MATH2019 PROBLEM CLASS
EXAMPLES 5
ORDINARY DIFFERENTIAL EQUATIONS

$\boxed{1998}$ 1. Use the substitution $y = z^{\frac{1}{3}}$ where y and z are both functions of x to transform the differential equation

$$3y' = e^x y^{-2} + y \tag{1}$$

into

$$z' = e^x + z$$

and hence find the general solution of (1).

Solution: Use the substitution $y = z^{1/3}$. Hence

$$y(x) = z^{1/3} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{3} z^{-2/3} \frac{dz}{dx}\,.$$

Then transform the ODE and solve, i.e.,

$$\begin{aligned}
3y' = e^x y^{-2} + y \quad &\Rightarrow \quad z^{-2/3}\frac{dz}{dx} = e^x z^{-2/3} + z^{1/3} \\
&\Rightarrow \quad \frac{dz}{dx} = e^x + z \qquad \text{(1st order linear)} \\
&\Rightarrow \quad \frac{dz}{dx} - z = e^x \qquad \text{(integrating factor } e^{-\int dx} = e^{-x}\text{)} \\
&\Rightarrow \quad \frac{d}{dx}\left(ze^{-x}\right) = 1 \\
&\Rightarrow \quad ze^{-x} = x + C \qquad \text{(integrating both sides)} \\
&\Rightarrow \quad z = xe^x + Ce^x \\
&\Rightarrow \quad y = (xe^x + Ce^x)^{1/3} \qquad \text{(using } y = z^{1/3}\text{)}\,.
\end{aligned}$$

$\boxed{1994}$ 2. A forced vibrating system is represented by

$$y'' + 5y' + 4y = 6\sin(2t)$$

where $6\sin(2t)$ is the driving force and y is the displacement from the equilibrium position. Find the motion of the system corresponding to the following initial displacement and velocity

$$y(0) = 1\,, \qquad y'(0) = 0\,.$$

Then find the steady state oscillations (i.e., the response of the system after a sufficiently long time).

Solution:

<u>y_H *(solution to homogeneous ODE):*</u>

$$\begin{aligned}
y'' + 5y' + 4y = 0 \quad &\Rightarrow \quad \lambda^2 + 5\lambda + 4 = (\lambda+1)(\lambda+4) = 0 \\
&\Rightarrow \quad \lambda = -1, -4 \\
&\Rightarrow \quad y_H = Ae^{-t} + Be^{-4t}\,.
\end{aligned}$$

<u>y_P *(particular solution):*</u> Given the forcing function $6\sin(2t)$ (on the RHS of the inhomogeneous ODE) we try

$$y_P = C\cos(2t) + D\sin(2t)\,.$$

Hence

$$\begin{aligned} y_P' &= -2C\sin(2t) + 2D\cos(2t)\,, \\ y_P'' &= -4C\cos(2t) - 4D\sin(2t) \;\; (= -4y_P)\,. \end{aligned}$$

Substituting the expressions for y_P, y_P' and y_P'' into the inhomogeneous ODE yields

$$\begin{aligned} y_P'' + 5y_P' + 4y_P &= -4C\cos(2t) - 4D\sin(2t) + 5\left(-2C\sin(2t) + 2D\cos(2t)\right) \\ &\qquad +4\left(C\cos(2t) + D\sin(2t)\right) \\ &= 10D\cos(2t) - 10C\sin(2t) \\ &= 6\sin(2t)\,. \end{aligned}$$

Comparing coefficients yields

$$\begin{aligned} 10D = 0 &\;\;\Rightarrow\;\; D = 0\,, \\ -10C = 6 &\;\;\Rightarrow\;\; C = -\frac{3}{5}\,. \end{aligned}$$

