Compensator Design

We show how to use Bode diagrams in order to design controllers such that the closed-loop system has the desired specifications. Three main types of controllersphase-lead, phase-lag, and phase-lag-lead controllers are considered.

1 Overshoot – Phase Margin

Consider the open-loop transfer function of a second-order system given by

$$G(j\omega)H(j\omega) = \frac{\omega_n^2}{(j\omega)(j\omega + 2\zeta\omega_n)}, \tag{1}$$

whose gain crossover frequency can be found from

$$|G(j\omega)H(j\omega)| = \frac{\omega_n^2}{\omega\sqrt{\omega^2 + 4\zeta^2\omega_n^2}} = 1, \tag{2}$$

leading to

$$\omega_{cg} = \omega_n\sqrt{\sqrt{1+4\zeta^2} - 2\zeta^2}. \tag{3}$$

The phase of (1) at the gain crossover frequency is

$$\angle\{G(j\omega_{cg})H(j\omega_{cg})\} = -90^\circ - \tan^{-1}\frac{\omega_{cg}}{2\zeta\omega_n}, \tag{4}$$

so that the corresponding phase margin becomes

$$\gamma = \tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^2} - 2\zeta^2}} = \gamma(\zeta). \tag{5}$$

It is a monotonically increasing function with respect to ζ; we therefore conclude that *the higher phase margin, the larger the damping ratio, which implies the smaller the overshoot.*

To see this, note that $\tan^{-1}(\theta)$ is an increasing function for $\theta > 0$. In the numerator of the $\gamma(\zeta)$ function, ζ is always positive. Furthermore, the denominator $\sqrt{1+4\zeta^2} - 2\zeta^2$ must be positive for $\zeta < 1$, i.e, $\sqrt{1+4} - 2 > 0$. In addition, when $\zeta > 1$, there is no overshoot. Thus, $\gamma(\zeta)$ is increasing with respect to ζ.

2 Phase-Lag Controller

The transfer function of a phase-lag controller is given by

$$G_{lag}(j\omega) = \frac{p}{z}\frac{z + j\omega}{p + j\omega} = \frac{1 + j\omega/z}{1 + j\omega/p}, \quad z > p. \tag{6}$$

Due to attenuation of the phase-lag controller at high frequencies, the frequency bandwidth of the compensated system (controller and system in series) is reduced. Thus, *the phase-lag controllers are used in order to decrease the system bandwidth (to slow down the system response)*. In addition, *they can be used to improve the stability margins (phase and gain) while keeping the steady state errors constant.*

To see this, since $z > p$, then $\omega/z < \omega/p$. Hence, the break frequency ω_z is smaller than ω/p. At higher frequencies, the denominator is larger than the numerator. The net result is a decrease of response at high frequencies.

3 Phase-Lead Controller

The transfer function of a phase-lead controller is

$$G_{lead}(j\omega) = \frac{p}{z}\frac{z + j\omega}{p + j\omega} = \frac{1 + j\omega/z}{1 + j\omega/p}, \quad p > z. \tag{7}$$

Due to phase-lead controller (compensator) amplification at higher frequencies, it increases the bandwidth of the compensated system. *The phase-lead controllers are used to improve the gain and phase stability margins and to increase the system bandwidth (decrease the system response rise time).*

The phase of a phase-lead controller is

$$\phi = \angle\{G_{lead}(j\omega) = \tan^{-1}(\omega/z) - \tan^{-1}(w/p), \tag{8}$$

so that

$$\frac{d}{d\omega}\angle\{G_{lead}(j\omega) = \frac{(p - z)(pz - \omega^2)}{(p^2 + \omega^2)(z^2 + \omega^2)} \tag{9}$$

Setting the derivative to zeros, we have

$$\omega_{max} = \sqrt{zp}, \quad p \neq z. \tag{10}$$

Substitute into (8), then

$$\phi = \tan^{-1}(\omega_{max}/z) - \tan^{-1}(\omega_{max}/p) = \tan^{-1}(\sqrt{p/z}) - \tan^{-1}(\sqrt{z/p}) \tag{11}$$

Consider the tangent of sum of angles, $\tan(A + B)$, it is equivalent to $\sin(A + B)/\cos(A + B)$. Now recall that $\sin(A + B) = \sin A \cos B + \cos A \sin B$, and $\cos(A + B) = \cos A \cos B - \sin A \cos B$, then

$$\tan(A + B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \cos B}, \tag{12}$$

then divide the numerator and denominator by $\cos A \cos B$, we have

$$\tan(A+B) = \frac{\sin A/\cos B + \sin B/\cos B}{1 - \sin A/\cos B \times \sin B/\cos B} = \frac{\tan A + \tan B}{1 - \tan A \tan B}. \tag{13}$$

