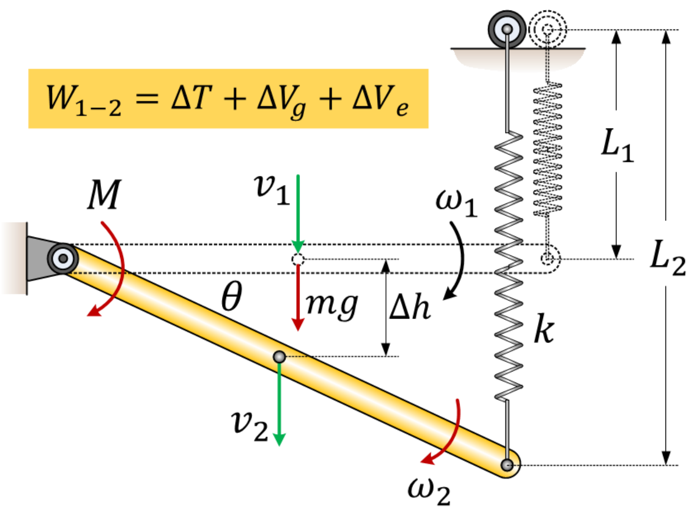
School of Mechanical and Manufacturing Engineering

MMAN1300 Engineering Mechanics 1

Dr. David C. Kellermann



Week 12, L1-2: Rigid Body Energy and Momentum

WORK ENERGY METHODS
- General equations of motion
- Translation, Fixed-axis rotation
- General plane motion

MOMENTUM METHODS
- Work-energy relations
- Acceleration from work-energy

Work Energy Methods for Rigid Bodies

Today's Objectives:

1. Define the various ways a force and couple do work.
2. Apply the principle of work and energy to a rigid body.

Topics:

- Applications
- Kinetic Energy
- Work of a Force or Couple
- Principle of Work and Energy

Quick Quiz

1. Kinetic energy due to rotation of the body is defined as

A) $\frac{1}{2}m(v_G)^2$

B) $\frac{1}{2}m(v_G)^2 + \frac{1}{2}I_G\omega^2$

C) $\frac{1}{2}I_G\omega^2$

D) $I_G\omega^2$

2. When calculating work done by forces, the work of an internal force does not have to be considered because ___________.

A) internal forces do not exist

B) the forces act in equal but opposite collinear pairs

C) the body is at rest initially

D) the body can deform

Applications

The work of the torque (or moment) developed by the driving gears on the two motors on the concrete mixer is transformed into the rotational kinetic energy of the mixing drum.

If the motor gear characteristics are known, how would you find the rotational velocity of the mixing drum?

Applications

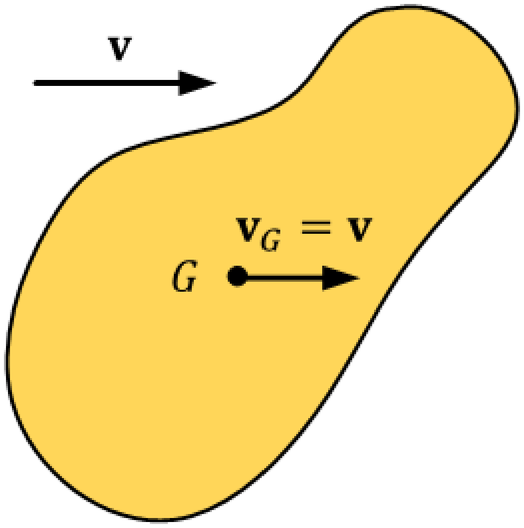
The work done by the soil compactor's engine is transformed into the translational kinetic energy of the frame and the translational and rotational kinetic energy of the roller and wheels (excluding the internal kinetic energy developed by the moving parts of the engine and drive train).

Are the kinetic energies of the frame and the roller related to each other? If so, how?

Kinetic energy

The kinetic energy of a rigid body can be expressed as the sum of its translational and rotational kinetic energies. In equation form, a body in general plane motion has kinetic energy given by:

$$\boxed{T = \frac{1}{2}m(v_G)^2 + \frac{1}{2}I_G\omega^2}$$



Translation

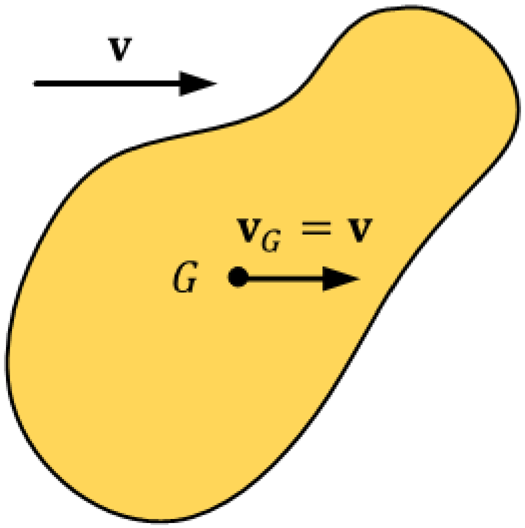
Kinetic energy

Several simplifications can occur.

1. Pure Translation:

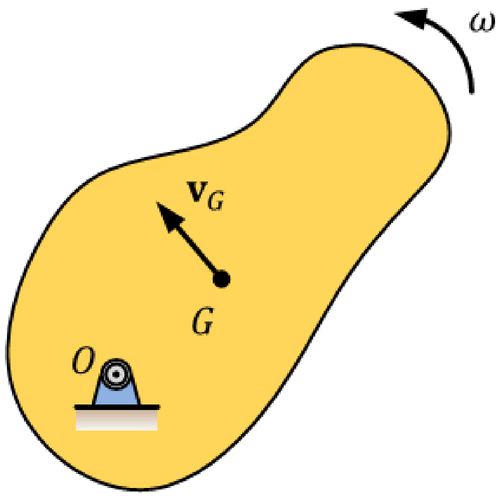
When a rigid body is subjected to only curvilinear or rectilinear translation, the rotational kinetic energy is zero ($\omega = 0$). Therefore,

$$\boxed{T = \frac{1}{2}m(v_G)^2}$$



Translation

Kinetic Energy



Rotation About a Fixed Axis

2. Pure Rotation:

When a rigid body is rotating about a fixed axis passing through point O, the body has both translational and rotational kinetic energy. Thus,

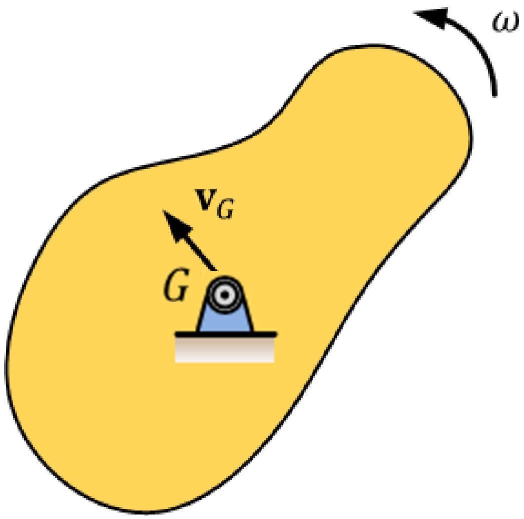
$$T = 0.5\ m\ (v_G)^2 + 0.5\ I_G\ \omega^2$$

Since $v_G = r_G\omega$, we can express the kinetic energy of the body as:

$$T = 0.5\ [\ I_G + m(r_G)^2\ ]\ \omega^2 = 0.5\ I_O\ \omega^2$$

Kinetic Energy About Mass Centre

If the rotation occurs about the mass center, G, then what is the value of $\mathbf{v}_G$?

