Fluid and Thermal System Models

As the most versatile medium for transmitting signals and power, fluids, liquids and gases have wide usage in industry. Liquids and gases can be distinguished basically by their relative incompressibilities and the fact that a liquid may have a free surface, whereas a gas expands to fill its vessel. In the engineering field the term pneumatic describes fluid systems that use air or gases and hydraulic applies to those using oil.

1 Liquid-Level Systems

In analyzing systems involving fluid flow, we find it necessary to divide flow regimes into laminar flow and turbulent flow, according to the magnitude of the Reynolds number. If the Reynolds number is greater than about 3000 to 4000, then the flow is turbulent. The flow is laminar if the Reynolds number is less than about 2000. In the laminar case, fluid flow occurs in streamlines with no turbulence. Systems involving laminar flow may be represented by linear differential equations.

Industrial processes often involve flow of liquids through connecting pipes and tanks. The flow in such processes is often turbulent and not laminar. Systems involving turbulent flow often have to be represented by nonlinear differential equations. If the region of operation is limited, however, such nonlinear differential equations can be linearized.

1.1 Resistance of Liquid-Level Systems

Consider the flow through a short pipe connecting two tanks. The resistance R for liquid flow in such a pipe or restriction is defined as the change in the level difference (the difference of the liquid levels of the two tanks) necessary to cause a unit change in flow rate; that is,

$$R = \frac{\text{change in level difference, } m}{\text{change in flow rate, } m^3/sec} \tag{1}$$

Consider the liquid-level system shown in Figure 1(a). In this system the liquid spouts through the load valve in the side of the tank. If the flow through this restriction is laminar, the relationship between the steady-state flow rate and steady-state head at the level of the restriction is given by

$$Q = KH \tag{2}$$

where Q = steady-state liquid flow rate, m^3/sec, K = coefficient, m^2/sec, H = steady-state head, m.

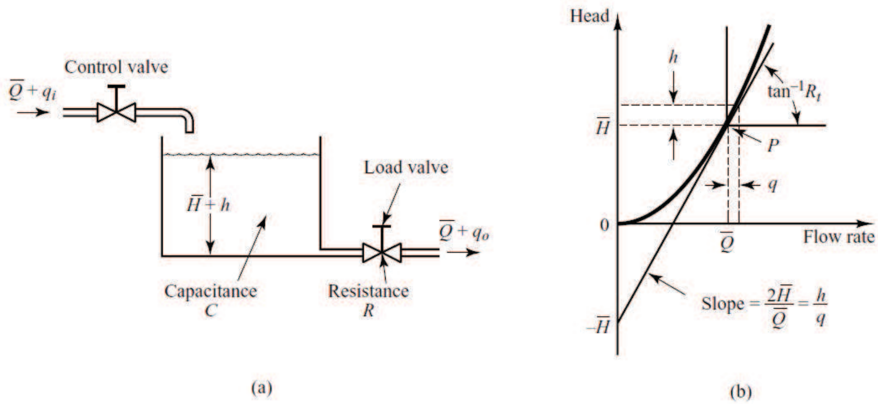

Figure 1: (a) Liquid-level system; (b) head versus-flow-rate curve.

For laminar flow, the resistance R_l is obtained as

$$R_l = \frac{dH}{dQ} = \frac{H}{Q} \tag{3}$$

The laminar-flow resistance is constant and is analogous to the electrical resistance.

If the flow through the restriction is turbulent, the steady-state flow rate is given by

$$Q = K\sqrt{H} \tag{4}$$

The resistance R_t for turbulent flow is obtained from

$$R_t = \frac{dH}{dQ} \tag{5}$$

Since from Equation (4) we obtain

$$dQ = \frac{K}{2\sqrt{H}} dH \tag{6}$$

we have

$$\frac{dH}{dQ} = \frac{2\sqrt{H}}{K} = \frac{2\sqrt{H}\sqrt{H}}{Q} = \frac{2H}{Q} \tag{7}$$

Thus

$$R_t = \frac{2H}{Q} \tag{8}$$

The value of the turbulent-flow resistance R_t depends on the flow rate and the head. The value of R_t, however, may be considered constant if the changes in head and flow rate are small.

By use of the turbulent-flow resistance, the relationship between Q and H can be given by

$$Q = \frac{2H}{R_t} \tag{9}$$

Such linearization is valid, provided that changes in the head and flow rate from their respective steady-state values are small.

In many practical cases, the value of the coefficient K in Equation (4), which depends on the flow coefficient and the area of restriction, is not known. Then the resistance may be determined by plotting the head-versus-flow-rate curve based on experimental data and measuring the slope of the curve at the operating condition. An example of such a plot is shown in Figure 1(b). In the figure, point P is the steady-state operating point. The tangent line to the curve at point P intersects the ordinate at point. Thus, the slope of this tangent line is $2\bar{H}/\bar{Q}$. Since the resistance R_t at the operating point P is given by $2\bar{H}/\bar{Q}$, the resistance R_t is the slope of the curve at the operating point.

