# State Space Modeling

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## Linear Time-invariant Systems

- Models of linear time-invariant (LTI) analog systems could be presented as linear differential equations with constant coefficients, and through transfer functions.
- By using the Laplace transform, the transfer function can be derived from the differential equations; a differential equation model can be derived from the transfer function using the inverse Laplace transform, as well.

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## State-Space Model

- Here, we consider another type of model: the state-space or state-variable model.
- This model is a differential equation model, but the equations are always written in a specific format (Matrix)

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- The state-variable model, or state-space model, is expressed as n first-order coupled differential equations.
- These equations preserve the system's input-output relationship (that of the transfer function); in addition, an internal model of the system is given.

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# Advantages

Computer-aided analysis and design of state models are performed more easily on digital computers for higher-order systems, while the transfer function approach tends to fail for these systems because of numerical problems.

(As usual, we choose the “domain” which is more convenient for us, to understand, model, design, simulate,..)

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# Advantages

Computer-aided analysis and design of state models are performed more easily on digital computers for higher-order systems, while the transfer function approach tends to fail for these systems because of numerical problems.

(We had not paid attention to this issue, in MMAN3200; because we worked in an analytical fashion, still not needing to deal with numerical simulations,  etc.; dealing with high order systems.)

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In Control applications: In state-variable design procedures, we feedback more information (internal variables) about the plant; hence, we can achieve a more complete control of the system than is possible with the transfer-function approach.

In Simulation applications: State-variable models are generally required for digital simulation (digital computer solution of differential equations).

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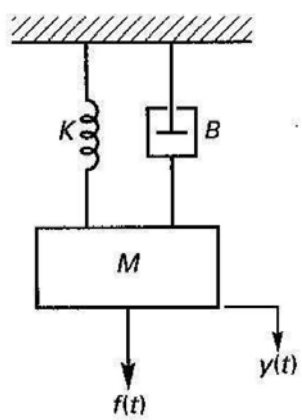
## Differential Equation Representation

Example. The system model used to illustrate state variables, a linear mechanical translational system, is given in Figure 1.

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The differential equation describing this system is given by

$$M\frac{d^2y(t)}{dt^2} = f(t) - B\frac{dy(t)}{dt} - Ky(t)$$

and the transfer function is given by

$$G(s) = \frac{Y(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

This equation gives a description of the position *y(t)* as a function of the force *f(t)*.

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Suppose that we also want information about the velocity.

Using the state-variable approach, we define the two state variables $x_1(t)$ and $x_2(t)$ as

$$x_1(t) = y(t)$$

$$x_2(t) = \frac{dy(t)}{dt} = \frac{dx_1(t)}{dt} = \dot{x}_1(t)$$

Thus, $x_1(t)$ is the position of the mass, and $x_2(t)$ is its velocity.

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$$x_1(t) = y(t)$$
$$x_2(t) = \frac{dy(t)}{dt} = \frac{dx_1(t)}{dt} = \dot{x}_1(t)$$

Then we may write

$$\frac{d^2y(t)}{dt^2} = \frac{dx_2(t)}{dt} = \dot{x}_2(t) = -\frac{B}{M}x_2(t) - \frac{K}{M}x_1(t) + \frac{1}{M}f(t)$$

The model is usually written in a specific format, which is given by rearranging the preceding equations as:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\dfrac{K}{M}\cdot x_1(t) - \dfrac{B}{M}\cdot x_2(t) + \dfrac{1}{M}\cdot f(t) \end{cases}$$

$$y(t) = x_1(t)$$

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## State-space Representation

Definition: The state of a system at a time $t_0$ is defined as the information at $t_0$ that, jointly with all inputs for $t \geq t_0$, uniquely determines the behavior of the system for all $t \geq t_0$.

This means that if we know the state of the system at certain time , $t_0$ , and we also know the inputs of the system from $t_0$ up to time t, we will be able to know (based on that) the state of the system at that time t, as well.

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Note: this may remind you about the need of knowing the initial conditions of y(t) and its first (n-1) time derivatives when solving an ODE of order n.

That ODE represents a SISO system.

This is because it is a particular case of a state representation of that system



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Usually, state equations are written in a vector-matrix format, since this allows the equations to be manipulated much more easily.

In this format, the preceding equations become

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\frac{K}{M}\cdot x_1(t) - \frac{B}{M}\cdot x_2(t) + \frac{1}{M}\cdot f(t) \end{cases}$$

$$y(t) = x_1(t)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) = \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix}\cdot\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix}\cdot f(t)$$

$$\mathbf{y}(t) = x_1(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\cdot\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$



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The standard form of the state equations, of a LTI analog system, is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t) + \mathbf{D} \cdot \mathbf{u}(t)$$

where the vector $\dot{\mathbf{x}}(t)$ is the time derivative of the vector $\mathbf{x}(t)$ .