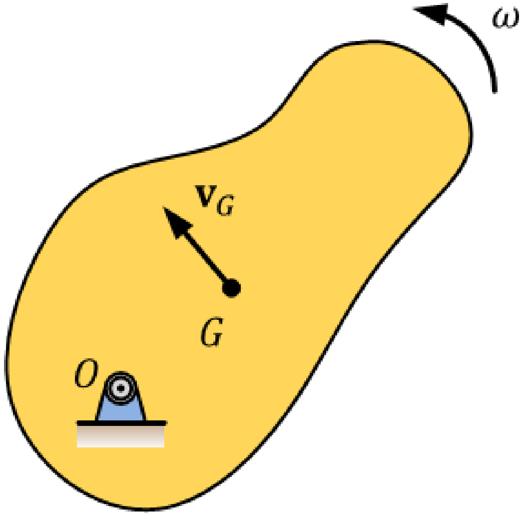


In this case, the velocity of the mass center is equal to zero.
So the kinetic energy equation reduces to:

$$\boxed{T = \frac{1}{2} I_G \omega^2}$$

Kinetic Energy About a Fixed Axis

If the rotation occurs about a fixed axis, O, then what is the value of $\mathbf{v}_O$?



Rotation About a Fixed Axis

In this case, the velocity of the mass center is not zero.
So the kinetic energy equation involves only rotation about the fixed axis:

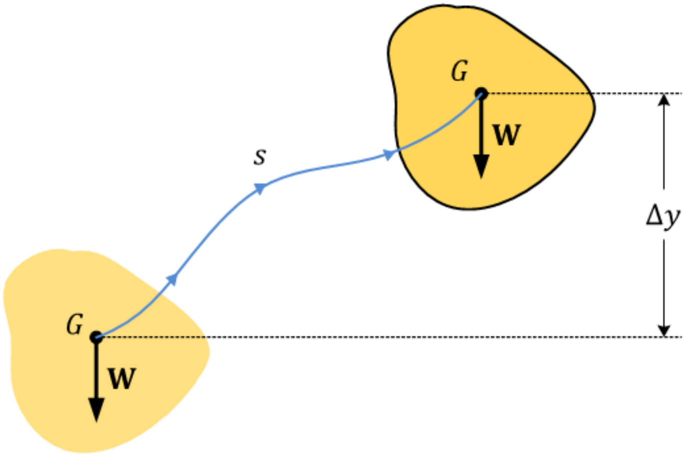
$$\boxed{T = \frac{1}{2} I_0 \omega^2}$$

The Work of aForce

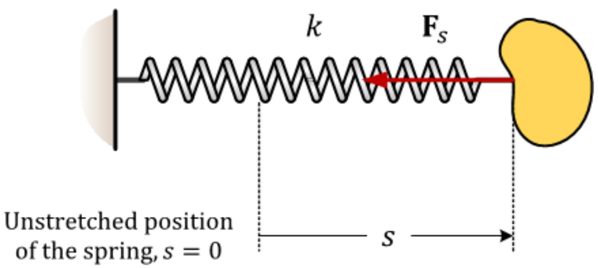
Recall that the work done by a force can be written as:

$$U_F = \int \boldsymbol{F} \bullet d\boldsymbol{r} = \int (F \cos \theta)\, ds.$$

When the force is constant, this equation reduces to $U_{Fc} = (F_c \cos \theta)\, s$ where $F_c \cos \theta$ represents the component of the force acting in the direction of the displacement, s.



The Work of Gravity / Springs



Work of a weight:

As before, the work can be expressed as

$$\boxed{U_w = -W\Delta y}$$

Remember, if the force and movement are in the same direction, the work is positive.

Work of a spring force:

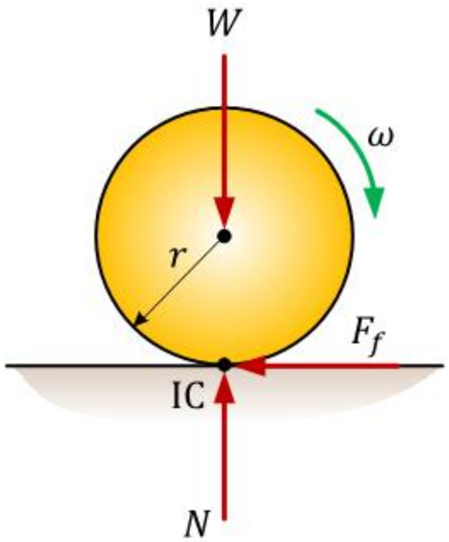
For a linear spring, the work is:

$$\boxed{U_s = -\frac{1}{2}\,k[(s_2)^2 - (s_1)^2]}$$

Forces that Do No Work

There are some external forces that do no work.

For instance, reactions at fixed supports do no work because the displacement at their point of application is zero.



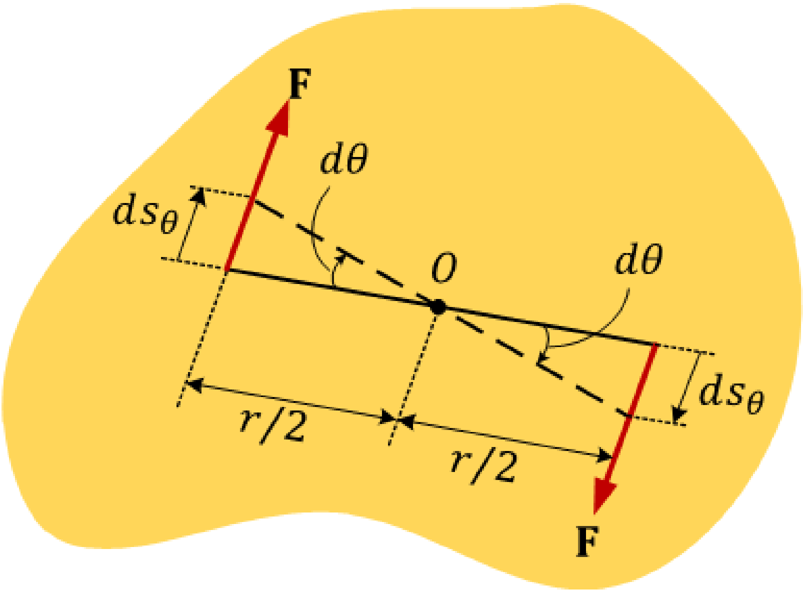
Normal forces and friction forces acting on bodies as they roll without slipping over a rough surface also do no work since there is no instantaneous displacement of the point in contact with ground (it is an instant center, IC).

Internal forces do no work because they always act in equal and opposite pairs. Thus, the sum of their work is zero.

The Work of a Couple

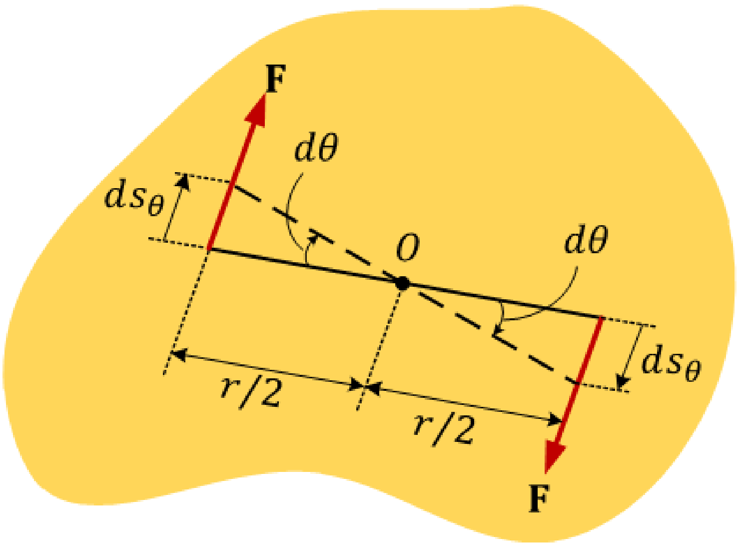
When a body subjected to a couple experiences general plane motion, the two couple forces do work only when the body undergoes rotation.