Consider the operating condition in the neighborhood of point P. Define a small deviation of the head from the steady-state value as h and the corresponding small change of the flow rate as q. Then the slope of the curve at point P can be given by

$$\text{Slope of curve at point } P = \frac{h}{q} = \frac{2\bar{H}}{\bar{Q}} = R_t \tag{10}$$

The linear approximation is based on the fact that the actual curve does not differ much from its tangent line if the operating condition does not vary too much.

1.2 Capacitance of Liquid-Level Systems

The capacitance C of a tank is defined to be the change in quantity of stored liquid necessary to cause a unit change in the potential (head). The potential is the quantity that indicates the energy level of the system.

$$C = \frac{\text{change in liquid stored, } m^3}{\text{change in head, } m} \tag{11}$$

It should be noted that the capacity (m^3) and the capacitance (m^2) are different. The capacitance of the tank is equal to its cross-sectional area. If this is constant, the capacitance is constant for any head.

1.3 Liquid-Level Systems

Consider the system shown in Figure 1(a). The variables are defined as follows:

- $\bar{Q}$ = steady-state flow rate (before any change has occurred), m^3/sec
- q_i = small deviation of inflow rate from its steady-state value, m^3/sec

- q_o = small deviation of outflow rate from its steady-state value, m^3/sec
- $\bar{H}$ = steady-state head (before any change has occurred), m
- h = small deviation of head from its steady-state value, m

A system can be considered linear if the flow is laminar. Even if the flow is turbulent, the system can be linearized if changes in the variables are kept small. Based on the assumption that the system is either linear or linearized, the differential equation of this system can be obtained as follows:

Since the inflow minus outflow during the small time interval dt is equal to the additional amount stored in the tank, we see that

$$C dh = (q_i - q_o) dt \tag{12}$$

From the definition of resistance, the relationship between q_o and h is given by

$$q_o = \frac{h}{R} \tag{13}$$

The differential equation for this system for a constant value of R becomes

$$RC\frac{dh}{dt} + h = Rq_i \tag{14}$$

Note that RC is the time constant of the system. Taking the Laplace transforms of both sides of Equation (14), assuming the zero initial condition, we obtain

$$(RCs + 1)H(s) = RQ_i(s) \tag{15}$$

where $H(s) = \mathfrak{L}[h]$, $Q_i(s) = \mathfrak{L}[q_i]$.

If q_i is considered the input and h the output, the transfer function of the system is

$$\frac{H(s)}{Q_i(s)} = \frac{R}{RCs + 1} \tag{16}$$

If, however, q_o is taken as the output, the input being the same, then the transfer function is

$$\frac{Q_o(s)}{Q_i(s)} = \frac{1}{RCs + 1} \tag{17}$$

where we have used the relationship

$$Q_o(s) = \frac{1}{R}H(s) \tag{18}$$

1.4 Liquid-Level Systems with Interaction

Consider the system shown in Figure 2. In this system, the two tanks interact. Thus the transfer function of the system is not the product of two first-order transfer functions. In the following, we shall assume only small variations of the

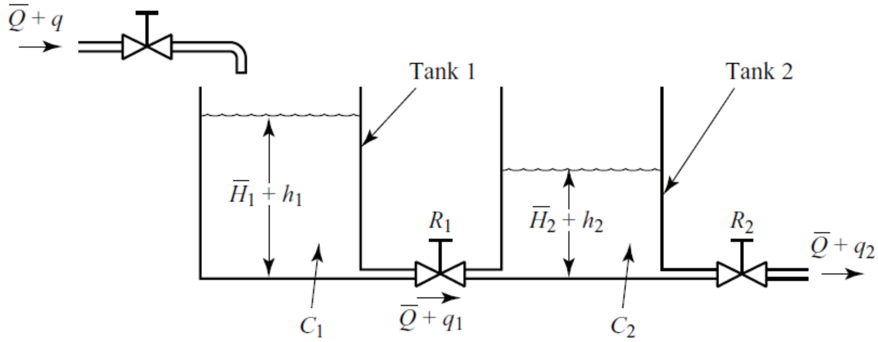

Figure 2: Liquid-level system with interaction.

variables from the steady-state values. Using the symbols as defined in Figure 2, we can obtain the following equations for this system:

$$\frac{h_1 - h_2}{R_1} = q_1 \tag{19}$$

$$C_1 \frac{dh_1}{dt} = q - q_1 \tag{20}$$

$$\frac{h_2}{R_2} = q_2 \tag{21}$$

$$C_2 \frac{dh_2}{dt} = q_1 - q_2 \tag{22}$$

If q is considered the input and q_2 the output, the transfer function of the system is

$$\frac{Q_2(S)}{Q(s)} = \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_2C_1)s + 1} \tag{23}$$

It is instructive to obtain Equation (23), the transfer function of the interacted system, by block diagram reduction. From Equations (19) through (22), we obtain the elements of the block diagram, as shown in Figure 3(a). By connecting signals properly, we can construct a block diagram, as shown in Figure 3(b). This block diagram can be simplified, as shown in Figure 3(c). Further simplifications result in Figures 3(d) and (e). Figure 3(e) is equivalent to Equation (23).

