MATH2019 PROBLEM CLASS
EXAMPLES 7
LAPLACE TRANSFORMS

$\boxed{\text{2014, S1}}$ 1. Using the table of Laplace transforms find

$$\mathcal{L}\left\{t+\sin(2t)+e^{-t}\right\}.$$

$\boxed{\text{2014, S1}}$ 2. i) By establishing an appropriate partial fraction decomposition find

$$\mathcal{L}^{-1}\left\{\frac{7s+1}{(s+1)(s-1)}\right\}.$$

ii) Hence or otherwise find

$$\mathcal{L}^{-1}\left\{\frac{7s+1}{(s+1)(s-1)}e^{-5s}\right\}.$$

$\boxed{\text{2014, S2}}$ 3. Find:

i) $\mathcal{L}\left\{e^t\cos(\pi t)+e^t\sin(\pi t)\right\}$.

ii) $\mathcal{L}^{-1}\left\{\dfrac{6}{s^2-4s+8}\right\}$.

$\boxed{\text{2014, S2}}$ 4. Use the Laplace transform method to solve the initial value problem

$$y''-y'=4u(t-2) \qquad \text{with} \qquad y(0)=1,\ y'(0)=1,$$

where $u(t-2)$ is a Heaviside step function.

$\boxed{\text{2015, S1}}$ 5. Find:

i) $\mathcal{L}\left\{t^3e^{\pi t}\right\}$.

ii) $\mathcal{L}^{-1}\left\{\dfrac{3-s}{s^2-4s+5}\right\}$.

$\boxed{\text{2015, S1}}$ 6. The function $f(t)$ is given by

$$f(t)=\begin{cases} 0 & \text{for } 0\le t<1, \\ t-1 & \text{for } 1\le t<3, \\ 2 & \text{for } t\ge 3. \end{cases}$$

i) Sketch the function $f(t)$ for $0\le t\le 4$.

ii) Write $f(t)$ in terms of the Heaviside step function $u(t-a)$.

iii) Hence, or otherwise, find the Laplace transform of $f(t)$.

$\boxed{\text{2015, S1}}$ 7. Use the Laplace transform method to solve the initial value problem

$$y''-4y=8u(t-1) \qquad \text{with} \qquad y(0)=1,\ y'(0)=2,$$

where $u(t-1)$ is a Heaviside step function.

2015, S2 8. Find:

i) $\mathcal{L}\left\{t^5 e^{3t}\right\}$.

ii) $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}s}{s^2+9}\right\}$.

2015, S2 9. The function $g(t)$ is given by

$$g(t) = \begin{cases} t & \text{for } 0 \leq t < 1 \\ e^t & \text{for } t \geq 1. \end{cases}$$

i) Sketch the function $g(t)$ for $0 \leq t \leq 2$.

ii) Write $g(t)$ in terms of the Heaviside step function.

iii) Hence, or otherwise, find the Laplace transform of $g(t)$.

2015, S2 10. i) Find the partial fraction decomposition of

$$\frac{30}{s(s+3)(s-2)}.$$

ii) Using the Laplace transform method and your answer in the previous part find the solution of the initial value problem

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 30u(t-4) \quad \text{with } y(0) = 0 \text{ and } y'(0) = 0,$$

where $u(t-4)$ is the Heaviside step function.

2016, S1 11. The Laplace transform of a function $f(t)$ is defined for $t \geq 0$ by

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_0^\infty f(t)e^{-st}dt.$$

i) Use Leibniz' rule to prove

$$\mathcal{L}\left\{tf(t)\right\} = -F'(s).$$

ii) Hence, or otherwise, find the following Laplace transform

$$\mathcal{L}\left\{t\sin(3t)\right\}.$$

2016, S1 12. Use the Laplace transform method to solve the initial value problem

$$y'' - y = u(t-1) \qquad \text{with} \qquad y(0) = 0,\ y'(0) = 1,$$

where $u(t-1)$ is a Heaviside step function.

2016, S2 13. Find:

i) $\mathcal{L}\left\{\mathrm{e}^{-3t}\sin(\pi t)\right\}$.

ii) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+3s-4}\right\}$.

2016, S2 14. The function $g(t)$ is given by

$$g(t) = \begin{cases} t & \text{for } 0 \le t < 2, \\ -1 & \text{for } t \ge 2. \end{cases}$$

i) Sketch the function $g(t)$ for $0 \le t \le 6$.

ii) Write $g(t)$ in terms of the Heaviside step function $u(t-a)$.

iii) Hence, or otherwise, show that the Laplace transform of $g(t)$ is given by

$$\mathcal{L}\{g(t)\} = \frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} + \frac{3}{s}\right).$$

2016, S2 15. A rocket is launched straight upwards at time $t = 0$ and its thrusters burn until $t = 2$. The vertical velocity $v(t)$ of the rocket satisfies the differential equation

$$\frac{dv}{dt} = g(t), \qquad v(0) = 0$$

where $g(t)$ is the function in the previous question.

i) Using Laplace transforms and the result from part iii) in the previous question, solve the differential equation above to find the velocity of the rocket $v(t)$ as a function of time.

ii) By writing your solution separately for times $0 \le t < 2$ and $t \ge 2$, or otherwise, sketch the velocity as a function of time for $0 \le t \le 6$.

iii) What is the maximum velocity of the rocket?

iv) At what time will the rocket reach its maximum height?

