THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

JUNE 2012

MATH2019
ENGINEERING MATHEMATICS 2E

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

TABLE OF LAPLACE TRANSFORMS AND THEOREMS

$g(t)$ is a function defined for all $t \geq 0$, and whose Laplace transform

$$G(s) = \mathcal{L}(g(t)) = \int_0^\infty e^{-st} g(t) dt$$

exists. The Heaviside step function u is defined to be

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ \frac{1}{2} & \text{for } t = a \\ 1 & \text{for } t > a \end{cases}$$

$g(t)$	$G(s) = \mathcal{L}[g(t)]$
1	$\dfrac{1}{s}$
t	$\dfrac{1}{s^2}$
$t^\nu, \nu > -1$	$\dfrac{\nu!}{s^{\nu+1}}$
$e^{-\alpha t}$	$\dfrac{1}{s+\alpha}$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$e^{-\alpha t} f(t)$	$F(s+\alpha)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$tf(t)$	$-F'(s)$

FOURIER SERIES

If $f(x)$ has period $p = 2L$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{aligned}$$

LEIBNIZ RULE FOR DIFFERENTIATING INTEGRALS

$$\frac{d}{dx}\int_u^v f(x,t)dt = \int_u^v \frac{\partial f}{\partial x}dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx}.$$

MULTIVARIABLE TAYLOR SERIES

$$\begin{aligned} f(x,y) &= f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\ &\quad + \frac{1}{2!}\left((x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) + (y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right) + \cdots \end{aligned}$$

SOME BASIC INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^{kx} dx = \frac{e^{kx}}{k} + C$$

$$\int a^x dx = \frac{1}{\ln a} a^x + C \quad \text{for } a \neq 1$$

$$\int \sin\, kx\, dx = -\frac{\cos kx}{k} + C$$

$$\int \cos kx\, dx = \frac{\sin kx}{k} + C$$

$$\int \sec^2 kx\, dx = \frac{\tan kx}{k} + C$$

$$\int \operatorname{cosec}^2 kx\, dx = -\frac{1}{k} \cot kx + C$$

$$\int \tan kx\, dx = \frac{\ln|\sec kx|}{k} + C$$

$$\int \sec kx\, dx = \frac{1}{k}(\ln|\sec kx + \tan kx|) + C$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx$$

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx$$

Answer question 1 in a separate book

1. i) Consider the double integral

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} 2y \; dy \; dx.$$

a) Sketch the region of integration.

b) Evaluate I using polar coordinates.

ii) Suppose that W is to be calculated from the formula

$$W = x^2 \ln(y)$$

using $x = 3$ and $y = 1$ with maximum possible absolute errors of $|\Delta x| = 0.04$ and $|\Delta y| = 0.05$. Find the maximum possible absolute error in the calculated value of W.

iii) Use Leibniz' theorem to find

$$\frac{d}{dt} \int_1^2 \frac{e^{xt}}{x} \; dx.$$

iv) Suppose that the atmospheric pressure P in a certain region of space is given by

$$P(x, y, z) = x^3 y^2 z.$$

a) Calculate grad(P) at the point $(1, 1, 2)$.

b) Find the rate of change of the pressure with respect to distance at the point $(1, 1, 2)$ in the direction of the vector $\mathbf{b} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

v) Consider the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors of A.

b) Hence write down the solution to the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= x + 2y \\ \frac{dy}{dt} &= 2x + y. \end{aligned}$$

Answer question 2 in a separate book

2. i) Given the vector field $\mathbf{F} = x\mathbf{i} + xyz\mathbf{j} - z\mathbf{k}$ calculate:

a) div $\mathbf{F}$.

b) curl $\mathbf{F}$.

ii) A thin triangular plate in the first quadrant is bounded by the line $x + y = 1$, and the coordinate axes. The plate has uniform density $\delta(x, y) = 1$.

a) Sketch the plate in the $x - y$ plane.

b) Without calculating any integrals write down the mass of the plate.

c) Find the coordinates of the centroid $(\bar{x}, \bar{y})$ of the plate by evaluating an appropriate double integral in Cartesian coordinates. (Note that by symmetry, $\bar{y} = \bar{x}$).

iii) Consider the differential equation

$$y'' + cy' + 9y = 0 \qquad (c > 0)$$

modeling a damped harmonic oscillator.

a) What values of the damping constant c produces underdamping?

b) Draw a possible sketch of the solution $y(t)$ for $t \geq 0$ if the system is underdamped. (Note that you are not being asked to solve the differential equation).

iv) By evaluating an appropriate line integral find the work done on a particle traveling in $\mathbb{R}^3$ through the vector field $\mathbf{F} = y\mathbf{i} + z^2\mathbf{j} + 5\mathbf{k}$ along the straight line from $(0, 0, 0)$ to $(1, 0, 4)$.

v) Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x, y) = 24x - 10y$ subject to the constraint $x^2 + y^2 = 1$.

