THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

JUNE 2013

MATH2019
ENGINEERING MATHEMATICS 2E

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

TABLE OF LAPLACE TRANSFORMS AND THEOREMS

$g(t)$ is a function defined for all $t \geq 0$, and whose Laplace transform

$$G(s) = \mathcal{L}(g(t)) = \int_0^\infty e^{-st} g(t)dt$$

exists. The Heaviside step function u is defined to be

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ \frac{1}{2} & \text{for } t = a \\ 1 & \text{for } t > a \end{cases}$$

$g(t)$	$G(s) = \mathcal{L}[g(t)]$
1	$\dfrac{1}{s}$
t	$\dfrac{1}{s^2}$
$t^\nu, \nu > -1$	$\dfrac{\nu!}{s^{\nu+1}}$
$e^{-\alpha t}$	$\dfrac{1}{s+\alpha}$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$e^{-\alpha t} f(t)$	$F(s+\alpha)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$tf(t)$	$-F'(s)$

FOURIER SERIES

If $f(x)$ has period $p = 2L$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{aligned}$$

LEIBNIZ RULE FOR DIFFERENTIATING INTEGRALS

$$\frac{d}{dx}\int_u^v f(x,t)dt = \int_u^v \frac{\partial f}{\partial x}dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx}.$$

MULTIVARIABLE TAYLOR SERIES

$$\begin{aligned} f(x,y) &= f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\ &\quad + \frac{1}{2!}\left((x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) + (y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b) \right) + \cdots \end{aligned}$$

SOME BASIC INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^{kx} dx = \frac{e^{kx}}{k} + C$$

$$\int a^x dx = \frac{1}{\ln a} a^x + C \quad \text{for } a \neq 1$$

$$\int \sin\, kx\, dx = -\frac{\cos kx}{k} + C$$

$$\int \cos kx\, dx = \frac{\sin kx}{k} + C$$

$$\int \sec^2 kx\, dx = \frac{\tan kx}{k} + C$$

$$\int \operatorname{cosec}^2 kx\, dx = -\frac{1}{k} \cot kx + C$$

$$\int \tan kx\, dx = \frac{\ln|\sec kx|}{k} + C$$

$$\int \sec kx\, dx = \frac{1}{k}(\ln|\sec kx + \tan kx|) + C$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx$$

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx$$

Answer question 1 in a separate book

1. i) Consider the double integral

$$I = \int_0^1 \int_x^1 3y \, dy \, dx.$$

a) Sketch the region of integration.

b) Evaluate I by first changing the order of integration.

ii) Determine the Taylor expansion of $x^2 \ln(y)$ about $(3, 1)$ up to and including quadratic terms.

iii) Ohm's law $V = IR$ relates the voltage V across a conductor to the current I and the resistance R. Suppose that both I and R are varying over time t. At a certain time you are given that

$$R = 100, \ \frac{dR}{dt} = 2, I = 0.04 \text{ and } \frac{dI}{dt} = -0.01.$$

Use the chain rule to find the rate of change $\dfrac{dV}{dt}$ of the voltage.

iv) Suppose that the temperature T in a certain region of space is given by

$$T(x, y, z) = xe^y \sin(z).$$

a) Calculate $\nabla T = \text{grad } T$ at the point $P(1, 0, \frac{\pi}{2})$.

b) Find the rate of change of the temperature with respect to distance at the point P in the direction of the vector $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

v) Consider the matrix $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors of A.

b) Note that $3x^2 + 4xy + 3y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

By considering the eigenvalues of A, write the curve $3x^2+4xy+3y^2 = 40$ in terms of its principal axes coordinates X and Y.

c) Sketch $3x^2 + 4xy + 3y^2 = 40$ in the $x - y$ plane, clearly labeling the principal axes.

Answer question 2 in a separate book

2. i) Given the vector field $\mathbf{F} = z\mathbf{i} + xyz\mathbf{j} - x\mathbf{k}$ calculate:

a) div $\mathbf{F}$.

b) curl $\mathbf{F}$.

ii) Find and classify the critical points of

$$f(x, y) = 2x^3 - 9x^2 + 12x + 3y^2 - 18y + 4.$$

iii) Consider the differential equation

$$y'' + cy' + 9y = 0 \qquad (c > 0)$$

modeling a damped harmonic oscillator.

a) What values of the damping constant c produce underdamping?

b) Draw a possible sketch of the solution $y(t)$ for $t \geq 0$ if the system is underdamped. (Note that you are not being asked to solve the differential equation).

iv) The semi-circular region bounded by $x = \sqrt{1 - y^2}$ and $x = 0$ has centroid $(\bar{x}, \bar{y})$.

a) Explain why $\bar{y} = 0$.

b) Find $\bar{x}$ by evaluating an appropriate double integral in polar coordinates.

v) Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be three non-coplanar vectors in $\mathbb{R}^3$.

