LECTURE 55
OTHER P.D.E'S

The method of Separation of Variables is extremely versatile and may be used in a variety of different circumstances. The technique itself varies little, however the implementation of boundary and initial conditions can feel quite different as you move from problem to problem.

Example 1 The steady state distribution $u(x, y)$ of heat in an infinite slab of width 4 is given by Laplace's Equation

$$\frac{\partial u^2}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with boundary conditions:

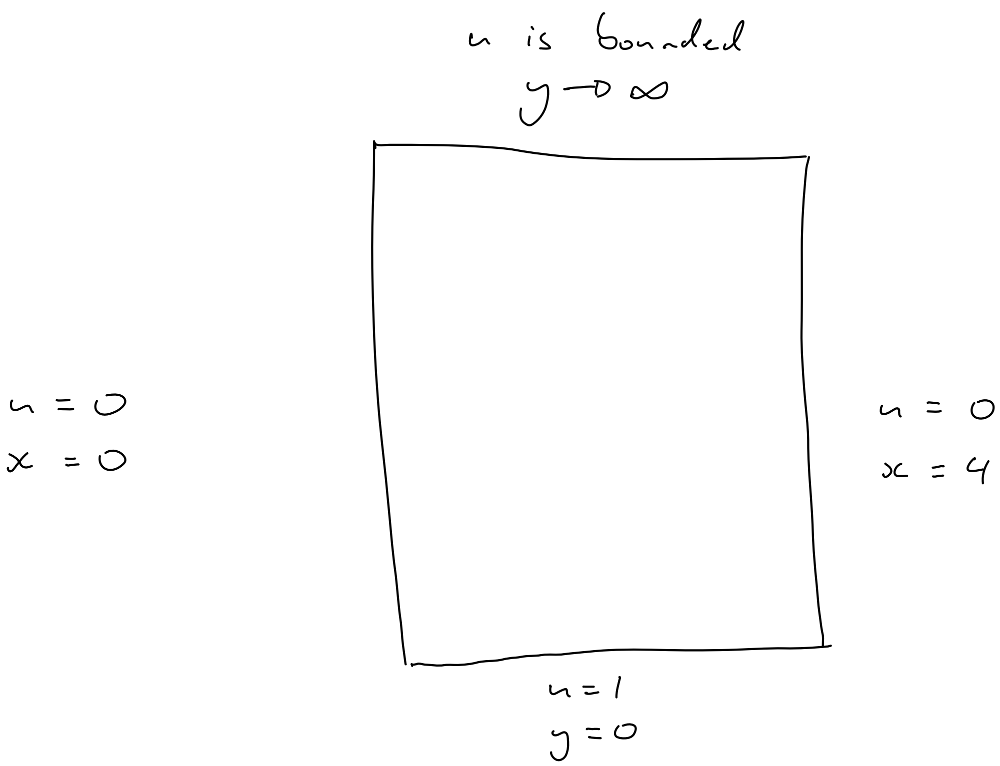
(i) $u = 0$ when $x = 0 \quad \forall\ y > 0$. (Note that $\forall$ means 'for all')

(ii) $u = 0$ when $x = 4 \quad \forall\ y > 0$.

(iii) u is bounded as $y \to \infty$.

(iv) $u = 1$ when $y = 0, \quad 0 < x < 4$.

DISCUSSION



i) By assuming a solution of the form $u(x,y) = F(x)G(y)$ show that

$$F'' - kF = 0$$

and

$$G'' + kG = 0$$

for k constant.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$F''G + FG'' = 0$$

$$\therefore \frac{F''}{F} = -\frac{G''}{G} = k$$

$$\therefore F'' - kF = 0 \quad , \quad G'' + kG = 0$$

ii) By implementing the boundary conditions $u(0,y) = u(4,y) = 0$ show that

$$F(0) = F(4) = 0$$

$$u(0,y) = F(0)G(y) = 0$$

$$\therefore F(0) = 0$$

$$u(4,y) = F(4)G(y) = 0$$

$$\therefore F(4) = 0$$

iii) By solving for F with $k = 0$ and $k > 0$ show that non-trivial solutions will only arise from $k < 0$. (We will say that $k = -\rho^2$).

