FAMILY NAME: ....
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THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Session I, 2012

MATH2089
Numerical Methods and Statistics

(1) TIME ALLOWED – 3 Hours

(2) TOTAL NUMBER OF QUESTIONS – 6

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) THIS PAPER MAY **NOT** BE RETAINED BY THE CANDIDATE

(6) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(7) STATISTICAL FORMULAE ARE ATTACHED AT END OF PAPER
STATISTICAL TABLES ARE ATTACHED AT END OF PAPER

Part A – Numerical Methods consists of questions 1 – 3

Part B – Statistics consists of questions 4 – 6

Both parts must be answered

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Part A – Numerical Methods

1. Answer in a separate book marked Question 1

The computational complexity of some common operations with n by n matrices are given in Table 1.1

Operation	Flops
Matrix multiplication	$2n^3$
LU factorization	$\frac{2n^3}{3} + O(n^2)$
Cholesky factorization	$\frac{n^3}{3} + O(n^2)$
Back/forward substitution	$n^2 + O(n)$
Tridiagonal solve	$8n + O(1)$

Table 1.1: Flops for some operations with n by n matrices

In each part a) to j) of this question a claim is made. For each claim, state whether you believe the claim is true or false ($\frac{1}{2}$ mark) and justify your answer ($1\frac{1}{2}$ marks).

a) **Claim:** The MATLAB expression

```
ans1 = 8 / 9 * pi
```

correctly evaluates the constant $\dfrac{8}{9\pi}$.

b) You are using a computer with IEEE floating point arithmetic.
Claim: The the value of

```
ans2 = 1/5
```

is stored with relative error 0.

c) **Claim:** The following MATLAB statements accurately estimate the derivative $f'(x)$ (assume the function `f` has been previously defined):

```
x = 200;
h = eps;
ans3 = (f(x+h) - f(x-h)) / (2*h)
ans3 =
     0
```

d) Consider an n by n double precision matrix which does not have any special structure and a computer with 2 Gb RAM.
Claim: The largest matrix that can be stored in RAM has $n \approx 16,000$.

e) You are given that for a large n by n matrix A with no special structure, calculating $B = A^2$ takes 1 hour.
Claim: Solving the linear system $A\mathbf{x} = \mathbf{b}$ will take 3 hours.

f) You are given that

- the Secant method for solving $f(x) = 0$ has order of convergence $\nu \approx 1.6$,
- the error $e_k = |x^* - x_k|$ on the kth iteration is $e_k = 1.02 \times 10^{-12}$.

Claim: The error on the next iteration will be $e_{k+1} \approx 3.96 \times 10^{-21}$.

g) Consider solving a single nonlinear equation $f(x) = 0$, where $x \in \mathbb{R}$.

Claim: Newton's method will **always** converge with a second order ($\nu = 2$) rate of convergence.

h) You are given the results of the following MATLAB statements:

```
r = A*x - b;
ans4 = norm(r, Inf)
ans4 =
   4.3857e-016
```

Claim: The variable `x` contains a solution to the system of linear equations $A\mathbf{x} = \mathbf{b}$.

i) You are given that

- A and $\mathbf{b}$ are computed to full double precision accuracy,
- $\kappa(A) = 1.12 \times 10^4$,

Claim: The computed solution to $A\mathbf{x} = \mathbf{b}$ has at least 11 significant figures.

j) Orthogonal matrices Q, which by definition satisfy $Q^T Q = I$, are widely used in numerical methods and least squares problems.

Claim: One reason for this is that orthogonal matrices have the ideal condition number $\kappa_2(Q) = 1$.

