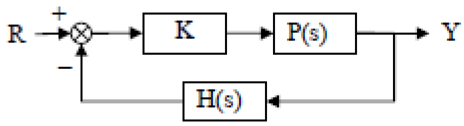
Lecture 11 - THE ROOT LOCUS TECHNIQUE

We now deal with one of several analysis techniques for studying the movement of *s* roots as functions of system variables in feedback loops.
Root locus concentrates on a SISO system using unity gain negative feedback. This scenario can cover many practical cases.

REMEMBER
Consider the SISO system using negative feedback with a proportional controller



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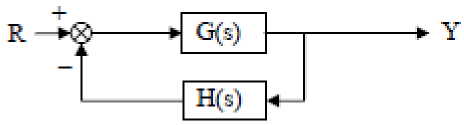


where $G(s) = KP(s)$.

What are the open-loop TF and the closed-loop TF?
The closed-loop TF is given by:

$$G_c(s) = \frac{G(s)}{1 + G(s)H(s)}$$

The stability of the closed-loop system can be discussed by solving the equation
$1 + F(s) = 0$
and finding the roots, where $F(s) = G(s)H(s) = KP(s)H(s)$ is the open-loop TF.
The equation is called the *characteristic equation (CE)*.

If the system has a unity gain negative feedback, i.e. $H(s) = 1$, and the open-loop TF is:

$$G(s) = \frac{n(s)}{d(s)},$$

then the characteristic equation of the closed-loop TF:

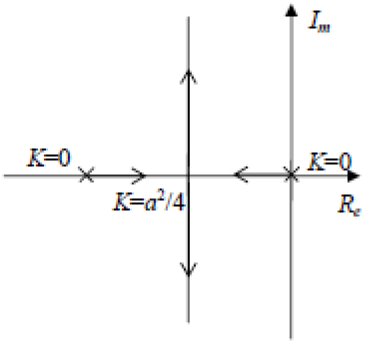
$d(s) + n(s) = 0$

Example of use

The simple feedback loop is applied to $G(s) = \frac{K}{s(s+a)}$, $H(s) = 1$.
Thus, the CE becomes

$1 + G(s) = 1 + \frac{K}{s(s+a)} = 0$, or $s^2 + as + K = 0$

We now draw a *root locus* by checking the movements of poles in the s-plane with respect to *K*



SOME PROPERTIES OF ROOTS

Let G(*s*) (open loop transfer function) be given by:

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

The CE of the closed-loop system is given by:

$$\prod(s+p_i) + K\prod(s+z_i) = 0$$

Concentrate on *K* and varying its magnitude. *K* is known as the multiplicative gain factor or proportional controller and is often the variable changed to alter control scenarios. We can deduce from the equation above:

1. The roots of *s* at *K* = 0 are at the poles of G(*s*) .
2. The roots of *s* at *K* = ∞ are at the zeros of G(*s*), some of which may be at infinity somewhere.
3. The roots of *s* lying on the real axis always lie in a section of the axis to the left of an odd number of singularities (poles or zeros of the open loop transfer function).
4. The number of loci (paths of roots as we vary 0 < *K* < ∞) is the greater of the number of poles or zeros. Usually $n_p > n_z$.
5. When $n_p > n_z$, then $n_p - n_z$ loci end up at infinity along prescribed asymptotes (as *K*→∞).
6. Loci proceed to infinity along asymptotes whose centroid is at

$\sigma_A = \frac{\sum(-p_i) - \sum(-z_i)}{n_p - n_z}$,

and asymptotic angles with the Re-axis:

$\varphi_A = \frac{2q+1}{n_p - n_z} \cdot 180$, q=0, 1, … $(n_p - n_z - 1)$

ROOT LOCUS EXAMPLE 1 (RULES 1-8)

Use of the six rules given so far (as an example)

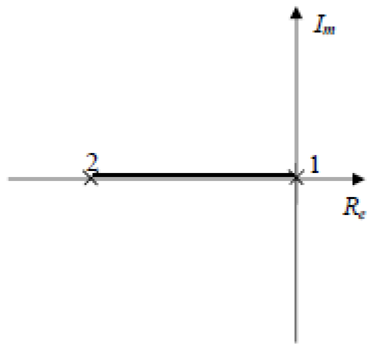
Take $1 + \frac{K}{s(s+a)} = 0$

and re-examine the equation.

Rule 1: Loci start at $-a$, 0 when K = 0 ($n_p = 2$)

Rule 2: Loci end at ∞, ∞ when K = ∞ ($n_z = 0$)

Rule 3: Place poles and zeros, number them from the right to the left, and draw a line from an odd pole (zero) to the first left pole (zero).



Rule 4: $n_p = 2, n_z = 0$
There are 2 loci.

Rule 5: $n_p - n_z = 2$
There are 2 roots that end up at infinity.

