THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

NOVEMBER 2016

MATH2019
ENGINEERING MATHEMATICS 2E

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED “UNSW APPROVED” STICKER MAY BE USED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

TABLE OF LAPLACE TRANSFORMS AND THEOREMS

$g(t)$ is a function defined for all $t \geq 0$, and whose Laplace transform

$$G(s) = \mathcal{L}(g(t)) = \int_0^\infty e^{-st} g(t) dt$$

exists. The Heaviside step function u is defined to be

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ \frac{1}{2} & \text{for } t = a \\ 1 & \text{for } t > a \end{cases}$$

$g(t)$	$G(s) = \mathcal{L}[g(t)]$
1	$\dfrac{1}{s}$
t	$\dfrac{1}{s^2}$
$t^\nu, \nu > -1$	$\dfrac{\nu!}{s^{\nu+1}}$
$e^{-\alpha t}$	$\dfrac{1}{s+\alpha}$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$e^{-\alpha t} f(t)$	$F(s+\alpha)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$tf(t)$	$-F'(s)$

FOURIER SERIES

If $f(x)$ has period $T = 2L$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}\, x\right) + b_n \sin\left(\frac{n\pi}{L}\, x\right) \right)$$

where

$$\begin{aligned}
a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)dx \\
a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L}\, x\right) dx \\
b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}\, x\right) dx
\end{aligned}$$

LEIBNIZ RULE FOR DIFFERENTIATING INTEGRALS

$$\frac{d}{dx} \int_u^v f(x,t)dt = \int_u^v \frac{\partial f}{\partial x} dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx}.$$

MULTIVARIABLE TAYLOR SERIES

$$\begin{aligned}
f(x,y) &= f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\
&\quad + \frac{1}{2!}\left((x-a)^2 \frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) + (y-b)^2 \frac{\partial^2 f}{\partial y^2}(a,b) \right) + \cdots
\end{aligned}$$

VARIATION OF PARAMETERS

Suppose that the second order differential equation

$$y'' + p(x)y' + q(x)y = f(x)$$

has homogeneous solution $y_h = Ay_1(x) + By_2(x)$. Then a particular solution is given by

$$y_P(x) = -y_1(x) \int \frac{y_2(x)f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)} dx$$

where $W(x) = \det \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}$.

Please see over ...

SOME BASIC INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^{kx} dx = \frac{e^{kx}}{k} + C$$

$$\int a^x dx = \frac{1}{\ln a} a^x + C \quad \text{for } a \neq 1$$

$$\int \sin\ kx\, dx = -\frac{\cos kx}{k} + C$$

$$\int \cos kx\, dx = \frac{\sin kx}{k} + C$$

$$\int \sec^2 kx\, dx = \frac{\tan kx}{k} + C$$

$$\int \mathrm{cosec}^2 kx\, dx = -\frac{1}{k} \cot kx + C$$

$$\int \tan kx\, dx = \frac{\ln|\sec kx|}{k} + C$$

$$\int \sec kx\, dx = \frac{1}{k}(\ln|\sec kx + \tan kx|) + C$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx$$

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx$$

Answer question 1 in a separate book

1. i) Suppose that the pressure ϕ in a region of space is given by the scalar field

$$\phi(x, y, z) = xy^2 z^3.$$

a) Calculate $\operatorname{grad} \phi$ at the point $A(1, 2, 1)$.

b) Find the rate of change of the pressure with respect to distance at the point $A(1, 2, 1)$ in the direction $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

c) Write down a unit normal to the level surface $\phi(x, y, z) = 4$ at the point $A(1, 2, 1)$.

ii) Estimate the maximum possible percentage error in calculating $V = \pi r^2 h$ if errors in r and h are at most 1%.

iii) Use Leibniz' theorem to find

$$\frac{d}{dt} \int_1^{\sin(t)} e^{1-x^2}\, dx.$$

iv) a) Use the method of Lagrange multipliers to find the minimum value of $x^2 + y^2$ subject to the constraint $x + y = 6$.

b) Using your solution in a) and making no further use of the method of Lagrange find the maximum value of xy subject to the constraint $x + y = 6$.

v) A thin plate in the first quadrant is bounded by the circle $x^2 + y^2 = 1$ and the coordinate axes. The plate has uniform density $\delta(x, y) = 1$.

a) Sketch the plate in the $x - y$ plane.

b) Without evaluating any integrals write down the mass of the plate.

c) Find the coordinates of the centroid $(\bar{x}, \bar{y})$ of the plate by evaluating an appropriate double integral in polar coordinates.
(Note that by symmetry, $\bar{y} = \bar{x}$).

