Tutorial—Inverse Laplace Transform (Complex Conjugate)

Complex zeros of $B(s)$, where the function of interest is $Y(s) = A(s)/B(s)$, always occur in pa1rs, and furthermore these zeros are always conjugates of one another. That is, they have the same real part but equal and opposite imaginary parts. Thus, if the polynomial $B(s)$ has a complex zero $a + jb$, the complex conjugate $a - jb$ will also be a zero of $B(s)$. The following analysis brings out more clearly the fact that a pair of complex conjugate zeros in $B(s)$ combine to introduce an a pair of term in $y(t)$.

A pair of complex conjugate zeros when multiplied together yield the following quadratic:

$$[(s-(a+jb)(s-(a-jb)] = s^2 - 2as + (a^2+b^2) \tag{1}$$

For any given quadratic term, the values of a and b may be computed by equating coefficients of like terms as follows: Consider the expression s^2+4s+9. The coefficient 4 of the s term is equal to $-2a$, so that $a = -2$. Similarly equating the constant terms gives $a^2 + b^2 = 9$ or $b = \sqrt{9-4} = \sqrt{5}$. Thus, the complex conjugate zeros are $a \pm jb = -2 \pm j\sqrt{5}$.

If in the determination of b it is found that b is an imaginary number, the two zeros are real and unequal rather than complex conjugates. For example, consider the quadratic $s^2 + 8s + 12$. The value of a is equal to -4, so that $b = \sqrt{12-16} = j2$. For this case, the zeros are $a \pm jb = -4 \pm (j^2 2) = -4 \mp 2 = -6$, -2. However, it is assumed in the following analysis that b is real, so that the zeros are complex conjugates.

For complex conjugate zeros $B(s)$ may be factored in the form

$$Y(s) = \frac{K_c}{s-a-jb} + \frac{K_{-c}}{s-a+jb} + \frac{K_1}{s-r_1} + \cdots \tag{2}$$

The inverse transform of Eq. 2 is

$$y(t) = K_c e^{(a+jb)t} + K_{-c} e^{(a-jb)t} + K_1 e^{-r_1 t} + \cdots \tag{3}$$

The constant K_c and K_{-c} associated with the complex conjugate zeros are evaluated as usual for distinct zeros. That is,

$$\begin{aligned}K_c &= \lim_{s\to a+jb}\left[(s-a-jb)\frac{A(s)}{(s-a-jb)(s-a+jb)(s-r_1)\cdots}\right]\\ &= \lim_{s\to a+jb}\left[\frac{1}{2jb}\frac{A(s)}{(s-r_1)\cdots}\right]\\ &= \frac{1}{2jb}K(a+jb)\end{aligned} \tag{4}$$

where

$$\begin{aligned}K(a+jb) &= \lim_{s\to a+jb}\frac{A(s)}{(s-r_1)\cdots}\\ &= \left[(s^2-2as+a^2+b^2)\frac{A(s)}{B(s)}\right]_{s=a+jb}\end{aligned} \tag{5}$$

Similarly, the constant K_{-c} is obtained as follows:

$$\begin{aligned}K_{-c} &= \lim_{s\to a-jb}\left[(s-a+jb)\frac{A(s)}{(s-a-jb)(s-a+jb)(s-r_1)\cdots}\right]\\ &= \lim_{s\to a-jb}\left[\frac{1}{-2jb}\frac{A(s)}{(s-r_1)\cdots}\right]\\ &= -\frac{1}{2jb}K(a-jb)\end{aligned} \tag{6}$$

where

$$\begin{aligned}K(a-jb) &= \lim_{s\to a-jb}\frac{A(s)}{(s-r_1)\cdots}\\ &= \left[(s^2-2as+a^2+b^2)\frac{A(s)}{B(s)}\right]_{s=a-jb}\end{aligned} \tag{7}$$

The constants $K(a+jb)$ and $K(a-jb)$ are complex conjugate numbers. They may be represented as

$$\begin{aligned}K(a+jb) &= |K(a+jb)|e^{j\alpha}\\ K(a-jb) &= |K(a-jb)|e^{-j\alpha}\end{aligned} \tag{8}$$

where $|K(a+jb)| = |K(a-jb)|$ is the length of either vector, α is the angle of vector $K(a+jb)$ and $-\alpha$ is the angle of the vector $K(a-jb)$.

The constants K_c and K_{-c} which are also complex conjugate numbers can be written in the form

$$\begin{aligned}K_c &= \frac{1}{2jb}\left|K(a+jb)\right|e^{j\alpha}\\ K_{-c} &= -\frac{1}{2jb}\left|K(a-jb)\right|e^{-j\alpha}\end{aligned} \tag{9}$$

Substitution of K_c and K_{-c} from Eq. 9 into Eq. 3 gives

$$y(t) = \frac{1}{b} |K(a + jb)| e^{at} \frac{e^{j(\alpha+bt)} - e^{-j(\alpha+bt)}}{2j} + K_1 e^{r_1 t} + \cdots \tag{10}$$

or

$$y(t) = \frac{1}{b} |K(a + jb)| e^{\alpha t} \sin(\alpha + bt) + K_1 e^{r_1 t} + \cdots \tag{11}$$

