Question 1 (Marks: 21)

(a) The position of a particle varies with time according to the vector equation

$$\vec{r} = 4.00\hat{i} - 5.00t^2\hat{j}$$

where $\vec{r}$ is measured in meters and t is measured in seconds.

(i) Determine an expression for the velocity of the particle as a function of time.

(ii) Determine the acceleration of the particle as a function of time.

(iii) Determine the position of the particle and its velocity at a time $t = 1.50s$.

(b) A mass weighing 3.00 kg is suspended from a spring that is hung vertically. The spring obeys Hooke's law. When the mass is attached, the spring stretches by 1.50cm.

(i) The 3.00kg mass is then removed and instead a 4.50kg mass is attached in its place. How far does the spring now stretch?

(ii) How much work is done on the spring in part (i)?

(c) A tennis player receives a ball that is travelling towards her horizontally at 50 m/s. She returns the ball (i.e. hits it back in the opposite direction) horizontally at a speed of 45 m/s. The ball has a mass of 60grams.

(i) Find the magnitude of the impulse delivered to the ball by the impact of the tennis racket.

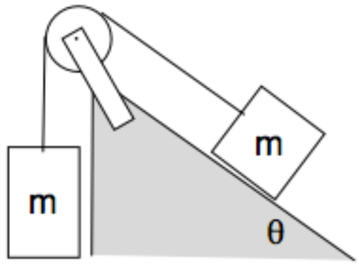
(ii) How much work does the racket do on the ball?

(d) The mass of the Earth is 5.972 x 10^{24} kg. The mass of the Moon is 7.348 x 10^{22} kg. The distance between the Earth and Moon is 384,400 km. Find the centre of mass of the Earth-Moon system as measured from the centre of the Earth.

(e) The Moon orbits the Earth with a period of 27.32 days. Using only this datum, the information in part (d) and your answer to it, calculate the magnitude of the gravitational force that the Moon exerts on the Earth. Show your working. Use only the data specified: do not use Newton's constant G.

Question 2 (Marks: 20)

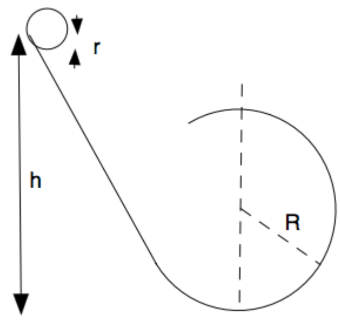
(a) The sketch shows a plane inclined at angle θ to the horizontal. Two equal masses (m) are connected via a light, inextensible string. One of the masses slides on the inclined plane: the coefficients of kinetic and static friction are μ_k and μ_s respectively. The string passes without slipping over a light frictionless pulley. When released, the mass on the left accelerates downwards with acceleration a.



(i) Draw free body diagrams for each of the masses m.

(ii) Hence or otherwise, derive an expression for a.

(b) A uniform, spherical ball, mass m and radius r, is released from rest and rolls without slipping on a track as shown (but not to scale). The curved section is an arc of a circle with radius R, and R is very much greater than r. The ball is released from rest at a height h above the lowest point of the track. Air resistance is negligible.



(i) When the ball reaches the highest point of the curved section of track, it has speed v. Derive an expression for v. State clearly any conservation principles you use, and explain why they are valid.

(ii) Derive an expression for the magnitude of the normal force that the track exerts on the ball at this point.

(iii) State the direction of the normal force when the ball is at this point.

[Hint: Remember that the ball is rolling, not sliding.]

Question 3 (Marks: 19)

(i) Define a conservative force.

(ii) Under what conditions is mechanical energy conserved?

(iii) For a symmetrical, spherical object, mass M and radius r, derive an expression for the escape speed v_e at its surface (also called escape velocity). (v_e is the minimal launch speed of an object that, with no forces other than gravity acting, will not fall back to the surface.) State any principles you use.

