LECTURE 24
HOMOGENEOUS SECOND ORDER DIFFERENTIAL EQUATIONS

To solve the homogeneous second order constant coefficient differential equation

$$ay'' + by' + cy = 0$$

first form the auxiliary (also called characteristic) equation

$$a\lambda^2 + b\lambda + c = 0.$$

The auxiliary equation is not a D.E., it is just a quadratic with two roots λ_1 and λ_2. Remarkably the solution to the D.E. is completely determined by the nature of λ_1 and λ_2!

- $\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 \neq \lambda_2 \to y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
- $\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 = \lambda_2 \to y = (Ax + B)e^{\lambda_1 x}$
- $\lambda_1, \lambda_2 \in \mathbb{C},\ \lambda_1, \lambda_2 = r \pm is \to y = e^{rx}(A\cos(sx) + B\sin(sx))$

We turn now to the theory of second order differential equations. These are D.E.'s where the second derivative also makes an appearance. These are in general more difficult to solve than their first order comrades. The theory is so tangled that we restrict our attention to only the very special case of linear D.E.s with constant coefficients. These D.E.s take the form $ay'' + by' + cy = RHS$ where $a, b, c \in \mathbb{R}$. In this lecture we will deal with the homogeneous case (RHS=0). In the next lecture we will broaden our scope a little and allow some simple functions to appear on the RHS.

Keep in mind that all second order D.E.s must have two independent arbitrary constants in solution and hence require a pair of i.c.'s to deal with the constants. The method of solution for second order problems is *totally* different from first order techniques. As a relief you will find that in most cases no integration is involved.

Proof of first bullet point above:

Example 1 Solve each of the following second order D.E.s:

a) $y'' - 3y' - 10y = 0$

b) $y'' - 8y' + 16y = 0$

c) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 58y = 0$

a) $\lambda^2 - 3\lambda - 10 = 0$

$(\lambda - 5)(\lambda + 2) = 0$

$\lambda = 5, -2$

$\therefore y_H = Ae^{5x} + Be^{-2x}$

b) $\lambda^2 - 8\lambda + 16 = 0$

$(\lambda - 4)^2 = 0$

$\therefore \lambda = 4, 4$

$\therefore y_H = (Ax + B)e^{4x}$

c) $\lambda^2 - 6\lambda + 58 = 0$

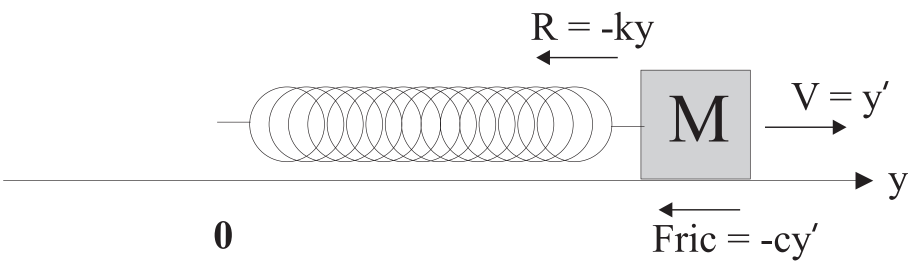
$\lambda = \dfrac{6 \pm \sqrt{6^2 - 4 \times 58}}{2}$

$= 3 \pm 7i$

$\therefore y_H = e^{3x}(A\cos(7x) + B\sin(7x))$

★ *a)* $y = Ae^{-2x} + Be^{5x}$ *b)* $y = (Ax + B)e^{4x}$ *c)* $y = e^{3x}(A\cos(7x) + B\sin(7x))$ ★

Note that is some texts, $y = e^{3x}(A\cos(7x)+B\sin(7x))$ will be written in the equivalent form $Re^{3x}\cos(7x - \delta)$. We will not do so here.



FREE OSCILLATIONS

Consider an object M of mass m attached to a spring oscillating up and down the y axis. At time t its velocity is $v = y' = \dfrac{dy}{dt}$ and its acceleration $a = y'' = \dfrac{d^2y}{dt^2}$. The total force $F = ma = my''$ acting upon the mass is the sum of two forces; $R = -ky$ the resistive force due to the spring and Fric$= -cy'$ the frictional force. Note that Fric is proportional to v and always points in a direction opposite to the velocity and R is proportional to y and always points opposite to the position.
Thus we have $F = R+$Fric implying that $my'' = -cy' - ky$. This leads to the D.E.

$$my'' + cy' + ky = 0$$

where m, c and k are all non negative. Such a system is **unforced** since the term on the right is 0.

