Lecture 12: Standing waves, and boundary conditions

Textbook reference: 16.7

https://youtu.be/ADdDvuAlEk
8

Last Lecture

The speed of a wave in a string with tension T and linear density μ is:

$$v = \sqrt{\frac{T}{\mu}} \qquad v = \sqrt{\frac{\text{elastic property}}{\text{inertial property}}}$$

Power transmitted by wave on a string:

$$P = \frac{1}{2}\mu v \omega^2 A^2$$

Last lecture...

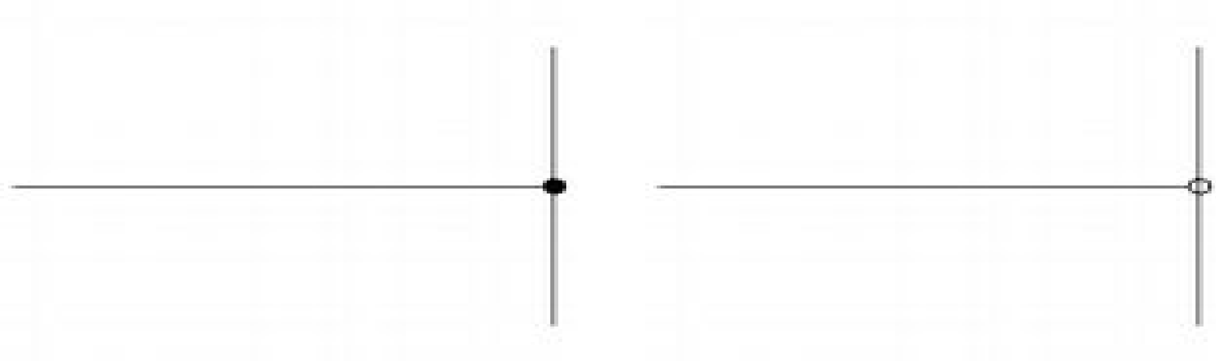
- Don't confuse the wave speed and the transverse speed of a wave!
  - **Wave speed**: how fast the wave travels along the string.

$$y(x,t)=f(x-vt)$$

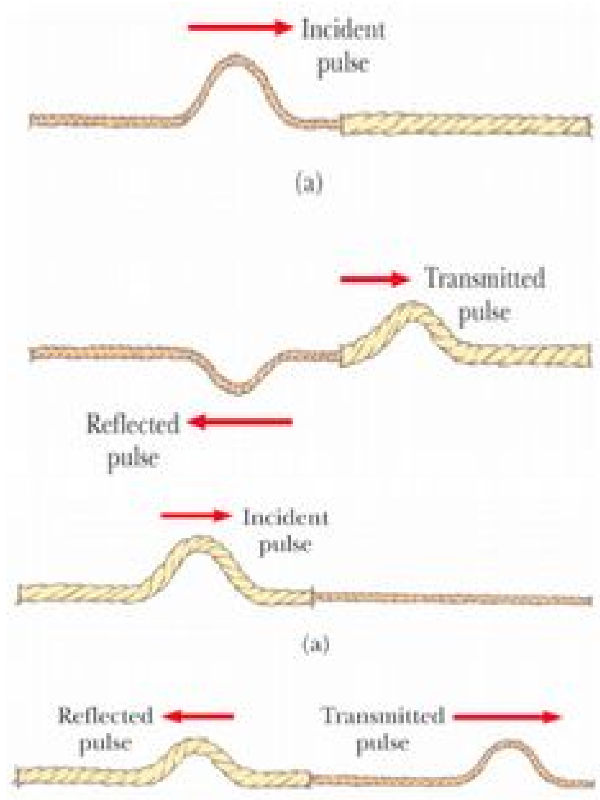
  - **Transverse speed**: how fast the perturbed elements of the medium are displaced up and down.

$$\frac{\partial y}{\partial t}=\frac{\partial f}{\partial t}\neq v$$

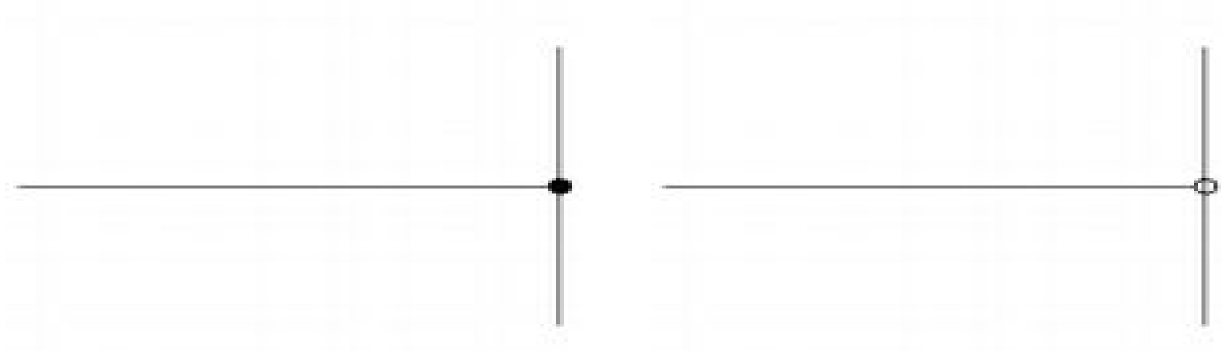
Last lecture...



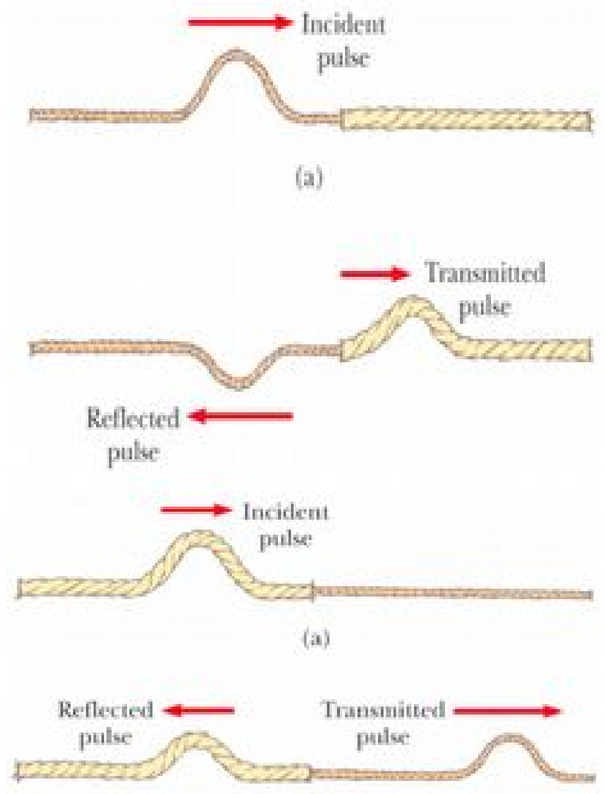
- When a pulse is **reflected** at a boundary (e.g., a end point, or an intersection between two media):
  - It is inverted if reflected at a fixed endpoint, or if the second medium is denser.
  - It is uninverted if reflected at a free endpoint, or if the second medium is less dense.



Last lecture...



- The wave can be **transmitted** from one medium to another.
  - Transmitted wave moves at a **new speed** determined by the property of the new medium.
  - Frequency does not change.
  - Wavelength changes. $v = f\lambda$



Last lecture...

