Root-Locus Method

1 Introduction

The basic characteristic of the transient response of a closed-loop system is closely related to the location of the closed-loop poles. If the system has a variable loop gain, then the location of the closed-loop poles depends on the value of the loop gain chosen. It is important, therefore, that the designer know how the closed-loop poles move in the s plane as the loop gain is varied.

From the design viewpoint, in some systems simple gain adjustment may move the closed-loop poles to desired locations. Then the design problem may become the selection of an appropriate gain value. If the gain adjustment alone does not yield a desired result, addition of a compensator to the system will become necessary.

The closed-loop poles are the roots of the characteristic equation. However, just finding the roots of the characteristic equation may be of limited value, because as the gain of the open-loop transfer function varies, the characteristic equation changes and the computations must be repeated.

A simple method for finding the roots of the characteristic equation has been developed by W. R. Evans and used extensively in control engineering. This method, called the root-locus method, is one in which the roots of the characteristic equation are plotted for all values of a system parameter.

The roots corresponding to a particular value of this parameter can then be located on the resulting graph. Note that the parameter is usually the gain, but any other variable of the open-loop transfer function may be used. Unless otherwise stated, we shall assume that the gain of the open-loop transfer function is the parameter to be varied through all values, from zero to infinity.

In designing a linear control system, we find that the root-locus method proves to be quite useful, since it indicates the manner in which the open-loop poles and zeros should be modified so that the response meets system performance specifications. This method is particularly suited to obtaining approximate results very quickly.

2 Root-Locus Plots

Angle and Magnitude Conditions. Consider the negative feedback system shown in Figure 1.

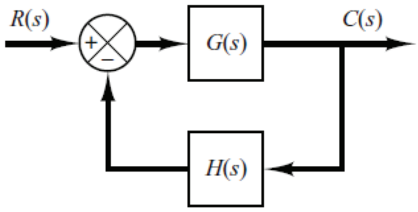

Figure 1: Control system.

The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{1}$$

The characteristic equation for this closed-loop system is obtained by setting the denominator of the right-hand side of Equation (1) equal to zero. That is,

$$1 + G(s)H(s) = 0, \ \Rightarrow G(s)H(s) = -1 \tag{2}$$

Here we assume that $G(s)H(s)$ is a ratio of polynomials in s. Since $G(s)H(s)$ is a complex quantity, Equation (2) can be split into two equations by equating the angles and magnitudes of both sides, respectively, to obtain the following:

Angle condition:

$$\angle G(s)H(s) = \pm 180^\circ(2k+1), \ k = 0, 1, 2, \cdots \tag{3}$$

Magnitude condition:

$$|G(s)H(s)| = 1 \tag{4}$$

The values of s that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles. A locus of the points in the complex plane satisfying the angle condition alone is the root locus. The roots of the characteristic equation (the closed-loop poles) corresponding to a given value of the gain can be determined from the magnitude condition.

In many cases, $G(s)H(s)$ involves a gain parameter K, and the characteristic equation may be written as

$$1 + \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \tag{5}$$

Then the root loci for the system are the loci of the closed-loop poles as the gain K is varied from zero to infinity.

Note that, because the open-loop complex-conjugate poles and complex-conjugate zeros, if any, are always located symmetrically about the real axis, the root loci are always symmetrical with respect to this axis. Therefore, we only need to construct the upper half of the root loci and draw the mirror image of the upper half in the lower-half s plane.

3 Plotting Root Loci With MATLAB

In this section we present the MATLAB approach to the generation of root-locus plots and finding relevant information from the root-locus plots.

3.1 Plotting Root Loci with MATLAB

In plotting root loci with MATLAB we deal with the system equation given in the form of Equation (5), which may be written as

$$1 + K\frac{num}{den} = 0 \tag{6}$$

where *num* is the numerator polynomial and *den* is the denominator polynomial. That is,

$$\begin{aligned} num &= (s + z_1)(s + z_2)\cdots(s + z_m) \\ &= s^m + (z_1 + z_2 + \cdots + z_m)s^{m-1} + \cdots + z_1 z_2 \cdots z_m \end{aligned} \tag{7}$$

$$\begin{aligned} den &= (s + p_1)(s + p_2)\cdots(s + p_n) \\ &= s^n + (p_1 + p_2 + \cdots + p_n)s^{n-1} + \cdots + p_1 p_2 \cdots p_n \end{aligned} \tag{8}$$

Both vectors *num* and *den* must be written in descending powers of s.

