LECTURE 6
EXTREME VALUES

Second Derivative Test

If f and all its first and second partial derivatives are continuous in the neighbourhood of (a, b) and $f_x(a, b) = f_y(a, b) = 0$ then

(i) f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $\mathcal{D} = f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

(ii) f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $\mathcal{D} = f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

(iii) f has a **saddle point** at (a, b) if $\mathcal{D} = f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).

(iv) If $\mathcal{D} = f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b) the second derivative test is **inconclusive**.

One of the most important applications of calculus of a single variable is the calculation and identification of local maxima and minima. With some minor modifications this theory extends quite naturally into higher dimensions.

Geometrical description of extrema for $z = f(x, y)$

Local maxima and minima for $z = f(x, y)$ are still calculated by setting the first derivative equal to zero. The trouble is that we now have more than one first derivative!! So we have to set both $\dfrac{\partial z}{\partial x} = 0$ and $\dfrac{\partial z}{\partial y} = 0$. This will generate a basket of points which then need to be classified using the tests above. Note that a saddle point is just a 3-D version of a point of inflection. We rarely look specifically for saddle points and they are usually just a nuisance as we search for the crucial extrema. The classification of extrema in higher dimensions is algebraically complicated and you need to take care not to lose solutions along the way. In particular remember to **never divide both sides of an equation by anything that could be zero!**

Example 1 Find and categorise the critical points of the following function:

$$f(x, y) = -x^2 - y^2 - 6x + 4y + 5$$

$\dfrac{\partial z}{\partial x} = -2x - 6 = 0 \qquad \therefore \quad x = -3$

$\dfrac{\partial z}{\partial y} = -2y + 4 = 0 \qquad \therefore \quad y = 2$

$\therefore \ (-3, 2, 18)$

$\dfrac{\partial^2 z}{\partial x^2} = -2, \quad \dfrac{\partial^2 z}{\partial y^2} = -2, \quad \dfrac{\partial^2 z}{\partial x \partial y} = 0$

$D = (-2)(-2) - 0^2 = 4 > 0$

$\therefore$ max or min

Since $\dfrac{\partial^2 z}{\partial x^2} < 0$, $\exists$ local max at $(-3, 2, 18)$

★ Local maximum at $(-3, 2, 18)$ ★

Example 2 Find and classify the critical points of the following function:

$$f(x,y) = 4xy - x^2y - xy^2$$

$$f_x = 4y - 2xy - y^2 = 0 \Rightarrow y(4-2x-y) = 0$$

$$f_y = 4x - x^2 - 2xy = 0 \Rightarrow x(4-x-2y) = 0$$

(I) $x=0,\ y=0 \Rightarrow (0,0)$

(II) $x=0,\ 4-2x-y=0 \Rightarrow (0,4)$

(III) $y=0,\ 4-x-2y=0 \Rightarrow (4,0)$

(IV) $4-2x-y=0,\ 4-x-2y=0 \Rightarrow \left(\frac{4}{3}, \frac{4}{3}\right)$

$$f_{xx} = -2y, \quad f_{yy} = -2x, \quad f_{xy} = 4-2x-2y$$

$$\begin{aligned} D &= f_{xx}f_{yy} - (f_{xy})^2 \\ &= (-2y)(-2x) - (4-2x-2y)^2 \\ &= 4xy - (4-2x-2y)^2 \end{aligned}$$

$D(0,0) = -16 < 0$ $\therefore$ saddle point

$D(0,4) = -16 < 0$ $\therefore$ saddle point

$D(4,0) = -16 < 0$ $\therefore$ saddle point

$D\left(\frac{4}{3}, \frac{4}{3}\right) = \frac{16}{3} > 0$ $\therefore$ local max or min

Since $f_{xx} = -2 \times \frac{4}{3} = -\frac{16}{3} < 0$ $\exists$ local max at $\left(\frac{4}{3}, \frac{4}{3}, \frac{64}{27}\right)$

★ Saddle points at $(0,0,0),(0,4,0),(4,0,0)$, local max at $(\frac{4}{3},\frac{4}{3},\frac{64}{27})$ ★

Example 3 Ben is an Olympic athlete and a drug cheat. He is soon to run in the 100m sprint final and is taking a mixture of steroids and beta-blockers in order to enhance his performance. He has found that his time T (in seconds) over 100 metres is given by

$$T(s, b) = 0.005s^2 + 0.005b^2 - 0.03s - 0.05b + 10.04$$

where s and b are the number of milligrams of steroids and beta-blockers (respectively) he injects daily.

1. What would his time be if he takes no drugs?
2. What would his time be if he takes 2mg of steroids and 4mg of beta-blockers?
3. How much of each drug should he take in order to minimise his time and what is this minimum time?

$T(0,0) = 10.04\,s$, $T(2,4) = 9.88\,s$

$\frac{\partial T}{\partial s} = 0.01s - 0.03 = 0 \quad \therefore s = 3\,mg$

$\frac{\partial T}{\partial b} = 0.01b - 0.05 = 0 \quad \therefore b = 5\,mg$

$D = (0.01)(0.01) - (0)^2 > 0 \quad \therefore \exists$ local max or min

$T(3,5) = 9.87\,s$

★ 10.04 sec, 9.88 sec, 9.87 sec with 3mg of steroids and 5mg of beta-blockers ★

[6] You can now do Q 36 to 38