# Work and Energy

- (the dot product)
- **definition of work**
- **definition of kinetic energy –>**

  work-energy theorem: a restatement of Newton's 2nd
- **conservative and non-conservative forces**
- **potential energy**

  *not trivial. Please be careful.*

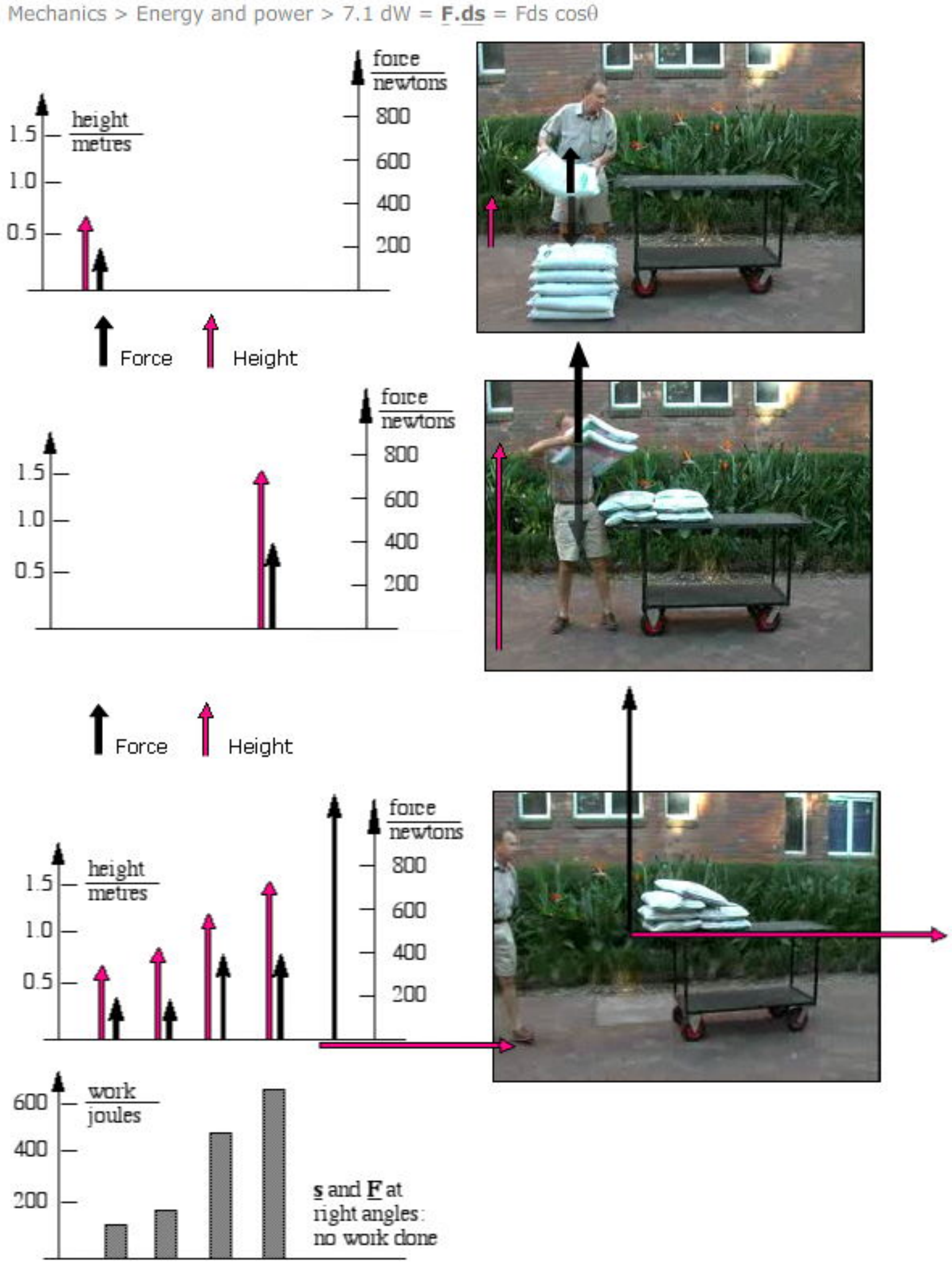
**Energy** occurs in different forms. Conversion is interesting – and employs lots of people.

**Example**

| | |
|---|---|
| Nuclear fusion in sun → | electrical and colour force energies |
| heats sun → | increases the kinetic energy of the ions in a plasma which |
| radiates light → | energy in electromagnetic radiation (at about 6000 kelvin) |
| evaporates ocean water → | latent heat |
| vapour rises → | molecular collisions providing gravitational potential energy |
| condenses as rain → | gives up latent heat, falls on mountain |
| drives hydroelectric power station → | gravitational potential energy to kinetic energy to electrical energy |
| transmitted as AC electrical energy → | electrical potential energy in transmission |
| used to charge battery → | electrochemical energy |

used to drive electric bike → electrochemical to electrical to mechanical work to kinetic energy

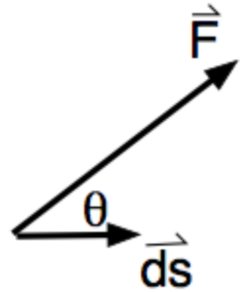
heats air via air resistance → thermal energy radiated into space (at about 300 kelvin)



*Video from chapter 6 of web stream, Chapter 7 of Physclips*

'Work' for physicists has some overlap with 'work' in everyday use. This bloke is working.

For a *small** displacement $\overrightarrow{ds}$, the work done $dW$ is proportional to $|\vec{F}|$, $|\overrightarrow{ds}|$ and cos θ. We define work thus:



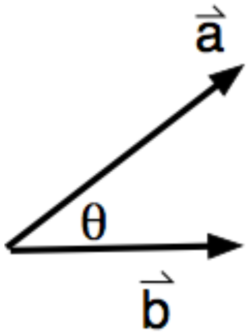

$$dW = |\vec{F}|\,|\overrightarrow{ds}|\,\cos\theta$$

Note that we define it by its <u>differential</u>, $dW$, which is a very small amount of work. In general, $F$ and $\theta$ vary with displacement $s$, so we must integrate to get the total work.

Only in the very special case of constant $F$ and $\theta$ (as at left) can we avoid the integral.

* Notation: $\Delta x$ means an increase in $x$. But $dx$ means a very small increase in $x$.
$\frac{\Delta y}{\Delta x}$ means the average slope, $\frac{dy}{dx}$ is the local slope.

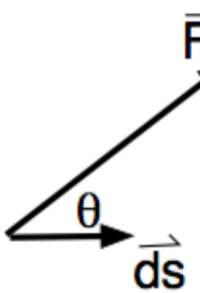
We introduce the **scalar product** of vectors *also called the dot product*




$\vec{a} \cdot \vec{b} = ab \cos\theta$ $(= \vec{b} \cdot \vec{a})$ *pronounced "a dot b"*

so that $dW = F\, ds \cos\theta = \vec{F} \cdot \overrightarrow{ds}$

**Why?** To quantify work, so we can write $dW = \vec{F} \cdot \overrightarrow{ds}$




dW = $|\vec{F}|\, |\overrightarrow{ds}| \cos\theta$ (*in Phys 1B: Voltage* $dV = -\vec{E} \cdot \overrightarrow{ds}$ )

$\vec{a} \cdot \vec{b} = ab \cos \theta$ $(= \vec{b} \cdot \vec{a})$ Apply to unit vectors:

**Unit vector·itself** $\mathbf{i}\cdot\mathbf{i} = 1\cdot 1 \cos 0° = 1 = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k}$

**Unit vector·another** $\mathbf{i}\cdot\mathbf{j} = 1\cdot 1 \cos 90° = 0 = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i}$

**Scalar product by components**

$\vec{a} \cdot \vec{b} = (a_x \,\mathbf{i} + a_y \,\mathbf{j} + a_z \,\mathbf{k}).(b_x \,\mathbf{i} + b_y \,\mathbf{j} + b_z \,\mathbf{k})$ definition

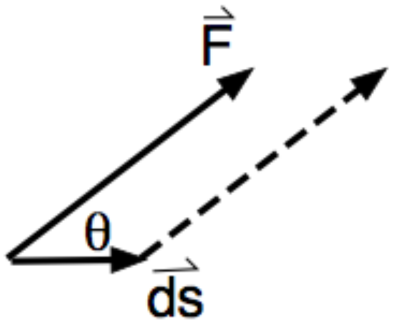
$= (a_x b_x)\, \mathbf{i}\cdot\mathbf{i} + (a_y b_y)\, \mathbf{j}\cdot\mathbf{j} + (a_z b_z)\, \mathbf{k}\cdot\mathbf{k}$

$+ (a_x b_y + a_y b_x)\, \mathbf{i}.\mathbf{j} + (..)\mathbf{j}.\mathbf{k} + (..)\, \mathbf{k}.\mathbf{i}$

$ab \cos \theta = \vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$ *easy!*

*Good way to find angle between two arbitrary vectors*

**Definition of work**




When force varies, use the differential displacement $\overrightarrow{ds}$ and integrate $dW$

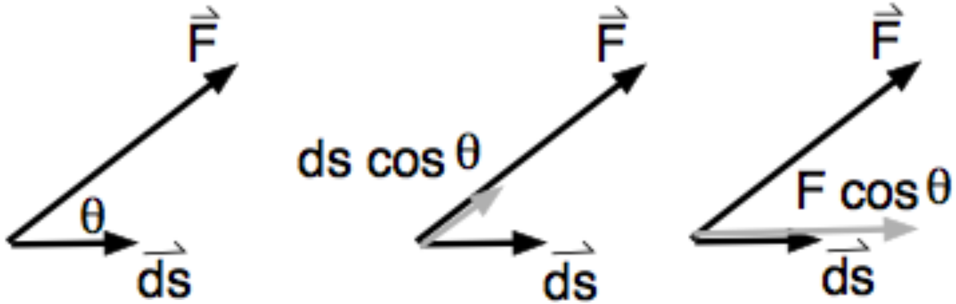
$$dW = F\, ds \cos\theta = \vec{F}\cdot\overrightarrow{ds}$$

*Think of this in either of two ways:*

*(F) (ds cosθ) –> F * component of ds // F, or*

*(F cosθ) (ds) –> ds * component of F // ds*

*where "//" means "parallel to"*



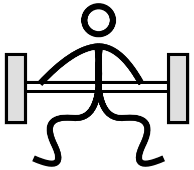

$$W = \int_{s=0}^{L} F \cos\theta$$

***if*** *F & θ are constant*, we get $W = FL\cos\theta$ *but this is a special case: don't count on it*

SI Unit: 1 newton x 1 metre = 1 joule

## SIMPLE MACHINES (pulleys, levers, screws, inclined planes etc)

**Example**. How much work is done by lifting 100 kg vertically by 1.8 m very slowly?



Slow $\therefore$ $F_{applied} \cong mg$

$W = mg\, d \cos 0°$ $\quad$ $(= mgh)$

$= 1.8$ kJ.

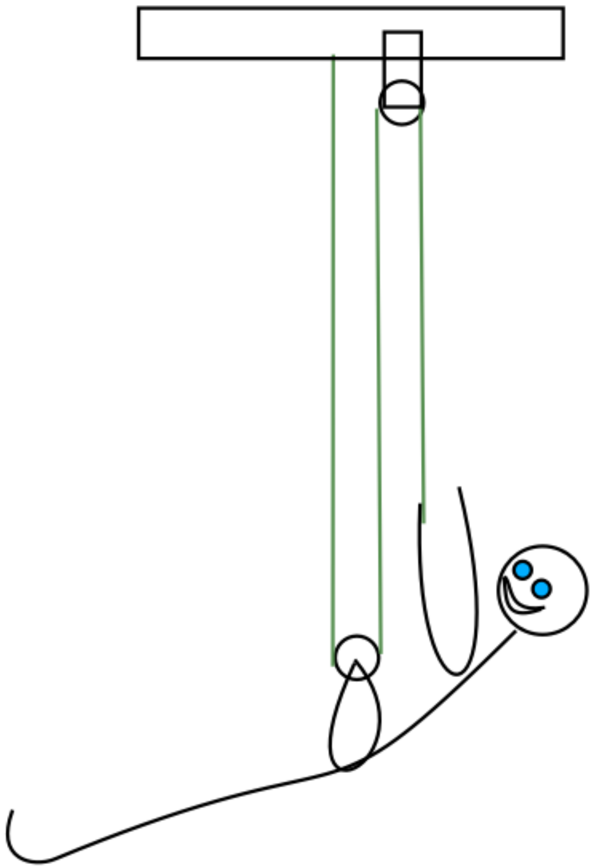
Not a lot, yet it is hard to do, because the force is inconveniently large.

