MATH2019 PROBLEM CLASS
EXAMPLES 3
DIV, GRAD, CURL AND LINE INTEGRALS

$\boxed{1996}$ 1. A moving particle has position vector

$$\mathbf{r}(t) = \cos(\omega t)\,\mathbf{i} + \sin(\omega t)\,\mathbf{j} + t\,\mathbf{k}$$

where ω is a positive constant and t is time.

i) Find the acceleration of the particle and show that it has constant magnitude.
Solution: The acceleration $\mathbf{a}$ is the second derivative of the displacement (position vector) $\mathbf{r}$ with respect to t, i.e.,

$$\begin{aligned}\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} &= \frac{d^2}{dt^2}\left(\cos(\omega t)\,\mathbf{i} + \sin(\omega t)\,\mathbf{j} + t\,\mathbf{k}\right)\\ &= -\omega^2\cos(\omega t)\,\mathbf{i} - \omega^2\sin(\omega t)\,\mathbf{j} + 0\,\mathbf{k}\,.\end{aligned}$$

Hence the magnitude of the acceleration $\mathbf{a}$ is given by

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{(-\omega^2\cos(\omega t))^2 + (-\omega^2\sin(\omega t))^2 + 0^2} = \omega^2\,.$$

which is independent of t, i.e., constant.

ii) Describe the path of the particle.
Solution: The path is a helix (spiral) in $\mathbb{R}^3$.

iii) Find $\displaystyle\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$ where $\mathcal{C}$ is the portion of the path of the particle between $t = 0$ and $t = 2\pi/\omega$ and

$$\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}\,.$$

[Hint: Show that $\mathbf{F} = \nabla(xyz)$.]
Solution: A line integral for $\mathbf{F}$ is path independent if $\mathbf{F} = \boldsymbol{\nabla}\phi$ for some scalar field ϕ, called a ***scalar potential***. In the hint it says to show (or verify) $\mathbf{F} = \nabla(xyz)$. This is done by the following calculation.

$$\boldsymbol{\nabla}(xyz) = \frac{\partial}{\partial x}(xyz)\,\mathbf{i}\frac{\partial}{\partial y}(xyz)\,\mathbf{j} + \frac{\partial}{\partial z}(xyz)\,\mathbf{k} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k} = \mathbf{F}\,.$$

Thus a scalar potential ϕ exists for $\mathbf{F}$ and is given by $\phi = xyz(+\,\text{constant})$. Before we start calculating the line integral we determine the coordinates in $\mathbb{R}^3$ for the position vector $\mathbf{r}(t)$ at the start and end of the curve $\mathcal{C}$, i.e.,

$$\begin{aligned}\mathbf{r}(0) &= \cos(\omega 0)\,\mathbf{i} + \sin(\omega 0)\,\mathbf{j} + 0\,\mathbf{k} = \mathbf{i}\,,\\ \mathbf{r}(2\pi/\omega) &= \cos\left(\omega\frac{2\pi}{\omega}\right)\,\mathbf{i} + \sin\left(\omega\frac{2\pi}{\omega}\right)\,\mathbf{j} + \frac{2\pi}{\omega}\,\mathbf{k} = \mathbf{i} + \frac{2\pi}{\omega}\,\mathbf{k}\,.\end{aligned}$$

Thus

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{C}} \boldsymbol{\nabla}\phi\cdot d\mathbf{r} &= \phi\left(x(t), y(t), z(t)\right)\Big|_{t=0}^{t=2\pi/\omega}\\ &= \phi\left(1, 0, \frac{2\pi}{\omega}\right) - \phi(1,0,0)\\ &= (1)(0)\left(\frac{2\pi}{\omega}\right) + C - ((1)(0)(0) + C)\\ &= 0\,.\end{aligned}$$

