MMAN2300

Engineering Mechanics 2

Part B: Vibration Analysis

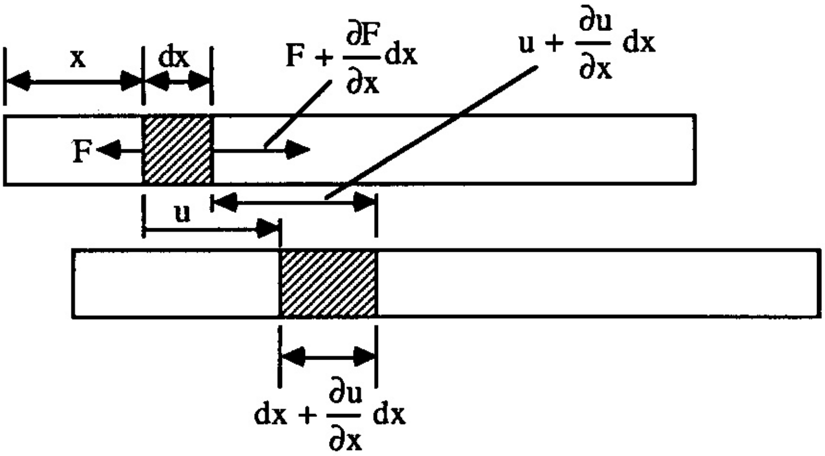
Continuous systems

Longitudinal vibration of bars

Longitudinal Vibration of a Uniform Bar or Rod

Section 8.3 Rao

Consider a homogeneous thin long bar of length *l* and cross-sectional area *A*, and subject to longitudinal vibrations due to an axial force *F*.



Longitudinal displacement of a bar

Take an element *dx* of the bar.

- $u(x,t)$ is the longitudinal displacement of the element *dx* at a position *x* along the bar
- *F* is the longitudinal force acting on the element *dx* at position *x*
- $F + \frac{\partial F}{\partial x} dx$ is an expansion of the force acting on the element *dx* at position $x + dx$

Under elastic strain, the element *dx* stretches by an amount $\frac{\partial u}{\partial x} dx$, where $\varepsilon = \frac{\partial u}{\partial x}$ is the elastic strain.

- *u* is the longitudinal displacement of the element *dx* at a position *x*
- $u + \frac{\partial u}{\partial x} dx$ is the longitudinal displacement of the element *dx* at a position $x + dx$

The mass of the element *dx* is *dm*, where $dm = \rho dV = \rho A dx$

Using $\sum F = ma$ and taking the sum of the forces in the axial direction, we get:

$$F + \frac{\partial F}{\partial x} dx - F = dm \frac{\partial^2 u}{\partial t^2}$$

Hooke's law states:

$$F = \sigma A = EA \frac{\partial u}{\partial x} \qquad \Rightarrow \qquad \frac{\partial F}{\partial x} = EA \frac{\partial^2 u}{\partial x^2}$$

Substituting $\frac{\partial F}{\partial x}$ and *dm* into the equation of motion:

$$\Rightarrow \qquad EA\frac{\partial^2 u}{\partial x^2} = \rho A\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \qquad \boxed{\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_L^2}\frac{\partial^2 u}{\partial t^2} = 0}$$ is the wave equation for longitudinal vibration of a bar or rod.

$$c_L = \sqrt{\frac{E}{\rho}}$$

c_L is the longitudinal wavespeed (m/s) and represents the speed of propagation of the longitudinal waves along the bar.

The wave equation for longitudinal vibration of a bar is similar to that for lateral vibration of a string, that is, it is a 2nd order partial differential equation.

Using the separation of variables technique: $u(x,t) = q(t)\phi(x)$, the general solution for $u(x,t)$ can also be written in the same way as for the lateral displacement of a string, that is:

$$u(x,t) = (A\sin\omega_n t + B\cos\omega_n t)(C\sin kx + D\cos kx)$$

where $k = \omega / c_L$ is the longitudinal wavenumber [1/m]

The following boundary conditions are used to determine the coefficients *C* and *D* (and hence the ω_n's and corresponding modeshapes)

