Tutorial Problems

MATH2019 Engineering Mathematics 2E

Term 1, 2019

Schedule

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Problems marked with a star ⋆ are core recommended exercises, to be attempted prior to the relevant tutorial.

1 Partial Differentiation

1. Verify that $w_{xy} = w_{yx}$.

(a) $w = \ln(2x + 3y)$

(b) $w = e^x \sinh y + \cos(2x - 3y)$

2⋆. Find $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$.

(a) $f(x, y) = \ln(x^2 + y^2)$

(b) $f(x, y) = x^2 y + \cos y + y \sin x$

(c) $f(x, y) = \tan^{-1} \dfrac{y}{x}$

3. Show that the following functions satisfy the Laplace equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

(a)⋆ $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z$

(b) $f(x, y) = \ln \sqrt{x^2 + y^2}$

(c)⋆ $f(x, y, z) = e^{3x+4y} \cos 5z$

4. Show that the following functions are solutions of the wave equation:

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}.$$

(a) $w = \cos(2x + 2ct)$

(b) $w = \ln(3x + 3ct)$

5. Use the chain rule to express $\dfrac{df}{dt}$ in terms of t. Then evaluate $\dfrac{df}{dt}$ at the given value of t.

(a) $f(x, y, z) = \ln(x + y + z)$ for $x = \cos^2 t$, $y = \sin^2 t$, $z = t$ at $t = \pi$.

(b) $f(x, y) = x^2 + y^2$ for $x = \cos t$, $y = \sin t$ at $t = \pi$.

6. Find $\dfrac{du}{dt}$ if $u = x^2 + e^{y^2}$, $x = \sin 2t$, and $y = \cos t^2$.

7★. Find $\dfrac{\partial z}{\partial v}$ if $z = x^2 + 2xy$, $x = u \cos v$, and $y = u \sin v$. (Express your answer in terms of x and y.)

8. Find $\dfrac{\partial z}{\partial u}$ when $u = 0$, $v = 1$ if $z = \sin xy + x \sin y$ for $x = u^2 + v^2$ and $y = uv$.

9. Find $\dfrac{\partial w}{\partial v}$ when $u = 0$, $v = 0$ if

$$w = (x^2 + y - 2)^4 + (x - y + 2)^3, \quad x = u - 2v + 1, \quad y = 2u + v - 2.$$

10★. Find $\dfrac{\partial w}{\partial x}$ at the point $(x, y, z) = (1, 1, 1)$ if

$$w = \cos uv, \quad u = xyz, \quad v = \frac{\pi}{4(x^2 + y^2)}.$$

11. If $z = f(t)$ and $t = \dfrac{x + y}{xy}$, show that $x^2 \dfrac{\partial z}{\partial x} = y^2 \dfrac{\partial z}{\partial y}$.

12. If a and b are constants, $w = f(u)$, and $u = ax + by$, show that $a \dfrac{\partial w}{\partial y} = b \dfrac{\partial w}{\partial x}$.

13★. If $w = f(u, v)$, $u = x + y$, and $v = x - y$, show that $\dfrac{\partial w}{\partial x} \dfrac{\partial w}{\partial y} = \left(\dfrac{\partial f}{\partial u}\right)^2 - \left(\dfrac{\partial f}{\partial v}\right)^2$.

14. If we substitute polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$ in a continuous function $w = f(x, y)$ that has continuous partial derivatives, show that

$$\frac{\partial w}{\partial r} = f_x \cos \theta + f_y \sin \theta.$$

Taylor expansion

$$\boxed{\begin{aligned} f(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &\quad + \frac{1}{2!}\left[f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2\right] + \cdots . \end{aligned}}$$

15. Determine the Taylor series expansion of $f(x) = \sin x$ about

(a) $x = 0$,

(b) $x = \pi/2$,

including the first two non-zero terms in each case. Sketch $f(x)$ and the two truncated expansions for $0 \leq x \leq \pi$.

16. Determine the Taylor series expansion of $f(x, y) = x^2 y$ about $(1, 2)$, including terms to 27th order.

17. Using a Taylor series in two variables, show that for small x and y we may make the following approximations.

(a) $e^x \sin y \sim y + xy$

(b) $e^x \ln(1 + y) \sim y + xy - \dfrac{y^2}{2}$

18. Expand $\cos(2x - y)$ in a Taylor series in two variables, including quadratic terms, about

(a) $(0, 0)$,

(b) $(0, -\pi/2)$.

19★. Determine the Taylor expansion of $e^{x+y} \cos y$ about the point $(1, 0)$, up to and including quadratic terms.

20. Expand $\cos(2x-y)$ about $(\pi/4, \pi/4)$ up to and including second order terms using Taylor's series for functions of two variables.

21★. Expand $\ln(x^2 + y^2)$ about $(1, 0)$ up to and including second order terms, using the Taylor series for functions of two variables. Then use your result to find an approximate value for $\ln(1.1^2 + 0.1^2)$.

22. Calculate the Taylor expansion up to and including second order terms of the function

$$z = F(x, y) = e^{-x} \sin y$$

about the point $(2, \pi/2)$. Use your result to estimate $F(1.92, \pi/2)$.

23. Find an approximate value for $\sqrt{(1.02)^3 + (1.97)^3}$ by using an appropriate Taylor series approximation.

24. Suppose T is to be found from the formula $T = x \cosh y$, where x and y are found to be 2 and $\ln 2$ with maximum possible errors of $|dx| = 0.04$ and $|dy| = 0.02$. Estimate the maximum possible error in the computed value of T.

25★. If $r = 5.0$ cm and $h = 12.0$ cm to the nearest millimetre, what should we expect the maximum percentage error in calculating $V = \pi r^2 h$ to be?

26. When an x-ohm and a y-ohm resistor are in parallel, the resistance R they produce will be calculated from the formula

$$\frac{1}{R} = \frac{1}{x} + \frac{1}{y}.$$

By what percentage will R change if x increases from 20 to 20.1 ohms and if y decreases from 25 to 24.9 ohms?

27. The specific gravity δ of a solid heavier than water is given by

$$\delta = \frac{W}{W - W_1},$$

where W and W_1 are its weight in air and water, respectively. If W and W_1 are observed to be 17.2 and 9.7 gm, find the maximum possible error in the calculated value of δ due to an error of 0.05 gm in each observation.

28. The pressure P, volume V and temperature T of a gas are related by the formula

$$PV = RT,$$

where R is a constant. If V is increased by 10%, and T decreased by 6%, find the percentage change in the pressure.

29★. When two resistances r_1 and r_2 are connected in parallel, the total resistance R (measured in ohms) is given by:

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}.$$

Suppose $r_1 = 6 \pm 0.1$ ohms and $r_2 = 9 \pm 0.03$ ohms.

(a) Calculate R.

(b) Show that $\dfrac{\partial R}{\partial r_1} = \dfrac{R^2}{r_1^2}$.

(c) Estimate the maximum possible error in the calculated value of R.

