Steady-State Response

1 Effects of Integral and Derivative Control Actions on System Performance

1.1 Integral Control Action

In the proportional control of a plant whose transfer function does not possess an integrator $1/s$, there is a steady-state error, or offset, in the response to a step input. Such an offset can be eliminated if the integral control action is included in the controller.

In the integral control of a plant, the control signal, the output signal from the controller, at any instant is the area under the actuating-error-signal curve up to that instant. The control signal $u(t)$ can have a nonzero value when the actuating error signal $e(t)$ is zero, as shown in Figure 1(a). This is impossible in the case of the proportional controller, since a nonzero control signal requires a nonzero actuating error signal. A nonzero actuating error signal at steady state means that there is an offset. Figure 1(b) shows the curve $e(t)$ versus t and the corresponding curve $u(t)$ versus t when the controller is of the proportional type.

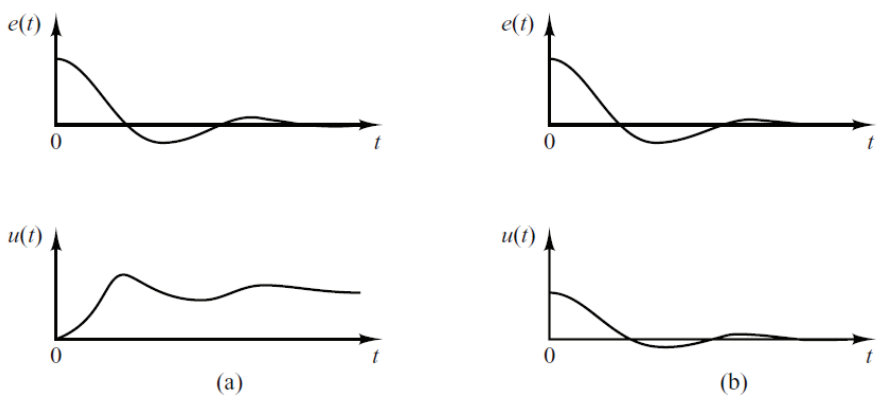

Figure 1: (a) Plots of $e(t)$ and $u(t)$ curves showing nonzero control signal when the actuating error signal is zero (integral control); (b) plots of $e(t)$ and $u(t)$ curves showing zero control signal when the actuating error signal is zero, (proportional control).

Note that integral control action, while removing offset or steady-state error, may lead to oscillatory response of slowly decreasing amplitude or even increasing amplitude, both of which are usually undesirable.

1.2 Proportional Control of Systems

We shall show that the proportional control of a system without an integrator will result in a steady-state error with a step input. We shall then show that such an error can be eliminated if integral control action is included in the controller.

Consider the system shown in Figure 2.

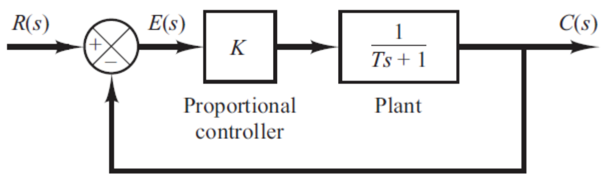

Figure 2: Proportional control system.

Let us obtain the steady-state error in the unit-step response of the system. Define

$$G(s) = \frac{K}{Ts+1} \tag{1}$$

Since

$$\frac{E(s)}{R(s)} = \frac{R(s) - C(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = \frac{1}{1+G(s)} \tag{2}$$

the error $E(s)$ is given by

$$E(s) = \frac{1}{1+G(s)}R(s) = \frac{1}{1+\frac{K}{Ts+1}}R(s) \tag{3}$$

For the unit-step input $R(s) = 1/s$, we have

$$E(s) = \frac{Ts+1}{Ts+1+K}\,\frac{1}{s} \tag{4}$$

The steady-state error is

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{Ts+1}{Ts+1+K} = \frac{1}{K+1} \tag{5}$$

Such a system without an integrator in the feedforward path always has a steady-state error in the step response. Such a steady-state error is called an offset. Figure 3 shows the unit-step response and the offset.

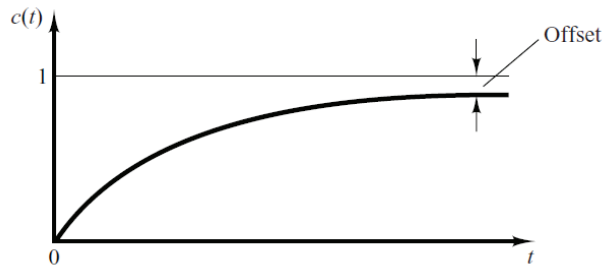

Figure 3: Unit-step response and offset.

