THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

JUNE 2016

MATH2019
ENGINEERING MATHEMATICS 2E

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

TABLE OF LAPLACE TRANSFORMS AND THEOREMS

$g(t)$ is a function defined for all $t \geq 0$, and whose Laplace transform

$$G(s) = \mathcal{L}(g(t)) = \int_0^\infty e^{-st} g(t) dt$$

exists. The Heaviside step function u is defined to be

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ \frac{1}{2} & \text{for } t = a \\ 1 & \text{for } t > a \end{cases}$$

$g(t)$	$G(s) = \mathcal{L}[g(t)]$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^\nu, \nu > -1$	$\frac{\nu!}{s^{\nu+1}}$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$e^{-\alpha t} f(t)$	$F(s+\alpha)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$tf(t)$	$-F'(s)$

Please see over ...

FOURIER SERIES

If $f(x)$ has period $p = 2L$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}\, x\right) + b_n \sin\left(\frac{n\pi}{L}\, x\right)\right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}\, x\right) dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}\, x\right) dx \end{aligned}$$

LEIBNIZ' RULE

$$\frac{d}{dx}\int_u^v f(x,t)dt = \int_u^v \frac{\partial f}{\partial x} dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx}$$

MULTIVARIABLE TAYLOR SERIES

$$\begin{aligned} f(x,y) &= f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) + \\ &\quad + \frac{1}{2!}\left((x-a)^2 \frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y} + (y-b)^2 \frac{\partial^2 f}{\partial y^2}(a,b)\right) \cdots \end{aligned}$$

SOME BASIC INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^{kx} dx = \frac{e^{kx}}{k} + C$$

$$\int a^x dx = \frac{1}{\ln a} a^x + C \quad \text{for } a \neq 1$$

$$\int \sin\, kx\, dx = -\frac{\cos kx}{k} + C$$

$$\int \cos kx\, dx = \frac{\sin kx}{k} + C$$

$$\int \sec^2 kx\, dx = \frac{\tan kx}{k} + C$$

$$\int \mathrm{cosec}^2 kx\, dx = -\frac{1}{k} \cot kx + C$$

$$\int \tan kx\, dx = \frac{\ln|\sec kx|}{k} + C$$

$$\int \sec kx\, dx = \frac{1}{k}(\ln|\sec kx + \tan kx|) + C$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx$$

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx$$

Answer question 1 in a separate book

1. a) The battery life of a mobile phone is given by

$$L = \frac{\alpha\, C}{b^2},$$

where C is the capacity of the battery, b is the width of the phone screen, and α is a positive constant. If the battery capacity C is increased by 20% and the screen size b is increased by 5%, use the chain rule to estimate the percentage change in the battery life of the phone. [Note that you do not need to know the value of α.]

b) Sketch the graph of the function

$$f(x, y) = 1 - x^2 - y^2.$$

Using the method of Lagrange multipliers, find the extreme value of $f(x, y)$ subject to the constraint $x + y = 1$. Explain why this extreme value is a maximum and not a minimum.

c) Given the scalar field $\phi(x, y, z) = xy\sin(z)$, calculate

i) grad $\phi = \nabla\phi$

ii) div (grad ϕ) $= \nabla \cdot (\nabla\phi)$

iii) curl (grad ϕ) $= \nabla \times (\nabla\phi)$.

d) Let $\mathcal{C}$ denote the path taken by a particle travelling in a straight line from the point $\mathrm{P}(1, 0, 0)$ to the point $\mathrm{Q}(1, 1, \frac{\pi}{2})$.

i) Write down a vector function $\mathbf{r}(t)$ that describes the path $\mathcal{C}$ and give the value of t at the start and the end of the path.

ii) If $\mathbf{F} = y\sin(z)\,\mathbf{i} + x\sin(z)\,\mathbf{j} + xy\cos(z)\,\mathbf{k}$, evaluate the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

[**Hint:** you can use the results of part (c) to solve this integral.]

Answer question 2 in a separate book

2. a) Use the substitution $v = x + y$ to solve the ordinary differential equation

$$(x+y)\,\frac{dy}{dx} = \frac{1}{x^2} - x - y, \qquad y(1) = 0.$$

b) Use the method of undetermined coefficients to solve the second order differential equation

$$y'' + 3y' + 2y = \mathrm{e}^{-2t} + 4t^2 + 2.$$

Also describe the long term steady state solution.

c) Consider the double integral

$$\int_0^1 \int_x^{\sqrt{3}\,x} \frac{x}{x^2+y^2}\; dy\; dx.$$

i) Sketch the region of integration.

ii) Evaluate the double integral using polar coordinates.

d) Consider the curve in the x-y plane

$$x^2 - 6xy + y^2 = 16.$$

i) Rewrite the equation for the curve in the form

$$\begin{pmatrix} x \\ y \end{pmatrix}^T \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix} = 16$$

where $\mathbf{A}$ is a symmetric 2×2 matrix. Find the eigenvalues and eigenvectors of $\mathbf{A}$.

ii) Write down the equation for the curve in terms of its principle axes X and Y. Hence find the closest distance from the origin to the curve.

iii) Find the x and y coordinates of the points on the curve closest to the origin.