Leibnitz formula for differentiating an integral

$$\boxed{\frac{d}{dx}\int_u^v f(x,y)\,dy = \int_u^v \frac{\partial f}{\partial x}\,dy + f(x,v)\,\frac{dv}{dx} - f(x,u)\,\frac{dx}{du}.}$$

30. Show that

(a) $\dfrac{d}{dx}\displaystyle\int_1^x t^2\,dt = x^2$,

(b)★ $\dfrac{d}{dt}\displaystyle\int_{t^3}^2 \ln(1+x^2)\,dx = -3t^2\ln(1+t^6)$,

(() c) $\dfrac{d}{du}\displaystyle\int_0^{\pi/(2u)} u\sin(ux)\,dx = 0$.

31. Given that

$$\int_0^\infty e^{-ax}\sin(kx)\,dx = \frac{k}{a^2+k^2},$$

evaluate

(a) $\displaystyle\int_0^\infty xe^{-ax}\sin(kx)\,dx,$

(b) $\displaystyle\int_0^\infty xe^{-ax}\cos(kx)\,dx.$

32★. Evaluate

$$\int_0^\infty \frac{dx}{\alpha^2+x^2},$$

and hence use Leibniz's theorem to deduce that

$$\int_0^\infty \frac{dx}{(\alpha^2+x^2)^2} = \frac{\pi}{4\alpha^3}.$$

33. Given that

$$\int_0^\pi \frac{d\theta}{\alpha-\cos\theta} = \frac{\pi}{\sqrt{\alpha^2-1}}, \quad \text{for } \alpha > 1,$$

use Leibniz's Rule (for differentiating integrals) to evaluate

$$\int_0^\pi \frac{d\theta}{(\alpha-\cos\theta)^2} \quad \text{for } \alpha > 1.$$

34★. Let $I(t) = \displaystyle\int_0^\pi \cos(tx)\,dx$. Show by simple integration that

$$I(t) = \frac{\sin t\pi}{t}$$

and then by differentiation that

$$\frac{dI}{dt} = \frac{\pi\cos t\pi}{t} - \frac{\sin t\pi}{t^2}.$$

Then use this result, together with Leibniz's rule for the differentiation of an integral, to evaluate

$$\int_0^\pi x\sin(tx)\,dx.$$

35. Evaluate $\displaystyle\frac{d}{dt}\int_{\sqrt{t}}^{t}\frac{\cos tx}{x}\,dx$ by using Leibniz's rule.

2 Extreme Values

36. Test the following functions for maxima, minima and saddle points. Find the function values at these points.

(a) $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

(b) $f(x,y) = x^2 + xy + 3x + 2y + 5$

(c)★ $f(x, y) = x^2 + xy + y^2 + 3y + 3$

(d) $f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$

(e)★ $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$

(f) $f(x, y) = 4xy - x^4 - y^4$

37. Find all critical points of the function

$$f(x, y) = x^3 + y^3 - 3xy + 15,$$

and classify each one as a relative maximum, relative minimum, or saddle point.

38. Determine and classify the critical points (extrema) of the following function

$$g(x, y) = x^2 - Axy + y^2 + 7,$$

where A is a positive constant. Discuss separately the cases $0 < A < 2$, $A > 2$ and $A = 2$.

39. Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x, y) = xy$ has its extreme values.

40★. Find the extreme values of $f(x, y) = x^2 y$ on the line $x + y = 3$.

41. Use the method of Lagrange multipliers to find

(a) the minimum value of $x + y$ subject to the constraints $xy = 16$, $x > 0$, $y > 0$.

(b) the maximum value of xy subject to the constraint $x + y = 16$.

42. Find the dimensions of the closed circular can of smallest surface area whose volume is 16π cm^3.

43★. The temperature at the point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centred at the origin. What are the highest and lowest temperatures encountered by the ant?

44★. Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1, 1, 1)$.

45. Find points on the surface $z^2 = xy + 4$ closest to the origin.

46. The temperature at any point (x, y, z) in space is $T = 400xyz^2$. Find the highest temperature on the unit sphere $x^2 + y^2 + z^2 = 1$.

47. Find the maximum value of the function $f(x, y, z) = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.

3 Vector Field Theory

48*. Let $\boldsymbol{u} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{v} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{w} = 2\boldsymbol{j} + 3\boldsymbol{k}$. Find

(a) $(\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w}$,

(b) $\boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w})$,

(c) $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$,

(d) $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$,

(e) $(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w}$,

(f) $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w})$.

49. Find the equations of the straight lines that satisfy each of the following sets of conditions.

(a) Passes through the points $P(3, 3, -5)$ and $Q(2, -6, 1)$.

(b) Passes through the point $P(1, -1, 1)$ and is perpendicular to the plane $2x + 3y - z = 4$.

50. Find the volume of the parallelepiped with one corner at P and with sides PQ, PR and PS.

(a) $P(0, 1, -6)$, $Q(-3, 1, 4)$, $R(1, 7, 2)$, $S(-3, 0, 4)$.

(b) $P(1, 1, 1)$, $Q(-2, 1, 6)$, $R(3, 5, 7)$, $S(0, 1, 6)$.

51. For any vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^3$, show that

(a) $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w}$,

(b) $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{v} \times (\boldsymbol{w} \times \boldsymbol{u}) + \boldsymbol{w} \times (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{0}$.

52. For each of the following, determine $(\boldsymbol{F} \cdot \boldsymbol{G})'$ and $(\boldsymbol{F} \times \boldsymbol{G})'$.

(a) $\boldsymbol{F} = (\cos 2t)\boldsymbol{i} + (\sin t)\boldsymbol{j} - e^{-t}\boldsymbol{k}$, $\boldsymbol{G} = 2t^2\boldsymbol{i} - 3t\boldsymbol{k}$

(b) $\boldsymbol{F} = 5t^2\boldsymbol{i} + t\boldsymbol{j} - t^3\boldsymbol{k}$, $\boldsymbol{G} = (\sin t)\boldsymbol{i} - (\cos t)\boldsymbol{j}$.

53*. A particle moves along a curve whose parametric equations are

$$x(t) = e^{-t}, \quad y(t) = 2\cos 3t, \quad z(t) = 2\sin 3t,$$

where t is the time.

(a) Determine its velocity vector and acceleration vector.

(b) Find the magnitudes of the velocity and acceleration at $t = 0$.

54. Compute $\nabla\psi$ and $\nabla\psi(P_0)$ for the given point P_0.

(a) $\psi = e^{xy} + z^2 x$, $P_0(0, 0, 4)$.

(b) $\psi = x^2 y - \sin(zx)$, $P_0(1, -1, \pi/4)$.

55. Find the tangent plane and normal line to the surface S at the point P_0.

(a) S: $x^2 + y^2 + z^2 = 4$, $P_0(1, 1, \sqrt{2})$.

(b)$^\bigstar$ S: $x^2 - 2y^2 + z^4 = 0$, $P_0(1, 1, 1)$.

56$^\bigstar$. The atmosphere pressure in a certain region of space is $P = xy^2 + yz^2 + zyx$. Find the rate of change of the pressure with respect to distance at the point $(1, 1, 4)$ in the region, in the direction of the vector $\boldsymbol{v} = \boldsymbol{j} - 3\boldsymbol{k}$.

