Chapter 4

Motion in Two and Three Dimensions

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4-1 Position and Displacement

Learning Objectives

4.01 Draw two-dimensional and three-dimensional position vectors for a particle, indicating the components along the axes of a coordinate system.

4.02 On a coordinate system, determine the direction and magnitude of a particle's position vector from its components, and vice versa.

4.03 Apply the relationship between a particle's displacement vector and its initial and final position vectors.

4-1 Position and Displacement

- A **position vector** locates a particle in space
  - Extends from a reference point (origin) to the particle

$$\vec{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}, \qquad \text{Eq. (4-1)}$$

Example

- Position vector (-3m, 2m, 5m)

$$\vec{r} = (-3\ \mathrm{m})\hat{\mathrm{i}} + (2\ \mathrm{m})\hat{\mathrm{j}} + (5\ \mathrm{m})\hat{\mathrm{k}}$$

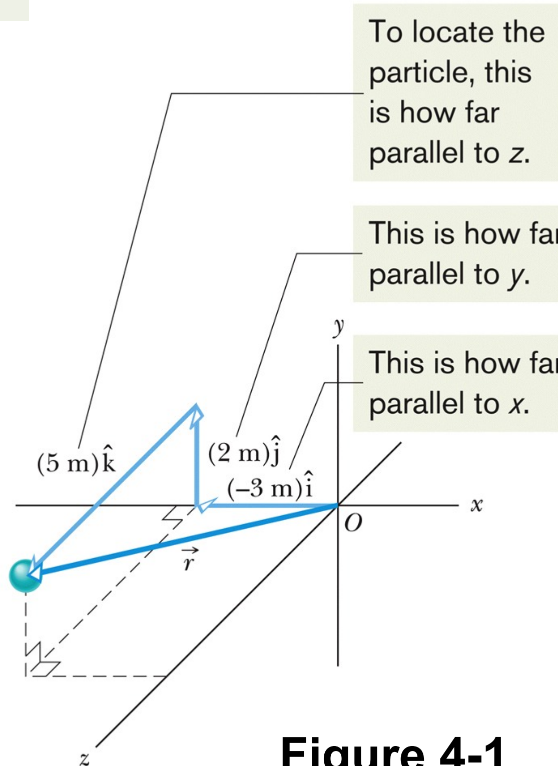

Figure 4-1

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4-1 Position and Displacement

- Change in position vector is a **displacement**

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1.$$ **Eq. (4-2)**

- We can rewrite this as:

$$\Delta\vec{r} = (x_2 - x_1)\hat{\mathrm{i}} + (y_2 - y_1)\hat{\mathrm{j}} + (z_2 - z_1)\hat{\mathrm{k}},$$ **Eq. (4-3)**

- Or express it in terms of changes in each coordinate:

$$\Delta\vec{r} = \Delta x\hat{\mathrm{i}} + \Delta y\hat{\mathrm{j}} + \Delta z\hat{\mathrm{k}}.$$ **Eq. (4-4)**

4-2 Average Velocity and Instantaneous Velocity

Learning Objectives

4.04 Identify that velocity is a vector quantity and thus has both magnitude and direction and also has components.

4.05 Draw two-dimensional and three-dimensional velocity vectors for a particle, indicating the components along the axes of the coordinate system.

4.06 In magnitude-angle and unit-vector notations, relate a particle's initial and final position vectors, the time interval between those positions, and the particle's average velocity vector.

4.07 Given a particle's position vector as a function of time, determine its (instantaneous) velocity vector.

4-2 Average Velocity and Instantaneous Velocity

- **Average velocity** is
  - A displacement divided by its time interval

$$\vec{v}_{avg} = \frac{\Delta \vec{r}}{\Delta t}.$$

Eq. (4-8)

- We can write this in component form:

$$\vec{v}_{avg} = \frac{\Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} + \frac{\Delta z}{\Delta t}\hat{k}.$$

Eq. (4-9)

Example

- A particle moves through displacement (12 m)i + (3.0 m)k in 2.0 s:

$$\vec{v}_{avg} = \frac{\Delta \vec{r}}{\Delta t} = \frac{(12\text{ m})\hat{i} + (3.0\text{ m})\hat{k}}{2.0\text{ s}} = (6.0\text{ m/s})\hat{i} + (1.5\text{ m/s})\hat{k}.$$

