} + P(x)y = Q(x).$$

If we evaluate the integrating factor

$$R(x) = e^{\int P(x)dx}$$

then the solution is given by

$$y = \frac{1}{R(x)} \int R(x)Q(x)\,dx$$

- The integrating factor should always be expressed in its simplest possible form.

- Note that the integrating factor $R(x)$ does **NOT** need a $+C$ but the final solution most certainly does.

Example 4 Solve the first order linear D.E.

$$\frac{dy}{dx} + \frac{2}{x}y = 4x$$

Observe that this D.E. is NOT separable and hence cannot be solved using the above techniques

$$\bigstar \quad y = x^2 + \frac{C}{x^2} \quad \bigstar$$

Proof of Linear Formulae

$\bigstar$

Example 5 Solve the initial value problem

$$x\frac{dy}{dx} - y = x^3\cos(x) \quad ; \quad y(\pi) = 0.$$

Check that your solution satisfies both the D.E. and the initial condition.

We MUST rewrite the equation:

$\bigstar \quad y = x^2\sin(x) + x\cos(x) + x \quad \bigstar$

Example 6 (HOMEWORK) Find the general solution of the differential equation

$$\frac{dy}{dx} + xy = x\,;$$

a) By treating the D.E. as linear.

b) By treating the D.E. as separable.

c) Check that the two solutions are equivalent.

$$\bigstar \quad y = 1 + De^{-\frac{x^2}{2}} \quad \bigstar$$

It is quite rare for a differential equation to be of two completely different types!

[22] You can now do Q 80 and 83

LECTURE 23
APPLICATIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS

When making substitutions into a D.E. remember to take care of $\dfrac{dy}{dx}$ as well as y

SEPARABLE

These are of the form

$$\frac{dy}{dx} = f(x)g(y).$$

and are solved by separating and integrating both sides of the equation.

LINEAR

The general first-order linear o.d.e. is

$$\frac{dy}{dx} + P(x)y = Q(x).$$

If we evaluate the integrating factor

$$R(x) = e^{\int P(x)dx}$$

then the solution is given by

$$y = \frac{1}{R(x)} \int R(x)Q(x)\, dx$$

We begin with a revision example on first order linear.

Example 1 Solve

$$\frac{dy}{dx} = \frac{xy+2}{1-x^2} \qquad \text{with} \qquad y(0) = 1.$$

.

$$\bigstar \quad y = \frac{1 + 2\sin^{-1}(x)}{\sqrt{1 - x^2}} \quad \bigstar$$

As with integration it is possible to clarify a D.E. by implementing a simple substitution.

Example 2 Solve

$$x^2 \frac{dy}{dx} = x^2 + y^2 + xy$$

by making the substitution $v = \dfrac{y}{x}$.

!!Observe that this D.E. is neither separable nor linear!!

$$\bigstar \quad y = x\tan(\ln(x) + C) \quad \bigstar$$

Example 3 Solve

$$(x+y)\frac{dy}{dx} = e^{3x} - x - y \qquad y(0) = 2,$$

by making the substitution $v = y + x$.

$\bigstar \quad 3(y+x)^2 = 2e^{3x} + 10 \quad \bigstar$

Sometimes you are not actually given a D.E. but rather are told various details about a situation and it is left up to you to construct the D.E. yourself. Keep in mind that D.E.'s are all about rates of change and thus your first task is to establish equations which govern the rates of change. The rate of change of any quantity is always equal the rate at which it increases minus the rate at which it decreases. Thus for example the rate of change of a population will be the rate of increase (births, immigration etc) minus the rate at which the population decreases (deaths etc). Also there is usually some information given as to initial conditions. Once the D.E. is formed there is still of course the problem of solving it to obtain formulae for the quantities of interest.

Example 4 The air in a 50 cubic metre room is initially clean. Chris lights up a cigarette introducing smoke into the room's air at a rate of 2 mg/minute. An air conditioning system exchanges the mixture of air and smoke with clean air at a rate of 5 cubic metres per minute. Assume that the smoke is mixed uniformly throughout the room and that Chris's cigarette lasts 4 minutes. Let S be the amount of smoke (in mg) present in the room at time t (in minutes).

a) Show that $\dfrac{dS}{dt} = 2 - \dfrac{S}{10}$
b) Hence show that $S(t) = 20 - 20e^{(-\frac{t}{10})}$
c) What is the level of smoke after 4 minutes?
d) What is the level of smoke after 14 minutes?

★ c) 6.6mg d) 2.43mg ★

[23] You can now do Q 81 and 82

LECTURE 24
HOMOGENEOUS SECOND ORDER DIFFERENTIAL EQUATIONS

To solve the homogeneous second order constant coefficient differential equation

$$ay'' + by' + cy = 0$$

first form the auxiliary (also called characteristic) equation

$$a\lambda^2 + b\lambda + c = 0.$$

The auxiliary equation is not a D.E., it is just a quadratic with two roots λ_1 and λ_2. Remarkably the solution to the D.E. is completely determined by the nature of λ_1 and λ_2!

- $\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 \neq \lambda_2 \to y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
- $\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 = \lambda_2 \to y = (Ax + B)e^{\lambda_1 x}$
- $\lambda_1, \lambda_2 \in \mathbb{C},\ \lambda_1, \lambda_2 = r \pm is \to y = e^{rx}(A\cos(sx) + B\sin(sx))$

We turn now to the theory of second order differential equations. These are D.E.'s where the second derivative also makes an appearance. These are in general more difficult to solve than their first order comrades. The theory is so tangled that we restrict our attention to only the very special case of linear D.E.s with constant coefficients. These D.E.s take the form $ay'' + by' + cy = RHS$ where $a, b, c \in \mathbb{R}$. In this lecture we will deal with the homogeneous case (RHS=0). In the next lecture we will broaden our scope a little and allow some simple functions to appear on the RHS.

Keep in mind that all second order D.E.s must have two independent arbitrary constants in solution and hence require a pair of i.c.'s to deal with the constants. The method of solution for second order problems is *totally* different from first order techniques. As a relief you will find that in most cases no integration is involved.

Proof of first bullet point above:

Example 1 Solve each of the following second order D.E.s:

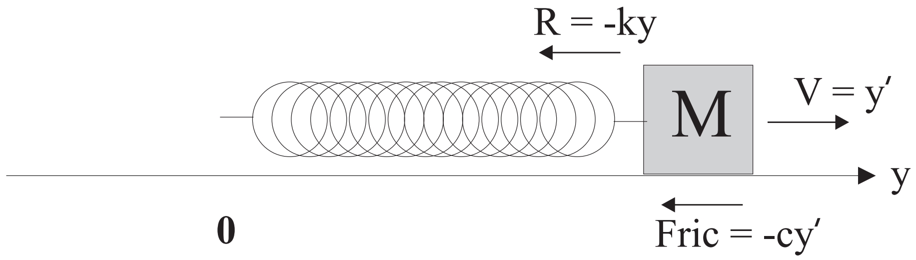
a) $y'' - 3y' - 10y = 0$

b) $y'' - 8y' + 16y = 0$

c) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 58y = 0$

★ *a)* $y = Ae^{-2x} + Be^{5x}$ *b)* $y = (Ax + B)e^{4x}$ *c)* $y = e^{3x}(A\cos(7x) + B\sin(7x))$ ★

Note that is some texts, $y = e^{3x}(A\cos(7x)+B\sin(7x))$ will be written in the equivalent form $Re^{3x}\cos(7x - \delta)$. We will not do so here.



FREE OSCILLATIONS

Consider an object M of mass m attached to a spring oscillating up and down the y axis. At time t its velocity is $v = y' = \dfrac{dy}{dt}$ and its acceleration $a = y'' = \dfrac{d^2y}{dt^2}$. The total force $F = ma = my''$ acting upon the mass is the sum of two forces; $R = -ky$ the resistive force due to the spring and Fric$= -cy'$ the frictional force. Note that Fric is proportional to v and always points in a direction opposite to the velocity and R is proportional to y and always points opposite to the position.
Thus we have $F = R+$Fric implying that $my'' = -cy' - ky$. This leads to the D.E.

$$my'' + cy' + ky = 0$$

where m, c and k are all non negative. Such a system is **unforced** since the term on the right is 0.

Seeking solutions $y = Ae^{\lambda t}$ gives the characteristic equation

$$m\lambda^2 + c\lambda + k = 0,$$

which has the solutions

$$\begin{aligned} \lambda_1 &= \frac{1}{2m}\left[-c + \sqrt{c^2 - 4mk}\right] \\ \lambda_2 &= \frac{1}{2m}\left[-c - \sqrt{c^2 - 4mk}\right]. \end{aligned}$$

Observe that $\sqrt{c^2 - 4mk} \leq \sqrt{c^2} = c$ implying that when λ_1 and λ_2 are real they will always be less than or equal to zero. **Thus all associated exponential function will decay!** The situation is governed by the relative magnitude of the frictional coefficient c. Three cases arise:

1. $c^2 > 4mk$. This is called "overdamping" since the damping or frictional coefficient c is large compared with $2\sqrt{mk}$. In this case λ_1 and λ_2 are both real and negative. The solution is $y = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ which decays to zero as $t \longrightarrow \infty$.

2. $c^2 = 4mk$. This is "critical damping" and $\lambda_1 = \lambda_2$ so the solution is

$$y = (A + Bt)e^{-ct/2m}.$$

The solution also decays to 0 as $t \longrightarrow \infty$. For critical and overdamping the friction is so large that the object does not get the opportunity to oscillate more than once. If however the frictional coefficient c is small enough we have underdamping and the system becomes unstable:

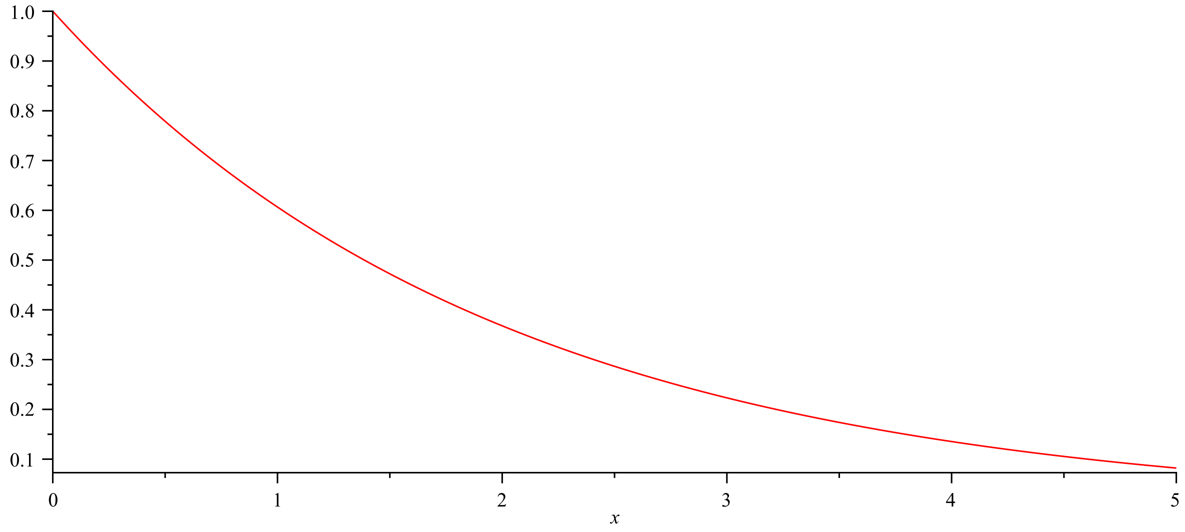


Figure 1: Overdamping (slower return to equilibrium, max of one oscillation)

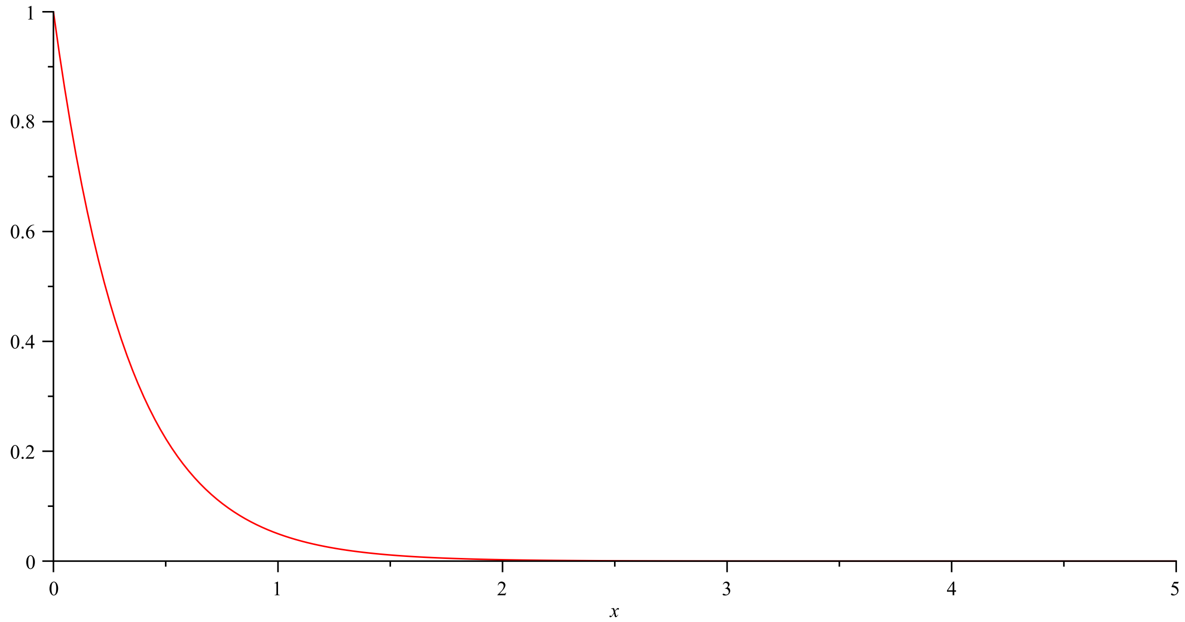


Figure 2: Critical damping (quick return to equilibrium, max of one oscillation. The system is approaching instability)

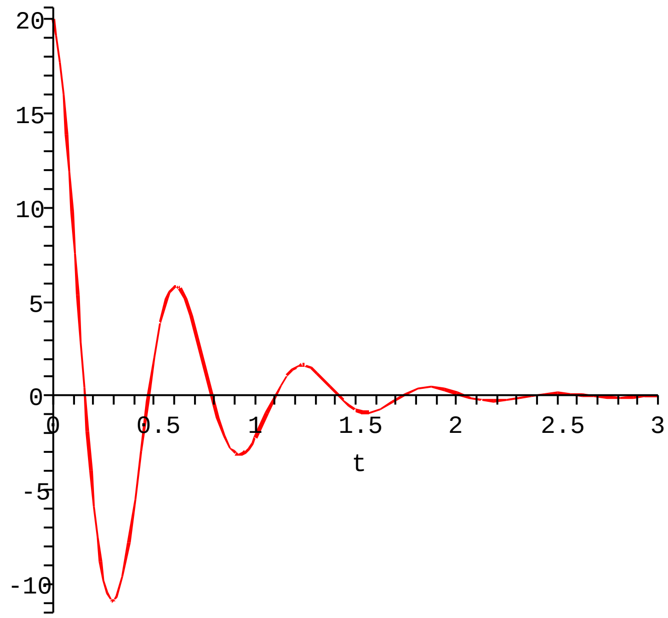


Figure 3: Underdamping. Many oscillations on return to equilibrium

3. $c^2 < 4mk$. This is called "underdamping" as c is smaller than $2\sqrt{mk}$. Then we have complex roots

$$\lambda_1 = r + is, \quad \lambda_2 = r - is$$

Thus

$$y = e^{rt}(A\cos st + B\sin st)$$

where $\quad r = -\frac{c}{2m}$ and $\quad s = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$

This is the first time trigonometric functions appear, and we now have a sequence of decaying oscillations. In the idealised case $c = 0$ (no friction), the exponential term is lost completely, and the motion becomes simple harmonic, oscillating forever. In reality, $c > 0$ and these oscillations are killed off by friction. Note that $c < 0$, that is negative friction, doesn't really make sense.

Keep in mind that the value of c which generates critical damping will always stem from repeated roots in the auxiliary equation, and hence can be found by forcing a zero discriminant. Overdamping then requires more friction, and underdamping less friction.

Example 2 Consider the differential equation

$$y'' + cy' + y = 0$$

(I) What value of the damping constant c produces:

a) Critical damping   b) OverDamping   c) Underdamping.

(II) Find and identify the solutions for $c = 1$ and $c = 3$. Compare these solutions with that obtained for no damping, $c = 0$, sketching possible graphs of all three cases.

$$\bigstar \quad Critical damping\ c=2,\ OverDamping\ c>2,\ Underdamping\ 0<c<2 \quad \bigstar$$

$$\bigstar \quad c=3 \rightarrow Overdamping\ and\ y \approx Ae^{-0.382t}+Be^{-2.618t} \quad \bigstar$$

$$\bigstar \quad c=1 \rightarrow Underdamping\ and\ y=e^{-\frac{1}{2}t}(A\cos(\frac{\sqrt{3}}{2}t)+B\sin(\frac{\sqrt{3}}{2}t)) \quad \bigstar$$

$$\bigstar \quad c=0 \rightarrow No\ damping\ and\ y=A\cos(t)+B\sin(t) \quad \bigstar$$

[24] You can now do Q 84

LECTURE 25
NON-HOMOGENEOUS SECOND ORDER DIFFERENTIAL EQUATIONS

To solve the non-homogeneous second order constant coefficient differential equation

$$ay'' + by' + cy = r(x)$$

we first solve the homogeneous problem $ay'' + by' + cy = 0$ to obtain a homogenous solution y_h. We then find a particular solution y_p by using the method of undetermined coefficients. The final solution is then $y = y_h + y_p$.

To find the long term steady state solution, consider the bahaviour of the solution as $t \to \infty$

We now deal with the constant coefficient second order D.E.'s of the previous lecture except that we will allow "nice" functions to appear on the RHS. This makes the D.E. non-homogeneous. We begin by finding the homogeneous solution y_h by using the methods of the previous lecture. We then find a particular solution y_p by essentially guessing an answer. The final solution is then $y = y_h + y_p$. It should be noted that this technique is prone to fail and will only work when the RHS is uncomplicated.

Example 1 Solve $y'' + 5y' + 6y = -6t + 25$. Describe the long term steady state solution to the differential equation.

★ $y = Ae^{-2t} + Be^{-3t} - t + 5$. Steady State Solution is $y = -t + 5$ ★

Why does this work?

The crucial question is of course how do we know what to guess for y_p? The general rule is that you guess an arbitrary representation of the RHS. The following table gives some RHS's and their associated guesses for y_p .

