NUMERICAL METHODS SAMPLE PAPER. NUMERICAL METHODS

2010

Calculator

Includes Solutions

Question 1.

Answer in a separate book marked Question 1.

Week 6
Least-squares regression
$y = ax + b$
matrix

(a) The heat-transfer coefficient (h) in a forced convection heat transfer in cross-flow past a cylinder at room temperature is found to vary with the velocity of the fluid (v) flowing past the cylinder as follows:

		0	1	2	3.
x	v_i (m/s)	2	4	6	8
y	h_i (W/m^{2}K)	6,000	10,000	13,000	15,000

Fit a linear equation between h and v.

least squares.

?

$f(x) = \cancel{6000} + \cancel{10000}(\cancel{x}-2) + \cancel{13000}\cancel{(x-6)} + \cancel{15000}(x-8)$

$= 6000 + 10000x - 20000 + 13000x - 78000 + 16000x - 120000$

$= 38000x - 212000$

(b) Compute forward difference approximation of $O(\Delta x)$ and central difference approximation of $O(\Delta x)^2$ for the first derivative of $f(x) = e^x + x$ at $x = 1$, using a value of $\Delta x = 0.25$.

Week 8

Calculate the percentage relative errors for each approximation by comparing with exact solution and discuss your results.

The exact solution is: $f'(1) = e^1 + 1 = 3.71828$

$f'(x) = xe^x + 1.$

$= 1 \times e^1 + 1.$

$[\;][\;] = [\;]$

Σx_i
Σx_i^2

Question 2.

Answer in a separate book marked Question 2.

The fourth-order Runge-Kutta method can be written as:

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

where

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}hk_1)$$
$$k_3 = f(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}hk_2)$$
$$k_4 = f(x_i + h, y_i + hk_3)$$

(a) Find the solution of the initial value problem:

$y' = 3y + 3t$ with $y(0) = 1$

at t=0.2

(i) Using Euler's method with $h = 0.2$.

(ii) Using the fourth-order Runge-Kutta method with $h = 0.2$.

(iii) Compare the results with the exact solution $y(t) = \frac{4}{3}e^{3t} - t - \frac{1}{3}$ and find the percentage errors for the results obtained in (i) and (ii).

$y(0.2) = 1.89616$

(iv) Why do you have improvement in the case of the Runge-Kutta method?

(b) Consider the second order differential equation

$$2x''(t) - 5x'(t) - 3x(t) = 45e^{2t}$$

Reformulate this equation as a system of two first order differential equations.

Week 9 Lecture Notes

$\boxed{y = x'(t)}$, $\therefore$ $y'(t) = x''(t)$.

1. hence 2. $\boxed{2y'(t) - 5y(t) - 3x(t) = 45e^{2t}}$ 1.

?

~~$2x''(t) = 45e^{2t} + 5x'(t) + 3x(t)$~~ 2

$$2x''(t) - 5x'(t) - 3x(t) = 45e^{2t}.$$

Let $y(t) = x'(t)$ 1 (i.e $y'(t) = x''(t)$)

$\therefore 2y'(t) - 5y(t) - 3x(t) = 45e^{2t}$ 2

Question 3.

Answer in a separate book marked Question 3.

The heat conduction equation which models the temperature in an insulated rod with ends held at constant temperatures can be written in the dimensionless form as:

Θ is just a random parameter eg. T, α etc.

$$\frac{\partial \Theta(x,t)}{\partial t} = \frac{\partial^2 \Theta(x,t)}{\partial x^2}$$

(a) Write a finite difference approximation of this equation using the Forward-Time, Central-Space (FTCS) scheme and rearrange it to be solved by an explicit method.

(b) Solve this equation and calculate the temperature $\Theta(x,t)$ at $t = 0.3$ and $x = 0.5$ if the initial condition is

$$\Theta(x,t=0) = 1 \qquad (0 < x \le 1)$$

and the boundary conditions at the ends of the rod are

$\phi(1,t)$

$\phi(0,t)$

$$\Theta(x=0,t) = 0; \ \Theta(x=1,t) = 1.$$

Use value of 0.5 for the step in space, Δx, and value of 0.1 for the time step, Δt.

