LECTURE 35
INVERSE LAPLACE TRANSFORMS AND THE HEAVISIDE FUNCTION

LAPLACE TRANSFORMS

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^m	$m!/s^{m+1}$
t^ν , $(\nu > -1)$	$\Gamma(\nu+1)/s^{\nu+1}$
e^{-at}	$1/(s+a)$
$\sin bt$	$b/(s^2+b^2)$
$\cos bt$	$s/(s^2+b^2)$
$\sinh bt$	$b/(s^2-b^2)$
$\cosh bt$	$s/(s^2-b^2)$
$\sin bt - bt\cos bt$	$2b^3/(s^2+b^2)^2$
$\sin bt + bt\cos bt$	$2bs^2/(s^2+b^2)^2$
$t\ \sin bt$	$2bs/(s^2+b^2)^2$
te^{-at}	$1/(s+a)^2$
$\mathrm{u}(t-c)$	e^{-cs}/s
$e^{-at}f(t)$	$F(s+a)$
$tf(t)$	$-F'(s)$
$f(t-c)\mathrm{u}(t-c)$	$e^{-cs}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f'''(t)$	$s^3F(s)-s^2f(0)-sf'(0)-f''(0)$
$\int_0^t f(\tau)d\tau$	$F(s)/s$

In the previous lecture we used the Laplace transform to change $f(t)$ to $F(s)$ (written $\mathcal{L}(f) = F$). We now want to undo this process and convert $F(s)$ back to $f(t)$. This is referred to as taking an inverse Laplace transform and we write $f = \mathcal{L}^{-1}(F)$. There is no integral formula for taking inverse Laplace transforms and we have no option but to just use the Laplace transform table together with a bag of interesting tricks.

Example 1 Find the inverse Laplace transform of each of the following functions:

a) $F(s) = \dfrac{4}{s-2} + \dfrac{1}{s^2+9}$;

b) $F(s) = \dfrac{3}{s} - \dfrac{12}{s^2} + \dfrac{2}{s^5}$.

a) $\mathcal{L}^{-1}\left(\dfrac{4}{s-2} + \dfrac{1}{s^2+9}\right) = 4e^{2t} + \dfrac{1}{3}\sin(3t)$

b) $\mathcal{L}^{-1}\left(\dfrac{3}{s} - \dfrac{12}{s^2} + \dfrac{2}{s^5}\right) = 3 - 12t + \dfrac{2}{4!}t^4$

$= 3 - 12t + \dfrac{t^4}{12}$

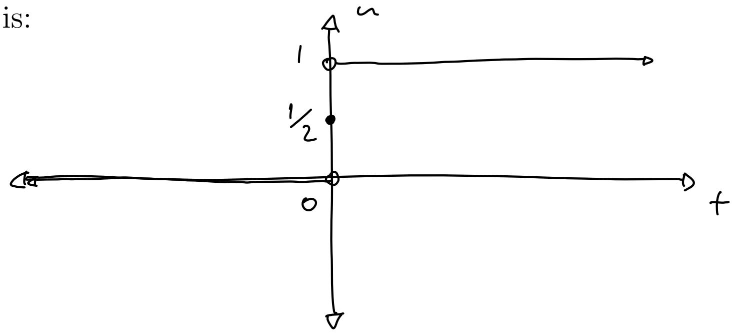
★ $a)$ $4e^{2t} + \dfrac{1}{3}\sin(3t)$ $b)$ $3 - 12t + \dfrac{t^4}{12}$ ★

Observe that we are now using the table backwards and traveling from the s variable back to the t variable. In order to be able to invert a wider class of objects we need a fancy little step function called the Heaviside function.

Definition: The Heaviside function $u(t)$ is given by

$$u(t) = \begin{cases} 0, & \text{t<0;} \\ \frac{1}{2}, & \text{t=0;} \\ 1, & \text{t>0.} \end{cases}$$

The graph of $u(t)$ is:



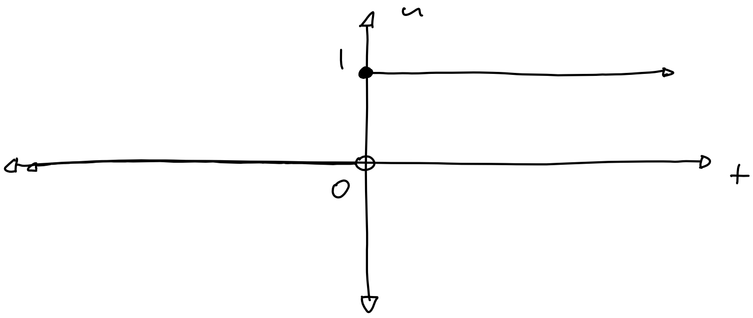
Note that the **name** of the above function is "u".

