MMAN2300
Engineering Mechanics 2

Part B: Vibration Analysis

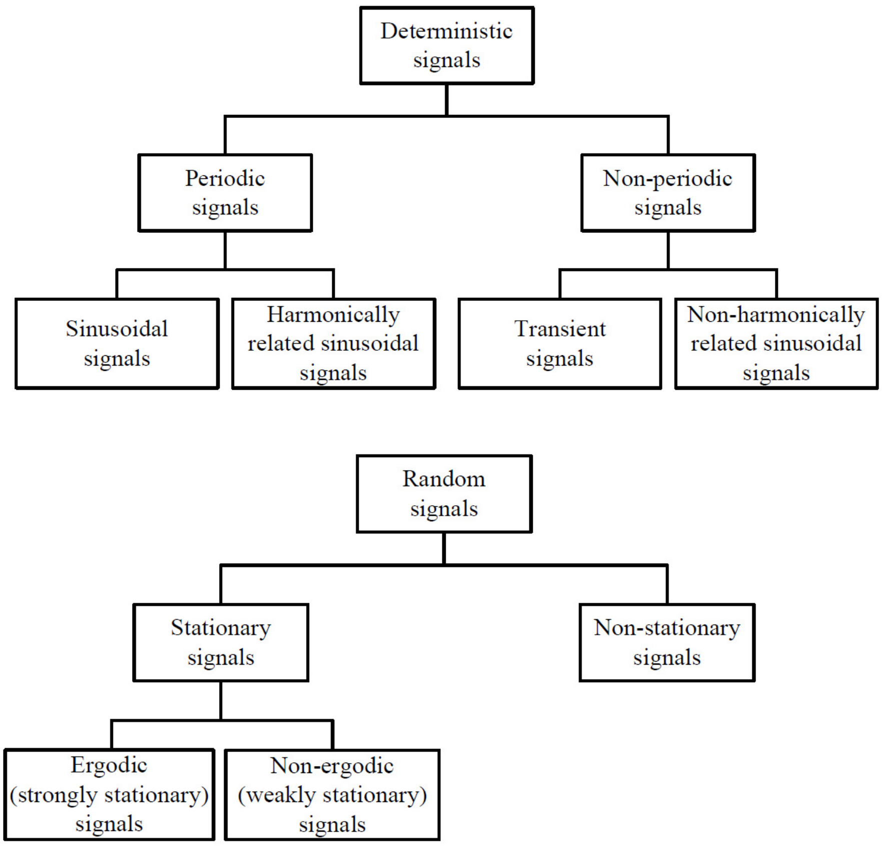
Forced Harmonic Vibration

Forced Harmonic Vibration

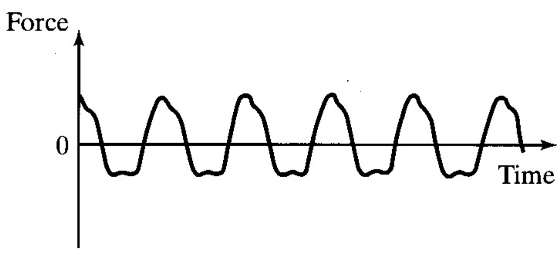
Some Classifications

Both the input and output of a system can be a range of functions, for example, force, displacement, velocity, acceleration, pressure, etc. The time histories of the input and output signals can be classified as either deterministic or random.

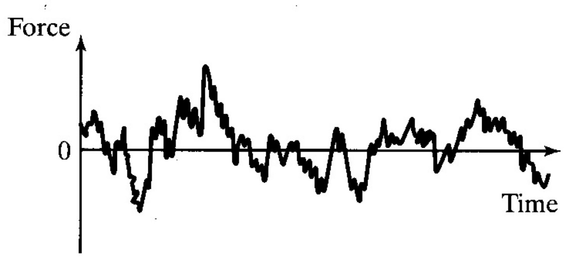
Deterministic signals can be expressed by mathematical relationships, but random signals have to be described in terms of probabilities and statistical averages. Deterministic signals may be either periodic or non-periodic. Some typical deterministic signals include vibration generated by rotating unbalance (electric motors, fans, rotors, piston-crankshaft mechanisms), vehicle vibration due to excitation of the suspension by a smooth road, vibration transmission in structures (e.g. from the diesel engines to a ship hull). Some typical random systems include turbulence, high speed gas flows in pipelines, response of a car travelling over a rough road.



Flow chart showing the different types of input and output signals.