57. Consider the scalar field

$$\phi(x, y, z) = -2xy + x\ln(y + z).$$

Determine:

(a) the direction and magnitude of the maximum rate of change of ϕ at $(1, 3, -2)$;

(b) the directional derivative of ϕ in the $(1, 3, -2)$ direction at $(1, 1, 0)$.

58$^\bigstar$. Consider the scalar field

$$\phi(x, y, z) = 2x^2 + 3y^2 + z^2.$$

(a) Calculate $\nabla\phi$.

(b) Using (a) calculate $\nabla\phi$ at the point $P(2, 1, 3)$.

(c) Find a unit normal to $\phi(x, y, z) = 20$ at the point $P(2, 1, 3)$;

(d) Calculate the directional derivative of ϕ at $P(2, 1, 3)$ parallel to the vector $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{k}$.

59. Compute $\nabla \cdot \boldsymbol{F}$ and $\nabla \times \boldsymbol{F}$, and verify that $\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$.

(a) $\boldsymbol{F} = x^2z\boldsymbol{i} - y\boldsymbol{j} + z^3\boldsymbol{k}$

(b) $\boldsymbol{F} = (\sinh x)\boldsymbol{i} + (\cosh y)\boldsymbol{j} - xyz\boldsymbol{k}$.

60. Compute $\nabla\psi$ and verify that $\nabla \times (\nabla\psi) = \boldsymbol{0}$.

(a) $\psi = \sin(xz) + \cos(yz)$

(b) $\psi = -4xy^3 + z^2x$

61. Evaluate $\displaystyle\int_{(1,1)}^{(4,2)} \big((x + y)\,dx + (y - x)\,dy\big)$ along the following curves.

(a) The parabola $y^2 = x$.

(b) The straight line segments from $(1, 1)$ to $(1, 2)$, and then to $(4, 2)$.

62★. Evaluate $\oint \big((2x - y + 4)\,dx + (5y + 3x - 6)\,dy\big)$ around

(a) a triangle in the xy-plane with vertices at $(0,0)$, $(3,0)$, $(3,2)$, traversed in the counterclockwise direction;

(b) a circle of radius 4 with centre at $(0,0)$, traversed counterclockwise.

63. Calculate the work done by the force field $\boldsymbol{F}$ along the curve C.

(a) $\boldsymbol{F} = 3xy\boldsymbol{i} - 2\boldsymbol{j}$ and C is the piece of the hyperbola $x^2 - y^2 = 1$, $z = 0$ from $(1,0,0)$ to $(2, \sqrt{3}, 0)$.

(b)★ $\boldsymbol{F} = x^3\boldsymbol{i} - z\boldsymbol{j} + 2xy\boldsymbol{k}$ and C is given by $x = t^2$, $y = z = \sqrt{t}$ and $2 \le t \le 4$.

4 Double Integrals

64. Evaluate the following double integrals.

(a) $\displaystyle\int_0^2 \int_1^3 x^3y^2\,dy\,dx$

(b) $\displaystyle\int_1^3 \int_2^3 (x^2 - 2xy + 2y^2)\,dy\,dx$

65. Use double integration to find the area of the following regions.

(a) The region bounded by $y = x^3$ and $y = x^2$.

(b)★ The region bounded by $y = \sqrt{x}$, $y = x$ and $y = \dfrac{x}{2}$.

66. Integrate the following functions f over the given regions Ω.

(a) $f(x,y) = xy$, Ω bounded by $y = 0$, $x = 2a$, and $x^2 = 4ay$.

(b) $f(x,y) = x^2y + y^3$, $\Omega = \{\,(x,y) : x^2 + y^2 \le 1, x \ge 0, y \ge 0\,\}$.

67. Evaluate the following integrals by first changing the order of integration.

(a) $\displaystyle\int_0^1 \int_{y^2}^1 2\sqrt{x}e^{x^2}\,dx\,dy$

(b)★ $\displaystyle\int_0^1 \int_y^1 e^{x^2}\,dx\,dy$

(c) $\displaystyle\int_0^1 \int_y^1 \sin(x^2)\,dx\,dy$

(d)★ $\displaystyle\int_{-1}^1 \int_{x^2}^{2-x^2} dy\,dx$

(e)★ $\displaystyle\int_0^1 \int_x^{2x} x^2e^x\,dy\,dx$

68. Evaluate using polar co-ordinates.

(a) $\displaystyle\int_0^2 \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} x^2 y^2 \, dy \, dx;$

(b)★ $\displaystyle\int_0^2 \int_{-\sqrt{(2y-y^2)}}^{\sqrt{(2y-y^2)}} \sqrt{(x^2+y^2)} \, dx \, dy.$

69. Use double integration to

(a)★ find the volume lying between the paraboloids $z = x^2 + y^2$ and $3z = 4 - x^2 - y^2$,

(b) find the volume lying inside both the sphere $x^2+y^2+z^2 = 2a^2$ and the cylinder $x^2+y^2 = a^2$, with $a > 0$.

70. Consider the integral

$$\int_0^4 \int_{3x}^{12} \sin(y^2) \, dy \, dx.$$

(a) Make a sketch of the region of integration.

(b) Express the integral with the reverse order of integration.

(c) Hence evaluate it.

71★. Consider

$$\int_0^1 \int_{\sqrt{x}}^1 \sqrt{1+y^3} \, dy \, dx.$$

(a) Make a sketch of the region of integration.

(b) Express the integral with the reverse order of integration.

(c) Hence evaluate the integral (leaving your answer in surd form).

72★. Find the centroid of the region in the first quadrant bounded by the x-axis, the parabola $y^2 = 2x$, and the line $x + y = 4$.

73★. Find the centroid of the region cut from the first quadrant by the circle $x^2 + y^2 = a^2$.

74. Find the centre of mass of a thin triangular plate bounded by the y-axis and the lines $y = x$ and $y = 2 - x$ if the density is $\delta(x, y) = 6x + 3y + 3$.

75★. Find the centre of mass and the moment of inertia about the x-axis of a thin plate bounded by the curves $x = y^2$ and $x = 2y - y^2$ if the density at the point (x, y) is given by $\delta(x, y) = y + 1$.

76. Find the centre of mass of a thin plate bounded by the semi-circle $y = \sqrt{a^2 - x^2}$, the lines $x = \pm a$ and the line $y = -a$ if the density $\delta(x, y)$ is given by

(a) k (some constant),

(b) $y + a$,

(c) $x + a$.

77. Calculate the volume of the tetrahedron bounded by the co-ordinate planes and the plane $z = 2 - 2x - y$.

78★. Find the volume inside the cylinder $x^2 + y^2 = 16$, cut off above by the plane $z = 5$ and below by the surface $z = \dfrac{x^2 + y^2}{8}$.

79. The solid S is bounded above by the sphere $z = \sqrt{2a^2 - x^2 - y^2}$ and below by the cone $z = \sqrt{x^2 + y^2}$. Sketch the solid and find its volume.

