MMAN3200

Frequency Domain - Fourier

We have seen that an LTI (Linear Time-Invariant) system can be represented in the Laplace Domain by its Transfer Function. Another useful transformation is the Fourier transform. Given a time signal, this transformation represents the original time signal in the frequency domain. Frequency analysis is usually very useful and, in some cases, even more convenient than working in the Laplace's domain.
To quickly understand this topic, we refresh our concept about the Fourier's Series, and then we generalize the concept to explain the Fourier transform, for finally introducing the concept of Frequency Response of LTI systems.

We, first, mention a very fundamental concept, about how functions can be interpreted as infinite dimensional vectors. There is a general theory about how functions can be expressed as linear combinations of more elementary functions which have certain properties and define a basis. A function is then understood to be a vector, in a space of infinite dimensions. As in other vector spaces (e.g. those of finite dimensionality), we can choose a basis for representing "points" in that space (e.g. in the same way we can choose a coordinate frame in 3D.)
The following equations express the concept,

$$f(x) = \sum_{i=0}^{+\infty} c_i \cdot \phi_i(x), \quad x \in [a,b]$$

$$c_i = \langle f(x), \phi_i(x) \rangle = \int_a^b w(x) \cdot f(x) \cdot \bar{\phi}_i(x) \cdot dx$$

$$\langle \phi_i(x), \phi_j(x) \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

In which, an inner product is defined $\langle \propto(x), \beta(x) \rangle$, and in which the components of the basis are orthonormal functions (as defined in the previous equation). The set of functions $\{\varphi_i(x)\}_{i=0}^{\infty}$ also satisfy other necessary properties, to guarantee they define a complete basis (a basis which can represent any possible vector in that vector space).
The inner product,

$$c_i = \langle f(x), \phi_i(x) \rangle = \int_a^b w(x) \cdot f(x) \cdot \bar{\phi}_i(x) \cdot dx$$

provides the projection of the function $f(x)$ into the direction of the "vector" $\varphi_i(x)$. Note that it involves the participation of $\bar{\varphi}_i(x)$ (complex conjugate of $\varphi_i(x)$), for the cases of functions in the complex domain. The weighting function $\mathrm{w}(x)$ depends on the basis. There is a beautiful theory about this; however, it is not our business in MMAN3200.
A function can then be uniquely expressed by its vector of coefficients (coordinates), $[c_0, c_1, c_2, c_3, \ldots]$. We see that function as a linear combination of the basis' functions, based on those coefficients; we will exploit this interpretation, later.

A number of bases have been defined/invented, each of them is useful for different applications. Some of them allow to work in domains which simplify analysis or design; some of others are good for approximating functions (in which the series are usually truncated) and for compressing information, etc.

This concept is also applied for functions of higher dimension domains, e.g. for a 2D function

$$f(x,y)=\sum_{i=0}^{+\infty}\sum_{k=0}^{+\infty}c_{ik}\cdot\phi_{ik}(x,y),\quad (x,y)\in\Omega,\qquad f:R^2\to R^1,$$

(e.g. good application examples of this are in the compression of images)

Now, we will focus on a very famous one, which is useful to us: FOURIER.

Fourier series:

Consider a periodic signal of the form

$$x(t)\quad t\in\left[-\frac{T}{2},\ \frac{T}{2}\right]$$
$$x(t)=x(t+n\cdot T),\quad\forall\quad n\in Z$$

in which $\boldsymbol{T}$ is the period of the signal. We can represent this periodic function through an expansion in Fourier's terms:

$$x(t)=\sum_{k=0}^{+\infty}a_k\cdot\cos\left(\frac{2\pi\cdot k}{T}\cdot t\right)+\sum_{k=1}^{+\infty}b_k\cdot\sin\left(\frac{2\pi\cdot k}{T}\cdot t\right)$$

In which the coefficients are defined as follows,

$$a_k=\frac{1}{T}\cdot\int_{-T/2}^{T/2}\cos\left(\frac{2\pi\cdot k}{T}\cdot t\right)\cdot x(t)\cdot dt\qquad,\qquad b_k=\frac{1}{T}\cdot\int_{-T/2}^{T/2}\sin\left(\frac{2\pi\cdot k}{T}\cdot t\right)\cdot x(t)\cdot dt$$

This is valid for certain type of functions (whose integrals in the interval $\left[-\frac{T}{2},+\frac{T}{2}\right]$ are finite, i.e. the integrals do converge).
The family of functions

$$1,\quad\left\{\cos\left(\frac{2\pi\cdot k}{T}\cdot t\right)\right\}_{k=1}^{\infty},\quad\left\{\sin\left(\frac{2\pi\cdot k}{T}\cdot t\right)\right\}_{k=1}^{\infty}$$

defines a basis, for periodic functions of that period of time, T. From our perspective, in MMAN3200, we see $x(t)$ as the composition of a high number (ideally infinite) of PURE tones (harmonics).
This series of cosine and sine components (harmonics) can be mathematically replaced by complex exponentials (it is just an equivalent representation),

$$e^{+j\cdot\left(\frac{2\pi\cdot k}{T}\cdot t\right)}=\cos\left(\frac{2\pi\cdot k}{T}\cdot t\right)+j\cdot\sin\left(\frac{2\pi\cdot k}{T}\cdot t\right)$$
$$e^{-j\cdot\left(\frac{2\pi\cdot k}{T}\cdot t\right)}=\cos\left(\frac{2\pi\cdot k}{T}\cdot t\right)-j\cdot\sin\left(\frac{2\pi\cdot k}{T}\cdot t\right)$$

