FAMILY NAME: ....Solution....
OTHER NAME(S):........................
STUDENT NUMBER: ....
SIGNATURE: ....

THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Example Class Test 4

MATH2089
Numerical Methods Example Class Test 4

(1) TIME ALLOWED – 50 minutes

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) THIS PAPER MAY **NOT** BE RETAINED BY THE CANDIDATE

(6) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(7) Write your answers on this test paper in the space provided.
Ask your tutor if you need more paper.

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

1. a) **[4 marks]** Give the results of the following MATLAB commands when executed on a computer and explain why those results are obtained:

i)
```
h = 1e-14;
z = 2001 + h > 2001
```
Answer:

$z = 0$ since $h < |z| \, eps \approx 4.4 \times 10^{-13}$

ii)
```
a = [-2  2];
b = (a-2)./(4 - a.^2)
```
Answer:

$b = ([-2 \;\; 2] - 2) \,./\, (4 - [4 \;\; 4])$

$= [-4 \;\; 0] \,./\, [0 \;\; 0]$

$= [-\mathrm{Inf} \;\; \mathrm{NaN}]$

b) **[3 marks]** A technician claims the amount of energy E used in a chemical reaction (in appropriate units) was measured to 2 decimal places. MATLAB gives

```
>> format long
>> E
  E =
       2.007358357045850e+02
```

i) Give an estimate of the absolute error in E.
Answer:

$\mathrm{abserr}(E) < 0.5 \times 10^{-2}$

ii) Give an estimate of the relative error in E.
Answer:

$\mathrm{relerr}(E) = \dfrac{\mathrm{abserr}(E)}{|E|} < \dfrac{0.5 \times 10^{-2}}{2.0074e+2}$

iii) Give the correctly rounded value for E.
Answer:

$E = \cancel{2.01}$

c) **[3 marks]** Estimate the size n of the largest n by n matrix that can be stored in 12Mb cache memory using IEEE double precision floating point arithmetic.

Answer:

$$8n^2 = 12 \times 10^6$$

$$n = [(12 \times 10^{+6})/8]^{1/2}$$

$$= 1224.7$$

$$\Rightarrow \quad n = 1225$$

2. a) A function f with $n+1$ continuous derivatives has Taylor series expansion

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\zeta).$$

where $\zeta \in [x, x+h]$. If $f \in C^3(\mathbb{R})$ then a central difference approximation of $O(h^2)$ to the first derivative is

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2). \tag{2.1}$$

i) **[4 marks]** Use Taylor series expansions of $O(h^3)$ for both $f(x+h)$ and $f(x-h)$ to derive equation (2.1).
Answer:

(1) $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + O(h$

(2) $f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) + O(h^3)$

take (1) - (2)

$f(x+h) - f(x-h) = 2hf'(x) + O(h^3)$

$\Rightarrow \quad f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$

ii) **[2 marks]** You are given that the optimal stepsize $h^* = O(\epsilon^{1/3})$, where ϵ is the relative machine precision. When using double precision floating point arithmetic, give an estimate for the optimal stepsize h^*.
Answer:

$h^* = (2.2 \times 10^{-16})^{1/3}$

b) **[4 marks]** Give MATLAB commands for **EITHER** an anonymous function `nrmc` **OR** a function M-file `nrmc.m` to calculate

$$f(x) = \sin(x)\, e^{-x^2}.$$

Your function should work for an array of inputs `x`, producing an array of output values of the same size.

Answer:

```
nrmc = @(x) sin(x) .* exp(-x.^2);
```

OR

```
function y = nrmc(x)
y = sin(x) .* exp(-x.^2);
```

3. Consider the problem of finding the sixth root $a^{1/6}$ of a real number $a > 1$.

a) **[1 mark]** Convert this into a problem of finding the zero x^* of a **polynomial** $p(x)$.

Answer:

$$p(x) = x^6 - a$$
$$p(x^*) = 0 \Leftrightarrow x^* = a^{1/6}.$$

b) **[2 marks]** Prove that p has at least one zero in the interval $(1, a)$

Answer:

$$p(1) = 1 - a < 0 \quad (\text{since } a > 1)$$
$$p(a) = a^6 - a = a(a^5 - 1) > 0$$

$$\left.\begin{array}{l} p \text{ is continuous} \\ p(1)\,p(a) < 0 \end{array}\right\} \Rightarrow p \text{ has at least } 1 \text{ zero in } (1, a)$$

c) **[2 marks]** Prove that p has at most one zero in the interval $(1, a)$

Answer:

$$p'(x) = 6x^5 > 0 \quad \text{in } (1, a)$$
$$\Rightarrow p \text{ is strictly increasing in } (1, a)$$
$$\Rightarrow p \text{ has at most } 1 \text{ zero in } (1, a)$$

d) **[4 marks]** Show that Newton's method for finding a zero of $p(x)$ can be written as

$$x_{k+1} = \frac{1}{6}\left(5x_k + \frac{a}{x_k^5}\right).$$

Answer:

$$x_{k+1} = x_k - \frac{p(x_k)}{p'(x_k)}$$

$$= x_k - \frac{x_k^6 - a}{6x_k^5}$$

$$= x_k - \frac{x_k}{6} + \frac{a}{6x_k^5}$$

$$= \frac{5}{6}x_k + \frac{a}{6x_k^5} = \frac{1}{6}\left(5x_k + \frac{a}{x_k^5}\right.$$

e) **[1 mark]** Let $e_k = |x_k - x^*|$ for $k = 0, 1, \ldots$ be the errors produced by Newton's method. If $e_8 = 2 \times 10^{-9}$, estimate e_9.