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- **x**(t) is the "state vector", an (n × 1) vector of the states of the system.
- **A** = (n × n), the "system matrix"
- **B** = (n × r) the "input matrix"
- **u**(t) = input vector = (r × 1) vector composed of the system input functions,
- **y**(t) is the "output vector" , a (p × 1) vector composed of the defined outputs,
- **C** , a matrix of size (p × n), is the "output matrix",
- **D** , a (p × r) matrix, to represent direct coupling between input and output.

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We refer to the two matrix equations of (9) as the state-variable equations of the system.
The first equation is called the state equation, and the second one is called the output equation.

$$\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t) + \mathbf{D} \cdot \mathbf{u}(t)$$

Note : in certain application areas, the state equation is called "process model"

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The signals, in expanded form, are given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix}, \quad \dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ .. \\ \dot{x}_n \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ .. \\ u_r \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ .. \\ y_p \end{bmatrix}$$

$$\mathbf{x} \in R^n; \quad \dot{\mathbf{x}} \in R^n; \quad \mathbf{u} \in R^r; \quad \mathbf{y} \in R^p;$$

(now, when we see those variables, we know they are vectors, of certain dimensionalities; which depend on the problem being described.)

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# State Equation

The state equation, is a first-order matrix differential equation, and the state vector, **x**(t), is its solution.

Given knowledge of **x**(t) and about the input vector **u**(t), the output equation yields the output **y**(t).

Comment: Usually, **y**(t) is what we see, which is available to us (e.g. what we can measure, what we "feel".)

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$$\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t) + \mathbf{D} \cdot \mathbf{u}(t)$$

Usually the matrix **D** is zero, since in physical systems, dynamics appear in all paths between the inputs and the outputs.

A nonzero value of **D** indicates at least one direct path between the inputs and the outputs, in which the path transfer function can be modeled as a pure gain.

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$$\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t) + \mathbf{D} \cdot \mathbf{u}(t)$$

In the equation for the state variables **x**(t), only the first derivatives of the state variables may appear on the left side of the equation, and no derivatives may appear on the right side.

No derivatives do appear in the output equation.

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For the special case in which the system is SISO, the original model is represented by an order-n ODE (Ordinary differential equation), about the 1D variable (scalar) **y**(t);

In the equivalent State Space representation, we model the same system by using an order-1 differential equation, about the variable **x**(t) which is of dimensionality =n (n-D, n dimensional variable)

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The general form of the state equations, just given, allows for multiple inputs and multiple outputs; (MIMO) these systems are called multi-variable systems.

For the case of one input, the matrix B is a column vector (nx1), and the vector **u**(t) is a scalar.

For the case of one output, the vector **y**(t) is a scalar, and the matrix C is a row vector (1xn).

However, there are systems which have more than one input, and/or more that one output. The state space representation can deal with those MIMO systems. (*)

(*) You can imagine the difficulty of treating those models using the traditional high order and couples diff. Equations)





# Example

Consider the system described by two coupled differential equations

$$\ddot{y}_1 + k_1 \cdot \dot{y}_1 + k_2 \cdot y_1 = u_1 + k_3 \cdot u_2$$
$$\dot{y}_2 + k_4 \cdot y_2 + k_5 \cdot \dot{y}_1 = k_6 \cdot u_1$$

In which $u_1$ and $u_2$ are inputs, $y_1$ and $y_2$ are outputs, and the set of coefficients $\{k_i\}$, i = 1, · · · , 6, are system parameters.

We may define the states as the outputs and, where necessary, certain derivatives of the outputs.

$$x_1 = y_1; \quad x_2 = \dot{y}_1; \quad x_3 = y_2$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \mathbf{y}(t) = \begin{bmatrix} x_1(t) \\ x_3(t) \end{bmatrix}$$

(The notational dependence of the variables on time has been omitted, in certain parts of the equation, for convenience.)

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$$\boxed{x_1 = y_1; \quad x_2 = \dot{y}_1; \quad x_3 = y_2} \quad \boxed{\begin{aligned} \ddot{y}_1 + k_1 \cdot \dot{y}_1 + k_2 \cdot y_1 &= u_1 + k_3 \cdot u_2 \\ \dot{y}_2 + k_4 \cdot y_2 + k_5 \cdot \dot{y}_1 &= k_6 \cdot u_1 \end{aligned}}$$

