Tutorial—Transient Response

Example 1. *Consider the system shown in Figure 1, where $\zeta = 0.6$ and $\omega_n = 5$ rad/sec. Obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time t_s when the system is subjected to a unit-step input.*

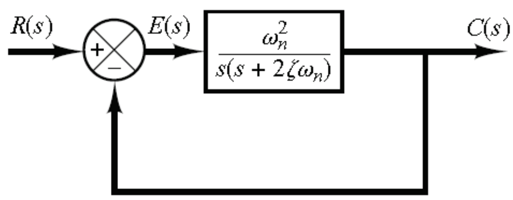

Figure 1: Second-order system.

Example 2. *For the system shown in Figure 2, determine the values of gain K and velocity-feedback constant K_h so that the maximum overshoot in the unit-step response is 0.2 and the peak time is 1 sec. With these values of K and K_h, obtain the rise time and settling time. Assume that $J = 1$ kg-m^2 and $B = 1$ N-m/rad/sec.*

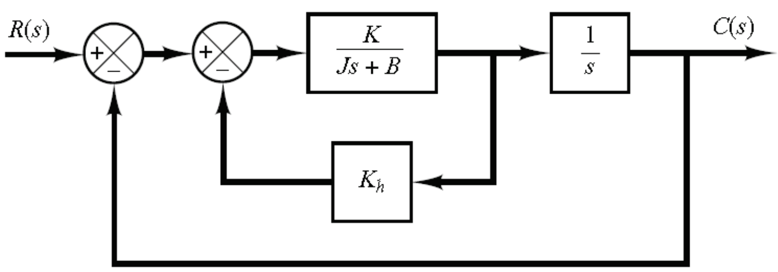

Figure 2: Block diagram of a servo system.

Example 3. *In the system of Figure 3, $x(t)$ is the input displacement and $\theta(t)$ is the output angular displacement. Assume that the masses involved are negligibly small and that all motions are restricted to be small; therefore, the system can be considered linear. The initial conditions for x and θ are zeros, or $x(0-) = 0$ and $\theta(0-) = 0$. Show that this system is a differentiating element. Then obtain the response $\theta(t)$ when $x(t)$ is a unit-step input.*

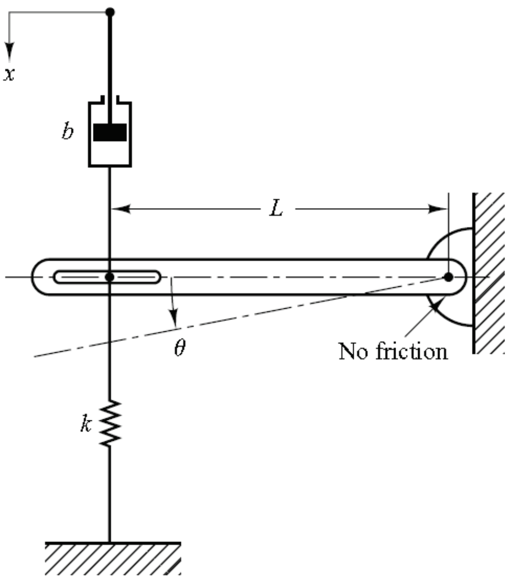

Figure 3: Mechanical system.

Example 4. *When the system shown in Figure 4(a) is subjected to a unit-step input, the system output responds as shown in Figure 4(b). Determine the values of K and T from the response curve.*

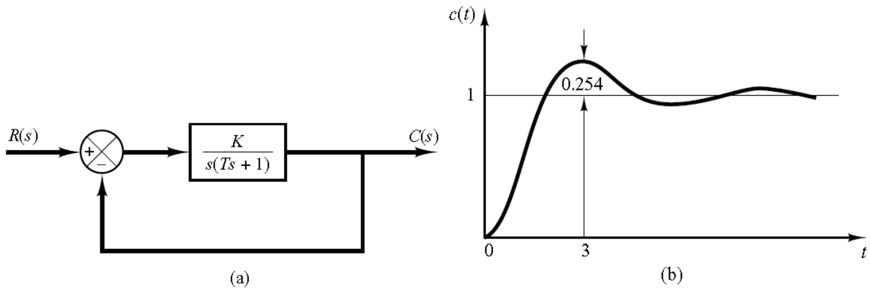

Figure 4: (a) Closed-loop system; (b) unit-step response curve.

Example 5. *Determine the values of K and k of the closed-loop system shown in Figure 5 so that the maximum overshoot in unit-step response is 25% and the peak time is 2 sec. Assume that $J = 1$ kg-m^2.*

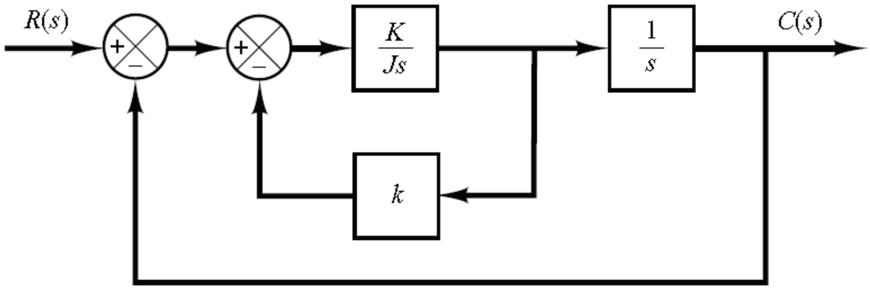

Figure 5: Closed-loop system.

