THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

JUNE 2014

MATH2019
ENGINEERING MATHEMATICS 2E

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

TABLE OF LAPLACE TRANSFORMS AND THEOREMS

$g(t)$ is a function defined for all $t \geq 0$, and whose Laplace transform

$$G(s) = \mathcal{L}(g(t)) = \int_0^\infty e^{-st} g(t) dt$$

exists. The Heaviside step function u is defined to be

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ \frac{1}{2} & \text{for } t = a \\ 1 & \text{for } t > a \end{cases}$$

$g(t)$	$G(s) = \mathcal{L}[g(t)]$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^\nu, \nu > -1$	$\frac{\nu!}{s^{\nu+1}}$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$e^{-\alpha t} f(t)$	$F(s+\alpha)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$tf(t)$	$-F'(s)$

Please see over ...

FOURIER SERIES

If $f(x)$ has period $T = 2L$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}\,x\right) + b_n \sin\left(\frac{n\pi}{L}\,x\right)\right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}\,x\right)dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}\,x\right)dx \end{aligned}$$

LEIBNIZ RULE FOR DIFFERENTIATING INTEGRALS

$$\frac{d}{dx}\int_u^v f(x,t)dt = \int_u^v \frac{\partial f}{\partial x}dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx}.$$

MULTIVARIABLE TAYLOR SERIES

$$\begin{aligned} f(x,y) &= f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\ &\quad + \frac{1}{2!}\left((x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) + (y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right) + \cdots \end{aligned}$$

VARIATION OF PARAMETERS

Suppose that the second order differential equation

$$y'' + p(x)y' + q(x)y = f(x)$$

has homogeneous solution $y_h = Ay_1(x) + By_2(x)$. Then a particular solution is given by

$$y_P(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx$$

where $W(x) = \det \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}$.

Please see over ...

SOME BASIC INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^{kx} dx = \frac{e^{kx}}{k} + C$$

$$\int a^x dx = \frac{1}{\ln a} a^x + C \quad \text{for } a \neq 1$$

$$\int \sin\, kx\, dx = -\frac{\cos kx}{k} + C$$

$$\int \cos kx\, dx = \frac{\sin kx}{k} + C$$

$$\int \sec^2 kx\, dx = \frac{\tan kx}{k} + C$$

$$\int \mathrm{cosec}^2 kx\, dx = -\frac{1}{k} \cot kx + C$$

$$\int \tan kx\, dx = \frac{\ln|\sec kx|}{k} + C$$

$$\int \sec kx\, dx = \frac{1}{k}(\ln|\sec kx + \tan kx|) + C$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx$$

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx$$

Answer question 1 in a separate book

1. i) Consider the double integral

$$I = \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} 3x \; dx \; dy.$$

a) Sketch the region of integration.

b) Evaluate I using polar coordinates.

ii) Suppose that the atmospheric pressure P in a certain region of space is given by

$$P(x, y, z) = x^2 + y^2 + z^2.$$

a) Calculate $\nabla P = \text{grad } P$ at the point $T(1, 2, 4)$.

b) Find the rate of change of the pressure with respect to distance at the point $T(1, 2, 4)$ in the direction of the vector $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$.

c) Give a geometrical description of the level surface L of P passing through the point $T(1, 2, 4)$.

d) Find a Cartesian equation of the tangent plane to the level surface L of P at the point $T(1, 2, 4)$.

iii) A thin triangular plate bounded by $y = 2x$, $y = 6$ and the y axis has non-uniform density given by $\rho(x, y) = 4xy$. Find the mass of the plate by evaluating an appropriate double integral in Cartesian coordinates.

iv) It is given that the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{pmatrix}$ has an eigenvalue $\lambda_1 = 1$

with an associated eigenvector $\mathbf{v_1} = \begin{pmatrix} 15 \\ 8 \\ -16 \end{pmatrix}$ and eigenvalue $\lambda_2 = 6$

with associated eigenvector $\mathbf{v_2} = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$.

a) Without calculating the characteristic polynomial explain why the remaining eigenvalue is $\lambda_3 = 7$.

b) Find an eigenvector $\mathbf{v_3}$ for the eigenvalue $\lambda_3 = 7$.

