Adiabatic processes PV-relation

From the ideal gas law:

$$PV = nRT$$

|| P and V are both functions of T.

Take the derivative with respect to T on both sides:

$$\frac{d}{dT}(PV) = \frac{d}{dT}(nRT)$$

$$\Rightarrow P\frac{dV}{dT} + V\frac{dP}{dT} = nR$$

$$\Rightarrow PdV + VdP = nRdT \quad (*)$$

|| Since $\frac{dV}{dT} = \lim_{\Delta T \to 0} \frac{\Delta V}{\Delta T}$, etc.

But

$$-PdV = nC_V dT \Rightarrow n\,dT = -\frac{P}{C_V}dV$$

Then, substituting into (*), we find

$$PdV + VdP = -\frac{PR}{C_V}dV$$

We also know that $C_P = C_V + R \Rightarrow R = C_P - C_V$.

$$\Rightarrow PdV + VdP = \left(\frac{C_V - C_P}{C_V}\right)PdV$$

$$= (1-\gamma)PdV$$

$$\Rightarrow VdP = -\gamma PdV$$

Now divide both sides by PV:

$$\Rightarrow \frac{dP}{P} = -\gamma\frac{dV}{V}$$

Now we can integrate from the initial state to the final state:

$$\int_{P_{ini}}^{P_{final}} \frac{dP}{P} = -\gamma \int_{V_{ini}}^{V_{final}} \frac{dV}{V}$$

$$\Rightarrow \ln \frac{P_{final}}{P_{ini}} = -\gamma \ln \frac{V_{final}}{V_{ini}} = \ln \left(\frac{V_{ini}}{V_{final}} \right)^{\gamma}$$

$$\Rightarrow \frac{P_{final}}{P_{ini}} = \left(\frac{V_{ini}}{V_{final}} \right)^{\gamma}$$

$$\Rightarrow P_{final} V_{final}^{\gamma} = P_{ini} V_{initial}^{\gamma}$$

$$\text{or} \quad \boxed{PV^{\gamma} = \text{constant}} \qquad (\star)$$

To get the T-V-relation, we can use the ideal gas law again:

$$P = \frac{nRT}{V}$$

Then substituting into $(\star)$ gives

$$nRT\,V^{\gamma-1} = \text{constant}$$

$$\Rightarrow \underline{\underline{T V^{\gamma-1} = \text{another constant}}}$$