UNIVERSITY OF NEW SOUTH WALES
School of Mathematics and Statistics

MATH2089 Numerical Methods and Statistics
Term 2, 2019

Numerical Methods Tutorial – Week 10

1. Use Euler's method, using the given step size h, to solve the following IVPs. Compute the actual errors at each time step.

   (a) $y' = te^{3t} - 2y, \quad t \in (0, 1], \quad y(0) = 0, \quad h = 0.5$
   - Solution: $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$

   (b) $y' = 1 + \dfrac{y}{t}, \quad t \in (1, 2], \quad y(1) = 2, \quad h = 0.25$
   - Solution: $y(t) = t\log(t) + 2t$

   (c) $y' = \dfrac{2}{t}y + t^2e^t, \quad t \in (1, 2], \quad y(1) = 0, \quad h = 0.1$
   - Solution: $y(t) = t^2(e^t - e)$

2. Consider the Initial Value Problem (IVP)

$$y' = f(t, y), \quad t > t_0, \qquad y(t_0) = y_0.$$

At step n we know t_n, $y_n \approx y(t_n)$ and $t_{n+1} = t_n + h$. A ν-stage explicit Runge-Kutta method with parameters a_{ij}, b_j, c_j is

$$\begin{aligned}
\xi_1 &= y_n \\
\xi_2 &= y_n + ha_{2,1}f(t_n + c_1h, \xi_1) \\
\xi_3 &= y_n + ha_{3,1}f(t_n + c_1h, \xi_1) + ha_{3,2}f(t_n + c_2h, \xi_2) \\
&\vdots \\
\xi_\nu &= y_n + h\sum_{i=1}^{\nu-1} a_{\nu,i}f(t_n + c_ih, \xi_i)
\end{aligned}$$

Then the next approximation $y_{n+1} \approx y(t_{n+1})$ is

$$y_{n+1} = y_n + h\sum_{j=1}^{\nu} b_jf(t_n + c_jh, \xi_j).$$

The classical fourth order four-stage Runge-Kutta method RK4 is defined by the following tableau

$$\begin{array}{c|c} \mathbf{c} & A \\ \hline & \mathbf{b}^T \end{array} \quad = \quad \begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \end{array}$$

   (a) Write down the formulae to define y_{n+1} from y_n for this method.

(b) Use the formulae obtained above (with $h = 0.2$) to compute an approximation to $y(0.2)$, where y satisfies the IVP

$$y' = \frac{y+t}{y-t}, \qquad y(0) = 1.$$

3. Consider Poisson's equation

$$\nabla^2 u(\mathbf{x}) = f(\mathbf{x})$$

on the rectangular domain

$$\Omega = \left\{ \mathbf{x} \in \mathbb{R}^2 : 0 \leq x \leq L_x, \quad 0 \leq y \leq L_y \right\}.$$

