LECTURE 11
APPLICATIONS OF GRAD

Given a scalar field ϕ, the directional derivative of ϕ in the direction of the vector $\mathbf{b}$ is given by $\quad (\text{grad}\,\phi)\cdot\hat{\mathbf{b}} \quad$ where $\hat{\mathbf{b}}$ is the unit vector in the direction of $\mathbf{b}$.

Given a scalar field ϕ at a point P the direction of maximum increase of ϕ from P is given by $\nabla\phi|_P$ with the magnitude of the increase being the magnitude of $\nabla\phi|_P$.

Given a scalar field ϕ at a point P the direction of maximum decrease of ϕ from P is given by $-\nabla\phi|_P$ with the magnitude of the decrease being the magnitude of $-\nabla\phi|_P$.

$\nabla\phi|_P$ points perpendicular to the level surface (or curve in 2-D) at P.

(**Note that $\nabla\phi|_P$ is just short hand for** grad ϕ **at** P)

Directional Derivatives

We have seen earlier that for a scalar field $\phi(x, y, z)$, we can easily find rates of change in the x, y and z directions by using the partial derivatives $\dfrac{\partial\phi}{\partial x}, \dfrac{\partial\phi}{\partial y}$ and $\dfrac{\partial\phi}{\partial z}$. But what if we are immersed in a scalar field and wish to determine the rate of change of the field in some other direction specified by a vector $\mathbf{b}$. This is called a **directional derivative**.

Given a scalar field ϕ, the directional derivative of ϕ in the direction of the vector $\mathbf{b}$ is given by $\quad (\text{grad}\,\phi)\cdot\hat{\mathbf{b}} \quad$ where $\hat{\mathbf{b}}$ is the unit vector in the direction of $\mathbf{b}$.

Proof: After example.

Example 1 Calculate the directional derivative of the scalar field $\phi(x, y, z) = x^2yz$ in the direction $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ at the point $P(-1, 1, 3)$.

$$\nabla\phi = 2xyz + x^2z + x^2y$$

$$\nabla\phi(-1,1,3) = \begin{pmatrix} -6 \\ 3 \\ 1 \end{pmatrix}$$

$$\vec{b} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\vec{b}\cdot\nabla\phi = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\cdot\begin{pmatrix} -6 \\ 3 \\ 1 \end{pmatrix}$$

$$= \frac{2}{3}$$

$$\bigstar \quad \frac{2}{3} \quad \bigstar$$

This means that if you sit at the point $P(-1, 1, 3)$ in the scalar field $\phi = x^2yz$ and head off in the direction $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ then the instantaneous rate of change of temperature with respect to distance is equal to $\frac{2}{3}$.

Proof of formula: We will prove the result in space. The argument in other dimensions is similar. Let the unit vector $\hat{\mathbf{b}}$ be given by $\hat{\mathbf{b}} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and define a straight-line path in the $\hat{\mathbf{b}}$ direction by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} s$. Then clearly for $s \geq 0$ the fact that $\hat{\mathbf{b}}$ is a unit vector implies that the magnitude of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is just s. Hence s may be interpreted as distance (in the $\hat{\mathbf{b}}$ direction).

Now $\phi(x, y, x)$ depends upon x, y and z which in turn depend upon s. Hence via the chain rule:

$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial s} = \frac{\partial\phi}{\partial x}b_1 + \frac{\partial\phi}{\partial y}b_2 + \frac{\partial\phi}{\partial z}b_3 = \begin{pmatrix} \dfrac{\partial\phi}{\partial x} \\ \\ \dfrac{\partial\phi}{\partial y} \\ \\ \dfrac{\partial\phi}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = (\text{grad}\phi)\cdot\hat{\mathbf{b}}$$

as required. $\bigstar$

Example 2 The pressure in a region of space is given by $P(x, y, z) = \ln(x)y^2e^z + 3xyz$. Calculate the rate of change of the pressure with respect to distance at the point (1,2,0) in the direction $2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

$$\nabla P(1,2,0) = \left.\begin{pmatrix} \frac{y^2e^z}{x} + 3yz \\ 2y\ln(x)e^z + 3xz \\ \ln(x)y^2e^z + 3xy \end{pmatrix}\right|_{(1,2,0)} = \begin{pmatrix} 4 \\ 0 \\ 6 \end{pmatrix}$$

$$\hat{b} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\hat{b} \cdot \nabla P = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 6 \end{pmatrix} = \frac{14}{\sqrt{6}}$$

$\bigstar \quad \frac{14}{\sqrt{6}} \quad \bigstar$

We can differentiate a scalar field in any direction. A crucial question we now need to answer is, given a point P in a scalar field ϕ, in which direction should we move in order to increase the scalar field as quickly as possible? In other words which direction $\mathbf{b}$ yields the maximal directional derivative? We have shown above that the directional derivative is given by $(\text{grad}\,\phi) \cdot \hat{\mathbf{b}}$, hence we wish to maximise $|(\text{grad}\,\phi) \cdot \hat{\mathbf{b}}|$.

