MATH2019 PROBLEM CLASS

EXAMPLES 1

PARTIAL DIFFERENTIATION, MULTIVARIABLE TAYLOR SERIES AND LEIBNIZ' RULE

1. Given $f(x,y) = e^{-x^2+y^2}$ and $x = r\cos\theta$, $y = r\sin\theta$. Calculate $\dfrac{\partial f}{\partial \theta}$ and evaluate $\dfrac{\partial f}{\partial \theta}$ when $x = 1$, $y = 0$.

   Solution: Note $x = r\cos\theta$ and $y = r\sin\theta$ is the representation of the cartesian coordinates x, y in terms of the polar coordinates r, θ. Use a chain rule to calculate $\dfrac{\partial f}{\partial \theta}$:

$$\begin{aligned}
\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \\
&= -2xe^{-x^2+y^2}\underbrace{(-r\sin\theta)}_{-y=-r\sin\theta} + 2ye^{-x^2+y^2}\underbrace{(r\cos\theta)}_{x=r\cos\theta} \\
&= -2xe^{-x^2+y^2}(-y) + 2ye^{-x^2+y^2}(x) \\
&= 4xye^{-x^2+y^2}
\end{aligned}$$

   Thus at $x = 1$ and $y = 0$, we have $\dfrac{\partial f}{\partial \theta} = 4(1)(0)e^{-(1^2)+0^2} = 0\,.$

   Note we decided to write $\dfrac{\partial f}{\partial \theta}$ in terms of x and y since we were given values for x and y. In general you should write a derivative in terms of one set of variables or the other - not a mixture. Let the left hand side (L.H.S.) derivative guide you.

$\boxed{1998}$ 2. For what values of n does

$$f(x,y,z) = \sin(3x)\cos(4y)e^{-nz}$$

   satisfy the Lapace equation $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$?

   Solution: First we calculate calculate the second order partial derivatives to use in the Laplace equation:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 3\cos(3x)\cos(4y)e^{-nz}, & \frac{\partial^2 f}{\partial x^2} &= -9\sin(3x)\cos(4y)e^{-nz} = -9f \\
\frac{\partial f}{\partial y} &= -4\sin(3x)\sin(4y)e^{-nz}, & \frac{\partial^2 f}{\partial y^2} &= -16\sin(3x)\cos(4y)e^{-nz} = -16f \\
\frac{\partial f}{\partial z} &= -n\sin(3x)\cos(4y)e^{-nz}, & \frac{\partial^2 f}{\partial z^2} &= n^2\sin(3x)\cos(4y)e^{-nz} = n^2 f
\end{aligned}$$

   For f to satisfy the Laplace equation we must have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = (-9 - 16 + n^2)f(x,y,z) = 0 \quad \forall\ (x,y,z) \in \mathbb{R}^3.$$

   Hence $n^2 = 25$, i.e., $n = \pm 5$.

$\boxed{1997}$ 3. Let f and g be twice-differentiable functions of a single variable. Show by direct substitution into the partial differential equation that

$$w(x,t) = f(x+t) + g(x-t)$$

is a solution of the wave equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2}\,.$$

Solution: A useful strategy is to introduce new variables. In this case

$$u \equiv u(x,t) = x+t, \qquad v \equiv v(x,t) = x-t\,.$$

Thus $w(x,t) = f\,(u(x,t)) + g\,(v(x,t)) = f(u) + g(v)$. Use a chain rule to calculate the first order partial derivatives and then second order partial derivatives to substitute into the wave equation. Hence

$$\begin{aligned}\frac{\partial w}{\partial x} \equiv \frac{\partial}{\partial x}(w) = \frac{\partial}{\partial u}(w)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}(w)\frac{\partial v}{\partial x} &= \frac{\partial}{\partial u}\left(f(u)+g(v)\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(f(u)+g(v)\right)\frac{\partial v}{\partial x} \\ &= \frac{df}{du}\frac{\partial u}{\partial x} + \frac{dg}{dv}\frac{\partial v}{\partial x} = \frac{df}{du}(1) + \frac{dg}{dv}(1) = \frac{df}{du} + \frac{dg}{dv}\,.\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial w}{\partial t} \equiv \frac{\partial}{\partial t}(w) = \frac{\partial}{\partial u}(w)\frac{\partial u}{\partial t} + \frac{\partial}{\partial v}(w)\frac{\partial v}{\partial t} &= \frac{\partial}{\partial u}\left(f(u)+g(v)\right)\frac{\partial u}{\partial t} + \frac{\partial}{\partial v}\left(f(u)+g(v)\right)\frac{\partial v}{\partial t} \\ &= \frac{df}{du}\frac{\partial u}{\partial t} + \frac{dg}{dv}\frac{\partial v}{\partial t} = \frac{df}{du}(1) + \frac{dg}{dv}(-1) = \frac{df}{du} - \frac{dg}{dv}\,.\end{aligned}$$