So, we often use **simple machines** to make work more convenient (not less, just more convenient)

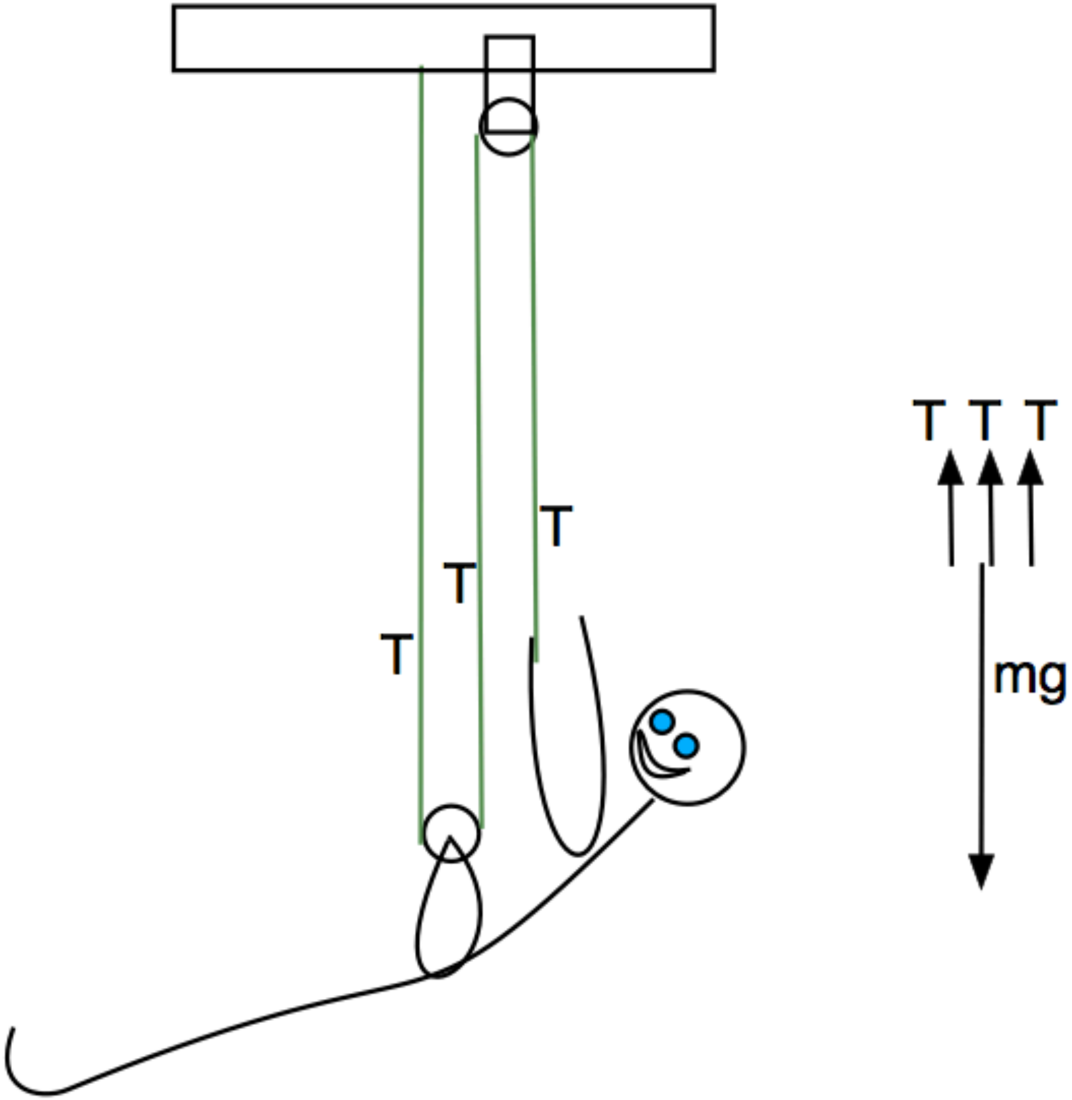
Examples: inclined plane, screw, pulleys.

Abseiling the East face of Burrows.

How much force does Joe need to apply to the end of the rope?



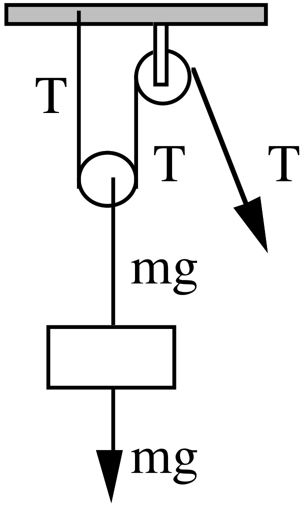
Neglecting acceleration:




$mg = 3T$, so $T = mg/3$

However, for him to rise a height $h$, three ropes must shorten by $h$, so he pulls on the rope over a distance $3h$. The work done by Joe is (force applied)*(distance of application) = $\frac{mg}{3} * 3h = mgh$.

How do you "pay for" the reduction in force?




If the rope and pulleys are light, and if the accelerations are negligible, then

Force on LH pulley

$$ma \cong 0 = 2T - mg$$

$$\therefore \quad T = mg/2$$

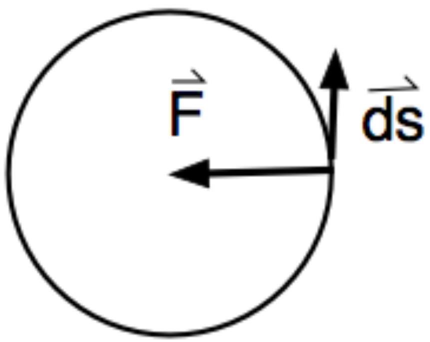
If mass rises by $D$, work done = $mgD$.

But rope shortens on both sides of rising pulley,

if mass rises by $D$, rope must be pulled $2D$, so

work done $= T * 2D = mgD$

**Example.** What is the work done by gravity in a circular orbit?



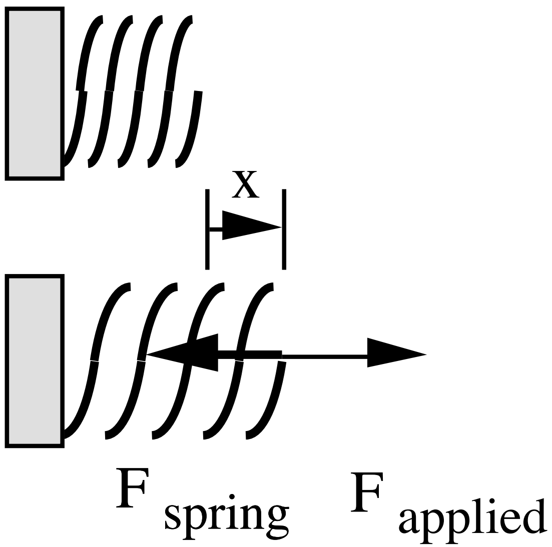

**Example.** What is the work done by gravity in a circular orbit?

$$W = \int F\, ds \cos\theta$$

$$= 0$$

*We'll see later that the gravitational potential energy at distance r from centre of a mass M is* $U = -G\dfrac{Mm}{r}$

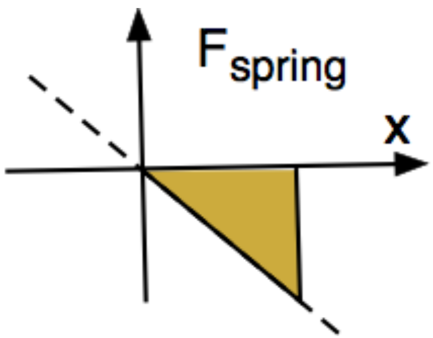
**Work to deform spring**




No applied force

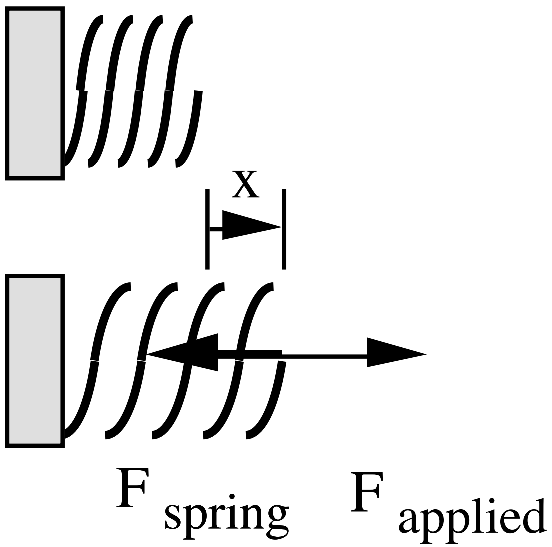
(x = 0)

Hooke's law:




Work done **by** spring = ?

**Work to deform spring**




No applied force

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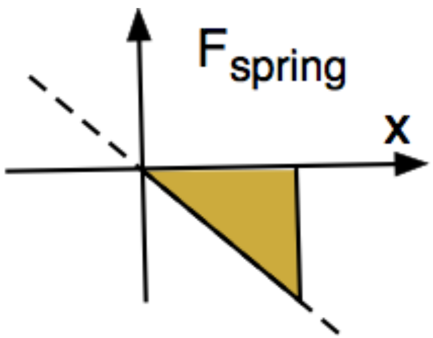
Hooke's law:




Work done **by** spring $= \int F_{spring}\, dx$

$$= \int -kx\, dx = -\frac{1}{2} kx^2 + 0$$

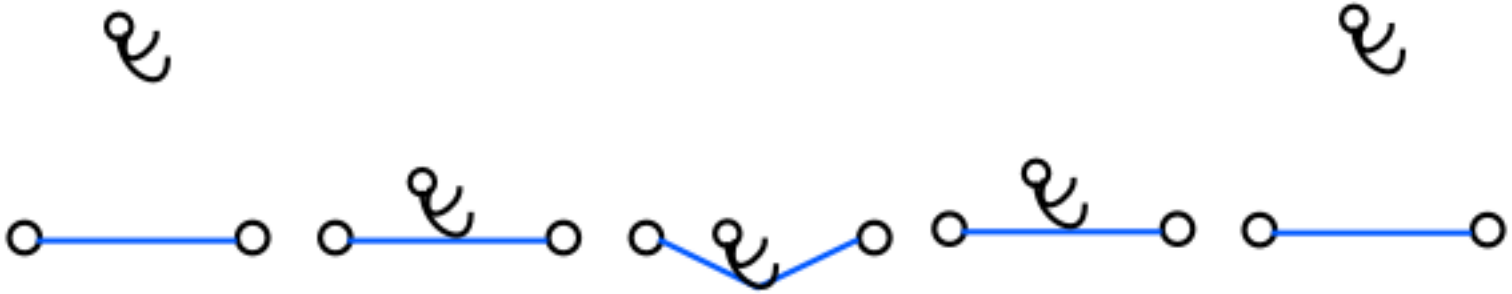
Work done **on** spring $= \int F_{applied}.dx$

$$= \int kx\, dx = \frac{1}{2} kx^2$$

(= energy "stored in spring")

**Question.** Use '+', '–' and 0 to indicate the sign of the entries

| | work done *by* gravity | work done *by* trampoline | work done *on* me | *(Change in my kinetic energy)* |
|---|---|---|---|---|
| I fall from 2nd storey window onto a trampoline | | | | |
| The trampoline deforms | | | | |
| The trampoline flattens | | | | |
| I fly up in the air | | | | |

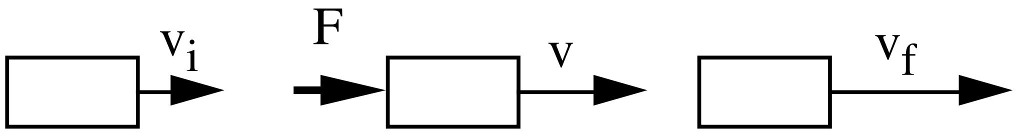


Use '+', '–' and 0 to indicate the sign of the entries. Black = displacement, red = mg blue = $F_{trampoline}$

| $\Delta s$ | mg | $F_{tramp}$ | | work done *by* gravity | work done *by* trampoline | work done *on* me | *(Change in my kinetic energy)* |
|---|---|---|---|---|---|---|---|
| $\downarrow$ | $\downarrow$ | | fall onto trampoline | + | 0 | + | + |
| $\downarrow$ | $\downarrow$ | $\uparrow$ | The trampoline deforms | (+) | – | – | – |
| $\uparrow$ | $\downarrow$ | $\uparrow$ | The trampoline flattens | (–) | + | + | + |
| $\uparrow$ | $\downarrow$ | | I fly up in the air | – | 0 | – | – |

## The work-energy theorem

Total external force F acts on mass m in x direction.



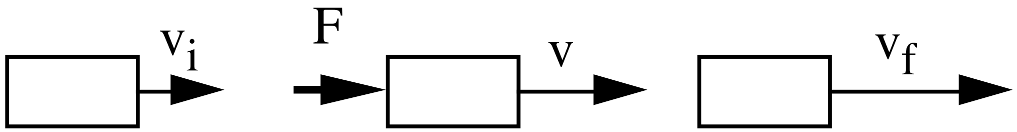

Work done by $F$: $\quad W = \int_{initial}^{final} F\,dx \quad$ *(then use F = ma)*

$$= \int_i^f m\frac{dv}{dt}\,dx$$

*How to do this integral?*

*change the order of* $\times$ *and* $\div$. *i.e. Use* $\quad \dfrac{a}{b}c = \dfrac{c}{b}a$

Total external force F acts on mass m in x direction.