$\Delta x = 0.5$, $\Delta t = 0.1$

(c) If the calculations in the previous part were repeated with $\Delta x = 0.1$ to reduce truncation error and Δt kept equal to 0.1, what difficulty would be encountered? Do not repeat the finite difference calculations to determine your answer.

$\Delta x = 0.1$

$\Delta t = 0.1$

stability: $\alpha\left[\frac{\Delta t}{\Delta x^2}\right] = \frac{0.1}{0.1^2} = 10$

This is greater than 0.5 so UNSTABLE.

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1. (a)

x	V (m/s)	2	4	6	8
y	h (W/m²k)	6000	10000	13000	15000

Linear Equation $\rightarrow$ $y = ax + b$

Can be found by solving matrix.

$$\begin{bmatrix} \sum x_i & n \\ \sum x_i^2 & \sum x_i \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$n = 4.$

$\sum x_i = 2 + 4 + 6 + 8 = \underline{20}$

$\sum x_i^2 = 2^2 + 4^2 + 6^2 + 8^2 = \underline{120}$

$\sum y_i = 6000 + 10000 + 13000 + 15000 = \underline{44000}$

$\sum x_i y_i = (2 \times 6000) + (4 \times 10000) + (6 \times 13000) + (8 \times 15000) = \underline{250000}$

$$\therefore \begin{bmatrix} 20 & 4 \\ 120 & 20 \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{bmatrix} 44000 \\ 250000 \end{bmatrix}$$

(I) $20a + 4b = 44000 \rightarrow a = \dfrac{44000 - 4b}{20}$

(II) $120a + 20b = 250000$

Sub (I) into (II)

$120\left(\dfrac{44000 - 4b}{20}\right) + 20b = 250000$

$264000 - 24b + 20b = 250000$

$-4b = -14000$

$\boxed{b = 3500}$

Sub into (I)

$\boxed{a = \dfrac{44000 - 4(3500)}{20} = 1500}$

$y = A + Bx$

$\therefore y = 1500x + 3500$

Zeb

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(1.) (b)

$$f(x) = e^{x} + x.$$

$$x = 1 \quad \gamma \quad \Delta x = 0.25.$$

$$\text{exact solution} = f'(1) = 3.71828.$$

Notation	x	$f(x)$
$i-1$	0.75	2.86700
i	1	3.71828
$i+1$	1.25	4.740343.

→ Forward Difference Approximation

1st order: $$f' = \frac{f_{i+1} - f_i}{\Delta x}$$

$$= \frac{4.740343 - 3.71828}{0.25}$$

$$= \underline{\underline{4.088252}}.$$

→ Central Difference Approximation.

2nd Order: $$f' = \frac{f_{i+1} - f_{i-1}}{2\Delta x}$$

$$= \frac{4.740343 - 2.86700}{0.5}$$

$$= \underline{\underline{3.746686}}$$

$$\frac{\text{Exact} - \text{found}}{\text{Exact}} \times 100$$

→ Percentage relative errors.

Forward Diff. $$= \frac{3.71828 - 4.088252}{3.71828} \times 100 = -9.95\%.$$

Central Diff. $$= \frac{3.71828 - 3.746686}{3.71828} \times 100 = -0.76\%$$

- It can be concluded that central diff is a far more accurate and faster converging method, as it is a higher order.

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2. (a)

$$y' = 3y + 3t \quad \text{at} \quad t = 0.2,$$
$$y_0 = 1.$$

(i) Euler's Method with $h = 0.2$.

i.e $t_0 = 0$, $y_0 = 1$.