Note that, by definition, the second order partial derivative, say $\dfrac{\partial^2 w}{\partial x^2}$, is the partial derivative of $\dfrac{\partial w}{\partial x}$ with respect to x. Hence, in general, the chain rule gives

$$\frac{\partial^2 w}{\partial x^2} \overset{\text{def}}{:=} \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial x}\right)\frac{\partial v}{\partial x}.$$

Noting that $\dfrac{\partial w}{\partial x} = \dfrac{df}{du} + \dfrac{dg}{dv}$ for the function w in consideration, and that f and g are functions of one variable, we have

$$\begin{aligned}\frac{\partial^2 w}{\partial x^2} &= \frac{\partial}{\partial u}\left(\frac{df}{du} + \frac{dg}{dv}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{df}{du} + \frac{dg}{dv}\right)\frac{\partial v}{\partial x} \\ &= \frac{d}{du}\left(\frac{df}{du}\right)\frac{\partial u}{\partial x} + \frac{d}{dv}\left(\frac{dg}{dv}\right)\frac{\partial v}{\partial x} \\ &= \frac{d^2 f}{du^2}(1) + \frac{d^2 g}{dv^2}(1) = \frac{d^2 f}{du^2} + \frac{d^2 g}{dv^2}\,.\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial^2 w}{\partial t^2} \overset{\text{def}}{:=} \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\right) &= \frac{\partial}{\partial u}\left(\frac{df}{du} - \frac{dg}{dv}\right)\frac{\partial u}{\partial t} + \frac{\partial}{\partial v}\left(\frac{df}{du} - \frac{dg}{dv}\right)\frac{\partial v}{\partial t} \\ &= \frac{d}{du}\left(\frac{df}{du}\right)\frac{\partial u}{\partial t} + \frac{d}{dv}\left(-\frac{dg}{dv}\right)\frac{\partial v}{\partial t} \\ &= \frac{d^2 f}{du^2}(1) - \frac{d^2 g}{dv^2}(-1) \\ &= \frac{d^2 f}{du^2} + \frac{d^2 g}{dv^2} \\ &= \frac{\partial^2 w}{\partial x^2}\,.\end{aligned}$$

Thus $w(x,t) = f(x+t) + g(x-t)$ does satisfy the wave equation $\dfrac{\partial^2 f}{\partial t^2} = \dfrac{\partial^2 f}{\partial x^2}$.

Note later in the course we will *derive* solutions to the wave equation. The form of the solution in this problem is called *D'Alembert's solution* to the wave equation.

4. Show that if $w = f(u,v)$ satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = 0$$

and if $u = \dfrac{x^2 - y^2}{2}$ and $v = xy$ then w satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0\,.$$

Solution: *Please make sure you understand the steps in the solution of the previous problem before attempting this problem.* Again we apply a chain rule