In this problem you could have determined the line integral directly, i.e.,

$$
\begin{aligned}
d\mathbf{r} &= \frac{d\mathbf{r}}{dt}dt = \left(-\omega\sin(\omega t)\,\mathbf{i} + \omega\cos(\omega t)\,\mathbf{j} + \mathbf{k}\right)\,dt = \begin{pmatrix} -\omega\sin(\omega t) \\ \omega\cos(\omega t) \\ 1 \end{pmatrix} dt\,, \\
\mathbf{F}\left(\mathbf{r}(t)\right) &= t\sin(\omega t)\,\mathbf{i} + t\cos(\omega t)\,\mathbf{j} + \cos(\omega t)\sin(\omega t)\,\mathbf{k} = \begin{pmatrix} t\sin(\omega t) \\ t\cos(\omega t) \\ \cos(\omega t)\sin(\omega t) \end{pmatrix}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi/\omega} \begin{pmatrix} t\sin(\omega t) \\ t\cos(\omega t) \\ \cos(\omega t)\sin(\omega t) \end{pmatrix} \cdot \begin{pmatrix} -\omega\sin(\omega t) \\ \omega\cos(\omega t) \\ 1 \end{pmatrix} dt \\
&= \int_0^{2\pi/\omega} \left[\omega t\left(\cos^2(\omega t) - \sin^2(\omega t)\right) + \cos(\omega t)\sin(\omega t)\right]\,dt \\
&= \int_0^{2\pi/\omega} \left(\omega t\cos(2\omega t) + \frac{1}{2}\sin(2\omega t)\right)\,dt \quad \text{using trig. identities} \\
&= \frac{1}{2}t\sin(2\omega t)\Big|_0^{2\pi/\omega} - \frac{1}{2}\int_0^{2\pi/\omega}\sin(2\omega t)\,dt + \frac{1}{2}\int_0^{2\pi/\omega}\sin(2\omega t)\,dt \quad \text{integrate by parts} \\
&= (0-0) \\
&= 0\,.
\end{aligned}
$$

Note: *In real life, you would like to know if a scalar potential ϕ exists for a vector field $\mathbf{F}$. Such a ϕ will exist if $\mathbf{F}$ is irrotational (or conservative), i.e., $curl\mathbf{F} = \mathbf{0}$. Thus for the vector field $\mathbf{F}$ we have*

$$
\begin{aligned}
curl\mathbf{F} &= \boldsymbol{\nabla}\times\mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz)\right)\mathbf{k} \\
&= (x-x)\mathbf{i} - (y-y)\mathbf{j} + (z-z)\mathbf{k} \\
&= \mathbf{0}\,.
\end{aligned}
$$

Thus $\mathbf{F}$ is irrotational (or conservative) and ϕ exists. Next you would like to determine an expression for ϕ. You would use the following proceedure to do this. Since $\mathbf{F} = \boldsymbol{\nabla}\phi$ for some unknown ϕ, we can use our knowledge of $\mathbf{F}$ and the definition of $\boldsymbol{\nabla}\phi$ to determine ϕ, i.e.,

$$
\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k} = \boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\,.
$$

Hence

$$
\begin{aligned}
\frac{\partial\phi}{\partial x} &= yz\,, && (1) \\
\frac{\partial\phi}{\partial y} &= xz\,, && (2) \\
\frac{\partial\phi}{\partial z} &= xy\,. && (3)
\end{aligned}
$$

Integrate (1) w.r.t. x yields

$$\phi(x,y,z) = xyz + g(y,z) \tag{4}$$

so that

$$\frac{\partial \phi}{\partial y} = xz + \frac{\partial g}{\partial y}\,. \tag{5}$$

Comparing (2) and (5) gives $\dfrac{\partial g}{\partial y} = 0$ *so that* $g(y,z) = h(z)$*. Thus (4) is updated to*

$$\phi(x,y,z) = xyz + h(z) \tag{6}$$

so that

$$\frac{\partial \phi}{\partial z} = xy + h'(z)\,. \tag{7}$$

Comparing (3) and (7) gives $h'(z) = 0$ *so that* $h(z) = C($*a constant*$)$*. Thus (6) is updated to*

$$\phi(x,y,z) = xyz + C\,. \tag{8}$$

Understanding the above procedure for determining ϕ *(up to an arbitrary constant* C*) allows the compact notation for determining* ϕ *below:*

$$\begin{array}{lcl}
\dfrac{\partial \phi}{\partial x} = yz & \Rightarrow & \phi = xyz + g(y,z) \\
 & & \Downarrow \\
\dfrac{\partial \phi}{\partial y} = xz & \Leftrightarrow & \dfrac{\partial \phi}{\partial y} = xz + \dfrac{\partial g}{\partial y} \\
 & & \qquad \Rightarrow \quad \dfrac{\partial g}{\partial y} = 0 \quad \Rightarrow \quad g(y,z) = h(z) \\
 & & \Downarrow \\
\dfrac{\partial \phi}{\partial z} = xy & \Leftrightarrow & xz + h'(z) \\
 & & \qquad \Rightarrow \quad h'(z) = 0 \quad \Rightarrow \quad h(z) = C\,(a\ constant) \\
 & & \Downarrow \\
 & & \phi = xyz + C\,.
\end{array}$$

In an exam, we could ask you to show (or verify) a vector field $\mathbf{F}$ *is irrotational (or conservative) and hence determine the scalar potential* ϕ *for vector field* $\mathbf{F}$ *OR ask you to show (or verify) a given* ϕ *is a scalar potential for a given vector field* $\mathbf{F}$*.*

2014, S1 2. Given the vector field $\mathbf{G} = yz^2\,\mathbf{i} + xz^2\,\mathbf{j} + 2xyz\,\mathbf{k}$ calculate:

i) div $\mathbf{G}$.