Then the Fourier expansion can be expressed as a series of complex exponentials

$$x(t)=\sum_{k=-\infty}^{+\infty} c_k \cdot e^{j\frac{2\pi\cdot k}{T}\cdot t}, \qquad c_k=\frac{1}{T}\cdot\int_{-T/2}^{+T/2} e^{-j\cdot\frac{2\pi\cdot k}{T}\cdot t}\cdot x(t)\cdot dt$$

In which the coefficients c_k are complex numbers, obtained by the inner product:

$$c_k=\left\langle x(t)\,,\,e^{j\cdot\frac{2\pi\cdot k}{T}\cdot t}\right\rangle=\frac{1}{T}\cdot\int_{-T/2}^{+T/2} e^{-j\cdot\frac{2\pi\cdot k}{T}\cdot t}\cdot x(t)\cdot dt$$

Fourier Transform

For non-periodic functions, those can be understood as functions evolving in the interval $\left[-\frac{T}{2},+\frac{T}{2}\right]$ for $T\to\infty$. For periods of infinite duration, in the limit, the Fourier's Series converge to an integral,

$$x(t)=\int_{w=-\infty}^{\infty} x[j\omega]\cdot e^{j\cdot\omega\cdot t}\cdot d\omega \qquad \leftrightarrow \qquad x[j\omega]=\frac{1}{2\pi}\int_{t=-\infty}^{\infty} e^{-j\cdot\omega\cdot t}\cdot x(t)\cdot dt$$

It can be demonstrated that as the period T increases, the infinite but discrete set of coefficients $\{c_k\}$ converge to such a dense set that, in the limit for $T\to\infty$, it is represented by a density function $x[j\omega]$ in the domain of ω consequently, the discrete summation tends to be an integration in that domain.

The function $x[j\omega]$ is called the Fourier Transform of $x(t)$. (Note its similarity with the Laplace's Transform, for $s=j\omega$).

The factors $x[j\omega]$ are complex numbers which are composed by real and imaginary terms; alternatively, they can be factorized as amplitude and phase ("polar representation") , $x[j\omega]=R[\omega]+j\cdot I[\omega]=M[\omega]\cdot e^{j\cdot\varphi[\omega]}$. Consequently,

$$x(t)=\int_{w=-\infty}^{\infty} x[j\omega]\cdot e^{j\cdot\omega\cdot t}\cdot d\omega=\int_{w=-\infty}^{\infty} M[\omega]\cdot e^{j\cdot(\omega\cdot t+\varphi[\omega])}\cdot d\omega$$

This means that $x(t)$ can be understood as the composition (in the limit) of infinite number of harmonic components, each of those components having frequency ω, having amplitude $M[\omega]\cdot d\omega$ and shifted by $\varphi[\omega]$.

There is an alternative definition which uses the frequency expressed in cycles/second, i.e. Hertz (HZ), in place of using radians/second (as in the case of ω)

$$\mathrm{f}=\frac{\omega}{2\pi}, \qquad x(t)=\int_{\mathrm{f}=-\infty}^{\infty} x[j\mathrm{f}]\cdot e^{j\cdot 2\pi\cdot \mathrm{f}\cdot t}\cdot d\mathrm{f}, \qquad x[j\mathrm{f}]=\int_{t=-\infty}^{\infty} e^{-j\cdot 2\pi\cdot \mathrm{f}\cdot t}\cdot x(t)\cdot dt$$

Frequency Response of a System

Now, as we can imagine time signals as being composed by a linear combination of harmonics, we can exploit that fact.

As in the Laplace case, if we have an LTI system, the ***Superposition Principle*** is valid. It can be demonstrated that if we apply an input $\mathrm{u}(t)$, that is equivalent to a linear combination of harmonics ($e^{j\omega t}$), on a **stable** LTI system that has a Laplace's Transfer Function $H[s]$, we obtain a system's output, $y(t)$, whose Fourier Transform $y[j\omega]$ is as follows,

$$u[j\omega]=\int_{t=-\infty}^{\infty} e^{-j\cdot\omega\cdot t}\cdot u(t)\cdot dt \qquad \Rightarrow \qquad y[j\omega]=H(s)\big|_{s=j\omega}\cdot u[j\omega]=H(j\omega)\cdot u[j\omega]$$

(As in the Laplace case, we can apply the inverse Fourier transform, for obtaining $y(t)$ from $y[j\omega]$.)

