LECTURE 30
SPECIAL MATRICES

A matrix A is said to be symmetric if $A = A^T$.

The eigenvectors from different eigenvalues of a symmetric matrix are mutually perpendicular.

A matrix Q is said to be orthogonal if $Q^TQ = I$ or equivalently $Q^{-1} = Q^T$.

The columns of an orthogonal matrix are an orthonormal set.

Let A be a **symmetric** matrix and Q the orthogonal matrix made up of unit eigenvectors of A. Then $Q^TAQ = D$ is an orthogonal diagonalisation of A with the matrix D being the diagonal matrix of corresponding eigenvalues of A.

$(AB)^{-1} = B^{-1}A^{-1}$.

$(AB)^T = B^TA^T$.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse in $\mathbb{R}^2$. $(++)$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a hyperbola in $\mathbb{R}^2$ $(+-)$.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is an ellipsoid in $\mathbb{R}^3$ $(+++)$.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ is a hyperboloid of one sheet in $\mathbb{R}^3$ $(++-)$. (Axis on the negative)

$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ is a hyperboloid of 2 sheets in $\mathbb{R}^3$ $(+--)$. (Axis on the positive)

In $\mathbb{R}^3$ the number of sheets is the number of $(-)$'s.

In this lecture we will first look at some special matrices and their properties. Then we will sketch some fundamental quadratic curves and quadric surfaces.

Definition: A square matrix A is said to be symmetric if $A = A^T$.

For a symmetric matrix the first row is the same as the first column, the second row is the same as the second column etc.

Definition: A square matrix Q is said to be orthogonal if $Q^TQ = I$ or equivalently $Q^{-1} = Q^T$.

An orthogonal matrix has **orthonormal** columns. That is, the columns are perpendicular to each other and have length equal to 1.

Eigenvectors of Symmetric Matrices

The crucial feature enjoyed by symmetric matrices is that their eigenanalysis is perfectly formed. Symmetric matrices are the best matrices money can buy.

Theorem: Let A be a real symmetric matrix. Then

I) All eigenvalues of A are real.

II) There is **always** a full set of eigenvectors.

III) Eigenvectors corresponding to different eigenvalues are orthogonal.

Property III) is especially significant. Lets take a look at the proof.

But first note that we will use the fact that dot products can always be expressed as matrix products.

That is $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$.

For example

$$\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}^T \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = 23. \text{ So}$$

Proof III Let A be a symmetric matrix and let $\mathbf{v}$ and $\mathbf{w}$ be two eigenvectors of A associated with distinct eigenvalues λ and μ respectively. Then

$$\begin{aligned}
\lambda(\mathbf{v} \cdot \mathbf{w}) &= (\lambda \mathbf{v}) \cdot \mathbf{w} \\
&= (A\mathbf{v}) \cdot \mathbf{w} \\
&= (A\mathbf{v})^T \mathbf{w} \\
&= \mathbf{v}^T A^T \mathbf{w} \\
&= \mathbf{v}^T A \mathbf{w} \qquad (\text{Since } A^T = A) \\
&= \mathbf{v}^T \mu \mathbf{w} \\
&= \mu(\mathbf{v}^T \mathbf{w}) \\
&= \mu(\mathbf{v} \cdot \mathbf{w})
\end{aligned}$$

Hence $\lambda(\mathbf{v} \cdot \mathbf{w}) = \mu(\mathbf{v} \cdot \mathbf{w}) \to (\lambda - \mu)(\mathbf{v} \cdot \mathbf{w}) = 0$. But $\lambda \neq \mu$ implies that $\mathbf{v} \cdot \mathbf{w} = 0$ and hence we have $\mathbf{v} \perp \mathbf{w}$.

★

Example 1 If A is an invertible matrix with eigenvector $\mathbf{v}$ and corresponding eigenvalue λ prove that $\mathbf{v}$ is also an eigenvector of A^{-1} and find the corresponding eigenvalue.

$$\lambda \vec{v} = A\vec{v}$$

$$A^{-1}\vec{v} = A^{-1}A\left(\frac{1}{\lambda}\right)\vec{v}$$

$$= \left(\frac{1}{\lambda}\right)\vec{v}$$

$$\bigstar \quad \frac{1}{\lambda} \quad \bigstar$$

Recall that an orthogonal matrix Q has columns which are both perpendicular and of unit length. An efficient way of saying this is that $Q^TQ = I$ since all the dot products of the columns are 0 when the columns are different and 1 when they are the same.

