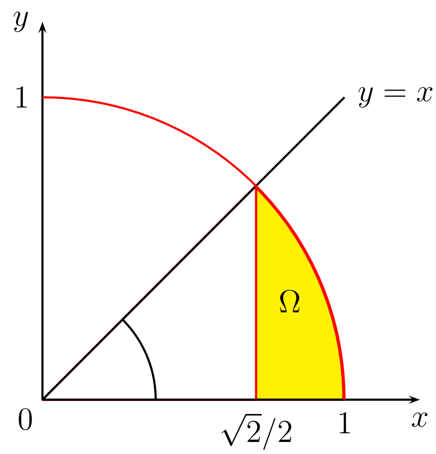
MATH2019 PROBLEM CLASS
EXAMPLES 4

DOUBLE INTEGRALS

$\boxed{1997}$ 1. Evaluate the following integral by changing to polar coordinates:

$$I = \int_{\sqrt{2}/2}^{1} \int_{0}^{\sqrt{1-x^2}} dy\,dx\,.$$

Solution: From the limits of integration in the double integral I we have for any x value between $\sqrt{2}/2$ and 1 the y value runs between 0 and the curve $y = \sqrt{1-x^2}$ (a circle of radius 1, centre $(0,0)$).



In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Considering the region of integration Ω, for any θ between 0 $(y=0)$ and $\frac{\pi}{4}$ $(y = x)$, the lower value of r (which measures the distance from the origin) will be at $x = \sqrt{2}/2 = \frac{1}{\sqrt{2}}$, i.e.,

$$x = \frac{1}{\sqrt{2}} = r\cos\theta \quad\Rightarrow\quad r = \frac{1}{\sqrt{2}\cos\theta}\,.$$

and the upper value at $r = 1$ (the circle of radius 1). Thus the region of integration Ω in polar coordinates is given by

$$\theta \in \left[0, \frac{\pi}{4}\right], \qquad r \in \left[\frac{1}{\sqrt{2}\cos\theta}, 1\right]\,.$$

Hence

$$\begin{aligned} I = \int_{\sqrt{2}/2}^{1} \int_{0}^{\sqrt{1-x^2}} dy\,dx = \int_{0}^{\frac{\pi}{4}} \int_{\frac{1}{\sqrt{2}\cos\theta}}^{1} r\,dr\,d\theta \quad &= \quad \int_{0}^{\frac{\pi}{4}} \left(\frac{1}{2} r^2 \Big|_{\frac{1}{\sqrt{2}\cos\theta}}^{1} \right) d\theta \\ &= \quad \frac{1}{2} \int_{0}^{\frac{\pi}{4}} \left(1 - \frac{1}{2}\sec^2\theta\right) d\theta \\ &= \quad \frac{1}{2}\left(\theta - \frac{1}{2}\tan\theta \Big|_{0}^{\frac{\pi}{4}}\right) = \frac{\pi}{8} - \frac{1}{4}\,. \end{aligned}$$

Note (as a check) the integral I is calculating the area of the region Ω (since the integrand is $f(x, y) = 1$). By inspection, this area is the area of the sector (radius 1 and angle $\frac{\pi}{4}$) minus the area of the triangle (base $\frac{1}{\sqrt{2}}$ and height $\frac{1}{\sqrt{2}}$), i.e.,

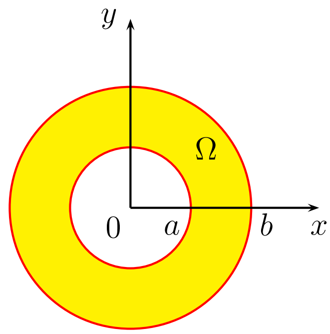
$$I = A_{\text{sector}} - A_{\text{triangle}} = \frac{1}{2}(1)^2\frac{\pi}{4} - \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{8} - \frac{1}{4}\,.$$

$\boxed{1998}$ 2. An annular washer of constant surface density δ occupies the region between the circles

$$x^2 + y^2 = a^2 \quad \text{and} \quad x^2 + y^2 = b^2 \quad \text{where} \quad b > a\,.$$

Find the moment of inertia of the washer about the x-axis.

Solution: Since the region of integration Ω (diagram below) is between two circles ($x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$) it is best to convert to polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$.



