LECTURE 36
THE SHIFTING THEOREMS

LAPLACE TRANSFORMS

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^m	$m!/s^{m+1}$
t^ν , $(\nu > -1)$	$\Gamma(\nu+1)/s^{\nu+1}$
e^{-at}	$1/(s+a)$
$\sin bt$	$b/(s^2+b^2)$
$\cos bt$	$s/(s^2+b^2)$
$\sinh bt$	$b/(s^2-b^2)$
$\cosh bt$	$s/(s^2-b^2)$
$\sin bt - bt\cos bt$	$2b^3/(s^2+b^2)^2$
$\sin bt + bt\cos bt$	$2bs^2/(s^2+b^2)^2$
$t\ \sin bt$	$2bs/(s^2+b^2)^2$
te^{-at}	$1/(s+a)^2$
$\mathrm{u}(t-c)$	e^{-cs}/s
$e^{-at}f(t)$	$F(s+a)$
$tf(t)$	$-F'(s)$
$f(t-c)\mathrm{u}(t-c)$	$e^{-cs}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f'''(t)$	$s^3F(s)-s^2f(0)-sf'(0)-f''(0)$
$\int_0^t f(\tau)d\tau$	$F(s)/s$

We turn now to two central theorems (both of which appear in your standard table) which will allow us to cope with shifts in both the s and the t variables. These results are handy when finding some particular Laplace transforms and are absolutely essential when dealing with inverse Laplace transforms.

First Shifting Theorem

$$\mathcal{L}(e^{-at}f(t)) = F(s+a)$$

Proof:

$$\mathcal{L}\left(e^{-at}f(t)\right) = \int_0^\infty e^{-st}\left(e^{-at}f(t)\right)\,dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\,dt$$

$$= F(s+a)$$

What the first shifting theorem is saying is that the impact of an e^{-at} in the t variable is to shift s to $s+a$ in the s variable. We use the first shifting theorem in both directions!

Example 1 Find $\mathcal{L}(e^{-8t}\cos(2t))$

$$\mathcal{L}\left(e^{-8t}\cos(2t)\right) = \frac{(s+8)}{(s+8)^2+2^2}$$

$$= \frac{s+8}{s^2+16s+68}$$

$$\bigstar \quad \frac{s+8}{s^2+16s+68} \quad \bigstar$$

Example 2 Find $\mathcal{L}^{-1}\left(\frac{6}{(s-7)^4}\right)$

$$\mathcal{L}^{-1}\left(\frac{6}{(s-7)^4}\right) = \frac{6t^3e^{7t}}{3!}$$

$$= t^3e^{7t}$$

★ t^3e^{7t} ★

Example 3 Find $\mathcal{L}^{-1}\left(\frac{10s-1}{s^2+6s+13}\right)$.

$$\mathcal{L}^{-1}\left(\frac{10s-1}{s^2+6s+13}\right) = \mathcal{L}^{-1}\left(\frac{10(s+3)-31}{(s+3)^2+4}\right)$$

$$= 10\cos(2t)e^{-3t} - \frac{31}{2}\sin(2t)e^{-3t}$$

★ $10e^{-3t}\cos(2t) - \frac{31}{2}e^{-3t}\sin(2t)$ ★

Second Shifting Theorem

$$\mathcal{L}(f(t-c)u(t-c)) = e^{-cs}F(s)$$

Proof:

$$\mathcal{L}\left(f(t-c)u(t-c)\right) = \int_0^\infty e^{-st} f(t-c)\,u(t-c)\,dt$$

