LECTURE NOTES FOR

MMAN 3200

MODULE A - LINEAR SYSTEMS ANALYSIS

Written by R. Willgoss
Modified by Z. Vulovic

These notes consist of lectures 1 to 10 covering the syllabus taught. There may be some differences of text and examples compared to what is offered in the lectures.

MECH 3211 – Linear Systems

LECTURE 1: BEGINNINGS

This subject attempts to help students to master the modelling and analysis of engineering systems in a linear mode. The subject is a preparatory course for MECH 3212 – Principles of Control.

Systems in engineering are normally composed of components:

Electrical – Servo motors, generators.
Mechanical – Mass, springs, dampers.
Thermal – Heat low.
Fluidic – Reservoir dynamics.

The analysis presented here is not limited to the above, we can have others:

Medicine – Blood flow.
Biology – Drug dispersal.
Economics – Money-go-round.
Ecology – Pollutant dispersal.

THERE ARE BASICALLY 3 PHASES TO STUDYING AN ENGINEERING SYSTEM:

1. Modelling – Can we find an acceptable approximation?
2. Analysis – Understand the dynamics.
3. Modification – Can we alter the behaviour to achieve our desired effect?

WHY DO WE CONCENTRATE ON LINEAR SYSTEMS?

1. Most systems are NON-LINEAR but can be made LINEAR in a limited range of interest.
2. Linear theory is very well understood and gives good results.
3. Many engineering systems are linear in isolation.
4. Once LINEARITY is assumed, we can use LAPLACE.

SOME PROPERTIES AND CONCEPTS WE RELY UPON:

1. Define a TRANSFER FUNCTION such that in LAPLACE, the TRANSFER FUNCTION is independent of the magnitude and time envelope of the input.
2. Superposition holds. Analyse INPUT 1 to get OUTPUT 1 and then INPUT 2 to get OUTPUT 2 separately on the same system. We can then claim that if we provide INPUT 1 + INPUT 2 as an input we will get OUTPUT 1 + OUTPUT 2 as an output.
3. If a periodic signal is applied to the input, as $t \to \infty$, the output will end up with the same frequency components, only phase-shifted. i.e.

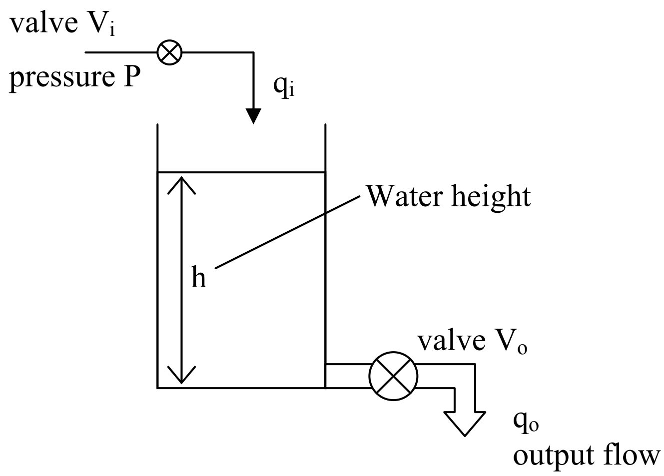
   Input: $A*\cos\omega t$; *Output*: $B*\cos(\omega t - \varphi)$

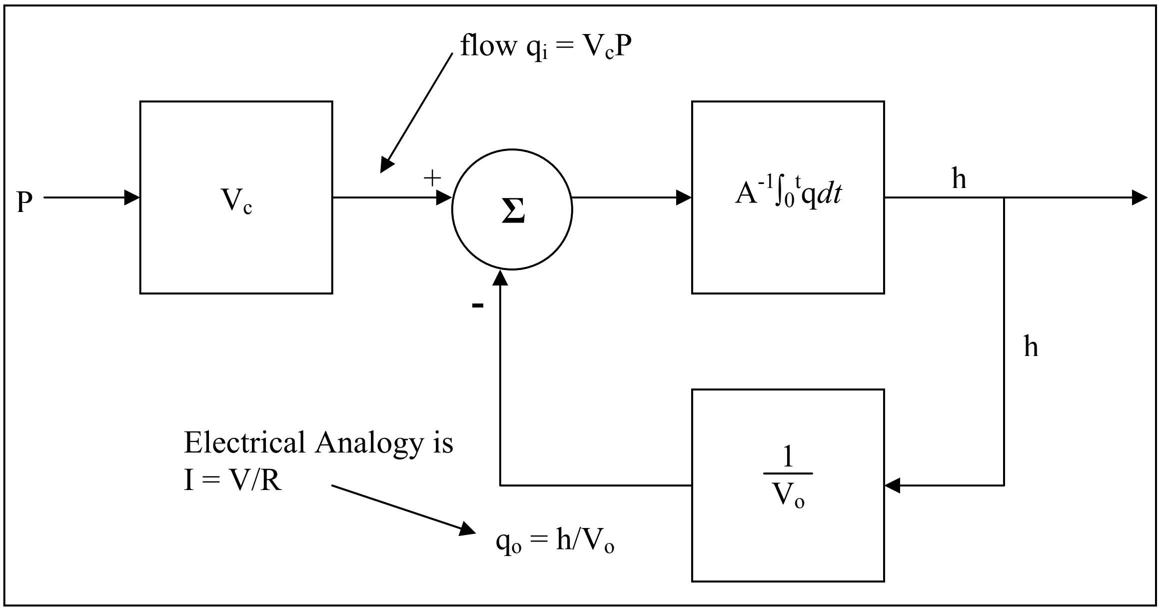
4. In order to mater linear systems analysis, we must learn to use BLOCK DIAGRAMS, REVISE LAPLACE ANALYSIS and derive equations for common engineering Components.