Work done by $F$: $W = \int_{initial}^{final} F\,dx$ *(then use F = ma)*

$$= \int_i^f m\frac{dv}{dt}\,dx$$

*change the order of × and ÷ . i.e. Use* $\frac{a}{b}c = \frac{c}{b}a$

$$= \int_i^f m\frac{dx}{dt}\,dv$$

$$= \int_i^f mv\,dv = \left[\frac{1}{2}\,mv^2\right]_i^f$$

Work done by $F$: $W = \frac{1}{2}\,mv_f^2 - \frac{1}{2}\,mv_i^2 = \Delta K$

where we define **kinetic energy** $K = \frac{1}{2}\,mv^2$

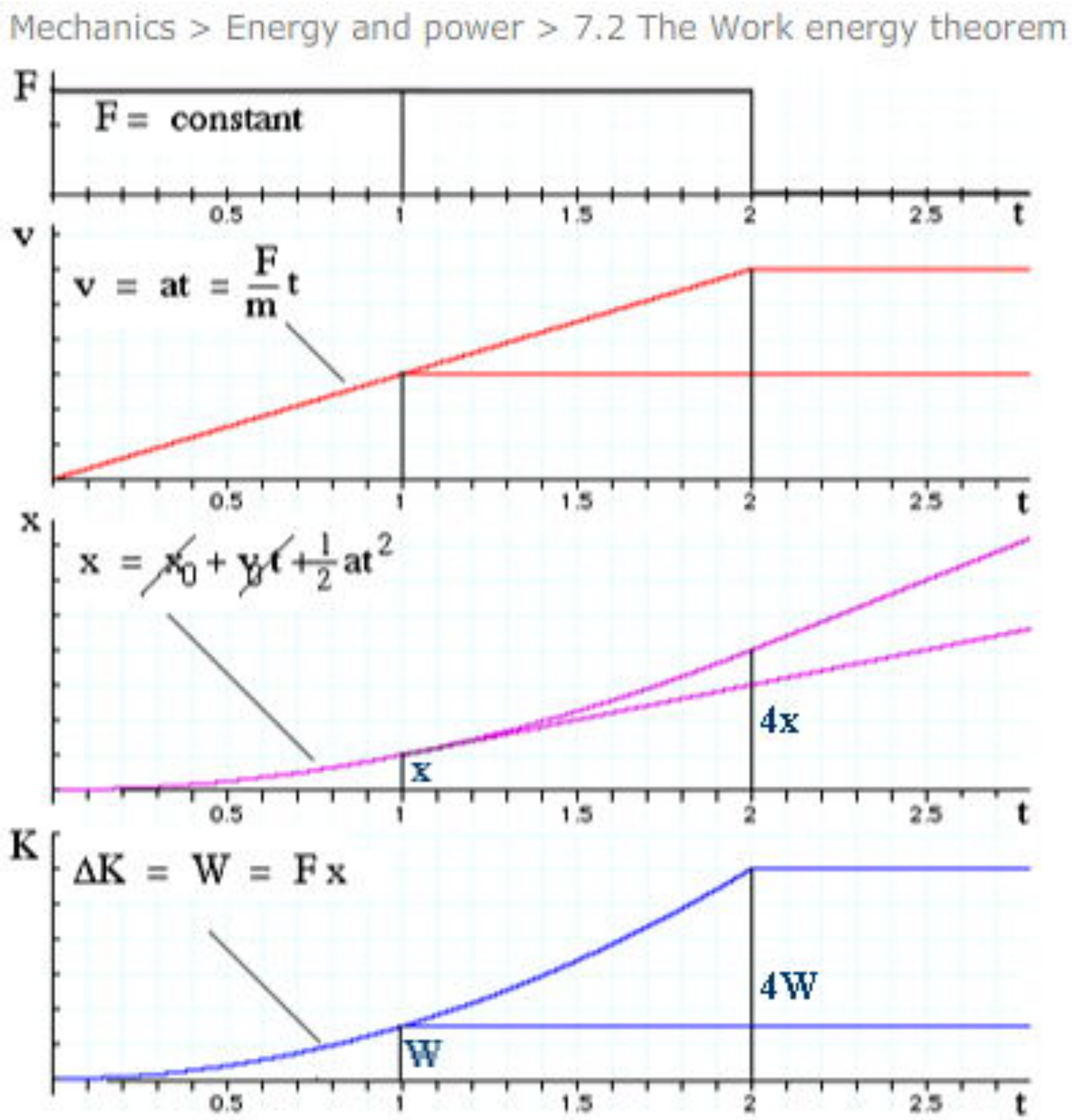
Increase in kinetic energy of body = work done by **total** force acting on it.

*This is a theorem, i.e. a tautology*

*because it is only true by definition of KE and by Newton 2.*

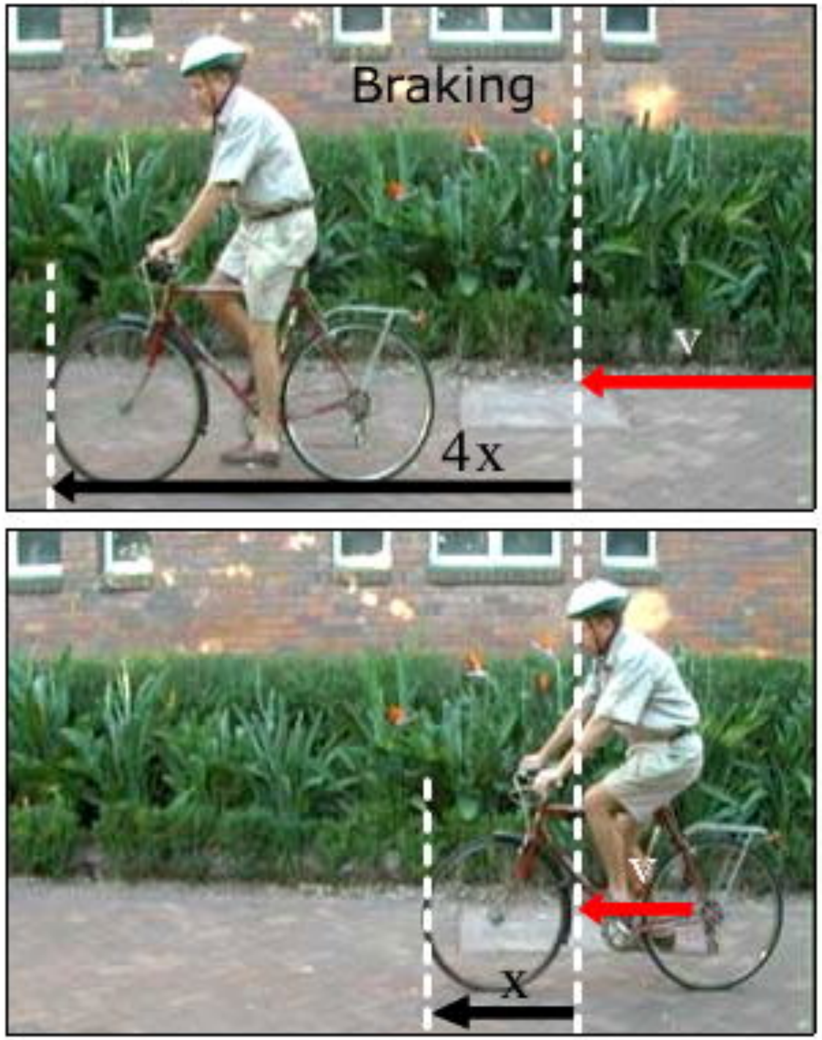
*∴ restatement of Newton 2 in terms of energy.* ***Not*** *a new law*

**Work energy theorem** (baby version limited to case with constant $F$)



**Road safety consequence**

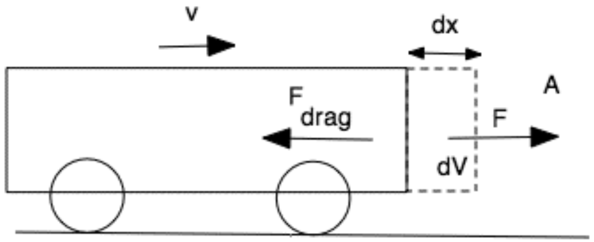
Energy and power > 7.2 The Work energy theorem




*v -> 2v*     *K -> 4K*

so same braking force must act over 4 times the distance

**Example: Air resistance.** A moving object at speed *v*, cross section *A*, pushes air (density = $\rho_{air}$ = mass/volume ~ 1.2 kg.m$^{-3}$), and accelerates a fraction ($C_{drag}$) of the air in front to *v*.




The volume accelerated is $C_{drag}Adx$.

This drag coefficient $C_{drag}$ is approximately 1 for a blunt object, about ½ for a ball, less for streamlined objects.

Set work done by drag force = increase in kinetic energy of the accelerated air gives

$$\text{Work done} = F_{drag} * dx = \frac{1}{2} m_{air} v^2 = \frac{1}{2} \rho_{air} (C_{drag} A dx) v^2 \qquad \text{cancel } dx$$

$$F_{drag} = \frac{1}{2} \rho_{air} C_{drag} A v^2$$

For dropped object reaches $v = v_{terminal}$, then weight $= F_{drag}$ so

$$v_{terminal} = \sqrt{\frac{2mg}{\rho_{air} C_{drag} A}}$$

If $v \ll v_{terminal}$, $F_{drag}$ is negligible compared to weight.

$v_{terminal}$ ~ 60 m.s$^{-1}$ for a person. For tennis ball ~ 30 m.s$^{-1}$; ping pong ball ~ 9 m.s$^{-1}$

**Power**. is the rate of doing work

Average power $\bar{P} = \frac{W}{\Delta t}$

Instantaneous power $P = \frac{dW}{dt}$ Definition of power

SI unit: 1 joule per second ≡ 1 watt (1 W) ***Caution***: *W used for work and for watt*

**Example** Jill ($m$ = 60 kg) climbs the stairs in Matthews Building and rises 50 m in 1 minute. How much work does she do against gravity? What is her average output power? (neglect accelerations)

$$W = \int dW = \int \vec{F} \cdot \overrightarrow{ds} = \int F_y dy$$

but $\bar{F}_y \cong mg$ if she ascends at constant speed

$$W = \int dW = \int F_y \, dy \cong \int mg \, dy = mg \int dy = mg\Delta y$$

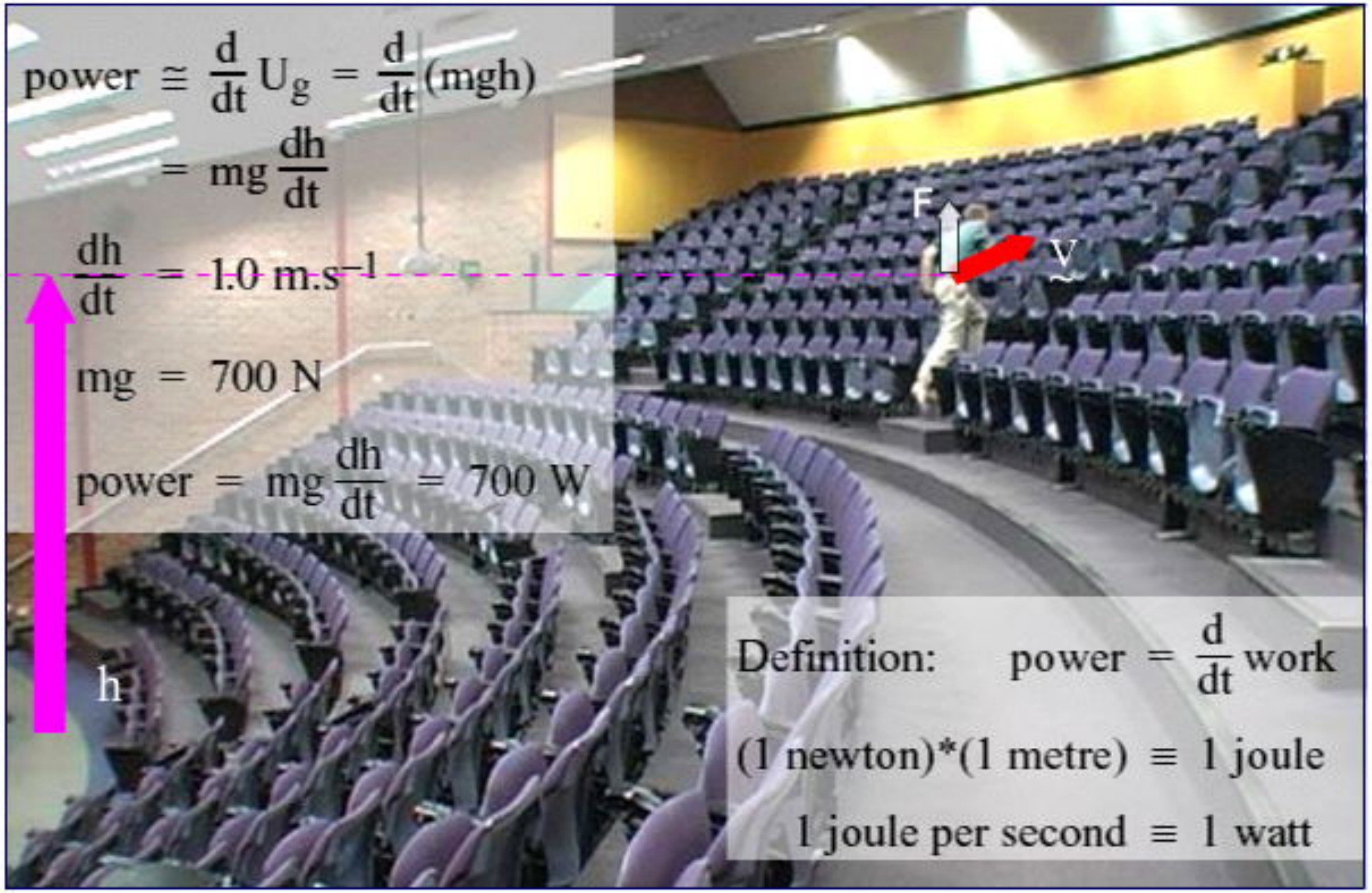
$= 29$ kJ *(compare with* $K = \frac{1}{2}mv^2 \sim 0.02$ to $0.04$ kJ. *So our approximation is justified)*