$$= \int_c^\infty f(t-c)\,e^{-st}\,dt$$

let $u = t-c$

$du = dt$

$t = c,\quad u = 0$

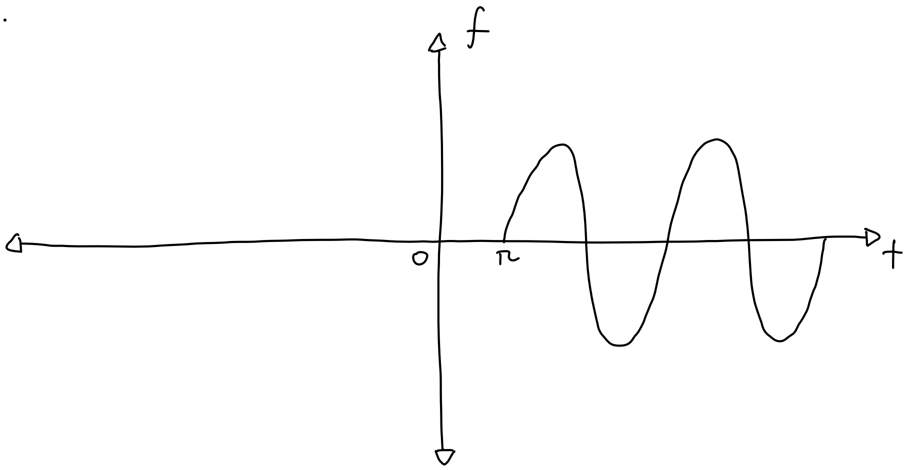
$t \to \infty,\quad u \to \infty$

$$= \int_0^\infty f(u)\,e^{-s(c+u)}\,du$$

$$= e^{-sc}F(s)$$

What the second shifting theorem is saying is that shifting and cutting in the t variable introduces an exponential function in the s variable.

Example 4 Let $f(t) = \sin(t-\pi)u(t-\pi)$. Sketch a graph of f and find its Laplace transform.

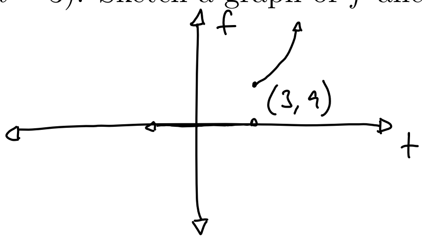


$$\mathcal{L}\left(\sin(t-\pi)u(t-\pi)\right) = \frac{e^{-\pi s}}{s^2+1}$$

$$\star \quad \frac{e^{-\pi s}}{1+s^2} \quad \star$$

Example 5 Let $f(t) = t^2 u(t-3)$. Sketch a graph of f and find its Laplace transform.

Sketch:



We have a Heaviside cut here but no shift ! So let's generate a 3 shift by writing t^2 as

$$t^2 + 0t + 0 = a(t-3)^2 + b(t-3) + c.$$

We have two different ways to find a, b and c :

Method 1: Blast away and compare coefficients of powers of t:

$$RHS = at^2 - (6a - b)t + 9a - 3b + c$$

$$\therefore a = 1$$
$$b = 6$$
$$c = 9$$

Method 2: First make a little substitution $w = t - 3$:

$$(w+3)^2 = aw^2 + bw + c$$

$$w^2 + 6w + 9 \equiv aw^2 + bw + c$$

$$a = 1$$
$$b = 6$$
$$c = 9$$

So

$$t^2 = (t-3)^2 + 6(t-3) + 9$$

and hence our problem is to find the Laplace transform of

$$\{(t-3)^2 + 6(t-3) + 9\}u(t-3) = (t-3)^2 u(t-3) + 6(t-3)u(t-3) + 9u(t-3).$$

Now:

$$\mathcal{L}\left(t^2 u(t-3)\right) = e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)$$

$\bigstar$ $e^{-3s}(\frac{9}{s} + \frac{6}{s^2} + \frac{2}{s^3})$ $\bigstar$

Example 6 Find $\mathcal{L}^{-1}(\frac{12e^{-7s}}{s^4})$ and sketch the inverse Laplace transform.

$$\mathcal{L}(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}$$

$$\mathcal{L}(2t^3) = \frac{12}{s^4}$$

$$\mathcal{L}\left(2(t-7)^3 u(t-7)\right) = \frac{12e^{-7s}}{s^4}$$

★ $2(t-7)^3u(t-7)$ ★

Example 7 Find the inverse Laplace transform of $F(s) = \dfrac{e^{-7s}}{1+(s+4)^2}$.

This is tough as it invloves **both** shifting theorems!

$$\mathcal{L}^{-1}\left(\frac{e^{-7s}}{1+(s+4)^2}\right)$$

$$\mathcal{L}^{-1}(\sin t) = \frac{1}{1+s^2}$$

$$\mathcal{L}^{-1}\left(\sin(t)\, e^{-4t}\right) = \frac{1}{1+(s+4)^2}$$

$$\mathcal{L}^{-1}\left(e^{-4(t-7)}\sin(t-7)\, u(t-7)\right) = \frac{e^{-7s}}{1+(s+4)^2}$$

★ $f(t) = e^{-4(t-7)}\sin(t-7)u(t-7)$ ★

[35] You can now do Q 96 e f g h, 98, 99 c d g h i