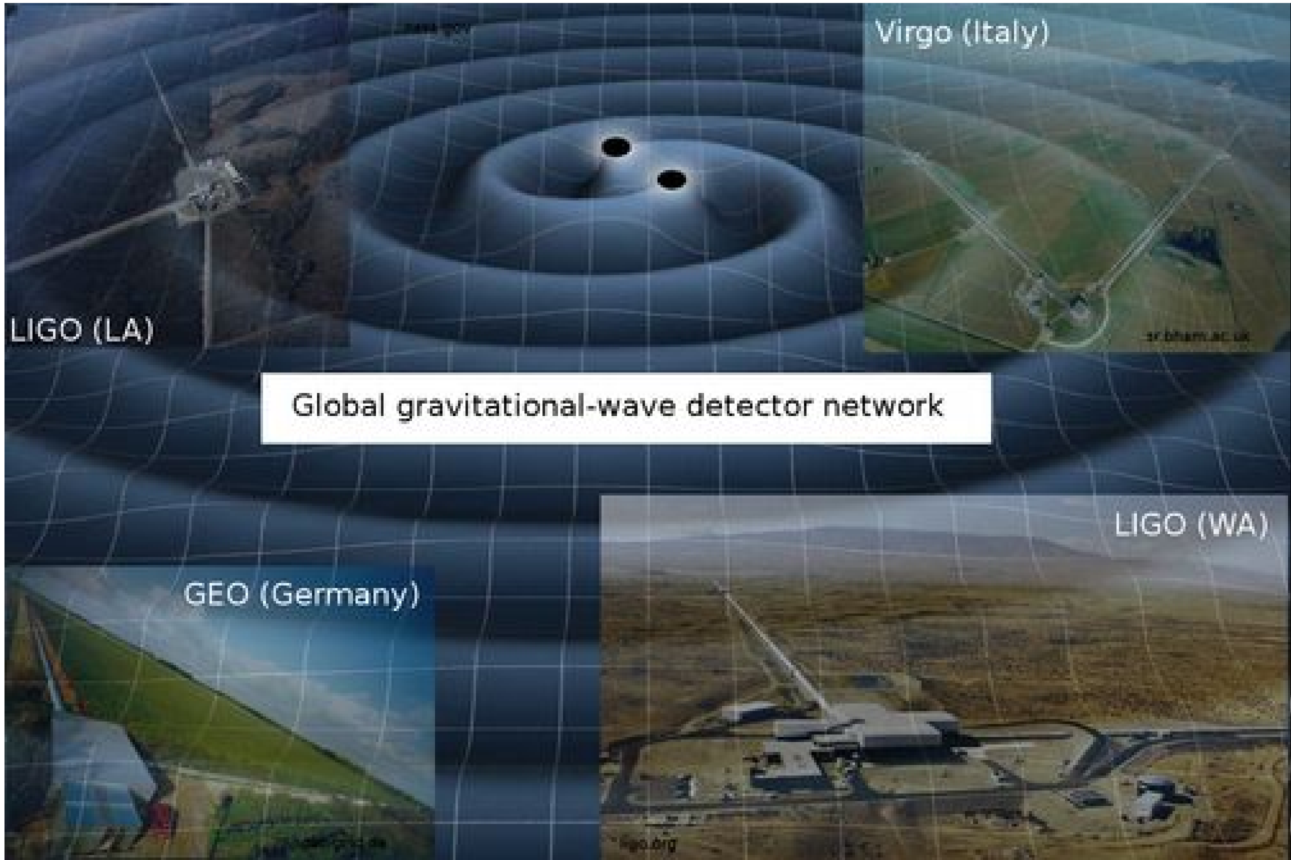
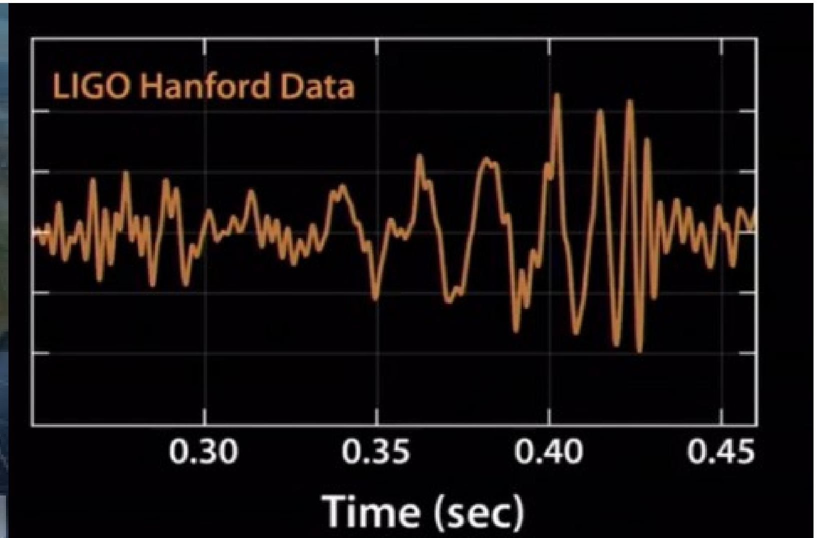
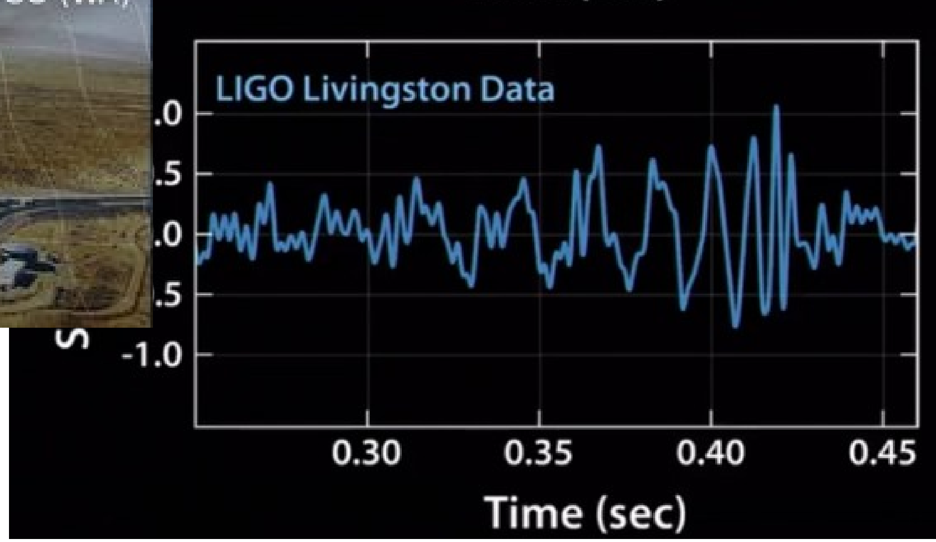
Waves and Oscillations

Lecture 9 – Pendulum, Damped & Forced Oscillations

Textbook reference: 15.1-15.6







LIGO

https://youtu.be/degD69wnZcY

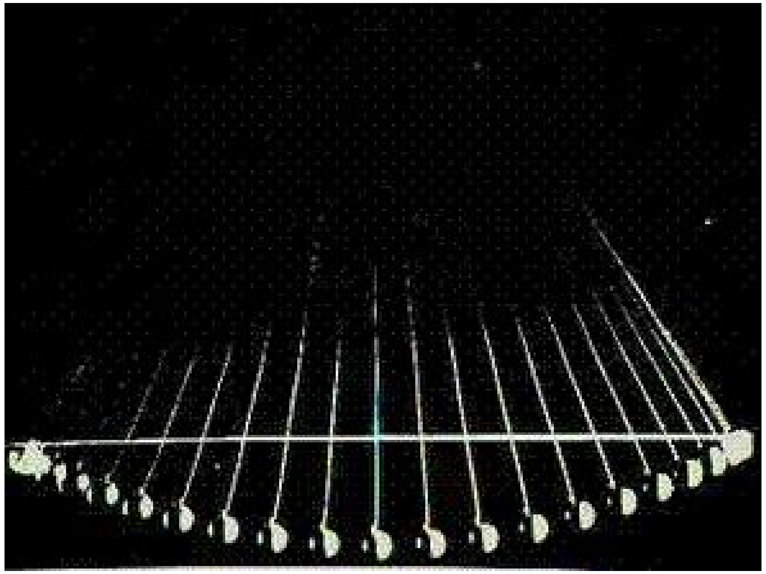
Last lecture...

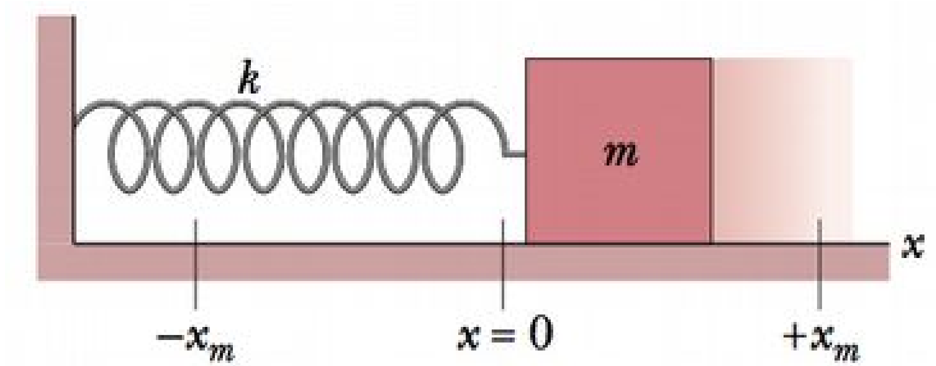
- **Simple harmonic motion** is a type of periodic motion described by

$$\frac{d^2 x}{d t^2} = -\omega^2 x$$

ω = Angular frequency

Displacement from the equilibrium (most relaxed) position (x = 0)





- Displacement:

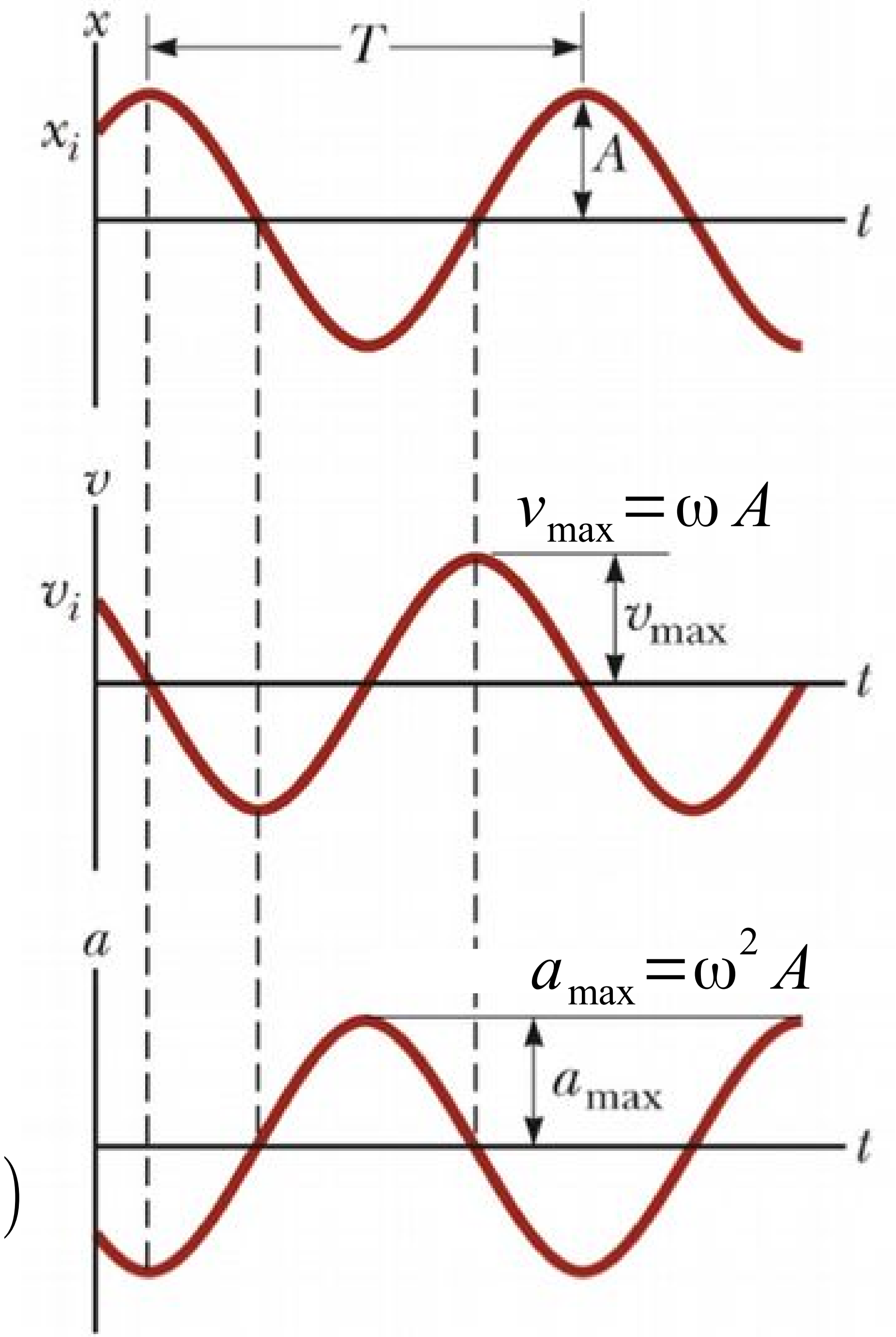
$$x(t) = A\cos(\omega t + \phi)$$

- Velocity:

