LECTURE 18
DOUBLE INTEGRALS IN POLAR COORDINATES

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

$$r = \sqrt{x^2 + y^2}$$

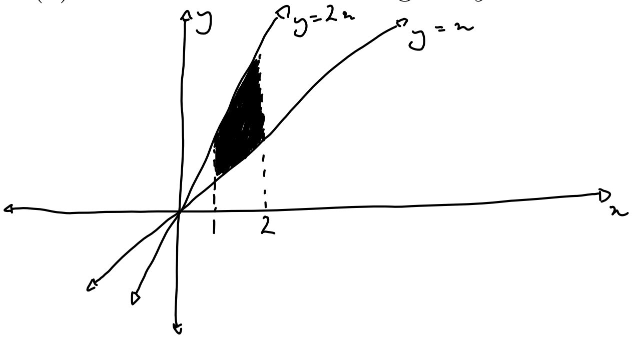
$$\tan(\theta) = \frac{y}{x}$$

$$dA = dxdy = dydx = rdrd\theta$$

First a revision example from the previous lecture.

Example 1 Evaluate $\displaystyle\int_1^2 \int_x^{2x} \frac{x}{y}\, dydx$ by first changing the order of integration.

This is an example we did at the start of the previous lecture using $dydx$ and easily got an answer of $\frac{3}{2}\ln(2)$. It is much harder using $dxdy$. First a sketch:



For homework evaluate the two integrals above and convince yourself that you still get an answer of $\frac{3}{2}\ln(2)$. The option $dxdy$ is poor. Always question your choice if you need to split regions.

$$\int_1^2 \int_x^{2x} \frac{x}{y}\, dy\, dx$$

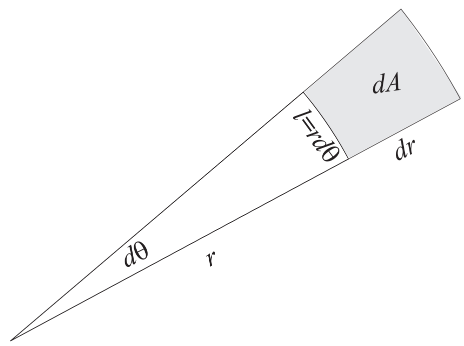
$$= \int_2^4 \int_{\frac{y}{2}}^{2} \frac{x}{y}\, dx\, dy + \int_1^2 \int_1^{y} \frac{x}{y}\, dx\, dy$$

★ $\frac{3}{2}\ln(2)$ ★

DOUBLE INTEGRALS IN POLAR COORDINATES

It is sometimes advantageous to abandon the rectangular coordinate system and replace it with polars. This is particularly true when dealing with circular objects or sums of squares.

Let us begin by defining polar coordinates and proving the transformation equations above



The shaded area dA is approximately a rectangle and hence its area is given by

$$dA \approx (r d\theta) dr = r dr d\theta$$

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Example 2 Find the polar coordinates of the point P whose Cartesian coordinates are $P(1, \sqrt{3})$.

$$r = \sqrt{1^2 + \sqrt{3}^2} = 2$$

$$\tan\theta = \frac{\sqrt{3}}{1}$$

$$\therefore \theta = \frac{\pi}{3}$$

$$\therefore \left(2, \frac{\pi}{3}\right)$$

★ $(2, \frac{\pi}{3})$ ★

Example 3 Find the Cartesian coordinates of the point Q whose polar coordinates are $Q(\sqrt{2}, \frac{3\pi}{4})$.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$= \sqrt{2}\cos\left(\frac{3\pi}{4}\right) \qquad = \sqrt{2}\sin\left(\frac{3\pi}{4}\right)$$

