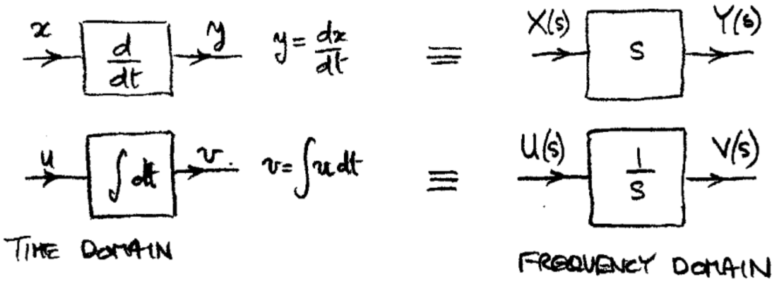
LECTURE 6 – ANALYSIS USING BLOCK DIAGRAMS

Block diagrams are a means to displaying graphically the connections between various components of a system. There are rules for reducing any block diagram down to just one box with an input and output. That box represents the transfer function of the system. Therefore we assume zero boundary conditions at all times when working with block diagrams. We can work either in the time or frequency domain as follows.

We recall

$$\mathcal{L}[f'(t)] = sF(s) \quad \text{and} \quad \mathcal{L}[\int f(t)dt] = \frac{F(s)}{s}$$

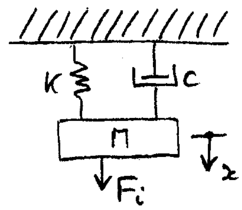
In the block diagram form we have



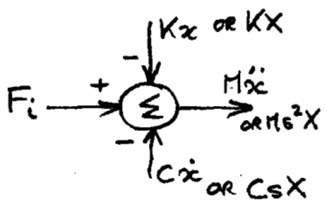
We see functions or multiplying factors are in boxes and variables are associated with the connecting lines.

<u>An illustrative example</u>

We want to find the transfer function of the mass/spring/damper system.

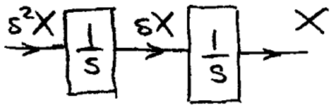


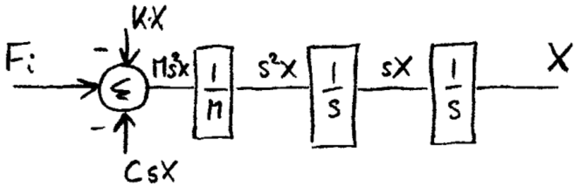
We know $F_i = Kx + c\dot{x} + M\ddot{x}$ so draw a summation junction making the highest differential the output.

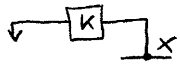


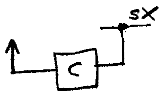
We will work in the <u>complex domain</u> for this solution using KX, CsX, Ms^2X etc.

The output is x with input F_i. So to obtain X alone we must divide the Ms^2X by M and then use the following:

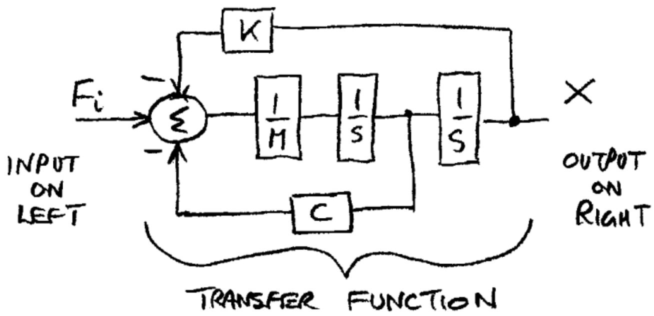




We also note we can derive KX using 

and CsX using 

So the complete block diagram is

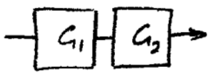


This has left us in the position of needing ways to reduce the block network down to one block that will contain the transfer function.