Define three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$ by:

$$\mathbf{a} = \frac{\mathbf{u} \times \mathbf{v}}{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})}, \quad \mathbf{b} = \frac{\mathbf{v} \times \mathbf{w}}{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})} \quad \text{and} \quad \mathbf{c} = \frac{\mathbf{w} \times \mathbf{u}}{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})}.$$

a) Explain why $\mathbf{a}$ is perpendicular to $\mathbf{u}$ and $\mathbf{v}$.

b) Let P_1 be the volume of the parallelepiped formed by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and P_2 be the volume of the parallelepiped formed by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Find a relationship between P_1 and P_2.

You may use the fact that for any three vectors $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{t}$ in $\mathbb{R}^3$, the vector triple product satisfies the equation $\mathbf{r} \times (\mathbf{s} \times \mathbf{t}) = (\mathbf{r} \cdot \mathbf{t})\mathbf{s} - (\mathbf{r} \cdot \mathbf{s})\mathbf{t}$ and the scalar triple product satisfies the equation $\mathbf{r} \cdot (\mathbf{s} \times \mathbf{t}) = \mathbf{s} \cdot (\mathbf{t} \times \mathbf{r}) = \mathbf{t} \cdot (\mathbf{r} \times \mathbf{s})$.

Answer question 3 in a separate book

3. i) A fluid's velocity field in a turbine of a hydroelectric generator is given in the plane by

$$\mathbf{F}(x, y) = -2y\mathbf{i} + 2x\mathbf{j}.$$

Let $\mathcal{C}$ be the unit circle with centre at (0,0) parametrized by

$$\mathbf{c}(t) = (\cos t, \sin t) \text{ from } t = 0 \text{ to } t = 2\pi.$$

Calculate the circulation of $\mathbf{F}$ around $\mathcal{C}$ by computing the line integral

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

ii) Let $f(x) = \begin{cases} x & 0 \leq x \leq 3\ ; \\ 0 & 3 < x < 4\,, \end{cases}$

with $f(x+4) = f(x)$ for all x.

a) What is the period of f ?

b) Sketch the graph of f over the domain $-4 \leq x \leq 4$.

c) To what value does the Fourier series of f converge at $x = 3$?

iii) Using the table of Laplace transforms find

$$\mathcal{L}(t^2 + \cos(t) + e^{5t}).$$

iv) By establishing an appropriate partial fraction decomposition find

$$\mathcal{L}^{-1}\left(\frac{8s - 42}{s^2 - 7s}\right).$$

v) Consider the function given by

$$f(t) = \begin{cases} 2, & 3 \leq t \leq 4; \\ 1, & \text{otherwise.} \end{cases}$$

a) Sketch f.

b) Express f in terms of Heaviside functions.

c) Hence write down $\mathcal{L}(f)$.

vi) By using Leibniz's rule or otherwise prove that $\mathcal{L}(tf(t)) = -F'(s)$, where $F(s) = \mathcal{L}(f(t))$.

Answer question 4 in a separate book

4. i) Define a function f by

$$f(x) = \begin{cases} 10 & 0 \le x \le \dfrac{\pi}{2} \\ 0 & \dfrac{\pi}{2} < x \le \pi. \end{cases}$$

a) Sketch the graph of f over the interval $0 \le x \le \pi$.

b) Sketch the even periodic extension of f over the interval $-3\pi \le x \le 3\pi$.

c) Find the Fourier cosine series of f.

ii) The temperature in a bar of length π satisfies the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ is the temperature in $^\circ C$. The ends of the bar are insulated, meaning

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \text{and} \quad \frac{\partial u}{\partial x}(\pi,t) = 0, \quad \text{for all} \quad t > 0.$$

a) Assuming a solution of the form

$$u(x,t) = F(x)G(t) \quad \text{show that}$$

$$\frac{1}{G}\frac{dG}{dt} = \frac{1}{F}\frac{d^2F}{dx^2} = k \quad \text{where } k \text{ is a constant.}$$

You are given that $k > 0$ yields only the trivial solution and that $k = 0$ generates the constant solution $u_0(x,t) = A_0$.