5 Ordinary differential equations

80. Find the general solution to the following 1st order differential equations.

(a)★ $2e^x + \dfrac{dy}{dx}(1 - e^x)\tan y = 0.$

(b) $\sec^2 x \tan y + \dfrac{dy}{dx} \sec^2 y \, \tan x = 0.$

(c)★ $(x^2 + 1)\dfrac{dy}{dx} + 2xy - 4x^2 = 0.$

(d) $x^3 - 2y + 3x^2 y - x^3 \dfrac{dy}{dx} = 0.$

81. Use the substitution $v = \dfrac{y}{x}$ to

(a)★ find the general solution to $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$,

(b) solve the initial value problem given by $yy' = x^3 + \dfrac{y^2}{x}$ and $y(2) = 6$.

82★. Use the substitution $v = y + x$ to find the general solution of $\dfrac{dy}{dx} = (y + x)^2$.

83. Solve the following differential equations.

(a) $y' = \dfrac{xy + 2}{1 - x^2}$ subject to $y(0) = 1$.

(b) $yy' = x^2 + \mathrm{sech}^2 x$ subject to $y(0) = 4$.

84★. Give the general solution $y = y(x)$ to each of the following 2nd order differential equations.

(a) $2y'' + y' - 6y = 0.$

(b) $y'' + 4y' + 53y = 0.$

(c) $y'' - 4y' + 4y = 0.$

85★. Solve

(a) $y'' + 3y' + 2y = 30e^{4x}$,

(b) $y'' - 4y' + 4y = xe^{3x}$,

(c) $y'' - 4y'(x) + 3y = 9x^2 + 2e^{3x}$.

86★. A forced vibrating system is represented by

$$y'' + 3y' + 2y = 5\cos t$$

where $r(t) = 5\cos t$ is the driving force and $y(t)$ is the displacement from the equilibrium position. Find the motion of the system corresponding to the following initial displacement and velocity

$$y(0) = \frac{1}{2}, \quad y'(0) = 1.$$

Then find the steady-state oscillations (that is, the response of the system after a sufficiently long time).

87. Solve

$$\frac{d^2x}{dt^2} + 9x = 12\sin 3t$$

subject to the initial conditions

$$x = 5 \quad \text{and} \quad \frac{dx}{dt} = 4 \quad \text{when } t = 0.$$

Variation of Parameters

Consider a second-order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x).$$

If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the the corresponding homogeneous equation, then a solution of the inhomogeneous equation is given by

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where

$$u_1(x) = -\int \frac{y_2(x)f(x)}{W(x)}\,dx, \qquad u_2(x) = \int \frac{y_1(x)f(x)}{W(x)}\,dx, \qquad W(x) = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}.$$

88. Use the method of variation of parameters to solve

(a)★ $\dfrac{d^2y}{dx^2} + y = \dfrac{1}{\cos x}$;

(b) $y'' - 4y' + 5y = \dfrac{e^{2x}}{\sin x}$;

(c)★ $y'' + y = \cot x$;

(d) $y'' + 2y' + y = e^{-x}\cos x$.

6 Matrices

89. Find the matrix products AB and BA if $A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$.

90. Find the characteristic equation, the eigenvalues and the associated eigenvectors for the following matrices.

(a) $A = \begin{pmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{pmatrix}$

(b)$^\bigstar$ $B = \begin{pmatrix} -6 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}$.

91$^\bigstar$. A curve has equation $x^2 + 8xy + 7y^2 = 36$. Find an orthogonal matrix P such that

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} X \\ Y \end{pmatrix}$$

will refer the equation to the principal axes of the curve, and hence write down the equation in terms of X and Y. Give the (x, y) coordinates of the points on the curve closest to the origin.

92$^\bigstar$. Let

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

(a) Show that the eigenvalues of the matrix are 1, -4 and 3 and then find the associated eigenvectors.

(b) Hence express the equation of the surface $x^2 - 2y^2 + z^2 + 6xy - 2yz = 16$ in terms of its principal axes X, Y and Z.

(c) Write out an orthogonal matrix P such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}.$$

(d) Deduce from your results the shortest distance from the origin to the surface described in (b).

93. Use eigenvalue methods to find the general solution to the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 7x + y + z, \\ \frac{dy}{dt} &= 3x + y + 2z, \\ \frac{dz}{dt} &= x + 3y + 2z. \end{aligned}$$

94★. Let

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{pmatrix}.$$

(a) By solving for the zeros of the characteristic polynomial show that the eigenvalues of the matrix are 0, 3 and 6.

(b) Find the eigenvectors corresponding to these eigenvalues.

(c) Find an orthogonal matrix P such that $D = P^{-1}AP$ is a diagonal matrix.

(d) Using the results of (a), (b), (c) find the solution to initial-value problem

$$\begin{aligned} \frac{dx_1}{dt} &= 3x_1 + 2x_2 + 2x_3, & x_1(0) &= 3, \\ \frac{dx_2}{dt} &= 2x_1 + 2x_2, & x_2(0) &= 1, \\ \frac{dx_3}{dt} &= 2x_1 + 4x_3, & x_3(0) &= 4. \end{aligned}$$

7 The Laplace Transform

95. Find, by direct integration, the Laplace transforms of the following functions.

(a)★ $5t + 3$

(b) $\cos(\omega t)$

96. Use tables to find the Laplace transforms of the following functions.

(a)★ $t^2 + 2t + 3$

(b)★ $\sin 5t$

(c)★ e^{3t-4}

(d)★ te^{2t}

(e) $t^6 e^{4t}$

(f) $e^t \sin t$

(g) $t \cos \omega t$

(h) $4t^2 e^t$

97★. Let $g(t) = 2t - 2u(t-1)$.

(a) Sketch the graph of g.

(b) Find $\mathcal{L}(g(t))$.

98. Find the Laplace transforms of the following functions.

(a) $(t-5)^3 u(t-5)$

(b) $\cos 3(t-4)\, u(t-4)$

99. Use tables to find the following inverse Laplace transforms.

(a)★ $\mathcal{L}^{-1}\left(\dfrac{1}{25s^2} + \dfrac{s}{s^2+25}\right)$

(b) $\mathcal{L}^{-1}\left(\dfrac{1}{s^2+(\pi/2)^2}\right)$

(c)★ $\mathcal{L}^{-1}\left(\dfrac{s-2}{s^2-4s+5}\right)$

(d) $\mathcal{L}^{-1}\left(\dfrac{\pi}{(s+\pi)^2}\right)$

(e)★ $\mathcal{L}^{-1}\left(\dfrac{1}{(s+3)(s+2)}\right)$

(f) $\mathcal{L}^{-1}\left(\dfrac{12s}{s^2+5s+4}\right)$

(g)★ $\mathcal{L}^{-1}\left(\dfrac{e^{-3s}}{s^2}\right)$

(h) $\mathcal{L}^{-1}\left(\dfrac{e^{-2s}s}{s^2+9}\right)$

(i) $\mathcal{L}^{-1}\left(\dfrac{3e^{-s}}{(s-2)^2}\right)$

100. Use Laplace transforms to solve the initial-value problem

$$y'' + 4y = 0, \qquad y(0) = 2, \qquad y'(0) = -8.$$

101. Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2} - 5\,\frac{dx}{dt} + 6x = 24e^{-t}$$

given that $x = 3$ and $\dfrac{dx}{dt} = 2$ when $t = 0$.

