Chapter 3

Vectors

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3-1 Vectors and Their Components

Learning Objectives

3.01 Add vectors by drawing them in head-to-tail arrangements, applying the commutative and associative laws.

3.02 Subtract a vector from a second one.

3.03 Calculate the components of a vector on a given coordinate system, showing them in a drawing.

3.04 Given the components of a vector, draw the vector and determine its magnitude and orientation.

3.05 Convert angle measures between degrees and radians.

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3-1 Vectors and Their Components

- Physics deals with quantities that have both size and direction
- A **vector** is a mathematical object with size and direction
- A **vector quantity** is a quantity that can be represented by a vector
  - Examples: position, velocity, acceleration
  - Vectors have their own rules for manipulation
- A **scalar** is a quantity that does not have a direction
  - Examples: time, temperature, energy, mass
  - Scalars are manipulated with ordinary algebra

3-1 Vectors and Their Components

- The simplest example is a **displacement vector**
- If a particle changes position from A to B, we represent this by a vector arrow pointing from A to B

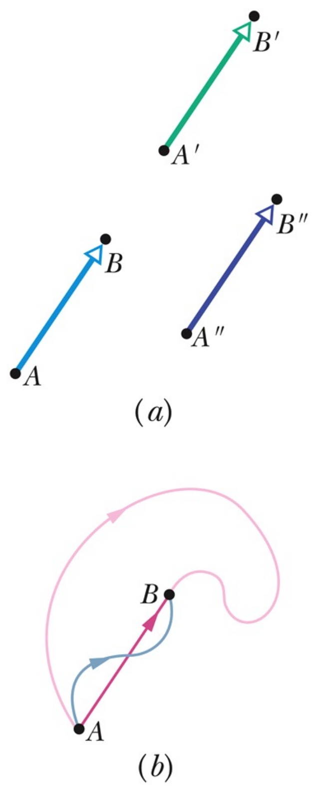


Figure 3-1

- In (a) we see that all three arrows have the same magnitude and direction: they are identical displacement vectors.
- In (b) we see that all three paths correspond to the same displacement vector. The vector tells us nothing about the actual path that was taken between A and B.

3-1 Vectors and Their Components

- The **vector sum**, or **resultant**
  - Is the result of performing vector addition
  - Represents the net displacement of two or more displacement vectors

$$\vec{s} = \vec{a} + \vec{b},$$

Eq. (3-1)

  - Can be added graphically as shown:

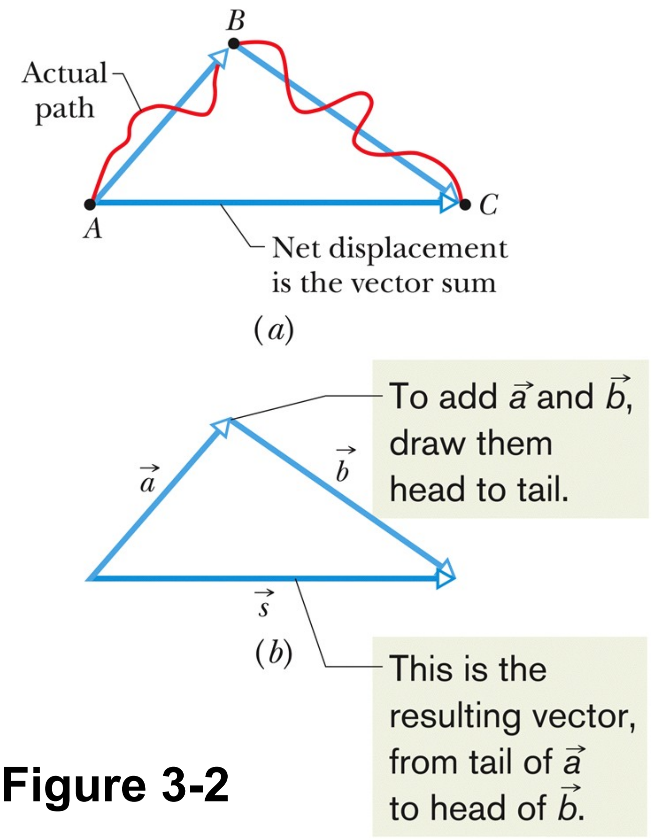


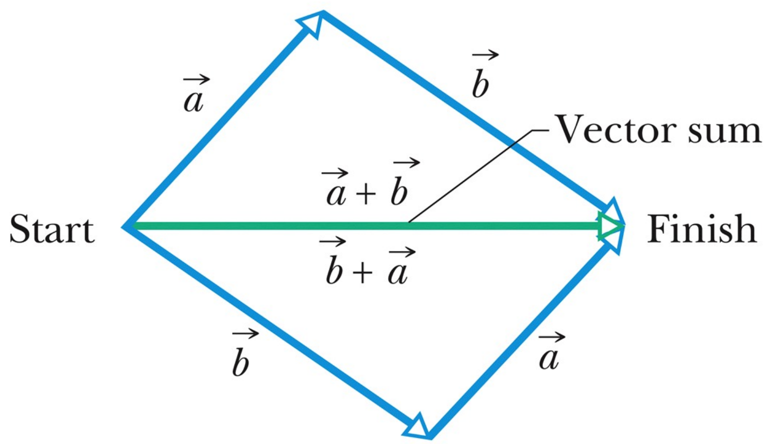
Figure 3-2

3-1 Vectors and Their Components

- Vector addition is **commutative**
  - We can add vectors in any order

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \text{(commutative law).}$$

Eq. (3-2)



You get the same vector result for either order of adding vectors.

Figure (3-3)

3-1 Vectors and Their Components

- Vector addition is **associative**
  - We can group vector addition however we like

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \qquad \text{(associative law).}$$

Eq. (3-3)

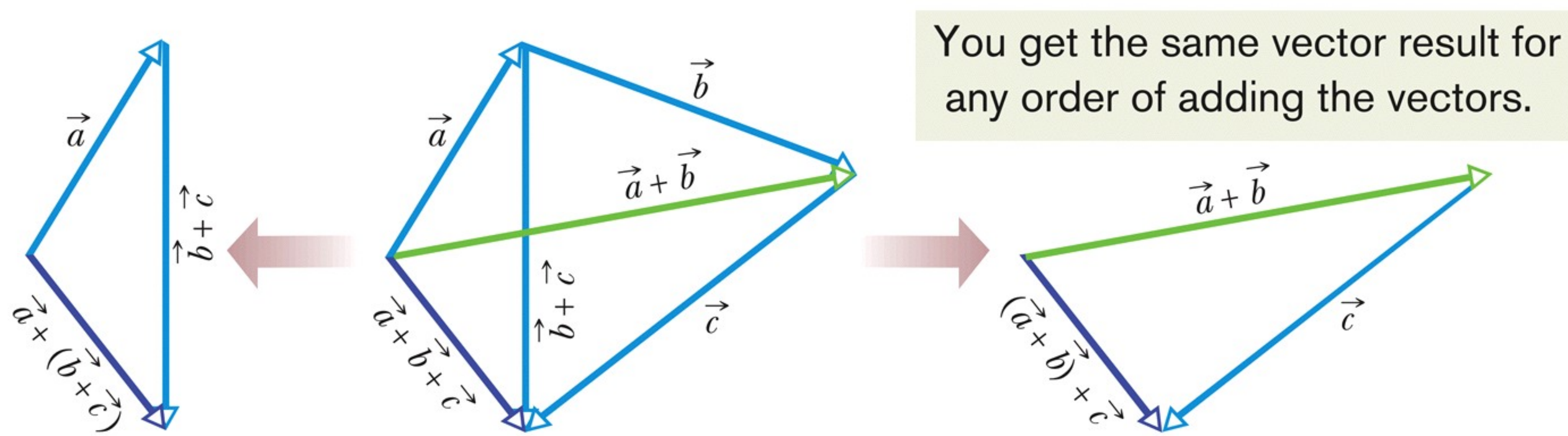


Figure (3-4)

3-1 Vectors and Their Components

- A negative sign reverses vector direction

$$\vec{b} + (-\vec{b}) = 0.$$

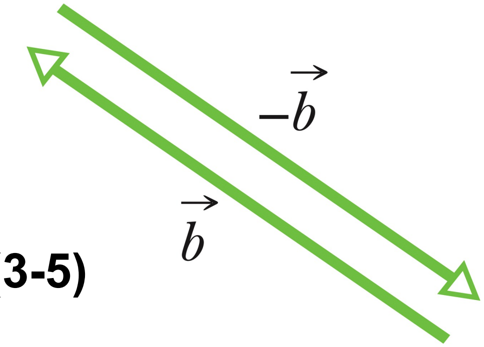


Figure (3-5)