If the body rotates through an angular displacement $d\theta$, the work of the couple moment, M, is:



$$\boxed{U_M = \int_{\theta_1}^{\theta_2} M \, d\theta}$$

The Work of a Couple



If the couple moment, M, is constant, then

$$\boxed{U_M = M(\theta_2 - \theta_2)}$$

The work is positive if M and $(\theta_2 - \theta_2)$ are in the same direction.

Principle of Work and Energy

Recall the statement of the principle of work and energy used earlier:

$$\boxed{T_1 + \sum U_{1-2} + T_2}$$

In the case of general plane motion, this equation states that the sum of the initial kinetic energy (both translational and rotational) and the work done by all external forces and couple moments equals the body's final kinetic energy (translational and rotational).

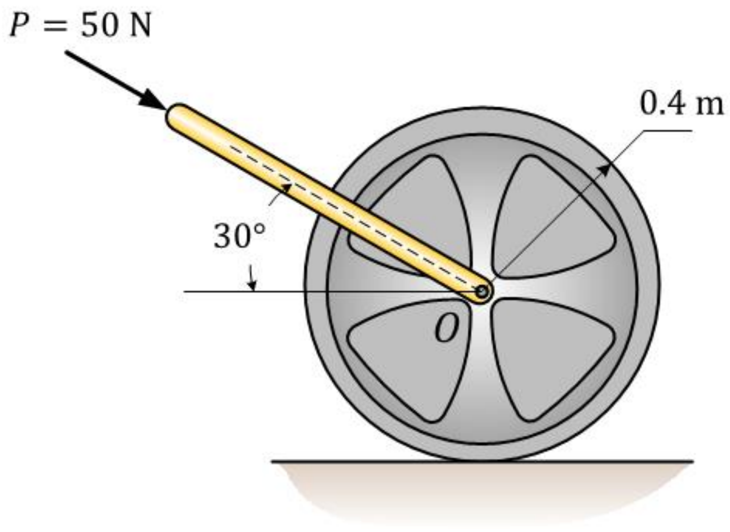
Principle of Work and Energy

This equation is a **scalar** equation.

$$\boxed{T_1 + \sum U_{1-2} + T_2}$$

It can be applied to a system of rigid bodies by summing contributions from all bodies.

Example 1



Given: The 50 kg wheel is subjected to a force of 50 N. The radius of gyration of the wheel about its mass center O is $k_o = 0.3$ m.

W12 Example 1 (Web view)

Find: The angular velocity of the wheel after it has rotated 10 revolutions. The wheel starts from rest and rolls without slipping.

Plan: Use the principle of work and energy to solve the problem since distance is the primary parameter. Draw a free body diagram of the wheel and calculate the work of the external forces.

Quick Quiz

1. If a rigid body rotates about its center of gravity, its translational kinetic energy is ___________ at all times.

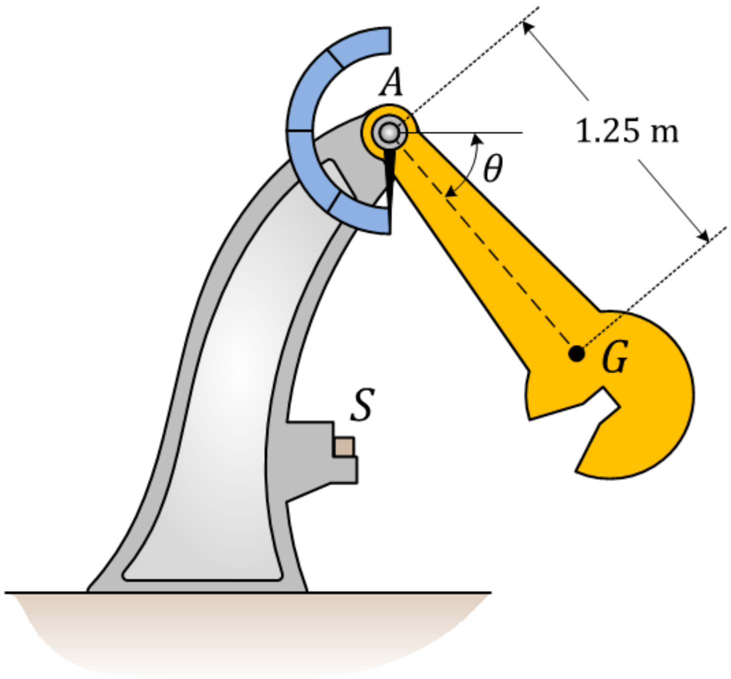
   A) constant

   B) equal to its rotational kinetic energy

   C) zero

   D) Cannot be determined

Example 2



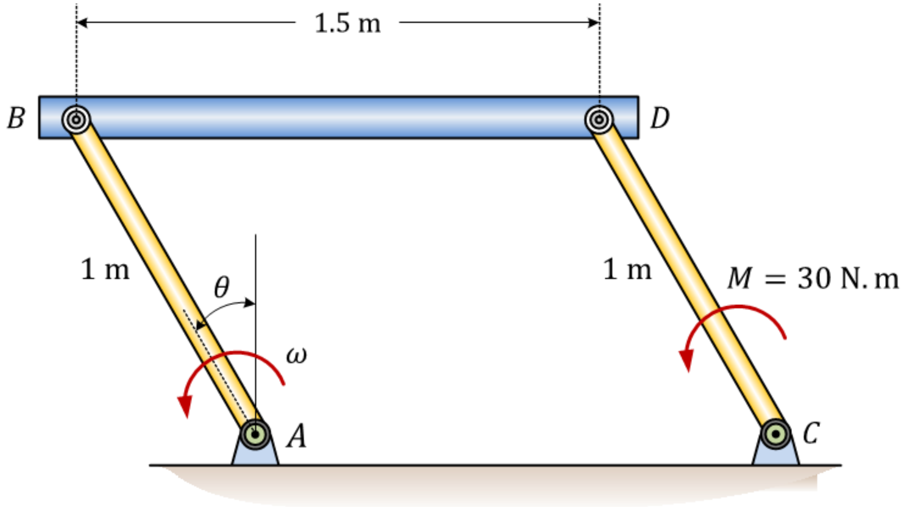
Given: The 50 kg pendulum of the Charpy impact machine is released from rest when $\theta = 0$. The radius of gyration is $k_A = 1.75$ m.

W12 Example 2 (Web view)

Find: The angular velocity of the pendulum when $\theta = 90^\circ$.

Plan: Since the problem involves distance, the principle of work and energy is an efficient solution method. The only force involved doing work is the weight, so only its work need be determined.

Example 3



W12 Example 3 (Web view)

Given: The linkage consists of two 6-kg rods AB and CD and a 20-kg bar BD. Rod CD is subjected to a couple moment M = 30 N·m. When $\theta = 0°$, rod AB is rotating with an angular velocity ω = 2 rad/s.

Find: The angular velocity ω when $\theta = 45°$.

Plan: Since the problem involves distance, the principle of work and energy is an efficient solution method.

