LECTURE 10
VECTOR AND SCALAR FIELDS

$$\nabla \phi = \text{grad } \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}.$$

grad : scalar to vector

$$\nabla \cdot \mathbf{F} = \text{div}\,\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

div : vector to scalar

$$\nabla \times \mathbf{F} = \text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

curl : vector to vector

Where the vector differential operator ∇ is given by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

We turn now to the theory of vector and scalar fields, a topic of immense application in Engineering for reasons which will soon become obvious. A scalar field assigns a scalar (think temperature) to each point in space. Imagine measuring the temperature at each point in the room. Each point would have a little tag designating it's temperature and as you moved around the space there would of course be subtle variations in the temperature. Similarly a vector field assigns a vector to each point in space. For example we could measure the wind velocity (speed plus direction) everywhere in the room. There would be big vectors near the AC ducts and small vectors in the corners of the room where there is little turbulence. Vector fields are the ideal tool for studying **flows** whether they be rivers, lava or weather patterns. These concepts may be applied in any dimension, however they are mostly used in $\mathbb{R}^2$ and $\mathbb{R}^3$. In this lecture we will simply get a feeling for how these fields are presented and manipulated. In the next two lectures we will focus on applications.

Example 1 Consider the scalar temperature field $T(x, y, z) = x^2 + y^2 + z^2$. Find the temperature at the point $P(3, 0, 4)$ and describe the surface of all points whose temperature is the same as that of P. (This is called a *level surface*).

$T(3,0,4) = 9 + 0 + 16 = 25^\circ c$

Level surface is all points on a sphere with radius 5

★ $T = 25$ and the level surface is all points on a sphere of radius 5 ★

Observe from the above that a scalar field is little more than a function from $\mathbb{R}^3$ or $\mathbb{R}^2$ into $\mathbb{R}$. Vector fields are a little trickier.

Example 2 Consider the vector field $\mathbf{F}(x, y, z) = (x^2 + y^2)\mathbf{i} + xyz\mathbf{j} - e^x z\mathbf{k}$. Find the vector at the point $Q(0, 2, 3)$.

$$\tilde{F}(0,2,3) = 4\hat{\imath} + 0\hat{\jmath} - e^0 3\hat{k}$$

$$= \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

★ $4\mathbf{i} - 3\mathbf{k} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$ ★

Observe how neatly vector fields assign vectors to points!

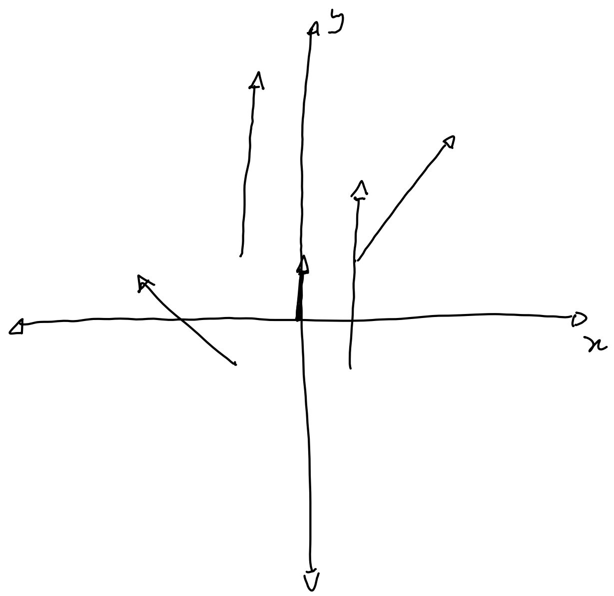
Drawing vector and scalar field is a messy and time consuming process best left to computers. Let's do one in two dimensions just to get a feeling for what they look like:

Example 3 Consider the vector field $\mathbf{T}(x, y) = (x + y)\mathbf{i} + (x^2 + y^2 + 1)\mathbf{j}$ in $\mathbb{R}^2$. Sketch the vectors of the field at the set of points $\{(0, 0), (1, 1), (-1, 1), (1, -1), (-1, -1)\}$.

