Topic and contents

UNSW, School of Mathematics and Statistics

MATH2089 – Numerical Methods

Week 05 – Interpolation by Polynomials

- MATLAB M-files
  - `finterp.m`
  - `polinterp1.m`
  - `lagpol.m`

Some motivations

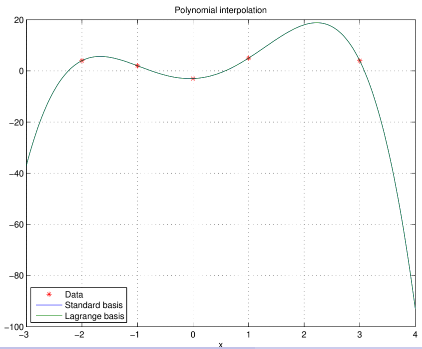
- In computer aided design, one needs to fit a curve through a finite number of points.
- The process is called interpolation.
- For example, run `hand.m`

Polynomial Interpolation example

- Interpolation $\iff$ exactly fit data
- Polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$, Degree n, $n+1$ coefficients

Example (Polynomial interpolation `polinterp1.m`)

- Data $\mathbf{x} = (-2, -1, 0, 1, 3)^T$, $\mathbf{y} = (4, 2, -3, 5, 4)^T$



- $m = 5$ data points
- Interpolation $p(x_i) = y_i, \quad i = 1, \ldots, m$
- Interpolating polynomial has degree $n = m - 1 = 5 - 1 = 4$
- Solve linear system for coefficients $a_k, k = 0, \ldots, 4$

Linear interpolation

Example (Linear interpolation)

Find the polynomial $p(x)$ that interpolates the data (x_0, y_0) and (x_1, y_1).

Solution

- *Two data points $\Longrightarrow$ interpolating polynomial has degree 1, i.e linear, that is equation of a line.*
- *A line that goes through (x_0, y_0) is of the form $p(x) = y_0 + a_1(x - x_0)$*
- *The slope of the line is a_1 given by*

$$a_1 = \frac{(y_1 - y_0)}{(x_1 - x_0)}$$

- *Linear interpolant*

$$p(x) = y_0 + \frac{(y_1 - y_0)}{(x_1 - x_0)}(x - x_0)$$

Another way

Solution

Let the polynomial be $p(x) = a_0 + a_1 x$. *Interpolation conditions imply*

$$\begin{aligned} p(x_0) &= a_0 + a_1 x_0 = y_0 \\ p(x_1) &= a_0 + a_1 x_1 = y_1 \end{aligned}$$

We write as a linear system

$$\begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

Solving the linear system, we get $a_0 = (x_1 y_0 - x_0 y_1)/(x_1 - x_0)$, $a_1 = (y_1 - y_0)/(x_1 - x_0)$. *Hence,*

$$p(x) = \frac{x_1 y_0 - x_0 y_1}{x_1 - x_0} + \frac{y_1 - y_0}{x_1 - x_0} x$$

Quadratic interpolation

Example (Quadratic interpolation)

Find the polynomial $p(x)$ that interpolates the data (x_0, y_0), (x_1, y_1) and (x_2, y_2).

Solution

Since there are 3 *data points, the polynomial is a quadratic of the form* $p(x) = a_0 + a_1 x + a_2 x^2$. *The interpolation conditions*

$$\begin{aligned} p(x_0) &= a_0 + a_1 x_0 + a_2 x_0^2 = y_0 \\ p(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 = y_1 \\ p(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 = y_2 \end{aligned}$$

Solution

(cont.) Write as a linear system

$$\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

Solving the linear system, we get the coefficients a_0, a_1, a_2.

Polynomial interpolation

- $n+1$ data points x_j for $j = 0, \dots, n$ distinct: $x_j \neq x_k$ for $j \neq k$
  - data values y_j at x_j for $j = 0, \dots, n$
- Interpolation conditions $p(x_j) = y_j$ for $j = 0, \dots, n$
  - The polynomial p goes exactly through data
- Polynomial interpolation $p(x) = \sum_{k=0}^{n} a_k x^k$
  - $n+1$ data points $\Longrightarrow$ polynomial of degree n
  - Interpolation conditions as a linear system
  $$\sum_{k=0}^{n} a_k x_j^k = y_j \quad j = 0, \dots, n$$
  - Basis functions x^k for $k = 0, \dots, n$
  - Coefficient matrix (Vandermonde matrix, MATLAB `vander`)
  $$A = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^n \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$
  - Distinct data $\Longrightarrow$ A nonsingular $\Longrightarrow$ unique solution $A\mathbf{a} = \mathbf{y}$ for any $\mathbf{y}$

Example

Find the interpolating polynomial $p(x)$ that interpolates $(5,1)$, $(-7,-23)$, $(-6,-54)$, and $(0,-954)$, using standard basis functions.

Solution

- *Using standard basis functions: Since there are 4 data points, $p(x)$ is of degree 3. The standard basis is $\{1, x, x^2, x^3\}$, and $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. We have*

$$\begin{aligned} a_0 + 5a_1 + 25a_2 + 125a_3 &= 1 \\ a_0 - 7a_1 + 49a_2 - 343a_3 &= -23 \\ a_0 - 6a_1 + 36a_2 - 216a_3 &= -54 \\ a_0 \qquad\qquad\qquad &= -954 \end{aligned}$$

Solution (cont.)

