MMAN2300
Engineering Mechanics 2

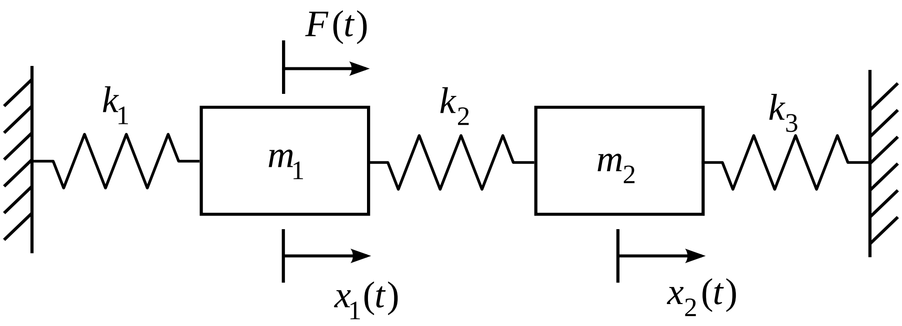
Part B: Vibration Analysis

Forced harmonic vibration of a 2DOF spring-mass system

Forced Harmonic Vibration of a 2DOF Spring-Mass System

Section 5.8 Rao

An external harmonic force $F(t) = F_o \sin \omega t$ is applied to mass m_1. Determine the steady-state amplitude of each mass as a function of excitation frequency.



Equations of motion

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2 (x_2 - x_1) + F_o \sin \omega t$$

$$m_2 \ddot{x}_2 = -k_2 (x_2 - x_1) - k_3 x_2$$

For $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$

$\omega_{n1} = \sqrt{k/m}$ and $\dfrac{A_1}{A_2} = 1$

$\omega_{n2} = \sqrt{3k/m}$ and $\dfrac{A_1}{A_2} = -1$

It is conventional to determine the non-dimensionalised steady-state amplitude for each mass

$$\frac{kA_1}{F_o}, \ \frac{kA_2}{F_o}$$

Using the following general solutions

$$x_1(t) = A_1 \sin \omega t \quad \rightarrow \quad \ddot{x}_1(t) = -\omega^2 A_1 \sin \omega t$$

$$x_2(t) = A_2 \sin \omega t \quad \rightarrow \quad \ddot{x}_2(t) = -\omega^2 A_2 \sin \omega t$$

the equations of motion become

$$(2k - m\omega^2) A_1 - kA_2 = F_o \qquad (1)$$

$$-kA_1 + (2k - m\omega^2) A_2 = 0 \qquad (2)$$

From (2), $$A_2 = \frac{kA_1}{2k - m\omega^2} \tag{3}$$

Substitute (3) in (1) gives $$(2k - m\omega^2)A_1 - \frac{k^2 A_1}{2k - m\omega^2} = F_o \tag{4}$$

Rearrange (4) $$\Rightarrow \quad (2k - m\omega^2)^2 A_1 - k^2 A_1 = (2k - m\omega^2)F_o$$

$$\Rightarrow \quad (4k^2 - 4km\omega^2 + m^2\omega^4 - k^2)A_1 = (2k - m\omega^2)F_o$$

$$\Rightarrow \quad \frac{A_1}{F_o} = \frac{2k - m\omega^2}{m^2\omega^4 - 4km\omega^2 + 3k^2} \tag{5}$$

Note: the denominator corresponds to the characteristic equation of the system.

To non-dimensionalise equation (5), multiply both sides by k

$$\frac{kA_1}{F_o} = \frac{2k^2 - mk\omega^2}{m^2\omega^4 - 4km\omega^2 + 3k^2} \tag{6}$$

In order to sketch the non-dimensionalised amplitude, divide by k^2/k^2 (for a 2DOF system), put $\omega_o = \sqrt{k/m}$ (ω_o is just a reference frequency) and expand into partial fractions as a function of the two ω_n's of the system. In equation (6), divide the right hand side by k^2/k^2 and put $\omega_o = \sqrt{k/m}$.

