Tutorial—Steady-state Response

Example 1. *Consider, for example, the system shown in Figure 1.*

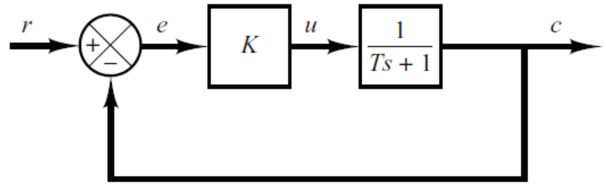

Figure 1: Control System.

Explain why the proportional control of a plant that does not possess an integrating property, which means that the plant transfer function does not include the factor $1/s$, suffers offset in response to step inputs.

Example 2. *The block diagram of Figure 2 shows a speed control system in which the output member of the system is subject to a torque disturbance.*

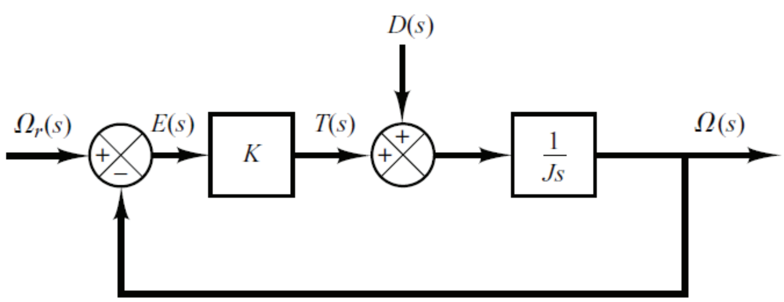

Figure 2: Speed control System.

In the diagram, $\Omega_r(s)$, $\Omega(s)$ and $D(s)$ are the Laplace transforms of the reference speed, output speed, driving torque, and disturbance torque, respectively. In the absence of a disturbance torque, the output speed is equal to the reference speed. Investigate the response of this system to a unit-step disturbance torque. Assume that the reference input is zero, or $\Omega_r(s) = 0$.

Example 3. *In the system considered in Example 2, it is desired to eliminate as much as possible the speed errors due to torque disturbances. Is it possible to cancel the effect of a disturbance torque at steady state so that a constant disturbance torque applied to the output member will cause no speed change at steady state?*

Example 4. *Consider the system shown in Figure 3(a). The steady-state error to a unit-ramp input is* $e_{ss} = 2\zeta/\omega_n$.

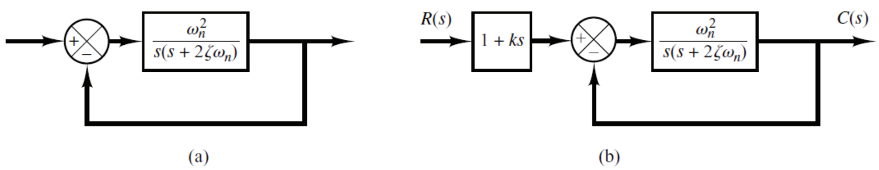

Figure 3: Speed control System.

Show that the steady-state error for following a ramp input may be eliminated if the input is introduced to the system through a proportional-plus-derivative filter, as shown in Figure 3(b), and the value of k *is properly set. Note that the error* $e(t)$ *is given by* $r(t) - c(t)$.

Example 5. *Consider a unity-feedback control system whose open-loop transfer function is*

$$G(s) = \frac{K}{s(Js + B)} \tag{1}$$

Discuss the effects that varying the values of K *and* B *has on the steady-state error in unit-ramp response.*

Solution 1. *At steady state, if c was equal to a nonzero constant r, then $e = 0$ and $u = Ke = 0$, resulting in $c = 0$, which contradicts the assumption that $c = r =$nonzero constant. A nonzero offset must exist for proper operation of such a control system. In other words, at steady state, if e was equal to $r/(1+K)$, then $u = Kr/(1+K)$ and $c = Kr/(1+K)$, which results in the assumed error signal $e = r/(1+K)$. Thus the offset of $r/(1+K)$ must exist in such a system.* □

Solution 2. *Figure 4 is a modified block diagram convenient for the present analysis.*

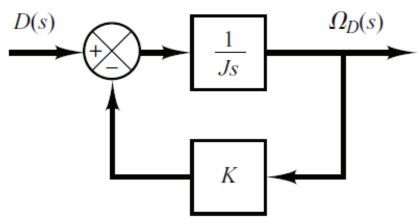

Figure 4: Block diagram of the speed control system of Figure 2 when $\Omega_r(s) = 0$.

The closed-loop transfer function is

$$\frac{\Omega_D(s)}{D(s)} = \frac{1}{Js + K} \tag{2}$$

where $\Omega_D(s)$ is the Laplace transform of the output speed due to the disturbance torque. For a unit-step disturbance torque, the steady-state output velocity is

$$\omega_D(\infty) = \lim_{s\to 0} s\Omega_D(s) = \lim_{s\to 0} \frac{s}{Js + K}\frac{1}{s} = \frac{1}{K} \tag{3}$$

From this analysis, we conclude that, if a step disturbance torque is applied to the output member of the system, an error speed will result so that the ensuing motor torque will exactly cancel the disturbance torque. To develop this motor torque, it is necessary that there be an error in speed so that nonzero torque will result. □

Solution 3. *Suppose that we choose a suitable controller whose transfer function is $G_c(s)$, as shown in Figure 5.*