Seeking solutions $y = Ae^{\lambda t}$ gives the characteristic equation

$$m\lambda^2 + c\lambda + k = 0,$$

which has the solutions

$$\begin{aligned}\lambda_1 &= \frac{1}{2m}\left[-c + \sqrt{c^2 - 4mk}\right]\\ \lambda_2 &= \frac{1}{2m}\left[-c - \sqrt{c^2 - 4mk}\right].\end{aligned}$$

Observe that $\sqrt{c^2 - 4mk} \leq \sqrt{c^2} = c$ implying that when λ_1 and λ_2 are real they will always be less than or equal to zero. **Thus all associated exponential function will decay!** The situation is governed by the relative magnitude of the frictional coefficient c. Three cases arise:

1. $c^2 > 4mk$. This is called "overdamping" since the damping or frictional coefficient c is large compared with $2\sqrt{mk}$. In this case λ_1 and λ_2 are both real and negative. The solution is $y = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ which decays to zero as $t \longrightarrow \infty$.

2. $c^2 = 4mk$. This is "critical damping" and $\lambda_1 = \lambda_2$ so the solution is

$$y = (A + Bt)e^{-ct/2m}.$$

The solution also decays to 0 as $t \longrightarrow \infty$. For critical and overdamping the friction is so large that the object does not get the opportunity to oscillate more than once. If however the frictional coefficient c is small enough we have underdamping and the system becomes unstable:

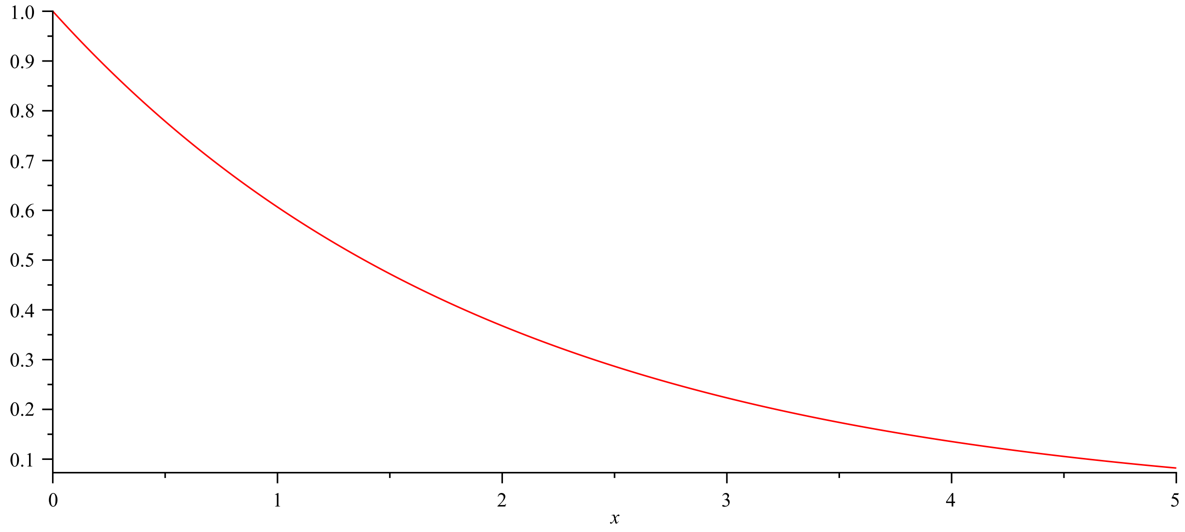


Figure 1: Overdamping (slower return to equilibrium, max of one oscillation)

