Answers

THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

NOVEMBER 2010

MATH2019
ENGINEERING MATHEMATICS 2E

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

TABLE OF LAPLACE TRANSFORMS AND THEOREMS

$g(t)$ is a function defined for all $t \geq 0$, and whose Laplace transform

$$G(s) = \mathcal{L}(g(t)) = \int_0^\infty e^{-st} g(t)\,dt$$

exists. The Heaviside step function u is defined to be

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ \frac{1}{2} & \text{for } t = a \\ 1 & \text{for } t > a \end{cases}$$

$g(t)$	$G(s) = \mathcal{L}[g(t)]$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^\nu, \nu > -1$	$\frac{\nu!}{s^{\nu+1}}$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$e^{-\alpha t} f(t)$	$F(s+\alpha)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$tf(t)$	$-F'(s)$

FOURIER SERIES

If $f(x)$ has period $p = 2L$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{aligned}$$

VARIATION OF PARAMETERS

Suppose that $y_h(x) = Ay_1(x) + By_2(x)$ is the general solution of the homogeneous differential equation

$$y'' + p(x)y' + q(x)y = 0,$$

where A and B are constants. Then a particular solution of the associated non-homogeneous equation

$$y'' + p(x)y' + q(x)y = f(x)$$

is given by

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx$$

where $W(x) = \det\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x)$.

SOME BASIC INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^{kx} dx = \frac{e^{kx}}{k} + C$$

$$\int a^x dx = \frac{1}{\ln a} a^x + C \quad \text{for } a \neq 1$$

$$\int \sin\, kx\, dx = -\frac{\cos kx}{k} + C$$

$$\int \cos kx\, dx = \frac{\sin kx}{k} + C$$

$$\int \sec^2 kx\, dx = \frac{\tan kx}{k} + C$$

$$\int \operatorname{cosec}^2 kx\, dx = -\frac{1}{k} \cot kx + C$$

$$\int \tan kx\, dx = \frac{\ln|\sec kx|}{k} + C$$

$$\int \sec kx\, dx = \frac{1}{k}(\ln|\sec kx + \tan kx|) + C$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx$$

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx$$

Answer question 1 in a separate book

1. a) For the scalar field

$$\phi(x, y, z) = 2x^2 + 3y^2 + z^2$$

find:

i) grad ϕ at the point $P(2, 1, 3)$.

ii) the directional derivative of ϕ at the point $P(2, 1, 3)$ in the direction of the vector $\mathbf{u} = \mathbf{i} - 2\mathbf{k}$.

iii) the maximum rate of change of ϕ at the point $P(2, 1, 3)$.

b) Find and classify the critical points of

$$h(x, y) = 6x^2 + 3y^2 - 2x^3 + 6xy.$$

Also give the function values at the critical points.

c) The matrix B is given by

$$B = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{pmatrix}.$$

i) Find two linearly independent eigenvectors of B corresponding to the eigenvalue $\lambda = 5$.

ii) Using part i) or otherwise, find the remaining eigenvalue of B.

d) The area A of a region R of the xy-plane is given by

$$A = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} dx\, dy.$$

i) Sketch the region R.

ii) When the order of integration is reversed the expression for A becomes

$$A = \int_{-1}^1 \int_{l_1}^{l_2} dy\, dx.$$

Find the limits l_1 and l_2.

iii) Hence, find the value of A.

Answer question 2 in a separate book

2. a) Find:

i) $\mathcal{L}(t^6 e^{4t})$.

ii) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2s+5}\right\}$.

b) The function $g(t)$ is given by

$$g(t) = \begin{cases} t & \text{for } 0 \leq t < 1 \\ 2-t & \text{for } t \geq 1. \end{cases}$$

i) Sketch the function $g(t)$ for $0 \leq t \leq 4$.

ii) Write $g(t)$ in terms of the Heaviside step function.

iii) Hence, or otherwise, find the Laplace transform of $g(t)$.

c) Use the Laplace transform method to solve the initial value problem

$$y'' + y' - 6y = 30u(t-4) \quad \text{with } y(0) = 0 \text{ and } y'(0) = 0,$$

where $u(t-4)$ is a Heaviside step function.