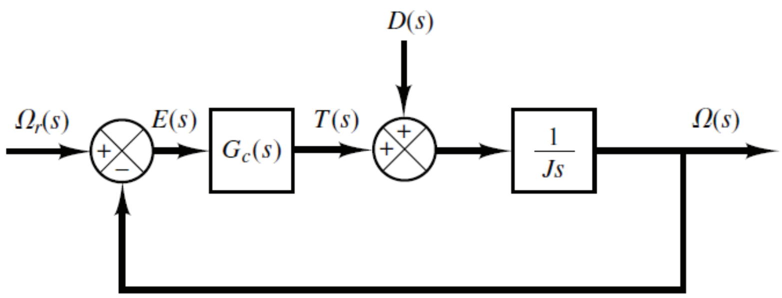

Figure 5: Block diagram of a speed control system.

Then in the absence of the reference input the closed-loop transfer function between the output velocity and the disturbance torque $D(s)$ is

$$\frac{\omega_D(s)}{D(s)} = \frac{\frac{1}{Js}}{1 + \frac{1}{Js}G_c(s)} = \frac{1}{Js + G_c(s)} \tag{4}$$

The steady-state output speed due to a unit-step disturbance torque is

$$\omega_D(\infty) = \lim_{s\to 0} s\Omega_D(s) = \lim_{s\to 0} \frac{s}{Js + G_c(s)}\frac{1}{s} = \frac{1}{Gc(s)} \tag{5}$$

To satisfy the requirement that $\omega_D(\infty) = 0$, we must choose $G_c(0) = \infty$. This can be realized if we choose

$$G_c(s) = \frac{K}{s} \tag{6}$$

Integral control action will continue to correct until the error is zero. This controller, however, presents a stability problem, because the characteristic equation will have two imaginary roots. One method of stabilizing such a system is to add a proportional mode to the controller or choose

$$G_c(s) = K_p + \frac{K}{s} \tag{7}$$

With this controller, the block diagram of Figure 5 in the absence of the reference input can be modified to that of Figure 6. The closed-loop transfer function becomes

$$\frac{\Omega_D(s)}{D(s)} = \frac{s}{Js^2 + K_p s + K} \tag{8}$$

For a unit-step disturbance torque, the steady-state output speed is

$$oemga_D(\infty) = \lim_{s\to 0} s\Omega_D(s) = \lim_{s\to 0} \frac{s^2}{Js^2 + K_p s + K}\frac{1}{s} = 0 \tag{9}$$

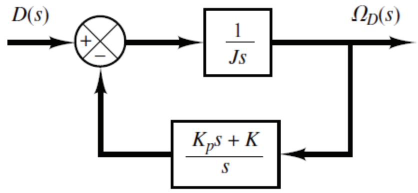

Figure 6: Block diagram of the speed control system of Figure 5 when $G_c(s) = K_p + (K/s)$ and $\Omega_r(s) = 0$.

Thus, we see that the proportional-plus-integral controller eliminates speed error at steady state. The use of integral control action has increased the order of the system by 1.

In the present system, a step disturbance torque will cause a transient error in the output speed, but the error will become zero at steady state. The integrator provides a nonzero output with zero error. The nonzero output of the integrator produces a motor torque that exactly cancels the disturbance torque.

Note that even if the system may have an integrator in the plant such as an integrator in the transfer function of the plant, this does not eliminate the steady-state error due to a step disturbance torque. To eliminate this, we must have an integrator before the point where the disturbance torque enters. □

Solution 4. *The closed-loop transfer function of the system shown in Figure 3(b) is*

$$\frac{C(s)}{R(s)} = \frac{(1+ks)\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{10}$$

Then

$$R(s) - C(s) = \left(\frac{s^2 + 2\zeta\omega_n s - \omega_n^2 ks}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) R(s) \tag{11}$$

If the input is a unit ramp, then the steady-state error is

$$e(\infty) = r(\infty) - c(\infty) = \lim_{s\to 0} s \left(\frac{s^2 + 2\zeta\omega_n s - \omega_n^2 ks}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \frac{1}{s^2} = \frac{2\zeta\omega_n - \omega_n^2 k}{\omega_n^2} \tag{12}$$

Therefore, if k is chosen as

$$k = \frac{2\zeta}{\omega_n} \tag{13}$$

then the steady-state error for following a ramp input can be made equal to zero. Note that, if there are any variations in the values of ζ and/or ω_n due to environmental changes or aging, then a nonzero steady-state error for a ramp response may result. □

Solution 5. *The closed-loop transfer function is*

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K} \tag{14}$$

For a unit-ramp input, $R(s) = 1/s^2$*, then*

$$\frac{E(s)}{R(s)} = \frac{R(s) - C(s)}{R(s)} = \frac{Js^2 + Bs}{Js^2 + Bs + K} \tag{15}$$

or

$$E(s) = \frac{Js^2 + Bs}{Js^2 + BS + K}\frac{1}{s^2} \tag{16}$$

The steady-state error is

$$e_{ss} = \lim_{s \to 0} sE(s) = \frac{B}{K} \tag{17}$$

We can reduce the steady-state error e_{ss} *by increasing the gain* K *or decreasing the viscous-friction coefficient* B*. However, it will cause the damping ration,* $\zeta = B/(2\sqrt{K/J})$*, to decrease, with the result that the transient response of the system will become more oscillatory.* □