Rotation

Rotational kinetic energy

Rotational inertia (rotational analogue of inertial mass)

Rolling

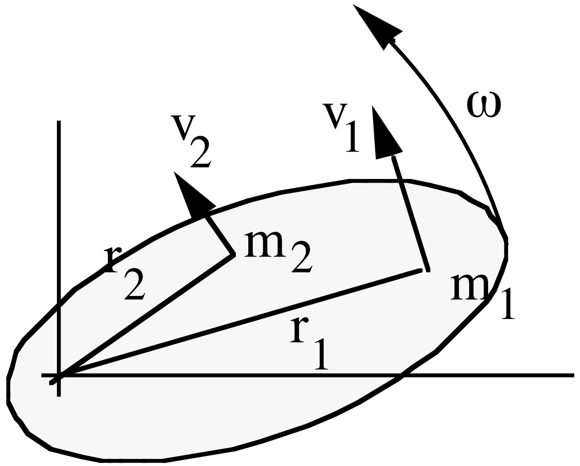
Kinematics (and its analogies with linear kinematics)

Torques and Newton's laws (more analogies with linear motion)

Angular momentum (more analogies with linear motion)

Chapter 10 in Web stream and also Physclips

Kinetic energy of a rotating body



Choose frame so that axis of rotation is at origin

$$K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \ldots$$

$$= \frac{1}{2}m_1(r_1\omega_1)^2 + \frac{1}{2}m_2(r_2\omega_2)^2 + \cdots$$

$$= \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

$K_{rot} = \frac{1}{2}I\omega^2$ *is the rotational kinetic energy*

(compare with $K_{trans} = \frac{1}{2}mv^2$*, the translational K)*

where I is the rotational analogue of mass:

Define the **Rotational inertia**

System of masses $I = \sum_i m_i r_i^2$

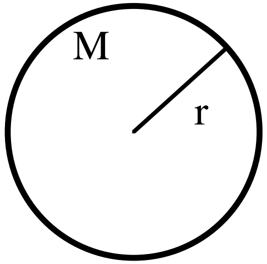
Continuous body $I = \int_{body} r^2 dm$

Units are kg.m^2

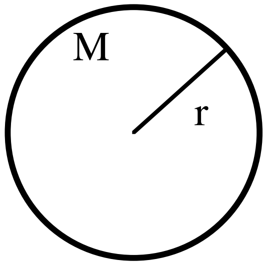
I depends on

- total mass,
- distribution of mass,
- shape and
- ***axis of rotation.***

How to integrate over dm?



Example What is I for a hoop about its axis?

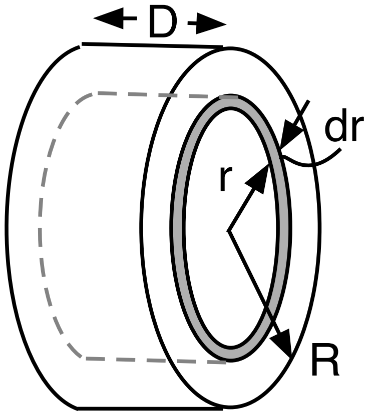


Example What is I for a hoop about its axis?

For a hoop, all the mass is at radius r, so

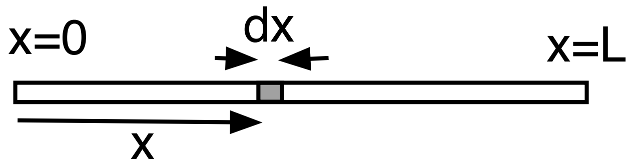
$$I = Mr^2$$

For a disc: $M = \rho V = \rho \pi R^2 D$



$dM =$

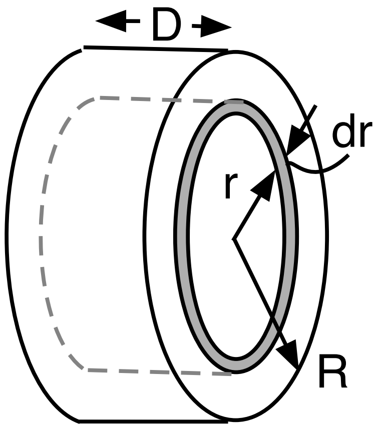
For a thin rod about one end:



mass M

$dM =$

Type equation here.



For a disc: $M = \rho V = \rho\pi R^2 D$

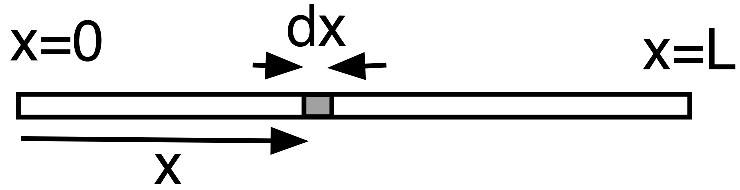
$dM = \rho dV = \rho D 2\pi r\, dr$

$I = \int_{body} r^2 dM = \int_{r=0}^{R} \rho D 2\pi r^3 dr$

$= 2\pi D\rho \int_{r=0}^{R} r^3 dr$

$= 2\pi D\rho \left[\frac{r^4}{4}\right]_{r=0}^{R} = \frac{1}{2}\rho(D\pi R^2)R^2$

$= \frac{1}{2}MR^2$



For a thin rod about one end:

$dM = M\frac{dx}{L}$

$I = \int_{body} r^2 dM = \int_{x=0}^{L} x^2 (M\frac{dx}{L})$

$= \frac{M}{L}\int_{x=0}^{L} x^2 dx$

$= \frac{M}{L}\left[\frac{x^3}{3}\right]_{x=0}^{L} = \frac{1}{3}ML^2$

For **a sphere** $I = \int_{body} r^2 dm = \cdots = \frac{2}{5}MR^2$

(longer derivation)

In general:

$$I = nMR^2 \qquad n \text{ is a number}$$

$$= M\left(\sqrt{n}R\right)^2 = Mk^2 \qquad \text{where } k = \sqrt{n}R$$

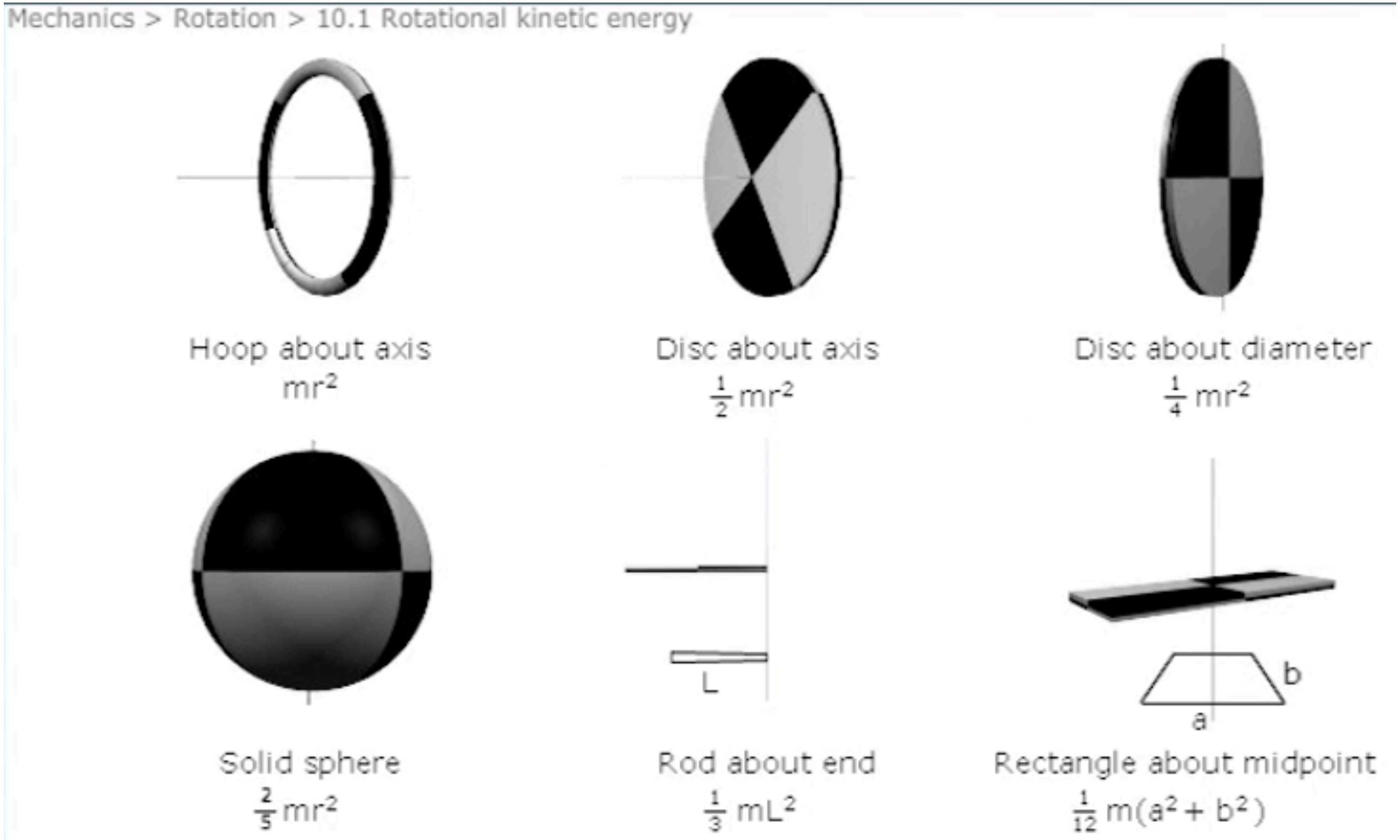
$I \equiv Mk^2$ defines the **radius of gyration** k

