Gravity

- context in physics (& history)
- Newton's law of gravity
- Cavendish measures G (and thus m_{Earth})
- the gravitational field
- Gravitational potential energy in a non-uniform field
- escape velocity
- Planetary motion

  Kepler's laws, and Newton's laws
- Orbits and energy
- Limitations to Newton’s laws

Gravity: where does it fit in?

Gravity [general relativity]	Electro-magnetism	'Weak' nuclear force	Colour force	mediated by:
Gravitons*?	Photons	W^+, W^-, Z	Gluons	

electro-weak unification

Grand Unified Theories

Theories of Everything (watch this space)

- Only gravity and electric force have macroscopic ("infinite") range.
- Gravity weakest, but dominates on large scale. *Why?*

* if we had a viable quantum theory of gravity

A very brief (occidental) history of gravity: (*not in our syllabus*)

Greeks to Galileo:

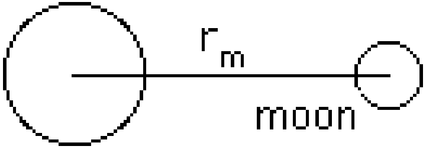
i) things fall to the ground because that's their 'natural' place

ii) planets etc move (variety of reasons)

but no connection (in fact, a contrast: natural *vs* supernatural). Contrast with:

Newton's calculation: the acceleration of the moon and of an "apple"

$$a_{moon} = r_m \omega_m{}^2$$



$$= r_m \left(\frac{2\pi}{period} \right)^2 = (3.8\ 10^8\ \mathrm{m}) \left(\frac{2\pi}{27.3\ 24\ 3600} \right)^2$$

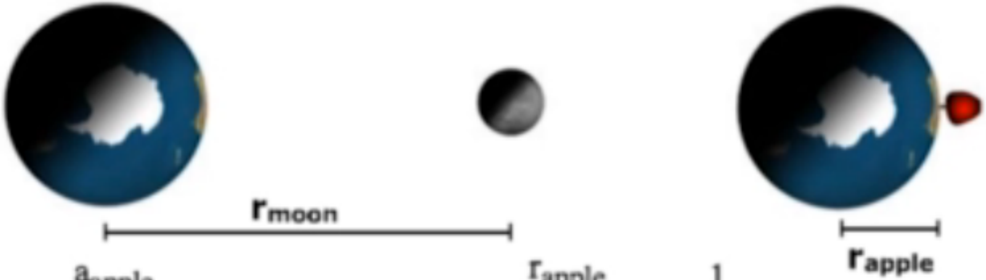
$$= 2.7 * 10^{-3}\ \mathrm{m.s}^{-2}$$

acceleration of apple = 9.8 ms^{-2}

$$\frac{a_{apple}}{a_{moon}} = 3600; \qquad \frac{r_m}{R_e} = \frac{385000\ \mathrm{km}}{6370\mathrm{km}} = 60;$$

$$\left(\frac{r_m}{R_e} \right)^2 = 3600$$

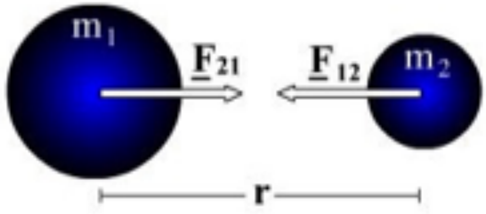
Newton: What if the same law applies to the apple and the moon? What if every body in the universe attracts every other via inverse square law?



$$\frac{a_{apple}}{a_{moon}} = 3600 \qquad \frac{r_{apple}}{r_{moon}} = \frac{1}{60}$$

$$\frac{F_{apple}}{F_{moon}} = \frac{m_{apple}a_{apple}}{m_{moon}a_{moon}} = \frac{m_{apple}}{m_{moon}}\left(\frac{r_{moon}}{r_{apple}}\right)^2$$

$$\frac{F_{apple}}{F_{moon}} = \frac{\dfrac{m_{apple}}{r_{apple}^2}}{\dfrac{m_{moon}}{r_{moon}^2}} \qquad F \propto \frac{m}{r^2}$$



$$F_{grav} = -G\,\frac{m_1 m_2}{r^2}$$ Negative sign means $\vec{F}$ is parallel to $-\vec{r}$

or $$\vec{F}_{grav} = -G\,\frac{m_1 m_2}{r^2}\,\hat{r}$$

Why is it linear in m_1 and m_2?

Why symmetric in m_1 and m_2 ?

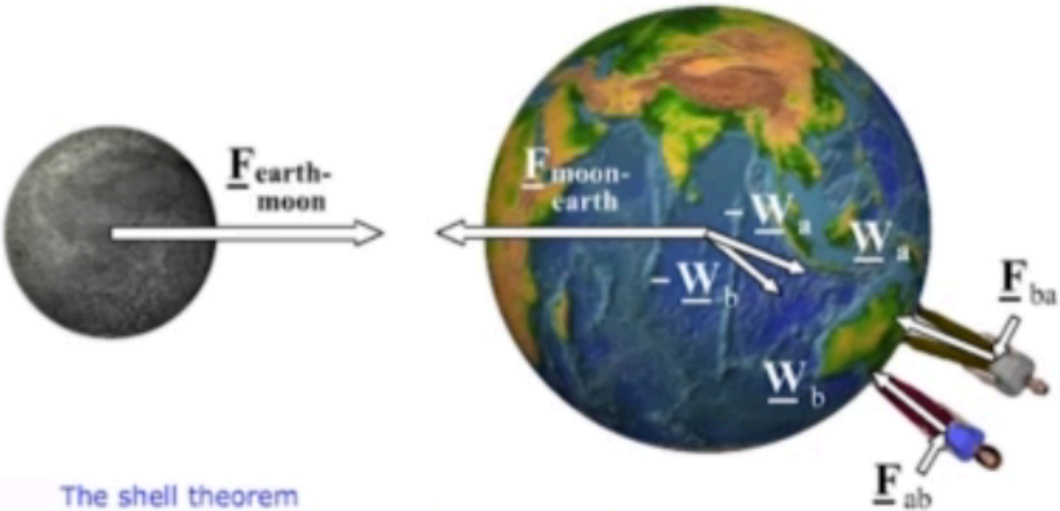
Why is it inverse square? Wait for Gauss' law in electricity.

Why is it attractive?

