UNIVERSITY OF NEW SOUTH WALES
School of Mathematics and Statistics

MATH2089 Numerical Methods and Statistics
Term 2, 2019

Numerical Methods Laboratory – Week 5

1. Let $\mathbf{x} = (5, -4, 0, -6)^T$.

   (a) Calculate $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$ and $\|\mathbf{x}\|_\infty$ by hand.

   (b) Use MATLAB to check your answers.

2. Let

$$A = \begin{bmatrix} 3 & -4 & 2 \\ 0 & 4 & -5 \\ 2 & -2 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

   (a) If possible, calculate $\|A\|_1$, $\|A\|_2$ and $\|A\|_\infty$ by hand.

   (b) Use MATLAB to check your answers.

   (c) Use MATLAB's backslash \ to solve the linear system $A\mathbf{x} = \mathbf{b}$.

   (d) Calculate the residual $\mathbf{r} = \mathbf{b} - A\mathbf{x}$. Is $\mathbf{r} = \mathbf{0}$?

   (e) Use the MATLAB function `lu` to calculate the LU factorization of A.

3. Consider the matrices

$$A = \begin{bmatrix} 0 & 3 & -2 \\ -1 & -4 & 2 \\ 5 & 14 & 26 \end{bmatrix}, \qquad B = \begin{bmatrix} -11/8 & -53/48 & -1/48 \\ 3/8 & 5/48 & 1/48 \\ 1/16 & 5/32 & 1/32 \end{bmatrix}$$

   (a) Verify that $B = A^{-1}$ by showing that $AB = I$. What is $||AB - I||_1$?

   (b) Calculate $\|A\|_1$, $\|A^{-1}\|_1$ and $\kappa_1(A)$ by hand and check using MATLAB.

   (c) Calculate $\|A\|_\infty$, $\|A^{-1}\|_\infty$ and $\kappa_\infty(A)$ by hand and check using MATLAB.

   (d) You want to calculate the 2-norm condition number $\kappa_2(A) = \|A\|_2 \, \|A^{-1}\|_2$.

      i. Use the MATLAB function `eig` to calculate the eigenvalues of A and A^{-1}

      ii. How are the eigenvalues of A and A^{-1} related?

      iii. Calculate $|\lambda_{\max}|/|\lambda_{\min}|$

      iv. Use MATLAB's `cond` to calculate $\kappa_2(A)$. Is this the same as in the previous part?

   (e) Use row operations to reduce A to row-echelon form (by hand or using MATLAB just to do each row operation)

   (f) The LU factorization produces a lower triangular matrix L and an upper triangular matrix U such that

$$PA = LU$$

   where P is a permutation matrix reordering the rows of A (equations in a linear system).

      i. Use the MATLAB function `lu` to calculate the matrices P, L, and U.

      ii. Calculate $E = PA - LU$ and $\|E\|$.

4. Consider the linear system $A\mathbf{x} = \mathbf{b}$, where you know A "exactly" and

```
norm(A-A') = 9.3e-16
rcond(A) = 1e-12
min(eig(A)) = 1.3e-7
max(eig(A)) = 2.7e+6
```

   (a) Is A symmetric?

   (b) Show that if A is symmetric and has an eigenvalue λ, then λ^2 is an eigenvalue of $A^T A$.

   (c) What is the condition number $\kappa_2(A)$ of A using the 2-norm?

   (d) Is this consistent with the given value of `rcond`?

   (e) The elements of $\mathbf{b}$ come from measurements which are accurate to 4 significant figures. Estimate the relative error in the computed solution to $A\mathbf{x} = \mathbf{b}$.

   (f) If you want a computed solution that is accurate to 4 significant figures, how accurate must the values of $\mathbf{b}$ be?

5. For a vector $\mathbf{x} \in \mathbb{R}^n$ the vector norms are related by

$$\begin{aligned} \|\mathbf{x}\|_2 &\le \|\mathbf{x}\|_1 \le \sqrt{n}\|\mathbf{x}\|_2, \\ \|\mathbf{x}\|_\infty &\le \|\mathbf{x}\|_2 \le \sqrt{n}\|\mathbf{x}\|_\infty, \\ \|\mathbf{x}\|_\infty &\le \|\mathbf{x}\|_1 \le n\|\mathbf{x}\|_\infty. \end{aligned}$$

   You are **not** required to prove these.

   (a) Let $\mathbf{e}_j \in \mathbb{R}^n$ be the jth unit vector (that is all elements are zero except the jth element which is 1). Calculate $\|\mathbf{e}_j\|_1$, $\|\mathbf{e}_j\|_2$ and $\|\mathbf{e}_j\|_\infty$. Hence show that the lower bounds above cannot be improved.

   (b) Let $\mathbf{e} \in \mathbb{R}^n$ be the vector with all elements equal to 1. Calculate $\|\mathbf{e}\|_1$, $\|\mathbf{e}\|_2$ and $\|\mathbf{e}\|_\infty$. Hence show that the upper bounds above cannot be improved.

   (c) The **relative error** in $\mathbf{x}$ measured using the ∞-norm is 3.4×10^{-7}.

      i. How many correct decimal places are there in $\mathbf{x}$ if $\|\mathbf{x}\|_\infty = 100$.

      ii. Does this apply to all elements of the vector $\mathbf{x}$?

6. Some suggestions in MATLAB to test if $A \in \mathbb{R}^{n\times n}$ is equal to the zero matrix are

```
Ans1 = A == 0
Ans2 = max(abs(A)) == 0
Ans3 = max(max(abs(A))) == 0
Ans4 = norm(A, 1) == 0
```

   (a) What do each of the above commands produce for the following matrices?

$$A = \begin{bmatrix} 3 & -2 \\ 0 & -4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 3 & 0 & -4 \\ 0 & 0 & -4 \\ 1 & 0 & 0 \end{bmatrix}.$$

   (b) Should the matrix `A = eps*randn(n,n)` be treated as zero?

   (c) Propose a test for two matrices $B, C \in \mathbb{R}^{n\times n}$ to be equal, taking into account the "size" of B and C.