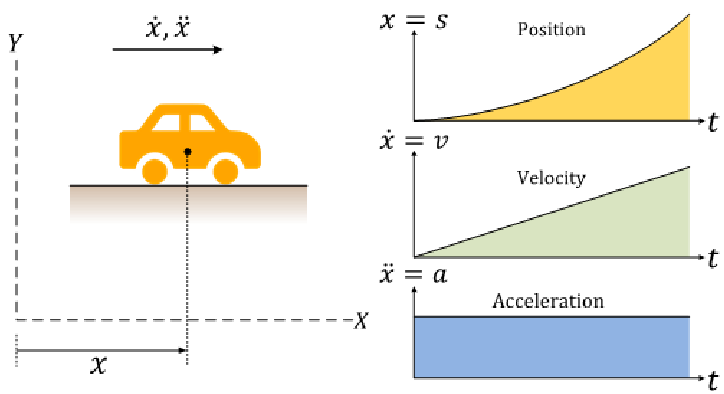
School of Mechanical and Manufacturing Engineering

MMAN1300 Engineering Mechanics 1

Dr. David C. Kellermann



Week 7, L1-2: Particle Kinematics

INTRODUCTION TO DYNAMICS

- Kinematics vs Kinetics

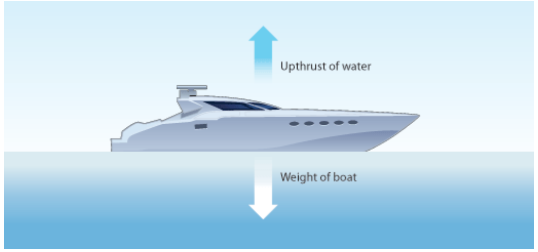
RECTILINEAR MOTION

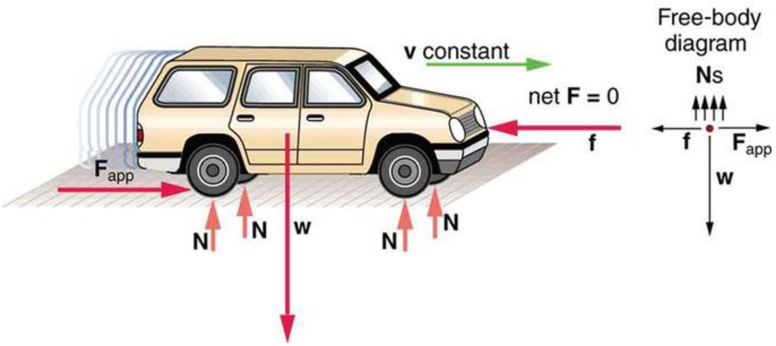
- Particle kinematics
- Displacement, velocity and acceleration
- Various function classes

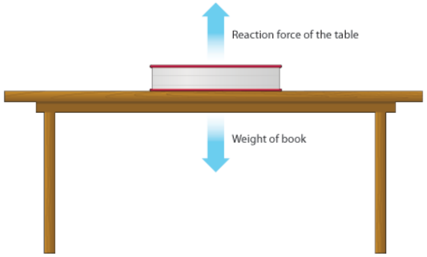
Equilibrium

If the net force acting on the body is zero, it is said to be in equilibrium. The acceleration of such objects will be zero according to Newton's Law. Zero acceleration implies a constant velocity.

Equilibrium: An object is said to be in equilibrium when the net force acting on an object is zero and if the object is not moving.
It is also in equilibrium when the net force acting on an object is zero but if the object is moving at a constant non-zero velocity.





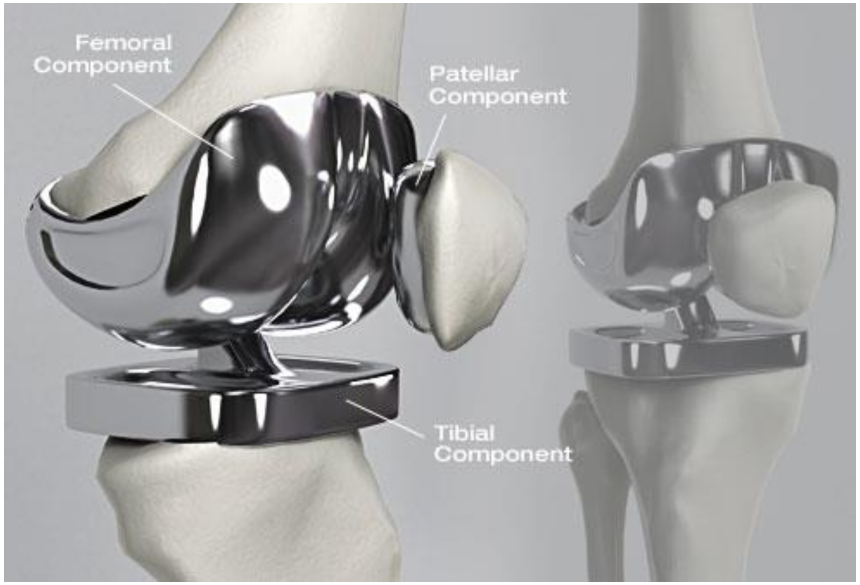


Dynamics Introduction

- Dynamics is concerned with the motion of objects.

  Only when: $\sum F \neq 0 \;\; and/or \sum M \neq 0$

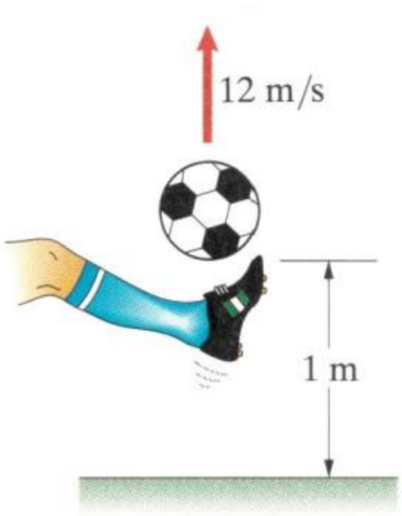
- Engineers use mathematical models to describe the dynamics of an object, for example, in the design of aircraft, cars, linkages, piston-crankshaft mechanisms, etc.



Dynamics- Kinematics vs. Kinetics

Dynamics can be roughly divided into 2 main areas:

1. **Kinematics**: describes the motion of an object.
2. **Kinetics**: is concerned with the forces acting on an object and the motion of the object.



Topics in Particle Dynamics

Kinematics of Particles

- Rectilinear motion
- Curvilinear motion

Kinetics of Particles

- Relative motion
- Kinetics of particles
- Work/energy methods
- Impulse & momentum

Topics in Rigid Body Dynamics

Rigid Body Mechanics

- Angular Motion
- Kinematics of Rigid Bodies
- Kinetics of rigid bodies
- Moment of inertia
- Work-energy methods for rigid bodies

Rectilinear Kinematics

Objectives:

- To gain knowledge of the fundamentals of engineering dynamics;
- To become familiar with common mechanisms;
- To learn techniques for analysing the performance of mechanisms;
- To develop and practise tactics for problem identification, formulation and solution.

Kinematics of Particles

Definition of Particles:

A particle is an approximate model of a body whose size is negligible compared with the other dimensions surrounding it.

A particle:

- can be modelled as a point
- its size can be ignored
- all forces (kinetics) act through a single point (which corresponds to its centre of mass)
  - Earth orbiting the sun
  - A car travelling on a freeway
  - A flying airplane

Important Considerations

Kinematics of particles examines the position (s), velocity (v) and acceleration (a) of particles.

The motion of particles can be described by using coordinates measured from fixed reference axes. Whilst the motion of a particle is three-dimensional, we initially will consider *plane* motion. Plane motion may be motion along a straight line (rectilinear motion) or motion along a curved path (curvilinear motion).

Rectilinear Motion

Rectilinear motion is the motion along a straight line.

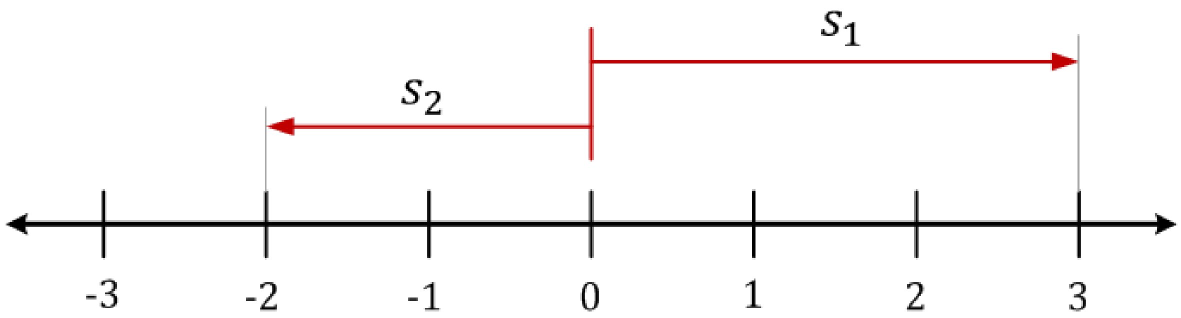
- position (s)
- velocity (v)
- acceleration (a)

mainly as functions of time.

In order to define the motion of a particle, we need to define a co-ordinate system and an origin.

Position (s)

We define position relative to a reference point (origin).



$s_1 = 3\ \text{m}$

$s_2 = -2\ \text{m}$

Note: Sign is important (it signifies the direction)

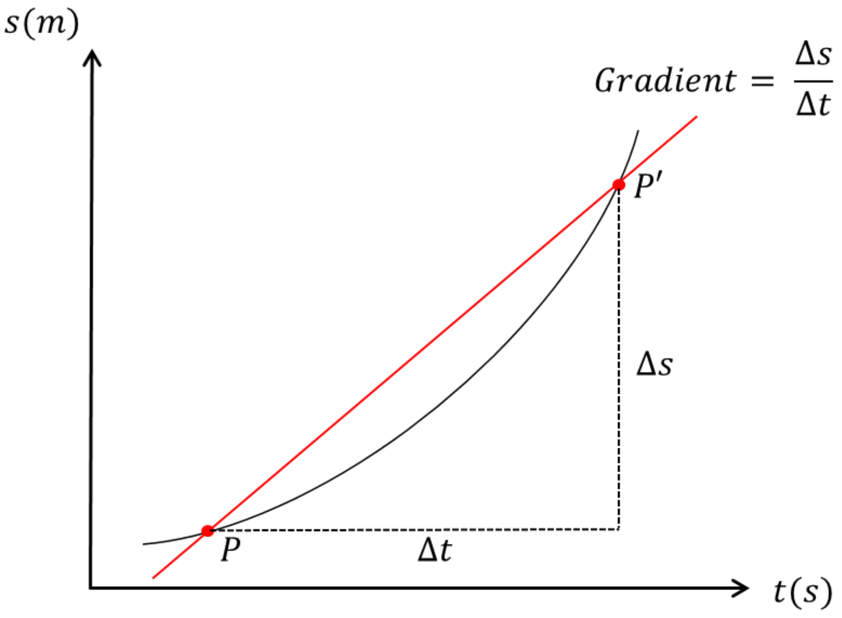
The displacement of P during an interval of time from t_0 to t is the change in position $s(t) - s(t_0)$, where $s(t)$ denotes the position at time t.

Velocity (v) – Average vs. Instantaneous

Average Velocity

The average velocity between two positions P and P' is defined as:

$$\bar{v} = \frac{change\ in\ position}{change\ in\ time} = \frac{\Delta s}{\Delta t}$$



Average vs. Instantaneous Velocity

Instantaneous Velocity

The instantaneous velocity at P can be determined by taking the limit as $\Delta t \rightarrow 0$

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = \dot{s}$$

Note the 'dot' in $\dot{s}$ indicates the derivative with respect to (w.r.t.) time.

Velocity ($v = \dot{s}$) is the rate of the change of position with respect to time (m/s).

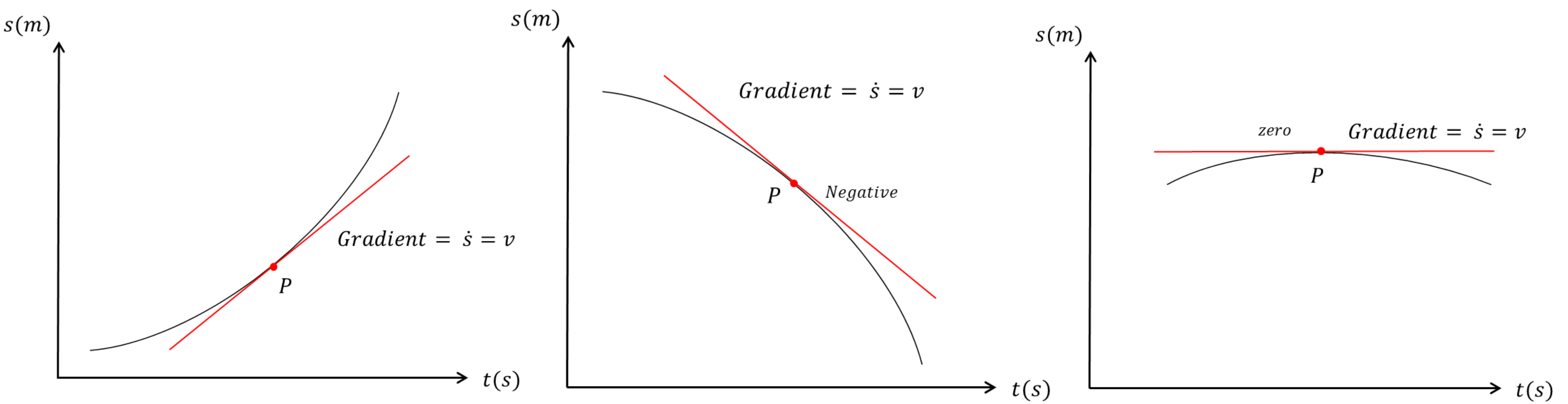
Average vs. Instantaneous Velocity

Instantaneous Velocity (v)

v is the gradient of the displacement vs time at P

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = \dot{s}$$

- Positive gradient = positive v
- Zero gradient = stationary point, may be a maxima or minima.



Integration of Velocity

We can integrate the velocity to establish a displacement equation,

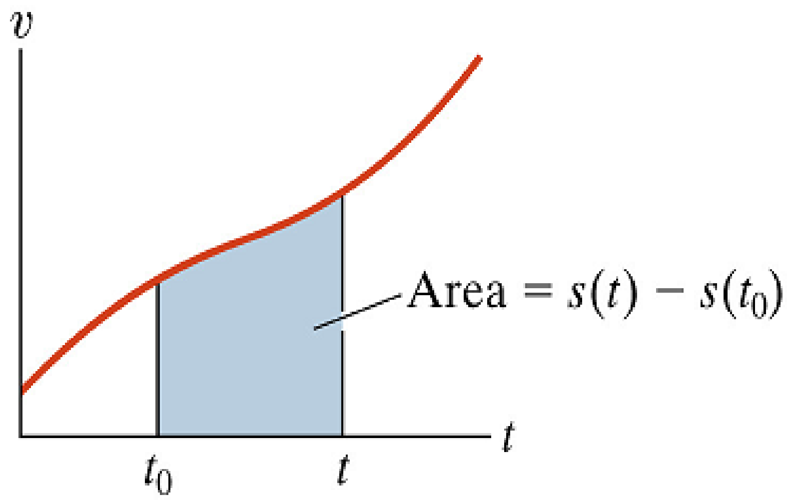
$$v = \frac{ds}{dt}$$

rearranging gives, $ds = vdt$

Integrating both sides

$$\int_{s_1}^{s_2} ds = \int_{v_1}^{v_2} v\,dt$$

$$\Longrightarrow \quad \boxed{s_2 - s_1 = \int_{v_1}^{v_2} v\,dt}$$



Area under the $v - t$ curve

Velocity as a function of Time *v(t)*

Velocity is often given as a function of time, eg.

$$v = 3t - 2t^3$$

The integration can then be performed mathematically.

