Mechanical and Electrical System Models

We consider mathematical modeling of a variety of mechanical systems and electrical systems that may appear in control systems. The fundamental law governing mechanical systems is Newton's second law. We apply this law to various mechanical systems and derive transfer function models and state-space models.

The basic laws governing electrical circuits are Kirchhoffs laws. We obtain transfer-function models and state-space models of various electrical circuits and operational amplifier systems that may appear in many control systems.

1 Mathematical Modeling of Mechanical Systems

1.1 Spring Systems

Let us obtain the equivalent spring constants for the systems shown in Figures 1(a) and 1(b), respectively.

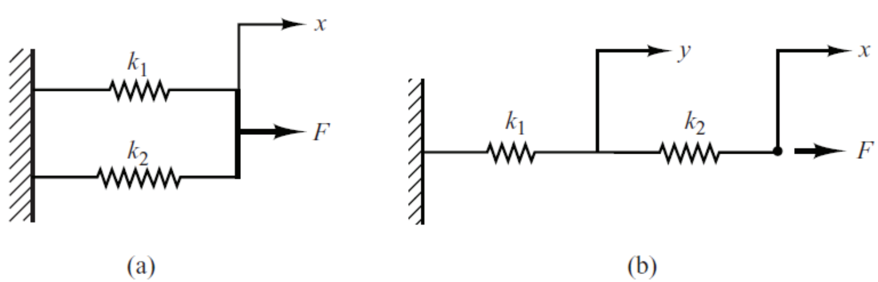

Figure 1: (a) System consisting of two springs in parallel; (b) system consisting of two springs in series.

For the springs in parallel, Figure 1(a), the equivalent spring constant k_{eq} is obtained from

$$k_1 x + k_2 x = F = k_{eq} x \Rightarrow k_{eq} = k_1 + k_2 \tag{1}$$

For the springs in series, Figure 1(b), the force in each spring is the same. Thus

$$k_1 y = F,\ \ k_2(x - y) = F \tag{2}$$

Elimination of y from these two equations results in

$$k_2 x = F + \frac{k_2}{k_1} F = \frac{k_1 + k_2}{k_1} F \tag{3}$$

The equivalent spring constant k_{eq} for this case is then found as

$$keq = \frac{F}{x} = \frac{k_1 k_2}{k_1 + k_2} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} \tag{4}$$

1.2 Damper Systems

Let us obtain the equivalent viscous-friction coefficient for each of the damper systems shown in Figures 2(a) and 2(b). An oil-filled damper is often called a dashpot. A dashpot is a device that provides viscous friction, or damping. It consists of a piston and oil-filled cylinder. Any relative motion between the piston rod and the cylinder is resisted by the oil because the oil must flow around the piston (or through orifices provided in the piston) from one side of the piston to the other. The dashpot essentially absorbs energy. This absorbed energy is dissipated as heat, and the dashpot does not store any kinetic or potential energy.

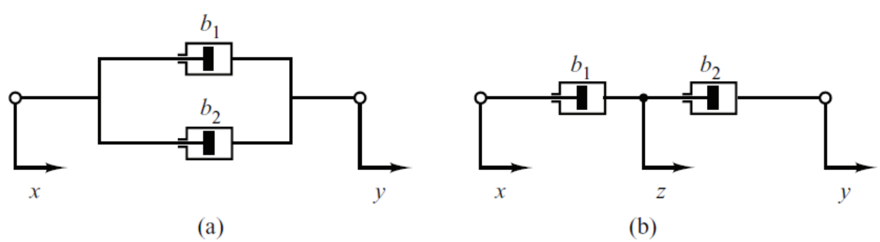

Figure 2: (a) Two dampers connected in parallel; (b) two dampers connected in series.

