FAMILY NAME: ....
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THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 2, 2016

MATH2089
Numerical Methods and Statistics

(1) TIME ALLOWED – 3 Hours

(2) TOTAL NUMBER OF QUESTIONS – 6

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) THIS PAPER MAY **NOT** BE RETAINED BY THE CANDIDATE

(6) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(7) STATISTICAL FORMULAE ARE ATTACHED AT END OF PAPER
STATISTICAL TABLES ARE ATTACHED AT END OF PAPER

Part A – Numerical Methods consists of questions 1 – 3

Part B – Statistics consists of questions 4 – 6

Both parts must be answered

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Part A – Numerical Methods

1. Answer in a separate book marked Question 1

a) [**6 marks**] You are given that

$$\begin{bmatrix} 2 & \text{-1} \\ \text{-1} & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},$$

$$\begin{bmatrix} 2 & \text{-1} \\ \text{-1} & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & \text{-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & \text{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

For the matrix

$$K = \begin{bmatrix} 2 & \text{-1} \\ \text{-1} & 2 \end{bmatrix}$$

find the following norms and condition numbers.

i) Find $||K||_1$

ii) Find $||K||_2$

iii) Find $||K||_\infty$

iv) Find $\kappa_1(K)$

v) Find $\kappa_2(K)$

vi) Find $\kappa_\infty(K)$

b) [**4 marks**] The computational complexity of some common operations with n by n matrices are given in Table 1.1

Operation	Flops
Matrix multiplication	$2n^3$
LU factorization	$\frac{2n^3}{3} + O(n^2)$
Cholesky factorization	$\frac{n^3}{3} + O(n^2)$
Back/forward substitution	$n^2 + O(n)$
Tridiagonal solve	$8n + O(1)$

Table 1.1: Flops for some operations with n by n matrices

i) On your computer, multiplying two large $n \times n$ matrices takes 1000 seconds. Estimate how long it will take to solve an $n \times n$ symmetric positive definite linear system for two different right-hand-side vectors.

c) **[6 marks]** You are given the QR factorization $A = QR$, and the condition number $\kappa_2(A) = 10^6$.

   i) Find $\kappa_2(A^T A)$.

   ii) Find $\kappa_2(R)$.

   iii) For this example, you solve a least squares problem. Would you use the normal equations or the QR factorization? Give a reason for your answer.

d) **[4 marks]** For each claim, state whether the claim is true or false (1 mark), and give a short reason for your answer (1 mark).

   i) You are given that
      - the Secant method for solving $f(x) = 0$ has order of convergence $\nu \approx 1.6$,
      - the error $e_k = |x^* - x_k|$ on the kth iteration is $e_k = 1.02 \times 10^{-12}$.

      Claim: The error on the next iteration will be $e_{k+1} \approx 3.96 \times 10^{-21}$.

   ii) You are given that
      - A and $\mathbf{b}$ are computed to full double precision accuracy,
      - $\kappa(A) = 1.12 \times 10^4$.

      Claim: The computed solution to $A\mathbf{x} = \mathbf{b}$ has at least 11 significant figures.

2. Answer in a separate book marked Question 2

a) **[11 marks]** Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \frac{8}{\pi}\sqrt{x(1-x)}.$$

The function f is a probability density function so

$$I(f) = \int_0^1 f(x)\,\mathrm{d}x = 1.$$

Approximations to $I(f)$ were calculated using the Trapezoidal rule, Simpson's rule and the Gauss-Legendre rule, giving the following table of errors $E_N(f) = I(f) - Q_N(f)$:

N	E_N Trapezoidal	E_N Simpson	E_N Gauss-Legendre
32	5.8377e-03	2.2902e-03	-1.5288e-05
64	2.0659e-03	8.0865e-04	-1.9545e-06
128	7.3076e-04	2.8571e-04	-2.4714e-07
256	2.5842e-04	1.0098e-04	-3.1072e-08
512	9.1377e-05	3.5695e-05	-3.8953e-09
1024	3.2309e-05	1.2619e-05	-4.8762e-10

i) Find a linear transformation $x = \alpha + \beta z$ that maps $z \in [-1,1]$ to $x \in [0,1]$.

ii) **Given** the nodes $z_j, j = 1,\ldots,N$ and weights $w_j, j = 1,\ldots,N$ for the Gauss-Legendre rule for the interval $[-1,1]$, how can you approximate $I(f)$?

iii) The error for the Trapezoidal rule satisfies

$$E_N^{\text{Trap}}(f) = O(N^{-2}), \tag{2.1}$$

provided that $f \in C^2([0,1])$. You do **not** need to prove this.

A) Use (2.1) to estimate the ratio

$$\frac{E_N^{\text{Trap}}(f)}{E_{2N}^{\text{Trap}}(f)}. \tag{2.2}$$

B) Use the table of errors to estimate the ratio (2.2) when $N = 512$.

