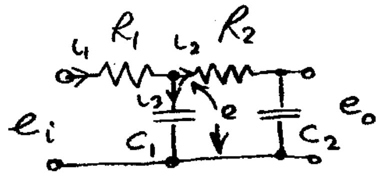
LECTURE 5 – MORE COMPLEX SYSTEMS

5. For the electrical circuit, derive the transfer function.



We have

$$e_i = i_1 R_1 + e \quad \rightarrow and \;\; i_1 R_1 = e_i - e$$
$$e = i_2 R_2 + e_0$$
$$i_2 = C_2 \dot{e_0}$$
$$i_3 = C_1 \dot{e}$$
$$i_1 = i_2 + i_3$$

$$\therefore i_1 = i_2 + i_3 = C_2 \dot{e_0} + C_1 \dot{e}$$

But $e = i_2 R_2 + e_0 = e_0 + C_2 R_2 \dot{e_0}$

$\therefore \dot{e} = \dot{e_0} + C_2 R_2 \ddot{e_0}$ by differentiation.

Since

$i_1 = \frac{e_i - e}{R_1} \equiv C_2 \dot{e_0} + C_1 \dot{e}$ all currents are eliminated.

Then

$$\frac{e_i}{R_1} - \frac{e_0}{R_1} - \frac{C_2 R_2}{R_1}\dot{e_0} = C_2 \dot{e_0} + C_1(\dot{e_0} + C_2 R_2 \ddot{e_0})$$

Rearranging

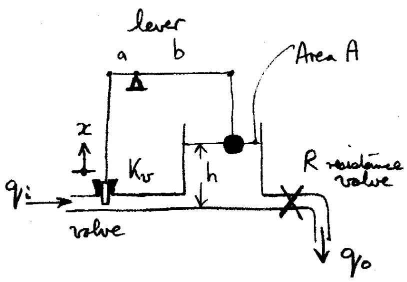
$$C_1 C_2 R_1 R_2 \ddot{e_0} + \big((C_1 + C_2)R_1 + C_2 R_2\big)\dot{e_0} + e_0 = e_i$$

Let

$T_1 = C_1 R_1,\; T_2 = C_2 R_2,$ and take Laplace with zero boundary conditions.

$$\therefore \frac{E_0(s)}{E_i(s)} = \frac{1}{T_1 T_2 s^2 + (T_1 + T_2 + R_1 C_2)s + 1}$$

6. For a <u>reservoir system</u> create a complete model and analysis (extended example!). This is an automatic water level control.



The net flow into the reservoir is $q = q_i - q_0$, $q_i = K_v x$ and $q_0 = \frac{h}{R}$ by definition. However x is linked to the reservoir height via the lever such that $x \to 0$ as $h \to h_d$ $\therefore x = \frac{a}{b}(h_d - h)$. Also we know that $q = A\dot{h}$ (net volume increase).

Thus we can write $A\dot{h} = K_v \frac{a}{b}(h_d - h) - \frac{h}{R}$, where $h = h(t)$.

We will now specify h_d as an input and h as output because we can drive the system to a particular h_d by arranging that $x \to 0$ as $h \to h_d$.

Rearranging:

$$A\dot{h} + h\left(\frac{1}{R} + K_v \frac{a}{b}\right) = K_v \frac{a}{b} h_d,$$

we get a first order DE expressing $h = f(h_d)$.

Take Laplace to get

$$AsH(s) - Ah(0) + H(s)\left(\frac{1}{R} + K_v \frac{a}{b}\right) = K_v \frac{a}{b} H_d(s)$$

We now specify the boundary conditions such that at $t = 0$, $h(0) = 0$. Also $H_d(s) = \frac{h_0}{s}$, demanding h_0 at $t = \infty$.

Simplifying, we get

$$H(s)(As + K) = \frac{K_1}{s} \text{ where } K = \frac{1}{R} + K_v \frac{a}{b},\ K_1 = K_v \frac{a}{b} h_0$$

$$\therefore H(s) = \frac{K_1}{s(As+K)} \equiv \frac{K_1/A}{s(s+K/A)} \text{ a standard form (normalised)}$$

$$\therefore h(t) = \frac{K_1}{A} \cdot \frac{A}{K}\left(1 - e^{\frac{-Kt}{A}}\right)$$

$$\therefore h(t) = \frac{h_0}{\left(1 + \frac{b}{K_v aR}\right)}\left(1 - e^{-\left(\frac{K_v a}{Ab} + \frac{1}{AR}\right)t}\right)$$