Thus the region of integration Ω in polar coordinates is given by

$$\theta \in [0, 2\pi]\,, \qquad r \in [a, b]\ .$$

Hence the moment of inertia about the x-axis I_x is given by

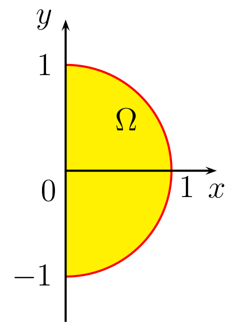
$$\begin{aligned}
I_x = \iint_\Omega y^2\delta(x,y)\,dA \quad &= \quad \delta\int_0^{2\pi}\int_a^b (r\sin\theta)^2\, r\, dr\, d\theta \quad \text{since } \delta \text{ is a constant} \\
&= \quad \delta\int_0^{2\pi} \sin^2\theta \left(\frac{1}{4}r^4\Big|_a^b\right) d\theta \\
&= \quad \delta\left(\frac{b^4 - a^4}{4}\right)\int_0^{2\pi}\sin^2\theta\, d\theta \\
&= \quad \delta\left(\frac{b^4 - a^4}{4}\right)\int_0^{2\pi}\frac{1}{2}\left(1 - \cos(2\theta)\right)\, d\theta \\
&= \quad \delta\left(\frac{b^4 - a^4}{8}\right)\left(\theta - \frac{1}{2}\sin(2\theta)\Big|_0^{2\pi}\right) \\
&= \quad \delta\left(\frac{b^4 - a^4}{8}\right)\left((2\pi - 0) - (0 - 0)\right) \\
&= \quad \frac{\pi\delta}{4}\left(b^4 - a^4\right)\ .
\end{aligned}$$

$\boxed{\text{2014, S1}}$ 3. Consider the double integral

$$I = \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} 3x \; dx \; dy.$$

i) Sketch the region of integration.

Solution: From the limits of integration in the double integral I we have for any y value between -1 and 1 the x value runs between 0 and the curve $x = \sqrt{1-y^2}$ (a circle of radius 1, centre $(0,0)$).



ii) Evaluate I using polar coordinates.

Solution: In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Considering shape of the region of integration Ω from part i) (which is a semicircle of radius 1 in quadrants 4 and 1) we have

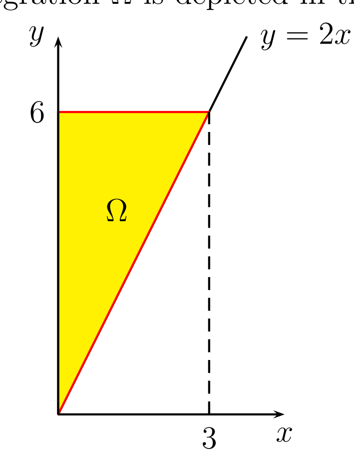
$$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \qquad r \in [0,1]\ .$$

Hence

$$\begin{aligned} I = \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} 3x \; dx \; dy &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{1} (3r\cos\theta)\, r \, dr \, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \left(r^3\Big|_0^1 \right) \, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \, d\theta \\ &= \left(\sin\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right) = (1-(-1)) = 2\,. \end{aligned}$$

$\boxed{\text{2014, S1}}$ 4. A thin triangular plate bounded by $y = 2x,\ y = 6$ and the y axis has non-uniform density given by $\rho(x,y) = 4xy$. Find the mass of the plate by evaluating an appropriate double integral in Cartesian coordinates.

Solution: The region of integration Ω is depicted in the following diagram.



$$
\begin{aligned}
M = \iint_\Omega \rho(x,y)\, dA = \int_0^3 \int_{2x}^6 4xy\, dy\, dx &= \int_0^3 \left(2xy^2\Big|_{2x}^6 \right) dx \\
&= \int_0^3 \left(72x - 8x^3\right) dx \\
&= 36x^2 - 2x^4\Big|_0^3 \\
&= 324 - 162 = 162\,.
\end{aligned}
$$

Note equally the calculation could be done with the order of integration reversed, i.e.,

