Chapter 15

Oscillations

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15-1 Simple Harmonic Motion

Learning Objectives

15.01 Distinguish simple harmonic motion from other types of periodic motion.

15.02 For a simple harmonic oscillator, apply the relationship between position x and time t to calculate either if given a value for the other.

15.03 Relate period T, frequency f, and angular frequency ω.

15.04 Identify (displacement) amplitude x_m, phase constant (or phase angle) ϕ, and phase $\omega t + \phi$.

15.05 Sketch a graph of the oscillator's position x versus time t, identifying amplitude x_m and period T.

15.06 From a graph of position versus time, velocity versus time, or acceleration versus time, determine the amplitude of the plot and the value of the phase constant ϕ.

15-1 Simple Harmonic Motion

15.07 On a graph of position x versus time t describe the effects of changing period T, frequency f, amplitude $x_{m,}$ or phase constant ϕ.

15.08 Identify the phase constant ϕ that corresponds to the starting time (t=0) being set when a particle in SHM is at an extreme point or passing through the center point.

15.09 Given an oscillator's position $x(t)$ as a function of time, find its velocity $v(t)$ as a function of time, identify the velocity amplitude v_m in the result, and calculate the velocity at any given time.

15.10 Sketch a graph of an oscillator's velocity v versus time t, identifying the velocity amplitude $v_{m.}$

15-1 Simple Harmonic Motion

15.11 Apply the relationship between velocity amplitude v_m, angular frequency ω, and (displacement) x_m.

15.12 Given an oscillator's velocity *v(t)* as a function of time, calculate its acceleration *a(t)* as a function of time, identify the acceleration amplitude a_m in the result, and calculate the acceleration at any given time.

15.13 Sketch a graph of an oscillator's acceleration *a* versus time *t*, identifying the acceleration amplitude a_m.

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15-1 Simple Harmonic Motion

Learning Objectives continued

15.14 Identify that for a simple harmonic oscillator the acceleration *a* at any instant is *always* given by the product of a negative constant and the displacement *x* just then.

15.15 For any given instant in an oscillation, apply the relationship between acceleration *a*, angular frequency ω, and displacement *x*.

15.16 Given data about the position *x* and velocity *v* at one instant, determine the phase $\omega t + \phi$ and phase constant ϕ.

Learning Objectives Continued

15.17 For a spring-block oscillator, apply the relationships between spring constant k and mass m and either period T or angular frequency ω.

15.18 Apply Hooke's law to relate the force F on a simple harmonic oscillator at any instant to the displacement x of the oscillator at that instant.

15-1 Simple Harmonic Motion

- The **frequency** of an oscillation is the number of times per second that it completes a full oscillation (cycle)
- Unit of hertz: 1 Hz = 1 oscillation per second
- The time in seconds for one full cycle is the **period**

$$T = \frac{1}{f}.$$ **Eq. (15-2)**

- Any motion that repeats regularly is called periodic
- **Simple harmonic motion** is periodic motion that is a sinusoidal function of time