Answer question 2 in a separate book

2. i) Use the substitution $v = y + x$ to find the general solution of

$$\frac{dy}{dx} = (y+x)^2.$$

ii) Consider the double integral

$$I = \int_0^2 \int_{x^2}^4 \frac{e^y}{\sqrt{y}}\, dy\, dx$$

a) Sketch the region of integration.

b) Evaluate I by first reversing the order of integration.

iii) Consider the vector field $\mathbf{F} = (27y - y^3)\mathbf{i} + (x^3)\mathbf{j} + (x - xz)\mathbf{k}$.

a) Calculate $\operatorname{curl} \mathbf{F}$.

b) Sketch the curve $\mathcal{C}$ in $\mathbb{R}^3$ for which $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

iv) Consider the matrix $A = \begin{pmatrix} 6 & 2 \\ -1 & 3 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors of A.

b) Hence solve the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 6x + 2y \\ \frac{dy}{dt} &= -x + 3y. \end{aligned}$$

v) a) Suppose that

$$\mathbf{r}_1(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$$

and

$$\mathbf{r}_2(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$$

are two curves in $\mathbb{R}^3$. Prove that

$$[\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)]' = \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t) + \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t).$$

b) Suppose that a particle P moves along a curve C in $\mathbb{R}^3$ in such a manner that its velocity vector is always perpendicular to its position vector. Using part(a) prove that the path C lies on the surface of a sphere whose centre is the origin.

Answer question 3 in a separate book

3. i) Use the method of undetermined coefficients to solve the second order differential equation

$$y'' - 4y = \mathrm{e}^{2t}.$$

ii) Find:

a)

$$\mathcal{L}\left\{\mathrm{e}^{-3t}\sin \pi t\right\}$$

b)

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s-4}\right\}$$

iii) The function $g(t)$ is given by

$$g(t) = \begin{cases} t & \text{for } 0 \le t < 2, \\ -1 & \text{for } t \ge 2. \end{cases}$$

a) Sketch the function $g(t)$ for $0 \le t \le 6$.

b) Write $g(t)$ in terms of the Heaviside step function $u(t-a)$.

c) Hence, or otherwise, show that the Laplace transform of $g(t)$ is given by

$$\mathcal{L}\left\{g(t)\right\} = \frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} + \frac{3}{s}\right).$$

iv) A rocket is launched straight upwards at time $t = 0$ and its thrusters burn until $t = 2$. The vertical velocity $v(t)$ of the rocket satisfies the differential equation

$$\frac{dv}{dt} = g(t), \qquad v(0) = 0$$

where $g(t)$ is the function in part iii).

a) Using Laplace transforms and the result from part iii) c), solve the differential equation above to find the velocity of the rocket $v(t)$ as a function of time.

b) By writing your solution separately for times $0 \le t < 2$ and $t \ge 2$, or otherwise, sketch the velocity as a function of time for $0 \le t \le 6$.

c) What is the maximum velocity of the rocket?

d) At what time will the rocket reach its maximum height?

Answer question 4 in a separate book

4. The temperature in a conducting metal bar of length L is described by the heat equation

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}\,,$$

where $u(x,t)$ is the temperature at position x and time t.

The ends of the metal bar are held at a constant temperature of 0°C so that

$$u(0,t) = u(L,t) = 0, \qquad \text{for all } t > 0.$$

i) Assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$\frac{G'(t)}{3G(t)} = \frac{F''(x)}{F(x)} = k$$

for some constant k.

ii) Write down the boundary conditions for $F(x)$.

iii) Apply the boundary conditions to show that the possible solutions for $F(x)$ are

$$F_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$$

where B_n are constants and $n = 1, 2, 3, \cdots$. You must consider all possible values of k.

iv) Find all possible solutions $G_n(t)$ for $G(t)$ and write down the general solution for $u_n(x,t) = F_n(x)G_n(t)$.

v) If the initial temperature distribution is $u(x,0) = 1$, find the solution $u(x,t)$ expressing your answer as an infinite series.

Math2018/2019 S2 2016 Question 1 Solutions

1 i) a) $\nabla\phi = \frac{\partial\phi}{\partial x}\underset{\sim}{i} + \frac{\partial\phi}{\partial y}\underset{\sim}{j} + \frac{\partial\phi}{\partial z}\underset{\sim}{k}$

$= y^2z^3\underset{\sim}{i} + 2xyz^3\underset{\sim}{j} + 3xy^2z^2\underset{\sim}{k}$

$= 4\underset{\sim}{i} + 4\underset{\sim}{j} + 12\underset{\sim}{k} = \underline{\begin{pmatrix}4\\4\\12\end{pmatrix}}$ @(1,2,1).

b) directional derivative = gradϕ. $\hat{\underset{\sim}{s}}$

$= \begin{pmatrix}4\\4\\12\end{pmatrix}\cdot\frac{1}{3}\begin{pmatrix}2\\1\\-2\end{pmatrix} = \frac{1}{3}(8+4-24)$

$= \underline{-4}$

c) $\frac{\pm 1}{\sqrt{176}}\begin{pmatrix}4\\4\\12\end{pmatrix} = \frac{\pm 1}{4\sqrt{11}}\begin{pmatrix}4\\4\\12\end{pmatrix} = \frac{\pm 1}{\sqrt{11}}\begin{pmatrix}1\\1\\3\end{pmatrix}$

(ii) $|\Delta V| \le \left|\frac{\partial V}{\partial r}\right||\Delta r| + \left|\frac{\partial V}{\partial h}\right||\Delta h|$

