mixed terms = rotation

# LECTURE 31
# QUADRIC SURFACES

When a standard quadric surface

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1$$

is rotated in space mixed terms $xy$, $xz$ and $yz$ appear in the equation. It is possible to express the resulting quadratic form in matrix form $\mathbf{x}^T A\mathbf{x} = 1$ where $A$ is a symmetric matrix. An analysis of the eigenvectors and eigenvalues of $A$ will reveal both the structure and the principal axes of the surface.

Consider the curve $2x^2 - 4xy + 5y^2 = 54$. The presence of the mixed term $xy$ indicates that this is a standard object (ellipse or hyperbola) which has been tilted to some degree so that its major and minor axes no longer point in the $x$ and $y$ directions. To understand the curve we need to apply a specific transformation which "untilts" the curve into standard form. Our first step is to rewrite the quadratic form in terms of matrices.

Claim: $2x^2 - 4xy + 5y^2 = 54$ is equivalent to $\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 54$

Proof:

shared

$= (x \;\; y)(2x - 2y \;\; -2x + 5y)$

$= 2x^2 - 2xy - 2xy + 5y^2$

$= 2x^2 - 4xy + 5y^2$

This gets us over into the arena of matrices where the theory of eigenvalues and eigenvectors may be brought into play! Observe that by its nature of construction, the matrix $A$ will be symmetric and thus its eigenvectors will be naturally orthogonal to each other. We now undertake a complete eigenanalysis of $A$.

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} = \begin{vmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix}$$

$$= (2-\lambda)(5-\lambda) - 4$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\therefore \lambda = 1, 6$$

$\underline{\lambda = 6:}$ $\left(\begin{array}{cc|c} 2-6 & -2 & 0 \\ -2 & 5-6 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -4 & -2 & 0 \\ 0 & 1 & 0 \end{array}\right)$

Let $y = t$ $\therefore x = -\frac{1}{2}t$

$\therefore$ for $\lambda = 6$, $\vec{v} = t_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $t_1 \in \mathbb{R}$

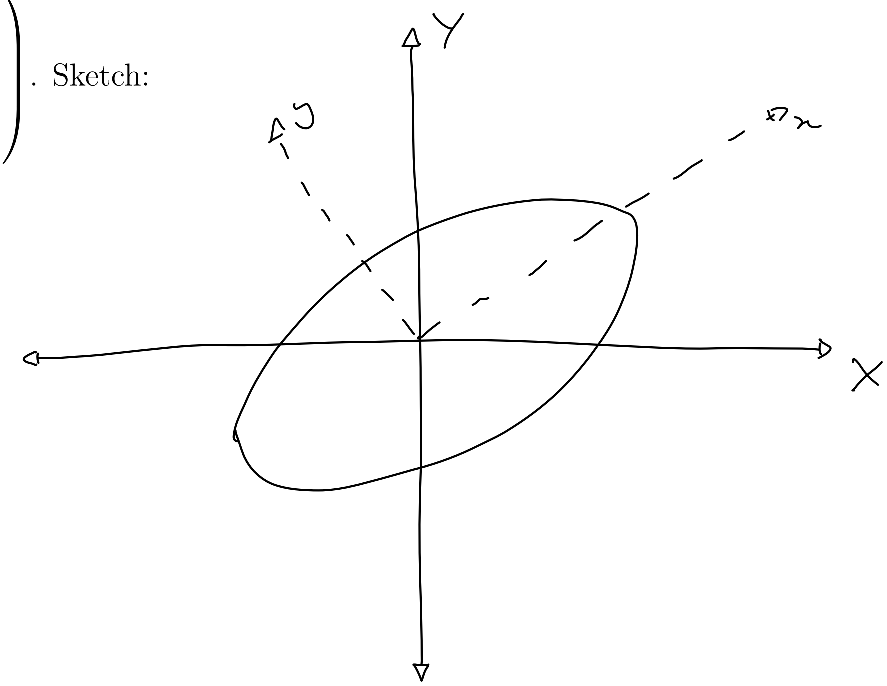
$\underline{\lambda = 1:}$ $\left(\begin{array}{cc|c} 2-1 & -2 & 0 \\ -2 & 5-1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\therefore \vec{v} = t_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for $\lambda = 1$

$\underline{\text{Check:}}$ $\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 0$

$\therefore \vec{v}_1 \perp \vec{v}_2$ i.e. forms principal axes.

