MMAN2300
Engineering Mechanics 2

Part B: Vibration Analysis

Two DOF spring-mass systems

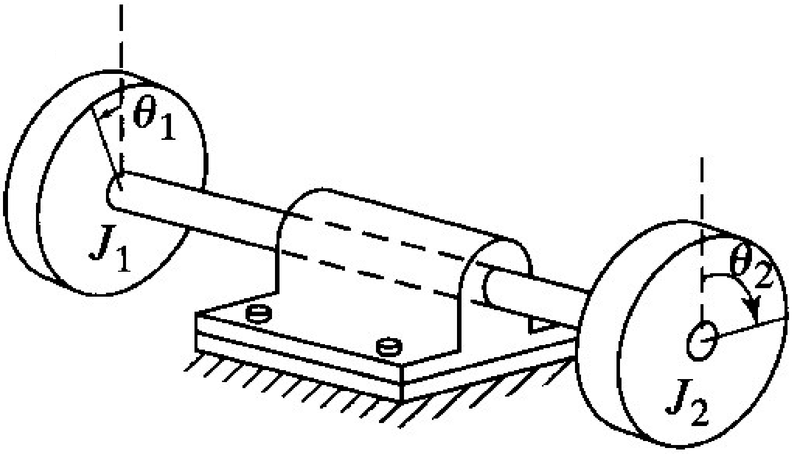
Torsional system

Coordinate coupling

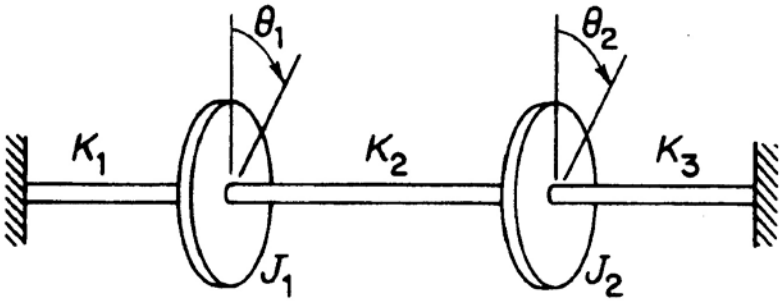
Free-free systems

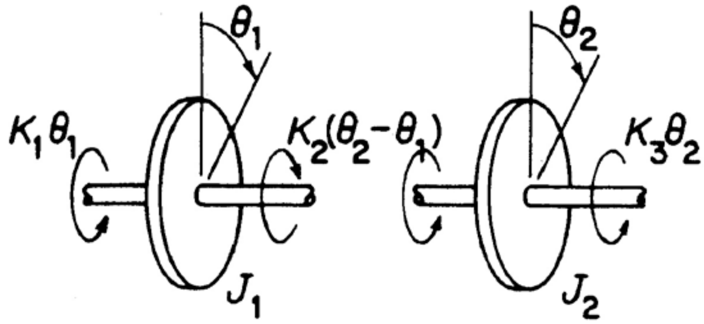
Torsional system   Section 5.4 Rao

The natural frequencies and modeshapes can also be obtained for a rotational system such as a system comprising of shafts and disks.



Consider the 2DOF system in the figure below consisting of 2 disks and 3 shafts.





- J represents the mass moment of inertia of the disk (kgm^2)
- K represents the torsional stiffness of the shafts (Nm/rad)
- θ is the angular displacement (rad)

The equations of motion are:

$$J_1\ddot{\theta}_1 = -K_1\theta_1 - K_2\theta_1 + K_2\theta_2 \qquad \text{or} \qquad J_1\ddot{\theta}_1 = -K_1\theta_1 + K_2(\theta_2 - \theta_1)$$

$$J_2\ddot{\theta}_2 = -K_2\theta_2 - K_3\theta_2 + K_2\theta_1 \qquad\qquad J_2\ddot{\theta}_2 = -K_2(\theta_2 - \theta_1) - K_3\theta_2$$

