THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

NOVEMBER 2018

MATH2018 / MATH2019
ENGINEERING MATHEMATICS 2D/E

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

TABLE OF LAPLACE TRANSFORMS AND THEOREMS

$g(t)$ is a function defined for all $t \geq 0$, and whose Laplace transform

$$G(s) = \mathcal{L}(g(t)) = \int_0^\infty e^{-st} g(t) dt$$

exists. The Heaviside step function u is defined to be

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ \frac{1}{2} & \text{for } t = a \\ 1 & \text{for } t > a \end{cases}$$

$g(t)$	$G(s) = \mathcal{L}[g(t)]$
1	$\dfrac{1}{s}$
t	$\dfrac{1}{s^2}$
$t^\nu, \nu > -1$	$\dfrac{\nu!}{s^{\nu+1}}$
$e^{-\alpha t}$	$\dfrac{1}{s+\alpha}$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$e^{-\alpha t} f(t)$	$F(s+\alpha)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$tf(t)$	$-F'(s)$

FOURIER SERIES

If $f(x)$ has period $T = 2L$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{aligned}$$

LEIBNIZ RULE FOR DIFFERENTIATING INTEGRALS

$$\frac{d}{dx}\int_u^v f(x,t)dt = \int_u^v \frac{\partial f}{\partial x}dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx}.$$

MULTIVARIABLE TAYLOR SERIES

$$\begin{aligned} f(x,y) &= f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\ &\quad + \frac{1}{2!}\left((x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) + (y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right) + \cdots \end{aligned}$$

VARIATION OF PARAMETERS

Suppose that the second order differential equation

$$y'' + p(x)y' + q(x)y = f(x)$$

has homogeneous solution $y_h = Ay_1(x) + By_2(x)$. Then a particular solution is given by

$$y_P(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx$$

where $W(x) = \det\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}$.

SOME BASIC INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^{kx} dx = \frac{e^{kx}}{k} + C$$

$$\int a^x dx = \frac{1}{\ln a} a^x + C \quad \text{for } a \neq 1$$

$$\int \sin\, kx\, dx = -\frac{\cos kx}{k} + C$$

$$\int \cos kx\, dx = \frac{\sin kx}{k} + C$$

$$\int \sec^2 kx\, dx = \frac{\tan kx}{k} + C$$

$$\int \operatorname{cosec}^2 kx\, dx = -\frac{1}{k}\cot kx + C$$

$$\int \tan kx\, dx = \frac{\ln|\sec kx|}{k} + C$$

$$\int \sec kx\, dx = \frac{1}{k}(\ln|\sec kx + \tan kx|) + C$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx$$

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx$$

Answer question 1 in a separate book

1. i) Suppose that the temperature T, at a point (x, y, z) in space is given by

$$T(x, y, z) = z - x^2 - y^2.$$

a) Sketch the level surface of all points with a temperature of zero.

b) Find $\text{grad}(T)$.

c) Calculate the rate of change of the temperature T at the point $P(1, 1, 0)$ in the direction of the vector $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$.

ii) Use the method of Lagrange multipliers to find the extreme values of

$$f(x, y) = 12 + 3x + 4y$$

subject to the constraint

$$g(x, y) = x^2 + y^2 - 1 = 0.$$

iii) The volume V of a circular cylinder with radius r and perpendicular height h is given by $V = \pi r^2 h$. Use a linear approximation to estimate the maximum percentage error in calculating V given that $r = 30$ metres and $h = 20$ metres, to the nearest metre.

iv) You are given that

$$\int_0^\infty \frac{1}{\alpha^2 + x^2}\,dx = \frac{\pi}{2}\alpha^{-1}.$$

Use Leibniz' theorem to find the following integral in terms of α

$$\int_0^\infty \frac{1}{(\alpha^2 + x^2)^2}\,dx.$$

v) Let $f(x) = \begin{cases} 0 & 0 \le x \le \dfrac{\pi}{2}\ ; \\ 1 & \dfrac{\pi}{2} < x \le \pi\ . \end{cases}$

a) Sketch the odd periodic extension of f over the domain $-\pi \le x \le \pi$.

b) Calculate the half range Fourier sine series of f.

c) To what value does the series in b) converge at $x = \dfrac{\pi}{2}$?

