MMAN2300

Engineering Mechanics 2

Part B: Vibration Analysis

Introduction to mechanical vibration

Single degree-of-freedom spring-mass-damper systems

Introduction to Mechanical Vibration

Spring-Mass-Damper Systems

The study of mechanical vibration is concerned with the oscillatory motions of bodies and the forces associated with them. All bodies possessing mass and elasticity are capable of vibration. Most engineering machines and structures experience some sort of vibration, and hence their design generally requires consideration of their oscillatory behaviour.

What is Vibration?

Vibration is mechanical oscillation about a reference position (generally called an equilibrium position). Vibration is an everyday phenomenon - we meet it in our homes, during transport and at work. Vibration is often a destructive and annoying side effect of a useful process, but is sometimes generated intentionally to perform a task.

Vibration is a result of dynamic forces in machines which have moving parts, and in structures which are connected to the machine. Different parts of the machine will vibrate with various frequencies and amplitudes. Vibration causes wear and fatigue. It is often responsible for the ultimate breakdown of the machine.

Vibration is now a very important factor in design, both from the point of view of the vibration themselves, and also because of noise which most often originates from mechanical vibrations (eg. noise from engines).

Vibration engineering is one of the most rapidly developing areas of mechanical engineering, because of two main developments:

1. Development of computer-based methods of analysis, eg. using finite element methods.
2. Development of experimental techniques which can be used to verify analytical models.

Some Classifications of Vibration   Section 1.5 Rao

Oscillatory systems can be broadly classified as **linear** or **non-linear**. For linear systems, the principle of superposition is used, which involves adding together the vibration generated by all the components of the system to obtain the total vibration. Mathematical techniques are available for most linear systems. Non-linear systems are very difficult to mathematically analyse. All systems tend to become non-linear with increasing amplitude of oscillation.

Vibrations may either be **free** or **forced**. Free vibration occurs when a system oscillates in the absence of external forces. In free vibration, the system will vibrate at one or more of its natural (resonance) frequencies. The resonance frequencies of a system are a property of the dynamic system, and are established by the mass and stiffness distribution within the system.

Forced vibration occurs when external forces are used to excite a system. When the excitation is oscillatory, the system is forced to vibrate at the excitation frequency. If the frequency of excitation coincides with one of the natural frequencies of the system, the system will vibrate at one of its resonances, and dangerously large amplitudes of vibration will result. In the past, the failure of major structures such as bridges, buildings, and even aircraft wings have occurred due to the system vibrating at one of its resonance frequencies. Hence, the calculation of the natural frequencies of a system is of major importance in the study of vibrations.

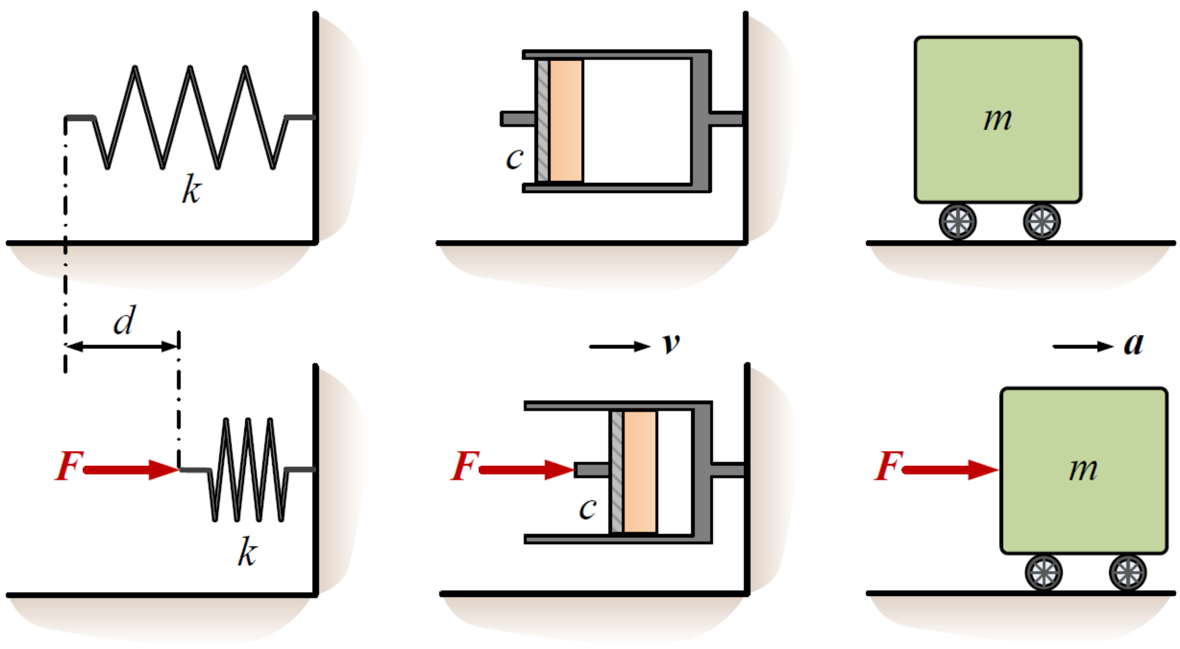
All vibrating systems are subject to **damping** to some degree, because vibrational energy is dissipated by friction and other resistances. If the damping of a system is small, it has very little influence on the natural frequencies of the system, and hence damping can be neglected in the calculations for the natural frequencies.

The number of independent co-ordinates required to describe the motion of a system is called the **degrees of freedom** of the system. For example, a free particle undergoing general motion in space will have three degrees of freedom (three components of translation), and a rigid body will have six degrees of freedom (that is, three components of translation and three components of rotation).

For spring-mass-damper systems, we are concerned with the motion of the <u>mass</u>. A single (rigid body) mass vibrating in one direction has a single degree-of-freedom (DOF). A single DOF system will have a single natural frequency (ω_n). In a multiple DOF system (eg. several vibrating masses joined via springs), or an N-DOF system, there are N number of coordinates required to describe the motion of the masses. An N-DOF system has N number of natural frequencies. For each ω_n, there is a corresponding mode of vibration (modeshape). The simplest N-DOF system is a two DOF system.

Spring, Mass, Damper Elements

All mechanical systems contain the three basic components: spring, damper and mass. When each of these in turn is exposed to a constant force they react with a constant displacement, a constant velocity and a constant acceleration respectively.