$$v_x(t) = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi)$$

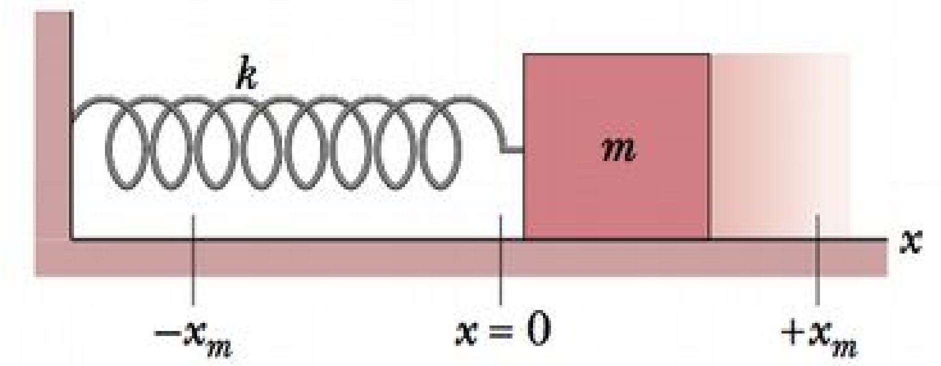
- Acceleration:

$$a_x(t) = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi)$$



Last lecture...

- Angular frequency to **period:** $$T = \frac{2\pi}{\omega}$$
- Angular frequency to **frequency**: $$f = \frac{\omega}{2\pi}$$
- For a block of mass of m attached to the end of a spring with spring constant k:



$$\omega = \sqrt{\frac{k}{m}}$$

This lecture...

- Energy
- SHM and circular motion
- Pendulum
- Damped and forced oscillations

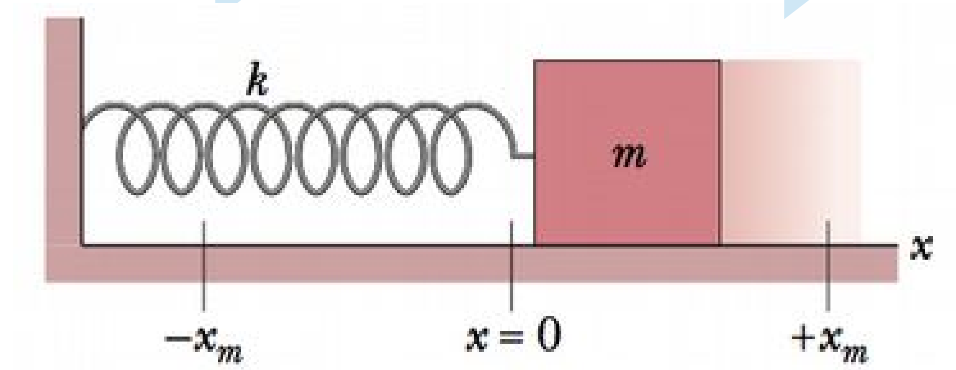
SHM>Energy...

- If there is no friction, i.e., the only force acting on the system is due to the spring, then the **total energy is conserved**.

Total energy = Potential energy + Kinetic energy

...stored in the spring

...of the moving block



$$x(t) = A\cos(\omega t + \phi) \qquad v_x(t) = -\omega A \sin(\omega t + \phi)$$

- **Kinetic energy** of the block:

$$\omega = \sqrt{\frac{k}{m}}$$

$$\mathrm{KE} = \frac{1}{2} m v^2 = \frac{1}{2} m \omega^2 A^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2} k A^2 \sin^2(\omega t + \phi)$$

- **Potential energy** stored in the spring:

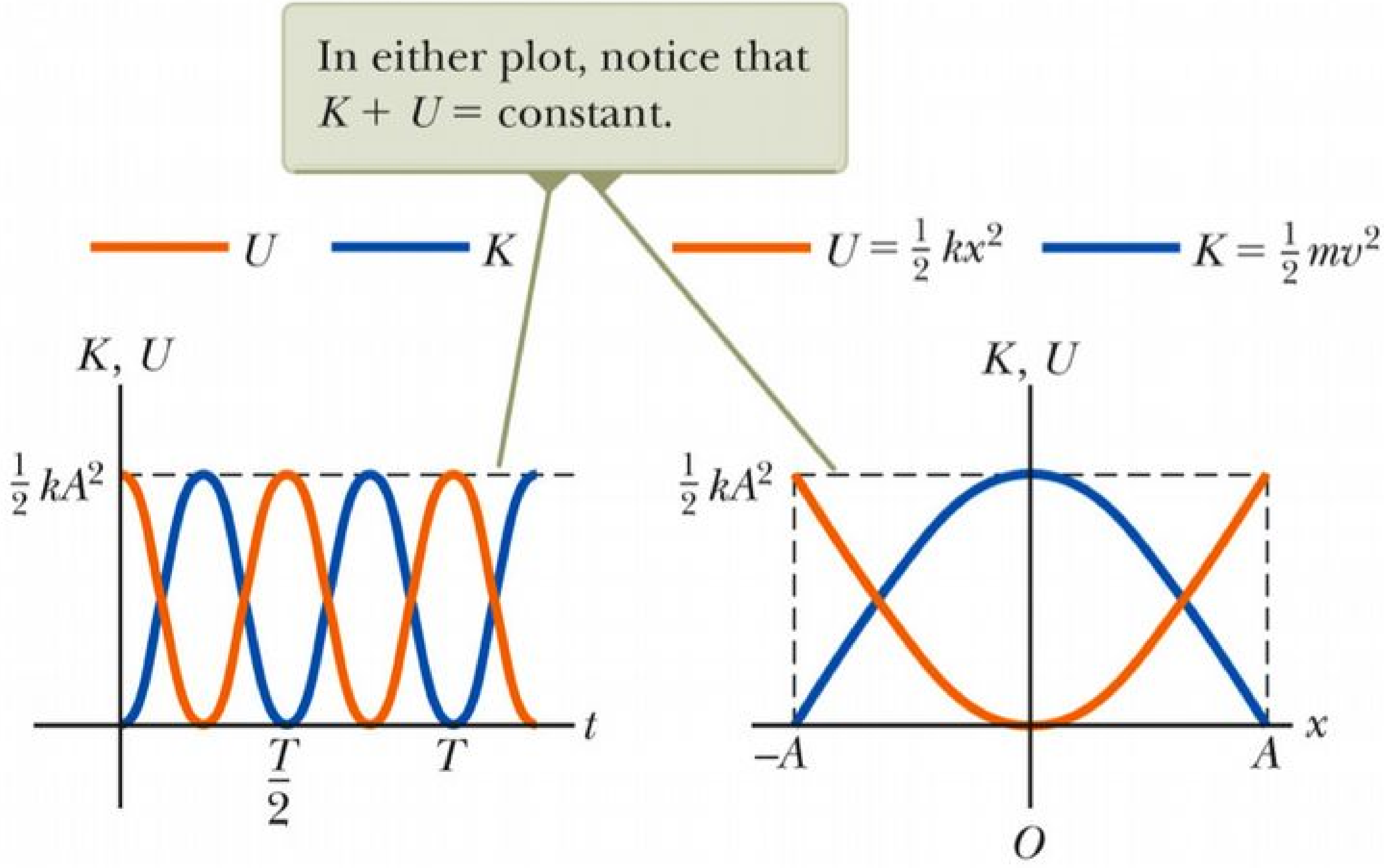
$$\mathrm{PE} = \frac{1}{2} k x^2 = \frac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

- **Total energy** (= PE + KE):

The total energy is constant.

$$\mathrm{Total} = \frac{1}{2} k A^2 \left[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\right] = \frac{1}{2} k A^2$$

SHM>Total Energy...



Conservation of energy demo

SHM>Velocity again...

- Conservation of energy means:

$$\frac{1}{2}k\,A^2=\frac{1}{2}m\,v^2+\frac{1}{2}k\,x^2$$

$$\Rightarrow \quad m\,v^2=k\left(A^2-x^2\right)$$

$$\Rightarrow \quad v=\pm\sqrt{\frac{k}{m}\left(A^2-x^2\right)}$$

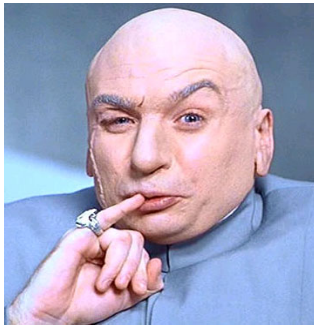
$$\boxed{\Rightarrow \quad v=\pm\omega\sqrt{\left(A^2-x^2\right)}}$$

Useful relation if you don't know the explicit time dependence of x.

Question

An object-spring system moving with simple harmonic motion has an amplitude A. When the kinetic energy of the object equals twice the potential energy stored in the spring, what is the position, *x,* of the object?

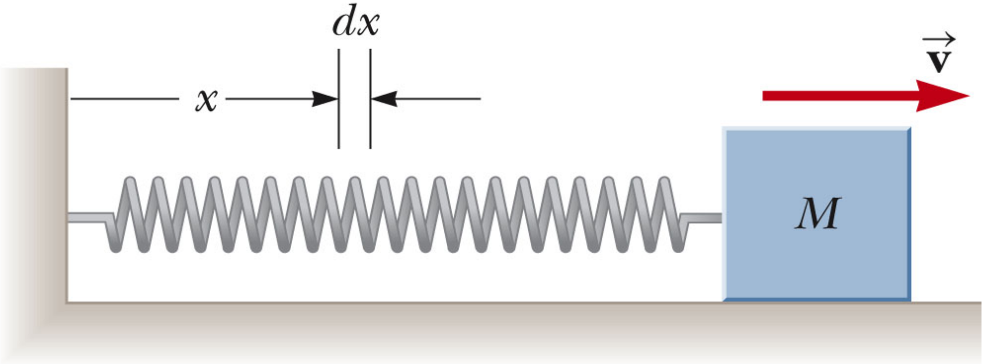
An Evil-looking Question



A block of mass M is connected to a spring of mass m and oscillates in simple harmonic motion on a frictionless, horizontal track. The force constant of the spring is k, and the equilibrium length is l. Assume all portions of the spring oscillate in phase and the velocity of a segment of spring of length dx is proportional to the distance x from the fixed end; that is, $v_x = (x/l)v$. Also notice that the mass of a segment of spring is $dm = (m/l)dx$. Find

(a) The kinetic energy of the spring when the block has velocity v and

(b) The period of oscillations.



SHM>Circular motion...

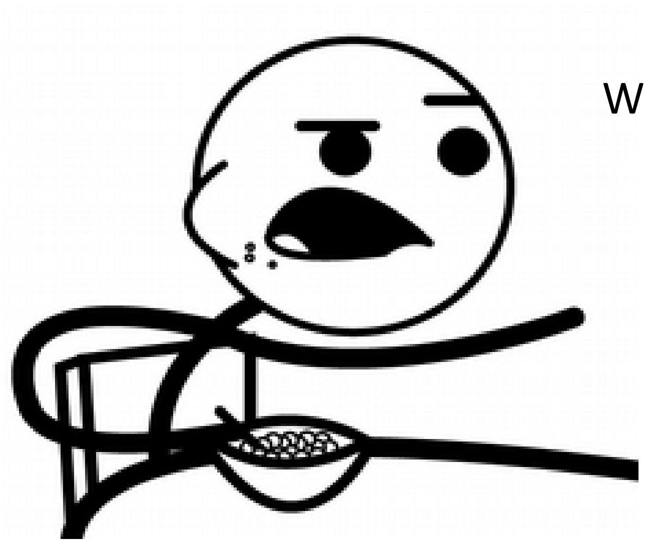
- The displacement during SHM is

$$x(t) = A\cos(\omega t + \phi)$$

where ω is called the **angular frequency**.

