LECTURE 13
GAUSSIAN ELIMINATION PART 1

> Any augmented matrix may be reduced to echelon form via the elementary row operations
>
> $$R_i = R_i \pm \alpha R_j \qquad \text{and} \qquad R_i \leftrightarrow R_j$$
>
> Once in echelon form the system may be solved via back-substitution.

We will now develop an extremely effective technique for solving linear equations called Gaussian Elimination, a truly wonderful algorithm which simply never fails to deliver.

We will only consider systems of **linear** equations, however we will allow any number of equations in any number of unknowns.

To motivate the study let us consider the problem of finding the point of intersection of 3 planes with Cartesian equations:

$$\begin{aligned} x + y + 3z &= 4 \quad \text{— plane} \qquad (1) \\ 2x + y + z &= 0 \quad \text{- plane} \qquad (2) \\ x + 3y - z &= 6 \quad \text{- plane.} \qquad (3) \end{aligned}$$

Discussion $\quad 0, 1, \infty$ solutions

A system of linear equations can have 0,1 or ∞ solutions. These are the only 3 options!

The first thing we do is dump the variables. They are just getting in the way and it doesn't really matter whether we have x, y and z or x_1, x_2 and x_3.

We do this my moving to what is called an augmented matrix $[A|\,b]$. A **matrix** is simply a rectangular block of numbers. (We will carefully examine matrices as abstract mathematical objects once we are finished with Gaussian Elimination). So we have have

$$[A|\,b] = \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 1 & 1 & 0 \\ 1 & 3 & -1 & 6 \end{array}\right) \longrightarrow \text{Echelon form.}$$

We then add multiples of one row to another ($R_i = R_i \pm \alpha R_j$) and if necessary swap rows ($R_i \leftrightarrow R_j$) to produce an equivalent system in what is called **Echelon form**. Echelon form is a specialised structure under which the system is particularly vulnerable to attack. The definition of Echelon form is

- Each successive row has more zeros injected from the left than the row above; and
- Complete rows of zeros sit at the bottom.

Example 1: The following augmented matrices are in echelon form:

a) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 8 & 2 \end{array}\right)$ b) $\left(\begin{array}{ccc|c} 5 & 0 & 0 & 14 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 8 & 7 \end{array}\right)$

c) $\left(\begin{array}{ccc|c} 0 & 5 & 1 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 5 \end{array}\right)$ d) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$

e) $\left(\begin{array}{ccccc|c} 3 & 5 & 1 & 0 & 2 & 4 \\ 0 & 0 & -1 & 8 & 1 & 6 \end{array}\right)$ f) $\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$

When a matrix is in echelon form the first non-zero entry in each row is called a leading (or pivot) element and the columns containing leading elements are called leading (or pivot) columns. It is OK for a leading element to be on the RHS.

Example 2: Circle the leading elements and indicate the leading columns in the augmented matrices in echelon form above.

Example 3: The following augmented matrices are **NOT** in echelon form:

a) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 3 & -1 & 6 \\ 0 & 2 & 8 & 0 \end{array}\right)$ b) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 8 & 0 \end{array}\right)$

c) $\left(\begin{array}{ccc|c} 3 & 0 & 0 & 4 \\ 7 & 6 & 0 & 6 \\ 1 & 6 & 2 & 5 \end{array}\right)$ d) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 1 & 0 & 0 & 0 \end{array}\right)$

A matrix in echelon form has very little structure in its base and it is this weakness that will be eventually exploited.

Example 4: Use elementary row operations to reduce our original system

$[A|\,b] = \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 1 & 1 & 0 \\ 1 & 3 & -1 & 6 \end{array}\right)$ to echelon form.

The method of reduction to echelon form (Gaussian elimination) boils down to using the diagonal elements to kill off all entries below the diagonal. It is required that you write down the elementary row operations you are using at each stage of the process.

Watch this carefully!

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 1 & 1 & 0 \\ 1 & 3 & -1 & 6 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & -5 & -8 \\ 0 & 2 & -4 & 2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & -5 & -8 \\ 0 & 0 & -14 & -14 \end{array}\right)$$

$R_2 = R_2 - 2R_1$
$R_3 = R_3 - R_1$

$R_3 = R_3 + 2R_1$

We have reached the echelon form:

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & -5 & -8 \\ 0 & 0 & -14 & -14 \end{array}\right)$$

It is important to realize that the elementary row operations we have used have not changed the solution to the system! After all we are just adding and subtracting multiples of rows to other rows. We now attack **from below** to solve the system via a process known as back-substitution.

$-14z = -14 \rightarrow z = 1.$

$-y - 5z = -8$

$-y - 5(1) = -8$

$-y = -8 + 5 \rightarrow y = 3.$

$x + y + 3z = 4$

$x + 3 + 3 = 4 \rightarrow x = -2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$

this is the intersection of three planes

$$\bigstar \quad \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \text{ is the unique point of intersection of the 3 planes} \quad \bigstar$$

Example 5: Check that the above solution is correct.

$$-2+3+3(1)=4 \checkmark$$
$$2(-2)+3+1=0 \checkmark$$
$$-2+3(3)-1=6 \checkmark$$

You will soon see that Gaussian Elimination (that is reduction to echelon form followed by back-substitution) is absolutely and completely bullet-proof. If a unique solution exists it will find it, if infinite solutions exist it will find the lot, and if no solution exists you will be given a clear signal by the algorithm that there are dramas.

Furthermore the algorithm simply doesn't care how many equations there are and how many unknowns! 17 equations in 22 variables?..........easy!

It is important to understand that Gaussian Elimination is **the** central tool of linear algebra and we will use it in the future to solve some very abstract mathematical problems. Get to know it well. We will examine all these features in great detail over the next few lectures. We close with another simple example.

Example 6: Solve the system of linear equations by reducing to echelon form and back-substituting. Check that your solution is correct and interpret your answer geometrically.

$$\begin{aligned} 2x+y+z &= 14 \\ 2x \quad +2z &= 20 \\ 4x-y+z &= 18 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 14 \\ 2 & 0 & 2 & 20 \\ 4 & -1 & 1 & 18 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 1 & 14 \\ 0 & -1 & 1 & 6. \\ 0 & -3 & -1 & -10 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 1 & 14 \\ 0 & -1 & 1 & 6 \\ 0 & 0 & -4 & -28 \end{array}\right)$$

$R_3 = R_3 - 2R_1$

$R_2 = R_2 - R_1$.

$R_3 = R_3 - 3R_2$.

$-4z = -28$

$z = 7$

$-y + z = 6$

$-y + 7 = 6$

$y = 1$

$2x + 1 + 7 = 14$

$2x = 6$

$x = 3$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$$

★ $\begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$ is the unique point of intersection of the 3 planes ★

LECTURE 14
GAUSSIAN ELIMINATION PART 2

> Any Augmented matrix may be reduced to echelon form via the row operations
>
> $$R_i = R_i \pm \alpha R_j \qquad \text{and} \qquad R_i \leftrightarrow R_j$$
>
> We can pivot off γ above to kill ϵ below by using $R_i = R_i - \frac{\epsilon}{\gamma} R_j$.
>
> Once in echelon form the system may be solved via back-substitution.
>
> An inconsistent equation at any stage of the process indicates that there is no solution and you may stop. Else
>
> The presence of a non-leading column on the LHS of the echelon form indicates infinite solutions with the non-leading variables serving as parameters.

We saw in the last lecture how reduction to echelon form followed by back-substitution could be used to solve 3 equations in 3 unknowns. We will see today that the number of equations and unknowns is of no particular consequence and Gaussian elimination glides along regardless.

We will also carefully examine the signals which indicate no solution and infinite solutions and how infinite solutions are presented. The nature of solution is not apparent until echelon form has been achieved but it is then quite obvious how many solutions there are and what you should do to find them.

Be aware that Gaussian elimination will never fail to work! This is one of the great algorithms of mathematics.

Example 1: Solve the following system of linear equations and interpret the problem geometrically.

$$\begin{aligned} 3x - y &= 4 && (1)\\ x + y &= 8 && (2)\\ x - y &= -2 && (3)\\ 6x - 3y &= 3 && (4) \end{aligned}$$

Remember that our technique is to present the system as an augmented matrix $[A|\,b]$ and then use row operations $R_i = R_i \pm \alpha R_j$ and $R_i \leftrightarrow R_j$ to reduce the system to echelon form. Recall that we can pivot off γ above to kill ϵ below by using $R_i = R_i - \frac{\epsilon}{\gamma} R_j$.

We will solve this system two different ways. Both are acceptable however the first is a little ham-fisted whereas in the second we get a little sneaky in order to keep the calculations under control. Moving to the augmented matrix we have:

$$\left(\begin{array}{cc|c} 3 & -1 & 4 \\ 1 & 1 & 8 \\ 1 & -1 & -2 \\ 6 & -3 & 3 \end{array}\right)$$

Method 1 (Blast away)

$$\begin{array}{l} \cancel{R_2 = R_2 - \tfrac{1}{3}R_1} \\ \cancel{R_3 = R_3 - \tfrac{1}{3}R_1} \\ \cancel{R_4 = R_4 - 2R_1} \end{array} \quad \begin{array}{c}\longrightarrow \\ \longrightarrow \\ \longrightarrow\end{array} \left(\begin{array}{cc|c} 3 & -1 & 4 \\ 0 & \frac{4}{3} & \frac{20}{3} \\ 0 & -\frac{2}{3} & -\frac{10}{3} \\ 0 & -1 & -5 \end{array}\right) \qquad \begin{array}{c}\longrightarrow \\ \longrightarrow\end{array} \left(\begin{array}{cc|c} 3 & -1 & 4 \\ 0 & \frac{4}{3} & \frac{20}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Method 2 Row swap first $R_1 \leftrightarrow R_2$

$$\left(\begin{array}{cc|c} 1 & 1 & 8 \\ 3 & -1 & 4 \\ 1 & -1 & -2 \\ 6 & -3 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & -4 & -20 \\ 0 & -2 & -10 \\ 0 & -9 & -45 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & 1 & 5 \\ 0 & -2 & -10 \\ 0 & -9 & -45 \end{array}\right)$$

$R_2 = R_2 - 3R_1$

$R_3 = R_3 - R_1$

$R_4 = R_4 - 6R_1$.

$R_2 = -\frac{1}{4}R_2$.

$R_3 = R_3 + 2R_1$

$R_4 = R_4 + 9R_1$

$$\left(\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

★ 4 lines in $\mathbb{R}^2$ intersecting at a common point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ ★

$\boxed{y = 5}$

$x + y = 8$

$x + 5 = 8$

$\boxed{x = 3}$

unique solution

Observe that it is always nicest to be pivoting off a 1!

Also note carefully from the above example that the presence of zero rows in the echelon form **DOES NOT** imply infinite solutions.

We have swapped rows to simplify our calculations. Sometimes we have no choice but to swap rows if we want echelon form.

Example 2: Solve the following system of linear equations.

$$\begin{aligned} x+y+z &= 6 \\ x+y+3z &= 14 \\ 2x+3y+4z &= 23 \\ -x+2y+z &= 11 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 1 & 3 & 14 \\ 2 & 3 & 4 & 23 \\ -1 & 2 & 1 & 11 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 1 & 2 & 11 \\ 0 & 3 & 2 & 17 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & 2 & 8 \\ 0 & 3 & 2 & 17 \end{array}\right)$$

$R_2 = R_2 - R_1$

$R_3 = R_3 - 2R_1$

$R_4 = R_4 + R_1$

$R_2 \longleftrightarrow R_3$

$R_4 = R_4 - 3R_2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & -4 & -16 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$R_4 = R_4 + 2R_2$

$2z = 8 \rightarrow z = 4$

$y + 2z = 11 \rightarrow y + 8 = 11 \rightarrow y = 3$

$x + y + z = 6 \rightarrow x + 3 + 4 = 6 \rightarrow x = -1$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$$

★ $\begin{pmatrix} x \\ y \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$. This is 4 planes intersecting at a common point in $\mathbb{R}^3$ ★

Instead of reducing to echelon form and back-substituting it is also possible to reduce much further to what is called **reduced** row echelon form where every leading element is 1 and only leading elements appear on the LHS of the echelon form. This is generally inefficient but it is true that the solution if unique will appear on the RHS.

Example 3: Carry on the above reduction till reduced echelon form is produced. Hence find the solution.

$R_3 = \frac{1}{2}R_3$ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_1 = R_1 - R_2]{} \left(\begin{array}{ccc|c} 0 & 0 & -1 & -5 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$R_1 = R_1 + R_2$

$R_2 = R_2 - 2R_3$

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Only do this ~~the~~ method if asked.

We turn now to the issue of systems which have no solution and systems which have an infinite number of solution. First note that it is not immediately obvious that we have trouble and that the true nature of the system is only apparent when we are looking at the echelon form.

The situation where there is no solution is simple. If at any stage you get zero equals something non-zero you can stop and claim no solution. If there is no solution the problem cannot hide forever and eventually the reduction will tease the issue out.

Example 4: Solve the following system of linear equations and interpret the problem geometrically.

$$\begin{aligned} 3x + 2y + 4z &= 1 \\ 5x - y + 3z &= 2 \\ 8x + y + 7z &= 4 \end{aligned}$$

We begin with a technical problem. The usual row operations we would use here are $R_2 = R_2 - \frac{5}{3}R_1$ and $R_3 = R_3 - \frac{8}{3}R_1$ (that is $R_i = R_i - \frac{kill}{use}R_j$). But all the resulting fractions are a disaster!