C) Is the table of errors consistent with the theoretical error estimate in (2.1)?

iv) You are **given** that for $x \in (0,1)$,

$$f'(x) = \frac{4}{\pi}\frac{1-2x}{\sqrt{x(1-x)}}.$$

Explain why $I(f)$ is difficult to approximate.

v) Suppose now that the function f above is replaced by the function $g : [0, 5] \to \mathbb{R}$

$$g(x) = \frac{8}{25\pi}\sqrt{x(5-x)}$$

and the integral I is replaced by

$$J(g) = \int_0^5 g(x)\,\mathrm{d}x,$$

and a new table of errors, $E_N(g) = J(g) - Q_N(g)$, is computed. Describe how you expect the new table of errors for g to compare to the old table of errors for f.

b) **[9 marks]** The motion of a damped mass-spring system is modelled by the initial value problem

$$my'' + cy' + ky = 0, \qquad y(1) = 1, \quad y'(1) = -1,$$

where $y(t)$ is the displacement of the block at time t, m is the mass of the block, c is the damping coefficient, and k is the spring constant. Consider here the case

$$m = 2, \quad c = 4, \quad \text{and} \quad k = 6.$$

i) What is the order of the differential equation?

ii) Convert this ordinary differential equation into a system

$$\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}), \quad \text{for } t > t_0,$$

of first order differential equations.

iii) What is the initial condition $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$?

iv) Write

- **EITHER** a MATLAB anonymous function `myode`
- **OR** a MATLAB function M-file `myode.m`

to evaluate the vector valued function $\boldsymbol{f}(t, \boldsymbol{x})$.

v) Use Euler's method with a step of $h = 0.1$ to estimate $\boldsymbol{x}(1.1)$.

3. Answer in a separate book marked Question 3

Fick's second law predicts how diffusion causes the concentration $u(x, y)$ of a chemical to change with position $(x, y) \in \Omega$. The steady state version of Fick's second law (without interior sources of the chemical) is Laplace's equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0. \tag{3.1}$$

Consider the rectangular domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \le x \le 2,\ 0 \le y \le 1\},$$

and discretize it using $h = 1/n$ and

$$\begin{cases} x_i = ih & \text{for } i = 0, 1, \ldots, 2n, \\ y_j = jh & \text{for } j = 0, 1, \ldots, n. \end{cases}$$

This is illustrated in Figure 3.1 for $n = 5$.

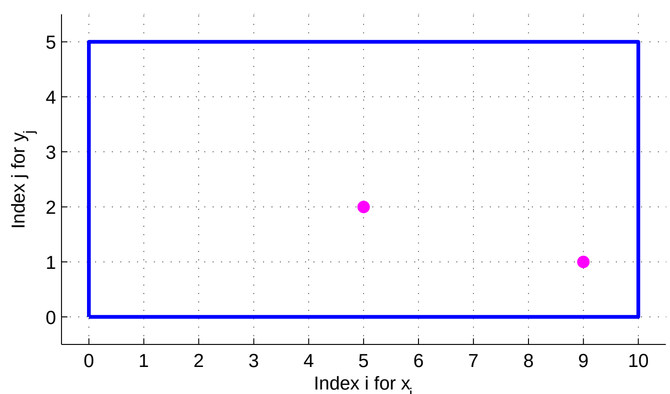


Figure 3.1: Discretization of domain for $n = 5$ and grid points for part d)

a) **[1 mark]** What extra information is needed to completely specify this problem?

b) **[4 marks]** You are **given** the following standard finite difference approximations for a function f of **one** variable:

$$\begin{aligned}
f'(x) &= \frac{f(x+h) - f(x)}{h} + O(h), \\
f'(x) &= \frac{f(x+h) - f(x-h)}{2h} + O(h^2), \\
f''(x) &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2).
\end{aligned}$$

Let $u_{i,j}$ denote the approximation to the value $u(x_i, y_j)$ of concentration at the grid point (x_i, y_j). Give central difference approximations of accuracy $O(h^2)$ to the following derivatives at the point (x_i, y_j)

i) $\dfrac{\partial^2 u(x,y)}{\partial x^2}$ ii) $\dfrac{\partial^2 u(x,y)}{\partial y^2}$

c) **[2 marks]** Using the finite difference approximations from the previous part, show that the equation (3.1) can be written as

$$\beta u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = 0, \tag{3.2}$$

and determine the value of β.

d) **[6 marks]** Given that, in appropriate units,

$$\begin{cases} u(x,0) = u(x,1) = 10x & \text{for } 0 \le x \le 2, \\ u(0,y) = 0 & \text{for } 0 \le y \le 1, \\ u(2,y) = 20 & \text{for } 0 \le y \le 1, \end{cases}$$

write down the equation (3.2) for a discretization with $n = 5$ at the grid points (marked in Figure 3.1)

i) (x_5, y_2) ii) (x_9, y_1)

e) **[7 marks]** Information about the coefficient matrix A using a row-ordering of the variables $u_{i,j}$ is given in Figure 3.2.

i) Calculate the sparsity of A.

ii) Why is calculating A^{-1} not a good idea?

iii) From Figure 3.2, give **two** reasons why A is positive definite?

iv) Explain how to use the Cholesky factorization $A = R^{\mathrm{T}}R$ to solve the linear system $A\boldsymbol{u} = \boldsymbol{b}$?

v) What other structure does A have that could make solving a linear system $A\boldsymbol{u} = \boldsymbol{b}$ more efficient?