So $\bar{P} = \frac{W}{\Delta t} = \frac{mg\Delta y}{\Delta t} = 490$ W

*(humans can produce 100s of W,*

*car engines several tens of kW,*

*1 horsepower ≡ 550 ft.lb.s$^{-1}$ = 0.76 kW)*



power ≅ d/dt U_g = d/dt (mgh)
= mg dh/dt
dh/dt = 1.0 m.s^-1
mg = 700 N
power = mg dh/dt = 700 W
F
v
h
Definition: power = d/dt work
(1 newton)*(1 metre) ≡ 1 joule
1 joule per second ≡ 1 watt

**Quiz: What is wrong with these statements:**

A power station (nuclear, solar, coal, your choice) requires more energy to build than it produces?

Boiling 1 kg of water takes more power than melting 1 kg of ice

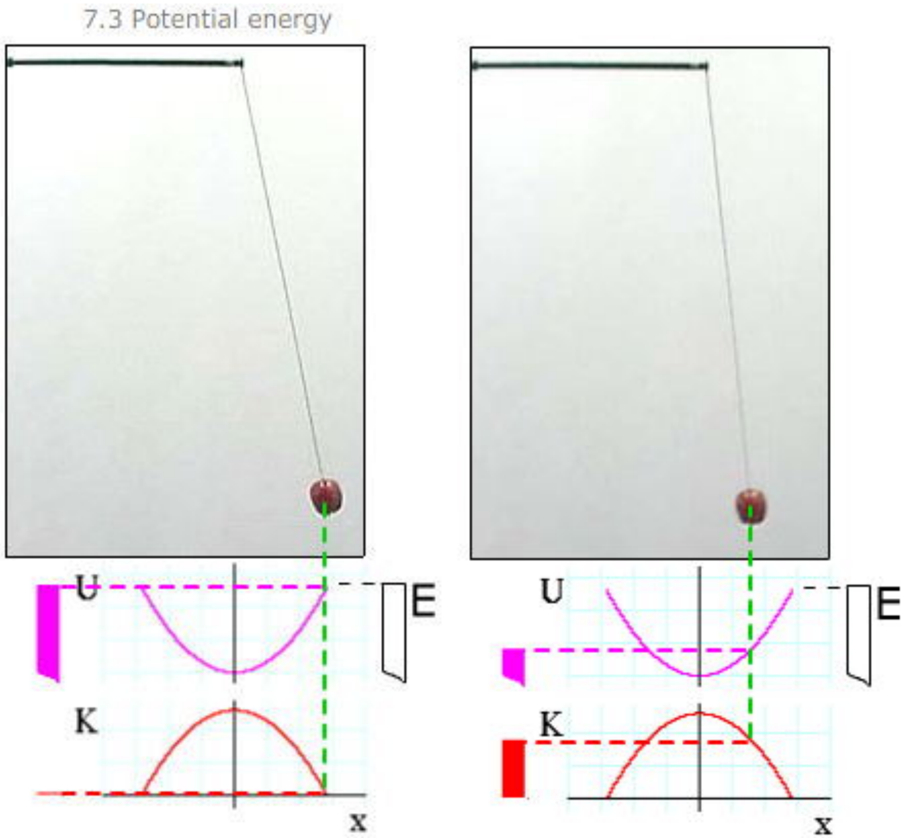
Electrical power costs in NSW are not high enough.

**Potential energy**.

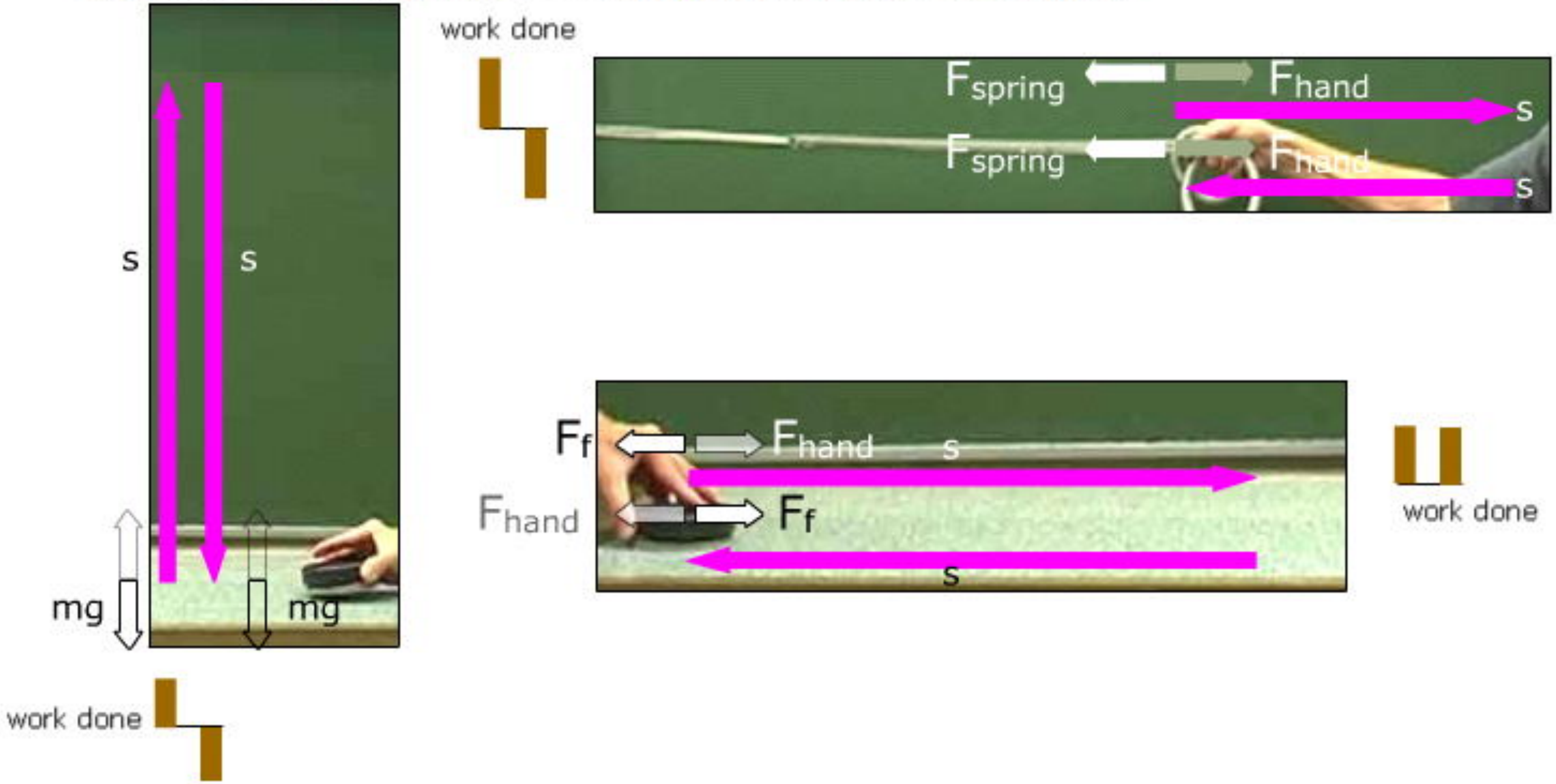
e.g. Compress **spring**, do $W$ on it, but get no $K$. Yet can get energy out: spring can expand and give $K$ to a mass. –> Idea of stored energy.

e.g. **Gravity**: lift object (slowly), do work but get no $K$. Yet object can fall back down and get $K$.

Recall $W_{\text{against grav}} = mg\,\Delta y$ i.e. $W = W(y)$



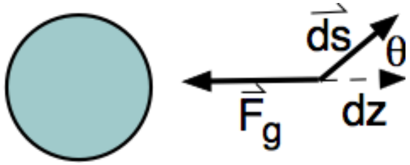
Mechanics > Energy and power > 7.4 Conservative and non conservative forces




**In contrast:** Slide mass slowly along a surface. Do work against **friction**, but can't recover this energy mechanically. **Not all forces "store" energy**

## Conservative and non-conservative forces

(same examples)




$$W_{against\ gravity} = \int_{initial}^{final} \vec{F} \cdot \overrightarrow{ds}$$

$$= \int_{initial}^{final} |F_{grav}| dz$$ where $z$ is the displacement measured in the vertical direction

In the approximation that the field is uniform (we'll fix this later), $|F_{grav}| = mg = constant$

$$\mathrm{W}_{\text{against grav}} = \int_{initial}^{final} mg dz$$

$$= mg(z_{final} - z_{initial})$$ *in a uniform field*

W is uniquely defined at all $\vec{r}$, *i.e.* $W = W(\vec{r})$

If $z_{final} = z_{initial}$, then $W = 0$.

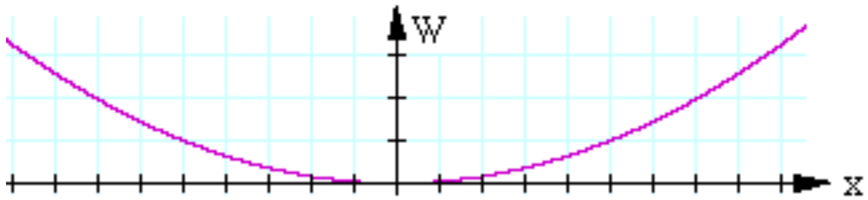
The definition of a **conservative force:** A conservative force does no work round a closed path.

So gravity is a **conservative force**

## Example: spring force and energy

$$W_{\text{against spring}} = -\int_{initial}^{final} F_{spring}\,dx = -\int_{initial}^{final} -kx\,dx$$

$$= \frac{1}{2}k(x_f^2 - x_i^2)$$




W is uniquely defined at all $x$, i.e. $W = W(x)$

So, if $x_f = x_i$ then $W = 0$.

$\therefore$ Work done round a closed path = 0 Therefore:

Spring force is a **conservative force** *so it has stored or potential energy: symbol U.*

## Potential energy

For a **conservative** force $\vec{F}$, *i.e.* one where work done against it $W = W(\vec{r})$, we can define potential energy U by $\Delta U = W_{against}$.

The **choice of zero for U is arbitrary.** Here U = 0 at x = 0 is convenient, so usually

$$U_{spring} = \frac{1}{2}kx^2$$

**Kinetic friction**

$dW_{\text{against fric}} = -\, F_f \, ds \cos\theta$

but kinetic friction $\vec{F}_f$ always has a component *opposite* $\overrightarrow{ds}$

$\therefore$ dW always $\geq 0$.   *(we never get work back: friction turns mechanical energy into heat)*

$\therefore$ cannot be zero round closed path, $\therefore$ $W \neq W(\vec{r})$

$\therefore$ friction is a **non-conservative force** ■

*Note that direction of friction (dissipative force) is always against motion.*

*Direction of spring force or weight doesn't change when we change sign of ds.*

**From energy to force:**

$U(s) = -\int \vec{F} \cdot \overrightarrow{ds} = -\int F\, ds$ where *ds* is in the direction parallel to *F*

**Question**

- If I give you $U(s)$, what is $F(s)$ ?
- If I give you $U(x,y)$, what can you tell me about $F(x,y)$ ?