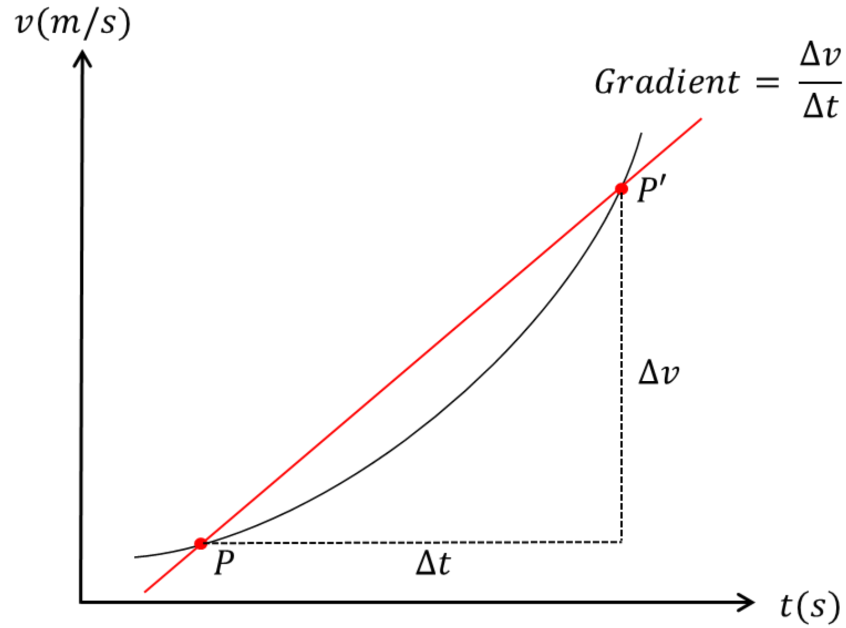
$$\int_{s_1}^{s_2} ds = \int_{v_1}^{v_2} v\, dt = \int_{t_1}^{t_2} (3t - 2t^3)\, dt$$

$$\Longrightarrow \qquad s_2 - s_1 = \left[\frac{3t^2}{2} - \frac{2t^4}{4} \right]_{t_1}^{t_2}$$

Acceleration

- The average acceleration is the gradient of the line joining P and P'.

- The average acceleration between two positions P and P' is defined as:



$$\bar{a} = \frac{change\ in\ velocity}{change\ in\ time} = \frac{\Delta v}{\Delta t}$$

Instantaneous Acceleration

The instantaneous acceleration at P can be determined by taking the limit as $\Delta t \to 0$

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \dot{v}$$

Acceleration $a = \dot{v}$ is the rate of the change of velocity w.r.t. time (m/s^2).

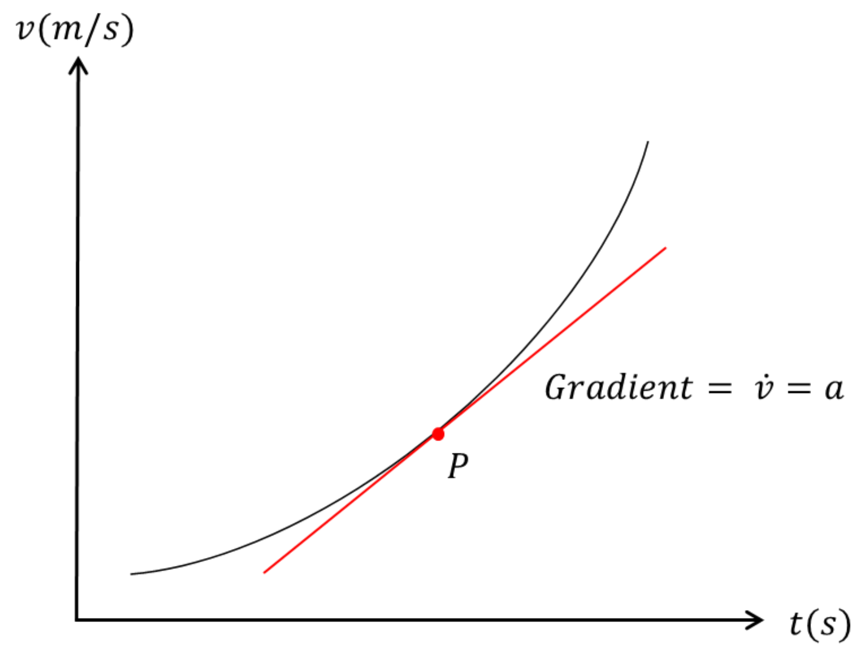
Acceleration as a Derivative

Since $v = \dot{s}$

then $a = \dot{v} = \ddot{s}$

That is

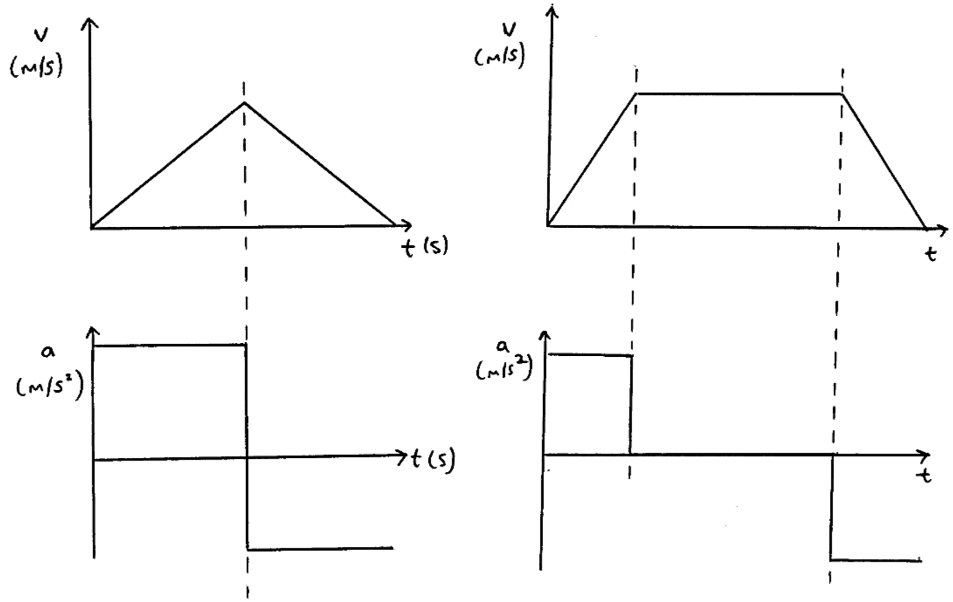
$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$



Graphically, a is the gradient (slope) of the graph at P

Graphical Method for Velocity/Acceleration

For discontinuous functions, a graphical method can be used

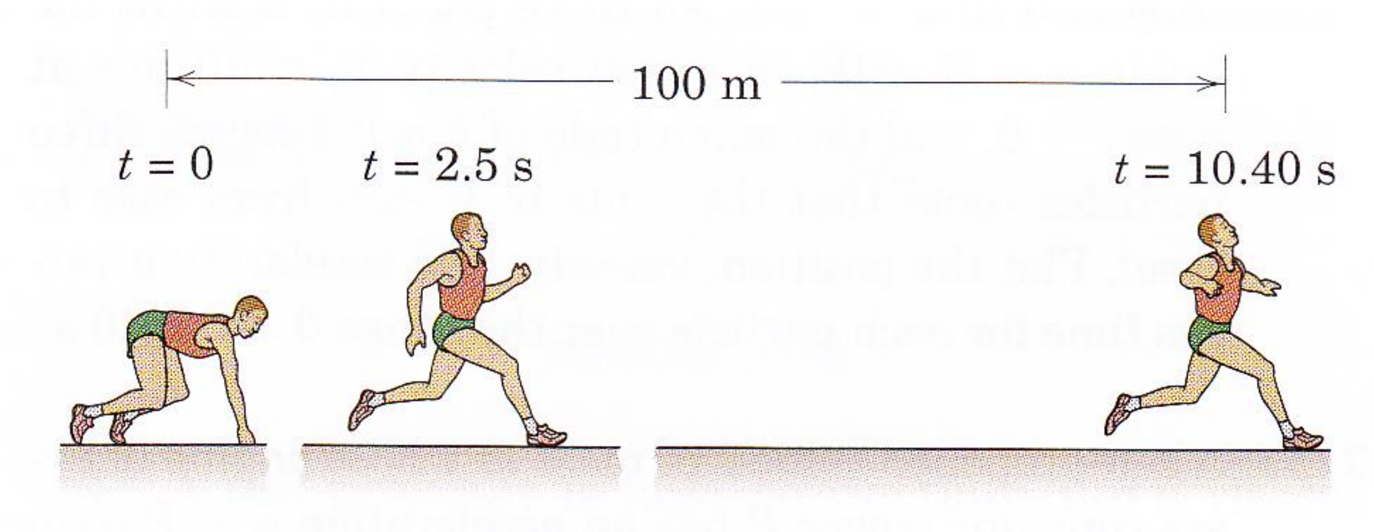


- In the first part of the graph, the velocity is increasing linearly → the acceleration is positive and constant.
  At max v, $a = 0$

- In the last part, v is decreasing linearly and a is negative and constant.

Example 1

A sprinter reaches his maximum speed v_{max} in 2.5 seconds from rest with constant acceleration. He then maintains that speed and finishes the 100 meters in the overall time of 10.40 seconds. Determine his maximum speed v_{max}.



W7 Example 1 (Web view)

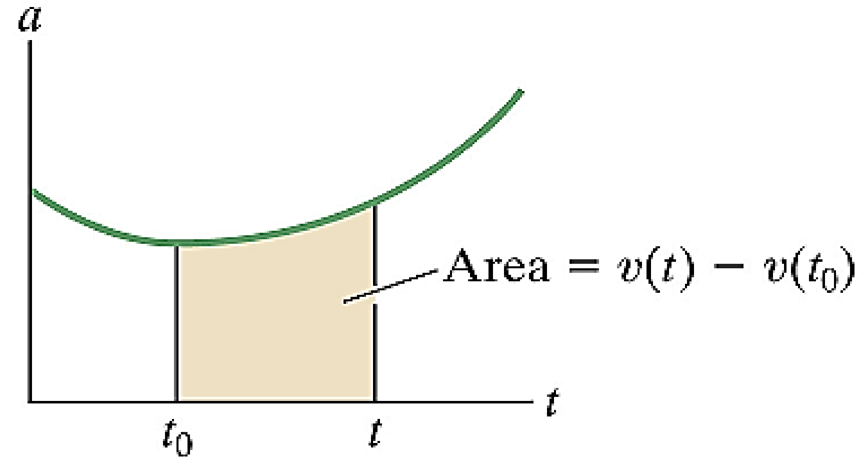
Integration of Acceleration

If acceleration is a function of time, velocity and displacement can be found through time integration

$$a = \frac{dv}{dt}$$

$$dv = a dt$$

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a dt$$



$$\Longrightarrow \quad v_2 - v_1 = \int_{t_1}^{t_2} a dt$$

= area under the $a - t$ curve

Acceleration as a Function of Time *a(t)*

Acceleration (a) is often given as a function of time, eg.

$$a = t^2 - 7t$$

The integration can then be performed mathematically.

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a\,dt = \int_{t_1}^{t_2} (t^2 - 7t)\,dt$$

$$\Rightarrow \qquad v_2 - v_1 = \left[\frac{t^3}{3} - \frac{7t^2}{2} \right]_{t_1}^{t_2}$$

Eliminating time t

$$v = \frac{ds}{dt}$$ v is the rate of change of position

$$a = \frac{dv}{dt}$$ a is the rate of change of velocity

We may eliminate dt from these equation

$$dt = \frac{ds}{v} = \frac{dv}{a}$$

$$\Rightarrow \frac{ds}{v} = \frac{dv}{a}$$

Acceleration as a Function of Displacement *a(s)*

Rearranging (cross-multiplying) gives:

$$v\,dv = a\,ds$$

Integrate both sides:

$$\int_{v_1}^{v_2} v\,dv = \int_{s_1}^{s_2} a\,ds$$

$$\Rightarrow \left[\frac{v^2}{2}\right]_{v_1}^{v_2} = \int_{s_1}^{s_2} a\,ds$$

Constant Acceleration Equations

We can use all the equations so far to derive (recall from Physics) the kinematic equations for constant acceleration:

$$v = v_0 + a_0(t - t_0)$$

$$s = s_0 + v_0(t - t_0) + \frac{1}{2}a_0(t - t_0)^2$$

$$v^2 = v_0^2 + 2a_0(s - s_0)$$

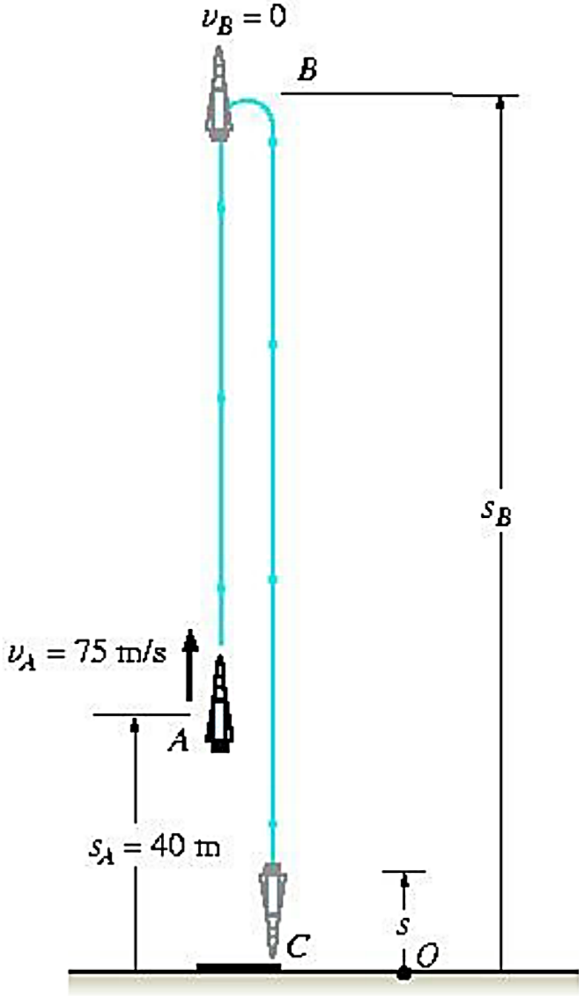
Example 2

A rocket travels upward at 75 m/s. When it is 40 m from the ground, the engine fails.

