MATH2019 PROBLEM CLASS
EXAMPLES 7
LAPLACE TRANSFORMS

2014, S1 1. Using the table of Laplace transforms find

$$\mathcal{L}\left\{t + \sin(2t) + e^{-t}\right\}.$$

Solution:

$$\begin{aligned}\mathcal{L}\left\{t + \sin(2t) + e^{-t}\right\} &= \mathcal{L}\{t\} + \mathcal{L}\{\sin(2t)\} + \mathcal{L}\left\{e^{-t}\right\} \\ &= \frac{1}{s^2} + \frac{2}{s^2+4} + \frac{1}{s+1}.\end{aligned}$$

2014, S1 2. i) By establishing an appropriate partial fraction decomposition find

$$\mathcal{L}^{-1}\left\{\frac{7s+1}{(s+1)(s-1)}\right\}.$$

Solution: Since the denominator has non-repeated linear factors using the "cover up method" yields the following partial fraction decomposition

$$\frac{7s+1}{(s+1)(s-1)} = \frac{3}{s+1} + \frac{4}{s-1}.$$

Hence, using the table of Laplace transforms, yields

$$\mathcal{L}^{-1}\left\{\frac{7s+1}{(s+1)(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+1} + \frac{4}{s-1}\right\} = 3e^{-t} + 4e^{t}.$$

ii) Hence, or otherwise, find

$$\mathcal{L}^{-1}\left\{\frac{7s+1}{(s+1)(s-1)}e^{-5s}\right\}.$$

Solution: Let $f(t) = 3e^{-t} + 4e^{t}$ from part i). Using the 2nd shifting theorem yields

$$\mathcal{L}^{-1}\left\{\frac{7s+1}{(s+1)(s-1)}e^{-5s}\right\} = u(t-5)f(t-5) = u(t-5)\left(3e^{-(t-5)} + 4e^{t-5}\right).$$

2014, S2 3. Find:

i) $\mathcal{L}\left\{e^{t}\cos(\pi t) + e^{t}\sin(\pi t)\right\}$.

Solution: Using the 1st shifting theorem with $F(s) = \mathcal{L}\{\cos(\pi t) + \sin(\pi t)\} = \dfrac{s}{s^2+\pi^2} + \dfrac{\pi}{s^2+\pi^2}$ (from table of Laplace transforms) yields

$$\begin{aligned}\mathcal{L}\left\{e^{t}\cos(\pi t) + e^{t}\sin(\pi t)\right\} &= \mathcal{L}\left\{e^{t}\left(\cos(\pi t) + \sin(\pi t)\right)\right\} \\ &= F(s-1) \\ &= \frac{s-1}{(s-1)^2+\pi^2} + \frac{\pi}{(s-1)^2+\pi^2}.\end{aligned}$$

ii) $\mathcal{L}^{-1}\left\{\dfrac{6}{s^2-4s+8}\right\}$.

Solution:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{6}{s^2-4s+8}\right\} &= \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^2+4}\right\} \quad \text{completing the square in denominator} \\
&= 3\mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+2^2}\right\} \\
&= 3\mathcal{L}^{-1}\left\{F(s-2)\right\} \quad \text{where } F(s)=\frac{2}{s^2+2^2}=\mathcal{L}\left\{\sin(2t)\right\} \\
&= 3e^{2t}\sin(2t) \quad \text{using 1st shifting theorem.}
\end{aligned}$$

2014, S2

4. Use the Laplace transform method to solve the initial value problem

$$y''-y'=4u(t-2) \qquad \text{with} \qquad y(0)=1,\ y'(0)=1,$$

where $u(t-2)$ is a Heaviside step function.

Solution: Let $Y(s)$ be the Laplace transform of $y(t)$, i.e., $\mathcal{L}\{y(t)\}=Y(s)$. Using the table of Laplace transforms to write down the Laplace transform of the ODE together with the initial conditions yields

$$\begin{aligned}
\left(s^2Y(s)-sy(0)-y'(0)\right)-\left(sY(s)-y(0)\right) &= \frac{4e^{-2s}}{s} \\
\Rightarrow \qquad \left(s^2-s\right)Y(s)-s-1+1 &= \frac{4e^{-2s}}{s} \\
\Rightarrow \qquad s(s-1)Y(s) &= s+\frac{4e^{-2s}}{s} \\
\Rightarrow \qquad Y(s) &= \frac{1}{s-1}+\frac{4e^{-2s}}{s^2(s-1)}.
\end{aligned}$$

We have a partial fraction decomposition to perform:

$$\begin{aligned}
\frac{4}{s^2(s-1)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1} \quad &\Rightarrow \quad 4=As(s-1)+B(s-1)+Cs^2 \\
&\Rightarrow \begin{cases} \underline{s=0}: & 0-B+0=4 \Rightarrow B=-4 \\ \underline{s=1}: & 0+0+C=4 \Rightarrow C=4 \\ \underline{s^2}: & A+C=0 \Rightarrow A=-4 \end{cases} \\
&\Rightarrow \quad \frac{4}{s^2(s-1)}=-\frac{4}{s}-\frac{4}{s^2}+\frac{4}{s-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
Y(s) &= \frac{1}{s-1}-e^{-2s}\left(\frac{4}{s}+\frac{4}{s^2}-\frac{4}{s-1}\right) \\
\Rightarrow \quad y(t) &= e^t-4u(t-2)\left(1+(t-2)-e^{t-2}\right) \\
&\qquad \text{using table of Laplace transforms} \\
&= e^t-4u(t-2)\left(t-1-e^{t-2}\right).
\end{aligned}$$

$\boxed{\text{2015, S1}}$ 5. Find:

i) $\mathcal{L}\left\{t^3 e^{\pi t}\right\}$.

