Frequency-Response Method

Contents

1 Introduction

Frequency-response methods were developed in 1930s and 1940s by Bode, Nyquist, Nichols, and many others. The frequency-response methods are most powerful in conventional control theory.

By the term frequency response, we mean the steady-state response of a system to a sinusoidal input. In frequency-response methods, we vary the frequency of the input signal over a certain range and study the resulting response.

An advantage of the frequency-response approach is that frequency-response tests are, in general, simple and can be made accurately by use of readily available sinusoidal signal generators and precise measurement equipment. Often the transfer functions of complicated components can be determined experimentally by frequency-response tests. In addition, the frequency-response approach has the advantages that a system may be designed so that the effects of undesirable noise are negligible and that such analysis and design can be extended to certain nonlinear control systems.

Although the frequency response of a control system presents a qualitative picture of the transient response, the correlation between frequency and transient responses is indirect, except for the case of second-order systems. In designing a closed-loop system, we adjust the frequency-response characteristic of the open-loop transfer function by using several design criteria in order to obtain acceptable transient-response characteristics for the system.

2 Obtaining Steady-State Outputs to Sinusoidal Inputs

We shall show that the steady-state output of a transfer function system can be obtained directly from the sinusoidal transfer function, that is, the transfer function in which s is replaced by $j\omega$, where ω is frequency.

Consider the stable, linear, time-invariant system shown in Figure 1.The input and output of the system, whose transfer function is $G(s)$, are denoted by $x(t)$ and $y(t)$, respectively. If the input $x(t)$ is a sinusoidal signal, the steady-state output will also be a sinusoidal signal of the same frequency, but with possibly different magnitude and phase angle.

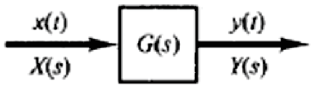

Figure 1: Stable, linear, time invariant system.

Let us assume that the input signal is given by

$$x(t) = X \sin \omega t. \tag{1}$$

Suppose that the transfer function $G(s)$ can be written as a ratio of two polynomials in s; that is,

$$G(s) = \frac{p(s)}{q(s)} = \frac{p(s)}{(s + s_1)(s + s_2) \cdots (s + s_n)}. \tag{2}$$

The Laplace-transformed output $Y(s)$ is then

$$Y(s) = G(s)X(s) = \frac{p(s)}{q(s)} X(s), \tag{3}$$

where $X(s)$ is the Laplace transform of the input $x(t)$.

The frequency response can be calculated by replacing s in the transfer function by $j\omega$. The steady-state response can be given by

$$G(j\omega) = Me^{j\phi} = M\angle\phi, \tag{4}$$

where M is the amplitude ratio of the output and input sinusoids and ϕ is the phase shift between the input sinusoid and the output sinusoid. In the frequency-response test, the input frequency ω is varied until the entire frequency range of interest is covered. The steady-state response of a stable, linear, time-invariant system to a sinusoidal input does not depend on the initial conditions.

Since $G(j\omega)$ is a complex quantity, it can be written in the following form:

$$G(j\omega) = |G(j\omega)|e^{j\phi}, \tag{5}$$

where $|G(j\omega)|$ presents the **magnitude** and ϕ represents the **phase angle** of $G(j\omega)$; that is,

$$\phi = \angle G(j\omega) = \tan^{-1}\left[\frac{Im\{G(j\omega)\}}{Re\{G(j\omega)\}}\right]. \tag{6}$$

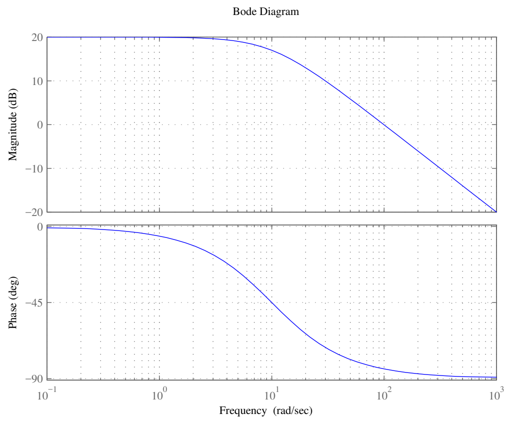

Figure 2: Bode diagram.

