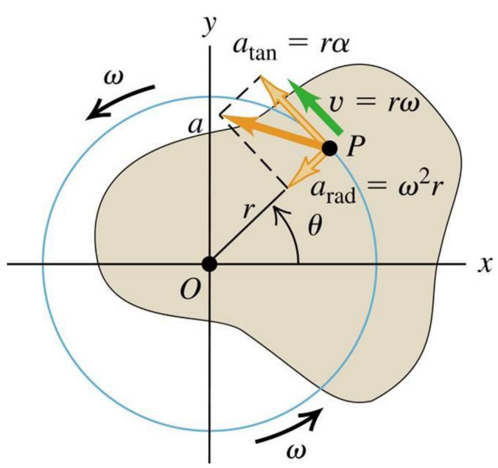
School of Mechanical and Manufacturing Engineering

MMAN1300 Engineering Mechanics 1

Dr. David C. Kellermann



Week 10- Rigid Body Kinematics

KINEMATICS OF RIGID BODIES

- Rotation of rigid bodies
- Angular displacement
- Angular velocity and acceleration

INSTANT CENTRES

- Instant Centres of rotation
- Relative velocity analysis

Kinematics of Rigid Bodies

- An object is a particle if it can be modeled as a single point.

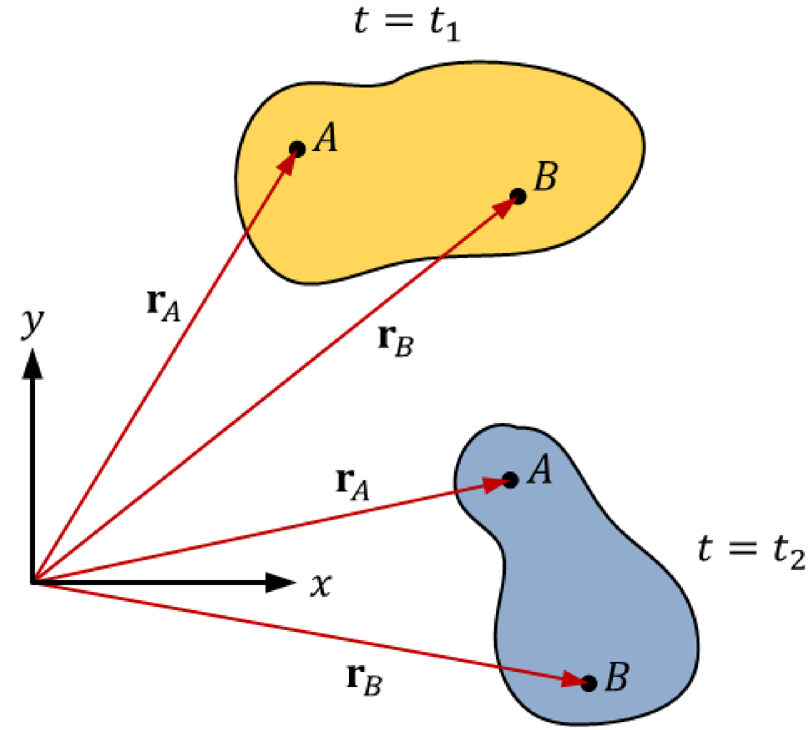
- An object is a rigid body if its size and shape are important.

We already know a bit about rigid bodies

- Rotation about a fixed axis can be treated analogously to rectilinear motion (as there is only one coordinate)
- This wind turbine is a good example
  - In particles, we didn't care about rotations
  - We can't describe the motion of this object without rotations

What are rigid bodies?

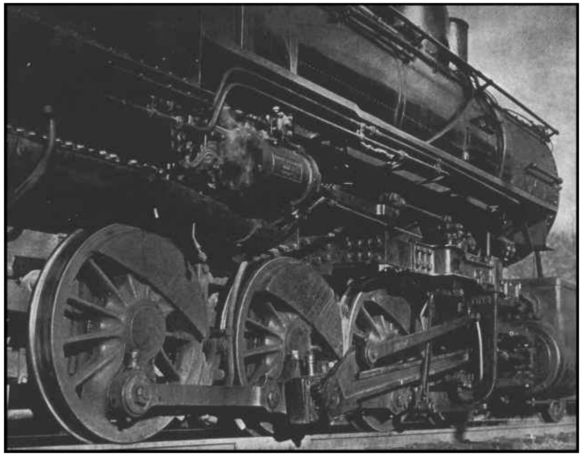
- A body is a collection of material points (particles)
- In a rigid body, the distance between *any* two particles in the body (*i.e.*, A and B at right) is constant
- $|\boldsymbol{r}_A - \boldsymbol{r}_B| = constant$



The Rigid Body Assumption

When do we make the assumption?

- When orientation and position are important
- When deflections of the body are small compared with its displacements



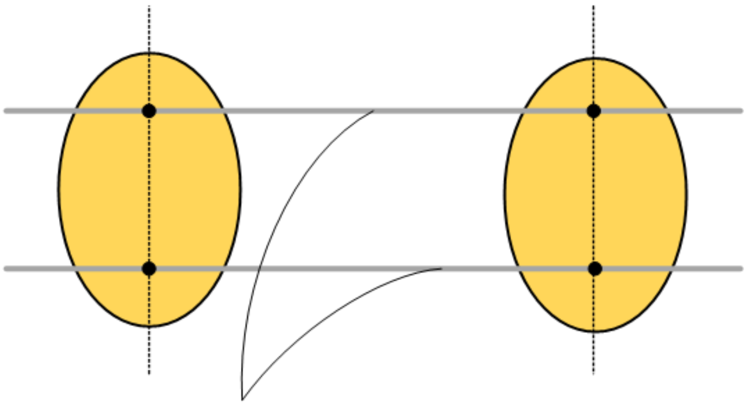
Planar Rigid Body Motion

- We are concerned in this course with Planar Rigid Body Motion
- When all the particles of a rigid body move along paths which are equidistant from a fixed plane, the body is said to undergo *planar motion*
  - Note: I mean that each point on the path is equidistant from a plane, not each path is equidistant from a plane
- There are 3 types of planar motion for a rigid body.

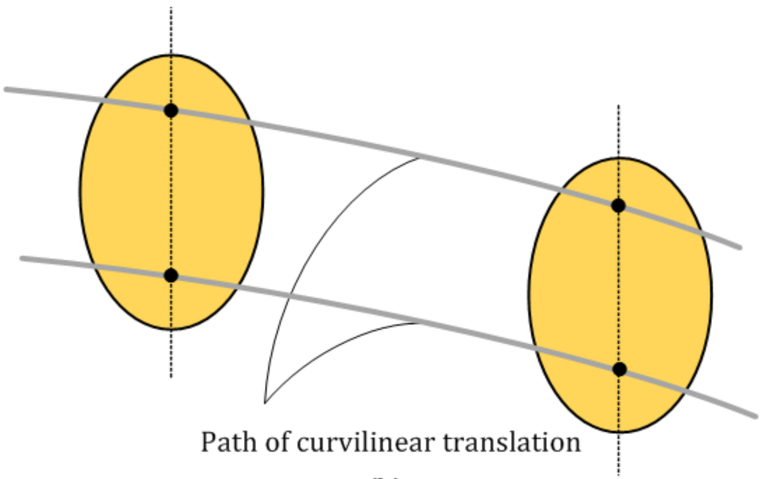
1. Translation
2. Rotation about a fixed axis
3. General plane motion (this is a combination of translation and rotation)

What do we mean by Translation?

