LECTURE 12
VECTOR CALCULUS

$$\nabla\phi = \text{grad}\,\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}.$$

grad : scalar to vector

$$\nabla\cdot\mathbf{F} = \text{div}\,\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

div : vector to scalar

$$\nabla\times\mathbf{F} = \text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

curl : vector to vector

Where the vector differential operator ∇ is given by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Before having a look at how the theories of calculus may be further applied to vectors, we have some final applications of grad, div and curl.

grad : · points to direction of max increase in scalar field.
· points perpendicular to level surface.

div : · net flow through a point.

curl : · rotation tendency at a point.

Example 1 Find the tangent plane and the normal line to the surface

$$x^4 + y^4 + 3z^4 = 20$$

at the point $P(2, 1, -1)$.

$$\operatorname{grad}(\phi) = \begin{pmatrix} 4x^3 \\ 4y^3 \\ 12z^3 \end{pmatrix}$$

$$\operatorname{grad}(\phi)\Big|_P = \begin{pmatrix} 32 \\ 4 \\ -12 \end{pmatrix} = \vec{n} = 4\begin{pmatrix} 8 \\ 1 \\ -3 \end{pmatrix}$$

$\therefore$ tangent plane: $32x + 4y - 12z = 32(2) + 4(1) - 12(-1)$ (sub P)

$$= 80$$

$$\therefore \{8x + y - 3z = 20\} \cap \left\{\vec{n} = \begin{pmatrix} 8 \\ 1 \\ -3 \end{pmatrix}\right\}$$

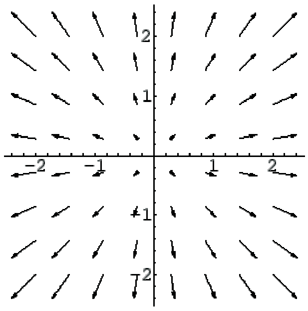
★ $32x + 4y - 12z = 80$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 32 \\ 4 \\ -12 \end{pmatrix} t : t \in \mathbb{R}$. ★

Physical Interpretation of Divergence $\nabla \cdot \mathbf{F} = \text{div}\mathbf{F}$

Imagine that $\mathbf{F}$ is a vector field representing the flow of a fluid and let B be a small wire cube immersed in the field. We wish to measure the rate R per unit volume at which the fluid flows through the cube across its six faces, at any given time. If we place the cube in a constant field (like a strongly flowing river with no turbulence) then $R = 0$, since all fluid that enters the cube also leaves at the same rate. Suppose however that a strong source of fluid is placed in the center of the cube (the end of a hose for example). Then R would be positive as water is leaving the cube on all of its faces. Alternatively if a drain hole is placed in the center of the cube then R would be negative since water would be entering the cube on all of its faces. This is what div measures! If we take the limit of this rate R as the size of the cube goes to zero we obtain

$$\nabla \cdot \mathbf{F} = \text{div}\,\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad \ldots.$$

a measure of the outward flow or expansion of the field from a point. Another way of saying this is that if the divergence is positive at a point P then a tiny cube placed around P would tend to explode whereas if the divergence is negative then the cube would tend to implode.



Positive divergence at the origin

Example 2 Calculate the divergence div($\mathbf{F}$) of the vector field $\mathbf{F} = (x^2)\mathbf{i} + (y^3 z)\mathbf{j} + (4z)\mathbf{k}$ at the origin and also the point $P(-10, 1, 5)$. How is the flow at P different from the flow at the origin?

$\text{div}(\vec{F}) = 2x + 3y^2 z + 4$

$\text{div}(\vec{F})\big|_{\vec{0}} = 4$ (outward flow)

$\text{div}(\vec{F})\big|_{P} = 2\times -10 + 3\times 1^2 \times 5 + 4 = -1$ (inward flow)

★ outward flow at the origin, inward flow at P. ★

curl gives axis of rotation

Physical Interpretion of $\nabla \times \mathbf{F} = \text{curl}\,\mathbf{F}$

Given a vector field $\mathbf{F}$, curl($\mathbf{F}$) measures infinitesimal rotations caused by the vector field. Consider a point P in the field and imagine a tiny sphere whose centre is fixed at P with the sphere still having the freedom of rotation in any direction. Then (the vector!!) curl($\mathbf{F}$) at P is the axis upon which the sphere would rotate under the action of the field (with the orientation of rotation given by the right hand rule). Furthermore the magnitude of the vector curl($\mathbf{F}$) measures the speed of that rotation.

Example 3 Consider the vector field $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j} + 4z\mathbf{k}$. Calculate curl($\mathbf{F}$) at the point $P(3,1,2)$ and hence show that this field induces a rotation at P about a vertical axis. Is the rotation clockwise or anticlockwise when viewed from above?

