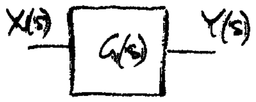
LECTURE 7 – ANALYSIS IN THE TIME DOMAIN

This section attempts to build up a consistent picture of how systems of various complexities will respond to standard inputs driving them.

We recall



where $\frac{Y(s)}{X(s)} = G(s) \equiv \frac{n(s)}{d(s)}$

$n(s)$ is the numerator polynomial in s and $d(s)$ the denominator. If we want to know what the transient response of the system might look like, it is $d(s)$ which determines all of the exponential terms that appear back in time domain solultions. To find these exponentials we equate $d(s) = 0$. This is known as the CHARACTERISTIC EQUATION and is fundamental to all studies in control. Note that

$$\frac{n(s)}{d(s)} = \frac{A}{(s+d_1)} + \frac{B}{(s+d_2)} \cdots \text{ etc}$$

and $\mathcal{L}^{-1}\{\frac{A}{s+d_1}\} = Ae^{-d_1 t}$ ... etc.

To discover the response of any system we must find $\mathcal{L}^{-1}\{G(s)X(s)\}$. We will now choose $G(s)$ and $X(s)$.

FOUR INPUTS $X(s)$

We will work with the following inputs

IDENTITY	$x(t)$	$X(s)$
Unit impulse	$\delta(t)$	1
Unit step	$u(t)$	$1/_s$
Unit ramp	t	$1/_{s^2}$
Unit cosine	$cos\omega t$	$\frac{s}{(s^2+\omega^2)}$

TWO SYSTEMS $G(s)$

We use two systems that are commonly called:

FIRST ORDER $\frac{K}{s+a}$, SECOND ORDER $\frac{K}{s^2+2\xi\omega_n s+\omega_n{}^2}$,

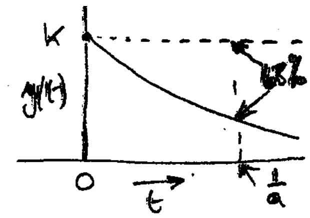
the order representing the multiplicity of differentiation. The form, with K, a, ξ, ω_n all being specific constants, will become clear later on.

[The simplest system/input combinations will be fully analysed. The more complex ones will be given by analogy.]

<u>TAKE THE SYSTEM</u> $\frac{K}{s+a}$

With $\delta(t)$

A decaying exponential

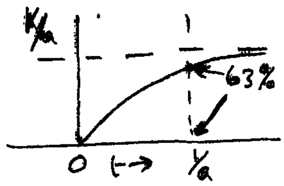


$$Y(s) = \frac{K}{s+a} \cdot 1 \ \therefore y(t) = \mathcal{L}^{-1}\{\frac{K}{s+a}\}$$

$$\therefore y(t) = Ke^{-at}$$

With u(t)

A rising exponential

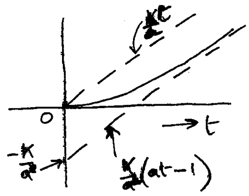


$$Y(s) = \frac{K}{s+a} \cdot \frac{1}{s} \ \therefore y(t) = \mathcal{L}^{-1}\{\frac{K}{s(s+a)}\}$$

$$\therefore y(t) = \frac{K}{a}(1 - e^{-at})$$

With t

A delayed ramp output



$$Y(s) = \frac{K}{(s+a)} \cdot \frac{1}{s^2} \ \therefore y(t) = \mathcal{L}^{-1}\{\frac{K}{s^2(s+a)}\}$$

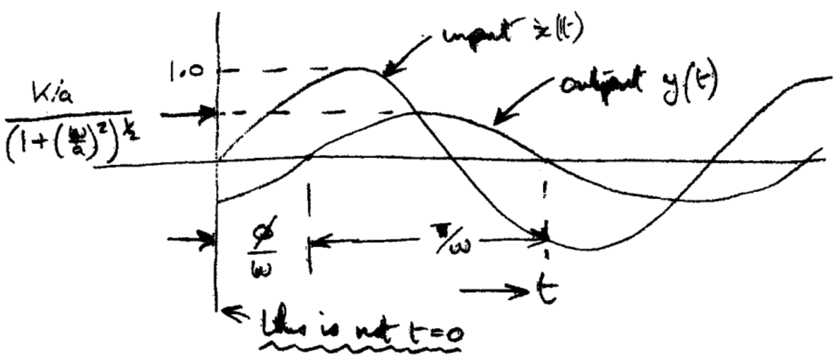
$$\therefore y(t) = \frac{K}{a^2}(e^{-at} + at - 1)$$

With $\frac{s}{s^2+\omega^2}$

$$Y(s) = \frac{K}{(s+a)} \cdot \frac{s}{(s^2+\omega^2)} \ \therefore y(t) = \mathcal{L}^{-1}\{\frac{K}{s+a} \cdot \frac{s}{s^2+\omega^2}\}$$

$$\therefore y(t) = \frac{\frac{K}{a}}{\left(1+\left(\frac{\omega}{a}\right)^2\right)} e^{-at} + \frac{\frac{K}{a}}{\left(1+\left(\frac{\omega}{a}\right)^2\right)^{\frac{1}{2}}} cos(\omega t - \varphi)$$

$\varphi = tan^{-1}\frac{\omega}{a}$. Notice the first term is a transient and dies out after about 5/a seconds. We will plot the second term that persists, against the input.



