Complex Numbers

1 Complex Numbers

Equations without real solutions, such as $x^2 = -1$ leads to the introduction of complex numbers. A complex number z is an ordered pair (x, y) of real numbers x and y, written $z = (x, y)$, x is called the **real** part and y the **imaginary** part of z, written $x = Re\ z$, $y = Im\ z$. $(0, 1)$ is called the **imaginary unit** and is denoted by i or j, $i = (0, 1)$. By definition, two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

2 Addition, Multiplication

Addition of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined by

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2). \tag{1}$$

Multiplication is defined by

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1). \tag{2}$$

In practice, complex numbers are written

$$z = x + jy. \tag{3}$$

If $x = 0$ then $z = y$ and is called **pure imaginary**. Also

$$j^2 = -1, \tag{4}$$

because, by the definition of multiplication, $j^2 = jj = (0, 1)(0, 1) = (-1, 0) = -1$.

For addition the notation gives

$$(x_1 + jy_1) + (x_2 + jy_2) = (x_1 + x_2) + j(y_1 + y_2). \tag{5}$$

For multiplication the notation gives the following very simple recipe. Multiply each term by each other term and use when it occurs

$$\begin{aligned} (x_1 + jy_1)(x_2 + jy_2) &= x_1 x_2 + jx_1 y_2 + jy_1 x_2 + j^2 y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1). \end{aligned} \tag{6}$$

3 Subtraction, Division

Subtraction and division are defined as the inverse operations of addition and multiplication, respectively. Thus the **difference** $z = z_1 - z_2$ is the complex number z for which $z_1 = z + z_2$. Hence

$$z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2). \tag{7}$$

The quotient $z = z_1/z_2(z_2 \neq 0)$ is the complex number z for which $z_1 = zz_2$. If we equate the real and the imaginary parts on both sides of this equation, setting we obtain $x_1 = x_2x - y_2y$, $y_1 = y_2x + x_2y$. The solution is

$$z = \frac{z_1}{z_2} = x + jy, \quad x = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \quad y = \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}. \tag{8}$$

The practical rule used to get this is by multiplying numerator and denominator of z_1/z_2 by $x_2 - jy_2$ and simplifying:

$$z = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + j\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}. \tag{9}$$

4 Complex Plane

We choose two perpendicular coordinate axes, the horizontal x-axis, called the **real axis**, and the vertical y-axis, called the **imaginary axis**. On both axes we choose the same unit of length (Fig. 1). This is called a **Cartesian coordinate system**. We now plot a given complex number $z = (x, y) = x + y$ as the

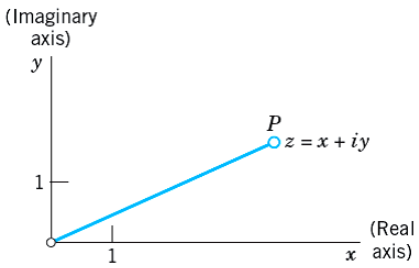

Figure 1: The complex plane

point P with coordinates x, y. The xy-plane in which the complex numbers are represented in this way is called the **complex plane**.

Addition and subtraction can now be visualized as illustrated in Figs. 2 and 3.

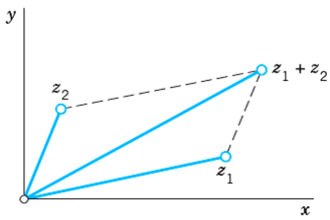

Figure 2: Addition of complex numbers

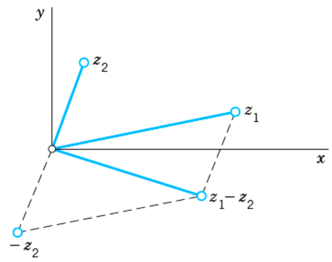

Figure 3: Subtraction of complex numbers

5 Complex Conjugate Numbers

The **complex conjugate** $\bar{z}$ of a complex number $z = x + jy$ is defined by

$$\bar{z} = x - jy. \tag{10}$$

It is obtained geometrically by reflecting the point z in the real axis. Figure 4 shows this for $z = 5{+}2$ and its conjugate $\bar{z} = 5{-}2$. By addition and subtraction,