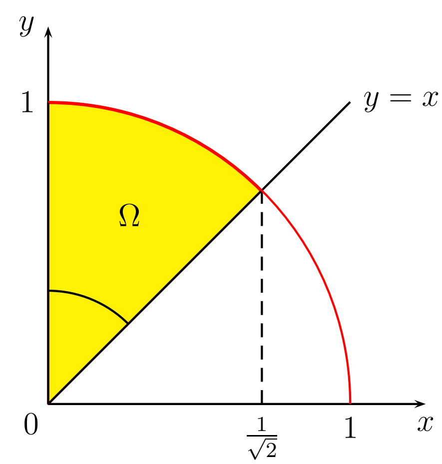
$$
\begin{aligned}
M = \iint_\Omega \rho(x,y)\, dA = \int_0^6 \int_0^{y/2} 4xy\, dx\, dy &= \int_0^6 \left(2yx^2\Big|_0^{y/2} \right) dy \\
&= \int_0^6 \frac{y^3}{2}\, dy \\
&= \frac{y^4}{8}\Big|_0^6 = 162\,.
\end{aligned}
$$

2014, S2 5. Consider the double integral

$$
\int_0^{\frac{1}{\sqrt{2}}} \int_x^{\sqrt{1-x^2}} 3x\ dy\ dx.
$$

i) Sketch the region of integration.

Solution: From the limits of integration in the double integral we have for any x value between 0 and $\frac{1}{\sqrt{2}}$ the y value runs between straight line $y = x$ and the curve $y = \sqrt{1-x^2}$ (a circle of radius 1, centre $(0,0)$).



ii) Evaluate the double integral by first converting to polar coordinates.

Solution: In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Considering shape of the region of integration Ω (which is a sector of radius 1 between $\frac{\pi}{4}$ and $\frac{\pi}{2}$) we have

$$
\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right], \qquad r \in [0,1]\ .
$$

Hence

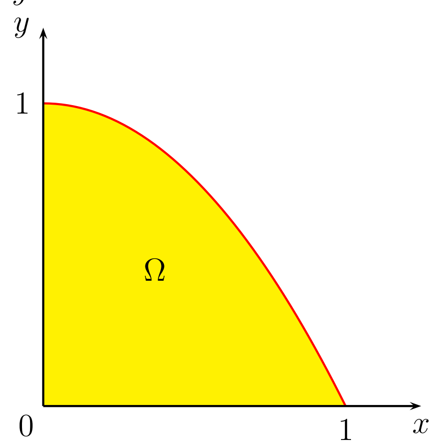
$$\begin{aligned}
\int_0^{\frac{1}{\sqrt{2}}}\int_x^{\sqrt{1-x^2}} 3x\ dy\ dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^1 (3r\cos\theta)\, r\, dr\, d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\theta \left(r^3\Big|_0^1 \right)\, d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\theta\, d\theta \\
&= \left(\sin\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right) = 1 - \frac{1}{\sqrt{2}}\,.
\end{aligned}$$

$\boxed{\text{2015, S1}}$ 6. Consider the double integral

$$\int_0^1\int_0^{1-x^2} \frac{y}{\sqrt{1-y}}\ dy\ dx.$$

i) Sketch the region of integration.

***Solution**:* From the limits of integration in the double integral we have for any x value between 0 and 1 the y value runs between 0 and the parabola $y = 1 - x^2$.



ii) Evaluate the double integral by first reversing the order of integration.

***Solution**:* Now we wish for any y value from 0 to 1 the x value runs between 0 and $x = \sqrt{1-y}$ (a rearrangement of $y = 1 - x^2$ with non-negative x the subject of the equation). Thus

$$\begin{aligned}
\int_0^1\int_0^{1-x^2} \frac{y}{\sqrt{1-y}}\, dy\, dx &= \int_0^1\int_0^{\sqrt{1-y}} \frac{y}{\sqrt{1-y}}\, dx\, dy \\
&= \int_0^1 \left(\frac{y}{\sqrt{1-y}} x\Big|_0^{\sqrt{1-y}} \right)\, dy \\
&= \int_0^1 \frac{y}{\sqrt{1-y}}\sqrt{1-y}\, dy \\
&= \int_0^1 y\, dy = \frac{1}{2}y^2\Big|_0^1 = \frac{1}{2}\,.
\end{aligned}$$

$\boxed{\text{2015, S2}}$ 7. The area A of a region R of the xy-plane is given by

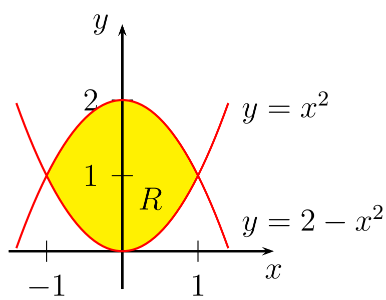
$$A = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} dx\,dy.$$

i) Sketch the region R.

