THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

JUNE 2008

MATH2019
ENGINEERING MATHEMATICS 2E

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) ONLY THE PROVIDED ELECTRONIC CALCULATORS MAY BE USED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

TABLE OF LAPLACE TRANSFORMS AND THEOREMS

$g(t)$ is a function defined for all $t \geq 0$, and whose Laplace transform

$$G(s) = \mathcal{L}(g(t)) = \int_0^\infty e^{-st} g(t)dt$$

exists. The Heaviside step function u is defined to be

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ \frac{1}{2} & \text{for } t = a \\ 1 & \text{for } t > a \end{cases}$$

$g(t)$	$G(s) = \mathcal{L}[g(t)]$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^\nu, \nu > -1$	$\frac{\nu!}{s^{\nu+1}}$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$e^{-\alpha t}f(t)$	$F(s+\alpha)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$tf(t)$	$-F'(s)$

Please see over ...

FOURIER SERIES

If $f(x)$ has period $p = 2L$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{aligned}$$

LEIBNIZ' THEOREM

$$\frac{d}{dx}\int_u^v f(x,t)dt = \int_u^v \frac{\partial f}{\partial x}dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx}$$

Please see over ...

SOME BASIC INTEGRALS

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^{kx} dx = \frac{e^{kx}}{k} + C$$

$$\int a^x dx = \frac{1}{\ln a} a^x + C \quad \text{for } a \neq 1$$

$$\int \sin\ kx\, dx = -\frac{\cos kx}{k} + C$$

$$\int \cos kx\, dx = \frac{\sin kx}{k} + C$$

$$\int \sec^2 kx\, dx = \frac{\tan kx}{k} + C$$

$$\int \text{cosec}^2 kx\, dx = -\frac{1}{k} \cot kx + C$$

$$\int \tan kx\, dx = \frac{\ln|\sec kx|}{k} + C$$

$$\int \sec kx\, dx = \frac{1}{k}(\ln|\sec kx + \tan kx|) + C$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx$$

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx$$

Answer question 1 in a separate book

1. a) The matrix A is given by

$$A = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} .$$

i) Show that the vector

$$\mathbf{v} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

is an eigenvector of the matrix A and find the corresponding eigenvalue.

ii) Given that the other two eigenvalues of A are 2 and -2, find the eigenvectors corresponding to these two eigenvalues.

b) Given the integral

$$\int_0^2 \int_{x^2}^4 \frac{x^3}{\sqrt{x^4 + y^2}}\, dy\, dx.$$

i) Make a clear sketch of the region of integration.

ii) Express the integral with the order of integration reversed.

iii) Evaluate the integral you found in (ii).

Answer question 2 in a separate book

2. a) Find

i) $\mathcal{L}(te^{-t}\sin 3t)$.

ii) $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2+4s+5}+\dfrac{e^{-2s}}{3s^4}\right\}$.

b) The function $f(t)$ is defined for $t \geq 0$ by

$$f(t)=\begin{cases}1 & 0 \leq t \leq 1\\ t-2 & 1 < t \leq 2\\ 0 & t > 2.\end{cases}$$

i) Express $f(t)$ in terms of the Heaviside function and hence or otherwise find $\mathcal{L}(f(t))$, the Laplace transform of $f(t)$.

c) Use the Laplace Transform method to solve the differential equation

$$y'' - 4y' + 4y = e^{2t} \quad , \; t > 0$$

subject to the initial condition $y(0) = 1,\; y'(0) = 0$.

Answer question 3 in a separate book

3. The odd periodic function $f(x)$ is defined by

$$f(x)=\begin{cases}-4-x & -4 \leq x \leq 0\\ 4-x & 0 < x < 4\end{cases} \tag{1}$$

with $f(x+8) = f(x)$ for all x.

a) Sketch $f(x)$ for $-12 \leq x \leq 8$.

b) Find the coefficients in the Fourier series for the function defined by equation (1) and write out the series, explicitly giving the first three non-zero terms in the series.

c) Find a particular solution of the ordinary differential equation

$$2\frac{d^2y}{dx^2} + 11y = f(x)$$

in terms of a Fourier series where $f(x)$ is given by equation (1) above. Hence, find the term(s) of largest magnitude in the particular solution and give evidence to justify your answer.