- Translation occurs if every line segment on the body remains parallel to its original direction during the motion
- *Rectilinear translation* occurs when the paths of motion for any two particles of the body are along equidistant straight lines
- *Curvilinear translation* occurs when the paths of motion are along curved lines which are equidistant



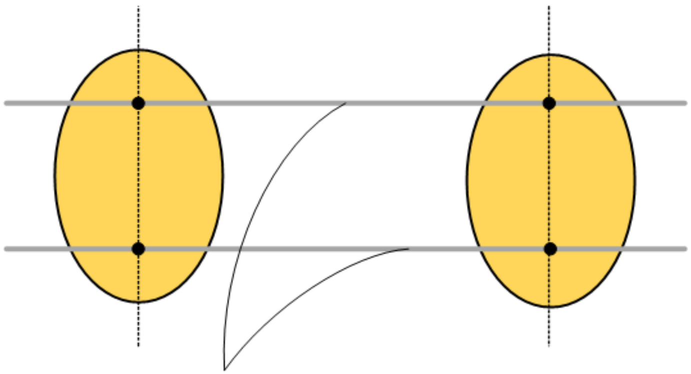
(a)



(b)

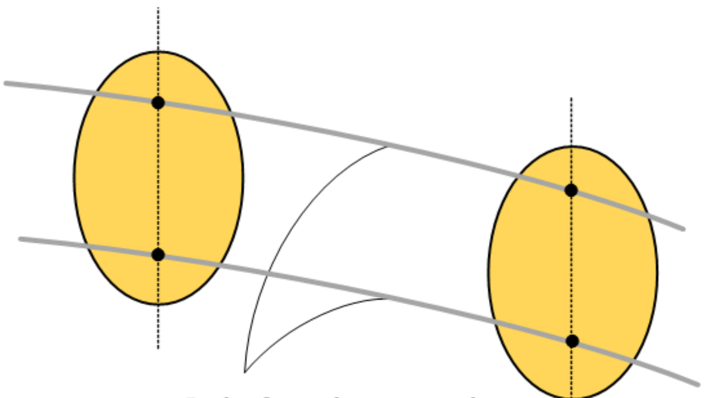
A body can translate around a curved path

Note: the motion is still translation if the body does not rotate or twist or turn



Path of rectilinear translation

(a)

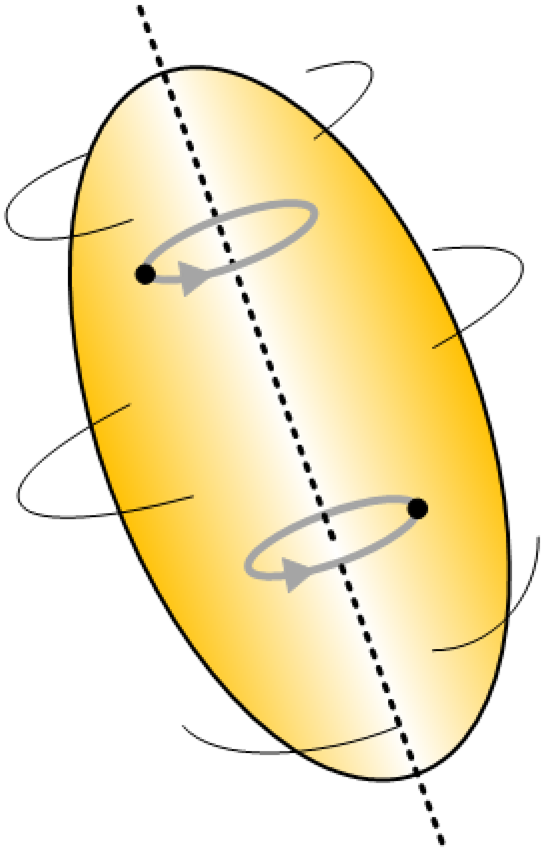


Path of curvilinear translation

(b)

Rotation about a fixed axis

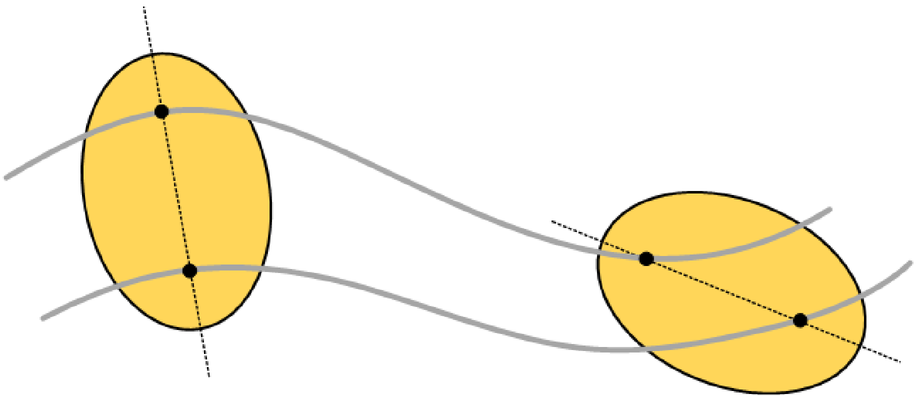
- When a rigid body rotates about a fixed axis, all the particles of the body, except those which lie on the axis of rotation, move along circular paths



Rotation about a fixed axis

(c)

General plane motion



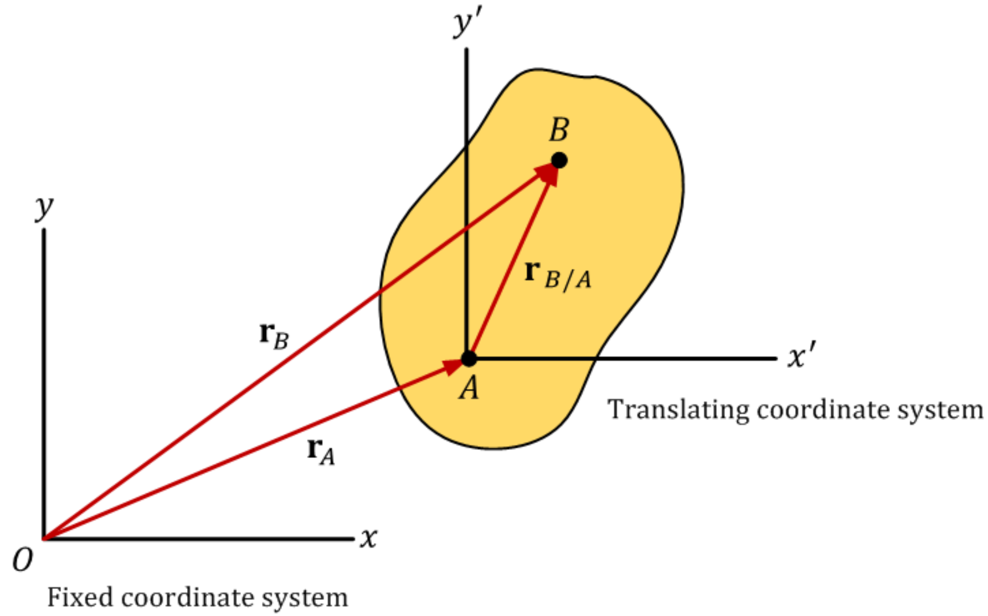
General plane motion

(d)

- When a body is subjected to general plane motion, it undergoes a combination of translation and rotation
- The translation occurs within a reference plane
- The rotation occurs about an axis perpendicular to the reference plane
- We'll use the principles we derived for relative motion to describe this case

Let's look at translation in more detail

- Consider a rigid body in translation in the $x - y$ plane
- The position of B with respect to A is denoted by the *relative-position vector* $\mathbf{r}_{B/A}$
- Hence: $\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$



Expression for velocity in translation

- Take the time derivative of $\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$ to get:

$$\mathbf{v}_B = \mathbf{v}_A$$

- Because $\mathbf{r}_{B/A}$ is constant
  - The magnitude is constant by the definition of rigidity
  - The direction is constant because there is no rotation

Expression for acceleration in translation

- Another time derivative gives

$$\mathbf{a}_B = \mathbf{a}_A$$

- The velocities and accelerations of every point in a translating body are the same

Recall rotation about a fixed axis

We already know how to treat rotation about a fixed axis

- Pure rotation is described by angular motion
- Angular displacement θ (rad)
- Angular velocity $\omega = d\theta/dt$ (rad/s)
- Angular acceleration $\alpha = d\omega/dt = d^2\theta/dt^2$ (rad/s^2)
- Angular motion can be treated in the same way as rectilinear motion

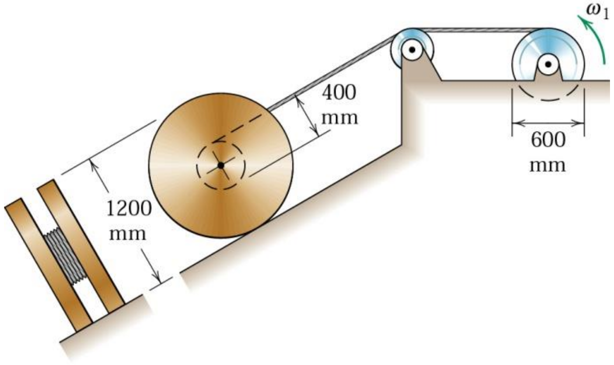
Translation & Rotation about a fixed axis…

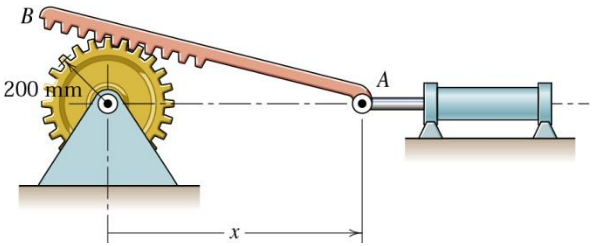
- Actually, we can already handle pure translational and pure rotational kinematics

- It’s the same as particle kinematics

- So what’s new? What are we missing?