The Heaviside function is also sometimes called the unit step function.

The complicated behaviour of $u(t)$ at $t = 0$ is somewhat cosmetic for our purposes and you may view the definition of the Heaviside function as being simply:

$$u(t) = \begin{cases} 0, & \text{t<0;} \\ 1, & \text{t}\geq\text{0.} \end{cases}$$

with graph

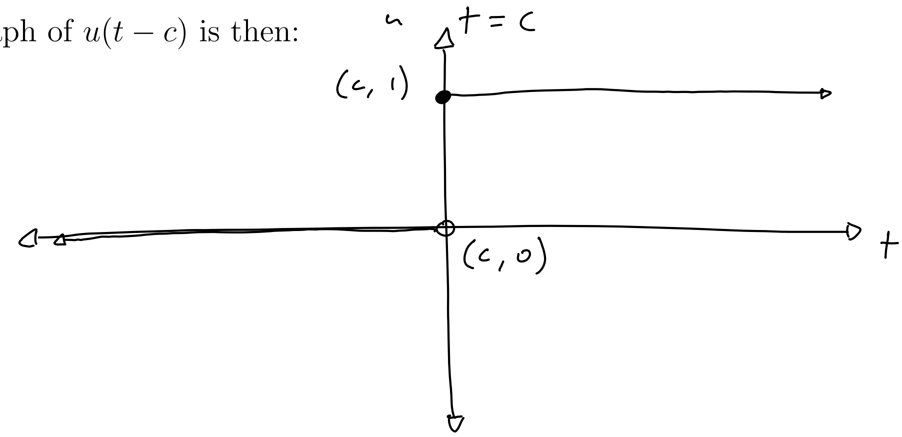


Actually the filled dots can pretty much go anywhere since the Laplace transform, being an integration process, smooths out all blemishes.

Of greater importance to us is the function $u(t - c)$.

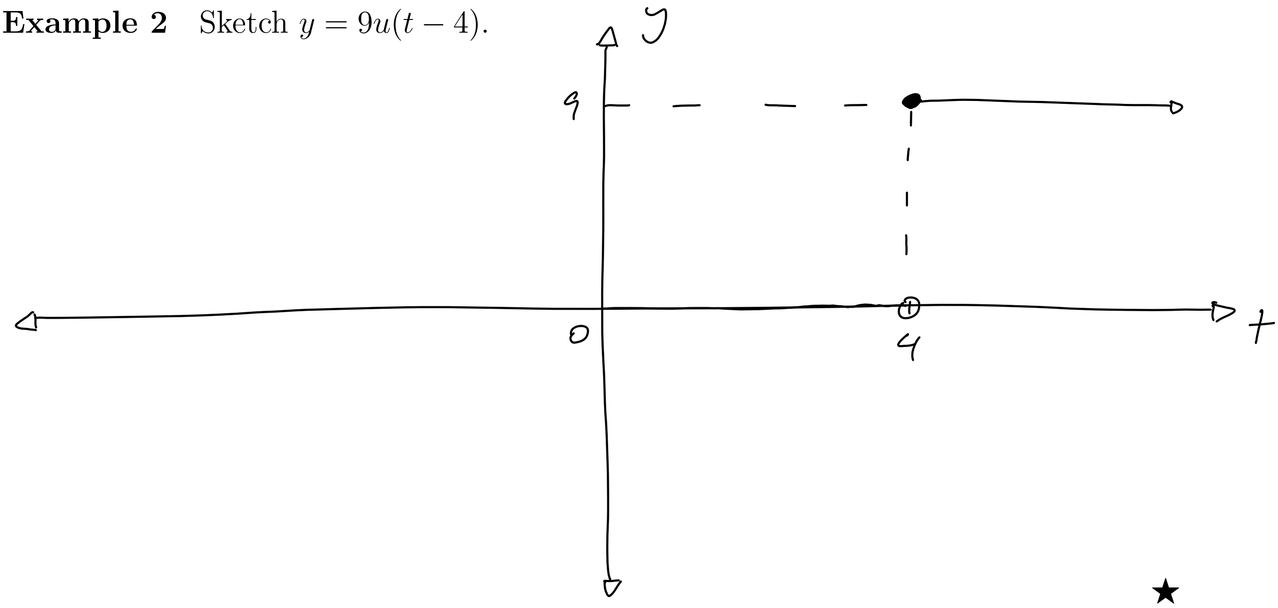
$$u(t-c) = \begin{cases} 0, & \text{t<c;} \\ 1, & \text{t}\geq\text{c.} \end{cases}$$