Example 2 Let $A = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{-5}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \end{pmatrix}$. Prove that A is orthogonal an hence evaluate A^{-1}.

$$AA^T = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{-5}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{30}} & -\frac{5}{\sqrt{30}} & \frac{2}{\sqrt{30}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Since $AA^T = I = AA^{-1}$

$$\therefore A^{-1} = A^T$$

$$\bigstar \quad A^{-1} = A^T \quad \bigstar$$

Example 3 Prove that for any square invertible matrices A and B of the same dimension

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(AB)^{-1}(AB) = I$$

$$(AB)^{-1}(AB)B^{-1} = B^{-1}$$

$$(AB)^{-1}AA^{-1} = B^{-1}A^{-1}$$

$$\therefore (AB)^{-1} = B^{-1}A^{-1}$$

★

Example 4 Prove the the determinant of a real orthogonal matrix Q is ± 1

$$Q^T Q = I$$

$$\det(Q^T Q) = \det(I) = 1$$

$$\det(Q^T) \cdot \det(Q) = 1$$

$$\det(Q)^2 = 1$$

$$\therefore \det(Q) = \pm 1$$

★

Example 5 Suppose that A, B and C are square matrices of the same dimension and that A is symmetric and B is orthogonal. Simplify $(ABC)^T(B^{-1}A)^{-1}$.

$$A^T = A \quad , \quad B^T B = I$$

$$(ABC)^T(B^{-1}A)^{-1} = C^T B^T A^T A^{-1} B$$

$$= C^T B^T A A^{-1} B$$

$$= C^T B^T B$$

$$= C^T$$

★ C^T ★

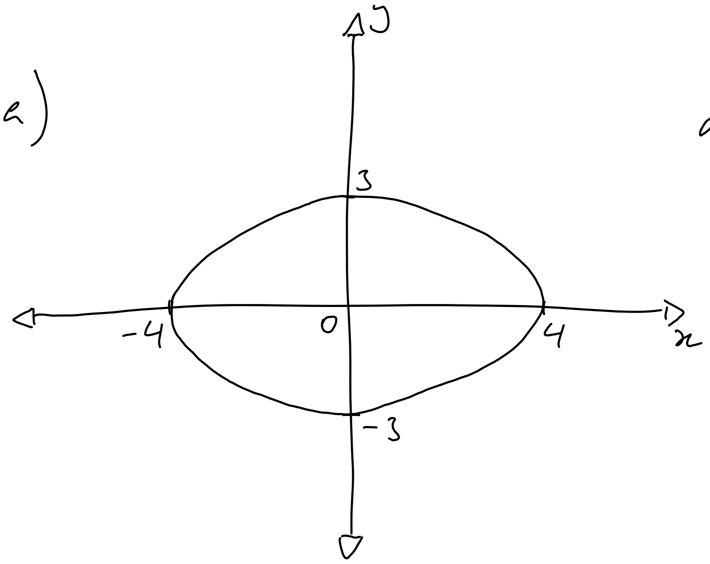
We now turn to the sketching of quadratic objects first in $\mathbb{R}^2$ and then in $\mathbb{R}^3$.

Example 6 Sketch each of the following curves. Find the smallest distance from the curve to the origin in each case and determine the point(s) on the curve where this minimal distance is achieved.

