# Mathematical Modeling

## 1 Introduction

A mathematical model of a dynamic system is defined as a set of equations that represents the dynamics of the system accurately, or at least fairly well. Note that a mathematical model is not unique to a given system. A system may be represented in many different ways and, therefore, may have many mathematical models, depending on one's perspective.

The dynamics of many systems, whether they are mechanical, electrical, thermal, economic, biological, and so on, may be described in terms of differential equations. Such differential equations may be obtained by using physical laws governing a particular system—for example, Newton's laws for mechanical systems and Kirchhoff's laws for electrical systems. We must always keep in mind that deriving reasonable mathematical models is the most important part of the entire analysis of control systems.

### 1.1 Mathematical Models

Mathematical models may assume many different forms. Depending on the particular system and the particular circumstances, one mathematical model may be better suited than other models.

### 1.2 Simplicity Versus Accuracy

In obtaining a mathematical model, we must make a compromise between the simplicity of the model and the accuracy of the results of the analysis. In deriving a reasonably simplified mathematical model, we frequently find it necessary to ignore certain inherent physical properties of the system.

### 1.3 Linear Systems

A system is called linear if the principle of superposition applies. The principle of superposition states that the response produced by the simultaneous application of two different forcing functions is the sum of the two individual responses. Hence, for the linear system, the response to several inputs can be calculated by treating one input at a time and adding the results. It is this principle that allows one to build up complicated solutions to the linear differential equation from simple solutions. In an experimental investigation of a dynamic system, if cause and effect are proportional, thus implying that the principle of superposition holds, then the system can be considered linear.

### 1.4 Linear Time-Invariant Systems and Linear Time-Varying Systems

A differential equation is linear if the coefficients are constants or functions only of the independent variable. Dynamic systems that are composed of linear time-invariant lumped-parameter components may be described by linear time-invariant differential equation—that is, constant-coefficient differential equations. Such systems are called linear time-invariant (or linear constant-coefficient) systems. Systems that are represented by differential equations whose coefficients are functions of time are called linear time-varying systems.

## 2 Transfer Function and Impulse-Response Function

In control theory, functions called transfer functions are commonly used to characterize the input-output relationships of components or systems that can be described by linear, time-invariant, differential equations.

### 2.1 Transfer Function

The transfer function of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero.

Consider the linear time-invariant system defined by the following differential equation:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 x^{(m)} + b_1 x^{(m-1)} + \cdots + b_{m-1}\dot{x} + b_n x, \quad n \geq m \tag{1}$$

where $y$ is the output of the system and $x$ is the input. The transfer function of this system is the ratio of the Laplace transformed output to the Laplace transformed input when all initial conditions are zero, or

$$\begin{aligned} G(s) &= \left.\frac{\mathfrak{L}[output]}{\mathfrak{L}[input]}\right|_{zero\ initial\ condition} \\ &= \frac{Y(s)}{X(s)} = \frac{b_0 s^{(m)} + b_1 s^{(m-1)} + \cdots + b_{m-1}s + b_m}{a_0 s^{(n)} + a_1 s^{(n-1)} + \cdots + a_{n-1}s + a_n} \end{aligned} \tag{2}$$

By using the concept of transfer function, it is possible to represent system dynamics by algebraic equations in $s$. If the highest power of $s$ in the denominator of the transfer function is equal to $n$, the system is called an $n$th-order system.

### 2.2 Convolution Integral

For a linear, time-invariant system the transfer function $G(s)$ is

$$G(s) = \frac{Y(s)}{X(s)} \tag{3}$$

where $X(s)$ is the Laplace transform of the input to the system and $Y(s)$ is the Laplace transform of the output of the system, where we assume that all initial conditions involved are zero. It follows that the output $Y(s)$ can be written as the product of $G(s)$ and $X(s)$, or

$$Y(s) = G(s)X(s) \tag{4}$$

Note that multiplication in the complex domain is equivalent to convolution in the time domain, so the inverse Laplace transform is given by the following convolution integral:

$$\begin{aligned} y(t) &= \int_0^t x(\tau)g(t-\tau)d\tau \\ &= \int_0^t g(\tau)x(t-\tau)d\tau \end{aligned} \tag{5}$$

where both $g(t)$ and $x(t)$ are 0 for $t < 0$.