c) Hence write down the general solution to the system of differential equations

$$\begin{aligned} y_1' &= y_1 \\ y_2' &= -8y_1 + 4y_2 - 6y_3 \\ y_3' &= 8y_1 + y_2 + 9y_3 \end{aligned}$$

Answer question 2 in a separate book

2. i) Use Leibniz' theorem to find

$$\frac{d}{dt}\int_1^{t^2} \frac{\sin(\sqrt{x}\,)}{x}\,dx.$$

ii) Given the vector field $\mathbf{G} = yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}$ calculate:

a) div $\mathbf{G}$.

b) curl $\mathbf{G}$.

iii) Find and classify the critical points of

$$f(x, y) = x^3 - 12xy + 8y^3.$$

iv) By evaluating an appropriate line integral calculate the work done on a particle traveling in $\mathbb{R}^3$ through the vector field $\mathbf{F} = -y\mathbf{i} + xyz\mathbf{j} + x^2\mathbf{k}$ along the straight line from $(1, 2, 3)$ to $(2, 2, 5)$.

v) Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be a path in space embedded within the surface $\phi(x, y, z) = 1$. Assuming that all relevant derivatives exist use the chain rule to show that grad ϕ is perpendicular to the velocity vector $\mathbf{v}(t)$ for all t.

Answer question 3 in a separate book

3. i) A cone with radius r and perpendicular height h has volume $V = \frac{1}{3}\pi r^2 h$.

Determine the maximum error in calculating V given that $r = 4$ cm and $h = 3$ cm to the nearest millimetre.

ii) Let $f(x) = \begin{cases} 1 & 0 < x < \pi \ ; \\ 0 & \pi \leq x \leq 2\pi \ ; \\ f(x+2\pi) & \text{otherwise.} \end{cases}$

a) What is the period of f ?

b) Sketch the graph of $y = f(x)$ over the interval $-2\pi \leq x \leq 2\pi$.

c) Sketch the graph of $y = f(x) - \frac{1}{2}$ over the interval $-2\pi \leq x \leq 2\pi$.

d) Hence or otherwise find the Fourier series of f, explicitly writing down the first 4 non-zero terms of the series.

e) By considering f and its series at $x = \frac{\pi}{2}$, show that

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

iii) Using the table of Laplace transforms find

$$\mathcal{L}(t + \sin(2t) + e^{-t}).$$

iv) a) By establishing an appropriate partial fraction decomposition find

$$\mathcal{L}^{-1}\left\{\frac{7s+1}{(s+1)(s-1)}\right\}.$$

b) Hence or otherwise find

$$\mathcal{L}^{-1}\left\{\frac{7s+1}{(s+1)(s-1)}e^{-5s}\right\}.$$

Answer question 4 in a separate book

4. i) Use the method of Lagrange multipliers to find the maximum and minimum values of $x + y$ on the circle $x^2 + y^2 - 1 = 0$.

ii) The temperature in a bar of length π metres satisfies the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

where $u(x, t)$ is the temperature in $^\circ C$, t is time in seconds and x is the distance in metres from the left hand end of the bar. Both ends of the bar are maintained at a temperature of $0^\circ C$. Hence

$$u(0, t) = u(\pi, t) = 0 \quad \text{for all} \quad t.$$

a) Assuming a solution of the form

$$u(x, t) = F(x)G(t) \quad \text{show that}$$

$$\frac{1}{G}\frac{dG}{dt} = \frac{1}{F}\frac{d^2F}{dx^2} = k \quad \text{where } k \text{ is a constant.}$$

b) You may assume that only $k < 0$ yields non-trivial solutions and set $k = -(p^2)$ for some $p > 0$.

Applying the initial conditions show that $p = n,\ n = 1, 2, 3, \ldots$ and that possible solutions for $F(x)$ are

$$F_n(x) = \sin(nx) \qquad n = 1, 2, 3, \ldots$$

c) Find all possible solutions $G_n(t)$ for $G(t)$.

d) Suppose now that the initial temperature distribution of the bar is

$$u(x, 0) = 2\sin(x) - 16\sin(2x).$$

Find the general solution $u(x, t)$.

e) Hence determine all points x along the bar with a temperature of $0^\circ C$ after $t = \ln(2)$ seconds.

iii) Suppose that a function $f(x)$ of period $T = 2L$ has a Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)$$

