FAMILY NAME: ......Solution......
OTHER NAME(S):..........................
STUDENT NUMBER: ....
SIGNATURE: ....

THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Example Class Test 3

MATH2089
Numerical Methods Example Class Test 3

(1) TIME ALLOWED – 50 minutes

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) THIS PAPER MAY **NOT** BE RETAINED BY THE CANDIDATE

(6) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(7) Write your answers on this test paper in the space provided.
Ask your tutor if you need more paper.

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

1. a) **[4 marks]** Give the results of the following MATLAB commands when executed on a computer:

```
a = [-1:1]
b = a./(a - a.^2)
```

Answer:

$a = [-1 \;\; 0 \;\; 1]$

$b = [-1 \; 0 \; 1]./([-1 \;\; 0 \;\; 1] - [1 \;\; 0 \;\; 1])$

$= [-1 \; 0 \; 1]./[-2 \;\; 0 \;\; 0]$

$= [0.5 \;\; \mathrm{NaN} \;\; \mathrm{Inf}]$

b) **[3 marks]** A value V with $|V| \approx 800$ is measured to 5 significant figures.

i) Give a bound on the relative error in V?
Answer:

$\text{rel-err}(V) < 0.5 \times 10^{-5}$

ii) Estimate the absolute error in V.
Answer:

$\text{abs.err}(V) = \text{rel-err}(V)|V|$

$< 0.5 \times 10^{-5} \times 800$

$= 400 \times 10^{-5}$

$= 0.4 \times 10^{-2}$

c) A function f with $n+1$ continuous derivatives has Taylor series expansion

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\zeta).$$

where $\zeta \in [x, x+h]$.

i) ([**1 mark**])

Write down the Taylor polynomial approximation including terms of $O(h^2)$ to $f(x-2h)$.

Answer:

$$f(x-2h) = f(x) - 2hf'(x) + \frac{(2h)^2}{2!}f''(x) + O(h^3)$$

Taylor polynomial

$$P = f(x) - 2hf'(x) + 2h^2 f''(x)$$

ii) [**2 marks**] A student tries to approximate the first derivative by

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

with $h = 10^{-16}$ and $x = 202$ and finds the derivative is zero. Is this accurate? Give reasons for your answer.

Answer:

No, since h is too small

$(h < eps)$

$$f(x+h) = f(x-h)$$

$$f' \approx 0$$

2. a) **[4 marks]** The computational complexity of some common operations with n by n matrices are given in the Table below.

Operation	Flops
Matrix multiplication	$2n^3$
LU factorization	$\frac{2n^3}{3} + O(n^2)$
Cholesky factorization	$\frac{n^3}{3} + O(n^2)$
Back/forward substitution	$n^2 + O(n)$
Tridiagonal solve	$8n + O(1)$

You have a 3GHz quad core computer where each core can do one floating point operation per clock cycle. Estimate the size n of the largest n by n linear system $A\mathbf{x} = \mathbf{b}$ with a symmetric positive definite coefficient matrix A that can be solved in one minute on this computer.

Answer:

In 1 minute, the computer can do

$$3\text{GHz} \times \underbrace{4}_{\text{quad core}} \times 1 \times \underbrace{60}_{1 \text{ min}}$$

$$= 12 \times 60 \times 10^9 \text{ flops}$$

A symmetric positive definite $\Rightarrow$ Cholesky factorization is used. Back/forward substitutions are used.

$$\frac{n^3}{3} + 2n^2 = 12 \times 60 \times 10^9$$

Ignore n^2

$$n^3/3 = 12 \times 60 \times 10^9$$

$$n = (3 \times 12 \times 60 \times 10^9)^{1/3} = 12926.1$$

Rounding to the nearest integer

Please see over ...

$$n = 12927$$

b) The matrix A has factorization

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 4 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}.$$

i) **[2 marks]** Show that A is symmetric (Hint: You do **not** need to find A)
Answer:

$A = R^T R \qquad R = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

$A^T = (R^T R)^T = R^T R^{TT} = R^T R = A$

A is symmetric.

ii) **[2 marks]** Is A positive definite? Give reasons.
Answer:

A has a Cholesky factorization
$\Rightarrow A$ is positive definite.

iii) **[2 marks]** Calculate the determinant of A.
Answer:

$$\begin{aligned} \det(A) &= \det(R^T)\det(R) \\ &= \det(R)\det(R) \\ &= [\det(R)]^2 = (4 \times 2 \times 3)^2 \\ &= 24^2 = 576 \end{aligned}$$

3. a) **[4 marks]** Give MATLAB commands for **EITHER** an anonymous function `myfun2` **OR** a function M-file `myfun2.m` to calculate

$$f(x) = \frac{1}{1 + e^{-x^2}}.$$

Your function should work for an array of inputs `x`, producing an array of output values of the same size.

Answer:

```
myfun2 = @(x) 1./(1 + exp(-x.^2));
```

OR

```
function y=myfun2(x)
y = 1./(1+ exp(-x.^2));
```

b) Consider the function $f(x) = -x^3 + 6x^2 - 11x + 6$.