AN INITIAL INTRODUCTION TO BLOCK DIAGRAMS

Block diagrams are a convenient link between the physical world and a mathematical model. They consist of boxes representing component functions linked together by lines that pass variables, sometimes using summing junctions to compare, add, and subtract values. A box is the smallest unit of a TRANSFER FUNCTION. For example:

Consider the simple reservoir on the right and how we might model this as a block diagram.

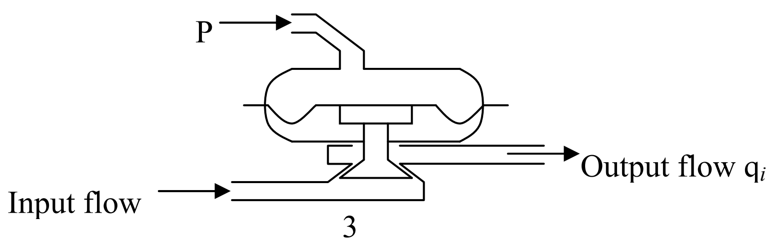




We say:

- h is the controlled variable
- V_c is the controlling variable [if P is constant]
- P is the controlling variable [if V_c is constant]

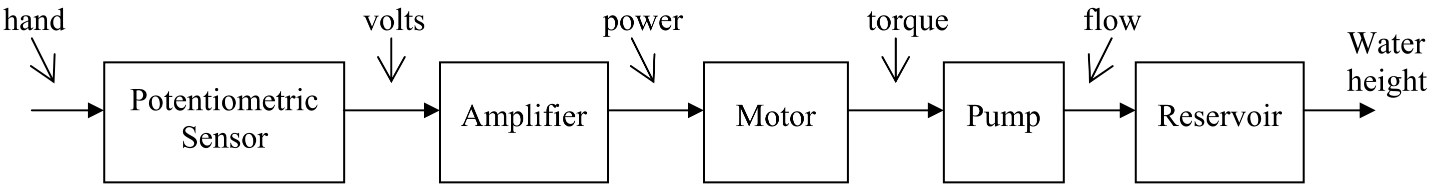
A design for the valve V_c could be as bellow:



When we exercise control in any system it is either OPEN LOOP or CLOSED LOOP.

OPEN LOOP CONTROL:

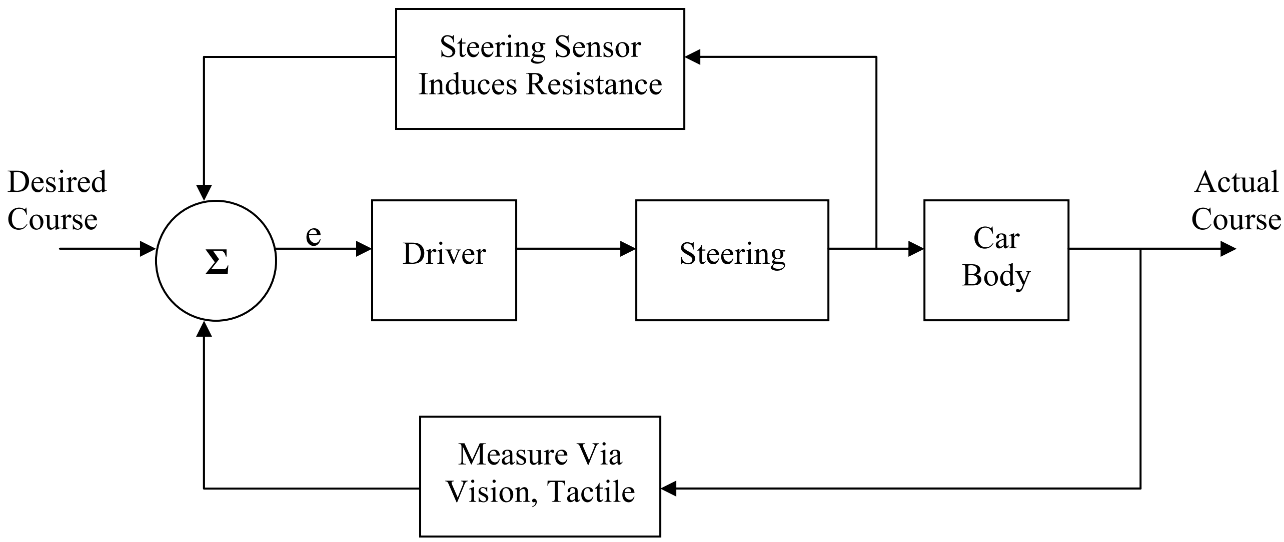
- No overt feedback path exists.
- The input is <u>not</u> readjusted with reference to the output.
- Often a series of boxes (transducing steps)



