UNIVERSITY OF NEW SOUTH WALES
School of Mathematics and Statistics

MATH2089 Numerical Methods and Statistics
Term 2, 2019

Numerical Methods Tutorial – Week 8 Solutions

1. For a function f the following data are known:

$$f(0) = 12.6, \quad f(1) = 6.7, \quad f(2) = 4.3, \quad f(3) = 2.7.$$

(a) What is the degree of the interpolating polynomial P for these data?

(b) Assume that we want to find P in the form

$$P(x) = a_0 + a_1 x + \cdots .$$

i. Write down the system of linear equations (in matrix form $A\mathbf{a} = \mathbf{b}$) you need to solve to obtain a_0, a_1, ...

ii. Use MATLAB to set up and solve this linear system using \ (backslash) to solve the linear system.

iii. Use MATLAB to plot both the data and your interpolating polynomial on the interval $[-1, 4]$.

(c) Now we are to find P in terms of the Lagrange polynomials $\ell_j(x)$ for $j = 0, 1, 2, 3$.

i. Write down the Lagrange polynomials $\ell_j(x)$ for $j = 0, 1, 2, 3$.

ii. Verify that

$$\ell_j(x_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad i, j = 0, 1, 2, 3.$$

iii. Write down the interpolating polynomial P using the Lagrange polynomials.

iv. Find an approximate value to $f(1.5)$ by hand. Check it agrees with your MATLAB code from the previous part.

Answer

(a) As there are 4 data values and a polynomial of degree n has $n + 1$ coefficients, the degree of the interpolating polynomial is $n = 4 - 1 = 3$.

(b) i. As the interpolating polynomial is a cubic (degree 3)

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

The interpolating polynomial agrees exactly with the data values, so

$$\begin{aligned} f(0) = 12.6 &\implies P(0) = a_0 = 12.6, \\ f(1) = 6.7 &\implies P(1) = a_0 + a_1 + a_2 + a_3 = 6.7, \\ f(2) = 4.3 &\implies P(2) = a_0 + 2a_1 + 4a_2 + 8a_3 = 4.3, \\ f(3) = 2.7 &\implies P(3) = a_0 + 3a_1 + 9a_2 + 27a_3 = 2.7. \end{aligned}$$

Writing this linear system in matrix form $A\mathbf{a} = \mathbf{f}$ gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 12.6 \\ 6.7 \\ 4.3 \\ 2.7 \end{bmatrix}$$

Note that each column of the coefficient matrix is x_i^k for $k = 0, 1, 2, 3$ where $\mathbf{x} = (0\ 1\ 2\ 3)^T$ is the column vector of points where data is known.
The interpolating polynomial is, after using MATLAB $\mathbf{a} = A\backslash\mathbf{f}$ or performing Gaussian elimination (row-operations) by hand,

