LECTURE 7
LAGRANGE MULTIPLIERS

The Method of Lagrange Multipliers

To find the local minima and maxima of $z = f(x, y)$ subject to the additional constraint that $g(x, y) = 0$, we find the values of x, y and λ that simultaneously satisfy the three equations

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \quad \text{(I)}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \quad \text{(II)}$$

$$g(x, y) = 0 \quad \text{(III)}$$

In reality we rarely have the opportunity to simply maximise or minimise a quantity. Usually additional constraints must also be met. For example the question 'Find the dimensions of the closed circular can of smallest surface area' is not really a sensible question. Much more interesting is the problem 'Find the dimensions of the closed circular can of smallest surface area whose volume is 16π cm^3. '

Situations where we need to maximise $f(x, y)$ subject to the side condition $g(x, y) = 0$ are best attacked using the method of Lagrange multipliers.

A geometrical view of the problem

We may justify the approach outlined above as follows:

Form the Lagrangian function $L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$. We refer to λ as a Lagrange multiplier, a dummy variable which is introduced to the system to generate solutions. Since $g(x, y) = 0$, L will have max/min exactly when f (constrained by g) has max/min. We now calculate the extrema of $L(x, y, \lambda)$ (a function of 3 variables) in the usual manner:

$$\frac{\partial L}{\partial x} = 0 \rightarrow \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0 \rightarrow \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial L}{\partial y} = 0 \rightarrow \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0 \rightarrow \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

$$\frac{\partial L}{\partial \lambda} = 0 \rightarrow -g(x, y) = 0 \rightarrow g(x, y) = 0$$

We will provide a more intuitive proof in Lecture 11.

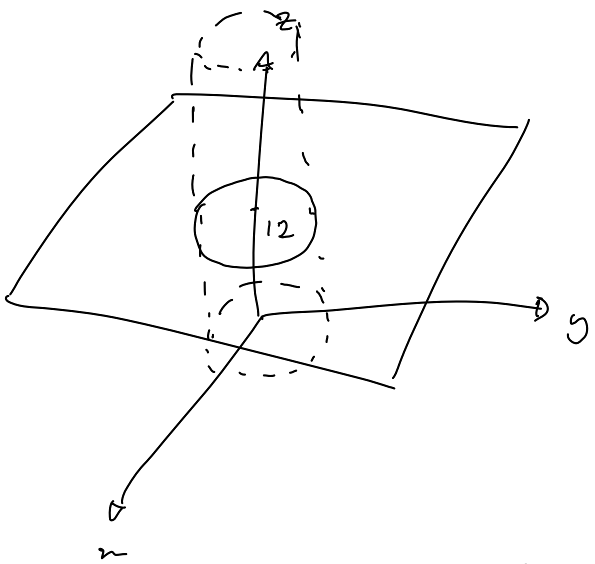
These equations, once carefully solved will generate a basket of points for consideration. We *DO NOT* test for max/min as in the previous lecture but rather simply check for max/min by direct evaluation into $f(x, y)$ and or physical considerations.

Example 1

Find the extreme values of $z = f(x, y) = 12 + 3x + 4y$ subject to the constraint

$$x^2 + y^2 - 1 = 0$$

Lets take a careful look at the problem geometrically:



Note that the question 'Find the extreme values of $z = f(x, y) = 12 + 3x + 4y$' makes no sense! A plane takes on neither a max or a min. The side condition $x^2 + y^2 = 1$ is essential here! Let's see how the Method of Lagrange Multipliers works.

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \qquad \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

$$3 = \lambda(2x) \quad (1) \qquad 4 = \lambda(2y) \quad (2)$$

(1) → (2): $4x = 3y$

$$y = \frac{4}{3}x \quad (3)$$

(3) → g: $x^2 + \left(\frac{4}{3}x\right)^2 = 1$

$$\therefore x = \pm\frac{3}{5}$$

$$\therefore \left(\frac{3}{5}, \frac{4}{5}\right), \left(-\frac{3}{5}, -\frac{4}{5}\right) \overset{\text{sub}}{\Rightarrow} g(x, y)$$

