(i

Q1 a)

i) The system has no structure, so LU factorization is recommended. to solve the system. (ignore substitutions)

flops in 1 second $= 3 \times 10^9$

flops in 1 hour $= 3 \times 10^9 \times 3600$

Since LU factorization takes $\frac{2n^3}{3} + O(n^2)$ flops

$$\frac{2n^3}{3} = 3 \times 10^9 \times 3600$$

$$n = \left\lceil \left(\frac{3}{2} \times 3 \times 10^9 \times 3600 \right)^{1/3} \right\rceil = 25\,303$$

So the claim is TRUE.

ii) Total error:

$$E(h) = R(h) + T(h) \quad \text{(rounding error + truncation error}$$

$$= c_1 \frac{\varepsilon}{h^2} + c_2 h^2$$

$$\frac{dE}{dh} = 0 \Rightarrow -\frac{2c_1 \varepsilon}{h^3} + 2c_2 h = 0$$

$$\Rightarrow h = O(\varepsilon^{1/4}).$$

$$\frac{d^2E}{dh^2} = \frac{6c_1 \varepsilon}{h^4} + 2c_2 > 0$$

$\Rightarrow E$ is minimized when $h = O(\varepsilon^{1/4})$

So the claim is TRUE.

iii) $p(x) = x^3 - a$

$p'(x) = 3x^2$

Newton's method is given by: start with an initial guess x_1, then

$$x_{k+1} = x_k - \frac{p(x_k)}{p'(x_k)} \qquad k \geq 1$$

$$= x_k - \frac{x_k^3 - a}{3x_k^2} = x_k - \frac{x_k}{3} + \frac{a}{3x_k^2}$$

$$= \frac{2}{3} x_k + \frac{a}{3x_k^2} = (2x_k + a/x_k^2)/3$$

So the program will find a solution for the equation $p(x) = 0$. The claim is TRUE.

iv) The matrix is symmetric, so

$$\kappa_2(A) = \frac{|\lambda_{max}|}{|\lambda_{min}|} = \frac{2.6e+6}{1.3e-7} = 2 \times 10^{13}$$

$$\text{rel\_err}(A) \approx \varepsilon \approx 2.2 \times 10^{-16}$$

$$\text{rel\_err}(\underset{\sim}{b}) < 0.5 \times 10^{-6}$$

$$\text{rel\_err}(\underset{\sim}{x}) \leq \kappa_2(A)(\text{Rel\_err}(A) + \text{rel\_err}(b))$$

$$\approx 2 \times 10^{13} (2.2 \times 10^{-16} + 0.5 \times 10^{-6})$$

$$\approx 4.4 \times 10^{-3} + 1 \times 10^{7}$$

No digits are significant.

So the claim is FALSE.

v) $A = R^T R$

$$A^T = (R^T R)^T = R^T R^{TT} = R^T R = A$$

$\Rightarrow$ A is symmetric

$$\underset{\sim}{x}^T A \underset{\sim}{x} = \underset{\sim}{x}^T R^T R \underset{\sim}{x} = \|R\underset{\sim}{x}\|_2^2 \geq 0$$

$R\underset{\sim}{x} = 0 \Rightarrow \underset{\sim}{x} = 0$ since R is non-singular

So A is positive definite.

The claim is TRUE.

vi) Q is an orthogonal matrix since it is the outp. from QR factorization.

$$\text{So } \|Q\underset{\sim}{x}\|_2^2 = (Q\underset{\sim}{x})^T Q\underset{\sim}{x} = \underset{\sim}{x}^T \underbrace{Q^T Q}\, \underset{\sim}{x}$$

$$= \underset{\sim}{x}^T I \underset{\sim}{x} = \underset{\sim}{x}^T \underset{\sim}{x}$$

$$= \|\underset{\sim}{x}\|_2^2$$

$$\Rightarrow \|Q\underset{\sim}{x}\| = \|x\|$$

$\Rightarrow$ The claim is TRUE.

b)

i)

A)
$$\frac{E_N^{Simp}(f)}{E_{2N}^{Simp}(f)} = \frac{C N^{-4}}{C(2N)^{-4}} = 2^4 = 16$$

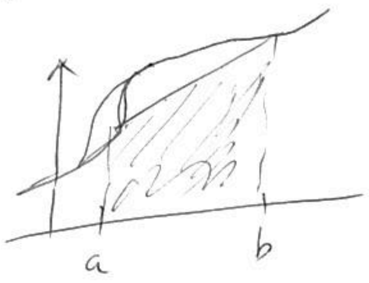
B)
$$\frac{E_{32}^{Simp}(f)}{E_{64}^{Simp}(f)} = \frac{4,48\ e-04}{1,59\ e-04} = 2.8176$$

C) No, it is not consistent.

D) The integrand $f = \sqrt{x}$ is not in $C^4([0,1])$, so the theoretical estimate doesn't apply.

ii) Degree of precision of Trapezoidal is 1.

Take trapezoidal rule with 1 interval:



$$Q(f) = \frac{1}{2}\left(f(a) + f(b)\right)(b-a)$$

$$E(1) = \int_a^b 1 - (1+1)\frac{(b-a)}{2} = 0$$

$$E(x) = \int_a^b x - (b+a)\frac{(b-a)}{2}$$

$$= \frac{b^2 - a^2}{2} - \frac{(b^2 - a^2)}{2} = 0$$

$$E(x^2) = \int_a^b x^2 - (b^2+a^2)\left(\frac{b-a}{2}\right) = \frac{b^3}{3} - \frac{a^3}{3}$$

$$+ (a^2+b^2)\left(\frac{b-a}{2}\right) \neq 0$$

So degree of precision of the Trapezoidal rule is 1.

