Waves and Oscillations

Lecture 8 – Simple Harmonic Motion

Textbook reference: 15.1-15.2

From 2001: A Space Odyssey

New topic:
Waves and oscillations

Lecture 8: Springs and simple harmonic motion

See also Joe Wolfe's physclips

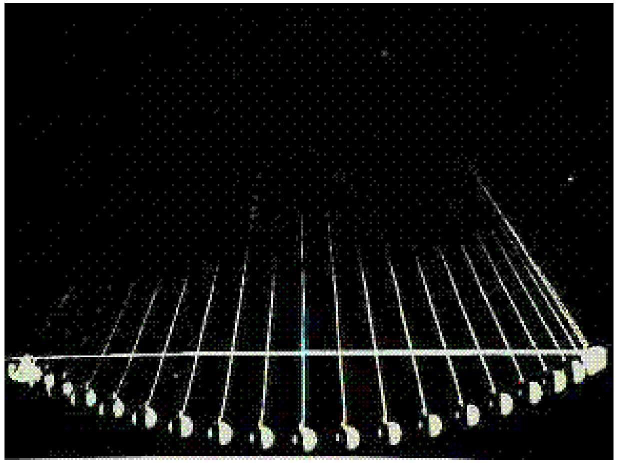
http://www.animations.physics.unsw.edu.au/mechanics/chapter4_simpleharmonicmotion.html

And https://youtu.be/klIlCfte0UM for MIT's open Courseware lecture.

Periodic motion...

- **Periodic motion**, or **harmonic motion**, is a motion of an object that regularly returns to a given position after a fixed time interval.

Guitar string



Motion of a pendulum

Simple harmonic motion (SHM)...

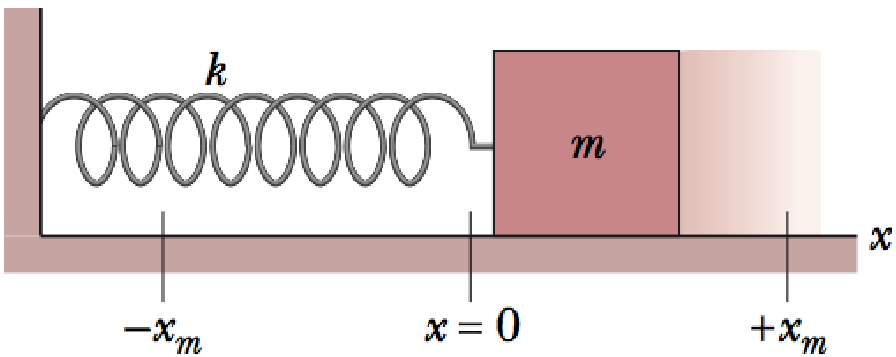
A type of periodic motion where the force is

– proportional to the displacement from the equilibrium (i.e., most relaxed) position, and

– directed towards the equilibrium position.

Hooke's law $$F_s = -kx$$ Displacement from the equilibrium position (x = 0).

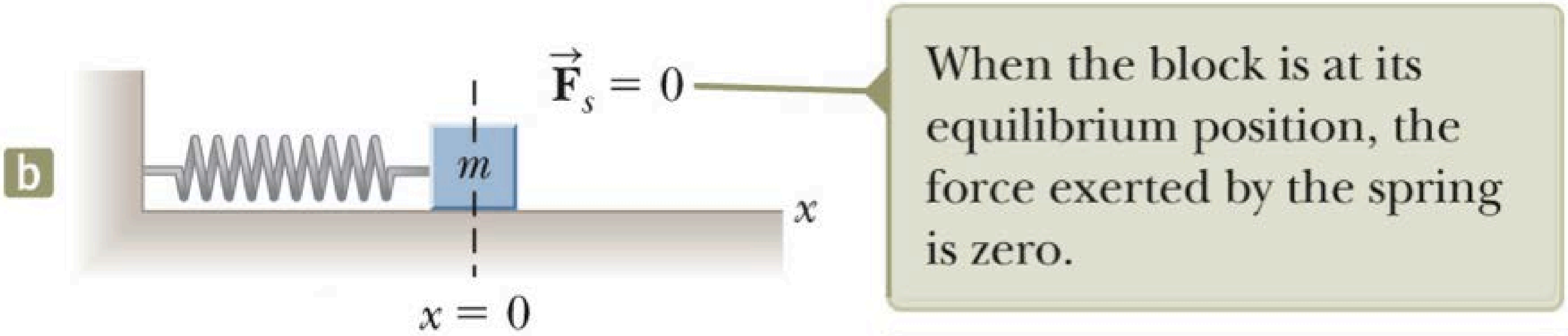
– e.g., motion of a spring, or a simple pendulum.



