# Bode Plots

## Contents



## 1 Bode Diagrams or Logarithmic Plots

A Bode diagram consists of two graphs: One is a plot of the logarithm of the magnitude of a sinusoidal transfer function; the other is a plot of the phase angle; both are plotted against the frequency on a logarithmic scale.

The use of Bode diagrams employing asymptotic approximations requires much less time than other methods that may be used for computing the frequency response of a transfer function. The ease of plotting the frequency response curves for a given transfer function and the ease of modification of the frequency-response curve as compensation is added are the main reasons why Bode diagrams are very frequently used in practice.

The standard representation of the logarithmic magnitude of $G(j\omega)$ is $20|\log G(j\omega)|$, where the base of the logarithm is 10. The unit used in this representation of the magnitude is the decibel, usually abbreviated dB. In the logarithmic representation, the curves are drawn on semilog paper, using the log scale for frequency and the linear scale for either magnitude (but in decibels) or phase angle (in degrees).

The main advantage of using the Bode diagram is that multiplication of magnitudes can be converted into addition. Furthermore, a simple method for sketching an approximate log-magnitude curve is available. It is based on asymptotic approximations. Such approximation by straight-line asymptotes is sufficient if only rough information on the frequency-response characteristics is needed. Should the exact curve be desired, corrections can be made easily to these basic asymptotic plots. Expanding the low-frequency range by use of a logarithmic scale for the frequency is highly advantageous, since characteristics at low frequencies are most important in practical systems.

# 2 Basic Factors of $G(j\omega)H(j\omega)$

The main advantage in using the logarithmic plot is the relative ease of plotting frequency-response curves. The basic factors that very frequently occur in an arbitrary transfer function $G(j\omega)H(j\omega)$ are

1. Gain $K$
2. Integral and derivative factors $(j\omega)^{\mp 1}$
3. First-order factors $(1 + j\omega T)^{\mp 1}$
4. Quadratic factors $[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2]^{\mp 1}$

It is possible to utilize them in constructing a composite logarithmic plot for any general form of $G(j\omega)H(j\omega)$ by sketching the curves for each factor and adding individual curves graphically, because adding the logarithms of the gains corresponds to multiplying them together.

## 2.1 The Gain

A number greater than unity has a positive value in decibels, while a number smaller than unity has a negative value. From Fig. 1, the decibel value of any number can be obtained from this line. As a number increases by a factor of 10, the corresponding decibel value increases by a factor of 20.

For example,

$$20 \log(K \times 10) = 20 \log K + 20, \tag{1}$$

$$20 \log(K \times 10^n) = 20 \log K + 20n. \tag{2}$$

Note that, when expressed in decibels, the reciprocal of a number differs from its value only in sign.

$$20 \log K = -20 \log \frac{1}{K}. \tag{3}$$

## 2.2 Integral and derivative factors

The logarithmic magnitude of $1/j\omega$ in decibels is

$$20 \log \left| \frac{1}{j\omega} \right| = -20 \log \omega \ dB, \tag{4}$$

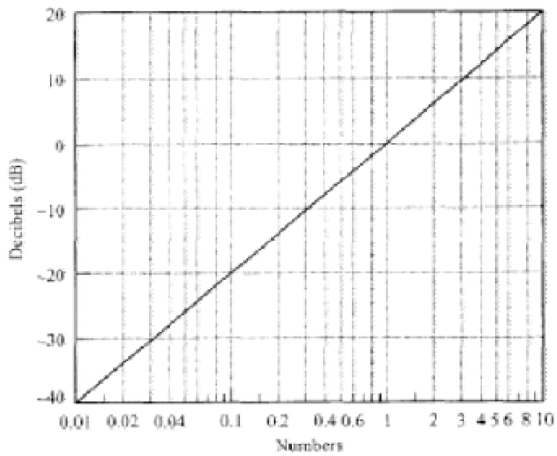


Figure 1: Number-decibel conversion line.

The phase angle of $1/j\omega$ is constant and equal to $-90°$.

In Bode diagrams, frequency ratios are expressed in terms of octaves or decades. An **octave** is a frequency band from $\omega_1$ to $2\omega_1$, where $\omega_1$ is any frequency value. A **decade** is a frequency band from $\omega_1$ to $10\omega_1$.

If the log magnitude $-20\log\omega$ dB is plotted against $\omega$ on a logarithmic scale, it is a straight line. Since $-20\log 10\omega = -20\log\omega - 20$, the slope of the line is $-20$ dB/decade or $-6$ dB/octave.

Similarly, the log magnitude of $j\omega$ in decibels is

$$20\log|j\omega| = 20\log\omega \ dB. \tag{5}$$

The phase angle of $j\omega$ is constant and equal to $90°$.The log-magnitude curve is a straight line with a slope of 20 dB/decade. Figure 2 shows frequency-response curves for $1/j\omega$ and $j\omega$, respectively.