A MATLAB command commonly used for plotting root loci is

```
rlocus(num,den)
```

Using this command, the root-locus plot is drawn on the screen. The gain vector K is automatically determined. The vector K contains all the gain values for which the closedloop poles are to be computed.

For the systems defined in state space, `rlocus(A,B,C,D)` plots the root locus of the system with the gain vector automatically determined. Note that commands

```
rlocus(num,den,K), rlocus(A,B,C,D,K)
```

use the user-supplied gain vector K. If it is desired to plot the root loci with marks 'o' or 'x', it is necessary to use the following command:

```
r = rlocus(num,den); plot(r,'o');

r = rlocus(num,den); plot(r,'x');
```

Plotting root loci using marks 'o' or 'x' is instructive, since each calculated closed-loop pole is graphically shown; in some portion of the root loci those marks are densely placed and in another portion of the root loci they are sparsely placed. MATLAB supplies its own set of gain values used to calculate a root-locus plot. It does so by an internal adaptive step size routine. Also, MATLAB uses the automatic axis-scaling feature of the plot command.

Example 1. *Consider the system shown in Figure 2.*

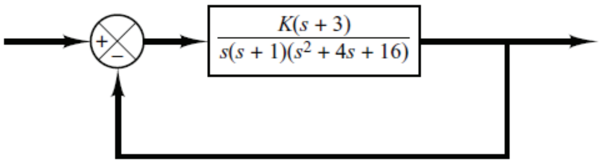

Figure 2: Control system.

Plot root loci with a square aspect ratio so that a line with slope 1 is a true 45° line. Choose the region of root-locus plot to be $-6 \leq x \leq 6$, $-6 \leq y \leq 6$, *where* x *and* y *are the real-axis coordinate and imaginary-axis coordinate, respectively.*

To set the given plot region on the screen to be square, enter the command

```
v = [-6 6 -6 6]; axis (v); axis('square');
```

With this command, the region of the plot is as specified and a line with slope 1 is at a true 45°, not skewed by the irregular shape of the screen.

Solution 1. *For this problem, the denominator is given as a product of first- and second-order terms. So we must multiply these terms to get a polynomial in* s. *The multiplication of these terms can be done easily by use of the convolution command, as shown next. Define*

```
a = s (s + 1): a = [1 1 0];
b = s^2 + 4s + 16: b = [1 4 16];
```

Then we use the following command:

```
c = conv(a, b);
```

Note that `conv(a, b)` *gives the product of two polynomials* a *and* b. *The denominator polynomial is thus found to be*

```
den = [1 5 20 16 0]
```

To find the complex-conjugate open-loop poles, the roots of $s^2 + 4s + 16 = 0$, *we may enter the roots command as follows:*

```
r = roots(b);
```

Thus, the system has the following open-loop zero and open-loop poles:

$$\begin{array}{ll} \textit{Open-loop zero:} & s = 3 \\ \textit{Open-loop poles:} & s = 0, s = 1, s = 2 \pm j3.4641 \end{array} \tag{9}$$

MATLAB commands below will plot the root-locus diagram for this system. The plot is shown in Figure 3.

```
num = [1 3];
den = [1 5 20 16 0];
rlocus(num,den);
v = [-6 6 -6 6];
axis(v); axis('square'); grid;
title ('Root-Locus Plot of G(s) = K(s + 3)/[s(s + 1)(s^2 + 4s + 16)]')
```

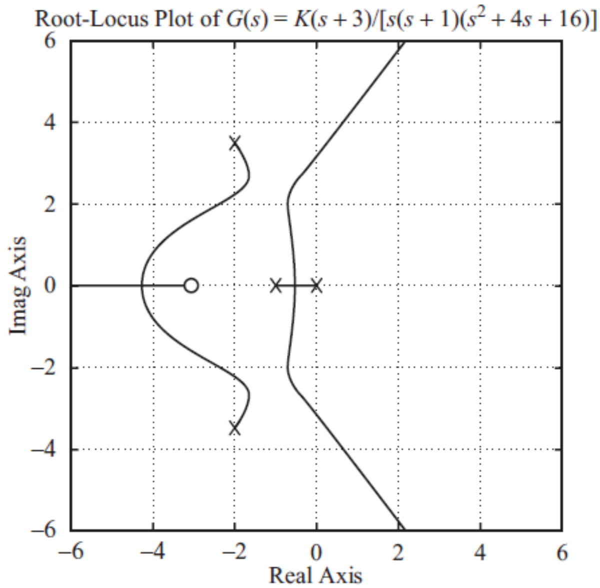

Figure 3: Root-locus plot.

