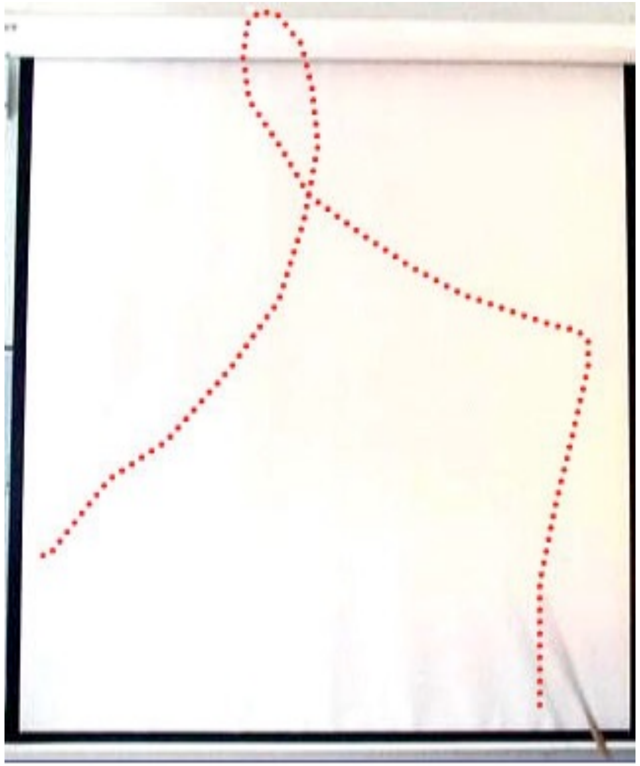
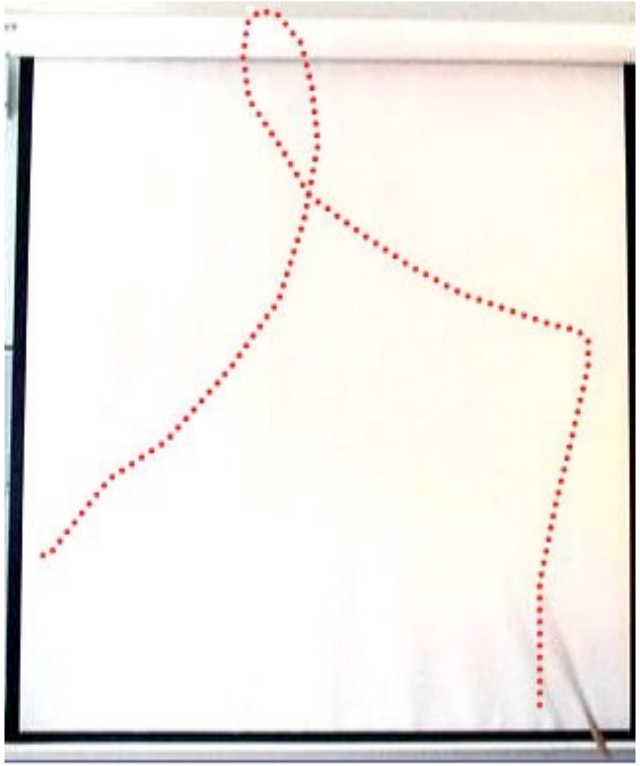
After mechanics of particles:

Mechanics for systems of particles and extended bodies

- **Centre of mass**
- **Momentum**
- **Collisions, elastic and inelastic**
- **Examples**



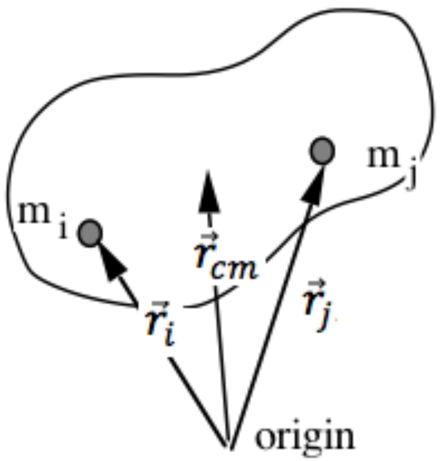
Left: trajectory of end of rod.



