LECTURE 44
FOURIER SERIES OVER AN ARBITRARY PERIOD

Suppose that a function f has period $T = 2L$. Then f may be approximated by the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) \tag{1}$$

where the Fourier coefficients a_0, a_n, and b_n are given by

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \dots) \\ b_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, \dots) \end{aligned}\right\} \tag{2}$$

- Odd functions have odd series $\qquad f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$
- Even functions have even series $\qquad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$
- $\int_{-a}^{a} \text{odd}\ dx = 0$
- $\int_{-a}^{a} \text{even}\ dx = 2\int_{0}^{a} \text{even}\ dx$
- If a periodic function f has a jump discontinuity at $x = a$ then its Fourier series at $x = a$ will converge to a y value which sits half way across the discontinuity.

We turn now to the Fourier series of function with arbitrary period T. To simplify the equations we define L to be half the period so that $T = 2L$ and express the equations in terms of L.

Observe from the equations above that if we replace L by π (and hence T by 2π) we obtain exactly the equations which we have been using over the last few lectures. The formulae above will be supplied to you in your final examination and they are all that you need to find all the different types of Fourier series. Do not commit anything else to memory.

All of the techniques and tricks from before still apply in this slightly more general setting.

Example 1 Let $f(x) = \begin{cases} |x| & -3 \le x < 3; \\ f(x+6) & \text{otherwise.} \end{cases}$

i) Sketch f over $-9 \le x < 9$.

ii) Find the Fourier series of f written in sigma notation.

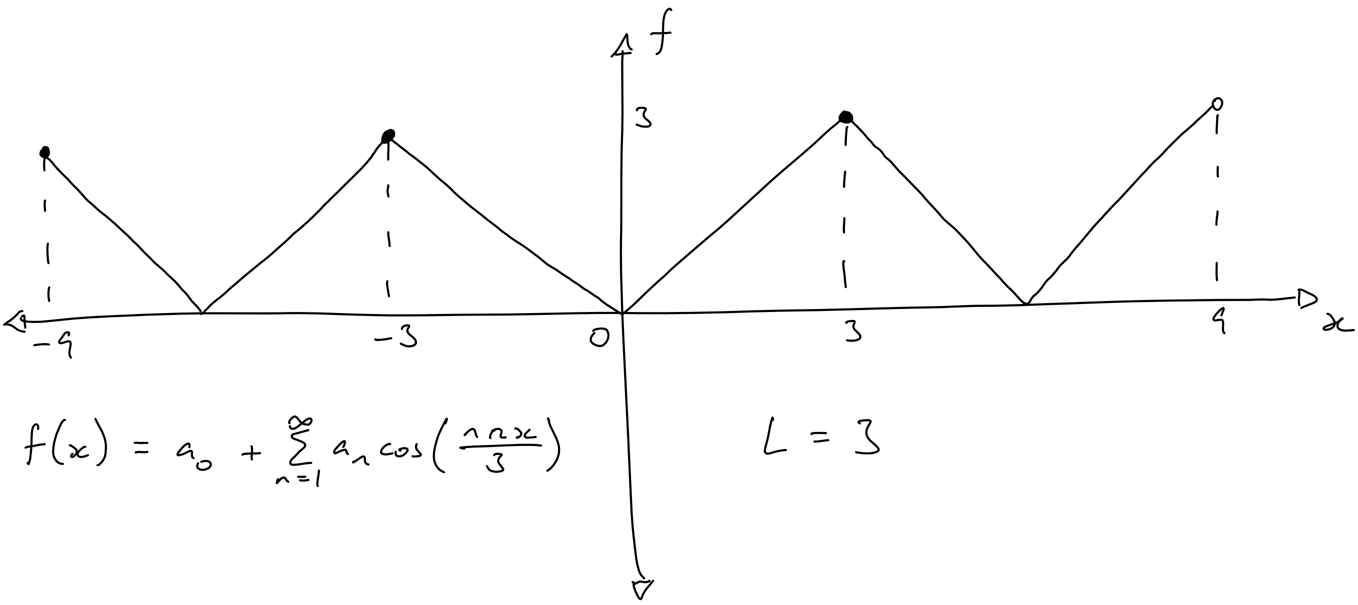
iii) Express the Fourier series of f as an explicit sum.

