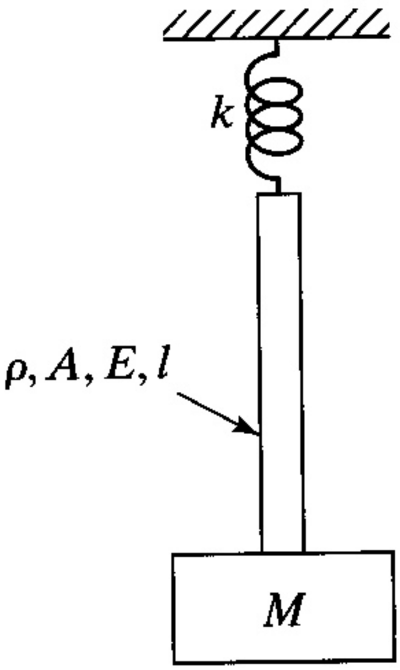
QUESTION

For the slender rod of length *l* shown in Figure Q6, identify the boundary conditions at each end of the rod for longitudinal vibration. Starting from the general solution to the wave equation for longitudinal vibration of a rod, derive the frequency equation, from which the natural frequencies may be derived. Discuss how the natural frequencies may be obtained.



Rod in longitudinal vibration

General solution

$$u(x,t) = (A\sin w_n t + B\cos w_n t)(C\sin kx + D\cos kx)$$

Boundary conditions

- At $x = 0$:
$$k_s u = EA\frac{du}{dx}$$

- At $x = L$:
$$M\frac{d^2u}{dt^2} = -EA\frac{du}{dx}$$

$$\frac{du}{dx} = q(t)(kC\cos kx - kD\sin kx)$$

<u>At $x=0$</u>

$$k_s D = EAkC$$

$\{$ Note: $k_s \rightarrow$ spring stiffness
$k \rightarrow$ wavenumber

$$\Rightarrow \quad D = \frac{EAkC}{k_s}$$

<u>At $x=L$</u>

$$\frac{d^2u}{dt^2} = -w_n^2\,(A\sin w_n t + B\cos w_n t)(C\sin kx + D\cos kx)$$

$$\Rightarrow -M\omega_n^2\left(C\sin kL + D\cos kL\right) = -EAk\left(C\cos kL - D\sin kL\right)$$

Group the sin and cos terms

$$\sin kL\left(EAkD + M\omega_n^2 C\right) = \cos kL\left(-M\omega_n^2 D + EAkC\right)$$

Substitute for $D = \dfrac{EAkC}{k_s}$

$$\Rightarrow \sin kL\left(\frac{E^2A^2k^2C}{k_s} + M\omega_n^2 C\right) = \cos kL\left(-\frac{M\omega_n^2 EAkC}{k_s} + EAkC\right)$$

$$\Rightarrow \sin kL\left(\frac{E^2A^2k^2 + M\omega_n^2 k_s}{k_s}\right) = \cos kL\left(\frac{EAk\left(k_s - M\omega_n^2\right)}{k_s}\right)$$

$$\Rightarrow \tan kL = \frac{EAk\left(k_s - M\omega_n^2\right)}{E^2A^2k^2 + M\omega_n^2 k_s}$$

We know $k = \dfrac{\omega}{c} \rightarrow \omega_n^2 = k^2 c_L^2$

where $c_L = \sqrt{E/\rho}$

$$\Rightarrow \tan kL = \frac{EAk\left(k_s - Mc_L^2 k^2\right)}{E^2A^2k^2 + Mc_L^2 k_s k^2}$$

$$\Rightarrow \tan kL = \frac{EA\left(k_s - Mc_L^2 k^2\right)}{k\left(E^2A^2 + Mc_L^2 k_s\right)}$$

We need an equation in terms of kL

Multiply the RHS by L^2/L^2

$$\tan kL = \frac{EA\left(k_s L^2 - Mc_L^2 (kL)^2\right)}{(kL)\left(E^2A^2 L + Mc_L^2 k_s L\right)}$$

The ω_n's for the system can be obtained by plotting $\tan x$ and $\dfrac{a - bx^2}{cx}$ against x,

where $x = kL = \dfrac{\omega_n L}{c_L}$. The points of intersection yield solutions for x and hence the ω_n's.