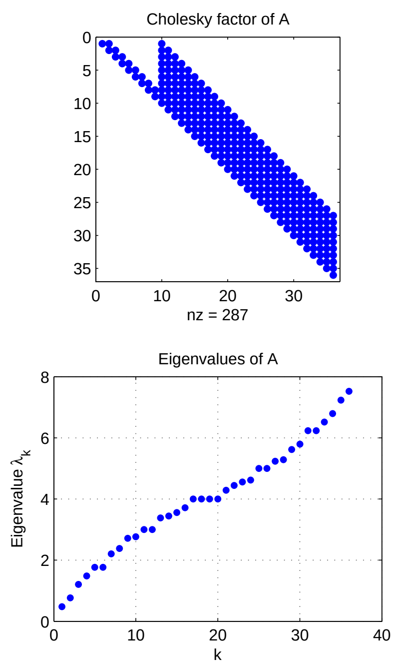


Figure 3.2: Spy plots of A, Cholesky factor R, inverse A^{-1}, and eigenvalues of A

Part B – Statistics

4. [**20 marks**] **Answer in a separate book marked Question 4**

a) [**7 marks**] "Spot or Save" is a game played by Australian school children. When one child spots a yellow car they declare "Spot or Save!" and then punch their friend on their arm until their friend says "Save!" The mean number of yellow cars passing a given school yard is 4.8 per hour.

i) [**2 marks**] What is the probability that at least one child can declare "Spot or Save!" in a 15 minute free time break?

ii) [**2 marks**] There are ten 15 minute free time breaks in one week. What is the probability that "Spot or Save!" can be declared in at least nine 15 minute free time breaks in one week?

iii) [**3 marks**] The school nurse has to spend 5 minutes consoling punched children for each 15 minute free time break period in which a child is punched by playing "Spot or Save". What is the expected number of minutes per week that the school nurse will spend consoling children because of this game? What is the standard deviation?

b) [**6 marks**] One measure of the impact of a twitter feed is the proportion of tweets that are retweeted within 10 minutes of posting. Out of 100 tweets in one week, one engineering researcher had 27 tweets that were retweeted within 10 minutes of posting.

i) [**3 marks**] Construct a two-sided 90% confidence interval for the true proportion of tweets that are retweeted within 10 minutes of posting for this researcher.

ii) [**3 marks**] State two assumptions you need to make in order to determine the above confidence interval. Explain whether each seems reasonable in this situation.

c) [**7 marks**] A battery manufacturer claims that their novel production process allows their batteries to produce more than 100 hours of continual high-power usage on average. A random sample of 50 batteries was tested, and their usable duration recorded. The sample mean duration was $\bar{x} = 105.3$ hours with a sample standard deviation of $s = 15.3$ hours.

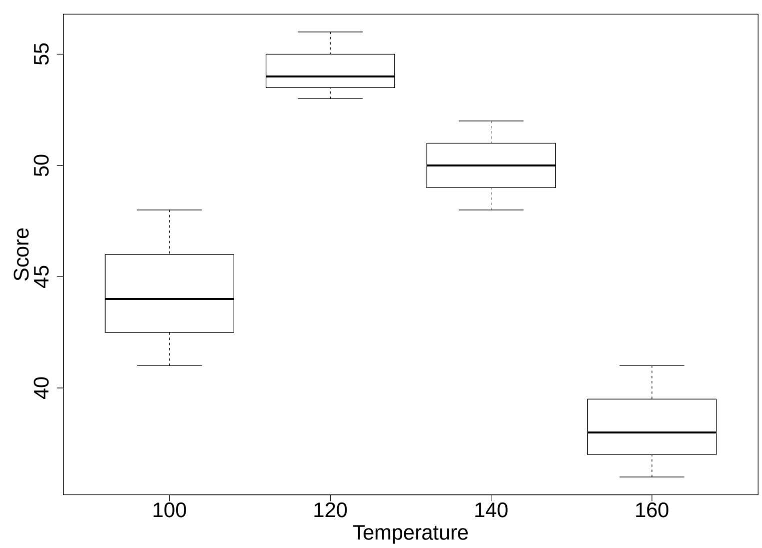
Do these data support the hypothesis (1% significance level) that the true mean duration of the manufacturers batteries is greater than 100 hours? (*Write the detail of the test: null and alternative hypotheses, the distribution of the test statistic under the null hypothesis, an expression for the p-value, and your conclusions in plain language. You may use bounds for the p-value. You may use a test statistic and rejection region.*)

5. [**20 marks**] **Answer in a separate book marked Question 5**

Following a study given in Bassett *et al.* (2000), an industrial plant was maintained at a sequence of increasing temperatures over four successive days. On each day three product samples were taken from the production process and analysed for quality. A score was awarded for each sample, and these are summarised in the table below:

Sample	Temperature (Day)			
	$100°C$ (1)	$120°C$ (2)	$140°C$ (3)	$160°C$ (4)
1	41	54	50	38
2	44	56	52	36
3	48	53	48	41
	$\bar{x}_1 = 44.33$	$\bar{x}_2 = 54.33$	$\bar{x}_3 = 50.00$	$\bar{x}_4 = 38.33$
	$s_1 = 3.51$	$s_2 = 1.53$	$s_3 = 2.00$	$s_4 = 2.52$

Comparative boxplots are given in the figure below.



a) [**3 marks**] What do the boxplots tell you about the scores for different temperatures? Comment on the location, spread and shape.

b) [**3 marks**] State three assumptions that need to be valid for an Analysis of Variance (ANOVA) to test whether there is a difference in mean scores among the four temperatures. Comment on the suitability of these assumptions here, where applicable.