Example 1. *Determine the inverse transform of the following equation:*

$$Y(s) = \frac{75}{(s^2 + 4s + 13)(s + 6)} \tag{12}$$

Solution 1. *Equating coefficients to obtain the value of* a *and* b *for the quadratic yields* $-2a = 4$*, or* $a = -2$*, and* $a^2 + b^2 = 13$*, or* $b = \sqrt{13 - 4} = 3$*. Equation of* $K(a + jb)$ *gives*

$$K(a + jb) = \left[(s^2 - 2as + a^2 + b^2)\frac{A(s)}{B(s)}\right]_{s=a+jb} = \left[\frac{75}{s + 6}\right]_{s=-2+j3} = \frac{75}{4 + j3} \tag{13}$$

The vector whose real part is 4 and whose imaginary part is 3 may be expressed in the polar form $4 + j3 = 5\underline{/36.9^\circ}$*. Hence, Eq. 13 becomes*

$$K(a + jb) = \frac{75}{5\underline{/36.9^\circ}} = 15\underline{/-36.9^\circ} \tag{14}$$

Thus $|K(a + jb)| = 15$ *and* $\alpha = -36.9^\circ$*. The general form of the inverse transformation is*

$$y(t) = \frac{1}{b}|K(a + jb)|e^{at} \sin(bt + \alpha) + K_1 e^{r_1 t} \tag{15}$$

Evaluating K_1 *gives*

$$K_1 = \lim_{s \to -6} \frac{75}{s^2 + 4s + 13} = \frac{75}{25} = 3 \tag{16}$$

Thus the desired result is

$$y(t) = 5e^{-2t} \sin(3t - 36.9^\circ) + 3e^{-6t} \tag{17}$$

Application of the relationship $\sin(A + B) = \sin A \cos B + \cos A \sin B$ *in which* $A = 3t$ *and* $B = -36.9^\circ$ *yields the alternate form*

$$\begin{aligned} y(t) &= 5e^{-2t}(\sin(3t) \times 0.7997 + \cos(3t) \times (-0.6004)) \\ &= e^{-2t}(4 \sin(3t) - 3 \cos(3t)) + 3e^{-6t} \end{aligned} \tag{18}$$

This result may also be obtained as follows.

$$Y(s) = \frac{75}{(s^2 + 4s + 13)(s + 6)} = \frac{C_1 s + C_2}{(s + 2)^2 + 3^2} + \frac{K_1}{s + 6} \tag{19}$$

Evaluation of the constants, multiply the numerators of each term by the denominator of the other term and equate to 75, yields $C_1 = -3$, $C_2 = 6$, and $K_1 = 3$. Hence

$$Y(s) = 4\frac{3}{(s+2)^2+3^2} - 3\frac{s+2}{(s+2)^2+3^3} + \frac{3}{s+6} \tag{20}$$

Inverting yields directly the preceding form for $y(t)$ in Eq. 18. □

Example 2. *Find inverse transform of*

$$F(s) = \frac{1}{s^2(s^2+4)} \tag{21}$$

Solution 2. *Put $F(s)$ in partial fractions*

$$F(s) = \frac{k_1}{s^2} + \frac{k_2}{s} + \frac{k_3}{s-j2} + \frac{k_4}{s+j2} \tag{22}$$

where $s^2+4 = (s-j2)(s+j2)$. Then we obtain the constants as

$$k_1 = \left.\frac{1}{s^2+4}\right|_{s=0} = \frac{1}{4} \tag{23}$$

$$k_2 = \left.\frac{d}{ds}\frac{1}{s^2+4}\right|_{s=0} = \frac{d(s^2+4)^{-1}}{d(s^2+4)}\frac{d(s^2+4)}{ds} = \frac{-2s}{s^2+4} = 0 \tag{24}$$

$$k_3 = \left.\frac{1}{s^2(s+j2)}\right|_{s=j2} = \frac{1}{-4(j4)} = \frac{-1}{j16} \tag{25}$$

$$k_4 = \left.\frac{1}{s^2(s-j2)}\right|_{s=-j2} = \frac{1}{-4(-j4)} = \frac{1}{j16} \tag{26}$$

Then

$$\begin{aligned} F(s) &= \frac{1}{4s^2} - \frac{1}{16j(s-j2)} + \frac{1}{16j(s+j2)} \\ &= \frac{1}{4}\left[\frac{1}{s^2} - \frac{1}{j4}\left(\frac{1}{s-j2} - \frac{1}{s+j2}\right)\right] \\ &= \frac{1}{4}\left[\frac{1}{s^2} - \frac{1}{j4}\left(\frac{s+j2-s+j2}{s^2+4}\right)\right] \\ &= \frac{1}{4}\left[\frac{1}{s^2} - \frac{1}{j4}\left(\frac{j4}{s^2+4}\right)\right] \\ &= \frac{1}{4}\left[\frac{1}{s^2} - \frac{1}{2}\left(\frac{2}{s^2+4}\right)\right] \end{aligned} \tag{27}$$

The inverse is

$$f(t) = \frac{1}{4}\left(t - \frac{1}{2}\sin 2t\right) \tag{28}$$

□