object	I	k
hoop	MR^2	R
disc	$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
solid sphere	$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$
rod about end	$\frac{1}{3}ML^2$	$\sqrt{\frac{1}{3}}L$

Interpretation:

i) k is the radius of a hoop with the same I as the object in question, or:

ii) if all the mass were k from the axis, it would have the same I.



In an exam, you may be asked to derive these, or they may be given.

Example. A car flywheel has a mass $m = 2$ kg, a radius $r = 18$ cm, and we'll approximate it as a uniform disc. What is its kinetic energy at 2000 r.p.m?

Caution: r.p.m = revolutions per minute

$$I = \frac{1}{2}mr^2 = 0.03\ kg.m^2$$

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2 = \frac{1}{2}\left(\frac{1}{2}(2\ kg)(0.18\ m)^2\right)\left(\frac{2000*2\pi}{60\ s}\right)^2 = 700\text{ J}$$

A baton (thin rod) has mass $m = 0.40$ kg and length 1.5 m. A bandmaster throws it at 3.0 m/s with a rotation rate of 1.0 revolution per second. What are its translational and rotational kinetic energies?

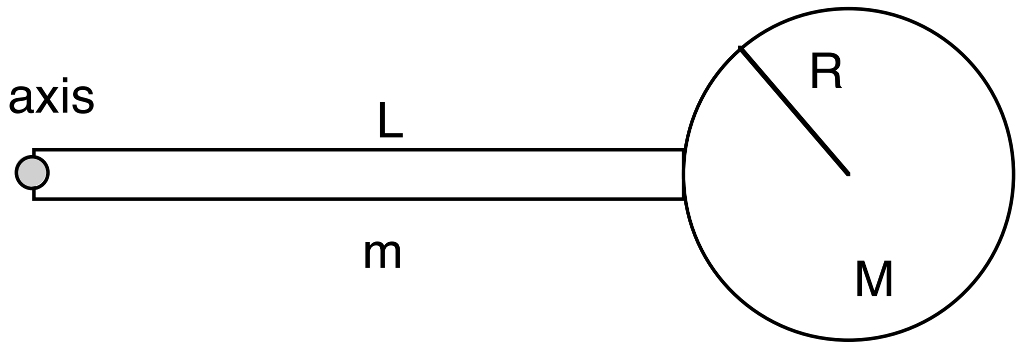
For rod about its centre, $I = \frac{1}{12}ML^2 = 0.19\text{ kgm}^2$.

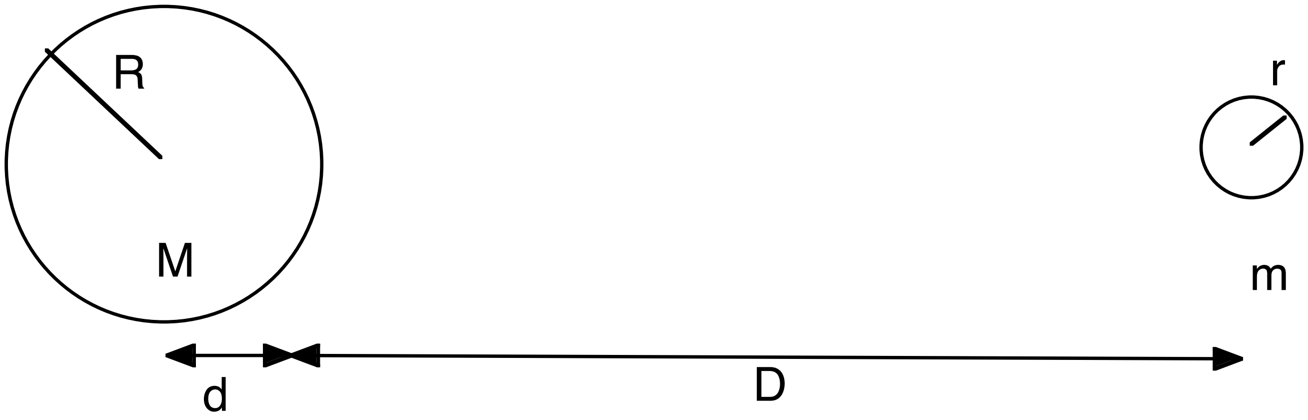
$$K_{trans} = \frac{1}{2}mv^2 = \frac{1}{2}(0.40\text{ kg})(2.0\text{ m/s})^2 = 1.8\text{ J.}$$

$$K_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{12}mL^2\right)\omega^2 = \frac{1}{2}\left(\frac{1}{12}(0.40\ kg)(1.5\ m)^2\right)\left(\frac{2\pi}{1\ s}\right)^2 = 1.5\text{ J}$$

Question What is the kinetic energy of the earth-moon system in its centre of mass frame, in terms of $M_{\text{earth}}, I_{\text{earth}}, M_{\text{moon}}, I_{\text{moon}}$, their distances from their common centre of mass and $\omega_{earth} = \frac{2\pi}{23.9 \text{ hours}}$ and $\omega_{moon} = \frac{2\pi}{27.3 \text{ days}}$?

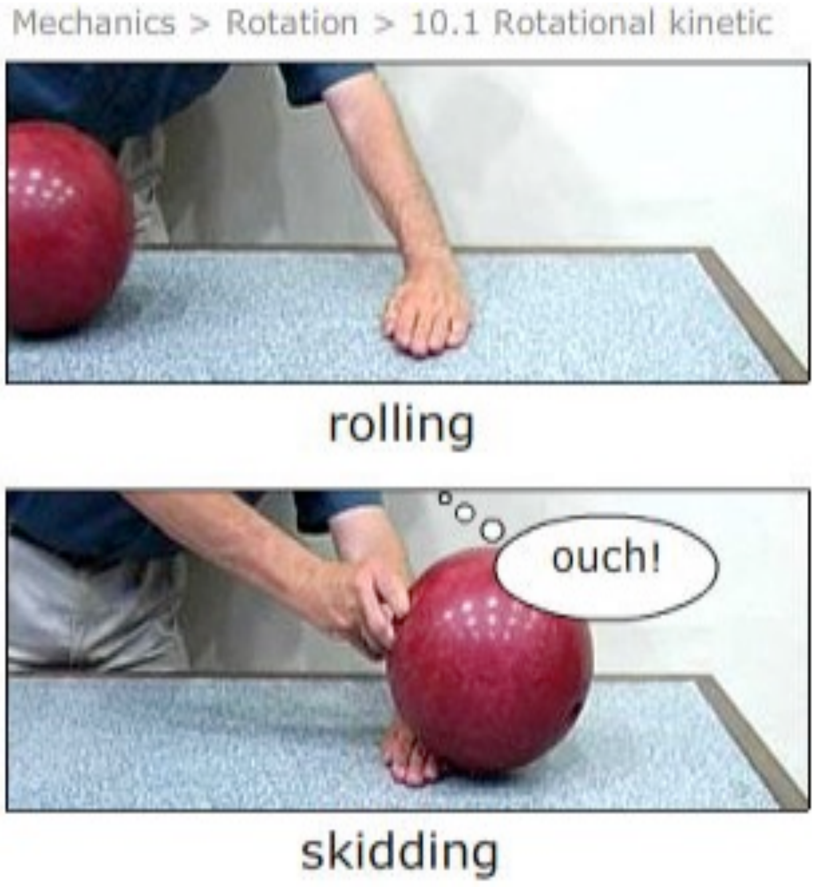
And for this (rigid) system of rod and sphere, rotating with angular frequency ω:





$$K_{total} = K_{trans,earth} + K_{rot,earth} + K_{trans,moon} + K_{rot,moon}$$

$$= \frac{1}{2}M_{earth}(\omega_{moon}d)^2 + \frac{1}{2}I_{earth}\omega_{earth}{}^2 + \frac{1}{2}M_{moon}(\omega_{moon}D)^2 + \frac{1}{2}I_{moon}\omega_{moon}{}^2$$