$F = kx$ $\quad$ $F = cv$ $\quad$ $F = ma$

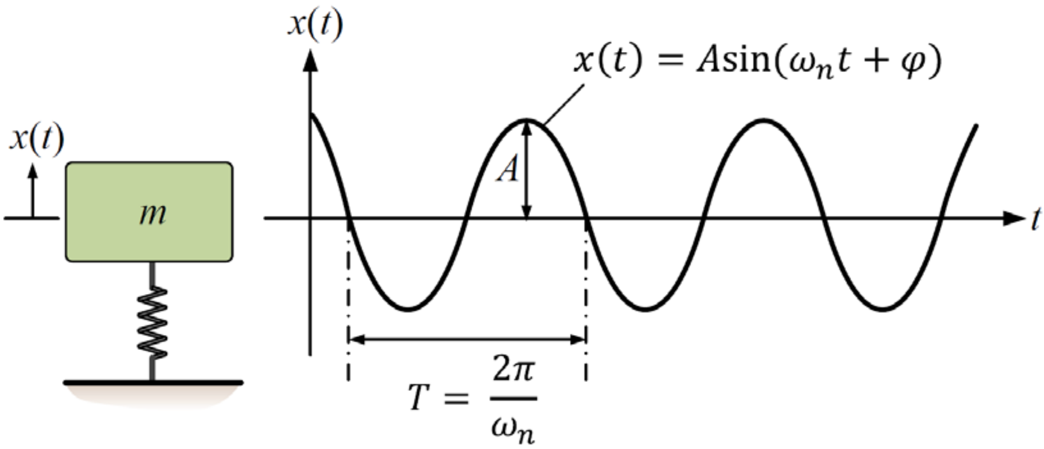
Single Degree-of-Freedom (DOF) Spring-Mass System in Free Vibration

Chapter 2 Rao

The simplest vibrating system consists of a rigid mass attached by a massless spring to a fixed abutment (wall). The spring has stiffness k (N/m). Once the spring-mass system is set in motion, it will continue this motion with constant frequency and amplitude. The system is said to oscillate with a sinusoidal waveform in harmonic motion.

Consider a mass suspended by a spring as shown in the figure below. When the mass is displaced from its rest position and released, it will oscillate up and down. The harmonic motion is in the form of a sine curve of maximum amplitude (A) and period (T). Frequency is defined as the number of cycles per second and is equal to the reciprocal of the period (f = 1/T). Frequency has units of hertz (Hz). By multiplying the frequency by 2π the angular frequency is obtained (ω) and has units of radians per second (rad/s).

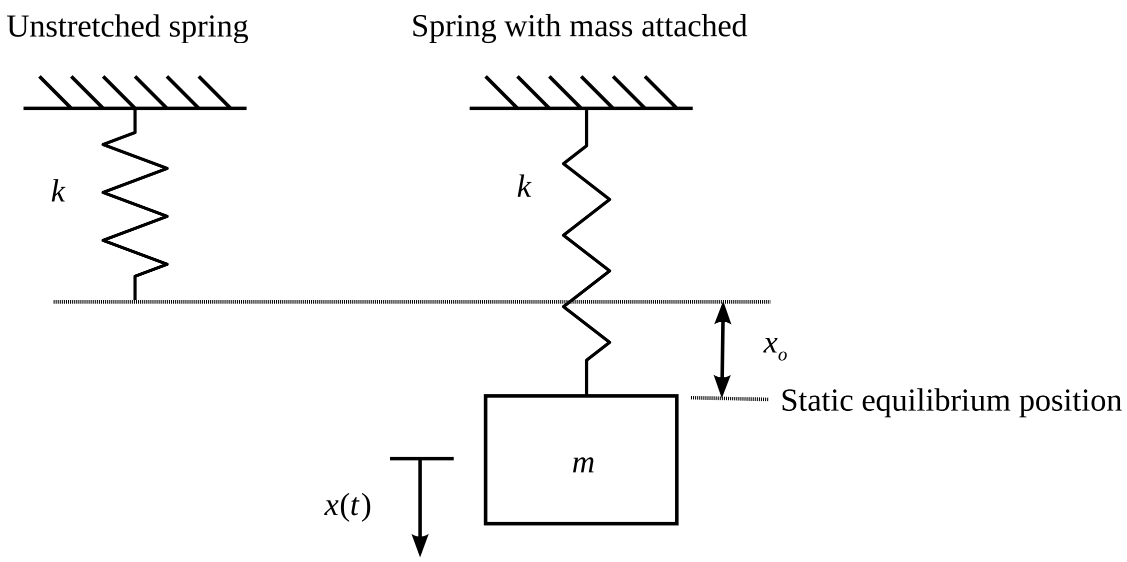
$$\omega = 2\pi f = \frac{2\pi}{T}$$

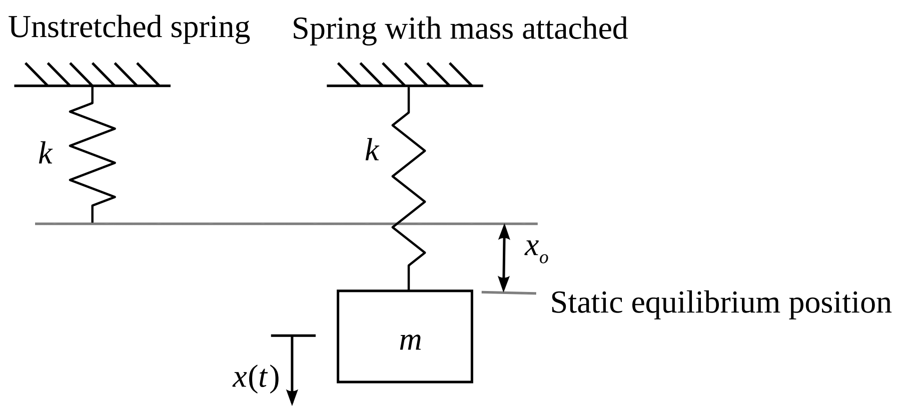


Equation of Motion

From the equation of motion we can obtain

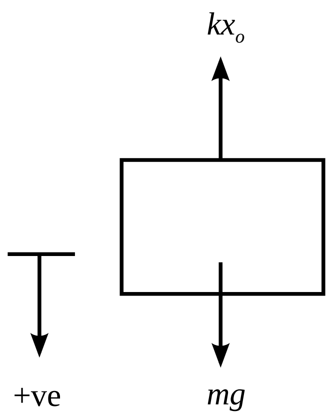
1) the natural frequencies of the system
2) the displacements as functions of time (and frequency with a Fast Fourier transform).



