MTRN4010

Advanced Autonomous Systems

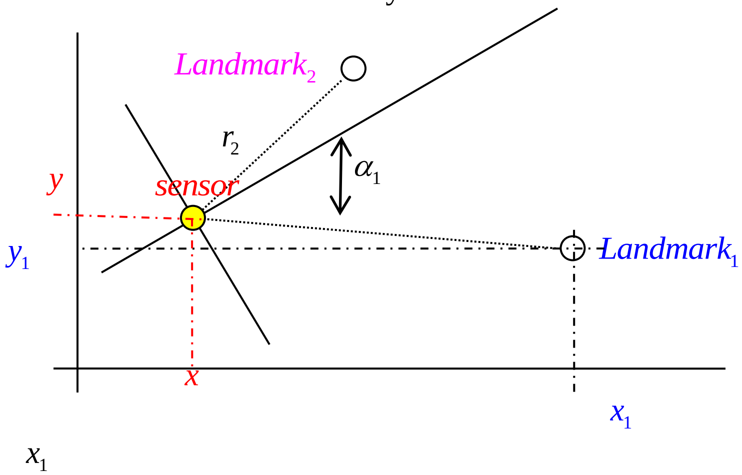
Basic Equations for Triangulation and Trilateration

This document describes the equations that are useful in many localization processes that extract information from ranges (distances) and/or angles (bearing) to landmarks whose positions are known (in a global coordinate frame), and which are detected by the on-board sensors (local CF)

Localization based on a known map

Suppose we move in a park and that we know the absolute positions of all the trees and poles in the park. Call those known objects the *Landmarks*. If we are able to measure our position respect to some of those landmarks, then we are able to estimate our position in the park. Measuring our relative position respect to some landmarks is equivalent to measuring the position of those landmarks in our local coordinate frame, i.e. our sensor's coordinate frame.
The measurements related to the relative position are usually Range and Bearing. These two variables are indicated in the next figure:

The figure shows two landmarks and a scanning sensor. The range and bearing values for each landmark is indicated by an instance of r and α .



Some sensors are able to provide both measurements, some others just range or bearing. For instance:

LiDAR:	Range and Bearing
Camera:	Bearing
Sonar:	Range
Depth Camera:	Range and Bearing

2D Trilateration based on Range measurements.

Localization can be performed through trilateration based on range measurements to a set of landmarks. Each range measurement of a known landmark introduces a constraint for the unknown variables (x,y).
If we have 2 constraints, there are two candidate solutions. If we have 3 constraints, there is a unique solution. This assumes that there is not uncertainty in the measurements and in the position of the landmarks.
Additional range measurements and proper processing allow a good localization even under the presence of noise in the measurements and errors in the map (landmarks' positions).
Three constraints, due to range measurements to three landmarks, would define a set of 3 equations for obtaining the position solution

$$\left\{ \begin{array}{l} r_1 - \sqrt{\left(x - x_1\right)^2 + \left(y - y_1\right)^2} = 0 \\ r_2 - \sqrt{\left(x - x_2\right)^2 + \left(y - y_2\right)^2} = 0 \\ r_3 - \sqrt{\left(x - x_3\right)^2 + \left(y - y_3\right)^2} = 0 \end{array} \right\}$$

When we have "redundant" constraints, there is not actual redundancy, as more equations allow to mitigate the effect of errors in the measurements of the ranges (those multiple sources of information, can be well processed by a Bayesian estimation process, as we will see later in that lecture).
Those redundant equations can still be used, to implement averaging of the solution, e.g. via a Least square approach.

In those cases, we will have a set of M equations,

$$\left\{ r_k - \sqrt{\left(x - x_k\right)^2 + \left(y - y_k\right)^2} = 0 \right\}_{k=1}^{N}$$

in which

r_k	:	range to landmark #k (measured in local or in global CF)
$\left(x_k, y_k\right)$	:	position of landmark #k (expressed in global CF)
$\left(x, y\right)$	:	position of sensor (expressed in global CF)

Trilateration does not allow to estimate heading. We can infer that from inspecting the equations: there is no involvement of the variable ϕ, the equations involve only the position variables, $\boldsymbol{x}$ and $\boldsymbol{y}$.

2D Triangulation based on Angular measurements.

Localization can be performed through triangulation to a set of landmarks and based on *bearing* measurements. In ideal situations, i.e. noise-free measurements and perfect map, three equations are enough for solving the three unknowns.

$$\begin{Bmatrix} \alpha_1 = \operatorname{atan2}\left(y_1 - \mathrm{y},\ x_1 - x \right) - \phi \\ \alpha_2 = \operatorname{atan2}\left(y_2 - \mathrm{y}, x_2 - x \right) - \phi \\ \alpha_3 = \operatorname{atan2}\left(y_3 - \mathrm{y}, x_3 - x \right) - \phi \end{Bmatrix}$$

It can be seen that the constraint introduced by a bearing observation involves three variables , i.e. *x*,*y* and heading, ϕ, of the sensor's body.

As in the localization based on range measurements, more equations can be used (i.e. more landmarks) for improving the solution under the presence of uncertainty in the measurements and map. We can show this situation as a set of constraints.

$$\left\{ \alpha_i = \operatorname{atan2}\left(y_i - y, x_i - x \right) - \phi \right\}_{i=1}^{N}$$

α_k : angle of position of landmark #1 (measured in local CF)

$\left(x_k, y_k \right)$: position of landmark #k (expressed in global CF)

$\left(x, y, \phi \right)$: pose of sensor (expressed in global CF)

2D Localization based on Range and Angular measurements.

In this case each observed landmark contributes with two constraints. Fewer landmarks are needed than in the previous cases. Two observed landmarks would provide four equations for solving 3 unknowns.

$$\left\{ \begin{matrix} r_i - \sqrt{\left(x - x_i \right)^2 + \left(y - y_i \right)^2} = 0 \\ \alpha_i = \operatorname{atan2}\left(y - y_i, x - x_i \right) - \phi \end{matrix} \right\}_{i=1}^{N}$$

Increasing the number of landmarks would allow improving the quality of the estimates in cases of uncertainty present in measurements and map knowledge.

The quality of our estimated position would depend on the number of landmarks used in the estimation, and on the geometrical distribution of those landmarks, the error in the surveyed positions of the landmarks, and on the error in the measurements (of ranges and bearing).

Alternative interpretation.

If a set of landmarks are detected in our Local CF, and if we know the positions (in a global coordinate frame) of those landmarks then we can estimate the position and heading of the sensor, in whose CF the landmarks are expressed locally.
That estimation problem is expressed as a set of equations, in which the positions of a set of points, expressed in two different coordinate frames, are related by a common transformation of coordinates. If the positions of the points in both CFs are known, then the set of equations can be used to obtain the unknown transformation, which in 2D means the 2D translation and the 1D angle of the rotation.

$$\left\{ p_k^a = R(\phi) \cdot p_k^b + T \right\}_{k=1}^{N}$$

$$R(\phi) = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

in which

p_k^b : position of object in coordinate frame CF_b (e.g. OOI in local CF)

p_k^a : position of object in coordinate frame CF_a (e.g. Landkmark, associated to OOI, expressed in global CF)

$R(\phi)$: rotation matrix associated to heading ϕ of the sensor in (in CF_a, e.g., global CF)

T : translation = position of the sensor (in CF_a)

We will use these constraints in the estimation approaches presented in MTRN4010. This document only introduces the equations.

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