2 Pneumatic Systems

The past decades have seen a great development in low pressure pneumatic controllers for industrial control systems, and today they are used extensively in industrial processes. Reasons for their broad appeal include an explosion proof character, simplicity, and ease of maintenance.

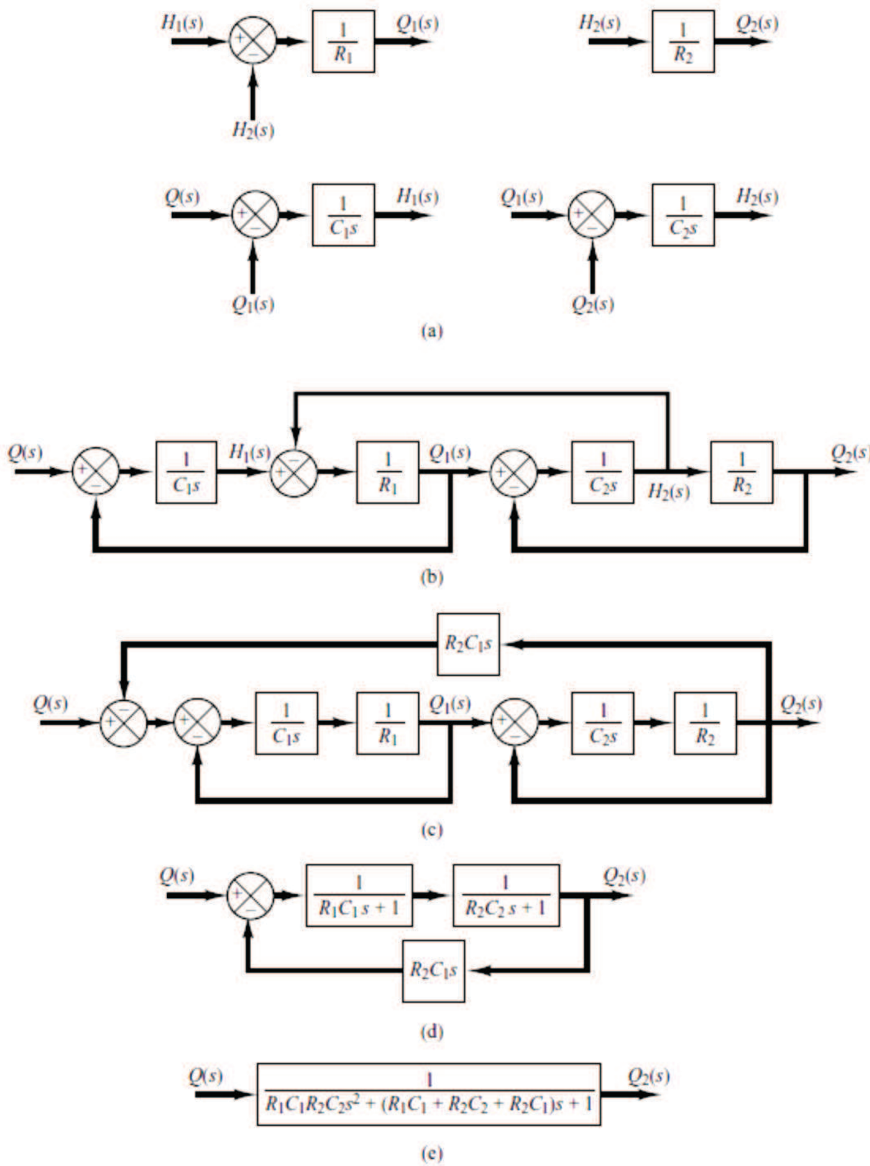

Figure 3: Liquid-level system with interaction.

2.1 Resistance of Pressure Systems

Many industrial processes and pneumatic controllers involve the flow of a gas or air through connected pipelines and pressure vessels. Consider the pressure system shown in Figure 4(a). The gas flow through the restriction is a function of the gas pressure difference $p_i - p_o$.