1.3 Integral Control of Systems

Consider the system shown in Figure 4. The controller is an integral controller. The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{K}{s(Ts+1)+K} \tag{6}$$

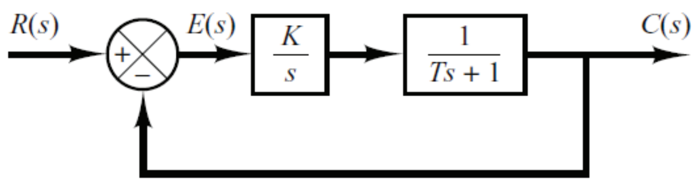

Figure 4: Integral control system.

Hence

$$\frac{E(s)}{R(s)} = \frac{R(s) - C(s)}{R(s)} = \frac{s(Ts+1)}{s(Ts+1)+K} \tag{7}$$

Since the system is stable, the steady-state error for the unit-step response can be obtained by applying the final-value theorem, as follows:

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s^2(Ts+1)}{Ts^2+s+K}\,\frac{1}{s} = 0 \tag{8}$$

Integral control of the system thus eliminates the steady-state error in the response to the step input. This is an important improvement over the proportional control alone, which gives offset.

1.4 Proportional Control of Systems with Inertia Load

Consider the system shown in Figure 5(a). The closed-loop transfer function is obtained as

$$\frac{C(s)}{R(s)} = \frac{K_p}{Js^2+K_p} \tag{9}$$

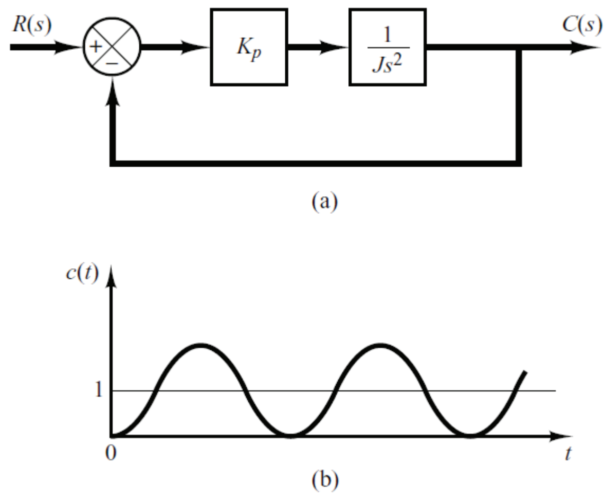

Figure 5: (a) Proportional control of a system with inertia load; (b) response to a unit-step input.

Since the roots of the characteristic equation

$$Js^2 + K_p = 0 \tag{10}$$

are imaginary, the response to a unit-step input continues to oscillate indefinitely, as shown in Figure 5(b). Control systems exhibiting such response characteristics are not desirable. We shall see that the addition of derivative control will stabilize the system.

1.5 Proportional-Plus-Derivative Control of a System with Inertia Load

Let us modify the proportional controller to a proportional-plus-derivative controller whose transfer function is $K_p(1 + T_d s)$. The torque developed by the controller is proportional to $K_p(e + T_d \dot{e})$. Derivative control is essentially anticipatory, measures the instantaneous error velocity, and predicts the large overshoot ahead of time and produces an appropriate counteraction before too large an overshoot occurs.

Consider the system shown in Figure 6(a). The closed-loop transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{K_p(1 + T_d s)}{Js^2 + K_p T s + K_p} \tag{11}$$

The characteristic equation

$$Js^2 + K_p T s + K_p = 0 \tag{12}$$

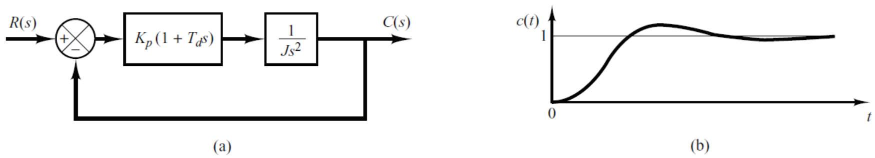

Figure 6: (a) Proportional-plus-derivative control of a system with inertia load; (b) response to a unit-step input.

now has two roots with negative real parts for positive values of J, K_p, and T_d. Thus derivative control introduces a damping effect. A typical response curve $c(t)$ to a unitstep input is shown in Figure 6(b). Clearly, the response curve shows a marked improvement over the original response curve shown in Figure 5(b).

