# LECTURE 4
# LEIBNIZ' THEOREM

**Leibniz Rule**

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)dt = \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}dt + f(x,v(x))\frac{dv}{dx} - f(x,u(x))\frac{du}{dx}.$$

**SOME BASIC INTEGRALS**

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int \frac{1}{x}dx = \ln|x| + C$$

$$\int e^{ax}dx = \frac{e^{ax}}{a} + C$$

$$\int \sin\, ax\, dx = -\frac{\cos ax}{a} + C$$

$$\int \cos ax\, dx = \frac{\sin ax}{a} + C$$

$$\int \sec^2 ax\, dx = \frac{\tan ax}{a} + C$$

Leibniz' Rule is one of the truly horrible equations in mathematics. It deals with the subtle problem of what happens when we start differentiating integrals of functions of several variables. Differentiation and Integration are of course opposing processes so it would seem reasonable to suspect that differentiating integrals would have some specific consequences! We will motivate the rule with a simple example in Example 2 but first a little revision on integration theory:

**Example 1** Evaluate each of the following integrals:

a) $\int e^{7x}dx$

b) $\int \sin(3x)dx$

c) $\int_0^{\frac{\pi}{2}} \cos(\frac{x}{2})dx$

d) $\int \frac{x}{x^2+1}dx$

(a) $\int e^{7x}\,dx = \frac{e^{7x}}{7} + C$

(b) $\int \sin(3x)\,dx = \frac{-\cos(3x)}{3} + C$

(c) $\int_0^{\frac{\pi}{2}} \cos\left(\frac{x}{2}\right) dx = 2\sin\left(\frac{x}{2}\right)\Big|_0^{\frac{\pi}{2}} = \sqrt{2}$

(d) $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln\left(x^2+1\right) + C$

★

**Example 2** Find

$$\frac{d}{dx}\int_1^{2x} x^6t^2dt$$

first directly and then using Leibniz' theorem.

$$\frac{d}{dx}\int_1^{2x} x^6 t^2 dt = \frac{d}{dx}\left(\left[\frac{x^6 t^3}{3}\right]_1^{2x}\right)$$

$$= \frac{d}{dx}\left(\frac{x^6(2x)^3}{3} - \frac{x^6}{3}\right)$$

$$= \frac{d}{dx}\left(\frac{8}{3}x^9 - \frac{x^6}{3}\right)$$

$$= 24x^8 - 2x^5$$

$$\frac{d}{dx}\int_1^{2x} x^6 t^2 dt = \int_1^{2x} 6x^5 t^2 dt + x^6(2x)^2(2) - x^6(1)^2(0)$$

$$= \left[2x^5 t^3\right]_1^{2x} + 8x^8$$

$$= 2x^5(8x^3 - 1) + 8x^8$$

$$= 24x^8 - 2x^5$$

★ $24x^8 - 2x^5$ ★

The problem with the direct approach above is that often the original integral is difficult or even impossible. Leibniz' rule then becomes the only alternative!

It is perhaps best to remember Leibniz in terms of words rather than symbols. It then runs:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)dt =$$

{bring the derivative inside the integral and make it partial}+
{replace variable of integration with the upper limit times the derivative of the upper} −
{replace variable of integration with the lower limit times the derivative of the lower}.

**Example 3** Use Leibniz' rule to find

$$\frac{d}{dx}\int_{\sqrt{x}}^{x} \frac{\sin(tx)}{t}dt$$

$$= \int_{\sqrt{x}}^{x} \cos(tx)\, dt + \frac{\sin(x^2)}{x} - \frac{\sin(x^{3/2})}{\sqrt{x}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$= \frac{\sin(tx)}{x}\Bigg|_{\sqrt{x}}^{x} + \frac{\sin(x^2)}{x} - \frac{\sin(x^{3/2})}{2x}$$

$$= \frac{2\sin(x^2)}{x} - \frac{3\sin(x^{3/2})}{2x}$$

$$\bigstar \quad 2\frac{\sin(x^2)}{x} - \frac{3}{2}\frac{\sin(x^{\frac{3}{2}})}{x} \quad \bigstar$$

It is important to still be able to implement the rule when the variables are different!!

**Example 4**

$$\frac{d}{d\alpha}\int_1^{\alpha^2} \ln(1+\beta^8)\,d\beta.$$

$$= \int_1^{\alpha^2} 0 \, d\beta \quad + (2\alpha) \ln(1+\alpha^{16}) - 0$$

$$= 2\alpha \ln(1+\alpha^{16})$$

$\bigstar \quad 2\alpha\ln(1+\alpha^{16}) \quad \bigstar$

**Example 5** You are given that $\displaystyle\int_0^{\pi} \frac{1}{\alpha-\cos(\theta)}\,d\theta = \frac{\pi}{\sqrt{\alpha^2-1}}$

Using Leibnitz' rule evaluate $\displaystyle\int_0^{\pi} \frac{1}{(\alpha-\cos(\theta))^2}\,d\theta.$

$$\frac{d}{d\alpha}\int_0^{\pi} \frac{1}{\alpha-\cos\theta}\,d\theta = \frac{d}{d\alpha}\left(\frac{\pi}{\sqrt{\alpha^2-1}}\right)$$

$$\int_0^{\pi} \frac{-1}{(\alpha-\cos\theta)^2}\,d\theta = -\frac{\pi}{2}\times 2\alpha\times(\alpha^2-1)^{-\frac{3}{2}}$$

$$\therefore \int_0^{\pi} \frac{1}{(\alpha-\cos\theta)^2}\,d\theta = \frac{\pi\alpha}{(\alpha^2-1)^{3/2}}$$

$\bigstar \quad \dfrac{\pi\alpha}{(\alpha^2-1)^{\frac{3}{2}}} \quad \bigstar$

[4]You can now do Q 30 to 35 and all of Problem Class 1