State Space Representation

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1 Simulation Diagrams

1.1 Overview

A simulation diagram is a certain type of either a block diagram or a flow graph that is constructed to have a specified transfer function or to model a specified set of differential equations. Given the transfer function, the differential equations, or the state equations of a system, we can construct a simulation diagram of the system. The simulation diagram is aptly named, since it is useful in constructing either digital computer or analog computer simulations of a system.

1.2 Simulation Diagrams

The basic element of the simulation diagram is the integrator. Figure 1 shows the block diagram of an integrating device.

In this figure, the variables and their Laplace transforms are

$$y(t) = \int x(t)dt, \; Y(s) = \frac{1}{s}X(s) \tag{1}$$

We are interested only in the transfer function; thus we have ignored the initial condition on $y(t)$. If the output of an integrator is labeled as $y(t)$, the input

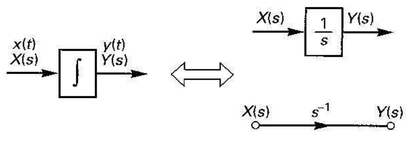
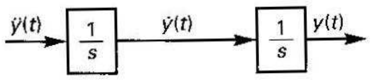
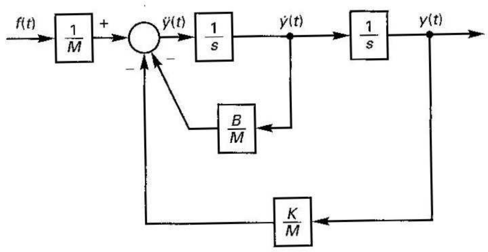


Figure 1: Integrating device.

to the integrator must be dy/dt. We use this characteristic of the integrator to aid us in constructing simulation diagrams. For example, two integrators are cascaded in Figure 2(a). If the output of the second integrator is $y(t)$, the



(a)



(b)

Figure 2: Simulation diagram.

input to this integrator must be $\dot{y}(t)$. In a like manner, the input to the first integrator must be $\ddot{y}(t)$, where $\ddot{y}(t) = d^2y(t)/dt^2$.

We can use these two integrators to construct a simulation diagram of the mechanical system of Figure 3. The input to the cascaded integrators in Figure 2(a) is $\ddot{y}(t)$, and the equation that $\ddot{y}(t)$ must satisfy for the mechanical system is obtained as

$$\ddot{y}(t) = -\frac{B}{M}\dot{y}(t) - \frac{K}{M}y(t) + \frac{1}{M}f(t) \tag{2}$$

Hence a summing junction and appropriate gains can be added to the block diagram of Figure 2(a) to satisfy this equation, as shown in Figure 2(b) which is called a *simulation diagram* of the mechanical system.

If the simulation diagram is constructed from the system differential equations, the simulation diagram will usually be unique. However, if the transfer function is used to construct the simulation diagram, the simulation diagram can be of many different forms, that is, the simulation diagram is not unique.

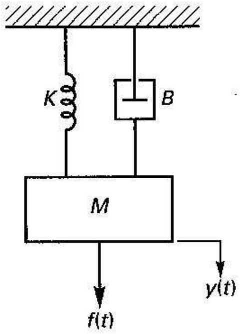


Figure 3: Mechanical translational system

1.3 General Procedure

Note that in the transfer function of (5), the order of the numerator must be at least one less than the order of the denominator. Of course, any coefficient a, or b, may be zero. The general transfer function of a physical system is always assumed to be of this form; the reason for this assumption will become evident when the frequency response of systems is covered. However, simulation diagrams can be constructed for transfer functions for which the order of the numerator is equal to that of the denominator.

Once a simulation diagram of a transfer function is constructed, a state model of the system is easily obtained. The procedure to do this has two steps:

1. Assign a state variable to the output of each integrator.

2. Write an equation for the input of each integrator and an equation for each system output. These equations are written as functions of the integrator outputs and the system inputs.

1.4 Example

Consider the dc motor model shown in Figure 4.

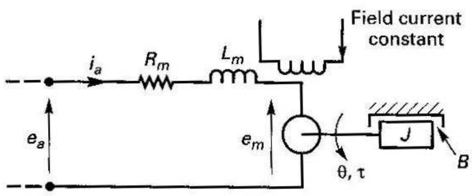


Figure 4: DC motor model.