Prove that

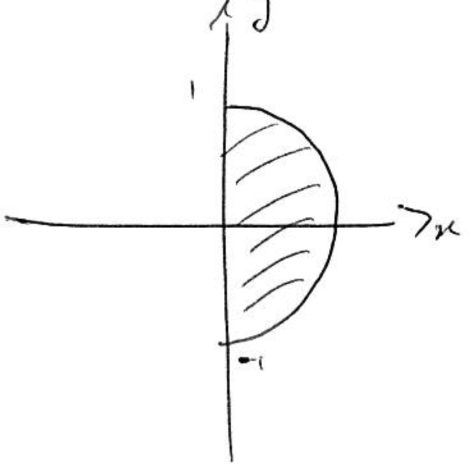
$$\frac{1}{L}\int_{-L}^{L} f(x)^2\,dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Math 2019 June 2014

Exam and Solutions

Question 1

i) a)



$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 3r\cos\theta \, r \, dr \, d\theta$$

$$= 3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r^2 \cos\theta \, dr \, d\theta$$

$$= 3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\tfrac{1}{3} r^3 \cos\theta\right]_0^1 d\theta$$

$$= 3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tfrac{1}{3}\cos\theta - 0 \, d\theta$$

$$= \left[\sin\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1 - -1 = \boxed{2}$$

ii) a) $\nabla P = 2x\underline{i} + 2y\underline{j} + 2z\underline{k}$

and at $T(1,2,4)$ we have $\nabla P = 2\underline{i} + 4\underline{j} + 8\underline{k}$

$$= \boxed{\begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}}.$$

b) $|\underline{b}| = \sqrt{9+16+144} = \sqrt{169} = 13$

$\therefore \ \hat{\underline{b}} = \frac{1}{13}\begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}.$

$$\text{Dir. Derivative} = \nabla P \cdot \hat{\underset{\sim}{b}} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} \cdot \frac{1}{13}\begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$$

$$= \frac{1}{13}(6 + 16 + 96) = \frac{1}{13} \cdot 118$$

$$= \boxed{118/13.}$$

c) We have $x^2+y^2+z^2 = 21$

This is a sphere, centre $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, radius $\sqrt{21}$.

d) $\nabla P \perp$ level surface. Hence the

equation is $2x + 4y + 8z = \beta$.

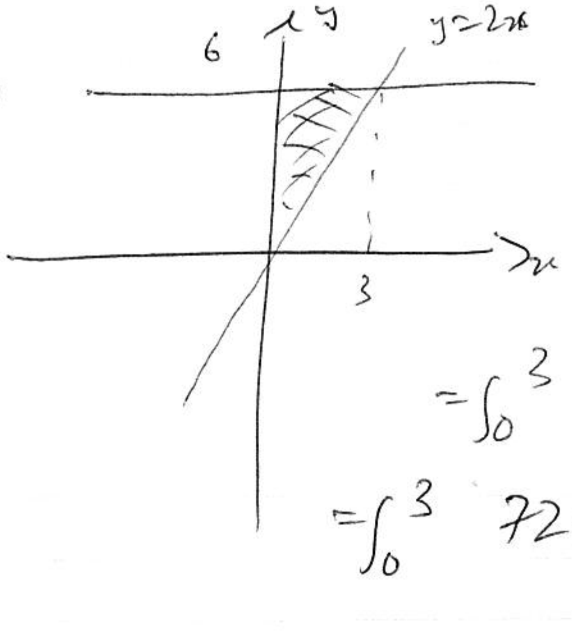
Sub in $T\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$: $2 + 8 + 32 = \beta$

$\rightarrow \beta = 42$

Hence $2x + 4y + 8z = 42$

$\rightarrow \boxed{x + 2y + 4z = 21}$.

iii)



$$\text{Mass} = \iint_R \rho(x,y)$$

$$= \int_0^3 \int_{2x}^6 4xy \, dy \, dx.$$

$$= \int_0^3 \left[2xy^2\right]_{2x}^6 dx$$

$$= \int_0^3 72x - 2x(4x^2) \, dx.$$

$$= \int_0^3 72x - 8x^3 \, dx.$$

$$= \left[36x^2 - 2x^4\right]_0^3$$

$$= (36 \times 9 - 2 \times 81) - 0.$$

$$= \boxed{162}.$$

a) Σ evals = Trace (A)