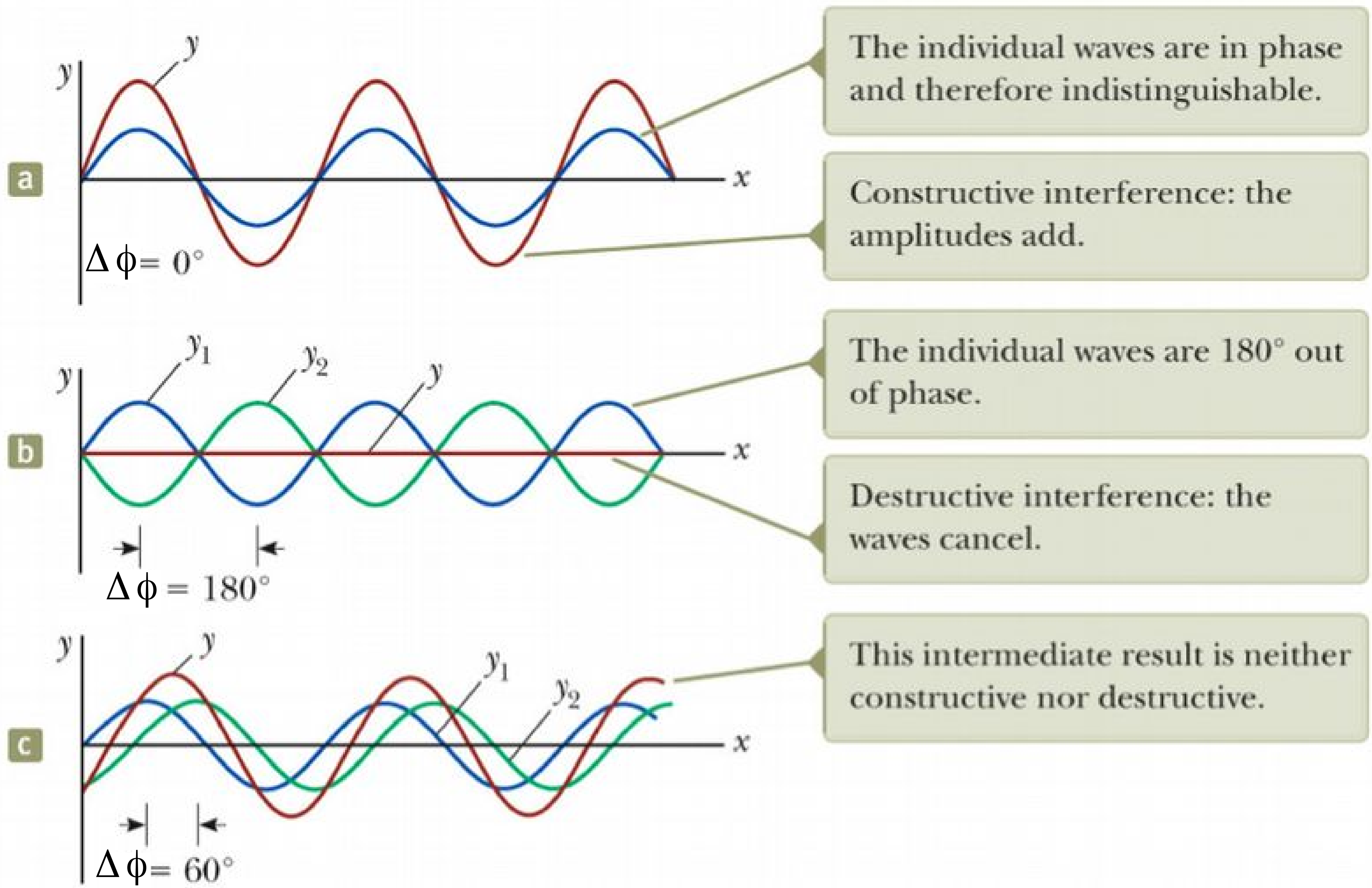
- The **principle of superposition**: If two or more waves are moving through the same medium, the resultant value of the wavefunction is the algebraic sum of the wavefunctions of the original waves.

- Example: two sine waves:

$$y_1 = A\sin(kx - \omega t + \phi_1)$$
$$y_2 = A\sin(kx - \omega t + \phi_2)$$

$$\Rightarrow \quad y_1 + y_2 = 2A\cos\left(\frac{\phi_1 - \phi_2}{2}\right)\sin\left(kx - \omega t + \frac{\phi_1 + \phi_2}{2}\right)$$

Phase difference $\Delta\phi \equiv \phi_1 - \phi_2$



The individual waves are in phase and therefore indistinguishable.
Constructive interference: the amplitudes add.
a
y
y
x
Δ ϕ= 0°
The individual waves are 180° out of phase.
Destructive interference: the waves cancel.
b
y
y1
y2
y
x
Δ ϕ = 180°
This intermediate result is neither constructive nor destructive.
c
y
y
y1
y2
x
Δ ϕ= 60°

This lecture...

- Path difference & phase difference
- Standing waves
- Boundary conditions

Phase difference vs path difference...

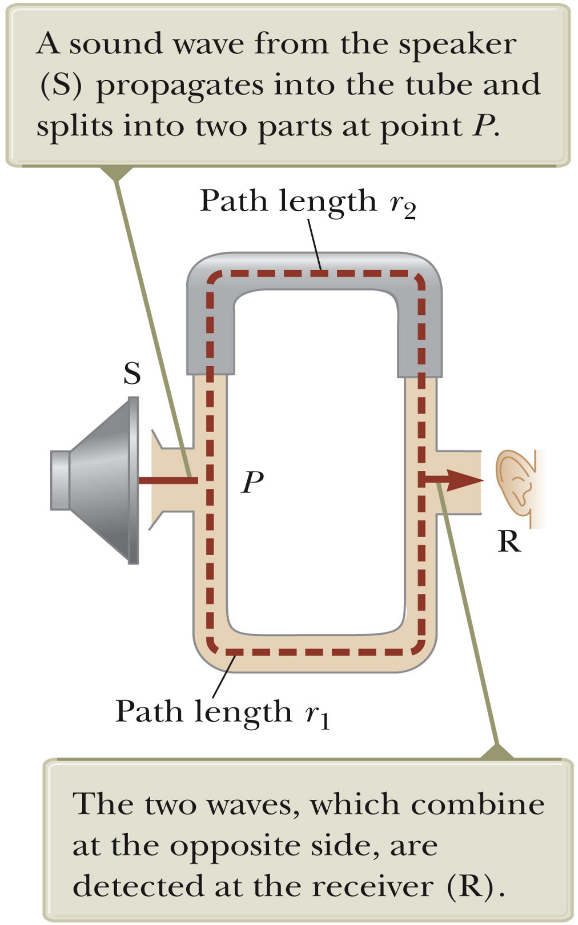
- Consider again 2 sine waves with a phase difference:

Phase difference

$$y_1(x,t) = A\sin(kx - \omega t + \phi_1)$$
$$y_2(x,t) = A\sin(kx - \omega t + \phi_1 + \Delta\phi)$$

Interference of sound waves

We have seen that waves interfere constructively if there is a phase difference of 0, 2π, 4π, 6π etc. between them. This is equivalent to having a "path length difference" of 0, λ, 2λ, 3λ etc. between them.



WHITEBOARD SKETCH

As an equation…

$$\frac{\text{path difference}}{\lambda} \times 2\pi = \text{phase difference}$$

Homework Set 6:

PHYS 1121:
1
PHYS 1131:
1, 8

Phase difference vs path difference...

- Consider again 2 sine waves with a phase difference:

Phase difference

$$y_1(x,t) = A\sin(kx - \omega t + \phi_1)$$
$$y_2(x,t) = A\sin(kx - \omega t + \phi_1 + \Delta\phi)$$

- We can also write $y_2(x, t)$ this way:

$$y_2(x,t) = A\sin\left[k\left(x + \frac{\Delta\phi}{k}\right) - \omega t + \phi_1\right]$$

Δx can be interpreted as the "extra distance" travelled by wave 2 compared with wave 1. It is also called the **path difference**.

$$\Delta x \equiv \frac{\Delta\phi}{k} = \frac{\lambda}{2\pi}\Delta\phi$$

Phase difference vs path difference...

- **Constructive interference** occurs when

Phase difference $\longrightarrow \Delta\phi = 0, 2\pi, 4\pi, \ldots 2n\pi$ $\quad n = 0, 1\ 2, 3, \ldots$

Path difference $\Rightarrow \Delta x \equiv \dfrac{\lambda}{2\pi}\Delta\phi = n\lambda$

Phase difference vs path difference...

- **Constructive interference** occurs when

Phase difference $\rightarrow$ $\Delta\phi = 0, 2\pi, 4\pi, \ldots 2n\pi$ $\quad n = 0, 1\ 2, 3, \ldots$

Path difference $\Rightarrow$ $\Delta x \equiv \frac{\lambda}{2\pi}\Delta\phi = n\lambda$