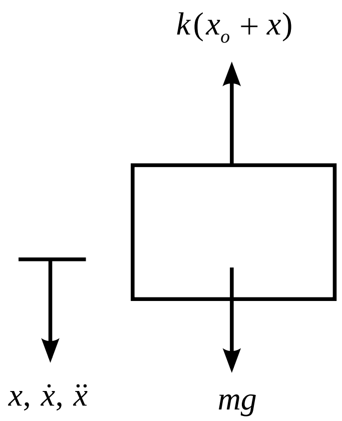


When the mass is attached to the spring, the spring has initially stretched by an amount x_o. The displacement of the mass $x(t)$ with respect to time t is measured from its static equilibrium position. After some time t, the mass has moved a distance x

Free body diagram (FBD) at t=0 (static equilibrium position)



FBD of mass at some time t



$$\sum F = ma = m\ddot{x}$$

$$\Rightarrow \quad mg - kx_o = 0$$

$$\Rightarrow \quad mg = kx_o$$

$$\sum F = m\ddot{x}$$

$$\Rightarrow \quad mg - k(x + x_o) = m\ddot{x}$$

Since $mg = kx_o$

$$\Rightarrow \quad \boxed{m\ddot{x} + kx = 0}$$

Work-Energy Method

The work-energy equation for a rigid body is

$$W_{1-2} = F(s_2 - s_1) + M(\theta_2 - \theta_1) = \Delta T + \Delta V_g + \Delta V_e$$

where

- F, M are external forces, moments
- T : kinetic energy
- V_g : gravitational potential energy
- V_e : elastic (strain) potential energy

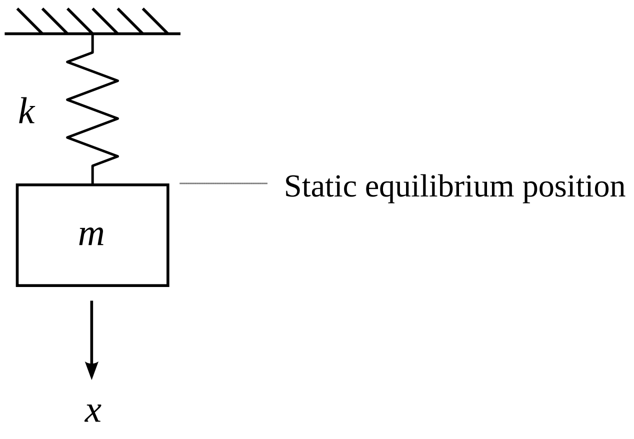
For a spring-mass system in free vibration (in the absence of any external forces and/or moments)

$$\Delta T + \Delta V_g + \Delta V_e = 0$$

$$\Delta T = \frac{1}{2}m(v^2 - v_o^2) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\dot{x}_o^2$$

$$\Delta V_g = mg(h - h_o) = -mgx$$

$$\Delta V_e = \frac{1}{2}k(x_2^2 - x_o^2)$$



x_o : amount the spring has initially stretched from its unstretched length

x_2 : amount the spring has finally stretched from its unstretched length

$\Rightarrow \quad x_2 = x + x_o$

$\Rightarrow \quad \Delta V_e = \frac{1}{2}k(x + x_o)^2 - \frac{1}{2}kx_o^2$ $\qquad$ k : linear spring stiffness (N/m)

The work-energy equation becomes

$$\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\dot{x}_o^2 - mgx + \frac{1}{2}k(x+x_o)^2 - \frac{1}{2}kx_o^2 = 0$$

Differentiate with respect to time

$$m\dot{x}\ddot{x} - mg\dot{x} + k(x+x_o)\dot{x} = 0$$

Group as

$$\dot{x}(m\ddot{x}+kx) + \dot{x}(kx_o - mg) = 0$$

We know $kx_o = mg$ from the static equilibrium position

$$\Rightarrow \qquad \dot{x}(m\ddot{x}+kx) = 0$$

Velocity $\dot{x}(t) \neq 0$ for any time t (except when the displacement $x(t)$ is a maximum or minimum)

$$\Rightarrow \qquad \boxed{m\ddot{x}+kx=0}$$

Solution to the equation of motion Section 2.2.4 Rao

The equation of motion is a second order linear differential equation with constant coefficients.

A general solution to the equation of motion is

$$x(t) = ae^{st} \qquad \Rightarrow \qquad \dot{x}(t) = sae^{st} \qquad \Rightarrow \qquad \ddot{x}(t) = s^2ae^{st}$$

Substituting x and $\ddot{x}$ into the equation of motion leads to

$$(ms^2+k)ae^{st} = 0$$

$$\Rightarrow \qquad ms^2+k=0 \qquad \Rightarrow \qquad s^2=-k/m \qquad \Rightarrow \qquad s=\pm j\sqrt{k/m} \qquad \Rightarrow \qquad s=\pm j\omega_n$$

where $j=\sqrt{-1}$ is the imaginary number.

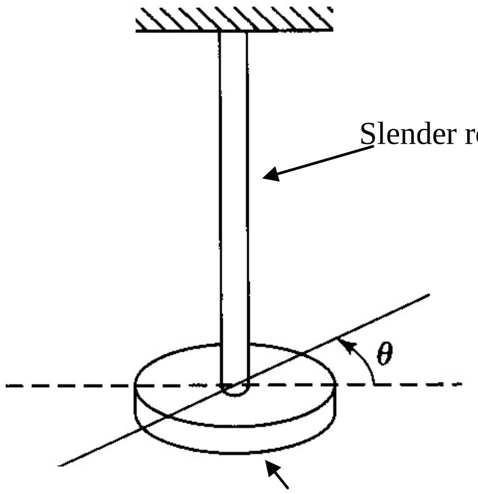
where $\omega_n = \sqrt{k/m}$ is the undamped natural frequency (rad/s) of the spring-mass system. The system will vibrate at its natural frequency when no external force is applied.

In cycles/s (Hz), the equivalent natural frequency is f_n.

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \text{(Hz)}$$

Torsional Single Degree-of-Freedom System   Section 2.3 Rao

Consider the torsional vibration of a shaft and disk (or torsion spring and disk).