Question A wheel or ball with radius r rolls one complete turn in period T.

How far does it travel in one complete turn?

What is its speed v?

What is its angular speed ω?

Relate v, r and ω

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Relate v, r and ω

In period T, wheel rotates 2π, so $\omega = 2\pi/T$.

vehicle goes forwards $2\pi r$ in T, so $v = 2\pi r/T = r\omega$. So

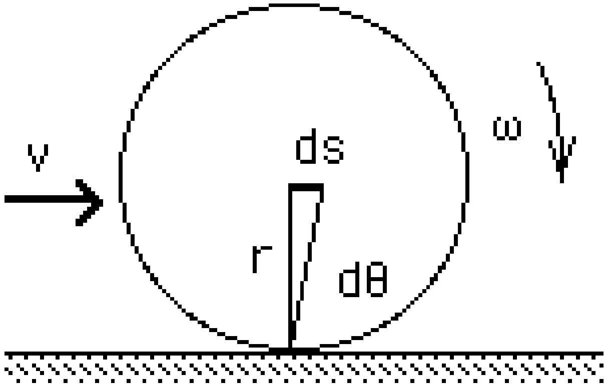
$$\omega = \frac{v}{r} \qquad \text{or} \qquad v = r\omega$$

Example A bicycle wheel has $r = 40$ cm. What is its angular velocity when the bicycle travels at 40 km.hr^{-1}?

Example A bicycle wheel has $r = 40$ cm. What is its angular velocity when the bicycle travels at 40 km.hr^{-1}?

$$\omega = \frac{v}{r} = \frac{40000 \text{ m}/3600 \text{ s}}{0.4 \text{ m}} = 28 \text{ rad.s}^{-1}$$ (= 4.4 turns/second = 270 r.p.m.)

Could also do it this way:

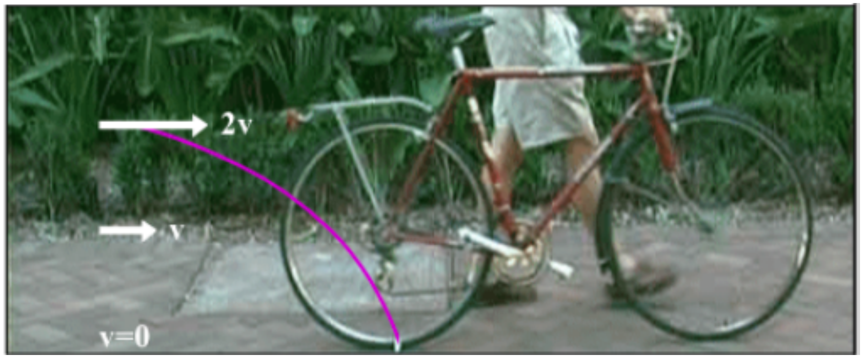


$$v = \frac{ds}{dt}$$

$$= \frac{rd\theta}{dt}$$

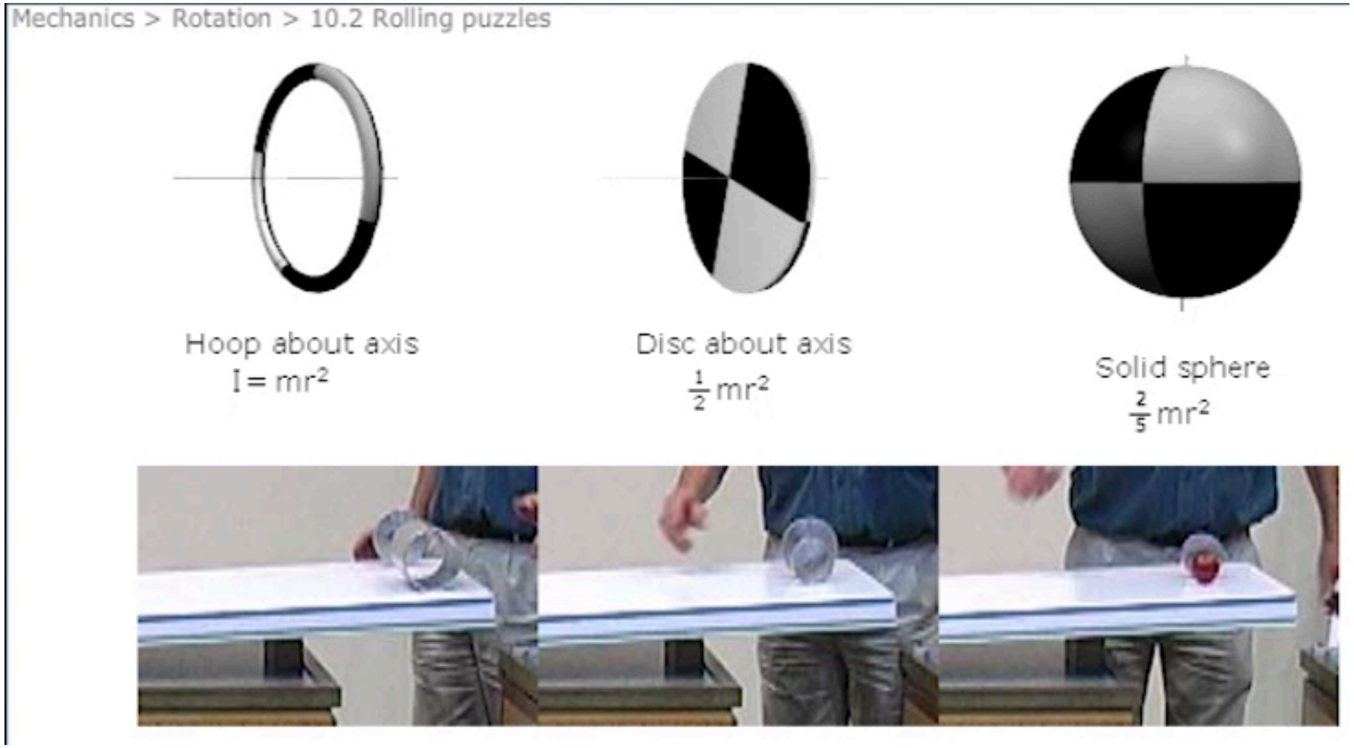
$$v = r\omega$$

- Point of contact stationary
- Axle travels at v
- Top of wheel travels $2v$, overtaking the bike at relative speed v and tracing a cycloid path in the frame of the ground.



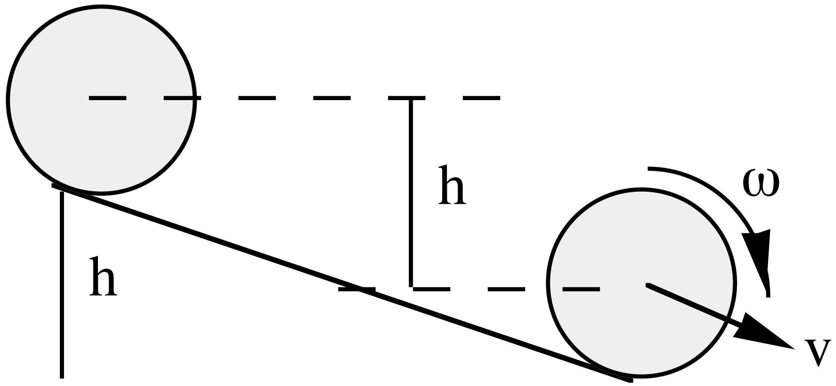
(see *Rolling* on *Physclips*)

Example. A hoop, a disc and a solid sphere roll down an inclined plane. Which travels fastest?



