LECTURE 9 – ANALYSIS IN THE FREQUENCY DOMAIN

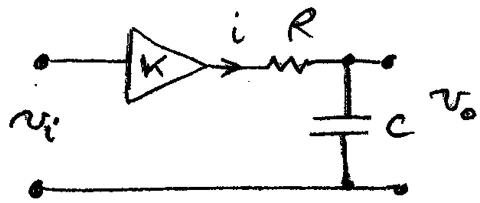
Working in the complex domain is a way of predicting what will happen in the time domain solution by looking at the positions of the roots of the characteristic equation. From now on we will refer to the more common phraseology of studying the s plane, that is, s root movement. We will:

- Take a few typical systems (electrical, mechanical etc.)
- Transform by Laplace
- State boundary conditions where necessary
- Monitor and analyse movement of the s roots in the s plane
- Relate the position of the roots in the s plane to solutions back in the time domain
- Introduce the concepts of CONTROL and SENSITIVITY

Recall the diagram in LECTURE 2 (2.1). We are looking at the top right hand box to predict the bottom right hand box.

EXAMPLE 1 – A simple RC circuit, first order electrical system

The system equation is



$$Kv_i = CR\dot{v}_0 + v_0$$

because $i = C\dot{v}_0$

Transform to get

$$KV_i = CRsV_0 - CRv_0(0) + V_0$$

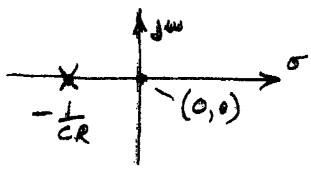
Choose the boundary condition $v_0(0) = 0$.

$\therefore \frac{V_0}{V_i} = \frac{K}{CRs+1}$ is the transfer function

Look at the denominator $CRs + 1$ and equate to zero to give the characteristic equation and derive roots of s .

$$\therefore s = -\frac{1}{CR}.$$

In the s plane we get



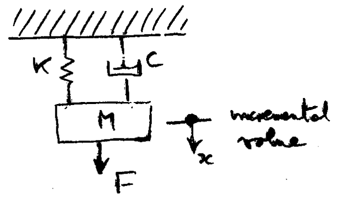
For this example we will continue to a time domain answer and relate it back to the s plane. Let the input to the system be $V_i = 1$, a unit impulse.

$$\therefore V_0(t) = \mathcal{L}^{-1}\left[\frac{K}{CRs+1}\right] = \frac{K}{CR} \cdot e^{-\frac{t}{CR}}$$

We discover that the root of s is equal to the reciprocal value of the TIME CONSTANT of the exponential decay. The student MUST understand this connection fully before going on to greater academic achievement. In terms of transient behaviour of the system, it is only the exponential that is important.

EXAMPLE 2 – A second order, mechanical system

Equating forces on the mass



$$M\ddot{x} + C\dot{x} + kx = F$$

In Laplace

$$Ms^2X - Msx(0) - M\dot{x}(0) + CsX - Cx(0) + kX = F$$

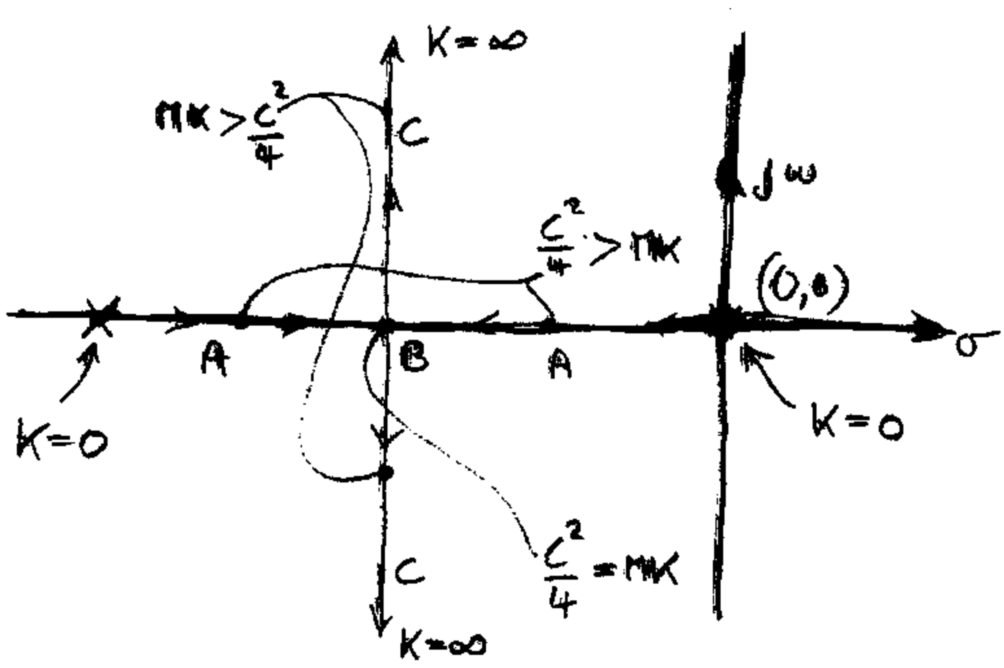
Let $\quad x(0) = \dot{x}(0) = 0$ (the system is at rest)

