Tutorial—Laplace Transform

Example 1. *Find the Laplace transform of hyperbolic sine:* $\sinh(bt)$ *and hyperbolic cosine:* $\cosh(bt)$. *Hint:* $\sinh(bt) = (e^{bt} - e^{-bt})/2$ *and* $\cosh(bt) = (e^{bt} + e^{-bt})/2$.

Example 2. *Find the Laplace transform of positive powers of* t^n. *Hint: apply integration by parts:* $\int uv'dx = uv - \int u'vdx$. *Let* $u = t^n$, $dv = e^{-st}dt$, *then* $du = nt^{n-1}dt$, *and* $v = \int e^{-st}dt$.

Example 3. *Find the Laplace transform of* $f(t) = Ae^{-at}\sin(bt + \theta)$. *Hint: (i) express* $f(t)$ *in exponential form, (ii) first find* $\mathfrak{L}[A\sin(bt + \theta)]$, *and then apply* $\mathfrak{L}[e^{-at}f(t)] = F(s + a)$.

Example 4. *Find the Laplace transform of the following waveforms in Fig. 1. Which waveform that the Shifting Theorem is applicable?*

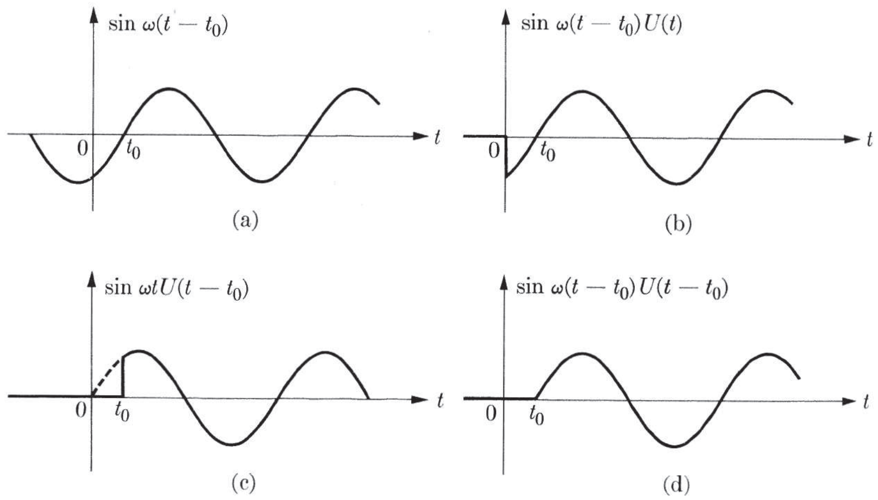

Figure 1: Four different waveforms

Example 5. *Find the Laplace transform of the sawtooth waveform shown in Fig. 2. Hint: de-composite the waveform into three components.*

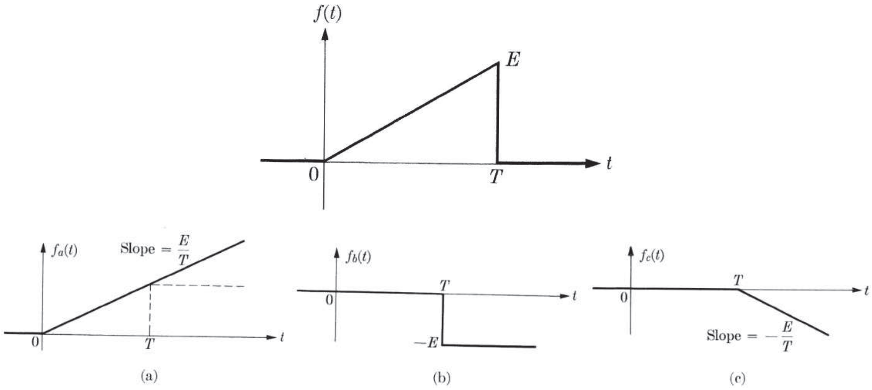

Figure 2: The sawtooth waveform and its compositions

Example 6. *Find the Laplace transform of the single half-cycle sine wave shown in Fig. 3*

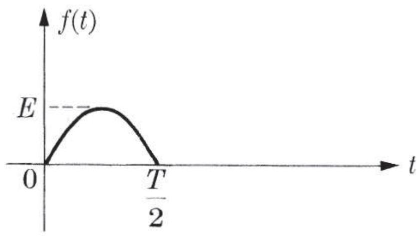

Figure 3: A single half-cycle sine wave

Example 7. *Find the Laplace transform of the sawtooth waveform shown in Fig. 2 by making use of a gate function* $G_0(t_0, T) = U(t - t_0) - U(t - t_0 - T)$*, see Fig. 4.*

