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UNSW, School of Mathematics and Statistics

MATH2089 – Numerical Methods

Week 07 – Ordinary Differential Equations I

1. Ordinary Differential Equations
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   - Higher order ODEs
   - First order systems
   - Existence and uniqueness
   - ODEs and Integration
   - Euler's method

- MATLAB M-files
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  - `euler.m`

Differential Equations

- Function defined by equations involving derivatives
- Ordinary Differential Equation (ODE)
  - Function of one variable $y(t)$
  - Variable often time t, but can represent anything
  - Concentration $C(t)$ of a chemical changing over time t
- Partial Differential Equation (PDE)
  - Function of more than one variable, eg $u(x, y)$, $u(x, t)$, $u(x, y, z)$
  - Variables: eg space (x, y, z) and time t
  - Temperature $u(x, y, t)$ at position (x, y) in a plate at time t
- Order is the highest derivative present
- Domain for variables: for example
  - $t \in [0, T]$, $\quad x \in [a, b]$, $\quad (x, y) \in \mathcal{D}$
- Initial conditions
  - Values at starting values for time variable, eg $y(0) = y_0$
- Boundary values
  - Values at boundary of space or time domain, eg $\mathcal{D} = [0, 1]$, $y(0) = y_0$, $y(1) = y_1$

Initial Value Problem – Example 1

Example (Newton's law of cooling)

Newton's law of cooling: time rate at which a body cools is proportional to the difference between the temperature of the body and the temperature of the surrounding medium.

- Let $U(t)$ denote temperature of body at time t
- Time domain $t \in [0, T]$
- ODE
$$\frac{dU}{dt} = k(U - u_M)$$
  - First order
  - Temperature of surrounding medium u_M
  - Rate constant $k < 0$ for cooling
- Initial condition $U(0) = u_0$, eg $u_0 = 37^o$ C for human body

Initial Value Problems – Example 2

Example (Vertical motion under gravity)

A free falling object close to the surface of the Earth accelerates at a constant rate g.

- Let $s(t)$ be vertical distance body travels, with upwards positive
- Time domain $[0, t_1]$, t_1 = time body hits ground
- ODE
$$\frac{d^2s}{dt^2} = -g$$
  - Second order
  - Positive direction up $\Longrightarrow$ gravity pulls down
- Initial conditions
  - Start at height $s(0) = s_0 \geq 0$
  - Initial velocity $s'(0) = v_0$

First order Initial Value Problem (IVP)

Definition (First order IVP)

Initial value problem for a first order ODE is

$$\frac{dy}{dt} = f(t,y) \quad \text{for } t > 0, \ \text{with } y(0) = y_0$$

- Unknown function $y(t)$ to be found
- Known function $f(t,y)$
- Known initial value y_0
- Actually

$$\frac{dy(t)}{dt} = f(t, y(t))$$

  - explicit dependence of y on t suppressed

Higher order ODEs

Definition

An nth order nonlinear ordinary differential equation is

$$\frac{d^n y}{dt^n} = g\left(t, y, \frac{dy}{dt}, \frac{d^2 y}{dt^2}, \ldots, \frac{d^{n-1}y}{dt^{n-1}}\right)$$

- Convert to a system of n first-order equations
- State vector

$$\mathbf{x} = \begin{bmatrix} y & \frac{dy}{dt} & \cdots & \frac{d^{n-1}y}{dt^{n-1}} \end{bmatrix}^T$$

- Differentiate kth component

$$\frac{dx_k}{dt} = \frac{d^k y}{dt^k} = x_{k+1}, \quad k = 1, \ldots, n-1$$

- Derivative of last component of state vector

$$\frac{dx_n}{dt} = \frac{d^n y}{dt^n} = g\left(t, y, \frac{dy}{dt}, \ldots, \frac{d^{n-1}y}{dt^{n-1}}\right) = g\left(t, x_1, x_2, \ldots, x_n\right)$$

First order system of ODEs

Definition (First order system)