$$P(x) = 12.6 - 8.55x + 3.1x^2 - 0.45x^3. \tag{1}$$

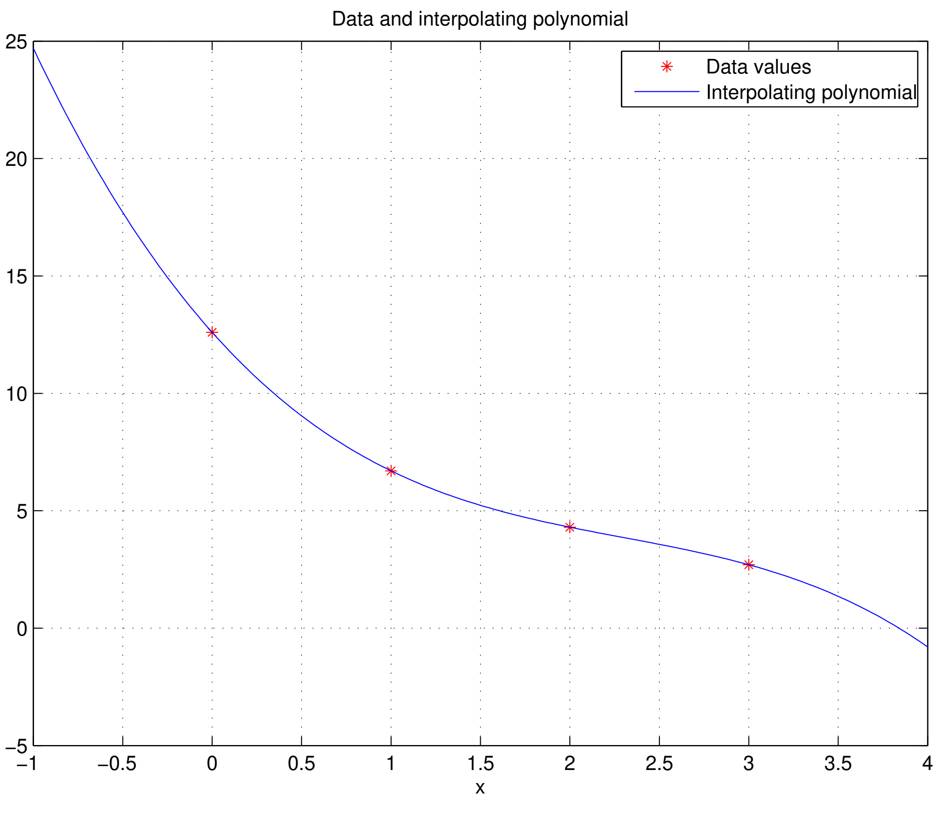

Figure 1: Data and interpolating cubic

(c) i. The Lagrange polynomials for the data points x_j for $j = 0, 1, \ldots, n$ are the degree n polynomials

$$\ell_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{(x - x_k)}{(x_j - x_k)}.$$

For the given data $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3$, this gives

$$\begin{aligned}
\ell_0(x) &= \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = \frac{-1}{6}(x-1)(x-2)(x-3) \\
\ell_1(x) &= \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \frac{1}{2}x(x-2)(x-3) \\
\ell_2(x) &= \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} = \frac{-1}{2}x(x-1)(x-3) \\
\ell_3(x) &= \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} = \frac{1}{6}x(x-1)(x-2)
\end{aligned}$$

Note that each Lagrange polynomial $\ell_j(x)$ is of degree 3.

ii. Direct substitution gives

$$\begin{aligned}
x = 0 &\implies \ell_0(0) = 1, \quad \ell_1(0) = 0, \ell_2(0) = 0, \ell_3(0) = 0, \\
x = 1 &\implies \ell_1(1) = 1, \quad \ell_0(1) = 0, \ell_2(1) = 0, \ell_3(1) = 0, \\
x = 2 &\implies \ell_2(2) = 1, \quad \ell_0(2) = 0, \ell_1(2) = 0, \ell_3(2) = 0, \\
x = 3 &\implies \ell_3(3) = 1, \quad \ell_0(3) = 0, \ell_1(3) = 0, \ell_2(3) = 0.
\end{aligned}$$

iii. Using the Lagrange polynomials there is no need to solve a linear system and the coefficients are simply the data values f_j, giving

$$\begin{aligned}
P(x) &= \sum_{j=0}^{n} f_j \ell_j(x) \\
&= 12.6\, \ell_0(x) + 6.7\, \ell_1(x) + 4.3\, \ell_2(x) + 2.7\, \ell_3(x) \\
&= \frac{-12.6}{6}(x-1)(x-2)(x-3) + \frac{6.7}{2}x(x-2)(x-3) + \\
&\quad \frac{-4.3}{2}x(x-1)(x-3) + \frac{2.7}{6}x(x-1)(x-2) \qquad (2)
\end{aligned}$$

Note that as the interpolating polynomial is unique, expanding the representation of $P(x)$ in equation (2) **must** give the same polynomial as in equation (1).

iv.

$$\begin{aligned}
P(1.5) &= 12.6\, \ell_1(1.5) + 6.7\, \ell_2(1.5) + 4.3\, \ell_3(1.5) + 2.7\, \ell_4(1.5) \\
&= 12.6 \times \left(\frac{-1}{16}\right) + 6.7\left(\frac{9}{16}\right) + 4.3\left(\frac{9}{16}\right) + 2.7\left(\frac{-1}{16}\right) \\
&= 5.23125,
\end{aligned}$$

which agrees with the value obtained from (1)

$$P(1.5) = 12.6 - 8.55 \times 1.5 + 3.1 \times (1.5)^2 - 0.45 \times (1.5)^3 = 5.23125.$$

2. The following table lists the values of $f(x) = \cos x$ (to 4 decimal places).

x	0.2	0.3	0.4	0.5	0.6
$f(x)$	0.9801	0.9553	0.9211	0.8776	0.8253

(a) Using the table, find an approximate value to $f(0.35)$.

i. Using linear interpolation

ii. Using quadratic interpolation

iii. Using Lagrange interpolation and all data values.