### 2.3 Impulse-Response Function

Consider the output (response) of a linear time-invariant system to a unit-impulse input when the initial conditions are zero. Since the Laplace transform of the unit-impulse function is unity, the Laplace transform of the output of the system is

$$Y(s) = G(s) \tag{6}$$

The inverse Laplace transform of the output gives the impulse response of the system. The inverse Laplace transform of $G(s)$, or

$$\mathfrak{L}^{-1}[G(s)] = g(t) \tag{7}$$

is called the impulse-response function. This function $g(t)$ is also called the weighting function of the system.

The impulse-response function $g(t)$ is thus the response of a linear time-invariant system to a unit-impulse input when the initial conditions are zero. The Laplace transform of this function gives the transfer function.

## 3 Automatic Control Systems

A control system may consist of a number of components. To show the functions performed by each component, in control engineering, we commonly use a diagram called the block diagram.

## 3.1 Block Diagrams

A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals. Such a diagram depicts the interrelationships that exist among the various components.

In a block diagram all system variables are linked to each other through functional blocks. The functional block or simply 'block' is a symbol for the mathematical operation on the input signal to the block that produces the output. The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals. Note that the signal can pass only in the direction of the arrows. Thus a block diagram of a control system explicitly shows a unilateral property.

Figure 1 shows an element of the block diagram. The arrowhead pointing toward the block indicates the input, and the arrowhead leading away from the block represents the output. Such arrows are referred to as signals.

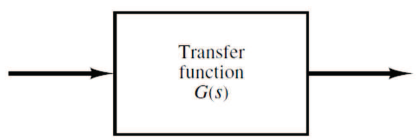


Figure 1: Element of a block diagram.

The advantages of the block diagram representation of a system are that it is easy to form the overall block diagram for the entire system by merely connecting the blocks of the components according to the signal flow and that it is possible to evaluate the contribution of each component to the overall performance of the system.

### 3.1.1 Summing Point

Referring to Figure 2, a circle with a cross is the symbol that indicates a summing operation. The plus or minus sign at each arrowhead indicates whether that signal is to be added or subtracted.

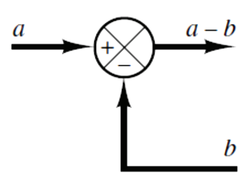


Figure 2: Summing point.

### 3.1.2 Branch Point

A branch point is a point from which the signal from a block goes concurrently to other blocks or summing points.

## 3.2 Block Diagram of a Closed-Loop System

Figure 3 shows an example of a block diagram of a closed-loop system. The output $C(s)$ is fed back to the summing point, where it is compared with the reference input $R(s)$. The closed-loop nature of the system is clearly indicated by the figure. The output of the block, $C(s)$ in this case, is obtained by multiplying the transfer function $G(s)$ by the input to the block, $E(s)$. Any linear control system may be represented by a block diagram consisting of blocks, summing points, and branch points.

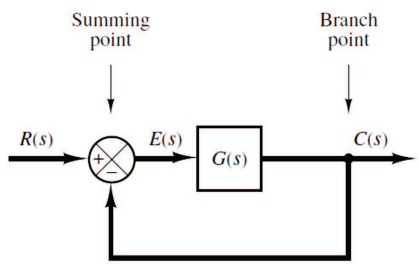


Figure 3: Block diagram of a closed-loop system.