Solution: Using the 1st shifting theorem with $F(s) = \mathcal{L}\left\{t^3\right\} = \dfrac{3!}{s^4}$ (from table of Laplace transforms) yields

$$\mathcal{L}\left\{t^3 e^{\pi t}\right\} = F(s-\pi) = \frac{3!}{(s-\pi)^4}\,.$$

ii) $\mathcal{L}^{-1}\left\{\dfrac{3-s}{s^2-4s+5}\right\}$.

Solution:

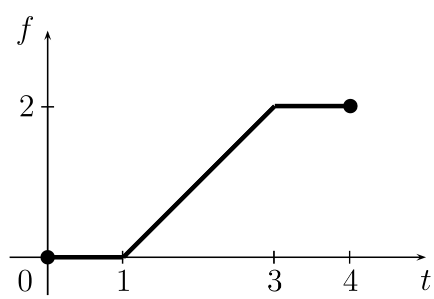
$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{3-s}{s^2-4s+5}\right\} &= \mathcal{L}^{-1}\left\{\frac{3-s}{(s-2)^2+1}\right\} \quad \text{completing the square in denominator} \\
&= \mathcal{L}^{-1}\left\{\frac{1-(s-2)}{(s-2)^2+1^2}\right\} \\
&= \mathcal{L}^{-1}\left\{F(s-2)\right\} \quad \text{where } F(s) = \frac{1}{s^2+1^2} - \frac{s}{s^2+1^2} = \mathcal{L}\left\{\sin t - \cos t\right\} \\
&= e^{2t}\left(\sin t - \cos t\right) \quad \text{using 1st shifting theorem.}
\end{aligned}$$

$\boxed{\text{2015, S1}}$ 6. The function $f(t)$ is given by

$$f(t) = \begin{cases} 0 & \text{for } 0 \le t < 1, \\ t-1 & \text{for } 1 \le t < 3, \\ 2 & \text{for } t \ge 3. \end{cases}$$

i) Sketch the function $f(t)$ for $0 \le t \le 4$.

Solution:



ii) Write $f(t)$ in terms of the Heaviside step function $u(t-a)$.

Solution:

$$\begin{aligned}
f(t) &= (t-1)\left(u(t-1) - u(t-3)\right) + 2u(t-3) \\
&= (t-1)u(t-1) + \left(2-(t-1)\right)u(t-3) \\
&= (t-1)u(t-1) - (t-3)u(t-3)\,.
\end{aligned}$$

iii) Hence, or otherwise, find the Laplace transform of $f(t)$.

Solution: Using the table of Laplace transforms yields

$$\mathcal{L}\left\{f(t)\right\} = \mathcal{L}\left\{(t-1)u(t-1) - (t-3)u(t-3)\right\} = \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s^2}\,.$$

2015, S1 7. Use the Laplace transform method to solve the initial value problem

$$y'' - 4y = 8u(t-1) \qquad \text{with} \qquad y(0) = 1,\ y'(0) = 2,$$

where $u(t-1)$ is a Heaviside step function.

Solution: Let $Y(s)$ be the Laplace transform of $y(t)$, i.e., $\mathcal{L}\{y(t)\} = Y(s)$. Using the table of Laplace transforms to write down the Laplace transform of the ODE together with the initial conditions yields

$$\begin{aligned}
\left(s^2 Y(s) - sy(0) - y'(0)\right) - 4Y(s) &= \frac{8e^{-s}}{s} \\
\Rightarrow \qquad \left(s^2 - 4\right) Y(s) - s - 2 &= \frac{8e^{-s}}{s} \\
\Rightarrow \qquad (s+2)(s-2)Y(s) &= \frac{8e^{-s}}{s} + (s+2) \\
\Rightarrow \qquad Y(s) &= \frac{8e^{-s}}{s(s+2)(s-2)} + \frac{1}{s-2} \\
&= \frac{1}{s-2} + e^{-s}\left(-\frac{2}{s} + \frac{1}{s+2} + \frac{1}{s-2}\right) \\
&\qquad \text{using “cover up method”} \\
\Rightarrow \qquad y(t) &= e^{2t} + u(t-1)\left(-2 + e^{-2(t-1)} + e^{-2(t-1)}\right) \\
&\qquad \text{using table of Laplace transforms} \\
&= e^{2t} + 2u(t-1)\left(\cosh\left(2(t-1)\right) - 1\right) .
\end{aligned}$$

2015, S2 8. Find:

i) $\mathcal{L}\{t^5 e^{3t}\}$.

Solution: Using the 1st shifting theorem with $F(s) = \mathcal{L}\{t^5\} = \dfrac{5!}{s^6}$ (from table of Laplace transforms) yields

$$\mathcal{L}\left\{t^5 e^{3t}\right\} = F(s-3) = \frac{5!}{(s-3)^6} .$$

ii) $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}s}{s^2+9}\right\}$.

