LECTURE 34
LAPLACE TRANSFORMS

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$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^m	$m!/s^{m+1}$
$t^\nu \ , (\nu > -1)$	$\Gamma(\nu+1)/s^{\nu+1}$
e^{-at}	$1/(s+a)$
$\sin bt$	$b/(s^2+b^2)$
$\cos bt$	$s/(s^2+b^2)$
$\sinh bt$	$b/(s^2-b^2)$
$\cosh bt$	$s/(s^2-b^2)$
$\sin bt - bt\cos bt$	$2b^3/(s^2+b^2)^2$
$\sin bt + bt\cos bt$	$2bs^2/(s^2+b^2)^2$
$t\ \sin bt$	$2bs/(s^2+b^2)^2$
te^{-at}	$1/(s+a)^2$
$\mathrm{u}(t-c)$	e^{-cs}/s
$e^{-at}f(t)$	$F(s+a)$
$tf(t)$	$-F'(s)$
$f(t-c)\mathrm{u}(t-c)$	$e^{-cs}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f'''(t)$	$s^3F(s)-s^2f(0)-sf'(0)-f''(0)$
$\int_0^t f(\tau)d\tau$	$F(s)/s$

LAPLACE TRANSFORMS

The Laplace transform changes a function $f(t)$ in a highly specific and very useful fashion into a different function $F(s)$. By transforming entire problems (for example differential equations) the nature of the problem may be fundamentally altered opening up unexpected avenues of attack. The Laplace transform is particularly effective in situations involving discontinuity and hence is extremely useful in applications of electrical engineering where switching plays such a central role.

We will spend a few lectures discussing the intricacies of finding Laplace transforms and also of course how to return home by finding inverse Laplace transforms. We will then focus on some classical applications. The definition on the Laplace transform is

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

That is, we integrate the function f against a decaying exponential function all the way from 0 to ∞.

We denote the original function by $f(t)$ and the transformed object $F(s)$. To keep track of where we are, we use t as the original variable and the transformed variable is s. Just like differentiation and integration, taking Laplace transforms is a linear process so

$$\mathcal{L}\left\{af(t) + bg(t)\right\} = a\mathcal{L}\left\{f(t)\right\} + b\mathcal{L}\left\{g(t)\right\}.$$

How do we find Laplace transforms? Well there is always the integral definition, but in reality we leave that behind and in most cases both Laplace transforms and their inverses are simply found by looking up an appropriate entry in a table of Laplace transforms. You will of course have this table supplied to you in your final exam.

But first a little revision on integration by parts and improper integrals.

Example 1 Evaluate each of the following definite integrals:

a) $\displaystyle\int_0^1 3te^{-7t}dt.$

b) $\displaystyle\int_0^\infty 3te^{-7t}dt.$

One thing to remember is that $\displaystyle\int e^{ax}dx = \frac{1}{a}e^{ax}$ implying that $\displaystyle\int e^{-st}dt = -\frac{1}{s}e^{-st}$. Also the exponential function goes to zero much faster than any polynomial goes to infinity!

a) $\displaystyle\int_0^1 3te^{-7t}\,dt$

$$= \left[\frac{3te^{-7t}}{-7}\right]_0^1 - \int_0^1 \frac{3}{-7}e^{-7t}\,dt$$

$$= \frac{-3}{7e^7} + \frac{3}{7}\left[\frac{e^{-7t}}{-7}\right]_0^1$$

$$= -\frac{3}{7e^7} - \frac{3}{7^2}\left(e^{-7} - 1\right)$$

$$= \frac{3}{49} - \frac{24}{49}e^{-7}$$

b) $\displaystyle\int_0^\infty 3te^{-7t}\,dt$

$$= \left[\frac{3te^{-7t}}{-7}\right]_0^\infty - \int_0^\infty -\frac{3}{7}e^{-7t}\,dt$$

$$= 0 + \frac{3}{7}\left[\frac{e^{-7t}}{-7}\right]_0^\infty$$

$$= \frac{3}{49}$$

★ $a)\ \ \dfrac{3}{49} - \dfrac{24}{49}e^{-7} \quad b)\ \ \dfrac{3}{49}$ ★

Example 2 Find the Laplace transform of $4t$, firstly via the definition and then by using the table of Laplace transforms.

