Tutorial—Mathematical Modeling

Example 1. *Simplify the block diagram shown in Figure 1.*

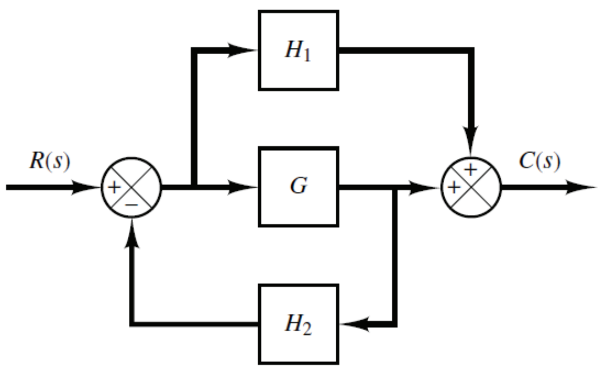

Figure 1: Block diagram of a system.

Example 2. *Simplify the block diagram shown in Figure 2. Obtain the transfer function relating* $C(s)$ *and* $R(s)$.

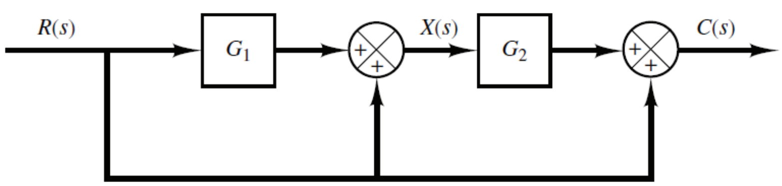

Figure 2: Block diagram of a system.

Example 3. *Simplify the block diagram shown in Figure 3. Then obtain the closed-loop transfer function $C(s)/R(s)$.*

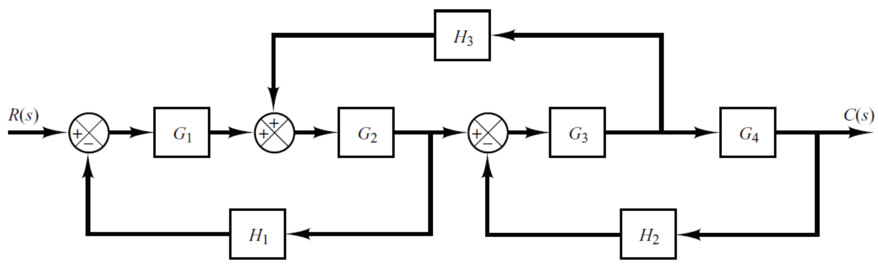

Figure 3: Block diagram of a system.

Example 4. *Obtain transfer functions $C(s)/R(s)$ and $C(s)/D(s)$ of the system shown in Figure 4.*

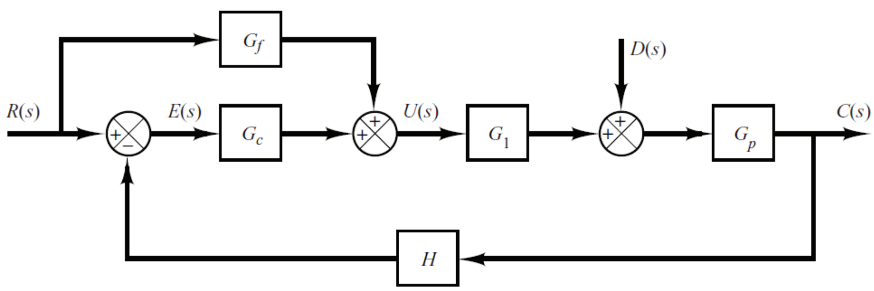

Figure 4: Control system with reference input and disturbance input.

Example 5. *Linearize the nonlinear equation*

$$z = x^2 + 4xy + 6y^2 \tag{1}$$

in the region defined by $8 \le x \le 10$, $2 \le y \le 4$.

Example 6. *Find a linearized equation for*

$$y = 0.2x^3 \tag{2}$$

about a point $x = 2$.