2. Answer in a separate book marked Question 2

a) Let

$$f(x) = \begin{cases} \dfrac{8}{9\pi}\sqrt{x(3-x)} & \text{for } 0 < x < 3; \\ 0 & \text{otherwise.} \end{cases}$$

The function f is a probability density function so

$$I(f) = \int_0^3 f(x)dx$$

must be equal to 1. Approximations to $I(f)$ were calculated using the Trapezoidal rule, Simpson's rule and the Gauss-Legendre rule, giving the following table of errors $E_N(f) = I(f) - Q_N(f)$:

N	E_N Trapezoidal	E_N Simpson	E_N Gauss-Legendre
2	3.6338e-01	1.5117e-01	-3.9596e-02
4	1.3036e-01	5.2688e-02	-6.0359e-03
8	4.6436e-02	1.8461e-02	-8.6510e-04
16	1.6480e-02	6.4946e-03	-1.1713e-04
32	5.8377e-03	2.2902e-03	-1.5288e-05
64	2.0659e-03	8.0865e-04	-1.9545e-06
128	7.3076e-04	2.8571e-04	-2.4714e-07
256	2.5842e-04	1.0098e-04	-3.1072e-08

i) Find a linear transformation $x = \alpha + \beta z$ that maps $z \in [-1, 1]$ to $x \in [0, 3]$.

ii) **Given** the nodes $z_j, j = 1, \ldots, N$ and weights $w_j, j = 1, \ldots, N$ for the Gauss-Legendre rule for the interval $[-1, 1]$, how can you approximate $I(f)$?

iii) Consider an integral over $[a, b]$ and let $h = (b - a)/N$. For Simpson's rule the error is

$$E_N(f) = -\frac{(b-a)h^4}{180} f^{(4)}(\eta), \quad \eta \in (a, b). \tag{2.1}$$

A) Use the formula (2.1) for the errors to estimate $E_N(f)/E_{2N}(f)$.

B) Use the table of numerical results to estimate $E_N(f)/E_{2N}(f)$ when $N = 128$.

C) Is the table of errors consistent with the formula (2.1) for the error?

iv) You are given that for $x \in (0, 3)$,

$$f'(x) = \frac{4}{9\pi} \frac{3 - 2x}{\sqrt{x(3-x)}}.$$

Explain why $I(f)$ is difficult to approximate.

b) Consider the initial value problem (IVP)

$$y'' + \frac{y'}{t} - 2 = 0, \qquad y(1) = 2, \quad y'(1) = -1.$$

i) What is the order of the differential equation?

ii) Convert this ordinary differential equation into a system

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \quad \text{for } t > t_0,$$

of first order differential equations.

iii) What is the initial condition $\mathbf{x}_0 = \mathbf{x}(t_0)$?

iv) Use Euler's method with a step of $h = 0.5$ to estimate $\mathbf{x}(1.5)$.

v) Write **EITHER** a MATLAB anonymous function `myode` **OR** a MATLAB function M-file `myode.m` to evaluate the vector valued function $\mathbf{f}(t, \mathbf{x})$.

3. Answer in a separate book marked Question 3

Fick's second law predicts how diffusion causes the concentration $c(x,t)$ to change with position x and time t:

$$\frac{\partial c(x,t)}{\partial t} = D\frac{\partial^2 c(x,t)}{\partial x^2}, \quad x \in (0,L), \quad t \in (0,T]. \tag{3.1}$$

The diffusion coefficient $D > 0$ is constant, while the concentration at $x = 0$ is maintained at $c(0,t) = c_0$ for $t \in (0,T]$.

Let c_j^ℓ be an approximation to $c(x_j, t_\ell)$ at the grid points

$$\begin{aligned} x_j &= j\Delta x \quad \text{for } j = 0,\ldots,n+1, \\ t_\ell &= \ell\Delta t \quad \text{for } \ell = 0,\ldots,m, \end{aligned}$$

where $L = (n+1)\Delta x$ and $T = m\Delta t$.

The space grid from a discretization with $n = 40$ is given in Figure 3.1.