Why G?

What is the weight of the Earth?

Newton's third law $\mathbf{F}_{12} = -\mathbf{F}_{21}$



The shell theorem
Why is gravitational force inverse square?

All are Newton pairs

Question

Consider an astronaut in the International Space Station at altitude 408 km. By how much is the gravitational force acting on her reduced compared with earth?

a) Not at all

b) Not much

c) Close to 100%

d) Exactly 100%

Is she weightless?

How big is G?

Consider an object mass m near the earth's surface.

Newton knew $|W| = mg \cong \frac{GM_{earth}m}{r_{earth}^2}$

Know m.

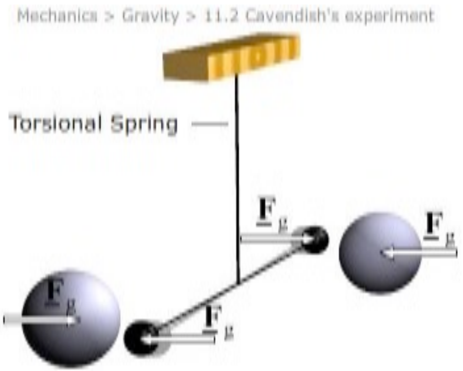
Know r_e.

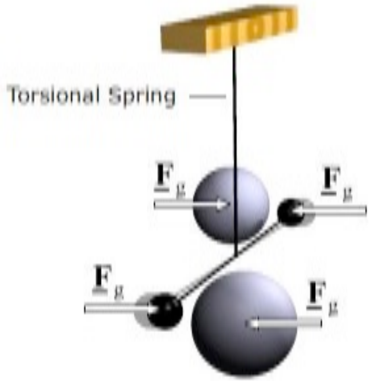
Therefore we know* $GM_{earth} = gr_{earth}^2$, as did Newton.

But how to know G and M_{earth} separately?

$* = 2.1 * 10^{14} \mathrm{m}^3.\mathrm{s}^{-2}$

Cavendish's experiment (1798)





$F = -G\,\frac{m_1 m_2}{r^2}$. From deflection and spring constant, calculate F, know m_1 and m_2, $\therefore$ can calculate G.

$G = 6.67\ 10^{-11}\ \mathrm{Nm^2kg^{-2}}$ *or* $6.67\ 10^{-11}\ m^3kg^{-1}s^{-2}$

Once we know *G*:

Weight of mass m: $mg \cong G\dfrac{mM_e}{r_e^2}$

∴ Cavendish first calculated the mass of the earth:

$$M_e \cong \frac{gr_e^2}{G} = \frac{9.8\ \mathrm{m.s^{-2}} \times (6.37\ 10^{6}\ \mathrm{m})^2}{6.67\ 10^{-11}\ \mathrm{Nm^2kg^{-2}}}$$

$$= 6.0*10^{24}\ \mathrm{kg}$$

Earth's orbit gives sun's mass ($1.99 * 10^{30}$kg*).*

Then get masses of other planets from the orbits of their moons

– except Venus and Mercury

Some numbers

What is force between two oil tankers at 100 m? Between two students in adjacent chairs?

$$F_{grav} = -G\,\frac{m_1 m_2}{r^2}$$

Oil tankers: $m \sim 10^8$ kg, $r \sim 100$ m –> 70 N

Students: $m \sim 70$ kg, $r \sim 0.4$ m –> 2 μN

Conclusion: usually can neglect gravity unless at least one of the bodies is of astronomical size.

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Conclusion 2: "*Gravitation cannot be held responsible for people falling in love*" Albert Einstein.

What happens when more there are ≥ 3 bodies?

Superposition principle. $\sum \quad \vec{F}$

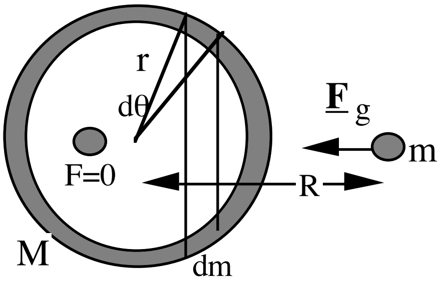
$$\vec{F}_{\text{all objects together}} = \sum \vec{F}_{\text{individual}}$$

or $\quad \vec{F}_1 = \sum_i \vec{F}_{i1}$ force on m_1 due to all masses m_{i}.

continuous body $\quad \vec{F}_{whole\ body} = \int_{body} \overrightarrow{dF}$

Shell theorem

A uniform shell of mass M causes the same gravitational force on a body outside it as does a point mass M located at the centre of the shell, and zero force on a body inside it.



Proof by integrating x components of F due to dm.

$$F_{grav} = \begin{cases} GMm/R^2 & \text{if outside} \\ 0 & \text{if inside} \end{cases}$$

derivation not in syllabus: see Physclips

Gravity becomes weaker above the Earth's surface

$$mg_o = F_{grav} = G\frac{M_e m}{r^2}$$ where g_o is acceleration in an inertial frame

$(i.e.\ without\ rotation:\ remember\ the\ a_{centrip} = 0.03\ m.s^{-2})$

$$g_o = G\frac{M_e}{r^2}$$

Usually we are close to the surface, $r \simeq R_e$, but

$$g_o = G\frac{M_e}{(R_e + h)^2} = g_s\left(\frac{R_e}{R_e + h}\right)^2$$ where h is altitude

$$= g_s\left(\frac{1}{1 + h/R_e}\right)^2$$ where g_s is g_o at surface

Other complications:

i) Earth is not uniform (especially the crust)

useful for prospecting

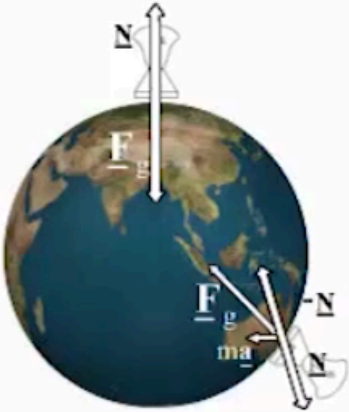
ii) Earth is not spherical

iii) Earth rotates

Question

Which direction is the centre of the earth (with respect to 'down')?

