FAMILY NAME: ....
OTHER NAME(S): ....
STUDENT NUMBER: ....
SIGNATURE: ....

THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

June 2011

MATH2089
Numerical Methods and Statistics

(1) TIME ALLOWED – 3 Hours

(2) TOTAL NUMBER OF QUESTIONS – 6

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) THIS PAPER MAY **NOT** BE RETAINED BY THE CANDIDATE

(6) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(7) STATISTICAL FORMULAE ARE ATTACHED AT END OF PAPER
STATISTICAL TABLES ARE ATTACHED AT END OF PAPER

Part A – Numerical Methods consists of questions 1 – 3

Part B – Statistics consists of questions 4 – 6

Both parts must be answered

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Part A – Numerical Methods

1. Answer in a separate book marked Question 1

a) What are the values produced by the following MATLAB expressions:

i)
```
x = -1e+200;
ans1 = log(exp(x))
```
ii)
```
v = [-1:1]
ans2 = v./v.^(1/2)
```
iii)
```
x = 200;
h = 1e-14;
ans3 = x + h >  x
```

b) The computational complexity of some common operations with n by n matrices are

Operation	Flops
Matrix multiplication	$2n^3$
LU factorization	$\frac{2n^3}{3} + O(n^2)$
Cholesky factorization	$\frac{n^3}{3} + O(n^2)$
Back/forward substitution	$n^2 + O(n)$
Tridiagonal solve	$8n + O(1)$

i) Estimate the size of the largest symmetric positive definite linear system that can be solved in one hour on a 2.5 GHz 6-core computer, where each core can do two floating point operations per clock cycle.

ii) A symmetric n by n matrix is determined by the $\frac{n(n+1)}{2}$ elements $A_{i,j}$ for $j = 1, \ldots, n$, $i = 1, \ldots, j$. Estimate the size of the largest n by n symmetric matrix that can be stored in a 6MB cache using double precision if only the elements on and above the diagonal are stored.

c) The coefficient matrix A is calculated "exactly" in MATLAB, while the right-hand-side vector $\mathbf{b}$ comes from measurements taken to 6 significant figures. The result of the MATLAB command `rcond` is

```
rc = rcond(A)
rc =
   1.0197e-04
```

i) What is the corresponding condition number $\kappa(A)$?

ii) Estimate how many significant figures there are in a computed solution to $A\mathbf{x} = \mathbf{b}$?

d) Explain how to efficiently solve the linear system $A\mathbf{x} = \mathbf{b}$, given the results of MATLAB's `lu` factorization:

```
[L, U, P] = lu(A);
```

2. Answer in a separate book marked Question 2

a) Two identical chemical tanks with the same cross-sectional area α contain a liquid with height $h_1(t)$ for tank 1 and height $h_2(t)$ for tank 2. The flow rate from tank 1 to tank 2 is

$$Q(t) = \frac{h_1(t) - h_2(t)}{R},$$

where R is a fixed constant. The liquid is withdrawn from tank 2 at a constant rate of Q_e. The concentration of chemical in tank 1 is fixed at the constant c_1, while the concentration in tank 2 is $c_2(t)$. The system is governed by the system of ordinary differential equations

$$\begin{aligned} \alpha \frac{dh_1(t)}{dt} &= -Q(t), \\ \alpha \frac{dh_2(t)}{dt} &= Q(t) - Q_e, \\ \alpha \frac{dc_2(t)}{dt} &= \frac{Q(t)}{h_2(t)}. \end{aligned}$$

Initially both tanks contain liquid with height h_0 and the concentration in tank 2 is c_0. You are asked to find the concentration of chemical in tank 2 over the first 2 seconds.

i) Write this as an initial value problem (IVP) in the standard form

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(t, \mathbf{x}), \quad t \in [t_0, t_f], \qquad \mathbf{x}(t_0) = \mathbf{x}_0.$$

Make sure you clearly define the vector of state variables $\mathbf{x}(t)$, the time domain, the right-hand-side function $\mathbf{f}(t, \mathbf{x})$ and the initial conditions.

ii) Write EITHER a MATLAB anonymous function OR a function M-file to define the function

```
ftanks(t, x)
```

to calculate the right-hand-side vector $\mathbf{f}(t, \mathbf{x})$.
For your chosen function, how can you set values for the problem data α, R, Q_e needed in your function?

b) Consider the integral

$$y(\tau) = \exp\left(\int_0^\tau \frac{2}{1 + e^t} dt \right), \qquad \tau > 0.$$

i) Calculate $y(1)$ using a numerical integration rule of your choice with 5 nodes.

ii) Find an initial value problem (IVP) satisfied by y.