If the transfer function contains the factor $(1/j\omega)^n$ or $(j\omega)^n$, the log magnitude becomes, respectively,

$$20\log\left|\frac{1}{(j\omega)^n}\right| = -n \times 20\log|j\omega| = -20n\log\omega dB \tag{6}$$

$$20\log|(j\omega)^n)| = n \times 20\log|j\omega| = 20n\log\omega dB. \tag{7}$$

The slopes of the log-magnitude curves for the factors $(1/j\omega)^n$ and $(j\omega)^n$ are thus $-20n$ dB/decade and $20n$ dB/decade, respectively.

The phase angle of $(1/j\omega)^n$ is equal to $-90° \times n$ over the entire frequency range, while that of $(j\omega)^n$ is equal to $90° \times n$ over the entire frequency range.

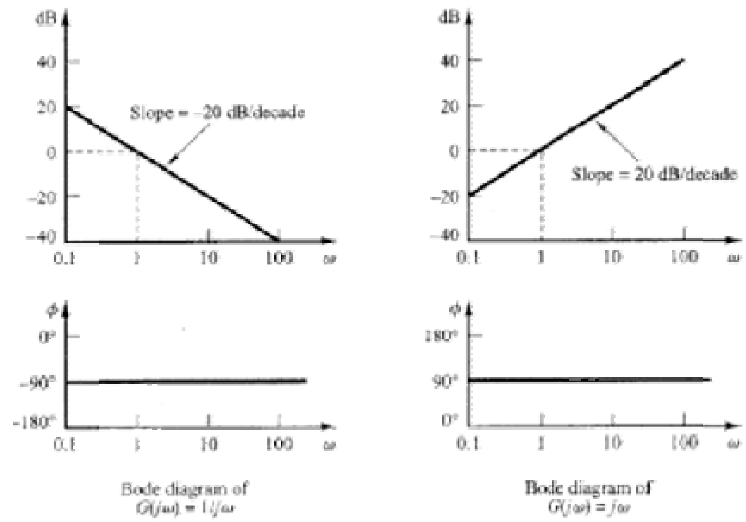


Figure 2: Bode diagram of $G(j\omega) = 1/j\omega$, and $G(j\omega) = j\omega$.

## 2.3 First-order factors

The log magnitude of the first-order factor $1/(1 + j\omega T)$ is

$$20 \log \left| \frac{1}{1 + j\omega T} \right| = -20 \log \sqrt{1 + \omega^2 T^2} \ dB. \tag{8}$$

For low frequencies, such that $\omega \ll 1/T$, the log magnitude may be approximated by

$$-20 \log \sqrt{1 + \omega^2 T^2} \doteqdot -20 \log 1 = 0 \ dB. \tag{9}$$

Thus, the log-magnitude curve at low frequencies is the constant 0-dB line.

For high frequencies, such that $\omega \gg 1/T$,

$$-20 \log \sqrt{1 + \omega^2 T^2} \doteqdot -20 \log \omega T \ dB. \tag{10}$$

This is an approximate expression for the high-frequency range. At $\omega = 1/T$, the log magnitude equals 0 dB; at $\omega = 10/T$, the log magnitude is $-20$ dB. Thus, the value of $-20 \log \omega T$ dB decreases by 20 dB for every decade of $\omega$.

For $\omega \gg 1/T$, the log-magnitude curve is thus a straight line with a slope of $-20$ dB/decade or $-6$ dB/octave.

The frequency at which the two asymptotes meet is called the **corner frequency** or **break frequency**. For the factor $1/(1 + j\omega T)$, the frequency $\omega = 1/T$ is the corner frequency since at $\omega = 1/T$ the two asymptotes have the same value. The corner frequency divides the frequency-response curve into two

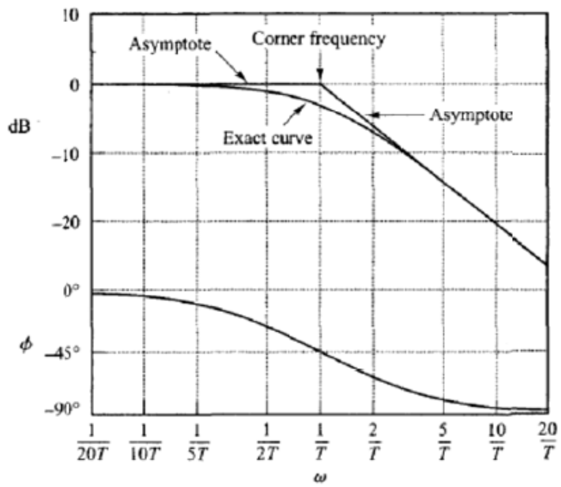


Figure 3: Log-magnitude curve, together with the asymptotes, and phase-angle curve of $1/(1 + j\omega T)$.

regions: a curve for the low-frequency region and a curve for the high-frequency region.

