NUMERICAL METHODS

Solutions

NAME OF CANDIDATE:.............................

STUDENT NUMBER:.................................

THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MECHANICAL AND MANUFACTURING ENGINEERING

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MECH3520 – PROGRAMMING AND NUMERICAL METHODS

1. Time allowed – ONE AND HALF (1.5) hours.
2. Reading time 10 minutes.
3. This examination paper has 4 pages.
4. Total number of questions - THREE (3).
5. Answer ALL questions.
6. Questions are of equal value, total marks = 60.
7. Answers must be written in ink. Except where they are expressly required, pencils may ONLY be used for drawing, sketching or graphical work.
8. Candidates may NOT bring their own calculators or computers to the examination.
9. The following material will be provided by the Examinations Unit:
   - CASIO fx-911W Calculator.
10. This paper may NOT be retained by the candidate.

Question 1 **(20 marks)**

Answer in a separate book marked Question 1.

(a) Write a function M-file

(i) myfun.m to calculate $f(x) = \sin(x)e^{-0.5x}$

Your function should work for an array of inputs x, producing an array of output values of the same size.

(ii) funplot.m to plot $f(x)$ above in the range $[0, 2\pi]$ at 101 equally spaced points.

(b) For the equation

$$f(x) = x^2 - 2x - 1 = 0$$

(i) write the Newton-Raphson formula to solve this equation,

(ii) employ the Newton-Raphson method to find a root of the equation with initial value of $x_o = 2.6$. For this perform three iterations and calculate estimates for x_1, x_2, x_3.

(iii) Calculate the errors $|x_n - x_{n-1}|$. Is the sequence of the estimates converging quadratically or linearly? Give an explanation for your answer.

(c) Consider the system of linear equations

$$\begin{aligned} x_1 - 5x_2 - x_3 &= -8 \\ 4x_1 + x_2 - x_3 &= 13 \\ 2x_1 - x_2 - 6x_3 &= -2 \end{aligned}$$

(i) Starting with $x_i^{(0)} = 0$, use Jacobi iteration to find $x_i^{(n)}$ for $n = 1, 2$.

(ii) Will Jacobi iteration converge to the solution? Give an explanation for your answer.

Question 2 **(20 marks)**

Answer in a separate book marked Question 2.

(a) The compact form of the Runge-Kutta method can be written as:

$$y_{i+1} = y_i + h\sum_{j=1}^{n} c_j k_j,$$

$$\text{where } k_1 = f(x_i, y_i) \text{ and } k_j = f(x_i + p_j h, y_i + \sum_{l=1}^{j-1} a_{jl} h k_l) \text{ for } j > 1$$

Show that 2nd order Runge-Kutta method with c_1=1/2, c_2=1/2, p_2=1 and a_{21}=1 becomes Heun's method.

(b) Find the solution of the problem

$$y' = -ty\ ;\ y(0) = 1.0$$

over the interval $0 \le x \le 0.4$.

(i) Using Euler's method with $h = 0.2$.

(ii) Using Heun's method with $h = 0.2$.

(iii) Compare the results with the exact solution $y(x) = e^{-t^2/2}$ and find the percentage errors for the results obtained in (a) and (b).

(iv) Why do you have improvement in the case of Heun's method?

(c) Consider the second order differential equation

$$x''(t) + 4x'(t) + 5x(t) = 0$$

Convert this into a system of first order differential equations.

Question 3 **(20 marks)**

Answer in a separate book marked Question 3.

Consider the dimensionless partial differential equation for transient conduction heat transfer.

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}$$

(a) Write a finite difference approximation of this equation using the Forward-Time, Central-Space (FTCS) scheme and rearrange it to be solved by an explicit method.

(b) What is the stability condition for the numerical solution of this equation using an explicit method?