(a) A deterministic (periodic) excitation



(b) A random excitation

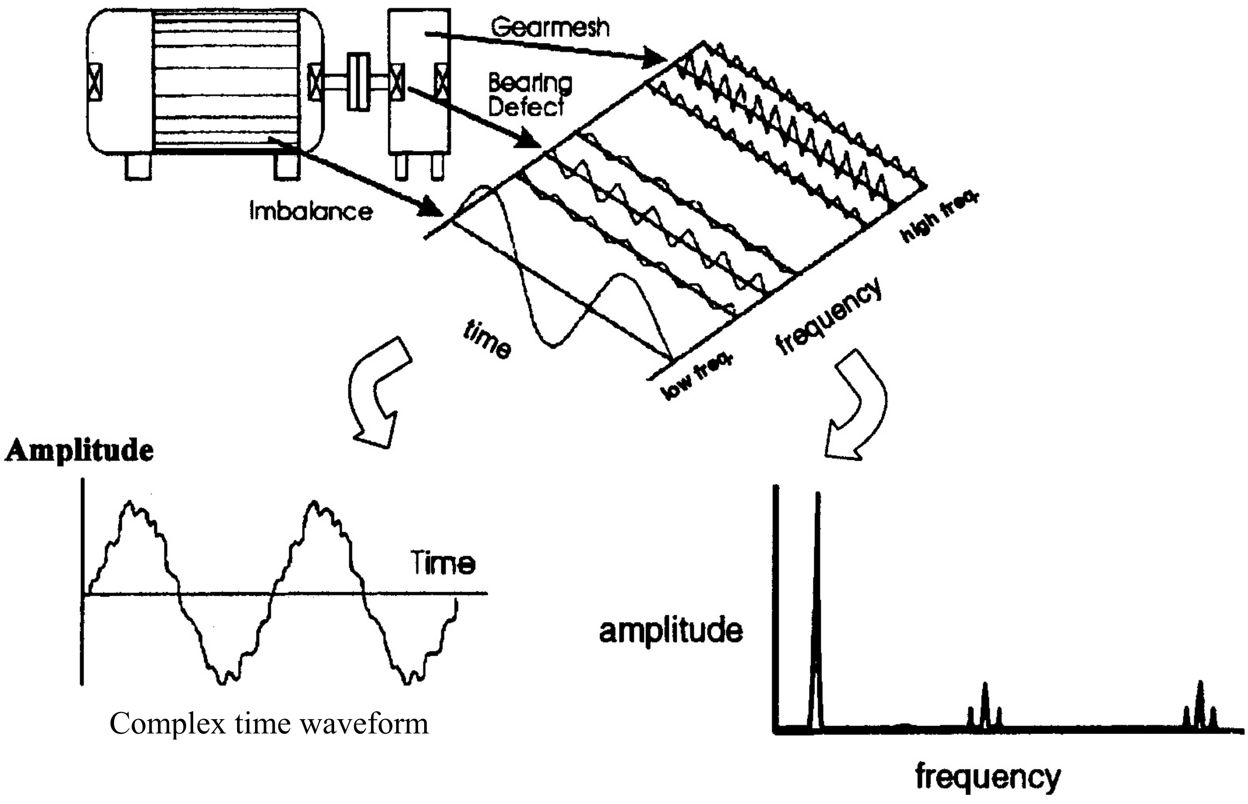
In this course, we are interested in periodic signals that result in sinusoidal or harmonic motion. When a system is subject to forced harmonic excitation, then the system is forced to vibrate at the same frequency as the external excitation frequency.

Common sources of harmonic motion are: unbalance of rotating machinery such as rotors, fans, pumps, electric motors, where a small mass unbalance can generate substantial vibration; response of a car travelling over a smooth road, etc. In these examples, only a few frequencies dominate the response. In most cases, these vibrations disturb the operation of the machines and the safety of the structure, especially if large amplitudes of vibration develop.

In the example below, individual components of the machine train (motor, bearings, gears) generate vibration signals unique to each component. Low frequency machinery problems can be attributed to shaft misalignment and imbalance. In addition, low frequency vibrations generally have high amplitudes. Higher frequency vibrations can be generated by bearing or gearmesh problems.

In the time waveform below, the individual sinusoidal vibration signals combine to form a complex time waveform showing overall vibration. The frequency spectrum is obtained by doing a Fast Fourier Transformation (FFT) analysis of the time response. The frequency spectrum shows vibration signals of the individual components at their respective frequencies.

The amplitude is the magnitude of the vibration signal. The time domain represents amplitude vs time. The frequency spectrum represents amplitude vs frequency. In vibration analysis of machinery, the vibration amplitude indicates the severity of the problem (the higher the amplitude, the bigger the problem). The frequency indicates the source of the problem (eg. imbalance, bearing fault, gear problem, etc). Resonance occurs when the excitation frequency matches the natural frequency or frequencies of the system ($\omega = \omega_n$). Resonance is to be avoided in most cases. In order to prevent large amplitudes from developing, dampers and absorbers are often used.