Conservation of Energy for Rigid Bodies

Today's Objectives:

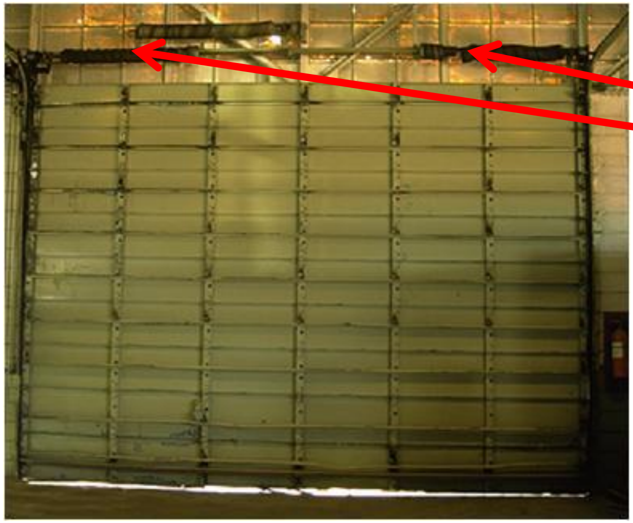
a) Determine the potential energy of conservative forces.

b) Apply the principle of conservation of energy.

Topics:

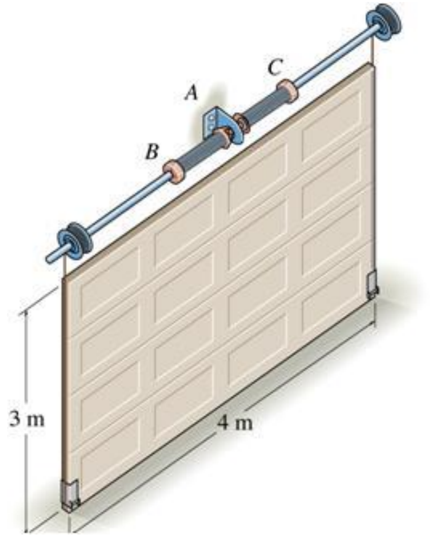
- Potential Energy
- Conservation of Energy

Applications



The torsion springs located at the top of the garage door wind up as the door is lowered.

When the door is raised, the potential energy stored in the spring is transferred into the gravitational potential energy of the door's weight, thereby making the door easy to open.



Are parameters such as the torsional spring stiffness and initial rotation angle of the spring important when you install a new spring?

Conservation of Energy

- The conservation of energy theorem is a “simpler” energy method (recall that the principle of work and energy is also an energy method) for solving problems.
- Once again, the problem parameter of distance is a key indicator for when conservation of energy is a good approach to solving a problem.
- If it is appropriate for the problem, conservation of energy is easier to use than the principle of work and energy.
- This is because the calculation of the work of a conservative force is simpler. But, what makes a force conservative?

Conservative Forces

- A force ***F*** is conservative if the work done by the force is independent of the path.

- In this case, the work depends only on the initial and final positions of the object with the path between the positions of no consequence.

- Typical conservative forces encountered in dynamics are gravitational forces (i.e., weight) and elastic forces (i.e., springs).

- What is a common force that is not conservative?

Conservation of Energy

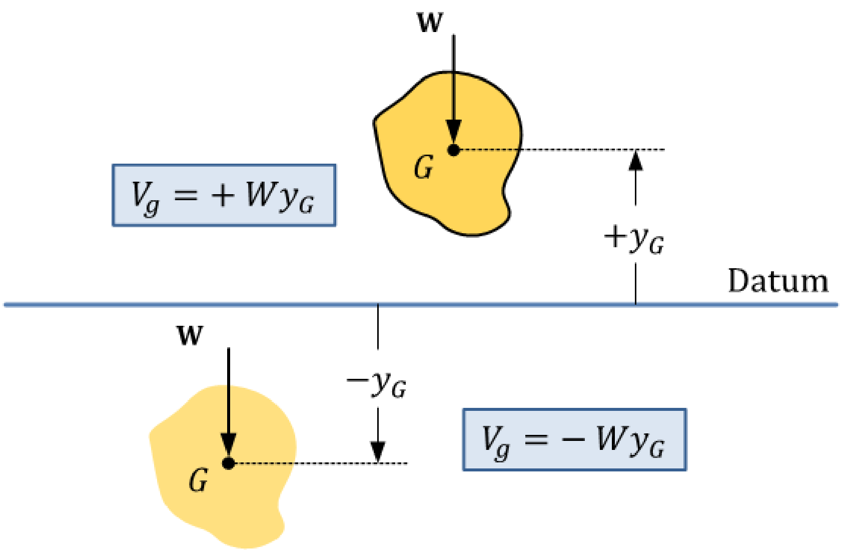
- When a rigid body is acted upon by a system of conservative forces, the work done by these forces is conserved. Thus, the sum of kinetic energy and potential energy remains constant.
  This principle is called conservation of energy and is expressed as:

$$\boxed{T_1 + V_1 = T_2 + V_2 = \text{Constant}}$$

In other words, as a rigid body moves from one position to another when acted upon by only conservative forces, kinetic energy is converted to potential energy and vice versa.

Gravitational Potential Energy

The gravitational potential energy of an object is a function of the height of the body's center of gravity above or below a datum.



Gravitational Potential Energy

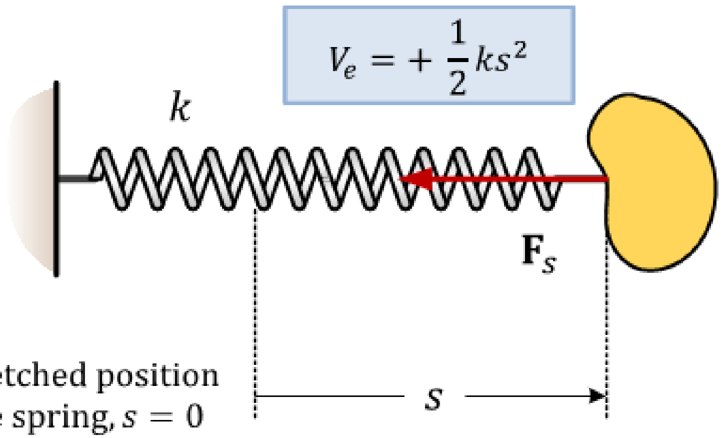
The gravitational potential energy of a body is found by the equation

$$\boxed{V_g = W y_G}$$

Gravitational potential energy is positive when y_G is positive, since the weight has the ability to do positive work (why is it positive?) when the body is moved back to the datum.

Elastic Potential Energy

Spring forces are also conservative forces.



Elastic Potential Energy

The potential energy of a spring force ($F = ks$) is found by the equation

$$\boxed{V_e = \frac{1}{2}ks^2}$$

Notice that elastic potential energy is always positive.

Procedure for Analysis

Problems involving velocity, displacement and conservative force systems can be solved using the conservation of energy equation.

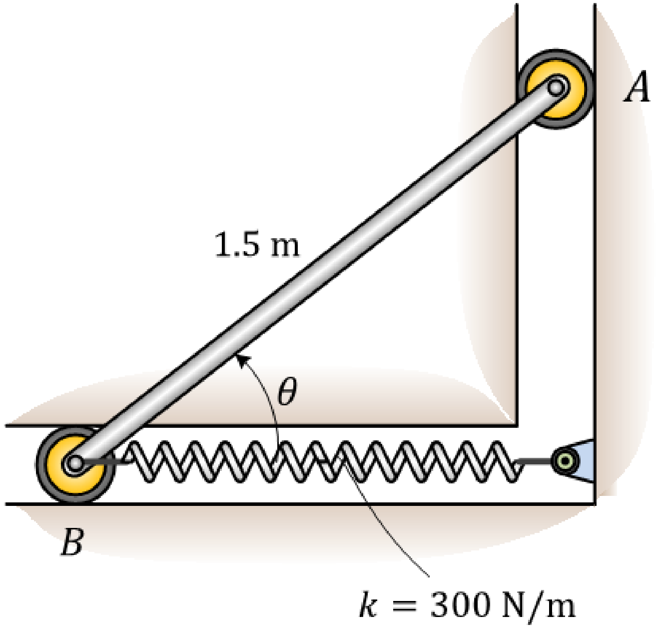
- Potential energy: Draw two diagrams: one with the body located at its initial position and one at the final position. Compute the potential energy at each position using

$$V = V_g + V_e \qquad \text{where} \qquad V_g = W y_G \qquad \text{and} \qquad V_e = \frac{1}{2} k s^2$$

- Kinetic energy: Compute the kinetic energy of the rigid body at each location. Kinetic energy has two components: translational kinetic energy, $\frac{1}{2} m (v_G)^2$, and rotational kinetic energy, $\frac{1}{2} I_G \omega^2$.