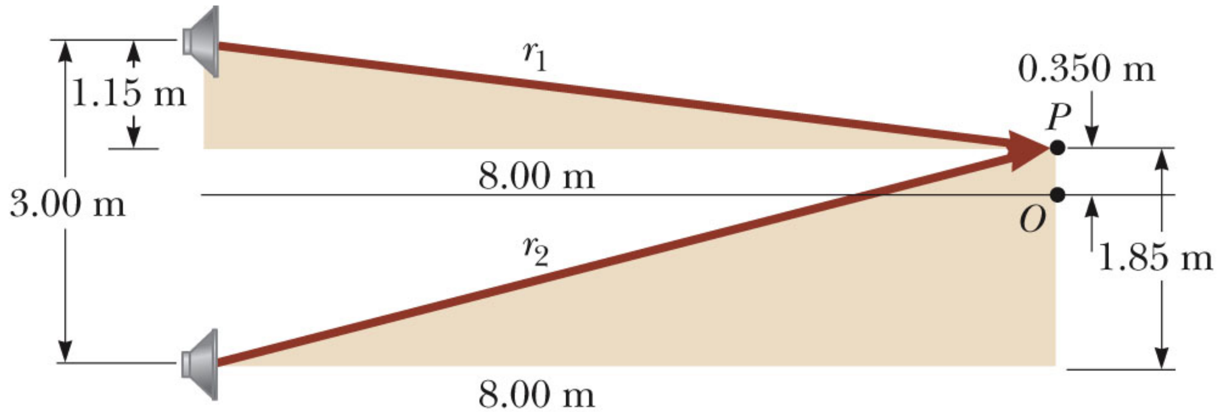
- **Destructive interference** occurs when

$$\Delta\phi = \pi, 3\pi, 5\pi, \ldots (2n+1)\pi$$

$$\Rightarrow \quad \Delta x = \left(n + \frac{1}{2}\right)\lambda$$

Question

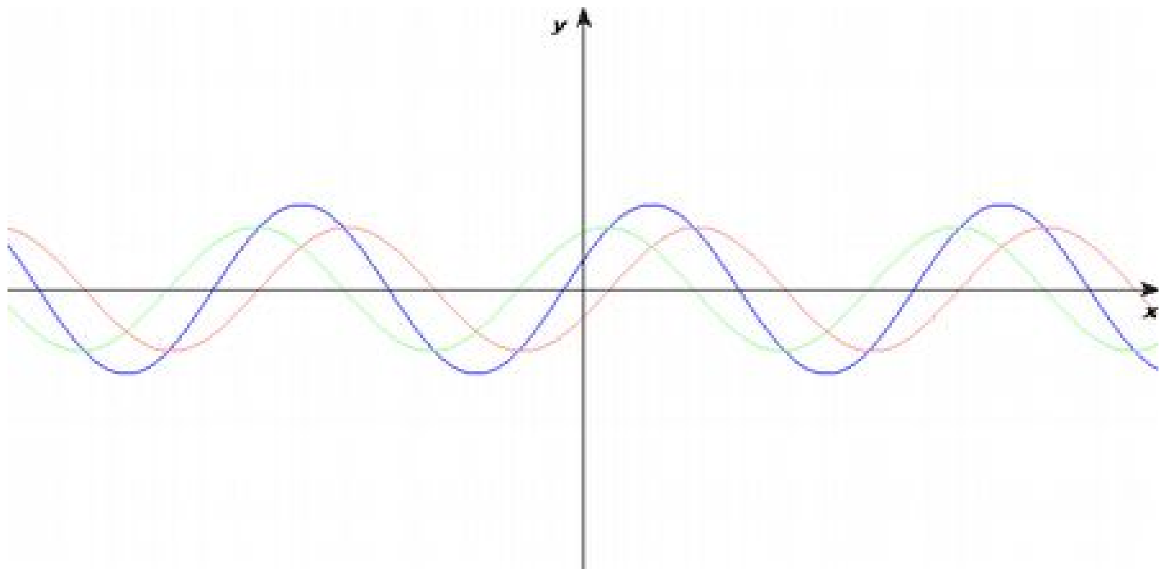
Two identical loud speakers placed 3.00 *m* apart are driven by the same oscillator. A listener is originally at point O, located 8.00 *m* from the center of the line connecting the two speakers. The listener then moves to point P, which is a perpendicular distance 0.350 *m* from O, and experiences the *first minimum* in sound intensity. What is the frequency of the oscillator?



The speed of sound in air is 343 m/s.

Standing waves...

- Consider two identical waves travelling in opposite directions.
- Interfere according to superposition principle:
  - Red wave →: $$y_1 = A\sin(kx - \omega t + \phi)$$
  - Green wave ←: $$y_2 = A\sin(kx + \omega t + \phi)$$
- The sum of the two waves (blue wave) is a **standing wave**, i.e., it does not move in the horizontal direction.

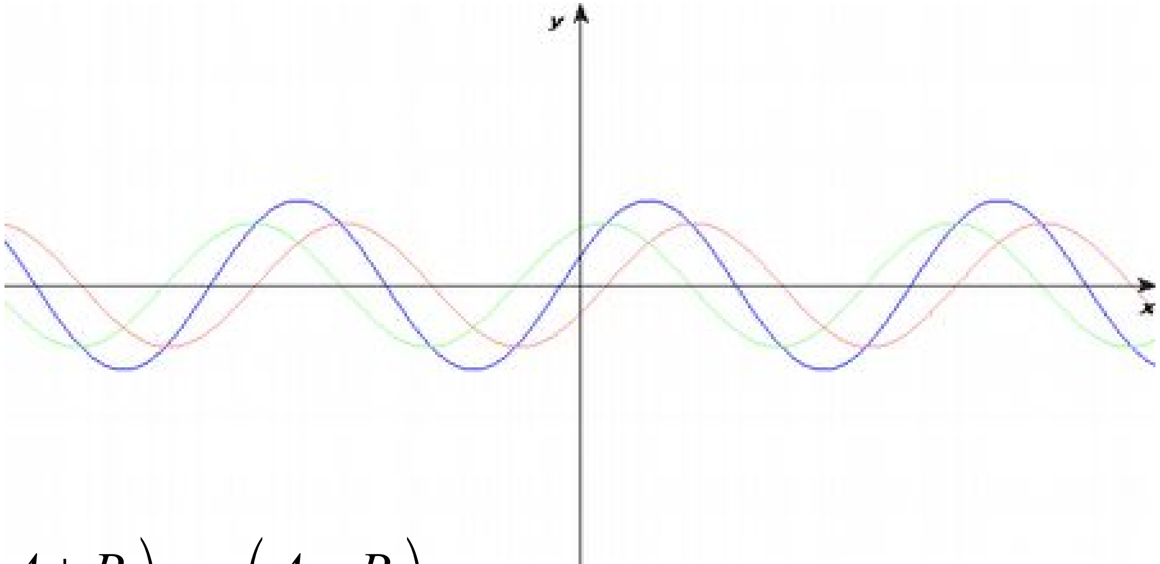


- Mathematical expression for a standing wave:

$$y = y_1 + y_2 = A\left[\sin(kx - \omega t + \phi) + \sin(kx + \omega t + \phi)\right]$$
$$= 2A\sin(kx + \phi)\cos(\omega t)$$

→ This wave function splits into a purely spatial part and a purely temporal part.

It is no longer in the form $f(kx - \omega t)$ and is therefore **not** a propagating wave.



Trigonometric identity

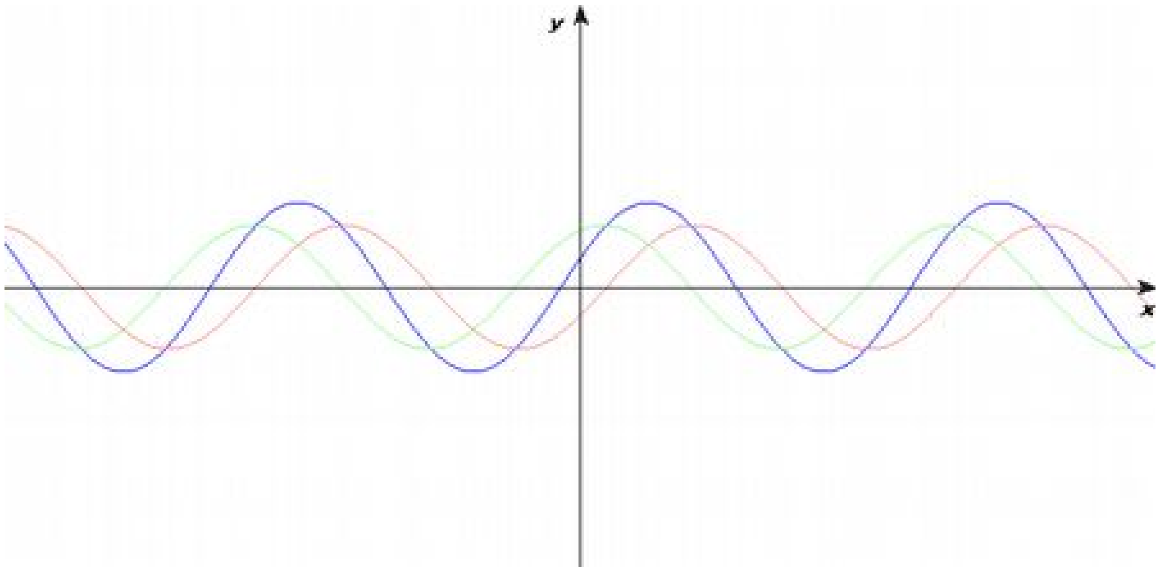
$$\sin A \pm \sin B = 2\sin\left(\frac{A \pm B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$

- Mathematical expression for a standing wave:

$$y = y_1 + y_2 = A\left[\sin(kx - \omega t + \phi) + \sin(kx + \omega t + \phi)\right]$$
$$= 2A\sin(kx + \phi)\cos(\omega t)$$

→ This wave function splits into a purely spatial part and a purely temporal part.

