QUESTION 1 (S2 - 2014 - MATH 2089)

a) i) ans1 = 0

ii) $v = [-2 : 2 : 2]$

$= [-2 \quad 0 \quad 2]$

$ans2 = v./v = [-2 \quad 0 \quad 2]./[-2 \quad 0 \quad 2]$

$= [1 \quad NaN \quad 1]$

ii) ans3 = Inf

b) i) In 1 hour, the computer can do

$\underbrace{3600}_{\#\text{ secs}} \times \underbrace{3 \times 10^9}_{3\text{ GHz}} \times \underbrace{4}_{\text{quad core}} \times \underbrace{2}_{2\text{ flops / core / clock cycle}} = 3600 \times 24 \times 10^9$ flops.

No special structure ⇒ use LU factorization, So

$$\frac{2n^3}{3} = 24 \times 10^9 \times 3600 = 86.4 \times 10^{12}$$

(ignoring $O(n^2)$ terms)

$$n = \left(\frac{3}{2} \times 84.6 \times 10^{12}\right)^{1/3} \approx 50\,252$$

=

The largest linear system the computer can solve in 1 hour has size about 50 252 × 50 252.

ii) $2n^3$ (matrix multiplication) flops take 1000 se

For solving a linear system via Cholesky factorizati it takes about $\frac{n^3}{3}$ flops, so it takes about 166 seconds.

c) i) A is symmetric since $\| A - A^T \|$ closes to ep.

ii) $\mathcal{K}_2(A) = \dfrac{\lambda_{max}(A)}{\lambda_{min}(A)} = \dfrac{100}{0.01} = 10^4$

iii) $\mathcal{K}_2(A^{-1}) = \mathcal{K}_2(A) = 10^4$

iv) $\text{rel-err}(b) \leq 0.5 \times 10^{-6}$

$$\text{rel-err}(\underset{\sim}{x}) \approx \mathcal{K}_2(A)\left[\text{rel-err}(A) + \text{rel-err}(\underset{\sim}{b})\right] \quad (*)$$

$$\leq 10^4 \left(2\times 10^{-16} + 0.5 \times 10^{-6}\right)$$

$$\leq 0.5 \times 10^{-2}$$

v) 2 significant figures.

vi) Suppose $\text{rel-err}(\underset{\sim}{x}) \approx 0.5 \times 10^{-5}$

From $(*)$ we have

$$0.5 \times 10^{-5} \approx 10^4 \left(2 \times 10^{-16} + \text{rel-err}(\underset{\sim}{b})\right)$$

$$\Rightarrow \text{rel-err}(b) \approx 0.5 \times 10^{-9}$$

$\Rightarrow$ 9 significant figures in $\underset{\sim}{b}$.

d) i) $\| A \|_2 = \sqrt{\lambda_{max}(A^T A)}$ true.

ii) Q orthogonal $\Rightarrow \mathcal{K}_2(Q) = 1$ true since

$$\| Q \| = \max_{x \neq 0} \frac{\| Q x \|}{\| x \|} = \max_{\underset{\sim}{x} \neq 0} \frac{\| x \|}{\| x \|} = 1$$

Here, we used $\| Q x \|^2 = (Q x)^T Q x = x^T Q^T Q x = x$

$= \| x \|^2$

$\| Q^T \| = \| Q^{-1} \| = \| Q \| = 1.$

Question 2 (S2-2014) MATH 2089

a) i) $$X_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 4 - \frac{4^2 - 9}{2 \times 4} = 4 - \frac{7}{8} = 3.125$$

ii) the order of convergence is quadratic since $e(k+1)/e(k)^2$ tends to a constant.

b) The vector $\underset{\sim}{x}$ which satisfies
$$A^T A \underset{\sim}{x} = A^T b$$
will minimize $\| A\underset{\sim}{x} - \underset{\sim}{b} \|_2^2$

c) $A = QR \qquad \kappa_2(A) = 10^6$

i) $\kappa_2(A^T A) = [\kappa_2(A)]^2 = 10^{12}$

ii) $\kappa_2(R) = \kappa_2(A) = 10^6$

iii) The normal equation has ill-conditioned matrix
In this case, to solve the least squares problem,
I would use the QR factorization

d) i)
```
A = [ ones(length(tdat),1) tdat];
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ii)
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x = A \ ydat;
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iii)
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r = A*x - ydat;
ans = sum(r.*r);
```

