LECTURE 15
DOUBLE INTEGRALS

For a region Ω in the $x-y$ plane and a surface $z = f(x,y)$ in $\mathbb{R}^3$ the double integral

$$\iint_{\Omega} f(x,y)dy\,dx.$$

evaluates the volume of the solid above Ω and below $z = f(x,y)$.

To get a feeling for how double integrals operate, let's compare single integrals $\int_a^b f(x)dx$ with double integrals $\iint_{\Omega} f(x,y)dy\,dx.$.

$\int_a^b f(x)dx$	$\iint_{\Omega} f(x,y)dy\,dx.$

When evaluating double integrals keep in mind that the extreme left hand integral must always have constant limits and that we attack all multiple integrals from the middle to the edges.

Example 1 Evaluate $\int_0^1 \int_0^2 12x^2y^3dy\,dx.$

$$= \int_0^1 \left[3x^2y^4\right]_0^2 dx$$

$$= \int_0^1 48x^2\,dx$$

$$= 48\left[\frac{x^3}{3}\right]_0^1$$

$$= 16$$

★ 16 ★

It is quite rare to have all four limits as constants and a more typical situation has the inner limits as functions.

Example 2 Evaluate $\displaystyle\int_0^1 \int_x^{x^2} 12x^2y^3dy\,dx$.

$$= \int_0^1 \left[3x^2y^4\right]_x^{x^2} dx$$

$$= 3\int_0^1 x^{10} - x^6 \quad dx$$

$$= 3\left[\frac{x^{11}}{11} - \frac{x^7}{7}\right]_0^1$$

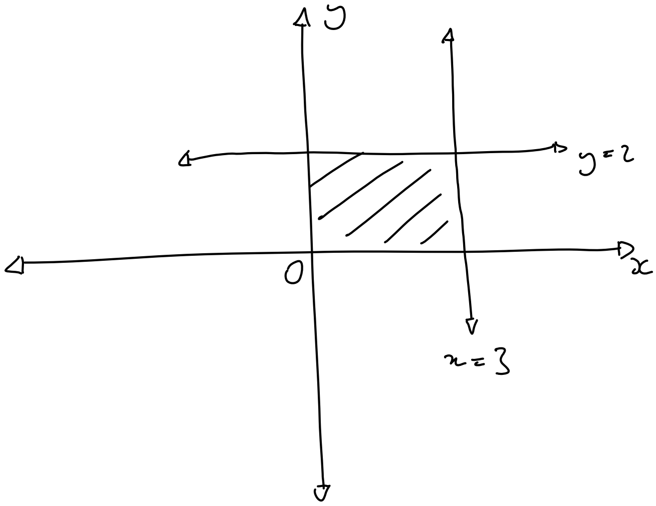
$$= -\frac{12}{77}$$

★ $-\frac{12}{77}$ ★

A crucial skill is the ability to create the limits through the consideration of the region Ω over which we are integrating. Interestingly we always have the choice as to whether we use $dA = dxdy$ or $dA = dydx$. When using $dA = dydx$ we are first slicing the cylinder over Ω parallel to the y axis and then accumulating those slices in the x direction, for $dA = dxdy$ it is similar, but the other way around. The result is the same either way, however it is often the case that one approach is significantly quicker than the other!

Example 3 Evaluate $\displaystyle\iint_\Omega 3y\ dA$ where $\Omega = \{(x, y) : 0 \leq x \leq 3 \text{ and } 0 \leq y \leq 2\}$

!!!!! YOU MUST SKETCH Ω IN THE $x - y$ PLANE!!!!!



We will use both $dA = dxdy$ and as a check $dA = dydx$.

$$\iint_\Omega 3y \, dx \, dy$$
$$= \int_0^2 \int_0^3 3y \, dx \, dy$$
$$= \int_0^2 \Big[3yx \Big]_0^3 \, dy$$
$$= \int_0^2 9y \, dy$$
$$= \left[\frac{9y^2}{2} \right]_0^2$$
$$= 18$$

$$\iint_\Omega 3y \, dy \, dx$$
$$= \int_0^3 \int_0^2 3y \, dy \, dx$$
$$= \int_0^3 \left[\frac{3y^2}{2} \right]_0^2 \, dx$$
$$= \int_0^3 6 \, dx$$
$$= 18$$

★ 18 ★

Question: What is the geometrical interpretation of this answer?

Volume of curve above Ω and below $z = f(x, y)$

Question: Can $\displaystyle\iint_\Omega f(x, y) \; dA < 0$?

Yes

In general the region Ω is much more complicated than a simple rectangle.

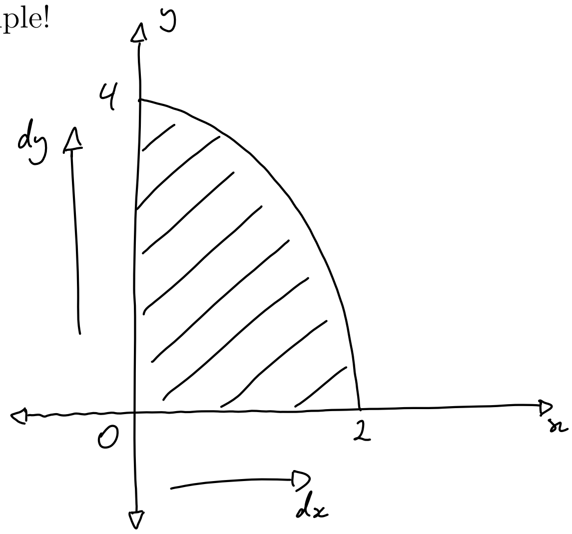
Example 4 Evaluate $\displaystyle\iint_\Omega x\, dA$

where Ω is the region in the first quadrant bounded by the parabola $y = 4 - x^2$ and the coordinate axes.