Answer question 2 in a separate book

2. i) Consider the following differential equation describing a vibrating system:

$$\frac{d^2y}{dt^2} + 4y = 8\cos(2\pi\omega t).$$

a) Find the solution y_h to the homogeneous equation.

b) For which value(s) of ω will the system exhibit resonance? Give reasons for your answer.
(Note that you are not being asked to find the particular solution y_p)

ii) Consider the double integral

$$I = \int_0^4 \int_{\sqrt{x}}^2 10x \; dydx.$$

a) Sketch the region of integration.

b) Evaluate I with the order of integration reversed.

iii) A quadratic curve is given by the equation $7x^2 + 6xy + 7y^2 = 200$.

a) Express the curve in the form

$$\mathbf{x}^T A\mathbf{x} = 200$$

where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, and A is a 2×2 symmetric matrix.

b) Find the eigenvalues and eigenvectors of the matrix A in part a).

c) Hence, or otherwise, find the shortest distance between the curve and the origin.

iv) Use the substitution $v = \dfrac{y}{x}$ to solve

$$xy' = y + 2x^3 \cos^2\left(\frac{y}{x}\right).$$

Answer question 3 in a separate book

3. i) Let Ω be the semi-circular region bounded by $y = \sqrt{1 - x^2}$ and $y = 0$. The region Ω is of uniform density and has centroid $(\bar{x}, \bar{y})$.

a) Sketch the region Ω and write down its area.

b) Explain why $\bar{x} = 0$.

c) Find $\bar{y}$ by evaluating an appropriate double integral expressed in polar coordinates.

ii) Find

a) $\mathcal{L}\{\sin(3t)\}$,

b) $\mathcal{L}\{e^{-7t}\sin(3t)\}$,

c) $\mathcal{L}^{-1}\left\{\dfrac{4s - 28}{(s-1)(s-9)}\right\}$.

iii) The function $g(t)$ is defined for $t \geq 0$ by

$$g(t) = \begin{cases} t^2, & 0 \leq t < 1, \\ e^{2t}, & t \geq 1. \end{cases}$$

a) Express $g(t)$ in terms of the Heaviside function.

b) Hence, or otherwise, show that the Laplace transform of $g(t)$ is

$$G(s) = \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) + \frac{e^{2-s}}{s-2}$$

Answer question 4 in a separate book

4. i) Consider the vector field

$$\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + (2xyz + 3)\mathbf{k}.$$

a) Calculate div($\mathbf{F}$).

b) Show that $\mathbf{F}$ is conservative by evaluating curl($\mathbf{F}$).

c) The path $\mathcal{C}$ in $\mathbb{R}^3$ starts at the point $(3, 4, 7)$ and subsequently travels anticlockwise four complete revolutions around the circle $x^2+y^2 = 25$ within the plane $z = 7$, returning to the starting point $(3, 4, 7)$. Using part b) or otherwise, evaluate the work integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

ii) A vibrating string of length π metres satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 25\frac{\partial^2 u}{\partial x^2},$$

where $u(x, t)$ is the transverse displacement of the string, at position x and time t. The ends of the string are held fixed so that

$$u(0, t) = u(\pi, t) = 0, \quad \text{for all time } t.$$

a) Assuming a solution of the form $u(x, t) = F(x)G(t)$, show that

$$\frac{1}{25G}\frac{d^2G}{dt^2} = \frac{1}{F}\frac{d^2F}{dx^2} = k$$

for some constant k.

b) You may assume that only $k < 0$ yields non-trivial solutions and set $k = -(p^2)$ for some $p > 0$.
Applying the boundary conditions, show that $p = n,\ n = 1, 2, 3, \ldots$ and that possible solutions for $F(x)$ are

$$F_n(x) = B_n \sin(nx)$$

where B_n are constants.

c) Find all possible solutions $G_n(t)$ for $G(t)$.

d) If the initial displacement and velocity of the string are

$$u(x, 0) = 2\sin(x) - \sin(2x) \qquad \text{and} \qquad u_t(x, 0) = 0,$$

find the general solution $u(x, t)$.

