MATH2019 PROBLEM CLASS

EXAMPLES 6

MATRICES

1991 & 1994

1. a) Find the eigenvalues and the corresponding eigenvectors of matrix

$$\mathsf{A} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{pmatrix}.$$

Solution: The eigenvalues of A are determined by solving $|\mathsf{A} - \lambda \mathsf{I}| = 0$, i.e.,

$$\begin{aligned} |\mathsf{A} - \lambda\mathsf{I}| = \begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 2-\lambda & 0 \\ 2 & 0 & 4-\lambda \end{vmatrix} &= -2\begin{vmatrix} 2 & 2 \\ 0 & 4-\lambda \end{vmatrix} + (2-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} \\ &= 4(\lambda-4) + (2-\lambda)\left[(\lambda-3)(\lambda-4) - 4\right] \\ &= 4(\lambda-4) + (2-\lambda)(\lambda^2 - 7\lambda + 8) \\ &= -\lambda^3 + 9\lambda^2 - 18\lambda \\ &= -\lambda(\lambda^2 - 9\lambda + 18) \\ &= -\lambda(\lambda-3)(\lambda-6) = 0\,. \end{aligned}$$

Hence $\lambda = 0, 3$ or 6.

[Check the sum of eigenvalues = trace(A), i.e., $0+3+6 = 3+2+4$. ✓]

We determine the set of eigenvectors for each λ by solving $(\mathsf{A} - \lambda\mathsf{I})\mathbf{v} = \mathbf{0}$, i.e.,

$$\underline{\lambda = 0:} \quad \left(\begin{array}{ccc|c} 3 & 2 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right) \quad \to \quad \mathbf{v}_{\lambda=0} = t\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}\,,$$

$$\underline{\lambda = 3:} \quad \left(\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ 2 & -1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right) \quad \to \quad \mathbf{v}_{\lambda=3} = t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}\,.$$

$$\underline{\lambda = 6:} \quad \left(\begin{array}{ccc|c} -3 & 2 & 2 & 0 \\ 2 & -4 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right) \quad \to \quad \mathbf{v}_{\lambda=6} = t\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}\,.$$

Note this real symmetric matrix A has distinct eigenvalues and the eigenvectors associated with distinct eigenvectors are orthogonal, i.e.,

$$\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 0.$$

b) Find an orthogonal matrix P such that

$$\mathsf{D} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P}$$

is a diagonal matrix and write down the matrix D.

Solution: The columns of orthogonal matrix P are the normalised eigenvectors associated with each of the eigenvalues of matrix A, i.e., $\mathsf{P} = \dfrac{1}{3}\begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{pmatrix}$ and

$$\mathsf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

c) Using your results from parts a) and b) find the solution of the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 3x + 2y + 2z\,, \\ \frac{dy}{dt} &= 2x + 2y\,, \\ \frac{dz}{dt} &= 2x + 4z\,, \end{aligned}$$

subject to the conditions

$$x(0) = 0\,, \quad y(0) = 0 \quad \text{and} \quad z(0) = 1\,.$$

Solution: The general solution to the set of differential equations is

$$\begin{aligned} \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} &= \alpha\,\widehat{\mathbf{v}}_{\lambda=0}\,e^{0t} + \beta\,\widehat{\mathbf{v}}_{\lambda=3}\,e^{3t} + \gamma\,\widehat{\mathbf{v}}_{\lambda=6}\,e^{6t} \\ &= P\begin{pmatrix} \alpha \\ \beta e^{3t} \\ \gamma e^{6t} \end{pmatrix} \end{aligned}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$.
To determine the constants α, β and γ we use the initial condition, i.e.,

$$\mathbf{x}(0) = \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathsf{P}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \mathsf{P}^T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}.$$

Hence the solution is given by

$$\begin{aligned} \mathbf{x}(t) &= -\frac{1}{3}\,\widehat{\mathbf{v}}_{\lambda=0} - \frac{2}{3}\,\widehat{\mathbf{v}}_{\lambda=3}\,e^{3t} + \frac{2}{3}\,\widehat{\mathbf{v}}_{\lambda=6}\,e^{6t} \\ &= -\frac{1}{9}\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} - \frac{2}{9}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} e^{3t} + \frac{2}{9}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{6t}\,. \end{aligned}$$

d) Express the quadric surface

$$3x^2 + 2y^2 + 4z^2 + 4xy + 4xz = 24$$

in terms of its principal axes X, Y and Z and write out an orthogonal matrix P such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathsf{P}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

What shape does this quadric surface represent?
Solution: In terms of the principal axes, the quadric surface is given by $0X^2 + 3Y^2 + 6Z^2 = 24$ (elliptic cylinder) with $\mathsf{P} = \frac{1}{3}\begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{pmatrix}$ (as determined in part b).

e) A and P are $n \times n$ matrices. A is symmetric and P is orthogonal. Prove that $\mathsf{P}^{-1}\mathsf{AP}$ is symmetric.

Solution: Note $\mathsf{A}^T = \mathsf{A}$ since A is symmetric and $\mathsf{P}^{-1} = \mathsf{P}^T$ since P is orthogonal. Wish to prove $\left(\mathsf{P}^{-1}\mathsf{AP}\right)^T = \mathsf{P}^{-1}\mathsf{AP}$. Consider the left hand side of this equation, i.e.,

$$\begin{aligned}
\left(\mathsf{P}^{-1}\mathsf{AP}\right)^T &= \mathsf{P}^T\mathsf{A}^T\left(\mathsf{P}^{-1}\right)^T \quad \text{since } (\mathsf{CB})^T = \mathsf{B}^T\mathsf{C}^T \\
&= \mathsf{P}^{-1}\mathsf{A}\left(\mathsf{P}^T\right)^T \quad \text{since } \mathsf{P}^{-1} = \mathsf{P}^T \text{ and } \mathsf{A}^T = \mathsf{A} \\
&= \mathsf{P}^{-1}\mathsf{AP} \quad \text{since } (\mathsf{B}^T)^T = \mathsf{B}\,.
\end{aligned}$$

$\boxed{1998}$ 2. Let

$$\mathsf{A} = \begin{pmatrix} -7 & 24 \\ 24 & 7 \end{pmatrix}$$

a) Find the eigenvalues and eigenvectors of A.