Answer question 3 in a separate book

3. i) Let $f(x) = \begin{cases} 0 & 0 < x < 1\ ; \\ 6 & 1 \leq x \leq 2\ ; \\ f(x+2) & \text{otherwise.} \end{cases}$

a) What is the period of f ?

b) Sketch a the graph of f over the domain $-2 \leq x \leq 2$.

c) To what value does the Fourier series of f converge at $x = 1$?

ii) By completing the square in the denominator and using the table of Laplace transforms find

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+2s+5}\right).$$

iii) Using the table of Laplace transforms find

$$\mathcal{L}(t + \sin(2t) + e^{-3t}).$$

iv) a) By implementing an appropriate partial fraction decomposition find

$$\mathcal{L}^{-1}\left(\frac{8}{s(s-4)}\right).$$

b) Sketch the graph of the Heaviside step function $u(t-7)$.

c) Find the Laplace transform of $u(t-7)$.

d) Using parts a) and c) solve the initial value problem

$$y' - 4y = 8u(t-7) \qquad \text{where} \quad y(0) = 1.$$

Answer question 4 in a separate book

4. i) Define a continuous function f by

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \pi. \end{cases}$$

a) Sketch the graph of f over the interval $0 \le x \le \pi$.

b) Sketch the odd periodic extension of f over the interval $-\pi \le x \le \pi$.

c) Find the Fourier sine series of f.

ii) The temperature in a bar of length π satisfies the heat equation

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ is the temperature in $°C$. The ends of the bar are maintained at a temperature of $0°C$. Hence

$$u(0,t) = u(\pi,t) = 0 \quad \text{for all} \quad t.$$

a) Assuming a solution of the form

$$u(x,t) = F(x)G(t) \quad \text{show that}$$

$$\frac{\dot{G}}{9G} = \frac{\ddot{F}}{F} = k \quad \text{where } k \text{ is a constant.}$$

b) You may assume that only $k < 0$ yields non-trivial solutions and set $k = -(p^2)$ for some $p > 0$.

Applying the initial conditions show that $p = n,\ n = 1, 2, 3, \ldots$ and that possible solutions for $F(x)$ are

$$F_n(x) = \sin(nx) \qquad n = 1, 2, \ldots$$

c) Find all possible solutions $G_n(t)$ for $G(t)$.

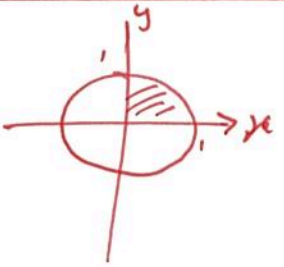
d) Suppose now that the initial temperature distribution of the bar is

$$u(x,0) = f(x)$$

where f is the function from part i). Using your series from part i) express the general solution $u(x,t)$ as a Fourier sine series.

Math 2019 June 2012 Answers

1) i) a)



b) $2/3$

ii) 0.45 iii) $\frac{1}{t}e^{2t} - \frac{1}{t}e^{t}$

iv) a) $6i+4j+k$ b) $-4/3$

v) a) $E_3 = \text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$, $E_{-1} = \text{span}\left\{\begin{pmatrix}-1\\1\end{pmatrix}\right\}$

b) $\begin{pmatrix}x\\y\end{pmatrix} = c_1 e^{3t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}-1\\1\end{pmatrix}$

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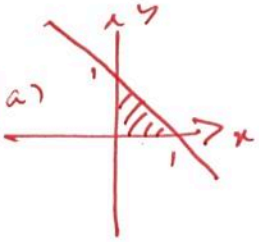
2) i) a) xz b) $\begin{pmatrix}-xy\\0\\yz\end{pmatrix}$

ii) a)



b) $\frac{1}{2}$ c) $\left(\frac{1}{3}, \frac{1}{3}\right)$

iii) a) $0 < c < 6$ b)

iv) 20 v) Max of 26 @ $\left(\frac{12}{13}, \frac{-5}{13}\right)$, Min of -26 @ $\left(-\frac{12}{13}, \frac{5}{3}\right)$

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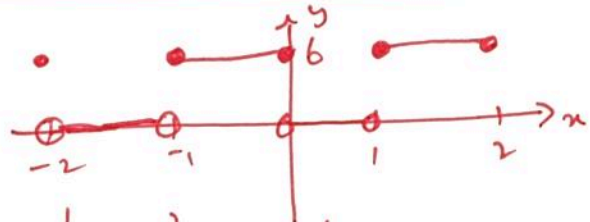
3) i) a) $T = 2$ b)



c) 3

ii) $\frac{1}{2}e^{-t}\sin(2t)$ iii) $\frac{1}{s^2} + \frac{2}{s^2+4} + \frac{1}{s+3}$

iv) a) $-2 + 2e^{4t}$ c) e^{-7s}/s d) $\left(-2+2e^{4(t-7)}\right)u(t-7) + e^{4t}$.

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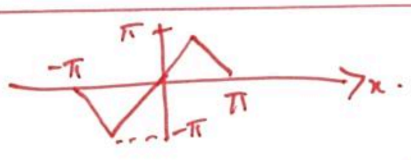
4) i) b)



$F(x) \simeq \sum_{n=1}^{\infty} \frac{4}{n^2\pi}\sin\left(\frac{n\pi}{2}\right)\sin(nx)$.

ii) c) $G_n(t) = B_n e^{-9n^2 t}$

d) $u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi}\sin\left(\frac{n\pi}{2}\right) e^{-9n^2 t}\sin(nx)$