But from first year we know that $|(\text{grad}\,\phi) \cdot \hat{\mathbf{b}}| = |(\text{grad}\,\phi)||\hat{\mathbf{b}}|\cos(\theta)$, where θ is the angle between $\text{grad}\,\phi$ and $\hat{\mathbf{b}}$. Now $\text{grad}\,\phi$ is fixed and $|\hat{\mathbf{b}}| = 1$, hence all we need to do is maximise $\cos(\theta)$ which occurs when $\theta = 0$. That is, when $\text{grad}\,\phi$ and $\mathbf{b}$ point in the same direction. So the direction $\mathbf{b}$ which maximises the rate of of change is just $\text{grad}\,\phi$. Furthermore the magnitude of this maximal directional derivative is just $|\text{grad}\phi| \times 1 \times 1 = |\text{grad}\phi|$. We therefore have:

> Given a scalar field ϕ at a point P the direction of maximum increase of ϕ from P is given by $\nabla\phi|_P$ with the magnitude of the increase being the magnitude of $\nabla\phi|_P$.
>
> Given a scalar field ϕ at a point P the direction of maximum decrease of ϕ from P is given by $-\nabla\phi|_P$ with the magnitude of the decrease being the magnitude of $-\nabla\phi|_P$.
>
> $\nabla\phi|_P$ points perpendicular to the level surface (or curve in 2-D) at P.
>
> (**Note that $\nabla\phi|_P$ is just short hand for** $\text{grad}\,\phi$ **at** P)

So grad ϕ always points in the direction of max increase of a scalar field. The last point above states that it is also true that grad ϕ is always orientated perpendicular to the level curves and surfaces. In other words the direction of maximal change in a scalar field is always perpendicular to the direction of no change. We will prove this at the end of the next lecture once we have introduced a little vector calculus.

Example 3 Consider the simple temperature field $T(x, y) = x^2 + y^2$ in $\mathbb{R}^2$. Find the direction and magnitude of maximum increase and maximum decrease of T at the point $P(3, 3)$. Describe the level curve at P and verify graphically that the direction of maximum increase of T at P is perpendicular the level curve through P.

$$T(3, 3) = 18$$

$$\nabla T\big|_P = \begin{pmatrix} 2x \\ 2y \end{pmatrix}\Big|_{(3,3)} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

$$|\text{max} \uparrow \text{ of } T| = \left|\nabla T\big|_P\right| = \sqrt{6^2+6^2} = 3\sqrt{2}$$

$$\nabla T\big|_Q = \begin{pmatrix} 2x \\ 2y \end{pmatrix}\Big|_{(-3,-3)} = \begin{pmatrix} -6 \\ -6 \end{pmatrix}$$

$$|\text{max.} \downarrow \text{ of } T| = \left|\nabla T\big|_Q\right| = 3\sqrt{2}$$

Level curve: $x^2+y^2=16$

Direction of max. increase is perpendicular to level curve at P.

★ $6\mathbf{i} + 6\mathbf{j}$, $-6\mathbf{i} - 6\mathbf{j}$, $\sqrt{72}$ ★

Example 4 Consider the scalar field $\phi(x, y) = \dfrac{5x^2}{y}$ in $\mathbb{R}^2$. Find the direction and magnitude of maximum increase of ϕ at the point $P(2, 20)$. Describe the level curve at P and verify graphically that the direction of maximum increase of ϕ at P is perpendicular the level curve through P.

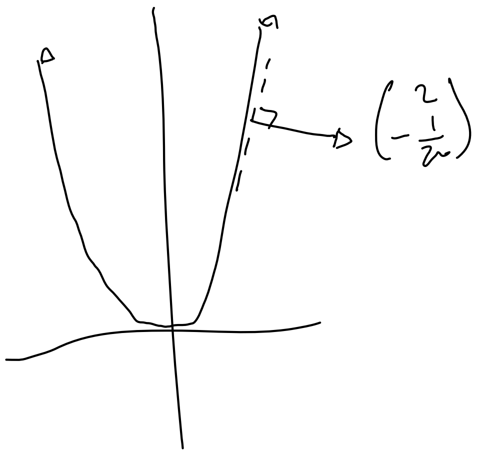
$$\phi(x, y) = 5x^2 y^{-1}$$

$$\nabla \phi \Big|_P = \begin{pmatrix} \dfrac{10x}{y} \\ -3x^2 y^{-2} \end{pmatrix} \Bigg|_P = \begin{pmatrix} 2 \\ -\dfrac{1}{20} \end{pmatrix}$$

$$\left| \nabla \phi \big|_P \right| = \left| \begin{pmatrix} 2 \\ -\frac{1}{20} \end{pmatrix} \right| = \sqrt{\frac{401}{400}}$$

$$\phi(2, 20) = \frac{5 \times 2^2}{20} = 1$$

level curve: $\dfrac{5x^2}{y} = 1 \implies y = 5x^2$



★ $\mathbf{i} - \frac{1}{20}\mathbf{j}, \sqrt{\frac{401}{400}}$ ★

Example 5 Consider the scalar field $\phi(x, y, z) = x^2z+2y^2-ye^{z^2}$. What is the magnitude and direction of the max rate of change of ϕ at $P(1, 2, 0)$?

$$\operatorname{grad}(\phi)\Big|_P = \begin{pmatrix} 2xz \\ 4y - e^{z^2} \\ x^2 - ye^{z^2}(2z) \end{pmatrix}\Bigg|_P = \begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix}$$

$$\left|\operatorname{grad}(\phi)\big|_P\right| = \left|\begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix}\right| = \sqrt{50}$$

★ $7\mathbf{j} + \mathbf{k}$, $\sqrt{50}$ ★

[11] You can now do Q 56,57,58