UNIVERSITY OF NEW SOUTH WALES
School of Mathematics and Statistics

MATH2089 Numerical Methods and Statistics
Term 2, 2019

Numerical Methods Tutorial – Week 10 Solutions

Initial Value Problems

1. Use Euler's method, using the given step size h, to solve the following IVPs. Compute the actual errors at each time step.

   (a) $y' = te^{3t} - 2y, \quad t \in (0, 1], \quad y(0) = 0, \quad h = 0.5$
   - Solution: $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$

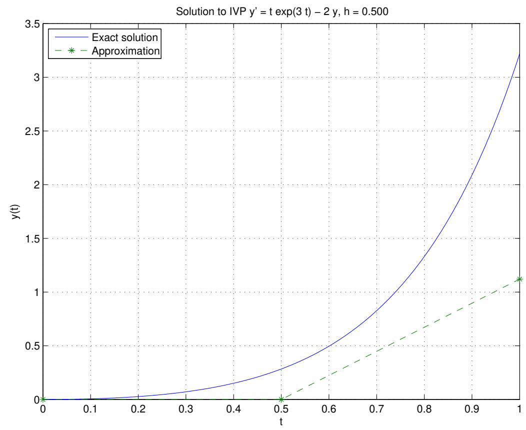
   (b) $y' = 1 + \dfrac{y}{t}, \quad t \in (1, 2], \quad y(1) = 2, \quad h = 0.25$
   - Solution: $y(t) = t\log(t) + 2t$

   (c) $y' = \dfrac{2}{t}y + t^2e^t, \quad t \in (1, 2], \quad y(1) = 0, \quad h = 0.1$
   - Solution: $y(t) = t^2(e^t - e)$

Answer

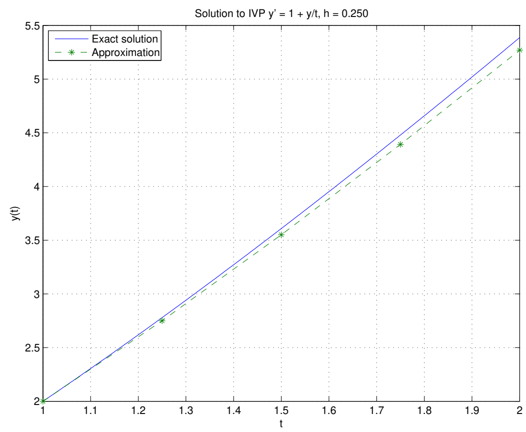
(a) With $t_0 = 0$, $t_f = 1$ and $h = 0.5$ we will have $N = 2$ steps of Euler's method $y_{n+1} = y_n + hf_n$ where $f_n = f(t_n, y_n)$. The errors are $E_n = y(t_n) - y_n$.

n	t_n	y_n	f_n	E_n
0	0.0	0.0000	0	0
1	0.5	0.0000	2.2408	0.2836
2	1.0	1.1204	17.845	2.0987



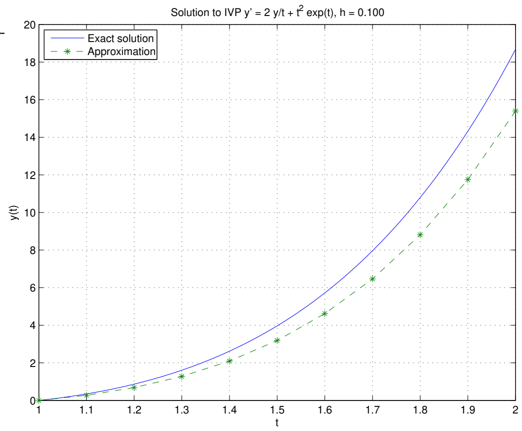
(b) With $t_0 = 1$, $t_f = 2$ and $h = 0.25$ we will have $N = 4$ steps of Euler's method. Note that the initial value is at $t_0 = 1$ in this example.

n	t_n	y_n	f_n	E_n
0	1.00	2.0000	3.0000	0
1	1.25	2.7500	3.2000	0.0289
2	1.50	3.5500	3.3667	0.0582
3	1.75	4.3917	3.5095	0.0877
4	2.00	5.2690	3.6345	0.1172