Hence a particular solution y_P is given by

$$y_P = -\frac{3}{5}\cos(2t)\,.$$

<u>*y_G (general solution):*</u> The general solution y_G to the ODE is given by

$$y_G = y_H + y_P = Ae^{-t} + Be^{-4t} - \frac{3}{5}\cos(2t)\,.$$

<u>*Apply initial conditions $y(0) = 1$ and $y'(0) = 0$:*</u> First calculate y_G', i.e.,

$$y_G' = -Ae^{-t} - 4Be^{-4t} - \frac{6}{5}\sin(2t)$$

Thus

$$\begin{aligned} y(0) = 1 &\;\;\Rightarrow\;\; 1 = A + B - \frac{3}{5} \\ &\;\;\Rightarrow\;\; A + B = \frac{8}{5} \\ y'(0) = 0 &\;\;\Rightarrow\;\; 0 = -A - 4B \\ &\;\;\Rightarrow\;\; A + 4B = 0 \end{aligned}$$

Hence

$$\left.\begin{aligned} A + B &= \tfrac{8}{5} \\ A + 4B &= 0 \end{aligned}\right\} \Rightarrow A = \frac{32}{15}, B = -\frac{8}{15}\,.$$

Thus overall

$$y = \frac{32}{15}e^{-t} - \frac{8}{15}e^{-4t} - \frac{3}{5}\cos(2t)\,.$$

As $t \to \infty$, y_H (solution to the homogenous ODE) tends to zero (due to the negative power in the exponentials) and thus $y \to -\frac{3}{5}\cos(2t)$ is the long term behaviour of the solution.

$\boxed{1997}$ 3. Consider the differential equation

$$\frac{1}{2}u'' + cu' + \frac{1}{2}u = 0$$

where c is a non-negative damping constant.

a) What damping constants c produce overdamping, critical damping, underdamping and no damping?

Solution:

y_H (solution to homogeneous ODE):

$$\begin{aligned} \frac{1}{2}u'' + cu' + \frac{1}{2}u = 0 \quad &\Rightarrow \quad \frac{1}{2}\lambda^2 + c\lambda + \frac{1}{2} = 0 \\ &\Rightarrow \quad \lambda = -c \pm \sqrt{c^2 - 1}\,. \end{aligned}$$

Hence

$$\begin{array}{rcl} \textit{Overdamping} \text{ - real distinct } \lambda & c^2 - 1 > 0 & \Rightarrow c > 1\,, \\ \textit{Critical damping} \text{ - repeated real } \lambda & c^2 - 1 = 0 & \Rightarrow c = 1\,, \\ \textit{Underdamping} \text{ - complex } \lambda & c^2 - 1 < 0,\ c \neq 0 & \Rightarrow 0 < c < 1\,, \\ \textit{No damping} & & \Rightarrow c = 0\,, \end{array}$$

b) Sketch an example of the solution $u(t)$ for the case of overdamping and for the case of underdamping.

Solution: See lecture notes

$\boxed{1999}$ 4. Consider the vibrating system

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = \sin(\omega t)\,.$$

Will the system in exhibit resonance for any choice of the forcing angular frequency ω? Give reasons for your answer.

Solution:

y_H (solution to homogeneous ODE):

$$\begin{aligned} y'' + 2y' + 2y = 0 \quad &\Rightarrow \quad \lambda^2 + 2\lambda + 2 = (\lambda + 1)^2 + 2 = 0 \\ &\Rightarrow \quad \lambda = -1 \pm i \\ &\Rightarrow \quad y_H = e^{-t}\left(A\cos t + B\sin t\right)\,. \end{aligned}$$

y_P (particular solution): Given the forcing function $\sin(\omega t)$ (on the RHS of the inhomogeneous ODE) we try

$$y_P = C\cos(\omega t) + D\sin(\omega t).$$

The system will NOT exhibit resonance since this initial guess for y_P is NOT proportional to y_H. If it was proportional then we would have had to multiply our guess by the independent variable t and resonance would occur.