3.2 Constant ζ Loci and Constant ω_n Loci

Recall that in the complex plane the damping ratio ζ of a pair of complex-conjugate poles can be expressed in terms of the angle ϕ, which is measured from the negative real axis, as shown in Figure 4(a) with

$$\zeta = \cos\phi \tag{10}$$

In other words, lines of constant damping ratio ζ are radial lines passing through the origin as shown in Figure 4(b). For example, a damping ratio of 0.5 requires that the complex-conjugate poles lie on the lines drawn through the origin making angles of $\pm 60°$ with the negative real axis. If the real part of a pair of complex-conjugate poles is positive, which means that the system is unstable, the corresponding ζ is negative.

The damping ratio determines the angular location of the poles, while the distance of the pole from the origin is determined by the undamped natural frequency ω_n. The constant ω_n loci are circles.

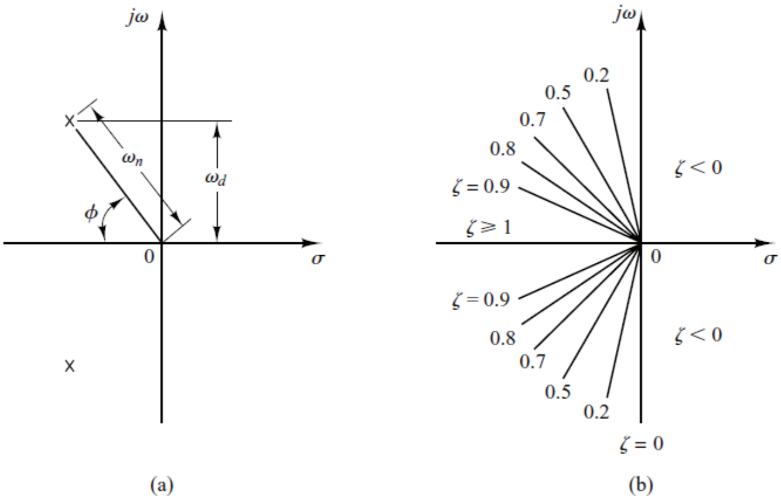

Figure 4: (a) Complex poles; (b) lines of constant damping ratio $zeta$.

To draw constant ζ lines and constant ω_n circles on the root-locus diagram with MATLAB, use the command `sgrid`. It overlays lines of constant damping ratio ($\zeta = 0 \sim 1$ with 0.1 increment) and circles of constant ω_n on the root-locus plot. The resulting diagram is shown in Figure 5.

If only particular constant ζ lines, such as the $\zeta = 0.5$ line and $\zeta = 0.707$ line, and particular constant ω_n circles, such as the $\omega_n = 0.5$ circle, $\omega_n = 1$ circle, and $\omega_n = 2$ circle, are desired, use the following command:

```
sgrid([0.5, 0.707], [0.5, 1, 2])
```

If we wish to overlay lines of constant ζ and circles of constant ω_n as given above to a root-locus plot of a negative feedback system with

```
num = [0 0 0 1]; den = [1 4 5 0];
```

Using the following commands, the resulting root-locus plot is shown in Figure 6.

```
num = [1];
den = [1 4 5 0];
K = 0:0.01:1000;
r = rlocus(num,den,K);
plot(r,'-'); v = [-3 1 -2 2]; axis(v); axis('square')
sgrid([0.5,0.707], [0.5,1,2])
grid
title('Root-Locus Plot with \zeta = 0.5 and 0.707 Lines...
and \omega_n = 0.5,1, and 2 Circles')
xlabel('Real Axis'); ylabel('Imag Axis')
```