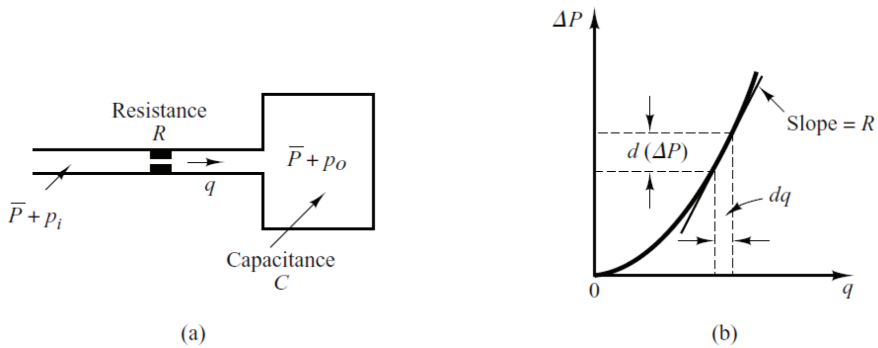

Figure 4: (a) Schematic diagram of a pressure system; (b) pressure difference versus flow-rate curve.

Such a pressure system may be characterized in terms of a resistance and a capacitance. The gas flow resistance R may be defined as follows:

$$R = \frac{\text{change in gas pressure difference, } lb_f/ft^2}{\text{change in gas flow rate, } lb/sec} = \frac{d(\Delta P)}{dq} \tag{24}$$

where $d(\Delta P)$ is a small change in the gas pressure difference and dq is a small change in the gas flow rate. Computation of the value of the gas flow resistance R may be quite time consuming. Experimentally, however, it can be easily determined from a plot of the pressure difference versus flow rate by calculating the slope of the curve at a given operating condition, as shown in Figure 4(b).

2.2 Capacitance of Pressure Systems

The capacitance of the pressure vessel may be defined by

$$C = \frac{\text{change in gas stored, } lb}{\text{change in gas pressure, } lb_f/ft^2} = \frac{dm}{dp} = V\frac{d\rho}{dp} \tag{25}$$

where C = capacitance, $lb\text{-}ft^2/lb_f$, m = mass of gas in vessel, lb, p = gas pressure, lb_f/ft^2, V = volume of vessel, ft^3, ρ = density, lb/ft^3.

The capacitance of the pressure system depends on the type of expansion process involved. The capacitance can be calculated by use of the ideal gas law. If the gas expansion process is polytropic* and the change of state of the gas is between isothermal and adiabatic†, then

$$p\left(\frac{V}{m}\right)^n = \frac{p}{\rho^n} = K = constant \tag{26}$$

*The term "polytropic" was originally coined to describe any reversible process on any open or closed system of gas or vapor which involves both heat and work transfer, such that a specified combination of properties were maintained constant throughout the process.

†An adiabatic process occurs without transfer of heat or mass of substances between a thermodynamic system and its surroundings. In an adiabatic process, energy is transferred to the surroundings only as work.

where n = polytropic exponent.

For ideal gases,

$$p\bar{v} = \bar{R}T, \ pv = \frac{\bar{R}}{M}T \tag{27}$$

where p = absolute pressure, lb_f/ft^2, $\bar{v}$ = volume occupied by 1 mole of a gas, $ft^3/lb\text{-}mole$, $\bar{R}$ = universal gas constant, $ft - lb_f/lb\text{-}mole^{\circ}R$, T = absolute temperature, $^{\circ}R$, v = specific volume of gas, ft^3/lb, M = molecular weight of gas per mole, $lb/lb\text{-}mole$. Thus

$$pv = \frac{p}{\rho} = \frac{\bar{R}}{M}T = R_{gas}T \tag{28}$$

where R_{gas} = gas constant, $ft\text{-}lb_f/lb^{\circ}R$.

The polytropic exponent n is unity for isothermal expansion. For adiabatic expansion, n is equal to the ratio of specific heats c_p/c_v, where c_p is the specific heat at constant pressure and c_v is the specific heat at constant volume. In many practical cases, the value of n is approximately constant, and thus the capacitance may be considered constant.

The value of $d\rho/d_p$ is obtained from Equations (26) and (28). From Equation (26) we have

$$dp = Kn\rho^{n-1}d\rho, \ \frac{d\rho}{dp} = \frac{1}{Kn\rho^{n-1}} = \frac{\rho^n}{pn\rho^{n-1}} = \frac{\rho}{pn} \tag{29}$$

Substituting Equation (28) into this last equation, we get

$$\frac{d\rho}{dp} = \frac{1}{nR_{gas}T} \tag{30}$$

The capacitance C is then obtained as

$$C = \frac{V}{nR_{gas}T} \tag{31}$$

2.3 Pressure System

Consider the system shown in Figure 4(a). If we assume only small deviations in the variables from their respective steady-state values, then this system may be considered linear. Let us define: $\bar{P}$ = gas pressure in the vessel at steady state (before changes in pressure have occurred), lb_f/ft^2, p_i = small change in inflow gas pressure, lb_f/ft^2, p_o = small change in gas pressure in the vessel, lb_f/ft^2, V = volume of the vessel, ft^3, m = mass of gas in the vessel, lb, q = gas flow rate, lb/sec, ρ = density of gas, lb/ft^3.