Left: trajectory of end of rod. Right: parabola.

Centre of mass

In a finite body, not all parts have the same acceleration. Not even if it is rigid. How to apply $\Sigma\vec{F} = m\vec{a}$ *?*



n particles, m_i at positions $\vec{r}_i$

where i goes from 1 to many

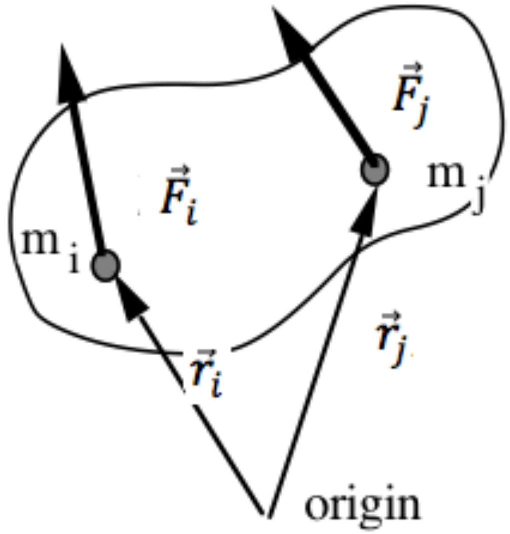
Total mass $M = \sum_{body} m_i$

Define the **centre of mass** as the *point* with displacement:

$$\vec{r}_{cm} = \frac{\Sigma_{body} m_i \vec{r}_i}{M}$$

Why this definition? Let $\vec{F}_i$ act on each and apply Newton's 2nd:

Why? n particles, m_i at positions $\underline{\mathbf{r}}_i$, $\underline{\mathbf{F}}_i$ acts on each. Total force acting on all particles:



$$\sum \vec{F}_i = \sum_{body} m_i \vec{a}_i$$

$$= \sum_{body} m_i \frac{d^2\vec{r}_i}{dt^2}$$ and, if masses constant:

$$= \sum_{body} \frac{d^2(m_i\vec{r}_i)}{dt^2}.$$ Use sum of derivatives = derivative of sum