$$x(t) = x_m \cos(\omega t + \phi)$$ **Eq. (15-3)**

15-1 Simple Harmonic Motion

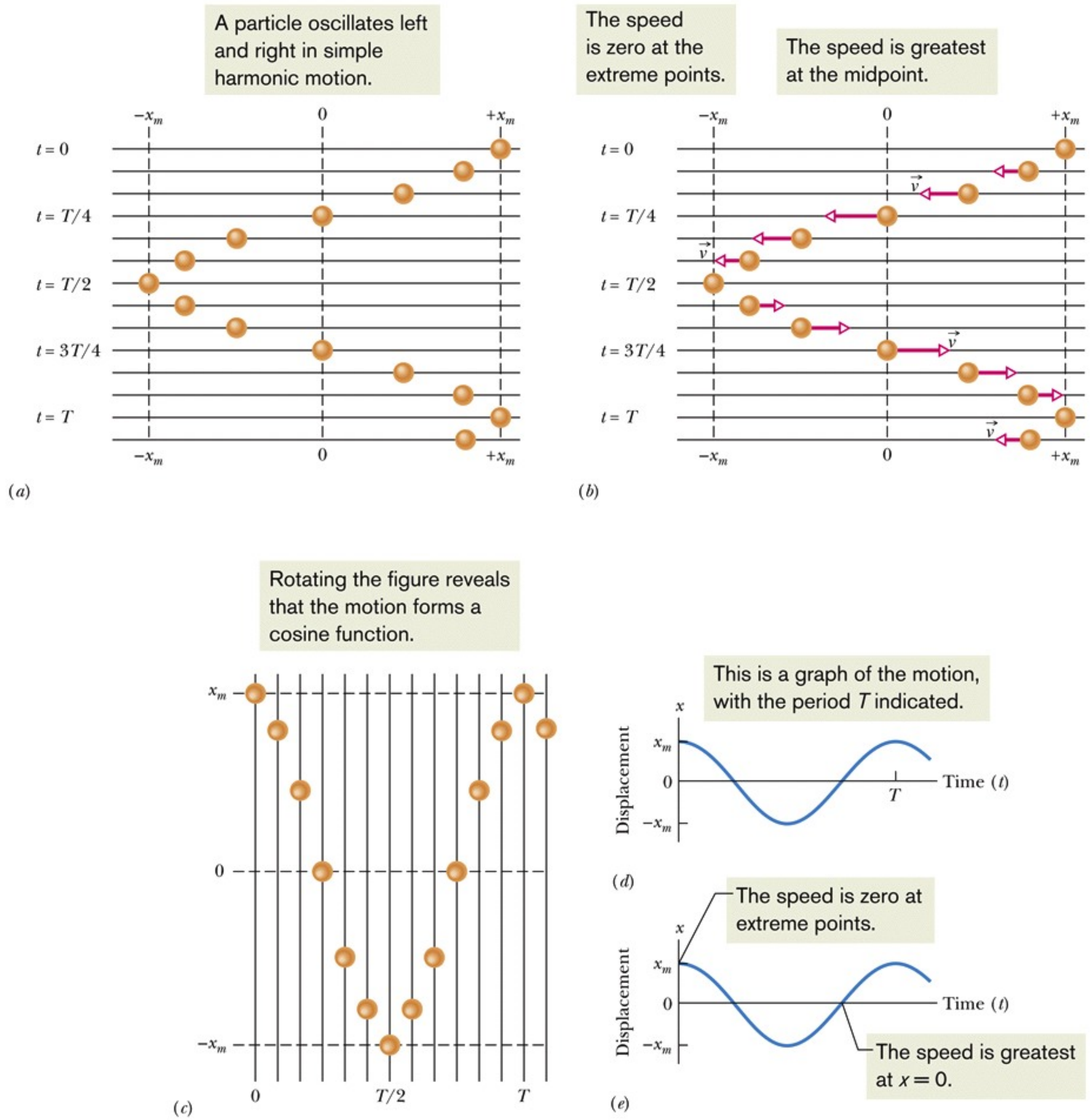

Figure 15-2

15-1 Simple Harmonic Motion

- The value written x_m is how far the particle moves in either direction: the **amplitude**
- The argument of the cosine is the **phase**
- The constant φ is called the **phase angle** or phase constant
- It adjusts for the initial conditions of motion at $t = 0$
- The **angular frequency** is written ω

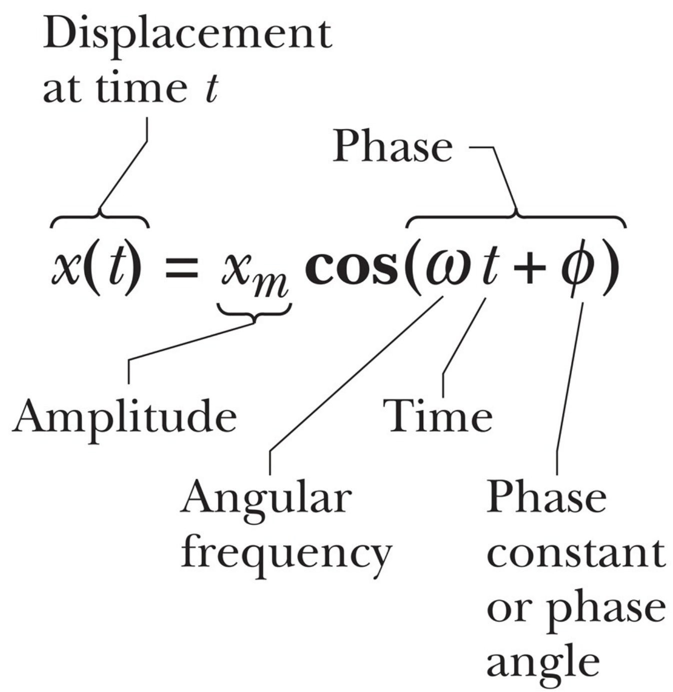

Figure 15-3

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15-1 Simple Harmonic Motion

- The angular frequency has the value:

$$\omega = \frac{2\pi}{T} = 2\pi f.$$ **Eq. (15-5)**

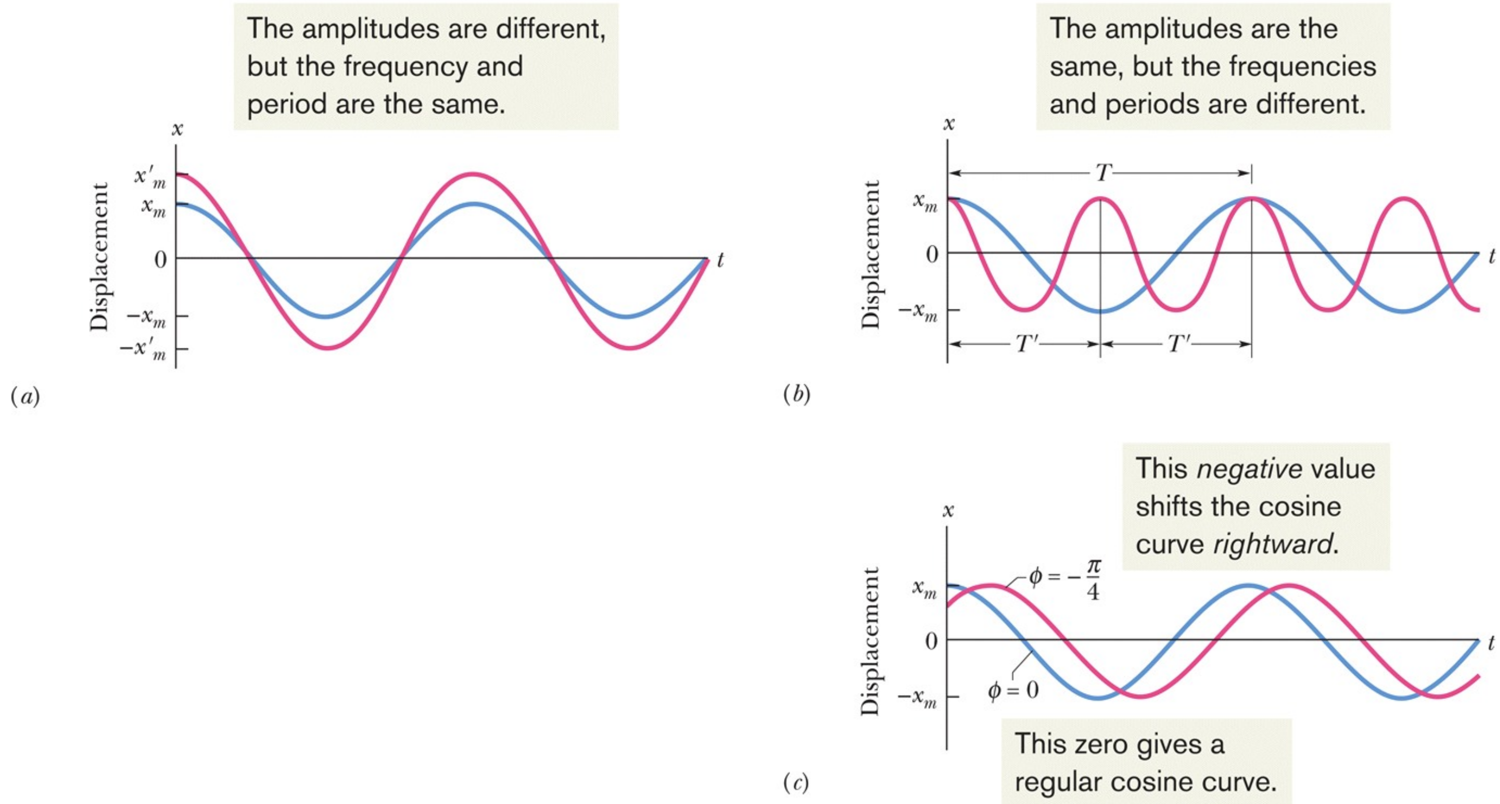


Figure 15-5

15-1 Simple Harmonic Motion

Checkpoint 1

A particle undergoing simple harmonic oscillation of period T (like that in Fig. 15-2) is at $-x_m$ at time $t = 0$. Is it at $-x_m$, at $+x_m$, at 0, between $-x_m$ and 0, or between 0 and $+x_m$ when (a) $t = 2.00T$, (b) $t = 3.50T$, and (c) $t = 5.25T$?

Answer: (a) at $-x_m$ (b) at x_m (c) at 0

- The velocity can be found by the time derivative of the position function:

$$v(t) = -\omega x_m \sin(\omega t + \phi)$$

Eq. (15-6)

- The value ωx_m is the **velocity amplitude** v_m

15-1 Simple Harmonic Motion

- The acceleration can be found by the time derivative of the velocity function, or 2nd derivative of position:

$$a(t) = -\omega^2 x_m \cos(\omega t + \phi)$$

Eq. (15-7)

- The value $\omega^2 x_m$ is the **acceleration amplitude** a_m
- Acceleration related to position:

$$a(t) = -\omega^2 x(t).$$

Eq. (15-8)

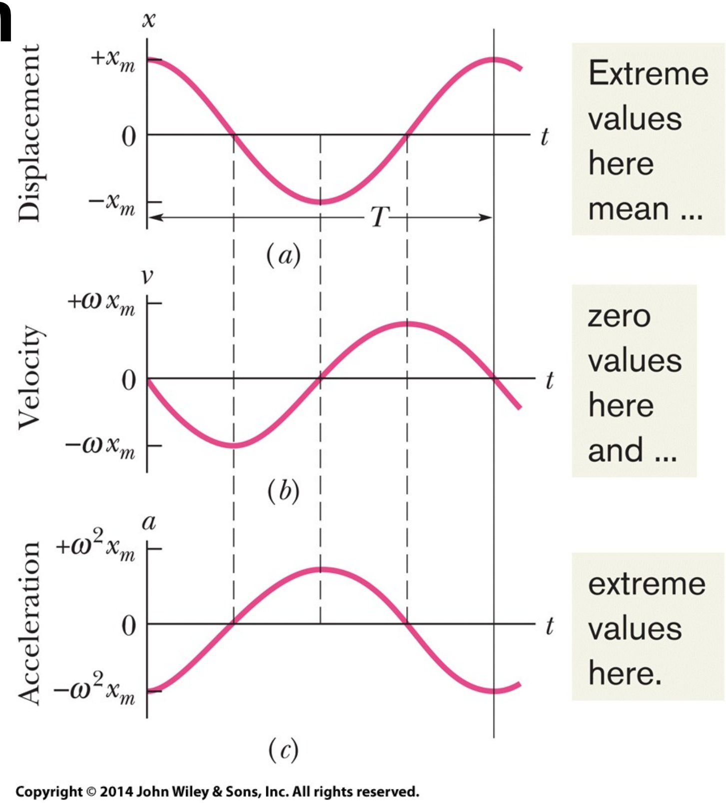


Figure 15-6

15-1 Simple Harmonic Motion

In SHM, the acceleration a is proportional to the displacement x but opposite in sign, and the two quantities are related by the square of the angular frequency ω.

Checkpoint 2

Which of the following relationships between a particle's acceleration a and its position x indicates simple harmonic oscillation: (a) $a = 3x^2$, (b) $a = 5x$, (c) $a = -4x$, (d) $a = -2/x$? For the SHM, what is the angular frequency (assume the unit of rad/s)?

Answer: (c) where the angular frequency is 2

15-1 Simple Harmonic Motion

- We can apply Newton's second law

$$F = ma = m(-\omega^2 x) = -(m\omega^2)x. \qquad \text{Eq. (15-9)}$$

- Relating this to Hooke's law we see the similarity

> Simple harmonic motion is the motion of a particle when the force acting on it is proportional to the particle's displacement but in the opposite direction.

