School of Mechanical and Manufacturing Engineering

MMAN1300 Engineering Mechanics 1

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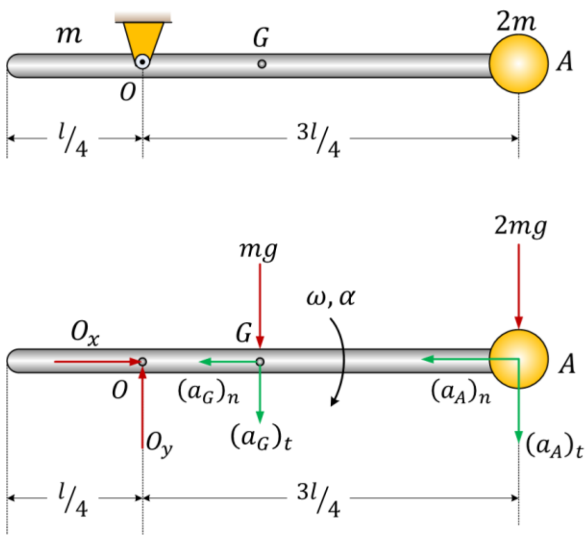


Week 11, L1-2: Rigid Body Kinetics

POLAR MOMENT OF INERTIA

- Mass moment of inertia
- Parallel axis theorem

RIGID BODY KINETICS

- Translation
- Fixed-axis rotation
- General plane motion

Kinetics of Rigid Bodies

- How do we deal with rotational acceleration of a body?
- What is the analog of inertia in a rotational system?

Topic Outline

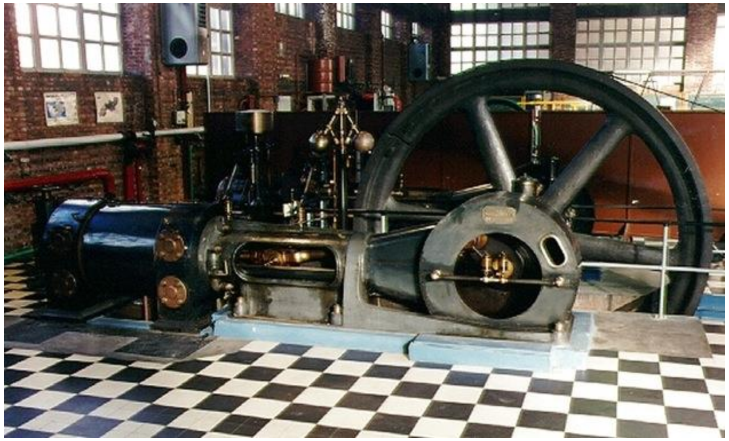
Polar Moment Of Inertia

- Mass moment of inertia
- Composite bodies
- Parallel axis theorem

Rigid Body Kinetics

- Kinetic theory of rigid bodies
- Translation
- Fixed-axis rotation
- General plane motion

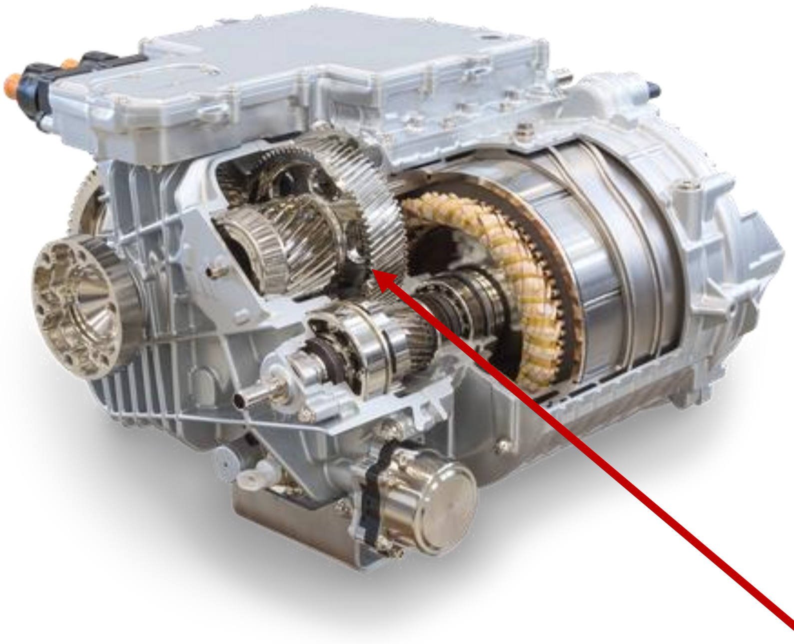
Applications: Flywheels



- The large flywheel in the first picture is connected to a single lower pressure steam piston.

- In modern IC engines, the flywheel is still present and serves the same fundamental purpose, though is much smaller.

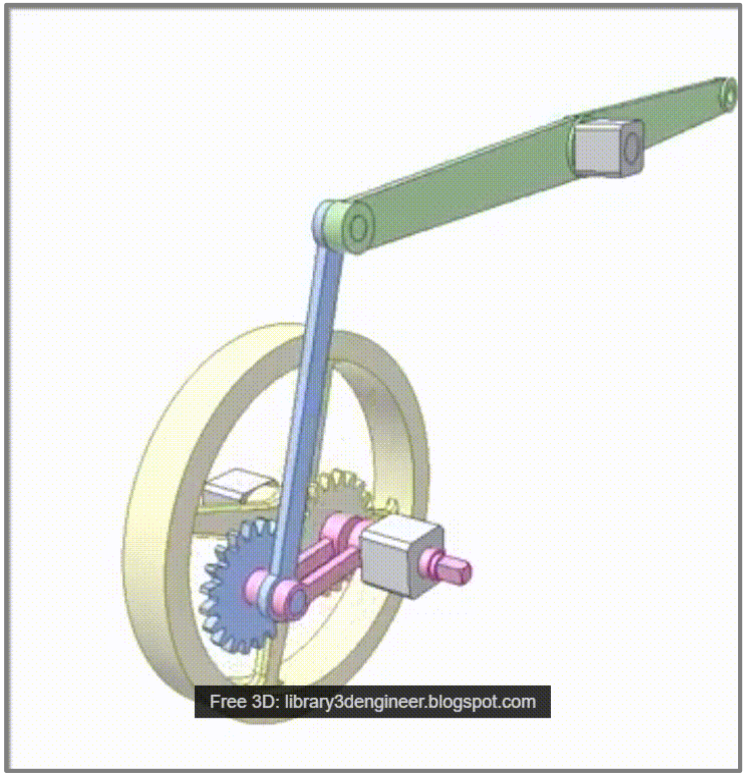
Applications: Flywheels



- But really, moment of inertia is an important design consideration of many devices involving rotating components

- Even in electric motors, MMI will significantly affect acceleration performance, and hence is minimised.

