Stability

1 Stability

The most important problem in linear control systems concerns stability. That is, under what conditions will a system become unstable? If it is unstable, how should we stabilize the system? A control system is stable if and only if all closed-loop poles lie in the left-half s-plane. Most linear closed-loop systems have closed-loop transfer functions of the form

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = \frac{B(s)}{A(s)} \tag{1}$$

where the a's and b's are constants and $m \leq n$. A simple criterion, known as Routh's stability criterion, enables us to determine the number of closed-loop poles that lie in the right-half s plane without having to factor the denominator polynomial.

1.1 Routh's Stability Criterion

Routh's stability criterion tells us whether or not there are unstable roots in a polynomial equation without actually solving for them. This stability criterion applies to polynomials with only a finite number of terms. When the criterion is applied to a control system, information about absolute stability can be obtained directly from the coefficients of the characteristic equation. The procedure in Routh's stability criterion is as follows:

1. Write the polynomial in s in the following form:

   $$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0 \tag{2}$$

   where the coefficients are real quantities.We assume that $a_n \neq 0$; that is, any zero root has been removed.

2. If any of the coefficients are zero or negative in the presence of at least one positive coefficient, a root or roots exist that are imaginary or that have positive real parts. Therefore, in such a case, the system is not stable.

   If we are interested in only the absolute stability, there is no need to follow the procedure further. Note that all the coefficients must be positive.This is a necessary condition, as may be seen from the following argument:

(a) A polynomial in s having real coefficients can always be factored into linear and quadratic factors, such as $(s + a)$ and $(s^2 + bs + c)$, where a, b, and c are real.

(b) The linear factors yield real roots and the quadratic factors yield complex-conjugate roots of the polynomial.

(c) The factor $(s^2 + bs + c)$ yields roots having negative real parts only if b and c are both positive.

(d) For all roots to have negative real parts, the constants a, b, c, and so on, in all factors must be positive.

(e) The product of any number of linear and quadratic factors containing only positive coefficients always yields a polynomial with positive coefficients.

It is important to note that the condition that all the coefficients be positive is not sufficient to assure stability. The necessary but not sufficient condition for stability is that the coefficients of Equation (2) all be present and all have a positive sign. If all a's are negative, they can be made positive by multiplying both sides of the equation by -1.

3. If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern:

$$\begin{array}{lllll}
s^n & a_0 & a_2 & a_4 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & \cdots \\
s^{n-4} & d_1 & d_2 & d_3 & \cdots \\
\vdots & \vdots & \vdots & & \\
s^2 & e_1 & e_2 & & \\
s^1 & f_1 & f_2 & & \\
s^0 & g_1 & & &
\end{array}$$

4. The process of forming rows continues until we run out of elements. That is, the total number of rows is $n + 1$. The coefficients b_1, b_2, b_3 , and so on, are evaluated as follows:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}, \; b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, \; b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}, \; \cdots \tag{3}$$

5. The evaluation of the b's is continued until the remaining ones are all zero. The same pattern of cross-multiplying the coefficients of the two previous rows is followed in evaluating the c's, d's, e's, and so on. That is,

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, \; c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}, \; c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}, \; \cdots \tag{4}$$

and

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1},\ d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1},\ \cdots \tag{5}$$

This process is continued until the nth row has been completed. The complete array of coefficients is triangular.

Routh's stability criterion states that:

1. The number of roots of Equation (2) with positive real parts is equal to the number of changes in sign of the coefficients of the first column of the array. It should be noted that the exact values of the terms in the first column need not be known; instead, only the signs are needed.

2. The necessary and sufficient condition that all roots of Equation (2) lie in the left-half s-plane is that all the coefficients of Equation (2) be positive and all terms in the first column of the array have positive signs.

1.1.1 Special Cases

1. If a first-column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then the zero term is replaced by a very small positive number ϵ and the rest of the array is evaluated. For example, consider the following equation:

$$s^3 + 2s^2 + s + 2 = 0 \tag{6}$$

The array of coefficients is

$$\begin{array}{lll} s^1 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & 0 \approx \epsilon & \\ s^0 & 2 & \end{array}$$

If the sign of the coefficient above the zero (ϵ) is the same as that below it, it indicates that there are a pair of imaginary roots. Actually, Equation (6) has two roots at $s = \pm j$.

