Bode Plots - Quadratic Terms

1 Quadratic Factors

Consider a general quadratic term in s-domain

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{1}$$

In the $j\omega$-domain, we have

$$\begin{aligned} G(j\omega) &= \frac{\omega_n^2}{(j\omega)^2 + j2\zeta\omega\omega_n + \omega_n^2} && (2) \\ &= \frac{\omega_n^2}{\omega_n^2 - \omega^2 + j2\zeta\omega\omega_n}. && (3) \end{aligned}$$

Divide numerator and denominator by ω_n^2, then

$$G(j\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j2\zeta\frac{\omega}{\omega_n}}. \tag{4}$$

The magnitude is

$$\begin{aligned} |G(j\omega)| &= \left|\frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j2\zeta\frac{\omega}{\omega_n}}\right| && (5) \\ &= -20\log_{10}\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}. && (6) \end{aligned}$$

The phase angle is

$$\theta = -\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}\right). \tag{7}$$

Now consider various frequency values. At $\omega << \omega_n$, then

$$\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 \approx 1, \;\; \left(2\zeta\frac{\omega}{\omega_n}\right)^2 \approx 0, \tag{8}$$

$$|G(j\omega)| \approx -20\log_{10}\sqrt{1} = 0B. \tag{9}$$

At $\omega >> \omega_n$, then

$$\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 \approx \left(\frac{\omega^2}{\omega_n^2}\right)^2 >> \left(2\zeta\frac{\omega}{\omega_n}\right)^2, \tag{10}$$

$$|G(j\omega)| \approx -20\log_{10}\sqrt{\left(\frac{\omega^2}{\omega_n^2}\right)^2} = -40\log_{10}\frac{\omega}{\omega_n} dB. \tag{11}$$

The Bode plot asymptote is thus $-40dB/$decade.

These two asymptotes meet at the break or corner frequency $\omega = \omega_n$, the magnitude depends on the damping factor ζ. Now

$$\omega = \omega_n,\ 1-\omega^2/\omega_n^2 = 0,\ 2\zeta\omega/\omega_n = 2\zeta, \tag{12}$$

$$|G(j\omega)| = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} = \frac{1}{2\zeta}. \tag{13}$$

At $\zeta = 0.5$, then

$$|G(j\omega)| = 1/(2\times 0.5) = 1 \Rightarrow -20\log_{10}(1) = 0dB. \tag{14}$$

At $\zeta = 0.05$, then

$$|G(j\omega)| = 1/(2\times 0.05) = 1/0.1 \Rightarrow -20\log_{10}(0.1) = 20dB. \tag{15}$$

At $\zeta = 1$, then

$$|G(j\omega)| = 1/(2\times 1) = 1/2 \Rightarrow -20\log_{10}(2) = -6dB. \tag{16}$$

At $\zeta = 2$, then

$$|G(j\omega)| = 1/(2\times 2) = 1/4 \Rightarrow -20\log_{10}(4) = -12dB. \tag{17}$$

Consider the phase angle. At $\omega << \omega_n$, we have

$$-\tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}\right) \approx -\tan^{-1}(0) = 0^\circ. \tag{18}$$

At $\omega >> \omega_n$, then

$$-\tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}\right) \approx -\tan^{-1}\left(\frac{1}{-\infty}\right) = -180^\circ. \tag{19}$$

At $\omega = \omega_n$, then

$$-\tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}\right) \approx -\tan^{-1}\left(\frac{2\zeta}{0}\right) = -90^\circ. \tag{20}$$

At $\omega/\omega_n = 0.5$, then

$$-\tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}\right) \approx -\tan^{-1}\left(\frac{\zeta}{1-0.5^2}\right) = -\tan^{-1}\left(\frac{\zeta}{0.75}\right), \tag{21}$$

and depends on ζ.

