MATH2019 PROBLEM CLASS

EXAMPLES 8

FOURIER SERIES

If f has period $2L$, then the Fourier series of f is given by

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right), \quad \text{with}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx\,, \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx\,.$$

2014, S1

1. Let $f(x) = \begin{cases} 1 & 0 < x < \pi\;; \\ 0 & \pi \le x \le 2\pi\;; \\ f(x \pm 2\pi) & \text{otherwise.} \end{cases}$

   i) What is the period of f ?

   ii) Sketch the graph of $y = f(x)$ over the interval $-2\pi \le x \le 2\pi$.

   iii) Sketch the graph of $y = f(x) - \dfrac{1}{2}$ over the interval $-2\pi \le x \le 2\pi$.

   iv) Hence, or otherwise, find the Fourier series of f, explicitly writing down the first four non-zero terms of the series.

2014, S1

2. Suppose that a function f of period $T = 2L$ has a Fourier series

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}\,x\right) + b_n \sin\left(\frac{n\pi}{L}\,x\right) \right).$$

   Prove that

$$\frac{1}{L}\int_{-L}^{L} (f(x))^2\,dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\,.$$

   [This is known as Parseval's identity.]

2014, S2

3. The function $f(x)$ is given by

$$f(x) = \begin{cases} 1 - x & \text{for } 0 \le x < 1, \\ 0 & \text{for } 1 \le x < 2. \end{cases}$$

   with $f(x \pm 2) = f(x)$.

   i) What is the period of $f(x)$?

   ii) Sketch the function $f(x)$ for $-4 \le x < 4$.

   iii) To what value does the Fourier series of f converge at $x = 2$? [Note that you are not being asked to evaluate the Fourier series.]

$\boxed{\text{2014, S2}}$ 4. Define the continuous function f by

$$f(x) = \begin{cases} x, & 0 \leq x < \dfrac{\pi}{2} \\ \pi - x, & \dfrac{\pi}{2} \leq x < \pi. \end{cases}$$

i) Sketch the odd periodic extension of f over the interval $-\pi \leq x \leq \pi$.

ii) Calculate the Fourier sine series of f. You can assume that

$$\int_a^b x \sin(nx)\, dx = \left[\frac{\sin(nx) - nx\cos(nx)}{n^2} \right]_a^b .$$

$\boxed{\text{2015, S1}}$ 5. Define the piecewise continuous function f by

$$f(x) = \begin{cases} 1, & 0 \leq x < \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} \leq x < \pi. \end{cases}$$

i) Sketch the even periodic extension of f on the interval $-2\pi \leq x \leq 2\pi$.

ii) Show that the Fourier cosine series of f is given by

$$\frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\left[(2k+1)\,x\right] .$$

iii) To what value will the Fourier cosine series converge at $x = \dfrac{\pi}{2}$?

$\boxed{\text{2015, S1}}$ 6. Consider the differential equation

$$\frac{d^2y}{dx^2} + 8y = f(x),$$

where $f(x)$ is the function defined in the previous question.

i) Find the homogeneous solution to the differential equation. What is the natural (or fundamental) angular frequency of the system?

ii) Find a particular solution of the differential equation in the form of a Fourier series.

iii) Calculate the numerical value of the first four non-zero terms of this particular solution and comment on which term dominates the solution.

$\boxed{\text{2015, S2}}$ 7. The function f is given by

$$f(x) = \begin{cases} -x & \text{for } -\pi \leq x \leq 0 \\ x & \text{for } 0 \leq x \leq \pi \end{cases}$$

with $f(x \pm 2\pi) = f(x)$ for all x.

i) Make a sketch of this function for $-4\pi \leq x \leq 4\pi$.

ii) Is $f(x)$ odd, even or neither?

iii) Find the Fourier series of f.

iv) By considering the value at $x = \pi$ in your answer for the Fourier series in part iii), find the sum of the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

2016, S1 — 8. Define the piecewise continuous function f by

$$f(x) = \begin{cases} 2, & 0 < x \le \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < x \le \pi. \end{cases}$$

i) Sketch the odd periodic extension of f on the interval $-2\pi \le x \le 2\pi$.

ii) Show that the first four non-zero terms of the Fourier sine series of f are given by

$$\frac{4}{\pi}\sin x + \frac{4}{\pi}\sin(2x) + \frac{4}{3\pi}\sin(3x) + \frac{4}{5\pi}\sin(5x)\,.$$

iii) By considering the Fourier series at the point $x = \pi/2$, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots\,.$$

2017, S1 — 9. Define the function f on the interval $[0, \pi)$ as

$$f(x) = \frac{x}{2}, \qquad \text{on} \qquad 0 \le x < \pi.$$

i) Sketch the odd periodic extension of f on the interval $-4\pi \le x \le 4\pi$.

ii) Show that the Fourier sine series of f is given by

$$\sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1}\sin(nx)\,.$$

iii) To what value does the Fourier sine series of f converge at $x = \pi$?

2017, S1 — 10. Consider the differential equation

$$\frac{d^2y}{dx} + 5y = f(x),$$

where $f(x)$ is the function defined in the previous question.

i) Find the homogeneous solution to the differential equation. What is the natural (or fundamental) angular frequency of the system?

ii) Find the particular solution of the differential equation in the form of a Fourier series.

iii) Write out the first four non-zero terms of this particular solution and comment on which term dominates the solution.

2017, S2 — 11. Define the piecewise continuous function f by

$$f(x) = \begin{cases} 2, & 0 \le x < \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} \le x < \pi. \end{cases}$$

i) Sketch the even periodic extension of $f(x)$ for $-3\pi \le x \le 3\pi$.

ii) Show that the Fourier cosine series of $f(x)$ is given by

$$1 + \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi}(-1)^{m+1}\cos\left[(2m-1)x\right].$$

iii) To what value does the Fourier cosine series in part ii) converge at $x = \dfrac{\pi}{2}$?

$\boxed{\text{2018, S1}}$ 12. The function $f = |x|$ is defined on the interval $[-\pi, \pi)$. We extend f to a periodic function of period 2π.

i) Sketch the function f on the interval $-3\pi \leq x \leq 3\pi$.

ii) Determine the Fourier coefficient a_0 for f.

iii) Determine the Fourier coefficient a_n for f, $n = 1, 2, 3, \ldots$.

iv) Determine the Fourier coefficient b_n for f, $n = 1, 2, 3, \ldots$.

$\boxed{\text{2018, S2}}$ 13. Let $f(x) = \begin{cases} 0 & 0 \leq x \leq \dfrac{\pi}{2}\ ; \\ 1 & \dfrac{\pi}{2} < x < \pi\ . \end{cases}$

i) Sketch the odd periodic extension of f over the domain $-\pi \leq x \leq \pi$.

ii) Calculate the half range Fourier sine series of f.

iii) To what value does the series in part ii) converge at $x = \dfrac{\pi}{2}$?