Hint: draw a free body diagram for a mass on a spring balance



At poles, $\underline{\mathbf{F}}_g + \underline{\mathbf{N}} = 0$

At latitude θ, $\underline{\mathbf{F}}_g + \underline{\mathbf{N}} = m\underline{\mathbf{a}}$

weight measured $= -\underline{\mathbf{N}} = \underline{\mathbf{F}}_g - m\underline{\mathbf{a}}$

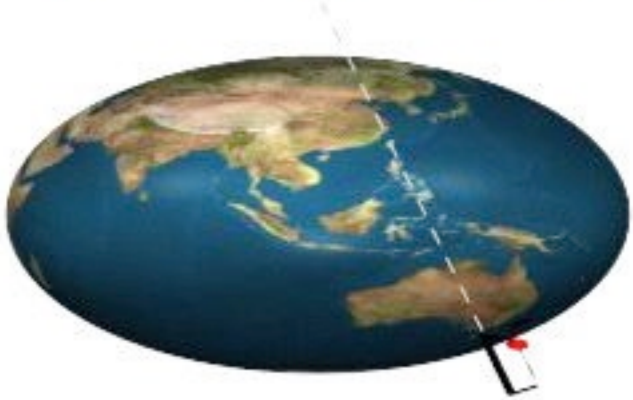
where $a = r\omega^2 = (R_e \cos\theta)\omega^2$

$= 0.034\ \mathrm{ms}^{-2}$ at equator

$= 0$ at poles

We often define $\vec{g} = -\frac{\vec{N}}{m} = -\frac{\vec{F} - m\vec{a}}{m}$ in equilibrium

So g is greatest at the poles, least at the equator, and does not (quite) point towards centre.

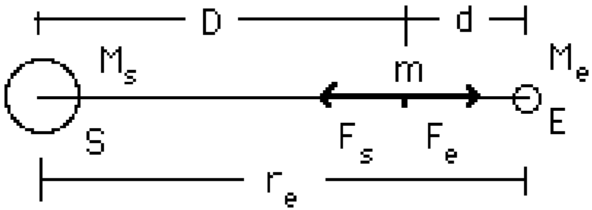


horizontal is at right angles to $\vec{g}$

Earth is flattened at poles

Puzzle: Save the moon

How far from the earth is the point at which the gravitational attractions towards the earth and towards the sun are equal and opposite? Compare with distance earth-moon (380,000 km)

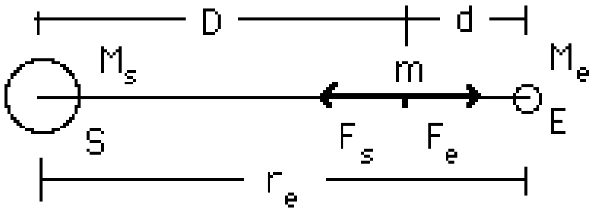


$$|F_e| = |F_s|$$

$$d = ?$$

Puzzle: Save the moon

How far from the earth is the point at which the gravitational attractions towards the earth and towards the sun are equal and opposite? Compare with distance earth-moon (380,000 km)



$$|F_e| = |F_s|$$

$$\frac{GM_e m}{d^2} = \frac{GM_s m}{(r_e - d)^2}$$

$$M_e(r_e - d)^2 = M_s d^2$$

$$r_e^2 - 2r_e d + d^2 = \frac{M_s}{M_e} d^2$$

$$\left(\frac{M_s}{M_e} - 1\right) d^2 + 2r_e d - r_e^2 = 0$$

$$\left(\frac{M_s}{M_e} \cancel{- 1}\right) d^2 \cancel{+ 2r_e d} - r_e^2 = 0$$

$$d \simeq \frac{r_e}{\sqrt{M_s/M_e}} = \frac{1.5\ 10^{11}\ \text{m}}{\sqrt{2.0\ 10^{30}\text{kg}/6.0\ 10^{24}}}$$

$= 260{,}000$ km

But the distance earth to moon = 380,000 km ?

Gravitational field. A field is ratio of the force on a particle to some property of the particle. For gravity, (gravitational) mass is the property:

$$\frac{\vec{F}_g}{m} = \vec{g} = \vec{g}(r) \qquad \text{is a vector field}$$

compare with electric field $\dfrac{\vec{F}_{electric}}{q} = \vec{E}(r)$ *(Phys 1b syllabus)*

Not important in our syllabus, but note the difference between these descriptions:

- *"Action at a distance": Earth's mass exerts an attractive force on every other mass in its environment, such as the moon, apple etc.*
- *Field description: Earth's mass creates a gravitational field in its environment. Objects in that environment are 'acted on' by the field.*
- *General relativity: Earth's mass distorts the spacetime around it, and this affects the motion of objects in its environment.*

Gravitational potential energy. Revision:

Potential energy

For a **conservative** force $\vec{F}$ ($i.e.$ one where work done against it $W = W(r)$, we can define potential energy U by $\Delta U = W_{\text{against}}$. i.e.

$$U = -\int_{initial}^{final} \vec{F}_{grav} \cdot \overrightarrow{ds}$$

***near** Earth's surface*, $\vec{F}_{grav} = -mg\hat{k} \cong$ uniform locally

$$U = -\int_{initial}^{final} (-mg\hat{k}) \cdot (dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k})$$

$$mg\ \hat{k} \cdot \hat{k} \int_{initial}^{final} dz$$

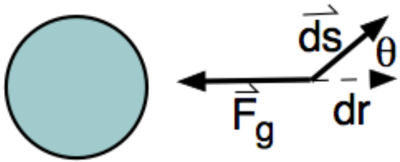
$$= \ mg\ (z_{final} - z_{initial})$$

choose reference $U = 0$ at a $= z_{initial}$, so

$U \cong mgz$ where z (or h) is altitude.

But only $\cong$ and only locally.

Gravitational potential energy of m *and* M.