- Apply the conservation of energy equation.

Example 4



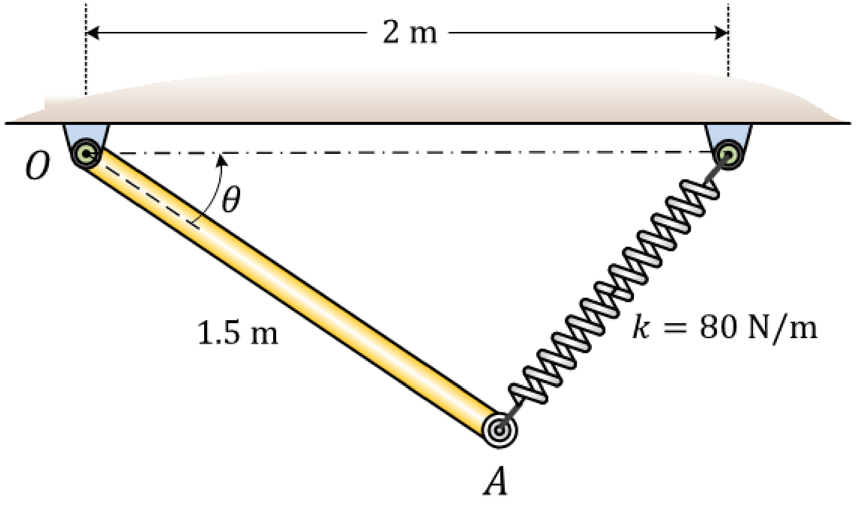
Given: The rod AB has a mass of 30 kg. The spring is un-stretched when $\theta = 45°$. The spring constant k is 300 N/m.

W12 Example 4 (Web view)

Find: The angular velocity of rod AB at $\theta = 0°$, if the rod is released from rest when $\theta = 45°$.

Plan: Use the energy conservation equation since all forces are conservative and distance is a parameter (represented here by θ). The potential energy and kinetic energy of the rod at states 1 and 2 will have to be determined.

Example 5



Given: The 30 kg rod is released from rest when $\theta = 0°$. The spring is unstretched when $\theta = 0°$.

Find: The angular velocity of the rod when $\theta = 30°$.

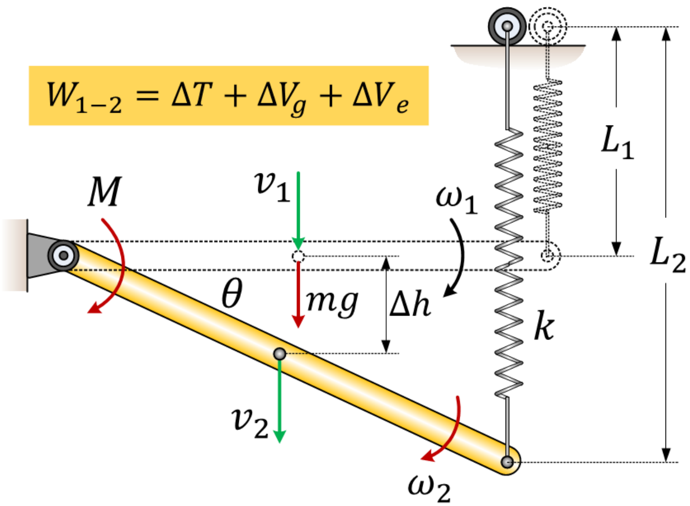
Plan:

Since distance is a parameter and all forces doing work are conservative, use conservation of energy. Determine the potential energy and kinetic energy of the system at both positions and apply the conservation of energy equation.

W12 Example 5 (Web view)

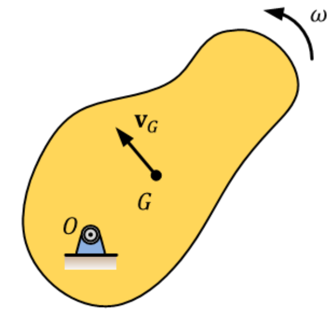
Summary of Rigid Body Work and Energy

$$W_{1-2} = \Delta T + \Delta V_g + \Delta V_e$$



- Angular kinetic energy

$$T = \frac{1}{2} I_0 \omega^2$$

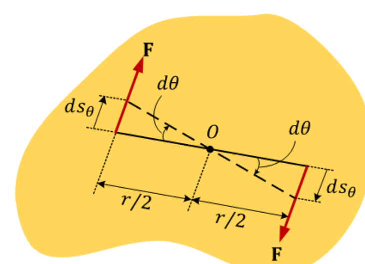


Rotation About a Fixed Axis

- The work of a moment

$$U_M = \int_{\theta_1}^{\theta_2} M \, d\theta$$

- Conservation of energy of a rigid body
  - $T_1 + V_1 = T_2 + V_2 = \text{Constant}$



Next Topic:

Momentum Methods for Rigid Bodies

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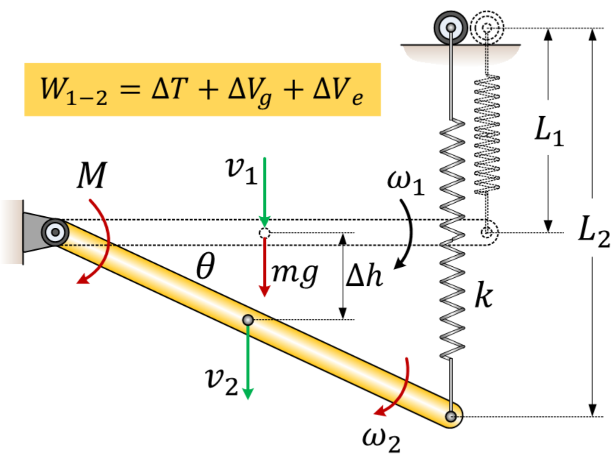
Week 12, L2- Momentum Method for Rigid Bodies



WORK ENERGY METHODS
- General equations of motion
- Translation, Fixed-axis rotation
- General plane motion

MOMENTUM METHODS
- Work-energy relations
- Acceleration from work-energy

Principle Of Angular Impulse And Momentum

Topics:

- Linear and Angular Momentum
- Principle of Impulse and Momentum
- Conservation of Linear and Angular Momentum

Principle Of Angular Impulse And Momentum

Objectives:

1. Develop formulations for the linear and angular momentum of a body.
2. Apply the principle of linear and angular impulse and momentum.
3. Understand necessary conditions for conservation of linear and angular momentum.
4. Use conservation of linear/ angular momentum for solving problems.