So the eigenvalues of $A$ are 1 and 6 with associated unit eigenvectors $\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$ respectively. Observe that the eigenvectors are orthogonal! These directions $X = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$ and $Y = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$ actually form the principal axes of the curve. That is the curve sits properly on the eigenvectors of $A$. We will prove this formally in the next lecture. Furthermore in the $\{X, Y\}$ system the equation of the curve is $1X^2 + 6Y^2 = 54$. (Note the use of the eigenvalues). We can now identify the curve as an ellipse whose closest point to the origin is 3 units in the $Y$ direction. Thus the closest points to the origin are $\pm \begin{pmatrix} -\frac{3}{\sqrt{5}} \\ \frac{6}{\sqrt{5}} \end{pmatrix}$ in the $\{x, y\}$ system. The transformation which interrelates the two coordinate systems is $\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} X \\ Y \end{pmatrix}$ where $P$ is the usual matrix of eigenvectors $\begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$. Sketch:




In summary:

The quadratic form $\mathbf{x}^T A \mathbf{x}$ where $A$ is a symmetric matrix has principal axes given by the orthogonal eigenvectors of $A$ and the associated quadratic curves and quadric surfaces may be transform into standard objects with the eigenvalues as coeffiecients.

**Example 1** Express the equation of the surface

$$x^2 + 2y^2 + 2z^2 + 4xy - 4xz + 6yz = 30$$

in terms of its principal axes $X$, $Y$ and $Z$ and hence determine the nature of the surface. Find an orthogonal matrix $P$ implementing the transformation through $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$.

Deduce the shortest distance from the surface to the origin and the $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ coordinates of these closest point(s).

---

The equation may be written as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}^T \begin{pmatrix} 1 & 2 & -2 \\ 2 & 2 & 3 \\ -2 & 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 30$

You are given that the matrix $A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 2 & 3 \\ -2 & 3 & 2 \end{pmatrix}$ has eigenvalues $-3$ and $5$ with associated eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Find the remaining eigenvalue and eigenvector and hence complete the question.

$$\begin{pmatrix} 1-\lambda & 2 & -2 \\ 2 & 2-\lambda & 3 \\ -2 & 3 & 2-\lambda \end{pmatrix}, \quad \lambda = 3$$

$$\rightarrow \left(\begin{array}{ccc|c} -2 & 2 & -2 & 0 \\ 2 & -1 & 3 & 0 \\ -2 & 3 & -1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} -2 & 2 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

let $z = t$

$\therefore y = -t$

$x = -2t$

$\therefore \vec{v} = t \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$

$-3 + 5 + \lambda = 1 + 2 + 2$

$\therefore \lambda = 3$

$$\left.\begin{aligned} X &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ Y &= \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \\ Z &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \end{aligned}\right\} \text{unit eig. vec}$$

wrt $\{X, Y, Z\}$ : $-3X^2 + 2Y^2 + 5Z^2 = 30$

(hyperboloid)

Shortest distance $= \sqrt{6}$ in $z$-direction : $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\therefore$ closest point: $\pm \begin{pmatrix} 0 \\ \sqrt{3} \\ \sqrt{3} \end{pmatrix}$

$$\bigstar \quad P = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \; -3X^2 + 3Y^2 + 5Z^2 = 30 \quad \bigstar$$

★ Hyperboloid of one sheet with closest distance to the origin of $\sqrt{6}$ at $\pm \begin{pmatrix} 0 \\ \sqrt{3} \\ \sqrt{3} \end{pmatrix}$ ★

We will start the next lecture with a formal proof of these algorithms and results.

[31] You can now do Q 91 and 92