e)

i)
$$\begin{cases} \dfrac{dx_1}{dt} = -k x_1 x_2 \\ \dfrac{dx_2}{dt} = -k x_1 x_2 \\ \dfrac{dx_3}{dt} = k x_1 x_2 \end{cases}$$

$$\Rightarrow \quad \underset{\sim}{x}' = f(t, \underset{\sim}{x}) \quad \text{with} \quad f(t, \underset{\sim}{x}) = \begin{bmatrix} -k x_1 x_2 \\ -k x_1 x_2 \\ k x_1 x_2 \end{bmatrix}$$

ii) Initial condition

$$\underset{\sim}{x}_0 = \underset{\sim}{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \quad \text{moles}/m^3$$

iii)
```
k = 2;
myode = @(t,x)  [- k * x(1) *  x(2);
                 - k * x(1) *  x(2);
                   k * x(1) * x(2)]
```

OR

```
function f = myode (t, x)
 k = 2;
 f(1) = - k * x(1) * x(2);
 f(2) = - k * x(1) * x(2);
 f(3) = k * x(1) * x(2)
```

iv)

$$\underset{\sim}{x}(0.1) = \underset{\sim}{x}(0) + h * f(0)$$

$$= \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + 0.1 \begin{bmatrix} -2 \times 3 \times 4 \\ -2 \times 3 \times 4 \\ 2 \times 3 \times 4 \end{bmatrix} = \begin{bmatrix} 3 - 2.4 \\ 4 - 2.4 \\ 0 + 2.4 \end{bmatrix} = \begin{bmatrix} 0. \\ 1. \\ 2. \end{bmatrix}$$

Question 3 (S2-2014) MATH 2089

a) Initial condition $u(x,0)$
Boundary conditions $u(0,t)$ and $u(L,t)$

b) $O(h)$ order 1

c) Implicit method, since the unknowns $\underset{\sim}{u}^{l+1}$ are in both left and right hand side.

d) Implicit method is more stable, doesn't require condition on $\Delta x / \Delta t$ or $\Delta t / (\Delta x)^2$

e) With central difference approximation for $\frac{\partial^2 u}{\partial x^2}$, (3.2) is reduced to

$$(*) \quad \frac{u_j^{l+1} - u_j^{l}}{\Delta t} = \frac{u_{j+1}^{l+1} - 2u_j^{l+1} + u_{j-1}^{l+1}}{(\Delta x)^2}, \quad j = 1, \dots, n.$$

when $j = 0$ $\quad u_0^l = 0$ since $u(x_0, t) = u(0,t) = 0$

$j = n+1$ $\quad u_{n+1}^l = 0$ since $u(x_{n+1}, t) = u(L,t) = 0$

i) For $j = 1$, $(*)$ reduces to

$$\frac{u_1^{l+1} - u_1^{l}}{\Delta t} = \frac{u_2^{l+1} - 2u_1^{l+1} + \cancelto{0}{u_0^{l+1}}}{(\Delta x)^2} = \frac{-2u_1^{l+1} + u_2^{l+1}}{(\Delta x)^2}$$

So the first row of A is

$$[2 \ \ -1 \ \ 0 \ \ 0 \ \dots \ 0]$$

ii) For $j = 2$, (*) reduces to

$$\frac{u_2^{l+1} - u_2^{l}}{\Delta t} = \frac{u_3^{l+1} - 2u_2^{l+1} + u_1^{l+1}}{(\Delta x)^2} = \frac{u_1^{l+1} - 2u_2^{l+1} + u_3^{l+1}}{(\Delta x)^2}$$

So the 2nd row of the matrix A is

$$[-1 \quad 2 \quad 1 \quad 0 \quad 0 \; \ldots \; 0].$$

iii) For $j = n$, (*) reduces to

$$\frac{u_n^{l+1} - u_n^{l}}{\Delta t} = \frac{\cancelto{0}{u_{n+1}^{l+1}} - 2u_n^{l+1} + u_{n-1}^{l+1}}{(\Delta x)^2} = \frac{+\,u_{n-1}^{l+1} - 2u_n^{l+1}}{(\Delta x)^2}$$

So the last row of the matrix A is

$$[0 \quad 0 \; \ldots \; -1 \quad 2]$$

iv) $v.\wedge 0 = [1 \; 1 \; 1 \; \ldots \; 1]$

$W = A * (v.\wedge 0)$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & & \\ -1 & 2 & 1 & 0 & \cdots & & \\ 0 & -1 & 2 & -1 & & & \\ & & & & & & \\ & & & & & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