$$= \frac{d^2}{dt^2}\sum_{body} m_i\vec{r}_i .$$ Then multiply by M/M

$$= M\frac{d^2}{dt^2}\left(\frac{\sum_{body} m_i\vec{r}_i}{M}\right)$$

But we defined $\vec{r}_{cm} = \frac{\sum_{body} m_i\vec{r}_i}{M}$

Then $\sum \vec{F}_i = M\frac{d^2}{dt^2}\vec{r}_{cm} = M\vec{a}_{cm}$

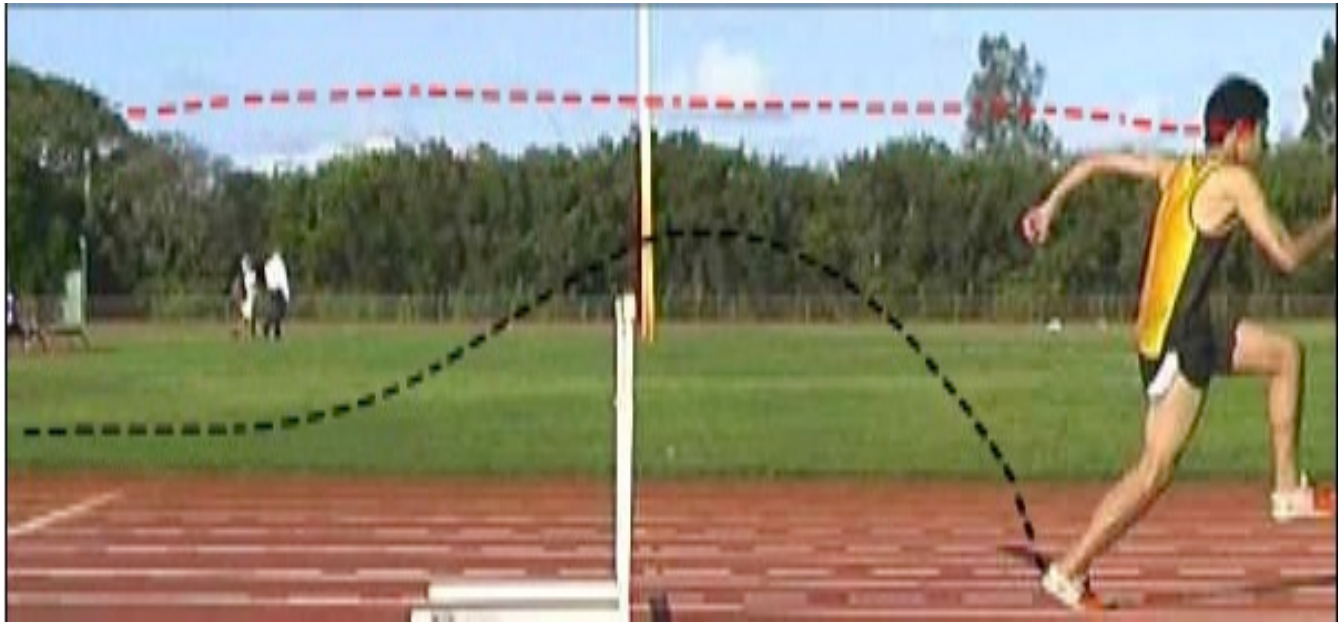
(total force) = (total mass)*(acceleration of centre of mass)   *but wait, there's more*

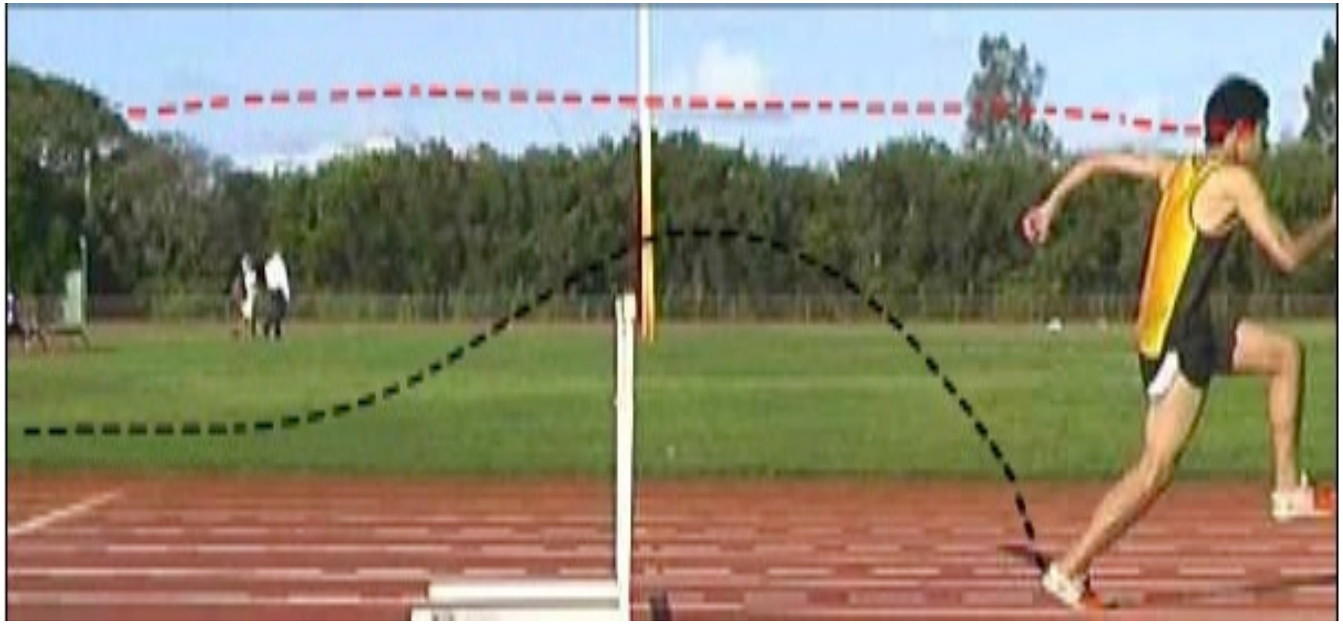
Each $\vec{F}_i$ is the sum of internal and external forces. $\sum \vec{F}_i = \sum \vec{F}_{i,internal} + \sum \vec{F}_{i,external}$

