LECTURE 38A: THE HEAVISIDE FUNCTION
LAPLACE TRANSFORMS

LAPLACE TRANSFORMS

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^m	$m!/s^{m+1}$
t^ν , $(\nu > -1)$	$\Gamma(\nu+1)/s^{\nu+1}$
e^{-at}	$1/(s+a)$
$\sin bt$	$b/(s^2+b^2)$
$\cos bt$	$s/(s^2+b^2)$
$\sinh bt$	$b/(s^2-b^2)$
$\cosh bt$	$s/(s^2-b^2)$
$\sin bt - bt\cos bt$	$2b^3/(s^2+b^2)^2$
$\sin bt + bt\cos bt$	$2bs^2/(s^2+b^2)^2$
$t\ \sin bt$	$2bs/(s^2+b^2)^2$
te^{-at}	$1/(s+a)^2$
$\mathrm{u}(t-c)$	e^{-cs}/s
$e^{-at}f(t)$	$F(s+a)$
$tf(t)$	$-F'(s)$
$f(t-c)\mathrm{u}(t-c)$	$e^{-cs}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f'''(t)$	$s^3F(s)-s^2f(0)-sf'(0)-f''(0)$
$\int_0^t f(\tau)d\tau$	$F(s)/s$

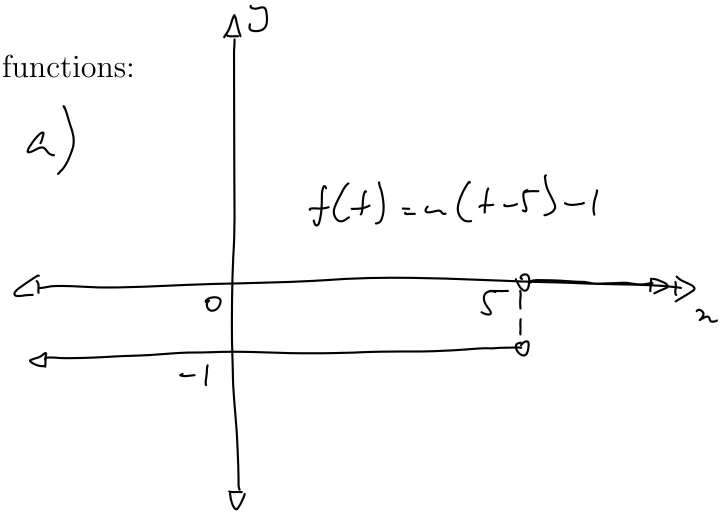
Example 1 Sketch each of the following functions:

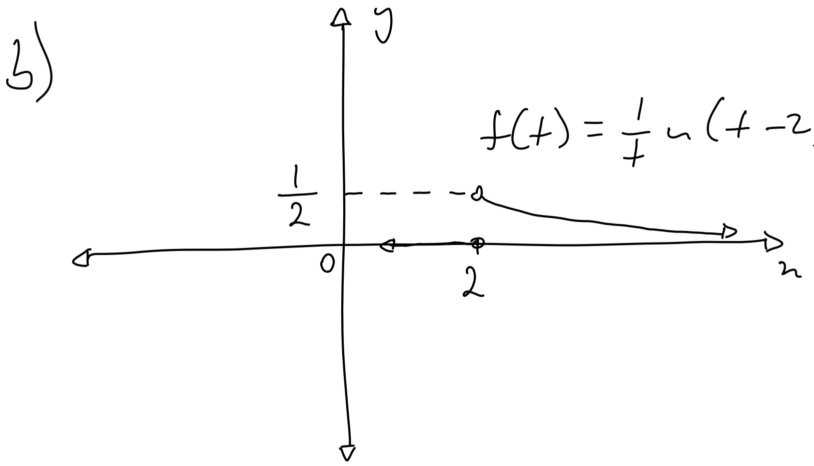
a) $f(t) = u(t-5) - 1$

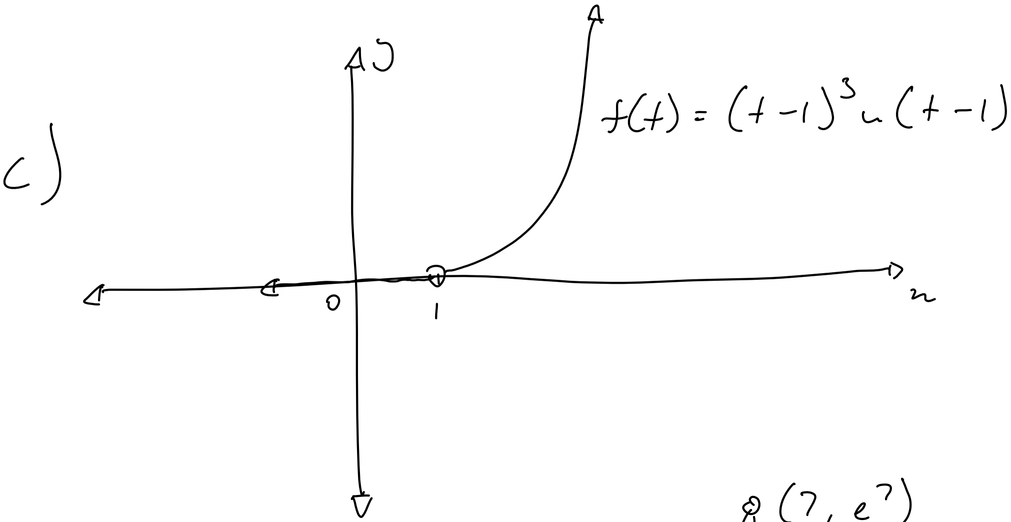
b) $f(t) = \dfrac{1}{t}u(t-2)$

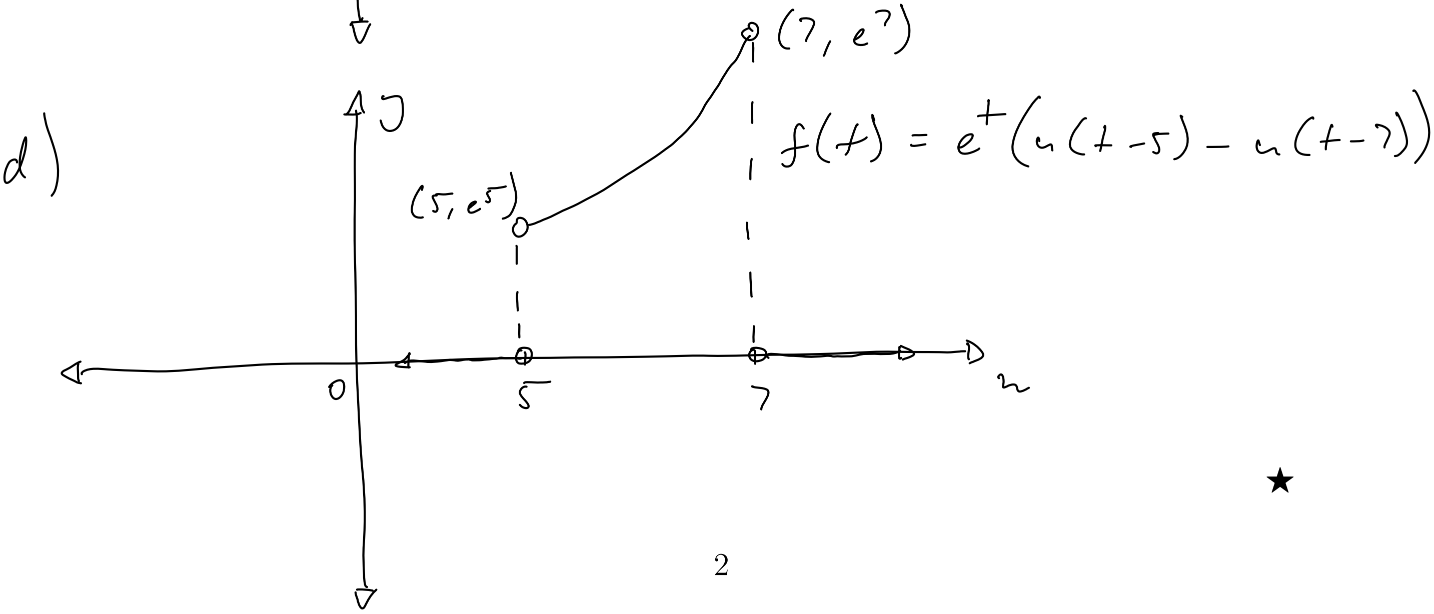
c) $f(t) = (t-1)^3 u(t-1)$

d) $f(t) = e^t\{u(t-5) - u(t-7)\}$









Example 2 Find the Laplace transform of $e^{4t}t^5$

$$\mathcal{L}(e^{-at}f(t)) = F(s+a)$$

We use the first shifting theorem whenever we are taking Laplace transforms and there is a rogue exponential function in the t variable OR we are taking inverse Laplace transforms and there is a shift in the s variable.

