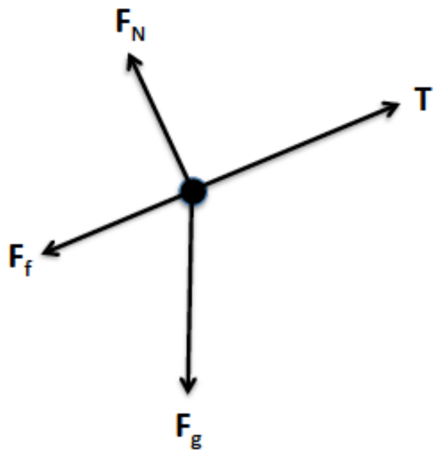
Solutions

(a)



Here $\mathbf{F}_g$ is the gravitational force, **T** is the tension in the string, $\mathbf{F}_f$ is the frictional force and $\mathbf{F}_N$ is the normal force.

(b)

Perpendicularly to the plane we still have:

$F_N = mg\cos\theta$

Along the plane:

$T - mg\sin\theta - \mu mg\cos\theta = ma$

where a is the acceleration. Hence:

$T = mg(\sin\theta + \mu\cos\theta) + ma$

$T = 5.1 \times 9.8 \times (\sin 30° + 0.4\cos 30°) + 5.1 \times 3.0 = 58N$

(c)

$W_g = (mg\sin\theta)d = 5.1 \times 9.8 \times \sin 30° \times 48 = 1199.5$ J

Make sure the sign is positive.

To two significant figures this is: 1200 J.

(d)

$W_f = -\mu(mg\cos\theta)d = -0.4 \times 5.1 \times 9.8 \times \cos 30° \times 48 = -831.1$ J

Make sure the sign is negative.

To two significant figures this is -830 J.

(e)

Either

Work done by spring + work done by gravity + work done by friction = 0 *or*

U gained by spring = U lost by gravity – work done *against* friction
1200 J - 830 J = 370 J.
When the mass hits the spring this is all kinetic energy, and if we ignore friction it will all be converted into potential energy stored in the spring, which we can call U.
Using:

$$U = \frac{1}{2}kx^2$$

x = compression of the spring

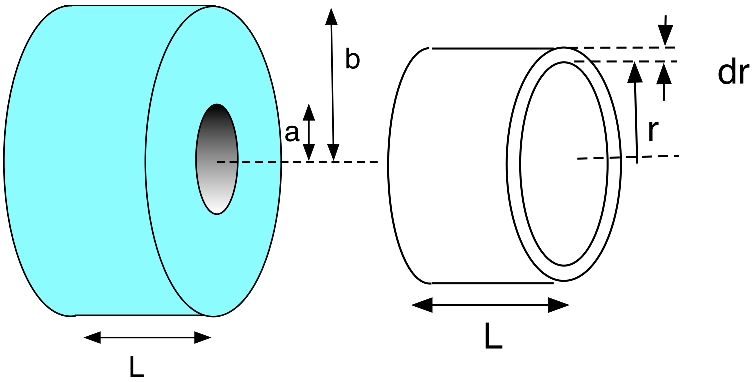
$$x = \sqrt{\frac{2U}{k}} = \sqrt{\frac{2 \times 370}{95000}} = 0.088 \text{ m or } 88 \text{ mm.}$$

Question 2.

i) The moment of inertia of a body about a particular axis is

$I = \int_{body} r^2\, dm$, where r is the radial displacement of each element of mass dm from the axis in question, and where the integration covers the whole of the body.

ii) For this body, consider the dm sketched, with radius r and thickness dr.



Here, $dm = \rho dV = \rho L*2\pi r*dr$

So $dI = r^2 dm = \rho L*2\pi r^2*dr$

This body lies entirely between $r = a$ and $r = b$. So those are the limits of integration:

$$I = 2\pi\rho L \int_{body} r^3\, dm = 2\pi\rho L(b^4/4 - a^4/4) = ½\, \pi\rho L(b^4 - a^4)$$

Using $(b^4 - a^4) = (b^2 - a^2)(b^2 + a^2)$ *and the total mass of the object, it's possible to simplify this further, but this was not sought here.*

b) $\tau = I\alpha$ where τ is the total (external) torque, I is the moment of inertia (or rotational inertia) and α is the angular acceleration.

c) On contact, there is relative motion, so the friction is kinetic and $F_f = \mu_k N$. So the torque about the axle is $\tau = rF \sin \Theta = r F_f = \mu_k rN$.

