MMAN2300
Engineering Mechanics 2

Part B: Vibration Analysis

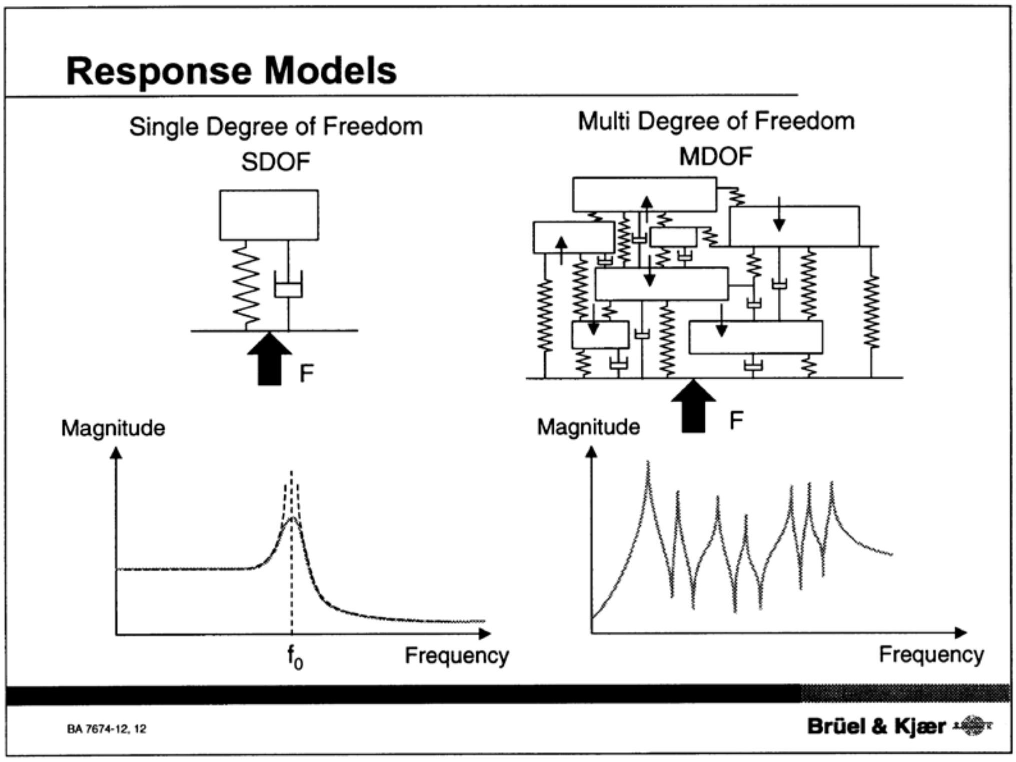
Two DOF Spring-Mass Systems

Two Degree-of-Freedom (2DOF) Systems

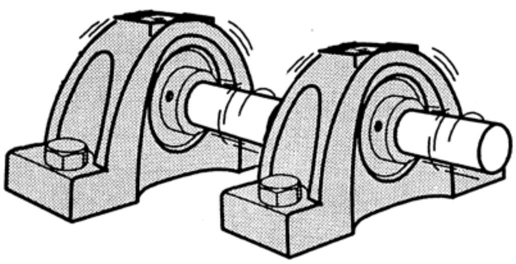
Chapter 5 Rao

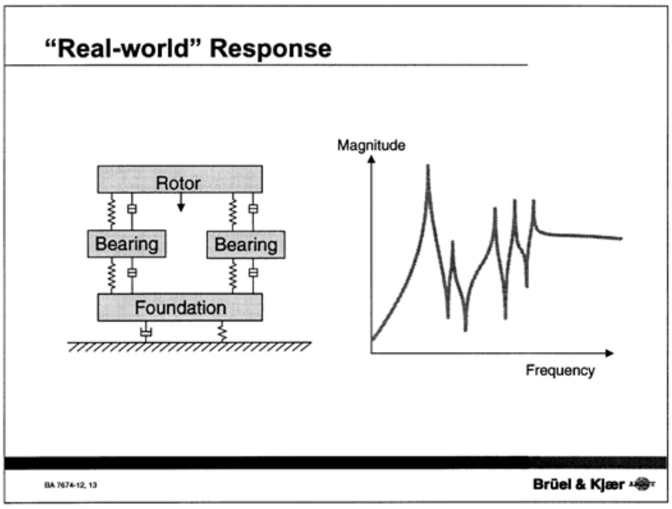
Real mechanical systems will normally be more complex than the single DOF systems. In a multi-DOF system, or an *N*-DOF system, the number of co-ordinates required to describe the motion is *N*. Also, an *N*-DOF system has *N* number of natural frequencies with a corresponding mode of vibration.