102★. Using Laplace transforms, solve $y'' - 4y' + 5y = 0$ if $y(0) = 1$ and $y'(0) = 2$.

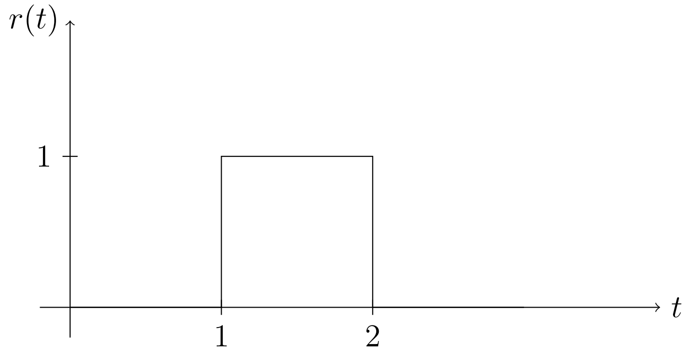
103. Use the Laplace transform method to find a solution to the system

$$\begin{aligned} \frac{dx}{dt} + 2y - x &= 0, & x(0) &= 1, \\ \frac{dy}{dt} - 2x - y &= 0, & y(0) &= 0. \end{aligned}$$

104★. Use the Laplace transform method to solve the initial value problem

$$y'' - 3y' + 2y = r(t), \qquad y(0) = 1, \qquad y'(0) = 3,$$

where $r(t)$ is as shown below.



105. A particle of mass m moves along the x-axis. At time $t = 0$ it is at the origin and moving with velocity V, when a constant force F is applied for a time a, after which it is removed.

Find the position of the particle as a function of t, using the Laplace transform method. That is, solve the following problem:

$$m\,\frac{d^2x}{dt^2} = \big(1 - u(t-a)\big)F \quad \text{for } t > 0, \quad \text{with } x = 0 \text{ and } \frac{dx}{dt} = V \text{ when } t = 0.$$

8 Fourier Series

If $f(x)$ has period $T = 2L$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

and, for $n \geq 1$,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

106. State which of the following functions of x are odd, even, both or neither.

(a) $|x|$

(b)★ $x\cos(nx)$

(c) $\sin(x) + \cos(x)$

(d) c, where c is a constant

(e) $\ln(1 + e^x) - x/2$

(f)★ $\sin^2 x$.

107. The following functions f are assumed to be periodic with period 2π. Sketch their graphs for $-4\pi \le x \le 4\pi$. Are they odd, even or neither?

(a) $f(x) = x|x|$ for $-\pi < x < \pi$

(b) $f(x) = e^{|x|}$ for $-\pi < x < \pi$

(c)★ $f(x) = \begin{cases} x, & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < x < \dfrac{3\pi}{2}. \end{cases}$

108★. Find the Fourier series for

$$f(x) = \begin{cases} 5, & -\pi < x \le 0 \\ 3, & 0 < x \le \pi. \end{cases}$$

109★. For the function g given by

$$g(x) = \begin{cases} 1, & 0 < x < 1 \\ 4 - 2x, & 1 \le x \le 2, \end{cases}$$

(a) sketch over $(-10, 10)$ the graph of the function represented by the half-range Fourier **sine** series;

(b) make a separate sketch over $(-10, 10)$ of the graph of the function represented by the half-range Fourier **cosine** series.

110. Find the Fourier series of $L(x) = 3(x+1)$ for $-2 < x < 2$. Find a result on an infinite series by considering your answer at $x = 5$.

111. Periodically extend the function

$$f(t) = e^{-t}, \qquad 0 < t < 1,$$

in an odd manner for $-1 < t < 0$ and find its Fourier series. Plot $f(t)$ for $-2 < t < 2$ and state the value of the Fourier series representation at $t = 0$.

112. Let

$$f(x) = e^{-x} \quad \text{for } 0 \le x \le 1,$$

and suppose that f is extended to an even function with period 2; thus

$$f(-x) = f(x) \quad \text{and} \quad f(x+2) = f(x) \quad \text{for all } x.$$

(a) Sketch the graph of $f(x)$ for $-2 \le x \le 2$.

(b) Find the coefficients in the cosine half-range expansion

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x.$$

You may assume that

$$\int e^{-x} \cos(bx)\, dx = \frac{e^{-x}(b \sin bx - \cos bx)}{1 + b^2} + \text{constant}.$$

(c) Compute the numerical values of a_0 and a_1, and use these to sketch the graph of

$$S_1(x) = a_0 + a_1 \cos \pi x$$

for $-2 \leq x \leq 2$. Are your graphs of $f(x)$ and $S_1(x)$ roughly similar in shape?

113★. A vibrating system is governed by the differential equation

$$\frac{d^2x}{dt^2} + 50x = F(t), \tag{1}$$

where t is the time, $x(t)$ is the displacement from equilibrium and $F(t)$ is the applied force function.

(a) When the function $F(t)$ is represented by the series

$$F(t) = b_0 + \sum_{n=1}^{\infty} b_n \sin nt \tag{2}$$

find a series which is a particular integral of the differential equation (1) given above.

(b) For the case of the following periodic force

$$F(t) = \begin{cases} 2, & 0 < t < \pi, \\ 0, & -\pi < t < 0, \end{cases}$$

with $F(t + 2\pi) = F(t)$ for all t, write down an integral formula for the b_n in the Fourier series (2) and evaluate the integral. Hence find an infinite series which is a particular integral of the differential equation (1).

(c) By tabulating the amplitudes of the various components of the input forcing function and the output displacement of the system, or otherwise, determine which component of the forcing function gives the largest contribution to the observed output displacement. What is this phenomenon called?

9 Partial Differential Equations

114. Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial y^2} = 0.$$

(a) Use D'Alembert's method to find a solution in terms of arbitrary functions.

(b) Determine the particular solution satisfying $u(x, 0) = 0$ and $u_y(x, 0) = 8 \sin 2x$.

115. Consider the partial differential equation

$$7\frac{\partial^2 u}{\partial x^2} - 8\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

(a) By finding the values of λ for which $u = F(x - \lambda y)$ is a solution of the PDE, find a solution in terms of two arbitrary functions.

(b) Determine the particular solution satisfying $u(x, 0) = 0$ and $u_y(x, 0) = 9e^{-x}$.

116★. A vibrating string of length π metres satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 25\frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ is the transverse displacement of the string, at position x and time t. The ends of the string are held fixed so that

$$u(0,t) = u(\pi,t) = 0, \quad \text{for all time } t.$$

(a) Assuming a solution of the form $u(x,t) = F(x)G(t)$, show that

$$\frac{1}{25G}\frac{d^2G}{dt^2} = \frac{1}{F}\frac{d^2F}{dx^2} = k \quad \text{for some constant } k.$$

(b) Show that only $k < 0$ yields non-trivial solutions and set $k = -(p^2)$ for some $p > 0$.

