Tutorial—Fluid and Thermal System Models

Example 1. *In the liquid-level system of Figure 1 assume that the outflow rate Q m^3/sec through the outflow valve is related to the head H m by $Q = K\sqrt{H} = 0.01\sqrt{H}$.*

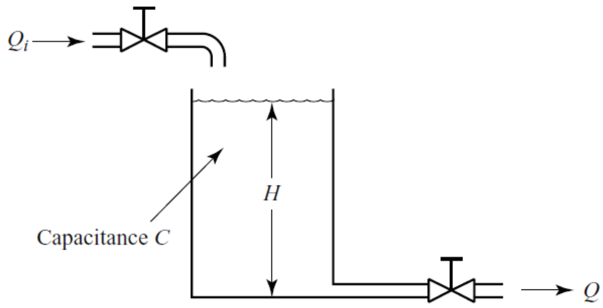

Figure 1: Liquid-level system.

Assume also that when the inflow rate Q_i is 0.015 m^3/sec the head stays constant. For $t < 0$ the system is at steady state, i.e, $Q_i = 0.015$ m^3/sec. At $t = 0$ the inflow valve is closed and so there is no inflow for $t \geq 0$. Find the time necessary to empty the tank to half the original head. The capacitance C of the tank is 2 m^2.

Example 2. *Consider the liquid-level system shown in Figure 2. In the system, $\bar{Q}_1$ and $\bar{Q}_2$ are steady-state inflow rates and $\bar{H}_1$ and $\bar{H}_2$ are steady-state heads. The quantities q_{i1}, q_{i2}, h_1, h_2, q_1, and q_o are considered small. Obtain a state-space representation for the system when h_1 and h_2 are the outputs and q_{i1} and q_{i2} are the inputs.*

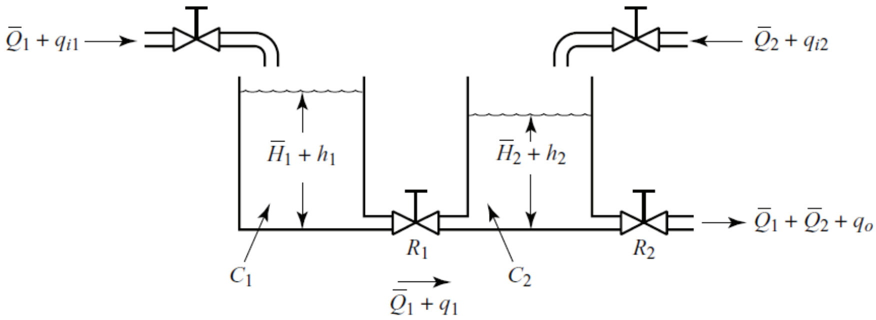

Figure 2: Liquid-level system.

Example 3. *Derive the transfer function $Z(s)/Y(s)$ of the hydraulic system shown in Figure 3. Assume that the two dashpots in the system are identical ones except the piston shafts.*

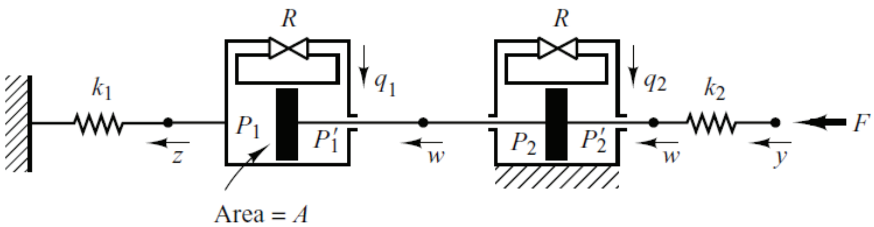

Figure 3: Hydraulic system.

Example 4. *Consider the thin, glass-wall, mercury thermometer system shown in Figure 4.*

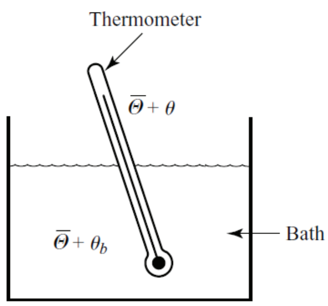

Figure 4: Thin, glass-wall, mercury thermometer system.