- **Linear simple harmonic oscillation** (*F* is proportional to *x* to the first power) gives:

$$\omega = \sqrt{\frac{k}{m}} \quad \text{(angular frequency).} \qquad \text{Eq. (15-12)}$$

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{(period).} \qquad \text{Eq. (15-13)}$$

15-1 Simple Harmonic Motion

Checkpoint 3

Which of the following relationships between the force F on a particle and the particle's position x gives SHM: (a) $F = -5x$, (b) $F = -400x^2$, (c) $F = 10x$, (d) $F = 3x^2$?

Answer: only (a) is simple harmonic motion

(note that b is harmonic motion, but nonlinear and not SHM)

15-2 Energy in Simple Harmonic Motion

Learning Objectives

15.19 For a spring-block oscillator, calculate the kinetic energy and elastic potential energy at any given time.

15.20 Apply the conservation of energy to relate the total energy of a spring-block oscillator at one instant to the total energy at another instant.

15.21 Sketch a graph of the kinetic energy, potential energy, and total energy of a spring-block oscillator, first as a function of time and then as a function of the oscillator's position.

15.22 For a spring-block oscillator, determine the block's position when the total energy is entirely kinetic energy and when it is entirely potential energy.

15-2 Energy in Simple Harmonic Motion

- Write the functions for kinetic and potential energy:

$$U(t) = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kx_m^2 \cos^2(\omega t + \phi).$$

Eq. (15-18)

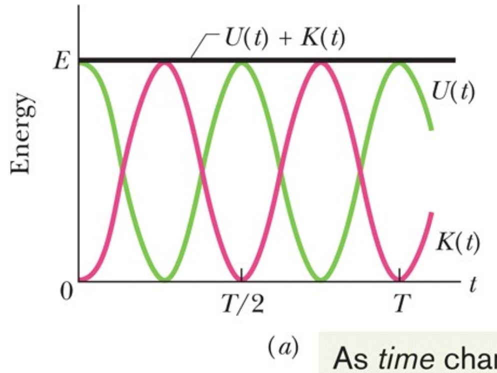
$$K(t) = \tfrac{1}{2}mv^2 = \tfrac{1}{2}kx_m^2 \sin^2(\omega t + \phi).$$

Eq. (15-20)

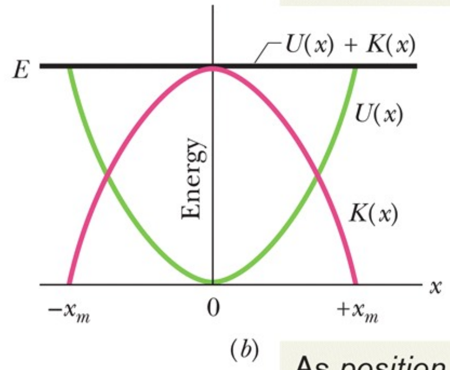
- Their sum is defined by:

$$E = U + K = \tfrac{1}{2}kx_m^2.$$

Eq. (15-21)



As *time* changes, the energy shifts between the two types, but the total is constant.



As *position* changes, the energy shifts between the two types, but the total is constant.

Figure 15-8

15-2 Energy in Simple Harmonic Motion

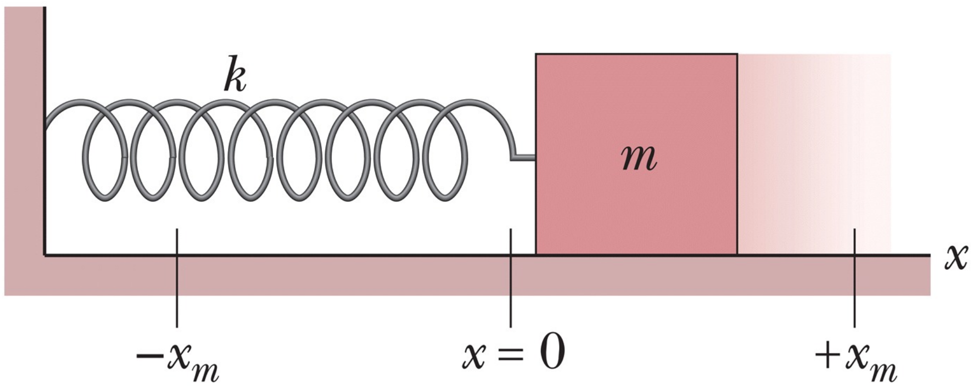


Figure 15-7

Checkpoint 4

In Fig. 15-7, the block has a kinetic energy of 3 J and the spring has an elastic potential energy of 2 J when the block is at $x = +2.0$ cm. (a) What is the kinetic energy when the block is at $x = 0$? What is the elastic potential energy when the block is at (b) $x = -2.0$ cm and (c) $x = -x_m$?

Answer: (a) 5 J (b) 2 J (c) 5 J

15-3 An Angular Simple Harmonic Oscillator

Learning Objectives

15.23 Describe the motion of an angular simple harmonic oscillator.

15.24 For an angular simple harmonic oscillator, apply the relationship between the torque τ and the angular displacement θ (from equilibrium).

15.25 For an angular simple harmonic oscillator, apply the relationship between the period T (or frequency f), the rotational inertia I, and the torsion constant κ.

15.26 For an angular simple harmonic oscillator at any instant, apply the relationship between the angular acceleration α, the angular frequency ω, and the angular displacement θ.