Determine:

1. Maximum height s_B reached by the rocket

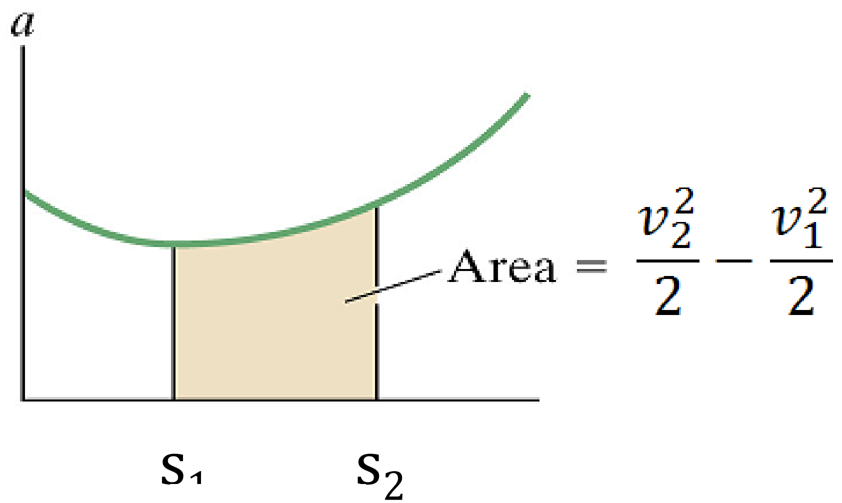
2. Its speed just before it hits the ground



[W7 Example 2](#) ([Web view](#))

Acceleration as a Function of Displacement *a(s)*

$$\frac{v_2^2}{2} - \frac{v_1^2}{2} = \int_{s_1}^{s_2} a ds \quad = \text{area under the } a - s \text{ graph}$$



Revision: Common Integration Formulae

$$\int_{x_1}^{x_2} dx = x\Big|_{x_1}^{x_2} = x_2 - x_1$$

$$\int_{x_1}^{x_2} x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_{x_1}^{x_2}$$

$$\int_{x_1}^{x_2} \frac{1}{x^n} dx = \int_{x_1}^{x_2} x^{-n} dx = \left[\frac{x^{-n+1}}{-n+1}\right]_{x_1}^{x_2}$$

Except when $n = 1$

when $n = 1$ $\rightarrow$ then we must use the natural logarithm, ln

$$\int_{x_1}^{x_2} \frac{1}{x} dx = \ln\frac{x_2}{x_1} = \ln x_2 - \ln x_1$$

Acceleration as a Function of Velocity *a(v)*

Using the separable variables method

$$\frac{dv}{dt} = a(v) \quad \Rightarrow \quad \frac{dv}{a(v)} = dt$$

and integrating to obtain,

$$\int_{v_0}^{v} \frac{dv}{a(v)} = \int_{t_0}^{t} dt$$

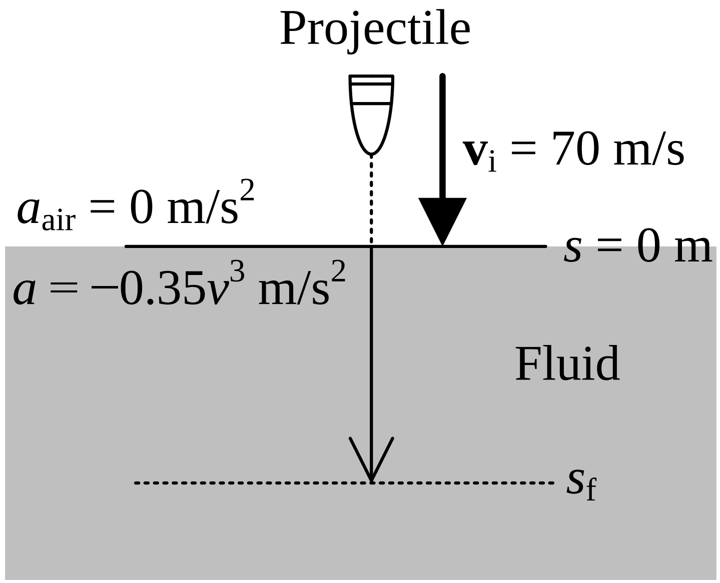
Solve for v(t) and calculate s(t) and a(t)

Example 3

A small projectile is initially fired with an initial velocity v_i of 70 m/s downwards in air with negligible resistance $a_{air} =$ 0 m/s. It then enters into a fluid medium. Due to the resistance of the fluid the projectile experiences a deceleration equal to:

$$a = (-0.35v^3)\ \mathrm{m/s^2}$$

where v is in m/s².



a) Find a function of velocity with respect to time.
b) Determine the projectile's velocity 4 seconds after it is fired.
c) Determine the projectile's position 4 seconds after it is fired

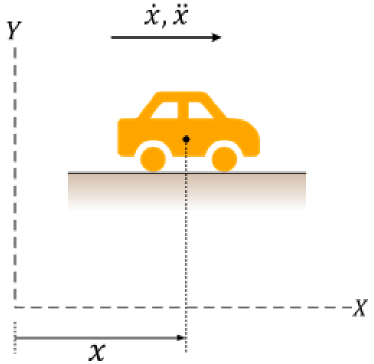
W7 Example 3 (Web view)

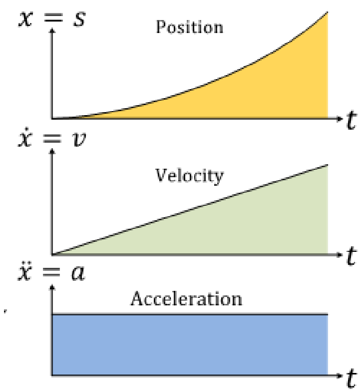
Summary of Rectilinear Kinematics

Concepts:

- Position and displacement
- Velocity
- Acceleration
- Rectilinear motion with Constant acceleration
- Acceleration specified as a function of time
- Acceleration specified as a function of velocity
- Acceleration specified as a function of position
- Graphical method

Summary





- Constant acceleration equations:

$$v = v_0 + a_0(t - t_0) \quad v^2 = v_0^2 + 2a_0(s - s_0)$$

$$s = s_0 + v_0(t - t_0) + \frac{1}{2}a_0(t - t_0)^2$$

- Acceleration as a function of time

$$v = \frac{ds}{dt} \quad a = \frac{dv}{dt}$$

- As a function of displacement $vdv = ads$

- As a function of velocity $\int_{v_0}^{v} \frac{dv}{a(v)} = \int_{t_0}^{t} dt$

Next Topic:

Curvilinear Kinematics

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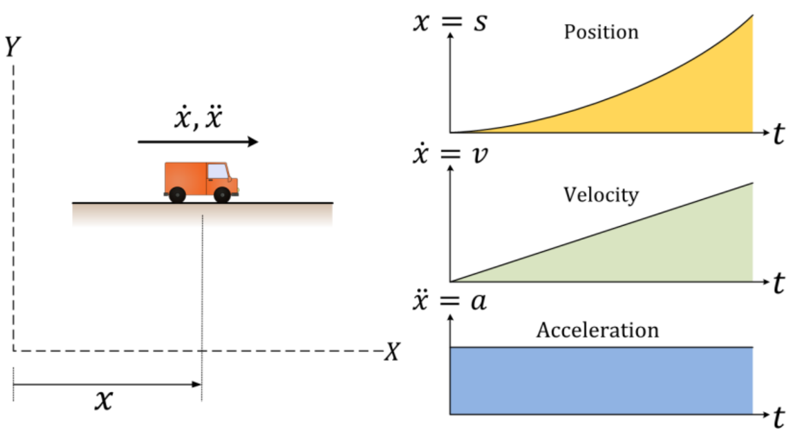


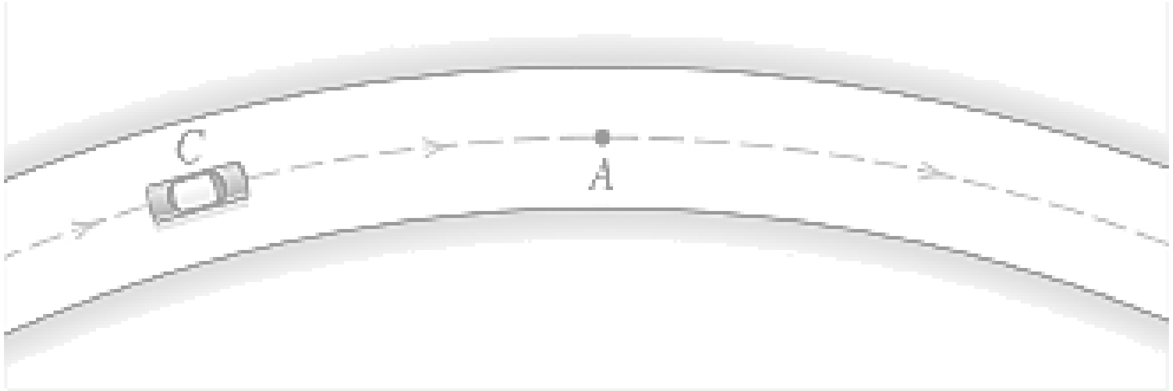
Week 7 L2– Curvilinear Motion

KINEMATICS OF PARTICLES

- Rectangular Coordinates (x-y)
- Projectile Motion
- Normal and Tangential Coordinates (n-t)
- Circular Motion
- Product rule in kinetics

Introduction

Curvilinear motion is the motion along a curved path. Although the path may be in 3-D, we are only concerned with motion in a plane (2-D motion).



We want to determine position, velocity and acceleration at any instant on the curvilinear path.

Topics: Curvilinear motion

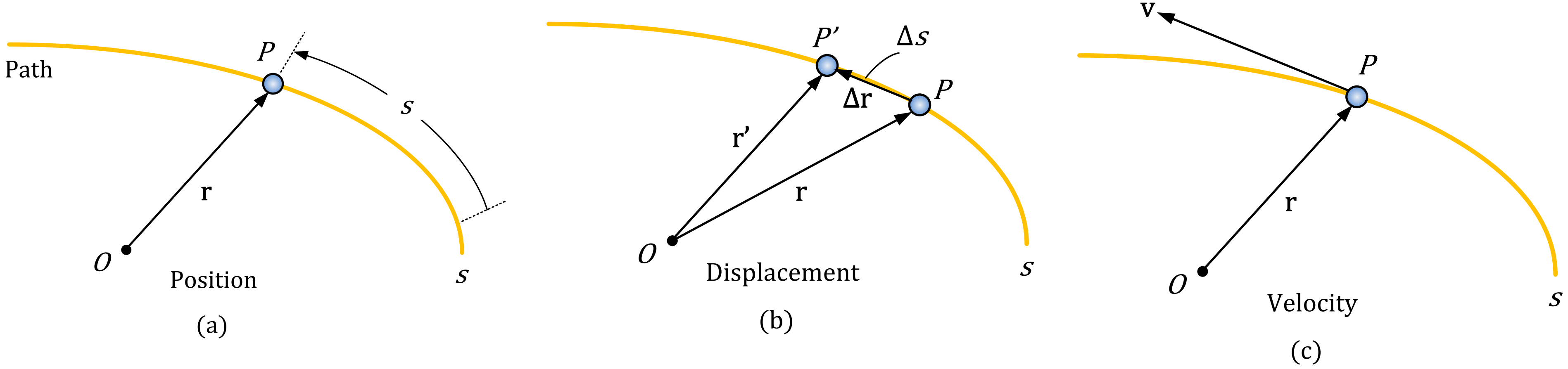
- Position
- Velocity
- Instantaneous velocity
- Acceleration
- Coordinate systems
- Curvilinear x/y coordinate system
- Projectile motion
- Curvilinear n/t coordinate system
- Circular motion

Plane Curvilinear Motion

Plane Curvilinear Motion describes the motion of a particle along a curved path that lies in a single plane.

Circular motion and **projectiles**:

Consider the continuous motion of a particle along a plane curve.

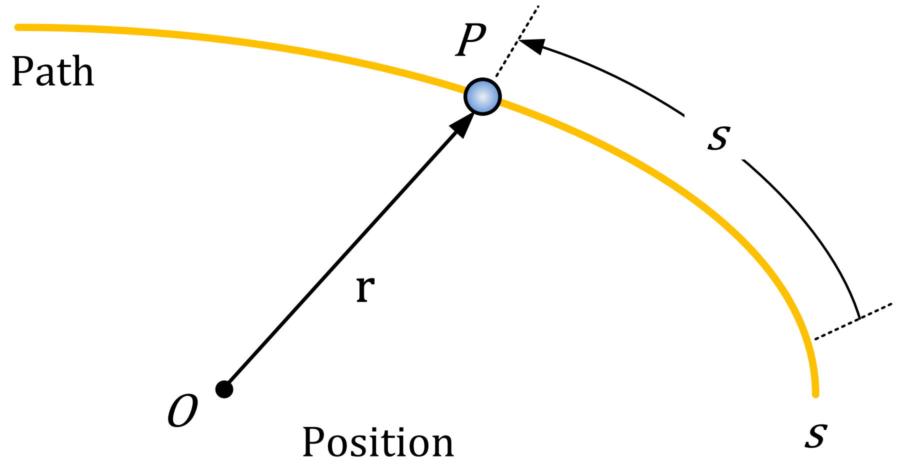


General Curvilinear Motion

Curvilinear motion

Occurs when the particle moves along a curved path

The position of the particle, measured from a fixed point O, is designated by the *position vector* **r** (a vector is written in **boldface**).



(a)

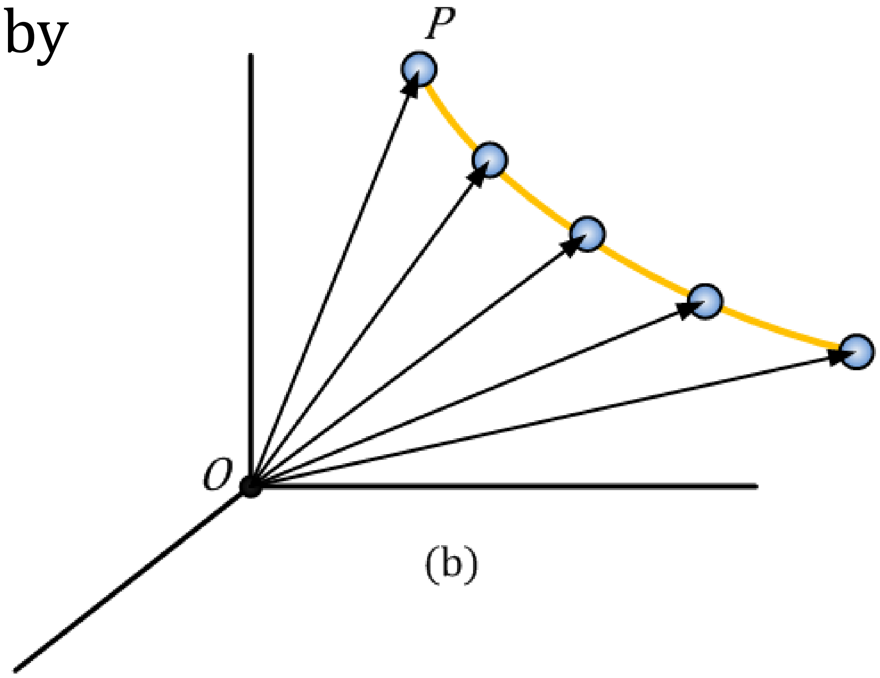
General Curvilinear Motion

Position

Suppose P is in a motion relative to the chosen reference frame, so the **r** is a function of time *t*.

We can express this motion by

$$r = r(t)$$

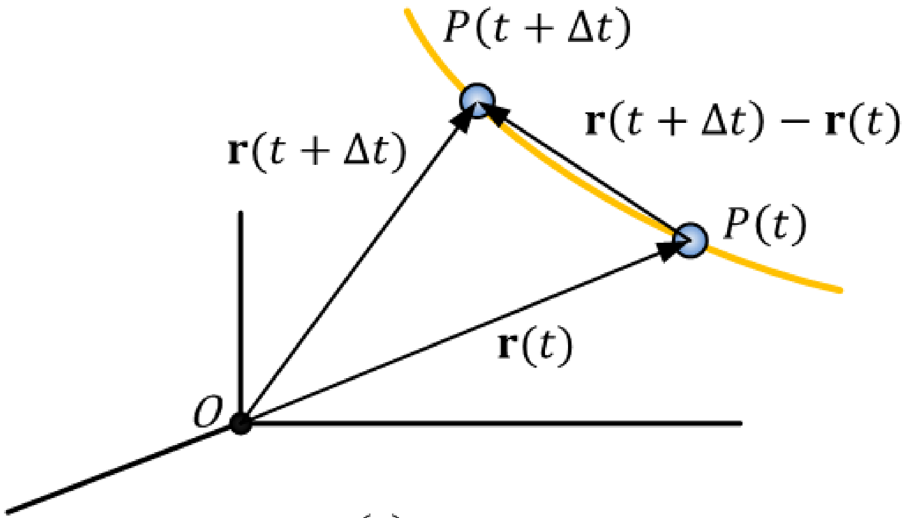


General Curvilinear Motion

Displacement

Suppose during a small time interval (t, $t+\Delta t$) the particle moves a distance Δs along the curve to a new position P', defined by $\mathbf{r}'=\mathbf{r}(t+\Delta t)$. The *displacement* $\Delta\boldsymbol{r}$ represents the change in the particle's position.

$$\Delta r = r(t+\Delta t) - r(t)$$



(c)