Answer question 3 in a separate book

3. a) The Laplace transform of a function $f(t)$ is defined for $t \geq 0$ by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt.$$

i) Use Leibniz' Rule to prove

$$\mathcal{L}\{tf(t)\} = -F'(s).$$

ii) Hence, or otherwise, find the following Laplace transform

$$\mathcal{L}\{t\sin 3t\}$$

b) Use the Laplace transform method to solve the initial value problem

$$y'' - y = u(t-1) \qquad \text{with} \qquad y(0) = 0,\ y'(0) = 1,$$

where $u(t-1)$ is a Heaviside step function.

c) Because of the effect of rotation, the Earth is not a perfect sphere but is slightly fatter at the equator than it is at the poles. A good approximation for the shape of the earth is an ellipsoid described by the formula

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1,$$

where z is the coordinate measured along the axis of rotation, $a = 6378$ km is the radius of the Earth at the equator and $b = 6357$ km is the radius of the Earth at the poles.

Calculate the volume of the Earth using an appropriate double integral.

Answer question 4 in a separate book

4. a) Define the piecewise continuous function f by

$$f(x) = \begin{cases} 2, & 0 < x \leq \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < x \leq \pi. \end{cases}$$

i) Sketch the odd periodic extension of f on the interval $-2\pi \leq x \leq 2\pi$.

ii) Show that the first four non-zero terms of the Fourier sine series of f are given by

$$f(x) = \frac{4}{\pi}\sin x + \frac{4}{\pi}\sin 2x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \cdots$$

iii) By considering the Fourier series at the point $x = \pi/2$, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

b) A stretched wire satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ is the displacement of the wire. The ends of the wire are held fixed so that

$$u(0,t) = u(\pi,t) = 0, \qquad \text{for all } t.$$

i) Assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$\frac{G''(t)}{4G(t)} = \frac{F''(x)}{F(x)} = k$$

for some constant k.

ii) Apply the boundary conditions to show that possible solutions for $F(x)$ are

$$F_n(x) = B_n \sin{(nx)}$$

where B_n are constants and $n = 1, 2, 3, \cdots$ You must consider all possible values of k.

iii) Find all possible solutions $G_n(t)$ for $G(t)$.

iv) If the initial displacement and velocity of the wire are

$$u(x,0) = 3\sin(x) + 4\sin(3x), \qquad \text{and} \qquad u_t(x,0) = 0,$$

find the general solution $u(x,t)$.

MATH 2019 EXAM JUNE 2016

QUESTION 1

a) $L = \dfrac{\alpha C}{b^2}$ $\qquad \dfrac{\Delta C}{C} = 0.2 \qquad \dfrac{\Delta b}{b} = 0.05$

$$\Delta L \simeq \frac{\partial L}{\partial C}\Delta C + \frac{\partial L}{\partial b}\Delta b$$

$$= \frac{\alpha}{b^2}\Delta C - \frac{2\alpha C}{b^3}\Delta b$$

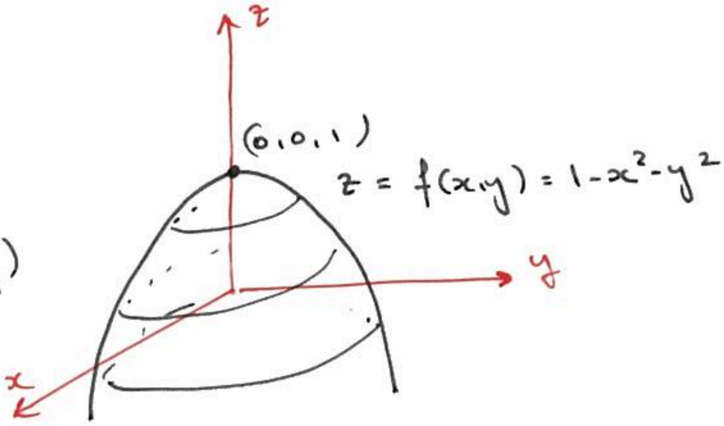
$$\frac{\Delta L}{L} \simeq \frac{\cancel{\alpha}\Delta C}{\cancel{b^2}} \times \frac{\cancel{b^2}}{\cancel{\alpha} C} - \frac{2\cancel{\alpha}\cancel{C}\Delta b}{b^3} \times \frac{\cancel{b^2}}{\cancel{\alpha}\cancel{C}}$$

$$= \frac{\Delta C}{C} - \frac{2\Delta b}{b} = 0.2 - 2(0.05) = 0.1$$

$\Rightarrow$ 10% increase in battery life.

b) $f(x,y) = 1 - x^2 - y^2$

$g(x,y) = x + y - 1$



$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$

$= 1 - x^2 - y^2 - \lambda(x + y - 1)$

$L_x = -2x - \lambda = 0 \quad \Rightarrow \quad x = -\lambda/2 \quad (1)$

$L_y = -2y - \lambda = 0 \quad \Rightarrow \quad y = -\lambda/2 \quad (2)$

$L_\lambda = 0 \quad \Rightarrow \quad x + y = 1 \quad (3)$

$(1,2) \longrightarrow (3) \quad \Rightarrow \quad -\dfrac{\lambda}{2} - \dfrac{\lambda}{2} = 1 \quad \Rightarrow \quad -\lambda = 1 \quad \Rightarrow \quad \boxed{\lambda = -1}$

