UNIVERSITY OF NEW SOUTH WALES
School of Mathematics and Statistics

MATH2089 Numerical Methods and Statistics
Term 2, 2019

Numerical Methods Tutorial – Week 4

1. For each of the following expressions, what is "Big O" $O()$ and what is "Little o" $o()$, as $h \to 0$

   (a) $f_1(h) = 945.2h^2 - 27.6h$

   (b) $f_2(h) = 3.5h^2 + 26.7\sqrt{h}$

2. (a) Give at least two examples of functions of n which are $o(n)$ as $n \to \infty$.

   (b) Give at least two example of functions of n that are $O(n^k)$ for a positive integer k.

3. The central difference approximation of $O(h^2)$ to the second derivative is

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2).$$

   (a) Use Taylor expansions of $O(h^4)$ to both $f(x+h)$ and $f(x-h)$ to derive this formula.

   (b) The rounding error in calculating the difference approximation is $O\left(\frac{\epsilon}{h^2}\right)$. Estimate the optimal value of h that will minimize the sum of the rounding error and the $O(h^2)$ truncation error.

4. Let $f(x) = x^3 - 6x^2 + 11x - 6$.

   (a) Prove that f has at least one zero on the interval $[0, 4]$.

   (b) Is this zero unique?

5. Consider the function

$$f(x) = \begin{cases} -e^{x+1} & \text{if } x < 0; \\ x^2 - x + \frac{1}{2} & \text{otherwise.} \end{cases}$$

   (a) Does f have a zero on $[-1, 1]$?

   (b) **[H]**Write an anonymous function `func` to calculate f for a vector of inputs `x`.
   Hint: What do the following MATLAB commands produce

```
x = linspace(-1, 1, 11)
ans1 = x < 0
ans2 = x >= 0
```

   (c) Plot the function f over $[-2, 2]$ using a grid on 2001 equally spaced points.

6. You want to calculate $a^{\frac{1}{3}}$ where $a > 1$ on a computer using only the basic arithmetic operations of addition, subtraction, multiplication and division.

   (a) Write this problem in the form of finding a zero to a **cubic** polynomial $p(x) = 0$.

   (b) Show that there exists at least one zero of p on $[1, a]$.

   (c) Show that there exists at most one zero of p on $[1, a]$.

   (d) Consider using Newton's method to solve $p(x) = 0$.

i. Show that the iterates can be written as

$$x_{k+1} = \frac{1}{3}\left(2x_k + \frac{a}{x_k^2}\right).$$

ii. What other information does Newton's method require?

iii. The errors in the iterates are $e_k = |x^* - x_k|$ where $x^* = a^{\frac{1}{3}}$. If $e_4 = 2 \times 10^{-4}$, estimate e_5. What if $e_4 = 2 \times 10^{-10}$?

7. For the function $f(x) = (x-1)^3$

(a) What is the zero of f and what is its multiplicity?

(b) Give an initial bracket on a zero?

(c) Perform 2 iterations of Newton's method starting from $x_1 = 2$.

(d) A MATLAB implementation of Newton's method produced

k	e(k)	e(k+1)/e(k)	e(k+1)/e(k)^2	e(k+1)/e(k)^3
1	1.00e+00	6.67e-01	6.67e-01	6.67e-01
2	6.67e-01	6.67e-01	1.00e-00	1.50e+00
3	4.44e-01	6.67e-01	1.50e+00	3.38e+00
4	2.96e-01	6.67e-01	2.25e+00	7.59e+00
5	1.98e-01	6.67e-01	3.38e+00	1.71e+01
6	1.32e-01	6.67e-01	5.06e+00	3.84e+01
7	8.78e-02	6.67e-01	7.59e+00	8.65e+01
8	5.85e-02	6.67e-01	1.14e+01	1.95e+02
9	3.90e-02	6.67e-01	1.71e+01	4.38e+02
10	2.60e-02			

Is the rate of convergence what you expect for Newton's method?

8. Consider using fixed point iteration to find a zero of $f(x) = 2x - \cos(x)$.

(a) Prove that $f(x)$ has a unique zero $[0, 1]$.

(b) Pose this as a fixed point problem $x = g(x)$ (there is one obvious and one slightly less obvious way).

(c) Prove that your fixed point iteration will converge for any starting point in $[0, 1]$.

(d) Perform 2 iterations of fixed point iteration starting from $x_1 = 1/2$.

(e) Write a simple MATLAB script to implement fixed point iteration. (**Hint:** Look at the script `nlog2n.m` discussed in lectures.)

9. For each of the tables of errors $e_k = |x^* - x_k|$ below, answer the following questions

(a) Is the method converging?

(b) What is the order of convergence (linear, superlinear, quadratic)?

(c) Can you trust the last few values?

- Method 1

k	e(k)	e(k+1)/e(k)	e(k+1)/e(k)^2	e(k+1)/e(k)^3
1	7.41e-02	1.15e-01	1.56e+00	2.10e+01
2	8.53e-03	8.41e-03	9.86e-01	1.16e+02
3	7.18e-05	7.41e-05	1.03e+00	1.44e+04
4	5.32e-09	7.34e-09	1.38e+00	2.60e+08
5	3.91e-17			

- Method 2

k	e(k)	e(k+1)/e(k)	e(k+1)/e(k)^2	e(k+1)/e(k)^3
1	1.00e-01	1.49e-01	1.49e+00	1.49e+01
2	1.49e-02	8.24e+00	5.52e+02	3.69e+04
3	1.23e-01	1.31e-01	1.07e+00	8.66e+00
4	1.61e-02	2.26e+01	1.40e+03	8.68e+04
5	3.65e-01	4.29e-01	1.18e+00	3.23e+00
6	1.57e-01	3.05e-01	1.95e+00	1.24e+01
7	4.78e-02	6.31e+00	1.32e+02	2.76e+03
8	3.01e-01	3.42e-01	1.14e+00	3.77e+00
9	1.03e-01	5.07e-02	4.92e-01	4.78e+00
10	5.23e-03	4.21e+01	8.04e+03	1.54e+06
11	2.20e-01	4.12e-01	1.88e+00	8.53e+00
12	9.07e-02	3.53e-01	3.89e+00	4.29e+01
13	3.20e-02	2.63e+00	8.21e+01	2.57e+03
14	8.40e-02	5.58e-01	6.65e+00	7.91e+01
15	4.69e-02	1.29e+00	2.76e+01	5.88e+02
16	6.07e-02	2.34e+00	3.85e+01	6.33e+02
17	1.42e-01	1.54e-01	1.09e+00	7.67e+00
18	2.19e-02	4.78e+00	2.18e+02	9.95e+03
19	1.05e-01	7.19e-02	6.87e-01	6.56e+00
20	7.53e-03			

- Method 3

k	e(k)	e(k+1)/e(k)	e(k+1)/e(k)^2	e(k+1)/e(k)^3
1	1.76e-01	1.84e+00	1.05e+01	5.95e+01
2	3.24e-01	2.91e-01	8.96e-01	2.77e+00
3	9.41e-02	1.41e+00	1.50e+01	1.59e+02
4	1.33e-01	8.86e-02	6.67e-01	5.03e+00
5	1.18e-02	9.82e-02	8.35e+00	7.11e+02
6	1.15e-03	1.26e-02	1.09e+01	9.43e+03
7	1.45e-05	1.20e-03	8.24e+01	5.68e+06
8	1.74e-08	1.50e-05	8.64e+02	4.98e+10
9	2.60e-13	1.50e-04	5.77e+08	2.22e+21
10	3.91e-17			