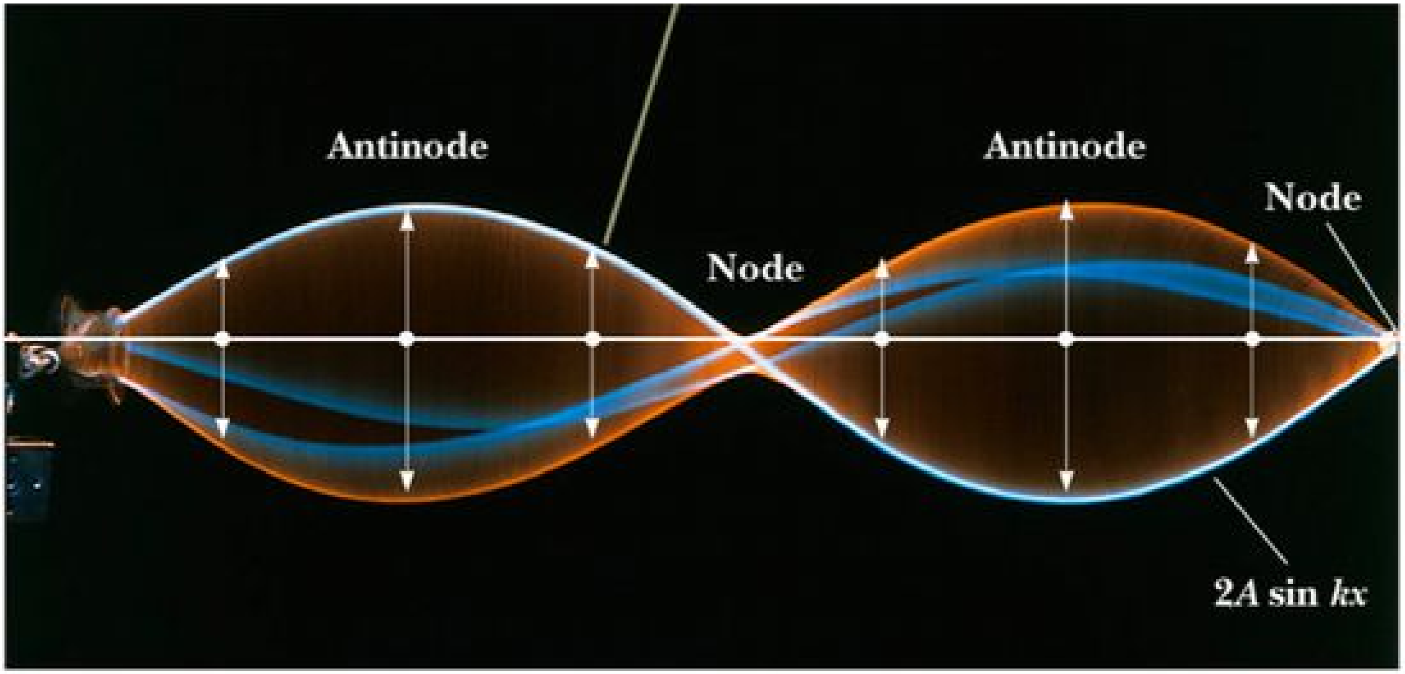
In observing a standing wave, there is no sense of motion in the direction of propagation of either of the original waves



What is longitudinal velocity of blue wave?
What is transverse velocity?

Standing waves on a string...

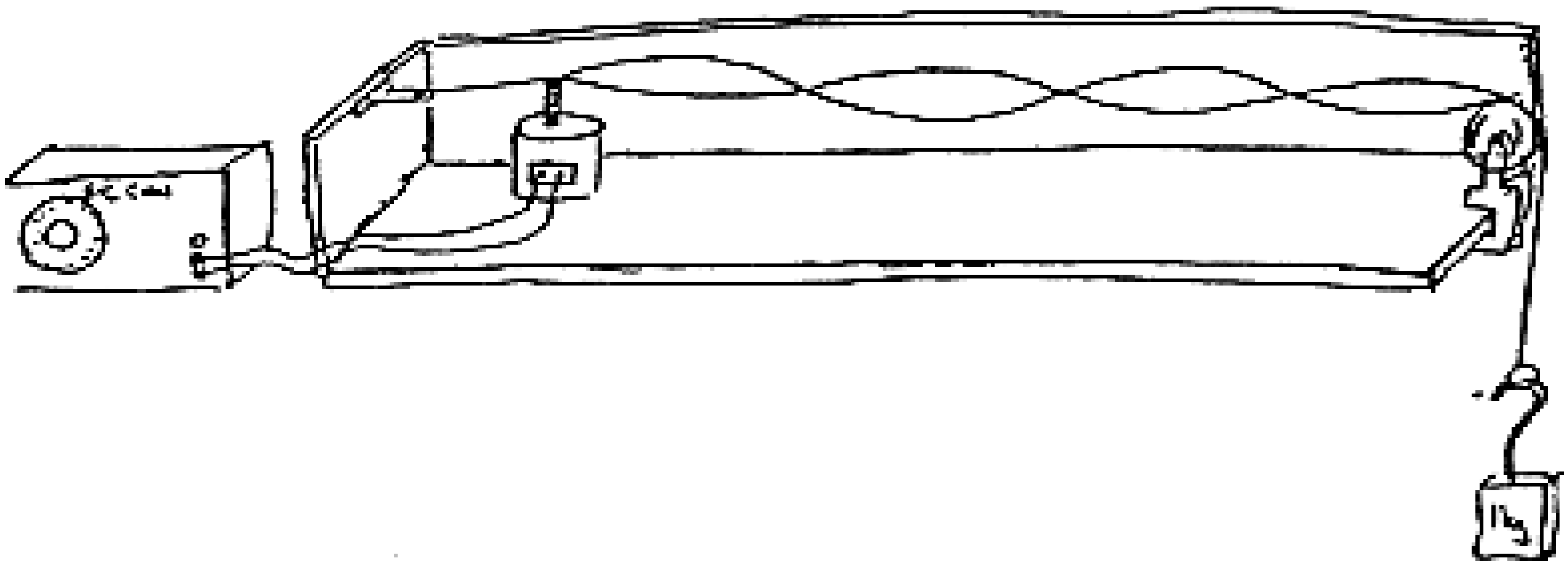
- We can set up standing waves on a string by a continuous superposition of waves incident on and reflected from two fixed ends of a string.



https://www.youtube.com/watch?v=no7ZPPqtZEg

Wa8: Standing Waves – vibration generator and wire

The different modes can be generated easily, and the dependence of the frequency on tension shown.



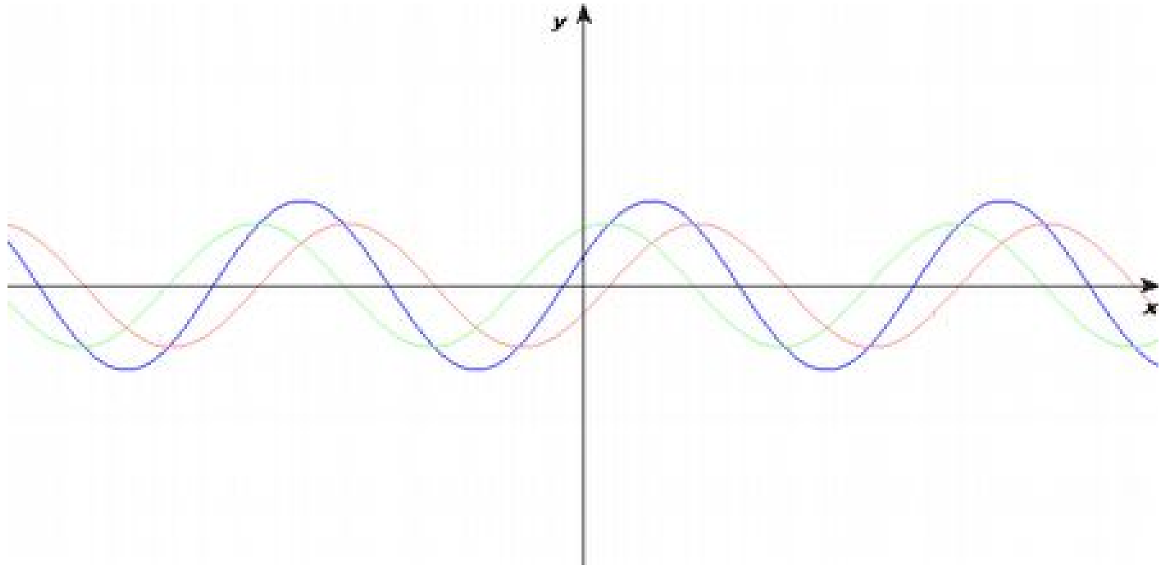
Standing waves and SHM...