$\rightarrow 1+6+\lambda_3 = 1+4+9$

$\rightarrow \lambda_3 = 7$ as required.

b) $A\underset{\sim}{v} = 7\underset{\sim}{v} \rightarrow (A-7I)\underset{\sim}{v} = \underset{\sim}{0}$

$$\rightarrow \left(\begin{array}{ccc|c} -6 & 0 & 0 & 0 \\ -8 & -3 & -6 & 0 \\ 8 & 1 & 2 & 0 \end{array}\right)$$

$R_1 = -\frac{1}{6}R_1$ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -8 & -3 & -6 & 0 \\ 8 & 1 & 2 & 0 \end{array}\right)$

$R_2 = R_2 + 8R_1$
$R_3 = R_3 - 8R_1$ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$

$R_3 = R_3 + \frac{1}{3}R_2$ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ ✓

Let $z = t$

$$-3y - 6z = 0 \rightarrow -3y = 6z = 6t$$
$$\rightarrow y = -2t.$$

$$x = 0.$$

$$\therefore \quad \underset{\sim}{v_3} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} t, \quad t \in \mathbb{R}, \quad t \neq 0.$$

and in particular $\boxed{\underset{\sim}{v_3} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}}$ will do.

c) The system is $\underset{\sim}{y}' = A\underset{\sim}{y}$. Hence the general solution is

$$\underset{\sim}{y} = c_1 e^{t} \begin{pmatrix} 15 \\ 8 \\ -16 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} + c_3 e^{7t} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

(5)

Question 2 June 2014 Math 2019.

2 i) $\frac{d}{dt}\int_1^{t^2} \frac{\sin(\sqrt{x})}{x}\,dx$

$= \int_1^{t^2} \frac{\partial}{\partial t} \frac{\sin(\sqrt{x})}{x}\,dx + \frac{\sin\sqrt{t^2}}{t^2}(2t) - 0$

$= \boxed{\frac{2\sin|t|}{t}}$.

ii) a) $\operatorname{div} G = \frac{\partial}{\partial x}(yz^2) + \frac{\partial}{\partial y}(xz^2) + \frac{\partial}{\partial z}(2xyz)$

$= 0 + 0 + 2xy$

$= \boxed{2xy}$.

b) $\nabla \times G = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xz^2 & 2xyz \end{vmatrix}$

$= \underset{\sim}{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^2 & 2xyz \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ yz^2 & 2xyz \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ yz^2 & xz^2 \end{vmatrix}$

$= \underset{\sim}{i}(2xz - 2xz) - \underset{\sim}{j}(2yz - 2yz) + \underset{\sim}{k}(z^2 - z^2)$

$= \boxed{\underset{\sim}{0}}$

(iii) $\frac{\partial f}{\partial x} = 3x^2 - 12y = 0 \qquad (I)$

$\frac{\partial f}{\partial y} = -12x + 24y^2 = 0. \qquad (II)$

$(II) \rightarrow \quad 12x = 24y^2 \rightarrow x = 2y^2.$

Sub in (I): $3(2y^2)^2 - 12y = 0$

$\rightarrow 3(4y^4) - 12y = 0$

$\rightarrow 12y^4 - 12y = 0$

$\rightarrow y^4 - y = 0$

$\rightarrow y(y^3 - 1) = 0 \rightarrow y = 0, 1.$

$y = 0 \rightarrow x = 2(0)^2 = 0 \rightarrow (0,0)$

$y = 1 \rightarrow x = 2(1)^2 = 2 \rightarrow (2,1).$

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = (6x)(48y) - (-12)^2$

$= 288xy - 144.$

$D(0,0) = -144 \rightarrow (0,0)$ is a saddle point.

$D(2,1) = 288(2) - 144 > 0 \rightarrow$ max or min

$f_{xx}(2,1) = 6(2) = 12 > 0 \quad \cup \rightarrow (2,1)$ is a local m.