Solution: Note $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}s}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{e^{-2s}\dfrac{s}{s^2+9}\right\}$. From the table of Laplace transforms $\mathcal{L}^{-1}\left\{e^{-2s}\dfrac{s}{s^2+9}\right\} = \cos(3t) = f(t)$. Hence, using the 2nd shifting theorem, yields

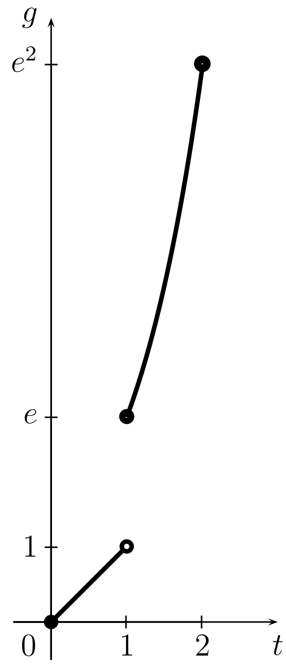
$$\mathcal{L}^{-1}\left\{e^{-2s}\frac{s}{s^2+9}\right\} = u(t-2)f(t-2) = u(t-2)\cos\left(3(t-2)\right) .$$

2015, S2 9. The function $g(t)$ is given by

$$g(t) = \begin{cases} t & \text{for } 0 \le t < 1 \\ e^t & \text{for } t \ge 1. \end{cases}$$

i) Sketch the function $g(t)$ for $0 \le t \le 2$.

Solution:



ii) Write $g(t)$ in terms of the Heaviside step function.

Solution:

$$\begin{aligned} g(t) &= t\,(1 - u(t-1)) + e^t u(t-1) \\ &= t + u(t-1)\left(e^t - t\right) \\ &= t + u(t-1)\left(ee^{t-1} - (t-1) - 1\right) \end{aligned}$$

iii) Hence, or otherwise, find the Laplace transform of $g(t)$.

Solution:

$$\begin{aligned} G(s) = \mathcal{L}\{g(t)\} &= \mathcal{L}\left\{t + u(t-1)\left(ee^{t-1} - (t-1) - 1\right)\right\} \\ &= \frac{1}{s^2} + e^{-s}\left(\frac{e}{s-1} - \frac{1}{s^2} - \frac{1}{s}\right) \end{aligned}$$

using table of Laplace transforms.

2015, S2 10. i) Find the partial fraction decomposition of

$$\frac{30}{s(s+3)(s-2)}.$$

Solution: Since the denominator has non-repeated linear factors using the "cover up method" yields the following partial fraction decomposition

$$\frac{30}{s(s+3)(s-2)} = -\frac{5}{s} + \frac{2}{s+3} + \frac{3}{s-2}.$$

ii) Using the Laplace transform method and your answer in the previous part find the solution of the initial value problem

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 30u(t-4) \quad \text{with } y(0) = 0 \text{ and } y'(0) = 0,$$

where $u(t-4)$ is the Heaviside step function.

Solution: Let $Y(s)$ be the Laplace transform of $y(t)$, i.e., $\mathcal{L}\{y(t)\} = Y(s)$. Using the table of Laplace transforms to write down the Laplace transform of the ODE together with the initial conditions yields

$$\begin{aligned}
\left(s^2Y(s) - sy(0) - y'(0)\right) + (sY(s) - y(0)) - 6Y(s) &= \frac{30e^{-4s}}{s} \\
\Rightarrow \qquad \left(s^2 + s - 6\right) Y(s) &= \frac{30e^{-4s}}{s} \\
\Rightarrow \qquad (s+3)(s-2)Y(s) &= \frac{30e^{-4s}}{s} \\
\Rightarrow \qquad Y(s) &= \frac{30e^{-4s}}{s(s+3)(s-2)} \\
&= e^{-4s}\left(-\frac{5}{s} + \frac{2}{s+3} + \frac{3}{s-2}\right) \\
&\quad \text{using part i)} \\
\Rightarrow \qquad y(t) &= u(t-4)\left(-5 + 2e^{-3(t-4)} + 3e^{2(t-4)}\right) \\
&\quad \text{using table of Laplace transforms.}
\end{aligned}$$

2016, S1

11. The Laplace transform of a function $f(t)$ is defined for $t \geq 0$ by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt.$$

i) Use Leibniz' rule to prove

$$\mathcal{L}\{tf(t)\} = -F'(s).$$

Solution: See lecture notes.

ii) Hence, or otherwise, find the following Laplace transform

$$\mathcal{L}\{t\sin(3t)\}\ .$$

Solution: From the table of Laplace transforms, let $F(s) = \mathcal{L}\{\sin(3t)\} = \dfrac{3}{s^2+9}$. Hence, using the result of part i), yields

$$\mathcal{L}\{t\sin(3t)\} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}\ .$$

$\boxed{\text{2016, S1}}$ 12. Use the Laplace transform method to solve the initial value problem

$$y'' - y = u(t-1) \qquad \text{with} \qquad y(0) = 0,\ y'(0) = 1,$$

where $u(t-1)$ is a Heaviside step function.

Solution: Let $Y(s)$ be the Laplace transform of $y(t)$, i.e., $\mathcal{L}\{y(t)\} = Y(s)$. Using the table of Laplace transforms to write down the Laplace transform of the ODE together with the initial conditions yields

$$\begin{aligned}
\left(s^2 Y(s) - sy(0) - y'(0)\right) - Y(s) &= \frac{e^{-s}}{s} \\
\Rightarrow \qquad \left(s^2 - 1\right) Y(s) - 1 &= \frac{e^{-s}}{s} \\
\Rightarrow \qquad (s+1)(s-1)Y(s) &= \frac{e^{-s}}{s} + 1 \\
\Rightarrow \qquad Y(s) &= \frac{e^{-s}}{s(s+1)(s-1)} + \frac{1}{(s+1)(s-1)} \\
&= \frac{1}{2(s-1)} - \frac{1}{2(s+1)} + e^{-s}\left(-\frac{1}{s} + \frac{1}{2(s+1)} + \frac{1}{2(s-1)}\right) \\
&\qquad \text{using “cover up method”} \\
\Rightarrow \qquad y(t) &= \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} + u(t-1)\left(-1 + \frac{1}{2}e^{t-1} + \frac{1}{2}e^{-(t-1)}\right) \\
&\qquad \text{using table of Laplace transforms} \\
&= \sinh t + u(t-1)\left(\cosh(t-1) - 1\right) .
\end{aligned}$$

$\boxed{\text{2016, S2}}$ 13. Find:

i) $\mathcal{L}\{\mathrm{e}^{-3t}\sin(\pi t)\}$.

