Question 1 (MATH 2089 - S1 - 2013)

a) i) ans1 = -Inf because $\begin{cases} \exp(-1e+20) = 0 \\ \log(0) = -\text{Inf} \end{cases}$

ii) v = [2 : -2 : -2]

v = [2 0 -2]

ans2 = v.^(1/2)./v

= [1.4142 0 +1.4142 i]./[2 0 -2]

= [0.7071 NaN -0.7071 i]

iii) ans3 = 0 because the absolute error in storing y is $eps * y = 2.2 \times 10^{-18}$

and $h = 10^{-16} > 2.2 \times 10^{-18}$.

So $y + h == y$ is false (0)

b) i) The number of flops the computer can do in 1 hour is

$$\underbrace{3600}_{\#\text{secs}} \times \underbrace{2.5}_{2.5\text{GHz}} \times 10^9 \times \underbrace{4}_{\text{quad core}} \times \underbrace{2}_{2\text{ flops/core/cycle}} = 7.2 \times 10^{13} \text{ flops.}$$

The matrix has no special structure, so LU factorization will be used. Ignoring back/forward substitutions (which are of order $O(n^2)$) we have

$$\frac{2n^3}{3} = 7.2 \times 10^{13} \text{ flops.}$$

$$\Rightarrow n = \left(\frac{3}{2} \times 7.2 \times 10^{13}\right)^{1/3} = 47622$$

ii) Multiplying 2 $n \times n$ matrices take $2n^3$ in 30 secs

So in 1 sec, the computer can do $2n^3/30$ flops.

When solving $n \times n$ symmetric positive definite linear system with multiple right hand side, the most ~~expensive~~ step is to compute the Cholesky factorization, which is $\frac{n^3}{3}$ flops (ignoring $O(n^2)$ terms). So, the time it takes is

$$\frac{(n^3/3) \text{ flops}}{\# \text{ flops / secs}} = \frac{\frac{n^3}{3}}{2n^3/30} = \frac{30}{6} = 5 \text{ secs}$$

c)

i) A is not symmetric since $\| A - A^T \| = 98.02$

ii) $$\kappa_1(A) = \frac{1}{\text{rcond}(A)} = \frac{1}{4.8677 \times 10^{-5}} = 2.0544 \times 10^4$$

iii) $$\text{rel\_err}(x) \approx \kappa_1(A) \left[\text{rel-err}(A) + \text{rel-err}(\underset{\sim}{b}) \right]$$

iii) Suppose $\text{rel-err}(x) < 0.5 \times 10^{-6}$

$$\Rightarrow \quad 0.5 \times 10^{-6} \simeq 2.0544 \times 10^4 \left[2.2 \times 10^{-16} + \text{rel-err}(b) \right]$$

$$\Rightarrow \quad \text{rel\_err}(b) \simeq 2.433 \times 10^{-11}$$

$$< 0.5 \times 10^{-10}$$

So $\underset{\sim}{b}$ needs to be accurate upto 10 significant figures.

d) $f(x) = e^x \sin(x) - 100.$

i) $f'(x) = e^x \sin x + e^x \cos x$

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

$$= x_n - \frac{e^{x_n}\sin(x_n) - 100}{e^{x_n}(\sin(x_n) + \cos(x_n))}$$

ii) Order of convergence for Newton's method = 2.

if the starting value near the zero.

iii)
$$e_1 = |x_1 - x^*| = 0.1 = 10^{-1}$$
$$e_2 = e_1^2 = 10^{-2}$$
$$e_3 = e_2^2 = e_1^4 = 10^{-4}$$
$$e_4 = e_3^2 = e_1^8 = 10^{-8}$$
$$e_5 = e_4^2 = e_1^{16} = 10^{-16}$$

So the number of iterations $k = 4$ is needed to achieve accuracy $e_{k+1} < 10^{-14}$.

Question 2 (MATH 2089 - S1 - 2013)

a)

i) $x = \alpha + \beta z$ maps $z \in [-1, 1]$ to $x \in [0, 5]$

At $x = 0$, $z = -1$ so

$$0 = \alpha - \beta \qquad (*)$$

At $x = 5$, $z = 1$ so

$$5 = \alpha + \beta \qquad (**)$$

From $(*)$ and $(**)$ $\alpha = \beta = 5/2$

ii)

$$I(f) = \int_0^5 f(x)\,dx = \int_{-1}^{1} f(\alpha + \beta z)\,\beta . dz$$

$$= \frac{5}{2}\int_{-1}^{1} f\left(\frac{5}{2} + \frac{5}{2}z\right) dz \approx \frac{5}{2}\sum_{j=1}^{N} f\left(\frac{5}{2} + \frac{5}{2}z_j\right) w$$

iii) $E_N^{Simp}(f) = O(N^{-4})$

A) $$\frac{E_N^{Simp}(f)}{E_{2N}^{Simp}(f)} = \frac{c\,N^{-4}}{c\,(2N)^{-4}} = 2^4 = 16$$

B) $$\frac{E_{512}^{Simp}(f)}{E_{1024}^{Simp}(f)} = \frac{3.5695 \times 10^{-5}}{1.2619 \times 10^{-5}} = 2.82867$$

C) NO,

iv) f' is not continuous at $x = 0$ and $x = 5$, so f does not belong to $C^4([0,5])$. Hence $I(f)$ is difficult to approximate.

