LECTURE 39
DE'S VIA LAPLACE TRANSFORMS PART 2

LAPLACE TRANSFORMS

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^m	$m!/s^{m+1}$
t^ν , $(\nu > -1)$	$\Gamma(\nu+1)/s^{\nu+1}$
e^{-at}	$1/(s+a)$
$\sin bt$	$b/(s^2+b^2)$
$\cos bt$	$s/(s^2+b^2)$
$\sinh bt$	$b/(s^2-b^2)$
$\cosh bt$	$s/(s^2-b^2)$
$\sin bt - bt\cos bt$	$2b^3/(s^2+b^2)^2$
$\sin bt + bt\cos bt$	$2bs^2/(s^2+b^2)^2$
$t\ \sin bt$	$2bs/(s^2+b^2)^2$
te^{-at}	$1/(s+a)^2$
$\mathrm{u}(t-c)$	e^{-cs}/s
$e^{-at}f(t)$	$F(s+a)$
$tf(t)$	$-F'(s)$
$f(t-c)\mathrm{u}(t-c)$	$e^{-cs}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f'''(t)$	$s^3F(s)-s^2f(0)-sf'(0)-f''(0)$
$\int_0^t f(\tau)d\tau$	$F(s)/s$

We will now look at DE's where discontinuous functions play a role. It needs to be kept in mind that many physical processes are in fact discontinuous by their very nature (for example activating switches or overnight gapping in the stock market). Laplace transforms are the tool of choice for these sorts of problems!

Example 1 Consider the differential equation

$$y'' - 5y' + 4y = r(t)$$

where

$$r(t) = \begin{cases} 24 & t \geq 7; \\ 0 & \text{otherwise.} \end{cases}$$

and the initial conditions are $y(0) = 0$ and $y'(0) = 3$.

i) Sketch $r(t)$ and express $r(t)$ in terms of the Heaviside function.

ii) By taking the Laplace transform of the differential equation show that

$$Y(s) = \frac{24e^{-7s}}{s(s-4)(s-1)} + \frac{3}{(s-4)(s-1)}$$

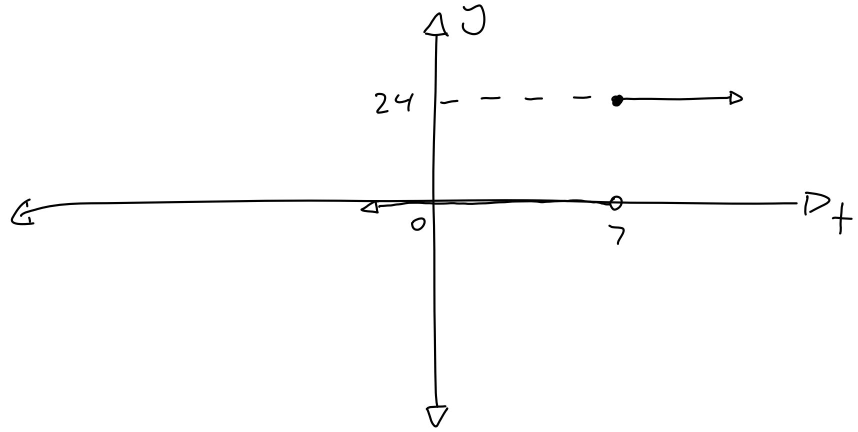
iii) Using partial fractions show that

$$Y(s) = e^{-7s}\left(\frac{6}{s} + \frac{2}{s-4} - \frac{8}{s-1}\right) + \frac{1}{s-4} - \frac{1}{s-1}$$

iv) Hence find the solution $y(t)$ in terms of the Heaviside function.

v) Express the solution without the Heaviside function.