RHS	Choice of y_P
$3e^{4x}$	Ce^{4x}
$x^3 - 7$	$\alpha x^3 + \beta x^2 + \gamma x + \delta$
$3\sin(4x)$	$\alpha\cos(4x) + \beta\sin(4x)$
$5e^{7x}\cos(2x)$	$e^{7x}(\alpha\cos(2x) + \beta\sin(2x))$
$9x^2e^{3x}$	$e^{3x}(\alpha x^2 + \beta x + \gamma)$

Note that if the RHS is too weird then no amount of guessing will save you.

Example 2 Solve the initial value problem

$$y'' + y = 55e^{2x} + 3x^2 + 14$$

where $y(0) = 20$ and $y'(0) = 28$.

★ $y = \cos(x) + 6\sin(x) + 11e^{2x} + 3x^2 + 8$ ★

VARIATION OF PARAMETERS

If the method of undetermined coefficients fails to produce a particular solution y_p, it is possible that technique of variation of parameters will do the job. This is a highly specialised process:

Variation of Parameters

Suppose that the second order differential equation

$$y'' + p(x)y' + q(x)y = f(x)$$

has homogeneous solution $y_h = Ay_1(x) + By_2(x)$. Then a particular solution is given by

$$y_P(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx$$

where $W(x) = \det\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}$.

A full proof of this formula is found in your printed notes.

Example 3 Use Variation of Parameters to solve $y'' + y = \sec(x)$.

$$\bigstar \quad y = A\cos(x) + B\sin(x) + \cos(x)\ln(|\cos(x)|) + x\sin(x) \quad \bigstar$$

[25] You can now do Q85 a and b, Q88

LECTURE 26
FORCED OSCILLATIONS AND RESONANCE

When simple periodic forcing is added to the mechanical or electrical system studied earlier, we have to solve an equation like

$$my'' + cy' + ky = F_0 \sin wt \tag{1}$$

where, as before, $m > 0, c > 0, k > 0$.

This models a mechanical system driven by periodic forces or an electrical system forced by a periodic voltage. Solutions to such a system may become unstable and start to resonate, a critical issue when dealing with physical systems.

We saw in the previous lecture that the method of undetermined coefficients only works now and then. Sometimes it fails in a very special way and needs to be patched up.

Example 1 Solve the second order differential equation

$$y'' + 4y' = 12e^{-4t}$$

and write down the long term steady state solution.

First y_h:

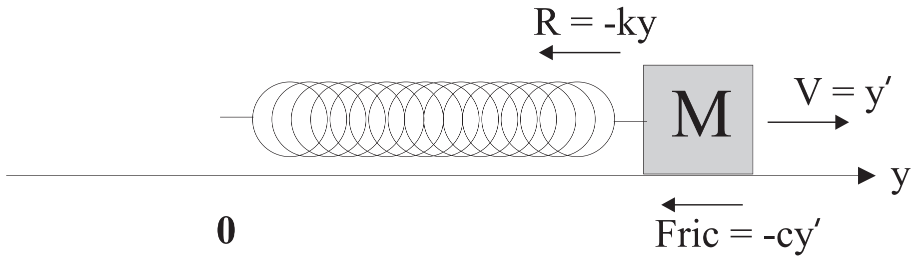
And now the guess for the particular solution:

Guess $y = \alpha e^{-4t}$

Guess $y = (\alpha t)e^{-4t}$

★ $y = A + Be^{-4t} - 3te^{-4t}$, Steady State Solution : $y = A$ ★

As a general rule, whenever your natural guess for y_p is already lurking inside y_h you must modify that guess by multiplying by the independent variable x. If this is still part of y_h then you must multiply by x again and keep on doing it till you get something new!



Forced Oscillations

We return now to the physical system described earlier. Recall that this system was governed by the D.E.

$$my'' + cy' + ky = 0$$

where m (mass),c (damping frictional coefficient) and k (spring constant) are all non negative. We now add what is called a periodic forcing function to the RHS which effectively drives the entire system with an additional periodic force. The D.E. is then

$$my'' + cy' + ky = F_0 \sin wt$$

Example 2 A forced vibrating system is represented by

$$y'' + 9y = 60 \sin(2t)$$

where $r(t) = \sin(2t)$ is the driving force and y is the displacement from the equilibrium position. Note the absence of any damping friction! Find a formula for the motion of the system by solving the differential equation.

★ $y = A\cos(3t) + B\sin(3t) + 12\sin(2t)$ ★

We now make a tiny modification to the forcing function which will have dramatic consequences.

Example 3 Solve the differential equation

$$y'' + 9y = 60\sin(3t)$$

All we have done is change the 2 to a 3! Observe that the damping frictional force is still zero thus the non-forced system will oscillate to infinity. The forcing function adds a layer of periodic behaviour which is unfortunately perfectly tuned to natural motion of the system. This leads to unstable resonance.

$$\bigstar \quad y = A\cos(3t) + B\sin(3t) - 10t\cos(3t) \quad \bigstar$$

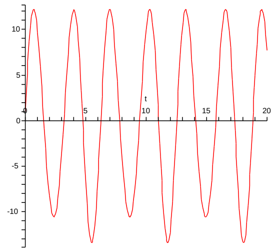


Figure 1: Example 2 No resonance

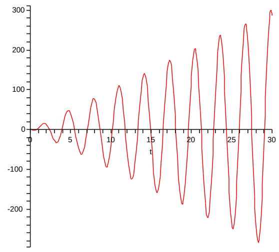


Figure 2: Example 3 With resonance

The solutions for the two previous examples are graphed above. Note the scale on the vertical axis!! What is happening here is that the system when not forced tends to naturally oscillate. The forcing function in Example 2 adds a periodic note of oscillation to the system which is different from the natural frequency and hence just produces a minor perturbation. But in Example 3 something critical happens! The forcing function is in harmony with the system and makes it resonate (it hums!!) The entire system becomes unstable and the amplitude goes through the roof. This is sometimes a good thing since you may want to amplify a signal...for example the amplification of a distant NASA signal. But in physical structures it is generally bad news and engineers have had to learn the hard way how disastrous simple resonance can be. Let's have a look at the Tacoma bridge disaster! This is more an example of aeroelastic flutter (like a flag in the wind) than resonance but it does display the immense importance of the use of appropriate damping in large structures.

$$http://www.youtube.com/watch?v=KVc7oBKzq9U$$

Resonance and flutter can be controlled with hydraulic dampers (bringing c into play) and fairing. The general equations governing resonance for forced oscillations are in your printed lecture notes.

IMPORTANT NOTE: Next lecture we will start on the theory of matrices and eigenvalues/eigenvectors. This will need a firm understanding of the theory of determinants and also Gaussian elimination of systems of linear equations. This material is covered in Math1131 linear algebra here at UNSW. If you are a rusty UNSW student or an overseas student who has not had seen this content before, please prepare by reading the revision first year algebra notes (available on Moodle).

[26] You can now do Q 85c,86,87

LECTURE 28
REVISION OF MATRIX THEORY

• If A is an m $\times$ n matrix and B is a p $\times$ q matrix then AB exists iff n=p and the product is m $\times$ q.

• The identity matrix I serves as the "1" of matrix theory.

• The transpose of A (denoted by A^T) has the columns of A as its rows. If A is $m \times n$ then A^T is $n \times m$.

• A matrix is said to be symmetric if $A = A^T$.

• Given a square matrix A the inverse of A (denoted by A^{-1}) is another matrix with the property that $AA^{-1} = I$.

• A^{-1} exists iff $\det(A) \neq 0$.

• If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

• For larger square matrices the inverse is calculated via row reduction using the row operations $R_i = R_i \pm \alpha R_j$ and $R_i \leftrightarrow R_j$.

This lecture will be a revision of the first year theory of matrices, inverses, determinants, systems of linear equations, Gaussian Elimination and row reduction.

If you are rusty on this material your are advised to have a look over your first year linear algebra notes on the above topics before moving on to the following lectures.

I have also posted a completed copy of my Math1131 Algebra lecture notes on Moodle if you need to revise the material in more detail.

Matrices are simply rectangular arrays of numbers which may be manipulated as mathematical objects.

Example 1 Let $A = \begin{pmatrix} 2 & 3 \\ 1 & -2 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 3 \\ 3 & 0 \\ 2 & 5 \end{pmatrix}$.

Find (if possible) $A+B$, $2A-C$, AB, BA and AC.

★ $2A - C = \begin{pmatrix} 3 & 3 \\ -1 & -4 \\ -2 & 3 \end{pmatrix}$, $AB = \begin{pmatrix} -1 & 19 & 6 \\ 3 & -8 & 3 \\ -4 & 20 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 4 & 11 \\ 3 & -13 \end{pmatrix}$

$A+B$ and AC are undefined ★

Some special objects are

row vector $\begin{pmatrix} 1 & 4 & -2 \end{pmatrix}$

column vector $\begin{pmatrix} 3 \\ 6 \\ -8 \\ 1 \end{pmatrix}$

zero matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (3×3 zero matrix)

upper triangular matrix $\begin{pmatrix} 4 & 7 & -2 \\ 0 & 5 & 8 \\ 0 & 0 & 2 \end{pmatrix}$

diagonal matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (2×2) identity and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (3×3) identity matrix.

The identity matrix serves as a "one" for matrix theory.

Example 2 Let $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 4 & 6 & 2 \end{pmatrix}$.

Find AI, IA and λI where $\lambda \in \mathbb{R}$.

★ $AI = IA = A$ and $\lambda I = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ ★

Matrix division is not quite properly defined, the closest we can manage is to use the concept of an inverse.

Given a square matrix A the inverse of A (denoted by A^{-1}) is another matrix with the property that $AA^{-1} = I$.

2×2 inverses are easy to find via the formula $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

But larger matrices are harder to invert. We usually use row reduction though other techniques are possible.

Example 3 Prove that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and find $\begin{pmatrix} 3 & 8 \\ 1 & 2 \end{pmatrix}^{-1}$.

★ $-\dfrac{1}{2} \begin{pmatrix} 2 & -8 \\ -1 & 3 \end{pmatrix}$ ★

Example 4 Let $A = \begin{pmatrix} 4 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Find $\det(A)$ and hence explain why A is invertible.

Find A^{-1} and hence solve the system of linear equations

$$\begin{aligned} 4x + 2y + z &= 21 \\ 3x + 2y + z &= 19 \\ x + y + z &= 11 \end{aligned}$$

$$\bigstar \quad A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} \quad \bigstar$$

Note that the above method for solving linear equations is NOT generally used. The calculation of the inverse is numerically inefficient and the method fails completely when there are infinite solutions. When finding eigenvalues and eigenvectors in the following chapters we will always use Guassian Elimination to Echelon Form.

We close the lecture with the definitions of transpose and symmetry for matrices.

Definition: The transpose A^T of a matrix A is defined as $[A^T]_{ij} = [A]_{ji}$.

A^T is quite simply the matrix whose rows are the columns of A (and vice versa).

Example 5 Find A^T if:

a) $A = \begin{pmatrix} 3 & 1 & -2 \\ 4 & 7 & 8 \end{pmatrix}$

b) $A = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$

★

Fact: If A is $m \times n$ then A^T is $n \times m$.

Transposes and inverses also behave in a similar fashion across matrix products:

Fact: $(AB)^T = B^T A^T$ and $(AB)^{-1} = B^{-1}A^{-1}$.

Definition: A matrix A is said to be **symmetric** if $A = A^T$.
The following are examples of symmetric matrices:

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 7 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 6 & 8 \\ 6 & -5 & -7 \\ 8 & -7 & 2 \end{pmatrix}$$

For a symmetric matrix the first row is the same as the first column, the second row is the same as the second column... etc.

Example 6 Show that the matrix C given by $C = A^T A$ is always symmetric regardless of the nature of A.

★

[28] You can now do Q 89

LECTURE 29
EIGENVALUES AND EIGENVECTORS

Given a square matrix A, a non-zero vector $\mathbf{v}$ is said to be an eigenvector of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$. The number λ is referred to as the associated eigenvalue of A.

We first find eigenvalues through the characteristic equation $\det(A-\lambda I) = 0$. The eigenvectors are then found via row reduction and back substitution.

The zero vector is **never** an eigenvector but it is OK to have a zero eigenvalue.

If an $n\times n$ matrix A has n linearly independent eigenvectors and P is the matrix of eigenvectors aligned vertically then $P^{-1}AP = D$ where D is the diagonal matrix of eigenvalues. The order of the eigenvalues in D must match the order of the eigenvectors in P. This is referred to as the diagonalization of A.

A matrix can be non-diagonalisable by coming up short on eigenvectors. The only general way to find out if a matrix has a full set of eigenvectors is to find them all.

A useful check is the fact that $\Sigma(\text{eigenvalues}) = \text{Trace}(A)$.

Eigenvectors from different eigenvalues are linearly independent.

Eigenvectors from different eigenvalues for symmetric matrices are perpendicular.

Establishing the eigenanalysis of a particular matrix gives you a clear vision of the internal workings of that matrix, and through diagonalisation the matrix may be transformed into a more workable diagonal structure.

Consider the matrix $A = \begin{pmatrix} 1 & 4 \\ -3 & 9 \end{pmatrix}$ and lets have a look at what A does to a random vector:

$\begin{pmatrix} 1 & 4 \\ -3 & 9 \end{pmatrix}\begin{pmatrix} \\ \\ \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}$......it's nothing special!

But now consider $\begin{pmatrix} 1 & 4 \\ -3 & 9 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} =$

Observe that A simply makes this vector 7 times as long! We say that $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector of A with associated eigenvalue $\lambda = 7$.

How do we find all the eigenvectors and eigenvalues of a matrix A? Well

$A\mathbf{v} = \lambda\mathbf{v} \to A\mathbf{v} = \lambda I\mathbf{v} \to A\mathbf{v} - \lambda I\mathbf{v} = 0 \to (A - \lambda I)\mathbf{v} = \mathbf{0}$.

Now $\mathbf{v} = 0$ is the trivial solution to the above matrix equation and we are seeking non-trivial solutions. Thus the matrix $A - \lambda I$ must be non-invertible and hence we demand that

$$\det(A - \lambda I) = 0.$$

This is called the characteristic equation and generates the eigenvalues. 2×2 matrices have a quadratic characteristic equation and 3×3 matrices will have a cubic characteristic equation. Once you have the eigenvalues you can then find the eigenvectors by solving $(A - \lambda I)\mathbf{v} = \mathbf{0}$ using row reduction.

Example 1 Find all the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 4 \\ -3 & 9 \end{pmatrix}$ and hence find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$\bigstar \quad P = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}, \; D = \begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix} \quad \bigstar$$

Example 2 Find all the eigenvalues and eigenvectors of $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$ and hence diagonalise A.

We start with the characteristic polynomial $\det(A - \lambda I) = 0$. If at all possible we will try to avoid the situation where we actually produce a cubic polynomial equation as these are difficult to solve.

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} 1-\lambda & -6 \\ -2 & -\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & -6 \\ -1 & -\lambda \end{vmatrix} - 3\begin{vmatrix} 2 & 1-\lambda \\ -1 & -2 \end{vmatrix}$$

$$= (-2-\lambda)\{-\lambda(1-\lambda) - 12\} - 2\{-2\lambda - 6\} - 3\{-4 + 1 - \lambda\}$$

$$= (-2-\lambda)\{\lambda^2 - \lambda - 12\} - 2\{-2\lambda - 6\} - 3\{-3 - \lambda\}$$

$$= (-2-\lambda)\{\lambda^2 - \lambda - 12\} + 4\lambda + 12 + 9 + 3\lambda$$

$$= (-2-\lambda)(\lambda - 4)(\lambda + 3) + 7\lambda + 21$$

$$= (-2-\lambda)(\lambda - 4)(\lambda + 3) + 7(\lambda + 3)$$

$$= (\lambda + 3)\{(-2-\lambda)(\lambda - 4) + 7\}$$

$$= (\lambda + 3)\{-\lambda^2 + 2\lambda + 15\}$$

$$= -(\lambda + 3)\{\lambda^2 - 2\lambda - 15\}$$

$$= -(\lambda + 3)(\lambda + 3)(\lambda - 5) = 0.$$

Thus $\lambda = -3, \ -3, \ 5.$

As a check $\quad -3 + -3 + 5 = -2 + 1 + 0.$

Note that the fact that $\lambda = -3$ has doubled up is certainly troubling but it does not imply that we necessarily will be short an eigenvector. Let's now find the eigenvectors, first for $\lambda = -3$:

$$\star \quad P = \begin{pmatrix} -1 & -2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \; D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad \star$$

What is happening with the process of diagonalisation?

When we think of $\mathbb{R}^3$ we like to use $\{\mathbf{i}\ , \mathbf{j}\ , \mathbf{k}\}$ as a basis. But these vectors mean nothing to A. If you were to ask A what would *it* prefer as a basis it would respond by saying "I'll have my eigenvectors thanks". A likes its eigenvectors since the action of A upon the eigenvectors is simply contraction and elongation. If we are prepared to abandon $\{\mathbf{i}, \mathbf{j}\ , \mathbf{k}\}$ and instead make A happy by using the coordinate system generated by its eigenvectors $\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\}$ then A transforms into the trivial matrix D. That is P transforms the complicated A into the very simple diagonal D via $P^{-1}AP = D$!

Proof of Diagonalisation formula

Let's prove the above claims in the 3×3 case. The proof in other dimensions is similar.

Suppose that A is a 3×3 matrix with a full set of linearly independent eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and associated eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$.

Let P be the matrix of eigenvectors $P = (\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3)$.

Then

$$AP = A\,(\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3) = (A\mathbf{v}_1|A\mathbf{v}_2|A\mathbf{v}_3) = (\lambda_1\mathbf{v}_1|\lambda_2\mathbf{v}_2|\lambda_3\mathbf{v}_3) = (\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3)\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$= PD$$

Thus $AP = PD \longrightarrow P^{-1}AP = D$ as required.

[29]You can now do Q 90

LECTURE 30
SPECIAL MATRICES

A matrix A is said to be symmetric if $A = A^T$.

The eigenvectors from different eigenvalues of a symmetric matrix are mutually perpendicular.

A matrix Q is said to be orthogonal if $Q^T Q = I$ or equivalently $Q^{-1} = Q^T$.

The columns of an orthogonal matrix are an orthonormal set.

Let A be a **symmetric** matrix and Q the orthogonal matrix made up of unit eigenvectors of A. Then $Q^T A Q = D$ is an orthogonal diagonalisation of A with the matrix D being the diagonal matrix of corresponding eigenvalues of A.

$(AB)^{-1} = B^{-1}A^{-1}$.

$(AB)^T = B^T A^T$.

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is an ellipse in $\mathbb{R}^2$. $(++)$

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is a hyperbola in $\mathbb{R}^2$ $(+-)$.

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ is an ellipsoid in $\mathbb{R}^3$ $(+++)$.

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ is a hyperboloid of one sheet in $\mathbb{R}^3$ $(++-)$. (Axis on the negative)

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ is a hyperboloid of 2 sheets in $\mathbb{R}^3$ $(+--)$. (Axis on the positive)

In $\mathbb{R}^3$ the number of sheets is the number of $(-)$'s.

In this lecture we will first look at some special matrices and their properties. Then we will sketch some fundamental quadratic curves and quadric surfaces.