For small values of p_i and p_o, the resistance R given by Equation (24) becomes constant and may be written as

$$R = \frac{p_i - p_o}{q} \tag{32}$$

The capacitance C is given by Equation (25), $C = dm/dp$. Since the pressure change dp_o times the capacitance C is equal to the gas added to the vessel during dt seconds, we obtain

$$Cdp_o = qdt,\ C\frac{dp_o}{dt} = \frac{p_i - p_o}{R} \tag{33}$$

which can be written as

$$RC\frac{dp_o}{dt} + p_o = p_i \tag{34}$$

If p_i and p_o are considered the input and output, respectively, then the transfer function of the system is

$$\frac{P_o(s)}{P_i(s)} = \frac{1}{RCs + 1} \tag{35}$$

where RC has the dimension of time and is the time constant of the system.

3 Thermal Systems

Thermal systems are those that involve the transfer of heat from one substance to another. Thermal systems may be analyzed in terms of resistance and capacitance, although the thermal capacitance and thermal resistance may not be represented accurately as lumped parameters, since they are usually distributed throughout the substance.

For precise analysis, distributed-parameter models must be used. Here, however, to simplify the analysis we shall assume that a thermal system can be represented by a lumped-parameter model, that substances that are characterized by resistance to heat flow have negligible heat capacitance, and that substances that are characterized by heat capacitance have negligible resistance to heat flow.

There are three different ways heat can flow from one substance to another: conduction, convection, and radiation. Here we consider only conduction and convection. For conduction or convection heat transfer,

$$q = K\Delta\theta \tag{36}$$

where q = heat flow rate, $kcal/sec$, $\Delta\theta$ temperature difference, $°C$, K = coefficient, $kcal/sec°C$. The coefficient K is given by

$$K = \begin{cases} \frac{kA}{\Delta X}, & \text{conduction} \\ HA, & \text{convection} \end{cases} \tag{37}$$

where k = thermal conductivity, $kcal/msec°C$, A = area normal to heat flow, m^2, X = thickness of conductor, m, H = convection coefficient, $kcal/m^2sec°C$.

3.1 Thermal Resistance

The thermal resistance R for heat transfer between two substances may be defined as follows:

$$R = \frac{\text{change in temperature difference, } {}^\circ C}{\text{change in heat flow rate, } kcal/sec} \tag{38}$$

The thermal resistance for conduction or convection heat transfer is given by

$$R = \frac{d(\Delta\theta)}{dq} = \frac{1}{K} \tag{39}$$

Since the thermal conductivity and convection coefficients are almost constant, the thermal resistance for either conduction or convection is constant.

3.2 Thermal Capacitance

The thermal capacitance C is defined by

$$C = \frac{\text{change in heat stored, } kcal}{\text{change in temperature, } {}^\circ C} = mc \tag{40}$$

where m = mass of substance considered, kg, c = specific heat of substance, $kcal/kg^\circ C$.

3.3 Thermal System

Consider the system shown in Figure 5(a). It is assumed that the tank is insulated to eliminate heat loss to the surrounding air. It is also assumed that there is no heat storage in the insulation and that the liquid in the tank is perfectly mixed so that it is at a uniform temperature. Thus, a single temperature is used to describe the temperature of the liquid in the tank and of the outflowing liquid.

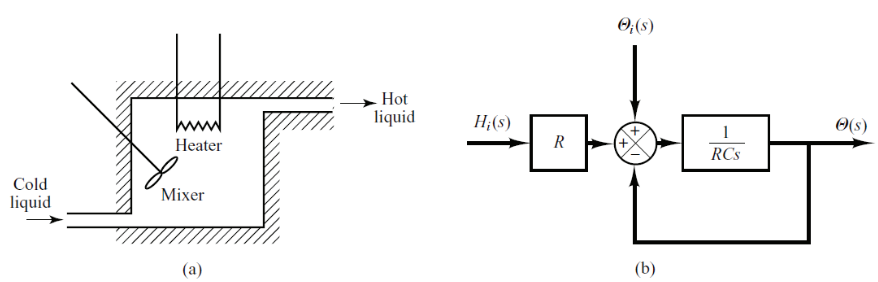

Figure 5: (a) Thermal system: (b) block diagram of the system.

Let us define, $\bar{\Theta}_i$ = steady-state temperature of inflowing liquid, $^\circ C$, $\bar{\Theta}_o$ = steady-state temperature of outflowing liquid, $^\circ C$, G = steady-state liquid flow rate, kg/sec, M = mass of liquid in tank, kg, c = specific heat of liquid, $kcal/kg°C$, R = thermal resistance, $°Csec/kcal$, C = thermal capacitance, $kcal/°C$, $\bar{H}$ = steady-state heat input rate, $kcal/sec$.