Answer question 4 in a separate book

4. The steady-state distribution of heat in a slab of width L is given by

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad \text{for } \ 0 < x < L, \ 0 < y < \infty$$

$$\begin{aligned} &U(0,y) = U(L,y) = 0, && \text{for } \ 0 < y < \infty \\ &U \text{ bounded} && \text{as } \ y \to +\infty \\ &U(x,0) = f(x), && \text{for } \ 0 \le x \le L. \end{aligned}$$

a) Draw a clear diagram of the slab indicating the temperature $U(x,y)$ on three sides of the slab.

b) Use the method of separation of variables to find the general solution $U(x,y)$, where any unknown constants are related to $U(x,0) = f(x)$. You must explicitly consider all possibilities for the separation constant in your working.

. We show $A\underset{\sim}{v} = \lambda \underset{\sim}{v}$ for some number λ.

Thus, we solve

$$\begin{pmatrix} 2 & -2 & 2 \\ -2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad \text{for } \lambda.$$

The third row gives

$4 - 1 + 1 = \lambda . 1$ and so

$\lambda = 4.$ is the e.value of interest.

(ii) If $\lambda = 2$ then we solve

$A\underset{\sim}{v} = \lambda \underset{\sim}{v}$ for e.vector $\underset{\sim}{v}$.

Thus we solve $(A - \lambda I)\underset{\sim}{v} = \underset{\sim}{0}$, ie

$$\begin{pmatrix} 0 & -2 & 2 \\ -2 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which gives $v_2 = v_3$ and $v_1 = 0$.

Choose $\underset{\sim}{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Similarly, for $\lambda = -2$ we solve

$$(A - \lambda I)\underset{\sim}{v} = \underset{\sim}{0}, \quad \text{ie}$$

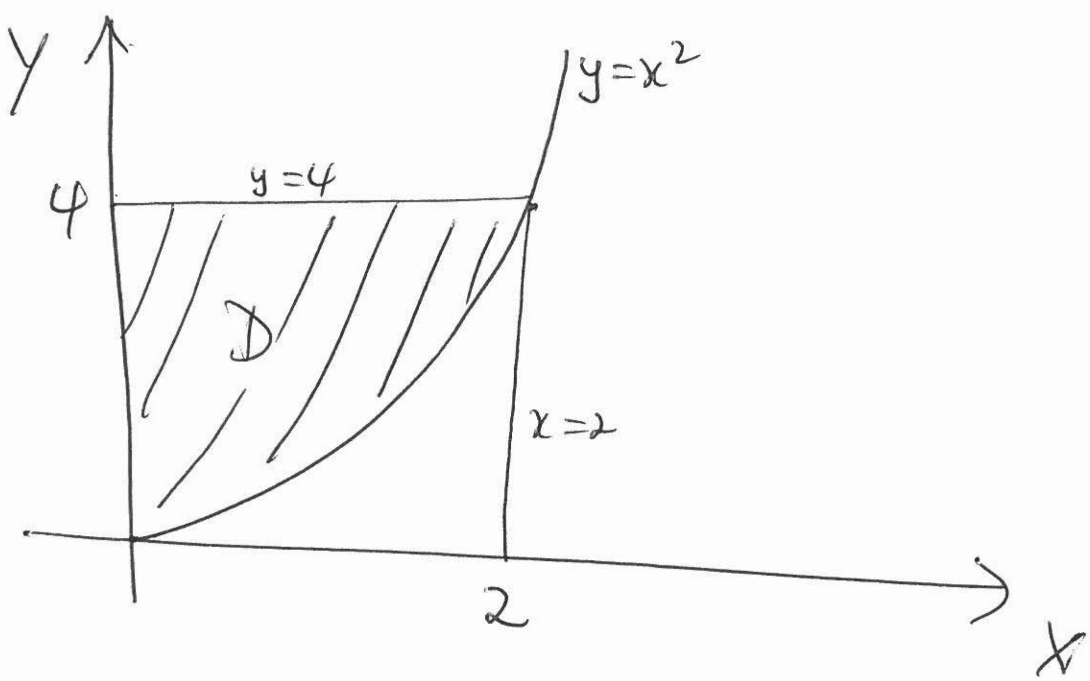
$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving, we obtain: $v_1 = -v_3$; $v_1 = v_2$