15-3 An Angular Simple Harmonic Oscillator

- A **torsion pendulum**: elasticity from a twisting wire
- Moves in **angular simple harmonic motion**

$$\tau = -\kappa\theta. \qquad \textbf{Eq. (15-22)}$$

- κ is called the torsion constant
- Angular form of Hooke's law
- Replace linear variables with their angular analogs and
- we find:

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \qquad \textbf{Eq. (15-23)}$$

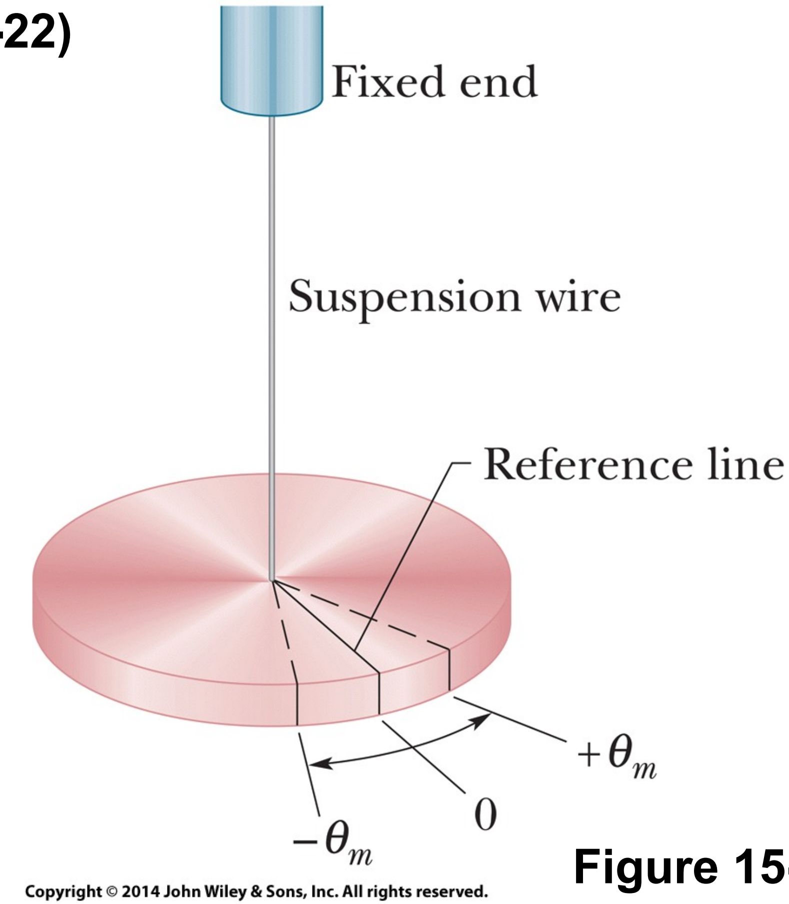


Figure 15-9

15-4 Pendulums, Circular Motion

Learning Objectives

15.27 Describe the motion of an oscillating simple pendulum.

15.28 Draw a free-body diagram.

15.29-31 Distinguish between a simple and physical pendulum, and relate their variables.

15.32 Find angular frequency from torque and angular displacement or acceleration and displacement.

15.33 Distinguish angular frequency from $d\theta/dt$.

15.34 Determine phase and amplitude.

15.35 Describe how free-fall acceleration can be measured with a pendulum.

15.36 For a physical pendulum, find the center of the oscillation.

15.37 Relate SHM to uniform circular motion.

15-4 Pendulums, Circular Motion

- A **simple pendulum**: a *bob* of mass *m* suspended from an unstretchable, massless string
- Bob feels a restoring torque:

$$\tau = -L(F_g \sin\theta), \qquad \text{Eq. (15-24)}$$

- Relating this to moment of inertia:

$$\alpha = -\frac{mgL}{I}\theta. \qquad \text{Eq. (15-26)}$$

- Angular acceleration proportional to position but opposite in sign

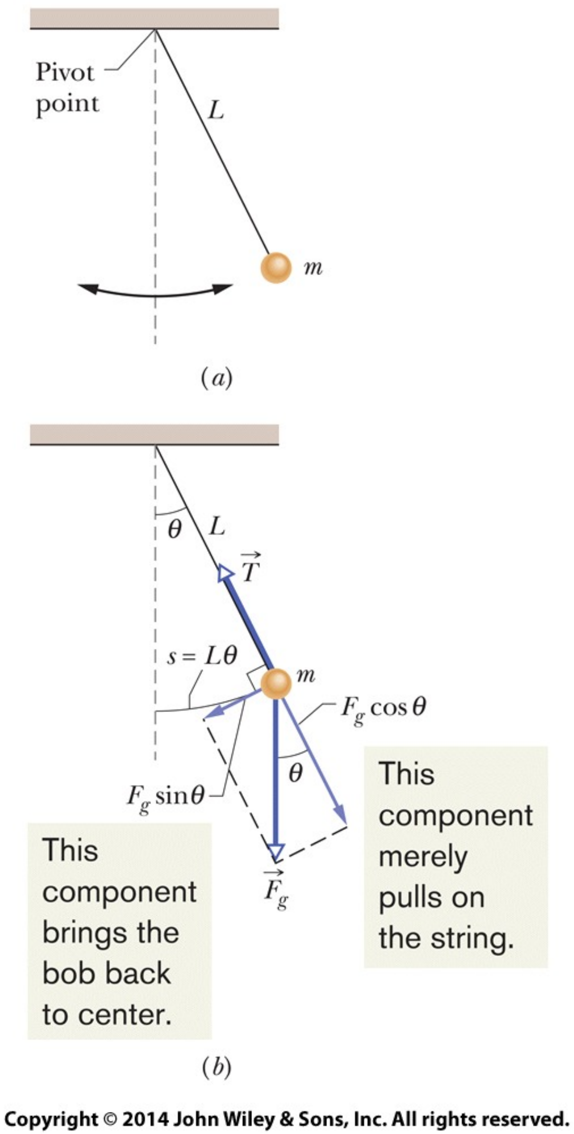


Figure 15-11

15-4 Pendulums, Circular Motion

- **Angular amplitude** θ_m of the motion must be small
- The angular frequency is:

$$\omega = \sqrt{\frac{mgL}{I}}.$$

- The period is (for simple pendulum, $I = mL^2$):

