THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

JUNE 2015

MATH2019
ENGINEERING MATHEMATICS 2E

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – **??**

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED “UNSW APPROVED” STICKER MAY BE USED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

TABLE OF LAPLACE TRANSFORMS AND THEOREMS

$g(t)$ is a function defined for all $t \geq 0$, and whose Laplace transform

$$G(s) = \mathcal{L}(g(t)) = \int_0^\infty e^{-st} g(t)dt$$

exists. The Heaviside step function u is defined to be

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ \frac{1}{2} & \text{for } t = a \\ 1 & \text{for } t > a \end{cases}$$

$g(t)$	$G(s) = \mathcal{L}[g(t)]$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^\nu, \nu > -1$	$\frac{\nu!}{s^{\nu+1}}$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$e^{-\alpha t}f(t)$	$F(s+\alpha)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$tf(t)$	$-F'(s)$

Please see over ...

FOURIER SERIES

If $f(x)$ has period $p = 2L$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}\,x\right) + b_n \sin\left(\frac{n\pi}{L}\,x\right)\right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}\,x\right)dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}\,x\right)dx \end{aligned}$$

LEIBNIZ' THEOREM

$$\frac{d}{dx}\int_u^v f(x,t)dt = \int_u^v \frac{\partial f}{\partial x}dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx}$$

MULTIVARIABLE TAYLOR SERIES

$$\begin{aligned} f(x,y) &= f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) + \\ &\quad + \frac{1}{2!}\left((x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right)\cdots \end{aligned}$$

SOME BASIC INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^{kx} dx = \frac{e^{kx}}{k} + C$$

$$\int a^x dx = \frac{1}{\ln a} a^x + C \quad \text{for } a \neq 1$$

$$\int \sin\ kx\, dx = -\frac{\cos kx}{k} + C$$

$$\int \cos kx\, dx = \frac{\sin kx}{k} + C$$

$$\int \sec^2 kx\, dx = \frac{\tan kx}{k} + C$$

$$\int \mathrm{cosec}^2 kx\, dx = -\frac{1}{k}\cot kx + C$$

$$\int \tan kx\, dx = \frac{\ln|\sec kx|}{k} + C$$

$$\int \sec kx\, dx = \frac{1}{k}(\ln|\sec kx + \tan kx|) + C$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx$$

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx$$

Please see over ...

Answer question 1 in a separate book

1. a) The power output of a photovoltaic cell with square cross-section is given by

$$P = \eta E l^2$$

where η is the efficiency of the cell, l is the length of each side of the cell, and E is the energy input from the sun (which is assumed to be constant). If η is increased by 15% and l is decreased by 5%, use the chain rule to estimate the percentage change in the power output of the cell. [Note that you do not need to know the value of E.]

b) The temperature T in a certain region of space is given by

$$T(x, y, z) = \sin(xyz)\,.$$

i) Calculate grad(T) at the point $(\frac{1}{2}, \frac{1}{2}, \pi)$.

ii) Find the rate of change of temperature with respect to distance at the point $(\frac{1}{2}, \frac{1}{2}, \pi)$ in the direction $\mathbf{b} = \mathbf{i} + \mathbf{j}$.

c) Use Leibniz' theorem to calculate

$$\frac{d}{dy} \int_{y^2}^{1} \frac{\sin(xy)}{x}\, dx.$$

d) Use the method of Lagrange multipliers to find the closest distance from the curve $5x^2 - 8xy + 5y^2 = 9$ to the origin.

Answer question 2 in a separate book

2. a) Given the vector field $\mathbf{F} = xz\ \mathbf{i} + y^2\ \mathbf{j} + yz\ \mathbf{k}$ calculate:

i) $\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$

ii) $\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$

iii) $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = \nabla \cdot (\nabla \times \mathbf{F})$

b) Let $\mathcal{C}$ denote the path taken by a particle travelling anticlockwise around the unit circle, starting *and* ending at the point $(1, 0)$ [i.e. the particle travels completely around the circle].

i) Write down a vector function $\mathbf{r}(t)$ that describes the path $\mathcal{C}$ and give the value of t at the start and the end of the path.