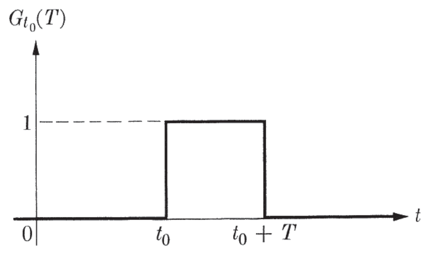

Figure 4: A gate function

Example 8. *Find the Laplace transform of the single half-cycle sine wave shown in Fig. 3 by making use of the gate function.*

Example 9. *Find the Laplace transform of the periodic half-cycle sine wave shown in Fig. 5.*

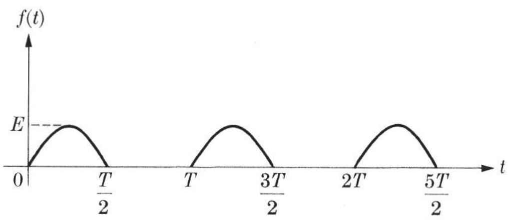

Figure 5: Rectified half-cycle sine waveform

Example 10. *Find the Laplace transform of the periodic sawtooth waveform shown in Fig. 6*

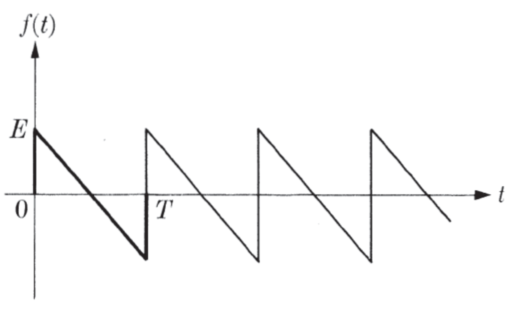

Figure 6: A periodic sawtooth waveform

Example 11. *Find* $\mathfrak{L}[t^2 \sin \omega t]$. *Hint: compute the differential of* $\mathfrak{L}[\sin \omega t]$.

Example 12. *Find* $\mathfrak{L}\left[\frac{1}{t} \sin \omega t\right]$. *Hint: compute the integral of* $\mathfrak{L}[\sin \omega t]$.

Solution 1. *From*

$$\mathfrak{L}[e^{-at}] = \frac{1}{s+a} \tag{1}$$

then

$$\mathfrak{L}[e^{bt}] = \frac{1}{s-b}, \ \mathfrak{L}[e^{-bt}] = \frac{1}{s+b} \tag{2}$$

and

$$\frac{1}{2}\left[\frac{1}{s-b} - \frac{1}{s+b}\right] = \frac{1}{2}\left[\frac{s+b-(s-b)}{(s-b)(s+b)}\right] = \frac{1}{2}\left[\frac{2b}{s^2-b^2}\right] \Rightarrow \mathfrak{L}[\sinh(bt)] = \frac{b}{s^2-b^2} \tag{3}$$

Similarly,

$$\frac{1}{2}\left[\frac{1}{s-b} + \frac{1}{s+b}\right] = \frac{1}{2}\left[\frac{s+b+(s-b)}{(s-b)(s+b)}\right] = \frac{1}{2}\left[\frac{2s}{s^2-b^2}\right] \Rightarrow \mathfrak{L}[\cosh(bt)] = \frac{s}{s^2-b^2} \tag{4}$$

□

Solution 2. *From $\mathfrak{L}[t^n] = \int t^n e^{-st} dt$, and integrate by parts*

$$\int_0^\infty t^n e^{-st} dt = -\frac{t^n}{s} e^{-st}\Big|_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt = \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt \tag{5}$$

It shows that

$$\mathfrak{L}[t^n] = \frac{n}{s}\mathfrak{L}[t^{n-1}] \tag{6}$$

By extending this process, we have

$$\mathfrak{L}[t^n] = \frac{n}{s}\mathfrak{L}[t^{n-1}] = \frac{n}{s}\frac{n-1}{s}\mathfrak{L}[t^{n-2}] = \frac{n}{s}\frac{n-1}{s}\frac{n-2}{s}\cdots\frac{1}{s}\mathfrak{L}[t^0] \tag{7}$$