Please see over ...

c) When using N equal intervals of width h, the error in estimating the integral

$$\int_0^\tau f(t)dt$$

by the Trapezoidal rule is $E_{trap} = O(h^2)$ and by Simpson's rule is $E_{Simp} = O(h^4)$.

i) For each of these numerical integration rules, what conditions are required on the integrand f so these error estimates are valid?

ii) Suppose that the error using $h = 5 \times 10^{-3}$ is $E_0 = 1.19 \times 10^{-4}$ when using either the Trapezoidal rule or Simpson's rule. For both rules, estimate the error if an interval width of $\hat{h} = 1 \times 10^{-3}$ is used.

iii) The MATLAB command

```
[z, w] = gauleg(N);
```

calculates the N Gauss-Legendre nodes `z` and weights `w` for the interval $[-1, 1]$. Show how `z` and `w` can be used to numerically calculate

$$\int_0^\tau f(t)dt.$$

3. Answer in a separate book marked Question 3

Fick's second law states how the concentration of a chemical changes with time because of diffusion. Let $c(x, y)$ denote the concentration at position $(x, y) \in \Omega$. The steady state version of Fick's second law (without interior sources of the chemical) is Laplace's equation

$$\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} = 0. \tag{3.1}$$

Consider a problem with domain

$$\Omega = \{(x, y) : 0 \le x \le 1, \quad 0 \le y \le 1.5\}.$$

Ω is discretised using $h = 1/n$ (where n is even) and

$$x_i = ih, \quad i = 0, \ldots, n \qquad y_j = jh, \quad j = 0, \ldots, \frac{3n}{2}.$$

The domain is illustrated in Figure 3.1 for $n = 6$. The variables $C_{i,j}$ approximate the values $c(x_i, y_j)$ of the concentration at the grid points (x_i, y_j).

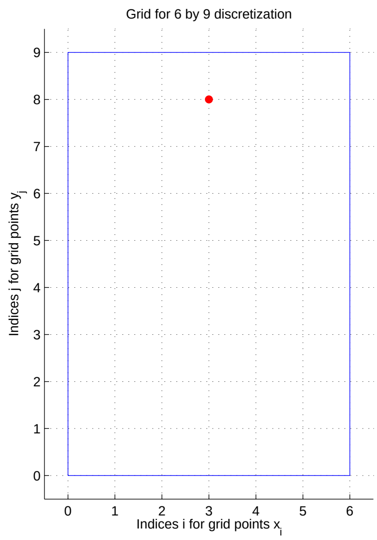


Figure 3.1: Discretization of domain Ω

Recall that standard finite difference approximations for a function f of **one** variable are

$$\begin{aligned} f'(x) &= \frac{f(x+h) - f(x)}{h} + O(h), \\ f'(x) &= \frac{f(x+h) - f(x-h)}{2h} + O(h^2), \\ f''(x) &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2). \end{aligned}$$

a) What additional information is needed to completely specify the problem?

b) At the point (x_i, y_j) give central difference approximations of accuracy $O(h^2)$ to $\frac{\partial^2 c}{\partial x^2}$ and $\frac{\partial^2 c}{\partial y^2}$.

c) Using the finite difference approximations from the previous part, show that equation (3.1) can be written as

$$C_{i,j} + \beta C_{i+1,j} + \beta C_{i-1,j} + \beta C_{i,j+1} + \beta C_{i,j-1} = 0 \tag{3.2}$$

and find β.

d) Suppose that $c(x, 1.5) = 0.6$ for $0 \le x \le 1$. For the discretization with $n = 6$, find the equation (3.2) at the grid point (x_3, y_8) marked in Figure 3.1. Clearly indicate what the unknowns are.

e) You are given the coefficient matrix A using a row-ordering of the variables $C_{i,j}$. Information about the coefficient matrix A is given in Figure 3.2.

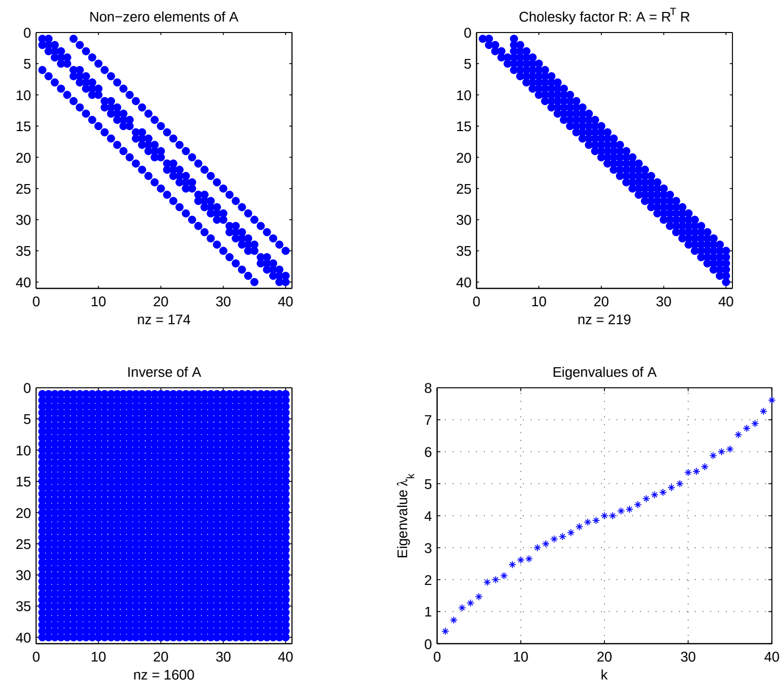


Figure 3.2: Spy plots of A, Cholesky factor R, inverse A^{-1} and eigenvalues of A

i) Calculate the sparsity of A.

