LECTURE 50 and LECTURE 51 (Double Lecture)
WAVE EQUATION AND SEPARATION OF VARIABLES

The equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

is called the one-dimensional wave equation and governs the vibration of an elastic string. The solution $u(x,t)$ describes the displacement of the string at position x at time t.

The boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \qquad \text{for all time } t$$

specify that the string is fixed at two endpoints $x = 0$ and $x = L$ for all time.

Initial conditions take the form:

$$\text{initial deflection} \qquad u(x,0) = f(x)$$

$$\text{initial velocity} \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

To solve we implement the process of separation of variables:

Step 1. Apply the **method of separation of variables** to obtain two ordinary differential equations.
Step 2. Determine the solution of these two equations that satisfy the boundary conditions.
Step 3. Combine these solutions so that the result will be a solution of the p.d.e. which also satisfies the initial conditions.

Solutions (eigenfunctions) take the form

$$u_n(x,t) = (C_n \cos \lambda_n t + D_n \sin \lambda_n t) \sin \frac{n\pi x}{L} \quad \text{for} \quad n = 1, 2, 3, \cdots$$

where the λ_n (eigenvalues) are given by $\lambda_n = cn\pi/L$.

The general solution is the superposition of all the eigenfunctions and takes the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (C_n \cos \lambda_n t + D_n \sin \lambda_n t) \sin \frac{n\pi x}{L}$$

The initial velocity and displacement are used in the calculation of C_n and D_n and will require the use of Fourier series and half range expansions when the initial conditions are non-sinusoidal.

Without doubt the major tool used when solving partial differential equations is separation of variables. Although several centuries old, this technique still enjoys a widespread use through all the major technical disciplines as well as all sectors of industry, from steel making to option pricing. A summary has been supplied at the start of the lecture and a more detailed development is in your printed notes. However you are encouraged not to simply memorise these results!

Separation of Variables is a *process* and the most effective approach is to carefully work through each problem as it is presented to you. Although there will be minor changes from example to example the overall technique does not change! We will do a variety of problems over the next few lectures. You will observe that in each case the question will be mechanically worked over without reference to external formulae. It is however a very technical and demanding method of solution and you must take care with each and every step. Lets start with a relatively simple example of the wave equation governing the vibration of an elastic string. You will see that a single example can easily take a whole lecture!

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Let's take a look at the evolution of the string for a few different initial displacements. This is the information that the solution of the wave equation provides.

https://academo.org/demos/1D-wave-equation/

In our first example below we have $L = \pi$ and $c = 1$.

Example 1 Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

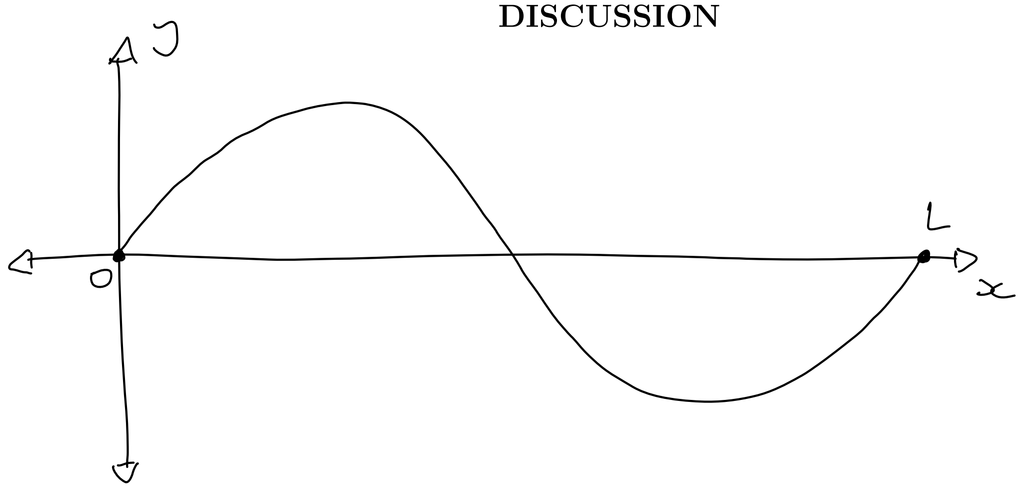
where

$u(0,t) = u(\pi,t) = 0 \qquad$ for all $t \qquad$ (tied down at 0 and π)