- We use this to define vector subtraction

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

Eq. (3-4)

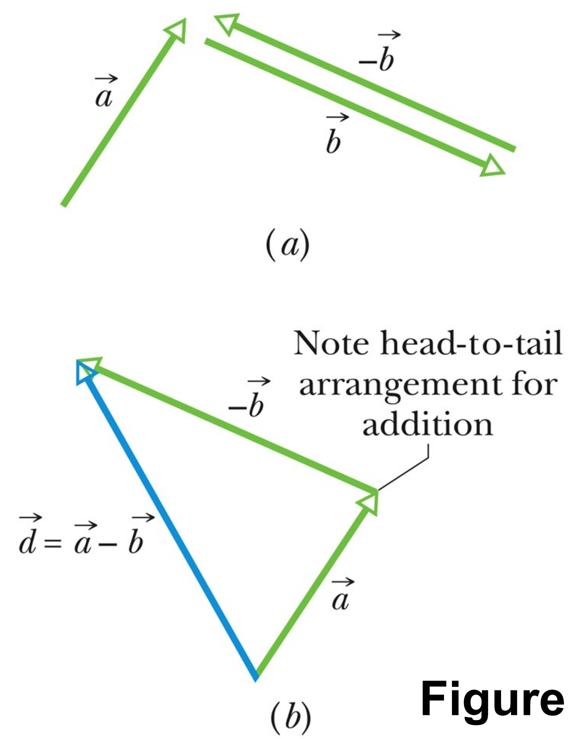


Figure (3-6)

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3-1 Vectors and Their Components

- These rules hold for all vectors, whether they represent displacement, velocity, etc.
- Only vectors of the same kind can be added
  - (*distance*) + (*distance*) makes sense
  - (*distance*) + (*velocity*) does not

Checkpoint 1

The magnitudes of displacements $\vec{a}$ and $\vec{b}$ are 3 m and 4 m, respectively, and $\vec{c} = \vec{a} + \vec{b}$. Considering various orientations of $\vec{a}$ and $\vec{b}$, what are (a) the maximum possible magnitude for $\vec{c}$ and (b) the minimum possible magnitude?

Answer:

(a) 3 m + 4 m = 7 m (b) 4 m - 3 m = 1 m

3-1 Vectors and Their Components

- Rather than using a graphical method, vectors can be added by **components**
  - A component is the projection of a vector on an axis
- The process of finding components is called **resolving the vector**
- The components of a vector can be positive or negative.
- They are unchanged if the vector is shifted in any direction (but not rotated).

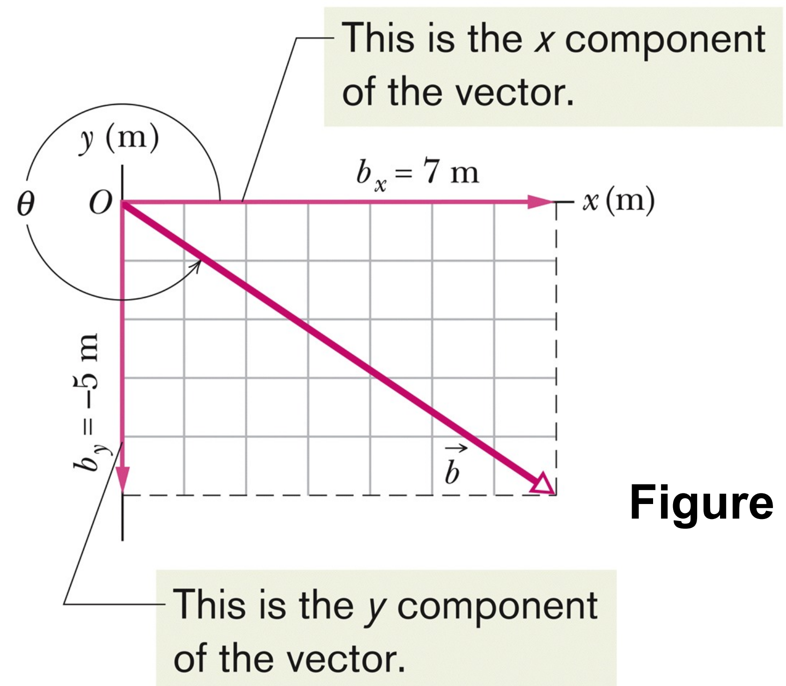


Figure (3-8)

