# LECTURE 20
# FURTHER VOLUMES

Recall that for a region $\Omega$ in the $x-y$ plane and a surface $z = f(x,y)$ in $\mathbb{R}^3$ the double integral

$$\iint_\Omega f(x,y)dy\,dx.$$

evaluates the volume of the solid above $\Omega$ and below $z = f(x,y)$.

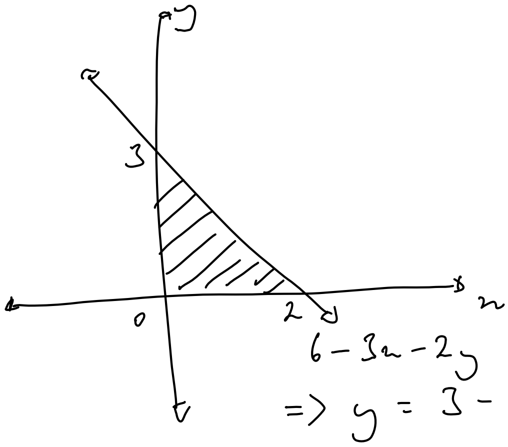
If the solid is pinned between two surfaces $z = f(x,y)$ and $z = g(x,y)$ with $f \geq g$ then its volume will be

$$\iint_\Omega [f(x,y) - g(x,y)]\;dy\,dx = \iint_\Omega [\text{top surface} - \text{bottom surface}]\;dydx.$$

where $\Omega$ is the projection of the solid back onto the $x-y$ plane.

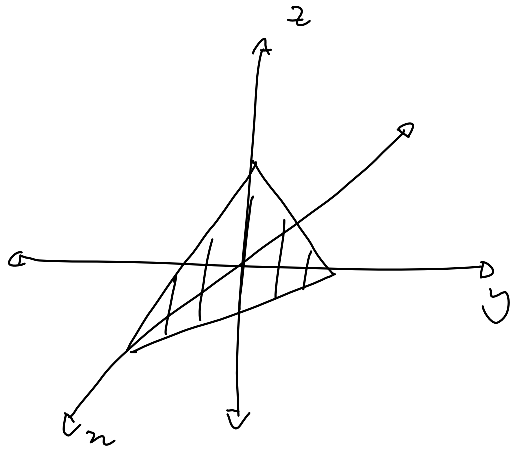
**Revision of the meaning of the double integral:**

**Example 1** Find the volume of the tetrahedron bounded by the plane $3x+2y+z=6$ and the coordinate planes.




$6-3x-2y$

$\Rightarrow y = 3 - \frac{3}{2}x$




$$= \iint_{\Omega} 6-3x-2y \; dA$$

$$= \int_0^2 \int_0^{3-\frac{3}{2}x} 6-3x-2y \; dy\, dx$$

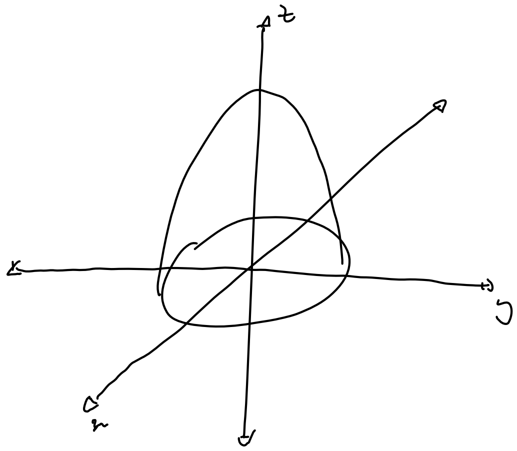
$$= \int_0^2 [6y-3xy-y^2]_0^{3-\frac{3}{2}x} dx = \int_0^2 6(3-\frac{3}{2}x)-3x(3-\frac{3}{2}x)-(3-\frac{3}{2}x)^2 dx$$

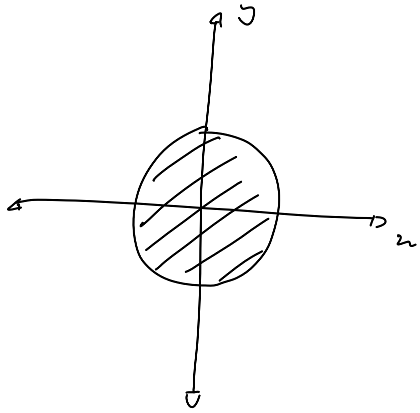
$$= \int_0^2 18-9x-9x+\frac{9}{2}x^2-(3-\frac{3}{2}x)^2 dx = \int_0^2 18-18x+\frac{9}{2}x^2-(9-9x+\frac{9}{4}x^2)dx$$

$$= \int_0^2 9-9x+\frac{9}{4}x^2 dx = \left[9x-\frac{9}{2}x^2+\frac{9}{12}x^3\right]_0^2 = (18-18+6)-(0) = 6.$$