1.6 Proportional-Plus-Derivative Control of Second-Order Systems

A compromise between acceptable transient-response behavior and acceptable steady-state behavior may be achieved by use of proportional-plus-derivative control action.

Consider the system shown in Figure 7.

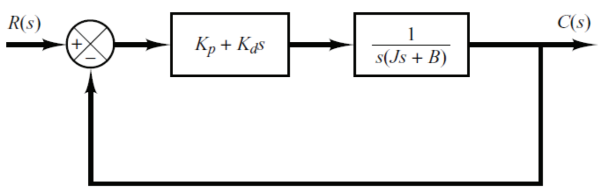

Figure 7: Control system.

The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K_p + K_d s}{Js^2 + (B + K_d)s + K_p} \tag{13}$$

The characteristic equation is

$$Js^2 + (B + K_d)s + K_p = 0 \tag{14}$$

The effective damping coefficient of this system is thus $B + K_d$ rather than B. Since the damping ratio ζ of this system is

$$\zeta = \frac{B + K_d}{2\sqrt{K_p J}} \tag{15}$$

It is possible to make both the steady-state error e_{ss} for a ramp input and the maximum overshoot for a step input small by making B small, K_p large, and K_d large enough so that ζ is between 0.4 and 0.7.

2 Steady-State Errors in Unity-Feedback Control Systems

Any physical control system inherently suffers steady-state error in response to certain types of inputs. A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input. The only way we may be able to eliminate this error is to modify the system structure. Whether a given system will exhibit steady-state error for a given type of input depends on the type of open-loop transfer function of the system, to be discussed in what follows.

2.1 Classification of Control Systems

Control systems may be classified according to their ability to follow step inputs, ramp inputs, parabolic inputs, and so on. This is a reasonable classification scheme, because actual inputs may frequently be considered combinations of such inputs. The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system.

Consider the unity-feedback control system with the following open-loop transfer function $G(s)$:

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1) \cdots (T_m s + 1)}{s^N (T_1 s + 1)(T_2 s + 1) \cdots (T_p s + 1)} \tag{16}$$

It involves the term s^N in the denominator, representing a pole of multiplicity N at the origin. The present classification scheme is based on the number of integrations indicated by the open-loop transfer function. A system is called type 0, type 1, type 2, $\cdots$, if $N = 0$, $N = 1$, $N = 2$, $\cdots$, respectively. Note that this classification is different from that of the order of a system. As the type number is increased, accuracy is improved; however, increasing the type number aggravates the stability problem. A compromise between steady-state accuracy and relative stability is always necessary. We shall see later that, if $G(s)$ is written so that each term in the numerator and denominator, except the term s^N, approaches unity as s approaches zero, then the open-loop gain K is directly related to the steady-state error.

2.2 Steady-State Errors

Consider the system shown in Figure 8.

The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \tag{17}$$

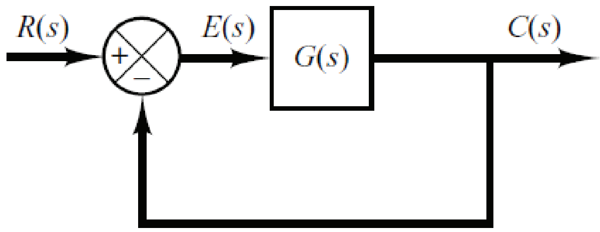

Figure 8: Control system.

The transfer function between the error signal $e(t)$ and the input signal $r(t)$ is

$$\frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = \frac{1}{1+G(s)} \tag{18}$$

where the error $e(t)$ is the difference between the input signal and the output signal. The final-value theorem provides a convenient way to find the steady-state performance of a stable system. Since $E(s)$ is

$$E(s) = \frac{1}{1+G(s)} R(s) \tag{19}$$

the steady-state error is

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} \tag{20}$$

The static error constants defined in the following are figures of merit of control systems. The higher the constants, the smaller the steady-state error. In a given system, the output may be the position, velocity, pressure, temperature, or the like. The physical form of the output, however, is immaterial to the present analysis. Therefore, in what follows, we shall call the output "position," the rate of change of the output "velocity," and so on. This means that in a temperature control system "position" represents the output temperature, "velocity" represents the rate of change of the output temperature, and so on.