$T(0,0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $T(1,1) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$T(-1,1) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, $T(1,-1) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$T(-1,-1) = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$



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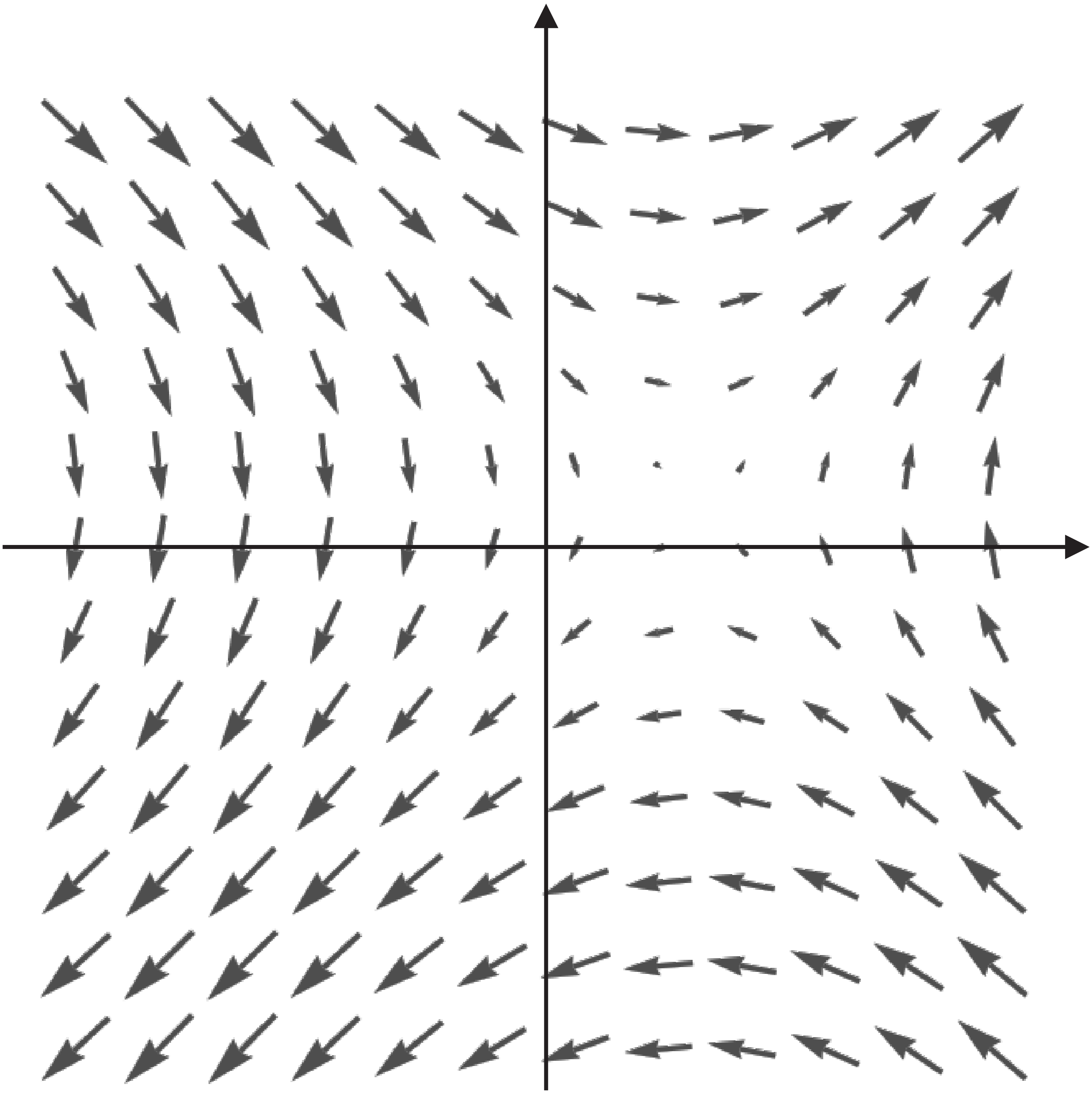
Please observe very carefully in the above analysis that we simply treat vectors as instruments that specify magnitude and direction. They are definitely not stuck at the origin. Indeed they are not fixed anywhere but rather are free to roam across space doing their work wherever it is necessary to point in a particularly direction. The x coordinate tells you how far to go left or right in the x direction and the y coordinate tells you how far to go up or down in the y direction. Where this actually happens is up to you.