Errors result in the desired output due to:

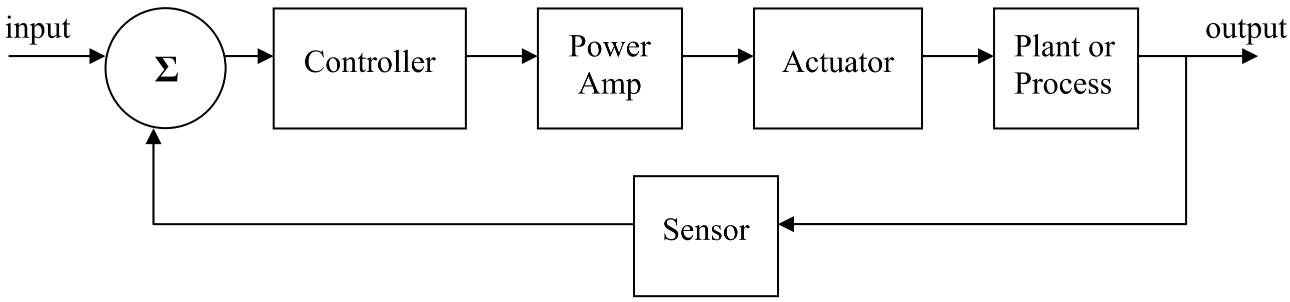
- Distortions; it rained, a pipe is blocked, power went low, etc.
- Parametric changes; the pump or motor is aging.

CLOSED LOOP CONTROL:



The error e tells the driver by how much the ACTUAL deviates from the Desired course.

A GENERAL FEEDBACK LOOP IN A SYSTEM (CLOSED LOOP):



Sensor	– Produces an analogue of the OUTPUT
Controller	– Compares INPUT with SENSOR and gives a low power OUTPUT
Power Amp	– Boosts the power available to the actuator
Actuator	– Transduces power into the desired parameter for the plant e.g. electrical to flow
Plant	– The main system we are controlling

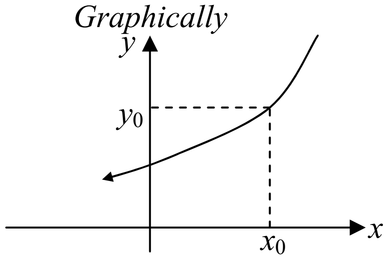
Linear systems Analysis is not directly concerned with control but being able to analyse the system first.

LINEARISATION

1. A mathematical procedure for creating linear equations from non-linear equations, including differentials.

2. The result is only a valid approximation of a model over a limited range of parameters used.

Graphically



Take any functional relationship.

$$y = f(x)$$

If we make a Taylor expansion around (x_0, y_0) we get:

$$y = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \ldots$$

Now simplify the expression down to a first order equation:

$$y = f(x_0) + (x - x_0)f'(x_0)$$

And we have a linearised approximation of y for $f(x)$ around x_0.

Example:

Linearise

$$q = K_1 v^2 + K_2 v \qquad \text{about } v_0$$

where

v = applied voltage
q = flow
K_1 & K_2 are constants

$$f(v_0) = K_1 v_0^2 + K_2 v_0$$

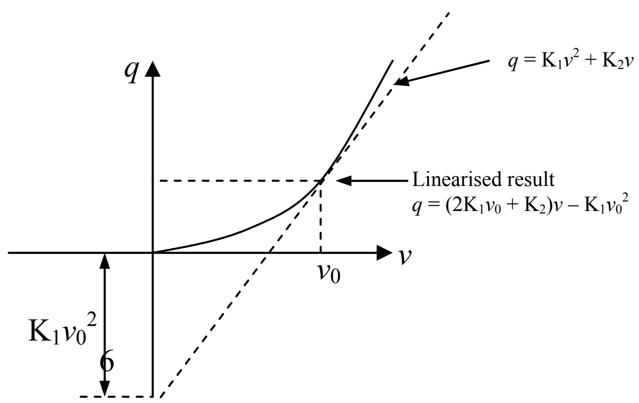
$$f'(v_0) = \frac{df(x)}{dx} = \frac{dq}{dv} = 2K_1 v_0 + K_2$$

$$q = K_1 v_0^2 + K_2 v_0 + (v - v_0)(2K_1 v_0 + K_2)$$

$$q = (2K_1 v_0 + K_2)v - K_1 v_0^2$$

which is linear.

