Matrix Algebra - Review

Contents

1 Overview

Matrices are rectangular arrays of elements. The elements of a matrix are referred to as *scalars* and will be denoted by lowercase letters, a, b, α, β, etc.

Boldface uppercase letters will be used to represent matrices, such as

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 8 & 1 \end{bmatrix}$$

Horizontal sets of entries such as (5 3) and (8 1) are called rows, whereas vertical sets of entries such as (5 8) and (3 1) are called columns. It will often be convenient to refer to the element in the ith row and jth column of $\mathbf{A}$ as a_{ij}. Rather than explicitly displaying all elements of $\mathbf{A}$, the shorthand notation $\mathbf{A} = [a_{ij}]$ will sometimes be used.

If $\mathbf{A}$ has m rows and n columns, it is said to be an $m \times n$ (or m by n) matrix. In that case, the indices i and j in the shorthand notation indicate collectively the range of values $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. In particular, when $m = n = 1$, the matrix has a single element and is just a scalar. The subscripts are then unnecessary.

If $n = 1$, the matrix has a single column and is called a *column matrix.* The column index j is then superfluous and is sometimes omitted. Similarly, when $m = 1$, the matrix is called a *row matrix.* Whenever $m = n$, the matrix is called a *square matrix.* In general, m and n can take on any finite integer values.

It is very often in control systems engineering that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are used as state variable matrices. The input vector is $\mathbf{u}$ and output vector is $\mathbf{y}$. The state vector is $\mathbf{x}$.

2 Algebraic Operations

2.1 Transpose

The operation of matrix transposition is the interchanging of each row with the column of the same index number. If $\mathbf{A} = [a_{ij}]$, then the *transpose* of $\mathbf{A}$ is $\mathbf{A}^T = [a_{ji}]$. The matrix $\mathbf{A}$ is said to be *symmetric* if $\mathbf{A} = \mathbf{A}^T$. If $\mathbf{A} = -\mathbf{A}^T$, then $\mathbf{A}$ is *skew-symmetric*. An important property of matrix transposition of products is illustrated by

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T,\ (\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T \tag{1}$$

2.2 Determinant

Given $\mathbf{A} = [a_{ij}]$, $|\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}$ for 2×2 matrix $\mathbf{A}$. $\mathbf{B} = [b_{ij}]$, $|\mathbf{B}| = b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31} - b_{12}b_{21}b_{33} - b_{11}b_{23}b_{32}$ for a 3×3 matrix $\mathbf{B}$.

Let

$$\mathbf{A} = \begin{bmatrix} 12 & 19 \\ 11 & 6 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 3 & 9 & 4 \\ 9 & 3 & 2 \\ 3 & 10 & 3 \end{bmatrix}$$

Find $|\mathbf{A}|$, $|\mathbf{B}|$.

Answer:

$$|\mathbf{A}| = -137,\ |\mathbf{B}| = 102$$

2.3 Rank

The rank of a matrix $\mathbf{A}$ may be defined as the size of the largest sub-matrix (may be the original matrix) such that the determinant of the sub-matrix is not zero.

Let

$$\mathbf{A} = \begin{bmatrix} 7 & 1 & 8 \\ 4 & 5 & 8 \\ 10 & 4 & 2 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 7 & 1 & 8 \\ 4 & 5 & 8 \\ 8 & 10 & 16 \end{bmatrix}$$

Find $|\mathbf{A}|$, $|\mathbf{B}|$ and determine their ranks if possible.

Answer:

$$|\mathbf{A}| = -354 \neq 0 \Rightarrow r(\mathbf{A}) = 3,\ |\mathbf{B}| = 0 \Rightarrow r(B) \neq 3$$

If $|\mathbf{B}| = 0$, try to find a non-zero determinant from a smaller size sub-matrix, e.g. $\mathbf{B}_{33} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and determine $|\mathbf{B}_{33}|$. If the result is not zero, the rank can be assigned as the size of matrix $\mathbf{B}$.

Answer:

$$r(\mathbf{B}) = 2$$

2.4 Minors

Minors are the determinant of sub-matrices obtained from ignoring the corresponding elements in the selected row and column. Given a 3×3 matrix $\mathbf{A} = [a_{ij}]$, the minor corresponding to the 1st row and 2nd column element is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$.

Let

$$\mathbf{A} = \begin{bmatrix} 3 & 9 & 4 \\ 9 & 3 & 2 \\ 3 & 10 & 3 \end{bmatrix}$$

Construct the minor $\mathbf{M}$.

Answer:

$$\mathbf{M} = \begin{bmatrix} -11 & 21 & 81 \\ -13 & -3 & 3 \\ 6 & -30 & -72 \end{bmatrix}$$

2.5 Cofactors

The cofactor is a matrix formed from the minor where elements are multiplied by power of (-1) raised to the sum of the row and column indices. $C_{ij} = (-1)^{i+j} M_{ij}$.