If the mechanical system consists of a number of interacting masses, springs and dampers, or it can move in more than one direction, it is called a multi-DOF system and the frequency spectrum will have one peak for each degree of freedom.

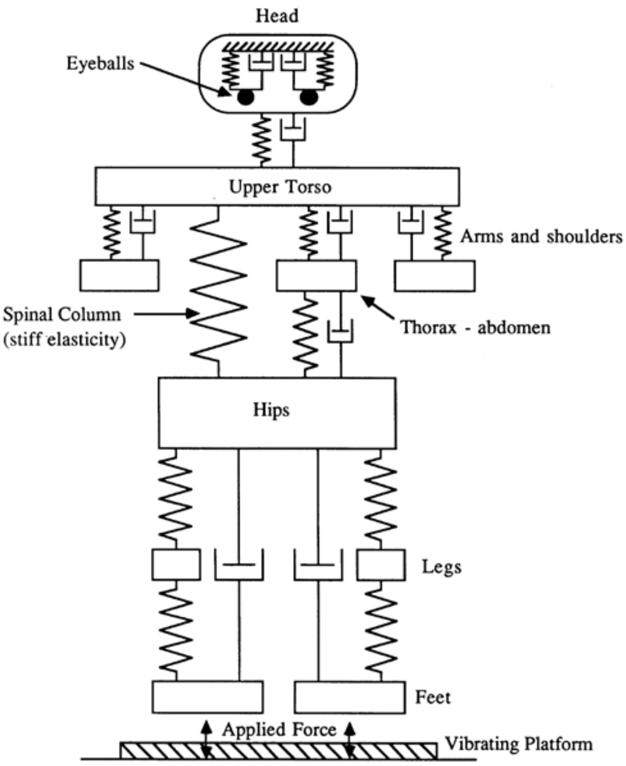


In most cases, even simple systems are to be considered multi-DOF systems, as illustrated here by a simple rotor in a couple of bearings.