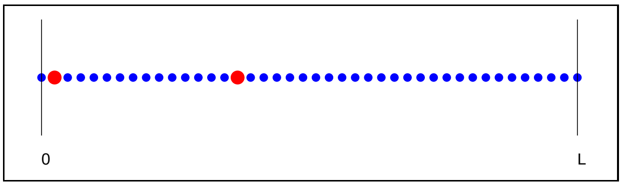

Figure 3.1: Space grid points for $n = 40$ and points for part e)

a) What extra information is needed to completely specify this problem?

b) You are **given** the following standard finite difference approximations for a function f of **one** variable:

$$\begin{aligned} f'(x) &= \frac{f(x+h) - f(x)}{h} + O(h), \\ f'(x) &= \frac{f(x+h) - f(x-h)}{2h} + O(h^2), \\ f''(x) &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2). \end{aligned}$$

At the space point x_j and the time step $t_{\ell+1}$

i) give a central difference approximation of $O(\Delta x^2)$ to

$$\frac{\partial^2 c(x,t)}{\partial x^2},$$

ii) give a backward difference approximation of $O(\Delta t)$ to

$$\frac{\partial c(x,t)}{\partial t}.$$

c) Using these finite difference approximations, derive the equations

$$-sc_{j-1}^{\ell+1} + (1+2s)c_j^{\ell+1} - sc_{j+1}^{\ell+1} = c_j^{\ell} \tag{3.2}$$

which determine the unknowns $c_j^{\ell+1}$ for $j = 1, \ldots, n$. Include an expression for s in your answer.

d) Explain whether this is an explicit method or an implicit method.

e) Given that, in appropriate units,

$$c(0,t) = 4, \quad c(L,t) = 1, \quad t \in (0,T)$$

write down the equation (3.2) for a discretization with $n = 40$ at the grid points x_j for

i) $j = 1$;

ii) $j = 15$.

Leave your answers in terms of s, but make sure you clearly identify what values are known.

f) It is claimed that for a method using (3.2), the time step Δt and the space step Δx must satisfy the stability condition

$$\frac{D\,\Delta t}{(\Delta x)^2} \le \frac{1}{2}.$$

Is this correct?

g) The coefficient matrix A for an implicit method $A\mathbf{c}^{\ell+1} = \mathbf{b}^{\ell}$ is illustrated in Figure 3.2.

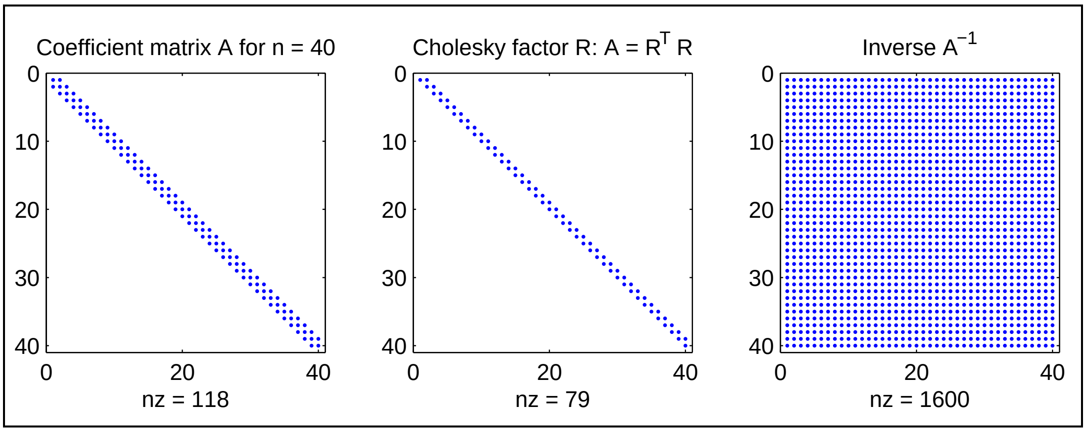

Figure 3.2: Coefficient matrix A for implicit method

i) Calculate the sparsity of A.

ii) Why is calculating A^{-1} not a good idea?

iii) What is the most efficient way to solve $A\mathbf{c}^{\ell+1} = \mathbf{b}^{\ell}$?