Solution:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(yz^2\,\mathbf{i} + xz^2\,\mathbf{j} + 2xyz\,\mathbf{k}\right) \\
&= \frac{\partial}{\partial x}(yz^2) + \frac{\partial}{\partial y}(xz^2) + \frac{\partial}{\partial z}(2xyz) \\
&= 0 + 0 + 2xy \\
&= 2xy\,.
\end{aligned}$$

ii) curl $\mathbf{G}$.

Solution:

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz^2 & xz^2 & 2xyz \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(2xyz) - \frac{\partial}{\partial z}(xz^2)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(2xyz) - \frac{\partial}{\partial z}(yz^2)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(xz^2) - \frac{\partial}{\partial y}(yz^2)\right)\mathbf{k} \\
&= (2xz - 2xz)\mathbf{i} - (2yz - 2yz)\mathbf{j} + (z^2 - z^2)\mathbf{k} \\
&= 0\,\mathbf{i} + 0\mathbf{j} + 0\,\mathbf{k} \\
&= \mathbf{0}\,.
\end{aligned}$$

2014, S1 3. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be a path in space embedded within the surface $\phi(x, y, z) = 1$. Assuming that all relevant derivatives exist use the chain rule to show that grad ϕ is perpendicular to the velocity vector $\mathbf{v}(t)$ for all t.

Solution: See lecture notes.

2014, S2 4. Given the vector field $\mathbf{F} = \sin x\,\mathbf{i} + \cos x\,\mathbf{j} + xyz\,\mathbf{k}$ calculate:

i) div $\mathbf{F}$.

Solution:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(\sin x\,\mathbf{i} + \cos x\,\mathbf{j} + xyz\,\mathbf{k}\right) \\
&= \frac{\partial}{\partial x}(\sin x) + \frac{\partial}{\partial y}(\cos x) + \frac{\partial}{\partial z}(xyz) \\
&= \cos x - 0 + xy \\
&= \cos x + xy\,.
\end{aligned}$$

ii) curl $\mathbf{F}$.

Solution:

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin x & \cos x & xyz \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(xyz) - \frac{\partial}{\partial z}(\cos x)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial z}(\sin x)\right)\mathbf{j} \\
&\qquad + \left(\frac{\partial}{\partial x}(\cos x) - \frac{\partial}{\partial y}(\sin x)\right)\mathbf{k} \\
&= (xz-0)\mathbf{i} - (yz-0)\mathbf{j} + (-\sin x - 0)\mathbf{k} \\
&= xz\,\mathbf{i} - yz\,\mathbf{j} - \sin x\,\mathbf{k}\,.
\end{aligned}$$

$\boxed{\text{2015, S1}}$ 5. Given the vector field $\mathbf{F} = xz\,\mathbf{i} + y^2\,\mathbf{j} + yz\,\mathbf{k}$ calculate:

i) div $\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F}$,

Solution:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(xz\,\mathbf{i} + y^2\,\mathbf{j} + yz\,\mathbf{k}\right) \\
&= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz) \\
&= z + 2y + y \\
&= z + 3y\,.
\end{aligned}$$

ii) curl $\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{F}$ and

Solution:

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & y^2 & yz \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(y^2)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial z}(xz)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(xz)\right)\mathbf{k} \\
&= (z-0)\mathbf{i} - (0-x)\mathbf{j} + (0-0)\mathbf{k} \\
&= z\,\mathbf{i} + x\,\mathbf{j}\,.
\end{aligned}$$

iii) div (curl $\mathbf{F}$) $= \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{F})$.

Solution:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{F}) &= \boldsymbol{\nabla}\cdot(z\,\mathbf{i} + x\,\mathbf{j} + 0\,\mathbf{k}) \\
&= \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(0) \\
&= 0 + 0 + 0 \\
&= 0\,.
\end{aligned}$$

2016, S2 6. i) Suppose that

$$\mathbf{r}_1(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$$

and

$$\mathbf{r}_2(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$$

are two curves in $\mathbb{R}^3$. Prove that

$$[\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)]' = \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t) + \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t).$$

Solution: Note

$$\mathbf{r}_1'(t) = x_1'(t)\mathbf{i} + y_1'(t)\mathbf{j} + z_1'(t)\mathbf{k} \quad \text{and} \quad \mathbf{r}_2'(t) = x_2'(t)\mathbf{i} + y_2'(t)\mathbf{j} + z_2'(t)\mathbf{k}\,.$$