$= |2\pi rh||\Delta r| + |\pi r^2||\Delta h|$

$\frac{|\Delta V|}{V} = \left|\frac{2\pi rh}{\pi r^2h}\right||\Delta r| + \left|\frac{\pi r^2}{\pi r^2 h}\right||\Delta h|$

$= 2\left|\frac{\Delta r}{r}\right| + \left|\frac{\Delta h}{h}\right|$

$\therefore\ \Delta V\% = 2\Delta r\% + \Delta h\%$

$= 2+1 = \underline{3\%}$

(iii) $\frac{d}{dt}\int_1^{\sin t} e^{1-x^2}\,dx = \int_1^{\sin t} \cancel{\frac{\partial}{\partial t} e^{1-x^2}\,dx}$

$+ e^{1-\sin^2 t}\cos t \quad - 0.$

$= \underline{e^{\cos^2(t)}\cos t}$

iv) a) Max/Min x^2+y^2 s.t. $x+y-6=0.$

$2x = \lambda(1) \Rightarrow x = \frac{\lambda}{2}$
$2y = \lambda(1) \Rightarrow y = \frac{\lambda}{2}.$

$\therefore \frac{\lambda}{2} + \frac{\lambda}{2} - 6 = 0 \rightarrow \underline{\lambda = 6}.$

$\therefore x = 3, \; y = 3.$

$\therefore$ Minimum value $9+9 = \boxed{\underline{18}}$

checking against $x=6, y=0 \rightarrow 36+0 = 36$

verifies that 18 is indeed a <u>min</u>

b) $(x+y)^2 = x^2 + 2xy + y^2$

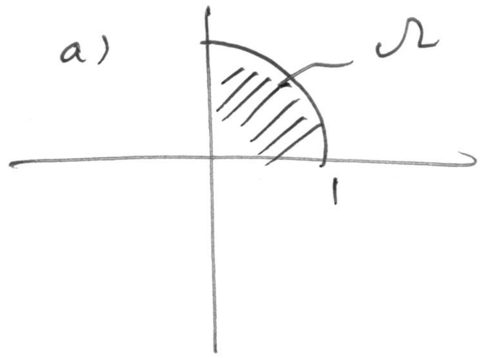
Thus $x^2+y^2 + (2xy) = 36$

Since minimum value of x^2+y^2 is 18

the maximum value of $2xy$ is also 18

Thus max of xy is $\boxed{9.}$

a)



v) b) $\text{Mass} = \iint_\Omega \rho(x,y)\, dA$

$= \text{area}(\Omega)$

$= \dfrac{\pi(1)^2}{4} = \underline{\dfrac{\pi}{4}}$

$$\iint x\rho(x,y)\,dA = \iint x\,dy\,dx = \int_0^{\frac{\pi}{2}}\int_0^1 r\cos\theta\, r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r^2\cos\theta\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left[\tfrac{1}{3}r^3\cos\theta\right]_0^1 d\theta$$

$$= \int_0^{\frac{\pi}{2}} \tfrac{1}{3}\cos\theta\,d\theta = \left[\tfrac{1}{3}\sin\theta\right]_0^{\frac{\pi}{2}} = \tfrac{1}{3} - 0 = \tfrac{1}{3}$$

$$\therefore\ \bar{x} = \bar{y} = \frac{\frac{1}{3}}{\left(\frac{\pi}{4}\right)} = \frac{4}{3\pi}$$

Question 2 Solutions

2 i) $v = y + x \Rightarrow \frac{dv}{dx} = \frac{dy}{dx} + 1 \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

Thus $\frac{dv}{dx} - 1 = v^2$

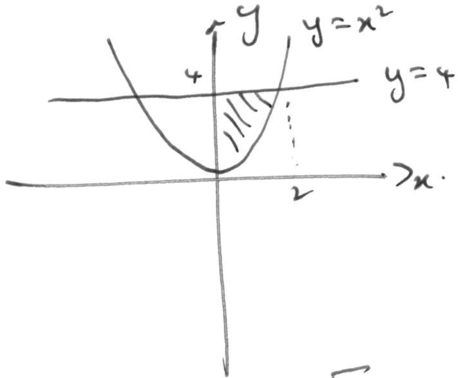
$\Rightarrow \frac{dv}{dx} = 1 + v^2$

$\Rightarrow \frac{dv}{1+v^2} = dx \Rightarrow \tan^{-1}(v) = x + c$

$\Rightarrow v = \tan^{-1}(x + c')$

$\Rightarrow y = \tan^{-1}(x + c') - x.$

ii) a)



b) $I = \int_0^4 \int_0^{\sqrt{y}} \frac{e^y}{\sqrt{y}} \, dx\,dy$

$= \int_0^4 \left[\frac{e^y}{\sqrt{y}} x \right]_0^{\sqrt{y}} dy = \int_0^4 e^y \, dy = \left[e^y \right]_0^4$

