Drawing Root-Locus

To begin sketching the root loci of a system by the root-locus method we must know the location of the poles and zeros of $G(s)H(s)$. The angles of the complex quantities originating from the open-loop poles and open-loop zeros to the test point s are measured in the counterclockwise direction.For example, if $G(s)H(s)$ is given by

$$G(s)H(s) = \frac{K(s+z_1)}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)} \tag{1}$$

where $-p_2$ and $-p_3$ are complex-conjugate poles, then the angle of $G(s)H(s)$ is

$$\angle G(s)H(s) = \phi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4 \tag{2}$$

where ϕ_1, θ_1, θ_2, $\theta 3$, and θ_4 are measured counterclockwise as shown in Figures 1(a) and (b).

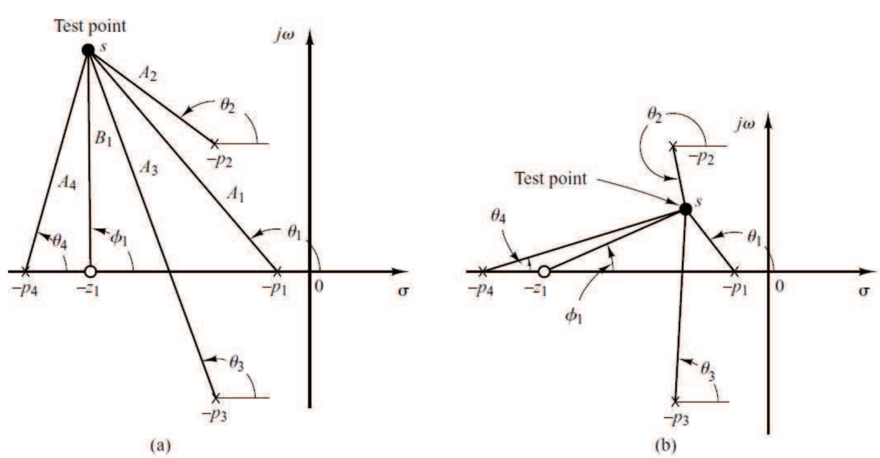

Figure 1: (a) and (b) Diagrams showing angle measurements from open-loop poles and open-loop zero to test point s.

The magnitude of $G(s)H(s)$ for this system is

$$|G(s)H(s)| = \frac{KB_1}{A_1A_2A_3A_4} \tag{3}$$

where A_1, A_2, A_3, A_4, and B_1 are the magnitudes of the complex quantities $s+p_1$, $s+p_2$, $s+p_3$, $s+p_4$, and $s+z_1$, respectively, as shown in Figure 1(a).

Note that, because the open-loop complex-conjugate poles and complex-conjugate zeros, if any, are always located symmetrically about the real axis, the root loci are always symmetrical with respect to this axis. Therefore, we only need to construct the upper half of the root loci and draw the mirror image of the upper half in the lower-half s plane.

Example 1. *Consider the negative feedback system shown in Figure 2.*

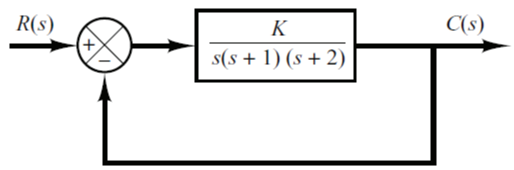

Figure 2: Control system.

We assume that the value of gain K is nonnegative. For this system, sketch the root-locus plot.

Solution 1. *For the given system, the angle condition becomes*

$$\begin{aligned} \angle G(s) &= \angle \frac{K}{s(s+1)(s+2)} \\ &= -\angle s - \angle s+1 - \angle s+2 \\ &= \pm 180^\circ(2k+1), \;\; k = 0, 1, 2, \cdots \end{aligned} \tag{4}$$

The magnitude condition is

$$|G(s)| = \left| \frac{K}{s(s+1)(s+2)} \right| = 1 \tag{5}$$

A typical procedure for sketching the root-locus plot is as follows:

Determine the root loci on the real axis. *The first step in constructing a root-locus plot is to locate the open-loop poles, $s = 0$, $s = -1$, and $s = -2$, in the complex plane. There are no open-loop zeros in this system. The locations of the open-loop poles are indicated by crosses. The locations of the open-loop zeros will be indicated by small circles. Note that the starting points of the root loci, the points corresponding to $K = 0$, are open-loop poles. The number of individual root loci for this system is three, which is the same as the number of open-loop poles.*

To determine the root loci on the real axis, we select a test point, s. If the test point is on the positive real axis, then

$$\angle s = \angle s+1 = \angle s+2 = 0^\circ \tag{6}$$

This shows that the angle condition cannot be satisfied. Hence, there is no root locus on the positive real axis.