The exact phase angle $\phi$ of the factor $1/(1 + j\omega T)$ is

$$\phi = -\tan^{-1} \omega T. \tag{11}$$

At zero frequency, the phase angle is $0°$. At the corner frequency, the phase angle is

$$\phi = -\tan^{-1} \frac{T}{T} = -\tan^{-1} 1 = -45°. \tag{12}$$

At infinity, the phase angle becomes $-90°$. Since the phase angle is given by an inverse tangent function, the phase angle is skew symmetric about the inflection point at $\phi = -45°$.

The maximum error occurs at the corner frequency and is approximately equal to $-3$ dB since

$$-20 \log \sqrt{1+1} + 20 \log 1 = -10 \log 2 = -3.03\ dB. \tag{13}$$

The error at the frequency one octave below the corner frequency, at $\omega = 1/(2T)$ is

$$-20 \log \sqrt{\frac{1}{4} + 1} + 20 \log 1 = -10 \log \frac{\sqrt{5}}{2} = -0.97\ dB. \tag{14}$$

The error at the frequency one octave above the corner frequency, at $\omega = 2/T$ is

$$-20 \log \sqrt{2^2 + 1} + 20 \log 2 = -10 \log \frac{\sqrt{5}}{2} = -0.97\ dB. \tag{15}$$

Thus, the error at one octave below or above the corner frequency is approximately equal to -1 dB. Similarly, the error at one decade below or above the corner frequency is approximately $-0.04$ dB.

Since the asymptotes are quite easy to draw and are sufficiently close to the exact curve, the use of such approximations in drawing Bode diagrams is convenient in establishing the general nature of the frequency-response characteristics quickly with a minimum amount of calculation and may be used for most preliminary design work.

An advantage of the Bode diagram is that for reciprocal factors, for example, the factor $1 + j\omega T$, the log-magnitude and the phase-angle curves need only be changed in sign, since

$$20 \log |1 + j\omega T| = -20 \log \left| \frac{1}{1 + j\omega T} \right|, \tag{16}$$

and

$$\angle\{1 + j\omega T\} = \tan^{-1} \omega T = -\angle\{\frac{1}{1 + j\omega T}\}. \tag{17}$$

To draw a phase curve accurately, we have to locate several points on the curve. The phase angles of $(1 + j\omega T)^{\mp}1$ are

$$\begin{gathered} \mp 45^\circ(\omega = 1/T),\ \mp 26.6^\circ(\omega = 1/(2T)),\ \mp 5.7^\circ(\omega = 1/(10T)), \\ \mp 63.4^\circ(\omega = 2/T),\ \mp 84.3^\circ(\omega = 10/T). \end{gathered} \tag{18}$$

# 3 General Procedure for Plotting Bode Diagrams

1. First rewrite the sinusoidal transfer function $G(j\omega)$ as a product of basic factors.
2. Then identify the corner frequencies associated with these basic factors.
3. Finally, draw the asymptotic log-magnitude curves with proper slopes between the corner frequencies.

The exact curve, which lies close to the asymptotic curve, can be obtained by adding proper corrections. The phase-angle curve of $G(j\omega)$ can be drawn by adding the phase-angle curves of individual factors.

# 4 Phase and Gain Margins

## 4.1 Gain Margin

The gain margin is the reciprocal of the magnitude $|G(j\omega)|$ at the frequency at which the phase angle is $-180^\circ$. Defining the phase crossover frequency $\omega_1$, to be the frequency at which the phase angle of the open-loop transfer function equals $-180^\circ$ gives the gain margin

$$K_g = \frac{1}{|G(j\omega_1)|} = 20 \log K_g = -20 \log |G(j\omega_1)|. \tag{19}$$

The gain margin expressed in decibels is positive if $K_g$ is greater than unity and negative if $K_g$ is smaller than unity. Thus, a **positive** gain margin (in decibels) means that the system is **stable**, and a **negative** gain margin (in decibels) means that the system is **unstable**.

For a **stable** minimum-phase system, the gain margin indicates how much the gain can be **increased** before the system becomes unstable. For an **unstable** system, the gain margin is indicative of how much the gain must be **decreased** to make the system stable.