A system of n first-order ODEs is

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$$

- Written as $\mathbf{x}' = f(t, \mathbf{x})$
- Need vector of n initial conditions $\mathbf{x}(0) = \mathbf{x}_0$
- nth order ODE:

$$\mathbf{x} = \begin{bmatrix} y \\ \frac{dy}{dt} \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}} \end{bmatrix}, \qquad \mathbf{f}(t, \mathbf{x}) = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ g\left(t, x_1, x_2, \ldots, x_n\right) \end{bmatrix}$$

IVP Example 2 cont

Example

Convert the vertical motion under gravity example into a first order system

Solution

- *Second order equation* $\Longrightarrow$ *state vector* $\mathbf{x}(t) = [s(t),\ s'(t)]^T \in \mathbb{R}^2$
- *ODE*

$$\mathbf{x}'(t) = \frac{d}{dt}\begin{bmatrix} s(t) \\ s'(t) \end{bmatrix} = \begin{bmatrix} s'(t) \\ s''(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -g \end{bmatrix} = \mathbf{f}(t, \mathbf{x})$$

- *Linear as*

$$\mathbf{f}(t, \mathbf{x}) = \begin{bmatrix} x_2(t) \\ -g \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -g \end{bmatrix} = A\mathbf{x} + \mathbf{b}$$

- *Initial values* $\mathbf{x}(0) = [s(0),\ s'(0)]^T = [s_0,\ v_0]^T$

Existence of solution to IVP

Proposition (Existence of solution)

IVP $y' = f(t, y)$ with initial condition $y(t_0) = y_0$.
f and $\partial f/\partial y$ continuous in the rectangle $|t - t_0| < \alpha, \quad |y - y_0| < \beta \Longrightarrow$ the IVP has a unique continuous solution in some interval $(t_0, t_0 + \epsilon)$

- If $|f(t, y)| \le M$ in the rectangle, then ϵ at least β/M
- Existence result, nothing about finding $y(t)$

Example

Consider the IVP $y' = 1 + y^2$ for $t > 0$, with $y(0) = 1$

1. Show the existence conditions are satisfied
2. Show the exact solution is $y(t) = \tan\left(t + \frac{\pi}{4}\right)$
3. Show that the exact solution is continuous for $0 \le t < \frac{\pi}{4}$, but blows up as $t \to \frac{\pi}{4}$

Existence – Example

Solution

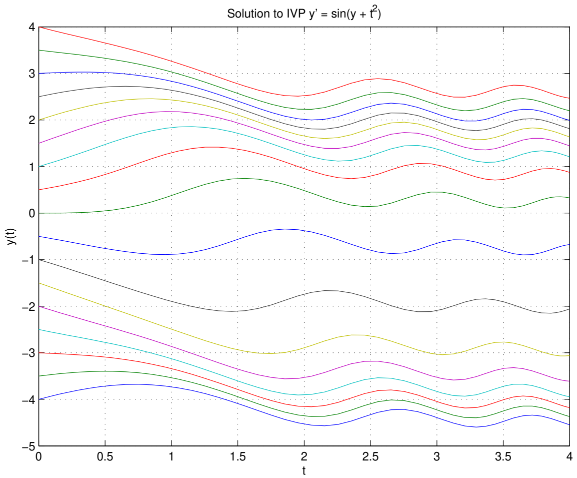
- $f(t, y) = 1 + y^2, \quad t_0 = 0, \quad y_0 = 1$
- $\frac{\partial f}{\partial y} = 2y$
- *f and $\partial f/\partial y$ are continuous and bounded on any rectangle* $|t - t_0| < \alpha, \quad |y - y_0| < \beta$
- *Check given $y(t)$ satisfies ODE*

$$y(t) = \tan\left(t + \frac{\pi}{4}\right) \Longrightarrow y'(t) = 1 + \tan^2\left(t + \frac{\pi}{4}\right) = 1 + y^2$$