Rule 6:
The centroid is:
$\sigma_A = \frac{-a-0}{2}$

Asymptotes:

- For q=0, $\varphi_A = \frac{2q+1}{n_p - n_z} \cdot 180 = 90^o$
- For q=1, $\varphi_A = \frac{2q+1}{n_p - n_z} \cdot 180 = 270^o$

Rule 7: Points of departure from the real axis (also called break-away points) can be obtained in the following way:
From the CE: G(s)+1=*KP*(*s*)*H*(*s*) +1= 0 , solve for *K* , i.e.,
$K = -\frac{1}{P(s)H(s)}$

It is known that the break-away point satisfies $\frac{d(-K)}{ds} = 0$

If $\mathrm{G(s)} = \frac{K}{s(s+a)} = -1$, then

$\frac{d(-K)}{ds} = \frac{d}{ds}(s^2 + as) = 2s + a = 0$

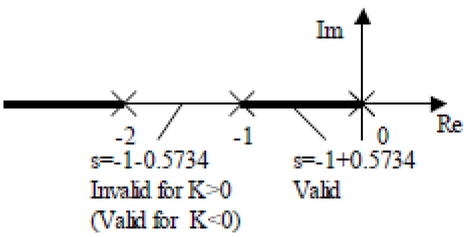
$s = -\frac{a}{2}$ is the break-away point as before.

Different example: If $G(s) = \frac{K}{s(s+1)(s+2)} = -1$

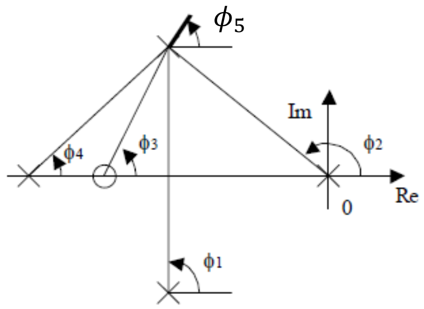
$$\frac{d(-K)}{ds} = \frac{d}{ds}(s^3 + 3s^2 + 2s) = 0$$

$$3s^2 + 6s + 2 = 0$$

$$s = -1 \pm 0.574$$



Rule 8: Construction of departure angles from poles that do not lie on the real axis using a geometric analogy. For any pole draw in the vectors from the other poles and zeros to a point very, very close to that pole, e.g.



The angle condition says that:

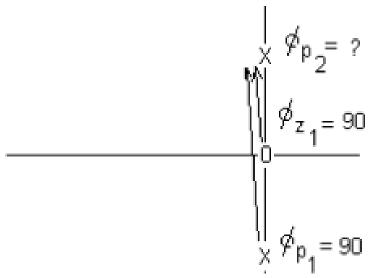
$$\sum_i \varphi_{z_i} - \sum_j \varphi_{p_j} = -180 - 360l$$

where *l* is an integer chosen such that the angle falls between –180 and 180.

$$\phi_3 - \phi_1 - \phi_2 - \phi_4 - \phi_5 = -180 - 360l$$

$\phi_5 = \phi_3 - \phi_1 - \phi_2 - \phi_4 + 180 + 360l$ degrees and is the direction of departure of the locus immediately leaving the pole.

An example: If $(s) = \frac{as}{s^2+b}$, the poles are $\pm\sqrt{-b}$. To find the departure angles from the poles $-j\sqrt{b}$ and $+j\sqrt{b}$, we draw the diagram:

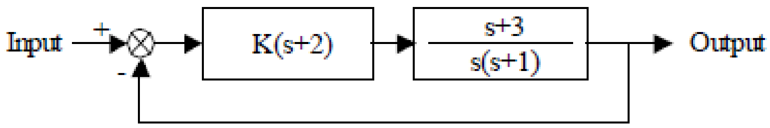


We see that $\sum \varphi_{z_i} = 90$ and $\sum \varphi_{p_j} = 90$

$$90 - 90 - \varphi_{p_2} = -180 - 360l$$

$\varphi_{p_2} = 180$ (and $\varphi_{p_1} = 180$ by symmetry)

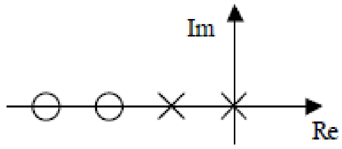
ROOT LOCUS EXAMPLE 2 (RULES 1-9)



RULE 1 Poles are at 0 and -1. $n_p = 2$

RULE 2 Zeros are at -2 and -3. $n_z = 2$

RULE 3 Loci positions on real axis are:



RULE 4 Number of loci = 2.

RULE 5 $n_p - n_z = 0$ (No loci end up at infinity)

RULE 6 N/A (No asymptotes)

RULE 7 From the CE,

$$-K = \frac{s(s+1)}{(s+2)(s+3)}$$

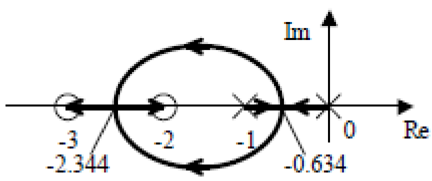
$$\frac{d(-K)}{ds} = \frac{(s+2)(s+3)(2s+1) - s(s+1)(2s+5)}{[(s+2)(s+3)]^2} = 0$$

$$4s^2 + 12s + 6 = 0$$

$$(s + 0.634)\ (s + 2.344) = 0$$

Breakaway points are at -0.634 and -2.344.