Solution: The eigenvalues of A are determined by solving $|\mathsf{A} - \lambda\mathsf{I}| = 0$, i.e.,

$$\begin{aligned}
|\mathsf{A} - \lambda\mathsf{I}| = \begin{vmatrix} -7-\lambda & 24 \\ 24 & 7-\lambda \end{vmatrix} &= \lambda^2 - 7^2 - 24^2 = 0 \\
&\Rightarrow \lambda = \pm 25\,.
\end{aligned}$$

Thus $\lambda = 25$ or -25.

[Check the sum of eigenvalues = trace(A), i.e., $25 + (-25) = (-7) + 7$. ✓]

Next we determine the set of eigenvectors for each λ by solving $(\mathsf{A} - \lambda\mathsf{I})\mathbf{v} = \mathbf{0}$, i.e.,

$$\begin{array}{llll}
\underline{\lambda = 25:} & \left(\begin{array}{cc|c} -32 & 24 & 0 \\ 24 & -18 & 0 \end{array}\right) & \rightarrow & \mathbf{v}_{\lambda=25} = t\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}\,, \\
\underline{\lambda = -25:} & \left(\begin{array}{cc|c} 18 & 24 & 0 \\ 24 & 32 & 0 \end{array}\right) & \rightarrow & \mathbf{v}_{\lambda=-25} = t\begin{pmatrix} 4 \\ -3 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}\,.
\end{array}$$

Note this real symmetric matrix A has distinct eigenvalues and the eigenvectors associated with distinct eigenvectors are orthogonal, i.e., $\begin{pmatrix} 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -3 \end{pmatrix} = 0$.

b) Normalise the eigenvectors to have length 1. Hence find an orthogonal matrix P such that

$$\mathsf{D} = \mathsf{P}^{-1}\mathsf{AP}$$

is a diagonal matrix. Evaluate both sides of this equation to show that it is satisfied by your P.

Solution: If we normalise the eigenvectors the orthogonal matrix P is given by $\mathsf{P} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$. Hence

$$\begin{aligned}
\mathsf{P}^{-1}\mathsf{AP} &= \mathsf{P}^T\mathsf{AP} \quad \text{since } \mathsf{P} \text{ is orthogonal, i.e., } \mathsf{P}^{-1} = \mathsf{P}^T \\
&= \mathsf{PAP} \quad \text{since } \mathsf{P}^T = \mathsf{P} \text{ in this case} \\
&= \frac{1}{25}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} -7 & 24 \\ 24 & 7 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \\
&= \frac{1}{25}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 75 & -100 \\ 100 & 75 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 25 & 0 \\ 0 & -25 \end{pmatrix} \\
&= \mathsf{D}\,.
\end{aligned}$$

c) For the system of differential equations

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x} \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

show (or verify) that the transformation

$$\mathbf{x} = \mathsf{P}\mathbf{z} \quad \text{where} \quad \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

yields the equation

$$\frac{d\mathbf{z}}{dt} = \mathsf{D}\mathbf{z}$$

where P and D are as in part b).

Solution: Substitute $\mathbf{x} = \mathsf{P}\mathbf{z}$ into $\dfrac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x}$ yields

$$\begin{aligned} \frac{d}{dt}\mathsf{P}\mathbf{z} = \mathsf{A}\mathsf{P}\mathbf{z} \quad &\Rightarrow \quad \mathsf{P}\frac{d\mathbf{z}}{dt} = \mathsf{A}\mathsf{P}\mathbf{z} \\ &\Rightarrow \quad \frac{d\mathbf{z}}{dt} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P}\mathbf{z} = \mathsf{D}\mathbf{z}\,. \end{aligned}$$

d) Hence solve the system of equations

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x}$$

if $x_1(0) = 1$, $x_2(0) = 0$.

Solution: First solve the system of differential equations $\dfrac{d\mathbf{z}}{dt} = \mathsf{D}\mathbf{z}$, i.e.,

$$\begin{aligned} \frac{d\mathbf{z}}{dt} = \begin{pmatrix} 25 & 0 \\ 0 & -25 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad &\Rightarrow \quad \begin{cases} \dfrac{dz_1}{dt} &= \quad 25z_1 \\[2ex] \dfrac{dz_2}{dt} &= \quad -25z_2 \end{cases} \\ &\Rightarrow \quad \begin{cases} z_1 &= \quad \alpha e^{25t} \\ z_2 &= \quad \beta e^{-25t} \end{cases} \\ &\Rightarrow \quad \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha e^{25t} \\ \beta e^{-25t} \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{R}\,. \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{x}(t) = \mathsf{P}\mathbf{z}(t) \quad &= \quad \mathsf{P}\begin{pmatrix} \alpha e^{25t} \\ \beta e^{-25t} \end{pmatrix} \\ &= \quad \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} \alpha e^{25t} \\ \beta e^{-25t} \end{pmatrix} \\ &= \quad \frac{\alpha}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} e^{25t} + \frac{\beta}{5}\begin{pmatrix} 4 \\ -3 \end{pmatrix} e^{-25t} \\ &= \quad \alpha\,\widehat{\mathbf{v}}_{\lambda=25}\, e^{25t} + \beta\,\widehat{\mathbf{v}}_{\lambda=-25}\, e^{-25t}\,. \end{aligned}$$

To determine the constants α and β we use the initial condition, i.e.,

$$\mathbf{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathsf{P}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \mathsf{P}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Hence the solution is given by

$$\begin{aligned} \mathbf{x}(t) &= \frac{3}{5}\,\widehat{\mathbf{v}}_{\lambda=25}\,e^{25t} + \frac{4}{5}\,\widehat{\mathbf{v}}_{\lambda=-25}\,e^{-25t} \\ &= \frac{3}{25}\begin{pmatrix} 3 \\ 4 \end{pmatrix} e^{25t} + \frac{4}{25}\begin{pmatrix} 4 \\ -3 \end{pmatrix} e^{-25t}. \end{aligned}$$

$\boxed{1999}$ 3. Let $\mathsf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors of A.