Definition: A square matrix A is said to be symmetric if $A = A^T$.

For a symmetric matrix the first row is the same as the first column, the second row is the same as the second column etc.

Definition: A square matrix Q is said to be orthogonal if $Q^T Q = I$ or equivalently $Q^{-1} = Q^T$.

An orthogonal matrix has **orthonormal** columns. That is, the columns are perpendicular to each other and have length equal to 1.

Eigenvectors of Symmetric Matrices

The crucial feature enjoyed by symmetric matrices is that their eigenanalysis is perfectly formed. Symmetric matrices are the best matrices money can buy.

Theorem: Let A be a real symmetric matrix. Then

I) All eigenvalues of A are real.

II) There is **always** a full set of eigenvectors.

III) Eigenvectors corresponding to different eigenvalues are orthogonal.

Property III) is especially significant. Lets take a look at the proof.

But first note that we will use the fact that dot products can always be expressed as matrix products.

That is $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$.

For example

$$\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}^T \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = 23. \text{ So}$$

Proof III Let A be a symmetric matrix and let $\mathbf{v}$ and $\mathbf{w}$ be two eigenvectors of A associated with distinct eigenvalues λ and μ respectively. Then

$$\begin{aligned}
\lambda(\mathbf{v} \cdot \mathbf{w}) &= (\lambda \mathbf{v}) \cdot \mathbf{w} \\
&= (A\mathbf{v}) \cdot \mathbf{w} \\
&= (A\mathbf{v})^T \mathbf{w} \\
&= \mathbf{v}^T A^T \mathbf{w} \\
&= \mathbf{v}^T A \mathbf{w} \qquad (\text{Since } A^T = A) \\
&= \mathbf{v}^T \mu \mathbf{w} \\
&= \mu(\mathbf{v}^T \mathbf{w}) \\
&= \mu(\mathbf{v} \cdot \mathbf{w})
\end{aligned}$$

Hence $\lambda(\mathbf{v} \cdot \mathbf{w}) = \mu(\mathbf{v} \cdot \mathbf{w}) \to (\lambda - \mu)(\mathbf{v} \cdot \mathbf{w}) = 0$. But $\lambda \neq \mu$ implies that $\mathbf{v} \cdot \mathbf{w} = 0$ and hence we have $\mathbf{v} \perp \mathbf{w}$.

★

Example 1 If A is an invertible matrix with eigenvector $\mathbf{v}$ and corresponding eigenvalue λ prove that $\mathbf{v}$ is also an eigenvector of A^{-1} and find the corresponding eigenvalue.

$$\star \quad \frac{1}{\lambda} \quad \star$$

Recall that an orthogonal matrix Q has columns which are both perpendicular and of unit length. An efficient way of saying this is that $Q^TQ = I$ since all the dot products of the columns are 0 when the columns are different and 1 when they are the same.

Example 2 Let $A = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{-5}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \end{pmatrix}$. Prove that A is orthogonal an hence evaluate A^{-1}.

$$\star \quad A^{-1} = A^T \quad \star$$

Example 3 Prove that for any square invertible matrices A and B of the same dimension

$$(AB)^{-1} = B^{-1}A^{-1}$$

★

Example 4 Prove the the determinant of a real orthogonal matrix Q is ± 1

★

Example 5 Suppose that A, B and C are square matrices of the same dimension and that A is symmetric and B is orthogonal. Simplify $(ABC)^T(B^{-1}A)^{-1}$.

★ C^T ★

We now turn to the sketching of quadratic objects first in $\mathbb{R}^2$ and then in $\mathbb{R}^3$.

Example 6 Sketch each of the following curves. Find the smallest distance from the curve to the origin in each case and determine the point(s) on the curve where this minimal distance is achieved.

$$a)\ \frac{x^2}{16} + \frac{y^2}{9} = 1.$$

$$b)\ \frac{x^2}{25} - \frac{y^2}{4} = 1.$$

★ a) Shortest distance of 3 at$(0, \pm 3)$ ★
★ b) Shortest distance of 5 at$(\pm 5, 0)$ ★

Example 7 Sketch each of the following quadric surfaces. Find the smallest distance from the surface to the origin in each case and determine the point(s) on the surface where this minimal distance is achieved..

$$a)\ \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1.$$

$$b)\ -\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1.$$

$$c)\ -\frac{x^2}{4} - \frac{y^2}{9} + \frac{z^2}{16} = 1.$$

$$d)\ -\frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{16} = 1.$$

$$e)\ 4x^2 + 9y^2 - z^2 = 36.$$

$$f)\ 5x^2 + 7z^2 = 70.$$

$\bigstar$ a) Shortest distance of 2 at$(\pm 2, 0, 0)$ $\bigstar$
$\bigstar$ b) Shortest distance of 3 at$(0, \pm 3, 0)$ $\bigstar$
$\bigstar$ c) Shortest distance of 4 at$(0, 0, \pm 4)$ $\bigstar$
$\bigstar$ d) Empty graph $\bigstar$
$\bigstar$ e) Shortest distance of 2 at$(0, \pm 2, 0)$ $\bigstar$
$\bigstar$ f) Shortest distance of $\sqrt{10}$ at$(0, 0, \pm\sqrt{10})$ $\bigstar$

In the next lecture we will generalise these surfaces and curves to situations where the axis of the object does not lie on one of the coordinate axes.

30

LECTURE 31
QUADRIC SURFACES

When a standard quadric surface

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1$$

is rotated in space mixed terms xy, xz and yz appear in the equation. It is possible to express the resulting quadratic form in matrix form $\mathbf{x}^T A\mathbf{x} = 1$ where A is a symmetric matrix. An analysis of the eigenvectors and eigenvalues of A will reveal both the structure and the principal axes of the surface.

Consider the curve $2x^2 - 4xy + 5y^2 = 54$. The presence of the mixed term xy indicates that this is a standard object (ellipse or hyperbola) which has been tilted to some degree so that its major and minor axes no longer point in the x and y directions. To understand the curve we need to apply a specific transformation which "untilts" the curve into standard form. Our first step is to rewrite the quadratic form in terms of matrices.

Claim: $2x^2 - 4xy + 5y^2 = 54$ is equivalent to $\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 54$

Proof:

This gets us over into the arena of matrices where the theory of eigenvalues and eigenvectors may be brought into play! Observe that by its nature of construction, the matrix A will be symmetric and thus its eigenvectors will be naturally orthogonal to each other. We now undertake a complete eigenanalysis of A.

So the eigenvalues of A are 1 and 6 with associated unit eigenvectors $\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$ respectively. Observe that the eigenvectors are orthogonal! These directions $X = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$ and $Y = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$ actually form the principal axes of the curve. That is the curve sits properly on the eigenvectors of A. We will prove this formally in the next lecture. Furthermore in the $\{X, Y\}$ system the equation of the curve is $1X^2 + 6Y^2 = 54$. (Note the use of the eigenvalues). We can now identify the curve as an ellipse whose closest point to the origin is 3 units in the Y direction. Thus the closest points to the origin are $\pm \begin{pmatrix} -\frac{3}{\sqrt{5}} \\ \frac{6}{\sqrt{5}} \end{pmatrix}$ in the $\{x, y\}$ system. The transformation which interrelates the two coordinate systems is $\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} X \\ Y \end{pmatrix}$ where P is the usual matrix of eigenvectors $\begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$. Sketch:

In summary:

The quadratic form $\mathbf{x}^T A \mathbf{x}$ where A is a symmetric matrix has principal axes given by the orthogonal eigenvectors of A and the associated quadratic curves and quadric surfaces may be transform into standard objects with the eigenvalues as coeffiecients.

Example 1 Express the equation of the surface

$$x^2 + 2y^2 + 2z^2 + 4xy - 4xz + 6yz = 30$$

in terms of its principal axes X, Y and Z and hence determine the nature of the surface. Find an orthogonal matrix P implementing the transformation through $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$.

Deduce the shortest distance from the surface to the origin and the $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ coordinates of these closest point(s).

The equation may be written as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}^T \begin{pmatrix} 1 & 2 & -2 \\ 2 & 2 & 3 \\ -2 & 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 30$

You are given that the matrix $A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 2 & 3 \\ -2 & 3 & 2 \end{pmatrix}$ has eigenvalues -3 and 5 with associated eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Find the remaining eigenvalue and eigenvector and hence complete the question.

$$\bigstar \quad P = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad -3X^2 + 3Y^2 + 5Z^2 = 30 \quad \bigstar$$

$\bigstar$ Hyperboloid of one sheet with closest distance to the origin of $\sqrt{6}$ at $\pm \begin{pmatrix} 0 \\ \sqrt{3} \\ \sqrt{3} \end{pmatrix}$ $\bigstar$

We will start the next lecture with a formal proof of these algorithms and results.

[31] You can now do Q 91 and 92

LECTURE 32
SYSTEMS OF DIFFERENTIAL EQUATIONS

Systems of differential equations $\mathbf{y}' = A\mathbf{y}$ may be easily solved by implementing the eigenvalues and eigenvectors of A.

If A is a 3×3 matrix with linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, and associated eigenvalues λ_1, λ_2 and λ_3, then the general solution to $\mathbf{y}' = A\mathbf{y}$ takes the form

$$\mathbf{y} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}$$

More complicated systems may be simplified through the transformation $\mathbf{y} = P\mathbf{z}$ where P is the usual matrix of eigenvectors of A.

We start this lecture by proving the validity of the algorithms used to analyse quadratic forms in the previous lecture.

Consider the quadratic form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}^T A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1$ where A is a symmetric matrix. Since A is symmetrix it admits a full set of orthogonal eigenvectors. Let P be the matrix of unit eigenvectors of A. The columns of P are orthogonal and also of unit length implying that P is an orthogonal matrix. Via the usual process of diagonalisation $P^{-1}AP = D$ where $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ is the diagonal matrix of eigenvalues. But since P is orthogonal we have $P^TAP = D$.

We now implement the orthogonal transformation (a rotation in space)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

The quadratic form becomes

$$\left(P\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}\right)^T AP \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 1 \text{ implying that}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}^T P^TAP \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 1 \text{ and hence we have}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}^T D \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 1 \to \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}^T \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 1$$

As promised in the last lecture this yields the simplified form (without mixed terms) with respect to the principal axes:

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = 1$$

We turn now to our second major application of eigenvectors, systems of differential equations.

SYSTEMS OF DIFFERENTIAL EQUATIONS

Example 1 Solve the system of differential equations.

$$\begin{aligned} y_1' &= 2y_1 + y_2 \\ y_2' &= -y_1 + y_3 \\ y_3' &= y_1 + y_2 + y_3 \end{aligned}$$

where $y_1(0) = 6$, $y_2(0) = -5$, and $y_3(0) = 7$.

We begin by noting that the system may be written in matrix form as

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

which is expressed as $\mathbf{y}' = A\mathbf{y}$. The usual eigenanalysis yields eigenvalues 0,1, and 2 with associated eigenvectors $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

The solution may now be simply written down as

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{0t} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{1t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}$$

Reading across the rows we have a general solution:

$y_1 = c_1 - c_2 e^t + c_3 e^{2t}$

$y_2 = -2c_1 + c_2 e^t$

$y_3 = c_1 + c_3 e^{2t}$

Before applying the initial conditions let's prove that this all works:

Claim: If A is a 3×3 matrix with linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, and associated eigenvalues λ_1, λ_2 and λ_3, then the general solution to $\mathbf{y}' = A\mathbf{y}$ takes the form

$$\mathbf{y} = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} + c_3\mathbf{v}_3e^{\lambda_3 t}$$

where $c_1,\ c_2$ and c_3 are arbitrary constants.

Proof:

Method 1: Assume a solution to $\mathbf{y}' = A\mathbf{y}$ of the form $\mathbf{y} = \mathbf{v}e^{\alpha t}$ where $\mathbf{v}$ is a vector and α is a number.

Method 2: Make the substitution $\mathbf{y} = P\mathbf{z}$ where P is the matrix of eigenvectors of A.

★

It is clear from the above that once we have the eigenvectors and eigenvalues of A the solution to the system $\mathbf{y}' = A\mathbf{y}$ is just one step away!

The i.c.'s are implemented at the last stage to evaluate the three arbitrary constants.

Recall that

$y_1 = c_1 - c_2e^t + c_3e^{2t}$

$y_2 = -2c_1 + c_2e^t$

$y_3 = c_1 + c_3e^{2t}$

and that $y_1(0) = 6,\ y_2(0) = -5$, and $y_3(0) = 7$.

So

So we have $c_1 = 3, \quad c_2 = 1, \quad c_3 = 4.$

Hence the final solution is

$y_1 = 3 - e^t + 4e^{2t}$

$y_2 = -6 + e^t$

$y_3 = 3 + 4e^{2t}$

These three functions satisfy both the system of differential equations and the i.c.'s. Lets check that the last equation $\quad y_3' = y_1 + y_2 + y_3 \quad$ is satisfied:

LHS $= y_3' =$

RHS $= y_1 + y_2 + y_3 =$

$$\bigstar \quad y_1 = 3 - e^t + 4e^{2t},\ y_2 = -6 + e^t,\ y_3 = 3 + 4e^{2t} \quad \bigstar$$

In more complicated examples $\mathbf{y}' = A\mathbf{y} + \mathbf{b}$ our approach is to actually implement the substitution $\mathbf{y} = P\mathbf{z}$ to yield $P\mathbf{z}' = AP\mathbf{z} + \mathbf{b}$. We then have $\mathbf{z}' = P^{-1}AP\mathbf{z} + P^{-1}\mathbf{b}$ implying $\mathbf{z}' = D\mathbf{z} + P^{-1}\mathbf{b}$. Since the diagonal matrix D has so little structure this final system when separated out, is trivial to solve using our standard first order linear theory.

[32]You can now do Q 93 and 94

LECTURE 34
LAPLACE TRANSFORMS

LAPLACE TRANSFORMS

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^m	$m!/s^{m+1}$
t^ν , $(\nu > -1)$	$\Gamma(\nu+1)/s^{\nu+1}$
e^{-at}	$1/(s+a)$
$\sin bt$	$b/(s^2+b^2)$
$\cos bt$	$s/(s^2+b^2)$
$\sinh bt$	$b/(s^2-b^2)$
$\cosh bt$	$s/(s^2-b^2)$
$\sin bt - bt\cos bt$	$2b^3/(s^2+b^2)^2$
$\sin bt + bt\cos bt$	$2bs^2/(s^2+b^2)^2$
$t\ \sin bt$	$2bs/(s^2+b^2)^2$
te^{-at}	$1/(s+a)^2$
$\mathrm{u}(t-c)$	e^{-cs}/s
$e^{-at}f(t)$	$F(s+a)$
$tf(t)$	$-F'(s)$
$f(t-c)\mathrm{u}(t-c)$	$e^{-cs}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f'''(t)$	$s^3F(s)-s^2f(0)-sf'(0)-f''(0)$
$\int_0^t f(\tau)d\tau$	$F(s)/s$

LAPLACE TRANSFORMS

The Laplace transform changes a function $f(t)$ in a highly specific and very useful fashion into a different function $F(s)$. By transforming entire problems (for example differential equations) the nature of the problem may be fundamentally altered opening up unexpected avenues of attack. The Laplace transform is particularly effective in situations involving discontinuity and hence is extremely useful in applications of electrical engineering where switching plays such a central role.

We will spend a few lectures discussing the intricacies of finding Laplace transforms and also of course how to return home by finding inverse Laplace transforms. We will then focus on some classical applications. The definition on the Laplace transform is

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

That is, we integrate the function f against a decaying exponential function all the way from 0 to ∞.

We denote the original function by $f(t)$ and the transformed object $F(s)$. To keep track of where we are, we use t as the original variable and the transformed variable is s. Just like differentiation and integration, taking Laplace transforms is a linear process so

$$\mathcal{L}\left\{af(t) + bg(t)\right\} = a\mathcal{L}\left\{f(t)\right\} + b\mathcal{L}\left\{g(t)\right\}.$$

How do we find Laplace transforms? Well there is always the integral definition, but in reality we leave that behind and in most cases both Laplace transforms and their inverses are simply found by looking up an appropriate entry in a table of Laplace transforms. You will of course have this table supplied to you in your final exam.

But first a little revision on integration by parts and improper integrals.

Example 1 Evaluate each of the following definite integrals:

a) $\displaystyle\int_0^1 3te^{-7t}dt.$

b) $\displaystyle\int_0^\infty 3te^{-7t}dt.$

One thing to remember is that $\displaystyle\int e^{ax}dx = \frac{1}{a}e^{ax}$ implying that $\displaystyle\int e^{-st}dt = -\frac{1}{s}e^{-st}$. Also the exponential function goes to zero much faster than any polynomial goes to infinity!

★ $a) \quad \dfrac{3}{49} - \dfrac{24}{49}e^{-7} \quad b) \quad \dfrac{3}{49}$ ★

Example 2 Find the Laplace transform of $4t$, firstly via the definition and then by using the table of Laplace transforms.

★ $\dfrac{4}{s^2}$ ★

You can see from the above example that the Laplace transform is an improper integral (since we are integrating all the way to infinity). What saves us is that e^{-st} converges to zero very aggressively and many of our standard functions (certainly all of the polynomials, sin and cos) are quickly defeated by e^{-st}. In general the Laplace transform of f will exist provided that f does not grow too rapidly as $t \to \infty$.

Observe also that the integral is with respect to t only, thus leaving the Laplace transform variable s as a residue. Notice also that we assume that $t \geq 0$. The variable t usually represents time so this quite natural. A nice thing about Laplace transforms (as opposed to derivatives for example) is that they do not get too worried when faced with discontinuities.

Example 3 Let $f(t) = \begin{cases} 9, & 3 \leq t \leq 4; \\ 0, & \text{otherwise.} \end{cases}$

Sketch the function and find its Laplace transform directly from the integral definition.

$$\bigstar \quad \frac{9(e^{-3s} - e^{-4s})}{s} \quad \bigstar$$

Example 4 Use the table to find the Laplace transform of

$$f(t) = 7t^2 + \cos(4t) - 5\sinh(3t) + 2$$

.

$$\bigstar \quad \frac{14}{s^3} + \frac{s}{s^2+16} - \frac{15}{s^2-9} + \frac{2}{s} \quad \bigstar$$

Example 5 Use the table to find the Laplace transform of

$$f(t) = 3e^{7t} - 6te^{-2t} + \sqrt{t}$$

.

$$\bigstar \quad \frac{3}{s-7} - \frac{6}{(s+2)^2} + \frac{\Gamma(\frac{3}{2})}{s^{\frac{3}{2}}} \quad \bigstar$$

Note that the gamma function Γ is an extension of the concept of factorials beyond the positive integers. Its definition is $\Gamma(\nu+1) = \displaystyle\int_0^\infty e^{-x} x^\nu \, dx$ and it is usually evaluated by computer or tables. If ν is a positive integer then $\Gamma(\nu+1) = \nu!$

Example 6 Prove that

$$\mathcal{L}\{te^{-t}\} = \frac{1}{(s+1)^2}\,.$$

Note that this result also appears in the table of Laplace transforms.