Does friction act?

Example. A solid sphere, a disc and a hoop roll down an inclined plane. Which travels fastest?



Rolling: point of application of friction stationary $\therefore$ non-conservative forces do no work $\therefore$

$$U_f + K_f = U_i + K_i$$

$$0 + \left(\frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2\right) = Mgh + 0$$

$\omega = \dfrac{v}{R}$ and write $I = Mk^2$ remember: $k = \sqrt{I/M}$ is the radius of gyration.

$$\frac{1}{2} Mv^2 + \frac{1}{2} Mk^2 \frac{v^2}{R^2} = Mgh$$

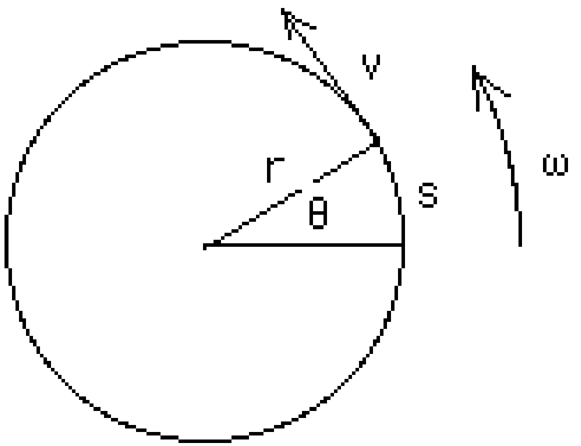
$$\frac{1}{2} v^2\left(1 + k^2/R^2\right) = gh$$

$$v = \sqrt{\frac{2gh}{1 + k^2/R^2}}$$

look at the size of the denominator:

$\dfrac{k_{sphere}}{R} = \sqrt{\dfrac{2}{5}} < \dfrac{k_{disc}}{R} = \sqrt{\dfrac{1}{2}} < \dfrac{k_{hoop}}{R} = 1$ $\quad\therefore\quad v_{\text{sphere}} > v_{\text{disc}} > v_{\text{hoop}}$ *independent of size*

Rotational kinematics:



r is constant

If θ measured in radians,

$$s \;=\; r\theta. \qquad\qquad (\textit{definition of angle})$$

$$\therefore \qquad v \;=\; \frac{ds}{dt} \;=\; r\frac{d\theta}{dt} \;\equiv\; r\omega$$

$$v \;=\; r\omega \qquad\qquad \left(or\ \omega \;=\; \frac{v}{r}\right)$$

compare with rolling

$$\therefore \qquad a \;=\; \frac{dv}{dt} \;=\; r\frac{d\omega}{dt} \;\equiv\; r\alpha \qquad\qquad where\ \alpha = \frac{d\omega}{dt} \qquad\qquad is\ the\ angular\ acceleration$$

$$a \;=\; r\alpha \qquad\qquad \left(or\ \alpha \;=\; \frac{a}{r}\right)$$

Motion with constant $\underline{\alpha}$.

Analogies	linear	angular
displacement	x	$\theta = s/r$
velocity	v	$\omega = v/r$
acceleration	a	$\alpha = a/r$

Kinematic equations

$$v_f = v_i + at \qquad \omega_f = \omega_i + \alpha t$$

$$\Delta x = v_i t + \frac{1}{2}\, at^2 \qquad \Delta\theta = \omega_i t + \frac{1}{2}\, \alpha t^2$$

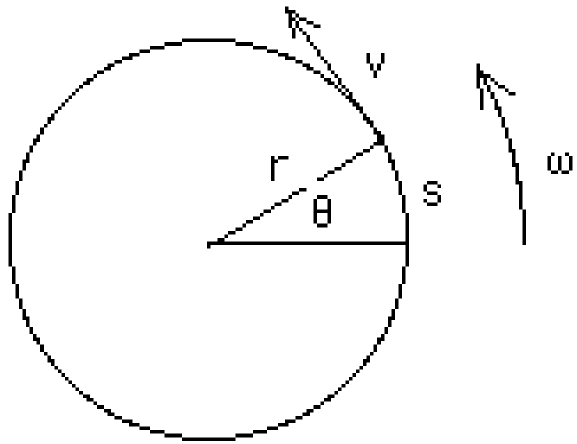
$$v_f^2 = v_i^2 + 2a\Delta x \qquad \omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$$

$$\Delta x = \frac{1}{2}(v_i + v_f)\, t \qquad \Delta\theta = \frac{1}{2}(\omega_i + \omega_f)\, t$$

Derivations identical - see previous

Need only remember one version

Example. Centrifuge, initially spinning at 5000 rpm, slows uniformly to rest over 30 s. (i) What is its angular acceleration? (ii) How far does it turn while slowing down? (iii) How far does it turn during the first second of deceleration? (rpm = revolutions per minute)



i) $\omega_f = \omega_i + \alpha t$ (*cf* $v_f = v_i + at$)

$$\alpha = \frac{\omega_f - \omega_i}{t}$$

$$= \frac{0 - \dfrac{5000*2\pi\ \text{rad}}{60\text{s}}}{30\text{s}}$$

$= -17\ \text{rad.s}^{-2}$ to 2 sig figs.

ii) $\Delta\theta = \frac{1}{2}(\omega_i + \omega_f)\, t$ (*cf* $\Delta x = \frac{1}{2}(v_i + v_f)\, t$)

$= \frac{1}{2}(0 + 5000\text{rpm}) *0.5\ \text{min}$

$= 1{,}300$ revolutions

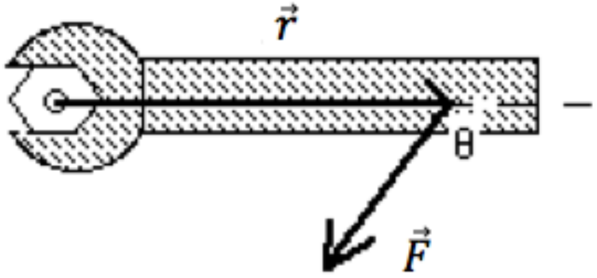
iii) $\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$ (*cf* $\Delta x = v_i t + \frac{1}{2}at^2$)

$$= \frac{5000*2\pi\ \text{rad}}{60\ \text{s}}(1\ \text{s}) - \frac{1}{2}(17.5\ \text{rad.s}^{-2}).(1\ \text{s})^2$$

$= 510$ rad (= 82 turns) to 2 sig figs

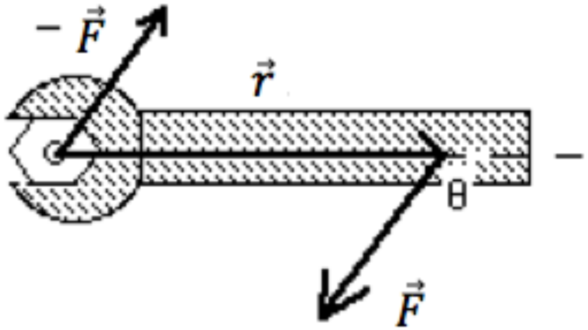
What causes angular acceleration?

Force $\vec{F}$ applied at point displaced $\vec{r}$ from axis of rotation.



(Note: if $\vec{F}$ were the only force $\Rightarrow$ acceleration:

To get an angular acceleration α without a linear acceleration $\vec{a}$ we need two forces that add to zero.