**From energy to force:**

$U(s) = -\int \vec{F} \cdot \overrightarrow{ds} = -\int F\ ds$ where $ds$ is in the direction parallel to $F$

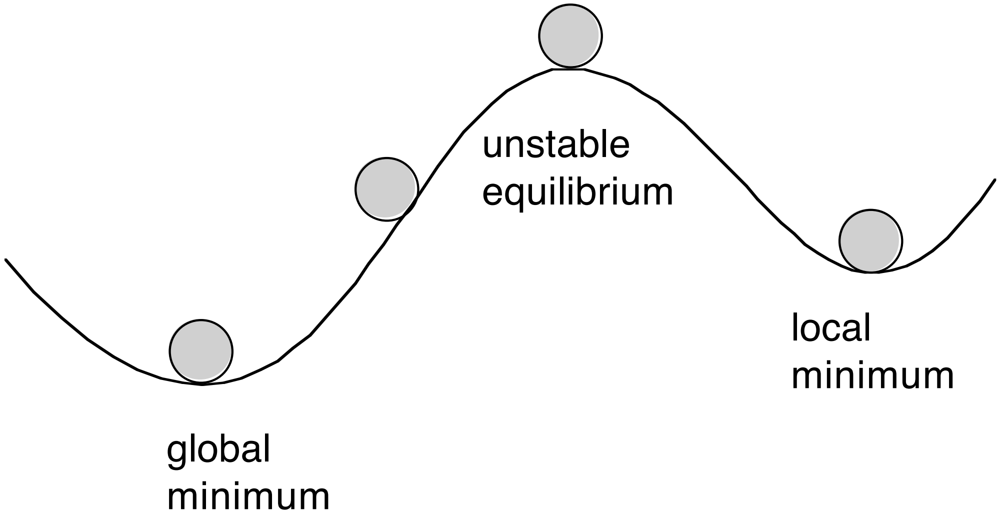
$$F = -\frac{dU}{ds}$$

*And* $F_x = -\frac{\partial U}{\partial x},\ F_y = -\frac{\partial U}{\partial y},\ F_z = -\frac{\partial U}{\partial z}$ *don't need partial derivatives in our syllabus*

Spring: $U_{\text{spring}} = \frac{1}{2} kx^2 \quad \therefore \quad F_{\text{spring}} = -kx$

Gravity: $U_{\text{g}} = mgz \quad \therefore \quad F_{\text{g}} = -\frac{dU}{dz} = -mg$ *In approximation that g is uniform. See later*

**Energy diagrams and equilibria:**




Treat this as $y(x)$ for a particle in a uniform gravitational field, we can see $U(x)$ and imagine the direction of force ($-dU/dx$).

Minima give stable equilibria: stable with respect to small perturbations.

Maxima give unstable equilibria.

*Similar diagrams in chemistry, materials science and elsewhere, sometimes metaphorical, e.g. reaction proceeds left to right.*

We defined work done by conservative forces $= -\Delta U$ (– change in potential energy)

Now define work done by non-conservative forces $= -\Delta U_{internal}$

$U_{internal}$ is the **internal energy**

*e.g.* I do muscular work; that work comes from my internal energy

A truck does mechanical work. The energy comes out of the internal energy of diesel oil

or I do work against friction, that work goes into the internal energy as heat.

The truck does work against air resistance, that work goes into the internal energy* as heat.

** At the quantum level, heat and chemical energy are kinetic and potential (Phys 1a and 1b), but that is often hard to quantify.*

Work energy theorem: Total work done by all forces $= \Delta K$, so

$$\begin{matrix}\text{work done by} \\ \text{non-conservative forces}\end{matrix} + \begin{matrix}\text{work done by} \\ \text{conservative forces}\end{matrix} = \Delta K$$

$$-\Delta U_{internal} - \Delta U = \Delta K$$

$$\begin{matrix}\text{loss in} \\ \text{internal energy}\end{matrix} + \begin{matrix}\text{loss in} \\ \text{potential energy}\end{matrix} = \Delta K$$

We define mechanical energy $E = U+K$, therefore

$$-\Delta U_{internal} = \Delta K + \Delta U = \Delta E$$

Work done by non-conservative forces $= \Delta K + \Delta U = \Delta E$

**If non-conservative forces do no work**, mechanical energy is conserved $\Delta E \equiv \Delta K + \Delta U = 0$

(*Note the '**if**'. This is not a law! You must justify before using.*)

*Equivalent to Newton 2, but useful for many mechanics problems where integration is difficult.*

*State the principle carefully!*

***Never, ever write***: ~~*"kinetic energy = potential energy"*~~

**Which are conservative, non-conservative or approximately conservative?**

Electric force between two electrons

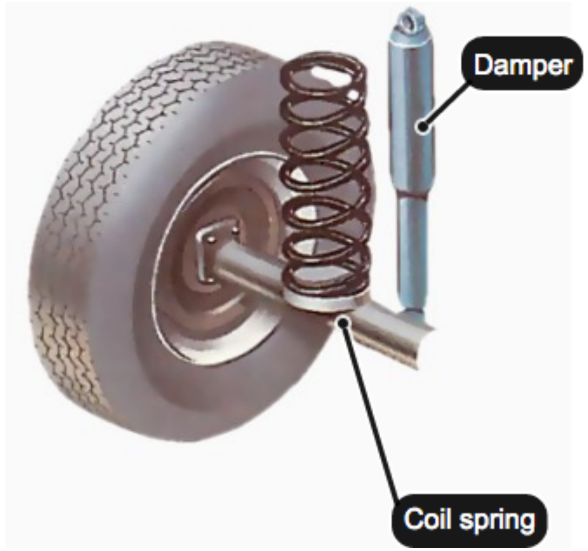
Force exerted by brake rubbers on rim of bike wheel

Force exerted by springs

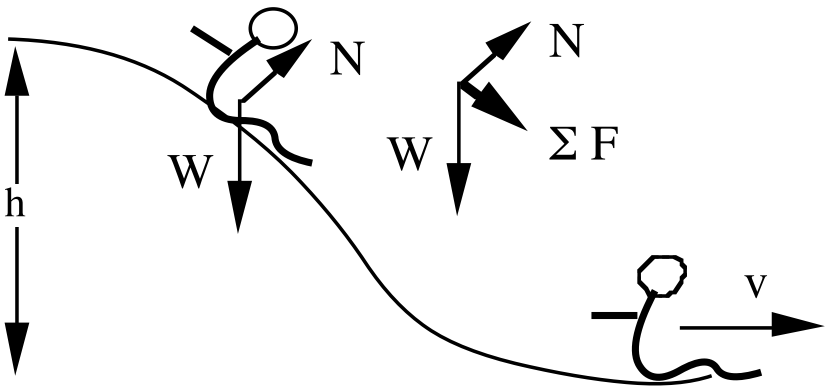
Gravitational force between two objects

Force exerted by muscles

Force exerted by a car's dampers (sometimes misleadingly called 'shock absorbers')



**Classic problem.** Child pushes off with $v_i$. How fast is the s/he going at the bottom of the water slide? Neglect friction (*a non-conservative force*).

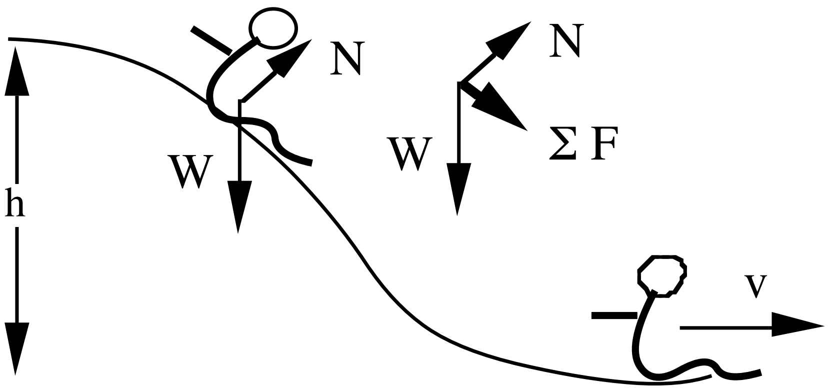



i) *By Newton 2 directly:*

$$v = \int_{top}^{bottom} a\,dt = \int_{top}^{bottom} \frac{F}{m}\,dt = \int_{top}^{bottom} g\cos\theta\,dt = \ldots\ldots \qquad \textit{difficult}$$

ii) *Using work energy theorem (Newton 2 indirectly):*

**Classic problem.** Child pushes off with $v_i$. How fast is the s/he going at the bottom of the slide? Neglect friction (*a non-conservative force*).




i) *By Newton 2 directly:*

$$v = \int_{top}^{bottom} a\, dt = \int_{top}^{bottom} \frac{F}{m}\, dt = \int_{top}^{bottom} g \cos\theta\, dt = .....$$

ii) *Using work energy theorem (Newton 2 indirectly):*

Non-conservative forces do no work, ∴ mechanical energy is conserved, i.e.

$$\Delta E = \Delta K + \Delta U = 0$$
$$K_f - K_i + U_f - U_i = 0$$

*or*

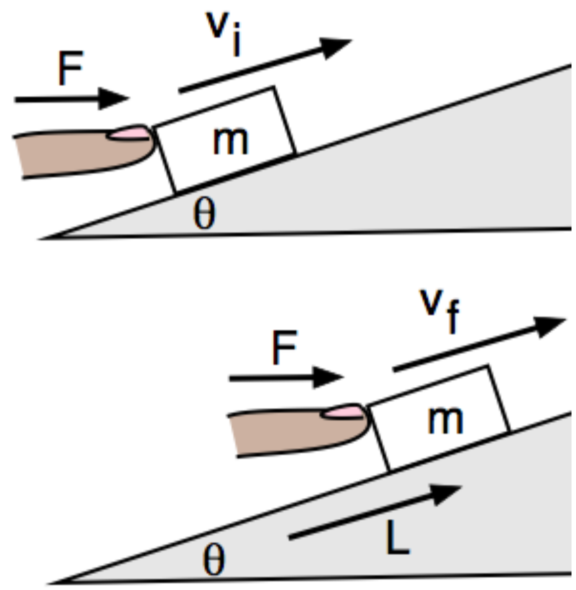
$$E_f = E_i$$
$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2} mv_f^2 - \frac{1}{2} mv_i^2 + mgy_f - mgy_i = 0$$

*rearrange –>* $v_f = \sqrt{v_i^2 + 2g(y_i - y_f)}$

*What would a complex* $v_f$ *mean?* (*Phys 1b*)

**Question**. A finger provides a constant horizontal force $F$ to a mass $m$, accelerating it from $v_i$ to $v_f$ over a distance $L$ up a plane inclined at $\theta$ to the horizontal. The coefficients of friction are $\mu_s$ and $\mu_k$. How much work is done by




Work done by $F$?

Work done by $Mg$?

Work done by normal force $N$?

Work done by friction?

Work done by all forces together?

What is the increase in kinetic energy?

*Here, all forces are constant. But there are several of them, so be careful.*

**Question**.

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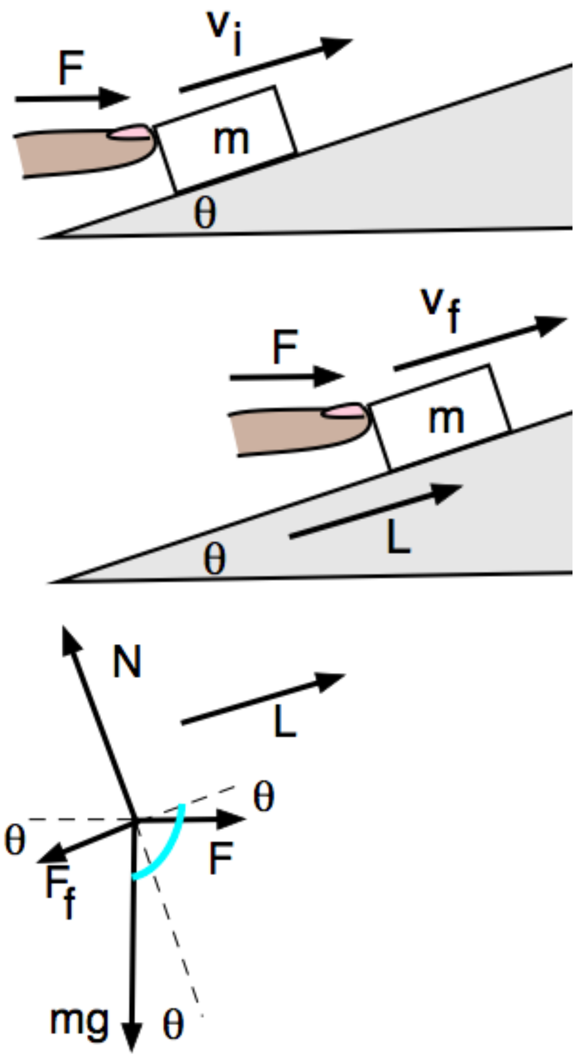


- Work done by $F$: $F$ constant, so $W_F = FL\cos\theta$
- Work done by $mg$?

$$W_{mg} = mg^{*}\,L \cos \text{ of angle between them} \qquad (angle\ )$$

$$W_{mg} = mg\cos(90° + \theta) = -mgL\sin\theta$$

- Work done by $N$? $W_N = NL\cos 90° = 0$

- Work done by friction? $F_f = \mu_k N,\ \ N = ?$

- Work done by all forces together
- What is the increase in kinetic energy?