$$\frac{kA_1}{F_o} = \frac{2k^2 - mk\omega^2}{m^2\omega^4 - 4km\omega^2 + 3k^2} = \frac{2 - \left(\frac{\omega}{\omega_o}\right)^2}{\left(\frac{\omega}{\omega_o}\right)^4 - 4\left(\frac{\omega}{\omega_o}\right)^2 + 3} \tag{7}$$

Now expand into partial fractions in terms of ω_{n1} and ω_{n2}, where $\omega_{n1} = \omega_o$ and $\omega_{n2} = \sqrt{3}\omega_o$.

$$\frac{kA_1}{F_o} = \frac{2 - \left(\frac{\omega}{\omega_o}\right)^2}{\left(\frac{\omega}{\omega_o}\right)^4 - 4\left(\frac{\omega}{\omega_o}\right)^2 + 3} = \frac{C_1}{1 - \left(\frac{\omega}{\omega_{n1}}\right)^2} + \frac{D_1}{1 - \left(\frac{\omega}{\omega_{n2}}\right)^2}$$

where C_1 and D_1 need to be determined.

Substitute $\omega_{n1} = \omega_o$, $\omega_{n2} = \sqrt{3}\omega_o$ and cross-multiply to get a common denominator, which must be the characteristic equation of the system!

$$\frac{kA_1}{F_o} = \frac{C_1}{1-\left(\frac{\omega}{\omega_{n1}}\right)^2} + \frac{D_1}{1-\left(\frac{\omega}{\omega_{n2}}\right)^2}$$

$$= \frac{C_1}{1-\left(\frac{\omega}{\omega_o}\right)^2} + \frac{D_1}{1-\left(\frac{\omega}{\sqrt{3}\omega_o}\right)^2}$$

$$= \frac{C_1}{1-\left(\frac{\omega}{\omega_o}\right)^2} + \frac{3D_1}{3-\left(\frac{\omega}{\omega_o}\right)^2}$$

$$= \frac{C_1\left[3-\left(\frac{\omega}{\omega_o}\right)^2\right] + 3D_1\left[1-\left(\frac{\omega}{\omega_o}\right)^2\right]}{\left[1-\left(\frac{\omega}{\omega_o}\right)^2\right]\left[3-\left(\frac{\omega}{\omega_o}\right)^2\right]}$$

$$= \frac{3C_1 + 3D_1 - (C_1 + 3D_1)\left(\frac{\omega}{\omega_o}\right)^2}{\left(\frac{\omega}{\omega_o}\right)^4 - 4\left(\frac{\omega}{\omega_o}\right)^2 + 3}$$

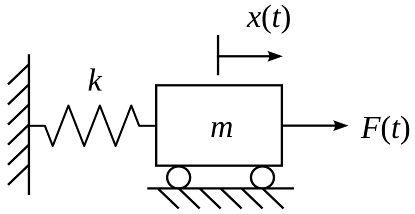
$$\Rightarrow \quad \frac{kA_1}{F_o} = \frac{3C_1 + 3D_1 - (C_1 + 3D_1)\left(\frac{\omega}{\omega_o}\right)^2}{\left(\frac{\omega}{\omega_o}\right)^4 - 4\left(\frac{\omega}{\omega_o}\right)^2 + 3} = \frac{2-\left(\frac{\omega}{\omega_o}\right)^2}{\left(\frac{\omega}{\omega_o}\right)^4 - 4\left(\frac{\omega}{\omega_o}\right)^2 + 3} \qquad \{\text{Eq. (7) from before}\}$$

In order for the two equations above to be the same, we must have

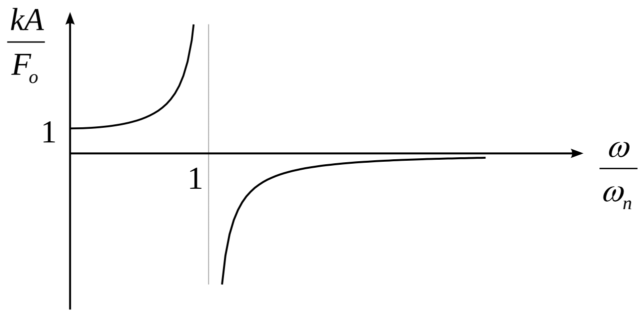
$3C_1 + 3D_1 = 2$ and $C_1 + 3D_1 = 1$ $\Rightarrow$ $C_1 = 0.5$, $D_1 = 0.167$

$$\Rightarrow \quad \frac{kA_1}{F_o} = \frac{2-\left(\frac{\omega}{\omega_o}\right)^2}{\left(\frac{\omega}{\omega_o}\right)^4 - 4\left(\frac{\omega}{\omega_o}\right)^2 + 3} = \frac{0.5}{1-\left(\frac{\omega}{\omega_{n1}}\right)^2} + \frac{0.167}{1-\left(\frac{\omega}{\omega_{n2}}\right)^2}$$

