LECTURE 53
HEAT EQUATION WITH INSULATED ENDS (ADIABATIC)

The equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

is called the one-dimensional heat equation. It governs the heat flow across a homogenous bar where c is determined by the thermal properties of the bar.

The adiabatic boundary conditions

$$u_x(0,t) = 0, \quad u_x(L,t) = 0 \qquad \text{for all time } t$$

maintain insulated endpoints at $x = 0$ and $x = L$ so that there is no heat flow across the ends of the rod. Eventually all the heat will be evenly distributed across the bar.

Initial conditions take the form:

$$\text{initial temperature distribution} \qquad u(x,0) = f(x)$$

Solutions (eigenfunctions) are obtained via separation of variables and take the form of the constant function A_0 together with

$$u_n(x,t) = A_n \cos \frac{n\pi x}{L} e^{-\lambda_n^2 t} \quad \text{for} \quad n = 1, 2, 3, \cdots$$

where the λ_n (eigenvalues) are given by $\lambda_n = cn\pi/L$.

The general solution is the superposition of all the eigenfunctions and takes the form

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-\lambda_n^2 t}$$

The initial temperature distribution is used to calculate the A_n $'s$ and will require the use of Fourier series and half range expansions when the initial temperature distribution is non-sinusoidal. The steady state temperature is A_0.

Example 1 The temperature in a bar of length π satisfies the heat equation

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ is the temperature. The bar is insulated so that the flux of heat at each end is zero. Hence:

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0 \qquad \text{for all } t.$$

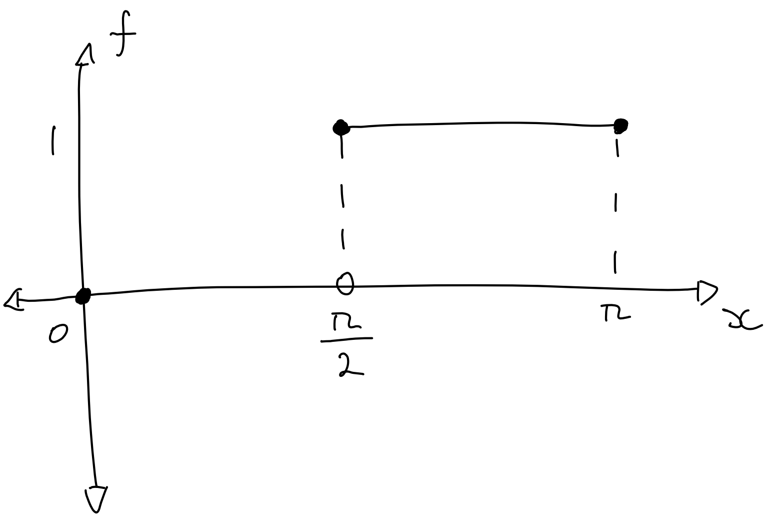
You are also given that the initial temperature distribution is given by

$$u(x,0) = f(x) = \begin{cases} 0 & 0 \le x < \frac{\pi}{2}; \\ 1 & \frac{\pi}{2} \le x \le \pi. \end{cases}$$

Express the general solution $u(x,t)$ as a Fourier cosine series and find the steady state temperature of the system.

DISCUSSION

A sketch of the initial temperature distribution is:



You need to be careful here, as $k = 0$ actually yields non-trivial solutions! Let us step carefully through the method of separation of variables.

i) By assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$F'' - kF = 0$$

and

$$G' - 9kG = 0$$

for k constant. (Note that the D.E. for G is first order!)

$$FG' = 9F''G$$

$$\frac{1}{9}\frac{G'}{G} = \frac{F''}{F} = k$$

$$\therefore \quad G' - 9kG = 0, \qquad F'' - kF = 0$$

ii) By implementing the boundary condition

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0 \qquad \text{for all } t.$$

show that

$$F'(0) = F'(\pi) = 0.$$

$$u_x(0,t) = F'(0)\,G(t) = 0$$

$$\therefore F'(0) = 0$$

$$u_x(\pi,t) = F'(\pi)\,G(t) = 0$$

$$\therefore F'(\pi) = 0$$

iii) Show that $k > 0$ (say $k = \rho^2$) yields the trivial solution for F.

