LECTURE 29
EIGENVALUES AND EIGENVECTORS

Given a square matrix A, a non-zero vector $\mathbf{v}$ is said to be an eigenvector of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$. The number λ is referred to as the associated eigenvalue of A.

We first find eigenvalues through the characteristic equation $\det(A - \lambda I) = 0$. The eigenvectors are then found via row reduction and back substitution.

The zero vector is **never** an eigenvector but it is OK to have a zero eigenvalue.

If an $n \times n$ matrix A has n linearly independent eigenvectors and P is the matrix of eigenvectors aligned vertically then $P^{-1}AP = D$ where D is the diagonal matrix of eigenvalues. The order of the eigenvalues in D must match the order of the eigenvectors in P. This is referred to as the diagonalization of A.

A matrix can be non-diagonalisable by coming up short on eigenvectors. The only general way to find out if a matrix has a full set of eigenvectors is to find them all.

A useful check is the fact that $\Sigma(\text{eigenvalues}) = \text{Trace}(A)$.

Eigenvectors from different eigenvalues are linearly independent.

Eigenvectors from different eigenvalues for symmetric matrices are perpendicular.

Establishing the eigenanalysis of a particular matrix gives you a clear vision of the internal workings of that matrix, and through diagonalisation the matrix may be transformed into a more workable diagonal structure.

Consider the matrix $A = \begin{pmatrix} 1 & 4 \\ -3 & 9 \end{pmatrix}$ and lets have a look at what A does to a random vector:

$\begin{pmatrix} 1 & 4 \\ -3 & 9 \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}$......it's nothing special!

But now consider $\begin{pmatrix} 1 & 4 \\ -3 & 9 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 21 \end{pmatrix} = 7 \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Observe that A simply makes this vector 7 times as long! We say that $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector of A with associated eigenvalue $\lambda = 7$.

How do we find all the eigenvectors and eigenvalues of a matrix A? Well

$A\mathbf{v} = \lambda\mathbf{v} \to A\mathbf{v} = \lambda I\mathbf{v} \to A\mathbf{v} - \lambda I\mathbf{v} = 0 \to (A - \lambda I)\mathbf{v} = \mathbf{0}$.

Now $\mathbf{v} = 0$ is the trivial solution to the above matrix equation and we are seeking non-trivial solutions. Thus the matrix $A - \lambda I$ must be non-invertible and hence we demand that

$$\det(A - \lambda I) = 0.$$

This is called the characteristic equation and generates the eigenvalues. 2×2 matrices have a quadratic characteristic equation and 3×3 matrices will have a cubic characteristic equation. Once you have the eigenvalues you can then find the eigenvectors by solving $(A - \lambda I)\mathbf{v} = \mathbf{0}$ using row reduction.

Example 1 Find all the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 4 \\ -3 & 9 \end{pmatrix}$ and hence find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$\det(A) = \begin{pmatrix} 1-\lambda & 4 \\ -3 & 9-\lambda \end{pmatrix} = (1-\lambda)(9-\lambda) + 12 = 0$$

$$\lambda^2 - 10\lambda + 21 = 0$$

$$(\lambda - 7)(\lambda - 3) = 0$$

$\therefore$ eigvals are $\lambda = 7, 3$

For $\lambda = 7$: $\begin{pmatrix} -6 & 4 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{0}$

by inspection: $v_1 = 2$, $v_2 = 3$

$\therefore$ eigvec is $\vec{v} = t\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $t \in \mathbb{R}$ for eigval $\lambda = 7$

For $\lambda = 3$: $\begin{pmatrix} -2 & 4 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{0}$

by inspection: $v_1 = 2$, $v_2 = 1$

$\therefore$ eigvec is $\vec{v} = t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$ for eigval $\lambda = 3$

$$A = \begin{pmatrix} 1 & 4 \\ -3 & 9 \end{pmatrix}$$

For $P^{-1}AP = D$: $P = \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}$ $\cap$ $D = \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix}$

$\bigstar \quad P = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}, \ D = \begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix} \quad \bigstar$

Example 2 Find all the eigenvalues and eigenvectors of $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$ and hence diagonalise A.

We start with the characteristic polynomial $\det(A - \lambda I) = 0$. If at all possible we will try to avoid the situation where we actually produce a cubic polynomial equation as these are difficult to solve.

