LECTURE 46
HALF RANGE EXPANSIONS

Suppose that a function f has period $T = 2L$. Then f may be approximated by the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) \tag{1}$$

where the Fourier coefficients a_0, a_n, and b_n are given by

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \ldots) \\ b_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \ldots) \end{aligned}\right\} \tag{2}$$

For a function f defined on the interval $(0, L)$ the Fourier **cosine** series of f is the Fourier series of the **even** periodic extension of f.

For a function f defined on the interval $(0, L)$ the Fourier **sine** series of f is the Fourier series of the **odd** periodic extension of f.

Fourier sine and cosine series are called Half-Range expansions.

In this lecture we will answer the critical question 'If a function is not periodic how do we find an applicable Fourier series?' If the function is defined over the whole real line then the answer is simple........you can't! However if the function is only defined over an interval $(0, L)$ then it is quite simple to fool the Fourier series into believing that it is looking at a periodic function by implementing what is called an odd or an even periodic extension.

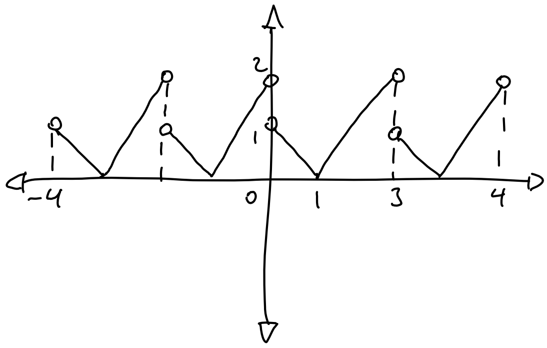
Example 1 Let $f(x) = |\, x - 1\, |$ for $0 < x < 3$;

i) Sketch the periodic extension of f and write down its period.

ii) Sketch the even periodic extension of f and write down its period.

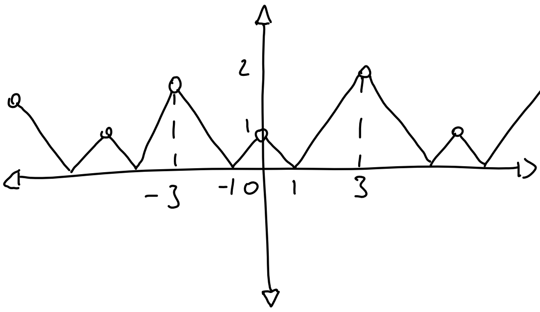
iii) Sketch the odd periodic extension of f and write down its period.

i)



$T = 3$

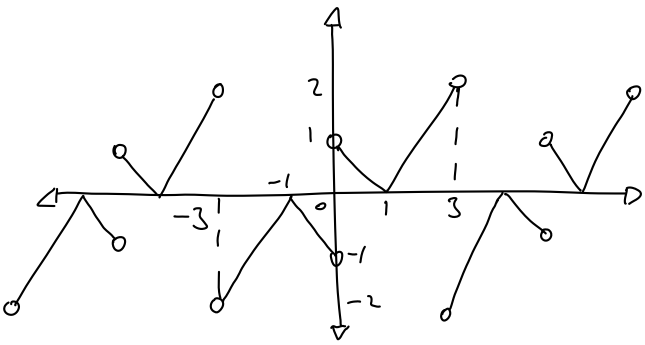
ii)



$T = 6$

"cos series"

iii)



$T = 6$

"sin series"