Case (a): The force f due to the dampers is

$$f = b_1(\dot{y} - \dot{x}) + b_2(\dot{y} - \dot{x}) = (b_1 + b_2)(\dot{y} - \dot{x}) \tag{5}$$

In terms of the equivalent viscous-friction coefficient b_{eq} , force f is given by

$$f = b_{eq}(\dot{y} - \dot{x}) \tag{6}$$

Hence

$$b_{eq} = b_1 + b_2 \tag{7}$$

Case (b): The force f due to the dampers is

$$f = b_1(\dot{z} - \dot{x}) = b_2(\dot{y} - \dot{z}) \tag{8}$$

where z is the displacement of a point between damper b_1 and damper b_2. Note that the same force is transmitted through the shaft. Then, we have

$$(b_1 + b_2)\dot{z} = b_2\dot{y} + b_1\dot{z} \Rightarrow \dot{z} = \frac{1}{b_1 + b_2}(b_2\dot{y} + b_1\dot{x}) \tag{9}$$

In terms of the equivalent viscous-friction coefficient b_{eq}, force f is given by

$$f = b_{eq}(\dot{y} - \dot{x}) \tag{10}$$

By substituting Equation (9) into Equation (8), we have

$$f = b_2(\dot{y} - \dot{z}) = b_2\left[\dot{y} - \frac{1}{b_1 + b_2}(b_2\dot{y} + b_1\dot{x})\right] = \frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x}) \tag{11}$$

Thus

$$f = b_{eq}(\dot{y} - \dot{x}) = \frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x}) \tag{12}$$

Hence

$$b_{eq} = \frac{b_1 b_2}{b_1 + b_2} = \frac{1}{\frac{1}{b_1} + \frac{1}{b_2}} \tag{13}$$

1.3 Spring-Mass-Dashpot System

Consider the spring-mass-dashpot system mounted on a massless cart as shown in Figure 3. Let us obtain mathematical models of this system by assuming that the cart is standing still for $t < 0$ and the spring-mass-dashpot system on the cart is also standing still for $t < 0$. In this system, $u(t)$ is the displacement of the cart and is the input to the system. At $t = 0$, the cart is moved at a constant speed, or $\ddot{u}$=constant. The displacement $y(t)$ of the mass is the output. The displacement is relative to the ground. In this system, m denotes the mass, b denotes the viscous-friction coefficient, and k denotes the spring constant. We assume that the friction force of the dashpot is proportional to and that the spring is a linear spring; that is, the spring force is proportional to $y - u$. For translational systems, Newton's second law states that

$$ma = \sum F \tag{14}$$

where m is a mass, a is the acceleration of the mass, and $\sum F$ is the sum of the forces acting on the mass in the direction of the acceleration a. Applying Newton's second law to the present system and noting that the cart is massless, we obtain

$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y - u) \tag{15}$$

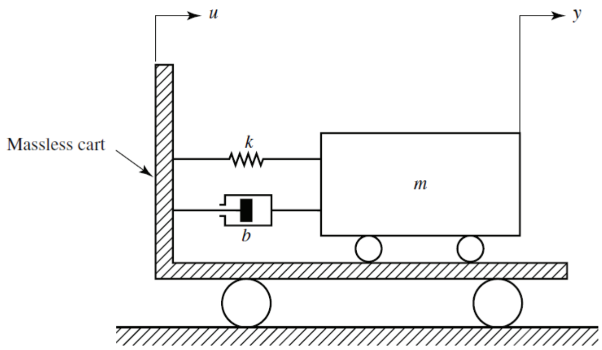

Figure 3: Spring-mass-dashpot system mounted on a cart.

$$m\frac{d^y}{dt^2} + b\frac{dt}{dt} + ky = b\frac{du}{dt} + ku \tag{16}$$

This equation represents a mathematical model of the system considered. Taking the Laplace transform of this last equation, assuming zero initial condition, gives

$$(ms^2 + bs + k)Y(s) = (bs + k)U(s) \tag{17}$$

Taking the ratio of $Y(s)$ to $U(s)$, we find the transfer function of the system to be

$$G(s) = \frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k} \tag{18}$$

Such a transfer-function representation of a mathematical model is used very frequently in control engineering.

1.3.1 State-Space Model

Next we shall obtain a state-space model of this system. We shall first compare the differential equation for this system

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{b}{m}\dot{u} + \frac{k}{m}u \tag{19}$$

with the standard form

$$\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u \tag{20}$$

and identify a_1, a_2, b_0, b_1, and b_2 as follows:

$$a_1 = \frac{b}{m},\ a_2 = \frac{k}{m},\ b_0 = 0,\ b_1 = \frac{b}{m},\ b_2 = \frac{k}{m} \tag{21}$$

Define

$$x_1 = y,\ x_2 = \dot{x}_1 - \frac{b}{m} \tag{22}$$

Then

$$\dot{x}_1 = x_2 + \frac{b}{m} \tag{23}$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{k}{m}x_2 + \left[\frac{k}{m} - \left(\frac{b}{m}\right)^2\right]u \tag{24}$$

and the output equation becomes

$$y = x_1 \tag{25}$$

then

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{b}{m}\\ \frac{k}{m} - \left(\frac{b}{m}\right)^2\end{bmatrix}u \tag{26}$$

and

$$y = [1\ 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{27}$$

Equations (26) and (27) give a state-space representation of the system. Note that this is not the only state-space representation. There are infinitely many state-space representations for the system.

