LECTURE 19
DENSITY, MASS AND CENTRE OF MASS

Consider a lamina Ω of varying composition (for example a thin sheet of metal) in the $x-y$ plane with density $\delta(x,y)$ at the point (x,y). Then

$$\text{Mass}(\Omega) = M = \iint_\Omega \delta(x,y)dA.$$

If the centre of mass of Ω is $(\bar{x},\bar{y})$ then

$$\bar{x} = \frac{\iint_\Omega x\ \delta(x,y)dA}{M} = \frac{M_y}{M}$$

and

$$\bar{y} = \frac{\iint_\Omega y\ \delta(x,y)dA}{M} = \frac{M_x}{M}$$

$M_y = \iint_\Omega x\ \delta(x,y)dA$ is called the first moment of the lamina about the y axis.

$M_x = \iint_\Omega y\ \delta(x,y)dA$ is called the first moment of the lamina about the x axis.

If the density is uniform with $\delta(x,y)=1$ then the centre of mass is referred to as a *centroid*. Note that the mass is then

$$\text{Mass}(\Omega) = M = \iint_\Omega 1dA = \text{Area}(\Omega).$$

When Isaac Newton was considering the gravitational forces between bodies a fundamental problem was how to measure the distance between two planets. Do we use the closest distance, the furthest distance or some average? Newton eventually understood that all of the mass of the planet could be treated as if it was concentrated at the centre of mass and thus the planets could be treated as abstract mathematical points. This simple idea greatly assisted the development of his theories.

So the centre of mass of an object is the unique point where we can pretend that all of the mass lies for our calculations. In this course will will only examine the centre of mass of laminas, that is flat sheets of material with negligible thickness. Think of a thin tile or plate. The centre of mass $(\bar{x},\bar{y})$ of such an object is the unique point where you could balance the plate on the tip of your finger.

Generally density $= \dfrac{\text{mass}}{\text{volume}}$, however since the lamina is assumed to have negligible thickness we can say that density $= \dfrac{\text{mass}}{\text{area}}$. Thus $\delta(x,y) = \dfrac{dm}{dA}$ implying that

$dm = \delta(x,y)dA$. Integrating to obtain the total mass we have

$\text{Mass}(\Omega) = M = \iint_{\Omega} \delta(x, y) dA$. That is, **mass is the integral of density**.

When calculating the x coordinate of the centre of mass via $\bar{x} = \dfrac{\iint_{\Omega} x\ \delta(x, y) dA}{M}$

we are simply taking a weighted average of all the $x's$ in Ω with the values of x of higher density making the greater contribution. Similarly for $\bar{y}$.

Please note that I will set up all the relevant double integrals in this lecture but will leave the actual evaluation to you. Answers are at the bottom of each example.

Note also that you should invoke symmetry wherever possible.

Other formulae of interest are:

Moments of Inertia

The moments of inertia of the above lamina about the x and y axes are

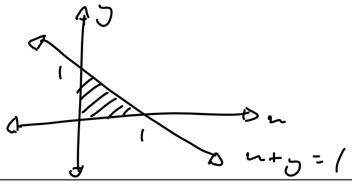
$$\begin{aligned} I_x &= \iint_{\Omega} y^2 \delta(x, y) dA \\ I_y &= \iint_{\Omega} x^2 \delta(x, y) dA. \end{aligned}$$

The polar moment of inertia about the origin is defined by

$$I_0 = I_x + I_y = \iint_{\Omega} (x^2 + y^2) \delta(x, y) dA.$$

Moments of inertia measure an objects resistance to spinning about a particular axis. A spinning ice skater with her arms out has a larger moment of inertia about the vertical axis than one with her arms tucked into her body.

Example 1 A thin plate Ω is the region in the first quadrant bounded by the coordinate axes and $x+y=1$. Find the mass M and centre of mass $(\bar{x},\bar{y})$ of the plate assuming



a) uniform density $\delta(x,y)=1$.

b) non uniform density given by $\delta(x,y)=xy$.

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a) (For uniform density the centre of mass is called the centroid)

$$M = \iint_\Omega \delta(x,y)\, dA = \iint_\Omega 1\, dA = \frac{1}{2}$$

$$M_x = \iint_\Omega \delta(x,y)\, x\, dA = \int_0^1\int_0^{1-x} x \, dy\, dx = \frac{1}{6}$$

$$M_y = \iint_\Omega \delta(x,y)\, y\, dA = \int_0^1\int_0^{1-x} y \, dy\, dx = \frac{1}{6}$$

$$\therefore (\bar{x},\bar{y}) = \left(\frac{1}{3},\frac{1}{3}\right)$$

b) $$M = \int_0^1\int_0^{1-x} xy\, dy\, dx = \frac{1}{24}$$

$$M_y = \int_0^1\int_0^{1-x} x(xy)\, dy\, dx = \frac{1}{60}$$

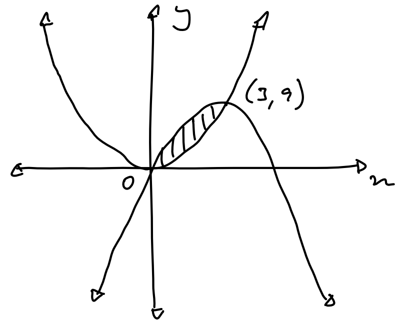
$$M_x = \int_0^1\int_0^{1-x} y(xy)\, dy\, dx = \frac{1}{60}$$

$$\therefore (\bar{x},\bar{y}) = \left(\frac{2}{5},\frac{2}{5}\right)$$

★ *a)* $M=\frac{1}{2},\ M_x=\frac{1}{6},\ M_y=\frac{1}{6},\ (\bar{x},\bar{y})=(\frac{M_y}{M},\frac{M_x}{M})=(\frac{1}{3},\frac{1}{3})$ ★