- Why angular??

SHM and Circular motion



Where does this "angular frequency" thing come from???

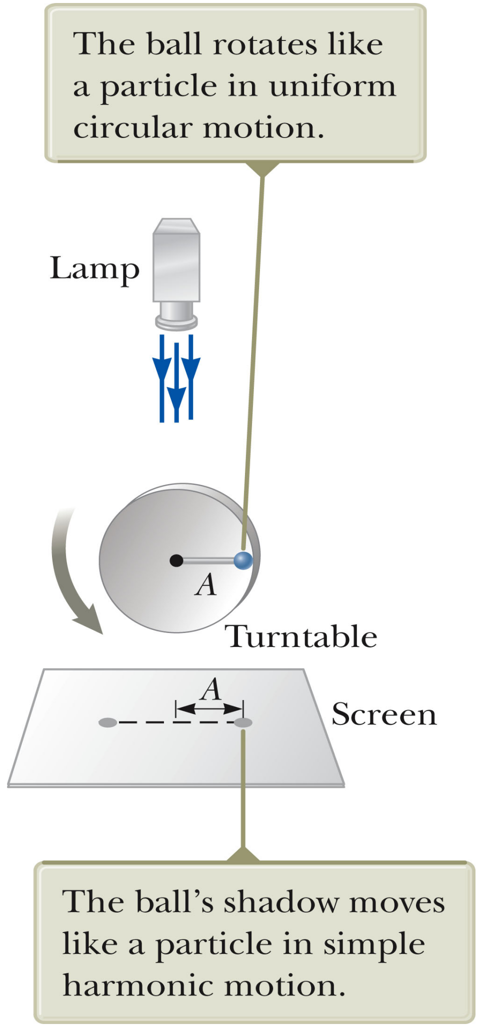
The physical meaning of ω, the angular frequency we defined in the last lecture, is something which drives many crazy in first year physics.

Now, we will draw analogies between circular motion and SHM to make the relation more clear.

SHM and Circular motion

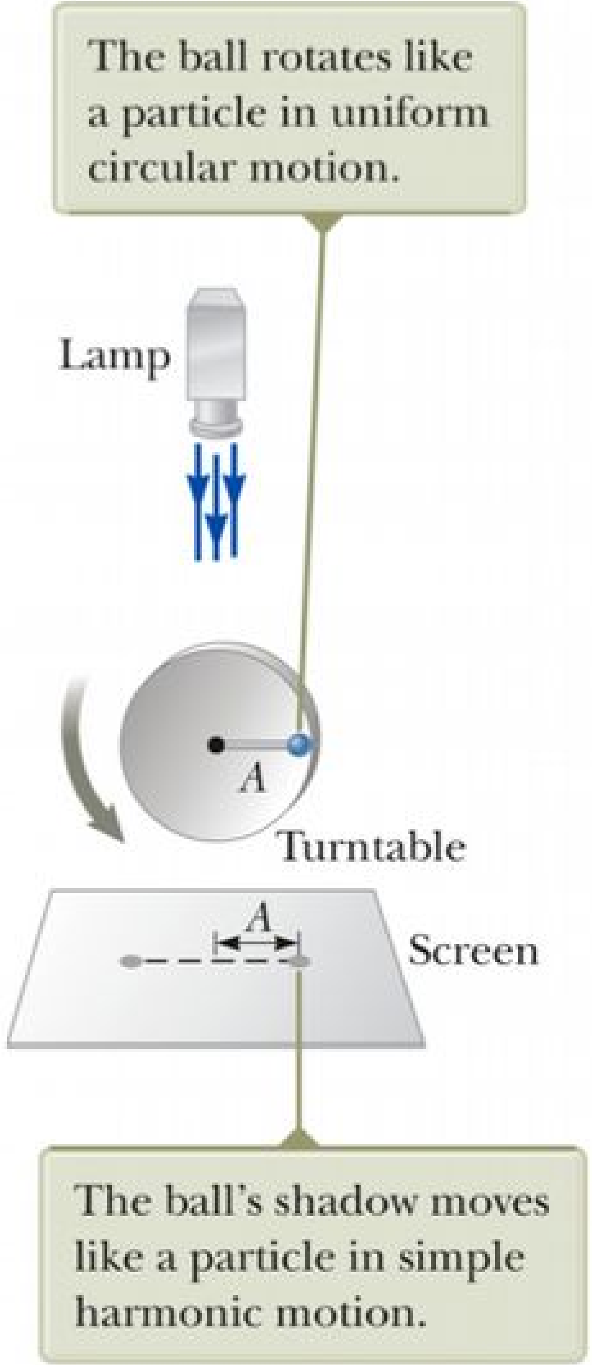
This is an overhead view of an experimental arrangement that shows the relationship between SHM and circular motion.

As the turntable rotates with constant angular speed, the ball's shadow moves back and forth in simple harmonic motion.

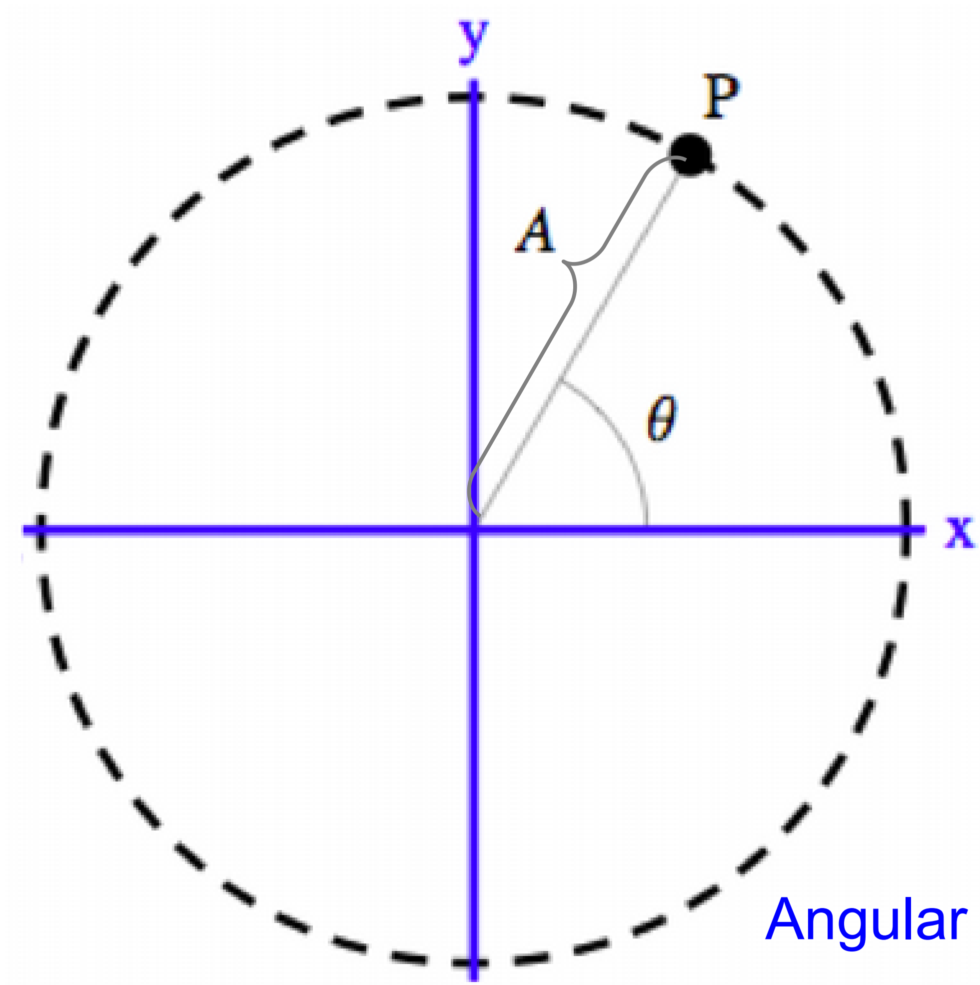


SHM>Circular motion...

- **Why angular?** Because simple harmonic motion is the projection of uniform circular motion on a diameter of the circle in which the circular motion occurs.



SHM>Circular motion...



- **Displacement** of ball:

$$x(t) = A\cos[\theta(t)]$$
$$y(t) = A\sin[\theta(t)]$$

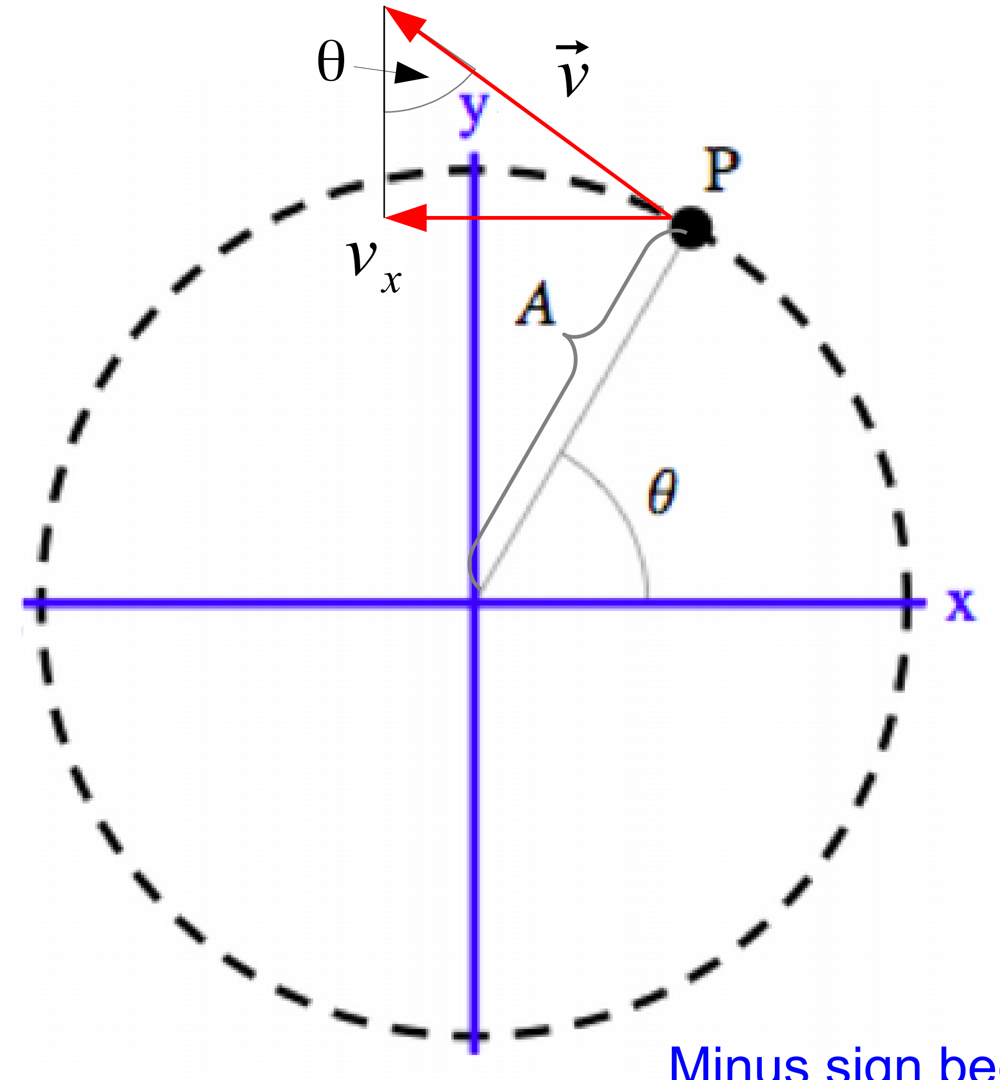
where the angle is a function of time:

$$\theta(t) = \omega t + \phi$$

Angular speed of ball

φ = Angle at time t = 0

SHM>Circular motion...