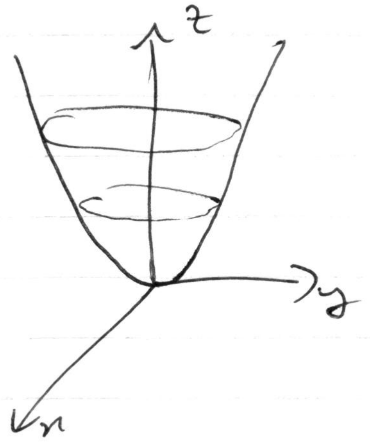
e) Hence determine the maximal transverse displacement of the string at time $t = \dfrac{\pi}{15}$ seconds.

END OF EXAMINATION

Math2019/2018 November 2018 Final Exam

Solutions

1 i) a) $z - x^2 - y^2 = 0 \Rightarrow z = x^2 + y^2$



Paraboloid.

b) $\text{grad}(T) = \frac{\partial T}{\partial x}\underset{\sim}{i} + \frac{\partial T}{\partial y}\underset{\sim}{j} + \frac{\partial T}{\partial z}\underline{k}$

$= -2x\underset{\sim}{i} - 2y\underset{\sim}{j} + \underset{\sim}{k} = \begin{pmatrix} -2x \\ -2y \\ 1 \end{pmatrix}$

c) at $P(1,1,0)$ we have $\text{grad}(T) = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$.

$|\underset{\sim}{b}| = \sqrt{9+16+144} = 13 \Rightarrow \hat{\underset{\sim}{b}} = \frac{1}{13}\begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$

$dd = \text{grad } T . \hat{\underset{\sim}{b}} = \frac{1}{13}\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$

$= \frac{1}{13}(-6 - 8 + 12) = \boxed{-\frac{2}{13}}$

(2)

ii) $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \Rightarrow 3 = \lambda 2x$

$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \Rightarrow 4 = \lambda 2y$

Now $\begin{array}{l} 3y = 2xy\lambda \\ 4x = 2xy\lambda \end{array} \Rightarrow 3y = 4x$

$\Rightarrow y = \frac{4}{3}x.$

Finally $x^2+y^2=1 \Rightarrow x^2 + \frac{16}{9}x^2 = 1$

$\Rightarrow 9x^2 + 16x^2 = 9 \Rightarrow 25x^2 = 9$

$\Rightarrow x^2 = \frac{9}{25} \Rightarrow x = \pm 3/5.$

$x = 3/5 \Rightarrow y = \frac{4}{3} \cdot \frac{3}{5} = 4/5$

$x = -3/5 \Rightarrow y = -4/5.$

So $(x,y) = (3/5, \frac{4}{5})$ or $(-3/5, -\frac{4}{5})$

$f(3/5, 4/5) = 12 + \frac{9}{5} + \frac{16}{5} = \boxed{17}$ (max)

$f(-3/5, -4/5) = 12 - \frac{9}{5} - \frac{16}{5} = \boxed{7}$ (min).

(iii) $\Delta V = \frac{\partial V}{\partial h}\Delta h + \frac{\partial V}{\partial r}\Delta r$

$= \pi r^2 \Delta h + 2\pi r h \Delta r$

$\left|\frac{\Delta V}{V}\right| \le \left|\frac{\pi r^2}{\pi r^2 h}\right| |\Delta h| + \left|\frac{2\pi r h}{\pi r^2 h}\right| |\Delta r|$

$$= \left|\frac{\Delta h}{h}\right| + 2\left|\frac{\Delta r}{r}\right|$$ ③

Now $h = 20$, $r = 30$, $|\Delta h| \le \frac{1}{2}$, $|\Delta r| \le \frac{1}{2}$