2.3 Static Position Error Constant K_p

The steady-state error of the system for a unit-step input is

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \frac{1}{s} = \frac{1}{1+G(0)} \tag{21}$$

The static position error constant K_p is defined by

$$K_p = \lim_{s\to 0} G(s) = G(0) \tag{22}$$

Thus, the steady-state error in terms of the static position error constant K_p is given by

$$e_{ss} = \frac{1}{1 + K_p} \tag{23}$$

For a type 0 system,

$$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1) \cdots}{(T_1 s + 1)(T_2 s + 1) \cdots} = K \tag{24}$$

For a type 1 or higher system,

$$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1) \cdots}{s^N (T_1 s + 1)(T_2 s + 1) \cdots} = \infty, \;\; N \geq 1 \tag{25}$$

Hence, for a type 0 system, the static position error constant K_p is finite, while for a type 1 or higher system, K_p is infinite.

For a unit-step input, the steady-state error ess may be summarized as follows:

$$e_{ss} = \begin{cases} \frac{1}{1+K}, & \text{type 0 system} \\ 0, & \text{type 1 or higher systems} \end{cases} \tag{26}$$

From the foregoing analysis, it is seen that the response of a feedback control system to a step input involves a steady-state error if there is no integration in the feedforward path. If small errors for step inputs can be tolerated, then a type 0 system may be permissible, provided that the gain K is sufficiently large. If the gain K is too large, however, it is difficult to obtain reasonable relative stability. If zero steady-state error for a step input is desired, the type of the system must be one or higher.

2.4 Static Velocity Error Constant K_v

The steady-state error of the system with a unit-ramp input is given by

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \, \frac{1}{s^2} = \lim_{s \to 0} \frac{1}{sG(s)} \tag{27}$$

The static velocity error constant K_v is defined by

$$K_v = \lim_{s \to 0} sG(s) \tag{28}$$

Thus, the steady-state error in terms of the static velocity error constant K_v is given by

$$e_{ss} = \frac{1}{K_v} \tag{29}$$

The term velocity error is used here to express the steady-state error for a ramp input. The dimension of the velocity error is the same as the system error. That is, velocity error is not an error in velocity, but it is an error in position due to a ramp input. For a type 0 system,

$$K_v = \lim_{s\to 0} \frac{sK(Tas+1)(T_b s+1)\cdots}{(T1s+1)(T_2 s+1)\cdots} = 0 \tag{30}$$

For a type 1 system,

$$K_v = \lim_{s\to 0} \frac{sK(Tas+1)(T_b s+1)\cdots}{s(T1s+1)(T_2 s+1)\cdots} = K \tag{31}$$

For a type 2 or higher system,

$$K_v = \lim_{s\to 0} \frac{sK(Tas+1)(T_b s+1)\cdots}{s^N(T1s+1)(T_2 s+1)\cdots} = \infty,\ N \geq 2 \tag{32}$$

The steady-state error e_{ss} for the unit-ramp input can be summarized as follows:

$$e_{ss} = \begin{cases} \frac{1}{K_v} = \infty, & \text{type 0 systems} \\ \frac{1}{K_v} = \frac{1}{K}, & \text{type 1 systems} \\ \frac{1}{K_v} = 0, & \text{type 2 or higher systems} \end{cases} \tag{33}$$

The foregoing analysis indicates that a type 0 system is incapable of following a ramp input in the steady state. The type 1 system with unity feedback can follow the ramp input with a finite error. In steady-state operation, the output velocity is exactly the same as the input velocity, but there is a positional error. This error is proportional to the velocity of the input and is inversely proportional to the gain K. Figure 9 shows an example of the response of a type 1 system with unity feedback to a ramp input. The type 2 or higher system can follow a ramp input with zero error at steady state.

2.5 Static Acceleration Error Constant K_a

The steady-state error of the system with a unit-parabolic input (acceleration input), which is defined by

$$r(t) = \begin{cases} \frac{t^2}{2}, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{34}$$

is given by

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \frac{1}{s^3} = \lim_{s\to 0} \frac{1}{s^2 G(s)} \tag{35}$$

The static acceleration error constant K_a is defined by the equation

$$K_a = \lim_{s\to 0} s^2 G(s) \tag{36}$$

The steady-state error is then

$$e_{ss} = \frac{1}{K_a} \tag{37}$$

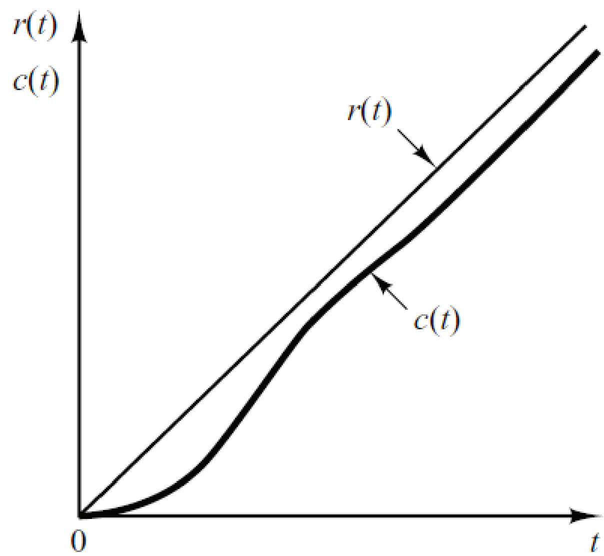

Figure 9: Response of a type 1 unity-feedback system to a ramp input.