Solution: Using the 1st shifting theorem with $F(s) = \mathcal{L}\{\sin(\pi t)\} = \dfrac{\pi}{s^2+\pi^2}$ (from table of Laplace transforms) yields

$$\mathcal{L}\left\{e^{-3t}\sin(\pi t)\right\} = F(s+3) = \frac{\pi}{(s+3)^2+\pi^2} .$$

ii) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+3s-4}\right\}$.

Solution:

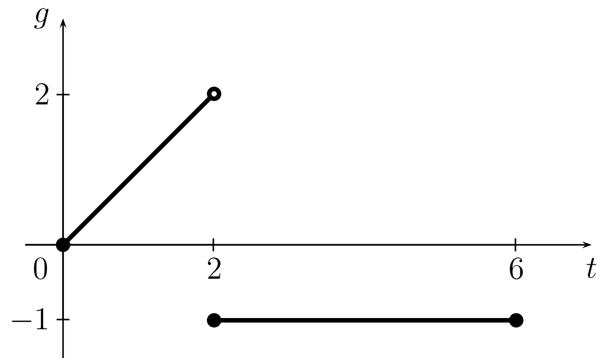
$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s-4}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s+4)(s-1)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{5(s-1)} - \frac{1}{5(s+4)}\right\} \quad \text{using “cover up method”} \\
&= \frac{1}{5}e^{t} - \frac{1}{5}e^{-4t} \quad \text{using table of Laplace transforms.}
\end{aligned}$$

$\boxed{\text{2016, S2}}$ 14. The function $g(t)$ is given by

$$g(t) = \begin{cases} t & \text{for } 0 \le t < 2, \\ -1 & \text{for } t \ge 2. \end{cases}$$

i) Sketch the function $g(t)$ for $0 \le t \le 6$.

Solution:



ii) Write $g(t)$ in terms of the Heaviside step function $u(t-a)$.

Solution:

$$\begin{aligned} g(t) &= t\,(1 - u(t-2)) - u(t-2) \\ &= t - tu(t-2) - u(t-2) \\ &= t - (t-2)u(t-2) - 3u(t-2)\,. \end{aligned}$$

iii) Hence, or otherwise, show that the Laplace transform of $g(t)$ is given by

$$\mathcal{L}\{g(t)\} = \frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} + \frac{3}{s}\right).$$

Solution:

$$\begin{aligned} \mathcal{L}\{g(t)\} &= \mathcal{L}\{t - (t-2)u(t-2) - 3u(t-2)\} \\ &= \mathcal{L}\{t\} - \mathcal{L}\{(t-2)u(t-2) + 3u(t-2)\} \\ &= \frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} + \frac{3}{s}\right) \quad \text{using table of Laplace transforms.} \end{aligned}$$

$\boxed{\text{2016, S2}}$ 15. A rocket is launched straight upwards at time $t = 0$ and its thrusters burn until $t = 2$. The vertical velocity $v(t)$ of the rocket satisfies the differential equation

$$\frac{dv}{dt} = g(t), \qquad v(0) = 0$$

where $g(t)$ is the function in the previous question.

i) Using Laplace transforms and the result part iii) from the previous question, solve the differential equation above to find the velocity of the rocket $v(t)$ as a function of time.

Solution: Let $V(s)$ be the Laplace transform of $v(t)$, i.e., $\mathcal{L}\{v(t)\} = V(s)$. Using the table of Laplace transforms to write down the Laplace transform of the ODE together with the initial condition yields

$$\begin{aligned} sV(s) - v(0) &= \frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} + \frac{3}{s}\right) \\ \Rightarrow \quad sV(s) - 0 &= \frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} + \frac{3}{s}\right) \\ \Rightarrow \quad V(s) &= \frac{1}{s^3} - e^{-2s}\left(\frac{1}{s^3} + \frac{3}{s^2}\right) \\ \Rightarrow \quad v(t) &= \frac{1}{2}t^2 - u(t-2)\left(\frac{1}{2}(t-2)^2 + 3(t-2)\right) \\ & \qquad \text{using table of Laplace transforms.} \end{aligned}$$

ii) By writing your solution separately for times $0 \le t < 2$ and $t \ge 2$, or otherwise, sketch the velocity as a function of time for $0 \le t \le 6$.

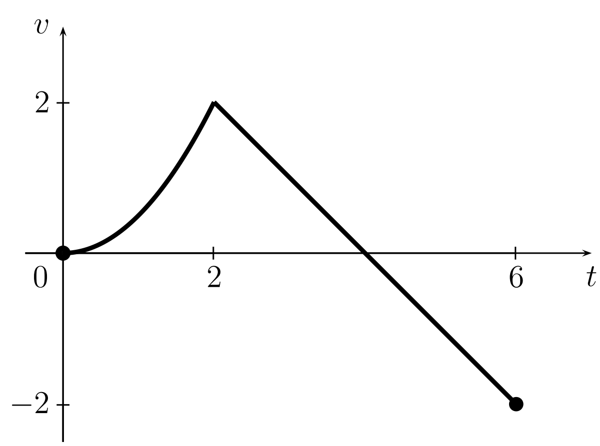
Solution: For $0 \le t < 2$, $v(t) = \dfrac{1}{2}t^2$ and for $t \ge 2$

$$\begin{aligned} v(t) &= \frac{1}{2}t^2 - \frac{1}{2}(t-2)^2 - 3(t-2) \\ &= \frac{1}{2}t^2 - \frac{1}{2}t^2 + 2t - 2 - 3t + 6 \\ &= 4 - t\,. \end{aligned}$$

Thus

$$v(t) = \begin{cases} \frac{1}{2}t^2 & 0 \le t < 2 \\ 4 - t & t \ge 2 \end{cases}$$

and



iii) What is the maximum velocity of the rocket?