ii) If $\mathbf{F} = -3y\ \mathbf{i} + 3x\ \mathbf{j}$ evaluate the line integral

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

c) Consider the double integral

$$\int_0^1 \int_0^{1-x^2} \frac{y}{\sqrt{1-y}}\ dy\ dx.$$

i) Sketch the region of integration.

ii) Evaluate the double integral by first reversing the order of integration.

d) Use the method of undetermined coefficients to solve the second order differential equation

$$y'' - 4y' + 4y = 5 \sin t.$$

Answer question 3 in a separate book

3. a) Find:

i) $\mathcal{L}\left(t^3e^{\pi t}\right)$.

ii) $\mathcal{L}^{-1}\left(\dfrac{3-s}{s^2-4s+5}\right)$.

b) The function $f(t)$ is given by

$$f(t) = \begin{cases} 0 & \text{for } 0 \le t < 1, \\ t-1 & \text{for } 1 \le t < 3, \\ 2 & \text{for } t \ge 3. \end{cases}$$

i) Sketch the function $f(t)$ for $0 \le t \le 4$.

ii) Write $f(t)$ in terms of the Heaviside step function $u(t-a)$.

iii) Hence, or otherwise, find the Laplace transform of $f(t)$.

c) Use the Laplace transform method to solve the initial value problem

$$y'' - 4y = 8u(t-1) \qquad \text{with} \qquad y(0) = 1,\ y'(0) = 2,$$

where $u(t-1)$ is a Heaviside step function.

d) The equations governing the response of a bridge to an earthquake are found to satisfy

$$\begin{aligned} \frac{dx}{dt} &= -x + ay, \\ \frac{dy}{dt} &= ax - y. \end{aligned}$$

where $a > 0$ is a parameter that depends on the material used for the bridge.

i) Express this set of differential equations in the form

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x}, \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

and find the eigenvalues and eigenvectors of the matrix A.

ii) Hence, or otherwise, write down a general solution for the problem.

iii) For what values of a will the solution grow with increasing t?

Answer question 4 in a separate book

4. a) Let $F(x)$ satisfy the differential equation

$$\frac{d^2F}{dx^2} = k\,F,$$

with boundary conditions $F(0) = 0$ and $F(\pi) = 0$. By considering separately the cases of $k > 0$, $k = 0$, $k < 0$, find the general solution for $F(x)$.

b) Define the piecewise continuous function f by

$$f(x) = \begin{cases} 1, & 0 \leq x < \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} \leq x < \pi. \end{cases}$$

i) Sketch the even periodic extension of f on the interval $-2\pi \leq x \leq 2\pi$.

ii) Show that the Fourier cosine series of f is given by

$$f(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2(-1)^k}{\pi\,(2k+1)} \cos\left[(2k+1)\,x\right].$$

iii) To what value will the Fourier series converge at $x = \frac{\pi}{2}$?

c) Consider the differential equation

$$\frac{d^2y}{dx^2} + 8y = f(x),$$

where $f(x)$ is the function defined in part (b).

i) Find the homogeneous solution to the differential equation. What is the natural (or fundamental) frequency of the system?

ii) Find a particular solution of the differential equation in the form of a Fourier series.

iii) Calculate the numerical value of the first three non-zero terms of this particular solution, and comment on which term dominates the solution.

MATH 2019 S1 2015 SOLUTIONS

Q1. a) $P = \eta E l^2$ $\qquad \frac{\Delta \eta}{\eta} = +0.15, \ \frac{\Delta l}{l} = -0.05$

$$\Delta P \simeq \frac{\partial P}{\partial \eta}\Delta\eta + \frac{\partial P}{\partial l}\Delta l$$
$$= El^2\,\Delta\eta + 2\eta E l\,\Delta l$$

$$\frac{\Delta P}{P} \simeq \frac{\cancel{El^2}\Delta\eta}{\eta \cancel{El^2}} + \frac{2\cancel{\eta E}\cancel{l}\Delta l}{\cancel{\eta E}l^2}$$
$$= \frac{\Delta\eta}{\eta} + \frac{2\Delta l}{l}$$
$$= 0.15 + 2(-0.05) = 0.05$$