Applications: Flywheels



A flywheel serves a clear mechanical purpose:

- What property of the flywheel is most important for this use?
- How can we determine a value for this property?
- Why is most of the mass of the flywheel located near the flywheel's circumference?

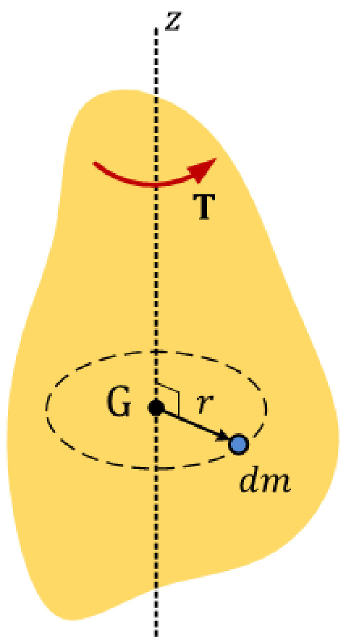
Deriving the Angular Kinetic Equation

Newton's second law is essential the Kinetic Equation. How can we adapt his equation for moments and angular acceleration? What is the rotational equivalent of mass?

$$\sum \mathbf{F} = m\mathbf{a}$$

Deriving Angular Kinetics (Web view)

Mass Moment Of Inertia

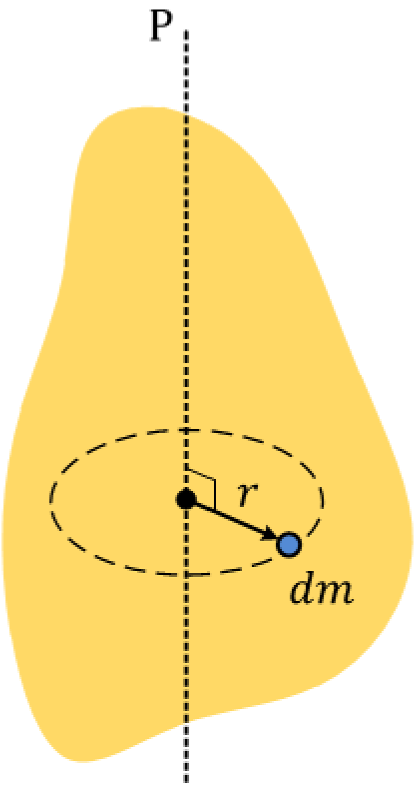


Consider a rigid body with a center of mass at G. It is free to rotate about the z axis, which passes through G. Now, if we apply a torque **T** about the z axis to the body, the body begins to rotate with an angular acceleration of $\boldsymbol{\alpha}$.

T and $\boldsymbol{\alpha}$ are related by the equation $\mathbf{T} = I\boldsymbol{\alpha}$. In this equation, I is the mass moment of inertia (MMI) about the z axis. **T** and **M** are interchangeable.

The MMI of a body is a property that measures the resistance of the body to angular acceleration. The MMI is often used when analyzing rotational motion.

Mass Moment Of Inertia



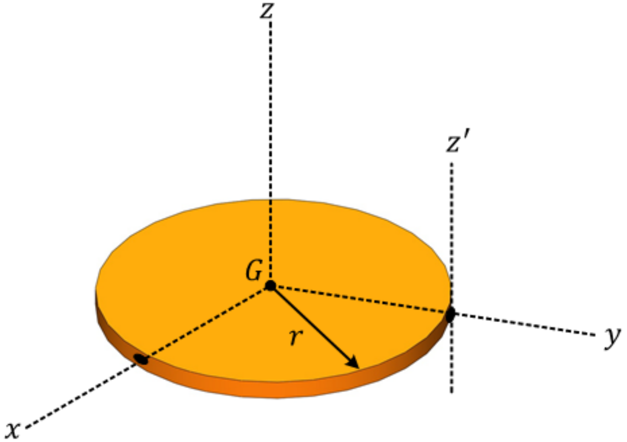
Consider a rigid body and the arbitrary axis P shown in the figure.

The MMI about the P axis is defined as
$I = \int_m r^2 \, dm,$
where r, the "moment arm," is the perpendicular distance from the axis to the arbitrary element dm.

The mass moment of inertia is always a positive quantity and has a unit of $kg \cdot m^2$.

Mass Moment Of Inertia

The mass moment of inertia formulations for the two shapes below are often used as the differential element being integrated over an entire body.