ii) Why is explicitly calculating A^{-1} not a good idea?

iii) From Figure 3.2, give **two** reasons why A is positive definite.

iv) Explain how to use the Cholesky factorization $A = R^T R$ to solve the linear system $A\mathbf{x} = \mathbf{b}$.

v) What other structure does A have that will make solving a linear system $A\mathbf{x} = \mathbf{b}$ more efficient?

Part B – Statistics

4. Answer in a separate book marked Question 4

a) In an air-pollution study, ozone concentrations were taken in a large California city at 5.00 P.M. The eight readings (in parts per million) were

7.9 11.3 6.9 12.7 13.2 8.8 9.3 10.6

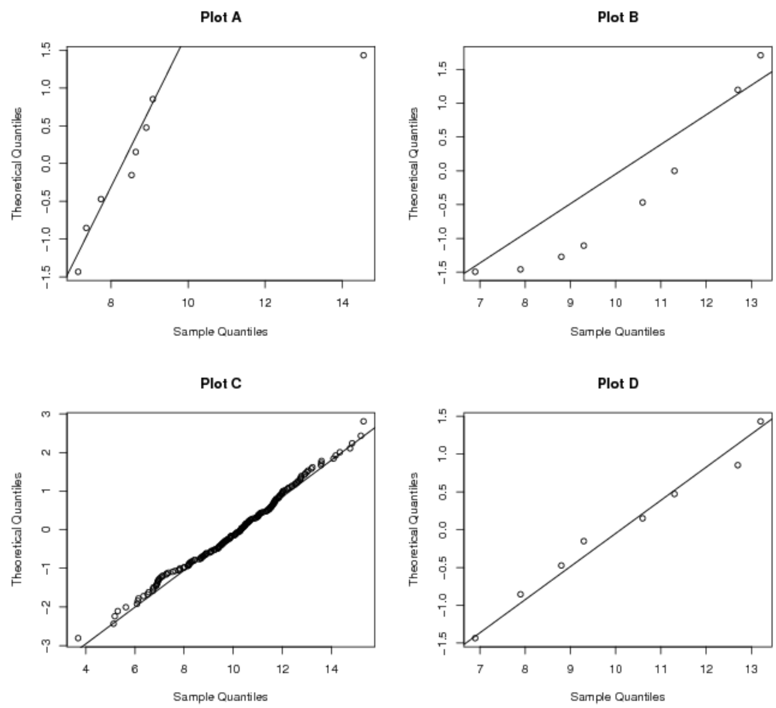
The sample mean and standard deviation are

$$\bar{x} = 10.0875 \text{ ppm} \qquad \text{and} \qquad s = 2.2510 \text{ ppm}$$

i) Determine the five-number summary for this sample.

ii) Draw a boxplot of the data and comment on its features.

iii) A normal quantile plot had been represented for this sample, however it has been mixed up with three other normal quantile plots for other data sets. Which of the 4 quantile plots presented below (A, B, C or D) is the normal quantile plot for this sample? Explain your choice.



iv) From the quantile plot you selected, what is a logical assumption about the underlying distribution of the data? Explain.

v) Based on this sample data, construct a two-sided 95% confidence interval for the true mean ozone concentration in that city.

vi) The mayor claims that mean ozone concentration in his city does not exceed 9 ppm. Can we contradict him? Carry out a one-sided hypothesis test at level $\alpha = 0.05$ and state your conclusion in plain language.

vii) Give bounds on the p-value associated to your test. Interpret these values.

b) An article in *The Engineer* reported the results of an investigation into wiring errors on commercial transport aircraft that may produce faulty information to the flight crew. Of 1600 randomly selected aircraft, eight were found to have wiring errors that could display incorrect information to the flight crew.

i) Find an approximate 99% two-sided confidence interval on the proportion of aircraft that have such wiring errors.

ii) How large a sample would be required if we wanted to be at least 99% confident that the observed sample proportion $\hat{p}$ differs from the true proportion p by at most 0.008, regardless of the value p?

