## Week 5 tutorial problems.

**Aim of this tutorial / Hands on tutorial:** By working on these problems, we get experience using Gaussian PDFs for finally using them in implementing estimators based on Kalman Filter (KF) and on Extended KF (EKF).

Some of these questions involve direct calculations; some others require implementing small programs, which you may modify and reuse in some parts of project 2.

Many of the questions, in this tutorial, are incremental, so that few complex questions are solved gradually.

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### Question 1.

Consider a Gaussian PDF, about $\mathbf{x} = [x_1 \quad x_2]^T$, whose expected value and covariance matrix are

$$\mathbf{x} \sim N(\mathbf{x}; \hat{\mathbf{x}}, \mathbf{P}_\mathbf{x}) \quad ; \quad \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{P}_\mathbf{x} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix},$$

Based on that joint PDF: what are the marginal PDFs $p_{x_1}(x_1)$ and $p_{x_2}(x_2)$? Are those PDFs Gaussian?

If so, what would be the parameters of those Gaussian PDFs?

Solution: Yes, both marginal PDFs are Gaussian, being their expressions as follows:

$$p_{x_1}(x_1) = N(x_1;1,4) \quad \text{and} \; p_{x_2}(x_2) = N(x_2;2,3)$$

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### Question 2.

Given random variable (RV) $\mathbf{x}$, which is Gaussian, and whose statistical properties are described by the joint PDF described in question 1.

1) What would be the PDF of the RV $\mathbf{w}$, which is the result of the transformation $\mathbf{w} = [2 \quad 1] \cdot \mathbf{x}$?
2) Now for the case $\mathbf{w} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \mathbf{x}$
3) Now for the case $\mathbf{w} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 3 & -1 \end{bmatrix} \cdot \mathbf{x}$

Solution:

$$\hat{\mathbf{w}} = \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \hat{\mathbf{x}}$$

2.1) $$\mathbf{P_w} = \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \mathbf{P_x} \cdot \begin{bmatrix} 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \text{(you complete this last calculation)}$$

2.2) Apply procedure used in (2.1). Resulting in an expected value, 2x1, and covariance matrix 2x2.

2.2) Apply procedure used in (2.1). Resulting in an expected value,3x1, and covariance matrix 3x3.

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**Question 3.**

For the RV $\mathbf{x}$, described in the previous problems,

1) would we be able to obtain a PDF about the RV $\mathbf{w}$, if we knew that $\mathbf{w} = 2 \cdot x_1 + (1 + 0.1 \cdot x_2)^3$

Answer: YES (approximately).

2) We assume that that PDF can be approximated by a Gaussian PDF. Based on that assumption, obtain that approximate PDF.

Solution: The expected value is $\hat{\mathbf{w}} = 2 \cdot 1 + (1 + 0.1 \cdot 2)^3$. The covariance is a scalar, obtained from:

$$\mathbf{P_w} = \mathbf{H} \cdot \mathbf{P_x} \cdot \mathbf{H}^T = \mathbf{H} \cdot \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \cdot \mathbf{H}^T$$

$$\mathbf{H} = \left[ 2 \;\vdots\; 3 \cdot (1 + 0.1 \cdot x_2)^2 \right]_{x_2 = 2} = \left[ 2 \;\vdots\; 3 \cdot (1.2)^2 \right] = \text{(you evaluate it)}$$

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**Question 4.**

Consider the random variables $x, \xi$, whose statistical descriptions are given by the following Gaussian PDFs

$$p_x(x) = N(\;x;\; 4,\; 1\;) \quad \text{and} \; p_\xi(\xi) = N(\;\xi;\; 0,\; 0.1)$$

We also know that both RVs are statistically independent.

What is the PDF for describing the RV $w$, considering that $w = x + \xi$

Solution: $$p_w(w) = N(w;\; 4,\; 1.1\;)$$

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**Question 5.**

Consider an LTI (Linear Time Invariant) system whose discrete time process model is

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.95 \end{bmatrix} \cdot \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

At time k=0 we have a belief about $\mathbf{x}(0)$, represented by the PDF

$$p_{\mathbf{x}(0)}(\mathbf{x}(0)) = N\left(\mathbf{x}(0);\ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

Assuming that the process model is perfectly accurate, what would be our belief about $\mathbf{x}(k)$ at time k=1?