We therefore have a saddle point at $\boxed{(0,0,0)}$

and a local minimum at $\boxed{(2,1,-8)}$

iv) A vector in the direction of the line is

$$\underset{\sim}{v} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

A parametric vector equation of the interval is then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} t \qquad t : 0 \to 1.$$

Hence
$$x = 1+t \to dx = dt$$
$$y = 2 \to dy = 0$$
$$z = 3+2t \to dz = 2dt.$$

$$\text{Work done} = \int_C \underset{\sim}{F} . dr = \int_C \begin{pmatrix} -y \\ xyz \\ x^2 \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

$$= \int_0^1 -y\,dx + \cancel{xyz\,dy} + x^2\,dz$$

$$= \int_0^1 (-2)(dt) + (1+t)^2\, 2\,dt$$

$$= \int_0^1 -2+2(1+t)^2\, dt$$

$$= \int_0^1 -2+2(1+2t+t^2)\, dt = \int_0^1 4t+2t^2\, dt$$

$$= \left[2t^2 + \tfrac{2}{3}t^3\right]_0^1$$

$$= 2 + \tfrac{2}{3} = \boxed{\tfrac{8}{3}}$$

v) Suppose $\phi(x, y, z) = 1$.

Then $\frac{d}{dt}\left(\phi(x,y,z)\right) = \frac{d}{dt}(1)$

$$\rightarrow \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \phi}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial \phi}{\partial z}\cdot\frac{\partial z}{\partial t} = 0.$$

$$\rightarrow \begin{pmatrix} \phi_x \\ \phi_y \\ \phi_z \end{pmatrix} \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = 0$$

$$\rightarrow (\text{grad}\, \phi) \cdot (\dot{\underset{\sim}{r}}(t)) = 0.$$

$$\rightarrow \text{grad}\, \phi \perp \dot{\underset{\sim}{r}}(t)$$

Question 3 June 2014 Math2019.

3 i) Since we are measuring to the nearest mm,

$|\Delta r| < 0.05$ cm and $|\Delta h| < 0.05$ cm.

Now $$\Delta V = \frac{\partial V}{\partial h}\Delta h + \frac{\partial V}{\partial r}\Delta r$$

$$= \frac{1}{3}\pi r^2 \Delta h + \frac{2}{3}\pi r h \Delta r.$$

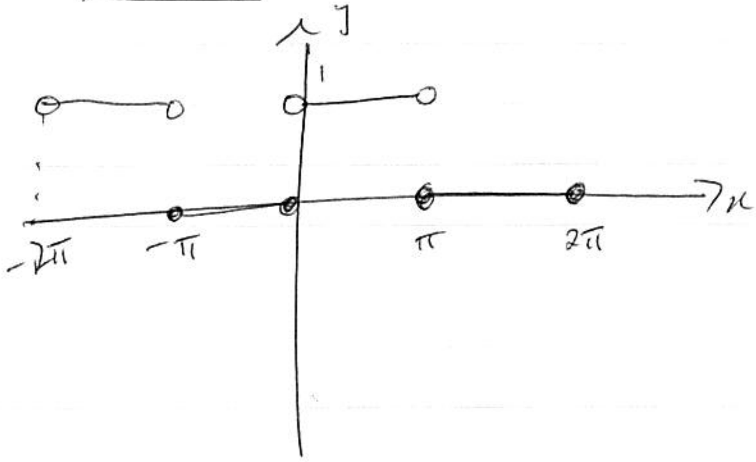
Hence

$$|\Delta V| \leq \frac{1}{3}\pi r^2 |\Delta h| + \frac{2}{3}\pi r h |\Delta r|$$

$$= \frac{1}{3}\pi (16)(0.05) + \frac{2}{3}\pi (4)(3)(0.05)$$

$$\cong \boxed{2.094 \text{ cm}^3}.$$

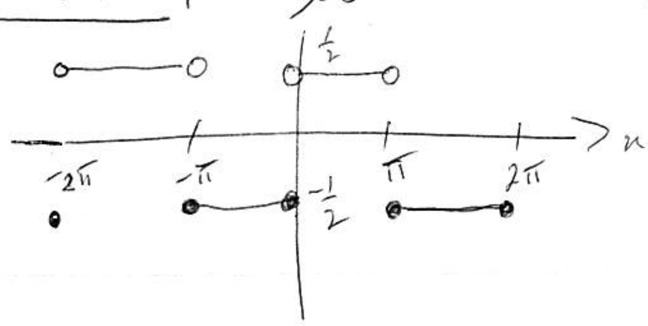
ii) b)



a) $\boxed{T = 2\pi}$

$L = \pi$

c)



(10)

d) Noting that $f(x)-\frac{1}{2}$ is an odd function, let $g(x)=f(x)-\frac{1}{2}$ and find the Fourier series of g.