Complex time waveform

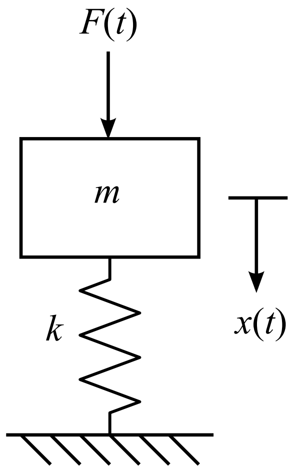
Response of an Undamped Spring-Mass System under Harmonic Force

Section 3.3 Rao

Forced harmonic excitation refers to a sinusoidal external force of a single excitation frequency applied to a system. Sinusoidal excitation comes from rotating machinery such as fans, electric motors and reciprocating machines. The external force $F(t)$ can be described as a sine function of the form:

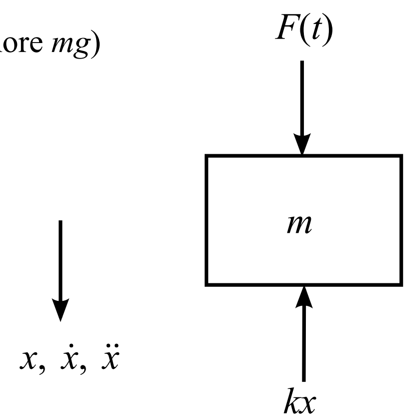
$$F(t) = F_o \sin \omega t$$

where F_o represents the maximum amplitude of the force, and ω is the excitation frequency of the applied force.



When a system is subject to forced harmonic excitation, it is forced to vibrate at the same frequency as the external excitation frequency.

Free body diagram of mass (ignore *mg*)



The equation of motion of the system under forced excitation becomes

$$m\ddot{x} + kx = F(t) \qquad \text{or} \qquad m\ddot{x} + kx = F_o \sin \omega t$$

The solution to this equation consists of two parts:

- the complementary function $x_c(t)$ which is the solution of the system in free vibration, and
- the particular function $x_p(t)$ which is the solution of the system under forced vibration.

Since we are dealing with linear systems, the total response of the system $x(t)$ is the superposition of the free and forced responses.

$$x(t) = x_p(t) + x_c(t)$$

For an undamped spring-mass system, the response of the mass in free vibration (complementary function) can be written as

$$x_c(t) = A\sin(\omega_n t + \phi_1)$$

where A and ϕ_1 are determined from the initial conditions for x and $\dot{x}$ at t=0.

A general solution for the forced response (particular solution) can be written as

$$x_p(t) = X\sin(\omega t - \phi)$$

Note that the forced response is a function of the external excitation frequency ω.

The forced response has a steady-state amplitude X and ϕ is the steady-state phase (the steady-state response $x_p(t)$ will lag the force $F(t)$ by ϕ).

Substituting $x_p(t)$ into the equation of motion results in the following particular solution

$$x_p(t) = \frac{F_o}{k} \cdot \frac{\sin(\omega t - \phi)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

The total response of the spring-mass system under forced excitation is

$$x(t) = x_p(t) + x_c(t) = \underbrace{\frac{F_o}{k} \cdot \frac{\sin(\omega t - \phi)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}}_{\text{Steady-state response}} + \underbrace{A\sin(\omega_n t + \phi_1)}_{\text{Transient response}}$$

The first part of the equation on the RHS represents the steady-state response (the particular solution) and the second part of the equation represents the transient or free response (the complementary solution).

In most cases of steady-state excitation, the transient response can be ignored. Transient vibration is of greater concern in situations where shock, impact and moving loads are involved.

Under steady-state excitation, we can generally ignore the free response. In the steady-state, we are interested in <u>amplitude</u> and <u>phase</u>. It is conventional to non-dimensionalise the amplitude (X) and phase (ϕ).

$$x_p(t) = X\sin(\omega t - \phi) = \frac{F_o}{k}.\frac{\sin(\omega t - \phi)}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