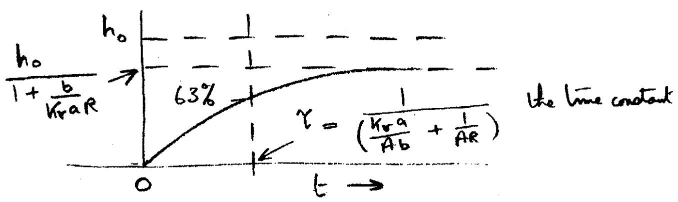
Analysing this result tells us:

a) As long as R is finite positive h never gets to h_0

b) If $R \to \infty$ $h = h_1\left(1 - e^{-\frac{K_v a}{Ab}t}\right)$ and we do reach h_0 at $t \to \infty$

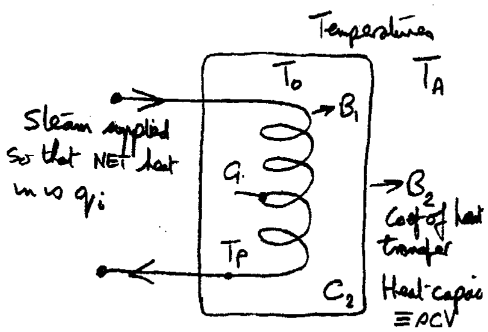
c) If $R \to 0$, there is no resistance to output flow and $h = h_0$ as $t \to \infty$.

In general,



Note that the model of the system is valid only in the region $0 < h < h_0$. When $h > h_0$, we should have completely different modelling. Note that h is always: $h \le \frac{h_0}{\left(1+\frac{b}{K_v a R}\right)}$.

7. For a <u>thermostatic tank</u>, we will look at the dynamics of how the tank temperature T_o changes as it is heated up by the net rate of heat q_i and loses heat to the atmosphere. The coil has significant heat capacity and has temperature T_p.



The increase in coil internal energy is

$$C_1 \dot{T}_p = q_i - B_1\left(T_p - T_0\right) \tag{1}$$

The increase in tank fluid internal energy is

$$C_2 \dot{T}_0 = B_1\left(T_p - T_0\right) - B_2(T_0 - T_A) \tag{2}$$

Substituting for $B_1\left(T_p - T_0\right)$ in 1 from 2 gives

$$q_i = C_1 \dot{T}_P + C_2 \dot{T}_0 + B_2(T_0 - T_A)$$

Remove T_P by using the differential of 2 (T_P is known as an intermediate variable):

$$B_1 \dot{T}_p = C_2 \ddot{T}_0 + B_1 \dot{T}_0 + B_2 \dot{T}_0$$

$$\therefore q_i = \frac{C_1}{B_1}\left(C_2 \ddot{T}_0 + (B_1 + B_2)\dot{T}_0\right) + C_2 \dot{T}_0 + B_2(T_0 - T_A)$$

$$= \frac{C_1 C_2}{B_1} \ddot{T}_0 + \left(\frac{C_1 B_2}{B_1} + C_1 + C_2\right) \dot{T}_0 + B_2(T_0 - T_A)$$

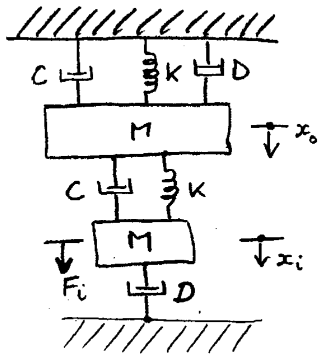
In order to simplify the working, change the variables to operate around T_A. This then becomes an incremented equation. Let $K_1 = \frac{C_1}{B_1}$, $K_2 = \frac{C_2}{B_2}$, $K_3 = \frac{C_1}{B_2}$

$$\therefore \frac{q_i}{B_2} = K_1 K_2 \ddot{T}_0 + (K_1 + K_2 + K_3)\dot{T}_0 + T_0$$

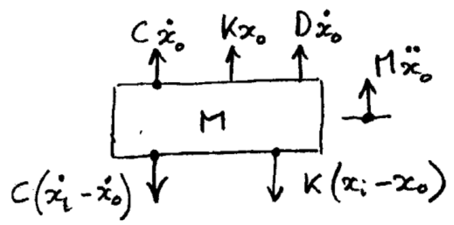
We assume zero boundary conditions and take Laplace.

$$\therefore \frac{T_{0(s)}}{Q_i(s)} = \frac{1/B_2}{K_1 K_2 s^2 + (K_1 + K_2 + K_3)s + 1}$$ This is non-dimensional form of the transfer function.

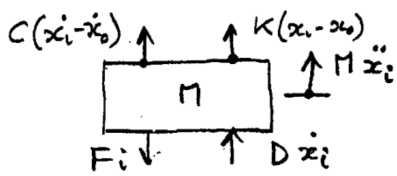
8. For a <u>mass/spring/damper system</u> find the transfer function relating F_i as an input and x_0 as an output. We assume the system to be in static equilibrium at the start, therefore the springs are already extended and the variables x_0, x_1 are incremental on those displacements.