We can see that a stable, linear, time-invariant system subjected to a sinusoidal input will, at steady state, have a sinusoidal output of the same frequency as the input. But the amplitude and phase of the output will, in general, be different from those of the input. In fact, the amplitude of the output is given by the product of that of the input and $|G(j\omega)|$, while the phase angle differs from that of the input by the amount $\phi = \angle G(j\omega)$. An example of input and output sinusoidal signals is shown in Figure 3.

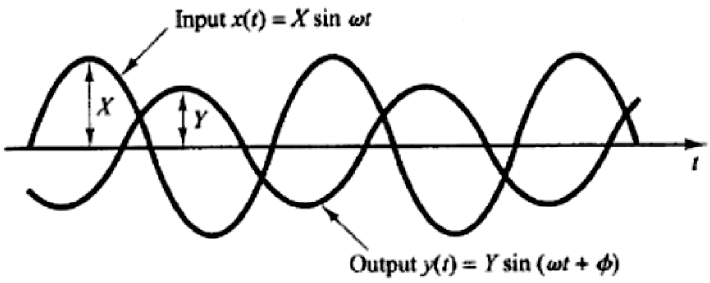

Figure 3: Input and output sinusoidal signals.

On the basis of this, we obtain this important result: For sinusoidal inputs,

$$|G(j\omega)| = \left|\frac{Y(j\omega)}{X(j\omega)}\right|, \tag{7}$$

which is the amplitude ratio of the output sinusoid to the input sinusoid, and

$$\angle G(j\omega) = \angle \frac{Y(j\omega)}{X(j\omega)}. \tag{8}$$

Hence, the steady-state response characteristics of a system to a sinusoidal input can be obtained directly from

$$\frac{Y(j\omega)}{X(j\omega)} = G(j\omega). \tag{9}$$

The function $G(j\omega)$ is called the sinusoidal transfer function. It is the ratio of $Y(j\omega)$ to $X(j\omega)$, is a complex quantity, and can be represented by the magnitude and phase angle with frequency as a parameter. The sinusoidal transfer function of any linear system is obtained by substituting $j\omega$ for s in the transfer function of the system.

A **positive phase angle** is called **phase lead**, and a **negative phase angle** is called **phase lag**. A network that has phase-lead characteristics is called a lead network, while a network that has phase-lag characteristics is called a lag network.

3 Examples

Example 3.1. Consider the system shown in Figure 4. The transfer function $G(s)$ is

$$G(s) = \frac{K}{Ts+1}. \tag{10}$$

For the sinusoidal input $x(t) = X \sin \omega t$, the steady-state output $y_{ss}(t)$ can be

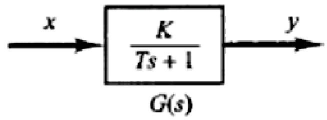

Figure 4: First-order system.

found as follows: Substituting $j\omega$ for s in $G(s)$ yields

$$G(j\omega) = \frac{K}{jT\omega + 1}. \tag{11}$$

The amplitude ratio of the output to the input is

$$|G(j\omega)| = \frac{K}{\sqrt{1+T^2\omega^2}}, \tag{12}$$

while the phase angle ϕ is

$$\phi = \angle G(j\omega) = -\tan^{-1}(T\omega). \tag{13}$$

Thus, for the input $x(t) = X \sin \omega t$, the steady-state output $y_{ss}(t)$ can be obtained from

$$y_{ss}(t) = \frac{XK}{\sqrt{1+T^2\omega^2}} \sin(\omega t - \tan^{-1}(T\omega)). \tag{14}$$

It can be seen that for small ω, the amplitude of the steady-state output $y_{ss}(t)$ is almost equal to K times the amplitude of the input. The phase shift of the output is small for small ω. For large ω, the amplitude of the output is small and almost inversely proportional to ω. The phase shift approaches $-90°$ as ω approaches infinity. This is a phase-lag network.

Example 3.2. Consider the network given by

$$G(s) = \frac{s + \frac{1}{T_1}}{s + \frac{1}{T_2}}, \tag{15}$$

determine whether this network is a lead network or lag network.