In matrix form ($A\mathbf{a} = \mathbf{b}$)

$$\begin{bmatrix} 1 & 5 & 25 & 125 \\ 1 & -7 & 49 & -343 \\ 1 & -6 & 36 & -216 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -23 \\ -54 \\ -954 \end{bmatrix}$$

Using the MATLAB backslash command $\mathbf{a} = A\backslash\mathbf{b}$ we obtain $(a_0, a_1, a_2, a_3) = (-954, -84, 35, 4)$, so that

$$p(x) = -954 - 84x + 35x^2 + 4x^3.$$

Lagrange polynomials

- Lagrange polynomials $\ell_j \in \mathbb{P}_n$ of degree n

$$\ell_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \left(\frac{x - x_k}{x_j - x_k} \right)$$

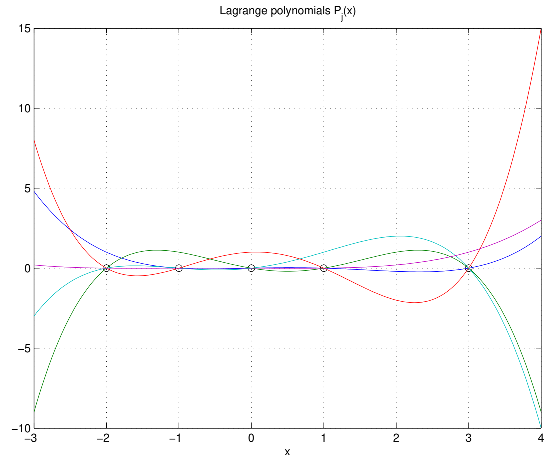
  - $\ell_j(x_i) = 0$ for $j \neq i$, $\quad \ell_j(x_i) = 1$ for $j = i$
  - Interpolating polynomial

$$p(x) = \sum_{j=0}^{n} y_j \ell_j(x)$$

  - No linear system to solve
  - p satisfies the interpolation conditions (Check!).
  - MATLAB `lagpol.m`

Lagrange polynomials

- Lagrange polynomials for points $\mathbf{x} = (-2, -1, 0, 1, 3)^T$
- Values at data points $\ell_j(x_i) = 0, i \neq j$, $\ell_j(x_j) = 1$



Polynomial Interpolation

- Polynomial $p_n(x)$ of degree n
- Interpolate the function f at distinct nodes $x_j \in [a,b]$ for $j = 0, \ldots, n$
- Need $n+1$ nodes for degree n
- Interpolation $p_n(x_j) = f(x_j)$ for $j = 0, \ldots, n$
- Lagrange polynomial

$$p_n(x) = \sum_{j=0}^{n} f(x_j)\ell_j(x), \qquad \ell_j(x) = \prod_{k=0,k\neq j}^{n} \frac{(x-x_k)}{(x_j-x_k)}$$

- Remainder term $R_n(x)$ such that $f(x) = p_n(x) + R_n(x)$
- $f \in C^{(n+1)}([a,b])$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{j=0}^{n}(x-x_j) \qquad \text{for some } \xi = \xi(x) \in [a,b]$$

Example

Find the interpolating polynomial $P(x)$ that interpolates $(5,1)$, $(-7,-23)$, $(-6,-54)$, and $(0,-954)$, using Lagrange polynomials.

Solution (cont.)

- *Recall the data:* $(5,1)$, $(-7,-23)$, $(-6,-54)$, *and* $(0,-954)$. *Using Lagrange polynomials:*

$$\ell_1(x) = \frac{(x+7)(x+6)x}{(5+7)(5+6)5} = \frac{1}{660}x(x+6)(x+7)$$

$$\ell_2(x) = \frac{(x-5)(x+6)x}{(-7-5)(-7+6)(-7)} = \frac{-1}{84}x(x-5)(x+6)$$

$$\ell_3(x) = \frac{(x-5)(x+7)x}{(-6-5)(-6+7)(-6)} = \frac{-1}{66}x(x-5)(x+7)$$

$$\ell_4(x) = \frac{(x-5)(x+7)(x+6)}{(0-5)(0+7)(0+6)} = \frac{-1}{210}(x-5)(x+6)(x+7)$$

we have

$$P(x) = \ell_1(x) - 23\ell_2(x) - 54\ell_3(x) - 954\ell_4(x).$$

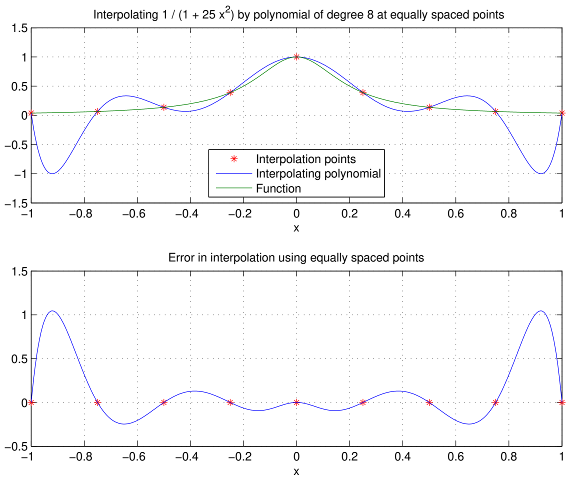
Equally spaced nodes

- Equally spaced nodes on $[a,b]$ (MATLAB `linspace`)