From the system differential equations, we write the model, in state space

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -k_2 \cdot x_1 - k_1 \cdot x_2 + u_1 + k_3 \cdot u_2$$

$$\dot{x}_3 = -k_5 \cdot x_2 - k_4 \cdot x_3 + k_6 \cdot u_1$$

with output equations

$$y_1 = x_1$$

$$y_2 = x_3$$

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$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -k_2 \cdot x_1 - k_1 \cdot x_2 + u_1 + k_3 \cdot u_2$$
$$\dot{x}_3 = -k_5 \cdot x_2 - k_4 \cdot x_3 + k_6 \cdot u_1$$

$$y_1 = x_1$$
$$y_2 = x_3$$

These equations may be written in matrix form as

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ -k_2 & -k_1 & 0 \\ 0 & -k_5 & -k_4 \end{bmatrix} \cdot \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & k_3 \\ k_6 & 0 \end{bmatrix} \cdot \mathbf{u}(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \mathbf{x}(t)$$

..btw, it follows the form:

$$\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t) + \mathbf{D} \cdot \mathbf{u}(t)$$

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Interesting comments, related to non-linear problems:

State Space representation does not necessarily need to be related to LTI systems. It can be very useful for modelling non-linear systems.

Let's see some examples:

Non-linear pendulum case.

Kinematic model of a car.

In some cases we can even linearize those, if they operate in a region of the state space, and if the equations are smooth enough to be linearly approximated in that region.

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(Case NOT included in EXAM)

Example: 2D Kinematic model of a car.

Suppose we can decide the values of the steering angle and about the longitudinal velocity of the platform:

$$\frac{d\mathbf{x}(t)}{dt} = f\left(\mathbf{x}(t), \mathbf{u}(t)\right)$$

$$\Downarrow$$

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \phi(t) \end{bmatrix} = \begin{bmatrix} v(t)\cdot\cos\left(\phi(t)\right) \\ v(t)\cdot\sin\left(\phi(t)\right) \\ \frac{v(t)}{L}\cdot\tan\left(\beta(t)\right) \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ \phi(t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} v(t) \\ \beta(t) \end{bmatrix}$$

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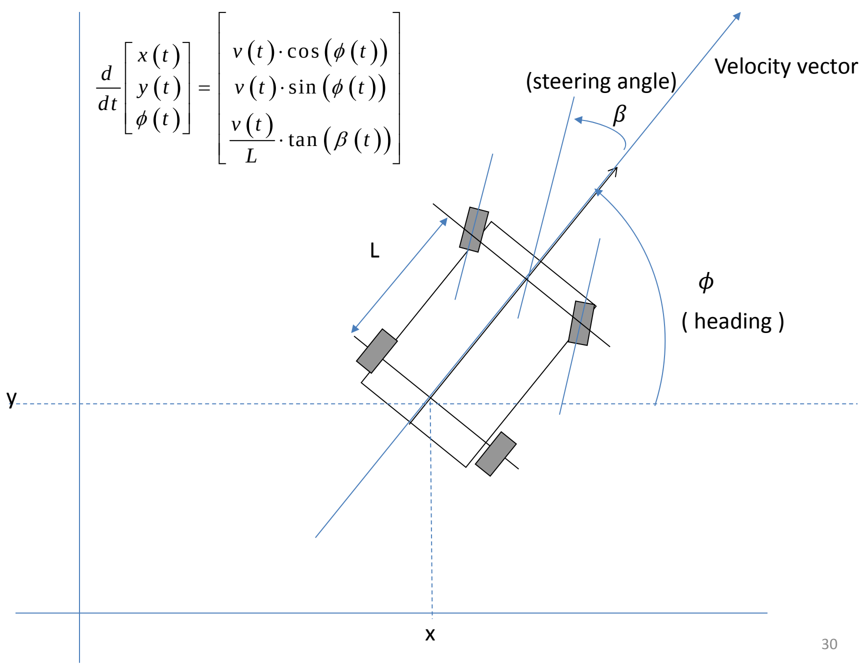
(Case NOT included in EXAM)

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \phi(t) \end{bmatrix} = \begin{bmatrix} v(t)\cdot\cos\left(\phi(t)\right) \\ v(t)\cdot\sin\left(\phi(t)\right) \\ \frac{v(t)}{L}\cdot\tan\left(\beta(t)\right) \end{bmatrix}$$

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Kinematic model of a car, a bit more sophisticated.

We operate the accelerator, which, we assume, introduces a force, longitudinal (in the direction of the heading of the platform, $\phi(t)$ . We also consider some opposition, which is due to friction/drag effects; so we propose some term which is proportional to the speed, v(t), (longitudinal velocity).

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \phi(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} v(t)\cdot\cos(\phi(t)) \\ v(t)\cdot\sin(\phi(t)) \\ \frac{v(t)}{L}\cdot\tan(\beta(t)) \\ a\cdot\tau(t) - b\cdot v(t) \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ \phi(t) \\ v(t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} \tau(t) \\ \beta(t) \end{bmatrix}$$

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Kinematic model of a car, a bit more sophisticated.