The factor $H(j\omega)$ is called the Frequency Response of the system. For a given ω, it implicitly describes how is the response of the system to an input which is a pure harmonic of frequency ω. $H(j\omega)$ is well defined for LTI stable systems.
As $H(j\omega)$ is a complex number parameterized in the angular frequency variable, ω, we can express it by using its magnitude and phase components.

$$H(j\omega)=R(j\omega)+j\cdot I(j\omega)=M[\omega]\cdot e^{j\cdot\varphi[\omega]}$$

$$M[w]=\|H(j\omega)\|_2, \quad \varphi[\omega]=\underline{/H(j\omega)}$$

Consequently, the product $H(j\omega)\cdot u[j\omega]$ can be understood as the original distribution of pure tones, $u[j\omega]$, selectively amplified and shifted, accordingly to the values of the magnitude $M[\omega]$and the phase $\varphi[\omega]$ of the system, respectively; i.e. each individual harmonic is affected in its amplitude and phase, depending on its frequency ω, and the associated system's frequency response, $M[\omega]$ and $\varphi[\omega]$, for that frequency.
The system $H(s)$ behaves as a ***filter***, attenuating (or amplifying) the pure tones according to a "selective gain" $M[\omega]$ (which is a function of the frequency ω). It also produces a change in the phases of each harmonic component, being this change in phase defined by $\varphi[\omega]$. This effect, is also called linear distortion.
Note: This is why certain systems are usually described in the signal processing community, as LOW PASS, HIGH PASS, ALL PASS or BAND PASS filters (to be discussed in the lecture). For instance, consider an LTI system which is modeled by a transfer function $H(s)$. If $H(j\omega)$ has a $M[\omega]$ that does attenuate the high frequencies, and that does not practically affect low frequencies, then the system is classified as a LOW PASS filter. If an input signal presents relevant high frequency Fourier components, those will show up on the system's output as highly attenuated versions, i.e. they are, in practical terms, "filtered out".

Some interesting digression:
When you define the shape of a Graphic Equalizer (e.g. in a MP3 Media Player or a radio tuner) you are defining the shape of certain $M[\omega]$. The program or audio device will then apply, the time domain, a version of a proper filter, $H(s)$ whose frequency response has an amplitude = $M[\omega]$ (or at least well approximated to the one you specify). In reality, because computers work in a time discrete fashion, the playback/radio tuner software applies a digital filter, i.e. a discrete time domain version of the TF $H(s)$. There is a version of the theory, for discrete time systems; in which there is an equivalent to the Laplace Transform (called Z-Transform), and also there is a discrete version for the Fourier Transform (DFT, Discrete Fourier Transform, from which the famous FFT, Fast Fourier Transform, is derived). But those matters are not our business in MMAN3200.

Experimental evaluation of the frequency response of a stable LTI system

If we want to obtain $H(j\omega)$, but we do not have the analytical expression $H(s)$, we can perform some experiment in order to estimate $H(j\omega)$. If we apply a pure harmonic input $u(t)=A\cdot\sin(\omega_0 t+\alpha)$ during "enough" time (ideally it should be from $t_{initial}\rightarrow-\infty$ to $t_{final}\rightarrow+\infty$; however, in practical terms, it can be a finite period of time, but sufficiently large, in which we measure the output of the system. We would find out that its response is

$y(t) = M_0 \cdot A \cdot \sin(\omega_0 t + \alpha + \varphi_0)$. The amplification and phase shift are just due to the frequency response of the system at the specific frequency $\omega = \omega_0$, i.e. $H(j\omega_0) = M_0 \cdot e^{j \cdot \varphi_0}$.
In practical terms, you apply the harmonic input during some finite time, long enough for the transient response to die out (this is possible because the system $H(s)$ is stable!).

To discuss in class. An alternative approach for estimating $H(j\omega)$:

Consider this: we apply an input signal $u(t)$ whose Fourier Transform is $u[j\omega]$, we measure the associated system's output $y(t)$, and we evaluate its Fourier equivalent $y[j\omega]$. Now we would be able to obtain, still in a naive way, $H(j\omega) = {}^{y[j\omega]}\!/_{u[j\omega]}$. For this naive approach to be feasible, in practical terms, the input signal $u(t)$ must be "rich", i.e. its Fourier transform must be such that $|u[j\omega]| > 0$ for all ω in the range of frequencies of interest. An approach like this one, would be faster that trying to excite the system using multiple harmonics, individually.
There are other faster and more reliable and efficient ways for obtaining $H(j\omega)$, provided that we know, through an experiment, the input and output signals. However, those approaches are not discussed in MMAN3200.
In the rest of the lectures we will assume that the TF of the system is always known; consequently, we will obtain the frequency response from the system's TF itself.

Ok, now after this marathonic "compressed" discussion about those fundamental facts, we can discuss about the methods in the frequency domain (e.g. Bode, Nichols and Nyquist), knowing that we know what the **Frequency Domain** is.

Note: this topic is briefly provided because the lecturer considers it convenient for a better understanding of the topic about BODE, which is part of MMAN3200. The material in this document is not included in the exam.