★

One final note is that the concept of an arbitrary constant C plays no role whatsoever in the theory of Laplace transforms!

[33]You can now do Q 95 and 96 a,b,c and d

LECTURE 35
INVERSE LAPLACE TRANSFORMS AND THE HEAVISIDE FUNCTION

LAPLACE TRANSFORMS

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^m	$m!/s^{m+1}$
t^ν , $(\nu > -1)$	$\Gamma(\nu+1)/s^{\nu+1}$
e^{-at}	$1/(s+a)$
$\sin bt$	$b/(s^2+b^2)$
$\cos bt$	$s/(s^2+b^2)$
$\sinh bt$	$b/(s^2-b^2)$
$\cosh bt$	$s/(s^2-b^2)$
$\sin bt - bt\cos bt$	$2b^3/(s^2+b^2)^2$
$\sin bt + bt\cos bt$	$2bs^2/(s^2+b^2)^2$
$t\ \sin bt$	$2bs/(s^2+b^2)^2$
te^{-at}	$1/(s+a)^2$
$\mathrm{u}(t-c)$	e^{-cs}/s
$e^{-at}f(t)$	$F(s+a)$
$tf(t)$	$-F'(s)$
$f(t-c)\mathrm{u}(t-c)$	$e^{-cs}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f'''(t)$	$s^3F(s)-s^2f(0)-sf'(0)-f''(0)$
$\int_0^t f(\tau)d\tau$	$F(s)/s$

In the previous lecture we used the Laplace transform to change $f(t)$ to $F(s)$ (written $\mathcal{L}(f) = F$). We now want to undo this process and convert $F(s)$ back to $f(t)$. This is referred to as taking an inverse Laplace transform and we write $f = \mathcal{L}^{-1}(F)$. There is no integral formula for taking inverse Laplace transforms and we have no option but to just use the Laplace transform table together with a bag of interesting tricks.

Example 1 Find the inverse Laplace transform of each of the following functions:

a) $F(s) = \dfrac{4}{s-2} + \dfrac{1}{s^2+9}$;

b) $F(s) = \dfrac{3}{s} - \dfrac{12}{s^2} + \dfrac{2}{s^5}$.

$$\bigstar \quad a)\ 4e^{2t} + \frac{1}{3}\sin(3t) \quad b)\ 3 - 12t + \frac{t^4}{12} \quad \bigstar$$

Observe that we are now using the table backwards and traveling from the s variable back to the t variable. In order to be able to invert a wider class of objects we need a fancy little step function called the Heaviside function.

Definition: The Heaviside function $u(t)$ is given by

$$u(t) = \begin{cases} 0, & \text{t<0;} \\ \frac{1}{2}, & \text{t=0;} \\ 1, & \text{t>0.} \end{cases}$$

The graph of $u(t)$ is:

Note that the **name** of the above function is "u".

The Heaviside function is also sometimes called the unit step function.

The complicated behaviour of $u(t)$ at $t = 0$ is somewhat cosmetic for our purposes and you may view the definition of the Heaviside function as being simply:

$$u(t) = \begin{cases} 0, & \text{t<0;} \\ 1, & \text{t}\geq\text{0.} \end{cases}$$

with graph

Actually the filled dots can pretty much go anywhere since the Laplace transform, being an integration process, smooths out all blemishes.

Of greater importance to us is the function $u(t - c)$.

$$u(t - c) = \begin{cases} 0, & \text{t<c;} \\ 1, & \text{t}\geq\text{c.} \end{cases}$$

The graph of $u(t - c)$ is then:

Always remember that the Heaviside function $u(t-c)$ is a single mathematical function which is asleep until c and then wakes up. It is then equal to 1 to infinity. The Heaviside function can be viewed as the simplest possible discontinuous function.

Example 2 Sketch $y = 9u(t-4)$.

★

Example 3 Prove that $\mathcal{L}\{u(t-c)\} = \dfrac{e^{-cs}}{s}$.

★

Note that the above result is part of your standard tables.

Example 4 Find the Laplace transform of $9u(t-4)$.

★

We use the Heaviside function to build up other discontinuous functions. Since taking the Laplace transform is a process of integration we may be sloppy on the definition of a function at endpoints as it has no impact on the final result. When sketching however it is a good idea to put in the dot on the discontinuities.

Example 5 Suppose that $f(t) = \begin{cases} 1, & a \le t \le b; \\ 0, & \text{otherwise.} \end{cases}$

Prove that $f(t) = u(t-a) - u(t-b)$.

So $u(t-a) - u(t-b)$ is the function that is asleep till $t = a$ wakes up for a little while until $t = b$ and then goes back to bed forever.

Example 6 Suppose that $f(t) = \begin{cases} 9, & 3 \le t \le 4; \\ 0, & \text{otherwise.} \end{cases}$

Use the Heaviside function to find $\mathcal{L}\{f\}$.

Recall that we have already answered this question via the integral definition in the previous lecture.

$$\bigstar \quad 9(\frac{e^{-3s}}{s} - \frac{e^{-4s}}{s}) \quad \bigstar$$

Example 7 Sketch each of the following functions and rewrite the function without the use of Heavisides:

a) $f(t) = 1 - u(t-3)$ (this is a flipped Heaviside)

b) $f(t) = t^2u(t-4)$ (This is a cut. Remember that when you multiply by a Heaviside half of the time you are multiplying by 0 and hence wiping the function away and the other half of the time you are multiplying by 1 and thus doing nothing at all!)

c) $f(t) = (t-4)^2u(t-4)$ (This is a shift plus cut. This structure is crucial in the next lecture)

d)$f(t) = (t-5)^2u(t-4)$ (These are less common)

e) $f(t) = t^2\{u(t-2) - u(t-7)\}$

★

Example 8 Express the function $f(t) = 3t + tu(t-1) + u(t-3) - 1$ without the use of the Heaviside and sketch.

★

Example 9 Suppose that $\mathcal{L}(f) = \dfrac{4}{s^2} + \dfrac{3e^{-9s}}{s}$. Find and sketch f.

★

[34] You can now do Q 97, 99 a, b

LECTURE 36
THE SHIFTING THEOREMS

LAPLACE TRANSFORMS

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = F(s)$$

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^m	$m!/s^{m+1}$
t^ν , $(\nu > -1)$	$\Gamma(\nu+1)/s^{\nu+1}$
e^{-at}	$1/(s+a)$
$\sin bt$	$b/(s^2+b^2)$
$\cos bt$	$s/(s^2+b^2)$
$\sinh bt$	$b/(s^2-b^2)$
$\cosh bt$	$s/(s^2-b^2)$
$\sin bt - bt\cos bt$	$2b^3/(s^2+b^2)^2$
$\sin bt + bt\cos bt$	$2bs^2/(s^2+b^2)^2$
$t\ \sin bt$	$2bs/(s^2+b^2)^2$
te^{-at}	$1/(s+a)^2$
$\mathrm{u}(t-c)$	e^{-cs}/s
$e^{-at}f(t)$	$F(s+a)$
$tf(t)$	$-F'(s)$
$f(t-c)\mathrm{u}(t-c)$	$e^{-cs}F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f'''(t)$	$s^3F(s) - s^2f(0) - sf'(0) - f''(0)$
$\int_0^t f(\tau)d\tau$	$F(s)/s$

We turn now to two central theorems (both of which appear in your standard table) which will allow us to cope with shifts in both the s and the t variables. These results are handy when finding some particular Laplace transforms and are absolutely essential when dealing with inverse Laplace transforms.

First Shifting Theorem

$$\mathcal{L}(e^{-at}f(t)) = F(s+a)$$

Proof:

What the first shifting theorem is saying is that the impact of an e^{-at} in the t variable is to shift s to $s+a$ in the s variable. We use the first shifting theorem in both directions!

Example 1 Find $\mathcal{L}(e^{-8t}\cos(2t))$

$$\star \quad \frac{s+8}{s^2+16s+68} \quad \star$$

Example 2 Find $\mathcal{L}^{-1}\left(\dfrac{6}{(s-7)^4}\right)$

$$\bigstar \quad t^3e^{7t} \quad \bigstar$$

Example 3 Find $\mathcal{L}^{-1}\left(\dfrac{10s-1}{s^2+6s+13}\right)$.

$$\bigstar \quad 10e^{-3t}\cos(2t) - \tfrac{31}{2}e^{-3t}\sin(2t) \quad \bigstar$$

Second Shifting Theorem

$$\mathcal{L}(f(t-c)u(t-c)) = e^{-cs}F(s)$$

Proof:

What the second shifting theorem is saying is that shifting and cutting in the t variable introduces an exponential function in the s variable.

Example 4 Let $f(t) = \sin(t-\pi)u(t-\pi)$. Sketch a graph of f and find its Laplace transform.

$$\bigstar \quad \frac{e^{-\pi s}}{1+s^2} \quad \bigstar$$

Example 5 Let $f(t) = t^2 u(t-3)$. Sketch a graph of f and find its Laplace transform.

Sketch:

We have a Heaviside cut here but no shift ! So let's generate a 3 shift by writing t^2 as

$$t^2 + 0t + 0 = a(t-3)^2 + b(t-3) + c.$$

We have two different ways to find a, b and c :

Method 1: Blast away and compare coefficients of powers of t:

Method 2: First make a little substitution $w = t - 3$:

So

$$t^2 = (t-3)^2 + 6(t-3) + 9$$

and hence our problem is to find the Laplace transform of

$$\{(t-3)^2 + 6(t-3) + 9\}u(t-3) = (t-3)^2 u(t-3) + 6(t-3)u(t-3) + 9u(t-3).$$

Now:

$$\bigstar \quad e^{-3s}\left(\frac{9}{s} + \frac{6}{s^2} + \frac{2}{s^3}\right) \quad \bigstar$$

Example 6 Find $\mathcal{L}^{-1}(\frac{12e^{-7s}}{s^4})$ and sketch the inverse Laplace transform.

$\bigstar \quad 2(t-7)^3u(t-7) \quad \bigstar$

Example 7 Find the inverse Laplace transform of $F(s) = \dfrac{e^{-7s}}{1+(s+4)^2}$.

This is tough as it invloves **both** shifting theorems!

$\bigstar \quad f(t) = e^{-4(t-7)}\sin(t-7)u(t-7) \quad \bigstar$

[35]You can now do Q 96 e f g h, 98, 99 c d g h i

LECTURE 37
PARTIAL FRACTIONS

LAPLACE TRANSFORMS

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^m	$m!/s^{m+1}$
t^ν , $(\nu > -1)$	$\Gamma(\nu+1)/s^{\nu+1}$
e^{-at}	$1/(s+a)$
$\sin bt$	$b/(s^2+b^2)$
$\cos bt$	$s/(s^2+b^2)$
$\sinh bt$	$b/(s^2-b^2)$
$\cosh bt$	$s/(s^2-b^2)$
$\sin bt - bt\cos bt$	$2b^3/(s^2+b^2)^2$
$\sin bt + bt\cos bt$	$2bs^2/(s^2+b^2)^2$
$t\ \sin bt$	$2bs/(s^2+b^2)^2$
te^{-at}	$1/(s+a)^2$
$\mathrm{u}(t-c)$	e^{-cs}/s
$e^{-at}f(t)$	$F(s+a)$
$tf(t)$	$-F'(s)$
$f(t-c)\mathrm{u}(t-c)$	$e^{-cs}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f'''(t)$	$s^3F(s)-s^2f(0)-sf'(0)-f''(0)$
$\int_0^t f(\tau)d\tau$	$F(s)/s$

We will now look at how the theory of partial fractions can be used to find the inverse Laplace transform of rational polynomials. This is a skill that you already have from integration theory and the methods transfer across without any change at all. But first a little revision on the Heaviside function and the shifting theorems.

Example 1 Suppose that $f(t) = \begin{cases} 0, & t < 1; \\ 7t, & 1 \leq t \leq 2; \\ 0, & 2 < t \leq 6; \\ 9, & t > 6. \end{cases}$

Sketch the function and find its Laplace transform.

$$\bigstar \quad \frac{7}{s^2}(e^{-s} - e^{-2s}) + \frac{1}{s}(7e^{-s} - 14e^{-2s} + 9e^{-6s}) \quad \bigstar$$

Example 2 is a lovely application of our earlier work to one of the table entries:

Example 2 Prove that $\mathcal{L}(tf(t)) = -F'(s)$.

We will do this without integration!

$\bigstar$

Example 3 Find the inverse Laplace transform of each of the following functions:

i) $F(s) = \dfrac{6s}{s^2 - 11s + 28}$

ii) $F(s) = \dfrac{7s^2 + s + 27}{(s^2 + 4)(s - 1)}$

iii) $F(s) = \dfrac{5s^2 - 36s + 23}{(s - 7)^2(s + 1)}$

All of these are partial fraction questions. Recall that the two crucial features we need for parfrac to work on a rational function is factors on the bottom and for the rational function to be bottom heavy. If the degree of the top is greater than **or equal to** the degree on the bottom we simply do a little long division first.

i) $\dfrac{6s}{s^2 - 11s + 28} = \dfrac{6s}{(s - 4)(s - 7)} = \dfrac{A}{s - 4} + \dfrac{B}{s - 7} = \dfrac{A(s - 7) + B(s - 4)}{(s - 4)(s - 7)}$. Thus

$$A(s - 7) + B(s - 4) \equiv 6s$$

ii) $\dfrac{7s^2+s+27}{(s^2+4)(s-1)} = \dfrac{As+B}{s^2+4} + \dfrac{C}{s-1} = \dfrac{(As+B)(s-1)+C(s^2+4)}{(s^2+4)(s-1)}$. Thus

$$(As+B)(s-1)+C(s^2+4) \equiv 7s^2+s+27$$

iii) For repeated factors we need to be very careful with both the decomposition and the recomposition.

$$\frac{5s^2-36s+23}{(s-7)^2(s+1)} = \frac{A}{(s-7)}+\frac{B}{(s-7)^2}+\frac{C}{(s+1)} = \frac{A(s-7)(s+1)+B(s+1)+C(s-7)^2}{(s-7)^2(s+1)}.$$

Thus

$$A(s-7)(s+1)+B(s+1)+C(s-7)^2 \equiv 5s^2-36s+23$$

$$\bigstar \quad i)\ 14e^{7t}-8e^{4t} \quad ii)\ \frac{1}{2}\sin(2t)+7e^{t} \quad iii)\ (2t+4)e^{7t}+e^{-t} \quad \bigstar$$

[37] You can now do Q 99 e f

LECTURE 38
DE'S VIA LAPLACE TRANSFORMS

LAPLACE TRANSFORMS

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^m	$m!/s^{m+1}$
t^ν , $(\nu > -1)$	$\Gamma(\nu+1)/s^{\nu+1}$
e^{-at}	$1/(s+a)$
$\sin bt$	$b/(s^2+b^2)$
$\cos bt$	$s/(s^2+b^2)$
$\sinh bt$	$b/(s^2-b^2)$
$\cosh bt$	$s/(s^2-b^2)$
$\sin bt - bt\cos bt$	$2b^3/(s^2+b^2)^2$
$\sin bt + bt\cos bt$	$2bs^2/(s^2+b^2)^2$
$t\ \sin bt$	$2bs/(s^2+b^2)^2$
te^{-at}	$1/(s+a)^2$
$\mathrm{u}(t-c)$	e^{-cs}/s
$e^{-at}f(t)$	$F(s+a)$
$tf(t)$	$-F'(s)$
$f(t-c)\mathrm{u}(t-c)$	$e^{-cs}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f'''(t)$	$s^3F(s)-s^2f(0)-sf'(0)-f''(0)$
$\int_0^t f(\tau)d\tau$	$F(s)/s$

The major application of Laplace transforms is to the solution of differential equations with discontinuous forcing functions. This physical system arises frequently particularly when switching is involved. The method is quite simple. We transform the entire equation to produce an algebraic equation in the s variable. This equation is usually trivially solved to produce a solution in the s variable. We then need to invert this solution to produce an answer in the t variable. It is this last step of inversion that is usually the toughest part of the process! First let us prove the required table entries linking the Laplace transform to the calculus.

Example 1 Prove that

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

It is also true that

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

The above two facts (which always appear in the Laplace transform tables) will prove extremely useful when solving D.E.'s.

Example 2 Solve $\dfrac{dy}{dt} + 4y = 8$ where $y(0) = 9$. First use your standard techniques and then implement Laplace transforms.

$$\bigstar \quad y = 2 + 7e^{-4t} \quad \bigstar$$

Example 3 Solve $y'' - 6y' + 8y = 0$ where $y(0) = 1$ and $y'(0) = 16$. First use your standard techniques and then implement Laplace transforms.

$$\bigstar \quad y = 7e^{4t} - 6e^{2t} \quad \bigstar$$

You will notice from the above that we really need initial conditions at $y(0)$ and $y'(0)$ before it becomes possible to use Laplace transforms effectively. However as a bonus the presence of discontinuous right hand sides may be easily dealt with! We will look at these more complicated differential equations in Lecture 39.

In preparation however, for homework please have a go at all the problems in next lecture (Lecture 38a) before we next meet. These examples explore the use of the Heaviside function in the theory of Laplace Transforms. The next lecture (Lecture 38a) will be a problem class working through these Heaviside examples.