Thus

$$\begin{aligned}
\frac{d}{dt}[\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)] &= \frac{d}{dt}[(x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}) \cdot (x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k})] \\
&= \frac{d}{dt}[x_1(t)x_2(t) + y_1(t)y_2(t) + z_1(t)z_2(t)] \\
&= x_1'(t)x_2(t) + x_1(t)x_2'(t) + y_1'(t)y_2(t) + y_1(t)y_2'(t) + z_1'(t)z_2(t) + z_1(t)z_2'(t) \\
&= (x_1'(t)x_2(t) + y_1'(t)y_2(t) + z_1'(t)z_2(t)) + (x_1(t)x_2'(t) + y_1(t)y_2'(t) + z_1(t)z_2'(t)) \\
&= \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t) + \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t)\,.
\end{aligned}$$

ii) Suppose that a particle P moves along a curve $\mathcal{C}$ in $\mathbb{R}^3$ in such a manner that its velocity vector is always perpendicular to its position vector. Using part i) prove that the path $\mathcal{C}$ lies on the surface of a sphere whose centre is the origin.

Solution: Let $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. The velocity vector $\mathbf{v}$ is the derivative of the position vector $\mathbf{r}$, i.e., $\mathbf{v} = \mathbf{r}'$. Since the velocity vector is always perpendicular to the position vector along $\mathcal{C}$ then $\mathbf{v} \cdot \mathbf{r} = \mathbf{r}' \cdot \mathbf{r} = 0$. From part i)

$$\frac{d}{dt}[\mathbf{r} \cdot \mathbf{r}] = \mathbf{r}' \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{r}' = 2\,\mathbf{r}' \cdot \mathbf{r} = 0\,.$$

But $\mathbf{r} \cdot \mathbf{r} = \|\mathbf{r}\|^2 = x^2 + y^2 + z^2$ and thus

$$\frac{d}{dt}[\mathbf{r} \cdot \mathbf{r}] = \frac{d}{dt}\left[x^2 + y^2 + z^2\right] = 0 \quad \Rightarrow \quad x^2 + y^2 + z^2 = C(\text{a constant})\,.$$

Hence $x^2 + y^2 + z^2 = C$ along $\mathcal{C}$ which is the equation of a sphere centred at the origin.

2016, S2 7. Consider the vector field $\mathbf{F} = (27y - y^3)\mathbf{i} + x^3\,\mathbf{j} + (x - xz)\mathbf{k}$.

i) Calculate curl $\mathbf{F}$.

Solution:

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 27y - y^3 & x^3 & x - xz \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(x - xz) - \frac{\partial}{\partial z}(x^3)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(x - xz) - \frac{\partial}{\partial z}(27y - y^3)\right)\mathbf{j} \\
&\qquad + \left(\frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial y}(27y - y^3)\right)\mathbf{k} \\
&= (0 - 0)\mathbf{i} - (1 - z - 0)\mathbf{j} + (3x^2 - (27 - 3y^2))\mathbf{k} \\
&= (z - 1)\,\mathbf{j} + 3(x^2 + y^2 - 9)\mathbf{k}\,.
\end{aligned}$$

ii) Sketch the curve $\mathcal{C}$ in $\mathbb{R}^3$ for which $\operatorname{curl}\mathbf{F}=\mathbf{0}$.

Solution:

$$\begin{aligned}
\operatorname{curl}\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{F} = \mathbf{0} \;&\Rightarrow\; (z-1)\,\mathbf{j} + 3(x^2-y^2-9)\mathbf{k} = \mathbf{0} \\
&\Rightarrow\; z-1=0 \quad \text{and} \quad x^2+y^2-9=0 \\
&\Rightarrow\; z=1 \quad \text{and} \quad x^2+y^2=9
\end{aligned}$$

Thus the curve $\mathcal{C}$ where $\operatorname{curl}\mathbf{F}=\mathbf{0}$ is the intersection of the plane $z=1$ and the cylinder $x^2+y^2=9$, i.e., a circle of radius 3 in the xy-plane, centred at $(0,0,1)$.

2017, S2

8. A vector field is given by

$$\mathbf{F}(x,y,z) = \sin x \sin y\;\mathbf{k}.$$

i) Calculate $\boldsymbol{\nabla}\times\mathbf{F}$.

Solution:

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & \sin x\sin y \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(\sin x\sin y) - \frac{\partial}{\partial z}(0)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(\sin x\sin y) - \frac{\partial}{\partial z}(0)\right)\mathbf{j} \\
&\qquad + \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(0)\right)\mathbf{k} \\
&= (\sin x\cos y - 0)\mathbf{i} - (\cos x\sin y - 0)\mathbf{j} + (0-0)\mathbf{k} \\
&= \sin x\cos y\,\mathbf{i} - \cos x\sin y\,\mathbf{j}\,.
\end{aligned}$$

ii) Calculate $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F})$.