Assume that the thermometer is at a uniform temperature $\bar{\theta}$ (ambient temperature) and that at $t = 0$ it is immersed in a bath of temperature $\bar{t}heta + \theta_b$ where θ_b is the bath temperature (which may be constant or changing) measured *from the ambient temperature $\bar{\theta}$. Define the instantaneous thermometer temperature by $\bar{\theta}+\theta$, so that θ is the change in the thermometer temperature satisfying the condition that $\theta(0) = 0$.*

Solution 1. *When the head is stationary, the inflow rate equals the outflow rate. Thus head H_o at $t = 0$ is obtained from $0.015 = 0.01\sqrt{H_o}$ or $H_o = 2.25$ m. The equation for the system for $t > 0$ is*

$$-CdH = Qdt \tag{1}$$

or

$$\frac{dH}{dt} = -\frac{Q}{C} = \frac{-0.01\sqrt{H}}{2} \tag{2}$$

Hence

$$\frac{dH}{\sqrt{H}} = -0.005dt \tag{3}$$

Assume that, at $t = t_1$, $H = 1.125$ m. Integrating both sides of this last equation, we obtain

$$\int_{2.25}^{1.125} \frac{dH}{\sqrt{H}} = \int_0^{t_1} -0.005dt = -0.005t_1 \tag{4}$$

It follows that

$$2\sqrt{H}\Big|_{2.25}^{1.125} = 2\sqrt{1.125} - 2\sqrt{2.25} = -0.005t_1 \tag{5}$$

or $t_1 = 175.7$. Thus, the head becomes half the original value, 2.25 m, in 175.7 sec. □

Solution 2. *The equations for the system are*

$$C_1dh_1 = (q_{i1} - q_1)dt \tag{6}$$

$$\frac{h_1 - h_2}{R_1} = q_1 \tag{7}$$

$$C_2dh_2 = (q_1 + q_{i2} - q_0)dt \tag{8}$$

$$\frac{h_2}{R_2} = q_0 \tag{9}$$

Elimination of q_1 from Equation (6) using Equation (7) results in

$$\frac{dh_1}{dt} = \frac{1}{C_1}\left(q_{i1} - \frac{h_1 - h_2}{R_1}\right) \tag{10}$$

Eliminating q_1 and q_o from Equation (8) by using Equations (7) and (9) gives

$$\frac{dh_2}{dt} = \frac{1}{C_2}\left(\frac{h_1 - h_2}{R_1} + q_{i2} - \frac{h_2}{R_2}\right) \tag{11}$$

Define state variables x_1 and x_2 by

$$x_1 = h_1,\ x_2 = h_2 \tag{12}$$

the input variables u_1 and u_2 by

$$u_1 = q_{i1}, \ u_2 = q_{i2} \tag{13}$$

and the output variables y_1 and y_2 by

$$y_1 = h_1, \ y_2 = h_2 \tag{14}$$

Then Equations (10) and (11) can be written as

$$\dot{x}_1 = -\frac{1}{R_1C_1}x_1 + \frac{1}{R_1C_1}x_2 + \frac{1}{C_1}u_1 \tag{15}$$

$$\dot{x}_2 = \frac{1}{R_1C_2}x_1 - \left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right)x_2 + \frac{1}{C_2}u_2 \tag{16}$$

In the form of the standard vector-matrix representation, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1C_1} & \frac{1}{R_1C_1} \\ \frac{1}{R_1C_2} & -\left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1} & 0 \\ 0 & \frac{1}{C_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{17}$$

which is the state equation, and

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{18}$$

which is the output equation. □

Solution 3. *In deriving the equations for the system, we assume that force F is applied at the right end of the shaft causing displacement y. All displacements y, w, and z are measured from respective equilibrium positions when no force is applied at the right end of the shaft. When force F is applied, pressure P_1 becomes higher than pressure P_1', i.e., $P_1 > P_1'$. Similarly, $P_2 > P_2'$. For the force balance, we have the following equation:*