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\pi}\right)$$

$$= \sum_{n=1}^{\infty} b_n \sin(nx).$$

$$b_n = \frac{1}{L}\int_{-L}^{L} g(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{g(x)}_{\text{odd}}\,\overbrace{\sin(nx)}^{\text{odd}}\,dx.$$ odd × odd = even

$$= \frac{2}{\pi}\int_{0}^{\pi} g(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \frac{1}{2}\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\cos(n\pi)+\frac{1}{n}\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}(-1)^n+\frac{1}{n}\right\}$$

$$= \frac{1}{n\pi}\left\{1-(-1)^n\right\}.$$

Thus $g(x) = \sum_{n=1}^{\infty} \frac{1}{n\pi}(1-(-1)^n)\sin(nx)$

$\rightarrow f(x) - \frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{n\pi}(1-(-1)^n)\sin(nx)$

$\rightarrow f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}(1-(-1)^n)\sin(nx)$

$= \frac{1}{2} + \frac{1}{\pi} \times 2\sin x + \frac{1}{3\pi} \cdot 2 \cdot \sin 3x.$

$+ \frac{1}{5\pi} \cdot 2\sin(5x) + \dots$

$= \frac{1}{2} + \frac{2}{\pi}\left\{ \sin x + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \dots \right.$

e) At $x = \frac{\pi}{2}$ the series converges to $y = 1$.

$\therefore \quad 1 = \frac{1}{2} + \frac{2}{\pi}\left\{ \sin\frac{\pi}{2} + \frac{\sin(\frac{3\pi}{2})}{3} + \frac{\sin(\frac{5\pi}{2})}{5} + \dots \right.$

$\frac{1}{2} = \frac{2}{\pi}\{ 1 - \frac{1}{3} + \frac{1}{5} \dots \}$

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots$

$\pi = 4\{ 1 - \frac{1}{3} + \frac{1}{5} \dots \}.$

(iii) $\mathcal{L}(t + \sin(2t) + e^{-t}) = \boxed{\frac{1}{s^2} + \frac{2}{s^2+4} + \frac{1}{s+1}}$

iv) a) $$\frac{7s+1}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1} = \frac{A(s-1)+B(s+1)}{(s+1)(s-1)}.$$

$$\therefore \quad A(s-1) + B(s+1) = 7s+1.$$

$\underline{s=1}$: $\quad 2B = 8 \rightarrow B = 4$

$s=-1$: $\quad -2A = -6 \rightarrow A = 3$

$$\therefore \frac{7s+1}{(s+1)(s-1)} = \frac{3}{s+1} + \frac{4}{s-1}$$

So $$\mathcal{L}^{-1}\left(\frac{7s+1}{(s+1)(s-1)}\right) = \mathcal{L}^{-1}\left(\frac{3}{s+1} + \frac{4}{s-1}\right)$$

$$= 3e^{-t} + 4e^{t}$$

b) Using the second shifting theorem:

$$\mathcal{L}^{-1}\left(\frac{7s+1}{(s+1)(s-1)} \times e^{-5s}\right) = u(t-5)\{3e^{-(t-5)} + 4e^{(t-5)}$$

Question 4 June 2014 Math2019.

4 i) $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \rightarrow 1 = 2x\lambda$ I

$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \rightarrow 1 = 2y\lambda$ II

Also $x^2+y^2=1$ III

So $2x\lambda = 2y\lambda \rightarrow x = y$ $(\lambda \neq 0)$

$\therefore x^2+x^2=1 \rightarrow 2x^2=1 \rightarrow x = \pm\frac{1}{\sqrt{2}}$

$x = \frac{1}{\sqrt{2}} \rightarrow y = \frac{1}{\sqrt{2}} \rightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$x = -\frac{1}{\sqrt{2}} \rightarrow y = -\frac{1}{\sqrt{2}} \rightarrow \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

At $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $x+y = \frac{2}{\sqrt{2}} = \boxed{\sqrt{2}}$ (max)

At $\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$ $x+y = -\frac{2}{\sqrt{2}} = \boxed{-\sqrt{2}}$ (min)

ii) a) Assume $u(x,t) = F(x)G(t)$.