- Demo Unit Mh1 and Mh11: Simple Harmonic Motion

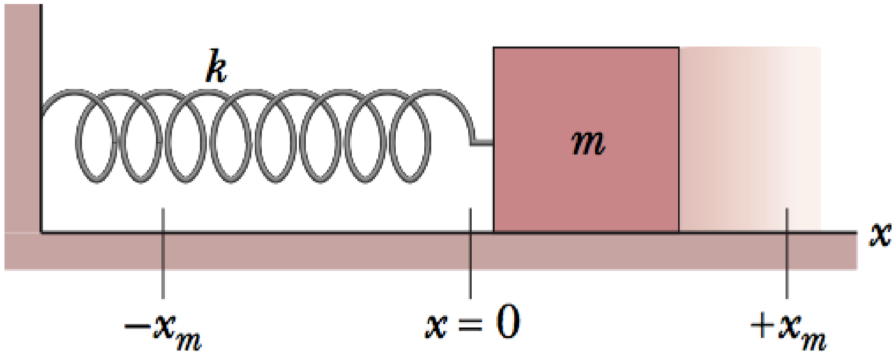
$$F_s = -kx$$

k = spring constant



SHM>Acceleration...

• Consider a block of mass m attached to the end of the spring.



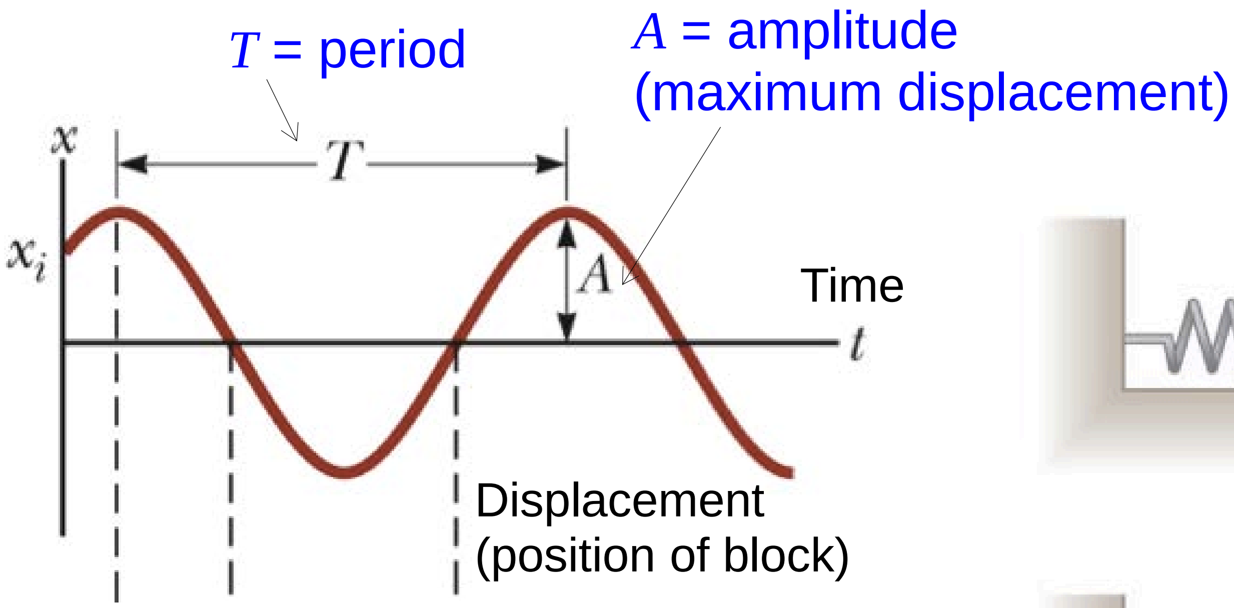
– The **acceleration** of the block is **not** constant during SHM.