Assume that the temperature of the inflowing liquid is kept constant and that the heat input rate to the system (heat supplied by the heater) is suddenly changed from $\bar{H}$ to $\bar{H}+h_i$, where h_i represents a small change in the heat input rate. The heat outflow rate will then change gradually from $\bar{H}$ to $\bar{H}+h_o$. The temperature of the outflowing liquid will also be changed from $\bar{\Theta}_o$ to $\bar{\Theta}_o+\theta$. For this case, h_o, C, and R are obtained, respectively, as

$$h_0 = Gc\theta,\ C = Mc,\ R = \frac{\theta}{h_o} = \frac{1}{Gc} \tag{41}$$

The heat-balance equation for this system is

$$Cd\theta = (h_i - h_o)dt,\ C\frac{d\theta}{dt} = h_i - h_o \tag{42}$$

which may be rewritten as

$$RC\frac{d\theta}{dt} + \theta = Rh \tag{43}$$

Note that the time constant of the system is equal to RC or M/G seconds. The transfer function relating θ and h_i is given by

$$\frac{\Theta(s)}{H_i(s)} = \frac{R}{RCs+1} \tag{44}$$

where $\Theta(s) = \mathfrak{L}[\theta(t)]$ and $H_i(s) = \mathfrak{L}[h_i(t)]$.

4 Hydraulic Systems

The widespread use of hydraulic circuitry in machine tool applications, aircraft control systems, and similar operations occurs because of such factors as positiveness, accuracy, flexibility, high horsepower-to-weight ratio, fast starting, stopping, and reversal with smoothness and precision, and simplicity of operations.

The operating pressure in hydraulic systems is somewhere between 145 and 5000 lb_f/in^2 (between 1 and 35 MPa). In some special applications, the operating pressure may go up to 10,000 lb_f/in^2 (70 MPa). For the same power requirement, the weight and size of the hydraulic unit can be made smaller by increasing the supply pressure. With high pressure hydraulic systems, very large force can be obtained. Rapid-acting, accurate positioning of heavy loads is possible with hydraulic systems. A combination of electronic and hydraulic systems is widely used because it combines the advantages of both electronic control and hydraulic power.

Appendix

A Hydraulic Servo System

Figure 6(a) shows a hydraulic servomotor. It is essentially a pilot-valve-controlled hydraulic power amplifier and actuator. The pilot valve is a balanced valve, in the sense that the pressure forces acting on it are all balanced. A very large power output can be controlled by a pilot valve, which can be positioned with very little power. In practice, the ports shown in Figure 6(a) are often made

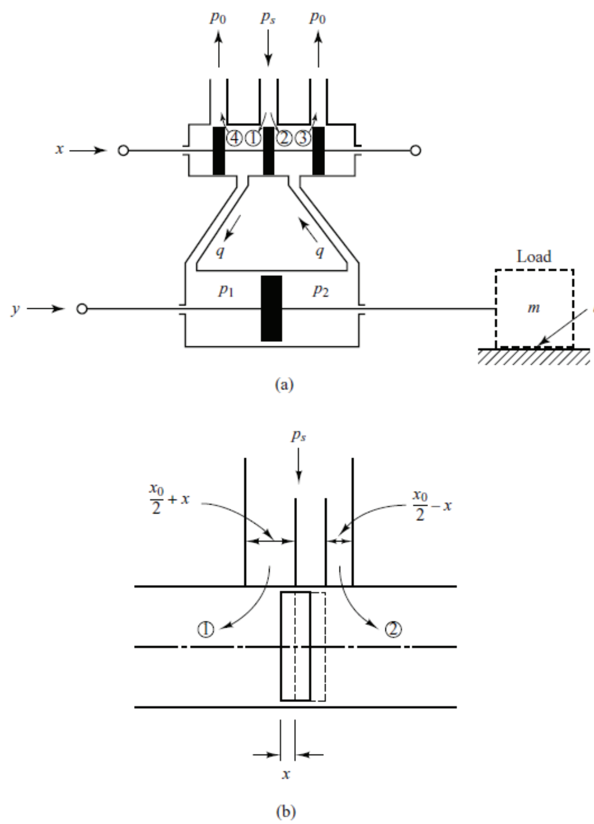

Figure 6: (a) Hydraulic servo system; (b) enlarged diagram of the valve orifice area.

wider than the corresponding valves. In such a case, there is always leakage through the valves. Such leakage improves both the sensitivity and the linearity of the hydraulic servomotor. In the following analysis we shall make the assumption that the ports are made wider than the valves, that is, the valves are underlapped.

We shall apply the linearization technique to obtain a linearized mathematical model of the hydraulic servomotor. We assume that the valve is underlapped and symmetrical and admits hydraulic fluid under high pressure into a power cylinder that contains a large piston, so that a large hydraulic force is established to move a load.

