# MMAN3200 LINEAR SYSTEMS AND CONTROL

## TEST 1 – March 2019

## Total Marks: 10

**Reading time: 5 minutes** **Time allowed: 60 minutes**

**No literature permitted**

## Question 1 (3 marks)

The expression for velocity of a particle in a polar co-ordinate system is:

$$v = \sqrt{\dot{r}^2 + (r\dot{\theta})^2} \qquad (1)$$

where r and θ are polar co-ordinates of the particle.

(a) Linearise the expression for velocity if the operating point is defined by the following values: $r_0 = 1m,\ \dot{r}_0 = 0\frac{m}{s},\ \theta_0 = 0\,rad,\ \dot{\theta}_0 = \frac{\pi}{4}\frac{rad}{s}$.

(b) Find the values of velocity using the expression derived in part **(a)** as well as equation (1), if the values of the variables are: $r = 2m,\ \dot{r} = 0\frac{m}{s},\ \theta = 0\,rad,\ \dot{\theta} = \frac{\pi}{2}\frac{rad}{s}$. Thus calculate the linearisation percentage error.

(c) What combination(s) of values of r and $\dot{\theta}$ will result in the linearisation error equal to zero (apart from the trivial case when $r = r_0$ and $\dot{\theta} = \dot{\theta}_0$)?

(d) (**Bonus question**) What is the form of the trajectory of the particle in the proximity of the operating point defined by: $r_0 = 1m,\ \dot{r}_0 = 0.5\frac{m}{s},\ \theta_0 = 0\,rad,\ \dot{\theta}_0 = 0\,\frac{rad}{s}$?

## Question 2 (1 marks)

Function f(t) consists of a train of impulses applied at 60 seconds intervals, the magnitude of each impulse being 15 $\cancel{\times 10^4 Ns}$. Volts If only three impulses are taken into account (**Figure 1**), describe f(t):

(a) in the time domain;

(b) in the s-domain.

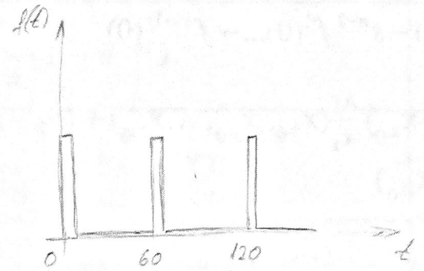


**Figure 1**

Please see over

## Question 3 (6 marks)

A solenoid, schematically presented in **Figure 2**, has resistance R and inductance L.

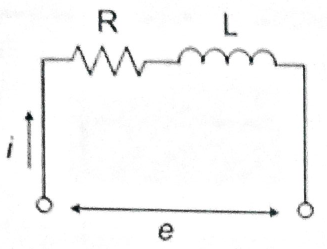


Figure 2

(a) Derive its transfer function I(s)/E(s), where **i** is current and **e** is voltage applied.
(b) If R=10 Ω and L = 0.1 H, determine the values of the time constants (if any) and the steady state gains (if any).
(c) Find the steady state value of the unit step response of the circuit.
(d) Derive the expression for i(t) if the input is a unit step function and all initial conditions are equal to zero.
(e) Derive the expression for i(t) if the value of current at the time the unit step is applied is $i_0 = 2A$.
(f) Derive the expression for i(t) if the input voltage has the same form as the function defined in **Question 2.** zero initial
(g) The solenoid generates an electromagnetic force, $F_{em} = \alpha i$, where $\alpha$ is a constant. What would be the expression for $F_{em}(s)$ under the scenario defined in **Part (d)** of this question?

**Important formulae:**

$$\lim_{s\to 0}[sF(s)] = \lim_{t\to\infty}[f(t)] \qquad L\{e^{-at}f(t)\} = F(s+a)$$

$$\lim_{s\to\infty}[sF(s)] = \lim_{t\to 0}[f(t)] \qquad L^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$$

$$L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0)... - f^{(n-1)}(0)$$

$$y = f(x_{1_0}, x_{2_0}, ...x_{n_0}) + (x_1 - x_{1_0})\frac{\partial f}{x_1}(x_{1_0}, x_{2_0}, ...x_{n_0}) + (x_2 - x_{2_0})\frac{\partial f}{x_2}(x_{1_0}, x_{2_0}, ...x_{n_0}) + ...(x_n - x_{n_0})\frac{\partial f}{x_n}(x_{1_0}, x_{2_0}, ...x_{n_0})$$

$$e = Ri \qquad e = L\frac{di}{dt} \qquad e = \frac{1}{C}\int idt \qquad e_b = K_b\dot{\theta}_m$$

$$\dot{Q} = q\rho c\Delta T \qquad \dot{Q} = (kA/d)\Delta T \qquad \dot{Q} = hA\Delta T \qquad \dot{Q} = \rho cV\dot{T}$$