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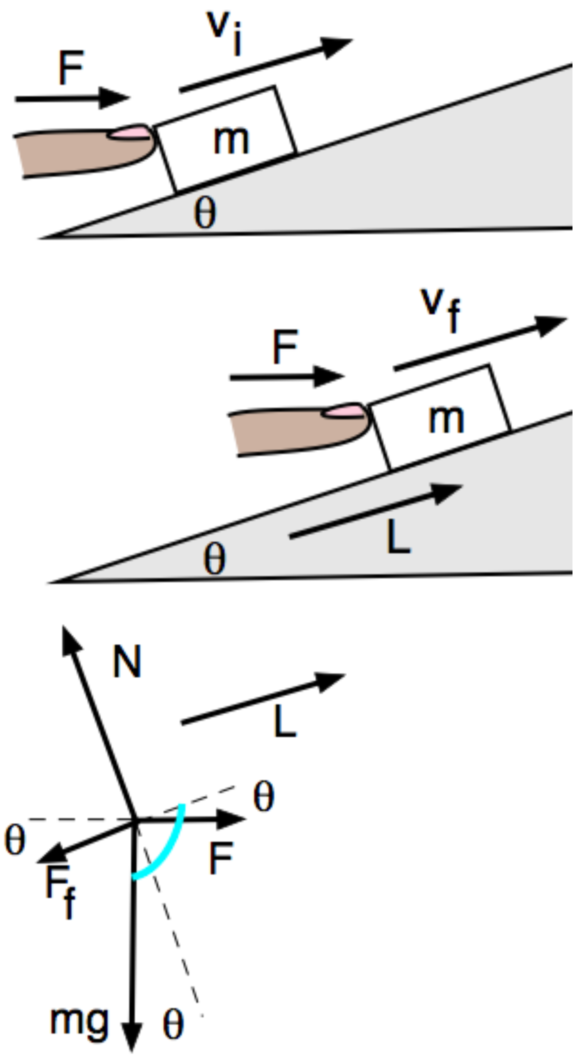


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- Work done by $mg$?

$$W_{mg} = mg * L \cos \text{ of angle between them} \quad (angle)$$

$$W_{mg} = mg\cos(90° + \theta) = -mgL\sin\theta$$

- Work done by $N$? $W_N = NL\cos 90° = 0$
- Work done by friction? $F_f = \mu_k N, \quad N = ?$

$$a_{normal} = 0. \quad \text{So } N = mg\cos\theta + F\sin\theta$$

$$W_f = \mu_k(mg\cos\theta + F\sin\theta) * L * \cos 180°$$

$$= -\mu_k(mg\cos\theta + F\sin\theta)L$$

- Work done by all forces together =
- What is the increase in kinetic energy?

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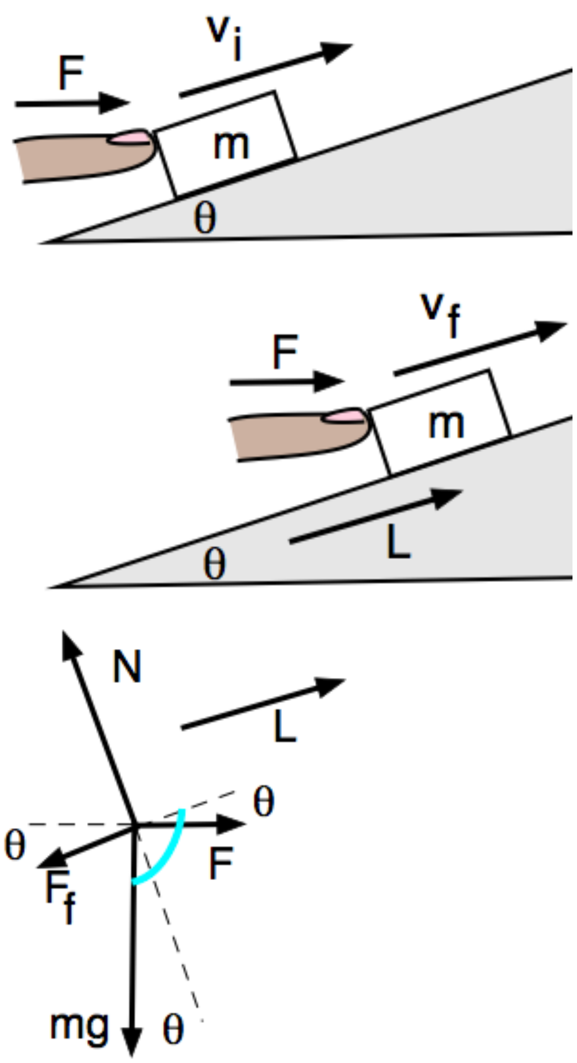


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$a_{normal} = 0$. So $N = mg\cos\theta + F\sin\theta$

$W_f = \mu_k(mg\cos\theta + F\sin\theta) * L * \cos 180°$

$= -\mu_k(mg\cos\theta + F\sin\theta)L$

- Work done by all forces together = sum of all the above =
- The increase in kinetic energy: $\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$

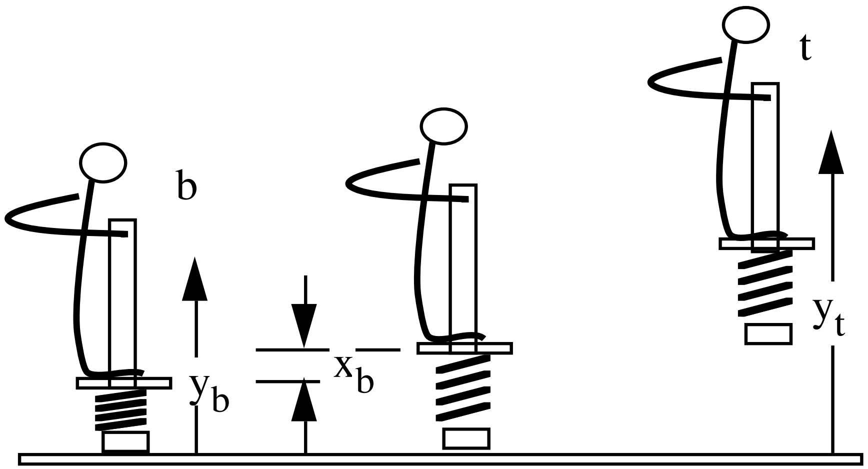
*Work-energy theorem $\Delta K$ = work done by all forces acting.*

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = L[F\cos\theta - mg\sin\theta - \mu_k(mg\cos\theta + F\sin\theta)]$$

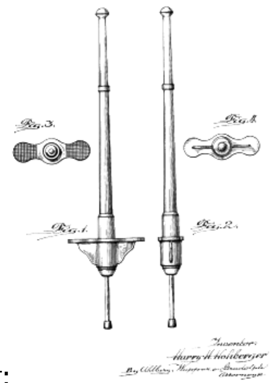
= $L$ times (sum of all force components up the plane)

**Example.**

Freda ($m$ = 60 kg) rides pogo stick ($m$ << 60 kg) with spring constant $k$ = 100 kN.m$^{-1}$. Neglecting friction, how far does spring compress if jumps are 50 cm high?




Patent extract:



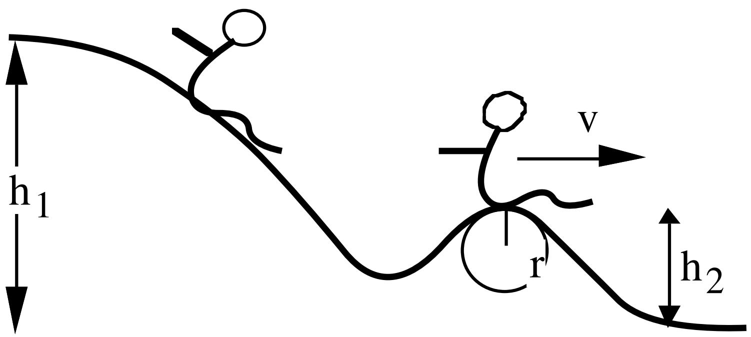
Non-conservative forces do no work, ∴ mechanical energy is conserved, i.e.

$$E_{bottom} = E_{top}$$

$$K_b + U_b = K_t + U_t \qquad (U = U_{grav} + U_{spring})$$

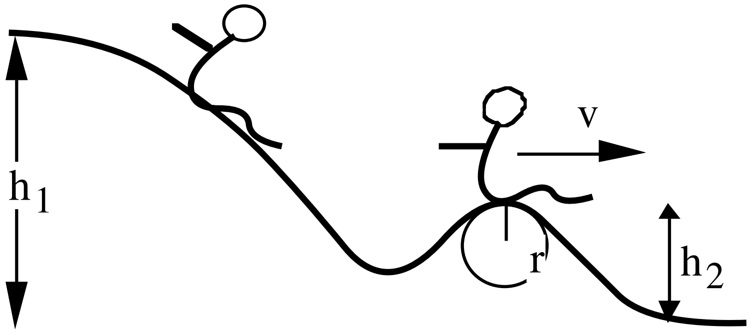
$$\frac{1}{2}mv_{horiz}^2 + (mgy_b + \frac{1}{2}kx_b^2) \cong \frac{1}{2}mv_{horiz}^2 + (mgy_t + \frac{1}{2}kx_t^2) \qquad (\text{at top \& bottom of jump, } v_{vert} = 0)$$

$$mg(y_t - y_b) \cong \frac{1}{2}kx_b^2 \qquad \therefore \quad x_b \cong \sqrt{\frac{2mg(y_t - y_b)}{k}} \cong 80 \text{ mm.}$$




**Example**. Slide starts at height $h_1$. Later there is a hump with height $h_2$ and (vertical) radius r. Starting from rest, what is the minimum value of $h_2 - h_1$ if slider is to become airborne? Neglect friction, air resistance. Treat slider as particle.

- What is the relation among $v_2$, $g$ and $r$ for becoming airborne?
- What is the minimum $(h_2 - h_1)$ for becoming airborne?




**Example**. Slide starts at height $h_1$. Later there is a hump with height $h_2$ and (vertical) radius r. Starting from rest, what is the minimum value of $h_2 - h_1$ if slider is to become airborne? Neglect friction, air resistance. Treat slider as particle.

Over hump, $a_c = \frac{v^2}{r}$ *(down)* . Airborne if $g < a_c$, i.e. if $v_2^2 > gr$.

No non-conservative forces act so

$$E_2 = E_1$$

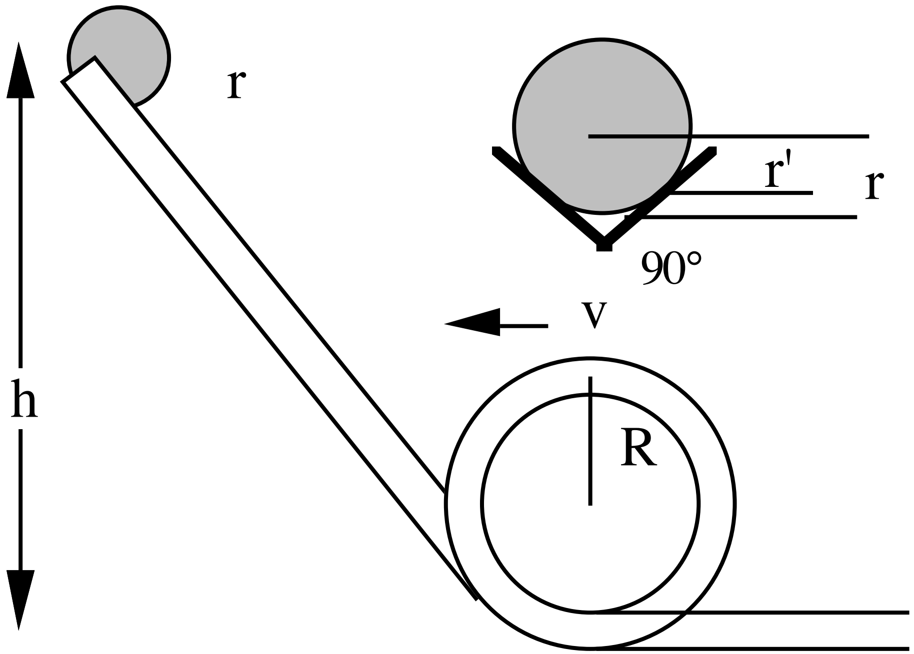
$$U_2 + K_2 = U_1 + K_1$$

$$mgy_2 + \frac{1}{2} mv_2^2 = mgy_1 + \frac{1}{2} mv_1^2$$

$$\frac{1}{2} mv_2^2 = mg(y_1 - y_2)$$

$$(y_1 - y_2) = \frac{v_2^2}{2g} > \frac{gr}{2g} = \frac{r}{2}$$

**Challenge puzzle**




How high should *h* be so that it can loop the loop?

