LECTURE 25
NON-HOMOGENEOUS SECOND ORDER DIFFERENTIAL EQUATIONS

To solve the non-homogeneous second order constant coefficient differential equation

$$ay'' + by' + cy = r(x)$$

we first solve the homogeneous problem $ay'' + by' + cy = 0$ to obtain a homogenous solution y_h. We then find a particular solution y_p by using the method of undetermined coefficients. The final solution is then $y = y_h + y_p$.

To find the long term steady state solution, consider the bahaviour of the solution as $t \to \infty$

We now deal with the constant coefficient second order D.E.'s of the previous lecture except that we will allow "nice" functions to appear on the RHS. This makes the D.E. non-homogeneous. We begin by finding the homogeneous solution y_h by using the methods of the previous lecture. We then find a particular solution y_p by essentially guessing an answer. The final solution is then $y = y_h + y_p$. It should be noted that this technique is prone to fail and will only work when the RHS is uncomplicated.

Example 1 Solve $y'' + 5y' + 6y = -6t + 25$. Describe the long term steady state solution to the differential equation.

Consider $\lambda^2 + 5\lambda + 6y = 0$

$\therefore \lambda = -2, -3$

$\therefore y_H = Ae^{-2t} + Be^{-3t}$

Try: $y_p = \alpha t + \beta$

$6(\alpha t + \beta) + 5(\alpha) + 0 = -6t + 25$

$6\alpha t + (6\beta + 5\alpha) = -6t + 25$

$\therefore \alpha = -1, \quad \beta = 5$

$y = y_H + y_p$

$\therefore y = Ae^{-2t} + Be^{-3t} - t + 5$

Steady State:

$\lim_{t \to \infty} y = -t + 5$

★ $y = Ae^{-2t} + Be^{-3t} - t + 5$. Steady State Solution is $y = -t + 5$ ★

Why does this work?

The crucial question is of course how do we know what to guess for y_p? The general rule is that you guess an arbitrary representation of the RHS. The following table gives some RHS's and their associated guesses for y_p .

RHS	Choice of y_P
$3e^{4x}$	Ce^{4x}
$x^3 - 7$	$\alpha x^3 + \beta x^2 + \gamma x + \delta$
$3\sin(4x)$	$\alpha\cos(4x) + \beta\sin(4x)$
$5e^{7x}\cos(2x)$	$e^{7x}(\alpha\cos(2x) + \beta\sin(2x))$
$9x^2e^{3x}$	$e^{3x}(\alpha x^2 + \beta x + \gamma)$

Note that if the RHS is too weird then no amount of guessing will save you.

Example 2 Solve the initial value problem

$$y'' + y = 55e^{2x} + 3x^2 + 14$$

where $y(0) = 20$ and $y'(0) = 28$.

Consider $\lambda^2 + 1 = 0$

$\therefore \lambda = \pm i$

$\therefore y_H = A\cos x + B\sin x$

Try: $y_P = \alpha e^{2x} + \beta x^2 + \gamma x + \delta$

$\alpha e^{2x} + \beta x^2 + \gamma x + \delta + (4\alpha e^{2x} + 2\beta) = 55e^{2x} + 3x^2 + 14$

$\therefore 5\alpha e^{2x} + \beta x^2 + \gamma x + (\delta + 2\beta) = 55e^{2x} + 3x^2 + 14$

$\therefore \alpha = 11, \ \beta = 3, \ \gamma = 0, \ \delta = 8$

$\therefore y = A\cos x + B\sin x + 11e^{2x} + 3x^2 + 8$

$$y(0) = A + 11 + 8 = 20$$

$$\therefore A = 1$$

$$y'(0) = B + 22 = 28$$

$$\therefore B = 6$$

$$\therefore y = \cos x + 6 \sin x + 11e^{2x} + 3x^2 + 8$$

$$\bigstar \quad y = \cos(x) + 6\sin(x) + 11e^{2x} + 3x^2 + 8 \quad \bigstar$$

VARIATION OF PARAMETERS

If the method of undetermined coefficients fails to produce a particular solution y_p, it is possible that technique of variation of parameters will do the job. This is a highly specialised process:

Variation of Parameters

Suppose that the second order differential equation

$$y'' + p(x)y' + q(x)y = f(x)$$

has homogeneous solution $y_h = Ay_1(x) + By_2(x)$. Then a particular solution is given by

$$y_P(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx$$

where $W(x) = \det \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}$.

A full proof of this formula is found in your printed notes.

Example 3 Use Variation of Parameters to solve $y'' + y = \sec(x)$.

Consider $\lambda^2 + 1 = 0$

$\therefore \lambda = \pm i$

$\therefore y_H = A\cos x + B\sin x$

Let $y_1 = \cos x$, $y_2 = \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$\int \frac{y_2 f}{W}\,dx = \int \sin x \sec x\,dx = -\ln|\cos x|$$

$$\int \frac{y_1 f}{W}\,dx = \int \cos x \sec x \, dx = x$$

$$y_p = -\cos x\left(-\ln|\cos x|\right) + \sin x\,(x)$$

$$\therefore y = A\cos x + B\sin x + \cos x \ln|\cos x| + x\sin x$$

$$\bigstar \quad y = A\cos(x) + B\sin(x) + \cos(x)\ln(|\cos(x)|) + x\sin(x) \quad \bigstar$$

[25] You can now do Q85 a and b, Q88