Tutorial—Mechanical and Electrical System Models

Example 1. *Obtain the transfer functions and of the mechanical system shown in Figure 1.*

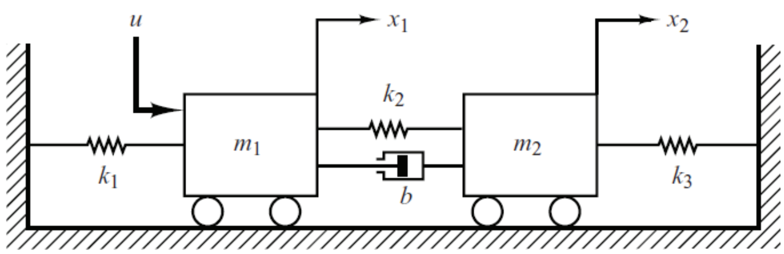

Figure 1: Mechanical system.

Example 2. *An inverted pendulum mounted on a motor-driven cart is shown in Figure 2(a). The objective of the control problem is to keep the pendulum rod in a vertical position. The inverted pendulum is unstable in that it may fall over any time in any direction unless a suitable control force is applied. Here we consider only a two-dimensional problem in which the pendulum moves only in the plane of the page. The control force u is applied to the cart. Assume that the center of gravity of the pendulum rod is at its geometric center. Obtain a mathematical model for the system.*

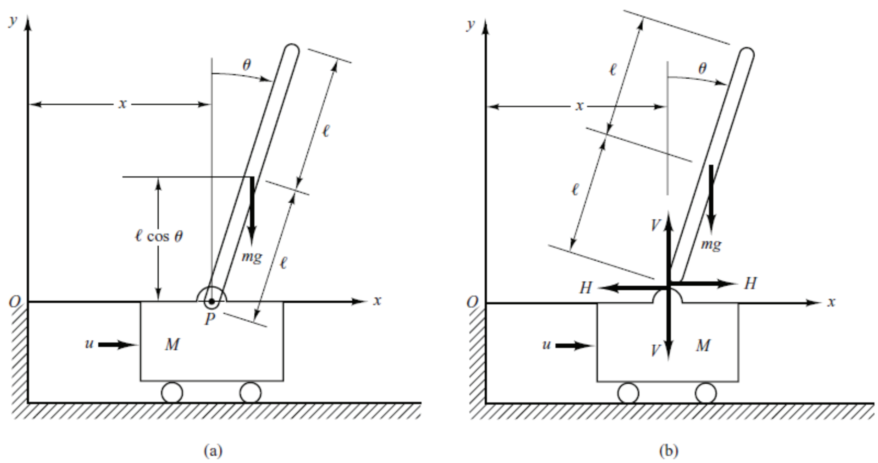

Figure 2: (a) Inverted pendulum system; (b) free-body diagram.

Example 3. *Obtain the transfer function $E_o(s)/E_i(s)$ of the op-amp circuit shown in Figure 3.*

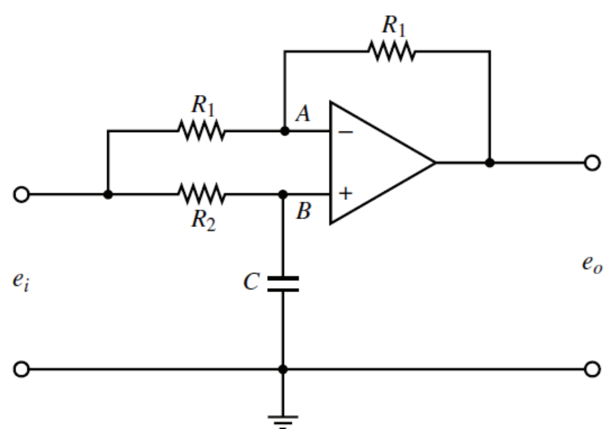

Figure 3: Operational-amplifier circuit.