Using Newton's second law of motion for rotation

$\tau = I\alpha$ so $\alpha = \tau/I = \mu_k rN/I$.

Torque and angular acceleration are constant, so the definition of ω and α give:

$\omega = \omega_0 + \alpha t$ (*compare with* $v = v_0 + at$ *for linear*)

where here the final condition is rolling, for which $\omega = v/r$, so

$v/r = 0 + (\mu_k rN/I)t$.

So $t = vI/\mu_k r^2 N = (48 \text{ m.s}^{-1} * 1.1 \text{ kg.m}^2)/(0.85 * (0.28 \text{ m})^2 * 960 \text{ N}) = 0.83$ s.

d) The runway exerts a torque to increase the ω of the wheel. The brakes exert a force to reduce the ω of the wheels. So the total accelerating torque on the wheel is less, so the wheels skid for longer.

Question 3.

a) The potential energy U is defined by $dU = -dW$ where $dW = \boldsymbol{F.ds}$ is the work done by a conservative force $\boldsymbol{F}$ over a displacement $\boldsymbol{ds}$. Here that force is $F = -\dfrac{GMm}{r^2}$.

$$U = \int \frac{GMm}{r^2} dr = -\frac{GMm}{r}$$ plus a constant of integration.

Conventionally, we set the zero of U at infinity, so

$$U = -\frac{GMm}{r}$$

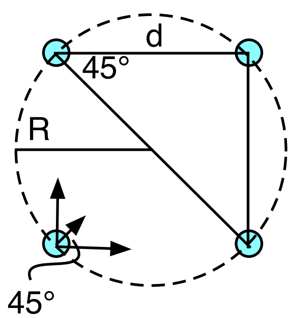
b) Newton's law of universal gravitation: $|F_{\text{grav}}| = \dfrac{GMm}{r^2}$

For m in circular orbit about m, write $|F_{\text{grav}}| = ma$, using Newton's 2nd law.
But the motion is uniform and circular so the acceleration is entirely centripetal, so $a = v^2/r$, so

$$\frac{GMm}{r^2} = m\frac{v^2}{r}$$ and cancelling r,

$$K = ½ mv^2 = \frac{GMm}{2r}$$

c)



i) Newton's law of gravitation: $F_{\text{grav}} = -\dfrac{GMm}{r^2}$ Forces shown on one star in sketch.

Here, the separation between closest stars is d = $\sqrt{2}\,R$ and between the furthest pairs, $2R$.

So $$|F_{\text{total}}| = \frac{Gm^2}{(2R)^2} + 2\frac{Gm^2}{(\sqrt{2}R)^2}\cos 45°$$

$$= \frac{Gm^2}{(2R)^2} + 2\frac{Gm^2}{(\sqrt{2}R)^2}\frac{\sqrt{2}}{2}$$

So $$|F_{\text{total}}| = \frac{Gm^2}{R^2}\left(\frac{1}{4} + \frac{1}{\sqrt{2}}\right) = 0.96\frac{Gm^2}{R^2}$$

ii) The acceleration is centripetal, so $|F_{\text{total}}| = ma = mR\omega^2$.
$\omega = 2\pi/T$, so

$T = 2\pi/\omega$ where, rearranging from above, $\omega = \sqrt{\dfrac{|F_{\text{total}}|}{mR}}$ so

$$T = 2\pi\sqrt{\frac{mR}{|F_{\text{total}}|}} = 2\pi\sqrt{\frac{mR}{0.96 * Gm^2/R^2}} = 2\pi\sqrt{\frac{R^3}{0.96 * Gm}}$$

iii) The stars are $d = \sqrt{2}\,R = 1$ light second apart. Rearranging the previous answer

$$\left(\frac{2\pi}{T}\right)^2 = 0.96\frac{Gm}{R^3} \quad \text{so}$$

$$m = \left(\frac{2\pi}{T}\right)^2 \frac{R^3}{0.96 * G} = \left(\frac{2\pi}{20*60\ \text{s}}\right)^2 \frac{(3.0*10^8\ \text{m}/\sqrt{2})^3}{0.96*6.67*10^{-11}\ \text{Nm}^2\text{kg}^{-2}} \quad (\textit{check units!})$$