Assume from now on that these assumptions are valid.

c) **[3 marks]** An ANOVA table was partially constructed to summarise the data:

Source	df	SS	MS	F
Treatment	**(1)**	**(2)**	**(3)**	23.16
Error	**(4)**	**(5)**	6.25	
Total	**(6)**	484.25		

Copy the ANOVA table in your answer booklet. Complete the table by determining the missing values (1)–(6) **without using the value of** $F = 23.16$, stating how you computed the missing entries. Confirm the value of $F = 23.16$ and explain how this is obtained from the above table.

d) **[5 marks]** Using a significance level of $\alpha = 0.05$, carry out the ANOVA F-test to determine whether there is a difference in mean scores among the four temperatures.

(*You can use the numerical values found in the above table; however, you are required to write the detail of the test: null and alternative hypotheses, rejection criterion or observed value of the test statistic and p-value, conclusion in plain language - you may use bounds for the p-value.*)

e) **[4 marks]** From the previous results, construct a 95% two-sided confidence interval on the difference between the "true" scores for temperatures $100°C$ and for $120°C$, that is, $\mu_1 - \mu_2$. Would you conclude that there is a significant difference between the "true" scores for temperatures $100°C$ and for $120°C$? Explain.

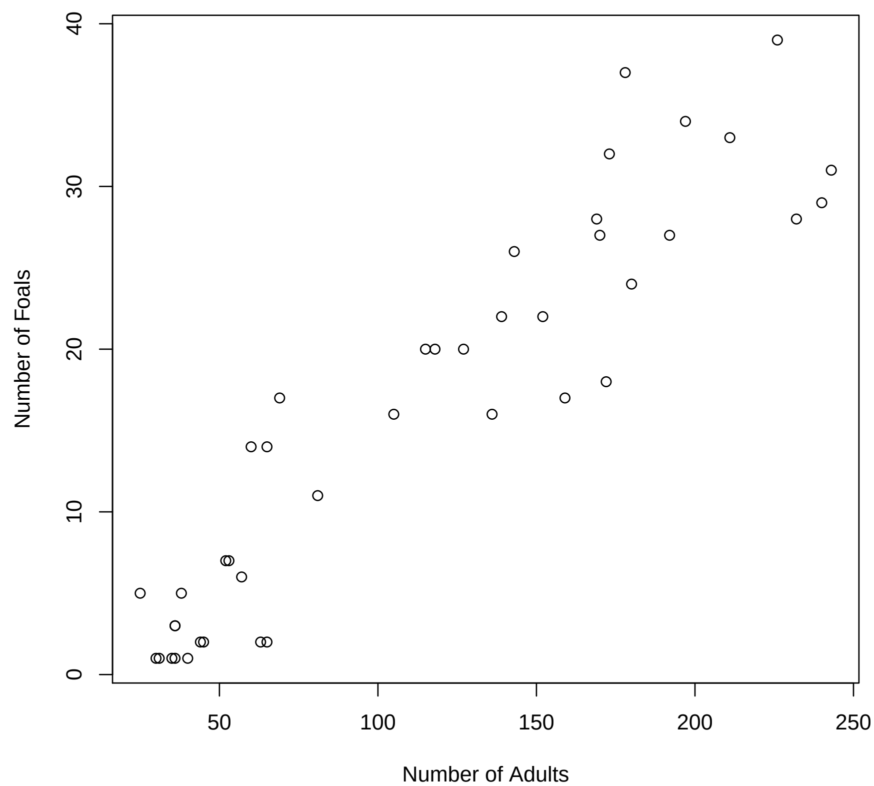
f) **[2 marks]** Six pairwise two-sample t-tests were carried out for comparing the "true" mean score for each temperature. The p-values were also obtained and given in the table below:

Pairwise comparison	p-value
$100°C$ vs $120°C$	0.00120
$100°C$ vs $140°C$	0.02407
$100°C$ vs $160°C$	0.01873
$120°C$ vs $140°C$	0.06653
$120°C$ vs $160°C$	0.00005
$140°C$ vs $160°C$	0.00045

Does simultaneously analysing the six pairwise comparisons (i.e., those given in the table above) allow you to come to the same conclusion as the ANOVA F-test in (d), at overall significance level $\alpha = 0.05$? Explain.

6. [20 marks] **Answer in a separate book marked Question 6**

Large herds of wild horses can become a problem on some federal lands in the West. Researchers hoping to improve the management of these herds collected data to see if they could predict the number of foals (young horses) that would be born based on the size of the current herd. They observed 42 herds and recorded how many adult horses and foals were born in each herd. A scatter plot of the `Number of Adults` and the `Number of Foals` is shown below.