$\Rightarrow$ 5% increase in power output.

b) $T = \sin(xyz)$ $\qquad P(\frac{1}{2}, \frac{1}{2}, \pi)$

i) $\text{grad}\, T = \underline{\nabla} T = T_x \underline{i} + T_y \underline{j} + T_z \underline{k}$

$$= yz\cos(xyz)\underline{i} + xz\cos(xyz)\underline{j} + xy\cos(xyz)\underline{k}$$

$$\underline{\nabla} T(\tfrac{1}{2}, \tfrac{1}{2}, \pi) = \frac{\pi}{2}\cos\left(\frac{\pi}{4}\right)\underline{i} + \frac{\pi}{2}\cos\left(\frac{\pi}{4}\right)\underline{j} + \frac{1}{4}\cos\left(\frac{\pi}{4}\right)\underline{k}$$

$$= \frac{\pi}{2\sqrt{2}}\underline{i} + \frac{\pi}{2\sqrt{2}}\underline{j} + \frac{1}{4\sqrt{2}}\underline{k}$$

ii) $\underline{b} = \underline{i} + \underline{j} \Rightarrow |\underline{b}| = \sqrt{1^2 + (+1)^2} = \sqrt{2}, \Rightarrow \hat{\underline{b}} = \frac{1}{\sqrt{2}}\underline{i} + \frac{1}{\sqrt{2}}\underline{j}$

$$\hat{\underline{b}} \cdot \underline{\nabla} T(\tfrac{1}{2}, \tfrac{1}{2}, \pi) = \frac{\pi}{2\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{\pi}{2\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

c) $$\frac{d}{dy}\int_{y^2}^{1} \frac{\sin(xy)}{x}\, dx$$

$$= \int_{y^2}^{1} \frac{\partial}{\partial y}\left\{\frac{\sin(xy)}{x}\right\} dx + \cancel{\frac{d}{dy}(1)\left[\frac{\sin(xy)}{x}\right]_{x=1}}$$

$$- \frac{d}{dy}(y^2)\left[\frac{\sin(xy)}{x}\right]_{x=y^2}$$

$$= \int_{y^2}^{1} \frac{\cancel{x}\cos(xy)}{\cancel{x}}\, dx - 2y\,\frac{\sin y^3}{y^2}$$

$$= \left[\frac{\sin xy}{y}\right]_{y^2}^{1} - \frac{2\sin y^3}{y}$$

$$= \frac{\sin y}{y} - \frac{\sin y^3}{y} - \frac{2\sin y^3}{y}$$

$$= \frac{\sin y}{y} - \frac{3\sin y^3}{y}$$

d) $f(x,y) = x^2 + y^2 =$ square distance

$g(x,y) = 5x^2 - 8xy + 5y^2 - 9 = 0$ (constraint)

$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$

$= x^2 + y^2 - \lambda(5x^2 - 8xy + 5y^2 - 9)$

$L_x = 2x - 10\lambda x + 8\lambda y = 0 \Rightarrow x(1-5\lambda) = -4\lambda y$ (1)

$L_y = 2y + 8\lambda x - 10\lambda y = 0 \Rightarrow y(1-5\lambda) = -4\lambda x$ (2)

$L_\lambda = 0 \Rightarrow 5x^2 - 8xy + 5y^2 = 9$ (3)

$(1) \times (2) \Rightarrow xy(1-5\lambda)^2 = 16\lambda^2 xy$

$xy(25\lambda^2 - 10\lambda + 1) = 16\lambda^2 xy$

$(9\lambda^2 - 10\lambda + 1)xy = 0$

$\Rightarrow xy = 0$ or $9\lambda^2 - 10\lambda + 1$

consider $xy = 0 \Rightarrow x = 0$ OR $y = 0$

But if $x = 0$ (1) $\Rightarrow y = 0$ and if $y = 0$ (2) $\Rightarrow x = 0$
but the point $(0,0)$ is not of the curve $g(x,y) = 0$

So $9\lambda^2 - 10\lambda + 1 = 0$

$(9\lambda - 1)(\lambda - 1) = 0 \Rightarrow \lambda = \frac{1}{9}$ or $\lambda = 1$.