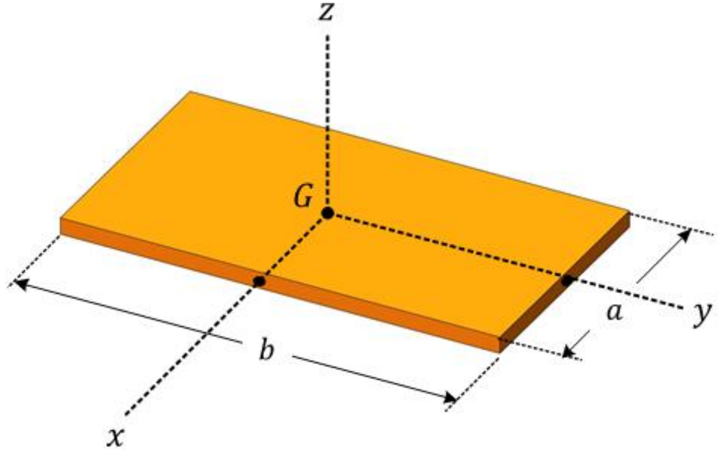


Thin circular disc

$$I_{xx} = I_{yy} = \frac{mr^2}{4}$$

$$I_{zz} = \frac{mr^2}{2}$$

$$I_{z'z'} = \frac{3mr^2}{2}$$



Thin plate

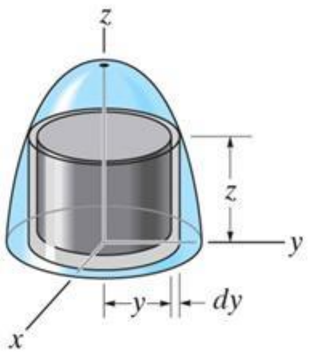
$$I_{xx} = \frac{mb^2}{12}$$

$$I_{yy} = \frac{ma^2}{12}$$

$$I_{zz} = \frac{1}{12}m(a^2 + b^2)$$

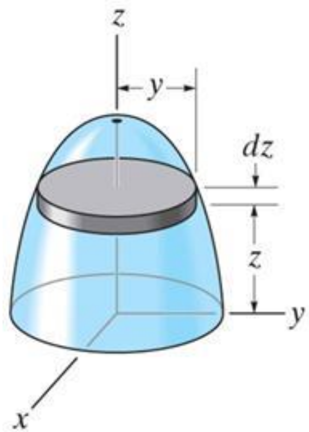
Procedure For Analysis

When using direct integration, only symmetric bodies having surfaces generated by revolving a curve about an axis will be considered here.



Shell element

- If a shell element having a height z, radius r = y, and thickness dy is chosen for integration, then the volume element is $dV = (2py)(z)dy$.
- This element may be used to find the moment of inertia I_z since the entire element, due to its thinness, lies at the same perpendicular distance y from the z-axis.



Disk element

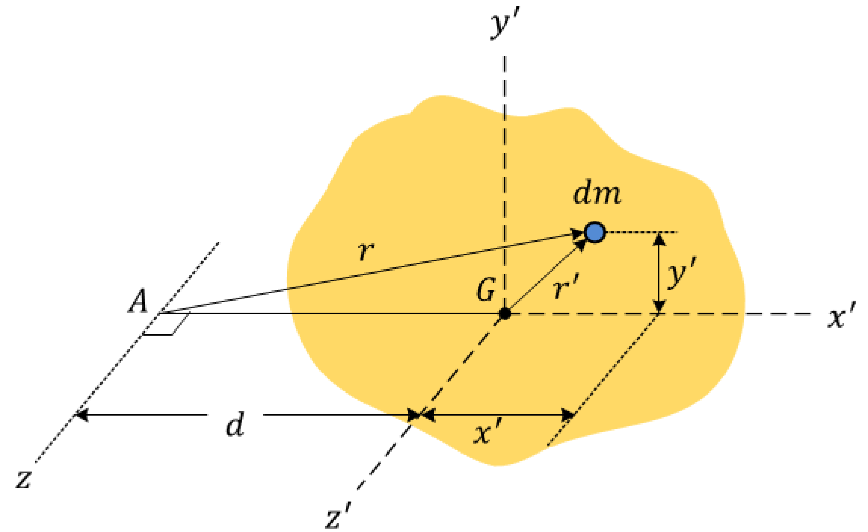
- If a disk element having a radius y and a thickness dz is chosen for integration, then the volume $dV = (py^2)dz$.
- Using the moment of inertia of the disk element, we can integrate to determine the moment of inertia of the entire body.

Parallel-axis Theorem

If the mass moment of inertia of a body about an axis passing through the body's mass center is known, then the moment of inertia about any other parallel axis may be determined by using the parallel axis theorem,

$$I = I_G + md^2$$

where I_G = mass moment of inertia about the body's mass center
m = mass of the body
d = perpendicular distance between the parallel axes



Radius of Gyration and Composite Bodies

Radius of Gyration

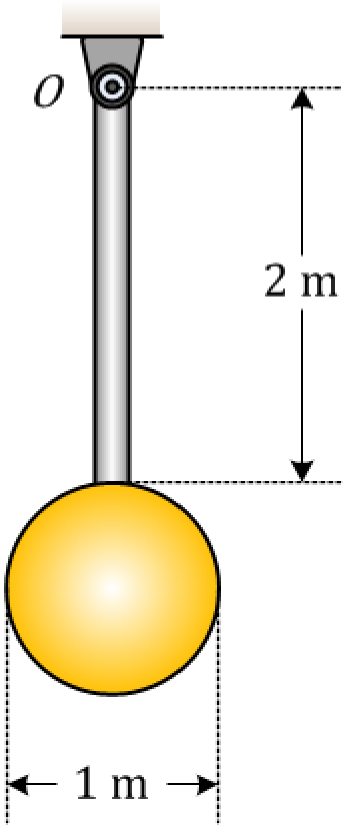
The mass moment of inertia of a body about a specific axis can be defined using the radius of gyration (k). The radius of gyration has units of length and is a measure of the distribution of the body's mass about the axis at which the moment of inertia is defined.

$$I = m\,k^2 \quad \text{or} \quad \mathrm{k} = \sqrt{(\mathrm{I}/\mathrm{m})}$$

Composite Bodies

If a body is constructed of a number of simple shapes, such as disks, spheres, or rods, the mass moment of inertia of the body about any axis can be determined by algebraically adding together all the mass moments of inertia, found about the same axis, of the different shapes.

Example 1



Given: The pendulum consists of a slender rod with a mass 2 kg and a circular plate with a mass of 4 kg.

Find: The pendulum's radius of gyration about an axis perpendicular to the screen and passing through point O.

Plan: Follow steps similar to finding the MoI for a composite area (as done in statics). The pendulum's can be divided into a slender rod (r) and a circular plate (p). Then, determine the radius of gyration.