The following summary statistics were obtained for the `Number of Adults`

$$\sum_{i=1}^{42} x_i = 4738 \quad \text{and} \quad s_{xx} = \sum_{i=1}^{42}(x_i - \bar{x})^2 = 199970.5$$

In order to study the herd growth, researchers attempted to fit a linear regression model given by

$$\texttt{Number of Foals} = \beta_0 + \beta_1(\texttt{Number of Adults}) + \epsilon.$$

Use the following regression output to answer the questions below.

```
Estimated Coefficients:

                        Estimate      SE        tStat       pValue
                       ---------   -------    -------    ----------

   (Intercept)          -1.91793    1.32703    -1.445      0.156
   Number of Adults      0.15862    0.01004    15.807      8.44e-19

Root Mean Squared Error: 4.487
R-squared: 0.862
```

a) **[2 marks]**

  i) **[1 mark]** What is the equation of the fitted linear regression line?

  ii) **[1 mark]** What proportion of variability in the response is explained by the predictor?

b) **[1 mark]** Estimate the true average change in the `Number of Foals` for 1 unit change in the `Number of Adults`?

c) **[1 mark]** Determine the observed sample correlation coefficient between the `Number of Foals` and the `Number of Adults`.

d) **[1 mark]** Assume σ is the standard deviation of the error term ϵ. Give an estimate of σ.

e) **[6 marks]** Perform a hypothesis test to determine whether the variable `Number of Adults` is significant in this model, at the 5% level of significance. (*You can use the numerical values found in the above output, however you are required to write the details of the test: null and alternative hypotheses; rejection criterion, or observed value of the test statistics and p-value (specify the degrees of freedom if applicable); conclusion in plain language.*)

f) **[2 marks]** Create a two-sided 90% confidence interval for β_1.

g) **[4 marks]** Suppose that a new herd with 120 adult horses is located.

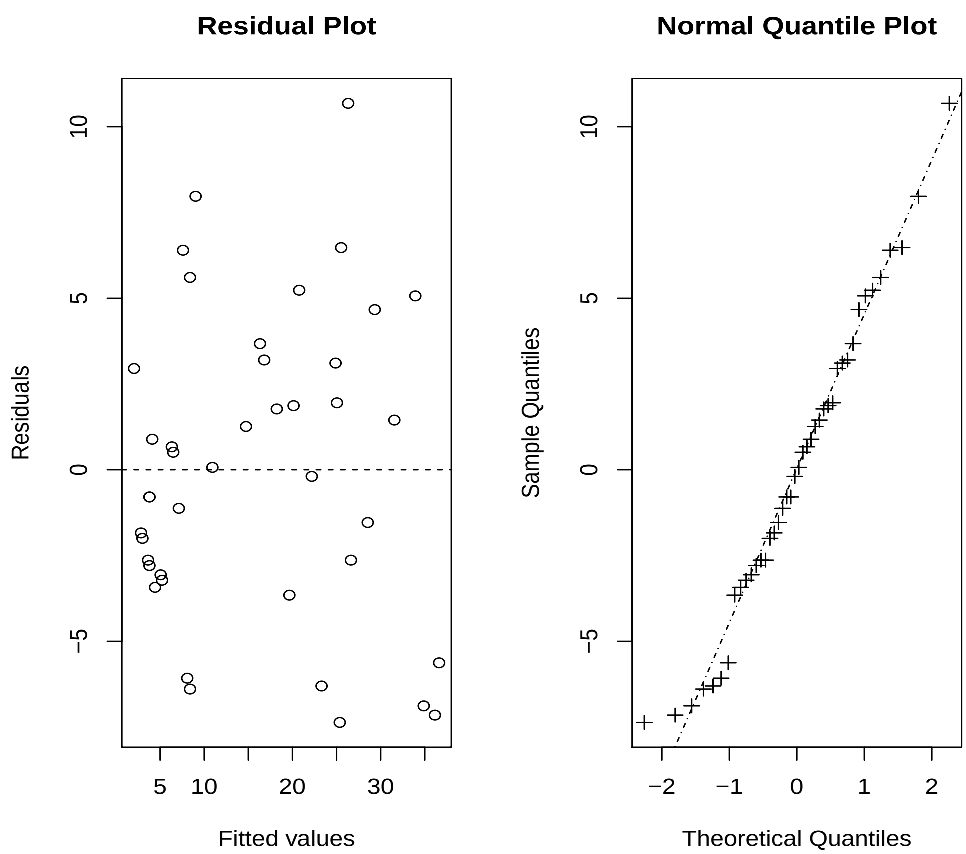
  i) **[1 mark]** What would be a point estimate of the true average `Number of Foals` that may be born?

  ii) **[3 marks]** Find a two-sided 99% prediction interval for the `Number of Foals`.

h) **[3 marks]**

i) **[2 marks]** For the above regression analysis to be valid, what are the three essential assumptions that the residuals must satisfy?

ii) **[1 mark]** Given the residual versus fitted values plot and the normal quantile plot below, explain why these assumptions are at least approximately valid.