$$\mathcal{L}(e^{4t}t^5) = \frac{5!}{(s-4)^6}$$

$$= \frac{120}{(s-4)^6}$$

$$\bigstar \quad \frac{120}{(s-4)^6} \quad \bigstar$$

Example 3 Find the inverse Laplace transform of $\dfrac{(s-12)}{(s-12)^2+1}$

$$\mathcal{L}(e^{-at}f(t)) = F(s+a)$$

$$\mathcal{L}^{-1}\left(\frac{s-12}{(s-12)^2+1}\right) = e^{12t}\cos(t)$$

$$\bigstar \quad e^{12t}\cos(t) \quad \bigstar$$

Example 4 Consider the function given by

$$f(t) = \begin{cases} 9, & 3 \leq t \leq 4; \\ 7, & \text{otherwise.} \end{cases}$$

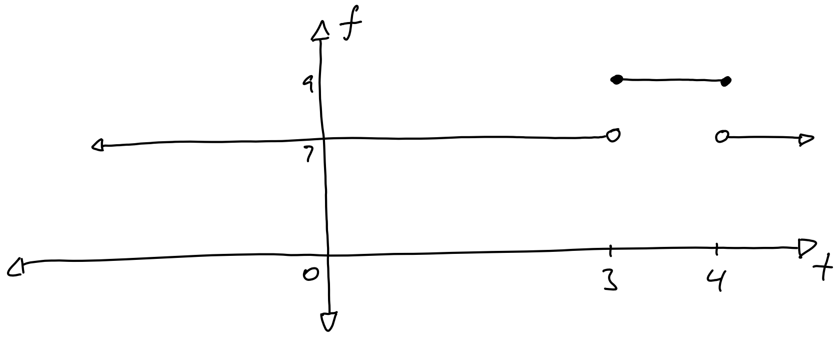
a) Sketch f.

b) Express f in terms of Heavisides.

c) Find $F(s)$.

$$\mathcal{L}(\mathrm{u}(t-c)) = \frac{e^{-cs}}{s}$$

a)



b) $f(t) = 7 + 2\left(u(t-3) - u(t-4)\right)$

c) $\mathcal{L}\left(f(t)\right) = \dfrac{7}{s} + \dfrac{2e^{-3s}}{s} - \dfrac{2e^{-4s}}{s}$

★ $\dfrac{2e^{-3s}}{s} - \dfrac{2e^{-4s}}{s} + \dfrac{7}{s}$ ★

Example 5 Consider the function

$$f(t) = 6tu(t-3) + 9tu(t-7) - 15u(t-12).$$

a) Write the function without the use of Heavisides.

b) Sketch the function.

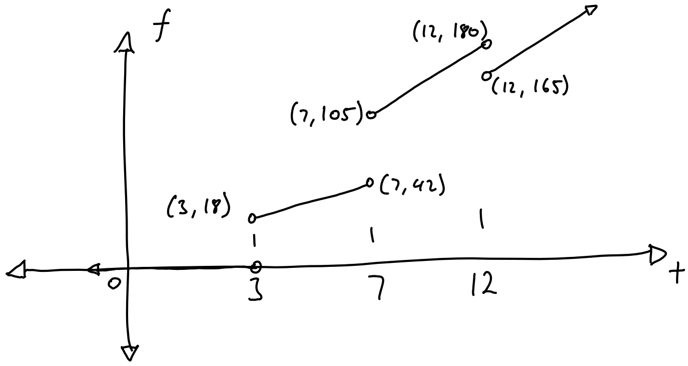
c) Find the Laplace transform of f.

$$\mathcal{L}(f(t-c)\mathrm{u}(t-c)) = e^{-cs}F(s)$$

We use the second shifting theorem whenever we are taking inverse Laplace transforms and there is a rogue exponential function in the s variable OR we are taking Laplace transforms and there is a shift and a cut in the t variable. If we have a cut in the t variable without a shift we need to manipulate the function so that a shift appears.