Let us explore the results:



Firstly: As $v \rightarrow v_0$, then q is exactly $K_1 v_0^2 + K_2 v_0$

Secondly: The approximation is valid only around v_0

Thirdly: If we could not have calculated this result, we could have generated it if given a graph

FUNCTIONS OF MORE THAN ONE VARIABLE:

The linearised expression can be expanded into many dependencies by using (not proving)

$$y = f(x_{1_0}, x_{2_0}, \ldots x_{n_0}) + (x_1 - x_{1_0}) \frac{\partial f}{\partial x_1}(x_{1_0}, x_{2_0}, \ldots x_{n_0}) + (x_2 - x_{2_0}) \frac{\partial f}{\partial x_2}(x_{1_0}, x_{2_0}, \ldots x_{n_0}) + \ldots (x_n - x_{n_0}) \frac{\partial f}{\partial x_n}(x_{1_0}, x_{2_0}, \ldots x_{n_0})$$

INCREMENTAL EQUATIONS:

Very often we are only interested in incremental changes because DC shifts are removed (the operating point is zero). Hence our equation reduces to:

$\Delta y = f'(x_0)\, \Delta x$ around the point x_0

So for the previous example:

$\Delta q = K_2\, \Delta v$ around the point v_0

The incremental linearisation for more than one variable becomes:

$$\Delta y = \Delta x_1 \frac{\partial f}{x_1}(x_{1_0}, x_{2_0}, \ldots x_{n_0}) + \Delta x_2 \frac{\partial f}{x_2}(x_{1_0}, x_{2_0}, \ldots x_{n_0}) + \ldots$$

So, if A=f(B,C)

$$\Delta A = \left.\frac{\partial f}{\partial B}\right|_{B_0, C_0} \Delta B + \left.\frac{\partial f}{\partial C}\right|_{B_0, C_0} \Delta C \quad \text{around the point } (B_0, C_0)$$

DIFFERENTIAL EXPRESSIONS:

Differentials can also be used as variables:

Take $\dot{z}\theta + z = 0$ and create $\dot{z} = f(z, \theta)$

$\therefore\ \dot{z} = f(z, \theta) = -z/\theta$ and we linearise around z_0, θ_0

$f(x_0, y_0) = -z_0/\theta_0$

$f_x'(x_0, y_0) = \frac{\partial}{\partial z}(-z/\theta)_{zo,\, \theta o} = -1/\theta_0$

$$f_y'(x_0, y_0) = \frac{\partial}{\partial\theta}\ (-z/\theta)_{zo,\ \theta o} \qquad = z_0/\theta_0^2$$

$$\therefore\ \dot{z} = -z_0/\theta_0 - (\mathrm{z} - \mathrm{z}_0)\ 1/\theta_0 + (\theta - \theta_0)\ z_0/\theta_0^2$$

$$\therefore\ \dot{z}\ \theta_0^2 + z_0\theta_0 + (z - z_0)\ \theta_0 - (\theta - \theta_0)\ z_0 = 0$$

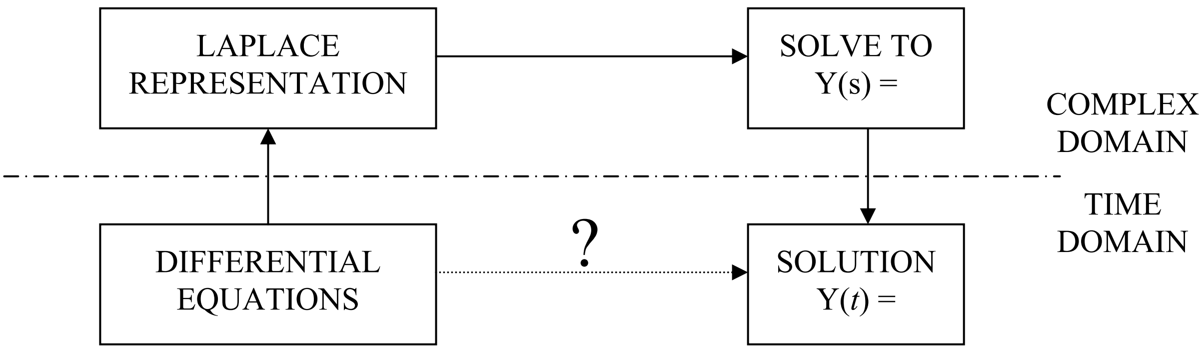
Rearranging

$\dot{z} = -z/\theta_0 + \theta z_0/\theta_0^2 - z_0/\theta_0$ is the linearised result

and is of the form

$y = \mathrm{m}_1 x + \mathrm{m}_2 y + (\mathrm{c}_1 + \mathrm{c}_2)$

LECTURE 2: LAPLACE TRANSFORMS



Transforming with Laplace is a means of simplifying the mathematics in solving differential equations so that we are doing only algebra, not dealing with differentiation. The "?" transition is often hard and sometimes impossible.

