Root-Locus Method – Examples

Example 1. *Consider the control system whose feedforward transfer function is*

$$G(s) = \frac{4}{s(s+0.5)} \tag{1}$$

This system has closed-loop poles at $s = -0.2500 \pm j1.9843$. The damping ratio is 0.125, the undamped natural frequency is 2 rad/sec, and the static velocity error constant is 8 sec^{-1}.

It is desired to make the damping ratio of the dominant closed-loop poles equal to 0.5 and to increase the undamped natural frequency to 5 rad/sec and the static velocity error constant to 80 sec^{-1}. Design an appropriate compensator to meet all the performance specifications.

Solution 1. *Let us assume that we use a lag-lead compensator having the transfer function*

$$G_c(s) = K_c \left(\frac{s+1/T_1}{s+\gamma/T_1}\right)\left(\frac{s+1/T_2}{s+1/(\beta T_2)}\right), \ \gamma > 1, \ \beta > 1 \tag{2}$$

where γ is not equal to β. Then the compensated system will have the open-loop transfer function

$$G_c(s)G(s) = K_c \left(\frac{s+1/T_1}{s+\gamma/T_1}\right)\left(\frac{s+1/T_2}{s+1/(\beta T_2)}\right) G(s) \tag{3}$$

From the performance specifications, the dominant closed-loop poles must be at $s = -2.50 \pm j4.33$. Recall that $\zeta = \cos\theta$, then $\theta = \cos^{-1}(0.5) = 60^\circ$ is the angle where the new pole should be located. Also from $\omega_n = 5$, then letting $s = \sigma + j\omega$, $\sigma = 5\cos(60) = 2.5$ and $j\omega = 5\sin(60) = j4.33$. Furthermore, since

$$\left. \underline{\Big/ \frac{4}{s(s+0.5)}} \right|_{s=-2.50\pm j4.33} = 125.2087^\circ \tag{4}$$

the phase-lead portion of the lag-lead compensator must contribute 54.7913°, as compared to the original system, so that the root locus passes through the desired location of the dominant closed-loop poles.

To design the phase-lead portion of the compensator, we first determine the location of the zero and pole that will give 54.7913° contribution. There are many possible choices, but we shall here choose the zero at $s = -0.5$ so that *this zero will cancel the pole of the plant. Once the zero is chosen, the pole can be located such that the angle contribution is 54.7913°. By simple calculation or graphical analysis, the pole must be located at $s = -5$. That is, from $(s + 1/T_1)/(s+P)$, the angle is*

$$\underline{/s+0.5} - \underline{/s+P} = 54.7913° \tag{5}$$

where $\underline{/s+0.5} = 114.7913°$ at $s = -2.50 \pm j4.33$. Hence $\underline{/s+P} = 114.7913 - 54.7913 = 60°$, and $P = 4.33/\tan^{-1}(59.7919) = 2.5$, therefore, $s = -5$. Thus, the phase-lead portion of the lag-lead compensator becomes

$$K_c \frac{s + 1/T_1}{s + \gamma/T_1} = K_c \frac{s+0.5}{s+5} \tag{6}$$

Thus

$$T_1 = 2,\ \gamma = \frac{5.02}{0.5} = 10 \tag{7}$$

Next we determine the value of K_c from the magnitude condition:

$$K_c \left| \frac{s+0.5}{s+5} \frac{4}{s(s+0.5)} \right|_{s=-2.50 \pm j4.33} = 1 \tag{8}$$

Hence,

$$K_c = \left| \frac{s(s+5)}{4} \right|_{s=-2.50 \pm j4.33} = 6.25 \tag{9}$$

The phase-lag portion of the compensator can be designed as follows: First the value of β is determined to satisfy the requirement on the static velocity error constant:

$$\begin{aligned} K_v &= \lim_{s\to 0} sG_c(s)G(s) = \lim_{s \to 0} sK_v \frac{\beta}{\gamma} G(s) \\ &= \lim_{s \to 0} s(6.25)\frac{\beta}{10}\frac{4}{s(s+0.5)} = 80 \end{aligned} \tag{10}$$

Hence, β is determined as $\beta = 16$.