W11 Example 1 (Web view)

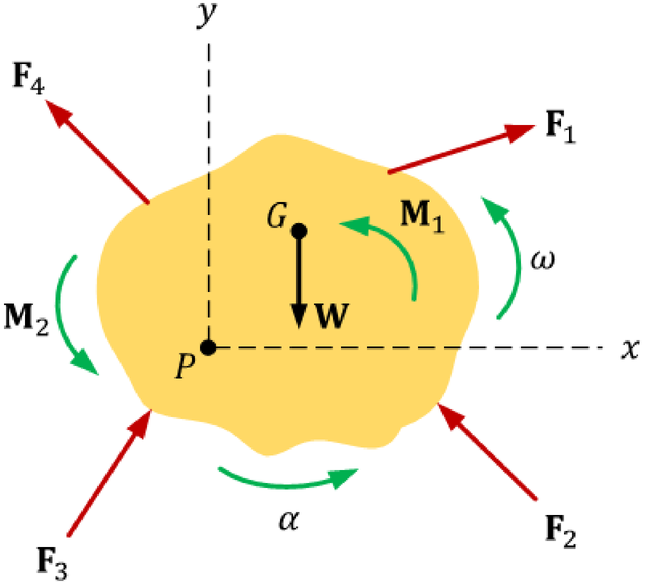
Planar Kinetic Equations of Motion

- We will limit our study of planar kinetics to rigid bodies that are symmetric with respect to a fixed reference plane.

- As discussed in Chapter 16, when a body is subjected to general plane motion, it undergoes a combination of translation and rotation.

- First, a coordinate system with its origin at an arbitrary point P is established.

  The x-y axes should not rotate but can either be fixed or translate with constant velocity.

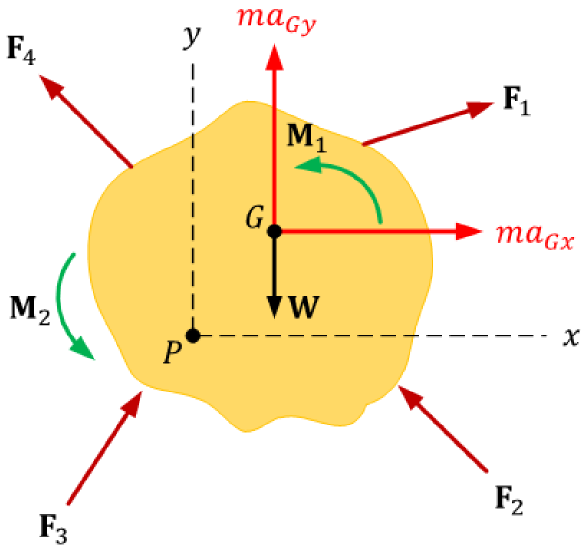


Equations of Translational Motion

- If a body undergoes translational motion, the equation of motion is $\Sigma \mathbf{F} = m\mathbf{a}_{\mathrm{G}}$. This can also be written in scalar form as

$$\Sigma F_x = m(a_{\mathrm{G}})_x \quad \text{and} \quad \Sigma F_y = m(a_{\mathrm{G}})_y$$

In other words:
The sum of all the external forces acting on the body is equal to the body's mass times the acceleration of its mass center.



Equations Of Rotational Motion

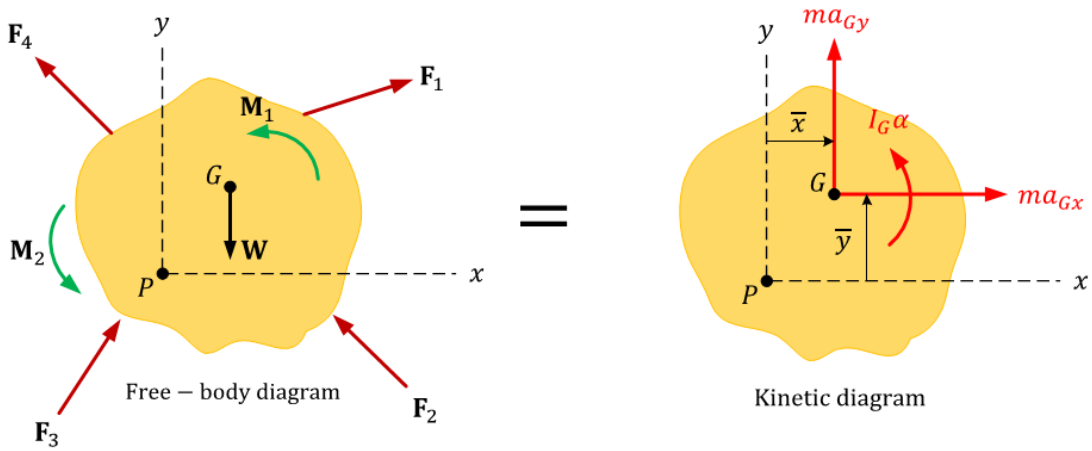
- We need to determine the effects caused by the moments of an external force system.
- The moment about point P can be written as:

$$\Sigma\,(\mathbf{r}_i \times \mathbf{F}_i) + \Sigma\,\mathbf{M}_i = \mathbf{r} \times m\mathbf{a}_G + I_G\alpha$$

$$\text{and } \Sigma\,\mathbf{M}_p = \Sigma(\mathbf{M}_k)_p$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ and $\Sigma\,\mathbf{M}_p$ is the resultant moment about P due to all the external forces.