To create a LAPLACE TRANSFORM of f(t) we apply:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)dt$$

This way a new complex variable, s, is introduced, $s = \sigma + \mathbf{j}\omega$

Using an electrical circuit as an example, where input voltage $v_i(t)$ generates output voltage $v_0(t)$, we have:

$V_i(s) = \mathcal{L}\{v_i(t)\}$ and $V_o(s) = \mathcal{L}\{v_o(t)\}$

The expression G(*s*) = $V_o(s)/V_i(s)$ is the TRANSFER FUNCTION.

If we can specify G(s) (Describe the system in the complex domain) and choose an input $V_i(s)$ we can find $v_o(t)$ as:

$v_o(t) = \mathcal{L}^{-1}\{G(s)V_i(s)\}$ where $\mathcal{L}^{-1}\{\ldots\}$ means INVERSE LAPLACE TRANSFORM

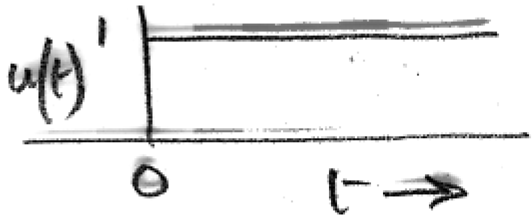
$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 1/(2\pi j)\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds,\ t > 0$$

Finding the inverse is normally achieved using tables and comparing the form of $G(s).V_i(s)$ with standard forms.

COMMON TRANSFORMS

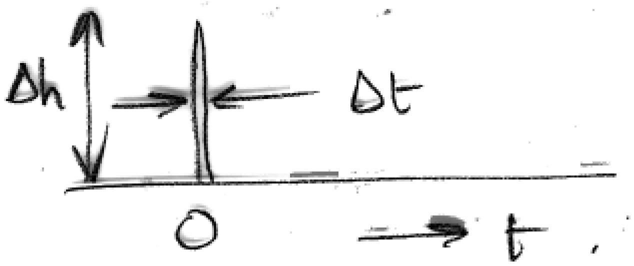
Often useful when describing an input to a system

1. Unit Step:



$$\mathcal{L}\{\mathrm{u}(t)\} = \int_0^\infty e^{-st}.1.dt \qquad = -e^{-st}/s\Big|_0^\infty \quad = 1/s$$

2. Unit Impulse:



Let $\Delta h\, \Delta t = 1$ (for a unit impulse)

$$\mathcal{L}\{\delta(t)\} \qquad = \int_0^{\Delta t} e^{-st}.\Delta h.dt$$

$$= -\Delta h.e^{-st}/s\Big|_0^{\Delta t}$$

$$= (-\Delta h\,/s)(e^{-s\Delta t} - 1)$$

Approximate

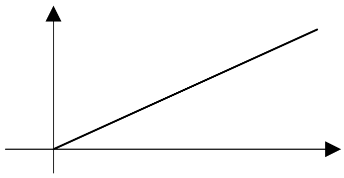
$$e^{-s\Delta t} = 1 - s\Delta t + (s\Delta t)^2/2! - \ldots \qquad = -\Delta h/s\,(1 - s\Delta t - 1)$$

$$= \Delta h \Delta t \qquad = 1$$

3. Ramp of slope A:

$$\mathcal{L}\{\mathrm{A}t\} = \int_0^\infty e^{-st}.\mathrm{A}t.dt \;=\; -\mathrm{A}t.e^{-st}/s\Big|_0^\infty \quad - (- \int_0^\infty \mathrm{A}.e^{-st}/s.dt)$$

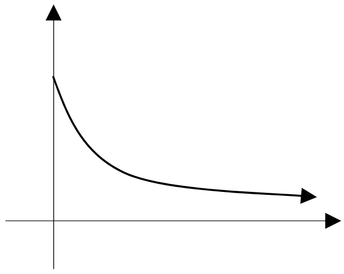
(remember partial integration: $\int u dv = uv - \int v du$)



$= -A/s\ (\infty \text{ x } 0 - 0 \text{ x } 1) - A/s^2\ (0 - 1)$

$= A/s^2$

4. Exponentials



$$\mathcal{L}\{Ae^{-at}\} = \int_0^\infty e^{-st}.\, Ae^{-at}.dt$$

$$= A \int_0^\infty e^{-(a+s)t} dt$$

$$= -[A/(s + a)]\ (e^{-(a+s)t}) \Big|_0^\infty$$

$$= A/(s + a)$$

n.b. 1. Don't go back to basics here but use tables.
2. Note differentiation has been transformed to algebra in *s*.
3. Tables are essential for finding inverses.

5. Derivatives

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t).dt$$

$$= e^{-st} f(t) \Big|_0^\infty + s \int_0^\infty e^{-st} f(t)dt$$

$$= -f(0) + s\,F(s)$$ where f(0) is the function evaluated at the point t=0

If the differential is more than first order, we can continue the process to get the general result.

$$\mathcal{L}\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$$

If all the initial conditions are zero i.e. f(0) = f '(0) = … = 0 then taking the transform of a differential equation is like multiplying by s instead of differentiating. E.g.