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u}(x) + \frac{\partial f}{\partial v}(y) = x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v}\,,\\
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}(-y) + \frac{\partial f}{\partial v}(x) = -y\frac{\partial f}{\partial u} + x\frac{\partial f}{\partial v}\,.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial}{\partial x}\left(x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v}\right)\\
&= \frac{\partial f}{\partial u} + x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right) + y\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right) \quad \text{using product rule}\\
&= \frac{\partial f}{\partial u} + x\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial v\,\partial u}\frac{\partial v}{\partial x}\right) + y\left(\frac{\partial^2 f}{\partial u\,\partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial x}\right)\\
&= \frac{\partial f}{\partial u} + x\left(\frac{\partial^2 f}{\partial u^2}(x) + \frac{\partial^2 f}{\partial v\,\partial u}(y)\right) + y\left(\frac{\partial^2 f}{\partial u\,\partial v}(x) + \frac{\partial^2 f}{\partial v^2}(y)\right)\\
&= \frac{\partial f}{\partial u} + x^2\frac{\partial^2 f}{\partial u^2} + xy\frac{\partial^2 f}{\partial v\,\partial u} + xy\frac{\partial^2 f}{\partial u\,\partial v} + y^2\frac{\partial^2 f}{\partial v^2}\,,\\
\frac{\partial^2 f}{\partial y^2} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial}{\partial y}\left(-y\frac{\partial f}{\partial u} + x\frac{\partial f}{\partial v}\right)\\
&= -\frac{\partial f}{\partial u} - y\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u}\right) + x\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial v}\right) \quad \text{using product rule}\\
&= -\frac{\partial f}{\partial u} - y\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v\,\partial u}\frac{\partial v}{\partial y}\right) + x\left(\frac{\partial^2 f}{\partial u\,\partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial y}\right)\\
&= -\frac{\partial f}{\partial u} - y\left(\frac{\partial^2 f}{\partial u^2}(-y) + \frac{\partial^2 f}{\partial v\,\partial u}(x)\right) + x\left(\frac{\partial^2 f}{\partial u\,\partial v}(-y) + \frac{\partial^2 f}{\partial v^2}(x)\right)\\
&= -\frac{\partial f}{\partial u} + y^2\frac{\partial^2 f}{\partial u^2} - xy\frac{\partial^2 f}{\partial v\,\partial u} - xy\frac{\partial^2 f}{\partial u\,\partial v} + x^2\frac{\partial^2 f}{\partial v^2}\,.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2} &= \frac{\partial f}{\partial u}+x^2\frac{\partial^2 f}{\partial u^2}+xy\frac{\partial^2 f}{\partial v\,\partial u}+xy\frac{\partial^2 f}{\partial u\,\partial v}+y^2\frac{\partial^2 f}{\partial v^2} \\
&\quad -\frac{\partial f}{\partial u}+y^2\frac{\partial^2 f}{\partial u^2}-xy\frac{\partial^2 f}{\partial v\,\partial u}-xy\frac{\partial^2 f}{\partial u\,\partial v}+x^2\frac{\partial^2 f}{\partial v^2} \\
&= (x^2+y^2)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right) \\
&= 0 \quad \text{since } \frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}=0.
\end{aligned}$$

Multivariable Taylor Series

$$\begin{aligned}
f(x,y) &= f(a,b)+(x-a)\frac{\partial f}{\partial x}(a,b)+(y-b)\frac{\partial f}{\partial y}(a,b) \\
&\quad +\frac{1}{2!}\left((x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b)+2(x-a)(y-b)\frac{\partial^2 f}{\partial x\partial y}(a,b)+(y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right)+\cdots
\end{aligned}$$

$\boxed{1994}$ 5. Calculate the Taylor series expansion up to and including second order terms of the function

$$z=F(x,y)=\ln x\cos y\,,$$

about the point $(1,\pi/4)$. Use your result to estimate $F(1.1,\pi/4)$.

Solution: We first calculate all the partial derivatives of F up to including second order terms at $(1,\pi/4)$.

$$\begin{aligned}
F(x,y) &= \ln x\cos y, & F(1,\pi/4) &= (\ln 1)\left(\cos\frac{\pi}{4}\right)=(0)\left(\frac{1}{\sqrt{2}}\right)=0\,, \\
\frac{\partial F}{\partial x} &= \frac{1}{x}\cos y, & F_x(1,\pi/4) &= \left(\frac{1}{1}\right)\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}\,, \\
\frac{\partial F}{\partial y} &= \ln x(-\sin y), & F_y(1,\pi/4) &= -(\ln 1)\left(\sin\frac{\pi}{4}\right)=-(0)(1/\sqrt{2})=0\,, \\
\frac{\partial^2 F}{\partial x^2} &= -\frac{1}{x^2}\cos y, & F_{xx}(1,\pi/4) &= \left(-\frac{1}{1^2}\right)\left(\frac{1}{\sqrt{2}}\right)=-\frac{1}{\sqrt{2}}\,, \\
\frac{\partial^2 F}{\partial x\,\partial y} &= \frac{1}{x}(-\sin y), & F_{yx}(1,\pi/4) &= -\left(\frac{1}{1}\right)\left(\frac{1}{\sqrt{2}}\right)=-\frac{1}{\sqrt{2}}\,, \\
\frac{\partial^2 F}{\partial y^2} &= \ln x(-\cos y), & F_{yy}(1,\pi/4) &= -(\ln 1)\left(\cos\frac{\pi}{4}\right)=-(0)(1/\sqrt{2})=0\,.
\end{aligned}$$

Thus

$$\begin{aligned}
z=F(x,y) &\approx F(1,\pi/4)+(x-1)F_x(1,\pi/4)+(y-\pi/4)F_y(1,\pi/4) \\
&\quad +\frac{1}{2!}\left((x-1)^2F_{xx}(1,\pi/4)+2(x-1)(y-\pi/4)F_{xy}(1,\pi/4)+(y-\pi/4)^2F_{yy}(1,\pi/4)\right) \\
&= \frac{x-1}{\sqrt{2}}-\frac{1}{2\sqrt{2}}\left((x-1)^2+2(x-1)(y-\pi/4)\right)\,.
\end{aligned}$$

