Tutorial—Inverse Laplace Transform

Example 1. *Find*

$$\mathfrak{L}^{-1}\left[\frac{s+4}{2s^2+5s+3}\right] \tag{1}$$

Hint: use partial fraction.

Example 2. *Find*

$$\mathfrak{L}^{-1}\left[\frac{se^{-2s}}{s^2+2s+5}\right] \tag{2}$$

Hint: apply the Shifting Theorem to the part without the exponential term. The exponent corresponds is a time shift.

Example 3. *Find*

$$\mathfrak{L}^{-1}\left[\frac{s+2}{s(s+1)^2(s+3)}\right] \tag{3}$$

Note: this equation has two simple roots and a double root (due to the $(s+1)^2$ *term).*

Example 4. *Find*

$$\mathfrak{L}^{-1}\left[\frac{1}{3s^2(s^2+4)}\right] \tag{4}$$

Note: this function has one double root and two conjugate roots.

Example 5. *A rectangular voltage pulse of unit height and duration* T *is applied to a series* R*-*C *combination at* $t = 0$*, see Fig. 1. Determine the voltage across the capacitor* C *as a function of time.*

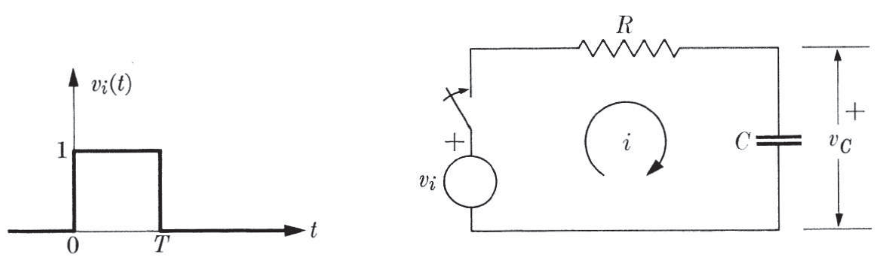

Figure 1: A rectangular pulse applied to an R-C circuit

Solution 1. *First, we factor the denominator*

$$B(s) = 2s^2 + 5s + 3 = 2\left(s^2 + \frac{5}{2}s + \frac{3}{2}\right) = 2(s+1)\left(s + \frac{3}{2}\right) \tag{5}$$

the distinct roots are $s = -1$ and $s = -3/2$. Next, we expand the given function of s into partial fractions and determine the constants.

$$\frac{s+4}{2s^2+5s+3} = \frac{1}{2}\left[\frac{K_1}{s+1} + \frac{K_2}{s+3/2}\right] \tag{6}$$

then

$$K_1 = 2\left[(s+1)\frac{s+4}{2(s+1)(s+3/2)}\right]_{s=-1} = \left[\frac{s+4}{s+3/2}\right]_{s=-1} = 6 \tag{7}$$

$$K_2 = 2\left[(s+3/2)\frac{s+4}{2(s+1)(s+3/2)}\right]_{s=-3/2} = \left[\frac{s+4}{s+1}\right]_{s=-3/2} = -5 \tag{8}$$

hence

$$\frac{s+4}{2s^2+5s+3} = \frac{1}{2}\left[\frac{6}{s+1} - \frac{5}{s+3/2}\right] \tag{9}$$

Finally, we find the inverse transform

$$\mathfrak{L}^{-1}\left[\frac{s+4}{2s^2+5s+3}\right] = \frac{1}{2}\mathfrak{L}^{-1}\left[\frac{6}{s+1} - \frac{5}{s+3/2}\right] = \frac{1}{2}\left(6e^{-t} - 5e^{-3t/2}\right) \tag{10}$$

□

Solution 2. *We know that multiplication by a factor e^{-2s} simply amounts to a shift in the independent variable from t to $t-2$. We can therefore first perform inverse Laplace transform without the shift and the re-apply it to the result.*

First, we factor the denominator

$$B(s) = s^2 + 2s + 5, \Rightarrow s = \frac{1}{2}\left(-2 \pm \sqrt{4-20}\right) = -1 \pm j2 \tag{11}$$