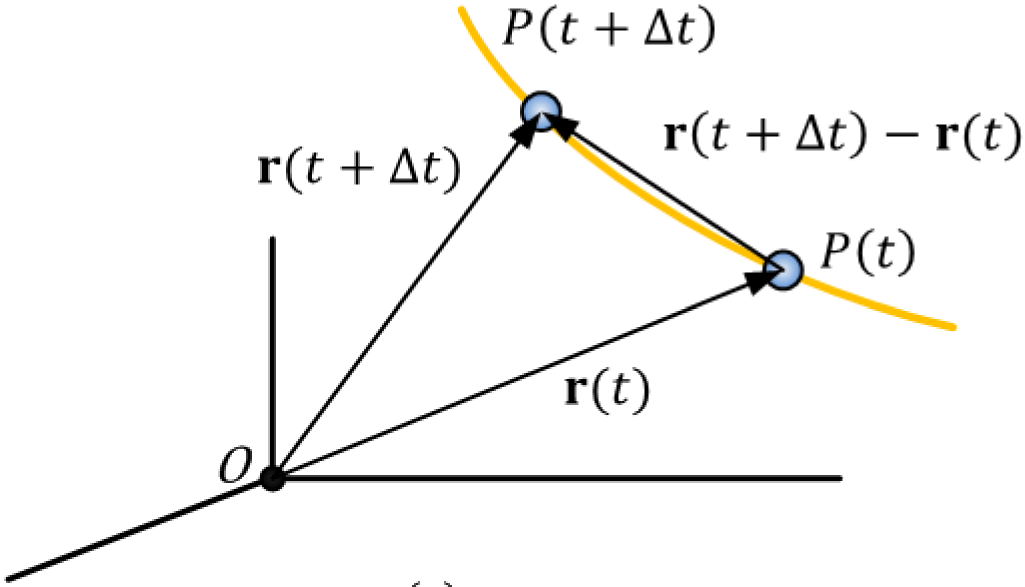
General Curvlinear Motion

Velocity

The *average velocity* of the particle is defined as $\mathbf{v}_{avg} = \dfrac{\Delta \mathbf{r}}{\Delta t}$

The instantaneous velocity is determined from this equation by letting $\Delta t \to 0$,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}$$

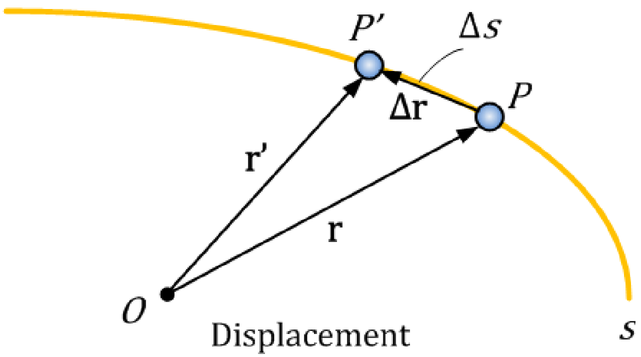


(c)

General Curvilinear Motion

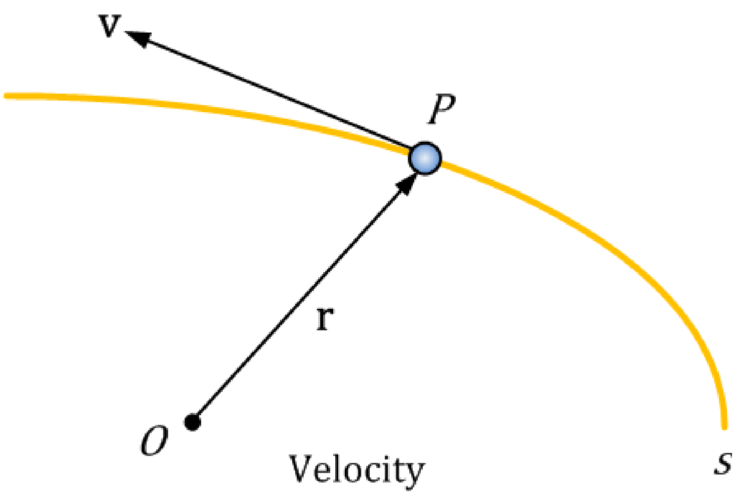
- *Direction* of ***v*** is *tangent to the curve*
- *Magnitude* of ***v*** is the *speed*, which may be obtained by noting the magnitude of the displacement Δr is the length of the straight line segment from P to P’.

$$v = \frac{ds}{dt}$$



Displacement

(b)



Velocity

(c)

General Curvilinear Motion

Acceleration

If the particle has a velocity ***v*** at time t and a velocity $\mathbf{v}' = \mathbf{v}(t + \Delta t)$ at time $t' = t + \Delta t$.

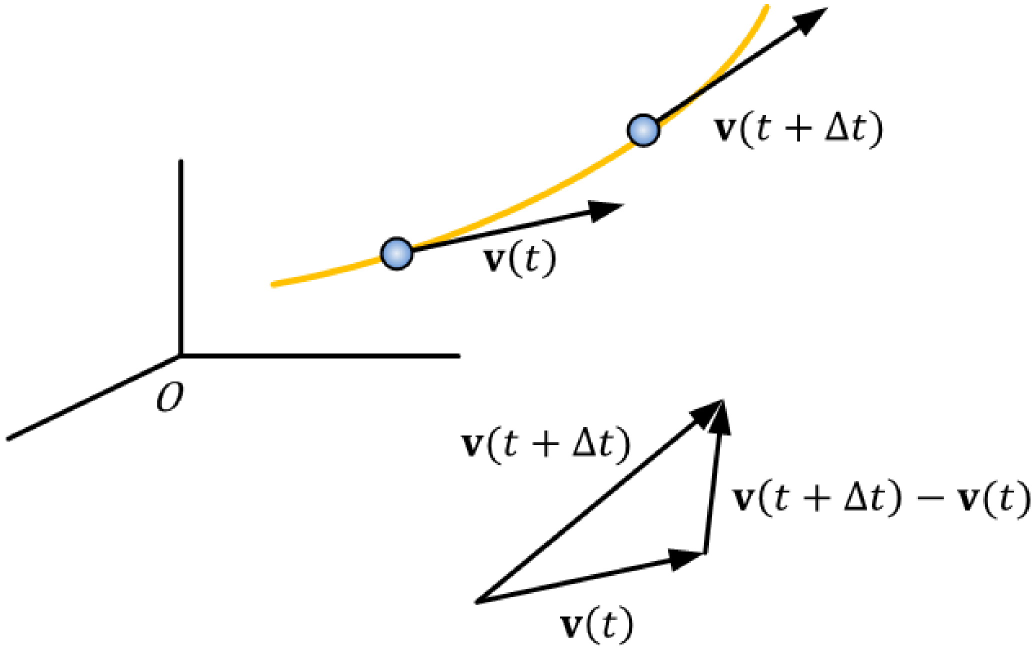
The increment of velocity is $\Delta\mathbf{v} = \mathbf{v}(t + \Delta t) - \mathbf{v}(t)$

The average acceleration

$$\mathbf{a}_{avg} = \frac{\Delta\mathbf{v}}{\Delta t}$$

Instantaneous *acceleration*

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{v} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{r}$$



We Need to Choose a Coordinate System!

- Three coordinate systems are used in problem solving in curvilinear motion. Each system has its advantages. When problem solving, it is best to select the most suitable coordinate system for the problem.

- Using a different coordinate system does NOT change the position, velocity and acceleration.

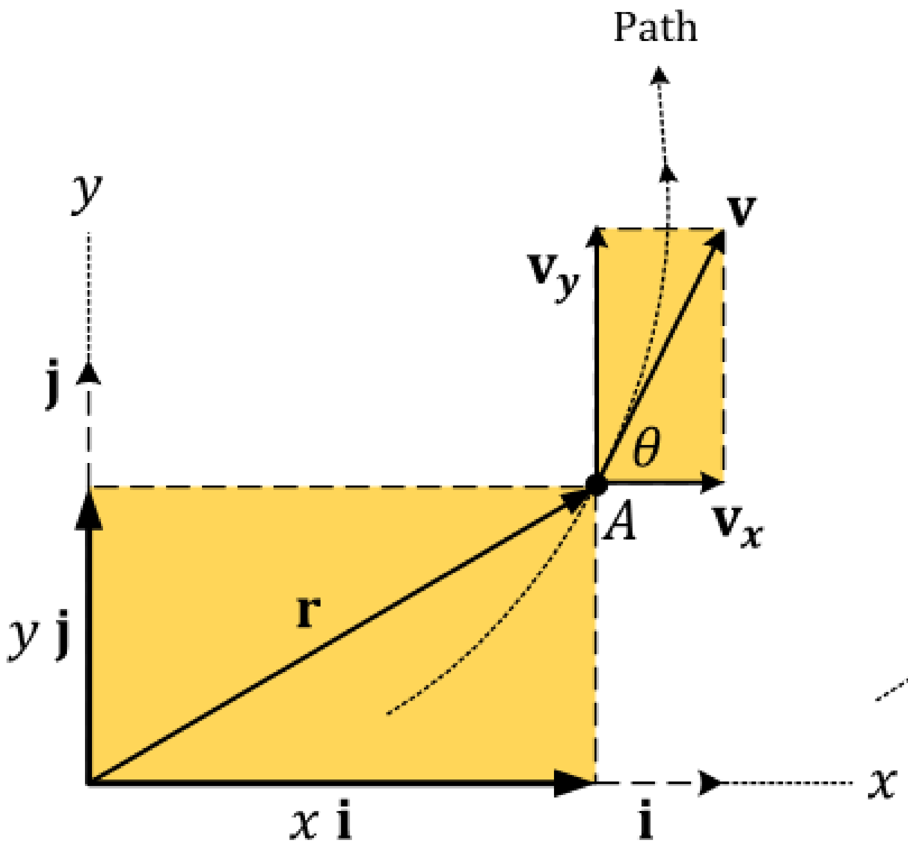
Coordinate Systems

- Three coordinate systems are used in problem solving in curvilinear motion. Each system has its advantages. When problem solving, it is best to select the most suitable coordinate system for the problem.
- Using a different coordinate system does NOT change the position, velocity and acceleration.

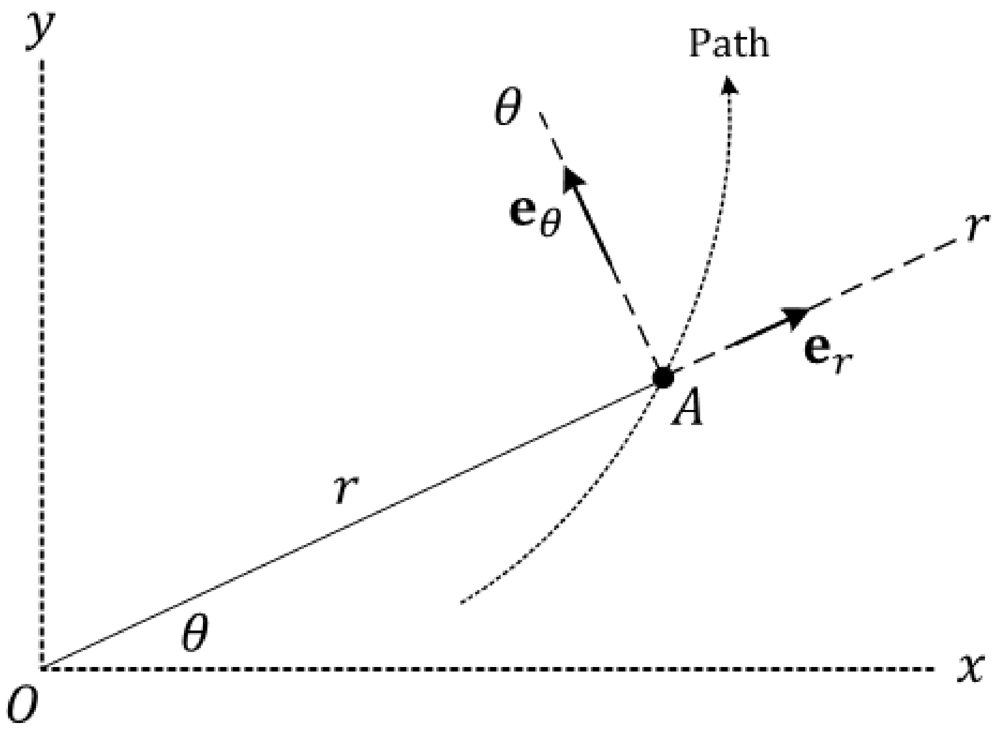
The 3 coordinate systems are:

1. Rectangular coordinates (x/y)
2. Normal and tangential coordinates (n/t)
3. Polar (Cylindrical) coordinates (r/θ)

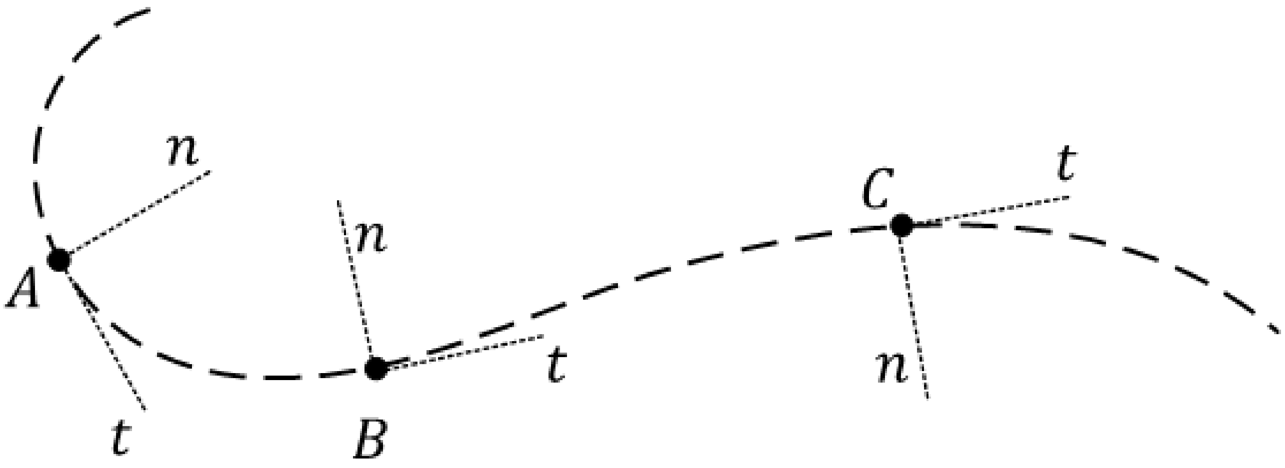
1) Rectangular coordinates (*x* - *y*)



2) Polar coordinates (r - θ)



3) Normal and tangential coordinates (n - t)



Curvilinear Motion: Rectangular Components

Position

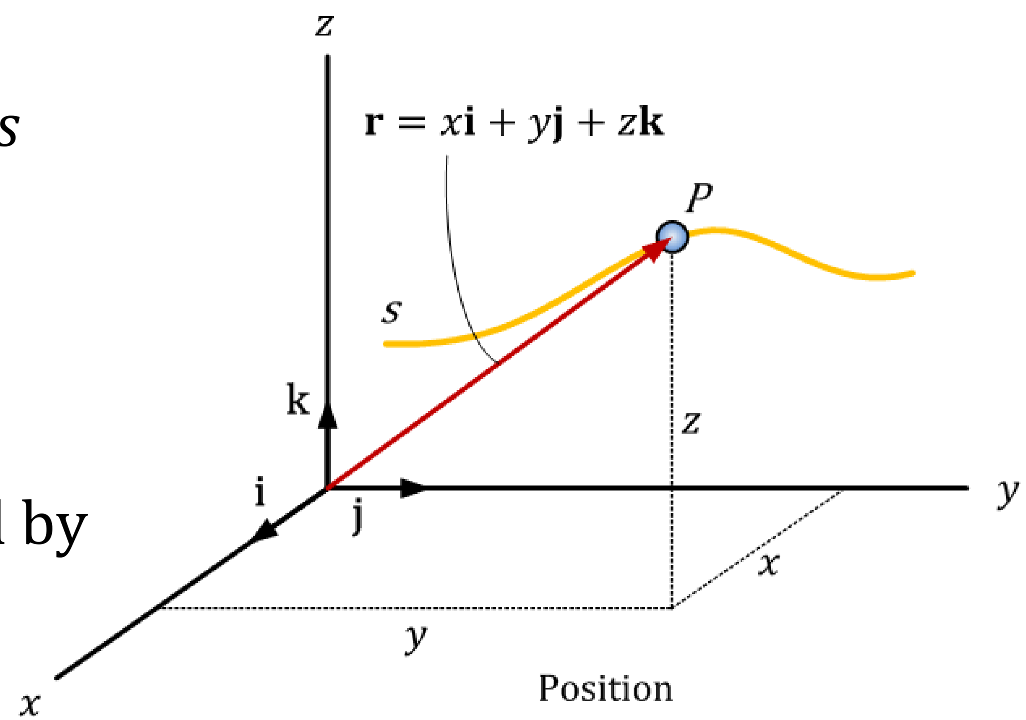
Position vector is defined by $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are fixed Eulerian unit vectors