$= 4.1 \times 10^{30}$ kg (*FYI about twice the mass of the sun*)

Question 4 PHYS1131/1141 T1 2017 Solutions

a) i) 3 translational degrees of freedom and 2 rotational degrees of freedom gives 5 degrees of freedom.

ii)

$$\begin{aligned}
&\Delta P = \tfrac{mg}{A} = \tfrac{5.00\times 9.80}{50.0\times 10^{-4}} = 9800Pa \\
&\Rightarrow P = 1.01 \times 1.013 \times 10^5 + 9800 \\
&= 112113Pa \\
&= 1.12 \times 10^5 Pa \text{ (3 sig fig)} \\
&\text{OR} \\
&1.11atm \text{ (3 sig fig)}
\end{aligned}$$

iii) As the mass is placed on quickly this is an adiabatic process.

$$P_A V_A^\gamma = P_B V_B^\gamma$$

In this case:

$$\gamma = \frac{C_P}{C_V} = \frac{f+2}{f} = \frac{7}{5}$$

We can also calculate the initial height (h_A: the height in state A):

$$h_A = \frac{V_A}{A} = \frac{1.00 \times 10^{-3}}{50.0 \times 10^{-4}} = 0.200m$$

So:

$$\begin{aligned}
&P_A h_A^\gamma A^\gamma = P_B h_B^\gamma A^\gamma \\
&\Rightarrow h_B^\gamma = \frac{P_A h_A^\gamma}{P_B} \\
&h_B^{1.4} = \frac{1.01 \times 0.200^{1.4}}{1.1067} \\
&\Rightarrow h_B = 0.187m = 18.7cm \text{ (3 sig fig)}
\end{aligned}$$

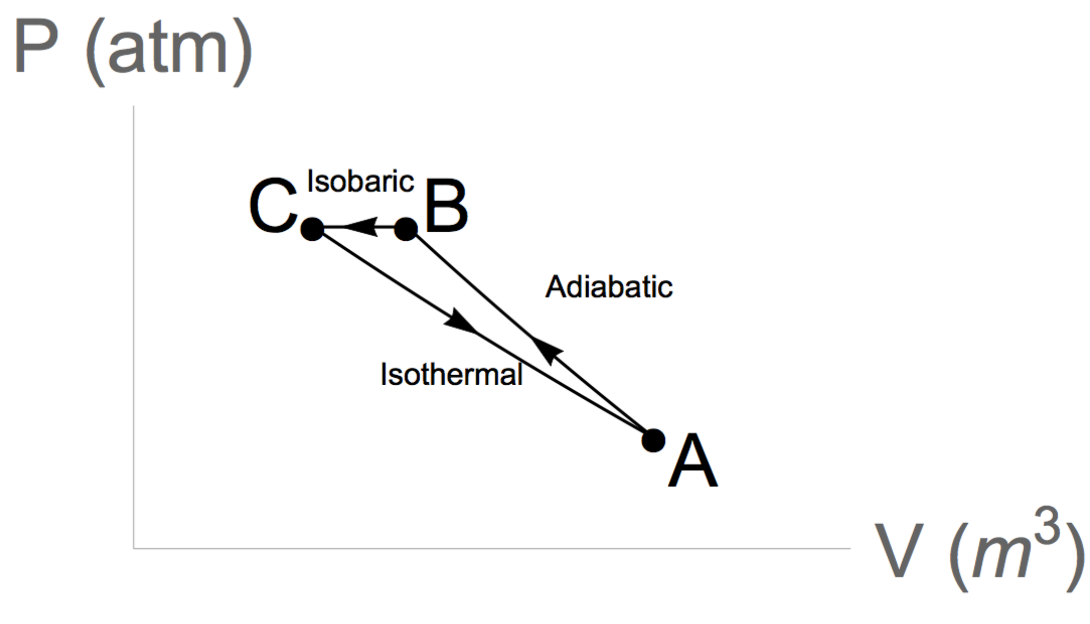
iv) Yes the height will change as the temperature changes going from state B to C.