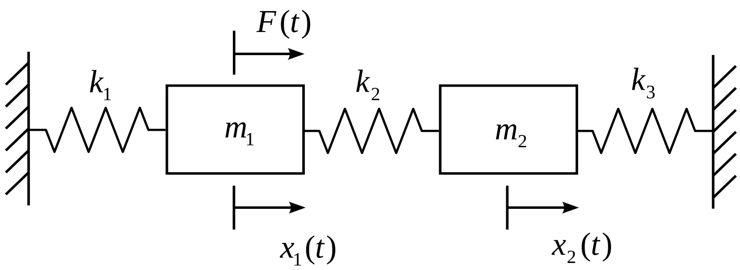
Recall the non-dimensionalised amplitude for a single degree-of-freedom (SDOF) spring-mass system in forced vibration



$$\frac{kA}{F_o} = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

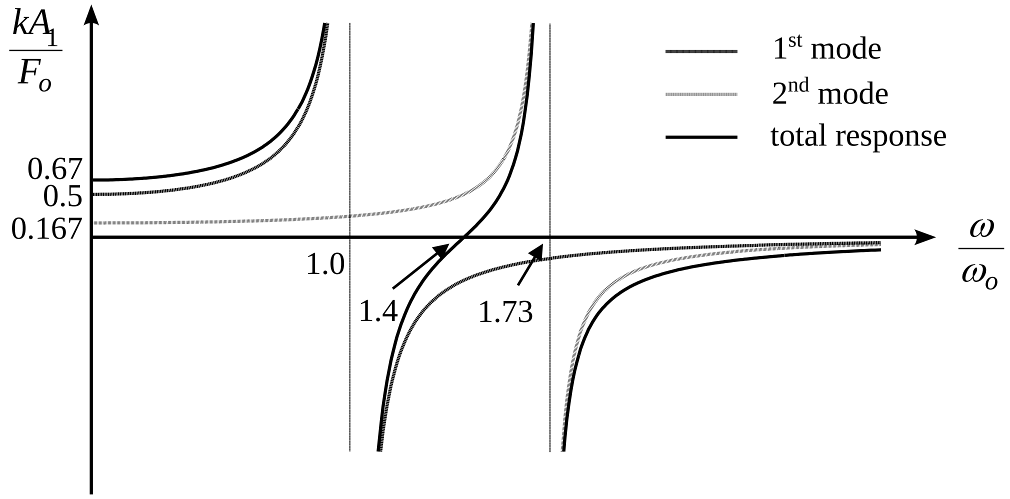


For the 2DOF spring-mass system



$$\frac{kA_1}{F_o} = \frac{0.5}{1-\left(\frac{\omega}{\omega_{n1}}\right)^2} + \frac{0.167}{1-\left(\frac{\omega}{\omega_{n2}}\right)^2}$$

This equation represents the linear superposition of the response of 2 SDOF systems with different natural frequencies (ω_{n1}, ω_{n2}). Note: $\omega_{n2} = \sqrt{3k/m} \approx 1.73\sqrt{k/m}$



In the absence of damping, the total response tends to infinity at the 2 ω_n's: $\omega_{n1} = \omega_o$ and $\omega_{n2} = \sqrt{3}\omega_o \approx 1.73\omega_o$.

$$\frac{kA_1}{F_o} = \frac{2-\left(\frac{\omega}{\omega_o}\right)^2}{\left(\frac{\omega}{\omega_o}\right)^4 - 4\left(\frac{\omega}{\omega_o}\right)^2 + 3} = \frac{0.5}{1-\left(\frac{\omega}{\omega_{n1}}\right)^2} + \frac{0.167}{1-\left(\frac{\omega}{\omega_{n2}}\right)^2}$$

The response is zero ($A_1 = 0$) when $(\omega/\omega_o)^2 = 2 \quad \Rightarrow \quad \omega = \sqrt{2}\omega_o \approx 1.41\omega_o$

Now to find the non-dimensionalised amplitude of the second mass.