For instance, at $\zeta = 0.75$, then

$$\theta = -\tan^{-1}(0.75/0.75) = -\tan^{-1}(1) = -45°. \tag{22}$$

Furthermore, at $\omega = 2\omega_n$, then the phase angle is

$$-\tan^{-1}(4 \times 0.75/(1-4)). \tag{23}$$

At $\zeta = 0.75$, then

$$-\tan^{-1}(3/-3) = -135°. \tag{24}$$

Other numerical values of ω and ζ that determine the magnitude and phase can be obtained in a similar manner. A plot of magnitude and phase is shown in Fig. 1.

2 Example

Example 2.1. Now consider the numerical example, where

$$G(s) = \frac{1}{s^2 + 3s + 25}. \tag{25}$$

We have the quadratic term

$$\frac{1}{s^2 + 3s + 25}, \tag{26}$$

which can be put into the form of second-order or quadratic equation as

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{27}$$

Note that we have the same form except a constant factor ω_n^2 in the numerator. Now put the expression in the $j\omega$ domain, we have

$$G(j\omega) = \frac{1}{(j\omega)^2 + j3\omega + 25} = \frac{1}{25 - \omega^2 + j3\omega}. \tag{28}$$

Comparing equations 26 and 27, we see that

$$\omega_n^2 = 25 \Rightarrow \omega_n = 5. \tag{29}$$

Now re-write eq. 27 as

$$\frac{\omega_n^2}{\omega_n^2 - \omega^2 + 2\zeta\omega\omega_n}, \tag{30}$$

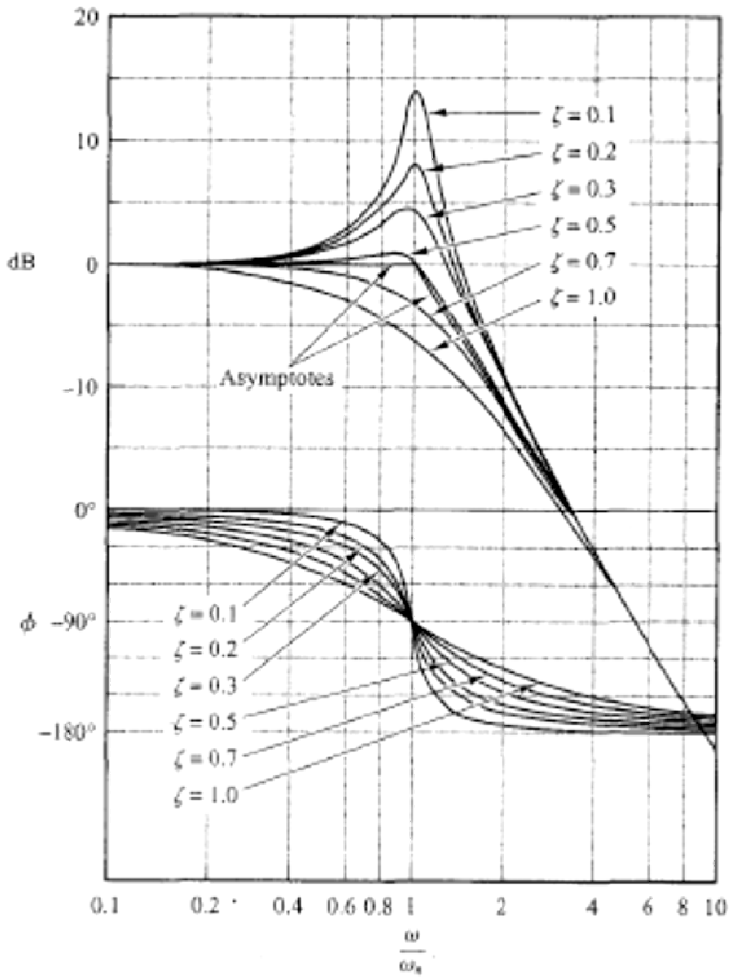

Figure 1: Log-magnitude curves, together with the asymptotes, and phase-angle curves of the quadratic transfer function.

