1. [**20 marks**]

(a) [2 marks] There are 16% of the observations before 5 hours (6% in $[3,4)$, 10% in $[4,5)$) and 27% before 6 hours (11% of the observations in $[5,6)$). Hence, q_1 must be in $[5,6)$ (1 mark). Similarly, $q_3 \in [8,9)$ (1 mark).

(b) [2 marks] $q_1 \in [5,6)$ and $q_3 \in [8,9)$, so the minimum iqr is 2 and conservative limits are given by $(6 - 1.5 \times 2, 8 + 1.5 \times 2) = (3, 11)$. Clearly, no outliers (1 mark for using a criterion like $(q_1 - 1.5\times \text{ iqr}, q_3 + 1.5\times \text{ iqr})$, 1 mark for conclusion using conservative limits)

(c) [3 marks]

i. [2 marks] Large sample interval: $[\bar{x} \pm z_{1-\alpha/2}\frac{s}{\sqrt{n}}] = [6.92 \pm 1.96 \times \frac{1.64}{\sqrt{100}}] = [6.60, 7.24]$ (Confidence interval using the t-quantile $[\bar{x} \pm t_{99,1-\alpha/2}\frac{s}{\sqrt{n}}] \simeq [\bar{x} \pm t_{100,1-\alpha/2}\frac{s}{\sqrt{n}}] = [6.92 \pm 1.984 \times \frac{1.64}{\sqrt{100}}] = [6.59, 7.25]$ also accepted) (1 mark for a correct expression of a CI, 1 mark for the correct CI)

ii. [1 mark] The previous large-sample confidence interval is based on the approximation $\bar{X} \overset{a}{\sim} \mathcal{N}$ (Central Limit Theorem). Here $n = 100$ is large enough for that approximation to be reliable whatever the initial distribution of the battery lifetime. Hence the non-normal appearance of the histogram is not a problem.

(d) [8 marks]

i. [1 mark] $\hat{p} = 0.28$

ii. [6 marks] Let π be the true proportion of produced batteries which last for more than 8 hours

- [1 mark] Hypotheses: $H_0 : \pi = 0.33$ against $H_a : \pi < 0.33$ (one-sided test)
- Rejection criterion: reject H_0 if $\hat{p} < \pi_0 - z_{1-\alpha}\sqrt{\frac{\pi_0(1-\pi_0)}{n}} = 0.33 - 1.645 \times \sqrt{\frac{0.33\times0.67}{100}} = 0.25265$ [1 mark] $\rightsquigarrow$ no reject of H_0, as here $\hat{p} = 0.28$ [1 mark]
- [1 mark] Observed value of the test statistic: $z_0 = \sqrt{n}\frac{\hat{p}-\pi_0}{\sqrt{\pi_0(1-\pi_0)}} = 10\times\frac{-0.05}{\sqrt{0.33\times0.67}} = -1.06$
- [1 mark] p-value: $p = \Phi(-1.06) = 0.1446$ (from table)
- [1 mark] Conclusion: the data do not show enough evidence to contradict the manufacturer (we cannot reject the claim that $\pi \geq 0.33$ at significance level 5%)

iii. [1 mark] The test is based on the normal approximation for $\hat{P}$. The rule-of-thumb for that approximation to be valid is $n\pi_0(1-\pi_0) > 5$. Here, $100 \times 0.33 \times 0.67 = 22.1$, so no problem. (Also accept $n\hat{p}(1-\hat{p}) > 5$).

(e) [5 marks] Let E_i = 'Inspector i detects a flaw', and F = 'the inspected battery has a flow'. Then, $\mathbb{P}(E_1|F) = 0.9$, $\mathbb{P}(E_2|F) = 0.7$ and E_1, E_2 are independent.

i. [2 marks: 1 mark for probability of a union, 1 mark for answer] $\mathbb{P}(E_1 \cup E_2|F) = \mathbb{P}(E_1|F) + \mathbb{P}(E_2|F) - \mathbb{P}(E_1 \cap E_2|F) = 0.9 + 0.7 - 0.9 \times 0.7 = 0.97$ (independence has been used)

ii. [3 marks: 1 mark for Bayes' rule, 1 mark for law of total probability, 1 mark for answer] $\mathbb{P}(E_1|F^c) = \mathbb{P}(E_2|F^c) = 0$, $\mathbb{P}(F) = 0.10$. Then,

$$\begin{aligned}
\mathbb{P}(F|(E_1 \cup E_2)^c) &= \mathbb{P}((E_1 \cup E_2)^c|F) \times \frac{\mathbb{P}(F)}{\mathbb{P}((E_1 \cup E_2)^c)} \\
&= \frac{\mathbb{P}((E_1 \cup E_2)^c|F)\mathbb{P}(F)}{\mathbb{P}((E_1 \cup E_2)^c|F)\mathbb{P}(F) + \mathbb{P}((E_1 \cup E_2)^c|F^c)\mathbb{P}(F^c)} \\
&= \frac{0.03 \times 0.10}{0.03 \times 0.10 + 1 \times 0.9} \\
&= 0.0033
\end{aligned}$$

2. [**20 marks**]

(a) [2 marks] Any two of the following will do:

- Range 3 is the most energy efficient (or range 1 is least efficient, or state the ordering).
- The variabilities for the three are similar
- The distributions are skewed
- There are not obvious outliers

(b) [1 mark] The variances for different groups are same/similar (either from summary statistics or boxplot).

