UNIVERSITY OF NEW SOUTH WALES
School of Mathematics and Statistics

MATH2089 Numerical Methods and Statistics
Term 2, 2019

Numerical Methods Tutorial – Week 8

1. For a function f the following data are known:

$$f(0) = 12.6, \quad f(1) = 6.7, \quad f(2) = 4.3, \quad f(3) = 2.7.$$

   (a) What is the (lowest) degree of the interpolating polynomial P for this data?

   (b) Assume that we want to find P in the form

$$P(x) = a_0 + a_1 x + \cdots .$$

      i. Write down the system of linear equations (in matrix form $A\mathbf{a} = \mathbf{f}$) you need to solve to obtain a_0, a_1, ...

      ii. Use MATLAB to set up and solve this linear system using \ (backslash) to solve the linear system.

      iii. Use MATLAB to plot both the data and your interpolating polynomial on the interval $[-1, 4]$.

   (c) Now we are to find P in terms of the Lagrange polynomials $\ell_j(x)$ for $j = 0, 1, 2, 3$.

      i. Write down the Lagrange polynomials $\ell_j(x)$ for $j = 0, 1, 2, 3$.

      ii. Verify that

$$\ell_j(x_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad i, j = 0, \ldots, 3.$$

      iii. Write down the interpolating polynomial P using the Lagrange polynomials.

      iv. Find an approximate value to $f(1.5)$ by hand. Check it agrees with your MATLAB code from the previous part.

2. The following table lists the values of $f(x) = \cos x$ (to 4 decimal places).

x	0.2	0.3	0.4	0.5	0.6
$f(x)$	0.9801	0.9553	0.9211	0.8776	0.8253

   (a) Using the table, find an approximate value to $f(0.35)$.

      i. Using linear interpolation

      ii. Using quadratic interpolation

      iii. Using Lagrange interpolation and all data values.

   (b) Determine a bound on the error involved.

   (c) Find approximate values to $f'(0.3)$ by using the forward difference and central difference. Find the exact value of $f'(0.3)$ and compare your answers with it. Which difference gives a better approximation?

   (d) Find an approximate value for $f''(0.4)$.

3. Some choices for basis functions for $\mathbb{P}_n$, the space of polynomials of degree at most n, are

- **Monomial basis:** x^k for $k = 0, \ldots, n$,
- **Chebyshev basis:** $T_k(x) = \cos(k \arccos(x))$, for $k = 0, \ldots, n$,

while choices for the (interpolation) points in $[-1, 1]$ are

- **Equally spaced points:** $x_j = -1 + jh$ for $j = 0, \ldots, n$ and $h = 2/n$,
- **Chebyshev points:** $x_j = \cos\left(\frac{\pi}{2}\frac{2j+1}{n+1}\right)$, for $j = 0, \ldots, n$.

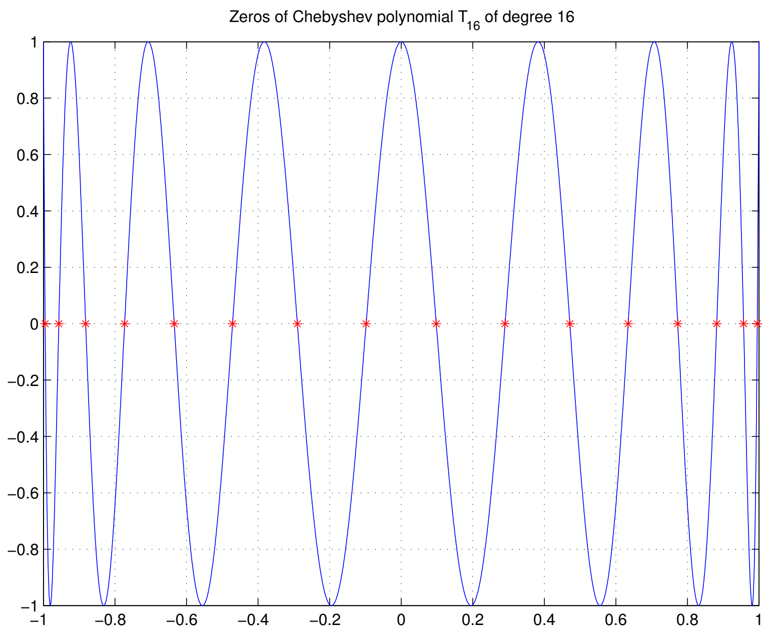

Figure 1: Zeros of Chebyshev polynomial of degree 16

(a) Show that the Chebyshev points are the zeros of the degree $n+1$ Chebyshev polynomial $T_{n+1}(x)$.

(b) The Vandermonde matrix for the basis $b_j(x), j = 0, \ldots, n$ and points $x_i, i = 0, \ldots, n$ is

$$A_{i+1,j+1} = b_j(x_i) \qquad i = 0, \ldots, n, \quad j = 0, \ldots, n.$$

The condition numbers of the Vandermonde matrix for choices of basis functions and point sets are given in the following table. For each combination of basis functions and points, what is the highest degree interpolating polynomial for which the computed solution to $A\mathbf{a} = \mathbf{f}$ has at least 4 significant figures when the function values f_j at x_j

Basis	Point set	$n = 5$	$n = 10$	$n = 15$	$n = 20$	n = 25
Monomial	Equally spaced	6.38e+01	1.40e+04	3.28e+06	8.31e+08	2.13e+11
Monomial	Chebyshev	4.52e+01	3.59e+03	2.83e+05	2.31e+07	1.86e+09
Chebyshev	Equally spaced	2.93e+00	2.37e+01	4.27e+02	8.64e+03	2.12e+05
Chebyshev	Chebyshev	1.41e+00	1.41e+00	1.41e+00	1.41e+00	1.41e+00

i. are known exactly.

ii. come from experimental measurements taken to 5 significant figures.