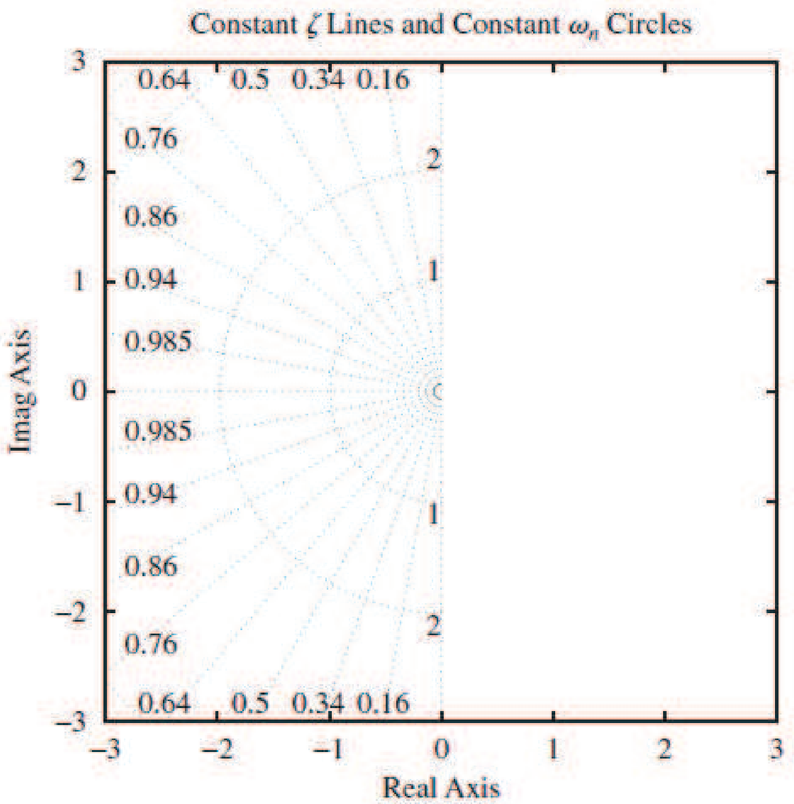

Figure 5: Constant ζ lines and constant ω_n circles.

```
gtext('\omega_n = 2')
gtext('\omega_n = 1')
gtext('\omega_n = 0.5')
% Place 'x' mark at each of 3 open-loop poles.
gtext('x')
gtext('x')
gtext('x')
```

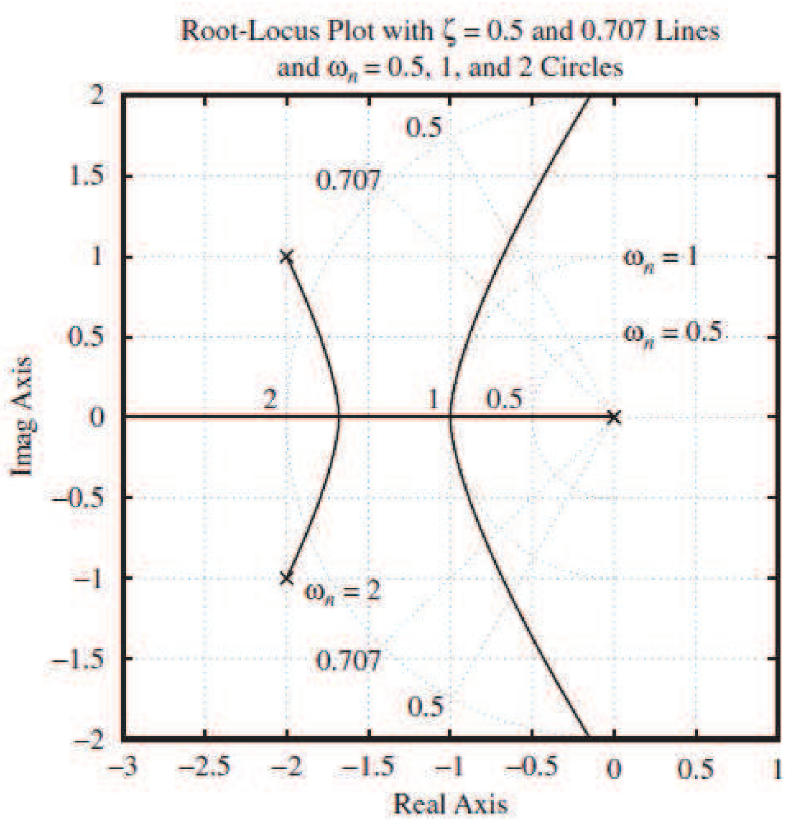

Figure 6: Constant ζ lines and constant ω_n circles superimposed on a root-locus plot.

3.3 Conditionally Stable Systems

Consider the negative feedback system shown in Figure 7.

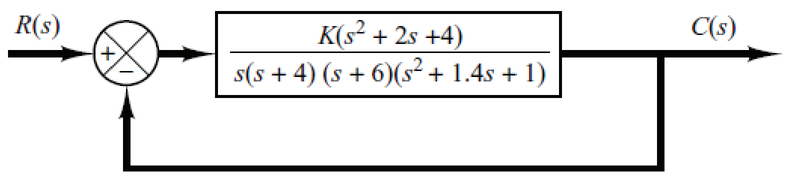

Figure 7: Control system.