$\underline{\lambda = 1}$: $(1) \Rightarrow x(1-5) = -4x = -4y \Rightarrow \boxed{x = y}$

$(3) \Rightarrow \quad 5x^2 - 8x^2 + 5x^2 = 2x^2 = 9$

$\Rightarrow \quad x = \pm \frac{3}{\sqrt{2}} \quad y = \pm \frac{3}{\sqrt{2}}$

$\text{distance} = \sqrt{x^2 + y^2} = \sqrt{\frac{9}{2} + \frac{9}{2}} = 3.$

$\underline{\lambda = \frac{1}{9}}$: $(1) \Rightarrow \quad x\left(1 - \frac{5}{9}\right) = \frac{4}{9}x = -\frac{4}{9}y \Rightarrow \boxed{x = -y}$

$(3) \Rightarrow \quad 5x^2 + 8x^2 + 5x^2 = 18x^2 = 9$

$\Rightarrow \quad x = \pm \frac{1}{\sqrt{2}}, \; y = \pm \frac{1}{\sqrt{2}}$

$\text{distance} = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$

$\Rightarrow$

closest distance to origin is 1.

Q2. a) $\underline{F} = xz\,\underline{i} + y^2\,\underline{j} + yz\,\underline{k}$

i) $\text{div}\,\underline{F} = \underline{\nabla}\cdot\underline{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

$= z + 2y + y = z + 3y$

ii) $\text{curl}\,\underline{F} = \underline{\nabla}\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & y^2 & yz \end{vmatrix}$

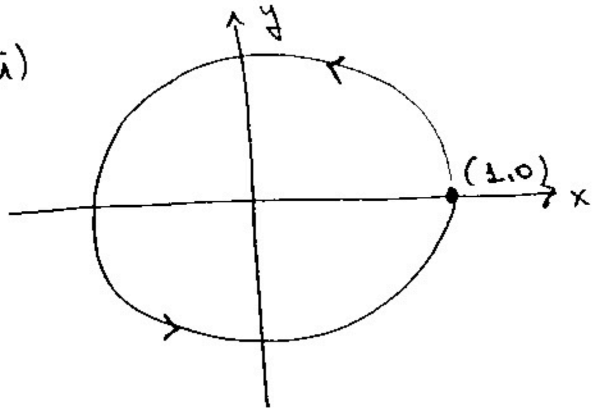
$= \underline{i}(z - 0) - \underline{j}(0 - x) + \underline{k}(0 - 0)$

$= z\,\underline{i} + x\,\underline{j}$

iii) $\text{div}\,(\text{curl}\,F) = \underline{\nabla}\cdot(\underline{\nabla}\times\underline{F}) = 0 + 0 = 0$

b) i) $\underline{r}(t) = (\cos t, \sin t)$ start : $t = 0$
end : $t = 2\pi$

ii)



$x = \cos t \Rightarrow dx = -\sin t\,dt$

$y = \sin t \Rightarrow dy = \cos t\,dt$

$F_1 = -3y = -3\sin t$

$F_2 = 3x = 3\cos t$

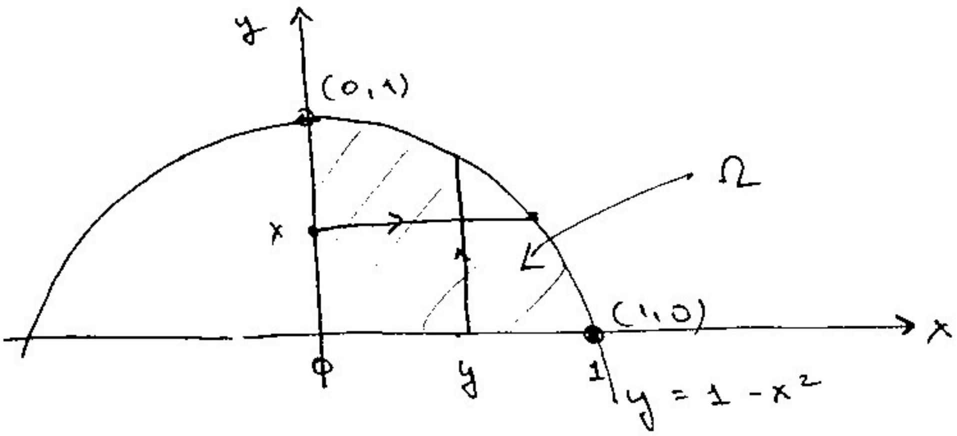
$\oint_c \underline{F}\cdot d\underline{r} = \int_0^{2\pi} (-3\sin t)(-\sin t\,dt) + (3\cos t)(\cos t\,dt)$