$K\,\theta_i\,(t) = A\,\ddot{\theta}_0\,(t) + B\,\dot{\theta}_0\,(t) + C\,\theta_0\,(t)$ becomes
$K\,\theta_i(s) = (As^2 + Bs + C)\,\theta_0(s)$

6. Integrals

Suppose that $g(t) = \int_0^t f(\tau)d\tau$, so that $g'(t) = f(t)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = sG(s) - g(0)$

However from the definition of g(t) above, g(0) → 0

$\mathcal{L}\{\int f(t).dt\} = \mathcal{L}\{g(t)\}$

$= G(s)$

$= 1/s\,\mathcal{L}\{g'(t)\}$

$= 1/s\,\mathcal{L}\{f(t)\}$

$= F(s)/s$

This integration is equivalent to dividing by s.

7. The Initial Value Theorem.

We can find f(0) by using:

$\lim_{t\to o} f(t) = \lim_{s\to\infty}[sF(s)] = f(0)$, a limiting value at the beginning.

8. The Final Value Theorem.

We can find f(∞) by using:

$\lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)] = f(\infty)$, which is also called a steady state value.

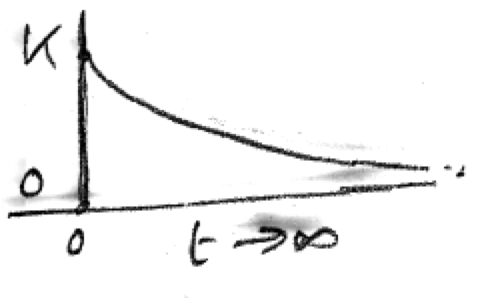
7. & 8. Can be used as a partial check on whether you have transformed correctly e.g.

if $f(t) = Ke^{-\alpha t}$ then $F(s) = K/(s + \alpha)$

Thus

$$\lim_{t \to 0} f(t) \quad = \lim_{s \to \infty} sK/(s + \alpha) \quad = \lim_{s \to \infty} K/(1 + \alpha/s) \quad = K \qquad \text{QED}$$

$$\lim_{t \to \infty} f(t) \quad = \lim_{s \to 0} sK/(s + \alpha) \quad = 0 \qquad \text{QED}$$



LAPLACE SHIFT THEOREMS

<u>Complex domain shift</u>: if the function in the time domain contains an exponential, we can simplify the transform by using:

$$\mathcal{L}\{e^{-at}.f(t)\} \quad = F(s + a)$$

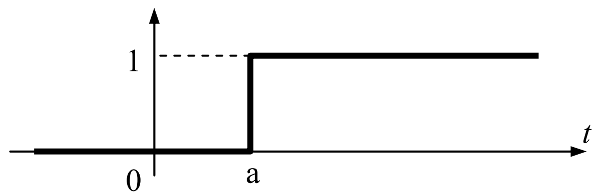
Similarly

$$\mathcal{L}\{e^{-bt}.\sin\omega t\} \;=\; \frac{\omega}{(s + b)^2 + \omega^2}$$

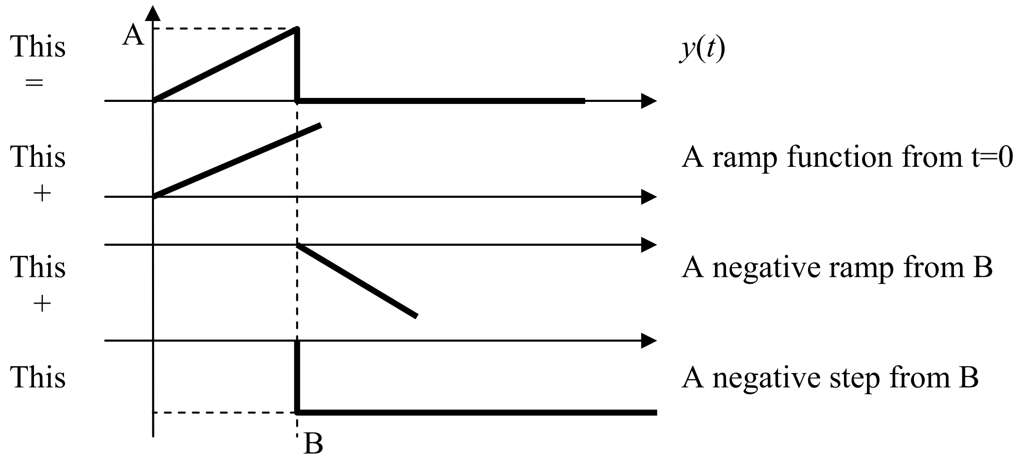
<u>Time domain shift</u>: if the function in the complex domain contains an exponential, we can simplify the inverse transform by using:

$$\mathcal{L}^{-1}\{e^{-as}.F(s)\} = u(t - a)\, f(t - a)$$

This says that the Laplace transform can be found for a signal that does not start at t = 0. Remember u(t – a) is:



We can combine wave shapes to generate a composite shape.