The magnitude of $\mathbf{r}$ is *always positive* and defined as

$$r = \sqrt{x^2 + y^2 + z^2}$$

The *direction* of r is specified by the components of the unit vector $\mathbf{e}_r = \mathbf{r}/r$



Position

(a)

Curvilinear Motion: Rectangular Components

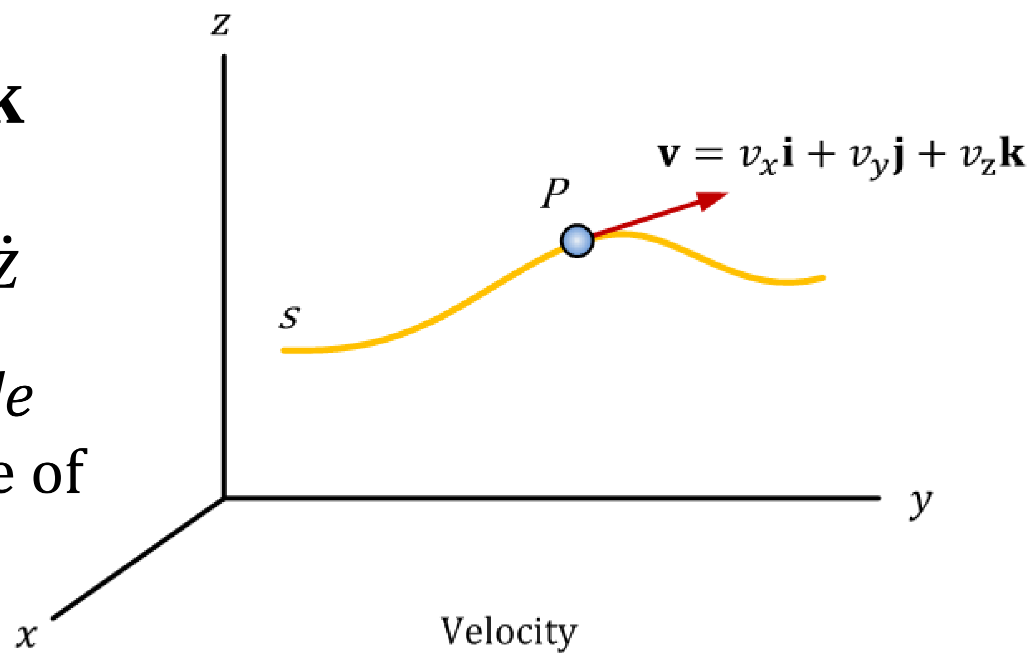
Velocity

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

$$v_x = \dot{x} \quad v_y = \dot{y} \quad v_z = \dot{z}$$

The velocity has a *magnitude* defined as the positive value of

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$



Velocity

(b)

and a *direction* that is specified by the components of the unit vector $\mathbf{e}_v = \mathbf{v}/v$ and is *always tangent to the path*.

Curvilinear Motion: Rectangular Components

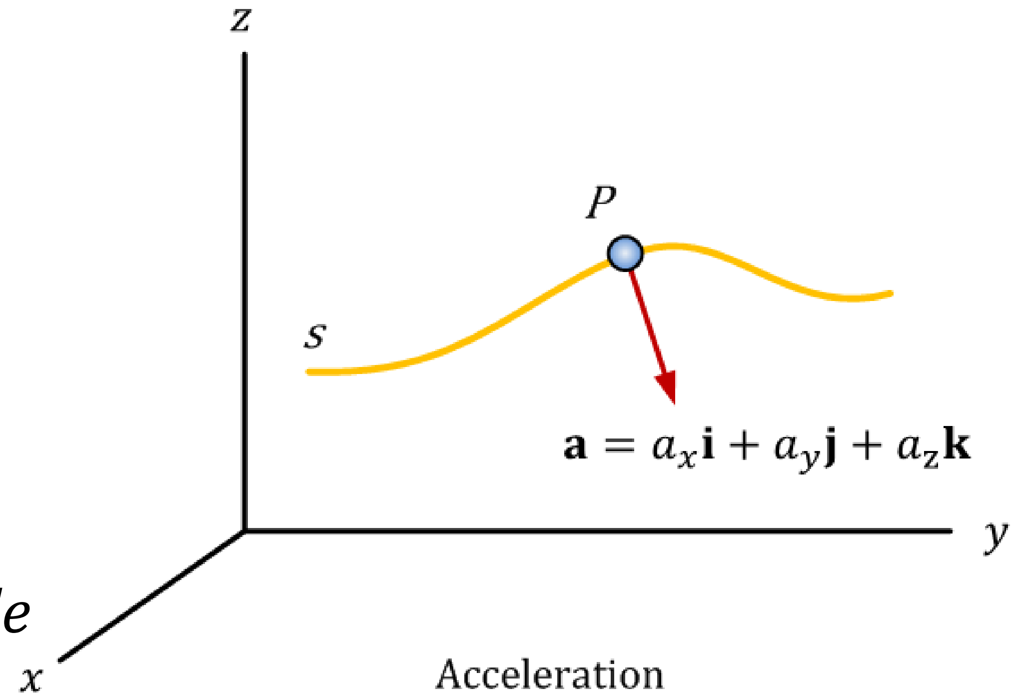
Acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

$$a_x = \dot{v}_x = \ddot{x}$$

$$a_y = \dot{v}_y = \ddot{y}$$

$$a_z = \dot{v}_z = \ddot{z}$$



Acceleration

(c)

Acceleration has a *magnitude*

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

Since **a** represents the time rate of *change* in velocity, **a** will *not* be tangent to the path.

Example 4: Curvilinear motion

The motion of a particle is defined by

$$x = 2t^2 - 4t - 4$$

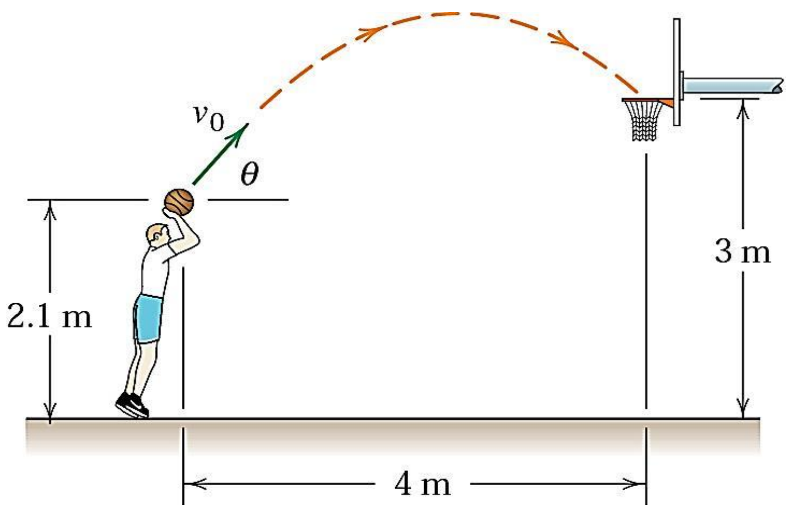
$$y = x^2 + 3$$

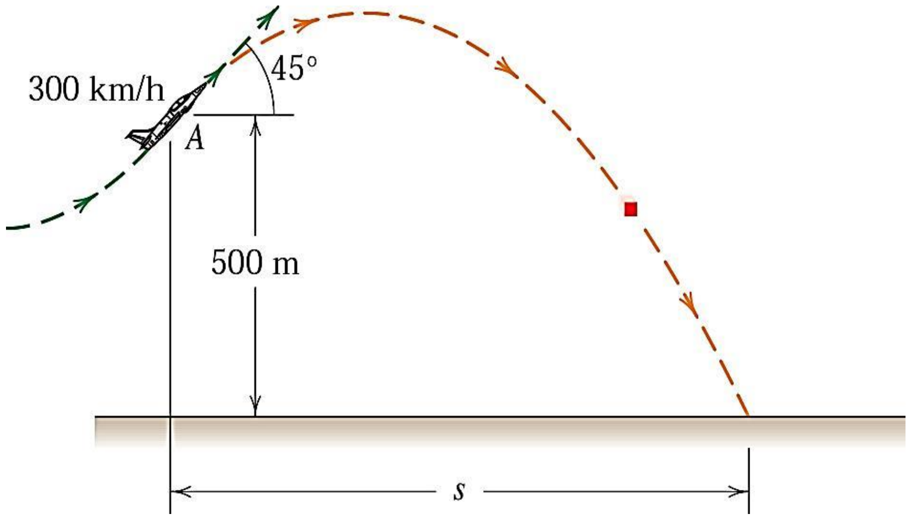
Where x and y are in metres and t is in seconds.
Find $|\boldsymbol{v}|$ and $|\boldsymbol{a}|$ of the particle when $t = 4.0\ s$.

W7 Example 4 (Web view)

Projectile Motion

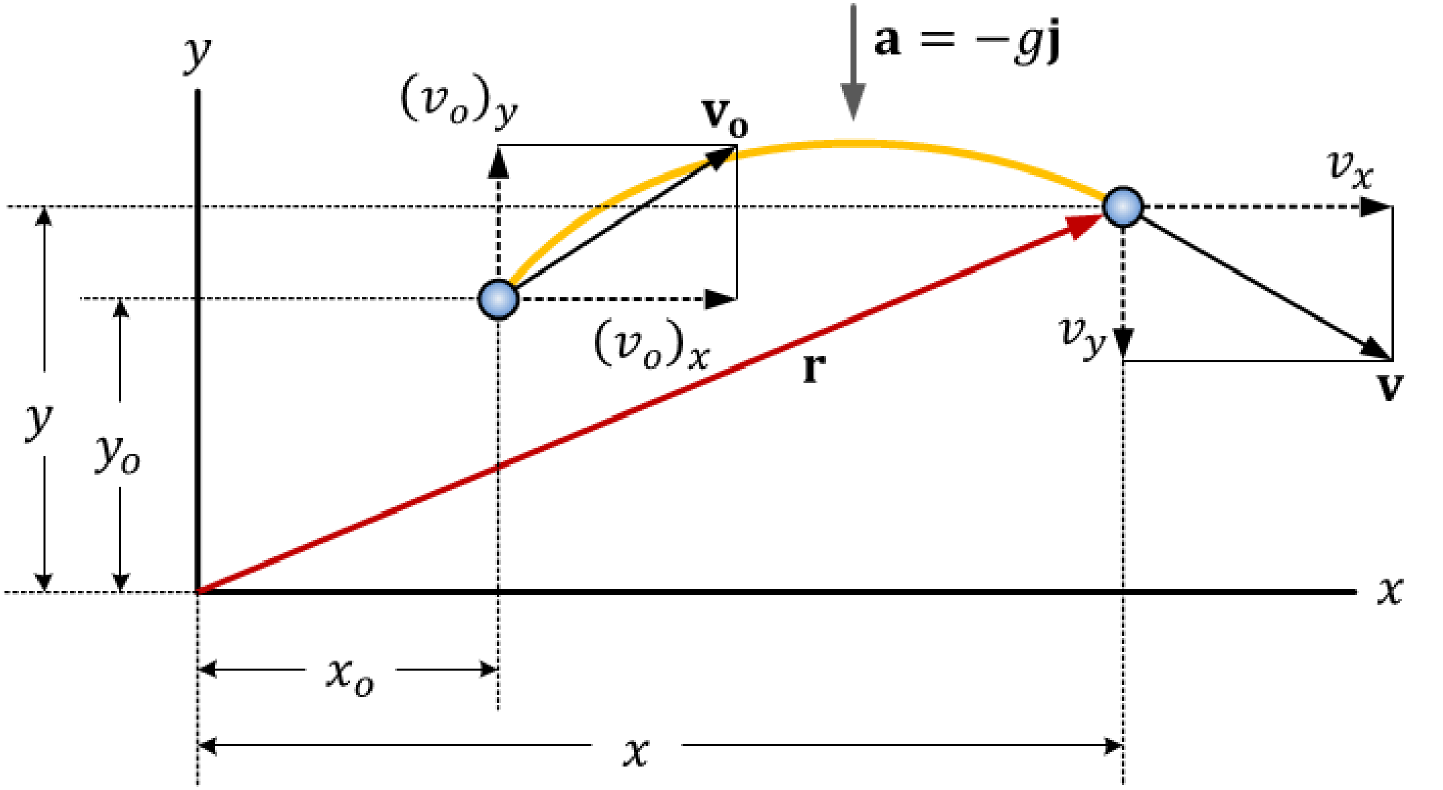
Rectangular coordinates are particularly useful for describing projectile motion. In projectile motion, we neglect aerodynamic drag and the curvature and rotation of the earth. We also assume that the altitude change is small enough so that the acceleration due to gravity can be considered constant.





Projectile Motion

An important application of two-dimensional kinematics theory is the problem of projectile motion

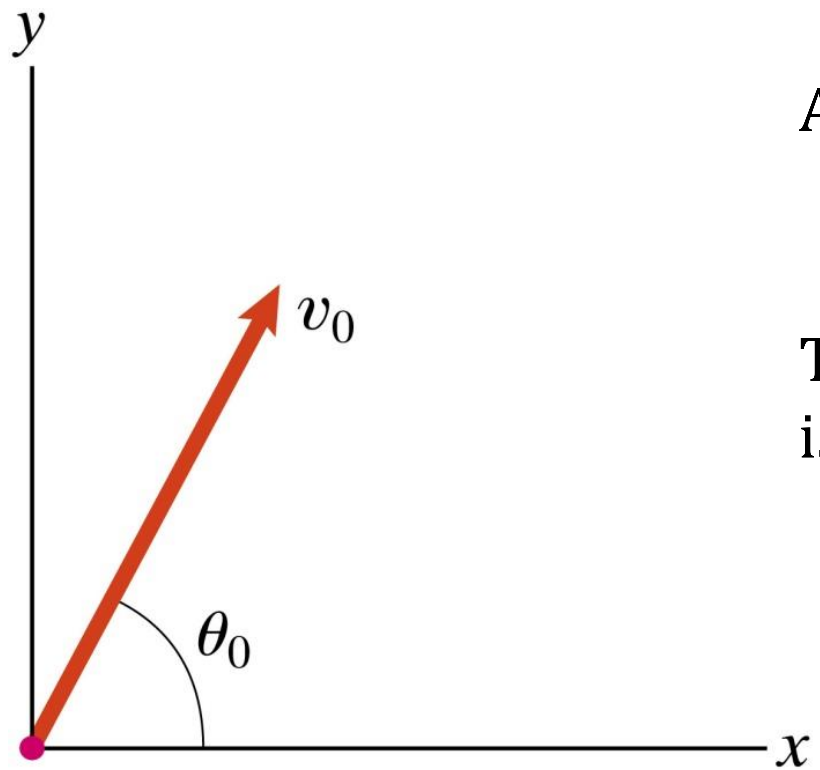


Motion of a Projectile

- Free-flight motion studied in terms of rectangular components since projectile's acceleration always act vertically
- Consider projectile launched at (x_0, y_0)
- Path defined in the *x-y* plane
- Air resistance neglected
- Only force acting on the projectile is its weight, resulting in constant downwards acceleration

$$a_y = g = -9.81 \text{ m/s}^2$$

Motion of a Projectile: Horizontal Component



At $t = 0$:

$x = 0$ and $v_x = v_0 \cos \theta_0$

The acceleration in the x-direction is zero i.e.

$$a_x = \frac{dv_x}{dt} = 0$$

Therefore v_x is constant and remains equal to its initial value:

$$v_x = \frac{dx}{dt} = v_0 \cos \theta_0$$

Motion of a Projectile: Horizontal Component

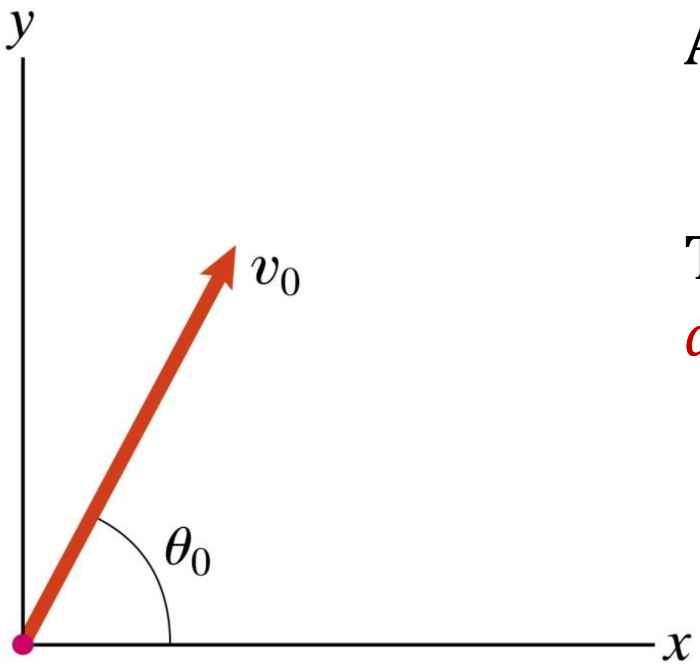
Integrating $dx = v_x dt = v_0 \cos\theta_0 dt$

$$\int_0^x dx = \int_0^t v_0 \cos\theta_0 \, dt$$

$$x = (v_0 \cos\theta_0)\, t$$

Thus, we have determined the position and velocity of the projectile in the x-direction as functions of time w/o considering the projectile's motion in the y-direction.

