Laplace Transform

1 Complex Variable and Function

A complex number has a real part and an imaginary part, both of which are constant. If the real part and/or imaginary part are variables, a complex quantity is called a complex variable. In the Laplace transformation we use the notation s as a complex variable; that is,

$$s = \sigma + j\omega \tag{1}$$

where σ is the real part and ω is the imaginary part.

A complex function $G(s)$, a function of s, has a real part and an imaginary part or

$$G(s) = G_x + jG_y \tag{2}$$

where G_x and G_y are real quantities. The magnitude $|G(s)|$ of $G(s)$ is $\sqrt{G_x^2 + G_y^2}$ and the angle θ of $G(s)$ is $\tan^{-1}(Gy/G_x)$. The angle is measured counterclockwise from the positive real axis. The complex conjugate of $G(s)$ is $\bar{G}(s) = G_x - jG_y$. Complex functions commonly encountered in linear control systems analysis are single-valued functions of s and are uniquely determined for a given value of s.

A complex function $G(s)$ is said to be *analytic* in a region if $G(s)$ and all its derivatives exist in that region. The derivative of an analytic function $G(s)$ is given by

$$\frac{d}{ds}G(s) = \lim_{\Delta s \to 0} \frac{G(s + \Delta s) - G(s)}{\Delta s} = \lim_{\Delta s \to 0} \frac{\Delta G}{\Delta s} \tag{3}$$

Note that the derivative of an analytic function can be obtained simply by differentiating $G(s)$ with respect to s. As an example,

$$\frac{d}{ds}\left(\frac{1}{s+1}\right) = \frac{d(s+1)^{-1}}{d(s+1)}\frac{d(s+1)}{ds} = -\frac{1}{(s+1)^2} \tag{4}$$

Points in the s plane at which the function $G(s)$ is analytic are called ordinary points, while points in the s plane at which the function $G(s)$ is not analytic are called singular points. Singular points at which the function $G(s)$ or its derivatives approach infinity are called *poles*. Singular points at which the function $G(s)$ equals zero are called *zeros*.

2 Laplace Transformation

Let us define

$$
\begin{aligned}
f(t) = &\quad \text{a function of time } t \text{ such that } f(t) = 0 \text{ for } t < 0 \\
s = &\quad \text{a complex variable} \\
\mathfrak{L} = &\quad \text{an operational symbol indicating that the quantity that it prefixes} \\
&\quad \text{is to be transformed by the Laplace integral } \textstyle\int_0^\infty e^{-st}dt \\
F(s) = &\quad \text{Laplace transform of } f(t)
\end{aligned}
\tag{5}
$$

Then the Laplace transform of $f(t)$ is given by

$$\mathfrak{L}[f(t)] = F(s) = \int_0^\infty e^{-st}dt[f(t)] = \int_0^\infty f(t)e^{-st}dt \tag{6}$$

The reverse process of finding the time function $f(t)$ from the Laplace transform $F(s)$ is called the inverse Laplace transformation. The notation for the inverse Laplace transformation is $\mathfrak{L}^{-1}$, and the inverse Laplace transform can be found from $F(s)$ by the following inversion integral:

$$\mathfrak{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds, \text{ for } t > 0 \tag{7}$$

where c, the abscissa of convergence, is a real constant and is chosen larger than the real parts of all singular points of $F(s)$. Thus, the path of integration is parallel to the $j\omega$ axis and is displaced by the amount c from it. This path of integration is to the right of all singular points.

3 Examples - Laplace Transform

3.1 Exponential Function

Consider the exponential function

$$f(t) = \begin{cases} 0, & t < 0 \\ Ae^{-\alpha t}, & t \geq 0 \end{cases} \tag{8}$$

where A and α are constants. The Laplace transform of this exponential function can be obtained as follows:

$$\mathfrak{L}\left[Ae^{-\alpha t}\right] = \int_0^\infty Ae^{-\alpha t}e^{-st}dt = A\int_0^\infty e^{-(\alpha+s)t}dt = \frac{A}{s+\alpha} \tag{9}$$

It is seen that the exponential function produces a pole in the complex plane.

3.2 Step Function

Consider the step function

$$f(t) = \begin{cases} 0, & t < 0 \\ A, & t > 0 \end{cases} \tag{10}$$

where A is a constant. Note that it is a special case of the exponential function $Ae^{-\alpha t}$, where $\alpha = 0$. The step function is undefined at $t = 0$. Its Laplace transform is given by

$$\mathfrak{L}[A] = \int_0^\infty Ae^{-st}dt = \frac{A}{s} \tag{11}$$

The step function whose height is unity (i.e., $A = 1$) is called unit-step function.