$= \underline{e^4 - 1}$

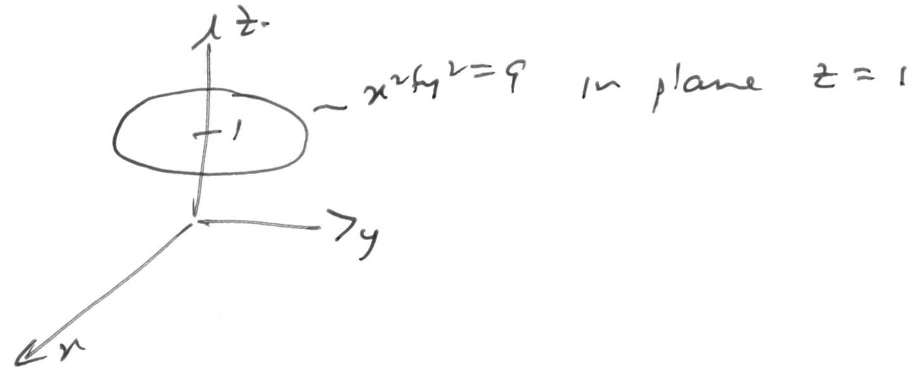
(iii) a) $\text{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 27y-y^3 & x^3 & x-xz \end{vmatrix}$

$= i \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & x-xz \end{vmatrix} - j \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 27y-y^3 & x-xz \end{vmatrix} + k \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 27y-y^3 & x^3 \end{vmatrix}$

$= i(0-0) - j(1-z \quad -0 \quad) + k(3x^2 - (27-3y^2))$

$= \begin{pmatrix} 0 \\ z-1 \\ 3x^2+3y^2-27 \end{pmatrix}$

b) $\text{curl}(F) = 0 \Rightarrow x^2+y^2=9$ and $z=1$



iv) $\begin{vmatrix} 6-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = (6-\lambda)(3-\lambda)+2$

$= 18 - 9\lambda + \lambda^2 + 2$

$= \lambda^2 - 9\lambda + 20 = 0$

$\therefore (\lambda-4)(\lambda-5) = 0 \Rightarrow \lambda = 4, 5.$

check $4+5 = 6+3$ ✓

<u>For $\lambda=4$:</u> $\left(\begin{array}{cc|c} 2 & 2 & 0 \\ -1 & -1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

Let $y=t$, $x=-t$ $\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t$ $\quad t \neq 0$.

For $\lambda=5$: $\left(\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & -2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

Let $y=t$, $x+2t=0 \rightarrow x=-2t$.

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} t \quad t \neq 0$.

check $\begin{pmatrix} 6 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \end{pmatrix} = 5\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ ✓

b) $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{4t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

2) a) $LHS = (r_1 . r_2)' = (x_1x_2 + y_1y_2 + z_1z_2)'$

$= x_1x_2' + x_2x_1' + y_1y_2' + y_2y_1' + z_1z_2' + z_2z_1'$

$RHS = r_1' . r_2 + r_1 . r_2' = \begin{pmatrix} x_1' \\ y_1' \\ z_1' \end{pmatrix} . \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} . \begin{pmatrix} x_2' \\ y_2' \\ z_2' \end{pmatrix}$

$= x_1'x_2 + y_1'y_2 + z_1'z_2 + x_1x_2' + y_1y_2' + z_1z_2' = LHS$.

b) Applying a) to the curve $r(t)$

$(r.r)' = \underline{r}'.\underline{r} + \underline{r}.\underline{r}' = \underline{v}.\underline{r} + \underline{r}.\underline{v} = 0+0 = 0$

(since $\underset{\sim}{v} \perp \underset{\sim}{r}$)

Thus $(r.r)'=0 \Rightarrow r.r = C \Rightarrow |r|^2 = C$

as required

Q3 i) $y'' - 4y = e^{2t}$

let $y = y_h + y_p$

where $y_h'' - 4y_h = 0$ (homogeneous equation)

let $y_h = e^{\lambda t}$ $\Rightarrow$ $\lambda^2 - 4 = 0$ $\Rightarrow$ $\lambda = \pm 2$

$\Rightarrow$ $y_h = Ae^{-2t} + Be^{2t}$

Particular solution: $y_p'' - 4y_p = e^{2t}$

Let $y_p = Cte^{2t}$

$\Rightarrow$ $y_p' = Ce^{2t} + 2Cte^{2t}$

$\Rightarrow$ $y_p'' = 2Ce^{2t} + 2Ce^{2t} + 4Cte^{2t}$
$= 4Ce^{2t} + 4Cte^{2t}$

Thus $y_p'' - 4y_p = e^{2t}$

$\Rightarrow$ $4Ce^{2t} + \cancel{4Cte^{2t}} - \cancel{4Cte^{2t}} = e^{2t}$

$\Rightarrow$ $4C = 1$ $\Rightarrow$ $C = \frac{1}{4}$

so

$y = Ae^{-2t} + Be^{2t} + \frac{1}{4}te^{2t}$

Q3 ii) a) $\mathcal{L}\{e^{-3t}\sin \pi t\} = \mathcal{L}\{e^{-\alpha t} f(t)\} = F(s+\alpha)$

(FIRST SHIFTING THEOREM)

