MMAN2300

Engineering Mechanics 2

Part B: Vibration Analysis

Continuous systems

Equivalent masses and springs

Transverse vibration of strings

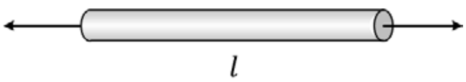
Modelling continuous systems as lumped spring-mass systems

In a lumped parameter spring-mass-damper system, the mass, spring and damper are treated as separate entities. In a continuous system (eg. a beam or a plate), the mass, stiffness and damping are uniformly distributed throughout the structure. In many simple cases, a continuous system can be modelled as a lumped parameter system. The equivalent lumped parameter system is dependent on:

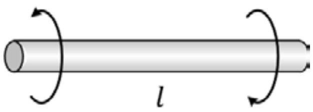
- the type of vibration the system is undergoing
- the boundary conditions
- location of load (external force or added mass).

There are three types of vibration that exist:

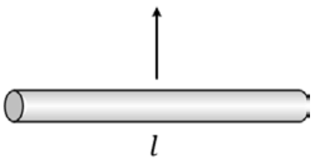
1) Longitudinal vibration (vibration in the direction of the length)



2) Torsional vibration (rotation about the length)



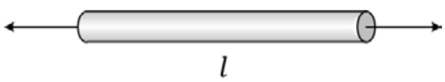
3) Bending (flexural, lateral, transverse) vibration (vibration normal to the length)



The stiffness of a bar k undergoing longitudinal vibration can be related to the material and geometric properties of the bar by

$$k = \frac{EA}{l}$$

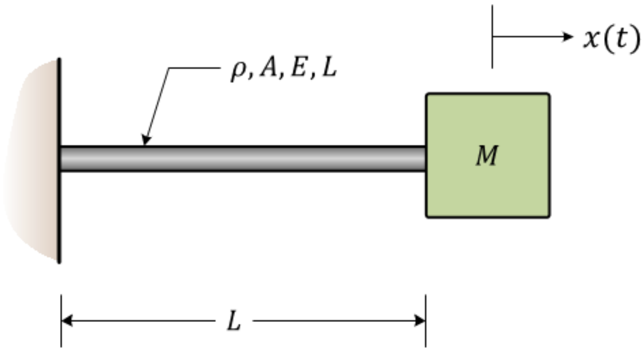
where: E is the Young's modulus of elasticity (N/m^2)
A is the cross-sectional area (m^2)
l is the length (m)



Rod under axial load

Example 1

Consider the longitudinal vibration of a cantilevered bar with a mass attached to one end.



The equivalent stiffness of a bar in longitudinal vibration is

$$k_{eq} = \frac{EA}{l}$$

In some cases, the mass of the bar is not negligible. It is possible to include the mass of the bar (m_{bar}) and still treat the system as a lumped spring-mass system.

For a bar in longitudinal vibration, the effective mass at the end of the bar is dynamically equivalent to adding $\frac{m_{bar}}{3}$ the main mass M

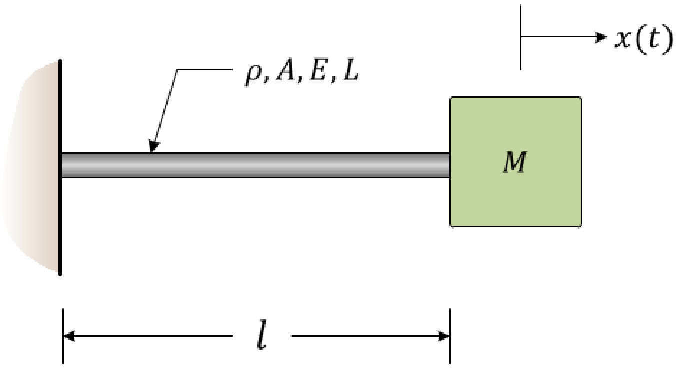
$$m_{eq} = M + m_{bar}/3$$

The natural frequency of the system is

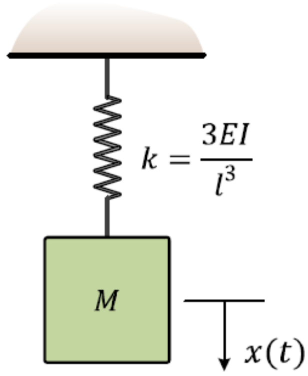
$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{EA}{l\left(M + \frac{m_{bar}}{3}\right)}}$$

Example 2

Consider the transverse vibration of a cantilevered beam with a mass attached to one end.