For the sinusoidal input $x(t) = X \sin \omega t$, the steady-state output $y_{ss}(t)$ can be found as follows: Since

$$G(j\omega) = \frac{j\omega + \frac{1}{T_1}}{j\omega + \frac{1}{T_2}} = \frac{T_2(1 + T_1 j\omega)}{T_1(1 + T_2 j\omega)}, \tag{16}$$

we have

$$|G(j\omega)| = \frac{T_2\sqrt{1+T_1^2\omega^2}}{T_1\sqrt{1+T_2^2\omega^2}} \tag{17}$$

and

$$\angle G(j\omega) = \tan^{-1} T_1\omega - \tan^{-1} T_2\omega. \tag{18}$$

Thus the steady-state output is

$$y_{ss}(t) = \frac{XT_2\sqrt{1+T_1^2\omega^2}}{T_1\sqrt{1+T_2^2\omega^2}} \sin(\omega t + \tan^{-1} T_1\omega - \tan^{-1} T_2\omega). \tag{19}$$

From this expression, we find that if $T_1 > T_2$, then $\tan^{-1} T_1\omega - \tan^{-1} T_2\omega > 0$. Thus, if $T_1 > T_2$, then the network is a lead network. If $T_1 < T_2$, then the network is a lag network.

4 Presenting Frequency-Response Characteristics in Graphical Forms

The sinusoidal transfer function, a complex function of the frequency ω, is characterized by its magnitude and phase angle, with frequency as the parameter. There are three commonly used representations of sinusoidal transfer functions:

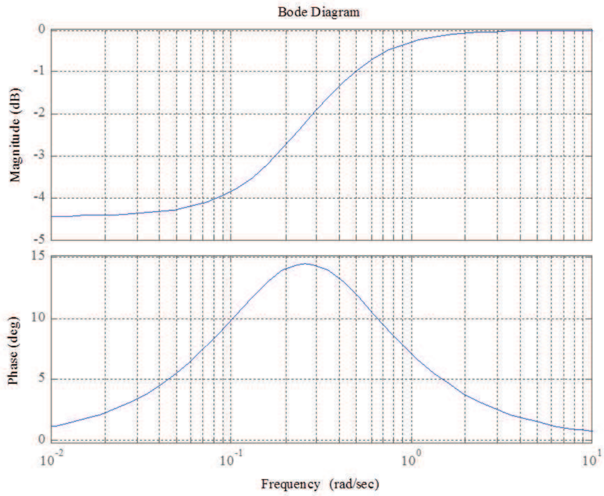

Figure 5: Lead network.

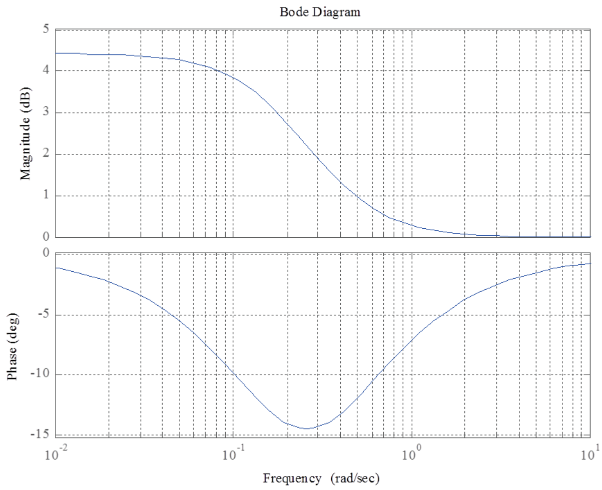

Figure 6: Lag network.