Note that the acceleration error, the steady-state error due to a parabolic input, is an error in position. The values of K_a are obtained as follows: For a type 0 system,

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b + 1) \cdots}{(T_1 s + 1)(T_2 s + 1) \cdots} = 0 \tag{38}$$

For a type 1 system,

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b + 1) \cdots}{s(T_1 s + 1)(T_2 s + 1) \cdots} = 0 \tag{39}$$

For a type 2 system,

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b + 1) \cdots}{s^2(T_1 s + 1)(T_2 s + 1) \cdots} = K \tag{40}$$

For a type 3 or higher system,

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b + 1) \cdots}{s^N(T_1 s + 1)(T_2 s + 1) \cdots} = \infty \tag{41}$$

Thus, the steady-state error for the unit parabolic input is

$$e_{ss} = \begin{cases} \infty, & \text{type 0 and type 1 systems} \\ \frac{1}{K}, & \text{type 2 systems} \\ 0, & \text{type 3 or higher systems} \end{cases} \tag{42}$$

Note that both type 0 and type 1 systems are incapable of following a parabolic input in the steady state. The type 2 system with unity feedback can follow a parabolic input with a finite error signal. Figure 10 shows an example of the response of a type 2 system with unity feedback to a parabolic input. The type 3 or higher system with unity feedback follows a parabolic input with zero error at steady state.

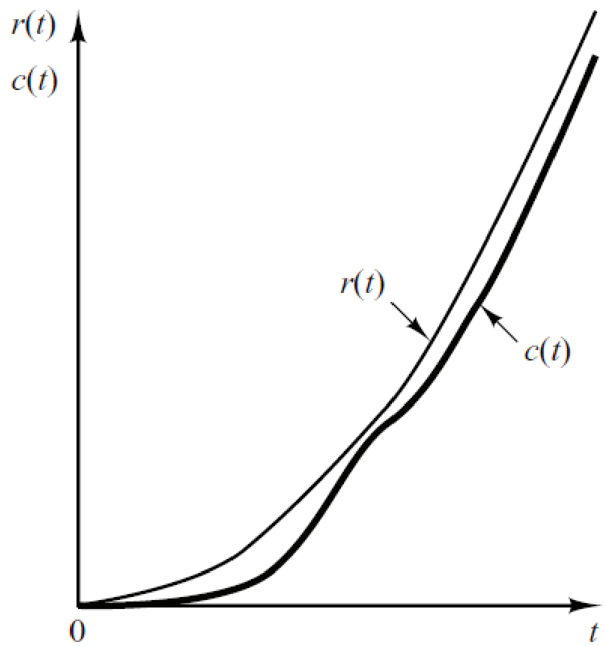

Figure 10: Response of a type 2 unity-feedback system to a parabolic input.

2.6 Summary

Table 1 summarizes the steady-state errors for type 0, type 1, and type 2 systems when they are subjected to various inputs. The finite values for steadystate errors appear on the diagonal line. Above the diagonal, the steady-state errors are infinity; below the diagonal, they are zero.

Table 1: Steady-State Error in Terms of Gain K

	Step Input $r(t) = 1$	Ramp Input $r(t) = t$	Acc Input $r(t) = t^2/2$
Type 0 system	$\frac{1}{1+K}$	∞	∞
Type 1 system	0	$\frac{1}{K}$	∞
Type 2 system	0	0	$\frac{1}{K}$

Remember that the terms position error, velocity error, and acceleration error mean steady-state deviations in the output position. A finite velocity error implies that after transients have died out, the input and output move at the same velocity but have a finite position difference.

The error constants K_p, K_v, and K_a describe the ability of a unity-feedback system to reduce or eliminate steady-state error. Therefore, they are indicative of the steady-state performance. It is generally desirable to increase the error constants, while maintaining the transient response within an acceptable range. It is noted that to improve the steadystate performance we can increase the type of the system by adding an integrator or integrators to the feedforward path. This, however, introduces an additional stability problem. The design of a satisfactory system with more than two integrators in series in the feedforward path is generally not easy.