$\frac{kA_2}{F_o} = \frac{kA_1}{F_o}.\frac{A_2}{A_1}$ = equation (6) * equation (3)

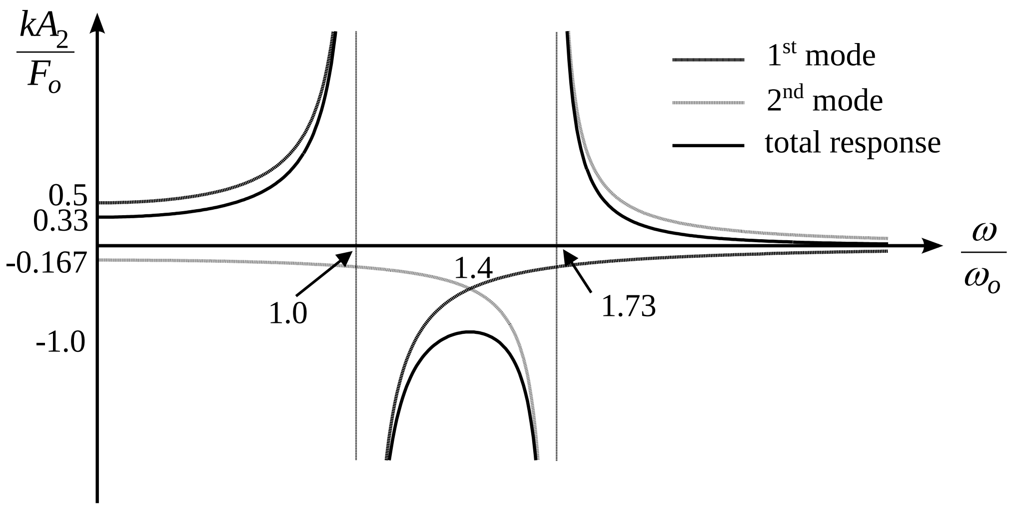
$$\frac{kA_2}{F_o} = \frac{kA_1}{F_o}.\frac{A_2}{A_1} = \frac{2k^2 - mk\omega^2}{m^2\omega^4 - 4km\omega^2 + 3k^2}.\frac{k}{2k - m\omega^2} = \frac{k^2}{m^2\omega^4 - 4km\omega^2 + 3k^2}$$

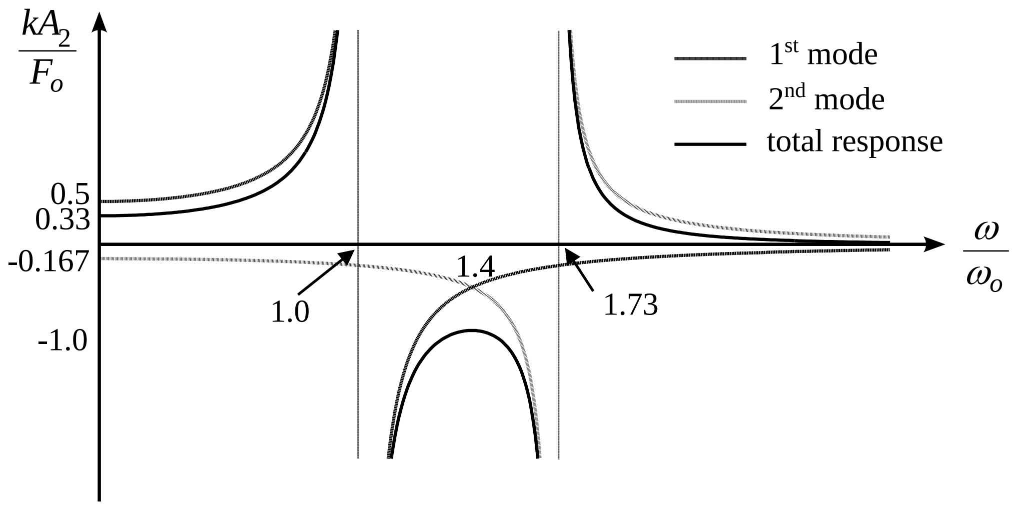
Again, divide by k^2/k^2, put $\omega_o = \sqrt{k/m}$ and expand into partial fractions in terms of ω_{n1} and ω_{n2}.