Hence

$$F(1.1,\pi/4)\approx\frac{0.1}{\sqrt{2}}-\frac{1}{2\sqrt{2}}\left((0.1)^2+2(0.1)(0)\right)=\frac{0.1}{\sqrt{2}}(1-0.05)=\frac{0.95}{\sqrt{2}}=0.0672\,.$$

2011, S1 6. Expand $f(x,y) = e^y \sin x$ about $(0,1)$ up to and including second-order terms, using Taylor series for functions of two variables.

Solution: We first calculate all the partial derivatives of f up to including second order terms at $(0,1)$.

$$\begin{aligned}
f(x,y) &= e^y \sin x, & f(0,1) &= e^1 \sin 0 = e^1(0) = 0\,,\\
\frac{\partial f}{\partial x} &= e^y \cos x, & f_x(0,1) &= e^1 \cos 0 = e^1(1) = e\,,\\
\frac{\partial f}{\partial y} &= e^y \sin x, & f_y(0,1) &= e^1 \sin 0 = e^1(0) = 0\,,\\
\frac{\partial^2 f}{\partial x^2} &= -e^y \sin x, & f_{xx}(0,1) &= -e^1(0) = 0\,,\\
\frac{\partial^2 f}{\partial x\,\partial y} &= e^y \cos x, & f_{yx}(0,1) &= e^1 \cos 0 = e^1(1) = e\,,\\
\frac{\partial^2 f}{\partial y^2} &= e^y \sin x, & f_{yy}(0,1) &= e^1 \sin 0 = e^1(0) = 0\,.
\end{aligned}$$

Thus

$$\begin{aligned}
f(x,y) &\approx f(0,1) + (x-0)f_x(0,1) + (y-1)f_y(0,1)\\
&\quad + \frac{1}{2!}\left((x-0)^2 f_{xx}(0,1) + 2(x-0)(y-1)f_{xy}(0,1) + (y-1)^2 f_{yy}(0,1)\right)\\
&= e\,(x + x(y-1))\\
&= exy\,.
\end{aligned}$$

A check for the answer is to determine the expansion using the Maclaurin series for e^t and $\sin t$ (since f is a product of functions), i.e.,

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \quad , \quad \sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\,.$$

Thus

$$\begin{aligned}
f(x,y) &= e^y \sin x\\
&= e^{1+(y-1)} \sin x \quad \text{since we wish to expand about } (0,1)\\
&= e\,e^{y-1} \sin x\\
&= e\left(1 + (y-1) + \frac{(y-1)^2}{2!} + \frac{(y-1)^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\\
&\approx e\,(1 + (y-1))\,x \quad \text{up to and including quadratic terms}\\
&= e\,(x + x(y-1))\\
&= exy\,.
\end{aligned}$$

2017, S2 7. i) Calculate the Taylor series expansion of the function $f(x,y) = \ln(x+y)$ about the point $(1,0)$ up to and including quadratic terms.

Solution: We first calculate all the partial derivatives of f up to including second order terms at $(1,0)$. We should note f is symmetric in x and y and this will reduce the amount of work in calculating the partial derivatives.

$$\begin{aligned}
f(x,y) &= \ln(x+y), & f(1,0) &= \ln(1+0) = \ln 1 = 0\\
\frac{\partial f}{\partial x} &= \frac{1}{x+y} = \frac{\partial f}{\partial y}, & f_x(1,0) &= f_y(1,0) = \frac{1}{1+0} = 1\\
\frac{\partial^2 f}{\partial x^2} &= -\frac{1}{(x+y)^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x\,\partial y}, & f_{xx}(1,0) &= f_{yy}(1,0) = f_{yx}(1,0) = -\frac{1}{(1+0)^2} = -1\,.
\end{aligned}$$

Thus

$$\begin{aligned}
f(x,y) &\approx f(1,0)+(x-1)f_x(1,0)+(y-0)f_y(1,0)\\
&\quad +\frac{1}{2!}\left((x-1)^2 f_{xx}(1,0)+2(x-1)(y-0)f_{xy}(1,0)+(y-0)^2 f_{yy}(1,0)\right)\\
&= (x-1)+y-\frac{1}{2}\left((x-1)2+2(x-1)y+y^2\right)\\
&= (x+y-1)-\frac{1}{2}(x+y-1)^2\,.
\end{aligned}$$