Solution: $p_{\mathbf{x}(1)}(\mathbf{x}(1))$ is Gaussian, having expected value and covariance matrix as follows:

$$\hat{\mathbf{x}}(1) = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.95 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{P}_{\mathbf{x}(1)} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.95 \end{bmatrix} \cdot \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.95 \end{bmatrix}^T$$

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**Question 6.**

For the same case described in question 5, we have the extra complication that the process model is not perfectly accurate, and whose uncertainty is modelled by a RV, $\xi(k)$.

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.95 \end{bmatrix} \cdot \mathbf{x}(k) + \begin{bmatrix} 1+\xi(k) \\ 0 \end{bmatrix}$$

$\xi(k)$ is known to behave as White Gaussian noise (**WGN**), having expected value =0 and standard deviation =0.4 (expressed in proper engineering units, so you do not need to care about scaling those values);

Obtain a sequence of predictions from k=0 up to k=10, being each prediction step reported via expected value and covariance matrix. (implement a small program in MATLAB, for that purpose)

Solution: To be explained in tutorial class. Hint: "**Q matrix**"

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**Question 7.**

For the same case described in question 6, we get some extra information to help in estimating the system state. Every 2 discrete time events (discrete times k), a sensor provides measurements of the "output variable" at that time, $\mathbf{w}(k) = [1 \quad 1] \cdot \mathbf{x}(k)$. The measurements are noisy, being the polluting noise assumed to be WGN, of variance 0.2 (expressed in adequate engineering units, so that you do not need to care about scaling the involved variables).

Propose the equations needed for implementing a Kalman Filter (KF) update.

Hints: "**H matrix, R matrix**"

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**Question 8.**

The system described in question 6, has been modified, having now a scalar input $\mathbf{u}(k)$,

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.95 \end{bmatrix} \cdot \mathbf{x}(k) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot \mathbf{u}(k) + \begin{bmatrix} 1 + \xi(k) \\ 0 \end{bmatrix}$$

The input $\mathbf{u}(k)$ is measured by a sensor, whose inaccuracies are modeled as WGN of standard deviation 0.1

Implement a small program for estimating $\mathbf{x}(k)$, at each time k, from k=0 to k=10, assuming that the measurements of the input signal $\mathbf{u}(k)$,are [1,1,1,0,0,0,1,2,3,2]; for k=0 to k=9;

Solution: to be released later. Now is time to think about solving it.

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**Question 9.**

Suppose you have a situation similar to that of question 7, but in which the state is the vehicle's pose $\mathbf{x} = [x \quad y \quad \phi]^T$, and in which the process model is your kinematic model used in Project 1, having a regular sample time usually close to 10 milliseconds (i.e. "small enough").

Consider that the measurements of speed and angular rate are noisy, having noises which behave like WGN, being the standard deviation of the speed noise = 0.01m/s, and that that of the gyroscope measurements is 0.5 degree/second.

Assume that the kinematic model does not introduce any other uncertainty (except that of the inputs' measurements)

Propose, and implement the code of a function, for performing prediction steps.

Your function must have the following interface (or a similar one):

***function [ X2, Q2] = MyPrediction( X1, Q1, u1, dt, sv, sw)***

In which the input arguments are:

***[X1, Q1]*** Expected value and covariance matrix before prediction, i.e. about X(k).

***[X2,Q2]*** Expected value and covariance matrix after prediction, i.e. about X(k+1).

***u1***: measured input u(k) (of speed and angular rate, at time k)

***[sv,sw]*** Assumed standard deviations of noises affecting speed and angular rates measurements.

***dt***: time step, as in Project 1 (Assumed to be short enough, so that the time discretization error is irrelevant.)

Solution: explained in tutorial class.( Watch videos of you tutorial session, or from other sessions).

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Some of these problems may be explained during lecture time.

Questions: Ask the lecturer (j.guivant@unsw.edu.au), or the teaching staff, via Teams or Moodle.

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(End of tutorial)