3.3 Ramp Function

Consider the ramp function

$$f(t) = \begin{cases} 0, & t < 0 \\ At, & t \geq 0 \end{cases} \tag{12}$$

where A is a constant. The Laplace transform of this ramp function is obtained as

$$\mathfrak{L}[At] = \int_0^\infty Ate^{-st}dt = At\left.\frac{e^{-st}}{-s}\right|_0^\infty - \int_0^\infty \frac{Ae^{-st}}{-s}dt = \frac{A}{s}\int_0^\infty e^{-st}dt = \frac{A}{s^2} \tag{13}$$

3.4 Sinusoidal Function

The Laplace transform of the sinusoidal function

$$f(t) = \begin{cases} 0, & t < 0 \\ A\sin(\omega t), & t \geq 0 \end{cases} \tag{14}$$

where A and ω are constants, is obtained as follows. Note that $\sin \omega t$ can be written as

$$\sin \omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \tag{15}$$

Hence

$$\mathfrak{L}[A\sin \omega t] = \frac{A}{2j}\int_0^\infty \left(e^{j\omega t} - e^{-j\omega t}\right)e^{-st}dt = \frac{A}{2j}\left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right) = \frac{A\omega}{s^2 + \omega^2} \tag{16}$$

Similarly, the Laplace transform of $A\cos \omega t$ can be derived by using $\cos \omega t = \left(e^{j\omega t} + e^{j\omega t}\right)/2$, hence

$$\mathfrak{L}[A\cos \omega t] = \frac{As}{s^2 + \omega^2} \tag{17}$$

3.5 Translated Function

Let us obtain the Laplace transform of the translated function $f(t-\alpha)1(t-\alpha)$, where $\alpha \geq 0$. This function is zero for $t < \alpha$. By definition, the Laplace transform is

$$\mathfrak{L}[f(t-\alpha)1(t-\alpha)] = \int_0^\infty f(t-\alpha)1(t-\alpha)e^{-st}dt \tag{18}$$

By changing the independent variable from t to τ where $\tau = t - \alpha$, we obtain

$$\int_0^{\infty} f(t-\alpha)1(t-\alpha)e^{-st}dt = \int_{-\alpha}^{\infty} f(\tau)1(\tau)e^{-s(\tau+\alpha)}d\tau \tag{19}$$

Since $f(t) = 0$ for $t < 0$, $f(\tau)1(\tau) = 0$ for $\tau < 0$, hence, we can change the lower limit of integration from $-\alpha$ to 0. Thus

$$\begin{aligned}\int_{-\alpha}^{\infty} f(\tau)1(\tau)e^{-s(\tau+\alpha)}d\tau &= \int_0^{\infty} f(\tau)1(\tau)e^{-s(\tau+\alpha)}d\tau \\ &= \int_0^{\infty} f(\tau)e^{-s\tau}e^{-s\alpha}d\tau \\ &= e^{-\alpha s}\int_0^{\infty} f(\tau)e^{-s\tau}d\tau \\ &= e^{-\alpha s}F(s)\end{aligned} \tag{20}$$

where

$$F(s) = \mathfrak{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}dt \tag{21}$$

It can be seen that the translation of the time function $f(t)1(t)$ by α (where $\alpha \geq 0$) corresponds to the multiplication of the transform $F(s)$ by $e^{-\alpha s}$.

3.6 Pulse Function

Consider the pulse function

$$f(t) = \begin{cases} A/t_0, & 0 < t < t_0 \\ 0, & t < 0, t_0 < t \end{cases} \tag{22}$$

where A and t_0 are constants. The pulse function here may be considered a step function of height A/t_0 that begins at $t = 0$ and that is superimposed by a negative step function of height A/t_0 beginning at $t = t_0$; that is,

$$f(t) = \frac{A}{t_0}1(t) - \frac{A}{t_0}1(t-t_0) \tag{23}$$

Then the Laplace transform of is obtained as

$$\mathfrak{L}[f(t)] = \mathfrak{L}\left[\frac{A}{t_0}1(t)\right] - \mathfrak{L}\left[\frac{A}{t_0}1(t-t_0)\right] = \frac{A}{t_0 s} - \frac{A}{t_0 s}e^{-st_0} = \frac{A}{t_0 s}\left(1 - e^{-st_0}\right) \tag{24}$$