5. Answer in a separate book marked Question 5

Suppose that a tensile ring is to be calibrated by measuring the deflection at various loads. In the following table, which gives the results for 12 measurements, the x_i's are the applied load forces in thousands of pounds and the y-values are the corresponding deflections in thousandths of an inch :

x_i	1	2	3	4	5	6	7	8	9	10	11	12
y_i	16	35	45	64	86	96	106	124	134	156	164	182

Elementary computations also yield

$$\bar{x} = 6.5 \qquad \text{and} \qquad s_{xx} = \sum_{i=1}^{12}(x_i - \bar{x})^2 = 143$$

The output that results from fitting a simple linear regression model to the data is shown below. The response variable Y is the deflection (in thousandth of an inch) and the predictor variable X is the load (in thousand of pounds). The fitted regression model is given by :

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

Use the following regression output to answer the questions below.

```
Regression Analysis: Y versus X

The regression equation is Y = 4.35 + 14.8 X


Predictor      Coef   SE Coef    T       P
Constant      4.348    2.244    1.94   0.081
X           14.8182   0.3049   48.60   0.000

S = 3.646   R-Sq = 99.6%   R-Sq(adj) = 99.5%
```

a) i) List three essentials assumptions that the error ε in the model must satisfy for the above regression analysis to be valid.

ii) Explain what plots (or other output) you would consider generating to assess how reasonable are these assumptions, and how you would use the output.

Assume from now on that these assumptions are valid.

b) What is the expected change in the deflection for a unit change in the load ?

c) What proportion of variation in the response is explained by the predictor?

d) What is the (sample) correlation between load and deflection? Interpret this value.

e) Give the estimated value of σ, the standard deviation of the error term ε.

f) Carry out a hypothesis test to determine whether the variable X is significant in this model, at significance level $\alpha = 0.05$. You can use the numerical values found in the above output, however you are asked to properly write the detail of the test (null and alternative hypothesis, rejection criterion, observed value of the test statistic, p-value, conclusion).

g) Determine a 95% two-sided confidence interval for β_1.

h) The value P associated to the 'Constant' predictor is seen to be equal to 0.081. Interpret this value.

i) Obtain a 95% two-sided prediction interval for the deflection when the load is set to 7.5 thousands of pounds.

6. Answer in a separate book marked Question 6

A manufacturer of paper used for making grocery bags is interested in improving the tensile strength of the product. Product engineering thinks that tensile strength is a function of the hardwood concentration in the pulp and that the range of hardwood concentrations of practical interest is between 5% and 15%. A team of engineers responsible for the study decides to investigate three levels of hardwood concentration : 5%, 10% and 15%. The decide to make up six test specimens at each concentration level. All 18 specimens are tested on a laboratory tensile tester, and the observed tensile strengths (in psi) are shown in the following table :

5%	10%	15%
7	12	14
8	17	18
15	13	19
11	18	17
9	19	16
10	15	18
$\bar{x}_1 = 10$	$\bar{x}_2 = 15.67$	$\bar{x}_3 = 17$
$s_1 = 2.8284$	$s_2 = 2.8048$	$s_3 = 1.7889$

a) What assumptions need to be valid for an Analysis of Variance to be an appropriate analysis here?

Assume from now on that these assumptions are valid.

b) An ANOVA table was partially constructed to summarise the data :

Source	df	SS	MS	F
Treatment	**(1)**	**(2)**	**(3)**	13.04
Error	**(4)**	95.333	**(5)**	
Total	**(6)**	261.111		

Complete the table by determining the missing values (1)-(6). (Copy the whole ANOVA table in your answer booklet).

c) Using a significance level of $\alpha = 0.05$, carry out the ANOVA F-test to determine whether the hardwood concentration significantly influences the tensile strength. You can use the numerical values found in the above table, however you are asked to properly write the detail of the test (null and alternative hypothesis, rejection criterion, observed value of the test statistic, p-value, conclusion - use bounds for the p-value).

d) Construct a 95% two-sided confidence interval on the difference between mean tensile strength at concentration 10% and mean tensile strength at concentration 15%, that is, $\mu_2 - \mu_3$. Would you say that there is a significant difference between these two means? Explain.

e) The engineers responsible for the study also carry out two two-sample t-tests to compare concentration 5% to concentration 10% and concentration 5% to concentration 15%, in terms of the mean tensile strength. They find p-values equal to 0.0059 and 0.0004, respectively. Does simultaneously analysing the three pairwise comparisons (these two t-tests and the confidence interval in d)) allow you to come to the same conclusion as the ANOVA F-test in c), at overall level $\alpha = 0.05$? Explain. *Hint :* recall the Bonferroni adjustment.

Formula sheet

1 Probability distributions

1.1 The Binomial distribution

Assume $\boxed{X \sim \text{Bin}(n, \pi)}$

- **domain of variation :**

$$S_X = \{0, 1, \dots, n\}$$

- **probability mass function (pmf) :**

$$p(x) = \binom{n}{x} \pi^x (1-\pi)^{n-x}, \qquad \text{for } x \in S_X$$

- **cumulative distribution function (cdf) :**

$$F(x) = \sum_{k=0}^{\lfloor x \rfloor} \binom{n}{k} \pi^k (1-\pi)^{n-k}$$

(where $\lfloor x \rfloor$ denotes the integer part of x).