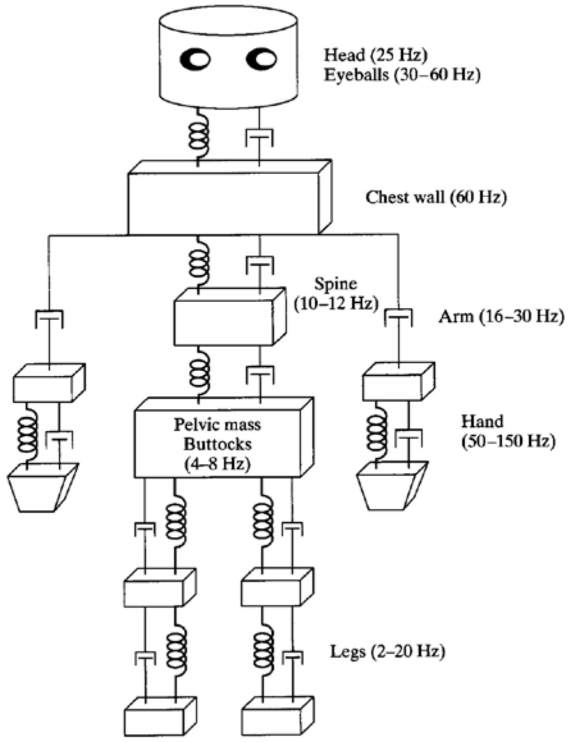




Even the human body can be approximated as a linear, lumped-parameter system for the analysis of low frequency (<200Hz) shock and vibration effects. A simplified multiple mass-spring-damper model of a human body standing on a vibration platform is shown in figure (a) on the left. Figure (b) on the right shows the vibration frequency sensitivity of different parts of the human body.

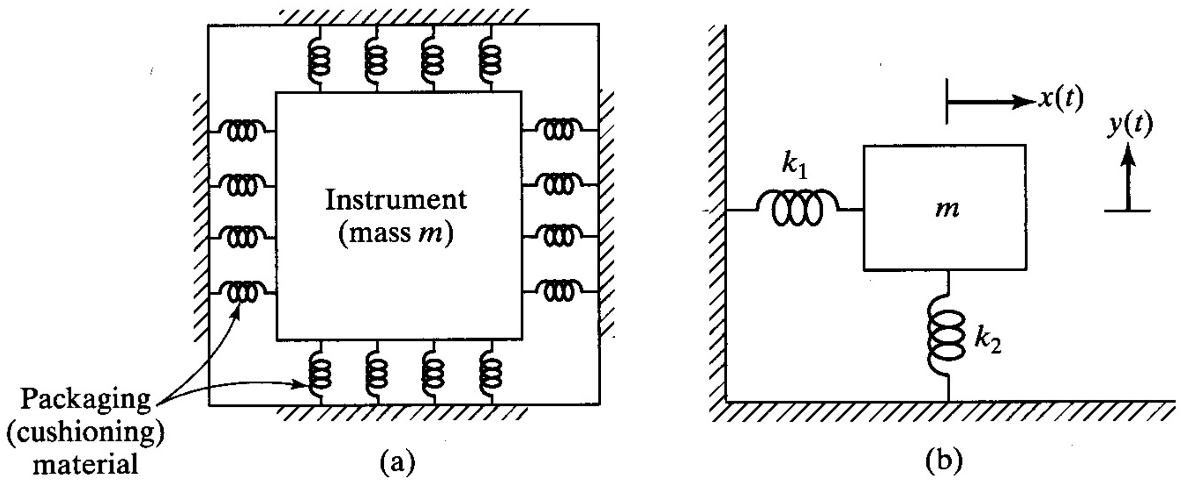


(a)



(b)

A single rigid body mass can have a maximum of 6 degrees of freedom – 3 components of translation and 3 components of rotation. The simplest *N*-DOF system is a 2-DOF system. A 2-DOF system requires 2 independent coordinates to describe its motion and has 2 natural frequencies. For example, the packaging of an instrument of mass *m* confines the motion of the instrument to the *xy* plane. The system is a 2-DOF system since there is a single mass with two types of motion – translation in the *x*-direction $x(t)$, and translation in the *y*-direction $y(t)$.



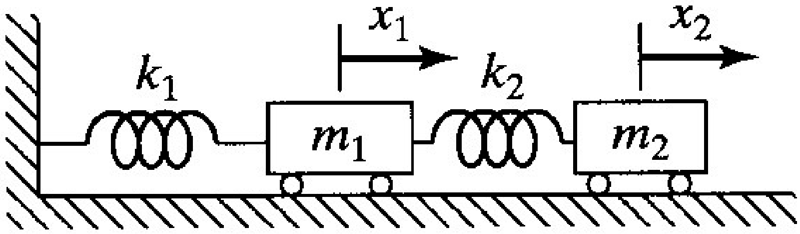
Packaging of an instrument (Figure 5.2 Rao)