For $k=0$:

$$F'' = 0$$
$$F = \alpha_1 x + \beta_1$$

$$F(0) = \beta_1 = 0 \qquad F(4) = 4\alpha_1 = 0$$
$$\therefore \alpha_1 = 0$$

$\therefore$ solutions are trivial

For $k=\rho^2>0$:

$$F'' - \rho^2 F = 0$$
$$\therefore F = \alpha_2 e^{\rho x} + \beta_2 e^{-\rho x}$$

$$F(0) = \alpha_2 + \beta_2 = 0 \qquad F(4) = \alpha_2 (e^{8\rho} - 1) = 0$$
$$\therefore \alpha_2 = -\beta_2$$

But $\rho \neq 0 \therefore \alpha_2 = 0 = \beta_2$

$\therefore$ solutions are trivial

iv) By implementing $F(0) = F(4) = 0$ with $k = -\rho^2$ show that

$$u_n(x,y) = B_n \sin(\frac{n\pi x}{4})e^{-\frac{n\pi}{4}y}$$

For $k = -\rho^2 < 0$: $\quad F'' + \rho^2 F = 0$

$$\therefore F = \alpha_3 \cos(\rho x) + \beta_3 \sin(\rho x)$$

$F(0) = \alpha_3 = 0 \qquad F(4) = \beta_3 \sin(4\rho) = 0$

$$\therefore \rho = \frac{n\pi}{4}, \quad n \in \mathbb{Z}$$

$$\therefore F_n = B_n \sin\left(\frac{n\pi}{4}\right)$$

$$G'' - \rho^2 G = 0$$

$$\therefore G = \gamma e^{\rho y} + \delta e^{-\rho y}$$

$$G(\infty) = \gamma e^{\rho\infty} + \delta e^{-\rho\infty} = 0$$

$$\therefore \gamma = 0$$

$$\therefore G_n = D_n e^{-\rho y} \quad (\text{ignore } D_n)$$

$$\therefore u_n(x,y) = B_n \sin\left(\frac{n\pi x}{4}\right) e^{-\frac{n\pi y}{4}}$$

v) Hence find the solution which also satisfies the final condition $u(x,0) = 1$.

$$u(x,0) = B_n \sin\left(\frac{n\pi x}{4}\right) = 1 = f(x)$$

$$B_n = \frac{1}{4}\int_{-4}^{4} f(x)\sin\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{1}{2}\int_{0}^{4} \sin\left(\frac{n\pi x}{4}\right) dx$$

$$= \frac{1}{2} \times \frac{-4}{n\pi}\left[\cos\left(\frac{n\pi x}{4}\right)\right]_0^4$$

$$= \frac{2}{n\pi}\left(1-(-1)^n\right)$$

$$\therefore u_n = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1-(-1)^n\right)\sin\left(\frac{n\pi x}{4}\right)e^{-\frac{n\pi y}{4}}$$

$$\bigstar \quad u(x,y) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\left(\frac{(2k+1)\pi x}{4}\right)e^{-\frac{(2k+1)\pi}{4}y} \quad \bigstar$$

SOME FINAL INFORMATION

1. Please check online that all your Math2019 marks are recorded correctly.

2. Read the school pages on additional assessment/special consideration so that you are fully aware of the rules that apply.

3. Note in particular that students with a final mark in the range 45-49 are automatically granted additional assessment and be aware of the strict dates for the additional assessment exams.

4. Past Math2019 final exams and solutions are available on Moodle.

5. The final exam is 2 hours long with 4 questions. Make sure you turn up at the right time in the right location. Check your exam timetable!!

6. Please start a new book for each of the 4 questions.

7. Make sure your calculator has a UNSW APPROVED sticker (available from the School of Mathematics office) or you will not be allowed to use it during the exam.

8. Please take the time to complete all surveys regarding the administration and teaching of the course.

9. A consultation roster will shortly be posted on Moodle.

Good Luck!

Milan Pahor

[55] You can now do Q 121