3.7 Impulse Function

The impulse function is a special limiting case of the pulse function. Consider the impulse function

$$f(t) = \begin{cases} \lim_{t_0 \to 0} A/t_0, & 0 < t < t_0 \\ 0, & t < 0, t_0 < t \end{cases} \tag{25}$$

Since the height of the impulse function is A/t_0 and the duration is t_0, the area under the impulse is equal to A. As the duration t_0 approaches zero, the height A/t_0 approaches infinity, but the area under the impulse remains equal to A. Note that the magnitude of the impulse is measured by its area. Hence,

$$\mathfrak{L}\left[f(t)\right] = \lim_{t_0 \to 0} \left[\frac{A}{t_0 s}\left(1 - e^{-st_0}\right)\right] = A \tag{26}$$

Thus the Laplace transform of the impulse function is equal to the area under the impulse. The impulse function whose area is equal to unity (i.e., $A = 1$) is called the unit-impulse function or the Dirac delta function.

3.8 Multiplication of $f(t)$ by $e^{-\alpha t}$

If $f(t)$ is Laplace transformable, its Laplace transform being $F(s)$, then the Laplace transform of $e^{-\alpha t} f(t)$ is obtained as

$$\mathfrak{L}\left[e^{-\alpha t} f(t)\right] = \int_0^\infty e^{-\alpha t} f(t) e^{-st} dt = F(s + \alpha) \tag{27}$$

We see that the multiplication of $f(t)$ by $e^{-\alpha t}$ has the effect of replacing s by $(s+\alpha)$ in the Laplace transform. Conversely, changing s to $(s+\alpha)$ is equivalent to multiplying $f(t)$ by $e^{-\alpha t}$. Note that α may be real or complex.

3.9 Change of Time Scale

If t is changed into t/α, where α is a positive constant, then the function $f(t)$ is changed into $f(t/\alpha)$. If we denote the Laplace transform of $f(t)$ by $F(s)$, then the Laplace transform of $f(t/\alpha)$ may be obtained as follows:

Letting $t/\alpha = t_1$ and $\alpha s = s_1$, we obtain

$$\mathfrak{L}\left[f\left(\frac{t}{\alpha}\right)\right] = \int_0^\infty f(t_1) e^{-s_1 t_1} d(\alpha t_1) = \alpha \int_0^\infty f(t_1) e^{-s_1 t_1} dt_1 = \alpha F(s_1) = \alpha F(\alpha s) \tag{28}$$

4 Laplace Transform Theorems

4.1 Real Differentiation Theorem

The Laplace transform of the derivative of a function $f(t)$ is given by

$$\mathfrak{L}\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0) \tag{29}$$

where $f(0)$ is the initial value of $f(t)$ evaluated at $t = 0$. Here we assumed $f(0-) = f(0+) = f(0)$.

To prove the real differentiation theorem, we proceed as follows. Integrating the Laplace integral by parts gives

$$\int_0^\infty f(t)e^{-st}dt = f(t)\left.\frac{e^{-st}}{-s}\right|_0^\infty - \int_0^\infty \left[\frac{d}{dt}f(t)\right]\frac{e^{-st}}{-s}dt \tag{30}$$

Hence

$$F(s) = \frac{f(0)}{s} + \frac{1}{s}\mathfrak{L}\left[\frac{d}{dt}f(t)\right] \tag{31}$$

It follows that

$$\mathfrak{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0) \tag{32}$$

Similarly, we obtain the following relationship for the second derivative of $f(t)$:

$$\mathfrak{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0)\dot{f}(0) \tag{33}$$

where $\dot{f}(0)$ is the value of $df(t)/dt$ evaluated at $t = 0$. To derive this equation, define $df(t)/dt = g(t)$, then

$$\begin{aligned}\mathfrak{L}\left[\frac{d^2}{dt^2}f(t)\right] &= \mathfrak{L}\left[\frac{d}{dt}g(t)\right] \\ &= s\mathfrak{L}\left[g(t)\right] - g(0) \\ &= s\mathfrak{L}\left[\frac{d}{dt}f(t)\right] - \dot{f}(0) \\ &= s^2F(s) - sf(0) - \dot{f}(0)\end{aligned} \tag{34}$$

Similarly, for the nth derivative of $f(t)$, we obtain

$$\mathfrak{L}\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - sf^{(n-1)}(0) \tag{35}$$

where $f(0)$, $\dot{f}(0)$, $\cdots$, $f^{(n-1)}(0)$ represent the values of $f(t)$, $df(t)/dt$, $\cdots$, $d^{n-1}f(t)/dt^{n-1}$, respectively, evaluated at $t = 0$.