But t^0 for $t > 0$ is the same as the unit step function $U(t)$, which has a Laplacec transform $1/s$. Hence

$$\mathfrak{L}[t^n] = \frac{n!}{s^{n+1}} \tag{8}$$

□

Solution 3. *(i) Express $f(t)$ in exponential form:*

$$\begin{aligned} f(t) &= \frac{A}{2j} e^{-at}\left[e^{j(bt+\theta)} - e^{-j(bt+\theta)}\right] \\ &= \frac{A}{2j}\left[e^{(-a+jb)t}e^{j\theta} - e^{(-a-jb)t}e^{-j\theta}\right] \end{aligned} \tag{9}$$

then

$$\begin{aligned}
\mathfrak{L}[f(t)] &= \frac{A}{2j}\left\{e^{j\theta}\mathfrak{L}[e^{(-a+jb)t}] - e^{-j\theta}\mathfrak{L}[e^{(-a-jb)t}]\right\} \\
&= \frac{A}{2j}\left\{\frac{e^{j\theta}}{s+a-jb} - \frac{e^{-j\theta}}{s+a+jb}\right\} \\
&= \frac{A}{2}\left\{\frac{(s+a+jb)(-j\cos\theta+\sin\theta)-(s+a-jb)(-j\cos\theta-\sin\theta)}{(s+a)^2+b^2}\right\} \\
&= A\left\{\frac{(s+a)\sin\theta+b\cos\theta}{(s+a)^2+b^2}\right\}
\end{aligned} \tag{10}$$

(ii) First we have

$$\begin{aligned}
\mathfrak{L}[A\sin(bt+\theta)] &= A\mathfrak{L}[\sin bt\cos\theta+\cos bt\sin\theta] \\
&= A\mathfrak{L}[\sin bt]\cos\theta + A\mathfrak{L}[\cos bt]\sin\theta \\
&= A\left[\frac{b}{s^2+b^2}\right]\cos\theta + A\left[\frac{s}{s^2+b^2}\right]\sin\theta \\
&= A\left[\frac{b\cos\theta+s\sin\theta}{s^2+b^2}\right]
\end{aligned} \tag{11}$$

From $\mathfrak{L}[e^{-at}f(t)] = F(s+a)$, *then*

$$\mathfrak{L}[Ae^{-st}\sin(bt+\theta)] = A\left[\frac{b\cos\theta+(s+a)\sin\theta}{(s+a)^2+b^2}\right] \tag{12}$$

$\square$

Solution 4. *For waveforms (a) and (b), they are equivalent because of the definition of Laplace transform. The integration starts from* $t=0$. *Then*

$$\mathfrak{L}[\sin\omega(t-t_0)] = \mathfrak{L}[\sin\omega t\cos\omega t_0 - \cos\omega t\sin\omega t_0] = \frac{\omega\cos\omega t_0 - s\sin\omega t_0}{s^2+\omega^2} \tag{13}$$

For waveform (c), we have

$$\begin{aligned}
\mathfrak{L}[\sin\omega t U(t-t_0)] &= \int_{t_0}^{\infty}\sin\omega t e^{-st}dt \\
&= \frac{1}{2j}\int_{t_0}^{\infty}\left[e^{(-s+j\omega)t} - e^{(-s-j\omega)t}\right]dt \\
&= \frac{1}{2j}\left[\frac{e^{(-s+j\omega)t_0}}{s-j\omega} - \frac{e^{(-s-j\omega)t_0}}{s+j\omega}\right] \\
&= e^{-st_0}\left[\frac{\omega\cos\omega t_0 + s\sin\omega t_0}{s^2+\omega^2}\right]
\end{aligned} \tag{14}$$

For waveform (d), we note that the start of the waveform coincides with the shifted unit function. Hence, we have a pure time-shifted sine wave. Then

$$\mathfrak{L}[\sin\omega(t-t_0)U(t-t_0)] = e^{-st_0}\mathfrak{L}[\sin\omega t] = e^{-st_0}\frac{\omega}{s^2+\omega^2} \tag{15}$$

Hence, the Shifting Theorem can be applied to waveform (d). □

Solution 5. *Before the Laplace transform can be found, it is necessary to write the expression for $f(t)$. It can be constructed from $f(t) = f_a(t) + f_b(t) + f_c(t)$, where*

$$f_a(t) = \frac{E}{T} tU(t),\ f_b(t) = -EU(t-T),\ f_c(t) = -\frac{E}{T}(t-T)U(t-T) \tag{16}$$

Now,

$$\mathfrak{L}[f_a(t)] = \frac{E}{Ts^2},\ \mathfrak{L}[f_b(t)] = -\frac{E}{s}e^{-sT},\ \mathfrak{L}[f_c(t)] = -\frac{E}{Ts^2}e^{-sT} \tag{17}$$