General Plane Motion

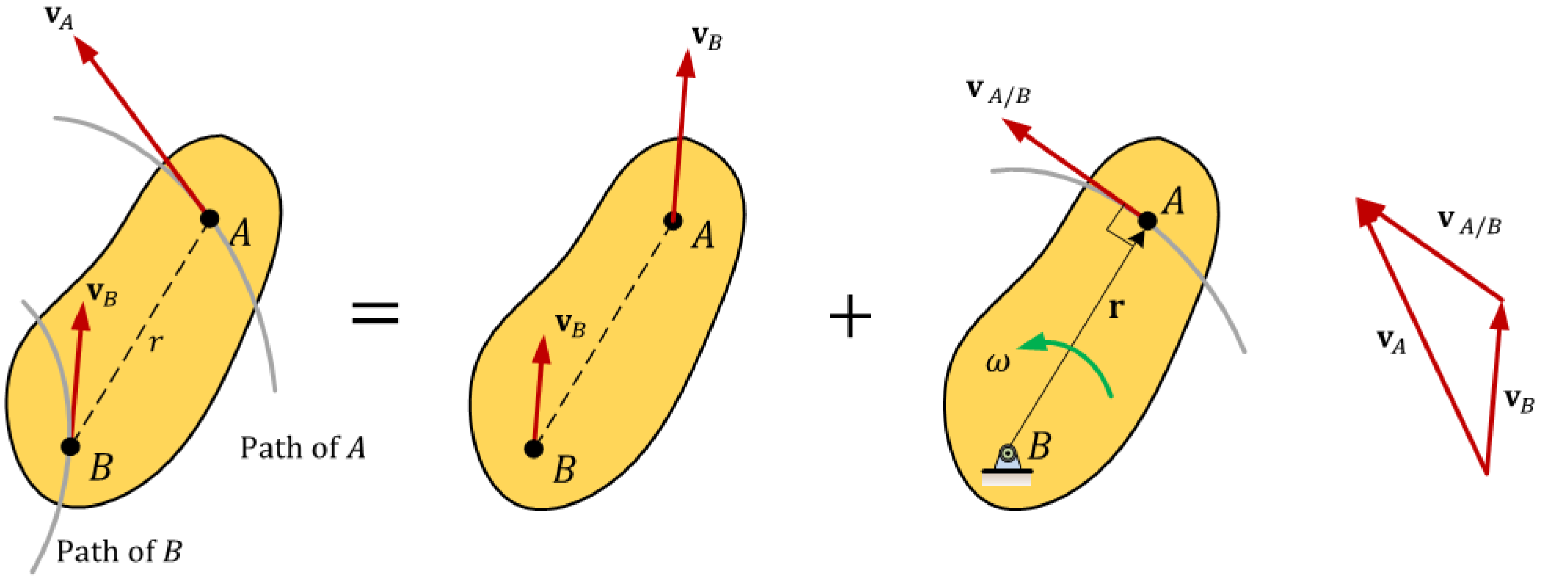




- We’ll use a relative motion approach
- An alternative approach that works for some simple problems is supplied in M&K(D) 5/3

The general plane motion of a rigid body

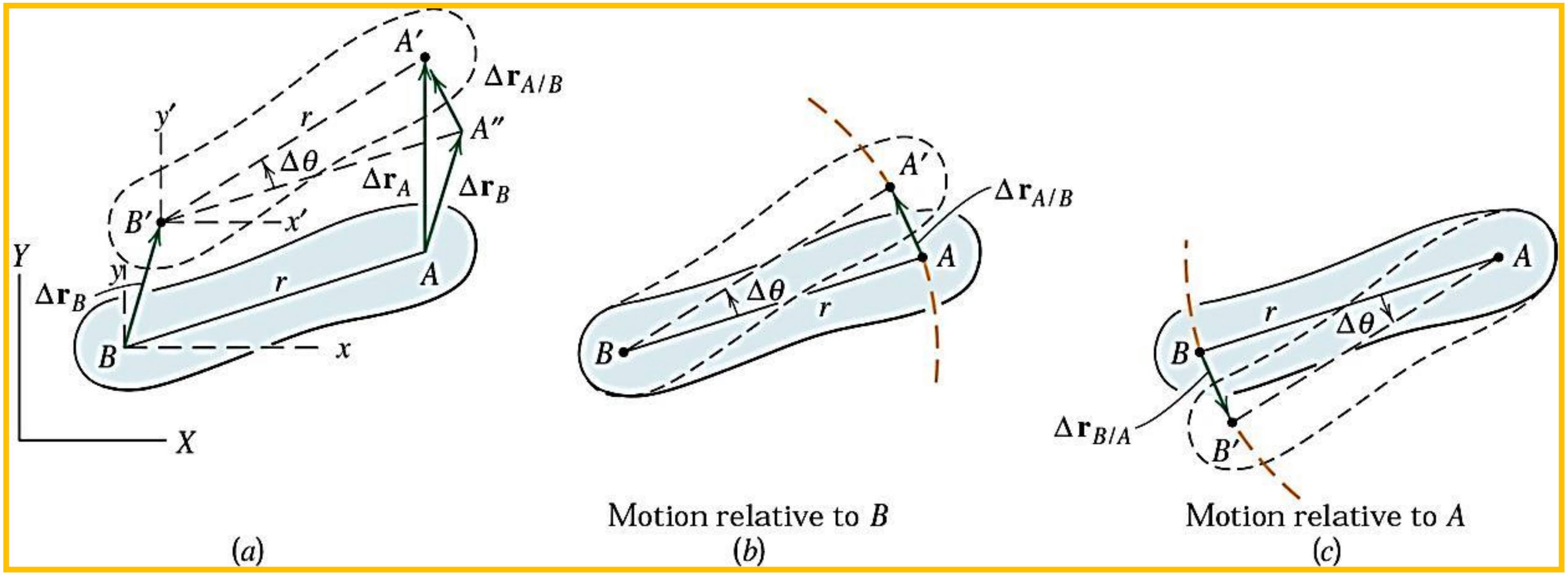
- The general plane motion of a rigid body can be described as a *combination* of translation and rotation



- To view these "component" motions *separately*, we use a *relative-motion analysis* involving two sets of coordinate axes

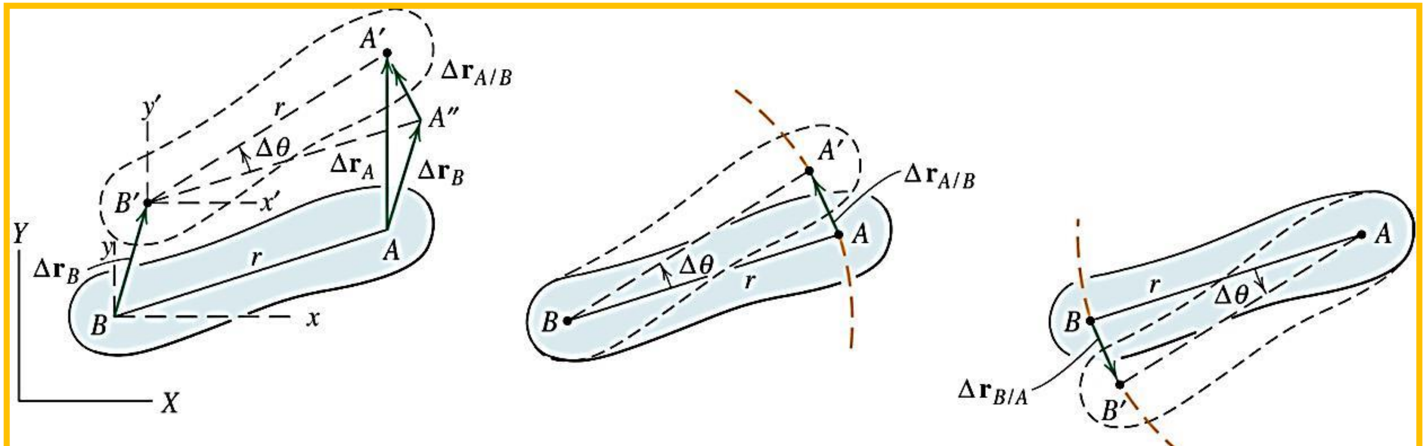
Relative velocity of 2 points on a rigid body

- From relative motion: $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$
- Now we choose A and B to be on the same rigid body
- The distance between A and B is <u>fixed</u> (by the assumption of rigidity), so the motion of one w.r.t. the other is circular