$$\therefore \left|\frac{\Delta V}{V}\right| \le \frac{\frac{1}{2}}{20} + \frac{2\,\frac{1}{2}}{30} = \frac{1}{40} + \frac{1}{30} = \frac{7}{120}$$

$$\therefore |\Delta V|\% \le \frac{700}{120} = \boxed{5\tfrac{5}{6}\,\%.}$$

iv)
$$\int_0^\infty \frac{dx}{\alpha^2+x^2} = \frac{\pi}{2}\alpha^{-1}$$

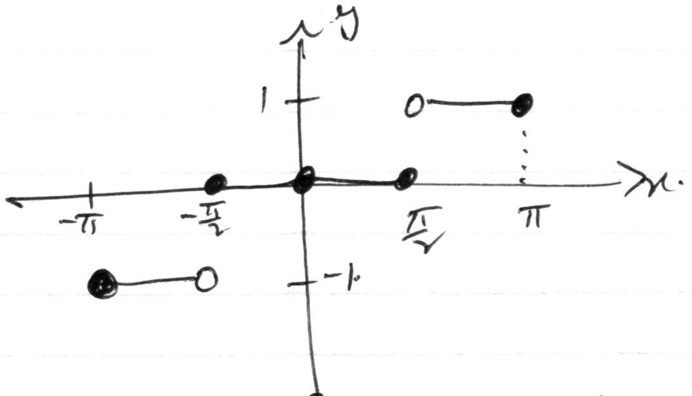
$$\Rightarrow \frac{d}{d\alpha}\int_0^\infty \frac{dx}{\alpha^2+x^2} = -\frac{\pi}{2}\alpha^{-2}$$

$$\Rightarrow \int_0^\infty \frac{\partial}{\partial\alpha}\left(\frac{1}{\alpha^2+x^2}\right)dx = -\frac{\pi}{2}\alpha^{-2}$$

$$\Rightarrow \int_0^\infty \frac{0 - 2\alpha\,dx}{(\alpha^2+x^2)^2} = -\frac{\pi}{2}\alpha^{-2}$$

$$\Rightarrow \int_0^\infty \frac{dx}{(\alpha^2+x^2)^2} = \boxed{\frac{\pi}{4}\alpha^{-3}}.$$

v) a)



$T = 2\pi$
$L = \pi$

b) $$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

$\sim$odd $\sim$odd

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \sin nx\,dx.$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\cos n\pi + \frac{1}{n}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{n\pi}\left\{\cos\frac{n\pi}{2} - (-1)^n\right\}.$$

c) Splitting the difference: $\frac{1}{2}$.

2 i) Aux equ^n: $m^2+4=0 \Rightarrow m^2=-4 \Rightarrow m=0\pm 2i$

$\therefore \quad y_h = e^{0t}\{A\cos 2t + B\sin 2t\}$

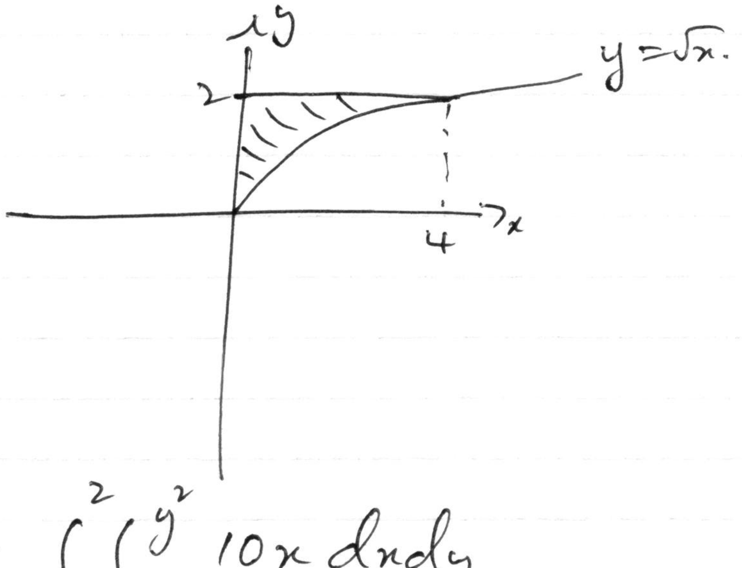
$= A\cos 2t + B\sin 2t$

b) For resonance we require

$$2 = 2\pi\omega$$

$$\Rightarrow \boxed{\omega = \frac{1}{\pi}}$$

ii) a)