Solution: The eigenvalues of A are determined by solving $|\mathsf{A} - \lambda\mathsf{I}| = 0$, i.e.,

$$\begin{aligned} |\mathsf{A} - \lambda\mathsf{I}| = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} &= (\lambda - 1)(\lambda + 2) - 4 \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda + 3)(\lambda - 2) = 0\,. \end{aligned}$$

Thus $\lambda = 2$ or -3.

[Check the sum of eigenvalues = trace(A), i.e., $2 + (-3) = 1 + (-2)$. ✓]

Next we determine the set of eigenvectors for each λ by solving $(\mathsf{A} - \lambda\mathsf{I})\mathbf{v} = \mathbf{0}$, i.e.,

$$\underline{\lambda = 2:} \quad \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 4 & -4 & 0 \end{array}\right) \quad \to \quad \mathbf{v}_{\lambda=2} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\},$$

$$\underline{\lambda = -3:} \quad \left(\begin{array}{cc|c} 4 & 1 & 0 \\ 4 & 1 & 0 \end{array}\right) \quad \to \quad \mathbf{v}_{\lambda=-3} = t\begin{pmatrix} 1 \\ -4 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}.$$

Note this real matrix A is not symmetric and hence we are NOT expecting the eigenvectors associated with distinct eigenvectors are orthogonal. Indeed $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \end{pmatrix} = -3 \neq 0$.

b) i) Find a matrix P such that $\mathsf{P}^{-1}\mathsf{A}\mathsf{P} = \mathsf{D}$ where D is a diagonal matrix.

Solution: The columns of matrix P are the eigenvectors associated with each of the eigenvalues of matrix A, i.e., $\mathsf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}$.

ii) Calculate $\mathsf{P}^{-1}\mathsf{A}\mathsf{P}$ to check this is indeed equal to D.

Solution: Using the formula for the inverse matrix for a 2×2 matrix we have

$$\begin{aligned} \mathsf{P}^{-1}\mathsf{A}\mathsf{P} &= \frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 2 & 12 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 10 & 0 \\ 0 & -15 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \\ &= \mathsf{D}\,. \end{aligned}$$

c) If $\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $\mathbf{f} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$ show that with the definition $\mathbf{x} = \mathsf{P}\mathbf{z}$, the system of differential equations

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x} + \mathbf{f} \tag{1}$$

becomes

$$\frac{d\mathbf{z}}{dt} = \mathsf{D}\mathbf{z} + \mathsf{P}^{-1}\mathbf{f}\,.$$

***Solution**:* Substitute $\mathbf{x} = \mathsf{P}\mathbf{z}$ into $\dfrac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x} + \mathbf{f}$ yields

$$\begin{aligned} \frac{d}{dt}\mathsf{P}\mathbf{z} = \mathsf{A}\mathsf{P}\mathbf{z} + \mathbf{f} \quad &\Rightarrow \quad \mathsf{P}\frac{d\mathbf{z}}{dt} = \mathsf{A}\mathsf{P}\mathbf{z} + \mathbf{f} \\ &\Rightarrow \quad \frac{d\mathbf{z}}{dt} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P}\mathbf{z} + \mathsf{P}^{-1}\mathbf{f} = \mathsf{D}\mathbf{z} + \mathsf{P}^{-1}\mathbf{f}\,. \end{aligned}$$

d) Using the result of c) find the general solution of (1) in the case when $f_1(t) = e^{2t}$ and $f_2(t) = 0$.

***Solution**:* First solve the system of differential equations $\dfrac{d\mathbf{z}}{dt} = \mathsf{D}\mathbf{z} + \mathsf{P}^{-1}\mathbf{f}$, i.e.,

$$\begin{aligned} \frac{d\mathbf{z}}{dt} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} \quad &\Rightarrow \quad \begin{cases} \dfrac{dz_1}{dt} = 2z_1 + \dfrac{4}{5}e^{2t} \\ \dfrac{dz_2}{dt} = -3z_2 + \dfrac{1}{5}e^{2t} \end{cases} \\ &\Rightarrow \quad \begin{cases} z_1 = \alpha e^{2t} + \dfrac{4}{5}te^{2t} \\ z_2 = \beta e^{-3t} + \dfrac{1}{25}e^{2t} \end{cases} \\ &\Rightarrow \quad \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha e^{2t} + \dfrac{4}{5}te^{2t} \\ \beta e^{-3t} + \dfrac{1}{25}e^{2t} \end{pmatrix}, \end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$. Hence

$$\begin{aligned} \mathbf{x}(t) = \mathsf{P}\mathbf{z}(t) &= \mathsf{P}\begin{pmatrix} \alpha e^{2t} + \dfrac{4}{5}te^{2t} \\ \beta e^{-3t} + \dfrac{1}{25}e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} \alpha e^{2t} + \dfrac{4}{5}te^{2t} \\ \beta e^{-3t} + \dfrac{1}{25}e^{2t} \end{pmatrix} \\ &= \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t} + \beta\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{-3t} + \begin{pmatrix} \dfrac{4}{5}te^{2t} + \dfrac{1}{25}e^{2t} \\ \dfrac{4}{5}te^{2t} - \dfrac{4}{25}e^{2t} \end{pmatrix}. \end{aligned}$$

$\boxed{2000}$ 4. Consider the quadric surface given by

$$x^2 + y^2 + 3z^2 + 4xz + 4yz = 5\,.$$

a) Express this equation in the form

$$\mathbf{v}^T \mathsf{A} \mathbf{v} = 5$$

where A is a real symmetric matrix and $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Solution: The quadric surface in matrix form is given by
$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5$ and therefore $\mathsf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$.

b) Show that the matrix A has an eigenvalue $\lambda = 1$ and two other distinct eigenvalues. What are the values of these other eigenvalues?