It is also OK to use $R_2 = 3R_2 - 5R_1$ and $R_3 = 3R_3 - 8R_1$. These row operations are a little dodgy and although they cause no problems here they will be inappropriate later on in the theory of determinants......but they are OK for Gaussian elimination and we tend to use them when there are no 1's to pivot off. So :

$$\left(\begin{array}{ccc|c} 3 & 2 & 4 & 1 \\ 5 & -1 & 3 & 2 \\ 8 & 1 & 7 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 3 & 2 & 4 & 1 \\ 0 & -13 & -11 & 1 \\ 0 & -13 & -11 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 3 & 2 & 4 & 1 \\ 0 & -13 & -11 & 1 \\ \hline 0 & 0 & 0 & 3 \end{array}\right)$$

$R_2 = 3R_2 - 5R_1$
$R_3 = 3R_3 - 8R_1$

$R_3 = R_3 - R_2$

$\underline{0 = 3}$
No Sol^n

★ No solution. Three planes in $\mathbb{R}^3$ with no common point. ★

The situation for infinite solutions is much trickier and very important for our later work. The signal for infinitely many solutions is the presence of non-leading columns on the LHS of the echelon form. The non-leading variables are assigned to be parameters before back-substituting.

Example 5: Solve the following system of linear equations and interpret the problem geometrically.

$$\begin{aligned} x + 3y + 5z &= 7 \\ x + 4y + 7z &= 11 \\ 2x + 7y + 12z &= 18 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 1 & 4 & 7 & 11 \\ 2 & 7 & 12 & 18 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 2 & 4 \end{array}\right) \longrightarrow \begin{array}{c} \begin{array}{ccc} \checkmark & \checkmark & \times \end{array} \\ \left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$R_2 = R_2 - R_1$

$R_3 = R_3 - 2R_1$

$R_3 = R_3 - R_2$

$0 = 0$

* Non pivot Column on LHS $\rightarrow \infty$ sol$^{\underline{n}}$

* Non-leading variable $\rightarrow$ paramete

Let $z = t$

$y + 2z = 4 \rightarrow y + 2t = 4 \rightarrow y = 4 - 2t$

$x + 3y + 5z = 7 \rightarrow x + 12 - 6t + 5t = 7 \rightarrow x = -5 + t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t$$

$$\star \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t \,;\ t \in \mathbb{R} \quad \star$$

★ Three planes in $\mathbb{R}^3$ intersecting along a common line. ★

Example 6: For the following systems in echelon form describe the nature of solution and identify which variables will need to be assigned as parameters when infinite solutions exist.

a) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 8 & 2 \end{array}\right)$

unique.

b) $\left(\begin{array}{ccc|c} 5 & 0 & 0 & 14 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 8 & 7 \end{array}\right)$

unique.

c) $\left(\begin{array}{ccc|c} 0 & 5 & 1 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 5 \end{array}\right)$

no solution.

d) $\begin{array}{c} \begin{array}{cccc} & x_2 & & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$

infinate $\quad x_2 = t$

e) $\begin{array}{c} \begin{array}{cccccc} & x_2 & & x_4 & x_5 & \end{array} \\ \left(\begin{array}{ccccc|c} 3 & 5 & 1 & 0 & 2 & 4 \\ 0 & 0 & -1 & 8 & 1 & 6 \end{array}\right) \end{array}$

Let $x_2 = \mu$

$x_4 = \lambda$

$x_5 = t$.

f) $\left(\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$

unique.

Recall that f) above was our first example today. Lots of zero rows but a **unique** solution since every column on the left leads !

Remember: !!! ZERO ROWS DO NOT IMPLY INFINITE SOLUTIONS !!!

★ a), b), f) unique solutions and c) no solution. ★

★ d) ∞ solutions with $x_2 = t$ e) ∞ solutions with $x_2 = \mu,\ x_4 = \lambda,\ x_5 = t$ ★

LECTURE 15

GAUSSIAN ELIMINATION PART 3 (INFINITE SOLUTIONS)

An inconsistent equation at any stage of reduction indicates that there is no solution and you may stop. Else

If every column on the LHS is a leading column then the solution is unique. Else

The presence of a non-leading column on the LHS of the echelon form indicates infinite solutions with the non-leading variables then serving as parameters.

If a system of equations $A\mathbf{x} = \mathbf{b}$ has infinite solutions then they will take the form
$$\mathbf{x} = \mathbf{x}_p + \lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2} + \cdots + \lambda_k\mathbf{v_k}$$

where

$\mathbf{x}_p$ is a particular solution of $A\mathbf{x} = \mathbf{b}$ and the vectors $\mathbf{v_k}$ are solutions to the homogeneous problem $A\mathbf{x} = \mathbf{0}$.

We have touched upon systems of equation with either no solution or infinite solutions.

Recall that if a ridiculous equation pops up (for eg. $\left[0\ 0\ 0\ 0\ \middle|\ 2\right]$ that is 0=2??) you may stop and claim no solution.

Otherwise infinite solutions are clearly signalled by the presence of a non-leading column on the LHS of the echelon form with the non-leading variables then serving as parameters. If infinite solutions exist you must find and describe them all.

Note that zero rows in the echelon form **DO NOT** imply infinite solutions.

Example 1: Find the line of intersection of the two planes

$3x + 5y + 8z = 1$ and $-7x - 11y - 18z = 3$.

$$\left(\begin{array}{ccc|c} 3 & 5 & 8 & 1 \\ -7 & -11 & -18 & 3 \end{array}\right).$$

Normally we would use the row operation $R_2 = R_2 + \frac{7}{3}R_1$. But the fractions are a bit scary so instead we use the slightly shonky $\underline{R_2 = 3R_2 + 7R_1}$

$$\begin{array}{cccc} \checkmark & \checkmark & \times & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 3 & 5 & 8 & 1 \\ 0 & 2 & 2. & 16 \end{array}\right)$$

Let $\boxed{z = t}$

$2y + 2t = 16 \Longrightarrow 2y = 16 - 2t \longrightarrow \boxed{y = 8 - t}$

$3x + 5(8-t) + 8t = 1 \longrightarrow 3x + 40 - 5t + 8t = 1$

$3x = -39 - 3t.$

$\boxed{x = -13 - t.}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -13 \\ 8 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} t.$$

$$\bigstar \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -13 \\ 8 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} t \; ; \; t \in \mathbb{R} \quad \bigstar$$

$\bigstar$ Two planes in $\mathbb{R}^3$ intersecting along a common line. $\bigstar$

Gaussian Elimination **ALWAYS WORKS**.

Example 2: Solve the system of equations $2x - 4y + z = 6$.

$$\begin{array}{cccc} & \times & \times & \\ \left(2\right. & -4 & 1 & \left.\middle|\, 6\right) \end{array}$$

Let $y = \mu$

$z = \lambda$

echelon form

$$2x - 4\mu + \lambda = 6$$

$$2x = 6 + 4\mu - \lambda \longrightarrow x = 3 + 2\mu - \tfrac{1}{2}\lambda$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\mu + \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}\lambda$$

★

Example 3: Solve the following system of linear equations.

$$\begin{aligned} 0x + 0y + 0z &= 0 \\ 0x + 0y + 0z &= 0 \\ 0x + 0y + 0z &= 0 \end{aligned}$$

$$\begin{array}{cccc} \times & \times & \times & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $x = \mu$, $y = \lambda$, $z = t$.

★

Let's take a look at a very messy problem:

Example 4: Solve the following system of equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 - x_5 &= 1 \\ 2x_1 + 2x_2 + 2x_4 - 6x_5 &= -4 \\ 6x_1 + 6x_2 + 4x_3 + 3x_4 - 10x_5 &= -3 \end{aligned}$$

This is 3 equations in 5 unknowns!

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -1 & 1 \\ 2 & 2 & 0 & 2 & -6 & -4 \\ 6 & 6 & 4 & 3 & -10 & -3 \end{array}\right)$$

$R_2 = R_2 - 2R_1$

$R_3 = R_3 - 6R_1$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -1 & 1 \\ 0 & 0 & -2 & 0 & -4 & -6 \\ 0 & 0 & -2 & -3 & -4 & -9 \end{array}\right)$$

$R_3 = R_3 - R_2$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -1 & 1 \\ 0 & 0 & -2 & 0 & -4 & -6 \\ 0 & 0 & 0 & -3 & 0 & -3 \end{array}\right)$$

Let $x_2 = \mu$

$x_5 = \lambda$

$-3x_4 = -3 \rightarrow \boxed{x_4 = 1}$

$-2x_3 - 4x_5 = -6$

$-2x_3 - 4\lambda = -6 \rightarrow -2x_3 = -6 + 4\lambda$

$\boxed{x_3 = 3 - 2\lambda}$

$x_1 + \mu + 3 - 2\lambda + 1 - \lambda = 1$

$\boxed{x_1 = -3 + 3\lambda - \mu}$

$$\bigstar \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mu + \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \lambda \; ; \; \mu, \lambda \in \mathbb{R} \quad \bigstar$$

Let us use the previous example to introduce some new notation.

We have been writing systems in the augmented matrix format $[A|\,\mathbf{b}] = \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -1 & 1 \\ 2 & 2 & 0 & 2 & -6 & -4 \\ 6 & 6 & 4 & 3 & -10 & -3 \end{array}\right)$

We can also use the matrix vector format $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ 2 & 2 & 0 & 2 & -6 \\ 6 & 6 & 4 & 3 & -10 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix}$$

A is called the coefficient matrix, $\mathbf{x}$ is the unknown vector and $\mathbf{b}$ is the RHS.

A lot of the systems we have been solving have infinite solutions. These infinite solutions always have the following structure.

FACT: If a system of equations $A\mathbf{x} = \mathbf{b}$ has infinite solutions then they will take the form

$$\mathbf{x} = \mathbf{x}_p + \lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2} + \cdots + \lambda_k\mathbf{v_k}$$

where $\mathbf{x}_p$ is a particular solution of $A\mathbf{x} = \mathbf{b}$, the λ_i's are scalars and the vectors $\mathbf{v_i}$ are solutions to the homogeneous problem $A\mathbf{x} = \mathbf{0}$.

In the above example $\mathbf{x}_p = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}$ solves $A\mathbf{x} = \mathbf{b}$.

$-3+0+3+1-0 = 1$ ✓

It is easily checked that

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 - x_5 &= 1 \\ 2x_1 + 2x_2 + 2x_4 - 6x_5 &= -4 \\ 6x_1 + 6x_2 + 4x_3 + 3x_4 - 10x_5 &= -3 \end{aligned}$$

$-1+1+0+0-0 = 0$ ✓

$-2+2+0-0 = 0$ ✓

$-6+6+0+6-0 = 0$ ✓

Also $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}$ solve the homogeneous problem $A\mathbf{x} = \mathbf{0}$. That is they satisfy:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 - x_5 &= 0 \\ 2x_1 + 2x_2 + 2x_4 - 6x_5 &= 0 \\ 6x_1 + 6x_2 + 4x_3 + 3x_4 - 10x_5 &= 0 \end{aligned}$$

Observe how $\mathbf{x}_p$ works hard to produce the RHS $\mathbf{b}$ while $\mathbf{v}_1$ and $\mathbf{v}_2$ quietly operate in the background by doing nothing. Brought together, $\mathbf{x} = \mathbf{x}_p + \lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2}$ provides the most general solution to $A\mathbf{x} = \mathbf{b}$.

Example 5:

a) Solve the following system of equations

$$\begin{aligned} x + 3y + 5z &= 7 \\ x + 4y + 7z &= 11 \\ 2x + 7y + 12z &= 18 \end{aligned}$$

b) Hence find a solution to the homogeneous problem

$$\begin{aligned} x + 3y + 5z &= 0 \\ x + 4y + 7z &= 0 \\ 2x + 7y + 12z &= 0 \end{aligned}$$

(a)

$$\left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 1 & 4 & 7 & 11 \\ 2 & 7 & 12 & 18 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 2 & 4 \end{array}\right) \rightarrow \begin{array}{c} \checkmark \quad\ \ \checkmark \quad\ \ \times \qquad \\ \left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$R_2 = R_2 - R_1$

$R_3 = R_3 - 2R_1$

$\boxed{\text{Let } z = t}$

$y + 2t = 4 \rightarrow \boxed{y = 4 - 2t}$

$x + 3(4-2t) + 5t = 7 \rightarrow x + 12 - 6t + 5t = 7$

$\boxed{x = -5 + t}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t$$

(b)

$$x + 3y + 5z = 0$$
$$x + 4y + 7z = 0$$
$$2x + 7y + 12z = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t$$

a) ★ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t \; ; \; t \in \mathbb{R}$ *b)* $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t \; ; \; t \in \mathbb{R}$ ★

Why does this work?? Consider $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 0 \end{pmatrix}$. This is a particular solution to the system and it is easily checked that these values of x,y and z satisfy all three equations in a) above. Now $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is a solution to the homogeneous system in b). It's job is to do nothing! Adding the two vectors we get $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}$ which remarkably is also a (new) particular solution to a). Check it! The reason that the new vector $\begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}$ is also a solution to a) is precisely because the vector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ does nothing!

LECTURE 16
GAUSSIAN ELIMINATION APPLICATIONS

> When setting up a modeling Gaussian Elimination question make sure that all your variables are declared.
> Remember that systems are most likely to have unique solutions but can also have no solution (inconsistent equation) or infinite solutions (non-leading columns).

