Topic and contents

UNSW, School of Mathematics and Statistics

MATH2089 – Numerical Methods

$\boxed{\text{Week 02 – Nonlinear equations - I}}$

- MATLAB M-files
  - nlog2n.m nlog2n_bisection.m
    nlog2n_fixedpoint.m pltsin.m

Nonlinear problems

- Many engineering problems are nonlinear with no analytic solution. For example,
  - Describe the relation between the height and velocity of water discharge from a reservoir.
  - Determine a location from GPS satellites' signals.
  - Determine the time when response of an electrical circuit is zero.
  - etc.
  - Analytic solution: Solve $ax^2 + bx + c = 0 \Longrightarrow x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
  - No analytic solution: Solve $x2^x = 10 \Longrightarrow x = ?$
- Can you establish existence of a solution x^*?
- Is the solution unique?
- What information do you have? data, function values, derivatives, ...

Iterative methods

- Iterates $x_k, k = 1, 2, 3, \ldots$ scalars, (or vectors or functions)
- Initial guess (starting point) x_1
- Based on simple approximation of nonlinear problem
- Converge to a (the) solution x^*

Desirable properties:

- Works reliably
- Easy to use
- Fast
- High accuracy, if desired
- Insensitive to choice of initial guess

Nonlinear Equations of one variable

- Single nonlinear equation, standard form

$$f(x) = 0, \qquad x \in \mathbb{R}$$

  - Assumed by most software packages, eg MATLAB `fzero`
- Rearrange if necessary to get in standard form

Example (Intersection of two functions)

Find the intersection of the functions $f_1(x)$ and $f_2(x)$.

Solution

$$f_1(x) = f_2(x) \iff f(x) = f_1(x) - f_2(x) = 0$$

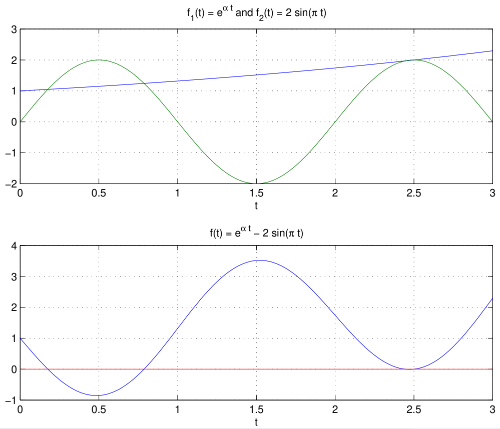
Example (Achieving a value)

Find the value of n such that $n\log_2(n) = 100$. MATLAB nlog2n.m

Intersection of two functions

Example (Intersection)

Find the point(s) of intersection of $f_1(x) = e^{\alpha x}$, where $\alpha = \log(4)/5$ and $f_2(x) = 2\sin(\pi x)$ for $x > 0$. MATLAB `nle1.m`



Inverse functions I

- Inverse functions

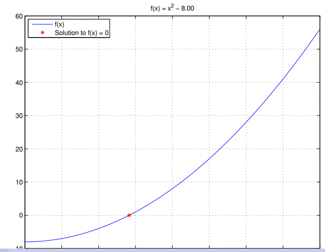
$$x = h^{-1}(y) \iff h(x) = y \iff f(x) = h(x) - y = 0$$

Example (n th root)

Let $n > 1$ be an integer and $a > 1$. Transform the problem of finding the nth root of a, into the solution of a polynomial equation $f(x) = 0$.

Solution (MATLAB `nthroot.m`)

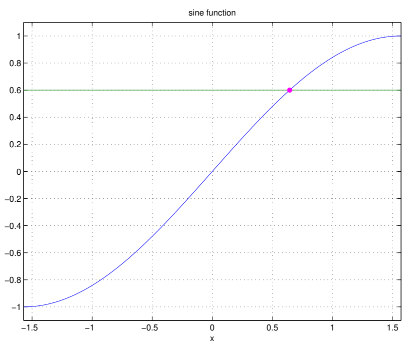
- $x = a^{\frac{1}{n}} \iff x^n = a \iff x^n - a = 0$, so $f(x) = x^n - a$

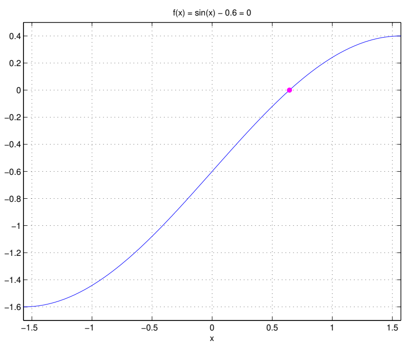


Inverse functions II

Example (arcsine or $\sin^{-1}(y)$)

- Given $y \in [-1, 1]$, find $x = \sin^{-1}(y)$
- Solve equation $f(x) = \sin(x) - y = 0$, $\quad x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
  MATLAB `pltsin.m`