Motion of a Projectile: Vertical Component



At $t = 0$;

$y = 0$ and $v_y = v_0 \sin \theta_0$

The acceleration in the y-direction is $a_y = -g$

$$a_y = \frac{dv_y}{dt} = -g$$

$$\int_{v_0 \sin \theta_0}^{v_y} dv_y = \int_0^t - g \; dt$$

By integrating, we obtain: $v_y = \dfrac{dy}{dt} = v_0 \sin \theta_0 - gt$

Motion of a Projectile: Vertical Component

Integrating v_y yields:

$$\int_0^y dy = \int_0^t (v_0 \sin\theta_0 - gt)\ dt$$

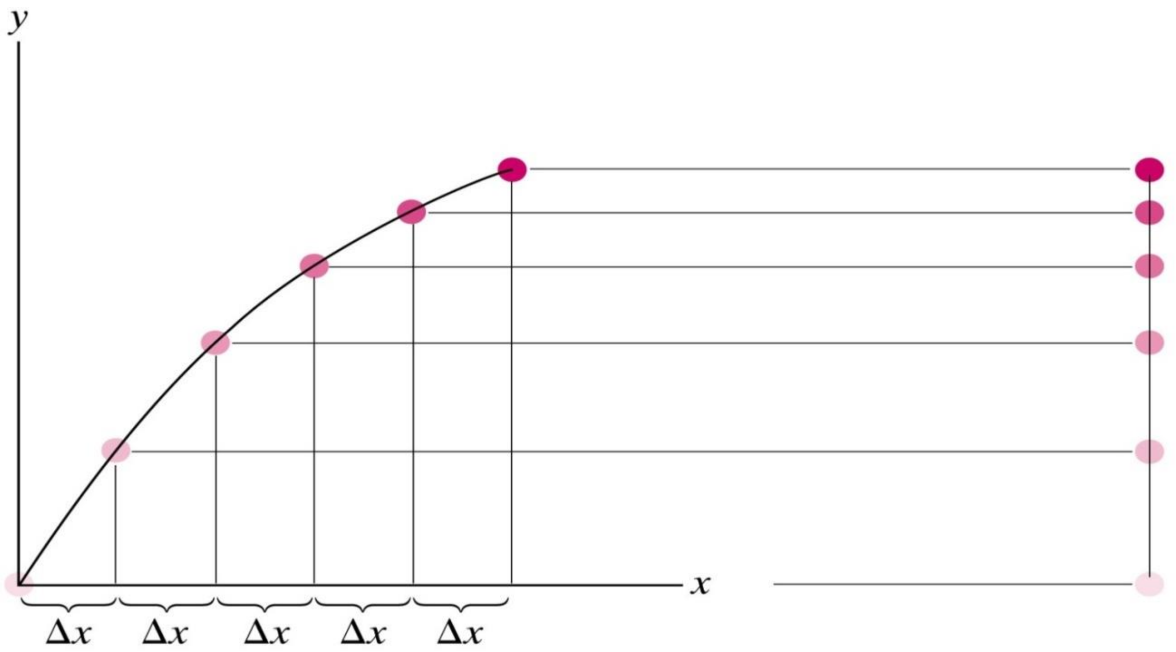
$$y = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$$

I.e. the vertical motion is independent of the horizontal motion.

Kinematics of a Projectile

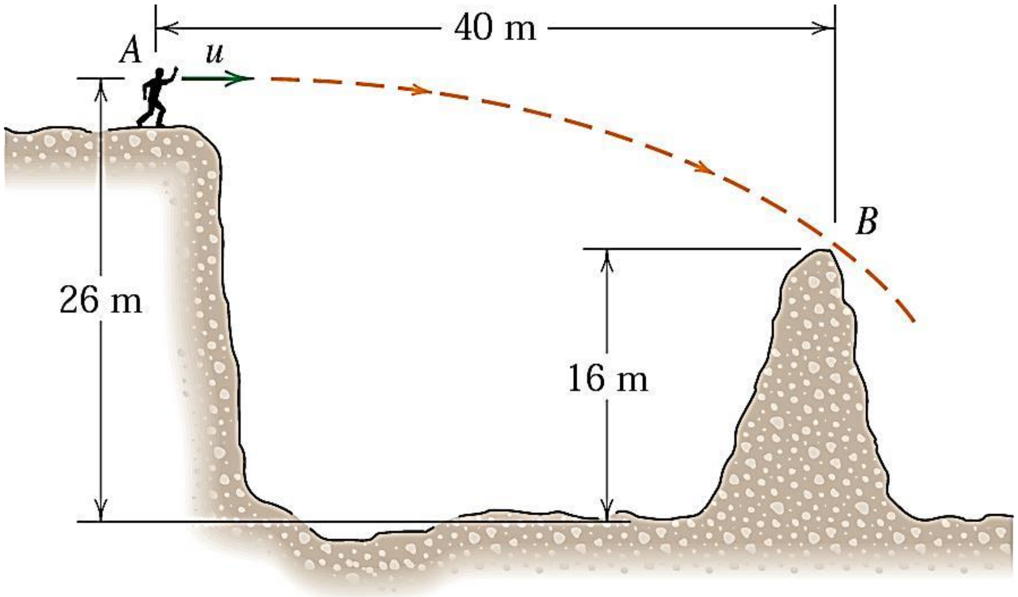
Eliminating time t from $x = (v_0 \cos\theta_0)\, t; \quad y = (v_0 \sin\theta_0)\, t - \frac{1}{2} g t^2$

Yields the parabolic trajectory of the projectile: $y = (\tan\theta_0)\, x - \dfrac{g}{2 v_0^2 \cos^2\theta_0} x^2$



Example 5

With what minimum horizontal velocity u can a boy throw a rock at A and have it just clear the obstruction at B?



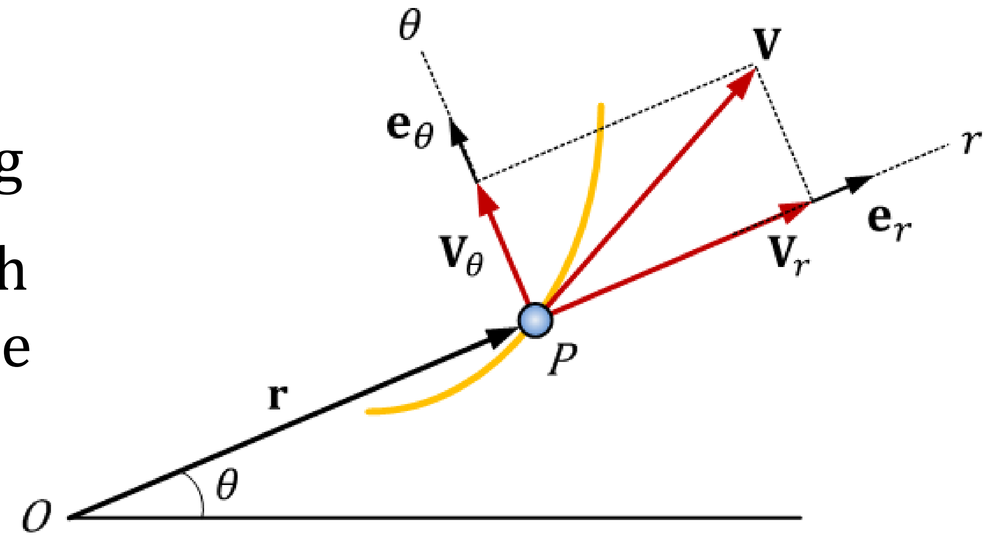
W7 Example 5 (Web view)

Curvilinear Motion: Polar Coordinates

Polar Coordinates

Specify the location of *P* using

1. *Radial coordinate r*, which extends outward from the fixed origin *O* to the particle and the
2. *Circumferential coordinate θ*, which is the counterclockwise angle between a fixed reference line and the *r* axis.



Position
(a)

$$r = \sqrt{x^2 + y^2}$$

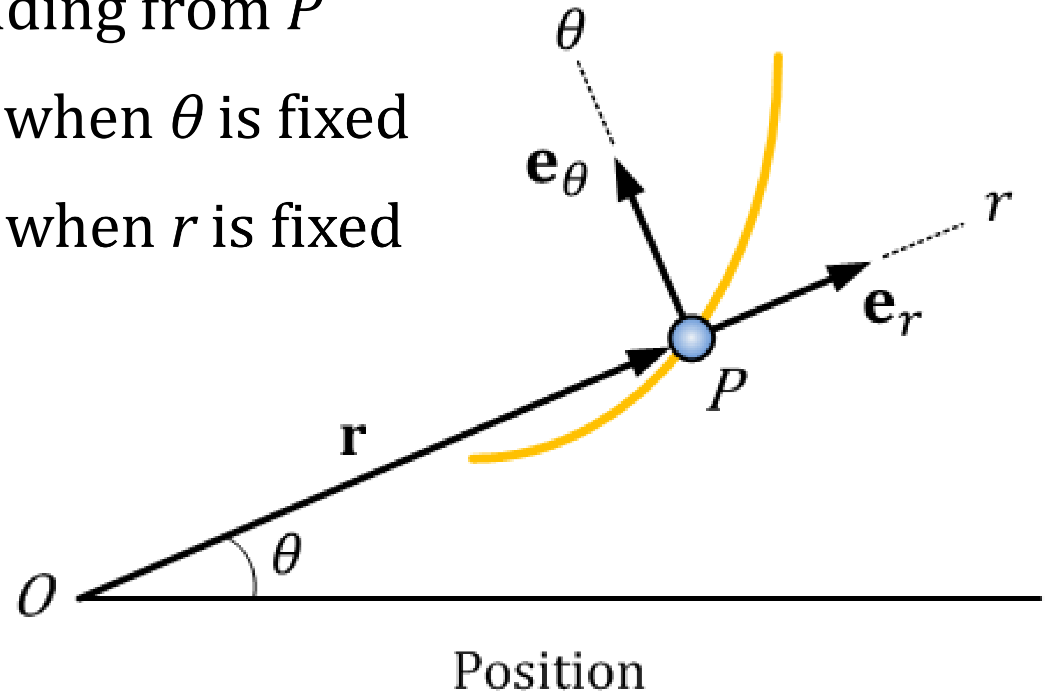
$$\theta = \tan^{-1}(y / x)$$

Curvilinear Motion: Cylindrical Coordinates

Basis Vectors

Vectors in polar coordinates are defined by using the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ extending from P

- $\mathbf{e}_r$: along increasing r, when θ is fixed
- $\mathbf{e}_\theta$: along increasing θ when r is fixed



Position

Position vector:

$\mathbf{r} = r\mathbf{e}_r \quad (\text{can be: } \vec{\mathrm{r}} = r\vec{\mathrm{e}}_r)$

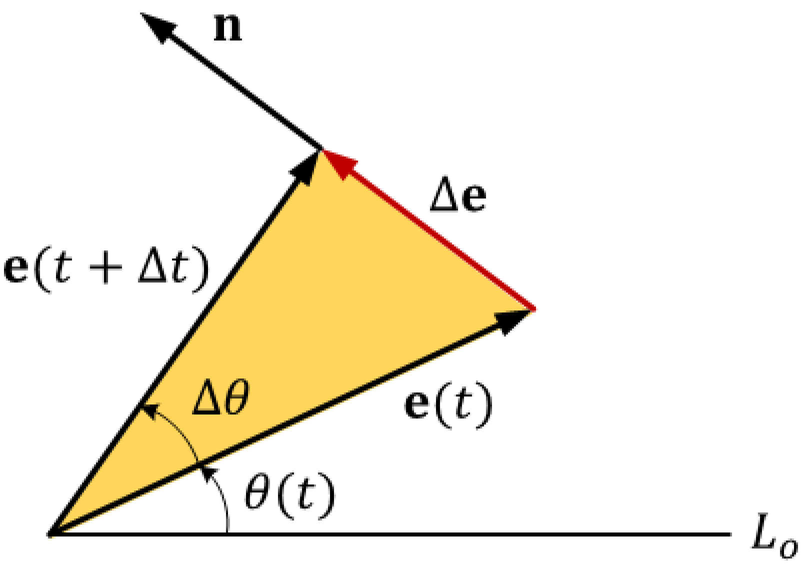
Note $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ change directions as P moves and are perpendicular to each other

Rotating Unit Vector (Basis Vectors)

If a unit vector **e** rotates at an angular velocity

$$\dot{\theta} = \frac{d\theta}{dt}$$

Then the rotation angle during Δt : $\Delta\theta = \theta(t + \Delta t) - \theta(t)$



The time derivative of **e** is defined by

$$\frac{d\mathbf{e}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{e}(t + \Delta t) - \mathbf{e}(t)}{\Delta t}$$

Rotating Unit Vector (Basis Vectors)

Lagrangian Basis Vectors

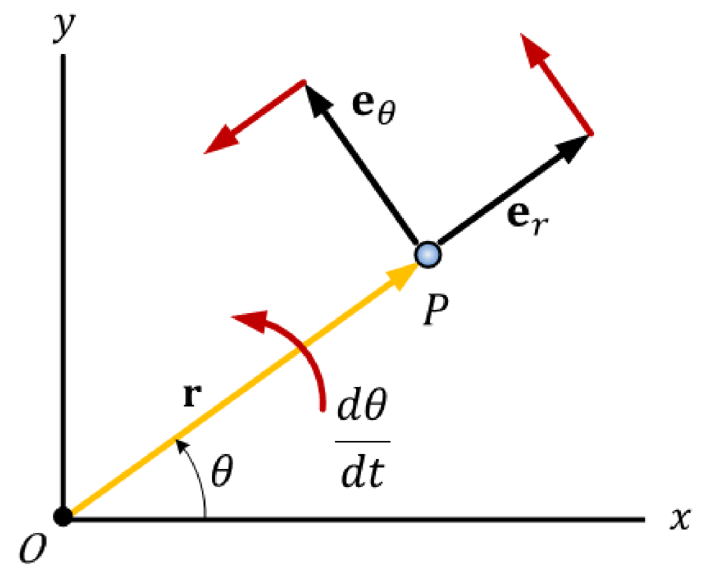
We need to introduce Lagrangian unit vectors $\mathbf{e}_i$ that move with the particle. These will replace the Eulerian basis vectors **i, j, k**

$\mathbf{e}_r$ and $\mathbf{e}_\theta$ change direction w.r.t. time

$$\frac{d\mathbf{e}_\theta}{dt} = -\frac{d\theta}{dt}\mathbf{e}_r \qquad \frac{d\mathbf{e}_r}{dt} = \frac{d\theta}{dt}\mathbf{e}_\theta$$

- To evaluate $\dot{\mathbf{e}}_r$ note that $\mathbf{e}_r$ changes its direction w.r.t. time although its magnitude = 1 (rotating of a unit vector)

$$\boxed{\dot{\mathbf{e}}_r = \dot{\theta}\,\mathbf{e}_\theta \qquad \dot{\mathbf{e}}_\theta = -\dot{\theta}\,\mathbf{e}_r}$$



