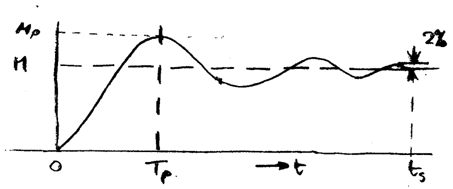
LECTURE 8 – PERFORMANCE PARAMETERS

One often used measure of how well a system responds to driving signals is to look at the response to a unit step input in terms of the shape of the output.

Here are the parameters that will be looked at.



T_p - the time to the first peak

$M_{\%}$ - the maximum percentage overshoot $M_{\%} = \frac{100(M_P - M)}{M}$

t_s - the settling time for the output to get within 2% of its final value.

These parameters only apply to a system that is essentially second order (perhaps with smaller higher order effects) and is UNDERDAMPED.

Hence we are looking at

$$y(t) = \mathcal{L}^{-1}\{\frac{1}{s} \cdot \frac{K}{s^2 + 2\xi\omega_n s + {\omega_n}^2}\}$$

$$= \frac{K}{{\omega_n}^2}\left[1 - \frac{1}{\sqrt{1-\xi^2}} \cdot e^{-\xi\omega_n t} \cdot sin\left(\omega_n\sqrt{1-\xi^2}t + \varphi\right)\right]$$

With $\varphi = tan^{-1}(\frac{\sqrt{1-\xi^2}}{\xi})$

ANALYSIS FOR T_p

To find T_p look for $\dot{y}(t) = 0$

First rewrite $y(t)$ as:

$$y(t) = \frac{K}{{\omega_n}^2}\left[1 - e^{-\xi\omega t}\left(cos\omega_n\sqrt{1-\xi^2}t + \frac{\xi}{\sqrt{1-\xi^2}} sin\left(\omega_n\sqrt{1-\xi^2}t\right)\right)\right]$$

Then differentiate it and equate $\dot{y}(t)$ with zero making $t = T_P$.

$$sin\left(\omega_n\sqrt{1-\xi^2}T_p\right) = 0$$

$\therefore \omega_n\sqrt{1-\xi^2} \cdot T_p = n\pi$ $\quad n = 1,2 \ldots$ and reselect $n = 1$ for the first rise.

$$\therefore T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

ANALYSIS FOR $M_{\%}$

M_P is obtained by back substitution of T_p in $y(t)$

$$M_p = \frac{K}{{\omega_n}^2}\left[1 + \frac{1}{\omega_n\sqrt{1-\xi^2}} \cdot e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \cdot sin(\pi - \varphi)\right]$$

But $\varphi = sin^{-1}\sqrt{1-\xi^2}$

$$M_p = \frac{K}{{\omega_n}^2}\left[1 + \frac{1}{\cancel{\omega_n\sqrt{1-\xi^2}}} \cdot e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \cdot \cancel{sin\left(sin^{-1}\omega_n\sqrt{1-\xi^2}\right)}\right]$$

Since $M = \frac{K}{{\omega_n}^2}$

and $\quad M_{\%} = \frac{M_p - M}{M} \times 100 = \frac{K}{{\omega_n}^2}\left[\frac{1+e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} - 1}{\frac{K}{{\omega_n}^2}}\right] \cdot 100$

$$\therefore M_{\%} = 100e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}$$

ANALYSIS FOR t_s

We chose a bandwidth within which the ripple is regarded as insignificant. 2% is often used. This means that the factor in $y(t)$ that multiplies the sin (the amplitude) must be reduced to that value (0.02) at the time of interest.

$$\therefore \frac{1}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t_s} = x(\equiv 0.02)$$

Take logs to get $t_s = \frac{1}{\xi\omega_n} \cdot \ln\left[\frac{1}{x\sqrt{1-\xi^2}}\right]$

Expand the log to get $t_s = \frac{1}{\xi\omega_n}\left[\ln\frac{1}{x} + \frac{1}{2}\ln\frac{1}{(1-\xi^2)}\right]$

If we recognise that the parameters of interest are usually associated with low damping where $\xi \leq 0.3$, we see that $\frac{1}{2}\ln\frac{1}{(1-\xi^2)} \rightarrow$ very small or zero.