Next, select a test point on the negative real axis between 0 and -1*. Then*

$$\angle s = 180^\circ, \angle s+1 = \angle s+2 = 0^\circ \tag{7}$$

thus

$$-\angle s - \angle s+1 - \angle s+2 = -180^\circ \tag{8}$$

and the angle condition is satisfied. Therefore, the portion of the negative real axis between 0 and -1 *forms a portion of the root locus.*

If a test point is selected between -1 *and* -2*, then*

$$\angle s = \angle s+1 = 180^\circ, \ \angle s+2 = 0^\circ \tag{9}$$

and

$$-\angle s - \angle s+1 - \angle s+2 = 0^\circ \tag{10}$$

It can be seen that the angle condition is not satisfied. Therefore, the negative real axis from -1 *to* -2 *is not a part of the root locus. Similarly, if a test point is located on the negative real axis from* -2 *to* $-\infty$*, the angle condition is satisfied. Thus, root loci exist on the negative real axis between 0 and* -1 *and between* -2 *and* $-\infty$*.*

Determine the asymptotes of the root loci. *The asymptotes of the root loci as s approaches infinity can be determined as follows: If a test point s is selected very far from the origin, then*

$$\lim_{s\to\infty} G(s) = \lim_{s\to\infty} \frac{K}{s(s+1)(s+2)} = \lim_{s\to\infty} \frac{K}{s^3} \tag{11}$$

and the angle condition becomes

$$-3\angle s = \pm 180^\circ(2k+1), \ k = 0, 1, 2, \cdots \tag{12}$$

Since the angle repeats itself as k is varied, the distinct angles for the asymptotes are determined as 60°, -60°, and 180°. Thus, there are three asymptotes. The one having the angle of 180° is the negative real axis.

Before we can draw these asymptotes in the complex plane, we must find the point where they intersect the real axis. Since

$$G(s) = \frac{K}{s(s+1)(s+2)} \tag{13}$$

if a test point is located very far from the origin, then $G(s)$ *may be written as*

$$G(s) = \frac{K}{s^3 + 3s^2 + \cdots} \tag{14}$$

For large values of s, this last equation may be approximated by

$$G(s) \approx \frac{K}{(s+1)^3} \tag{15}$$

A root-locus diagram of $G(s)$ given by Equation (4) consists of three straight lines. This can be seen as follows: The equation of the root locus is

$$\underline{\Big/ \frac{K}{(s+1)^3}} = \pm 180^\circ (2k+1) \tag{16}$$

which can be written as

$$-3\angle s+1 = \pm 180^\circ (2k+1) \Rightarrow \angle s+1 = \pm 60^\circ (2k+1) \tag{17}$$

By substituting $s = \sigma + j\omega$ into this last equation, we obtain

$$\underline{/ \sigma + j\omega + 1} = \pm 60^\circ (2k+1) \tag{18}$$

or

$$\tan^{-1} \frac{\omega}{\sigma + 1} = 60^\circ,\ -60^\circ,\ 0^\circ \tag{19}$$

Taking the tangent of both sides of this last equation,

$$\frac{\omega}{\sigma+1} = \sqrt{3},\ -\sqrt{3},\ 0 \tag{20}$$

which can be written as

$$\sigma + 1 - \frac{\omega}{\sqrt{3}} = 0,\ \sigma + 1 + \frac{\omega}{\sqrt{3}} = 0,\ \omega = 0 \tag{21}$$

These three equations represent three straight lines, as shown in Figure 3.

The three straight lines shown are the asymptotes. They meet at point $s = -1$. Thus, the abscissa of the intersection of the asymptotes and the real axis is obtained by setting the denominator of the right-hand side of Equation (15) equal to zero and solving for s. The asymptotes are almost parts of the root loci in regions very far from the origin.