Curvilinear Motion: Cylindrical Coordinates

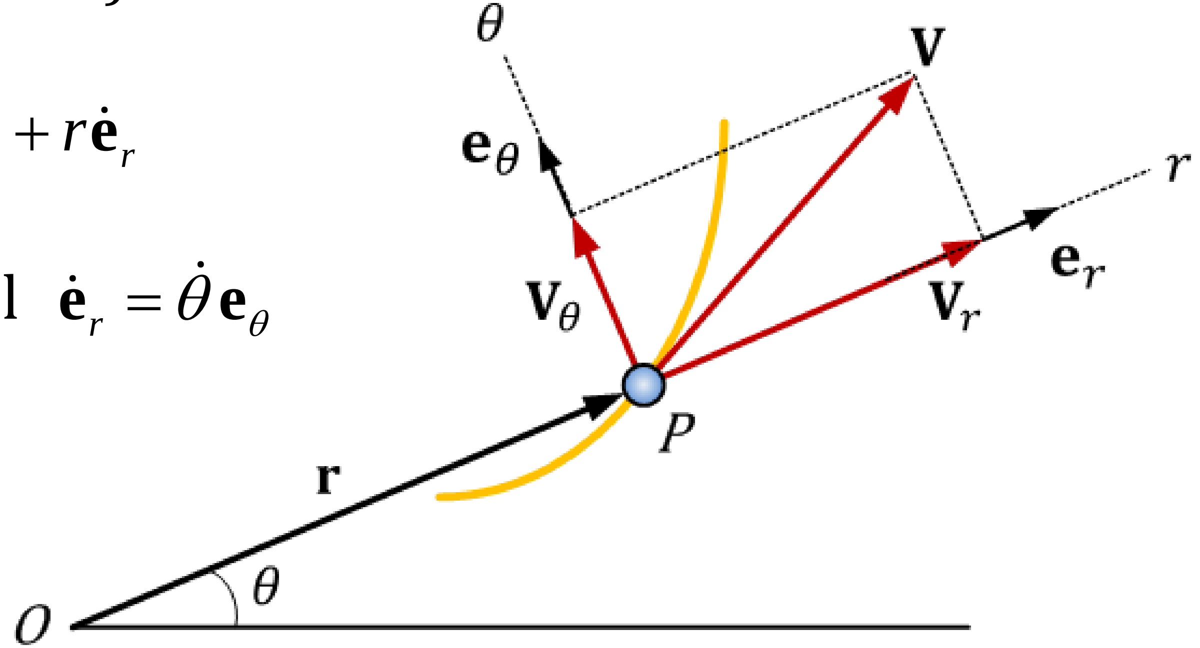
Velocity

- Instantaneous velocity **v** is obtained by the time derivative of **r** (using product rule)

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d(r\mathbf{e}_r)}{dt} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r$$

- To evaluate $\dot{\boldsymbol{e}}_r$ recall $\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta$

$$\therefore \quad \mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$



Position

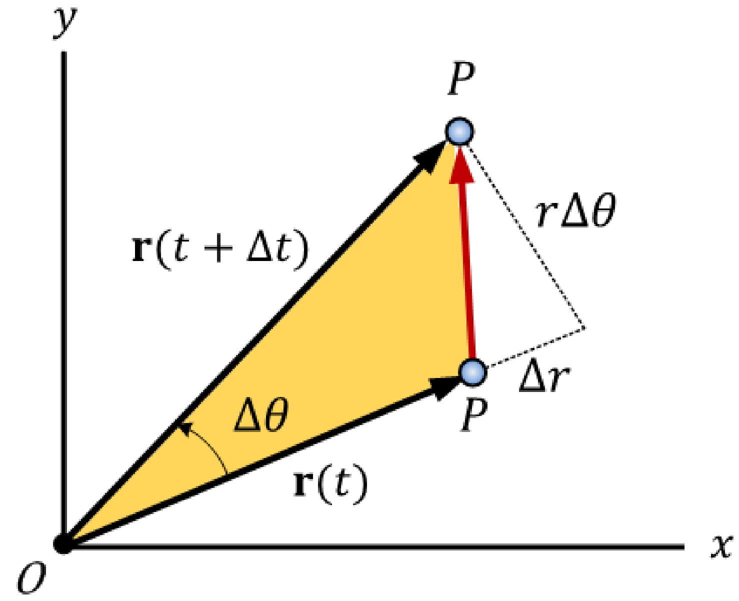
(a)

Curvilinear Motion: Cylindrical Coordinates

Instantaneous velocity **v**

$$\mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta$$
$$v_r = \dot{r}$$
$$v_\theta = r\dot{\theta}$$

e



- *Radial component* $\mathbf{v}_r$ is a measure of the rate of increase or decrease in the length of the radial coordinate
- *Circumferential component* $\mathbf{v}_\theta$ is the rate of motion along the circumference of a circle having a radius *r*

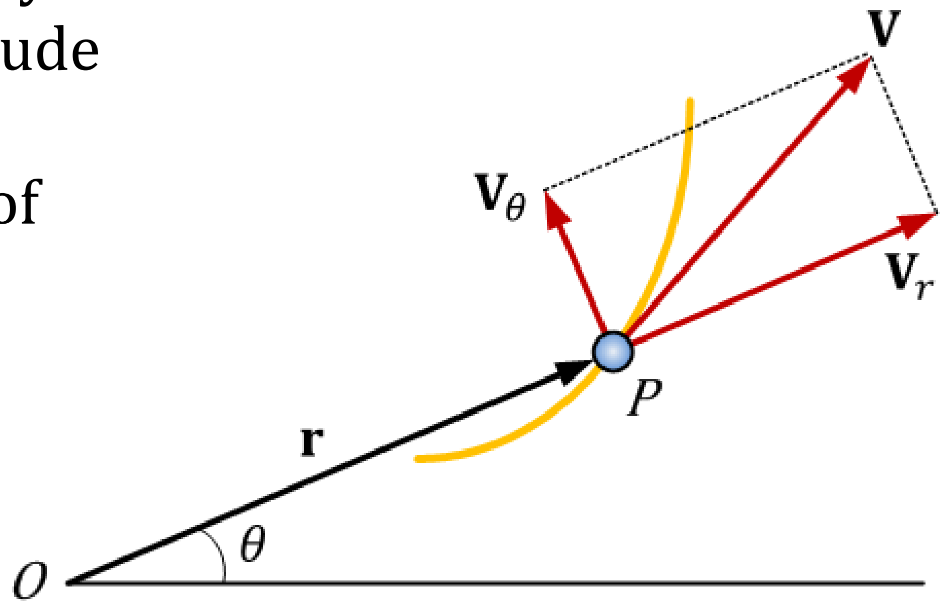
Curvilinear Motion: Cylindrical Coordinates

Speed

- Since $\mathbf{v}_r$ *and* $\mathbf{v}_\theta$ are mutually perpendicular, the magnitude of the velocity or speed is simply the positive value of

$$v = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2}$$

- Direction of **v** is always tangent to the path at P



Velocity

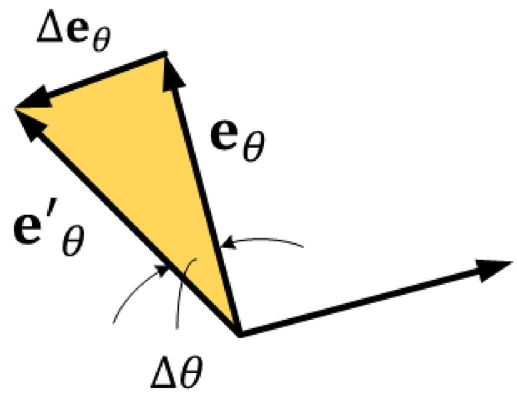
Curvilinear Motion: Cylindrical Coordinates

Acceleration

Let's take the rate derivative of our expression for **v**

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{d}{dt}(\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta)$$

$$= \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta$$



- Recall our expressions for the rate derivative of $\mathbf{e}_r$ and $\mathbf{e}_\theta$

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta \qquad \dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r$$

$$\therefore \quad \mathbf{a} = \ddot{r}\mathbf{e}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e}_r$$

Curvilinear Motion: Cylindrical Coordinates

Generalised acceleration

Instant acceleration has two components:

$$\mathbf{a} = a_r \mathbf{e}_r + a_\theta \mathbf{e}_\theta$$

$$a_r = \ddot{r} - r\dot{\theta}^2$$

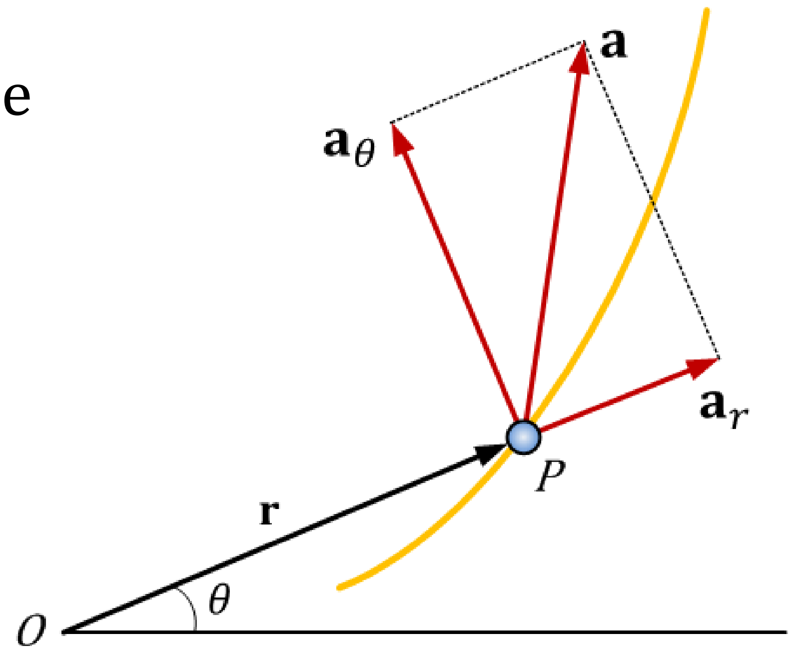
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

- The term $\ddot{\theta} = d^2\theta / dt^2$ is called the *angular acceleration* since it measures the change made in the angular velocity during an instant of time, use unit rad/s^2

Curvilinear Motion: Cylindrical Coordinates

- Since $\mathbf{a}_r$ and $\mathbf{a}_\theta$ are always perpendicular, the *magnitude* of the acceleration is simply the positive value of

$$a = \sqrt{\left(\ddot{r} - r\dot{\theta}^2\right)^2 + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)^2}$$

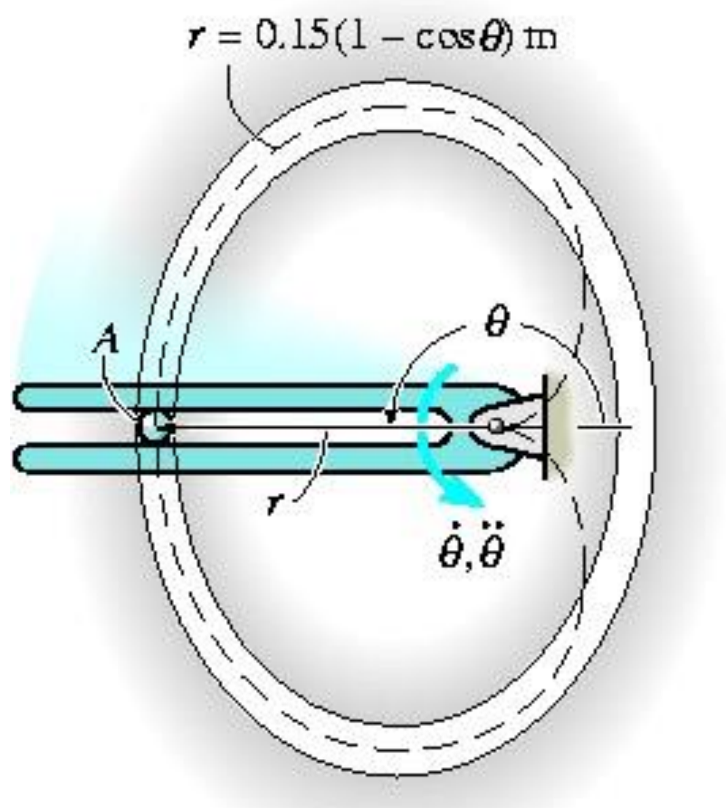


Acceleration

- *Direction* is determined from the vector addition of its components
- Acceleration is not tangent to the path

Example 6

Due to the rotation of the forked rod, ball *A* travels across the slotted path, a portion of which is in the shape of a cardioids, $r = 0.15(1 - \cos\theta)$m where θ is in radians. If the ball's velocity is v = 1.2m/s and its acceleration is 9m/s^2 at instant $\theta = 180°$, determine the angular velocity and angular acceleration of the fork.



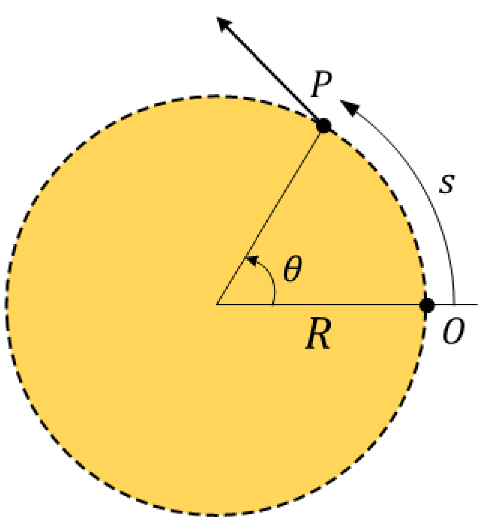
(a)

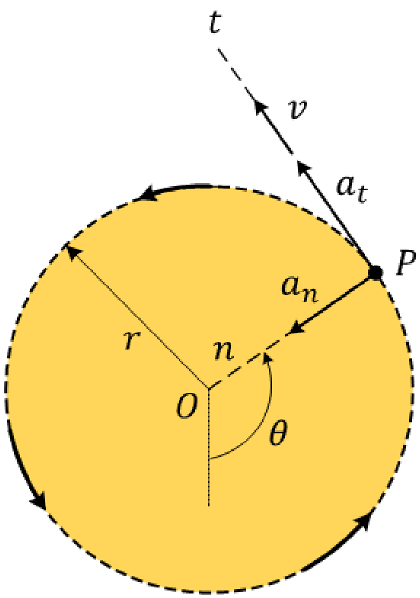
W7 Example 6 (Web view)

Circular Motion

Circular motion is an important special case of plane curvilinear motion where the radius of curvature ρ becomes a constant radius of the circle.

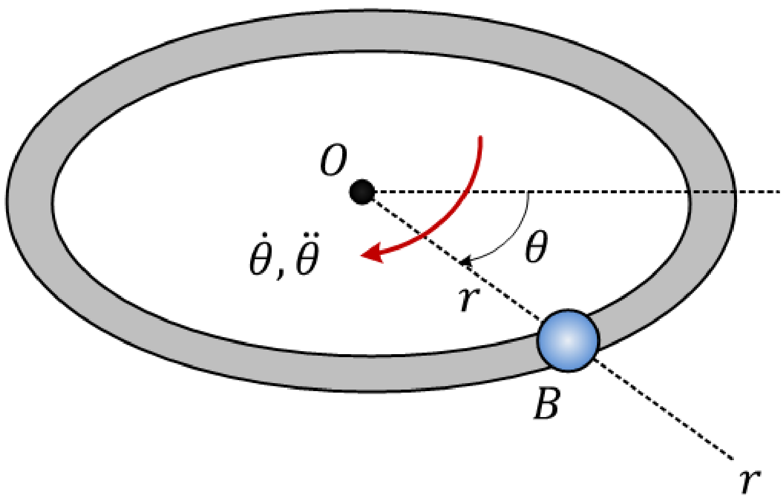
Circular motion is commonly associated with n/t coordinates. Circular motion is the motion of a point moving around in a circular path, eg. linkages, tip of a turbine blade.