In this final lecture on Gaussian Elimination we will look at a host of applications. Keep in mind that you should apply the reduction algorithm strictly and don't wander off into your own methods of solution.

Example 1: For the following system of equations, find conditions on the RHS vector $\mathbf{b}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ which ensure that the system is consistent (that is has a solution).

$$\begin{aligned} x+3y+5z &= b_1 \\ x+4y+7z &= b_2 \\ 2x+7y+12z &= b_3 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 5 & b_1 \\ 1 & 4 & 7 & b_2 \\ 2 & 7 & 12 & b_3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 5 & b_1 \\ 0 & 1 & 2 & b_2-b_1 \\ 0 & 1 & 2. & b_3-2b_1 \end{array}\right)$$

$R_2 = R_2 - R_1.$

$R_3 = R_3 - 2R_1$

$\downarrow$ $R_3 = R_3 - R_2$

$$\left(\begin{array}{ccc|c} 1 & 3 & 5 & b_1 \\ 0 & 1. & 2 & b_2-b_1 \\ 0 & 0 & 0 & b_3-b_1-b_2 \end{array}\right)$$

$\text{Sol}^{\underline{n}}$ exists $\rightarrow$ $b_3 - b_1 - b_2 = 0$

★ $b_3 = b_1 + b_2$ ★

Example 2: For the following system of equations determine which values of λ (if any) will yield: a) no solution b) infinite solutions c) a unique solution.

$$\begin{aligned} x+y+z &= 4 \\ x+\lambda y+2z &= 5 \\ 2x+(\lambda+1)y+(\lambda^2-1)z &= \lambda+7 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & \lambda & 2 & 5 \\ 2 & 1+\lambda & \lambda^2-1 & \lambda+7 \end{array}\right)$$

$R_2 = R_2 - R_1$
$R_3 = R_3 - 2R_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & \lambda-1 & 1 & 1 \\ 0 & \lambda-1 & \lambda^2-3 & \lambda-1 \end{array}\right)$$

$R_3 = R_3 - R_2$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & \lambda-1 & 1 & 1 \\ 0 & 0 & \lambda^2-4 & \lambda-2 \end{array}\right)$$

$(\lambda+2)(\lambda-2)$

∞ sol^{ns} : $\lambda = 2$.

no sol^n : $\lambda = -2$.

Bottom row $0\ 0\ 0\ -4$.

$\underline{\lambda = 1}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -1 \end{array}\right)$$

$R_3 = R_3 + 3R_2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

no sol^n

★ *a*) No solution $\lambda = -2, 1$ *b*) $\lambda = 2$, ∞ solutions *c*) All other λ, unique solution ★

Example 3: Consider the following system of equations:

$$\begin{aligned} x+2y+z &= 8 \\ 4x+3y-z &= 7 \\ 3x+y-2z &= -1 \end{aligned}$$

a) Find the general solution.

b) Given that x, y and z must all be non-negative find the maximum value of y.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 4 & 3 & -1 & 7 \\ 3 & 1 & -2 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -5 & -5 & -25 \\ 0 & -5 & -5 & -25 \end{array}\right)$$

$R_2 = R_2 - 4R_1$.

$R_3 = R_3 - 3R_1$

$R_3 = R_3 - R_2$.

$$\downarrow$$

$$\begin{array}{cccc} \checkmark & \checkmark & \times & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -5 & -5 & -25 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = t$.

$-5y - 5t = -25$

$5y = 25 - 5t$.

$\boxed{y = 5 - t.}$

$x + 2(5-t) + t = 8$

$x + 10 - 2t + t = 8$

$\boxed{x = t - 2}$

★ $a)\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} t\ ;\ t \in \mathbb{R}$ $\quad b)\ y = 3$ ★

$x \geq 0 \rightarrow -2 + t \geq 0 \longrightarrow t \geq 2$

$y \geq 0 \rightarrow 5 - t \geq 0 \rightarrow t \leq 5$

$z > 0 \rightarrow t \geq 0$

3

$2 \leq t \leq 5$

$y = 5 - t$

Max $y = 3$.

Exam type Q.

Example 4: A forest contains three different types of gum trees: The red gum, the blue gum and the swamp gum. Each red gum houses 1 koala bear and 3 bandicoots. Each blue gum houses 2 koala bears and 1 bandicoot and each swamp gum houses 3 koala bears and 2 bandicoots. Conservationists are planning to occupy the forest to protect it from logging. They have assigned one person to each red gum, two people to each blue gum and 4 people to each swamp gum. Given that the forest holds a total of 1700 koala bears, 1200 bandicoots and 1900 conservationists set up and solve a system of linear equations to determine how many of each type of tree the forest contains.

R 1K, 3B, 1C
B 2K, 1B, 2C
S 3K, 2B; 4C

1700 K
1200 B.
1900 C

Let x = number of red Gums

y = " " blue "

z = " " Swamp "

$x + 2y + 3z = 1700$

$3x + y + 2z = 1200$

$x + 2y + 4z = 1900$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1700 \\ 3 & 1 & 2 & 1200 \\ 1 & 2 & 4 & 1900 \end{array}\right)$$

$R_2 = R_2 - 3R_1$

$R_3 = R_3 - R_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1700 \\ 0 & -5 & -7 & -3900 \\ 0 & 0 & 1 & 200 \end{array}\right)$$

$\boxed{z = 200}$

$-5y - 1400 = -3900$

$-5y = -2500$

$\boxed{y = 500}$

$x + 1000 + 600 = 1700$

$\boxed{x = 100}$

★ 100 red gums, 500 blue gums and 200 swamp gums. ★

Let us finish this topic by returning to our original motivating question.

Example 5: Determine where the line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} t\ ;\ \ t \in \mathbb{R}$$

meets the plane

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \mu + \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \lambda\ ;\quad \mu, \lambda \in \mathbb{R}.$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} t - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \mu - \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \lambda = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 0 & -3 & 4 & -10 \\ 0 & -3 & 8 & -14 \end{array}\right)$$

$R_2 = R_2 - 2R_1$

$R_3 = R_3 - 3R_1$

$R_3 = R_3 - R_2.$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 0 & -3 & 4 & -10 \\ 0 & 0 & 4 & -4 \end{array}\right)$$

$4\lambda = -4 \to \boxed{\lambda = -1}$

$-3\mu - 4 = -10$

$-3\mu = -6$

$\boxed{\mu = 2}$

$\star \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \star$

$t + 2 + 2 = 5 \to \boxed{t = 1}$

$t = 1$

$\mu = 2$

$\lambda = -1$

$$\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

LECTURE 17
MATRIX OPERATIONS

> An $m \times n$ matrix is a rectangular array of real numbers with m rows and n columns.
>
> Matrices are added and subtracted using simple pointwise operations.
>
> If A is $m \times n$ and B is $p \times q$ then the product AB exists if and only if $n = p$ and the resulting matrix is $m \times q$.
>
> In general $AB \neq BA$

A matrix is simply a mathematical object comprising a rectangular block of real numbers.

Example 1: Examples of matrices are

$$\text{a)} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 7 & 8 \\ 3 & 5 & 9 \end{pmatrix} \quad b) \begin{pmatrix} \frac{1}{3} & \frac{3}{11} \\ -\frac{1}{4} & -\frac{5}{9} \\ \frac{2}{7} & \frac{7}{11} \end{pmatrix} \quad \text{and} \quad \text{c)} \begin{pmatrix} \pi & -1 \\ \sqrt{2} & 3\pi \end{pmatrix}$$

3×3 3×2 2×2.

Observe that the matrix can be any rectangular shape and that the numbers (called entries) can be any real number. Later on we will also look at complex matrices where the entries are complex numbers!

We classify matrices according to their size (also called dimension) with a matrix of size $m \times n$ having m rows and n columns. We always specify rows first. Thus the above three matrices are of size 3×3, 3×2 and 2×2.

Note that if a matrix has as many rows as columns (and hence is $n \times n$) we call it a square matrix.

We denote the set all all real $m \times n$ matrices by $M_{mn}(\mathbb{R})$.

Note that the vector $\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$ can also be viewed as a 3×1 matrix and all vectors (row or column) can be interpreted as matrices.

We denote by $[A]_{ij}$ the ijth entry of A.

Example 2: If $A = \begin{pmatrix} 4 & 6 & -2 & 7 \\ -4 & -1 & 3 & -9 \\ -5 & 9 & -3 & -7 \\ 1 & 8 & -6 & 5 \end{pmatrix}$ then A is a 4×4 matrix with

$[A]_{24} = -9$, $[A]_{13} = -2$ and $[A]_{41} = 1$

We sometimes also write $[A]_{ij}$ as a_{ij}.

Matrices are useful as a notational tool (for example we made great use of $[A|b]$ in Gaussian Elimination) and also as abstract mathematical objects.

We would like to be able to manipulate matrices just as if they were numbers! The operations of addition, subtraction and scalar multiplication are trivial.

Example 3: Let $A = \underset{}{\overset{3\times 2}{\begin{pmatrix} 2 & 3 \\ 1 & -2 \\ 0 & 4 \end{pmatrix}}}$, $B = \overset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 0 \end{pmatrix}}$ and $C = \overset{3\times 2}{\begin{pmatrix} 1 & 3 \\ 3 & 0 \\ 2 & 5 \end{pmatrix}}$.

Find (if possible) $A+C$, $A+B$, $4B$ and $2A-C$.

$$A + C = \begin{pmatrix} 2 & 3 \\ 1 & -2 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 3 & 0 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 4 & -2 \\ 2 & 9 \end{pmatrix}$$

$$4B = 4\begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 12 \\ -4 & 20 & 0 \end{pmatrix}$$

$$A+B = \underset{3\times 2}{\begin{pmatrix} 2 & 3 \\ 1 & -2 \\ 0 & 4 \end{pmatrix}} + \underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 0 \end{pmatrix}}$$

* is undefined
not compatible

$$2A - C$$

$$2\begin{pmatrix} 2 & 3 \\ 1 & -2 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 3 & 0 \\ 2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 6 \\ 2 & -4 \\ 0 & 8 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 3 & 0 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -1 & -4 \\ -2 & 3 \end{pmatrix}$$

★ $A + C = \begin{pmatrix} 3 & 6 \\ 4 & -2 \\ 2 & 9 \end{pmatrix}$ $\quad 4B = \begin{pmatrix} 4 & 8 & 12 \\ -4 & 20 & 0 \end{pmatrix}$ $\quad 2A - C = \begin{pmatrix} 3 & 3 \\ -1 & -4 \\ -2 & 3 \end{pmatrix}$

$A + B$ is undefined ★

Observe that matrices can only be added or subtracted when they are of the exactly the same size.

35+11

The definition of matrix addition and subtraction is easy to implement. The definition of matrix multiplication is however quite bizarre and not what you would expect. We will see in the second session why this definition is needed (and in fact quite natural). Let us take a look at a simple example.

Example 4: Let $A = \begin{pmatrix} 7 & 2 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 5 \\ 6 & 8 \end{pmatrix}$. Find AB.

$$\begin{pmatrix} 7 & 2 \\ 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 33 & 51 \\ 27 & 37 \end{pmatrix}$$

$7\times3+2\times6$ $\quad$ $7\times5+2\times8$

$1\times3+4\times6$ $\quad$ $1\times5+4\times8$

$BA =$

$$\begin{pmatrix} 3 & 5 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 26 & 26 \\ 50 & 44 \end{pmatrix}$$

What about BA?? Have a go now:

$$BA = \begin{pmatrix} 3 & 5 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 26 & 26 \\ 50 & 44 \end{pmatrix}$$

$\star$ $AB = \begin{pmatrix} 33 & 51 \\ 27 & 37 \end{pmatrix}$, $BA = \begin{pmatrix} 26 & 26 \\ 50 & 44 \end{pmatrix}$ $\star$

!!Observe that $AB \neq BA$!!

We say that matrix multiplication is non commutative.

The formal definition of matrix multiplication may be found in your notes and also at the end of this lecture. Generally if you want the $(ij)^{th}$ entry of AB you take the i^{th} row of A and the j^{th} column of B and run along them both multiplying and adding as you go.

1412. 6 10 16

14 24

A remarkable feature of matrix multiplication is that matrices of different sizes can sometimes be multiplied together!

Example 5: Find AB if $A = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & 7 \\ 4 & 8 \end{pmatrix}$.

A is a 2×3 matrix and B is 3×2. Strangely (more on this soon) AB is 2×2.

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 7 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 17 & 38 \\ 32 & 67 \end{pmatrix}$$

2×3 $\quad$ 3×2

35 22 $\quad$ 14 + 24

$$\star \quad AB = \begin{pmatrix} 17 & 38 \\ 32 & 67 \end{pmatrix} \quad \star$$

Note that if A is $m \times n$ and B is $p \times q$ then AB exists if and only if $n = p$ and the resulting matrix is $m \times q$.

Example 6: Find BA for the matrices of the previous question.

$$BA = \begin{pmatrix} 1 & 0 \\ 2 & 7 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 44 & 39 & 34 \\ 52 & 48 & 44 \end{pmatrix}$$

3×2 $\quad$ 2×3

$\rightarrow (3 \times 3)$

$\star$

Observe that in general $AB \neq BA$!! That is matrix multiplication is non-commutative. In fact AB and BA can even be of different sizes.