Solution: The maximum velocity is 2 at $t = 2$.

iv) At what time will the rocket reach its maximum height?

Solution: Maximum height occurs when $v = 0$. This is at $t = 4$.

$\boxed{2017,\ \mathrm{S1}}$ 16. Find:

i) $\mathcal{L}\{tu(t-2)\}$.
Solution:

$$\begin{aligned}\mathcal{L}\{tu(t-2)\} = \mathcal{L}\{(t-2+2)u(t-2)\} &= \mathcal{L}\{(t-2)u(t-2)+2u(t-2)\} \\ &= e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right) \quad \text{using table of Laplace transforms.}\end{aligned}$$

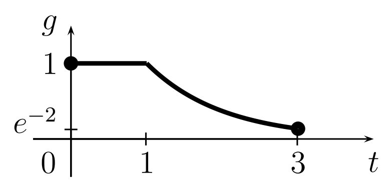
ii) $\mathcal{L}^{-1}\left\{\dfrac{3s}{s^2-2s+10}\right\}$.
Solution:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3s}{s^2-2s+10}\right\} &= \mathcal{L}^{-1}\left\{\frac{3s}{(s-1)^2+9}\right\} \quad \text{completing the square in denominator} \\ &= \mathcal{L}^{-1}\left\{\frac{3(s-1)+3}{(s-1)^2+3^2}\right\} \\ &= \mathcal{L}^{-1}\{F(s-1)\} \\ &\qquad \text{where } F(s) = 3\frac{s}{s^2+3^2}+\frac{3}{s^2+3^2} = \mathcal{L}\{3\cos(3t)+\sin(3t)\} \\ &= e^t\left(3\cos(3t)+\sin(3t)\right) \quad \text{using 1st shifting theorem.}\end{aligned}$$

$\boxed{2017,\ \mathrm{S1}}$ 17. The function $g(t)$ is given by

$$g(t) = \begin{cases} 1 & \text{for } 0 \le t < 1, \\ \mathrm{e}^{-t+1} & \text{for } t \ge 1. \end{cases}$$

i) Sketch the function $g(t)$ for $0 \le t \le 3$.
Solution:



ii) Write $g(t)$ in terms of the Heaviside step function $u(t-a)$.
Solution:

$$g(t) = 1\left(1-u(t-1)\right)+e^{-(t-1)}u(t-1) = 1-u(t-1)+e^{-(t-1)}u(t-1)\,.$$

iii) Hence, or otherwise, show that the Laplace transform of $g(t)$ is

$$\mathcal{L}\{g(t)\} = \frac{1}{s}-\mathrm{e}^{-s}\left(\frac{1}{s}-\frac{1}{s+1}\right).$$

Solution:

$$\begin{aligned}\mathcal{L}\{g(t)\} &= \mathcal{L}\left\{1-u(t-1)+e^{-(t-1)}u(t-1)\right\} \\ &= \frac{1}{s}-\frac{e^{-s}}{s}+\frac{e^{-s}}{s+1} \quad \text{using table of Laplace transforms} \\ &= \frac{1}{s}-e^{-s}\left(\frac{1}{s}-\frac{1}{s+1}\right).\end{aligned}$$

$\boxed{\text{2017, S1}}$ 18. Use the Laplace transform method to solve the initial value problem

$$y'' - y' = g(t), \qquad y(0) = -1,\ y'(0) = 0,$$

where $g(t)$ is the function from the previous question.

Solution: Let $Y(s)$ be the Laplace transform of $y(t)$, i.e., $\mathcal{L}\{y(t)\} = Y(s)$. Using the table of Laplace transforms to write down the Laplace transform of the ODE together with the initial conditions yields

$$\begin{aligned}
\left(s^2Y(s) - sy(0) - y'(0)\right) - (sY(s) - y(0)) &= \frac{1}{s} - e^{-s}\left(\frac{1}{s} - \frac{1}{s+1}\right) \\
\Rightarrow \qquad \left(s^2 - s\right)Y(s) + s - 0 - 1 &= \frac{1}{s} - e^{-s}\left(\frac{1}{s} - \frac{1}{s+1}\right) \\
\Rightarrow \qquad s(s-1)Y(s) &= \frac{1}{s} - (s-1) - e^{-s}\left(\frac{1}{s} - \frac{1}{s+1}\right) \\
\Rightarrow \qquad Y(s) &= \frac{1}{s^2(s-1)} - \frac{1}{s} - e^{-s}\left(\frac{1}{s^2(s-1)} - \frac{1}{s(s-1)(s+1)}\right) \\
&= \frac{1}{s^2(s-1)} - \frac{1}{s} \\
&\qquad -e^{-s}\left(\frac{1}{s^2(s-1)} + \frac{1}{s} - \frac{1}{2(s-1)} - \frac{1}{2(s+1)}\right) \\
&\qquad \text{using “cover up method”}
\end{aligned}$$

We have another partial fraction decomposition to perform:

$$\begin{aligned}
\frac{1}{s^2(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} \quad &\Rightarrow \quad 1 = As(s-1) + B(s-1) + Cs^2 \\
&\Rightarrow \begin{cases} \underline{s=0}: & 0 - B + 0 = 1 \Rightarrow B = -1 \\ \underline{s=1}: & 0 + 0 + C = 1 \Rightarrow C = 1 \\ \underline{s^2}: & A + C = 0 \Rightarrow A = -1 \end{cases} \\
&\Rightarrow \quad \frac{1}{s^2(s-1)} = -\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s-1}\,.
\end{aligned}$$

