MATH2019 PROBLEM CLASS
EXAMPLES 6
MATRICES

1991 & 1994

1. a) Find the eigenvalues and the corresponding eigenvectors of matrix

$$\mathsf{A} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{pmatrix}.$$

b) Find an orthogonal matrix P such that

$$\mathsf{D} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P}$$

is a diagonal matrix and write down the matrix D.

c) Using your results from parts a) and b) find the solution of the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 3x + 2y + 2z\,, \\ \frac{dy}{dt} &= 2x + 2y\,, \\ \frac{dz}{dt} &= 2x + 4z\,, \end{aligned}$$

subject to the conditions

$$x(0) = 0\,, \quad y(0) = 0 \quad \text{and} \quad z(0) = 1\,.$$

d) Express the quadric surface

$$3x^2 + 2y^2 + 4z^2 + 4xy + 4xz = 24$$

in terms of its principal axes X, Y and Z and write out an orthogonal matrix P such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathsf{P} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

What shape does this quadric surface represent?

e) A and P are $n \times n$ matrices. A is symmetric and P is orthogonal. Prove that $\mathsf{P}^{-1}\mathsf{A}\mathsf{P}$ is symmetric.

1998

2. Let

$$\mathsf{A} = \begin{pmatrix} -7 & 24 \\ 24 & 7 \end{pmatrix}$$

a) Find the eigenvalues and eigenvectors of A.

b) Normalise the eigenvectors to have length 1. Hence find an orthogonal matrix P such that

$$\mathsf{D} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P}$$

is a diagonal matrix. Evaluate both sides of this equation to show that it is satisfied by your P.

c) For the system of differential equations

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x} \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

show (or verify) that the transformation

$$\mathbf{x} = \mathsf{P}\mathbf{z} \quad \text{where} \quad \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

yields the equation

$$\frac{d\mathbf{z}}{dt} = \mathsf{D}\mathbf{z}$$

where P and D are as in part b).

d) Hence solve the system of equations

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x}$$

if $x_1(0) = 1$, $x_2(0) = 0$.

$\boxed{1999}$ 3. Let $\mathsf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors of A.

b) i) Find a matrix P such that $\mathsf{P}^{-1}\mathsf{A}\mathsf{P} = \mathsf{D}$ where D is a diagonal matrix.

ii) Calculate $\mathsf{P}^{-1}\mathsf{A}\mathsf{P}$ to check this is indeed equal to D.

c) If $\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $\mathbf{f} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$ show that with the definition $\mathbf{x} = \mathsf{P}\mathbf{z}$, the system of differential equations

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x} + \mathbf{f} \tag{1}$$

becomes

$$\frac{d\mathbf{z}}{dt} = \mathsf{D}\mathbf{z} + \mathsf{P}^{-1}\mathbf{f}\,.$$

d) Using the result of c) find the general solution of (1) in the case when $f_1(t) = e^{2t}$ and $f_2(t) = 0$.

$\boxed{2000}$ 4. Consider the quadric surface given by

$$x^2 + y^2 + 3z^2 + 4xz + 4yz = 5\,.$$

a) Express this equation in the form

$$\mathbf{v}^T\mathsf{A}\mathbf{v} = 5$$

where A is a real symmetric matrix and $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

b) Show that the matrix A has an eigenvalue $\lambda = 1$ and two other distinct eigenvalues. What are the values of these other eigenvalues?

c) Write down the equation of the quadric surface in terms of its principal axes X, Y and Z. Then sketch the surface relative to principal axes, clearly labelling the (X, Y, Z) coordinates of the points where the surface intersects the principal axes.

d) Find the eigenvectors of A and hence find an orthogonal matrix, P, which relates $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$. Write down this relationship.

e) Write down the points of intersection of the quadric surface with its principal axes in terms of the (x, y, z) coordinate system.

2014, S1 5. It is given that the matrix $\mathsf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{pmatrix}$ has an eigenvalue $\lambda_1 = 1$

with an associated eigenvector $\mathbf{v}_{\lambda=1} = \begin{pmatrix} 15 \\ 8 \\ -16 \end{pmatrix}$ and eigenvalue $\lambda_2 = 6$

with associated eigenvector $\mathbf{v}_{\lambda=6} = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$.

a) Without calculating the characteristic polynomial explain why the remaining eigenvalue is $\lambda_3 = 7$.

b) Find an eigenvector $\mathbf{v}_{\lambda=7}$ for the eigenvalue $\lambda_3 = 7$.

c) Hence write down the general solution to the system of differential equations

$$\begin{aligned} y_1' &= y_1 \\ y_2' &= -8y_1 + 4y_2 - 6y_3 \\ y_3' &= 8y_1 + y_2 + 9y_3 \end{aligned}$$

2014, S2 6. Let

$$\mathsf{A} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

a) Find the eigenvalues and eigenvectors of the matrix A.

b) By considering the eigenvalues of A, write the curve

$$2x^2 + 6xy + 2y^2 = 45$$

in terms of principle axes coordinates X and Y. Sketch the curve in the XY-plane.

c) Find the distance from the curve $2x^2 + 6xy + 2y^2 = 45$ to the origin.