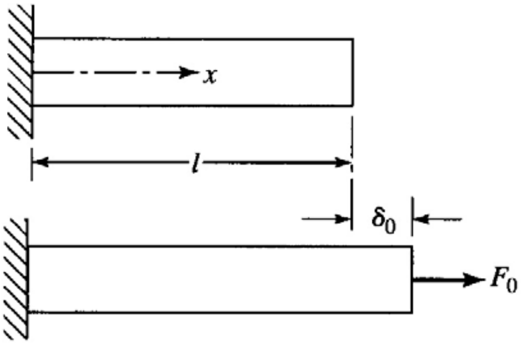
<u>Boundary Conditions</u>	$x = 0$	$x = L$
Fixed	$u = 0$	$u = 0$
Free	$\frac{\partial u}{\partial x} = 0$	$\frac{\partial u}{\partial x} = 0$
Spring load	$k_s u = EA\frac{\partial u}{\partial x}$	$k_s u = -EA\frac{\partial u}{\partial x}$
Inertia load	$M\frac{\partial^2 u}{\partial t^2} = EA\frac{\partial u}{\partial x}$	$M\frac{\partial^2 u}{\partial t^2} = -EA\frac{\partial u}{\partial x}$

Example 1

A bar is fixed at one end and free at the other. An axial force F_o is applied to the free end and released at *t*=0. Find an expression for the motion $u(x,t)$ of the bar.

The general solution for the longitudinal displacement of the bar is:

$$u(x,t) = (A\sin\omega_n t + B\cos\omega_n t)(C\sin kx + D\cos kx)$$



Boundary conditions

- Fixed at $x = 0 \quad \Rightarrow \quad u(0,t) = 0$
- Free at $x = l \quad \Rightarrow \quad \dfrac{du(l,t)}{dx} = 0$

At $x = 0$

$$u(0,t) = (A\sin\omega_n t + B\cos\omega_n t)D = 0 \qquad \Rightarrow \qquad D = 0$$

Hence

$$u(x,t) = (A\sin\omega_n t + B\cos\omega_n t)C\sin kx$$

$$\frac{du(x,t)}{dx} = (A\sin\omega_n t + B\cos\omega_n t)kC\cos kx$$

The second boundary condition requires

$$\frac{du(l,t)}{dx} = (A\sin\omega_n t + B\cos\omega_n t)kC\cos kl = 0$$

$$\Rightarrow \qquad kC\cos kl = 0$$

Now *k* and *C* are not zero – otherwise trivial solution!

$$\Rightarrow \qquad \cos kl = 0$$

$$\Rightarrow \qquad kl = \frac{n\pi}{2}, \qquad n = 1, 3, 5 \ldots \text{ etc}$$

$$\Rightarrow \qquad k = \frac{n\pi}{2l}, \qquad n = 1, 3, 5 \ldots \text{ etc}$$

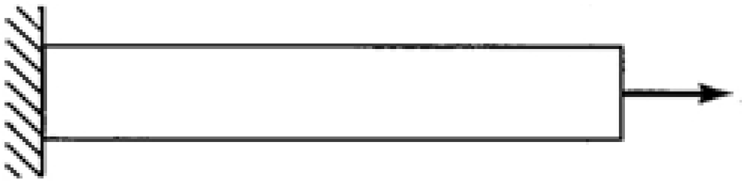
$$\Rightarrow \qquad \omega_n = kc_L = \frac{n\pi c_L}{2l}, \qquad n = 1, 3, 5 \ldots \text{ etc}$$

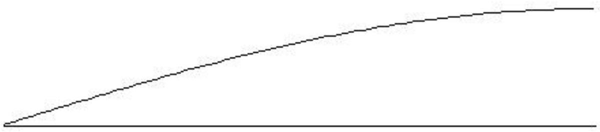
$$\Rightarrow \qquad \omega_n = kc_L = \frac{(2n-1)\pi c_L}{2l}, \qquad n = 1, 2, 3 \ldots \infty$$

ω_n ($n = 1, 2, 3 \ldots \infty$) are the natural frequencies of the bar.

The corresponding modeshapes for the bar are (note: D=0)

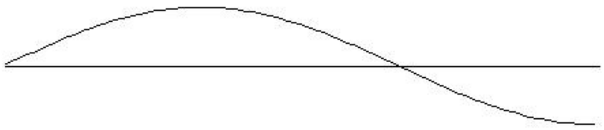
$$\phi(x) = \sin kx = \sin\frac{(2n-1)\pi x}{2l}, \qquad n = 1, 2, 3 \ldots \infty$$