Solution 1. *From the given values of* ζ *and* ω_n, *we obtain and* $\omega_d = \omega_n\sqrt{1-\zeta^2} = 4$ *and* $\sigma = \zeta\omega_n = 3$.

The rise time is

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - \beta}{4} \tag{1}$$

where β *is given by*

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{4}{3} = 0.93\ rad \tag{2}$$

the rise time t_r *is thus*

$$t_r = \frac{\pi - 0.93}{\omega_d} = 0.55\ sec \tag{3}$$

The peak time is

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785\ sec \tag{4}$$

The maximum overshoot is

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4)\pi} = 0.095 = 9.5\% \tag{5}$$

For the 2% criterion, the settling time is

$$t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33\ sec \tag{6}$$

For the 5% criterion, the settling time is

$$t_s = \frac{3}{\sigma} = \frac{3}{3} = 1\ sec \tag{7}$$

□

Solution 2. *The maximum overshoot* M_p *is given by*

$$M_p = e^{-\zeta\sqrt{1-\zeta^2}\pi} \tag{8}$$

This value must be 0.2.Thus,

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 1.61 \tag{9}$$

which yields $\zeta = 0.456$.

The peak time t_p *is specified as 1 sec; therefore,*

$$t_p = \frac{\pi}{\omega_d} = 1, \quad \Rightarrow \omega_d = 3.14 \tag{10}$$

Since $\zeta = 0.456$, *then*

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 3.53 \tag{11}$$

Since the natural frequency is $\omega_n = \sqrt{K/J}$*, then*

$$K = J\omega_n^2 = \omega_n^2 = 12.5 \tag{12}$$

Then K_h *is*

$$K_h = \frac{2\sqrt{K/J}\zeta - B}{K} = \frac{2\sqrt{K}\zeta - 1}{K} = 0.178 \tag{13}$$

The rise time is

$$t_r = \frac{\pi - \beta}{\omega_d} \tag{14}$$

where

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1} 1.95 = 1.1 \tag{15}$$

Thus $t_r = 0.65$ *sec.*

For the 2% criterion

$$t_s = \frac{4}{\sigma} = 2.48 \tag{16}$$

For the 5% criterion

$$t_s = \frac{3}{\sigma} = 1.86 \tag{17}$$

□

Solution 3. *The equation for the system is*

$$b(\dot{x} - L\dot{\theta}) = kL\theta \tag{18}$$

or

$$L\dot{\theta} + \frac{k}{b}L\theta = \dot{x} \tag{19}$$

The Laplace transform of this last equation, using zero initial conditions, gives

$$\left(sL + \frac{k}{b}L\right)\Theta(s) = sX(s) \tag{20}$$

And so

$$\frac{\Theta(s)}{X(s)} = \frac{1}{L}\frac{s}{s + k/b} \tag{21}$$

Thus the system is a differentiating system (the term $1/(s + k/b)$ *is multiplied with* s*).*

For the unit-step input $X(s) = 1/s$*, the output becomes*

$$\Theta(s) = \frac{1}{L}\frac{1}{s + k/b} \tag{22}$$

The inverse Laplace transform of $\Theta(s)$ *gives*

$$\theta(t) = \frac{1}{L}e^{-kt/b} \tag{23}$$

□

Solution 4. *The maximum overshoot of 25.4% corresponds to* $\zeta = 0.4$*, because* $M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$*. From the response curve we have* $t_p = 3$*. Consequently*

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_n\sqrt{1-0.4^2}} = 3 \tag{24}$$

It follows that $\omega_n = 1.14$*. From the block diagram we have*

$$\frac{C(s)}{R(s)} = \frac{K}{s^2T + s + K} \tag{25}$$

from which

$$\omega_n = \sqrt{K/T},\ 2\zeta\omega_n = 1/T \tag{26}$$

Therefore, the values of T *and* K *are determined as*

$$T = \frac{1}{2\zeta\omega_n} = \frac{1}{2 \times 0.4 \times 1.14} = 1.09 \tag{27}$$

$$K = \omega_n^2 T = 1.14^2 \times 1.09 = 1.42 \tag{28}$$

□

Solution 5. *The closed-loop transfer function is*

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Kks + K} \tag{29}$$

By substituting $J = 1$ *kg-m*^{2} *into this last equation, we have*

$$\frac{C(s)}{R(s)} = \frac{K}{s^2 + Kks + K} \tag{30}$$

Note that in this problem

$$\omega_n = \sqrt{K}, \quad 2\zeta\omega_n = Kk \tag{31}$$

The maximum overshoot M_p *is*

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \tag{32}$$

which is specified as 25%. Hence

$$e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.25 \tag{33}$$

from which

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 1.386, \quad \Rightarrow \zeta = 0.404 \tag{34}$$

The peak time t_p *is specified as 2 sec. And so*

$$t_p = \frac{\pi}{\omega_d} = 2, \quad \omega_d = 1.57 \tag{35}$$

Then the undamped natural frequency ω_n is

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{1.57}{\sqrt{1-0.404^2}} = 1.72 \tag{36}$$

Therefore, we obtain

$$K = \omega_n^2 = 1.72^2 = 2.95 \tag{37}$$

$$k = \frac{2\zeta\omega_n}{K} = \frac{2 \times 0.404 \times 1.72}{2.95} = 0.471 \tag{38}$$

□