$\therefore \frac{X}{F} = \frac{1}{Ms^2+Cs+k}$, the transfer function.

From the characteristic equation the roots of $s = -\frac{C}{2M} \pm \sqrt{\frac{c^2}{4M^2} - \frac{k}{M}}$

There are a range of possibilities for the nature of the roots (see Lecture 7 (7.4).

Plotting the possibilities on the s plane gives:



Whatever the input to this system, there are three distinct regions of interest: A, B & C.

REGION A –

the s roots are real and different and the form of the output is $Ae^{-\alpha t} + Be^{-\beta t}$, a difference of two decaying exponentials; $\frac{C^2}{4} > Mk$.

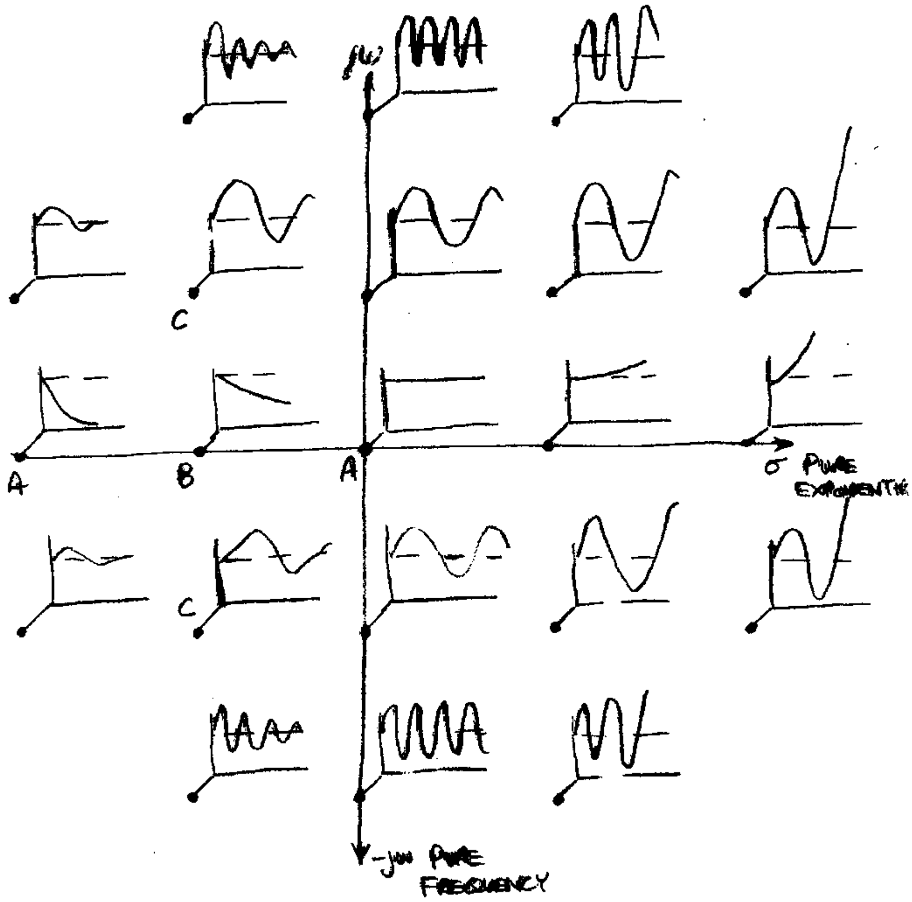
REGION B –

the s roots are real and equal and the form of the outout is $(K_1 + K_2 t)e^{-\gamma t}$ a single optimum exponential decay; $\frac{C^2}{4} = Mk$.