The graph of $u(t - c)$ is then:



Always remember that the Heaviside function $u(t-c)$ is a single mathematical function which is asleep until c and then wakes up. It is then equal to 1 to infinity. The Heaviside function can be viewed as the simplest possible discontinuous function.

Example 2 Sketch $y = 9u(t-4)$.



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Example 3 Prove that $\mathcal{L}\{u(t-c)\} = \dfrac{e^{-cs}}{s}$.

$$\mathcal{L}\{u(t-c)\} = \int_0^\infty e^{-st} u(t-c)\, dt$$

$$= \int_c^\infty e^{-st}\, dt$$

$$= \frac{e^{-cs}}{s}$$

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Note that the above result is part of your standard tables.

Example 4 Find the Laplace transform of $9u(t-4)$.

$$\mathcal{L}\left(9u(t-4)\right) = \frac{9e^{-4s}}{s}$$

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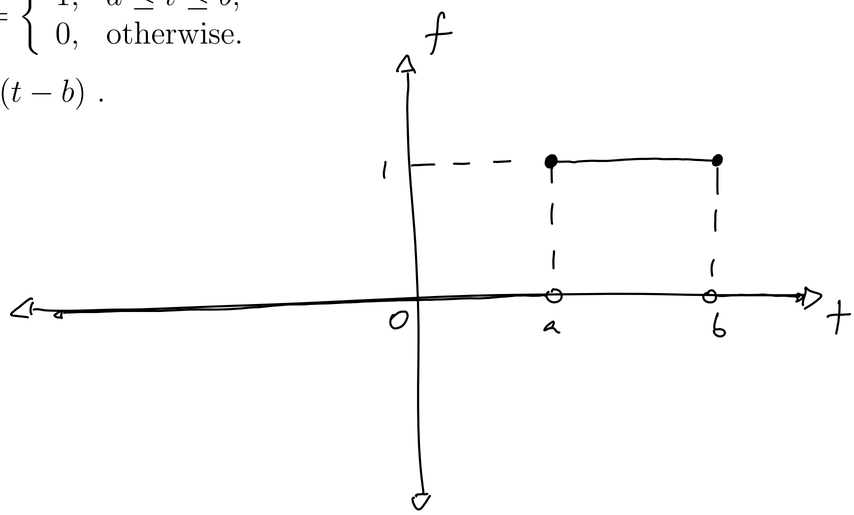
We use the Heaviside function to build up other discontinuous functions. Since taking the Laplace transform is a process of integration we may be sloppy on the definition of a function at endpoints as it has no impact on the final result. When sketching however it is a good idea to put in the dot on the discontinuities.

Example 5 Suppose that $f(t) = \begin{cases} 1, & a \leq t \leq b; \\ 0, & \text{otherwise.} \end{cases}$

Prove that $f(t) = u(t-a) - u(t-b)$.



$$u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$$

$$u(t-b) = \begin{cases} 1, & t \geq b \\ 0, & t < b \end{cases}$$

$$u(t-a) - u(t-b) = \begin{cases} 1, & t \geq a \cap t \leq b \\ 0, & t < a \cup t > b \end{cases}$$

$$= \begin{cases} 1, & a \leq t \leq b \\ 0, & \text{otherwise} \end{cases}$$

$$= f(t)$$

So $u(t-a) - u(t-b)$ is the function that is asleep till $t = a$ wakes up for a little while until $t = b$ and then goes back to bed forever.