Solution 1. *Let*

$$C = B + D,\ D = AH_1,\ B = AG \tag{3}$$

then

$$C = AG + AH_1 = AG\left(1 + \frac{H_1}{G}\right) \tag{4}$$

First, move the branch point of the path involving H_1 outside the loop involving H_2, as shown in Figure 5(a). Also let

$$A = R - E,\ E = BH_2,\ B = AE \tag{5}$$

then

$$B = RG - EG = RG - BGH_2,\ \Rightarrow B(1 + GH_2) = RG,\ \Rightarrow B = \frac{G}{1 + GH_2}R \tag{6}$$

Then eliminating two loops results in Figure 5(b). Combining two blocks into one gives Figure 5(c). □

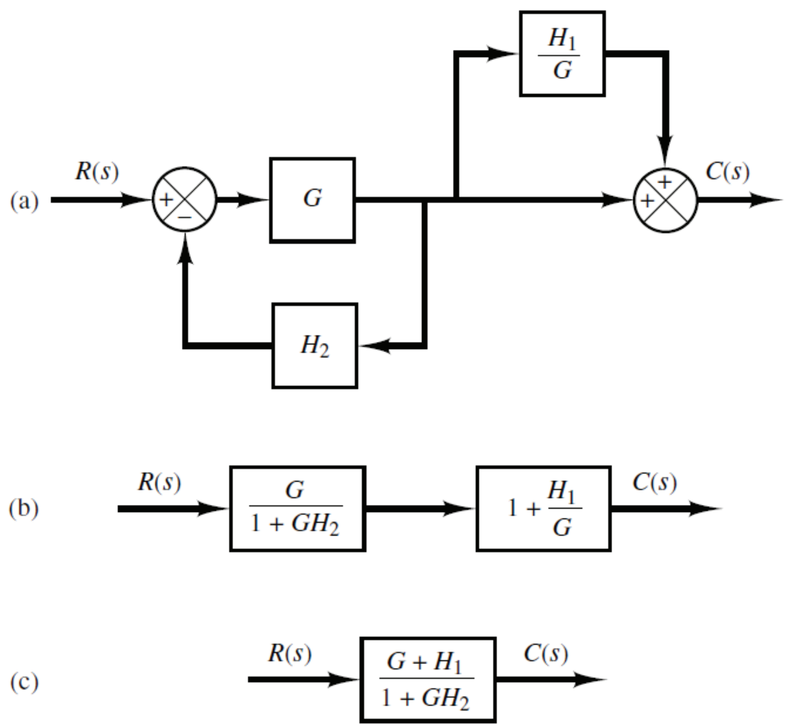

Figure 5: Simplified block diagrams for the system shown in Figure 1.

Solution 2. *The block diagram of Figure 2 can be modified to that shown in Figure 6(a). Eliminating the minor feedforward path, we obtain Figure 6(b), which can be simplified to Figure 6(c).*

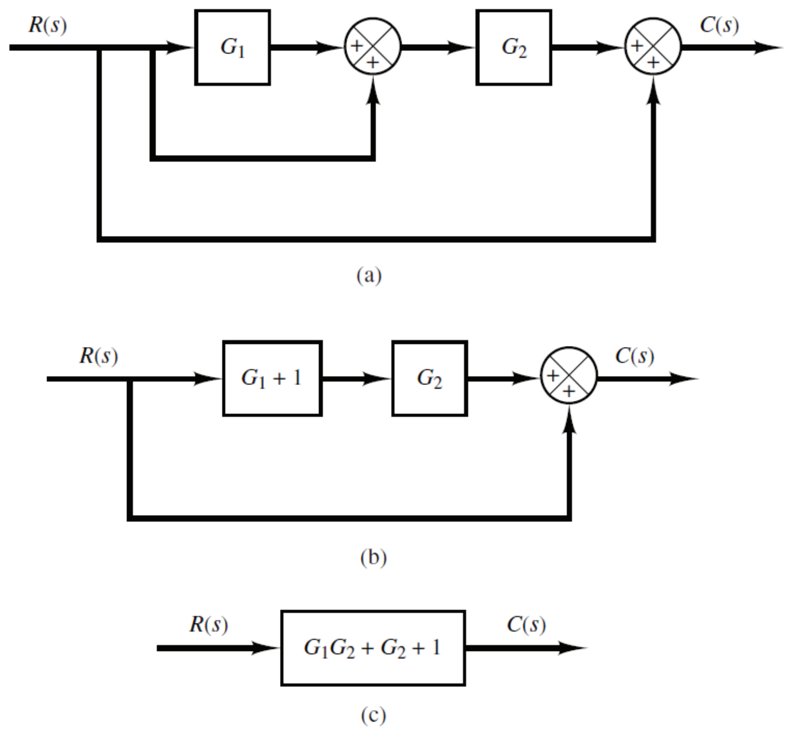

Figure 6: Simplified block diagrams for the system shown in Figure 2.