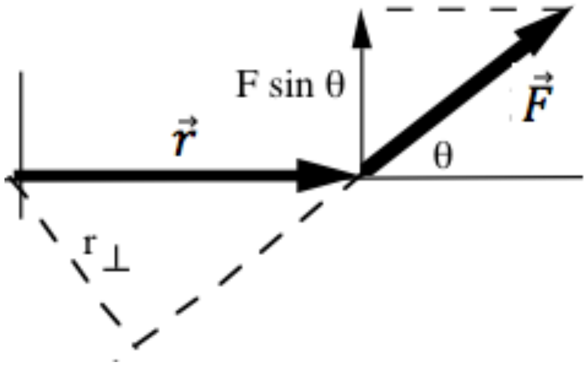


$-\vec{F}$ *does not contribute to the turning about axis*.

How does the 'turning tendency' depend on F? r? θ?

Torque. *(rotational analogue of force)*

Consider rotation about z axis

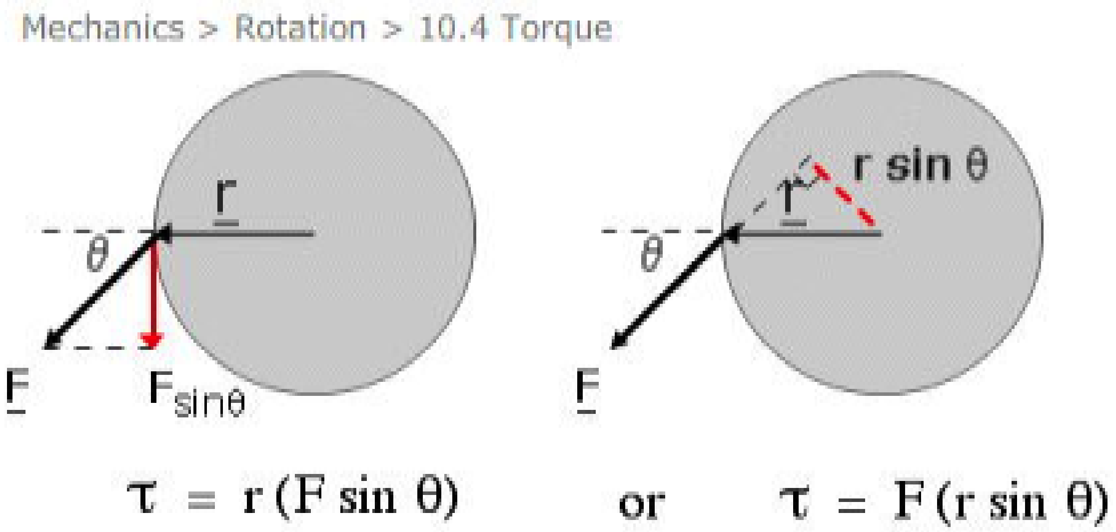


Only the component $F \sin \theta$ tends to turn, so the **torque** τ is

$\tau = r\,(F \sin \theta)$ (r * *perpendicular component of* F)

or $= F\,(r \sin \theta) = F\,r_\perp$ (F * *perpendicular component of* r)

where $r_\perp$ is called the moment arm



Example What is the maximum torque I apply by standing on a wheel spanner 300 mm long?

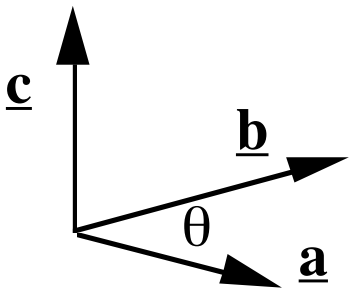
$\tau = r\,(F \sin \theta)$

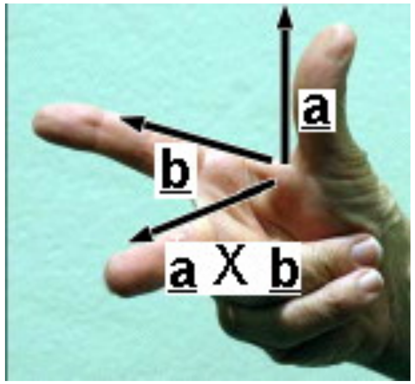
max $\tau \;=\; r\,F \sin 90°$

$= \;0.3\text{ m} * 700\text{ N} \;=\; 200\text{ Nm}$

if it still doesn't move: lift, use both hands to pull ***up*** *or jump on it*

The vector product (or cross product). We define it so that we can write: $\vec{\tau} = \vec{r} \times \vec{F}$





Define $|\vec{a} \times \vec{b}| = ab \sin\theta$

$\vec{a} \times \vec{b}$ is at right angles to both $\vec{a}$ and $\vec{b}$ in the right-hand sense

pronounced "a cross b"

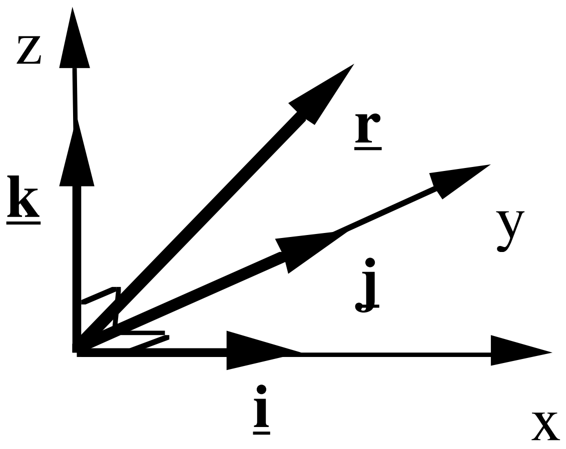
For **right hand**

$\overrightarrow{thumb} \times \overrightarrow{index} = \overrightarrow{middle}$ (*remember TIM*)

(or $\overrightarrow{North} \times \overrightarrow{East} = \overrightarrow{down}$ *remember NED*)

Turn screwdriver from $\vec{a}$ to $\vec{b}$ and a right-handed screw moves in the direction of $\vec{a} \times \vec{b}$

$\vec{\tau} = \vec{r} \times \vec{F}$ Note that direction of the torque is often in the direction of the axis of rotation



$\left|\vec{a}\times\vec{b}\right| = ab\sin\theta$ Apply to unit vectors:

$|i\times i| = 1 \,.\, 1 \sin 0° = 0 = j\times j = k\times k$

$|\,i\times j\,| = 1 \,.\, 1 \sin 90° = 1 = |\,j\times k\,| = |\,k\times i\,|$

$i\times j = k \qquad j\times k = i \quad k\times i = j$

but $j\times i = -k \qquad k\times j = -i \qquad i\times k = -j$

Try these:

$North\times East = down \qquad East\times down = North \qquad down\times North = East$

but

$East\times North = up \qquad etc$

Usually we'll evaluate by $|\vec{a}\times\vec{b}| \equiv ab \sin\theta$

but **Vector product by components** is neat

$$\vec{a}\times\vec{b} = (a_x\,\boldsymbol{i} + a_y\,\boldsymbol{j} + a_z\,\boldsymbol{k}) \times (b_x\,\boldsymbol{i} + b_y\,\boldsymbol{j} + b_z\,\boldsymbol{k})$$

*3 terms * 3 terms -> 9 terms:*

$$= (a_x b_x)\, i\times i + (a_y b_y)\, j\times j + (a_z b_z)\, k\times k$$

$$+ \; (a_x b_y)\, i\times j + (a_y b_z)\, j\times k + (a_z b_x)\, k\times i$$

$$+ \; (a_y b_x)\, j\times i + (a_z b_y)\, k\times j + (a_x b_z)\, i\times k$$

$$\vec{a}\times\vec{b} = (a_x b_y - a_y b_x)k + (a_y b_z - a_z b_y)i + (a_z b_x - a_x b_z)j$$

$$\begin{array}{cccc} a_x & a_y & a_z & a_x \\ b_x & b_y & b_z & b_x \\ \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} & \underline{\mathbf{i}} \end{array}$$

Example.