iv) Express the Fourier series of f as a sum over the odd integers.

v) By considering the Fourier series at $x = 0$ and using part iii), show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \dots$$

i)



$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right) \qquad L = 3$$

ii)

$$a_0 = \frac{1}{6}\int_{-3}^{3} f(x)\,dx = \frac{1}{3}\int_0^3 x\,dx = \frac{1}{3}\left[\frac{x^2}{2}\right]_0^3 = \frac{3}{2}$$

$$a_n = \frac{1}{3}\int_{-3}^{3} f(x)\cos\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\int_0^3 x\cos\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left(\left[\frac{x\sin\left(\frac{n\pi x}{3}\right)}{\frac{n\pi}{3}}\right]_0^3 - \frac{3}{n\pi}\left[\frac{\cos\left(\frac{n\pi x}{3}\right)}{-\frac{n\pi}{3}}\right]_0^3\right)$$

$$= \frac{6}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$\therefore\ f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{6}{n^2\pi^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi x}{3}\right)$$

iii) $$f(x) = \frac{3}{2} - \frac{12}{\pi^2}\cos\left(\frac{\pi x}{3}\right) - \frac{4}{3\pi^2}\cos(\pi x)$$

$$- \frac{12}{25\pi^2}\cos\left(\frac{5\pi x}{3}\right) - \dots$$

iv) For $n = 2k+1$:

$$f(x) = \frac{3}{2} + \sum_{k=1}^{\infty} \frac{-12}{(2k+1)^2\pi^2}\cos\left(\frac{(2k+1)\pi x}{3}\right)$$

v) $$f(0) = \frac{3}{2} - \frac{12}{\pi^2} - \frac{12}{9\pi^2} - \frac{12}{25\pi^2} - \dots = 0$$

$$\therefore\ \frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \dots$$

$$\star \quad ii) \quad f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{6((-1)^n - 1)}{n^2\pi^2} \cos(\frac{n\pi x}{3}) \quad \star$$

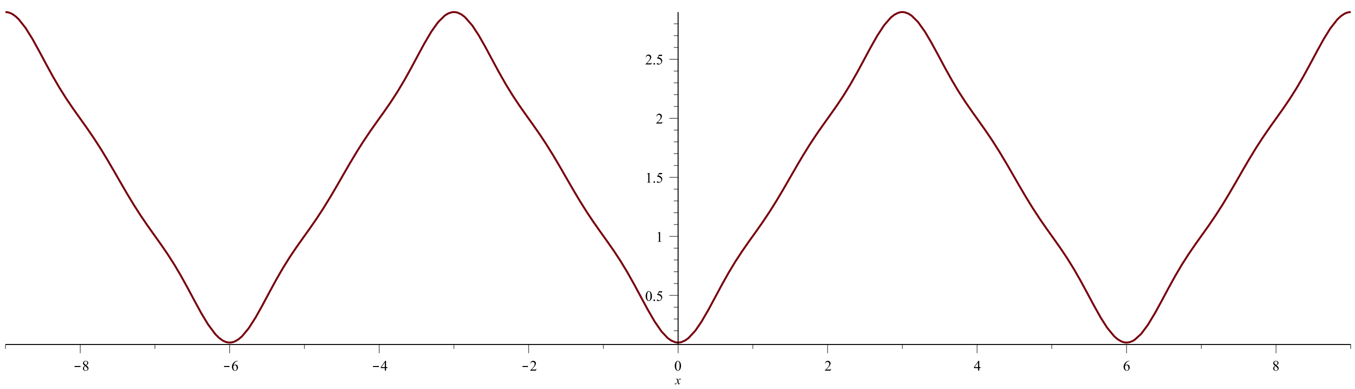
$$\star \quad iii) \quad f(x) = \frac{3}{2} - \frac{12}{\pi^2} \left(\frac{\cos(\frac{\pi x}{3})}{1} + \frac{\cos(\frac{3\pi x}{3})}{9} + \frac{\cos(\frac{5\pi x}{3})}{25} + \frac{\cos(\frac{7\pi x}{3})}{49} + \dots \right) \quad \star$$

$$\star \quad iv) \quad f(x) = \frac{3}{2} - \frac{12}{\pi^2} \sum_{k=0}^{\infty} \frac{\cos\left(\frac{(2k+1)\pi x}{3}\right)}{(2k+1)^2} \quad \star$$

