MATH2019 LECTURE 16
CHANGING THE ORDER OF INTEGRATION AND AREAS

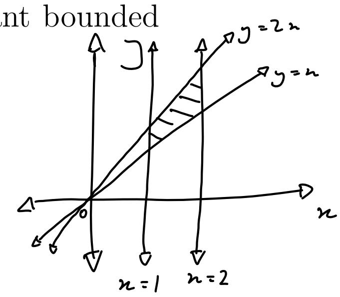
It is important to be able to convert $\iint_\Omega f(x, y)\; dxdy$ into $\iint_\Omega f(x, y)\; dydx$ and vice versa. This should always be done via the production and consideration of the region Ω over which the integration takes place.

$$\iint_\Omega 1\; dA = \text{ area}(\Omega).$$

First a revision example from the last lecture.

Example 1: Evaluate $\iint_\Omega \frac{x}{y}\; dA$ where Ω is the region in the first quadrant bounded by the four lines:

$$y = x \quad y = 2x \quad x = 1 \text{ and } x = 2.$$



$$\int_1^2\int_x^{2x} \frac{x}{y}\; dy\; dx = \int_1^2 \left[x \ln y\right]_x^{2x} dx$$

$$= \int_1^2 x\ln(2)\; dx$$

$$= \left[\frac{x^2\ln(2)}{2}\right]_1^2$$

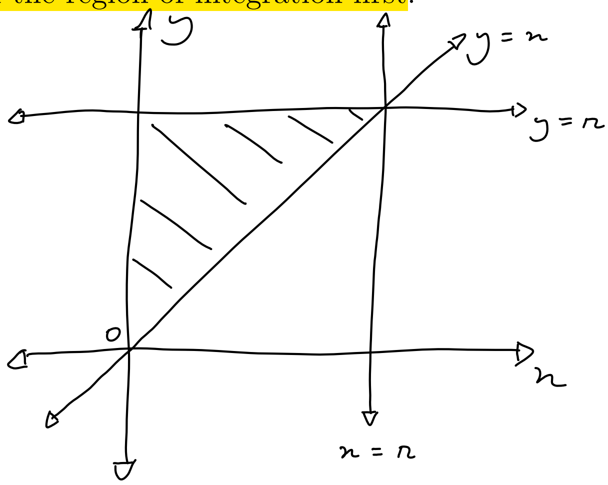
$$= \frac{3}{2}\ln(2)$$

Observe that using $dxdy$ would be a bit of a disaster here as we would no longer have clear curves of entry and exit. We will have a go at the $dxdy$ version at the start of the next lecture.

★ $\frac{3}{2}\ln(2)$ ★

Example 2: Evaluate $\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\, dydx$ by first changing the order of integration.

Note firstly that the integral is impossible to evaluate directly! When changing the order of integration it is absolutely essential to sketch the region of integration first.



$$\int_0^\pi \int_0^y \frac{\sin(y)}{y}\, dx\, dy$$

$$= \int_0^\pi \left[\frac{\sin(y)}{y}\, x\right]_0^y dy$$

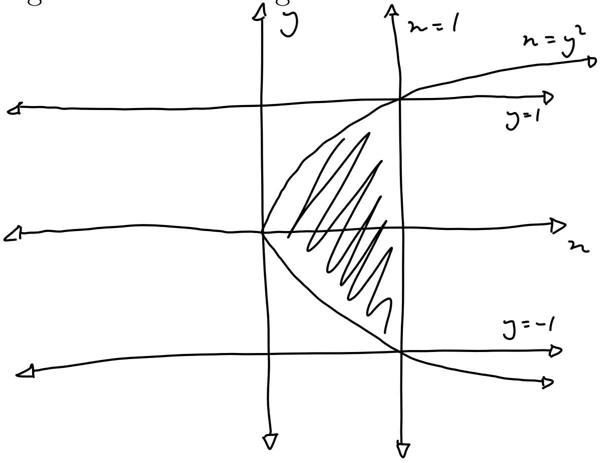
$$= \int_0^\pi \sin(y)\, dy$$

$$= \left[-\cos(y)\right]_0^\pi$$

$$= 2$$

★ 2 ★

Example 3: Evaluate $\int_{-1}^{1}\int_{y^2}^{1} 2\sqrt{x}e^{x^2}\, dxdy$ by first changing the order of integration.



$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} 2\sqrt{x}\, e^{x^2}\, dy\, dx$$

$$= \int_0^1 \left[2\sqrt{x}\, e^{x^2} y \right]_{-\sqrt{x}}^{\sqrt{x}} dx$$

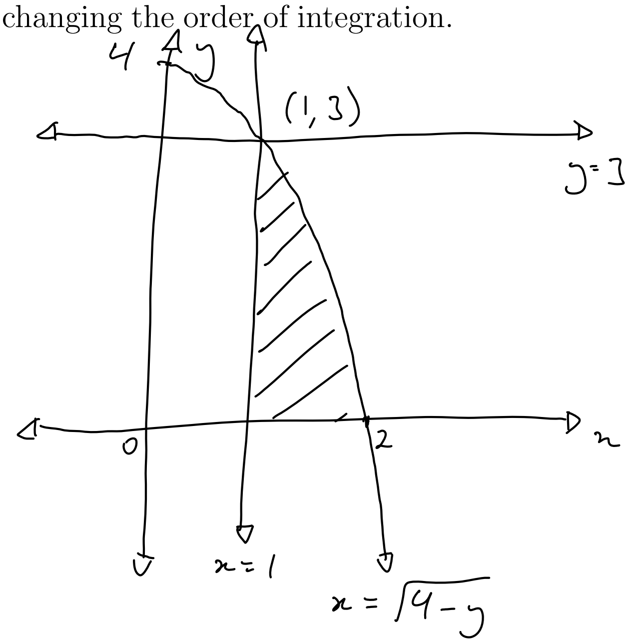
$$= \int_0^1 4x e^{x^2}\, dx$$

$$= 2\left[e^{x^2} \right]_0^1$$

$$= 2e - 2$$

★ $2e-2$ ★

Example 4: Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\ dxdy$ by first changing the order of integration.