$= \int_0^{2\pi} (3\sin^2 t + 3\cos^2 t)\,dt$

$= \int_0^{2\pi} 3\,dt = 3\big[t\big]_0^{2\pi} = 6\pi$

c) $$\int_0^1 \int_0^{1-x^2} \frac{y}{\sqrt{1-y}} \, dy \, dx$$

i) $x : 0 \longrightarrow 1$, $y : 0 \longrightarrow 1 - x^2$



ii) $y : 0 \longrightarrow 1$, $x : 0 \longrightarrow \sqrt{1-y}$

Note that $y = 1 - x^2 \Rightarrow x = \pm\sqrt{1+y}$
but clearly here need positive root.

$$\int_0^1 \int_0^{\sqrt{1-y}} \frac{y}{\sqrt{1-y}} \, dx \, dy = \int_0^1 \left[\frac{y \cdot x}{\sqrt{1-y}} \right]_0^{\sqrt{1-y}} dy$$

$$= \int_0^1 y \, dy = \left[\frac{y^2}{2} \right]_0^1 = \frac{1}{2}$$

d) $$y'' - 4y' + 4y = 5\sin t$$

$y = y_h + y_p$: $y_h'' - 4y_h' + 4y_h = 0$

let $y_h \sim e^{\lambda t}$ $\Rightarrow$ $\lambda^2 - 4\lambda + 4 = 0$

$\Rightarrow$ $(\lambda - 2)^2 = 0$ $\Rightarrow$ $\lambda = 2$ double root

$\Rightarrow$ $y_h = (At + B)e^{2t}$

guess $y_p = C\sin t + D\cos t$

$y_p' = C\cos t - D\sin t$

$y_p'' = -C\sin t - D\cos t$

$\therefore$ $$y_p'' - 4y_p' + 4y_p = (-C + 4D + 4C)\sin t + (-D - 4C + 4D)\cos t$$

$$= (4D + 3C)\sin t + (3D - 4C)\cos t$$

$$= 5\sin t$$

$\Rightarrow$ $4D + 3C = 5$ and $3D - 4C = 0$ $\Rightarrow$ $D = \frac{4}{3}C$

$\Rightarrow$ $\frac{16}{3}C + 3C = \frac{25C}{3} = 5$ $\Rightarrow$ $C = \frac{3}{5}$, $D = \frac{4}{5}$

so $$y(t) = (At + B)e^{2t} + \frac{3}{5}\sin t + \frac{4}{5}\cos t$$

Q3. a) i) $\mathcal{L}\left\{t^3 e^{\pi t}\right\} = \dfrac{3!}{(s-\pi)^4}$

ii) $\mathcal{L}^{-1}\left\{\dfrac{3-s}{s^2-4s+5}\right\}$

complete the squares $s^2-4s+5 = (s-A)^2 + B$
$= s^2 - 2As + A^2 + B$

$\Rightarrow A = 2 \;,\; A^2 + B = 5 \quad \Rightarrow \quad B = 5 - 2^2 = 1$

$\therefore \; s^2 - 4s + 5 = (s-2)^2 + 1$

so $\dfrac{3-s}{s^2-4s+5} = \dfrac{1-(s-2)}{(s-2)^2+1}$

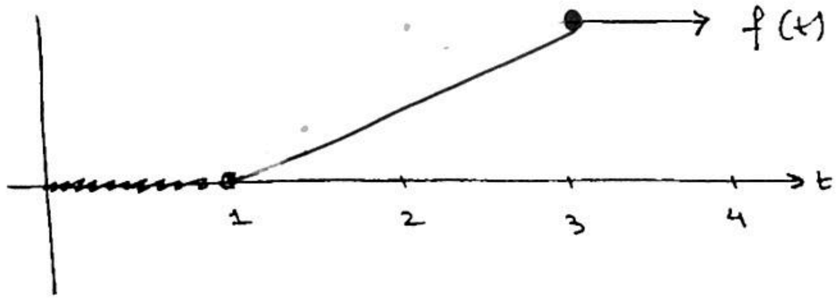
$$\mathcal{L}^{-1}\left\{\frac{1-(s-2)}{(s-2)^2+1^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+1^2}\right\} - \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+1^2}\right\}$$