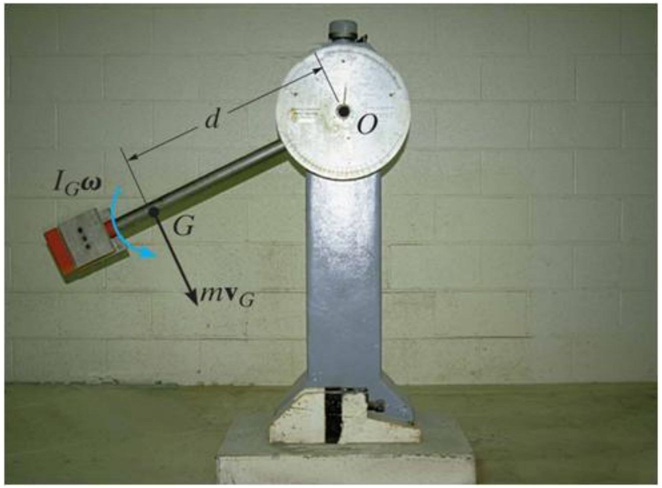
Applications

A swing bridge opens and closes by turning, using a motor located at A under the center of the deck that applies a torque **M** to the bridge.

If the bridge was supported by and rotated about at its end B, would the same torque open the bridge in the same time, or would it open slower or faster?

What are the benefits of making the bridge with the variable depth (thickness) substructure shown?

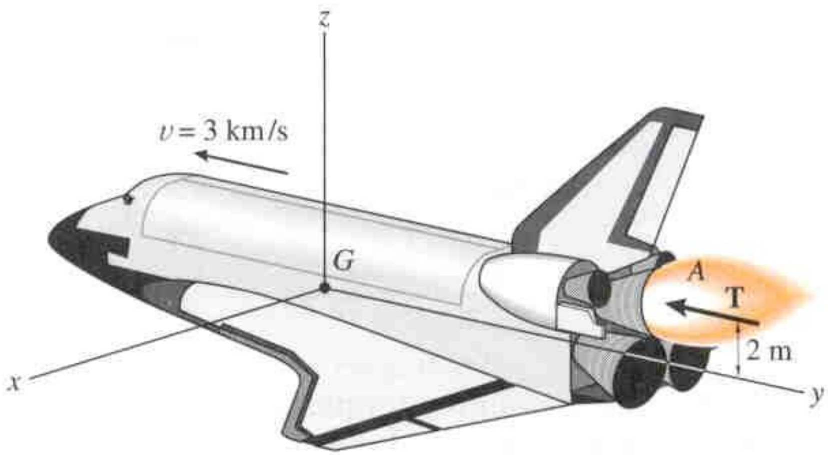
Applications



As the pendulum of the Charpy tester swings downward, its angular momentum and linear momentum both increase. By calculating its momenta in the vertical position, we can calculate the impulse the pendulum exerts when it hits the test specimen.

As the pendulum rotates about point O, what is its angular momentum about point O?

Applications



The space shuttle, now retired from NASA's fleet, has several engines that exerted thrust on the shuttle when they were fired. By firing different engines, the pilot could control the motion and direction of the shuttle.

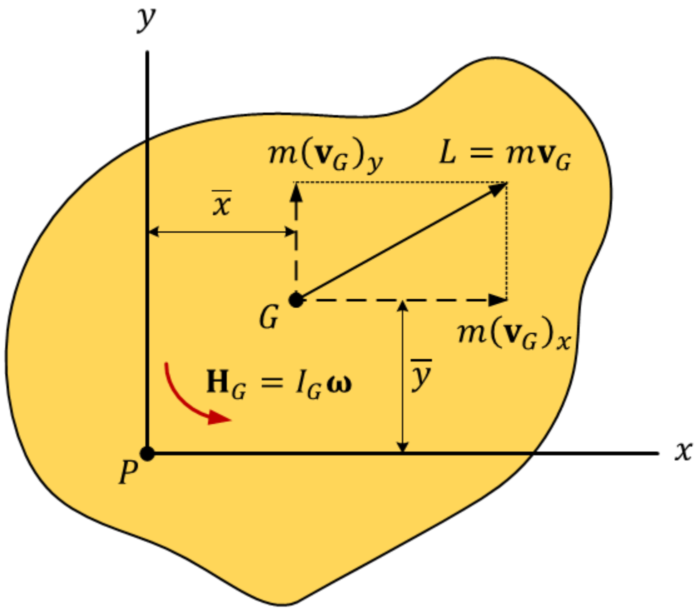
If only engine A was fired, about which axis did the shuttle tend to rotate? What would the resultant impulse be?

Linear and Angular Momentum

The **linear momentum** of a rigid body is defined as

$$L = m\mathbf{v}_G$$

This equation states that the linear momentum vector **L** has a magnitude equal to $(m\mathbf{v}_G)$ and a direction defined by $\mathbf{v}_G$.



Body Momentum Diagram

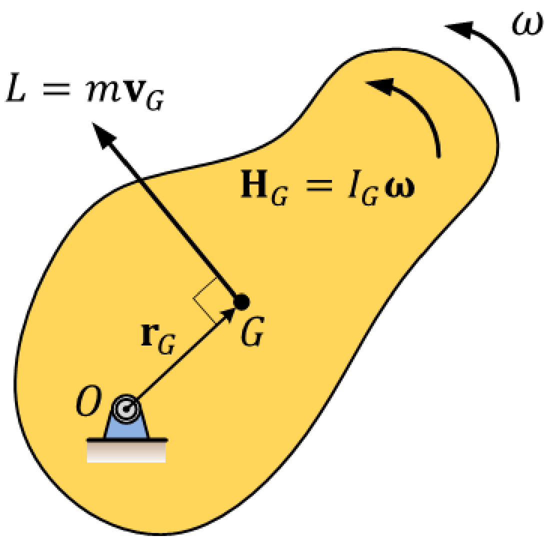
The **angular momentum** of a rigid body is defined as

$$\mathbf{H}_G = I_G\boldsymbol{\omega}$$

Remember that the direction of $\mathbf{H}_G$ is perpendicular to the plane of rotation.

Linear and Angular Momentum

Rotation about a fixed axis



Rotation About a Fixed Axis

When a rigid body is rotating about a fixed axis passing through point O, the body's linear momentum and angular momentum about G are:

$$\mathbf{L} = m\mathbf{v}_G$$

$$\mathbf{H}_G = I_G\boldsymbol{\omega}$$

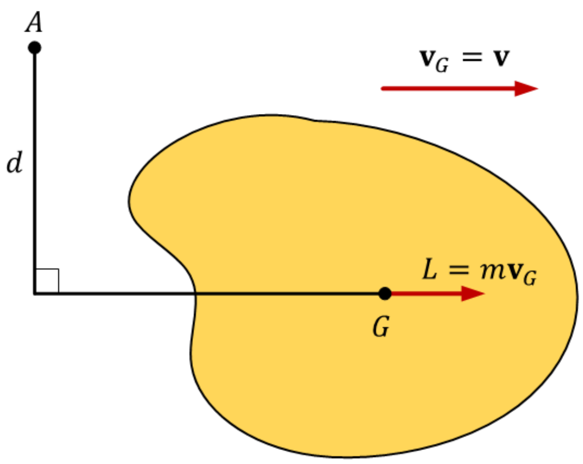
To compute the angular momentum of the body about the center of rotation O.

$$\boxed{\mathbf{H}_O = (\mathbf{r}_G \times m\mathbf{v}_G) + I_G\boldsymbol{\omega} = I_O\boldsymbol{\omega}}$$

Linear and Angular Momentum

Translation

When a rigid body undergoes rectilinear or curvilinear translation, its angular momentum is zero because $\omega = 0$.



