LECTURE 23
APPLICATIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS

When making substitutions into a D.E. remember to take care of $\frac{dy}{dx}$ as well as y

SEPARABLE

These are of the form

$$\frac{dy}{dx} = f(x)g(y).$$

and are solved by separating and integrating both sides of the equation.

LINEAR

The general first-order linear o.d.e. is

$$\frac{dy}{dx} + P(x)y = Q(x).$$

If we evaluate the integrating factor

$$R(x) = e^{\int P(x)dx}$$

then the solution is given by

$$y = \frac{1}{R(x)} \int R(x)Q(x)\, dx$$

We begin with a revision example on first order linear.

Example 1 Solve

$$\frac{dy}{dx} = \frac{xy+2}{1-x^2} \qquad \text{with} \qquad y(0) = 1.$$

.

$$\frac{dy}{dx} - \frac{xy}{1-x^2} = \frac{2}{1-x^2}$$

$$e^{\int -\frac{x}{1-x^2}dx} = e^{\frac{\ln(1-x^2)}{2}} = \sqrt{1-x^2}$$

$$y = \frac{1}{\sqrt{1-x^2}} \int \sqrt{1-x^2} \; \frac{2}{1-x^2} \, dx$$

$$y(0) = \frac{2\sin^{-1}(0)}{\sqrt{1-0^2}} + C = 1$$

$$\therefore y = \frac{1 + 2\sin^{-1}(x)}{\sqrt{1-x^2}}$$

$$\bigstar \quad y = \frac{1 + 2\sin^{-1}(x)}{\sqrt{1-x^2}} \quad \bigstar$$

As with integration it is possible to clarify a D.E. by implementing a simple substitution.

Example 2 Solve

$$x^2 \frac{dy}{dx} = x^2 + y^2 + xy$$

by making the substitution $v = \dfrac{y}{x}$.

$$\frac{dv}{dx} = -\frac{y}{x^2}$$

!!Observe that this D.E. is neither separable nor linear!!

$$\frac{dy}{dx} = 1 + \frac{y^2}{x^2} + \frac{y}{x}$$

$$\bigstar \quad y = x\tan(\ln(x) + C) \quad \bigstar$$

Example 3 Solve

$$(x+y)\frac{dy}{dx} = e^{3x} - x - y \qquad y(0) = 2,$$

by making the substitution $v = y + x$. $\frac{dv}{dx} = \frac{dy}{dx} + 1$

$$v\left(\frac{dv}{dx} - 1\right) = e^{3x} - v$$

$$v\frac{dv}{dx} = e^{3x}$$

$$\frac{v^2}{2} = \frac{e^{3x}}{3} + C_0$$

$$\therefore (y+x)^2 = \frac{2}{3}e^{3x} + C_1$$

$$(2+0)^2 = \frac{2}{3}e^{3\times 0} + C_1 \Rightarrow C_1 = 10$$

$$\therefore 3(y+x)^2 = 2e^{3x} + 10$$

★ $3(y+x)^2 = 2e^{3x} + 10$ ★

Sometimes you are not actually given a D.E. but rather are told various details about a situation and it is left up to you to construct the D.E. yourself. Keep in mind that D.E.'s are all about rates of change and thus your first task is to establish equations which govern the rates of change. The rate of change of any quantity is always equal the rate at which it increases minus the rate at which it decreases. Thus for example the rate of change of a population will be the rate of increase (births, immigration etc) minus the rate at which the population decreases (deaths etc). Also there is usually some information given as to initial conditions. Once the D.E. is formed there is still of course the problem of solving it to obtain formulae for the quantities of interest.

Example 4 The air in a 50 cubic metre room is initially clean. Chris lights up a cigarette introducing smoke into the room's air at a rate of 2 mg/minute. An air conditioning system exchanges the mixture of air and smoke with clean air at a rate of 5 cubic metres per minute. Assume that the smoke is mixed uniformly throughout the room and that Chris's cigarette lasts 4 minutes. Let S be the amount of smoke (in mg) present in the room at time t (in minutes).

a) Show that $\dfrac{dS}{dt} = 2 - \dfrac{S}{10}$

b) Hence show that $S(t) = 20 - 20e^{(-\frac{t}{10})}$

c) What is the level of smoke after 4 minutes?

d) What is the level of smoke after 14 minutes?

a) $\dfrac{dS}{dt} = \text{smoke}_{in} - \text{smoke}_{out}$

$= 2 - \dfrac{S}{50} \cdot 5$

$\therefore \dfrac{dS}{dt} = 2 - \dfrac{S}{10}$

b) $\dfrac{1}{20-S}\, dS = \dfrac{1}{10}\, dt$

$-\ln(20-S) = \dfrac{t}{10} + C$

at $t=0$, $S=0$ $\therefore C = -\ln(20)$

$\dfrac{20}{20-S} = e^{\frac{t}{10}}$

$\therefore S = 20 - 20e^{-\frac{t}{10}}$

c) $S(4) = 20 - 20e^{-\frac{4}{10}}$

$= 6.59$ mg

d) cig. lasts only 4 minutes

$\therefore \dfrac{dS}{dt} = -\dfrac{S}{10}$

$\ln(S) = -\dfrac{t}{10} + C$

at $t=4$, $S = 6.59$ $\therefore C = \ln(6.59)$

$\therefore S = 6.59e^{-\frac{t}{10}}$

$S(14-4) = 6.59e^{-\frac{10}{10}}$

$= 2.42$ mg

★ c) 6.6mg d) 2.43mg ★

[23] You can now do Q 81 and 82