LECTURE 45
SOME HARDER PROBLEMS

Suppose that a function f has period $T = 2L$. Then f may be approximated by the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}) \tag{1}$$

where the Fourier coefficients a_0, a_n, and b_n are given by

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \ldots) \\ b_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \ldots) \end{aligned}\right\} \tag{2}$$

Example 1 Suppose that $f(x) = \begin{cases} x + 4, & -1 \leq x < 1; \\ f(x+2), & \text{otherwise.} \end{cases}$

i) By considering the graph of f and without integration show that a_n=0 and explain why the Fourier series of f takes the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

ii) Find a_0 by inspection and b_n using Example 2 from the last lecture.

iii) By considering the **graph** of f, determine the y value to which the Fourier series converges at $x = 1$?

iii) By considering the **series itself**, determine the y value to which the Fourier series converges at $x = 1$?

★ $f(x) = 4 + \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1}\sin(n\pi x)$ Converges to $y = 4$ at $x = 1$ ★

The next example will involve some very tricky integration. We will need the following results:

I) $\sin(\frac{\pi}{2} + \theta) = \cos(\theta)$

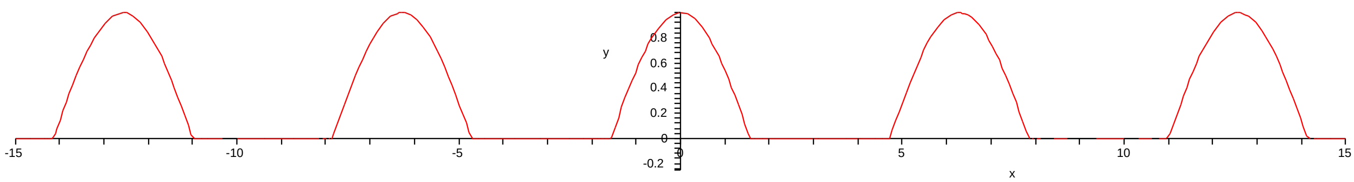
II) $\cos(A)\cos(B) = \frac{1}{2}(\cos(A + B) + \cos(A - B))$

Proof:

$\bigstar$

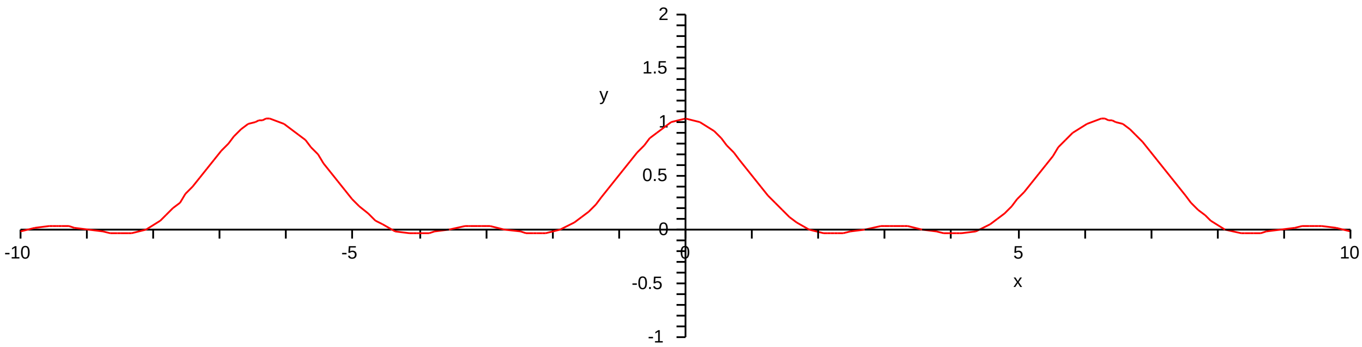
Example 2 A periodic voltage $\cos(t)$ is passed through a half-wave rectifier which clips the negative portion of the wave. Find the Fourier series of the resulting periodic function.

The function is a "clipped" cosine curve and looks like:



$$\bigstar \quad f(x) = \frac{1}{\pi} + \frac{1}{2}\cos(x) + \sum_{n=2}^{\infty} \frac{2\cos(\frac{n\pi}{2})}{\pi(1-n^2)}\cos(nx) \quad \bigstar$$

The graph below is the Fourier series terminated at $n = 10$.



Observe again how nicely the original function is being approximated, despite its strange definition.

Example 3 Find the Fourier series of $f(x) = \sin^2(x) + 7\sin(13x) + 7.5$.

This is a trick question! Make sure you do not spend half an hour on this problem.

★

We close with the observation that Fourier series may also be developed from a complex viewpoint via

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

with

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

This is then the **complex form** of the Fourier series for $f(x)$. The c_n are the complex Fourier coefficients. Students are referred to the printed notes for a proof of the above results. We tend to focus on the real case in Math2019.