Part B – Statistics

4. Answer in a separate book marked Question 4

a) Compression strength (in psi) was measured on a sample of 58 specimens of a new alloy being developed for aircraft construction. The data is shown below on a stem-and-leaf plot.
In this plot 66|4 denotes 66.4×10^3 psi.

66	4											
67	7											
68	0	0	3	4	6	8	9					
69	0	1	2	3	3	5	5	6	7	8	8	9
70	0	0	1	2	3	3	4	5	6	6	8	9
71	0	1	2	3	3	5	6	6	7	8	8	9
72	1	2	3	4	6	7	9					
73	1	3	5									
74	2	5										
75	3											

i) Comment on the shape of the distribution of the data as displayed in the stem-and-leaf plot.

ii) The mean of the sample is 70.7×10^3 psi and the standard deviation of the sample is 1.78×10^3 psi.

A) Determine a 99% confidence interval for the true mean compression strength of the alloy.

B) State any assumptions you make to determine this confidence interval. Explain whether you have sufficient information to justify making these assumptions here.

iii) It is of interest to determine the chance that a specimen of the alloy has a compression strength less than 70×10^3 psi.

A) What proportion of the sample values are less than 70×10^3 psi?

B) Determine a 95% confidence interval for the true proportion of specimens of this alloy which would have a compression strength of less than 70×10^3 psi.

C) State any assumptions you make to determine this confidence interval. Explain whether they seem reasonable in this situation.

b) Suppose $X \sim Exp(\lambda)$, with $\lambda = 1$, so X has density function $f(x)$ where

$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{elsewhere.} \end{cases}$$

i) Determine the c.d.f. for X. That is determine $F(x) = P(X \leq x)$.
Sketch $F(x)$ for $-\infty < x < \infty$.

ii) Calculate $P(X \leq 1)$.

iii) Calculate the conditional probability $P(X \geq 1 \,|\, X \leq 2)$.

5. Answer in a separate book marked Question 5

a) An experiment was conducted to compare the degree of soiling for fabric copolymerized with three different mixtures of methacrylic acid. For each of the three mixtures, 5 samples of cotton/polyester fabric were treated and the degree of soiling was determined as a ratio (comparing the soiled and unsoiled fabric). The data are displayed below. We would like to know whether the true mean degree of soiling is the same for all mixtures.

Mixture 1	Mixture 2	Mixture 3
0.56	0.72	0.62
1.12	0.69	1.08
0.90	0.87	1.07
1.07	0.78	0.99
0.94	0.91	0.93
$\bar{x}_1 = 0.918$	$\bar{x}_2 = 0.794$	$\bar{x}_3 = 0.938$
$s_1 = 2.8284$	$s_2 = 2.8048$	$s_3 = 1.7889$

i) What assumptions need to be valid for an Analysis of Variance to be an appropriate analysis here?

Assume from now on that these assumptions are valid.

ii) An ANOVA table was partially constructed to summarise the data :

Source	df	SS	MS	F
Treatment	**(1)**	0.0608	**(2)**	0.99
Error	**(3)**	**(4)**	**(5)**	
Total	**(6)**	0.4309		

Copy the ANOVA table in your answer booklet. Complete the table by determining the missing values (**1**)–(**6**).

iii) Using a significance level of $\alpha = 0.05$, carry out the ANOVA F-test to determine whether the mixture significantly influences the degree of soiling. (*You may use the numerical values found in the above table, however your answer should include: a statement of the null and alternative hypotheses; the observed value and distribution of the test statistic; a mathematical expression for the p-value; the p-value, as accurately as can be determined from reading the tables; and the conclusion of the test stated in plain language.*)

b) The sediment density (g/cm^3) of a random specimen from a certain dam is normally distributed with mean 2.65 and standard deviation 0.85.

i) A random sample of 25 specimens is selected. What is the probability that the sample average specimen density is at most 3.00 g/cm^3?

ii) Suppose now that we can choose the sample size, but we require a probability of at least 0.99 that the sample average specimen density is at most 3.00 g/cm^3. How large a sample size must be taken?

6. Answer in a separate book marked Question 6

The relationship between applied stress (independent variable X, in kg/mm^2) and time to failure (the dependent variable Y, in hrs) for '18-8 stainless steel' under uniaxial tensile stress is important for many applications. In one study ten different settings of applied stress were used under fixed conditions of temperature and alkalinity.

The resulting data values are given in the table below giving y_i, recorded observations of failure times (in hrs), for different applied stress levels x_i (in kg/mm^2). A scatterplot is given in the left subplot of Figure 6.1.

i	1	2	3	4	5	6	7	8	9	10
x_i	2.5	5	10	15	17.5	20	25	30	35	40
y_i	63	58	55	61	62	37	38	45	46	19

The regression model to be fitted is given by

$$\mathrm{Y} = \beta_0 + \beta_1 \mathrm{X} + \epsilon.$$

A scatterplot of the residuals versus fitted values is provided in the right subplot of Figure 6.1.

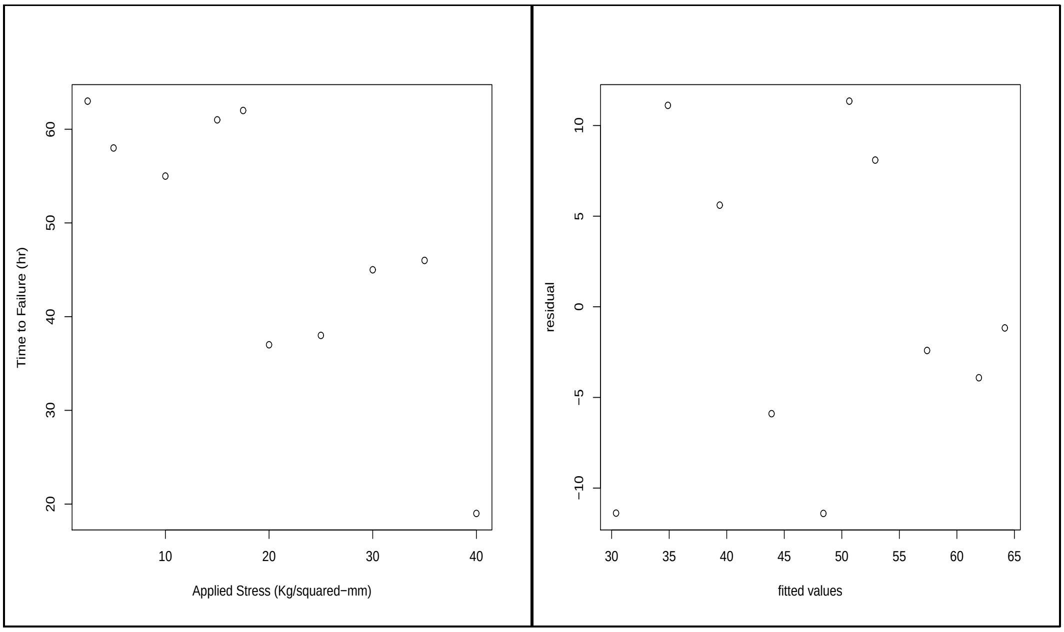

Figure 6.1: Left Subplot: Plot of Failure Times (hrs) vs Applied Stress (kg/mm^2). Right Subplot: Plot of regression residuals vs fitted values.

Some regression output and sample statistics are given for use in your answers:

```
Regression Analysis: Y versus X

The regression equation is Y = 66.4177  -0.9009 X

Predictor   Coef      SE Coef        T          P
Constant    66.4177    5.6481       11.759      2.5e-06
X           -0.9009    0.2428       -3.711      0.00595

Residual standard error: 9.124 on 8 degrees of freedom
R-squared: 0.6325   Adjusted R-squared: 0.5866
F-statistic: 13.77 on 1 and 8 DF,  p-value: 0.00595
```

$\bar{x} = 20;$ $s_x = 12.5277;$ $S_{xx} = 1412.5;$ $\bar{y} = 48.4;$ $s_y = 14.1908.$

a) i) List three essential assumptions that the error ϵ in the model must satisfy for the above regression analysis to be valid.

ii) Some of the assumptions in (i) can be checked using the plots provided in Figure 6.1. Comment on the suitability of these statistical assumptions for this model, where applicable.

b) i) What proportion of variation in the failure time Y is explained by the linear relationship with the applied stress X?

ii) Determine the (sample) correlation between failure time and applied stress.

c) Carry out a hypothesis test to determine whether the applied stress is significant in this model. (*Your answer should include: a statement of the null and alternative hypotheses; the observed value and distribution of the test statistic; a mathematical expression for the p-value; the p-value, as accurately as can be determined from reading the tables; and the conclusion of the test stated in plain language.*)

d) Determine a 95% confidence interval for β_1.

e) For a particular application, it is of interest to know about the likely failure times for a stress level of 20 kg/mm^2.

i) Calculate a 95% confidence interval for the mean time to failure at a stress level of 20 kg/mm^2.

ii) Calculate a 95% prediction interval for the time to failure at a stress level of 20 kg/mm^2.

iii) Explain when it would be appropriate to use the prediction interval in (i) or the confidence interval in (ii) to understand the failure times which are likely to be experienced at a stress level of 20 kg/mm^2.

Formula sheet

a) Let $\bar{X}$ be the sample average from a random sample of size n from a population with mean μ and standard deviation σ.
Under appropriate conditions

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1) \text{ (either approximately or exactly).}$$

b) Let $\bar{X}$ and S be the sample average and standard deviation obtained from a random sample of size n from a population with mean μ. Under appropriate conditions

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}.$$

c) For two independent samples of size n_1 and n_2 from two populations with means μ_1 and μ_2 respectively, let $\bar{X}_1$ and $\bar{X}_2$, and S_1 and S_2 be the sample average and sample standard deviations respectively and S_p be the pooled sample standard deviation, $S_p = \sqrt{\frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}}$. Under appropriate conditions

$$\frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}.$$

d) Let p be the sample proportion of 'successes' in n trials where the true probability of a success is π. Under appropriate conditions

$$\frac{p - \pi}{\sqrt{\pi(1-\pi)/n}} \sim N(0,1) \text{ (approximately).}$$

e) Consider the simple linear regression model:

$$\text{for} \quad i = 1, 2, \ldots, n, \qquad y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

where the ϵ_i are normally distributed $N(0, \sigma^2)$ for constant σ.
The least squares estimators b_0 and b_1 of β_0 and β_1 are

$$b_1 = \frac{S_{xy}}{S_{xx}} \qquad b_0 = \bar{y} - b_1\bar{x}$$

where

$$S_{xy} = \sum_i (x_i - \bar{x})(y_i - \bar{y}) \qquad S_{xx} = \sum_i (x_i - \bar{x})^2.$$

An unbiased estimator of σ^2 is

$$s^2 = \frac{\sum_i (y_i - b_0 - b_1 x_i)^2}{n-2}.$$

Please see over ...

Under the simple linear regression model:

$$\frac{b_1 - \beta_1}{s/\sqrt{S_{xx}}} \text{ comes from a } t_{n-2} \text{ distribution}$$

$$\frac{b_0 - \beta_0}{s\sqrt{\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}}} \text{ comes from a } t_{n-2} \text{ distribution.}$$

Let x_0 denote the predictor value for a response yet to be observed:

i) a $100(1-\alpha)\%$ confidence interval for the mean response at x_0 is

$$\hat{y}(x_0) \pm s\, t_{\alpha/2;n-2}\sqrt{\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}}$$

where $\hat{y}(x_0) = b_0 + b_1 x_0$ and $t_{\alpha/2;n-2}$ denotes the upper $100\alpha/2$ percentage point of the t-density with $n-2$ degrees of freedom;

ii) and a $100(1-\alpha)\%$ prediction interval for the response at x_0 is

$$\hat{y}(x_0) \pm s\, t_{\alpha/2;n-2}\sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}}.$$

Statistical Tables

t distribution critical values

Key: Table entry for p and C is the critical value t^* with probability p lying to its right and probability C lying between $-t^*$ and t^*.

	Upper tail probability p											
df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.000	1.376	1.963	3.078	6.314	12.71	15.89	31.82	63.66	127.3	318.3	636.6
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33	31.60
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21	12.92
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.610	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
	.50	.60	.70	0.80	.90	.95	.96	.98	.99	.995	.998	.999
	Probability C											

Standard normal probabilities

Key: Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

χ^2 distribution critical values

Key: Table entry for p is the critical value with probability p lying to its right.

	Upper tail probability p									
df	.995	.99	.975	.95	.90	.10	.05	.025	.01	.005
1	0.000039	0.00016	0.00098	0.0039	0.0158	2.71	3.84	5.02	6.63	7.88
2	0.0100	0.0201	0.0506	0.1026	0.2107	4.61	5.99	7.38	9.21	10.60
3	0.0717	0.115	0.216	0.352	0.584	6.25	7.81	9.35	11.34	12.84
4	0.207	0.297	0.484	0.711	1.064	7.78	9.49	11.14	13.28	14.86
5	0.412	0.554	0.831	1.15	1.61	9.24	11.07	12.83	15.09	16.75
6	0.676	0.872	1.24	1.64	2.20	10.64	12.59	14.45	16.81	18.55
7	0.989	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09	21.95
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.72	26.76
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	21.06	23.68	26.12	29.14	31.32
15	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.09	24.77	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	34.38	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	36.74	40.11	43.19	46.96	49.64
28	12.46	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	51.81	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38	91.95
80	51.17	53.54	57.15	60.39	64.28	96.58	101.88	106.63	112.33	116.32
100	67.33	70.06	74.22	77.93	82.36	118.50	124.34	129.56	135.81	140.17

F distribution critical values

Key: p=Upper tail probability p, df_n=degrees of freedom in numerator, df_d=degrees of freedom in denominator,
$*$ Multiply by 10, $\dagger$ Multiply by 100.

	df_n	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
df_d	p																			
1	.05	161	200	216	225	230	234	237	239	241	242	244	246	248	249	250	251	252	253	254
	.025	648	800	864	900	922	937	948	957	963	969	977	986	993	997	1001	1006	1010	1014	1018
	.01	405^{*}	500^{*}	540^{*}	563^{*}	576^{*}	586^{*}	593^{*}	598^{*}	602^{*}	606^{*}	611^{*}	616^{*}	621^{*}	624^{*}	626^{*}	629^{*}	631^{*}	634^{*}	637^{*}
	.005	$162^{\dagger}$	$200^{\dagger}$	$216^{\dagger}$	$225^{\dagger}$	$231^{\dagger}$	$234^{\dagger}$	$237^{\dagger}$	$239^{\dagger}$	$241^{\dagger}$	$242^{\dagger}$	$244^{\dagger}$	$246^{\dagger}$	$248^{\dagger}$	$249^{\dagger}$	$250^{\dagger}$	$251^{\dagger}$	$253^{\dagger}$	$254^{\dagger}$	$255^{\dagger}$
2	.05	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
	.025	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48	39.49	39.50
	.01	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50
	.005	199	199	199	199	199	199	199	199	199	199	199	199	199	200	200	200	200	200	200
3	.05	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
	.025	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04	13.99	13.95	13.90
	.01	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13
	.005	55.55	49.80	47.47	46.19	45.39	44.84	44.43	44.13	43.88	43.69	43.39	43.08	42.78	42.62	42.47	42.31	42.15	41.99	41.83
4	.05	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
	.025	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66	8.56	8.51	8.46	8.41	8.36	8.31	8.26
	.01	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
	.005	31.33	26.28	24.26	23.15	22.46	21.97	21.62	21.35	21.14	20.97	20.70	20.44	20.17	20.03	19.89	19.75	19.61	19.47	19.32
5	.05	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
	.025	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.28	6.23	6.18	6.12	6.07	6.02
	.01	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
	.005	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.38	13.15	12.90	12.78	12.66	12.53	12.40	12.27	12.14
6	.05	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
	.025	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.12	5.07	5.01	4.96	4.90	4.85
	.01	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
	.005	18.63	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.03	9.81	9.59	9.47	9.36	9.24	9.12	9.00	8.88
7	.05	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
	.025	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.41	4.36	4.31	4.25	4.20	4.14
	.01	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
	.005	16.24	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.51	8.38	8.18	7.97	7.75	7.64	7.53	7.42	7.31	7.19	7.08
8	.05	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
	.025	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.95	3.89	3.84	3.78	3.73	3.67
	.01	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
	.005	14.69	11.04	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.01	6.81	6.61	6.50	6.40	6.29	6.18	6.06	5.95
9	.05	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
	.025	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.61	3.56	3.51	3.45	3.39	3.33
	.01	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
	.005	13.61	10.11	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.23	6.03	5.83	5.73	5.62	5.52	5.41	5.30	5.19
10	.05	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
	.025	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.37	3.31	3.26	3.20	3.14	3.08
	.01	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
	.005	12.83	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.66	5.47	5.27	5.17	5.07	4.97	4.86	4.75	4.64
12	.05	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
	.025	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.02	2.96	2.91	2.85	2.79	2.72
	.01	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
	.005	11.75	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.91	4.72	4.53	4.43	4.33	4.23	4.12	4.01	3.90
15	.05	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
	.025	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.70	2.64	2.59	2.52	2.46	2.40
	.01	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
	.005	10.80	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.25	4.07	3.88	3.79	3.69	3.58	3.48	3.37	3.26
20	.05	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
	.025	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.41	2.35	2.29	2.22	2.16	2.09
	.01	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
	.005	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.68	3.50	3.32	3.22	3.12	3.02	2.92	2.81	2.69
24	.05	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
	.025	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.27	2.21	2.15	2.08	2.01	1.94
	.01	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
	.005	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.42	3.25	3.06	2.97	2.87	2.77	2.66	2.55	2.43
30	.05	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
	.025	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.14	2.07	2.01	1.94	1.87	1.79
	.01	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
	.005	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.18	3.01	2.82	2.73	2.63	2.52	2.42	2.30	2.18
40	.05	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
	.025	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.18	2.07	2.01	1.94	1.88	1.80	1.72	1.64
	.01	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
	.005	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	2.95	2.78	2.60	2.50	2.40	2.30	2.18	2.06	1.93
60	.05	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
	.025	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.88	1.82	1.74	1.67	1.58	1.48
	.01	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
	.005	8.49	5.79	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.74	2.57	2.39	2.29	2.19	2.08	1.96	1.83	1.69
120	.05	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
	.025	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.05	1.94	1.82	1.76	1.69	1.61	1.53	1.43	1.31
	.01	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
	.005	8.18	5.54	4.50	3.92	3.55	3.28	3.09	2.93	2.81	2.71	2.54	2.37	2.19	2.09	1.98	1.87	1.75	1.61	1.43
∞	.05	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00
	.025	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.94	1.83	1.71	1.64	1.57	1.48	1.39	1.27	1.00
	.01	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00
	.005	7.88	5.30	4.28	3.72	3.35	3.09	2.90	2.74	2.62	2.52	2.36	2.19	2.00	1.90	1.79	1.67	1.53	1.36	1.00