$$a)\ \frac{x^2}{16} + \frac{y^2}{9} = 1.$$

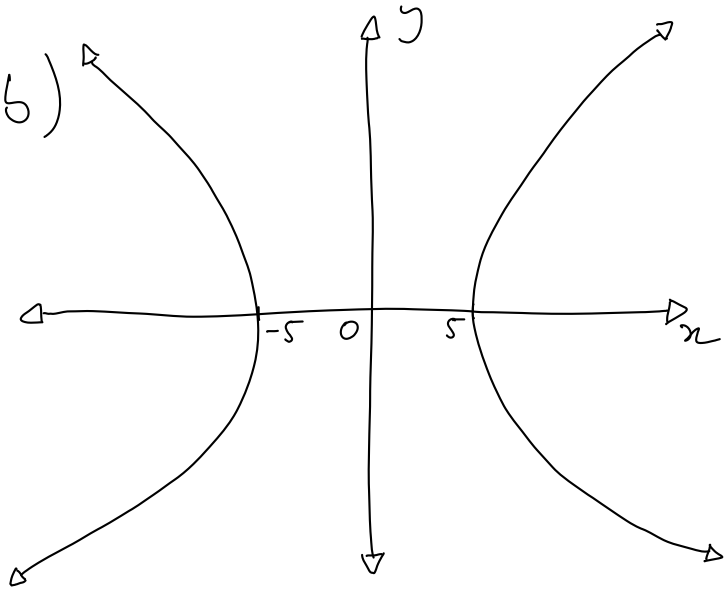
$$b)\ \frac{x^2}{25} - \frac{y^2}{4} = 1.$$

a)



$d_{min} = 3$ at $(0, \pm 3)$

b)



$d_{min} = 5$ at $(\pm 5, 0)$

★ a) Shortest distance of 3 at$(0, \pm 3)$ ★
★ b) Shortest distance of 5 at$(\pm 5, 0)$ ★

Example 7 Sketch each of the following quadric surfaces. Find the smallest distance from the surface to the origin in each case and determine the point(s) on the surface where this minimal distance is achieved..

$$a)\ \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1.$$

$$b)\ -\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1.$$

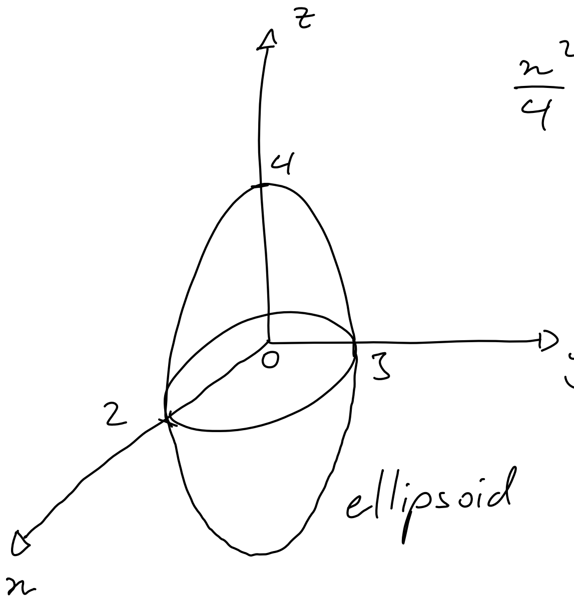
$$c)\ -\frac{x^2}{4} - \frac{y^2}{9} + \frac{z^2}{16} = 1.$$

$$d)\ -\frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{16} = 1.$$

$$e)\ 4x^2 + 9y^2 - z^2 = 36.$$

$$f)\ 5x^2 + 7z^2 = 70.$$

a)

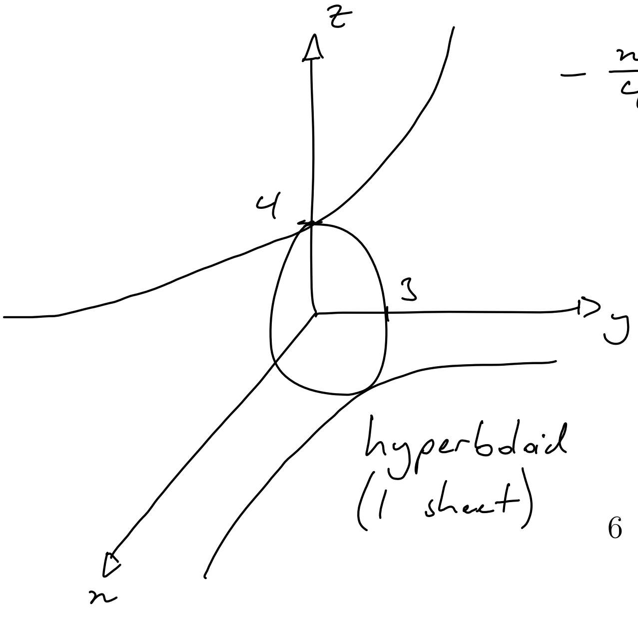


$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

$\therefore d_{min} = 2$

at $(\pm 2, 0, 0)$

b)