2 Mathematical Modeling of Electrical Systems

Basic laws governing electrical circuits are Kirchhoff's current law and voltage law. Kirchhoff's current law (node law) states that the algebraic sum of all currents entering and leaving a node is zero. This law can also be stated as follows: The sum of currents entering a node is equal to the sum of currents leaving the same node. Kirchhoff's voltage law (loop law) states that at any given instant the algebraic sum of the voltages around any loop in an electrical circuit is zero. This law can also be stated as follows: The sum of the voltage drops is equal to the sum of the voltage rises around a loop. A mathematical model of an electrical circuit can be obtained by applying one or both of Kirchhoff's laws to it.

2.1 LRC Circuit

Consider the electrical circuit shown in Figure 4. The circuit consists of an inductance L (henry), a resistance R (ohm), and a capacitance C (farad). Applying Kirchhoff's voltage law to the system, we obtain the following equations:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = e_i \tag{28}$$

$$\frac{1}{C}\int idt = e_o \tag{29}$$

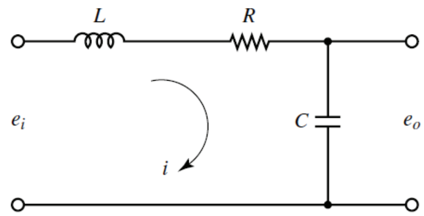

Figure 4: Electrical circuit.

A transfer-function model of the circuit can also be obtained as follows: Taking the Laplace transforms of Equations (28) and (29), assuming zero initial conditions, we obtain

$$LsI(s) + RI(s) + \frac{1}{sC}I(s) = E_i(s) \tag{30}$$

$$\frac{1}{sC}I(s) = E_0(s) \tag{31}$$

If e_i is assumed to be the input and e_o the output, then the transfer function of this system is found to be

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1} \tag{32}$$

A state-space model of the system shown in Figure 4 may be obtained as follows: First, note that the differential equation for the system can be obtained from Equation (32) as

$$\ddot{e}_o + \frac{R}{L}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{LC}e_i \tag{33}$$

Then by defining state variables by

$$x_1 = e_o, \; x_2 = \dot{e}_o \tag{34}$$

and the input and output variables by

$$u = e_i, \; y = e_o = x_1 \tag{35}$$

we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{C} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u \tag{36}$$

and

$$y = [1 \; 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{37}$$

These two equations give a mathematical model of the system in state space.

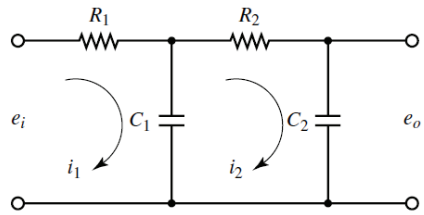

Figure 5: Cascaded Electrical system.

2.2 Transfer Functions of Cascaded Elements

Many feedback systems have components that load each other. Consider the system shown in Figure 5. Assume that e_i is the input and e_o is the output. The capacitances C_1 and C_2 are not charged initially. The second stage of the circuit (R_2C_2 portion) produces a loading effect on the first stage (R_1C_1 portion). The equations for this system are

$$\frac{1}{C_1}\int (i_1 - i_2)dt + R_1 i_1 = e_1 \tag{38}$$

and

$$\frac{1}{C_1}\int (i_1 - i_2)dt + R_2 i_2 + \frac{1}{C_2}\int i_2 dt = 0 \tag{39}$$

$$\frac{1}{C_2}\int i_2 dt = e_o \tag{40}$$

Taking the Laplace transforms of Equations (38) through (40), respectively, using zero initial conditions, we obtain

$$\frac{1}{C_1 s}[I_1(s) - I_2(s)] + R_1 I_1(s) = E_i(S) \tag{41}$$

$$\frac{1}{C_1 s}[I_1(s) - I_2(s)] + R_2 I_2(s) + \frac{1}{sC_2} I_2(s) = 0 \tag{42}$$