(c) Applying the boundary conditions, show that $p = n$ for $n = 1, 2, 3, \ldots$ and that possible solutions for $F(x)$ are

$$F_n(x) = B_n \sin(nx)$$

where B_n are constants.

(d) Find all possible solutions $G_n(t)$ for $G(t)$.

(e) If the initial displacement and velocity of the string are

$$u(x,0) = 2\sin(x) - \sin(2x) \qquad \text{and} \qquad u_t(x,0) = 0,$$

find the general solution $u(x,t)$.

(f) Hence determine the maximal transverse displacement of the string at time $t = \dfrac{\pi}{15}$ seconds.

117★. The temperature in a conducting metal bar of length π units is described by the heat equation

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ is the temperature at position x and time t. The ends of the metal bar are held at a constant temperature of 0°C so that

$$u(0,t) = u(\pi,t) = 0 \quad \text{for all } t > 0.$$

(a) Assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$\frac{G'(t)}{3G(t)} = \frac{F''(x)}{F(x)} = k \quad \text{for some constant } k.$$

(b) Write down the boundary conditions for $F(x)$.

(c) Apply the boundary conditions to show that the possible solutions for $F(x)$ are

$$F_n(x) = B_n \sin(nx)$$

where B_n are constants and $n = 1, 2, 3, \ldots$. You must consider all possible values of k.

(d) Find all possible solutions $G_n(t)$ for $G(t)$ and write down the general solution for $u_n(x,t) = F_n(x)G_n(t)$.

(e) If the initial temperature distribution is $u(x,0) = 1$, find the solution $u(x,t)$, expressing your answer as an infinite series.

118. The temperature $u(x,t)$ in a bar of unit length obeys the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}.$$

The bar has its ends maintained at zero temperature, that is,

$$u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0 \quad \text{for all } t \geq 0.$$

The initial temperature distribution is

$$u(x,0) = \sin 2\pi x - \frac{1}{3}\sin 4\pi x \quad \text{for } 0 \leq x \leq 1.$$

Obtain the solution $u(x,t)$ by using the method of separation of variables.

119. Solve the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{for } 0 < x < L \text{ and } t > 0,$$

subject to the homogeneous boundary conditions

$$u = 0 \quad \text{at } x = 0 \text{ and at } x = L, \text{ for all } t > 0,$$

together with the initial conditions

$$u = k \sin\frac{4\pi x}{L} \quad \text{and} \quad \frac{\partial u}{\partial t} = 0 \quad \text{for } 0 < x < L, \text{ when } t = 0.$$

120. The temperature in a bar of unit length obeys the heat equation

$$\frac{\partial v}{\partial t} = \frac{1}{4}\frac{\partial^2 v}{\partial x^2},$$

where $v(x,t)$ is the temperature. The bar has an initial temperature distribution

$$v(x,0) = \begin{cases} \alpha, & 0 \leq x < 1/2, \\ 0, & 1/2 \leq x \leq 1, \end{cases}$$

and is insulated so that the flux of heat at each end is zero:

$$\frac{\partial v}{\partial x} = 0 \quad \text{at } x = 0 \text{ and at } x = 1, \text{ for all } t > 0.$$

Using the method of separation of variables, obtain the solution $v(x,t)$. Plot the temperature distribution at $t = 0$ and as $t \to \infty$. Explain why v does not tend to zero as $t \to \infty$.

121. The steady-state distribution of heat in a semi-infinite slab of height h is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for } x > 0 \text{ and } 0 < y < h,$$

with boundary conditions

$$\begin{aligned} &u(x,0) = u(x,h) = 0 && \text{for all } x > 0, \\ &u(x,y) \to 0 && \text{as } x \to \infty, \text{ for } 0 < y < h, \\ &u(0,y) = 1 && \text{for } 0 < y < h. \end{aligned}$$

Use the method of separation of variables to find the solution $u(x,y)$.

Standard integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{dx}{x} = \ln|x| + C$$

$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a \neq 1)$$

$$\int \sin kx\,dx = -\frac{\cos kx}{k} + C \qquad \int \cos kx\,dx = \frac{\sin kx}{k} + C$$

$$\int \sec^2 kx\,dx = \frac{\tan kx}{k} + C \qquad \int \mathrm{cosec}^2 kx\,dx = -\frac{\cot kx}{k} + C$$

$$\int \tan kx\,dx = \frac{\ln|\sec kx|}{k} + C \qquad \int \sec kx\,dx = \frac{1}{k}(\ln|\sec kx + \tan kx|) + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C \qquad \int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C$$

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx \qquad \int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx$$

Table of Laplace Transforms

The function $g(t)$ is defined for all $t \geq 0$, and its Laplace transform

$$G(s) = \mathcal{L}[g(t)] = \int_0^\infty e^{-st} g(t) dt.$$

exists. The Heaviside step function u is defined by

$$u(t-a) = \begin{cases} 0 & , t < a, \\ \frac{1}{2}, & t = a, \\ 1, & t > a. \end{cases}$$

$g(t)$	$G(s) = \mathcal{L}[g(t)]$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^m	$\frac{m!}{s^{m+1}}$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$e^{-\alpha t} f(t)$	$F(s+\alpha)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$tf(t)$	$-F'(s)$

Answers to Selected Questions

1 Partial Differentiation

2. (a) $f_{xx} = -f_{yy} = -2(x^2 - y^2)/(x^2 + y^2)^2$, $f_{xy} = -4xy/(x^2 + y^2)^2$ (b) $f_{xx} = 2y - y\sin x$, $f_{yy} = -\cos y$, $f_{xy} = 2x + \cos x$ (c) $2xy/(x^2 + y^2)^2$, $-2xy/(x^2 + y^2)^2$, $(y^2 - x^2)/(x^2 + y^2)^2$.
5. (a) $1/(1 + \pi)$ (b) 0. **6**. $4x\cos 2t - 4yte^{y^2}\sin t^2$. **7**. $2x^2 - 2xy - 2y^2$. **8**. 2.
9. 99. **10**. 0. **15**. (a) $x - x^3/6$ (b) $1 - (x - \pi/2)^2/2$.
18. (a) $1 - 2x^2 + 2xy - y^2/2 + \cdots$ (b) $-2x + (y + \pi/2) + \cdots$.
19. $e\big[1 + (x - 1) + y + \frac{1}{2}(x - 1)^2 + (x - 1)y + \cdots\big]$.
20. $\frac{1}{\sqrt{2}} - \sqrt{2}(x - \frac{\pi}{4}\Big) + \frac{1}{\sqrt{2}}(y - \frac{\pi}{4}) - \sqrt{2}(x - \frac{\pi}{4})^2 + \sqrt{2}(x - \frac{\pi}{4})(y - \frac{\pi}{4}) - \frac{1}{2\sqrt{2}}(y - \frac{\pi}{4})^2$.
21. $2(x - 1) - (x - 1)^2 + y^2$, 0.2. **22**. $e^{-2}[1 - (x - 2) + \frac{1}{2}(x - 2)^2 - \frac{1}{2}(y - \frac{\pi}{2})^2]$, 0.14659.
23. 2.95. **24**. $|dT| \le 8/100$. **25**. $|dv/v| \le 0.0242 = 2.42\%$. **26**. 0.1%.
27. 0.024. **28**. 16%. **29**. (a) 3.6 (c) 0.0408.
31. (a) $2ka/(a^2 + k^2)^2$ (b) $(a^2 - k^2)/(a^2 + k^2)^2$. **33**. $\pi\alpha/(\alpha^2 - 1)^{3/2}$.
34. $\dfrac{\sin t\pi}{t^2} - \dfrac{\pi\cos t\pi}{t}$. **35**. $\dfrac{2\cos t^2}{t} - \dfrac{3\cos t^{3/2}}{2t}$.