- **Speed** of ball around the circle:

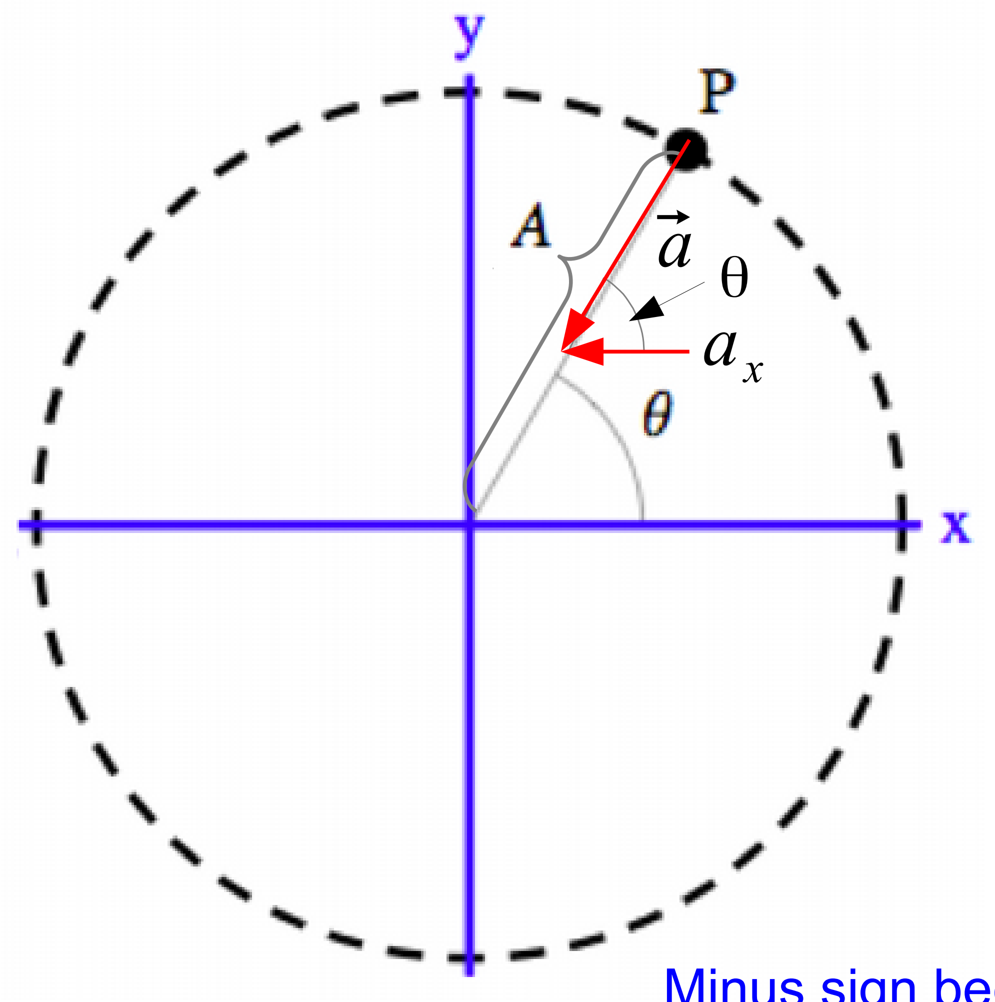
$$v = \omega A$$

→ the x-velocity is the projection of the velocity vector onto the x-axis:

$$v_x(t) = -\omega A \sin[\theta(t)]$$
$$= -\omega A \sin(\omega t + \phi)$$

Minus sign because v_x points in the negative x direction when x is positive.

SHM>Circular motion...



- **Acceleration** of ball:

$$a = \frac{v^2}{A} = \omega^2 A$$

directed towards the centre of the circle.

→ Projection onto x-axis:

$$a_x(t) = -\omega^2 A \cos[\theta(t)]$$
$$= -\omega^2 A \cos(\omega t + \phi)$$

Minus sign because a_x points in the negative x direction when x is positive.

SHM>Circular motion...

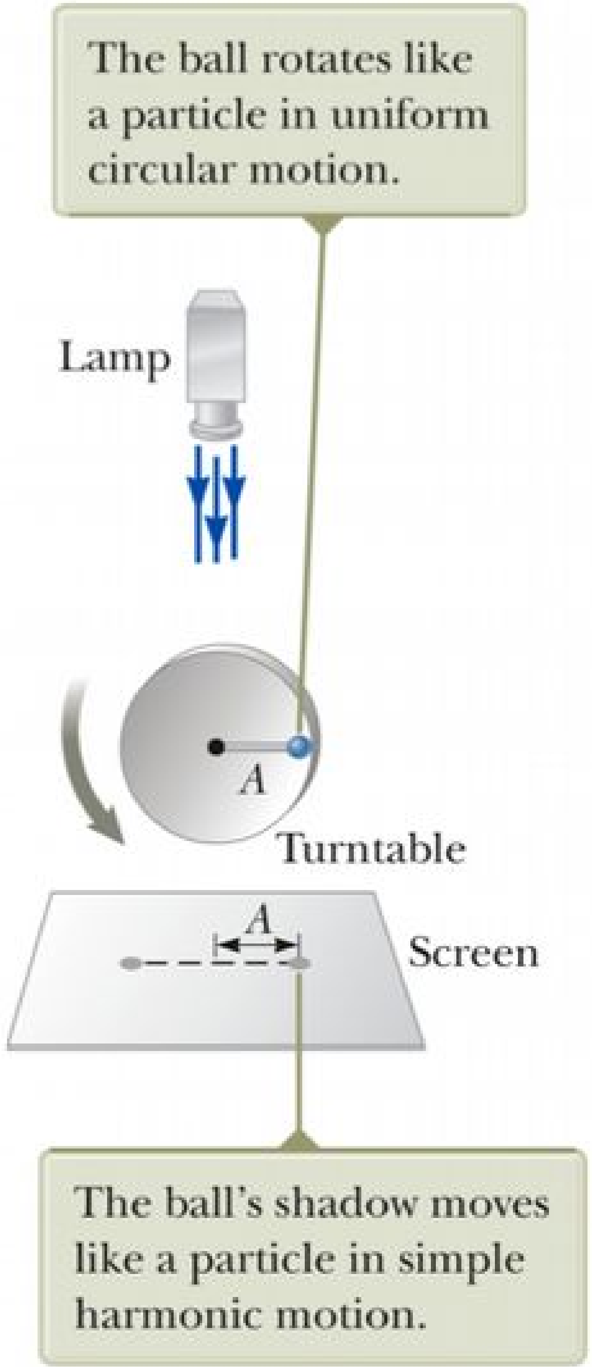
- Motion of the ball's shadow:

$$x(t) = A\cos(\omega t + \phi)$$

$$v_x(t) = -\omega A\sin(\omega t + \phi)$$

$$a_x(t) = -\omega^2 A\cos(\omega t + \phi)$$

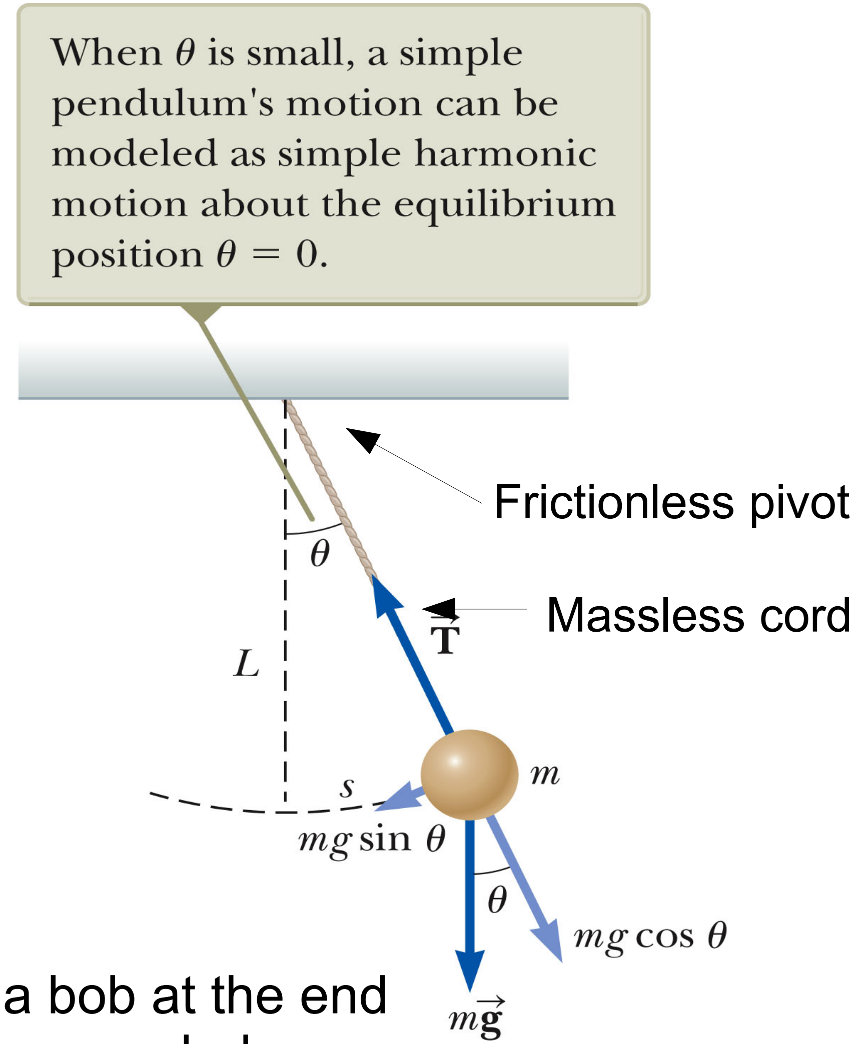
→ These are exactly the equations for simple harmonic motion, and ω is the angular speed of the ball.



The Pendulum

A pendulum also undergoes periodic motion. For small angles (θ << 1 radian; i.e. <10^{o}) it can be considered to undergo SHM.

Simple Pendulum



Simple pendulum = a bob at the end of a massless cord suspended from a frictionless pivot.

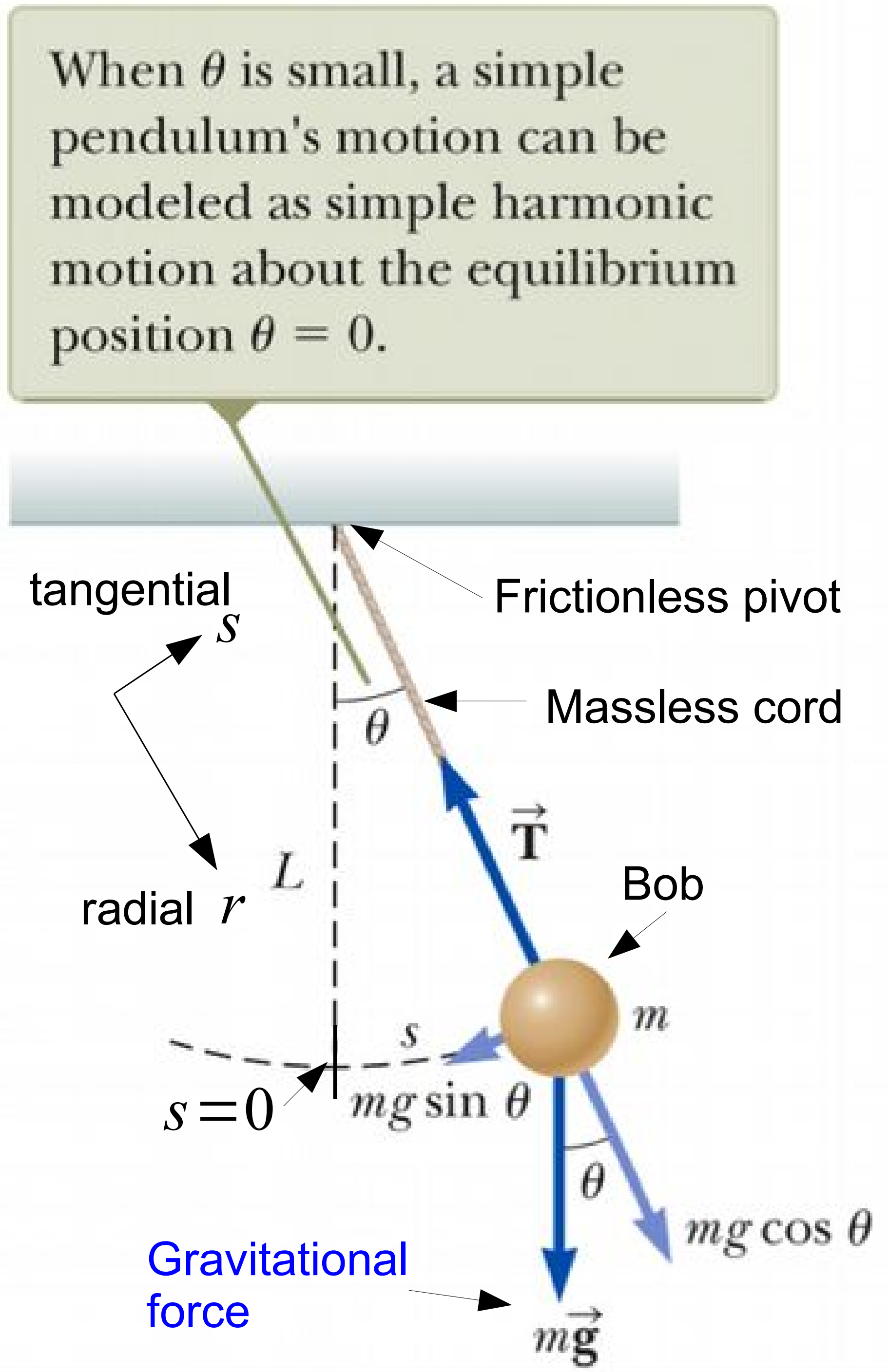
SHM>Pendulum...