i) $r(t) = 24u(t-7)$



ii) $\mathcal{L}(y''-5y'+4y) = \mathcal{L}(24u(t-7))$

$$s^2Y(s) - 3 - 5(sY(s)) + 4Y(s) = \frac{24e^{-7s}}{s}$$

$$\therefore Y(s) = \frac{24e^{-7s}}{s(s-1)(s-4)} + \frac{3}{(s-1)(s-4)}$$

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iii) $Y(s) = 24e^{-7s}\left(\frac{1}{4s} - \frac{1}{3(s-1)} + \frac{1}{12(s-4)}\right) + 3\left(-\frac{1}{3(s-1)} + \frac{1}{3(s-4)}\right)$

$$= e^{-7s}\left(\frac{6}{s} - \frac{8}{s-1} + \frac{2}{s-4}\right) + \frac{1}{s-4} - \frac{1}{s-1}$$

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iv) $y(t) = \mathcal{L}^{-1}(Y(s))$

$$= u(t-7)(6-8e^{t}+2e^{4t}) + e^{4t} - e^{t}$$

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v) $y(t) = \begin{cases} e^{4t} - e^{t}, & 0 \le t \le 7 \\ 6-8e^{t-7}+2e^{4t-28} + e^{4t} - e^{t}, & t \ge 7 \end{cases}$

★ $y(t) = u(t-7)(6+2e^{4t-28} - 8e^{t-7}) + e^{4t} - e^{t}$ ★

★ $y(t) = \begin{cases} e^{4t} - e^{t}, & 0 \le t \le 7; \\ 6 + 2e^{4t-28} - 8e^{t-7} + e^{4t} - e^{t}, & t \ge 7. \end{cases}$ ★

Example 2 Consider the system of differential equations

$$\begin{aligned} \frac{dx}{dt} + 4x + 10y &= 0 \\ \frac{dy}{dt} - 5x - 11y &= 0 \end{aligned}$$

where $x(0) = -1$ and $y(0) = 0$.

i) By taking Laplace transforms of both equations show that:

$$\begin{aligned} (s+4)X + 10Y &= -1 & (1) \\ -5X + (s-11)Y &= 0 & (2) \end{aligned}$$

ii) Hence show that:

$$\begin{aligned} 5(s+4)X + 50Y &= -5 & (3) \\ -5(s+4)X + (s+4)(s-11)Y &= 0 & (4) \end{aligned}$$

i) Show that adding (3) and (4) yields

$$Y(s) = \frac{-5}{s^2 - 7s + 6}.$$

ii) Hence show that $y(t) = e^t - e^{6t}$.

iii) Without inverting again find $x(t)$.

i) $\mathcal{L}\left(\frac{dx}{dt} + 4x + 10y\right) = 0$

$sX(s) + 1 + 4X(s) + 10Y(s) = 0$

$\therefore (s+4)X(s) + 10Y(s) = -1$ — ①

$\mathcal{L}\left(\frac{dy}{dt} - 5x - 11y\right) = 0$

$sY(s) - 5X(s) - 11Y(s) = 0$

$\therefore (s-11)Y(s) - 5X(s) = 0$ — ②

ii) $5 \times ①$: $5(s+4)X(s) + 50Y(s) = -5$ — ③

$(s+4) \times ②$: $-5(s+4)X(s) + (s+4)(s-11)Y(s) = 0$ — ④

iii) ③ + ④: $Y(s)(50 + s^2 - 7s - 44) = -5$

$$\therefore Y(s) = \frac{-5}{s^2 - 7s + 6} = \frac{-5}{(s-6)(s-1)}$$

iv) $\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{1}{s-1} - \frac{1}{s-6}\right)$

$$\therefore y(t) = e^t - e^{6t}$$

v) $$\frac{dy}{dt} - 5x - 11y = 0$$

$$e^t - 6e^{6t} - 5x - 11(e^t - e^{6t}) = 0$$

$$\therefore x(t) = -2e^t + e^{6t}$$

★ $x(t) = -2e^t + e^{6t}$ ★

[39] You can now do Q 103 104 105