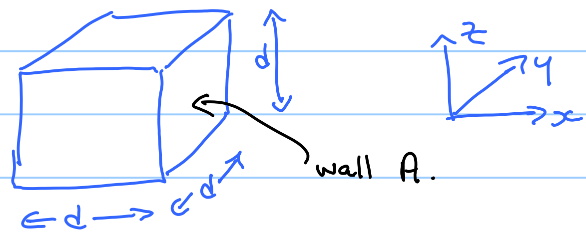
Note Title 22/04/2016

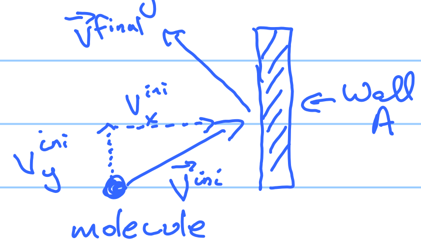
Kinetic Theory – Proof for PHYS 11M1 (without mistakes)

Consider a box of volume
$V = d^3$



It contains N identical molecules of an ideal gas.

Consider one molecule of mass m colliding with wall A.



elastic collision: elastic collision conserves energy. $\frac{1}{2}mv^2 = const.$ But sign of v can change.

Because the collision is elastic, it only flips the sign of v_x:

$$v_x^{final} = -v_x^{ini}$$

There is $\underline{\underline{\text{no}}}$ change to the y component

$\Rightarrow$ Change in molecule's momentum is

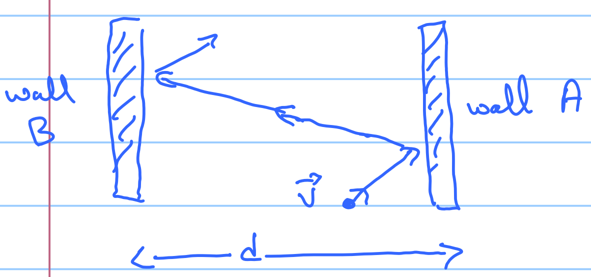
$$\Delta p_x^{molecule} = m\, v_x^{final} - m\, v_x^{ini}$$
$$= -2m\, v_x^{ini}$$

$\Rightarrow$ momentum gained by wall A is

$$\Delta p_x^{wall} = +2m\, v_x^{ini} = +2m\, v_x$$

(drop "ini" superscript)

After the collision, molecule travels to opposite side of the box, and collides with the opposite wall.



The time the molecule takes to go from wall A to wall B and back is:

$$\Delta t_{\text{round trip}} = \frac{\text{distance}}{\text{speed}} = \frac{2d}{v_x}$$

$\Rightarrow$ Average rate at which momentum is transferred to the wall A is:

$$\frac{\Delta p_x^{wall}}{\Delta t} = \frac{\cancel{2} m v_x}{\cancel{2} d / v_x} = \frac{m v_x^2}{d}$$

Now Newton II states $F = d\bar{p}/dt$, so $F_x = dp_x/dt$; $F_y = dp_y/dt$

$\Rightarrow$ Force due to this one molecule on wall A is

$$F_x = \frac{m v_x^2}{d}$$

But there are N identical molecules in the box.

$\therefore$ Total force due to all N molecules

$$F_x^{total} = \sum_{i=1}^{N} \frac{m}{d} \left(v_{x_i}\right)^2$$

x-velocity of molecule i

total force

label for each molecule

$$F_x^{total} = \frac{m}{d} \sum_{i=1}^{N} \left(v_{x_i}\right)^2$$

$$= \frac{Nm}{d} \underbrace{\left(\frac{1}{N} \sum_{i=1}^{N} \left(v_{x_i}\right)^2\right)}$$

Average velocity squared in the x-direction

$\equiv \overline{v_x^2}$

$$\Rightarrow F_x = \frac{Nm}{d}\overline{v_x^2} \quad (1)$$

Watch out: $\overline{v_x^2} \neq (\overline{v_x})^2$

The average of v_x^2 is not the same as the (average velocity)^2.

In fact, the average velocity of the gas $= 0$ – it isn't moving! $(\overline{v_x})^2 = 0$.

So far we only considered the x-direction.
But molecules move in y and z directions as well.

$$\therefore v_i^2 = v_{x_i}^2 + v_{y_i}^2 + v_{z_i}^2$$

squared velocity of molecule i

This means that the average values are related by

$$\overline{v^2} = \overline{v_{x_i}^2} + \overline{v_{y_i}^2} + \overline{v_{z_i}^2}$$

Now statistically molecules are just as likely to be moving in the x, y or z directions. So

$$\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2}$$

$$\Rightarrow \overline{v^2} = 3\overline{v_x^2}$$

Substitute back into (1):

$$F_x = \frac{Nm}{d}\overline{v_x^2} = \frac{1}{3}\frac{Nm}{d}\overline{v^2}$$

Now we can calculate the pressure on wall A:

$$P = \frac{F_x}{\underset{\text{area}}{A}} = \frac{F_x}{d^2} = \frac{1}{3}\frac{Nm\overline{v^2}}{d^3} = \frac{1}{3}\frac{Nm\overline{v^2}}{\underset{\text{volume}}{V}}$$

Recall that KE of a particle is $\frac{1}{2} m v^2$,
then the <u>average</u> KE of a collection of gas molecules is
$\frac{1}{2} m \overline{v^2}$

$$\therefore \text{ Write } \boxed{P = \frac{2}{3} \frac{N}{V} \left(\frac{1}{2} m \overline{v^2} \right)}$$

Rearrange:

$$PV = \frac{2}{3} N \left(\frac{1}{2} m \overline{v^2} \right)$$

c.f. ideal gas law: $PV = N k_B T$

$$\therefore \quad k_B T = \frac{2}{3} \left(\frac{1}{2} m \overline{v^2} \right)$$

$$\Rightarrow \boxed{T = \frac{2}{3 k_B} \left(\frac{1}{2} m \overline{v^2} \right)}$$

(Note: I wrote this wrong in one of the lectures!)

i.e. temperature is a direct measure of the average molecular kinetic energy of a gas!