But $\sum \vec{F}_{i,internal} = 0$ from Newton's 3rd law, so

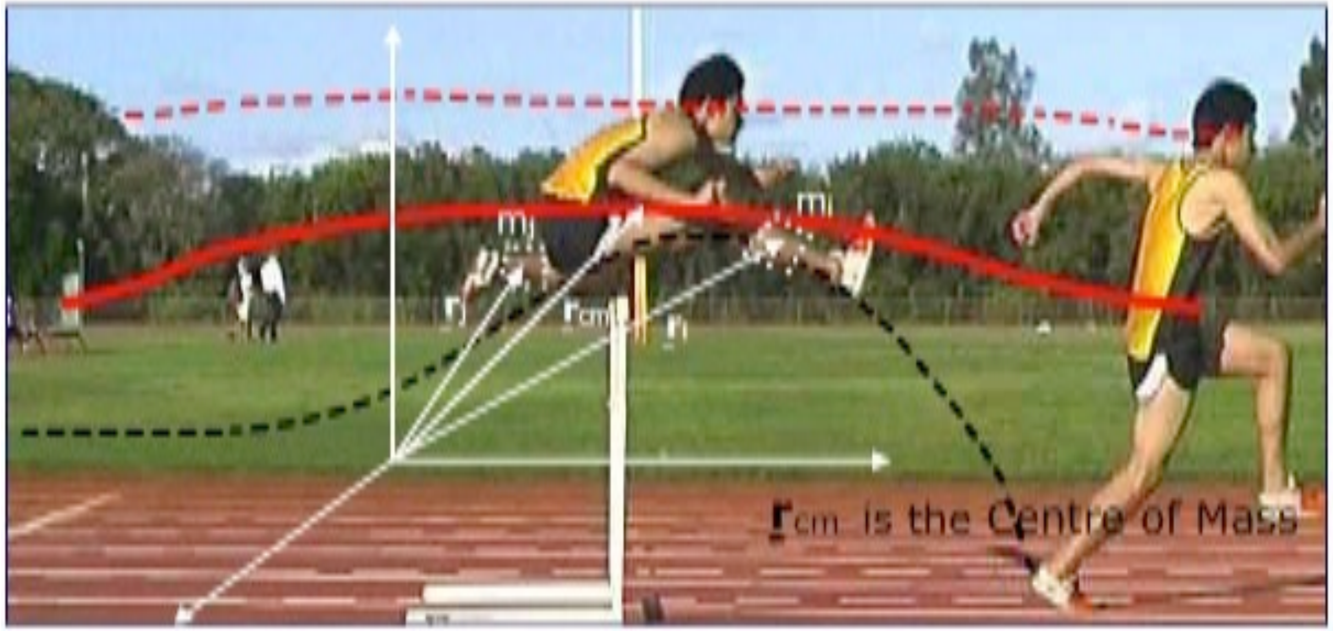
$$\sum \vec{F}_i = \sum \vec{F}_{i,external} = M \frac{d^2}{dt^2} \vec{r}_{cm} = M \vec{a}_{cm} \qquad \blacksquare$$

(total *external* force) = (total mass)*(acceleration of centre of mass)





Mechanics > Centre of mass > 8.6 Newton's laws and centre of mass



For *n* discrete particles, **centre of mass** at

$$\vec{r}_{cm} = \frac{\sum_{body} m_i \vec{r}_i}{M} \qquad \text{(i)}$$

For a continuous body, elements of mass dm at $\vec{r}$

$$\vec{r}_{cm} = \frac{\int_{body} \vec{r}\, dm}{M} \qquad \text{(ii)}$$

We can rearrange (i):

$$\sum_{body} m_i \vec{r}_i = M\vec{r}_{cm} = \left(\sum_{body} m_i\right) \vec{r}_{cm}$$