# MMAN3200 - LINEAR SYSTEMS ANALYSIS
## Mid-semester test
## July 2019

**Time allowed: 1 hour 40 minutes**
**Reading time: 5 minutes**

**No literature permitted**
**Total marks: 25**

## Question 1 (12 marks)

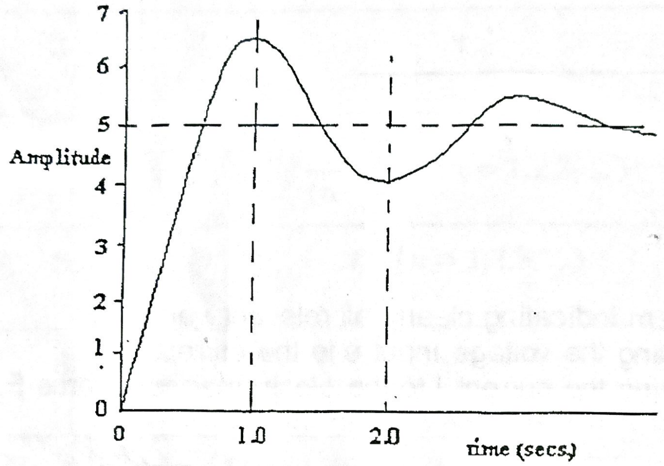


**Figure 1**

The graph in **Figure 1** is the response y(t) of a mechanical system to a step input of magnitude 10.

a) What is the order of the system?
b) Estimate the values of the relevant parameters off the graph and determine the transfer function of the system.
c) If the values of K and $\omega_n$ are the same as determined in b), what would be the minimum value of $\xi$ that results in a zero percentage overshoot?
d) If the values of $\xi$ and K are the same as determined in b), and $\omega_n$ is twice as high, what would be the new value of the percentage overshoot?
e) Find the settling time if the settling criterion is 2% of the steady state value.
f) Determine the initial and final values of the **unit impulse** response of the system.
g) Use the transfer function determined in b) to derive the time domain mathematical model with x(t) and y(t) being the input and the output respectively.
h) If function x(t)=3sin(2t) is applied as the input, sketch the **steady state component** of the output.

## Question 2 (13 marks)

A railway point's actuator has been constructed as shown in **Figure 2**. The linear movement on the changeover tray is accomplished by movement of the solenoid S of inductance, L and series internal resistance, R. Two springs of coefficient K return the tray to a central position if no force is generated from the solenoid. The damper C is installed to provide viscous resistance to the motion of the tray. The mass of the solenoid core is M [kg] and the force derived from current flow is $\alpha i$.