$\therefore$ max of 17 at $\left(\frac{3}{5}, \frac{4}{5}\right)$

and min of 7 at $\left(-\frac{3}{5}, -\frac{4}{5}\right)$

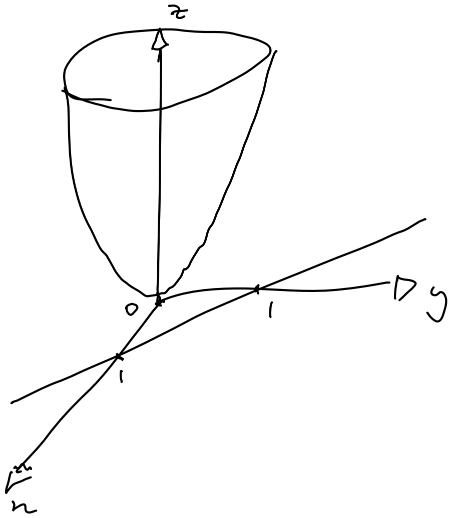
★ Maximum of 17 at $(\frac{3}{5}, \frac{4}{5})$. Minimum of 7 at $(-\frac{3}{5}, -\frac{4}{5})$ ★

Observe that we **DO NOT** run any tests here as in the last lecture. We simply see which point yields the biggest value of $f(x, y)$ and which point yields the smallest value of $f(x, y)$.

Example 2 Find the minimum value of $z = f(x,y) = x^4 + y^4$ subject to the condition

$$g(x,y) = x + y - 1 = 0$$

Geometrically:



$$\frac{\partial z}{\partial x} = \lambda \frac{\partial g}{\partial x} \qquad \frac{\partial z}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

$$4x^3 = \lambda(1) \qquad 4y^3 = \lambda(1)$$

$$\therefore x = y$$

$$x + y - 1 = 0$$

$$2y - 1 = 0$$

$$\therefore x = y = \frac{1}{2}$$

$$z = f\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^4 + \left(\tfrac{1}{2}\right)^4$$

$$\therefore z = \frac{1}{8}$$

test: $z(1,0) = 1 > \frac{1}{8}$

$\therefore$ min of $\frac{1}{8}$ at $\left(\frac{1}{2}, \frac{1}{2}\right)$

★ Minimum of $\dfrac{1}{8}$ at $(\frac{1}{2}, \frac{1}{2})$ ★

How do we know this is a min and not a max?

Example 3 A closed cylindrical soup can needs to hold 16π cm^3 of soup. If we wish to make the can using a minimal amount of sheet metal, find the radius and the height of the can. Is there a maximal amount of sheet metal for this can?

Surface area $= S = 2\pi r^2 + 2\pi rh$.

Volume$= \pi r^2 h = 16\pi$.

Hence the Lagrange question is to minimise

$$S = 2\pi r^2 + 2\pi rh$$

subject to the constraint that

$$g(r, h) = \pi r^2 h - 16\pi = 0$$

$$\frac{\partial S}{\partial r} = \lambda \frac{\partial g}{\partial r} \quad \to \quad 4\pi r + 2\pi h = \lambda 2\pi rh \quad \to \quad 2r + h = \lambda rh. \qquad (\mathbf{I})$$

$$\frac{\partial S}{\partial h} = \lambda \frac{\partial g}{\partial h} \quad \to \quad 2\pi r = \lambda \pi r^2 \quad \to \quad 2 = \lambda r. \qquad (\mathbf{II}) \quad (\text{Note } r \neq 0)$$

We also have:

$$g(r, h) = 0 \quad \to \quad \pi r^2 h - 16\pi = 0 \quad \to \quad r^2 h = 16. \qquad (\mathbf{III})$$

We have our three equations in three unknowns! Now:

(**II**) $\to \lambda = \dfrac{2}{r}$. Substituting in (**I**) yields $2r + h = \dfrac{2}{r} rh \to 2r + h = 2h \to 2r = h$.

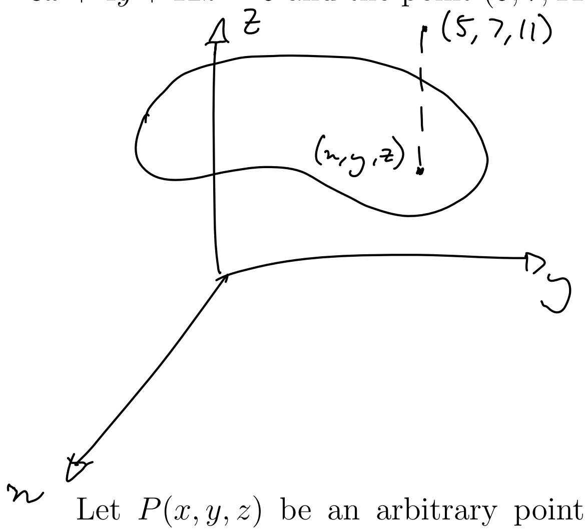
Finally we sub into (**III**) to get $r^2(2r) = 16 \to r^3 = 8 \to r = 2 \to h = 4$.