(b) Determine a bound on the error involved.

(c) Find approximate values to $f'(0.3)$ by using the forward difference and central difference. Use a calculator to find $f'(0.3)$ and compare your answers with it. Which difference gives a better approximation?

(d) Find an approximate value to $f''(0.4)$.

Answer

(a) i. Linear interpolation (degree 1) requires two data values, so it is natural to use the values surrounding $x = 0.35$, namely $x_2 = 0.3$ and $x_3 = 0.4$. A convenient form of the linear interpolating polynomial is

$$P_1(x) = m(x - x_2) + c.$$

The interpolation conditions then give

$$\begin{aligned} P_1(x_2) = y_2 &\implies c = y_2 = 0.9553 \\ P_1(x_3) = y_3 &\implies c + m(x_3 - x_2) = y_3 = 0.9211 \end{aligned}$$

The slope of the linear interpolant is

$$m = \frac{y_3 - y_2}{x_3 - x_2} = \frac{0.9211 - 0.9553}{0.4 - 0.3} = -0.342$$

giving

$$P_1(x) = y_2 + \frac{y_3 - y_2}{x_3 - x_2}(x - x_2)$$

so

$$P(0.35) = 0.9553 - 0.342(0.35 - 0.3) = 0.9382.$$

The exact value (to 14 decimal places) is $\cos(0.35) = 0.939372712847379$.

ii. A quadratic (degree 2) has three parameters, so we require three sets of data values to get an interpolating quadratic. As we want the value at $x = 0.35 \in (x_2, x_3)$, the two most natural choices are the values at x_2, x_3 and x_4 or the values at x_1, x_2 and x_3. We will again use a polynomial based at x_2, namely

$$q(x) = a(x - x_2)^2 + b(x - x_2) + c.$$

- Using x_2, $x_3 = x_2 + h$ and $x_4 = x_2 + 2h$ the interpolation conditions are

$$\begin{aligned} q(x_2) = y_2 &\implies c = y_2 \\ q(x_3) = y_3 &\implies ah^2 + bh + c = y_3 \\ q(x_4) = y_4 &\implies a(2h)^2 + b(2h) + c = y_4 \end{aligned}$$

Solving these equations gives

$$\begin{aligned} c &= y_2 \\ b &= \frac{-3y_2 + 4y_3 - y_4}{2h} \\ a &= \frac{y_2 - 2y_3 + y_4}{2h^2} \end{aligned}$$

and an approximation $q(0.35) = 0.9393625$ (giving 0.9394 rounded to 4 decimal places).

- Using $x_1 = x_2 - h$, x_2 and $x_3 = x_2 + h$ the interpolation conditions are

$$\begin{aligned} q(x_1) = y_1 &\implies a(-h)^2 + b(-h) + c = y_1 \\ q(x_2) = y_2 &\implies c = y_2 \\ q(x_3) = y_3 &\implies ah^2 + bh + c = y_3 \end{aligned}$$

Solving these equations gives

$$\begin{aligned} c &= y_2 \\ b &= \frac{y_3 - y_1}{2h} \\ a &= \frac{y_1 - 2y_2 + y_3}{2h^2} \end{aligned}$$

and an approximation $q(0.35) = 0.939375$ (giving 0.9394 rounded to 4 decimal places).

iii. As there are 5 data values a quartic (degree = 4) polynomial can interpolate all data values. One way of obtaining this is using Lagrange interpolation, where

$$P(x) = 0.9801\ell_0(x) + 0.9553\ell_1(x) + 0.9211\ell_2(x) + 0.8776\ell_3(x) + 0.8253\ell_4(x),$$

$$\ell_j(x) = \prod_{\substack{k=0 \\ k\neq j}}^{n} \frac{(x - x_k)}{(x_j - x_k)},$$

and $x_j = 0.2 + ih$ for $j = 0, \ldots, 4$ and $h = 0.1$. This gives a value $P(0.35) = 0.9396125$ or 0.9396 rounded to 4 decimal places.