Answer question 3 in a separate book

3. a) Use the method of variation of parameters to find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^3}.$$

b) The function f is given by

$$f(x) = \begin{cases} -x & \text{for } -\pi \leq x \leq 0 \\ x & \text{for } 0 \leq x \leq \pi \end{cases}$$

with $f(x+2\pi) = f(x)$ for all x.

i) Make a sketch of this function for $-4\pi \leq x \leq 4\pi$.

ii) Is $f(x)$ odd, even or neither?

iii) Find the Fourier series of $f(x)$.

iv) By considering the value at $x = \pi$ in your answer for the Fourier series in iii), find the sum of the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Answer question 4 in a separate book

4. The tempertaure in a bar of unit length satisfies the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ is the temperature. The bar has its ends maintained at zero temperature. Hence,

$$u(0,t) = 0 \text{ and } u(1,t) = 0, \text{ for all } t.$$

a) Assuming a solution of the form $u(x,t) = X(x)Y(t)$ show that

$$X'' - 4kX = 0 \quad \text{and} \quad Y' - kY = 0,$$

for some constant k.

b) Applying the boundary conditions (and considering all possibilities for the constant k) show that

$$k = -p^2 \qquad (p > 0)$$

and that possible solutions for $X(x)$ are

$$X_n(x) = \sin(n\pi x), \quad \text{for } n = 1, 2, \ldots.$$

c) Find all possible solutions $Y_n(t)$ for $Y(t)$.

d) Suppose now that the initial temperature distribution is given by

$$u(x,0) = \sin(2\pi x) - \frac{1}{5}\sin(4\pi x).$$

Using b) and your answer in c) find the solution $u(x,t)$.

1 a) Let $\phi(x,y,z) = 2x^2+3y^2+z^2$. Then

i) $\text{grad}\,\phi = 4x\underset{\sim}{i} + 6y\underset{\sim}{j} + 2z\underset{\sim}{k}$.

$\text{grad}\,\phi\big|_{(2,1,3)} = 8\underset{\sim}{i} + 6\underset{\sim}{j} + 6\underset{\sim}{k}$.

ii) $D_{\hat{u}}\phi(2,1,3) = \underset{\sim}{\nabla}\phi(2,1,3)\cdot\hat{u}$, where

$$\underset{\sim}{\hat{u}} = \frac{1\underset{\sim}{i} + 0\underset{\sim}{j} - 2\underset{\sim}{k}}{\sqrt{1^2+0^2+(-2)^2}} = \frac{1}{\sqrt{5}}(1,0,-2).$$

Then, $D_{\hat{u}}\phi(2,1,3) = (8,6,6)\cdot\frac{1}{\sqrt{5}}(1,0,-2)$

$= 2/\sqrt{5}$

iii) The maximum rate of change of ϕ at $P(2,1,3)$ is $|\nabla\phi(2,1,3)| = \sqrt{8^2+6^2+6^2} = \sqrt{136}$.

b) Let $h(x,y) = 6x^2 + 3y^2 - 2x^3 + 6xy$.

Then $h_x = 12x - 6x^2 + 6y$

and

$$h_y = 6y + 6x$$

Solving $h_x = 0 = h_y$, gives $y = -x$

and

$$h_x = 12x - 6x^2 - 6x = 6x - 6x^2$$
$$= 6x(1-x) = 0$$

$x = 0$ or 1. Then the critical points are

$$\begin{bmatrix}0\\0\end{bmatrix} \text{ and } \begin{bmatrix}1\\-1\end{bmatrix}.$$

Differentiating h_x & h_y again

$h_{xx} = 12 - 12x$, $h_{xy} = 6$ and $h_{yy} = 6$.

Then

$$D\begin{pmatrix}x\\y\end{pmatrix} = h_{xx}h_{yy} - h_{xy}^2 = (12 - 12x)6 - 36$$

At $\begin{bmatrix}0\\0\end{bmatrix}$, $D\begin{bmatrix}0\\0\end{bmatrix} = 12 \cdot 6 - 36 > 0$ and

$$h_{xx}\begin{pmatrix}0\\0\end{pmatrix} = 12 > 0.$$

Using the second derivative test, $\begin{bmatrix}0\\0\end{bmatrix}$ is a local minimizer.