$u(x,0) = 5\sin(x) - 3\sin(7x) \qquad$ (initial displacement)

$u_t(x,0) = 0 \qquad$ (initial velocity)

DISCUSSION



To solve we implement the process of separation of variables:

Step 1. Assume the variables separate to convert the p.d.e. into two ordinary differential equations.
Step 2. Determine the solution of these two equations that satisfy the boundary conditions.
Step 3. Combine these solutions so that the result will be a solution of the p.d.e. which also satisfies the initial conditions.

i) By assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$F'' - kF = 0$$

and

$$G'' - kG = 0$$

for k constant.

$$\frac{\partial^2 u}{\partial t^2} = F \cdot G'' \qquad \frac{\partial^2 u}{\partial x^2} = F'' \cdot G$$

$$F \cdot G'' = F'' \cdot G$$

$$\frac{G''}{G} = \frac{F''}{F}$$

$$\text{So,} \quad \frac{G''}{G}(t) = \frac{F''}{F}(x)$$

LHS and RHS must be constants

$$\text{Let} \quad \frac{G''}{G} = \frac{F''}{F} = k$$

$$\therefore \; G'' - kG = 0, \quad F'' - kF = 0$$

ii) By implementing the boundary condition $u(0,t) = u(\pi,t) = 0$ show that

$$F(0) = F(\pi) = 0$$

$$u(0,1) = F(0) \cdot G(t) = 0$$

$$\therefore F(0) = 0 \quad \forall t$$

$$u(\pi,t) = F(\pi) \cdot G(t) = 0$$

$$\therefore F(\pi) = 0 \quad \forall t$$

iii) By solving for F with $k = 0$ and $k > 0$ show that non-trivial solutions will only arise from $k < 0$. (We will say that $k = -\rho^2$).

For $k = 0$: $\quad F'' = 0 \implies F' = \alpha$

$$\therefore F = \alpha_1 x + \beta_1$$

$F(0) = \beta_1 = 0 \qquad\qquad F(\pi) = \pi \alpha_1 = 0$

$$\therefore \alpha_1 = 0$$

$$\therefore F = 0$$

For $k = \rho^2 > 0$: $\quad F'' - \rho^2 F = 0 \quad (2^{nd}\ ODE)$

$$\therefore F = \alpha_2 e^{\rho x} + \beta_2 e^{-\rho x} = 0$$

$F(0) = \alpha_2 + \beta_2 = 0 \qquad\qquad F(\pi) = \alpha_2 (e^{2\rho\pi} - 1) = 0$

$$\therefore \beta_2 = -\alpha_2$$

$$\{\alpha_2 = 0,\ \beta_2 = 0,\ F = 0\} \cup \{\rho = 0,\ k = 0,\ F = 0\}$$

$\therefore$ solutions for $k \geq 0$ are trivial.

iv) By implementing $F(0) = F(\pi) = 0$ with $k = -\rho^2$ show that

$$u_n(x,t) = \sin(nx)\{C_n \cos(nt) + D_n \sin(nt)\}$$

For $k = -\rho^2 < 0$: $\quad F'' + \rho^2 F = 0$

$$\therefore F = \alpha_3 \cos(\rho x) + \beta_3 \sin(\rho x)$$

$F(0) = \alpha_3 = 0$ $\qquad$ $F(\pi) = \beta_3 \sin(\pi\rho) = 0$

$$\therefore \rho = n, \quad n \in \mathbb{Z}^+, \quad \beta_3 \neq 0$$

$$\therefore F_n(x) = \beta_3 \sin(nx)$$

$G'' + \rho^2 G = 0 \quad$ (2nd ODE)