Translation

Therefore,

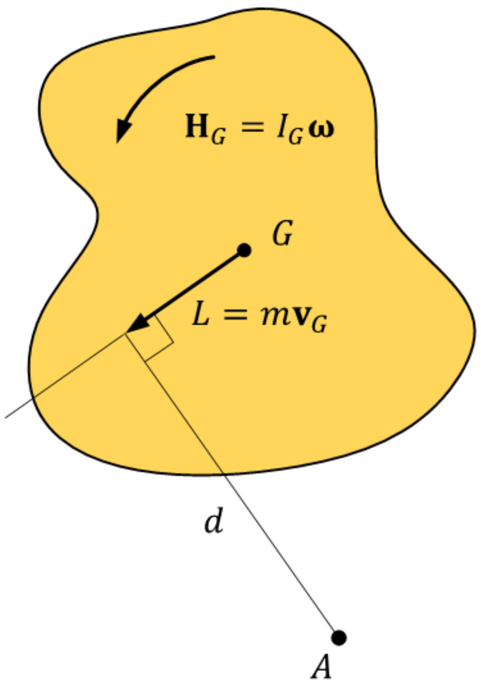
$$\mathbf{L} = m\mathbf{v}_G$$

And

$$\mathbf{H}_G = 0$$

Linear and Angular Momentum

General plane motion



General Plane Motion

When a rigid body is subjected to general plane motion, both the linear momentum and the angular momentum computed about G are required.

$$\mathbf{L} = m\mathbf{v}_G$$

$$\mathbf{H}_G = I_G\boldsymbol{\omega}$$

The angular momentum about point A is

$$\mathbf{H}_A = I_G\boldsymbol{\omega} + m\mathbf{v}_G(d)$$

Principle of Impulse and Momentum

As in the case of particle motion, the principle of impulse and momentum for a rigid body is developed by combining the equation of motion with kinematics. The resulting equations allow a *direct solution to problems involving force, velocity, and time.*

Principle of Impulse and Momentum

Linear impulse-linear momentum equation:

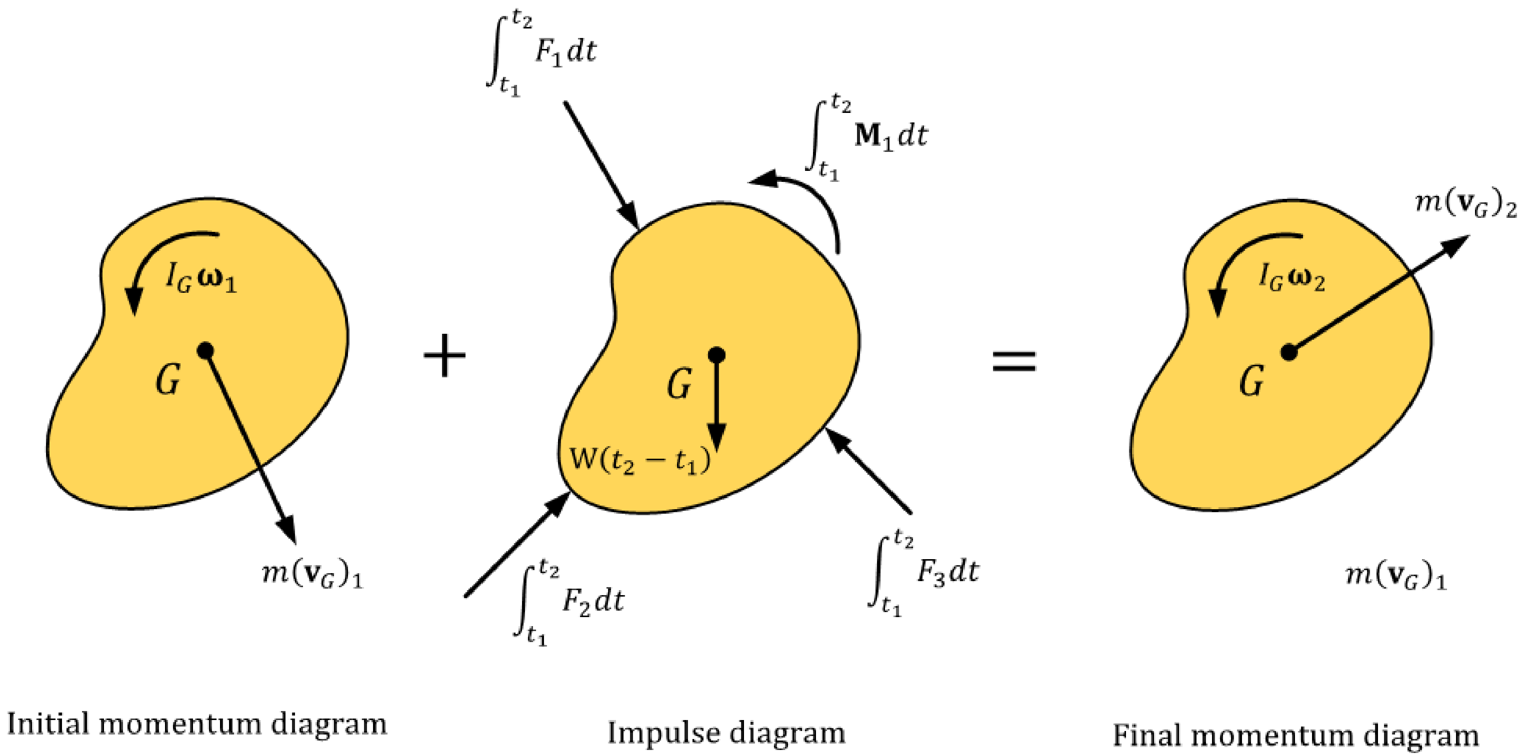
$$\mathbf{L}_1 + \sum \int_{t_1}^{t_2} \mathbf{F}\, dt = \mathbf{L}_2 \qquad \text{or} \qquad (m\mathbf{v}_G)_1 + \sum \int_{t_1}^{t_2} \mathbf{F}\, dt = (m\mathbf{v}_G)_2$$

Angular impulse-angular momentum equation:

$$(\mathbf{H}_G)_1 + \sum \int_{t_1}^{t_2} \mathbf{M}_G\, dt = (\mathbf{H}_G)_2 \qquad \text{or} \qquad I_G \boldsymbol{\omega}_1 + \sum \int_{t_1}^{t_2} \mathbf{M}_G\, dt = I_G\, \boldsymbol{\omega}_2$$

Principle of Impulse and Momentum

The previous relations can be represented graphically by drawing the **impulse-momentum diagram.**



Principle of Impulse and Momentum

To summarize, if motion is occurring in the x-y plane, the linear impulse-linear momentum relation can be applied to the x and y directions and the angular momentum-angular impulse relation is applied about a z-axis passing through any point (i.e., G). Therefore, the principle yields three scalar equations describing the planar motion of the body.

Procedure for Analysis

- Establish the x, y, z-inertial frame of reference.

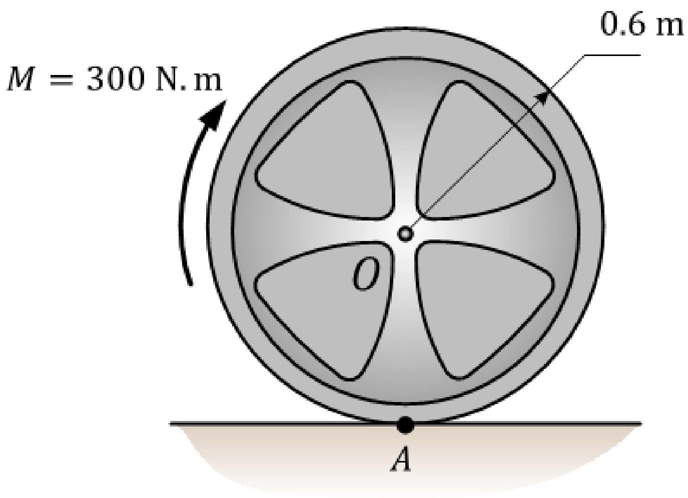
- Draw the impulse-momentum diagrams for the body.