Finally, we choose the value T_2 such that the following two conditions are satisfied:

$$\left| \frac{s + 1/T_2}{s + 1/(16T_2)} \right|_{s=-2.50 \pm j4.33} \approx 1,\ -5° < \underline{\left/ \frac{s + 1/T_2}{s + 1/(16T_2)} \right.}\Bigg|_{s=-2.50 \pm j4.33} < 0° \tag{11}$$

We may choose several values for T_2 and check if the magnitude and angle conditions are satisfied. After simple calculations we find for $T_2 = 5$

$$1 > \textit{magnitude}{=}0.9818 > 0.98,\ -5° < \textit{angle}{=}-1.9° < 0° \tag{12}$$

Since $T_2 = 5$ satisfies the two conditions, then we may choose it.

Now the transfer function of the designed lag-lead compensator is given by

$$\begin{aligned} G_c(s) &= 6.25\left(\frac{s+1/2}{s+10/2}\right)\left(\frac{s+1/5}{s+1/(16\times 5)}\right) \\ &= 6.25\left(\frac{s+0.5}{s+5}\right)\left(\frac{s+0.2}{s+0.0125}\right) \end{aligned} \tag{13}$$

The compensated system will have the open-loop transfer function

$$G_c(s)G(s) = \frac{25(s+0.2)}{s(s+5)(s+0.0125)} \tag{14}$$

With the cancellation of the $(s = -0.5)$ terms, the compensated system is a third-order system. Because the angle contribution of the phase lag portion of the lag-lead compensator is quite small, there is only a small change in the location of the dominant closed-loop poles from the desired location $s = -2.50 \pm j4.33$, The characteristic equation for the compensated system is

$$\begin{aligned} &s^3 + 5.0125s^2 + 25.06s + 5 \\ =&(s+2.4024+j4.2770)(s+2.4024-j4.2770)(s+0.2078) = 0 \end{aligned} \tag{15}$$

Hence the new closed-loop poles are located at $s = -2.4024 \pm j4.2770$.

The new damping ratio is $\zeta = 0.4897$ obtained from the location of the closed-loop pole. Therefore the compensated system meets all the required performance specifications. The third closed-loop pole of the compensated system is located at $s = -0.2078$. Since this closed-loop pole is very close to the zero at $s = -0.2$ the effect of this pole on the response is small. The root locus plots before and after compensation are shown in Figure 1 and the step responses are shown in Figure 2.

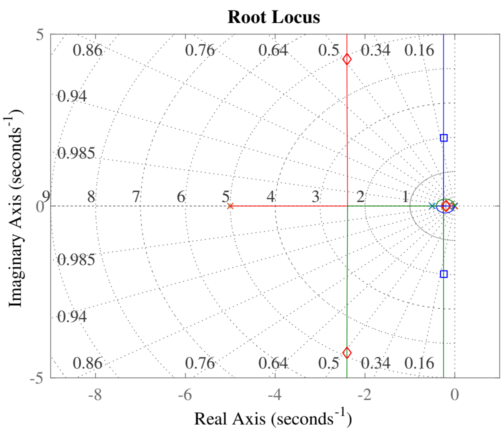

Figure 1:

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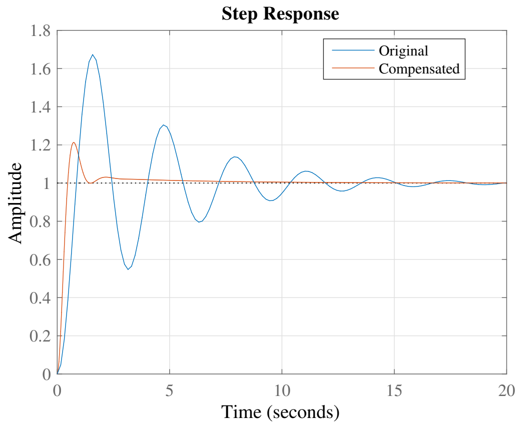

Figure 2:

Example 2. *Consider the control system of Example 1 again. Suppose that we use a lag-lead compensator of the form*

$$G_c(s) = K_c \frac{(s+1/T_1)(s+1/T_2)}{(s+\beta/T_1)(s+1/(\beta T_2))}, \;\; \beta > 1, \;\; \gamma = \beta \tag{16}$$

Assuming the specifications are the same as those given in Example 1, design a compensator $G_c(s)$.