(c) Solve this equation for $T(x,t)$ at $t = 0.15$ and $x = 0.5$ when the initial condition is

$$T(x, t=0) = 0 \qquad (0 \le x < 1)$$

and the boundary conditions are

$$T(x=0,t) = 0\,;\; T(x=1,t) = 1\,.$$

Use value of 0.5 for the step in space, Δx, and value of 0.05 for the time step, Δt.

(d) If the calculations in the previous part were repeated with $\Delta x = 0.1$ to reduce truncation error and Δt kept equal to 0.05, what difficulty would be encountered? Do not repeat the finite difference calculations to determine your answer.

$\frac{\Delta t}{\Delta x^2} \le 0.5.$

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2 (a)

$$y_{i+1} = y_i + h \sum_{j=1}^{n} c_j k_j$$

where $k_1 = f(x_i, y_i)$

$$k_j = f\left(x_i + p_j h, y_i + \sum_{l=1}^{j-1} a_{jl} h k_l\right)$$

2nd Order Runge-Kutta $\longrightarrow$ Heun's Method.

$n = 2$
$c_1 = \frac{1}{2}$
$c_2 = \frac{1}{2}$
$p_2 = 1$
$a_{21} = 1.$

$$y_{i+1} = y_i + h(c_1 k_1 + c_2 k_2)$$

$$= y_i + h\left(\frac{1}{2} f(x_i, y_i) + \frac{1}{2} k_2\right)$$

$$= y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_i + p_2 h, y_i + a_{21} h k_1)\right]$$

$$= y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_i + h, y_i + h k_1)\right]$$

$\Rightarrow$ Heun's Methods.

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(b) $y' = -ty \qquad y(0) = 1.$

$0 \le t \le 0.4.$

(i) Use Euler's Method with $h=0.2.$

$t_0 = 0 \qquad y_0 = 1.$

$t_1 = 0.2 \qquad y_1 = ?$

$t_2 = 0.4 \qquad y_2 = ?$

$$y_{i+1} = y_i + h f(t_i, y_i)$$

$$\therefore \; y_1 = y_0 + h f(t_0, y_0)$$
$$= 1 + 0.2(-0 \times 1)$$
$$= \underline{\underline{1}}$$

$$y_2 = y_1 + h f(t_1, y_1)$$
$$= 1 + 0.2(-0.2 \times 1)$$
$$= \underline{\underline{0.96}}$$

(ii) Using Heun's Method with $h=0.2.$

For y_1.

(I) $f(t_0, y_0) = -0 \times 1.$

$= \underline{0}$

(II) $y_1^{(p)} = y_0 + h[f(t_0, y_0)]$

$= 1 + 0.2 \times 0.$

$= \underline{1}$ (predictor)

(III) $f(t_1, y_1^{(p)}) = -0.2 \times 1.$

$= \underline{\underline{-0.2}}$

(IV) $y_1 = y_0 + \frac{h}{2}\left[f(t_0, y_0) + f(t_1, y_1^{(p)})\right]$

$= 1 + \frac{0.2}{2}[0 + -0.2]$

$= \underline{\underline{0.98}}$ (corrector)

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2.(b)(ii) cont.

for y_2.

(I) $f(t_1, y_1) = -0.2 \times 0.98$
$= -0.196.$

(II) $y_2^{(P)} = y_1 + h\left[f(t_1, y_1)\right]$
$= 0.98 + 0.2\left[-0.196\right]$
$= \underline{0.9408}$ (predictor)

(III) $f(t_2, y_2^{(P)}) = -0.4 \times 0.9408$
$= -0.37632$

(IV) $y_2 = y_1 + \frac{h}{2}\left[f(t_1, y_1) + f(t_2, y_2^{(P)})\right]$

$= 0.98 + \frac{0.2}{2}\left[-0.196 + -0.37632\right]$

$= \underline{\underline{0.922768}}$ (corrector.

(iii) Exact solution: $y(t) = e^{-t^2/2}$.

$y(0.4) = e^{-0.4^2/2} = \underline{\underline{0.923116}}$

Percentage Errors.