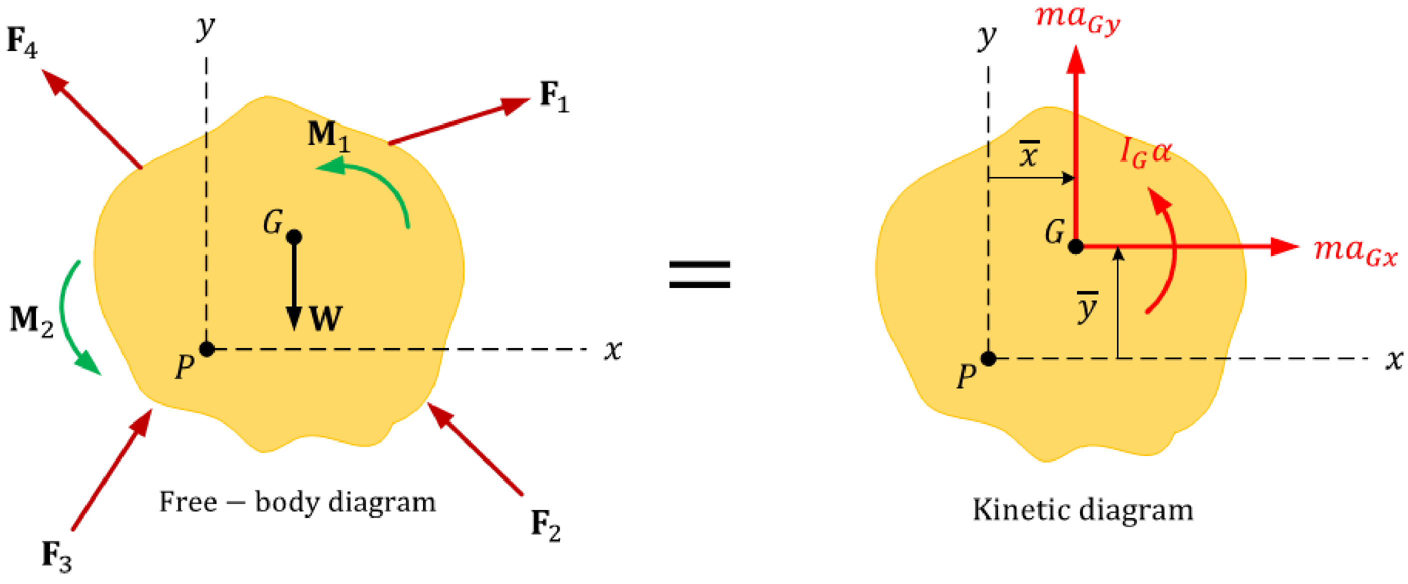
The term $\Sigma(\mathrm{M}_k)_p$ is called the kinetic moment about point P.



The Kinetic Diagram

The Kinetic Diagram is an accelerating FBD:

$$\Sigma \mathbf{F}_i = m\mathbf{a} \qquad \text{and} \qquad \Sigma \mathbf{M}_i = I_G \boldsymbol{\alpha}$$



Equations of Rotational Motion

If point P coincides with the mass center G, this equation reduces to the scalar equation of $\Sigma M_G = I_G \alpha$.

In words: the resultant (summation) moment about the mass center due to all the external forces is equal to the moment of inertia about G times the angular acceleration of the body.

Thus, three independent scalar equations of motion may be used to describe the general planar motion of a rigid body. These equations are:

$$\Sigma F_x = m(a_G)_x$$

$$\Sigma F_y = m(a_G)_y$$

$$\text{and} \quad \Sigma M_G = I_G \alpha \quad \text{or} \quad \Sigma M_p = \Sigma (M_k)_p$$

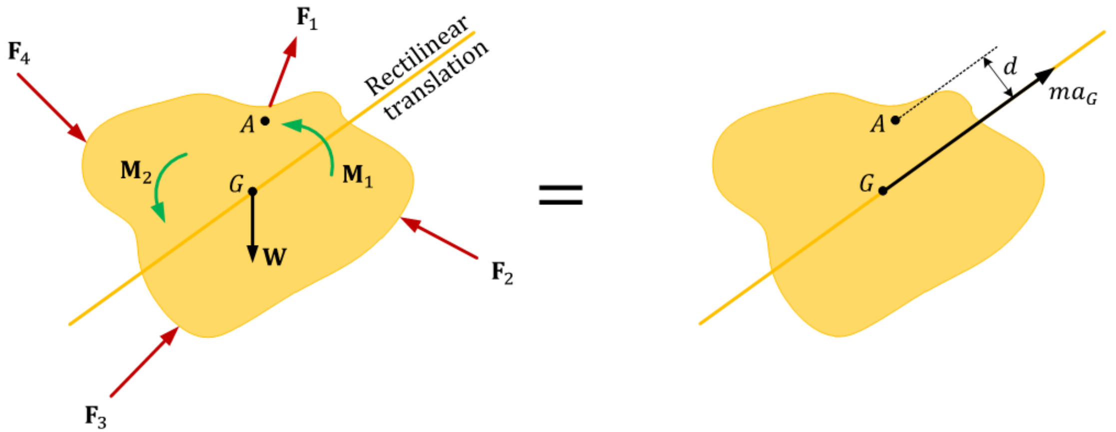
Equations of Motion: Translation

When a rigid body undergoes only translation, all the particles of the body have the same acceleration so $a_G = a$ and $\alpha = 0$. The equations of motion become:

$$\Sigma F_x = m(a_G)_x$$

$$\Sigma F_y = m(a_G)_y$$

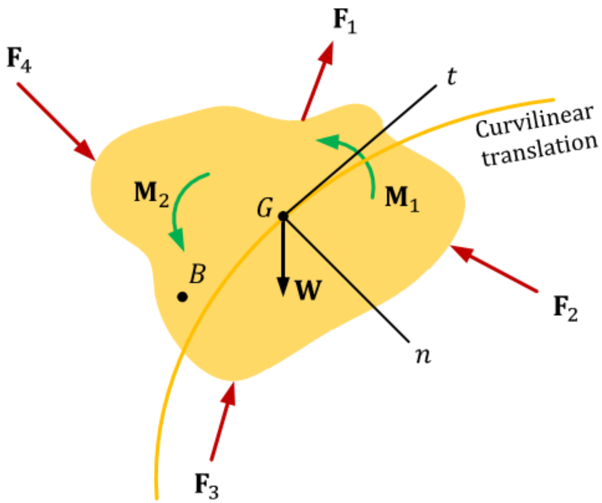
$$\Sigma M_G = 0$$



Note that, if it makes the problem easier, the moment equation can be applied about another point instead of the mass center. For example, if point A is chosen,

$$\Sigma M_A = (m\, a_G)\, d$$

Equations of Motion: Translation



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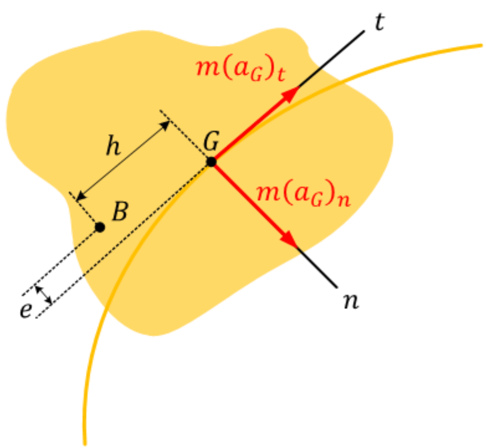
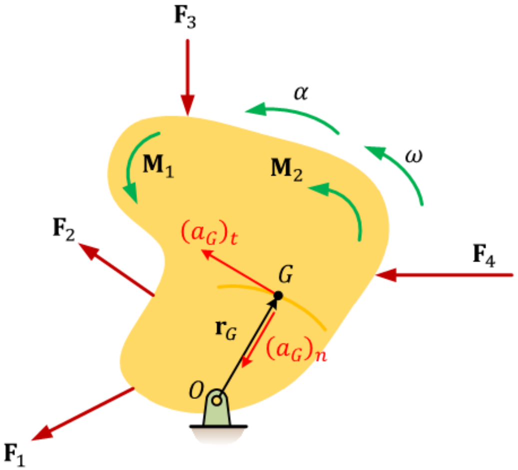