1. Bode diagram or logarithmic plot
2. Nyquist plot or polar plot
3. Nichols plots or log-magnitude-versus-phase plot.

4.1 Matlab Implementation

```
% FreqRespAnal.m
clc; clear all; close all;
% G(jw)=K/(1+jTw)
K=10; T=0.1;
numG=K; denG=[T 1];
figure; bode(tf(numG,denG)); grid on;
figure; nyquist(tf(numG,denG));
figure; nichols(tf(numG,denG));
% G(jw)=(jw+1/T1)/(jw+1/T2)=T2/T1*(1+T1jw)/(1+T2jw)
T1=5; T2=3;
numG=T2/T1*[T1 1]; denG=[T2 1];
figure; bode(tf(numG,denG)); grid on;
T1=3; T2=5;
numG=T2/T1*[T1 1]; denG=[T2 1];
figure; bode(tf(numG,denG)); grid on;
```

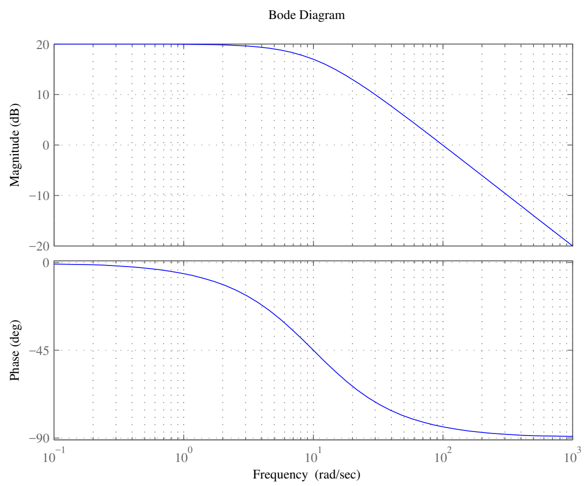

Figure 7: Bode diagram.

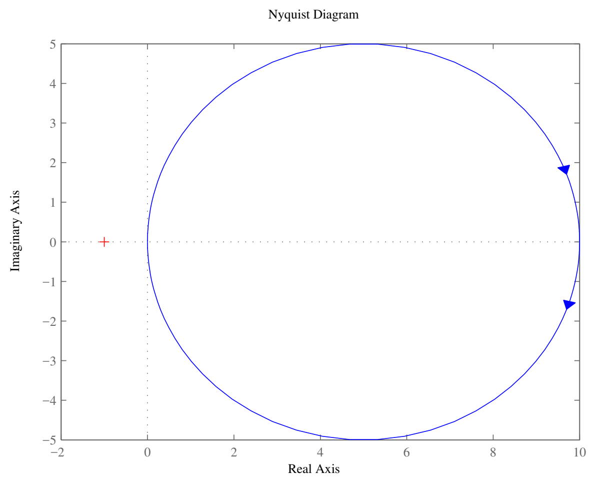

Figure 8: Nyquist plot.

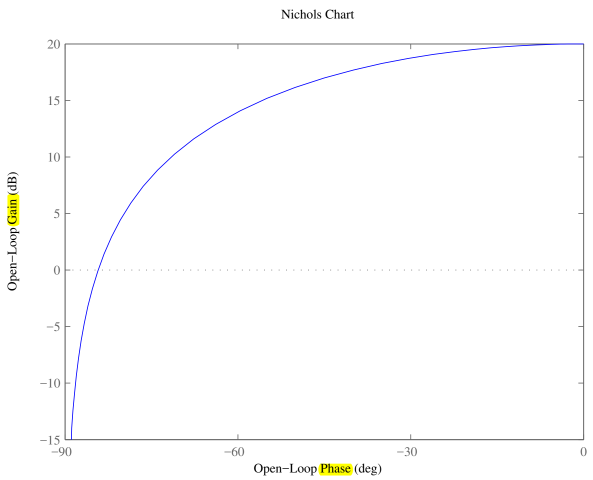

Figure 9: Nichols plot.

4.2 Commands

BODE(SYS) draws the Bode plot of the LTI model SYS (created with either TF, ZPK, SS, or FRD). The frequency range and number of points are chosen automatically.

[MAG,PHASE] = BODE(SYS,W) and [MAG,PHASE,W] = BODE(SYS) return the response magnitudes and phases in degrees (along with the frequency vector W if unspecified). No plot is drawn on the screen. If SYS has NY outputs and NU inputs, MAG and PHASE are arrays of size [NY NU LENGTH(W)] where MAG(:,:,k) and PHASE(:,:,k) determine the response at the frequency W(k). To get the magnitudes in dB, type MAGDB = 20*log10(MAG).