- **expectation :**

$$\mathbb{E}(X) = n\pi$$

- **variance :**

$$\mathbb{V}\text{ar}(X) = n\pi(1-\pi)$$

1.2 The Poisson distribution

Assume $\boxed{X \sim \mathcal{P}(\lambda)}$

- **domain of variation :**

$$S_X = \{0, 1, 2, \dots\}$$

- **probability mass function (pmf) :**

$$p(x) = e^{-\lambda} \frac{\lambda^x}{x!}, \qquad \text{for } x \in S_X$$

- **cumulative distribution function (cdf) :**

$$F(x) = e^{-\lambda} \sum_{k=0}^{\lfloor x \rfloor} \frac{\lambda^k}{k!}$$

(where $\lfloor x \rfloor$ denotes the integer part of x).

- **expectation :**

$$\mathbb{E}(X) = \lambda$$

- **variance :**

$$\mathbb{V}\text{ar}(X) = \lambda$$

1.3 The Uniform distribution

Assume $\boxed{X \sim U_{[\alpha,\beta]}}$

- **domain of variation :**

$$S_X = [\alpha, \beta]$$

- **probability density function (pdf) :**

$$f(x) = \frac{1}{\beta - \alpha}, \qquad \text{for } x \in S_X$$

- **cumulative distribution function (cdf) :**

$$F(x) = \frac{x - \alpha}{\beta - \alpha}, \qquad \text{for } x \in S_X$$

- **expectation :**

$$\mathbb{E}(X) = \frac{\alpha + \beta}{2}$$

- **variance :**

$$\mathbb{V}\text{ar}(X) = \frac{(\beta - \alpha)^2}{12}$$

1.4 The Exponential distribution

Assume $\boxed{X \sim \text{Exp}(\lambda)}$

- **domain of variation :**

$$S_X = [0, +\infty)$$

- **probability density function (pdf) :**

$$f(x) = \lambda e^{-\lambda x}, \qquad \text{for } x \in S_X$$

- **cumulative distribution function (cdf) :**

$$F(x) = 1 - e^{-\lambda x}, \qquad \text{for } x \in S_X$$

- **expectation :**

$$\mathbb{E}(X) = \frac{1}{\lambda}$$

- **variance :**

$$\mathbb{V}\text{ar}(X) = \frac{1}{\lambda^2}$$

1.5 The Normal distribution

Assume $\boxed{X \sim \mathcal{N}(\mu, \sigma)}$

- **domain of variation :**

$$S_X = (-\infty, +\infty)$$

- **probability density function (pdf) :**

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}, \qquad \text{for } x \in S_X$$

- **cumulative distribution function (cdf) :**

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}\, dy, \qquad \text{for } x \in S_X$$

  (no closed form)

- **expectation :**

$$\mathbb{E}(X) = \mu$$

- **variance :**

$$\mathbb{V}\text{ar}(X) = \sigma^2$$

1.6 The Standard Normal distribution

Assume $\boxed{X \sim \mathcal{N}(0, 1)}$

- **domain of variation :**

$$S_X = (-\infty, +\infty)$$

- **probability density function (pdf) :**

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \qquad \text{for } x \in S_X$$

- **cumulative distribution function (cdf) :**

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy, \qquad \text{for } x \in S_X$$

(no closed form, see also the attached Standard Normal table)

- **expectation :**

$$\mathbb{E}(X) = 0$$

- **variance :**

$$\mathbb{V}\text{ar}(X) = 1$$

- **standardisation** : If $X \sim \mathcal{N}(\mu, \sigma)$, then

$$Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$

1.7 The Student's t distribution

Assume $\boxed{X \sim t_\nu}$

- **domain of variation :**

$$S_X = (-\infty, +\infty)$$

- **probability density function (pdf) :**

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \qquad \text{for } x \in S_X$$

- **cumulative distribution function (cdf) :**

$$F(x) = \int_{-\infty}^{x} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{y^2}{\nu}\right)^{-\frac{\nu+1}{2}} dy, \qquad \text{for } x \in S_X$$

(no closed form, see also the attached t-distribution critical values table)

- **expectation :**

$$\mathbb{E}(X) = 0$$

- **variance :**

$$\mathbb{V}\text{ar}(X) = \frac{\nu}{\nu - 2} \qquad \text{for } \nu > 2$$

1.8 The χ^2-distribution

Assume $\boxed{X \sim \chi^2_\nu}$

- **domain of variation :**
$$S_X = [0, +\infty)$$
- **probability density function (pdf) :**
$$f(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}, \qquad \text{for } x \in S_X$$
- **cumulative distribution function (cdf) :**
$$F(x) = \int_{-\infty}^{x} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} y^{\nu/2-1} e^{-y/2}\, dy, \qquad \text{for } x \in S_X$$
(no closed form, see also the attached χ^2-distribution critical values table)
- **expectation :**
$$\mathbb{E}(X) = \nu$$
- **variance :**
$$\mathbb{V}\mathrm{ar}(X) = 2\nu$$