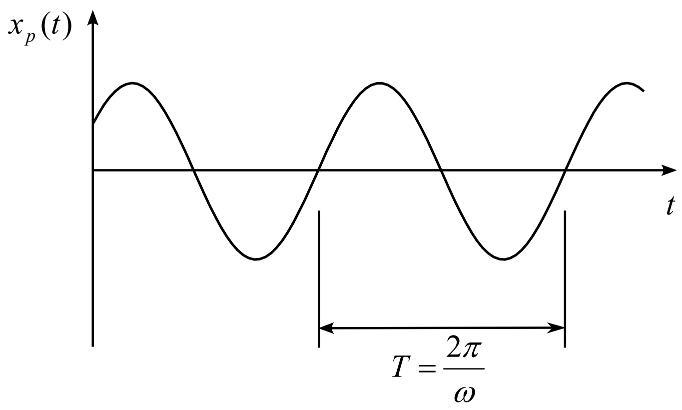
The non-dimensionalised amplitude becomes $\frac{kX}{F_o} = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}$

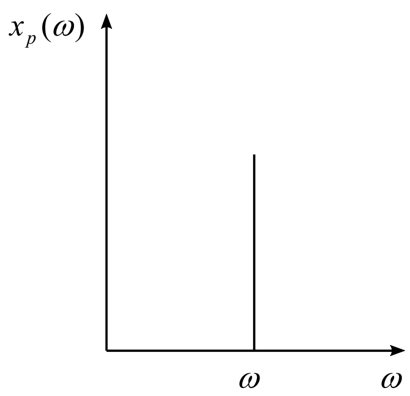
$\frac{kX}{F_o}$ is also represented by $\frac{X}{\delta_{st}}$ where $\delta_{st} = \frac{F_o}{k}$ is the static deflection if F_o is a static force.

$\frac{X}{\delta_{st}}$ represents the ratio of the dynamic to the static amplitude, and is called the 'dynamic amplification factor'.

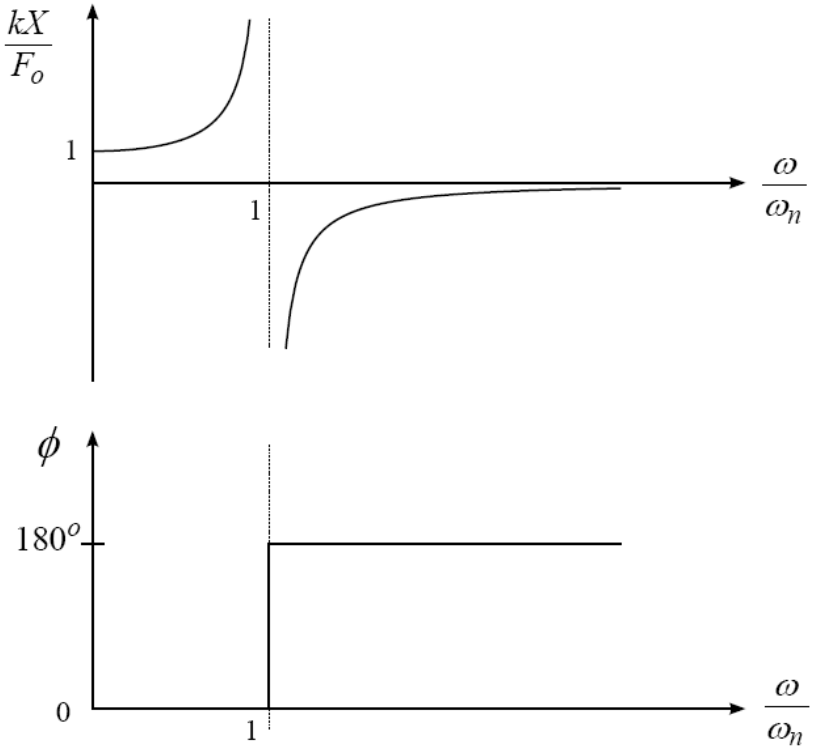
$\frac{kX}{F_o}$ is only a function of the frequency ratio $\frac{\omega}{\omega_n}$

For a single driving frequency ω, the time and frequency responses for x_p are





We can also plot $\frac{kX}{F_o}$ as a function of the frequency ratio $\frac{\omega}{\omega_n}$