Eulers Method $= \frac{0.923116 - 0.96}{0.923116} \times 100 = \underline{\underline{-3.996\%}}$

Heun's Method: $\frac{0.923116 - 0.922768}{0.923116} \times 100 = \underline{\underline{0.0377\%}}$

(iv) In Heun's Method, the slope is computed at two points, and their averages are used to achieve an improvement.

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2. (c)

$$x''(t) + 4x'(t) + 5x(t) = 0 \qquad \text{I}$$

Let $y(t) = x'(t)$

$\therefore \; y'(t) = x''(t)$

Subbing into I

$$y'(t) + 4y(t) + 5x(t) = 0$$

Therefore the two corresponding first order equations are:

$$y(t) = x'(t)$$

$$y'(t) + 4y(t) = -5x(t)$$

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3.
$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}.$$

(a) FTCS approximation for explicit method.

FTCS gives:

$$\frac{T_{i,j+1} - T_{i,j}}{\Delta t} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2}.$$

Rearranging for explicit method.

$$T_{i,j+1} = \left(\frac{\Delta t}{\Delta x^2}\right)T_{i+1,j} + \left(\frac{1-2\Delta t}{\Delta x^2}\right)T_{i,j} + \left(\frac{\Delta t}{\Delta x^2}\right)T_{i-1,j}$$

(b) Stability condition for an explicit method?

if solution is stable, then.

$$\frac{\Delta t}{\Delta x^2} \leq 0.5.$$

(c) Solve $T(x,t)$ at $t = 0.15$ and $x = 0.5$.

$\Delta x = 0.5$ $\Delta t = 0.05$.

Checking for stability first:

$$\frac{\Delta t}{\Delta x^2} = \frac{0.05}{0.5^2} = 0.2 \leq 0.5 \quad \therefore \underline{\text{STABLE}}$$

$$\frac{1-2\Delta t}{\Delta x^2} = \frac{1-2\times 0.05}{0.5^2} = \underline{3.6}$$

Hence equation can be written as.

$$T_{i,j+1} = 0.2\,T_{i+1,j} + 3.6T_{i,j} + 0.2T_{i-1,j}$$

PTO.

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3. (c) cont.

We are asked to find $T(x,t)$ when $t = 0.15$ and $x = 0.5$.

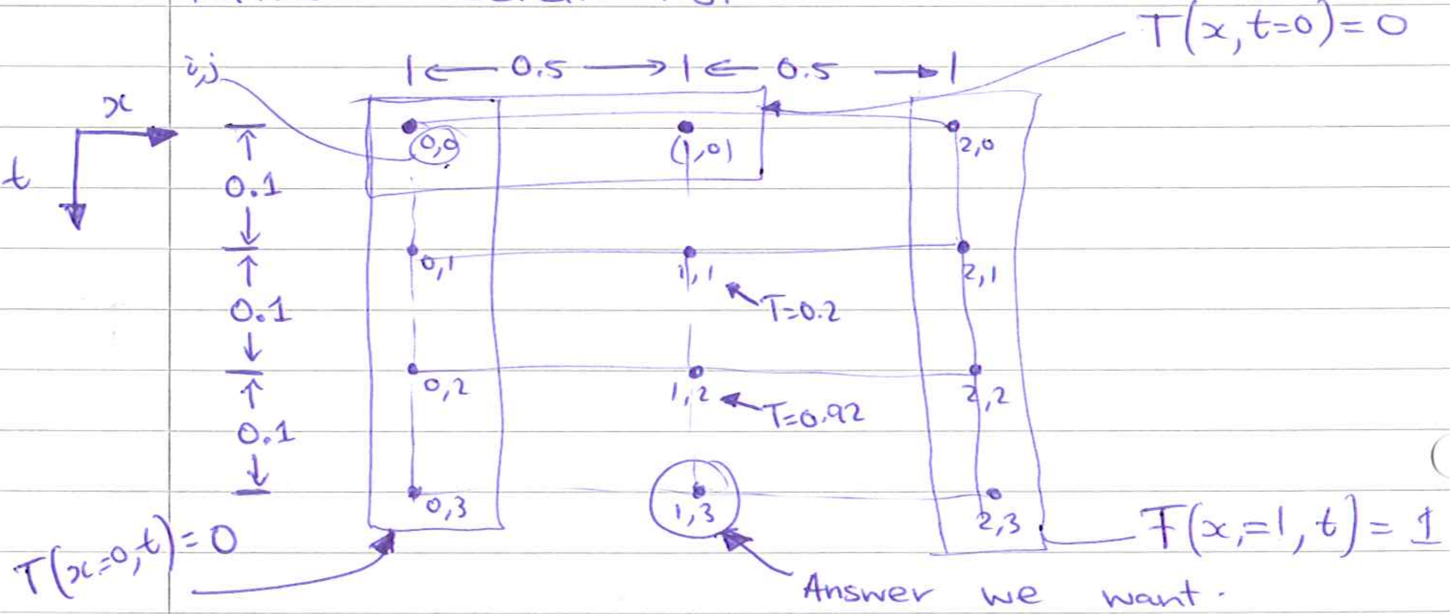
converting to i & j notation.