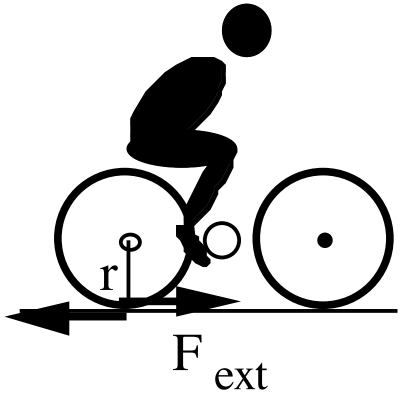
$\vec{F} = (3\boldsymbol{i} + 5j)\text{N}$, $\vec{r} = (4j + 6k)\text{m}$; $\vec{\tau} = ?$

$\vec{\tau} = \vec{r}\times\vec{F}$

$= (r_x F_y - r_y F_x)k + (r_y F_z - r_z F_y)i + (r_z F_x - r_x F_z)j$

$= (0 - 4\text{ m}.3\text{ N})k + (0 - 6\text{ m}.5\text{ N})i + (6\text{ m}.3\text{ N} - 0)j$

$= -30i + 18j - 12k$ Nm



Example: A bicycle and rider (m = 80 kg) accelerate at 2.0 ms^{-2}. Wheel with r = 40 cm (2 sig figs). What is the torque at the wheel?

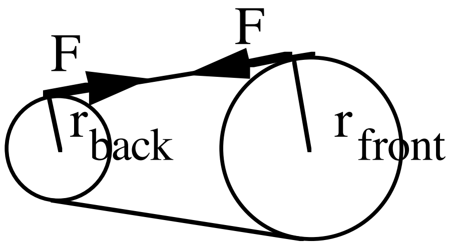
$$F_{ext} = ma$$

$$\tau = rF_{ext} \sin\theta = rF$$

$$= rma = \ldots = 64 \text{ Nm.}$$

What is the direction of the torque?

Front sprocket has 50 teeth, rear has 25, what is the torque applied by the legs?



bottom chain is slack

$$F_{front} = F_{back} \qquad \frac{r_{front}}{r_{back}} = \frac{50}{25}$$

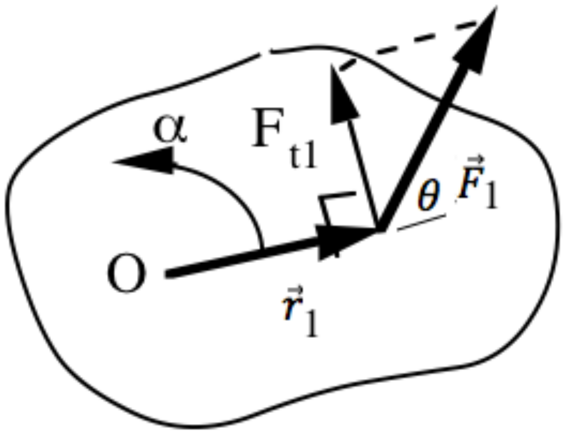
$$\frac{\tau_{front}}{\tau_{back}} = \frac{r_{front}F_{front}}{r_{back}F_{back}} = 2.$$

$$\tau_{front} = 130 \text{ Nm} \quad \text{horizontal}$$

why larger?

Newton's law for rotation

Rigid system of particles, m_i, all rotating with same ω and α about same axis. r_i is the distance of each m_i from the axis of rotation.



$\vec{\tau} = \vec{r} \times \vec{F}$ OR just use $\tau_i = r_i F_i \sin\theta$

$\tau_i = r_i F_{ti}$ where F_{ti} is the tangential component of F_i

and use Newton's 2nd and a_i is tangential

$\tau_i = r_i m_i a_i$ but $a_i = r_i \alpha_i$

$= r_i m_i (r_i \alpha_i)$ and reorganise:

$= (m_i r_i^2)\alpha_i$ and sum over the body:

$\sum_{body} \tau_i = \sum_{body} (m_i r_i^2)\alpha_i$ but if it is rigid, all $\alpha_i = \alpha$

so $\tau_{total} = I\alpha$ and τ and α are on the axis

Newton's law for rotation

$\tau_{total} = I\alpha$ *cf* $F_{total} = ma$

Example. What constant torque would be required to stop the earth's rotation in one revolution? (Assume earth uniform (although it isn't, of course).)

Know: $M, R, \omega_i, \omega_f, \Delta\theta$. *Need* τ. *How can we do this?*

Example. What constant torque would be required to stop the earth's rotation in one revolution? (Assume earth uniform (although it isn't, of course).)

Know: $M, R, \omega_i, \omega_f, \Delta\theta$. *Need* τ.

Use $\tau = I\alpha$, *where* $\omega_i, \omega_f, \Delta\theta \rightarrow \alpha$

$$\omega_f = 0, \; \omega_i = \frac{2\pi}{23\text{h}56\text{min}} = 7.3\ 10^{-5}\ \text{rad.s}^{-1}$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta \qquad \text{(cf } v_f^2 = v_i^2 + 2a\Delta x\text{)}$$

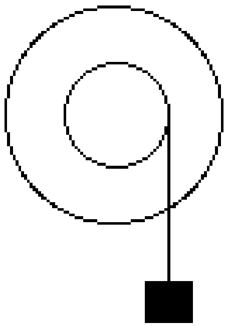
$$\alpha = \frac{\omega_f^2 - \omega_i^2}{2\Delta\theta}$$

$$\tau = I\alpha = \frac{2}{5} MR^2 \frac{\omega_f^2 - \omega_i^2}{2\Delta\theta}$$

$$= \ldots$$

$$= 4 \times 10^{28}\ \text{Nm}$$

overestimate. Density higher near centre



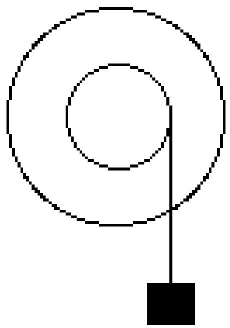
Classic example Mass m on string on drum radius r on a wheel with radius of gyration k and mass M. How long does it take to turn 10 turns?

solve for a or α, use kinematic equations.

N2 for m:

N2 for wheel:

Which kinematic equation?



Example Mass m on string on drum radius r on a wheel with radius of gyration k and mass M. How long does it take to turn 10 turns?

solve for a or α, use kinematic equations.

Let's take a downwards and α clockwise

N2 for m (vertical): $mg - T = ma$ (1)

N2 for wheel: $\tau = I\alpha$ (2)

(2) –> $rT = Mk^2 . \frac{a}{r}$

$$T = Ma\left(\frac{k}{r}\right)^2$$

substitute in (1) $mg - Ma\left(\frac{k}{r}\right)^2 = ma$ rearrange to have

$$a = \frac{mg}{m + M\left(\frac{k}{r}\right)^2}$$

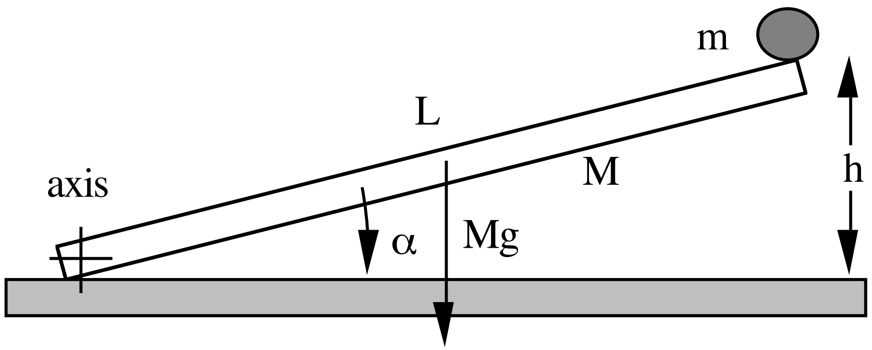
then we need kinematics:

$$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$$

cf $\Delta x = v_i t + \frac{1}{2} a t^2$

$\omega_i = 0$. $\Delta\theta = 20$ rad. Rearrange to have $t = \sqrt{\frac{2\Delta\theta}{\alpha}} = \sqrt{\frac{2(20\pi \text{ rad})\left(1+\frac{M}{m}\left(\frac{k}{r}\right)^2\right)r}{g}}$

Example. Rod rotates about one end. Which reaches bottom first: *m* or the end of the rod?