RULES FOR REDUCING BLOCK DIAGRAMS

Combinations of boxes can be reduced to one box using the following rules.

RULE

1 

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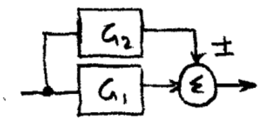
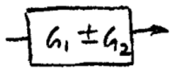
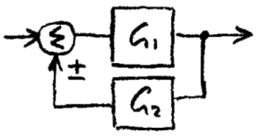
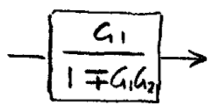
Multiplication (Serial)

2 

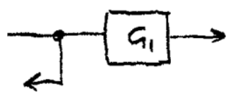
→ 

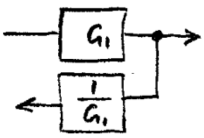
Addition/Subtraction (Parallel)

3 

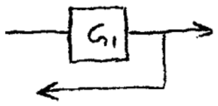
→ 

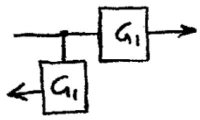
Feedback loop (positive or negative)

4 

→ 

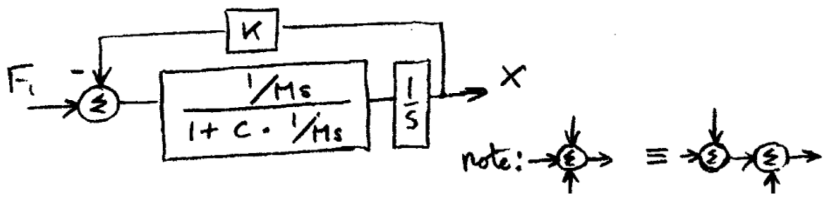
Moving a junction on

5 

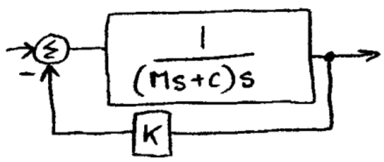
→ 

Moving a junction back

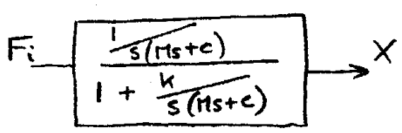
Now returning to our example, using Rule 3 gives



Using Rule 1 gives



Finally using Rule 3 gives



Simplifying the contents of the box gives

$\frac{X}{F_i} = \frac{1}{s(Ms+C)+K}$, the transfer function.

Just to complete the analysis, say that F_i was a step function of magnitude F. We will find out what the displacement x looks like.

$$\therefore X = \frac{F}{Ms\left(s^2 + \frac{C}{M}s + \frac{K}{M}\right)} = \frac{F}{K} \cdot \frac{K/_M}{s\left(s^2 + \frac{C}{M}s + \frac{K}{M}\right)}$$

To illustrate the inversion, we use

$$\frac{F}{K} = \frac{{\omega_n}^2}{s(s^2 + 2\xi\omega_n s + {\omega_n}^2)}$$

We made $\omega_n = \sqrt{\frac{K}{M}}$, $\xi = \frac{C}{2\sqrt{KM}}$; ω_n is called **natural frequency**, ξ is called **damping ratio**.

Hence using tables, the result is

$$x(t) = \frac{F}{K}\left(1 + \frac{1}{(1-\xi^2)^{1/2}} \cdot e^{-\xi\omega t} \cdot sin\left(\omega_n(1-\xi^2)^{1/2} \cdot t + \emptyset\right)\right)$$

with

$$\emptyset = tan^{-1}\frac{(1-\xi^2)^{1/2}}{\xi}$$

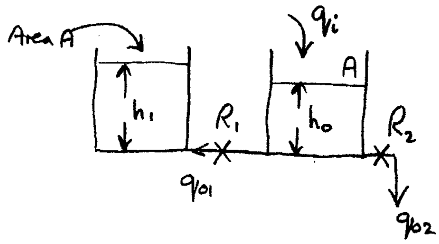
Such an expression says we have oscillations that are decaying away. So finally, what value does $x(t)$, $t \to \infty$ settle at?