A delayed cosine of different magnitude

TAKE THE SYSTEM $\frac{K}{s^2+2\xi\omega_n s+\omega_n{}^2}$

Before we can proceed, it has to be noted that the characteristic equation of this system gives rise to roots that are in some ways not fully defined. We study the implications of this point before proceeding to a full analysis. Look again at the possible roots of the denominator and note the significance of the parameter ξ known as the DAMPING RATIO (or DAMPING COEFFICIENT).

Case 1 $\frac{K}{(s+\alpha)(s+\beta)}$ The roots are both real and different. Therefore, this system is known to be OVERDAMPED and $\xi > 1$.

Case 2 $\frac{K}{(s+\gamma)^2}$ The roots are real and the same. Therefore, this system is known to be CRITICALLY DAMPED and $\xi = 1$.

Case 3 $\frac{K}{s^2+2\xi\omega_n s+\omega_n{}^2}$ The roots are a complex conjugate pair. Therefore the system is known to be UNDERDAMPED and $0 < \xi < 1$.

Case 4 $\frac{K}{s^2+\omega^2}$ Here $\xi = 0$ and the roots are imaginary. Therefore the system is totally UNDAMPED, that is, it will oscillate at all times.

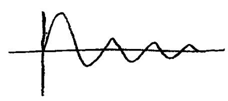
A selection of system cases with inputs will now be analysed.

WITH $\boldsymbol{\delta(t)}$ $y(t) = \mathcal{L}^{-1}\{\frac{K}{s^2+2\xi\omega_n s+\omega_n{}^2}\}$ OR one of the other forms above

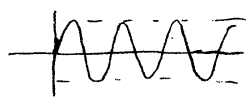
IF $\xi > 1$, $\mathrm{y(t)} = \frac{\mathrm{K}}{\beta-\alpha}\left(\mathrm{e}^{-\alpha t} - \mathrm{e}^{-\beta t}\right)$

IF $\xi = 1$, $y(t) = Kte^{-\gamma t}$

IF $\xi < 1$, $\mathrm{y(t)} = \frac{\mathrm{K}}{\omega} \cdot \frac{1}{\sqrt{1-\xi^2}} \cdot \mathrm{e}^{-\xi\omega t} \cdot \sin[\omega_n\sqrt{1-\xi^2}\,\mathrm{t}]$



IF $\xi = 0$, $y(t) = \frac{K}{\omega} \cdot sin\omega t$



Hence we see the bigger ξ is the more damped the output is.

WITH $u(t)$

Here is the analysis for an UNDERDAMPED system:

Case 3 $\quad y(t) = \mathcal{L}^{-1}\{\frac{K}{s(s^2+2\xi\omega_n s+\omega_n{}^2)}\}$

$$= \frac{K}{\omega_n{}^2}\left(1 - \frac{1}{\sqrt{1-\xi^2}} \cdot e^{-\xi\omega_n t} \cdot sin\left(\omega_n\sqrt{1-\xi^2}\,t + \varphi\right)\right)$$

with $\quad \varphi = tan^{-1}(\frac{\sqrt{1-\xi^2}}{\xi})$

To see what this looks like graphically, we reveal

$sin(A+B) = sinAcosB + cosAsinB,$ with $A = \omega_n\sqrt{1-\xi^2}\,t\,, b = \varphi$

Hence we rewrite our output as

$$y(t) = \frac{K_1}{\omega_n{}^2}\left(1 - \frac{\xi}{\sqrt{1-\xi^2}} \cdot e^{-\xi\omega_n t} \cdot sin\left(\omega_n\sqrt{1-\xi^2}\,t\right) - e^{-\xi\omega_n t} \cdot cos\left(\omega_n\sqrt{1-\xi^2}\,t\right)\right)$$