If, however, the sign of the coefficient above the zero (ϵ) is opposite that below it, it indicates that there is one sign change. For example, for the equation

$$s^3 - 3s + 2 = (s-1)^2(s+2) = 0 \tag{7}$$

the array of coefficients is

$$\begin{array}{lll} s^3 & 1 & \text{-3} \\ s^2 & 0 \approx \epsilon & 2 \\ s^1 & -3 - 2/\epsilon & \\ s^0 & 2 & \end{array}$$

There are two sign changes of the coefficients in the first column. So there are two roots in the right-half s-plane. This agrees with the correct result indicated by the factored form of the polynomial equation.

2. If all the coefficients in any derived row are zero, it indicates that there are roots of equal magnitude lying radially opposite in the s-plane, that is, two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots.

   In such a case, the evaluation of the rest of the array can be continued by forming an *auxiliary polynomial* with the coefficients of the last row and by using the coefficients of the derivative of this polynomial in the next row.

   Such roots with equal magnitudes and lying radially opposite in the s-plane can be found by solving the auxiliary polynomial, which is always even.

   For a $2n$-degree auxiliary polynomial, there are n pairs of equal and opposite roots. For example, consider the following equation:

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0 \tag{8}$$

   The array of coefficients is

$$\begin{array}{cccc} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \\ s^3 & 0 & 0 & \end{array}$$

   The terms in the s^3 row are all zero. Note that such a case occurs only in an odd numbered row. The auxiliary polynomial is then formed from the coefficients of the s^4 row. The auxiliary polynomial $P(s)$ is

$$P(s) = 2s^4 + 48s^2 - 50 \tag{9}$$

   which indicates that there are two pairs of roots of equal magnitude and opposite sign, that is, two real roots with the same magnitude but opposite signs or two complex-conjugate roots on the imaginary axis). These pairs are obtained by solving the auxiliary polynomial equation $P(s) = 0$. The derivative of $P(s)$ with respect to s is

$$\frac{dP(s)}{ds} = 8s^3 + 96s \tag{10}$$

   The terms in the s^3 row are replaced by the coefficients of the last equation, that is, 8 and 96. The array of coefficients then becomes

s^5	1	24	-25
s^4	2	48	-50
s^3	8	96	
s^2	24	-50	
s^1	112.7	0	
s^0	-50		

We see that there is one change in sign in the first column of the new array. Thus, the original equation has one root with a positive real part. By solving for roots of the auxiliary polynomial equation,

$$2s^4 + 48s^2 - 50 = 0 \tag{11}$$

we obtain

$$s^2 = 1,\ s^2 = 25,\ \Rightarrow s = \pm 1,\ s = \pm j5 \tag{12}$$

These two pairs of roots of $P(s)$ are a part of the roots of the original equation. As a matter of fact, the original equation can be written in factored form as follows:

$$(s+1)(s-1)(s+j5)(s-j5)(s+2) = 0 \tag{13}$$

Clearly, the original equation has one root with a positive real part.

1.2 Application of Routh's Stability Criterion to Control-System Analysis

It is possible to determine the effects of changing one or two parameters of a system by examining the values that cause instability. In the following, we shall consider the problem of determining the stability range of a parameter value.

Consider the system shown in Figure 1.

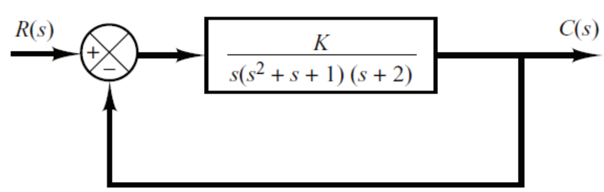

Figure 1: Control system.