When the output is fed back to the summing point for comparison with the input, it is necessary to convert the form of the output signal to that of the input signal. For example, in a temperature control system, the output signal is usually the controlled temperature. The output signal, which has the dimension of temperature, must be converted to a force or position or voltage before it can be compared with the input signal. This conversion is accomplished by the feedback element whose transfer function is $H(s)$, as shown in Figure 4.

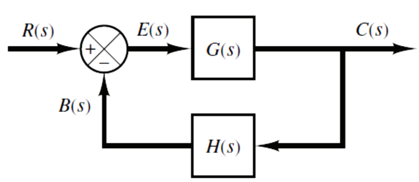


Figure 4: Closed-loop system.

The role of the feedback element is to modify the output before it is compared with the input. In the present example, the feedback signal that is fed back to the summing point for comparison with the input is $B(s) = H(s)C(s)$.

### 3.3 Open-Loop Transfer Function and Feed-forward Transfer Function

Referring to Figure 4, the ratio of the feedback signal $B(s)$ to the actuating error signal $E(s)$ is called the open-loop transfer function. That is,

$$\text{Open-loop transfer function } = \frac{B(s)}{E(s)} = G(s)H(s) \tag{8}$$

The ratio of the output $C(s)$ to the actuating error signal $E(s)$ is called the feed-forward transfer function, so that

$$\text{Feed-forward transfer function } = \frac{C(s)}{E(s)} = G(s) \tag{9}$$

If the feedback transfer function $H(s)$ is unity, then the open-loop transfer function and the feed-forward transfer function are the same.

### 3.4 Closed-Loop Transfer Function

For the system shown in Figure 4, the output $C(s)$ and input $R(s)$ are related as follows: since

$$\begin{aligned} C(s) &= G(s)E(s) \\ E(s) &= R(s) - B(s) \\ &= R(s) - H(s)C(s) \end{aligned} \tag{10}$$

eliminating $E(s)$ from these equations gives

$$C(s) = G(s)[R(s) - H(s)C(s)] \tag{11}$$

or

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{12}$$

The transfer function relating $C(s)$ to $R(s)$ is called the closed-loop transfer function. It relates the closed-loop system dynamics to the dynamics of the feed-forward elements and feedback elements. $C(s)$ is given by

$$C(s) = \frac{G(s)}{1 + G(s)H(s)} R(s) \tag{13}$$

Thus the output of the closed-loop system clearly depends on both the closed-loop transfer function and the nature of the input.

### 3.5 Block Diagram Reduction

It is important to note that blocks can be connected in series only if the output of one block is not affected by the next following block. If there are any loading effects between the components, it is necessary to combine these components into a single block.

Any number of cascaded blocks representing non-loading components can be replaced by a single block, the transfer function of which is simply the product of the individual transfer functions.

## 3.6 Procedures for Drawing a Block Diagram

To draw a block diagram for a system, first write the equations that describe the dynamic behavior of each component. Then take the Laplace transforms of these equations, assuming zero initial conditions, and represent each Laplace-transformed equation individually in block form. Finally, assemble the elements into a complete block diagram.

As an example, consider the RC circuit shown in Figure 5(a). The equations for this circuit are