(a) Actual system



(b) Single DOF system

The equivalent stiffness of a cantilever beam with an end load is

$$k_{eq} = \frac{3EI}{l^3}$$

E: Young's modulus of elasticity (N/m^2)
I: second moment of area of cross-section (m^4)
l: length (m)

For a rectangular beam in bending vibration, the second moment of area of cross-section is

$$I = \frac{bh^3}{12}$$

where b is the width and h is the height. For a beam of negligible mass, we can treat the system as a single degree-of-freedom spring-mass system. The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{3EI}{ml^3}}$$

If the mass of the beam is not negligible, it can be shown that the effective mass at the end of the beam is dynamically equivalent to adding $\frac{33m_{beam}}{140}$ to the main mass

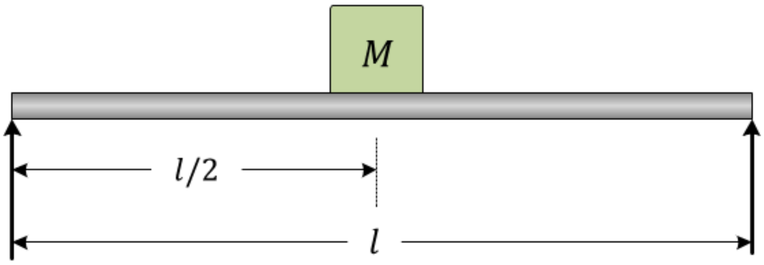
$$m_{eq} = m + \frac{33m_{beam}}{140}$$

The natural frequency of the system becomes

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{3EI}{\left(m + \frac{33m_{beam}}{140}\right)l^3}}$$

Example 3

Consider the transverse vibration of a simply supported beam carrying a mass M in the middle.



The equivalent stiffness of a simply supported beam with a load at the middle is

$$k_{eq} = \frac{48EI}{l^3}$$

The equivalent mass of the simply supported beam of mass m carrying a concentrated mass M at the middle is

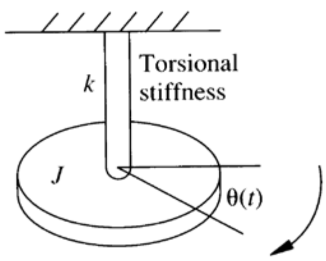
$$m_{eq} = M + 0.5m$$

The natural frequency of the system can be described by

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{48EI}{(M + 0.5m)l^3}}$$

Example 4

Consider the torsional vibration of a shaft and disk. The disk has mass moment of inertia J (kgm^2). The shaft has length l (m), diameter d (m), and second moment of area of the cross-section J_p (m^4).



For a solid circular shaft, the second moment of area of cross-section is $J_p = \frac{\pi d^4}{32}$

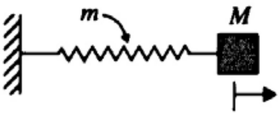
The torsional stiffness of the shaft (k_t) can be written as $k_t = \frac{GJ_p}{l}$

where G is the shear modulus of elasticity (N/m^2). The natural frequency of this system is

$$\omega_n = \sqrt{\frac{k_t}{J}} = \sqrt{\frac{GJ_p}{lJ}} = \sqrt{\frac{\pi G d^4}{32lJ}}$$

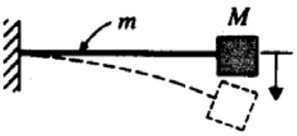
Equivalent Masses (inside cover of Rao text)