Let us determine the range of K for stability. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2+s+1)(s+2)+K} \tag{14}$$

The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0 \tag{15}$$

The array of coefficients becomes

$$\begin{array}{llll}
s^4 & 1 & 3 & K \\
s^3 & 3 & 2 & 0 \\
s^2 & \frac{7}{3} & K & \\
s^1 & \frac{2\times 7-9K}{7} & & \\
s^0 & \text{K} & &
\end{array}$$

For stability, K must be positive, and all coefficients in the first column must be positive. Therefore,

$$\frac{14}{9} > K > 0 \tag{16}$$

When $K = 14/9$ the system becomes oscillatory and, mathematically, the oscillation is sustained at constant amplitude. Note that the ranges of design parameters that lead to stability may be determined by use of Routh's stability criterion.

Example 1. *Consider the following characteristic equation:*

$$s^4 + Ks^3 + s^2 + s + 1 = 0 \tag{17}$$

Determine the range of K for stability.

Solution 1. *The Routh array of coefficients is*

$$\begin{array}{llll}
s^4 & 1 & 1 & 1 \\
s^3 & K & 1 & 0 \\
s^2 & \frac{K-1}{K} & 1 & \\
s^1 & 1-\frac{K^2}{K-1} & & \\
s^0 & 1 & &
\end{array}$$

For stability, we require that

$$K > 0,\ \frac{K-1}{K} > 0,\ 1 - \frac{K^2}{K-1} > 0 \tag{18}$$

From the first and second conditions, K must be greater than 1. For $K > 1$, notice that the term $1 - [K^2/(K-1)]$ is always negative, since

$$\frac{K-1-K^2}{K-1} = \frac{-1+K(1-K)}{K-1} < 0 \tag{19}$$

Thus, the three conditions cannot be fulfilled simultaneously.Therefore, there is no value of K that allows stability of the system.

Example 2. *Determine the range of K for stability of a unity feedback control system whose open-loop transfer function is*

$$G(s) = \frac{K}{s(s+1)(s+2)} \tag{20}$$

Solution 2. *The closed-loop transfer function is*

$$\frac{C(s)}{R(s)} = \frac{K}{s(s+1)(s+2)+K} \tag{21}$$

The characteristic equation is

$$s^3 + 3s^2 + 2s + K = 0 \tag{22}$$

The Routh array becomes

$$\begin{array}{lll} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \frac{6-K}{3} & \\ s^0 & \text{K} & \end{array}$$

For stability, we require $6 > K$ *and* $K > 0$*, or* $6 > K > 0$.

Example 3. *Consider the following characteristic equation:*

$$s^4 + 2s^3 + (4+K)s^2 + 9s + 25 = 0 \tag{23}$$

Using the Routh stability criterion, determine the range of K *for stability.*

Solution 3. *The Routh array is*

$$\begin{array}{llll} s^4 & 1 & 4+K & 25 \\ s^3 & 2 & 9 & 0 \\ s^2 & \frac{2K-1}{2} & 25 & \\ s^1 & \frac{18K-109}{2K-1} & 0 & \\ s^0 & 25 & & \end{array}$$

For stability, we require

$$\frac{2K-1}{2} > 0, \ \frac{18K-109}{2K-1} > 0 \tag{24}$$

that is

$$K > 0.5, \ 18K > 109 \tag{25}$$

Hence, $K > 109/18 = 6.056$

Example 4. *Consider a unity-feedback system, the open-loop transfer function is*

$$G(s) = \frac{K(s-2)}{(s+1)(s^2+6s+25)}, \ K > 0 \tag{26}$$

Determine the range of K *for stability.*

Solution 4. *The closed-loop transfer function is*

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{K(s-2)}{(s+1)(s^2+6s+25)+K(s-2)} \\ &= \frac{K(s-2)}{s^3+7s^2+(31+K)s+(25-2K)}\end{aligned} \tag{27}$$

The Routh array is

$$\begin{array}{lll} s^3 & 1 & 31+K \\ s^2 & 7 & 25-2K \\ s^1 & \frac{192+9K}{7} & 0 \\ s^0 & \text{25-2K} & \end{array}$$

Since we had assumed $K > 0$*, for stability, we require* $25-2K > 0$*, i.e.,* $12.5 > K > 0$*.*