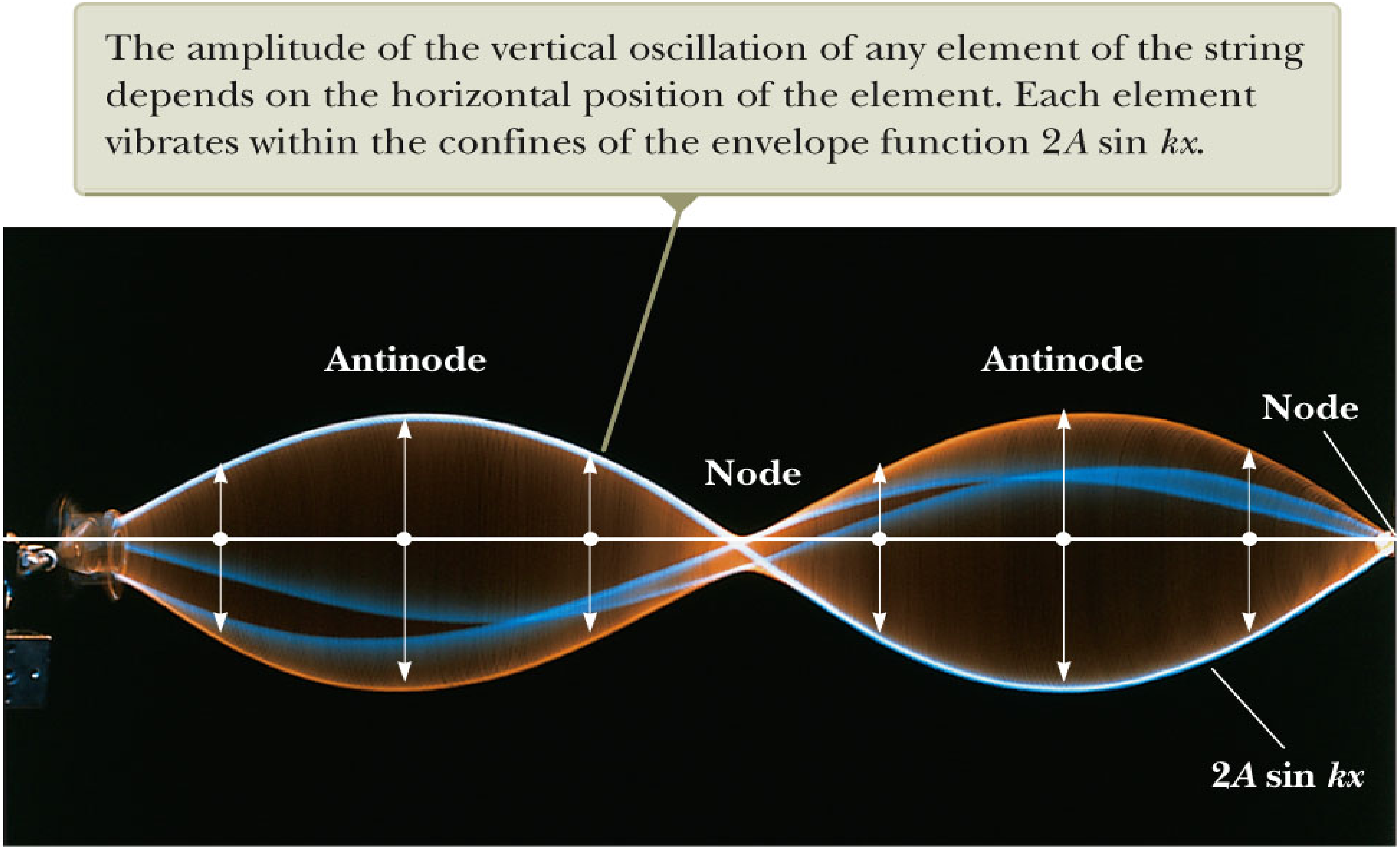
- A standing wave:

$$y(x,t) = 2A\sin(kx+\phi)\cos(\omega t)$$

- Every element in the medium oscillates in **simple harmonic motion** with the same frequency ω.

- However the **amplitude** of the SHM depends on the location of the element.




The amplitude of the vertical oscillation of any element of the string depends on the horizontal position of the element. Each element vibrates within the confines of the envelope function $2A \sin kx$.
Antinode
Antinode
Node
Node
$2A \sin kx$

Standing waves>Nodes...

- A **node** on a standing wave is a point x_n at which displacement of the medium has **zero amplitude**.

$$y(x_n, t) = 2A\sin(k x_n)\cos(\omega t) = 0$$

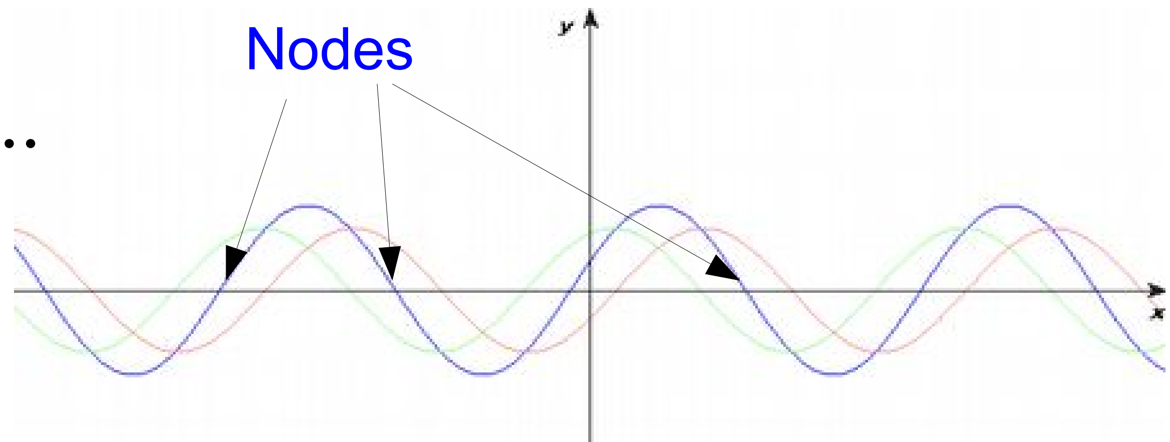
Assuming a zero phase constant

$$\Rightarrow \quad \sin(k x_n) = 0$$

$$\Rightarrow \quad k x_n = 0, \pi, 2\pi, 3\pi, \ldots$$

$k = 2\pi/\lambda$

$$\Rightarrow \quad \boxed{x_n = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2,} \ldots}$$



The separation between two adjacent nodes is $\lambda/2$.

Standing waves>Antinodes...

- A **antinode** on a standing wave is a point x_a at which the **maximum displacement** occurs.

$$y(x_a, t) = 2A\sin(k x_a)\cos(\omega t) = y_{max}$$

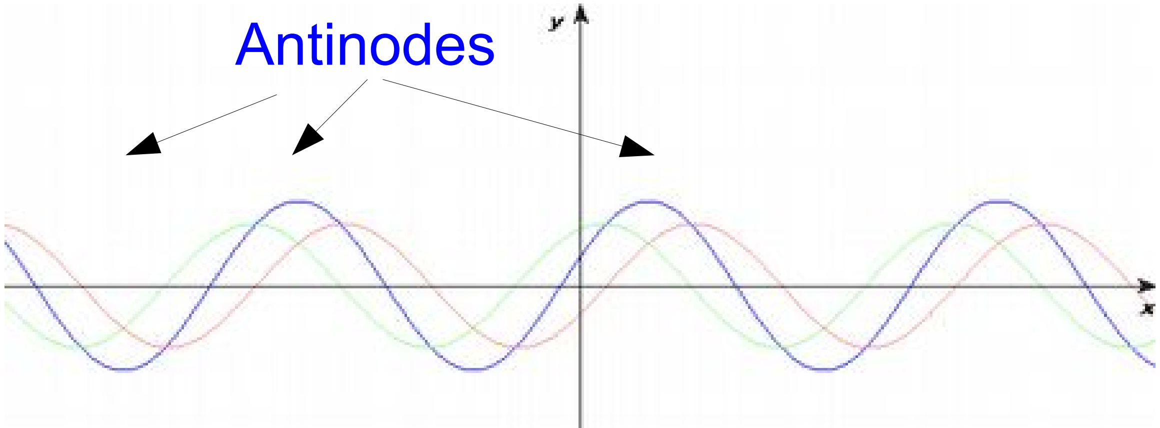
Assuming a zero phase constant

$$\Rightarrow \quad \sin(k x_a) = \pm 1$$

$$\Rightarrow \quad k x_a = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$k = 2\pi/\lambda$$

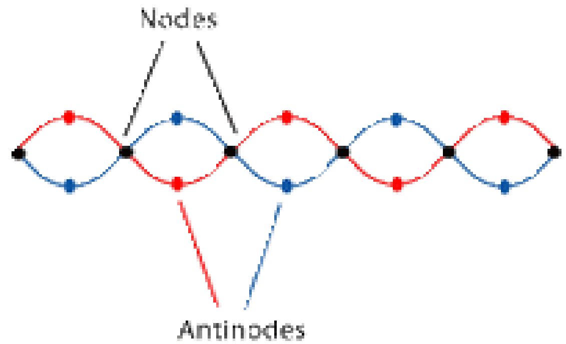
$$\Rightarrow \quad \boxed{x_a = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots}$$



The separation between two adjacent antinodes is $\lambda/2$.