4-2 Average Velocity and Instantaneous Velocity

- **Instantaneous velocity** is
  - The velocity of a particle at a single point in time
  - The limit of avg. velocity as the time interval shrinks to 0

$$\vec{v} = \frac{d\vec{r}}{dt}.$$

Eq. (4-10)

- Visualize displacement and instantaneous velocity:

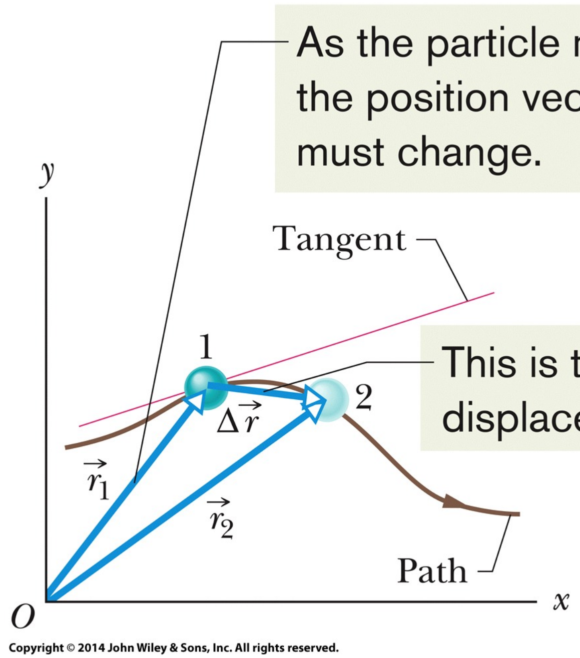


Figure 4-3

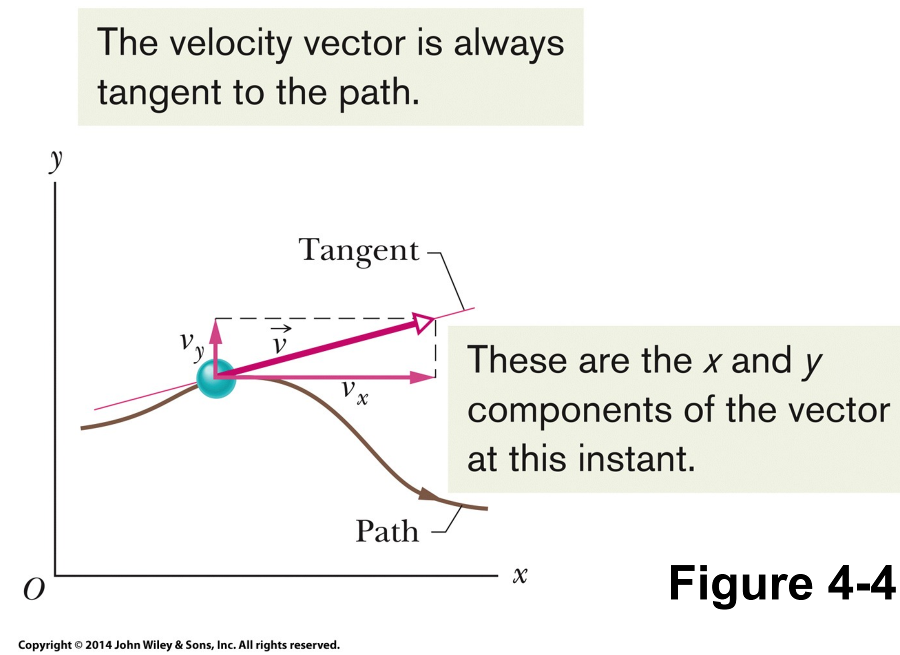


Figure 4-4

4-2 Average Velocity and Instantaneous Velocity

The direction of the instantaneous velocity $\vec{v}$ of a particle is always tangent to the particle's path at the particle's position.