$\Rightarrow \quad x = \dfrac{1}{2}, \; y = \dfrac{1}{2}, \; f\left(\dfrac{1}{2}, \dfrac{1}{2}\right) = 1 - \dfrac{1}{4} - \dfrac{1}{4} = \dfrac{1}{2}$

This is a maximum value because $f(x,y)$ has no minimum.

c) $f(x,y,z) = xy\sin z$

i) $\underline{\nabla} f = \dfrac{\partial f}{\partial x}\,\underline{i} + \dfrac{\partial f}{\partial y}\,\underline{j} + \dfrac{\partial f}{\partial z}\,\underline{k}$

$= y\sin z\,\underline{i} + x\sin z\,\underline{j} + xy\cos z\,\underline{k}$

ii) $\underline{\nabla}\cdot(\underline{\nabla} f) = \dfrac{\partial}{\partial x}\cancel{\left(y\sin z\right)} + \dfrac{\partial}{\partial y}\cancel{\left(x\sin z\right)} + \dfrac{\partial}{\partial z}\left(xy\cos z\right)$

$= -xy\sin z$

iii) $\underline{\nabla}\times(\underline{\nabla} f) = \underline{0}$ for any f.

OR show by direct calculation:

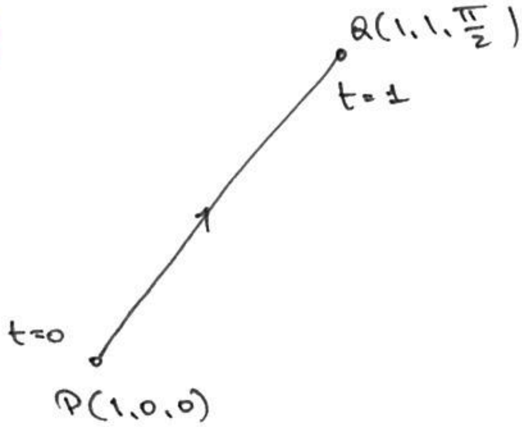
$$\underline{\nabla}\times(\underline{\nabla} f) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y\sin z & x\sin z & xy\cos z \end{vmatrix}$$

$= \underline{i}\left(x\cos z - x\cos z\right) - \underline{j}\left(y\cos z - y\cos z\right) + \underline{k}\left(\sin z - \sin z\right)$

$= 0\,\underline{i} + 0\,\underline{j} + 0\,\underline{k}$

$= \underline{0}$

d) i)



$\underline{r}(t) = \vec{OP} + t\,\vec{PQ}$

$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1-1 \\ 1-0 \\ \frac{\pi}{2}-0 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ t \\ \frac{\pi}{2}t \end{pmatrix}$ $\quad t: 0 \to 1$

ii) $\underline{F} = y \sin z \, \underline{i} + x \sin z \, \underline{j} + xy \cos z \, \underline{k}$

$= \underline{\nabla} \phi$

$\Rightarrow \underline{\nabla} \times \underline{F} = \underline{\nabla} \times (\underline{\nabla} \phi) = 0$

$\Rightarrow \underline{F}$ is a conservative force.

Thus

$$\int_C \underline{F} \cdot d\underline{r} = \phi \Big|_Q - \phi \Big|_P \quad , \phi = xy \sin z .$$

$$= (1)(1) \sin\left(\frac{\pi}{2}\right) - (1)(0) \sin(0)$$

$$= 1 - 0 = 1$$

OR show by direct calculation:

$x(t) = 1 \Rightarrow dx = 0$

$y(t) = t \Rightarrow dy = dt$

$z(t) = \frac{\pi}{2} t \Rightarrow dz = \frac{\pi}{2} dt$

$F_1 = y \sin z = t \sin \frac{\pi}{2} t$

$F_2 = x \sin z = \sin \frac{\pi}{2} t$

$F_3 = xy \cos z = t \cos \frac{\pi}{2} t$

$$\int_C \underline{F} \cdot d\underline{r} = \int_P^Q F_1 \, dx + F_2 \, dy + F_3 \, dz$$

$$= \int_0^1 \cancel{\left(t \sin \frac{\pi}{2} t\right)(0)} + \left(\sin \frac{\pi}{2} t\right)(dt) + \left(t \cos \frac{\pi}{2} t\right)\left(\frac{\pi}{2} dt\right)$$

$$= \int_0^1 \sin\left(\frac{\pi}{2} t\right) + \frac{\pi t}{2} \cos\left(\frac{\pi}{2} t\right) \, dt$$

$$\int_0^1 \sin\left(\frac{\pi t}{2}\right) dt = \left[-\frac{2}{\pi}\cos\left(\frac{\pi t}{2}\right)\right]_0^1 = -\frac{2}{\pi}\left\{0 - 1\right\} = \frac{2}{\pi}$$