(5

iii) $x = \alpha + \beta z \qquad z \in [-1, 1] \rightarrow x \in [0, 1]$

$$\left.\begin{aligned} 0 &= \alpha - \beta \\ 1 &= \alpha + \beta \end{aligned}\right\} \Rightarrow \alpha = \beta = \frac{1}{2}.$$

$$x = \frac{1}{2} + \frac{1}{2} z \qquad \Rightarrow dx = \frac{1}{2} dz$$

iv) $$I(f) = \int_0^1 f(x)\,dx \approx \sum_{j=1}^{N} f\left(\frac{1}{2} + \frac{1}{2} z_j\right) \frac{w_j}{2}$$

Q2

i) Order = 2

ii) $$\begin{cases} x_1 = y \\ x_2 = y' \end{cases} \qquad my'' + cy' + ky = 0$$

$$\Rightarrow y'' = -\frac{c}{m} y' - \frac{k}{m} y$$

$$\Rightarrow x_2' = -\frac{c}{m} x_2 - k x_1$$

So the system of first order ODEs is

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{c}{m} x_2 - \frac{k}{m} x_1 \end{cases} \qquad \begin{cases} x_1(1) = 2 \\ x_2(1) = y'(1) = 1 \end{cases}$$

iii) $$\underset{\sim}{x_0} = x(1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

iv) m = 4; c = 3; k = 5;

myode = @(t,x) [x(2); -c*~~x~~(2)/m - k*x(1)/m];

OR

```
function y = myode(t, x)
    m = 4; k = 5; c = 3;
    y = [x(2); -c * x(2)/m - k * x(1)/m];
end
```

v) At $\underset{\sim}{x}(1.1)$ since $h = 0.1$ we need to compute only $\underset{\sim}{x}_1$.

$$\underset{\sim}{z}_1 = \underset{\sim}{x}_0 + h f(t_0, \underset{\sim}{x}_0) = \underset{\sim}{x}_0 + h \begin{bmatrix} x_0(2) \\ -\frac{3\, x_0(2)}{4} - \frac{5}{4} x_0(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (0.1) \begin{bmatrix} 1 \\ -\frac{3}{4} - \frac{5}{4} \times 2 \end{bmatrix} = \begin{bmatrix} 2 + 0.1 \\ 1 - 0.325 \end{bmatrix} = \begin{bmatrix} 2.1 \\ 0.675 \end{bmatrix}$$

$$\underset{\sim}{X}_1 = \underset{\sim}{X}_0 + \frac{h}{2}\left[f(t_0, \underset{\sim}{x}_0) + f(t_1, z_1) \right]$$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{0.1}{2} \left(\begin{bmatrix} 0.1 \\ -0.325 \end{bmatrix} + \begin{bmatrix} 0.675 \\ -\frac{3 \times 0.675}{4} - \frac{5}{4} \times 2.1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 2.08375 \\ 0.6809375 \end{bmatrix}$$

Q2 b)

i) Boundary conditions

A)

$$\frac{\partial^2 u}{\partial x^2} \simeq \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}$$

At $x = x_i \quad y = y_j$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_i, y_j)} \simeq \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$\frac{\partial^2 u}{\partial y^2} \simeq \frac{u(x, y+h) - 2u(x,y) + u(x, y-h)}{h^2}$$

At $x = x_i, \quad y = y_j$

$$\frac{\partial^2 u}{\partial y^2} \simeq \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}$$

iii) In (2.1), using finite difference approximation

$$u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = 0$$

$$\Leftrightarrow \quad 4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = 0$$

$$\Rightarrow \quad \beta = 4.$$

iv) At (x_5, y_2) $\quad i = 5, \ j = 2$. It is an interior point.

$$4u_{5,2} - u_{6,2} - u_{4,2} - u_{5,3} - u_{5,1} = 0.$$

At (x_9, y_1)

$$4u_{9,1} - u_{10,1} - u_{8,1} - u_{9,2} - u_{9,0} = 0$$

Using boundary conditions for $u_{9,0}$ and $u_{10,1}$

$$u_{9,0} = 2x_9 = 2 \times x_9 = 2 \times 1.8 = 3.6$$

$$u_{10,1} = 2$$

$$4u_{9,1} - 2 - u_{8,1} - u_{9,2} - 3.6 = 0$$

$$\Rightarrow \quad 4u_{9,1} - u_{8,1} - u_{9,2} = 5.6$$

v) Use Cholesky factorization.

$$A = R^T R$$

$$A\underset{\sim}{u} = \underset{\sim}{b} \quad \Rightarrow \quad R^T \underbrace{Ru}_{} = b$$

$$R^T \underset{\sim}{v} = \underset{\sim}{b}$$

with $R\underset{\sim}{u} = \underset{\sim}{v}$

<u>Step 1</u>

Solve $R^T\underset{\sim}{v} = \underset{\sim}{b}$ by forward substitution.

<u>Step 2</u>

Solve $R\underset{\sim}{u} = \underset{\sim}{v}$ by back substitution.