MATH2019 PROBLEM CLASS

EXAMPLES 9

PARTIAL DIFFERENTIAL EQUATIONS

2014, S1

1. The temperature in a bar of length π metres satisfies the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ is the temperature in $°C$, t is time in seconds and x is the distance in metres from the left hand end of the bar. Both ends of the bar are maintained at a temperature of $0°C$. Hence

$$u(0,t) = u(\pi,t) = 0 \quad \text{for all} \quad t.$$

i) Assuming a solution of the form

$$u(x,t) = F(x)G(t) \quad \text{show that}$$

$$\frac{1}{G}\frac{dG}{dt} = \frac{1}{F}\frac{d^2F}{dx^2} = k \quad \text{where } k \text{ is a constant.}$$

ii) You may assume that only $k < 0$ yields non-trivial solutions and set $k = -p^2$ for some $p > 0$.

Applying the initial conditions show that $p = n,\ n = 1, 2, 3, \ldots$
and that possible solutions for $F(x)$ are

$$F_n(x) = b_n \sin(nx)$$

where b_n are constants and $n = 1, 2, 3, \ldots$

iii) Find all possible solutions $G_n(t)$ for $G(t)$.

iv) Suppose now that the initial temperature distribution of the bar is

$$u(x,0) = 2\sin(x) - 16\sin(2x).$$

Find the general solution $u(x,t)$.

v) Hence determine all points x along the bar with a temperature of $0°C$ after $t = \ln(2)$ seconds.

2014, S2

2. The temperature in a bar of length π satisfies the heat equation

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ is the temperature in degrees Celsius. The ends of the bar are held at a constant temperature of 0°C so that

$$u(0,t) = u(\pi,t) = 0 \qquad \text{for all } t.$$

i) Assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$\frac{G'(t)}{5G(t)} = \frac{F''(x)}{F(x)} = k$$

for some constant k.

ii) You can assume that the only non-trivial solutions are given by $k < 0$. Apply the boundary conditions to show that possible solutions for $F(x)$ are

$$F_n(x) = b_n \sin(nx)$$

where b_n are constants and $n = 1, 2, 3, \ldots$.

iii) Find all possible solutions $G_n(t)$ for $G(t)$.

iv) If the initial temperature distribution of the bar is

$$u(x, 0) = 5\sin(2x) - 3\sin(4x),$$

find the general solution $u(x, t)$.

v) What is the equilibrium temperature in the bar as $t \to \infty$?

2015, S1

3. Let $F(x)$ satisfy the differential equation

$$F''(x) = k\,F,$$

with boundary conditions $F(0) = 0$ and $F(\pi) = 0$. By considering separately the cases of $k > 0$, $k = 0$, $k < 0$, find the general solution for $F(x)$.

2015, S2

4. The tempertaure in a bar of unit length satisfies the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}$$

where $u(x, t)$ is the temperature. The bar has its ends maintained at zero temperature. Hence,

$$u(0, t) = 0 \text{ and } u(1, t) = 0, \text{ for all } t.$$

i) Assuming a solution of the form $u(x, t) = X(x)Y(t)$ show that

$$X'' - 4kX = 0 \quad\text{and}\quad Y' - kY = 0,$$

for some constant k.

ii) Applying the boundary conditions (and considering all possibilities for the constant k) show that

$$4k = -p^2 \qquad\qquad (p > 0)$$

and that possible solutions for $X(x)$ are

$$X_n(x) = b_n \sin(n\pi x)$$

where b_n are constants and $n = 1, 2, 3, \ldots$.

iii) Find all possible solutions $Y_n(t)$ for $Y(t)$.

iv) Suppose now that the initial temperature distribution is given by

$$u(x, 0) = \frac{1}{2}\sin(2\pi x) - \frac{1}{4}\sin(4\pi x).$$

Using ii) and your answer in iii) find the solution $u(x, t)$.

v) Let P be the point on the bar where $x = \frac{1}{4}$. Determine the time where the temperature at P is half the initial temperature at P.

$\boxed{2016, S1}$ 5. A stretched wire satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ is the displacement of the wire. The ends of the wire are held fixed so that

$$u(0,t) = u(\pi,t) = 0, \qquad \text{for all } t.$$

i) Assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$\frac{G''(t)}{4G(t)} = \frac{F''(x)}{F(x)} = k$$

for some constant k.

ii) Apply the boundary conditions to show that possible solutions for $F(x)$ are

$$F_n(x) = b_n \sin(nx)$$

where b_n are constants and $n = 1, 2, 3, \ldots$ You must consider all possible values of k.

iii) Find all possible solutions $G_n(t)$ for $G(t)$.

iv) If the initial displacement and velocity of the wire are

$$u(x,0) = 3\sin(x) + 4\sin(3x), \qquad \text{and} \qquad u_t(x,0) = 0,$$

find the general solution $u(x,t)$.

$\boxed{2016, S2}$ 6. The temperature in a conducting metal bar of length L is described by the heat equation

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ is the temperature at position x and time t.