Example 7: Let $C = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ -1 & 0 & -2 \end{pmatrix}$ and $D = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$. Find (if possible) CD and DC

3×3 $\quad$ 3×1

$$CD \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ -1 & 0 & -2 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} \longrightarrow \begin{pmatrix} -14 \\ -4 \\ 7 \end{pmatrix}$$

$-1-4+9$

$-1-3$

$1+6$

$3\times 3 \quad 3\times 1$

$\rightarrow 3\times 1$

$$DC \quad \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ -1 & 0 & -2 \end{pmatrix}$$

$3\times 1 \quad 3\times 3$

Does not exist

Algarith doesn't work.

$\star$ $CD = \begin{pmatrix} -14 \\ -4 \\ 7 \end{pmatrix}$ $\qquad$ DC does not exist $\star$

We have two special square matrices, the identity matrix I and the zero matrix 0. These are

$$2\times 2: \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \quad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$3\times 3: \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \quad 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

These serve as the 'one' and the 'zero' of matrix theory.

Example 8: Let $A = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$. Find AI, IA, $A0$ and $0A$ where the identity and zero matrices are of the appropriate size.

AI $\begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$

IA $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$

★ $AI = IA = A, \quad A0 = 0A = 0$ ★

Example 9: Let $D = \begin{pmatrix} 3 & 5 & 7 \\ 2 & 4 & 8 \end{pmatrix}$. Show that $DI = ID = D$.

$$\underset{2\times 2}{\begin{pmatrix} I \end{pmatrix}}\underset{2\times 3}{\begin{pmatrix} 3 & 5 & 7 \\ 2 & 4 & 8 \end{pmatrix}}\underset{3\times 3}{\begin{pmatrix} I \end{pmatrix}}$$

★

eg $\quad A \quad B \quad C$

$(3\times4)\,(4\times7)\,(7\times8)$

result would be

3×8

Properties of Matrix Multiplication

Suppose that$A$ and B are matrices and that the relevant products exist. Then:

i) $A(BC) = (AB)C$ (associativity)

ii) $A(B+C) = AB + AC$ (distributivity)

iii) $A(\lambda B) = \lambda AB$ for $\lambda \in \mathbb{R}$

iv) $AI = IA = A$

v) In general $AB \neq BA$

vi) Is it true that $(A+B)^2 = A^2 + 2AB + B^2$??

The proofs of these results come down to the formal definition of matrix multiplication and may be found in your lecture notes.

We have been multiplying matrices algorithmically however a formal definition does exist:

Definition If A is an $m \times n$ matrix and B is a $n \times q$ matrix then

$$\boxed{[AB]_{ij} = \sum_{k=1}^{n} [A]_{ik}[B]_{kj}}$$

formal definition

Example 10: If $A = \underset{}{\overset{2\times3}{\begin{pmatrix} * & * & * \\ 4 & 6 & 9 \end{pmatrix}}}$ and $B = \underset{3\times2}{\begin{pmatrix} 3 & * \\ 5 & * \\ 8 & * \end{pmatrix}}$ find $[AB]_{21}$ using the formal definition of matrix multiplication.

$$[AB]_{21} = \sum_{K=1}^{3} [A]_{2K}[B]_{K}$$

$K = 1$ to n

$i=2$
$j=1$

$$= A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31}$$

$$= 4(3) + 6\times5 + 9\times8$$

★

We have not mentioned matrix division! This is because it doesn't work properly and the closest we can come to division is the concept of an inverse, which we leave to the next lecture.

LECTURE 18
TRANSPOSES, INVERSES AND DETERMINANTS

If A is $m \times n$ and B is $p \times q$ then the product AB exists if and only if $n = p$ and the resulting matrix is $m \times q$.

In general $AB \neq BA$.

A^T (the transpose of A) is quite simply the matrix whose rows are the columns of A (and vice versa).

If A is $m \times n$ then A^T is $n \times m$.

$(AB)^T = B^T A^T$.

A matrix is said to be **symmetric** if $A = A^T$.

Given a square $n \times n$ matrix A the inverse of A (denoted by A^{-1}) is another $n \times n$ matrix with the property that $AA^{-1} = I$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A^{-1} exists if and only if $\det(A) \neq 0$.

In the last lecture we spent some time defining matrix multiplication. We start with some harder examples on multiplication and follow with the definition and properties of matrix transposes.

Common Exam, test Q

Example 1: Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix}$. Show that $A^2 = 2A + 5I$ and hence express A^4 as a linear combination of A and I.

LHS $= A^2 = \begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 2 \\ 10 & 5 \end{pmatrix}$

RHS $= 2A + 5I = \begin{pmatrix} 4 & 2 \\ 10 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 9 & 2 \\ 10 & 5 \end{pmatrix} =$ LHS

★ $A^4 = 28A + 45I$ ★

$A^4 = A^2A^2 = (2A+5I)(2A+5I)$

$= 4A^2 + 10AI + 10IA + 25I^2 \rightarrow = 4A^2 + 10A + 10A + 25I$

$= 4A^2 + 20A + 25I$

$4\{2A+5I\} + 20A + 25I$

$= 8A + 20I + 20A + 25I$

$= 28A + 45I$

Example 2: Let $B = \begin{pmatrix} 3 & 1 & -5 & 4 \\ 8 & 7 & 2 & 2 \\ 9 & -1 & -2 & -7 \end{pmatrix}$.

a) Find a vector $\mathbf{v}$ such that $B\mathbf{v}$ is the third column of B.

b) Find a vector $\mathbf{w}$ such that $\mathbf{w}B$ is the second row of B.

(a) This is done by inspection.

$$B = \underset{(3\times 4)}{\begin{pmatrix} 3 & 1 & -5 & 4 \\ 8 & 7 & 2 & 2 \\ 9 & -1 & -2 & -7 \end{pmatrix}} \underset{(4\times 1)}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}} = \begin{pmatrix} -5 \\ 2 \\ -2 \end{pmatrix}$$

(b)

$$\underset{(1\times 3)}{\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}} \underset{(3\times 4)}{\begin{pmatrix} 3 & 1 & -5 & 4 \\ 8 & 7 & 2 & 2 \\ 9 & -1 & -2 & -7 \end{pmatrix}} = \underset{(1,4)}{\begin{pmatrix} 8 & 7 & 2 & 2 \end{pmatrix}}$$

★ *a)* $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ **b)** $\mathbf{w} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$ ★

Definition: The transpose A^T of a matrix A is defined as $[A^T]_{ij} = [A]_{ji}$.

A^T is quite simply the matrix whose rows are the columns of A (and vice versa).

Example 3: Find A^T if:

a) $A = \underset{2\times 3}{\begin{pmatrix} 3 & 1 & -2 \\ 4 & 7 & 8 \end{pmatrix}} \longrightarrow A^T = \underset{3\times 2}{\begin{pmatrix} 3 & 4 \\ 1 & 7 \\ -2 & 8 \end{pmatrix}}$

b) $A = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \longrightarrow A^T = \begin{pmatrix} 3 & 6 & 9 \end{pmatrix}$

★

Fact: If A is $m \times n$ then A^T is $n \times m$.

Fact: $(AB)^T = B^T A^T$

Definition: A matrix A is said to be **symmetric** if $A = A^T$.

The following are examples of symmetric matrices:

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 7 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 6 & 8 \\ 6 & -5 & -7 \\ 8 & -7 & 2 \end{pmatrix}$$

For a symmetric matrix the first row is the same as the first column, the second row is the same as the second column... etc.

Example 4: Show that the matrix C given by $C = A^T A$ is always symmetric regardless of the nature of A. Find C when A is the matrix of example 3a).

Let $A \equiv$ random, $C = A^T A$

Prove C is symmetric

Pf//

$$C^T = (A^T A)^T = A^T {A^T}^T = A^T A = C.$$

C is symmetric

$$A = \begin{pmatrix} 3 & 1 & -2 \\ 4 & 7 & 8 \end{pmatrix}$$

$$C = A^T A = \underbrace{\begin{pmatrix} 3 & 4 \\ 1 & 7 \\ -2 & 8 \end{pmatrix}}_{3\times 2} \underbrace{\begin{pmatrix} 3 & 1 & -2 \\ 4 & 7 & 8 \end{pmatrix}}_{2\times 3}$$

$$= \begin{pmatrix} 25 & 31 & 26 \\ 31 & 50 & 54 \\ 26 & 54 & 68 \end{pmatrix}$$

★

$3\left(\frac{1}{3}\right) = 1$

MATRIX INVERSES

We turn now to the problem of matrix division. This is poorly defined for matrices and the best we can do is the concept of an inverse. We never even speak of matrix division! The trouble with the inverse is that for some matrices it does not exist and hence for those matrices division is impossible.

Definition: Given a square $n \times n$ matrix A the inverse of A (denoted by A^{-1}) is another $n \times n$ matrix with the property that $AA^{-1} = I$.

The problem with A^{-1} is that it may or may not exist and even if it does exist it can be very hard to find. If it exists we can use A^{-1} as a tool for pseudo-division by A. It is easy to find 2×2 inverses but 3×3 is tough. Note that inverses only apply to square matrices.

In the 2×2 case we have a lovely little formula for the job:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Proof:

$$\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\frac{1}{ad - bc} \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{I}.$$

Example 5: Find the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$ and check that your answer is correct. Hence solve the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} X = \begin{pmatrix} 5 & 7 & -1 \\ 19 & 21 & -5 \end{pmatrix} \text{ for } X.$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} \rightarrow A^{-1} = \frac{1}{1 \times 8 - 2 \times 3} \begin{pmatrix} 8 & -2 \\ -3 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 8 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

CHECK

$$\begin{pmatrix} 4 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$A^{-1}Ax = A^{-1}B$

Solve Matrix Equation $\rightarrow$ Multiply on left by inverse

$$\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} X = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}^{-1} \begin{pmatrix} 5 & 7 & -1 \\ 19 & 21 & -5 \end{pmatrix}$$

$\frac{1}{2} \begin{pmatrix} 8 & -2 \\ -3 & 1 \end{pmatrix}$

$$IX = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}^{-1} \begin{pmatrix} 5 & 7 & -1 \\ 19 & 21 & -5 \end{pmatrix}$$

$$X = \begin{pmatrix} 4 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 5 & 7 & -1 \\ 19 & 21 & -5 \end{pmatrix} \rightarrow = \begin{pmatrix} 1 & 7 & 1 \\ 2 & 0 & -1 \end{pmatrix}$$

$$\bigstar \quad X = \begin{pmatrix} 1 & 7 & 1 \\ 2 & 0 & -1 \end{pmatrix} \quad \bigstar$$

Observe in the above example that we have effectively divided both sides of the equation by $\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$.

Note also that if $ad - bc = 0$ then the matrix has no inverse and we say it is non-invertible (or singular).

DETERMINANTS

The quantity $ad - bc = 0$ is so important to us that we give it a special name (the *determinant*) and denote it by $\det(A)$ or $|A|$.

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

It is clearly true from the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

that for 2×2 matrices, A^{-1} exists if and only if $\det(A) \neq 0$

Example 6: Find the determinant of each of the following matrices and hence determine whether they are invertible. Find the inverse if it exists and check your answer by evaluating AA^{-1}

a) $\begin{pmatrix} 3 & 9 \\ 2 & 6 \end{pmatrix}$ *b)* $\begin{pmatrix} -3 & -1 \\ 2 & 6 \end{pmatrix}$.

(a) $\begin{vmatrix} 3 & 9 \\ 2 & 6 \end{vmatrix} = 3\times 6 - 2\times 9$ $\underline{ad - bc}$

$= 0$

(b) $\begin{vmatrix} -3 & -1 \\ 2 & 6 \end{vmatrix} = -3\times 6 - 2(-1)$

$= -16$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

★ *a)* 0   *b)* -16 Only b) is invertible ★

In summary we have a formula to establish the inverse of a 2×2 matrix and an effective test for invertibility, with A^{-1} existing if and only if $\det(A) \neq 0$. Unfortunately these techniques for inverting and calculating determinants of square matrices do not extend naturally to larger matrices. In the next lecture we will see how to extend the theory to 3×3 matrices and beyond. We close the lecture with some simple properties of the inverse.

a) $(AB)^{-1} = B^{-1}A^{-1}$.

b) $(A^{-1})^T = (A^T)^{-1}$

Proof a): $(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1}$

$= AIA^{-1}$

$= AA^{-1}$

$= I$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

Example 7: Simplify $B^2(AB)^{-1}A(B^{-1})^T(AB)^T$.

Test or exam type Q

$B^2(AB)^{-1}A(B^{-1})^T(AB)^T$

$= B^2B^{-1}A^{-1}A(B^{-1})^T B^TA^T$

$= B(B^T)^{-1}(B^T)A^T$

$= BA^T$

★ BA^T ★

LECTURE 19
INVERSES AND DETERMINANTS FOR LARGE MATRICES

> To find A^{-1} reduce $[A|I]$ to $[I|A^{-1}]$
>
> A^{-1} exists if and only if $\det(A) \neq 0$

In the last lecture we presented the theory for 2×2 inverses and determinants. Recall that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and that

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Inverses allow us to "sort of" divide both sides of an equation by a matrix. Determinants (which are a single number extracted from the matrix) enable us to check whether or not an inverse exists, with A^{-1} existing if and only if $\det(A) \neq 0$. In this lecture we will extend these concepts beyond 2×2 but be warned that the algorithms become quite nasty. We will develop the theory through examples.