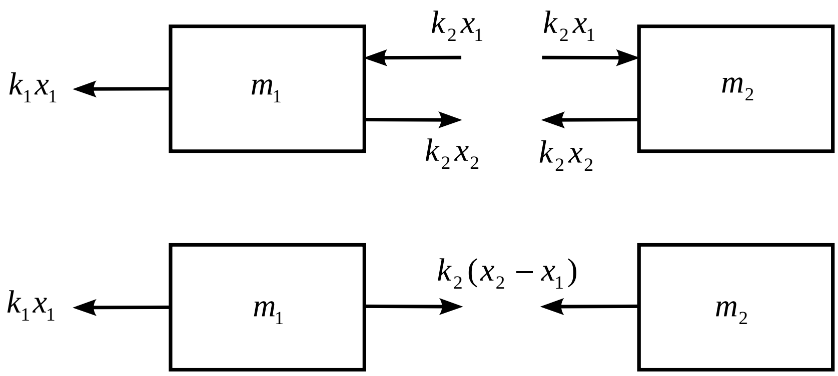
Two Degree-of-freedom (2DOF) System in Free Vibration

We will now consider a 2DOF system comprising two masses connected by a spring. As in the SDOF spring-mass system, the natural frequencies and corresponding modes of vibration are determined from the free vibration of the system.

Translational system

For the simple 2DOF system shown below, two masses are constrained to move in one direction only (in translational motion). The displacement of each mass with respect to time is measured from the static equilibrium position.





The equations of motion for the two masses are ($\sum F = m\ddot{x}$)

$$m_1\ddot{x}_1 = -k_1 x_1 - k_2 x_1 + k_2 x_2 \qquad m_1\ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1)$$

$$m_2\ddot{x}_2 = -k_2 x_2 + k_2 x_1 \qquad m_2\ddot{x}_2 = -k_2(x_2 - x_1)$$

and rearranging gives

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0$$

$$m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$$

Each mass undergoes harmonic motion of the same frequency. Hence, we can assume a general solution for each mass in sinusoidal motion of the form:

$$x_1(t) = A_1 \sin\omega t \qquad \rightarrow \qquad \ddot{x}_1(t) = -\omega^2 A_1 \sin\omega t$$

$$x_2(t) = A_2 \sin\omega t \qquad \rightarrow \qquad \ddot{x}_2(t) = -\omega^2 A_2 \sin\omega t$$

Substituting the general solutions into the equations of motion yields

$$-m_1\omega^2 A_1 + (k_1 + k_2)A_1 - k_2 A_2 = 0$$

$$-m_2\omega^2 A_2 - k_2 A_1 + k_2 A_2 = 0$$

Rearranging in matrix form yields a mass and a stiffness matrix:

$$-\omega^2 \underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{\text{mass matrix}} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} + \underbrace{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}}_{\text{stiffness matrix}} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Combining the mass and stiffness matrices yields

$$\begin{bmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The determinant of the mass and stiffness matrix results in the characteristic equation of the system

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 0$$

$$\begin{vmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\omega^2 \end{vmatrix} = (k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2 = 0$$

For this example, let $m_1 = m_2 = m$ and $k_1 = k_2 = k$. The characteristic equation becomes

$$(2k - m\omega^2)(k - m\omega^2) - k^2 = 0 \qquad \Rightarrow \qquad m^2\omega^4 - 3km\omega^2 + k^2 = 0$$

If *k* and *m* are known, we can solve directly for the natural frequencies. If *k* and *m* are not known, divide by k^2 (divide by k^N for an *N*-DOF system) and set $\omega_o = \sqrt{k/m}$. ω_o is simply a <u>reference</u> frequency. The solutions for ω from the characteristic equation yield the natural frequencies of the system (in terms of $\sqrt{k/m}$).

$$\frac{m^2}{k^2}\omega^4 - \frac{3m}{k}\omega^2 + 1 = 0 \qquad \Rightarrow \qquad \left(\frac{\omega}{\omega_o}\right)^4 - 3\left(\frac{\omega}{\omega_o}\right)^2 + 1 = 0$$