2 Extreme Values

36. (a) Minimum of -5 at $(-3, 3)$. (b) Saddle point at $(-2, 1)$. (c) Minimum of 0 at $(1, -2)$. (d) Minimum of -6 at $(2, -1)$. (e) Saddle point at $(1, -1)$, minimum of 0 at $(0, 0)$. (f) Saddle point at $(0, 0)$, maximum of 2 at $(1, 1)$ and $(-1, -1)$. **37**. Saddle point at $(0, 0)$. Minimum of 14 at $(1, 1)$. **38**. For $0 < A < 2$, local minimum at $(0, 0, 7)$. For $A > 2$, saddle point at $(0, 0, 7)$. For $A = 2$, local minimum on line $x = y$. **39**. Maximum at $(\pm 1/\sqrt{2}, \pm 1/2)$, minimum at $(-1/\sqrt{2}, 1/2)$ and $(1/\sqrt{2}, -1/2)$. **40**. Local minimum of 0 at $(0, 3)$, local maximum of 4 at $(2, 1)$. **41**. (a) Minimum of 8 at $(4, 4)$. (b) maximum of 64 at $(8, 8)$. **42**. $r = 2\,\text{cm}$, $h = 4\,\text{cm}$. **43**. $T = 0$ at $(\sqrt{5}, 2\sqrt{5})$ or $(-\sqrt{5}, -2\sqrt{5})$, $T = 125$ at $(2\sqrt{5}, -\sqrt{5})$ or $(-2\sqrt{5}, \sqrt{5})$. **44**. $(3/2, 2, 5/2)$. **45**. $(0, 0, \pm 2)$ are each closest to the origin. **46**. 50. **47**. $x = 2/3$, $y = 4/3$, $z = -4/3$.

3 Vector Field Theory

48. (a) $-10\boldsymbol{j} - 15\boldsymbol{k}$ (b) $6\boldsymbol{i} - 6\boldsymbol{j} + 3\boldsymbol{k}$ (c) 30 (d) 30 (e) $-7\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$ (f) $-\boldsymbol{i} + 7\boldsymbol{j} + 16\boldsymbol{k}$.
49. (a) $x = 3 - t$, $y = 3 - 9t$, $z = -5 + 6t$ (b) $\boldsymbol{r} = (1 + 2t)\boldsymbol{i} + (-1 + 3t)\boldsymbol{j} + (1 - t)\boldsymbol{k}$.
50. (a) 34 (b) 40.
52. (a) $(-3\sin t - 3t\cos t)\boldsymbol{i} + (3\cos 2t - 6t\sin 2t - 4te^{-t} + 2t^2e^{-t})\boldsymbol{j} + (-4t\sin t - 2t^2\cos t)\boldsymbol{k}$ (b) $(-3t^2\cos t + t^3\sin t)\boldsymbol{i} - (3t^2\sin t + t^3\cos t)\boldsymbol{j} + (5t^2\sin t - \sin t - 11t\cos t)\boldsymbol{k}$.
53. (a) $\boldsymbol{v} = -e^{-t}\boldsymbol{i} - 6\sin 3t\boldsymbol{j} + 6\cos 3t\boldsymbol{k}$, $\boldsymbol{a} = e^{-t}\boldsymbol{i} - 18\cos 3t\boldsymbol{j} - 18\sin 3t\boldsymbol{k}$ (b) $\sqrt{37}, \sqrt{325}$.
54. (a) $16\boldsymbol{i}$ (b) $(-2-\pi\sqrt{2}/8)\boldsymbol{i}+\boldsymbol{j}-\sqrt{2}/2\boldsymbol{k}$. **55**. (a) $x = 1+2t$, $y = 1+2t$, $z = \sqrt{2}+2\sqrt{2}\,t$. (b) $x = 1 + 2t$, $y = 1 - 4t$, $z = 1 + 4t$. **56**. $-5/\sqrt{10}$. **57**. (a) $\sqrt{38}$ (b) $-7/\sqrt{14}$.
61. (a) $11\frac{1}{3}$ (b) 14. **62**. (a) 12 (b) 64π. **63**. (a) $\sqrt{3}$ (b) 49013/3.

4 Double Integrals

64. (a) 104/3 (b) 14. **65**. (a) 1/12 (b) 7/6. **66**. (a) $a^4/3$ (b) 1/5.
67. (a) $(e - 1)$ (b) $(e - 1)/2$ (c) $(1 - \cos 1)/2$ (d) 8/3 (e) $6 - 2e$. **68**. (a) $4\pi/3$ (b) 32/9. **69**. (a) $2\pi/3$ (b) $4\pi a^3(2\sqrt{2} - 1)/3$. **70**. $(1 - \cos 144)/6$.
71. $(2/9)(2\sqrt{2} - 1)$. **72**. $(64/35, 5/7)$. **73**. $(4a/(3\pi), 4a/(3\pi))$. **74**. $(3/8, 17/16)$.
75. $(8/15, 8/15)$, $I_x = 1/6$. **78**. 64π.

5 Ordinary differential equations

80. (a) $\sec y = C(e^x - 1)^2$ (b) $\tan x \tan y = C$ (c) $y = \dfrac{4x^3}{3} + \dfrac{c}{x^2+1}$
(d) $y = x^3/2 + Cx^3 e^{1/x^2}$. **83**. (a) $y\sqrt{1-x^2} = 2\sin^{-1} x + 1$ (b) $3y^2 = 2x^3 + 6\tanh x + 48$.
84. (a) $y = Ae^{3x/2} + Be^{-2x}$ (b) $y = e^{-2x}(A\cos 7x + B\sin 7x)$ (c) $y = (C_1 + C_2 x)e^{2x}$
85. (a) $y = Ae^{-x} + Be^{-2x} + e^{4x}$ (b) $y = (A + Bx)e^{2x} + (x-2)e^{3x}$
(c) $y = Ae^x + (x+B)e^{3x} + 3x^2 + 8x + 26/3$. **86**. $y = (\cos t + 3\sin t)/2$.
87. $x = (5-2t)\cos 3t + 2\sin 3t$. **88**. (a) $y = A\cos x + B\sin x + \cos x \ln|\cos x| + x\sin x$
(b) $y = e^{2x}(A\cos x + B\sin x - x\cos x + \ln|\sin x|\sin x)$
(c) $y = A\cos x + B\sin x - \sin x\cos x + \sin x(\ln|\mathrm{cosec}x - \cot x| + \cos x)$
(d) $y = e^{-x}(A + Bx - \cos x)$.

