

FAMILY NAME:
OTHER NAME(S):
STUDENT NUMBER:
SIGNATURE:

THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF MATHEMATICS AND STATISTICS

June 2010

MATH2089
Numerical Methods and Statistics

- (1) TIME ALLOWED – 3 Hours
- (2) TOTAL NUMBER OF QUESTIONS – 6
- (3) ANSWER ALL QUESTIONS
- (4) THE QUESTIONS ARE OF EQUAL VALUE
- (5) THIS PAPER MAY **NOT** BE RETAINED BY THE CANDIDATE
- (6) **ONLY** CALCULATORS WITH AN AFFIXED “UNSW APPROVED” STICKER
MAY BE USED
- (7) STATISTICAL FORMULAE ARE ATTACHED AT END OF PAPER
STATISTICAL TABLES ARE ATTACHED AT END OF PAPER

Part A – Numerical Methods consists of questions 1 – 3

Part B – Statistics consists of questions 4 – 6

Both parts must be answered

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Part A – Numerical Methods

1. Answer in a separate book marked Question 1

- a) What are the values produced by the following MATLAB expressions:
- i) `x = 1e+200;`
`ans1 = exp(x^3)`
 - ii) `v = [-2:2:2];`
`ans2 = v./sqrt(v)`
 - iii) `x = 0.1;`
`h = 1e-18;`
`ans3 = x + h <= x`
- b) The computational complexity of some common operations with n by n matrices are

| Operation | Flops |
|---------------------------|---------------------------|
| Matrix multiplication | $2n^3$ |
| LU factorization | $\frac{2n^3}{3} + O(n^2)$ |
| Cholesky factorization | $\frac{n^3}{3} + O(n^2)$ |
| Back/forward substitution | $\frac{n^2}{2} + O(n)$ |
| Tridiagonal solve | $8n + O(1)$ |

- i) Estimate the size of the largest linear system that can be solved in one minute on a 3 GHz quad core computer, where each core can do two floating point operations per clock cycle. Assume that the coefficient matrix has no special structure.
 - ii) If multiplying two n by n matrices takes 12 seconds, estimate how long it will take to solve an n by n symmetric positive definite linear system.
- c) The coefficient matrix A and the right-hand-side vector \mathbf{b} are known to 8 significant figures, and

$$\|A\| = 1.9 \times 10^1, \quad \|A^{-1}\| = 2.2 \times 10^3.$$

- i) What is the condition number $\kappa(A)$?
- ii) How many significant figures are there in a computed solution to $A\mathbf{x} = \mathbf{b}$?

- d) You want to find an intersection of the functions

$$f_1(x) = \frac{1}{1+x^2} \quad \text{and} \quad f_2(x) = \log(|x|).$$

- i) Define a function f to reformulate this as the problem of solving the nonlinear equation $f(x) = 0$.
- ii) Write a MATLAB anonymous function `myfun` to evaluate f . Your function must work for a vector of input values, as well as a scalar input argument.
- iii) The Secant method produces iterates x_k with errors $e_k = |x^* - x_k|$, where $x^* = 1.4013216221938154421$ has been found to high accuracy by another method. The errors satisfy

| k | e(k) | e(k+1)/e(k) | e(k+1)/e(k)^2 | e(k+1)/e(k)^3 |
|---|----------|-------------|---------------|---------------|
| 1 | 4.01e-01 | 1.49e+00 | 3.72e+00 | 9.26e+00 |
| 2 | 5.99e-01 | 1.71e-01 | 2.85e-01 | 4.76e-01 |
| 3 | 1.02e-01 | 2.53e-01 | 2.48e+00 | 2.43e+01 |
| 4 | 2.58e-02 | 4.37e-02 | 1.69e+00 | 6.55e+01 |
| 5 | 1.13e-03 | 1.11e-02 | 9.80e+00 | 8.67e+03 |
| 6 | 1.25e-05 | 4.84e-04 | 3.87e+01 | 3.09e+06 |
| 7 | 6.06e-09 | 5.35e-06 | 8.83e+02 | 1.46e+11 |
| 8 | 3.24e-14 | | | |

Giving reasons, answer the following questions.

- A) Is the method converging?
- B) Is the order of convergence linear, super-linear or quadratic?

2. Answer in a separate book marked Question 2

- a) You have measurements at times t_i of the concentration $c_i = c(t_i)$ of a chemical which is being consumed, but not produced, in a reaction. The data is given in Table 2.1 and plotted in Figure 2.1.

| i | 0 | 1 | 2 | 3 | 4 |
|-------|--------|--------|--------|--------|--------|
| t_i | 0 | 0.5 | 1 | 1.5 | 2 |
| c_i | 1.0000 | 0.4283 | 0.5297 | 0.1344 | 0.0549 |

Table 2.1: Measured data values

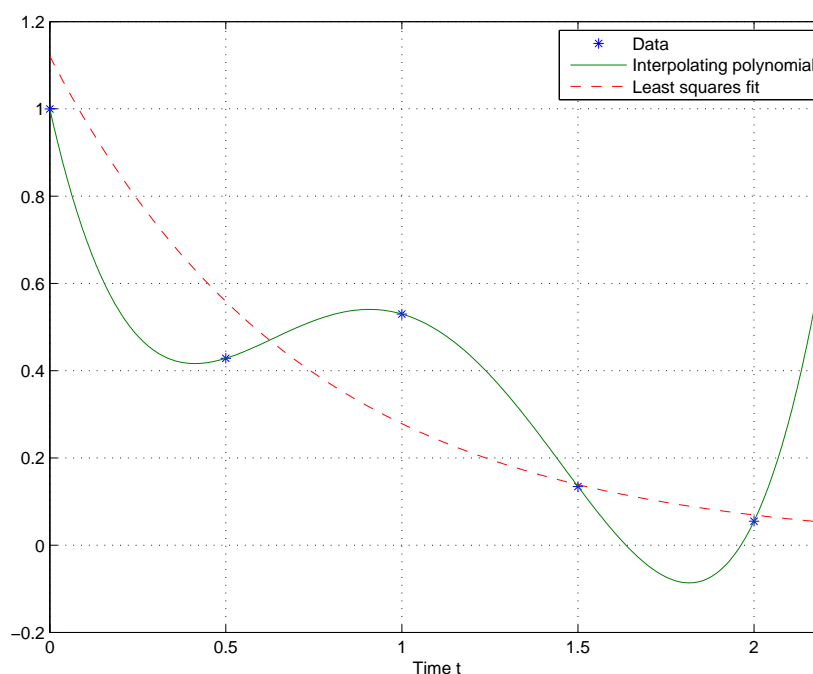


Figure 2.1: Data, polynomial interpolant and least squares approximation

- i) You are asked to approximate $c(t)$ by the interpolating polynomial, which is illustrated in Figure 2.1.
- What is the degree of the interpolating polynomial?
 - Give **two** reasons why this is not a sensible approximation.
- ii) As the chemical is being consumed, an approximation of the form

$$c(t) = \alpha e^{-\beta t}$$

is thought to be more appropriate. Give MATLAB commands to calculate the least squares approximation of this form, assuming the data is in the column vectors `tdata` and `cdata`. You do **not** need to calculate values for α and β .

Please see over ...

- iii) Consider the problem of numerically approximating the integral

$$\int_a^b f(x)dx$$

using the values $f_i = f(x_i)$, $i = 0, \dots, N$ at the equally spaced points $x_i = a + ih$, $i = 0, \dots, N$ where $h = (b - a)/N$. The Trapezoidal rule and Simpson's rule are, assuming N is even,

$$Q_N^{Trap}(f) = h \left(\frac{f_0}{2} + f_1 + f_2 + \dots + f_{N-1} + \frac{f_N}{2} \right) + O(h^2)$$

$$Q_N^{Simp}(f) = \frac{h}{3} (f_0 + 4f_1 + 2f_2 + 4f_3 + \dots + 4f_{N-1} + f_N) + O(h^4)$$

- A) Use the data in Table 2.1 and Simpson's rule to estimate

$$\int_0^2 c(t)dt. \quad (2.1)$$

- B) If additional data values at $t = 0.25, 0.75, 1.25, 1.75$ are measured, estimate how much the error in Simpson's method will decrease.
 C) Can a Gauss-Legendre rule be used to estimate the integral (2.1) using just the data values in Table 2.1?
- b) Consider the initial value problem (IVP)

$$y''' + 2y'' - (\pi^2 + 1)y = \pi(\pi^2 + 1)e^{-t} \sin(\pi t)$$

$$y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 1 - \pi^2.$$

- i) Reformulate the IVP as a system of first-order differential equations

$$\mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x})$$

with the appropriate initial condition.