It is easiest to calculate the new height considering states A and C which are at the same temperature (though it is possible to calculate T_B = 306.8 K and use that, it takes longer though….).

States B and C have the same pressure as the weight has not changed.

$$\begin{aligned}
&P_A V_A = P_C V_C \\
&\Rightarrow P_A h_A = P_C h_C \\
&\Rightarrow h_C = \frac{P_A h_A}{P_C} = \frac{1.01 \times 0.200}{1.1067} = 0.183m = 18.3cm \text{ (3 sig fig)}
\end{aligned}$$

v) and vi)



Where P_A = 1.01 atm, V_A = 1.00 L, P_B = 1.11 atm, V_B = 0.937 L, P_C = 1.11 atm, V_C = 0.913 L.

vii) Need to use:

$$W = -\int P \cdot dV$$

where:

$$PV = P_A V_A \Rightarrow P = \frac{P_A V_A}{V}$$

Which results in:

$$\begin{aligned}
W &= -\int_{V_C}^{V_A} \frac{P_A V_A}{V} \cdot dV = -P_A V_A \int_{V_C}^{V_A} \frac{dV}{V} \\
&= -1.01 \times 1.013 \times 10^5 \times 1.00 \times 10^{-3} \times \left[\ln V\right]_{0.913\times10^{-3}}^{1.00\times10^{-3}} \\
&= -102.313 \ln\left(\frac{1.00}{0.913}\right) \\
&= -9.31J \text{ (3 sig fig)}
\end{aligned}$$

b) i) Use:

$$\begin{aligned}
&\Delta V = 3\alpha V_i \Delta T \\
&= 3 \times 17 \times 10^{-6} \times (5.00 \times 10^{-2})^3 \times (275 - 25) \\
&= 1.6 \times 10^{-6} m^3 \text{ (2 sig fig)}
\end{aligned}$$

ii) Need to know the mass of copper:

$$m_C = \rho_C \times V_C = 8.96\text{g}/\text{cm}^3 \times (0.0500)^3\text{cm}^3 = 1120g = 1.120kg$$

Then use:

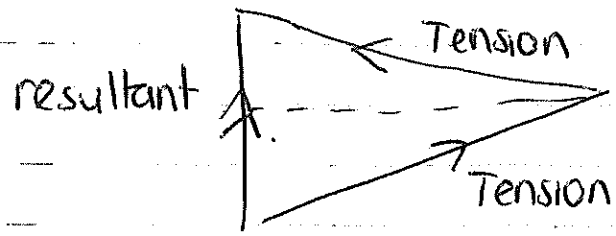
$$
\begin{aligned}
&0 = m_w c_w (T_f - T_{iw}) + m_C c_C (T_f - T_{iC}) + m_A c_A (T_f - T_{iw}) \\
&T_f (m_w c_w + m_C c_C + m_A c_A) = m_w c_w T_{iw} + m_C c_C T_{iC} + m_A c_A T_{iw} \\
&T_f = \frac{m_w c_w T_{iw} + m_C c_C T_{iC} + m_A c_A T_{iw}}{m_w c_w + m_C c_C + m_A c_A} \\
&= \frac{50.0 \times 4186 \times 20.0 + 1.12 \times 390 \times 275 + 0.500 \times 910 \times 20.0}{50.0 \times 4186 + 1.12 \times 390 + 0.500 \times 910} \\
&= 20.5^o C \text{ (3 sig fig)}
\end{aligned}
$$

Question 5

(a)

(i) T is upwards.

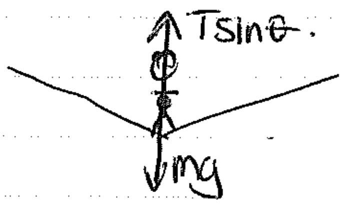
Consider two components opposite each other.

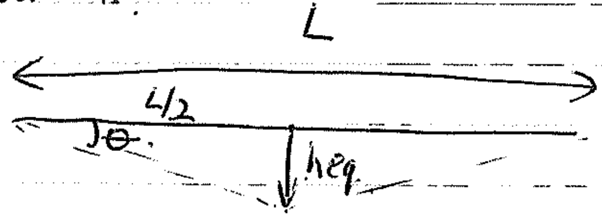


The horizontal components cancel and the vertical components add. All around the trampoline we can split it into pairs such as this.