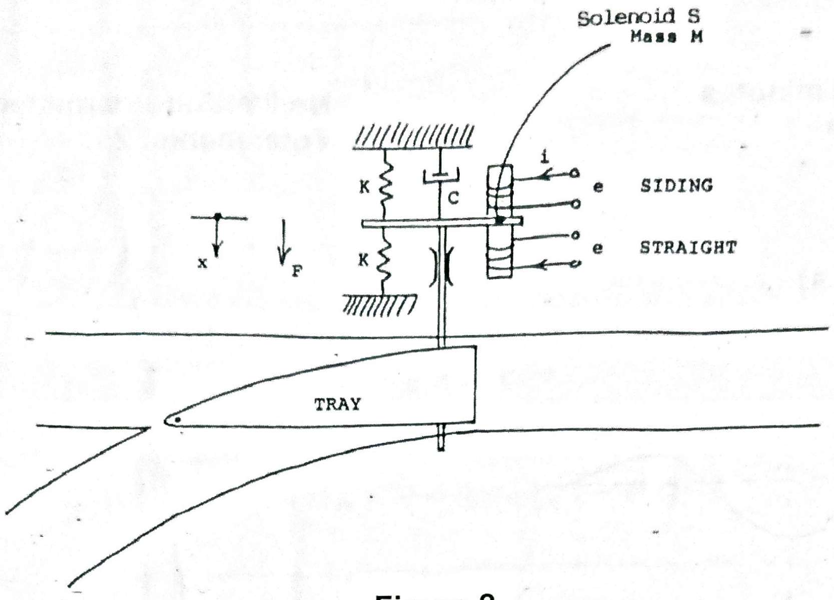


Figure 2

a) Draw a block diagram of the system indicating clearly all relevant variables.
b) Derive a differential equation relating the voltage input e to the current i.
c) Derive a differential equation relating the current I to the electromagnetic force F.
d) Derive a differential equation relating the electromagnetic force F to the movement x of the core of the solenoid.
e) Using the method of your choice derive the transfer function X(s)/E(s) of the whole system.
f) Derive a relationship between M, C and K if the mechanical part of the circuit is to be critically damped.
g) If the denominator of the transfer function of the mechanical part has a pair of complex-conjugate roots, what would be effect on the functioning of the whole system? Explain in up to 30 words.
h) The normal separation of tray rail to fixed rail is 0.2m, the spring coefficients are rated at 1500 $N.m^{-1}$ each, the attractive force coefficient of the solenoid, $\alpha$ is 15 N/A, and its internal resistance, R is 0.5Ω. What is the magnitude of the voltage step required if the design criterion on the force holding the tray to the fixed rail is chosen to be 100 N?

## Important Formulae

$$\Delta e = Ri \qquad \Delta e = L\frac{di}{dt} \qquad \Delta e = \frac{1}{C}\int i\,dt \qquad e_b = K_b\dot{\theta}_m$$

$$F = m\ddot{x} \qquad F = kx \qquad F = c\dot{x} \qquad F = \alpha i$$

$$T = J\ddot{\theta} \qquad T = k_t\theta \qquad P = Fv \qquad T = K_t i$$

$$Rq = \Delta h \qquad \sum q = A\dot{h} \qquad q = Kx \qquad \dot{Q} = q\rho c\Delta T$$

$$\dot{Q} = (kA/d)\Delta T \qquad \dot{Q} = hA\Delta T \qquad \dot{Q} = \rho cV\dot{T}$$

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \xi = c/2\sqrt{km} \qquad T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} \qquad t_s = \frac{3.91}{\xi\omega_n}$$

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# PART A

**QUESTION 1** (15 marks)

The transfer function of an open loop system is:

$$G(s) = \frac{K}{(s+1)(s+2)(s+5)}$$

a) Determine the differential equation that describes the system in the time domain with x(t) as the input and y(t) as the output. (1 mark)

b) What is the order of the open loop system G($s$)? What is its type? (1 mark)

c) Determine the steady state error of the open loop system if the input is a unit step function. (0.5 marks)

d) Determine the steady state error of the closed loop system if the input is (i) an impulse of magnitude 5, and (ii) a unit ~~ramp~~ step function. Assume H(s)=1. (2 marks)

e) Sketch the root locus of the closed loop system for 0<K<∞ indicating clearly the starting and ending points (if any), asymptotes (if any), intersections between the root locus and the imaginary axis (if any), break-away and break-in points (if any) as well as the departure angles from the open-loop poles (if any). (5 marks)

f) Comment on the types of stability (stable/unstable, oscillatory/exponential) of the closed loop system that are achieved for different positive values of K. (2 marks)

g) What are the values of the poles of the closed loop system, as well as the values of K, when closed loop system is critically ~~stable?~~ damped. (1 mark)

h) Assume a zero at s=0 is added to the numerator of the open loop transfer function. How would the stability of the closed loop change in this case? (Note: You don't have to perform detailed calculations, only a simple sketch and a general discussion are required.) (2.5 marks)