(b) From lectures, the error in the Lagrange interpolant is

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j) \qquad \text{for some } \xi = \xi(x) \in [a, b].$$

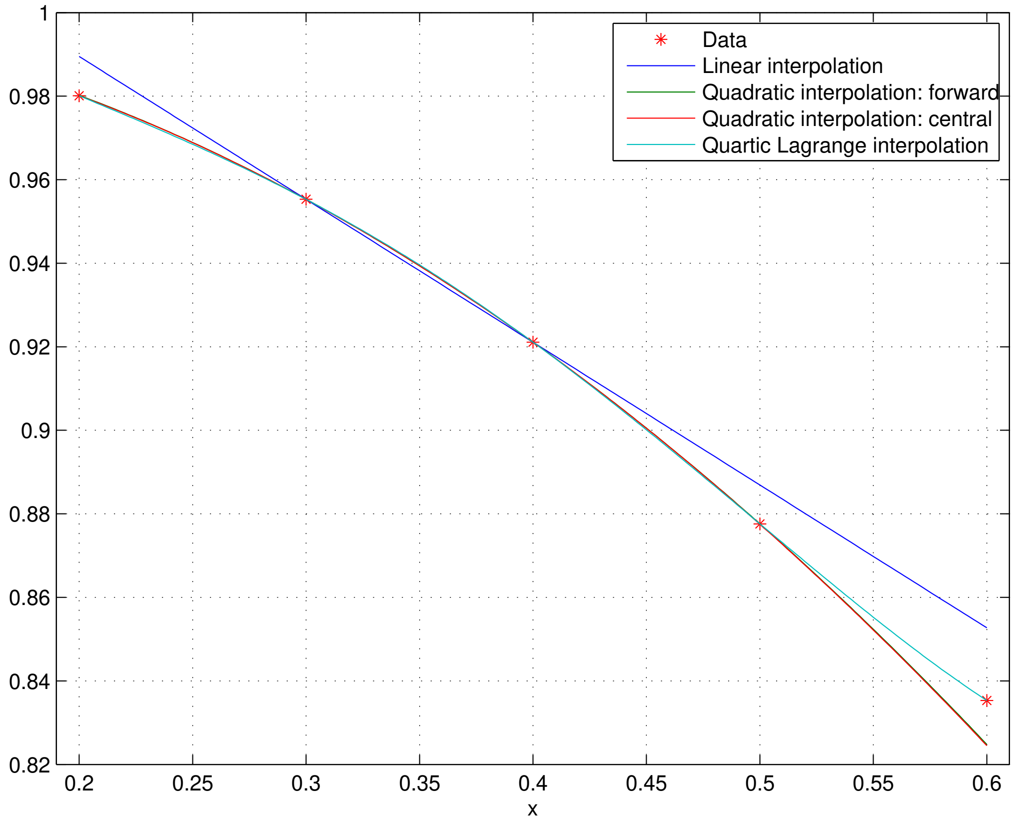

Figure 2: Interpolation by linear, quadratic and quartic polynomials'

Here $n = 4$ and we only know that $\xi \in [0.2,\ 0.6]$. However all the derivatives of $\cos(x)$ are $\pm\sin(x)$ or $\pm\cos(x)$, so $|f^{(n+1)}(\xi)| \leq 1$. Thus

$$
\begin{aligned}
|R_4(0.35)| &\leq \frac{1}{5!}\prod_{j=0}^{4}(0.35 - x_j) \\
&= \frac{1}{120}|(0.35-0.2)(0.35-0.3)(0.35-0.4)(0.35-0.5)(0.35-0.6)| \\
&= 1.2 \times 10^{-7}
\end{aligned}
$$

This assumes the data values are `exact`, at least up to machine precision. But here they were rounded to 4 decimal places implying an error of up to 0.5×10^{-4} in each function value. Thus this error estimate is far too optimistic and we cannot expect better than the 4 decimal places of accuracy in the function values.