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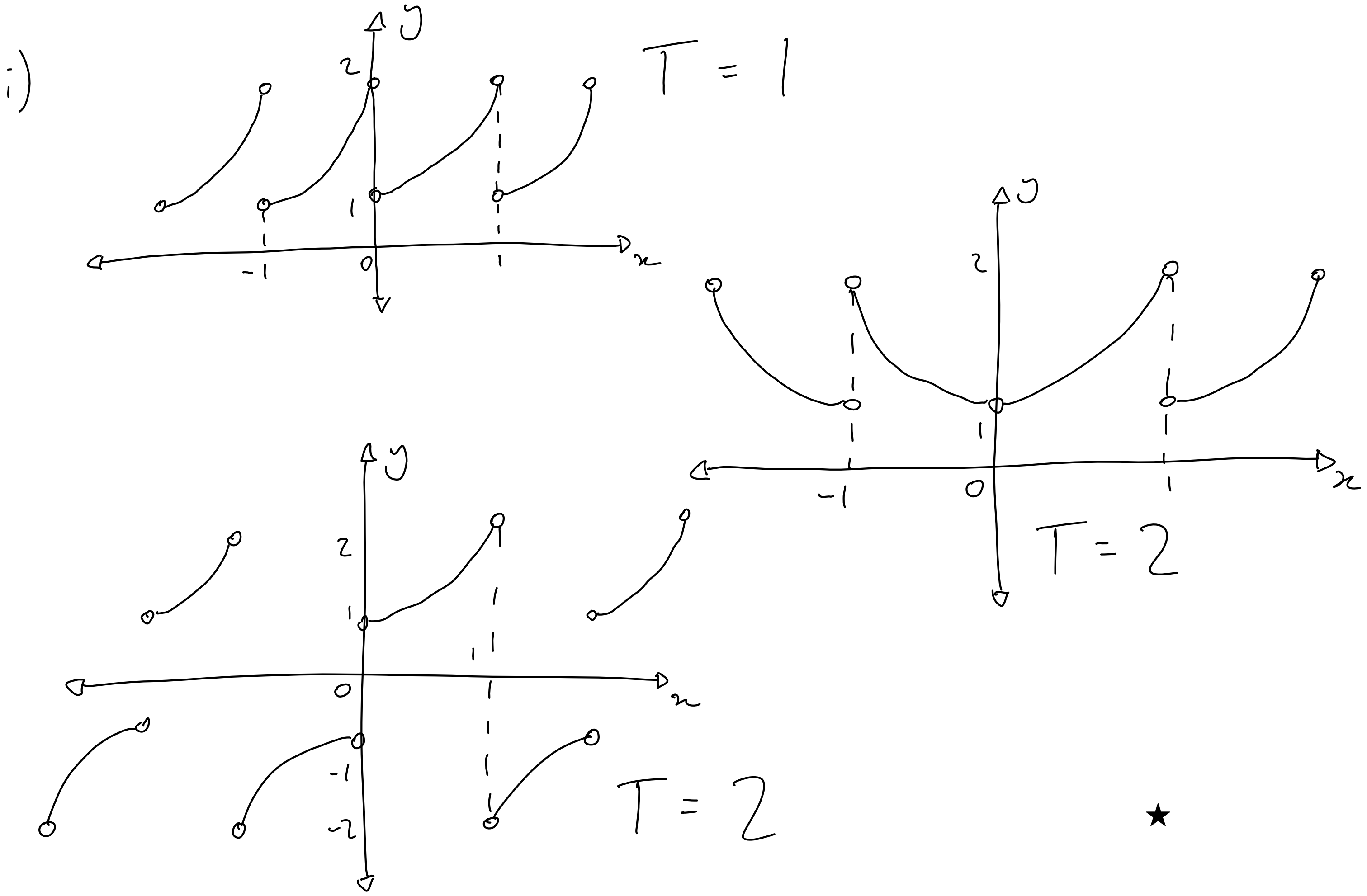
It is important to note in the above example that the original function f is definitely non-periodic. We are however taking that small definition over $(0, 3)$ and embedding it within a periodic environment in a number of different ways. Observe in each of the three graphs above you can still *see* the original f.

Example 2 Let $f(x) = x^2 + 1 \qquad$ for $0 < x < 1$;

i) Sketch the periodic extension of f and write down its period.

ii) Sketch the even periodic extension of f and write down its period.

iii) Sketch the odd periodic extension of f and write down its period.



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Now if a function is only defined over $(0, L)$ we can easily extend it to a periodic function using the trick above and then find its Fourier series! We are particularly interested in the odd and even periodic extensions since the calculations for these functions are a little less horrible and the pure sine and cosine series that are produced are easier to work with.