In parts b) and c) assume that $k < 0$ and set $k = -(p^2)$ for some $p > 0$.

b) Applying the boundary conditions show that $p = n,\ n = 1, 2, 3, \ldots$ and that possible solutions for $F(x)$ are

$$F_n(x) = A_n \cos(nx) \qquad n = 1, 2, \ldots$$

c) Find all possible solutions $G_n(t)$ for $G(t)$.

d) Suppose now that the initial temperature distribution of the bar is

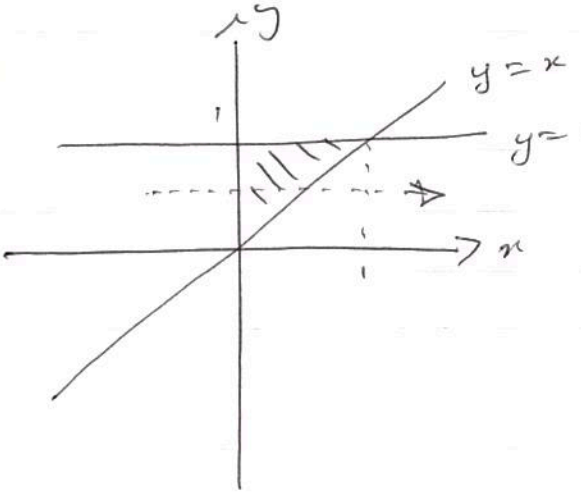
$$u(x,0) = f(x)$$

where f is the function from Question 4 part i). Using your series from part i), find the general series solution $u(x,t)$.

e) What is the equilibrium temperature as $t \longrightarrow \infty$?

Math 2019 June 2013 Question 1 Solutions

1 i)



$$I = \int_0^1 \int_0^y 3y \, dx\, dy$$

$$= \int_0^1 \left[3xy\right]_0^y \, dy$$

$$= \int_0^1 3y^2 - 0 \, dy$$

$$= \left[y^3\right]_0^1 = 1 - 0$$

$$= \underline{1}$$

ii) $f(x,y) = x^2 \ln(y) \rightarrow f(3,1) = 9\ln(1) = 0.$

$f_x(x,y) = 2x \ln y \rightarrow f_x(3,1) = 6\ln 1 = 0.$

$f_y(x,y) = \dfrac{x^2}{y} \rightarrow f_y(3,1) = \dfrac{9}{1} = 9.$

$f_{xx}(x,y) = 2\ln(y) \rightarrow f_{xx}(3,1) = 2\ln(1) = 0$

$f_{xy}(x,y) = \dfrac{2x}{y} \rightarrow f_{xy}(3,1) = \dfrac{6}{1} = 6$

$f_{yy}(x,y) = -\dfrac{x^2}{y^2} \rightarrow f_{yy}(3,1) = -\dfrac{9}{1} = -9$

$$\therefore f(x,y) = 9(y-1) + \frac{1}{2!}\left\{2(x-3)(y-1)(6) + (-9)(y-1)^2\right\}$$

$$\underline{= 9(y-1) + 6(x-3)(y-1) - \tfrac{9}{2}(y-1)^2}$$

iii) $$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\cdot\frac{dR}{dt}$$

$$= R\frac{dI}{dt} + I\frac{dR}{dt}$$

$$= 100(-0.01) + 0.04(2)$$

$$= \underline{-0.92}.$$

iv) a) $$\nabla T = \frac{\partial T}{\partial x}\underset{\sim}{i} + \frac{\partial T}{\partial y}\underset{\sim}{j} + \frac{\partial T}{\partial z}\underset{\sim}{k}$$

$$= e^y \sin z\,\underset{\sim}{i} + xe^y\sin(z)\underset{\sim}{j} + xe^y\cos(z)\underset{\sim}{k}$$

$$= e^0\sin\frac{\pi}{2}\underset{\sim}{i} + 1e^0\sin\frac{\pi}{2}\underset{\sim}{j} + 1e^0\cancel{\cos}\left(\frac{\pi}{2}\right)\underset{\sim}{k}$$

$$= \underset{\sim}{i} + \underset{\sim}{j} + 0\underset{\sim}{k}$$

b) $$\hat{\underset{\sim}{b}} = \frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix}.$$