Solution: To determine the region of integration R we consider the double integrals for A separately. In the first one, from the limits we have for any y value between 0 and 1 the x value runs between the curve $x = -\sqrt{y}$ and curve $x = \sqrt{y}$ (which both can be rearranged to $y = x^2$). In the second double integral, from the limits we have for any y value between 1 and 2 the x value runs between the curve $x = -\sqrt{2-y}$ and curve $x = \sqrt{2-y}$ (which both can be rearranged to $y = 2 - x^2$). Next we determine the points of intersection of the curves $y = x^2$ and $y = 2 - x^2$, i.e.,

$$\begin{aligned} y = x^2 = 2 - x^2 \quad &\Rightarrow \quad x^2 = 1 \\ &\Rightarrow \quad x = \pm 1\,. \end{aligned}$$

Thus the curves intersect at $(-1, 1)$ and $(1, 1)$. Putting it all together we have the following sketch.



ii) When the order of integration is reversed the expression for A becomes

$$A = \int_{-1}^{1} \int_{l_1(x)}^{l_2(x)} dy\,dx.$$

Find the limits $l_1(x)$ and $l_2(x)$.

Solution: From the diagram above, for any x from -1 to 1 the y value will run from the lower curve $y = x^2$ to the upper curve $y = 2 - x^2$. Hence

$$l_1(x) = x^2, \qquad l_2(x) = 2 - x^2\,.$$

iii) Hence, find the value of A.

Solution:

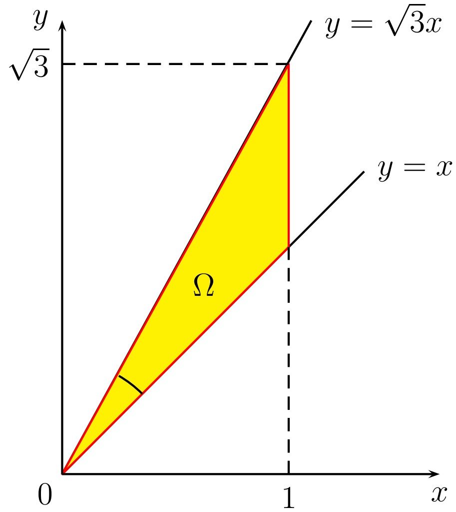
$$\begin{aligned} A = \int_{-1}^{1} \int_{x^2}^{2-x^2} dy\,dx = \int_{-1}^{1} \left(y\Big|_{x^2}^{2-x^2} \right) dx \quad &= \quad \int_{-1}^{1} \left(2 - 2x^2\right) dx \\ &= \quad 2\int_0^1 \left(2 - 2x^2\right) dx \\ &= \quad 4\left(x - \frac{1}{3}x^3 \Big|_0^1 \right) \\ &= \quad 4\left(1 - \frac{1}{3}\right) = \frac{8}{3}\,. \end{aligned}$$

$\boxed{\text{2016, S1}}$ 8. Consider the double integral

$$\int_0^1 \int_x^{\sqrt{3}\,x} \frac{x}{x^2+y^2}\, dy\, dx.$$

i) Sketch the region of integration.

Solution: From the limits of integration in the double integral we have for any x value between 0 and 1 the y value runs between straight lines $y = x$ and $y = \sqrt{3}x$.



ii) Evaluate the double integral using polar coordinates.

Solution: In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Considering the region of integration Ω, for any θ between $\frac{\pi}{4}$ and $\frac{\pi}{3}$, the upper value of r will be at $x = 1$, i.e.,

$$x = 1 = r\cos\theta \quad \Rightarrow \quad r = \frac{1}{\cos\theta}.$$

and thus the region of integration Ω in polar coordinates is given by

$$\theta \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right], \qquad r \in \left[0, \frac{1}{\cos\theta}\right].$$

Hence

$$\begin{aligned}
I = \int_0^1 \int_x^{\sqrt{3}\,x} \frac{x}{x^2+y^2}\, dy\, dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \int_0^{\frac{1}{\cos\theta}} \frac{r\cos\theta}{r^2}\, r\, dr\, d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \int_0^{\frac{1}{\cos\theta}} \cos\theta\, dr\, d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(r\cos\theta \Big|_0^{\frac{1}{\cos\theta}} \right) d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos\theta} \cos\theta\, d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\theta = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.
\end{aligned}$$

$\boxed{\text{2016, S1}}$ 9. Because of the effect of rotation, the Earth is not a perfect sphere but is slightly fatter at the equator than it is at the poles. A good approximation for the shape of the earth is an ellipsoid described by the formula

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1,$$

where z is the coordinate measured along the axis of rotation, $a = 6378$ km is the radius of the Earth at the equator and $b = 6357$ km is the radius of the Earth at the poles.