2014, S2 5. Use the method of undetermined coefficients to solve the second order differential equation

$$y'' + 2y' + 5y = -25x^2.$$

Solution:

y_H *(solution to homogeneous ODE):*

$$\begin{aligned} y'' + 2y' + 5y = 0 \quad &\Rightarrow \quad \lambda^2 + 2\lambda + 5 = (\lambda+1)^2 + 4 = 0 \\ &\Rightarrow \quad \lambda = -1 \pm 2i \\ &\Rightarrow \quad y_H = e^{-x}\left(A\cos(2x) + B\sin(2x)\right) . \end{aligned}$$

y_P *(particular solution):* Given the forcing function $-25x^2$ (on the RHS of the inhomogeneous ODE) we try

$$y_P = Cx^2 + Dx + E, .$$

Hence

$$\begin{aligned} y_P' &= 2Cx + D\,, \\ y_P'' &= 2C\,. \end{aligned}$$

Substituting the expressions for y_P, y_P' and y_P'' into the inhomogeneous ODE yields

$$\begin{aligned} y_P'' + 2y_P' + 5y_P &= 2C + 2(2Cx + D) + 5(Cx^2 + Dx + E) \\ &= 5Cx^2 + (4C + 5D)x + 2C + 2D + 5E \\ &= -25x^2\,. \end{aligned}$$

Comparing coefficients yields

$$\begin{aligned} 5C = -25 \quad &\Rightarrow \quad C = -5\,, \\ 4C + 5D = 0 \quad &\Rightarrow \quad D = -\frac{4}{5}C = 4\,, \\ 2C + 2D + 5E = 0 \quad &\Rightarrow \quad E = -\frac{2}{5}(C + D) = \frac{2}{5}\,. \end{aligned}$$

Hence a particular solution y_P is given by

$$y_P = -5x^2 + 4x + \frac{2}{5}\,.$$

y_G *(general solution):* The general solution y_G to the ODE is given by

$$y_G = y_H + y_P = e^{-x}\left(A\cos(2x) + B\sin(2x)\right) - 5x^2 + 4x + \frac{2}{5}\,.$$

2015, S1 6. Use the method of undetermined coefficients to solve the second order differential equation

$$y'' - 4y' + 4y = 5\sin t.$$

Solution:

y_H *(solution to homogeneous ODE):*

$$\begin{aligned} y'' - 4y' + 4y = 0 \quad &\Rightarrow \quad \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0 \\ &\Rightarrow \quad \lambda = 2, 2 \\ &\Rightarrow \quad y_H = (A + Bt)e^{2t}\,. \end{aligned}$$

$\underline{y_P \text{ (particular solution):}}$ Given the forcing function $5\sin t$ (on the RHS of the inhomogeneous ODE) we try

$$y_P = C\cos t + D\sin t\,.$$

Hence

$$\begin{aligned} y_P' &= -C\sin t + D\cos t\,, \\ y_P'' &= -C\cos t - D\sin t \; (= -y_P)\,. \end{aligned}$$

Substituting the expressions for y_P, y_P' and y_P'' into the inhomogeneous ODE yields

$$\begin{aligned} y_P'' - 4y_P' + 4y_P &= -C\cos t - D\sin t - 4\,(-C\sin t + D\cos t) + 4\,(C\cos t + D\sin t) \\ &= (3C - 4D)\cos t + (3D + 4C)\sin t \\ &= 5\sin t\,. \end{aligned}$$

Comparing coefficients yields

$$\begin{aligned} 3C - 4D = 0 \;&\Rightarrow\; C = \frac{4}{3}D\,, \\ 3D + 4C = 5 \;&\Rightarrow\; 3D + \frac{16}{3}D = 5 \\ &\Rightarrow\; D = \frac{3}{5} \\ &\Rightarrow\; C = \frac{4}{5}\,. \end{aligned}$$

Hence a particular solution y_P is given by

$$y_P = \frac{4}{5}\cos t + \frac{3}{5}\sin t\,.$$

$\underline{y_G \text{ (general solution):}}$ The general solution y_G to the ODE is given by

$$y_G = y_H + y_P = (A + Bt)e^{2t} + \frac{4}{5}\cos t + \frac{3}{5}\sin t\,.$$

$\boxed{\text{2015, S2}}$ 7. Use the method of variation of parameters to find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 35e^x x^{3/2}.$$

Solution:

$\underline{y_H \text{ (solution to homogeneous ODE):}}$

$$\begin{aligned} y'' - 2y' + y = 0 \;&\Rightarrow\; \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \\ &\Rightarrow\; \lambda = 1, 1 \\ &\Rightarrow\; y_H = (A + Bx)e^x = A\underbrace{e^x}_{y_1} + B\underbrace{xe^x}_{y_2}\,. \end{aligned}$$

$\underline{y_P \text{ (particular solution):}}$ We first calculate the Wronskian of the two functions y_1 and y_2, i.e.,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^x(e^x + xe^x) - xe^{2x} = e^{2x}\,.$$

Given the forcing function $f = 35e^x x^{3/2}$ (on the RHS of the inhomogeneous ODE) a particular solution y_P is given by

$$\begin{aligned} y_P &= -y_1 \int \frac{y_2 f}{W}\, dx + y_2 \int \frac{y_1 f}{W}\, dx \\ &= -e^x \int \frac{xe^x(35e^x x^{3/2})}{e^{2x}}\, dx + xe^x \int \frac{e^x(35e^x x^{3/2})}{e^{2x}}\, dx \\ &= -35e^x \int x^{5/2}\, dx + 35xe^x \int x^{3/2}\, dx \\ &= -10e^x x^{7/2} + 14e^x x^{7/2} \\ &= 4e^x x^{7/2}\,. \end{aligned}$$

Hence a particular solution y_P is given by

$$y_P = 4e^x x^{7/2}\,.$$

y_G *(general solution):* The general solution y_G to the ODE is given by

$$y_G = y_H + y_P = (A + Bx)e^x + 4e^x x^{7/2} = \left(A + Bx + 4x^{7/2}\right) e^x\,.$$

$\boxed{\text{2016, S1}}$ 8. Use the substitution $v = x + y$ to solve the ordinary differential equation

$$(x + y)\,\frac{dy}{dx} = \frac{1}{x^2} - x - y, \qquad y(1) = 0.$$

Solution: Use the substitution $y(x) = v(x) - x$. Hence

$$y(x) = v(x) - x \;\Rightarrow\; \frac{dy}{dx} = \frac{dv}{dx} - 1\,.$$

Then transform the ODE and solve, i.e.,

$$\begin{aligned} (x + y)\,\frac{dy}{dx} = \frac{1}{x^2} - x - y \;&\Rightarrow\; v\left(\frac{dv}{dx} - 1\right) = \frac{1}{x^2} - v \\ &\Rightarrow\; v\frac{dv}{dx} = \frac{1}{x^2} \qquad \text{(1st order separable)} \\ &\Rightarrow\; \int v\, dv = \int \frac{dx}{x^2} \\ &\Rightarrow\; \frac{1}{2}v^2 = C - \frac{1}{x} \\ &\Rightarrow\; \frac{1}{2}(x + y)^2 = C - \frac{1}{x} \qquad \text{(using } v = y + x\text{)}\,. \end{aligned}$$

Now apply the initial condition, $y(1) = 0$, i.e.,

$$\begin{aligned} y(1) = 0 \;&\Rightarrow\; \frac{1}{2}(0 + 1)^2 = C - \frac{1}{1} \\ &\Rightarrow\; C = \frac{3}{2} \end{aligned}$$

Thus the solution to the initial value problem (IVP) is $(x + y)^2 = 3 - \dfrac{2}{x}$.

$\boxed{\text{2016, S1}}$ 9. Use the method of undetermined coefficients to solve the second order differential equation

$$y'' + 3y' + 2y = \mathrm{e}^{-2t} + 4t^2 + 2.$$

Also describe the long term steady state solution.