$t_{0.2} = 0.2$, $y_{0.2} = ?$

$$y_{0.2} = y_0 + h\left[f(t_0, y_0)\right]$$
$$= 1 + 0.2\left[3y_0 + 3t_0\right].$$
$$= 1 + 0.2\left[3 \times 1 + 3 \times 0\right].$$
$$= 1.6$$

(ii) Fourth-order Runge-Kutta Method.

$$y_{0.2} = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

$$k_1 = f(t_0, y_0) = 3$$

$$k_2 = f\left(t_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}hk_1\right)$$
$$= f\left(0 + 0.1, 1 + \tfrac{1}{2} \times 0.2 \times 3\right)$$
$$= f(0.1, 1.3)$$
$$= (3 \times 1.3) + (3 \times 0.1) = 4.2$$

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(2)(a)(ii) cont.

$$k_3 = f\left(t_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}hk_2\right)$$
$$= f\left(0 + 0.1,\ 1 + \tfrac{1}{2}\times 0.2\times 4.2\right)$$
$$= f\left(0.1,\ 1.42\right)$$
$$= (3\times 1.42) + (3\times 0.1) = \underline{\underline{4.56}}$$

$$k_4 = f\left(t_0 + h,\ y_0 + hk_3\right)$$
$$= f\left(0 + 0.2,\ 1 + 0.2\times 4.56\right)$$
$$= f\left(0.2,\ 1.912\right)$$
$$= (3\times 1.912) + (3\times 0.2) = \underline{\underline{6.336}}.$$

$$\therefore y_{0.2} = y_0 + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$
$$= \cancel{1 + \left(\tfrac{1}{30}\times 3\right) + (2\times 4.2) + (2\times 4.56) + 6.336}$$
$$\cancel{= 24.956} ??$$
$$= 1 + \frac{0.2}{6}\left(3 + (2\times 4.2) + (2\times 4.56) + 6.336\right)$$
$$= 1 + \frac{1}{30}\left(26.856\right)$$
$$= \underline{\underline{1.8952}}$$

(iii) Exact Solution: $y(t) = \frac{4}{3}e^{3t} - t - \frac{1}{3}$.

$$y(0.2) = \frac{4}{3}e^{0.6} - 0.2 - \frac{1}{3}$$
$$= \underline{1.89616}$$

Percentage errors:

Eulers $\rightarrow$ $\dfrac{1.89616 - 1.6}{1.89616}\times 100 = \underline{\underline{-15.62\%}}$

Rugge-Kutta $\rightarrow$ $\dfrac{1.89616 - 1.8952}{1.89616}\times 100 = \underline{\underline{-0.05\%}}$

(iv)

ignore.

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(3) (a) $$\frac{d\Theta(x,t)}{\partial t} = \frac{\partial^2 \Theta(x,t)}{dx^2}$$

Applying Forward-Time, Central-Space (FTCS)

$$\frac{\phi_{i,j+1} - \phi_{i,j}}{\Delta t} = \left[\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2}\right]$$

Rearranging to be solved by an explicit method:

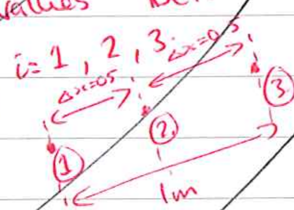
$$\phi_{i,j+1} = \left(\frac{\Delta t}{\Delta x^2}\right)\phi_{i+1,j} + \left(\frac{1-2\Delta t}{\Delta x^2}\right)\phi_{i,j} + \left(\frac{\Delta t}{\Delta x^2}\right)\phi_{i-1,j}$$

(b). Solve this equation. check stability

- Determine $\Theta(x,t)$ at $t=0.3$ and $x=0.5$

$\Delta t = 0.1$, $\Delta x = 0.5$. note: indicates 3 i values between boundaries

$$\frac{\Delta t}{\Delta x^2} = \frac{0.1}{0.5^2} = \underline{0.4} < 0.5 \quad \therefore \text{stable.}$$



$$\frac{1-2\Delta t}{\Delta x^2} = \frac{1-2\times0.1}{0.5^2} = \underline{3.2}.$$

$$\therefore \phi_{i,j+1} = 0.4\,\phi_{i+1,j} + 3.2\,\phi_{i,j} + 0.4\,\phi_{i-1,j}$$

Initial conditions.