Now make use of the identities $\sin(-B) = -\sin(B)$, and $\cos(-B) = \cos(B)$, then

$$\tan(A-B) = \frac{\sin A/\cos B - \sin B/\cos B}{1 + \sin A/\cos B \times \sin B/\cos B} = \frac{\tan A - \tan B}{1 + \tan A \tan B}. \tag{14}$$

Further let $A = \tan^{-1} X$ and $B = \tan^{-1} Y$, then

$$\tan(A-B) = \tan(\tan^{-1} X - \tan^{-1} Y) \tag{15}$$

$$= \frac{\tan(\tan^{-1} X) - \tan(\tan^{-1} Y)}{1 + \tan\tan^{-1} X \tan\tan^{-1} Y} = \frac{X-Y}{1+XY}, \tag{16}$$

thus

$$\tan^{-1} X - \tan^{-1} Y = \tan^{-1}\left(\frac{X-Y}{1+XY}\right). \tag{17}$$

Now $X = \sqrt{p/z}$, $Y = \sqrt{z/p}$, then

$$X - Y = \sqrt{\frac{p}{z}} - \sqrt{\frac{z}{p}} = \frac{\sqrt{p}\sqrt{p} - \sqrt{z}\sqrt{z}}{\sqrt{zp}} = \frac{p-z}{\sqrt{zp}}, \tag{18}$$

and $xy = \sqrt{p/z}\sqrt{z/p} = 1$. Thus

$$\tan^{-1}\sqrt{p/z} - \tan^{-1}\sqrt{z/p} = \frac{p-z}{2\sqrt{zp}} = \frac{p-z}{2\omega_{max}}. \tag{19}$$

Put a design parameter $a = p/z$, then

$$\tan^{-1}\sqrt{p/z} - \tan^{-1}\sqrt{z/p} = \frac{z(a-1)}{2\sqrt{zp}} = \frac{a-1}{2\sqrt{p/z}} = \frac{a-1}{2\sqrt{a}}. \tag{20}$$

Recall that this is the maximum phase angle ϕ of the phase-lead controller (compensator). It can be regarded as the sine angle of a right-angle triangle with opposite side equal to $a-1$ and the adjacent side equal to $2\sqrt{a}$, and the hypothenuse is $\sqrt{4a + (a-1)^2} = a+1$. We can represent ϕ as

$$\sin\phi = \frac{a-1}{a+1}, \tag{21}$$

giving

$$a\sin\phi + \sin\phi = a - 1, \quad a(1-\sin\phi) = 1 + \sin\phi, \quad \Rightarrow a = \frac{1+\sin\phi}{1-\sin\phi}. \tag{22}$$

Using parameter a and determining the corresponding frequency in the Bode plot would allow us to design the compensator characteristics, ie. setting p and z appropriately.

4 Phase-Lag-Lead Controller

The phase-lag-lead controller has the features of both phase-lag and phase-lead controllers and can be used to improve both the transient response and steady state errors. However, its design is more complicated than the design of either phase-lag or phase-lead controllers. The frequency transfer function of the phase lag-lead controller is given by

$$G_{LL}(j\omega) = \frac{(z_1 + j\omega)(z_2 + j\omega)}{(p_1 + j\omega)(p_2 + j\omega)}, \quad z_1 z_2 = p_1 p_2, \quad p_2 > z_2 > z_1 > p_1. \tag{23}$$

5 Phase-Lead Compensator Design

Consider the following open-loop frequency transfer function

$$G(j\omega)H(j\omega) = \frac{K(j\omega + 6)}{(j\omega + 1)(j\omega + 2)(j\omega + 3)}. \tag{24}$$

Let the design requirements be set such that the steady state error due to a unit step is less than 2% and the phase margin is at least 45°. Since

$$e_{ss} = \frac{1}{1 + K_B}, \quad K_B = \frac{K \times 6}{1 \times 2 \times 3} = K, \tag{25}$$

where K_B is the open-loop gain obtained from the Bode plot. Note that at

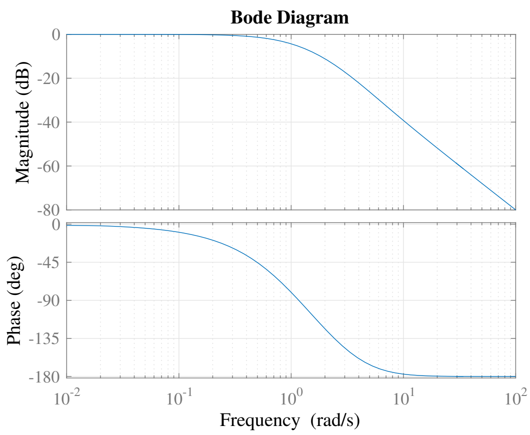

Figure 1: Bode plot when $K = 1$.

steady state response to a step function, it refers to the low-frequency (zero frequency). From the Bode plot, it is 0 dB. That is, a unity gain depending on the numerical value of K.

we conclude that $K \geq 50$ will satisfy the steady state error requirement of being less than 2%. We know from the root locus technique that high static gains can damage system stability, and so for the rest of this design problem we take $K = 50$. To see this, the steady-state error is $e_{ss} = 1/(1 + 50) = 1/51 < 0.02$.

