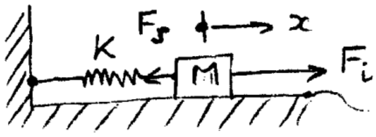
LECTURE 4 – THE DERIVATION OF SYSTEM EQUATIONS

To MODEL a physical system into a LINEAR system, we must describe each of the components of the system with its governing mathematical equation. Then we combine the equations into a complete system specifying the input and output.

Then we ANALYSE the LINEAR system by discovering its behaviour as it goes from one state to another.

<u>An example with a simple system</u>:



(no friction)

The mass m is initially at rest. Assume the spring is neither compressed nor extended. Put a disturbing force F_i on the mass but unspecified in $F_i(t)$ and allow the mass to displace by $x(t)$.

Newton's 2nd Law says $F_i - F_s = M\ddot{x}$ (Ignore (t) in F(t) and x(t))

However, the spring element has an equation $F_s = kx$

$\therefore M\ddot{x} + kx = F_i$, a second order differential equation where F_i is the input and x is the output.

Take LAPLACE TRANSFORM of the differential equation:

$M\left(s^2X(s) - sx(o) - \dot{x}(o)\right) + KX(s) = F_i(s)$ into the complex domain.
We know $x(o) = 0$ (not compressed), $\dot{x}(o) = 0$ (not moving)

$\therefore X(s)(Ms^2 + K) = F_i(s)$ or

$\frac{X(s)}{F_i(s)} = \frac{1}{Ms^2+K} \equiv G(s)$ This is the TRANSFER FUNCTION of the system given.

Note the relationship is independent of magnitude of the input $F_i(s)$.

We are ready to analyse the system but must specify our input to see what it does to the output.

Imagine F_i is a unit impulse (you give M a quick flick)

$$\therefore F_i(s) = \mathcal{L}\{\delta(\mathrm{t})\} = 1 \quad \therefore X(s) = \frac{1}{Ms^2+K} \cdot 1$$

$$\therefore x(t) = \mathcal{L}^{-1}\left\{\frac{\frac{1}{M}}{s^2+\frac{K}{M}}\right\} = \mathcal{L}^{-1}\{\frac{\frac{1}{\sqrt{MK}}\cdot\sqrt{\frac{K}{M}}}{s^2+\frac{K}{M}}\} = \frac{1}{\sqrt{MK}}\cdot sin\left(\sqrt{\frac{K}{M}}\cdot t\right),$$

which is taken from the standard form:

$$\mathcal{L}^{-1}\{\frac{\omega}{s^2+\omega^2}\} = sin\omega t$$

Notice we fed in boundary conditions at $t = 0$. They happened to be all zero but this does not always occur.

<u>A closer look at the TRANSFER FUNCTION</u>

There are basically two ways to arrange the transfer function terms:

1. Non-dimensional form, e.g. $\frac{1/K}{\frac{M}{K}s^2+1}$

2. Normalised form, e.g. $\frac{1/M}{s^2+\frac{K}{M}}$

From the normalised form, the roots are obvious $s = \pm\sqrt{-\frac{K}{M}}$

From the non dimensional form we can deduce a few properties. Take $G(S) = \frac{A}{(s+a)(s+b)}$ as an example (normalised). Non-dimensionally $G(S) = \frac{A/ab}{\left(\frac{s}{a}+1\right)\left(\frac{s}{b}+1\right)}$. From this form, we can pick at the following:

A/ab is the DC gain (or steady state gain) of the system (the gain at zero frequency). $\frac{1}{a}, \frac{1}{b}$ are the TIME CONSTANTS of the system (the exponential coefficients that will appear in time domain answers).

To prove A/ab is the DC gain, let us use the FINAL VALUE THEOREM to find this gain, i.e. when s$\rightarrow 0$ or $t \rightarrow \infty$. We must specify what the input is first, i.e. input = $\frac{1}{S}$, a unit step.

$$\therefore \lim_{t\rightarrow\infty} G(S)\cdot\frac{1}{s} = \lim_{s\rightarrow 0}\left(s\cdot\frac{A}{(s+a)(s+b)}\cdot\frac{1}{s}\right) = \frac{A}{ab}\ QED.$$

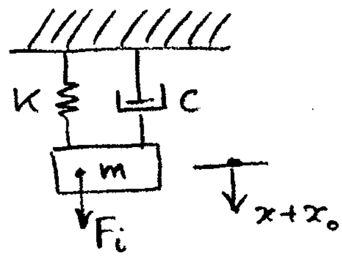
HELPFUL HINTS TO GET TO A TRANSFER FUNCTION

1. Specify exactly what the start $(t = 0)$ conditions are for the system; the boundary conditions. Make as many of them equate to zero as possible by moving datum.