$\therefore t_s \simeq \frac{1}{\xi\omega_n}\ln\left(\frac{1}{x}\right)$ and if we make $x = 0.02$

$$\therefore t_s \simeq \frac{3.91}{\xi\omega_n}$$

EXAMPLE

If a system transfer function is $G(s) = \frac{1}{0.01s^2+0.05s+1}$ and a unit step is applied to the input, find the three performance parameters: $T_p,\ M_{\%}$ and t_s .

Rewrite

$$G(s) = \frac{100}{s^2+5s+100}$$

Find the characteristic equation roots from

$$s^2 + 5s + 100 = 0 \quad \therefore s = -2.5 \pm j\sqrt{93.75}$$

Therefore the system is UNDERDAMPED and appropriate for analysis.

We can also equate $2\xi\omega_n = 5,\ {\omega_n}^2 = 100\ \ \therefore \omega_n = 10$

$$\therefore \xi = \frac{5}{2.10} = 0.25$$

$$\therefore T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{3.14}{10\sqrt{1-0.25^2}} = 0.324 \text{ seconds}$$

$$\therefore M_{\%} = 100e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} = 100e^{\frac{-0.25\cdot\pi}{\sqrt{1-0.25^2}}} = 44.4\%$$

$$\therefore t_s = \frac{3.91}{\xi\omega_n} = \frac{3.91}{0.25\cdot 10} = 1.56 \text{ seconds}$$

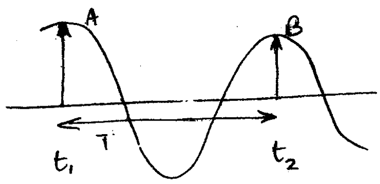
The student should now try this exercise with

$$G(s) = \frac{1/16}{\frac{s^2}{16}+\frac{10}{16}s+1} \text{ and be careful.}$$

THE LOGARITHMIC DECREMENT

If we can capture just a few oscillations of the output of a system that has been perturbed in some way, we can still find the damping coefficient.

Consider the waveform:



We recall the transient is of the form:

$$\frac{K}{\omega_n{}^2\sqrt{1-\xi^2}} \cdot e^{-\xi\omega_n t} \cdot sin\left(\omega_n\sqrt{1-\xi^2}t\right)$$

If we assume A & B are measured at the peak of the waveform, then

$$A \propto e^{-\xi\omega_n t_1} \text{ and } B \propto e^{-\xi\omega_n t_2}$$

The ratio $\frac{A}{B} = \frac{e^{-\xi\omega_n t_1}}{e^{-\xi\omega_n t_2}} = e^{-\xi\omega_n(t_1-t_2)} \equiv R$

R can be measured. $T = (t_2 - t_1)$ can be measured.

Take logs to get $-\xi\omega_n(t_1 - t_2) = \ln R$

$$\therefore \xi = \frac{1}{\omega_n T}\ln R$$

But T is the period of waveform $= \frac{2\pi}{\omega_n\sqrt{1-\xi^2}} \equiv T$

If we assume $\xi \ll 1$, then $T \simeq \frac{2\pi}{\omega_n}$

$$\therefore \xi \simeq \frac{1}{\omega_n} \cdot \frac{\omega_n}{2\pi}\ln R = \frac{1}{2\pi} \cdot \ln R$$

Hence just by measuring R above we can get ξ .

EXAMPLE

A technician measures $A = 2.4cm$ and $B = 1.9cm$ on a scope for successive peaks of a waveform output. What is the damping coeficient in the system?

$$R = \frac{2.4}{1.9} = 1.263 \quad \therefore \xi = \frac{1}{2\pi}\ln 1.263 = \frac{0.233}{6.283} = 0.037$$