Solution: The eigenvalues of A are determined by solving $|\mathsf{A} - \lambda \mathsf{I}| = 0$, i.e.,

$$\begin{aligned} |\mathsf{A} - \lambda \mathsf{I}| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 1-\lambda & 2 \\ 2 & 2 & 3-\lambda \end{vmatrix} &= (1-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} + 2 \begin{vmatrix} 0 & 2 \\ 1-\lambda & 2 \end{vmatrix} \\ &= (1-\lambda)\left[(\lambda-3)(\lambda-1) - 4\right] - 4(1-\lambda) \\ &= (1-\lambda)\left(\lambda^2 - 4\lambda - 5\right) \\ &= -(\lambda-1)(\lambda-5)(\lambda+1) = 0\,. \end{aligned}$$

Thus $\lambda = 1, 5$ or -1.
[Check the sum of eigenvalues = trace(A), i.e., $1 + 5 + (-1) = 1 + 1 + 3$. ✓]

c) Write down the equation of the quadric surface in terms of its principal axes X, Y and Z. Then sketch the surface relative to principal axes, clearly labelling the (X, Y, Z) coordinates of the points where the surface intersects the principal axes.

Solution: The quadric surface, relative to the principal axes, is given by

$$X^2 + 5Y^2 - Z^2 = 5\,.$$

This surface, a hyperboloid of 1-sheet, doesn't intersect the Z-axis but intercepts the X-axis at $(\pm\sqrt{5}, 0, 0)$ and the Y-axis at $(0, \pm 1, 0)$.

d) Find the eigenvectors of A and hence find an orthogonal matrix, P, which relates $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$. Write down this relationship.

Solution: We determine the set of eigenvectors for each λ by solving $(\mathsf{A} - \lambda \mathsf{I})\mathbf{v} = \mathbf{0}$, i.e.,

$$\underline{\lambda = 1:} \qquad \left(\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right) \quad \rightarrow \quad \mathbf{v}_{\lambda=1} = t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}\,,$$

$$\underline{\lambda = 5:} \qquad \left(\begin{array}{ccc|c} -4 & 0 & 2 & 0 \\ 0 & -4 & 2 & 0 \\ 2 & 2 & -2 & 0 \end{array}\right) \quad \rightarrow \quad \mathbf{v}_{\lambda=5} = t \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}\,.$$

$$\underline{\lambda = -1:} \qquad \left(\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 2 & 4 & 0 \end{array}\right) \quad \rightarrow \quad \mathbf{v}_{\lambda=-1} = t \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}\,.$$

Note this real symmetric matrix A has distinct eigenvalues and the eigenvectors associated with distinct eigenvectors are orthogonal, i.e.,

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0.$$

The columns of orthogonal matrix P are the normalised eigenvectors associated with each of the eigenvalues of matrix A, i.e., $\mathsf{P} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & 1 & \sqrt{2} \\ -\sqrt{3} & 1 & \sqrt{2} \\ 0 & 2 & -\sqrt{2} \end{pmatrix}$. Note

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathsf{P}\mathbf{X} = \mathsf{P} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

e) Write down the points of intersection of the quadric surface with its principal axes in terms of the (x, y, z) coordinate system.

Solution:

intersection X-axis: $\quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathsf{P} \begin{pmatrix} \pm\sqrt{5} \\ 0 \\ 0 \end{pmatrix} = \pm\sqrt{5}\widehat{\mathbf{v}}_{\lambda=1} = \pm\sqrt{\frac{5}{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$

intersection Y-axis: $\quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathsf{P} \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix} = \pm\widehat{\mathbf{v}}_{\lambda=5} = \pm\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$

$\boxed{\text{2014, S1}}$ 5. It is given that the matrix $\mathsf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{pmatrix}$ has an eigenvalue $\lambda_1 = 1$

with an associated eigenvector $\mathbf{v}_{\lambda=1} = \begin{pmatrix} 15 \\ 8 \\ -16 \end{pmatrix}$ and eigenvalue $\lambda_2 = 6$

with associated eigenvector $\mathbf{v}_{\lambda=6} = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$.

a) Without calculating the characteristic polynomial explain why the remaining eigenvalue is $\lambda_3 = 7$.

Solution: A general result is the sum of the eigenvalues of a matrix is equal to the sum of the diagonal elements of the matrix, called the trace of the matrix. In this case the trace of A is $1 + 4 + 9 = 14$. The sum of the eigenvalues is then $\lambda_1 + \lambda_2 + \lambda_3 = 1 + 6 + \lambda_3 = 14$. Hence $\lambda_3 = 7$.

b) Find an eigenvector $\mathbf{v}_{\lambda=7}$ for the eigenvalue $\lambda_3 = 7$.