Equivalent masses



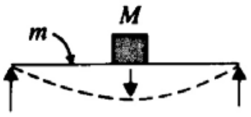
Mass (M) attached at end of spring of mass m

$$m_{eq} = M + \frac{m}{3}$$



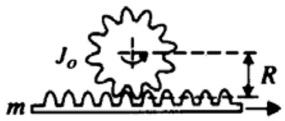
Cantilever beam of mass m carrying an end mass M

$$m_{eq} = M + 0.23\,m$$



Simply supported beam of mass m carrying a mass M at the middle

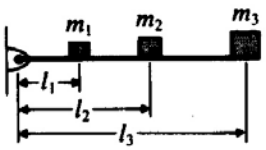
$$m_{eq} = M + 0.5\,m$$



Coupled translational and rotational masses

$$m_{eq} = m + \frac{J_0}{R^2}$$

$$J_{eq} = J_0 + mR^2$$

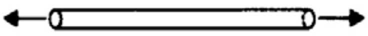


Masses on a hinged bar

$$m_{eq_1} = m_1 + \left(\frac{l_2}{l_1}\right)^2 m_2 + \left(\frac{l_3}{l_1}\right)^2 m_3$$

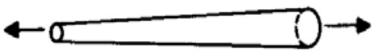
Equivalent Springs (inside cover of Rao text)

Equivalent springs



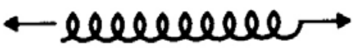
Rod under axial load
(l = length, A = cross sectional area)

$$k_{eq} = \frac{EA}{l}$$



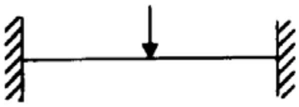
Tapered rod under axial load
(D, d = end diameters)

$$k_{eq} = \frac{\pi EDd}{4l}$$



Helical spring under axial load
(d = wire diameter, D = mean coil diameter, n = number of active turns)

$$k_{eq} = \frac{Gd^4}{8nD^3}$$

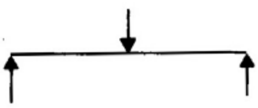


Fixed-fixed beam with load at the middle

$$k_{eq} = \frac{192EI}{l^3}$$

Cantilever beam with end load

$$k_{eq} = \frac{3EI}{l^3}$$

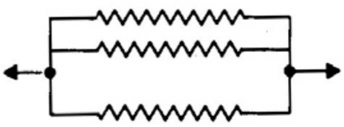


Simply supported beam with load at the middle

$$k_{eq} = \frac{48EI}{l^3}$$

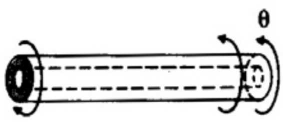
Springs in series

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}$$



Springs in parallel

$$k_{eq} = k_1 + k_2 + \cdots + k_n$$



Hollow shaft under torsion
(l = length, D = outer diameter, d = inner diameter)

$$k_{eq} = \frac{\pi G}{32l}(D^4 - d^4)$$

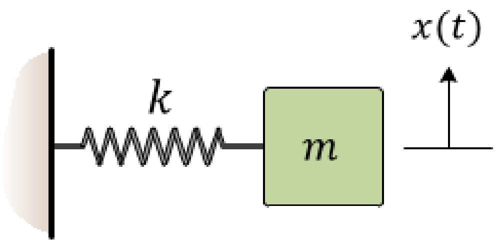
Continuous Systems (Distributed Parameter Systems)

Chapter 8 Rao

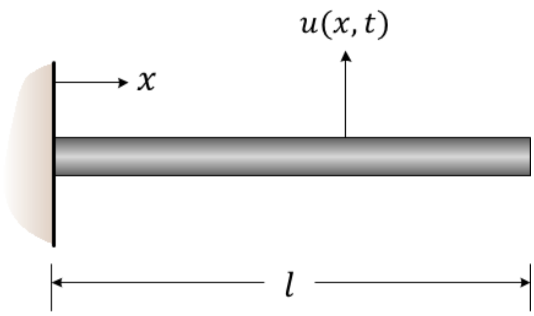
In continuous systems, such as beams, plates, shells and cylinders, which are commonly found in aircraft structures, ship hulls, pipelines, panels, etc, the mass, stiffness and damping are uniformly distributed throughout the structure as a series of infinitely small elements. When a structure vibrates, each of these infinite number of elements move relative to each other, and hence an infinitely large number of coordinates is required to describe their motion.