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} 1-\lambda & -6 \\ -2 & -\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & -6 \\ -1 & -\lambda \end{vmatrix} - 3\begin{vmatrix} 2 & 1-\lambda \\ -1 & -2 \end{vmatrix}$$

$$= (-2-\lambda)\{-\lambda(1-\lambda) - 12\} - 2\{-2\lambda - 6\} - 3\{-4 + 1 - \lambda\}$$

$$= (-2-\lambda)\{\lambda^2 - \lambda - 12\} - 2\{-2\lambda - 6\} - 3\{-3 - \lambda\}$$

$$= (-2-\lambda)\{\lambda^2 - \lambda - 12\} + 4\lambda + 12 + 9 + 3\lambda$$

$$= (-2-\lambda)(\lambda - 4)(\lambda + 3) + 7\lambda + 21$$

$$= (-2-\lambda)(\lambda - 4)(\lambda + 3) + 7(\lambda + 3)$$

$$= (\lambda + 3)\{(-2-\lambda)(\lambda - 4) + 7\}$$

$$= (\lambda + 3)\{-\lambda^2 + 2\lambda + 15\}$$

$$= -(\lambda + 3)\{\lambda^2 - 2\lambda - 15\}$$

$$= -(\lambda + 3)(\lambda + 3)(\lambda - 5) = 0.$$

Thus $\lambda = -3, \ -3, \ 5$.

As a check $\quad -3 + -3 + 5 = -2 + 1 + 0.$

Note that the fact that $\lambda = -3$ has doubled up is certainly troubling but it does not imply that we necessarily will be short an eigenvector. Let's now find the eigenvectors, first for $\lambda = -3$:

For $\lambda = -3$: $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{0}$

$\therefore \; v_1 + 2v_2 - 3v_3 = 0$

By inspection: $v_1 = 1, \quad v_2 = 1, \quad v_3 = 1$

$v_1 = 3, \quad v_2 = 0, \quad v_3 = 1$

$\therefore$ eigvecs are $\vec{v} = t_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; t_1 \in \mathbb{R}$, $\vec{v} = t_2 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \; t_2 \in \mathbb{R}$

for eigval $\lambda = -3$.

For $\lambda = 5$: $\begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{0}$

$-7v_1 + 2v_2 - 3v_3 = 0, \quad v_1 - 2v_2 - 3v_3 = 0, \quad -v_1 - 2v_2 - 5v_3 = 0$

$$-7v_1 + (v_1 - 3v_3) - 3v_3 = 0$$
$$-6v_1 - 6v_3 = 0$$
$$\therefore v_1 = -v_3$$

$$(-2v_2 - 5v_3) - 2v_2 - 3v_3 = 0$$
$$4v_2 + 8v_3 = 0$$
$$\therefore v_2 = -2v_3$$

let $v_1 = 1 \quad \therefore \quad v_3 = -1, \quad v_2 = 2$

$\therefore$ eigvec is $\vec{v} = t\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}$

$$\therefore D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

$$\bigstar \quad P = \begin{pmatrix} -1 & -2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad \bigstar$$

What is happening with the process of diagonalisation?

When we think of $\mathbb{R}^3$ we like to use $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ as a basis. But these vectors mean nothing to A. If you were to ask A what would *it* prefer as a basis it would respond by saying "I'll have my eigenvectors thanks". A likes its eigenvectors since the action of A upon the eigenvectors is simply contraction and elongation. If we are prepared to abandon $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and instead make A happy by using the coordinate system generated by its eigenvectors $\left\{\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}\right\}$ then A transforms into the trivial matrix D. That is P transforms the complicated A into the very simple diagonal D via $P^{-1}AP = D$!

Proof of Diagonalisation formula

Let's prove the above claims in the 3×3 case. The proof in other dimensions is similar.

Suppose that A is a 3×3 matrix with a full set of linearly independent eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and associated eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$.

Let P be the matrix of eigenvectors $P = (\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3)$.

Then

$$AP = A\,(\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3) = (A\mathbf{v}_1|A\mathbf{v}_2|A\mathbf{v}_3) = (\lambda_1\mathbf{v}_1|\lambda_2\mathbf{v}_2|\lambda_3\mathbf{v}_3) = (\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3)\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$= PD$$

Thus $AP = PD \longrightarrow P^{-1}AP = D$ as required.

[29] You can now do Q 90