b) $I = \int_0^2 \int_0^{y^2} 10x \, dx\, dy$

$= \int_0^2 \left[5x^2\right]_0^{y^2} dy$

$= \int_0^2 5y^4\, dy = \left[y^5\right]_0^2 = 2^5 = \boxed{32}$

(iii) a) $$(x\ y)\begin{pmatrix}7 & 3\\ 3 & 7\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=200$$

b) $$\begin{vmatrix}7-\lambda & 3\\ 3 & 7-\lambda\end{vmatrix}=0$$

$$\Rightarrow (7-\lambda)^2-9=0 \Rightarrow 7-\lambda=\pm 3$$

$$\Rightarrow \lambda = 7\mp 3 = 10,\ 4$$

check $10+4=7+7$ ✓

$\underline{\underline{\lambda=4}}$: $$\left(\begin{array}{cc|c}3 & 3 & 0\\ 3 & 3 & 0\end{array}\right) \Rightarrow \left(\begin{array}{cc|c}3 & 3 & 0\\ 0 & 0 & 0\end{array}\right)$$

Let $y=t$, $x=-t$ $\Rightarrow \begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-1\\ 1\end{pmatrix}t$

e-vec: $\begin{pmatrix}-1\\ 1\end{pmatrix}$, unit e-vec $\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ 1\end{pmatrix}$.

$\lambda=10$: $$\left(\begin{array}{cc|c}-3 & 3 & 0\\ 3 & -3 & 0\end{array}\right) \Rightarrow \left(\begin{array}{cc|c}-3 & 3 & 0\\ 0 & 0 & 0\end{array}\right)$$

Let $y=t$, $x=t$ $\Rightarrow \begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}t$

e-vec: $\begin{pmatrix}1\\ 1\end{pmatrix}$, unit e-vec $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$.

c) Choosing princ. axes $X=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ 1\end{pmatrix}$, $Y=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$

the eq^n becomes $4X^2+10Y^2=200$.

$\therefore$ Shortest distance: $\sqrt{\frac{200}{10}}=\boxed{\sqrt{20}}$ at $\frac{\pm\sqrt{20}}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$.

7

iv) $v = y/x \Rightarrow y = xv \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}.$

sub in:

$$x\left(v + x\frac{dv}{dx}\right) = xv + 2x^3\cos^2(v)$$

$$v + x\frac{dv}{dx} = v + 2x^2\cos^2(v)$$

$$x\frac{dv}{dx} = 2x^2\cos^2(v)$$

$$\frac{dv}{dx} = 2x\cos^2(v)$$

$$\Rightarrow dv(\sec^2 v) = 2x\,dx.$$

$$\Rightarrow \int \sec^2 v\,dv = \int 2x\,dx.$$

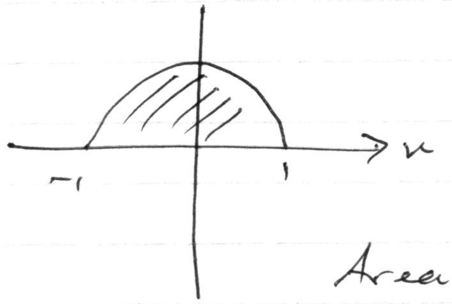
$$\Rightarrow \tan v = x^2 + c$$

$$\Rightarrow \tan(y/x) = x^2 + c$$

$$\Rightarrow y/x = \tan^{-1}(x^2+c)$$

$$\Rightarrow \boxed{y = x\tan^{-1}(x^2+c)}$$

3 i) a)



Area $= \frac{1}{2}\pi r^2 = \frac{1}{2}\pi$

b) Since the density is uniform and the semicircle is symmetric about the y-axis, $\bar{x} = 0$.