3-1 Vectors and Their Components

- Components in two dimensions can be found by:

$$a_x = a\cos\theta \quad \text{and} \quad a_y = a\sin\theta, \qquad \textbf{Eq. (3-5)}$$

- Where θ is the angle the vector makes with the positive x axis, and a is the vector length
- The length and angle can also be found if the components are known

$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\theta = \frac{a_y}{a_x} \qquad \textbf{Eq. (3-6)}$$

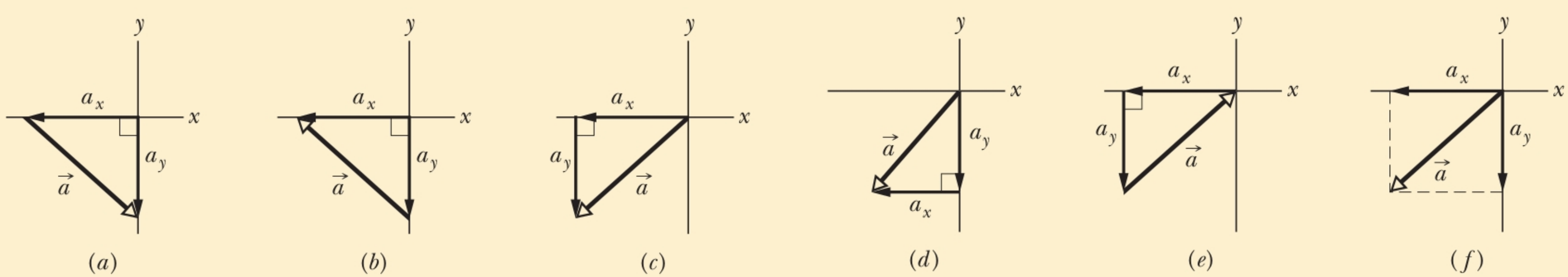
- Therefore, components fully define a vector

3-1 Vectors and Their Components

- In the three dimensional case we need more components to specify a vector
  - (a,θ,φ) or (a_x,a_y,a_z)

Checkpoint 2

In the figure, which of the indicated methods for combining the x and y components of vector $\vec{a}$ are proper to determine that vector?



Answer: choices (c), (d), and (f) show the components properly arranged to form the vector

3-1 Vectors and Their Components

- Angles may be measured in degrees or radians
- Recall that a full circle is 360˚, or 2π rad

$$40^\circ \frac{2\pi \text{ rad}}{360^\circ} = 0.70 \text{ rad.}$$

- Know the three basic trigonometric functions

$$\sin\theta = \frac{\text{leg opposite } \theta}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{leg adjacent to } \theta}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{leg opposite } \theta}{\text{leg adjacent to } \theta}$$

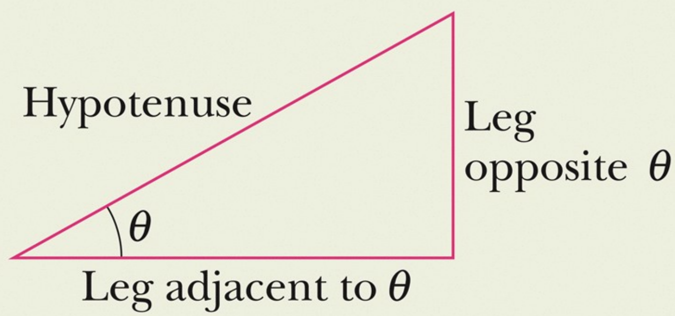


Figure (3-11)

3-2 Unit Vectors, Adding Vectors by Components

Learning Objectives

3.06 Convert a vector between magnitude-angle and unit-vector notations.

3.07 Add and subtract vectors in magnitude-angle notation and in unit-vector notation.

3.08 Identify that, for a given vector, rotating the coordinate system about the origin can change the vector's components, but not the vector itself.

3-2 Unit Vectors, Adding Vectors by Components

- A **unit vector**
  - Has magnitude 1
  - Has a particular direction
  - Lacks both dimension and unit
  - Is labeled with a hat: ^
- We use a **right-handed coordinate system**
  - Remains right-handed when rotated

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} \quad \textbf{Eq. (3-7)}$$

$$\vec{b} = b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}}. \quad \textbf{Eq. (3-8)}$$

The unit vectors point along axes.