Solution:

$$\begin{aligned}
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin x\cos y & -\cos x\sin y & 0 \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-\cos x\sin y)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(\sin x\cos y)\right)\mathbf{j} \\
&\qquad + \left(\frac{\partial}{\partial x}(-\cos x\sin y) - \frac{\partial}{\partial y}(\sin x\cos y)\right)\mathbf{k} \\
&= (0-0)\mathbf{i} - (0-0)\mathbf{j} + (\sin x\sin y + \sin x\sin y)\mathbf{k} \\
&= 2\sin x\sin y\;\mathbf{k} \\
&= 2\,\mathbf{F}\,.
\end{aligned}$$

iii) Hence, or otherwise, evaluate $\nabla \times (\nabla \times (\nabla \times (\nabla \times \mathbf{F})))$.

Solution: Using the result from part ii), i.e., $\nabla \times (\nabla \times \mathbf{F}) = 2\,\mathbf{F}$ we have

$$\begin{aligned}
\nabla \times (\nabla \times \underbrace{(\nabla \times (\nabla \times \mathbf{F}))}_{2\,\mathbf{F}}) &= 2\,\underbrace{\nabla \times (\nabla \times \mathbf{F})}_{2\,\mathbf{F}} \\
&= 4\,\mathbf{F} \\
&= 4 \sin x \sin y\, \mathbf{k}\,.
\end{aligned}$$

$\boxed{\text{2014, S1}}$ 9. By evaluating an appropriate line integral calculate the work done on a particle traveling in $\mathbb{R}^3$ through the vector field $\mathbf{F} = -y\mathbf{i} + xyz\mathbf{j} + x^2\mathbf{k}$ along the straight line from $(1,2,3)$ to $(2,2,5)$.

Solution: Let $\mathcal{C}$ denote the straight line path and the work done along path $\mathcal{C}$ is given by $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$. First determine a vector parametric form for the straight line path $\mathcal{C}$.

$$\mathbf{r}(t) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\left(\begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad t \in [0,1]\,.$$

Thus

$$\begin{aligned}
d\mathbf{r} &= \frac{d\mathbf{r}}{dt}dt = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} dt\,, \\
\mathbf{F}\,(\mathbf{r}(t)) &= -2\,\mathbf{i} + (1+t)(2)(3+2t)\,\mathbf{j} + (1+t)^2\,\mathbf{k} = \begin{pmatrix} -2 \\ 2(1+t)(3+2t) \\ (1+t)^2 \end{pmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}\,(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\,dt &= \int_0^1 \begin{pmatrix} -2 \\ 2(1+t)(3+2t) \\ (1+t)^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} dt \\
&= 2\int_0^1 \left(-1 + (1+t)^2\right) dt \\
&= 2\left(-t + \frac{1}{3}(1+t)^3\Big|_0^1\right) \\
&= 2\left(-1 + \frac{8}{3} - \left(0 + \frac{1}{3}\right)\right) \\
&= \frac{8}{3}\,.
\end{aligned}$$

$\boxed{\text{2014, S2}}$ 10. Let $\mathcal{C}$ denote the path taken by a particle travelling in a straight line from point $P(-2,3,0)$ to point $Q(-2,0,3)$.

i) Write down a vector function $\mathbf{r}(t)$ that describes the path $\mathcal{C}$ and give the value of t at the start and the end of the path.

Solution:

$$\mathbf{r}(t) = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + t\left(\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}, \quad t \in [0,1]\,.$$

ii) If $\mathbf{F} = y^2\,\mathbf{i} + xyz\,\mathbf{j} - z^2\,\mathbf{k}$ evaluate the line integral $\displaystyle\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$.

Solution: Using $\mathbf{r}(\mathbf{t})$ from part i) we have

$$\begin{aligned} d\mathbf{r} &= \frac{d\mathbf{r}}{dt}dt = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix} dt\,, \\ \mathbf{F}\left(\mathbf{r}(t)\right) &= (3-3t)^2\,\mathbf{i} + (-2)(3-3t)(3t)\,\mathbf{j} - (3t)^2\,\mathbf{k} = \begin{pmatrix} 9(1-t)^2 \\ -18(t-t^2) \\ -9t^2 \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}\left(\mathbf{r}(t)\right)\cdot\frac{d\mathbf{r}}{dt}\,dt &= \int_0^1 \begin{pmatrix} 9(1-t)^2 \\ -18(t-t^2) \\ -9t^2 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix} dt \\ &= 27\int_0^1 \left(2t - 2t^2 - t^2\right) dt \\ &= 27\int_0^1 \left(2t - 3t^2\right) dt \\ &= 27\left(t^2 - t^3\big|_0^1\right) \\ &= 27\left(1 - 1 - (0-0)\right) \\ &= 0\,. \end{aligned}$$

