LECTURE 10 – STEADY STATE ERRORS

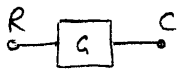
There is a need to know how closely the output of a system follows the input driving function. So we will

- Calculate offsets that occur
- Comment on trends that are inherent in a system
- Look for common characteristics expressed by reference to the form of the transfer function

DEFINITION

The steady state error (SSE) is the difference in magnitude between an input R and output C of the full open loop in any circut that may contain feedback.

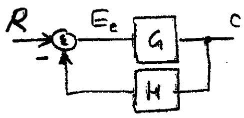
OPEN LOOP CASE



$E_o = R - C = R - GR$

$\therefore E_o = R(1-G)$

CLOSED LOOP CASE



$E_c = R - CH = R - GE_cH$

$\therefore E_c = \dfrac{R}{1+GH}$

We desire to know the values for E_0 and E_c at $t \to \infty$, that is, where they settle down to. So invoke the final value theorem to get

$$e_o(\infty) = \lim_{s\to 0} sE_o(s) \quad \text{and} \quad e_c(\infty) = \lim_{s\to 0} sE_c(s)$$

To process the analysis, we will discover responses to our standard inputs, in particular by using the unit step input.

Hence

$$e_0(\infty) = \lim_{s\to 0} s\left(\frac{1}{s} - \frac{G}{s}\right) = 1 - G(0)$$

and

$$e_c(\infty) = \lim_{s\to 0} s\left(\frac{1/s}{1+GH}\right) = \frac{1}{1+G(0)H(0)}$$

The value $G(0)$ is known as the *steady state gain* or *DC gain* (gain at zero frequency) and is normally greater than 1. Therefore $e_0(\infty)$ can be a significant SSE.

$H(0)$, the zero frequency feedback component, is normally around 1 and is exactly 1 in a unity gain, negative feedback loop. Assume $H(0) = 1$ from now on.

$\therefore e_c(\infty) = \frac{1}{1+G(0)} \ll 1$, so the closed loop SSE is much reduced on the open loop SSE. We could try reducing $e_0(\infty) = R - C$ to zero by design. However, the system would require continual recalibration to maintain this accuracy. With $e_c(\infty) = \frac{1}{1+G(0)}$, the value is always small no matter if $G(0)$ drifts.

GENERAL CASES FOR SSE

We categorise a system transfer function into type according to the number of roots occurring in the characteristic equation that are equal to zero. These roots are called POLES. The roots of the numerator are called ZEROS, because they give the transfer function a value of zero.

Express the OPEN LOOP transfer function in the form

$$G(s) = \frac{K(a_n s^n + a_{n-1} s^{n-1} \ldots\ldots + 1)}{s^{\ell}(b_m s^m + b_{m-1} s^{m-1} \ldots\ldots + 1)} \quad \text{(this is NOT the closed loop form!)}$$

K is a constant and ℓ gives the type; $\ell = 1$ gives a TYPE 1. We can now use this form to obtain $G(0)$ for substitution in the expressions for SSE already obtained. Set aside $e_0(\infty)$ because it is invariable worse than $e_c(\infty)$ and now study the value of $e_c(\infty)$ for various system types with unity gain negative feedback and standard inputs.

Hence

$$e_c(\infty) = \lim_{s\to 0} s\left(\frac{R}{1+G}\right), H = 1.$$

In order to draw up a table of SSE values, let us explore some particular cases to fill in the matrix.

CASE 1 – A type 0 system driven by a unit step

$$e_c(\infty) = \lim_{s\to 0} s\left(\frac{\frac{1}{s}}{1+\frac{K(0\cdots+1)}{1(0\cdots+1)}}\right) = \frac{1}{1+K}$$

see the general expression
for G(s) above

CASE 2 – A type 2 system driven by a unit ramp

$$e_c(\infty) = \lim_{s\to 0} s\left(\frac{\frac{1}{s^2}}{1+\frac{K(0\cdots+1)}{S^2(0\cdots+1)}}\right) = 0$$

$$s\cdot\frac{1/s^2}{1+\frac{K}{S^2}} = \frac{s}{s^2}\cdot\frac{s^2}{s^2+K} = \frac{s}{s^2+K} \quad \text{evaluated at } s = 0$$

CASE 3 – A type 1 system driven by an impulse

$$e_c(\infty) = \lim_{s\to 0} s\left(\frac{1}{1+\frac{K(0\cdots+1)}{s(0\cdots+1)}}\right) = 0$$

Now we will construct a full table of SSE values against type number and standard input.

STEADY STATE ERRORS

DRIVING FUNCTION		*SYSTEM TYPE*			
		0	*1*	*2*	*3*
IMPULSE	$\delta(t)$	0	0 *	0	0
STEP	$u(t)$	$\frac{1}{1+K}$ *	0	0	0
RAMP	t	∞	$\frac{1}{K}$	0 *	0
ACCELERATION	t^2	∞	∞	$\frac{1}{K}$	0

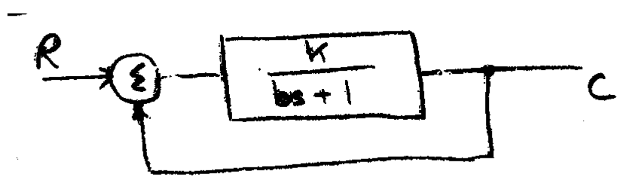
The stared entries have been derived above.

We see all system types pass an impulse without retaining a SSE. Type 0 systems have finite SSE error for a step but go unstable for ramps and above. Type 2 system can control our standard inputs but have finite SSE for accelerations. In general, the more dynamic the input the higher type number is needed to restore a finite SSE.

FULL ANALYSES OF AN SSE

Say we have $G = \frac{K}{(bs+1)}$ and driven by a unit ramp in closed loop form.

CASE 4

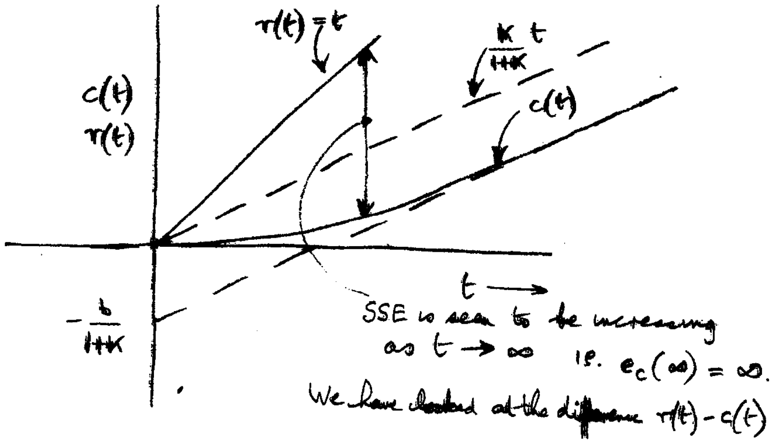


The closed loop transfer function $G = \frac{K}{bs+1+K}$

$$\therefore c(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{K}{bs+1+K}\right\} =$$

$$\therefore c(t) = \frac{K}{1+K}\left[t - \frac{b}{1+K}\left(1 - e^{-\frac{(K+1)t}{b}}\right)\right]$$

Graphically:



CASE 5

Say we have $G = \frac{K}{s(bs+1)}$, a type 1 system, and driven by a ramp in closed loop form.

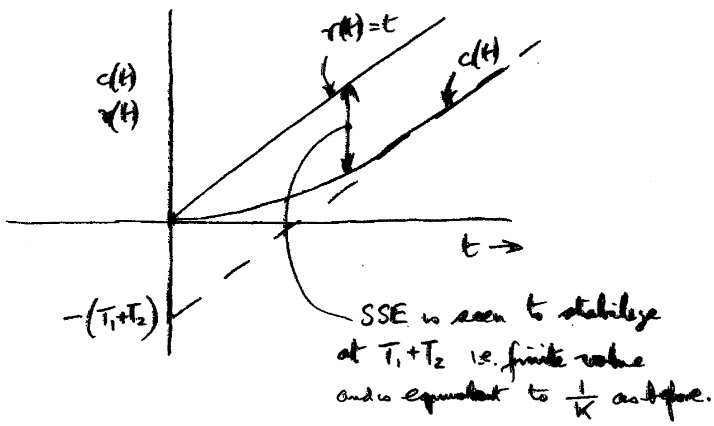
$$G_c = \frac{K}{s(bs+1)+K} = \frac{1}{\frac{s(bs+1)}{K}+1}$$

$$c(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{\frac{s(bs+1)}{K}+1}\right\}$$ Now rewrite in the form

$$c(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\cdot\frac{1}{(\alpha s+1)(\beta s+1)}\right]$$ where α, β are easily calculated.

$$c(t) = t - \frac{\alpha+\beta}{\alpha\beta} + \frac{\alpha\beta}{\beta-\alpha}\left(\frac{\beta}{\alpha}e^{-\alpha t} - \frac{\alpha}{\beta}e^{-\beta t}\right)$$

Graphically:



Where $T_1 = \alpha$, $T_2 = \beta$

QUESTON 1 – What sort of SSE does an integrator produce when driven by (a) unit impulse; (b) unit step; (c) unit ramp; (d) unit acceleration?

ANSWER – an integrator has the form $\frac{K}{s}$ and is type 1. Therefore looking at the tables: (a) zero; (b) zero; (c) $\frac{1}{K}$; (d) never settles (∞).

QUESTION 2 – What is the closed loop SSE of a system whose open loop transfer function is $G = \frac{500}{(S+9)(s+5)}$ and driven by step of height 10?

$$e_c(\infty) = \lim_{s\to 0} s\,\frac{R}{1+G} = \lim_{s\to 0} s\left(\frac{\frac{10}{s}}{1+\frac{500}{(s+9)(s+5)}}\right) = \frac{10\times 45}{45+500} = \frac{450}{545}$$

Using tables $e_c(\infty) = 10\cdot\left(\frac{1}{1+K}\right)$ where $K = \frac{500}{9\times 5}$

$$\therefore e_c(\infty) = 10\cdot\frac{1}{1+\frac{500}{45}} = \frac{10\times 45}{45+500} = \frac{450}{545}\ QED$$