The two distinct roots are $s_1 = -1 + j2$ and $s_2 = -1 - j2$. Next, we expand into partial fractions. Because of the conjugate roots, we have

$$\frac{d}{ds}B(s) = 2s + 2 = 2(s+1) \tag{12}$$

then

$$K_1 = \left.\frac{s}{2(s+1)}\right|_{s=s_1=-1+j2} = \frac{-1+j2}{j4} = \frac{1}{4}(2+j) \tag{13}$$

$$K_2 = \left.\frac{s}{2(s+1)}\right|_{s=s_1=-1-j2} = \frac{-1-j2}{-j4} = \frac{1}{4}(2-j) \tag{14}$$

so we have

$$\frac{s}{s^2+2s+5} = \frac{1}{4}\left[\frac{2+j}{s+1-j2} + \frac{2-j}{s+1+j2}\right] \tag{15}$$

Now we find

$$\begin{aligned}\mathfrak{L}^{-1}\left[\frac{s}{s^2+2s+5}\right] &= \frac{1}{4}\mathfrak{L}^{-1}\left[\frac{2+j}{s+1-j2} + \frac{2-j}{s+1+j2}\right] \\ &= \frac{1}{4}\left[(2+j)e^{(-1+j2)t} + (2-j)e^{(-1-j2)t}\right] \\ &= \frac{1}{2}e^{-t}(2\cos 2t - \sin 2t)\end{aligned} \tag{16}$$

Finally, apply time shifting to obtain

$$\mathfrak{L}^{-1}\left[\frac{se^{-2s}}{s^2+2s+5}\right] = \frac{1}{2}e^{-(t-2)}\left[2\cos 2(t-2) - \sin 2(t-2)\right]U(t-2) \tag{17}$$

□

Solution 3. *The denominator has four roots: simple roots $s = 0$, $s = -3$, double root $s = -1$. Using partial fraction, we have*

$$\frac{A(s)}{B(s)} = \frac{s+2}{s(s+1)^2(s+3)} = \frac{K_1}{s} + \left(\frac{K_{22}}{(s+1)^2} + \frac{K_{21}}{s+1}\right) + \frac{K_3}{s+3} \tag{18}$$

Then

$$K_1 = s\frac{A(s)}{B(s)}\bigg|_{s=0} = \frac{s+2}{(s+1)^2(s+3)}\bigg|_{s=0} = \frac{2}{3} \tag{19}$$

$$K_{22} = (s+1)^2\frac{A(s)}{B(s)}\bigg|_{s=-1} = \frac{s+2}{s(s+3)}\bigg|_{s=-1} = -\frac{1}{2} \tag{20}$$

$$K_{21} = \frac{d}{ds}\left(\frac{s+2}{s(s+3)}\right)\bigg|_{s=-1} = \frac{s(s+3)-(s+2)(2s+3)}{s^2(s+3)^2}\bigg|_{s=-1} = -\frac{3}{4} \tag{21}$$

$$K_3 = (s+3)\frac{A(s)}{B(s)}\bigg|_{s=-3} = \frac{s+2}{s(s+1)^2}\bigg|_{s=-3} = \frac{1}{12} \tag{22}$$

Hence

$$\frac{s+2}{s(s+1)^2(s+3)} = \frac{2}{3s} - \frac{1}{2(s+1)^2} - \frac{3}{4(s+1)} + \frac{1}{12(s+3)} \tag{23}$$

and

$$\mathfrak{L}^{-1}\left[\frac{s+2}{s(s+1)^2(s+3)}\right] = \frac{2}{3} - \frac{1}{2}\left(t+\frac{3}{2}\right)e^{-t} + \frac{1}{12}e^{-3t} \tag{24}$$

□

Solution 4. *The denominator has four roots: one double root $s = 0$, two conjugate roots $s = j2$ and $s = -j2$. Furthermore, we keep the conjugate roots together as a unit. We write the partial fraction expansion as follows.*

$$\frac{1}{3s^2(s^2+4)} = \frac{1}{3}\left[\frac{K_{12}}{s^2} + \frac{K_{11}}{s} + \frac{C_1 s + C_2}{s^2+4}\right] \tag{25}$$