The transfer function $C(s)/R(s)$ is thus given by

$$\frac{C(s)}{G(s)} = G_1G_2 + G_2 + 1 \tag{7}$$

The same result can also be obtained by proceeding as follows: Since signal $X(s)$ is the sum of two signals $G_1R(s)$ and $R(s)$, we have

$$X(s) = G_1(s)R(s) + R(s) \tag{8}$$

The output signal $C(s)$ is the sum of $G_2X(s)$ and $R(s)$. Hence

$$C(s) = G_2(s)X(s) + R(s) = G_2(s)\left[G_1(s)R(s) + R(s)\right] + R(s) \tag{9}$$

And we have the same result as before:

$$\frac{C(s)}{G(s)} = G_1G_2 + G_2 + 1 \tag{10}$$

□

Solution 3. *First move the branch point between G_3 and G_4 to the right-hand side of the loop containing G_3, G_4, and H_2. Then move the summing point* *between G_1 and G_2 to the left-hand side of the first summing point. See Figure 7(a). By simplifying each loop, the block diagram can be modified as shown in Figure 7(b).*

The two blocks between the summation, let it be denoted as B, and $C(s)$ can be expressed as

$$\begin{aligned} C &= \frac{G_1G_2G_3G_4}{(1+G_1G_1H_1)(1+G_3G_4H_2)}B \\ &= \frac{G_1G_2G_3G_4}{1+G_1G_2H_1+G_3G_4H_2+G_1G_2G_3G_4H_1H_2}B \end{aligned} \tag{11}$$

Further, let $B = R + A$, where $A = H_3C/(G_1G_4)$, then $B = R + CH_3/(G_1G_4)$. Then

$$\begin{aligned} & C\left[\frac{1+G_1G_2H_1+G_3G_4H_2+G_1G_2G_3G_4H_1H_2}{G_1G_2G_3G_4} - \frac{H_3}{G_1G_4}\right] \\ &= C\left[\frac{1+G_1G_2H_1+G_3G_4H_2+G_1G_2G_3G_4H_1H_2-G_2G_3H_1}{G_1G_2G_3G_4}\right] = R \end{aligned} \tag{12}$$

Further simplification results in Figure 7(c).

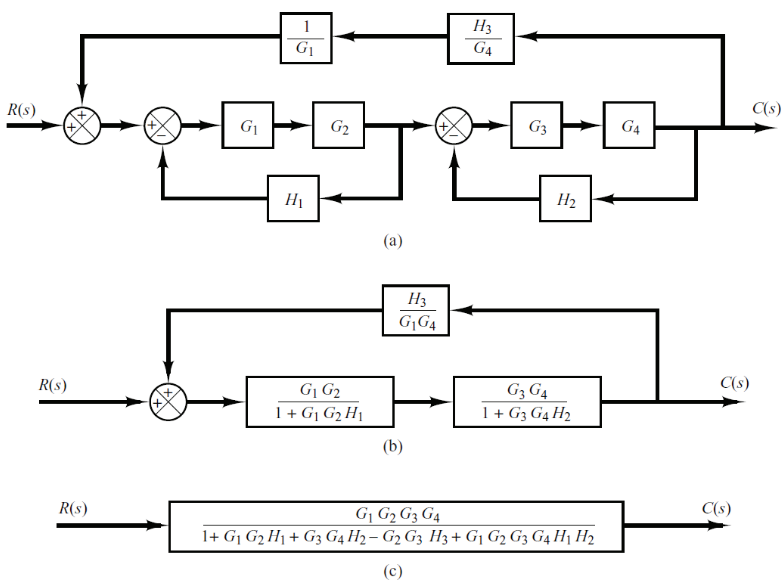

Figure 7: Simplified block diagrams for the system shown in Figure 3.