- Forces acting on the bob.
  - In the radial direction:

Tension of cord

$$F_r = m g \cos\theta - T = 0$$

Zero because the bob is stationary in this direction.



SHM>Pendulum...

- Forces acting on the bob.
  - In the radial direction:

Tension of cord

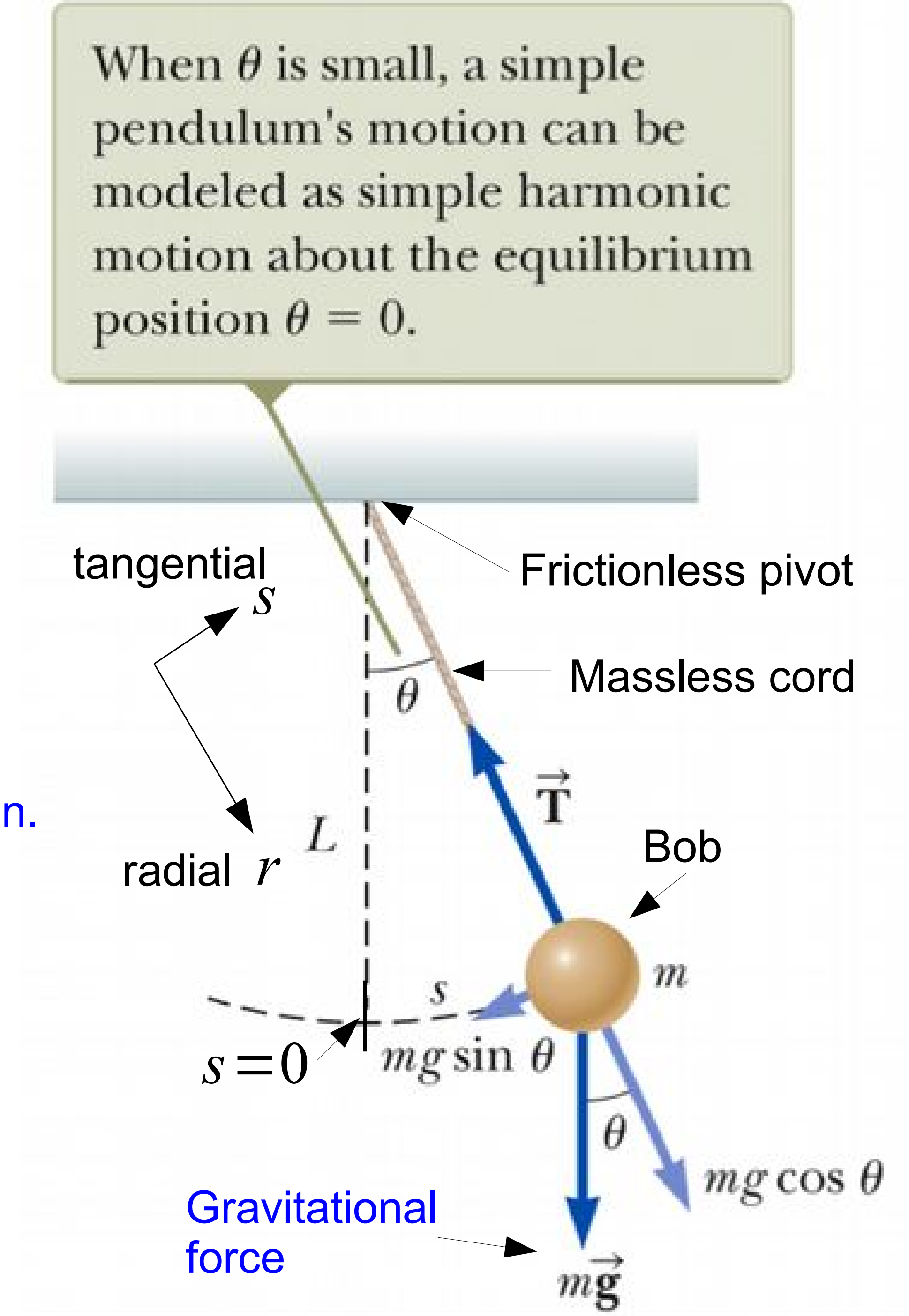
$$F_r = m\,g\cos\theta - T = 0$$

Zero because the bob is stationary in this direction.

  - In the tangential direction, there is a net force:

$$F_s = -m\,g\sin\theta$$

Note minus sign!



- The tangential motion is then described by:

$$m a_s = m \frac{d^2 s}{d t^2} = -m g \sin\theta$$

$s = L\theta$

Small angle approximation

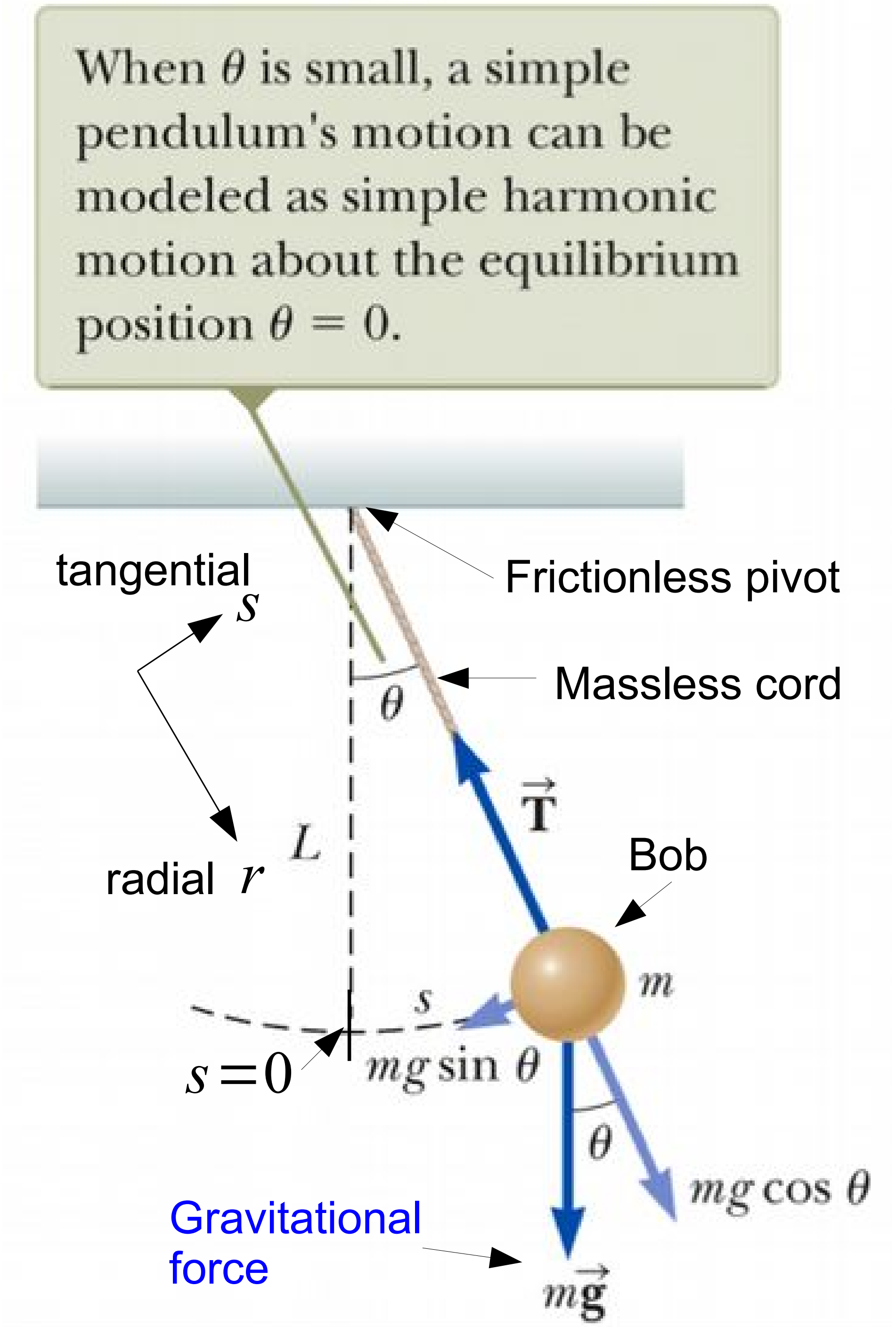
$\sin\theta \approx \theta$

if θ << 1 radian

$$m L \frac{d^2\theta}{d t^2} \qquad -m g \theta$$

$$\boxed{\frac{d^2\theta}{d t^2} = -\frac{g}{L}\theta}$$

SHM!



SHM>Simple pendulum...

- Compare
  - What we just derived:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

  - General equation of motion for SHM in terms of the angular frequency:

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta$$

There is no dependence on the mass of the bob!

Angular frequency of an oscillating pendulum

$$\boxed{\omega = \sqrt{\frac{g}{L}}}$$

Acceleration due to gravity

Length of cord

Period

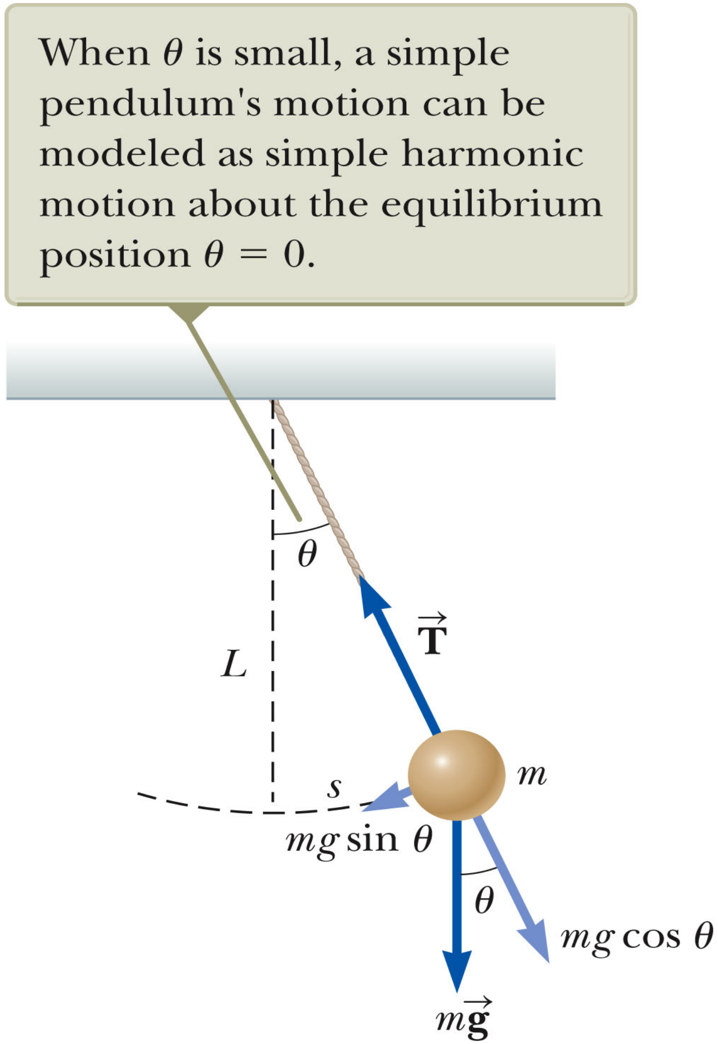
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

Demo

- Check period: L=2.7m

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

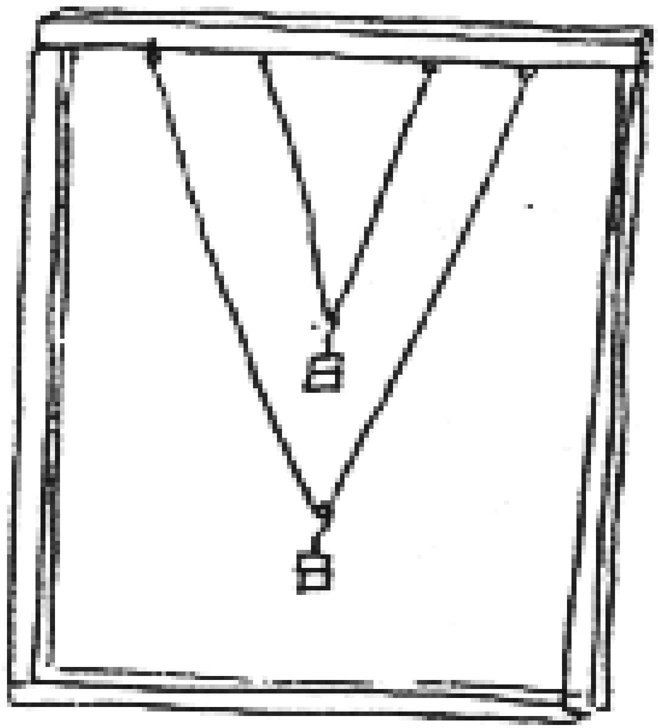
- Does it depend on mass?