Example 6 Suppose that $f(t) = \begin{cases} 9, & 3 \leq t \leq 4; \\ 0, & \text{otherwise.} \end{cases}$

Use the Heaviside function to find $\mathcal{L}\{f\}$.

$$\mathcal{L}\{f\} = \mathcal{L}\left(9\left(u(t-3) - u(t-4)\right)\right)$$
$$= 9\left(\frac{e^{-3s}}{s} - \frac{e^{-4s}}{s}\right)$$

Recall that we have already answered this question via the integral definition in the previous lecture.

$$\bigstar \quad 9\left(\frac{e^{-3s}}{s} - \frac{e^{-4s}}{s}\right) \quad \bigstar$$

Example 7 Sketch each of the following functions and rewrite the function without the use of Heavisides:

a) $f(t) = 1 - u(t-3)$ (this is a flipped Heaviside)

b) $f(t) = t^2 u(t-4)$ (This is a cut. Remember that when you multiply by a Heaviside half of the time you are multiplying by 0 and hence wiping the function away and the other half of the time you are multiplying by 1 and thus doing nothing at all!)

c) $f(t) = (t-4)^2 u(t-4)$ (This is a shift plus cut. This structure is crucial in the next lecture)

d)$f(t) = (t-5)^2 u(t-4)$ (These are less common)

e) $f(t) = t^2\{u(t-2) - u(t-7)\}$

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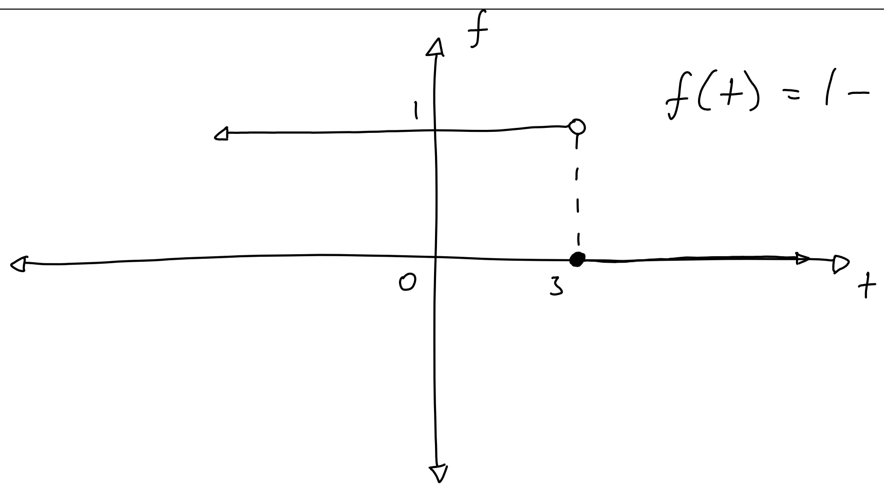
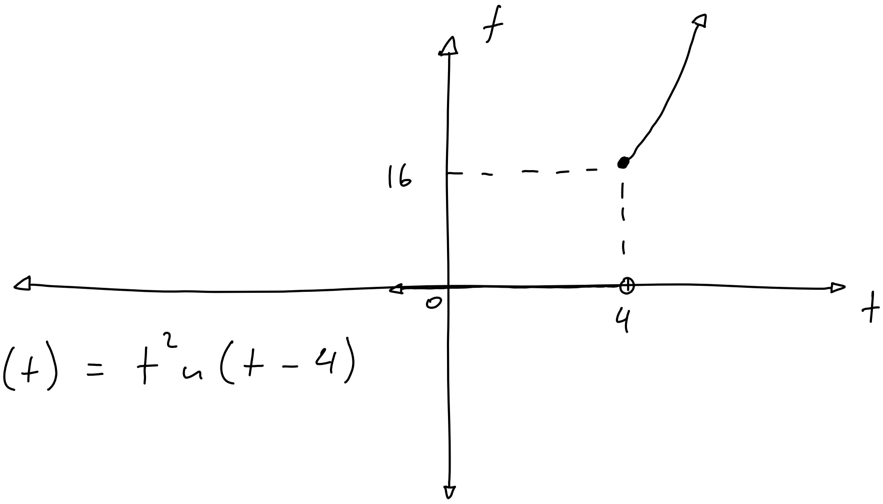
a)