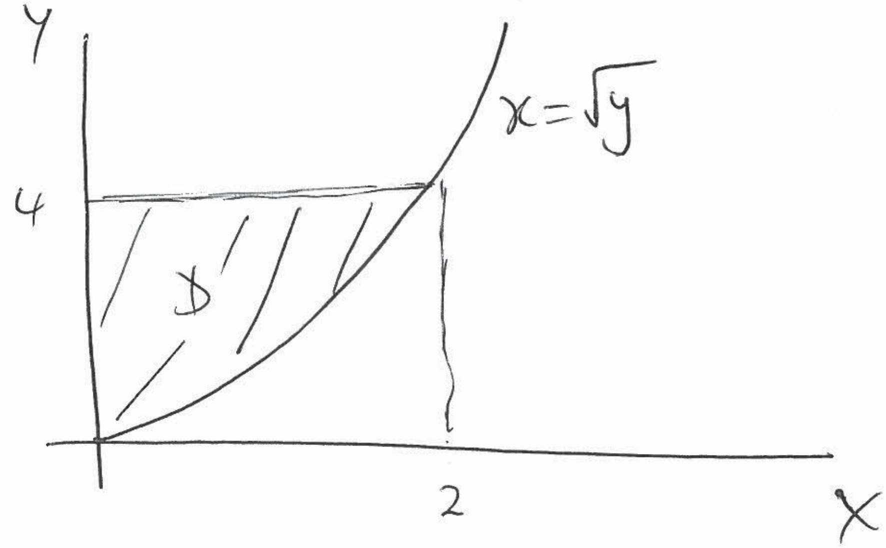
and so choose $\underset{\sim}{v} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

(b) Let $I := \int_0^2 \int_{x^2}^4 \frac{x^3}{\sqrt{x^4+y^2}} \, dy \, dx$.

(i) $D := \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 2, \ x^2 \leq y \leq 4\}$



(ii) Redescribe D:



$$D := \{(x,y) : 0 \le x \le \sqrt{y},\ 0 \le y \le 4\}.$$

(iii) $$I = \iint_D \frac{x^3}{\sqrt{x^4+y^2}}\, dy\, dx$$

$$= \int_0^4 \left[\int_0^{\sqrt{y}} \frac{x^3}{\sqrt{x^4+y^2}}\, dx \right] dy$$

$$= \int_0^4 \left[\frac{1}{2}\left(x^4+y^2\right)^{\frac{1}{2}} \right]_{x=0}^{x=\sqrt{y}} dy$$

$$= \frac{1}{2} \int_0^4 \sqrt{2y^2} - \sqrt{y^2}\, dy = \frac{\sqrt{2}-1}{2} \int_0^4 y\, dy$$

$$= \frac{\sqrt{2}-1}{2} \left[\frac{1}{2} y^2 \right]_0^4 = 4\left[\sqrt{2}-1\right].$$

(2) (a)

(i) $\mathcal{L}\{t\, \overbrace{e^{-t}\sin 3t}^{f(t)}\} = -F'(s)$ (from table)

where $F(s) = \mathcal{L}\{f(t)\}$.

Now $\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t}\sin 3t\} = \mathcal{L}\{e^{-\alpha t} g(t)\}$

$= G(s+\alpha)$ with $\alpha = +1$, $g(t) = \sin t$ (First shift theorem)

where $G(s) = \mathcal{L}\{g(t)\}$

$= \mathcal{L}\{\sin 3t\} = \dfrac{1}{s^2+3^2}$ (table).

Thus $G(s+\alpha) = G(s+1) = \dfrac{1}{(s+1)^2+3^2}$.

Finally, $F(s) = \dfrac{1}{(s+1)^2+3^2}$.