- ii) Write a MATLAB function M-file **or** anonymous function
`myode(t, x)`
 to specify the function **f** in your first order system.

3. Answer in a separate book marked Question 3

Fick's second law, which predicts how the concentration of a chemical changes with time under the influence of diffusion, is

$$\frac{\partial c}{\partial t} = D \left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right), \quad (3.1)$$

where (x, y) is the position, t is the time, $c(x, y, t)$ is the concentration at position (x, y) and time t , and D is the constant diffusion coefficient. The

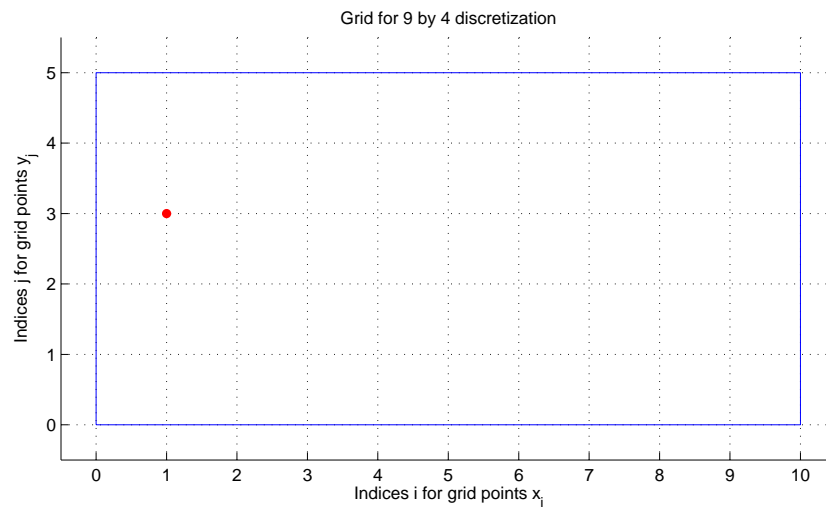


Figure 3.1: Discretization of space domain

space domain

$$\Omega = \{(x, y) : 0 \leq x \leq 2, \quad 0 \leq y \leq 1\}$$

is discretised using $h = 1/n$ and

$$x_i = ih, \quad i = 0, \dots, 2n \quad y_j = jh, \quad j = 0, \dots, n.$$

This is illustrated in Figure 3.1 for $n = 5$. The time steps $t_\ell = \ell \Delta t$, for $\ell = 0, 1, 2, \dots$ are used to calculate the approximate concentrations

$$c_{i,j}^\ell \approx c(x_i, y_j, t_\ell).$$

Standard finite difference approximations for a function f of **one** variable are

$$\begin{aligned} f'(x) &= \frac{f(x+h) - f(x)}{h} + O(h), \\ f'(x) &= \frac{f(x+h) - f(x-h)}{2h} + O(h^2), \\ f''(x) &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2). \end{aligned}$$

Please see over ...

- a) What additional information is needed to completely specify the problem?
- b) What is the steady-state version of Fick's second law (3.1)?
- c) At the point (x_i, y_j) and time $t_{\ell+1}$ give a backward difference approximation of $O(\Delta t)$ to $\frac{\partial c}{\partial t}$.
- d) At the point (x_i, y_j) and time $t_{\ell+1}$ give central difference approximations of $O(h^2)$ to $\frac{\partial^2 c}{\partial x^2}$ and $\frac{\partial^2 c}{\partial y^2}$.
- e) Using the finite difference approximations from the previous two parts, show that Fick's second law (3.1) is approximated by

$$\alpha c_{i,j}^{\ell+1} + \beta c_{i+1,j}^{\ell+1} + \beta c_{i-1,j}^{\ell+1} + \beta c_{i,j+1}^{\ell+1} + \beta c_{i,j-1}^{\ell+1} = c_{i,j}^{\ell}$$

and find α, β in terms of $s = (D\Delta t)/h^2 > 0$.

- f) Suppose that $c(0, y, t) = 6$ for $0 < y < 1$ and $t > 0$, and $c(x, y, 0) = 3$ for all $(x, y) \in \Omega$. Find the equation at the grid point (x_1, y_3) marked in Figure 3.1 and at time t_1 . Clearly indicate what the unknowns are.
- g) Is this an explicit method or an implicit method? Briefly discuss the relative advantages and disadvantages of an implicit method compared to an explicit method.
- h) You are given that using a row-ordering of the variables $c_{i,j}^{\ell}$ produces the coefficient matrix A whose non-zero entries are illustrated in Figure 3.2.

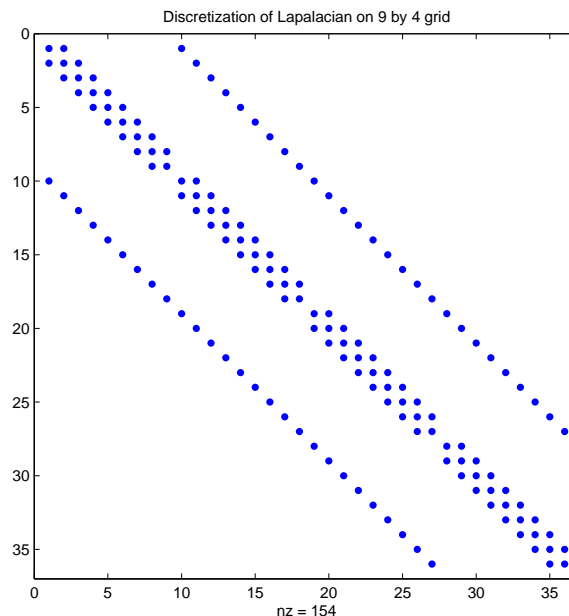


Figure 3.2: Spy plot of coefficient matrix

- i) Calculate the sparsity of A .
- ii) Why is explicitly calculating A^{-1} not a good idea?
- iii) What other structure does A have that will make solving a linear system $A\mathbf{x} = \mathbf{b}$ more efficient?

Please see over ...

Part B – Statistics

4. Answer in a separate book marked Question 4

- a) Assessment of air quality and level of pollution often includes data on the *pollutant standards index*, PSI. A sample of PSI data from the city of Atlanta was obtained on $n = 10$ different days. The sample mean and standard deviation were $\bar{x} = 12.2$ and $s = 6.8$. A boxplot of the data and normal quantile plot of the data are given in figure 4.1.

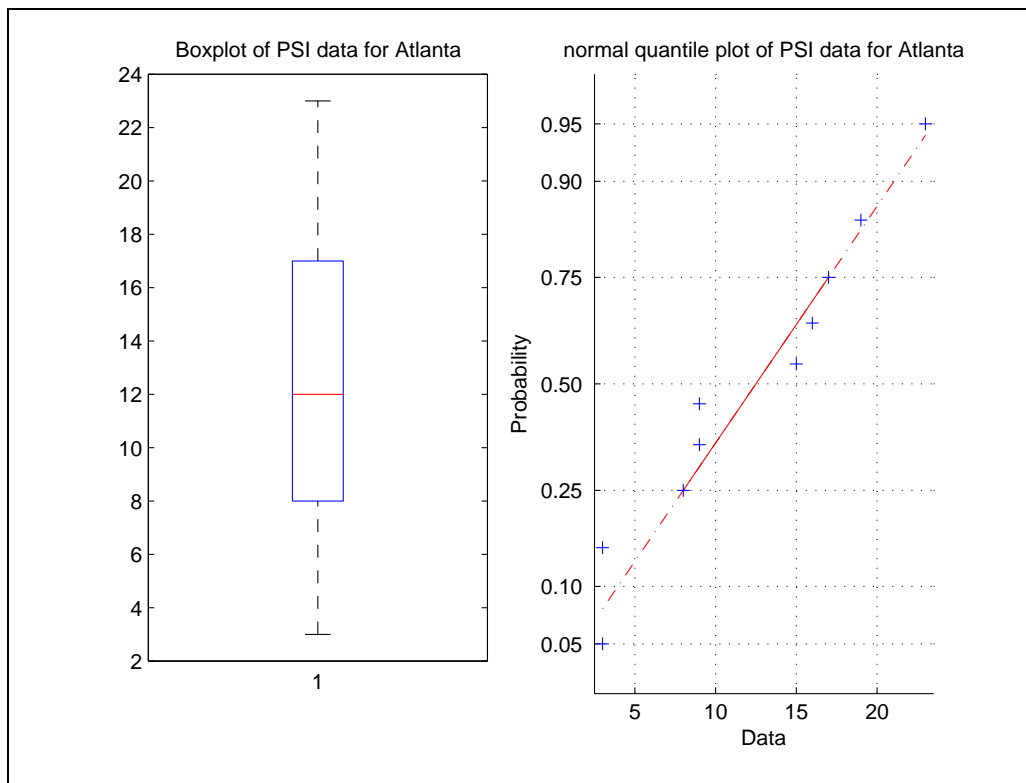


Figure 4.1: PSI data for Atlanta.