$$\frac{kA_2}{F_o} = \frac{1}{\left(\frac{\omega}{\omega_o}\right)^4 - 4\left(\frac{\omega}{\omega_o}\right)^2 + 3} = \frac{C_2}{1-\left(\frac{\omega}{\omega_{n1}}\right)^2} + \frac{D_2}{1-\left(\frac{\omega}{\omega_{n2}}\right)^2} = \frac{3C_2 + 3D_2 - (C_2 + 3D_2)\left(\frac{\omega}{\omega_o}\right)^2}{\left(\frac{\omega}{\omega_o}\right)^4 - 4\left(\frac{\omega}{\omega_o}\right)^2 + 3}$$

$$\Rightarrow \quad 3C_2 + 3D_2 = 1 \quad \text{and} \quad C_2 + 3D_2 = 0 \quad \Rightarrow \quad C_2 = 0.5, \quad D_2 = -0.167$$

$$\frac{kA_2}{F_o} = \frac{1}{\left(\frac{\omega}{\omega_o}\right)^4 - 4\left(\frac{\omega}{\omega_o}\right)^2 + 3} = \frac{0.5}{1-\left(\frac{\omega}{\omega_{n1}}\right)^2} - \frac{0.167}{1-\left(\frac{\omega}{\omega_{n2}}\right)^2}$$





Note, when $\omega = \sqrt{2}\omega_o \approx 1.41\omega_o$, $\dfrac{kA_2}{F_o} = -1$

For each mode of vibration, we can also check the modeshapes of the system $\dfrac{A_1}{A_2}$

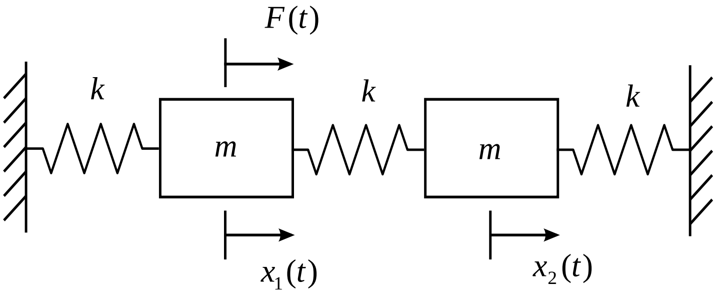
For $\quad \omega_{n1} = \omega_o = \sqrt{\dfrac{k}{m}}$, we know $\dfrac{A_1}{A_2} = 1$

$$\Rightarrow \quad \frac{A_1}{A_2} = \frac{kA_1}{F_o}.\frac{F_o}{kA_2} = \frac{0.5}{1-\left(\omega/\omega_{n1}\right)^2}.\frac{1-\left(\omega/\omega_{n1}\right)^2}{0.5} = 1$$

For $\quad \omega_{n2} = \sqrt{3}\omega_o = \sqrt{\dfrac{3k}{m}}$, we know $\dfrac{A_1}{A_2} = -1$

$$\Rightarrow \quad \frac{A_1}{A_2} = \frac{kA_1}{F_o}.\frac{F_o}{kA_2} = \frac{0.167}{1-\left(\omega/\omega_{n2}\right)^2}.\frac{1-\left(\omega/\omega_{n2}\right)^2}{(-0.167)} = -1$$

Discussion



The non-dimensionalised steady-state amplitude for each mass has been determined in terms of its modes of vibration (2 modes for a 2DOF system).