Note finally that we know this yields minimum surface area since there is clearly no maximum surface area. That is, we can make these cans as huge as we wish and still have them hold only 16π cubic centimetres!

★ $r = 2$ cm, $h = 4$ cm ★

The method of Lagrange extends naturally to functions of three variables instead of just 2 variables. The following example is a common quiz question.

Example 4 Use Lagrange multipliers to find the minimum distance between the plane $3x + 4y + 12z = 6$ and the point $(5, 7, 11)$.



Let $P(x, y, z)$ be an arbitrary point in space. The distance from the fixed point $(5, 7, 11)$ to $P(x, y, z)$ is then given by

$$\sqrt{(x-5)^2 + (y-7)^2 + (z-11)^2}$$

Thus we wish to minimise $\sqrt{(x-5)^2 + (y-7)^2 + (z-11)^2}$ subject to the constraint that $g(x, y, z) = 3x + 4y + 12z - 6 = 0$.

That is, we wish to minimize the distance from $(5, 7, 11)$ to $P(x, y, z)$ subject to the condition that the point P lies on the given plane. We have an extra dimension here however the method of Lagrange still works in much the same way except that we will now have 4 equations in 4 unknowns x, y, z and λ !

Note first that we can simplify our calculations by dropping the square root and dealing instead with the equivalent problem:

Minimise $s(x, y, z) = (x-5)^2 + (y-7)^2 + (z-11)^2$ subject to the constraint that $g(x, y, z) = 3x + 4y + 12z - 6 = 0$.

Our four equations in 4 unknowns are:

$$\begin{aligned}
\frac{\partial s}{\partial x} &= \lambda \frac{\partial g}{\partial x} &\rightarrow\quad & 2(x-5) = 3\lambda. \\
\frac{\partial s}{\partial y} &= \lambda \frac{\partial g}{\partial y} &\rightarrow\quad & 2(y-7) = 4\lambda. \\
\frac{\partial s}{\partial z} &= \lambda \frac{\partial g}{\partial z} &\rightarrow\quad & 2(z-11) = 12\lambda. \\
&\text{and the side condition} &\rightarrow\quad & 3x + 4y + 12z = 6.
\end{aligned}$$

For homework please read through the solution on the next page carefully.

$$\frac{\partial s}{\partial x} = \lambda\frac{\partial g}{\partial x} \quad \to 2(x-5) = 3\lambda \quad \to x-5 = \frac{3}{2}\lambda \quad \to x = 5+\frac{3}{2}\lambda.$$
$$\frac{\partial s}{\partial y} = \lambda\frac{\partial g}{\partial y} \quad \to 2(y-7) = 4\lambda \quad \to y-7 = \frac{4}{2}\lambda \quad \to y = 7+2\lambda.$$
$$\frac{\partial s}{\partial z} = \lambda\frac{\partial g}{\partial z} \quad \to 2(z-11) = 12\lambda \quad \to z-11 = \frac{12}{2}\lambda \quad \to z = 11+6\lambda.$$

Substituting into $3x+4y+12z=6$ yields $15+\frac{9}{2}\lambda+28+8\lambda+132+72\lambda=6$ and hence $84.5\lambda = -169 \to \lambda = -2$. It follows that $x = 5+\frac{3}{2}(-2) = 2$, $y = 7+2(-2) = 3$ and $z = 11+6(-2) = -1$. The closest point on the plane is therefore $(2,3,-1)$ and the shortest distance is $\sqrt{(2-5)^2+(3-7)^2+(-1-11)^2} = \sqrt{169} = 13$.

Note that this must be the shortest distance, as the greatest distance is unbounded.

★ shortest distance of 13 at the point on the plane $(2,3,-1)$ ★

The next lecture (Lecture 8) is a problem class and the lecture after that (Lecture 9) will be a revision of first year linear algebra. In preparation for Lecture 9 please revise all of your first year material on vectors, dot and cross products of vectors, the Cartesian equation of a plane in space and the parametric vector equation of a lines and planes in space. You may wish to look at the revision first year linear algebra notes on Math2019 Moodle pp 49-84.

[7] You can now do Q 39 to 47