(c) [3 marks, 1 mark for (1)-(3), 1 mark for (4) and (5) and 1 mark for (6)]

(1) $k - 1 = 2$

(2) $n - k = 18 - 3 = 15$.

(3) $n - 1 = 18 - 1 = 17$.

(4) $SS_{Tr} = 6\sum_{k=1}^{3}(x_k - \bar{\bar{X}})^2 = 10.598$.

(5) $SS_{Er} = 5(s_1^2 + s_2^2 + s_3^2) = 28.279$. (either (4) or (5) can also use the fact that they sum to 38.88)

(6) $MS_{Tr} = SS_{Tr}/2 = 5.299$ (or use $2.81 \times 1.885 = 5.299$)

Source	df	SS	MS	F
Treatment	(1) 2	(4) 10.598	(6) 5.299	2.811
Error	(2) 15	(5) 28.279	1.885	
Total	(3) 17	38.88		

(d) [6 marks]

- [1 mark] $H_0 : \mu_1 = \mu_2 = \mu_3$ vs. H_a : not all three means are equal
- [2 marks, 1 for degree of freedom and 1 for the rest] For rejection region we use $P(F > f_{2,15,0.95}) = 0.05$ and $f_{2,15,0.95} = 3.68$ the rejection region is $(3.68, \infty)$.
- [2 marks, 1 for stating the p-value as probability and 1 for the final numerical value] The p-value is equal to $P(F_{2,15} > 2.811)$. From table $P(F_{2,15} > 3.68) = 0.05$ and thus p-value is larger than 0.05.
- [1 mark]conclusion: we do not reject the null.

(e) [6 marks]

- [1 mark] $H_0 : \mu_1 = \mu_2$ vs $H_a : \mu_1 \neq \mu_2$
- [2 marks, 1 for degree of freedom and 1 for the rest] We reject if $|\bar{x}_1 - \bar{x}_2| > t_{15,0.975}\sqrt{MS_{Er}/3} = 2.131\sqrt{MS_{Er}/3} = 1.689$ (or $|\bar{x}_1 - \bar{x}_2|/\sqrt{MS_{Er}/3} > t_{15,0.975} = 2.131$).
- [2 marks, 1 for test statistic and 1 for p-value] The test statistics is $\frac{\bar{x}_1 - \bar{x}_2}{\sqrt{MS_{Er}/3}} = 0.759$ and p-value is $2P(t_{15} > 0.759) = 0.46$ (from statistical table it can only be seen to be between 0.4 and 0.5).
- [1 mark] We do not reject the null.

(f) [2 marks]

- [1 mark] The p-values should be compared with $0.05/3 = 0.0167$.
- [1 mark] All p-values are bigger than this and we cannot reject the null hypothesis.

3. [**20 marks**]

(a) i. $\epsilon \sim N(0, \sigma)$ [1 mark], also take full marks for simply "normal"

ii. qq plot not perfect but results are approximately correct by Central Limit Theorem [1 mark]

(b) [1 mark] $\hat{\beta}_1 = 0.0383$

(c) [1 mark] $r^2 = 33.5\%$

(d) [1 mark] $r = \sqrt{r^2} = \sqrt{0.335} = 0.579$

(e) [1 mark] $s = \hat{\sigma} = 9.96$

(f) [7 marks total: 2 marks for hypotheses, 1 mark for test statistic, 1 mark for df, 1 mark for p-value (or rejection region), 1 mark for decision, 1 mark for conclusion]

- $H_0 : \beta_1 = 0$ vs. $H_a : \beta_1 \neq 0$ [2 marks: 1 mark for H_0 and 1 mark for H_a]
- $t = \dfrac{0.038321}{0.0076395} = 5.0161$, $df = 50$
- $p < 0.0001$ as reported, could also use $t_{50,0.975} = 2.009$
- Reject H_0
- `HIC` is significantly associated with `CTI` (or something similar that ties original problem with statistical results)

(g) [2 marks] $z_{0.975} = 1.96$

$$0.038321 \pm 1.96 \times 0.0076395 = (0.02334758, 0.05329442)$$

(h) [5 marks, 1 mark for $\hat{y}(x_0)$, 1 mark for t quantile, 1 mark for correct equation, 1 mark for correct interval, 1 mark for explanation]
$t_{50,0.995} = 2.678$
$\hat{y}(x_0) = 68.518 + 0.038321(1000) = 106.839$

$$\begin{aligned} & \hat{y}(x_0) \pm s t_{n-2,1-\alpha/2} \sqrt{\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{s_{XX}}} \\ = \; & 106.839 \pm 9.96 \times 2.678 \sqrt{\frac{1}{52} + \frac{(1000 - 33023/52)^2}{1698115}} \\ = \; & [98.50354, 115.17446] \end{aligned}$$

Prediction interval for Y at $X = x$ will be wider than the corresponding CI for the mean, as it takes into account extra variability associated with a single observation, in addition to the uncertainty inherent in the estimation of the mean response.