LECTURE 47
FORCED OSCILLATIONS AND FOURIER SERIES

Suppose that a function f has period $T = 2L$. Then f may be approximated by the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) \tag{1}$$

where the Fourier coefficients a_0, a_n, and b_n are given by

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \dots) \\ b_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \dots) \end{aligned}\right\} \tag{2}$$

Let us now start using Fourier series to solve some special problems. Fourier series will prove an essential tool in solving partial differential equations for the remainder of the course. But today we will use Fourier series to help us solve ordinary differential equations

$$my'' + cy' + ky = f(t)$$

governing oscillating systems where the forcing function $f(t)$ is exotic and periodic. We have already covered this theory when $f(t)$ is a standard function. First some revision:

Example 1 Find the general solution of the differential equation

$$y'' + 9y = 60\sin(2t)$$

$$\bigstar \quad y = A\sin(3t) + B\cos(3t) + 12\sin(2t) \quad \bigstar$$

Note in the above that since the y' term is missing from the D.E. we can drop the cos term from the guess for y_p and make a simpler guess of $y_p = \alpha\sin(2t)$ instead.

We can solve any differential equation of the form $y'' + 9y = 60\sin(\omega t)$ in the same manner, noting that in the special case where $\omega = 3$ we will need to modify our guess for y_p to $y_p = \{\alpha\sin(3t) + \beta\cos(3t)\}\,t$ and the system will then suffer from resonance.

We now turn to a similar situation except that the forcing function is no longer sinusoidal but rather is simply a random periodic function. The method of undertermined coefficients will probably no longer work as the guess for y_p is no longer obvious. Laplace transforms are an option but we have seen that they can get messy. Our method of attack using Fourier series is simple.

We decompose the function $f(t)$ into an infinite sum of sines and or cosines using the theory of Half Range expansions. We then solve the differential equation with the RHS $f(t)$ replaced by the nth term of its Fourier series. The final solution is then expressed as a series of the individual smaller solutions. An issue of particular concern is which Fourier component of the solution is closest to resonance and hence provides the largest contribution to the particular solution.

This is best explained via an example. There will also be a few more in the problem class.

Example 2 Suppose that $f(x) = \begin{cases} 1-x & 0 < x < 2\ ; \\ f(x+2) & \text{otherwise.} \end{cases}$

and consider the differential equation

$$y'' + 484y = f(x).$$

(a) Show that the Fourier series of f is

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(n\pi x).$$

(b) Show that a particular solution to $y'' + 484y = \dfrac{2}{n\pi}\sin(n\pi x)$ is

$$y_n = \frac{2}{n\pi(484 - (n\pi)^2)} \sin(n\pi x).$$

(c) Hence show that the solution to $y'' + 484y = f(x)$ is

$$y = A\cos(22x) + B\sin(22x) + \sum_{n=1}^{\infty} \frac{2}{n\pi(484 - (n\pi)^2)} \sin(n\pi x)$$

(d) Explain why the seventh term in the above expansion will dominate the series solution for y_p. Find the sixth, seventh and eighth coefficients in the series for y_p.

(a) We **ALWAYS** start with a sketch and hope for symmetry.

★ $n = 7$ comes closest to resonance , $B_6 = .00082$, $B_7 = 0.23$, $B_8 = -0.00054$ ★

[47] You can now do Q 113