State Space Modeling

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1 Overview

1.1 Linear Time-invariant Systems

Models of linear time-invariant (LTI) analog systems could be presented as linear differential equations with constant coefficients and transfer functions. By use of the Laplace transform, the transfer function can be derived from the differential equations, and a differential equation model can be derived from the transfer function using the inverse Laplace transform.

1.2 State-Space Model

Here, we consider another type of model: the state-space or state-variable model. This model is a differential equation model, but the equations are always written in a specific format. The state-variable model, or state-space model, is expressed as n first-order coupled differential equations. These equations preserve the system's input-output relationship (that of the transfer function); in addition, an internal model of the system is given.

1.3 Advantages

Some additional reasons for developing the state model are as follows:

1. Computer-aided analysis and design of state models are performed more easily on digital computers for higher-order systems, while the transfer-function approach tends to fail for these systems because of numerical problems.

2. In state-variable design procedures, we feedback more information (internal variables) about the plant; hence, we can achieve a more complete control of the system than is possible with the transfer-function approach.

3. State-variable models are generally required for digital simulation (digital computer solution of differential equations).

2 State-Space Modeling

2.1 Differential Equation Representation

The system model used to illustrate state variables, a linear mechanical translational system, is given in Figure 1. The differential equation describing this

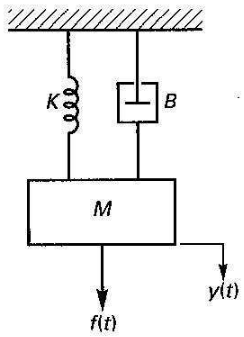

Figure 1: Mechanical translational system

system is given by

$$M\frac{d^2y(t)}{dt^2} = f(t) - B\frac{dy(t)}{dt} - Ky(t) \tag{1}$$

and the transfer function is given by

$$G(s) = \frac{Y(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K} \tag{2}$$

This equation gives a description of the position $y(t)$ as a function of the force $f(t)$. Suppose that we also want information about the velocity. Using the state-variable approach, we define the two state variables $x_1(t)$ and $x_2(t)$ as

$$x_1(t) = y(t) \tag{3}$$

$$x_2(t) = \frac{dy(t)}{dt} = \frac{dx_1(t)}{dt} = \dot{x}_1(t) \tag{4}$$

Thus $x_1(t)$ is the position of the mass and $x_2(t)$ is its velocity. Then, we may write

$$\frac{d^2y(t)}{dt^2} = \frac{dx_2(t)}{dt} = \dot{x}_2(t) = -\frac{B}{M}x_2(t) - \frac{K}{M}x_1(t) + \frac{1}{M}f(t) \tag{5}$$

The model is usually written in a specific format, which is given by rearranging the preceding equations as

$$\dot{x}_1(t) = x_2(t),\ \dot{x}_2(t) = -\frac{B}{M}x_2(t) - \frac{K}{M}x_1(t) + \frac{1}{M}f(t) \tag{6}$$

$$y(t) = x_1(t) \tag{7}$$

2.2 State-space Representation

Definition: *The* ***state*** *of a system at any time t_0 is defined as the amount of information at t_0 that, together with all inputs for $t \leq t_0$, uniquely determines the behavior of the system for all $t \leq t_0$.*

Usually, state equations are written in a vector-matrix format, since this allows the equations to be manipulated much more easily. In this format the preceding equations become

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} f(t) \tag{8}$$

The standard form of the state equations of a LTI analog system is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{9}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{10}$$

where the vector $\dot{\mathbf{x}}(t)$ is the time derivative of the vector $\mathbf{x}(t)$. In these equations,

- $\mathbf{x}(t) = \textit{state vector} = (n \times 1)$ vector of the states of an nth-order system,
- $\mathbf{A} = (n \times n)\textit{system matrix}$,
- $\mathbf{B} = (n \times r)\textit{input matrix}$,
- $u(t) = \textit{input vector} = (r \times 1)$ vector composed of the system input functions,
- $y(t) = \textit{outputvector} = (p \times 1)$ vector composed of the defined outputs,
- $\mathbf{C} = (p \times n)\textit{output matrix}$,
- $\mathbf{D} = (p \times r)$ matrix to represent direct coupling between input and output.

We refer to the two matrix equations of (9) as the state-variable equations of the system. The first equation is called the state equation, and the second one is called the output equation. The signals in expanded form are given by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}, \ \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \ \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}, \ \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix} \tag{11}$$

2.3 State Equation

The state equation, is a first-order matrix differential equation, and the state vector, $\mathbf{x}(t)$, is its solution. Given knowledge of $\mathbf{x}(t)$ and the input vector $u(t)$, the output equation yields the output $y(t)$. Usually the matrix $\mathbf{D}$ is zero, since in physical systems, dynamics appear in all paths between the inputs and the outputs. A nonzero value of $\mathbf{D}$ indicates at least one direct path between the inputs and the outputs, in which the path transfer function can be modeled as a pure gain.

In the equation for the state variables $\mathbf{x}(t)$, only the first derivatives of the state variables may appear on the left side of the equation, and no derivatives may appear on the right side. No derivatives may appear in the output equation.

The general form of the state equations just given allows for more than one input and output; these systems are called *multi-variable systems.* For the case of one input, the matrix $\mathbf{B}$ is a column vector, and the vector $\mathbf{u}(t)$ is a scalar. For the case of one output, the vector $y(t)$ is a scalar, and the matrix $\mathbf{C}$ is a row vector. In this book, we do not differentiate notationally between a column vector and a row vector. The use of the vector implies the type.

2.4 Example

Consider the system described by the coupled differential equations

$$\ddot{y}_1 + k_1\dot{y}_1 + k_2 y_1 = u_1 + k_3 u_2 \tag{12}$$

$$\dot{y}_2 + k_4 y_2 + k_5\dot{y}_1 = k_6 u_1 \tag{13}$$

where u_1 and u_2 are inputs, y_1 and y_2 are outputs, and $k_i, i = 1, \cdots, 6$ are system parameters. The notational dependence of the variables on time has been omitted for convenience. We may define the states as the outputs and, where necessary, the derivatives of the outputs.

$$x_1 = y_1, \ x_2 = \dot{y}_1 = \dot{x}_1, \ x_3 = y_2 \tag{14}$$

From the system differential equations, we write

$$\dot{x}_1 = x_2 \tag{15}$$

$$\dot{x}_2 = -k_2 x_1 - k_1 x_2 + u_1 + k_3 u_2 \tag{16}$$

$$\dot{x}_3 = -k_5 x_2 - k_4 x_3 + k_6 u_1 \tag{17}$$

with output equations

$$y_1 = x_1,\ y_2 = x_3 \tag{18}$$

These equations may be written in matrix form as

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -k_2 & -k_1 & 0 \\ 0 & -k_5 & -k_4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 1 & k_3 \\ k_6 & 0 \end{bmatrix} \mathbf{u} \tag{19}$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} \tag{20}$$