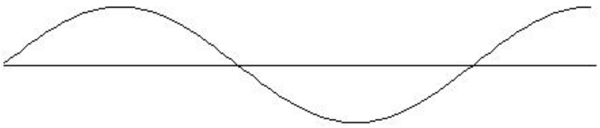
Mode 1

$$\omega_1 = \frac{\pi c_L}{2l} \qquad \phi_1(x) = \sin\frac{\pi x}{2l}$$



Mode 2

$$\omega_2 = \frac{3\pi c_L}{2l} \qquad \phi_2(x) = \sin\frac{3\pi x}{2l}$$



Mode 3

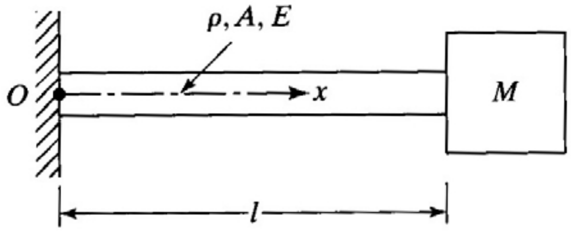
$$\omega_3 = \frac{5\pi c_L}{2l} \qquad \phi_3(x) = \sin\frac{5\pi x}{2l}$$

The solution for the longitudinal displacement of the fixed-free bar becomes

$$u(x,t) = \sum_{n=1}^{\infty}\left(A\sin(\frac{(2n-1)\pi c_L t}{2l} + B\cos(\frac{(2n-1)\pi c_L t}{2l}\right)\sin\frac{(2n-1)\pi x}{2l}$$

Example 2

Determine the ω_n's of a uniform bar of length l undergoing longitudinal vibration. The bar is fixed at one end and has a concentrated mass M attached to the other end.



The general solution for the longitudinal displacement of the bar is

$$u(x,t) = (A\sin\omega_n t + B\cos\omega_n t)(C\sin kx + D\cos kx)$$

Boundary conditions

- Fixed at $x = 0$ $\Rightarrow$ $u(0,t) = 0$
- Inertial load at $x = l$ $\Rightarrow$ $M\dfrac{\partial^2 u(l,t)}{\partial t^2} = -EA\dfrac{\partial u(l,t)}{\partial x}$

At $x = 0$

$$u(0,t) = (A\sin\omega_n t + B\cos\omega_n t)D = 0 \qquad \Rightarrow \qquad D = 0$$

$$\Rightarrow \qquad u(x,t) = (A\sin\omega_n t + B\cos\omega_n t)C\sin kx$$

$$\frac{du(x,t)}{dx} = (A\sin\omega_n t + B\cos\omega_n t)kC\cos kx$$

$$\frac{d^2u(x,t)}{dt^2} = -\omega_n^2(A\sin\omega_n t + B\cos\omega_n t)C\sin kx$$

The second boundary condition requires

$$M\frac{\partial^2 u(l,t)}{\partial t^2} = -EA\frac{\partial u(l,t)}{\partial x}$$

$$\Rightarrow \qquad -M\omega_n^2(A\sin\omega_n t + B\cos\omega_n t)C\sin kl = -EA(A\sin\omega_n t + B\cos\omega_n t)kC\cos kl$$

$$\Rightarrow \qquad M\omega_n^2\sin kl = EAk\cos kl$$

$$\Rightarrow \qquad \tan kl = \frac{EAk}{M\omega_n^2}$$

We want the RHS of the above equation to be a function of *kl*.

From $k = \omega / c_L$ and $c_L = \sqrt{E/\rho}$

$$\Rightarrow \qquad \omega_n^2 = k^2 c_L^2 = k^2 \frac{E}{\rho}$$

$$\Rightarrow \qquad \tan kl = \frac{EAk}{Mk^2 c_L^2} = \frac{\rho A}{M}\frac{1}{k}$$