We operate the accelerator, and we feed the desired steering angle, but is filtered by a "pole", at s=A, to make it smooth.

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \phi(t) \\ v(t) \\ \beta(t) \end{bmatrix} = \begin{bmatrix} v(t)\cdot\cos(\phi(t)) \\ v(t)\cdot\sin(\phi(t)) \\ \frac{v(t)}{L}\cdot\tan(\beta(t)) \\ a\cdot\tau(t) - b\cdot v(t) \\ A\cdot(\beta_{desired}(t) - \beta(t)) \end{bmatrix},$$

$$\mathbf{X}(t) = \begin{bmatrix} x(t) & y(t) & \phi(t) & v(t) & \beta(t) \end{bmatrix}^T,$$

$$\mathbf{u}(t) = \begin{bmatrix} \tau(t) & \beta_{desired}(t) \end{bmatrix}^T$$

(we included some low pass filter, for the steering input: $\beta[s] = \beta_{desired}[s]\cdot\frac{A}{s+A}$)

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Kinematic model of a car, a bit more sophisticated.

We operate the accelerator, and we feed the desired steering angle, but is filtered by a “pole”, to make it smooth. We also specify some limits on the steering angle.

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \phi(t) \\ v(t) \\ \beta(t) \end{bmatrix} = \begin{bmatrix} v(t)\cdot\cos(\phi(t)) \\ v(t)\cdot\sin(\phi(t)) \\ \frac{v(t)}{L}\cdot\tan(\beta(t)) \\ a\cdot\tau(t) - b\cdot v(t) \\ A\cdot\left(\min\left(25^o, \max\left(-25^o, \beta_{desired}(t)\right)\right) - \beta(t)\right) \end{bmatrix},$$

$$\mathbf{x}(t) = \begin{bmatrix} x(t) & y(t) & \phi(t) & v(t) & \beta(t) \end{bmatrix}^T,$$

$$\mathbf{u}(t) = \begin{bmatrix} \tau(t) & \beta_{desired}(t) \end{bmatrix}^T$$

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We may make our model even more complex, considering other dynamic aspects.

We can also add some other features, such as limiting the desired steering angle, by using limits which could depend on the speed at which we are travelling (a safety protection)

However, those are not topics of MMAN3200.

(so, the example case about a 2D Kinematic model of a car is not included in tutorials and exam)

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(Topic **Included** in Exam)

## Linearization

Linear approximation, obtained in state space, based on a multivariate first order Taylor's approximation.

This concept is similar to the linearization approach you used in the first weeks, in MMAN3200.

We now generalize it, for being applied to models which are "native" in state space representation.

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Linear approximation, obtained in state space, based on a multivariate first order Taylor's approximation, at certain "point of operation" ($\mathbf{x}_0,\mathbf{u}_0$).

$$\dot{\mathbf{x}}(t) = f\left(\mathbf{x}(t),\mathbf{u}(t)\right) \cong$$

$$\cong f\left(\mathbf{x}_0,\mathbf{u}_0\right) + \mathbf{A}\cdot\left(\mathbf{x}(t) - \mathbf{X}_0\right) + \mathbf{B}\cdot\left(\mathbf{u}(t) - \mathbf{u}_0\right)$$

$$\mathbf{A} = \left[\frac{\partial}{\partial \mathbf{x}} f\left(\mathbf{x},\mathbf{u}\right)\right]_{\substack{\mathbf{x}=\mathbf{x}_0,\\ \mathbf{u}=\mathbf{u}_0}}$$

$$\mathbf{B} = \left[\frac{\partial}{\partial \mathbf{u}} f\left(\mathbf{x},\mathbf{u}\right)\right]_{\substack{\mathbf{x}=\mathbf{x}_0,\\ \mathbf{u}=\mathbf{u}_0}}$$

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## The "Jacobian" matrixes,

$$f(\mathbf{x},\mathbf{u}) = f(x_1, x_2, .., x_n,\ u_1, u_2, ..u_r) = \begin{bmatrix} f_1(x_1, x_2, .., x_n, u_1, u_2, ..u_r) \\ f_2(x_1, x_2, .., x_n, u_1, u_2, ..u_r) \\ \cdots \\ f_n(x_1, x_2, .., x_n, u_1, u_2, ..u_r) \end{bmatrix}$$

$$\left[\frac{\partial}{\partial \mathbf{x}} f(\mathbf{x},\mathbf{u})\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & .. & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & .. & \frac{\partial f_2}{\partial x_n} \\ .. & .. & .. & .. \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & & \frac{\partial f_n}{\partial x_n} \end{bmatrix},$$