Solution:

<u>y_H *(solution to homogeneous ODE):*</u>

$$\begin{aligned} y'' + 3y' + 2y = 0 \ \ &\Rightarrow \ \ \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) = 0 \\ &\Rightarrow \ \ \lambda = -1, -2 \\ &\Rightarrow \ \ y_H = Ae^{-t} + Be^{-2t}\,. \end{aligned}$$

<u>y_P *(particular solution):*</u> Given the forcing function $e^{-2t} + 4t^2 + 2$ (on the RHS of the inhomogeneous ODE) we try

$$y_P = Ce^{-2t} + Dt^2 + Et + F\,.$$

But this guess for y_P is proportional to the homogenous solution y_H. We update the guess by multiplying by the independent variable t in the exponential part of y_P, i.e.,

$$y_P = Cte^{-2t} + Dt^2 + Et + F\,.$$

Hence

$$\begin{aligned} y_P' &= Ce^{-2t} - 2Cte^{-2t} + 2Dt + E\,, \\ y_P'' &= -2Ce^{-2t} - 2Ce^{-2t} + 4Cte^{-2t} + 2D \\ &= -4Ce^{-2t} + 4Cte^{-2t} + 2D\,. \end{aligned}$$

Substituting the expressions for y_P, y_P' and y_P'' into the inhomogeneous ODE yields

$$\begin{aligned} y_P'' + 3y_P' + 2y_P &= -4Ce^{-2t} + 4Cte^{-2t} + 2D + 3\left(Ce^{-2t} - 2Cte^{-2t} + 2Dt + E\right) \\ &\quad +2\left(Cte^{-2t} + Dt^2 + Et + F\right) \\ &= -Ce^{-2t} + 2Dt^2 + (6D + 2E)t + 2D + 3E + 2F \\ &= e^{-2t} + 4t^2 + 2\,. \end{aligned}$$

Comparing coefficients yields

$$\begin{aligned} -C = 1 \ \ &\Rightarrow \ \ C = -1\,, \\ 2D = 4 \ \ &\Rightarrow \ \ D = 2\,, \\ 6D + 2E = 0 \ \ &\Rightarrow \ \ E = -3D = -6\,, \\ 2D + 3E + 2F = 2 \ \ &\Rightarrow \ \ F = 1 - D - \frac{3}{2}E = 8\,. \end{aligned}$$

Hence a particular solution y_P is given by

$$y_P = -te^{-2t} + 2t^2 - 6t + 8\,.$$

<u>y_G *(general solution):*</u> The general solution y_G to the ODE is given by

$$y_G = y_H + y_P = Ae^{-t} + Be^{-2t} - te^{-2t} + 2t^2 - 6t + 8\,.$$

As $t \to \infty$, y_H (solution to the homogenous ODE) and $-te^{-2t}$ tend to zero (due to the negative power in the exponentials) and thus $y_G \to 2t^2 - 6t + 8$ is the long term behaviour of the solution.

$\boxed{\text{2016, S2}}$ 10. Use the substitution $v = y + x$ to find the general solution of

$$\frac{dy}{dx} = (y + x)^2.$$

Solution: Use the substitution $y(x) = v(x) - x$. Hence

$$y(x) = v(x) - x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dv}{dx} - 1\,.$$

Then transform the ODE and solve, i.e.,

$$\begin{aligned}
\frac{dy}{dx} = (y+x)^2 \quad &\Rightarrow \quad \frac{dv}{dx} - 1 = v^2 \\
&\Rightarrow \quad \frac{dv}{dx} = 1 + v^2 \qquad \text{(1st order separable)} \\
&\Rightarrow \quad \int \frac{dv}{1+v^2} = \int dx \\
&\Rightarrow \quad \tan^{-1} v = x + C \\
&\Rightarrow \quad v = \tan\left(x + C\right) \\
&\Rightarrow \quad y = -x + \tan\left(x + C\right) \qquad \text{(using } v = y + x\text{)}\,.
\end{aligned}$$

$\boxed{\text{2016, S2}}$ 11. Use the method of undetermined coefficients to solve the second order differential equation

$$y'' - 4y = \mathrm{e}^{2t}.$$

Solution:

<u>y_H *(solution to homogeneous ODE):*</u>

$$\begin{aligned}
y'' - 4y = 0 \quad &\Rightarrow \quad \lambda^2 - 4 = 0 \\
&\Rightarrow \quad \lambda = \pm 2 \\
&\Rightarrow \quad y_H = Ae^{2t} + Be^{-2t}\,.
\end{aligned}$$