Solution: We determine the set of eigenvectors for $\lambda_3 = 7$ by solving $(\mathsf{A} - 7I)\mathbf{v} = \mathbf{0}$, i.e.,

$$\underline{\lambda = 7:} \qquad \left(\begin{array}{ccc|c} -6 & 0 & 0 & 0 \\ -8 & -3 & -6 & 0 \\ 8 & 1 & 2 & 0 \end{array}\right) \quad \rightarrow \quad \mathbf{v}_{\lambda=7} = t \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}.$$

Check $\mathsf{A}\mathbf{v}_{\lambda=7} = 7\,\mathbf{v}_{\lambda=7}$.

c) Hence write down the general solution to the system of differential equations

$$\begin{aligned} y_1' &= y_1 \\ y_2' &= -8y_1 + 4y_2 - 6y_3 \\ y_3' &= 8y_1 + y_2 + 9y_3 \end{aligned}$$

Solution: The general solution to the system of differential equations is given by

$$\begin{aligned} \mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} &= \alpha\, \mathbf{v}_{\lambda=1}\, e^{t} + \beta\, \mathbf{v}_{\lambda=6}\, e^{6t} + \gamma\, \mathbf{v}_{\lambda=7}\, e^{7t} \\ &= \alpha \begin{pmatrix} 15 \\ 8 \\ -16 \end{pmatrix} e^{t} + \beta \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} e^{6t} + \gamma \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} e^{7t}, \quad \alpha, \beta, \gamma \in \mathbb{R}\,. \end{aligned}$$

2014, S2

6. Let

$$\mathsf{A} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

a) Find the eigenvalues and eigenvectors of the matrix A.

Solution: The eigenvalues of A are determined by solving $|\mathsf{A} - \lambda \mathsf{I}| = 0$, i.e.,

$$\begin{aligned} |\mathsf{A} - \lambda \mathsf{I}| = \begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} &= (\lambda - 2)^2 - 9 = 0 \\ &\Rightarrow \lambda - 2 = \pm 3\,. \end{aligned}$$

Thus $\lambda = 5$ or -1.

[Check the sum of eigenvalues = trace(A), i.e., $5 + (-1) = 2 + 2$. ✓]

Next we determine the set of eigenvectors for each λ by solving $(\mathsf{A} - \lambda \mathsf{I})\mathbf{v} = \mathbf{0}$, i.e.,

$$\underline{\lambda = 5:} \quad \left(\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right) \quad \to \quad \mathbf{v}_{\lambda=5} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}\,,$$

$$\underline{\lambda = -1:} \quad \left(\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right) \quad \to \quad \mathbf{v}_{\lambda=-1} = t \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}\,.$$

Note this real symmetric matrix A has distinct eigenvalues and the eigenvectors associated with distinct eigenvectors are orthogonal, i.e., $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$.

b) By considering the eigenvalues of A, write the curve

$$2x^2 + 6xy + 2y^2 = 45$$

in terms of principle axes coordinates X and Y. Sketch the curve in the XY-plane.

Solution: The quadric curve, relative to the principal axes, is given by

$$5X^2 - Y^2 = 45\,.$$

This curve, a hyperbola, doesn't intersect the Y-axis but intercepts the X-axis at $X = \pm 3$.

c) Find the distance from the curve $2x^2 + 6xy + 2y^2 = 45$ to the origin.

Solution: Hence the points $\pm 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (on the curve) are closest to the origin, relative to the principal axes. Thus the distance from the origin to the curve is 3.

$\boxed{\text{2015, S1}}$ 7. The equations governing the response of a bridge to an earthquake are found to satisfy

$$\begin{aligned} \frac{dx}{dt} &= -x + ay, \\ \frac{dy}{dt} &= ax - y. \end{aligned}$$

where $a > 0$ is a parameter that depends on the material used for the bridge.

a) Express this set of differential equations in the form

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x}, \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

and find the eigenvalues and eigenvectors of the matrix A.

Solution: The set of differential equations in matrix form is given by $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & a \\ a & -1 \end{pmatrix} \mathbf{x}$ and therefore $\mathsf{A} = \begin{pmatrix} -1 & a \\ a & -1 \end{pmatrix}$.

The eigenvalues of A are determined by solving $|\mathsf{A} - \lambda\mathsf{I}| = 0$, i.e.,

$$\begin{aligned} |\mathsf{A} - \lambda\mathsf{I}| = \begin{vmatrix} -1-\lambda & a \\ a & -1-\lambda \end{vmatrix} &= (\lambda+1)^2 - a^2 = 0 \\ &\Rightarrow \lambda + 1 = \pm a\,. \end{aligned}$$

Thus $\lambda = -1 + a$ or $-1 - a$.

[Check the sum of eigenvalues = trace(A), i.e., $(-1+a) + (-1-a)) = (-1) + (-1)$. ✓]

Next we determine the set of eigenvectors for each λ by solving $(\mathsf{A} - \lambda\mathsf{I})\mathbf{v} = \mathbf{0}$, i.e.,

$$\underline{\lambda = -1 + a:} \quad \left(\begin{array}{cc|c} -a & a & 0 \\ a & -a & 0 \end{array}\right) \quad \to \quad \mathbf{v}_{\lambda=-1+a} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\},$$

$$\underline{\lambda = -1 - a:} \quad \left(\begin{array}{cc|c} a & a & 0 \\ a & a & 0 \end{array}\right) \quad \to \quad \mathbf{v}_{\lambda=-1-a} = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}.$$

Note this real symmetric matrix A has distinct eigenvalues and the eigenvectors associated with distinct eigenvectors are orthogonal, i.e., $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$.

b) Hence, or otherwise, write down a general solution for the problem.

Solution: The general solution to the system of differential equations is given by

$$\begin{aligned} \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \alpha\,\mathbf{v}_{\lambda=-1+a}\,e^{-(1-a)t} + \beta\,\mathbf{v}_{\lambda=-1-a}\,e^{-(1+a)t} \\ &= \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-(1-a)t} + \beta\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-(1+a)t}, \quad \alpha, \beta \in \mathbb{R}\,. \end{aligned}$$

c) For what values of a will the solution grow with increasing t?

Solution: Consider the exponent of each of the exponentials, i.e., the eigenvalues. Since $a > 0$ the eigenvalue $\lambda = -1 - a < 0$ for all $a > 0$. But the eigenvalue $\lambda = -1 + a > 0$ if $a > 1$. Thus the solution $\mathbf{x}(t)$ will grow if $a > 1$.