To evaluate general double integrals where the limits still need to be produced:

- Draw Ω.
- Run a tracer parallel to the inner variable to determine the inner limits.
- Consider the degree of freedom in your tracer to determine the outer limits.
- Remember the outer limits must be constants.

We will once again use both $dA = dxdy$ and $dA = dydx$. Note that in practice, only one of these (the easy one!) should be evaluated. We will see in the next lecture that one of the two may well be impossible while the other is quite simple!



$$\iint_\Omega x\, dA$$

$$= \int_0^2 \int_0^{4-x^2} x \; dy \; dx$$

$$= \int_0^2 \Big[xy \Big]_0^{4-x^2} dx$$

$$= \int_0^2 4x - x^3 \; dx$$

$$= \left[2x^2 - \frac{x^4}{4} \right]_0^2$$

$$= 4$$

$$\iint_{\Omega} x \, dA$$

$$= \int_0^4 \int_0^{\sqrt{4-y}} x \, dx \, dy$$

$$= \int_0^4 \left[\frac{x^2}{2} \right]_0^{\sqrt{4-y}}$$

$$= \int_0^4 \frac{4-y}{2} \, dy$$

$$= \int_0^4 2 - \frac{y}{2} \, dy$$

$$= \left[2y - \frac{y^2}{4} \right]_0^4$$

$$= 8 - \frac{4^2}{4}$$

$$= 4$$

★ 4 ★

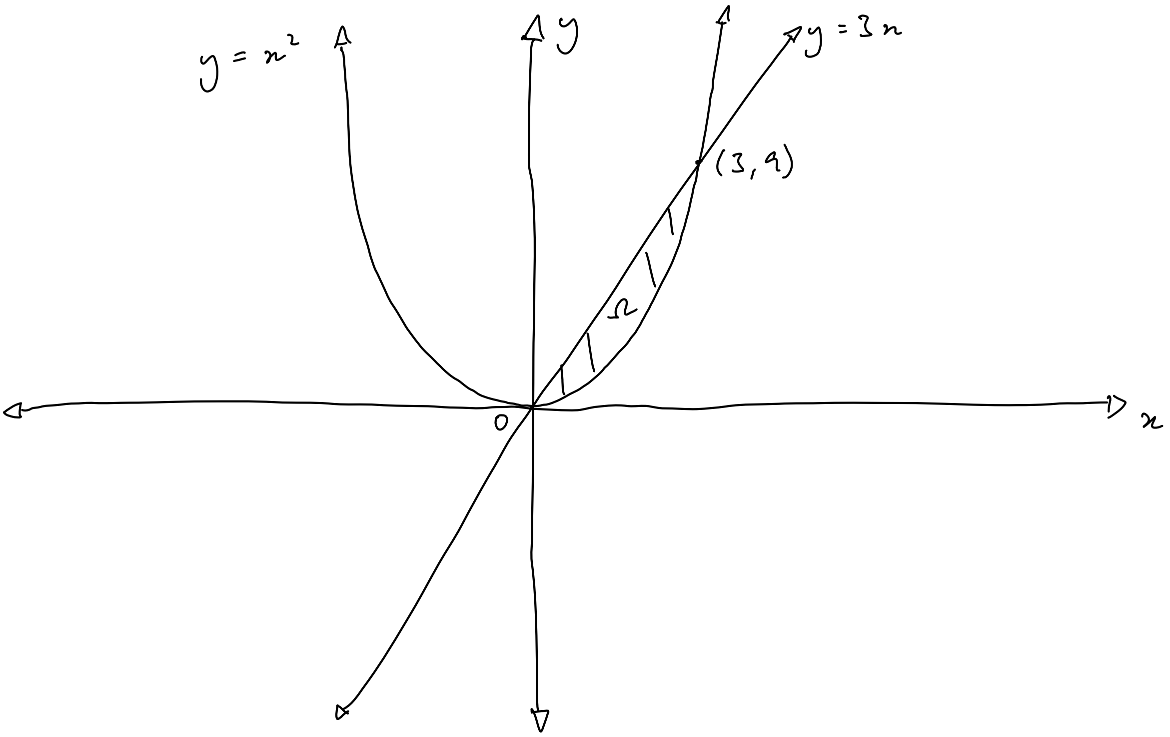
Example 5 Evaluate $\displaystyle\iint_{\Omega} 8xy\ dA$

where Ω is the region in the $x-y$ plane bounded by the two curves $y = 3x$ and $y = x^2$.

Recall that to evaluate general double integrals where the limits still need to be produced:

- Draw Ω.
- Run a tracer parallel to the inner variable to determine the inner limits.
- Consider the degree of freedom in your tracer to determine the outer limits.
- Remember the outer limits must be constants.

We will once again use both $dA = dxdy$ and $dA = dydx$. Note that in practice, only one of these (the easy one!) should be evaluated. We will see in the next lecture that one of the two may well be impossible while the other is quite simple!



$$\iint_{\Omega} 8xy \; dA$$

$$= \int_0^3 \int_{3x}^{x^2} 8xy \; dy \, dx$$ (expect −ve answer)

$$= \int_0^3 \left[4xy^2 \right]_{3x}^{x^2} dx$$

$$= \int_0^3 4x^5 - 36x^3 \; dx$$

$$= \left[\frac{2x^6}{3} - 9x^4 \right]_0^3$$

$$= \frac{2 \times 3^6}{3} - 9 \times 3^4$$

$$= -243$$

★ 243 ★

[15] You can now do Q 64 and 66