<u>For $k = \rho^2 > 0$:</u> $F'' - \rho^2 F = 0$

$$\therefore F = \alpha_1 e^{\rho x} + \beta_1 e^{-\rho x}$$

$$F' = \alpha_1 \rho e^{\rho x} - \beta_1 \rho e^{-\rho x}$$

$F'(0) = \rho(\alpha_1 - \beta_1) = 0$ $\qquad$ $F'(\pi) = \alpha_1 \rho (e^{2\rho\pi} - 1) = 0$

$\therefore \alpha_1 = \beta_1, \ \rho \neq 0$ $\qquad$ but $\rho \neq 0$

$$\therefore \alpha_1 = \beta_1 = 0 \quad \text{for } F' = 0$$

iv) Show that $k = 0$ yields a constant solution for both F and G and hence that

$$u_0(x, t) = A_0.$$

<u>For $k = 0$:</u> $F'' = 0$

$$F' = \alpha_2 = 0$$

$$\therefore F = \beta_2 \text{ which is constant}$$

$$G' = 0$$

$$\therefore G = \gamma_2 \text{ which is constant}$$

$$\therefore u_0(x, t) = \beta_2 \cdot \gamma_2 = A_0$$

v) Show that $k < 0$ (say $k = -\rho^2$) yields the solution

$$u_n(x,t) = A_n \cos(nx) e^{-9n^2 t}.$$

For $k = -\rho^2 < 0$: $\quad F'' + kF = 0$

$$\therefore F = \alpha_3 \cos(\rho x) + \beta_3 \sin(\rho x)$$

$$F' = -\alpha_3 \rho \sin(\rho x) + \beta_3 \rho \cos(\rho x)$$

$F'(0) = \beta_3 \rho = 0$ $\qquad$ $F'(\pi) = -\beta_3 \rho = 0$

$\therefore \beta_3 = 0, \ \rho = n, \ n \in \mathbb{Z}$ $\qquad$ $\therefore \beta_3 = 0, \ \rho = n, \ n \in \mathbb{Z}$

$$\therefore F' = -\alpha_3 \rho \sin(\rho x)$$

$$\therefore F = A_n \cos(nx)$$

$G' - 9kG = 0$ $\qquad$ $e^{\int 9n^2 \, dt}$

$G' + 9n^2 G = 0$

$$G = e^{-9n^2 t} \int 0 \, dt$$

$$\therefore G = e^{-9n^2 t}$$

$$\therefore u_n(x,t) = A_n \cos(nx) e^{-9n^2 t}$$

Taking the sum of all possible solutions we have a general solution given by

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) e^{-9n^2 t}$$

We now apply the initial temperature distribution. This is tricky and will involve Fourier cosine series:

Recall that the initial temperature distribution is given by

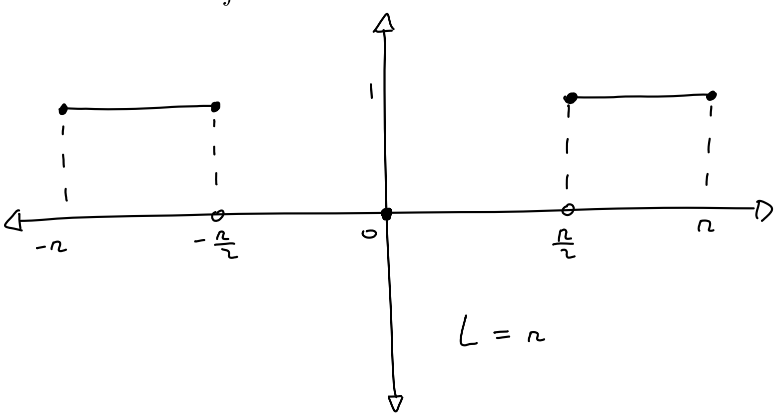
$$u(x,0) = f(x) = \begin{cases} 0 & 0 \le x < \dfrac{\pi}{2}; \\ \\ 1 & \dfrac{\pi}{2} \le x \le \pi. \end{cases}$$

Thus setting $t = 0$ we have

$$A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) = f(x).$$

This means that A_0 and A_n are the Fourier cosine coefficients of f.

The even periodic extension of f has sketch:



$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overset{\text{even}}{f(x)}\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \left(\int_0^{\frac{\pi}{2}} 0\, dx + \int_{\frac{\pi}{2}}^{\pi} 1\, dx \right)$$

$$= \frac{\pi\left(1 - \frac{1}{2}\right)}{\pi}$$

$$\therefore a_0 = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{f(x)\cos\left(\frac{n\pi x}{\pi}\right)}^{\text{even}}\, dx$$

$$= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} 0\, dx + \int_{\frac{\pi}{2}}^{\pi} \cos(nx)\, dx\right)$$

$$= \frac{2}{\pi}\left[\frac{\sin(nx)}{n}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{n\pi}\left(\sin(\pi x) - \sin\left(\frac{n\pi}{2}\right)\right)$$

$$\therefore a_n = \frac{-2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Our final solution is therefore

$$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos(nx)e^{-9n^2t} = \frac{1}{2} - \frac{2}{\pi}\left\{\cos(x)e^{-9t} - \frac{\cos(3x)}{3}e^{-81t} + \frac{\cos(5x)}{5}e^{-225t}\right\}$$

vi) Discuss the behaviour of the $u(x,t)$ as $t \to \infty$.

$$u(x, \infty) = \frac{1}{2}$$

[53] You can now do Q 120