$\alpha = 3 \qquad f(t) = \sin \pi t$

$\Rightarrow F(s) = \mathcal{L}\{\sin \pi t\} = \dfrac{\pi}{s^2+\pi^2}$

and $F(s+\alpha) = \dfrac{\pi}{(s+3)^2+\pi^2}$

b) $\dfrac{1}{s^2+3s-4} = \dfrac{1}{(s+4)(s-1)}$

(PARFRAC) $= \dfrac{A}{s+4} + \dfrac{B}{s-1} = \dfrac{A(s-1)+B(s+4)}{(s+4)(s-1)}$

so $A(s-1) + B(s+4) = 1$

at $s = -4$: $-5A = 1 \Rightarrow A = -1/5$

at $s = 1$: $5B = 1 \Rightarrow B = 1/5$

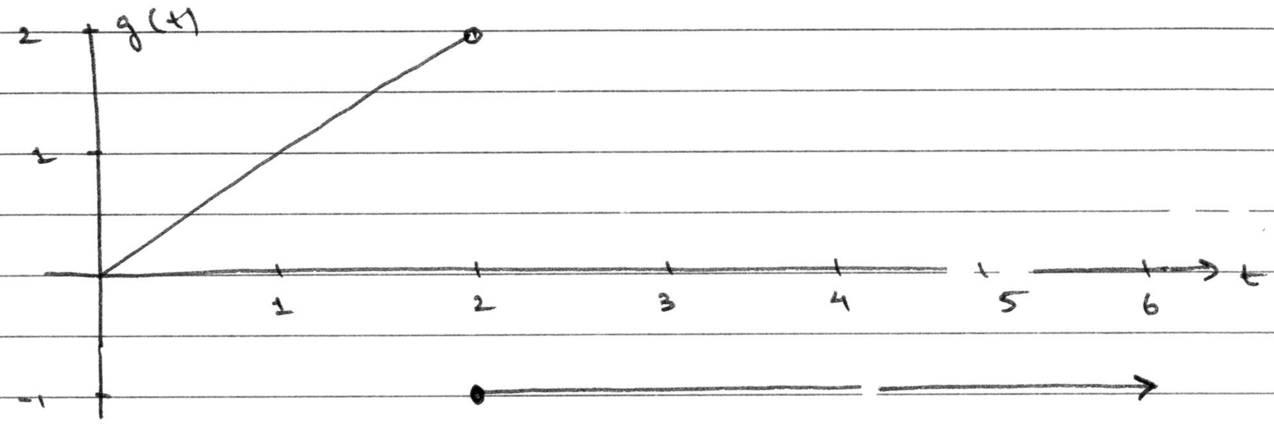
so $\dfrac{1}{s^2+3s-4} = -\dfrac{1}{5}\dfrac{1}{s+4} + \dfrac{1}{5}\dfrac{1}{s-1}$

and $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+3s-4}\right\} = -\dfrac{1}{5}\mathcal{L}^{-1}\left\{\dfrac{1}{s+4}\right\} + \dfrac{1}{5}\mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\}$

$= -\dfrac{1}{5}e^{-4t} + \dfrac{1}{5}e^{t}$

Q3 iii) $g(t) = \begin{cases} t & \text{for} \quad 0 \le t < 2 \\ -1 & \text{for} \quad t \geq 2 \end{cases}$

a)



b) $g(t) = t\left[1 - u(t-2)\right] - 1\left[u(t-2)\right]$

$$= t - t\,u(t-2) - u(t-2)$$

c) $g(t) = t - (t-2+2)\,u(t-2) - u(t-2)$

$$= t - (t-2)\,u(t-2) - 3u(t-2)$$

$$\mathcal{L}\{g(t)\} = \frac{1}{s^2} - \frac{e^{-2s}}{s^2} - \frac{3e^{-2s}}{s}$$

from the second shifting theorem.

Thus

$$\mathcal{L}\{g(t)\} = \frac{1}{s^2} - e^{-2s}\left\{\frac{1}{s^2} + \frac{3}{s}\right\}$$

Q3 iv) $\frac{dv}{dt} = g(t)$ $\qquad v(0) = 0$

a) $\mathcal{L}\{LHS\} = sF(s) - \cancel{v(0)}^{\,0} = sF(s)$

where $F(s) = \mathcal{L}\{v(t)\}$.

$\mathcal{L}\{RHS\} = \frac{1}{s^2} - e^{-2s}\left\{\frac{1}{s^2} + \frac{3}{s}\right\}$

so $F(s) = \frac{1}{s^3} - \frac{e^{-2s}}{s^3} - \frac{3e^{-2s}}{s^2}$

$\Rightarrow v(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{t^2}{2!} - u(t-2)\left\{\frac{(t-2)^2}{2!} + 3(t-2)\right\}$