$$\frac{1}{sC_2} I_2(s) = E_o(s) \tag{43}$$

Eliminating $I_1(s)$ from Equations (41) and (42) and writing $E_i(s)$ in terms of $I_2(s)$, we find the transfer function between $E_o(s)$ and $E_i(s)$ to be

$$\begin{aligned}\frac{E_o(s)}{E_i(s)} &= \frac{1}{R_1C_1s + 1(R_2C_2s + 1) + R_1C_2s} \\ &= \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_1C_2)s + 1}\end{aligned} \tag{44}$$

The term R_1C_2s in the denominator of the transfer function represents the interaction of two simple RC circuits. Since the two roots of the denominator of Equation (44) are real.

The present analysis shows that, if two RC circuits are connected in cascade so that the output from the first circuit is the input to the second, the overall transfer function is not the product of $1/(R_1C_1s+1)$ and $1/(R_2C_2s+1)$.

The reason for this is that, when we derive the transfer function for an isolated circuit, we implicitly assume that the output is unloaded. In other words, the load impedance is assumed to be infinite, which means that no power is being withdrawn at the output. When the second circuit is connected to the output of the first, however, a certain amount of power is withdrawn, and thus the assumption of no loading is violated. Therefore, if the transfer function of this system is obtained under the assumption of no loading, then it is not valid. The degree of the loading effect determines the amount of modification of the transfer function.

3 Electronic Controllers

In what follows we shall discuss electronic controllers using operational amplifiers. We begin by deriving the transfer functions of simple operational amplifier circuits. Then we derive the transfer functions of some of the operational-amplifier controllers.

3.1 Operational Amplifiers

Operational amplifiers, often called op amps, are frequently used to amplify signals in sensor circuits. Op amps are also frequently used in filters used for compensation purposes. Figure 6 shows an op amp.

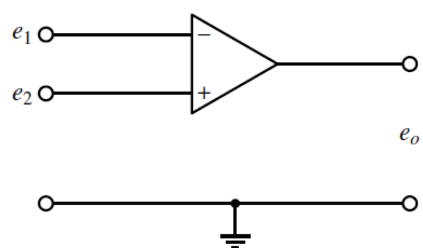

Figure 6: Operational amplifier.

It is a common practice to choose the ground as 0 volt and measure the input voltages e_1 and e_2 relative to the ground. The input e_1 to the minus terminal of the amplifier is inverted, and the input e_2 to the plus terminal is not inverted. The total input to the amplifier thus becomes $e_2 - e_1$. Hence, for the circuit shown in Figure 6, we have

$$e_0 = K(e_2 - e_1) = -K(e_1 - e_2) \tag{45}$$

where the inputs e_1 and e_2 may be dc or ac signals and K is the differential gain (voltage gain). The magnitude of K is approximately $10^5 \sim 10^6$ for dc signals and ac signals with frequencies less than approximately 10 Hz. The differential gain K decreases with the signal frequency and becomes about unity for frequencies of 1 MHz $\sim$ 50 MHz. Note that the op amp amplifies the difference in voltages e_1 and e_2. Such an amplifier is commonly called a differential amplifier. Since the gain of the op amp is very high, it is necessary to have a negative feedback from the output to the input to make the amplifier stable.

In the ideal op amp, no current flows into the input terminals, and the output voltage is not affected by the load connected to the output terminal. In other words, the input impedance is infinity and the output impedance is zero. In an actual op amp, a very small (almost negligible) current flows into an input terminal and the output cannot be loaded too much. In our analysis here, we make the assumption that the op amps are ideal.

3.1.1 Inverting Amplifier

Consider the operational-amplifier circuit shown in Figure 7. Let us obtain the output voltage e_o.

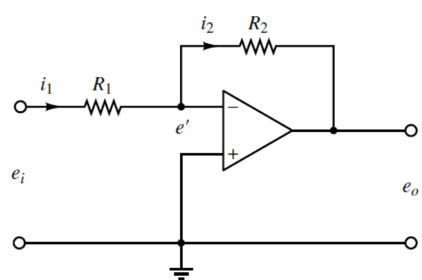

Figure 7: Inverting amplifier.

