LECTURE 52
HEAT EQUATION

The equation
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$
is called the one-dimensional heat equation. It governs the heat flow across a homogenous bar where c is determined by the thermal properties of the bar.

The boundary conditions

$$u(0, t) = 0, \quad u(L, t) = 0 \qquad \text{for all time } t$$

specify that the bar has length L and that the temperature is maintained at zero on the ends of the bar for all time. Eventually all the heat will be drawn out of the bar.

Initial conditions take the form:

$$\text{initial temperature distribution} \qquad u(x, 0) = f(x)$$

Solutions (eigenfunctions) are obtained via separation of variables and take the form

$$u_n(x, t) = B_n \sin \frac{n\pi x}{L} e^{-\lambda_n^2 t} \quad \text{for} \quad n = 1, 2, 3, \cdots$$

where the λ_n (eigenvalues) are given by $\lambda_n = cn\pi/L$.

The general solution is the superposition of all the eigenfunctions and takes the form

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\lambda_n^2 t}$$

The initial temperature distribution is used to calculate the $B_n\ 's$ and will require the use of Fourier series and half range expansions when the initial distribution is non-sinusoidal.

Example 1 Solve the heat equation

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$$

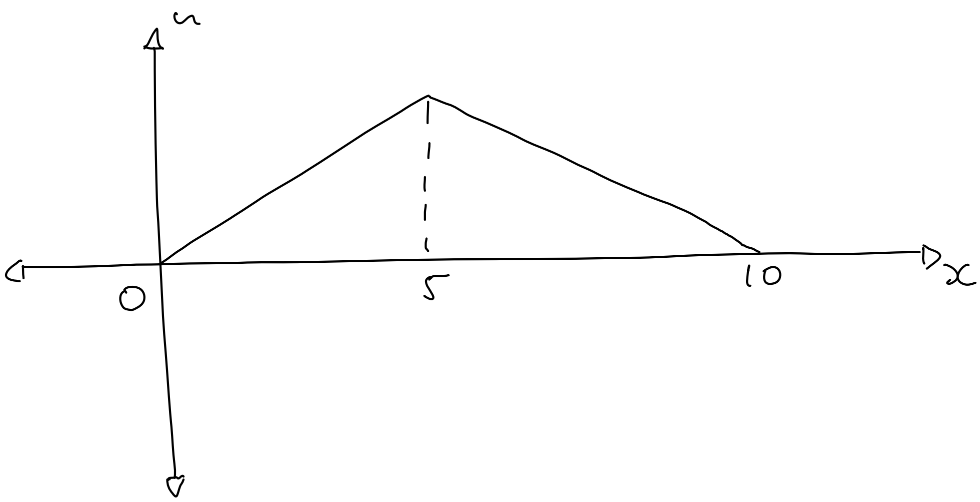
where

$u(0,t) = u(10,t) = 0$ for all t (maintained at zero degrees on the ends)

With initial temperature distribution given by

$$u(x,0) = h(x) = \begin{cases} x, & 0 \le x < 5; \\ 10 - x, & 5 \le x \le 10. \end{cases}$$

DISCUSSION



i) By assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$F'' - kF = 0$$

and

$$G' - 4kG = 0$$

for k constant. (Note that the D.E. for G is first order!)

$$\frac{\partial^2 u}{\partial x^2} = F''G \qquad \frac{\partial u}{\partial t} = FG'$$

$$FG' = 4F''G$$

$$\frac{1}{4}\frac{G'}{G} = \frac{F''}{F} = k$$

$$\therefore \quad G' - 4kg = 0, \quad F'' - kF = 0$$

ii) By implementing the boundary condition $u(0,t) = u(10,t) = 0$ show that

$$F(0) = F(10) = 0$$

$$u(0,t) = F(0)G(t) = 0$$

$$\therefore F(0) = 0 \quad \forall t$$

$$u(10,t) = F(10)G(t) = 0$$

$$\therefore F(10) = 0 \quad \forall t$$

iii) By solving for F with $k = 0$ and $k > 0$ show that non-trivial solutions will only arise from $k < 0$. (We will say that $k = -\rho^2$).