**Example** Bicycle and rider (80 kg), travelling at 20 m.s$^{-1}$, stop without skidding. $\mu_S$ = 1.1. What is minimum stopping distance? How much work done by friction between tire and road? Between brake pad and rim? Wheel rim is ~300 g with specific heat $c$ ~ 1 kJ.kg$^{-1}$, how hot does it get?

*How would you do this problem?*

**Example** Bicycle and rider (80 kg), travelling at 20 m.s$^{-1}$, stop without skidding. $\mu_s = 1.1$. What is minimum stopping distance? How much work done by friction between tire and road? Between brake pad and rim? Wheel rim is ~300 g with specific heat $c$ ~ 1 kJ.kg$^{-1}$, how hot would it get if no loss?

*friction –> deceleration –> stopping distance –> work* *OR*

*work energy theorem –> change in K = work done etc*

$$|a| = \frac{F_f}{m} \le \frac{\mu_s N}{m} = \mu_s g$$

$a = -\mu_s g$ in the direction of travel

$$v_f^2 - v_i^2 = 2as \;\text{–>}\; s = \frac{v_f^2 - v_i^2}{2a}$$

$$s \ge \left| \frac{-v_i^2}{-2\mu_s g} \right| = 19 \text{ m}$$

Work done by friction between tire and road?

No skidding, ∴ no relative motion, ∴ locally, $W = 0$. *But there is also centre-of-mass work which is the work done by the total force considered to act at the C of M.*

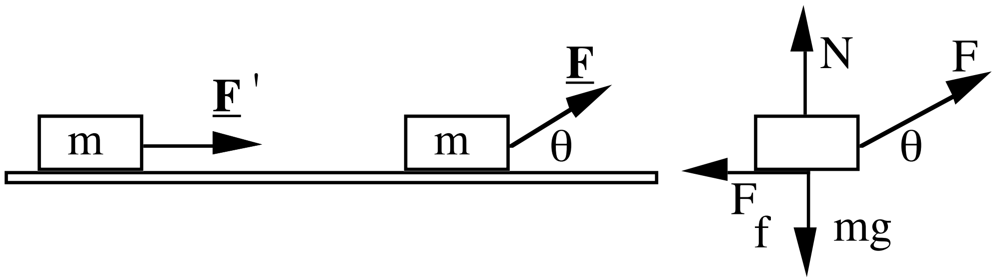
Between pad and rim? Here there is relative motion.

All $K$ of bike & rider –> heat in rim and pad *In practice, heat would be lost to the air.*

$W = \Delta K = K_f - K_i = -16$ kJ $\Delta U_{int}$ is heat, $Q$

$Q = mC\Delta T$ ...... $\Delta T \sim 50$ °C (*Not yet defined: do this later in Heat*)

**Example** Which way is it easier to drag an object?

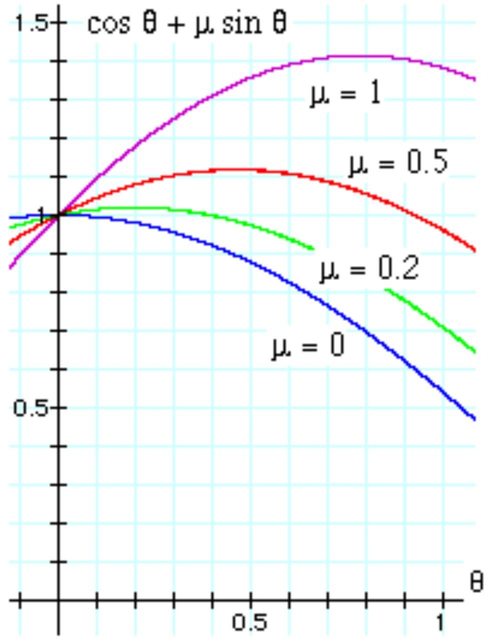



*Suppose we move at steady speed, a = 0. Which requires less F? Which requires less work?*

mechanical equilibrium –> horizontal $F \cos\theta = F_f$

vertical $N + F \sin\theta = mg$

$W = \mu_k N s = \mu_k s(mg - F \sin\theta)$ *decreases* with θ




sliding –> $F_f = \mu_k N$

$$F \cos\theta = \mu_k N$$

eliminate N –> $F \cos\theta = \mu_k (mg - F \sin\theta)$

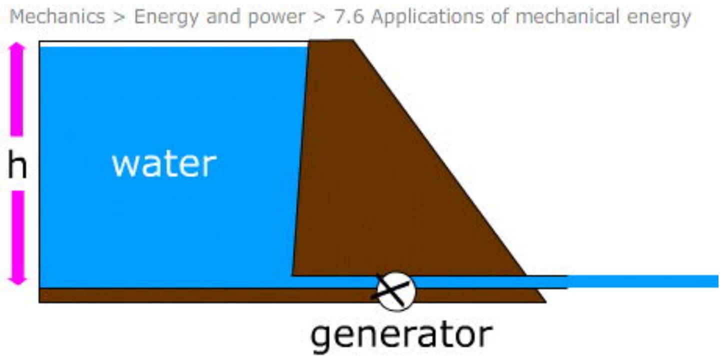
$$F = \frac{\mu_k mg}{\cos\theta + \mu_k \sin\theta} \quad \textit{(see graph)}$$

when θ = 0, $F' = \mu_k mg$

$F < F'$ *if* $\cos\theta + \mu_k \sin\theta > 1$, i.e. if $\mu_k$ large & θ small

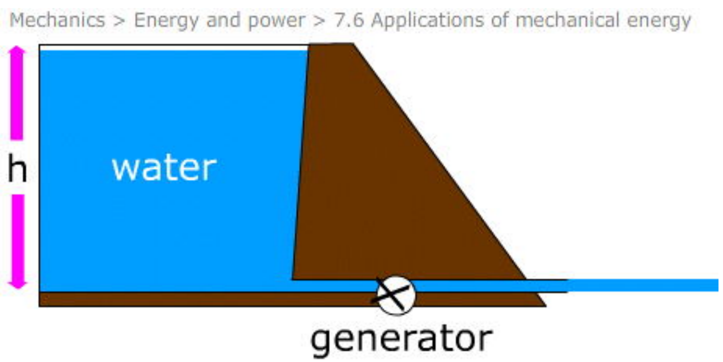
Work done = $Fs \cos\theta = F_f s$

**Example.** A hydroelectric dam is 100 m tall. Assuming that the turbines and generators are 100% efficient, and neglecting friction, calculate the flow of water required to produce 10 MW of power. The output pipes have a cross section of 5 $m^2$.




Assume that a small mass of stationary water *dm* is lost from the top of the dam but appears at the bottom with speed v, meanwhile having done work *dW* on the turbine.

- Write an equation for *dW* in this problem, and use it to obtain an expression for the power P.




Nett effect: ~ stationary water lost from ***top*** of dam, water appears with speed *v* at bottom.

$$\textit{power}\ldots\frac{dW}{dt} \quad \textit{time derivative, define } \text{flow} = \frac{dm}{dt} \;.$$

$$dW \equiv \begin{matrix}\text{work done}\\ \text{by water}\end{matrix}$$

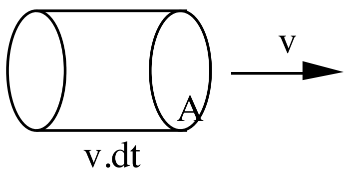
$$= -\begin{matrix}\text{Work done}\\ \text{on water}\end{matrix} = -\begin{matrix}\text{energy increase}\\ \text{of water}\end{matrix}$$

$$dW = -dE = -dK - dU$$

$$= -\left(\frac{1}{2}dmv^2 - 0\right) - (0 - dm.gh) = dm\left(gh - \frac{v^2}{2}\right)$$

$$P = \frac{dW}{dt} = \frac{dm}{dt}\left(gh - \frac{v^2}{2}\right)$$

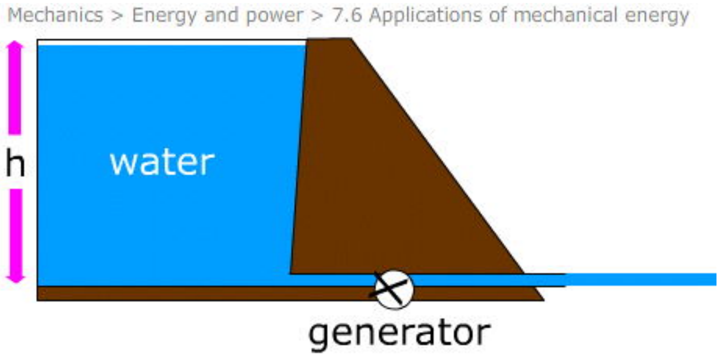
Now, how to relate *dm/dt* to the velocity *v*?




$$\frac{dV}{dt} = \frac{A.(v.dt)}{dt} = Av$$

Density: $\rho \equiv \frac{mass}{volume} = \frac{m}{V} \quad so\ m = \rho V$

$$\frac{dm}{dt} =$$




Nett effect: ~ stationary water lost from ***top*** of dam, water appears with speed $v$ at bottom.

*power*... $\frac{dW}{dt}$ *time derivative, define* flow = $\frac{dm}{dt}$ .

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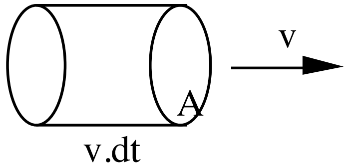
$$P = \frac{dW}{dt} = \frac{dm}{dt}\left(gh - \frac{v^2}{2}\right)$$

Problem: $v$ depends on $\frac{dm}{dt}$



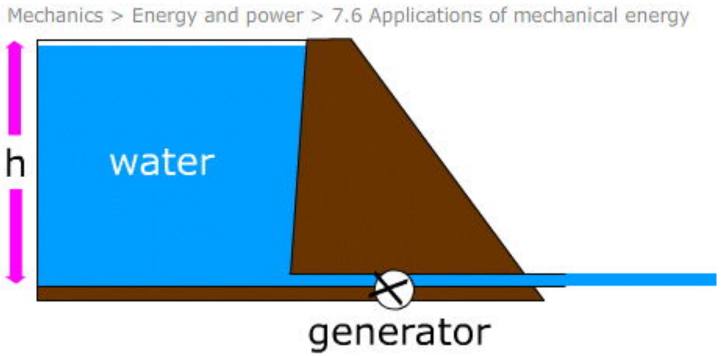

$$\frac{dV}{dt} = \frac{A.(v.dt)}{dt} = Av$$

Density: $\rho \equiv \frac{mass}{volume} = \frac{m}{V}$ so $m = \rho V$

$$\frac{dm}{dt} = \rho\frac{dV}{dt} = \rho Av$$

$$P = \rho Av\left(gh - \frac{v^2}{2}\right)$$

$$v^3 - (2gh)v + \frac{2P}{\rho A} = 0$$ *Ugh! solve cubic?*




Nett effect: ~ stationary water lost from ***top*** of dam, water appears with speed $v$ at bottom.

*power...* $\frac{dW}{dt}$ *time derivative, define* flow = $\frac{dm}{dt}$ .

$dW \equiv$ work done by water

$=-$ Work done on water $=-$ energy increase of water

$$dW = -dE = -dK - dU$$

$$= -\left(\frac{1}{2}dmv^2 - 0\right) - (0 - dm.gh) = dm\left(gh - \frac{v^2}{2}\right)$$

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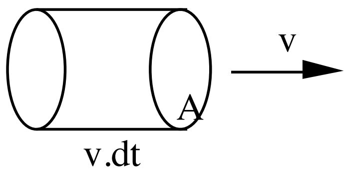
Problem: $v$ depends on $\frac{dm}{dt}$




$$\frac{\mathrm{dV}}{\mathrm{dt}} = \frac{A.(v.dt)}{dt} = Av$$

Density: $\rho \equiv \frac{mass}{volume} = \frac{m}{V}$ so $m = \rho V$

$$\frac{dm}{dt} = \rho\frac{dV}{dt} = \rho Av$$

$$P = \rho Av\left(gh - \frac{v^2}{2}\right)$$

$v^3 - (2gh)v + \frac{2P}{\rho A} = 0$ *can solve cubic, but messy*

Neglect $v^3$ –> $v = \frac{P}{gh\rho A}$ = 2 m/s

Flow = $vA$ = 10 m$^3$/s substitute: it ***is*** a solution.

**Example:**

How much work is required to accelerate a car

i) from 0 to 10 km/hr?

ii) from 100 to 110 km/hr?