Use the final value theorem on the definition of X.

$\lim_{t\to\infty}[x(t)] = \lim_{s\to 0}[s\left(\frac{1}{S}\cdot\frac{F}{Ms^2+Cs+K}\right)] = \frac{F}{K}$, in other words the resulting extension is that due to a static force F extending the spring, which is what we would expect, i.e. $F = Kx$.

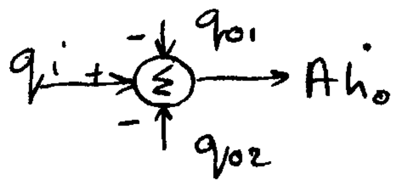
<u>Block diagrams for reservoirs</u>

We will analyse the rhs diagram treating q_i as an input and h_0 as an output.

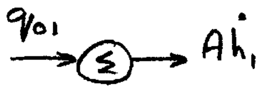


The net flow into the right-hand reservoir:

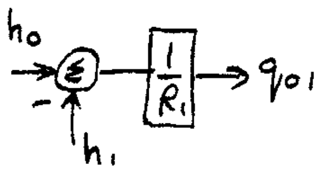
$A\dot{h}_0 = q_i - q_{01} - q_{02}$, keeping the sign convention which becomes:

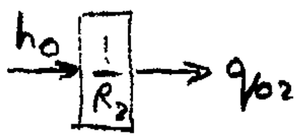


whereas the net flow into the left-hand reservoir is directly:

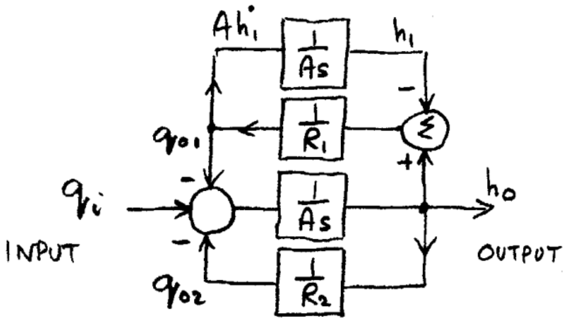


However, q_{01} is made up of (by superposition) $\frac{h_0}{R_1} - \frac{h_1}{R_1}$ and $q_{02} = \frac{h_0}{R_2}$ which gives



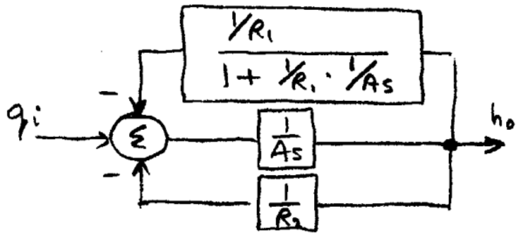


Combining all the sub block diagrams we get:

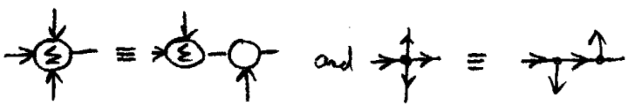


All variables have been written for the time domain just for a change.

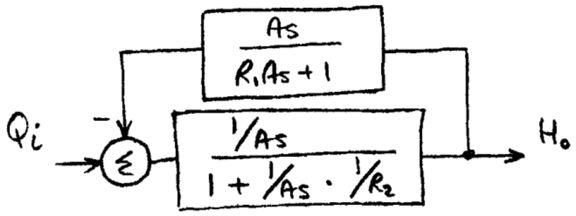
Treat the top part as a feedback loop and apply Rule 3 to get:



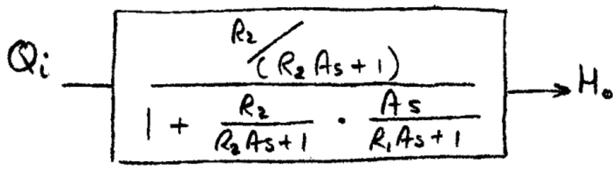
Remember:



Now apply Rule 3 to the bottom loop to get:



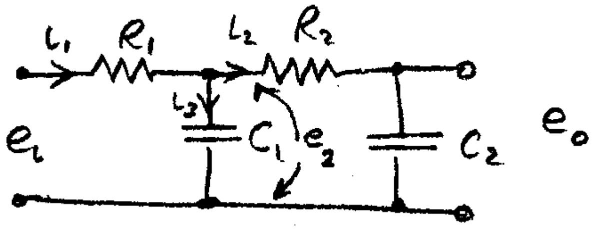
which reduces to one block using Rule 3 to get:



$$\therefore \frac{H_0}{Q_i} = \frac{R_2(R_1As+1)}{(R_2As+1)(R_1As+1)+R_2As}$$

Block diagrams for electrical circuits

We will analyse the rh diagram treating e_i as an input and e_0 as an output.



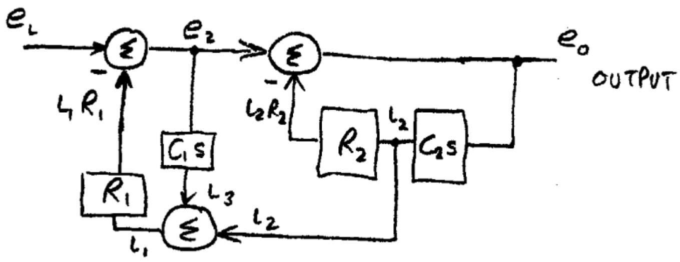
Applying Kirchhoff's Laws, we know

$$e_i = i_1 R_1 + e_2 \ , \ e_2 = i_2 R_2 + e_0 \ , \ e_0 = \frac{1}{c_2}\int i_2 dt$$

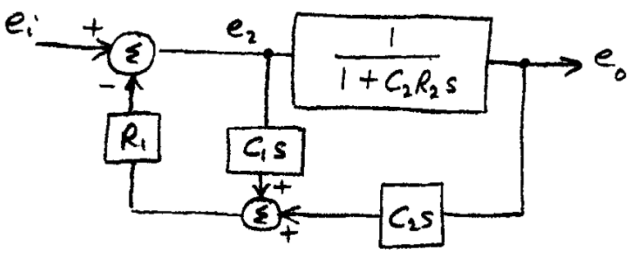
and

$$e_2 = \frac{1}{c_1}\int i_3 dt \quad \therefore \quad i_2 = c_2 \dot{e}_0 \ \text{ and } \ i_3 = c_1 \dot{e}_2$$

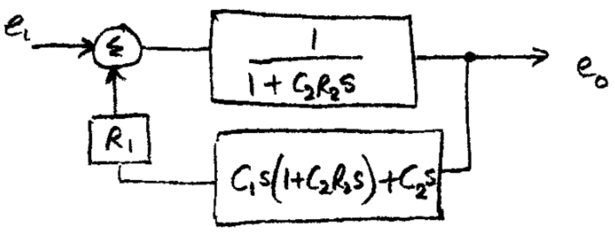
Hence we can draw, knowing $i_1 = i_2 + i_3$



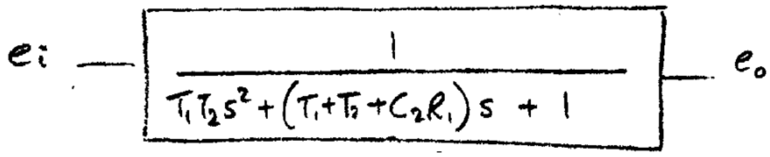
Next step is to move the junction at i_2 to e_0 using Rule 5 then apply Rule 3 to the right-hand loop to get:



Now move the top point of $C_1 s$ box to the output using Rule 4 and sum all the boxes in the feedback path using Rule 2 to get:



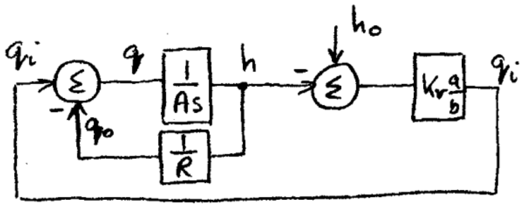
Finally, use Rule 3 to make one box, using $T_1 = C_1R_1$, $T_2 = C_2R_2$, to get:



This is the same result as in the example in LECTURE 4.4.

<u>Block diagrams for reservoir systems</u>

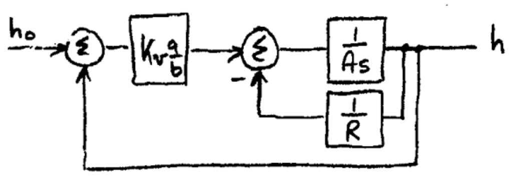
We will remake Example 6 in LECTURE 5 and show the full block diagram.



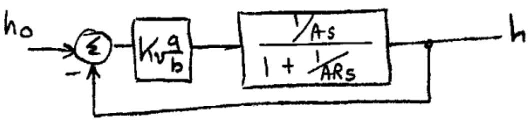
Using
$$q = q_i - q_0$$
$$q = A\dot{h}$$
$$q_i = K_v \frac{a}{b}(h_0 - h)$$
$$q_o = \frac{h}{R}$$

We can clearly see now that h_0 can be treated as an input like a demanded height to achieve; h remains the output. Redraw the diagram to put h_0 at left and h at right.

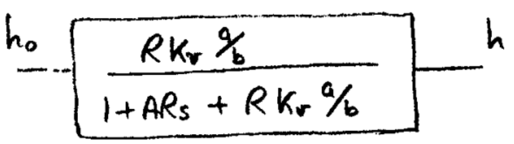
We get:



and using Rule 3 on the inner loop to get



and using Rule 1 to combine boxes then Rule 3 again gives



Hence

$$\frac{H(S)}{H_0(s)} = \frac{K}{\tau s+1} \quad \text{where} \quad K = \frac{RK_v{}^{a}/_{b}}{1+RK_v{}^{a}/_{b}} \equiv \frac{1}{1+\frac{b}{K_v aR}} \quad \text{and} \quad \tau = (1 + RK_v\frac{a}{b})^{-1}$$

These constants are the same as the previous solution. QED.

FINALLY! The general case:

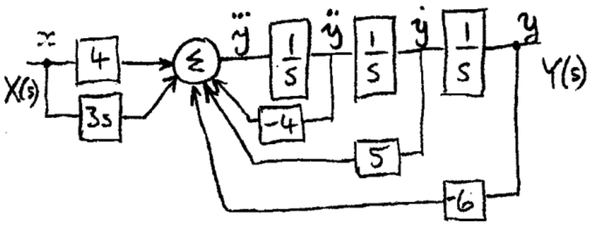
Say we have a general differential equation relating input to output. How does it translate itno a block diagram?

Take the equation:

$$\dddot{y} + 4\ddot{y} - 5\dot{y} + 6y = 3\dot{x} + 4x \;\; [y = f(x)],$$

rewrite as:

$$\dddot{y} = -4\ddot{y} + 5\dot{y} - 6y + 3\dot{x} + 4x$$, and start by creating all the right-hand terms.



Integrate $\dddot{y}$ in stages to y.
Put x on the left-hand side and differentiate. Then feedback each term needed with coefficients.

If this block diagram is reduced to one box it will generate the transfer function

$$\frac{Y}{X} = \frac{3s+4}{s^3+4s^2-5s+6}$$