$$\frac{kA_1}{F_o} = \boxed{\frac{0.5}{1-\left(\dfrac{\omega}{\omega_{n1}}\right)^2}} + \boxed{\frac{0.167}{1-\left(\dfrac{\omega}{\omega_{n2}}\right)^2}}$$

$$\frac{kA_2}{F_o} = \underbrace{\boxed{\frac{0.5}{1-\left(\dfrac{\omega}{\omega_{n1}}\right)^2}}}_{\text{1st mode of vibration}} - \underbrace{\boxed{\frac{0.167}{1-\left(\dfrac{\omega}{\omega_{n2}}\right)^2}}}_{\text{2nd mode of vibration}}$$

Each mode of vibration has its own natural frequency (ω_n) and corresponding modeshape (A_1 / A_2). Also, each mode of vibration acts as a single DOF system. Hence, the total response of a 2DOF system can be obtained by the superposition of the responses of 2 single DOF systems.

The response of an *N*-DOF system can be obtained by the superposition of the responses of *N* single DOF systems.

For the example given of the 2DOF system, mass m_1 does not vibrate when the excitation frequency is $\sqrt{2}\omega_o \approx 1.41\omega_o$. That is, $kA_1 / F_o = 0$ when $\omega = \sqrt{2}\omega_o \approx 1.41\omega_o$. Mass m_1 has been *detuned*, and $\omega = \sqrt{2}\omega_o$ is called the detuned frequency. This is an important concept in vibration absorption.

Vibration Absorbers <u>Section 9.11 Rao</u>

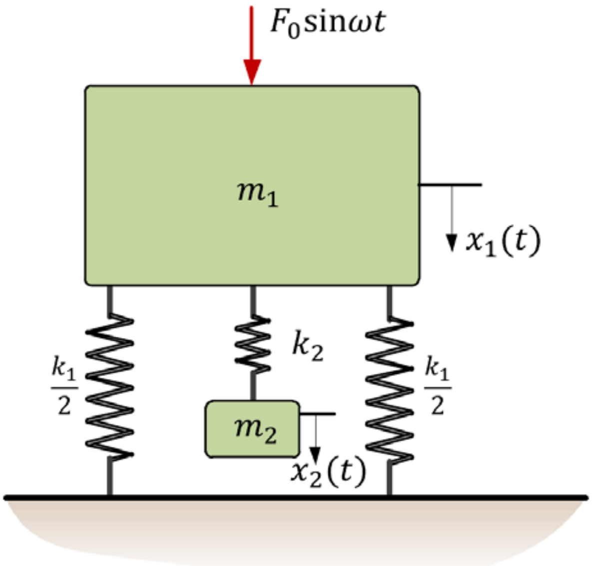
A machine or system may experience excessive vibration if it is acted upon by a force whose excitation frequency nearly coincides with a natural frequency of the machine or system. In such cases, the vibration of the machine or system can be reduced by using a dynamic vibration absorber, which is simply another spring-mass system. A vibration absorber protects the machine from steady-state harmonic excitation at a constant frequency. Absorbers are often used on machines that run at constant speed, for example, sanders, compactors and reciprocating tools. Vibration absorbers are also used in high-rise buildings to absorb the vibration due to air-conditioners.

The vibration absorber consists of a second mass-spring combination added to the primary machine or system to protect it from vibrating. The major effect of adding the second spring-mass system is to change from a single (or *N*) degree-of-freedom system to a 2 (or *N*+1) DOF system. The new system now has 2 (or *N*+1) natural frequencies. The values of the absorber mass and stiffness are chosen such that the motion of the original mass or system is a minimum at its frequency of operation.

The success of the absorber depends on a number of factors:

- The frequency of the harmonic excitation must be known and must be fairly constant (If the driving frequency drifts to one of the natural frequencies of the combined system, failure will occur).
- Damping can defeat the effectiveness of the absorber.
- Size and geometry constraints on the absorber reduces its effectiveness.