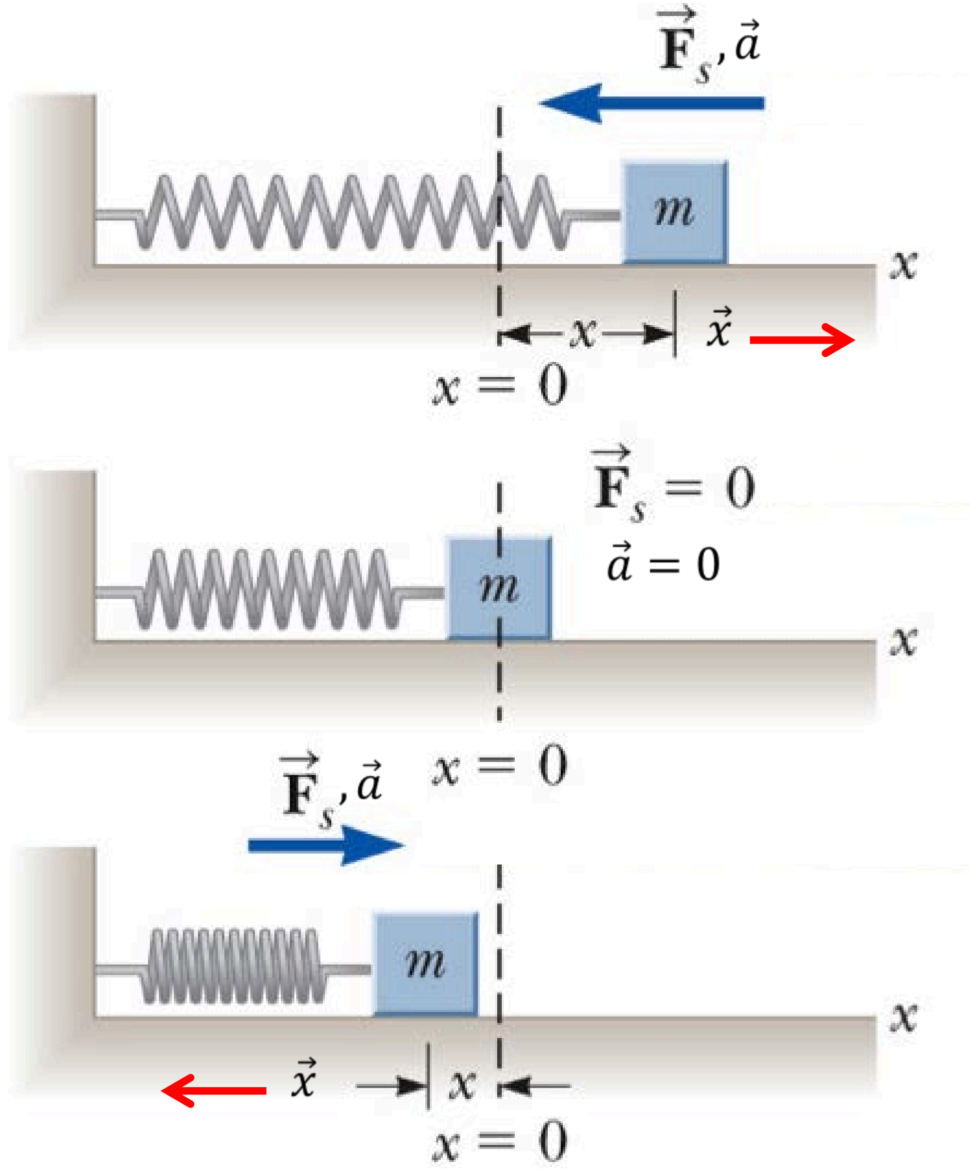
$$F_x = ma_x = -kx$$

Acceleration of the block in the x-direction.

$$\Rightarrow a_x = -\frac{k}{m}x$$

The acceleration is maximum when the displacement of the block is maximum; note opposite sign!