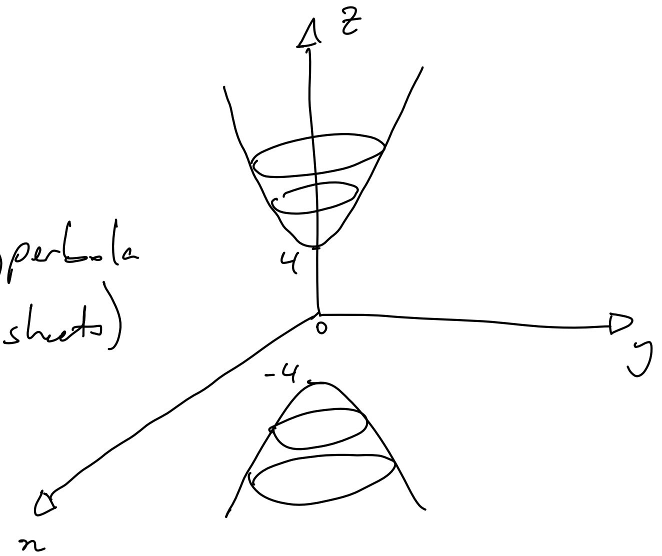
$$-\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

$\therefore d_{min} = 3$

at $(0, \pm 3, 0)$

c)

hyperbola
(2 sheets)

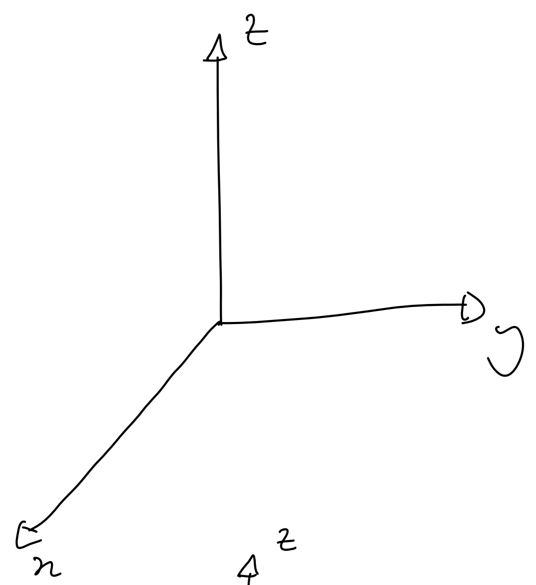


$$-\frac{x^2}{4}-\frac{y^2}{9}+\frac{z^2}{16}=1$$

$\therefore d_{min} = 4$

at $(0,0,\pm 4)$

d)