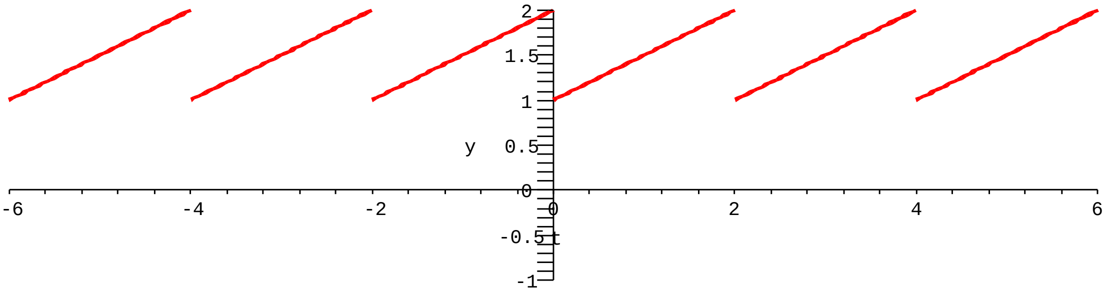
Note that if we take an even periodic extension then the Fourier series will be a Fourier cosine series but if we take the odd periodic extension then we will end up with a Fourier sine series. These are called Half Range expansions. Admittedly the sine and cosine series will be defined over the whole real line rather than just $(0, L)$, but that is OK as we can choose to implement the series only over the restricted domain $(0, L)$. In fact everything beyond $(0, L)$ is just a mirage! It is crucial to note that you DO NOT NEED special formulae for half range expansions and the usual equations will serve you perfectly well, provided you use a few simple tricks. Note also that we do not need to be too fussy about what happens at endpoints and discontinuities since the Fourier series will always make its own decision anyway and converge to a point which splits the gap.

Example 3 Let

$$f(x) = \left\{ \begin{array}{ll} \frac{x}{2} + 1 & 0 < x < 2; \end{array} \right.$$

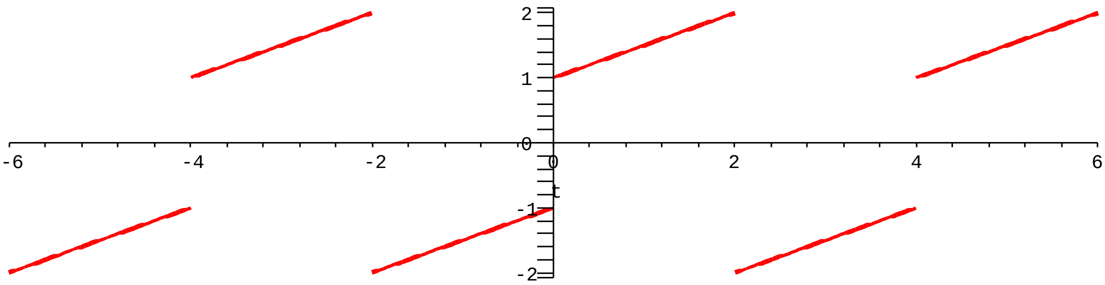
Find the Fourier **sine** series of f expressing your answer in sigma and expanded form.

Note that we could simply extend f to a periodic function of period 2 in a trivial fashion as follows:



But this periodic function is neither odd nor even and hence the Fourier series will be messy. We prefer the sine series.

The first step in calculating the Fourier sine series of f is to draw a careful sketch of the odd periodic extension of f:



$T = 4, \quad L = 2$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} \underbrace{f(x)}_{\text{odd}} \times \underbrace{\sin\left(\frac{n\pi x}{2}\right)}_{\text{odd}} = \text{even} \; dx$$

$$= \int_0^2 \left(1 + \frac{x}{2}\right) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \left[\frac{\cos\left(\frac{n\pi x}{2}\right)}{-\frac{n\pi}{2}}\right]_0^2 + \frac{1}{2}\left(\left[\frac{x\cos\left(\frac{n\pi x}{2}\right)}{-\frac{n\pi}{2}}\right]_0^2 + \left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right)\right]_0^2\right)$$

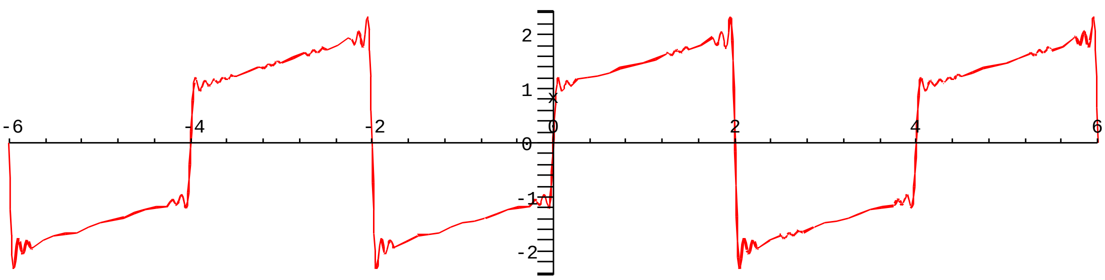
$$= \frac{-2\cos(n\pi)}{n\pi} + \frac{2}{n\pi} - \frac{2\cos(n\pi)}{n\pi}$$

$$= \frac{2 - 4(-1)^n}{n\pi}$$

$$\bigstar \quad f(x) = \sum_{n=1}^{\infty} \frac{2 - 4(-1)^n)}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \quad \bigstar$$

$$\bigstar \quad f(x) = \frac{6}{\pi}\sin\left(\frac{\pi x}{2}\right) - \frac{1}{\pi}\sin(\pi x) + \frac{2}{\pi}\sin\left(\frac{3\pi x}{2}\right)\ldots \quad \bigstar$$