$$= e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\} - e^{2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1^2}\right\}$$

$$= e^{2t}(\sin t - \cos t)$$

b) $$f(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t < 1 \\ t-1 & \text{for} \quad 1 \leq t < 3 \\ 2 & \text{for} \quad t \geq 3 \end{cases}$$

i)



ii)

$$f(t) = (t-1)\left[u(t-1) - u(t-3)\right] + 2\,u(t-3)$$

$$= (t-1)\,u(t-1) + (2-t+1)\,u(t-3)$$

$$= (t-1)\,u(t-1) - (t-3)\,u(t-3)$$

iii) $$\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s^2}$$

c) $y'' - 4y = 8u(t-1) \qquad y(0) = 1,\ y'(0) = 2$

$$s^2 Y(s) - s(1) - 2 - 4Y(s) = \frac{8e^{-s}}{s}$$

$$(s^2 - 4)Y(s) = s + 2 + \frac{8e^{-s}}{s}$$

$$\uparrow$$
$$(s-2)(s+2)$$

$$Y(s) = \frac{\cancel{s+2}}{(s-2)\cancel{(s+2)}} + \frac{8e^{-s}}{s(s-2)(s+2)}$$
$$= \frac{1}{s-2} + \frac{8e^{-s}}{s(s-2)(s+2)}$$

$$\frac{8}{s(s-2)(s+2)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+2}$$
$$= \frac{A(s-2)(s+2) + Bs(s+2) + Cs(s-2)}{s(s-2)(s+2)}$$

$$\Rightarrow \quad A(s-2)(s+2) + Bs(s+2) + Cs(s-2) = 8$$

at $s = 0 \qquad -4A = 8 \qquad \Rightarrow \quad A = -2$

at $s = 2 \qquad 8B = 8 \qquad \Rightarrow \quad B = 1$

at $s = -2 \qquad 8C = 8 \qquad \Rightarrow \quad C = 1$

$$\therefore \quad Y(s) = \frac{1}{s-2} + e^{-s}\left\{ -\frac{2}{s} + \frac{1}{s-2} + \frac{1}{s+2} \right\}$$

$$\Rightarrow \quad y(t) = e^{2t} + u(t-1)\left\{ -2 + e^{2(t-1)} + e^{-2(t-1)} \right\}$$

d) i) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & a \\ a & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$$A = \begin{pmatrix} -1 & a \\ a & -1 \end{pmatrix}$$

$$|A - \lambda \mathbb{1}| = \begin{vmatrix} -1-\lambda & a \\ a & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 - a^2 = 1 + 2\lambda + \lambda^2 - a^2 = 0$$

$$\lambda^2 + 2\lambda + 1 - a^2 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4(1-a^2)}}{2} = -1 \pm \sqrt{a^2} = -1 \pm a$$

$\underline{\lambda_1 = -1 + a}$: $\begin{array}{cc|c} -a & a & 0 \\ a & -a & 0 \end{array} \longrightarrow \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}$

$\Rightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = -1 - a}$: $\begin{array}{cc|c} a & a & 0 \\ a & a & 0 \end{array} \longrightarrow \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}$

$\Rightarrow \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

ii) $\underline{x} = A\,\underline{v}_1 e^{\lambda_1 t} + B\underline{v}_2 e^{\lambda_2 t} = A\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{(-1+a)t} + B\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{(-1-a)t}$

iii) for $a > 1$, there is a growing solution and the bridge collapses!