Consider a vibration absorber of mass m_2 and stiffness k_2 attached to a machine of mass m_1 and total stiffness k_1



Equations of motion

$$m_1\ddot{x}_1 = -\frac{k_1}{2}x_1 - \frac{k_1}{2}x_1 - k_2x_1 + k_2x_2 + F_o\sin\omega t$$

$$m_2\ddot{x}_2 = -k_2x_2 + k_2x_1$$

For harmonic motion, use general solutions of the form

$$x_1(t) = A_1 \sin \omega t \qquad \rightarrow \qquad \ddot{x}_1(t) = -\omega^2 A_1 \sin \omega t$$

$$x_2(t) = A_2 \sin \omega t \qquad \rightarrow \qquad \ddot{x}_2(t) = -\omega^2 A_2 \sin \omega t$$

Substituting the general solutions into the equations of motion results in

$$(k_1 + k_2 - m_1\omega^2)A_1 - k_2 A_2 = F_o \qquad (1)$$

$$-k_2 A_1 + (k_2 - m_2\omega^2)A_2 = 0 \qquad (2)$$

From (2), $$A_2 = \frac{k_2 A_1}{k_2 - m_2\omega^2} \qquad (3)$$

Substitute (3) in (1) gives $$(k_1 + k_2 - m_1\omega^2)A_1 - \frac{k_2^2 A_1}{k_2 - m_2\omega^2} = F_o$$

$$\Rightarrow \qquad \frac{A_1}{F_o} = \frac{k_2 - m_2\omega^2}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2} \qquad (4)$$

We are interested to reduce the amplitude of the machine of mass m_1 (that is, to reduce A_1). From (4) we see that $A_1 = 0$ when $k_2 - m_2\omega^2 = 0$

$$\Rightarrow \qquad \omega^2 = \frac{k_2}{m_2}$$

In addition, before the addition of the vibration absorber, the machine of mass m_1 operates near its resonance condition, that is, $\omega \approx \omega_n = \sqrt{k_1 / m_1}$. Hence, if the absorber is designed such that

$$\boxed{\omega^2 = \frac{k_2}{m_2} = \frac{k_1}{m_1}}$$

then the amplitude of vibration of the machine while operating at its original resonant frequency ($\omega \approx \omega_n = \sqrt{k_1 / m_1}$ for the SDOF system) will be zero.

Assume $m_2 = 0.2 m_1$. Sketch the non-dimensionalised amplitudes of the machine and absorber.

If $m_2 = 0.2 m_1$, $\quad k_2 = \dfrac{k_1}{m_1} m_2 = 0.2 k_1$

The natural frequencies of the combined system are found from the characteristic equation:

$$(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2 = 0$$

$$(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2 = 0$$

$$\Rightarrow \quad (k_1 + 0.2k_1 - m_1\omega^2)(0.2k_1 - 0.2m_1\omega^2) - (0.2k_1)^2 = 0$$

$$\Rightarrow \quad 0.2m_1^2\omega^4 - 0.44k_1m_1\omega^2 + 0.2k_1^2 = 0$$

$$\Rightarrow \quad m_1^2\omega^4 - 2.2k_1m_1\omega^2 + k_1^2 = 0$$

$$\Rightarrow \quad \omega_{n1} = 0.8\sqrt{\frac{k_1}{m_1}}, \qquad \omega_{n2} = 1.25\sqrt{\frac{k_1}{m_1}}$$

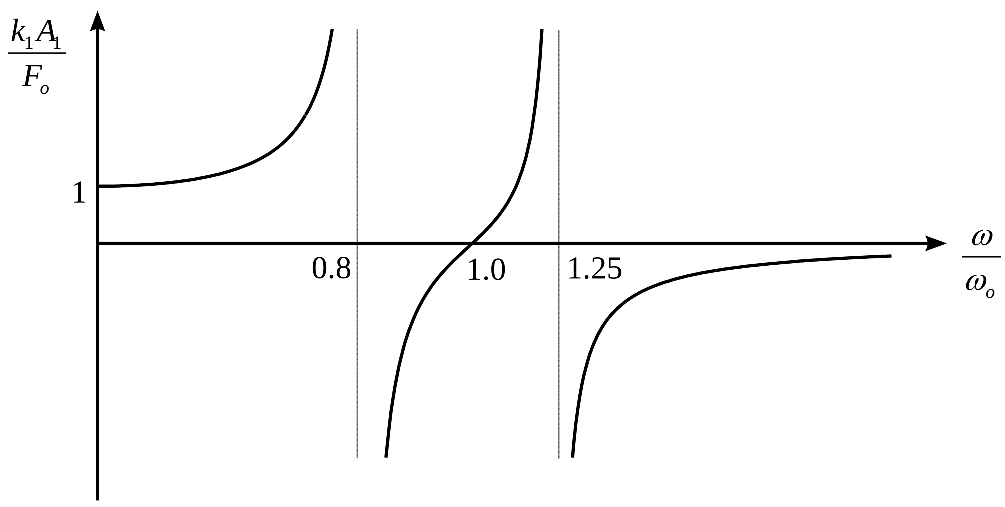
From (4)