2015, S2 8. The matrix B is given by

$$\mathsf{B} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

a) Show that the vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

is an eigenvector of the matrix B and find the corresponding eigenvalue.

Solution: If $\mathbf{v}$ is an eigenvector of matrix B then $\mathsf{B}\mathbf{v} = \lambda\mathbf{v}$ will have a solution. Hence

$$\mathsf{B}\mathbf{v} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Thus $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is an eigenvector associated with eigenvalue $\lambda = 1$.

b) Given that the other two eigenvalues of B are -1 and 2, find the eigenvectors corresponding to these two eigenvalues.

Solution: We determine the set of eigenvectors for each λ by solving $(\mathsf{B} - \lambda\mathsf{I})\mathbf{v} = \mathbf{0}$, i.e.,

$$\underline{\lambda = -1:} \quad \left(\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right) \quad \rightarrow \quad \mathbf{v}_{\lambda=-1} = t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\},$$

$$\underline{\lambda = 2:} \quad \left(\begin{array}{rrr|r} -2 & -1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \rightarrow \quad \mathbf{v}_{\lambda=2} = t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}.$$

Note this real symmetric matrix B has distinct eigenvalues and the eigenvectors associated with distinct eigenvectors are orthogonal, i.e.,

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0.$$

2016, S1 9. Consider the curve in the xy-plane

$$x^2 - 6xy + y^2 = 16.$$

a) Rewrite the equation for the curve in the form

$$\begin{pmatrix} x \\ y \end{pmatrix}^T \mathsf{A} \begin{pmatrix} x \\ y \end{pmatrix} = 16$$

where A is a real symmetric 2×2 matrix. Find the eigenvalues and eigenvectors of A.

Solution: The curve in matrix form is given by $\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 16$ and therefore $\mathsf{A} = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$.

The eigenvalues of A are determined by solving $|\mathsf{A} - \lambda\mathsf{I}| = 0$, i.e.,

$$|\mathsf{A} - \lambda\mathsf{I}| = \begin{vmatrix} 1-\lambda & -3 \\ -3 & 1-\lambda \end{vmatrix} \begin{aligned} &= (\lambda-1)^2 - 9 = 0 \\ &\Rightarrow \lambda - 1 = \pm 3\,. \end{aligned}$$

Thus $\lambda = 4$ or -2.
[Check the sum of eigenvalues = trace(A), i.e., $4 + (-2) = 1 + 1$. ✓]
Next we determine the set of eigenvectors for each λ by solving $(\mathsf{A} - \lambda\mathsf{I})\mathbf{v} = \mathbf{0}$, i.e.,

$$\underline{\lambda = 4:} \quad \left(\begin{array}{cc|c} -3 & -3 & 0 \\ -3 & -3 & 0 \end{array}\right) \quad \rightarrow \quad \mathbf{v}_{\lambda=4} = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}\,,$$

$$\underline{\lambda = -2:} \quad \left(\begin{array}{cc|c} 3 & -3 & 0 \\ -3 & 3 & 0 \end{array}\right) \quad \rightarrow \quad \mathbf{v}_{\lambda=-2} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}\,.$$

Note this real symmetric matrix A has distinct eigenvalues and the eigenvectors associated with distinct eigenvectors are orthogonal, i.e., $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$.

b) Write down the equation for the curve in terms of its principle axes X and Y. Hence find the closest distance from the origin to the curve.
Solution: The quadric curve, relative to the principal axes, is given by

$$4X^2 - 2Y^2 = 16\,.$$

This curve, a hyperbola, doesn't intersect the Y-axis but intercepts the X-axis at $X = \pm 2$. Hence the points $\pm 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (on the curve) are closest to the origin, relative to the principal axes. Thus the distance from the origin to the curve is 2.

c) Find the x and y coordinates of the points on the curve closest to the origin.
Solution: The points on the curve closest to the origin, relative the original axes, are given by the position vector

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \pm 2\widehat{\mathbf{v}}_{\lambda=4} = \pm 2\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \pm\begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix}.$$

2016, S2

10. Consider the matrix $\mathsf{A} = \begin{pmatrix} 6 & 2 \\ -1 & 3 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors of A.
Solution: The eigenvalues of A are determined by solving $|\mathsf{A} - \lambda\mathsf{I}| = 0$, i.e.,

$$|\mathsf{A} - \lambda\mathsf{I}| = \begin{vmatrix} 6-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} \begin{aligned} &= (\lambda-6)(\lambda-3) + 2 \\ &= \lambda^2 - 9\lambda + 20 \\ &= (\lambda-4)(\lambda-5) = 0\,. \end{aligned}$$

Thus $\lambda = 4$ or 5.
[Check the sum of eigenvalues = trace(A), i.e., $4 + 5 = 6 + 3$. ✓]
Next we determine the set of eigenvectors for each λ by solving $(\mathsf{A} - \lambda\mathsf{I})\mathbf{v} = \mathbf{0}$, i.e.,

$$\underline{\lambda = 4:} \quad \left(\begin{array}{cc|c} 2 & 2 & 0 \\ -1 & -1 & 0 \end{array}\right) \quad \rightarrow \quad \mathbf{v}_{\lambda=4} = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}\,,$$

$$\underline{\lambda = 5:} \quad \left(\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & -2 & 0 \end{array}\right) \quad \rightarrow \quad \mathbf{v}_{\lambda=5} = t\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}\,.$$

Note this real matrix A is not symmetric and hence we are NOT expecting the eigenvectors associated with distinct eigenvectors are orthogonal. Indeed $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 3 \neq 0$.

b) Hence solve the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 6x + 2y \\ \frac{dy}{dt} &= -x + 3y. \end{aligned}$$

Solution: The general solution to the system of differential equations is given by

$$\begin{aligned} \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \alpha\, \mathbf{v}_{\lambda=4}\, e^{4t} + \beta\, \mathbf{v}_{\lambda=5}\, e^{5t} \\ &= \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + \beta \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{5t}, \quad \alpha, \beta \in \mathbb{R}\,. \end{aligned}$$

$\boxed{\text{2017, S1}}$ 11. Consider the set of differential equations

$$\begin{aligned} \frac{dx}{dt} &= -x + y, \\ \frac{dy}{dt} &= x - y, \end{aligned}$$

with initial conditions $x(0) = 1$, $y(0) = 0$.

a) Express this set of differential equations in the form

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x}, \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

and find the eigenvalues and eigenvectors of the matrix A.