We can plot the root loci for this system by applying the general rules and procedure for constructing root loci, or use MATLAB to get root-locus plots. The MATLAB program below will plot the root-locus diagram for the system. The plot is shown in Figure 8.

```
num = [1 2 4];
den = conv(conv([1 4 0],[1 6]), [1 1.4 1]);
rlocus(num, den)
v = [-7 3 -5 5]; axis(v); axis('square')
grid
title('Root-Locus Plot of ...
G(s) = K(s^2 + 2s + 4)/[s(s + 4)(s + 6)(s^2 + 1.4s + 1)]')
text(1.0, 0.55,'K = 12')
text(1.0,3.0,'K = 73')
text(1.0,4.15,'K = 154')
```

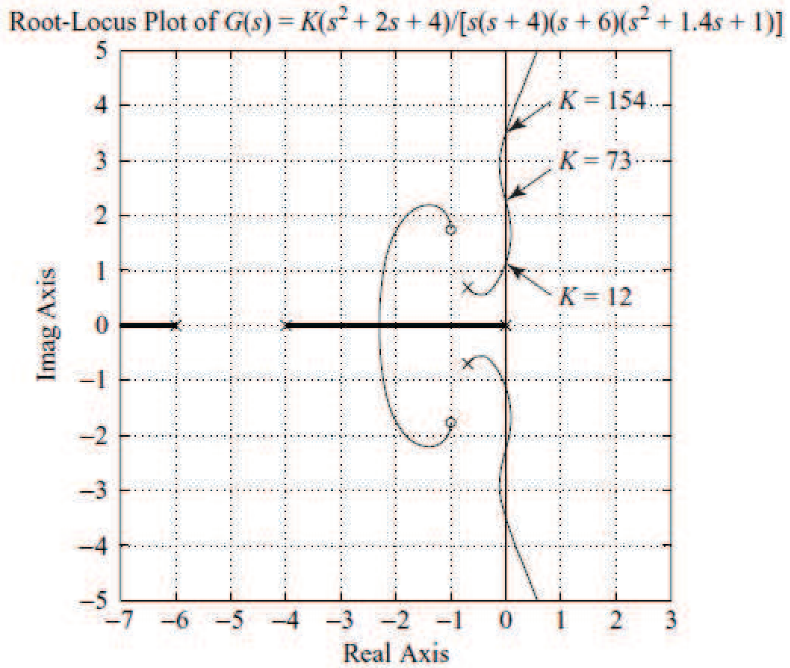

Figure 8: Control system.

It can be seen from the root-locus plot of Figure 8 that this system is stable only for limited ranges of the value of K, that is, $0 < K < 12$ and $73 < K < 154$. The system becomes unstable for $12 < K < 73$ and $154 > K$. If K assumes a value corresponding to unstable operation, the system may break down or may become nonlinear due to a saturation nonlinearity that may exist. Such a system is called conditionally stable.

In practice, conditionally stable systems are not desirable. Conditional stability is dangerous but does occur in certain systems, in particular, a system that has an unstable feedforward path. Such an unstable feedforward path may occur if the system has a minor loop. It is advisable to avoid such conditional stability since, if the gain drops beyond the critical value for any reason, the system becomes unstable. Note that the addition of a proper compensating network will eliminate conditional stability. An addition of a zero will cause the root loci to bend to the left. Hence conditional stability may be eliminated by adding proper compensation.

4 Root-Locus Approach to Control-Systems Design

In building a control system, we know that proper modification of the plant dynamics may be a simple way to meet the performance specifications. This, however, may not be possible in many practical situations because the plant may be fixed and not modifiable. Then we must adjust parameters other than those in the fixed plant.

In practice, the root-locus plot of a system may indicate that the desired performance cannot be achieved just by the adjustment of gain or some other adjustable parameter. In fact, in some cases, the system may not be stable for all values of gain or other adjustable parameter. Then it is necessary to reshape the root loci to meet the performance specifications.

4.1 Effects of the Addition of Poles

The addition of a pole to the open-loop transfer function has the effect of pulling the root locus to the right, tending to lower the system's relative stability and to slow down the settling of the response. Remember that the addition of integral control adds a pole at the origin, thus making the system less stable. Figure 9 shows examples of root loci illustrating the effects of the addition of a pole to a single-pole system and the addition of two poles to a single-pole system.