$$i = \frac{e_i - e_o}{R}, \quad e_o = \frac{\int i dt}{C} \tag{14}$$

The Laplace transforms with zero initial condition, become

$$I(s) = \frac{E_i(s) - E_o(s)}{R}, \quad E_o(s) = \frac{I(s)}{sC} \tag{15}$$

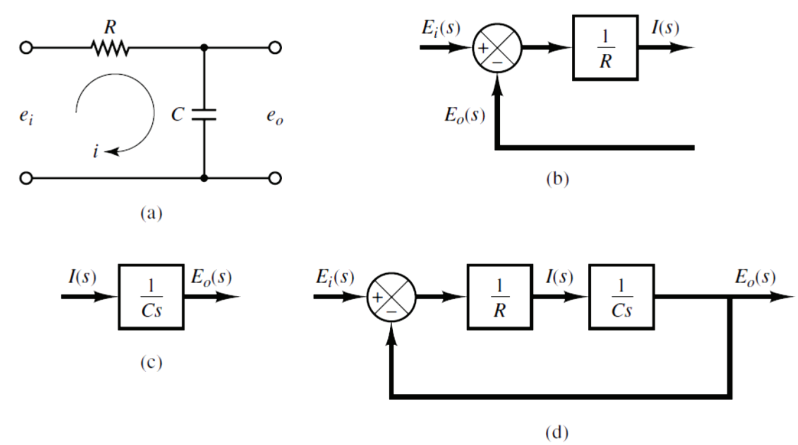


Figure 5: (a) *RC* circuit; (b) block diagram representing $I(s)$; (c) block diagram representing $E_o(s)$; (d) block diagram of the *RC* circuit.

The equation for $I(s)$ represents a summing operation, and the corresponding diagram is shown in Figure 5(b). $E_o(s)$ represents the block as shown in Figure 5(c). Assembling these two elements, we obtain the overall block diagram for the system as shown in Figure 5(d).

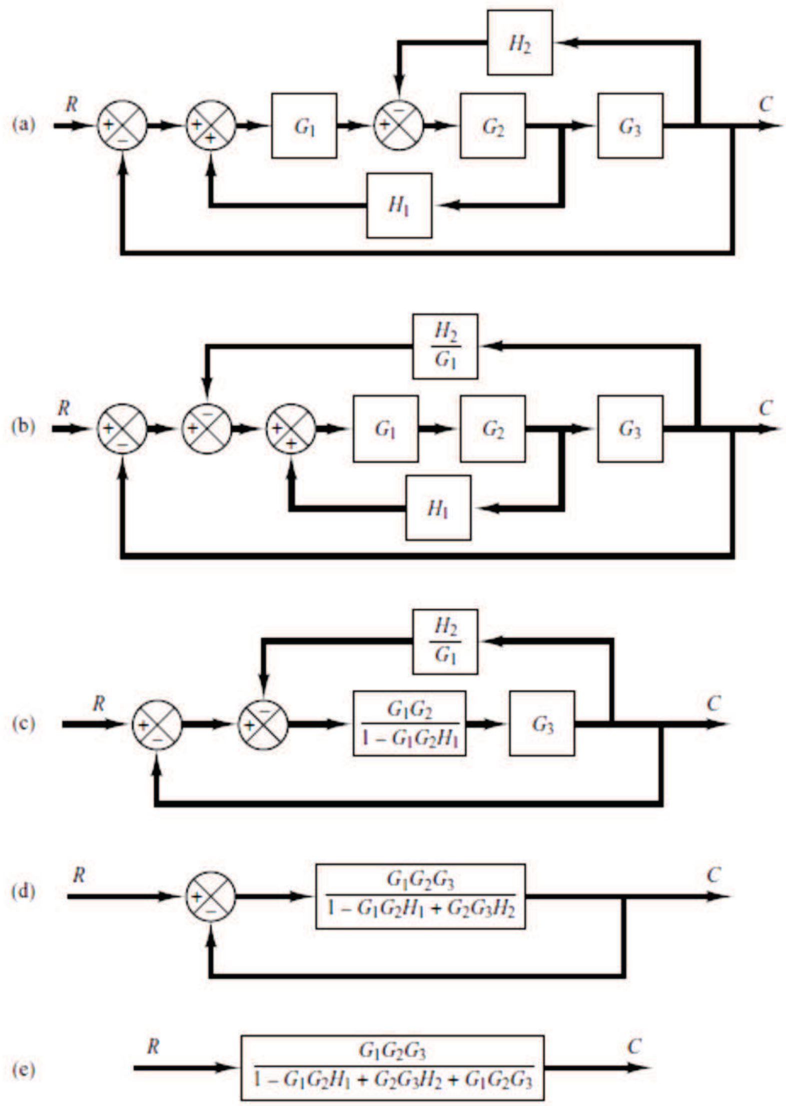


Figure 6: (a) Multiple-loop system; (b)–(e) successive reductions of the block diagram shown in (a).