We can express relative velocity as circular motion

- The magnitude is given by $\mathbf{v}_{A/B} = \omega\mathbf{r}$
- The direction is tangent to the relative motion
  - That is to say perpendicular to $\mathbf{r}_{A/B}$
  - In the direction of the relative motion
  - $\mathrm{v}_{A/B} = \boldsymbol{\omega}\, x\, \mathbf{r}_{A/B}$ where $\boldsymbol{\omega} = \omega\mathbf{k}$ (right-hand rule)

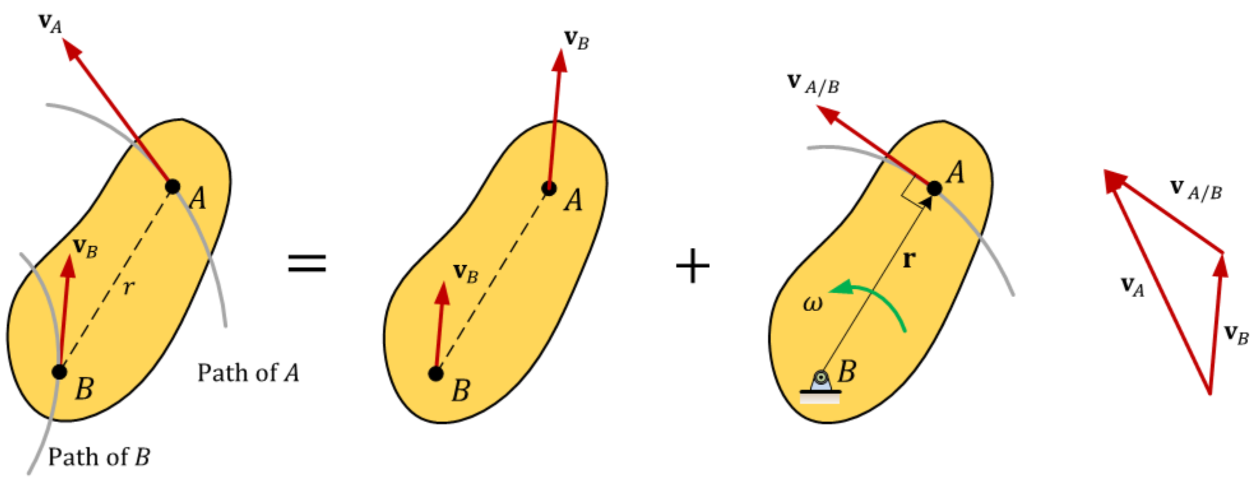


Motion relative to B

Motion relative to A

(a) (b) (c)

Relative Velocity



- The velocity of A is the *vector* sum of
  1. The translational portion $\mathbf{v}_B$
  2. The rotational portion $\mathbf{v}_{A/B} = \boldsymbol{\omega} \times \mathbf{r}$ which is perpendicular to the line between the two points
- Drawing a diagram is often helpful
- *Every* point on the body has the same angular velocity $\boldsymbol{\omega}$

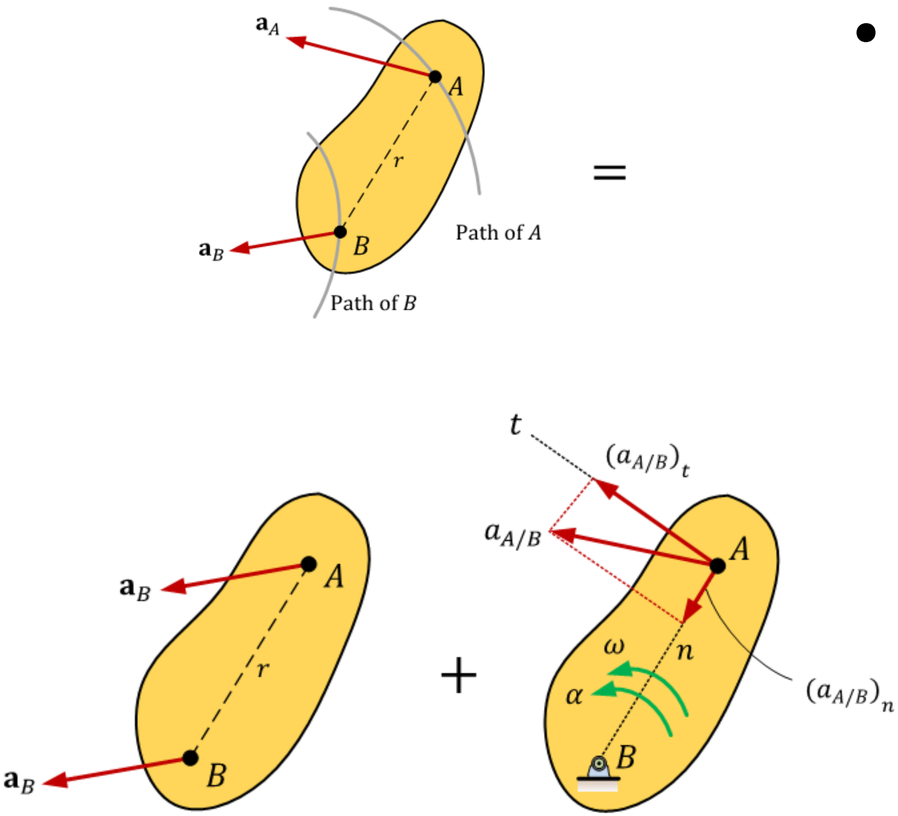
We can also examine the relative acceleration

- From relative motion, we have $\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$
- Once again, we note that particles A and B move in circles w.r.t. each other

- So combining relative motion and circular motion, we get

$$\mathbf{a}_{A/B} = \left(\mathbf{a}_{A/B}\right)_n + \left(\mathbf{a}_{A/B}\right)_t$$

Relative Acceleration

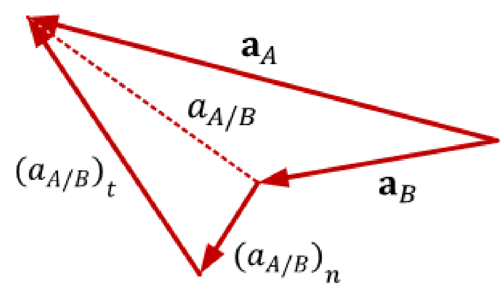


- The magnitudes of the relative acceleration components are given by

$$(\mathbf{a}_{A/B})_n = \boldsymbol{v}_{A/B}{}^2/r = \omega^2 r$$
$$(\mathbf{a}_{A/B})_t = \alpha r$$

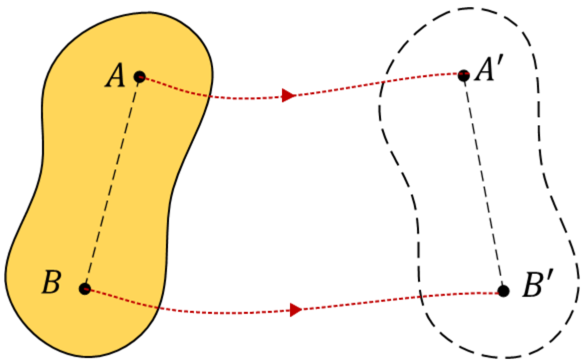
In vector form:

$$(\mathbf{a}_{A/B})_t = \boldsymbol{\alpha} \, x \, \mathbf{r}_{A/B}$$
$$(\mathbf{a}_{A/B})_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B}) = -\omega^2 \mathbf{r}_{A/B}$$