Construct the cofactor of matrix $\mathbf{A}$ given above.

Answer:

$$\mathbf{C} = \begin{bmatrix} -11 & -21 & 81 \\ 13 & -3 & -3 \\ 6 & 30 & -72 \end{bmatrix}$$

2.6 Inversion

The inversion of a matrix is given by $\mathbf{A}^{-1} = \frac{adj(\mathbf{A})}{det(\mathbf{A})}$. Where the adjoint matrix $adj(\mathbf{A})$ is obtained as the transpose of the cofactor matrix $\mathbf{C}^T = [C_{ij}]^T$. However, the inversion does not exist if $det(\mathbf{A}) = 0$.

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 8 \\ 4 & 4 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 7 & 1 \\ 4 & 5 \end{bmatrix}$$

Find $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$.

Answer:

$$\mathbf{A}^{-1} = \begin{bmatrix} -0.1429 & 0.2857 \\ 0.1429 & -0.0357 \end{bmatrix}, \ \mathbf{B}^{-1} = \begin{bmatrix} 0.1613 & -0.0323 \\ -0.1290 & 0.2258 \end{bmatrix}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 8 & 2 \\ 4 & 4 & 7 \\ 2 & 6 & 3 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 7 & 1 & 8 \\ 4 & 5 & 8 \\ 17 & 5 & 10 \end{bmatrix}$$

Find $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$.

Answer:

$$\mathbf{M} = \begin{bmatrix} \begin{vmatrix} 4 & 7 \\ 6 & 3 \end{vmatrix} & \begin{vmatrix} 4 & 7 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 4 & 4 \\ 2 & 6 \end{vmatrix} \\ \\ \begin{vmatrix} 8 & 2 \\ 6 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 8 \\ 2 & 6 \end{vmatrix} \\ \\ \begin{vmatrix} 8 & 2 \\ 4 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 8 \\ 4 & 4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -30 & -2 & 16 \\ 12 & -1 & -10 \\ 48 & -1 & -28 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} -30 & 2 & 16 \\ -12 & -1 & 10 \\ 48 & 1 & -28 \end{bmatrix}, \ adj(\mathbf{A}) = \begin{bmatrix} -30 & -12 & 48 \\ 2 & -1 & 1 \\ 16 & 10 & -28 \end{bmatrix}$$

$det(\mathbf{A}) = (1)(4)(3) + (8)(7)(2) + (2)(4)(6) - (2)(4)(2)_(8)(4)(3) - (1)(7)(6) = 18$

$$\mathbf{A}^{-1} = \frac{adj(\mathbf{A})}{det(\mathbf{A})} = \begin{bmatrix} -1.6667 & -0.6667 & 2.6667 \\ 0.1111 & -0.0556 & 0.0556 \\ 0.8889 & 0.5556 & -1.5556 \end{bmatrix}, \ \mathbf{B}^{-1} = \begin{bmatrix} -0.0282 & -0.0847 & 0.0904 \\ -0.2712 & 0.1864 & 0.0678 \\ 0.1836 & 0.0508 & -0.0876 \end{bmatrix}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 8 \\ 4 & 4 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 7 & 1 \\ 4 & 5 \end{bmatrix} \ \mathbf{C} = \begin{bmatrix} 5 & 8 \\ 4 & 2 \end{bmatrix}$$

Find $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$, $\mathbf{C}^{-1}$, $(\mathbf{ABC})^{-1}$, $(\mathbf{CBA})^{-1}$. Using the results from above, find also $\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$ and $\mathbf{A}^{-1}\mathbf{B}^{-1}\mathbf{C}^{-1}$. What conclusion can be drawn on the order of multiplication among the matrix inversions?

Answer:

$$\mathbf{A}^{-1} = \begin{bmatrix} -0.1429 & 0.2857 \\ 0.1429 & -0.0357 \end{bmatrix}, \ \mathbf{B}^{-1} = \begin{bmatrix} 0.1613 & -0.0323 \\ -0.1290 & 0.2258 \end{bmatrix}, \ \mathbf{C}^{-1} = \begin{bmatrix} -0.0909 & 0.3636 \\ 0.1818 & -0.2273 \end{bmatrix}$$

$$(\mathbf{ABC})^{-1} = \begin{bmatrix} 0.0209 & -0.0206 \\ -0.0165 & 0.0188 \end{bmatrix}, \ (\mathbf{CBA})^{-1} = \begin{bmatrix} 0.0180 & -0.0375 \\ -0.0048 & 0.0129 \end{bmatrix}$$

$$\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} 0.0209 & -0.0206 \\ -0.0165 & 0.0188 \end{bmatrix}, \ \mathbf{A}^{-1}\mathbf{B}^{-1}\mathbf{C}^{-1} = \begin{bmatrix} 0.0180 & -0.0375 \\ -0.0048 & 0.0129 \end{bmatrix}$$