LECTURE 32
SYSTEMS OF DIFFERENTIAL EQUATIONS

Systems of differential equations $\mathbf{y}' = A\mathbf{y}$ may be easily solved by implementing the eigenvalues and eigenvectors of A.

If A is a 3×3 matrix with linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, and associated eigenvalues λ_1, λ_2 and λ_3, then the general solution to $\mathbf{y}' = A\mathbf{y}$ takes the form

$$\mathbf{y} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}$$

More complicated systems may be simplified through the transformation $\mathbf{y} = P\mathbf{z}$ where P is the usual matrix of eigenvectors of A.

We start this lecture by proving the validity of the algorithms used to analyse quadratic forms in the previous lecture.

Consider the quadratic form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}^T A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1$ where A is a symmetric matrix. Since A is symmetrix it admits a full set of orthogonal eigenvectors. Let P be the matrix of unit eigenvectors of A. The columns of P are orthogonal and also of unit length implying that P is an orthogonal matrix. Via the usual process of diagonalisation $P^{-1}AP = D$ where $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ is the diagonal matrix of eigenvalues. But since P is orthogonal we have $P^T AP = D$.

We now implement the orthogonal transformation (a rotation in space)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

The quadratic form becomes

$$\left(P\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}\right)^T AP \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 1 \text{ implying that}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}^T P^T AP \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 1 \text{ and hence we have}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}^T D \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 1 \rightarrow \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}^T \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 1$$

As promised in the last lecture this yields the simplified form (without mixed terms) with respect to the principal axes:

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = 1$$

We turn now to our second major application of eigenvectors, systems of differential equations.

SYSTEMS OF DIFFERENTIAL EQUATIONS

Example 1 Solve the system of differential equations.

$$\begin{aligned} y_1' &= 2y_1 + y_2 \\ y_2' &= -y_1 + y_3 \\ y_3' &= y_1 + y_2 + y_3 \end{aligned}$$

where $y_1(0) = 6,\ y_2(0) = -5$, and $y_3(0) = 7$.

We begin by noting that the system may be written in matrix form as

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

which is expressed as $\mathbf{y}' = A\mathbf{y}$. The usual eigenanalysis yields eigenvalues 0,1, and 2 with

associated eigenvectors $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

The solution may now be simply written down as

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{0t} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{1t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}$$

Reading across the rows we have a general solution:

$y_1 = c_1 - c_2 e^t + c_3 e^{2t}$

$y_2 = -2c_1 + c_2 e^t$

$y_3 = c_1 + c_3 e^{2t}$

Before applying the initial conditions let's prove that this all works:

Claim: If A is a 3×3 matrix with linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, and associated eigenvalues λ_1, λ_2 and λ_3, then the general solution to $\mathbf{y}' = A\mathbf{y}$ takes the form

$$\mathbf{y} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}$$

where $c_1,\ c_2$ and c_3 are arbitrary constants.

Proof:

Method 1: Assume a solution to $\mathbf{y}' = A\mathbf{y}$ of the form $\mathbf{y} = \mathbf{v}e^{\alpha t}$ where $\mathbf{v}$ is a vector and α is a number.

$\vec{y}' = \alpha \vec{v} e^{\alpha t}$ sub into $\vec{y}' = A\vec{y}$

$$\alpha \vec{v} e^{\alpha t} = A\vec{v}e^{\alpha t}$$
$$\alpha \vec{v} = A\vec{v}$$

$\therefore$ $\vec{v}$ is eigvec with eigval α

Method 2: Make the substitution $\mathbf{y} = P\mathbf{z}$ where P is the matrix of eigenvectors of A.

Let $P = (v_1 | \cdots | v_n)$ be matrix of eigvec.

Then $P^{-1}AP = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Let $\vec{y} = P\vec{z} \Rightarrow \vec{y}' = P\vec{z}'$

Sub into $\vec{y}' = A\vec{y}$

$$P\vec{z}' = AP\vec{z}$$
$$\vec{z}' = P^{-1}AP\vec{z}$$
$$= D\vec{z}$$

$z_1' = \lambda_1 z_1 \Rightarrow z_1 = c_1 e^{\lambda_1 t}$

$z_2' = \lambda_2 z_2 \Rightarrow z_2 = c_2 e^{\lambda_2 t}$

$z_3' = \lambda_3 z_3 \Rightarrow z_3 = c_3 e^{\lambda_3 t}$

$$\therefore z = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

$$y = P\vec{z}$$
$$= (\vec{v}_1 | \ldots | v_n) \begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

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It is clear from the above that once we have the eigenvectors and eigenvalues of A the solution to the system $\mathbf{y}' = A\mathbf{y}$ is just one step away!

The i.c.'s are implemented at the last stage to evaluate the three arbitrary constants.

Recall that

$y_1 = c_1 - c_2e^t + c_3e^{2t}$

$y_2 = -2c_1 + c_2e^t$

$y_3 = c_1 + c_3e^{2t}$

and that $y_1(0) = 6$, $y_2(0) = -5$, and $y_3(0) = 7$.

So $\quad \tilde{y} = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + c_3 e^{\lambda_3 t} v_3$

$y_1(0) = 6 \Rightarrow c_1 - c_2 + c_3 = 6$

$y_2(0) = -5 \Rightarrow -2c_1 + c_2 = -5$

$y_3(0) = 7 \Rightarrow c_1 + c_3 = 7$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ -2 & 1 & 0 & -5 \\ 1 & 0 & 1 & 7 \end{array}\right)$$

$\therefore c_1 = 3, \quad c_2 = 1, \quad c_3 = 4$

So we have $c_1 = 3, \quad c_2 = 1, \quad c_3 = 4.$

Hence the final solution is

$y_1 = 3 - e^t + 4e^{2t}$

$y_2 = -6 + e^t$

$y_3 = 3 + 4e^{2t}$

These three functions satisfy both the system of differential equations and the i.c.'s. Lets check that the last equation $\quad y_3' = y_1 + y_2 + y_3 \quad$ is satisfied:

LHS $= y_3' =$ $8e^{2t}$

RHS $= y_1 + y_2 + y_3 =$ $8e^{2t}$ $=$ LHS

$$\bigstar \quad y_1 = 3 - e^t + 4e^{2t}, \; y_2 = -6 + e^t, \; y_3 = 3 + 4e^{2t} \quad \bigstar$$

In more complicated examples $\mathbf{y}' = A\mathbf{y} + \mathbf{b}$ our approach is to actually implement the substitution $\mathbf{y} = P\mathbf{z}$ to yield $P\mathbf{z}' = AP\mathbf{z} + \mathbf{b}$. We then have $\mathbf{z}' = P^{-1}AP\mathbf{z} + P^{-1}\mathbf{b}$ implying $\mathbf{z}' = D\mathbf{z} + P^{-1}\mathbf{b}$. Since the diagonal matrix D has so little structure this final system when separated out, is trivial to solve using our standard first order linear theory.

[32]You can now do Q 93 and 94