Calculate the volume of the Earth using an appropriate double integral.

Solution: We can calculate the volume of the Earth by considering the two halves of the "hemisphere" (actually ellipsoid), i.e., $z = \pm b\sqrt{1-\dfrac{x^2+y^2}{a^2}}$. The volume is twice the volume under one "hemisphere" $z = b\sqrt{1-\dfrac{x^2+y^2}{a^2}}$. The region of integration Ω for the double integral is the projection of z onto the xy-plane, i.e.,

$$z = 0 = b\sqrt{1-\frac{x^2+y^2}{a^2}} \quad\Rightarrow\quad x^2+y^2=a^2 \quad \text{(circle of radius } a\text{, centre at the origin)}\,.$$

Since the region of integration is (part of) a circle we use polar coordinates ($x = r\cos\theta, y = r\sin\theta$) to calculate the double integral for the volume V. In polar coordinates the region Ω is given by

$$r \in [0, a], \qquad \theta \in [0, 2\pi]\,.$$

Thus the volume V of the Earth is given by

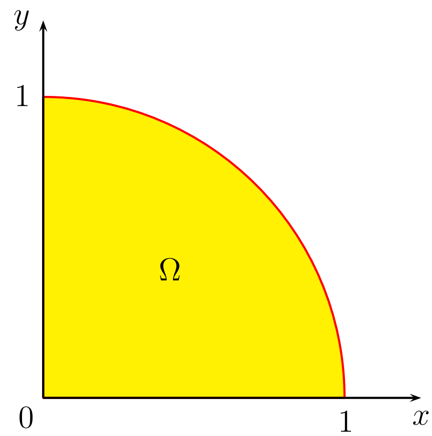
$$\begin{aligned}
V = 2\iint_\Omega b\sqrt{1-\frac{x^2+y^2}{a^2}}\,dA &= 2b\int_0^{2\pi}\int_0^a \sqrt{1-\frac{r^2}{a^2}}\,r\,dr\,d\theta \\
&= 2b\int_0^{2\pi}\left(-\frac{a^2}{3}\left(1-\frac{r^2}{a^2}\right)^{\frac{3}{2}}\Bigg|_0^a\right)d\theta \\
&= -\frac{2a^2b}{3}\int_0^{2\pi}(0-1)\,d\theta \\
&= \frac{2a^2b}{3}\int_0^{2\pi}d\theta \\
&= \frac{4\pi}{3}a^2b\,.
\end{aligned}$$

With $a = 6378$ km and $b = 6357$ km then $V = \dfrac{4\pi}{3}(6378)^2(6357) = 1.083\times10^{12}$ km^3.

$\boxed{\text{2016, S2}}$ 10. A thin plate in the first quadrant is bounded by the circle $x^2 + y^2 = 1$ and the coordinate axes. The plate has uniform density $\delta(x, y) = 1$.

i) Sketch the plate in the $x - y$ plane.

Solution:



ii) Without evaluating any integrals write down the mass of the plate.

Solution: Mass $M=$ density δ $\times$ area A. In this case the density δ is constant, i.e., $\delta = 1$. Since the region Ω is a quarter circle of radius 1 then the area is $\frac{\pi}{4}$. Hence the mass $M = 1 \times \frac{\pi}{4} = \frac{\pi}{4}$.

iii) Find the coordinates of the centroid $(\bar{x}, \bar{y})$ of the plate by evaluating an appropriate double integral in polar coordinates.
(Note that by symmetry, $\bar{y} = \bar{x}$).