(c) The forward difference approximation of $O(h)$ is

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

Using $x = x_1 = 0.3$ and $x + h = x_2 = 0.4$ this gives

$$f'(0.3) \approx \frac{f(0.4) - f(0.3)}{0.1} = \frac{0.9211 - 0.9553}{0.1} = -0.342.$$

The central difference approximation of $O(h^2)$ is

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

Using $x = x_1 = 0.3$, $x + h = x_2 = 0.4$ and $x - h = x_1 = 0.2$ this gives

$$f'(0.3) \approx \frac{f(0.4) - f(0.2)}{2 \times 0.1} = \frac{0.9211 - 0.9801}{0.2} = -0.295.$$

As $f'(x) = -\sin(x)$ the exact value is $f'(0.3) = -\sin(0.3) = -0.295520$, so the absolute error in the forward differences estimate is approximately 4.6×10^{-2}, while the absolute error in the central difference approximation is -5.2×10^{-4}.

(d) The central difference approximation of $O(h^2)$ is

$$f''(x) \approx \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}.$$

Thus using $x = x_2 = 0.4$, $x - h = x_2 - h = x_1 = 0.3$ and $x + h = x_2 + h = x_3 = 0.5$, so

$$f''(0.4) \approx \frac{f(0.3) - 2f(0.4) + f(0.5)}{0.1^2} = \frac{0.9553 - 2 * 0.9211 + 0.8776}{0.01} = -0.9300$$

As $f''(x) = -\cos(x)$ the exact value is $f''(0.4) = -0.92106....$ Again the estimate of the second derivative has errors arising from

- The truncation error $O(h^2)$ in the central difference approximation
- The approximate function values (rounded to 4 decimal places), which will be amplified in the calculation of the difference quotient.

3. Some choices for basis functions for $\mathbb{P}_n$, the space of polynomials of degree at most n, are

- **Monomial basis:** x^k for $k = 0, \dots, n$,
- **Chebyshev basis:** $T_k(x) = \cos(k \arccos(x))$, for $k = 0, \dots, n$,

while choices for the (interpolation) points in $[-1, 1]$ are

- **Equally spaced points:** $x_j = -1 + jh$ for $j = 0, \dots, n$ and $h = 2/n$,
- **Chebyshev points:** $x_j = \cos\left(\frac{\pi}{2}\frac{2j+1}{n+1}\right)$, for $j = 0, \dots, n$.

(a) Show that the Chebyshev points are the zeros of the degree $n + 1$ Chebyshev polynomial $T_{n+1}(x)$.

(b) The Vandermonde matrix for the basis $b_k(x), k = 0, \dots, n$ and points $x_j, j = 1, \dots, n + 1$ is

$$A_{i+1,j+1} = b_j(x_i) \qquad i = 0, \dots, n, \quad j = 0, \dots, n.$$

The condition numbers of the Vandermonde matrix for choices of basis functions and point sets are given in the following table. For each combination of basis functions and points, what is the highest degree interpolating polynomial for which the computed solution to $A\mathbf{a} = \mathbf{f}$ has at least 4 significant figures when the function values f_j at x_j