Now,
$$F'(s) = \frac{-2(s+1)}{((s+1)^2+3^2)^2}.$$

Hence
$$\mathcal{L}\{te^{-t}\sin t\} = -F'(s)$$
$$= \frac{2(s+1)}{((s+1)^2+3^2)^2}.$$

(ii) Alternatively:

$$\mathcal{L}\{e^{-t}\, t \sin 3t\} = \mathcal{L}\{e^{-\alpha t} f(t)\}$$

$$= F(s+\alpha) \qquad \text{(first shifting theorem)} \quad (\alpha = 1)$$

where $F(s) = \mathcal{L}\{f(t)\}$

$$= \mathcal{L}\{t \sin 3t\} = \mathcal{L}\{t \cdot g(t)\}$$

$$= -G'(s) \qquad \text{(table)}$$

where $G(s) = \mathcal{L}\{g(t)\}$

$$= \mathcal{L}\{\sin 3t\}$$

$$= \frac{3}{s^2 + 3^2}.$$

thus $G'(s) = \dfrac{-6s}{(s^2+3^2)^2}$ and

so $F(s) = -G'(s) = \dfrac{6s}{(s^2+3^2)^2}$.

$$\mathcal{L}\{e^{-t}\, t \sin 3t\}$$

$$= F(s+\alpha) = F(s+1)$$

$$= \frac{6(s+1)}{((s+1)^2 + 3^2)^2}$$

②(a) (ii) P6

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+5} + e^{-2s}\cdot\frac{1}{3s^4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s+2)\ -1}{(s+2)^2+1} + e^{-2s}\cdot\frac{1}{3s^4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2+1} - \frac{1}{(s+2)^2+1} + e^{-2s}\cdot\frac{1}{3s^4}\right\}.$$

$$= e^{-2t}\cdot\mathcal{L}\left\{\frac{s}{s^2+1^2}\right\} - e^{-2t}\cdot\mathcal{L}\left\{\frac{1}{s^2+1^2}\right\}$$

(First shift thm)

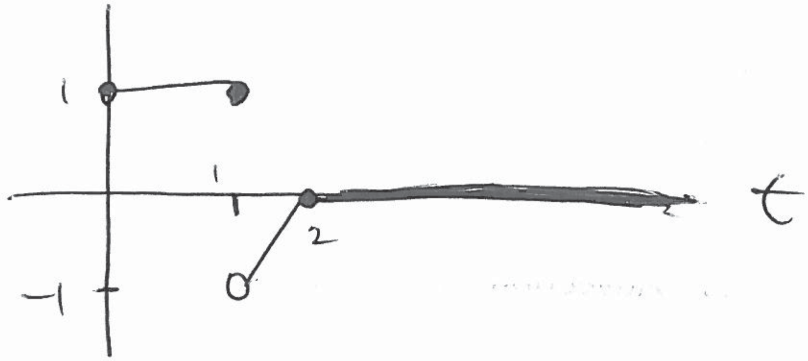
$$+ \ f(t-2)\cdot u(t-2) \quad \text{where } f(t) = \mathcal{L}^{-1}\left\{\frac{1}{3s^4}\right\} = \frac{1}{3}\frac{t^3}{3!}$$

(Second shift thm)

$$= e^{-2t}\cdot\cos t - e^{-2t}\sin t + \frac{1}{3}\cdot\frac{1}{3!}(t-2)^3\cdot u(t-2)$$

from tables.

(2) b)



$$f(t) = 1 - u(t-1) + (t-2)u(t-1) - (t-2)u(t-2).$$

$$= 1 - 2u(t-1) + (t-1)u(t-1) - (t-2)u(t-2).$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s} - 2 \cdot \frac{e^{-s}}{s} \qquad \text{(tables)}$$

$$+ e^{-s} \cdot \mathcal{L}\{t\} - e^{-2s} \cdot \mathcal{L}\{t\} \qquad \text{(2nd shift thm)}$$

$$= \frac{1}{s} - 2\frac{e^{-s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}.$$

② (c) $\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{e^{2t}\}$

p8

and so from tables (transf. of derivs.)