$$\frac{\pi}{2}\int_0^1 t\cos\left(\frac{\pi t}{2}\right) dt \underset{\text{parts}}{\overset{\text{by}}{=}} \frac{\pi}{2}\left[t \cdot \frac{2}{\pi}\sin\left(\frac{\pi t}{2}\right)\right]_0^1 - \frac{\pi}{2}\int_0^1 \frac{2}{\pi}\sin\left(\frac{\pi t}{2}\right) dt$$

$$= \left(1 \cdot \sin\frac{\pi}{2} - 0\right) - \left[-\frac{2}{\pi}\cos\left(\frac{\pi t}{2}\right)\right]_0^1$$

$$= 1 + \frac{2}{\pi}\left\{0 - 1\right\} = 1 - \frac{2}{\pi}$$

Thus $$\int_C \underline{F} \cdot d\underline{r} = \int_0^1 \sin\frac{\pi t}{2}\, dt + \frac{\pi}{2}\int_0^1 t\cos\left(\frac{\pi t}{2}\right) dt$$

$$= \frac{2}{\pi} + 1 - \frac{2}{\pi} = 1.$$

QUESTION 2

a) $(x+y)\frac{dy}{dx} = \frac{1}{x^2} - x - y \qquad y(1) = 0$

$v = x + y \Rightarrow y = v - x \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

$\Rightarrow (\cancel{x} + v - \cancel{x})\left(\frac{dv}{dx} - 1\right) = \frac{1}{x^2} - \cancel{x} - (v - \cancel{x})$

$v\frac{dv}{dx} - \cancel{v} = \frac{1}{x^2} - \cancel{v}$

$\int v\,dv = \int \frac{dx}{x^2}$

$\frac{v^2}{2} = -\frac{1}{x} + C$

$\frac{(x+y)^2}{2} = -\frac{1}{x} + C$

at $x = 1$, $y = 0 \Rightarrow \frac{(1+0)^2}{2} = \frac{1}{2} = -1 + C$

$\Rightarrow C = \frac{3}{2}$

so $\frac{(x+y)^2}{2} = -\frac{1}{x} + \frac{3}{2}$

10) $y'' + 3y' + 2y = e^{-2t} + 4t^2 + 2$

let $y = y_h + y_p$ where $y_h'' + 3y_h' + 2y_h = 0$

let $y_h = e^{\lambda t} \Rightarrow y_h' = \lambda e^{\lambda t} \Rightarrow y_h'' = \lambda^2 e^{\lambda t}$

$\therefore \quad \lambda^2 + 3\lambda + 2 = 0$

$(\lambda + 2)(\lambda + 1) = 0 \Rightarrow \lambda = -2, -1$

and $y_h = Ae^{-t} + Be^{-2t}$

Choose for trial solution:

$$y_p = Cte^{-2t} + Dt^2 + Et + F$$
$$y_p' = Ce^{-2t} - 2Cte^{-2t} + 2Dt + E$$
$$y_p'' = -4Ce^{-2t} + 4Cte^{-2t} + 2D$$

Sub into ODE:

$$-4Ce^{-2t} + \cancel{4Cte^{-2t}} + 2D + 3\{Ce^{-2t} - \cancel{2Cte^{-2t}} + 2Dt + E\}$$
$$2\{\cancel{Cte^{-2t}} + Dt^2 + Et + F\}$$

$$= -Ce^{-2t} + 2Dt^2 + \{6D + 2E\}t + \{2D + 3E + 2F\}$$

$$= e^{-2t} + 4t^2 + 2$$

$\Rightarrow \quad -C = 1 \quad \Rightarrow \quad C = -1$

$2D = 4 \quad \Rightarrow \quad D = 2$

$6D + 2E = 0 \quad \Rightarrow \quad E = -3D = -6$

$2D + 3E + 2F = 2 \Rightarrow F = 1 - D - \frac{3}{2}E = 8$

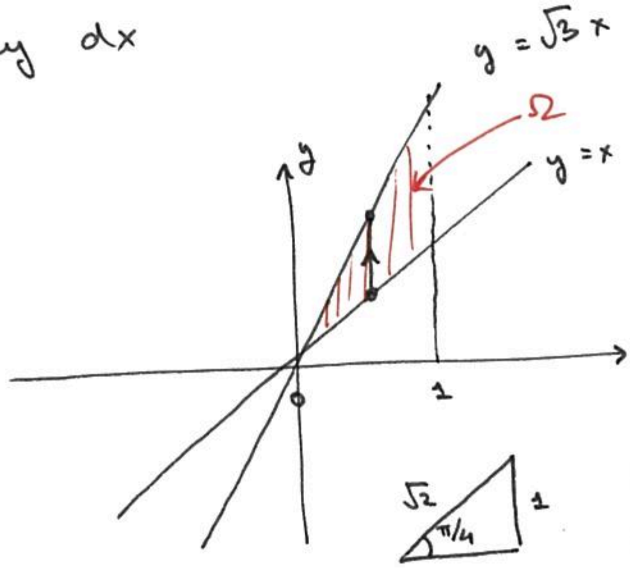
So $y = y_h + y_p = Ae^{-t} + Be^{-2t} - te^{-2t} + 2t^2 - 6t + 8$

