LECTURE 28
REVISION OF MATRIX THEORY

• If A is an m $\times$ n matrix and B is a p $\times$ q matrix then AB exists iff n=p and the product is m $\times$ q.

• The identity matrix I serves as the "1" of matrix theory.

• The transpose of A (denoted by A^T) has the columns of A as its rows. If A is $m \times n$ then A^T is $n \times m$.

• A matrix is said to be symmetric if $A = A^T$.

• Given a square matrix A the inverse of A (denoted by A^{-1}) is another matrix with the property that $AA^{-1} = I$.

• A^{-1} exists iff $\det(A) \neq 0$.

• If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

• For larger square matrices the inverse is calculated via row reduction using the row operations $R_i = R_i \pm \alpha R_j$ and $R_i \leftrightarrow R_j$.

This lecture will be a revision of the first year theory of matrices, inverses, determinants, systems of linear equations, Gaussian Elimination and row reduction.

If you are rusty on this material your are advised to have a look over your first year linear algebra notes on the above topics before moving on to the following lectures.

I have also posted a completed copy of my Math1131 Algebra lecture notes on Moodle if you need to revise the material in more detail.

Matrices are simply rectangular arrays of numbers which may be manipulated as mathematical objects.

Example 1 Let $A = \begin{pmatrix} 2 & 3 \\ 1 & -2 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 3 \\ 3 & 0 \\ 2 & 5 \end{pmatrix}$.

Find (if possible) $A+B$, $2A-C$, AB, BA and AC.

$$2A - C = 2\begin{pmatrix} 2 & 3 \\ 1 & -2 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 3 & 0 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -1 & -4 \\ -2 & 3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 & 3 \\ 1 & -2 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 19 & 6 \\ 3 & -8 & 3 \\ -4 & 20 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & -2 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 11 \\ 3 & -13 \end{pmatrix}$$

★ $2A - C = \begin{pmatrix} 3 & 3 \\ -1 & -4 \\ -2 & 3 \end{pmatrix}$, $AB = \begin{pmatrix} -1 & 19 & 6 \\ 3 & -8 & 3 \\ -4 & 20 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 4 & 11 \\ 3 & -13 \end{pmatrix}$

$A+B$ and AC are undefined ★

Some special objects are

row vector $\begin{pmatrix} 1 & 4 & -2 \end{pmatrix}$

column vector $\begin{pmatrix} 3 \\ 6 \\ -8 \\ 1 \end{pmatrix}$

zero matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (3×3 zero matrix)

upper triangular matrix $\begin{pmatrix} 4 & 7 & -2 \\ 0 & 5 & 8 \\ 0 & 0 & 2 \end{pmatrix}$

diagonal matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (2×2) identity and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (3×3) identity matrix.

The identity matrix serves as a "one" for matrix theory.

Example 2 Let $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 4 & 6 & 2 \end{pmatrix}$.

Find AI, IA and λI where $\lambda \in \mathbb{R}$.

$AI = IA = A$

$\lambda I = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$

★ $AI = IA = A$ and $\lambda I = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ ★

Matrix division is not quite properly defined, the closest we can manage is to use the concept of an inverse.

Given a square matrix A the inverse of A (denoted by A^{-1}) is another matrix with the property that $AA^{-1} = I$.

2×2 inverses are easy to find via the formula $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

But larger matrices are harder to invert. We usually use row reduction though other techniques are possible.

Example 3 Prove that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and find $\begin{pmatrix} 3 & 8 \\ 1 & 2 \end{pmatrix}^{-1}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$$

$$ae + bg = 1, \quad af + bh = 0, \quad ce + dg = 0, \quad cf + dh = 1$$

$$ae + b\left(-\frac{ce}{d}\right) = 1 \qquad cf + d\left(-\frac{af}{b}\right) = 1$$

$$\therefore e = \frac{d}{ad-bc} \qquad \therefore f = \frac{-b}{ad-bc}$$

$$c\left(\frac{d}{ad-bc}\right) + dg = 0 \qquad a\left(\frac{-b}{ad-bc}\right) + bh = 0$$

$$\therefore g = \frac{-c}{ad-bc} \qquad \therefore h = \frac{a}{ad-bc}$$

$$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} 3 & 8 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{6-8}\begin{pmatrix} 2 & -8 \\ -1 & 3 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 2 & -8 \\ -1 & 3 \end{pmatrix}$$

★ $-\frac{1}{2}\begin{pmatrix} 2 & -8 \\ -1 & 3 \end{pmatrix}$ ★

Example 4 Let $A = \begin{pmatrix} 4 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Find $\det(A)$ and hence explain why A is invertible.

Find A^{-1} and hence solve the system of linear equations

$$\begin{aligned} 4x + 2y + z &= 21 \\ 3x + 2y + z &= 19 \\ x + y + z &= 11 \end{aligned}$$

$$\det(A) = \begin{vmatrix} 4 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 4(2-1) - 2(3-1) + (3-2) = 1$$

Since $\det(A) \neq 0$, $AA^{-1} = I$

$$A^{-1} = \left(\begin{array}{ccc|ccc|c} 4 & 2 & 1 & 1 & 0 & 0 & 21 \\ 3 & 2 & 1 & 0 & 1 & 0 & 19 \\ 1 & 1 & 1 & 0 & 0 & 1 & 11 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc|c} 1 & 1 & 1 & 0 & 0 & 1 & 11 \\ 0 & -1 & -2 & 0 & 1 & -3 & -14 \\ 0 & -2 & -3 & 1 & 0 & -4 & -23 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc|c} 1 & 0 & -1 & 0 & 1 & -2 & -3 \\ 0 & 1 & 2 & 0 & -1 & 3 & 14 \\ 0 & 0 & 1 & 1 & -2 & 2 & 5 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & -2 & 3 & -1 & 4 \\ 0 & 0 & 1 & 1 & -2 & 2 & 5 \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

$$\bigstar \quad A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} \quad \bigstar$$

Note that the above method for solving linear equations is NOT generally used. The calculation of the inverse is numerically inefficient and the method fails completely when there are infinite solutions. When finding eigenvalues and eigenvectors in the following chapters we will always use Guassian Elimination to Echelon Form.

We close the lecture with the definitions of transpose and symmetry for matrices.

Definition: The transpose A^T of a matrix A is defined as $[A^T]_{ij} = [A]_{ji}$.

A^T is quite simply the matrix whose rows are the columns of A (and vice versa).

Example 5 Find A^T if:

a) $A = \begin{pmatrix} 3 & 1 & -2 \\ 4 & 7 & 8 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 3 & 4 \\ 1 & 7 \\ -2 & 8 \end{pmatrix}$

b) $A = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 3 & 6 & 9 \end{pmatrix}$

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Fact: If A is $m \times n$ then A^T is $n \times m$.

Transposes and inverses also behave in a similar fashion across matrix products:

Fact: $(AB)^T = B^T A^T$ and $(AB)^{-1} = B^{-1}A^{-1}$.

Definition: A matrix A is said to be **symmetric** if $A = A^T$.
The following are examples of symmetric matrices:

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 7 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 6 & 8 \\ 6 & -5 & -7 \\ 8 & -7 & 2 \end{pmatrix}$$

For a symmetric matrix the first row is the same as the first column, the second row is the same as the second column... etc.

Example 6 Show that the matrix C given by $C = A^T A$ is always symmetric regardless of the nature of A.

$C = A^T A$

$C^T = (A^T A)^T = A^T A$

$\therefore C = C^T$

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[28] You can now do Q 89