(it is an n by n matrix)

$$\left[\frac{\partial}{\partial \mathbf{u}} f(\mathbf{x},\mathbf{u})\right] = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & .. & \frac{\partial f_1}{\partial u_r} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & .. & \frac{\partial f_2}{\partial u_r} \\ .. & .. & .. & .. \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & & \frac{\partial f_n}{\partial u_r} \end{bmatrix}$$

(it is an n by r matrix)

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## We can express the state and input vectors, relative to the point of operation, $(X_0,u_0)$

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)) \cong$$
$$\cong f(\mathbf{x}_0, \mathbf{u}_0) + \mathbf{A}\cdot(\mathbf{x}(t) - \mathbf{x}_0) + \mathbf{B}\cdot(\mathbf{u}(t) - \mathbf{u}_0)$$

$$\mathbf{x}^*(t) = \mathbf{x}(t) - \mathbf{x}_0,$$
$$\mathbf{u}^*(t) = \mathbf{u}(t) - \mathbf{u}_0$$
$$\Downarrow$$
$$\dot{\mathbf{x}}^*(t) = f(\mathbf{x}_0, \mathbf{u}_0) + \mathbf{A}\cdot\mathbf{x}^*(t) + \mathbf{B}\cdot\mathbf{u}^*(t)$$

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In many cases, we choose a point of operation, ($X_0$,$u_0$), at which the system would be in equilibrium, i.e. $f(\mathbf{x}_0,\mathbf{u}_0)=0$,

$$\dot{\mathbf{x}}(t) = f\left(\mathbf{x}(t),\mathbf{u}(t)\right) \cong$$
$$\cong f\left(\mathbf{x}_0,\mathbf{u}_0\right) + \mathbf{A}\cdot\left(\mathbf{x}(t)-\mathbf{x}_0\right) + \mathbf{B}\cdot\left(\mathbf{u}(t)-\mathbf{u}_0\right)$$

$$\mathbf{x}^*(t) = \mathbf{x}(t)-\mathbf{x}_0,\quad \mathbf{u}^*(t) = \mathbf{u}(t)-\mathbf{u}_0,\quad f\left(\mathbf{x}_0,\mathbf{u}_0\right) = \bar{\mathbf{0}}$$
$$\mathbf{A} = \left[\frac{\partial}{\partial \mathbf{x}} f\left(\mathbf{x},\mathbf{u}\right)\right]_{\substack{\mathbf{x}=\mathbf{x}_0,\\ \mathbf{u}=\mathbf{u}_0}},\quad \mathbf{B} = \left[\frac{\partial}{\partial \mathbf{u}} f\left(\mathbf{x},\mathbf{u}\right)\right]_{\substack{\mathbf{x}=\mathbf{x}_0,\\ \mathbf{u}=\mathbf{u}_0}}$$
$$\Downarrow$$
$$\dot{\mathbf{x}}^*(t) = \mathbf{A}\cdot\mathbf{x}^*(t) + \mathbf{B}\cdot\mathbf{u}^*(t)$$

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## Non-linear Output Equation

$$\mathbf{y}(t) = c\left(\mathbf{x}(t)\right) \cong c\left(\mathbf{x}_0\right) + \mathbf{C}\cdot\left(\mathbf{x}(t)-\mathbf{x}_0\right) = \mathbf{y}_0 + \mathbf{C}\cdot\left(\mathbf{x}(t)-\mathbf{x}_0\right)$$

$$\mathbf{x}_r(t) = \mathbf{x}(t)-\mathbf{x}_0,\quad \mathbf{y}_r(t) = \mathbf{y}(t)-\mathbf{y}_0,$$
$$\mathbf{C} = \left[\frac{\partial}{\partial \mathbf{x}} c\left(\mathbf{x}\right)\right]_{\mathbf{x}=\mathbf{x}_0,}$$
$$\Downarrow$$
$$\mathbf{y}_r(t) = \mathbf{C}\cdot\mathbf{x}_r(t)$$

$$\left[\frac{\partial}{\partial \mathbf{x}} c\left(\mathbf{x}\right)\right] = \begin{bmatrix} \frac{\partial c_1}{\partial x_1} & \frac{\partial c_1}{\partial x_2} & .. & \frac{\partial c_1}{\partial x_n} \\ \frac{\partial c_2}{\partial x_1} & \frac{\partial c_2}{\partial x_2} & .. & \frac{\partial c_2}{\partial x_n} \\ .. & .. & .. & .. \\ \frac{\partial c_p}{\partial x_1} & \frac{\partial c_p}{\partial x_2} & & \frac{\partial c_p}{\partial x_n} \end{bmatrix}$$