At $\begin{bmatrix}1\\-1\end{bmatrix}$, $D\begin{bmatrix}1\\-1\end{bmatrix} = (12 - 12)6 - 36$
$= -36 < 0$.

So, $\begin{bmatrix}1\\-1\end{bmatrix}$ is a saddle point.

At $\begin{pmatrix}1\\1\end{pmatrix}$, $f\begin{pmatrix}1\\1\end{pmatrix} = 6 + 3 - 2 + 6 = 13$ &

At $\begin{pmatrix}0\\0\end{pmatrix}$, $f\begin{pmatrix}0\\0\end{pmatrix} = 0$

d) Let

$$A = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{pmatrix}.$$

~~The~~ To find eigenvectors corresponding to the eigenvalue $\lambda = 5$ of A, solve

$$(A - 5I)\underset{\sim}{u} = \underset{\sim}{0}$$

$$\begin{pmatrix} 0 & 4 & 4 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$u_2 + u_3 = 0$$

$$u_2 = -u_3.$$

The eigen vectors are

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ -u_3 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -u_3 \\ u_3 \end{pmatrix}$$

$$= u_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + u_3\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

where u_1 and u_2 are arbitrary with $(u_1, u_3) \neq (0,0)$.

The two linearly independent eigenvectors are $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$.

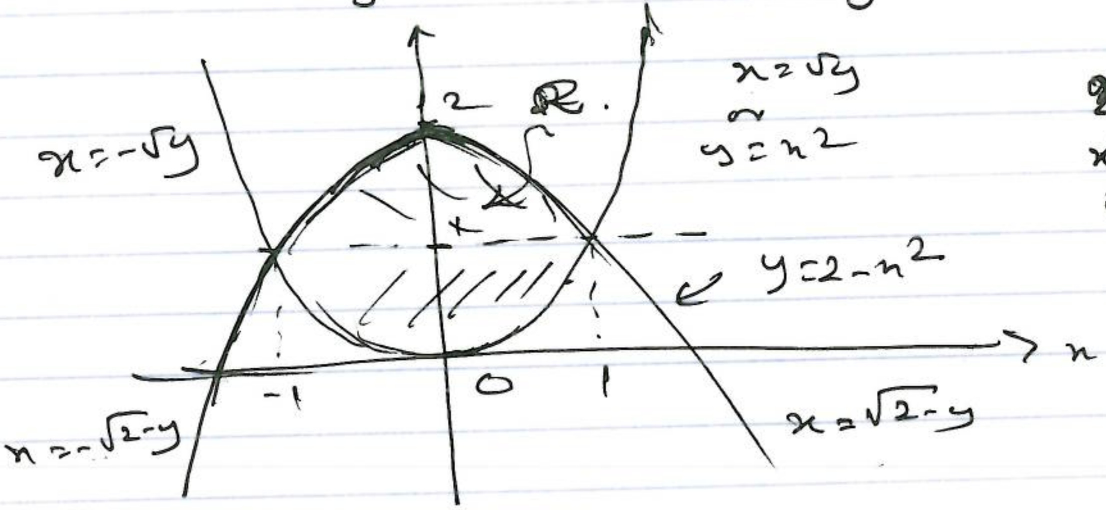
ii) As $\lambda = 5$ is a repeated eigenvalue,

$$5 + 5 + \lambda = 5 + 3 + 3 = 11 \implies \lambda = 1.$$

The remaining eigenvalue of A is $\lambda = 1$.

c) $$A = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} dx\,dy$$

i)



$x=\sqrt{2-y}$
$x^2=2-y$
$y=2-x^2$

ii) When the order of integration is reversed,

$$M = \int_{-1}^{1} \int_{l_1}^{l_2} dy\,dx = \int_{-1}^{1} \int_{x^2}^{2-x^2} dy\,dx$$

where $l_1 = x^2$ and $l_2 = 2-x^2$

iii) $$A = \int_{-1}^{1} \left[y \right]_{x^2}^{2-x^2} dx = \int_{-1}^{1} (2-2x^2)\,dx$$

$$= \left[2x - \frac{2x^3}{3} \right]_{-1}^{1} = 2.\left(2-\tfrac{2}{3}\right) = 2.\frac{4}{3} = 8/3$$