- From the graph(s), write down the five number summary for the Atlanta PSI data (approximately).
- Comment on the features of the data as shown in the boxplot and normal quantile plot.
- Determine a 99% confidence interval for the true (population) mean PSI for Atlanta.
- State any assumptions you need to make to determine the confidence interval in part aiii). Is it possible to determine whether the assumptions are reasonable in this case? Explain whether any verifiable assumption seems reasonable.

- b) Samples of PSI data were collected for many cities in the USA. Data for Chicago, Atlanta and Houston are displayed in side-by-side (comparative) boxplots in figure 4.2.

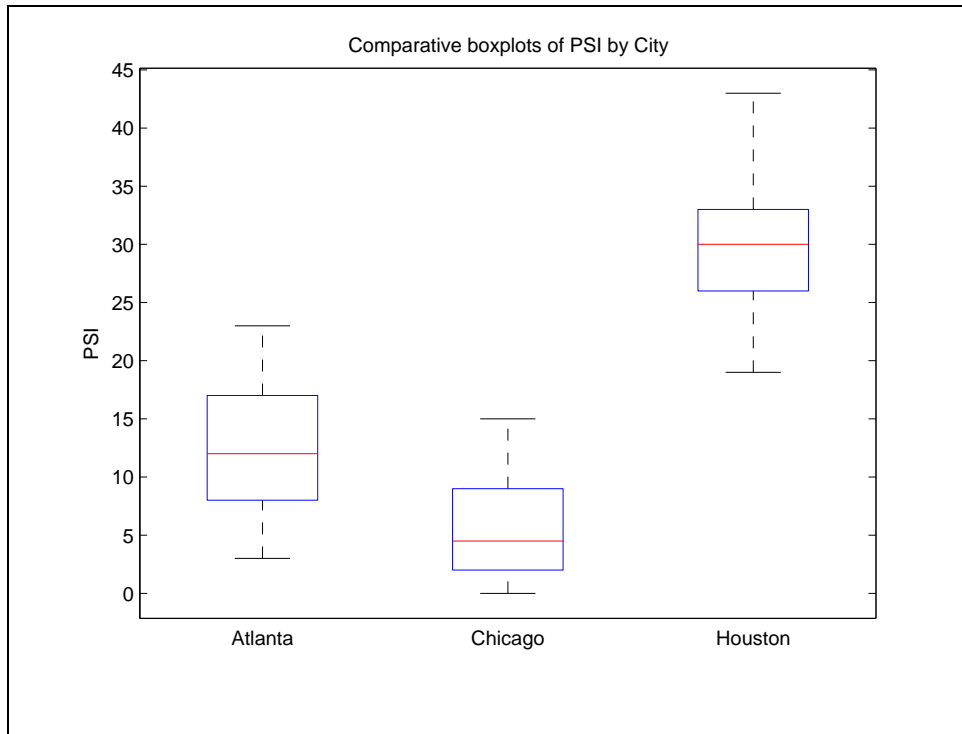


Figure 4.2: PSI levels in three cities.

In two or three sentences, comment on the data from these three cities as shown in these plots. Include a comparison of the symmetry (or otherwise), centre and spread of the data for the different cities.

- c) A new material is tested for use in the construction of car bumper bars. In controlled crash testing, damage occurs in the bumper bars of 12 out of 96 cars tested.
- Determine a 98% confidence interval for the true (population) proportion of car bumper bars constructed from the new material which would incur damage in such a crash.
 - What assumptions did you have to make for your confidence interval to be valid?

5. Answer in a separate book marked Question 5

- a) In an experiment to investigate the performance of four different types of spark plugs intended for use on a motorcycle, five plugs of each type were tested and the number of driving km (at a constant speed) until plug failure was recorded. The following ANOVA table summarises the data, but it has two missing entries indicated by **A** and **B**.

| Source | df | SS | MS | F |
|--------|----|--------|----------|-----|
| Factor | 3 | 75081 | A | 1.7 |
| Error | 16 | 235424 | B | |
| Total | 19 | 310505 | | |

- i) Calculate the values represented by **A** and **B**.
 - ii) Set up suitable notation, and state the null and alternative hypotheses for the ANOVA F-test. What assumptions need to be made for an Analysis of Variance to be an appropriate analysis here?
 - iii) Using a significance level of $\alpha = 0.01$, (and making the necessary assumptions) complete the ANOVA F-test. Include in your answer: the observed value and distribution of the test statistic; a mathematical expression for the p -value; the range of values within which the p -value falls (and a statement of how these are obtained); and the conclusion of the test stated in plain language.
- b) The automatic opening device of a military cargo parachute is designed to open at 200 metres above the ground. Equipment is damaged if the parachute opens at an altitude of less than 100 metres. Assume the opening altitude actually has a normal distribution with mean $\mu = 200$ metres and standard deviation $\sigma = 35$ metres.
- i) In what proportion of drops does the parachute open at an altitude of less than 100 metres?
 - ii) Five parachutes are dropped independently. What is the probability that at least one of them opens at an altitude of less than 100 metres?
- c) Let X and Y be independent random variables with $X \sim \text{Poisson}(2)$ and $Y \sim \text{Exp}(1)$.
- i) Calculate $P(X = 1)$.
 - ii) Calculate $P(Y < 1)$.
 - iii) Calculate $P(X = 1 \text{ or } Y < 1)$.

6. Answer in a separate book marked Question 6

Concrete is vulnerable to shock vibrations, which may cause hidden damage to the material. In a study of vibration phenomena, an experiment is carried out and data is reported: including the variables **ppv** — peak particle velocity (mm/sec), and **Ratio** — ratio of ultrasonic pulse velocity after impact to that before impact in concrete prisms. Investigators fit the simple linear regression model:

$$\text{Ratio} = \beta_0 + \beta_1 \times \text{ppv} + \epsilon. \quad (\star)$$

Using the regression analysis output (which includes residual plots and a fitted line plot) at the end of the question, answer the following questions.

- a)
 - i) Write the null and alternative hypotheses to test whether the variable **ppv** is significant in predicting the variable **Ratio**.
 - ii) Carry out the test at the 1% significance level. Include:
 - (1) the observed value and distribution of the test statistic;
 - (2) a mathematical expression for the p -value;
 - (3) the conclusion of the test stated in plain language.
- b) Determine the (sample) correlation between the variables **Ratio** and **ppv**.
- c) Determine a 95% confidence interval for the change in the mean of **Ratio** for an increase of 1 mm/sec in **ppv**.
- d) At the end of the regression output is the section about predicted values for new observations when the peak particle velocity is 750 mm/sec.

| New | | | | |
|-----|----------|----------|----------------------|----------------------|
| Obs | Fit | SE Fit | 95% CI | 95% PI |
| 1 | 0.988947 | 0.000625 | (0.987666, 0.990228) | (0.982370, 0.995523) |

- i) Explain what the 95% CI (0.987666, 0.990228) represents.
 - ii) Explain why the 95% PI is different to the 95% CI.
 - iii) Each of the intervals (0.987666, 0.990228) and (0.982370, 0.995523) is centred at 0.988947. Explain why this is the case. Show how the value 0.988947 is determined from the equation of the regression line.
- e) List three essential assumptions that the error ϵ in Model (\star) must satisfy for the above statistical inferences to be valid. Which of these three can be checked by considering the accompanying regression analysis output? Explain whether these verifiable assumptions are supported, with specific reference to the regression analysis output.

