LECTURE 49
PARTIAL DIFFERENTIAL EQUATIONS

The equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

is called the one-dimensional wave equation.

It models the oscillations of a tightly stretched string.

The solution $u(x,t)$ describes the displacement of the string at position x and time t.

The constant c is determined by the physical characteristics of the string.

The general solution to the one dimensional wave equation is

$$u = \phi(x+ct) + \psi(x-ct) \tag{2}$$

where ϕ and ψ are arbitrary functions. (This is D'Alembert's solution of the wave equation)

If we have an initial displacement of

$$u(x,0) = f(x)$$

then D'Alembert's solution to the one dimensional wave equation is

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

If we have an initial displacement of

$$u(x,0) = f(x) \quad \text{and an initial velocity of} \quad \frac{\partial u(x,0)}{\partial t} = g(x)$$

then D'Alembert's solution to the one dimensional wave equation is

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

Basic concepts

An equation involving one or more partial derivatives of an (unknown) function of two or more independent variables is called a **partial differential equation**.

- The order of the highest derivative is called the **order** of the equation.
- A p.d.e (partial differential equation) is **linear** if it is linear in all terms involving u and its partial derivatives
- A **solution** of a p.d.e. is a function of several variables which satisfies the equation.

Some examples

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ one dimensional wave equation

(2) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ one dimensional heat equation

(3) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ two dimensional Laplace equation

(4) $$(\frac{\partial u}{\partial t})^2 = \frac{\partial u}{\partial x}$$

(1)-(3) are second order, linear. (4) is first order non-linear.

We may, depending on the problem, have **boundary conditions** (the solution has some given value on the boundary of some domain) or **initial conditions** (where the value of the solution will be given at some initial time, *e.g.* $t = 0$).

There are few general methods for solving p.d.e's. Sometimes simple partial integration will work and we look at some elementary techniques in this lecture. However the main tool is a process called separation of variable which (together with the associated use of Fourier series) will be examined in detail for the remainder of the course.

Example 1 Solve the partial differential equation

$$\frac{\partial^2 u}{\partial x \partial y} = \cos(y).$$

Check that your solution is correct.

$$\frac{\partial^2 u}{\partial x \, \partial y} = \cos(y)$$

$$\frac{\partial u}{\partial x} = \int \cos(y) \, dy$$

$$= \sin(y) + f(x)$$

$$u = \int \sin(y) + f(x) \, dx$$

$$\therefore u(x,y) = x \sin(y) + F(x) + G(x)$$

$$\bigstar \quad u(x,t) = x\sin(y) + F(x) + G(y) \quad \bigstar$$

We see from the above example that there is a lot of freedom (probably too much) in the solution of a p.d.e. In the above solution F and G can be **any** functions! This is why boundary and initial conditions play such a central role in the theory of p.d.e's.

Example 2 Verify that $u(x,t) = e^{-t}\sin(3x)$ is a solution to the one dimensional heat equation
$$\frac{\partial u}{\partial t} = \frac{1}{9}\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = -e^{-t}\sin(3x)$$

$$\therefore -\frac{\partial u}{\partial t} = u \quad \text{— (1)}$$

$$\frac{\partial u}{\partial x} = 3e^{-t}\cos(3x)$$

$$\frac{\partial^2 u}{\partial t^2} = -9e^{-t}\sin(3x)$$

$$\therefore -\frac{1}{9}\frac{\partial^2 u}{\partial t^2} = u \quad \text{— (2)}$$

$(1) \to (2)$: $$\frac{\partial u}{\partial t} = \frac{1}{9}\frac{\partial^2 u}{\partial t^2}$$

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Homework: Verify that $u(x,t) = e^{-100t}\sin(30x)$ is also a solution to the same p.d.e..

$$\frac{\partial u}{\partial t} = \frac{1}{9}\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = -100\,e^{-100t}\sin(30x)$$

$$\frac{\partial^2 u}{\partial x^2} = -900\,e^{-100t}\sin(30x)$$

$$\therefore \frac{\partial u}{\partial t} = \frac{1}{9}\frac{\partial^2 u}{\partial x^2}$$

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This is the the problem with p.d.e's, there are so many different looking solutions!

Example 3 Consider the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = 25\frac{\partial^2 u}{\partial x^2}$$

with initial displacment

$$u(x,0) = 6e^x \text{ and an initial velocity of } \frac{\partial u(x,0)}{\partial t} = 10\cos(x)$$

Show that D'Alembert's solution is

$$u(x,t) = 3e^{x+5t} + 3e^{x-5t} + \sin(x+5t) - \sin(x-5t).$$

Verify that both the p.d.e. and the initial conditions are satisfied.

$$u(x,t) = \phi(x+5t) + \psi(x-5t)$$

$$\therefore u(x,0) = \phi(x) + \psi(x) = 6e^x \quad \text{—} ①$$

$$\frac{\partial u(x,t)}{\partial t} = 5\frac{\partial \phi}{\partial t}(x+5t) - 5\frac{\partial \psi}{\partial t}(x-5t)$$

$$\therefore \frac{\partial u(x,0)}{\partial t} = 5\phi'(x) - 5\psi'(x) = 10\cos(x)$$

$$\phi(x) - \psi(x) = 2\sin(x) \quad \text{—} ②$$

$$① + ② : \quad \phi(x) = 3e^x + \sin(x)$$

$$\therefore \phi(x+5t) = 3e^{x+5t} + \sin(x+5t)$$

$$① - ② : \quad \psi(x) = 3e^x - \sin(x)$$

$$\therefore \psi(x-5t) = 3e^{x-5t} - \sin(x-5t)$$

$$\therefore u(x,t) = 3e^{x+5t} + 3e^{x-5t} + \sin(x+5t) - \sin(x-5t)$$

$$\therefore u(x,0) = 3e^{x} + 3e^{x} + \sin x - \sin x = 6e^{x}$$

$$u_t = 15e^{x+5t} - 15e^{x-5t} + 5\cos(x+5t) + 5\cos(x-5t)$$

$$\therefore u_t(x,0) = 15e^{x} - 15e^{x} + 5\cos x + 5\cos x = 10\cos x$$

$$u_{tt} = 75e^{x+5t} + 75e^{x-5t} - 25\sin(x+5t) + 25\sin(x-5t) \quad — (3)$$

$$u_{xx} = 3e^{x+5t} + 3e^{x-5t} - \sin(x+5t) + \sin(x-5t) \quad — (4)$$

(3) $\to$ (4): $$u_{xx} = \frac{1}{25} u_{tt}$$

$$\therefore u_{tt} = 25 u_{xx}$$

The full proof for D'Alembert's solution (making extensive use of the chain rule) is in your printed notes. D'Alembert's solution **only** works for the wave equation and will not be examined.

In Math2019 we instead focus on the more general technique of separation of variables. We will apply separation of variables not only to the wave equation but also to a host of other p.d.e's. The theory of separation of variables is our last topic in Math2019 and usually appears as a complete question (one out of four) in the final examination.

[49]You can now do Q 114 115