★ *b)* $M=\frac{1}{24},\ M_x=\frac{1}{60},\ M_y=\frac{1}{60},\ (\bar{x},\bar{y})=(\frac{M_y}{M},\frac{M_x}{M})=(\frac{2}{5},\frac{2}{5})$ ★

Example 2 Find the mass and the centre of mass of a thin plate bounded by the curves $y = x^2$ and $y = 6x - x^2$ with variable density given by $\delta(x, y) = 10x + 10$.
Also find the moment of inertia I_y about the $y-$ axis. Recall that $I_y = \iint_{\Omega} x^2 \delta(x, y) dA$.



$$M = \iint_{\Omega} \delta(x, y)\, dA$$

$$= \int_0^3 \int_{x^2}^{6x - x^2} 10x + 10 \, dy\, dx$$

$$= 225$$

$$M_y = \int_0^3 \int_{x^2}^{6x - x^2} x(10x + 10)\, dy\, dx = 378$$

$$M_x = \int_0^3 \int_{x^2}^{6x - x^2} y(10x + 10)\, dy\, dx = 1134$$

$$(\bar{x}, \bar{y}) = (1.68, 5.04)$$

$$I_y = \iint_{\Omega} x^2 \delta(x, y)\, dA$$

$$= \int_0^3 \int_{x^2}^{6x - x^2} x^2 (10x + 10)\, dy\, dx$$

$$= 729$$

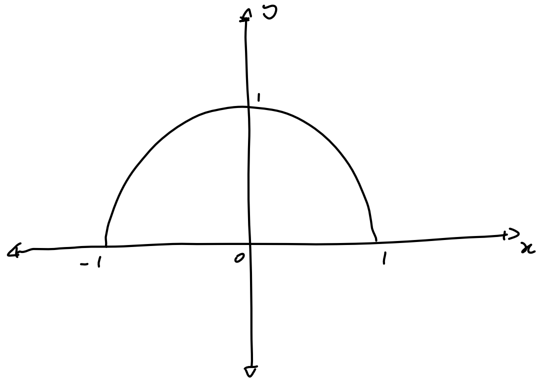
★ $M = 225,\ M_x = 1134,\ M_y = 378,\ (\bar{x}, \bar{y}) = (\frac{M_y}{M}, \frac{M_x}{M}) = (\frac{42}{25}, \frac{126}{25}),\ I_y = 729$ ★

Example 3 Find the mass and the centre of mass of a lamina bounded by $y = \sqrt{1-x^2}$ and the x axis assuming:

a) uniform density $\delta(x, y) = 1$.

b) variable density given by $\delta(x, y) = y$.

c) variable density given by $\delta(x, y) = r$.



a) $M = \iint_\Omega 1 \, dA = \frac{\pi}{2}$

$M_x = \iint y \, dy \, dx = \int_0^\pi \int_0^1 r \sin\theta (r \, dr \, d\theta) = \frac{2}{3}$

$\therefore \left(0, \frac{4\pi}{3}\right)$

b) $M = \iint_\Omega y \, dA = \int_0^\pi \int_1^1 r \sin\theta (r \, dr \, d\theta) = \frac{2}{3}$

$M_x = \iint y(y) \, dA = \int_0^\pi \int_0^1 r^2 \sin^2\theta \, r \, dr \, d\theta = \frac{\pi}{8}$

$\therefore \left(0, \frac{3\pi}{16}\right)$

c) $M = \iint_\Omega r \, dA = \int_0^\pi \int_0^1 r \, r \, dr \, d\theta = \frac{\pi}{3}$

$M_x = \iint y r \, dA = \int_0^\pi \int_0^1 r \sin\theta \, r \, r \, dr \, d\theta = \frac{1}{2}$

$\therefore \left(0, \frac{3\pi}{2}\right)$

★ *a)* $M = \frac{\pi}{2},\ M_x = \frac{2}{3},\ M_y = 0,\ (\bar{x}, \bar{y}) = (\frac{M_y}{M}, \frac{M_x}{M}) = (0, \frac{4}{3\pi})$ ★

★ *b)* $M = \frac{2}{3},\ M_x = \frac{\pi}{8},\ M_y = 0,\ (\bar{x}, \bar{y}) = (\frac{M_y}{M}, \frac{M_x}{M}) = (0, \frac{3\pi}{16})$ ★

★ *c)* $M = \frac{\pi}{3},\ M_x = \frac{1}{2},\ M_y = 0,\ (\bar{x}, \bar{y}) = (\frac{M_y}{M}, \frac{M_x}{M}) = (0, \frac{3}{2\pi})$ ★

[19] You can now do Q 72 to 76