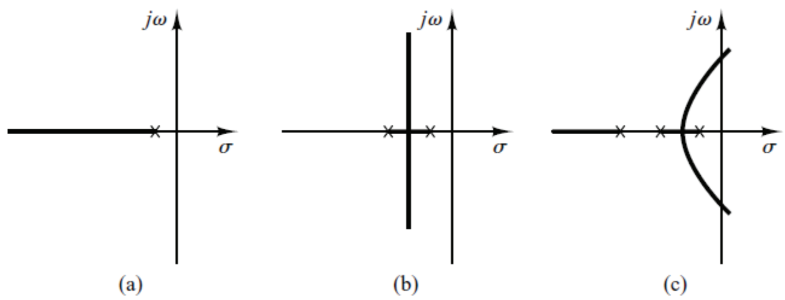

Figure 9: (a) Root-locus plot of a single-pole system; (b) root-locus plot of a two-pole system; (c) root-locus plot of a three-pole system.

4.2 Effects of the Addition of Zeros

The addition of a zero to the open-loop transfer function has the effect of pulling the root locus to the left, tending to make the system more stable and to speed up the settling of the response. Physically, the addition of a zero in the feedforward transfer function means the addition of derivative control to the system. The effect of such control is to introduce a degree of anticipation into the system and speed up the transient response.

Figure 10(a) shows the root loci for a system that is stable for small gain but unstable for large gain. Figures 10(b), (c), and (d) show root-locus plots for the system when a zero is added to the open-loop transfer function. Notice that when a zero is added to the system of Figure 10(a), it becomes stable for all values of gain.

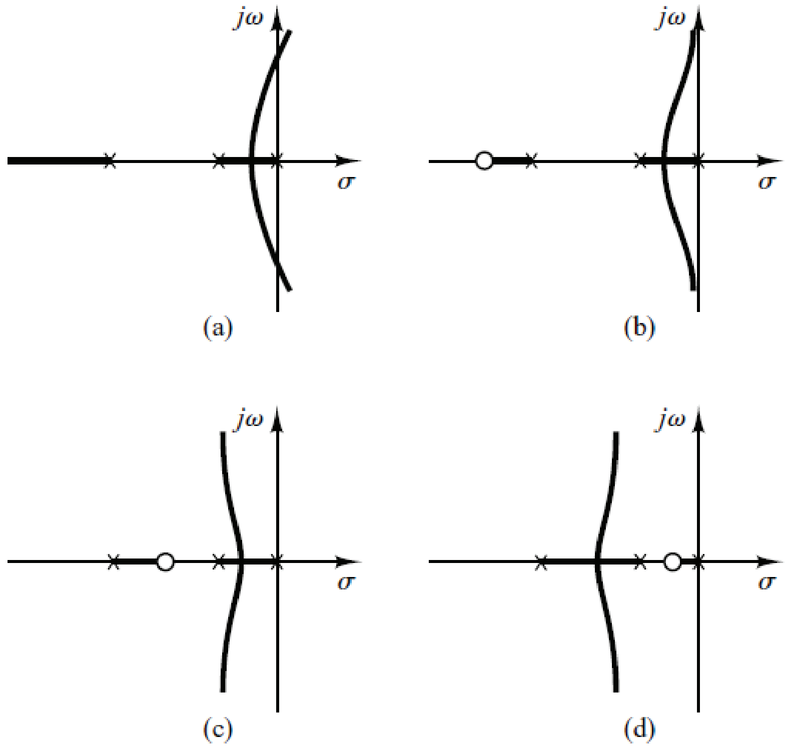

Figure 10: (a) Root-locus plot of a three-pole system; (b), (c), and (d) root-locus plots showing effects of addition of a zero to the three-pole system.

5 Lag-Lead Compensation

If improvements in both transient response and steady-state response are desired, then both a lead compensator and a lag compensator may be used simultaneously. Rather than introducing both a lead compensator and a lag compensator as separate units, however, it is economical to use a single lag-lead compensator.

Lag-lead compensation combines the advantages of lag and lead compensations. Since the lag-lead compensator possesses two poles and two zeros, such a compensation increases the order of the system by 2, unless cancellation of pole(s) and zero(s) occurs in the compensated system.

5.1 Electronic Lag-Lead Compensator Using Operational Amplifiers

Figure 11 shows an electronic lag-lead compensator using operational amplifiers.