4.2 Final-Value Theorem

The final-value theorem relates the steady-state behavior of $f(t)$ to the behavior of $sF(s)$ in the neighborhood of $s = 0$. This theorem, however, applies if and only if $\lim_{t\to\infty} f(t)$ exists, which means that $f(t)$ settles down to a definite value for $t \to \infty$.

The final-value theorem may be stated as follows. If $f(t)$ and $df(t)/dt$ are Laplace transformable, if $F(s)$ is the Laplace transform of $f(t)$, and if $\lim_{t\to\infty} f(t)$ exists, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \tag{36}$$

To prove the theorem, we let s approach zero in the equation for the Laplace transform of the derivative of $f(t)$ or

$$\lim_{s\to 0}\int_0^\infty \left[\frac{d}{dt}f(t)\right] e^{-st}dt = \lim_{s\to 0}\left[sF(s) - f(0)\right] \tag{37}$$

Since $\lim_{s\to 0} e^{-st} = 1$, we obtain

$$\int_0^\infty \left[\frac{d}{dt}f(t)\right] dt = f(t)|_0^\infty = f(\infty) - f(0) = \lim_{s\to 0} sF(s) - f(0) \tag{38}$$

from which

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \tag{39}$$

The final-value theorem states that the steady-state behavior of $f(t)$ is the same as the behavior of $sF(s)$ in the neighborhood of $s = 0$. Thus, it is possible to obtain the value of $f(t)$ at $t = \infty$ directly from $F(s)$.

5 Initial-Value Theorem

The initial-value theorem is the counterpart of the final-value theorem. By using this theorem, we are able to find the value of $f(t)$ at $t = 0+$ directly from the Laplace transform of $f(t)$. The initial-value theorem does not give the value of $f(t)$ at exactly $t = 0$ but at a time slightly greater than zero.

The initial-value theorem may be stated as follows: If $f(t)$ and $df(t)/dt$ are both Laplace transformable and if $\lim_{s\to\infty} sF(s)$ exists, then

$$f(0+) = \lim_{s\to\infty} sF(s) \tag{40}$$

To prove this theorem, we use the equation for the $\mathfrak{L}_+$, transform of $df(t)/dt$:

$$\mathfrak{L}_+\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0+) \tag{41}$$

For the time interval $0+ \le t \le \infty$, as s approaches infinity, e^{-st} approaches zero. Note that we must use $\mathfrak{L}+$, rather than $\mathfrak{L}-$ for this condition. So

$$\lim_{s\to\infty}\int_0^\infty \left[\frac{d}{dt}f(t)\right] e^{-st}dt = \lim_{s\to\infty}\left[sF(s) - f(0+)\right] = 0 \tag{42}$$

or

$$f(0+) = \lim_{s\to\infty} sF(s) \tag{43}$$

In applying the initial-value theorem, we are not limited as to the locations of the poles of $sF(s)$. Thus the initial-value theorem is valid for the sinusoidal function. It should be noted that the initial-value theorem and the final-value theorem provide a convenient check on the solution, since they enable us to predict the system behavior in the time domain without actually transforming functions in s back to time functions.

5.1 Real-Integration Theorem

If $f(t)$ is of exponential order and $f(0-) = f(0+) = f(0)$, then the Laplace transform of $\int f(t)dt$ exists and is given by

$$\mathfrak{L}\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s} \tag{44}$$

where $F(s) = \mathfrak{L}[f(t)]$ and $f^{-1}(0) = \int f(t)dt$ evaluated at $t = 0$. The real-integration theorem can be proved in the following way. Integration by parts yields

$$\begin{aligned}\mathfrak{L}\left[\int f(t)dt\right] &= \int_0^\infty \left[\int f(t)dt\right] e^{-st}dt \\ &= \left[\int f(t)dt\right] \frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty f(t)\frac{e^{-st}}{-s}dt \\ &= \frac{1}{s}\int f(t)dt\bigg|_0 + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt \\ &= \frac{f^{(-1)}(0)}{s} + \frac{F(s)}{s}\end{aligned} \tag{45}$$

We see that integration in the time domain is converted into division in the s domain. If the initial value of the integral is zero, the Laplace transform of the integral of $f(t)$ is given by $F(s)/s$.