Example 4. *Obtain the transfer function of the operational-amplifier circuit shown in Figure 4.*

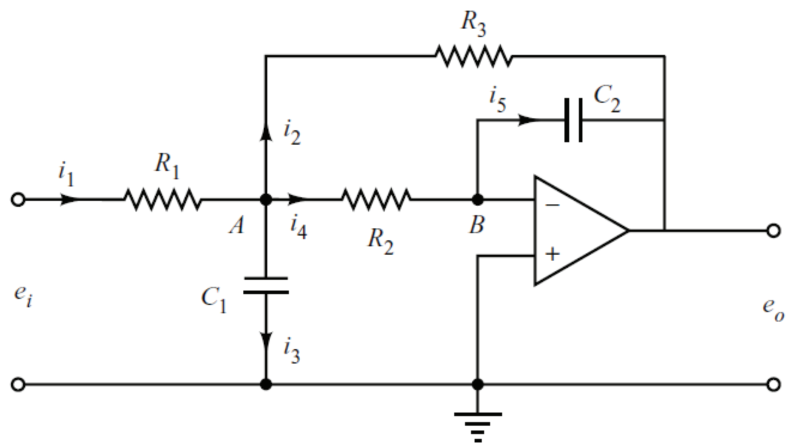

Figure 4: Operational-amplifier circuit.

Example 5. *Consider the servo system shown in Figure 5. The motor shown is a servomotor, a dc motor designed specifically to be used in a control system.The operation of this system is as follows:A pair of potentiometers acts as an error-measuring device. They convert the input and output positions into proportional electric signals. The command input signal determines the angular position r of the wiper arm of the input potentiometer. The angular position r is the reference input to the system, and the electric potential of the arm is proportional to the angular position of the arm. The output shaft position determines the angular position c of the wiper arm of the output potentiometer. The difference between the input angular position r and the output angular position c is the error signal e, or $e = r - c$.*

The potential difference is $e_r - e_c = e_v$ the error voltage, where e_r is proportional to r and e_c is proportional to c; that is, $e_r = K_0 r$ and $e_c = K_0 c$, where K_0 is a proportionality constant. The error voltage that appears at the potentiometer terminals is amplified by the amplifier whose gain constant is K_1. The output voltage of this amplifier is applied to the armature circuit of the dc motor. A fixed voltage is applied to the field winding. If an error exists, the motor develops a torque to rotate the output load in such a way as to reduce the error to zero. For constant field current, the torque developed by the motor is $T = K_2 i_a$, where K_2 is the motor torque constant and i_a is the armature current.

When the armature is rotating, a voltage proportional to the product of the flux and angular velocity is induced in the armature. For a constant flux, the induced voltage e_b is directly proportional to the angular velocity $d\theta/dt$ or $e_b = K_3 d\theta/dt$, where e_b is the back emf, K_3 is the back emf constant of the motor, and θ is the angular displacement of the motor shaft.

Obtain the transfer function between the motor shaft angular displacement θ and the error voltage e_v.

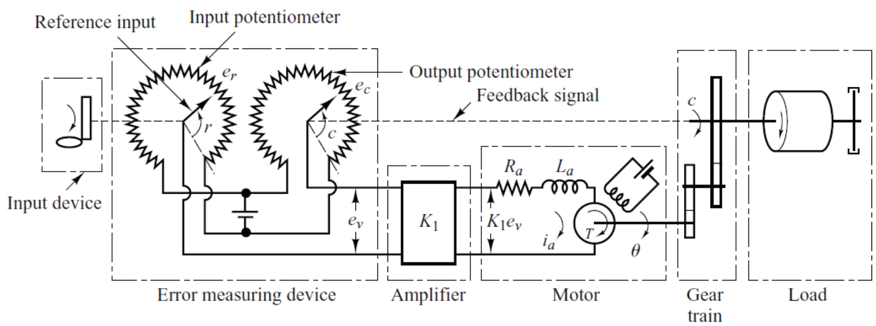

Figure 5: Schematic diagram of servo system.

Solution 1. *The equations of motion for the system shown in Figure 1 are*

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) + u \\ m_2\ddot{x}_2 &= -k_3x_2 - k_2(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1) \end{aligned} \tag{1}$$

Simplifying, we obtain

$$\begin{aligned} m_1\ddot{x}_1 + b\dot{x}_1 + (k_1 + k_2)x_1 &= b\dot{x}_2 + k_2x_2 + u \\ m_2\ddot{x}_2 + b\dot{x}_2 + (k_2 + k_3)x_2 &= b\dot{x}_1 + k_2x_1 \end{aligned} \tag{2}$$