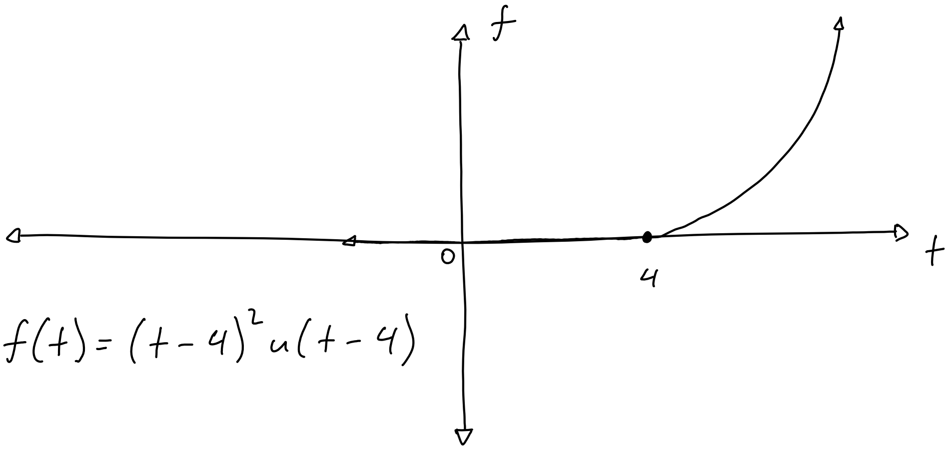
$f(t) = 1 - u(t-3)$

b)



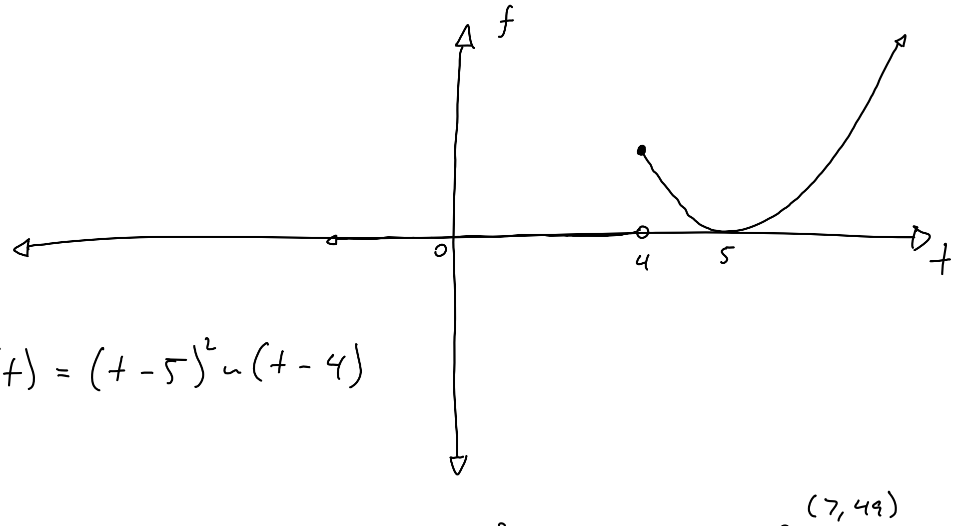
$f(t) = t^2 u(t-4)$

c)



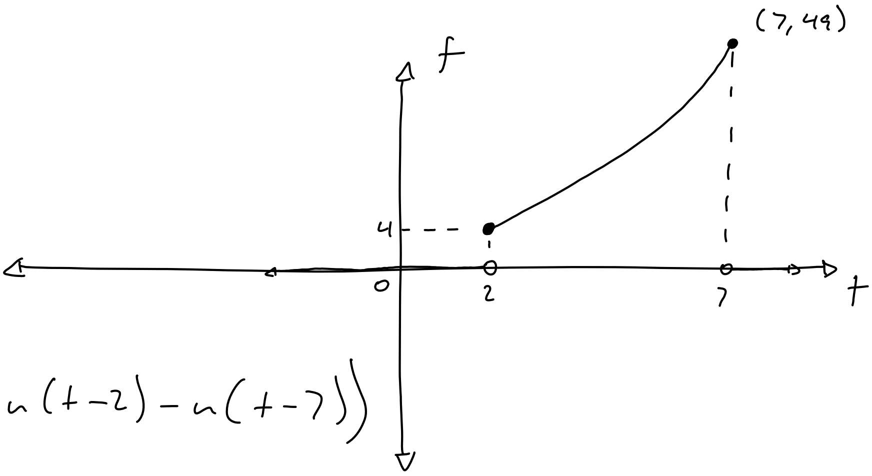
$f(t) = (t-4)^2 u(t-4)$

d)



$f(t) = (t-5)^2 u(t-4)$

e)