Rotational systems are analogous to translational systems

- force ⇔ torque (moment)
- mass ⇔ mass moment of inertia
- linear displacement ⇔ angular displacement θ
- linear velocity ⇔ angular velocity $\dot{\theta}$
- linear acceleration ⇔ angular acceleration $\ddot{\theta}$

The rod of torsion stiffness K (Nm/rad) will resist the motion of the disk.

Using $\sum M = J\alpha = J\ddot{\theta}$

$$\Rightarrow \quad -K\theta = J\ddot{\theta} \quad \Rightarrow \quad \boxed{J\ddot{\theta} + K\theta = 0}$$

The natural frequency of the system is $\omega_n = \sqrt{K/J}$ (rad/s)

Units / Dimensions

Linear stiffness k (N/m); mass (kg): $\sqrt{\frac{k}{m}} \quad \Rightarrow \quad \sqrt{\frac{\text{N}}{\text{m}}.\frac{1}{\text{kg}}} = \sqrt{\frac{\text{kgm}}{\text{s}^2}.\frac{1}{\text{m}}.\frac{1}{\text{kg}}} = \frac{1}{\text{s}}$

Torsion stiffness K (Nm/rad); J (kgm^2): $\sqrt{\frac{K}{J}} \quad \Rightarrow \sqrt{\frac{\text{Nm}}{\text{rad}}.\frac{1}{\text{kgm}^2}} = \sqrt{\frac{\text{kgm}}{\text{s}^2}.\frac{\text{m}}{\text{rad}}.\frac{1}{\text{kgm}^2}} = \frac{1}{\text{s}}$

Harmonic Motion   Section 1.10 Rao

The motion of a spring-mass system is called oscillatory motion, or harmonic motion. Simple harmonic motion (for an undamped spring-mass system) is sinusoidal motion. Many exciting forces are sinusoidal (for example, rotating unbalance) or can be interpreted as sums of sinusoidal signals (for example, by Fourier analysis).

The following parameters define a sinusoid

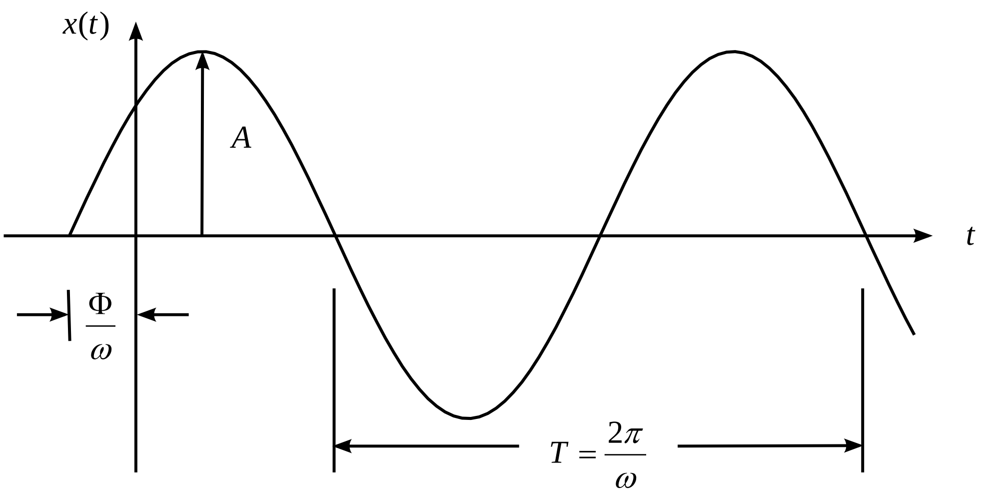
1) peak amplitude (A)

2) period of oscillation (T) – time taken to complete one cycle of motion

3) frequency (f or ω) – f (Hz) is the number of cycles per second; ω (rad/s) is the angular velocity (also called the radian frequency)

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

4) phase Φ (rad) – phase angle at time zero



Harmonic (sinusoidal) motion can be written in several mathematically equivalent ways:

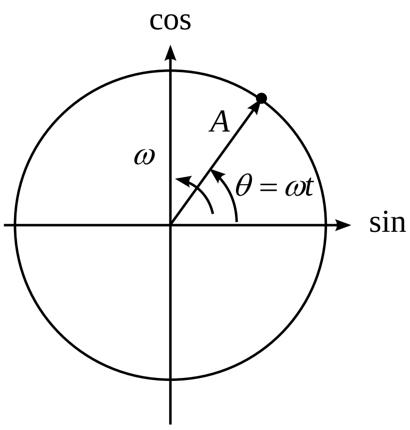
1) $x(t) = a_1 e^{j\omega_n t} + a_2 e^{-j\omega_n t}$

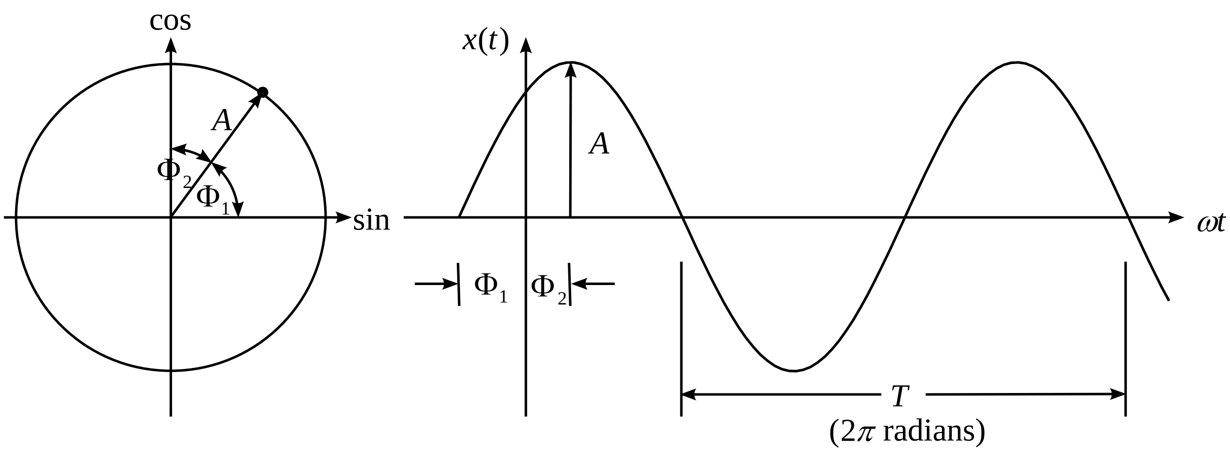
2) $x(t) = A\sin(\omega t + \Phi_1)$

3) $x(t) = A\cos(\omega t - \Phi_2)$

4) $x(t) = C\sin\omega t + D\cos\omega t$

Harmonic motion can also be represented as a vector of magnitude (amplitude) A rotating with a constant angular velocity ω





$$x(t) = A\sin(\omega t + \Phi_1) = A\cos(\omega t - \Phi_2) = C\sin\omega t + D\cos\omega t \qquad (A = \sqrt{C^2 + D^2})$$