Relative Acceleration

Therefore

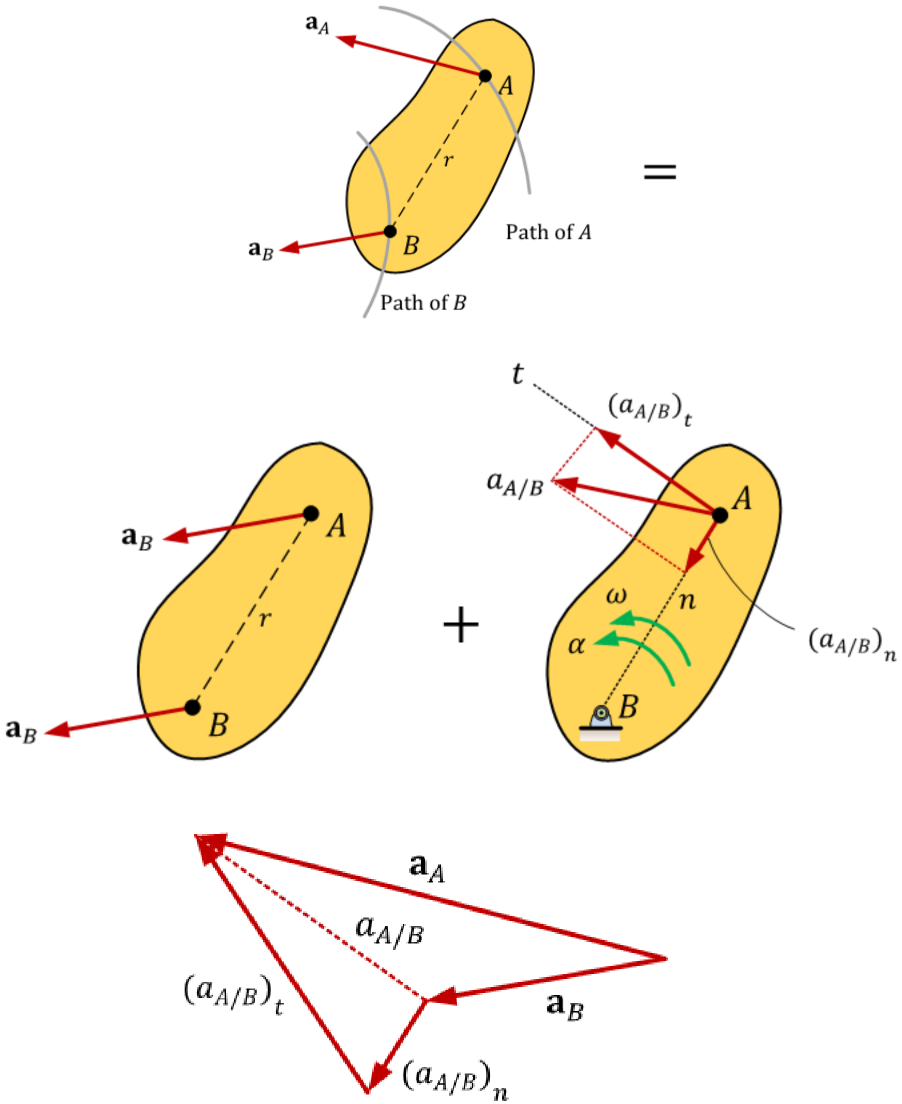


$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

$$= \vec{a}_A + (\vec{a}_{B/A_n} + \vec{a}_{B/A_t})$$

$$= \underbrace{\vec{a}_A}_{\text{Translation}} + (\underbrace{\omega^2 \overline{AB}\boldsymbol{e_n} + \alpha \overline{AB}\boldsymbol{e_t}}_{\text{Rotation}})$$

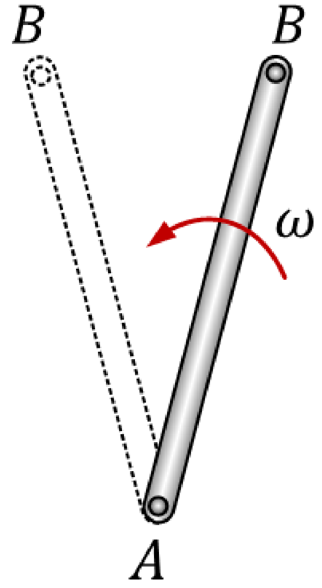
Every point on the body has the same α and ω

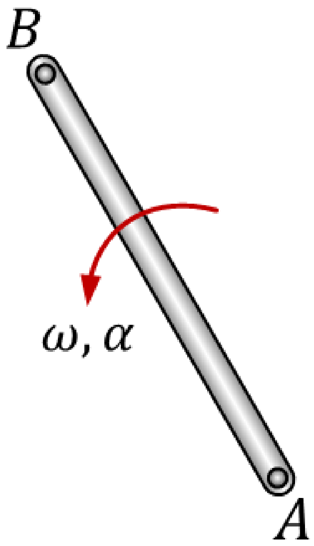


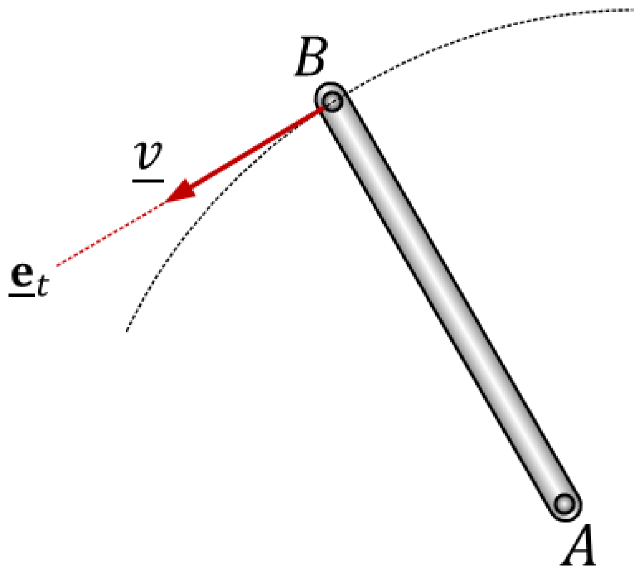
- α and ω are absolute quantities (not relative)
- Often a sketch is helpful
- It is sometimes helpful to pretend that the reference point is fixed when finding relative velocity and acceleration

Relative motion of rigid links

When examining the rotation of rigid links, and we want to examine the rotation of point B relative to point A, imagine A is fixed and B is rotating about A. Hence, we are dealing with circular motion.

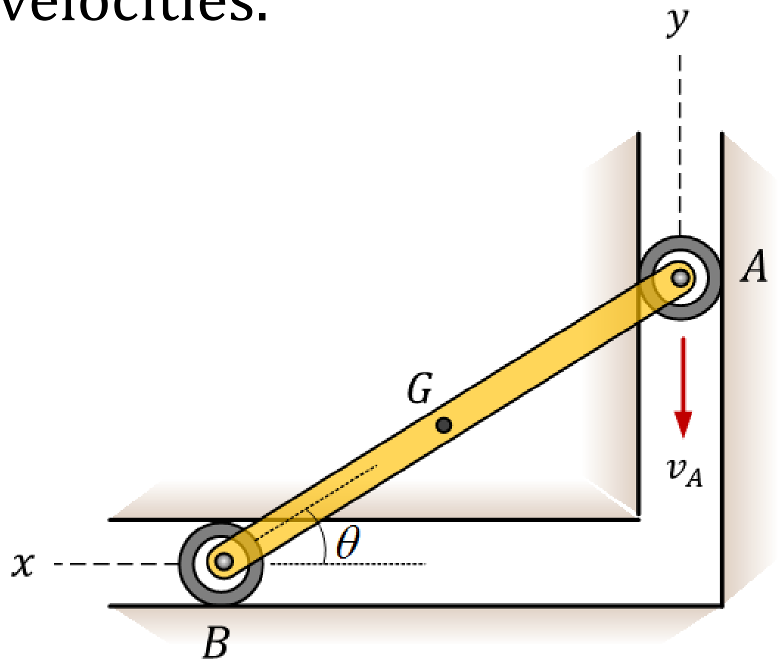






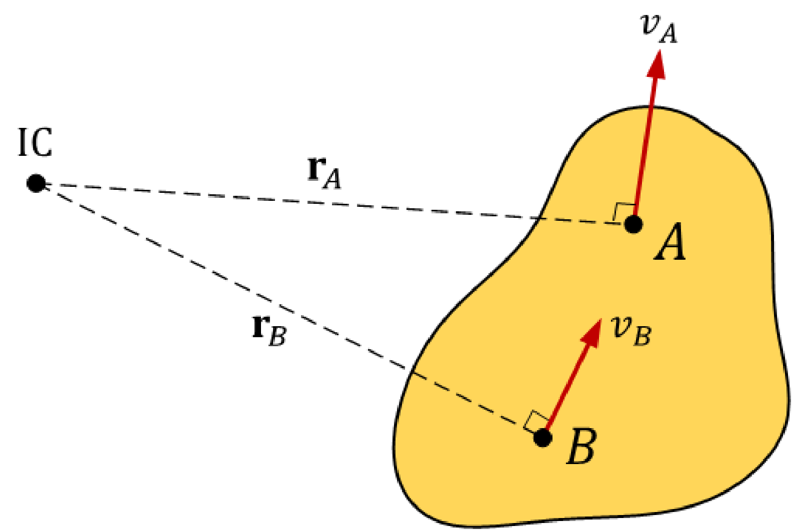
Example 1: Relative motion

A rigid link $\overline{AB}$ is 225 mm long and has a roller at each end. The rollers are constrained to move in the guides. The end A has a constant velocity of 2.2 m/s in the direction shown. At the instant when $\theta = 35°$, find the angular velocity of $\overline{AB}$ using relative velocities.