- In unit-vector form, we write:

$$\vec{v} = \frac{d}{dt}(x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}) = \frac{dx}{dt}\hat{\mathrm{i}} + \frac{dy}{dt}\hat{\mathrm{j}} + \frac{dz}{dt}\hat{\mathrm{k}}.$$

- Which can also be written:

$$\vec{v} = v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}}, \qquad \text{Eq. (4-11)}$$

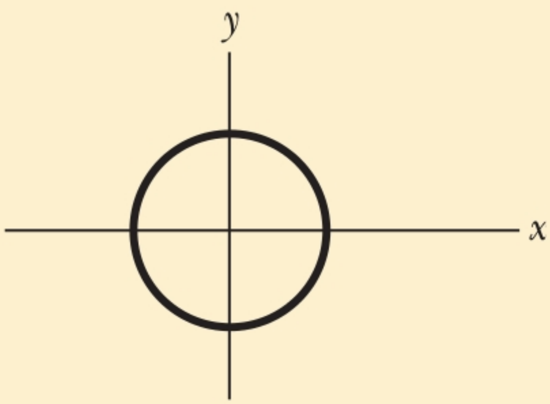
$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad \text{and} \quad v_z = \frac{dz}{dt}. \qquad \text{Eq. (4-12)}$$

- Note: a velocity vector does *not* extend from one point to another, only shows direction and magnitude

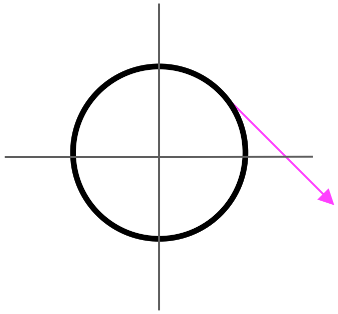
4-2 Average Velocity and Instantaneous Velocity

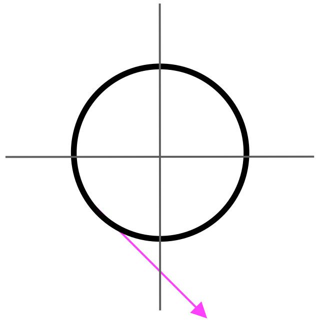
Checkpoint 1

The figure shows a circular path taken by a particle. If the instantaneous velocity of the particle is $\vec{v} = (2\text{ m/s})\hat{i} - (2\text{ m/s})\hat{j}$, through which quadrant is the particle moving at that instant if it is traveling (a) clockwise and (b) counterclockwise around the circle? For both cases, draw $\vec{v}$ on the figure.



Answer: (a) Quadrant I (b) Quadrant III





4-3 Average Acceleration and Instantaneous Acceleration

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Learning Objectives

4.08 Identify that acceleration is a vector quantity, and thus has both magnitude and direction.

4.09 Draw two-dimensional and three-dimensional acceleration vectors for a particle, indicating the components.

4.10 Given the initial and final velocity vectors of a particle and the time interval, determine the average acceleration vector.

4.11 Given a particle's velocity vector as a function of time, determine its (instantaneous) acceleration vector.

4.12 For each dimension of motion, apply the constant-acceleration equations (Chapter 2) to relate acceleration, velocity, position, and time.

4-3 Average Acceleration and Instantaneous Acceleration

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- **Average acceleration** is
  - A change in velocity divided by its time interval

$$\vec{a}_{\text{avg}} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta \vec{v}}{\Delta t}. \quad \textbf{Eq. (4-15)}$$

- **Instantaneous acceleration** is again the limit $t \rightarrow 0$:

$$\vec{a} = \frac{d\vec{v}}{dt}. \quad \textbf{Eq. (4-16)}$$

- We can write Eq. 4-16 in unit-vector form:

$$\vec{a} = \frac{d}{dt}(v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}})$$

$$= \frac{dv_x}{dt}\hat{\mathrm{i}} + \frac{dv_y}{dt}\hat{\mathrm{j}} + \frac{dv_z}{dt}\hat{\mathrm{k}}.$$

4-3 Average Acceleration and Instantaneous Acceleration

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- We can rewrite as:

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}},$$ **Eq. (4-17)**

$$a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad \text{and} \quad a_z = \frac{dv_z}{dt}.$$ **Eq. (4-18)**

- To get the components of acceleration, we differentiate the components of velocity with respect to time