- Compute I_G, as necessary.

- Apply the equations of impulse and momentum (one vector and one scalar or the three scalar equations).

- If more than three unknowns are involved, kinematic equations relating the velocity of the mass center G and the angular velocity ω should be used to furnish additional equations.

Example 6



Given: The 300 kg wheel has a radius of gyration about its mass center O of $k_O = 0.4$ m. The wheel is subjected to a couple moment of 300 N·m.

W12 Example 6 (Web view)

Find: The angular velocity after 6 seconds if it starts from rest and no slipping occurs.

Plan: Time as a parameter should make you think Impulse and Momentum! Since the body rolls without slipping, point A is the center of rotation. Therefore, applying the angular impulse and momentum relationships along with kinematics should solve the problem.

Applications

A skater spends a lot of time either spinning on the ice or rotating through the air. To spin fast, or for a long time, the skater must develop a large amount of angular momentum.

If the skater's angular momentum is constant, can the skater vary her rotational speed? How?

The skater spins faster when the arms are drawn in and slower when the arms are extended. Why?

Conservation of Linear Momentum

Recall that the linear impulse and momentum relationship is

$$\mathbf{L}_1 + \cancelto{0}{\sum \int_{t_1}^{t_2} \mathbf{F}\, \mathrm{d}t} = \mathbf{L}_2 \quad \text{or} \quad (m\, \mathbf{v}_{\mathrm{G}})_1 + \cancelto{0}{\sum \int_{t_1}^{t_2} \mathbf{F}\, \mathrm{d}t} = (m\, \mathbf{v}_{\mathrm{G}})_2$$

If the sum of all the linear impulses acting on the rigid body (or a system of rigid bodies) is zero, all the impulse terms are zero. Thus, the linear momentum for a rigid body (or system) is constant, or conserved. So $\mathbf{L}_1 = \mathbf{L}_2$.

Conservation of Angular Momentum

Similarly, if the sum of all the angular impulses due to external forces acting on the rigid body (or a system of rigid bodies) is zero, all the impulse terms are zero. Thus, angular momentum is conserved

$$(\mathbf{H}_G)_1 + \sum \cancelto{\mathbf{0}}{\int_{t_1}^{t_2} \mathbf{M}_G \, dt} = (\mathbf{H}_G)_2 \quad \text{or} \quad I_G \boldsymbol{\omega}_1 + \sum \cancelto{\mathbf{0}}{\int_{t_1}^{t_2} \mathbf{M}_G \, dt} = I_G \boldsymbol{\omega}_2$$

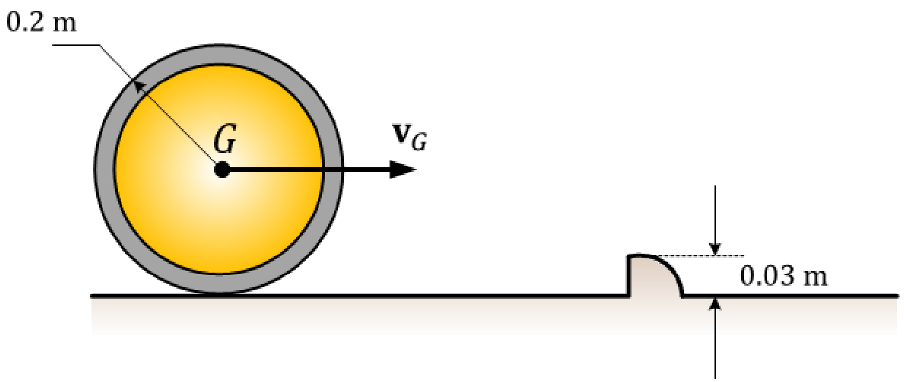
The resulting equation is referred to as the **Conservation of angular momentum** or

$$\boxed{(\mathbf{H}_G)_1 = (\mathbf{H}_G)_2}$$

Procedure for Analysis

- Establish the x, y, z inertial frame of reference and draw FBDs.
- Write the conservation of linear momentum equation.
- Write the conservation of angular momentum equation about a fixed point or at the mass center G.
- Solve the conservation of linear or angular momentum equations in the appropriate directions.
- If the motion is complicated, use of kinematic equations that relate the velocity of the mass center, G, and the angular velocity, ω, may be necessary.

Example 7



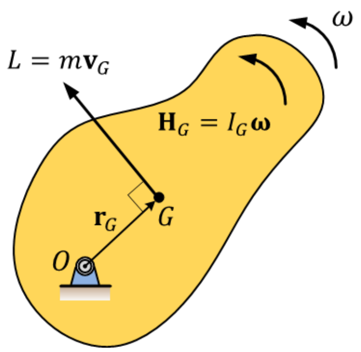
Given: A 10 kg wheel (I_G = 0.156 kg·m^2) rolls without slipping and does not rebound.

W12 Example 7 (Web view)

Find: The minimum velocity, v_G, the wheel must have to just roll over the obstruction at A.

Plan: Since no slipping or rebounding occurs, the wheel pivots about point A. The force at A is much greater than the weight, and since the time of impact is very short, the weight can be considered non-impulsive. The reaction force at A is a problem as we don't know either its direction or magnitude. This force can be eliminated by applying the conservation of angular momentum equation about A.

Summary of Rigid Body Impulse and Momentum



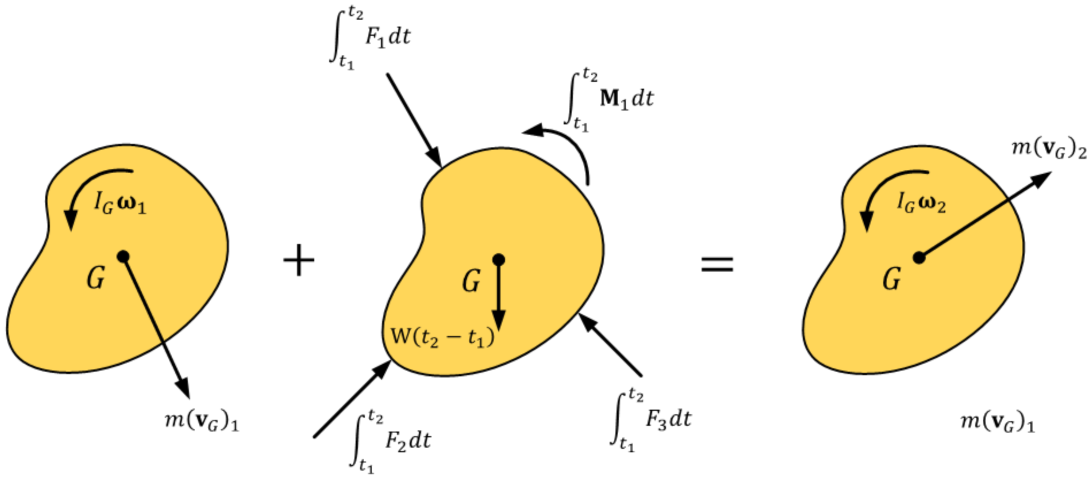
Rotation About a Fixed Axis

- Angular momentum

$$\mathbf{H}_O = (\mathbf{r}_G \times m\mathbf{v}_G) + I_G\boldsymbol{\omega} = I_O\boldsymbol{\omega}$$

- Angular impulse-angular momentum equation:

$$(\mathbf{H}_G)_1 + \sum\int_{t_1}^{t_2} \mathbf{M}_G\, dt = (\mathbf{H}_G)_2 \quad \text{or} \quad I_G\boldsymbol{\omega}_1 + \sum\int_{t_1}^{t_2} \mathbf{M}_G\, dt = I_G\,\boldsymbol{\omega}_2$$



That's it!

Thank you all for your hard work.

-Dr Dave