1.9 The Fisher F-distribution

Assume $\boxed{X \sim \mathbf{F}_{d_1,d_2}}$

- **domain of variation :**
$$S_X = [0, +\infty)$$
- **probability density function (pdf) :**
$$f(x) = \frac{\Gamma((d_1+d_2)/2)(d_1/d_2)^{d_1/2} x^{d_1/2-1}}{\Gamma(d_1/2)\Gamma(d_2/2)((d_1/d_2)x+1)^{(d_1+d_2)/2}}, \qquad \text{for } x \in S_X$$
- **cumulative distribution function (cdf) :**
$$F(x) = \int_{-\infty}^{x} \frac{\Gamma((d_1+d_2)/2)(d_1/d_2)^{d_1/2} y^{d_1/2-1}}{\Gamma(d_1/2)\Gamma(d_2/2)((d_1/d_2)y+1)^{(d_1+d_2)/2}}\, dy, \qquad \text{for } x \in S_X$$
(no closed form, see also the attached F-distribution critical values table)
- **expectation :**
$$\mathbb{E}(X) = \frac{d_2}{d_2-2} \qquad \text{for } d_2 > 2$$
- **variance :**
$$\mathbb{V}\mathrm{ar}(X) = \frac{2d_2^2(d_1+d_2-2)}{d_1(d_2-2)^2(d_2-4)} \qquad \text{for } d_2 > 4$$

2 Sampling distributions

2.1 Sample mean

2.1.1 known variance

Let $\bar{X}$ be the sample average from a random sample of size n from a population with mean μ and standard deviation σ. Under appropriate conditions,

$$Z = \sqrt{n}\,\frac{\bar{X}-\mu}{\sigma} \sim \mathcal{N}(0,1)$$

(exact result if the population distribution is normal, approximate result if the population distribution is not normal but $n > 30$)

2.1.2 unknown variance

Let $\bar{X}$ and S be the sample average and standard deviation from a random sample of size n from a normal population with mean μ. Under appropriate conditions,

$$T = \sqrt{n}\,\frac{\bar{X}-\mu}{S} \sim t_{n-1}$$

If the population is not normal but n is large enough ($n > 40$), we can also write

$$T = \sqrt{n}\,\frac{\bar{X}-\mu}{S} \sim \mathcal{N}(0,1)$$

approximately

2.2 Sample proportion

Let $\hat{p}$ be the sample proportion of 'successes' where the number of trials is n and the true probability of a success is π. Under appropriate conditions,

$$\sqrt{n}\,\frac{\hat{p}-\pi}{\sqrt{\pi(1-\pi)}} \sim N(0,1)$$

approximately when $n\pi(1-\pi) > 5$

2.3 Sample variance

Let S^2 be the sample variance from a random sample of size n from a normal population with variance σ. Under appropriate conditions,

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

Please see over ...

2.4 Difference in sample means

2.4.1 variances σ_1^2 and σ_2^2 known

For two independent samples of size n_1 and n_2 from two populations with means μ_1 and μ_2 and standard deviations σ_1 and σ_2 respectively, let $\bar{X}_i$ be the sample average of sample i for $i = 1$ and 2. Under appropriate conditions,

$$\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim \mathcal{N}(0, 1)$$

(exact result if both population distributions are normal, approximate result if they are not but $n_1, n_2 > 30$)

2.4.2 variances σ_1^2 and σ_2^2 unknown; $\sigma_1^2 = \sigma_2^2$

For two independent samples of size n_1 and n_2 from two normal populations with means μ_1 and μ_2 respectively and common standard deviation σ, let $\bar{X}_i$ and S_i be the sample average and sample standard deviation of sample i for $i = 1$ and 2. Under appropriate conditions,

$$\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2},$$

where S_p is the pooled sample standard deviation,

$$S_p = \sqrt{\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}}.$$

2.4.3 variances σ_1^2 and σ_2^2 unknown; $\sigma_1^2 \neq \sigma_2^2$

For two independent samples of size n_1 and n_2 from two normal populations with means μ_1 and μ_2 and standard deviations σ_1 and σ_2 respectively, let $\bar{X}_i$ and S_i be the sample average and sample standard deviation of sample i for $i = 1$ and 2. Under appropriate conditions,

$$\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \sim t_\nu,$$

where

$$\nu = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{\frac{(s_1^2/n_1)^2}{n_1-1} + \frac{(s_2^2/n_2)^2}{n_2-1}}$$

(rounded down to the nearest integer)

Please see over . . .