- *Check initial condition* $y(0) = \tan\left(\frac{\pi}{4}\right) = 1$
- *As* $t \to \frac{\pi}{4}$, $y \to \tan(\frac{\pi}{2}) = \infty$
- *Solution only exists for* $0 \le t < \frac{\pi}{4}$

Simple IVP using MATLAB

- $f(t, y) = \sin(y + t^2), \quad t_0 = 0, \quad y_0 \in [-4, 4]$
- MATLAB script `ivpex1.m` using `ode45`



ODEs and Integration

- Fundamental theorem of calculus

$$\frac{d}{dx}\int_a^x y(t)dt = y(x)$$

- Integral

$$\int_a^b f(t, y(t))dt$$

- Equivalent IVP: Find $y(b)$ where

$$\frac{dy}{dt} = f(t, y), \qquad y(a) = 0$$

Time discretization

- Time interval $[t_0, t_{\text{max}}]$
- Discretization: grid points

$$t_0 < t_1 < t_2 < \cdots < t_N = t_{\text{max}}$$

- Equally spaced points

$$t_n = t_0 + nh, \quad n = 0, 1, \ldots, N, \qquad h = \frac{t_{\text{max}} - t_0}{N}$$

- Approximate solution $y_h(t)$
- Approximate values at grid points

$$y_n = y_h(t_n)$$

- In between grid point must interpolate
  eg $t \in (t_k, t_{k+1})$, linearly interpolate (t_k, y_k), (t_{k+1}, y_{k+1})
- Errors $E_n = y(t_n) - y_n$

Explicit Euler Method

- Euler's method (1768): Linear approximation at left hand end
- At (t_n, y_n) make approximation

$$f(t, y(t)) \approx f(t_n, y_n), \quad t \in [t_n, t_{n+1}]$$

- ODE gives

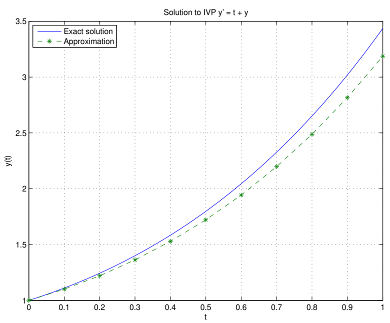
$$\begin{aligned} y_{n+1} \approx y(t_{n+1}) &= y(t_n) + \int_{t_n}^{t_{n+1}} y'(t)dt \\ &\approx y_n + \int_{t_n}^{t_{n+1}} f(t, y(t))dt \\ &\approx y_n + \int_{t_n}^{t_{n+1}} f(t_n, y_n)dt \\ &\approx y_n + hf(t_n, y_n) \end{aligned}$$

- Simple and easy to implement, but not computationally efficient

Euler's method – Example 1

- MATLAB `euler.m`
- $f(t, y) = t + y, \quad y(0) = 1$
- Exact solution $y(t) = 2e^t - 1 - t$
- Euler's method, $h = 0.1$

n	t_n	$y(t_n)$	y_n	E_n
0	0.00	1.0000	1.0000	0
1	0.10	1.1103	1.1000	1.03×10^{-2}
2	0.20	1.2428	1.2200	2.28×10^{-2}
3	0.30	1.3997	1.3620	3.77×10^{-2}
4	0.40	1.5836	1.5282	5.54×10^{-2}
5	0.50	1.7974	1.7210	7.64×10^{-2}



Euler's method – Example 2

- MATLAB `euler.m`
- $f(t, y) = \frac{1}{1+t^2} - 2y^2, \quad y(0) = 0$
- Exact solution $y(t) = \frac{1}{1+t^2}$
- Euler's method, $h = 0.1$

n	t_n	$y(t_n)$	y_n	E_n
0	0.00	1.0000	1.0000	0
1	0.10	0.0990	0.1000	-9.90×10^{-4}
2	0.20	0.1923	0.1970	-4.70×10^{-3}
3	0.30	0.2752	0.2854	-1.02×10^{-2}
4	0.40	0.3448	0.3609	-1.60×10^{-2}
5	0.50	0.4000	0.4210	-2.10×10^{-2}