Statistical Formulae

1. Calculation formulae

For a sample $x_1, x_2, \ldots, x_n$

- Sample mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

- Sample variance

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right)$$

2. The Binomial distribution

Assume $\boxed{X \sim \text{Bin}(n, \pi)}$

- **domain of variation :** $S_X = \{0, 1, \ldots, n\}$
- **probability mass function (pmf) :**

$$p(x) = \binom{n}{x}\pi^x(1-\pi)^{n-x}, \qquad \text{for } x \in S_X$$

  Note that $\binom{n}{x} = {}^nC_x = \frac{n!}{x!(n-x)!}$.
- **cumulative distribution function (cdf) :**

$$F(x) = \sum_{k=0}^{\lfloor x \rfloor}\binom{n}{k}\pi^k(1-\pi)^{n-k}$$

  (where $\lfloor x \rfloor$ denotes the integer part of x).
- **expectation :**

$$\mathbb{E}(X) = n\pi$$

- **variance :**

$$\mathbb{V}\text{ar}(X) = n\pi(1-\pi)$$

3. The Poisson distribution

Assume $\boxed{X \sim \mathcal{P}(\lambda)}$

- **domain of variation :** $S_X = \{0, 1, 2, \ldots\}$
- **probability mass function (pmf) :**

$$p(x) = e^{-\lambda}\frac{\lambda^x}{x!}, \qquad \text{for } x \in S_X$$

- **cumulative distribution function (cdf) :**

$$F(x) = e^{-\lambda}\sum_{k=0}^{\lfloor x \rfloor}\frac{\lambda^k}{k!}$$

  (where $\lfloor x \rfloor$ denotes the integer part of x). See also the attached Poisson table.
- **expectation :**

$$\mathbb{E}(X) = \lambda$$

- **variance :**

$$\mathbb{V}\text{ar}(X) = \lambda$$

4. The Uniform distribution

Assume $\boxed{X \sim U_{[\alpha,\beta]}}$

- **domain of variation :** $S_X = [\alpha, \beta]$
- **probability density function (pdf) :**

$$f(x) = \frac{1}{\beta - \alpha}, \qquad \text{for } x \in S_X$$

- **cumulative distribution function (cdf) :**

$$F(x) = \frac{x - \alpha}{\beta - \alpha}, \qquad \text{for } x \in S_X$$

- **expectation :**

$$\mathbb{E}(X) = \frac{\alpha + \beta}{2}$$

- **variance :**

$$\mathbb{V}\text{ar}(X) = \frac{(\beta - \alpha)^2}{12}$$

5. The Exponential distribution

Assume $\boxed{X \sim \text{Exp}(\mu)}$

- **domain of variation :** $S_X = [0, +\infty)$
- **probability density function (pdf) :**

$$f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}, \qquad \text{for } x \in S_X$$

- **cumulative distribution function (cdf) :**

$$F(x) = 1 - e^{-\frac{x}{\mu}}, \qquad \text{for } x \in S_X$$

- **expectation :**

$$\mathbb{E}(X) = \mu$$

- **variance :**

$$\mathbb{V}\text{ar}(X) = \mu^2$$

6. The Normal distribution

Assume $\boxed{X \sim \mathcal{N}(\mu, \sigma)}$

- **domain of variation :** $S_X = (-\infty, +\infty)$
- **probability density function (pdf) :**

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}, \qquad \text{for } x \in S_X$$

- **cumulative distribution function (cdf) :**

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}\, dy, \qquad \text{for } x \in S_X$$

  (no closed form)
- **expectation :**

$$\mathbb{E}(X) = \mu$$

- **variance :**

$$\mathbb{V}\text{ar}(X) = \sigma^2$$

7.1. **Sample mean.**

7.1.1. *known variance.* Let $\bar{X}$ be the sample average from a random sample of size n from a population with mean μ and standard deviation σ. Under appropriate conditions,

$$Z = \sqrt{n}\,\frac{\bar{X}-\mu}{\sigma} \sim \mathcal{N}(0,1)$$

(exact result if the population distribution is normal, approximate result if the population distribution is not normal but $n > 30$)

7.1.2. *unknown variance.* Let $\bar{X}$ and S be the sample average and standard deviation from a random sample of size n from a normal population with mean μ. Under appropriate conditions,

$$T = \sqrt{n}\,\frac{\bar{X}-\mu}{S} \sim t_{n-1}$$

If the population is not normal but n is large enough ($n > 40$), we can also write

$$T = \sqrt{n}\,\frac{\bar{X}-\mu}{S} \sim \mathcal{N}(0,1)$$

approximately

7.2. **Sample proportion.**

Let $\hat{p}$ be the sample proportion of 'successes' where the number of trials is n and the true probability of a success is π. Under appropriate conditions,

$$\sqrt{n}\,\frac{\hat{p}-\pi}{\sqrt{\pi(1-\pi)}} \sim N(0,1)$$

approximately when $n\pi(1-\pi) > 5$

7.3. **Sample variance.**

Let S^2 be the sample variance from a random sample of size n from a normal population with variance σ. Under appropriate conditions,