$= \frac{t^2}{2} - u(t-2)\left\{\frac{(t-2)^2}{2} + 3(t-2)\right\}.$

b) for $0 \le t < 2$, $v(t) = t^2/2$

for $t \geq 2$,

$v(t) = \frac{t^2}{2} - \frac{(t-2)^2}{2} - 3(t-2)$

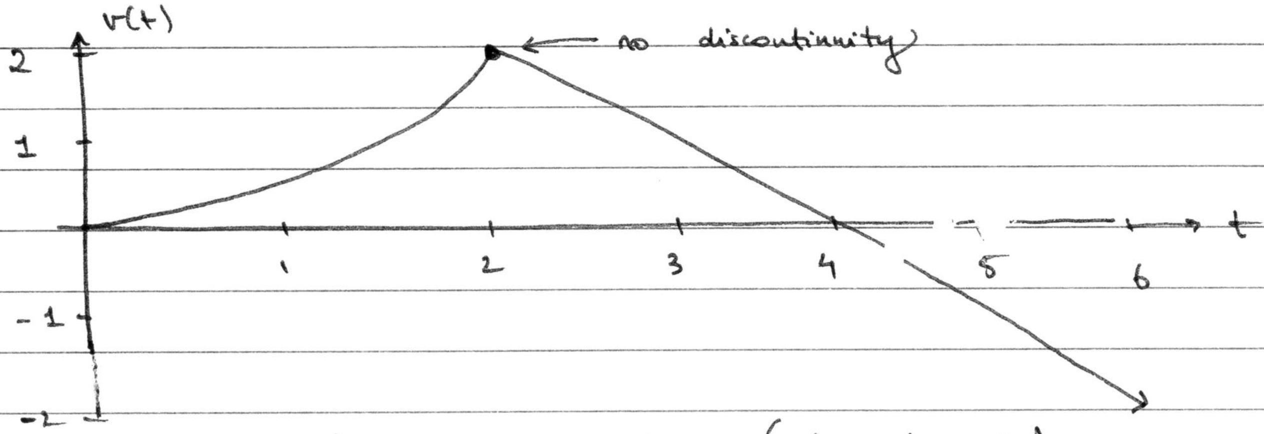
$= \frac{t^2}{2} - \frac{1}{2}\left(t^2 - 4t + 4\right) - 3t + 6$

$= \cancel{\frac{t^2}{2}} - \cancel{\frac{t^2}{2}} + 2t - 2 - 3t + 6$

$= -t + 4$

Q3 iv) continued :

$$v(t) = \begin{cases} t^2/2 & 0 \le t < 2 \\ -t + 4 & t \geq 2 \end{cases}$$



c) maximum velocity $v = 2$ (at $t = 2$)

d) maximum height occurs when $v = 0$. This happens at $t = 4$.

Q4. i) $u(x,t) = F(x)\,G(t)$

$u_t = F(x)\,G'(t)$ $\qquad G'(t) = \dfrac{dG}{dt}$

$u_{xx} = F''(x)\,G(t)$ $\qquad F''(x) = \dfrac{d^2F}{dx^2}$

so $u_t = 3u_{xx}$ becomes

$$F(x)\,G'(t) = 3F''(x)\,G(t)$$

$$\Rightarrow \underbrace{\frac{G'(t)}{3G(t)}}_{\substack{\text{function}\\ \text{of } t \text{ only}}} = \underbrace{\frac{F''(x)}{F(x)}}_{\substack{\text{function}\\ \text{of } x \text{ only}}}$$

The only way these can be equal is if they are both constants, so

$$\frac{G'(t)}{3G(t)} = \frac{F''(x)}{F(x)} = k$$

for some k = constant.

ii) $u(0,t) = F(0)\,G(t) = 0$

$G(t)$ can't be zero (trivial solution) so $F(0) = 0$

$u(L,t) = F(L)\,G(t) = 0$

$G(t)$ can't be zero (trivial solution) so $F(L) = 0$

Q4 iii) $F''(x) - k F(x) = 0$

<u>case 1</u> : $k = 0 \Rightarrow F''(x) = 0$

$\Rightarrow F = Ax + B$

$F(0) = 0 = B \Rightarrow \boxed{B = 0}$

$F(L) = 0 = A \cdot L \Rightarrow \boxed{A = 0}$

So $F(x) = 0 \Rightarrow$ trivial solution.

<u>case 2</u> : $k = \phi^2 > 0 \Rightarrow F''(x) - \phi^2 F(x) = 0$

if $F = e^{\lambda x} \Rightarrow \lambda^2 - \phi^2 = 0 \Rightarrow \lambda = \pm\phi$

so $F(x) = Ae^{-\phi x} + Be^{\phi x}$

$F(0) = 0 = A + B \Rightarrow \boxed{A = -B}$

$F(L) = 0 = A\left(e^{-\phi L} - e^{\phi L}\right)$

$\Rightarrow \boxed{A = 0}$ (trivial solution)

OR $e^{-\phi L} - e^{\phi L} = 0 \Rightarrow e^{\phi L} = e^{-\phi L}$

$\Rightarrow e^{2\phi L} = 1 \Rightarrow \phi = 0$ but this is a contradiction $(\phi > 0)$

case 3 : $k = -p^2 < 0 \Rightarrow F''(x) + p^2 F(x) = 0$

if $F = e^{\lambda x} \Rightarrow \lambda^2 + p^2 = 0 \Rightarrow \lambda = \pm ip$

so $F(x) = A\cos px + B\sin px$

$F(0) = 0 = A \Rightarrow \boxed{A = 0}$

$F(L) = 0 = B\sin pL$

$\Rightarrow \boxed{B = 0}$ (trivial solution)