(As asked, this is work done by *total* force: it includes *negative* work done by air resistance)

Work energy theorem

$$W_{total} = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2$$

i) $\frac{1}{2}(1000\ \mathrm{kg})\left(\frac{10\ 000\mathrm{m}}{3\ 600\ \mathrm{s}}\right)^2 - 0 \qquad = 4\ \mathrm{kJ}$

ii) $\frac{1}{2}(1000\mathrm{kg})\left(\frac{110\ 000\mathrm{m}}{3600\mathrm{s}}\right)^2 - \frac{1}{2}(1000\mathrm{kg})\left(\frac{100\ 000\mathrm{m}}{3600\mathrm{s}}\right)^2 = 80\ \mathrm{kJ}$

$$dW = dK = d\left(\frac{1}{2} m v^2\right) = mv\, dv$$

*Imagine doing this over the same time. Acceleration and* ***total*** *force would be the same but distance is greater at higher speed.*

**Energy density:**

**Small rechargeable NiCad**:

600 mA.hr and 1.25 V

–> (0.6 A)(3600 s)(1.25 V) = 3 kJ

(3 kJ)/(20 g) = 150 kJ/kg = 0.15 MJ/kg

**Car (lead acid) battery**:

Up to 100 Amp hours @ 12 V –> 4 MJ

< 0.5 MJ/kg

**Lithium ion:**

100 W.hour/kg -> 0.36 MJ/kg (some quote up to 0.9 MJ/kg)

**Hydroelectric dam**, 100 m high

$$\frac{mgh}{m} = gh = 0.001 \text{ MJ/kg}$$

| | MJ/litre | MJ/kg |
|---|---|---|
| | | *Not counting oxygen* |
| **Petrol** | 34 | 46 |
| **LPG** | 26 | 46 |
| **Ethanol** | 24 | 30 |
| **Diesel** | 36 | 48 |

*Which is why we're addicted to fossil fue*

**Carbohydrate or protein** 17

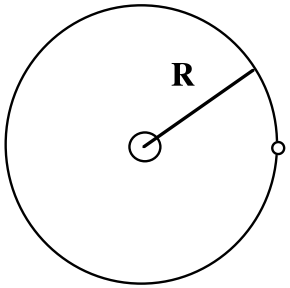
**Nuclear fuels** $\sim 8\text{x}10^7$ MJ/kg

**Speeding bullet**

$$\frac{\frac{1}{2}mv^2}{m} = \frac{1}{2}v^2 \sim \frac{1}{2}(500\text{ m/s})^2 = 0.1 \text{ MJ/kg}$$

*not our syllabus, but interesting*

**Example:** What is intensity of solar radiation? $P_{sun} = 3.9\ 10^{26}$ W. Earth is 150 million km from sun.

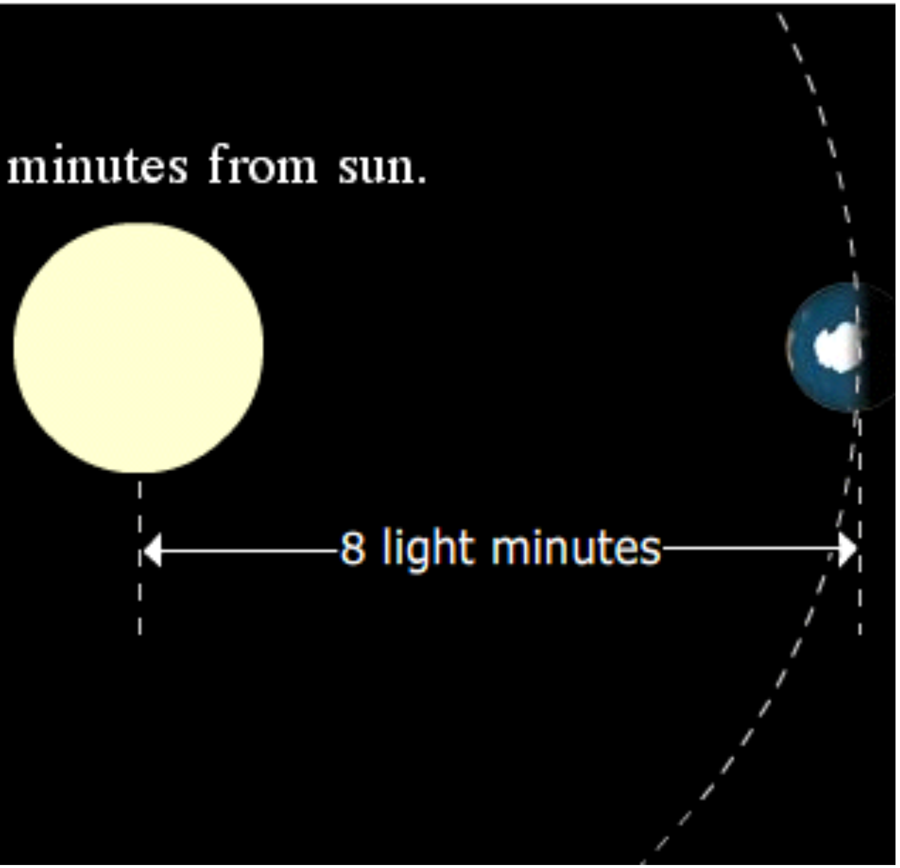



$$\text{Intensity} \equiv \frac{P}{4\pi r^2} = .... = 1.38\ \text{kWm}^{-2}$$

*called 'solar constant', above atmosphere, at right angles to radiation*

*common, practical solar cells are ~ 10% efficient.*

Example: $I_{solar} = 1.4\ \mathrm{kW.m^{-2}}$ at earth, 8 light minutes from sun.



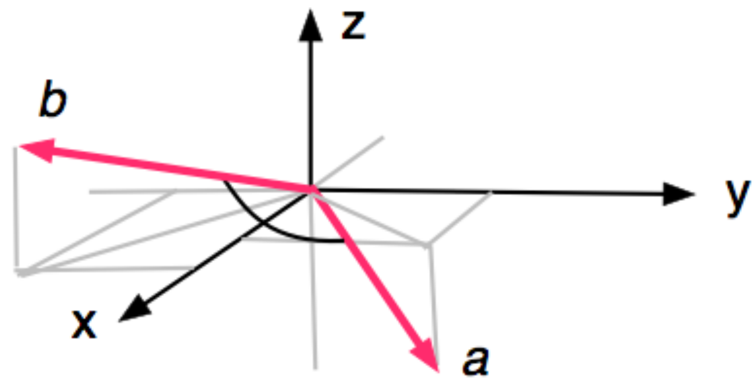

Power of sun $= I.\ 4\pi r^2$

$= (1.4\ \mathrm{kW.m^{-2}})4\pi(8\ \text{minutes} * \text{speed of light})^2$

$= 4 \times 10^{26}$ Watts

$$I = \frac{P}{4\pi r^2} \quad \rightarrow \quad I \propto \frac{1}{r^2}$$

Appendix. A geometry **problem**. Find the angle between




$\vec{a} = \ 4\,\mathbf{i} - 3\,\mathbf{j} + 2\,\mathbf{k}$

$\vec{b} = \ 2\,\mathbf{i} + 5\,\mathbf{j} - 3\,\mathbf{k}$

Tricky to do by geometry but:

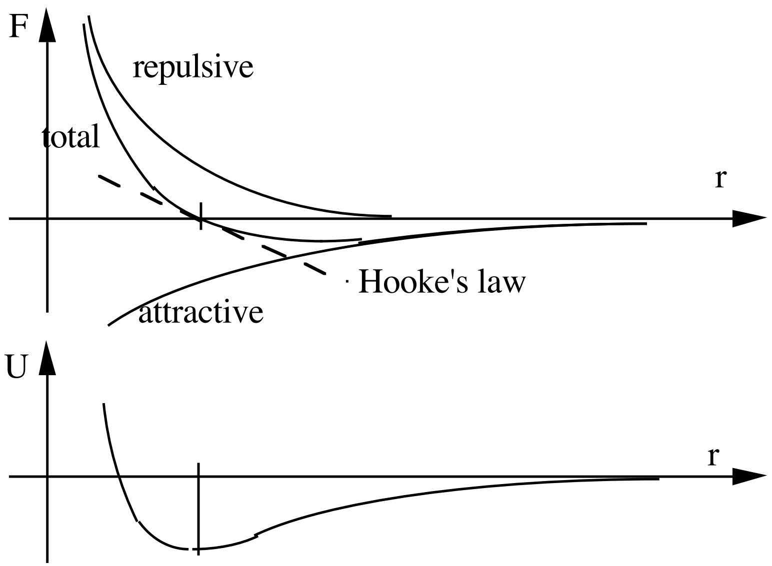
Use two different expressions for $\vec{a} \cdot \vec{b}$ : the definition and the result we found above

$ab \cos\theta = \vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$ Rearrange to get

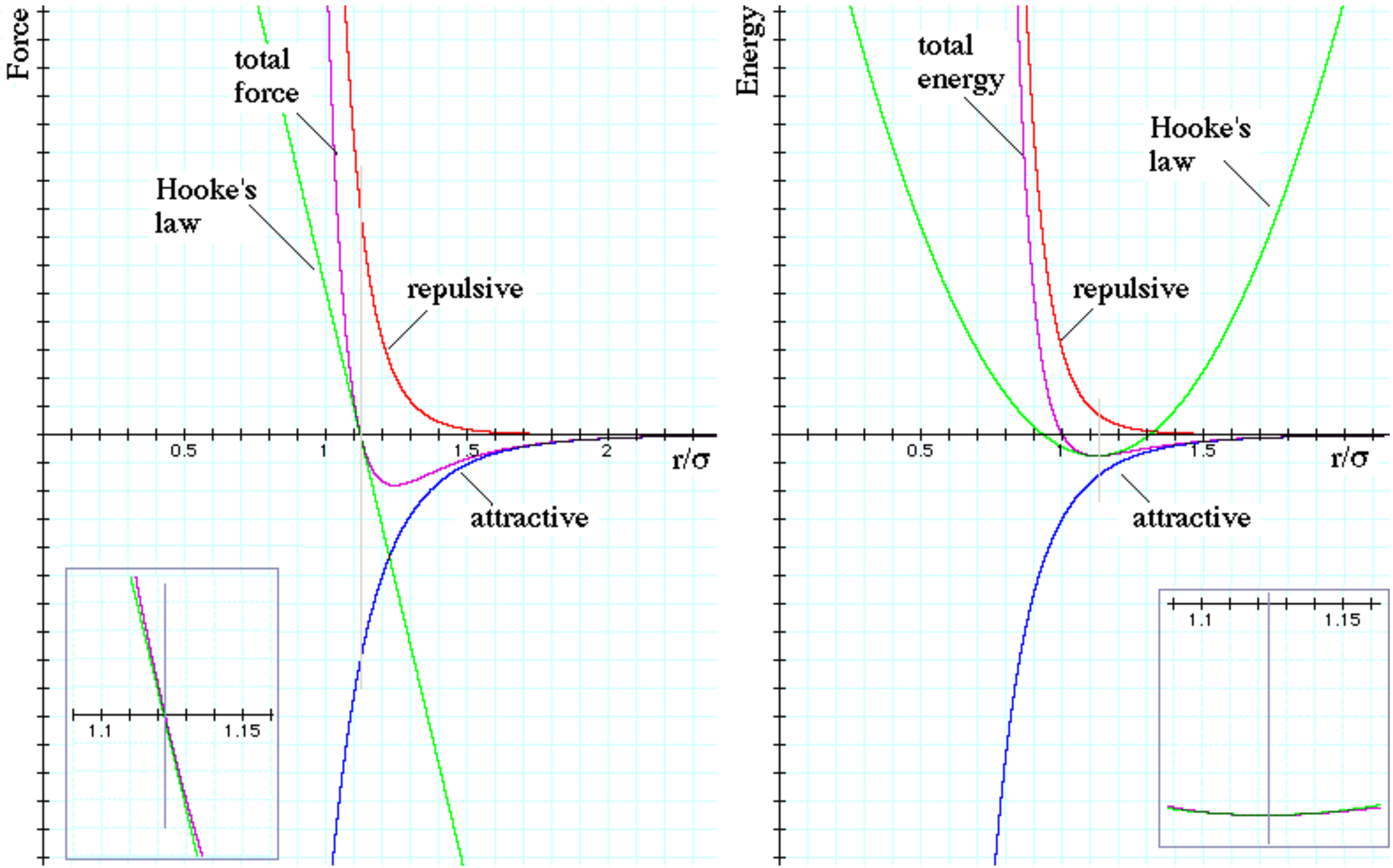
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{ab} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{b_x^2 + b_y^2 + b_z^2}} = \frac{4*2-3*5-2*3}{\sqrt{4^2+3^2+2^2}\sqrt{2^2+5^2+3^2}} = -0.39$$

So $\theta = 113°$

Remember that we described Hooke's law in terms of molecular/atomic interaction: now we can relate forces to energies




or, if we calculate it with a standard model for the interactions (see Q11, homework set 2) :




Hooke's law is in green: a linear approximation *F(x)* , which is a parabolic approximation to *U(x).*