Solution to the Equation of Motion of the Spring-Mass System

Section 2.2.5 Rao

$m\ddot{x} + kx = 0$

Using $x(t) = ae^{st}$ $\Rightarrow$ $s = \pm j\omega_n$ (from before)

where $\omega_n = \sqrt{k/m}$ is the undamped natural frequency of the system (rad/s).

The free response of the system (in the absence of any external forces) can be written as

$x(t) = a_1 e^{j\omega_n t} + a_2 e^{-j\omega_n t}$

For non-zero initial conditions (at t=0),

$$x(t=0) = x_o, \quad \dot{x}(t=0) = \dot{x}_o$$

the response (displacement) of the mass can be written as

$$\boxed{x(t) = \left(x_o + \frac{\dot{x}_o}{j\omega_n} \right) \frac{e^{j\omega_n t}}{2} + \left(x_o - \frac{\dot{x}_o}{j\omega_n} \right) \frac{e^{-j\omega_n t}}{2}}$$

Or, using $x(t) = A\sin(\omega t + \Phi_1)$

$$\Rightarrow \quad x(0) = A\sin\Phi_1 = x_o, \qquad \dot{x}(0) = \omega_n A\cos\Phi_1 = \dot{x}_o$$

$$\Rightarrow \quad A = \frac{x_o}{\sin\Phi_1} = \frac{\dot{x}_o}{\omega_n \cos\Phi_1} \qquad \Rightarrow \qquad \Phi_1 = \tan^{-1}\left(\frac{\omega_n x_o}{\dot{x}_o} \right) \quad \text{and} \quad A = \frac{\sqrt{x_o^2 \omega_n^2 + \dot{x}_o^2}}{\omega_n}$$

$$\Rightarrow \quad \boxed{x(t) = \frac{\sqrt{x_o^2 \omega_n^2 + \dot{x}_o^2}}{\omega_n} \sin\left(\omega_n t + \tan^{-1}\left(\frac{\omega_n x_o}{\dot{x}_o} \right) \right)}$$

Or, using $x(t) = C\sin\omega t + D\cos\omega t$

$$\Rightarrow \quad x(0) = D = x_o, \qquad \dot{x}(0) = \omega_n C = \dot{x}_o \qquad \Rightarrow \qquad C = \frac{\dot{x}_o}{\omega_n}$$

$$\Rightarrow \quad \boxed{x(t) = \frac{\dot{x}_o}{\omega_n} \sin\omega t + x_o \cos\omega t}$$

Damping Elements

Real vibration systems have a source of energy dissipation. Vibrational energy can be converted to heat or sound. Due to the reduction in the energy of the vibrating system, the response of the system (such as the displacement) gradually decreases. Although the damping of a system may be small, damping can be important for the accurate prediction of the vibrating response of a system. It is difficult to determine the causes of damping in practical systems. There are three main types of damping: (1) viscous damping, (2) Coulomb or dry friction damping, and (3) structural hysteretic damping.

Viscous damping: Viscous damping is the most commonly used damping mechanism in vibration analysis. When mechanical systems vibrate in a fluid medium such as air, gas, water or oil, the resistance by the fluid of the moving body causes energy to be dissipated. In viscous damping, the damping force is proportional to the velocity of the vibrating body: $F = -c\dot{x}(t)$

Typical examples of viscous damping include:

- fluid film between sliding surfaces
- fluid flow around a piston in a cylinder
- fluid flow through an orifice
- fluid film around a journal in a bearing.

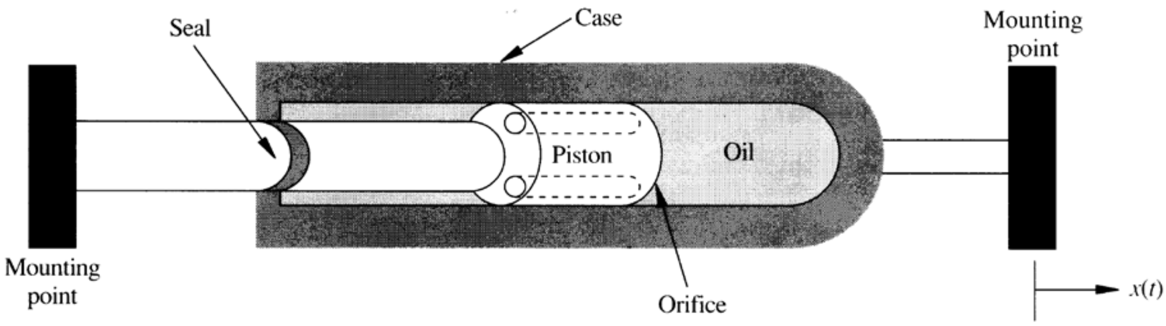
Consider a plate sliding on another plate with a viscous liquid in between (see Rao section 1.9.1). The damping coefficient c (kg/s) is a function of the fluid viscosity μ, surface area of the moving plate A, and fluid thickness h:

$$c = \mu A / h$$

Coulomb or dry friction damping: Coulomb or dry friction damping is caused by the friction between two rubbing surfaces that are either dry or have insufficient lubrication. In this case, the damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body.

Hysteretic or structural damping: Hysteretic damping is from internal energy losses in a structure. When materials are deformed, energy is absorbed and dissipated by the material.

In this course we are interested in viscous damping, which is represented this by a massless viscous damper. The damper produces a drag force opposing the motion, and which is dependent on the velocity of the mass. The most common example of a damper (also called a dashpot) is a piston moving in oil. The damping coefficient c is related to the oil viscosity. In most cases, c is a constant, and is related to the fluid or material properties (e.g. oil or rubber).