Hence

$$\mathfrak{L}[f(t)] = \mathfrak{L}[f_a(t)] + \mathfrak{L}[f_b(t)] + \mathfrak{L}[f_c(t)] = \frac{E}{Ts^2}\left[1 - (sT+1)e^{-sT}\right] \tag{18}$$

□

Solution 6. *We can de-composite the waveform as the addition of a continuous sine wave and a shifted sine wave that cancels the rest of the continuous waveform. We can write*

$$f(t) = f_a(t) + f_b(t) = E\sin\frac{\pi}{T}tU(t) + E\sin\frac{2\pi}{T}\left(t - \frac{T}{2}\right)U\left(t - \frac{T}{2}\right) \tag{19}$$

Since we known the Laplace transform of the sine function, with the aid of the Shifting Theorem we can write

$$\begin{aligned}\mathfrak{L}[f(t)] = \mathfrak{L}[f_a(t)] + \mathfrak{L}[f_b(t)]&\\ &= \frac{E2\pi/T}{s^2 + (2\pi/T)^2)} + \frac{E2\pi/T}{s^2 + (2\pi/T)^2)}e^{-sT/2}\\ &= \frac{E2\pi/T}{s^2 + (2\pi/T)^2)}\left(1 + e^{-sT/2}\right)\end{aligned} \tag{20}$$

□

Solution 7. *We can note that*

$$f(t) = \frac{E}{T}tG(T) = \frac{E}{T}t\left[U(t) - U(t-T)\right] \tag{21}$$

then no graphical composition is necessary. We have

$$\begin{aligned}\mathfrak{L}[f(t)] &= \frac{E}{T}\left\{\mathfrak{L}[tU(t)] - \mathfrak{L}[tU(t-T)]\right\}\\ &= \frac{E}{T}\left\{\frac{1}{s^2} - \mathfrak{L}[(t-T)+T]U(t-T)\right\}\\ &= \frac{E}{T}\left\{\frac{1}{s^2} - \left[\frac{1}{s^2} + \frac{T}{s}\right]e^{-sT}\right\}\end{aligned} \tag{22}$$

□

Solution 8. *Using a gate function $G_0(T/2)$, we have*

$$f(t) = E\sin\left(\frac{2\pi}{T}t\right)G_0\left(\frac{T}{2}\right) = E\sin\left(\frac{2\pi}{T}t\right)\left[U(t) - U\left(t - \frac{T}{2}\right)\right] \tag{23}$$

Then

$$\mathfrak{L}[f(t)] = E\mathfrak{L}\left[\sin\left(\frac{2\pi}{T}t\right)U(t)\right] - E\mathfrak{L}\left[\sin\left(\frac{2\pi}{T}t\right)U\left(t - \frac{T}{2}\right)\right] = F_1(s) - F_2(s) \tag{24}$$

where

$$F_1(s) = E\mathfrak{L}\left[\sin\left(\frac{2\pi}{T}t\right)U(t)\right] = \frac{E2\pi/T}{s^2 + (2\pi/T)^2} \tag{25}$$

$$\begin{aligned} F_2(s) &= E\mathfrak{L}\left[\sin\left(\frac{2\pi}{T}t\right)U\left(t - \frac{T}{2}\right)\right] \\ &= E\mathfrak{L}\left\{\sin\frac{2\pi}{T}\left[\left(t - \frac{T}{2}\right) + \frac{T}{2}\right]U\left(t - \frac{T}{2}\right)\right\} \\ &= E\mathfrak{L}\left\{\sin\left[\frac{2\pi}{T}\left(t - \frac{T}{2}\right) + \pi\right]U\left(t - \frac{T}{2}\right)\right\} \\ &= E\mathfrak{L}\left\{-\sin\frac{2\pi}{T}\left(t - \frac{T}{2}\right)U\left(t - \frac{T}{2}\right)\right\} \\ &= -\frac{E2\pi/T}{s^2 + (2\pi/T)^2}e^{-sT/2} \end{aligned} \tag{26}$$

then

$$\mathfrak{L}[f(t)] = \frac{E2\pi/T}{s^2 + (2\pi/T)^2}\left(1 + e^{-sT/2}\right) \tag{27}$$

□

Solution 9. *Let $f(t)$ be a periodic function with period T_1 and $f_1(t)$, $f_2(t)$, $\cdots$ be the function describing the first, second cycle (time shifted), $\cdots$. Then*

$$f(t) = f_1(t) + f_1(t - T)U(t - T) + f_1(t)(t - 2T)U(t - 2T) + \cdots \tag{28}$$