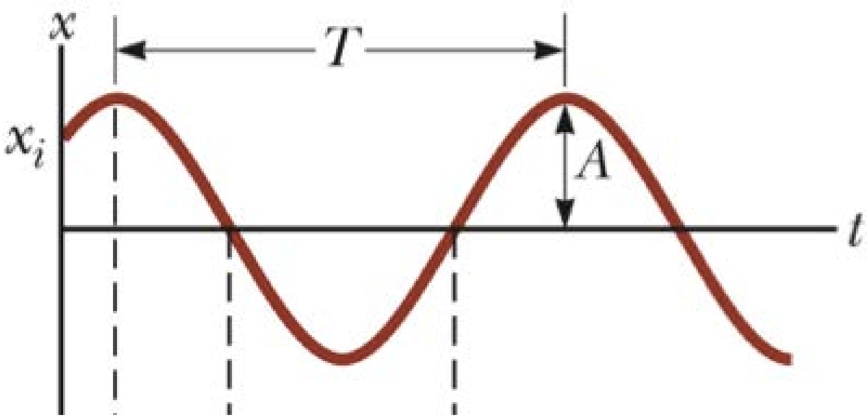


In the absence of friction, the block goes on oscillating between $x = +A$ and $x = -A$ forever...

SHM>Motion of the block...

- The displacement x of the block is a **sinusoidal function** of time t , i.e.,

$$x(t) = A\cos(\omega t + \phi)$$



Or...

$$x(t) = A\sin(\omega t + \phi') = A\sin(\omega t + \phi + \pi/2)$$

– A = amplitude

– ω = angular frequency (units: rad/s)

– φ, φ' = phase constants

Why sinusoidal??

Quick questions...

- A body oscillates with SHM between x = 2.0 and x = 10.0 m. What is the amplitude of the motion?

- What is the equilibrium position?

- What is the acceleration of the block at the equilibrium position?

We need an equation….

$$F = m\, a_x = -k\, x$$

$$a_x = \frac{d^2x}{dt^2} = \frac{-k}{m}x$$

We need an equation….

$x(t)$ must satisfy the **equation of motion** for SHM:

$$a_x = \frac{d^2x}{dt^2} = \frac{-k}{m}x$$

What expression will satisfy this equation?

$$x(t) = A\cos(\omega t + \phi)$$

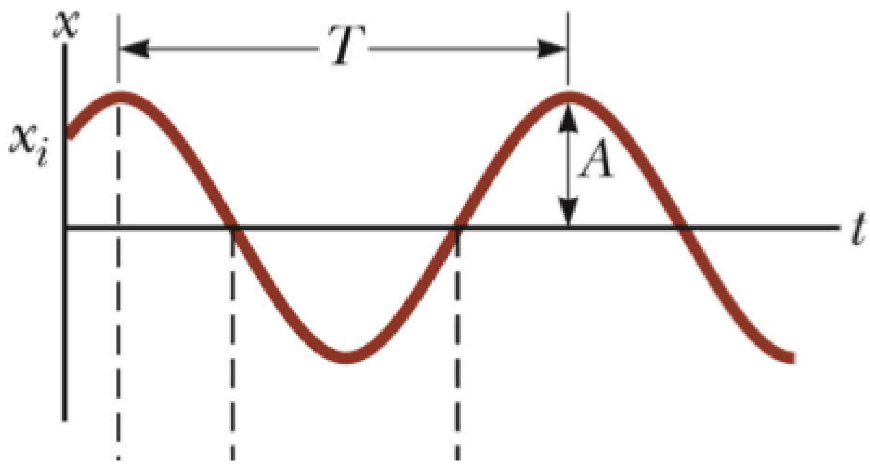
$$\frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \phi)$$

$$= -\omega^2 x(t)$$

- Displacement:

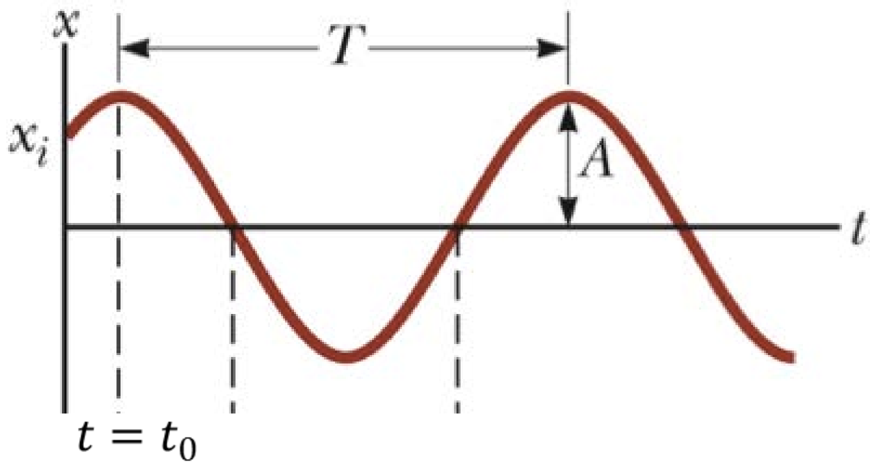
$$x(t) = A\cos(\omega t + \phi)$$



SHM>Phase constant...

- What is the meaning of φ in $x(t) = A\cos(\omega t + \varphi)$?

– The phase constant φ is generally nonzero because at $t = 0$ the displacement x is **not necessarily** maximum.



– Another way to write $x(t)$:

$$x(t) = A\cos[\omega(t - t_0)] \Rightarrow \phi = -\omega t_0$$

A reference time at which the displacement is maximum.

What is ω?

$$F = m\, a_x = -k\, x$$

$$x(t) = A\cos(\omega t + \phi)$$

$$a_x = \frac{-k}{m} x = -\omega^2 x$$

$$\Rightarrow \omega^2 = \frac{k}{m}$$

$$\Rightarrow \omega = \sqrt{\frac{k}{m}}$$

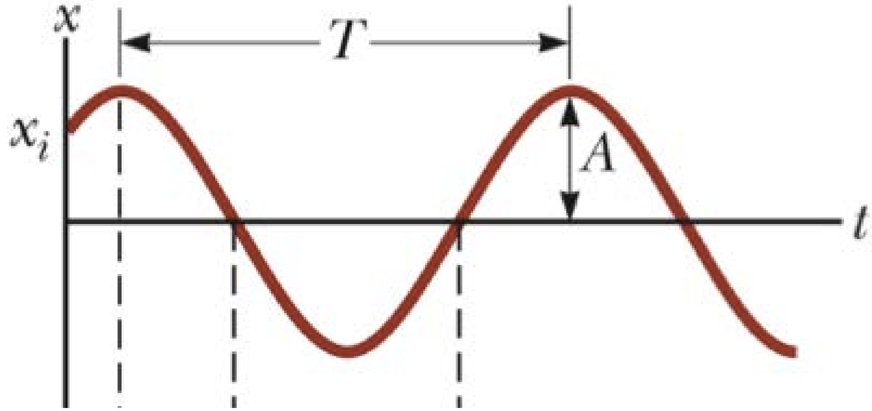
ω is an angular frequency (required by argument of cosine).

What is the relationship between ω and the period T?

SHM>Period, frequency, etc...

- The **period** T is the time interval required for the block's motion to go through one full cycle.

$$x(t) = x(t+T)$$



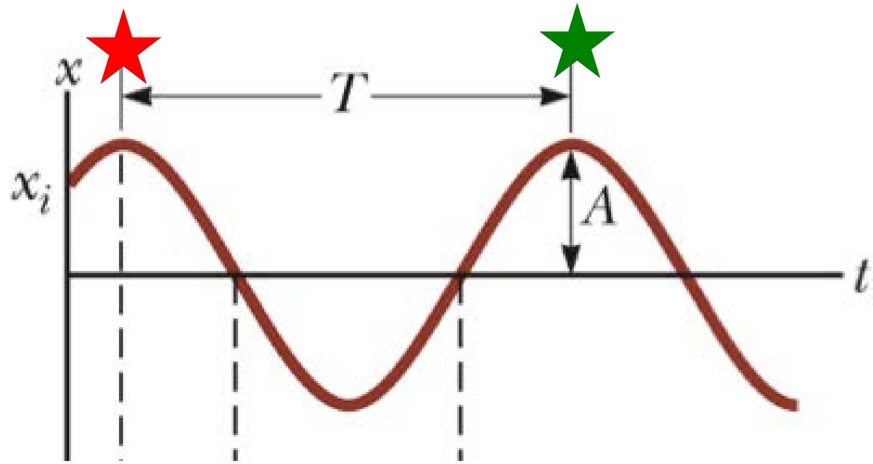
- The **frequency** f is the inverse of the period.

$$f = \frac{1}{T}$$

- To relate the **period** T to the **angular frequency** ω, consider two adjacent maxima.