Now determine the phase and gain margins and the crossover frequencies using $K = 50$. The phase and gain margins are obtained as $G_m = \infty$, $\gamma_m =$

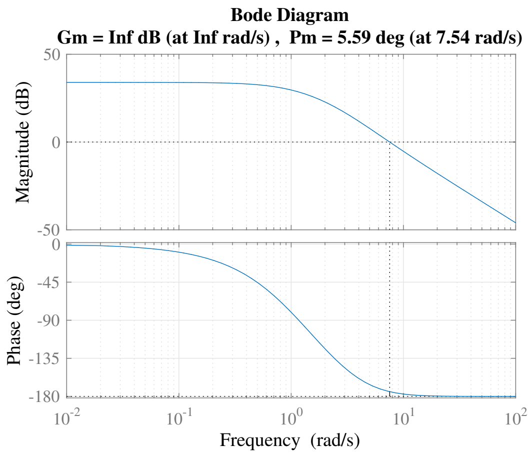

Figure 2: Bode plot when $K = 50$.

$5.59°$, and crossover frequencies are $\omega_g = 7.54$ rad/s and $\omega_p = \infty$.

Since the desired phase is well above the desired one, say $45°$, the phase-lead controller must make up for $45° - 5.59° = 39.41°$. We add $10°$ to the required make up phase, in a practice sense, to give $\phi_{max} = 49.41°$. This is due to the fact that we have to give an estimate of a new gain crossover frequency, which can not be determined very accurately. We then calculate a parameter

$$a = \frac{1 + \sin \phi_{max}}{1 - \sin \phi_{max}}, \quad a = \frac{1 + \sin(49.41°)}{1 - \sin(49.41°)} = 7.31. \tag{26}$$

In order to obtain an estimate for the new gain crossover frequency we first find the controller amplification at high frequencies, which is equal to $20 \log_{10}(a) = \Delta G = 17.28$ dB.

This value of gain is added by the compensator to the open-loop system. The close-loop system will be at the onset of instability when the compensated gain is at 0 dB. The magnitude Bode diagram need to be increased increased by 17.28 dB and the frequency is read from the Bode plot at approx. 19 rad/s. We guess (estimate) the value for p as $p = 19\sqrt{a} \approx 51$ rad/s. Then the phase-lead compensator transfer function is

$$G_{lead}(j\omega) = \frac{as + p}{s + p} = \frac{51 + 7.31 j\omega}{51 + j\omega}. \tag{27}$$

Now put the compensator $G_{lead}(j\omega)$ in series with the system $G(j\omega)$, and draw a Bode plot, shown in Fig. 3. We see that the phase margin is now $46.4°$ satisfying

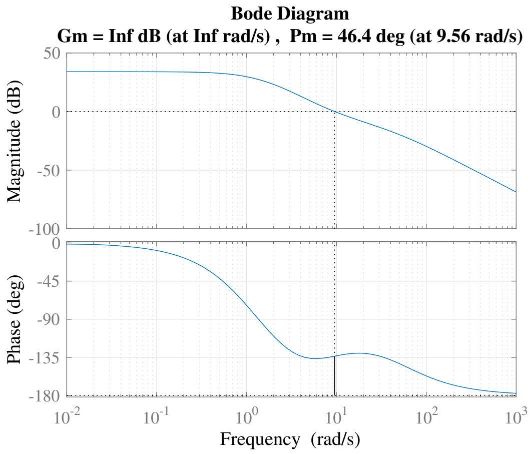

Figure 3: Bode plot of compensated system.

the specification. The unit-step responses, when the systems are configured in unity-feedback, are shown in Fig. 4. It can be observed that oscillations appear in the original system, and they are reduced when the system is compensated.

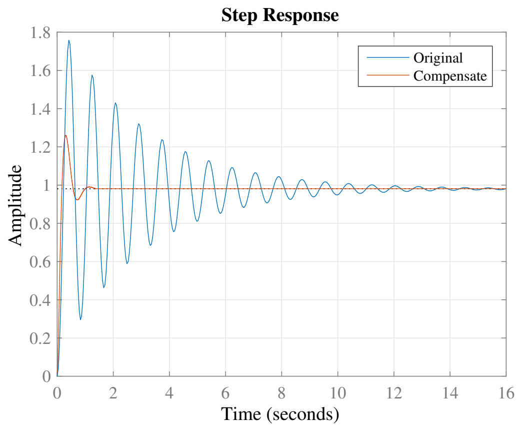

Figure 4: Responses to unit-step function.