Thee closed-loop transfer function $C(s)/R(s)$ is obtained as

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_4G_4}{1+G_1G_1H_1+G_3G_4H_2-G_2G_3H_3+G_1G_2G_3G_4H_1H_2} \tag{13}$$

$\square$

Solution 4. *From Figure 4 we have*

$$U(s) = G_f(s)R(s) + G_c(S)E(s) \tag{14}$$

$$C(s) = G_p(s)\left[D(s) + G_1(s)U(s)\right] \tag{15}$$

$$E(s) = R(s) - HC(s) \tag{16}$$

By substituting Equation (14) into Equation (15), we get

$$C(s) = G_p(s)D(s) + G_1(s)G_p(s)\left[G_f(s)R(s) + G_cE(s)\right] \tag{17}$$

By substituting Equation (16) into Equation (17), we obtain

$$C(s) = G_pD(s) + G_1G_p\left\{G_fR(s) + G_c\left[R(s) - HC(s)\right]\right\} \tag{18}$$

Solving this last equation for $C(s)$, we get

$$C(s) + G_1G_pG_cHC(s) = G_pD(s) + G_1G_p\left(G_f + G_c\right)R(s) \tag{19}$$

Hence

$$C(s) = \frac{G_pD(s) + G_1G_p\left(G_f + G_c\right)R(s)}{1 + G_1G_pG_cH} \tag{20}$$

Note that Equation (20) gives the response $C(s)$ when both reference input $R(s)$ and disturbance input $D(s)$ are present. To find transfer function $C(s)/R(s)$, we let $D(s) = 0$ in Equation (20). Then we obtain

$$\frac{C(s)}{R(s)} = \frac{G_1G_p\left(G_f + G_c\right)}{1 + G_1G_pG_cH} \tag{21}$$

Similarly, to obtain transfer function $C(s)/D(s)$, we let $R(s) = 0$ in Equation (20). Then $C(s)/D(s)$ can be given by

$$\frac{C(s)}{D(s)} = \frac{G_p}{1 + G_1G_pG_cH} \tag{22}$$

□

Solution 5. *Define $f(x,y) = z = x^2 + 4xy + 6y^2$, then*

$$z = f(x,y) = f(\bar{x},\bar{y}) + \left[\frac{\partial f}{\partial x}(x-\bar{x}) + \frac{\partial f}{\partial y}(y-\bar{y})\right]_{x=\bar{x},y=\bar{y}} + \cdots \tag{23}$$

where we choose $\bar{x} = 9$, $\bar{y} = 3$. Since the higher-order terms in the expanded equation are small, neglecting these higher-order terms, we obtain

$$z = \bar{z} + K_1(x-\bar{x}) + K_2(y-\bar{y}) \tag{24}$$

where

$$K_1 = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x},y=\bar{y}} = 2\bar{x} + 4\bar{y} = 30 \tag{25}$$

$$K_2 = \left.\frac{\partial f}{\partial y}\right|_{x=\bar{x},y=\bar{y}} = 4\bar{x} + 12\bar{y} = 72 \tag{26}$$

$$\bar{z} = \bar{x}^2 + 4\bar{x}\bar{y} + 6\bar{y}^2 = 243 \tag{27}$$

thus

$$z = 243 + 30(x-9) + 72(y-3) \tag{28}$$

□

Solution 6. *Define* $y = 0.2x^3 = f(x)$, $\bar{x} = 2$. *Then*

$$y = f(x) = f(\bar{x}) + \frac{\partial f}{\partial x}(x - \bar{x}) + \cdots \tag{29}$$

Since the higher-order terms in this equation are small, neglecting these terms, we obtain

$$y - f(\bar{x}) = 0.6\bar{x}^2(x-\bar{x}), \; \Rightarrow y - 0.2 \times 2^3 = 0.6 \times 2^2(x-2) \tag{30}$$

Thus, linear approximation of the given nonlinear equation near the operating point is

$$y = 2.4x - 3.2 \tag{31}$$

□