Regression analysis output for Question 6

Regression Analysis: Ratio versus ppv

The regression equation is

Ratio = 1.00 - 0.000015 ppv

| Predictor | Coef | SE Coef | T | P |
|-----------|-------------|------------|--------|-------|
| Constant | 1.00007 | 0.00131 | 761.91 | 0.000 |
| ppv | -0.00001484 | 0.00000190 | -7.80 | 0.000 |

S = 0.00314906 R-Sq = 68.5% R-Sq(adj) = 67.4%

Analysis of Variance

| Source | DF | SS | MS | F | P |
|----------------|----|------------|------------|-------|-------|
| Regression | 1 | 0.00060380 | 0.00060380 | 60.89 | 0.000 |
| Residual Error | 28 | 0.00027766 | 0.00000992 | | |
| Total | 29 | 0.00088147 | | | |

Unusual Observations

| Obs | ppv | Ratio | Fit | SE Fit | Residual | St Resid |
|-----|-----|----------|----------|----------|-----------|----------|
| 10 | 486 | 0.985000 | 0.992864 | 0.000629 | -0.007864 | -2.55R |

R denotes an observation with a large standardized residual.

Predicted Values for New Observations

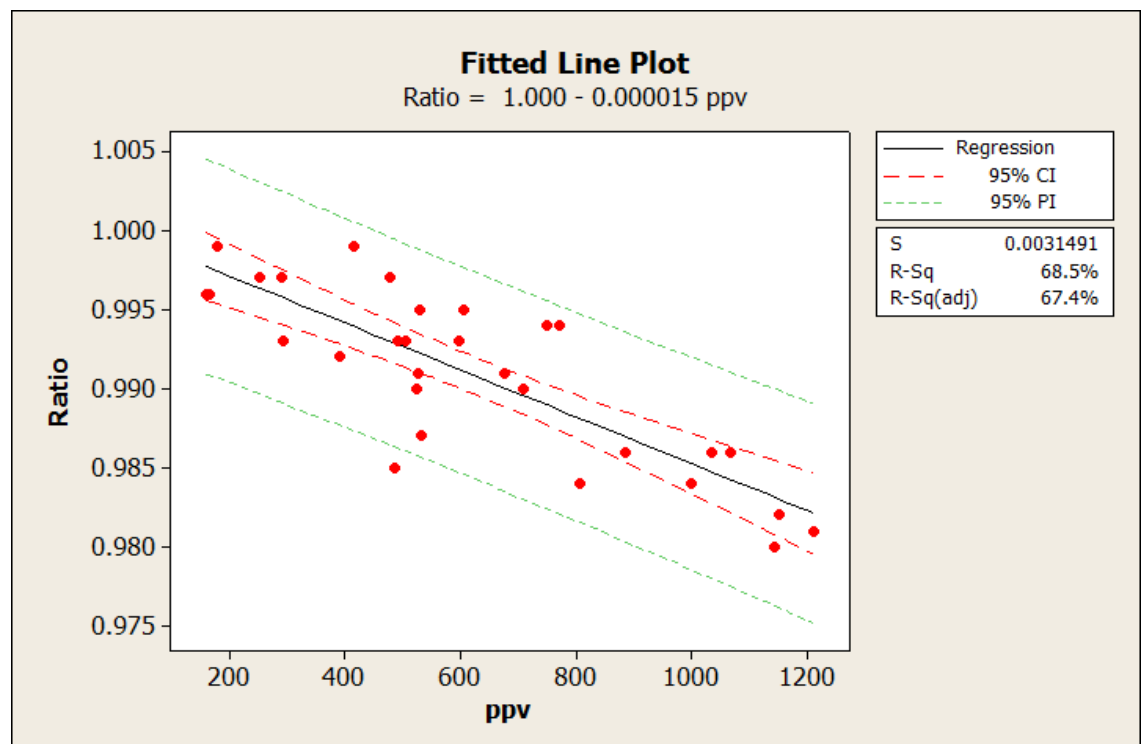
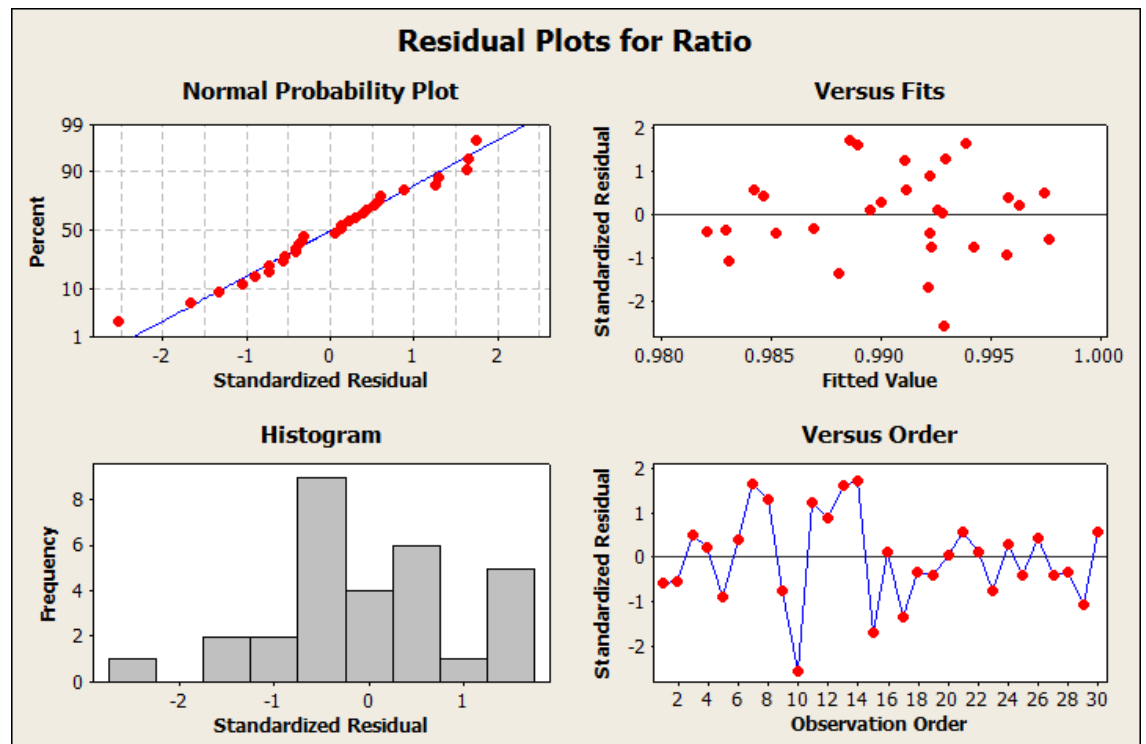
| New Obs | Fit | SE Fit | 95% CI | 95% PI |
|---------|----------|----------|----------------------|----------------------|
| 1 | 0.988947 | 0.000625 | (0.987666, 0.990228) | (0.982370, 0.995523) |

Values of Predictors for New Observations

| New Obs | ppv |
|---------|-----|
| 1 | 750 |

Note: Regression analysis continues on the next page with residual plots and a fitted line plot.

Please see over ...



Please see over ...

Some Statistical Formulae

1. Let p be the sample proportion of ‘successes’ in a binomial setting where the number of trials is n and the true probability of a success is π . Under appropriate conditions

$$\frac{p - \pi}{\sqrt{\pi(1 - \pi)/n}} \sim N(0, 1) \text{ (approximately).}$$

2. Let \bar{X} be the sample average from a random sample of size n from a population with mean μ and standard deviation σ .

Under appropriate conditions

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1) \text{ (either approximately or exactly).}$$

3. Let \bar{X} and S be the sample average and standard deviation obtained from a random sample of size n from a population with mean μ .

Under appropriate conditions

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{(n-1)}.$$

4. For two independent samples of size n_1 and n_2 from two populations with means μ_1 and μ_2 respectively, let \bar{X}_i and S_i be the sample average and sample standard deviation of sample i for $i = 1$ and 2 .

Under appropriate conditions

$$\frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{(n_1+n_2-2)},$$

where S_p is the pooled sample standard deviation,

$$S_p = \sqrt{\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}}.$$

5. The exponential variable $X \sim \text{Exp}(\lambda)$ has density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

This exponential variable has mean $\mu = \frac{1}{\lambda}$ and variance $\sigma^2 = \frac{1}{\lambda^2}$.