NYQUIST(SYS) draws the Nyquist plot of the LTI model SYS (created with either TF, ZPK, SS, or FRD). The frequency range and number of points are chosen automatically.

NICHOLS(SYS) draws the Nichols plot of the LTI model SYS (created with either TF, ZPK, SS, or FRD). The frequency range and number of points are chosen automatically.

SYS = TF(NUM,DEN) creates a continuous-time transfer function SYS with numerator(s) NUM and denominator(s) DEN. The output SYS is a TF object.

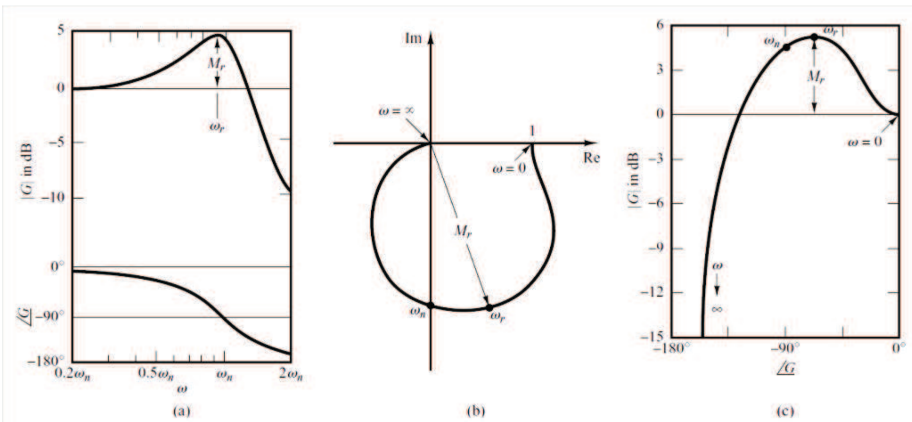

Figure 10: (a) Bode, (b) Nyquist, (c) Nichols plots.

4.3 Polar Plot

The polar plot of a sinusoidal transfer function $G(j\omega)$ is a plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as ω is varied from zero to infinity.

Thus, the polar plot is the locus of vectors $|G(j\omega)|\angle G(j\omega)$ as ω is varied from zero to infinity. Note that in polar plots a positive (negative) phase angle is measured counterclockwise (clockwise) from the positive real axis. The polar plot is often called the Nyquist plot.

Each point on the polar plot of $G(j\omega)$ represents the terminal point of a vector at a particular value of ω. In the polar plot, it is important to show the frequency graduation of the locus. The projections of $G(j\omega)$ on the real and imaginary axes are its real and imaginary components.

An advantage in using a polar plot is that it depicts the frequency-response characteristics of a system over the entire frequency range in a single plot. One disadvantage is that the plot does not clearly indicate the contributions of each individual factor of the open-loop transfer function.

4.4 Nichols Plot

Another approach to graphically portraying the frequency-response characteristics is to use the log-magnitude-versus-phase plot, which is a plot of the logarithmic magnitude in decibels versus the phase angle or phase margin for a frequency range of interest. The phase margin is the difference between the actual phase angle ϕ and $-180°$; that is, $\phi - (-180°) = 180° + \phi$. The curve is graduated in terms of the frequency ω. Such log-magnitude-versus-phase plots are commonly called Nichols plots.

4.5 Bode Diagram

In the Bode diagram, the frequency-response characteristics of $G(j\omega)$ are shown on semilog paper by two separate curves, the log-magnitude curve and the phase-angle curve, while in the log-magnitude-versus-phase plot, the two curves in the Bode diagram are combined into one. In the manual approach the log-magnitude-versus-phase plot can easily be constructed by reading values of the log magnitude and phase angle from the Bode diagram. Notice that in the log-magnitude-versus-phase plot a change in the gain constant of $G(j\omega)$ merely shifts the curve up (for increasing gain) or down (for decreasing gain), but the shape of the curve remains the same.

Advantages of the log-magnitude-versus-phase plot are that the relative stability of the closed-loop system can be determined quickly and that compensation can be worked out easily.