(c) With $t_0 = 1$, $t_f = 2$ and $h = 0.1$ we will have $N = 10$ steps of Euler's method.

n	t_n	y_n	f_n
0	1.0	0.0000e+00,	2.7183e+00
1	1.1	2.7183e-01,	4.1293e+00
2	1.2	6.8476e-01,	5.9222e+00
3	1.3	1.2770e+00,	8.1657e+00
4	1.4	2.0935e+00,	1.0939e+01
5	1.5	3.1874e+00,	1.4334e+01
6	1.6	4.6208e+00,	1.8456e+01
7	1.7	6.4664e+00,	2.3427e+01
8	1.8	8.8091e+00,	2.9389e+01
9	1.9	1.1748e+01,	3.6502e+01
10	2.0	1.5398e+01,	4.4954e+01



See the various examples in the MATLAB M-file `ivpmain.m` available from the course web page. You can also try other methods, Heun, RK4, by changing which lines are commented out.

2. Consider the Initial Value Problem (IVP)

$$y' = f(t, y), \quad t > t_0, \qquad y(t_0) = y_0.$$

At step n we know t_n, $y_n \approx y(t_n)$ and $t_{n+1} = t_n + h$. A ν-stage explicit Runge-Kutta method with parameters a_{ij}, b_j, c_j is

$$\begin{aligned}
\xi_1 &= y_n \\
\xi_2 &= y_n + ha_{2,1} f(t_n + c_1 h, \xi_1) \\
\xi_3 &= y_n + ha_{3,1} f(t_n + c_1 h, \xi_1) + ha_{3,2} f(t_n + c_2 h, \xi_2) \\
&\vdots \\
\xi_\nu &= y_n + h \sum_{i=1}^{\nu-1} a_{\nu,i} f(t_n + c_i h, \xi_i)
\end{aligned}$$

Then the next approximation $y_{n+1} \approx y(t_{n+1})$ is

$$y_{n+1} = y_n + h \sum_{j=1}^{\nu} b_j f(t_n + c_j h, \xi_j).$$

The classical fourth order four-stage Runge-Kutta method RK4 is defined by the following tableau

$$\begin{array}{c|c} \mathbf{c} & A \\ \hline & \mathbf{b}^T \end{array} \quad = \quad \begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \end{array}$$

(a) Write down the formulae to define y_{n+1} from y_n for this method.

(b) Use the formulae obtained above (with $h = 0.2$) to compute an approximation to $y(0.2)$, where y satisfies the IVP

$$y' = \frac{y + t}{y - t}, \qquad y(0) = 1.$$

Answer

(a) RK4 is a $\nu = 4$ stage method. Using the given A, $\mathbf{b}$, $\mathbf{c}$ gives

$$\begin{aligned}
\xi_1 &= y_n \\
\xi_2 &= y_h + h\,\frac{1}{2}\,f(t_n, \xi_1) \\
\xi_3 &= y_n + h\,\frac{1}{2}\,f(t_n + \frac{h}{2}, \xi_2) \\
\xi_4 &= y_n + h\,1\,f(t_n + \frac{h}{2}, \xi_3) \\
y_{n+1} &= y_n + h\left(\frac{1}{6}\,f(t_n, \xi_1) + \frac{2}{6}\,f(t_n + \frac{h}{2}, \xi_2) + \frac{2}{6}\,f(t_n + \frac{h}{2}, \xi_3) + \frac{1}{6}\,f(t_n + h, \xi_4)\right)
\end{aligned}$$

Note that the function values $f(t_n, \xi_1)$, $f(t_n + \frac{h}{2}, \xi_2)$, $f(t_n + \frac{h}{2}, \xi_3)$ used in calculating the ξ_i values are **also** used in calculating y_{n+1}.

