FAMILY NAME: ....
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UNSW SYDNEY

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 2, 2018

MATH2089
Numerical Methods and Statistics

(1) TIME ALLOWED – 2 Hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) THIS PAPER MAY **NOT** BE RETAINED BY THE CANDIDATE

(6) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(7) STATISTICAL FORMULAE ARE ATTACHED AT END OF PAPER

Part A – Numerical Methods consists of questions 1 – 2

Part B – Statistics consists of questions 3 – 4

Both parts must be answered

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Part A – Numerical Methods

1. Answer in a separate book marked Question 1

a) **[20 marks]** The computational complexity of some common operations with n by n matrices are given in Table 1.1

Operation	Flops
Matrix multiplication	$2n^3$
LU factorization	$\frac{2n^3}{3} + O(n^2)$
Cholesky factorization	$\frac{n^3}{3} + O(n^2)$
Back/forward substitution	$n^2 + O(n)$
Tridiagonal solve	$8n + O(1)$

Table 1.1: Flops for some operations with n by n matrices

In each of the remaining parts of this question, a claim is made. For each claim, state whether the claim is true or false (1 mark), and give a short reason for your answer (2 or 3 marks).

i) **Claim:** On a 3GHz single core PC which can do one flop per clock cycle, the largest $n \times n$ linear systems it can solve in 1 hour is of size $n = 25,303$.

ii) The central difference approximation to the second derivative of a smooth function f is

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2).$$

Claim: The optimal value of h that will minimize the sum of rounding error and the truncation error is $O(\varepsilon^{1/4})$, where ε is the smallest machine number so that $1 + \varepsilon > 1$.

iii) You are given the following short Matlab program.

```
x = 1;
a = 27;
for k=1:20
   x = (2*x + a/x^2)/3;
end
```

Claim: The above program will find the solution to the equation $p(x) = 0$ where $p(x) = x^3 - a$ using Newton's method.

iv) You are given that

```
norm(A-A') = 9.3e-16
min(eig(A)) = 1.3e-7
max(eig(A)) = 2.6e+6
```

- A is computed to full double precision accuracy,
- $\mathbf{b}$ is computed up to 6 significant figures.

Claim: The computed solution to $A\mathbf{x} = \mathbf{b}$ has at least 10 significant figures.

v) You are given the results of the following Matlab commands:

```
>> R = chol(A)

R  =
      1.4142  0.7071
           0  1.2247

>> A - R'*R

ans =
  1.0e-15 *
  -0.4441           0
        0      0.2220
```

Claim: The matrix A is symmetric and positive definite.

vi) You are given that

```
>> size(A)

ans =
    5    5

>> [Q,R] = qr(A);
```

Claim: For every vector $\mathbf{x} \in \mathbb{R}^5$, we have $\|Q\mathbf{x}\|_2 = \|\mathbf{x}\|_2$.

b) **[10 marks]** Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$. The exact integral is given by

$$I(f) = \int_0^1 f(x)\,\mathrm{d}x = \frac{2}{3}.$$

Approximations to $I(f)$ were calculated using the trapezoidal rule and Simpson's rule giving the following table of errors $E_N(f) = I(f) - Q_N(f)$:

i) **[3 marks]** The error for the Simpson's rule satisfies

$$E_N^{\mathrm{Simp}}(f) = O(N^{-4}), \tag{1.1}$$

provided that $f \in C^4([0,1])$. You do **not** need to prove this.

A) Use (1.1) to estimate the ratio

$$\frac{E_N^{\mathrm{Simp}}(f)}{E_{2N}^{\mathrm{Simp}}(f)}. \tag{1.2}$$

	Trapezoidal rule		Simpson's rule	
N	$Q_N(f)$	$E_N(f)$	$Q_N(f)$	$E_N(f)$
2	0.6035533906	6.31e-02	0.6380711875	2.86e-02
4	0.6432830462	2.34e-02	0.6565262648	1.01e-02
8	0.6581302216	8.54e-03	0.6630792801	3.59e-03
16	0.6635811969	3.09e-03	0.6653981886	1.27e-03
32	0.6655589363	1.11e-03	0.6662181827	4.48e-04
64	0.6662708114	3.96e-04	0.6665081031	1.59e-04

B) Use the table of errors to estimate the ratio (1.2) when $N = 32$.

C) Is the table of errors consistent with the theoretical error estimate in (1.1)?

D) Give reasons for your answer in C).

ii) **[2 marks]** What is the degree of precision of the trapezoidal rule? Give reasons for your answer.

iii) **[1 mark]** Find a linear transformation $x = \alpha + \beta z$ that maps $z \in [-1, 1]$ to $x \in [0, 1]$.

iv) **[2 marks]** **Given** the nodes $z_j, j = 1, \ldots, N$ and weights $w_j, j = 1, \ldots, N$ for the Gauss-Legendre rule for the interval $[-1, 1]$, how can you approximate $I(f)$?