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

7.4. **Difference in sample means.**

7.4.1. *variances σ_1^2 and σ_2^2 known.* For two independent samples of size n_1 and n_2 from two populations with means μ_1 and μ_2 and standard deviations σ_1 and σ_2 respectively, let $\bar{X}_i$ be the sample average of sample i for $i = 1$ and 2. Under appropriate conditions,

$$\frac{(\bar{X}_1-\bar{X}_2)-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} \sim \mathcal{N}(0,1)$$

(exact result if both population distributions are normal, approximate result if they are not but $n_1, n_2 > 30$)

7.4.2. *variances σ_1^2 and σ_2^2 unknown; $\sigma_1^2 = \sigma_2^2$.* For two independent samples of size n_1 and n_2 from two normal populations with means μ_1 and μ_2 respectively and common standard deviation σ, let $\bar{X}_i$ and S_i be the sample average and sample standard deviation of sample i for $i = 1$ and 2. Under appropriate conditions,

$$\frac{(\bar{X}_1-\bar{X}_2)-(\mu_1-\mu_2)}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \sim t_{n_1+n_2-2},$$

where S_p is the pooled sample standard deviation,

$$S_p = \sqrt{\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}}.$$

7.4.3. *variances σ_1^2 and σ_2^2 unknown; $\sigma_1^2 \neq \sigma_2^2$.* For two independent samples of size n_1 and n_2 from two normal populations with means μ_1 and μ_2 and standard deviations σ_1 and σ_2 respectively, let $\bar{X}_i$ and S_i be the sample average and sample standard deviation of sample i for $i = 1$ and 2. Under appropriate conditions,

$$\frac{(\bar{X}_1-\bar{X}_2)-(\mu_1-\mu_2)}{\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}} \sim t_\nu,$$

where

$$\nu = \frac{(s_1^2/n_1+s_2^2/n_2)^2}{\frac{(s_1^2/n_1)^2}{n_1-1}+\frac{(s_2^2/n_2)^2}{n_2-1}}$$

(rounded down to the nearest integer)

7.5. **Ratio of sample variances.** Let S_1^2 and S_2^2 be the sample variances from two independent random samples of size n_1 and n_2 from normal populations with variances σ_1^2 and σ_2^2 respectively. Under appropriate conditions,

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim \mathbf{F}_{n_1-1,n_2-1}$$

8. Simple linear regression

Consider the simple linear regression model $\quad Y = \beta_0 + \beta_1 X + \epsilon \quad$ where $\epsilon \sim \mathcal{N}(0, \sigma)$

The least squares estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ of β_0 and β_1 are

$$\hat{\beta}_1 = \frac{S_{XY}}{S_{XX}} \qquad \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

where

$$S_{XY} = \sum_i (X_i - \bar{X})(Y_i - \bar{Y}) \qquad S_{XX} = \sum_i (X_i - \bar{X})^2.$$

An estimator of σ is

$$S = \sqrt{\frac{\sum_i (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2}{n-2}}.$$

Under fixed design :

$$\sqrt{s_{xx}}\frac{\hat{\beta}_1 - \beta_1}{S} \sim t_{n-2}$$

$$\frac{\hat{\beta}_0 - \beta_0}{S\sqrt{\frac{1}{n} + \frac{\bar{x}^2}{s_{xx}}}} \sim t_{n-2}$$

Let x_0 denote the predictor value for a response yet to be observed :

i) a $100 \times (1-\alpha)\%$ confidence interval for the mean response at x_0 is

$$\left[\hat{y}(x_0) \pm s\, t_{n-2;1-\alpha/2}\sqrt{\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{s_{xx}}}\right]$$

where $\hat{y}(x_0) = \hat{b}_0 + \hat{b}_1 x_0$;

ii) a $100 \times (1-\alpha)\%$ prediction interval for the response at x_0 is

$$\left[\hat{y}(x_0) \pm s\, t_{n-2;1-\alpha/2}\sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{s_{xx}}}\right]$$

9. ANOVA

- Total sum of squares :

$$SS_{\text{Tot}} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \bar{\bar{X}})^2$$

- Treatment sum of squares :

$$SS_{\text{Tr}} = \sum_{i=1}^{k} n_i(\bar{X}_i - \bar{\bar{X}})^2$$

- Error sum of squares :

$$SS_{\text{Er}} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2$$

- Under the assumption of equality of means in each of the k groups of a one-way Analysis of Variance,

$$F = \frac{\text{MS}_{\text{Tr}}}{\text{MS}_{\text{Er}}} \sim \mathbf{F}_{k-1,n-k},$$

where n is the total number of observations.

Statistical Tables

t distribution critical values

Key: Table entry for p and C is the critical value t^* with probability p lying to its right and probability C lying between $-t^*$ and t^*.

