Topic and contents

UNSW, School of Mathematics and Statistics

MATH2089 – Numerical Methods

Week 03 – Nonlinear equations - II

- MATLAB M-files
  - nlog2n_newton.m nthroot.m nle1.m nle2.m pltsin.m

Newton's method

- Requires function value $f(x)$ and its derivative $f'(x)$
- First order Taylor series approximation of f about x_k gives

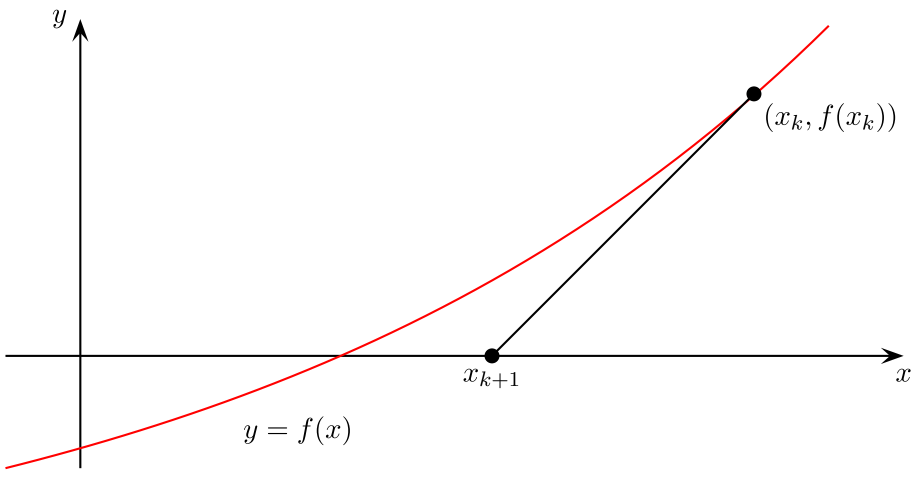
$$f(x) \approx f(x_k) + (x - x_k)f'(x_k).$$

- Choose the next point x_{k+1} to make right-hand-side zero
- Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

  - Assumes that the first derivatives $f'(x_k) \neq 0$
- Fast rate of convergence in the neighbourhood of a simple zero.
- MATLAB M-file nlog2n_newton.m

Geometrical interpretation



Example

Formulate Newton's method for solving $x\log_2(x) = c$ for some constant $c > 0$.

Example

Formulate Newton's method for finding the $n$root of $a > 1$ for some integer $n > 1$.

Secant method

- Difficulties with Newton's method
  - need to calculate derivative $f'(x)$
  - possibly dividing by zero if $f'(x_k) = 0$ for some k.
- Derivatives
  - Symbolic algebra packages: MAPLE, MUPAD, MATHEMATICA
  - Automatic differentiation: computer program: input your program to evaluate $f(x)$; output a program to evaluate the derivative $f'(x)$
  - Finite difference approximation using values at x_k, x_{k-1}

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}.$$

- Secant condition

$$f'(x_k)(x_k - x_{k-1}) = f(x_k) - f(x_{k-1})$$

- Secant method Replace $f'(x_k)$ in Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)\,(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

Secant method cont.

- Secant method

$$x_{k+1} = \frac{f(x_k)x_{k-1} - f(x_{k-1})x_k}{f(x_k) - f(x_{k-1})}$$

  - Only requires function values
  - Equivalent to approximating function by line through

$$(x_{k-1}, f(x_{k-1}))\,, \quad \text{and} \quad (x_k, f(x_k))\,.$$

  - Requires two starting points x_0, x_1

Question Which method is the fastest, which method is the slowest ?

Iterative methods and convergence

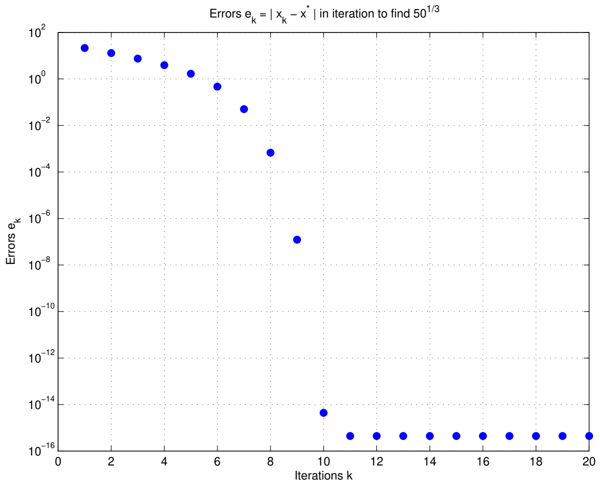
- Iterative methods
  - Starting point x_1
  - Generate a sequence of iterates x_k, $k = 2, 3, \ldots$
  - Notation: $f_k \equiv f(x_k)$
- Convergence $\lim_{k\to\infty} x_k = x^*$ where $f(x^*) = 0.$
- f is continuous then