$\Rightarrow$ resultant tension is upwards.

(ii)





It will be in equilibrium when the forces are balanced

$$\Rightarrow \quad mg = T\sin\theta$$

$$\sin\theta = \frac{mg}{T} \simeq \theta$$

$$\tan\theta = \frac{h_{eq}}{L/2} \quad \text{(from diagram} \uparrow\text{)}$$

In small angle approximation $\tan\theta \approx \sin\theta \approx \theta$.

$$\Rightarrow \quad \frac{mg}{T} = \frac{h_{eq}}{L/2} \Rightarrow h_{eq} = \frac{Lmg}{2T}$$

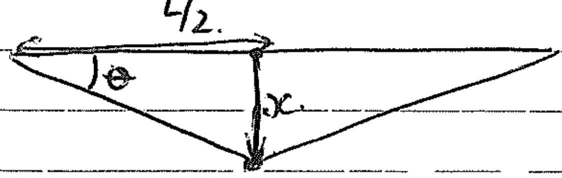
(iii) Simple harmonic motion occurs when the force is proportional to the distance from equilibrium and directed back towards the equilibrium position.

From Newton's second law:

$$ma = \frac{md^2h}{dt^2} = -T\sin\theta + mg$$

(taking down as +ve and up as negative)

$$\sin\theta \approx \frac{x}{L/2}$$



x = distance from top of trampoline (downwards)
$= h_{eq} + h'$ ← distance from equilibrium (downwards).

$$m\frac{d^2h}{dt^2} = -\frac{2Tx}{L} + mg$$

$$= -\frac{2T(h_{eq}+h')}{L} + mg$$

this term is mg

$$= -\frac{2Th_{eq}}{L} - \frac{2Th'}{L} + mg.$$

$$= -\frac{2Th'}{L}.$$

$$\Rightarrow \frac{d^2h}{dt^2} = -\frac{2Th'}{Lm}$$

This is proportional to distance from equilibrium h' and opposite.

(iv) $a = -\omega^2 x.$

$$\Rightarrow \omega^2 = \frac{2T}{Lm}.$$

$$\omega = \sqrt{\frac{2T}{Lm}} \quad \text{or} \quad f = \frac{1}{2\pi}\sqrt{\frac{2T}{Lm}} \text{ is also fine.}$$

(v) At equilibrium position all energy is KE

$$\tfrac{1}{2}kA^2 = \tfrac{1}{2}mv^2.$$

we need to find v at equilibrium.

$$k = m\omega^2 = \frac{\cancel{m}2T}{L\cancel{m}} = \frac{2T}{L}.$$

Distance fallen $= H - h_0 + h_{eq}$

$$v^2 = v_0^2 + 2ay$$

$$= 2g(H - h_0 + h_{eq})$$

$$\Rightarrow A^2 = \frac{mv^2}{k} = \frac{\cancel{2}mg(H-h_0+h_{eq})L}{\cancel{2}T}$$

$$= \frac{mgL(H-h_0+h_{eq})}{T}$$

$$A = \sqrt{\frac{mgL}{T}\left(H-h_0+\frac{Lmg}{2T}\right)}$$

(b) $f = 6.11\ kHz$ $v = 343\ m/s$ $\xrightarrow{10.7 m/s}$ nest

(i) nest contains observers

$$\Rightarrow f' = f\left(\frac{v}{v - v_s}\right)$$

$$= 6.11 \times \left(\frac{343}{343 - 10.7}\right)$$

$$= 6.3067\ kHz$$

$$= 6.31\ kHz \quad (3 \text{ sig fig}).$$

(ii) Now consider swiftlet flying towards the wall as the observer: the wall is the source

$$\Rightarrow f'' = f'\left(\frac{v + v_0}{v}\right).$$

$$= 6.3067 \times \left(\frac{343 + 10.7}{343}\right)$$

$$= 6.50348\ kHz$$

$$= 6.50\ kHz \quad (3 \text{ sig fig.})$$

(iii) $f''' = f\left(\dfrac{v + v_0}{v - v_s}\right)$ s $\xrightarrow{10.7 m/s}$ $\xleftarrow{14.1 m/s}$ o

$$= 6.11 \times \left(\frac{343 + 14.1}{343 - 10.7}\right).$$

$$= 6.5660\ kHz$$

$$= 6.57\ kHz \quad (3 \text{ sig fig})$$

(c)(i) $f_{beat} = 4.00\ Hz = |f_1 - f_2|$

For beats $f = 444\ Hz$ or $436\ Hz$.

but it is too tight $\Rightarrow$ Tension is high $v = \sqrt{\frac{T}{\mu}}$.

v is high $\Rightarrow$ f is high.