Solution 2. *The desired locations for the dominant closed-loop poles are at $s = -2.50 \pm j4.33$. The open-loop transfer function of the compensated system is*

$$G_c(s)G(s) = K_c \frac{(s+1/T_1)(s+1/T_2)}{(s+\beta/T_1)(s+1/(\beta T_2))} \frac{4}{s(s+0.5)} \tag{17}$$

Since the requirement on the static velocity error constant K_v is 80 sec^{-1}, we have

$$K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} K_c \frac{4}{0.5} = 8K_c = 80 \tag{18}$$

Thus $K_c = 10$. The time constant T_1 and the value of β are determined from

$$\begin{aligned} \left|\frac{s+1/T_1}{s+\beta/T_1}\right| \left|\frac{40}{s(s+0.5)}\right|_{s=-2.50\pm j4.33} &= \left|\frac{s+1/T_1}{s+\beta/T_1}\right| 1.68 = 1 \\ \angle \left.\frac{s+1/T_1}{s+\beta/T_1}\right|_{s=-2.50\pm j4.33} &= 55^\circ \end{aligned} \tag{19}$$

Referring to Figure 3, we start with the angle for the desired pole, i.e., from $-2.50 + j4.33$ then $\theta = \tan^{-1}(4.33/(-2.50)) = 120^\circ$. Furthermore, compute the angle that bisets the angle between lines AP and PO, i.e, half the angle of the

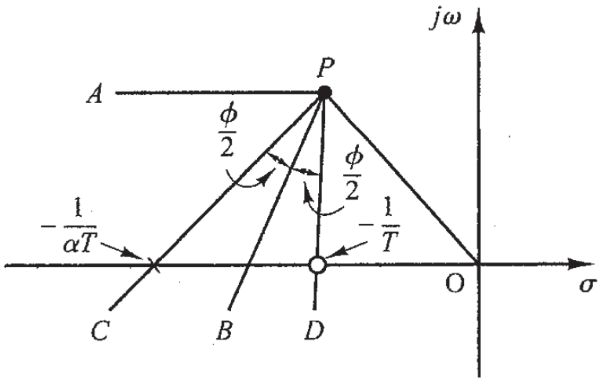

Figure 3: Determination of the desired pole-zero location.

desired pole measured from line AP. The next step is to find the angles from D and C to the pole P, which are half the angle of deficiency ($\phi = 54.7913^\circ/2$), see previous example. They can be obtained by noting that the height of pole P is the imaginary part of $-2.50 + j4.33$ (because the points are drawn on the s plane). The vertical line (not shown in the figure but should be realizable) and the lines DP and CP form right angle triangles. The horizontal distances from the vertical projection of P, say it is the point X, to points D and C are computed from

$$XD = \frac{4.33}{\tan\theta_D}, \ XC = \frac{4.33}{\tan\theta_C} \tag{20}$$

where $\theta_D = \theta/2\phi/2 = 60 + 27.4565 = 87.4565^\circ$ and $\theta_c = \theta/2 - \phi/2 = 60 - 27.456 = 32.5435^\circ$.

As shown in the Figure, point D and C correspond to the zero $s = -1/T$ and pole $s = -1/(\beta T)$ respectively. From these values, we check the magnitude and angle conditions given in Eq. 19. If the results are not satisfactory, we can adjust the angles θ_D and θ_C recursively to obtain the best values.

The result, after some adjustments on θ_D and θ_C in this example, we have

$$T_1 = 0.4176, \ \beta = 3.4786 \tag{21}$$

where the corresponding magnitude and angles are 1.0004 and 54.7913° respectively.