Note: $ax^2 + bx + c = 0 \qquad \Rightarrow \qquad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\Rightarrow \qquad \left(\frac{\omega}{\omega_o}\right)^2 = 0.382,\ 2.618 \qquad \Rightarrow \qquad \left(\frac{\omega}{\omega_o}\right) = 0.618,\ 1.618$$

$$\Rightarrow \qquad \omega_{n1} = 0.618\omega_o = 0.618\sqrt{k/m}, \qquad \omega_{n2} = 1.618\omega_o = 1.618\sqrt{k/m}$$

From the matrix expression, the ratio of the amplitudes are found to be

$$\frac{A_1}{A_2} = \frac{k_2}{k_1 + k_2 - m_1\omega^2} = \frac{k}{2k - m\omega^2} \qquad \text{OR} \qquad \frac{A_1}{A_2} = \frac{k_2 - m_2\omega^2}{k_2} = \frac{k - m\omega^2}{k}$$

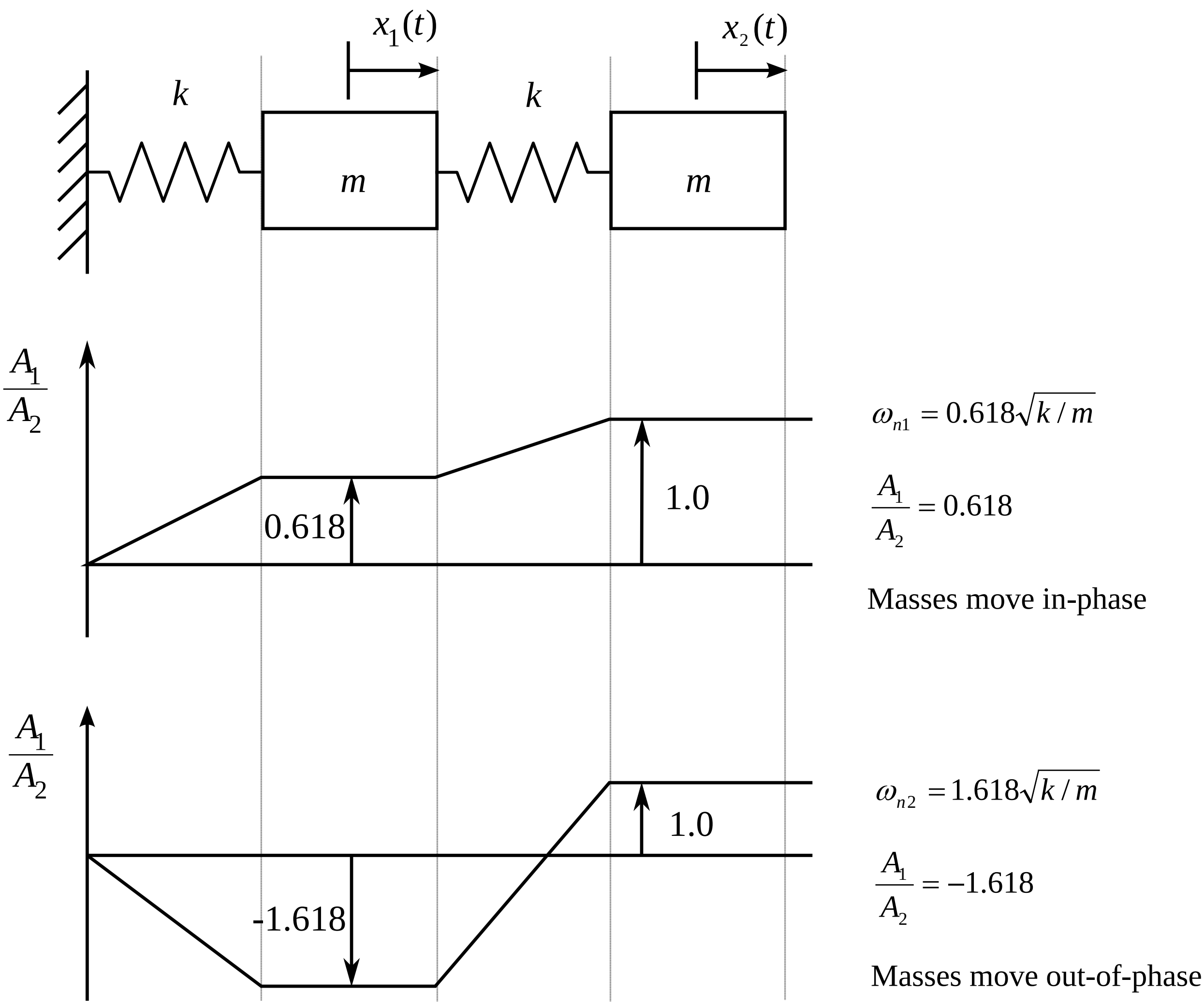
For each natural frequency, there is a corresponding modeshape (mode of vibration). Substitution of the natural frequencies in either one of the above equations for the amplitude ratio gives us the modeshapes of the system. That is, for spring-mass systems, the modeshape is simply the amplitude ratio at the natural frequencies.

