# LECTURE 43
# FOURIER SERIES PART II: SOME PROOFS

Suppose that a function $f$ has period $T = 2\pi$. Then $f$ may be approximated by the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where the Fourier coefficients $a_0$, $a_n$, and $b_n$ are given by

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad (n = 1, 2, \dots) \\ b_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad (n = 1, 2, \dots) \end{aligned}\right\}$$

- Odd functions have odd series $\qquad f(x) = \sum_{n=1}^{\infty} b_n \sin nx$
- Even functions have even series $\qquad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

In this lecture we will prove some of the claims of the last lecture and also develop some shortcuts which can sometimes be used the cut down on the laborious calculations often required when finding Fourier series. We will still be assuming throughout this lecture that $f$ has period $2\pi$. This will be generalised in the next lecture however it should be noted at this stage that we only need to make small modifications to our formulae to deal with arbitrary periodic functions.

**Example 1** Fully describe each of the following series

i) $\{\sin(n\pi)\}_{n=1}^{\infty}$ $= 0$

ii) $\{\cos(n\pi)\}_{n=1}^{\infty}$ $= 1, -1$

iii) $\{\sin(\frac{n\pi}{2})\}_{n=1}^{\infty}$ $= 0, 1, -1$

iv) $\{\cos(\frac{n\pi}{2})\}_{n=1}^{\infty}$ $= 0$

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**Example 2** Prove that

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)dx = 0$$

for all $m = 1, 2, 3 \ldots$ and $n = 1, 2, 3 \ldots$

$$\int_{-\pi}^{\pi} \text{odd} \times \text{even} = \int_{-\pi}^{\pi} \text{odd} = 0$$

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**Example 3** Prove that

$$\int_{-\pi}^{\pi} \cos(mx)dx = 0$$

for all $m = 1, 2, 3 \ldots$

$$\int_{-\pi}^{\pi} \cos(mx)\,dx = \left[\frac{\sin(mx)}{m}\right]_{-\pi}^{\pi} = \frac{\sin(m\pi)}{m} - \frac{\sin(-m\pi)}{m} = 0$$

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**Example 4** Prove that

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \begin{cases} 0, & \text{If } m \neq n; \\ \pi, & \text{If } m = n. \end{cases}$$

$\underline{m = n}$

$$\int_{-\pi}^{\pi} \cos^2(nx)\,dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2} - \frac{\cos(n\pi x)}{2}\,dx$$

$$= \int_{0}^{\pi} 1 - \cos(n\pi x)\,dx$$

$$= x - \frac{\sin(n\pi x)}{n}\Bigg|_{0}^{\pi}$$

$$= \pi$$

$\underline{m \neq n}$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(nx - mx) + \cos(nx + mx)\,dx$$

$$= \int_{0}^{\pi} \cos((n-m)x) + \cos((n+m)x)\,dx$$

$$= \frac{\sin((n-m)x)}{n-m} + \frac{\sin((n+m)x)}{n+m}\Bigg|_{0}^{\pi}$$

$$= 0$$

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We say that $\cos(nx)$ and $\cos(mx)$ are orthogonal. We are now in a position to prove the formula for $a_n$.

**Example 5** Suppose that a function $f$ has period $T = 2\pi$ and assume that

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

By multiplying both sides of this equation by $\cos(mx)$ and integrating from $-\pi$ to $\pi$ show that

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos mx \, dx \qquad (m = 1, 2, \ldots)$$

$$\int_{-\pi}^{\pi} f(x) \cos(mx)\, dx = \int_{-\pi}^{\pi} a_0 \cos(mx)\, dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n \cos(nx) \cos(mx)\, dx$$

$$+ \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} b_n \sin(nx) \cos(mx)\, dx$$

$$= 0 + a_n \times \pi + 0$$

$$\therefore a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx)\, dx$$

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The other formulae for $a_0$ and $b_n$ are similarly verified. Note that when we calculate $a_n$ all we really end up doing is integrating the function $f$ against $\cos(nx)$ which is of course the very term for which $a_n$ is the coefficient!

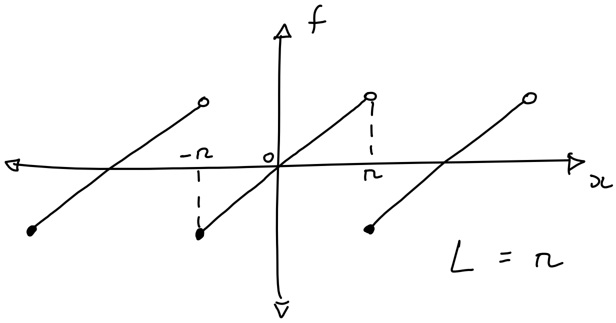
Another crucial observation to make is that since $f$, $\cos(nx)$, and $\sin(nx)$ all repeat every $2\pi$ units, the integrals in the Fourier series formulae may be taken over any interval of length $2\pi$. So the equations could also be written as

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2\pi}\int_0^{2\pi} f(x)\, dx \\ a_n &= \tfrac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx \qquad (n = 1, 2, \ldots) \\ b_n &= \tfrac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx \qquad (n = 1, 2, \ldots) \end{aligned}\right\}$$

We close with an example which will display quite clearly that when you start with and **even** or an **odd** function $f$ the amount of work required to calculate the Fourier series can be greatly reduced just by using a few simple tricks.

**Example 6** Suppose that $f(x) = \begin{cases} x & -\pi \le x < \pi; \\ f(x+2\pi) & \text{otherwise.} \end{cases}$

Sketch $f$ and find its Fourier series.




$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

odd × odd = even

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left(\left[\frac{x\cos(nx)}{-n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right)$$

$$= \frac{2}{\pi}\left(\frac{-\pi(-1)^n}{n}\right)$$

$$b_n = \frac{2(-1)^{n+1}}{n}$$

$$a_0 = a_n = 0$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nx)$$

$$= 2\sin(x) - \sin(2x) + \frac{2}{3}\sin\left(\frac{2}{3}x\right) - \ldots$$

$$\bigstar \quad b_n = \frac{2(-1)^{n+1}}{n} \qquad a_0 = a_n = 0 \quad \bigstar$$

$$\bigstar \quad f(x) = 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - \ldots.. \quad \bigstar$$

Note that for all odd functions $a_0 = a_n = 0$ and that for all even functions $b_n = 0$. In other words an odd function will have an odd Fourier series made up entirely of sin's and an even function will have an even Fourier series consisting of a constant $a_0$ and cos's. If a function is neither odd nor even then you are up for a lot of work!

The following is an interesting applet where we can add in the terms of a Fourier Series one by one and see the effect. The example above is the sawtooth button:

http://www.intmath.com/fourier-series/fourier-graph-applet.php