Hence

$$\begin{aligned}
Y(s) &= -\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s-1} - \frac{1}{s} \\
&\qquad -e^{-s}\left(-\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s-1} + \frac{1}{s} - \frac{1}{2(s-1)} - \frac{1}{2(s+1)}\right) \\
&= -\frac{2}{s} - \frac{1}{s^2} + \frac{1}{s-1} + e^{-s}\left(\frac{1}{s^2} - \frac{1}{2(s-1)} + \frac{1}{2(s+1)}\right) \\
\Rightarrow \quad y(t) &= -2 - t + e^t + (t-1)u(t-1) - \frac{1}{2}e^{t-1}u(t-1) + \frac{1}{2}e^{-(t-1)}u(t-1) \\
&\qquad \text{using table of Laplace transforms} \\
&= e^t - 2 - t + (t-1)u(t-1) - u(t-1)\sinh(t-1)\,.
\end{aligned}$$

$\boxed{\text{2017, S2}}$ 19. i) The Laplace transform of a function $f(t)$ is defined for $t \geq 0$ by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt.$$

Prove directly from the above definition that

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

where $a > 0$ and $u(t-a)$ is the Heaviside function.

Solution: See lecture notes.

ii) Find:

α) $\mathcal{L}\{e^t\,u(t-3)\}$.

Solution:

$$\begin{aligned}
\mathcal{L}\left\{e^t\,u(t-3)\right\} &= \mathcal{L}\left\{e^3e^{t-3}\,u(t-3)\right\}\\
&= e^3\mathcal{L}\left\{e^{t-3}\,u(t-3)\right\}\\
&= e^3e^{-3s}\frac{1}{s-1} \quad \text{using 2nd shifting theorem}\\
&= \frac{e^{-3(s-1)}}{s-1}.
\end{aligned}$$

β) $\mathcal{L}^{-1}\left\{\dfrac{s+2}{s^2+2s+5}\right\}$.

Solution:

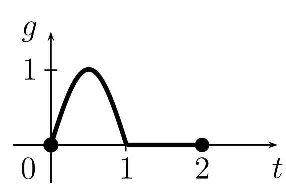
$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+2s+5}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+1)^2+4}\right\} \quad \text{completing the square in denominator}\\
&= \mathcal{L}^{-1}\left\{\frac{s+1+1}{(s+1)^2+2^2}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}+\frac{1}{2}\frac{2}{(s+1)^2+2^2}\right\}\\
&= \mathcal{L}^{-1}\left\{F(s+1)\right\}\\
&\qquad \text{where } F(s) = \frac{s}{s^2+2^2}+\frac{1}{2}\frac{2}{s^2+2^2} = \mathcal{L}\left\{\cos(2t)+\frac{1}{2}\sin(2t)\right\}\\
&= e^{-t}\left(\cos(2t)+\frac{1}{2}\sin(2t)\right) \quad \text{using 1st shifting theorem.}
\end{aligned}$$

$\boxed{\text{2017, S2}}$ 20. The function $g(t)$ is given by

$$g(t) = \begin{cases} \sin(\pi t) & \text{for } 0 \leq t < 1,\\ 0 & \text{for } t \geq 1.\end{cases}$$

i) Sketch the function $g(t)$ for $0 \leq t \leq 2$.

Solution:



ii) Write $g(t)$ in terms of the Heaviside step function $u(t-a)$.

Solution:

$$\begin{aligned}
g(t) &= \sin(\pi t)\,(1-u(t-1)) \\
&= \sin(\pi t) - \sin(\pi t)u(t-1) \\
&= \sin(\pi t) - \sin\left((\pi(t-1)+\pi\right)u(t-1) \\
&= \sin(\pi t) + \sin\left(\pi(t-1)\right)u(t-1) \quad \text{using } \sin(A+\pi) = -\sin A.
\end{aligned}$$

iii) Hence, or otherwise, show that the Laplace transform of $g(t)$ is

$$\mathcal{L}\{g(t)\} = \frac{\pi}{s^2+\pi^2}\left(1+e^{-s}\right).$$

[Hint: You can use $\sin(A+\pi) = -\sin A$.]

Solution:

$$\begin{aligned}
\mathcal{L}\{g(t)\} &= \mathcal{L}\{\sin(\pi t) + \sin\left(\pi(t-1)\right)u(t-1)\} \\
&= \frac{\pi}{s^2+\pi^2} + e^{-s}\frac{\pi}{s^2+\pi^2} \quad \text{using table of Laplace transforms} \\
&= \frac{\pi}{s^2+\pi^2}\left(1+e^{-s}\right).
\end{aligned}$$