The equation for this circuit can be obtained as follows: Define

$$i_1 = \frac{e_i - e'}{R_1}, \; i_2 = \frac{e' - e_o}{R_2} \tag{46}$$

Since only a negligible current flows into the amplifier, the current i_1 must be equal to current i_2. Thus

$$\frac{e_i - e'}{R_1} = \frac{e' - e_o}{R_2} \tag{47}$$

Since $K(0 - e') = E_o$ and $K >> 1$, e' must be almost zero. Hence we have

$$\frac{e_i}{R_1} = -\frac{e_o}{R_2}, \; e_o = -\frac{R_2}{R_1} e_i \tag{48}$$

Thus the circuit shown is an inverting amplifier. If $R_1 = R_2$, then the op-amp circuit shown acts as a sign inverter.

3.1.2 Noninverting Amplifier

Figure 8(a) shows a noninverting amplifier. A circuit equivalent to this one is shown in Figure 8(b). For the circuit of Figure 8(b), we have

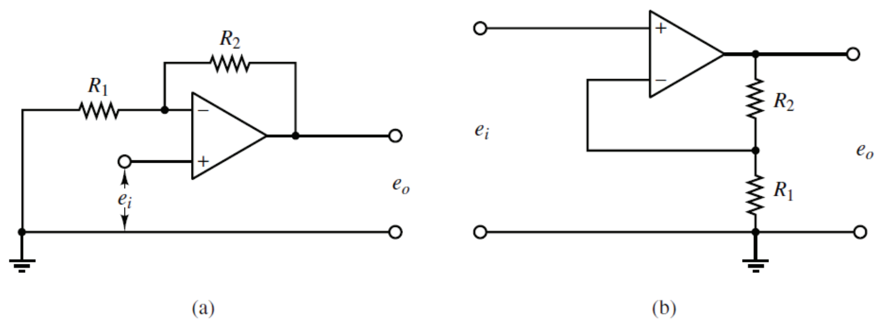

Figure 8: (a) Noninverting operational amplifier; (b) equivalent circuit.

$$e_o = K\left(ei - \frac{R_2}{R_1 + R_2}e_o\right) \tag{49}$$

where K is the differential gain of the amplifier. From this last equation, we get

$$e_i = \left(\frac{R_1}{R_1 + R_2} + \frac{1}{K}\right)e_o \tag{50}$$

Since $K >> 1$, if $R_1/(R_1 + R_2) >> 1/K$, then

$$e_o = \left(1 + \frac{R_2}{R_1}\right)e_i \tag{51}$$

This equation gives the output voltage e_o. Since e_o and e_i have the same signs, the op-amp circuit shown in Figure 8(a) is noninverting.

3.1.3 First-order lag circuit using operational amplifier

Figure 9 shows an electrical circuit involving an operational amplifier. Let us define

$$i_1 = \frac{e_i - e'}{R_1},\ i_2 = C\frac{d(e' - e_o)}{dt},\ i_3 = \frac{e' - e_o}{R_2} \tag{52}$$

Noting that the current flowing into the amplifier is negligible, we have $i_1 = i_2 + i_3$, hence

$$\frac{e_i - e'}{R_1} = C\frac{d(e' - e_o)}{dt} + \frac{e; - e_o}{R_2} \tag{53}$$

Since $e' \approx 0$, we have

$$\frac{e_i}{R_1} = -C\frac{de_o}{dt} - \frac{e_o}{R_2} \tag{54}$$

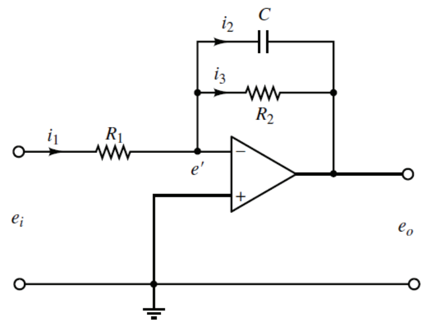

Figure 9: First-order lag circuit using operational amplifier.