When a rigid body is subjected to curvilinear translation, it is best to use an n-t coordinate system. Then apply the equations of motion, as written below, for n-t coordinates.

$$\Sigma F_n = m(a_G)_n$$
$$\Sigma F_t = m(a_G)_t$$

$$\Sigma M_G = 0 \qquad \text{or}$$
$$\Sigma M_B = e[m(a_G)_t] - h[m(a_G)_n]$$

Rotation About a Fixed Axis



When a rigid body rotates about a fixed axis perpendicular to the plane of the body at point O, the body's center of gravity G moves in a circular path of radius r_G. Thus, the acceleration of point G can be represented by:

- a tangential component $(a_G)_t = r_G\, a$, and
- a normal component $(a_G)_n = r_G\, \omega^2$.

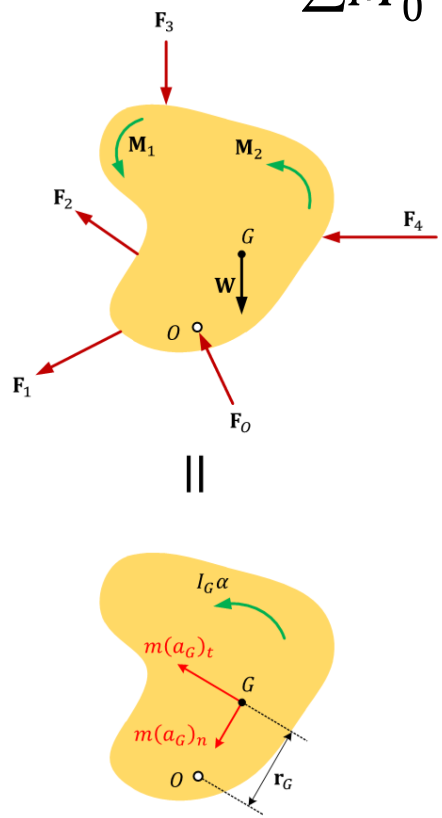
Since the body experiences an angular acceleration, its inertia creates a moment of magnitude, $I_g \alpha$, equal to the moment of the external forces about point G. Thus, the scalar equations of motion can be stated as:

$$\sum F_n = m\,(a_G)_n = m\, r_G\, \omega^2$$
$$\sum F_t = m\,(a_G)_t = m\, r_G\, a$$
$$\sum M_G = I_G\, a$$

Equations of Motion

Note that the $\sum M_G$ moment equation may be replaced by a moment summation about any arbitrary point. Summing the moment about the center of rotation O yields

$$\sum M_0 = I_G\alpha + r_G m(a_G)_t = [I_G + m(r_G)^2]\alpha$$



From the parallel axis theorem, $I_O = I_G + m(r_G)^2$, therefore the term in parentheses represents I_O.

Consequently, we can write the three equations of motion for the body as:

$$\sum F_n = m(a_G)_n = m r_G \omega^2$$

$$\sum F_t = m(a_G)_t = m r_G \alpha$$

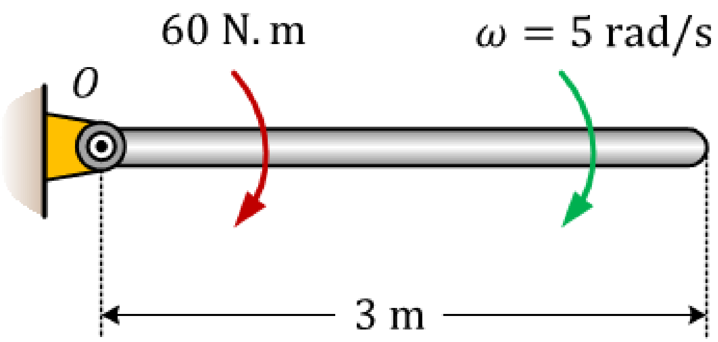
$$\sum M_O = I_O \alpha$$

Procedure For Analysis

Problems involving the kinetics of a rigid body rotating about a fixed axis can be solved using the following process.

1. Establish an inertial coordinate system and specify the sign and direction of $(a_G)_n$ and $(a_G)_t$.
2. Draw a free body diagram accounting for all external forces and couples. Show the resulting inertia forces and couple (typically on a separate kinetic diagram).
3. Compute the mass moment of inertia I_G or I_O.
4. Write the three equations of motion and identify the unknowns. Solve for the unknowns.
5. Use kinematics if there are more than three unknowns (since the equations of motion allow for only three unknowns).

Example 2



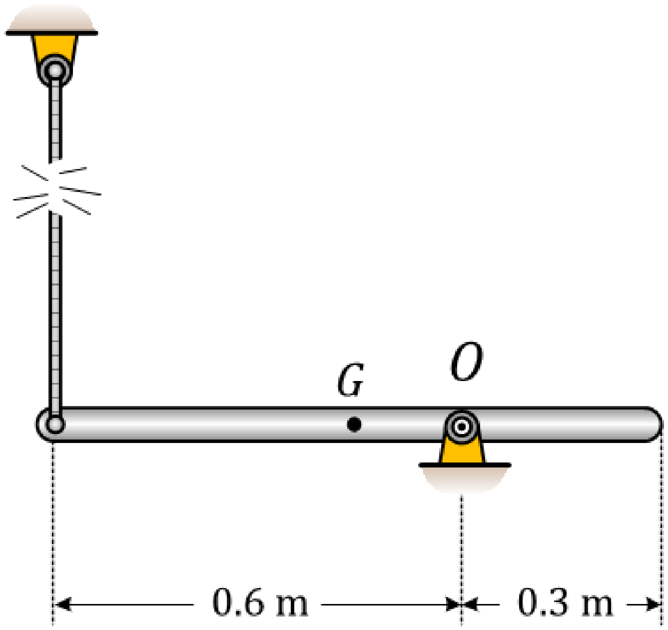
Given: A rod with mass of 20 kg is rotating at 5 rad/s at the instant shown. A moment of 60 N·m is applied to the rod.