and factorize ω_n^2 from the denominator, we have

$$\frac{1/\omega_n^2}{1-\frac{\omega^2}{\omega_n^2}+j2\zeta\frac{\omega}{\omega_n}}, \tag{31}$$

and the numerator is normalized as compared with eq. 27, giving

$$\frac{1}{1-\frac{\omega^2}{\omega_n^2}+j2\zeta\frac{\omega}{\omega_n}}, \tag{32}$$

and the normalization factor $1/\omega_n^2$ will be restored as an off-set to all magnitude values calculated. From the term involved with the j-term, we see that

$$2\zeta\omega_n = 3 \Rightarrow \zeta = 3/(2\times 5) = 0.3. \tag{33}$$

Consider the magnitude response. At $\omega << \omega_n$, then from eq. 9, we have

$$|G(j\omega)| = 0dB. \tag{34}$$

For $\omega >> \omega_n$, from eq. 11, we have

$$|G(j\omega)| = -40dB/dec. \tag{35}$$

For the phase angle, at $\omega << \omega_n$, and according to eq. 18, then

$$-\tan^{-1}(0) = 0^\circ. \tag{36}$$

For $\omega >> \omega_n$, and from eq. 19, we have

$$-\tan^{-1}\left(\frac{1}{-\infty}\right) = -180^\circ. \tag{37}$$

Further consider the case when $\omega = \omega_n$ and damping $\zeta = 0.3$. The magnitude is

$$|G(j\omega)| = -20\log_{10}\sqrt{(1-1)^2+(2\times 0.3\times 1)^2} = -20\log_{10}\sqrt{0.6^2} = 4.4dB. \tag{38}$$

The phase is

$$\theta = -\tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}\right) = -\tan^{-1}\left(\frac{0.6}{0}\right) = -90^\circ. \tag{39}$$

In addition, we can also calculate the magnitude and phase for other cases. When $\omega = 0.5\omega_n$, the magnitude is

$$|G(j\omega)| = -20\log_{10}\sqrt{(1-0.5^2)^2+(2\times 0.3/2)^2} = 1.85dB. \tag{40}$$

The phase is

$$\theta = -\tan^{-1}\left(\frac{2\times 0.3\times 0.5}{1-0.5^2}\right) = -22^\circ. \tag{41}$$

At $\omega = 2\omega_n$, then the magnitude is

$$|G(j\omega)| = -20\log_{10}\sqrt{(1-2^2)^2 + (2\times 0.3\times 2)^2} = -10dB. \tag{42}$$

The phase is

$$\theta = -\tan^{-1}\left(\frac{2\times 0.3\times 2}{1-2^2}\right) = -158°. \tag{43}$$

The Bode plot is shown in Fig. 2. Recall that all magnitude calculated had been scaled by the factor $1/\omega_n^2$. This factor is numerically

$$20\log_{10}(1/25) = -28dB. \tag{44}$$

Hence, all magnitudes are scaled by $-28dB$ for the original system. It should also be noted that the phase angle is independent of the scale factor, hence there is no scaling needed.