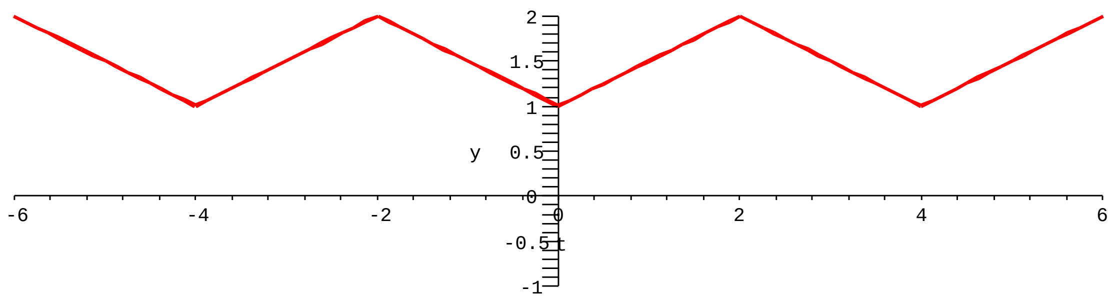
The graph of the sine series (taking the first 20 terms) is:



Note that we are really only interested in $[0, 2]$ since that is where the original function was defined! Note also that we have managed to approximate f using nothing but the sine curve.

Example 4 Find the Fourier **cosine** series of the same function f above. Express your answer in sigma notation, as a sum over the odd integers and also in expanded form.

First lets draw the even periodic extension:



$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

even

$$a_0 = \frac{1}{4}\int_{-2}^{2} 1 + \frac{x}{2} \, dx$$

$$= \frac{1}{2}\left[x + \frac{x^2}{4}\right]_0^2$$

$$= \frac{3}{2}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} \left(1 + \frac{x}{2}\right) \cos\left(\frac{n\pi x}{2}\right) dx$$

$\vdots$ int. by parts

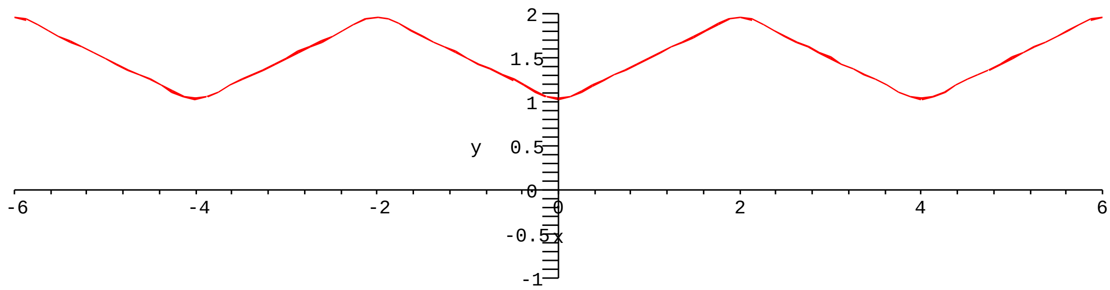
$$\therefore a_n = \frac{2\left((-1)^n - 1\right)}{(n\pi)^2}$$

$$\therefore f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi x}{2}\right)$$

$$\star \quad f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{n^2\pi^2}\cos(\frac{n\pi x}{2}) = \frac{3}{2} + \sum_{k=0}^{\infty} \frac{-4}{(2k+1)^2\pi^2}\cos(\frac{(2k+1)\pi x}{2}) \quad \star$$

$$\star \quad f(x) = \frac{3}{2} - \frac{4}{\pi^2}\cos(\frac{\pi x}{2}) - \frac{4}{9\pi^2}\cos(\frac{3\pi x}{2}) - \frac{4}{25\pi^2}\cos(\frac{5\pi x}{2})\ldots \quad \star$$

The graph of the cosine series (taking the first 6 terms) is:



Note that we have now approximated f entirely with cosines!

[46] You can now do Q 109 111 112