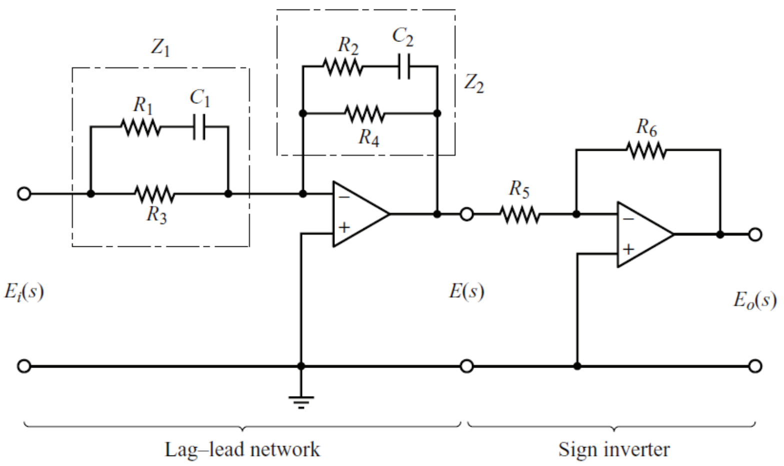

Figure 11: Lag-lead compensator.

The transfer function for this compensator may be obtained as follows: The complex impedance Z_1 is given by

$$\frac{1}{Z_1} = \frac{1}{R_1 + \frac{1}{C_1 s}} + \frac{1}{R_3} \quad \Rightarrow Z_1 = \frac{(R_1 C_1 s + 1) R_3}{(R_1 + R_3) C_1 s + 1} \tag{11}$$

Similarly, complex impedance Z_2 is given by

$$Z_2 = \frac{(R_2 C_2 s + 1) R_4}{(R_2 + R_4) C_2 s + 1} \tag{12}$$

Hence, we have

$$\frac{E(s)}{E_i(s)} = -\frac{Z_2}{Z_1} = -\frac{R_4}{R_3}\left[\frac{(R_1+R_3)C_1 s+1}{R_1 C_1 s+1}\right]\left[\frac{R_2 C_2 s+1}{(R_2+R_4)C_2 s+1}\right] \tag{13}$$

The sign inverter has the transfer function

$$\frac{E_o(s)}{E(s)} = -\frac{R_6}{R_5} \tag{14}$$

Thus the transfer function of the compensator shown in Figure 10 is

$$\begin{aligned}\frac{E_o(s)}{E_i(s)} &= \frac{E_o(s)}{E(s)}\frac{E(s)}{E_i(s)}\\ &= \frac{R_4 R_6}{R_3 R_5}\left[\frac{(R_1+R_3)C_1 s+1}{R_1 C_1 s+1}\right]\left[\frac{R_2 C_2 s+1}{(R_2+R_4)C_2 s+1}\right]\end{aligned} \tag{15}$$

Let us define

$$\begin{aligned}T_1 &= (R_1+R_3)C_1,\ T_1/\gamma = R_1 C_1\\ T_2 &= R_2 C_2,\ \beta T_2 = (R_2+R_4)C_2\end{aligned} \tag{16}$$

Then Equation (15) becomes

$$\begin{aligned}\frac{E_o(s)}{E_i(s)} &= K_c\frac{\beta}{\gamma}\left(\frac{T_1 s+1}{T_1 s/\gamma+1}\right)\left(\frac{T_2 s+1}{\beta T_2 s+1}\right)\\ &= K_c\frac{(s+1/T_1)(s+1/T_2)}{(s+\gamma/T_1)(s+1/\beta T_2)}\end{aligned} \tag{17}$$

where

$$\gamma = \frac{R_1+R_3}{R_1} > 1,\ \beta = \frac{R_2+R_4}{R_2} > 1,\ K_c = \frac{R_2 R_4 R_6}{R_1 R_3 R_5}\frac{(R_1+R_3)}{(R_2+R_4)} \tag{18}$$

Note that γ is often chosen to be equal to β.

5.2 Lag-lead Compensation Techniques Based on the Root-Locus Approach

Consider the system shown in Figure 12. Assume that we use the lag-lead compensator given in Eq. (17), where $\beta > 1$, $\gamma > 1$ and K_c belong to the lead portion of the lag-lead compensator. In designing lag-lead compensators, consider two cases where $\gamma \neq \beta$ and $\gamma = \beta$.

