LECTURE 3
TAYLOR SERIES AND LINEAR APPROXIMATION

The Taylor Series of $f(x, y)$ about the point (a, b) is

$$f(x,y) = f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b)$$

$$+\frac{1}{2!}\left\{(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b)\right.$$

$$\left. +(y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right\} + \text{higher-order terms.}$$

You have already seen in first year how Taylor and Maclaurin series are used to approximate complicated functions with simple polynomials. Polynomials are very easy to work with and approximations of this sort are often used in engineering and the sciences to simplify complex structures. For example $\sin(x)$ is often approximated by x when x is small.

TAYLOR SERIES FOR A SINGLE VARIABLE

Example 1 Find the Maclaurin series for $g(x, y) = e^{x-2y}$ by considering the series above.

$$g(x,y) = e^0 + \frac{(e^0 x + -2e^0 y)}{1!} + \frac{(e^0 x^2 - 2xy + 4y^2)}{2!} + \dots$$

$$= 1 + \frac{(x-2y)}{1!} + \frac{(x-2y)^2}{2!} + \dots$$

$\bigstar 1 + \frac{(x-2y)}{1!} + \frac{(x-2y)^2}{2!} + \cdots \bigstar$

But this simple approach will not always work!

When we jump up a dimension to an arbitrary function $z = f(x, y)$ the theory of Taylor Series is still workable though a little more complicated. We are now approximating complicated surfaces $z = f(x, y)$ with simpler polynomial surfaces. Things to keep in mind are that:

- The production of a Taylor series will involve substantial partial differentiation of the function in question

- A Taylor series about (a, b) will work best near (a, b)

- The more terms you take the better your approximation will be.

Example 2 Calculate the Taylor series of $f(x, y) = e^x \cos(y)$ about $(a, b) = (0, 0)$ up to and including quadratic terms. Compare the value of $f(-0.1, 0.2)$ with its Taylor approximation.

$f(0,0) = 1$

$f_x(0,0) = e^x \cos(y)\big|_{(0,0)} = 1$

$f_y(0,0) = -e^x \sin(y)\big|_{(0,0)} = 0$

$f_{xx}(0,0) = e^x \cos(y)\big|_{(0,0)} = 1$

$f_{xy}(0,0) = -e^x \sin(y)\big|_{(0,0)} = 0$

$f_{yy}(0,0) = -e^x \cos(y)\big|_{(0,0)} = -1$

$f(x,y) \approx 1 + x + \frac{1}{2}(x^2 - y^2)$

$f(-0.1, 0.2) = 0.885$

$e^x \cos y\big|_{(-0.1,\, 0.2)} = 0.88680091\ldots$

★ $1 + x + \frac{1}{2}(x^2 - y^2)$,0.88680091 vs 0.885 ★

Example 3 Find the Taylor series of $f(x, y) = x^4y^3$ about the point $(1, 2)$ up to and including quadratic terms.

$$f(1,2) = 8$$
$$f_x(1,2) = 4x^3y^3\Big|_{(1,2)} = 32$$
$$f_y(1,2) = 3x^4y^2\Big|_{(1,2)} = 12$$
$$f_{xx}(1,2) = 12x^2y^3\Big|_{(1,2)} = 96$$
$$f_{xy}(1,2) = 12x^3y^2\Big|_{(1,2)} = 48$$
$$f_{yy}(1,2) = 6x^4y\Big|_{(1,2)} = 12$$

$$f(x,y) = 8 + 32(x-1) + 12(y-2) + \frac{1}{2!}\left(96(x-1)^2 + 2\times 48(x-1)(y-2) + 12(y-2)^2\right)$$
$$= 8 + 32(x-1) + 12(y-2) + 48(x-1)^2 + 48(x-1)(y-2) + 6(y-2)^2$$

★ $8 + 32(x - 1) + 12(y - 2) + 48(x - 1)^2 + 48(x - 1)(y - 2) + 6(y - 2)^2$★

ERROR ESTIMATES

It is often the case that the independent variables x and y for $z = f(x, y)$ have a degree of uncertainty in their values. That is we need to cope with small errors Δx and Δy in x and y respectively. We may rewrite the linear approximation

$$f(x, y) \approx f(a, b) + (x - a)\frac{\partial f}{\partial x}(a, b) + (y - b)\frac{\partial f}{\partial y}(a, b)$$

as

$$f(x, y) - f(a, b) \approx (x - a)\frac{\partial f}{\partial x}(a, b) + (y - b)\frac{\partial f}{\partial y}(a, b)$$

and hence

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

This lovely little equation gives you the error in f in terms of the errors in x and y.

Example 4 Suppose that temperature T is given by $T(x,y) = x^2 + y^3$. Find the approximate error in T if $x = 1$, $y = 2$, $\Delta x = 0.1$ and $\Delta y = -0.3$.

$$\frac{\partial T}{\partial x}(1,2) = 2x\Big|_{(1,2)} = 2$$

$$\frac{\partial T}{\partial y}(1,2) = 3y^2\Big|_{(1,2)} = 12$$

$$\Delta T(1,2) = 2(0.1) + 12(-0.3) = -3.4$$

★ -3.4 ★

It is rare that we know the exact value and the exact direction of an error! Usually the error could be $\pm$ up to some maximum. We then use

$$|\Delta f| \leq |\frac{\partial f}{\partial x}||\Delta x| + |\frac{\partial f}{\partial y}||\Delta y|.$$

to generate the maximum possible error under given conditions.

Example 5 Suppose that temperature T is given by $T(x,y) = x^2 + y^3$. Find the maximum possible a) absolute error, b) relative absolute error and c) percentage absolute error in T if $x = 1$, $y = 2$, $|\Delta x| \leq 0.1$ and $|\Delta y| \leq 0.3$.

$$\begin{aligned} |\Delta T| &\leq \left|\frac{\partial T}{\partial x}\right||\Delta x| + \left|\frac{\partial T}{\partial y}\right||\Delta y| \\ &\leq |2||0.1| + |12||0.3| \\ &\leq 3.8 \end{aligned}$$

$$T(1,2) = 9$$

$$\frac{\Delta T}{T} \leq \frac{3.8}{9} \approx 0.42 = 42\%$$

★$3.8, \frac{3.8}{9} \approx 0.42, 42\%$★

Example 6 The volume V of a cone with radius r and perpendicular height h is given by $V = \dfrac{1}{3}\pi r^2 h$. Determine the maximum absolute error and the maximum percentage error in calculating V given that $r = 5$ cm and $h = 3$ cm to the nearest millimetre.

$$\frac{\partial V}{\partial r}(5, 3) = \frac{2}{3}\pi(5)(3) = 10\pi$$

$$\frac{\partial V}{\partial h}(5, 3) = \frac{1}{3}\pi(5)^2 = \frac{25\pi}{3}$$

$$\Delta r = \Delta h = 0.1 \text{ cm}, \quad V(5,3) = 78.54 \text{ cm}^3$$

$$|\Delta V| \le \left|\frac{\partial V}{\partial r}\right||\Delta r| + \left|\frac{\partial V}{\partial h}\right||\Delta h|$$

$$\le |10\pi||0.05| + \left|\frac{25\pi}{3}\right||0.05|$$

$$\le 2.87$$

$$\frac{\Delta V}{V} = \frac{2.87}{78.54} \approx 3.7\%$$

★$2.87, \dfrac{2.87}{78.54} \times 100 \approx 3.7\%$★

[3] You can now do Q 15 to 29