Find: The angular acceleration α and the reaction at pin O when the rod is in the horizontal position.

Plan: Since the mass center moves in a circle of radius 1.5 m, its acceleration has a normal component toward O and a tangential component acting downward and perpendicular to r_G.

W11 Example 2 (Web view)

Example 3



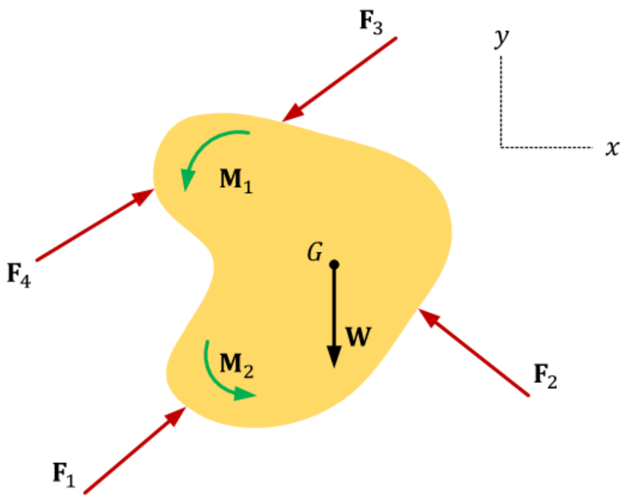
Given: The uniform slender rod has a mass of 15 kg and its mass center is at point G.

Find: The reactions at the pin O and the angular acceleration of the rod just after the cord is cut.

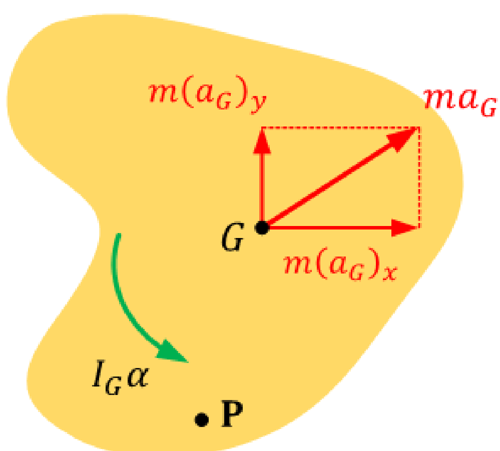
Plan: Since the mass center, G, moves in a circle of radius 0.15 m, it's acceleration has a normal component toward O and a tangential component acting downward and perpendicular to r_G.

W11 Example 3 (Web view)

General Plane Motion



When a rigid body is subjected to external forces and couple-moments, it can undergo both translational motion and rotational motion. This combination is called general plane motion.



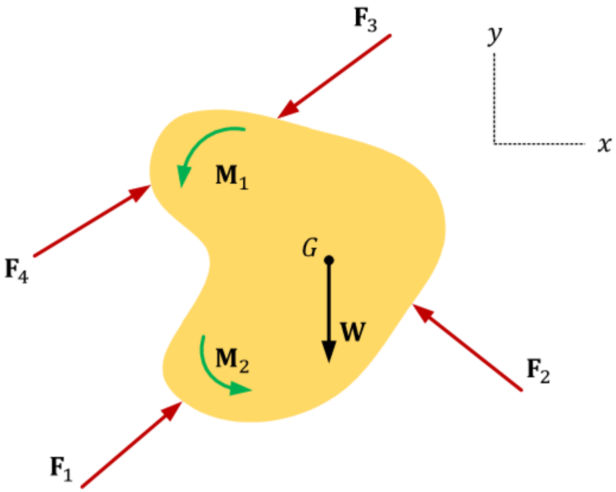
Using an x-y inertial coordinate system, the scalar equations of motions about the center of mass, G, may be written as:

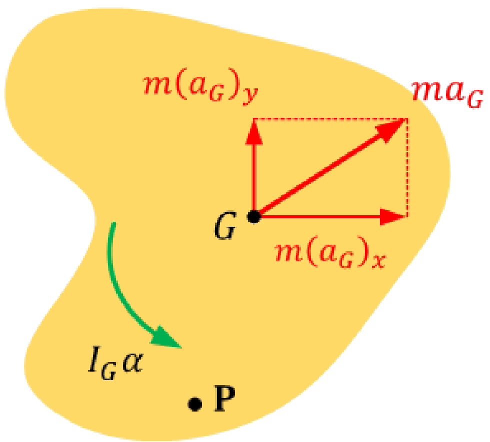
$$\sum F_x = m\,(a_G)_x$$

$$\sum F_y = m\,(a_G)_y$$

$$\sum M_G = I_G\alpha$$

Equations of Motion





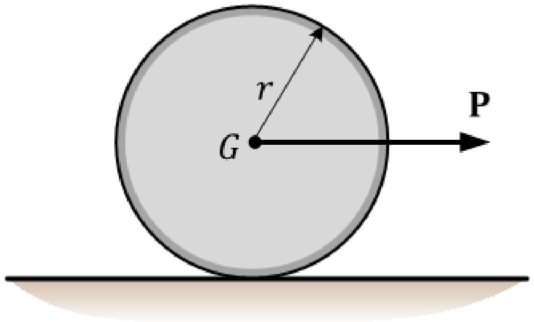
Sometimes, it may be convenient to write the moment equation about a point P, rather than G. Then the equations of motion are written as follows:

$$\sum F_x = m\,(a_G)_x$$
$$\sum F_y = m\,(a_G)_y$$
$$\sum M_P = \sum\,(M_k)_P$$

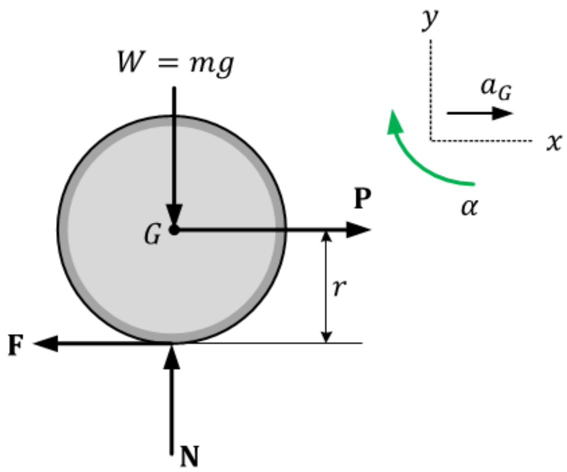
In this case, $\sum(M_k)_P$ represents the sum of the moments of $I_G\alpha$ and $m\mathbf{r} \times \mathbf{a}_G$ about point P.