6. The binomial variable $X \sim B(n, \pi)$ has probability mass function

$$Pr(x) = \binom{n}{x} \pi^x (1 - \pi)^{n-x} \text{ for } x = 0, 1, \dots, n.$$

This binomial variable has mean $\mu = n\pi$ and variance $\sigma^2 = n\pi(1 - \pi)$.

7. The Poisson variable $X \sim \text{Poisson}(\lambda)$ has probability mass function

$$Pr(x) = \frac{e^{-\lambda} \lambda^x}{x!} \text{ for } x = 0, 1, 2, \dots$$

This Poisson variable has mean $\mu = \lambda$ and variance $\sigma^2 = \lambda$.

Statistical Tables

t distribution critical values

Key: Table entry for p and C is the critical value t^* with probability p lying to its right and probability C lying between $-t^*$ and t^* .

| Upper tail probability p | | | | | | | | | | | | |
|----------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| df | .25 | .20 | .15 | .10 | .05 | .025 | .02 | .01 | .005 | .0025 | .001 | .0005 |
| 1 | 1.000 | 1.376 | 1.963 | 3.078 | 6.314 | 12.71 | 15.89 | 31.82 | 63.66 | 127.3 | 318.3 | 636.6 |
| 2 | 0.816 | 1.061 | 1.386 | 1.886 | 2.920 | 4.303 | 4.849 | 6.965 | 9.925 | 14.09 | 22.33 | 31.60 |
| 3 | 0.765 | 0.978 | 1.250 | 1.638 | 2.353 | 3.182 | 3.482 | 4.541 | 5.841 | 7.453 | 10.21 | 12.92 |
| 4 | 0.741 | 0.941 | 1.190 | 1.533 | 2.132 | 2.776 | 2.999 | 3.747 | 4.604 | 5.598 | 7.173 | 8.610 |
| 5 | 0.727 | 0.920 | 1.156 | 1.476 | 2.015 | 2.571 | 2.757 | 3.365 | 4.032 | 4.773 | 5.893 | 6.869 |
| 6 | 0.718 | 0.906 | 1.134 | 1.440 | 1.943 | 2.447 | 2.612 | 3.143 | 3.707 | 4.317 | 5.208 | 5.959 |
| 7 | 0.711 | 0.896 | 1.119 | 1.415 | 1.895 | 2.365 | 2.517 | 2.998 | 3.499 | 4.029 | 4.785 | 5.408 |
| 8 | 0.706 | 0.889 | 1.108 | 1.397 | 1.860 | 2.306 | 2.449 | 2.896 | 3.355 | 3.833 | 4.501 | 5.041 |
| 9 | 0.703 | 0.883 | 1.100 | 1.383 | 1.833 | 2.262 | 2.398 | 2.821 | 3.250 | 3.690 | 4.297 | 4.781 |
| 10 | 0.700 | 0.879 | 1.093 | 1.372 | 1.812 | 2.228 | 2.359 | 2.764 | 3.169 | 3.581 | 4.144 | 4.587 |
| 11 | 0.697 | 0.876 | 1.088 | 1.363 | 1.796 | 2.201 | 2.328 | 2.718 | 3.106 | 3.497 | 4.025 | 4.437 |
| 12 | 0.695 | 0.873 | 1.083 | 1.356 | 1.782 | 2.179 | 2.303 | 2.681 | 3.055 | 3.428 | 3.930 | 4.318 |
| 13 | 0.694 | 0.870 | 1.079 | 1.350 | 1.771 | 2.160 | 2.282 | 2.650 | 3.012 | 3.372 | 3.852 | 4.221 |
| 14 | 0.692 | 0.868 | 1.076 | 1.345 | 1.761 | 2.145 | 2.264 | 2.624 | 2.977 | 3.326 | 3.787 | 4.140 |
| 15 | 0.691 | 0.866 | 1.074 | 1.341 | 1.753 | 2.131 | 2.249 | 2.602 | 2.947 | 3.286 | 3.733 | 4.073 |
| 16 | 0.690 | 0.865 | 1.071 | 1.337 | 1.746 | 2.120 | 2.235 | 2.583 | 2.921 | 3.252 | 3.686 | 4.015 |
| 17 | 0.689 | 0.863 | 1.069 | 1.333 | 1.740 | 2.110 | 2.224 | 2.567 | 2.898 | 3.222 | 3.646 | 3.965 |
| 18 | 0.688 | 0.862 | 1.067 | 1.330 | 1.734 | 2.101 | 2.214 | 2.552 | 2.878 | 3.197 | 3.610 | 3.922 |
| 19 | 0.688 | 0.861 | 1.066 | 1.328 | 1.729 | 2.093 | 2.205 | 2.539 | 2.861 | 3.174 | 3.579 | 3.883 |
| 20 | 0.687 | 0.860 | 1.064 | 1.325 | 1.725 | 2.086 | 2.197 | 2.528 | 2.845 | 3.153 | 3.552 | 3.850 |
| 21 | 0.686 | 0.859 | 1.063 | 1.323 | 1.721 | 2.080 | 2.189 | 2.518 | 2.831 | 3.135 | 3.527 | 3.819 |
| 22 | 0.686 | 0.858 | 1.061 | 1.321 | 1.717 | 2.074 | 2.183 | 2.508 | 2.819 | 3.119 | 3.505 | 3.792 |
| 23 | 0.685 | 0.858 | 1.060 | 1.319 | 1.714 | 2.069 | 2.177 | 2.500 | 2.807 | 3.104 | 3.485 | 3.768 |
| 24 | 0.685 | 0.857 | 1.059 | 1.318 | 1.711 | 2.064 | 2.172 | 2.492 | 2.797 | 3.091 | 3.467 | 3.745 |
| 25 | 0.684 | 0.856 | 1.058 | 1.316 | 1.708 | 2.060 | 2.167 | 2.485 | 2.787 | 3.078 | 3.450 | 3.725 |
| 26 | 0.684 | 0.856 | 1.058 | 1.315 | 1.706 | 2.056 | 2.162 | 2.479 | 2.779 | 3.067 | 3.435 | 3.707 |
| 27 | 0.684 | 0.855 | 1.057 | 1.314 | 1.703 | 2.052 | 2.158 | 2.473 | 2.771 | 3.057 | 3.421 | 3.690 |
| 28 | 0.683 | 0.855 | 1.056 | 1.313 | 1.701 | 2.048 | 2.154 | 2.467 | 2.763 | 3.047 | 3.408 | 3.674 |
| 29 | 0.683 | 0.854 | 1.055 | 1.311 | 1.699 | 2.045 | 2.150 | 2.462 | 2.756 | 3.038 | 3.396 | 3.659 |
| 30 | 0.683 | 0.854 | 1.055 | 1.310 | 1.697 | 2.042 | 2.147 | 2.457 | 2.750 | 3.030 | 3.385 | 3.646 |
| 40 | 0.681 | 0.851 | 1.050 | 1.303 | 1.684 | 2.021 | 2.123 | 2.423 | 2.704 | 2.971 | 3.307 | 3.551 |
| 50 | 0.679 | 0.849 | 1.047 | 1.299 | 1.676 | 2.009 | 2.109 | 2.403 | 2.678 | 2.937 | 3.261 | 3.496 |
| 60 | 0.679 | 0.848 | 1.045 | 1.296 | 1.671 | 2.000 | 2.099 | 2.390 | 2.660 | 2.915 | 3.232 | 3.460 |
| 80 | 0.678 | 0.846 | 1.043 | 1.292 | 1.664 | 1.990 | 2.088 | 2.374 | 2.639 | 2.887 | 3.195 | 3.416 |
| 100 | 0.677 | 0.845 | 1.042 | 1.290 | 1.660 | 1.984 | 2.081 | 2.364 | 2.626 | 2.871 | 3.174 | 3.390 |
| 1000 | 0.675 | 0.842 | 1.037 | 1.282 | 1.646 | 1.962 | 2.056 | 2.330 | 2.581 | 2.813 | 3.098 | 3.300 |
| | .50 | .60 | .70 | 0.80 | .90 | .95 | .96 | .98 | .99 | .995 | .998 | .999 |
| Probability C | | | | | | | | | | | | |