Determine the breakaway point. *To plot root loci accurately, we must find the breakaway point, where the root-locus branches originating from the poles at 0 and -1 break away, as K is increased, from the real axis and move into the complex plane. The breakaway point corresponds to a point in the s plane where multiple roots of the characteristic equation occur.*

Let us write the characteristic equation as

$$f(s) = B(s) + KA(s) = 0 \tag{22}$$

where $A(s)$ and $B(s)$ do not contain K. Note that $f(s) = 0$ has multiple roots at points where $df(s)/ds = 0$. This can be seen as follows: Suppose that $f(s)$ has multiple roots of order r, where $r \geq 2$. Then $f(s)$ may be written as

$$f(s) = (s - s_1)^r (s - s_2) \cdots (s - s_n) \tag{23}$$

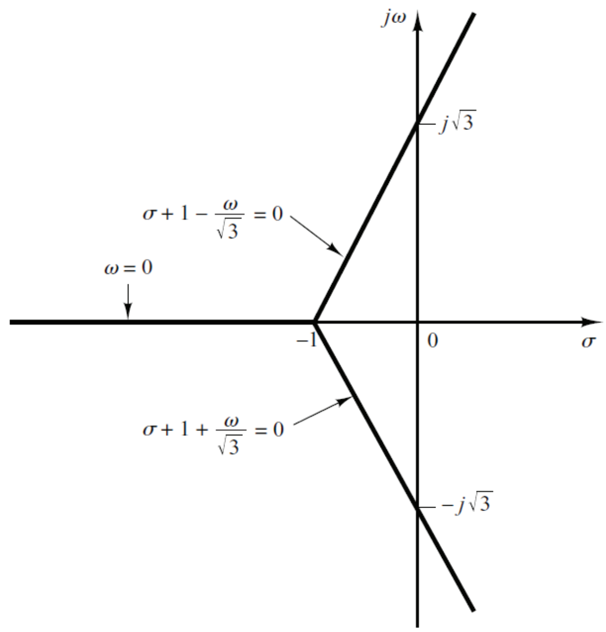

Figure 3: Three asymptotes.

Now we differentiate this equation with respect to s and evaluate $df(s)/ds$ at $s = s_1$. Then we get

$$\left.\frac{df(s)}{ds}\right|_{s=s_1} = 0 \tag{24}$$

This means that multiple roots of $f(s)$ will satisfy Equation (24). From Equation (22), we obtain

$$\frac{df(s)}{ds} = B'(s) + KA'(s) = 0 \tag{25}$$

where $A'(s) = dA(s)/ds$ and $B'(s) = dB(s)/ds$. The particular value of K that will yield multiple roots of the characteristic equation is obtained from Equation (25) as

$$K = -\frac{B'(s)}{A'(s)} \tag{26}$$

If we substitute this value of K into Equation (22), we get

$$f(s) = B(s) - \frac{B'(s)}{A'(s)}A(s) = 0 \tag{27}$$

or

$$B(s)A'(s) - B'(s)A(s) = 0 \tag{28}$$

If Equation (28) is solved for s, the points where multiple roots occur can be obtained. On the other hand, from Equation (22) we obtain $K = -B(s)/A(s)$

and

$$\frac{dK}{ds} = -\frac{B'(s)A(s) - B(s)A'(s)}{A^2(s)} \tag{29}$$

If dK/ds is set equal to zero, we get the same equation as Equation (28). Therefore, the breakaway points can be simply determined from the roots of $dK/ds = 0$.

It should be noted that not all the solutions of Equation (28) or of $dK/ds = 0$ correspond to actual breakaway points. If a point at which $dK/ds = 0$ is on a root locus, it is an actual breakaway or break-in point. Stated differently, if at a point at which $dK/ds = 0$ the value of K takes a real positive value, then that point is an actual breakaway or break-in point.

For the present example, the characteristic equation $G(s)+1 = 0$ is given by

$$\frac{K}{s(s+1)(s+2)} + 1 = 0 \tag{30}$$

or

$$K = -(s^3 + 3s^2 + 2s) \tag{31}$$

By setting $dK/ds = 0$, we obtain

$$\frac{dK}{ds} = -(3s^2 + 6s + 2) = 0, \; \Rightarrow s = -0.4226, \; s = -1.5774 \tag{32}$$

Since the breakaway point must lie on a root locus between 0 and -1, it is clear that $s = -0.4226$ corresponds to the actual breakaway point. Point $s = -1.5774$ is not on the root locus. Hence, this point is not an actual breakaway or break-in point. In fact, evaluation of the values of K corresponding to $s = -0.4226$ and $s = -1.5774$ yields $K = 0.3849$ for $s = -0.4226$ and $K = -0.3849$ for $s = -1.5774$.