$\boxed{\text{2017, S2}}$ 21. Use the Laplace transform method to solve the initial value problem

$$y'' - y' - 2y = 6u(t-1), \qquad y(0) = 1,\ y'(0) = 2.$$

Solution: Let $Y(s)$ be the Laplace transform of $y(t)$, i.e., $\mathcal{L}\{y(t)\} = Y(s)$. Using the table of Laplace transforms to write down the Laplace transform of the ODE together with the initial conditions yields

$$\begin{aligned}
\left(s^2Y(s) - sy(0) - y'(0)\right) - (sY(s) - y(0)) - 2Y(s) &= 6\frac{e^{-s}}{s} \\
\Rightarrow \qquad \left(s^2 - s - 2\right)Y(s) - s - 2 + 1 &= 6\frac{e^{-s}}{s} \\
\Rightarrow \qquad (s-2)(s+1)Y(s) &= 6\frac{e^{-s}}{s} + s + 1 \\
\Rightarrow \qquad Y(s) &= e^{-s}\frac{6}{s(s-2)(s+1)} + \frac{1}{s-2} \\
&= \frac{1}{s-2} + e^{-s}\left(-\frac{3}{s} + \frac{1}{s-2} + \frac{2}{s+1}\right) \\
&\qquad \text{using “cover up method”} \\
\Rightarrow \qquad y(t) &= e^{2t} + u(t-1)\left(-3 + e^{2(t-1)} + 2e^{-(t-1)}\right) \\
&\qquad \text{using table of Laplace transforms.}
\end{aligned}$$

$\boxed{\text{2018, S1}}$ 22. Find

i) $\mathcal{L}\{te^{-t}\sin(3t)\}$;

Solution: Let $f(t)=\sin(3t)$. Then

$$F(s)=\mathcal{L}\{f(t)\}=\frac{3}{s^2+9}.$$

Using the 1st shifting theorem, $\mathcal{L}\{e^{-\alpha t}f(t)\}=F(s+\alpha)$, from the table of Laplace transforms, we have

$$\mathcal{L}\left\{e^{-t}\sin(3t)\right\}=F(s+1)=\frac{3}{(s+1)^2+9}=G(s).$$

Finally using $\mathcal{L}\{tg(t)\}=-G'(s)$ from table of Laplace transforms with $g(t)=e^{-t}\sin(3t)$ we have

$$\mathcal{L}\left\{te^{-t}\sin(3t)\right\}=-G'(s)=-\frac{d}{ds}\left(\frac{3}{(s+1)^2+9}\right)=\frac{6(s+1)}{[(s+1)^2+9]^2}.$$

OR

Let $f(t)=\sin(3t)$. Then

$$F(s)=\mathcal{L}\{f(t)\}=\frac{3}{s^2+9}.$$

Now let $g(t)=tf(t)=t\sin(3t)$. Then using $\mathcal{L}\{tf(t)\}=-F'(s)$ from table of Laplace transforms we have

$$G(s)=\mathcal{L}\{g(t)\}=-F'(s)=-\frac{d}{ds}\left(\frac{3}{s^2+9}\right)=\frac{6s}{(s^2+9)^2}.$$

Finally, using the 1st shifting theorem and the result for $G(s)$ we have

$$\mathcal{L}\left\{te^{-t}\sin(3t)\right\}=\mathcal{L}\left\{e^{-t}g(t)\right\}=G(s+1)=\frac{6(s+1)}{[(s+1)^2+9]^2}.$$

ii) $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2+4s+5}\right\}$.

Solution: We first write (partial fractions)

$$\begin{aligned}\frac{s+1}{s^2+4s+5}=\frac{s+1}{(s+2)^2+1}&=\frac{s+2}{(s+2)^2+1}-\frac{1}{(s+2)^2+1}\\&=F(s+2)-G(s+2)\end{aligned}$$

where

$$F(s)=\frac{s}{s^2+1}\quad\text{and}\quad G(s)=\frac{1}{s^2+1}.$$

Hence, by the 1st shifting theorem and table of Laplace transform we have

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+5}\right\}=e^{-2t}\big(f(t)-g(t)\big)=e^{-2t}\big(\cos t-\sin t\big).$$

$\boxed{\text{2018, S1}}$ 23. The function $f(t)$ is defined for $t \geq 0$ by

$$f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ 0, & t \geq 1. \end{cases}$$

i) Express $f(t)$ in terms of the Heaviside function.
Solution: $f(t) = u(t) - u(t-1) = 1 - u(t-1)$.

ii) Hence or otherwise find $\mathcal{L}\{f(t)\}$, the Laplace transform of $f(t)$.
Solution: $\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{u(t-1)\} = \dfrac{1}{s} - \dfrac{e^{-s}}{s}$ using table of Laplace transforms.

$\boxed{\text{2018, S1}}$ 24. Solve the differential equation

$$y'' - 4y' + 4y = f(t), \quad t > 0,$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = 0$, where $f(t)$ is given in the previous question.

Solution: Taking the Laplace transform on both sides yields

$$[s^2Y(s) - \underbrace{sy(0)}_{=s} - \underbrace{y'(0)}_{=0}] - 4[sY(s) - \underbrace{y(0)}_{=1}] + 4Y(s) = \underbrace{\frac{1}{s} - \frac{e^{-s}}{s}}_{\mathcal{L}\{f(t)\}}$$

$$\Rightarrow \quad (s^2 - 4s + 4)Y(s) = s - 4 + \frac{1}{s} - \frac{e^{-s}}{s}$$

$$\Rightarrow \quad (s-2)^2Y(s) = s - 4 + \frac{1}{s} - \frac{e^{-s}}{s}$$

$$\Rightarrow \quad Y(s) = \frac{1}{s-2} - \frac{2}{(s-2)^2} + \frac{1}{s(s-2)^2} - \frac{e^{-s}}{s(s-2)^2}.$$

Hence

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s(s-2)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s-2)^2}\right\} \\ &= \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}}_{=T_1} - 2\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}}_{=T_2} + \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s(s-2)^2}\right\}}_{=T_3} - u(t-1)\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s(s-2)^2}\right\}(t-1)}_{=T_4}. \end{aligned}$$

Now

$$\begin{aligned} T_1 &= \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t} \\ T_2 &= \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = te^{2t}. \end{aligned}$$