$$= -1 \qquad = 1$$

$$\therefore (-1, 1)$$

★ $(-1, 1)$ ★

Example 4 Express the equation $r = 2\cos(\theta)$ in Cartesian form

$$r = 2\cos\theta$$

$$r^2 = 2r\cos\theta = 2x$$

$$x^2 + y^2 = 2x$$

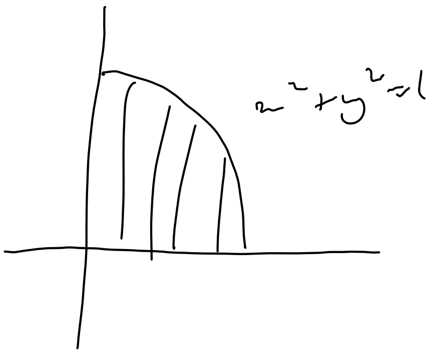
$$x^2 - 2xy - y^2 = 0$$

$$(x-1)^2 - y^2 = 1$$

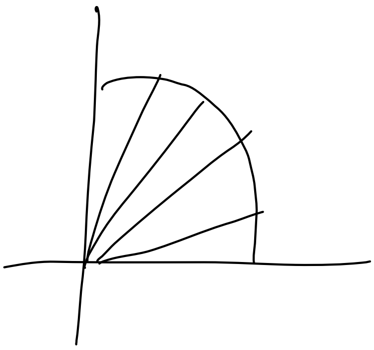
★ $(x - 1)^2 + y^2 = 1$ ★

When converting or presenting a double integral in polar form it is crucial to carefully consider a sketch of the underlying region Ω and to always remember that dA or $dydx$ or $dxdy$ are NOT replaced by $drd\theta$ bit rather by $rdrd\theta$. Unlike the Cartesian situation we almost never reverse the order and use $rd\theta dr$. It's *always* $rdrd\theta$.

Example 5 Evaluate $\displaystyle\iint_\Omega 3x\ dydx$ where Ω is the region in the first quadrant given by $\Omega = \{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$. That is Ω is the interior of a quarter circle centre the origin. First evaluate the integral in Cartesian coordinates and then in polars.



$$\int_0^1\int_0^{\sqrt{1-x^2}} 3x\ dy\,dx = \int_0^1 \Big[3xy\Big]_0^{\sqrt{1-x^2}} dx$$
$$= \int_0^1 3x\sqrt{1-x^2}\ dx$$
$$= \int_1^0 \frac{3\sqrt{u}}{-2}\ du$$
$$= \frac{3}{2}\left[\frac{2u^{3/2}}{3}\right]_0^1$$
$$= 1$$

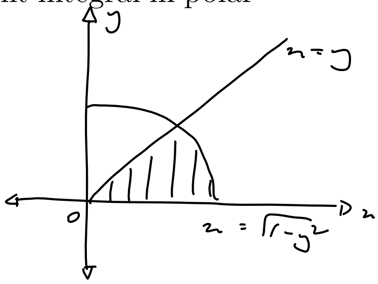


$$\iint_\Omega 3x\,dy\,dx = \iint 3r\cos\theta\,(r\,dr\,d\theta)$$
$$= \int_0^{\frac{\pi}{2}}\int_0^1 3r^2\cos\theta\ dr\ d\theta$$
$$= \int_0^{\frac{\pi}{2}} \Big[r^3\cos\theta\Big]_0^1\ d\theta$$
$$= \int_0^{\frac{\pi}{2}} \cos\theta\ d\theta$$
$$= 1$$

★ 1 ★

Example 6 Convert the following Cartesian integral into an equivalent integral in polar coordinates and evaluate:

$$\int_0^{\frac{1}{\sqrt{2}}} \int_y^{\sqrt{1-y^2}} x \; dxdy$$



$$= \int_0^{\frac{\pi}{4}} \int_0^{1} r\cos\theta \; r\, dr\, d\theta$$

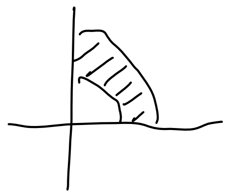
$$= \int_0^{\frac{\pi}{4}} \int_0^{1} r^2 \cos\theta \; dr\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[\frac{r^3}{3} \cos\theta \right]_0^1 d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{3}\cos(\theta) d\theta = \frac{1}{3}[\sin(\theta)]_0^{\frac{\pi}{4}} = \frac{1}{3\sqrt{2}}$$

$$\star \quad \frac{1}{3\sqrt{2}} \quad \star$$

Example 7 Evaluate $\iint_\Omega 2xy \; dydx$ where Ω is the region in the first quadrant between the circles of radius 2 and radius 5 centered at the origin.



$$= \int_0^{\frac{\pi}{2}} \int_2^{5} 2(r\cos\theta)(r\sin\theta)\cdot r \; dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_2^{5} r^3 (2\sin\theta\cos\theta) \; dr \cdot d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4} \sin 2\theta \right]_2^5 d\theta = \frac{1}{4} \int_0^{\frac{\pi}{2}} (5^4 - 2^4) \sin 2\theta \; d\theta$$

$$= \frac{609}{4} \int_0^{\frac{\pi}{2}} \sin(2\theta) d\theta = \frac{609}{4}[-\tfrac{1}{2}\cos(2\theta)]_0^{\frac{\pi}{2}} = \frac{609}{4}[-\tfrac{1}{2}(-1) + \tfrac{1}{2}] = \frac{609}{4}$$

$$\star \quad \frac{609}{4} \quad \star$$

Example 8 Suppose that Ω is the finite region bounded by the curve $r = 2\cos(\theta)$.

a) Sketch the region Ω in the plane.

b) Use polar coordinates to evaluate $\displaystyle\iint_\Omega \frac{3}{\sqrt{x^2+y^2}}\, dA$.