<u>y_P *(particular solution):*</u> Given the forcing function e^{2t} (on the RHS of the inhomogeneous ODE) we try

$$y_P = Ce^{2t}\,.$$

But this guess for y_P is proportional to the homogenous solution y_H. We update the guess by multiplying y_P by the independent variable t, i.e.,

$$y_P = Cte^{2t}\,.$$

Hence

$$\begin{aligned}
y_P' &= Ce^{2t} + 2Cte^{2t}\,, \\
y_P'' &= 2Ce^{2t} + 2Ce^{2t} + 4Cte^{2t} \\
&= 4Ce^{2t} + 4Cte^{2t}\,.
\end{aligned}$$

Substituting the expressions for y_P and y_P'' into the inhomogeneous ODE yields

$$\begin{aligned}
y_P'' - 4y_P &= 4Ce^{2t} + 4Cte^{2t} - 4Cte^{2t} \\
&= 4Ce^{2t} \\
&= e^{2t}\,.
\end{aligned}$$

Comparing coefficients yields

$$4C = 1 \quad \Rightarrow \quad C = \frac{1}{4}\,.$$

Hence a particular solution y_P is given by

$$y_P = \frac{1}{4} t e^{2t}\,.$$

<u>*y_G (general solution):*</u> The general solution y_G to the ODE is given by

$$y_G = y_H + y_P = Ae^{2t} + Be^{-2t} + \frac{1}{4} t e^{2t}\,.$$

$\boxed{\text{2017, S1}}$ 12. Use the substitution $v = \dfrac{y}{x}$ to solve the ordinary differential equation

$$x^2 \frac{dy}{dx} = 2x^2 + xy + 2y^2.$$

Solution: Use the substitution $y(x) = xv(x)$. Hence

$$y(x) = xv(x) \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}\,.$$

Then transform the ODE and solve, i.e.,

$$\begin{aligned}
x^2 \frac{dy}{dx} = 2x^2 + xy + 2y^2 \quad &\Rightarrow \quad x^2\left(v + x\frac{dv}{dx}\right) = 2x^2 + x^2 v + 2x^2 v^2 \\
&\Rightarrow \quad v + x\frac{dv}{dx} = 2 + v + 2v^2 \qquad (x \neq 0) \\
&\Rightarrow \quad x\frac{dv}{dx} = 2(1+v^2) \qquad \text{(1st order separable)} \\
&\Rightarrow \quad \int \frac{dv}{1+v^2} = \int \frac{2}{x}\,dx \\
&\Rightarrow \quad \tan^{-1} v = C + \ln x^2 \\
&\Rightarrow \quad v = \tan\left(C + \ln x^2\right) \\
&\Rightarrow \quad y = x\tan\left(C + \ln x^2\right) \qquad (\text{using } y = xv)\,.
\end{aligned}$$

$\boxed{\text{2017, S1}}$ 13. Use the method of undetermined coefficients to solve the second order differential equation

$$y'' + 9y = 6\cos(3t) + 5\mathrm{e}^t.$$

Solution:

<u>*y_H (solution to homogeneous ODE):*</u>

$$\begin{aligned}
y'' + 9y = 0 \quad &\Rightarrow \quad \lambda^2 + 9 = 0 \\
&\Rightarrow \quad \lambda = \pm 3i \\
&\Rightarrow \quad y_H = A\cos(3t) + B\sin(3t)\,.
\end{aligned}$$

<u>*y_P (particular solution):*</u> Given the forcing function $6\cos(3t) + 5e^t$ (on the RHS of the inhomogeneous ODE) we try

$$y_P = C\cos(3t) + D\sin(3t) + Ee^t\,.$$

But this guess for y_P is proportional to the homogenous solution y_H. We update the guess by multiplying by the independent variable t in the trigonometric part of y_P, i.e.,

$$y_P = Ct\cos(3t) + Dt\sin(3t) + Ee^t\,.$$

Hence

$$\begin{aligned}
y_P' &= C\cos(3t) - 3Ct\sin(3t) + D\sin(3t) + 3Dt\cos(3t) + Ee^t\,,\\
y_P'' &= -3C\sin(3t) - 3C\sin(3t) - 9Ct\cos(3t) + 3D\cos(3t) + 3D\cos(3t) - 9Dt\sin(3t) + Ee^t\\
&= -6C\sin(3t) - 9Ct\cos(3t) + 6D\cos(3t) - 9Dt\sin(3t) + Ee^t\,.
\end{aligned}$$