Example 1: Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$ find A^{-1} and check your solution through multiplication.

The method is simple to explain but horrible to implement. We simply write down an augmented system $[A|I]$ and magically reduce it to $[I|A^{-1}]$.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix} \quad \to \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$R_2 = R_2 - 2R_1$
$R_3 = R_3 - R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -4 & -2 & 1 & 0 \\ 0 & -3 & -2 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 3 & 0 & 1 & -1 & 2 & 0 \\ 0 & -3 & -4 & -2 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right)$$

$R_3 = R_3 - R_2$

$R_1 = 3R_1 + 2R_2$

over page
continues

$R_1 = 2R_1 - R_3$
$R_2 = R_2 + 2R_3.$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 6 & 0 & 0 & -3 & 5 & -1 \\ 0 & -3 & 0 & 0 & -1 & 2 \\ 0 & 0 & 2. & 1 & -1 & 1 \end{array}\right)$$

$R_1 = \frac{1}{6} R_1$
$R_2 = -\frac{1}{3} R_2$
$R_3 = \frac{1}{2} R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{5}{6} & -\frac{1}{6} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \quad A^{-1}$$

$$A^{-1} = \frac{1}{6}\begin{pmatrix} -3 & 5 & -1 \\ 0 & 2 & -4 \\ 3 & -3 & 3 \end{pmatrix}$$

$\frac{1}{6}$?

$$\star \quad A^{-1} = \frac{1}{6}\begin{pmatrix} -3 & 5 & -1 \\ 0 & 2 & -4 \\ 3 & -3 & 3 \end{pmatrix} \quad \star$$

You can imagine the torture of calculating a 5×5 inverse! In application we often have matrices as large as 1000×1000 and of course these calculations are usually done via computer. Recall that

a) A^{-1} may or may not exist.

b) If A^{-1} exists it is unique.

c) $(A^{-1})^{-1} = A$.

d) $(AB)^{-1}=B^{-1}A^{-1}$

Once we have the inverse we can use it in interesting ways.

Example 2: Use the inverse in Example 1 to solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 14 \\ 2x + y + 2z &= 10 \\ x - y + z &= 2 \end{aligned}$$

$$\underset{3\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{3\times 1.}{\begin{pmatrix} 14 \\ 10 \\ 2 \end{pmatrix}}$$

$$A^{-1}A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 14 \\ 10 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 14 \\ 10 \\ 2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -3 & 5 & -1 \\ 0 & 2 & -4 \\ 3 & -3 & 3 \end{pmatrix}\begin{pmatrix} 14 \\ 10 \\ 2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 6 \\ 12 \\ 18 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

$$\star \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \star$$

This example perhaps clarifies why matrix multiplication is defined as it is, so that systems of equations can be rewritten as matrix equations.

Note that we never solve systems of equations using inverses. Elimination to echelon form followed by back-substitution is much more efficient. However once you have the inverse of the coefficient matrix (after much work) it is then easy to implement! Also a technique involving the inverse will be of little use when the inverse fails to exist (infinite solutions or no solutions).

How does the above method fail when A^{-1} doesn't exist?

Example 3: Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Show that A^{-1} does not exist.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right)$$

$R_2 = R_2 - 4R_1$
$R_3 = R_3 - 7R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right)$$

$R_3 = R_3 - 2R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & & & \end{array}\right)$$

does not exist
Row of zeros

Find determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} =$$

★

So whenever you get a zero row on the left you can stop! No A^{-1}.

Note that if we have a square system of equations $A\mathbf{x} = \mathbf{b}$ then

- If A^{-1} exists the unique solution is $\mathbf{x} = A^{-1}\mathbf{b}$
- If A^{-1} does not exist then the system is either inconsistent or has infinite solutions.

DETERMINANTS

We now turn to the problem of finding the determinant of large matrices. We have two approaches. The first is to blast away using a fundamental algorithm. In the following lecture we will also make use of row operations to simplify the calculations.

Definition: For a square matrix A, the ij minor (denoted by $|A_{ij}|$) is the determinant obtained from A by deleting the i^{th} row and the j^{th} column.

Example 4: For $A = \begin{pmatrix} 1 & 4 & 6 \\ 1 & -8 & 7 \\ 5 & 9 & 0 \end{pmatrix}$ we have $|A_{23}| = \begin{vmatrix} 1 & 4 \\ 5 & 9 \end{vmatrix} = -11.$

cross out

We can now formally define the determinant of any matrix but we tend not to use the definition directly but rather just approach the problem as an algorithm.

Definition: For an $n \times n$ matrix A

$$\det(A) = a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}| - \cdots (-1)^n a_{1n}|A_{1n}|$$

Example 5: Find $\det \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{pmatrix} 6 & 8 & 9 \\ 5 & 2 & 1 \\ 7 & 4 & 3 \end{pmatrix} \end{array}$

Our method is to reduce the 3×3 determinant into 3 smaller 2×2 determinants. We will first do this by expanding into minors along the top row.

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ always use this grid.

$$\det \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{pmatrix} 6 & 8 & 9 \\ 5 & 2 & 1 \\ 7 & 4 & 3 \end{pmatrix} \end{array} = 6\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} - 8\begin{vmatrix} 5 & 1 \\ 7 & 3 \end{vmatrix} + 9\begin{vmatrix} 5 & 2 \\ 7 & 4 \end{vmatrix}$$

$$= 6\{(6-4)\} - 8(15-7) + 9(20-14)$$

$$= 12 - 64 + 54$$

$$= 2$$

Remarkably we can actually expand along any row or column as long as we make sure at all times we get the sign of the minors correct.

Example 6: Find the determinant of the matrix above by expanding along the second column instead.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\det \begin{pmatrix} 6 & 8 & 9 \\ 5 & 2 & 1 \\ 7 & 4 & 3 \end{pmatrix} = -8 \begin{vmatrix} 5 & 1 \\ 7 & 3 \end{vmatrix} + 2 \begin{vmatrix} 6 & 9 \\ 7 & 3 \end{vmatrix} - 4 \begin{vmatrix} 6 & 9 \\ 5 & 1 \end{vmatrix}$$

$$= -8(15-7) + 2(18-63) - 4(6-45)$$

$$\rightarrow = -64 - 90 + 156 = \underline{\underline{2}}.$$

Example 7: (To do now) Find the determinant of the matrix above by expanding along the bottom row instead.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\begin{pmatrix} 6 & 8 & 9 \\ 5 & 2 & 1 \\ 7 & 4 & 3 \end{pmatrix} = 7 \begin{vmatrix} 8 & 9 \\ 2 & 1 \end{vmatrix} - 4 \begin{vmatrix} 6 & 9 \\ 5 & 1 \end{vmatrix} + 3 \begin{vmatrix} 6 & 8 \\ 5 & 2 \end{vmatrix}$$

$$= 7\{8-18\} - 4\{6-45\} + 3\{12-40\}$$

$$= -70 + 156 - 84$$

$$= \underline{\underline{2}}$$

How do we evaluate a 4×4 determinant? Well we reduce it to 4 smaller 3×3's and each of those 3×3's is reduced to 3 smaller 2×2's. But keep your eyes open, there may be shortcuts by making a good choice of row or column over which to expand.

Example 8: Determine if $A = \begin{pmatrix} 4 & 5 & 8 & -1 \\ 2 & 0 & -2 & -3 \\ 0 & 0 & 2 & 0 \\ 8 & 0 & -4 & -5 \end{pmatrix}$ is invertible.

2nd column

$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$

$$\det = (-5) \begin{array}{ccc} + & - & + \end{array} \begin{vmatrix} 2 & -2 & -3 \\ 0 & 2 & 0 \\ 8 & -4 & -5 \end{vmatrix} \pm 0$$

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$$(-5)(+2) \begin{vmatrix} 2 & -3 \\ 8 & -5 \end{vmatrix} = -10\ (-10 + 24) = -140$$

★ det(A)$= -140 \neq 0$. Hence invertible. ★

LECTURE 20
DETERMINANTS AND VECTOR GEOMETRY

> The determinant is unchanged by the row and column operations $R_i = R_i \pm \alpha R_j$ and $C_i = C_i \pm \alpha C_j$
>
> Swapping rows or columns multiplies the determinant by -1.
>
> The determinant of a matrix in echelon form is the product of the diagonal elements of the echelon form.
>
> $$\text{For } \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ v_n \end{pmatrix} \text{ in } \mathbb{R}^n, \quad |\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + \dots v_n^2}.$$
>
> For any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ the triangle inequality asserts that
>
> $$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|.$$

PROPERTIES OF DETERMINANTS

1) A^{-1} exists if and only if $\det(A) \neq 0$.

<<[

2) $\det(AB) = \det(A)\det(B)$.

3) $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

<<|x, y, z||A, b, c|>

4) In general $\det(A+B) \neq \det(A) + \det(B)$.

5) $\det(A) = \det(A^T)$.

6) If a matrix has a zero row or column its determinant is 0.

7) Swapping rows or columns multiplies the determinant by -1.

8) Multiplying a row or column by α multiplies the determinant by α. Note that this means that we can pull common factors out of rows and/or columns when evaluating determinants.

Proof 3): Claim $\det(A^{-1}) = \dfrac{1}{\det(A)}$

Pf $AA^{-1} = I$

$\det(AA^{-1}) = \det(I)$

$\det A \det A^{-1} = 1$

$\det(A^{-1}) = \dfrac{1}{\det(A)}$

Example 1: A particular 3×3 matrix A has determinant equal to 5. Find the determinant of B if B is defined to be:

a) A^T. 5

b) A with the first two rows swapped. -5

c) A with the first two rows swapped and the last two columns swapped. 5

d) A with the second row multiplied by 3. 15

e) $2A$. $\det(2A) = 40$

f) A^{-1}. $\frac{1}{5}$

$$\star \quad 5, -5, 5, 15, 40, \frac{1}{5} \quad \star$$

We turn now to the problem of finding the determinant of a large or complicated matrix. The most efficient way of doing this is to use row operations! The following facts will be of use.

PROPERTIES

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1) The determinant is unchanged by the row and column operations $R_i = R_i \pm \alpha R_j$ and $C_i = C_i \pm \alpha C_j$ (but NOT $R_i = \beta R_i \pm \alpha R_j$ and $C_i = \beta C_i \pm \alpha C_j$). This is a spot where your row and column operations need to be formal.

2) Swapping rows or columns multiplies the determinant by -1.

3) The determinant of a matrix in echelon form is the product of the diagonal elements of the echelon form.

Discussion of 3):

eg

$$\begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 7 \\ 0 & 0 & 5 \end{pmatrix}$$

multiply $3 \times 2 \times 5$
$= 30$

ad - bc

Example 2: Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 8 \end{pmatrix}$.

a) Find $\det(A)$.

b) Display property 1) using $R_2 = R_2 - 5R_1$

c) Display property 1) using $C_2 = C_2 + 4C_1$

d) Show that $R_2 = 3R_2 + 4R_1$ affects the determinant.

e) Verify property 2) by swapping the rows of A.

f) Reduce A to echelon form and verify property 3).

~~2-5=-3~~

~~8-15=-7~~

(a) $\det(A) = 1 \times 8 - 2 \times 3 = 8 - 6 = 2.$

(b) $R_2 = R_2 - 5R_1$ $\begin{pmatrix} 1 & 3 \\ -3 & 7 \end{pmatrix} \xrightarrow{\det} 1 \times 7 - 3(-3)$

$= -7 + 9 = \underline{2}.$

(c) $C_2 = C_2 + 4C_1$ $\begin{pmatrix} 1 & 7 \\ 2 & 16 \end{pmatrix} \xrightarrow{\det} 16 - 14 = \underline{2}.$

(d) Homework.

(e) $\begin{pmatrix} 2 & 8 \\ 1 & 3 \end{pmatrix} \to \det. = 2 \times 3 - 1 \times 8 = \underline{-2}$

(f) $\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \xrightarrow{\det} 1 \times 2 - 0 \times 3$

$= \underline{2}$

★

This opens the door for us the evaluate large and complicated determinants. We simply reduce to echelon form using only the formal row operations (and keeping track of row swaps!) and then find the determinant of the final echelon form matrix by simply multiplying the diagonal elements.

Example 3: Find $\begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & -2 & 6 & -1 \\ 4 & -2 & 1 & 7 \\ 3 & 5 & -7 & 2 \end{vmatrix}$.

$$\xrightarrow[R_4 = R_4 - 3R_1]{R_2 = R_2 - 2R_1,\ R_3 = R_3 - 4R_1} \begin{vmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 6 & -7 \\ 0 & 2 & 1 & -5 \\ 0 & 8 & -7 & -7 \end{vmatrix}$$

$$R_2 \longleftrightarrow R_3 \quad \begin{vmatrix} 1 & -1 & 0 & 3 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & 6 & -7 \\ 0 & 8 & -7 & -7 \end{vmatrix}$$

$$\xrightarrow{R_4 = R_4 - 4R_2} \begin{vmatrix} 1 & -1 & 0 & 3 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & 6 & -7 \\ 0 & 0 & -11 & 13 \end{vmatrix}$$

$$R_4 = R_4 + \tfrac{11}{6}R_3 . \quad \begin{vmatrix} 1 & -1 & 0 & 3 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & 6 & -7 \\ 0 & 0 & 0 & \tfrac{1}{6} \end{vmatrix}$$

$$\det = -\left(1 \times 2 \times 6 \times \tfrac{1}{6}\right) = -2$$

row swap

★ -2 ★

Example 4: Find $\begin{vmatrix} 1 & 5 & 4 & 3 \\ 3 & -2 & 12 & -2 \\ 1 & -3 & 4 & 3 \\ 2 & -11 & 8 & 1 \end{vmatrix}$. Hint: Think about the columns.