OR $\boxed{\sin pL = 0} \Rightarrow pL = n\pi \qquad n = 1, 2, 3, \ldots$

thus $p = \frac{n\pi}{L}$ and $F_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$

iv) $\frac{G'(t)}{3G(t)} = k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$

thus $G'(t) = -\frac{3n^2\pi^2}{L^2} G(t)$

$\Rightarrow G(t) = C e^{-3n^2\pi^2/L^2 \cdot t}$

Thus $u_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right) e^{-3n^2\pi^2 t/L^2}$

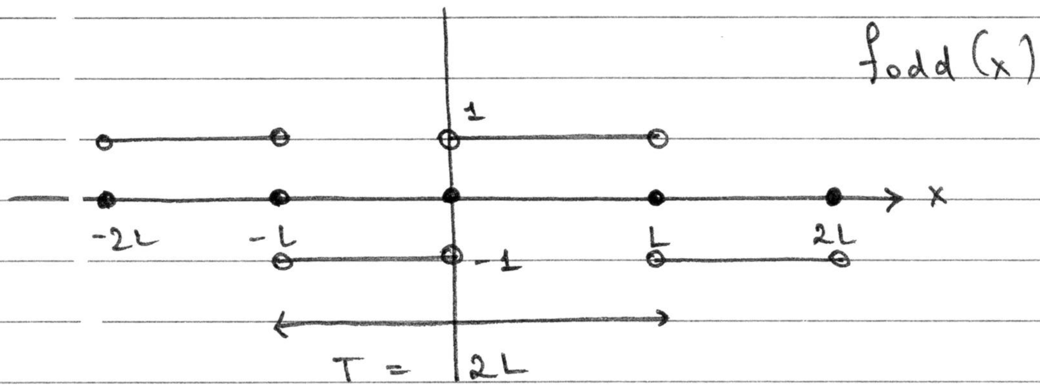
absorbs C_n

and $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-3n^2\pi^2 t/L^2}$

Q4 iv) Initial condition $u(x,0) = 1$

$$\Rightarrow \quad \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = 1.$$

Thus, B_n are the coefficients of the Fourier series of the odd periodic extension of 1:



$$f_{odd}(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n = \frac{1}{L}\int_{-L}^{L} \underset{\text{ODD}}{f_{odd}(x)} \times \underset{\text{ODD}}{\sin\left(\frac{n\pi x}{L}\right)}\, dx \quad = \text{EVEN}$$

$$= \frac{2}{L}\int_{0}^{L} 1 \cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\,\frac{L}{n\pi}\left[-\cos\frac{n\pi x}{L}\right]_0^L$$

$$= \frac{2}{n\pi}\left\{-\cos n\pi + \cos 0\right\} = \frac{2}{n\pi}\left\{1 - (-1)^n\right\}$$

for $n = 2m$ = even $\qquad m = 1, 2, 3, \dots$

$$B_{2m} = \frac{2}{2m\pi}\left\{1 - (-1)^{2m}\right\} = \frac{1}{m\pi}\left\{1 - 1\right\} = 0$$

for $n = 2m+1$ = odd $\qquad m = 0, 1, 2, \dots$

$$B_{2m+1} = \frac{2}{(2m+1)\pi}\left\{1 - (-1)^{2m+1}\right\} = \frac{4}{(2m+1)\pi}$$

Thus

$$u(x,t) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} \sin\left(\frac{(2m+1)\pi x}{L}\right) e^{-3(2m+1)^2\pi^2 t/L^2}$$

Q4. i) $u(x,t) = F(x)\,G(t)$

$u_t = F(x)\,G'(t)$ $\qquad G'(t) = \dfrac{dG}{dt}$

$u_{xx} = F''(x)\,G(t)$ $\qquad F''(x) = \dfrac{d^2F}{dx^2}$

so $u_t = 3u_{xx}$ becomes

$$F(x)\,G'(t) = 3F''(x)\,G(t)$$

$$\Rightarrow \underbrace{\frac{G'(t)}{3G(t)}}_{\text{function of } t \text{ only}} = \underbrace{\frac{F''(x)}{F(x)}}_{\text{function of } x \text{ only}}$$

The only way these can be equal is if they are both constants, so

$$\frac{G'(t)}{3G(t)} = \frac{F''(x)}{F(x)} = k$$

for some k = constant.

ii) $u(0,t) = F(0)\,G(t) = 0$

$G(t)$ can't be zero (trivial solution) so $F(0) = 0$

$u(L,t) = F(L)\,G(t) = 0$

$G(t)$ can't be zero (trivial solution) so $F(L) = 0$

Q4 iii) $F''(x) - k F(x) = 0$

case 1 : $k = 0 \Rightarrow F''(x) = 0$

$\Rightarrow F = Ax + B$

$F(0) = 0 = B \Rightarrow \boxed{B = 0}$

$F(L) = 0 = A \cdot L \Rightarrow \boxed{A = 0}$

So $F(x) = 0 \Rightarrow$ trivial solution.