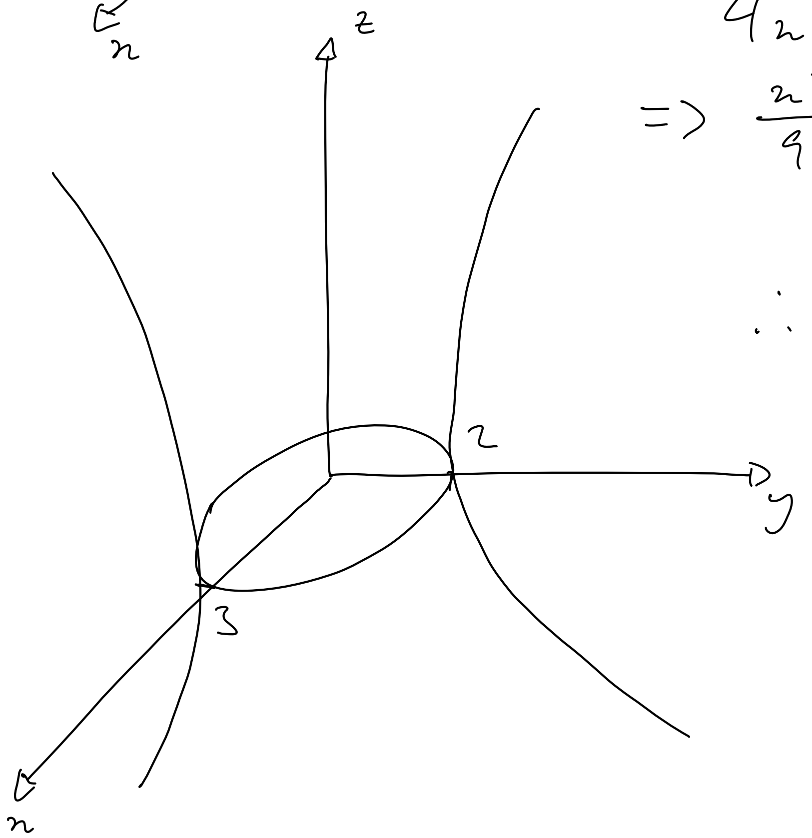


$$-\frac{x^2}{4}-\frac{y^2}{9}-\frac{z^2}{16}=1$$

has no curve

i.e. no real solutions.

e)



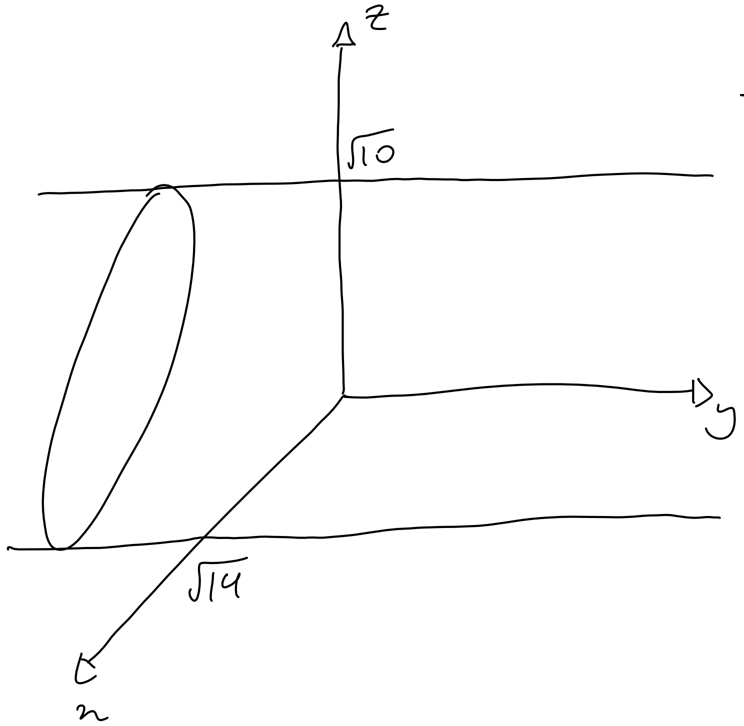
$$4x^2+9y^2-z^2=36$$

$$\Rightarrow \frac{x^2}{9}+\frac{y^2}{4}-\frac{z^2}{36}=1$$

$\therefore d_{min} = 2$

at $(0,\pm 2,0)$

f)



$$5x^2 + 7z^2 = 70$$

$$\Rightarrow \frac{x^2}{14} + \frac{z^2}{10} = 1$$

$$\therefore d_{min} = \sqrt{10}$$

at $(0, 0, \pm\sqrt{10})$

★ a) Shortest distance of 2 at$(\pm 2, 0, 0)$ ★
★ b) Shortest distance of 3 at$(0, \pm 3, 0)$ ★
★ c) Shortest distance of 4 at$(0, 0, \pm 4)$ ★
★ d) Empty graph ★
★ e) Shortest distance of 2 at$(0, \pm 2, 0)$ ★
★ f) Shortest distance of $\sqrt{10}$ at$(0, 0, \pm\sqrt{10})$ ★

In the next lecture we will generalise these surfaces and curves to situations where the axis of the object does not lie on one of the coordinate axes.

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