Remember that vector fields are an attempt to mathematically analyse flows. An example of a more complicated vector field is:



We now turn to three operators grad, div and curl. These transform vector fields into scalar fields and vice versa. Today we will only look at the technicalities of their calculations.

Example 4 Let $\phi(x,y,z) = x^2y^3z + 2y$ be a scalar field. Calculate grad(ϕ).

$$\nabla\phi = \text{grad}\,\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}. \qquad \textbf{grad : scalar to vector}$$

Note that the differential operator $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ can be used in one way or another to define each of grad, div, and curl. For grad we simply allow ∇ to operate on ϕ.

$$\text{grad}(\phi) = \nabla\phi = \begin{pmatrix} 2xy^3z \\ 3x^2y^2z+2 \\ x^2y^3 \end{pmatrix}$$

$$\bigstar \quad 2xy^3z\mathbf{i} + (3x^2y^2z+2)\mathbf{j} + x^2y^3\mathbf{k} = \begin{pmatrix} 2xy^3z \\ 3x^2y^2z+2 \\ x^2y^3 \end{pmatrix} \quad \bigstar$$

Example 5 Let $\mathbf{F}(x,y,z) = (x^2+y^2)\mathbf{i} + xyz\mathbf{j} - e^xz\mathbf{k} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ be a vector field. Find div($\mathbf{F}$). (the divergence of the vector field $\mathbf{F}$)

$$\nabla\cdot\mathbf{F} = \text{div}\,\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \qquad \textbf{div : vector to scalar}$$

Here we are just taking the scalar product of ∇ and $\mathbf{F}$

$$\text{div}(\vec{F}) = \nabla\vec{F} = 2x + xz - e^x$$

$$\bigstar \quad 2x + xz - e^x \quad \bigstar$$

Example 6 Given the vector field $\mathbf{F}(x,y,z) = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = 3y\mathbf{i} + xz\mathbf{j} - x^3\mathbf{k}$ find curl($\mathbf{F}$).

$$\nabla \times \mathbf{F} = \operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

curl : vector to vector

degree of twirling / spinning

For curl we take the cross product of ∇ and $\mathbf{F}$ and thus we need to find a determinant.

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & xz & -x^3 \end{vmatrix}$$

$$= \begin{pmatrix} -x \\ 3x^2 \\ z-3 \end{pmatrix}$$

$$\bigstar \quad -x\mathbf{i} + 3x^2\mathbf{j} + (z-3)\mathbf{k} = \begin{pmatrix} -x \\ 3x^2 \\ z-3 \end{pmatrix} \quad \bigstar$$

We will fully explore the applications of these processes in the next few lectures . We close however with some special relations between grad, div and curl.

We close with an example of a simple proof:

Theorem $\nabla \times (\nabla\phi) = \mathbf{0}$. (That is curl(grad($\phi$))=**0**)

Proof: Let $\phi(x, y, z)$ be a scalar field. Then

$$\text{grad}(\phi) = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\text{curl}(\text{grad}(\phi)) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \begin{pmatrix} \frac{\partial^2\phi}{\partial z\partial y} - \frac{\partial^2\phi}{\partial y\partial z} \\ \frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x} \\ \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \vec{0}$$

[10]You can now do Q 54,59,60