A check for the answer is to determine the expansion using the Maclaurin series for $\ln(1+t)$, i.e.,

$$\begin{aligned}
\ln(1+t)=\int_0^t \frac{1}{1+x}\,dx=\int_0^t \frac{1}{1-(-x)}\,dx &= \int_0^t\left(\sum_{k=0}^{\infty}(-x)^k\right)dx\\
&= \sum_{k=0}^{\infty}(-1)^k\int_0^t x^k\,dx\\
&= \sum_{k=0}^{\infty}(-1)^k\frac{t^{k+1}}{k+1}\\
&= t-\frac{t^2}{2}+\frac{t^3}{3}-\cdots\,,\ \ |t|<1\,.
\end{aligned}$$

Thus

$$\begin{aligned}
f(x,y) &= \ln(x+y)\\
&= \ln(1+(x-1)+y) \quad \text{since we wish to expand about } (1,0)\\
&= \ln(1+(x+y-1))\\
&= (x+y-1)-\frac{(x+y-1)^2}{2}+\frac{(x+y-1)^3}{3}-\cdots\\
&\approx (x+y-1)-\frac{1}{2}(x+y-1)^2 \quad \text{up to and including quadratic terms.}
\end{aligned}$$

ii) Use your solution to find an approximate value for $\ln(1.1)$.

Solution: Note to approximate $\ln(1.1)$ we will set $x+y=1.1$ in f, i.e.,

$$\ln(1.1)=f(1.1)\approx(1.1-1)-\frac{1}{2}(1.1-1)^2=0.1-0.005=0.095\,.$$

2014, S1

8. A cone with radius r and perpendicular height h has volume $V=\frac{1}{3}\pi r^2 h$.

Determine the maximum error in calculating V given that $r=4$ cm and $h=3$ cm to the nearest millimetre.

Solution: Note the expression to the nearest millimetre means $\Delta r=\Delta h=\pm 0.5\,\text{mm}=\pm 0.05\,\text{cm}$. Using a linear approximation we have

$$\begin{aligned}
|\Delta V| &\le \left|\frac{\partial V}{\partial r}(4,3)\right||\Delta r|+\left|\frac{\partial V}{\partial h}(4,3)\right||\Delta h|\\
&= \left(\frac{2}{3}\pi r h\Big|_{r=4,h=3}+\frac{1}{3}\pi r^2\Big|_{r=4,h=3}\right)(0.05)\\
&= \frac{\pi}{3}\left(2(4)(3)+(4)^2\right)(0.05)\\
&= \frac{2\pi}{3}=2\,\text{cm}^3\,.
\end{aligned}$$

Thus the maximum error in V is $2\,\mathrm{cm}^3$. Note the answer has units!

$\boxed{\text{2014, S2}}$ 9. The pressure P of a gas in a reactor is given by

$$P = r\rho T,$$

where ρ is the density, T is the temperature, and r is a constant. If the pressure in the reactor decreases by 5% and the temperature increases by 7%, what is the percentage change in the density of the gas inside the reactor? [Note that you do not need to know the value of r.]

Solution: Note in the problem we are given the percentage change in quantities so must derive an expression in terms $\dfrac{\Delta P}{P}$ and $\dfrac{\Delta T}{T}$. Using a linear approximation we have

$$\begin{aligned} \Delta P &\approx \frac{\partial P}{\partial \rho}\Delta\rho + \frac{\partial P}{\partial T}\Delta T \\ &= rT\Delta\rho + r\rho\Delta T \\ &= r\rho T\frac{\Delta\rho}{\rho} + r\rho T\frac{\Delta T}{T} \\ \Rightarrow \quad \frac{\Delta P}{P} &\approx \frac{\Delta\rho}{\rho} + \frac{\Delta T}{T} \quad \text{since } P = r\rho T \\ \Rightarrow \quad \frac{\Delta\rho}{\rho} &\approx \frac{\Delta P}{P} - \frac{\Delta T}{T} = -5\% - 7\% = -12\%\,. \end{aligned}$$

The density ρ *decreases* approximately by 12%.

$\boxed{\text{2016, S1}}$ 10. The battery life of a mobile phone is given by

$$L = \frac{\alpha\, C}{b^2},$$

where C is the capacity of the battery, b is the width of the phone screen, and α is a positive constant. If the battery capacity C is increased by 20% and the screen size b is increased by 5%, use the chain rule to estimate the percentage change in the battery life of the phone. [Note that you do not need to know the value of α.]

Solution: Note in the problem we are given the percentage change in quantities so must derive an expression in terms $\dfrac{\Delta C}{C}$ and $\dfrac{\Delta b}{b}$. Using a linear approximation we have

$$\begin{aligned} \Delta L &\approx \frac{\partial L}{\partial C}\Delta C + \frac{\partial L}{\partial B}\Delta b \\ &= \frac{\alpha}{b^2}\Delta C - 2\frac{\alpha C}{b^3}\Delta b \\ &= \frac{\alpha\, C}{b^2}\frac{\Delta C}{C} - 2\frac{\alpha\, C}{b^2}\frac{\Delta b}{b} \\ \Rightarrow \quad \frac{\Delta L}{L} &\approx \frac{\Delta C}{C} - 2\frac{\Delta b}{b} \quad \text{since } L = \frac{\alpha\, C}{b^2} \\ &= +20\% - 2(5\%) = 10\%\,. \end{aligned}$$

The battery life L *increases* approximately by 10%.

$\boxed{\text{2017, S1}}$ 11. A metal cylinder contains a volume of liquid given by

$$V = \pi r^2 h,$$

where r is the radius of the cylinder and h is the height of the cylinder. Small variations in the manufacturing process can result in errors in the cylinder radius of 1% and the cylinder height of 2%. What is the maximum percentage error in the volume of the cylinder?

Solution: Note in the problem we are given the percentage change in quantities so must derive an expression in terms $\dfrac{\Delta r}{r}$ and $\dfrac{\Delta h}{h}$. Using a linear approximation we have

$$\begin{aligned}
|\Delta V| &\leq \left|\frac{\partial V}{\partial r}\right| |\Delta r| + \left|\frac{\partial V}{\partial h}\right| |\Delta h| \\
&= 2\pi r h |\Delta r| + \pi r^2 |\Delta h| \\
&= 2\pi r^2 h \frac{|\Delta r|}{r} + \pi r^2 h \frac{|\Delta h|}{h} \\
\Rightarrow \frac{|\Delta V|}{V} &\leq 2\frac{|\Delta r|}{r} + \frac{|\Delta h|}{h} \quad \text{since } V = \pi r^2 h \\
&= 2(1\%) + 2\% = 4\%\,.
\end{aligned}$$

Thus the maximum percentage error in V is 4%.

$\boxed{\text{2018, S2}}$ 12. The volume V of a circular cylinder with radius r and perpendicular height h is given by $V = \pi r^2 h$. Use a linear approximation to estimate the maximum percentage error in calculating V given that $r = 30$ metres and $h = 20$ metres, to the nearest metre.

Solution: Note in the problem we are given the absolute change in quantities (to the nearest metre), i.e., $|\Delta r| = |\Delta h| = 0.5$ metre with $r = 30$ metres and $h = 20$ metres. Using a linear approximation we have

$$\begin{aligned}
|\Delta V| &\leq \left|\frac{\partial V}{\partial r}\right| |\Delta r| + \left|\frac{\partial V}{\partial h}\right| |\Delta h| \\
&= 2\pi r h |\Delta r| + \pi r^2 |\Delta h| \\
&= 2\pi r^2 h \frac{|\Delta r|}{r} + \pi r^2 h \frac{|\Delta h|}{h} \\
\Rightarrow \frac{|\Delta V|}{V} &\leq 2\frac{|\Delta r|}{r} + \frac{|\Delta h|}{h} \quad \text{since } V = \pi r^2 h \\
&= 2\left(\frac{0.5}{30}\right) + \frac{0.5}{20} \\
&= \frac{1}{30} + \frac{1}{40} = \frac{7}{120} \\
&= \frac{35}{6}\% = 5\frac{5}{6}\%\,.
\end{aligned}$$

Thus the maximum percentage error in V is $5\frac{5}{6}\%$.

Leibniz' Rule for Differentiation of Integrals

$$\frac{d}{dx}\int_u^v f(x,t)dt = \int_u^v \frac{\partial f}{\partial x} dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx}$$

$\boxed{\text{2013, S2}}$ 13. Use Leibniz' rule to find

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}\, dx$$

given that

$$\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}.$$

Solution: We differentiate the given integral with respect to the parameter (variable) a, i.e.,

$$\begin{aligned} \frac{d}{da}\left(\int_{-\infty}^{\infty} e^{-ax^2}\, dx\right) &= \frac{d}{da}\left(\sqrt{\frac{\pi}{a}}\right) \\ \Rightarrow \quad \int_{-\infty}^{\infty} \frac{\partial}{\partial a}\left(e^{-ax^2}\right) dx &= -\frac{\sqrt{\pi}}{2a^{3/2}} \quad \text{using Leibniz' rule on L.H.S.} \\ \Rightarrow \quad -\int_{-\infty}^{\infty} x^2 e^{-ax^2}\, dx &= -\frac{\sqrt{\pi}}{2a^{3/2}} \\ \Rightarrow \quad \int_{-\infty}^{\infty} x^2 e^{-ax^2}\, dx &= \frac{\sqrt{\pi}}{2a^{3/2}}\,. \end{aligned}$$

2014, S1

14. Use Leibniz' rule to find

$$\frac{d}{dt}\int_1^{t^2} \frac{\sin(\sqrt{x}\,)}{x}\, dx.$$

Solution: Note for this problem Leibniz' rule reduces to the Fundamental Theorem of Calculus (MATH 1131/41), i.e.,

$$\frac{d}{dt}\int_1^{t^2} \frac{\sin(\sqrt{x}\,)}{x}\, dx = \frac{\sin(\sqrt{t^2})}{t^2}\frac{d}{dt}(t^2) = \frac{\sin(|t|)}{t^2}2t = 2\frac{\sin(|t|)}{t}\,.$$

2014, S2

15. Use Leibniz' rule to find

$$\int_0^{\infty} xe^{-bx}\sin x\, dx$$

given that

$$\int_0^{\infty} e^{-bx}\sin x\, dx = \frac{1}{1+b^2}.$$

Solution: We differentiate the given integral with respect to the parameter (variable) b, i.e.,

$$\begin{aligned} \frac{d}{db}\left(\int_0^{\infty} e^{-bx}\sin x\, dx\right) &= \frac{d}{db}\left(\frac{1}{1+b^2}\right) \\ \Rightarrow \quad \int_0^{\infty} \frac{\partial}{\partial b}\left(e^{-bx}\sin x\right) dx &= -\frac{2b}{(1+b^2)^2} \quad \text{using Leibniz' rule on L.H.S.} \\ \Rightarrow \quad -\int_0^{\infty} xe^{-bx}\sin x\, dx &= -\frac{2b}{(1+b^2)^2} \\ \Rightarrow \quad \int_0^{\infty} xe^{-bx}\sin x\, dx &= \frac{2b}{(1+b^2)^2}\,. \end{aligned}$$

2015, S1

16. Use Leibniz' rule to calculate

$$\frac{d}{dy}\int_{y^2}^{1} \frac{\sin(xy)}{x}\, dx.$$

Solution:

$$\begin{aligned}
\frac{d}{dy}\int_{y^2}^{1}\frac{\sin(xy)}{x}\,dx &= \int_{y^2}^{1}\frac{\partial}{\partial y}\left(\frac{\sin(xy)}{x}\right)dx - \frac{\sin(y^2y)}{y^2}\frac{d}{dy}(y^2) \quad \text{using Leibniz' rule on L.H.S.}\\
&= \int_{y^2}^{1}\cos(xy)dx - 2\frac{\sin(y^3)}{y}\\
&= \left(\frac{1}{y}\sin(xy)\bigg|_{y^2}^{1}\right) - 2\frac{\sin(y^3)}{y}\\
&= \frac{\sin(y)}{y} - 3\frac{\sin(y^3)}{y}\,.
\end{aligned}$$

$\boxed{\text{2016, S2}}$ 17. Use Leibniz' rule to find

$$\frac{d}{dt}\int_{1}^{\sin t} e^{1-x^2}\,dx.$$

Solution: Note for this problem Leibniz' rule reduces to the Fundamental Theorem of Calculus (MATH 1131/41) since the integrand is a function only of the integration variable. Hence

$$\frac{d}{dt}\int_{1}^{\sin t} e^{1-x^2}\,dx = e^{1-\sin^2 t}\frac{d}{dt}(\sin t) = e^{\cos^2 t}\cos t\,.$$

$\boxed{\text{2017, S1}}$ 18. You are given the following integral,

$$\int_0^\infty \sqrt{x}\mathrm{e}^{-tx}\,dx = \frac{\sqrt{\pi}}{2t^{3/2}}.$$

Use Leibniz' rule to evaluate

$$\int_0^\infty x^{3/2}\mathrm{e}^{-tx}\,dx.$$

Solution: We differentiate the given integral with respect to the parameter (variable) t, i.e.,

$$\begin{aligned}
\frac{d}{dt}\left(\int_0^\infty \sqrt{x}\mathrm{e}^{-tx}\,dx\right) &= \frac{d}{dt}\left(\frac{\sqrt{\pi}}{2t^{3/2}}\right)\\
\Rightarrow \quad \int_0^\infty \frac{\partial}{\partial t}\left(\sqrt{x}\mathrm{e}^{-tx}\right)dx &= -\frac{3\sqrt{\pi}}{4t^{5/2}} \quad \text{using Leibniz' rule on L.H.S.}\\
\Rightarrow \quad -\int_0^\infty x\sqrt{x}\mathrm{e}^{-tx}\,dx &= -\frac{3\sqrt{\pi}}{4t^{5/2}}\\
\Rightarrow \quad \int_0^\infty x^{3/2}\mathrm{e}^{-tx}\,dx &= \frac{3\sqrt{\pi}}{4t^{5/2}}\,.
\end{aligned}$$