**Example 1.** *Consider the system shown in Figure 6(a). Simplify this diagram.*

*By moving the summing point of the negative feedback loop containing $H_2$ outside the positive feedback loop containing $H_1$, we obtain Figure 6(b). Eliminating the positive feedback loop, we have Figure 6(c). The elimination of the loop containing $H_2/G_1$ gives Figure 6(d). Finally, eliminating the feedback loop results in Figure 6(e).*

*Notice that the numerator of the closed-loop transfer function $C(s)/R(s)$ is the product of the transfer functions of the feed-forward path. The denominator of $C(s)/R(s)$ is equal to*

$$\begin{aligned} &1+\sum \textit{product of the transfer functions around each loop} \\ &= 1+(-G_1G_2H_1+G_2G_3H_2+G_1G_2G_3) \\ &= 1-G_1G_2H_1+G_2G_3H_2+G_1G_2G_3 \end{aligned} \tag{16}$$

# 4 Linear Approximation of Nonlinear Mathematical Models

To obtain a linear mathematical model for a nonlinear system, we assume that the variables deviate only slightly from some operating condition. Consider a system whose input is $x(t)$ and output is $y(t)$. The relationship between is given by

$$y = f(x) \tag{17}$$

If the normal operating condition corresponds to $\bar{x}$, $\bar{y}$, then Equation (17) may be expanded into a Taylor series about this point as follows:

$$y = f(\bar{x}) + \frac{df}{dx}(x-\bar{x}) + \frac{1}{2!}\frac{d^2f}{dx^2}(x-\bar{x})^2 + \cdots \tag{18}$$

where the derivatives $df/dx$, $d^2f/dx^2$, $\cdots$ are evaluated at $x = \bar{x}$. If the variation is small, we may neglect the higher-order terms in Equation (18) may be written as

$$y = \bar{y} + K(x-\bar{x}) \tag{19}$$

where

$$\bar{y} = f(\bar{x}), \quad K = \left.\frac{df}{dx}\right|_{x=\bar{x}} \tag{20}$$

Equation (19) may be rewritten as

$$y - \bar{y} = K(x-\bar{x}) \tag{21}$$

which indicates that $y-\bar{y}$ is proportional to $x-\bar{x}$. Equation (21) gives a linear mathematical model for the nonlinear system given by Equation (17) near the operating point $x=\bar{x}$, $y=\bar{y}$.

Next, consider a nonlinear system whose output $y$ is a function of two inputs $x_1$ and $x_2$, so that

$$y = f(x_1, x_2) \tag{22}$$

To obtain a linear approximation to this nonlinear system, we may expand Equation (22) into a Taylor series about the normal operating point, then Equation (22) becomes

$$y = f(\bar{x}_1, \bar{x}_2) + \left[\frac{df}{dx_1}(x_1-\bar{x}_1) + \frac{df}{dx_2}(x_2-\bar{x}_2)\right] + \cdots \tag{23}$$

where the partial derivatives are evaluated at $x_1 = \bar{x}_1$, $x_2 = \bar{x}_2$. Near the normal operating point, the higher-order terms may be neglected. The linear mathematical model of this nonlinear system in the neighborhood of the normal operating condition is then given by

$$y - \bar{y} = K_1(x_1 - \bar{x}_1) + K_2(x_2 - \bar{x}_2) \tag{24}$$

where

$$\bar{y} = f(\bar{x}_1, \bar{x}_2), \quad K_1 = \left.\frac{\delta f}{\delta x_1}\right|_{x_1=\bar{x}_1, x_2=\bar{x}_2}, \quad K_2 = \left.\frac{\delta f}{\delta x_2}\right|_{x_1=\bar{x}_1, x_2=\bar{x}_2} \tag{25}$$