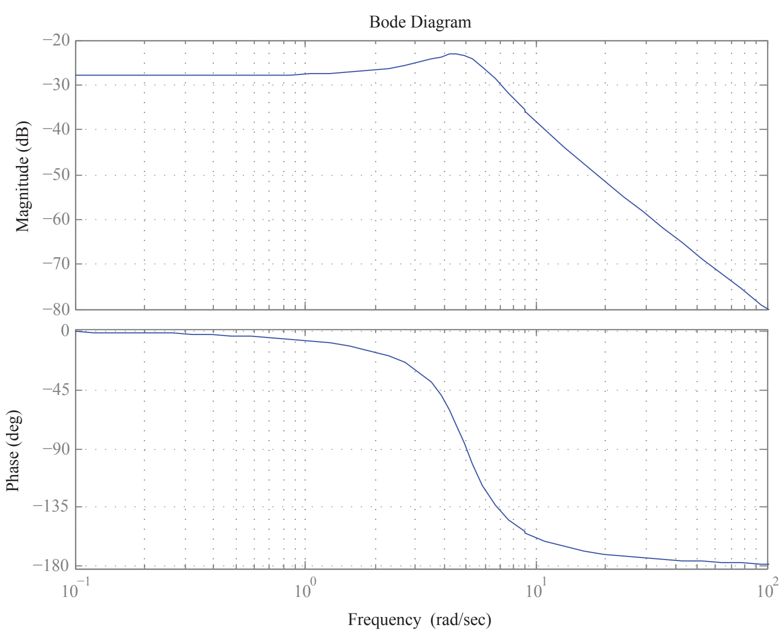

Figure 2: Bode Plot

By inspecting the Bode plot, we see that the system response is below $0dB$ for all frequencies considered. Therefore, the phase margin is undefined. Also observe that the phase is above $-180°$ for all frequencies, hence, the gain margin is also undefined.

Example 2.2. Consider another system

$$G(s) = \frac{5}{(1+2s)(s^2+3s+25)}, \tag{45}$$

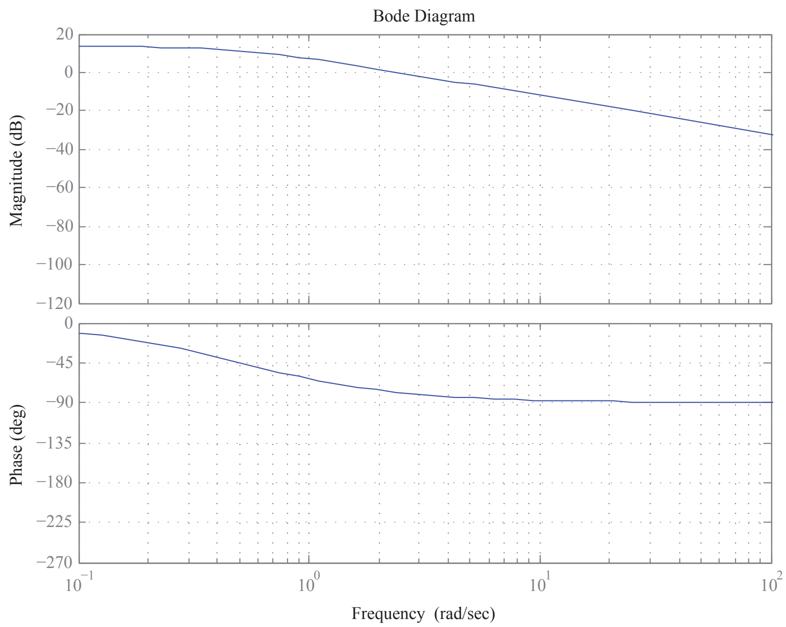

Figure 3: Bode Plot - gain plus simple-lag

where a gain term and simple lag term is included. The gain plus simple-lag term $\frac{5}{1+2s}$ gives a Bode plot as Fig. 3 and together with the quadratic term $\frac{1}{s^2+3s+25}$, Fig. 4, constitute the overall system. The Bode plot becomes as shown in Fig. 5. In this system, the gain is again below $0dB$ for all frequencies, thus there is no phase margin. It is observed that the phase passes through $-180°$ when the gain is $-30dB$ at $\omega \approx 5$. Hence, the margin margin is $30dB$. If the gain of the system is increased by $30dB$ and the input signal is at $5rad/sec$, then the system becomes oscillating and unstable.