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Example:

A SISO case, pendulum

Its usual (and simplified) model is a (2nd order ODE, which is non linear)

$$\frac{d^2\varphi(t)}{dt^2} = -2\cdot\sin\left(\varphi(t)\right) - 3\cdot\frac{d\varphi(t)}{dt} + u(t)$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \varphi \\ \dfrac{d\varphi}{dt} \end{bmatrix}$$

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x},\mathbf{u}) = f(x_1, x_2, u) = \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{bmatrix} = \begin{bmatrix} x_2 \\ -2\cdot\sin(x_1) - 3\cdot x_2 + u(t) \end{bmatrix}$$

We want to obtain a linearized / approximated model of the type: $\dot{\mathbf{x}}(t) = \mathbf{A}\cdot\mathbf{x}(t) + \mathbf{B}\cdot\mathbf{u}(t)$

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(pendulum example)

$$\frac{d^2\varphi(t)}{dt^2} = -2\cdot\sin\left(\varphi(t)\right) - 3\cdot\frac{d\varphi(t)}{dt} + u(t)$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \varphi \\ \dfrac{d\varphi}{dt} \end{bmatrix}$$

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -2\cdot\sin(x_1) - 3\cdot x_2 + u(t) \end{bmatrix}$$

We want to obtain a linearized / approximated model of the type:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\cdot\mathbf{x}(t) + \mathbf{B}\cdot\mathbf{u}(t)$$

valid for certain region of operation.
We consider certain equilibrium point given by input *u=1* and $x_1 = \pi/6, \quad x_2 = 0$
(you can verify that dx/dt-0 for those values)

We choose that point for obtaining a Taylor approximation, of order 1.

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Concept: Equilibrium point

Given a process model, in state space,

$$\frac{d\mathbf{x}}{dt} = f\left(\mathbf{x},\mathbf{u}\right)$$

A point

$$\left(\mathbf{x}_0,\mathbf{u}_0\right)$$

which satisfies

$$\frac{d\mathbf{x}}{dt} = f\left(\mathbf{x}_0,\mathbf{u}_0\right) = \overline{0} = \begin{bmatrix} 0 \\ 0 \\ .. \\ 0 \end{bmatrix}$$

is said to be "an equilibrium point" (EP)

In certain cases it makes sense linearizing the state equation, for describing the dynamics of the system, for a region surrounding an EP, if we expect the system will be operating in that region of the state space and for inputs close to that nominal input value.

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(.. example, pendulum)

$$\dot{\mathbf{x}} = f\left(\mathbf{x},\mathbf{u}\right) = f\left(x_1,x_2,u\right) = \begin{bmatrix} f_1\left(x_1,x_2,u\right) \\ f_2\left(x_1,x_2,u\right) \end{bmatrix} = \begin{bmatrix} x_2 \\ -2\cdot\sin\left(x_1\right) - 3\cdot x_2 + u\left(t\right) \end{bmatrix}$$

$$\mathbf{x}_0 = \begin{bmatrix} \pi/6 \\ 0 \end{bmatrix}, \quad \mathbf{u}_0 = 1$$

$$\mathbf{A} = \left[\frac{\partial}{\partial\mathbf{x}} f\left(\mathbf{x},\mathbf{u}\right)\right]_{\mathbf{x}=\mathbf{x}_0,\mathbf{u}=\mathbf{u}_0} = \left[\begin{bmatrix} 0 & 1 \\ -2\cdot\cos\left(x_1\right) & -3 \end{bmatrix}\right]_{\mathbf{x}=\mathbf{x}_0,\mathbf{u}=\mathbf{u}_0} = \begin{bmatrix} 0 & 1 \\ -2\cdot\cos\left(\pi/6\right) & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1.7321 & -3 \end{bmatrix}$$

$$\mathbf{B} = \left[\frac{\partial}{\partial\mathbf{u}} f\left(\mathbf{x},\mathbf{u}\right)\right]_{\mathbf{x}=\mathbf{x}_0,\mathbf{u}=\mathbf{u}_0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{x}^* = \mathbf{x} - \mathbf{x}_0 = \begin{bmatrix} x_1 - \pi/6 \\ x_2 - 0 \end{bmatrix}$$

$$u^* = u - 1$$

(done!)

$$\dot{\mathbf{x}}^*\left(t\right) = \mathbf{A}\cdot\mathbf{x}^*\left(t\right) + \mathbf{B}\cdot\mathbf{u}^*\left(t\right)$$

(remember to verify all these equations, so you learn and ,also , correct possible mistakes in my slides)