For $\omega_{n1} = 0.618\sqrt{k/m}$, $\quad \dfrac{A_1}{A_2} = \dfrac{k}{2k - m\omega^2} = \dfrac{k}{2k - m0.618^2 k/m} = 0.618$

OR $\quad \dfrac{A_1}{A_2} = \dfrac{k - m\omega^2}{k} = \dfrac{k - m0.618^2 k/m}{k} = 0.618$

Hence, if $A_2 = 1$, then $A_1 = 0.618$

For $\omega_{n2} = 1.618\sqrt{k/m}$, $\quad \dfrac{A_1}{A_2} = -1.618$



The modeshapes of the system are shown in the figure above. In the first modeshape, the masses move in phase (that is, they are moving in the same direction), but in the second modeshape, the masses are moving out of phase with each other (in opposite directions).

The 2DOF spring-mass system has two natural frequencies and corresponding modeshapes.

The displacements of masses m_1 and m_2 are respectively given by:

$$x_1(t) = A_{11}\sin(\omega_{n1}t + \phi_1) + A_{12}\sin(\omega_{n2}t + \phi_2)$$

$$x_2(t) = A_{21}\sin(\omega_{n1}t + \phi_1) + A_{22}\sin(\omega_{n2}t + \phi_2)$$

where $\omega_{n1} = 0.618\sqrt{k/m}$ and $\omega_{n2} = 1.618\sqrt{k/m}$

$$\frac{A_{11}}{A_{21}} = 0.618 \qquad \frac{A_{12}}{A_{22}} = -1.618$$

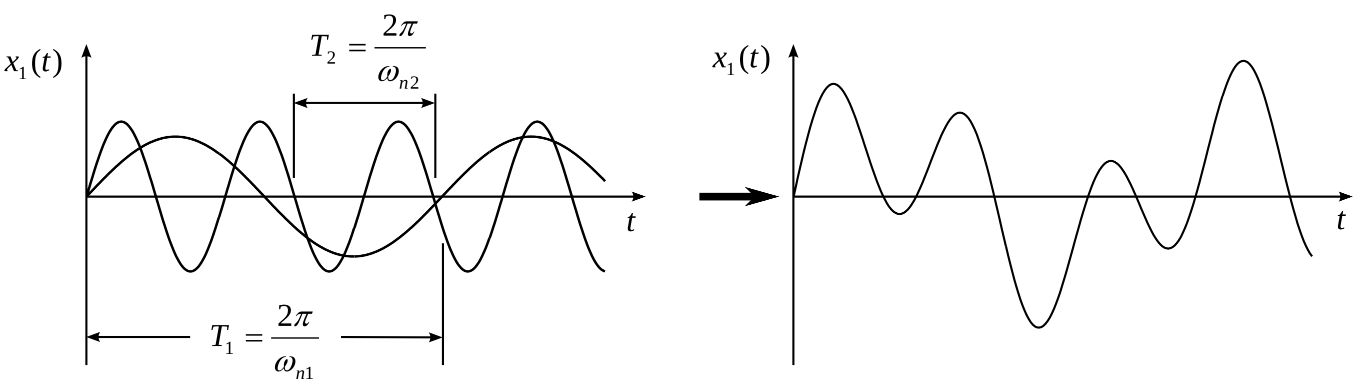
Hence we have

$$x_1(t) = 0.618A_{21}\sin(0.618\sqrt{k/m}t + \phi_1) - 1.618A_{22}\sin(1.618\sqrt{k/m}t + \phi_2)$$

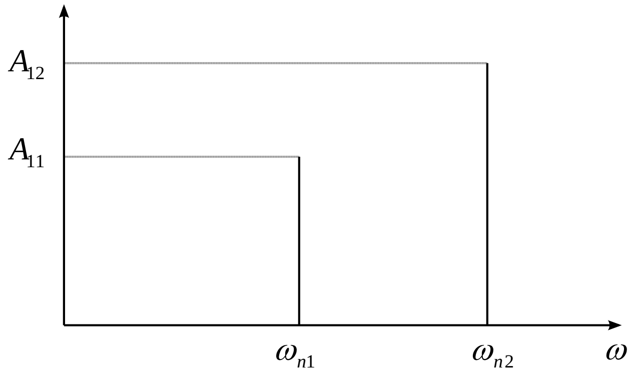
$$x_2(t) = A_{21}\sin(0.618\sqrt{k/m}t + \phi_1) + A_{22}\sin(1.618\sqrt{k/m}t + \phi_2)$$

where A_{21} and A_{22} (or A_{11} and A_{12}) and ϕ_1, ϕ_2 are found from the initial conditions given by $x_1(0)$, $\dot{x}_1(0)$, $x_2(0)$, $\dot{x}_2(0)$.

What is physically happening? For zero initial displacement of the first mass, that is, $x_1(0) = 0$, we can plot the time and frequency responses.