The real-integration theorem can be modified slightly to deal with the definite integral of $f(t)$. If $f(t)$ is of exponential order, the Laplace transform of the definite integral $\int_0^t f(t)dt$ is given by

$$\mathfrak{L}\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s} \tag{46}$$

where $F(s) = \mathfrak{L}[f(t)]$. This is also referred to as the real-integration theorem.

6 Inverse Laplace Transformation

6.1 Partial-Fraction Expansion when $F(s)$ Involves Distinct Poles Only

Consider $F(s)$ written in the factored form

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}, \text{ for } m < n \tag{47}$$

where p_1, p_2, $\cdots$, p_n and z_1, z_2, $\cdots$, z_m are either real or complex quantities, but for each complex p_i or z_j there will occur the complex conjugate of p_i or z_j, respectively. If $F(s)$ involves distinct poles only, then it can be expanded into a sum of simple partial fractions as follows:

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2} + \cdots + \frac{a_n}{s+p_n} \tag{48}$$

where a_k $(k = 1, 2, \cdots, n)$ are constants. The coefficient a_k is called the *residue* at the pole at $s = -p_k$. The value of a_k can be found by multiplying both sides of Equation 48 by $s + p_k$ and letting $s = -p_k$, which gives

$$\begin{aligned}\left[(s+p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k} &= \frac{a_1}{s+p_1}(s+p_k) + \cdots + \frac{a_k}{s+p_k}(s+p_k) \\ &\quad + \cdots + \frac{a_n}{s+p_n}(s+p_k) = a_k\end{aligned} \tag{49}$$

We see that all the expanded terms drop out with the exception of a_k.Thus the residue a_k is found from

$$a_k = \left[(s+p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k} \tag{50}$$

Note that, since $f(t)$ is a real function of time, if p_1 and p_2 are complex conjugates, then the residues a_1 and a_2 are also complex conjugates. Only one of the conjugates, a_1 or a_2, needs to be evaluated, because the other is known automatically.

6.2 Partial-Fraction Expansion when $F(s)$ Involves Multiple Poles

Instead of discussing the general case, we shall use an example to show how to obtain the partial fraction expansion of $F(s)$.

Consider the following F(s):

$$F(s) = \frac{s^2 + 2s + 3}{(s+1)^3} \tag{51}$$

The partial-fraction expansion of this $F(s)$ involves three terms,

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_1}{s+1} + \frac{b_2}{(s+1)^2} + \frac{b_3}{(s+1)^3} \tag{52}$$

where b_1, b_2, and b_3 are determined as follows. By multiplying both sides of this last equation by $(s+1)^3$, we have

$$(s+1)^3\frac{B(s)}{A(s)} = b_1(s+1)^2 + b_2(s+1) + b_3 \tag{53}$$

Then letting $s = -1$, Equation 53 gives

$$\left[(s+1)^3\frac{B(s)}{A(s)}\right]_{s=-1} = b_3 \tag{54}$$

Also, differentiation of both sides of Equation 53 with respect to s yields

$$\frac{d}{ds}\left[(s+1)^3\frac{B(s)}{A(s)}\right] = b_2 + 2b_1(s+1) \tag{55}$$

If we let $s = -1$ in Equation 55, then

$$\frac{d}{ds}\left[(s+1)^3\frac{B(s)}{A(s)}\right]_{s=-1} = b_2 \tag{56}$$

By differentiating both sides of Equation 55 with respect to s, the result is

$$\frac{d^2}{ds^2}\left[(s+1)^3\frac{B(s)}{A(s)}\right] = 2b_1 \tag{57}$$

From the preceding analysis it can be seen that the values of b_3, b_2, and b_1 are found systematically as follows:

$$b_3 = \left[(s+1)^3\frac{B(s)}{A(s)}\right]_{s=-1} = (s^2+2s+3) = 2 \tag{58}$$

$$b_2 = \left\{\frac{d}{ds}\left[(s+1)^3\frac{B(s)}{A(s)}\right]\right\} s = -1 = \left[\frac{d}{ds}(s^2+2s+3)\right]_{s=-1} = (2s+2) = 0 \tag{59}$$

$$b_1 = \frac{1}{2!}\left\{\frac{d^2}{ds^2}\left[(s+1)^3\frac{B(s)}{A(s)}\right]\right\}_{s=-1} = \frac{1}{2!}\left[\frac{d^2}{ds^2}(s^2+2s+3)\right]_{s=-1} = \frac{1}{2}\times 2 = 1 \tag{60}$$