$C_3 = C_3 - 4C_1$

$$\begin{vmatrix} 1 & 5 & 0 & 3 \\ 3 & -2 & 0 & -2 \\ 1 & -3 & 0 & 3 \\ 2 & -11 & 0 & 1 \end{vmatrix}$$

★ 0 ★

Example 5: Suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has determinant equal to 4. Find the determinant of $\begin{pmatrix} 2a & 200b \\ c & 100d \end{pmatrix}$.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 4.$

$$\begin{vmatrix} 2a & 200b \\ c & 100d \end{vmatrix} = 200ad - 200bc.$$
$$= 200(ad - bc).$$
$$= 200 \times 4 = 800$$

★ 800 ★

Example 6: Show that $\det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = (b-a)(c-a)(c-b)$

$R_2 = R_2 - R_1$
$R_3 = R_3 - R_1$

$$\begin{vmatrix} 1 & a & a^2 \\ 0 & (b-a) & (b-a)(b+a). \\ 0 & (c-a) & (c-a)(c+a) \end{vmatrix}$$

pull factors to the front

$$(b-a)(c-a)\begin{vmatrix} 1 & a & a^2. \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$R_3 = R_3 - R_2$. $= (b-a)(c-a)$.

$$\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$$

$(b-a)(c-a) \times 1 \times 1$

$\Downarrow$

$(b-a)(c-a)(c-b)$

We close the lecture with some revision from earlier in the course in preparation for vector geometry.

Definition Given a vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ v_n \end{pmatrix}$ in $\mathbb{R}^n$, the length $|\mathbf{v}|$ of $\mathbf{v}$, (also referred to as the magnitude of $\mathbf{v}$) is given by

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + \ldots v_n^2}$$

Properties

1) $|\mathbf{v}|$ is a real number.

2) $|\mathbf{v}| \geq 0$.

3) $|\mathbf{v}| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

4) $|\lambda \mathbf{v}| = |\lambda||\mathbf{v}|$ for $\lambda \in \mathbb{R}$.

Example 7: Let $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 6 \end{pmatrix}$. Find $|\mathbf{v}|$ and hence produce two different unit vectors parallel to $\mathbf{v}$.

Note that a unit vector is simply a vector of length 1.

$$|\underset{\sim}{v}| = \sqrt{1 + 9 + 4 + 36}$$

$$= \sqrt{50}$$

unit vector – vector divided by it's own magnitude

$$\pm \frac{1}{\sqrt{50}} \begin{pmatrix} 1 \\ 3 \\ 2 \\ 6 \end{pmatrix}$$

$$\star \quad \pm \frac{1}{\sqrt{50}} \begin{pmatrix} 1 \\ 3 \\ 2 \\ 6 \end{pmatrix} \quad \star$$

Example 8: Find a vector of length 7 parallel to the vector $\mathbf{w} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$.

$$|\underset{\sim}{w}| = \sqrt{4 + 1 + 25}$$

$$= \sqrt{30}$$

$$= \frac{1}{\sqrt{30}} \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \quad \rightarrow \quad \frac{7}{\sqrt{30}} \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}.$$

$$\star \quad \frac{7}{\sqrt{30}} \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \quad \star$$

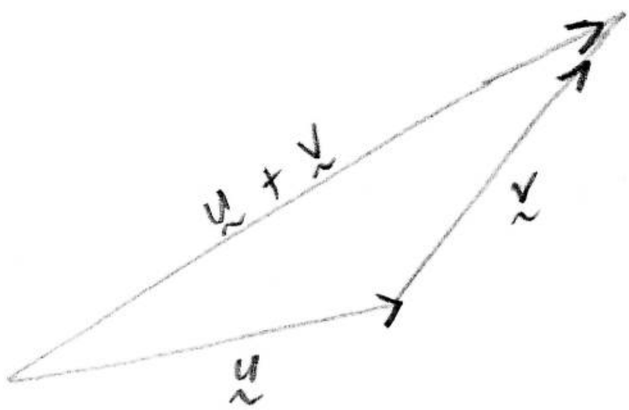
The Triangle Inequality

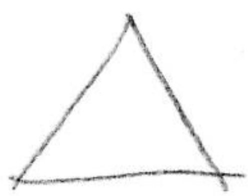
For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$

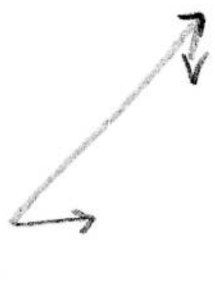
$$|\,\mathbf{u} + \mathbf{v}| \leq |\,\mathbf{u}| + |\,\mathbf{v}|.$$

this has to be longer

By considering the following diagram we can see that the triangle inequality is simply saying that any side of a triangle must be shorter than the sum of the other 2 sides. Makes sense! A full proof may be found in your printed notes.







LECTURE 21
DOT PRODUCT

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + \ldots a_nb_n$$

$$\cos(\theta) = \frac{\mathbf{a}\cdot\mathbf{b}}{|\,\mathbf{a}||\,\mathbf{b}|}$$

$$\mathbf{u}\perp\mathbf{v} \text{ if and only if } \mathbf{u}\cdot\mathbf{v} = 0.$$

$$\mathrm{Proj}_{\mathbf{v}}\mathbf{u} = (\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}})\mathbf{v}$$

We now look at two different types of multiplication for vectors, today the dot product and in the next lecture the cross product. The dot product (also called the scalar product) is a very simple way of multiplying two vectors together in any dimension. Applications are deep and elegant.

Given two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{pmatrix}$ in n dimensional space $\mathbb{R}^n$ the dot product of $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + \cdots a_nb_n$$

It is important to note that the dot product of two vectors is a number, **NOT** a vector. In the next lecture we will look at cross products which do in fact produce a vector as an answer.

Example 1: Find the dot product of each of the following pairs of vectors:

a) $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 7 \\ 2 \end{pmatrix}$ in $\mathbb{R}^3$ b) $\mathbf{u} = \begin{pmatrix} -1 \\ 7 \\ -3 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3 \\ 1 \\ 6 \\ -2 \end{pmatrix}$ in $\mathbb{R}^4$.

(a)

$a . b \quad \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} . \begin{pmatrix} 4 \\ 7 \\ 2 \end{pmatrix}$

$= 1 \times 4 + 3 \times 7 + 8 \times 2$

$= 4 + 21 + 16$

$= 41$

(b) $u . v$

$= \begin{pmatrix} -1 \\ 7 \\ -3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 6 \\ -2 \end{pmatrix}$

$= -1 \times -3 + 7 \times 1 + -3 \times 6 + 2 \times -2$

$= 3 + 7 - 18 - 4$

$= -12.$

★ *a)* 41 *b)* − 12 ★

Observe that the dot product works nicely in all dimensions and that its value may be positive, negative or zero.

Example 2: Find the dot product $\mathbf{u} = \begin{pmatrix} 8 \\ 1 \\ 5 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ 2 \\ 9 \end{pmatrix}$.

undefined will not work

★ Undefined ★

Properties

For all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$:

1) $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$.

2) $\mathbf{u} \cdot \mathbf{v} \in \mathbb{R}$. (always a real number)

3) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$. (commutative.

4) $\mathbf{u} \cdot (\lambda \mathbf{v}) = \lambda(\mathbf{u} \cdot \mathbf{v})$ for $\lambda \in \mathbb{R}$.

5) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

6) $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$.

Proof : 1) $\quad ||u|| = \sqrt{u.v}$

Let $\underset{\sim}{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n.$

RHS $\sqrt{\begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix}}$

$= \sqrt{u_1^2 + u_2^2 + u_3^2 \ldots} \quad = |\underset{\sim}{u}| = LHS.$

Proof : 6)

$u.v = u^T v$

RHS, Let $\underset{\sim}{u} = \begin{pmatrix} u_1 \\ \vdots \\ \vdots \\ u_n \end{pmatrix} \quad \underset{\sim}{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

$u^T v \quad (u_1 \ldots \ldots v_n) \begin{pmatrix} v_1 \\ \vdots \\ \vdots \\ v_1 \end{pmatrix}$

$(1 \times n)(n \times 1) = u_1 v_1 + u_2 v_2 + \ldots \ldots u_n v_n.$

$= u.v \quad = LHS.$

The main application for dot products lies in the determination of the orientation of one vector with respect to another. We can easily find the angle between two vectors **a** and **b** in any dimension via the formula

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Example 3: Find the angle (to the nearest minute) between $\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$.

$a . b$

$$\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} = 1\times3 + 4\times2 + 5\times6$$

$$= 3 + 8 + 30$$

$$= \underline{41}$$

$$|a| = \sqrt{1^2 + 4^2 + 5^2}$$

$$= \sqrt{1+16+25}$$

$$= \sqrt{42}$$

$$|b| = \sqrt{(3)^2 + (2)^2 + (6)^2}$$

$$= \sqrt{9+4+36}$$

$$= \sqrt{49}$$

$$= 7.$$

$$\cos(\theta) = \frac{41}{\sqrt{42} \times 7}$$

$$\theta = \cos^{-1}\left(\frac{41}{7\sqrt{42}}\right)$$

$$= 25°20'$$

★ $\theta = \cos^{-1}(\frac{41}{7\sqrt{42}}) \approx 25°20'$ ★

Remarkably we can just as easily find the angle between two vectors in 5 dimensional space!

The following crucial test is easily verified from the above formula.

FACT:

Non zero vectors **u** and **v** in $\mathbb{R}^n$ are perpendicular to each other if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Proof:

$$\underset{\sim}{u} \cdot \underset{\sim}{v} \longrightarrow \theta = 90° \longrightarrow \cos(\theta) = 0$$

$$\longrightarrow \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = 0 \longleftrightarrow \underset{\sim}{u} \cdot \underset{\sim}{v} = 0$$

★

Example 4: Prove that $\begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$ in $\mathbb{R}^3$.

$u \cdot v$

$\begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$ $\quad -3 \times 5 + 4 \times 2 + 1 \times 7$

$= -15 + 8 + 7$

$= 0$

$\therefore$ u and v $\perp$

★

It is interesting to consider the impact of the above equation in $\mathbb{R}^2$. The vectors $\mathbf{v_1} = \begin{pmatrix} 1 \\ m_1 \end{pmatrix}$ and and $\mathbf{v_2} = \begin{pmatrix} 1 \\ m_2 \end{pmatrix}$ have gradients m_1 and m_2 respectively. Now $\mathbf{v_1} \cdot \mathbf{v_2} = 0$ implies that $1 + m_1 m_2 = 0$ and hence that $m_1 m_2 = -1$!

Example 5: Let $\mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ -2 \\ 5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 9 \\ -6 \\ \beta \end{pmatrix}$. Find β if $\mathbf{u}$ and $\mathbf{v}$ are:

a) Parallel. $\quad \beta = 15$

b) Perpendicular.

$u \cdot v$

$3 + 27 + 12 + 5\beta = 0$

$5\beta = -42$

$\beta = -\frac{42}{5}$

★ $a)$ 15 $\quad b)$ $-\dfrac{42}{5}$ ★

If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are perpendicular we say that they are orthogonal and often write $\mathbf{u} \perp \mathbf{v}$. An orthogonal set of vectors is a set where all the vectors are mutually perpendicular and if they are also unit vectors (length 1) we say the set is orthonormal.

The classic orthonormal set in $\mathbb{R}^3$ is $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

We close the lecture by defining projections and then using them to find distances between points and lines in space.

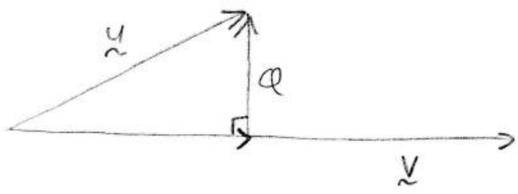
Projections

The projection of $\mathbf{u}$ onto $\mathbf{v}$ is essentially the shadow that $\mathbf{u}$ casts in the direction of $\mathbf{v}$. It is also often referred to as the component of $\mathbf{u}$ in the direction of $\mathbf{v}$. Geometrically, it is the perfect scalar multiple of $\mathbf{v}$ which generates a right angled triangle with $\mathbf{u}$ and is given by

$$\text{Proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v}$$

Discussion and Proof:

projection of u onto v



proj^{u}_{v}

let $\text{proj}_v\, u = \alpha v$

$a + \alpha v = u$

$a = u - \alpha v$

But $a \perp v \Rightarrow a \cdot v = 0$

$\therefore (u - \alpha v) \cdot v = 0$

$\Rightarrow u \cdot v \bullet \alpha v \cdot v = 0$

$\alpha\, v \cdot v = u \cdot v$

$\Rightarrow \alpha = \dfrac{u \cdot v}{v\ v}$

$\text{proj}_v u = \alpha v \quad = \left(\dfrac{u \cdot v}{v . v}\right) v$

Example 6: Find the projection of $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ onto $\begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}$ in $\mathbb{R}^3$.