Using a Fast Fourier Transform (FFT), the frequency response can be obtained.



The motion of a 2-DOF system can be described as the *superposition* of the motion of 2 single DOF systems.

Hence, the motion of an *N*-DOF system can be described as the superposition of the motion of *N* single DOF systems.

At each natural frequency there is a corresponding natural mode of vibration

$$\omega_{n1} \quad \rightarrow \quad \frac{A_{11}}{A_{21}} \qquad \text{and} \qquad \omega_{n2} \quad \rightarrow \quad \frac{A_{12}}{A_{22}}$$

These modes of vibration (amplitude ratios) determine the shape at which the system vibrates at its natural frequencies. The modes can be represented by vectors in the form of column matrices:

$$\{A\}_1 = \begin{Bmatrix} A_{11} \\ A_{21} \end{Bmatrix}, \qquad \{A\}_2 = \begin{Bmatrix} A_{12} \\ A_{22} \end{Bmatrix}$$

The vectors $\{A\}_1$ and $\{A\}_2$ are called the *modal vectors*. The natural frequency ω_{n1} and the modal vector $\{A\}_1$ constitute what is termed the first mode of vibration, and ω_{n2} and $\{A\}_2$ constitute the second mode of vibration. For a 2DOF system there are two modes of vibration.

The natural modes of vibration, that is, the natural frequencies and the modal vectors represent a property of the system, and they are unique for a given system except for the magnitude of the modal vectors.

The magnitudes of the modal vectors will vary for different initial conditions, but the mode shape will not be affected. It is often conventional to assign a unity value to one of the components of the modal vector – this is known as *normalization*.