Substituting the expressions for y_P and y_P'' into the inhomogeneous ODE yields

$$\begin{aligned}
y_P'' + 9y_P &= -6C\sin(3t) - 9Ct\cos(3t) + 6D\cos(3t) - 9Dt\sin(3t) + Ee^t\\
&\qquad +9\left(Ct\cos(3t) + Dt\sin(3t) + Ee^t\right)\\
&= 6D\cos(3t) - 6C\sin(3t) + 10Ee^t\\
&= 6\cos(3t) + 5e^t\,.
\end{aligned}$$

Comparing coefficients yields

$$\begin{aligned}
6D = 6 &\;\Rightarrow\; D = 1\,,\\
-6C = 0 &\;\Rightarrow\; C = 0\,,\\
10E = 5 &\;\Rightarrow\; E = \frac{1}{2}\,.
\end{aligned}$$

Hence a particular solution y_P is given by

$$y_P = t\sin(3t) + \frac{1}{2}e^t\,.$$

y_G (general solution): The general solution y_G to the ODE is given by

$$y_G = y_H + y_P = A\cos(3t) + B\sin(3t) + t\sin(3t) + \frac{1}{2}e^t\,.$$

2017, S2

14. Use the method of undetermined coefficients to solve the second order ordinary differential equation

$$y'' - 2y' - 8y = 8 + 5e^t\cos t.$$

Solution:

y_H (solution to homogeneous ODE):

$$\begin{aligned}
y'' - 2y' - 8y = 0 &\;\Rightarrow\; \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2) = 0\\
&\;\Rightarrow\; \lambda = -2, 4\\
&\;\Rightarrow\; y_H = Ae^{-2t} + Be^{4t}\,.
\end{aligned}$$

y_P (particular solution): Given the forcing function $8 + 5e^t\cos t$ (on the RHS of the inhomogeneous ODE) we try

$$y_P = C + De^t\cos t + Ee^t\sin t\,.$$

Hence

$$\begin{aligned} y_P' &= De^t\cos t - De^t\sin t + Ee^t\sin t + Ee^t\cos t\,, \\ &= (D+E)\,e^t\cos t + (E-D)\,e^t\sin t \\ y_P'' &= De^t\cos t - De^t\sin t - De^t\sin t - De^t\cos t + Ee^t\sin t + Ee^t\cos t + Ee^t\cos t - Ee^t\sin t \\ &= 2Ee^t\cos t - 2De^t\sin t\,. \end{aligned}$$

Substituting the expressions for y_P, y_P' and y_P'' into the inhomogeneous ODE yields

$$\begin{aligned} y_P'' - 2y_P' - 8y_P &= -2De^t\sin t + 2Ee^t\cos t - 2\left((D+E)\,e^t\cos t + (E-D)\,e^t\sin t\right) \\ &\qquad -8\left(C + De^t\cos t + Ee^t\sin t\right) \\ &= -8C - 10De^t\cos t - 10Ee^t\sin t \\ &= 8 + 5e^t\cos t\,. \end{aligned}$$

Comparing coefficients yields

$$\begin{aligned} -8C = 8 &\;\Rightarrow\; C = -1\,, \\ -10D = 5 &\;\Rightarrow\; D = -\frac{1}{2}\,, \\ -10E = 0 &\;\Rightarrow\; E = 0\,. \end{aligned}$$

Hence a particular solution y_P is given by

$$y_P = -1 - \frac{1}{2}e^t\cos t\,.$$

<u>y_G *(general solution):*</u> The general solution y_G to the ODE is given by

$$y_G = y_H + y_P = Ae^{-2t} + Be^{4t} - 1 - \frac{1}{2}e^t\cos t\,.$$

2018, S1 15. An inhomogeneous Euler–Cauchy ordinary differential equation (ODE) is given by

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = 2x^2\,, \qquad x > 0\,.$$

You are **given** that $y_1 = x$ and $y_2 = x^3$ are solutions to the corresponding **homogeneous** Euler–Cauchy ODE. You **do not** have to check this.

i) Calculate the Wronskian of y_1 and y_2.