We thus obtain

$$\begin{aligned} f(t) &= \mathfrak{L}^{-1}[F(s)] \\ &= \mathfrak{L}^{-1}\left[\frac{1}{s+1}\right] + \mathfrak{L}^{-1}\left[\frac{0}{(s+1)^2}\right] + \mathfrak{L}^{-1}\left[\frac{2}{(s+1)^3}\right] \\ &= e^{-t} + 0 + t^2e^{-t} \\ &= (1+t^2)e^{-t} \end{aligned} \tag{61}$$

7 Example - Inverse Laplace Transform

7.1 Degree of denominator greater than numerator

Find the inverse Laplace transform of

$$F(s) = \frac{s+3}{(s+1)(s+2)} \tag{62}$$

The partial-fraction expansion of $F(s)$ is

$$F(s) = \frac{s+3}{(s+1)(s+2)} = \frac{a_1}{s+1} + \frac{a_2}{s+2} \tag{63}$$

where a_1 and a_2 are found as

$$\begin{aligned} a_1 &= \left[(s+1)\frac{s+3}{(s+1)(s+2)}\right]_{s=-1} = \left[\frac{s+3}{s+2}\right]_{s=-1} = 2 \\ a_2 &= \left[(s+2)\frac{s+3}{(s+1)(s+2)}\right]_{s=-2} = \left[\frac{s+3}{s+1}\right]_{s=-2} = -1 \end{aligned} \tag{64}$$

Thus

$$\begin{aligned} f(t) &= \mathfrak{L}^{-1}\left[F(s)\right] \\ &= \mathfrak{L}^{-1}\left[\frac{2}{s+1}\right] - \mathfrak{L}^{-1}\left[\frac{1}{s+2}\right] \\ &= 2e^{-t} - e^{-2t} \end{aligned} \tag{65}$$

7.2 Degree of denominator less than numerator

Obtain the inverse Laplace transform of

$$G(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s+1)(s+2)} \tag{66}$$

Here, since the degree of the numerator polynomial is higher than that of the denominator polynomial, we must divide the numerator by the denominator.

$$G(s) = s + 2 + \frac{s+3}{(s+1)(s+2)} \tag{67}$$

Note that the Laplace transform of the unit-impulse function $\delta(t)$ is 1 and that the Laplace transform of $d\delta(t)/dt$ is s. The third term on the right-hand side of this last equation is $F(s)$ in the previous example. So the inverse Laplace transform of $G(s)$ is given as

$$g(t) = \frac{d}{dt}\delta(t) + 2\delta(t) + 2e^{-t} - e^{-2t} \tag{68}$$

7.3 Complex roots

Find the inverse Laplace transform of

$$F(s) = \frac{2s + 12}{s^2 + 2s + 5} \tag{69}$$

Notice that the denominator polynomial can be factored as

$$s^2 + 2s + 5 = (s + 1 + j2)(s + 1 - j2) \tag{70}$$

If the function $F(s)$ involves a pair of complex-conjugate poles, it is convenient not to expand $F(s)$ into the usual partial fractions but to expand it into the sum of a damped sine and a damped cosine function.

Noting that $s^2+2s+5=(s+1)^2+2^2$ and referring to the Laplace transforms of $e^{-\alpha t}\sin\omega t$ and $e^{-\alpha t}\cos\omega t$, rewritten thus,

$$\begin{aligned}\mathfrak{L}\left[e^{-\alpha t}\sin\omega t\right] &= \frac{\omega}{(s+\alpha)^2+\omega^2}\\ \mathfrak{L}\left[e^{-\alpha t}\cos\omega t\right] &= \frac{s+\alpha}{(s+\alpha)^2+\omega^2}\end{aligned} \tag{71}$$

the given $F(s)$ can be written as a sum of a damped sine and a damped cosine function:

$$\begin{aligned}F(s) &= \frac{2s+12}{s^2+2s+5}\\ &= \frac{10+2(s+1)}{(s+1)^2+2^2}\\ &= 5\frac{2}{(s+1)^2+2^2}+2\frac{s+1}{(s+1)^2+2^2}\end{aligned} \tag{72}$$

It follows that

$$\begin{aligned}f(t) &= \mathfrak{L}^{-1}\left[F(s)\right]\\ &= 5\mathfrak{L}^{-1}\left[\frac{2}{(s+1)^2+2^2}\right]+2\mathfrak{L}^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right]\\ &= \left(5e^{-t}\sin 2t+2e^{-t}\cos 2t\right)e^{-t}\end{aligned} \tag{73}$$