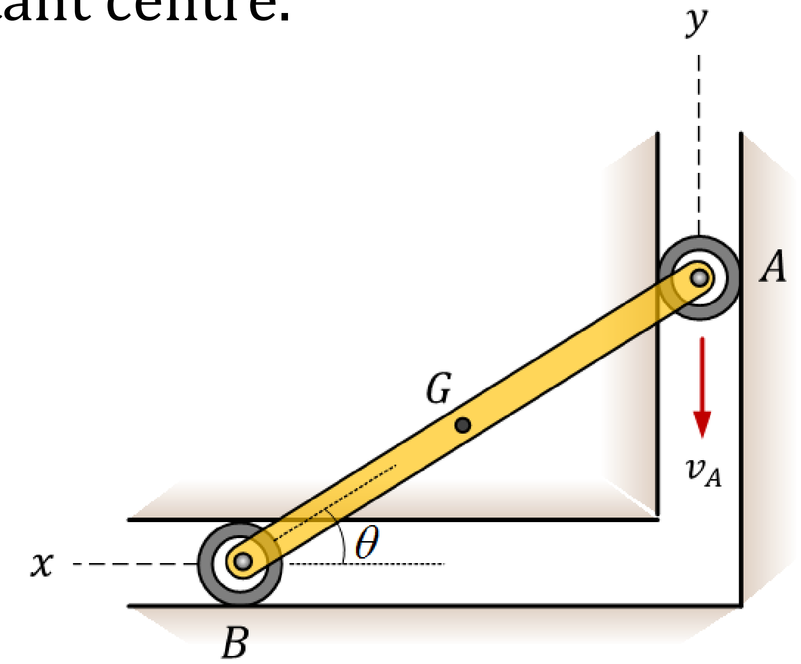
Instant Centres

Every kind of motion involving angular change has an *Instantaneous Centre*



Example 2: Instant Centre

A rigid link $\overline{AB}$ is 225 mm long and has a roller at each end. The rollers are constrained to move in the guides. The end A has a constant velocity of 2.2 m/s in the direction shown. At the instant when $\theta = 35°$, find the angular velocity of $\overline{AB}$ using the instant centre.



Instant Centres in rigid body kinematics

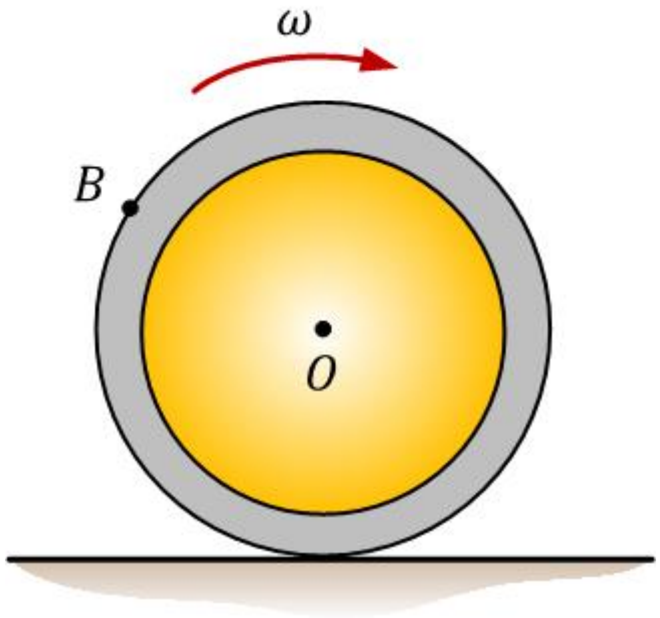
- The velocity of any point B located on a rigid body can be obtained in a very direct way if one chooses the base point A to be a point that has *zero velocity* at the instant considered

- If $\boldsymbol{v}_A = 0, \boldsymbol{v}_B$ has magnitude $v_B = \omega r_{B/A}$ and is in a direction perpendicular to the line from A to B
  ($\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/A}$)

- Point A is called the *instantaneous centre of zero velocity* (IC) and it lies on the *instantaneous axis of zero velocity*

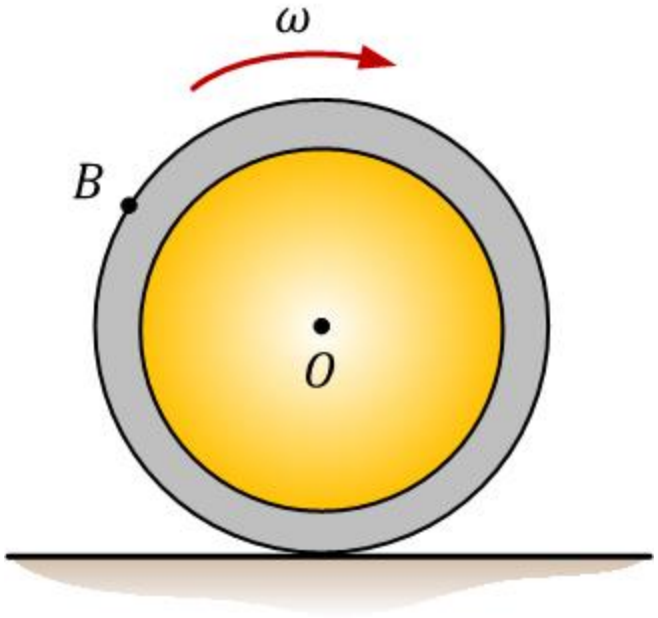
Consider the wheel as shown

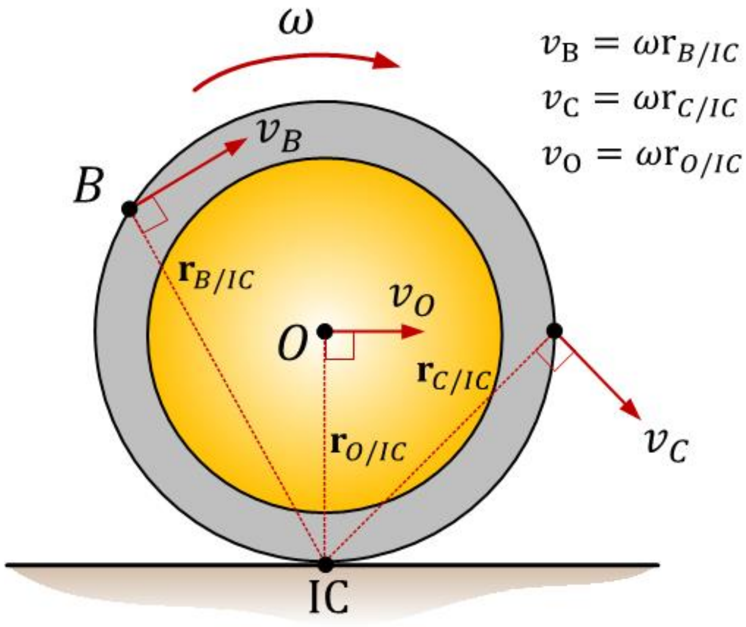
- If it rolls **without slipping**, then the point of contact with the ground has **zero velocity**
- Hence this point represents the IC (Instant Centre) for the wheel



Instant Centre of a Wheel

- **Recall, at this instant the point labelled *IC* below has zero velocity**
- If it is imagined that the wheel is momentarily pinned at this point, the velocities of points *B, C, O* and so on, can be found using $v = \omega r$



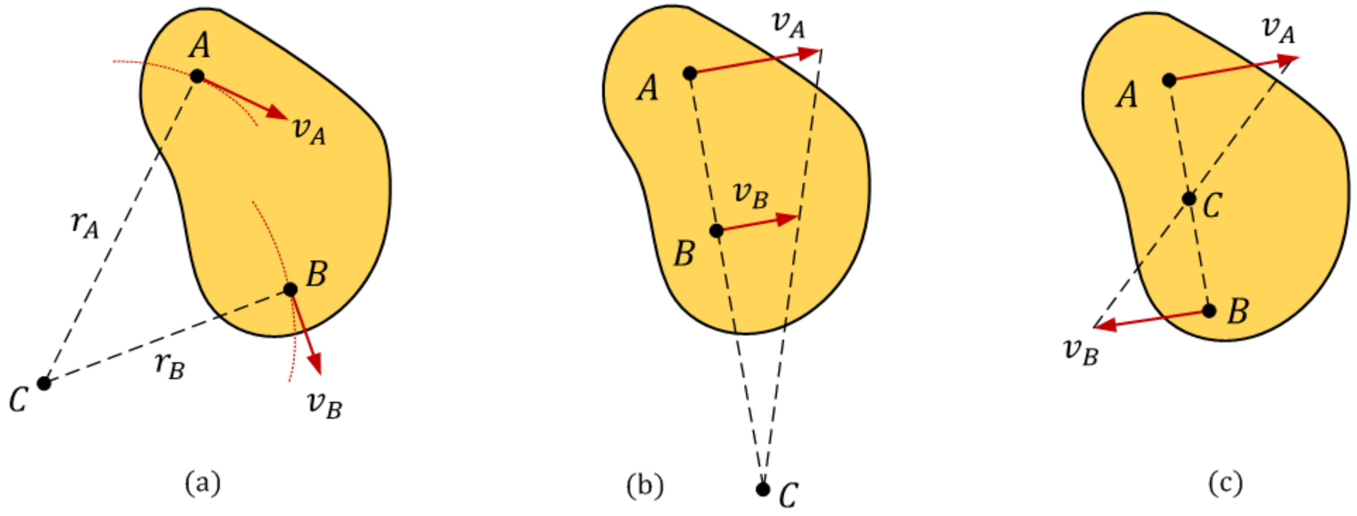


Is there always such a point?