Mh12: The Pendulum – period and length

Half class time top pendulum, half class time bottom pendulum.

Is T $\propto\sqrt{L}$?



SHM>Simple pendulum>Summary...

- The period and frequency of a simple pendulum depend only on the length of the string and the acceleration due to gravity.

- They are independent of the mass of the bob.

- All simple pendula that are of equal length and are at the same location oscillate with the same frequency.

Question...

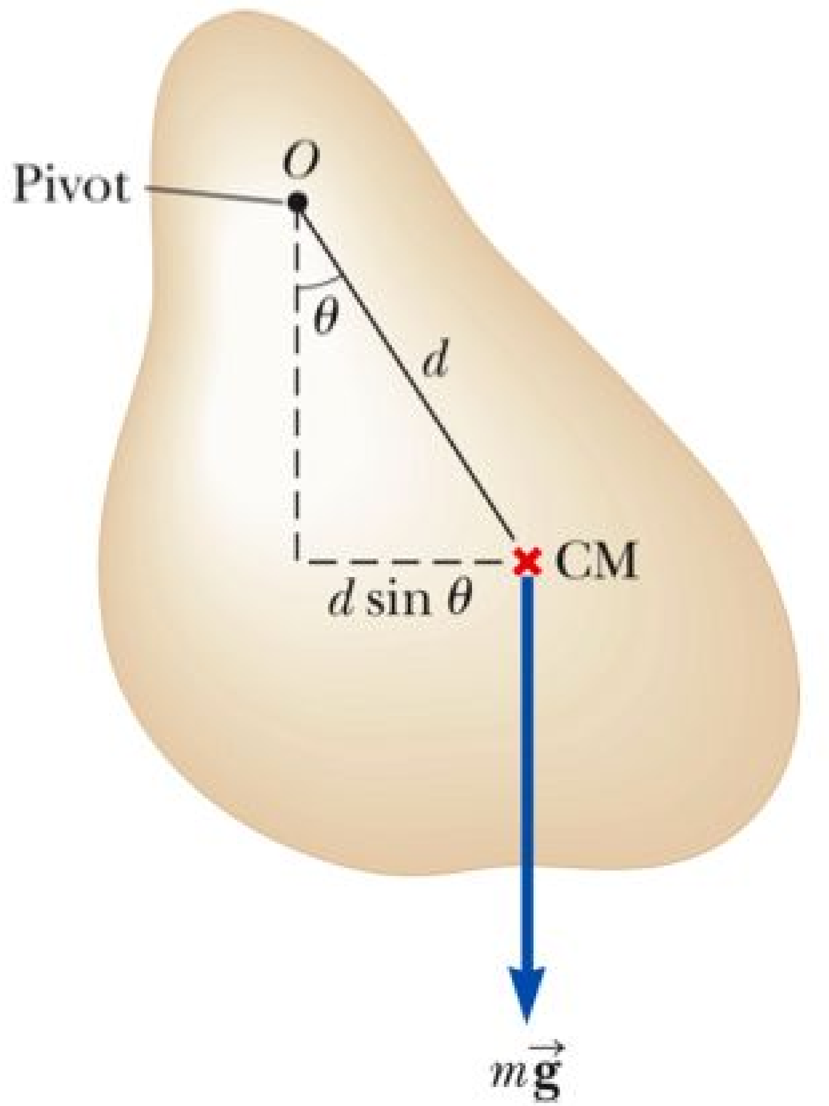
A grandfather clock depends on the period of a pendulum to keep correct time. Suppose a grandfather clock is calibrated correctly and then a mischievous child slides the bob of the pendulum downward on the oscillating rod. The grandfather clock will run:

1. Slow
2. Correctly
3. Fast

https://goo.gl/forms/oheaXMAKP5uJVfop1

SHM>Physical pendulum...

- A physical pendulum is an object that swings about some pivot point O that is not at the centre-of-mass (CM) of the object.
  - Its motion is however still SHM (for θ << 1 radian).



d = Distance from O to CM
m = Total mass of object

$$\left|\vec{a}\times\vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right|\sin\theta$$

- The torque about the pivot O has magnitude

$$\tau = \left|\vec{\tau}\right| = \left|\vec{r}\times\vec{F}\right|$$
$$= d\, m\, g \sin\theta$$

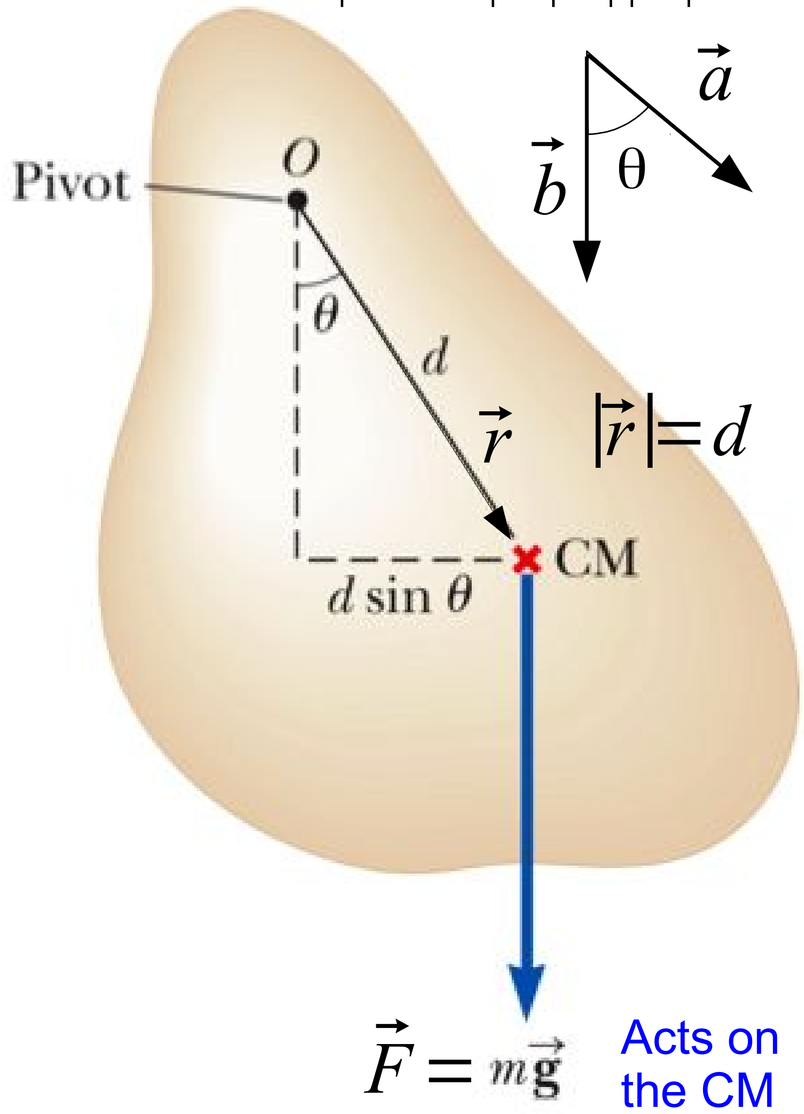
Distance to the CM

and points **into the page** (use the right-hand rule).

- From Newton's 2nd law of rotation, a net torque about O leads to angular acceleration:

$$\tau_{\text{net}} = I\alpha = I\frac{d^2\theta}{dt^2}$$

Moment of inertia about the axis through O



- In this scenario, the torque points into the page → the acceleration is in the **negative θ direction** (right-hand curl rule).

$$\Rightarrow \quad I\frac{d^2\theta}{dt^2} = -d\,m\,g\sin\theta$$

Small angle approximation

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{d\,m\,g}{I}\theta = -\omega^2\theta$$

Angular frequency and period

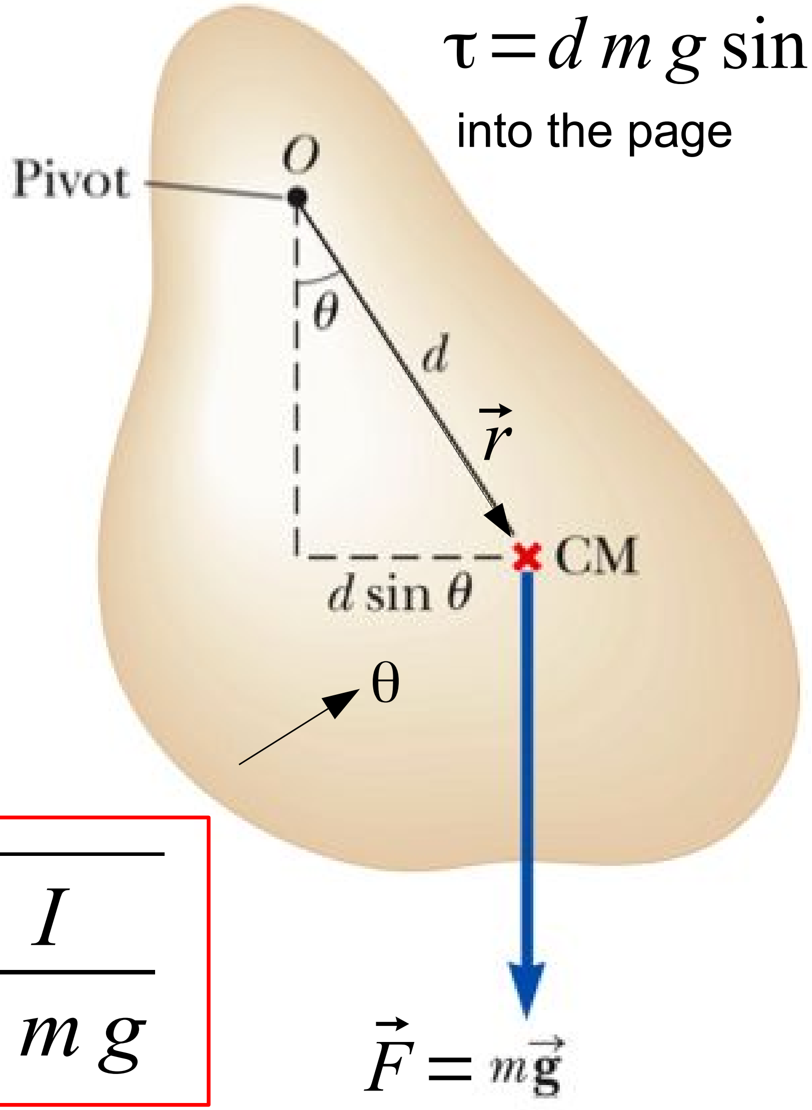
$$\Rightarrow \quad \boxed{\omega = \sqrt{\frac{d\,m\,g}{I}}, \quad T = 2\pi\sqrt{\frac{I}{d\,m\,g}}}$$