Assuming sinusoidal motion of the form

$$\theta_1(t) = A_1 \sin \omega t \qquad \rightarrow \qquad \ddot{\theta}_1(t) = -\omega^2 A_1 \sin \omega t$$

$$\theta_2(t) = A_2 \sin \omega t \qquad \rightarrow \qquad \ddot{\theta}_2(t) = -\omega^2 A_2 \sin \omega t$$

the equations of motion can be written as

$$-J_1\omega^2 A_1 + (K_1 + K_2)A_1 - K_2 A_2 = 0$$

$$-J_2\omega^2 A_2 - K_2 A_1 + (K_2 + K_3)A_2 = 0$$

and then arranging in matrix form by:

$$\begin{bmatrix} K_1 + K_2 - J_1\omega^2 & -K_2 \\ -K_2 & K_2 + K_3 - J_2\omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Again, the characteristic equation is obtained by taking the determinant of the mass moment of inertia and stiffness matrix:

$$\begin{vmatrix} K_1 + K_2 - J_1\omega^2 & -K_2 \\ -K_2 & K_2 + K_3 - J_2\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow \qquad (K_1 + K_2 - J_1\omega^2)(K_2 + K_3 - J_2\omega^2) - K_2^2 = 0$$

For $J_1 = J_2 = J$ and $K_1 = K_2 = K$, the characteristic equation becomes

$$(2K - J\omega^2)(2K - J\omega^2) - K^2 = 0 \qquad \Rightarrow \qquad J^2\omega^4 - 4JK\omega^2 + 3K^2 = 0$$

Divide by K^2 and put $\omega_o = \sqrt{K/J}$.

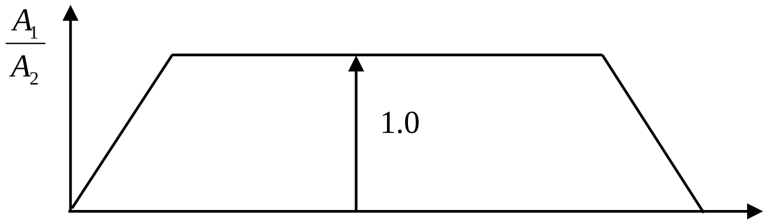
$$\left(\frac{\omega}{\omega_o}\right)^4 - 4\left(\frac{\omega}{\omega_o}\right)^2 + 3 = 0$$

$$\Rightarrow \qquad \omega_{n1} = \omega_o = \sqrt{K/J}\,, \quad \omega_{n2} = \sqrt{3}\omega_o = \sqrt{3K/J}$$

From the matrix expression, the ratio of the amplitudes are found to be

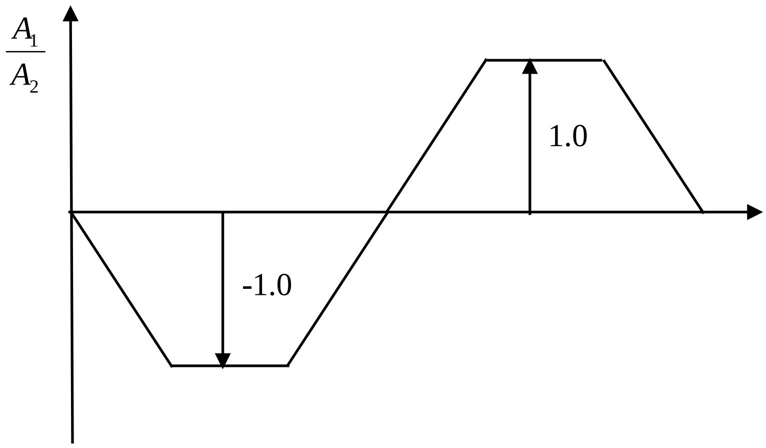
$$\frac{A_1}{A_2} = \frac{K_2}{K_1 + K_2 - J_1\omega^2} = \frac{K}{2K - J\omega^2}$$