$\boxed{\text{2015, S1}}$ 11. Let $\mathcal{C}$ denote the path taken by a particle travelling anticlockwise around the unit circle, starting *and* ending at the point $(1,0)$ [i.e., the particle travels completely around the circle].

i) Write down a vector function $\mathbf{r}(t)$ that describes the path $\mathcal{C}$ and give the value of t at the start and the end of the path.

Solution: Since the path is a circle, use polar coordinates to parametrise the path $\mathcal{C}$, i.e.,

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}\,,\quad t\in[0,2\pi]\ .$$

Note the start of start of path $\mathcal{C}$ is when $t=0$ ($\mathbf{r}(0) = (1,0)$) and the end is when $t = 2\pi$ ($\mathbf{r}(2\pi) = (1,0)$).

ii) If $\mathbf{F} = -3y\,\mathbf{i} + 3x\,\mathbf{j}$ evaluate the line integral $\displaystyle\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$.

Solution: Using $\mathbf{r}(t)$ from part i) we have

$$\begin{aligned} d\mathbf{r} = &= \frac{d\mathbf{r}}{dt}\,dt = \left(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j}\right) dt = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt\,, \\ \mathbf{F}\left(\mathbf{r}(t)\right) &= -3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} = \begin{pmatrix} -3\sin t \\ 3\cos t \end{pmatrix}. \end{aligned}$$

Hence

$$\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}\left(\mathbf{r}(t)\right)\cdot\frac{d\mathbf{r}}{dt}\,dt \;=\; \int_0^{2\pi}\begin{pmatrix}-3\sin t\\ 3\cos t\end{pmatrix}\cdot\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}dt$$
$$= 3\int_0^{2\pi}\left(\sin^2 t+\cos^2 t\right)dt$$
$$= 3\int_0^{2\pi} dt$$
$$= 6\pi\,.$$

$\boxed{\text{2015, S2}}$ 12. Given a vector field

$$\mathbf{F} = 8e^{-x}\,\mathbf{i} + \cosh z\,\mathbf{j} - y^2\,\mathbf{k}$$

i) Compute $\boldsymbol{\nabla}\cdot\mathbf{F}$ (i.e., div $\mathbf{F}$) and $\boldsymbol{\nabla}\times\mathbf{F}$ (i.e., curl $\mathbf{F}$).

Solution:

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(8e^{-x}\,\mathbf{i}+\cosh z\,\mathbf{j}-y^2\,\mathbf{k}\right)\\ &= \frac{\partial}{\partial x}(8e^{-x})+\frac{\partial}{\partial y}(\cosh z)+\frac{\partial}{\partial z}(-y^2)\\ &= -8e^{-x}+0+0\\ &= -8e^{-x}\,,\end{aligned}$$

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 8e^{-x} & \cosh z & -y^2\end{vmatrix}\\ &= \left(\frac{\partial}{\partial y}(-y^2)-\frac{\partial}{\partial z}(\cosh z)\right)\mathbf{i}-\left(\frac{\partial}{\partial x}(-y^2)-\frac{\partial}{\partial z}(8e^{-x})\right)\mathbf{j}\\ &\qquad+\left(\frac{\partial}{\partial x}(\cosh z)-\frac{\partial}{\partial y}(8e^{-x})\right)\mathbf{k}\\ &= (-2y-\sinh z)\mathbf{i}-(0-0)\mathbf{j}+(0-0)\mathbf{k}\\ &= -(2y+\sinh z)\mathbf{i}\,.\end{aligned}$$

ii) Calculate the line integral $\int_{\mathcal{C}}\mathbf{F}\cdot d\mathbf{r}$ where $\mathcal{C}$ is the straight line path from $A(0,1,0)$ to $B(\ln(2)\ 1,\ 2)$.

