LECTURE 26
FORCED OSCILLATIONS AND RESONANCE

When simple periodic forcing is added to the mechanical or electrical system studied earlier, we have to solve an equation like

$$my'' + cy' + ky = F_0 \sin wt \tag{1}$$

where, as before, $m > 0, c > 0, k > 0$.

This models a mechanical system driven by periodic forces or an electrical system forced by a periodic voltage. Solutions to such a system may become unstable and start to resonate, a critical issue when dealing with physical systems.

We saw in the previous lecture that the method of undetermined coefficients only works now and then. Sometimes it fails in a very special way and needs to be patched up.

Example 1 Solve the second order differential equation

$$y'' + 4y' = 12e^{-4t}$$

and write down the long term steady state solution.

First y_h:

$$y_H = A + Be^{-4t}$$

And now the guess for the particular solution:

Guess $y = \alpha e^{-4t}$

$$4(-4\alpha e^{-4t}) + (16\alpha e^{-4t}) = 0$$

$y_p = \alpha e^{-4t}$ is already a solution of y_H.

Guess $y = (\alpha t)e^{-4t}$

$$4(\alpha e^{-4t} - 4\alpha t e^{-4t}) + (-4\alpha e^{-4t} - 4\alpha e^{-4t} + 16\alpha t e^{-4t})$$

$$= -4\alpha e^{-4t} = 12e^{-4t}$$

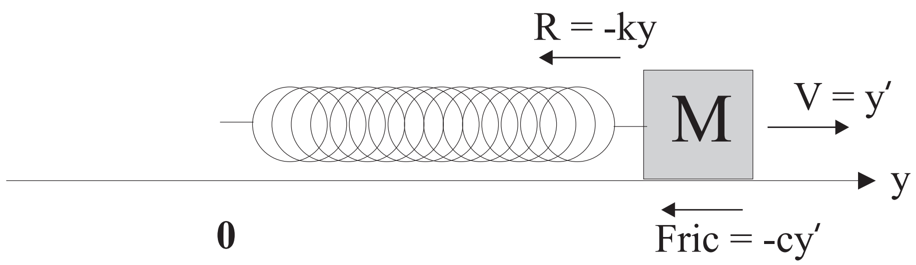
$$\therefore \alpha = -3$$

$$\therefore y_p = -3te^{-4t}$$

$$\therefore y = A + Be^{-4t} - 3te^{-4t}$$

★ $y = A + Be^{-4t} - 3te^{-4t}$, Steady State Solution : $y = A$ ★

As a general rule, whenever your natural guess for y_p is already lurking inside y_h you must modify that guess by multiplying by the independent variable x. If this is still part of y_h then you must multiply by x again and keep on doing it till you get something new!



Forced Oscillations

We return now to the physical system described earlier. Recall that this system was governed by the D.E.

$$my'' + cy' + ky = 0$$

where m (mass),c (damping frictional coefficient) and k (spring constant) are all non negative. We now add what is called a periodic forcing function to the RHS which effectively drives the entire system with an additional periodic force. The D.E. is then

$$my'' + cy' + ky = F_0 \sin wt$$

Example 2 A forced vibrating system is represented by

$$y'' + 9y = 60 \sin(2t)$$

where $r(t) = \sin(2t)$ is the driving force and y is the displacement from the equilibrium position. Note the absence of any damping friction! Find a formula for the motion of the system by solving the differential equation.

$$\lambda^2 + 9 = 0$$
$$\lambda = \pm 3i$$

$$\therefore y_H = A\cos 3t + B \sin 3t$$

$$\text{Try: } y_P = \alpha \sin 2t + \beta \cos 2t$$

$$9(\alpha \sin 2t + \beta \cos 2t) + (-4\alpha \sin 2t - 4\beta \cos 2t) = 60 \sin 2t$$

$$\therefore 5\alpha = 60 \quad , \quad \beta = 0$$
$$\alpha = 12$$

$$\therefore y_P = 12 \sin 2t$$

$$\therefore y = A\cos 3t + B\sin 3t + 12 \sin 2t$$

$$\star \quad y = A\cos(3t) + B\sin(3t) + 12\sin(2t) \quad \star$$

We now make a tiny modification to the forcing function which will have dramatic consequences.

Example 3 Solve the differential equation

$$y'' + 9y = 60\sin(3t)$$

All we have done is change the 2 to a 3! Observe that the damping frictional force is still zero thus the non-forced system will oscillate to infinity. The forcing function adds a layer of periodic behaviour which is unfortunately perfectly tuned to natural motion of the system. This leads to unstable resonance.

$y_H = A\cos 3t + B\sin 3t$

Try $y_P = \alpha t\cos 3t$

$9(\alpha t\cos 3t) - 6\alpha\sin 3t - 9\alpha t\cos 3t = 60\sin 3t$

$\therefore \alpha = -10$

$\therefore y_P = -10t\cos 3t$

$\therefore y = A\cos 3t + B\sin 3t - 10t\cos 3t$

★ $y = A\cos(3t) + B\sin(3t) - 10t\cos(3t)$ ★

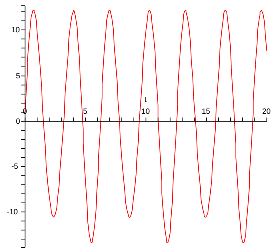


Figure 1: Example 2 No resonance

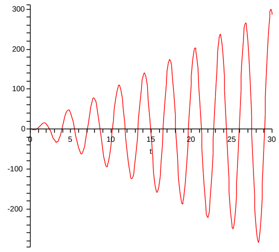


Figure 2: Example 3 With resonance

The solutions for the two previous examples are graphed above. Note the scale on the vertical axis!! What is happening here is that the system when not forced tends to naturally oscillate. The forcing function in Example 2 adds a periodic note of oscillation to the system which is different from the natural frequency and hence just produces a minor perturbation. But in Example 3 something critical happens! The forcing function is in harmony with the system and makes it resonate (it hums!!) The entire system becomes unstable and the amplitude goes through the roof. This is sometimes a good thing since you may want to amplify a signal...for example the amplification of a distant NASA signal. But in physical structures it is generally bad news and engineers have had to learn the hard way how disastrous simple resonance can be. Let's have a look at the Tacoma bridge disaster! This is more an example of aeroelastic flutter (like a flag in the wind) than resonance but it does display the immense importance of the use of appropriate damping in large structures.

$$http://www.youtube.com/watch?v = KVc7oBKzq9U$$

Resonance and flutter can be controlled with hydraulic dampers (bringing c into play) and fairing. The general equations governing resonance for forced oscillations are in your printed lecture notes.

IMPORTANT NOTE: Next lecture we will start on the theory of matrices and eigenvalues/eigenvectors. This will need a firm understanding of the theory of determinants and also Gaussian elimination of systems of linear equations. This material is covered in Math1131 linear algebra here at UNSW. If you are a rusty UNSW student or an overseas student who has not had seen this content before, please prepare by reading the revision first year algebra notes (available on Moodle).

[26] You can now do Q 85c,86,87