For $\omega_{n1} = \sqrt{K/J}$, $\dfrac{A_1}{A_2} = \dfrac{K}{2K - JK/J} = 1$



The disks rotate in phase with each other

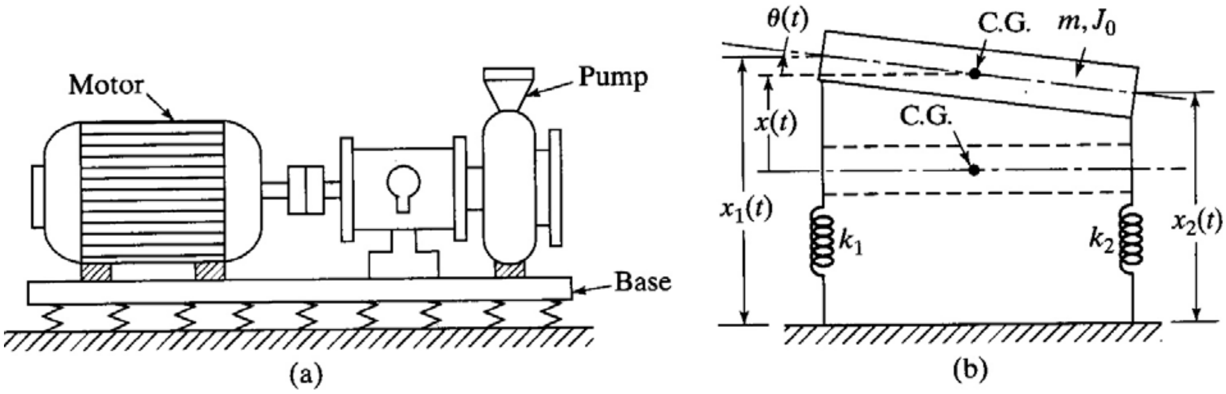
For $\omega_{n2} = \sqrt{3K/J}$, $\dfrac{A_1}{A_2} = \dfrac{K}{2K - J3K/J} = -1$



The disks vibrate with equal amplitude but out of phase with each other

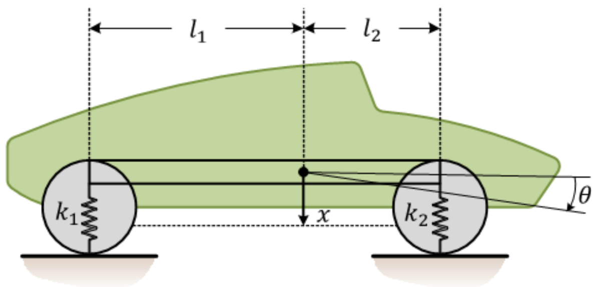
Coordinate Coupling   Section 5.5 Rao

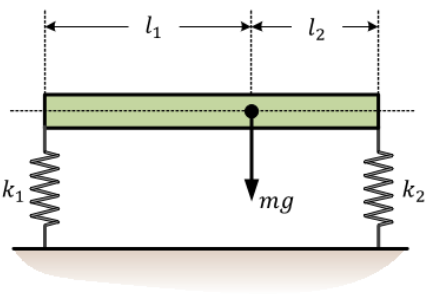
The simplest N-DOF system is the 2-DOF system. A 2-DOF system requires 2 independent co-ordinates to describe its motion, and has 2 natural frequencies. Note that a single rigid body mass can have a maximum of 6 degrees of freedom – 3 components of translation and 3 components of rotation. The motor-pump system on springs shown below is modelled as a single rigid body mass, with translation $x(t)$ and rotation $\theta(t)$ about its centre of gravity (centre of mass). Hence, it is a 2-DOF system.

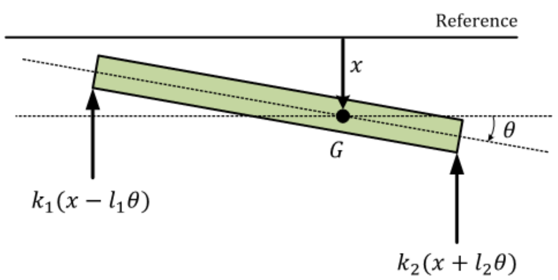


Motor-pump system on springs (Figure 5.1 Rao)

An automobile can be modelled as a 2DOF system consisting of a rigid bar supported by two springs, k_1 and k_2. There is translational motion which is specified by a linear displacement $x(t)$ of the centre of gravity of the mass. There is also rotational motion which is specified by an angular displacement $\theta(t)$, to describe the rotation of the mass about its centre of gravity.







There are two equations of motion which are applied to the centre of gravity of the rigid mass:

$\sum F = m\ddot{x}$, $\sum M = J\ddot{\theta}$. Note: for small θ, $\sin\theta \approx \theta$, $\cos\theta \approx 1$, $\tan\theta \approx \theta$.