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Eq. (15-28)

- A **physical pendulum** has a complicated mass distribution

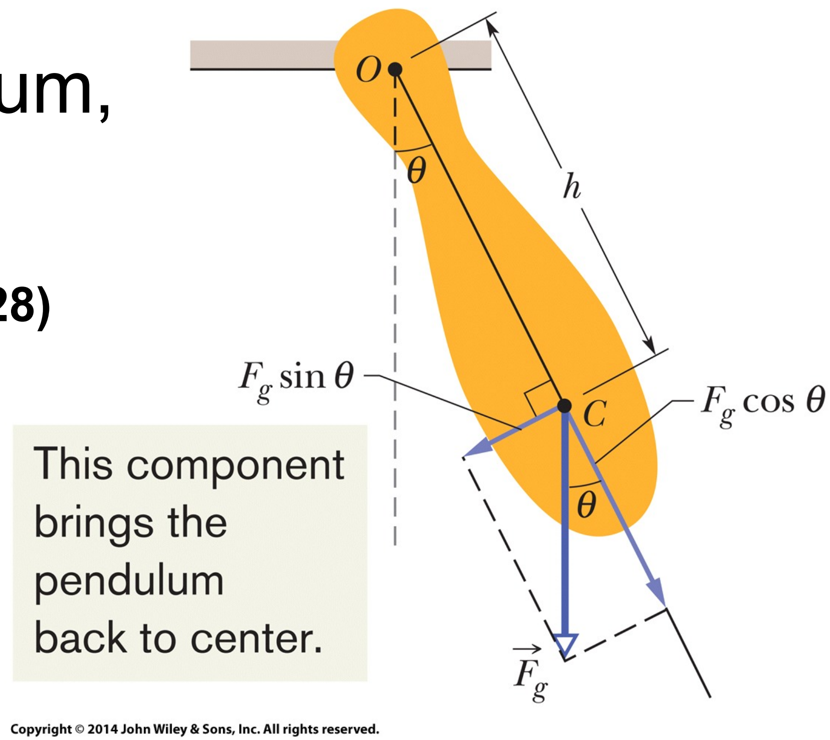


Figure 15-12

15-4 Pendulums, Circular Motion

- An analysis is the same except rather than length *L* we have distance *h* to the com, and *I* will be particular to the mass distribution
- The period is:

$$T = 2\pi\sqrt{\frac{I}{mgh}}$$

Eq. (15-29)

- A physical pendulum will not show SHM if pivoted about its com
- The *center of oscillation* of a physical pendulum is the length L_0 of a simple pendulum with the same period

15-4 Pendulums, Circular Motion

- A physical pendulum can be used to determine free-fall acceleration *g*
- Assuming the pendulum is a uniform rod of length L:

$$I = I_{\text{com}} + mh^2 = \tfrac{1}{12}mL^2 + m(\tfrac{1}{2}L)^2 = \tfrac{1}{3}mL^2.$$

Eq. (15-30)

- Then solve Eq. 15-29 for *g*:

$$g = \frac{8\pi^2 L}{3T^2}.$$

Eq. (15-31)

Checkpoint 5

Three physical pendulums, of masses m_0, $2m_0$, and $3m_0$, have the same shape and size and are suspended at the same point. Rank the masses according to the periods of the pendulums, greatest first.

Answer: All the same: mass does not affect the period of a pendulum

15-4 Pendulums, Circular Motion

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- Simple harmonic motion is circular motion viewed edge-on

> Simple harmonic motion is the projection of uniform circular motion on a diameter of the circle in which the circular motion occurs.

- Figure 15-15 shows a reference particle moving in uniform circular motion
- Its angular position at any time is $\omega t + \varphi$