Solution: First determine a vector parametric form for the straight line path $\mathcal{C}$.

$$\mathbf{r}(t) = \begin{pmatrix}0\\1\\0\end{pmatrix}+t\left(\begin{pmatrix}\ln 2\\1\\2\end{pmatrix}-\begin{pmatrix}0\\1\\0\end{pmatrix}\right)=\begin{pmatrix}0\\1\\0\end{pmatrix}+t\begin{pmatrix}\ln 2\\0\\2\end{pmatrix}\,,\quad t\in[0,1]\,.$$

Thus

$$d\mathbf{r} = \frac{d\mathbf{r}}{dt}dt = \begin{pmatrix}\ln 2\\0\\2\end{pmatrix}dt\,,$$

$$\mathbf{F}\left(\mathbf{r}(t)\right) = 8e^{-t\ln 2}\,\mathbf{i}+\cosh 2t\,\mathbf{j}-\mathbf{k} = \begin{pmatrix}8e^{-t\ln 2}\\ \cosh 2t\\ -1\end{pmatrix}.$$

Hence

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}\left(\mathbf{r}(t)\right)\cdot \frac{d\mathbf{r}}{dt}\,dt \;=\; \int_0^1 \begin{pmatrix} 8e^{-t\ln 2} \\ \cosh 2t \\ -1 \end{pmatrix}\cdot\begin{pmatrix} \ln 2 \\ 0 \\ 2 \end{pmatrix} dt$$
$$= \int_0^1 \left(8\ln 2 e^{-t\ln 2} - 2\right) dt$$
$$= -8e^{-t\ln 2} - 2t\Big|_0^1$$
$$= 8e^{-\ln 2} - 2 - (-8 - 0)$$
$$= 6 - 8e^{-\ln 2}\,.$$

2017, S1

13. A charged particle moves in an electric field given by

$$\mathbf{F}(x, y) = 3y\mathbf{i} - 3x\mathbf{j}.$$

Let $\mathcal{C}$ denote the path taken by the particle travelling anticlockwise around the unit circle, starting at $(1, 0)$ and ending at $(0, 1)$.

i) Write down a vector function $\mathbf{r}(\theta)$ that describes the path $\mathcal{C}$ and give the values of θ at the start and the end of the path.

Solution: Since the path is a circle, use polar coordinates to parametrise the path $\mathcal{C}$, i.e.,

$$\mathbf{r}(\theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}\,, \quad \theta \in \left[0, \frac{\pi}{2}\right]\,.$$

Note the start of start of path $\mathcal{C}$ is when $\theta = 0$ ($\mathbf{r}(0) = (1, 0)$) and the end is when $\theta = \frac{\pi}{2}$ ($\mathbf{r}\left(\frac{\pi}{2}\right) = (0, 1)$).

ii) Calculate the work done on the particle as it moves along the path $\mathcal{C}$ by evaluating the line integral $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$.

Solution: We need to determine $\mathbf{F}$ along $\mathcal{C}$, i.e., $\mathbf{F}\left(\mathbf{r}(\theta)\right)$ and $d\mathbf{r}$. Thus from part i)

$$d\mathbf{r} = \;=\; \frac{d\mathbf{r}}{d\theta}\,d\theta = \left(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\right)d\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} d\theta\,,$$
$$\mathbf{F}\left(\mathbf{r}(\theta)\right) \;=\; 3\sin\theta\,\mathbf{i} - 3\cos\theta\,\mathbf{j} = \begin{pmatrix} 3\sin\theta \\ -3\cos\theta \end{pmatrix}\,.$$

Hence

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_0^{\frac{\pi}{2}} \mathbf{F}\left(\mathbf{r}(\theta)\right)\cdot\frac{d\mathbf{r}}{d\theta}\,d\theta \;=\; \int_0^{\frac{\pi}{2}} \begin{pmatrix} 3\sin\theta \\ -3\cos\theta \end{pmatrix}\cdot\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} d\theta$$
$$= -3\int_0^{\frac{\pi}{2}} \left(\sin^2\theta + \cos^2\theta\right) d\theta$$
$$= -3\int_0^{\frac{\pi}{2}} d\theta$$
$$= -\frac{3\pi}{2}\,.$$

2018, S1

14. Consider the scalar field
$$\phi(x,y,z) = xe^{z-1} + \cos y$$
and let $\mathbf{F} = \boldsymbol{\nabla}\phi$.

i) Calculate $\mathbf{F}$.

Solution: The important thing to note in this problem since $\mathbf{F} = \boldsymbol{\nabla}\phi$ then $\mathbf{F}$ is ***conservative***.
$$\mathbf{F} = \boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} = e^{z-1}\,\mathbf{i} - \sin y\,\mathbf{j} + xe^{z-1}\,\mathbf{k}\,.$$

ii) What is $\boldsymbol{\nabla} \times \mathbf{F}$?