★ 6 ★

**Example 2** Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.




$$V = \iint_{\Omega} 1 - x^2 - y^2 \, dA$$

$$= \int_0^{\pi} \int_0^1 (1 - r^2) \, r \, dr \, d\theta$$

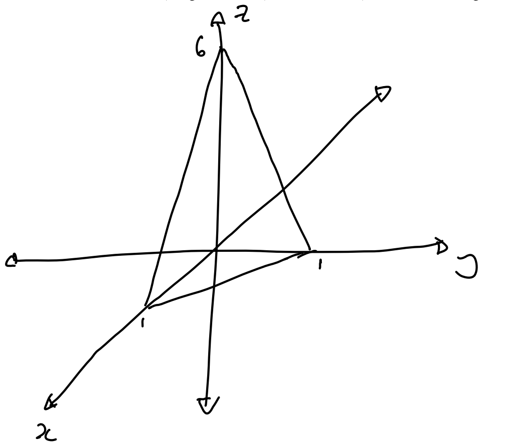
$$= \int_0^{\pi} \left[ \frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 d\theta$$

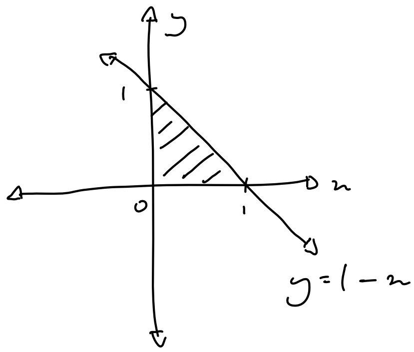
$$= \int_0^{\pi} \frac{1}{2} \, d\theta$$

$$= \frac{\pi}{2}$$

★ $\frac{\pi}{2}$ ★

**Example 3** Find the volume of the solid containing the origin, bounded by the 5 planes $x = 0,\ y = 0,\ z = 0,\ 3x + 2y + z = 6$ and $x + y = 1$ .




$$V = \iint_{\Omega} 6 - 3x - 2y \; dA$$

$$= \int_0^1 \int_0^{1-x} 6 - 3x - 2y \; dy \, dx$$

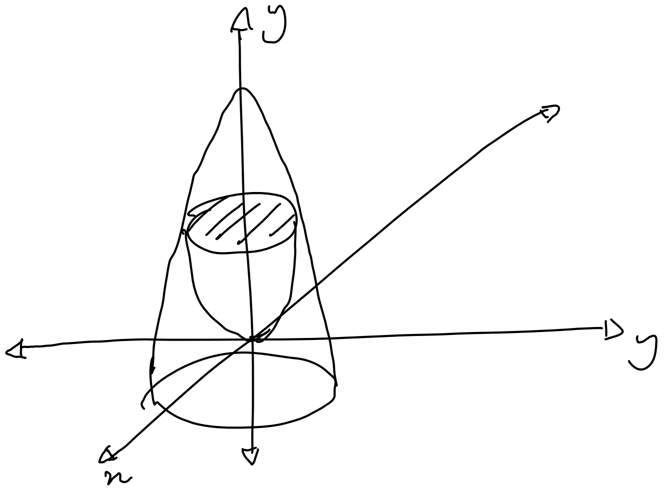
$$= \int_0^1 \Big[ 6y - 3xy - y^2 \Big]_0^{1-x} \, dx$$

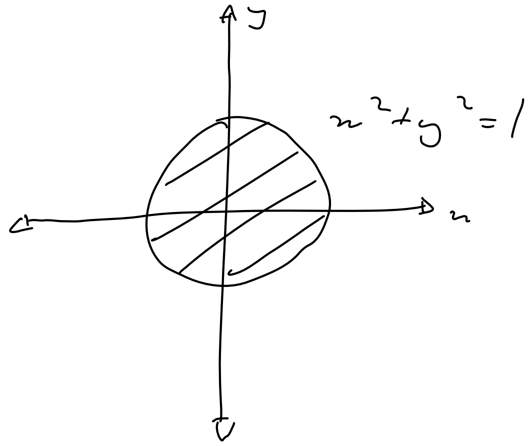
$$\int_0^1 6(1-x) - 3x(1-x) - (1-x)^2 \, dx = \int_0^1 6 - 6x - 3x + 3x^2 - (1 - 2x + x^2) \, dx$$

$$= \int_0^1 5 - 7x + 2x^2 \, dx = \left[ 5x - \frac{7}{2}x^2 + \frac{2}{3}x^3 \right]_0^1 = 5 - \frac{7}{2} + \frac{2}{3} = \frac{13}{6}.$$