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & 4z \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2x - 2y \end{pmatrix}$$

$$\text{curl}(\vec{F})\big|_P = \begin{pmatrix} 0 \\ 0 \\ 2\times 3 - 2\times 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

Tendency for P to rotate about $\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ anticlockwise

Since curl($\mathbf{F}$) points straight up the positive $\mathbf{k}$ axis at P we have an anticlockwise rotation (when viewed from above) about a vertical axis!

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Example 4 Find a point Q where the field in the previous example induces no rotation.

$$\text{curl}(\vec{F})\big|_Q = \begin{pmatrix} 0 \\ 0 \\ 2x - 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore Q(0, 0, 0)$

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Vector Calculus

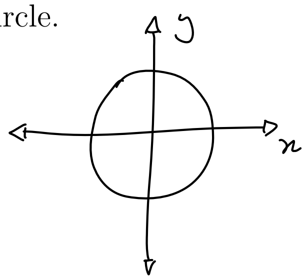
By establishing a time dependent position vector of a particle

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

it is possible to not only specify paths in $\mathbb{R}^3$ but to also to use calculus to analyse the velocity $\mathbf{v}(t) = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} + \dot{z}(t)\mathbf{k}$ and acceleration $\mathbf{a}(t) = \ddot{x}(t)\mathbf{i} + \ddot{y}(t)\mathbf{j} + \ddot{z}(t)\mathbf{k}$ of the particle's motion along the path. Note that the velocity vector is tangential to the path and that by taking the magnitude of the velocity vector we obtain the speed of the particle.

Example 5 Consider the path in $\mathbb{R}^2$ given by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$

Show that the path is a circle and prove that the velocity vector is always tangential to the circle and the acceleration of the particle is always directed to the centre of the circle.



$$\vec{r}(t) \cdot \vec{v}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \cdot \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} = 0$$

$$\therefore \vec{r}(t) \perp \vec{v}(t)$$

$$\vec{v}(t) \cdot \vec{a}(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} \cdot \begin{pmatrix} -\cos(t) \\ -\sin(t) \end{pmatrix} = 0$$

$$\therefore \vec{a}(t) \perp \vec{v}(t)$$

$x = \cos(t)$, $y = \sin(t)$

$\sin^2(t) + \cos^2(t) = 1$

$\therefore x^2 + y^2 = 1$

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Example 6 Consider the path in $\mathbb{R}^2$ given by $\mathbf{r}(t) = \cos(3t)\mathbf{i} + \sin(3t)\mathbf{j}$. How is this motion different from the previous example?

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Example 7 Suppose that a particle moves through space along the path

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + (t^2)\mathbf{k}. \qquad (\text{helix})$$

Describe the path and determine the speed and magnitude of the acceleration of the particle after 3 seconds.

$$\underset{\sim}{v}(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 2t \end{pmatrix}$$

$$\therefore \left|\underset{\sim}{v}(3)\right| = \sqrt{\sin^2 3 + \cos^2 3 + 6^2} = \sqrt{37}$$

$$\underset{\sim}{a}(t) = \begin{pmatrix} -\cos(t) \\ -\sin(t) \\ 2 \end{pmatrix}$$

$$\therefore \left|\underset{\sim}{a}(3)\right| = \sqrt{\cos^2 3 + \sin^2 3 + 2^2} = \sqrt{5}$$

★ *speed* $= \sqrt{37}$, *acceleration* $= \sqrt{5}$ ★

Example 8 Suppose that

$$\mathbf{r}_1(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$$

and that

$$\mathbf{r}_2(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$$

are two curves in space.

Prove that $(\mathbf{r}_1 \cdot \mathbf{r}_2)' = \mathbf{r}_1' \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \mathbf{r}_2'$

$$(\vec{r}_1 \cdot \vec{r}_2)' = \left(\begin{pmatrix} x_1(t) \\ y_1(t) \\ z_1(t) \end{pmatrix} \cdot \begin{pmatrix} x_2(t) \\ y_2(t) \\ z_2(t) \end{pmatrix} \right)'$$

$$= \left(x_1(t)x_2(t) + y_1(t)y_2(t) + z_1(t)z_2(t) \right)'$$

$$= \left(x_1'(t)x_2(t)\right) + x_1(t)x_2'(t) + \left(y_1'(t)y_2(t)\right) + y_1(t)y_2'(t) + \left(z_1'(t)z_2(t)\right) + z_1(t)z_2'(t)$$

$$= \vec{r}_1'(t) \cdot \vec{r}_2(t) + \vec{r}_1(t) \cdot \vec{r}_2'(t)$$

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[12] You can now do Q52,53,55