Q4 a) $\dfrac{d^2F}{dx^2} = k F$ $\qquad F(0) = 0 = F(\pi)$

case (i) $k = p^2 > 0 \Rightarrow F'' = p^2 F$ $\qquad$ let $F \sim e^{\lambda x}$

$\rightarrow \; (\lambda^2 - p^2) = 0 \quad \rightarrow \quad \lambda = \pm p$

$\rightarrow \; F = A e^{px} + B e^{-px}$

boundary conditions : $F(0) = A + B = 0 \rightarrow A = -B$
$F(\pi) = A(e^{p\pi} - e^{-p\pi}) = 0$

$\rightarrow$ $A = 0$ (trivial solution) or $e^{2p\pi} = 1 \rightarrow p = 0$ (contradiction

case (ii) $k = 0 \rightarrow F'' = 0$

$\rightarrow F = A + Bx$

boundary conditions : $F(0) = A = 0$
$F(\pi) = B\pi = 0 \rightarrow B = 0$ (trivial solution

case (iii) $k = -p^2 < 0 \rightarrow F'' = -p^2 F$ $\qquad$ let $F \sim e^{\lambda x}$

$\rightarrow \; \lambda^2 + p^2 = 0 \rightarrow \lambda = \pm ip$

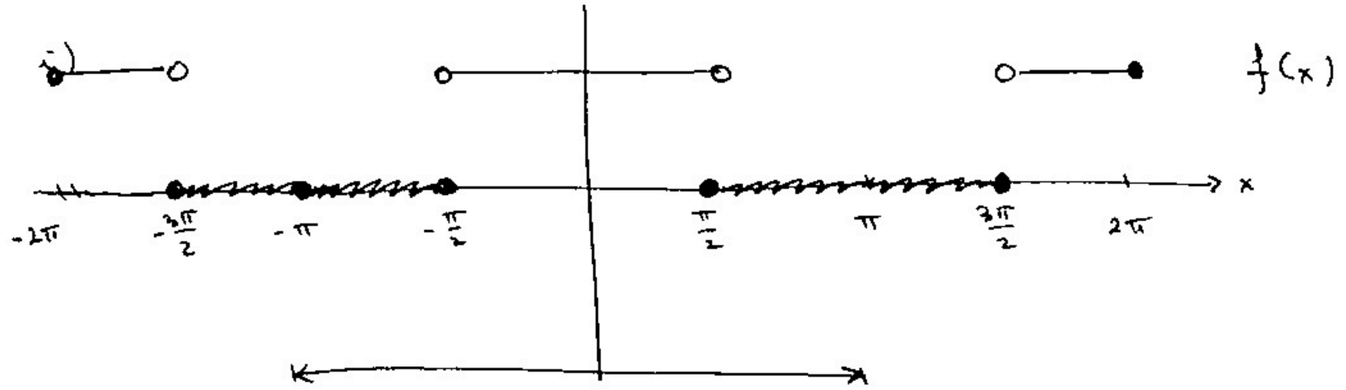
$\rightarrow \; F = A \cos px + B \sin px$

boundary conditions: $F(0) = A = 0$
$F(\pi) = B \sin p\pi = 0 \rightarrow B = 0$ (trivial solution)
or $p = n = 1, 2, 3, \dots$

Thus $\quad F(x) = \sum_{n=1}^{\infty} B_n \sin nx$

Q4 b) $f = \begin{cases} 1 , & 0 \le x < \frac{\pi}{2} \\ 0 , & \frac{\pi}{2} \le x < \pi \end{cases}$

i)



$T = 2L = 2\pi$

ii) $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$ ($b_n = 0$, even funct