- YES!
  - Though it may not actually lie on the body
- So how do we find it?

How do we find *IC*?

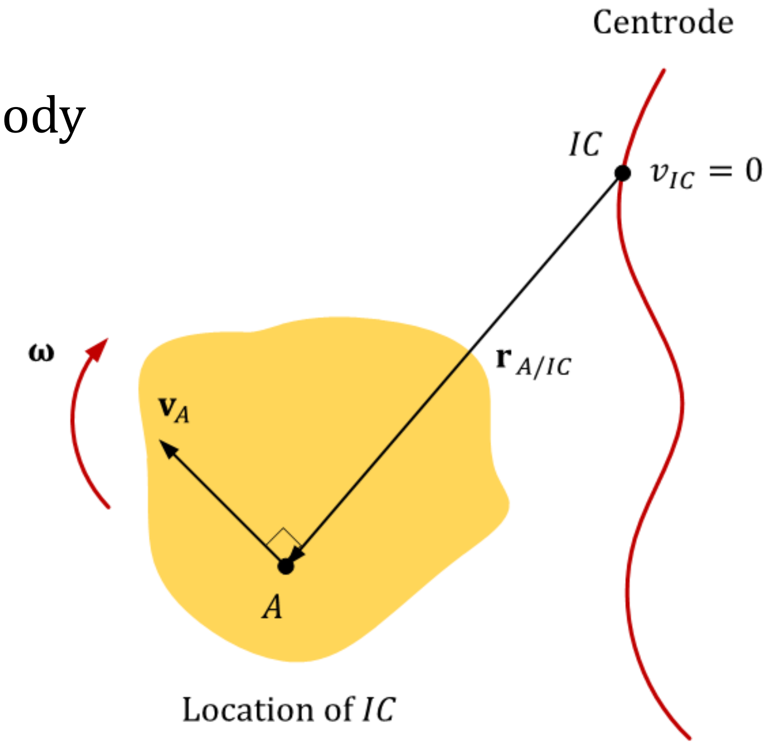
- To locate the *IC*, we use the fact that the **absolute velocity** of a point on the body is **always perpendicular** to the **relative-position vector** extending from the *IC* to the point
- Several possibilities exist:



Velocity and Angular Acceleration known:

Case 1:
Given the velocity $\mathbf{v}_A$ of a point A on the body and the angular velocity ω of the body

- In this case, the IC is located along the line drawn perpendicular to $\mathbf{v}_A$ at A, such that the distance from A to the IC is $r_{A/IC} = v_A/\omega$
- Note that the IC lies up and to the right of A since $\mathbf{v}_A$ must cause a clockwise angular velocity ω about the IC

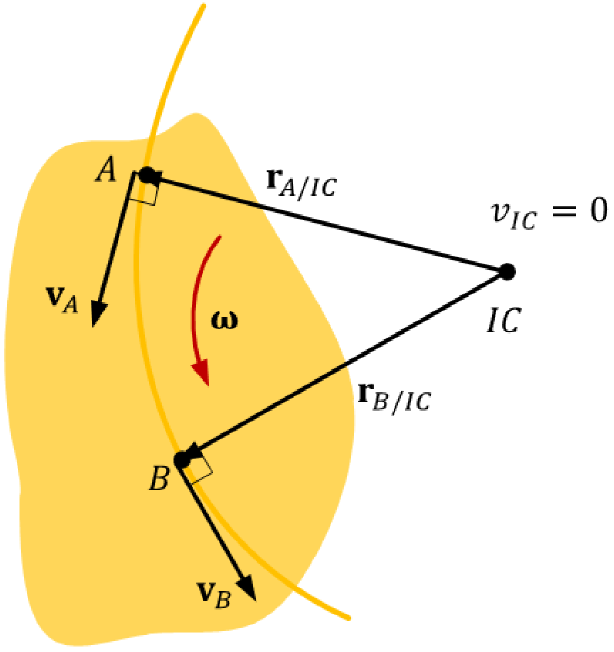


Location of IC
Knowing $\mathbf{v}_A$ and $\boldsymbol{\omega}$

Two Velocities Known:

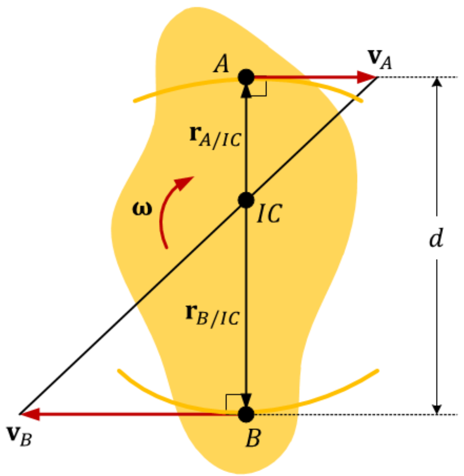
Case 2:
Given the line of action of two nonparallel velocities $\mathbf{v}_A$ and $\mathbf{v}_B$

- Construct at points A and B line segments that are perpendicular to $\mathbf{v}_A$ and $\mathbf{v}_B$
- Extending these perpendicular to their point of intersection as shown locates the IC at the instant considered



Location of IC
Knowing the lines of action of $\mathbf{v}_A$ and $\mathbf{v}_B$

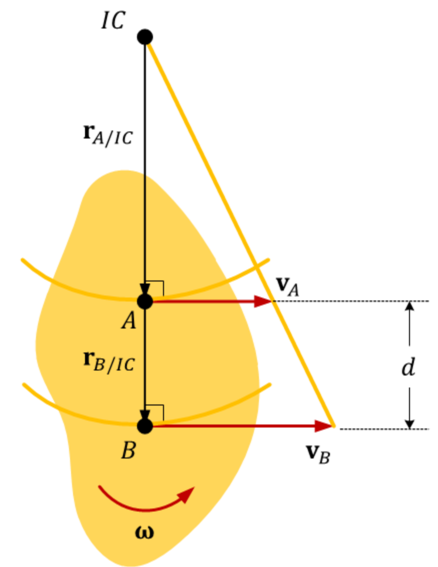
Parallel Velocities



Case 3:
Given the magnitude and direction of two parallel velocities $\mathbf{v}_A$ and $\mathbf{v}_B$

- Here the location of the IC is determined by proportional triangles

- The magnitude of the velocity increases linearly with distance from the instant centre



Location of IC
Knowing $\mathbf{v}_A$ and $\mathbf{v}_B$

Notes on Instant Centres

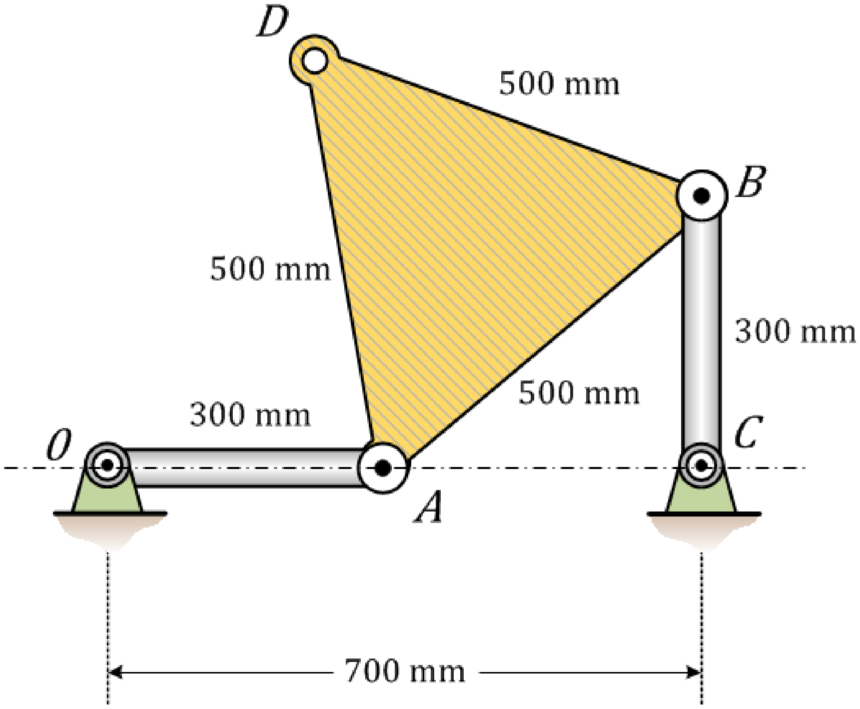
- If the body is translating, then $\mathbf{v}_A = \mathbf{v}_B$, and the IC would be located at infinity
- The point determined as the instantaneous center of zero velocity for the body can only be used for an instant of time
  - Because the body changes its position from one instant to the next
  - The locus of ICs in space is the space centrode and the locus of ICs on the body is the body centrode