Taking the Laplace transforms of these two equations, assuming zero initial conditions, we obtain

$$\left[m_1s^2 + bs + (k_1 + k_2)\right] X_1(s) = (bs + k_2)X_2(s) + U(s) \tag{3}$$

$$\left[m_2s^2 + bs + (k_2 + k_3)\right] X_2(s) = (bs + k_2)X_1(s) \tag{4}$$

Solving Equation (4) for $X_2(s)$ and substituting it into Equation (3) and simplifying, we get

$$\begin{aligned} \left[\left(m_1s^2 + bs + k_1 + k_2\right)\left(m_2s^2 + bs + k_2 + k_3\right) - (bs + k_2)^2\right] X_1(s) \\ = \left(m_2s^2 + bs + k_2 + k_3\right) U(s) \end{aligned} \tag{5}$$

from which we obtain

$$\frac{X_1(s)}{U(s)} = \frac{m_2s^2 + bs + k_2 + k_3}{(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2} \tag{6}$$

From Equations (4) and (6) we have

$$\frac{X_2(s)}{U(s)} = \frac{bs + k_2}{(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2} \tag{7}$$

$\square$

Solution 2. *Define the angle of the rod from the vertical line as θ. Define also the (x, y) coordinates of the center of gravity of the pendulum rod as (x_G, y_G). Then*

$$x_G = x + l\sin\theta, \; y_G = l\cos\theta \tag{8}$$

To derive the equations of motion for the system, consider the free-body diagram shown in Figure 2(b). The rotational motion of the pendulum rod about its center of gravity can be described by

$$I\ddot{\theta} = Vl\sin\theta - Hl\cos\theta \tag{9}$$

where I is the moment of inertia of the rod about its center of gravity. The horizontal motion of center of gravity of pendulum rod is given by

$$m\frac{d^2}{dt^2}(x + l\sin\theta) = H \tag{10}$$

The vertical motion of center of gravity of pendulum rod is

$$m\frac{d^2}{dt^2}\left(l\cos\theta\right) = V - mg \tag{11}$$

The horizontal motion of cart is described by

$$M\frac{d^2x}{dt^2} = u - H \tag{12}$$

Since we must keep the inverted pendulum vertical, we can assume that $\theta(t)$ and $\dot{\theta}(t)$ are small quantities such that $\sin\theta \approx \theta$, $\cos\theta \approx 1$, and $\theta\dot{\theta} \approx 0$. Then, Equations (9) through (11) can be linearized. The linearized equations are

$$I\ddot{\theta} = Vl\theta - lH \tag{13}$$

$$m(\ddot{x} + l\ddot{\theta}) = H \tag{14}$$

$$0 = V - mg \tag{15}$$

From Equations (12) and (14), we obtain

$$(M+m)\ddot{x} + ml\ddot{\theta} = u \tag{16}$$

From Equations (13), (14), and (15), we have

$$I\ddot{\theta} = mgl\theta - lH = mgl\theta - l(m\ddot{x} + ml\ddot{\theta}) = (I + ml^2)\ddot{\theta} + ml\ddot{x} = mgl\theta \tag{17}$$

□

Solution 3. *The voltage at point A is*

$$e_a = \frac{1}{2}\left(e_i - e_o\right) + e_o \tag{18}$$

The Laplace-transformed version of this last equation is

$$E_A(s) = \frac{1}{2}\left[E_i(s) + E_o(s)\right] \tag{19}$$

The voltage at point B is

$$E_B(s) = \frac{1/(Cs)}{R_2 + 1/(Cs)}E_i(s) = \frac{1}{R_2Cs + 1}E_i(s) \tag{20}$$

Since $\left[E_B(s) - E_A(s)\right]K = E_o(s)$ and $K >> 1$ we must have $E_A(s) = E_B(s)$. Thus

$$\frac{1}{2}\left[E_i(s) + E_o(s)\right] = \frac{1}{R_2Cs + 1}E_i(s) \tag{21}$$

hence

$$\frac{E_o(s)}{E_i(s)} = -\frac{R_2Cs - 1}{R_2Cs + 1} = -\frac{s - 1/(R_2C)}{s + 1/(R_2C)} \tag{22}$$

□

Solution 4. *We will first obtain currents i_1, i_2, i_3, i_4, and i_5. Then we will use node equations at nodes A and B.*

$$i_1 = \frac{e_i - e_A}{R_1},\ i_2 = \frac{e_A - e_o}{R_3},\ i_3 = C_1\frac{de_A}{dt},\ i_4 = \frac{e_A}{R_2},\ i_5 = C_s\frac{-de_o}{dt} \tag{23}$$

At node A, we have $i_1 = i_2 + i_3 + i_4$, or

$$\frac{e_i - e_A}{R_1} = \frac{e_A - e_o}{R_3} + C_1\frac{de_A}{dt} + \frac{e_A}{R_2} \tag{24}$$