6 Matrices

89. $AB = 0$ and $BA = \begin{bmatrix} -11 & 6 & 1 \\ -22 & 12 & -2 \\ -11 & 6 & -1 \end{bmatrix}$. **90**. (a) $\lambda_1 = 4$, $\boldsymbol{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$,
$\lambda_2 = 2$, $\boldsymbol{v}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$, $\lambda_3 = -2$, $\boldsymbol{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. (b) $\lambda_1 = -7$, $\boldsymbol{v}_1 = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$,
$\lambda_2 = -1$, $\boldsymbol{v}_2 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$, $\lambda_3 = 3$, $\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.
91. $\dfrac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$, $\dfrac{X^2}{4} - \dfrac{Y^2}{36} = 1$, $\left(\pm\dfrac{2}{\sqrt{5}}, \pm\dfrac{4}{\sqrt{5}}\right)$. **92**. (b) $X^2 + 3Y^2 - 4Z^2 = 16$
(c) $P = \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{14} & -3/\sqrt{35} \\ 0 & -2/\sqrt{14} & 5/\sqrt{35} \\ 3/\sqrt{10} & 1/\sqrt{14} & 1/\sqrt{35} \end{bmatrix}$ (c) $4/\sqrt{3}$.
93. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ae^{-t}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + be^{3t}\begin{pmatrix} 4 \\ -5 \\ -11 \end{pmatrix} + ce^{8t}\begin{pmatrix} 9 \\ 5 \\ 4 \end{pmatrix}$. **94**. (b) $\lambda_1 = 0$, $\boldsymbol{v}_1 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$,
$\lambda_2 = 3$, $\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, $\lambda_3 = 6$, $\boldsymbol{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ (b) $\dfrac{1}{3}\begin{pmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{pmatrix}$
(d) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \dfrac{1}{3}\left[e^{3t}\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} + e^{6t}\begin{pmatrix} 10 \\ 5 \\ 10 \end{pmatrix}\right]$.

7 The Laplace Transform

95. (a) $\dfrac{5}{s^2} + \dfrac{3}{s}$ (b) $\dfrac{s}{s^2+\omega^2}$. **96**. (a) $\dfrac{2}{s^3} + \dfrac{2}{s^2} + \dfrac{3}{s}$ (b) $\dfrac{5}{w^2+25}$ (c) $\dfrac{e^{-4}}{s-3}$ (d) $\dfrac{1}{(s-2)^2}$
(e) $\dfrac{6!}{(s-4)^7}$ (f) $\dfrac{1}{1+(s-1)^2}$ (g) $\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$ (h) $\dfrac{8}{(s-1)^3}$. **97**. (b) $\dfrac{2}{s^2} - \dfrac{2e^{-s}}{s}$.
98. (a) $\dfrac{3!e^{-5s}}{s^4}$ (b) $\dfrac{se^{-4s}}{s^2+9}$. **99**. (a) $\dfrac{t}{25} + \cos 5t$ (b) $\dfrac{2}{\pi}\sin\dfrac{\pi t}{2}$ (c) $e^{2t}\cos t$ (d) $\pi t e^{-\pi t}$
(e) $-e^{-3t} + e^{-2t}$ (f) $16e^{-4t} - 4e^{-t}$ (g) $(t-3)u(t-3)$ (h) $\cos 3(t-2)\,u(t-2)$
(i) $3e^{2(t-1)}(t-1)u(t-1)$. **100**. $y = 2\cos 2t - 4\sin 2t$. **101**. $x = 2e^{-t} + 2e^{3t} - e^{2t}$.
102. $y = e^{2t}\cos t$. **103**. $x = e^t\cos 2t$, $y = e^t \sin 2t$.
104. $y = u(t-1)[\frac{1}{2} + \frac{1}{2}e^{2(t-1)} - e^{t-1}] - u(t-2)[\frac{1}{2} + \frac{1}{2}e^{2(t-2)} - e^{t-2}] + [2e^{2t} - e^t]$.

105. $x = \dfrac{Ft^2}{2m} + Vt$ for $t < a$, and $x = \dfrac{Ft^2}{2m} - \dfrac{F}{2m}(t-a)^2 + Vt$ for $t > a$.

8 Fourier Series

108. $4 + \displaystyle\sum_{n=1}^{\infty} \frac{2}{n\pi}(\cos n\pi - 1)\sin nx$. **110**. $3 + \dfrac{12}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{2}$, $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$. **111**. $e^{-t} = \displaystyle\sum_{n=1}^{\infty} \frac{e-(-1)^n}{e} \frac{2n\pi}{1+n^2\pi^2} \sin n\pi t$ **112**. $a_0 = 1 - e^{-1}$, $a_n = \dfrac{2}{1+n^2\pi^2}[1-(-1)^n e^{-1}]$. **113**. (a) $\dfrac{b_0}{50} + \displaystyle\sum_{n=1}^{\infty} \frac{b_n}{50-n^2} \sin nt$ (b) $\dfrac{1}{50} + \displaystyle\sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi\big(50-(2k-1)^2\big)} \sin(2k-1)t$ (c) $k = 4$.

9 Partial Differential Equations

114. $u = -4\cos(2x+y) + 4\cos(2x-y)$. **115**. $u = -\frac{3}{2}e^{-(x+7y)} + \frac{3}{2}e^{-(x+y)}$.

116. (d) $G_n(t) = C\cos(5nt) + D\sin(5nt)$ (e) $2\sin(x)\cos(5t) - \sin(2x)\cos(10t)$ (f) $\dfrac{3\sqrt{3}}{4}$.

117. (b) $F(0) = F(\pi) = 0$ (d) $u_n(x,t) = B_n \sin(nx)e^{-3n^2t}$ (e) $u(x,t) = \displaystyle\sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi} \sin(nx)e^{-3n^2t}$. **118**. $u = e^{-\pi^2 t}\sin 2\pi x - \frac{1}{3}e^{-4\pi^2 t}\sin 4\pi x$.

119. $u = k \sin \dfrac{4\pi x}{L} \cos \dfrac{4\pi ct}{L}$.

120. $v = \dfrac{\alpha}{2} + \dfrac{2\alpha}{\pi}\displaystyle\sum_{k=0}^{\infty} \frac{1}{2k+1} \sin \frac{(2k+1)\pi}{2} e^{-(2k+1)^2\pi^2 t/4} \cos(2k+1)\pi x$.

121. $u = \dfrac{4}{\pi}\displaystyle\sum_{k=0}^{\infty} \frac{e^{-(2k+1)\pi x/h}}{2k+1} \sin(2k+1)\pi\frac{y}{h}$