In the limit of $t \longrightarrow \infty$

$y \longrightarrow 2t^2 - 6t + 8$

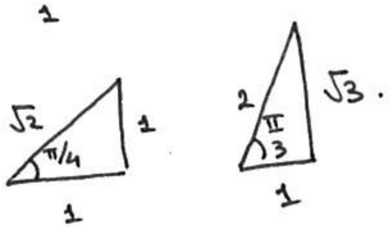
c) $$\int_0^1 \int_x^{\sqrt{3}x} \frac{x}{x^2+y^2}\, dy\, dx$$

i) Sketch:



$x : 0 \to 1$

$y : x \to \sqrt{3}\,x$



Polar coordinates:

$\theta : \frac{\pi}{4} \to \frac{\pi}{3}$

$r : 0 \to \frac{1}{\cos\theta}$

on $x = 1$, $r\cos\theta = 1$

$\Rightarrow r = \frac{1}{\cos\theta}$

$$\therefore \int_{\pi/4}^{\pi/3} \int_0^{1/\cos\theta} \frac{r\cos\theta}{r^2} \cdot r\, dr\, d\theta$$

$$= \int_{\pi/4}^{\pi/3} \int_0^{1/\cos\theta} \cos\theta\, dr\, d\theta$$

$$= \int_{\pi/4}^{\pi/3} \Big[r\cos\theta \Big]_0^{1/\cos\theta} d\theta$$

$$= \int_{\pi/4}^{\pi/3} 1\, d\theta = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

d) i) $x^2 - 6xy + y^2 = 16$

$$(x \quad y)\begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 16$$

$$A = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$$

$$|A - \lambda 1| = \begin{vmatrix} 1-\lambda & -3 \\ -3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 - 9 = 0$$

$$\lambda^2 - 2\lambda - 8 = 0$$

$$(\lambda - 4)(\lambda + 2) = 0 \quad \Rightarrow \quad \lambda = -2,\ \lambda = +4$$

$\underline{\lambda = -2}$:

$$\left.\begin{matrix} +3 & -3 \\ -3 & +3 \end{matrix}\,\right|\,\begin{matrix} 0 \\ 0 \end{matrix} \rightarrow \left.\begin{matrix} 1 & -1 \\ 0 & 0 \end{matrix}\,\right|\,\begin{matrix} 0 \\ 0 \end{matrix} \rightarrow \begin{matrix} v_1 - v_2 = 0 \\ v_2 = t \in \mathbb{R} \end{matrix}$$

$$\Rightarrow \quad \underline{v} = \begin{pmatrix} t \\ t \end{pmatrix} \quad \text{normalize} \Rightarrow \quad \underline{v} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\underline{\lambda = +4}$:

$$\left.\begin{matrix} -3 & -3 \\ -3 & -3 \end{matrix}\,\right|\,\begin{matrix} 0 \\ 0 \end{matrix} \rightarrow \left.\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}\,\right|\,\begin{matrix} 0 \\ 0 \end{matrix} \rightarrow \begin{matrix} v_1 + v_2 = 0 \\ v_2 = t \in \mathbb{R} \end{matrix}$$

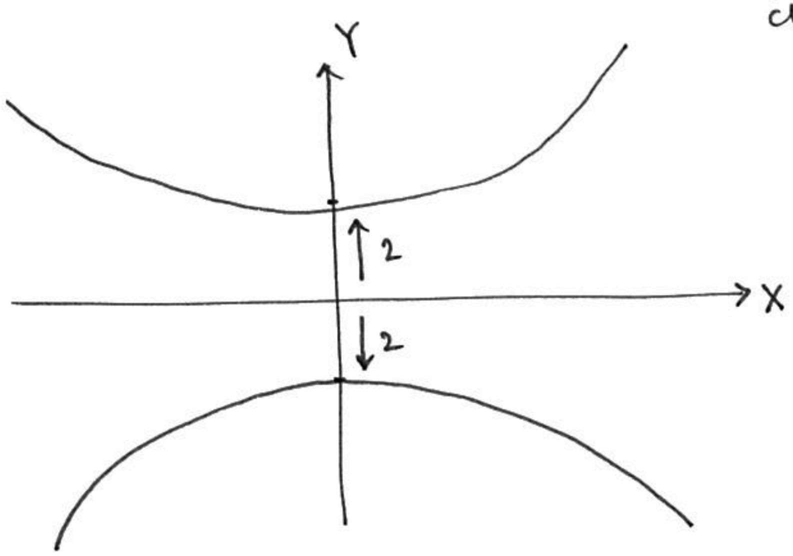
$$\Rightarrow \quad \underline{v} = \begin{pmatrix} -t \\ t \end{pmatrix} \quad \text{normalize} \Rightarrow \quad \underline{v} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

ii) in terms of principle axes X, Y have

$$-2X^2 + 4Y^2 = 16$$

$$-\frac{X^2}{8} + \frac{Y^2}{4} = 1 \qquad \text{paraboloid.}$$

closest distance $= 2$.