At node B, we get $i_4 = i_5$, or

$$\frac{e_A}{R_2} = C_s\frac{-de_o}{dt} \tag{25}$$

By rewriting Equation (24), we have

$$C_1\frac{de_A}{dt} + \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)e_A = \frac{e_i}{R_1} + \frac{e_o}{R_3} \tag{26}$$

From Equation (25), we get

$$e_A = -R_2C_2\frac{de_o}{dt} \tag{27}$$

By substituting Equation (27) into Equation (26), we obtain

$$C_1\left(-R_2C_2\frac{d^2e_o}{dt^2}\right) - R_2C_2\frac{de_o}{dt}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) = \frac{e_i}{R_1} + \frac{e_o}{R_3} \tag{28}$$

Taking the Laplace transform of this last equation, assuming zero initial conditions, we obtain

$$-C_1C_2R_2s^2E_o(s) - R_2C_2\frac{de_o}{dt}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)E_o(s) - \frac{1}{R_3}E_o(s) = \frac{E_i(s)}{R_1} \tag{29}$$

from which we get the transfer function as follows:

$$\frac{E_o(s)}{E_i(s)} = -\frac{1}{R_1C_1R_2C_2s^2 + [R_2C_2 + R_1C_2 + (R_1/R_3)R_2C_2]\,s + R_1/R_3} \tag{30}$$

□

Solution 5. *The speed of an armature-controlled dc servomotor is controlled by the armature voltage e_a. The armature voltage $e_a = K_1e_v$ is the output of the amplifier. The differential equation for the armature circuit is*

$$L_a\frac{di_a}{dt} + R_ai_a + e_b = e_a,\ \Rightarrow L_a\frac{di_a}{dt} + R_ai_a + K_3\frac{d\theta}{dt} = K_1e_v \tag{31}$$

The equation for torque equilibrium is

$$J_0 \frac{d^2\theta}{dt^2} + b_0 \frac{d\theta}{dt} + T = K_2 i_a \tag{32}$$

where J_0 is the inertia of the combination of the motor, load, and gear train referred to the motor shaft and b_0 is the viscous-friction coefficient of the combination of the motor, load, and gear train referred to the motor shaft.

By eliminating i_a from Equations (31) and (32), we obtain

$$\frac{\Theta(s)}{E_o(s)} = \frac{K_1 K_2}{s\left(L_a s + R_a\right)\left(J_0 s + b_0\right) + K_1 K_3 s} \tag{33}$$

We assume that the gear ratio of the gear train is such that the output shaft rotates n times for each revolution of the motor shaft. Thus,

$$C(s) = n\Theta(s) \tag{34}$$

The relationship among $E_v(s)$, $R(s)$, and $C(s)$ is

$$E_v(s) = K_0 \left[R(s) - C(s)\right] = K_0 E(s) \tag{35}$$

The transfer function in the feedforward path of this system is

$$G(s) = \frac{C(s)}{\Theta(s)} \frac{\Theta(s)}{E_v(s)} \frac{E_v(s)}{E(s)} = \frac{K_0 K_1 K_2 n}{s\left[\left(L_a s + R_a\right)\left(J_0 s + b_0\right) + K_2 K_3\right]} \tag{36}$$

If L_a is small, it can be neglected, and the transfer function $G(s)$ in the feedforward path becomes

$$G(s) = \frac{K_0 K_1 K_2 n}{s\left[R_a\left(J_0 s + b_0\right) + K_2 K_3\right]} = \frac{K_0 K_1 K_2 n / R_a}{J_0 s^2 + s\left(b_0 + \frac{K_2 K_3}{R_a}\right)} \tag{37}$$

The term $s\left(b_0 + K_2 K_3 / R_a\right)$ indicates that the back emf of the motor effectively increases the viscous friction of the system. The inertia J_0 and viscous friction coefficient $b_0 + K_2 K_3 / R_a$ are referred to the motor shaft. When J_0 and are multiplied by $1/n^2$, the inertia and viscous-friction coefficient are expressed in terms of the output shaft.

To see the effect of the gear ratio, let $J = J_0/n^2$, $B = \left[b_0 + \left(K_2 K_3 / R_a\right)\right]/n^2$, $K = K_0 K_1 K_2 / (n R_a)$, the transfer function $G(s)$ can be simplified, yielding

$$G(s) = \frac{K}{J s^2 + B s} = \frac{K_m}{s(T_m s + 1)} \tag{38}$$

where $K_m = K/B$, $T_m = J/B = R_a J_0 / (R_a b_0 + K_2 K_3)$ $\square$