Determine the points where the root loci cross the imaginary axis. *These points can be found by use of Routh's stability criterion as follows: Since the characteristic equation for the present system is s^3+3s^2+2s+K, the Routh array becomes*

$$\begin{array}{ccc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & (6-K)/3 & 0 \\ s^0 & K & \end{array}$$

The value of K that makes the s_1 term in the first column equal zero is $K = 6$. The crossing points on the imaginary axis can then be found by solving the auxiliary equation obtained from the s_2 row; that is,

$$3s^2 + K = 3s^2 + 6 = 0, \; \Rightarrow s = \pm j\sqrt{2} \tag{33}$$

The frequencies at the crossing points on the imaginary axis are thus $\omega = \pm\sqrt{2}$. The gain value corresponding to the crossing points is $K = 6$.

An alternative approach is to let $s = j\omega$ in the characteristic equation, equate both the real part and the imaginary part to zero, and then solve for ω and K. For the present system, the characteristic equation, with $s = j\omega$, is

$$(j\omega)^3 + 3(j\omega)^2 + 2(j\omega) + K = 0, \Rightarrow (K - 3\omega^2) + j(2\omega - \omega^2) = 0 \tag{34}$$

Equating both the real and imaginary parts of this last equation to zero, respectively, we obtain

$$K - 3\omega^2 = 0, \; 2\omega - \omega^3 = 0 \tag{35}$$

from which

$$\omega = \pm\sqrt{2}, \; K = 6, \; \textit{or } \omega = 0, \; K = 0 \tag{36}$$

Thus, root loci cross the imaginary axis at $\omega = \pm\sqrt{2}$ and the value of K at the crossing points is 6. Also, a root-locus branch on the real axis touches the imaginary axis at $\omega = 0$. The value of K is zero at this point.

Draw the root loci. *Choose a test point in the broad neighborhood of the $j\omega$ axis and the origin, as shown in Figure 4, and apply the angle condition.*

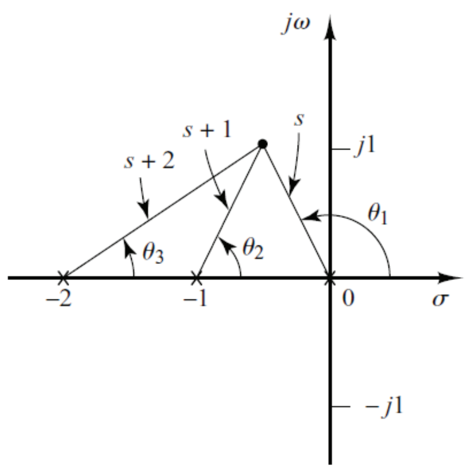

Figure 4: Construction of root locus.

If a test point is on the root loci, then the sum of the three angles, $\theta_1+\theta_2+\theta_3$, must be 180°. If the test point does not satisfy the angle condition, select another test point until it satisfies the condition. The sum of the angles at the test point will indicate the direction in which the test point should be moved. Continue this process and locate a sufficient number of points satisfying the angle condition. Based on the information obtained in the foregoing steps, as shown in Figure 5.

□

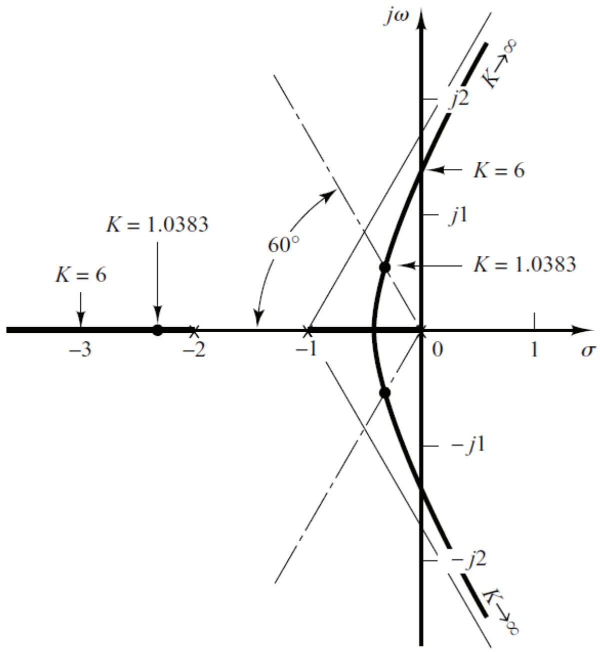

Figure 5: Root-locus plot.