# LECTURE 38
# DE'S VIA LAPLACE TRANSFORMS

## LAPLACE TRANSFORMS

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

| $f(t)$ | $F(s)$ |
|---|---|
| $1$ | $1/s$ |
| $t$ | $1/s^2$ |
| $t^m$ | $m!/s^{m+1}$ |
| $t^\nu$ , $(\nu > -1)$ | $\Gamma(\nu+1)/s^{\nu+1}$ |
| $e^{-at}$ | $1/(s+a)$ |
| $\sin bt$ | $b/(s^2+b^2)$ |
| $\cos bt$ | $s/(s^2+b^2)$ |
| $\sinh bt$ | $b/(s^2-b^2)$ |
| $\cosh bt$ | $s/(s^2-b^2)$ |
| $\sin bt - bt\cos bt$ | $2b^3/(s^2+b^2)^2$ |
| $\sin bt + bt\cos bt$ | $2bs^2/(s^2+b^2)^2$ |
| $t\ \sin bt$ | $2bs/(s^2+b^2)^2$ |
| $te^{-at}$ | $1/(s+a)^2$ |
| $\mathrm{u}(t-c)$ | $e^{-cs}/s$ |
| $e^{-at}f(t)$ | $F(s+a)$ |
| $tf(t)$ | $-F'(s)$ |
| $f(t-c)\mathrm{u}(t-c)$ | $e^{-cs}F(s)$ |
| $f'(t)$ | $sF(s)-f(0)$ |
| $f''(t)$ | $s^2F(s)-sf(0)-f'(0)$ |
| $f'''(t)$ | $s^3F(s)-s^2f(0)-sf'(0)-f''(0)$ |
| $\int_0^t f(\tau)d\tau$ | $F(s)/s$ |

The major application of Laplace transforms is to the solution of differential equations with discontinuous forcing functions. This physical system arises frequently particularly when switching is involved. The method is quite simple. We transform the entire equation to produce an algebraic equation in the $s$ variable. This equation is usually trivially solved to produce a solution in the $s$ variable. We then need to invert this solution to produce an answer in the $t$ variable. It is this last step of inversion that is usually the toughest part of the process! First let us prove the required table entries linking the Laplace transform to the calculus.

**Example 1** Prove that

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

$$\mathcal{L}(f') = \int_0^\infty f'(t)\, e^{-st}\, dt$$

$$= \Big[e^{-st} f(t)\Big]_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt$$

$$= sF(s) - f(0)$$

It is also true that

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

The above two facts (which always appear in the Laplace transform tables) will prove extremely useful when solving D.E.'s.

**Example 2** Solve $\dfrac{dy}{dt} + 4y = 8$ where $y(0) = 9$. First use your standard techniques and then implement Laplace transforms.

$$e^{\int 4\,dx} = e^{4x}$$

$$y = \frac{1}{e^{4x}} \int e^{4x}\, 8 \, dx$$

$$= \frac{1}{e^{4x}} \left( 2e^{4x} + C \right)$$

$$y(0) = 9 \quad \therefore \quad C = 7$$

$$\therefore y = 2 + 7e^{-4x}$$

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$$y' + 4y = 8$$

$$\mathcal{L}(y') + 4\mathcal{L}(y) = \frac{8}{s}$$

$$s\mathcal{L}(y) - y(0) + 4\mathcal{L}(y) = \frac{8}{s}$$

Let $Y = \mathcal{L}(y)$: $\quad sY - 9 + 4Y = \dfrac{8}{s}$

$$Y(s+4) = 9 + \frac{8}{s}$$

$$Y = \frac{9s+8}{s(s+4)}$$

$$\therefore y = \mathcal{L}^{-1}\left(\frac{9s+8}{s(s+4)}\right) = \mathcal{L}^{-1}\left(\frac{2}{s} + \frac{7}{s+4}\right)$$

$$\therefore y = 2 + 7e^{-4t}$$

$\bigstar \quad y = 2 + 7e^{-4t} \quad \bigstar$

**Example 3** Solve $y'' - 6y' + 8y = 0$ where $y(0) = 1$ and $y'(0) = 16$. First use your standard techniques and then implement Laplace transforms.

$$\lambda^2 - 6\lambda + 8 = 0 \implies \lambda = 2, 4$$

$$\therefore y_H = Ae^{2t} + Be^{4t}$$

For $y(0) = 1$, $y'(0) = 16$: $A = -6$, $B = 7$

$$\therefore y_H = -6e^{2t} + 7e^{4t}$$

$$y'' - 6y' + 8y = 0$$

$$\mathcal{L}(y'') - 6\mathcal{L}(y') + 8\mathcal{L}(y) = 0$$

$$\left(s^2\mathcal{L}(y) - s\,y(0) - y'(0)\right) - 6\left(s\,\mathcal{L}(y) - y'(0)\right) + 8\mathcal{L}(y) = 0$$

$$s^2Y - s - 16 - 6Y + 6 + 8Y = 0$$

$$Y = \frac{s+10}{(s-4)(s-5)}$$

$$y = \mathcal{L}^{-1}\left(\frac{7}{s-4} - \frac{6}{s-5}\right)$$

$$= 7e^{4t} - 6e^{2t}$$

★ $y = 7e^{4t} - 6e^{2t}$ ★

You will notice from the above that we really need initial conditions at $y(0)$ and $y'(0)$ before it becomes possible to use Laplace transforms effectively. However as a bonus the presence of discontinuous right hand sides may be easily dealt with! We will look at these more complicated differential equations in Lecture 39.

In preparation however, for homework please have a go at all the problems in next lecture (Lecture 38a) before we next meet. These examples explore the use of the Heaviside function in the theory of Laplace Transforms. The next lecture (Lecture 38a) will be a problem class working through these Heaviside examples.

[38] You can now do Q 100 101 102