In Figure 6(b) we have an enlarged diagram of the valve orifice area. Let us define the valve orifice areas of ports 1, 2, 3, 4 as A_1, A_2, A_3, A_4, respectively. Also, define the flow rates through ports 1, 2, 3, 4 as q_1, q_2, q_3, q_4, respectively. Note that, since the valve is symmetrical, $A_1 = A_3$ and $A_2 = A_4$. Assuming the displacement x to be small, we obtain

$$A_1 = A_3 = k\left(\frac{x_0}{2} + x\right),\ A_2 = A_4 = k\left(\frac{x_0}{2} - x\right) \tag{A.1}$$

where k is a constant. Furthermore, we shall assume that the return pressure p_o in the return line is small and thus can be neglected. Then, referring to Figure 6(a), flow rates through valve orifices are

$$q_1 = c_1 A_1 \sqrt{\frac{2g}{\gamma}(p_s - p_1)} = C_1\sqrt{p_s - p_1}\left(\frac{x_o}{2} + x\right) \tag{A.2}$$

$$q_2 = c_2 A_2 \sqrt{\frac{2g}{\gamma}(p_s - p_2)} = C_2\sqrt{p_s - p_2}\left(\frac{x_o}{2} - x\right) \tag{A.3}$$

$$q_3 = c_1 A_3 \sqrt{\frac{2g}{\gamma}(p_2 - p_o)} = C_1\sqrt{p_2 - p_o}\left(\frac{x_o}{2} + x\right) = C_1\sqrt{p_2}\left(\frac{x_o}{2} + x\right) \tag{A.4}$$

$$q_4 = c_2 A_4 \sqrt{\frac{2g}{\gamma}(p_1 - p_o)} = C_2\sqrt{p_1 - p_o}\left(\frac{x_o}{2} - x\right) = C_2\sqrt{p_1}\left(\frac{x_o}{2} - x\right) \tag{A.5}$$

where $C_1 = c_1 k\sqrt{2g/\gamma}$ and $C_2 = c_2 k\sqrt{2g/\gamma}$, and γ is the specific weight and is given by $\gamma = \rho g$, where ρ is mass density and g is the acceleration of gravity. The flow rate q to the left-hand side of the power piston is

$$q = q_1 - q_4 = C_1\sqrt{p_s - p_1}\left(\frac{x_o}{2} + x\right) - C_2\sqrt{p_1}\left(\frac{x_o}{2} - x\right) \tag{A.6}$$

The flow rate from the right-hand side of the power piston to the drain is the same as this q and is given by

$$q = q_3 - q_2 = C_1\sqrt{p_2}\left(\frac{x_o}{2} + x\right) - C_2\sqrt{p_s - p_2}\left(\frac{x_o}{2} - x\right) \tag{A.7}$$

In the present analysis we assume that the fluid is incompressible. Since the valve is symmetrical, we have $q_1 = q_3$ and $q_2 = q_4$. By equating q_1 and q_3, we obtain

$$p_s - p_1 = p_2,\ p_s = p_1 + p_2 \tag{A.8}$$

If we define the pressure difference across the power piston as $\Delta p = p_1 - p_2$, then

$$p_1 = \frac{p_s + \Delta p}{2},\ p_2 = \frac{p_s - \Delta p}{2} \tag{A.9}$$

For the symmetrical valve shown in Figure 6(a), the pressure in each side of the power piston is $(1/2)p_s$ when no load is applied, or $\Delta p = 0$. As the spool valve is displaced, the pressure in one line increases as the pressure in the other line decreases by the same amount.

In terms of p_s and we can rewrite the flow rate q given by Equation (A.6) as

$$q = q_1 - q_4 = C_1\sqrt{\frac{p_s - \Delta p}{2}}\left(\frac{x_0}{2} + x\right) - C_2\sqrt{\frac{p_s + |deltap}{2}}\left(\frac{x_0}{2} - x\right) \tag{A.10}$$

Noting that the supply pressure p_s is constant. The flow rate q can be written as a function of the valve displacement x and pressure difference or

$$q = C_1\sqrt{\frac{p_s - \Delta p}{2}}\left(\frac{x_0}{2} + x\right) - C_2\sqrt{\frac{p_s + \Delta p}{2}}\left(\frac{x_0}{2} - x\right) = f(x, \Delta p) \tag{A.11}$$