$-1 \times -1 = 1$

$-1 \times -1 = 1.$

$-2 \times -2 = 4.$

$$\text{prog}_v u = \left(\frac{u \cdot v}{v \cdot v} \right) v$$

$$= \left(\frac{-1 - 2 - 8}{1 + 1 + 4} \right) \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}$$

$$= \frac{-11}{6} \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} \longrightarrow \star \begin{pmatrix} \frac{11}{6} \\ \frac{11}{6} \\ \frac{11}{3} \end{pmatrix} \star$$

Observe that $\text{Proj}_{\mathbf{v}}\mathbf{u}$ is always a **vector** parallel to $\mathbf{v}$.

Questions:

1) What is $\text{Proj}_{\mathbf{v}}\mathbf{u}$ if $\mathbf{u} \perp \mathbf{v}$?

0

2) What is $\text{Proj}_{\mathbf{v}}\mathbf{u}$ if $\mathbf{u}$ is parallel to $\mathbf{v}$?

u

We now turn to one of our major applications on this topic. Make sure you fully understand the following question!

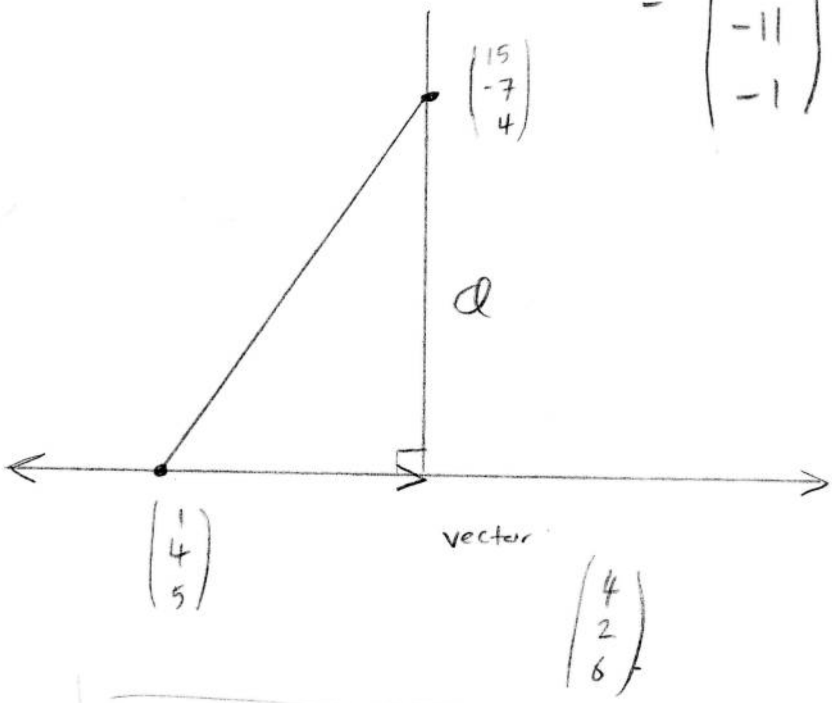
Example 7: Find the shortest distance between the point $\begin{pmatrix} 15 \\ -7 \\ 4 \end{pmatrix}$ in $\mathbb{R}^3$ and the line in space:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} t \; ; \; t \in \mathbb{R}$$

$$\begin{pmatrix} 15 \\ -7 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 14 \\ -11 \\ -1 \end{pmatrix}$$

Also find the point on the given line which is closest to $\begin{pmatrix} 15 \\ -7 \\ 4 \end{pmatrix}$.

$\perp$ is the shortest distance



$$\underset{\sim}{V} = \begin{pmatrix} 14 \\ -11 \\ -1 \end{pmatrix}$$

$$\mathrm{proj}_{\begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}} V = \mathrm{proj}_{\begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}} \begin{pmatrix} 14 \\ -11 \\ -1 \end{pmatrix}$$

$$= \frac{\begin{pmatrix} 14 \\ -11 \\ -1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}}{\begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}} \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$$

$$= \left(\frac{56 - 22 - 6}{16 + 4 + 36} \right) \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$$

$$= \frac{28}{56} \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \underset{\sim}{d} = \underset{\sim}{V}$$

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \underset{\sim}{d} = \begin{pmatrix} 14 \\ -11 \\ -1 \end{pmatrix}$$

$$d = \begin{pmatrix} 12 \\ -12 \\ -4 \end{pmatrix}$$

$$= \sqrt{304}$$
$$= 17.4$$

★ shortest distance is $\sqrt{304} \approx 17.4$ and the closest point is $\begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}$ ★

$$\boxed{\mathrm{proj}_v u = \left(\frac{u \cdot v}{v \cdot v} \right) v}$$

closest pt.

$$\Rightarrow \begin{pmatrix} 15 \\ -7 \\ 4 \end{pmatrix} - \begin{pmatrix} 12 \\ -12 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}.$$

LECTURE 22
THE CROSS PRODUCT

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$$

$$\mathbf{a} \perp (\mathbf{a} \times \mathbf{b}) \text{ and } \mathbf{b} \perp (\mathbf{a} \times \mathbf{b})$$

$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

The area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ is equal to the magnitude of $\mathbf{a} \times \mathbf{b}$.

Given $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$, the scalar triple product is the number $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

The absolute value of the scalar triple product is the volume of the parallelepiped formed by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

We turn now to the vector cross product which is a completely different way of multiplying vectors together. Unlike the dot product, the cross product of two vectors is actually a vector. The cross product (denoted $\times$) is designed for $\mathbb{R}^3$ only, and is evaluated via determinants.

Definition: If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are two vectors in $\mathbb{R}^3$ then the cross product $\mathbf{a} \times \mathbf{b}$ is given by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The technique is best presented through a simple example, but first some revision on notation:

Example 1:

$$2\mathbf{i} - 7\mathbf{j} + 12\mathbf{k} = \begin{pmatrix} 2 \\ -7 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 9 \\ -4 \end{pmatrix} = 2i + 9j - 4k.$$

Example 2: Find $\mathbf{u} \times \mathbf{v}$ if $\mathbf{u} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$.

$$\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} = \begin{vmatrix} i & j & k \\ 3 & 0 & 2 \\ 1 & 4 & 5 \end{vmatrix}$$

$$= i \begin{vmatrix} 0 & 2 \\ 4 & 5 \end{vmatrix} - j \begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} + k \begin{vmatrix} 3 & 0 \\ 1 & 4 \end{vmatrix}$$

$$= i(0-8) - j(15-2) + k(12-0)$$

$$\begin{pmatrix} -8 \\ -13 \\ 12 \end{pmatrix}$$

$$\star \quad \begin{pmatrix} -8 \\ -13 \\ 12 \end{pmatrix} \quad \star$$

Observe that unlike the dot product, the cross product is a vector. Not a random vector, but rather a highly specialised vector with the following crucial property:

Fact: $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ with its orientation given by the right hand rule.

This give us a simple and effective tool for generating a vector orthogonal to two given vectors!

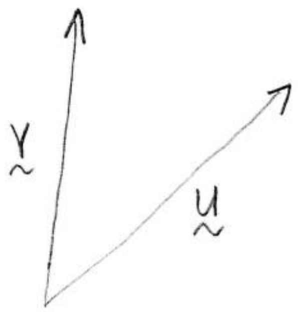
Example 3: For the example above check that $\mathbf{u} \perp (\mathbf{u} \times \mathbf{v})$ and $\mathbf{v} \perp (\mathbf{u} \times \mathbf{v})$.

check $\begin{pmatrix} -8 \\ -13 \\ 12 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = -24 + 0 + 24 = 0$

$$\begin{pmatrix} -8 \\ -13 \\ 12 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} = -8 - 52 + 60 = 0$$

$\star$

Discussion of the Right Hand Rule



black board

$\underset{\sim}{u} \times \underset{\sim}{v}$ out of board

$\underset{\sim}{v} \times \underset{\sim}{u}$ into ~~bd~~ board.

Example 4: Find $\mathbf{v} \times \mathbf{u}$ where $\mathbf{u}$ and $\mathbf{v}$ are the vectors in the previous example.

HW

$$\underset{\sim}{v} \times \underset{\sim}{u} \quad \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} \times \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \\ -12 \end{pmatrix}$$

$$\begin{vmatrix} i & j & k \\ 1 & 4 & 5 \\ 3 & 0 & 2 \end{vmatrix}$$

$$\star \quad \begin{pmatrix} 8 \\ 13 \\ -12 \end{pmatrix} \quad \star$$

The displays the following property:

Fact: $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

Note that this is a direct result of the row swap property of determinants!

Example 5: Find a unit vector perpendicular to $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$.

$$\underset{\sim}{u} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \underset{\sim}{v} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{u} \times \underset{\sim}{v} = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ -1 & 3 & 0 \end{vmatrix}$$

$$i \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} - j \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} + k \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix}$$

$$-6\underset{\sim}{i} - 2\underset{\sim}{j} + 4k = \begin{pmatrix} -6 \\ -2 \\ 4 \end{pmatrix} \qquad \frac{1}{\sqrt{56}} \begin{pmatrix} -6 \\ -2 \\ 4 \end{pmatrix}.$$

$$\star \quad \frac{1}{\sqrt{56}} \begin{pmatrix} -6 \\ -2 \\ 4 \end{pmatrix} \quad \star$$

Cross products enjoy the following properties:

1) $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

2) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

3) $\mathbf{a} \times (\lambda \mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b})$ where $\lambda \in \mathbb{R}$.

4) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$.

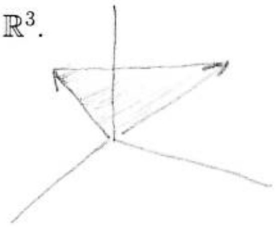
5) $\mathbf{a} \perp (\mathbf{a} \times \mathbf{b})$ and $\mathbf{b} \perp (\mathbf{a} \times \mathbf{b})$.

6) $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

See your printed notes for proofs.

Note that 6) implies that the area of the triangle formed by $\mathbf{a}$ and $\mathbf{b}$ is equal to half the magnitude of $\mathbf{a} \times \mathbf{b}$.

Example 6: Find the area of the triangle formed by the two vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ in $\mathbb{R}^3$.



$$\begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= i \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} - j \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix}$$

$$= i(-2-0) - j(-1-3) + k(0-2).$$

$$= -2\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$$

$$\left\| \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} \right\| = \sqrt{4+16+4} = \sqrt{24} = \sqrt{4}\sqrt{6} \to 2\sqrt{6}$$

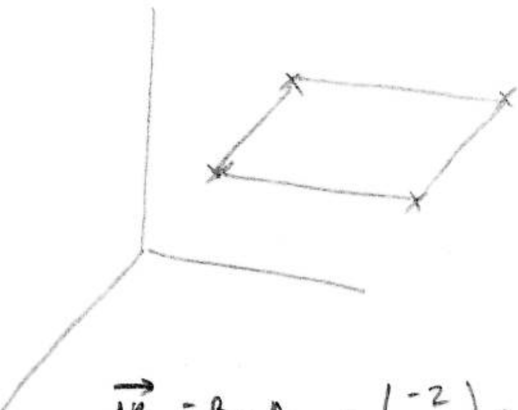
$\therefore$ Area $\Delta = \frac{1}{2}(2\sqrt{6})$
$= \sqrt{6}$

★ $\sqrt{6}$ ★

6) also implies that the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ is equal to the magnitude of $\mathbf{a} \times \mathbf{b}$.

Example 7: Find the area of the parallelogram $ABCD$ with vertices at the points

$$A\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, B\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}, \text{ and } D\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix},$$



$$\vec{AB} = B - A = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

$$\vec{AD} = D - A = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} i & j & k \\ -3 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= i \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} - j \begin{vmatrix} -3 & 2 \\ 2 & 3 \end{vmatrix} + k \begin{vmatrix} -3 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= i + 13j - 5k = \begin{pmatrix} 1 \\ 13 \\ \sqrt{5} \end{pmatrix}$$

$\underline{\text{mag}}$

$$= \sqrt{1 + 69 + 25}$$

$$= \sqrt{195}$$

★ $\sqrt{195}$ ★

The Scalar Triple Product

The scalar triple product combines the dot and cross products in a single numerical expression.

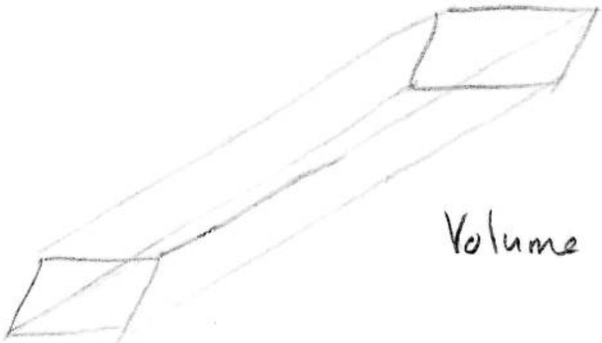
Given $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$, the scalar triple product is the number $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

The absolute value of the scalar triple product is the volume of the parallelepiped formed by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. (See printed notes for a proof).