[38] You can now do Q 100 101 102

LECTURE 38A: THE HEAVISIDE FUNCTION LAPLACE TRANSFORMS

LAPLACE TRANSFORMS

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^m	$m!/s^{m+1}$
t^ν , $(\nu > -1)$	$\Gamma(\nu+1)/s^{\nu+1}$
e^{-at}	$1/(s+a)$
$\sin bt$	$b/(s^2+b^2)$
$\cos bt$	$s/(s^2+b^2)$
$\sinh bt$	$b/(s^2-b^2)$
$\cosh bt$	$s/(s^2-b^2)$
$\sin bt - bt\cos bt$	$2b^3/(s^2+b^2)^2$
$\sin bt + bt\cos bt$	$2bs^2/(s^2+b^2)^2$
$t\ \sin bt$	$2bs/(s^2+b^2)^2$
te^{-at}	$1/(s+a)^2$
$\mathrm{u}(t-c)$	e^{-cs}/s
$e^{-at}f(t)$	$F(s+a)$
$tf(t)$	$-F'(s)$
$f(t-c)\mathrm{u}(t-c)$	$e^{-cs}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f'''(t)$	$s^3F(s)-s^2f(0)-sf'(0)-f''(0)$
$\int_0^t f(\tau)d\tau$	$F(s)/s$

Example 1 Sketch each of the following functions:

a) $f(t) = u(t-5) - 1$

b) $f(t) = \dfrac{1}{t} u(t-2)$

c) $f(t) = (t-1)^3 u(t-1)$

d) $f(t) = e^t \{u(t-5) - u(t-7)\}$

★

Example 2 Find the Laplace transform of $e^{4t}t^5$

$$\mathcal{L}(e^{-at}f(t)) = F(s+a)$$

We use the first shifting theorem whenever we are taking Laplace transforms and there is a rogue exponential function in the t variable OR we are taking inverse Laplace transforms and there is a shift in the s variable.

$$\bigstar \quad \frac{120}{(s-4)^6} \quad \bigstar$$

Example 3 Find the inverse Laplace transform of $\dfrac{(s-12)}{(s-12)^2+1}$

$$\mathcal{L}(e^{-at}f(t)) = F(s+a)$$

$$\bigstar \quad e^{12t}\cos(t) \quad \bigstar$$

Example 4 Consider the function given by

$$f(t) = \begin{cases} 9, & 3 \leq t \leq 4; \\ 7, & \text{otherwise.} \end{cases}$$

a) Sketch f.

b) Express f in terms of Heavisides.

c) Find $F(s)$.

$$\mathcal{L}(\mathrm{u}(t-c)) = \frac{e^{-cs}}{s}$$

$$\bigstar \quad \frac{2e^{-3s}}{s} - \frac{2e^{-4s}}{s} + \frac{7}{s} \quad \bigstar$$

Example 5 Consider the function

$$f(t) = 6tu(t-3) + 9tu(t-7) - 15u(t-12).$$

a) Write the function without the use of Heavisides.

b) Sketch the function.

c) Find the Laplace transform of f.

$$\mathcal{L}(f(t-c)\mathrm{u}(t-c)) = e^{-cs}F(s)$$

We use the second shifting theorem whenever we are taking inverse Laplace transforms and there is a rogue exponential function in the s variable OR we are taking Laplace transforms and there is a shift and a cut in the t variable. If we have a cut in the t variable without a shift we need to manipulate the function so that a shift appears.

$$\bigstar \quad 6\frac{e^{-3s}}{s^2} + 9\frac{e^{-7s}}{s^2} + 18\frac{e^{-3s}}{s} + 63\frac{e^{-7s}}{s} - 15\frac{e^{-12s}}{s} \quad \bigstar$$

Example 6 Find and sketch the inverse Laplace transform of $\dfrac{e^{-3s}}{s^2+1}$.

$$\mathcal{L}(f(t-c)\mathrm{u}(t-c)) = e^{-cs}F(s)$$

$\bigstar \quad \sin(t-3)u(t-3) \quad \bigstar$

Example 7 Find the Laplace transform of $e^{7t}u(t-4)$.

This is a rare example where both shifting theorems may be used !

$\mathcal{L}(f(t-c)\mathrm{u}(t-c)) = e^{-cs}F(s)$

$\mathcal{L}(e^{-at}f(t)) = F(s+a)$

$\bigstar \quad \dfrac{e^{28-4s}}{s-7} \quad \bigstar$

Example 8 Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{(s-7)^4}\right\}$.

$\bigstar \quad \dfrac{1}{6}u(t-2)e^{7(t-2)}(t-2)^3 \quad \bigstar$

Example 9 (Challenge Problem)

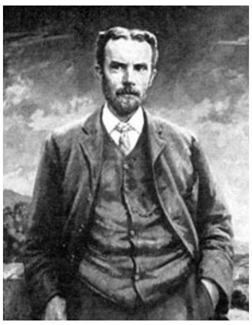
Suppose that $f(t) = \begin{cases} 1, & 0 < t \leq 1; \\ -1, & 1 < t \leq 2; \\ f(t-2), & \text{otherwise.} \end{cases}$

Sketch the function and find its Laplace transform.

Hint: You may need the theory of limiting sums of G.P.'s.

Note that the condition $f(t) = f(t-2)$ simply forces the function to repeat every 2 units. That is, it forces a periodicity of 2.

$$\star \quad \frac{1}{s}\left(\frac{e^s - 1}{e^s + 1}\right) \quad \star$$



Oliver Heaviside

$u(t-1850) - u(t-1925)$

LECTURE 39
DE'S VIA LAPLACE TRANSFORMS PART 2

LAPLACE TRANSFORMS

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = F(s)$$

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^m	$m!/s^{m+1}$
$t^\nu \ , (\nu > -1)$	$\Gamma(\nu+1)/s^{\nu+1}$
e^{-at}	$1/(s+a)$
$\sin bt$	$b/(s^2+b^2)$
$\cos bt$	$s/(s^2+b^2)$
$\sinh bt$	$b/(s^2-b^2)$
$\cosh bt$	$s/(s^2-b^2)$
$\sin bt - bt \cos bt$	$2b^3/(s^2+b^2)^2$
$\sin bt + bt \cos bt$	$2bs^2/(s^2+b^2)^2$
$t \ \sin bt$	$2bs/(s^2+b^2)^2$
te^{-at}	$1/(s+a)^2$
$\mathrm{u}(t-c)$	e^{-cs}/s
$e^{-at} f(t)$	$F(s+a)$
$tf(t)$	$-F'(s)$
$f(t-c)\mathrm{u}(t-c)$	$e^{-cs}F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f'''(t)$	$s^3F(s) - s^2f(0) - sf'(0) - f''(0)$
$\int_0^t f(\tau)d\tau$	$F(s)/s$

We will now look at DE's where discontinuous functions play a role. It needs to be kept in mind that many physical processes are in fact discontinuous by their very nature (for example activating switches or overnight gapping in the stock market). Laplace transforms are the tool of choice for these sorts of problems!

Example 1 Consider the differential equation

$$y'' - 5y' + 4y = r(t)$$

where

$$r(t) = \begin{cases} 24 & t \geq 7; \\ 0 & \text{otherwise.} \end{cases}$$

and the initial conditions are $y(0) = 0$ and $y'(0) = 3$.

i) Sketch $r(t)$ and express $r(t)$ in terms of the Heaviside function.

ii) By taking the Laplace transform of the differential equation show that

$$Y(s) = \frac{24e^{-7s}}{s(s-4)(s-1)} + \frac{3}{(s-4)(s-1)}$$

iii) Using partial fractions show that

$$Y(s) = e^{-7s}(\frac{6}{s} + \frac{2}{s-4} - \frac{8}{s-1}) + \frac{1}{s-4} - \frac{1}{s-1}$$

iv) Hence find the solution $y(t)$ in terms of the Heaviside function.

v) Express the solution without the Heaviside function.

$$\star \quad y(t) = u(t-7)(6 + 2e^{4t-28} - 8e^{t-7}) + e^{4t} - e^{t} \quad \star$$

$$\star \quad y(t) = \begin{cases} e^{4t} - e^{t}, & 0 \le t \le 7; \\ 6 + 2e^{4t-28} - 8e^{t-7} + e^{4t} - e^{t}, & t \ge 7. \end{cases} \quad \star$$

Example 2 Consider the system of differential equations

$$\begin{aligned} \frac{dx}{dt} + 4x + 10y &= 0 \\ \frac{dy}{dt} - 5x - 11y &= 0 \end{aligned}$$

where $x(0) = -1$ and $y(0) = 0$.

i) By taking Laplace transforms of both equations show that:

$$\begin{aligned} (s+4)X + 10Y &= -1 && (1) \\ -5X + (s-11)Y &= 0 && (2) \end{aligned}$$

ii) Hence show that:

$$\begin{aligned} 5(s+4)X + 50Y &= -5 && (3) \\ -5(s+4)X + (s+4)(s-11)Y &= 0 && (4) \end{aligned}$$

i) Show that adding (3) and (4) yields

$$Y(s) = \frac{-5}{s^2 - 7s + 6}.$$

ii) Hence show that $y(t) = e^t - e^{6t}$.

iii) Without inverting again find $x(t)$.

$$\bigstar \quad x(t) = -2e^{t} + e^{6t} \quad \bigstar$$

[39] You can now do Q 103 104 105

LECTURE 41
PERIODICITY

- A function f is said to have a period of T if $f(x+T) = f(x)$ for all x.
- $\sin(x)$ and $\cos(x)$ have period 2π.
- $\sin(nx)$ and $\cos(nx)$ have period $\dfrac{2\pi}{n}$.
- A function f is said to be odd if $f(-x) = -f(x)$ for all x.
- A function f is said to be even if $f(-x) = f(x)$ for all x.
- $\sin(-x) = -\sin(x)$ (sin is an odd function).
- $\cos(-x) = \cos(x)$ (cos is an even function).
- Odd$\times$ Odd=Even, Odd$\times$ Even=Odd, Even$\times$ Even=Even.
- Odd$\pm$Odd=Odd, Even$\pm$Even=Even.
- $\displaystyle\int_{-a}^{a} \text{Odd } dx = 0 \qquad \int_{-a}^{a} \text{Even } dx = 2\int_{0}^{a} \text{Even } dx$

This lecture will prepare you for the theory of Fourier series by investigating various issues and definitions surrounding the concept of periodicity. It is quite common for physical systems to exhibit strong repetitive characteristics. For example consider the motion of a piston or the timing on spark plugs. Fourier series provide us with the tools to analyse these systems effectively by making specialised use of the sine and cosine functions which are of course the fundamental periodic objects in mathematics. But first we need to get a feel for how repetition is dealt with at a technical level.

Definition: A function f is said to have a period of T if $f(x+T) = f(x)$ for all x. This means that the function repeats itself every T units.

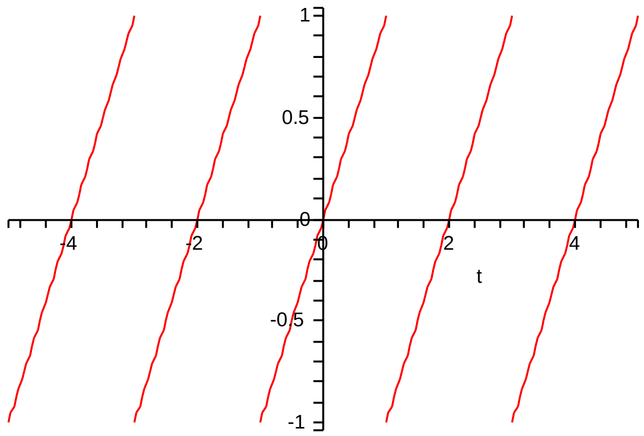
Example 1 Find the period of each of the following functions:

i) $f(x) = \sin(x)$

ii) $f(x) = 4\cos(7x)$

iii) $f(x) = x^2$

iv) f has the following graph:



$\bigstar \quad 2\pi, \ \dfrac{2\pi}{7}, \ \text{no period}, \ 2 \quad \bigstar$

Note that $\sin(nx)$ and $\cos(nx)$ both have a period of $\dfrac{2\pi}{n}$.

Example 2 Suppose that $f(t) = \begin{cases} t^2, & 0 \le t < 3; \\ f(t+3), & \text{otherwise.} \end{cases}$

Note that the above definition *forces* a periodicity of 3.

Sketch a graph of f for $-6 \le t \le 6$. What is the value of $f(11)$?

$\bigstar$

$\text{odd} \times \text{ odd} \rightarrow \text{even}$

$\text{odd} \times \text{ even} \rightarrow \text{odd}$

$\text{even} \times \text{ even} \rightarrow \text{even}$

$\dfrac{\text{odd}}{\text{even}} \rightarrow \text{odd}$

$\dfrac{\text{even}}{\text{odd}} \rightarrow \text{odd}$

$\dfrac{\text{even}}{\text{even}} \rightarrow \text{even}$

$\dfrac{\text{odd}}{\text{odd}} \rightarrow \text{even}$

$\text{odd} \pm \text{odd} = \text{odd}$

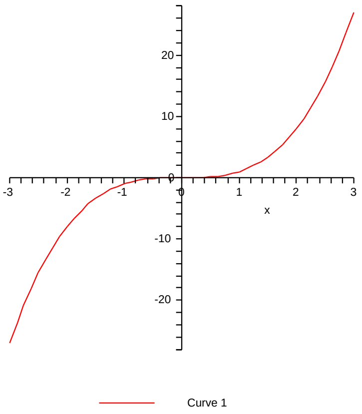
$\text{even} \pm \text{even} = \text{even}$

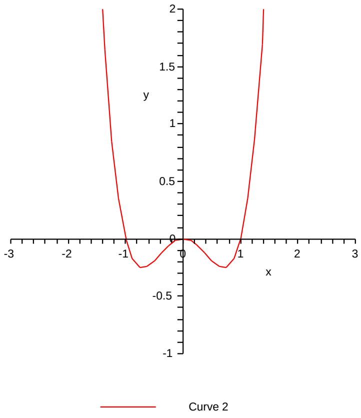
$\text{even} \pm \text{odd} = \text{nothing}$

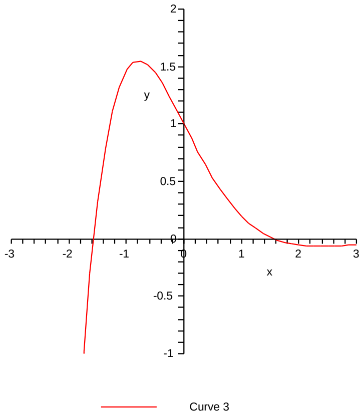
$\displaystyle\int_{-a}^{a} \text{Odd } dx = 0$

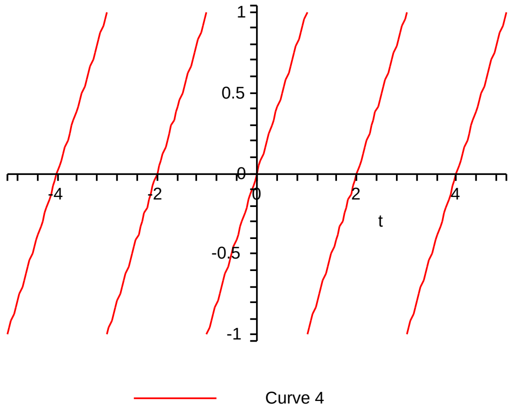
$\displaystyle\int_{-a}^{a} \text{Even } dx = 2\int_{0}^{a} \text{Even } dx$

Example 3 Identify each of the following functions as odd, even or neither:









★

Example 4 Identify each of the following functions as odd, even or neither:

i) $f(x) = \cos(3x)$

ii) $f(x) = x\cos(x)$

iii) $f(x) = \sin^2(x)$

iv) Prove your answer in iii) from the definition.

★

Example 5 Evaluate $\displaystyle\int_{-3}^{3} \frac{x^2\cos(7x)\sin(2x)}{1+x^2+x^4}$

★

Example 6 Prove that the derivative of an even function is an odd function.

★

Example 7 Suppose that $f(t) = \begin{cases} 2t, & 0 \le t < 1; \\ 2, & 1 \le t < 2. \end{cases}$

i) Sketch f over its domain.

ii) Sketch g, the **periodic extension** of f over $-8 \le t \le 8$.
What is the period of g? Evaluate $g(2)$ and $g(8.3)$.

iii) Sketch h, the **even periodic extension** of f over $-8 \le t \le 8$.
What is the period of h? Evaluate $h(15.8)$.

iv) Sketch j, the **odd periodic extension** of f over $-8 \le t \le 8$.
What is the period of j? Evaluate $j(14.7)$.

v) Sketch the derivative of j over $-8 \le t \le 8$.

★

One last comment regarding a very common error when sketching the even periodic extension h. Consider the following (incorrect) sketch of h:

This is certainly **even** but it is **not** periodic! For a periodic function we need to have a single fundamental component graph repeated endlessly.

[41] You can now do Q 106 107

LECTURE 42
FOURIER SERIES PART I

Suppose that a function f has period $T = 2\pi$. Then f may be approximated by the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1}$$

where the Fourier coefficients a_0, a_n, and b_n are given by

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad (n = 1, 2, \dots) \\ b_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad (n = 1, 2, \dots) \end{aligned}\right\} \tag{2}$$

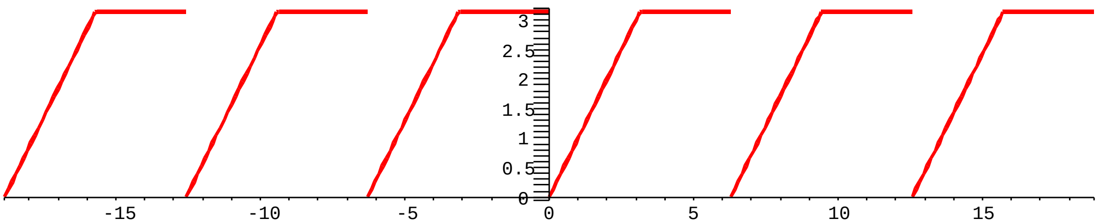
Suppose that we are dealing with a periodic function f of period 2π. Periodic functions are complex in their own special way. We saw earlier that the infinite repetition made the calculation of their Laplace transforms quite a drama. By using Fourier series we can bust such functions up as a sum of much simpler periodic trigonometric components $\sin(nx)$ and $\cos(nx)$. This is of enormous value when dealing with both differential and partial differential equations.

In this lecture we will start off with the simpler case of functions with period $T = 2\pi$. Later on the theory will be generalised to functions of arbitrary period T.