c) $\iint_{\Omega} y\,\delta(x,y)\,dA = \iint_{\Omega} r\sin\theta\,(1)\,r\,dr\,d\theta$

$= \int_0^{\pi}\int_0^1 r^2\sin\theta\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{3}r^3\sin\theta\right]_0^1 d\theta$

$= \int_0^{\pi}\frac{1}{3}\sin\theta\,d\theta = \left[-\frac{1}{3}\cos\theta\right]_0^{\pi}$

$= -\frac{1}{3}(-1) + \frac{1}{3} = 2/3$

$\therefore \bar{y} = \dfrac{2/3}{\pi/2} = \boxed{\dfrac{4}{3\pi}}$

(ii)a) Using the table $\mathcal{L}(\sin 3t) = \boxed{\dfrac{3}{s^2+9}}$.

b) $\mathcal{L}(e^{-at}f(t)) = (\mathcal{L}f)(s+a)$

$\Rightarrow \mathcal{L}(e^{-7t}\sin 3t) = \dfrac{3}{(s+7)^2+9} = \dfrac{3}{s^2+14s+58}$

c) $$\frac{4s-28}{(s-1)(s-9)} = \frac{A}{s-1} + \frac{B}{s-9}$$

$$= \frac{A(s-9)+B(s-1)}{(s-1)(s-9)}.$$

$\therefore$ $A(s-9) + B(s-1) = 4s-28$

$s=9$: $8B = 36-28 \Rightarrow B = 1$

$s=1$: $-8A = -24 \Rightarrow A = 3.$

$$\therefore \quad \frac{4s-28}{(s-1)(s-9)} = \frac{3}{s-1} + \frac{1}{s-9}$$

So $$\mathcal{L}^{-1}\left(\frac{4s-28}{(s-1)(s-9)}\right) = \boxed{3e^{t} + e^{9t}}.$$

$u(t-0)$

(iii) a) $g(t) = t^2(1 - u(t-1)) + e^{2t}u(t-1).$

$$= t^2 - t^2u(t-1) + e^{2t}u(t-1)$$

b) Suppose $t^2 = \alpha(t-1)^2 + \beta(t-1) + \gamma$

Let $w = t-1$

$$(w+1)^2 = \alpha w^2 + \beta w + \gamma$$
$$= w^2 + 2w + 1$$

$\therefore$ $\alpha = 1,\ \beta = 2,\ \gamma = 1.$

So $\quad t^2 = (t-1)^2 + 2(t-1) + 1.$

$\therefore \mathcal{L}(g(t)) = \mathcal{L}\left(t^2 - ((t-1)^2 + 2(t-1) + 1)u(t-1) + e^{2t}u(t-1)\right)$

$\mathcal{L}(t^2) = \frac{2}{s^3}$

$\mathcal{L}((t-1)^2 u(t-1)) = \frac{2e^{-s}}{s^3}$

$\mathcal{L}(2(t-1)u(t-1)) = \frac{2e^{-s}}{s^2}$

$\mathcal{L}(u(t-1)) = e^{-s}/s$

$$\mathcal{L}(e^{2t}u(t-1)) = \mathcal{L}(e^{2(t-1)}u(t-1)e^2)$$

$$= e^2 \frac{1}{(s-2)} e^{-s} \qquad (\mathcal{L}(e^{2t}) = \tfrac{1}{s-2})$$

$$= \frac{e^{2-s}}{s-2}$$

$$\therefore \mathcal{L}(g(t)) = 2/s^3 - \frac{2e^{-s}}{s^3} - \frac{2e^{-s}}{s^2} - \frac{e^{-s}}{s} + \frac{e^{2-s}}{s-2}$$

$$= 2/s^3 - e^{-s}\left\{\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right\} + \frac{e^{2-s}}{s-2}$$

as required

4(i) a) $\operatorname{div}(\underset{\sim}{F}) = 0 + 0 + 2xy = \boxed{2xy}$

b) $\operatorname{curl}(\underset{\sim}{F}) = \begin{vmatrix} \overset{\oplus}{i} & \overset{\ominus}{j} & \overset{\oplus}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xz^2 & 2xyz+3 \end{vmatrix}$

$$= i \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^2 & 2xyz+3 \end{vmatrix} - j \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ yz^2 & 2xyz+3 \end{vmatrix} + k \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ yz^2 & xz^2 \end{vmatrix}$$

$$= i(2xz - 2xy) - j(2yz - 2yz) + \underset{\sim}{k}(z^2 - z^2)$$

$$= \boxed{\underline{0}.}$$

c) Since the curve is closed $\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \boxed{0}$.