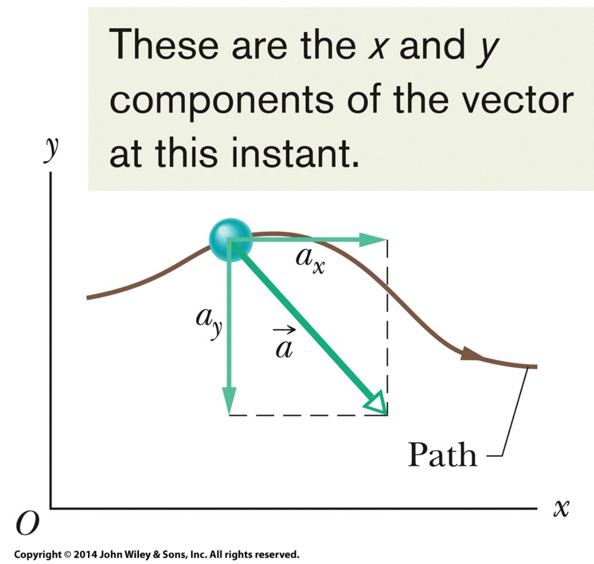

Figure 4-6

4-3 Average Acceleration and Instantaneous Acceleration

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- Note: as with velocity, an acceleration vector does *not* extend from one point to another, only shows direction and magnitude

Checkpoint 2

Here are four descriptions of the position (in meters) of a puck as it moves in an xy plane:

(1) $x = -3t^2 + 4t - 2$ and $y = 6t^2 - 4t$ (3) $\vec{r} = 2t^2\hat{\mathrm{i}} - (4t + 3)\hat{\mathrm{j}}$

(2) $x = -3t^3 - 4t$ and $y = -5t^2 + 6$ (4) $\vec{r} = (4t^3 - 2t)\hat{\mathrm{i}} + 3\hat{\mathrm{j}}$

Are the x and y acceleration components constant? Is acceleration $\vec{a}$ constant?

Answer: (1) x:yes, y:yes, a:yes (3) x:yes, y:yes, a:yes

(2) x:no, y:yes, a:no (4) x:no, y:yes, a:no

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4-4 Projectile Motion

Learning Objectives

4.13 On a sketch of the path taken in projectile motion, explain the magnitudes and directions of the velocity and acceleration components during the flight.

4.14 Given the launch velocity in either magnitude-angle or unit-vector notation, calculate the particle's position, displacement, and velocity at a given instant during the flight.

4.15 Given data for an instant during the flight, calculate the launch velocity.

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4-4 Projectile Motion

- A **projectile** is
  - A particle moving in the vertical plane
  - With some initial velocity
  - Whose acceleration is always free-fall acceleration (g)
- The motion of a projectile is **projectile motion**
- Launched with an initial velocity v_0

$$\vec{v}_0 = v_{0x}\hat{\mathrm{i}} + v_{0y}\hat{\mathrm{j}}.$$ **Eq. (4-19)**

$$v_{0x} = v_0 \cos\theta_0 \quad \text{and} \quad v_{0y} = v_0 \sin\theta_0.$$ **Eq. (4-20)**

In projectile motion, the horizontal motion and the vertical motion are independent of each other; that is, neither motion affects the other.

4-4 Projectile Motion

- Therefore we can decompose two-dimensional motion into 2 one-dimensional problems

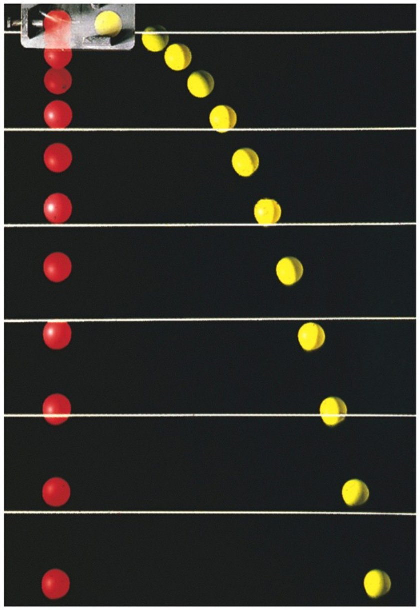

Figure 4-10

Richard Megna/Fundamental Photographs

Checkpoint 3

At a certain instant, a fly ball has velocity $\vec{v} = 25\hat{\mathrm{i}} - 4.9\hat{\mathrm{j}}$ (the x axis is horizontal, the y axis is upward, and $\vec{v}$ is in meters per second). Has the ball passed its highest point?

Answer: Yes. The y-velocity is negative, so the ball is now falling.