	Upper tail probability p											
df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.000	1.376	1.963	3.078	6.314	12.71	15.89	31.82	63.66	127.3	318.3	636.6
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33	31.60
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21	12.92
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.610	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
	.50	.60	.70	0.80	.90	.95	.96	.98	.99	.995	.998	.999
	Probability C											

Standard normal probabilities

Key: Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

F distribution critical values

Key: p=Upper tail probability p, df_n=degrees of freedom in numerator, df_d=degrees of freedom in denominator,
∗ Multiply by 10, † Multiply by 100.

df_d	df_n / p	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	.05	161	200	216	225	230	234	237	239	241	242	244	246	248	249	250	251	252	253	254
	.025	648	800	864	900	922	937	948	957	963	969	977	986	993	997	1001	1006	1010	1014	1018
	.01	405∗	500∗	540∗	563∗	576∗	586∗	593∗	598∗	602∗	606∗	611∗	616∗	621∗	624∗	626∗	629∗	631∗	634∗	637∗
	.005	162†	200†	216†	225†	231†	234†	237†	239†	241†	242†	244†	246†	248†	249†	250†	251†	253†	254†	255†
2	.05	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
	.025	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48	39.49	39.50
	.01	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50
	.005	199	199	199	199	199	199	199	199	199	199	199	199	199	200	200	200	200	200	200
3	.05	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
	.025	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04	13.99	13.95	13.90
	.01	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13
	.005	55.55	49.80	47.47	46.19	45.39	44.84	44.43	44.13	43.88	43.69	43.39	43.08	42.78	42.62	42.47	42.31	42.15	41.99	41.83
4	.05	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
	.025	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66	8.56	8.51	8.46	8.41	8.36	8.31	8.26
	.01	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
	.005	31.33	26.28	24.26	23.15	22.46	21.97	21.62	21.35	21.14	20.97	20.70	20.44	20.17	20.03	19.89	19.75	19.61	19.47	19.32
5	.05	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
	.025	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.28	6.23	6.18	6.12	6.07	6.02
	.01	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
	.005	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.38	13.15	12.90	12.78	12.66	12.53	12.40	12.27	12.14
6	.05	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
	.025	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.12	5.07	5.01	4.96	4.90	4.85
	.01	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
	.005	18.63	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.03	9.81	9.59	9.47	9.36	9.24	9.12	9.00	8.88
7	.05	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
	.025	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.41	4.36	4.31	4.25	4.20	4.14
	.01	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
	.005	16.24	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.51	8.38	8.18	7.97	7.75	7.64	7.53	7.42	7.31	7.19	7.08
8	.05	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
	.025	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.95	3.89	3.84	3.78	3.73	3.67
	.01	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
	.005	14.69	11.04	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.01	6.81	6.61	6.50	6.40	6.29	6.18	6.06	5.95
9	.05	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
	.025	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.61	3.56	3.51	3.45	3.39	3.33
	.01	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
	.005	13.61	10.11	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.23	6.03	5.83	5.73	5.62	5.52	5.41	5.30	5.19
10	.05	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
	.025	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.37	3.31	3.26	3.20	3.14	3.08
	.01	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
	.005	12.83	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.66	5.47	5.27	5.17	5.07	4.97	4.86	4.75	4.64
12	.05	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
	.025	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.02	2.96	2.91	2.85	2.79	2.72
	.01	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
	.005	11.75	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.91	4.72	4.53	4.43	4.33	4.23	4.12	4.01	3.90
15	.05	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
	.025	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.70	2.64	2.59	2.52	2.46	2.40
	.01	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
	.005	10.80	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.25	4.07	3.88	3.79	3.69	3.58	3.48	3.37	3.26
20	.05	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
	.025	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.41	2.35	2.29	2.22	2.16	2.09
	.01	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
	.005	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.68	3.50	3.32	3.22	3.12	3.02	2.92	2.81	2.69
24	.05	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
	.025	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.27	2.21	2.15	2.08	2.01	1.94
	.01	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
	.005	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.42	3.25	3.06	2.97	2.87	2.77	2.66	2.55	2.43
30	.05	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
	.025	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.14	2.07	2.01	1.94	1.87	1.79
	.01	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
	.005	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.18	3.01	2.82	2.73	2.63	2.52	2.42	2.30	2.18
40	.05	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
	.025	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.18	2.07	2.01	1.94	1.88	1.80	1.72	1.64
	.01	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
	.005	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	2.95	2.78	2.60	2.50	2.40	2.30	2.18	2.06	1.93
60	.05	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
	.025	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.88	1.82	1.74	1.67	1.58	1.48
	.01	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
	.005	8.49	5.79	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.74	2.57	2.39	2.29	2.19	2.08	1.96	1.83	1.69
120	.05	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
	.025	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.05	1.94	1.82	1.76	1.69	1.61	1.53	1.43	1.31
	.01	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
	.005	8.18	5.54	4.50	3.92	3.55	3.28	3.09	2.93	2.81	2.71	2.54	2.37	2.19	2.09	1.98	1.87	1.75	1.61	1.43
∞	.05	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00
	.025	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.94	1.83	1.71	1.64	1.57	1.48	1.39	1.27	1.00
	.01	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00
	.005	7.88	5.30	4.28	3.72	3.35	3.09	2.90	2.74	2.62	2.52	2.36	2.19	2.00	1.90	1.79	1.67	1.53	1.36	1.00