(b) Starting at $n = 0$, $t_0 = 0$, $y_0 = 1$, a step of $h = 0.2$ gives $t_1 = t_0 + h = 1.2$. The RHS of the ODE is given by $f(t, y) = \frac{y+t}{y-t}$, so

$$\begin{aligned}
\xi_1 &= y_0 = 1 \\
f_1 &= f(t_0, \xi_1) = f(0, 1) = \left(\frac{1+0}{1-0}\right) = 1 \\
\xi_2 &= y_0 + \frac{h}{2} f(t_0, \xi_1) = 1 + 0.1\ f_1 = 1.1 \\
f_2 &= f(t_0 + \frac{h}{2}, \xi_2) = f(0.1, 1.1) = \left(\frac{1.1 + 0.1}{1.1 - 0.1}\right) = 1.2 \\
\xi_3 &= y_0 + \frac{h}{2} f(t_0 + \frac{h}{2}, \xi_2) = 1 + 0.1\ f_2 = 1.12 \\
f_3 &= f(t_0 + \frac{h}{2}, \xi_3) = f(0.1, 1.12) = \left(\frac{1.12 + 0.1}{1.12 - 0.1}\right) = 1.19607843\ldots \\
\xi_4 &= y_0 + h f(t_0 + \frac{h}{2}, \xi_3) = 1 + 0.2\ f_3 = 1.23921568\ldots \\
f_4 &= f(t_0 + h, \xi_4) = f(0.2, 1.23921568) = 1.38490566\ldots \\
y_1 &= y_0 + \frac{h}{6}\left(f(t_0, \xi_1) + 2f(t_0 + \frac{h}{2}, \xi_2) + 2f(t_0 + \frac{h}{2}, \xi_3) + f(t_0 + h, \xi_4)\right) \\
&= 1 + \frac{0.2}{6}\,(f_1 + 2f_2 + 2f_3 + f_4) \\
&= 1.2392354\ldots
\end{aligned}$$

Partial Differential Equations

3. Consider Poisson's equation

$$\nabla^2 u(\mathbf{x}) = f(\mathbf{x})$$

on the rectangular domain

$$\Omega = \left\{\mathbf{x} \in \mathbb{R}^2 : 0 \le x \le L_x, \quad 0 \le y \le L_y\right\}.$$

Divide the x interval $[0, L_x]$ into $m + 1$ equal length subintervals

$$0 = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m < x_{m+1} = L_x,$$

$$x_i = i\,h_x, \quad i = 0, \ldots, m+1, \qquad h_x = \frac{L_x}{m+1}.$$

Divide the y interval $[0, L_y]$ into $n+1$ equal length subintervals

$$0 = y_0 < y_1 < y_2 < \cdots < y_{n-1} < y_n < y_{n+1} = L_y,$$

$$y_j = j\ h_y, \quad j = 0, \ldots, n+1, \qquad h_y = \frac{L_y}{n+1}.$$

Let $u_{ij} \approx u(x_i, y_j)$

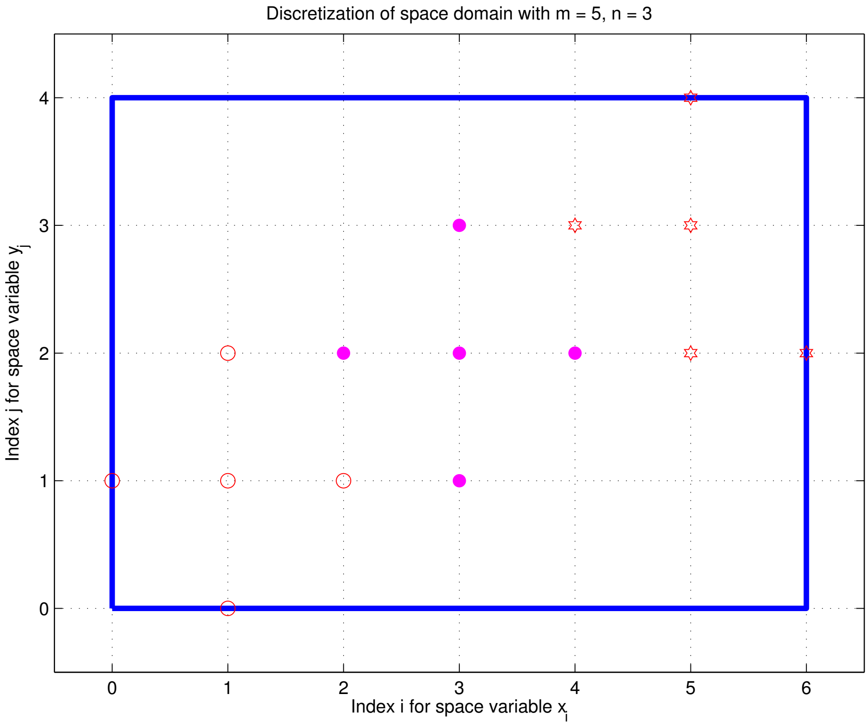


Figure 1: Grid for $m = 5$ and $n = 3$

(a) Is this an elliptic, parabolic or hyperbolic PDE?
Answer The governing partial differential equation is Poisson's equation in two space variables $\mathbf{x} = (x, y)^T$, that is