To evaluate T_3 and T_4, we use partial fractions:

$$\frac{1}{s(s-2)^2} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2} = \frac{1}{4s} - \frac{1}{4}\frac{1}{s-2} + \frac{1}{2}\frac{1}{(s-2)^2}.$$

Hence

$$\begin{aligned}
T_3 &= \mathcal{L}^{-1}\left\{\frac{1}{s(s-2)^2}\right\} \\
&= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} \\
&= \frac{1}{4} - \frac{1}{4}T_1 + \frac{1}{2}T_2 \\
&= \frac{1}{4} - \frac{1}{4}e^{2t} + \frac{1}{2}te^{2t}, \\
T_4 &= \mathcal{L}^{-1}\left\{\frac{1}{s(s-2)^2}\right\}(t-1) = \frac{1}{4} - \frac{1}{4}e^{2(t-1)} + \frac{1}{2}(t-1)e^{2(t-1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(t) &= T_1 - 2T_2 + T_3 - u(t-1)T_4 \\
&= e^{2t} - 2te^{2t} + \frac{1}{4} - \frac{1}{4}e^{2t} + \frac{1}{2}te^{2t} - u(t-1)\Big(\frac{1}{4} - \frac{1}{4}e^{2(t-1)} + \frac{1}{2}(t-1)e^{2(t-1)}\Big) \\
&= \frac{1}{4} + \frac{3}{4}e^{2t} - \frac{3}{2}te^{2t} - u(t-1)\Big(\frac{1}{4} - \frac{1}{4}e^{2(t-1)} + \frac{1}{2}(t-1)e^{2(t-1)}\Big) \\
&= \begin{cases} \frac{1}{4} + \frac{3}{4}e^{2t} - \frac{3}{2}te^{2t}, & 0 \le t < 1, \\ \frac{3}{4}e^{2t} - \frac{3}{2}te^{2t} + \frac{1}{4}e^{2(t-1)} - \frac{1}{2}(t-1)e^{2(t-1)}, & t \ge 1. \end{cases}
\end{aligned}$$

2018, S2

25. Find

i) $\mathcal{L}\{\sin(3t)\}$,

Solution: Reading directly off the table of Laplace Transforms we have

$$\mathcal{L}\{\sin(3t)\} = \frac{3}{s^2+9}.$$

ii) $\mathcal{L}\{e^{-7t}\sin(3t)\}$,

Solution: Using the 1st shifting theorem $\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$ and the result from part i) we have

$$\mathcal{L}\left\{e^{-7t}\sin(3t)\right\} = \frac{3}{(s+7)^2+9} = \frac{3}{s^2+14s+58}.$$

iii) $\mathcal{L}^{-1}\left\{\frac{4s-28}{(s-1)(s-9)}\right\}$.

Solution:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{4s-28}{(s-1)(s-9)}\right\} &= \mathcal{L}^{-1}\left\{\frac{3}{s-1} + \frac{1}{s-9}\right\} && \text{using the “cover up method”} \\
&= 3e^t + e^{9t} && \text{using table of Laplace transforms.}
\end{aligned}$$

$\boxed{\text{2018, S2}}$ 26. The function $g(t)$ is defined for $t \geq 0$ by i

$$g(t) = \begin{cases} t^2, & 0 \leq t < 1, \\ e^{2t}, & t \geq 1. \end{cases}$$

i) Express $g(t)$ in terms of the Heaviside function.

Solution:

$$g(t) = t^2 \left[u(t) - u(t-1)\right] + e^{2t}u(t-1) = t^2 - t^2u(t-1) + e^{2t}u(t-1)\,.$$

ii) Hence, or otherwise, show that the Laplace transform of $g(t)$ is

$$G(s) = \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) + \frac{e^{2-s}}{s-2}\,.$$

Solution: To determine the Laplace transform of $g(t)$ we need to write the 2nd and 3rd parts of our answer to part i) as functions of $t-1$, i.e.,

$$\begin{aligned} g(t) &= t^2 - t^2u(t-1) + e^{2t}u(t-1) \\ &= t^2 - \big((t-1)^2 + 2(t-1) + 1\big)\,u(t-1) + e^2e^{2(t-1)}u(t-1)\,. \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{L}\left\{g(t)\right\} &= \mathcal{L}\left\{t^2 - \big((t-1)^2 + 2(t-1) + 1\big)\,u(t-1) + e^2e^{2(t-1)}u(t-1)\right\} \\ &= \mathcal{L}\left\{t^2\right\} - \mathcal{L}\left\{(t-1)^2u(t-1)\right\} - 2\mathcal{L}\left\{(t-1)u(t-1)\right\} - \mathcal{L}\left\{u(t-1)\right\} \\ &\quad + e^2\mathcal{L}\left\{e^{2(t-1)}u(t-1)\right\} \\ &= \frac{2}{s^3} - e^{-s}\frac{2}{s^3} - 2e^{-s}\frac{1}{s^2} - e^{-s}\frac{1}{s} + e^2e^{-s}\frac{1}{s-2} \\ &= \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) + \frac{e^{2-s}}{s-2} \quad \text{as required.} \end{aligned}$$

TABLE OF LAPLACE TRANSFORMS AND THEOREMS
on next page

TABLE OF LAPLACE TRANSFORMS AND THEOREMS

$g(t)$ is a function defined for all $t \geq 0$, and whose Laplace transform

$$G(s) = \mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} g(t) dt$$

exists. The Heaviside step function u is defined to be

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t > a \end{cases}$$

$g(t)$	$G(s) = \mathcal{L}\{g(t)\}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^m, m = 0, 1, \ldots$	$\frac{m!}{s^{m+1}}$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$e^{-\alpha t} f(t)$	$F(s+\alpha)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$tf(t)$	$-F'(s)$