4-4 Projectile Motion

- Horizontal motion:
  - No acceleration, so velocity is constant (recall Eq. 2-15):

$$x - x_0 = v_{0x}t.$$

$$x - x_0 = (v_0 \cos \theta_0)t. \qquad \textbf{Eq. (4-21)}$$

- Vertical motion:
  - Acceleration is always -g (recall Eqs. 2-15, 2-11, 2-16):

$$y - y_0 = v_{0y}t - \tfrac{1}{2}gt^2$$

$$= (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2, \qquad \textbf{Eq. (4-22)}$$

$$v_y = v_0 \sin \theta_0 - gt \qquad \textbf{Eq. (4-23)}$$

$$v_y^2 = (v_0 \sin \theta_0)^2 - 2g(y - y_0). \qquad \textbf{Eq. (4-24)}$$

4-4 Projectile Motion

- The projectile's **trajectory** is
  - Its path through space (traces a parabola)
  - Found by eliminating time between Eqs. 4-21 and 4-22:

$$y = (\tan \theta_0)x - \frac{gx^2}{2(v_0 \cos \theta_0)^2}$$ **Eq. (4-25)**

- The **horizontal range** is:
  - The distance the projectile travels in *x* by the time it returns to its initial height

$$R = \frac{v_0^2}{g} \sin 2\theta_0.$$ **Eq. (4-26)**

The horizontal range R is maximum for a launch angle of 45°.

4-4 Projectile Motion

- In these calculations we assume air resistance is negligible
- In many situations this is a poor assumption:

Table 4-1 Two Fly Balls[a]

	Path I (Air)	Path II (Vacuum)
Range	98.5 m	177 m
Maximum height	53.0 m	76.8 m
Time of flight	6.6 s	7.9 s

[a]See Fig. 4-13. The launch angle is 60° and the launch speed is 44.7 m/s.

Table 4-1

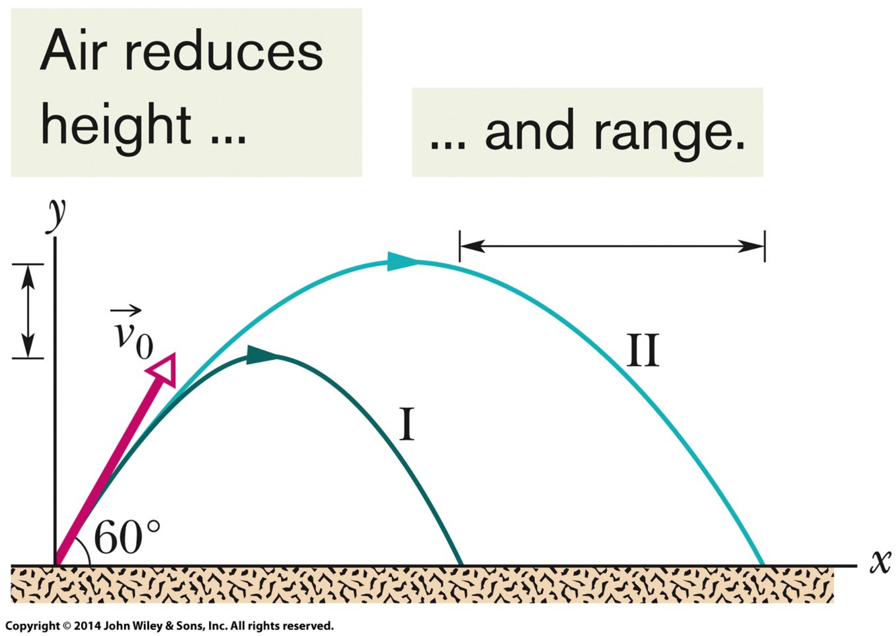

Figure 4-13

4-4 Projectile Motion

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Checkpoint 4

A fly ball is hit to the outfield. During its flight (ignore the effects of the air), what happens to its (a) horizontal and (b) vertical components of velocity? What are the (c) horizontal and (d) vertical components of its acceleration during ascent, during descent, and at the topmost point of its flight?

Answer: (a) is unchanged (b) decreases (becomes negative)

(c) 0 at all times (d) -g (-9.8 m/s^2) at all times

4-5 Uniform Circular Motion

Learning Objectives

4.16 Sketch the path taken in uniform circular motion and explain the velocity and acceleration vectors (magnitude and direction) during the motion.

4.17 Apply the relationships between the radius of the circular path, the period, the particle's speed, and the particle's acceleration magnitude.

