MATH2019 PROBLEM CLASS
EXAMPLES 1

PARTIAL DIFFERENTIATION, MULTIVARIABLE TAYLOR SERIES AND LEIBNIZ' RULE

1. Given $f(x,y) = e^{-x^2+y^2}$ and $x = r\cos\theta$, $y = r\sin\theta$. Calculate $\dfrac{\partial f}{\partial \theta}$ and evaluate $\dfrac{\partial f}{\partial \theta}$ when $x = 1$, $y = 0$.

1998 2. For what values of n does

$$f(x,y,z) = \sin(3x)\cos(4y)e^{-nz}$$

satisfy the Lapace equation $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$?

1997 3. Let f and g be twice-differentiable functions of a single variable. Show by direct substitution into the partial differential equation that

$$w(x,t) = f(x+t) + g(x-t)$$

is a solution of the wave equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2}\,.$$

4. Show that if $w = f(u,v)$ satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = 0$$

and if $u = \dfrac{x^2-y^2}{2}$ and $v = xy$ then w satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0\,.$$

Multivariable Taylor Series

$$\begin{aligned} f(x,y) &= f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\ &\quad + \frac{1}{2!}\left((x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) + (y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right) + \cdots \end{aligned}$$

1994 5. Calculate the Taylor series expansion up to and including second order terms of the function

$$z = F(x,y) = \ln x \cos y\,,$$

about the point $(1, \pi/4)$. Use your result to estimate $F(1.1, \pi/4)$.

2011, S1 6. Expand $f(x,y) = e^y \sin x$ about $(0,1)$ up to and including second-order terms, using Taylor series for functions of two variables.

2017, S2 7. i) Calculate the Taylor series expansion of the function $f(x,y) = \ln(x+y)$ about the point $(1,0)$ up to and including quadratic terms.

ii) Use your solution to find an approximate value for $\ln(1.1)$.

2014, S1 8. A cone with radius r and perpendicular height h has volume $V = \frac{1}{3}\pi r^2 h$.

Determine the maximum error in calculating V given that $r = 4$ cm and $h = 3$ cm to the nearest millimetre.

2014, S2 9. The pressure P of a gas in a reactor is given by

$$P = r\rho T,$$

where ρ is the density, T is the temperature, and r is a constant. If the pressure in the reactor decreases by 5% and the temperature increases by 7%, what is the percentage change in the density of the gas inside the reactor? [Note that you do not need to know the value of r.]

2016, S1 10. The battery life of a mobile phone is given by

$$L = \frac{\alpha\, C}{b^2},$$

where C is the capacity of the battery, b is the width of the phone screen, and α is a positive constant. If the battery capacity C is increased by 20% and the screen size b is increased by 5%, use the chain rule to estimate the percentage change in the battery life of the phone. [Note that you do not need to know the value of α.]

2017, S1 11. A metal cylinder contains a volume of liquid given by

$$V = \pi r^2 h,$$

where r is the radius of the cylinder and h is the height of the cylinder. Small variations in the manufacturing process can result in errors in the cylinder radius of 1% and the cylinder height of 2%. What is the maximum percentage error in the volume of the cylinder?

2018, S2 12. The volume V of a circular cylinder with radius r and perpendicular height h is given by $V = \pi r^2 h$. Use a linear approximation to estimate the maximum percentage error in calculating V given that $r = 30$ metres and $h = 20$ metres, to the nearest metre.

Leibniz' Rule for Differentiation of Integrals

$$\frac{d}{dx}\int_u^v f(x,t)dt = \int_u^v \frac{\partial f}{\partial x}dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx}$$

2013, S2 13. Use Leibniz' rule to find

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}\, dx$$

given that

$$\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}.$$

$\boxed{\text{2014, S1}}$ 14. Use Leibniz' rule to find

$$\frac{d}{dt}\int_1^{t^2} \frac{\sin(\sqrt{x}\,)}{x}\,dx.$$

$\boxed{\text{2014, S2}}$ 15. Use Leibniz' rule to find

$$\int_0^\infty xe^{-bx}\sin x\;dx$$

given that

$$\int_0^\infty e^{-bx}\sin x\;dx = \frac{1}{1+b^2}.$$

$\boxed{\text{2015, S1}}$ 16. Use Leibniz' rule to calculate

$$\frac{d}{dy}\int_{y^2}^{1} \frac{\sin(xy)}{x}\,dx.$$

$\boxed{\text{2016, S2}}$ 17. Use Leibniz' rule to find

$$\frac{d}{dt}\int_1^{\sin t} e^{1-x^2}\,dx.$$

$\boxed{\text{2017, S1}}$ 18. You are given the following integral,

$$\int_0^\infty \sqrt{x}\mathrm{e}^{-tx}\;dx = \frac{\sqrt{\pi}}{2t^{3/2}}.$$

Use Leibniz' rule to evaluate

$$\int_0^\infty x^{3/2}\mathrm{e}^{-tx}\;dx.$$

$\boxed{\text{2017, S2}}$ 19. You are given the following integral,

$$\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\;dx = \sinh^{-1}(1)\,.$$

Use Leibniz' rule to evaluate

$$\int_0^a \frac{1}{(x^2+a^2)^{3/2}}\;dx.$$

$\boxed{\text{2018, S1}}$ 20. Consider the following ordinary differential equation

$$\frac{dy}{dx}+\frac{2}{x}y = \frac{\sin(x^2)}{x^2}, \quad x>0\,,$$

with initial condition $y(\sqrt{\frac{\pi}{2}}) = 0$. A student solves this ordinary differential equation and writes the solution in an integral form, i.e.,

$$y = \frac{1}{x^2}\int_{\sqrt{\frac{\pi}{2}}}^{x} \sin(t^2)\,dt\,.$$

i) **Verify** that this function y satisfies the initial condition.

ii) Use Leibniz' rule to **verify** that y satisfies the differential equation.

$\boxed{\text{2018, S2}}$ 21. You are given that

$$\int_0^\infty \frac{1}{\alpha^2 + x^2}\,dx = \frac{\pi}{2}\alpha^{-1}.$$

Use Leibniz' rule to find the following integral in terms of α

$$\int_0^\infty \frac{1}{(\alpha^2 + x^2)^2}\,dx.$$