2015, S1 7. The equations governing the response of a bridge to an earthquake are found to satisfy

$$\begin{aligned} \frac{dx}{dt} &= -x + ay, \\ \frac{dy}{dt} &= ax - y. \end{aligned}$$

where $a > 0$ is a parameter that depends on the material used for the bridge.

a) Express this set of differential equations in the form

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x}, \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

and find the eigenvalues and eigenvectors of the matrix A.

b) Hence, or otherwise, write down a general solution for the problem.

c) For what values of a will the solution grow with increasing t?

2015, S2

8. The matrix B is given by

$$\mathsf{B} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

a) Show that the vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

is an eigenvector of the matrix B and find the corresponding eigenvalue.

b) Given that the other two eigenvalues of B are -1 and 2, find the eigenvectors corresponding to these two eigenvalues.

2016, S1

9. Consider the curve in the xy-plane

$$x^2 - 6xy + y^2 = 16.$$

a) Rewrite the equation for the curve in the form

$$\begin{pmatrix} x \\ y \end{pmatrix}^T \mathsf{A} \begin{pmatrix} x \\ y \end{pmatrix} = 16$$

where A is a real symmetric 2×2 matrix. Find the eigenvalues and eigenvectors of A.

b) Write down the equation for the curve in terms of its principle axes X and Y. Hence find the closest distance from the origin to the curve.

c) Find the x and y coordinates of the points on the curve closest to the origin.

2016, S2

10. Consider the matrix $\mathsf{A} = \begin{pmatrix} 6 & 2 \\ -1 & 3 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors of A.

b) Hence solve the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 6x + 2y \\ \frac{dy}{dt} &= -x + 3y. \end{aligned}$$

2017, S1

11. Consider the set of differential equations

$$\begin{aligned} \frac{dx}{dt} &= -x + y, \\ \frac{dy}{dt} &= x - y, \end{aligned}$$

with initial conditions $x(0) = 1$, $y(0) = 0$.

a) Express this set of differential equations in the form

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x}, \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

and find the eigenvalues and eigenvectors of the matrix A.

b) Hence, or otherwise, write down the solution for the problem using the initial conditions.

2017, S2

12. A quadric curve is given by the equation $2x^2 + 4xy - y^2 = 1$.

a) Express the curve in the form

$$\mathbf{x}^T\mathsf{A}\mathbf{x} = 1, \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix},$$

and find the eigenvalues and eigenvectors of the matrix A.

b) Hence, or otherwise, find the distance from the curve to the origin. Write down the x and y coordinates of the points on the curve closest to the origin.

2018, S1

13. A **real symmetric** 3×3 matrix A has eigenvalues denoted by λ_1, λ_2 and λ_3. We define a quadric surface

$$\mathbf{x}^T\mathsf{A}\mathbf{x} = 12 \quad \text{where } \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

A student is given the following extra information about matrix A:

- $\text{trace}(\mathsf{A}) = 0$,
- $\lambda_1 = 2$ and $\lambda_3 = 4$ with associated eigenvectors, respectively,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

a) What is the value of the remaining eigenvalue, namely λ_2?

b) Write down the equation of the quadric surface, relative to the principal axes of the surface.

c) Write down a vector $\mathbf{v}_2$ that is orthogonal to **both** eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_3$.

d) What is the relationship between λ_2 and $\mathbf{v}_2$? Give reasons for your answer.

e) Hence determine an **orthogonal** matrix P which diagonalises the matrix A such that $\mathsf{P}^{-1}\mathsf{A}\mathsf{P} = \mathsf{D}$ where D is a 3×3 diagonal matrix.

f) Hence determine the matrix A.

2018, S2

14. A quadric curve is given by the equation $7x^2 + 6xy + 7y^2 = 200$.

i) Express the curve in the form

$$\mathbf{x}^T\mathsf{A}\mathbf{x} = 200$$

where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and A is a 2×2 real symmetric matrix.

ii) Find the eigenvalues and eigenvectors of the matrix A in part i).

iii) Hence, or otherwise, find the shortest distance between the curve and the origin.