4-5 Uniform Circular Motion

- A particle is in **uniform circular motion** if
  - It travels around a circle or circular arc
  - At a constant speed
- Since the velocity changes, the particle is accelerating!
- Velocity and acceleration have:
  - Constant magnitude
  - Changing direction

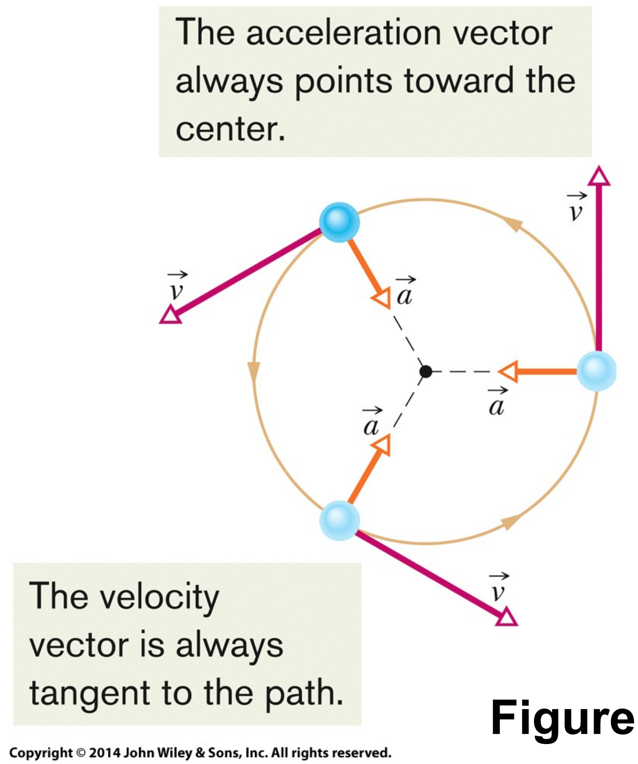

Figure 4-16

4-5 Uniform Circular Motion

- Acceleration is called **centripetal acceleration**
  - Means "center seeking"
  - Directed radially inward

$$a = \frac{v^2}{r}$$

Eq. (4-34)

- The **period of revolution** is:
  - The time it takes for the particle go around the closed path exactly once

$$T = \frac{2\pi r}{v}$$

Eq. (4-35)

4-5 Uniform Circular Motion

Checkpoint 5

An object moves at constant speed along a circular path in a horizontal xy plane, with the center at the origin. When the object is at $x = -2$ m, its velocity is $-(4 \text{ m/s})\hat{\text{j}}$. Give the object's (a) velocity and (b) acceleration at $y = 2$ m.

Answer: (a) -(4 m/s)i (b) -(8 m/s^2)j

4-6 Relative Motion in One Dimension

Learning Objectives

4.18 Apply the relationship between a particle's position, velocity, and acceleration as measured from two reference frames that move relative to each other at a constant velocity and along a single axis.

4-6 Relative Motion in One Dimension

- Measures of position and velocity depend on the **reference frame** of the measurer
  - How is the observer moving?
  - Our usual reference frame is that of the ground

- Read subscripts "PA", "PB", and "BA" as "P as measured by A", "P as measured by B", and "B as measured by A"
- Frames A and B are each watching the movement of object P

Frame *B* moves past frame *A* while both observe *P*.

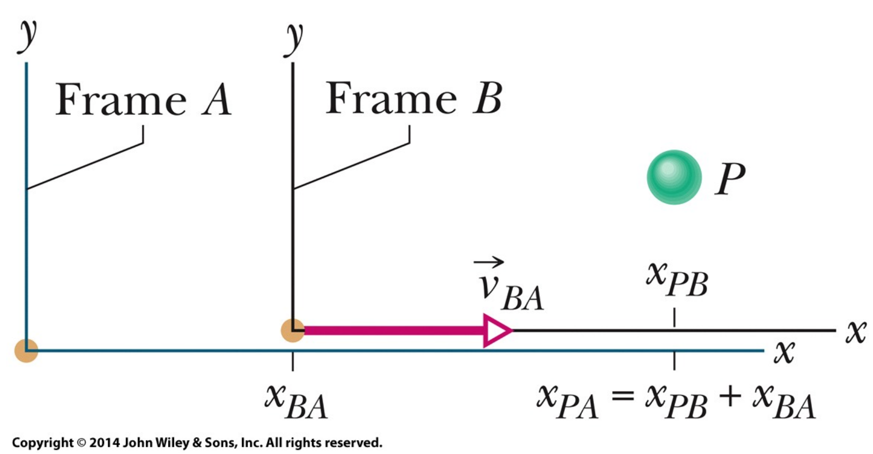

Figure 4-18

4-6 Relative Motion in One Dimension

- Positions in different frames are related by:

$$x_{PA} = x_{PB} + x_{BA}.$$ **Eq. (4-40)**

- Taking the derivative, we see velocities are related by:

$$\frac{d}{dt}(x_{PA}) = \frac{d}{dt}(x_{PB}) + \frac{d}{dt}(x_{BA}).$$

$$v_{PA} = v_{PB} + v_{BA}.$$ **Eq. (4-41)**

- But accelerations (for non-accelerating reference frames, a_{BA} = 0) are related by

$$\frac{d}{dt}(v_{PA}) = \frac{d}{dt}(v_{PB}) + \frac{d}{dt}(v_{BA}).$$

$$a_{PA} = a_{PB}.$$ **Eq. (4-42)**

4-6 Relative Motion in One Dimension

Observers on different frames of reference that move at constant velocity relative to each other will measure the same acceleration for a moving particle.

Example

Frame A: $x = 2$ m, $v = 4$ m/s
Frame B: $x = 3$ m, $v = -2$ m/s
P as measured by A: $x_{PA} = 5$ m, $v_{PA} = 2$ m/s, $a = 1$ m/s^2

So P as measured by B:

- $x_{PB} = x_{PA} + x_{AB} = 5$ m + (2m – 3m) = 4 m
- $v_{PB} = v_{PA} + v_{AB} = 2$ m/s + (4 m/s – -2m/s) = 8 m/s
- $a = 1$ m/s^2

4-7 Relative Motion in Two Dimensions

Learning Objectives

4.19 Apply the relationship between a particle's position, velocity, and acceleration as measured from two reference frames that move relative to each other at a constant velocity and in two dimensions.

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4-7 Relative Motion in Two Dimensions

- The same as in one dimension, but now with vectors:
- Positions in different frames are related by:

$$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}.$$ **Eq. (4-43)**

- Velocities:

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}.$$ **Eq. (4-44)**

- Accelerations (for non-accelerating reference frames):