RULE 8 Departure (or return) angles away from (0, 0) are always at right angles to the real axis.



Some questions which arise.
Is the system stable?
Are there any problem parts of the loci?
What value of *K* arises when *s* = −0.634 ?

Take $1 + \frac{K(s+2)(s+3)}{s(s+1)} = 0$

Substituting *s=-0.634* gives *K* = 0.0718
Similarly, when *s* = −2.344, *K* =14 .
What is the best (most stable and responsive) value to choose?

ROOT LOCUS EXAMPLE 3 (RULES 1-9)

$$G(s) = \frac{K}{s(s^2 + 2s + 4)}$$

RULE 1: The poles of *s* are 0 and $-1 \pm j\sqrt{3}$.

RULE 2: The zeros of *s* are at infinity (three of them).

RULE 3: Segment of the Re-axis where the locus exists: to the left from the origin.

RULE 4: There are 3 loci.

RULE 5: 3-0 loci end up at infinity.

RULE 6: Asymptotes:
Centroid: $\sigma_A = \frac{0-1+j\sqrt{3}-1+j\sqrt{3}}{3} = -\frac{2}{3}$
Angles: $\varphi_A = \frac{2q+1}{3} \cdot 180 = 60{,}180, 300$

RULE 7: No break away points.

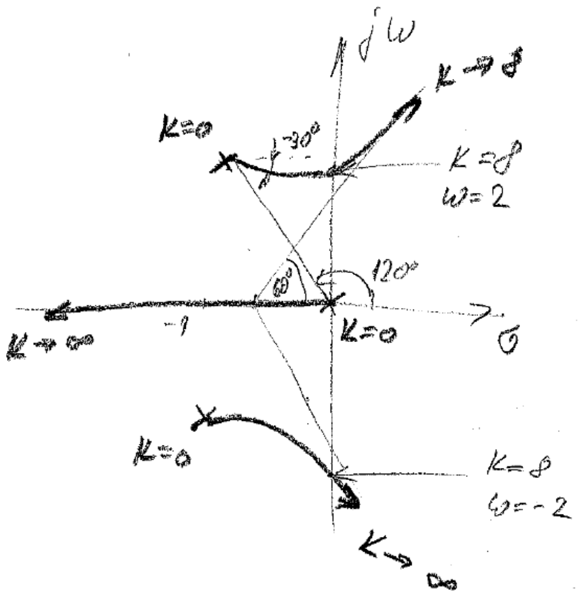
RULE 8: Departure angles:

$$tan^{-1}\frac{\sqrt{3}}{1} = 60^o$$

$$0 - (120 + 90 + \phi_3) = -180$$

$$\phi_3 = -30^o$$

Therefore we can sketch:



Now a final rule:

RULE 9: To detect a locus crossing the imaginary axis i.e. crossing the stable/unstable divide, we invoke ROUTH'S CRITERION (alternatively make $s = j\omega$)

First arrange G(s) +1 = 0 to the form

$$\sum_{i=0}^{n} a_{n-i}s^i = a_0 s^n + a_1 s^{n-1} + \cdots + a_n s^0$$

(Keep $a_0 = 1$)

Now create an array in the form below using "*a*" coefficients

s^n	a_0	a_2	a_4	...
s^{n-1}	a_1	a_3	a_5	...
s^{n-2}	b_1	b_2	b_3	...
s^{n-3}	c_1	c_2	c_3	...
$\vdots$	$\vdots$			
s^0				

where:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}; \quad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}; \quad \ldots$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}; \quad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}; \quad \ldots$$

$\vdots$

putting in numerical values and variables where given.

<u>ROUTH'S CRITERION states that the number of roots of the equation with positive real parts is equal to the number of sign changes in the first column of the array.</u>

$s^3+2s^2+4s+K=0$

$$\begin{array}{c|cc}
s^3 & 1 & 4 \\
s^2 & 2 & K \\
s^1 & \frac{8-K}{2} & 0 \\
s^0 & K &
\end{array}
\qquad b_1 = \frac{8-K}{2} \qquad c_1 = K$$

Change of sign happens when:

$\frac{8-K}{2}=0$, $K=8$

(Also when $K=0$)

REVISION QUESTIONS

- **ROUTH CRITERION EXAMPLE 1**

Let $G(s) = \sum_{i=0}^{n} a_{n-i}s^i = s^4 + 5s^3 + 20s^2 + 40s + 50$

How many unstable poles does this system have?

- **ROUTH CRITERION EXAMPLE 2**

Take $s^5 + 2s^4 + 3s^3 + 6s^2 + 10s + 15 = 0$
N.B. To avoid divisions by 0, introduce $\varepsilon \to 0$.

- **ROOT LOCUS EXAMPLE 4**

Find the root locus of $\frac{K}{(s+2)(s+4)}$. Use all or as many rules as necessary to make a complete root locus.

- **ROOT LOCUS EXAMPLE 5**

Find the root locus of $\frac{K}{s^2+4s+10}$