Taking the Laplace transform of this last equation, assuming the zero initial condition, we have

$$\frac{E_i(s)}{R_1} = -\frac{R_2Cs + 1}{R_2}E_o(s) \tag{55}$$

which can be written as

$$\frac{E_o(s)}{E_i(s)} = -\frac{R_2}{R_1}\frac{1}{R_2Cs + 1} \tag{56}$$

3.2 PID Controller Using Operational Amplifiers

Figure 10 shows an electronic proportional-plus-integral-plus-derivative controller (a PID controller) using operational amplifiers. The transfer function

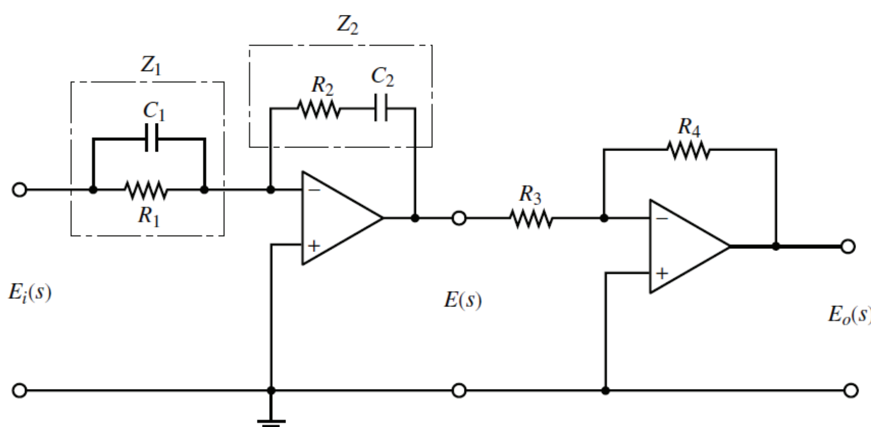

Figure 10: Electronic PID controller.

$E(s)/E_i(s)$ is given by

$$\frac{E(s)}{E_i(s)} = -\frac{Z_2}{Z_1} \tag{57}$$

where

$$Z_1 = \frac{R_1}{R_1 C_1 s + 1}, \; Z_2 = \frac{R_2 C_2 s + 1}{C_2 s} \tag{58}$$

Thus

$$\frac{E(s)}{E_i(s)} = \left(\frac{R_2 C_2 s + 1}{C_2 s}\right)\left(\frac{R_1 C_1 s + 1}{R_1}\right) \tag{59}$$

Noting that

$$\frac{E_o(s)}{E(s)} = -\frac{R_4}{R_3} \tag{60}$$

we have

$$\begin{aligned}
\frac{E_o(s)}{E_1(s)} &= \frac{E_o(s)}{E(s)} \frac{E(s)}{E_1(s)} \\
&= \frac{R_4 R_2}{R_3 R_1} \frac{(R_1 C_1 s + 1)(R_2 C_2 + 1)}{R_2 C_2 s} \\
&= \frac{R_4 R_2}{R_3 R_1} \left(\frac{R_1 C_1 + R_2 C_2}{R_2 C_2} + \frac{1}{R_2 C_2 s} + R_1 C_1 s\right) \\
&= \frac{R_4 (R_1 C_1 + R_2 C_2)}{R_3 R_1 C_2} \left[1 + \frac{1}{(R_1 C_1 + R_2 C_2) s} + \left(\frac{R_1 C_1 R_2 C_2 s}{R_1 C_1 + R_2 C_2}\right)\right]
\end{aligned} \tag{61}$$

Notice that the second operational-amplifier circuit acts as a sign inverter as well as a gain adjuster.

When a PID controller is expressed as

$$\frac{E_o(s)}{E_i(s)} = K_p \left(1 + \frac{T_i}{s} + T_d s\right) \tag{62}$$

K_p is called the proportional gain, T_i is called the integral time, and T_d is called the derivative time. From Equation (61) we obtain the proportional gain K_p, integral time and derivative time to be

$$K_p = \frac{R_4 (R_1 C_1 + R_2 C_2)}{R_3 R_1 C_2}, \; T_i = \frac{1}{(R_1 C_1 + R_2 C_2) s}, \; T_d = \frac{R_1 C_1 R_2 C_2 s}{R_1 C_1 + R_2 C_2} \tag{63}$$

When a PID controller is expressed as

$$\frac{E_o(s)}{E_i(s)} = K_p + \frac{K_i}{s} + K_d s \tag{64}$$

For this controller

$$K_p = \frac{R_4 (R_1 C_1 + R_2 C_2)}{R_3 R_1 C_2}, \; K_i = \frac{R_4}{R_3 R_1 C_2}, \; K_d = \frac{R_4 R_2 C_1}{R_3} \tag{65}$$