$$s^2 Y(s) - s y(0) - y'(0)$$
$$-4\left[sY(s) - y(0) \right] + 4Y(s) = \frac{1}{s-2}$$

which, incorporating I.C.s, gives

$$s^2 Y(s) - s - 4sY(s) + 4 + 4Y(s) = \frac{1}{s-2}.$$

Thus
$$(s-2)^2 Y(s) = \frac{1}{s-2} + s - 4$$

and so
$$Y(s) = \frac{1}{(s-2)^3} + \frac{s-4}{(s-2)^2}$$

$$= \frac{1}{(s-2)^3} + \frac{(s-2)-2}{(s-2)^2}$$

$$= \frac{1}{(s-2)^3} + \frac{1}{s-2} - \frac{2}{(s-2)^2}.$$

<u>Soln</u>

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

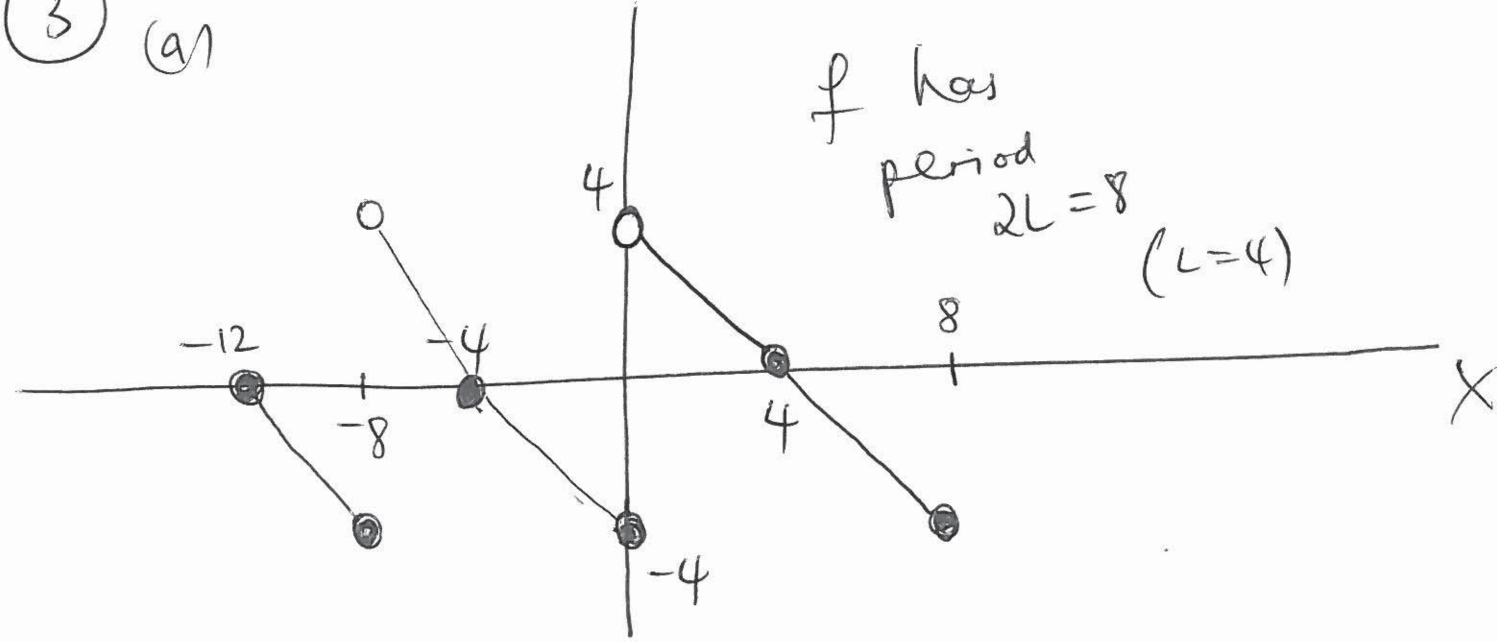
$$= \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^3} + \frac{1}{s-2} - \frac{2}{(s-2)^2}\right\}$$

$$= e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} + e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

(first shift thm)

$$= e^{2t}\cdot\frac{t^2}{2} + e^{2t} - 2e^{2t}\cdot t.$$

(3) (a)



f has period $2L = 8$ $(L=4)$

(b) See that f is <u>odd</u> so $a_0 = 0 = a_k$

and $b_k = 2 \cdot \frac{1}{L} \int_0^L f(x) \cdot \sin \frac{k\pi x}{L} \, dx$

$$= \frac{1}{2} \int_0^4 \underbrace{(4-x)}_{u} \underbrace{\sin \frac{k\pi x}{4}}_{v'} \, dx$$