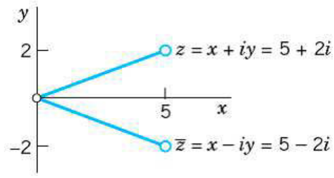

Figure 4: Complex conjugate numbers

$z + \bar{z} = 2x$, $z - \bar{z} = j2y$. We thus obtain for the real part x and the imaginary part y of the important formulas

$$Re\ z = x = \frac{1}{2}(z + \bar{z}), \quad Im\ z = y = \frac{1}{j2}(z - \bar{z}). \tag{11}$$

Working with conjugates is easy, since we have

$$\begin{aligned} \overline{z_1 + z_2} &= \bar{z}_1 + \bar{z}_2 \\ \overline{z_1 - z_2} &= \bar{z}_1 - \bar{z}_2 \\ \overline{z_1 z_2} &= \bar{z}_1 \bar{z}_2 \\ \overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\bar{z}_1}{\bar{z}_2} \end{aligned} \tag{12}$$

6 Polar Form of Complex Numbers – Powers and Roots

We gain further insight into the arithmetic operations of complex numbers if, in addition to the xy-coordinates in the complex plane, we also employ the usual polar coordinates r, θ defined by

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{13}$$

We see that then takes the so-called polar form

$$z = r(\cos\theta + j\sin\theta), \tag{14}$$

r is called the **absolute value** or **modulus** of z and is denoted by $|z|$. Hence

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}. \tag{15}$$

Geometrically, $|z|$ is the distance of the point z from the origin (Fig. 5). Similarly, $|z_1 - z_2|$ is the distance betweenz_1 and z_2 (Fig. 6).

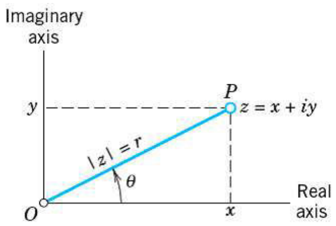

Figure 5: Complex plane, polar form of a complex number

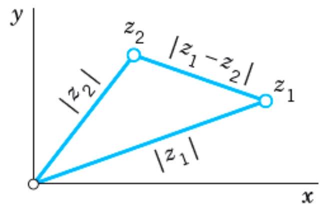

Figure 6: Distance between two points in the complex plane

θ is called the argument of z and is denoted by arg z. Thus $\theta = arg\ z$ and

$$\tan\theta = \frac{y}{x}. \tag{16}$$

Geometrically, θ is the directed angle from the positive x-axis to OP in Fig. 5. Here, all angles are measured in radians and positive in the counterclockwise sense.

For a given $z \neq 0$, the angle is determined only up to integer multiples of 2π since cosine and sine are periodic with period 2π. But one often wants to specify a unique value of arg z of a given $z \neq 0$. For this reason one defines the principal value Arg z (with capital A!) of arg z by the double inequality

$$-\pi < Arg\ z \leq \pi. \tag{17}$$

7 Multiplication and Division in Polar Form

Let

$$z_1 = r_1(\cos\theta_1 + j\sin\theta_1), \quad z_2 = r_2(\cos\theta_2 + j\sin\theta_2). \tag{18}$$

7.1 Multiplication

The product is

$$z_1z_2 = r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]. \tag{19}$$

From the addition rules for the sine and cosine, now yield

$$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)]. \tag{20}$$

Taking absolute values on both sides, we see that *the absolute value of a product equals the product of the absolute values of the factors*,

$$|z_1z_2| = |z_1||z_2|. \tag{21}$$

Taking arguments shows that *the argument of a product equals the sum of the arguments of the factors*,

$$arg(z_1z_2) = arg\ z_1 + arg\ z_2. \quad \text{up to multiples of } 2\pi. \tag{22}$$

7.2 Division

We have $z_1 = (z_1/z_2)z_2$. Hence $|z_1| = |(z_1/z_2)z_2| = |z_1/z_2||z_2|$ and by division by $|z_2|$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}. \tag{23}$$

Similarly, $arg\ z_1 = arg\ [(z_1/z_2)z_2] = arg\ (z_1/z_2) + arg\ z_2$ and by subtraction of $arg\ z_2$

$$arg\frac{z_1}{z_2} = arg\ z_1 - arg\ z_2, \quad \text{up to multiples of } 2\pi. \tag{24}$$

Then we have

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + j\sin(\theta_1-\theta_2)]. \tag{25}$$

Note that it is the polar form of a complex number of absolute value r_1/r_2 and argument $\theta_1 - \theta_2$.