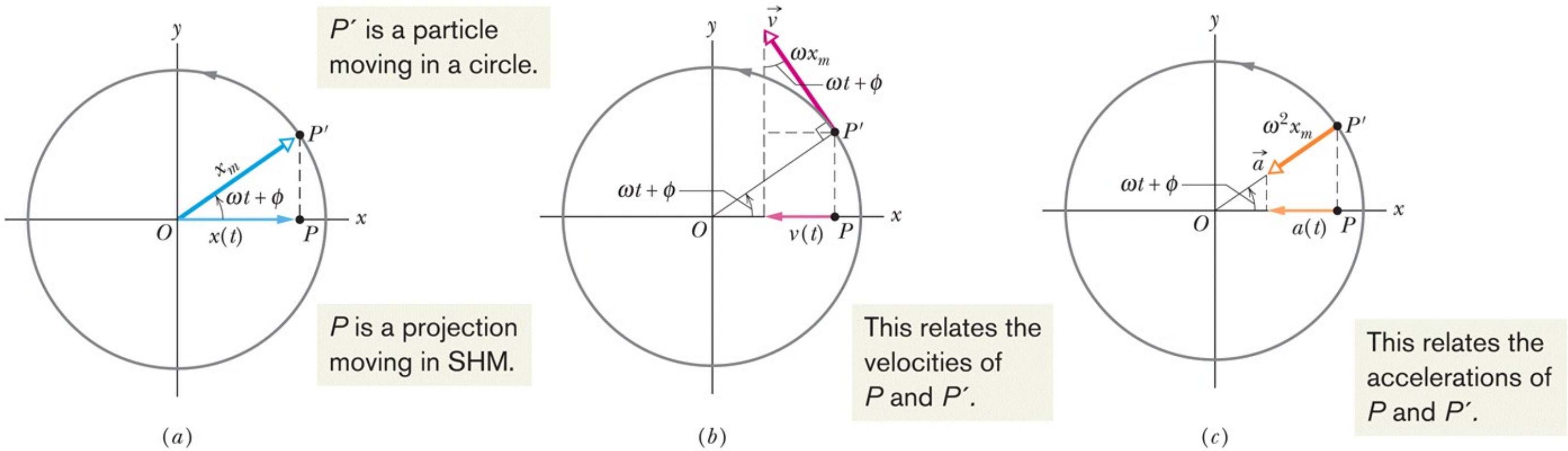

Figure 15-15

15-4 Pendulums, Circular Motion

- Projecting its position onto *x*:

$$x(t) = x_m \cos(\omega t + \phi), \quad \text{Eq. (15-36)}$$

- Similarly with velocity and acceleration:

$$v(t) = -\omega x_m \sin(\omega t + \phi), \quad \text{Eq. (15-37)}$$

$$a(t) = -\omega^2 x_m \cos(\omega t + \phi), \quad \text{Eq. (15-38)}$$

- We indeed find this projection is simple harmonic motion

15-5 Damped Simple Harmonic Motion

Learning Objectives

15.38 Describe the motion of a damped simple harmonic oscillator and sketch a graph of the oscillator's position as a function of time.

15.39 For any particular time, calculate the position of a damped simple harmonic oscillator.

15.40 Determine the amplitude of a damped simple harmonic oscillator at any given time.

15.41 Calculate the angular frequency of a damped simple harmonic oscillator in terms of the spring constant, the damping constant, and the mass, and approximate the angular frequency when the damping constant is small.

15.42 Apply the equation giving the (approximate) total energy of a damped simple harmonic oscillator as a function of time.

15-5 Damped Simple Harmonic Motion

- When an external force reduces the motion of an oscillator, its motion is **damped**
- Assume the liquid exerts a **damping force** proportional to velocity (accurate for slow motion)

$$F_d = -bv, \quad \textbf{Eq. (15-39)}$$

- *b* is a damping constant, depends on the vane and the viscosity of the fluid

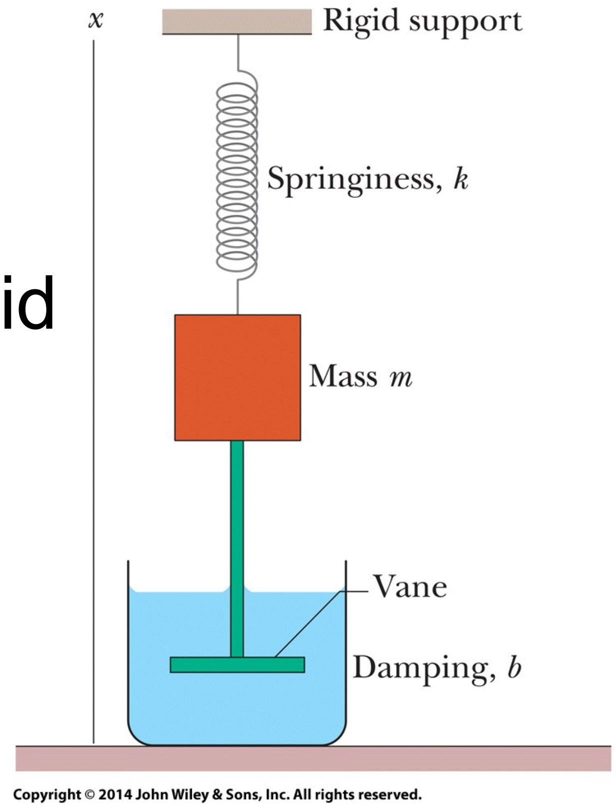

Figure 15-16

15-5 Damped Simple Harmonic Motion

- We use Newton's second law and rearrange to find:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$$ **Eq. (15-41)**

- The solution to this differential equation is:

$$x(t) = x_m e^{-bt/2m}\cos(\omega' t + \phi),$$ **Eq. (15-42)**

- With angular frequency:

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.$$ **Eq. (15-43)**

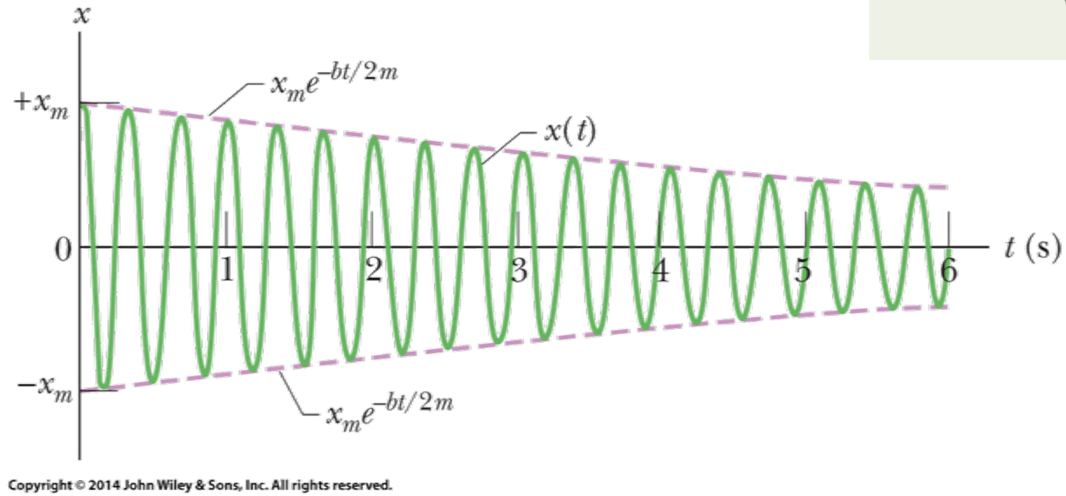

Figure 15-17

15-5 Damped Simple Harmonic Motion

- If the damping constant is small, $\omega' \approx \omega$
- For small damping we find mechanical energy by substituting our new, decreasing amplitude:

$$E(t) \approx \tfrac{1}{2}kx_m^2 e^{-bt/m}, \qquad \text{Eq. (15-44)}$$

Checkpoint 6

Here are three sets of values for the spring constant, damping constant, and mass for the damped oscillator of Fig. 15-16. Rank the sets according to the time required for the mechanical energy to decrease to one-fourth of its initial value, greatest first.

Set 1	$2k_0$	b_0	m_0
Set 2	k_0	$6b_0$	$4m_0$
Set 3	$3k_0$	$3b_0$	m_0

Answer: 1,2,3

15-6 Forced Oscillations and Resonance

Learning Objectives

15.43 Distinguish between natural angular frequency and driving angular frequency.

15.44 For a forced oscillator, sketch a graph of the oscillation amplitude versus the ratio of the driving angular frequency to the natural angular frequency, identify the approximate location of resonance, and indicate the effect of increasing the damping.

15.45 For a given natural angular frequency, identify the approximate driving angular frequency that gives resonance.

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15-6 Forced Oscillations and Resonance

- Forced, or driven, oscillations are subject to a periodic applied force
- A forced oscillator oscillates at the angular frequency of its driving force:

$$x(t) = x_m \cos(\omega_d t + \phi),$$ **Eq. (15-45)**

- The displacement amplitude is a complicated function of ω and ω_0
- The velocity amplitude of the oscillations is greatest when:

$$\omega_d = \omega$$ **Eq. (15-46)**

15-6 Forced Oscillations and Resonance

- This condition is called **resonance**
- This is also approximately when the displacement amplitude is largest
- Resonance has important implications for the stability of structures
- Forced oscillations at resonant frequency may result in rupture or collapse

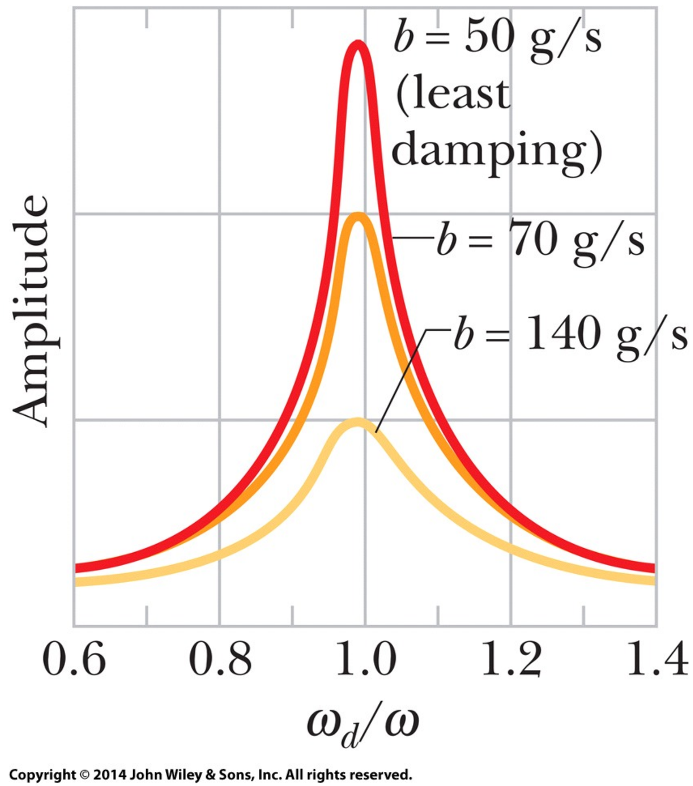

Figure 15-18

15 Summary

Frequency

- 1 Hz = 1 cycle per second

Period

$$T = \frac{1}{f}.$$

Eq. (15-2)

The Linear Oscillator

$$\omega = \sqrt{\frac{k}{m}}$$

Eq. (15-12)

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Eq. (15-13)

Simple Harmonic Motion

- Find *v* and *a* by differentiation

$$x(t) = x_m \cos(\omega t + \phi)$$

Eq. (15-3)

$$\omega = \frac{2\pi}{T} = 2\pi f.$$

Eq. (15-5)

Energy

- Mechanical energy remains constant as *K* and *U* change
- K = ½ mv^2, U = ½ kx^2

15 Summary

Pendulums

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$ **Eq. (15-23)**

$$T = 2\pi\sqrt{\frac{L}{g}}$$ **Eq. (15-28)**

$$T = 2\pi\sqrt{\frac{I}{mgh}}$$ **Eq. (15-29)**

Simple Harmonic Motion and Uniform Circular Motion

- SHM is the projection of UCM onto the diameter of the circle in which the UCM occurs

Damped Harmonic Motion

$$x(t) = x_m e^{-bt/2m} \cos(\omega' t + \phi),$$ **Eq. (15-42)**

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.$$ **Eq. (15-43)**

Forced Oscillations and Resonance

- The velocity amplitude is greatest when the driving force is related to the natural frequency by:

$$\omega_d = \omega$$ **Eq. (15-46)**