d = Distance from O to CM
m = Total mass of object

Torque about O

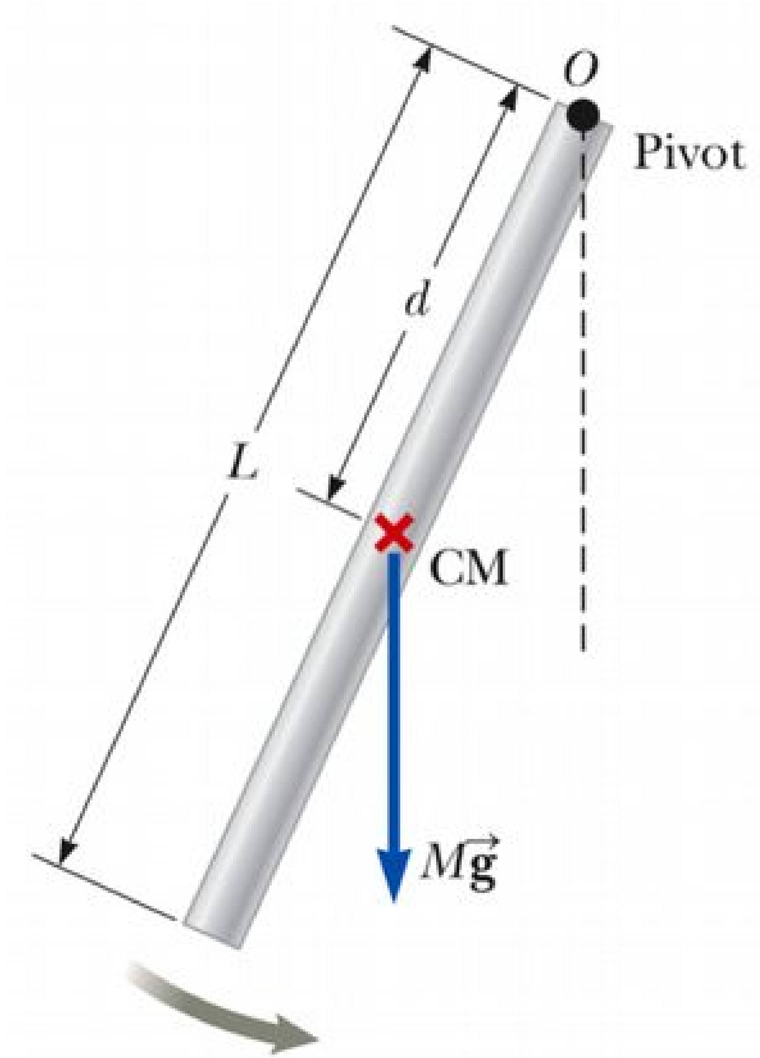
$$\tau = d\,m\,g\sin\theta$$

into the page



Question...

- A uniform rod of mass M and length L is pivoted about one end and oscillates in a vertical plane. Find the period of the oscillations assuming the oscillation amplitude is small.



BLACKBOARD

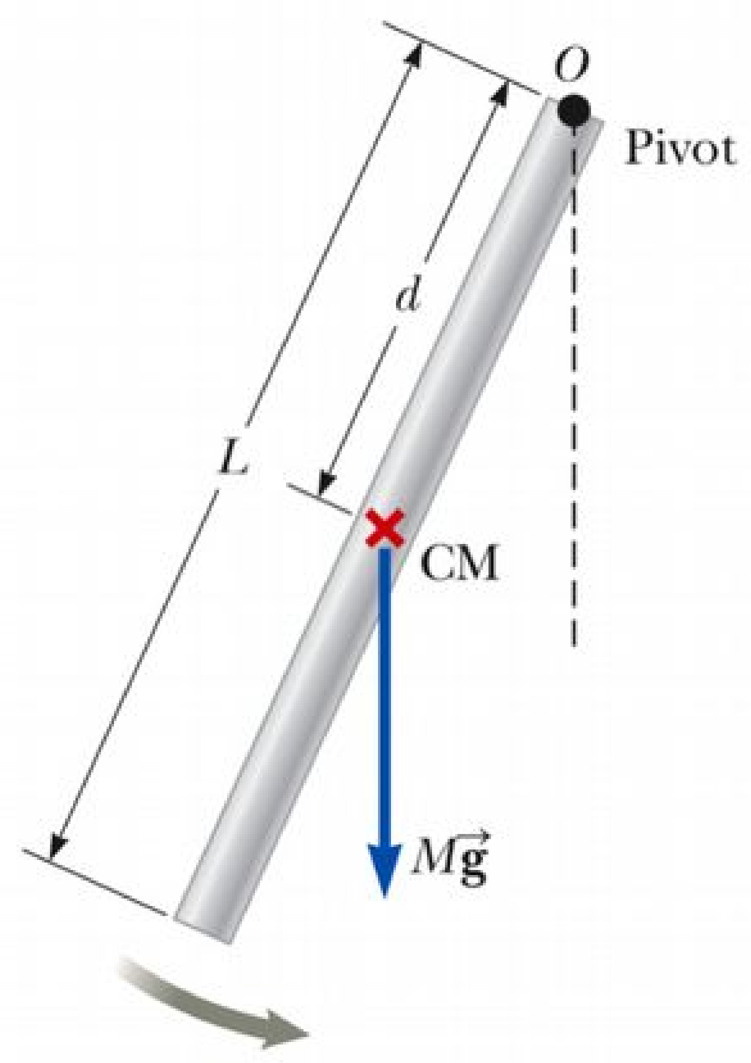
Question...

- A uniform rod of mass M and length L is pivoted about one end and oscillates in a vertical plane. Find the period of the oscillations assuming the oscillation amplitude is small.

Answer:

$$T = 2\pi\sqrt{\frac{2L}{3g}}$$

Independent of the mass of the rod!

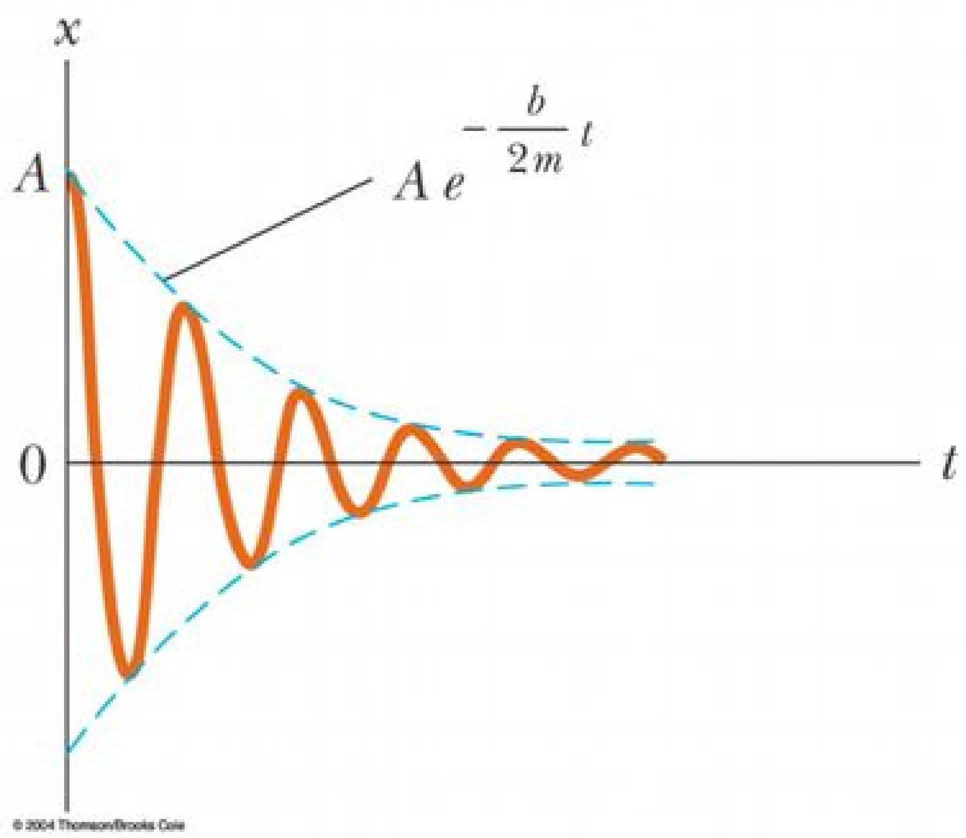


Damped oscillations...

- In realistic systems, **non-conservative forces** such as friction and air resistance are usually present.

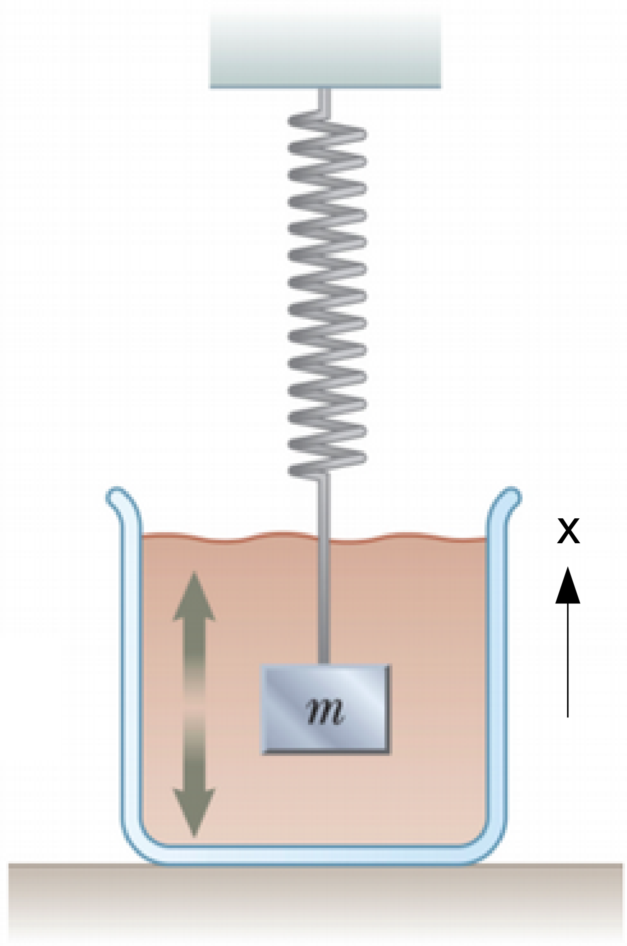
- In such cases, the mechanical energy of the system diminishes in time, and the motion of the oscillator is said to be **damped**.

- Need to know qualitatively… i.e. be able to discuss, explain relative sizes, signs, sketch graphs



Damped oscillations>Example...

- **A spring in a viscous fluid**. The fluid opposes the motion of the block by exerting a "retarding force" proportional to its velocity:



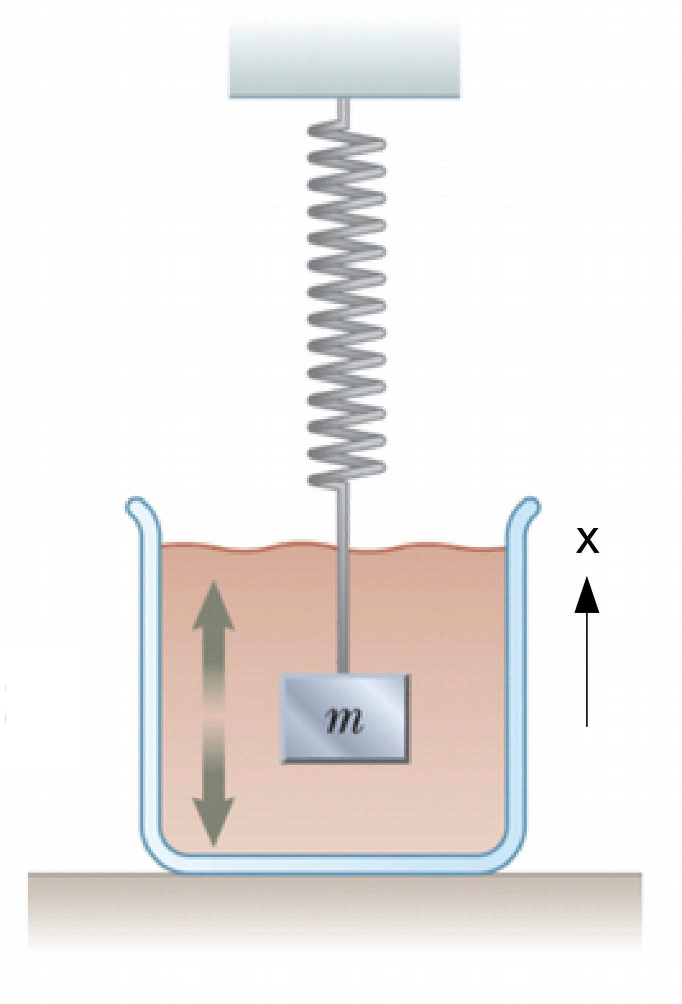
$$\vec{F} = -\underbrace{b}_{\text{Damping constant}}\vec{v}$$