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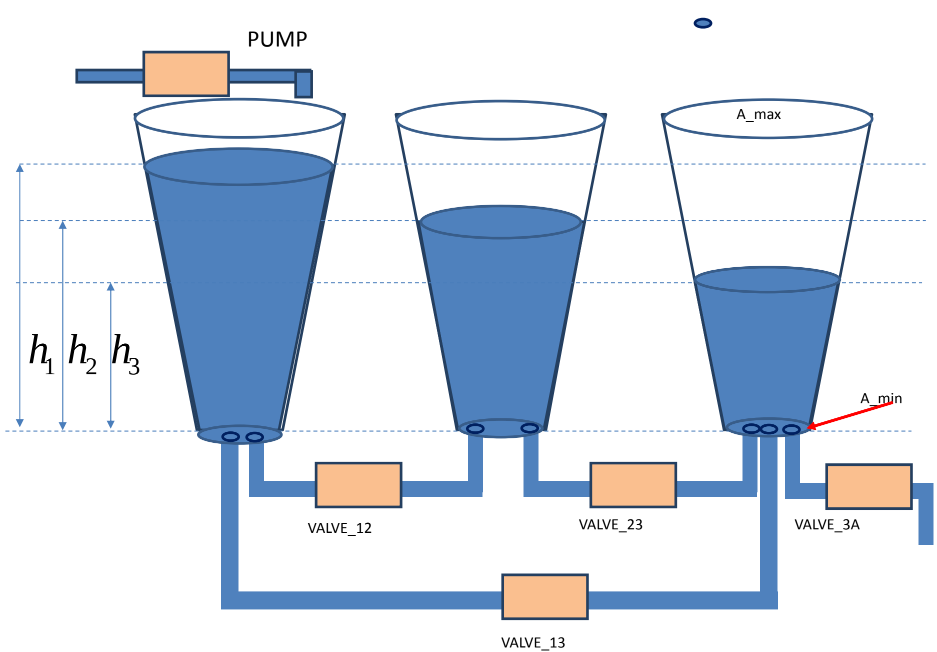
A student asked me "Why do we need to use state space? We already know how to model systems."

The approach we learnt before is for SISO systems. State space allows to deal with MIMO systems and with other complex systems which cannot be represented using ODEs.

Example: MIMO case. Dynamic model of a system of 3 interconnected tanks.







Because I know that each individual tank has a dynamics of order-1, and the rest of components in this system are pure algebraic models, I would need 3 states.

I choose,

$$\mathbf{x} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

Inputs: at least the pump's flow, q(t)

I may consider others, e.g. the aperture of certain valve.

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Outputs:

In general, output variables are functions of the states.

We chose them, according to application's needs.

E.g. { levels of tank 1 and 2, and outcoming flow of last tank}

OR e{levels of tank 1, 2 and 3 }

OR { level of tank 2 and the flow between tanks 2 and 3.

$$\mathbf{y} = g(\mathbf{x}) = \begin{bmatrix} h_2 \\ c_{23} \cdot sign(h_2 - h_3) \cdot \sqrt{|h_2 - h_3|} \end{bmatrix}$$

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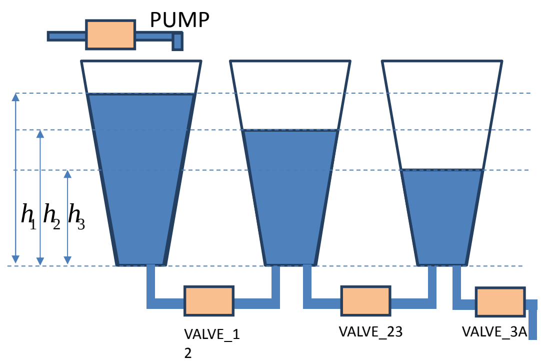
For the sake of simplicity, in this example, we consider the case of cylindrical tanks) i.e.

$$A_{\min} = A_{\max} = A$$



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We express the tank's volume as a function of its liquid level.

We also obtain their rates.

$$V(h) = h \cdot A$$

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = A \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = A^{-1} \cdot \frac{dV}{dt}$$

We express the valves' flows as functions of their differential pressures, and then as functions of their liquid levels.

We express volumes' rates based on incoming and outcoming flows, for each tank.