Divide the x interval $[0, L_x]$ into $m+1$ equal length subintervals

$$0 = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m < x_{m+1} = L_x,$$

$$x_i = i\, h_x, \quad i = 0, \ldots, m+1, \qquad h_x = \frac{L_x}{m+1}.$$

Divide the y interval $[0, L_y]$ into $n+1$ equal length subintervals

$$0 = y_0 < y_1 < y_2 < \cdots < y_{n-1} < y_n < y_{n+1} = L_y,$$

$$y_j = j\, h_y, \quad j = 0, \ldots, n+1, \qquad h_y = \frac{L_y}{n+1}.$$

Let $u_{ij} \approx u(x_i, y_j)$

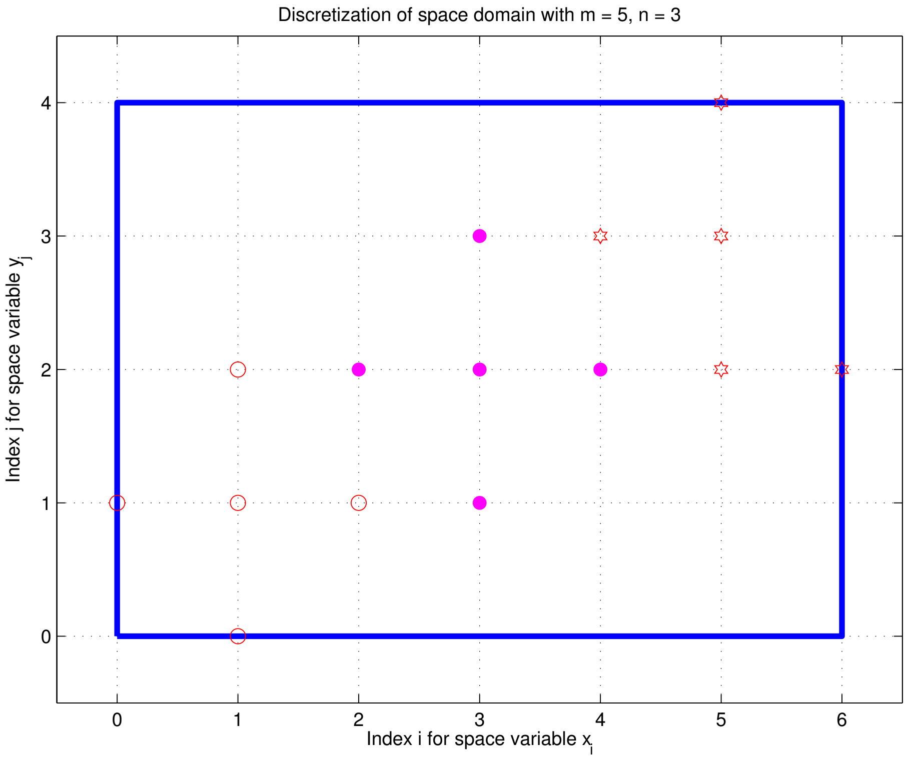

Figure 1: Grid for $m = 5$ and $n = 3$

(a) Is this an elliptic, parabolic or hyperbolic PDE?

(b) What problem could this model?

(c) What else is needed to completely specify the problem?

(d) At the grid point x_i, y_j use central difference approximations of $O(h^2)$ to the second derivatives to derive an approximation to Poisson's equation.

(e) Consider a problem with Dirichlet boundary conditions $u(\mathbf{x}) = 20$ for $\mathbf{x} \in \partial\Omega$. For $L_x = 3, m = 5$ and $L_y = 2$, $n = 3$

i. Give the equation at the grid point (x_3, y_2)

ii. Give the equation at the grid point (x_5, y_3)

iii. generate the linear system $A\mathbf{u} = \mathbf{b}$ corresponding to a row-ordering of the variables.

iv. Write the coefficient matrix A as the block matrix

$$A = \begin{bmatrix} B & -I & & & & & \\ -I & B & -I & & & & \\ & -I & B & -I & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -I & B & -I & \\ & & & & -I & B & -I \\ & & & & & -I & B \end{bmatrix}$$

What are the matrices B and I.

v. Say what you can about the structure of the coefficient matrix A that could make solving the linear system $A\mathbf{u} = \mathbf{b}$ more efficient.

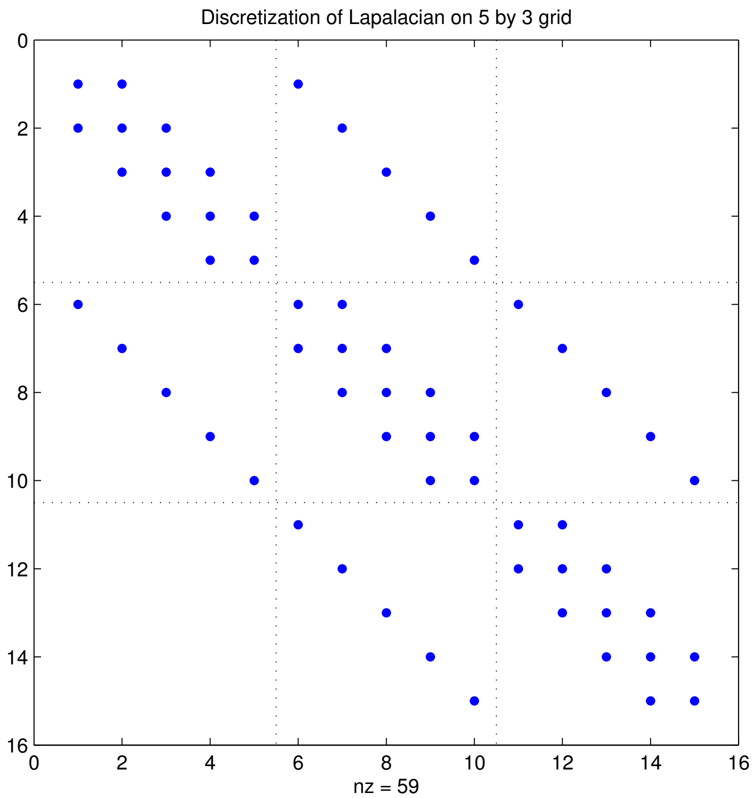

Figure 2: Spy plot of coefficient matrix for $m = 5$ and $n = 3$