$$\int_1^2 \int_0^{4-x^2} (x+y)\ dy\ dx$$

$$= \int_1^2 \left[xy + \frac{y^2}{2} \right]_0^{4-x^2} dx$$

$$= \int_1^2 4x - x^3 + \frac{16 - 8x^2 + x^4}{2}\ dx$$

$$= \int_1^2 \frac{x^4}{2} - x^3 - 4x^2 + 4x + 8\ dx$$

$$= \left[\frac{x^5}{10} - \frac{x^4}{4} - \frac{4x^3}{3} + 2x^2 + 8x \right]_1^2$$

$$= \frac{241}{60}$$

$$= \int_1^2 4x - x^3 + 8 - 4x^2 + \frac{1}{2}x^4\, dx = \left[2x^2 - \frac{1}{4}x^4 + 8x - \frac{4}{3}x^3 + \frac{1}{10}x^5 \right]_1^2$$

$$= (8 - 4 + 16 - \frac{32}{3} + \frac{32}{10}) - (2 - \frac{1}{4} + 8 - \frac{4}{3} + \frac{1}{10}) = \frac{188}{15} - \frac{511}{60} = \frac{241}{60}.$$

★ $\frac{241}{60}$ ★

Areas Via Double Integrals

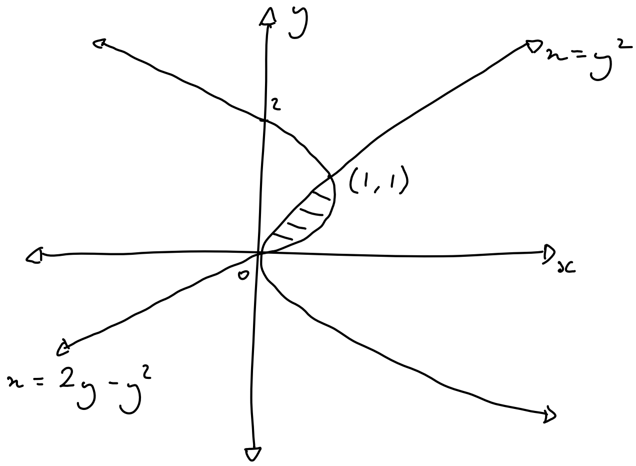
Although designed to evaluate volumes the double integral may be tricked into the evaluation of areas by simply replacing $f(x, y)$ with 1. That is

$$\iint_\Omega 1\; dA = \text{ area}(\Omega).$$

This works since $\iint_\Omega 1\; dA$ is the volume above Ω below the horizontal plane $z = 1$, which is in turn equal to the (area of Ω) $\times$ 1

Example 5: Use double integration to find the area bounded by $x = y^2$ and $x = 2y - y^2$.

$$\int_0^1 \int_{y^2}^{2y-y^2} 1 \quad dx\; dy$$



$$= \int_0^1 2y - 2y^2 \; dy$$

$$= \left[y^2 - \frac{2y^3}{3} \right]_0^1$$

$$= \frac{1}{3}$$

★ $\frac{1}{3}$ square unit ★

Observe that use of $dydx$ would be a disaster here!

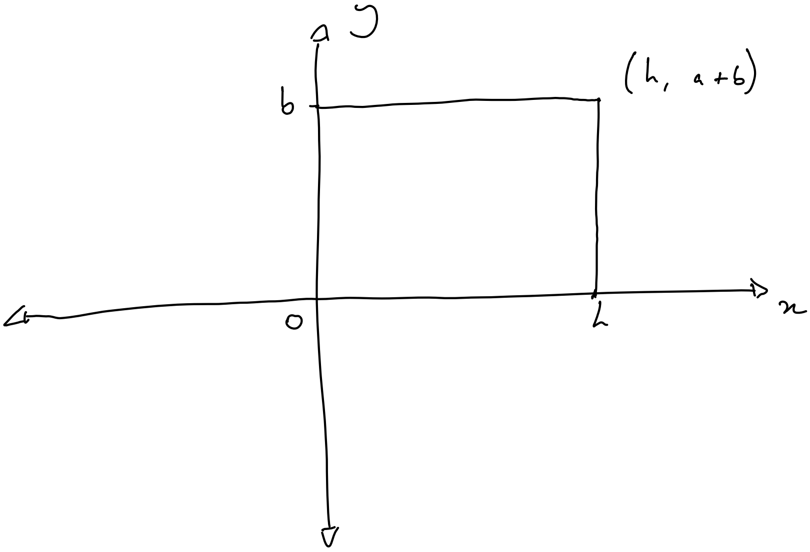
Example 6: Prove that standard formula for the area of a parallelogram

$$A = bh$$

using double integrals.

Let's construct a parallelogram with vertices at $(0,0), (0,b), (h,a)$ and $(h,a+b)$ where $a,b,h>0$.

Sketch:



The two interesting lines are $y = \frac{a}{h}x$ and $y = \frac{a}{h}x + b$. Thus

$$\text{Area} = A = \int_0^h \int_{\frac{a}{h}x}^{\frac{a}{h}x+b} 1 \, dy \, dx$$

$$= \int_0^h b \, dx$$

$$= bh$$

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[16] You can now do Q 65, 67, 70 and 71