$\Theta(x,0) = 1$ for $0 < x \leq 1$.

$\Theta(0,t) = 0$

$\Theta(1,t) = 1$.

PTO.

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3.
$$\frac{\partial \phi(x,t)}{\partial t} = \frac{\partial^2 \phi(x,t)}{\partial x^2}.$$

(a) By applying the Forward-Time, Central-Space Scheme (FTCS) we get.

$$\frac{\phi_{i,j+1} - \phi_{i,j}}{\Delta t} = \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2}.$$

By rearranging to allow explicit solving we get:

$$\phi_{i,j+1} = \left(\frac{\Delta t}{\Delta x^2}\right)\phi_{i+1,j} + \left(1 - \frac{2\Delta t}{\Delta x^2}\right)\phi_{i,j} + \left(\frac{\Delta t}{\Delta x^2}\right)\phi_{i-1,j}$$

(b) Solve $\phi(x,t)$ at $t = 0.3$ and $x = 0.5$.
Initial Condition: $\phi(x,0) = 1 \qquad 0 < x \leq 1.$
Boundary Conditions: $\phi(0,t) = 0$
$\phi(1,t) = 1.$

$\Delta x = 0.5$ and $\Delta t = 0.1$
First, we must check for stability.

$$\frac{\Delta t}{\Delta x^2} = \frac{0.1}{0.5^2} = 0.4 \leq 0.5 \quad \therefore \text{ stable } \checkmark$$

$$1 - \frac{2\Delta t}{\Delta x^2} = 1 - \frac{2 \times 0.1}{0.5^2} = 0.2$$

We can write our initial equation as:

$$\phi_{i,j+1} = 0.4\phi_{i+1,j} + 0.2\phi_{i,j} + 0.4\phi_{i-1,j}$$

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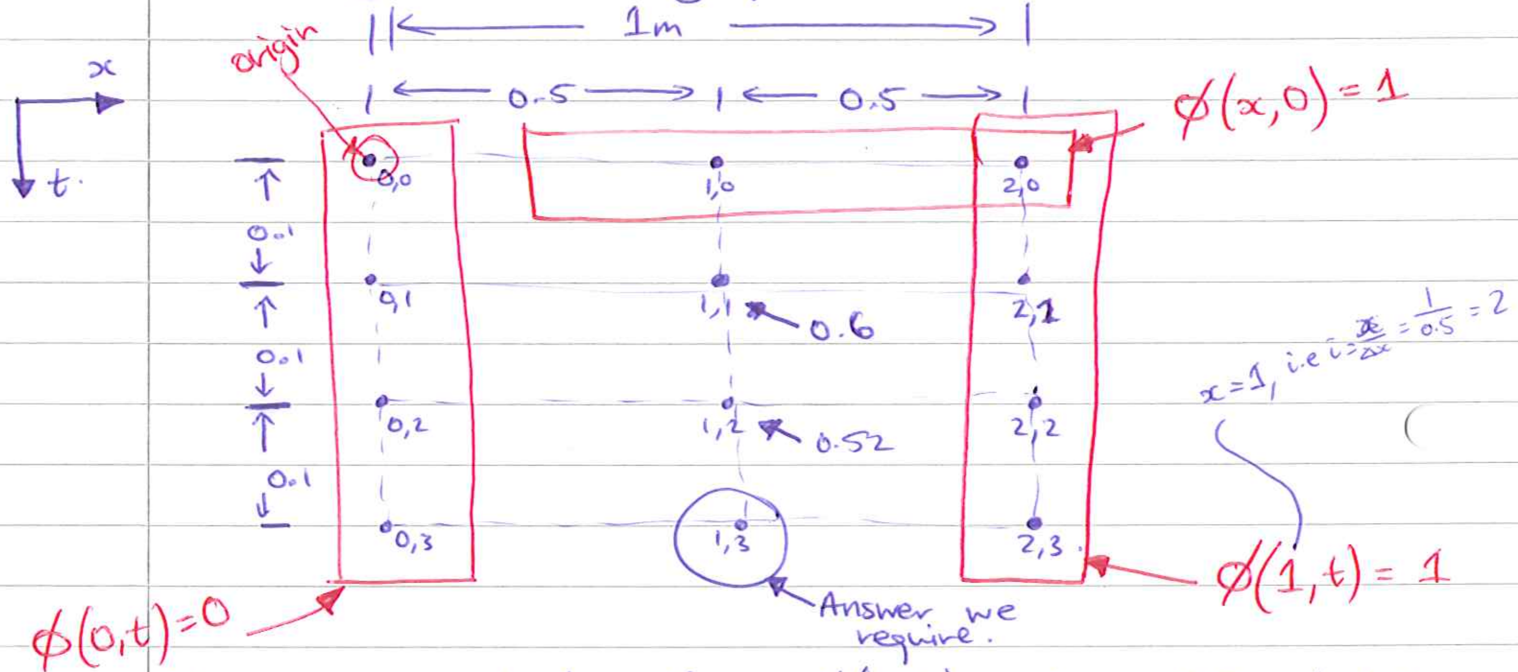
(3.) (b) cont.