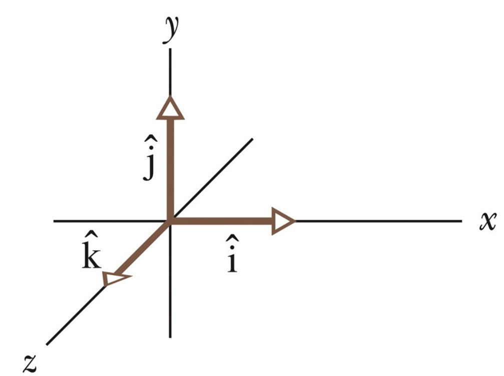


Figure (3-13)

3-2 Unit Vectors, Adding Vectors by Components

- The quantities a_x**i** and a_y**j** are **vector components**

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}}$$ **Eq. (3-7)**

$$\vec{b} = b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}}.$$ **Eq. (3-8)**

- The quantities a_x and a_y alone are **scalar components**
  - Or just "components" as before
- Vectors can be added using components

Eq. (3-9) $$\vec{r} = \vec{a} + \vec{b}, \quad \longrightarrow$$

$$r_x = a_x + b_x$$ **Eq. (3-10)**

$$r_y = a_y + b_y$$ **Eq. (3-11)**

$$r_z = a_z + b_z.$$ **Eq. (3-12)**

3-2 Unit Vectors, Adding Vectors by Components

- To subtract two vectors, we subtract components

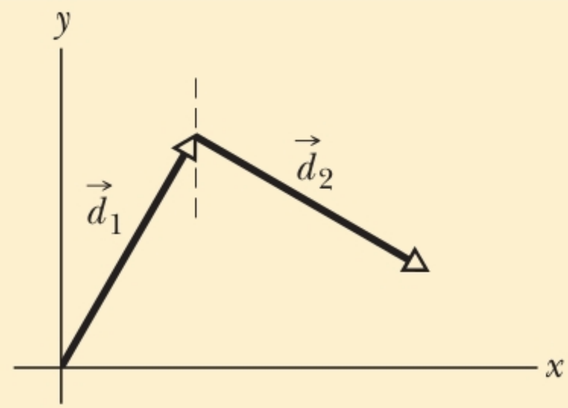
$$d_x = a_x - b_x, \quad d_y = a_y - b_y, \quad \text{and} \quad d_z = a_z - b_z,$$

Eq. (3-13)

$$\vec{d} = d_x\hat{\mathrm{i}} + d_y\hat{\mathrm{j}} + d_z\hat{\mathrm{k}}.$$

Checkpoint 3

(a) In the figure here, what are the signs of the x components of $\vec{d}_1$ and $\vec{d}_2$? (b) What are the signs of the y components of $\vec{d}_1$ and $\vec{d}_2$? (c) What are the signs of the x and y components of $\vec{d}_1 + \vec{d}_2$?



Answer: (a) positive, positive (b) positive, negative

(c) positive, positive

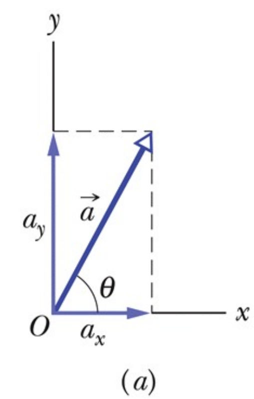
3-2 Unit Vectors, Adding Vectors by Components

- Vectors are independent of the coordinate system used to measure them
- We can rotate the coordinate system, without rotating the vector, and the vector remains the same

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{a_x'^2 + a_y'^2} \qquad \textbf{Eq. (3-14)}$$

$$\theta = \theta' + \phi. \qquad \textbf{Eq. (3-15)}$$

- All such coordinate systems are equally valid



Rotating the axes changes the components but not the vector.