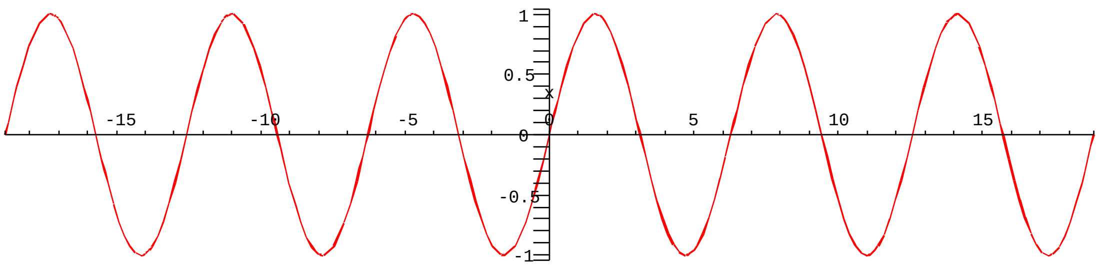
We begin with the periodic function

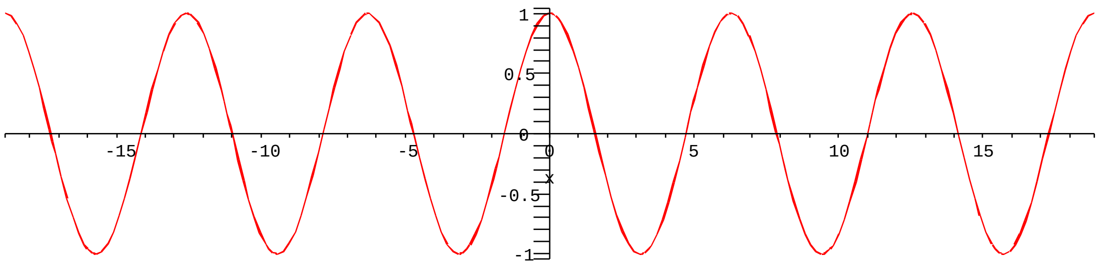
$$f(x) = \begin{cases} x, & 0 \le x < \pi; \\ \pi, & \pi \le x < 2\pi. \\ f(x + 2\pi) & \text{otherwise} \end{cases}$$

This is (by definition) a function of period 2π and the graph over $-6\pi \le x \le 6\pi$ looks like

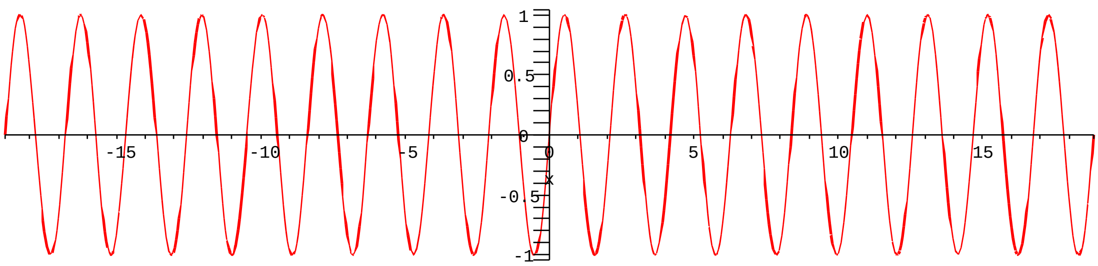


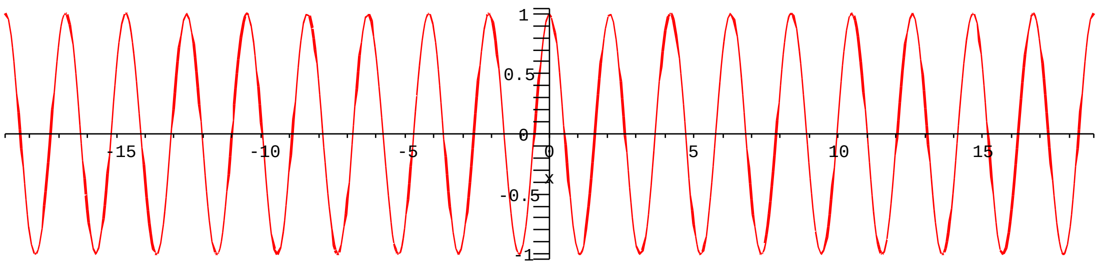
The graphs of $\sin(x)$ and $\cos(x)$ are also of period $T = 2\pi$:





It is fairly clear that there is no way that f could possibly be approximated by taking linear combinations of $\sin(x)$ and $\cos(x)$....... f looks nothing at all like these two functions! But we have a trick up our sleeves! We can make the trig functions busier (by reducing their periods). Consider the graphs of $\sin(3x)$ and $\cos(3x)$:





(Note that in general the period of $\sin(nx)$ and $\cos(nx)$ is $T = \dfrac{2\pi}{n}$).

What we then do to create a Fourier series is to use **all** possible functions $\sin(nx)$ and $\cos(nx)$ together with a constant term a_0 to approximate f as an infinite series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

The coefficients a_0,a_n and b_n are referred to as the Fourier coefficients. The bad news is that their calculation is usually a gruesome process involving the integral formulae

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad (n = 1, 2, \dots) \\ b_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad (n = 1, 2, \dots) \end{aligned}\right\}$$

Note that the above equations only apply when the period of the function f is 2π. Minor modifications need to be made for functions of arbitrary period T. We will prove the above formulae in the next lecture but for today lets look at what needs to be done to calculate the Fourier series of our function.

Example 1 Find the Fourier series of the function f above.

Some equations that you must have at your fingertips for Fourier series are:

$$\int \cos(nx)\,dx = \frac{1}{n}\sin(nx)$$
$$\int \sin(nx)\,dx = \frac{-1}{n}\cos(nx)$$
$$\sin(n\pi) = 0$$
$$\cos(n\pi) = (-1)^n$$
$$\sin(-x) = -\sin(x)$$
$$\cos(-x) = \cos(x)$$
$$\sin(0) = 0$$
$$\cos(0) = 1$$

$$\star \quad a_0 = \frac{3\pi}{4} \qquad a_n = \frac{(-1)^n - 1}{n^2\pi} \qquad b_n = \frac{-1}{n} \quad \star$$

$$\star \quad f(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty} \{\frac{(-1)^n - 1}{n^2\pi}\} \cos(nx) + \{\frac{-1}{n}\} \sin(nx) \quad \star$$

$$\star \quad f(x) = \frac{3\pi}{4} + \frac{-2}{\pi} \cos(x) - \sin(x) - \frac{1}{2} \sin(2x) - \frac{2}{9\pi} \cos(3x) - \frac{1}{3} \sin(3x) + \ldots \quad \star$$

Observe that both a_n and b_n tend to zero as $n \to \infty$. This always happens under normal circumstances and guarantees that the Fourier series will converge for standard functions.

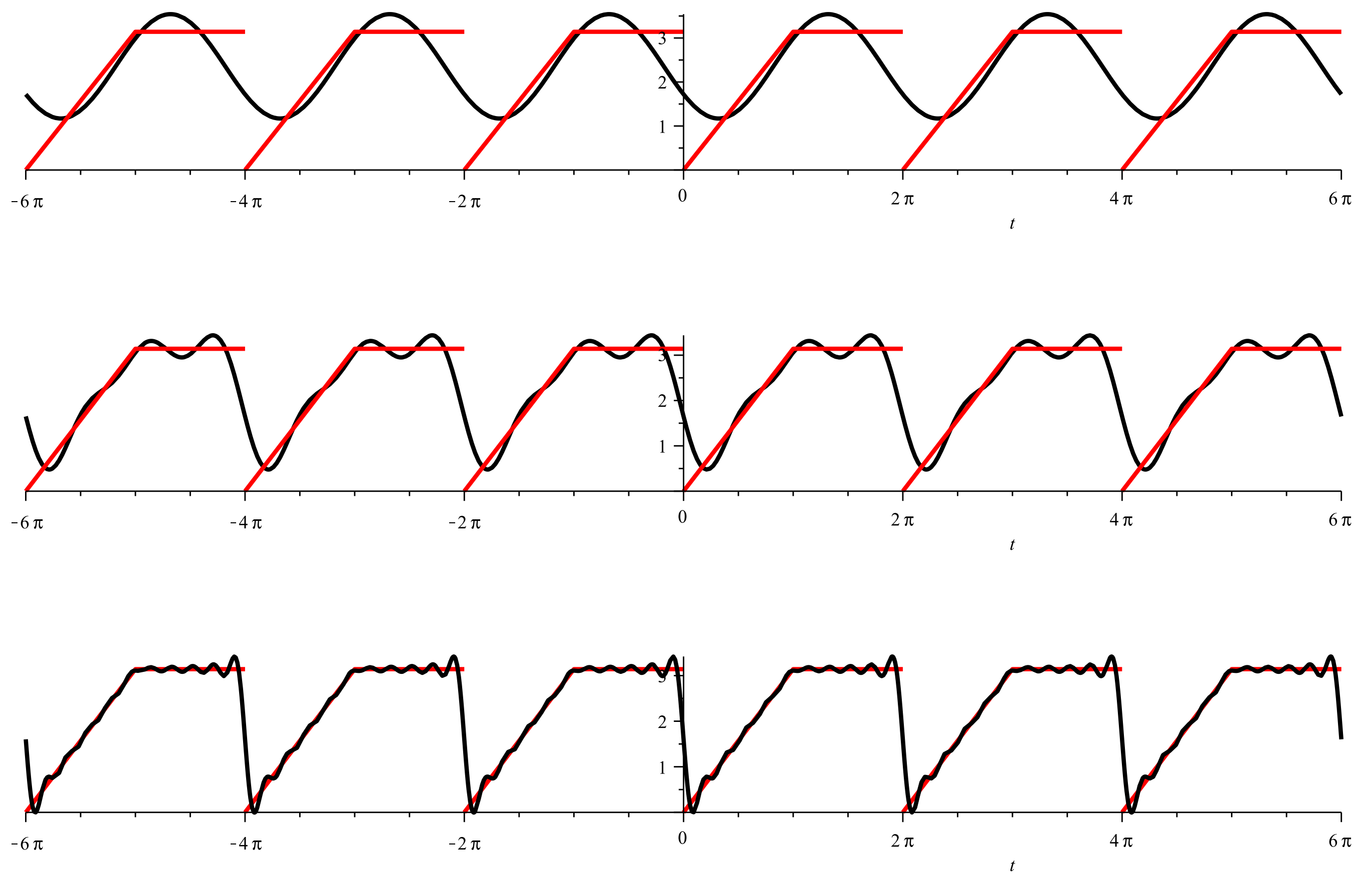
It is fascinating to look at how the Fourier series steps up in accuracy as the number of terms increases. The three graphs below show the first partial sum

$$\frac{3\pi}{4} + \frac{-2}{\pi}\cos(x) - \sin(x)$$

the third partial sum

$$\frac{3\pi}{4} + \frac{-2}{\pi}\cos(x) - \sin(x) - \frac{1}{2}\sin(2x) - \frac{2}{9\pi}\cos(3x) - \frac{1}{3}\sin(3x)$$

and finally the tenth partial sum.



If we were to take infinitely many terms then the Fourier series would sit right over the original function. Observe that the nth partial sums are continuous and yet they do a fine job of approximating discontinuous objects. We will examine carefully at a later stage what actually happens at the points of discontinuity, a situation known as the Gibb's phenomenon.

It is fair to say that the construction of a Fourier series is a major undertaking, but the payoffs are huge. In the next lecture we will still restrict our attention to functions with a period of 2π but will develop some shortcuts which will help us out on occasion.

[42]You can now do Q 108

LECTURE 43
FOURIER SERIES PART II: SOME PROOFS

Suppose that a function f has period $T = 2\pi$. Then f may be approximated by the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where the Fourier coefficients a_0, a_n, and b_n are given by

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad (n = 1, 2, \dots) \\ b_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad (n = 1, 2, \dots) \end{aligned}\right\}$$

- Odd functions have odd series $\qquad f(x) = \sum_{n=1}^{\infty} b_n \sin nx$
- Even functions have even series $\qquad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

In this lecture we will prove some of the claims of the last lecture and also develop some shortcuts which can sometimes be used the cut down on the laborious calculations often required when finding Fourier series. We will still be assuming throughout this lecture that f has period 2π. This will be generalised in the next lecture however it should be noted at this stage that we only need to make small modifications to our formulae to deal with arbitrary periodic functions.

Example 1 Fully describe each of the following series

i) $\{\sin(n\pi)\}_{n=1}^{\infty}$

ii) $\{\cos(n\pi)\}_{n=1}^{\infty}$

iii) $\{\sin(\frac{n\pi}{2})\}_{n=1}^{\infty}$

iv) $\{\cos(\frac{n\pi}{2})\}_{n=1}^{\infty}$

★

Example 2 Prove that

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)dx = 0$$

for all $m = 1, 2, 3\ldots$ and $n = 1, 2, 3\ldots$

★

Example 3 Prove that

$$\int_{-\pi}^{\pi} \cos(mx)dx = 0$$

for all $m = 1, 2, 3\ldots$

★

Example 4 Prove that

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \begin{cases} 0, & \text{If } m \neq n; \\ \pi, & \text{If } m = n. \end{cases}$$

$\underline{m = n}$	$\underline{m \neq n}$

★

We say that $\cos(nx)$ and $\cos(mx)$ are orthogonal. We are now in a position to prove the formula for a_n.

Example 5 Suppose that a function f has period $T = 2\pi$ and assume that

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

By multiplying both sides of this equation by $\cos(mx)$ and integrating from $-\pi$ to π show that

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\, dx \qquad (m = 1, 2, \ldots)$$

★

The other formulae for a_0 and b_n are similarly verified. Note that when we calculate a_n all we really end up doing is integrating the function f against $\cos(nx)$ which is of course the very term for which a_n is the coefficient!

Another crucial observation to make is that since f, $\cos(nx)$, and $\sin(nx)$ all repeat every 2π units, the integrals in the Fourier series formulae may be taken over any interval of length 2π. So the equations could also be written as

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2\pi}\int_0^{2\pi} f(x)\, dx \\ a_n &= \tfrac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx \qquad (n = 1, 2, \ldots) \\ b_n &= \tfrac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx \qquad (n = 1, 2, \ldots) \end{aligned}\right\}$$

We close with an example which will display quite clearly that when you start with and **even** or an **odd** function f the amount of work required to calculate the Fourier series can be greatly reduced just by using a few simple tricks.

Example 6 Suppose that $f(x) = \begin{cases} x & -\pi \leq x < \pi; \\ f(x+2\pi) & \text{otherwise.} \end{cases}$

Sketch f and find its Fourier series.

$$\bigstar \quad b_n = \frac{2(-1)^{n+1}}{n} \qquad a_0 = a_n = 0 \quad \bigstar$$

$$\bigstar \quad f(x) = 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - ..... \quad \bigstar$$

Note that for all odd functions $a_0 = a_n = 0$ and that for all even functions $b_n = 0$. In other words an odd function will have an odd Fourier series made up entirely of sin's and an even function will have an even Fourier series consisting of a constant a_0 and cos's. If a function is neither odd nor even then you are up for a lot of work!

The following is an interesting applet where we can add in the terms of a Fourier Series one by one and see the effect. The example above is the sawtooth button:

http://www.intmath.com/fourier-series/fourier-graph-applet.php

LECTURE 44
FOURIER SERIES OVER AN ARBITRARY PERIOD

Suppose that a function f has period $T = 2L$. Then f may be approximated by the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) \tag{1}$$

where the Fourier coefficients a_0, a_n, and b_n are given by

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \dots) \\ b_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \dots) \end{aligned}\right\} \tag{2}$$

- Odd functions have odd series $\quad f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$
- Even functions have even series $\quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$
- $\int_{-a}^{a}$ odd $dx = 0$
- $\int_{-a}^{a}$ even $dx = 2\int_{0}^{a}$ even dx
- If a periodic function f has a jump discontinuity at $x = a$ then its Fourier series at $x = a$ will converge to a y value which sits half way across the discontinuity.

We turn now to the Fourier series of function with arbitrary period T. To simplify the equations we define L to be half the period so that $T = 2L$ and express the equations in terms of L.

Observe from the equations above that if we replace L by π (and hence T by 2π) we obtain exactly the equations which we have been using over the last few lectures. The formulae above will be supplied to you in your final examination and they are all that you need to find all the different types of Fourier series. Do not commit anything else to memory.

All of the techniques and tricks from before still apply in this slightly more general setting.

Example 1 Let $f(x) = \begin{cases} |x| & -3 \le x < 3; \\ f(x+6) & \text{otherwise.} \end{cases}$

i) Sketch f over $-9 \le x < 9$.

ii) Find the Fourier series of f written in sigma notation.

iii) Express the Fourier series of f as an explicit sum.

iv) Express the Fourier series of f as a sum over the odd integers.

v) By considering the Fourier series at $x = 0$ and using part iii), show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \dots$$

$$\bigstar \quad ii) \quad f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{6((-1)^n - 1)}{n^2\pi^2} \cos(\frac{n\pi x}{3}) \quad \bigstar$$

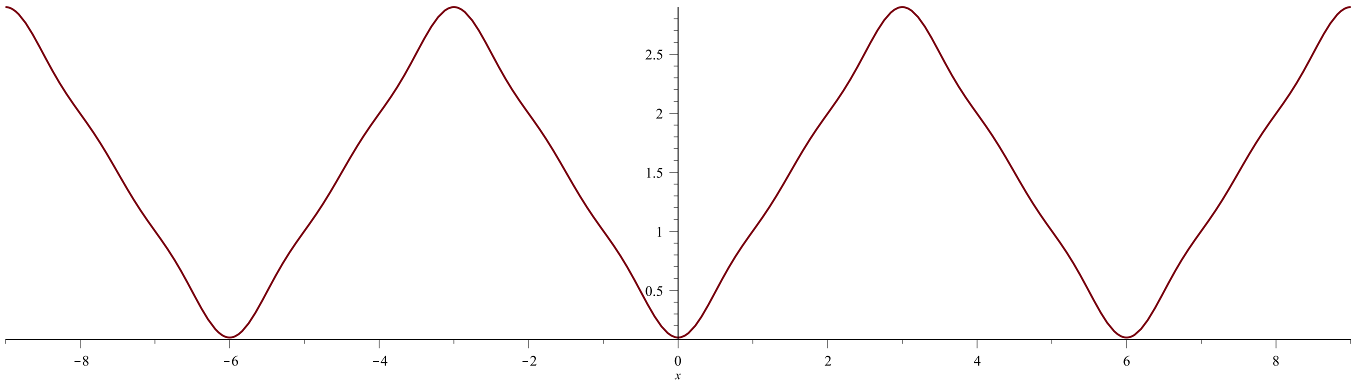
$$\bigstar \quad iii) \quad f(x) = \frac{3}{2} - \frac{12}{\pi^2} \left(\frac{\cos(\frac{\pi x}{3})}{1} + \frac{\cos(\frac{3\pi x}{3})}{9} + \frac{\cos(\frac{5\pi x}{3})}{25} + \frac{\cos(\frac{7\pi x}{3})}{49} + \dots \right) \quad \bigstar$$

$$\bigstar \quad iv) \quad f(x) = \frac{3}{2} - \frac{12}{\pi^2} \sum_{k=0}^{\infty} \frac{\cos\left(\frac{(2k+1)\pi x}{3}\right)}{(2k+1)^2} \quad \bigstar$$

The sketch below shows the graph of the Fourier series truncated at $n = 4$:

$$\frac{3}{2} - \frac{12}{\pi^2} \left(\frac{\cos(\frac{\pi x}{3})}{1} + \frac{\cos(\frac{3\pi x}{3})}{9} + \frac{\cos(\frac{5\pi x}{3})}{25} + \frac{\cos(\frac{7\pi x}{3})}{49} \right)$$

Observe how nicely the Fourier series accomplishes its approximating duties.



Example 2 Let $f(x) = \begin{cases} x & -1 \leq x < 1; \\ f(x+2) & \text{otherwise.} \end{cases}$

i) Sketch f over $-3 \leq x < 3$.

ii) Find the Fourier series of f.

iii) To what value does the Fourier series converge at $x = 1$?

iv) By considering the series at $x = \dfrac{1}{2}$ show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

$$\bigstar \quad f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1}\sin(n\pi x) = \frac{2}{\pi}\left(\frac{\sin(\pi x)}{1} - \frac{\sin(2\pi x)}{2} + \frac{\sin(3\pi x)}{3} - \frac{\sin(4\pi x)}{4} + \dots\right) \quad \bigstar$$

$\bigstar$ Converges to $y = 0$ at $x = 1$ $\bigstar$

Observe from the above example that if a periodic function f has a jump discontinuity at $x = a$ then its Fourier series at $x = a$ will converge to a y value which sits half way across the discontinuity.