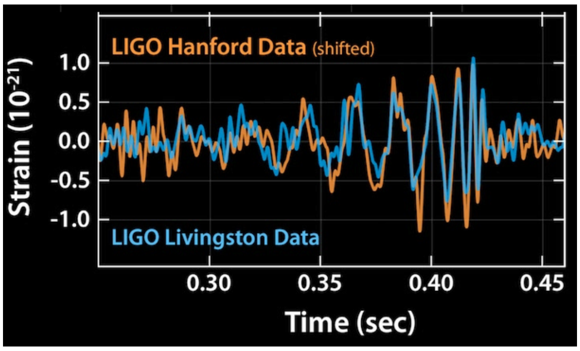
(iv) The event horizon of a black hole has a radius r_{BH}: from inside of this, even light cannot escape. Use your previous result to derive an expression for r_{BH} for a black hole, using only Newtonian mechanics. (Surprisingly, this will give the same answer as using General Relativity.)

(v) Two black holes, each with mass M, are separated by a distance $2R$ and are both in a circular orbit about a point midway between them. Again using only Newtonian mechanics, derive an expression for E, the total orbital mechanical energy of their orbits, in terms of R and M. (Hints: Because they have equal mass, it's not the same situation as a planet and sun. Use Newton's second law to obtain an expression for Mv^2 in terms of G, M, and R.)

(vi) Consider two black holes in very close circular orbit: in fact, assume that the radius R of the orbit equals twice the radius of the event horizon: $R = 2r_{BH}$. Although this would not be true, assume that the two black holes are still spherically symmetrical. Once again, neglect relativistic effects (i.e. use only Newtonian mechanics). Use your previous answers to obtain an expression for E in terms of M and physical constants, without including R or r_{BH}.

(vii) In February, 2016, the LIGO team announced the discovery of gravitational waves caused by two black holes orbiting a common centre. Their masses were roughly the same, at approximately $M = 6\text{x}10^{31}$ kg, and we'll assume (for now) that the orbit was a constant circle. Calculate the mechanical energy associated with their orbits when $R = 2r_{BH}$.

(viii) Now suppose that the orbit shrinks by radiating energy (in the form of gravitational waves) that reduces the mechanical energy of the orbits. Suppose that (as the published signal* approximately suggests) the orbital energy changes from $E/2$ to E in a time of roughly 10 ms. Calculate the power radiated as gravitational waves over this interval.



* *The published curves are included here for interest*

(ix) Now calculate the ratio of power radiated in (viii) to the total power radiated by all the stars in the observable universe. Assume that there are about 10^{11} galaxies, each with about 10^{11} stars, and that each star has roughly the same power as our sun, $4\text{x}10^{26}$ W.

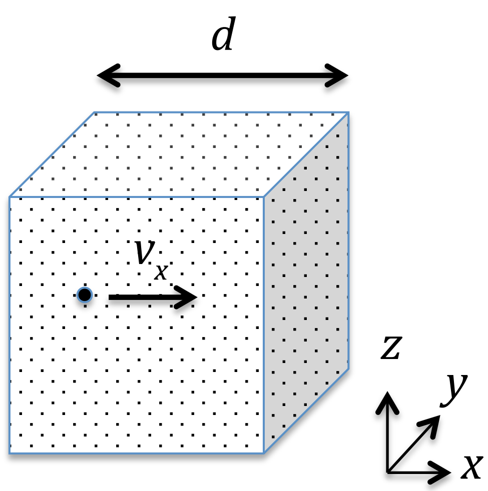
Question 4 (Marks: 30)

(a) Soon after the Earth formed, heat released by the decay of radioactive elements raised the average internal temperature from 300K to 3,000K, about the value it remains at today. Assuming an average coefficient of volume expansion of 3.2×10^{-5} K^{-1}, by what percentage has the radius of the Earth increased since its formation due to the release of this heat? You may assume that the Earth is a uniform sphere.