$\therefore$ directional derivative is $\nabla T\cdot\hat{b}$

$$= \begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\left(\frac{1}{3}\right)\begin{pmatrix}2\\1\\2\end{pmatrix}$$

$$= \frac{1}{3}(2+1+0)$$

$$= \underline{1.}$$

v) $\begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 4 = 0$

$\Rightarrow (3-\lambda)^2 = 4$

$\Rightarrow 3-\lambda = \pm 2$

$\Rightarrow \lambda = 3 \mp 2 = 5, 1.$

<u>For $\lambda = 1$</u>: $\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

Let $y = t \Rightarrow 2x = -2t \Rightarrow x = -t$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t$

In particular $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

For $\lambda = 5$: $\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

Let $y = t \Rightarrow -2x + 2t = 0 \Rightarrow x = t$.

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t$ and in particular

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

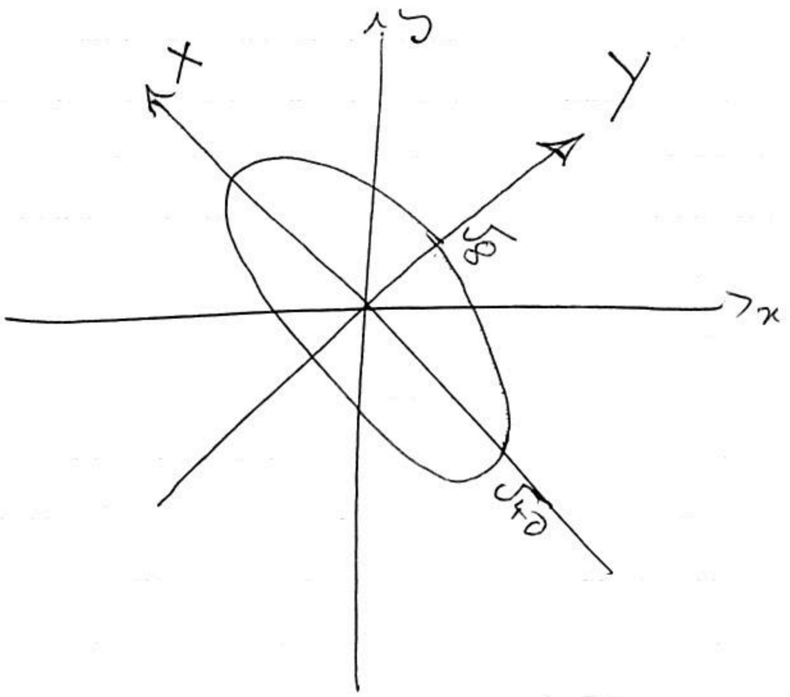
b) Let $X = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$, $Y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ be the principle axes. With respect to this new co-ord system the equation becomes:

$$X^2 + 5Y^2 = 40$$

(can ellipse ++)

(Note X and Y could be swapped)

c)



Math 2019 June 2013 Q2 Solutions.

2i) a) $div(F) = 0 + xz + 0 = \underline{xz}$.

b)
$$\text{curl}F = \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & xyz & -x \end{vmatrix}$$

$$= \underset{\sim}{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & -x \end{vmatrix} - \underline{j} \cdot \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & -x \end{vmatrix} + \underset{\sim}{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & xyz \end{vmatrix}$$

$$= i(0 - xy) - j(-1 - 1) + k(yz - 0)$$

$$= \underline{\begin{pmatrix} -xy \\ 2 \\ yz \end{pmatrix}}.$$

(ii) $f_x = 6x^2 - 18x + 12 = 0$

$\rightarrow x^2 - 3x + 2 = 0 \rightarrow (x-2)(x-1) = 0$

$\rightarrow x = 1, 2.$

$f_y = 6y - 18 = 0 \rightarrow y = 3.$

we therefore have two critical points

$(1,3)$ and $(2,3)$.

$$D = f_{xx} f_{yy} - (f_{xy})^2$$
$$= (12x-18)6 - 0^2 = 6(12x-18).$$

$D(1,3) = 6(-6) < 0$

$\rightarrow$ $(1,3)$ is a saddle point.

$D(2,3) = 6(6) > 0 \rightarrow$ max/min.