$$k_2(y - w) = A(P_1 - p_1') + A(P_2 - p_2') \tag{19}$$

Since

$$k_1 z = A(P_1 - P_1') \tag{20}$$

and

$$q_1 = \frac{P_1 - P_1'}{R} \tag{21}$$

we have

$$k_1 z = ARq_1 \tag{22}$$

Also, since

$$q_1 dt = A(dw - dz)\rho \tag{23}$$

we have

$$q_1 = A(\dot{w} - \dot{z})\rho \tag{24}$$

or

$$\dot{w} - \dot{z} = \frac{k_1 z}{A^2 R\rho} \tag{25}$$

Define $A^2R\rho = B$, the viscous-friction coefficient, then

$$\dot{w} - \dot{z} = \frac{k_1}{B} z \tag{26}$$

Also, for the right-hand-side dashpot we have

$$q_2 dt = A\rho dw \tag{27}$$

Since $q_2 = (P_2 - P_2')/R$, we obtain

$$\dot{w} = \frac{q_2}{A\rho} = \frac{A(P_2 - P_2')}{A^2 R\rho} \tag{28}$$

or

$$A(P_2 - P_2') = B\dot{w} \tag{29}$$

Substituting Equations (20) and (29) into Equation (19), we have

$$k_2 y - k_2 w = k_1 z + B\dot{w} \tag{30}$$

Taking the Laplace transform of this last equation, assuming zero initial condition, we obtain

$$k_2 Y(s) = (k_2 + Bs)W(s) + k_1 Z(s) \tag{31}$$

Taking the Laplace transform of Equation (26), assuming zero initial condition, we obtain

$$W(s) = \frac{k_1 + Bs}{Bs} Z(s) \tag{32}$$

By using Equation (32) to eliminate $W(s)$ from Equation (31), we obtain

$$k_2 Y(s) = (k_2 + Bs)\frac{k_1 + Bs}{Bs} Z(s) + k_1 Z(s) \tag{33}$$

from which we obtain the transfer function $Z(s)/Y(s)$ to be

$$\frac{Z(s)}{Y(s)} = \frac{k_2 s}{Bs^2 + (2k_1 + k_2)s + \frac{k_1 k_2}{B}} \tag{34}$$

Multiplying $B/(k_1k_2)$ to both the numerator and denominator of this last equation, we get

$$\frac{Z(s)}{Y(s)} = \frac{\frac{B}{k_1} s}{\frac{B^2}{k_1 k_2} s^2 + \left(\frac{2B}{k_2} + \frac{B}{k_1}\right) s + 1} \tag{35}$$

Define $B/k_1 = T_1$, $B/k_2 = T_2$. Then the transfer function $Z(s)/Y(s)$ becomes as follows:

$$\frac{Z(s)}{Y(s)} = \frac{T_1 s}{T_1 T_2 s^2 + (2T_1 + T_2)s + 1} \tag{36}$$

$\square$

Solution 4. *A mathematical model for the system can be derived by considering heat balance as follows: The heat entering the thermometer during dt sec is qdt, where q is the heat flow rate to the thermometer. This heat is stored in the thermal capacitance C of the thermometer, thereby raising its temperature by $d\theta$. Thus the heat-balance equation is*

$$C d\theta = q dt \tag{37}$$

Since thermal resistance R may be written as

$$R = \frac{d\Delta\theta}{dq} = \frac{\Delta\theta}{q} \tag{38}$$

heat flow rate q may be given, in terms of thermal resistance R, as

$$q = \frac{(\bar{\theta} + \theta_b) - (\bar{\theta} + \theta)}{R} = \frac{\theta_b - \theta}{R} \tag{39}$$

where $\bar{\theta} + \theta_b$ is the bath temperature and $\bar{\theta} + \theta$ is the thermometer temperature. Hence, we can rewrite Equation (37) as

$$\begin{aligned} C\frac{d\theta}{dt} &= \frac{\theta_b - \theta}{R} \\ RC\frac{d\theta}{dt} + \theta &= \theta_b \end{aligned} \tag{40}$$

which is a mathematical model of the thermometer system. □