MATH2019 PROBLEM CLASS

EXAMPLES 2

EXTREMA, METHOD OF LAGRANGE MULTIPLIERS AND DIRECTIONAL DERIVATIVES

2014, S1 1. Find and classify the critical points of

$$f(x,y) = x^3 - 12xy + 8y^3.$$

2014, S2 2. Find and classify the critical points of

$$f(x,y) = 2x^3 - 15x^2 + 36x + y^2 + 4y - 16.$$

Also give the function value at the critical points.

2015, S2 3. Find and classify the critical points of

$$h(x,y) = 2x^3 + 3x^2y + y^2 - y.$$

Also give the function value at the critical points.

2017, S1 4. You are given the function $f(x,y) = ax^2 + y^2 - 2y$, where a is a constant not equal to zero. This function has one critical point.

i) Find the critical point of the function.

ii) Find the value of the function at the critical point.

iii) State whether the critical point can be a maximum, a minimum, or a saddle point. Write down the values of a (if they exist) for each case.

1995 5. A rectangular box without a lid is to be made from 12 m^2 of sheet metal.

i) If the length, width and height of the box are given by x, y and z metres respectively, show that the constraint function for this problem is given by:

$$g(x,y,z) = 2xz + 2yz + xy - 12 = 0\,.$$

ii) Use the method of Lagrange multipliers and the constraint function given in part i) to determine the maximum possible volume of the box.

2014, S1 6. Use the method of Lagrange multipliers to find the maximum and minimum values of $x+y$ on the circle $x^2 + y^2 - 1 = 0$.

2014, S2 7. Use the method of Lagrange multipliers to find the maximum value of the function $f(x,y) = xy$ on the curve $x^2 + y^2 = 1$.

2015, S1 8. Use the method of Lagrange multipliers to find the distance from the origin to the curve $5x^2 - 8xy + 5y^2 = 9$.

2016, S1 9. Consider the function

$$f(x, y) = 1 - x^2 - y^2.$$

i) Sketch the graph of the function f.

ii) Using the method of Lagrange multipliers, find the extreme value of $f(x, y)$ subject to the constraint $x + y = 1$.

iii) Explain why this extreme value is a maximum and not a minimum.

2016, S2 10. i) Use the method of Lagrange multipliers to find the minimum value of $x^2 + y^2$ subject to the constraint $x + y = 6$.

ii) Using your solution in i) and making no further use of the method of Lagrange multipliers find the maximum value of xy subject to the constraint $x + y = 6$.

2017, S2 11. The temperature in a region of space is given by $T(x, y) = x^2 + y^2$. A sensor measures temperature along a curve given by the equation $xy = 1$.

i) Why does the sensor measure no maximum value of the temperature?

ii) Use the method of Lagrange multipliers to find the minimum temperature measured by the sensor.

2018, S1 12. A student wants to use the method of Lagrange multipliers to find the point on the surface

$$x^2 - xy + y^2 - z^2 = 1$$

nearest to the origin. Write down the algebraic equations the student needs to solve in order to find this point. You **do not** have to solve these equations.

2018, S2 13. Use the method of Lagrange multipliers to find the extreme values of

$$f(x, y) = 12 + 3x + 4y$$

subject to the constraint

$$g(x, y) = x^2 + y^2 - 1 = 0.$$

2014, S1 14. Suppose that the atmospheric pressure P in a certain region of space is given by

$$P(x, y, z) = x^2 + y^2 + z^2.$$

i) Calculate $\nabla P = \operatorname{grad} P$ at the point $T(1, 2, 4)$.

ii) Find the rate of change of the pressure with respect to distance at the point $T(1, 2, 4)$ in the direction of the vector $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$.

iii) Give a geometrical description of the level surface L of P passing through the point $T(1, 2, 4)$.

iv) Find a Cartesian equation of the tangent plane to the level surface L of P at the point $T(1, 2, 4)$.

2014, S2 15. Suppose the atmospheric pressure P in a certain region of space is given by

$$P(x, y, z) = e^z(x^3 + y).$$

i) Calculate $\operatorname{grad} P$ at the point $(1, -2, 0)$.

ii) Find the rate of change of pressure with respect to distance at the point $(1, -2, 0)$ in the direction $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

$\boxed{\text{2015, S1}}$ 16. The temperature T in a certain region of space is given by

$$T(x,y,z) = \sin(xyz)\,.$$

i) Calculate grad T at the point $(\frac{1}{2}, \frac{1}{2}, \pi)$.

ii) Find the rate of change of temperature with respect to distance at the point $(\frac{1}{2}, \frac{1}{2}, \pi)$ in the direction $\mathbf{b} = \mathbf{i} + \mathbf{j}$.

$\boxed{\text{2015, S2}}$ 17. For the scalar field

$$\phi(x,y,z) = x^2 + 3y^2 + 4z^2$$

find:

i) gradϕ at the point $P(1,0,1)$,

ii) the directional derivative of ϕ at the point $P(1,0,1)$ in the direction of the vector $\mathbf{u} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$ and

iii) the maximum rate of change of ϕ at the point $P(1,0,1)$.

$\boxed{\text{2016, S2}}$ 18. Suppose that the pressure ϕ in a region of space is given by the scalar field

$$\phi(x,y,z) = xy^2z^3.$$

i) Calculate grad ϕ at the point $A(1,2,1)$.

ii) Find the rate of change of the pressure with respect to distance at the point $A(1,2,1)$ in the direction $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

iii) Write down a unit normal to the level surface $\phi(x,y,z) = 4$ at the point $A(1,2,1)$.

$\boxed{\text{2017, S1}}$ 19. Suppose the temperature in a region of space is given by the scalar field

$$T(x,y,z) = x^4 + y^4 + z^4.$$

i) Calculate the gradient of T at the point $P(1,1,1)$.

ii) Find the rate of change of temperature with respect to distance at the point $P(1,1,1)$ in the direction $\mathbf{i} + \mathbf{j}$.

iii) Write down the equation of the tangent plane to the surface $T(x,y,z) = 3$ at the point $P(1,1,1)$.

$\boxed{\text{2017, S2}}$ 20. Consider the scalar field

$$\phi(x,y,z) = x^2 - y^2 + z^2.$$

i) Calculate the gradient of ϕ at the point $P(1,1,0)$.

ii) Find the direction and magnitude of the maximum rate of increase of ϕ at $P(1,1,0)$.

iii) Write down any non-zero vector $\mathbf{b}$ that is perpendicular to the gradient of ϕ at the point $P(1,1,0)$.

iv) What is the rate of change of ϕ at the point $P(1,1,0)$ in the direction $\mathbf{b}$ found in part iii)?

$\boxed{\text{2018, S1}}$ 21. Consider the function

$$f(x,y) = 2e^{y-1}\sin x\,.$$

i) Calculate the Taylor series expansion of f about the point $\left(\frac{\pi}{6}, 1\right)$ up to and including **linear** terms.

ii) Determine the **direction** from the point $\left(\frac{\pi}{6}, 1\right)$ for which the change in f with distance

α) is a minimum;

β) is zero.

$\boxed{\text{2018, S2}}$ 22. Suppose that the temperature T, at a point (x, y, z) in space is given by

$$T(x, y, z) = z - x^2 - y^2.$$

i) Sketch the level surface of all points with a temperature of zero.

ii) Find $\operatorname{grad} T$.

iii) Calculate the rate of change of the temperature T at the point $P(1, 1, 0)$ in the direction of the vector $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$.