Acceleration of ***end*** of rod is

$$a = L\alpha$$

For rod, $\tau = I\alpha$

Combine -> $a = L.\dfrac{\tau}{I}$

For rod about an end, $I = \frac{1}{3}ML^2$

Mg acts at c.m. so $\tau = Mg\dfrac{L}{2}$ when it is close to horizontal

$$a = L.\frac{Mg.\frac{L}{2}}{\frac{1}{3}ML^2}$$

$$= \frac{3}{2}\ g$$

Falling chimneys break. Falling trees sometimes break in fall

Example A car is doing work at a rate of 20 kW and travelling at 100 kph. Wheels have r = 30 cm. What is the (total) torque applied by the drive wheels?

$$P = Fv, \textit{ so by analogy}: \quad P = \tau\omega$$

Wheels are rolling so $\omega = \dfrac{v}{r}$

$$\therefore \quad \tau = \frac{P}{\omega} = \frac{Pr}{v}$$

$$= \frac{2\ 10^4 \text{ W } 0.3 \text{ m}}{10^5 \text{ m}/3600 \text{ s}}$$

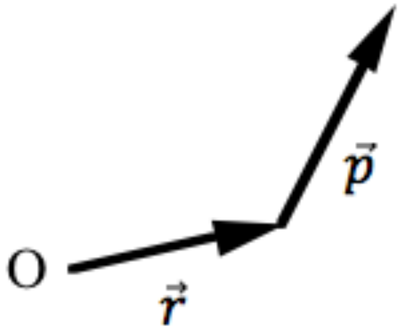
$$= 220 \text{ Nm}$$

not a big torque, but big power at that ω

(not equal to torque on tail shaft or at flywheel)

Total force ?

Angular momentum



For a *particle* of mass m at position $\vec{r}$ and with momentum $\vec{p}$ relative to origin O of an inertial reference frame, we define angular momentum (w.r.t.) O

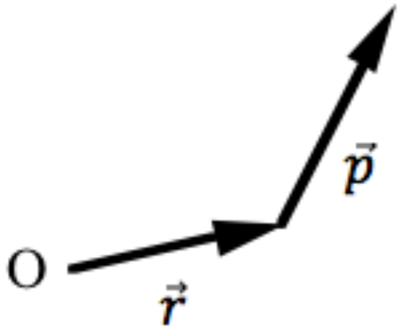
$$\vec{L} = \vec{r} \times \vec{p}$$

or $L = rp \sin \theta$

Example What is the direction of the angular momentum of the moon about the earth?

$L = |\vec{r} \times \vec{p}|$ Direction = ?

Angular momentum



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$$\vec{L} = \vec{r} \times \vec{p}$$

or $L = rp \sin \theta$

Example What is the angular momentum of the moon about the earth?

$$L = |\vec{r} \times \vec{p}| \quad = \text{rp} \sin \theta$$

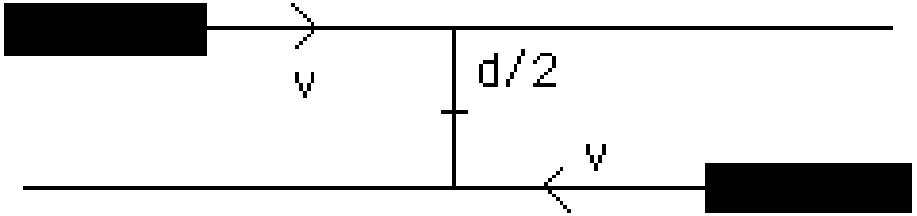
$$\cong rmv \sin 90°$$

$$= mr^2\omega$$

$$= (7.4\ 10^{22}\ \text{kg})\ (3.8\ 10^{8}\ \text{m})^2\ \frac{2\pi}{27.3\ 24\ 3600\ \text{s}}$$

$$= 2.8\ 10^{34}\ \text{kg m}^2\ \text{s}^{-1}$$

Direction is North



Example Two trains mass m approach at same speed v, travelling antiparallel, on tracks separated by distance d. What is their total angular momentum, as a function of separation, about a point halfway between them?

$$L_1 = |\vec{r}_1 \times \vec{p}_1| = |\vec{r}_1 \times m\vec{v}_2|$$

$|L_1| = (d/2)mv$ (clockwise on my diagram)

$|L_2| = (d/2)mv$ (also clockwise)

$\Sigma L = dmv$ independent of separation along the track

Just to show that you don't need circular motion to have angular momentum

Newton 2 for angular momentum:

$$\Sigma\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d}{dt}\vec{p}$$

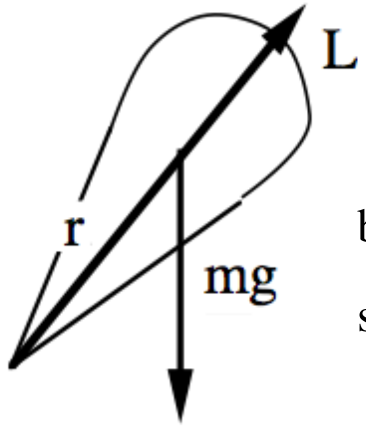
$$\frac{d}{dt}\vec{L} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$= \left(\frac{d}{dt}\vec{r}\right) \times \vec{p} + \vec{r} \times \left(\frac{d}{dt}\vec{p}\right)$$ *order is important!*

$$= \vec{v} \times m\vec{v} + \Sigma\vec{\tau}$$

$$\Sigma\vec{\tau} = \frac{d}{dt}\vec{L}$$ Newton's 2nd law for rotation *cf* $\Sigma\vec{F} = \frac{d}{dt}\vec{p}$

Question: A rapidly spinning (large *L*) top balances on a sharp point. Why doesn't it fall over? *(Qualitative treatment only.)*



$$\Sigma\vec{\tau} = \frac{d}{dt}\vec{L}$$

So $\overrightarrow{dl}$ is parallel to $\vec{\tau}$

but $\vec{r} \times m\vec{g}$ is horizontal

so $\overrightarrow{dl}$ is approx perpendicular to $\vec{g}$

Also explains the stability of boomerangs, frisbees, gyroscopes, satellites

Systems of particles

Total angular momentum $\vec{L}$

$$\vec{L} = \Sigma\, \vec{r}_i \times \vec{p}_i$$

$$\frac{d}{dt}\vec{L} = \Sigma \frac{d}{dt}\left(\vec{r}_i \times \vec{p}_i\right)$$

$$= \Sigma \vec{\tau}_i$$

$$= \Sigma \vec{\tau}_{i\ internal} + \Sigma \vec{\tau}_{i\ external}$$

Internal torques cancel in pairs (Newton 3)

$$\Sigma \vec{\tau}_{i\ external} = \frac{d}{dt}\vec{L} \qquad cf \qquad \Sigma \vec{F}_{i\ external} = \frac{d}{dt}(\Sigma \vec{p}_i)$$

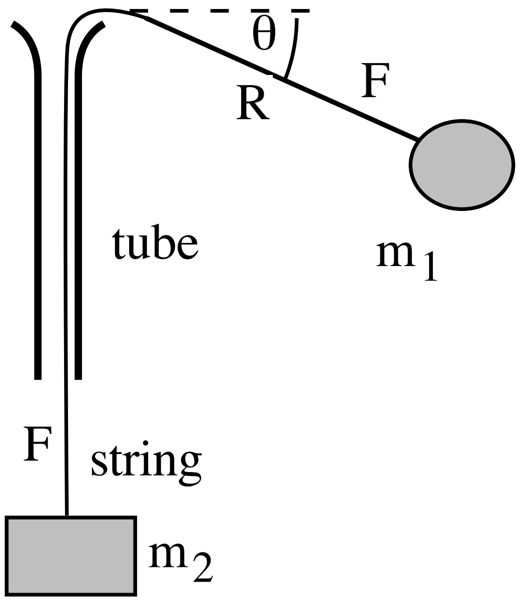
(This equation derived for ***inertial frames*** *but it is also true for other frames if centre of mass is taken as origin.)*

Consequence:

If $\Sigma \vec{\tau}_{i\ external} = 0, \qquad \frac{d}{dt}\vec{L} = 0.$

Conservation of angular momentum of isolated system

Example Circular motion of ball on string. What happens to the speed of the ball as the string is shortened? (Neglect air resistance).



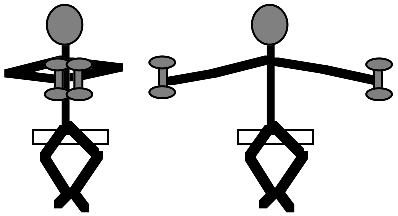
Tension *does* do work, but it doesn't exert torque because tension is parallel to $\vec{r}$.