a) $$f(t) = 6t\big(u(t-3) - u(t-7)\big) + 15t\big(u(t-7) - u(t-12)\big) + (15t - 15)u(t-12)$$

b)



c) $$\mathcal{L}\Big(6tu(t-3) + 9tu(t-7) - 15u(t-12)\Big)$$

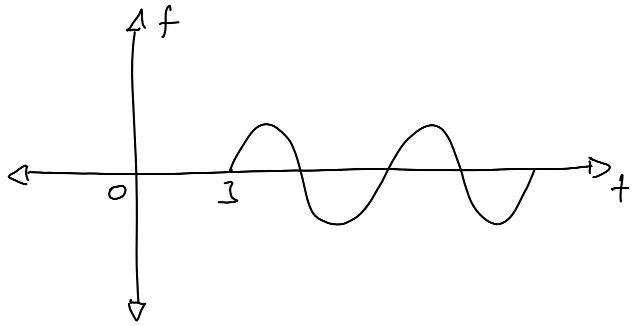
$$= \frac{6e^{-3s}}{s^2} + \frac{9e^{-7s}}{s^2}$$

$$\bigstar \quad 6\frac{e^{-3s}}{s^2} + 9\frac{e^{-7s}}{s^2} + 18\frac{e^{-3s}}{s} + 63\frac{e^{-7s}}{s} - 15\frac{e^{-12s}}{s} \quad \bigstar$$

Example 6 Find and sketch the inverse Laplace transform of $\frac{e^{-3s}}{s^2+1}$.

$$\mathcal{L}(f(t-c)\mathrm{u}(t-c)) = e^{-cs}F(s)$$

$$\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2+1}\right) = \sin(t-3)\,u(t-3)$$



★ $\sin(t-3)u(t-3)$ ★

Example 7 Find the Laplace transform of $e^{7t}u(t-4)$.

This is a rare example where both shifting theorems may be used !

$$\mathcal{L}(f(t-c)\mathrm{u}(t-c)) = e^{-cs}F(s)$$

$$\mathcal{L}\left(e^{7t}u(t-4)\right) = \frac{e^{-4(s-7)}}{s-7}$$

$a = -7, \quad c = 4$

$$= \frac{e^{28-4s}}{s-7}$$

$$e^{-at} \longrightarrow \frac{1}{s+a}$$

$$\mathcal{L}(e^{-at}f(t)) = F(s+a)$$

$$\mathcal{L}\left(e^{7t}u(t-4)\right) = \frac{e^{-4(s-7)}}{s-7}$$

$$= \frac{e^{28-4s}}{s-7}$$

$a = -7$
$c = 4$

★ $\frac{e^{28-4s}}{s-7}$ ★

$$u(t-c) \longrightarrow \frac{e^{-cs}}{s}$$

Example 8 Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{(s-7)^4}\right\}$.

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{(s-7)^4}\right) = \frac{1}{3!}u(t-2)e^{7(t-2)}(t-2)^3$$

$$t^n \leftarrow \frac{n!}{s^{n+1}}$$

$$u(t-c) \leftarrow \frac{e^{-cs}}{s}$$

$$e^{-at}f(t) \leftarrow F(s+a)$$

$$a = -7$$

$$n = 3$$

$$c = 2$$

$\bigstar \quad \dfrac{1}{6}u(t-2)e^{7(t-2)}(t-2)^3 \quad \bigstar$

Example 9 (Challenge Problem)

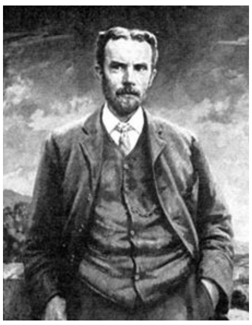
Suppose that $f(t) = \begin{cases} 1, & 0 < t \leq 1; \\ -1, & 1 < t \leq 2; \\ f(t-2), & \text{otherwise.} \end{cases}$

Sketch the function and find its Laplace transform.

Hint: You may need the theory of limiting sums of G.P.'s.

Note that the condition $f(t) = f(t-2)$ simply forces the function to repeat every 2 units. That is, it forces a periodicity of 2.

$$\star \quad \frac{1}{s}\left(\frac{e^s - 1}{e^s + 1}\right) \quad \star$$



Oliver Heaviside
$u(t-1850) - u(t-1925)$