As a result, a continuous system has an infinite number of degrees-of-freedom, and hence an infinite number of natural frequencies and corresponding modeshapes.

A continuous system has to be modelled such that the motion of *each point* in the system can be specified as a function of time.



Lumped spring-mass system



Continuous system

The vibration of a continuous system is described in terms of both space and time coordinates. The resulting equation of motion is a partial differential equation.

The partial differential equations of continuous systems are called *wave equations*.

Wave equations are used to describe the propagation of waves in solids (beams, plates, shells), and in fluids (air, water).

Transverse Vibration of a String or Cable

Section 8.2 Rao

The simplest example of a continuous system is the transverse vibration of a flexible taut string (eg. a guitar string, cable between two electricity transmission towers). The string has mass per unit length m_L and is stretched under tension *T*. The lateral deflection of the string (u) is a function of both time t and position along the string x.

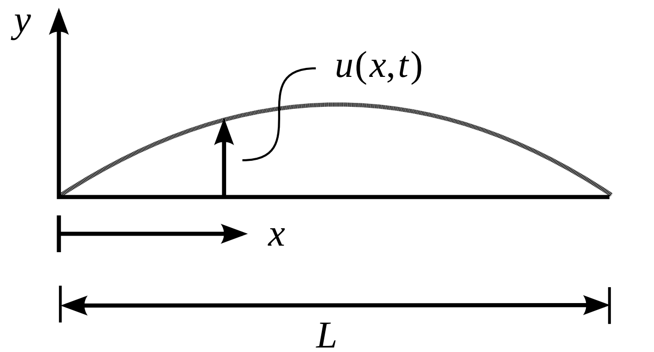
The lateral deflection u is assumed to be small, and the change in tension with deflection is negligible.

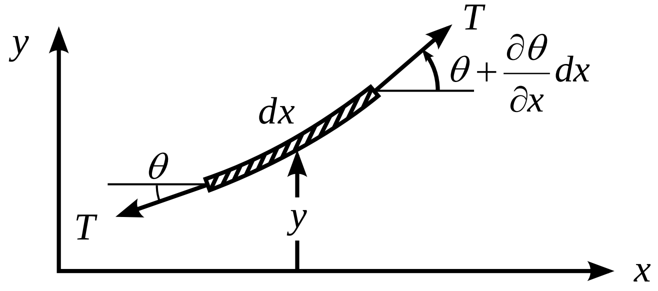
The equation of motion for transverse vibration in the y-direction is obtained by using Newton's 2nd law and assuming small deflections (u) and slopes (θ).

$$\sum F_y = ma_y$$

$$T\sin\left(\theta + \frac{\partial\theta}{\partial x}dx\right) - T\sin\theta = m_L dx \frac{\partial^2 u}{\partial t^2}$$

- θ is the slope of the string and is defined as $\theta = \dfrac{\partial u}{\partial x} \quad \Rightarrow \quad \dfrac{\partial\theta}{\partial x} = \dfrac{\partial^2 u}{\partial x^2}$
- $\theta + \dfrac{\partial\theta}{\partial x}dx$ is the expansion of the slope θ at position $x + dx$
- $m_L dx$ is the mass of the element dx (m_L is the mass per unit length)
- $\dfrac{\partial^2 u}{\partial t^2}$ is the acceleration of the string in the y-direction





Free body diagram of an element dx of the string

For small slopes, $\sin\theta \approx \theta$

$$\Rightarrow \qquad T\theta + T\frac{\partial\theta}{\partial x}dx - T\theta = m_L dx\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \qquad T\frac{\partial\theta}{\partial x}dx = m_L dx\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \qquad T\frac{\partial\theta}{\partial x} = m_L\frac{\partial^2 u}{\partial t^2} \qquad \text{and using} \qquad \frac{\partial\theta}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \qquad \boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c_s^2}\frac{\partial^2 u}{\partial t^2}} \quad \text{is the wave equation for a taut string}$$

where $c_s = \sqrt{\dfrac{T}{m_L}}$ is the wavespeed (m/s) of the waves propagating along the string.

The wave equation is a one-dimensional 2nd order partial differential equation (pde). The solution to the wave equation $u(x,t)$ can be obtained using the Separation of Variables technique.