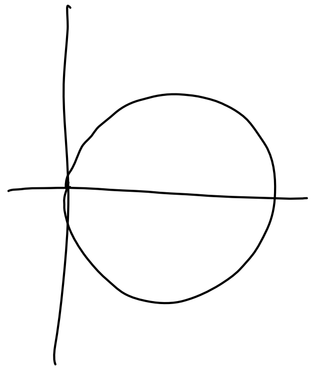
We saw in Example 3 that $r = 2\cos(\theta)$ is a circle centre at $(1,0)$ with a radius of 1.

$$\iint_\Omega \frac{3}{\sqrt{x^2+y^2}}\, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} \frac{3}{r}\cdot r\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} 3\, dr\, d\theta$$

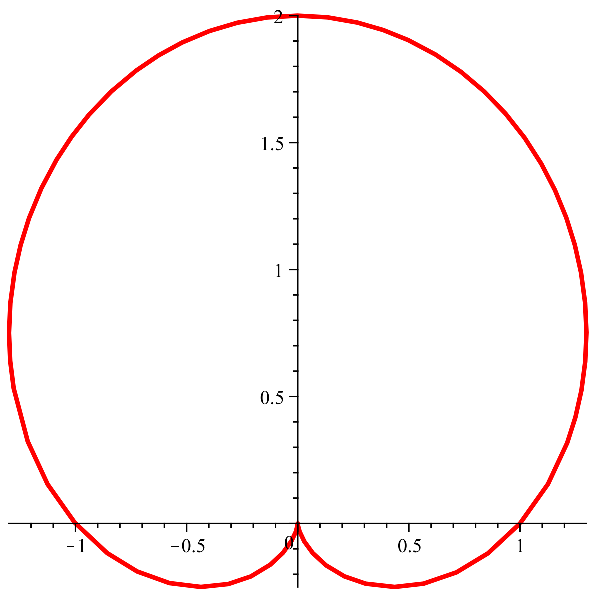
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 6\cos\theta\, d\theta$$

$$= \Big[6\sin\theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 12$$



★ 12 ★

Example 9 Find the area of the region in the first two quadrants bounded by the cardioid $r = 1+\sin(\theta)$.



$= \frac{1}{2}\int_0^\pi (1+\sin(\theta))^2 d\theta = \frac{1}{2}\int_0^\pi 1+2\sin(\theta)+\sin^2(\theta) d\theta = \frac{1}{2}\int_0^\pi 1+2\sin(\theta)+\frac{1}{2}(1-\cos(2\theta))d\theta$

$= \frac{1}{2}\int_0^\pi \frac{3}{2}+2\sin(\theta)-\frac{1}{2}\cos(2\theta)d\theta = \frac{1}{2}[\frac{3}{2}\theta - 2\cos(\theta) - \frac{1}{4}\sin(2\theta)]_0^\pi$

$= \frac{1}{2}\{(\frac{3\pi}{2} - 2(-1) - 0) - (0-2-0)\} = \frac{1}{2}(\frac{3\pi}{2}+4)$.

★ $\dfrac{3\pi}{4}+2$ ★

Jacobian Transformation (Optional)

The equation $dA = dxdy = rdrd\theta$ is a special case of a Jacobian transformation. If $x = x(u, v)$, $y = y(u, v)$ is a change of variable in a double integral then

$$\iint_{\Omega} f(x, y)dx\,dy = \iint_{\Omega^*} f\left(x(u, v), y(u, v)\right)|J|du\,dv$$

where Ω^* is the region in the (u, v) plane corresponding to Ω in the (x, y) plane and J is the Jacobian Determinant

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

If $x = x(r, \theta) = r\cos(\theta)$ and $y = y(r, \theta) = r\sin(\theta)$ then:

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} = \cos(\theta)r\cos(\theta) + r\sin(\theta)\sin(\theta) = r(\cos^2(\theta) + \sin^2(\theta)) = r.$$

Hence

$$\iint_{\Omega} f(x, y)dx\,dy = \iint_{\Omega^*} f\left(x(r, \theta), y(r, \theta)\right)r dr\,d\theta$$

[18] You can now do Q 68