Solution: In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Considering shape of the region of integration Ω from part i) (which is a quarter-circle of radius 1 in quadrant 1) we have

$$\theta \in \left[0, \frac{\pi}{2}\right], \qquad r \in [0, 1] \,.$$

Hence

$$\begin{aligned}
M_y = \iint_\Omega x\,\delta(x, y)\,dA &= \int_0^{\frac{\pi}{2}} \int_0^1 \underbrace{r\cos\theta}_{x}\,\underbrace{1}_{\delta(x,y)}\,\underbrace{r\ dr\,d\theta}_{dA} \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \cos\theta\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \left(\frac{1}{3} r^3 \cos\theta \Big|_0^1 \right) d\theta \\
&= \frac{1}{3} \int_0^{\frac{\pi}{2}} \cos\theta\,d\theta \\
&= \frac{1}{3} \sin\theta \big|_0^{\frac{\pi}{2}} = \frac{1}{3} \,.
\end{aligned}$$

Hence the coordinates of the centroid are $\bar{x} = \bar{y} = \dfrac{M_y}{M} = \dfrac{1/3}{\pi/4} = \dfrac{4}{3\pi}$.

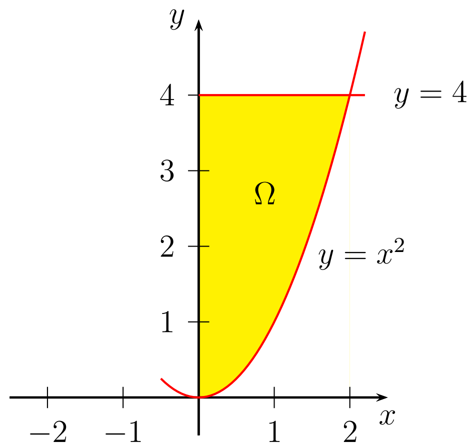
2016, S2

11. Consider the double integral

$$I = \int_0^2 \int_{x^2}^4 \frac{e^y}{\sqrt{y}}\, dy\, dx$$

i) Sketch the region of integration.

Solution: From the limits of integration in the double integral I we have for any x value between 0 and 2 the y value runs between parabola $y = x^2$ and the straight line $y = 4$.



ii) Evaluate I by first reversing the order of integration.

Solution: Now we wish for any y value from 0 to 4 the x value runs between 0 and $x = \sqrt{y}$ (a rearrangement of $y = x^2$ to make x the subject). Thus

$$\begin{aligned} I = \int_0^2 \int_{x^2}^4 \frac{e^y}{\sqrt{y}}\, dy\, dx &= \int_0^4 \int_0^{\sqrt{y}} \frac{e^y}{\sqrt{y}}\, dx\, dy \\ &= \int_0^4 \left(\frac{e^y}{\sqrt{y}} x \Big|_0^{\sqrt{y}} \right) dy \\ &= \int_0^4 \frac{e^y}{\sqrt{y}} (\sqrt{y} - 0)\, dy = \int_0^4 e^y\, dy = e^4 - 1\,. \end{aligned}$$

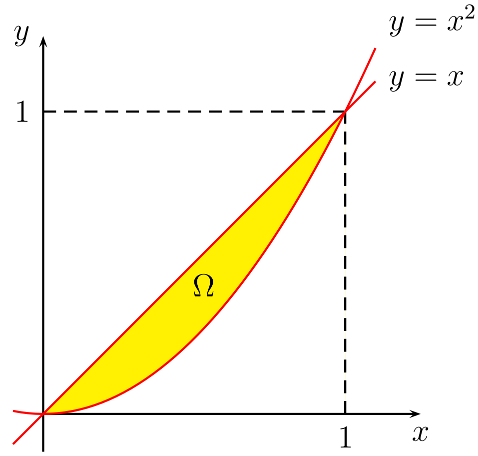
2017, S1

12. Use double integration to find the area bounded by $y = x$ and $y = x^2$.

Solution: We first determine the points of intersection of the curves $y = x$ and $y = x^2$, i.e.,

$$\begin{aligned} y = x = x^2 \quad &\Rightarrow \quad x(x-1) = 0 \\ &\Rightarrow \quad x = 0, 1\,. \end{aligned}$$

Thus the curves intersect at $(0, 0)$ and $(1, 1)$. Hence we have the following sketch for the region bounded by the two curves.



$$A = \iint_\Omega dA = \int_0^1 \int_{x^2}^x 1\, dy\, dx = \int_0^1 \left(y|_{x^2}^x \right) dx \;=\; \int_0^1 (x - x^2)\, dx \quad \text{i.e., “Top curve” - “Bottom curve”}$$