Solution: The Wronskian of the two functions $y_1 = x$ and $y_2 = x^3$ is given by

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3\,.$$

ii) Use the method of Variation of Parameters to determine a particular solution y_P for the inhomogeneous Euler–Cauchy ODE.

Solution: Using the forcing function $f = 2$ (note we divided the ODE by $x^2 > 0$ to transform to standard form) then a particular solution y_P is given by

$$\begin{aligned} y_P = -y_1\int\frac{y_2 f}{W}\,dx + y_2\int\frac{y_1 f}{W}\,dx &= -x\int\frac{2x^3}{2x^3}\,dx + x^3\int\frac{2x}{2x^3}\,dx \\ &= -x\int 1\,dx + x^3\int\frac{1}{x^2}\,dx \\ &= -x^2 - x^2 = -2x^2\,. \end{aligned}$$

Hence a particular solution y_P is given by $y_P = -2x^2$.

$\boxed{\text{2018, S2}}$ 16. Consider the following differential equation describing a vibrating system:

$$\frac{d^2y}{dt^2} + 4y = 8\cos(2\pi f t).$$

i) Find the solution y_H to the homogeneous equation.

***Solution**:*

<u>*y_H (solution to homogeneous ODE):*</u> The homogeneous ODE i.e., $y'' + 4y = 0$ describes SHM. Thus we could write down the solution as $y_H = A\cos(2t) + B\sin(2t)$ or otherwise

$$\begin{aligned} y'' + 4y = 0 \quad &\Rightarrow \quad \lambda^2 + 4 = 0 \\ &\Rightarrow \quad \lambda = \pm 2i \\ &\Rightarrow \quad y_H = A\cos(2t) + B\sin(2t)\,. \end{aligned}$$

ii) For which value(s) of f will the system exhibit resonance? Give reasons for your answer.

(Note that you are not being asked to find the particular solution y_P.)

***Solution**:* Resonance will occur when the angular frequency of the forcing function is equal (matches) to the natural angular frequency of the SHM (homogeneous) system, i.e.,

$$2\pi f = 2 \quad \Rightarrow \quad f = \frac{1}{\pi}\,.$$

Hence $f = \dfrac{1}{\pi}$ is the only value for which the system will exhibit resonance.

This problem is a "classic" class test question.

$\boxed{\text{2018, S2}}$ 17. Use the substitution $v = \dfrac{y}{x}$ to solve

$$xy' = y + 2x^3\cos^2\left(\frac{y}{x}\right).$$

***Solution**:* Note $x \neq 0$. Use the substitution $y(x) = xv(x)$. Hence

$$y(x) = xv(x) \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}\,.$$

Then transform the ODE and solve, i.e.,

$$\begin{aligned} x\frac{dy}{dx} = y + 2x^3\cos^2\left(\frac{y}{x}\right) \quad &\Rightarrow \quad x\left(v + x\frac{dv}{dx}\right) = xv + 2x^3\cos^2 v \\ &\Rightarrow \quad x^2\frac{dv}{dx} = 2x^3\cos^2 v \qquad (x \neq 0) \\ &\Rightarrow \quad \frac{dv}{dx} = 2x\cos^2 v \qquad \text{(1st order separable)} \\ &\Rightarrow \quad \int \sec^2 v\,dv = \int 2x\,dx \\ &\Rightarrow \quad \tan v = x^2 + C \\ &\Rightarrow \quad v = \tan^{-1}\left(x^2 + C\right) \\ &\Rightarrow \quad y = x\tan^{-1}\left(x^2 + C\right) \qquad (\text{using } y = xv)\,. \end{aligned}$$