$$\frac{\partial u}{\partial t} = F(x)\frac{dG}{dt}$$

$$\frac{\partial u}{\partial x^2} = \frac{d^2F}{dx^2}G(t)$$

Thus $F(x)\frac{dG}{dt} = \frac{d^2F}{dx^2} G(t)$

$$\rightarrow \frac{dG}{dt}\Big/ G = \frac{\frac{d^2F}{dx^2}}{F(x)} = k \quad \text{(say)}$$

b) $\quad u(0,t)=0 \rightarrow F(0)G(t)=0 \rightarrow F(0)=0$

$u(\pi,t)=0 \rightarrow F(\pi)G(t)=0 \rightarrow F(\pi)=0.$

Also $\frac{d^2F}{dx^2} = kF \rightarrow F'' - kF = 0.$

We are given that $k = -p^2$

Thus $\ddot{F} + p^2F = 0.$

Aux. Eqn : $m^2+p^2=0 \rightarrow m^2 = -p^2$

$\rightarrow m = 0 \pm ip.$

So $F(x) = A\cos(px) + B\sin(px)$

$F(0)=0 \rightarrow \quad 0 = A.1 + B.0$

$\rightarrow A = 0.$

$\therefore F(x) = B\sin(px)$

$F(\pi)=0 \rightarrow B\sin(p\pi)=0 \qquad (B \neq 0)$

$\rightarrow p = n \qquad n = 1, 2, 3, \ldots$

We then have $F_n(x) = B_n \sin(nx)$.

c) $G' = kG \rightarrow G' = -p^2 G$

$$\rightarrow G(t) = e^{-p^2 t}$$

$$\rightarrow G_n(t) = e^{-n^2 t}$$

Hence $u_n(x,t) = F_n(x) G_n(t)$

$$= B_n \sin(nx) e^{-n^2 t}$$

are possible solutions.

d) Taking $u(x,t) = u_1(x,t) + u_2(x,t)$

$$= B_1 \sin x e^{-t} + B_2 \sin(2x) e^{-4t}$$

$u(x,0) = B_1 \sin x + B_2 \sin(2x)$

$$\equiv 2\sin x - 16 \sin 2x.$$

$$\longrightarrow B_1 = 2 \text{ and } B_2 = -16.$$

Therefore $u(x,t) = 2\sin(x) e^{-t} - 16 \sin 2x \, e^{-4t}$

e) When $t = \ln(2)$

$$u(x,t) = 2\sin(x)e^{-\ln 2} - 16\sin(2x)e^{-4\ln}$$

$$= 2\sin(x)\, e^{\ln 2^{-1}} - 16\sin(2x)\, e^{\ln(2^{-4})}$$

$$= 2\sin(x)\left(\tfrac{1}{2}\right) - 16\sin(2x)\left(\tfrac{1}{16}\right)$$

$$= \sin x - \sin 2x = 0.$$

Thus $\sin x = \sin 2x$.

$$\sin x = 2\sin x\cos x.$$

$\swarrow \qquad \searrow$

$\sin x = 0$
$\rightarrow x = 0, \pi$

$1 = 2\cos x$
$\rightarrow \cos(x) = \frac{1}{2}$
$\rightarrow x = \pi/3$ m.

Temperature is zero at $\boxed{x = \pi/3}$ m as well as the endpoints.

(iii) LHS $= \frac{1}{L}\int_{-L}^{L} f(x)^2\, dx$

$$= \frac{1}{L}\int_{-L}^{L} f(x)\, f(x)\, dx.$$

$$= \frac{1}{L}\int_{-L}^{L} f(x)\left\{ a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right\} dx$$

$$= \frac{1}{L}\int_{-L}^{L} f(x)\, a_0\, dx + \frac{1}{L}\int_{-L}^{L} \sum_{n=1}^{\infty} a_n f(x) \cos\left(\frac{n\pi x}{L}\right) + b_n f(x) \sin\left(\frac{n\pi x}{L}\right)$$

$$= a_0 \frac{1}{L}\int_{-L}^{L} f(x)\, dx + \sum_{n=1}^{\infty} a_n \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) + b_n \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)$$

$$= a_0(2a_0) + \sum_{n=1}^{\infty} a_n(a_n) + b_n(b_n)$$

$$= 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

= RHS.