If we call $\mathfrak{L}[f_1(t)] = F_1(s)$, then by the Shifting Theorem, we have

$$\mathfrak{L}[f(t)] = (1 + e^{-sT} + e^{-2sT} + \cdots)F_1(s) = \frac{1}{1 - e^{-sT}}F_1(s) \tag{29}$$

Now, $F_1(s) = \frac{E2\pi/T}{s^2+(2\pi/T)^2}\left(1 + e^{-sT/2}\right)$, then

$$\begin{aligned} \mathfrak{L}[f(t)] &= \frac{1 + e^{-sT/2}}{1 - e^{-sT}}\frac{E2\pi/T}{s^2 + (2\pi/T)^2} \\ &= \frac{1}{1 - e^{-sT/2}}\frac{E2\pi/T}{s^2 + (2\pi/T)^2} \end{aligned} \tag{30}$$

□

Solution 10. *We first determine $f_1(t)$ for the first period (it is simplest to use the gate function).*

$$
\begin{aligned}
f_1(t) &= -\frac{2E}{T}\left(t-\frac{T}{2}\right)[U(t)-U(t-T)] \\
&= -\frac{2E}{T}U\left(t-\frac{T}{2}\right)+\frac{2E}{T}\left(t-T+\frac{T}{2}\right)U(t-T) \\
&= -\frac{2E}{T}U\left(t-\frac{T}{2}\right)+\left[\frac{2E}{T}(t-T)+E\right]U(t-T)
\end{aligned}
\tag{31}
$$

We now find $F_1(s)$ from $f_1(t)$

$$
\begin{aligned}
F_1(s) = \mathfrak{L}[f_1(t)] &= -\frac{2E}{T}\left(\frac{1}{s^2}-\frac{T}{2s}\right)+\left(\frac{2E}{Ts^2}+\frac{E}{s}\right)e^{-sT} \\
&= \frac{2E}{Ts}\left[\frac{T}{2}\left(1+e^{-sT}\right)-\frac{1}{s}\left(1-e^{-sT}\right)\right]
\end{aligned}
\tag{32}
$$

Finally, we obtain $F(s)$ from $F_1(s)$

$$
\begin{aligned}
F(s) &= \frac{F_1(s)}{1-e^{-sT}} \\
&= \frac{2E}{Ts}\left[\frac{T}{2}\left(\frac{1+e^{-sT}}{1-e^{-sT}}\right)-\frac{1}{s}\right]
\end{aligned}
\tag{33}
$$

□

Solution 11. *From $\mathfrak{L}[f(t)] = \int_0^\infty e^{-st}f(t)dt$, then*

$$
\frac{d}{ds}\mathfrak{L}[f(t)] = \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt = \int_0^\infty \frac{d}{ds}\left(e^{-st}\right)f(t)dt = \int_0^\infty -te^{-st}f(t)dt
\tag{34}
$$

then

$$
-\frac{d}{ds}\mathfrak{L}[f(t)] = \int_0^\infty tf(t)e^{-st}dt = \mathfrak{L}[tf(t)]
\tag{35}
$$

hence

$$
\mathfrak{L}[tf(t)] = -\frac{d}{ds}F(s)
\tag{36}
$$

Now

$$
\begin{aligned}
\mathfrak{L}[t^2\sin\omega t] &= (-1)^2\frac{d^2}{ds^2}\left\{\mathfrak{L}[\sin\omega t]\right\} \\
&= \frac{d^2}{ds^2}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{2\omega(3s^2-\omega^2)}{(s^2+\omega^2)^3}
\end{aligned}
\tag{37}
$$

□

Solution 12. *Consider*

$$\begin{aligned}\int_s^\infty F(v)dv &= \int_s^\infty \int_0^\infty e^{-st} f(t)dtdv = \int_0^\infty \int_s^\infty e^{-st} f(t)dvdt \\ &= \int_0^\infty -\frac{1}{t} e^{-vt} f(t) \mid_s^\infty dt = \int_0^\infty e^{-st} \frac{f(t)}{t} dt = \mathfrak{L}\left[\frac{f(t)}{t}\right]\end{aligned} \tag{38}$$

then

$$\begin{aligned}\mathfrak{L}\left[\frac{\sin \omega t}{t}\right] &= \int_s^\infty \mathfrak{L}[\sin \omega t] ds = \int_s^\infty \frac{\omega}{s^2+\omega^2} ds \\ &= \tan^{-1}\left(\frac{s}{\omega}\right)\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{\omega}\right) = \tan^{-1}\left(\frac{s}{\omega}\right)\end{aligned} \tag{39}$$

□