For $k=0$: $F'' = 0$

$$F = \alpha_1 x + \beta_1 = 0$$

$F(0) = \beta_1 = 0$ $F(10) = 10\alpha_1 = 0$

$\therefore \alpha_1 = 0$

$\therefore F(x) = 0 \quad \forall x$

For $k = \rho^2 > 0$: $F'' - \rho^2 F = 0$

$$\therefore F = \alpha_2 e^{\rho x} + \beta_2 e^{-\rho x}$$

$F(0) = \alpha_2 + \beta_2 = 0$ $F(10) = \alpha_2(e^{20\rho} - 1) = 0$

$\therefore \alpha_2 = -\beta_2$

$\therefore \{\alpha_2 = 0, \beta_2 = 0, F = 0\} \cup \{\rho = 0, k = 0, F = 0\}$

$\therefore$ solutions for $k \geq 0$ are trivial.

iv) By implementing $F(0) = F(10) = 0$ with $k = -\rho^2$ show that

$$u_n(x,t) = B_n \sin(\frac{n\pi x}{10}) e^{-\frac{n^2\pi^2}{25}t}$$

<u>For $k = -\rho^2 < 0$:</u> $F'' + \rho^2 F = 0$

$$\therefore F = \alpha_3 \cos(\rho x) + \beta_3 \sin(\rho x)$$

$F(0) = \alpha_3 = 0$ $\qquad$ $F(10) = \beta_3 \sin(10x) = 0$

$$\therefore \rho = \frac{n\pi}{10}, \quad \beta \neq 0$$

$$\therefore F_n(x) = \beta_n \sin\left(\frac{n\pi}{10} x\right)$$

$G' = 4kG = -4\rho^2 G = -4\left(\frac{n\pi}{10}\right)^2 G$ $\qquad$ (1^{st} ODE)

$$G = \gamma e^{-4\left(\frac{n\pi}{10}\right)^2 t}$$

$$\therefore G_n(x) = \gamma_n e^{-\frac{n^2\pi^2}{25}t}$$

$$\therefore u_n(x,t) = \beta_n \sin\left(\frac{n\pi}{10}x\right) \cdot \gamma_n e^{-\frac{n^2\pi^2}{25}t}$$

$$= B_n \sin\left(\frac{n\pi}{10}x\right) e^{-\frac{n^2\pi^2}{25}t}$$

v) Verify that this solution in iv) is obtainable through the equations presented at the start of the lecture. (Note that the examiners will not look kindly upon students who simply memorise formulae. You must prove your results!)

$$u(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}$$

$$L = 10, \quad \lambda_n^2 = \left(\frac{c n \pi}{L}\right)^2 = \left(\frac{2 n \pi}{10}\right)^2 = \frac{n^2 \pi^2}{25}$$

$$\therefore u(x,t) = B_n \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{n^2\pi^2}{25} t}$$

vi) Discuss the behaviour of the u_n as $t \to \infty$

As $t \to \infty$: $u_n(x,t) \to B_n \sin\left(\frac{n\pi x}{10}\right)$

vii)Assuming a final solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{10}) e^{-\frac{n^2\pi^2}{25}t}$$

express B_n in integral form.

$$B_n = \frac{1}{L} \int_{-L}^{L} h(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$= \frac{1}{10} \int_{-10}^{10} h(x) \sin\left(\frac{n\pi}{10} x\right) dx$$

$$\therefore B_n = \frac{1}{5} \int_{0}^{10} h(x) \sin\left(\frac{n\pi x}{10}\right) dx$$

$$\star \quad B_n = \frac{1}{5}\int_0^{10} h(x) \sin(\frac{n\pi x}{10}) dx \quad \star$$

Lets take a look at the evolution of the temperature for different initial temperature distributions:

http://mathlets.org/mathlets/heat-equation/

[52]You can now do Q 117,118