  - The **total force** on the block is:

$$F_{\text{total}} = \underbrace{-b\,v_x}_{\text{From fluid}} \underbrace{-k\,x}_{\text{From spring}}$$

- **Equation of motion** for the block: $F_{total} = m a_x = -b v_x - k x$

$$\Rightarrow \quad \frac{d^2 x}{d t^2} = -\frac{b}{m}\frac{d x}{d t} - \frac{k}{m} x$$



- **Equation of motion** for the block: $ma_x = -bv_x - kx$

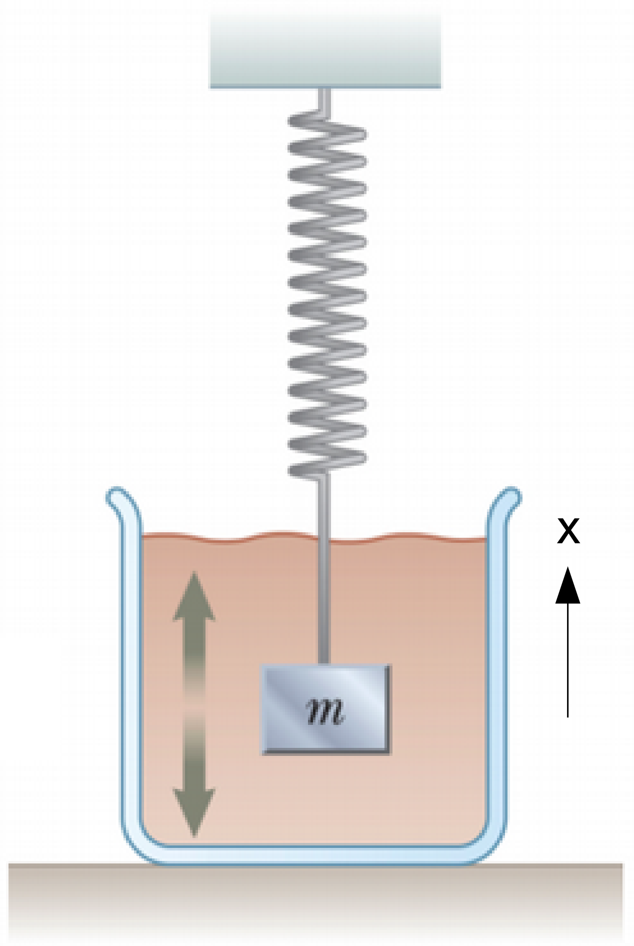
$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{b}{m}\frac{dx}{dt} - \frac{k}{m}x$$

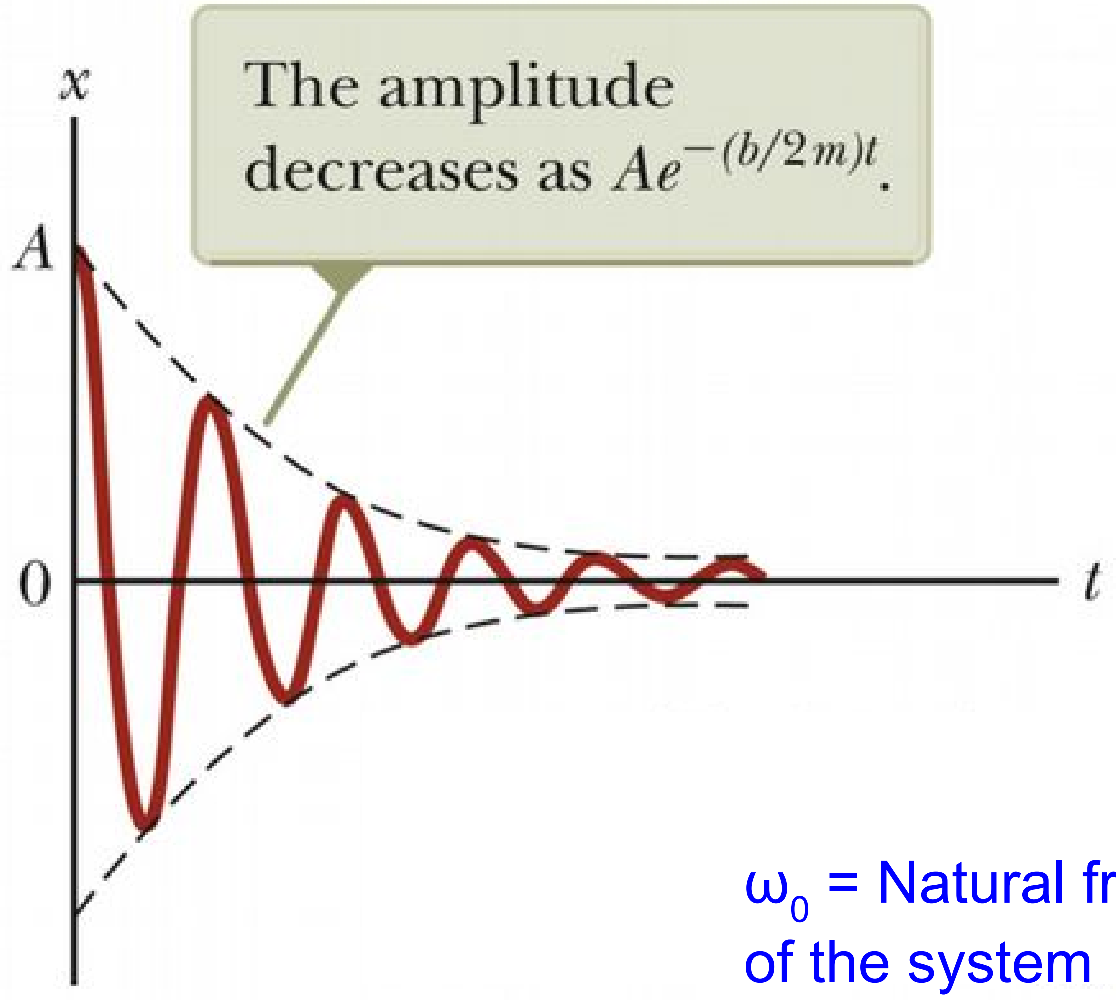
- If the damping constant b is small (I'll tell you what "small" means in a moment), then the **solution** looks like this:

$$x(t) = A e^{-\frac{b}{2m}t} \cos(\omega t + \phi)$$

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

Angular frequency





$$x(t) = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi)$$

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

- This solution holds when

$$\omega_0^2 \equiv \frac{k}{m} > \left(\frac{b}{2m}\right)^2$$

"Small" damping constant

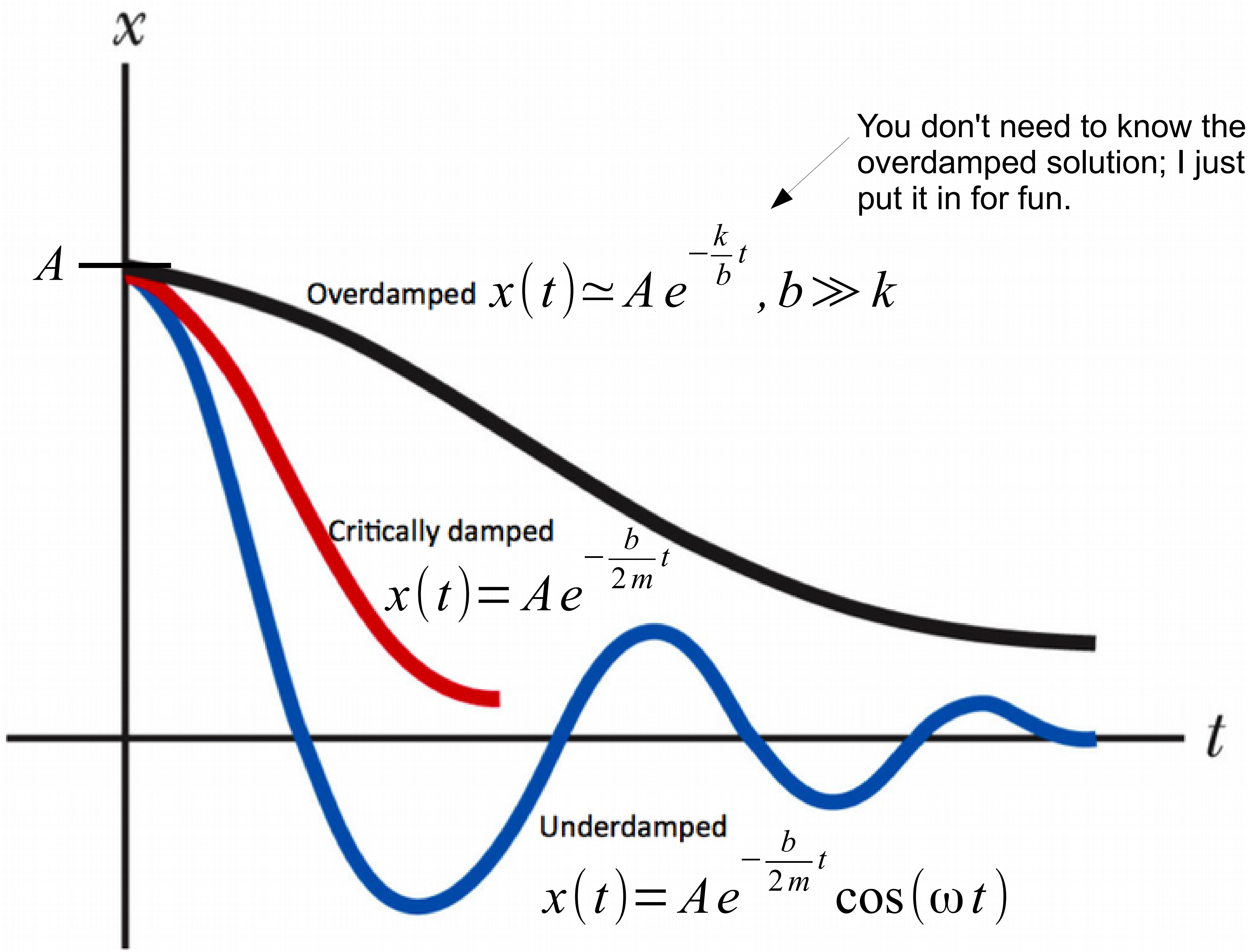
ω_0 = Natural frequency of the system

- When the retarding force is small, the oscillatory character of the motion is preserved, but the amplitude decreases exponentially with time. What if the retarding force is large?

Types of damping...

- In general, there are **three types** damping:
  - Underdamping (we've just seen it): $\omega_0^2 > \left(\frac{b}{2m}\right)^2 \Rightarrow \omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} = \text{Real}$
  - Critical damping: $\omega_0^2 = \left(\frac{b}{2m}\right)^2 \Rightarrow \omega = 0$
  - Overdamping: $\omega_0^2 < \left(\frac{b}{2m}\right)^2 \Rightarrow \omega = \text{Imaginary}$

No oscillations



x
A
t
You don't need to know the overdamped solution; I just put it in for fun.
Overdamped $x(t) \simeq A e^{-\frac{k}{b}t}, b \gg k$
Critically damped
$x(t) = A e^{-\frac{b}{2m}t}$
Underdamped
$x(t) = A e^{-\frac{b}{2m}t} \cos(\omega t)$

Forced oscillations...

- In a damped oscillator, the system loses mechanical energy.
- It is also possible to add energy to the system by doing work in each cycle.
  - For example:

$$\underbrace{F_0 \sin \omega_d t}_{\text{A periodic driving force}} \overbrace{- b \frac{dx}{dt} - kx = m \frac{d^2 x}{dt^2}}^{\text{Damped oscillator}}$$

  - This is very hard to solve...

- …but it will reach a steady state when the energy added each cycle is equal to the energy lost . Then,

$$x(t)=A\cos(\omega_d t+\phi)$$

Angular frequency of driving force

$$A=\frac{F_0/m}{\sqrt{(\omega_d^2-\omega_0^2)^2+b^2/(4m^2)}}$$

Natural frequency of the undamped system

→ **Resonance** happens when ω_d is close to ω_0 → Dramatic increase in the amplitude.