(b) The atmospheric pressure on the surface of the Earth is 101 kPa. Determine the mass of the Earth's atmosphere, and express your answer as a fraction of the total mass of the Earth. You may assume that the Earth is a sphere of radius R = 6,400 km and has mass $M = 6.0 \times 10^{24}$ kg.

(c) Describe three of the assumptions associated with the model for an ideal gas enclosed within a container.

(d) Consider the motion of N identical particles of mass m in an ideal gas moving inside a box of side d. Their average speed in the x-direction is v_x, as in the diagram.
Derive an expression for the force exerted on a wall oriented in the y-z plane in terms of the variables give in the question.



(e) Hence derive an expression for the pressure, P, of the gas in the box. Use V, volume of the box and $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$ the average speed of the particles in your expression along with any of the other variables you need given in this question.

(f) Using the equation for an ideal gas, $PV = nRT$, where n is the number of moles and R the gas constant, determine the total kinetic energy of the gas in terms of its temperature T.

(g) Calculate the average speed of nitrogen molecules (which are comprised of two nitrogen atoms) in air at room temperature (T=20°C, mass of nitrogen atom = 14.0 u).

Question 5 (Marks: 30)

(a) The A-string of a violin is 32cm long between two fixed points, and has a mass per unit length of 0.58 g/m. It is tuned to give a natural (fundamental) note at 440Hz.

(i) What is the wave speed in the string?

(ii) What is the tension in the string?

(iii) Our violinist now presses a finger onto the string, near one end, to change the note from an A to a D (f=587 Hz). How far from the fixed end of the string should they place their finger?

(iv) The violin has another string of the same length, with a natural (fundamental) note at D. If the tension in the A and D strings is the same and they are made from the same material, what is the ratio of the diameter of the A string to the D string?

(b) Bungee jumping is the curious practice of paying somebody rather a lot of money to attach a long elastic cord to your legs, and then jumping off a tall bridge. A tourist in New Zealand jumps from a bridge h=100.0 m above an icy cold river. The cord is carefully chosen so that the tourist's head just touches the water.

The bungee cord is made of a massless elastic compound of the type only found in first year physics exams. It is l_0 metres long, has a spring constant k, and is carefully attached to the jumper's feet. Assume the bungee jumper has a mass of m kg, and is a point mass of zero height.

(i) Write down an expression for the initial energy of the bungee jumper before they jump, if we define the height h=0 as the river surface.

(ii) Using conservation of energy, derive a general expression for the spring constant k of the bungee cord needed if the jumper's head is to just touch the water before they bounce back up.

(iii) If the jumper has a mass m=60.0 kg and l_0 =70.0 metres, how large is k ?

For parts (iv)-(x), assume their motion is simple harmonic after touching the water:

(iv) What is the peak-to-peak magnitude of the oscillations

(v) What is the maximum velocity in km/h?

(vi) There is a device on the bridge that makes a continuous loud sound with a frequency of 600 Hz. As the bungee jumper oscillates what is the maximum frequency he hears? The speed of the sound in air is 343 m/s.

(vii) If the bungee jumper screams with a frequency of 600 Hz, what is the maximum frequency a person directly above the bungee jumper will detect when the bungee jumper is oscillating?

(viii) If two bungee jumpers with identical weight and cords jump off the bridge simultaneously, and one screams with a frequency of 600 Hz what is the maximum frequency the other will detect?

(ix) What is the period of their oscillations?

(x) Assuming the motion is simple harmonic, sketch a graph showing the jumper's vertical position (in a solid line) and their velocity (with a dashed line) as a function of time. Ignore damping. Fully label the graph. Mark on the graph the vertical position the jumper oscillates around, and the times of maximum amplitude and velocity.

(xi) Is the motion really simple harmonic?

(xii) Sketch and label the jumper's real vertical displacement vs time (ignore damping), and mark on it the region of simple harmonic motion.

(xiii) Using the time it takes for the jumper to reach the water, calculate real period of the oscillations.