LECTURE 2
CHAIN RULE

If $z = f(x, y)$ and $x = x(t)$ and $y = y(t)$ then

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

If $z = f(x, y)$ and $x = x(u, v)$ and $y = y(u, v)$ then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

A common situation is that z is a function of x and y with x and y themselves functions of other variables......say u and v. It is then true that z is *ultimately* a function of u and v and thus it makes sense to ask the question "What is $\frac{\partial z}{\partial u}$"? The chain rule enables us to answer this question without actually ever having to produce z as an explicit function of u and v. The chain rule comes in many different flavours, two of which are presented above. However all that needs to be remembered is that you keep on differentiating z with respect to what you can and then always fudge your answer back to what you want.

Note that it is usually more effective to use the chain rule than to explicitly detail the structure of the new function.

Example 1 Suppose that $z = x^2 + 4y$ where $x = u^3 \ln(v)$ and $y = uv^2$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= (2x)(3u^2 \ln(v)) + (4)(v^2)$$

$$= 6u^5 \ln(v)^2 + 4v^2$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= (2x)\left(\frac{u^3}{v}\right) + (4)(2uv)$$

$$= \frac{2u^6 \ln(v)}{v} + 8uv$$

$$\star \quad \frac{\partial z}{\partial u} = 6u^5(\ln(v))^2 + 4v^2, \quad \frac{\partial z}{\partial v} = \frac{2u^6\ln(v)}{v} + 8uv \quad \star$$

Example 2 Given that $z = x^2y^3$ with $x = 2t$ and $y = \sin(4t)$ use the chain rule to find $\frac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= 2xy^3 \cdot 2 + 3x^2y^2 \cdot 4\cos(4t)$$

$$= 4(2t)(\sin(4t))^3 + 12(2t)^2(\sin(4t))^2\cos(4t)$$

$$= 8t\sin^3(4t) + 48t^2\sin^2(4t)\cos(4t)$$

$$\star \quad 8t\sin^3(4t) + 48t^2\sin^2(4t)\cos(4t) \quad \star$$

Example 3 If $z = a^2 + b + c^5 + d^7$ where $a = uv,\ b = 2u + 3v,\ c = u^2$ and $d = v^2$ use the chain rule to find $\frac{\partial z}{\partial u}$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial a} \cdot \frac{\partial a}{\partial u} + \frac{\partial z}{\partial b} \cdot \frac{\partial b}{\partial u} + \frac{\partial z}{\partial c} \cdot \frac{\partial c}{\partial u} + \frac{\partial z}{\partial d} \cdot \frac{\partial d}{\partial u}$$

$$= 2a \cdot v + 1 \cdot 2 + 5c^4 \cdot 2u + 7d^6 \cdot 0$$

$$= 2uv^2 + 10u^9 + 2$$

★ $2uv^2 + 10u^9 + 2$ ★

Example 4 If $w = a^2 - ab^3$ with $a = e^{uv}$ and $b = 3u + 2v$ use the chain rule to find $\frac{\partial w}{\partial u}$ when $u = 0$ and $v = 1$.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial a} \cdot \frac{\partial a}{\partial u} + \frac{\partial w}{\partial b} \cdot \frac{\partial b}{\partial u}$$

$$= (2a - b^3)(ve^{uv}) + (-3b^2)(3)$$

$$= (2e^{uv} - (3u + 2v))(ve^{uv}) - 9(3u+2v)^2$$

$$\frac{\partial w}{\partial u}(0, 1) = (2e^0 - (3 \times 0 + 2 \times 1))(1e^0) - 9(3 \times 0 + 2 \times 1)^2$$

$$= -42$$

★ -42 ★

Example 5 Suppose that temperature in the plane is given by $T(x,y) = x^2 + y^2$ and that a particle is traveling along a path C defined parametrically by $x = t - 1$ $y = t^3 - 3t^2 + 3t + 1$ where t is time in seconds. Show that $T(t)$ has a stationary point when $t = 1$.

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial T}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= 2x(1) + 2y \cdot (3t^2 - 6t + 3)$$

$$= 2(t-1) + 2(t^3 - 3t^2 + 3t + 1)(3t^2 - 6t + 3)$$

$$\frac{\partial T}{\partial t}(1) = 2(0) + 2(1 - 3 + 3 + 1)(3 - 6 + 3)$$

$$= 0$$

Example 6 If $w = f(u,v)$ with $u = x + y$ and $v = x - y$ show that

$$\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = (\frac{\partial w}{\partial u})^2 - (\frac{\partial w}{\partial v})^2$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}$$

$$\frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} = \left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}\right)\left(\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}\right)$$

$$= \left(\frac{\partial w}{\partial u}\right)^2 - \left(\frac{\partial w}{\partial v}\right)^2$$

Example 7 Suppose that $z = f(x, y)$ where x and y are expressed in polar coordinates

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

Prove that

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial^2 z}{\partial x^2}\cos^2\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta + \frac{\partial^2 z}{\partial x \partial y}\sin(2\theta)$$

This is going to be tough since f is a random function.

To help with the algebra we will denote $\frac{\partial^2 z}{\partial r^2}$ simply by z_{rr}, and $\frac{\partial^2 z}{\partial x \partial y}$ by z_{xy} ...etc. Now

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta = z_x\cos\theta + z_y\sin\theta$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(z_x\cos\theta + z_y\sin\theta\right) = \frac{\partial}{\partial r}(z_x\cos\theta) + \frac{\partial}{\partial r}(z_y\sin\theta)$$

$$= \cos\theta\frac{\partial}{\partial r}(z_x) + \sin\theta\frac{\partial}{\partial r}(z_y) \qquad \text{(Why?)}$$

$\cos\theta$ is the coefficient of z_x
$\sin\theta$ is the coefficient of z_y

Now the bad news. The partial derivatives z_x and z_y are functions of x and y so we will need to use the chain rule again! Thus

$$\frac{\partial}{\partial r}(z_x) = \frac{\partial}{\partial x}(z_x)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}(z_x)\frac{\partial y}{\partial r} = z_{xx}\frac{\partial x}{\partial r} + z_{yx}\frac{\partial y}{\partial r} = z_{xx}\cos\theta + z_{yx}\sin\theta$$

Similarly

$$\frac{\partial}{\partial r}(z_y) = \frac{\partial}{\partial x}(z_y)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}(z_y)\frac{\partial y}{\partial r} = z_{xy}\cos\theta + z_{yy}\sin\theta$$

Noting that $z_{xy} = z_{yx}$ and putting it all together we have

$$\frac{\partial^2 z}{\partial r^2} = \cos\theta\left(z_{xx}\cos\theta + z_{yx}\sin\theta\right) + \sin\theta\left(z_{xy}\cos\theta + z_{yy}\sin\theta\right)$$

$$= z_{xx}\cos^2\theta + z_{yy}\sin^2\theta + z_{xy}\sin 2\theta$$

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[2]You can now do Q 5 to 14