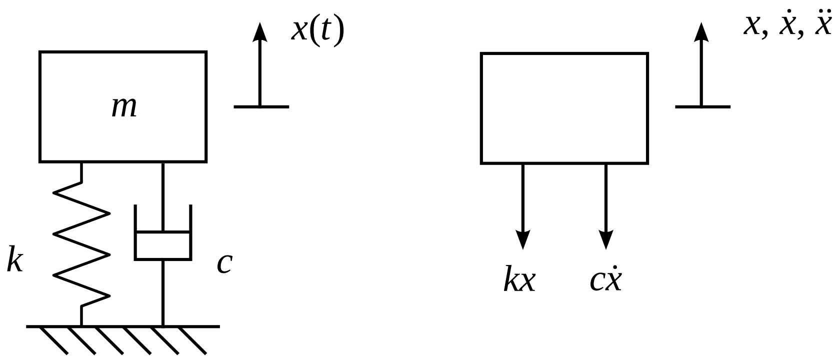


When a damper is added to the system, it results in a decrease in amplitude with time. The frequency of oscillation known as the damped natural frequency is constant and almost the same as the natural frequency.

Free Vibration of a Spring-Mass-Damper System Section 2.6 Rao

Consider a spring-mass system with viscous damping. The damper produces a drag force to oppose the motion.

Free body diagram (FBD)



The equation of motion becomes $-kx - c\dot{x} = m\ddot{x}$ $\Rightarrow$ $\boxed{m\ddot{x} + c\dot{x} + kx = 0}$

Divide the equation of motion by m gives

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$\omega_n = \sqrt{k/m}$ is the undamped natural frequency (rad/s)

$\zeta = \dfrac{c}{2\sqrt{km}}$ is the damping ratio (dimensionless)

c is the damping coefficient (kg/s) or (Ns/m)

Solution for the response of the spring-mass-damper system

Section 2.6.2 Rao

Assume a general solution of the form

$$x(t) = ae^{st} \quad \Rightarrow \quad \dot{x}(t) = sae^{st} \quad \Rightarrow \quad \ddot{x}(t) = s^2ae^{st}$$

Substituting x, $\dot{x}$ and $\ddot{x}$ into the equation of motion results in the following two solutions for s

$$s = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \quad \Rightarrow \quad s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

The damping ratio ζ will determine whether s is complex or real, that is, $\zeta < 1 \Rightarrow$ complex s. This in turn determines the nature of the response of the system. There are three cases of damping: under-damped ($\zeta < 1$), critically damped ($\zeta = 1$) and over-damped ($\zeta > 1$).

Case 1: Under-damped motion ($\zeta < 1$)

Under-damped motion ($\zeta < 1$) results in complex values of s

$$s_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$

$$s_2 = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$$

We can rewrite as

$$s_1 = -\zeta\omega_n + j\omega_d\,, \qquad s_2 = -\zeta\omega_n - j\omega_d$$

$\omega_n = \sqrt{k/m}$ is the undamped natural frequency (rad/s)

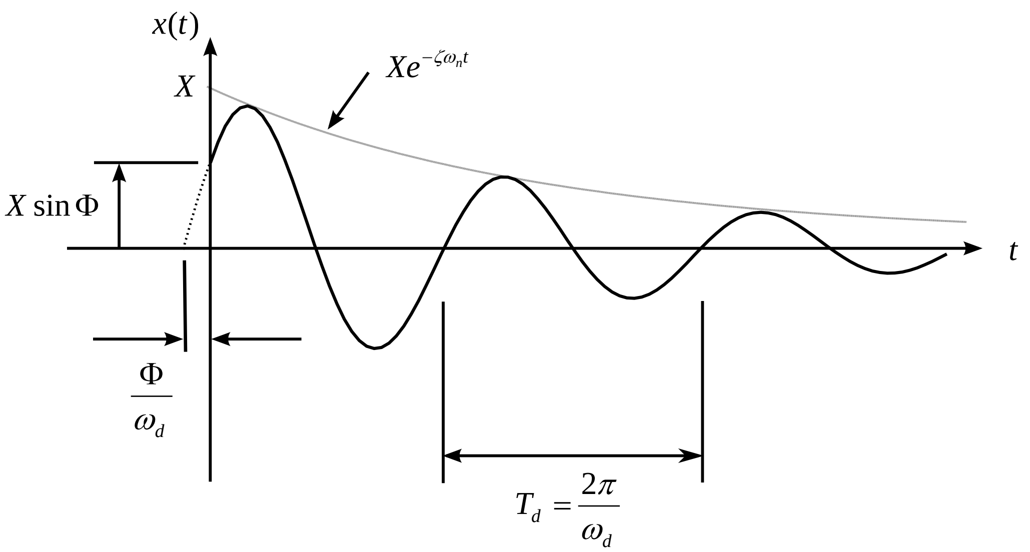
$\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency (rad/s)

Note: when $\zeta < 1$, $\sqrt{\zeta^2 - 1} \Rightarrow j\sqrt{1-\zeta^2}$ where $j = \sqrt{-1}$ is the complex number.

The solution for $x(t)$ can be written in several mathematically equivalent ways:

1) $x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{j\omega_d t} + a_2 e^{-j\omega_d t}\right)$
2) $x(t) = Xe^{-\zeta\omega_n t}\sin(\omega_d t + \Phi)$
3) $x(t) = e^{-\zeta\omega_n t}(B\sin\omega_d t + C\cos\omega_d t)$

When the system is under-damped, the motion is oscillatory.



Note that ζ determines the rate of decay. The constants (a_1, a_2), (X, Φ) or (B, C) can be determined from the initial conditions at time $t = 0$, $x(t=0) = x_o$, $\dot{x}(t=0) = \dot{x}_o$

For initial conditions x_o , $\dot{x}_o$ we have

$$x(t) = e^{-\zeta\omega_n t}(B\sin\omega_d t + C\cos\omega_d t) = e^{-\zeta\omega_n t}\left(\frac{\dot{x}_o + \zeta\omega_n x_o}{\omega_d}\sin\omega_d t + x_o\cos\omega_d t\right)$$

Similarly, for (X, Φ) we have $X = \sqrt{B^2 + C^2}$, $\Phi = \tan^{-1}\left(\frac{C}{B}\right)$

<u>Case 2: Over-damped motion ($\zeta > 1$)</u>

Over-damped motion ($\zeta > 1$) results in real values of s

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

The response $x(t)$ can be expressed as $x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{(\omega_n\sqrt{\zeta^2-1})t} + a_2 e^{-(\omega_n\sqrt{\zeta^2-1})t}\right)$

This response represents non-oscillatory motion. The over-damped system doesn't oscillate but instead returns to its rest position exponentially (and slowly).