Drawing a graphical interpretation



We are asked for $\phi(x,t)$ at $x=0.5$, $t=0.3$

In terms of i & j notation we have

IMPORTANT

$$i=\frac{x}{\Delta x}=\frac{0.5}{0.5}=1.$$

$$j=\frac{t}{\Delta t}=\frac{0.3}{0.1}=3.$$

In other Words, we require $\phi_{i,j}=\underline{\phi_{1,3}}$. (see above)

First We must find $\phi_{1,1}$

Using $i=1$ (displacement) and $j=0$ (time).

$$\phi_{1,1}=0.4\underbrace{\phi_{2,0}}_{1}+0.2\underbrace{\phi_{1,0}}_{1}+0.4\underbrace{\phi_{0,0}}_{0}$$

Applying Initial Conditions.

$$\phi_{1,1}=0.4+0.2=\underline{\underline{0.6}}$$

PTO.

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3. (b) cont.

Next, we need to find $\phi_{1,2}$

Using $i = 1$ and $j = 1$.

$$\phi_{1,2} = 0.4\underbrace{\phi_{2,1}}_{1} + 0.2\underbrace{\phi_{1,1}}_{0.6} + 0.4\underbrace{\phi_{0,1}}_{0}$$

Applying initial / boundary conditions.

$$\phi_{1,2} = 0.4 + 0.12 = \underline{\underline{0.52}}$$

Now we can determine $\phi_{1,3}$.

Using $i = 1$ and $j = 2$.

$$\phi_{i,j+1} = 0.4\phi_{i+1,j} + 0.2\phi_{i,j} + 0.4\phi_{i-1,j}$$

$$\phi_{1,3} = 0.4\underbrace{\phi_{2,2}}_{1} + 0.2\underbrace{\phi_{1,2}}_{0.52} + 0.4\underbrace{\phi_{0,2}}_{0}.$$

Applying initial / boundary conditions

$$\phi_{1,3} = 0.4 + 0.104 = \underline{\underline{0.504}}$$

In conclusion, $\phi(0.5, 0.3) = \phi_{1,3} = \underline{\underline{0.504}}$

(c) If wee were to set $\Delta x = 0.1$ and $\Delta t = 0.1$.

$$\frac{\Delta t}{\Delta x^2} = \frac{0.1}{0.1^2} = 10 \geqslant 0.5.$$

As this value is greater than 0.5, $\underline{\underline{\text{instability}}}$ will result.