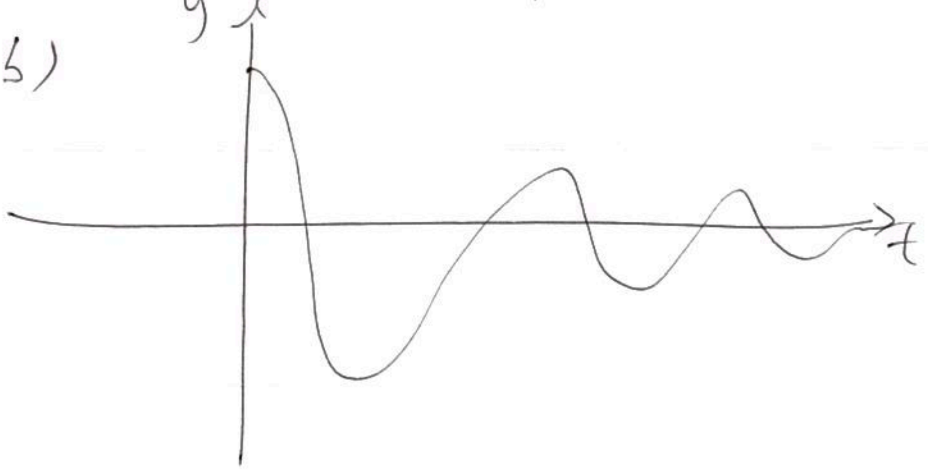
$f_{xx}(2,3) = 6 > 0$

$\rightarrow$ $(2,3)$ is a local minimum.

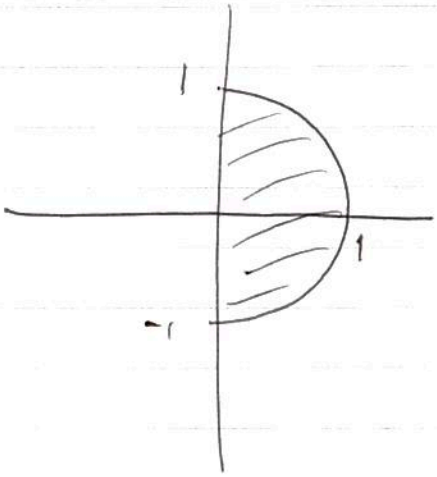
iii) $\Delta = c^2 - 4.1.9 = c^2 - 36$

a) For underdamping $0 < c < 6$.

b)



iv)



a) By symmetry $\bar{y} = 0$.

b) $\text{Mass} = \iint_\Omega \rho(x,y)\,dA$

$= \iint_\Omega 1 \, dA$

$= \text{area}(\Omega) = \frac{\pi(1)^2}{2} = \frac{\pi}{2}.$

Also $M_y = \iint_\Omega x\rho(x,y)\,dA$

$= \iint_\Omega x \cdot 1 \, dy\,dx.$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 r\cos\theta \; r\,dr\,d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 r^2\cos\theta \, dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{3}r^3\cos\theta\right]_0^1 d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3}\cos\theta\,d\theta = \left[\frac{1}{3}\sin(\theta)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{1}{3}\{1 - -1\} = 2/3.$

$\therefore \quad \bar{x} = \frac{M_y}{M} = \frac{2/3}{\pi/2} = 2/3 \times \frac{2}{\pi} = \underline{\frac{4}{3\pi}}$

V) a) $\mathbf{a}$ is a non-zero scalar multiple of the cross product $\mathbf{u}\times\mathbf{v}$ and is hence perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

P_1 = magnitude of $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$

P_2 = magnitude of $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$.

$$P_2 = \frac{(\mathbf{u}\times\mathbf{v})\cdot\left[(\mathbf{v}\times\mathbf{w})\times(\mathbf{w}\times\mathbf{u})\right]}{\alpha^3}$$

where $\alpha = (\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}))^3$

Now $(\mathbf{u}\times\mathbf{v})\cdot\left[(\mathbf{v}\times\mathbf{w})\times(\mathbf{w}\times\mathbf{u})\right]$

$= (\mathbf{u}\times\mathbf{v})\cdot\left[((\mathbf{v}\times\mathbf{w})\cdot\mathbf{u})\mathbf{w} - \cancel{((\mathbf{v}\times\mathbf{w})\cdot\mathbf{w})\mathbf{u}}\right]$

$= ((\mathbf{v}\times\mathbf{w})\cdot\mathbf{u})(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$

$= (\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}))(\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}))$

$= (\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}))(\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})) = \alpha^2$

$\therefore\ P_2 = \dfrac{\alpha^2}{\alpha^3} = \dfrac{1}{\alpha}$

Thus $P_2 = \dfrac{1}{P_1}$

P_1 and P_2 are reciprocals of each other.