Amplitudes relevant to describing wave motion

- Three types of amplitudes exist:
  - The amplitude of the individual waves, A
  - The amplitude of the simple harmonic motion of the elements in the medium, $2A \sin kx$
  - The amplitude of the standing wave, $2A$
    - A given element in a standing wave vibrates within the constraints of the envelope function $2A\sin kx$, where x is the position of the element in the medium



https://goo.gl/forms/FQak2zRLiJviaMaQ2

Question 1...

- Two waves travelling in opposite directions produce a standing wave. The individual wavefunctions are:

$$y_1(x,t) = 4.0\sin(3.0x - 2.0t)$$
$$y_2(x,t) = 4.0\sin(3.0x + 2.0t)$$

where x and y are measured in cm and t is in s.

1. Find the amplitude of the SHM of the element of the medium located at $x = 2.3\ \text{cm}$.

2. Find the position of the nodes and antinodes if one end of the string is at $x = 0$.

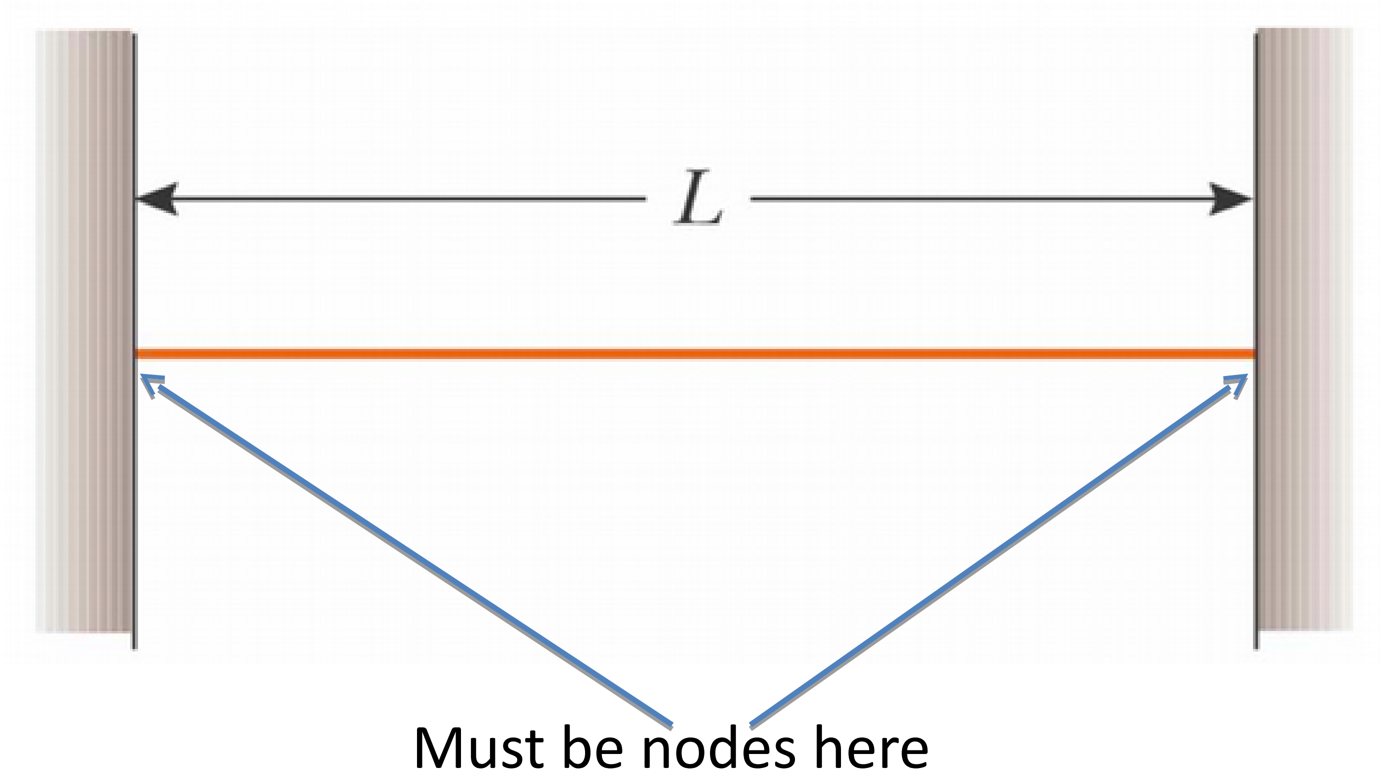
BLACKBOARD

Ruben’s Tube with MOAR FIRE

When you have enough gas pressure in the tube, it is more impressive

Standing waves on a string, again...

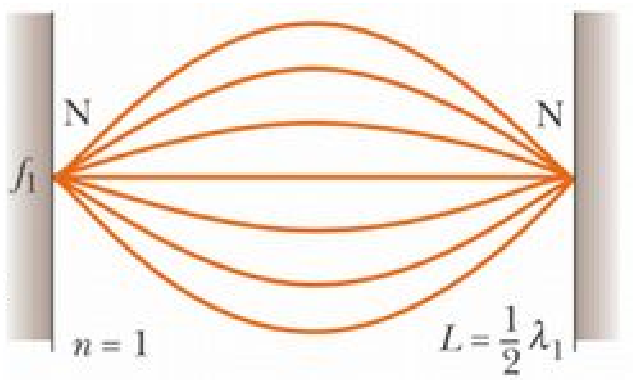
- Consider a string fixed at both ends.



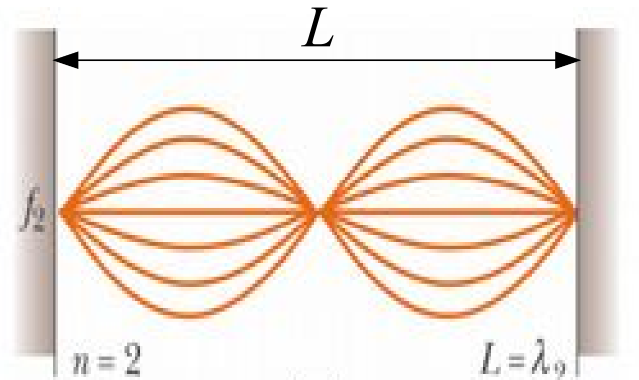
- **Boundary condition**: Any standing wave set up on this string must have nodes at the two ends.

Standing waves>What are the possibilities?

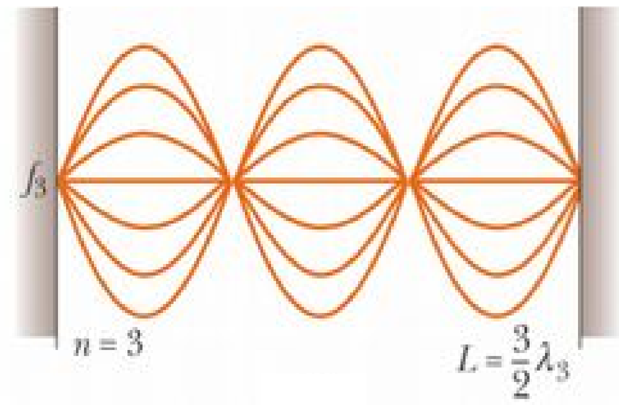
- Those oscillation modes that satisfy the boundary condition are called the **normal modes**, and there's literally an infinite number of them...



Fundamental,
or first harmonic



Second harmonic



Third harmonic

… and so on...

Standing waves>Normal modes...