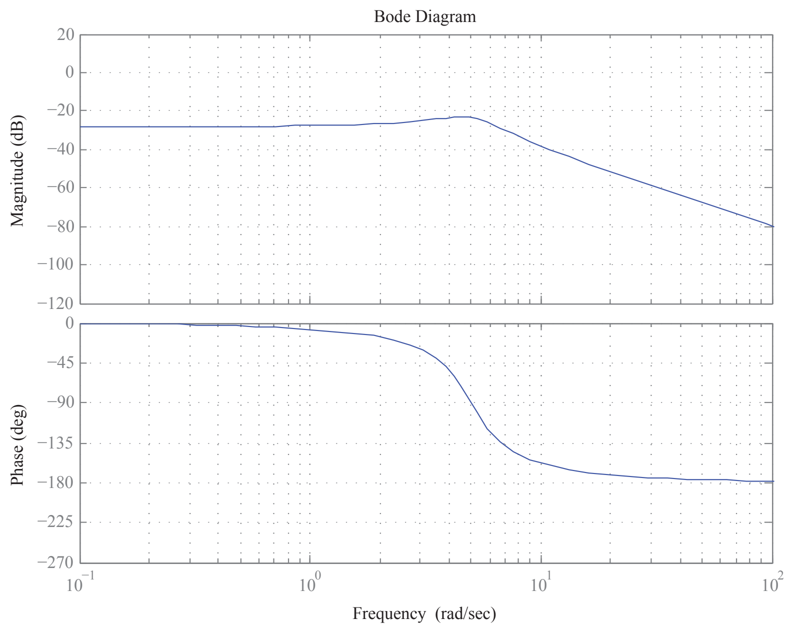

Figure 4: Bode Plot - quadratic term

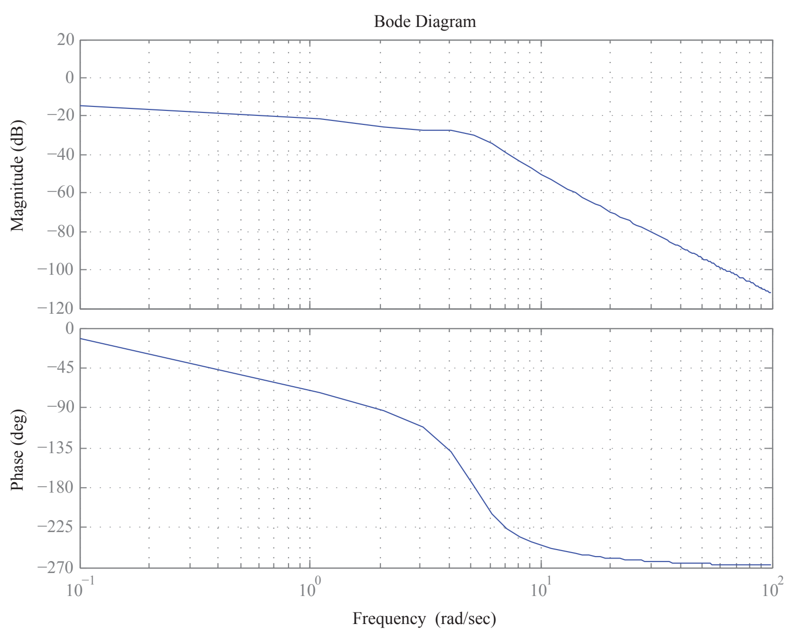

Figure 5: Bode Plot - overall system

3 Matlab Code

```
% BodePlot Tutorial

clc; clear all; close all;

w=logspace(-1,2,100);
num=1;
den=[1 3 25];
sys=tf(num,den)
figure; bode(sys,w); grid on; hold on;
margin(sys);
bandwidth(sys)

num=5;
den=[2 1];
sys=tf(num,den)
figure; bode(sys,w); grid on; hold on;
den=[1 3 25];
sys=tf(num,den)
figure; bode(sys,w); grid on; hold on;
den=conv([2 1],[1 3 25]);
sys=tf(num,den)
figure; bode(sys,w); grid on; hold on;
margin(sys)
bandwidth(sys)
```