iii) closest points are $(X, Y) = (0, 2)$ and $(0, -2)$

$$\begin{pmatrix} x \\ y \end{pmatrix} = Q \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

Thus for $(0,2)$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

and for $(0,-2)$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix}$$

QUESTION 3

a) i) $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left\{e^{-st} f(t)\right\} dt + \cancel{\frac{d}{ds}(\infty)}\left[e^{-st} f(t)\right]_{t=\infty}$$

$$- \cancel{\frac{d}{ds}(0)}\left[e^{-st} f(t)\right]_{t=0}$$ using Leibniz' Rule

$$= \int_0^\infty -t e^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\left(t f(t)\right) dt = -\mathcal{L}\{t f(t)\}$$

Thus $\mathcal{L}\{t f(t)\} = -F'(s)$

ii) $\mathcal{L}\{t \sin 3t\} = ?$ Let $f(t) = \sin 3t$

$$\Rightarrow F(s) = \frac{3}{s^2+9}$$

$$\Rightarrow F'(s) = \frac{-3}{(s^2+9)^2}(2s) = -\frac{6s}{(s+9)^2}$$

Thus $\mathcal{L}\{t \sin 3t\} = -F'(s) = \dfrac{6s}{(s+9)^2}$

b) $y'' - y = u(t-1) \qquad y(0) = 0 \ , \ y'(0) = 1$

$$\mathcal{L}\{y'' - y\} = s^2 F(s) - s\,y(0) - y'(0) - F(s)$$

$$= (s^2 - 1) F(s) - 1$$

$$\mathcal{L}\{u(t-1)\} = \frac{e^{-s}}{s}$$

Thus $\quad (s^2 - 1) F(s) - 1 = \dfrac{e^{-s}}{s}$

OR $\quad F(s) = \dfrac{1}{s^2 - 1} + \dfrac{e^{-s}}{s(s^2-1)}$

$$= \frac{1}{(s-1)(s+1)} + e^{-s} \frac{1}{s(s-1)(s+1)}$$

Partial fractions:

* $\dfrac{1}{(s-1)(s+1)} = \dfrac{A}{s-1} + \dfrac{B}{s+1} = \dfrac{A(s+1) + B(s-1)}{(s-1)(s+1)}$

$\Rightarrow \quad A(s+1) + B(s-1) = 1$

at $\quad s = 1 \quad : \quad 2A = 1 \quad \Rightarrow \quad A = \frac{1}{2}$

at $\quad s = -1 \quad : \quad -2B = 1 \quad \Rightarrow \quad B = -\frac{1}{2}$

$\Rightarrow \quad \dfrac{1}{(s-1)(s+1)} = \dfrac{1}{2} \dfrac{1}{s-1} - \dfrac{1}{2} \dfrac{1}{s+1}$

$$* \quad \frac{1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$$

$$= \frac{A(s-1)(s+1) + Bs(s+1) + Cs(s-1)}{s(s-1)(s+1)}$$

$$\Rightarrow \quad A(s-1)(s+1) + Bs(s+1) + Cs(s-1) = 1$$

at $s = 0$ $\quad -A = 1 \quad \Rightarrow \quad A = -1$

at $s = 1$ $\quad 2B = 1 \quad \Rightarrow \quad B = \frac{1}{2}$

at $s = -1$ $\quad 2C = 1 \quad \Rightarrow \quad C = \frac{1}{2}$

$$\Rightarrow \quad \frac{1}{s(s-1)(s+1)} = -\frac{1}{s} + \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}$$

Thus

$$F(s) = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} + e^{-s}\left\{-\frac{1}{s} + \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}\right\}$$

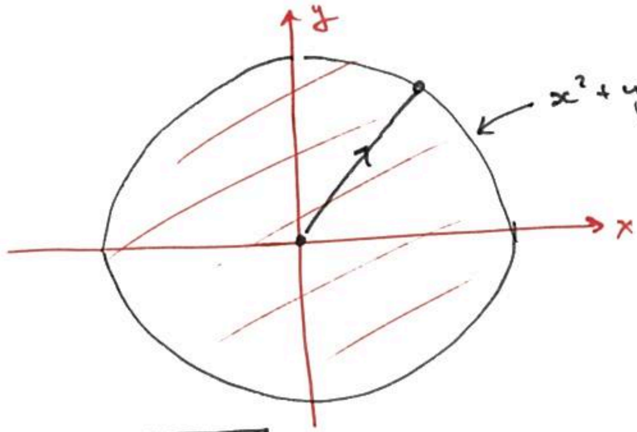
$$y(t) = \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} + u(t-1)\left\{-1 + \frac{1}{2}e^{t-1} + \frac{1}{2}e^{-(t-1)}\right\}.$$

c) $$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1$$ $a = 6378$ km, $b = 6357$ km

Vol = 2 × volume under one "hemisphere"

$$z = b\sqrt{1 - \frac{x^2+y^2}{a^2}}$$

Region of integration:



$r : 0 \to a$

$\theta : 0 \to 2\pi$

$$\text{Vol} = 2\int_0^{2\pi}\int_0^a b\sqrt{1 - r^2/a^2}\; r\,dr\,d\theta$$

let $u = 1 - r^2/a^2$

$\Rightarrow du = -\frac{2r}{a^2}\,dr$

$u(0) = 1$, $u(a) = 0$

$$= 2b\int_0^{2\pi}\int_0^a \sqrt{1 - r^2/a^2}\; r\,dr\,d\theta$$

$$= 2b\int_0^{2\pi}\int_1^0 -\frac{a^2}{2}\,u^{1/2}\,du\,d\theta$$

$$= + a^2 b\int_0^{2\pi}\int_0^1 u^{1/2}\,du\,d\theta$$

$$= a^2 b\int_0^{2\pi}\left[\frac{u^{3/2}}{3/2}\right]_0^1 d\theta$$

$$= 2\,\frac{a^2 b}{3}\int_0^{2\pi} d\theta$$

$$= \frac{4}{3}\pi a^2 b.$$

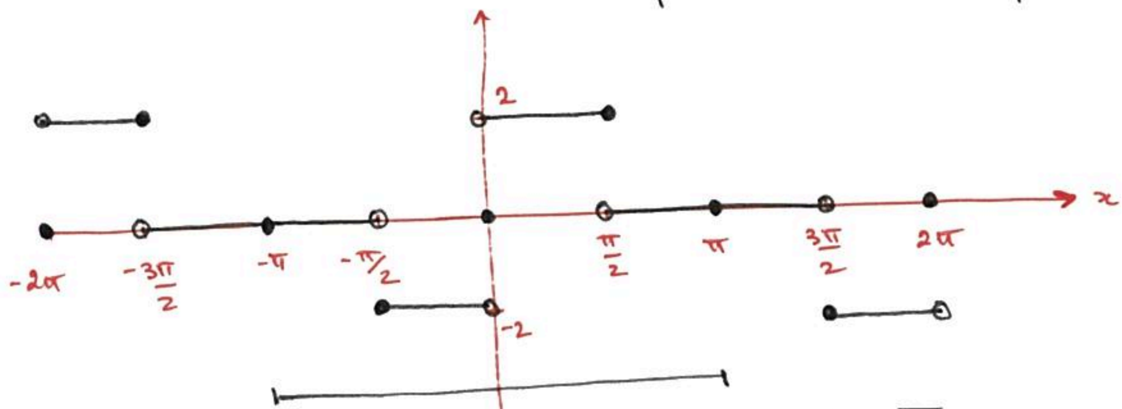
Thus, Volume of Earth $= \frac{4\pi}{3}(6378)^2(6357)$

$= 1.083 \times 10^{12}$ km^3.

QUESTION 4

odd periodic extension $f_{odd}(x)$

a) i)



$P = 2L = 2\pi \Rightarrow L = \pi$

ii) $$f_{odd}(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin nx$$ Fourier sine series

$$B_n = \frac{1}{L}\int_{-L}^{L} f_{odd}(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f_{odd}(x)}_{odd}\ \underbrace{\sin nx}_{odd}\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi/2} 2\sin nx\, dx + \cancel{\frac{2}{\pi}\int_{\pi/2}^{\pi} 0 \cdot \sin nx\, dx}$$

$$= \frac{4}{n\pi}\Big[-\cos nx\Big]_0^{\pi/2} = \frac{4}{n\pi}\left\{-\cos\frac{n\pi}{2} + \cos 0\right\}$$

$$= \frac{4}{n\pi}\left\{1 - \cos\frac{n\pi}{2}\right\}.$$

let $n = 2k+1$ = odd $\quad (k = 0, 1, 2, \dots)$

$$B_{2k+1} = \frac{4}{(2k+1)\pi}\left\{1 - \cancelto{0}{\cos\left(k+\tfrac{1}{2}\right)\pi}\right\} = \frac{4}{\pi(2k+1)}$$

let $n = 2k$ = even $\quad (k = 1, 2, 3 \dots)$

$$B_{2k} = \frac{4}{2k\pi}\left\{1 - \cos k\pi\right\} = \frac{2}{\pi k}\left(1 - (-1)^k\right).$$

optional.

Thus

$$B_1 = \frac{4}{\pi(2(0)+1)} = \frac{4}{\pi}$$

$$B_2 = \frac{2}{\pi(1)}\left(1-(-1)^1\right) = \frac{4}{\pi}$$

$$B_3 = \frac{4}{\pi(2(1)+1)} = \frac{4}{3\pi}$$

$$B_4 = \frac{2}{\pi(2)}\left(1-(-1)^2\right) = 0$$

$$B_5 = \frac{4}{\pi(2(2)+1)} = \frac{4}{5\pi}$$

OR from $B_n = \frac{4}{n\pi}\left\{1-\cos\frac{n\pi}{2}\right\}$

$$B_1 = \frac{4}{\pi}\{1-0\} = \frac{4}{\pi}$$

$$B_2 = \frac{4}{2\pi}\{1-(-1)\} = \frac{4}{\pi}$$

$$B_3 = \frac{4}{3\pi}\{1-0\} = \frac{4}{3\pi}$$

$$B_4 = \frac{4}{4\pi}\{1-1\} = 0$$

$$B_5 = \frac{4}{5\pi}\{1-0\} = \frac{4}{5\pi}$$

$$\Rightarrow \quad f(x) = \frac{4}{\pi}\sin x + \frac{4}{\pi}\sin 2x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \cdots$$

iii) at $x = \frac{\pi}{2}$ the Fourier series of $f(x)$ converges to

$$f\left(\frac{\pi}{2}\right) = \frac{f\left(\frac{\pi}{2}^-\right) + f\left(\frac{\pi}{2}^+\right)}{2} = 1$$