$$\frac{A_1}{F_o} = \frac{k_2 - m_2\omega^2}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2}$$

$$= \frac{0.2k_1 - 0.2m_1\omega^2}{0.2m_1^2\omega^4 - 0.44k_1m_1\omega^2 + 0.2k_1^2}$$

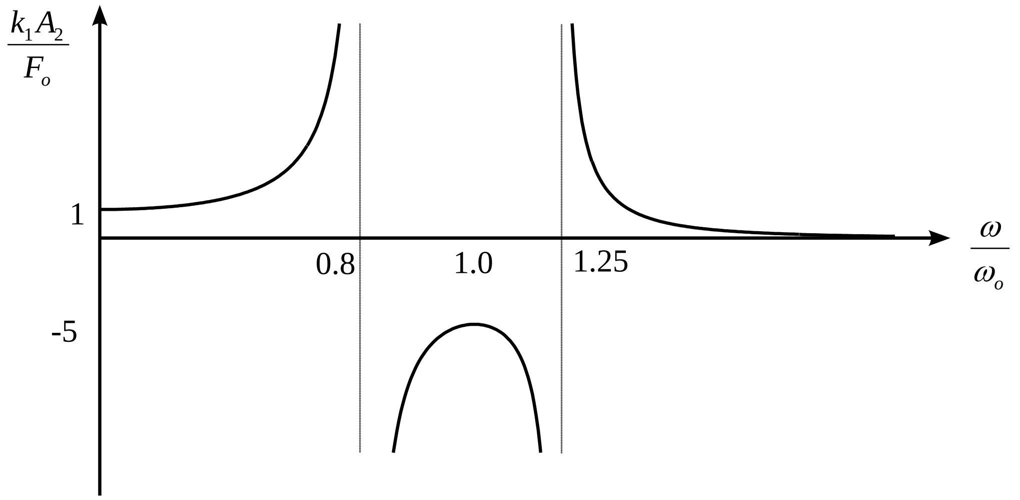
$$= \frac{k_1 - m_1\omega^2}{m_1^2\omega^4 - 2.2k_1m_1\omega^2 + k_1^2}$$

$$\frac{k_1A_1}{F_o} = \frac{1 - \left(\dfrac{\omega}{\omega_o}\right)^2}{\left(\dfrac{\omega}{\omega_o}\right)^4 - 2.2\left(\dfrac{\omega}{\omega_o}\right)^2 + 1} \qquad \text{where } \omega_o = \sqrt{\frac{k_1}{m_1}}$$



Similarly we can find

$$\frac{k_1 A_2}{F_o} = \frac{k_1 A_1}{F_o} \cdot \frac{A_2}{A_1} = \frac{1}{\left(\frac{\omega}{\omega_o}\right)^4 - 2.2\left(\frac{\omega}{\omega_o}\right)^2 + 1}$$



When $\frac{\omega}{\omega_o} = 1, \quad \frac{k_1 A_2}{F_o} = -5 \quad \Rightarrow \quad \frac{k_2 A_2}{F_o} = -1$

Discussion

- The original SDOF system of mass m_1 and stiffness k_1 is operating at near resonance conditions ($\omega \approx \omega_n = \sqrt{k_1 / m_1}$).
- A vibration absorber is attached, changing the SDOF system to a 2DOF system.
- The new system now has 2 natural frequencies.
- The original machine is no longer operating at its natural frequency and hence resonance is avoided.
- The values of the absorber (m_2, k_2) have been properly selected so that the vibration of the machine at its operating speed is minimised. The absorber values are chosen such that

$$\boxed{\omega^2 = \frac{k_2}{m_2} = \frac{k_1}{m_1}}$$