So $w(j) = w(2) = 0$ since $j = 2$.

ans $0 = 0$

$v.\wedge 1 = [1 \; 2 \; 3 \; 4 \; \ldots \;]$

$$W = A * (v.\wedge 1)$$

$$= \begin{bmatrix} +2 & -1 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & \\ 0 & -1 & 2 & -1 & \\ & & \cdots & & \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ 20 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \end{bmatrix}$$

$$\Rightarrow \quad w(j) = w(2) = 0$$

$$\text{ans1} = 0$$

$$W = A * (V.\wedge 2)$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & \cdots \\ & 0 & -1 & 2 & -1 \\ & & & & \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 9 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ \\ \\ \end{bmatrix}$$

$-1\times 1 + 2\times 4 - 1\times 9$

$$\Rightarrow \quad w(j) = w(2) = -2$$

$$\text{ans2} = -2.$$

v) $u(x,0) = 1 - \cos\left(x\frac{2\pi}{L}\right)$

(3.2) at $l = 0$

$$\frac{\underset{\sim}{u}^1 - \underset{\sim}{u}^0}{\Delta t} = -A\,\underset{\sim}{u}^1 / (\Delta x)^2$$

where

$$\underset{\sim}{u}^0 = \left[1 - \cos\left(x_j \frac{2\pi}{L}\right)\right]_{j=1}^{n}$$

$$= \left[1 - \cos\left(j\frac{L}{n+1}\frac{2\pi}{L}\right)\right]_{j=1}^{n} = \left[1 - \cos\left(\frac{2\pi j}{n+1}\right)\right]$$

f) For $n = 20$

$$A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \ddots & \ddots & \\ & & & & -1 & 2 \end{bmatrix} \Bigg\} \text{ 20 rows.}$$

Number of non-zero elements of A

$$nnz(A) = \underset{\substack{\uparrow \\ \text{first row}}}{2} + \underbrace{3 \times 18}_{\substack{\text{middle} \\ \text{rows}}} + \underset{\substack{\uparrow \\ \text{last row}}}{2} = 58$$

Number of elements of $A = 20^2 = 400$

Sparsity of $A = \dfrac{58}{400} = 14.5\%$

g) No, A^{-1} is not very sparse.

h) $A = B^T B$

$$\underset{\sim}{x}^T A \underset{\sim}{x} = \underset{\sim}{x}^T B^T B \underset{\sim}{x} = (B\underset{\sim}{x})^T B \underset{\sim}{x} = \|Bx\|^2 \geq 0$$

If $B\underset{\sim}{x} = 0$ then $\underset{\sim}{x} = 0$ since B has independent columns.

So A is positive definite.

i) $A = R^T R$

$$\Rightarrow A\underset{\sim}{x} = \underset{\sim}{b} \Leftrightarrow R^T R \underset{\sim}{x} = \underset{\sim}{b}.$$

Step 1: solve $R^T \underset{\sim}{y} = \underset{\sim}{b}$. by forward substituti

Step 2: solve $R \underset{\sim}{x} = \underset{\sim}{y}$ by back-substituti j).

$$(\underset{\sim}{u}^{l+1} - \underset{\sim}{u}^{l})/\Delta t = -A\underset{\sim}{u}^{l+1}/(\Delta x)^2$$

$$\Leftrightarrow \quad \underset{\sim}{u}^{l+1}/\Delta t + A\,\underset{\sim}{u}^{l+1}/(\Delta x)^2 = \underset{\sim}{u}^{l}/\Delta t.$$

$$\Leftrightarrow \Delta t\left(\frac{1}{\Delta t} I + \frac{1}{(\Delta x)^2} A\right)\underset{\sim}{u}^{l+1} = \underset{\sim}{u}^{l}.$$

$$\Leftrightarrow \quad \underbrace{\left(I + \frac{\Delta t}{(\Delta x)^2} A\right)}_{K}\underset{\sim}{u}^{l+1} = \underset{\sim}{u}^{l}$$

$$K \quad \underset{\sim}{u}^{l+1} = \underset{\sim}{u}^{l}.$$

With $\Delta t = 0.1, \quad \Delta x = 0.5$

$$K = I + \frac{1}{2.5} A$$