[44]You can now do Q 110

LECTURE 45
SOME HARDER PROBLEMS

Suppose that a function f has period $T = 2L$. Then f may be approximated by the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) \tag{1}$$

where the Fourier coefficients a_0, a_n, and b_n are given by

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \ldots) \\ b_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \ldots) \end{aligned}\right\} \tag{2}$$

Example 1 Suppose that $f(x) = \begin{cases} x+4, & -1 \le x < 1; \\ f(x+2), & \text{otherwise.} \end{cases}$

i) By considering the graph of f and without integration show that a_n=0 and explain why the Fourier series of f takes the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

ii) Find a_0 by inspection and b_n using Example 2 from the last lecture.

iii) By considering the **graph** of f, determine the y value to which the Fourier series converges at $x = 1$?

iii) By considering the **series itself**, determine the y value to which the Fourier series converges at $x = 1$?

★ $f(x) = 4 + \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1}\sin(n\pi x)$ Converges to $y = 4$ at $x = 1$ ★

The next example will involve some very tricky integration. We will need the following results:

I) $\sin(\frac{\pi}{2} + \theta) = \cos(\theta)$

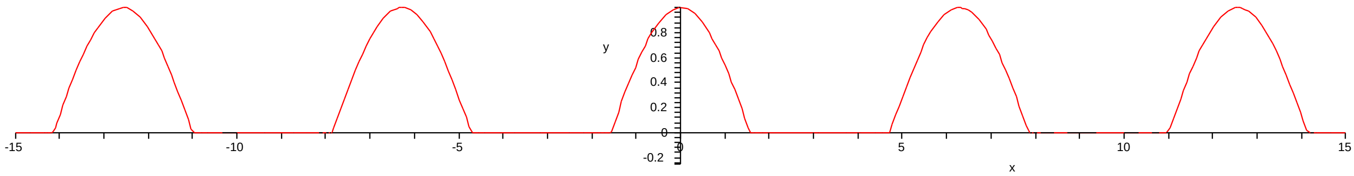
II) $\cos(A)\cos(B) = \frac{1}{2}(\cos(A + B) + \cos(A - B))$

Proof:

$\bigstar$

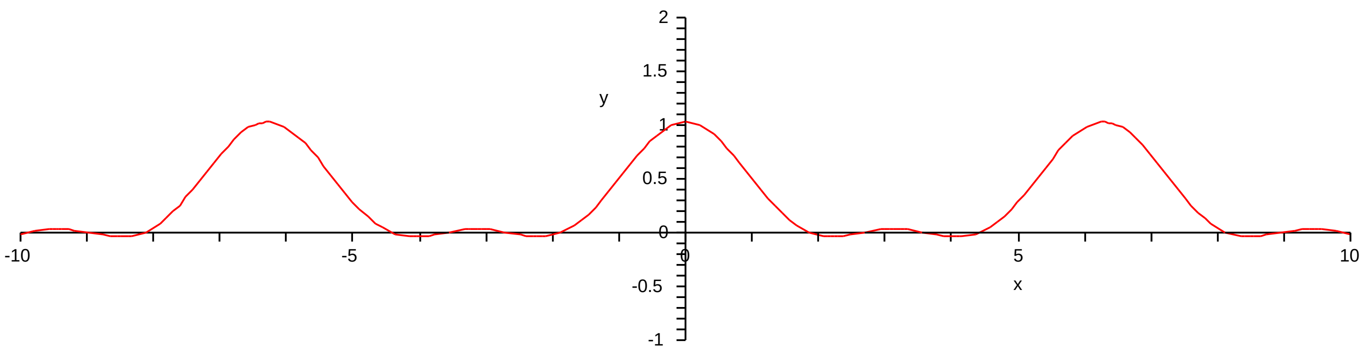
Example 2 A periodic voltage $\cos(t)$ is passed through a half-wave rectifier which clips the negative portion of the wave. Find the Fourier series of the resulting periodic function.

The function is a "clipped" cosine curve and looks like:



$$\bigstar \quad f(x) = \frac{1}{\pi} + \frac{1}{2}\cos(x) + \sum_{n=2}^{\infty} \frac{2\cos(\frac{n\pi}{2})}{\pi(1-n^2)}\cos(nx) \quad \bigstar$$

The graph below is the Fourier series terminated at $n = 10$.



Observe again how nicely the original function is being approximated, despite its strange definition.

Example 3 Find the Fourier series of $f(x) = \sin^2(x) + 7\sin(13x) + 7.5$.

This is a trick question! Make sure you do not spend half an hour on this problem.

★

We close with the observation that Fourier series may also be developed from a complex viewpoint via

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

with

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

This is then the **complex form** of the Fourier series for $f(x)$. The c_n are the complex Fourier coefficients. Students are referred to the printed notes for a proof of the above results. We tend to focus on the real case in Math2019.

LECTURE 46
HALF RANGE EXPANSIONS

Suppose that a function f has period $T = 2L$. Then f may be approximated by the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) \tag{1}$$

where the Fourier coefficients a_0, a_n, and b_n are given by

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \dots) \\ b_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \dots) \end{aligned}\right\} \tag{2}$$

For a function f defined on the interval $(0, L)$ the Fourier **cosine** series of f is the Fourier series of the **even** periodic extension of f.

For a function f defined on the interval $(0, L)$ the Fourier **sine** series of f is the Fourier series of the **odd** periodic extension of f.

Fourier sine and cosine series are called Half-Range expansions.

In this lecture we will answer the critical question 'If a function is not periodic how do we find an applicable Fourier series?' If the function is defined over the whole real line then the answer is simple........you can't! However if the function is only defined over an interval $(0, L)$ then it is quite simple to fool the Fourier series into believing that it is looking at a periodic function by implementing what is called an odd or an even periodic extension.

Example 1 Let $f(x) = |\,x - 1\,|$ for $0 < x < 3$;

i) Sketch the periodic extension of f and write down its period.

ii) Sketch the even periodic extension of f and write down its period.

iii) Sketch the odd periodic extension of f and write down its period.

★

It is important to note in the above example that the original function f is definitely non-periodic. We are however taking that small definition over $(0, 3)$ and embedding it within a periodic environment in a number of different ways. Observe in each of the three graphs above you can still *see* the original f.

Example 2 Let $f(x) = x^2 + 1 \qquad$ for $0 < x < 1$;

i) Sketch the periodic extension of f and write down its period.

ii) Sketch the even periodic extension of f and write down its period.

iii) Sketch the odd periodic extension of f and write down its period.

★

Now if a function is only defined over $(0, L)$ we can easily extend it to a periodic function using the trick above and then find its Fourier series! We are particularly interested in the odd and even periodic extensions since the calculations for these functions are a little less horrible and the pure sine and cosine series that are produced are easier to work with.

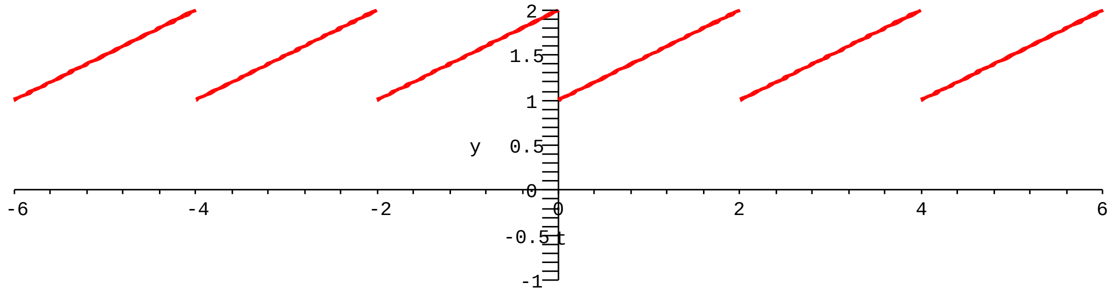
Note that if we take an even periodic extension then the Fourier series will be a Fourier cosine series but if we take the odd periodic extension then we will end up with a Fourier sine series. These are called Half Range expansions. Admittedly the sine and cosine series will be defined over the whole real line rather than just $(0, L)$, but that is OK as we can choose to implement the series only over the restricted domain $(0, L)$. In fact everything beyond $(0, L)$ is just a mirage! It is crucial to note that you DO NOT NEED special formulae for half range expansions and the usual equations will serve you perfectly well, provided you use a few simple tricks. Note also that we do not need to be too fussy about what happens at endpoints and discontinuities since the Fourier series will always make its own decision anyway and converge to a point which splits the gap.

Example 3 Let

$$f(x) = \left\{ \begin{array}{ll} \dfrac{x}{2} + 1 & 0 < x < 2; \end{array} \right.$$

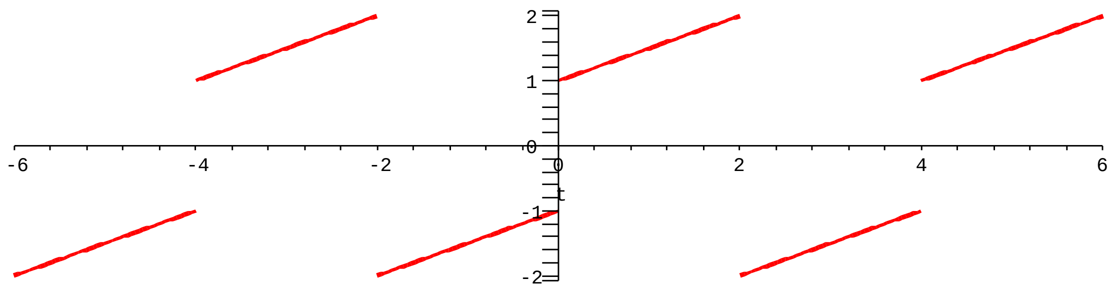
Find the Fourier **sine** series of f expressing your answer in sigma and expanded form.

Note that we could simply extend f to a periodic function of period 2 in a trivial fashion as follows:



But this periodic function is neither odd nor even and hence the Fourier series will be messy. We prefer the sine series.

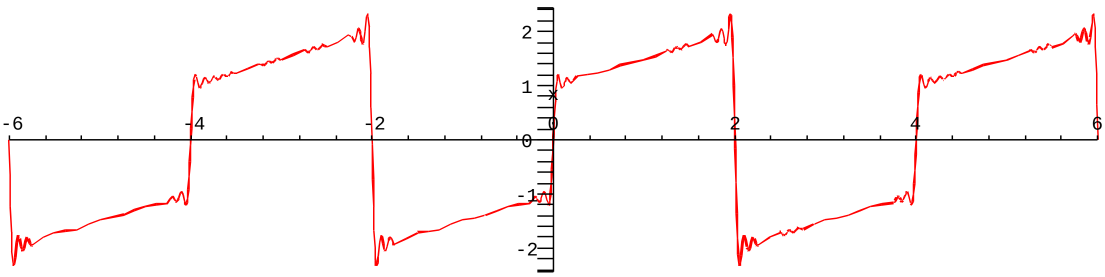
The first step in calculating the Fourier sine series of f is to draw a careful sketch of the odd periodic extension of f:



$$\bigstar \quad f(x) = \sum_{n=1}^{\infty} \frac{2-4(-1)^n}{n\pi} \sin(\frac{n\pi x}{2}) \quad \bigstar$$

$$\bigstar \quad f(x) = \frac{6}{\pi}\sin(\frac{\pi x}{2}) - \frac{1}{\pi}\sin(\pi x) + \frac{2}{\pi}\sin(\frac{3\pi x}{2}) \dots \quad \bigstar$$

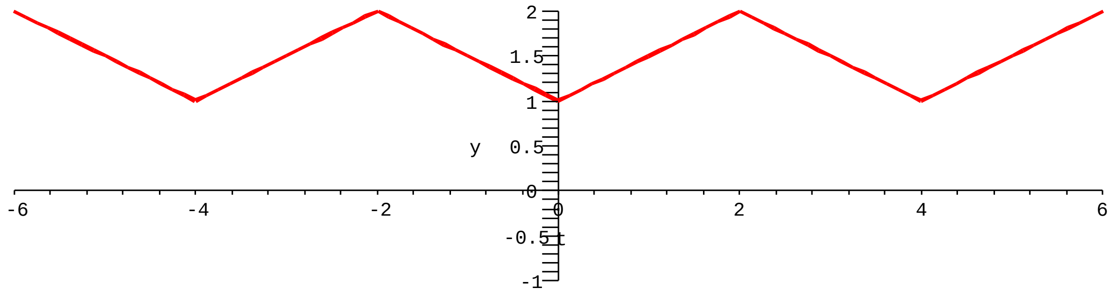
The graph of the sine series (taking the first 20 terms) is:



Note that we are really only interested in $[0, 2]$ since that is where the original function was defined! Note also that we have managed to approximate f using nothing but the sine curve.

Example 4 Find the Fourier **cosine** series of the same function f above. Express your answer in sigma notation, as a sum over the odd integers and also in expanded form.

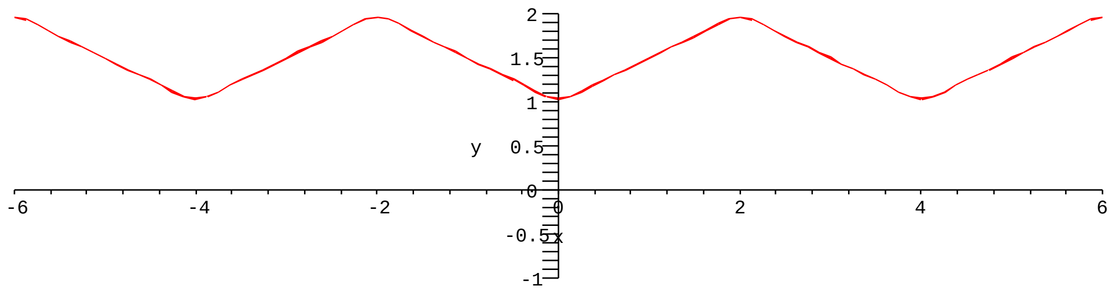
First lets draw the even periodic extension:



$$\star \quad f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{n^2\pi^2} \cos(\frac{n\pi x}{2}) = \frac{3}{2} + \sum_{k=0}^{\infty} \frac{-4}{(2k+1)^2\pi^2} \cos(\frac{(2k+1)\pi x}{2}) \quad \star$$

$$\star \quad f(x) = \frac{3}{2} - \frac{4}{\pi^2}\cos(\frac{\pi x}{2}) - \frac{4}{9\pi^2}\cos(\frac{3\pi x}{2}) - \frac{4}{25\pi^2}\cos(\frac{5\pi x}{2}) \dots \quad \star$$

The graph of the cosine series (taking the first 6 terms) is:



Note that we have now approximated f entirely with cosines!

[46]You can now do Q 109 111 112

LECTURE 47
FORCED OSCILLATIONS AND FOURIER SERIES

Suppose that a function f has period $T = 2L$. Then f may be approximated by the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) \tag{1}$$

where the Fourier coefficients a_0, a_n, and b_n are given by

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \dots) \\ b_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \dots) \end{aligned}\right\} \tag{2}$$

Let us now start using Fourier series to solve some special problems. Fourier series will prove an essential tool in solving partial differential equations for the remainder of the course. But today we will use Fourier series to help us solve ordinary differential equations

$$my'' + cy' + ky = f(t)$$

governing oscillating systems where the forcing function $f(t)$ is exotic and periodic. We have already covered this theory when $f(t)$ is a standard function. First some revision:

Example 1 Find the general solution of the differential equation

$$y'' + 9y = 60\sin(2t)$$

$$\star \quad y = A\sin(3t) + B\cos(3t) + 12\sin(2t) \quad \star$$

Note in the above that since the y' term is missing from the D.E. we can drop the cos term from the guess for y_p and make a simpler guess of $y_p = \alpha\sin(2t)$ instead.

We can solve any differential equation of the form $y'' + 9y = 60\sin(\omega t)$ in the same manner, noting that in the special case where $\omega = 3$ we will need to modify our guess for y_p to $y_p = \{\alpha\sin(3t) + \beta\cos(3t)\}\, t$ and the system will then suffer from resonance.

We now turn to a similar situation except that the forcing function is no longer sinusoidal but rather is simply a random periodic function. The method of undertermined coefficients will probably no longer work as the guess for y_p is no longer obvious. Laplace transforms are an option but we have seen that they can get messy. Our method of attack using Fourier series is simple.

We decompose the function $f(t)$ into an infinite sum of sines and or cosines using the theory of Half Range expansions. We then solve the differential equation with the RHS $f(t)$ replaced by the nth term of its Fourier series. The final solution is then expressed as a series of the individual smaller solutions. An issue of particular concern is which Fourier component of the solution is closest to resonance and hence provides the largest contribution to the particular solution.

This is best explained via an example. There will also be a few more in the problem class.

Example 2 Suppose that $f(x) = \begin{cases} 1 - x & 0 < x < 2\ ; \\ \\ f(x+2) & \text{otherwise.} \end{cases}$

and consider the differential equation

$$y'' + 484y = f(x).$$

(a) Show that the Fourier series of f is

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(n\pi x).$$

(b) Show that a particular solution to $y'' + 484y = \dfrac{2}{n\pi}\sin(n\pi x)$ is

$$y_n = \frac{2}{n\pi(484 - (n\pi)^2)} \sin(n\pi x).$$

(c) Hence show that the solution to $y'' + 484y = f(x)$ is

$$y = A\cos(22x) + B\sin(22x) + \sum_{n=1}^{\infty} \frac{2}{n\pi(484 - (n\pi)^2)} \sin(n\pi x)$$

(d) Explain why the seventh term in the above expansion will dominate the series solution for y_p. Find the sixth, seventh and eighth coefficients in the series for y_p.

(a) We **ALWAYS** start with a sketch and hope for symmetry.

★ $n = 7$ comes closest to resonance , $B_6 = .00082,\ B_7 = 0.23,\ B_8 = -0.00054$ ★

[47]You can now do Q 113

LECTURE 49
PARTIAL DIFFERENTIAL EQUATIONS

The equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

is called the one-dimensional wave equation.

It models the oscillations of a tightly stretched string.

The solution $u(x,t)$ describes the displacement of the string at position x and time t.

The constant c is determined by the physical characteristics of the string.

The general solution to the one dimensional wave equation is

$$u = \phi(x+ct) + \psi(x-ct) \tag{2}$$

where ϕ and ψ are arbitrary functions. (This is D'Alembert's solution of the wave equation)

If we have an initial displacement of

$$u(x,0) = f(x)$$

then D'Alembert's solution to the one dimensional wave equation is

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

If we have an initial displacement of

$$u(x,0) = f(x) \quad \text{and an initial velocity of} \quad \frac{\partial u(x,0)}{\partial t} = g(x)$$

then D'Alembert's solution to the one dimensional wave equation is

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

Basic concepts

An equation involving one or more partial derivatives of an (unknown) function of two or more independent variables is called a **partial differential equation**.

- The order of the highest derivative is called the **order** of the equation.
- A p.d.e (partial differential equation) is **linear** if it is linear in all terms involving u and its partial derivatives
- A **solution** of a p.d.e. is a function of several variables which satisfies the equation.

Some examples

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ one dimensional wave equation

(2) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ one dimensional heat equation

(3) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ two dimensional Laplace equation

(4) $$(\frac{\partial u}{\partial t})^2 = \frac{\partial u}{\partial x}$$

(1)-(3) are second order, linear. (4) is first order non-linear.