Draw free body diagrams for both masses.



$$\therefore M\ddot{x_0} + (2C + D)\dot{x_0} + 2Kx_0 = C\dot{x_\iota} + Kx_i \tag{1}$$



$$\therefore M\ddot{x_1} + (C + D)\dot{x_\iota} + Kx_i = F_i + C\dot{x_0} + Kx_0 \tag{2}$$

Take Laplace of 1 and 2 to get:

$$(Ms^2 + s(2C + D) + 2K)X_0(s) = (Cs + K)X_i(s)$$

$$(Ms^2 + s(C + D) + K)X_i(s) = F_i(s) + (Cs + K)X_0(s)$$

To simplify, rewrite as:

$f_1(s)X_0 = f_2X_i$ and $f_3(s)X_i = F_i + f_2(s)X_0$

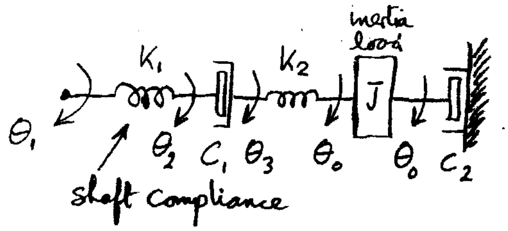
Then by rearranging:

$$\frac{X_0(s)}{F_i(s)} = \left(f_3(s)\left(\frac{f_1(s)}{f_2(s)} - \frac{f_2(s)}{f_3(s)}\right)\right)^{-1}$$

we can work out all the f functions and put them in to get:

$$= \frac{Cs+K}{(Ms^2+(2C+D)S+K)(Ms^2+(C+D)S+K)+(Cs+K)^2} \equiv \frac{P(s)}{Q(s)}$$

9. For a rotational drive shaft system we wish to know the relationship between angular displacements θ_1 and θ_0 as the shaft flexes in use.



Working from the left,

$$K_1(\theta_1 - \theta_2) = C_1(\dot{\theta}_2 - \dot{\theta}_3) = K_2(\theta_3 - \theta_0) = J\ddot{\theta}_0 + C_2\dot{\theta}_0$$

Without removing the intermediate variables θ_2 and θ_3 we rearrange and take Laplace to get:

$$\begin{array}{cccc}
(C_1s + K_1)\theta_2 & -C_1s\theta_3 & & = K_1\theta_1 \\
-C_1s\theta_2 & +(C_1s + K_2)\theta_3 & -K_2\theta_0 & = 0 \\
0 & -K_2\theta_3 & +(Js^2 + C_2s + K_2)\theta_0 & = 0
\end{array}$$

If $\bar{\theta} = \begin{bmatrix}\theta_2\\ \theta_3\\ \theta_0\end{bmatrix}$ we can rewrite as a matrix equation

$$\underline{A}\bar{\theta} = \begin{bmatrix}K_1\theta_1\\ 0\\ 0\end{bmatrix}, \text{ hence } \bar{\theta} = \underline{A}^{-1}\begin{bmatrix}K_1\theta_1\\ 0\\ 0\end{bmatrix}$$

It is possible to get any transfer function required that involves θ_2, θ_3 or θ_0 with θ_1 as the input, depending on what we extract from $\underline{A}^{-1}$.

To get $\underline{A}^{-1}$, the first method is $\underline{A}^{-1} = \frac{ADJOINT\ of\ \underline{A}}{DETERMINANT\ of\ \underline{A}} \equiv \frac{adj\underline{A}}{det\underline{A}}$

Finding adjoints and determinants is bookwork.

The second method works as follows.

If we define a set of column vectors

$$\underline{A}\bar{\theta} \equiv \begin{bmatrix} \bar{A}_1 \bar{A}_2 \dots \bar{A}_n \end{bmatrix} \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix} = \begin{bmatrix} K_1 \\ \vdots \\ K_n \end{bmatrix} \equiv \begin{bmatrix} K_1\theta_1 \\ 0 \\ 0 \end{bmatrix}$$

We can extract the equation for θ_i by doing this:

$$\theta_i = \frac{det\begin{bmatrix} \bar{A}_1 \dots \begin{bmatrix} K_1 \\ \vdots \\ K_n \end{bmatrix} \dots \bar{A}_n \end{bmatrix}}{det\underline{A}}$$ substitution of the ith vector with the r.h.s.