REGION C –

the s roots are complex pair and the form of the output is $e^{-\alpha t} \cdot sin(\beta t + \emptyset)$, a decaying sinusoid $\frac{C^2}{4} < Mk$.

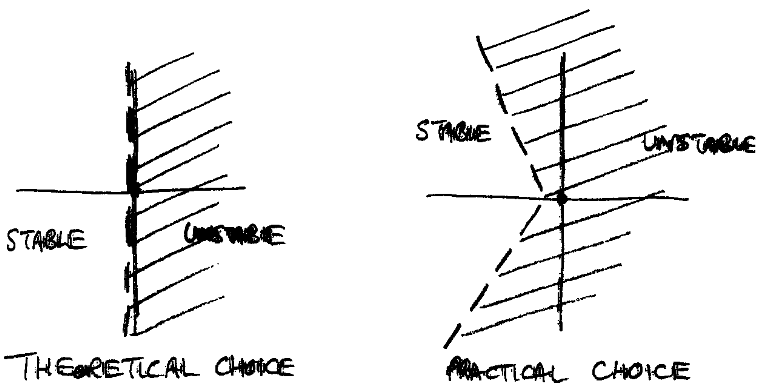
Using this information we can sketch the type of output to be expected by inserting small graphs over the domain of the s plane.



We see $s = \sigma \pm j\omega$, where σ is a real component and $j\omega$ is imaginary. Study this diagram carefully and master the concepts it illustrates. Note any root on the zero horizontal axis gives rise to exponentials. Any roots off this axis will generate oscillations. All roots to the left of the zero vertical axis decay, all to the right get bigger indefinitely. Think of the significance of the right hand part of the s plane.

THE s PLANE STABILITY CRITERION

The implications of the previous analysis mean that <u>all</u> solutions on or to the right of the zero vertical axis must be UNSTABLE, because the output keeps on increasing in magnitude as a function of time.

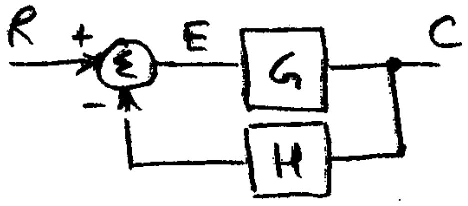


THE CONCEPT OF CONTROL

This is an introduction as to what happens when we attempt to apply feedback to alter the characteristics of our system.

A transfer function is of the form $G(s) = \frac{C(s)}{R(s)}$ where $C(s)$ is the output and $R(s)$ is the input.

If the transfer function does not behave as desired, we have an option to create control paths to modify $G(s)$ into a more appropriate response. The most commonly used modification is to feedback a function of the output C (drop the s) to the input and compare. Our new block diagram becomes:



which has been seen as rule 3 in LECTURE 6 (6.3).

It can be seen that $E = R - HC,\ C = GE$

$$\therefore \frac{C}{R} = \frac{G}{1+GH}$$ is a new transfer function (simply rule 3).

We now refer to G as the OPEN LOOP transfer function and to $\frac{G}{1+GH}$ as the CLOSED LOOP transfer function. Loops such as that proposed here, are necessary in practical situtions because of uncertainties in the component values in a system due to ageing, wear, changing environment, temperature etc. The signal E is effectively the resultant error signal that is needed to modify C in the light of C not being the value you want it to be. Now let us check the effect on the output of the transfer function changing (for whatever reason).

In the OPEN LOOP case:

Let $\quad G \to G + \Delta G$

$$\therefore C_0 + \Delta C_0 = RG + R\Delta G \quad \text{Hence} \quad \Delta C_0 = R\Delta G$$

In the CLOSED LOOP case:

$$C_c + \Delta C_c = \frac{(G+\Delta G)R}{1+(G+\Delta G)H}$$

$$\therefore \Delta C_c = \frac{R\Delta G}{1+GH+\cancel{\Delta G \cdot H}}$$ We assume $GH \gg 1 \gg \Delta GH$ a small movement in G .

Look at the ratio

$$\frac{\Delta C_c}{\Delta C_o} = \frac{R \cdot \Delta G}{1+GH} \cdot \frac{1}{R\Delta G} = \frac{1}{1+GH} \ .$$

We see the shift due to changes in components is significantly reduced when closing the loop.