We may, depending on the problem, have **boundary conditions** (the solution has some given value on the boundary of some domain) or **initial conditions** (where the value of the solution will be given at some initial time, *e.g.* $t = 0$).

There are few general methods for solving p.d.e's. Sometimes simple partial integration will work and we look at some elementary techniques in this lecture. However the main tool is a process called separation of variable which (together with the associated use of Fourier series) will be examined in detail for the remainder of the course.

Example 1 Solve the partial differential equation

$$\frac{\partial^2 u}{\partial x \partial y} = \cos(y).$$

Check that your solution is correct.

$\bigstar \quad u(x,t) = x\sin(y) + F(x) + G(y) \quad \bigstar$

We see from the above example that there is a lot of freedom (probably too much) in the solution of a p.d.e. In the above solution F and G can be **any** functions! This is why boundary and initial conditions play such a central role in the theory of p.d.e's.

Example 2 Verify that $u(x,t) = e^{-t}\sin(3x)$ is a solution to the one dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{9}\frac{\partial^2 u}{\partial x^2}$$

★

Homework: Verify that $u(x,t) = e^{-100t}\sin(30x)$ is also a solution to the same p.d.e..

$$\frac{\partial u}{\partial t} = \frac{1}{9}\frac{\partial^2 u}{\partial x^2}$$

★

This is the the problem with p.d.e's, there are so many different looking solutions!

Example 3 Consider the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = 25\frac{\partial^2 u}{\partial x^2}$$

with initial displacment

$$u(x,0) = 6e^x \quad \text{and an initial velocity of} \quad \frac{\partial u(x,0)}{\partial t} = 10\cos(x)$$

Show that D'Alembert's solution is

$$u(x,t) = 3e^{x+5t} + 3e^{x-5t} + \sin(x+5t) - \sin(x-5t).$$

Verify that both the p.d.e. and the initial conditions are satisfied.

The full proof for D'Alembert's solution (making extensive use of the chain rule) is in your printed notes. D'Alembert's solution **only** works for the wave equation and will not be examined.

In Math2019 we instead focus on the more general technique of separation of variables. We will apply separation of variables not only to the wave equation but also to a host of other p.d.e's. The theory of separation of variables is our last topic in Math2019 and usually appears as a complete question (one out of four) in the final examination.

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[49] You can now do Q 114 115

LECTURE 50 and LECTURE 51 (Double Lecture)
WAVE EQUATION AND SEPARATION OF VARIABLES

The equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

is called the one-dimensional wave equation and governs the vibration of an elastic string. The solution $u(x,t)$ describes the displacement of the string at position x at time t.

The boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \qquad \text{for all time } t$$

specify that the string is fixed at two endpoints $x = 0$ and $x = L$ for all time.

Initial conditions take the form:

$$\text{initial deflection} \qquad u(x,0) = f(x)$$

$$\text{initial velocity} \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

To solve we implement the process of separation of variables:

Step 1. Apply the **method of separation of variables** to obtain two ordinary differential equations.
Step 2. Determine the solution of these two equations that satisfy the boundary conditions.
Step 3. Combine these solutions so that the result will be a solution of the p.d.e. which also satisfies the initial conditions.

Solutions (eigenfunctions) take the form

$$u_n(x,t) = (C_n \cos\lambda_n t + D_n \sin\lambda_n t)\sin\frac{n\pi x}{L} \quad \text{for} \quad n = 1, 2, 3, \cdots$$

where the λ_n (eigenvalues) are given by $\lambda_n = cn\pi/L$.

The general solution is the superposition of all the eigenfunctions and takes the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (C_n \cos\lambda_n t + D_n \sin\lambda_n t)\sin\frac{n\pi x}{L}$$

The initial velocity and displacement are used in the calculation of C_n and D_n and will require the use of Fourier series and half range expansions when the initial conditions are non-sinusoidal.

Without doubt the major tool used when solving partial differential equations is separation of variables. Although several centuries old, this technique still enjoys a widespread use through all the major technical disciplines as well as all sectors of industry, from steel making to option pricing. A summary has been supplied at the start of the lecture and a more detailed development is in your printed notes. However you are encouraged not to simply memorise these results!

Separation of Variables is a *process* and the most effective approach is to carefully work through each problem as it is presented to you. Although there will be minor changes from example to example the overall technique does not change! We will do a variety of problems over the next few lectures. You will observe that in each case the question will be mechanically worked over without reference to external formulae. It is however a very technical and demanding method of solution and you must take care with each and every step. Lets start with a relatively simple example of the wave equation governing the vibration of an elastic string. You will see that a single example can easily take a whole lecture!

WAVE EQUATION AND SEPARATION OF VARIABLES

The equation

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The boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \qquad \text{for all time } t$$

specify that the string is fixed at two endpoints $x = 0$ and $x = L$ for all time.

Initial conditions take the form:

$$\begin{aligned} &\text{initial deflection} && u(x,0) = f(x) \\ &\text{initial velocity} && \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \end{aligned}$$

Let's take a look at the evolution of the string for a few different initial displacements. This is the information that the solution of the wave equation provides.

https://academo.org/demos/1D-wave-equation/

In our first example below we have $L = \pi$ and $c = 1$.

Example 1 Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

where

$u(0,t) = u(\pi,t) = 0$ for all t (tied down at 0 and π)

$u(x,0) = 5\sin(x) - 3\sin(7x)$ (initial displacement)

$u_t(x,0) = 0$ (initial velocity)

DISCUSSION

To solve we implement the process of separation of variables:

Step 1. Assume the variables separate to convert the p.d.e. into two ordinary differential equations.
Step 2. Determine the solution of these two equations that satisfy the boundary conditions.
Step 3. Combine these solutions so that the result will be a solution of the p.d.e. which also satisfies the initial conditions.

i) By assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$F'' - kF = 0$$

and

$$G'' - kG = 0$$

for k constant.

ii) By implementing the boundary condition $u(0,t) = u(\pi,t) = 0$ show that

$$F(0) = F(\pi) = 0$$

iii) By solving for F with $k = 0$ and $k > 0$ show that non-trivial solutions will only arise from $k < 0$. (We will say that $k = -\rho^2$).

iv) By implementing $F(0) = F(\pi) = 0$ with $k = -\rho^2$ show that

$$u_n(x,t) = \sin(nx)\{C_n \cos(nt) + D_n \sin(nt)\}$$

v) Verify that this solution in iv) is consistent with the formula

$$u_n(x,t) = (C_n \cos \lambda_n t + D_n \sin \lambda_n t) \sin \frac{n\pi x}{L} \quad \text{for} \quad n = 1, 2, 3, \cdots$$

where the λ_n (eigenvalues) are given by $\lambda_n = cn\pi/L$. (Note that the examiners will not look kindly upon students who simply memorise formulae. You must prove your results!)

vi) Using the initial velocity $u_t(x, 0) = 0$ show that

$$u_n(x,t) = C_n \sin(nx) \cos(nt)$$

vi) By summing appropriate $u_n's$ and applying the initial displacement

$$u(x,0) = 5\sin(x) - 3\sin(7x)$$

show that the final solution is

$$u(x,t) = 5\sin(x)\cos(t) - 3\sin(7x)\cos(7t)$$

Note that we always apply the initial displacement last.

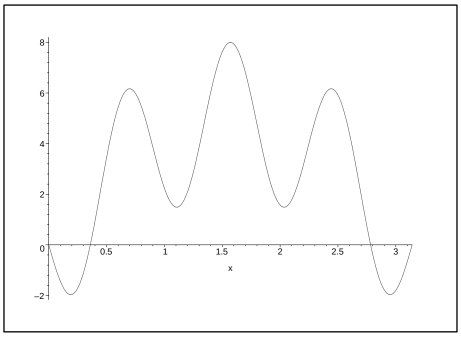
We can use our solution $u(x,t) = 5\sin(x)\cos(t) - 3\sin(7x)\cos(7t)$ in several interesting ways.

Suppose that we substitute $t = 0.15$ into our solution. That is, suppose that we focus on what is happening after $t = 0.15$ seconds. Then

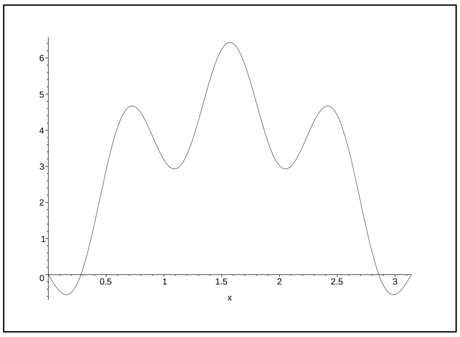
$$u(x, 0.15) = 5\sin(x)\cos(0.15) - 3\sin(7x)\cos(1.05) \approx 4.94\sin(x) - 1.49\sin(7x).$$

This is then the shape of the string after $t = 0.15$ seconds:

t=0



t=0.15



If we further specify that we are positioned half way along the string so that $x = \dfrac{\pi}{2}$ then

$$u(\frac{\pi}{2}, 0.15) = 5\sin(\frac{\pi}{2})\cos(0.15) - 3\sin(\frac{7\pi}{2})\cos(1.05) \approx 5.44$$

which can be interpreted as the displacement half way along the string after $t = 0.15$ seconds.

★

We turn now to a new issue. What if the initial displacement is not sinusoidal?

Example 2 Solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

where

$u(0,t) = u(\pi,t) = 0$ for all t (tied down at 0 and π)

$$u(x,0) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2}; \\ \pi - x, & \frac{\pi}{2} \le x \le \pi. \end{cases}$$

$u_t(x,0) = 0$ (initial velocity)

Sketch:

This is identical to before except that we have a different initial displacement.

Thus we will again reach the point

$$u_n(x,t) = C_n \sin(nx)\cos(nt)$$

Clearly the initial displacement is not sinusoidal so no finite linear combination of the $u_n's$ will do the job. So what we do is take the lot of them! That is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin(nx)\cos(nt)$$

Something very interesting then happens:

$$\bigstar \quad u(x,t) = \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi} \sin(\frac{n\pi}{2}) \right) \sin(nx) \cos(nt) \quad \bigstar$$

[5051] You can now do Q 116,119

LECTURE 52
HEAT EQUATION

The equation
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$
is called the one-dimensional heat equation. It governs the heat flow across a homogenous bar where c is determined by the thermal properties of the bar.

The boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \qquad \text{for all time } t$$

specify that the bar has length L and that the temperature is maintained at zero on the ends of the bar for all time. Eventually all the heat will be drawn out of the bar.

Initial conditions take the form:

$$\text{initial temperature distribution} \qquad u(x,0) = f(x)$$

Solutions (eigenfunctions) are obtained via separation of variables and take the form

$$u_n(x,t) = B_n \sin \frac{n\pi x}{L} e^{-\lambda_n^2 t} \quad \text{for} \quad n = 1, 2, 3, \cdots$$

where the λ_n (eigenvalues) are given by $\lambda_n = cn\pi/L$.

The general solution is the superposition of all the eigenfunctions and takes the form

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\lambda_n^2 t}$$

The initial temperature distribution is used to calculate the $B_n\,'s$ and will require the use of Fourier series and half range expansions when the initial distribution is non-sinusoidal.

Example 1 Solve the heat equation

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$$

where

$$u(0,t) = u(10,t) = 0 \qquad \text{for all } t \qquad \text{(maintained at zero degrees on the ends)}$$

With initial temperature distribution given by

$$u(x,0) = h(x) = \begin{cases} x, & 0 \le x < 5; \\ \\ 10 - x, & 5 \le x \le 10. \end{cases}$$

DISCUSSION

i) By assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$F'' - kF = 0$$

and

$$G' - 4kG = 0$$

for k constant. (Note that the D.E. for G is first order!)

ii) By implementing the boundary condition $u(0,t) = u(10,t) = 0$ show that

$$F(0) = F(10) = 0$$

iii) By solving for F with $k = 0$ and $k > 0$ show that non-trivial solutions will only arise from $k < 0$. (We will say that $k = -\rho^2$).

iv) By implementing $F(0) = F(10) = 0$ with $k = -\rho^2$ show that

$$u_n(x,t) = B_n \sin(\frac{n\pi x}{10})e^{-\frac{n^2\pi^2}{25}t}$$

v) Verify that this solution in iv) is obtainable through the equations presented at the start of the lecture. (Note that the examiners will not look kindly upon students who simply memorise formulae. You must prove your results!)

vi) Discuss the behaviour of the u_n as $t \to \infty$

vii)Assuming a final solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{10}) e^{-\frac{n^2\pi^2}{25}t}$$

express B_n in integral form.

$$\bigstar \quad B_n = \frac{1}{5}\int_0^{10} h(x)\sin(\frac{n\pi x}{10})dx \quad \bigstar$$

Lets take a look at the evolution of the temperature for different initial temperature distributions:

http://mathlets.org/mathlets/heat-equation/

[52] You can now do Q 117,118

LECTURE 53
HEAT EQUATION WITH INSULATED ENDS (ADIABATIC)

The equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

is called the one-dimensional heat equation. It governs the heat flow across a homogenous bar where c is determined by the thermal properties of the bar.

The adiabatic boundary conditions

$$u_x(0,t) = 0, \quad u_x(L,t) = 0 \qquad \text{for all time } t$$

maintain insulated endpoints at $x = 0$ and $x = L$ so that there is no heat flow across the ends of the rod. Eventually all the heat will be evenly distributed across the bar.

Initial conditions take the form:

$$\text{initial temperature distribution} \qquad u(x,0) = f(x)$$

Solutions (eigenfunctions) are obtained via separation of variables and take the form of the constant function A_0 together with

$$u_n(x,t) = A_n \cos \frac{n\pi x}{L} e^{-\lambda_n^2 t} \quad \text{for} \quad n = 1, 2, 3, \cdots$$

where the λ_n (eigenvalues) are given by $\lambda_n = cn\pi/L$.

The general solution is the superposition of all the eigenfunctions and takes the form

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-\lambda_n^2 t}$$

The initial temperature distribution is used to calculate the $A_n\,'s$ and will require the use of Fourier series and half range expansions when the initial temperature distribution is non-sinusoidal. The steady state temperature is A_0.

Example 1 The temperature in a bar of length π satisfies the heat equation

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ is the temperature. The bar is insulated so that the flux of heat at each end is zero. Hence:

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0 \qquad \text{for all } t.$$

You are also given that the initial temperature distribution is given by

$$u(x,0) = f(x) = \begin{cases} 0 & 0 \le x < \dfrac{\pi}{2}; \\ 1 & \dfrac{\pi}{2} \le x \le \pi. \end{cases}$$

Express the general solution $u(x,t)$ as a Fourier cosine series and find the steady state temperature of the system.

DISCUSSION

A sketch of the initial temperature distribution is:

You need to be careful here, as $k = 0$ actually yields non-trivial solutions! Let us step carefully through the method of separation of variables.

i) By assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$F'' - kF = 0$$

and

$$G' - 9kG = 0$$

for k constant. (Note that the D.E. for G is first order!)

ii) By implementing the boundary condition

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0 \qquad \text{for all } t.$$

show that

$$F'(0) = F'(\pi) = 0.$$

iii) Show that $k > 0$ (say $k = \rho^2$) yields the trivial solution for F.

iv) Show that $k = 0$ yields a constant solution for both F and G and hence that

$$u_0(x, t) = A_0.$$

v) Show that $k < 0$ (say $k = -\rho^2$) yields the solution

$$u_n(x,t) = A_n \cos(nx) e^{-9n^2 t}.$$

Taking the sum of all possible solutions we have a general solution given by

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) e^{-9n^2 t}$$

We now apply the initial temperature distribution. This is tricky and will involve Fourier cosine series:

Recall that the initial temperature distribution is given by

$$u(x,0) = f(x) = \begin{cases} 0 & 0 \le x < \dfrac{\pi}{2}; \\ 1 & \dfrac{\pi}{2} \le x \le \pi. \end{cases}$$

Thus setting $t = 0$ we have

$$A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) = f(x).$$

This means that A_0 and A_n are the Fourier cosine coefficients of f.

The even periodic extension of f has sketch:

Our final solution is therefore

$$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2}{n\pi} \sin(\frac{n\pi}{2}) \cos(nx) e^{-9n^2 t} = \frac{1}{2} - \frac{2}{\pi} \left\{ \cos(x) e^{-9t} - \frac{\cos(3x)}{3} e^{-81t} + \frac{\cos(5x)}{5} e^{-225t} \right\}$$

vi) Discuss the behaviour of the $u(x,t)$ as $t \to \infty$.

[53] You can now do Q 120

LECTURE 55
OTHER P.D.E'S

The method of Separation of Variables is extremely versatile and may be used in a variety of different circumstances. The technique itself varies little, however the implementation of boundary and initial conditions can feel quite different as you move from problem to problem.

Example 1 The steady state distribution $u(x, y)$ of heat in an infinite slab of width 4 is given by Laplace's Equation

$$\frac{\partial u^2}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with boundary conditions:

(i) $u = 0$ when $x = 0 \quad \forall\ y > 0$. (Note that $\forall$ means 'for all')

(ii) $u = 0$ when $x = 4 \quad \forall\ y > 0$.

(iii) u is bounded as $y \to \infty$.

(iv) $u = 1$ when $y = 0, \quad 0 < x < 4$.

DISCUSSION

i) By assuming a solution of the form $u(x,y) = F(x)G(y)$ show that

$$F'' - kF = 0$$

and

$$G'' + kG = 0$$

for k constant.

ii) By implementing the boundary conditions $u(0,y) = u(4,y) = 0$ show that

$$F(0) = F(4) = 0$$

iii) By solving for F with $k = 0$ and $k > 0$ show that non-trivial solutions will only arise from $k < 0$. (We will say that $k = -\rho^2$).

iv) By implementing $F(0) = F(4) = 0$ with $k = -\rho^2$ show that

$$u_n(x, y) = B_n \sin(\frac{n\pi x}{4})e^{-\frac{n\pi}{4}y}$$

v) Hence find the solution which also satisfies the final condition $u(x,0) = 1$.

$$\bigstar \quad u(x,y) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\left(\frac{(2k+1)\pi x}{4}\right)e^{-\frac{(2k+1)\pi}{4}y} \quad \bigstar$$

SOME FINAL INFORMATION

1. Please check online that all your Math2019 marks are recorded correctly.

2. Read the school pages on additional assessment/special consideration so that you are fully aware of the rules that apply.

3. Note in particular that students with a final mark in the range 45-49 are automatically granted additional assessment and be aware of the strict dates for the additional assessment exams.

4. Past Math2019 final exams and solutions are available on Moodle.

5. The final exam is 2 hours long with 4 questions. Make sure you turn up at the right time in the right location. Check your exam timetable!!

6. Please start a new book for each of the 4 questions.

7. Make sure your calculator has a UNSW APPROVED sticker (available from the School of Mathematics office) or you will not be allowed to use it during the exam.

8. Please take the time to complete all surveys regarding the administration and teaching of the course.

9. A consultation roster will shortly be posted on Moodle.

Good Luck!

Milan Pahor

[55] You can now do Q 121