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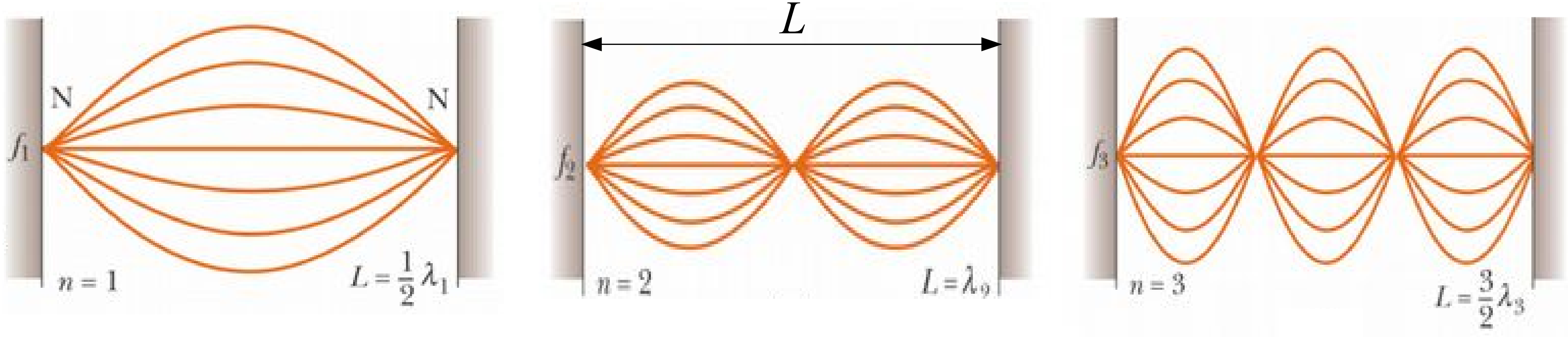


Fundamental,
or first harmonic

Second harmonic

Third harmonic

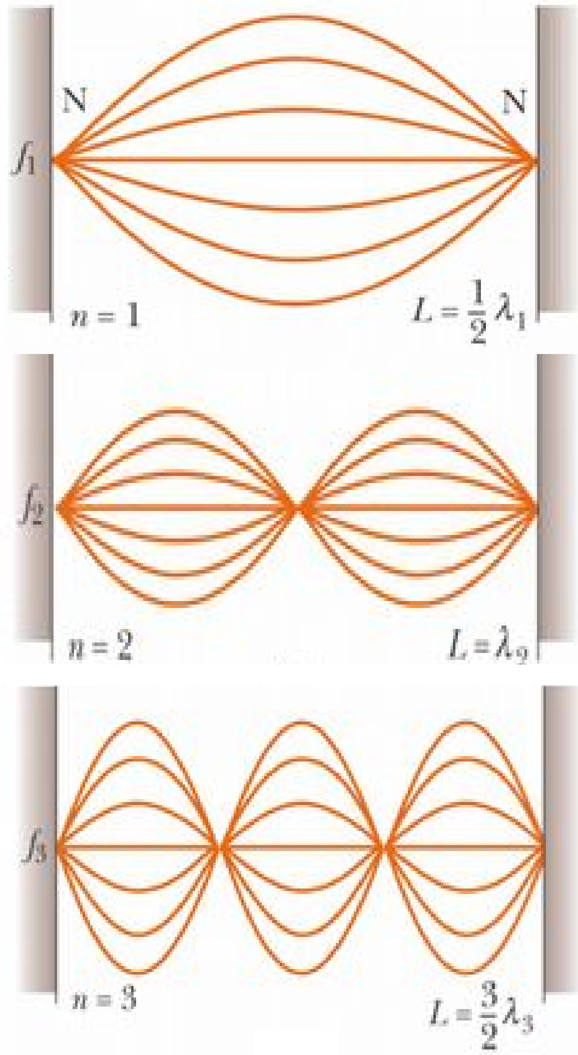
Wavelengths: $\lambda_1 = 2L$ $\qquad \lambda_2 = L$ $\qquad \lambda_3 = \frac{2L}{3}$

The longest possible for this string

Standing waves>Normal modes...

- The nth harmonic has **wavelength**:

$$\lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3, \ldots$$



Standing waves>Normal modes...

- The nth harmonic has **wavelength**:

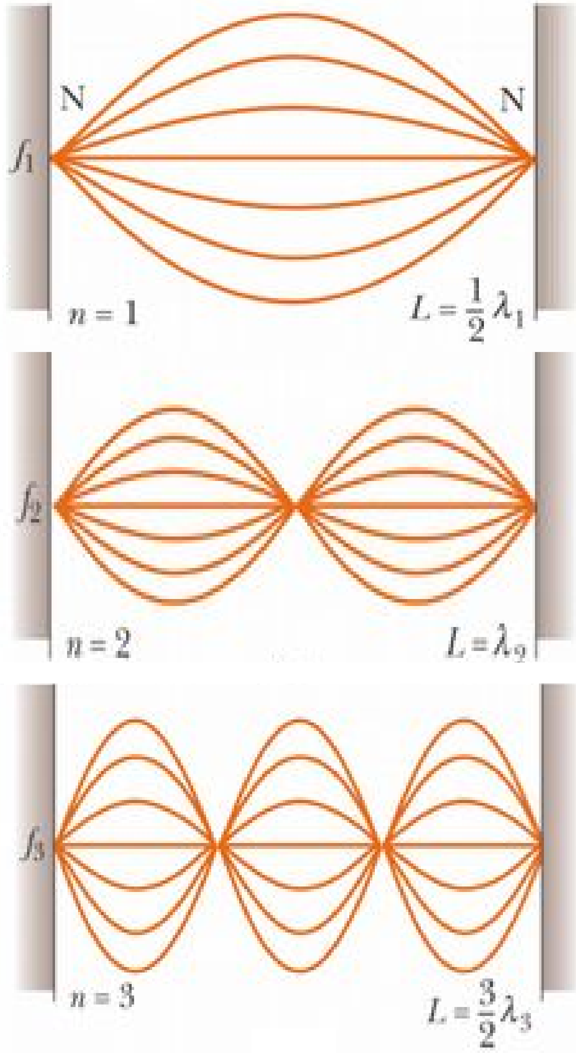
$$\lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3, \ldots$$

and **frequency**:

Wave speed on string

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{2L} = n f_1 \qquad v = \sqrt{\frac{T}{\mu}}$$

Fundamental frequency



Standing waves>Normal modes...

- The nth harmonic has **wavelength**:

$$\lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3, \ldots$$

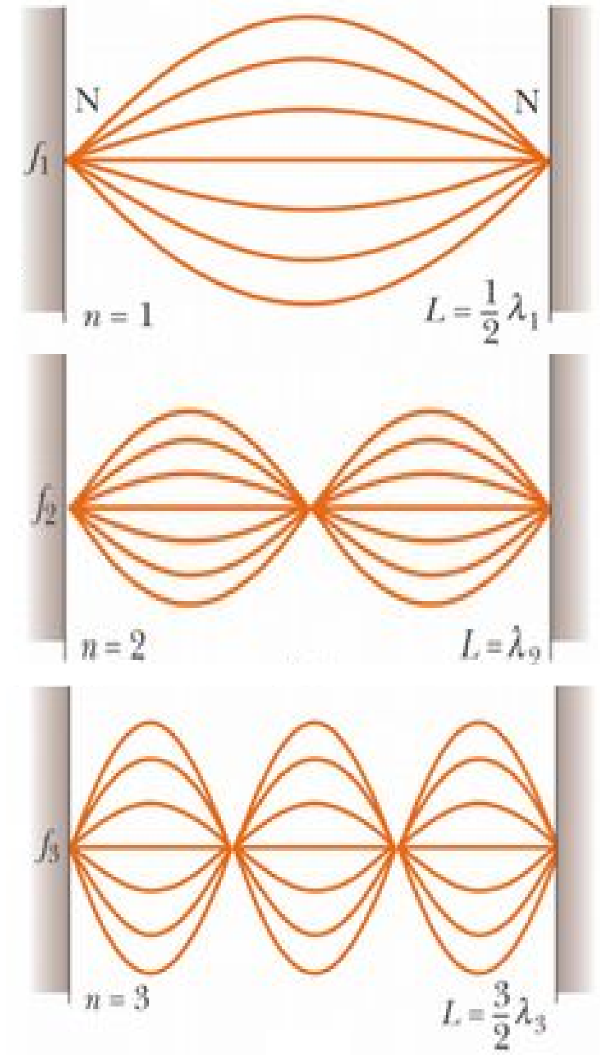
and **frequency**:

Wave speed on string

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{2L} = n f_1 \qquad v = \sqrt{\frac{T}{\mu}}$$