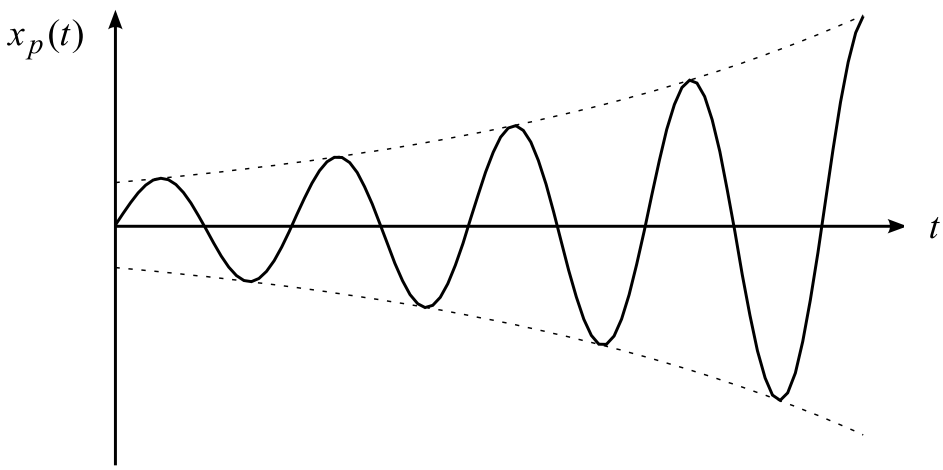


When $0 < \omega/\omega_n < 1$, the response of the system $x_p(t)$ is in phase with the external force. As $\omega \to 0$, the amplitude X approaches F_o/k (ie. $kX/F_o \to 1$).

When $\omega/\omega_n > 1$, the amplitude decreases. As $\omega/\omega_n \to \infty$, $X \to 0$. Thus the response of the system to a harmonic force of very high frequency is close to zero.

When $\omega/\omega_n = 1$, the amplitude X tends to infinity. $\omega = \omega_n$ is called resonance.

The response of an undamped system at resonance grows without bounds. It can be shown that the amplitude of the steady-state response builds up linearly. Note: even with zero damping, it is possible to pass through a resonance frequency as long as the frequency sweep rate is sufficiently high.



Response of a Spring-Mass-Damper System under Harmonic Force

Section 3.4 Rao

Now consider a single DOF system with viscous damping under forced excitation. We are only interested in under-damped systems ($\zeta < 1$). The equation of motion now becomes:

$$m\ddot{x} + c\dot{x} + kx = F_o \sin \omega t$$

The total response of the system is the super-position of the particular solution $x_p(t)$ (forced response) and the complementary function $x_c(t)$ (response in free vibration):

$$x(t) = x_p(t) + x_c(t)$$

For an under-damped system, the general solution in free vibration can be written as:

$$x_c(t) = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1)$$

where A and ϕ_1 are determined from the initial conditions at t = 0. A general solution for the forced response is:

$$x_p(t) = X \sin(\omega t - \phi)$$

Substituting the particular solution into the equation of motion results in:

$$X = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}, \qquad \phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

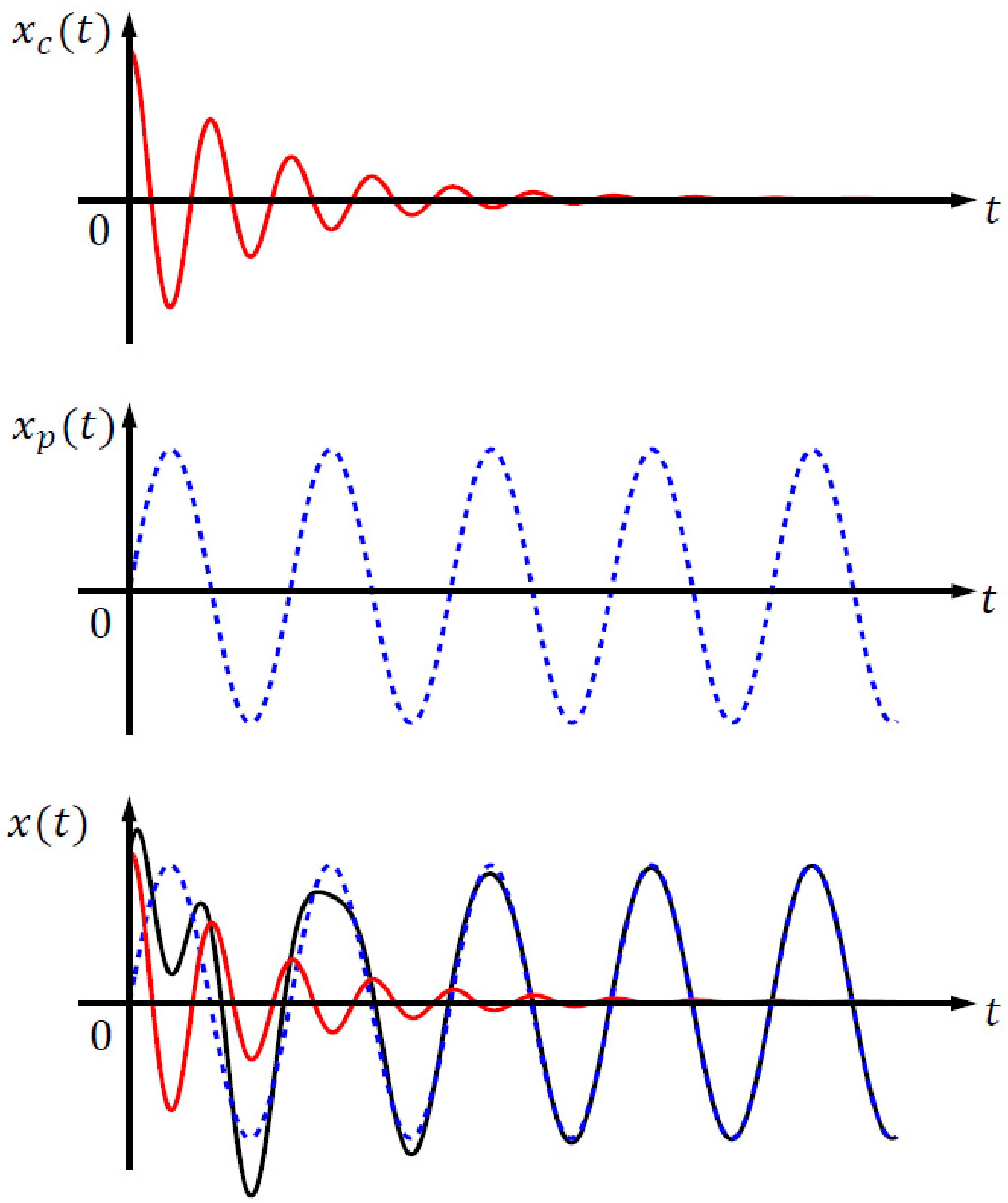
Dividing throughout by k, and putting $\omega_n = \sqrt{k/m}$ and $2\zeta \dfrac{\omega}{\omega_n} = \dfrac{c\omega}{k}$ results in:

$$X = \frac{F_o / k}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta \frac{\omega}{\omega_n}\right]^2}}, \qquad \phi = \tan^{-1}\left(\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$$

Hence, the total response of the forced, under-damped single DOF system is:

$$x(t) = \underbrace{\frac{F_o}{k} \frac{\sin(\omega t - \phi)}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta \frac{\omega}{\omega_n}\right]^2}}}_{\text{Steady-state response}} + \underbrace{Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1)}_{\text{Transient response}}$$

Again, the first part of the equation on the RHS represents the steady-state response, and the second part of the equation represents the transient (free) response. In many cases, the transient response can be ignored, depending also on the value of the damping ratio ζ. For large damping, the term $e^{-\zeta\omega_n t}$ causes the transient response to die out quickly. If the system is lightly damped ($\zeta << 1$), the transient response may be significant. Transient vibration is of greater concern in situations where shock, impact and moving loads are involved. Steady-state vibration exists long after free vibration has died away. It is generally associated with the continuous operation of machinery. If mechanical failure occurs, it is usually due to fatigue after a long period of time.



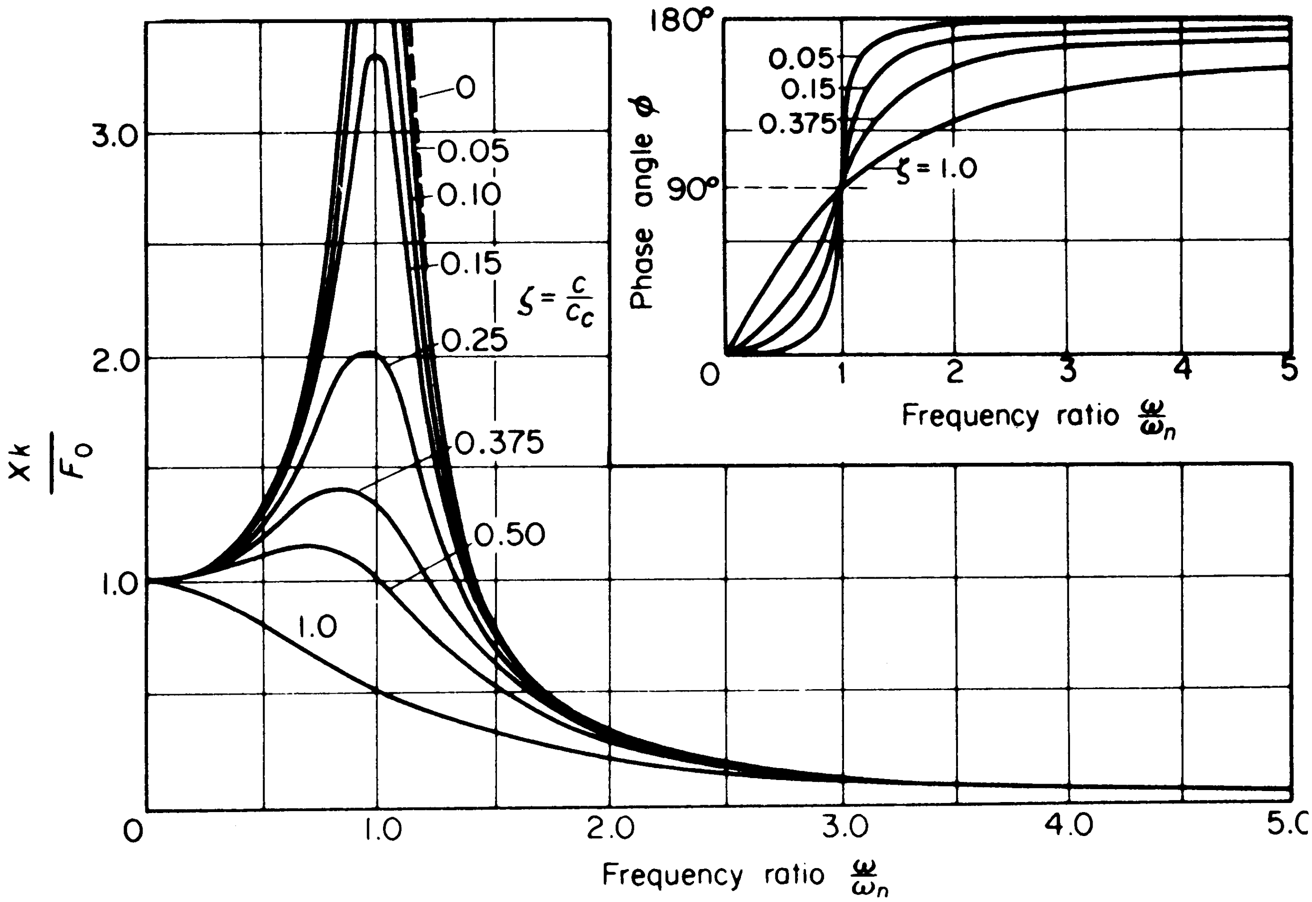
$$x(t) = x_p(t) + x_c(t)$$

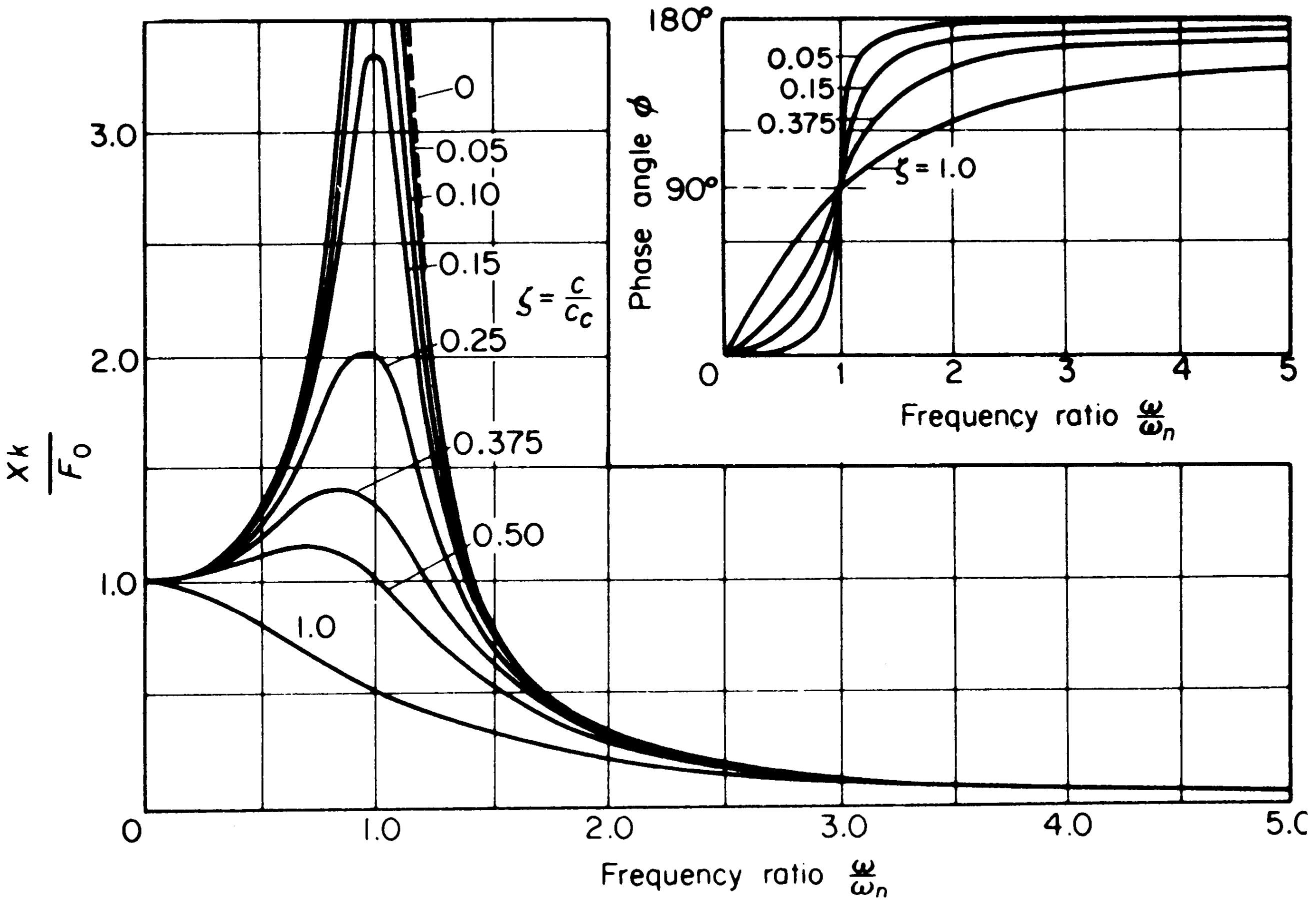
The non-dimensionalised steady-state amplitude and phase can be written as:

$$\frac{kX}{F_o} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}}, \qquad \phi = \tan^{-1}\left(\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$$

These equations show that the non-dimensional amplitude kX/F_o and the phase ϕ are only functions of the frequency ratio ω/ω_n and the damping ratio ζ. Both the amplitude and phase can be plotted as a function of the frequency ratio ω/ω_n, for various values of damping ratio ζ.

The curves for the under-damped case shows that the damping ratio ζ has a large influence on the amplitude and phase angle in the frequency region near resonance.





At low frequencies ($\omega/\omega_n << 1$), the amplitude X approaches F_o/k, which is the static deflection if F_o is the static force.

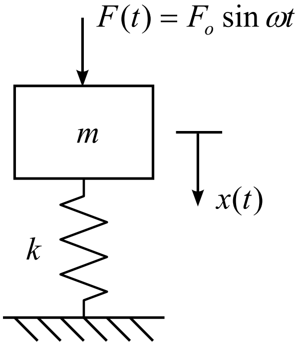
For $\omega/\omega_n = 1$, the excitation frequency is equal to the natural frequency of the system, and the resonance condition occurs. This results in a very large amplitude of vibration. The corresponding phase angle is 90°, and there is a phase shift from 0° to 180°. A phase shift from 0° to 180° (and vice versa) represents resonance.

For very large frequencies ($\omega/\omega_n >> 1$), the amplitude decreases and the motion of the system becomes very small.

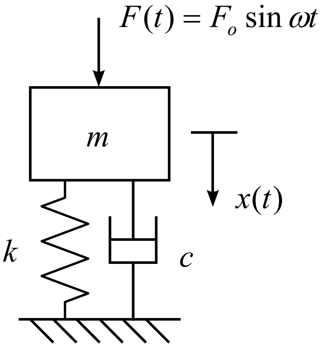
Note: in the case of a machine designed to operate at a speed above its resonance speed ($\omega > \omega_n$), no great difficulty is experienced in passing through the resonance condition, provided the transition is made quickly. However, if a vibrating system is allowed to reach steady-state just below resonance, it becomes difficult to accelerate the machine through the resonance condition. Additional power supplied tends to increase the amplitude of vibration rather than the running speed.

Summary of Forced Harmonic Vibration of Undamped / Damped Systems

Undamped system



Damped system



Equation of motion (undamped)

$$m\ddot{x} + kx = F_o \sin \omega t$$

Equation of motion (damped)

$$m\ddot{x} + c\dot{x} + kx = F_o \sin \omega t$$

General solution for the steady-state (forced) response

$$x_p(t) = X \sin(\omega t - \phi) \qquad\qquad x_p(t) = X \sin(\omega t - \phi)$$

Steady-state amplitude X and non-dimensionalised amplitude

$$X = \frac{F_o}{k - m\omega^2} \qquad\qquad X = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$\frac{kX}{F_o} = \frac{1}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2} \qquad\qquad \frac{kX}{F_o} = \frac{1}{\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta \dfrac{\omega}{\omega_n}\right]^2}}$$