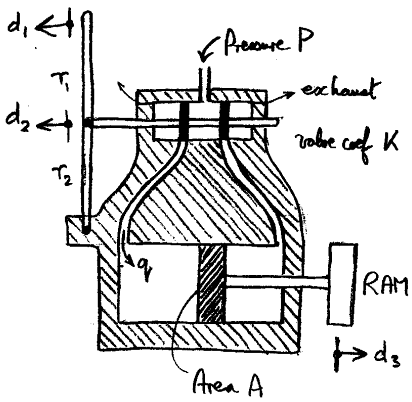
By way of example, to get the expression for θ_0 we substitute the 3rd column with the r.h.s to get

$$\theta_0 = \frac{det\begin{bmatrix} C_1 s + K_1 & -C_1 s & K_1\theta_1 \\ -C_1 s & C_1 s + K_2 & 0 \\ 0 & -K_2 & 0 \end{bmatrix}}{det\underline{A}}$$

$$= \frac{C_1 K_1 K_2 s \theta_1}{C_1(K_1+K_2)Js^3 + (C_1 C_2(K_1+K_2) + K_1 K_2)s^2 + C_1 K_1 K_2 s + K_1 K_2^2}$$

from which the transfer function $\frac{\theta_0}{\theta_1}$ is obtained. Note we have shortened $\theta_0(s)$ to θ_0 etc, s understood.

10. For a <u>servo hydraulic system</u> we will discover the transfer function between movement d_1 of the lever on the control valve to movement d_3 of the ram.



On the lever $d_2 = d_1 \frac{r_2}{r_1 + r_2}$

In the piston $A\dot{d}_3 = q$

where

$q = Kd_2$ by definition.

Remove intermediate variables d_2 and q to get

$A\dot{d}_3 = q = Kd_2 = Kd_1 \frac{r_2}{r_1+r_2}$. Hence $\dot{d}_3 = \frac{Kr_2}{A(r_1+r_2)} \cdot d_1$

and by taking Laplace (with zero boundary conditions) we get

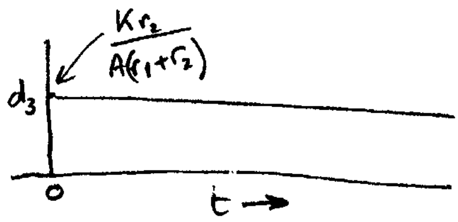
$\frac{D_{3(s)}}{D_1(s)} = \frac{Kr_2}{A(r_1+r_2)s} \equiv \frac{K_1}{s}$ which is by definition called an <u>integrator</u>.

We will look at what happens to movement of the ram under two cases of playing with the input lever at d_1.

CASE 1 – put an impulse on the lever, flick on and off quickly.

$$\therefore D_3 = \frac{Kr_2}{A(r_1+r_2)} \cdot 1 \quad \text{for a unit impulse}$$

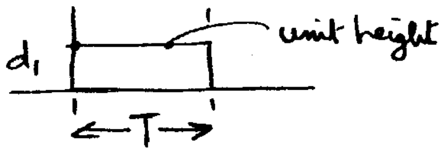
$$\therefore d_3 = \frac{Kr_2}{A(r_1+r_2)}$$



The ram instantly jumps a set distance and then stops.

CASE 2 – put a finite deflection on the lever for a limited period of time.

$$d_3 = \mathcal{L}^{-1}\left[\frac{Kr_2}{A(r_1+r_2)s} \cdot D_1(s)\right], D_1(s) = \frac{1}{s} - e^{-sT}\frac{1}{s}$$



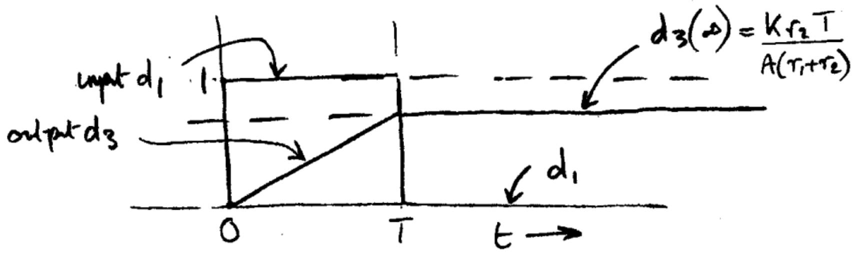
Using the shift theorems:

$$\therefore d_3 = \frac{Kr_2}{A(r_1+r_2)}\mathcal{L}^{-1}\left[\frac{1}{s^2} - \frac{e^{-sT}}{s^2}\right]$$

$$= \frac{Kr_2}{A(r_1+r_2)}[t - (t-T)u(t-T)]$$

Two ramps one positive at $t = 0$ and one negative at $t = T$.

Graphically we have



This second case shows clearly how the general earth moving hydraulic equipment produces movement by altering the position of the control lever.