For the phase-lag portion, we choose T_2 (this is a user choice) such that it satisfies the conditions

$$\left|\frac{s + 1/T_2}{s + 1/(3.4786T_2)}\right|_{s=-2.50\pm j4.33} \approx 1, \ -5^\circ < \left.\angle\frac{s + 1/T_2}{s + 1/(3.4786T_2)}\right|_{s=-2.50\pm j4.33} < 0^\circ \tag{22}$$

By simple calculations, we find that if we choose $T_2 = 10$, then

$$1 > \textit{magnitude=0.9930} > 0.99, \ -1^\circ < \textit{angle= -0.7162} < 0^\circ \tag{23}$$

Thus, the lag-lead compensator becomes

$$G_c(s) = 10\left(\frac{s+2.38}{s+8.34}\right)\left(\frac{s+0.1}{s+0.02875}\right) \tag{24}$$

The compensated system will have the open-loop transfer function

$$G_c(s)G(s) = \frac{40(s+2.395)(s+0.1)}{(s+8.33)(s+0.02875)(s+0.5)s} \tag{25}$$

No cancellation occurs in this case, and the compensated system is of fourth order. Because the angle contribution of the phase lag portion of the lag-lead network is quite small, the dominant closed-loop poles are located very near the desired location. In fact, the location of the dominant closed-loop poles can be found from the characteristic equation as follows: The characteristic equation of the compensated system is

$$(s+8.34)(s+0.0285)(s+0.5)s + 40(s+2.38)(s+0.1) = 0 \tag{26}$$

which can be simplified to

$$\begin{aligned} &s^4 + 8.858s^3 + 44.42s^2 + 99.9s + 9.578 \\ &= (s+2.4382+j4.3205)(s+2.4382-j4.3205)(s+0.1003)(s+3.8816) = 0 \end{aligned} \tag{27}$$

The dominant closed-loop poles are located at $s = -2.4382 \pm j4.3205$, *other closed-loop poles are located at* $s = -0.1003$ *and* $s = -3.8816$.

Since the closed-loop pole at $s = -0.1003$ *is very close to a zero at* $s = -0.1$ *they almost cancel each other. Thus, the effect of this closed-loop pole is very small. The remaining closed-loop pole* $s = -3.8816$ *does not quite cancel the zero at* $s = -2.3945$. *The effect of this zero is to cause a larger overshoot in the step response than a similar system without such a zero.*

The root locus plots before and after compensation are shown in Figure 4 and the step responses are shown in Figure 5.

The maximum overshoot in the step response of the compensated system is approximately 38%. This is much larger than the maximum overshoot of 21% in the design presented in Example 1. It is possible to decrease the maximum overshoot by a small amount from 38%, but not to 20% if $\gamma = \beta$ *is required, as in this example. Note that by not requiring* $\gamma = \beta$, *we have an additional parameter to play with and thus can reduce the maximum overshoot.*

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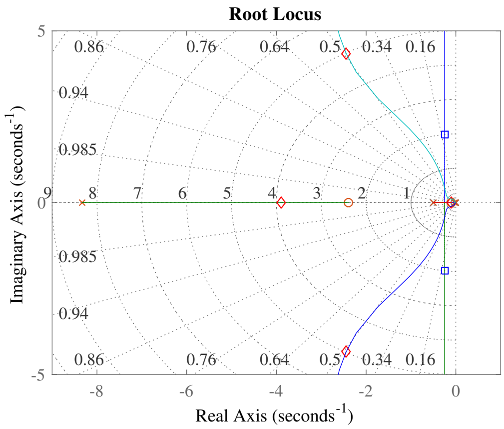

Figure 4:

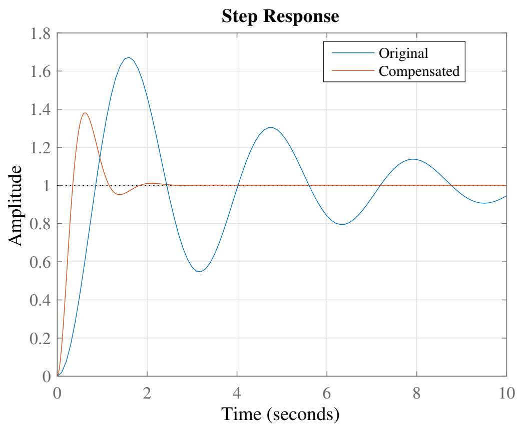

Figure 5: