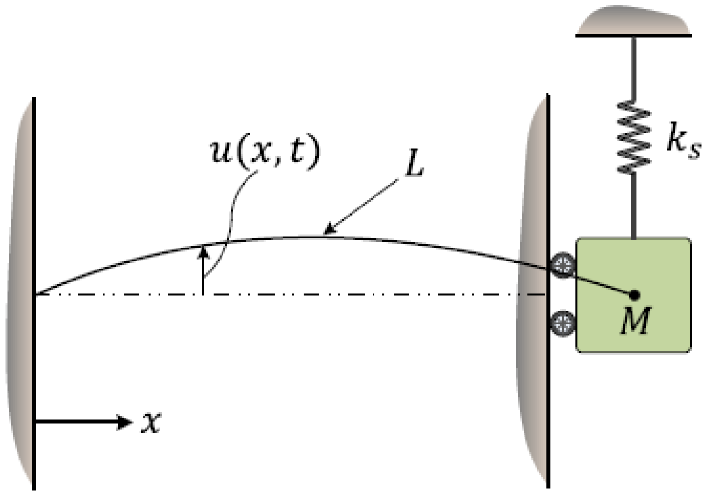
**QUESTION**

A cord of length $L$ and mass per unit length $m_L$ is under tension $T$ with the left end fixed. The right-hand end is attached to a spring-mass system as shown in Figure Q2. In the static equilibrium condition, the ends of the cord are along the centreline as shown. The spring-mass system is constrained to move in the vertical direction. Assume small slopes.

(i) Draw a free body diagram of the mass and show that the boundary condition at $x = L$ is

$$T\frac{du}{dx} - k_s u - M\frac{d^2u}{dt^2} = 0$$

(ii) Derive the natural frequency equation of the system.



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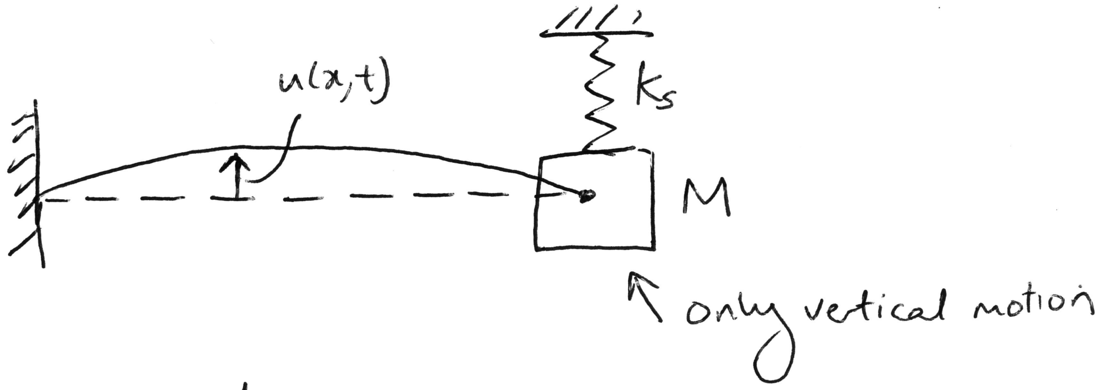
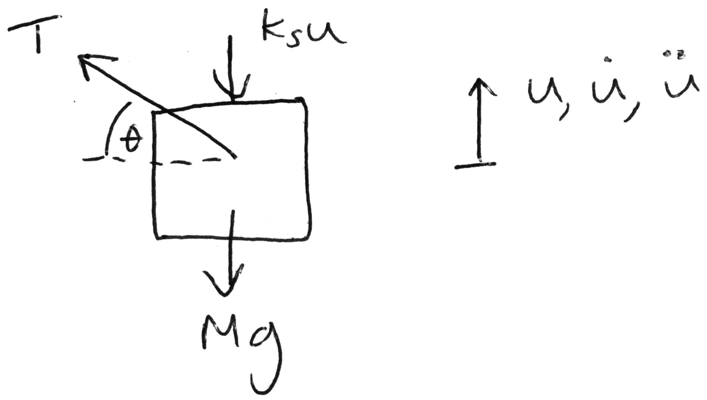



FBD of mass M




Take $\Sigma F$ in the vertical direction

$$T\sin\theta - k_s u = M\ddot{u}$$

(Note: Mg is cancelled with the static spring deflection)

For small $\theta$, $\sin\theta \approx \theta$

Also, $\theta = \frac{du}{dx}$ is the slope

The boundary condition at $x = L$ becomes

$$T\frac{du}{dx} - k_s u - M\frac{d^2u}{dt^2} = 0$$

(2)

General solution for the transverse displacement of a string

$$u(x,t) = (A\sin\omega_n t + B\cos\omega_n t)(C\sin kx + D\cos kx)$$

Fixed at $x = 0$ $\Rightarrow$ $u(0,t) = 0$

$$u(0,t) = (A\sin\omega_n t + B\cos\omega_n t)D = 0$$

$$\Rightarrow D = 0$$

$$\Rightarrow u(x,t) = (A\sin\omega_n t + B\cos\omega_n t)C\sin kx$$

At $x = L$

$$T\frac{du}{dx} - k_s u - M\frac{d^2u}{dt^2} = 0$$

$$\frac{du}{dx} = (A\sin\omega_n t + B\cos\omega_n t)kC\cos kx$$

$$\frac{d^2u}{dt^2} = -\omega_n^2(A\sin\omega_n t + B\cos\omega_n t)C\sin kx$$

$$\Rightarrow Tk\cos kL - k_s\sin kL + \omega_n^2 M\sin kL = 0$$

3)

Group the sin and cos terms

$$\sin kL \left( k_s - M\omega_n^2 \right) = \cos kL \,(Tk)$$

$$\Rightarrow \quad \tan kL = \frac{Tk}{k_s - M\omega_n^2}$$

Since $k = \frac{\omega_n}{c_s} \Rightarrow \omega_n^2 = k^2 c_s^2 = k^2 \frac{T}{M_L}$

where $c_s$ is the wavespeed

$$\Rightarrow \quad \tan kL = \frac{Tk}{k_s - M\frac{T}{M_L}k^2}$$

Rearrange and multiply the RHS by $L^2/L^2$

$$\tan kL = \frac{T M_L L (kL)}{k_s M_L L^2 - TM(kL)^2}$$

is the transcendental natural frequency equation of the string