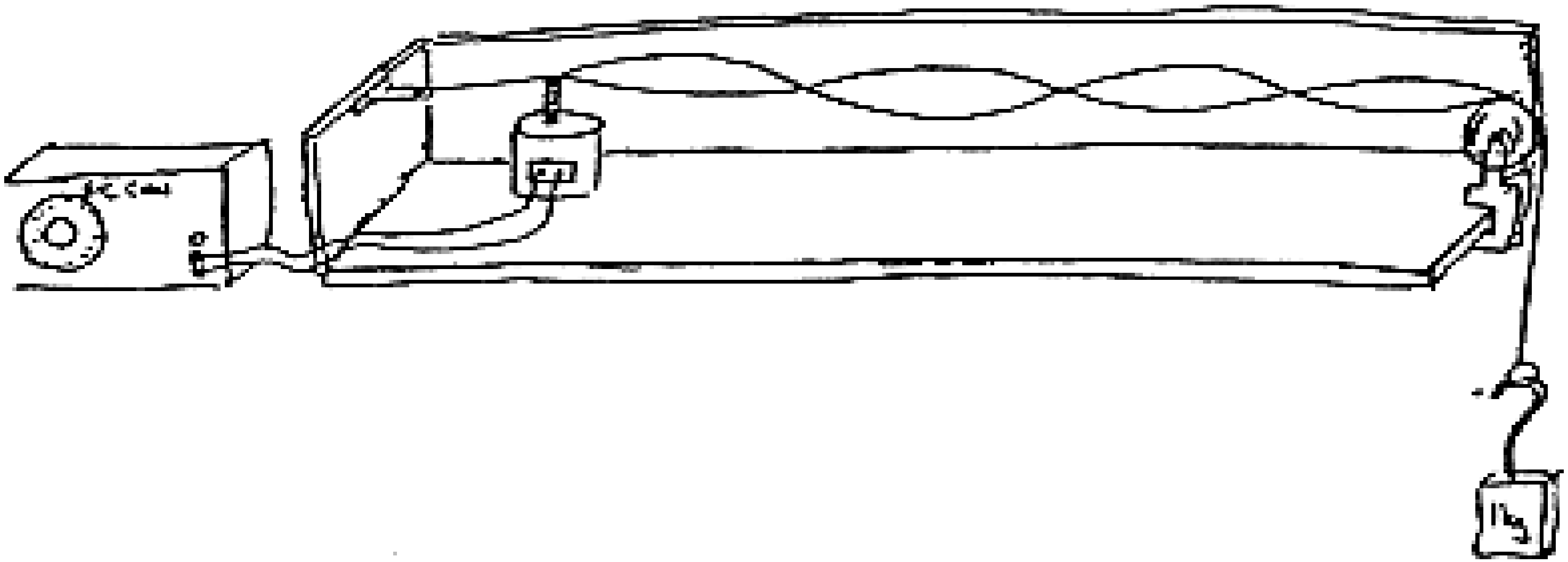
Fundamental frequency

→ The frequencies of all other allowed modes are **integer multiples** of the fundamental frequency.



Wa8: Standing Waves – vibration generator and wire

The different modes can be generated easily, and the dependence of the frequency on tension shown.



Definitions

Normal modes: describes the way in which a string fixed at both ends can vibrate, the fundamental mode, second harmonic, third harmonic etc.

Quantization: When only certain frequencies of oscillations are allowed we say a system is quantized.

Quantisation is a common occurrence when waves are subject to boundary conditions → the basis of Quantum Mechanics https://goo.gl/forms/FQak2zRLiJviaMaQ2

Quick quiz...

- When a standing wave is set up on a string fixed at both ends, which of the following statements is true?

1. The number of nodes is equal to the number of antinodes.
2. The wavelength is equal to the length of the string divided by an integer.
3. The frequency is equal to the number of nodes times the fundamental frequency.
4. The shape of the string at any instant shows a symmetry about the midpoint of the string.

Question 2...

BLACKBOARD

- A standing wave pattern is observed in a thin wire with a length 3.00 m. The wavefunction is

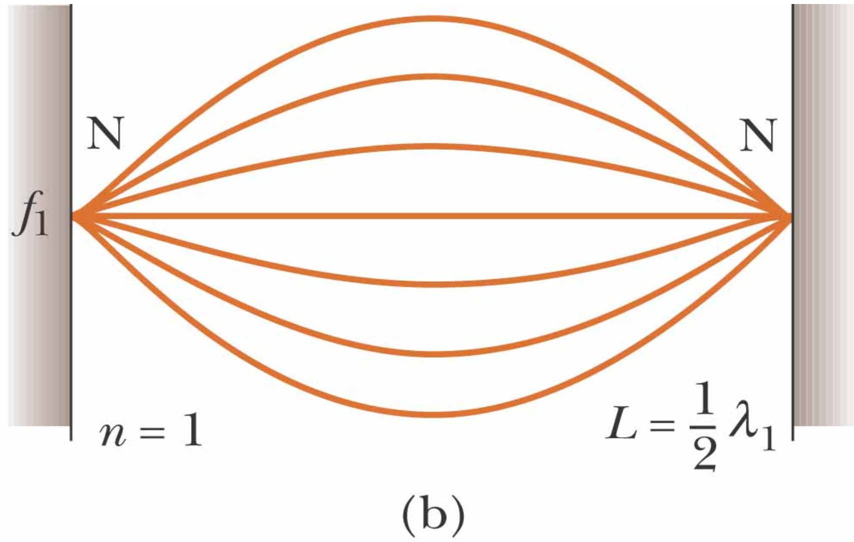
$$y(x,t)=0.00200\sin(\pi x)\cos(100\pi t)$$

where x and y are measured in m and t is in s.

1. How many antinodes does this pattern exhibit?
2. What is the fundamental frequency of vibration of the wire?
3. If the original frequency is held constant and the tension in the wire is increased by a factor of 9, how many antinodes are present in the new pattern?

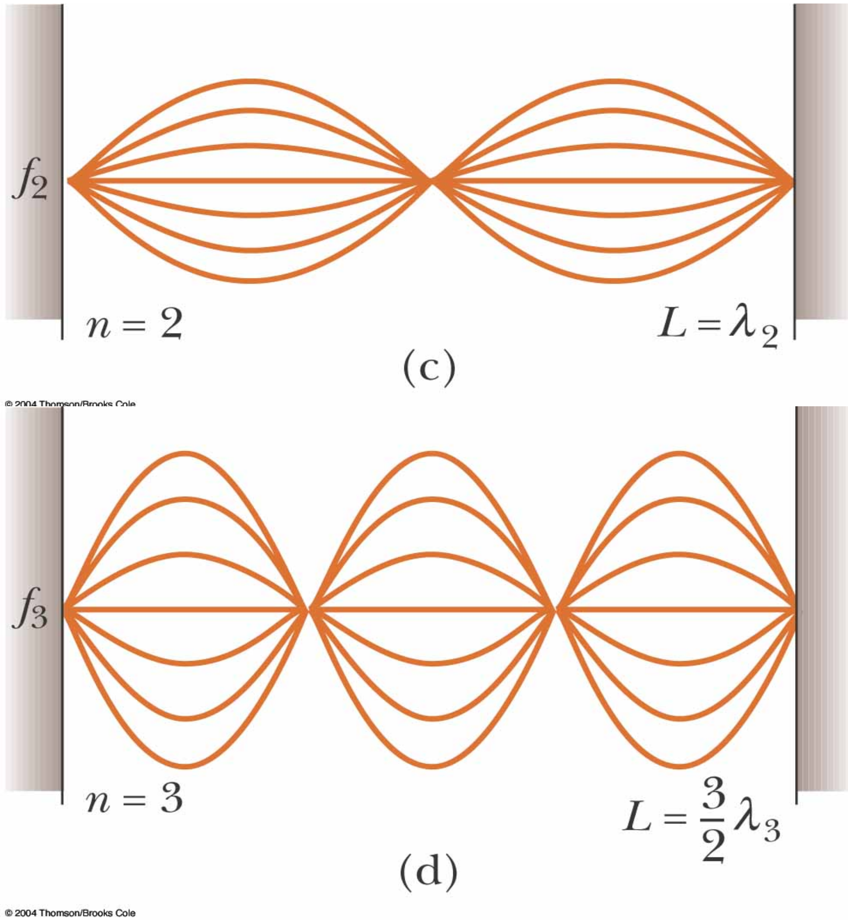
Standing Waves in a String: II

- This is the first normal mode that is consistent with the boundary conditions
- There are nodes at both ends
- There is one antinode in the middle
- This is the longest wavelength mode
  - ½λ = L so λ = 2L



Standing Waves in String: III

- For consecutive normal modes add an antinode at each step
  - i.e. second mode (c) corresponds to to $\lambda = L$
  - i.e. third mode (d) corresponds to $\lambda = 2L/3$



Standing Waves on String: IV

- The wavelengths of the normal modes for a string of length *L* fixed at both ends are

  $\lambda_n = 2L / n \qquad n = 1, 2, 3, \ldots$

  – *n* is the n^{th} normal mode of oscillation
- Since $v=f\lambda$, the natural frequencies are

$$f_n = n\frac{v}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\mu}} = nf_1$$