2.5 Ratio of sample variances

Let S_1^2 and S_2^2 be the sample variances from two independent random samples of size n_1 and n_2 from normal populations with variances σ_1^2 and σ_2^2 respectively. Under appropriate conditions,

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim \mathbf{F}_{n_1-1,n_2-1}$$

3 Simple linear regression

Consider the simple linear regression model

$$Y = \beta_0 + \beta_1 X + \epsilon$$

where $\epsilon \sim \mathcal{N}(0, \sigma)$
The least squares estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ of β_0 and β_1 are

$$\hat{\beta}_1 = \frac{S_{XY}}{S_{XX}} \qquad \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

where

$$S_{XY} = \sum_i (X_i - \bar{X})(Y_i - \bar{Y}) \qquad S_{XX} = \sum_i (X_i - \bar{X})^2.$$

An estimator of σ is

$$S = \sqrt{\frac{\sum_i (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2}{n-2}}.$$

Under the simple linear regression model :

$$\frac{\hat{\beta}_1 - \beta_1}{S/\sqrt{S_{XX}}} \sim t_{n-2}$$

$$\frac{\hat{\beta}_0 - \beta_0}{S\sqrt{\frac{1}{n} + \frac{\bar{X}^2}{S_{XX}}}} \sim t_{n-2}$$

Let x_0 denote the predictor value for a response yet to be observed :

i) a $100(1-\alpha)\%$ confidence interval for the mean response at x_0 is

$$\hat{y}(x_0) \pm s\, t_{n-2;1-\alpha/2} \sqrt{\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{s_{xx}}}$$

where $\hat{y}(x_0) = \hat{b}_0 + \hat{b}_1 x_0$;

ii) and a $100(1-\alpha)\%$ prediction interval for the response at x_0 is

$$\hat{y}(x_0) \pm s\, t_{n-2;1-\alpha/2} \sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{s_{xx}}}$$

4 ANOVA

- Total sum of squares :

$$SS_{\text{Tot}} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{\bar{X}})^2$$

- Treatment sum of squares :

$$SS_{\text{Tr}} = \sum_{i=1}^{k} n_i (\bar{X}_i - \bar{\bar{X}})^2$$

- Error sum of squares :

$$SS_{\text{Er}} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2$$

- Under the assumption of equality of means in each of the k groups of a one-way Analysis of Variance,

$$F = \frac{\text{MS}_{\text{Tr}}}{\text{MS}_{\text{Er}}} \sim \mathbf{F}_{k-1,n-k},$$

  where n is the total number of observations.

Statistical Tables

t distribution critical values

Key: Table entry for p and C is the critical value t^* with probability p lying to its right and probability C lying between $-t^*$ and t^*.

	Upper tail probability p											
df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.000	1.376	1.963	3.078	6.314	12.71	15.89	31.82	63.66	127.3	318.3	636.6
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33	31.60
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21	12.92
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.610	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
	.50	.60	.70	0.80	.90	.95	.96	.98	.99	.995	.998	.999
	Probability C											

Standard normal probabilities

Key: Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

χ^2 distribution critical values

Key: Table entry for p is the critical value with probability p lying to its right.

	Upper tail probability p									
df	.995	.99	.975	.95	.90	.10	.05	.025	.01	.005
1	0.000039	0.00016	0.00098	0.0039	0.0158	2.71	3.84	5.02	6.63	7.88
2	0.0100	0.0201	0.0506	0.1026	0.2107	4.61	5.99	7.38	9.21	10.60
3	0.0717	0.115	0.216	0.352	0.584	6.25	7.81	9.35	11.34	12.84
4	0.207	0.297	0.484	0.711	1.064	7.78	9.49	11.14	13.28	14.86
5	0.412	0.554	0.831	1.15	1.61	9.24	11.07	12.83	15.09	16.75
6	0.676	0.872	1.24	1.64	2.20	10.64	12.59	14.45	16.81	18.55
7	0.989	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09	21.95
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.72	26.76
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	21.06	23.68	26.12	29.14	31.32
15	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.09	24.77	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	34.38	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	36.74	40.11	43.19	46.96	49.64
28	12.46	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	51.81	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38	91.95
80	51.17	53.54	57.15	60.39	64.28	96.58	101.88	106.63	112.33	116.32
100	67.33	70.06	74.22	77.93	82.36	118.50	124.34	129.56	135.81	140.17

F distribution critical values

Key: p=Upper tail probability p, df_n=degrees of freedom in numerator, df_d=degrees of freedom in denominator,
$*$ Multiply by 10, † Multiply by 100.