Math 2019 June 2013 Question 3 Solutions

3 i) $x = \cos t \rightarrow dx = -\sin t \, dt$ $\quad t: 0 \rightarrow 2\pi.$
$y = \sin t \rightarrow dy = \cos t \, dt$

$$\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \oint_C \begin{pmatrix} -2y \\ 2x \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} = \oint_C -2y\,dx + 2x\,dy$$

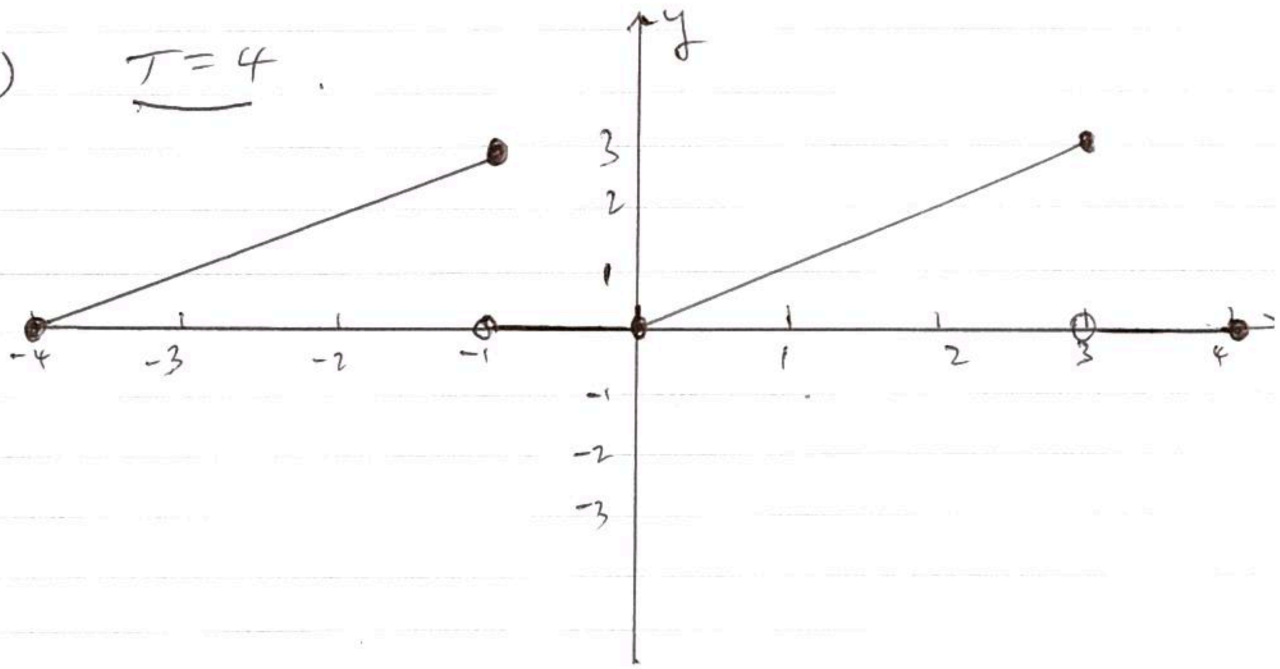
$$= \int_0^{2\pi} -2\sin t\,(-\sin t\,dt) + 2\cos t\,(\cos t\,dt)$$

$$= \int_0^{2\pi} 2\sin^2 t + 2\cos^2 t \; dt$$

$$= \int_0^{2\pi} 2\,dt = \Big[2t\Big]_0^{2\pi} = \underline{4\pi}.$$

ii) a) $\underline{T = 4}$.

b)



c) The Fourier series will converge to a point half-way across the discontinuity. That is $y = 3/2$.