  - The first maximum occurs when

$$\cos(\omega t + \phi) = 1$$
$$\Rightarrow \omega t + \phi = 0 \quad ★$$



  - The next maximum will be when

$$\omega(t + T) + \phi = 2\pi \quad ★$$

  - ★ minus ★ $\quad \Rightarrow \omega T = 2\pi$

$$\boxed{\omega = \frac{2\pi}{T}}$$

- Period to angular frequency: $$\omega = \frac{2\pi}{T}$$

- Frequency to angular frequency: $$\omega = 2\pi f$$

:

Quick Quiz

- An object of mass *m* is hung from a spring and set into oscillation.
- The period of the oscillation is measured and recorded as T.
- The object of mass *m* is removed and replaced with an object of mass *2m*.
- When this object is set into oscillation, what is the period of the motion?

$$(a) 2T \ (b) \sqrt{2}T \ (c) T \ (d) \frac{T}{\sqrt{2}} \ (e) \frac{T}{2}$$

https://goo.gl/forms/jZaEYQm68QntuBis2

- Period to angular frequency: $$\omega = \frac{2\pi}{T}$$

- Frequency to angular frequency: $$\omega = 2\pi f$$

- Spring constant and mass to period:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

- Spring constant and mass to frequency:

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Pro-tip: set your calculators to RADIANS not degrees

Question

A 0.500 kg object attached to a spring with a force constant of 8.00N/m vibrates in simple harmonic motion with an amplitude of 10.0 cm. Calculate the maximum value of its

a) Speed and

b) Acceleration,

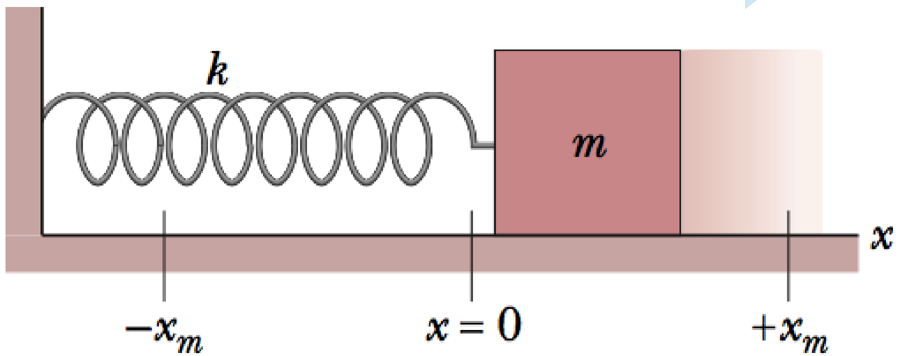
SHM>Energy...

If there is no friction, i.e., the only force acting on the system is due to the spring, then the **total energy is conserved**.

Total energy = Potential energy + Kinetic energy

...stored in the spring

...of the moving block



$$\omega = \sqrt{\frac{k}{m}}$$

- **Kinetic energy** of the block:

$$\mathrm{KE} = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi)$$
$$= \frac{1}{2}kA^2\sin^2(\omega t + \phi)$$