By definition:

$$\int_0^\infty e^{-st} \cdot 4t \, dt$$

$$= \left[-\frac{4te^{-st}}{s} \right]_0^\infty + \frac{4}{s} \int_0^\infty e^{-st} \, dt$$

$$= \frac{4}{s^2}$$

From table:

$$\mathcal{L}(4t) = \frac{4}{s^2}$$

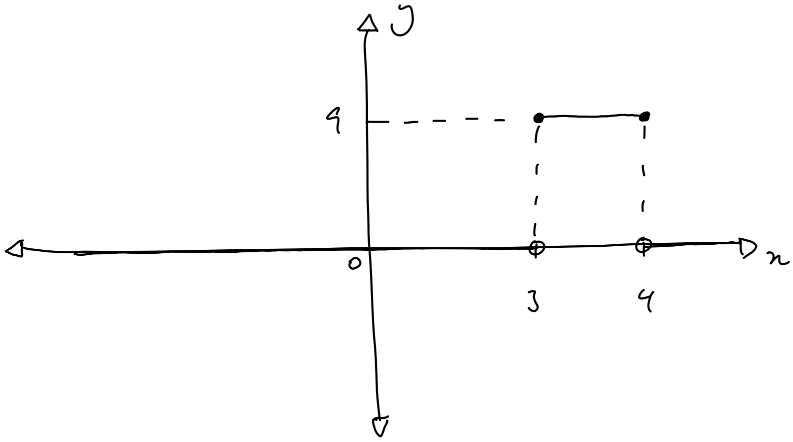
★ $\frac{4}{s^2}$ ★

You can see from the above example that the Laplace transform is an improper integral (since we are integrating all the way to infinity). What saves us is that e^{-st} converges to zero very aggressively and many of our standard functions (certainly all of the polynomials, sin and cos) are quickly defeated by e^{-st}. In general the Laplace transform of f will exist provided that f does not grow too rapidly as $t \to \infty$.

Observe also that the integral is with respect to t only, thus leaving the Laplace transform variable s as a residue. Notice also that we assume that $t \geq 0$. The variable t usually represents time so this quite natural. A nice thing about Laplace transforms (as opposed to derivatives for example) is that they do not get too worried when faced with discontinuities.

Example 3 Let $f(t) = \begin{cases} 9, & 3 \leq t \leq 4; \\ 0, & \text{otherwise.} \end{cases}$

Sketch the function and find its Laplace transform directly from the integral definition.



$$\mathcal{L}(f(t)) = \mathcal{L}\left(u(t-3) - u(t-4)\right) \times \mathcal{L}(9)$$

$$= \frac{9}{s}\left(e^{-3s} - e^{-4s}\right)$$

$$\bigstar \quad \frac{9(e^{-3s} - e^{-4s})}{s} \quad \bigstar$$

Example 4 Use the table to find the Laplace transform of

$$f(t) = 7t^2 + \cos(4t) - 5\sinh(3t) + 2$$

.

$$\mathcal{L}(f(t)) = \mathcal{L}\left(7t^2 + \cos(4t) - 5\sinh(3t) + 2\right)$$

$$= 7 \times \frac{2!}{s^3} + \frac{s}{s^2+4^2} - 5 \times \frac{3}{s^2-3^2} + \frac{2}{s}$$

$$= \frac{14}{s^3} + \frac{s}{s^2+16} - \frac{15}{s^2-9} + \frac{2}{s}$$

$$\bigstar \quad \frac{14}{s^3} + \frac{s}{s^2+16} - \frac{15}{s^2-9} + \frac{2}{s} \quad \bigstar$$

Example 5 Use the table to find the Laplace transform of

$$f(t) = 3e^{7t} - 6te^{-2t} + \sqrt{t}$$

.

$$\mathcal{L}(f(t)) = 3 \times \frac{1}{s-7} - 6 \times \frac{1}{(s+2)^2} + \frac{\Gamma(\frac{3}{2})}{s^{\frac{3}{2}}}$$

$$= \frac{3}{s-7} - \frac{6}{(s+2)^2} + \frac{\Gamma(\frac{3}{2})}{s^{\frac{3}{2}}}$$

$$\bigstar \quad \frac{3}{s-7} - \frac{6}{(s+2)^2} + \frac{\Gamma(\frac{3}{2})}{s^{\frac{3}{2}}} \quad \bigstar$$

Note that the gamma function Γ is an extension of the concept of factorials beyond the positive integers. Its definition is $\Gamma(\nu + 1) = \displaystyle\int_0^\infty e^{-x} x^\nu \, dx$ and it is usually evaluated by computer or tables. If ν is a positive integer then $\Gamma(\nu + 1) = \nu!$

Example 6 Prove that

$$\mathcal{L}\{te^{-t}\} = \frac{1}{(s+1)^2}\,.$$

Note that this result also appears in the table of Laplace transforms.

$$\mathcal{L}\{te^{-t}\} = \int_0^\infty e^{-st}\, t e^{-t}\, dt$$

$$= \int_0^\infty t e^{-(s+1)t}\, dt$$

$$= \left[\frac{t e^{-(s+1)t}}{-(s+1)}\right]_0^\infty + \frac{1}{s+1}\int_0^\infty e^{-(s+1)t}\, dt$$

$$= \frac{1}{s+1} \times -\left(-\frac{1}{s+1}\right)$$

$$= \frac{1}{(s+1)^2}$$

★

One final note is that the concept of an arbitrary constant C plays no role whatsoever in the theory of Laplace transforms!

[33]You can now do Q 95 and 96 a,b,c and d