By applying the linearization technique to this case, the linearized equation about point $x = \bar{x}$, $\Delta p = \Delta\bar{p}$, $q = \bar{q}$ is

$$q - \bar{q} = a(x - \bar{x}) + b(\Delta p - \Delta\bar{p}) \tag{A.12}$$

where $\bar{q} = f(\bar{x}, \Delta\bar{p})$, and

$$a = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}.\Delta p=\Delta\bar{p}} = C_1\sqrt{\frac{p_s - \Delta\bar{p}}{2}} + C_2\sqrt{\frac{p_s + \Delta\bar{p}}{2}} \tag{A.13}$$

$$b = \left.\frac{\partial f}{\partial \Delta p}\right|_{x=\bar{x}.\Delta p=\Delta\bar{p}} = -\left[\frac{C_1}{2\sqrt{2}\sqrt{p_s - \Delta\bar{p}}}\left(\frac{x_0}{2} + \bar{x}\right) + \frac{C_1}{2\sqrt{2}\sqrt{p_s + \Delta\bar{p}}}\left(\frac{x_0}{2} - \bar{x}\right)\right] < 0 \tag{A.14}$$

Coefficients a and b here are called valve coefficients. Equation (A.12) is a linearized mathematical model of the spool valve near an operating point The values of valve coefficients a and b vary with the operating point. Note that $\partial f/\partial\Delta p$ is negative and so b is negative.

Since the normal operating point is the point where $\bar{x} = 0$, $\Delta\bar{p} = 0$, $\bar{q} = 0$ near the normal operating point Equation (A.12) becomes

$$q = K_1 x - L_2 \Delta p \tag{A.15}$$

where

$$K_1 = (C_1 + C_2)\sqrt{\frac{p_s}{2}} > 0,\ K_2 = (C_1 + C_2)\frac{x_0}{4\sqrt{2p_s}} > 0 \tag{A.16}$$

Equation (A.15) is a linearized mathematical model of the spool valve near the origin Note that the region near the origin, $\bar{x} = 0$, $\Delta\bar{p} = 0$, $\bar{q} = 0$, is most important in this kind of system, because the system operation usually occurs near this point.

In the present analysis we assume that the load reactive forces are small, so that the leakage flow rate and oil compressibility can be ignored. Referring to Figure 6(a), we see that the rate of flow of oil q times dt is equal to the power-piston displacement dy times the piston area A times the density of oil ρ. Thus, we obtain

$$A\rho dy = qdt \tag{A.17}$$

Notice that for a given flow rate q the larger the piston area A is, the lower will be the velocity dy/dt. Hence, if the piston area A is made smaller, the other variables remaining constant, the velocity dy/dt will become higher. Also, an increased flow rate q will cause an increased velocity of the power piston and will make the response time shorter. Equation (A.15) can now be written as

$$\Delta P = \frac{1}{K_2}\left(K_1 x - A\rho\frac{Dy}{dt}\right) \tag{A.18}$$

The force developed by the power piston is equal to the pressure difference ΔP times the piston area A or

$$\text{Force developed by the power piston } = A\Delta P = \frac{A}{K_2}\left(K_1 x - A\rho\frac{dy}{dt}\right) \tag{A.19}$$

For a given maximum force, if the pressure difference is sufficiently high, the piston area, or the volume of oil in the cylinder, can be made small. Consequently, to minimize the weight of the controller, we must make the supply pressure sufficiently high.

Assume that the power piston moves a load consisting of a mass and viscous friction. Then the force developed by the power piston is applied to the load mass and friction, and we obtain

$$m\ddot{y} + b\dot{y} = \frac{A}{K_2}\left(K_1 x - A\rho\dot{y}\right),\ m\ddot{y} + \left(b + \left(K_1 x - A\rho\dot{y}\right)\right) = \frac{AK_1}{K_2}x \tag{A.20}$$

where m is the mass of the load and b is the viscous-friction coefficient.

Assuming that the pilot-valve displacement x is the input and the power-piston displacement y is the output, we find that the transfer function for the hydraulic servomotor is, from Equation (A.20)

$$\frac{Y(s)}{X(s)} = \frac{1}{s\left[\left(\frac{mK_2}{AK_1}\right)s + \frac{bK_2}{AK_1} + \frac{A\rho}{K_1}\right]} = \frac{K}{s(Ts+1)} \tag{A.21}$$

where

$$K = \frac{1}{\frac{bK_2}{AK_1} + \frac{A\rho}{K_1}},\ T = \frac{mK_2}{bK_2 + A^2\rho} \tag{A.22}$$

From Equation (A.21) we see that this transfer function is of the second order. If the ratio $mK_2/(bK_2 + A^2\rho)$ is negligibly small or the time constant T is negligible, the transfer function $Y(s)/X(s)$ can be simplified to give

$$\frac{Y(s)}{X(s)} = \frac{K}{s} \tag{A.23}$$

It is noted that a more detailed analysis shows that if oil leakage, compressibility (including the effects of dissolved air), expansion of pipelines, and the like are taken into consideration, the transfer function becomes

$$\frac{Y(s)}{X(s)} = \frac{K}{s(T_1 s + 1)(T_2 s + 1)} \tag{A.24}$$

where T_1 and T_2 are time constants. As a matter of fact, these time constants depend on the volume of oil in the operating circuit. The smaller the volume, the smaller the time constants.