Standard normal probabilities

Key: Table entry for z is the area under the standard normal curve to the left of z

| z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| −3.4 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0002 |
| −3.3 | 0.0005 | 0.0005 | 0.0005 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0003 |
| −3.2 | 0.0007 | 0.0007 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0005 | 0.0005 | 0.0005 |
| −3.1 | 0.0010 | 0.0009 | 0.0009 | 0.0009 | 0.0008 | 0.0008 | 0.0008 | 0.0008 | 0.0007 | 0.0007 |
| −3.0 | 0.0013 | 0.0013 | 0.0013 | 0.0012 | 0.0012 | 0.0011 | 0.0011 | 0.0011 | 0.0010 | 0.0010 |
| −2.9 | 0.0019 | 0.0018 | 0.0018 | 0.0017 | 0.0016 | 0.0016 | 0.0015 | 0.0015 | 0.0014 | 0.0014 |
| −2.8 | 0.0026 | 0.0025 | 0.0024 | 0.0023 | 0.0023 | 0.0022 | 0.0021 | 0.0021 | 0.0020 | 0.0019 |
| −2.7 | 0.0035 | 0.0034 | 0.0033 | 0.0032 | 0.0031 | 0.0030 | 0.0029 | 0.0028 | 0.0027 | 0.0026 |
| −2.6 | 0.0047 | 0.0045 | 0.0044 | 0.0043 | 0.0041 | 0.0040 | 0.0039 | 0.0038 | 0.0037 | 0.0036 |
| −2.5 | 0.0062 | 0.0060 | 0.0059 | 0.0057 | 0.0055 | 0.0054 | 0.0052 | 0.0051 | 0.0049 | 0.0048 |
| −2.4 | 0.0082 | 0.0080 | 0.0078 | 0.0075 | 0.0073 | 0.0071 | 0.0069 | 0.0068 | 0.0066 | 0.0064 |
| −2.3 | 0.0107 | 0.0104 | 0.0102 | 0.0099 | 0.0096 | 0.0094 | 0.0091 | 0.0089 | 0.0087 | 0.0084 |
| −2.2 | 0.0139 | 0.0136 | 0.0132 | 0.0129 | 0.0125 | 0.0122 | 0.0119 | 0.0116 | 0.0113 | 0.0110 |
| −2.1 | 0.0179 | 0.0174 | 0.0170 | 0.0166 | 0.0162 | 0.0158 | 0.0154 | 0.0150 | 0.0146 | 0.0143 |
| −2.0 | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| −1.9 | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.0250 | 0.0244 | 0.0239 | 0.0233 |
| −1.8 | 0.0359 | 0.0351 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0301 | 0.0294 |
| −1.7 | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| −1.6 | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| −1.5 | 0.0668 | 0.0655 | 0.0643 | 0.0630 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0571 | 0.0559 |
| −1.4 | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0721 | 0.0708 | 0.0694 | 0.0681 |
| −1.3 | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| −1.2 | 0.1151 | 0.1131 | 0.1112 | 0.1093 | 0.1075 | 0.1056 | 0.1038 | 0.1020 | 0.1003 | 0.0985 |
| −1.1 | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.1230 | 0.1210 | 0.1190 | 0.1170 |
| −1.0 | 0.1587 | 0.1562 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |
| −0.9 | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.1660 | 0.1635 | 0.1611 |
| −0.8 | 0.2119 | 0.2090 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1922 | 0.1894 | 0.1867 |
| −0.7 | 0.2420 | 0.2389 | 0.2358 | 0.2327 | 0.2296 | 0.2266 | 0.2236 | 0.2206 | 0.2177 | 0.2148 |
| −0.6 | 0.2743 | 0.2709 | 0.2676 | 0.2643 | 0.2611 | 0.2578 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| −0.5 | 0.3085 | 0.3050 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.2810 | 0.2776 |
| −0.4 | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.3300 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| −0.3 | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.3520 | 0.3483 |
| −0.2 | 0.4207 | 0.4168 | 0.4129 | 0.4090 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| −0.1 | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| −0.0 | 0.5000 | 0.4960 | 0.4920 | 0.4880 | 0.4840 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| 3.1 | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| 3.2 | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| 3.3 | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| 3.4 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |

χ^2 distribution critical values

Key: Table entry for p is the critical value with probability p lying to its right.

| df | Upper tail probability p | | | | | | | | | |
|-----|----------------------------|---------|---------|--------|--------|--------|--------|--------|--------|--------|
| | .995 | .99 | .975 | .95 | .90 | .10 | .05 | .025 | .01 | .005 |
| 1 | 0.000039 | 0.00016 | 0.00098 | 0.0039 | 0.0158 | 2.71 | 3.84 | 5.02 | 6.63 | 7.88 |
| 2 | 0.0100 | 0.0201 | 0.0506 | 0.1026 | 0.2107 | 4.61 | 5.99 | 7.38 | 9.21 | 10.60 |
| 3 | 0.0717 | 0.115 | 0.216 | 0.352 | 0.584 | 6.25 | 7.81 | 9.35 | 11.34 | 12.84 |
| 4 | 0.207 | 0.297 | 0.484 | 0.711 | 1.064 | 7.78 | 9.49 | 11.14 | 13.28 | 14.86 |
| 5 | 0.412 | 0.554 | 0.831 | 1.15 | 1.61 | 9.24 | 11.07 | 12.83 | 15.09 | 16.75 |
| 6 | 0.676 | 0.872 | 1.24 | 1.64 | 2.20 | 10.64 | 12.59 | 14.45 | 16.81 | 18.55 |
| 7 | 0.989 | 1.24 | 1.69 | 2.17 | 2.83 | 12.02 | 14.07 | 16.01 | 18.48 | 20.28 |
| 8 | 1.34 | 1.65 | 2.18 | 2.73 | 3.49 | 13.36 | 15.51 | 17.53 | 20.09 | 21.95 |
| 9 | 1.73 | 2.09 | 2.70 | 3.33 | 4.17 | 14.68 | 16.92 | 19.02 | 21.67 | 23.59 |
| 10 | 2.16 | 2.56 | 3.25 | 3.94 | 4.87 | 15.99 | 18.31 | 20.48 | 23.21 | 25.19 |
| 11 | 2.60 | 3.05 | 3.82 | 4.57 | 5.58 | 17.28 | 19.68 | 21.92 | 24.72 | 26.76 |
| 12 | 3.07 | 3.57 | 4.40 | 5.23 | 6.30 | 18.55 | 21.03 | 23.34 | 26.22 | 28.30 |
| 13 | 3.57 | 4.11 | 5.01 | 5.89 | 7.04 | 19.81 | 22.36 | 24.74 | 27.69 | 29.82 |
| 14 | 4.07 | 4.66 | 5.63 | 6.57 | 7.79 | 21.06 | 23.68 | 26.12 | 29.14 | 31.32 |
| 15 | 4.60 | 5.23 | 6.26 | 7.26 | 8.55 | 22.31 | 25.00 | 27.49 | 30.58 | 32.80 |
| 16 | 5.14 | 5.81 | 6.91 | 7.96 | 9.31 | 23.54 | 26.30 | 28.85 | 32.00 | 34.27 |
| 17 | 5.70 | 6.41 | 7.56 | 8.67 | 10.09 | 24.77 | 27.59 | 30.19 | 33.41 | 35.72 |
| 18 | 6.26 | 7.01 | 8.23 | 9.39 | 10.86 | 25.99 | 28.87 | 31.53 | 34.81 | 37.16 |
| 19 | 6.84 | 7.63 | 8.91 | 10.12 | 11.65 | 27.20 | 30.14 | 32.85 | 36.19 | 38.58 |
| 20 | 7.43 | 8.26 | 9.59 | 10.85 | 12.44 | 28.41 | 31.41 | 34.17 | 37.57 | 40.00 |
| 21 | 8.03 | 8.90 | 10.28 | 11.59 | 13.24 | 29.62 | 32.67 | 35.48 | 38.93 | 41.40 |
| 22 | 8.64 | 9.54 | 10.98 | 12.34 | 14.04 | 30.81 | 33.92 | 36.78 | 40.29 | 42.80 |
| 23 | 9.26 | 10.20 | 11.69 | 13.09 | 14.85 | 32.01 | 35.17 | 38.08 | 41.64 | 44.18 |
| 24 | 9.89 | 10.86 | 12.40 | 13.85 | 15.66 | 33.20 | 36.42 | 39.36 | 42.98 | 45.56 |
| 25 | 10.52 | 11.52 | 13.12 | 14.61 | 16.47 | 34.38 | 37.65 | 40.65 | 44.31 | 46.93 |
| 26 | 11.16 | 12.20 | 13.84 | 15.38 | 17.29 | 35.56 | 38.89 | 41.92 | 45.64 | 48.29 |
| 27 | 11.81 | 12.88 | 14.57 | 16.15 | 18.11 | 36.74 | 40.11 | 43.19 | 46.96 | 49.64 |
| 28 | 12.46 | 13.56 | 15.31 | 16.93 | 18.94 | 37.92 | 41.34 | 44.46 | 48.28 | 50.99 |
| 29 | 13.12 | 14.26 | 16.05 | 17.71 | 19.77 | 39.09 | 42.56 | 45.72 | 49.59 | 52.34 |
| 30 | 13.79 | 14.95 | 16.79 | 18.49 | 20.60 | 40.26 | 43.77 | 46.98 | 50.89 | 53.67 |
| 40 | 20.71 | 22.16 | 24.43 | 26.51 | 29.05 | 51.81 | 55.76 | 59.34 | 63.69 | 66.77 |
| 50 | 27.99 | 29.71 | 32.36 | 34.76 | 37.69 | 63.17 | 67.50 | 71.42 | 76.15 | 79.49 |
| 60 | 35.53 | 37.48 | 40.48 | 43.19 | 46.46 | 74.40 | 79.08 | 83.30 | 88.38 | 91.95 |
| 80 | 51.17 | 53.54 | 57.15 | 60.39 | 64.28 | 96.58 | 101.88 | 106.63 | 112.33 | 116.32 |
| 100 | 67.33 | 70.06 | 74.22 | 77.93 | 82.36 | 118.50 | 124.34 | 129.56 | 135.81 | 140.17 |