8 Roots of Complex Numbers

If $z = w^n$, $n = 1, 2, \cdots$, then to each value of w there corresponds one value of z. We shall immediately see that, conversely, to a given $z \neq 0$ there correspond precisely n distinct values of w. Each of these values is called an nth root of z, and we write

$$w = \sqrt[n]{z}. \tag{26}$$

We write z and w in polar form

$$z = r(\cos\theta + \sin\theta), \quad w = R(\cos\phi + j\sin\phi) \tag{27}$$

Then the equation $w^n = z$ becomes, by De Moivre's formula

$$w^n = R^n(\cos n\phi + j\sin n\phi) = z = r(\cos\theta + \sin\theta). \tag{28}$$

The absolute values on both sides must be equal; thus $R^n = r$, so that $R = \sqrt[n]{r}$ where $\sqrt[n]{r}$ is positive real and thus uniquely determined. Equating the arguments $n\phi$ and θ and recalling that θ is determined only up to integer multiples of 2π, we obtain

$$n\phi = \theta + 2k\pi, \quad \phi = \frac{\theta}{n} + \frac{2k\pi}{n}, \tag{29}$$

where k is an integer. For $k = 0, 1, \cdots$ we get n distinct values of w.

Consequently, $\sqrt[n]{z}$ for $z \neq 0$, has the n distinct values

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + j\sin\frac{\theta + 2k\pi}{n}\right). \tag{30}$$

These n values lie on a circle of radius $\sqrt[n]{r}$ with center at the origin and constitute the vertices of a regular polygon of n sides. The value of obtained by taking the principal value of arg z is called the **principal value** of $w = \sqrt[n]{z}$.

Taking $z = 1$, we have $|z| = r = 1$ and Arg $z = 0$. Then

$$\sqrt[n]{1} = \cos\frac{\theta + 2k\pi}{n} + j\sin\frac{\theta + 2k\pi}{n}, \quad k = 0, 1, \cdots. \tag{31}$$

These n values are called the n**th roots of unity**. They lie on the circle of radius 1 and center 0, briefly called the **unit circle**.

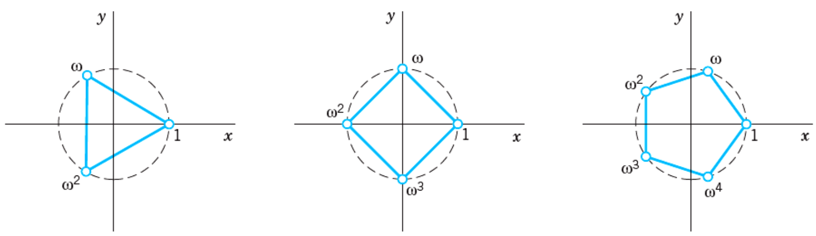

Figure 7: Unit circles: $\sqrt[3]{1}$, $\sqrt[4]{1}$, $\sqrt[5]{1}$

9 Exercises

1. Let $z_1 = 8 + j3$, $z_2 = 9 - j2$, find Real Part, Imaginary Part, Sum and Product.
2. For $z_1 = 8 + j3$, $z_2 = 9 - j2$ find Difference and Quotient.
3. Let $z_1 = 4 + j3$ and $z_2 = 2 + j5$, verify Eq. 11, Eq. 12.
4. Let $z_1 = 1 + j$, $z_2 = -2 - j3$, find Polar Form, Principal Value Arg z.
5. Let $z_1 = -2 + j2$, $z_2 = j3$, verify Eq. 21, 22, 23, and 24.