$f(t) = t^2\left(u(t-2) - u(t-7)\right)$

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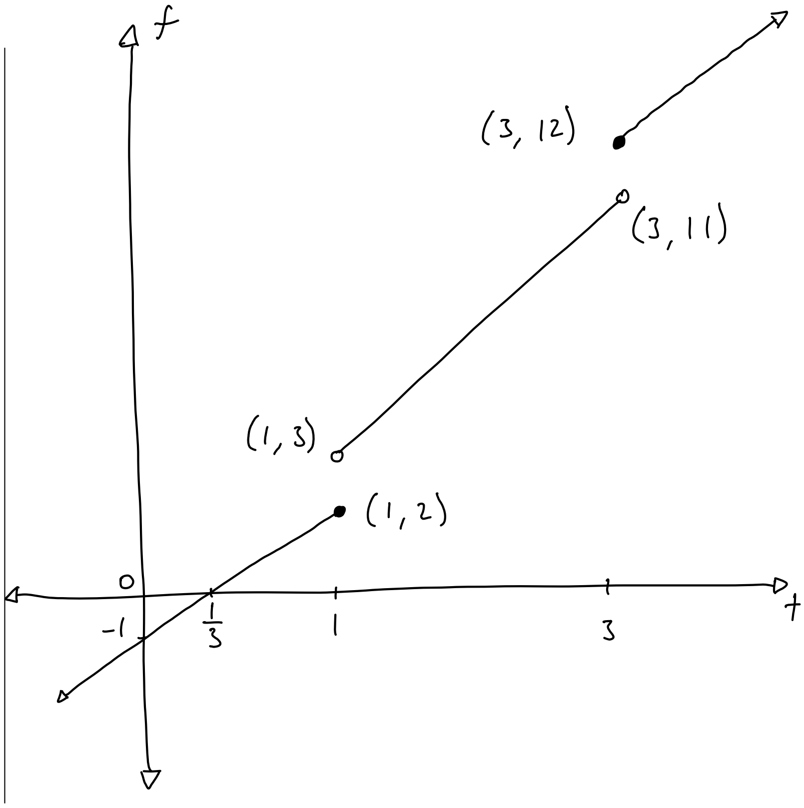
Example 8 Express the function $f(t) = 3t + tu(t-1) + u(t-3) - 1$ without the use of the Heaviside and sketch.

$$f(t) = \begin{cases} 3t - 1, & t \leq 1 \\ 3t + t - 1, & 1 < t < 3 \\ 3t + t - 1 + 1, & t \geq 3 \end{cases}$$

$$= \begin{cases} 3t - 1, & t \leq 1 \\ 4t - 1, & 1 < t < 3 \\ 4t, & t \geq 3 \end{cases}$$



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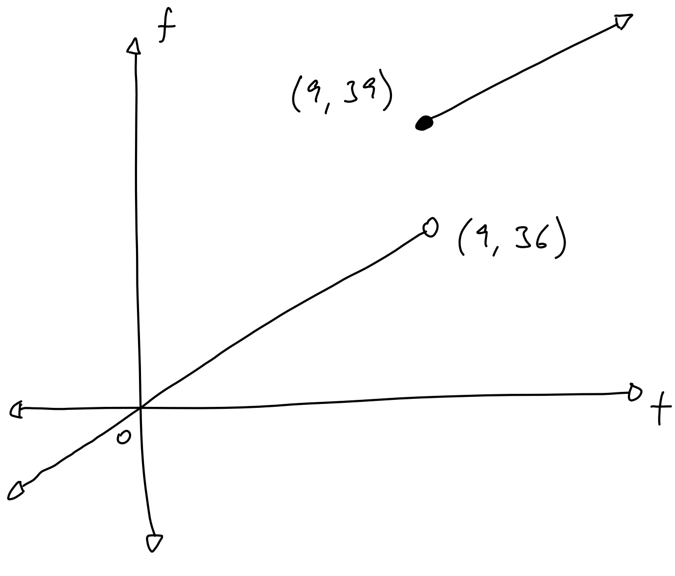
Example 9 Suppose that $\mathcal{L}(f) = \dfrac{4}{s^2} + \dfrac{3e^{-9s}}{s}$. Find and sketch f.

$$\mathcal{L}^{-1}\left(\frac{4}{s^2} + \frac{3e^{-9s}}{s}\right) = f$$

$$f = 4t + 3u(t-9)$$



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[34]You can now do Q 97, 99 a, b