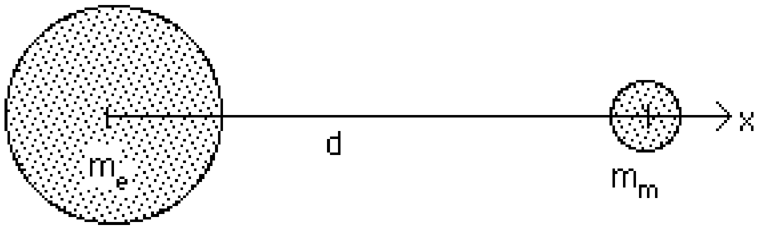
$$\sum_{body} m_i (\vec{r}_i - \vec{r}_{cm}) = 0$$

The 'law of the see saw':

sum of (mass(distance from c.m.)) = 0*

or (ii) –> $\int_{body} (\vec{r}_i - \vec{r}_{cm}) dm = 0$

Example. Where is the c.m. of the earth-moon system?



$$\vec{r}_{cm} = \frac{\Sigma_{body} m_i \vec{r}_i}{M}$$

Take origin at centre of earth. *Symmetry: must be on the axis*

$$x_{cm} = \frac{m_{earth}x_{earth} + m_{moon}x_{moon}}{m_{earth} + m_{moon}}$$

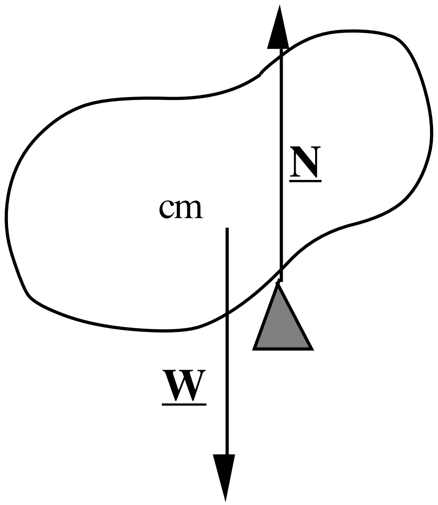
Measure x from centre of earth:

$$= \frac{0 + m_{moon}d}{m_{earth} + m_{moon}} \quad \sim \frac{380{,}000 \text{ km}}{81}$$

$\sim$ 4,600 km i.e. inside the earth.

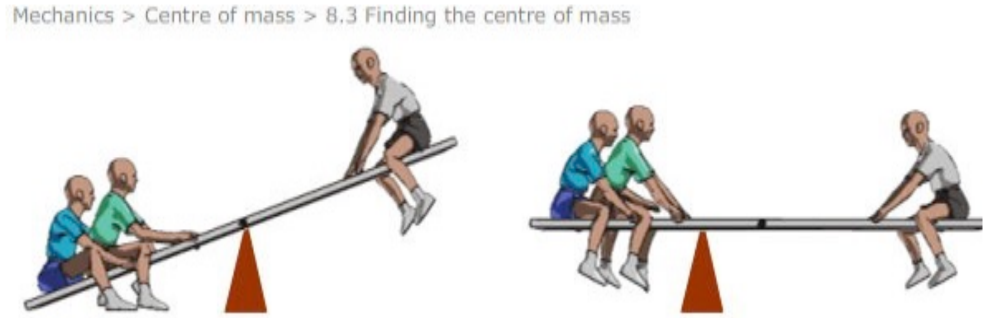
recall: we derived the centre of rotation of this system