The output equation is $y(t) = d\theta(t)/dt = x_1(t)$. Hence the state equations are

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -\frac{B}{J} & \frac{K_\tau}{J} \\ -\frac{K_m}{L_m} & -\frac{R_m}{L_m} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{L_m} \end{bmatrix} u(t) \tag{3}$$

$$y(t) = [1\ 0]\mathbf{x}(t) \tag{4}$$

where the input $u(t) = e_a(t)$. The simulation diagram, in an alternative form, for these equations is shown in Figure 5, where $\Omega(s) = L[d\theta(t)/dt]$. This example develops a state model that is used in practical speed-control design; the model is neither of the canonical forms presented earlier.

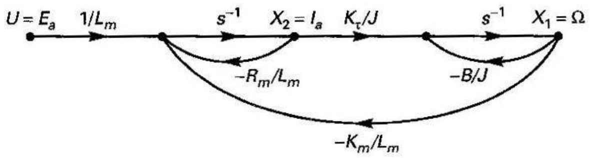


Figure 5: Simulation diagram.

2 Representation Forms

The two different simulation diagrams to be given below, realize the general transfer function

$$G(s) = \frac{b_{n-1}s^{-1} + b_{n-2}s^{-2} + \cdots + b_0 s^{-n}}{1 + a_{n-1}s^{-1} + a_{n-2}s^{-2} + \cdots + a_0 s^{-n}} \tag{5}$$

$$= \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0} \tag{6}$$

2.1 Control Canonical Form

The first simulation diagram, called the control canonical form, is given in Figure 6 for the case $n = 3$, that is, for

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} \tag{7}$$

This procedure yields the following state equations for the control canonical form of Figure 6, where the states are identified in the figure. The state

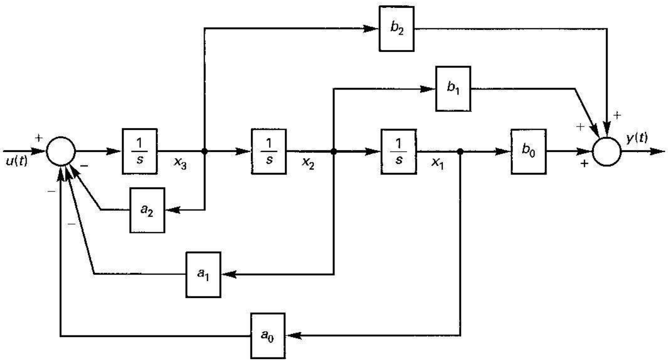


Figure 6: Control canonical form.

equations are, from Figure 6,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{8}$$

$$y = [b_0 \ b_1 \ b_2]\mathbf{x} \tag{9}$$

2.2 Example

Consider again the mechanical system of Figures 3 and 2(b)). In the simulation diagram of Figure 2(b), which is repeated in Figure 7, a state has been assigned to each integrator output. With this assignment, the input to the rightmost

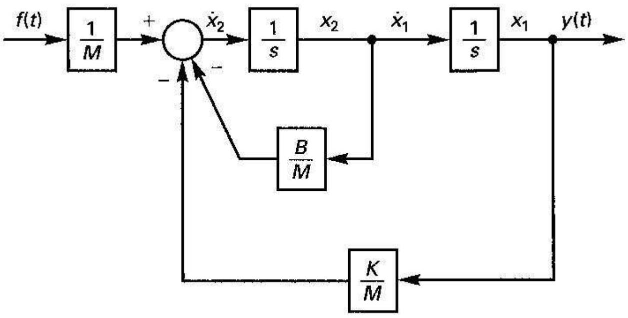


Figure 7: Simulation diagram.

integrator is $\dot{x}_l$; thus the equation for this integrator input is $\dot{x}_1 = x_2$. For the other integrator the input is x_2. Thus the equation for this integrator input is

$$\dot{x}_2 = -\frac{K}{M}x_1 - \frac{B}{M}x_2 + \frac{1}{M}f \tag{10}$$

and the system output equation is

$$y = x_1 \tag{11}$$

These equations may be written in the standard state-variable matrix format as

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} f(t) \tag{12}$$

3 Summary

We have examined some specific ways to represent a system with state variables. These methods are by no means exhaustive, since the selection of the state-variables is not a unique process. The number of state variables required to model a system will always be equal to the order of the system. While this number is apparent for single-input, single-output systems described by a single nth-order differential equation or by a transfer function, such is sometimes not the case for *multi-variable* systems