F distribution critical values

Key: p =Upper tail probability p , df_n =degrees of freedom in numerator, df_d =degrees of freedom in denominator,
 * Multiply by 10, † Multiply by 100.

| | df _n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 15 | 20 | 24 | 30 | 40 | 60 | 120 | ∞ |
|------|-----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1 | p | | | | | | | | | | | | | | | | | | | |
| | .05 | 161 | 200 | 216 | 225 | 230 | 234 | 237 | 239 | 241 | 242 | 244 | 246 | 248 | 249 | 250 | 251 | 252 | 253 | 254 |
| | .025 | 648 | 800 | 864 | 900 | 922 | 937 | 948 | 957 | 963 | 969 | 977 | 986 | 993 | 997 | 1001 | 1006 | 1010 | 1014 | 1018 |
| | .01 | 405* | 500* | 540* | 563* | 576* | 586* | 593* | 598* | 602* | 606* | 611* | 616* | 621* | 624* | 626* | 629* | 631* | 634* | 637* |
| | .005 | 162† | 200† | 216† | 225† | 231† | 234† | 237† | 239† | 241† | 242† | 244† | 246† | 248† | 249† | 250† | 251† | 253† | 254† | 255† |
| 2 | .05 | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.33 | 19.35 | 19.37 | 19.38 | 19.40 | 19.41 | 19.43 | 19.45 | 19.45 | 19.46 | 19.47 | 19.48 | 19.49 | 19.50 |
| | .025 | 38.51 | 39.00 | 39.17 | 39.25 | 39.30 | 39.33 | 39.36 | 39.37 | 39.39 | 39.40 | 39.41 | 39.43 | 39.45 | 39.46 | 39.46 | 39.47 | 39.48 | 39.49 | 39.50 |
| | .01 | 98.50 | 99.00 | 99.17 | 99.25 | 99.30 | 99.33 | 99.36 | 99.37 | 99.39 | 99.40 | 99.42 | 99.43 | 99.45 | 99.46 | 99.47 | 99.47 | 99.48 | 99.49 | 99.50 |
| | .005 | 199 | 199 | 199 | 199 | 199 | 199 | 199 | 199 | 199 | 199 | 199 | 199 | 199 | 200 | 200 | 200 | 200 | 200 | 200 |
| | 3 | .05 | 10.13 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.89 | 8.85 | 8.81 | 8.79 | 8.74 | 8.70 | 8.66 | 8.64 | 8.62 | 8.59 | 8.57 | 8.55 |
| .025 | | 17.44 | 16.04 | 15.44 | 15.10 | 14.88 | 14.73 | 14.62 | 14.54 | 14.47 | 14.42 | 14.34 | 14.25 | 14.17 | 14.12 | 14.08 | 14.04 | 13.99 | 13.95 | 13.90 |
| .01 | | 34.12 | 30.82 | 29.46 | 28.71 | 28.24 | 27.91 | 27.67 | 27.49 | 27.35 | 27.23 | 27.05 | 26.87 | 26.69 | 26.60 | 26.50 | 26.41 | 26.32 | 26.22 | 26.13 |
| .005 | | 55.55 | 49.80 | 47.47 | 46.19 | 45.39 | 44.84 | 44.43 | 44.13 | 43.88 | 43.69 | 43.39 | 43.08 | 42.78 | 42.62 | 42.47 | 42.31 | 42.15 | 41.99 | 41.83 |
| 4 | | .05 | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.09 | 6.04 | 6.00 | 5.96 | 5.91 | 5.86 | 5.80 | 5.77 | 5.75 | 5.72 | 5.69 | 5.66 |
| | .025 | 12.22 | 10.65 | 9.98 | 9.60 | 9.36 | 9.20 | 9.07 | 8.98 | 8.90 | 8.84 | 8.75 | 8.66 | 8.56 | 8.51 | 8.46 | 8.41 | 8.36 | 8.31 | 8.26 |
| | .01 | 21.20 | 18.00 | 16.69 | 15.98 | 15.52 | 15.21 | 14.98 | 14.80 | 14.66 | 14.55 | 14.37 | 14.20 | 14.02 | 13.93 | 13.84 | 13.75 | 13.65 | 13.56 | 13.46 |
| | .005 | 31.33 | 26.28 | 24.26 | 23.15 | 22.46 | 21.97 | 21.62 | 21.35 | 21.14 | 20.97 | 20.70 | 20.44 | 20.17 | 20.03 | 19.89 | 19.75 | 19.61 | 19.47 | 19.32 |
| | 5 | .05 | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.88 | 4.82 | 4.77 | 4.74 | 4.68 | 4.62 | 4.56 | 4.53 | 4.50 | 4.46 | 4.43 | 4.40 |
| .025 | | 10.01 | 8.43 | 7.76 | 7.39 | 7.15 | 6.98 | 6.85 | 6.76 | 6.68 | 6.62 | 6.52 | 6.43 | 6.33 | 6.28 | 6.23 | 6.18 | 6.12 | 6.07 | 6.02 |
| .01 | | 16.26 | 13.27 | 12.06 | 11.39 | 10.97 | 10.67 | 10.46 | 10.29 | 10.16 | 10.05 | 9.89 | 9.72 | 9.55 | 9.47 | 9.38 | 9.29 | 9.20 | 9.11 | 9.02 |
| .005 | | 22.78 | 18.31 | 16.53 | 15.56 | 14.94 | 14.51 | 14.20 | 13.96 | 13.77 | 13.62 | 13.38 | 13.15 | 12.90 | 12.78 | 12.66 | 12.53 | 12.40 | 12.27 | 12.14 |
| 6 | | .05 | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 | 4.15 | 4.10 | 4.06 | 4.00 | 3.94 | 3.87 | 3.84 | 3.81 | 3.77 | 3.74 | 3.70 |
| | .025 | 8.81 | 7.26 | 6.60 | 6.23 | 5.99 | 5.82 | 5.70 | 5.60 | 5.52 | 5.46 | 5.37 | 5.27 | 5.17 | 5.12 | 5.07 | 5.01 | 4.96 | 4.90 | 4.85 |
| | .01 | 13.75 | 10.92 | 9.78 | 9.15 | 8.75 | 8.47 | 8.26 | 8.10 | 7.98 | 7.87 | 7.72 | 7.56 | 7.40 | 7.31 | 7.23 | 7.14 | 7.06 | 6.97 | 6.88 |
| | .005 | 18.63 | 14.54 | 12.92 | 12.03 | 11.46 | 11.07 | 10.79 | 10.57 | 10.39 | 10.25 | 10.03 | 9.81 | 9.59 | 9.47 | 9.36 | 9.24 | 9.12 | 9.00 | 8.88 |
| | 7 | .05 | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 | 3.73 | 3.68 | 3.64 | 3.57 | 3.51 | 3.44 | 3.41 | 3.38 | 3.34 | 3.30 | 3.27 |