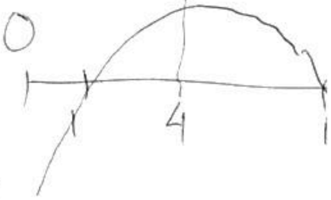
i) **[2 marks]** Prove that f has at least one zero in the interval $[0, 4]$?
Answer:

- $f(0) = 6 > 0$
- $f(4) = -4^3 + 6 \times 4^2 - 11 \times 4 + 6 = -6 < 0$

$\left.\begin{array}{l} f \text{ is continuous} \\ f(0)\, f(4) < 0 \end{array}\right\} \Rightarrow f$ has at least 1 zero in $[0,4]$

ii) **[2 marks]** Is f strictly increasing or strictly decreasing on $[0, 4]$?
Answer:

$$f' = -3x^2 + 12x - 11$$
$$= -(x-1)(x-11)$$



f' change the sign in $[0,4]$

f is neither strictly incr nor decr in $[0,4]$

c) **[2 marks]** You are asked to compare using Newton's method and the Secant method to solve a very complex nonlinear equation $f(x) = 0$.

i) Give one reason why Newton's method may be preferable.
Answer:

• quadratic convergence.

ii) Give one reason why the Secant method may be preferable.
Answer:

• secant does not require derivatives.

4. You are given the results of the following MATLAB commands and the spy plots in Figure 4.1.

```
size(A)
ans =
  60     20
B = A'*A;
Bev = eig(B);
Bevmin = min(Bev)
Bevmin =
  0.2424
Bevmax = max(Bev)
Bevmax =
  5.7770

[Q, R] = qr(A);
```

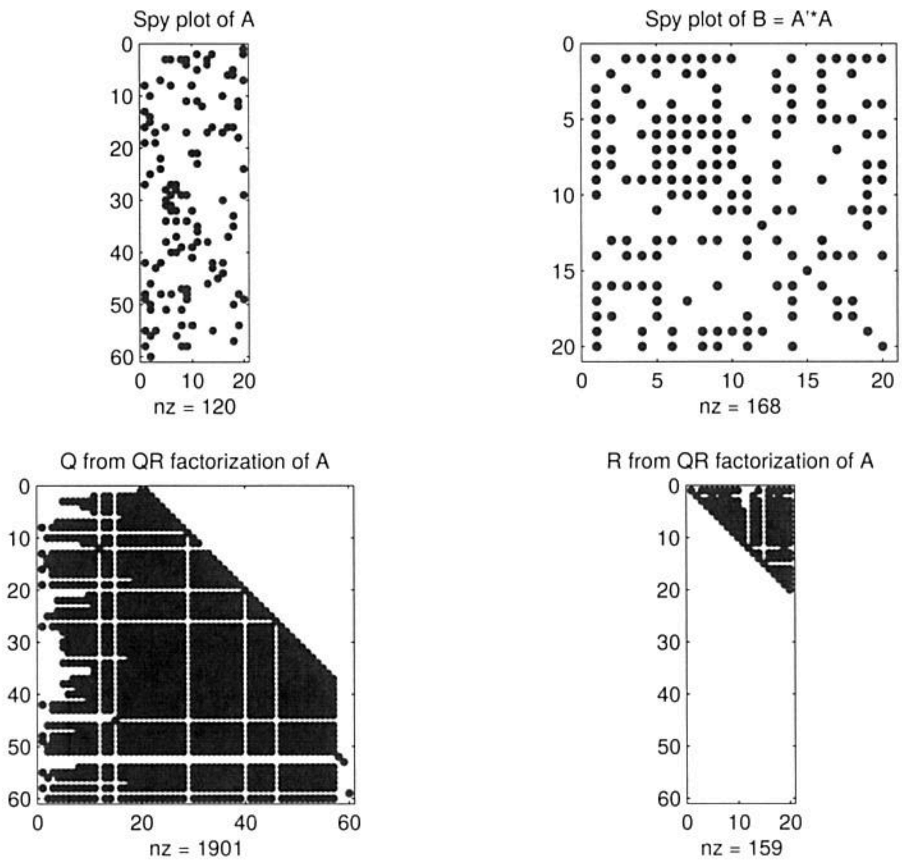


Figure 4.1: Spy plot of A, $B = A^T A$ and matrices Q, R from QR factorization

a) **[2 marks]** A student claims the matrix $B = A^T A$ is not symmetric. Justify or refute this claim.

Answer:

$$B^T = (A^T A)^T = A^T A^{TT} = A^T A = B$$

$\Rightarrow$ B is symmetric

b) **[1 mark]** Is B positive definite? Give reasons for your answer.

Answer:

$\lambda_{min}(B) = 0.2424 > 0$

$\Rightarrow$ all eigenvalues are positive

$\Rightarrow$ B positive definite.

c) **[2 marks]** Calculate the condition number $\kappa_2(B)$.

Answer:

B is symmetric

$$\kappa_2(B) = \frac{|\lambda_{max}(B)|}{|\lambda_{min}(B)|} = \frac{5.777}{0.2424}$$

$$= 23.8325$$

d) **[3 marks]** You want to guarantee 6 significant figures in the computed solution $\mathbf{x}$ to $B\mathbf{x} = \mathbf{b}$, where $B = A^T A$ has been calculated "exactly". Estimate the relative error in the right-hand-side vector $\mathbf{b}$ that would be required to achieve this.

Answer:

$$\text{rel-err}(x) \approx \kappa_2(B)\left[\text{rel-err}(B) + \text{rel-err}(b)\right]$$

$$0.5 \times 10^{-6} \approx 23.83\ [2.2 \times 10^{-16} + \text{rel-err}(b)$$

$$\Rightarrow \text{rel-err}(b) \approx \frac{0.5 \times 10^{-6}}{23.83} \approx 0.21 \times 10$$

e) **[2 marks]** The least squares solution to $A\mathbf{x} = \mathbf{b}$ can be calculated using either the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$ or using the QR factorization.

i) Give one advantage of using the QR factorization.

ii) Give one disadvantage of using the QR factorization.

Answer:

i) $\kappa(Q) = 1$, $\|A\|_2 = \|QA\|_2$

good condition number in solving the linear system.

ii) QR is hard to compute.