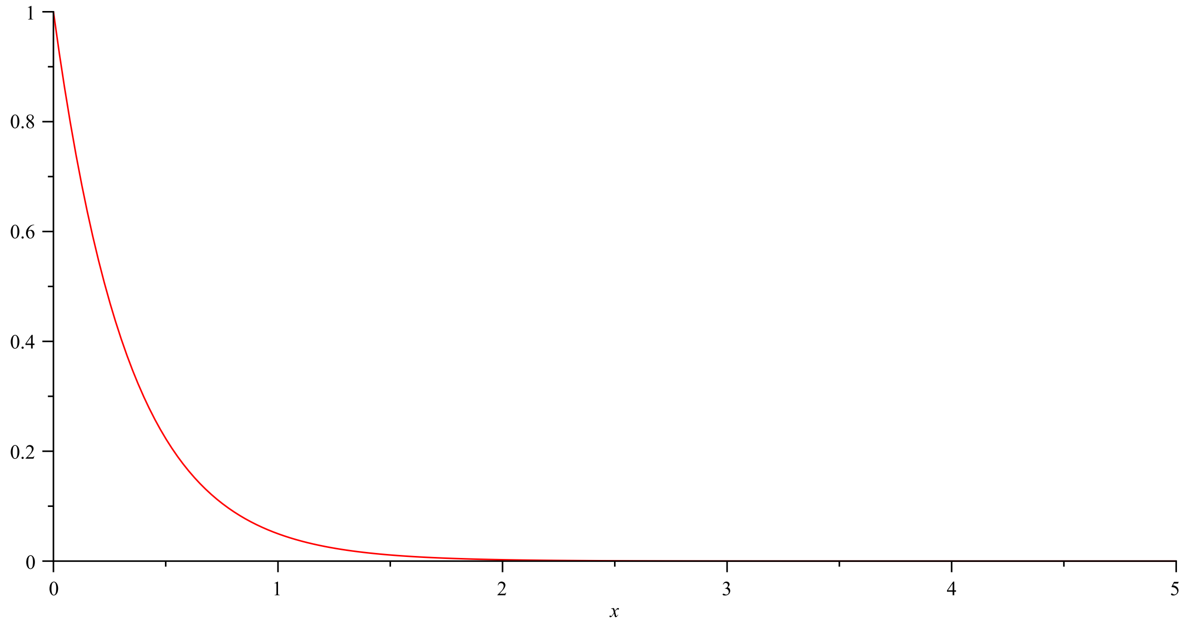


Figure 2: Critical damping (quick return to equilibrium, max of one oscillation. The system is approaching instability)

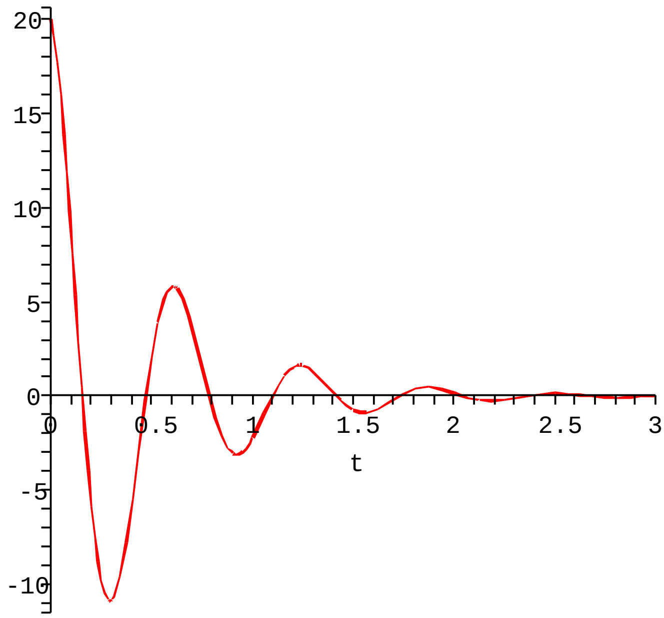


Figure 3: Underdamping. Many oscillations on return to equilibrium

3. $c^2 < 4mk$. This is called "underdamping" as c is smaller than $2\sqrt{mk}$. Then we have complex roots

$$\lambda_1 = r + is, \quad \lambda_2 = r - is$$

Thus

$$y = e^{rt}(A\cos st + B\sin st)$$

where $$r = -\frac{c}{2m} \text{ and } \quad s = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

This is the first time trigonometric functions appear, and we now have a sequence of decaying oscillations. In the idealised case $c = 0$ (no friction), the exponential term is lost completely, and the motion becomes simple harmonic, oscillating forever. In reality, $c > 0$ and these oscillations are killed off by friction. Note that $c < 0$, that is negative friction, doesn't really make sense.

Keep in mind that the value of c which generates critical damping will always stem from repeated roots in the auxiliary equation, and hence can be found by forcing a zero discriminant. Overdamping then requires more friction, and underdamping less friction.

Example 2 Consider the differential equation

$$y'' + cy' + y = 0$$

(I) What value of the damping constant c produces:

a) Critical damping b) OverDamping c) Underdamping.

(II) Find and identify the solutions for $c = 1$ and $c = 3$. Compare these solutions with that obtained for no damping, $c = 0$, sketching possible graphs of all three cases.

I) $\lambda^2 + c\lambda + 1 = 0$

$\Delta = c^2 - 4$

a) critical damping : $c = 2$

b) overdamping : $c > 2$

c) underdamping : $0 < c < 2$

II) $c = 1$ $\therefore$ underdamping

$$\lambda = \frac{-1 \pm \sqrt{3}\, i}{2}$$

$$\therefore y_H = e^{-\frac{1}{2}t}\left(A\cos\left(\frac{\sqrt{3}}{2}t\right) + B\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

$c = 3$ $\therefore$ overdamping

$$\lambda = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$$

$$\therefore y_H = Ae^{\frac{-3+\sqrt{5}}{2}t} + Be^{\frac{-3-\sqrt{5}}{2}t}$$

$c = 0 \therefore$ no damping

$\lambda = \pm i$

$\therefore y_H = A\cos t + B\sin t$

★ $Critical damping\ c = 2,\ OverDamping\ c > 2,\ Underdamping\ 0 < c < 2$ ★

★ $c = 3 \to Overdamping\ and\ y \approx Ae^{-0.382t} + Be^{-2.618t}$ ★

★ $c = 1 \to Underdamping\ and\ y = e^{-\frac{1}{2}t}(A\cos(\frac{\sqrt{3}}{2}t) + B\sin(\frac{\sqrt{3}}{2}t))$ ★

★ $c = 0 \to No\ damping\ and\ y = A\cos(t) + B\sin(t)$ ★

[24] You can now do Q 84