ΔU = *work done* ***against*** *gravity*

$$\Delta U = -\int_{initial}^{final} \vec{F}_g \cdot \vec{ds}$$

$$= \int_{initial}^{final} F_g ds \cos\theta$$

$$= \int_{initial}^{final} F_g dr$$

$$= \int_{initial}^{final} G\frac{Mm}{r^2} dr$$

$$\Delta U = -GMm\left[\frac{1}{r_{final}} - \frac{1}{r_{initial}}\right]$$

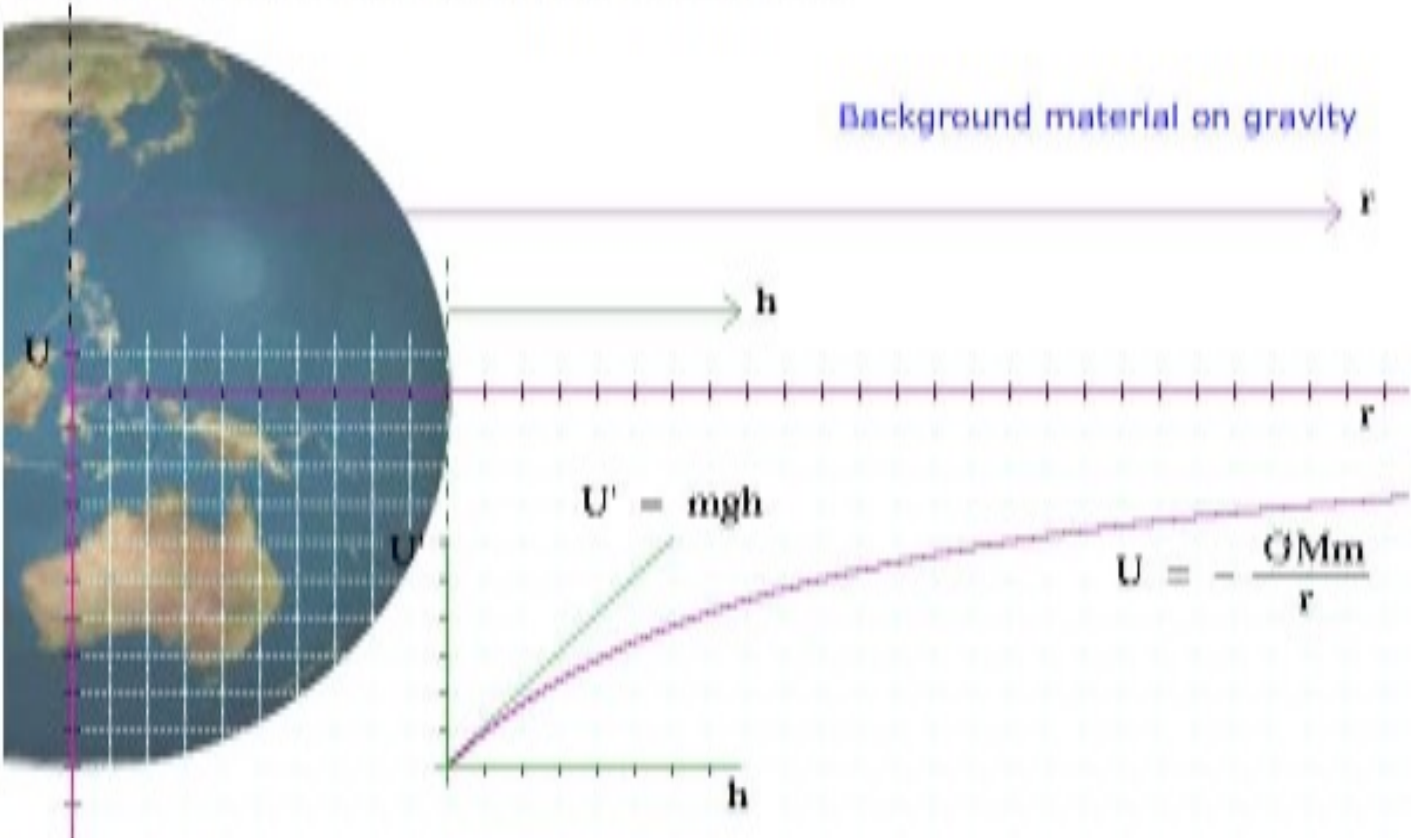
Convention: take $r_i = \infty$ as reference for $U = 0$.

$$U(r) = -\frac{GMm}{r}$$

U = work to move one mass from ∞ to r in the field of the other. U is always negative.

Usually one mass >> other, we talk of U of one in the field of the other (*e.g.* U of a projectile near Earth), but formally it is always U of both.

Background material on gravity

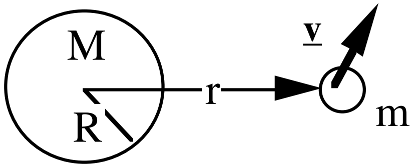


Escape speed ("escape velocity").

*"What goes up **sometimes** comes down"*

Escape "velocity" is **minimum** speed v_e required to escape, i.e. to get to a large distance ($r \to \infty$).

Newton's calculation:



Projectile in space: no non-conservative forces so conservation of mechanical energy

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2} m v_e^2 - \frac{GMm}{R} = 0 + 0$$

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

For Earth:

$$v_{esc} = \sqrt{\frac{2 \ 6.67 \ 10^{-11} \ \mathrm{m^3 kg^{-1} s^{-2}} \ 5.98 \ 10^{24} \ \mathrm{kg}}{6.37 \ 10^{6} \ \mathrm{m}}}$$

$$= 11.2 \ \mathrm{km.s^{-1}} = 40{,}000 \ \mathrm{k.p.h.}$$

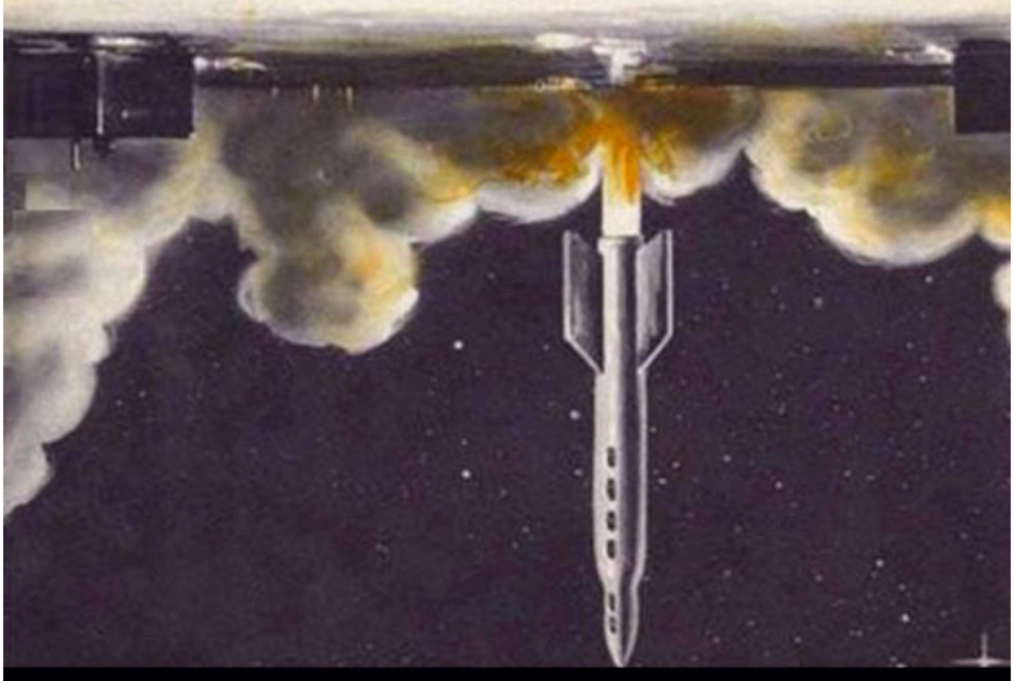
Put launch sites near equator:

$$v_{eq} = R_e \omega_e = 0.47 \ \mathrm{km.s^{-1}}$$

DID YOU KNOW?

It doesn't take as much fuel to launch a rocket into space when it is launched from Australia.

Scientists simply untie the rocket, and it falls into outer space, where it is then guided by its engines.



Question: What is the relation between M and R such that $v_{escape} = c$?

$$\text{Escape speed} = c = \sqrt{\frac{2GM}{R}}$$

$$R_{BH} = \frac{2GM}{c^2}$$ (*radius of Newtonian black hole*. Mitchell, 1783)

General relativity gives the same result.

But G is small and c is large so R is small so

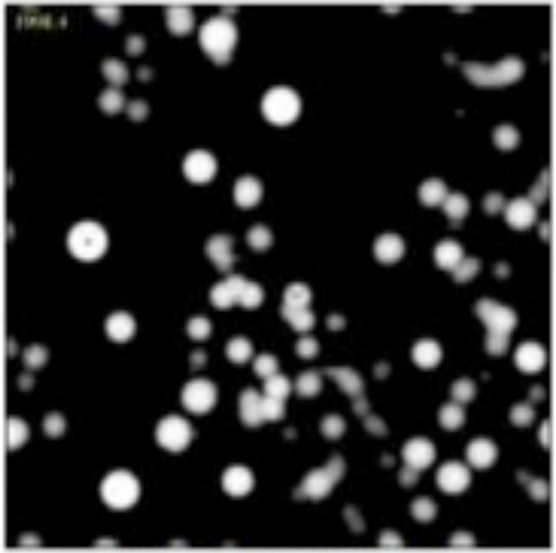
Earth would become a black hole if compressed to:

$$R_{BH} = \frac{2 * 6.67\ 10^{-11}\ m^3kg^{-1}s^{-2} * 5.98\ 10^{24}\ kg}{(3\ 10^8\ m/s)^2} = 9\ mm$$

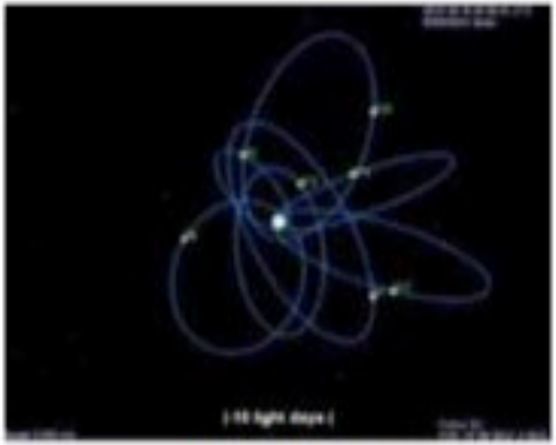
For the sun

$$R_{BH} = \frac{2 * 6.67\ 10^{-11}\ m^3kg^{-1}s^{-2} * 1.99\ 10^{30}\ kg}{(3\ 10^8\ m/s)^2} = 3\ km$$

Stars near the centre of our galaxy



Telescope images



Animation

courtesy of Max-Planck-Institut für extraterrestrische Physik

Planetary motion

"The music of the spheres" - Plato

Leucippus & Democritus C5 B.C.: heliocentric universe

Hipparchus (C2 BC) & Ptolemy (C2 AD) geocentric universe

Tycho Brahe (1546-1601) – very many, very careful, naked eye observations.

Johannes Kepler joined him. He fitted the data to these *empirical* laws:

Kepler's laws:

1 All planets move in elliptical orbits, with the sun at one focus.

For most planets, these ellipses are roughly circles

$M_{sun} >> m_{planet}$, *so sun is $\simeq$ the centre of mass*

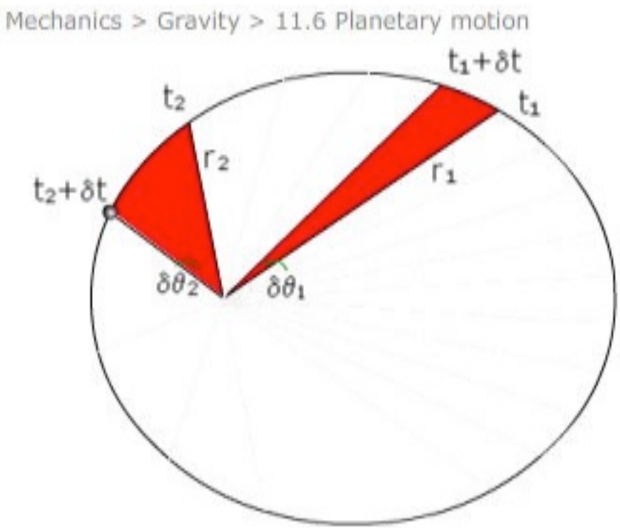
2 A line joining the planet to the sun sweeps out equal areas in equal time.