Note that the scalar triple product is essentially symmetric in all three vectors so

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$$

Discussion:



Volume $= |\underset{\sim}{a} \cdot (b \times c)|$

Example 8: Find the volume of the parallelepiped spanned by the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$

Method 1 $\quad a \times b = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -2 & 4 & -1 \end{vmatrix}$

$$= i \begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} - j \begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix}$$

$$= i(-2-12) - j(-1+6) + k(4+4)$$

$$= -14i - 5j + 8k = \begin{pmatrix} -14 \\ -5 \\ 8 \end{pmatrix}$$

Then $\begin{pmatrix} -14 \\ -5 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = -42 - 25 + 8 = -59$

$= 59u^3$

Method 2 $\quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\begin{array}{ccc} + & - & + \end{array}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & -1 \\ 3 & 5 & 1 \end{pmatrix}$$

$$1 \begin{vmatrix} 4 & -1 \\ 5 & 1 \end{vmatrix} - 2 \begin{vmatrix} -2 & -1 \\ 3 & 1 \end{vmatrix} + 3 \begin{vmatrix} -2 & 4 \\ 3 & 5 \end{vmatrix}$$

$$= 1 \times 9 - 2(1) + 3(-22).$$

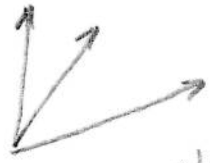
★ $59\ u^3$ ★

$$= 9 - 2 - 56$$

$$= -59 \longrightarrow 59u^3$$

Example 9: Prove that the vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 9 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$ are coplanar.

$$\underset{\sim}{a} \cdot \underset{\sim}{b} \cdot \underset{\sim}{c} \text{ coplanar} \longleftrightarrow \underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = 0$$



don't generate volume. (exist within a plane)

$$\underset{\sim}{a}(\underset{\sim}{b} \times \underset{\sim}{c}) = \begin{vmatrix} 1 & 9 & 0 \\ -2 & 4 & 1 \\ 3 & 5 & 1 \end{vmatrix}$$

$$1\begin{vmatrix} 4 & -1 \\ 5 & 1 \end{vmatrix} - 9\begin{vmatrix} -2 & -1 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} -2 & 4 \\ 3 & 5 \end{vmatrix}$$

$$= 4 + 5 - 9(-2+3)$$

$$= 9 - 9 \times 1$$

$$= \underline{\underline{0}}$$

★

Example 10: Give an interpretation of $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$. Note this is NOT a scalar triple product!

★

LECTURE 23
PLANES REVISITED

If a plane contains a point P and has a normal vector $\mathbf{n}$ then the point-normal form of the plane is given by $\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - P\right) \cdot \mathbf{n} = 0.$

The plane $ax + by + cz + d = 0$ has normal vector $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

The shortest distance from $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ to the plane $ax + by + cz + d = 0$ is

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

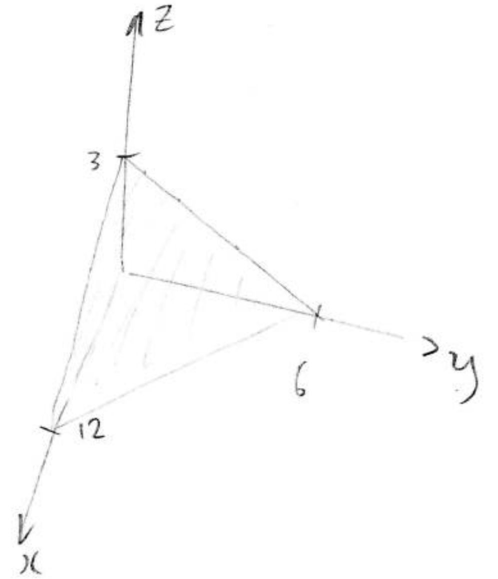
In this last lecture we will return to the theory of planes in $\mathbb{R}^3$ fully armed now with the theory of vector dot and cross products.

Recall that the Cartesian equation of a plane in $\mathbb{R}^3$ is given by

$$ax + by + cz + d = 0$$

where $a, b, c, d \in \mathbb{R}$.

Example 1: Sketch the plane $x + 2y + 4z = 12$ and find one point on the plane.



$\begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix}$

You will recall that a point on a plane and two vectors parallel to the plane is sufficient data to pin a plane down.

A moments consideration should convince you that a point on the plane and a perpendicular vector to the plane is also enough! We call the perpendicular vector a **normal** vector.

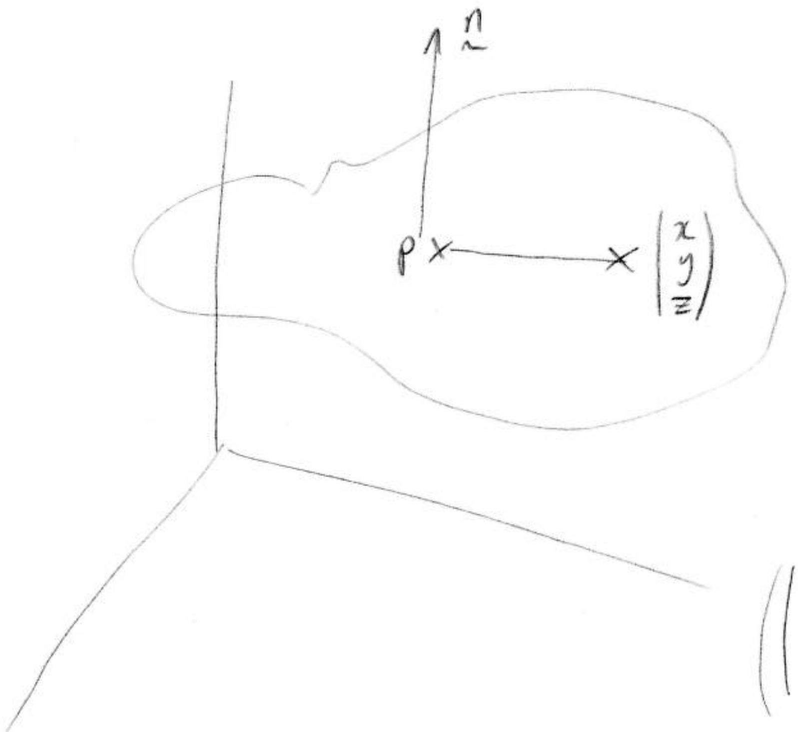
If a plane contains a point P and has a normal vector $\mathbf{n}$ then the point-normal form of the plane is given by

$$\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - P\right) \cdot \mathbf{n} = 0.$$

$n = \perp$ vector

$p =$ point

Discussion and proof:



$$\underset{\sim}{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - P$$

$$\underset{\sim}{n} \perp \underset{\sim}{v} \rightarrow \underset{\sim}{n} \cdot \underset{\sim}{v} = 0$$

$$\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - P\right) \underset{\sim}{n} = 0$$

★

Example 2: Find the point-normal equation of the plane passing through $P = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ and perpendicular to the vector $\mathbf{n} = \begin{pmatrix} 3 \\ 5 \\ -7 \end{pmatrix}$. Hence find the Cartesian equation of the plane.

$$\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \right) \cdot \begin{pmatrix} 3 \\ 5 \\ -7 \end{pmatrix} = 0$$

$$\begin{pmatrix} x-4 \\ y-1 \\ z+2 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} = 0 \rightarrow 3(x-4) + 5(y-1) + 7(z+2) = 0$$

$$3x - 12 + 5y - 5 + 7z + 14 = 0$$

$$3x + 5y - 7z = 31$$

$$\bigstar \quad \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \right) \cdot \begin{pmatrix} 3 \\ 5 \\ -7 \end{pmatrix} = 0, \quad 3x + 5y - 7z = 31 \quad \bigstar$$

!!Observe carefully how the normal vector appears as the coefficient of x, y and z!!

Fact: The plane $ax + by + cz = d$ has normal vector $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Example 3: Find a unit vector normal to the plane $6x + 2y + 3z = 8$.

$\underset{\sim}{n} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$ $\quad \hat{\underset{\sim}{n}} = \frac{1}{7}\begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$

$= \sqrt{36+4+9}$

$= \sqrt{49}$

$= 7$.

divide by it's own length.-

$$\bigstar \quad \begin{pmatrix} \frac{6}{7} \\ \frac{2}{7} \\ \frac{3}{7} \end{pmatrix} \quad \bigstar$$

Note also, that if all we need is a Cartesian equation we can find it without the point-normal form.

Example 4: Find a Cartesian equation of the plane passing through $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ and perpendicular to the vector $\begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$.

$P \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ $\quad \underset{\sim}{n} = \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$

$4x - 2y + 6z = \#$

$4(1) - 2 \times 2 + 6 \times 5 = \#$

$4x - 2y + 6z = 30$

$$\bigstar \quad 2x - y + 3z = 15 \quad \bigstar$$

Converting between Parametric and Cartesian forms

Example 5: Find a parametric vector equation of $x + 2y + 4z = 12$. Hence find a point on the plane and two vectors parallel to the plane.

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 12 \end{array}\right)$$

Let $y = \mu$, $z = \lambda$.

$x = 12 - 2\mu - 4\lambda$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix}}_{P} + \overbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}^{V_1}\mu + \overbrace{\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}}^{V_2}\lambda$$

$\mu, \lambda \in \mathbb{R}$.

★ Point: $\begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix}$ Vectors: $\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ ★

Example 6: Find a Cartesian equation of the plane

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}\mu + \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}\lambda\,; \quad \mu, \lambda \in \mathbb{R}.$$

use cross product

$$\underset{\sim}{V_1} \times \underset{\sim}{V_2} = \begin{vmatrix} i & j & k \\ 3 & 5 & 1 \\ 1 & -1 & 5 \end{vmatrix}$$

$$= i\begin{vmatrix} 5 & 1 \\ -1 & 5 \end{vmatrix} - j\begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} + k\begin{vmatrix} 3 & 5 \\ 1 & -1 \end{vmatrix} = 26\underset{\sim}{i} - 14j - 8\underset{\sim}{k}.$$

$$\begin{pmatrix} 26 \\ -14 \\ -8 \end{pmatrix} = \underset{\sim}{n}$$

$26x - 14y - 8z = \#$ $\leftarrow P\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$

$52 - 14 - 32 = \#$

★ $13x - 7y - 4z = 3$ ★

$\# = 6$

$26x - 14y - 8z = 6$

$\rightarrow \underline{13x - 7y - 4z = 3}$.

Common exam Q

Example 7: Find a Cartesian equation of the plane passing through the points A, B and C with position vectors $\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$ respectively.

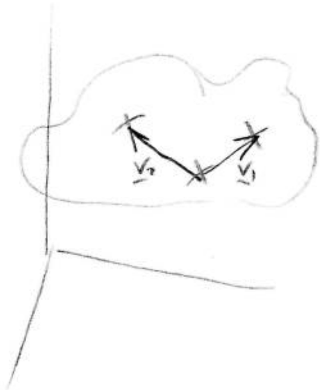
Note that this question could be more naturally asked as:

Find a Cartesian equation of the plane passing through the points $A\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$, $B\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$ and $C\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$.

$$\underset{\sim}{v_1} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

$$\underset{\sim}{v_2} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

vectors in the plane



$$\underset{\sim}{v_1} \times \underset{\sim}{v_2} = \begin{vmatrix} i & j & k \\ 0 & -1 & -1 \\ 2 & 1 & 3 \end{vmatrix}$$

$$i\begin{vmatrix} -1 & -1 \\ 1 & 3 \end{vmatrix} - j\begin{vmatrix} 0 & -1 \\ 2 & 3 \end{vmatrix} + k\begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= -2i - 2j + 2k = \underset{\sim}{n}$$

$$-2x - 2y + 2z = \#$$

$$-2 - 2 - 6 = -10$$

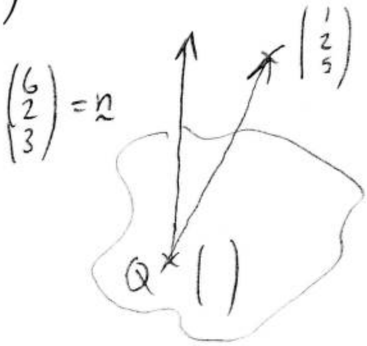
$$-2x - 2y + 2z = -10$$

★ $x + y - z = 5$ ★

$$x + y - z = 5$$

Example 8: Find the shortest distance between the plane $6x + 2y + 3z = 4$ and the point $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$.

Method 1: Using projections and noting that $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ is a point on the plane and $\begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$ is a normal to the plane.



$6x + 2y + 3z = 4.$

$P = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$

Find Q on plane $Q = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

$\underset{\sim}{v} = \overrightarrow{QP} = P - Q = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$

$\text{Proj}_{\underset{\sim}{n}} \underset{\sim}{v} = \dfrac{\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}}{\begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}} \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} = \dfrac{21}{49} \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$

$d = \left| \dfrac{21}{49} \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} \right| = \dfrac{21}{49} \left| \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} \right| = \dfrac{21}{49} \sqrt{36+4+9} = \dfrac{21}{49} \cdot 7$

$= 3.$

★ 3 ★

Method 2: The shortest distance from $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ to the plane $ax + by + cz + d = 0$ is

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$6x + 2y + 3z - 4 = 0$

$P = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$

$6(1) + 2(2) + 3 \times 5 - 4$

$= \sqrt{36 + 4 + 9}$

$= 21$

$= \frac{21}{7} \quad \to 7 \quad = 3.$

★ 3 ★

Example 9: Find the point on the plane in the previous example which is closest to $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} - \frac{21}{49} \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} - \frac{3}{7} \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} -11/7 \\ 8/7 \\ 26/7 \end{pmatrix}$

★ $\begin{pmatrix} -\frac{11}{7} \\ \frac{8}{7} \\ \frac{26}{7} \end{pmatrix}$ ★

100