2. Try to write equations that deal with a change of state rather than going from state 1 to state 2 e.g. look for a change of temperature on ambient, not T_1 to T_2.

Some examples of system modelling to a transfer function

1. For the mass-spring- damper system, derive the differential equation relating displacement x to force F_i. Assume the equipment is in equilibrium before F_i is applied and x_o is the extension of the spring due to the dead weight of the mass m.



$\therefore F_i + mg = K(x + x_o) + C(\dot{x} + \dot{x_0}) + m(\ddot{x} + \ddot{x_0})$

$\therefore F_i = Kx + Kx_o - mg + m\ddot{x} + C\dot{x}$

But $Kx_o = mg \quad \therefore m\ddot{x} + C\dot{x} + Kx = F_i$

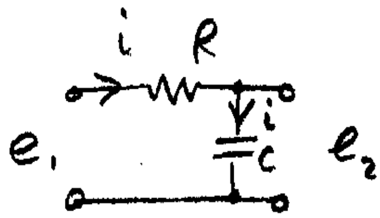
Here we could have assumed x was the incremental extension.

Now take Laplace to get $X(s)(ms^2 + Cs + K) = F_i(s)$ where $x(o) = \dot{x}(o) = 0$.

$$\therefore \frac{X(s)}{F_i(s)} = \frac{1}{ms^2 + Cs + K} \equiv \frac{\frac{1}{m}}{s^2 + \frac{C}{m}s + \frac{K}{m}}$$

Hence $\frac{1}{K}$ is the DC gain, i.e. if F_i is not changing, $x = \frac{1}{K} \cdot F_i$ QED

2. For the electrical circuit, derive the transfer function.



Assume zero boundary conditions
$\therefore e_1 = iR + e_2,\ i = C\dot{e}_2$

$\therefore RC\dot{e}_2 + e_2 = e_1$

e_2 is output and e_1 is input.

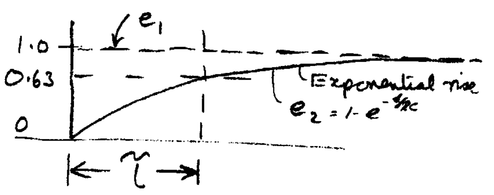
$\therefore E_2(s)(RCs + 1) = E_1(s) \quad \therefore \frac{E_2(s)}{E_1(s)} = \frac{1}{RCs+1}$

We see RC is the time constant of this system and its DC gain is 1. For a unit step input:

$$E_2(s) = \frac{1}{s(RCs+1)} = \frac{\frac{1}{RC}}{s(s+\frac{1}{RC})}$$

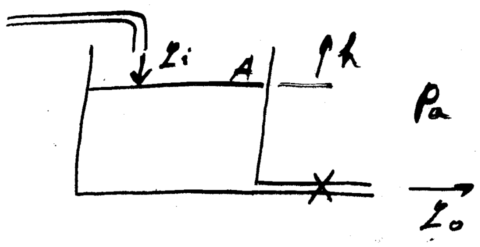
$$e_2 = 1 - e^{-\frac{t}{RC}}$$

Graphically:



$\tau = RC$ is the time taken to get 63% towards its final value at $t \to \infty$.

3. <u>Water tank</u> system:
Create a model in the time domain. Derive the transfer function. Analyse the unit step response.



Flow through the valve:

$$\begin{aligned} q_0 &= K\Delta p \\ &= K[(p_a + \rho g h) - p_a] \\ &= K\rho g h \end{aligned}$$

Differential equations:

$$q_0 = \frac{h}{R} \qquad (1)$$

$$q_i - q_0 = A\dot{h} \qquad (2)$$

Combining (1) and (2) yields: $\boldsymbol{A\dot{h} + \frac{h}{R} = q_i}$

Laplace transform:

$$\left(As + \frac{1}{R}\right)H(s) = Q_i(s)$$

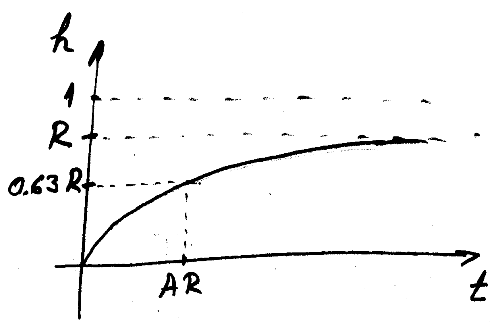
$$\frac{H(s)}{Q_i(s)} = \frac{1}{As+\frac{1}{R}} = \frac{R}{ARs+1} = \frac{\frac{1}{A}}{s+\frac{1}{AR}}$$