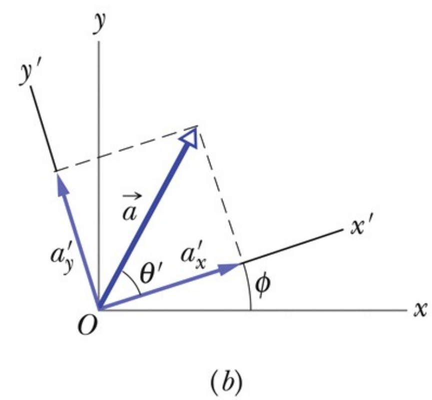


Figure (3-15)

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3-3 Multiplying Vectors

Learning Objectives

3.09 Multiply vectors by scalars.

3.10 Identify that multiplying a vector by a scalar gives a vector, the dot product gives a scalar, and the cross product gives a perpendicular vector.

3.11 Find the dot product of two vectors.

3.12 Find the angle between two vectors by taking their dot product.

3.13 Given two vectors, use the dot product to find out how much of one vector lies along the other.

3.14 Find the cross product of two vectors.

3.15 Use the right-hand rule to find the direction of the resultant vector.

3.16 In nested products, start with the innermost product and work outward.

3-3 Multiplying Vectors

- Multiplying a vector ***z*** by a scalar *c*
  - Results in a new vector
  - Its magnitude is the magnitude of vector ***z*** times |*c*|
  - Its direction is the same as vector ***z***, or opposite if *c* is negative
  - To achieve this, we can simply multiply each of the components of vector ***z*** by *c*
- To divide a vector by a scalar we multiply by 1/*c*

Example Multiply vector ***z*** by 5

- $\boldsymbol{z} = -3\,\mathbf{i} + 5\,\mathbf{j}$
- $5\,\boldsymbol{z} = -15\,\mathbf{i} + 25\,\mathbf{j}$

3-3 Multiplying Vectors

- Multiplying two vectors: the **scalar product**
  - Also called the **dot product**
  - Results in a scalar, where *a* and *b* are magnitudes and *φ* is the angle between the directions of the two vectors:

$$\vec{a} \cdot \vec{b} = ab \cos \phi,$$

Eq. (3-20)

- The commutative law applies, and we can do the dot product in component form

$$\vec{a} \cdot \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \cdot (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}),$$

Eq. (3-22)

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}.$$

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z.$$

Eq. (3-23)

3-3 Multiplying Vectors

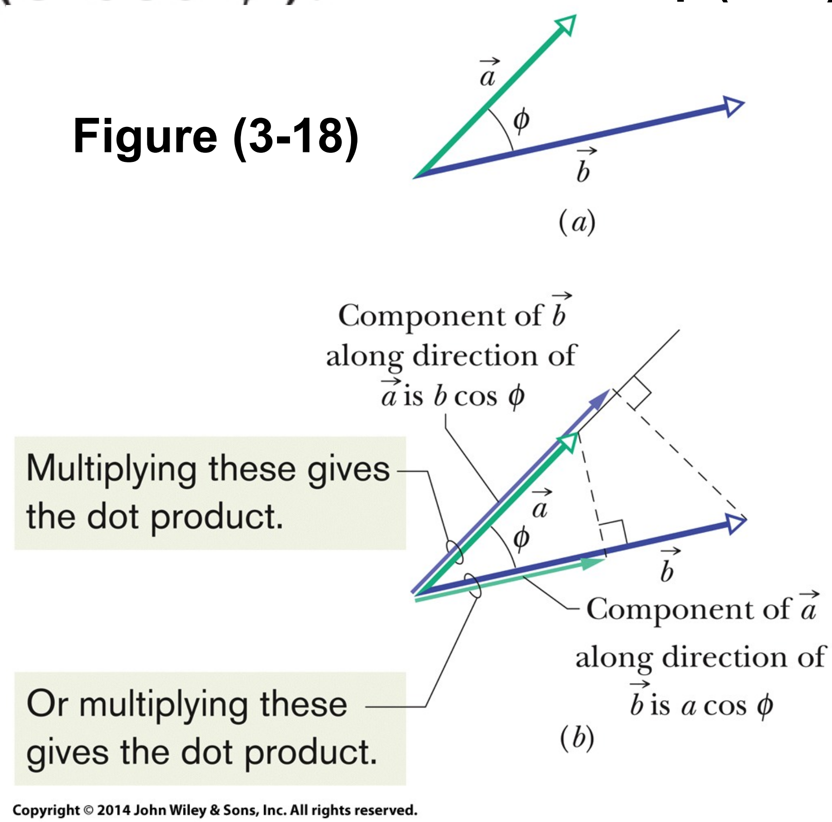
- A dot product is: the product of the magnitude of one vector times the scalar component of the other vector in the direction of the first vector

$$\vec{a} \cdot \vec{b} = (a \cos \phi)(b) = (a)(b \cos \phi).$$

Eq. (3-21)

- Either projection of one vector onto the other can be used
- To multiply a vector by the projection, multiply the magnitudes

Figure (3-18)



3-3 Multiplying Vectors

If the angle ϕ between two vectors is $0°$, the component of one vector along the other is maximum, and so also is the dot product of the vectors. If, instead, ϕ is $90°$, the component of one vector along the other is zero, and so is the dot product.