2. Answer in a separate book marked Question 2

a) **[14 marks]** The motion of a damped mass-spring system is modelled by the initial value problem

$$my'' + cy' + ky = 0, \qquad y(1) = 2, \quad y'(1) = 1,$$

where $y(t)$ is the displacement of the block at time t, m is the mass of the block, c is the damping coefficient, and k is the spring constant. Consider here the case

$$m = 4, \quad c = 3, \quad \text{and} \quad k = 5.$$

i) What is the order of the differential equation?

ii) Convert this ordinary differential equation into a system

$$\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}), \quad \text{for } t > t_0,$$

of first order differential equations.

iii) What is the initial condition $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$?

iv) Write

- **EITHER** a MATLAB anonymous function `myode`
- **OR** a MATLAB function M-file `myode.m`

to evaluate the vector valued function $\boldsymbol{f}(t, \boldsymbol{x})$. Explain how to set values for the parameters m, c, and k.

v) Heun's method for solving the initial value problem $u' = g(t, u)$ with $u(t_0) = u_0$ can be summarized by the following formula: for $n = 0, 1, \ldots, N-1$, compute:

$$\begin{aligned} z_{n+1} &= u_n + hg(t_n, u_n) \\ u_{n+1} &= u_n + \frac{h}{2}[g(t_n, u_n) + g(t_{n+1}, z_{n+1})]. \end{aligned}$$

Use Heun's method with a step size of $h = 0.1$ to estimate $\boldsymbol{x}(1.1)$ for the initial value problem in question ii).

b) **[16 marks]** Fick's second law predicts how diffusion causes the concentration $u(x, y)$ of a chemical to change with position $(x, y) \in \Omega$. The steady state version of Fick's second law (without interior sources of the chemical) is Laplace's equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0. \tag{2.1}$$

Consider the rectangular domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \le x \le 2, \ 0 \le y \le 1\},$$

and discretize it using $h = 1/n$ and

$$\begin{cases} x_i = ih & \text{for } i = 0, 1, \dots, 2n, \\ y_j = jh & \text{for } j = 0, 1, \dots, n. \end{cases}$$

This is illustrated in Figure 2.1 for $n = 5$.

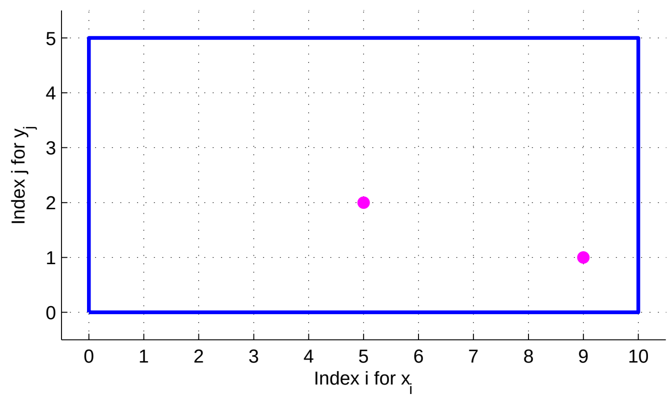

Figure 2.1: Discretization of the domain for $n = 5$ and grid points for part iv)

i) What extra information is needed to completely specify this problem?

ii) You are **given** the following standard finite difference approximations for a function f of **one** variable:

$$\begin{aligned} f'(x) &= \frac{f(x+h) - f(x)}{h} + O(h), \\ f'(x) &= \frac{f(x+h) - f(x-h)}{2h} + O(h^2), \\ f''(x) &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2). \end{aligned}$$

Let $u_{i,j}$ denote the approximation to the value $u(x_i, y_j)$ of concentration at the grid point (x_i, y_j). Give central difference approximations of accuracy $O(h^2)$ to the following derivatives at the point (x_i, y_j)

$A)$ $\dfrac{\partial^2 u(x,y)}{\partial x^2}$ $B)$ $\dfrac{\partial^2 u(x,y)}{\partial y^2}$

iii) Using the finite difference approximations from the previous part, show that the equation (2.1) can be approximated by

$$\beta u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = 0, \tag{2.2}$$

and determine the value of β.

iv) Given that, in appropriate units,

$$\begin{cases} u(x,0) = u(x,1) = 2x & \text{for } 0 \le x \le 2, \\ u(0,y) = 0 & \text{for } 0 \le y \le 1, \\ u(2,y) = 2 & \text{for } 0 \le y \le 1, \end{cases}$$

write down the equation (2.2) for a discretization with $n = 5$ at the grid points (marked in Figure 2.1)

A) (x_5, y_2) *B*) (x_9, y_1)

v) You are given that the coefficient matrix A is symmetric positive definite. Outline an effective way to solve the linear system $A\boldsymbol{u} = \boldsymbol{b}$.

Part B – Statistics

3. Answer in a separate book marked Question 3

4. **Answer in a separate book marked Question 4**