Separation of Variables Technique

Section 8.2.3 Rao

Using the separation of variables technique, let $u(x,t)$ be the product of two separate functions, one of space x and the other of time t

$$\Rightarrow \qquad u(x,t) = \phi(x)q(t)$$

$$\Rightarrow \qquad \frac{d^2u}{dx^2} = \frac{d^2\phi(x)}{dx^2}q(t) \quad \text{and} \quad \frac{d^2u}{dt^2} = \phi(x)\frac{d^2q(t)}{dt^2}$$

Substituting the two equations above into the wave equation results in

$$\frac{d^2\phi(x)}{dx^2}q(t) = \frac{1}{c_s^2}\phi(x)\frac{d^2q(t)}{dt^2}$$

Divide both sides by $u(x,t)$ (that is, divide both sides by $\phi(x)q(t)$)

$$\Rightarrow \qquad \frac{1}{\phi(x)}\frac{d^2\phi(x)}{dx^2} = \frac{1}{c_s^2}\frac{1}{q(t)}\frac{d^2q(t)}{dt^2}$$

and rearrange as

$$\frac{c_s^2}{\phi(x)}\frac{d^2\phi(x)}{dx^2} = \frac{1}{q(t)}\frac{d^2q(t)}{dt^2}$$

The left hand side is independent of time t. The right hand side is independent of space x. Since both sides are a function of a different variable, for the equation to be valid, both sides have to be equal to a constant. It has been pre-determined (by 18th century mathematicians – Euler and Bernoulli) that this constant is $-\lambda_n$, where $\lambda_n = \omega_n^2$ is the eigenvalue and ω_n is the natural frequency.

$$\Rightarrow \qquad \frac{c_s^2}{\phi(x)}\frac{d^2\phi(x)}{dx^2} = \frac{1}{q(t)}\frac{d^2q(t)}{dt^2} = -\omega_n^2$$

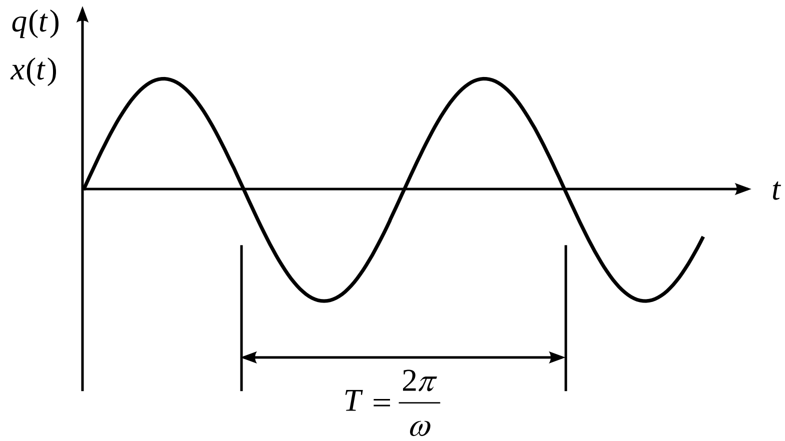
$$\Rightarrow \qquad \boxed{\frac{d^2q(t)}{dt^2} + \omega_n^2 q(t) = 0} \quad \text{and} \quad \boxed{\frac{d^2\phi(x)}{dx^2} + \frac{\omega_n^2}{c_s^2}\phi(x) = 0}$$

$k = \dfrac{\omega_n}{c_s}$ is the wavenumber and is the spatial equivalence of frequency. k has units [1/m]