Equations of motion

$$m\ddot{x} = -k_1(x - l_1\theta) - k_2(x + l_2\theta)$$

$$J\ddot{\theta} = k_1(x - l_1\theta)l_1 - k_2(x + l_2\theta)l_2$$

Rearrange as

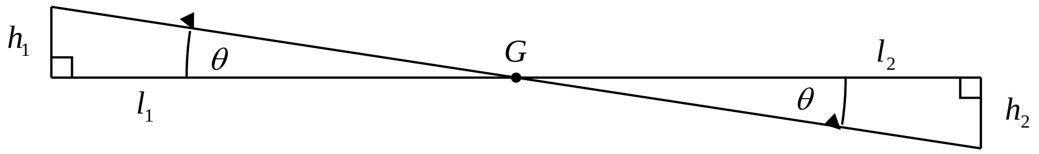
$$m\ddot{x} + x(k_1 + k_2) + \theta(-k_1 l_1 + k_2 l_2) = 0$$

$$J\ddot{\theta} + x(-k_1 l_1 + k_2 l_2) + \theta(k_1 l_1^2 + k_2 l_2^2) = 0$$

For harmonic motion, use general solutions of the form

$$x(t) = X\sin\omega t \qquad \rightarrow \qquad \ddot{x}(t) = -\omega^2 X\sin\omega t$$

$$\theta(t) = \Theta\sin\omega t \qquad \rightarrow \qquad \ddot{\theta}(t) = -\omega^2\Theta\sin\omega t$$

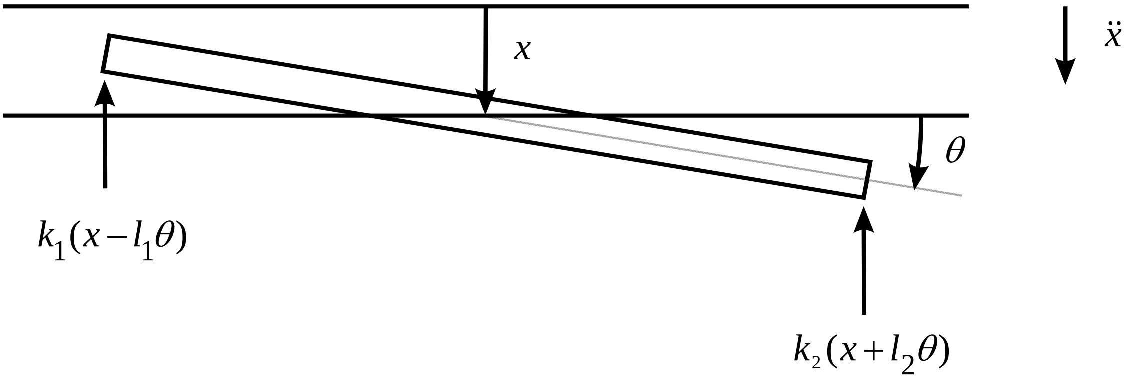


$$\tan\theta = \frac{h_1}{l_1} = \frac{h_2}{l_2} \qquad \Rightarrow \qquad h_1 = l_1\tan\theta \approx l_1\theta \quad \text{and} \quad h_2 = l_2\tan\theta \approx l_2\theta$$

The linear displacement $x(t)$ is defined as positive down ($x \downarrow$)