Frictional Rolling Problems

When analyzing the rolling motion of wheels, cylinders, or disks, it may not be known if the body rolls without slipping or if it slips/slides as it rolls.



For example, consider a disk with mass m and radius r, subjected to a known force P.



The equations of motion will be:

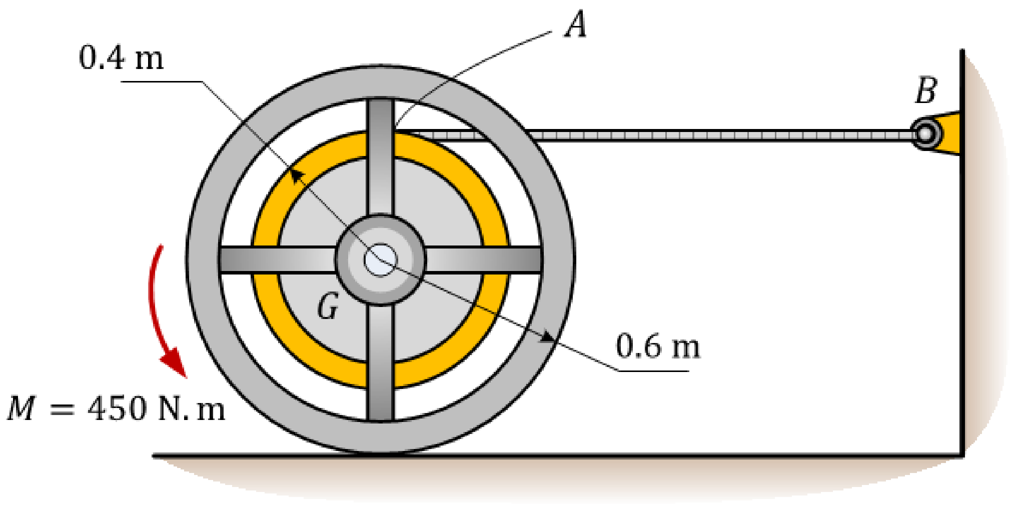
$$\sum F_x = m\,(a_G)_x \;\Rightarrow\; P - F = m\,a_G$$

$$\sum F_y = m\,(a_G)_y \;\Rightarrow\; N - mg = 0$$

$$\sum M_G = I_G\alpha \;\Rightarrow\; Fr = I_G\alpha$$

There are 4 unknowns ($F, N, \alpha,$ and a_G) in these three equations.

Example 4

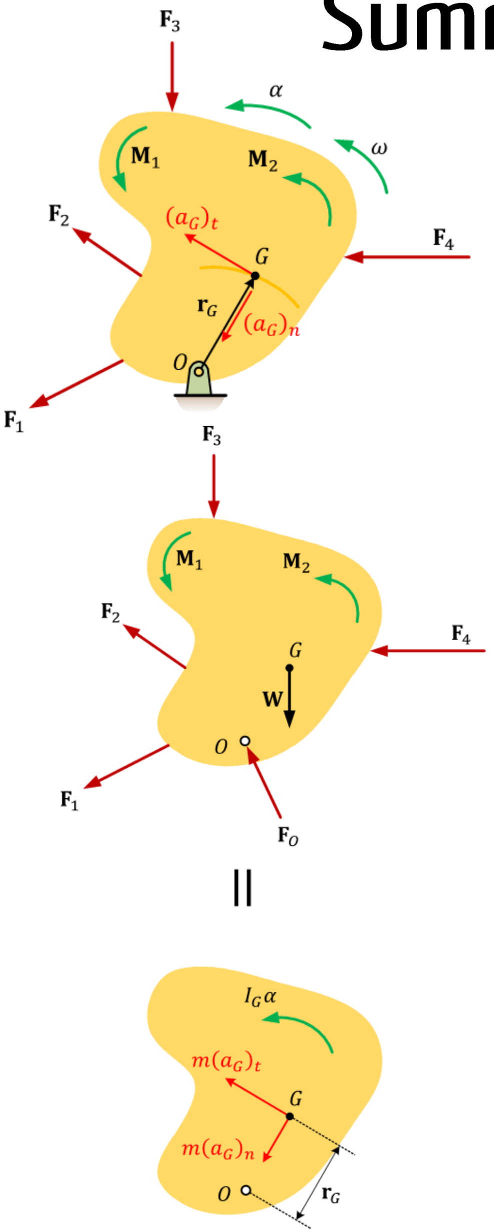


Given: A spool has a mass of 200 kg and a radius of gyration (k_G) of 0.3 m. The coefficient of kinetic friction between the spool and the ground is $\mu_k = 0.1$.

Find: The angular acceleration (α) of the spool and the tension (T) in the cable.

[W11 Example 4]() ([Web view]())

Summary of Rigid Body Kinetics



Mass moment of inertia:

$$I = \int_m r^2 \, dm$$

Parallel axis theorem:

$$I = I_G + md^2$$

Using and instant centre or fixed axis

$$\sum M_0 = I_G\alpha + r_G m(a_G)_t = [I_G + m(r_G)^2]\alpha$$

n-t components for kinetics

$$\sum F_n = m(a_G)_n = m r_G \omega^2$$

$$\sum F_t = m(a_G)_t = m r_G \alpha$$

$$\sum M_O = I_O \alpha$$

Next Topic:

Rigid Body Energy and Momentum