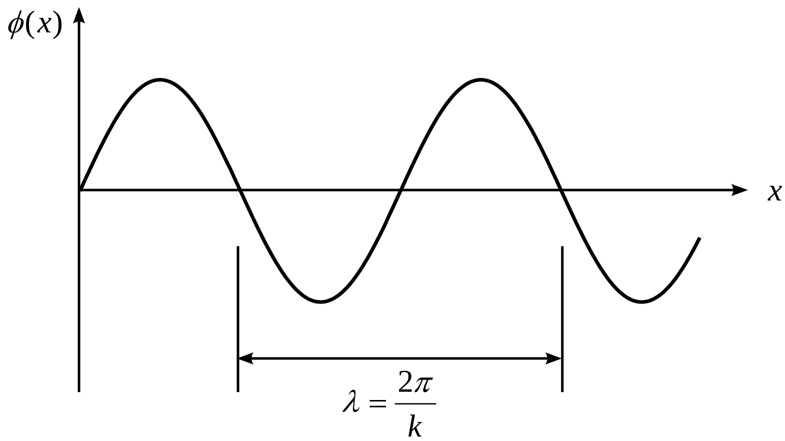
$$\Rightarrow \qquad \boxed{\frac{d^2\phi(x)}{dx^2} + k^2\phi(x) = 0}$$

The 2[nd] order partial differential equation now becomes two separate linear differential equations of motion (similar to the equation of motion for a single DOF spring-mass system $\ddot{x}+\omega_n^2 x=0$).

Frequency tells us how often a wave of a certain period T occurs in time



Wavenumber tells us how often a wave of a certain wavelength λ occurs in space



General solutions to the linear differential equations of motion are

$$q(t)=A\sin\omega_n t+B\cos\omega_n t$$

$$\phi(x)=C\sin kx+D\cos kx$$

The complete solution for $u(x,t)$ now becomes

$$u(x,t)=q(t)\phi(x)$$

$$\Rightarrow \quad \boxed{u(x,t)=(A\sin\omega_n t+B\cos\omega_n t)(C\sin kx+D\cos kx)}$$

The constants A and B are determined from the initial conditions at time $t=0$

$\Rightarrow \quad u(x,t=0)$ and $\dot{u}(x,t=0)$

The constants C and D are determined from the boundary conditions.

For a string stretched between two fixed points, the string has no displacement at its two ends.

The boundary conditions are

$$\Rightarrow \qquad u(x=0,t) = u(x=L,t) = 0$$

$$\Rightarrow \qquad \phi(0) = 0 \qquad \text{and} \qquad \phi(L) = 0$$

Since $\phi(x) = C\sin kx + D\cos kx$, the first boundary condition results in

$$\phi(0) = D = 0$$

$$\Rightarrow \qquad \phi(x) = C\sin kx$$

The second boundary condition results in

$$\phi(L) = C\sin kL = 0$$

Now $C \neq 0$, otherwise we have a trivial solution

$$\Rightarrow \qquad \sin kL = 0$$

$$\Rightarrow \qquad kL = n\pi, \qquad n = 1,\ 2,\ 3.....\infty$$

$$\Rightarrow \qquad k = \frac{n\pi}{L} = \frac{\omega_n}{c_s}$$

$$\Rightarrow \qquad \boxed{\omega_n = \frac{n\pi c_s}{L}} \qquad n = 1,\ 2,\ 3.....\infty$$

The string has an infinite number of natural frequencies. The spatial function $\phi(x)$ now becomes