$G = \gamma \cos(nt) + \delta \sin(nt)$

$\therefore G_n(t) = C_n \cos(nt) + D_n \sin(nt)$

v) Verify that this solution in iv) is consistent with the formula

$$u_n(x,t) = (C_n \cos \lambda_n t + D_n \sin \lambda_n t) \sin \frac{n\pi x}{L} \quad \text{for} \quad n = 1, 2, 3, \cdots$$

where the λ_n (eigenvalues) are given by $\lambda_n = cn\pi/L$. (Note that the examiners will not look kindly upon students who simply memorise formulae. You must prove your results!)

vi) Using the initial velocity $u_t(x, 0) = 0$ show that

$$u_n(x,t) = C_n \sin(nx) \cos(nt)$$

vi) By summing appropriate $u_n's$ and applying the initial displacement

$$u(x,0) = 5\sin(x) - 3\sin(7x)$$

show that the final solution is

$$u(x,t) = 5\sin(x)\cos(t) - 3\sin(7x)\cos(7t)$$

Note that we always apply the initial displacement last.

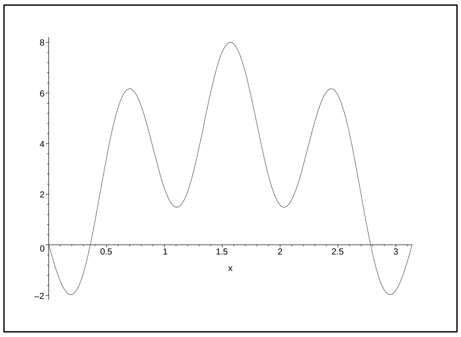
We can use our solution $u(x,t) = 5\sin(x)\cos(t) - 3\sin(7x)\cos(7t)$ in several interesting ways.

Suppose that we substitute $t = 0.15$ into our solution. That is, suppose that we focus on what is happening after $t = 0.15$ seconds. Then

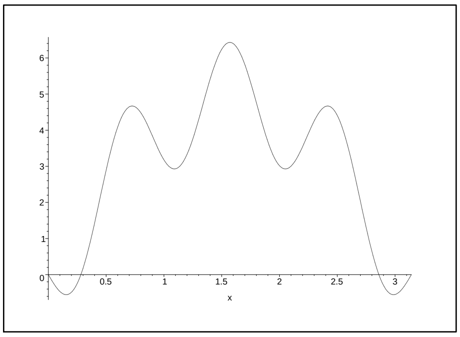
$$u(x, 0.15) = 5\sin(x)\cos(0.15) - 3\sin(7x)\cos(1.05) \approx 4.94\sin(x) - 1.49\sin(7x).$$

This is then the shape of the string after $t = 0.15$ seconds:

t=0



t=0.15



If we further specify that we are positioned half way along the string so that $x = \dfrac{\pi}{2}$ then

$$u(\frac{\pi}{2}, 0.15) = 5\sin(\frac{\pi}{2})\cos(0.15) - 3\sin(\frac{7\pi}{2})\cos(1.05) \approx 5.44$$

which can be interpreted as the displacement half way along the string after $t = 0.15$ seconds.

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We turn now to a new issue. What if the initial displacement is not sinusoidal?

Example 2 Solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

where

$$u(0,t) = u(\pi,t) = 0 \qquad \text{for all } t \qquad \text{(tied down at 0 and } \pi)$$

$$u(x,0) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2}; \\ \pi - x, & \frac{\pi}{2} \le x \le \pi. \end{cases}$$

$$u_t(x,0) = 0 \qquad \text{(initial velocity)}$$

Sketch:

This is identical to before except that we have a different initial displacement.

Thus we will again reach the point

$$u_n(x,t) = C_n \sin(nx)\cos(nt)$$

Clearly the initial displacement is not sinusoidal so no finite linear combination of the $u_n's$ will do the job. So what we do is take the lot of them! That is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin(nx)\cos(nt)$$

Something very interesting then happens:

$$\bigstar \quad u(x,t) = \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi} \sin(\frac{n\pi}{2}) \right) \sin(nx)\cos(nt) \quad \bigstar$$

[5051] You can now do Q 116,119