$\bigstar \quad \frac{13}{6} \quad \bigstar$

**Example 4** Find the volume of the solid lying between the paraboloids $z = x^2 + y^2$ and $3z = 4 - x^2 - y^2$.




$$V = \iint_\Omega \frac{4}{3} - \frac{1}{3}(x^2+y^2) - (x^2+y^2)\, dA$$

$$= \int_0^{\pi}\int_0^1 \left(\frac{4}{3} - \frac{4}{3}r^2\right) r\, dr\, d\theta$$

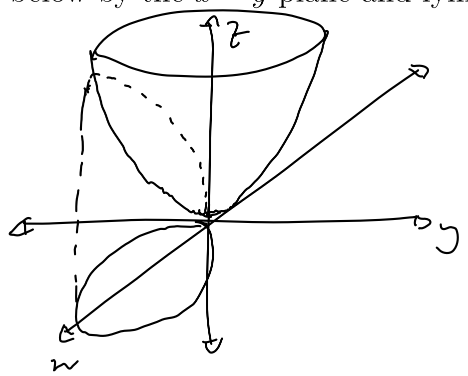
$$= \int_0^{\pi} \left[\frac{2}{3}r^2 - \frac{1}{3}r^4\right]_0^1 d\theta$$

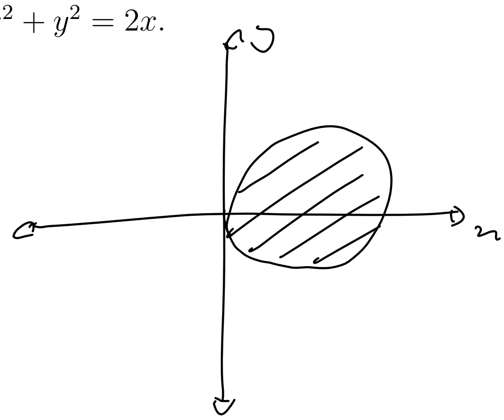
$$= \int_0^{\pi} \frac{1}{3}\, d\theta$$

$$= \frac{\pi}{3}$$

★ $\frac{2\pi}{3}$ ★

**Example 5** Find the volume of the solid bounded above by the paraboloid $z = x^2 + y^2$, below by the $x - y$ plane and lying inside the cylinder $x^2 + y^2 = 2x$.




$$V = \iint_{\Omega} x^2 + y^2 \, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 \, r \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{4} r^4\right]_0^{2\cos\theta} d\theta$$

$$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(\theta) d\theta.$$

Noting that cos is an even function and that $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ we have:

$$= 8\int_0^{\frac{\pi}{2}} \cos^4(\theta) d\theta = 8\int_0^{\frac{\pi}{2}} \cos^2(\theta)\cos^2(\theta) d\theta = 8\int_0^{\frac{\pi}{2}} \{\frac{1}{2}(1 + \cos(2\theta))\}^2 d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (1 + \cos(2\theta))^2 d\theta = 2\int_0^{\frac{\pi}{2}} 1 + 2\cos(2\theta) + \cos^2(2\theta) d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 1 + 2\cos(2\theta) + \frac{1}{2}(1 + \cos(4\theta)) d\theta = 2\int_0^{\frac{\pi}{2}} \frac{3}{2} + 2\cos(2\theta) + \frac{1}{2}\cos(4\theta) d\theta$$

$$= 2\left[\tfrac{3}{2}\theta + \sin(2\theta) + \tfrac{1}{8}\sin(4\theta)\right]_0^{\frac{\pi}{2}} = 2\left[\frac{3}{2}\frac{\pi}{2}\right] = \frac{3\pi}{2}.$$

★ $\frac{3\pi}{2}$ ★

**HOMEWORK:** Attempt the polar integrals in this lecture using Cartesian coordinates instead. Some may be almost impossible using $dydx$.

**IMPORTANT NOTE:** In 5 lectures time, we will start on the theory of eigenvalues and eigenvectors. This will need a firm understanding of the theory of determinants and also Gaussian elimination of systems of linear equations. This material is covered in Math1131 linear algebra here at UNSW. If you are a rusty UNSW student or an overseas student who has not had seen this content before please read the revision first year algebra notes (available on Moodle) before Lecture 28.

[20] You can now do Q 69,77,78,79