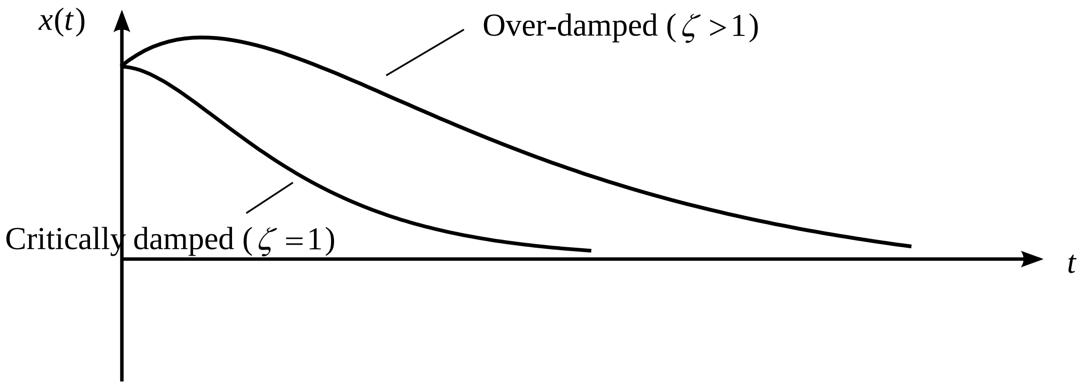
In an over-damped system, the system returns to rest very gradually. Since there is no oscillation, over-damped systems are not of interest in vibration studies.

<u>Case 3: Critically damped motion ($\zeta = 1$)</u>

When $\zeta = 1$, this separates the oscillatory motion from the non-oscillatory motion. In this case, the roots s_1 and s_2 are equal; $s_1 = s_2 = -\omega_n$

Because of the repeated roots, the solution is given by $x(t) = (a_1 + a_2 t)e^{-\omega_n t}$

The critically damped system comes to rest quickly and without oscillation.

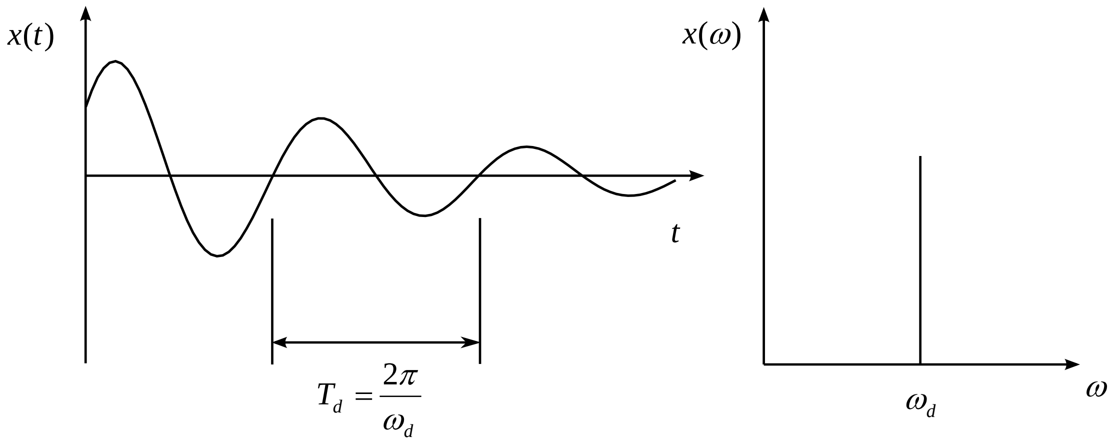


Logarithmic Decrement   Section 2.6.3 Rao

Under-damped vibration is the most common and of the most interest in mechanical engineering problems. The damping ratio ζ is generally very small ($\zeta << 1$). Also, the damping ratio ζ or damping coefficient *c* are difficult to measure in practice. It is possible to measure ζ from a measured decay trace of the free response of the system (time response in free vibration).

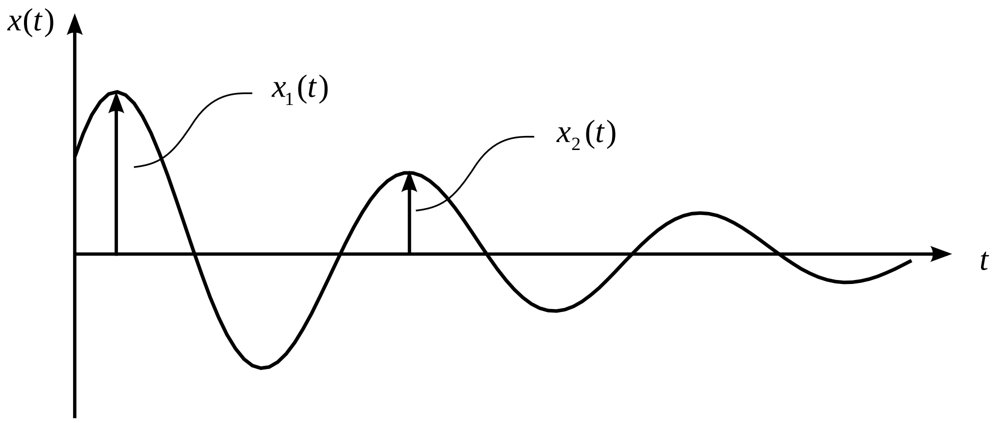
Note: the larger the damping, the greater will be the rate of decay.

The time and frequency responses of a single DOF spring-mass-damper system are



From the time response of the system in free vibration, the logarithmic decrement δ is the natural logarithm of any two successive amplitudes

$$\delta = \ln\left(\frac{x_1(t)}{x_2(t)}\right)$$



For an under-damped system ($\zeta < 1$), the general solution in free vibration is

$$x(t) = Xe^{-\zeta\omega_n t}\sin(\omega_d t + \Phi)$$

Substituting the general solution into the equation for δ gives

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \ln\left(\frac{Xe^{-\zeta\omega_n t}\sin(\omega_d t + \Phi)}{Xe^{-\zeta\omega_n (t+T_d)}\sin(\omega_d t + \omega_d T_d + \Phi)}\right)$$

$T_d = \dfrac{2\pi}{\omega_d} = \dfrac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$ is the damped period of oscillation.