$$\nabla^2 u(\mathbf{x}) = \frac{\partial^2 u(\mathbf{x})}{\partial x^2} + \frac{\partial^2 u(\mathbf{x})}{\partial y^2} = f(\mathbf{x}).$$

Thus in the notation from lectures the coefficients of the highest order derivatives are $A = 1$, $2B = 0$ (as there are no mixed derivatives $\frac{\partial^2 u(\mathbf{x})}{\partial x \partial y}$) and $C = 1$, giving $AC - B^2 = 1 \times 1 - 0 = 1 > 0$, so this is an elliptic equation.

(b) What problem could this model?
Answer The heat equation for two space variables is

$$\frac{\partial u(x,y,t)}{\partial t} = D\left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right).$$

The steady state version, with no time dependence, so $\frac{\partial u}{\partial t} = 0$, gives the 2-D Laplace's equation

$$\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} = 0.$$

Here, instead of a right-hand-side of 0, we have a forcing term $f(\mathbf{x})$ which depends on the position $\mathbf{x} \in \Omega$.

(c) What else is needed to completely specify the problem?
Answer To fully specify a problem you need to specify

- The space domain Ω and the time domain, typically $[0, T]$;
- The governing partial differential equation;
- The boundary conditions on $\partial\Omega$.
- The initial conditions $u(\mathbf{x}, 0)$ at $t = 0$.

This is a steady state problem with no time dependence, so the time domain and initial conditions are not required.
The space domain Ω and the PDE have been specified, but you also need the boundary conditions on

$$\partial\Omega = \{\mathbf{x} \in \mathbb{R}^2 : x = 0 \text{ or } x = L_x, \text{ or } y = 0 \text{ or } L_y\}.$$

(d) At the grid point x_i, y_j use central difference approximations of $O(h^2)$ to the second derivatives to derive an approximation to Poisson's equation.
Answer At the grid point x_i, y_j

$$\begin{aligned} \left.\frac{\partial^2 u(\mathbf{x})}{\partial x^2}\right|_{x_i, y_j} &= \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2} + O(h_x^2), \\ \left.\frac{\partial^2 u(\mathbf{x})}{\partial y^2}\right|_{x_i, y_j} &= \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2} + O(h_y^2). \end{aligned}$$

Ignoring the $O(h_x^2)$ and $O(h_y^2)$ terms and substituting in the governing Poisson equation gives the approximation

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2} = f_{i,j}$$

where $f_{i,j} = f(\mathbf{x})$ evaluated at the grid point $\mathbf{x} = (x_i, y_j)^T$. A very common mistake is to forget that the forcing term $f(\mathbf{x})$ must be evaluated at the grid point where the approximation is being made.

(e) Consider a problem with Dirichlet boundary conditions $u(\mathbf{x}) = 20$ for $\mathbf{x} \in \partial\Omega$. For $L_x = 3, m = 5$ and $L_y = 2$, $n = 3$

i. Give the equation at the grid point (x_3, y_2)

ii. Give the equation at the grid point (x_5, y_3)

iii. generate the linear system $A\mathbf{u} = \mathbf{b}$ corresponding to a row-ordering of the variables.

iv. Write the coefficient matrix A as the block matrix

$$A = \begin{bmatrix} B & -I & & & & & \\ -I & B & -I & & & & \\ & -I & B & -I & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -I & B & -I & \\ & & & & -I & B & -I \\ & & & & & -I & B \end{bmatrix}$$

What are the matrices B and I.

v. Say what you can about the structure of the coefficient matrix A that could make solving the linear system $A\mathbf{u} = \mathbf{b}$ more efficient.