case 2 : $k = \phi^2 > 0 \Rightarrow F''(x) - \phi^2 F(x) = 0$

if $F = e^{\lambda x} \Rightarrow \lambda^2 - \phi^2 = 0 \Rightarrow \lambda = \pm \phi$

so $F(x) = A e^{-\phi x} + B e^{\phi x}$

$F(0) = 0 = A + B \Rightarrow \boxed{A = -B}$

$F(L) = 0 = A\left(e^{-\phi L} - e^{\phi L}\right)$

$\Rightarrow \boxed{A = 0}$ (trivial solution)

OR $e^{-\phi L} - e^{\phi L} = 0 \Rightarrow e^{\phi L} = e^{-\phi L}$

$\Rightarrow e^{2\phi L} = 1 \Rightarrow \phi = 0$ but this is a contradiction $(\phi > 0)$

<u>case 3</u> : $k = -p^2 < 0 \Rightarrow F''(x) + p^2 F(x) = 0$

if $F = e^{\lambda x} \Rightarrow \lambda^2 + p^2 = 0 \Rightarrow \lambda = \pm ip$

so $F(x) = A\cos px + B\sin px$

$F(0) = 0 = A \Rightarrow \boxed{A = 0}$

$F(L) = 0 = B\sin pL$

$\Rightarrow \boxed{B = 0}$ (trivial solution)

OR $\boxed{\sin pL = 0} \Rightarrow pL = n\pi \qquad n = 1, 2, 3, \cdots$

thus $p = \dfrac{n\pi}{L}$ and $F_n(x) = B_n \sin\left(\dfrac{n\pi x}{L}\right)$

iv) $\dfrac{G'(t)}{3G(t)} = k = -p^2 = -\left(\dfrac{n\pi}{L}\right)^2$

thus $G'(t) = -\dfrac{3n^2\pi^2}{L^2} G(t)$

$\Rightarrow G(t) = C e^{-3n^2\pi^2/L^2 \cdot t}$

Thus $u_n(x,t) = B_n \sin\left(\dfrac{n\pi x}{L}\right) e^{-3n^2\pi^2 t/L^2}$

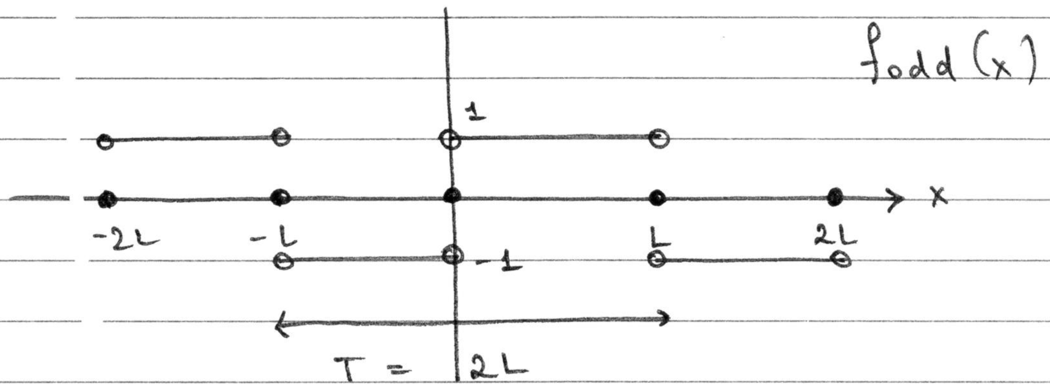
absorb C_n

and $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\dfrac{n\pi x}{L}\right) e^{-3n^2\pi^2 t/L^2}$

Q4 iv) Initial condition $u(x,0) = 1$

$$\Rightarrow \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = 1.$$

Thus, B_n are the coefficients of the Fourier series of the <u>odd</u> periodic extension of 1:



$$f_{odd}(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f_{odd}(x)}_{ODD} \times \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{ODD} dx \quad = \text{EVEN}$$

$$= \frac{2}{L}\int_{0}^{L} 1 \cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\,\frac{L}{n\pi}\left[-\cos\frac{n\pi x}{L}\right]_0^L$$

$$= \frac{2}{n\pi}\left\{-\cos n\pi + \cos 0\right\} = \frac{2}{n\pi}\left\{1 - (-1)^n\right\}$$

for $n = 2m$ = even $\qquad m = 1, 2, 3, \dots$

$$B_{2m} = \frac{2}{2m\pi}\left\{1 - (-1)^{2m}\right\} = \frac{1}{m\pi}\left\{1 - 1\right\} = 0$$

for $n = 2m+1$ = odd $\qquad m = 0, 1, 2, \dots$

$$B_{2m+1} = \frac{2}{(2m+1)\pi}\left\{1 - (-1)^{2m+1}\right\} = \frac{4}{(2m+1)\pi}$$

Thus

$$u(x,t) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} \sin\left(\frac{(2m+1)\pi x}{L}\right) e^{-3(2m+1)^2\pi^2 t/L^2}$$