We know $\quad T_d = \dfrac{2\pi}{\omega_d} \Rightarrow \quad \omega_d T_d = 2\pi$

and $\quad \sin(\omega_d t + 2\pi + \Phi) = \sin(\omega_d t + \Phi)$

$$\Rightarrow \qquad \delta = \ln\left(\frac{e^{-\zeta\omega_n t}}{e^{-\zeta\omega_n t}e^{-\zeta\omega_n T_d}}\right) = \ln e^{\zeta\omega_n T_d} = \zeta\omega_n T_d$$

Now $\quad T_d = \dfrac{2\pi}{\omega_d} = \dfrac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$

$$\Rightarrow \qquad \delta = \zeta\omega_n \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

Rearranging gives the damping ratio

$$\boxed{\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}}$$

The damping coefficient can then be obtained by $\quad c = 2\zeta\sqrt{km}$

Hence, from any experimentally measured plot of the time response in free vibration, it is possible to measure two successive peaks. Using the above equation, we can determine the damping ratio and coefficient.

If the response is measured after *n* cycles have elapsed, the logarithmic decrement is given by:

$$\delta = \frac{1}{n}\ln\frac{x_1}{x_{n+1}}$$

Summary for a Single DOF Spring-Mass-Damper System

The equation of motion for a single degree-of-freedom spring-mass-damper system is

$$m\ddot{x} + c\dot{x} + kx = 0$$

It is useful to divide by mass *m*, resulting in

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

- $\omega_n = \sqrt{\dfrac{k}{m}}$ is the undamped natural frequency

- $\zeta = \dfrac{c}{2\sqrt{km}}$ is called the viscous damping ratio

- $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency.

The damping ratio ζ determines the nature of the response of the system. There are three cases of damping: under-damped ($\zeta < 1$), over-damped ($\zeta > 1$), and critically damped ($\zeta = 1$).

- In an under-damped system, the motion is oscillatory. The damping ratio ζ determines the rate of decay.
- In an over-damped system, the motion is non-oscillatory. The system returns to rest very gradually (non-periodic motion).
- In a critically damped system, the system comes to rest quickly and without oscillation.

In vibration analysis, we are generally interested in under-damped systems due to the oscillatory nature of the response. For an under-damped system, the response of the mass, $x(t)$, is given by

$$x(t) = Xe^{-\zeta\omega_n t}\sin(\omega_d t + \Phi)$$

where

- X is the peak amplitude

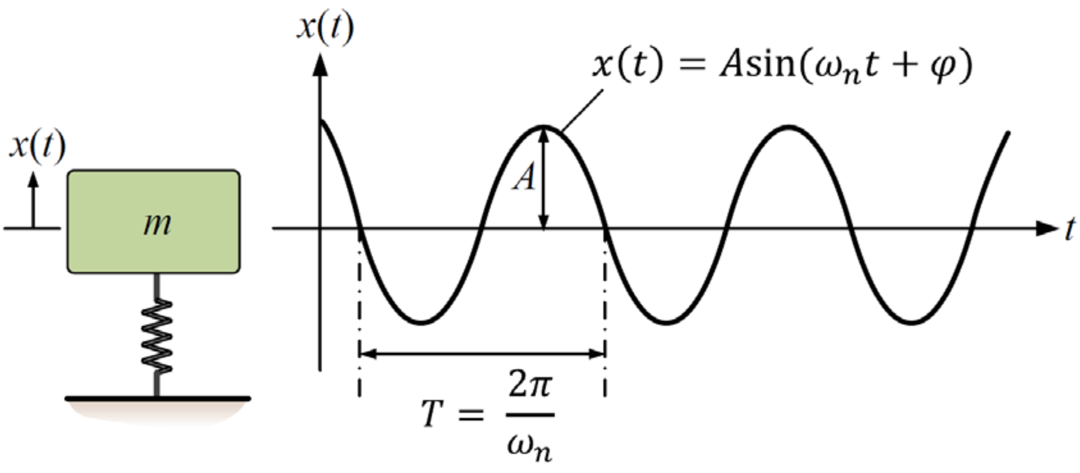
- ζ is the viscous damping ratio

- ω_n is the undamped natural frequency

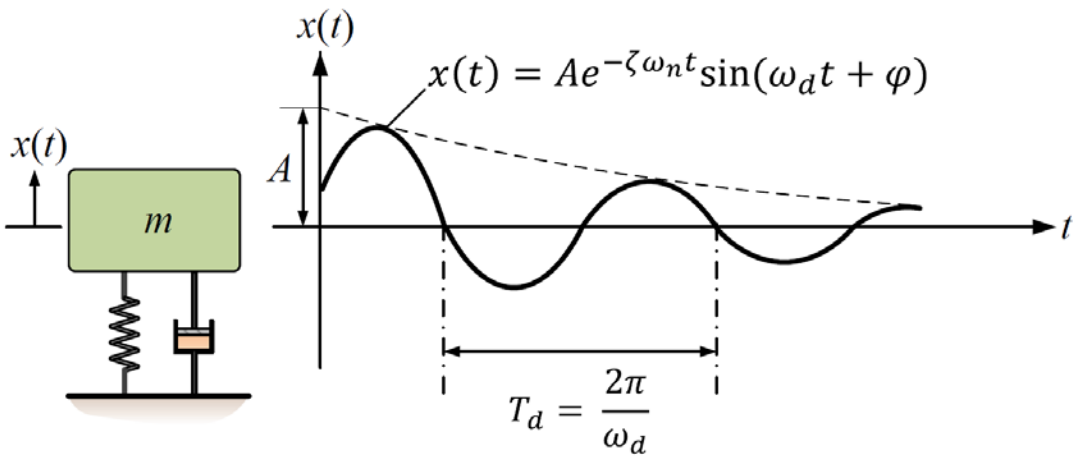
- ω_d is the damped natural frequency

- Φ is the phase

Response of an undamped spring-mass system



Response of a spring-mass-damper system



Logarithmic decrement $\delta = \ln\left(\frac{x_1(t)}{x_2(t)}\right)$ or $\delta = \frac{1}{n}\ln\frac{x_1}{x_{n+1}}$

Damping ratio $\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$

Damping coefficient $c = 2\zeta\sqrt{km}$