Solution: The set of differential equations in matrix form is given by $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{x}$ and therefore $\mathsf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$.

The eigenvalues of A are determined by solving $|\mathsf{A} - \lambda \mathsf{I}| = 0$, i.e.,

$$\begin{aligned} |\mathsf{A} - \lambda \mathsf{I}| = \begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} &= (\lambda+1)^2 - 1 = 0 \\ &\Rightarrow \lambda + 1 = \pm 1\,. \end{aligned}$$

Thus $\lambda = 0$ or -2.

[Check the sum of eigenvalues = trace(A), i.e., $0 + 2 = (-1) + (-1)$. ✓]

Next we determine the set of eigenvectors for each λ by solving $(\mathsf{A} - \lambda \mathsf{I})\mathbf{v} = \mathbf{0}$, i.e.,

$$\underline{\lambda = 0:} \quad \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \quad \to \quad \mathbf{v}_{\lambda=0} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}\,,$$

$$\underline{\lambda = -2:} \quad \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \quad \to \quad \mathbf{v}_{\lambda=-2} = t \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}\backslash\{0\}\,.$$

Note this real symmetric matrix A has distinct eigenvalues and the eigenvectors associated with distinct eigenvectors are orthogonal, i.e., $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$.

b) Hence, or otherwise, write down the solution for the problem using the initial conditions.

Solution: Since the eigenvectors associated with distinct eigenvalues are orthogonal in this case will take advantage of that when constructing the general solution to the system of differential equations, i.e.,

$$\begin{aligned}\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \alpha\, \widehat{\mathbf{v}}_{\lambda=0}\, e^{0\,t} + \beta\, \widehat{\mathbf{v}}_{\lambda=-2}\, e^{-2t} \\ &= \frac{\alpha}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\beta}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}, \quad \alpha, \beta \in \mathbb{R}\,.\end{aligned}$$

Using initial conditions, $x(0) = 1$, $y(0) = 0$, we can determine α and β, i.e.,

$$\begin{aligned}\mathbf{x}(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \frac{\alpha}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\beta}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= P \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\end{aligned}$$

where P is an orthogonal matrix (since the eigenvectors associated with distinct eigenvalues are orthogonal) and hence

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = P \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \;\Rightarrow\; \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = P^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = P \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence the solution to the set of differential equations is given by

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}.$$

2017, S2

12. A quadric curve is given by the equation $2x^2 + 4xy - y^2 = 1$.

a) Express the curve in the form

$$\mathbf{x}^T \mathsf{A} \mathbf{x} = 1, \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix},$$

and find the eigenvalues and eigenvectors of the matrix A.

Solution: The curve in matrix form is given by $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1$ and therefore $\mathsf{A} = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$.

The eigenvalues of A are determined by solving $|\mathsf{A} - \lambda \mathsf{I}| = 0$, i.e.,

$$\begin{aligned}|\mathsf{A} - \lambda \mathsf{I}| = \begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} &= (\lambda - 2)(\lambda + 1) - 4 \\ &= \lambda^2 - \lambda - 6 \\ &= (\lambda - 3)(\lambda + 2) = 0\,.\end{aligned}$$

Thus $\lambda = 3$ or -2.

[Check the sum of eigenvalues = trace(A), i.e., $3 + (-2) = 2 + (-1)$. ✓]

Next we determine the set of eigenvectors for each λ by solving $(\mathsf{A}-\lambda\mathsf{I})\mathbf{v}=\mathbf{0}$, i.e.,

$$\underline{\lambda=3:}\qquad \left(\begin{array}{cc|c} -1 & 2 & 0\\ 2 & -4 & 0\end{array}\right) \quad\to\quad \mathbf{v}_{\lambda=3}=t\begin{pmatrix}2\\1\end{pmatrix},\quad t\in\mathbb{R}\backslash\{0\},$$

$$\underline{\lambda=-2:}\qquad \left(\begin{array}{cc|c} 4 & 2 & 0\\ 2 & 1 & 0\end{array}\right) \quad\to\quad \mathbf{v}_{\lambda=-2}=t\begin{pmatrix}1\\-2\end{pmatrix},\quad t\in\mathbb{R}\backslash\{0\}.$$

Note this real symmetric matrix A has distinct eigenvalues and the eigenvectors associated with distinct eigenvectors are orthogonal, i.e., $\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\end{pmatrix}=0.$

b) Hence, or otherwise, find the distance from the curve to the origin. Write down the x and y coordinates of the points on the curve closest to the origin.

Solution: The quadric curve, relative to the principal axes, is given by

$$3X^2-2Y^2=1\,.$$

This curve, a hyperbola, doesn't intersect the Y-axis but intercepts the X-axis at $X=\pm\dfrac{1}{\sqrt{3}}$. Hence the points $\pm\dfrac{1}{\sqrt{3}}\begin{pmatrix}1\\0\end{pmatrix}$ (on the curve) are closest to the origin, relative to the principal axes. Thus the distance from the origin to the curve is $\dfrac{1}{\sqrt{3}}$.