$$x_j = a + jh, \quad j = 0, \ldots, n, \qquad h = \frac{(b-a)}{n}$$

- Carl Runge (1856 - 1927) example $f(x) = 1/(1+25x^2)$ on $[-1,1]$
  - MATLAB script `finterp.m`: error grows as degree increases!

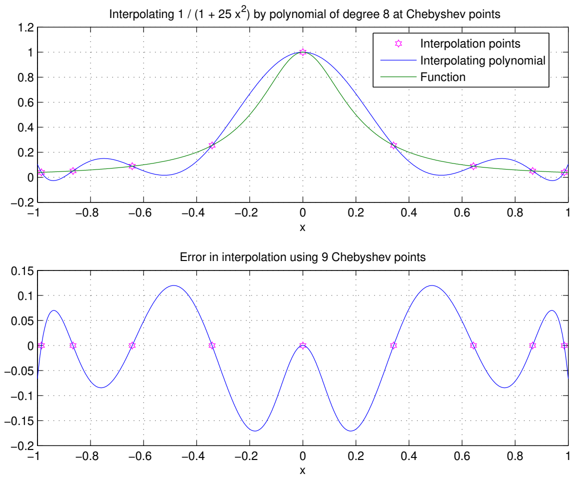
Interpolation at equally spaced nodes



Chebyshev nodes

- Choose nodes x_j to minimize $\displaystyle \max_{x\in[-1,1]} \prod_{j=0}^{n}(x-x_j)$
- Chebyshev nodes
  - $t_j = \cos\left(\frac{2n+1-2j}{2n+2}\pi\right) \in [-1,1]$ for $j = 0,\ldots,n$
  - On $[a,b] \implies x_j = \frac{a+b}{2} + \frac{b-a}{2}t_j$ for $j = 0,\ldots,n$

Interpolation at Chebyshev nodes



Chebyshev polynomials

- Chebyshev polynomial of degree n on $[-1,1]$

$$T_n(x) = \cos(n\arccos(x)) \quad n = 0,1,\ldots$$

- Three term recurrence

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

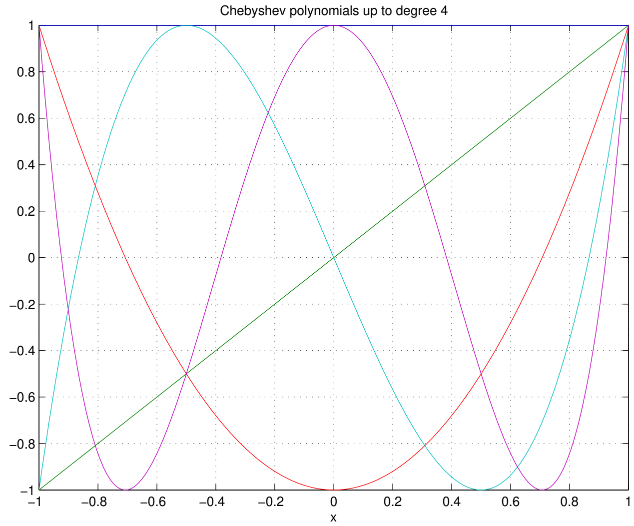
  - Basis for $\mathbb{P}_n$ space of all polynomials of degree at most n
  - $T_n(x)$ has leading coefficient 2^{n-1}, $n \geq 1$
- Chebyshev polynomial $T_n(x)$, $n \geq 1$ has n simple zeros in $[-1,1]$ at

$$\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right), k = 1,\ldots,n$$

- Extrema are attained at

$$\bar{x}'_k = \cos\left(\frac{k\pi}{n}\right), \quad T_n(\bar{x}'_k) = (-1)^k, \quad k = 0,1,\ldots,n$$

Chebyshev polynomials cont



Key Concepts

- Consider case when data values known “exactly”
- Interpolation: exactly match data values
- Example: approximating a function
  - How should you measure quality of approximation? Maximum norm.
  - Choice of points x_j to interpolate function values $y_j = f(x_j)$.
    - Equally spaced points
    - Chebyshev points
    - ....
- Choice of interpolating function
  - Example: Degree n polynomial $\Longrightarrow n+1$ parameters
- Choice of representation (basis) for interpolating function
  - Example: polynomials
    - Monomial basis: x^k for $k = 0, 1, \ldots, n$
    - Lagrange polynomials: $\ell_j(x_i) = 0, i \neq j$, $\ell_j(x_j) = 1$, $j = 0, \ldots, n$
    - Chebyshev polynomials: $T_k(x) = \cos(k \arccos(x))$, $k = 0, 1, \ldots, n$
    - ....