b)

i) Order of the differential eqn $= 2$

ii) Let $\begin{cases} x_1 = y \\ x_2 = y' \end{cases}$

then $\begin{cases} x_1' = y' = x_2 \\ x_2' = y'' = -\frac{c}{m} y' - \frac{k}{m} y = -\frac{c}{m} x_2 - \frac{k}{m} x_1 \end{cases}$

which is of the form $\underset{\sim}{x}' = f(t, \underset{\sim}{x})$ with

$$f(t, \underset{\sim}{x}) = \begin{bmatrix} x_2 \\ -\frac{c}{m} x_2 - \frac{k}{m} x_1 \end{bmatrix}.$$

iii) $\underset{\sim}{x_0} = \underset{\sim}{x}(t_0) = \begin{bmatrix} y(t_0) \\ y'(t_0) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

iv)
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m = 2; c = 1; k = 1;
myode = @(t, x)  [x(2); -(c/m)* x(2) - (k/m)* x(1
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OR

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function f = myode (t, x)
m = 2; c = 1; k = 1;
f(1) = x(2);
f(2) = (-c/m)* x(2) - (k/m)* x(1);
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v)

$$\underset{\sim}{x}(1.2) = \underset{\sim}{x}(1.0) + h\, f\left(1.0,\ \underset{\sim}{x}(1.0)\right)$$

$$= \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 0.2 \begin{bmatrix} -1 \\ \left(-\frac{1}{2}\right)(-1) - \frac{1}{2}(2) \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 0.2 \\ -1 + 0.2\left(\frac{1}{2} - 1\right) \end{bmatrix} = \begin{bmatrix} 1.8 \\ -1.1 \end{bmatrix}$$

N.B The general formula for Euler's method in this case is

$$\underset{\sim}{x}_{n+1} = \underset{\sim}{x}_n + h\, f(t_n,\ \underset{\sim}{x}_n)$$

Here with $h = 0.2$; $\underset{\sim}{x}(1.2)$ is $\underset{\sim}{x}_1$ and
$\underset{\sim}{x}(1.0)$ is $\underset{\sim}{x}_0$ which is given.

Question 3 (MATH 2089 - S1 - 2013)

a) Boundary conditions.

b) i)

$$\left.\frac{\partial^2 u(x,y)}{\partial x^2}\right|_{(x_i, y_j)} = \frac{u(x_i+h, y_j) - 2u(x_i, y_j) + u(x_i-h,}{h^2}$$

$$+ O(h^2)$$

ii)

$$\left.\frac{\partial^2 u(x,y)}{\partial y^2}\right|_{(x_i, y_j)} = \frac{u(x_i, y_j+h) - 2u(x_i, y_j) + u(x_i, y_j-h}{h^2}$$

$$+ O(h^2)$$

c) The approximations:

$$\begin{cases} u_{ij} \approx u(x_i, y_j) \\ x_i + h = x_{i+1} \; ; \; x_i - h = x_{i-1} \\ y_j + h = x_{j+1} \; ; \; y_j - h = y_{j-1} \end{cases}$$

Combining b) and the approximations into the PDE,

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = ($$

Multiplying by h^2, re-arrange:

$$4u_{ij} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = 0$$

(so $\beta = 4$).

d) i) at (x_5, y_2) $\quad i = 5, \ j = 2$

equation (3.2) becomes

$$4u_{5,2} - u_{6,2} - u_{4,2} - u_{5,3} - u_{5,1} = 0$$

ii) at (x_9, y_1) $\quad i = 9, \ j = 1$

$$4u_{9,1} - u_{10,1} - u_{8,1} - u_{9,2} - u_{9,0} = 0$$

Using the boundary conditions, we have

$$u_{10,1} = u(x_{10}, y_1) = u(2, y_1) = 20$$

$$u_{9,0} = u(x_9, y_0) = u(x_9, 0) = 10x_9$$

$$= 10 \times 1.8 = 18$$

So the equation at (x_9, y_1) is simplified to

$$4u_{9,1} - 20 - u_{8,1} - u_{9,2} - 18 = 0$$

$$\Leftrightarrow \quad 4u_{9,1} - u_{8,1} - u_{9,2} = 38$$

e) i) Sparsity of $A = \dfrac{nnz(A)}{\text{total numbers of elements}(A)}$

$$= \frac{154}{36 \times 36} = \frac{154}{1296} = 11,88\%$$

N.B. There are 36 interior points on the discretized grid, hence there are 36 unknowns, and the matrix A is of size 36^2.

ii) A^{-1} is a dense matrix, so calculating A^{-1} is not a good idea, it increases the computational cost.

iii) A is positive definite because

α) A has a Cholesky factorization.

β) All eigenvalues of A are positive.

iiv) $A = R^T R$. So, $A\underset{\sim}{u} = \underset{\sim}{b}$ is equivalent to

$$R^T R \underset{\sim}{u} = \underset{\sim}{b}. \qquad (*)$$

We solve $(*)$ in 2 steps:

1) solve $R^T \underset{\sim}{y} = \underset{\sim}{b}$ by forward substitutio

2) solve $R\underset{\sim}{u} = y$ by backward substituti

v) The matrix A is sparse, has non-zero elements arranged on the diagonals.