LECTURE 37
PARTIAL FRACTIONS

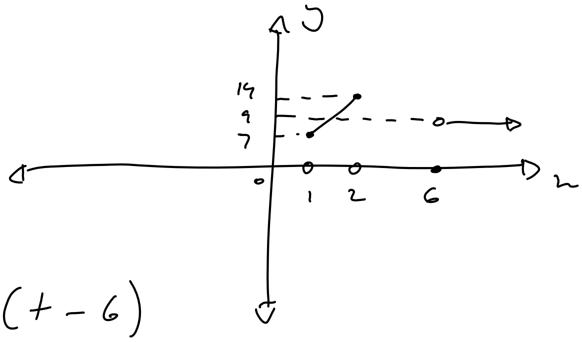
LAPLACE TRANSFORMS

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^m	$m!/s^{m+1}$
t^ν , $(\nu > -1)$	$\Gamma(\nu+1)/s^{\nu+1}$
e^{-at}	$1/(s+a)$
$\sin bt$	$b/(s^2+b^2)$
$\cos bt$	$s/(s^2+b^2)$
$\sinh bt$	$b/(s^2-b^2)$
$\cosh bt$	$s/(s^2-b^2)$
$\sin bt - bt\cos bt$	$2b^3/(s^2+b^2)^2$
$\sin bt + bt\cos bt$	$2bs^2/(s^2+b^2)^2$
$t\ \sin bt$	$2bs/(s^2+b^2)^2$
te^{-at}	$1/(s+a)^2$
$\mathrm{u}(t-c)$	e^{-cs}/s
$e^{-at}f(t)$	$F(s+a)$
$tf(t)$	$-F'(s)$
$f(t-c)\mathrm{u}(t-c)$	$e^{-cs}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f'''(t)$	$s^3F(s)-s^2f(0)-sf'(0)-f''(0)$
$\int_0^t f(\tau)d\tau$	$F(s)/s$

We will now look at how the theory of partial fractions can be used to find the inverse Laplace transform of rational polynomials. This is a skill that you already have from integration theory and the methods transfer across without any change at all. But first a little revision on the Heaviside function and the shifting theorems.

Example 1 Suppose that $f(t) = \begin{cases} 0, & t < 1; \\ 7t, & 1 \leq t \leq 2; \\ 0, & 2 < t \leq 6; \\ 9, & t > 6. \end{cases}$



Sketch the function and find its Laplace transform.

$$f(t) = 7t\big(u(t-1) - u(t-2)\big) + 9u(t-6)$$

$$= 7tu(t-1) - 7tu(t-2) + 9u(t-6)$$

$$= 7(t-1)u(t-1) + 7u(t-1) - 7(t-2)u(t-2)$$

$$- 14u(t-2) + 9u(t-6)$$

$$= \frac{7}{s^2}e^{-s} + \frac{7}{s}e^{-s} - \frac{7}{s^2}e^{-2s} - \frac{14}{s}e^{-2s} + \frac{9}{s}e^{-6s}$$

$$\bigstar \quad \frac{7}{s^2}(e^{-s} - e^{-2s}) + \frac{1}{s}(7e^{-s} - 14e^{-2s} + 9e^{-6s}) \quad \bigstar$$

Example 2 is a lovely application of our earlier work to one of the table entries:

Example 2 Prove that $\mathcal{L}(tf(t)) = -F'(s)$.

We will do this without integration!

$$\mathcal{L}(t) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = \frac{d}{ds}F(s)$$

$$= \int_0^\infty -te^{-st} f(t)\,dt = F'(s)$$

$$\therefore \mathcal{L}(tf(t)) = -F'(s)$$

$\bigstar$

Example 3 Find the inverse Laplace transform of each of the following functions:

i) $F(s) = \dfrac{6s}{s^2 - 11s + 28}$

ii) $F(s) = \dfrac{7s^2 + s + 27}{(s^2 + 4)(s - 1)}$

iii) $F(s) = \dfrac{5s^2 - 36s + 23}{(s - 7)^2(s + 1)}$

All of these are partial fraction questions. Recall that the two crucial features we need for parfrac to work on a rational function is factors on the bottom and for the rational function to be bottom heavy. If the degree of the top is greater than **or equal to** the degree on the bottom we simply do a little long division first.

i) $\dfrac{6s}{s^2 - 11s + 28} = \dfrac{6s}{(s-4)(s-7)} = \dfrac{A}{s-4} + \dfrac{B}{s-7} = \dfrac{A(s-7) + B(s-4)}{(s-4)(s-7)}$. Thus

$$A(s-7) + B(s-4) \equiv 6s$$

$s = 7$: $B = 14$

$s = 4$: $A = -8$

$$\therefore \mathcal{L}^{-1}\left(\frac{-8}{s-4} + \frac{14}{s-7}\right) = -8e^{4t} + 14e^{7t}$$

ii) $\dfrac{7s^2+s+27}{(s^2+4)(s-1)} = \dfrac{As+B}{s^2+4} + \dfrac{C}{s-1} = \dfrac{(As+B)(s-1)+C(s^2+4)}{(s^2+4)(s-1)}$. Thus

$$(As+B)(s-1)+C(s^2+4) \equiv 7s^2+s+27$$

$s=1$: $\quad C = 7$

$s=2i$: $\quad B + 2Ai = 1$

$\therefore B = 1, \quad A = 0$

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^2+4} + \frac{7}{s-1}\right) = \frac{1}{2}\sin 2t + 7e^{t}$$

iii) For repeated factors we need to be very careful with both the decomposition and the recomposition.

$$\frac{5s^2-36s+23}{(s-7)^2(s+1)} = \frac{A}{(s-7)}+\frac{B}{(s-7)^2}+\frac{C}{(s+1)} = \frac{A(s-7)(s+1)+B(s+1)+C(s-7)^2}{(s-7)^2(s+1)}.$$

Thus

$$A(s-7)(s+1)+B(s+1)+C(s-7)^2 \equiv 5s^2-36s+23$$

$s = 7: \quad B = 2$

$s = -1: \quad C = 1$

$s = 0: \quad A = 4$

$$\therefore \mathcal{L}^{-1}\left(\frac{4}{s-7}+\frac{2}{(s-7)^2}+\frac{1}{s+1}\right)$$

$$= 4e^{7t} + 2te^{7t} + e^{-t}$$

$\bigstar$ *i)* $14e^{7t}-8e^{4t}$ *ii)* $\frac{1}{2}\sin(2t)+7e^{t}$ *iii)* $(2t+4)e^{7t}+e^{-t}$ $\bigstar$

[37]You can now do Q 99 e f