The ends of the metal bar are held at a constant temperature of 0°C so that

$$u(0,t) = u(L,t) = 0, \qquad \text{for all } t > 0.$$

i) Assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$\frac{G'(t)}{3G(t)} = \frac{F''(x)}{F(x)} = k$$

for some constant k.

ii) Write down the boundary conditions for $F(x)$.

iii) Apply the boundary conditions to show that the possible solutions for $F(x)$ are

$$F_n(x) = b_n \sin\left(\frac{n\pi x}{L}\right)$$

where b_n are constants and $n = 1, 2, 3, \ldots$. You must consider all possible values of k.

iv) Find all possible solutions $G_n(t)$ for $G(t)$ and write down the general solution for $u_n(x,t) = F_n(x)G_n(t)$.

v) If the initial temperature distribution is $u(x,0) = 1$, find the solution $u(x,t)$ expressing your answer as an infinite series.

2017, S1

7. Consider the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ is the displacement at position x and time t. D'Alembert's solution to this wave equation is

$$u(x,t) = \phi(x+3t) + \psi(x-3t),$$

for arbitrary functions ϕ and ψ. If the initial displacement of the wave is $u(x,0) = g(x)$ and the initial velocity is $u_t(x,0) = 0$, prove that

$$u(x,t) = \frac{1}{2}\left[g(x+3t) + g(x-3t)\right].$$

2017, S2

8. The temperature in a conducting metal bar of length π is described by the heat equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ is the temperature at position x and time t. The ends of the metal bar are insulated so that

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0, \qquad \text{for all } t > 0.$$

You are additionally given that the initial temperature distribution in the metal bar is

$$u(x,0) = U_0(x).$$

i) Assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$\frac{G'(t)}{2G(t)} = \frac{F''(x)}{F(x)} = k$$

for some constant k.

ii) Write down the boundary conditions for $F(x)$.

iii) Separately consider the cases of $k > 0$, $k = 0$, and $k < 0$. Apply the boundary conditions to find all non-trivial solutions for $F(x)$.

iv) For each non-trivial solution for $F(x)$ found in (iii), find the corresponding solutions for $G(t)$. Hence, write down the general solution for $u(x,t)$.

v) The initial temperature distribution in the metal bar is

$$U_0(x) = f(x),$$

where $f(x)$ is the function considered in Q11, Problem Set 8. Find the solution $u(x,t)$, expressing your answer as an infinite series.

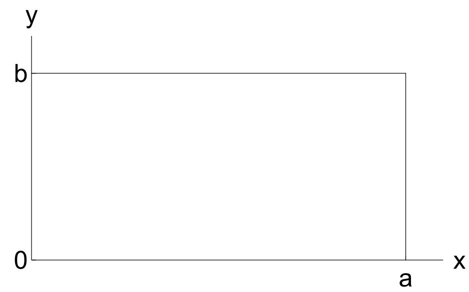
2018, S1

9. The steady state temperature $u(x,y)$ in a slab of length a and width b satisfies the partial differential equation equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for} \quad 0 < x < a, \quad 0 < y < b,$$

with boundary conditions

$$\begin{aligned} &u(0,y) = u(a,y) = 0, && 0 < y < b, \\ &u(x,0) = 0, \quad u(x,b) = \pi, && 0 < x < a. \end{aligned}$$



i) Assuming a solution of the form $u(x,y) = X(x)Y(y)$, show that

$$\frac{X''}{X} = -\frac{Y''}{Y} = k$$

for some constant k.

ii) Write down the ordinary differential equations and the associated boundary conditions for $X(x)$ and $Y(y)$.

iii) Consider all values of k, that is, $k = p^2 > 0, k = 0, k = -p^2 < 0$ and solve for $X(x)$.

iv) With the values of k obtained in part iii) which yield non-trivial solutions for $X(x)$, solve for $Y(y)$.

v) Write down the general solution of $u(x,y)$.

2018, S2

10. A vibrating string of length π metres satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 25\frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ is the transverse displacement of the string, at position x and time t. The ends of the string are held fixed so that

$$u(0,t) = u(\pi,t) = 0, \quad \text{for all time } t.$$

i) Assuming a solution of the form $u(x,t) = F(x)G(t)$, show that

$$\frac{1}{25G}\frac{d^2G}{dt^2} = \frac{1}{F}\frac{d^2F}{dx^2} = k$$

for some constant k.

ii) You may assume that only $k < 0$ yields non-trivial solutions and set $k = -p^2$ for some $p > 0$.
Applying the boundary conditions, show that $p = n,\ n = 1, 2, 3, \ldots$
and that possible solutions for $F(x)$ are

$$F_n(x) = b_n \sin(nx)$$

where b_n are constants.

iii) Find all possible solutions $G_n(t)$ for $G(t)$.

iv) If the initial displacement and velocity of the string are

$$u(x,0) = 2\sin(x) - \sin(2x) \qquad \text{and} \qquad u_t(x,0) = 0,$$

find the general solution $u(x,t)$.

v) Hence determine the maximal transverse displacement of the string at time $t = \dfrac{\pi}{15}$ seconds.