## 4.2 Phase Margin

The phase margin is that amount of additional phase lag at the gain crossover frequency required to bring the system to the verge of instability. The gain crossover frequency is the frequency at which $|G(j\omega)|$, the magnitude of the open-loop transfer function, is unity. The phase margin $\gamma$ is 180° plus the phase angle $\phi$ of the open-loop transfer function at the gain crossover frequency, or

$$\gamma = 180^\circ + \phi. \tag{20}$$

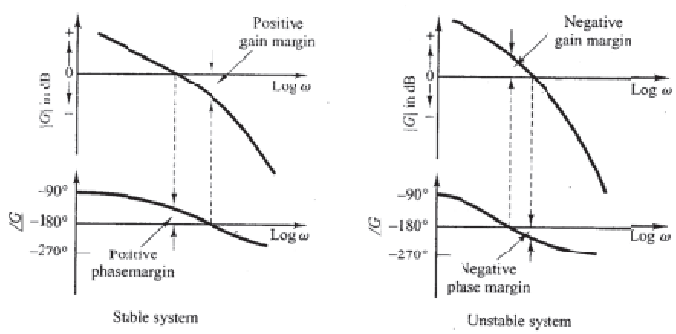


Figure 4: Phase and gain margins of stable and unstable systems.

## 4.3 Cutoff Frequency and Bandwidth

Referring to Figure 5, the frequency $\omega_b$ at which the magnitude of the closed-loop frequency response is 3 dB below its zero-frequency value is called the cutoff frequency. Thus

$$\left|\frac{C(j\omega)}{R(j\omega)}\right| < \left|\frac{C(j0)}{R(j0)}\right| - 3dB, \quad \text{for } \omega > \omega_b. \tag{21}$$

The frequency range $0 \leq \omega \leq \omega_b$ which the magnitude of is greater than -3 dB is called the **bandwidth** of the system. The bandwidth indicates the frequency where the gain starts to fall off from its low-frequency value. Thus, the bandwidth indicates how well the system will track an input sinusoid. Note that for a given $\omega$, the rise time increases with increasing damping ratio $\zeta$. On the other hand, the bandwidth decreases with the increase in $\zeta$. Therefore, the **rise time** and the bandwidth are inversely proportional to each other.

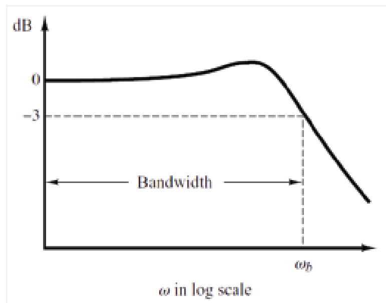


Figure 5: Plot of a closed-loop frequency response curve showing cutoff frequency $\omega_b$ and bandwidth.

### 4.4 Minimum-Phase Systems and Nonminimum-Phase Systems

Transfer functions having neither poles nor zeros in the right-half $s$-plane are **minimum-phase** transfer functions, whereas those having poles and/or zeros in the right-half $s$-plane are **nonminimum-phase** transfer functions. Systems with minimum-phase transfer functions are called minimum-phase systems, whereas those with nonminimum-phase transfer functions are called nonminirnum-phase systems.

For systems with the same magnitude characteristic, the range in phase angle of the minimum-phase transfer function is minimum among all such systems, while the range in phase angle of any nonminimum-phase transfer function is greater than this minimum. It is noted that for a minimum-phase system, the transfer function can be uniquely determined from the magnitude curve alone.

Consider as an example the two systems whose sinusoidal transfer functions are, respectively,

$$G_1(j\omega) = \frac{1 + j\omega T}{1 + j\omega T_1}, \; G_2(j\omega) = \frac{1 - j\omega T}{1 + j\omega T_2}, \; 0 < T < T_1. \tag{22}$$

The two sinusoidal transfer functions have the same magnitude characteristics, but they have different phase-angle characteristics, as shown in Figure 6. These two systems differ from each other by the factor

$$\frac{G_2(j\omega)}{G_1(j\omega)} = \frac{1 - j\omega T}{1 + j\omega T}. \tag{23}$$

The magnitude of the factor $(1-j\omega T)/(1+j\omega T)$ is always unity. But the phase angle equals $-2\tan^{-1}\omega T$ and varies from $0°$ to $-180°$ as $\omega$ is increased from zero to infinity.

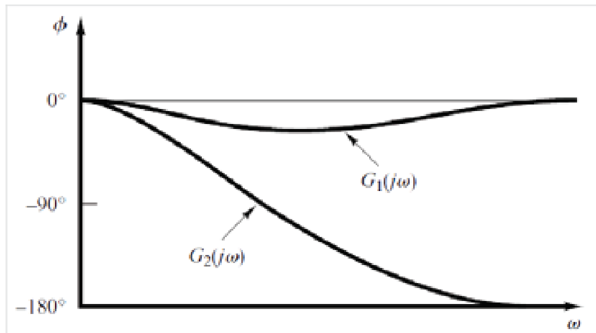


Figure 6: Phase-angle characteristics of the systems $G_1(j\omega)$ and $G_2(j\omega)$.