2 a) i) $\mathcal{L}\{t^6 e^{4t}\} = \mathcal{L}\{t^6\}\big|_{s\to s-4}$

$$= \left\{\frac{6!}{s^{6+1}}\right\}\Big|_{s\to s-4} = \frac{720}{(s-4)^7}.$$

ii) $$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+2^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s+1)-1}{(s+1)^2+2^2}\right\}$$

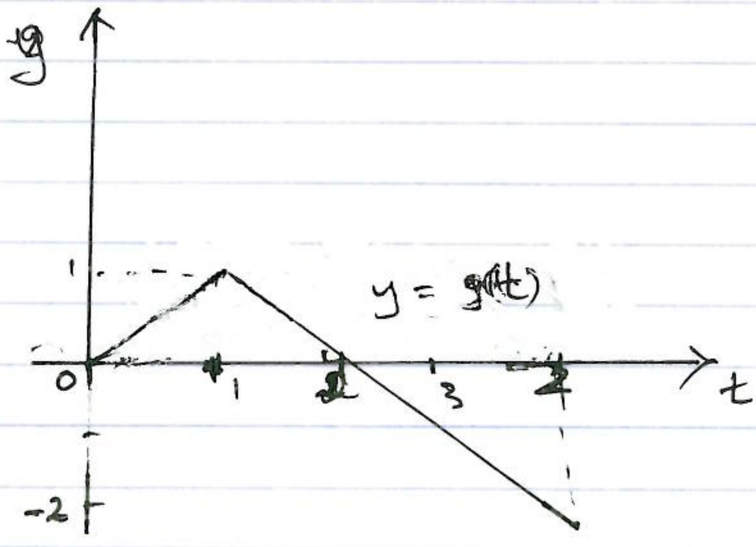
$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}\Big|_{s\to s+1} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}\Big|_{s\to s+1}$$

$$= e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t$$

$$= \frac{e^{-t}}{2}(2\cos 2t - \sin 2t)$$

b) i)



ii) $$g(t) = t(1-u(t-1)) + (2-t)u(t-1)$$
$$= t + (2-2t)u(t-1)$$

iii) $\mathcal{L}\{g(t)\} = \mathcal{L}\{t\} + \mathcal{L}\{(2-2t)u(t-1)\} = \frac{1}{s^2} + \mathcal{L}\{f(t-1)u(t-1)\}$,
where $f(t-1) = 2-2t$. Then, $f(t) = -2t$. So,
$\mathcal{L}\{g(t)\} = 1/s^2 + e^{-s}(-2/s^2) = (1-2e^{-s})/s^2$

c) $y'' + y' - 6y = 30\,\mathcal{U}(t-4)$, $y(0)=0$, $y'(0)=0$.

Let $\mathcal{L}\{y(t)\} = Y(s)$

Taking Laplace transforms each side,

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - 6\,\mathcal{L}\{y\} = 30\,\mathcal{L}\{\mathcal{U}(t-4)\}$$

$$s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} + sY(s) - \underbrace{y(0)}_{0} - 6Y(s) = 30e^{-4s}/s$$

$$(s^2+s-6)Y(s) = 30e^{-4s}/s.$$

So,

$$Y(s) = 30e^{-4s}/s(s^2+s-6) = \frac{30e^{-4s}}{s(s+3)(s-2)}$$

Applying partial fractions,

$$\frac{30}{s(s+3)(s-2)} = A/s + B/s+3 + C/s-2$$

$$= \frac{30/-6}{s} + \frac{30/-3\cdot-5}{s+3} + \frac{30/2\cdot5}{s-2}$$

$$= -5/s + 2/s+3 + 3/s-2$$

So,

$$\mathcal{L}^{-1}\left\{\frac{30}{s(s+3)(s-2)}\right\} = -\mathcal{L}^{-1}\{5/s\} + \mathcal{L}^{-1}\{2/s+3\} + \mathcal{L}^{-1}\{3/s-2\}$$

$$= -5 + 2e^{-3t} + 3e^{-2t}$$

Now,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{30e^{-4s}}{s(s+3)(s-2)}\right\} = \mathcal{L}^{-1}\{e^{-4s}F(s)\}$$

$$= f(t-4)\,\mathcal{U}(t-4),$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{30/s(s+3)(s-2)\} = -5 + 2e^{-3t} + 3e^{-2t}$.