DC gain: R; time constant: AR

Assume:

$$Q_i(s) = \frac{1}{s}$$

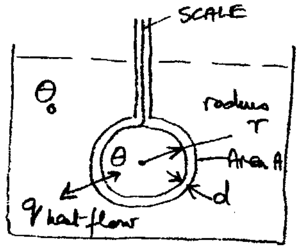
$$H(s) = \frac{\frac{1}{A}}{s\left(s+\frac{1}{AR}\right)}$$



$$h(t) = \mathcal{L}^{-1}\{H(s)\}$$

$$h(t) = R\left(1 - e^{-\frac{t}{AR}}\right)$$

4. For a thermometer (thermal system) we will look at the dynamics of the temperature indicated on the scale. The bulb is glass of conductivity K containing mercury of characteristics $\rho c V$. The bulb is at θ_1 prior to being placed in the bath at θ_0. Let us find out how the temperature indicated changes with time.



Conductivity Law:

$$q_C = -\frac{KA}{d} \cdot \frac{d\theta}{dx} \equiv \frac{KA}{d}(\theta_0 - \theta) \qquad (1)$$

Internal Energy Law:

$$q_I = C_t \frac{d\theta}{dt} \equiv \rho c V \dot{\theta} \qquad C_t \text{ is heat capacity } \rho c V \qquad (2)$$

Balancing 1 against 2 gives

$$q_C - q_I = \frac{KA}{d}(\theta_0 - \theta) - \rho c V \dot{\theta} \equiv 0$$

$$\therefore \dot{\theta} - \frac{KA}{\rho c V d}(\theta_0 - \theta) = 0 \quad \text{and let } \frac{A}{V} \cong \frac{4\pi r^2}{{}^4\!/_3 \pi r^3} = \frac{3}{r}$$

Rearranging to treat θ_0 as the input and θ as output

$$\dot{\theta} + \alpha\theta = \alpha\theta_0 \text{ , where } \alpha = \frac{3K}{\rho c d r}$$

Take Laplace to give

$$s\theta(s) - \theta(0) + \alpha\theta(s) = \alpha\theta_0(s)$$

The trick is to treat the system as for incremental changes by making $\theta(0) = 0$, in which case $\theta(s)$ is the difference over ambient, not an absolute value.

$$\therefore \text{ The transfer function is } \frac{\theta(s)}{\theta_0(s)} = \frac{\alpha}{s+\alpha}$$

Now specify the driving function $\theta_0(s)$ as a step function starting at $t = 0$ of magnitude θ_B (difference between ambient and fluid temperatures). This is like saying at t=0 we plunge the bulb into a water bath and swish it about all the time. $\therefore \theta(s) \equiv \frac{\theta_B}{s}$

$$\therefore \theta(s) = \frac{\alpha\theta_B}{s(s+\alpha)} \equiv \frac{\theta_B}{s(Ts+1)} \quad \text{where} \quad T = \frac{1}{\alpha} = \frac{\rho c d r}{3K}$$

With inverse Laplace, we get

$\theta(t) = \theta_B(1 - e^{-\alpha t})$, a normal exponential change.

Finally, let us calculate α or T for a typical thermometer.

For mercury, $\rho = 1.35 \times 10^3 Kgm^{-3}$, $c = 1.37 \times 10^2\ Jkg^{-1}K^{-1}$.

Let $d = 0.5\ mm$, $K = 1.1\ Js^{-1}m^{-1}K^{-1}$, $r = 2.0\ mm$.

$$T = \frac{\rho c d r}{3K} = \frac{1.35 \cdot 10^4 \cdot 1.37 \cdot 10^2 \cdot 5.10 \cdot 10^{-4} \cdot 2 \cdot 10^{-3}}{3.1^{-1}} = 5.6 \cdot 10^{-1} s$$

The time constant is ~½ sec, so if we wait ~5 time constants, i.e. 2.5 seconds, the reading should be virtually at its final value. In other words, this thermometer stabilises very quickly.