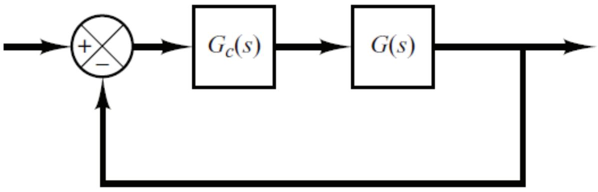

Figure 12: Control system.

5.2.1 Case 1: $\gamma \neq \beta$

In this case, the design process is a combination of the design of the lead compensator and that of the lag compensator. The design procedure for the lag-lead compensator follows:

1. From the given performance specifications, determine the desired location for the dominant closed-loop poles.

2. Using the uncompensated open-loop transfer function $G(s)$, determine the angle deficiency ϕ if the dominant closed-loop poles are to be at the desired location. The phase-lead portion of the lag-lead compensator must contribute this angle ϕ.

3. Assuming that we later choose T_2 sufficiently large so that the magnitude of the lag portion
$$\left| \frac{s_1 + 1/T_2}{s_1 + 1/(\beta T_2)} \right| \tag{19}$$
is approximately unity, where $s = s_1$ is one of the dominant closed-loop poles, choose the values of T_1 and γ from the requirement that
$$\angle \frac{s_1 + 1/T_1}{s_1 + \gamma/T_1} = \phi \tag{20}$$
The choice of T_1 and γ is not unique. Infinitely many sets of T_1 and γ are possible. Then determine the value of K_c from the magnitude condition:
$$\left| K_c \frac{s_1 + 1/T_1}{s_1 + \gamma/T_1} G(s_1) \right| = 1 \tag{21}$$

4. If the static velocity error constant K_v is specified, determine the value of β to satisfy the requirement for K_v. The static velocity error constant K_v is given by
$$\begin{aligned} K_v &= \lim_{s\to 0} sG_c(s)G(s) \\ &= \lim_{s\to 0} sK_c \left(\frac{s_1 + 1/T_1}{s_1 + \gamma/T_1} \right) \left(\frac{s_1 + 1/T_2}{s_1 + 1/(\beta T_2)} \right) G(s) \\ &= \lim_{s\to 0} sK_c \frac{\beta}{\gamma} G(s) \end{aligned} \tag{22}$$

where K_c and γ are already determined in step 3. Hence, given the value of K_v, the value of β can be determined from this last equation. Then, using the value of β thus determined, choose the value of β such that

$$\left|\frac{s_1 + 1/T_2}{s_1 + 1/(\beta T_2)}\right| \approx 1, \ -5^\circ < \angle \frac{s_1 + 1/T_2}{s_1 + 1/(\beta T_2)} < 0^\circ \tag{23}$$

5.2.2 Case 2: $\gamma = \beta$

If $\gamma = \beta$ is required, then the preceeding design procedure for the lag-lead compensator may be modified as follows:

1. From the given performance specifications, determine the desired location for the dominant closed-loop poles.

2. The lag-lead compensator is modified to

$$\begin{aligned} G_c(s) &= K_c \frac{(T_1 s + 1)(T_2 s + 1)}{(T_1 s/\beta + 1)(\beta T_2 s + 1)} \\ &= K_c \frac{(s + 1/T_1)(s + 1/T_2)}{(s + \beta/T_1)(s + 1/(\beta T_2))} \end{aligned} \tag{24}$$

   where $\beta > 1$. The open-loop transfer function of the compensated system is $G_c(s)G(s)$. If the static velocity error constant K_v is specified, determine the value of constant K_c from the following equation:

$$K_v = \lim_{s \to 0} sG_c(s)G(s) = \lim_{s \to 0} sK_c G(s) \tag{25}$$

3. To have the dominant closed-loop poles at the desired location, calculate the angle contribution ϕ needed from the phase-lead portion of the lag-lead compensator.

4. For the lag-lead compensator, we later choose T_2 sufficiently large so that

$$\left|K_c \left(\frac{s_1 + 1/T_1}{s_1 + \beta/T_1}\right) G(s_1)\right| = 1, \ \angle \frac{s_1 + 1/T_1}{s_1 + \beta/T_1} = \phi \tag{26}$$

5. Using the value of β just determined, choose T_2 so that

$$\left|K_c \left(\frac{s_1 + 1/T_1}{s_1 + \beta/T_1}\right) G(s_1)\right| \approx 1, \ -5^\circ < \angle \frac{s_1 + 1/T_1}{s_1 + \beta/T_1} < 0^\circ \tag{27}$$

   The value of βT_2, the largest time constant of the lag-lead compensator, should not be too large to be physically realized.