Slow at apogee (distant), fast at perigee (close)

3 The square of the period $\propto$ the cube of the semi-major axis

Slow for distant planet, fast for close

Newton's explanations: *derivation not required*



Area of triangle $= \frac{1}{2}\ r.r\delta\theta$

i.e. for same δt, $\frac{1}{2}\ r^2\delta\theta =$ constant

No torques so: conservation of angular momentum $\vec{L}$. Sun at centre of mass

$$\therefore \quad \left|\vec{L}\right| = |\vec{r}\times\vec{p}| = |\vec{r}\times m\vec{v}|$$

($\vec{p}$ = *momentum*)

$$= r\ mv_{tangential}$$

$$= mr.r\omega \ = \ mr^2\frac{\delta\theta}{\delta t}$$

$$= \frac{m}{\delta t}\ r^2\delta\theta = \text{constant.}$$

Conservation of $\vec{L} \Rightarrow$ Kepler 2

Law of periods: *(we consider only circular orbits)*

Kepler's 3rd:	$T^2 \propto r^3$	*We'll derive it from*
Newton's 2nd:	$F = ma$	(a is centripetal)

Law of periods: *(we consider only circular orbits)*

Kepler's 3rd: $T^2 \propto r^3$ *We'll derive it from*

Newton's 2nd: $F = ma$ (a is centripetal)

$$F = m\, r\omega^2$$

$$G\frac{Mm}{r^2} = mr\left(\frac{2\pi}{T}\right)^2$$

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$ *(works for ellipses with semi-major axis a instead of r)*

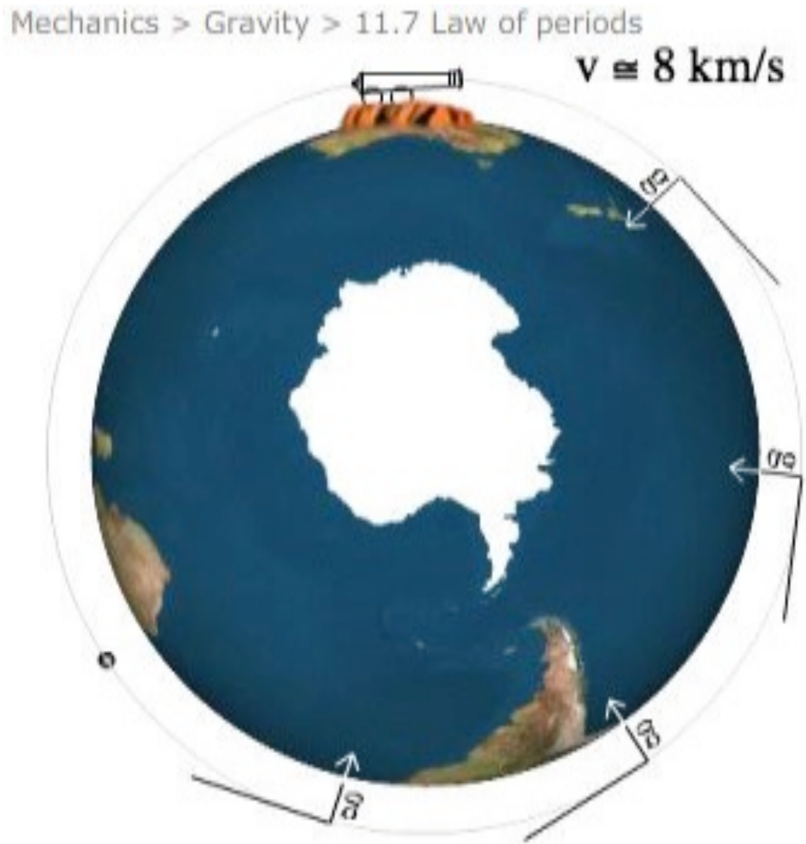
Newton 2 &
Newton's gravity $\Rightarrow$ Kepler 3

Newton 2 &
Newton's gravity also $\Rightarrow$ Kepler 1

but the algebra is long. See e.g. Newton "Principia Mathematica" or Bradbury "Theoretical mechanics" Wiley 1968

Newton’s cannon

Newton’s cannon



Example What is the period of the smallest earth orbit? ($r \cong R_e$) What is the period of the moon? ($r_{moon} = 3.82\ 10^8$ m)

$$T_1 = \sqrt{\left(\frac{4\pi^2}{GM}\right) r^3} = ...$$

$$= \sqrt{\frac{4\ \pi^2}{6.67\ 10^{-11}\ 5.98\ 10^{24}}\ (6.37\ 10^6)^3}\ \ \mathrm{s}$$

$$= 84\ \mathrm{min}$$

Kepler 3: $T^2 \propto r^3$

$$\frac{T_2}{T_1} = \left(\frac{r_2}{r_1}\right)^{3/2} = \left(\frac{3.82\ 10^8}{6.37\ 10^6}\right)^{3/2} = 464$$

$$T_2 = 464\ T_1 = 27.2\ \mathrm{days}$$

For other planets: most have moons, so the mass of the planet can be calculated from $$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

Orbits and energy

No non-conservative forces do work, so mechanical energy is constant:

$$E = K + U$$

$$= \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Let's remove v. Consider circular orbit, use centripetal acceleration

and Newton's 2nd:

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Let's remove v. Consider circular orbit, use centripetal acceleration

$$m\frac{v^2}{r} = ma_c = |F| = \frac{GMm}{r^2}$$

$$\therefore \quad \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r} \qquad (1)$$

$E = K + U$ *(& substitute (1))*

$$= \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r}$$

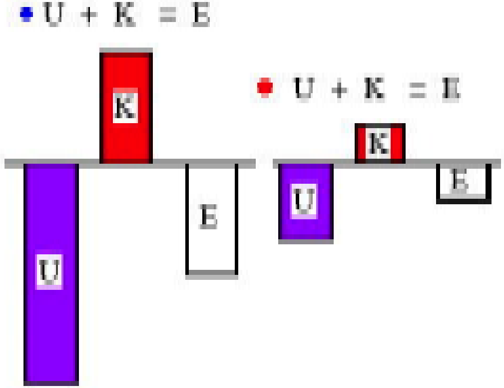
$$= -\frac{GMm}{2r}$$

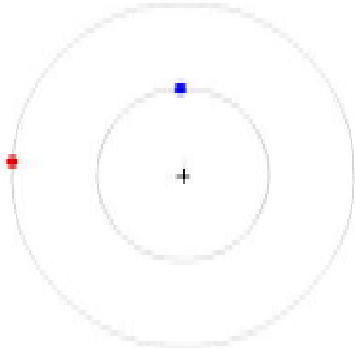
$i.e.\ E = \frac{1}{2}U,$ or $K = -\frac{1}{2}U$ or

$$K = -E$$

Small $r \Rightarrow U$ very negative, so K large.