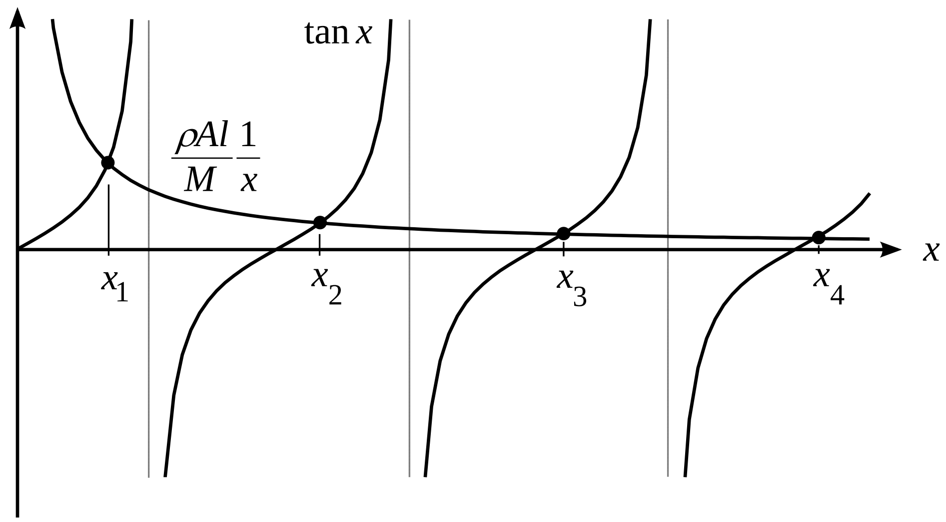
Multiply the RHS by $\frac{l}{l}$

$$\Rightarrow \qquad \tan kl = \frac{\rho Al}{M}\frac{1}{kl}$$

This frequency equation is called a *transcendental* equation in terms of *kl*. A transcendental equation is also called a natural frequency equation or a characteristic equation.

The ω_n's for the system can be obtained from the transcendental equation by plotting $\tan x$ and $\frac{\rho Al}{M}\frac{1}{x}$ against *x*, where $x = kl = \frac{\omega_n l}{c_L}$.

The points of intersection yield solutions for *x* and hence the ω_n's.



In this example, two special cases arise

- $\rho Al \ll M$
- $\rho Al \gg M$

It is possible to approximate the ω_n's from the transcendental frequency equation in these two special cases.

Case 1 $\rho Al << M$

$$\tan kl = \frac{\rho Al}{M}\frac{1}{kl} \qquad \Rightarrow \qquad kl\tan kl = \frac{\rho Al}{M}$$

The RHS of the above equation will be very small. Hence, the LHS must also be very small.

$\Rightarrow \quad kl\tan kl$ is very small $\quad \Rightarrow \quad kl$ is small

Using $\sin\theta \approx \theta$, $\cos\theta \approx 1$ and $\tan\theta \approx \theta$ for small θ

$$\Rightarrow \quad kl\tan kl \approx (kl)^2 = \left(\frac{\omega_n l}{c_L}\right)^2$$

$$\Rightarrow \quad \left(\frac{\omega_n l}{c_L}\right)^2 = \frac{\rho Al}{M}$$

$$\Rightarrow \quad \omega_n^2 = \frac{\rho Al}{M}\frac{c_L^2}{l^2} = \frac{\rho A}{M}\frac{E}{\rho l} = \frac{EA}{lM}$$

$$\Rightarrow \quad \omega_n = \sqrt{\frac{EA}{lM}}$$

In this case, there is only one ω_n and it is equivalent to that of a single DOF bar-mass system, where the bar in longitudinal vibration has equivalent stiffness $k_{eq} = \frac{EA}{l}$.

Case 2 $\rho Al >> M$

$$\Rightarrow \quad kl\tan kl = \frac{\rho Al}{M}$$

$$\Rightarrow \quad \frac{\rho Al}{M} \to \infty$$

$$\Rightarrow \quad \tan kl \to \infty$$

$$\Rightarrow \quad kl = \frac{n\pi}{2}, \qquad n = 1, 3, 5 \dots \text{ etc}$$

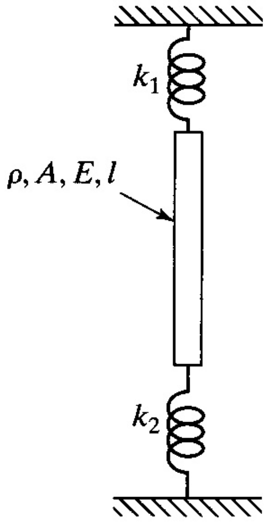
$$\Rightarrow \quad \omega_n = kc_L = \frac{n\pi c_L}{2l}, \qquad n = 1, 3, 5 \dots \text{ etc}$$

$$\Rightarrow \quad \omega_n = kc_L = \frac{(2n-1)\pi c_L}{2l}, \qquad n = 1, 2, 3 \dots \infty$$

This is the same ω_n equation of a fixed-free bar in longitudinal vibration, that is, the attached mass M becomes negligible.

Example 3

Find the natural frequency equation for longitudinal vibration of the bar shown in the figure below. Note that k_1 and k_2 are spring stiffness and are not to be confused with wavenumber *k*.