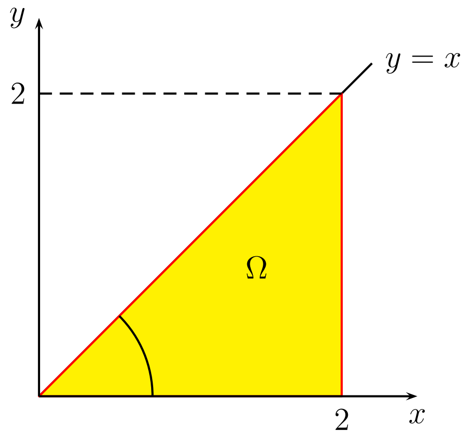
$$= \left(\frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_0^1 \right) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}\,.$$

$\boxed{\text{2017, S2}}$ 13. Consider the double integral

$$I = \int_0^2 \int_0^x \frac{x}{x^2 + y^2}\, dy\, dx.$$

i) Sketch the region of integration.

Solution: From the limits of integration in the double integral I we have for any x value between 0 and 2 the y value runs between 0 and the straight line $y = x$. The region of integration Ω is a triangular region.



ii) Evaluate I by first changing to polar coordinates.

Solution: In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Considering the region of integration Ω, for any θ between 0 and $\frac{\pi}{4}$, the upper value of r will be at $x = 2$, i.e.,

$$x = 2 = r\cos\theta \;\;\Rightarrow\;\; r = \frac{2}{\cos\theta}\,.$$

Thus the region of integration Ω in polar coordinates is given by

$$\theta \in \left[0, \frac{\pi}{4}\right], \qquad r \in \left[0, \frac{2}{\cos\theta}\right]\,.$$

Hence

$$I = \int_0^2 \int_0^x \frac{x}{x^2+y^2}\, dy\, dx = \int_0^{\frac{\pi}{4}} \int_0^{\frac{2}{\cos\theta}} \frac{r\cos\theta}{r^2}\, r\, dr\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} \int_0^{\frac{2}{\cos\theta}} \cos\theta\, dr\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} \left(r\cos\theta \Big|_0^{\frac{2}{\cos\theta}} \right) d\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{2}{\cos\theta} \cos\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} 2\, d\theta = 2\frac{\pi}{4} = \frac{\pi}{2}.$$

$\boxed{\text{2017, S2}}$ 14. Find the volume of the solid bounded above by the surface $z = 1 - x^2 - y^2$ and below by the plane $z = 0$.

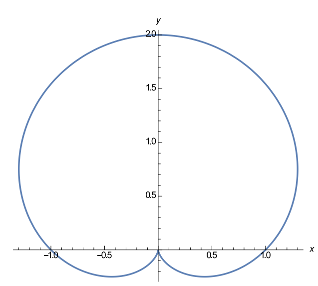
***Solution**:* The boundary of the integration region Ω is the intersection of the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$, i.e., $x^2 + y^2 = 1$ at $z = 0$. Since the region of integration is (part of) a circle then we use polar coordinates ($x = r\cos\theta, y = r\sin\theta$) to calculate the double integral for the volume V. In polar coordinates the region Ω is given by

$$r \in [0,1], \qquad \theta \in [0, 2\pi].$$

Thus the volume of the solid bounded above by the surface $z = 1 - x^2 - y^2$ and below by the plane $z = 0$ is given by

$$V = \iint_\Omega f(x,y)\, dA = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^1 (r - r^3)\, dr\, d\theta$$
$$= 2\pi \left(\frac{1}{2} r^2 - \frac{1}{4} r^4 \Big|_0^1 \right)$$
$$= 2\pi \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\pi}{2}.$$

$\boxed{\text{2018, S1}}$ 15. Consider the polar curve $r = 1 + \sin\theta$ whose figure is given below.



Determine the area of the region enclosed by the curve by using a suitable double integral.

Solution: Noting the symmetry of the figure about the vertical axis we have the area A of the region Ω enclosed by the curve (a cardioid) is given by

$$
\begin{aligned}
A = \iint_\Omega dA &= 2\int_{-\pi/2}^{\pi/2}\int_0^{1+\sin\theta} r\,dr\,d\theta \\
&= \int_{-\pi/2}^{\pi/2}(1+\sin\theta)^2 d\theta \\
&= \int_{-\pi/2}^{\pi/2}\Big(1+2\sin\theta+\sin^2\theta\Big)d\theta \\
&= \int_{-\pi/2}^{\pi/2}\Big(1+2\sin\theta+\frac{1}{2}(1-\cos(2\theta))\Big)d\theta \\
&= \int_{-\pi/2}^{\pi/2}\Big(\frac{3}{2}+2\sin\theta-\frac{1}{2}\cos(2\theta)\Big)d\theta \\
&= \frac{3}{2}\theta-2\cos\theta-\frac{1}{4}\sin((2\theta))\Big|_{-\pi/2}^{\pi/2} \\
&= \frac{3\pi}{2}.
\end{aligned}
$$