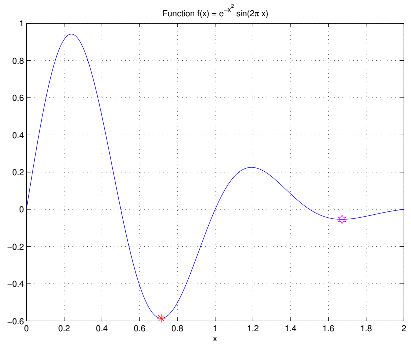


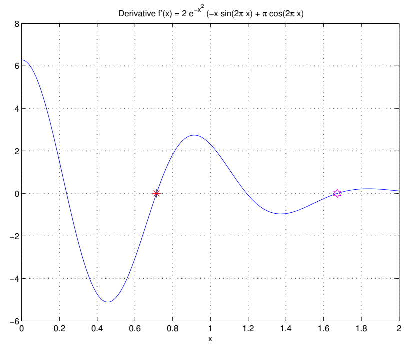
Optimization

- Maximum/minimum of $F(x) \implies$ stationary points $f(x) = F'(x) = 0$

Example (Minimum)

Find the minimum of $f(x) = e^{-x^2}\sin(2\pi x)$ on $[0, 2]$. MATLAB `nle2.m`





Simple vs multiple roots

Definition (Simple and multiple roots)

Let $f^{(k)}(x)$ denote the kth derivative of f with respect to x.

- $f(x^*) = 0$ and $f'(x^*) \neq 0 \iff x^*$ is a simple root of f
- $f(x^*) = 0, f'(x^*) = 0, \ldots, f^{(k-1)}(x^*) = 0$ and $f^{(k)}(x^*) \neq 0 \iff$ x^* is a root of multiplicity k.
- A root with multiplicity greater than 1 is called a multiple root.

Example (Simple root)

For $a > 1$ and integer $n > 1$ show that $f(x) = x^n - a$ has a simple root.

Solution

Existence and uniqueness

Proposition (Existence, Uniqueness)

Let $f : \mathbb{R} \to \mathbb{R}$.

- *$f \in C([a, b])$ and $f(a)f(b) < 0 \implies$ there exists at least one zero of f on (a, b) (the interval $[a, b]$ brackets a root)*
- *f is strictly monotone (either strictly increasing or strictly decreasing) on the interval $[a, b] \implies$ there exists at most one zero of f on $[a, b]$.*

- Strictly monotone: f differentiable and
  - $f'(x) > 0$ for all $x \in (a, b) \implies f$ strictly increasing on $[a, b]$
  - $f'(x) < 0$ for all $x \in (a, b) \implies f$ strictly decreasing on $[a, b]$
- Combine: If both
  - f is continuous on $[a, b]$, $f(a)(f(b) < 0$ and
  - f is strictly increasing or strictly decreasing

  then there exists a unique root of f on (a, b)

Existence of unique root

Example

Show the following problems have a unique zero on the given interval, or explain why the theory does not apply.

1. $f(x) = x^5 - 10$ on $[0, 2]$.
2. $f(x) = 1/(x - 1)$ on $[0, 2]$.
3. $f(x) = x + 3 - 4/(1 + x^2)$ on $[-3, 1]$.
4. $f(t) = e^{\alpha t} - 2\sin(\pi t)$, $\alpha = \log(4)/5$ on $[\frac{1}{2}, 1]$.
5. $f(x) = \sin(x) - y$ for fixed $y \in [-1, 1]$

Unique zero – partial solutions

Solution

Bisection

- Suppose $[a, b]$ brackets a root (f continuous, $f(a)f(b) < 0$)
- Midpoint $x_{\text{mid}} = \dfrac{a + b}{2}$
- New bracket
  - If $f(a)f(x_{\text{mid}}) < 0 \implies [a, x_{\text{mid}}]$ brackets root
  - If $f(x_{\text{mid}})f(b) < 0 \implies [x_{\text{mid}}, b]$ brackets root
- If $f \in C([a, b])$ and $[a, b]$ brackets a root, then bisection converges to a root $x^* \in (a, b)$.
- $[a, b]$ brackets root, then estimate of root is x_{mid}
  - Maximum error is $\frac{b-a}{2}$
  - Each bisection step reduces bracket length by 2
  - MATLAB `nlog2n_bisection.m`

Fixed point iteration

Definition (Fixed point)

A fixed point x^* of a function $g(x)$ satisfies $x^* = g(x^*)$

Definition (Fixed point iteration)

Fixed point iteration for a function $g(x)$, given a starting point x_1, is

$$x_{k+1} = g(x_k)$$

MATLAB `nlog2n_fixedpoint.m`

Example

Formulate a fixed point iteration method for solving $x \log_2(x) = c$ for some constant $c > 0$.