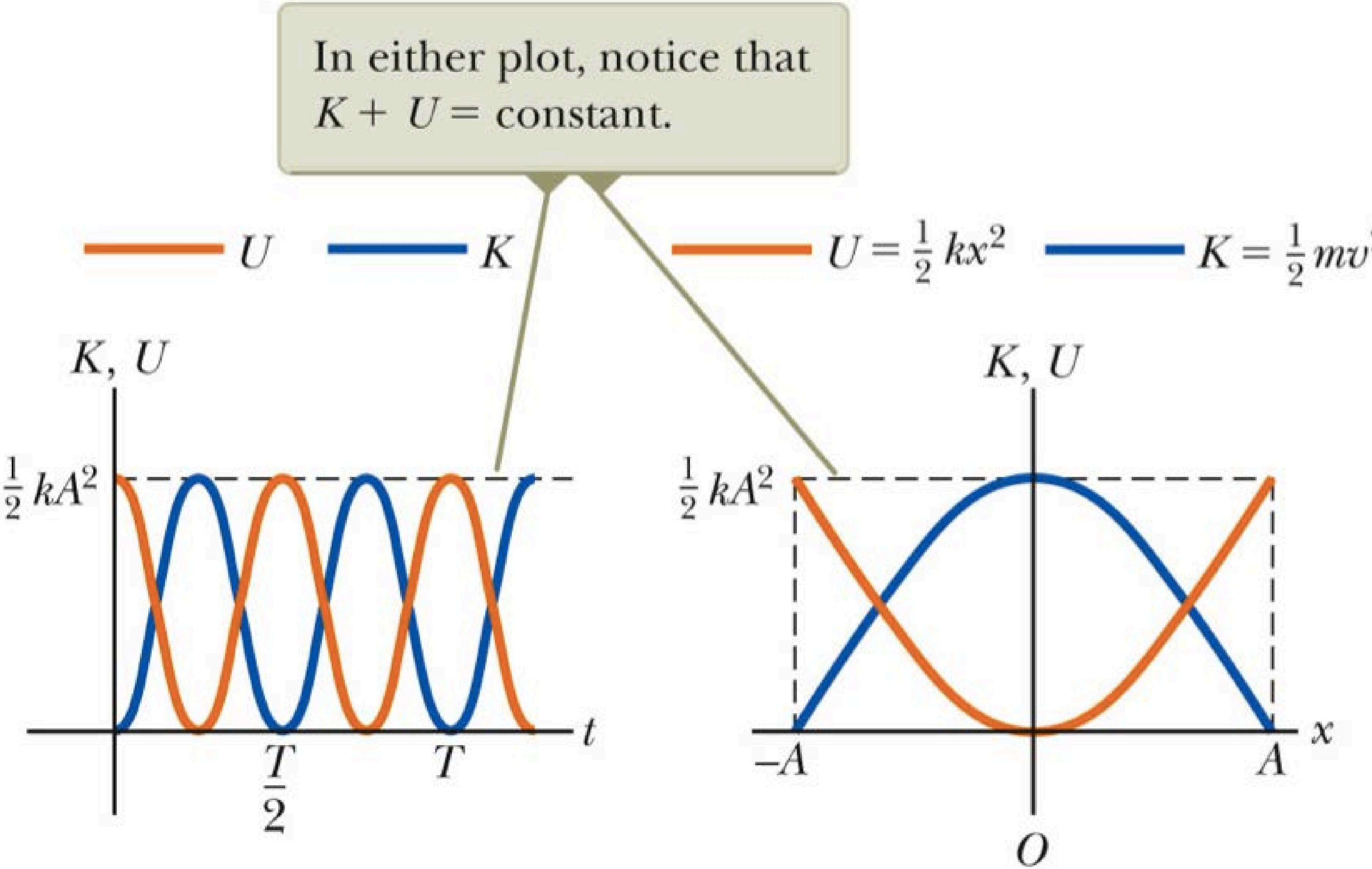
- **Potential energy** stored in the spring:

$$\mathrm{PE} = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$

- **Total energy** (= PE + KE):

The total energy is constant.

$$\mathrm{Total} = \frac{1}{2}kA^2[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)] = \frac{1}{2}kA^2$$


In either plot, notice that $K + U =$ constant.
U
K
$U = \frac{1}{2}kx^2$
$K = \frac{1}{2}mv^2$
K, U
$\frac{1}{2}kA^2$
t
$\frac{T}{2}$
T
K, U
$\frac{1}{2}kA^2$
x
$-A$
A
O

Velocity at a given position

Total energy

$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Solve for v

$$\Rightarrow mv^2 = k(A^2 - x^2)$$

$$\Rightarrow v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)}$$

$$\boxed{\Rightarrow v = \pm\omega\sqrt{(A^2 - x^2)}}$$

Useful relation if you don't know
the explicit time dependence of x.

Question

A 420 g object is attached to a spring and executes simple harmonic motion with a period of 0.250s. If the total energy of the system is 5.83 J, find

a) The maximum speed of the object

b) The force constant of the spring, and

c) The amplitude of the motion