(iii) $\mathcal{L}(t^2 + \cos t + e^{5t}) = \underline{\frac{2}{s^3} + \frac{s}{1+s^2} + \frac{1}{s-5}}$

iv) $$\frac{8s-42}{s(s-7)} = \frac{A}{s} + \frac{B}{s-7} = \frac{A(s-7)+Bs}{s(s-7)}$$

$\therefore \quad A(s-7) + Bs = 8s - 42.$

$\underline{s=0}$: $\quad -7A = -42 \rightarrow \quad A = 6$

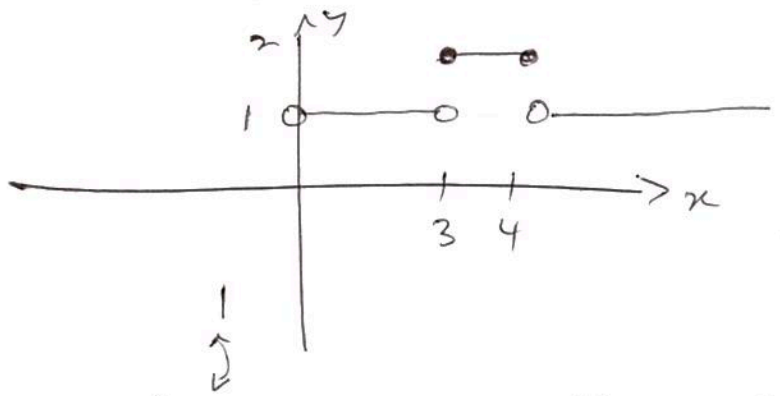
$s=7$: $\quad 7B = 14 \quad \rightarrow \quad B = 2.$

So $$\frac{8s-42}{s(s-7)} = \underline{\frac{6}{s} + \frac{2}{s-7}}.$$

The inverse Laplace transform is therefore.

$$\underline{6 + 2e^{7t}}.$$

v) a)



b) $f(t) = [u(t-0) - u(t-3)]1 + [u(t-3) - u(t-4)] \times 2$

$+ \; u(t-4).$

$= 1 + u(t-3) - u(t-4).$

c) $\mathcal{L}(f) = \frac{1}{s} + \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s}$.

vi) $\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

$\Rightarrow \frac{d}{ds} \int_0^\infty e^{-st} f(t)\,dt = \frac{d}{ds} F(s)$

via Leibniz $\Rightarrow \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\,dt = F'(s)$

$\Rightarrow \int_0^\infty -t e^{-st} f(t)\,dt = F'(s)$

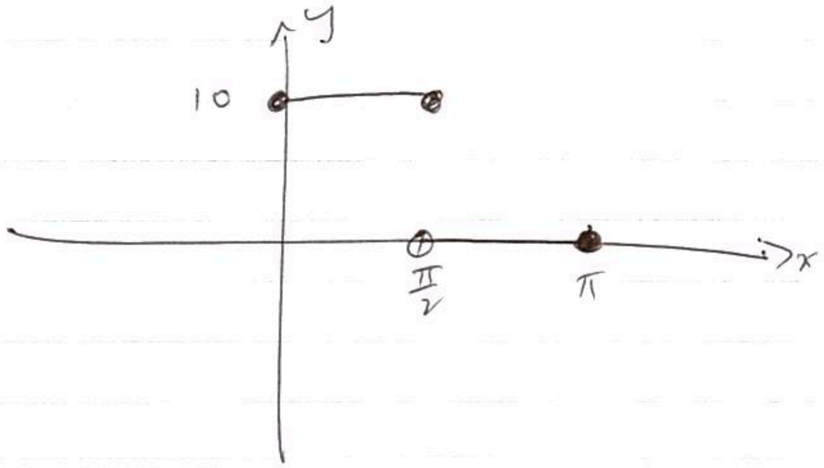
$\Rightarrow \int_0^\infty e^{-st} (t f(t)) = -F'(s)$

$\Rightarrow \mathcal{L}(t f(t)) = -F'(s)$

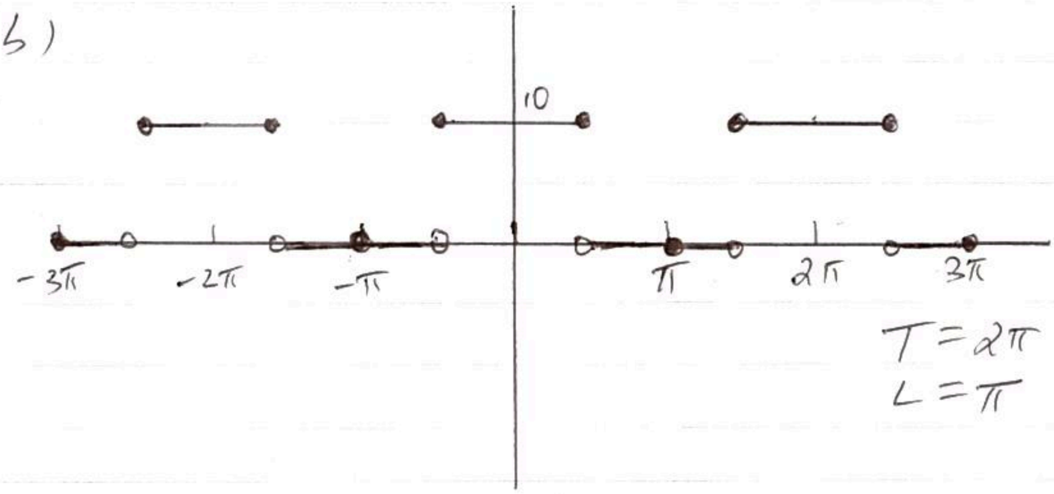
as required.