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{even}} \, dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_{0}^{\pi/2} 1 \cdot dx + \cancel{\frac{1}{\pi} \int_{\pi/2}^{\pi} 0 \, dx}$$

$$= \frac{1}{\pi} \Big[x \Big]_{0}^{\pi/2} = \frac{1}{\pi} \, \frac{\pi}{2} = \frac{1}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f \underbrace{(x) \cos(nx)}_{\text{even}} \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos(nx) \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi/2} 1 \cos(nx)\, dx + \cancel{\frac{2}{\pi} \int_{\pi/2}^{\pi} 0 \cdot \cos nx \, dx}$$

$$= \frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi/2}$$

$$= \frac{2}{\pi n}\sin\frac{n\pi}{2}$$

let $n = \text{even} = 2k$: $\qquad (k = 1, 2, 3, \cdots)$

$$a_{2k} = \frac{2}{\pi(2k)}\sin k\pi = 0$$

let $n = \text{odd} = 2k+1$: $\qquad (k = 0, 1, 2, 3, \cdots)$

$$a_{2k+1} = \frac{2}{\pi(2k+1)}\sin\frac{(2k+1)\pi}{2}$$

$$= \frac{2}{\pi(2k+1)}\sin\left(k+\tfrac{1}{2}\right)\pi$$

$$= \frac{2(-1)^k}{\pi(2k+1)}$$

so $\quad f(x) = \dfrac{1}{2} + \displaystyle\sum_{k=0}^{\infty}\frac{2(-1)^k}{\pi(2k+1)}\cos(2k+1)x$

iii) at $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = \frac{1}{2} + \sum_{k=0}^{\infty}\frac{2(-1)^k}{\pi(2k+1)}\cancelto{0}{\cos\left(k+\tfrac{1}{2}\right)\pi} = \frac{1}{2}$$

or $\quad f\left(\frac{\pi}{2}\right) = \dfrac{f\left(\frac{\pi}{2}^+\right) - f\left(\frac{\pi}{2}^-\right)}{2} = \dfrac{1-0}{2} = \dfrac{1}{2}$

Q4 c) $\dfrac{d^2y}{dx^2} + 8y = f(x)$

i) $\dfrac{d^2y_h}{dx^2} + 8y_h = 0$ let $y_h \sim e^{\lambda x}$

$\lambda^2 + 8 = 0 \Rightarrow \lambda = \pm i\sqrt{8}$

$y_h = A\cos\sqrt{8}\,x + B\sin\sqrt{8}\,x$

$\sqrt{2}$ = natural frequency

ii) let $y_p = C_0 + \sum_{k=0}^{\infty} C_{2k+1}\cos(2k+1)x + \sum_{k=0}^{\infty} D_{2k+1}\sin(2k+1)x$

$\Rightarrow y_p'' = -\sum_{k=0}^{\infty}(2k+1)^2 C_{2k+1}\cos(2k+1)x - \sum_{k=0}^{\infty}(2k+1)D_{2k+1}\sin(2k+1)x$

Then $y_p'' + 8y_p = 8C_0 + \sum_{k=0}^{\infty}\left[8-(2k+1)^2\right]C_{2k+1}\cos(2k+1)x + \sum_{k=0}^{\infty}\left[8-(2k+1)^2\right]D_{2k+1}\sin(2k+1)x$

$= \frac{1}{2} + \sum_{k=0}^{\infty}\dfrac{2(-1)^k}{\pi(2k+1)}\cos(2k+1)x$

$\Rightarrow D_{2k+1} = 0, \quad C_0 = \frac{1}{16}$

$C_{2k+1} = \dfrac{1}{8-(2k+1)^2}\,\dfrac{2(-1)^k}{\pi(2k+1)}$

$y_p = \frac{1}{16} + \sum_{k=0}^{\infty}\dfrac{1}{8-(2k+1)^2}\,\dfrac{2(-1)^k}{\pi(2k+1)}\cos(2k+1)x$

iii) $$C_0 = \frac{1}{16} = 0.0625$$

$k=0$ $$C_1 = \frac{2}{\pi}\frac{(-1)^0}{1}\frac{1}{8-1^2} = \frac{2}{7\pi} = 0.0909$$

$k=1$ $$C_3 = \frac{2}{\pi}\frac{(-1)^1}{3}\frac{1}{8-9} = \frac{2}{3\pi} = 0.212$$

$k=2$ $$C_5 = \frac{2}{\pi}\frac{(-1)^2}{5}\cdot\frac{1}{8-25} = -\frac{2}{85\pi} = -0.00749$$

Thus, the dominant term in the series is $k=1$ $(n=3)$. This frequency is closest to the natural frequency $\sqrt{8} = 2.828$ [i.e. they are nearly resonant].