| .025 | | 8.07 | 6.54 | 5.89 | 5.52 | 5.29 | 5.12 | 4.99 | 4.90 | 4.82 | 4.76 | 4.67 | 4.57 | 4.47 | 4.41 | 4.36 | 4.31 | 4.25 | 4.20 | 4.14 |
| .01 | | 12.25 | 9.55 | 8.45 | 7.85 | 7.46 | 7.19 | 6.99 | 6.84 | 6.72 | 6.62 | 6.47 | 6.31 | 6.16 | 6.07 | 5.99 | 5.91 | 5.82 | 5.74 | 5.65 |
| .005 | | 16.24 | 12.40 | 10.88 | 10.05 | 9.52 | 9.16 | 8.89 | 8.68 | 8.51 | 8.38 | 8.18 | 7.97 | 7.75 | 7.64 | 7.53 | 7.42 | 7.31 | 7.19 | 7.08 |
| 8 | | .05 | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 | 3.44 | 3.39 | 3.35 | 3.28 | 3.22 | 3.15 | 3.12 | 3.08 | 3.04 | 3.01 | 2.97 |
| | .025 | 7.57 | 6.06 | 5.42 | 5.05 | 4.82 | 4.65 | 4.53 | 4.43 | 4.36 | 4.30 | 4.20 | 4.10 | 4.00 | 3.95 | 3.89 | 3.84 | 3.78 | 3.73 | 3.67 |
| | .01 | 11.26 | 8.65 | 7.59 | 7.01 | 6.63 | 6.37 | 6.18 | 6.03 | 5.91 | 5.81 | 5.67 | 5.52 | 5.36 | 5.28 | 5.20 | 5.12 | 5.03 | 4.95 | 4.86 |
| | .005 | 14.69 | 11.04 | 9.60 | 8.81 | 8.30 | 7.95 | 7.69 | 7.50 | 7.34 | 7.21 | 7.01 | 6.81 | 6.61 | 6.50 | 6.40 | 6.29 | 6.18 | 6.06 | 5.95 |
| | 9 | .05 | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 | 3.23 | 3.18 | 3.14 | 3.07 | 3.01 | 2.94 | 2.90 | 2.86 | 2.83 | 2.79 | 2.75 |
| .025 | | 7.21 | 5.71 | 5.08 | 4.72 | 4.48 | 4.32 | 4.20 | 4.10 | 4.03 | 3.96 | 3.87 | 3.77 | 3.67 | 3.61 | 3.56 | 3.51 | 3.45 | 3.39 | 3.33 |
| .01 | | 10.56 | 8.02 | 6.99 | 6.42 | 6.06 | 5.80 | 5.61 | 5.47 | 5.35 | 5.26 | 5.11 | 4.96 | 4.81 | 4.73 | 4.65 | 4.57 | 4.48 | 4.40 | 4.31 |
| .005 | | 13.61 | 10.11 | 8.72 | 7.96 | 7.47 | 7.13 | 6.88 | 6.69 | 6.54 | 6.42 | 6.23 | 6.03 | 5.83 | 5.73 | 5.62 | 5.52 | 5.41 | 5.30 | 5.19 |
| 10 | | .05 | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.14 | 3.07 | 3.02 | 2.98 | 2.91 | 2.85 | 2.77 | 2.74 | 2.70 | 2.66 | 2.62 | 2.58 |
| | .025 | 6.94 | 5.46 | 4.83 | 4.47 | 4.24 | 4.07 | 3.95 | 3.85 | 3.78 | 3.72 | 3.62 | 3.52 | 3.42 | 3.37 | 3.31 | 3.26 | 3.20 | 3.14 | 3.08 |
| | .01 | 10.04 | 7.56 | 6.55 | 5.99 | 5.64 | 5.39 | 5.20 | 5.06 | 4.94 | 4.85 | 4.71 | 4.56 | 4.41 | 4.33 | 4.25 | 4.17 | 4.08 | 4.00 | 3.91 |
| | .005 | 12.83 | 9.43 | 8.08 | 7.34 | 6.87 | 6.54 | 6.30 | 6.12 | 5.97 | 5.85 | 5.66 | 5.47 | 5.27 | 5.17 | 5.07 | 4.97 | 4.86 | 4.75 | 4.64 |
| | 12 | .05 | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 | 2.85 | 2.80 | 2.75 | 2.69 | 2.62 | 2.54 | 2.51 | 2.47 | 2.43 | 2.38 | 2.34 |
| .025 | | 6.55 | 5.10 | 4.47 | 4.12 | 3.89 | 3.73 | 3.61 | 3.51 | 3.44 | 3.37 | 3.28 | 3.18 | 3.07 | 3.02 | 2.96 | 2.91 | 2.85 | 2.79 | 2.72 |
| .01 | | 9.33 | 6.93 | 5.95 | 5.41 | 5.06 | 4.82 | 4.64 | 4.50 | 4.39 | 4.30 | 4.16 | 4.01 | 3.86 | 3.78 | 3.70 | 3.62 | 3.54 | 3.45 | 3.36 |
| .005 | | 11.75 | 8.51 | 7.23 | 6.52 | 6.07 | 5.76 | 5.52 | 5.35 | 5.20 | 5.09 | 4.91 | 4.72 | 4.53 | 4.43 | 4.33 | 4.23 | 4.12 | 4.01 | 3.90 |
| 15 | | .05 | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.71 | 2.64 | 2.59 | 2.54 | 2.48 | 2.40 | 2.33 | 2.29 | 2.25 | 2.20 | 2.16 | 2.11 |
| | .025 | 6.20 | 4.77 | 4.15 | 3.80 | 3.58 | 3.41 | 3.29 | 3.20 | 3.12 | 3.06 | 2.96 | 2.86 | 2.76 | 2.70 | 2.64 | 2.59 | 2.52 | 2.46 | 2.40 |
| | .01 | 8.68 | 6.36 | 5.42 | 4.89 | 4.56 | 4.32 | 4.14 | 4.00 | 3.89 | 3.80 | 3.67 | 3.52 | 3.37 | 3.29 | 3.21 | 3.13 | 3.05 | 2.96 | 2.87 |
| | .005 | 10.80 | 7.70 | 6.48 | 5.80 | 5.37 | 5.07 | 4.85 | 4.67 | 4.54 | 4.42 | 4.25 | 4.07 | 3.88 | 3.79 | 3.69 | 3.58 | 3.48 | 3.37 | 3.26 |
| | 20 | .05 | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.51 | 2.45 | 2.39 | 2.35 | 2.28 | 2.20 | 2.12 | 2.08 | 2.04 | 1.99 | 1.95 | 1.90 |
| .025 | | 5.87 | 4.46 | 3.86 | 3.51 | 3.29 | 3.13 | 3.01 | 2.91 | 2.84 | 2.77 | 2.68 | 2.57 | 2.46 | 2.41 | 2.35 | 2.29 | 2.22 | 2.16 | 2.09 |
| .01 | | 8.10 | 5.85 | 4.94 | 4.43 | 4.10 | 3.87 | 3.70 | 3.56 | 3.46 | 3.37 | 3.23 | 3.09 | 2.94 | 2.86 | 2.78 | 2.69 | 2.61 | 2.52 | 2.42 |
| .005 | | 9.94 | 6.99 | 5.82 | 5.17 | 4.76 | 4.47 | 4.26 | 4.09 | 3.96 | 3.85 | 3.68 | 3.50 | 3.32 | 3.22 | 3.12 | 3.02 | 2.92 | 2.81 | 2.69 |
| 24 | | .05 | 4.26 | 3.40 | 3.01 | 2.78 | 2.62 | 2.51 | 2.42 | 2.36 | 2.30 | 2.25 | 2.18 | 2.11 | 2.03 | 1.98 | 1.94 | 1.89 | 1.84 | 1.79 |