The sketch below shows the graph of the Fourier series truncated at $n = 4$:

$$\frac{3}{2} - \frac{12}{\pi^2} \left(\frac{\cos(\frac{\pi x}{3})}{1} + \frac{\cos(\frac{3\pi x}{3})}{9} + \frac{\cos(\frac{5\pi x}{3})}{25} + \frac{\cos(\frac{7\pi x}{3})}{49} \right)$$

Observe how nicely the Fourier series accomplishes its approximating duties.



Example 2 Let $f(x) = \begin{cases} x & -1 \le x < 1; \\ f(x+2) & \text{otherwise.} \end{cases}$

i) Sketch f over $-3 \le x < 3$.

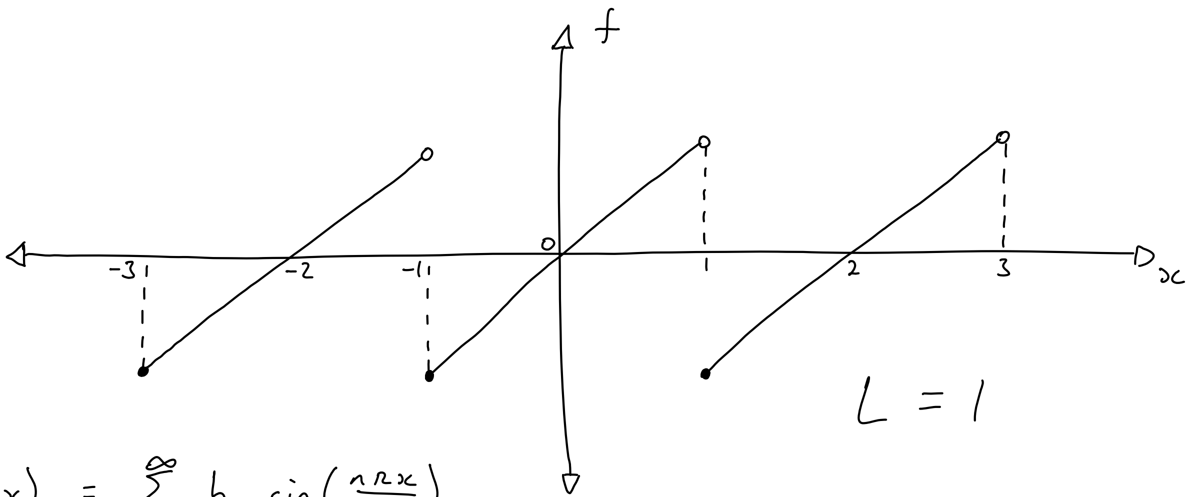
ii) Find the Fourier series of f.

iii) To what value does the Fourier series converge at $x = 1$?

iv) By considering the series at $x = \frac{1}{2}$ show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

i)



$L = 1$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

ii) $$b_n = \int_{-1}^{1} f(x) \sin(n\pi x)\, dx$$

$$= 2\int_0^1 x \sin(n\pi x)\, dx$$

$$= 2\left(\left[\frac{x\cos(n\pi x)}{-n\pi}\right]_0^1 + \frac{1}{n\pi}\left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1\right)$$

$$= \frac{2(-1)^{n+1}}{n\pi}$$

$$\therefore\ f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

iii) $f(1) =$

$$\bigstar \quad f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1}\sin(n\pi x) = \frac{2}{\pi}\left(\frac{\sin(\pi x)}{1} - \frac{\sin(2\pi x)}{2} + \frac{\sin(3\pi x)}{3} - \frac{\sin(4\pi x)}{4} + \dots\right) \quad \bigstar$$

★ Converges to $y = 0$ at $x = 1$ ★

Observe from the above example that if a periodic function f has a jump discontinuity at $x = a$ then its Fourier series at $x = a$ will converge to a y value which sits half way across the discontinuity.

[44] You can now do Q 110