The composite function in Laplace is:

$$\underset{t=0\text{ to }\infty}{\mathcal{L}\{{}^{A}/_{B}t\}} \qquad -\underset{t=B\text{ to }\infty}{\mathcal{L}\{{}^{A}/_{B}t\}} \qquad -\underset{t=B\text{ to }\infty}{\mathcal{L}\{A\}}$$

$$= \frac{A}{Bs^2} - \frac{e^{-Bs}A}{Bs^2} - \frac{e^{-Bs}A}{s}$$

$$\therefore \quad \mathcal{L}\{y(t)\} \qquad = \frac{A}{Bs^2}\left[1 - e^{-bs}(1 + Bs)\right]$$

PARTIAL FRACTIONS

Once a function in the Laplace domain has been obtained, we need to consider specifying it in its simplest form to aid inversion. Creating partial fractions usually achieves this aim.

Note $\mathcal{L}\{a\,f(t) + b\,g(t)\} \qquad = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}$

In words this means that if the constituent parts of a system add together, we can transform them independently. Similarly for the inverse transform. $\mathcal{L}$ is a linear operator!

Any complex-domain function has the form N(s)/D(s) where D(s) has components like $(s + d_1)(s + d_2)(s^2 + d_3s + d_4)$ etc. in which case the partial fraction expansion would look like:

$$\frac{N(s)}{D(s)} = \frac{A}{(s + d_1)} + \frac{B}{(s + d_2)} + \frac{Cs + D}{s^2 + d_3s + d_4}$$

If the root is repeated n times, we must include the terms

$$\frac{A_1}{(s + d_1)} + \frac{A_2}{(s + d_1)^2} + \dots + \frac{A_n}{(s + d_1)^n}$$

Examples:

1. Determine the Partial Fraction of $F(s) = (s + 3)/(s^2 + 3s + 2)$

$= (s + 3)/[(s + 2)(s + 1)]$

$\therefore \quad F(s) = A/(s + 2) + B/(s + 1)$

$\therefore \quad (s + 3)/ [(s + 2)(s + 1)] = [A(s + 1) + B(s + 2)] /[(s + 2)(s + 1)]$

$\therefore \quad (s + 3) = A(s + 1) + B(s + 2)$

Now we can equate the coefficients with s^1 or s^0 to obtain two equations and solve them for A and B.

An alternative way:

$s = -1 \quad \therefore \quad 2 = B$

$s = -2 \quad \therefore \quad 1 = -A$

$\therefore \quad F(s) = 2/(s + 1) - 1/(s + 2)$

$\therefore \quad f(t) = (2e^{-t} - e^{-2t})\,u(t)$

2. Determine the Partial Fraction of $F(s) = 2(s + 6)/[s(s + 2)(s + 4)]$

$$F(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4} = \frac{A(s+2)(s+4)+Bs(s+4)+Cs(s+2)}{s(s+2)(s+4)}$$

Equate the numerators $\quad 2(s + 6) = A(s + 2)(s + 4) + Bs(s + 4) + Cs(s + 2)$

For $s = 0 \quad 12 = 8A \quad \therefore A = 1.5$

$s = -2 \quad 8 = -4B \quad \therefore B = -2$

$s = -4 \quad 4 = 8C \quad \therefore C = 0.5$

$F(s) = 1.5/s - 2/(s + 2) + 0.5/(s + 4)$

Now find $f(t)$.

3. What is the unit step response from a system where the transfer function is:

$G(s) = 5(1 + 0.4s)/[(s + 1)(0.2s + 1)] \quad = (10s + 25)/[(s + 1)(s + 5)]$

$$G(s) = \frac{V_0(s)}{V_i(s)}$$

The output is $V_o(s)$ and the input is: $V_i(s) = \mathcal{L}\{u(t)\} = 1/s$

$$V_o = G(s)\,V_i(s) = (10s + 25)/[s(s + 1)(s + 5)]$$

$$= A/s + B/(s + 1) + C/(s + 5)$$

$$10s + 25 = A(s + 1)(s + 5) + Bs(s + 5) + Cs(s + 1)$$

Make $s = 0$ $\quad 25 = 5A \quad A = 5$

$s = -1 \quad 15 = -4B \quad B = -3.75$

$s = -5 \quad -25 = 20C \quad C = -1.25$

$$V_o(s) = 5/s - 3.75/(s + 1) - 1.25/(s + 5)$$

$$v(t) = (5 - 3.75e^{-t} - 1.25e^{-5t})\,u(t)$$

4. Calculate the response to a decaying exponential input $v_i(t) = e^{-t}$ of a system whose transfer function is

$$G(s) = 3(s^2 + 9s + 18)/(s + 4)(s + 2)$$

We know $V_i(s) = \mathcal{L}\{e^{-t}\} = 1/(s + 1)$

$$V_o(s) = G(s)V_i(s)$$

$$V_o(s) = 3(s^2 + 9s + 18)/[(s+1)(s + 4)(s + 2)]$$

Into partial fractions:

$$3(s^2 + 9s + 18) = A(s + 4)(s + 2) + B(s + 1)(s + 2) + C(s + 1)(s + 4)$$

$s = -1 \quad 30 = 3A \quad A = 10$

$s = -4 \quad -6 = 6B \quad B = -1$

$s = -2 \quad 12 = -2C \quad C = -6$

$$V(s) = 10/(s + 1) - 1/(s + 4) - 6/(s + 2)$$

$$v_o(t) = (10e^{-t} - e^{-4t} - 6e^{-2t})\,u(t)$$

5. If a function is given as: $Y(s) = (2s^2 + 6s + 6)/[(s + 2)^2(s^2 + 2s + 2)]$ find the partial fraction expansion.

We must generate:

$$Y(s) = A/(s + 2) + B/(s + 2)^2 + (Cs + D)/(s^2 + 2s + 2)$$

$$2s^2 + 6s + 6 = A(s + 2)(s^2 + 2s + 2) + B(s^2 + 2s + 2) + (Cs + D)(s + 2)^2$$

Equate the coefficients:

of s^3,	$0 = A + C$
of s^2,	$2 = 2A + 2A + B + 4C + D$
of s^1,	$6 = 2A + 4A + 2B + 4C + 4D$
of s^0,	$6 = 4A + 2B + 4D$

Four equations with four unknowns yields: A = 0, B = 1, C = 0, D = 1

Hence $Y(s) = 1/(s + 2)^2 + 1/(s^2 + 2s + 2)$

case 7 case 22

We can find the time domain equivalent

$$Y(s) = 1/(s + 2)^2 + ½\ 2/[s^2 + 2\ \surd 2(1/\surd 2)\ s + (\surd 2)^2]$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = (te^{-2t} + e^{-t}\sin t)\ u(t)$$

Solutions Using the Method of Residues

The residue R_i, the i^{th} numerator of the partial fraction expansion of a function F(s) is given by:

$$F(s).(s + a_i)\Big|_{s=-a_i}$$ where $-a_i$ is the root of the denominator.

Example:

Say $$F(s) = \frac{3s^2 + 2s + 1}{(s + 1)(s^2 + 12s + 32)}$$

$$= \frac{3s^2 + 2s + 1}{(s + 1)(s + 4)(s + 8)}$$

This will expand into:

$$F(s) = \frac{R_1}{(s + 1)} + \frac{R_2}{(s + 4)} + \frac{R_3}{(s + 8)}$$

$$\therefore \quad R_1 = \frac{(3s^2 + 2s + 1)\cancel{(s + 1)}}{\cancel{(s + 1)}(s + 4)(s + 8)}\Bigg|_{s=-1} = \frac{3 - 2 + 1}{3 \times 7} = \frac{2}{21}$$

$$\therefore \quad R_2 = \frac{(3s^2 + 2s + 1)}{(s + 1)(s + 8)}\Bigg|_{s=-4} = \frac{48 - 8 + 1}{-3 \times 4} = \frac{-41}{12}$$

$$\therefore \quad R_3 = \left.\frac{(3s^2 + 2s + 1)}{(s+1)(s+4)}\right|_{s=-8} = \frac{(3 \text{ x } 64) - 16 + 1}{-7 \text{ x } -4} = \frac{177}{28}$$

$$\therefore \quad F(s) = \frac{2}{21(s+1)} - \frac{41}{12(s+4)} + \frac{177}{28(s+8)}$$

The Heaviside's Expansion Theorem

To find $\mathcal{L}^{-1}\{P(s)/Q(s)\}$ we use:

$$\sum_{i=1}^{n} \left.\frac{P(s)}{Q'(s)}\right|_{s=\alpha_i} \cdot e^{\alpha_i t}$$

Example:

If $F(s) = \dfrac{(s+2)}{(s+3)(s+1)} = \dfrac{P(s)}{Q(s)}$

$$Q'(s) = \frac{d}{ds}(s^2 + 4s + 3) = 2s + 4$$

$$\therefore \quad \mathcal{L}^{-1}\{F(s)\} = \left.\frac{s+2}{2s+4}\right|_{s=-3} .e^{-3t} + \left.\frac{s+2}{2s+4}\right|_{s=-1} .e^{-t} = ½\, e^{-3t} + ½\, e^{-t}$$

Let's confirm this result by using inverse tables. We find:

$$\mathcal{L}^{-1}\left\{\frac{s + c}{(s+a)(s+b)}\right\} = \frac{(c - a).e^{-at} - (c - b).e^{-bt}}{b - a}$$

With a = 3, b = 1, c = 2

$$f(t) = \frac{-1e^{-3t} - 1e^{-t}}{-2} = (½\, e^{-3t} + ½\, e^{-t})\, u(t) \qquad \text{QED}$$