$\Rightarrow$ spring k_1 compresses by an amount $x - l_1\theta$

spring k_2 compresses by an amount $x + l_2\theta$

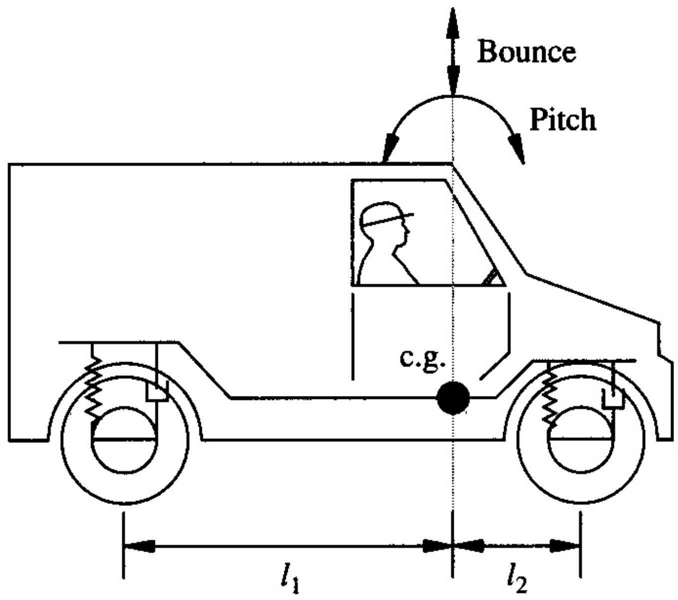


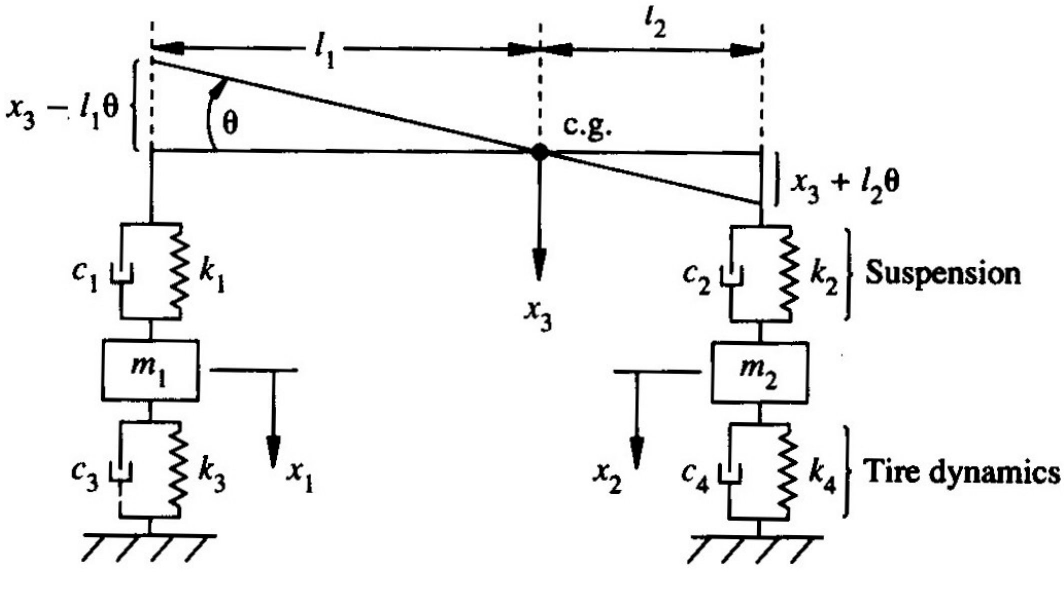
Substitution of the general solutions into the equations of motion and arrange in matrix form

$$\begin{bmatrix} k_1 + k_2 - m\omega^2 & -k_1 l_1 + k_2 l_2 \\ -k_1 l_1 + k_2 l_2 & k_1 l_1^2 + k_2 l_2^2 - J\omega^2 \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

With given numbers, the natural frequencies ω_{n1} and ω_{n2} can be determined, as well as the corresponding modeshapes X/Θ.

In general, vehicles are modelled as a 4DOF system (including tyre dynamics and suspension).



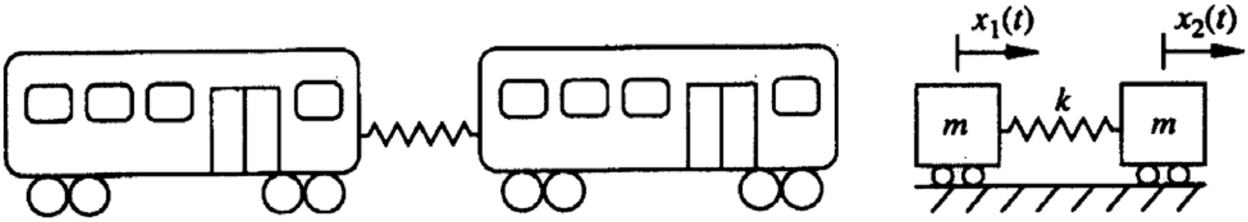


Free-Free Systems (Semi-Definite Systems) Section 5.7 Rao

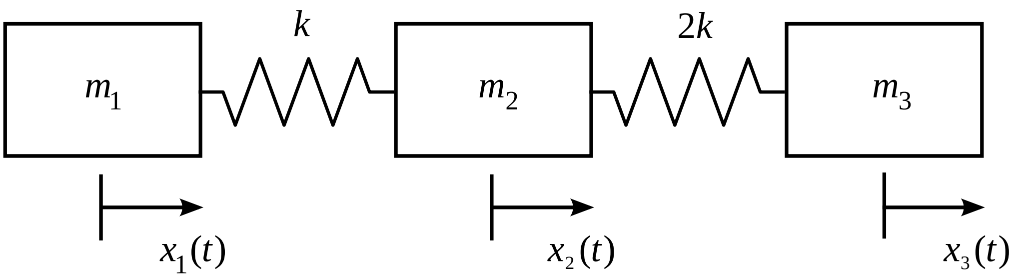
A free-free system is a vibrational system that is unrestrained. It is free to move as a rigid body, as well as vibrate. An airplane in flight, aerospace structure, ship or a moving train are examples of unrestrained (free-free) systems. Free-free systems are also known as semi-definite systems, or degenerate systems.