$$\lim_{k\to\infty} f_k = 0$$

- Practical convergence: tolerances τ
  - $|x_k - x^*| < \tau$ Don't know x^*
  - $|x_{k+1} - x_k| < \tau_x$ convergence $\implies |x_{k+1} - x_k| \to 0$
  - $|f_k| < \tau_f$
  - Maximum number of iterations

Errors

- Errors $e_k = |x_k - x^*|$.
- Convergence as $k \to \infty$: $x_k \to x^* \iff e_k \to 0$



Rate of convergence

- How quickly does $x_k \to x^*$? (or equivalently $e_k \to 0$)

Definition (Order of convergence)

The order of convergence is the largest ν such that

$$\lim_{k\to\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^{\nu}} < \infty$$

- Linear or first order convergence $\iff \nu = 1$

$$\lim_{k\to\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|} = \beta, \text{ where } 0 < \beta < 1$$

  - Rate constant $0 < \beta < 1$ is critical
  - β close to 1, eg $\beta = 0.99$ very slow
  - $\beta = 0.1 \implies$ reduce error by 10 on each iteration.
- Super-linear convergence order $\iff 1 < \nu < 2$
- Quadratic convergence order $\iff \nu = 2$
- MATLAB M-file `nlog2n.m`

Rates of convergence II

- Convergence $e_k \to 0$
- Linear convergence, rate β (order of convergence $\nu = 1$)

$$\frac{e_{k+1}}{e_k} \to \beta \in (0,1), \qquad \frac{e_{k+1}}{(e_k)^2} \to \infty$$

- Super-linear convergence (order of convergence $1 < \nu < 2$)

$$\frac{e_{k+1}}{e_k} \to 0, \qquad \frac{e_{k+1}}{(e_k)^2} \to \infty$$

- Quadratic convergence (order of convergence $\nu = 2$)

$$\frac{e_{k+1}}{e_k} \to 0, \qquad \frac{e_{k+1}}{(e_k)^2} \to K \in (0,\infty), \qquad \frac{e_{k+1}}{(e_k)^3} \to \infty$$

- Example: MATLAB `nlog2n.m`

Convergence of fixed point iteration

Proposition

- *$g \in C([a,b])$ and $g(x) \in [a,b]$ for all $x \in [a,b]$ $\Longrightarrow$ g has a fixed point in $[a,b]$*
- *$g \in C^1((a,b))$ and $\exists K : 0 < K < 1$ with $|g'(x)| \leq K \;\forall\, x \in (a,b)$ $\Longrightarrow$ fixed point iteration converges linearly for any $x_1 \in [a,b]$*

Convergence of Newton's method

Proposition (Convergence of Newton's method)

f twice continuously differentiable, $x^* : f(x^*) = 0$ and $f'(x^*) \neq 0$
x_1 sufficiently close to x^* $\Longrightarrow$ Newton's method is well-defined and converges to x^* with a second order rate of convergence.

Convergence of secant method

Proposition

Suppose f is continuously differentiable, and f has a simple zero x^*, i.e. $f(x^*) = 0$ and $f'(x^*) \neq 0$.

- *The secant method has a super-linear rate of convergence.*
- *Order of convergence is $\nu = (1+\sqrt{5})/2 \approx 1.618$*

Rates of convergence III

Example (Rate of convergence)

If $|x_1 - x^*| \approx 0.1$ estimate how many iterations it will take to get $|x_{k+1} - x^*| < 10^{-14}$ using a method

1. linearly convergent with rate $\beta = 0.99$
2. linearly convergent with rate $\beta = 0.1$
3. super-linearly convergent with order $\nu = 1.5$
4. quadratically convergent
5. third order method

Solution

Example – rates of convergence cont

Solution