Math 2019 June 2013 Question 4 Solutions.

4 i) a)



b)



$T = 2\pi$
$L = \pi$

c) $$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi}\right)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos(nx).$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$= \frac{1}{2\pi} \times 10 \times \pi = 5.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{even}}\underbrace{\cos(nx)}_{\text{even}}\,dx$$

$$= \frac{1}{\pi} \times 2 \int_0^{\pi} f(x) \cos(nx)\, dx.$$

$$= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} 10 \cos(nx)\, dx.$$

$$= \frac{20}{\pi} \left[\frac{1}{n} \sin(nx) \right]_0^{\frac{\pi}{2}}$$

$$= \frac{20}{\pi} \cdot \frac{1}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$= \frac{20}{n\pi} \sin\left(\frac{n\pi}{2}\right).$$

$$\therefore \underline{f(x) = 5 + \sum_{n=1}^{\infty} \frac{20}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(nx)}$$

a) $u(x,t) = F(x)G(t)$

$$\frac{\partial u}{\partial t} = F(x) G'(t)$$

$$\frac{\partial^2 u}{\partial x^2} = F''(x) G(t)$$

So $FG' = F''G$

$$\rightarrow \frac{F''}{F} = \frac{G'}{G} = k.$$

b) Assume that $k = -(p^2)$.

$$\frac{F''}{F} = -p^2 \Rightarrow F'' + p^2 F = 0$$

Aux eqn $m^2 + p^2 = 0$

$$\Rightarrow m = 0 \pm ip.$$

Hence $F(x) = e^{0x}\{A\cos px + B\sin(px)\}$

$= A\cos px + B\sin px.$

$\frac{\partial v}{\partial x}(0,t) = 0 \Rightarrow F'(0)G(t) = 0 \rightarrow F'(0) = 0$

$\frac{\partial v}{\partial x}(\pi,t) = 0 \Rightarrow F'(\pi)\, G(t) = 0 \rightarrow F'(\pi) = 0.$

Now $F'(x) = -pA\sin px + pB\cos px.$

$F'(0) = 0 \Rightarrow 0 = -pA.0 + pB.1$

$\Rightarrow pB = 0 \Rightarrow B = 0.$

$\therefore F(x) = A\cos(px) \rightarrow F'(x) = -Ap\sin(px)$

$F'(\pi) = 0 \Rightarrow -Ap\sin(p\pi) = 0 \Rightarrow \sin(p\pi) = 0$

$\Rightarrow p = n = 1, 2, 3, \ldots$

$\therefore F(x) = F_n(x) = A_n \cos(nx)$

are possible solutions for F.

c) $\frac{G'}{G} = -(p^2) \Rightarrow \frac{G'}{G} = -(n^2)$

$\Rightarrow G' = -(n^2)G \Rightarrow G(t) = Ce^{-n^2 t}$

without loss of generality set $C = 1$.

Thus $u_n(x,t) = A_n \cos(nx)\, e^{-n^2 t}$

and taking linear combinations of all possible solutions yields

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx)\, e^{-n^2 t}.$$

d)

$$u(x,0) = f(x) \Rightarrow A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) = f(x)$$

$\Rightarrow A_0$ & A_n are the Fourier cosine series co-efficients of f. That is

$$A_0 = 5 \quad \text{and} \quad A_n = \frac{20}{n\pi} \sin\left(\frac{n\pi}{2}\right).$$

Thus $u(x,t) = 5 + \sum_{n=1}^{\infty} \frac{20}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(nx)\, e^{-n^2}$

e) $\lim_{t \to \infty} u(x,t) = \underline{\underline{5}}$