Checkpoint 4

Vectors $\vec{C}$ and $\vec{D}$ have magnitudes of 3 units and 4 units, respectively. What is the angle between the directions of $\vec{C}$ and $\vec{D}$ if $\vec{C} \cdot \vec{D}$ equals (a) zero, (b) 12 units, and (c) -12 units?

Answer: (a) 90 degrees (b) 0 degrees (c) 180 degrees

3-3 Multiplying Vectors

- Multiplying two vectors: the **vector product**
  - The **cross product** of two vectors with magnitudes *a* & *b*, separated by angle *φ*, produces a vector with magnitude:

$$c = ab \sin \phi, \qquad \text{Eq. (3-24)}$$

  - And a direction perpendicular to both original vectors
- Direction is determined by the **right-hand rule**
- Place vectors tail-to-tail, sweep fingers from the first to the second, and thumb points in the direction of the resultant vector

If $\vec{a}$ and $\vec{b}$ are parallel or antiparallel, $\vec{a} \times \vec{b} = 0$. The magnitude of $\vec{a} \times \vec{b}$, which can be written as $|\vec{a} \times \vec{b}|$, is maximum when $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

3-3 Multiplying Vectors

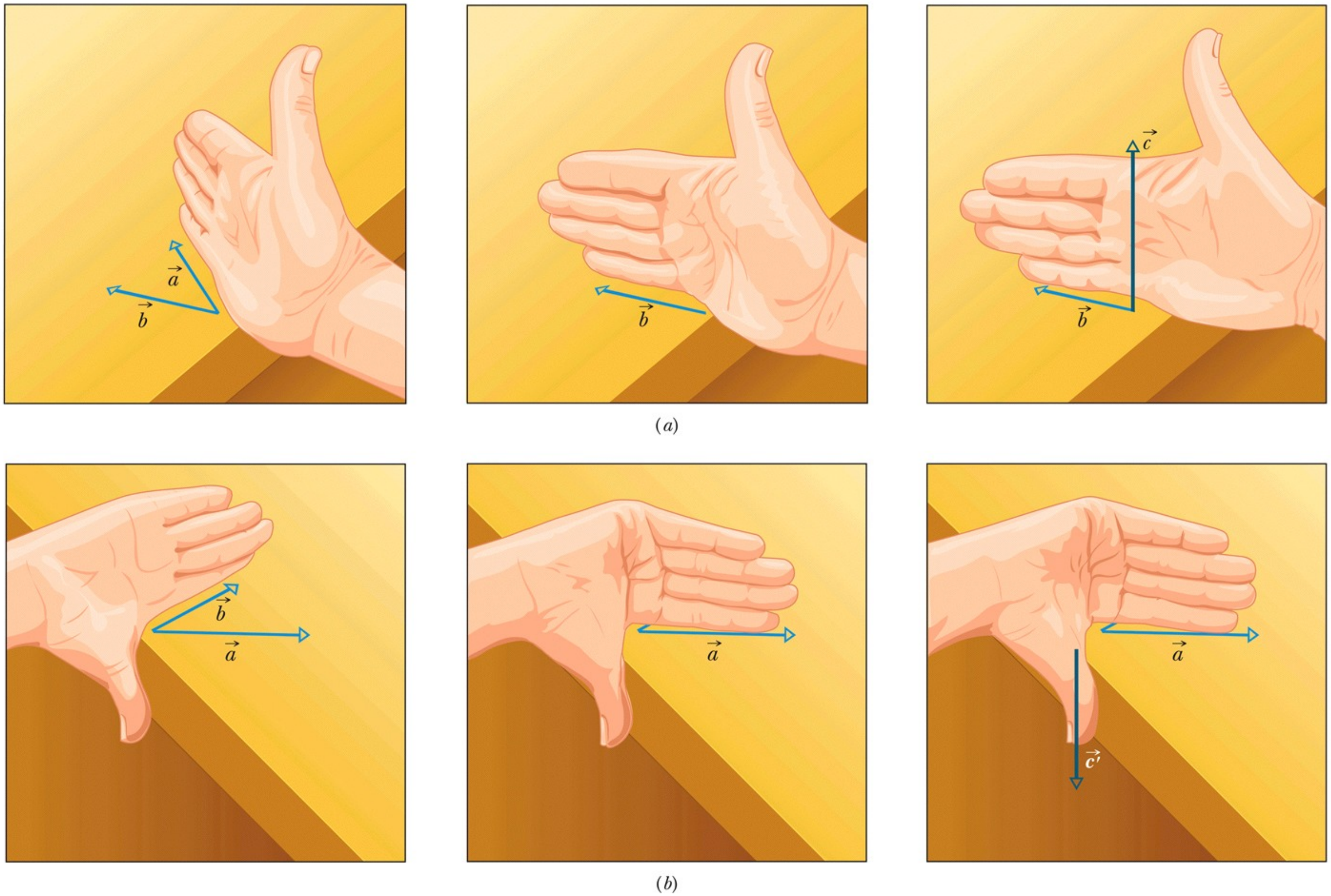


Figure (3-19)

The upper shows vector *a* cross vector *b*, the lower shows vector *b* cross vector *a*

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3-3 Multiplying Vectors

- The cross product is not commutative

$$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b}). \qquad \textbf{Eq. (3-25)}$$

- To evaluate, we distribute over components:

$$\vec{a} \times \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \times (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}), \qquad \textbf{Eq. (3-26)}$$

$$a_x\hat{\mathrm{i}} \times b_x\hat{\mathrm{i}} = a_x b_x(\hat{\mathrm{i}} \times \hat{\mathrm{i}}) = 0,$$

$$a_x\hat{\mathrm{i}} \times b_y\hat{\mathrm{j}} = a_x b_y(\hat{\mathrm{i}} \times \hat{\mathrm{j}}) = a_x b_y\hat{\mathrm{k}}.$$

- Therefore, by expanding (3-26):