First term

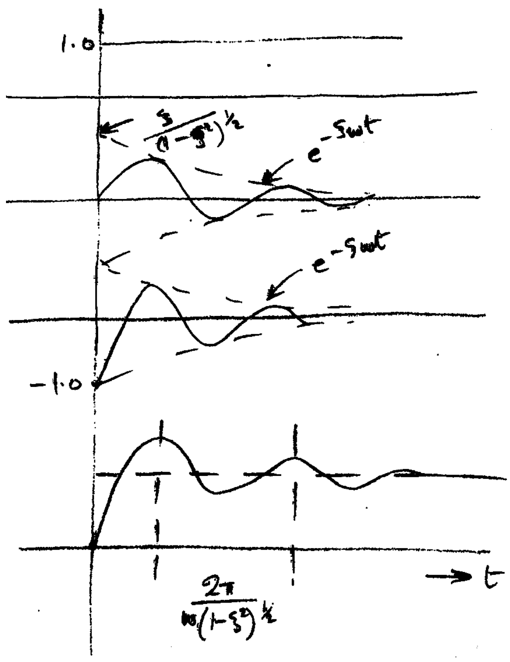
\-

Second term

\+

Third term

Gives this output



$\frac{K}{\omega_n{}^2}$ is just scaling factor

A step function with decaying oscillations

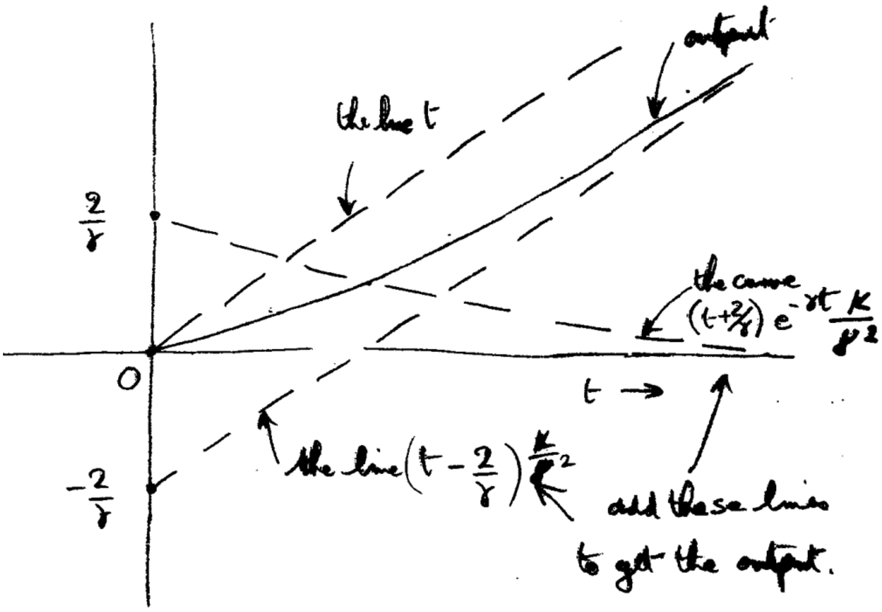
WITH t

Here is the analysis for a CRITICALLY DAMPED system:

Case 2 $\quad y(t) = \mathcal{L}^{-1}\{\frac{K}{s^2(s+\gamma)^2}\}$

$$= \frac{K}{\gamma^2}\left[t - \frac{2}{\gamma} + \left(t + \frac{2}{\gamma}\right) e^{-\gamma t}\right]$$

Let us look at the output graphically.



WITH $\boldsymbol{cos\omega t}$

Case 1 Complete the analysis for an OVERDAMPED system:

$$y(t) = \mathcal{L}^{-1}\{\frac{s}{s^2+\omega^2} \cdot \frac{K}{(s+\alpha)(s+\beta)}\}$$

HINT: Use the partial fractions method.

Case 3 UNDERDAMPED

$$y(t) = \mathcal{L}^{-1}\{\frac{s}{(s^2+\omega^2)} \cdot \frac{K}{s^2+2\xi\omega_n s+{\omega_n}^2}\}$$

According to the partial fractions rules:

$$Y(s) = \frac{As+B}{s^2+\omega^2} + \frac{Cs+D}{s^2+2\xi\omega_n s+{\omega_n}^2},$$

which results in:

$$Y(s) = \frac{K\omega}{K_1}\left[\frac{1}{\omega} sin(\omega t + \varphi_1) + \frac{1}{\omega_n\sqrt{1-\xi^2}} \cdot e^{-\xi\omega_n t} \cdot sin\left(\omega_n\sqrt{1-\xi^2}t + \varphi_2\right)\right],$$

with: $K_1 = \sqrt{4\xi{\omega_n}^2\omega^2 + {\omega_n}^2 - \omega^2}$,

$$\varphi_1 = tan^{-1}(-\frac{2\xi\omega_n\omega}{\omega_n{}^2-\omega^2})\ ,\ \varphi_2 = tan^{-1}[-\frac{2\xi\omega_n{}^3(1-\xi^2)}{\omega_n{}^2-\omega^2}]$$

[Alternatively: $y(t) = \mathcal{L}^{-1}\{\frac{1}{(s^2+\omega^2)}\cdot\frac{K}{(s+\alpha)^2+b^2}\}$

$$= \frac{K\omega}{K_1}\left[\frac{1}{\omega}sin(\omega t+\varphi_1)+\frac{1}{b}\cdot e^{-at}\cdot sin(bt+\varphi_2)\right],$$

with $K_1 = [4a^2\omega^2+(a^2+b^2-\omega^2)]^{1/2}$,

$$\varphi_1 = tan^{-1}(-\frac{2a\omega}{a^2+b^2-\omega^2}),\quad \varphi_2 = tan^{-1}(-\frac{2ab}{a^2+b^2-\omega^2})$$

Again it is best to look at this graphically. There are just two terms in the solution.
Ignore $\frac{K\omega}{K_1}$ as a scaling constant.