$$i = \frac{x}{\Delta x} = \frac{0.5}{0.5} = 1.$$

$$j = \frac{t}{\Delta t} = \frac{0.15}{0.05} = 3.$$

In other words, we are find $T_{1,3}$.

Doing a graphical interpretation, and applying initial conditions.



Before we can determine $T_{1,3}$ we must first find $T_{1,1}$ and $T_{1,2}$.

for $T_{1,1}$, using $i=1$, $j=0$

$$T_{i,j+1} = 0.2\,T_{i+1,j} + 3.6\,T_{i,j} + 0.2\,T_{i-1,j}$$

$$T_{1,1} = 0.2\,\underbrace{T_{2,0}}_{1.} + 3.6\,\underbrace{T_{1,0}}_{0} + 0.2\,\underbrace{T_{0,0}}_{0}$$

Using initial condition from above.

$$T_{1,1} = 0.2(1) + 3.6(0) + 0.2(0) = \underline{\underline{0.2}}$$

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3. (c) cont.

For $T_{1,2}$, using $i=1$, $j=1$.

$T_{i,j+1} = 0.2T_{i+1,j} + 3.6T_{i,j} + 0.2T_{i-1,j}$

$T_{1,2} = 0.2\underbrace{T_{2,1}}_{1} + 3.6\underbrace{T_{1,1}}_{0.2} + 0.2\underbrace{T_{0,1}}_{0}$.

Using initial conditions (see graph)

$T_{1,2} = 0.2(1) + 3.6(0.2) + 0.2(0)$

$= \underline{\underline{0.92}}$

We can now determine $T_{1,3}$, using $i=1$, $j=2$

$T_{i,j+1} = 0.2T_{i+1,j} + 3.6T_{i,j} + 0.2T_{i-1,j}$

$T_{1,3} = 0.2\underbrace{T_{2,2}}_{1} + 3.6\underbrace{T_{1,2}}_{0.92} + 0.2\underbrace{T_{0,2}}_{0}$.

Using initial conditions (see graph)

$T_{1,3} = 0.2(1) + 3.6(0.92) + 0.2(0)$

$= \underline{\underline{3.512}}$

$\therefore$ $T(x,t)$ at $t=0.15$ & $x=0.5$ is $\underline{\underline{3.512}}$

(d) If $\Delta x = 0.1$ and $\Delta t = 0.05$ were used, what difficulty would be encountered?

$$\frac{\Delta t}{\Delta x^2} = \frac{0.05}{0.1^2} = 5.$$

Since $5 > 0.5$ the method would not work, as the numerical solution is unstable.