The general solution for the longitudinal displacement of the bar is

$$u(x,t) = (A\sin\omega_n t + B\cos\omega_n t)(C\sin kx + D\cos kx)$$

Boundary conditions

- Spring load at $x = 0$ $\quad\Rightarrow\quad$ $k_1 u(0,t) = EA\dfrac{\partial u(0,t)}{\partial x}$
- Spring load at $x = l$ $\quad\Rightarrow\quad$ $k_2 u(l,t) = -EA\dfrac{\partial u(l,t)}{\partial x}$

where $\dfrac{du(x,t)}{dx} = (A\sin\omega_n t + B\cos\omega_n t)(kC\cos kx - kD\sin kx)$

At $x = 0$

$$k_1(A\sin\omega_n t + B\cos\omega_n t)D = EA(A\sin\omega_n t + B\cos\omega_n t)kC$$

$$\Rightarrow \qquad D = \frac{EAk}{k_1}C$$

At $x = l$

$$k_2(A\sin\omega_n t + B\cos\omega_n t)(C\sin kl + D\cos kl) = -EA(A\sin\omega_n t + B\cos\omega_n t)(kC\cos kl - kD\sin kl)$$

$$\Rightarrow \qquad k_2(C\sin kl + D\cos kl) = -EAk(C\cos kl - D\sin kl)$$

Group the sin and cos terms

$$\Rightarrow \qquad \sin kl\left(k_2 C - EAkD\right) = \cos kl\left(-EAkC - k_2 D\right)$$

Substitute for $D = \dfrac{EAk}{k_1}C$

$$\Rightarrow \qquad \sin kl\left(k_2 C - \frac{E^2A^2k^2}{k_1}C\right) = \cos kl\left(-EAkC - \frac{EAkk_2}{k_1}C\right)$$

$$\Rightarrow \qquad \sin kl\left(\frac{k_1k_2 - E^2A^2k^2}{k_1}\right) = \cos kl\left(-\frac{EAk(k_1 + k_2)}{k_1}\right)$$

$$\Rightarrow \qquad \tan kl = -\frac{EAk(k_1 + k_2)}{k_1k_2 - E^2A^2k^2}$$

We need an equation in terms of *kl*. Multiply the RHS by $\dfrac{l^2}{l^2}$

$$\Rightarrow \qquad \tan kl = -\frac{EAl(k_1 + k_2)(kl)}{k_1k_2l^2 - E^2A^2(kl)^2}$$

This is the transcendental equation of a bar spring-loaded at both ends in longitudinal vibration, from which the natural frequencies can be obtained.

The ω_n's for the system can be obtained from the transcendental equation by plotting $\tan x$ and $\dfrac{-ax}{b - cx^2}$ against *x*, where $x = kl = \dfrac{\omega_n l}{c_L}$.

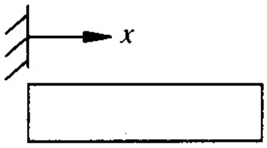
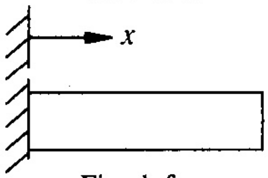
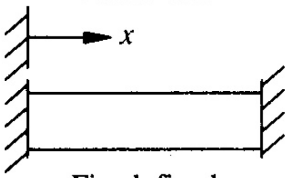
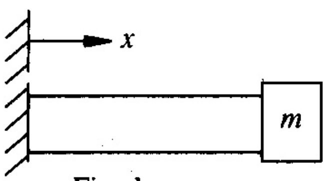
The points of intersection yield solutions for *x* and hence the ω_n's.

Various configurations of a uniform bar or rod of length *l* in longitudinal vibration, illustrating the natural frequencies and modeshapes

Table 6.1 Inman reference text

In the table below, k is the spring stiffness (not wavenumber)

$\lambda_n = kl$ where k is the wavenumber for the product kl

Configuration	Frequency (rad/s) or characteristic equation	Mode shape
 Free–free	$\omega_n = \dfrac{n\pi c}{l},\ n = 0,\ 1,\ 2, \ldots$	$\cos \dfrac{n\pi x}{l}$
 Fixed–free	$\omega_n = \dfrac{(2n-1)\pi c}{2l}, n = 1, 2, \ldots$	$\sin \dfrac{(2n-1)\pi x}{2l}$
 Fixed–fixed	$\omega_n = \dfrac{n\pi c}{l}, n = 1, 2, \ldots$	$\sin \dfrac{n\pi x}{l}$
 Fixed–spring	$\lambda_n \cot \lambda_n = -\left(\dfrac{kl}{EA}\right)$ $\omega_n = \dfrac{\lambda_n c}{l}$	$\sin \dfrac{\lambda_n x}{l}$
 Fixed–mass	$\cot \lambda_n = \left(\dfrac{m}{\rho A l}\right) \lambda_n$ $\omega_n = \dfrac{\lambda_n c}{l}$	$\sin \dfrac{\lambda_n x}{l}$