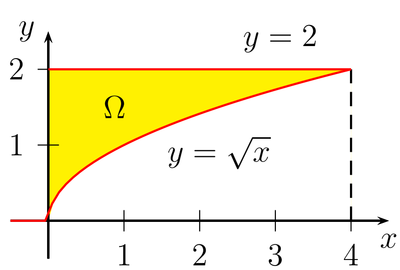
Thus the area of region enclosed by the cardioid is $3\pi/2$ square unit.

$\boxed{\text{2018, S2}}$ 16. Consider the double integral

$$I = \int_0^4\int_{\sqrt{x}}^2 10x\ dydx.$$

i) Sketch the region of integration.

Solution: From the limits of integration in the double integral we have for any x value between 0 and 4 the y value runs between $y=\sqrt{x}$ and the parabola $y=2$.



ii) Evaluate I with the order of integration reversed.

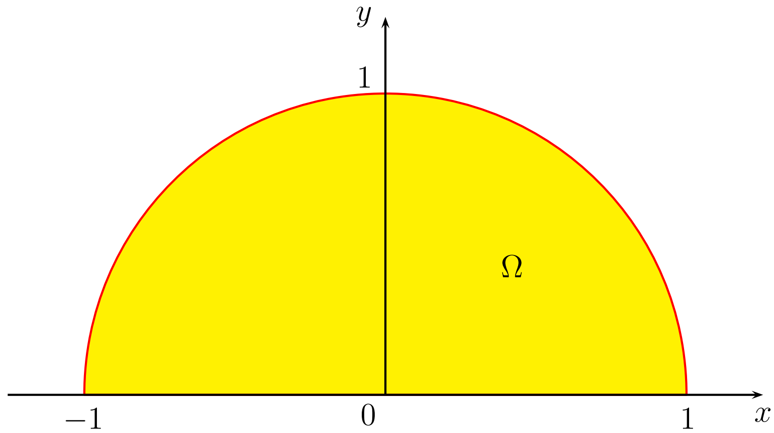
Solution: Now we wish for any y value from 0 to 2 the x value runs between 0 and $x=y^2$ (a rearrangement of $y=\sqrt{x}$ to make x the subject). Thus

$$
\begin{aligned}
I = \int_0^4\int_{\sqrt{x}}^2 10x\,dydx &= \int_0^2\int_0^{y^2} 10x\,dxdy \\
&= \int_0^2\left(5x^2\big|_0^{y^2}\right)\,dy \\
&= \int_0^2 5y^4\,dy \\
&= y^5\big|_0^2 \\
&= 32\,.
\end{aligned}
$$

$\boxed{\text{2018, S2}}$ 17. Let Ω be the semi-circular region bounded by $y = \sqrt{1 - x^2}$ and $y = 0$. The region Ω is of uniform density and has centroid $(\bar{x}, \bar{y})$.

i) Sketch the region Ω and write down its area.

Solution:



Area of Ω is $\dfrac{\pi}{2}$ (area of semicircle, radius 1).

ii) Explain why $\bar{x} = 0$.

Solution: Since BOTH the density (constant in this case) and region Ω are symmetric about the y-axis then $\bar{x} = 0$.

iii) Find $\bar{y}$ by evaluating an appropriate double integral expressed in polar coordinates.

Solution:

$$\begin{aligned} \bar{y} = \frac{1}{A}\iint_{\Omega} y\delta(x,y)\,dA &= \frac{2}{\pi}\int_0^{\pi}\int_0^1 r\sin\theta r\,dr\,d\theta \\ &= \frac{2}{\pi}\int_0^{\pi}\left(\frac{1}{3}r^3\Big|_0^1\right)\sin\theta\,d\theta \\ &= \frac{2}{3\pi}\int_0^{\pi}\sin\theta\,d\theta \\ &= \frac{2}{3\pi}\left(-\cos\theta|_0^{\pi}\right) \\ &= \frac{4}{3\pi}\,. \end{aligned}$$