Solution: Could quote the result curl grad $\phi = \boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \mathbf{0}$, i.e., the curl of a conservative vector field is zero and hence $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$ OR explicitly calculate (and a lot more work)
$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{z-1} & -\sin y & xe^{z-1} \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(xe^{z-1}) - \frac{\partial}{\partial z}(-\sin y)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(xe^{z-1}) - \frac{\partial}{\partial z}(e^{z-1})\right)\mathbf{j} \\
&\qquad + \left(\frac{\partial}{\partial x}(-\sin y) - \frac{\partial}{\partial y}(e^{z-1})\right)\mathbf{k} \\
&= (0-0)\,\mathbf{i} - \left(e^{z-1} - e^{z-1}\right)\,\mathbf{j} + (0-0)\,\mathbf{k} \\
&= \mathbf{0}\,.
\end{aligned}$$

iii) Hence, or otherwise, calculate the line integral $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ along the straight line path $\mathcal{C}$ from $(1,0,1)$ to $(5,\pi,1)$.

Solution: Since $\mathbf{F}$ is conservative ANY line integral is ***path independent***. Thus ANY line integral for a conservative vector field $\mathbf{F} = \boldsymbol{\nabla}\phi$ depends ONLY on the value of ϕ at the end points, i.e.,
$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \phi(5,\pi,1) - \phi(1,0,1) &= (5e^{1-1} + \cos\pi) - (1e^{1-1} + \cos 0) \\
&= (5-1) - (1+1) = 2\,.
\end{aligned}$$

A student who doesn't realise $\mathbf{F}$ is ***conservative*** can still solve the problem with first determining a vector parametric form for the straight line path $\mathcal{C}$, i.e.,
$$\mathbf{r}(t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\left(\begin{pmatrix} 5 \\ \pi \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 4 \\ \pi \\ 0 \end{pmatrix}, \quad t \in [0,1]\,.$$

Thus

$$d\mathbf{r} = \frac{d\mathbf{r}}{dt}dt = \begin{pmatrix} 4 \\ \pi \\ 0 \end{pmatrix} dt\,,$$

$$\mathbf{F}\left(\mathbf{r}(t)\right) = \boldsymbol{\nabla}\phi\left(\mathbf{r}(t)\right) = 1\,\mathbf{i} - \sin(t\pi)\,\mathbf{j} + (1+4t)\mathbf{k} = \begin{pmatrix} 1 \\ -\sin(t\pi) \\ 1+4t \end{pmatrix}.$$

Hence

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}\left(\mathbf{r}(t)\right)\cdot\frac{d\mathbf{r}}{dt}\,dt &= \int_0^1 \begin{pmatrix} 1 \\ -\sin(t\pi) \\ 1+4t \end{pmatrix}\cdot\begin{pmatrix} 4 \\ \pi \\ 0 \end{pmatrix} dt \\
&= \int_0^1 \left(4 - \pi\sin(t\pi)\right) dt \\
&= 4t + \cos(t\pi)\big|_0^1 \\
&= (4-1) - (0+1) \\
&= 2\,.
\end{aligned}$$

Obviously this second method is more work (and not the point of the question).

$\boxed{\text{2018, S2}}$ 15. Consider the vector field

$$\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + (2xyz+3)\mathbf{k}.$$

i) Calculate div $\mathbf{F}$.

Solution:

$$\text{div }\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial}{\partial x}(yz^2) + \frac{\partial}{\partial y}(xz^2)\frac{\partial}{\partial z}(2xyz+3) = 0 + 0 + 2xy = 2xy\,.$$

ii) Show that $\mathbf{F}$ is conservative by evaluating curl $\mathbf{F}$.

Solution: $\mathbf{F}$ is conservative if curl $\mathbf{F} = \mathbf{0}$.

$$\begin{aligned}
\text{curl }\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz^2 & xz^2 & 2xyz+3 \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(2xyz+3) - \frac{\partial}{\partial z}(xz^2)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(2xyz+3) - \frac{\partial}{\partial z}(yz^2)\right)\mathbf{j} \\
&\qquad + \left(\frac{\partial}{\partial x}(xz^2) - \frac{\partial}{\partial y}(yz^2)\right)\mathbf{k} \\
&= (2xz - 2xz)\,\mathbf{i} - (2yz - 2yz)\,\mathbf{j} + (z^2 - z^2)\,\mathbf{k} \\
&= \mathbf{0}\,.
\end{aligned}$$

Hence $\mathbf{F}$ is conservative.

iii) The path $\mathcal{C}$ in $\mathbb{R}^3$ starts at the point $(3,4,7)$ and subsequently travels anticlockwise four complete revolutions around the circle $x^2+y^2=25$ within the plane $z=7$, returning to the starting point $(3,4,7)$. Using part ii) or otherwise, evaluate the work integral $\displaystyle\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$.

Solution: Since $\mathbf{F}$ is conservative and $\mathcal{C}$ is a closed curve then $\displaystyle\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = 0\,.$