Note that from

$$\frac{K_2}{s-j2} + \frac{K_3}{s+j2} = \frac{(K_2+K_3)s + j2(K_2-K_3)}{s^2+4} \tag{26}$$

we can put $C_1 = K_2 + K_3$ and $C_2 = j2(K_2 - K_3)$. Then re-write Eq. 25 as

$$\begin{aligned} 1 &= K_{12}(s^2+4) + K_{11}s(s^2+4) + (C_1 s + C_2)s^2 \\ &= (K_{11}+C_1)s^3 + (K_{12}+C_2)s^2 + 4K_{11}s + 4K_{12} \end{aligned} \tag{27}$$

Thus $K_{11} + C_1 = 0$, $K_{12} + C_2 = 0$, $4K_{11} = 0$, $4K_{12} = 1$, and $K_{12} = 1/4$, $K_{11} = 0$, $C_1 = 0$, $C_2 = -1/4$. Hence, from Eq. 25

$$\frac{1}{3s^2(s^2+4)} = \frac{1}{3}\left[\frac{1}{4s^2} - \frac{1}{4(s^2+4)}\right] \tag{28}$$

and then

$$\mathfrak{L}^{-1}\left[\frac{1}{3s^2(s^2+4)}\right] = \frac{1}{12}\left(t - \frac{1}{2}\sin 2t\right) \tag{29}$$

□

Solution 5. *First, let us write the expression for the input rectangular voltage pulse*

$$v_i(t) = U(t) - U(t-T) \tag{30}$$

Next, we write the differential equation of this simple one-loop circuit using Kirchoff's voltage law. Since the desired response is $v_c = q/C$ and $i = dq/dt = C(dv_c/dt)$, we write the equation with the dependent variable

$$RC\frac{dv_c}{dt} + v_c = v_i(t) = U(t) - U(t-T) \tag{31}$$

We now apply Laplace transform $\mathfrak{L}[v_c(t)] = V_c(s)$, then

$$RC\left[sV_c(s) - v_c(0^+)\right] + V_c(s) = \frac{1}{s}\left(1 - e^{-sT}\right) \tag{32}$$

With an initially uncharged C, $v_c(0^+) = 0$, hence

$$(RCs+1)V_c(s) = \frac{1}{s}\left(1 - e^{-sT}\right), \Rightarrow V_c(s) = \frac{1 - e^{-sT}}{s(RCs+1)} \tag{33}$$

Now we find the inverse transform of $V_c(s)$. Note that the factor e^{-sT} amounts to a simple shift in time T in the inverse transformation, we have

$$\frac{1}{s(RCs+1)} = \frac{1}{RCs(s+1/RC)} = \frac{1}{RC}\left[\frac{K_1}{s} + \frac{K_2}{s+1/RC}\right] \tag{34}$$

where

$$K_1 = \left[\frac{1}{s+1/RC}\right]_{s=0} = RC,\ K_2 = \left[\frac{1}{s}\right]_{s=-1/RC} = -RC \tag{35}$$

Hence

$$\mathfrak{L}\left[\frac{1}{s} - \frac{1}{s+1/RC}\right] = 1 - e^{-t/RC} \tag{36}$$

Finally, by considering the rectangular pulse (ceased after a time shift), then

$$\begin{aligned} v_c(t) &= \mathfrak{L}^{-1}\left[\frac{1-e^{-sT}}{s(RCs+1)}\right] \\ &= \left(1-e^{-t/RC}\right)U(t) - \left(1-e^{-(t-T)/RC}\right)U(t-T) \end{aligned} \tag{37}$$

A plot of $v_c(t)$ is shown in Fig. 2, the exact shape of $v_c(t)$ depends upon the relative values of T, R and C. □

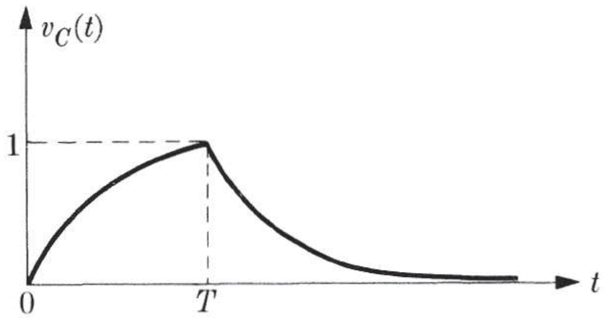

Figure 2: Output voltage of the R-C circuit