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## PART B

**(Answer in a separate booklet)**

**QUESTION 2 (20 marks)**

a) Given a system whose transfer function (TF) is the following one,

$$G(s) = \frac{\mathbf{10}}{\boldsymbol{s \cdot (1 + 0.1 \cdot s)}}$$

Using straight line approximations, draw a Bode diagram (magnitude and phase plots) for the range of angular frequencies $\boldsymbol{\omega}$ from 0.1 to 100 rad/s. (3 marks)

b) Calculate the actual magnitude and phase at $\boldsymbol{\omega}$=1 rad/s. Evaluate the error of the straight line approximations at that frequency. (3 marks)

c) By inspecting the result in (a), obtain, approximately, the Gain Margin (GM) and the Phase Margin (PM). (3 marks).

d) Explain the Gain Margin obtained in (c) in terms of what you see in its Root Locus (you simply need to sketch the RL, and confirm why you got that GM value. We are not asking you to fully draw the RL in detail; we expect you to briefly sketch it, to show the shape of the RL, and to interpret it to answer the question. What you see in the RL should be consistent with the GM that you infer from the Bode plot.). (6 marks)

e) In general, for evaluating the Phase Margin (PM) of a system $\boldsymbol{G(s)}$, we find the frequency $\boldsymbol{\omega}$ at which the magnitude of the system TF is =**0 db** ( i.e. $\boldsymbol{M(j\omega) = 20 \cdot log_{10}(\|G(j\omega)\| = 0)}$, and then we evaluate the phase at that specific frequency $\boldsymbol{\omega}$; Given that phase value we evaluate is the amount of additional phase (which we name *phase margin*) that would make the total phase equal to certain value, i.e.

$$\boldsymbol{phase(G(j\omega)) - PM = A.}$$

What value is $\boldsymbol{A}$? (Give its value, indicating the engineering units in which you are expressing it). Why are we doing that? (E.g. you may give your explanation about what would occur for an OLTF having, at that frequency, M=0db and phase = **A**. (5 marks) (0)1(0

**QUESTION 3 (10 marks)**

a) Consider the system whose dynamics is represented by the following linear differential equation

$$\frac{d^3y}{dt^3} = 2 \cdot \frac{d^2y}{dt^2} - 3 \cdot \frac{dy}{dt} + y(t) + 4 \cdot u(t)$$

In which the variable *u(t)* is the input of the system, and *y(t)* its output.

You are asked to propose a state space vector, **X**, and the resulting system's state equation, in the form: $\frac{dX}{dt} = \boldsymbol{A \cdot X + B \cdot u}$. You need to clearly show the components and the size of the vector **X,** and then obtain the values of the matrixes **A** and **B**.

tol:11

b) Would this type of state space representation (**X**, proposed by you in (a)) be adequate for the following system? (Note: we are asking if the state space representation you have chosen is adequate for being used in this case. We are not asking about using the same state equation.)

$$\frac{d^3y}{dt^3} = 2\cdot\frac{d^2y}{dt^2} - 3\cdot\frac{dy}{dt} + y(t) + 4\cdot u(t) + 3\cdot\frac{du}{dt}(t)$$

c) And for this one?

$$\frac{d^3y}{dt^3} = 11\cdot\frac{d^2y}{dt^2} - 5\cdot\frac{dy}{dt} + y(t) + 7\cdot u(t)$$

d) How many states (i.e. dimension of vector **X**) would be needed for representing the following system

$$\frac{d^3y}{dt^3} = y(t) + 4\cdot u(t)$$

( (a) : 6 marks, (b): 1 mark, (c) : 1 mark, (d):2 marks )

10:30

**END OF EXAMINATION PAPER**

**Useful formulae:**

$$\lim_{t\to 0}[f(t)] = \lim_{s\to\infty}[sF(s)] \qquad \lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[sF(s)] \qquad \lim_{t\to\infty}[e_c(t)] = \lim_{s\to 0}\left[\frac{sR(s)}{1+G(s)H(s)}\right]$$

$$L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0)\ldots - f^{(n-1)}(0)$$

$$\sigma_A = \frac{\Sigma(-p_i)-\Sigma(-z_i)}{n_p-n_z}, \qquad \varphi_A = \frac{2q+1}{n_p-n_z}\cdot 180,\ q=0, 1, \ldots (n_p - n_z - 1)$$

$$\frac{d(-K)}{ds} = 0, \qquad \Sigma_i\varphi_{z_i} - \Sigma_j\varphi_{p_j} = -180 - 360l$$

*Please see over for Laplace transform table*