$\boxed{\text{2017, S2}}$ 19. You are given the following integral,

$$\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,dx = \sinh^{-1}(1)\,.$$

Use Leibniz' rule to evaluate

$$\int_0^a \frac{1}{(x^2+a^2)^{3/2}}\,dx.$$

Solution: We differentiate the given integral with respect to the parameter (variable) a, i.e.,

$$\begin{aligned}
\frac{d}{da}\left(\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,dx\right) &= \frac{d}{da}\left(\sinh^{-1}(1)\right)\\
\Rightarrow \quad \int_0^a \frac{\partial}{\partial a}\left(\frac{1}{(x^2+a^2)^{1/2}}\right)dx + \frac{1}{\sqrt{a^2+a^2}}(1) &= 0 \quad \text{using Leibniz' rule on L.H.S.}\\
\Rightarrow \quad -\int_0^a \frac{a}{(x^2+a^2)^{3/2}}\,dx + \frac{1}{|a|\sqrt{2}} &= 0\\
\Rightarrow \quad \int_0^a \frac{1}{(x^2+a^2)^{3/2}}\,dx &= \frac{1}{a|a|\sqrt{2}}\,.
\end{aligned}$$

$\boxed{\text{2018, S1}}$ 20. Consider the following ordinary differential equation

$$\frac{dy}{dx}+\frac{2}{x}y=\frac{\sin(x^2)}{x^2}, \quad x>0\,,$$

with initial condition $y(\sqrt{\frac{\pi}{2}})=0$. A student solves this ordinary differential equation and writes the solution in an integral form, i.e.,

$$y=\frac{1}{x^2}\int_{\sqrt{\frac{\pi}{2}}}^{x}\sin(t^2)\,dt\,.$$

i) **Verify** that this function y satisfies the initial condition.

Solution: $y(\sqrt{\frac{\pi}{2}})=\frac{1}{x^2}\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\frac{\pi}{2}}}\sin(t^2)\,dt=0$ using the integral property $\int_a^a f(x)\,dx=0$. Hence the integral form of the solution y satisfies the initial condition $y(\sqrt{\frac{\pi}{2}})=0$.

ii) Use Leibniz' rule to **verify** that y satisfies the differential equation.

Solution: We can rewrite the function y as $y=\int_{\sqrt{\frac{\pi}{2}}}^{x}\frac{1}{x^2}\sin(t^2)\,dt$ and consider $\frac{dy}{dx}+\frac{2}{x}y$, i.e.,

$$\begin{aligned}
\frac{dy}{dx}+\frac{2}{x}y &= \int_{\sqrt{\frac{\pi}{2}}}^{x}\frac{\partial}{\partial x}\left(\frac{1}{x^2}\sin(t^2)\right)dt+\frac{1}{x^2}\sin(x^2)+\frac{2}{x}\int_{\sqrt{\frac{\pi}{2}}}^{x}\frac{1}{x^2}\sin(t^2)\,dt\\
&\qquad \text{using Leibniz' rule for the derivative on L. H. S.}\\
&= -\int_{\sqrt{\frac{\pi}{2}}}^{x}\frac{2}{x^3}\sin(t^2)\,dt+\frac{\sin(x^2)}{x^2}+\int_{\sqrt{\frac{\pi}{2}}}^{x}\frac{2}{x^3}\sin(t^2)\,dt\\
&= \frac{\sin(x^2)}{x^2} \quad \text{since 1st \& 3rd terms cancel.}
\end{aligned}$$

Thus the student's integral form solution satisfies the differential equation $\frac{dy}{dx}+\frac{2}{x}y=\frac{\sin(x^2)}{x^2}$.

$\boxed{\text{2018, S2}}$ 21. You are given that

$$\int_0^\infty \frac{1}{\alpha^2+x^2}\,dx=\frac{\pi}{2}\alpha^{-1}.$$

Use Leibniz' rule to find the following integral in terms of α

$$\int_0^\infty \frac{1}{(\alpha^2 + x^2)^2}\, dx.$$

Solution: Note this is essentially Tutorial Problem Q32.