Hence, $f(t-4) = -5 + 2e^{-3(t-4)} + 3e^{-2(t-4)}$ &

$$y(t) = \left(-5 + 2e^{-3(t-4)} + 3e^{-2(t-4)}\right)\mathcal{U}(t-4).$$

3a) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x/x^3$

The characteristic equation for $y'' - 2y' + y = 0$ is

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0 \Rightarrow \lambda = 1, 1.$$

The solution to the homogeneous equation is $y_h(x) = c_1 \underset{y_1}{e^x} + c_2 \underset{y_2}{xe^x}$.

The particular solution of the given equation is

$$y_p(x) = -e^x \int \frac{xe^x \cdot e^x/x^3}{\begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix}} dx$$

$$+ xe^x \int \frac{e^x \; e^x/x^3}{\begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix}}$$

$$= -e^x \int \frac{e^{2x}/x^2}{e^{2x}} dx + xe^x \int \frac{e^{2x}/x^3}{e^{2x}} dx$$

$$= -e^x \int x^{-2} dx + xe^x \int x^{-3} dx$$

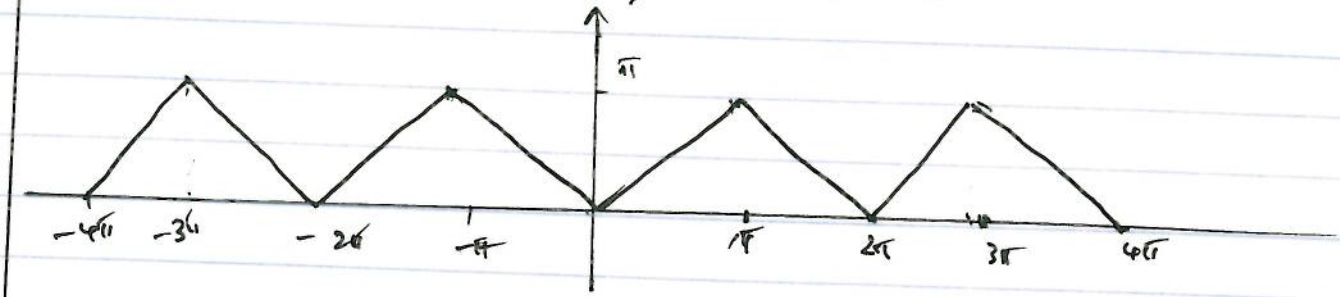
$$= -e^x \left(\frac{x^{-1}}{-1}\right) + xe^x \left(\frac{x^{-2}}{-2}\right).$$

$$= +e^x/x - e^x/2x = + e^x/2x.$$

The general solution is

$$y = c_1 e^x + c_2 xe^x + e^x/2x.$$

3 b) Let $f(x) = \begin{cases} -x, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \end{cases}$



i) f is an even function.

As f is an even periodic function with period $2L = 2\pi$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{even}}\ \underbrace{\sin(nx)}_{\text{odd}}}_{\text{odd}}\,dx$$

$$= 0.$$

So, the Fourier Series of f is given by

$$F(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos(nx),$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} -x\,dx + \int_{0}^{\pi} x\,dx\right)$

$$= \frac{1}{2\pi}\left(\left[-\frac{x^2}{2}\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}\right) = \frac{1}{2\pi}\left(-\tfrac{1}{2}(0-\pi^2) + \frac{\pi^2}{2}\right)$$

$$= \frac{1}{2\pi}\pi^2 = \pi/2.$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{even}}\ \underbrace{\cos(nx)}_{\text{even}}\,dx = \frac{1}{\pi}\left\{2\int_{0}^{\pi} x\cos(nx)\,dx\right\}$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\left\{ \left[x\frac{\sin(nx)}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n}\,dx \right\}$$

$$= \frac{2}{\pi}\left\{ 0 - \frac{1}{n}\left[-\frac{\cos nx}{n} \right]_0^{\pi} \right\}$$

$$= \frac{2}{n^2\pi}\left(\cos(n\pi) - 1\right)$$

$$= \frac{2}{n^2\pi}\left((-1)^n - 1\right)$$

$$= \begin{cases} 0, & n = 2k \\ -\frac{4}{(2k+1)^2\pi}, & n = 2k+1, \quad k = 0, 1, 2, \ldots \end{cases}$$