$q_{ik}$: flow from tank$_i$ to tank$_k$ $= c_{ik} \cdot sign(h_i - h_k) \cdot \sqrt{|h_i - h_k|}$

$$\frac{dV_1}{dt} = A \cdot \frac{dh_1}{dt} = q - q_{12} - q_{13} = q - c_{12} \cdot sign(h_1 - h_2) \cdot \sqrt{|h_1 - h_2|} - c_{13} \cdot sign(h_1 - h_3) \cdot \sqrt{|h_1 - h_3|}$$

$$\frac{dV_2}{dt} = A \cdot \frac{dh_2}{dt} = +q_{12} - q_{23} = +c_{12} \cdot sign(h_1 - h_2) \cdot \sqrt{|h_1 - h_2|} - c_{23} \cdot sign(h_2 - h_3) \cdot \sqrt{|h_2 - h_3|}$$

$$\frac{dV_3}{dt} = A \cdot \frac{dh_3}{dt} = +q_{13} + q_{23} - q_{3A} = c_{13} \cdot sign(h_1 - h_3) \cdot \sqrt{|h_1 - h_3|} - c_{23} \cdot sign(h_2 - h_3) \cdot \sqrt{|h_2 - h_3|} - c_{3A} \cdot \sqrt{h_3}$$



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Note: Actually, we do not even need to express V as function of h. If we at least have the section of the tank as function of h, A(h).

We also obtain their rates.

$$V(h) = \int_{h_0}^{h} A(z) \cdot dz \quad \Rightarrow \quad \frac{dV}{dh} = A(h)$$

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = A(h) \cdot \frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{1}{A(h)} \cdot \frac{dV}{dt}$$

Again, we express the valves' flows as functions of their differential pressures, and then as functions of their liquid levels (because valves' flows are function of tanks' pressures (which are functions of the liquid levels).

We express volumes' rates based on incoming and outcoming flows, for each tank.





Expressing the time derivatives of the states as function of the states and the inputs:

$$\frac{dh_1}{dt} =$$

$$= A^{-1} \cdot \left( q(t) - c_{12} \cdot sign(h_1 - h_2) \cdot \sqrt{|h_1 - h_2|} - c_{13} \cdot sign(h_1 - h_3) \cdot \sqrt{|h_1 - h_3|} \right)$$

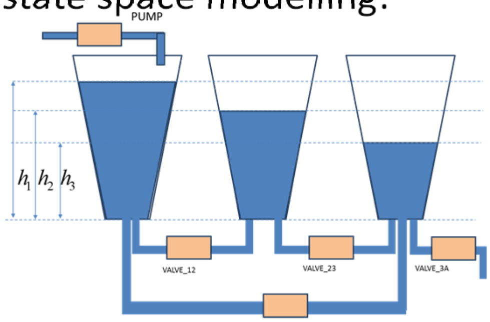
$$\frac{dh_2}{dt} =$$

$$= A^{-1} \cdot \left( c_{12} \cdot sign(h_1 - h_2) \cdot \sqrt{|h_1 - h_2|} - c_{23} \cdot sign(h_2 - h_3) \cdot \sqrt{|h_2 - h_3|} \right)$$

$$\frac{dh_3}{dt} =$$

$$= A^{-1} \cdot \left( c_{13} \cdot sign(h_1 - h_3) \cdot \sqrt{|h_1 - h_3|} + c_{23} \cdot sign(h_2 - h_3) \cdot \sqrt{|h_2 - h_3|} - c_{3A} \cdot \sqrt{h_3} \right)$$

# A case which needs state space modelling:




$$\mathbf{x} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

$$f(\mathbf{x}, \mathbf{u}) = f(x_1, x_2, x_3, u_1) = \begin{bmatrix} f_1(x_1, x_2, x_3, u_1) \\ f_2(x_1, x_2, x_3, u_1) \\ f_3(x_1, x_2, x_3, u_1) \end{bmatrix}$$

$$\left[\frac{\partial}{\partial \mathbf{x}} f(\mathbf{x}, \mathbf{u})\right] = ?,$$

$$\left[\frac{\partial}{\partial \mathbf{u}} f(\mathbf{x}, \mathbf{u})\right] = ?$$

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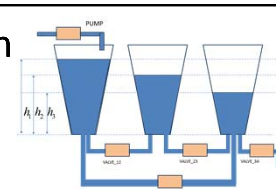
# We consider some point of operation




Consider the equilibrium point given by having he valves with fixed apertures such as :

$$c_{12} = c_{13} = c_{23} = 0.5, \quad c_{3A} = 1.21 \sim 0.5 \cdot \left(\sqrt{2} + 1\right)$$

And a constant incoming pump flow $q(t) = q_0 = 0.5 \cdot \left(\sqrt{2} + 1\right) m^3 / \sec$

Being that equilibrium point $h_1 = 3\,\mathrm{m}, h_2 = 2\,\mathrm{m}, h_3 = 1m$

$$\dot{h}_1 = \dot{h}_2 = \dot{h}_3 = 0$$

(you may verify that under those conditions of operation , the time derivatives of the system's states are NULL).

Obtain a linearized process model, for a region of operation surrounding that point of operation. The model should consider small variations of the pump flow, and the variations of the tanks' levels, respect to the nominal operation values.

(Linearization, to be solved at home)

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