df_d	df_n / p	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	.05	161	200	216	225	230	234	237	239	241	242	244	246	248	249	250	251	252	253	254
	.025	648	800	864	900	922	937	948	957	963	969	977	986	993	997	1001	1006	1010	1014	1018
	.01	405^*	500^*	540^*	563^*	576^*	586^*	593^*	598^*	602^*	606^*	611^*	616^*	621^*	624^*	626^*	629^*	631^*	634^*	637^*
	.005	$162^\dagger$	$200^\dagger$	$216^\dagger$	$225^\dagger$	$231^\dagger$	$234^\dagger$	$237^\dagger$	$239^\dagger$	$241^\dagger$	$242^\dagger$	$244^\dagger$	$246^\dagger$	$248^\dagger$	$249^\dagger$	$250^\dagger$	$251^\dagger$	$253^\dagger$	$254^\dagger$	$255^\dagger$
2	.05	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
	.025	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48	39.49	39.50
	.01	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50
	.005	199	199	199	199	199	199	199	199	199	199	199	199	199	200	200	200	200	200	200
3	.05	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
	.025	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04	13.99	13.95	13.90
	.01	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13
	.005	55.55	49.80	47.47	46.19	45.39	44.84	44.43	44.13	43.88	43.69	43.39	43.08	42.78	42.62	42.47	42.31	42.15	41.99	41.83
4	.05	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
	.025	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66	8.56	8.51	8.46	8.41	8.36	8.31	8.26
	.01	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
	.005	31.33	26.28	24.26	23.15	22.46	21.97	21.62	21.35	21.14	20.97	20.70	20.44	20.17	20.03	19.89	19.75	19.61	19.47	19.32
5	.05	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
	.025	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.28	6.23	6.18	6.12	6.07	6.02
	.01	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
	.005	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.38	13.15	12.90	12.78	12.66	12.53	12.40	12.27	12.14
6	.05	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
	.025	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.12	5.07	5.01	4.96	4.90	4.85
	.01	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
	.005	18.63	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.03	9.81	9.59	9.47	9.36	9.24	9.12	9.00	8.88
7	.05	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
	.025	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.41	4.36	4.31	4.25	4.20	4.14
	.01	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
	.005	16.24	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.51	8.38	8.18	7.97	7.75	7.64	7.53	7.42	7.31	7.19	7.08
8	.05	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
	.025	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.95	3.89	3.84	3.78	3.73	3.67
	.01	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
	.005	14.69	11.04	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.01	6.81	6.61	6.50	6.40	6.29	6.18	6.06	5.95
9	.05	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
	.025	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.61	3.56	3.51	3.45	3.39	3.33
	.01	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
	.005	13.61	10.11	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.23	6.03	5.83	5.73	5.62	5.52	5.41	5.30	5.19
10	.05	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
	.025	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.37	3.31	3.26	3.20	3.14	3.08
	.01	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
	.005	12.83	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.66	5.47	5.27	5.17	5.07	4.97	4.86	4.75	4.64
12	.05	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
	.025	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.02	2.96	2.91	2.85	2.79	2.72
	.01	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
	.005	11.75	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.91	4.72	4.53	4.43	4.33	4.23	4.12	4.01	3.90
15	.05	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
	.025	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.70	2.64	2.59	2.52	2.46	2.40
	.01	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
	.005	10.80	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.25	4.07	3.88	3.79	3.69	3.58	3.48	3.37	3.26
20	.05	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
	.025	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.41	2.35	2.29	2.22	2.16	2.09
	.01	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
	.005	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.68	3.50	3.32	3.22	3.12	3.02	2.92	2.81	2.69
24	.05	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
	.025	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.27	2.21	2.15	2.08	2.01	1.94
	.01	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
	.005	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.42	3.25	3.06	2.97	2.87	2.77	2.66	2.55	2.43
30	.05	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
	.025	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.14	2.07	2.01	1.94	1.87	1.79
	.01	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
	.005	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.18	3.01	2.82	2.73	2.63	2.52	2.42	2.30	2.18
40	.05	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
	.025	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.18	2.07	2.01	1.94	1.88	1.80	1.72	1.64
	.01	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
	.005	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	2.95	2.78	2.60	2.50	2.40	2.30	2.18	2.06	1.93
60	.05	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
	.025	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.88	1.82	1.74	1.67	1.58	1.48
	.01	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
	.005	8.49	5.79	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.74	2.57	2.39	2.29	2.19	2.08	1.96	1.83	1.69
120	.05	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
	.025	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.05	1.94	1.82	1.76	1.69	1.61	1.53	1.43	1.31
	.01	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
	.005	8.18	5.54	4.50	3.92	3.55	3.28	3.09	2.93	2.81	2.71	2.54	2.37	2.19	2.09	1.98	1.87	1.75	1.61	1.43
∞	.05	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00
	.025	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.94	1.83	1.71	1.64	1.57	1.48	1.39	1.27	1.00
	.01	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00
	.005	7.88	5.30	4.28	3.72	3.35	3.09	2.90	2.74	2.62	2.52	2.36	2.19	2.00	1.90	1.79	1.67	1.53	1.36	1.00