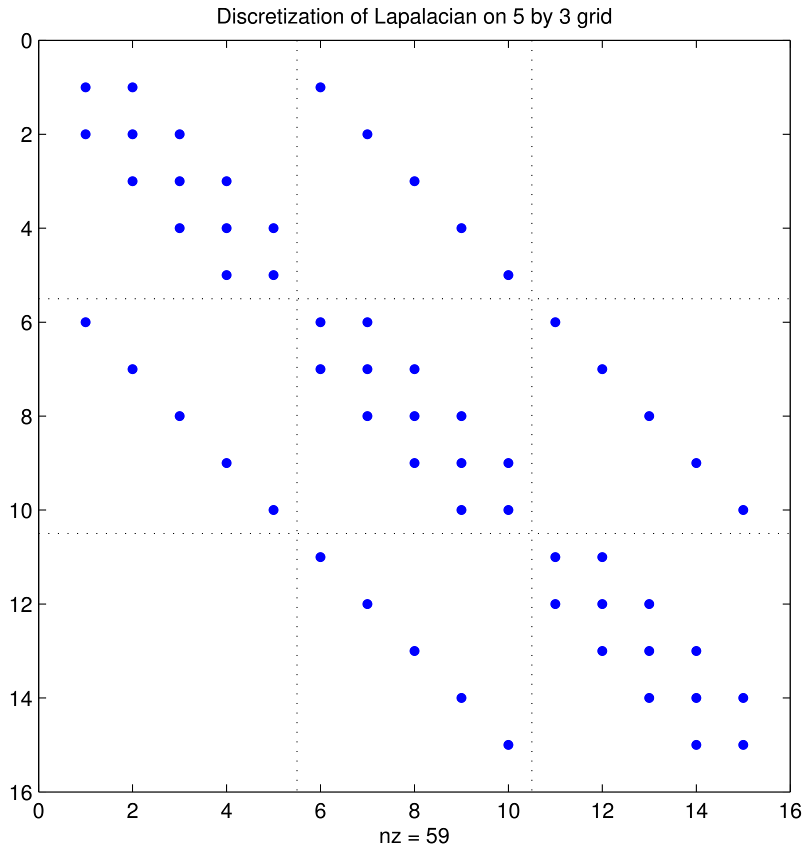


Figure 2: Spy plot of coefficient matrix for $m = 5$ and $n = 3$

Answer For $L_x = 3, m = 5$ and $L_y = 2$, $n = 3$, which is illustrated in Figure 1, the grid spacings are

$$h_x = \frac{L_x}{m+1} = \frac{3}{5+1} = \frac{1}{2}, \qquad h_y = \frac{L_y}{n+1} = \frac{2}{3+1} = \frac{1}{2},$$

so let $h = \frac{1}{2}$ be the common grid spacing. The finite difference approximation to Poisson's equation then simplifies, after multiplying through by $-h^2$, to

$$4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = -h^2 f_{i,j}.$$

i. At the grid point (x_3, y_2), so $i = 3$ and $j = 2$, the five point stencil for the function values in the finite difference approximation lies in the interior of the domain Ω (see Figure 1), thus the approximation is

$$4u_{3,2} - u_{4,2} - u_{2,2} - u_{3,3} - u_{3,1} = -h^2 f_{3,2}.$$

ii. At the grid point (x_5, y_3), so $i = 5$ and $j = 3$, the approximation is

$$4u_{5,3} - u_{6,3} - u_{4,3} - u_{5,4} - u_{5,2} = -h^2 f_{5,3}.$$

However values with $i = 0$ or $i = 6$ or $j = 0$ or $j = 4$ lie on the boundary of the domain, so are determined by the boundary conditions. Thus $u_{6,2} = 20$ and $u_{5,4} = 20$, giving

$$4u_{5,3} - u_{4,3} - u_{5,2} = -h^2 f_{5,3} + 40.$$

iii. A row ordering, so the index i varies first, of the variables gives the vector of unknowns $\mathbf{u} \in \mathbb{R}^{15}$

$$\mathbf{u} = (u_{1,1}, u_{2,1}, u_{3,1}, u_{4,1}, u_{5,1}, u_{1,2}, u_{2,2}, u_{3,2}, u_{4,2}, u_{5,2}, u_{1,3}, u_{2,3}, u_{3,3}, u_{4,3}, u_{5,3})^T.$$