$$\vec{a}_{PA} = \vec{a}_{PB}.$$ **Eq. (4-45)**

- Again, observers in different frames will see the same acceleration

4-7 Relative Motion in Two Dimensions

- Frames A and B are both observing the motion of P

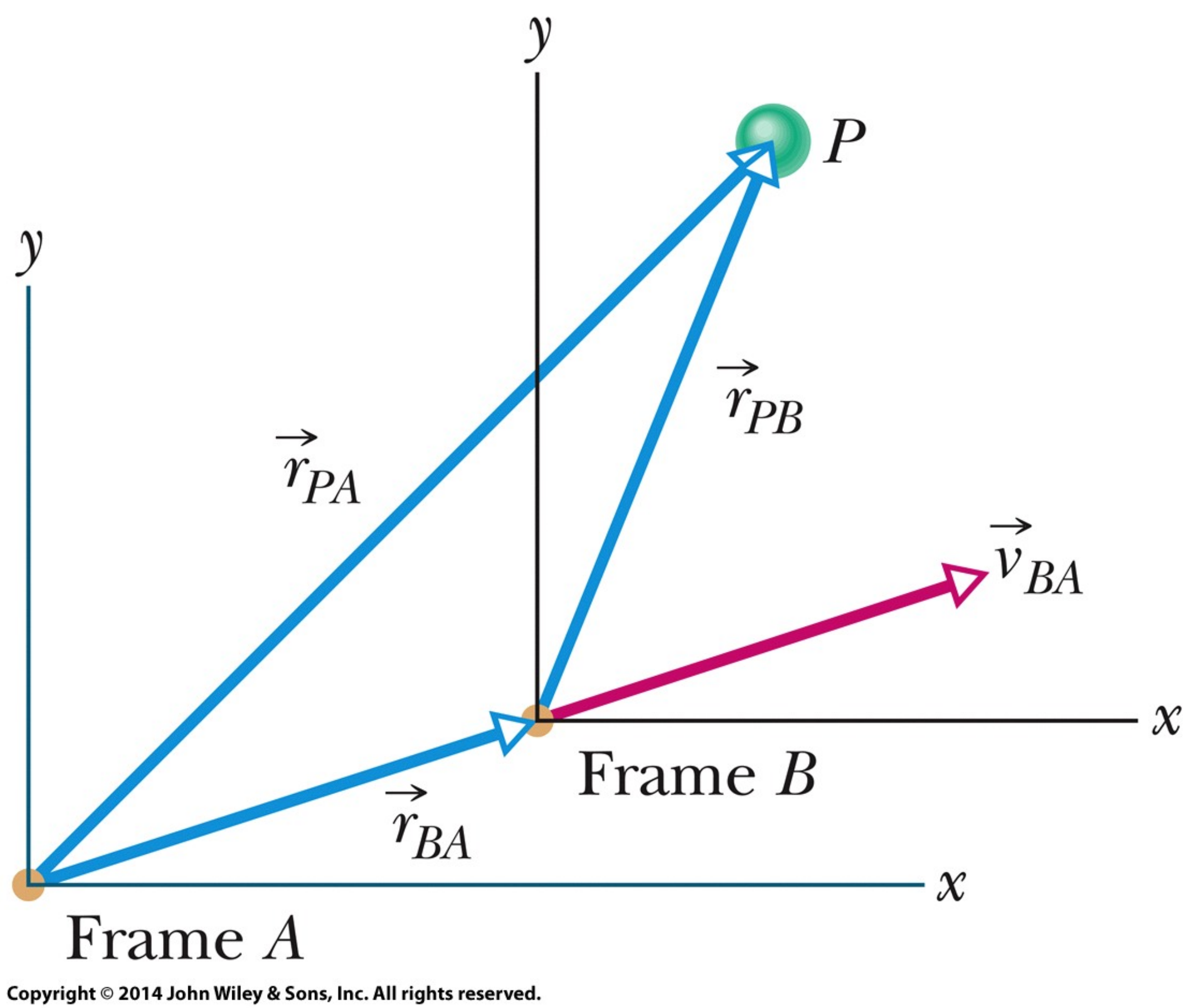

Figure 4-19

4 Summary

Position Vector

- Locates a particle in 3-space

$$\vec{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}},$$

Eq. (4-1)

Displacement

- Change in position vector

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1.$$

Eq. (4-2)

$$\Delta\vec{r} = (x_2 - x_1)\hat{\mathrm{i}} + (y_2 - y_1)\hat{\mathrm{j}} + (z_2 - z_1)\hat{\mathrm{k}},$$

Eq. (4-3)

$$\Delta\vec{r} = \Delta x\hat{\mathrm{i}} + \Delta y\hat{\mathrm{j}} + \Delta z\hat{\mathrm{k}}.$$

Eq. (4-4)

Average and Instantaneous Velocity

$$\vec{v}_{\text{avg}} = \frac{\Delta\vec{r}}{\Delta t}.$$

Eq. (4-8)

$$\vec{v} = \frac{d\vec{r}}{dt}.$$

Eq. (4-10)

Average and Instantaneous Accel.

$$\vec{a}_{\text{avg}} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta\vec{v}}{\Delta t}.$$

Eq. (4-15)

$$\vec{a} = \frac{d\vec{v}}{dt}.$$

Eq. (4-16)

4 Summary

Projectile Motion

- Flight of particle subject only to free-fall acceleration (g)

$$y - y_0 = v_{0y}t - \tfrac{1}{2}gt^2 = (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2,$$ **Eq. (4-22)**

$$v_y = v_0 \sin \theta_0 - gt$$ **Eq. (4-23)**

- Trajectory is parabolic path

$$y = (\tan \theta_0)x - \frac{gx^2}{2(v_0 \cos \theta_0)^2}$$ **Eq. (4-25)**

- Horizontal range:

$$R = \frac{v_0^2}{g} \sin 2\theta_0.$$ **Eq. (4-26)**

Uniform Circular Motion

- Magnitude of acceleration:

$$a = \frac{v^2}{r}$$ **Eq. (4-34)**

- Time to complete a circle:

$$T = \frac{2\pi r}{v}$$ **Eq. (4-35)**

Relative Motion

- For non-accelerating reference frames

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}.$$ **Eq. (4-44)**

$$\vec{a}_{PA} = \vec{a}_{PB}.$$ **Eq. (4-45)**