$\Rightarrow f = 444\ Hz.$

(ii) $y_1 = A\sin(kx - \omega_1 t + \phi)$
$y_2 = A\sin(kx - \omega_2 t + \phi)$

Trigonometric identities are easier if these are cosine functions.
Lets choose:

$y_1 = A\cos(kx - \omega_1 t)$
$y_2 = A\cos(kx - \omega_2 t)$

$$y_1 + y_2 = 2A\cos\left(\frac{kx - \omega_1 t + kx - \omega_2 t}{2}\right)\cos\left(\frac{kx - \omega_1 t - kx + \omega_2 t}{2}\right)$$

$$= 2A\cos\left(\frac{2kx - (\omega_1 + \omega_2)t}{2}\right)\cos\left(\frac{(\omega_2 - \omega_1)t}{2}\right)$$

Sub in $\omega_1 = 2\pi f_1$, $\omega_2 = 2\pi f_2$ $\Rightarrow$

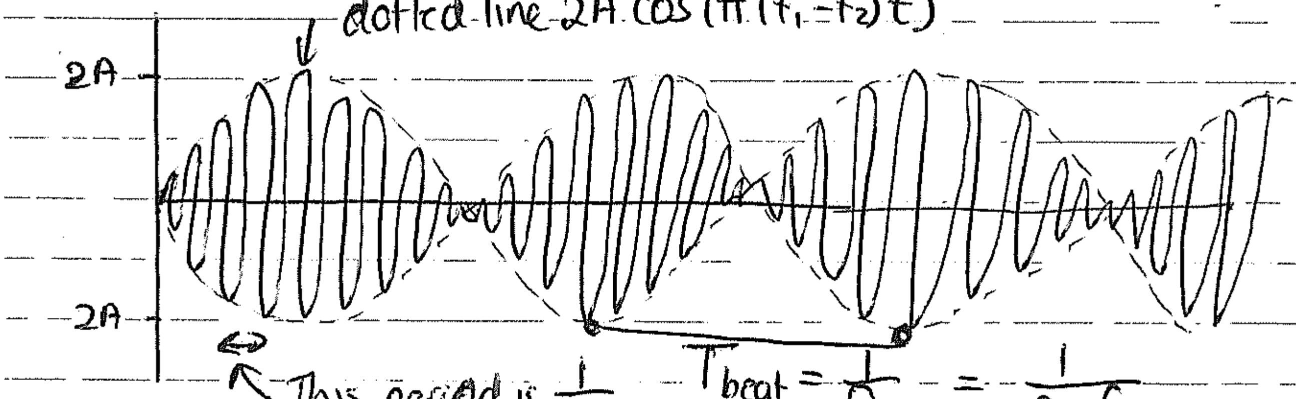
$$= \underbrace{2A\cos(kx - \pi(f_1 + f_2)t)}_{\text{varies rapidly with time}}\ \underbrace{\cos(\pi(f_2 - f_1)t)}_{\text{varies slowly with time}}$$

varies slowly with time $\Downarrow$ gives beats.

$f_{beat} = f_2 - f_1$ but f_2 and f_1 are interchangable.
$\Rightarrow f_{beat} = |f_2 - f_1|$

(iii) Choose any $x \Rightarrow x = 0$ to make it easy.

dotted line $2A\cos(\pi(f_1 - f_2)t)$



This period is $\frac{1}{f_1 + f_2}$ $T_{beat} = \frac{1}{f_{beat}} = \frac{1}{f_1 - f_2}$.

The $\cos(\pi(f_2 - f_1)t)$ term slowly varies between 1 and -1 giving the beats. The other term is a rapid oscillation. This is too quick for us to detect, we hear the loud/soft coming from the difference in the two frequencies.