Taking special care with the boundary values, the system of equations is, with $h = \frac{1}{2}$,

$$\begin{aligned}
4u_{1,1} - u_{2,1} - u_{1,2} &= -h^2 f_{1,1} + 40 \\
4u_{2,1} - u_{3,1} - u_{1,1} - u_{2,2} &= -h^2 f_{2,1} + 20 \\
4u_{3,1} - u_{4,1} - u_{2,1} - u_{3,2} &= -h^2 f_{3,1} + 20 \\
4u_{4,1} - u_{5,1} - u_{3,1} - u_{4,2} &= -h^2 f_{4,1} + 20 \\
4u_{5,1} - u_{4,1} - u_{5,2} &= -h^2 f_{5,1} + 40 \\
4u_{1,2} - u_{2,2} - u_{1,3} - u_{1,1} &= -h^2 f_{1,2} + 20 \\
4u_{2,2} - u_{3,2} - u_{1,2} - u_{2,3} - u_{2,1} &= -h^2 f_{2,2} \\
4u_{3,2} - u_{4,2} - u_{2,2} - u_{3,3} - u_{3,1} &= -h^2 f_{3,2} \\
4u_{4,2} - u_{5,2} - u_{3,2} - u_{4,3} - u_{4,1} &= -h^2 f_{4,2} \\
4u_{5,2} - u_{4,2} - u_{5,3} - u_{5,1} &= -h^2 f_{5,2} + 20 \\
4u_{1,3} - u_{2,3} - u_{1,2} &= -h^2 f_{1,3} + 40 \\
4u_{2,3} - u_{3,3} - u_{1,3} - u_{2,2} &= -h^2 f_{2,3} + 20 \\
4u_{3,3} - u_{4,3} - u_{2,3} - u_{3,2} &= -h^2 f_{3,3} + 20 \\
4u_{4,3} - u_{5,3} - u_{3,3} - u_{4,2} &= -h^2 f_{4,3} + 20 \\
4u_{5,3} - u_{4,3} - u_{5,2} &= -h^2 f_{5,3} + 40.
\end{aligned}$$

The four equations with the $+40$ on the right-hand-side correspond to the nodes in the corners where both the side and top/bottom boundary values have an effect.

Thus the coefficient matrix $A \in \mathbb{R}^{15\times 15}$ is

$$A = \begin{bmatrix}
4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4
\end{bmatrix}$$

The spy plot of the coefficient matrix A is given in Figure 2. The right-hand-side vector $\mathbf{b} \in \mathbb{R}^{15}$ is

$$\begin{aligned}
\mathbf{b} \;=\; & (-h^2 f_{1,1} + 40,\ -h^2 f_{2,1} + 20,\ -h^2 f_{3,1} + 20,\ -h^2 f_{4,1} + 20,\ -h^2 f_{5,1} + 40, \\
& -h^2 f_{1,2} + 20,\ -h^2 f_{2,2},\ -h^2 f_{3,2},\ -h^2 f_{4,2},\ -h^2 f_{5,2} + 20, \\
& -h^2 f_{1,3} + 40,\ -h^2 f_{2,3} + 20,\ -h^2 f_{3,3} + 20,\ -h^2 f_{4,3} + 20,\ -h^2 f_{5,3} + 40)^T.
\end{aligned}$$

iv. The matrices that make up the 3 by 3 block matrix

$$A = \begin{bmatrix} B & -I & \\ -I & B & -I \\ & -I & B \end{bmatrix}$$

are the 5 by 5 tridiagonal matrix B and the 5 by 5 identity matrix I given by

$$B = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(f) The coefficient matrix A has the following structure

- symmetric, as $A^T = A$ as the central difference stencil is symmetric.
- banded, with lower bandwidth m_ℓ = upper bandwidth $m_u = 5$ (the number of unknowns across a row).
- sparse, as only 59 of the possible $15^2 = 225$ elements are non-zero, for a sparsity of $100 \times 59/225 = 26.2\%$.
- positive definite, although this is not immediate from what we have done in the course. You could check using the Cholesky factorization or looking at the eigenvalues.
- not Toeplitz, as the $+1$ and -1 diagonals are not constant with most values equal to -1 but some values 0, corresponding to the start or end of a row in the discretization of the domain.