So, the Fourier Series of f is

$$F(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{-4}{(2k+1)^2\pi} \cos(2k+1)\cancel{x}$$

(iii) At $x = \pi$, $F(\pi) = f(\pi)$, as f is continuous at $x = \pi$. ~~Hence,~~

$$\pi = F(\pi) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos(2k+1)\pi$$

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

$$= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

A $\dfrac{\partial u}{\partial t} = \dfrac{1}{4}\dfrac{\partial^2 u}{\partial x^2}$, $u(0,t)=0$, $u(1,t)=0$, $t \geq 0$

a Let $u(x,t) = X(x)\,Y(t)$. Then,

$\dfrac{\partial u}{\partial t} = X(x)\,Y'(t)$, $\dfrac{\partial u}{\partial x} = X'(x)\,Y(t)$, $\dfrac{\partial^2 u}{\partial x^2} = X''(x)\,Y(t)$

Substituting into the equation.

$$X(x)\,Y'(t) = \frac{1}{4}X''(x)\,Y(t).$$

$$\frac{Y'(t)}{Y(t)} = \frac{1}{4}\,\frac{X''(x)}{X(x)} = k$$

$$Y'(t) - kY(t) = 0 \quad \text{— (1)}$$

$$X''(x) - 4k\,X(x) = 0 \quad \text{— (2)}$$

$u(0,t)=0 \Rightarrow X(0)\,Y(t)=0 \Rightarrow X(0)=0$

$u(1,t)=0 \Rightarrow X(1)\,Y(t)=0 \Rightarrow X(1)=0.$

Case 1 : $k=0$.

Then, $X''(x)=0 \Rightarrow X(x)=ax+b.$

$X(0)=0=X(1) \Rightarrow a=0=b$, So, $X(x)\equiv 0.$

Case 2 : $k>0$.

Let $k=\mu^2$. Then,

$$X'' - 4\mu^2 X = 0$$

Solution is $X(x) = Ae^{2\mu x} + Be^{-2\mu x}$

$X(0)=X(1)=0 \Rightarrow A=0=B.$

i.e. $X(x)\equiv 0$

So, $k<0$. Let $k=-p^2$, $p>0$.

Then, $X''+4p^2X=0$. The solution of this equation is

$$X(x)=B\cos(2px)+C\sin(2px).$$

$X(0)=0 \Rightarrow B=0$

$X(1)=0 \Rightarrow 0=C\sin(2p)$

$\sin(2p)=0 \Rightarrow 2p=n\pi$, $p=n\pi/2$.

$X(x)=C\sin(n\pi x)$, where C is arbitrary.

Take $C=1$. So, possible solutions are

$X_n(x)=\sin(n\pi x)$, $n=1,2,\dots$

Now, ① becomes.

$$Y'(t)+(n\pi/2)^2Y(t)=0$$

$$Y(t)=De^{-(n\pi/2)^2t}$$

So the ~~possib~~ possible solutions for $Y(t)$ are

$$Y_n(t)=D_ne^{-\frac{n^2\pi^2}{4}t}$$

Hence the solution the PDE is

$$u(x,t)=X_n(x)Y_n(t)=D_ne^{-\frac{n^2\pi^2}{4}t}\sin(n\pi x).$$

d) Let $$u(x,t)=\sum_{n=1}^{\infty}D_ne^{-\frac{n^2\pi^2}{4}t}\sin(n\pi x)$$

$$U(x,0) = \sum_{n=1}^{\infty} D_n \sin(n\pi x)$$

$$= \sin(2\pi x) - \frac{1}{5}\sin(4\pi x)$$

Comparing solutions.

$D_2 = 1$ and $D_4 = -1/5$, $D_k = 0$, for all $k \neq 1, 4$.

Hence the solution is

$$U(x,t) = D_2 \cancel{}\, e^{-\pi^2 t} \sin(2\pi x) + D_4\, e^{-4\pi^2 t} \sin(4\pi x)$$

$$= e^{-\pi^2/t} \sin(2\pi x) - \frac{1}{5} e^{-4\pi^2 t} \sin(4\pi x)$$