$u' = -1$

$v = \frac{-4}{k\pi} \cos \frac{k\pi x}{4}.$

$$= \frac{1}{2} \left(\left[(4-x)\left(-\frac{4}{k\pi} \cos \frac{k\pi x}{4}\right) \right]_{x=0}^{x=4} - \frac{4}{k\pi} \int_0^4 \cos \frac{k\pi x}{4} \, dx \right)$$

$$= \frac{1}{2} \left[\frac{16}{k\pi} - \frac{16}{k^2\pi^2} \underbrace{\left[\sin \frac{k\pi x}{4} \right]_{x=0}^{x=4}}_{= 0} \right)$$

$$= \frac{8}{k\pi}$$

Thus

$$Sf(x) = \sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{2}$$

$$= \sum_{k=1}^{\infty} \frac{8}{k\pi} \cdot \sin \frac{k\pi x}{4}.$$

First three (non-zero) terms are

$$\frac{8}{\pi} \left\{ \sin \frac{\pi x}{4} + \frac{1}{2} \sin \frac{\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{4} \right\}.$$

We try particular solution

$$y_p = \sum_{k=1}^{\infty}{}' C_k \sin \frac{k\pi x}{4}.$$

Thus $y_p' = \sum_{k=1}^{\infty}{}' \frac{k\pi}{4} C_k \cos \frac{k\pi x}{4}$ and

$$y_p'' = \sum_{k=1}^{\infty}{}' -\left(\frac{k\pi}{4}\right)^2 C_k \sin \frac{k\pi x}{4}.$$

Substitution into our ODE yields

$$\cancel{2} \sum_{k=1}^{\infty}{}' \left[-2\left(\frac{k\pi}{4}\right)^2 C_k + 11\, C_k\right] \sin \frac{k\pi x}{4}$$

$$= \sum_{k=1}^{\infty}{}' \frac{8}{k\pi} \cdot \sin \frac{k\pi x}{4}.$$

Thus

$$C_k \left[-\frac{k^2\pi^2}{8} + 11\right] = \frac{8}{k\pi}$$

and so

$$C_k = \frac{\frac{8}{k\pi}}{-\frac{k^2\pi^2}{8} + 11}.$$

Finally

$$y_p = \sum_{k=1}^{\infty}{}' \left[\frac{\frac{8}{k\pi}}{-\frac{k^2\pi^2}{8} + 11}\right] \sin \frac{k\pi x}{4}.$$

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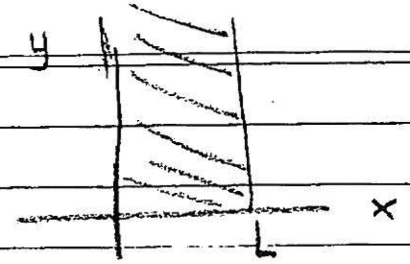
4)



let $u = X(x)Y(y)$

$$X''Y + XY'' = 0 \quad 2$$

$$\frac{X''}{X} = \frac{-Y''}{Y} = \lambda, \quad 2 \text{ constant}$$

since LHS function of x, RHS of y.

$$u(0, y) = u(L, y) = 0 \Rightarrow X(0) = X(L) = 0 \quad (i) \quad 2$$

Solve $X'' - \lambda X = 0$ subject to (i)

Non-trivial solutions only for $\lambda = -k^2$, $k > 0$

$$\Rightarrow X(x) = A\cos kx + B\sin kx \quad 2$$

$$X(0) = A = 0 \quad 1$$

$$X(x) = B\sin kx$$

$$X(L) = 0 \Rightarrow kL = n\pi$$

$$k = \frac{n\pi}{L} \quad 1$$

$$\therefore X(x) = B_n \sin\frac{n\pi}{L}x$$

$$-Y'' = -k^2 Y \Rightarrow Y(y) = Ce^{ky} + De^{-ky} \quad 2$$

since u bounded as $y \to +\infty$, $C = 0$ 2

$$\therefore Y(y) = D_n e^{-\frac{n\pi y}{L}}$$

and $$u(x,y) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{L} e^{-\frac{n\pi y}{L}} \quad 2$$

$$u(x,0) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{L} = f(x) \quad 1$$

From Fourier series : $$E_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad 1$$