Do not use the *IC* to find any accelerations

- Although the *IC* may be used to determine the velocity of any point in a body, it generally does not have zero acceleration
- Therefore it should not be used for finding the accelerations of points on a body
- There is no instantaneous centre of zero acceleration (in general)

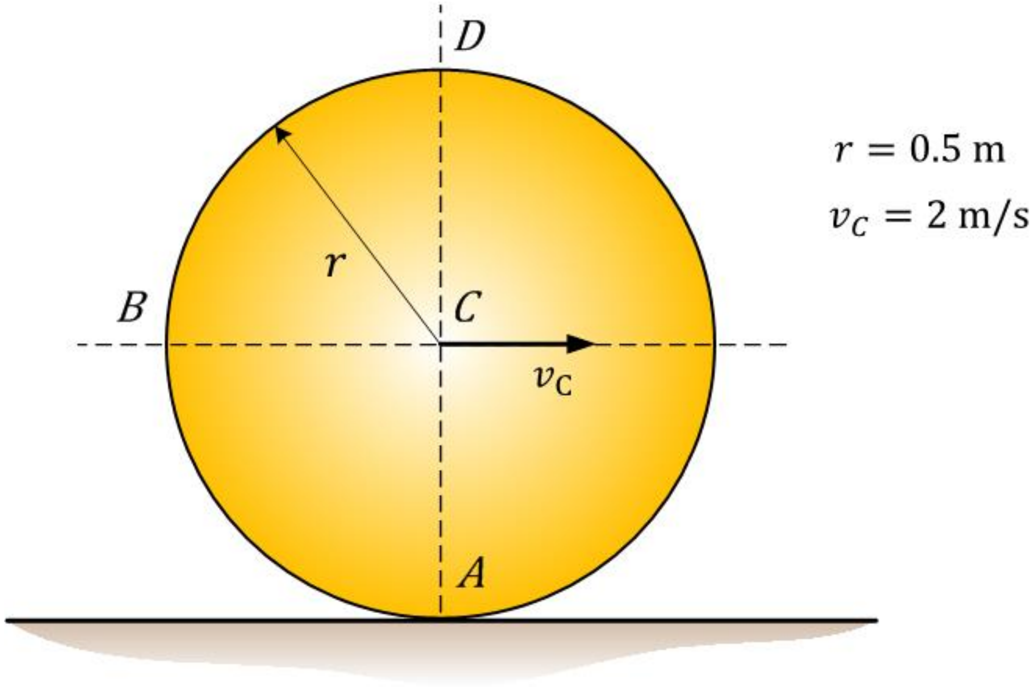
Example 3

At the instant shown, the triangular plate ABD has a clockwise angular velocity of 3 rad/s. For this instant determine the angular velocity of link BC



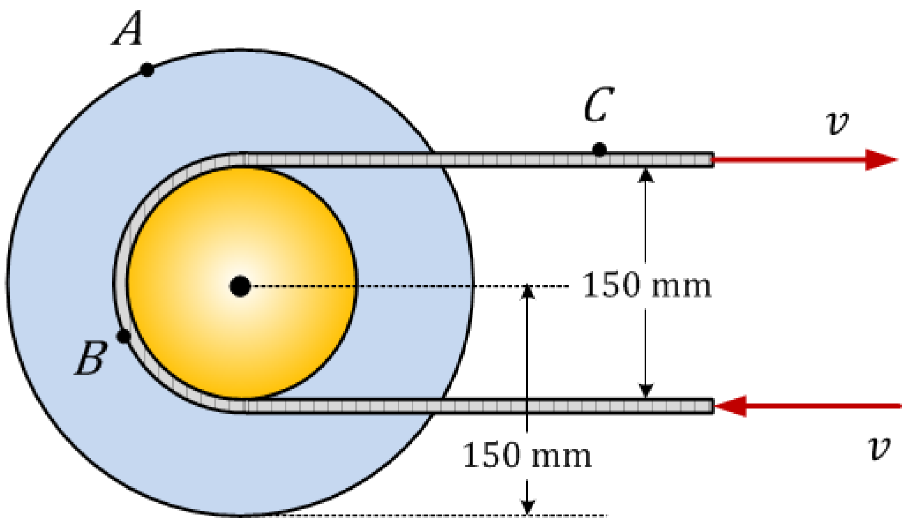
Example 4

A ball rolls without slipping on flat surface. What are the velocities of points *A*, *B* and *D*?

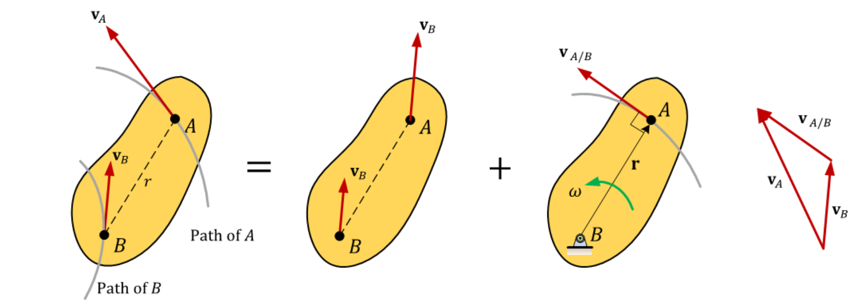


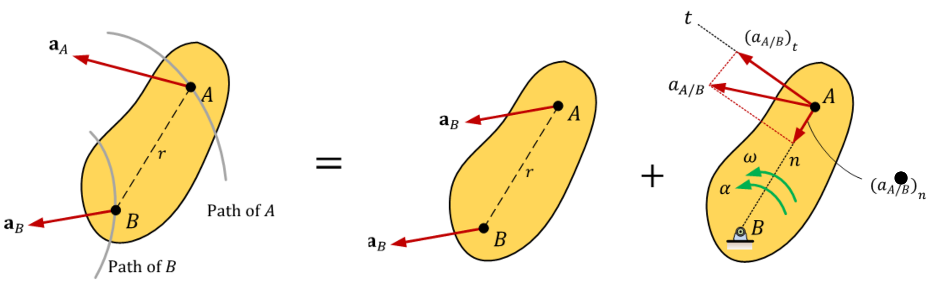
Example 5

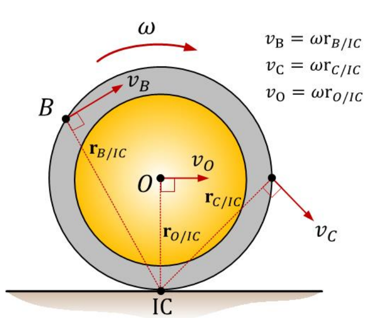
The belt-driven pulley and attached disk are rotating with increasing angular velocity. At a certain instant the speed of the belt is 1.5 m/s and the total acceleration of point A is 75 m/s^2. For this instant determine (a) the angular acceleration of the pulley and disk, (b) the total acceleration of point B and (c) the acceleration of point C on the belt.



Summary of Rigid Body Kinematics







- In a rigid body, the distance between *any* two particles in the body is constant

$$|\boldsymbol{r}_A - \boldsymbol{r}_B| = constant$$

- The velocity of A is the *vector* sum of the translational portion $\mathbf{v}_B$ and the rotational portion $\mathbf{v}_{A/B} = \boldsymbol{\omega} \times \mathbf{r}$

- From relative motion, the acceleration is $\mathbf{a}_A = \mathbf{a}_B + \left(\mathbf{a}_{A/B}\right)_n + \left(\mathbf{a}_{A/B}\right)_t$, where

$$(\mathbf{a}_{A/B})_n = \boldsymbol{v}_{A/B}{}^2/r = \omega^2 r$$

$$(\mathbf{a}_{A/B})_t = \alpha r$$

- The method of instant centres is very efficient for solving velocity and angular acceleration of complex motion

Next Topic:

Rigid Body Kinetics