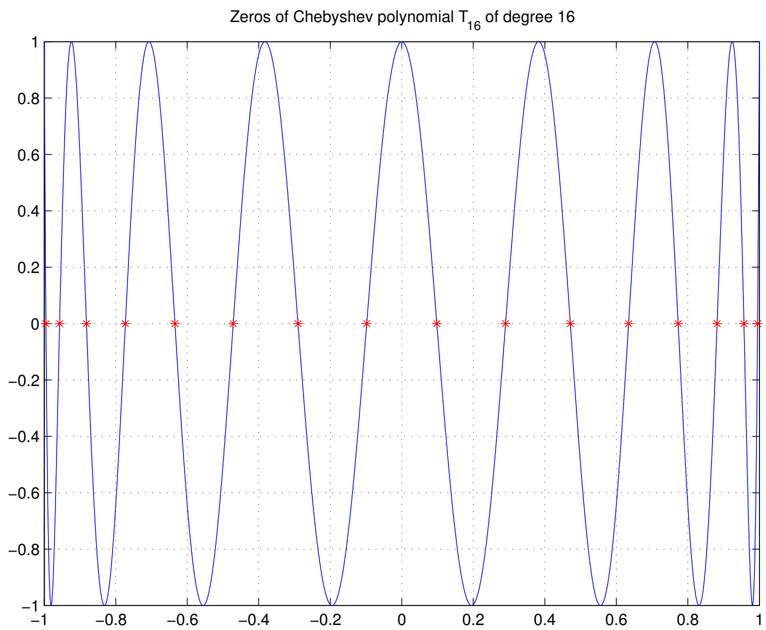

Figure 3: Zeros of Chebyshev polynomial of degree 16

Basis	Point set	$n = 5$	$n = 10$	$n = 15$	$n = 20$	n = 25
Monomial	Equally spaced	6.38e+01	1.40e+04	3.28e+06	8.31e+08	2.13e+11
Monomial	Chebyshev	4.52e+01	3.59e+03	2.83e+05	2.31e+07	1.86e+09
Chebyshev	Equally spaced	2.93e+00	2.37e+01	4.27e+02	8.64e+03	2.12e+05
Chebyshev	Chebyshev	1.41e+00	1.41e+00	1.41e+00	1.41e+00	1.41e+00

i. are known exactly.

ii. come from experimental measurements taken to 5 significant figures.

Answer

(a) Using the definition of the Chebyshev polynomial $T_{n+1}(x) = \cos((n+1)\arccos(x))$ and Chebyshev points $x_j = \cos\left(\frac{\pi}{2}\frac{2j+1}{n+1}\right)$ for $j = 0, \dots, n$

$$T_{n+1}(x_j) = \cos\left((n+1)\arccos(x_j)\right) = \cos\left(\frac{\pi}{2}(2j+1)\right) = 0.$$

Here we have used the property of the inverse function that $\arccos(\cos(x)) = x$ and that $\cos((2j+1)\pi/2) = 0$ for $j = 0, \dots, n$ as the cosine function is zero for odd multiples of $\pi/2$.

(b) The relative error $\mathrm{re}\,(\mathbf{a})$ in the computed polynomial coefficients $\mathbf{a}$ is estimated by

$$\mathrm{re}\,(\mathbf{a}) \leq \kappa(A)\left[\mathrm{re}\,(A) + \mathrm{re}\,(\mathbf{f})\right]$$

To get at least 4 significant figures in the computed coefficients $\mathbf{a}$ we want $\mathrm{re}\,(\mathbf{a}) \leq 0.5 \times 10^{-5}$. For this to apply to all the coefficients of the vector $\mathbf{a}$ we should be using $\|\mathbf{a}\|_\infty$ and the corresponding condition number $\kappa_\infty(A)$. However it is not clear from the provided information which norm has been used in calculating the condition numbers in the Table. Ignoring this issue, and assuming the coefficient matrix A is known "exactly" so $\mathrm{re}\,(A) = \epsilon = 2.2 \times 10^{-16}$, we want

$$\kappa(A)\,[\epsilon + \mathrm{re}\,(\mathbf{f})] \leq 0.5 \times 10^{-4} \tag{3}$$

i. If the elements of $\mathbf{f}$ are known exactly, then the values stored on the computer will have a relative error of ϵ. In this case (3) becomes

$$\kappa(A) \leq \frac{0.5 \times 10^{-4}}{2\epsilon} = \frac{0.5 \times 10^{-4}}{4.4 \times 10^{-16}} \approx 1.1 \times 10^{11}$$

From the Table, this is satisfied for all bases, point sets and degrees, except for degree $n = 25$ with Monomial basis and equally spaced points.

ii. If the elements of $\mathbf{f}$ come from experimental measurements taken to 5 significant figures then the relative error in the elements of $\mathbf{f}$ is at most 0.5×10^{-5}. Then (3) gives

$$\kappa(A) \leq \frac{0.5 \times 10^{-4}}{0.5 \times 10^{-5}} = 10^1.$$

From the Table, this is **only** satisfied for the Chebyshev basis with the Chebyshev points (for which $\kappa_2(A) = \sqrt{2} \approx 1.4142$ for all degrees) and the Chebyshev basis and equally spaced points of degree 5.