The points on the curve closest to the origin, relative to the original axes, are given by the position vector

$$\mathbf{x}=\begin{pmatrix}x\\y\end{pmatrix}=\pm\frac{1}{\sqrt{3}}\widehat{\mathbf{v}}_{\lambda=3}=\pm\frac{1}{\sqrt{3}}\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}=\pm\frac{1}{\sqrt{15}}\begin{pmatrix}2\\1\end{pmatrix}.$$

$\boxed{\text{2018, S1}}$ 13. A **real symmetric** 3×3 matrix A has eigenvalues denoted by λ_1, λ_2 and λ_3. We define a quadric surface

$$\mathbf{x}^T\mathsf{A}\mathbf{x}=12\quad\text{where } \mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}.$$

A student is given the following extra information about matrix A:

- trace$(\mathsf{A})=0$,
- $\lambda_1=2$ and $\lambda_3=4$ with associated eigenvectors, respectively,

$$\mathbf{v}_1=\begin{pmatrix}1\\0\\1\end{pmatrix}\quad\text{and}\quad\mathbf{v}_3=\begin{pmatrix}1\\0\\-1\end{pmatrix}.$$

a) What is the value of the remaining eigenvalue, namely λ_2?

Solution: Using the result, the sum of the eigenvalues is equal to trace(A), we have

$$\text{trace}(\mathsf{A})=0=\lambda_1+\lambda_2+\lambda_3=2+\lambda_2+4\quad\Rightarrow\quad\lambda_2=-6\,.$$

It is important to note the matrix A has (real) distinct eigenvalues and is symmetric.

b) Write down the equation of the quadric surface, relative to the principal axes of the surface.

Solution: Relative to the principal axes, the quadric surface (one of 6 different equations) is given by

$$2X^2-6Y^2+4Z^2=12\,,\quad\text{Hyperboloid of 1-sheet}\,.$$

c) Write down a vector $\mathbf{v}_2$ that is orthogonal to **both** eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_3$.

Solution:

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} .$$

Check $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ and $\mathbf{v}_2 \cdot \mathbf{v}_3 = 0$.

d) What is the relationship between λ_2 and $\mathbf{v}_2$? Give reasons for your answer.

Solution: Since the matrix A is a real symmetric matrix with distinct eigenvalues then eigenvectors associated with distinct eigenvalues are orthogonal to each other. Noting $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$ it must be the case

$$\mathsf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2 ,$$

i.e., $\mathbf{v}_2$ is an eigenvector of matrix A associated with the eigenvalue $\lambda_2 = -6$.

e) Hence determine an **orthogonal** matrix P which diagonalises the matrix A such that $\mathsf{P}^{-1}\mathsf{A}\mathsf{P} = \mathsf{D}$ where D is a 3×3 diagonal matrix.

Solution:

$$\mathsf{P} = \begin{pmatrix} \widehat{\mathbf{v}}_1 & \widehat{\mathbf{v}}_2 & \widehat{\mathbf{v}}_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \qquad \text{(one of 6 possible answers)} .$$

f) Hence determine the matrix A.

Solution: Since the matrix P is orthogonal and in this case symmetric, then $P^{-1} = P^T = P$. Hence

$$\begin{aligned} \mathsf{A} = \mathsf{PDP} &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & -6\sqrt{2} & 0 \\ 4 & 0 & -4 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 6 & 0 & -2 \\ 0 & -12 & 0 \\ -2 & 0 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 & -1 \\ 0 & -6 & 0 \\ -1 & 0 & 3 \end{pmatrix} . \end{aligned}$$

Note A is a real symmetric matrix and trace$(\mathsf{A}) = 0$!

2018, S2

14. A quadric curve is given by the equation $7x^2 + 6xy + 7y^2 = 200$.

i) Express the curve in the form

$$\mathbf{x}^T \mathsf{A} \mathbf{x} = 200$$

where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and A is a 2×2 real symmetric matrix.

Solution: The quadric curve in matrix form is given by $\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 200$ and therefore $\mathsf{A} = \begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix}$.

ii) Find the eigenvalues and eigenvectors of the matrix A in part i).

Solution: The eigenvalues of A are determined by solving $|\mathsf{A}-\lambda\mathsf{I}|=0$, i.e.,

$$\begin{aligned}|\mathsf{A}-\lambda\mathsf{I}| = \begin{vmatrix} 7-\lambda & 3 \\ 3 & 7-\lambda \end{vmatrix} &= (\lambda-7)^2-9=0 \\ &\Rightarrow \lambda-7=\pm 3\,.\end{aligned}$$

Thus $\lambda=4$ or 10.

[Check the sum of eigenvalues = trace(A), i.e., $4+10=7+7$. ✓]

Next we determine the set of eigenvectors for each λ by solving $(\mathsf{A}-\lambda\mathsf{I})\mathbf{v}=\mathbf{0}$, i.e.,

$$\begin{aligned}
\underline{\lambda=4:} \qquad & \left(\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right) \quad \rightarrow \quad \mathbf{v}_{\lambda=4} = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t\in\mathbb{R}\backslash\{0\}\,, \\
\underline{\lambda=10:} \qquad & \left(\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right) \quad \rightarrow \quad \mathbf{v}_{\lambda=10} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t\in\mathbb{R}\backslash\{0\}\,.
\end{aligned}$$

Note this real symmetric matrix A has distinct eigenvalues and the eigenvectors associated with distinct eigenvectors are orthogonal, i.e., $\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}=0$.

iii) Hence, or otherwise, find the shortest distance between the curve and the origin.

Solution: The quadric curve, relative to the principal axes, is given by

$$4X^2+10Y^2=200\,.$$

This curve, an ellipse, intersects the Y-axis at $Y=\pm 2\sqrt{5}$ and intercepts the X-axis at $X=\pm 5\sqrt{2}$. Hence the points $\pm 2\sqrt{5}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (on the curve) are closest to the origin (since $2\sqrt{5}<5\sqrt{2}$), relative to the principal axes. Thus the distance from the origin to the curve is $2\sqrt{5}$.