A free-free system will always have a zero natural frequency. At this frequency, the system moves as a single rigid body, that is, the system moves off as one object without any oscillation between the masses.

The figure below shows a vibration model of two railway cars connected by a coupling device modelled as a massless spring.



Consider the following 3DOF free-free system. Find the natural frequencies and corresponding modeshapes.



Equations of motion

$$m_1\ddot{x}_1 = -kx_1 + kx_2$$

$$m_2\ddot{x}_2 = -kx_2 - 2kx_2 + kx_1 + 2kx_3$$

$$m_3\ddot{x}_3 = -2kx_3 + 2kx_2$$

For harmonic motion, assume general solutions of the form

$$x_1(t) = A_1 \sin \omega t\,,\quad x_2(t) = A_2 \sin \omega t\,,\quad x_3(t) = A_3 \sin \omega t$$

Substitute the general solutions into the equations of motion and arrange in matrix form

$$\begin{bmatrix} k - m_1\omega^2 & -k & 0 \\ -k & 3k - m_2\omega^2 & -2k \\ 0 & -2k & 2k - m_3\omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To find the characteristic equation, we need to take the determinant of the 3×3 matrix. The determinant of a 3×3 matrix is given by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = 0$$

$$\Rightarrow \qquad (k - m_1\omega^2)\left[(3k - m_2\omega^2)(2k - m_3\omega^2) - 4k^2\right] - (-k)(-k)(2k - m_3\omega^2) = 0$$

For this example, let $m_1 = m_3 = m$, $m_2 = 2m$

$$\Rightarrow \qquad (k - m\omega^2)\left[(3k - 2m\omega^2)(2k - m\omega^2) - 4k^2\right] - k^2(2k - m\omega^2) = 0$$

$$\Rightarrow \qquad -2m^3\omega^6 + 9km^2\omega^4 - 8k^2m\omega^2 = 0$$

Divide by k^3 (because it is a 3DOF system) and put $\omega_o = \sqrt{k/m}$

$$-2\left(\frac{\omega}{\omega_o}\right)^6 + 9\left(\frac{\omega}{\omega_o}\right)^4 - 8\left(\frac{\omega}{\omega_o}\right)^2 = 0$$

$$\Rightarrow \qquad \left(\frac{\omega}{\omega_o}\right)^2\left[-2\left(\frac{\omega}{\omega_o}\right)^4 + 9\left(\frac{\omega}{\omega_o}\right)^2 - 8\right] = 0$$

The first natural frequency is zero, $\omega_{n1} = 0$ (called a zero natural frequency). A free-free system <u>always</u> has a zero natural frequency. The other two ω_n's can be found by

$$\left(\frac{\omega}{\omega_o}\right)^2 = \frac{-9 \pm \sqrt{9^2 - (4)(-2)(-8)}}{(2)(-2)} = 1.22,\ 3.28$$

$$\Rightarrow \qquad \omega_{n2} = \sqrt{1.22}\,\omega_o = 1.10\sqrt{k/m} \qquad \text{and} \qquad \omega_{n3} = \sqrt{3.28}\,\omega_o = 1.81\sqrt{k/m}$$

We now need to determine the corresponding modeshapes. Choose $\frac{A_1}{A_2}$, $\frac{A_3}{A_2}$

From the matrix expression (1st row)

$$\frac{A_1}{A_2} = \frac{k}{k - m\omega^2}$$

From the matrix (3rd row)

$$\frac{A_3}{A_2} = \frac{2k}{2k - m\omega^2}$$

For $\omega_{n1} = 0 \quad \Rightarrow \quad \dfrac{A_1}{A_2} = \dfrac{A_3}{A_2} = 1$

For $\omega_{n2} = 1.10\sqrt{k/m} \quad \Rightarrow \quad \dfrac{A_1}{A_2} = -4.55, \quad \dfrac{A_3}{A_2} = 2.56$

For $\omega_{n3} = 1.81\sqrt{k/m} \quad \Rightarrow \quad \dfrac{A_1}{A_2} = -0.44, \quad \dfrac{A_3}{A_2} = -1.56$

Modeshapes for the 3DOF free-free system. At the zero natural frequency, the system is not oscillating but instead moves as a whole without any relative motion between the masses (rigid body translation).