Answer:

$$e_8 = 2 \times 10^{-9}$$

$$e_9 = e_8^2 = 4 \times 10^{-18}$$

4. Consider the data values y_j measured at the times t_j for $j = 1, 2, 3, 4, 5, 6$, given in Table 4.1. The data, which is in column vectors `tdat` and `ydat`,

j	1	2	3	4	5	6
t_j	0	0.2	0.4	0.6	0.8	1.0
y_j	0.91	0.88	0.84	0.45	0.32	0.12

Table 4.1: Data

produces the approximation obtained with the following MATLAB commands

```
A = [ones(size(tdat)) tdat];
[m, n] = size(A);
x1 = (A'*A) \ (A'*ydat)
x1 =
    1.0167
   -0.8600
x2 = A \ ydat
x2 =
    1.0167
   -0.8600
```

The data and the approximation are plotted in Figure 4.1.

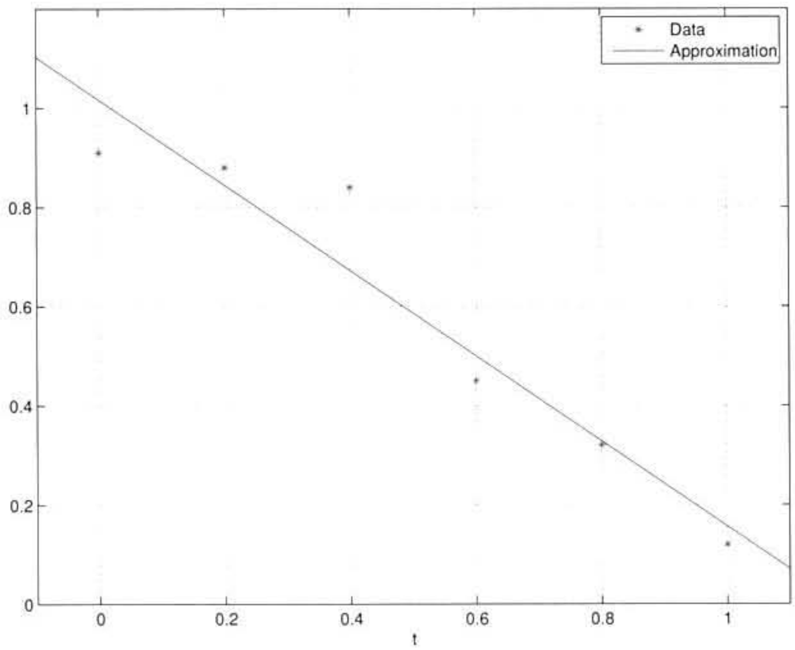

Figure 4.1: Data and approximation

a) **[2 marks]** What are the values of `m` and `n` for this example?
Answer:

m = # rows of the matix A
= 6

n = # colums of the matix A
= 2

b) **[1 mark]** It is claimed that the solution $\mathbf{x}$ to the linear system $A\mathbf{x} = \mathbf{y}$ is given by $\mathbf{x} = A^{-1}\mathbf{y}$. Why is this not correct for this example?
Answer:

A is not a square matrix
A^{-1} doesn't exist.

c) **[1 mark]** Why are the values of the variables `x1` and `x2` the same?
Answer:

x1 is the sol^n of the normal equ.
x2 is the sol^n of the least squar problem by MATLAB \

$\Rightarrow$ x1 and x2 are the same.

d) **[2 marks]** Write down the approximation obtained.
Answer:

$y = 1.0167 + (-0.86)t$

e) **[2 marks]** If A has condition number $\kappa_2(A) \approx 3 \times 10^4$, estimate the condition number of $A^T A$.

Answer:

$$\kappa_2(A^TA) \simeq (\kappa_2(A))^2 = (3 \times 10^4)^2$$
$$= 9 \times 10^8$$

f) **[2 marks]** The QR factorization of an m by n matrix A with $m > n$ gives

$$A = Q \begin{bmatrix} R \\ 0 \end{bmatrix},$$

where Q is an m by m orthogonal matrix and R is an n by n upper triangular matrix. Let Y be the m by n orthogonal matrix obtained from the first n columns of Q, so $A = YR$.

How can the QR factorization be used to calculate the least squares solution to $A\mathbf{x} = \mathbf{b}$?

Answer:

Normal equ $\quad A^TA x = A^T y$

$$A^TA = (YR)^T(YR) = R^T \underbrace{Y^TY}_{} R$$
$$= R^TR$$

Normal equ becomes

$$\cancel{R^T} R x = (YR)^T y = \cancel{R^T} Y^T y$$
$$Rx = Y^T y$$

Solve by substitution to get x