ii) a) Assume $u = F(x)G(t)$

$$\therefore \quad FG'' = 25F''G$$

$$\Rightarrow \frac{G''}{25G} = \frac{F''}{F} = k \quad (\text{say})$$

b) $u(0,t) = 0 \Rightarrow F(0)G(t) = 0 \Rightarrow F(0) = 0$
$u(\pi,t) = 0 \Rightarrow F(\pi)G(t) = 0 \Rightarrow F(\pi) = 0.$

Let $k = -(p^2)$

$$F'' = kF \Rightarrow F'' = -p^2F$$

$$\Rightarrow F'' + p^2F = 0.$$

Aux equ^n $m^2+p^2=0$

$\Rightarrow m = 0 \pm ip$

$\Rightarrow F(x) = A\cos px + B\sin px.$

$F(0)=0 \Rightarrow \quad A + 0B = 0 \Rightarrow A = 0$

$\therefore F(x) = B\sin(px)$

$F(\pi)=0 \Rightarrow B\sin(p\pi)=0 \Rightarrow p = 1, 2, 3, \dots$

$\Rightarrow p = n \quad n = 1, 2, 3, \dots$

$\therefore F_n(x) = B_n \sin(nx).$

c) $G'' = 25kG = -25p^2 G = -25n^2 G$

So $G'' + 25n^2 G = 0$

Aux equ^n $m^2 + 25n^2 = 0$

$\Rightarrow m = 0 \pm 5ni$

$\therefore G(t) = C\cos(5nt) + D\sin(5nt).$

d) $u_t(x,0) = 0 \Rightarrow G'(0) = 0$

$G'(t) = -5nC\sin(5nt) + 5nD\cos(5nt)$

$G'(0) = -5nC(0) + 5nD(1) = 0$

$\Rightarrow D = 0$

$\therefore\ G_n(t) = C_n \cos(5nt)$ (set $C_n = 1$)

General solution:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx)\cos(5nt)$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin(nx)$$

$$= 2\sin x - \sin 2x.$$

$\therefore\ B_1 = 2,\ B_2 = -1$, all other B_n's $= 0$.

So $u(x,t) = \boxed{2\sin x\cos 5t - \sin 2x\cos 10t.}$

c) when $t = \frac{\pi}{15}$

$$u\left(x, \tfrac{\pi}{15}\right) = 2\sin x\cos \pi/3 - \sin 2x\cos\left(\tfrac{10\pi}{15}\right)$$

$$= 2\sin x\left(\tfrac{1}{2}\right) - \sin 2x\left(-\tfrac{1}{2}\right)$$

$$= \sin x + \tfrac{1}{2}\sin 2x. = W(x) \text{ say.}$$

$W' = \cos(x) + \cos 2x = 0$ (for max/min)

$$\therefore\ \cos(2x) = -\cos x$$
$$2\cos^2 x - 1 = -\cos x$$
$$2\cos^2 x + \cos x - 1 = 0$$

$$\Rightarrow (2\cos x - 1)(\cos x + 1) = 0$$

$$\Rightarrow \cos x = \frac{1}{2}, \quad \cos x = -1.$$

$$\Rightarrow x = \pi/3, \quad \pi$$

$$W(0) = W(\pi) = 0 \quad (\text{min}).$$

$$W(\pi/3) = \sin \pi/3 + \frac{1}{2} \sin 2\pi/3$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2} \frac{\sqrt{3}}{2}$$

$$= \boxed{\frac{3\sqrt{3}}{4}} = \underline{\underline{\text{max}}}.$$