$$\vec{a} \times \vec{b} = (a_y b_z - b_y a_z)\hat{\mathrm{i}} + (a_z b_x - b_z a_x)\hat{\mathrm{j}} + (a_x b_y - b_x a_y)\hat{\mathrm{k}}.$$

Eq. (3-27)

3-3 Multiplying Vectors

Checkpoint 5

Vectors $\vec{C}$ and $\vec{D}$ have magnitudes of 3 units and 4 units, respectively. What is the angle between the directions of $\vec{C}$ and $\vec{D}$ if the magnitude of the vector product $\vec{C} \times \vec{D}$ is (a) zero and (b) 12 units?

Answer: (a) 0 degrees (b) 90 degrees

3 Summary

Scalars and Vectors

- Scalars have magnitude only
- Vectors have magnitude and direction
- Both have units!

Adding Geometrically

- Obeys commutative and associative laws

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \quad \textbf{Eq. (3-2)}$$

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}). \quad \textbf{Eq. (3-3)}$$

Vector Components

- Given by

$$a_x = a \cos \theta \quad \text{and} \quad a_y = a \sin \theta, \quad \textbf{Eq. (3-5)}$$

- Related back by

$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan \theta = \frac{a_y}{a_x} \quad \textbf{Eq. (3-6)}$$

Unit Vector Notation

- We can write vectors in terms of unit vectors

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}, \quad \textbf{Eq. (3-7)}$$

3 Summary

Adding by Components

- Add component-by-component

$$r_x = a_x + b_x$$

$$r_y = a_y + b_y$$

Eqs. (3-10) - (3-12) $$r_z = a_z + b_z.$$

Scalar Times a Vector

- Product is a new vector
- Magnitude is multiplied by scalar
- Direction is same or opposite

Scalar Product

- Dot product

$$\vec{a} \cdot \vec{b} = ab \cos \phi,$$

Eq. (3-20)

$$\vec{a} \cdot \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \cdot (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}),$$

Eq. (3-22)

Cross Product

- Produces a new vector in perpendicular direction
- Direction determined by right-hand rule

$$c = ab \sin \phi,$$

Eq. (3-24)