So $$f\left(\frac{\pi}{2}\right) = 1 = \frac{4}{\pi}\sin\frac{\pi}{2} + \cancel{\frac{4}{\pi}\sin\pi} + \frac{4}{3\pi}\sin\frac{3\pi}{2} + \frac{4}{5\pi}\sin\frac{5\pi}{2} + \cdots$$

$$= \frac{4}{\pi} - \frac{4}{3\pi} + \frac{4}{5\pi} - \cdots$$

$$= \frac{4}{\pi}\left\{1 - \frac{1}{3} + \frac{1}{5} - \cdots\right\}$$

Thus

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

10) $$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} \qquad u(0,t) = u(\pi,t) = 0$$

i) let $u(x,t) = F(x)\,G(t)$

$$u_{tt}(x,t) = F(x)\,G''(t)$$

$$u_{xx}(x,t) = F''(x)\,G(t)$$

$$\Rightarrow F(x)\,G''(t) = 4F''(x)\,G(t)$$

$$\Rightarrow \underbrace{\frac{F''(x)}{F(x)}}_{\text{function of } x} = \underbrace{\frac{G''(t)}{4G(t)}}_{\text{function of } t} = k = \text{constant.}$$

ii) $F''(x) = k\,F(x)$

case 1 : $k = 0 \Rightarrow F''(x) = 0$

$\Rightarrow F(x) = Ax + B$

$$\left.\begin{array}{ll} F(0) = 0 \Rightarrow & B = 0 \\ F(\pi) = 0 \Rightarrow & A = 0 \end{array}\right\} \text{ no non-trivial solution}$$

case 2 : $k = p^2 > 0$

$\Rightarrow F''(x) = p^2 F(x)$ if $F(x) = e^{\lambda x}$

$\Rightarrow \lambda^2 = p^2 \Rightarrow \lambda = \pm p$

$$F(x) = Ae^{px} + Be^{-px}$$

$F(0) = 0 = A + B \Rightarrow B = -A$

$F(\pi) = 0 = Ae^{+p\pi} + Be^{-p\pi} = A(e^{p\pi} - e^{-p\pi})$

$\Rightarrow A = 0$ (trivial solution) or

$e^{p\pi} - e^{-p\pi} = 0 \Rightarrow e^{2p\pi} = 1$

no solution because $p > 0$

case 3 : $k = -p^2 < 0$.

$F''(x) = -p^2 F(x)$ if $F(x) = e^{\lambda x}$

$\lambda^2 = -p^2 \Rightarrow \lambda = \pm ip$

$F(x) = A \cos px + B \sin px$

$F(0) = 0 = A \Rightarrow A = 0$

$F(\pi) = 0 = B \sin p \cdot \pi$

$\Rightarrow B = 0$ (trivial solution) OR

$p\pi = n\pi \quad n = 1, 2, 3, \ldots$

Thus, nontrivial solution is

$$F_n(x) = B_n \sin nx$$

ii) given $k = -p^2 = -n^2 \quad n = 1, 2, 3 \ldots$

$G_n''(t) = -4n^2 G_n(t)$ let $G_n = e^{\lambda t}$

$\Rightarrow \lambda^2 = -4n^2 \Rightarrow \lambda = \pm 2ni$

$G_n(t) = C_n \cos 2nt + D_n \sin 2nt$

iii) $u(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos 2nt + D_n \sin 2nt \right) \sin nx$

Absorb B_n into C_n & D_n.

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[-2nC_n \sin 2nt + 2nD_n \cos 2nt \right] \sin nx$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \left\{ -2nC_n(0) + 2nD_n(1) \right\} \sin nx$$

$$= \sum_{n=1}^{\infty} 2nD_n \sin nx = 0$$

$\Rightarrow D_n = 0$ all n.

Thus $\quad u(x,t) = \sum_{n=1}^{\infty} C_n \cos 2nt \cdot \sin nx$

$$u(x,0) = \sum_{n=1}^{\infty} C_n (1) \sin nx = \sum_{n=1}^{\infty} C_n \sin nx$$

$$= 3 \sin x + 4 \sin 3x$$

Thus $\quad C_1 = 3 \,, \; C_3 = 4 \,, \; C_n = 0 \quad$ all other n.

Thus

$$u(x,t) = 3 \cos 2t \sin x + 4 \cos 4t \cdot \sin 2x$$