(inner planets fast, outer slow)

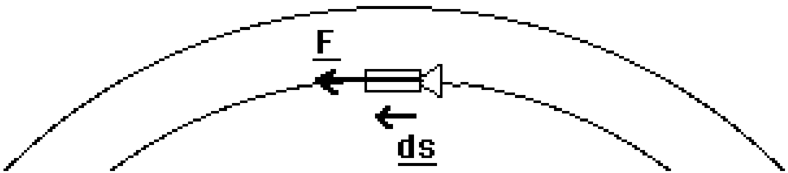




Small r → U very negative, K large.

$$E = \frac{1}{2}U, \quad \text{or} \quad K = -\frac{1}{2}U, \quad \text{or} \quad K = -E.$$

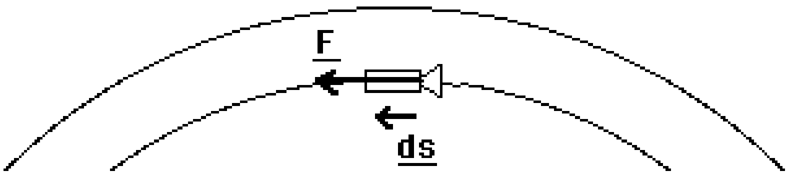
Example A spacecraft in orbit fires rockets while pointing forward.



- Does the rocket engine do positive or negative work* on the ship?
- Does its new orbit have higher or lower energy?
- Is the new orbit higher or lower?
- Is its new orbit faster or slower?

* Is the force exerted by the engines conservative or non-conservative?

Example A spacecraft in orbit fires rockets while pointing forward. Is its new orbit faster or slower?



$\vec{F} // \overrightarrow{ds}$ $\therefore$ Work done *on* craft

$$W = \vec{F} \cdot \overrightarrow{ds} > 0.$$

$\therefore$ Total mechanical energy

$E = -\dfrac{GMm}{2r}$ increases, *i.e.* it becomes less negative. (r is larger).

$K = -E$, $\therefore$ K smaller, so it goes to a higher, slower orbit

It travels *more slowly*. *called "Speeding down"*

Engines do positive work to get to a high, slow orbit.

Manœuvring in orbit.

To catch up, trailing craft fires engines *backwards*, and loses energy. It thus falls to a lower orbit where it travels faster, until it catches up. It then fires its engines *forwards* in order to slow down (it climbs back to the original, slower orbit).

Example: In what orbit does a satellite remain above the same point on the equator? *Called the Clarke Geosynchronous Orbit*

i) Period of orbit = period of earth's rotation

ii) Must be circular so that ω constant

$$T \quad = \quad 23.9 \text{ hours}$$

Kepler: $$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

$$r = \sqrt[3]{\frac{GMT^2}{4\pi^2}} \quad = \quad ..... \quad = \; 42{,}000 \text{ km}$$ popular orbit!

Question

To take a satellite with mass m from orbit with radius r_1 to orbit with radius r_2, how much work must the engines do? $\left(\text{Recall: } U = -\frac{GMm}{r}\right)$

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$$E = U + K \qquad K = -\frac{U}{2} \qquad E = \frac{U}{2} = -\frac{GMm}{2r}$$

$$\Delta E = \frac{GMm}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Caution: There are limits to Newtonian mechanics:

Newtonian gravity accurate if $\frac{|U_{grav}|}{mc^2} \ll 1$

otherwise use General Relativity.

At earth's surface

$$\frac{|U_{grav}|}{mc^2} = 7 \times 10^{-10}$$

Newtonian mechanics accurate if $\frac{v}{c} \ll 1$

otherwise use Special Relativity.

For a jet airliner

$$\frac{v}{c} < 10^{-6}$$

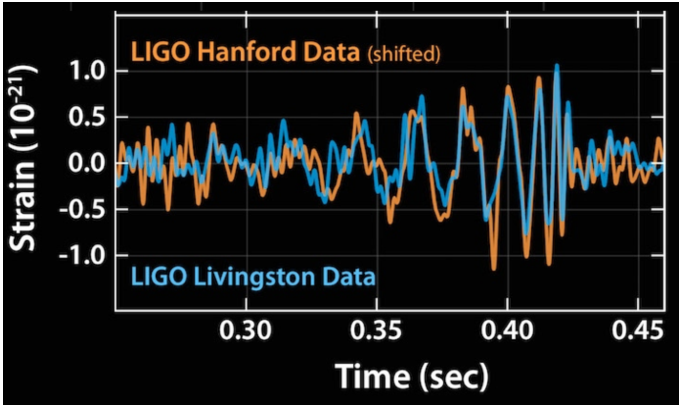
Newtonian mechanics accurate if

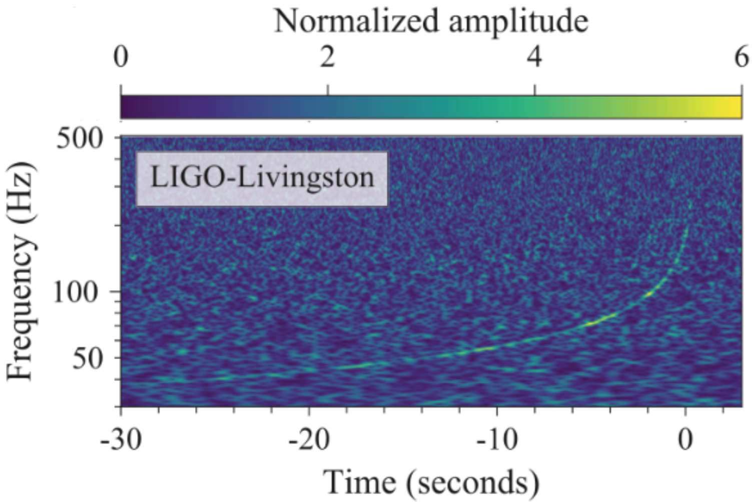
$$\frac{\text{momentum} * \text{size}}{\text{Planck's constant}} \gg 1$$

otherwise, use Quantum Mechanics.

