MATH2019 PROBLEM CLASS

EXAMPLES 4

DOUBLE INTEGRALS

$\boxed{1997}$ 1. Evaluate the following integral by changing to polar coordinates:

$$I = \int_{\sqrt{2}/2}^{1} \int_{0}^{\sqrt{1-x^2}} dy\, dx\,.$$

$\boxed{1998}$ 2. An annular washer of constant surface density δ occupies the region between the circles

$$x^2 + y^2 = a^2 \quad \text{and} \quad x^2 + y^2 = b^2 \quad \text{where} \quad b > a\,.$$

Find the moment of inertia of the washer about the x-axis.

$\boxed{2014, S1}$ 3. Consider the double integral

$$I = \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} 3x \; dx \; dy.$$

i) Sketch the region of integration.

ii) Evaluate I using polar coordinates.

$\boxed{2014, S1}$ 4. A thin triangular plate bounded by $y = 2x,\ y = 6$ and the y axis has non-uniform density given by $\rho(x, y) = 4xy$. Find the mass of the plate by evaluating an appropriate double integral in Cartesian coordinates.

$\boxed{2014, S2}$ 5. Consider the double integral

$$\int_{0}^{\frac{1}{\sqrt{2}}} \int_{x}^{\sqrt{1-x^2}} 3x \; dy \; dx.$$

i) Sketch the region of integration.

ii) Evaluate the double integral by first converting to polar coordinates.

$\boxed{2015, S1}$ 6. Consider the double integral

$$\int_{0}^{1} \int_{0}^{1-x^2} \frac{y}{\sqrt{1-y}} \; dy \; dx.$$

i) Sketch the region of integration.

ii) Evaluate the double integral by first reversing the order of integration.

$\boxed{2015, S2}$ 7. The area A of a region R of the xy-plane is given by

$$A = \int_{0}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_{1}^{2} \int_{-\sqrt{2-y}}^{\sqrt{2-y}} dx\, dy.$$

i) Sketch the region R.

ii) When the order of integration is reversed the expression for A becomes

$$A = \int_{-1}^{1} \int_{l_1(x)}^{l_2(x)} dy\, dx.$$

Find the limits $l_1(x)$ and $l_2(x)$.

iii) Hence, find the value of A.

2016, S1 8. Consider the double integral

$$\int_0^1 \int_x^{\sqrt{3}\,x} \frac{x}{x^2+y^2}\, dy\ dx.$$

i) Sketch the region of integration.

ii) Evaluate the double integral using polar coordinates.

2016, S1 9. Because of the effect of rotation, the Earth is not a perfect sphere but is slightly fatter at the equator than it is at the poles. A good approximation for the shape of the earth is an ellipsoid described by the formula

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1,$$

where z is the coordinate measured along the axis of rotation, $a = 6378$ km is the radius of the Earth at the equator and $b = 6357$ km is the radius of the Earth at the poles.

Calculate the volume of the Earth using an appropriate double integral.

2016, S2 10. A thin plate in the first quadrant is bounded by the circle $x^2 + y^2 = 1$ and the coordinate axes. The plate has uniform density $\delta(x, y) = 1$.

i) Sketch the plate in the $x - y$ plane.

ii) Without evaluating any integrals write down the mass of the plate.

iii) Find the coordinates of the centroid $(\bar{x}, \bar{y})$ of the plate by evaluating an appropriate double integral in polar coordinates.
(Note that by symmetry, $\bar{y} = \bar{x}$).

2016, S2 11. Consider the double integral

$$I = \int_0^2 \int_{x^2}^4 \frac{e^y}{\sqrt{y}}\, dy\, dx$$

i) Sketch the region of integration.

ii) Evaluate I by first reversing the order of integration.

2017, S1 12. Use double integration to find the area bounded by $y = x$ and $y = x^2$.

2017, S2 13. Consider the double integral

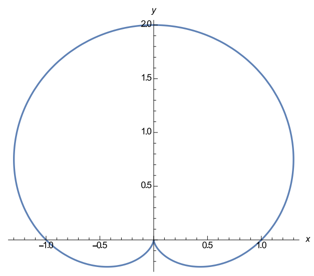
$$I = \int_0^2 \int_0^x \frac{x}{x^2+y^2}\, dy\ dx.$$

i) Sketch the region of integration.

ii) Evaluate I by first changing to polar coordinates.

2017, S2 14. Find the volume of the solid bounded above by the surface $z = 1 - x^2 - y^2$ and below by the plane $z = 0$.

2018, S1 15. Consider the polar curve $r = 1 + \sin\theta$ whose figure is given below.



Determine the area of the region enclosed by the curve by using a suitable double integral.

2018, S2 16. Consider the double integral

$$I = \int_0^4 \int_{\sqrt{x}}^2 10x \; dydx.$$

i) Sketch the region of integration.

ii) Evaluate I with the order of integration reversed.

2018, S2 17. Let Ω be the semi-circular region bounded by $y = \sqrt{1 - x^2}$ and $y = 0$. The region Ω is of uniform density and has centroid $(\bar{x}, \bar{y})$.

i) Sketch the region Ω and write down its area.

ii) Explain why $\bar{x} = 0$.

iii) Find $\bar{y}$ by evaluating an appropriate double integral expressed in polar coordinates.