Therefore angular momentum is conserved.

$$\vec{L} = \vec{r} \times \vec{p}$$

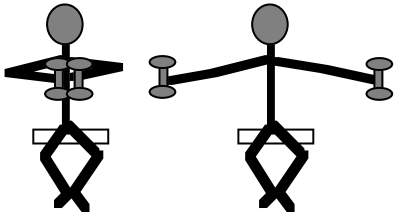
$$L = rp$$

$$= rmv$$ so if r is reduced, v should increase.



Example: Person on rotating seat holds two 2.2 kg masses at arms' length. Draws masses in to chest. What is $\Delta\omega$? Is K conserved?

Rough estimates: k_{person} about long axis ~ 15 cm



Example: Person on rotating seat holds two 2.2 kg masses at arms' length. Draws masses in to chest. What is $\Delta\omega$? Is K conserved?

Rough estimates: k_{person} about long axis ~ 15 cm

$$I_p = Mk^2 = \sim 70\text{ kg.}\,(.15\text{ m})^2$$ include moving part of chair

$$I_p \sim 1.6\text{ kgm}^2$$

$$I_m = mr^2 \cong 2.2\text{ kg.}\,(0.8\text{ m})^2$$

$$I_m \cong 1.4\text{ kgm}^2 \qquad \text{(arms extended)}$$

$$I_m' = mr'^2 \cong 2.2\text{ kg.}\,(0.2\text{ m})^2$$

$$\cong 0.1\text{ kgm}^2 \qquad \text{(arms in)}$$

No external torques $\Rightarrow$ $L_i = L_f$

$$(I_p + 2I_m)\omega_i = (I_p + 2I'_m)\omega_f$$

$$\frac{\omega_f}{\omega_i} = \frac{I_p + 2I_m}{I_p + 2I'_m} \sim 2.4$$

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}(I_p + 2I_m)\omega_f^2}{\frac{1}{2}(I_p + 2I'_m)\omega_f^2} = 2.4$$

Arms do work: $Fds = ma_{centrip}.ds$

Questions

Can a docking spacecraft rotate without using rockets?

Can a cat, initially with $L = 0$, rotate while falling so as to land on its feet?

Analogies: linear and rotational kinematics

Linear		**Angular**		
displacement	x	angular displacement	θ	$= s/r$
velocity	v	angular velocity	ω	$= v/r$
acceleration	a	angular acceleration	α	$= a/r$

kinematic equations

$v_f = v_i + at$	$\omega_f = \omega_i + \alpha t$
$\Delta x = v_i t + \frac{1}{2} at^2$	$\Delta\theta = \omega_i t + \frac{1}{2} \alpha t^2$
$v_f^2 = v_i^2 + 2a\Delta x$	$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$
$\Delta x = \frac{1}{2}(v_i + v_f)\, t$	$\Delta\theta = \frac{1}{2}(\omega_i + \omega_f)\, t$

Analogies: linear and rotational mechanics

mass m — rotational inertia I

$I = \Sigma m_i r_i^2 \qquad I = \int r^2 dm$

Work & energy

$W = \int \vec{F}.\overrightarrow{ds} \qquad W = \int \tau . d\theta$

$K = \frac{1}{2} Mv^2 \qquad K = \frac{1}{2} I\omega^2$

force $\vec{F}$ — torque $\vec{\tau} = \vec{r} \times \vec{F}$

momentum — angular momentum

$\vec{p} = m\vec{v} \qquad \vec{L} = m\vec{r} \times \vec{v}$

Newton 2:

$\vec{F} = \frac{d}{dt}\vec{p} = m\vec{a} \qquad \vec{\tau} = \frac{d}{dt}\vec{L} = I\vec{\alpha}$

if m const — if I const

Momentum $\vec{p} = m\vec{v}$

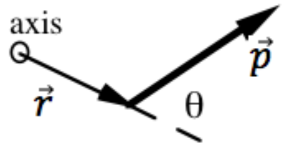
Newton 1&2 $\vec{F}_{ext} = \frac{d}{dt}\vec{p}$

Conservation law:

If no external forces act momentum conserved

If m constant, $\vec{F}_{ext} = \frac{d}{dt}\vec{p} = \frac{d}{dt}(m\vec{v}) = m\vec{a}$

Angular momentum $L \equiv (r \sin\theta)\, mv$



$\tau_{ext} = \frac{d}{dt} L$ *only consider one axis*

Newton for rotation $F_{ext}\,(r \sin\theta) = \frac{d}{dt}(p\, r \sin\theta)$

Conservation law:

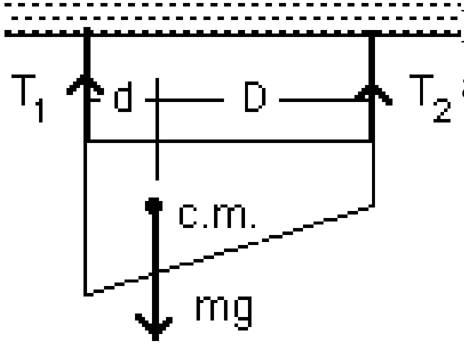
If no external *torques* act *angular* momentum is conserved

If I constant, $\tau_{ext} = \frac{d}{dt} L = \frac{d}{dt} I\omega = I\alpha$

Conservation of $\vec{p}$ (linear momentum) and $\vec{L}$ (angular momentum):

If no external $\begin{pmatrix}\text{forces}\\ \text{torques}\end{pmatrix}$ act on a system, its $\begin{pmatrix}\text{momentum}\\ \text{angular momentum}\end{pmatrix}$ is conserved.

Conservation of mechanical energy: if non-conservative forces and torques do no work, mechanical energy is conserved



Example. Object mass m suspended by two strings as shown. Find T_1 and T_2.

It's not accelerating vertically so

N2 –> $\Sigma F_y = ma_y = 0$

$\therefore \quad T_1 + T_2 - mg = 0 \qquad$ (i)

It's not accelerating horizontally so

N2 –> $\Sigma F_x = ma_x = 0$

$\therefore \quad 0 = 0$ *not enough equations*

It's not **rotationally accelerating** so:

N2 –> $\Sigma \tau = I\alpha = 0$

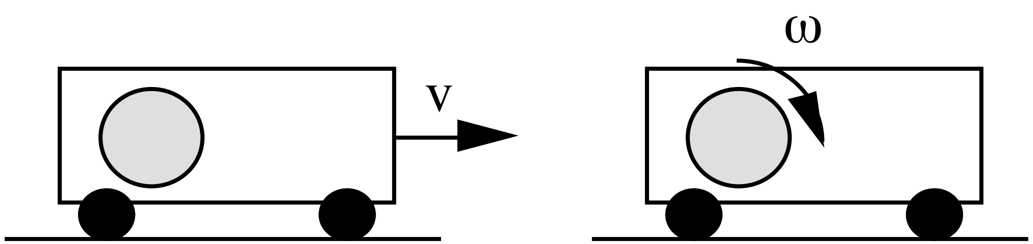
τ about c.m. clockwise $\quad \therefore \quad \tau_1 + \tau_2 = T_2 D - T_1 d = 0$

$$T_1 + \frac{d}{D} T_1 - mg = 0$$

$$T_1 = \frac{mg}{1 + d/D} \qquad T_2 = \frac{mg}{1 + D/d}$$

Example Use a flywheel to store the K of a bus at stops. Disc R = 80 cm, M = 1 tonne. How fast must it turn to store all the kinetic energy of a 10 t. bus at 60 km.hr^{-1}?

Moving (subscript m), stopped (subscript s)



$v_m = 60$ km.hr^{-1} $\qquad v_s = 0$ *not* rolling

$\omega_m = 0$ $\qquad \omega_s = ?$ rev.s^{-1}

$$K_m = K_s$$

$$\frac{1}{2} M_{bus} v_m^2 = \frac{1}{2} I_{disc} \omega_s^2$$

$$M_{bus} v_m^2 = \frac{1}{2} M_{disc} R^2 \omega_s^2$$

$$\omega_s = \frac{v_m}{R}\sqrt{\frac{2M_{bus}}{M_{disc}}}$$

$$= 90 \text{ rad.s}^{-1} = 900 \text{ rpm}$$