For a small molecule at room temperature

$$\frac{\text{momentum} * \text{size}}{\text{Planck's constant}} \gtrsim 10$$





Gravity recent news

February 2016

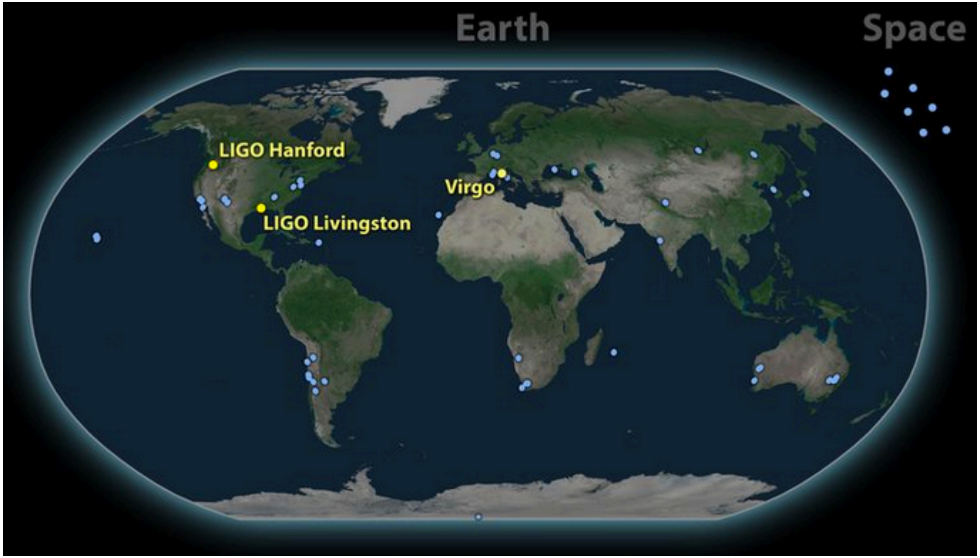
Two black holes, each ~ 30 * mass of sun.

Fuse; create gravity waves.

October 2017

Two neutron stars, each ~ 1.5 *mass of sun.

Fuse; create gravity waves, supernova, optical and radio signals, heavy elements…



Yellow dots: gravitational wave observatories involved.
Blue dots: optical telescopes involved, including (bottom of map) UNSW's observatory in Antarctica.
Search 'fusing neutron stars' or ask Michael Ashley in Astrophysics (one of the authors on the paper).

Gravity (revision)

$$F = -\frac{Gm_1m_2}{r^2}$$

Shell theorem

field *inside* a uniform, hollow shell = 0

field *outside* a uniform sphere is the same as if all mass were located at the centre

Near the Earth's surface, $g \simeq GM/R_e{}^2$

"≅" because of inhomogeneity, because it's not an inertial frame, change in field strength with altitude

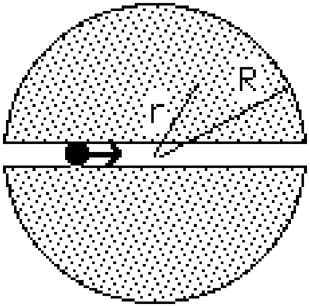
Fields and energy

Integrate force to get energy

Set $\frac{GMm}{r^2} = |F| = ma = \frac{mv^2}{r}$ to get K.

Set $\frac{GMm}{r^2} = |F| = ma = mr\omega^2$ to relate r and T.

Example of the shell theorem. If ρ_{earth} were uniform (it's not), how long would it take for a mass to fall through a hole in the earth to the other side?



$$M_r = \rho . \frac{4}{3} \pi r^3$$

$$\therefore \; F_r = -G \frac{m\rho . \frac{4}{3} \pi r^3}{r^2} = -Kr$$

where $K = Gm\rho\pi . \frac{4}{3}$

$\therefore$ motion is simple harmonic motion (Session 2) with $\omega = \sqrt{\frac{K}{m}}$

$$T = \frac{2\pi}{\omega} = T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{K}}$$

$$= 2\pi\sqrt{\frac{m}{Gm\rho\frac{4}{3}\pi}} = 2\pi\sqrt{\frac{R^3}{G\rho\frac{4}{3}\pi R^3}} = 2\pi\sqrt{\frac{R^3}{GM}}$$ *Units: G -> $m^3 kg^{-1} s^{-2}$, ρ -> $kg.m^{-3}$*

= .... = 84 minutes

$\therefore$ falls through (one half cycle) in 42 minutes *(actually faster: $\rho(r)$ varies)*

Appendix. Why did Newton expect $F \propto \frac{1}{r^2}$? He knew Kepler's empirical law:

For planets with circular orbit, $r^3 \propto T^2$ (r = orbit radius, T = period)

Now if $r\omega^2 = a_{centrip} \propto F \propto \frac{1}{r^2}$ so $r(\frac{2\pi}{T})^2 \propto \frac{1}{r^2}$ so $r^3 \propto T^2$

Planet	r from sun	T	ω	$r\omega^2$
	million km	Ms	rad.s^{-1}	ms^{-2}
Mercury	58	7.62	8.25 10^{-7}	3.95 10^{-5}
Venus	108	19.4	3.23 10^{-7}	1.13 10^{-5}
Earth	150	31.6	1.99 10^{-7}	5.94 10^{-6}

so calculate $r^3\omega^2$:

mercury	1.31 10^{20} m^{3}s^{-2}
venus	1.32 10^{20} m^{3}s^{-2}
earth	1.33 10^{20} m^{3}s^{-2}

Question In Jules Verne's "From the Earth to the Moon", the heroes' spaceship is fired from a cannon*. If the barrel were 100 m long, what would be the average acceleration in the barrel to get to v_{esc}?

$$v_f^2 - v_i^2 = 2as$$

$$a = \frac{v_e^2 - 0}{2s} = \frac{(1.12\ 10^4\ \mathrm{ms^{-2}})^2}{2 \times 100\ \mathrm{m}}$$

$$= 630{,}000\ \mathrm{ms^{-2}} = 64{,}000\ \mathrm{g}$$

* why? If you burn all the fuel on the ground, you don't have to accelerate it and to lift it. *Much* more efficient.

Question. How can one buy and sell gold at different latitudes so as to make a profit?

Equator: $g = 9.78\ \mathrm{ms}^{-2}$ so 1 kg weighs 9.78 N

Poles: $g = 9.83\ \mathrm{ms}^{-2}$ so 1 kg weighs 9.83 N

Use spring balance or electric balance, not scales