Tutorial—Stability

Example 1. *Consider the following characteristic equation:*

$$s^4 + Ks^3 + s^2 + 1 = 0 \tag{1}$$

Determine the range of K for stability.

Example 2. *Consider the equation*

$$2s^4 + s^3 + 3s^2 + 5s + 10 = 0 \tag{2}$$

Determine if the system is stable or not.

Example 3. *Consider the characteristic equation of a linear system*

$$s^4 + s^3 + 2s^2 + 2s + 3 = 0 \tag{3}$$

Determine if the system is stable or not.

Example 4. *Consider the following equation*

$$s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0 \tag{4}$$

Determine if the system is stable or not.

Example 5. *Consider that a third-order control system has the characteristic equation*

$$s^3 + 3408.3s^2 + 1204000s + 1.5 \times 10^7 K = 0 \tag{5}$$

Determine the critical value of K for stability, that is, the value of K for which at least one root will lie on the $j\omega$-axis and none in the right-half s-plane.

Solution 1. *The Routh array of coefficients is*

$$\begin{array}{cccc}
s^4 & 1 & 1 & 1 \\
s^3 & K & 1 & 0 \\
s^2 & (K-1)/K & 1 & \\
s^1 & 1-K^2/(K-1) & & \\
s^0 & 1 & &
\end{array}$$

For stability, we require no change of sign in the first column, that is

$$K > 0,\ \frac{K-1}{K},\ 1-\frac{K^2}{K-1} \tag{6}$$

From the first and second conditions, K must be greater than 1. For $K > 1$, notice that the term $1-K^2/(K-1)$ is always negative, since

$$\frac{K-1-K^2}{K-1} = \frac{-1+K(1-K)}{K-1} < 0 \tag{7}$$

Thus, the three conditions cannot be fulfilled simultaneously. Therefore, there is no value of K that allows stability of the system. □

Solution 2. *Routh's tabulation is made as follows:*

$$\begin{array}{cccc}
s^4 & 2 & 3 & 10 \\
s^3 & 1 & 5 & 0 \\
s^2 & (1\times 3-2\times 5)/1=-7 & 10 & 0 \\
s^1 & (-7\times 5-1\times 10)/(-7)=6.43 & 0 & 0 \\
s^0 & 10 & 0 & 0
\end{array}$$

Because there are two sign changes ($s^3 \to s^2$ and $s^2 \to s^1$) in the first column of the tabulation, the system is unstable. □

Solution 3. *Routh's tabulation is carried out as follows:*

$$\begin{array}{cccc}
s^4 & 1 & 2 & 3 \\
s^3 & 1 & 2 & 0 \\
s^2 & 0 & 3 &
\end{array}$$

Because the first element of the s^2 row is zero, the elements in the s^1 row would all be infinite. To overcome this difficulty, we replace the zero in the s^2 row with a small positive number ϵ, and then proceed with the tabulation. Starting with the s^2 row, the results are as follows:

$$\begin{array}{ccc}
s^2 & \epsilon & 0 \\
s^1 & (2\epsilon-3)/\epsilon \approx -3/\epsilon & 0 \\
s^0 & 3 & 0
\end{array}$$

Because there are two sign changes ($s^2 \to s^2$ and $s^1 \to s^0$) in the first column of Routh's tabulation, the system is not stable. □

Solution 4. *Routh's tabulation is*

$$\begin{array}{llll} s^5 & 1 & 8 & 7 \\ s^4 & 4 & 8 & 4 \\ s^3 & 6 & 6 & 0 \\ s^2 & 4 & 4 & \\ s^1 & 0 & 0 & \end{array}$$

Because a row of zeros appears prematurely, we form the auxiliary equation using the coefficients of the s^2 row:

$$A(s) = 4s^2 + 4 = 0 \tag{8}$$

The derivative of $A(s)$ with respect to s is

$$\frac{dA(s)}{ds} = 8s = 0 \tag{9}$$

from which the coefficients 8 and 0 replace the zeros in the s^1 row of the original tabulation. The remaining portion of the Routh's tabulation is

$$\begin{array}{llll} s^1 & 8 & 4 & \text{coefficients of } dA(s)/ds \\ s^0 & 4 & & \end{array}$$

Because there are no sign changes in the first column of the entire Routh's tabulation, the equation in Eq. (9) does not have any root in the right-half s-plane, hence, the system is stable. Solving the auxiliary equation in Eq. (8), we get the two roots at $s = j$ and $s = -j$, which are also two of the roots of Eq. (4). Thus. the equation has two roots on the $j\omega$-axis, and the system is marginally stable. These imaginary roots caused the initial Routh's tabulation to have the entire row of zeros in the s^1 row. □

Solution 5. *Routh's tabulation is made as follows:*

$$\begin{array}{lcc} s^3 & 1 & 1204000 \\ s^2 & 3408.3 & 1.5 \times 10^7 K \\ s^1 & \frac{410.36\times10^7 - 1.5\times10^7 K}{3408.3} & 0 \\ s^0 & 1.5 \times 10^7 K & \end{array}$$

For the system to be stable, all the roots of Eq. (5) must be in the left-half s-plane, thus, all the coefficients in the first column of Routh's tabulation must have the same sign. This leads to the fo1lowing conditions:

$$\frac{410.36 \times 10^7 - 1.5 \times 10^7 K}{3408.3} > 0 \tag{10}$$

and

$$1.5 \times 10^7 K > 0 \tag{11}$$

From the inequality of Eq. (10). we have $K < 273.57$. and the condition in Eq. (11) gives $K > 0$. Therefore, the condition of K for the system to be stable is

$$0 < K < 273.57 \tag{12}$$

If we let $K = 273.57$, then Eq. (10) becomes zero, the characteristic equation in Eq. (5) will have two roots on the $j\omega$-axis. To find these roots, we substitute $K = 273.57$ in the auxiliary equation, which is obtained from Routh's tabulation by using the coefficients of the s^2 row. Thus,

$$A(s) = 3408.3s^2 + 4.1036 \times 10^9 = 0 \tag{13}$$

which has roots at $s = j1097$ and $s = -j1097$, and the corresponding value of K at these roots is 273.57. Also, if the system is operated with $K = 273.57$, the system will be an undamped sinusoid with a frequency of 1097 rad/sec. □