2017, S1 16. Find:

i) $\mathcal{L}\{t\,u(t-2)\}$.

ii) $\mathcal{L}^{-1}\left\{\dfrac{3s}{s^2 - 2s + 10}\right\}$.

2017, S1 17. The function $g(t)$ is given by

$$g(t) = \begin{cases} 1 & \text{for } 0 \le t < 1, \\ \mathrm{e}^{-t+1} & \text{for } t \ge 1. \end{cases}$$

i) Sketch the function $g(t)$ for $0 \le t \le 3$.

ii) Write $g(t)$ in terms of the Heaviside step function $u(t-a)$.

iii) Hence, or otherwise, show that the Laplace transform of $g(t)$ is

$$\mathcal{L}\{g(t)\} = \frac{1}{s} - \mathrm{e}^{-s}\left(\frac{1}{s} - \frac{1}{s+1}\right).$$

2017, S1 18. Use the Laplace transform method to solve the initial value problem

$$y'' - y' = g(t), \qquad y(0) = -1,\ y'(0) = 0,$$

where $g(t)$ is the function from the previous question.

2017, S2 19. i) The Laplace transform of a function $f(t)$ is defined for $t \geq 0$ by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt.$$

Prove directly from the above definition that

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

where $a > 0$ and $u(t-a)$ is the Heaviside function.

ii) Find:

α) $\mathcal{L}\{e^t\,u(t-3)\}$.

β) $\mathcal{L}^{-1}\left\{\dfrac{s+2}{s^2+2s+5}\right\}$.

2017, S2 20. The function $g(t)$ is given by

$$g(t) = \begin{cases} \sin(\pi t) & \text{for } 0 \leq t < 1, \\ 0 & \text{for } t \geq 1. \end{cases}$$

i) Sketch the function $g(t)$ for $0 \leq t \leq 2$.

ii) Write $g(t)$ in terms of the Heaviside step function $u(t-a)$.

iii) Hence, or otherwise, show that the Laplace transform of $g(t)$ is

$$\mathcal{L}\{g(t)\} = \frac{\pi}{s^2+\pi^2}\left(1+e^{-s}\right).$$

[Hint: You can use $\sin(A+\pi) = -\sin A$.]

2017, S2 21. Use the Laplace transform method to solve the initial value problem

$$y'' - y' - 2y = 6u(t-1), \qquad y(0) = 1,\ y'(0) = 2.$$

2018, S1 22. Find

i) $\mathcal{L}\{te^{-t}\sin(3t)\}$;

ii) $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2+4s+5}\right\}$.

2018, S1 23. The function $f(t)$ is defined for $t \geq 0$ by

$$f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ 0, & t \geq 1. \end{cases}$$

i) Express $f(t)$ in terms of the Heaviside function.

ii) Hence or otherwise find $\mathcal{L}\{f(t)\}$, the Laplace transform of $f(t)$.

2018, S1 24. Solve the differential equation

$$y'' - 4y' + 4y = f(t), \quad t > 0,$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = 0$, where $f(t)$ is given in the previous question.

$\boxed{\text{2018, S2}}$ 25. Find

i) $\mathcal{L}\{\sin(3t)\}$,

ii) $\mathcal{L}\{e^{-7t}\sin(3t)\}$,

iii) $\mathcal{L}^{-1}\left\{\dfrac{4s-28}{(s-1)(s-9)}\right\}$.

$\boxed{\text{2018, S2}}$ 26. The function $g(t)$ is defined for $t \geq 0$ by

$$g(t) = \begin{cases} t^2, & 0 \leq t < 1, \\ e^{2t}, & t \geq 1. \end{cases}$$

i) Express $g(t)$ in terms of the Heaviside function.

ii) Hence, or otherwise, show that the Laplace transform of $g(t)$ is

$$G(s) = \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) + \frac{e^{2-s}}{s-2}.$$

TABLE OF LAPLACE TRANSFORMS AND THEOREMS
on next page

TABLE OF LAPLACE TRANSFORMS AND THEOREMS

$g(t)$ is a function defined for all $t \geq 0$, and whose Laplace transform

$$G(s) = \mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} g(t) dt$$

exists. The Heaviside step function u is defined to be

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t > a \end{cases}$$

$g(t)$	$G(s) = \mathcal{L}\{g(t)\}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^m, m = 0, 1, \ldots$	$\frac{m!}{s^{m+1}}$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$e^{-\alpha t} f(t)$	$F(s+\alpha)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$tf(t)$	$-F'(s)$