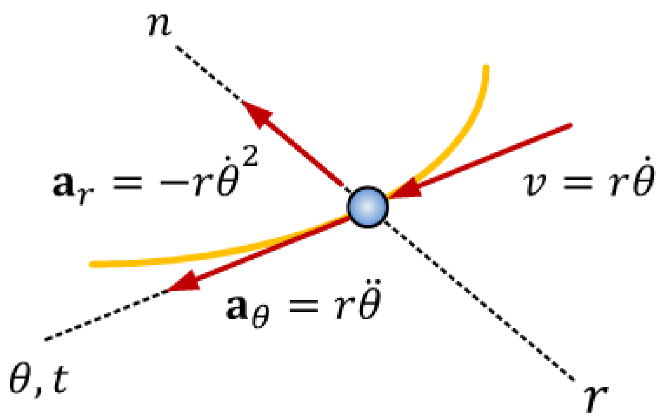




Circular Motion

For circular motion, radius *r* is constant for all θ.





Coordinate System. Polar coordinates

Velocity and Acceleration. Since *r* is constant,

$$r = r \quad \dot{r} = 0 \quad \ddot{r} = 0$$

Reduction to Circular Motion

Velocity and Acceleration

If we take the generalized formulation and reduce it to circular motion, do we get our classical equation?

if $r = \text{constant} \quad \dot{r} = 0 \quad \ddot{r} = 0$

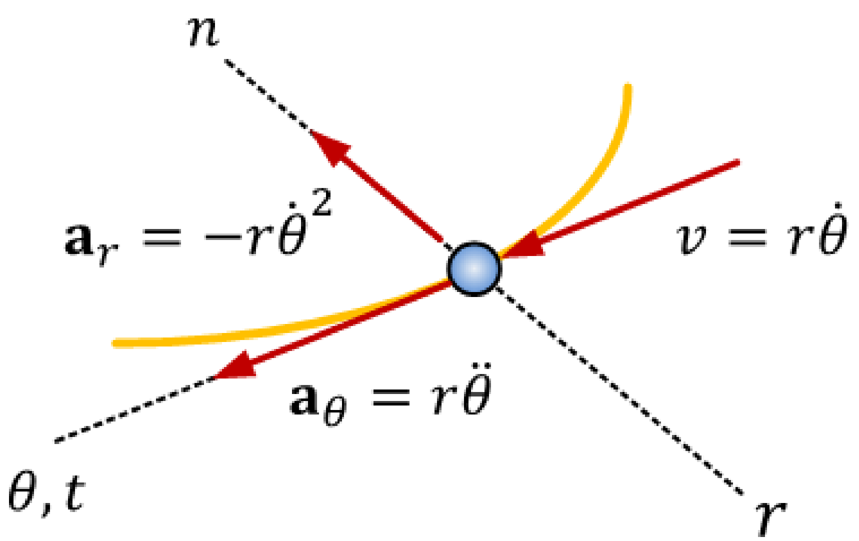
$$v_r = \dot{r} = 0$$

$$v_\theta = r\dot{\theta}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 \Rightarrow -r\dot{\theta}^2$$

$$= -\frac{v_\theta^2}{r}$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \Rightarrow r\ddot{\theta}$$



Circular Motion

For circular motion with a constant angular velocity

$$\dot{\theta} = const.\,(\ddot{\theta} = 0)$$

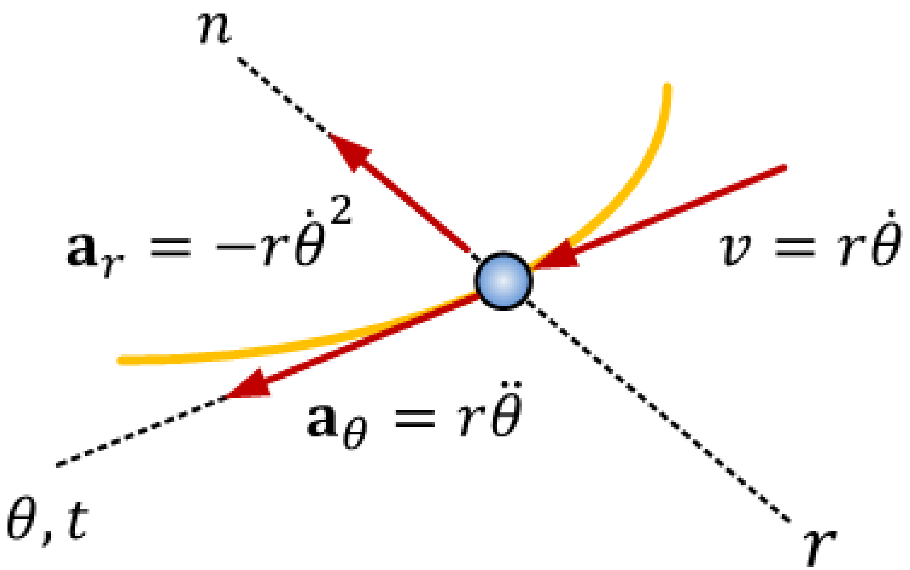
$$v_r = \dot{r} = 0$$

$$v_\theta = r\dot{\theta}$$

$$a_r = -r\dot{\theta}^2$$

$$= -\frac{v_\theta^2}{r}$$

$$a_\theta = 0$$



Acceleration is towards the centre of the circle

Example 7: Circular Motion

A car rounds a bend which has a radius of 200 m. At a particular instant, the car has a velocity of 60 km/h. The speed of the car is increasing at a rate of 5 m/s^2. Determine

(i) the normal and tangential components of the acceleration of the car,

(ii) the magnitude and direction of the resultant acceleration,

(iii) how far around the bend the car will travel before its speed has increased to 20 m/s.

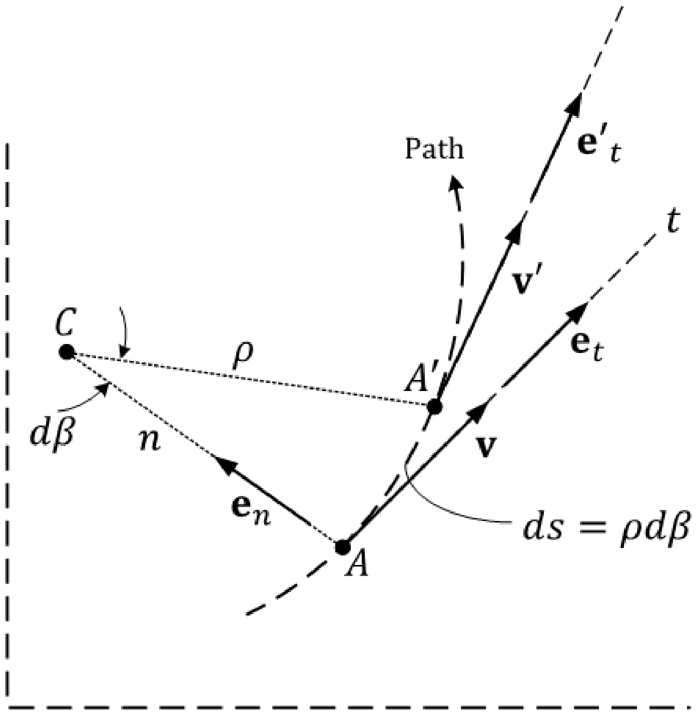
W7 Example 7 (Web view)

Normal and tangential coordinates (n - t)

Normal–Tangential Coordinates

For the $n - t$ coordinate system:

- $\mathbf{e}_{\boldsymbol{n}}$ is a unit vector in the normal direction;
- $\mathbf{e}_{\boldsymbol{t}}$ is a unit vector in the tangential direction



Curvilinear *n* - *t* coordinate system

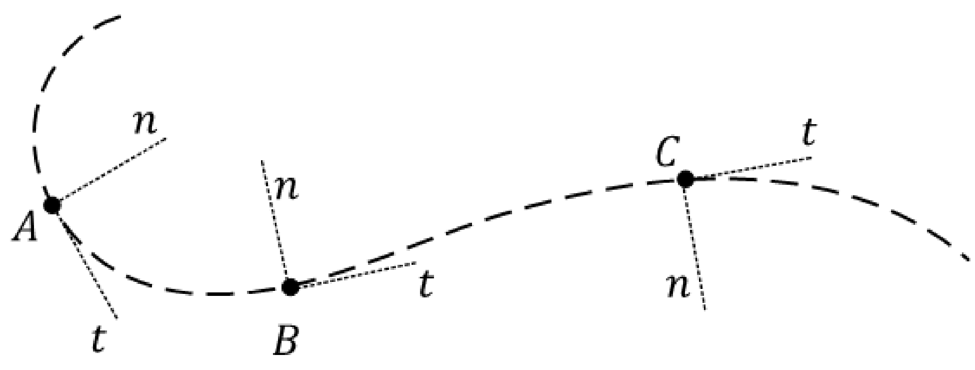
The tangential axis is defined to be in the direction tangential to the path, that is, in the direction of velocity. The direction of this axis thus changes with time, and

$$\boldsymbol{v} = v\boldsymbol{e}_t$$

where v is the speed and $\mathbf{e}_t$ is a unit vector in the tangential direction.

The normal axis is perpendicular to the tangential axis and is towards the centre of curvature of the path.

The direction of the normal axis also changes with time. $\mathbf{e}_n$ is a uni vector in the normal direction.

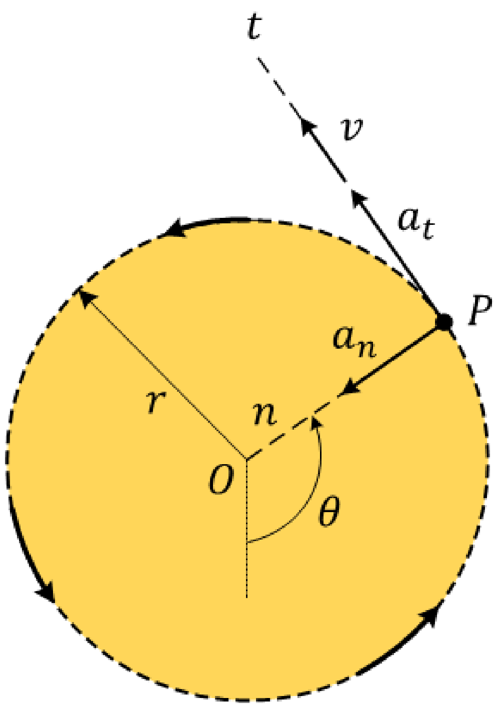


Curvilinear *n - t* coordinate system

n/t coordinates are not normally used to define position, but are commonly used to define acceleration.

We know

$$\boldsymbol{v} = v\boldsymbol{e}_t$$



Acceleration becomes $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d(v\boldsymbol{e}_t)}{dt} = \dot{v}\boldsymbol{e}_t + v\dot{\boldsymbol{e}}_t$

Since both v and $\mathbf{e}_t$ vary with time.

It can be shown that $\dot{\boldsymbol{e}}_t = \frac{v}{\rho}\boldsymbol{e}_n$

where ρ is the instantaneous centre of curvature.

Thus $v\dot{\boldsymbol{e}}_t = \frac{v^2}{\rho}\boldsymbol{e}_n$

Curvilinear *n* - *t* coordinate system

$$\boldsymbol{a} = \dot{v}\boldsymbol{e}_t + \frac{v^2}{\rho}\boldsymbol{e}_n = a_t\boldsymbol{e}_t + a_n\boldsymbol{e}_n$$

$a_t = \dot{v}$ is the acceleration in the tangential direction

$a_n = \dfrac{v^2}{\rho}$ is the acceleration in the normal direction

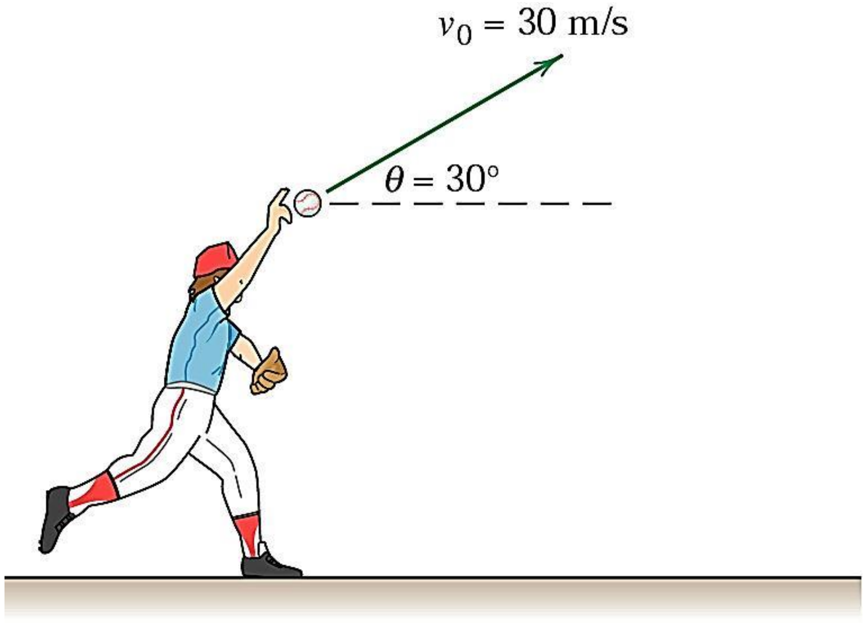
The magnitude of acceleration is $|\boldsymbol{a}| = a = \sqrt{a_n^2 + a_t^2}$

Note: $|\boldsymbol{a}| = a = \sqrt{a_x^2 + a_y^2} = \sqrt{a_n^2 + a_t^2}$

The acceleration does NOT change using a different coordinate system.

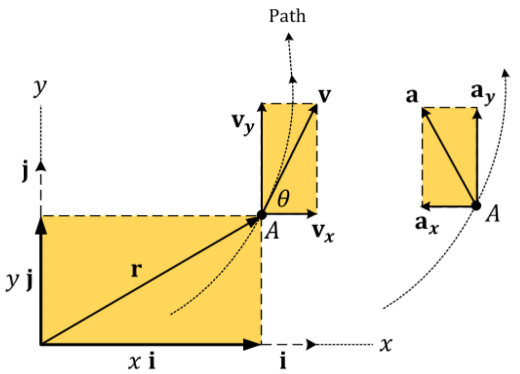
Example 8: Curvilinear *n/t*

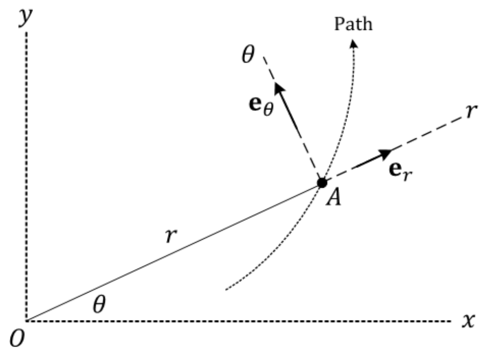
A baseball player releases a ball with the initial conditions shown. Determine the radius of curvature of the trajectory (a) just after release and (b) at the highest point of the ball. For each case, determine the time rate of change of the speed.

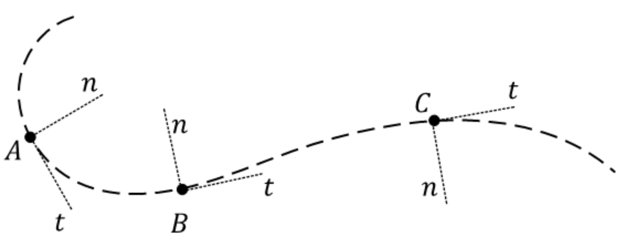


W7 Example 8 (Web view)

Summary – Curvilinear Kinematics







- Curvilinear x-y including projectile motion

$$v_x = (v_x)_o = v_o \cos\theta \qquad v_y = (v_y)_o + a_y t = v_o \sin\theta - gt$$

- Polar coordinates for circular motion

$$\mathbf{r} = r\mathbf{e}_r$$

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

$$\mathbf{a} = \ddot{r}\mathbf{e}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta + \dot{r}\dot{\theta}\,\mathbf{e}_\theta + r\ddot{\theta}\,\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e}_r$$

- Curvlinear n-t coordinates for paths

$$\boldsymbol{v} = v\boldsymbol{e}_t$$

$$\boldsymbol{a} = \dot{v}\boldsymbol{e}_t + \frac{v^2}{\rho}\boldsymbol{e}_n = a_t\boldsymbol{e}_t + a_n\boldsymbol{e}_n$$

Next Week:

Particle Kinetics