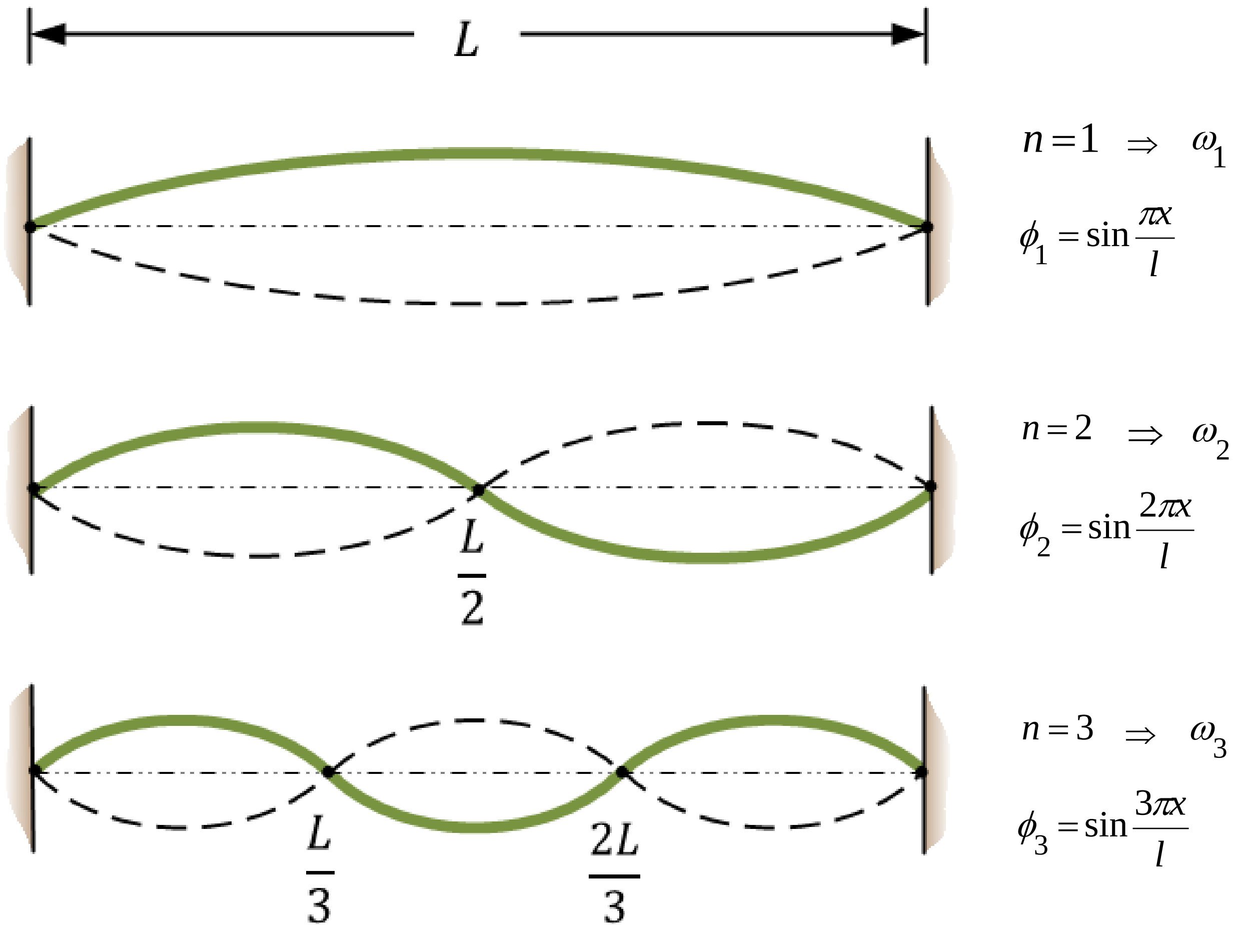
$$\phi(x) = \sin kx = \sin\frac{n\pi x}{L}, \qquad n = 1,\ 2,\ 3.....\infty$$

$\phi(x)$ represents the modeshape function (eigenfunction) for a continuous system.

For each mode number *n*, the string has a natural frequency ω_n and corresponding modeshape.

The complex vibration of a string can be described by an infinite number of simple modeshapes (in terms of half sine waves for a string). The summation of all of the modeshapes results in the total wave motion. The total string displacement is

$$u(x,t) = \sum_{n=1}^{\infty}\left(A\sin\omega_n t + B\cos\omega_n t\right)\sin\frac{n\pi x}{L}$$



Summary to obtain the Response of a Continuous System

- The displacement of a continuous system is described by both space (x) and time (t) coordinates $\Rightarrow\ u(x,t)$
- Take an element dx of the structure
- Draw a free body diagram of the element dx, showing all forces and/or moments acting on the element
- For a continuous system undergoing a particular type of vibration (longitudinal, torsional, bending),
- Use Newton's 2nd law ($\sum F = ma$, $\sum M = J\alpha$) to obtain the wave equation (a partial differential equation)
- Use the Separation of Variables technique: $u(x,t) = q(t)\phi(x)$ to obtain two linear differential equations (one in x and the other in t)
- General solutions to the linear differential equations
- Use the boundary conditions to get the ω_n's and modeshapes
- Use the initial conditions for u and $\dot{u}$ at t=0 to get the time constants