Later, when doing rotation, we'll consider torques

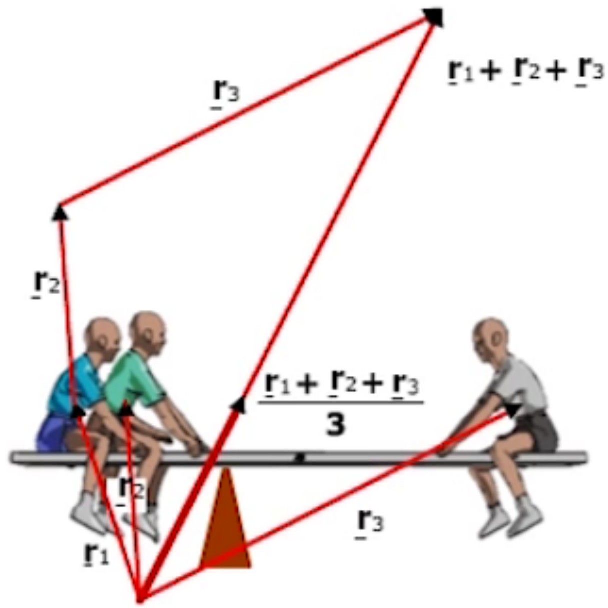


centre of gravity (balance point) is usually the c. of m.

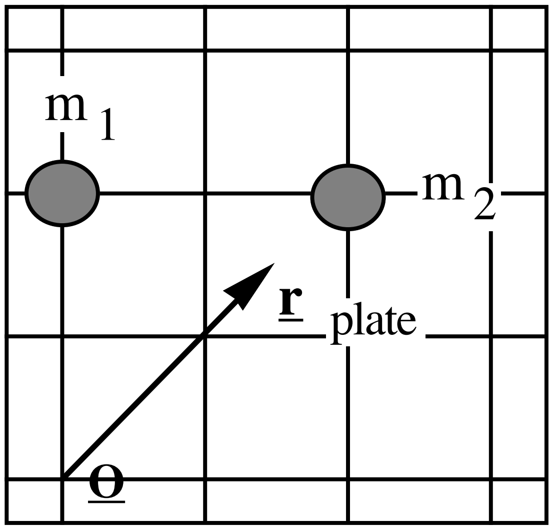
Example: Apply the equation directly



$$\underline{\mathbf{r}}_{cm} = \frac{m\underline{\mathbf{r}}_1 + m\underline{\mathbf{r}}_2 + m\underline{\mathbf{r}}_3}{m + m + m} = \frac{\underline{\mathbf{r}}_1 + \underline{\mathbf{r}}_2 + \underline{\mathbf{r}}_3}{3}$$



Example



$$\vec{r}_{cm} = \frac{\Sigma_{body} m_i \vec{r}_i}{M}$$

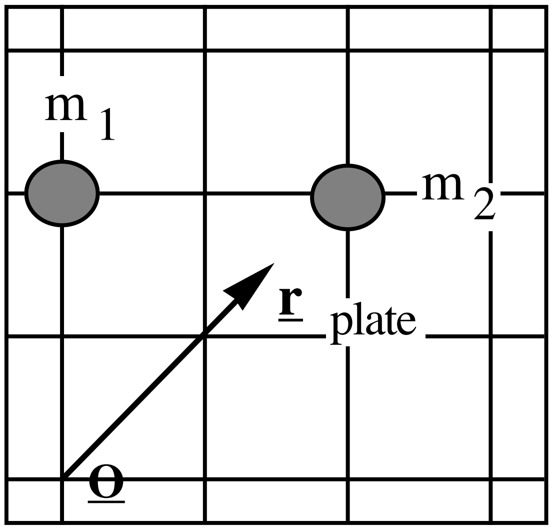
On a square plate (mass m_{p}), we place m_1 and m_2 as indicated.

m_{p} = 135 g, m_1 = 100 g and m_2 = 50 g

Where is the cm of the system?

(*Use length 'units' of the grid*)

Example



On a square plate (mass m_p), we place m_1 and m_2 as indicated.

m_p = 135 g, m_1 = 100 g and m_2 = 50 g

Where is the cm of the system?

$$\vec{r}_{cm} = \frac{\sum_{body} m_i \vec{r}_i}{M}$$

$$= \frac{m_p(1.5\underline{\mathbf{i}} + 1.5\underline{\mathbf{j}}) + m_1(2.0\underline{\mathbf{j}}) + m_2(2.0\underline{\mathbf{i}} + 2.0\underline{\mathbf{j}})}{m_p + m_1 + m_2}$$

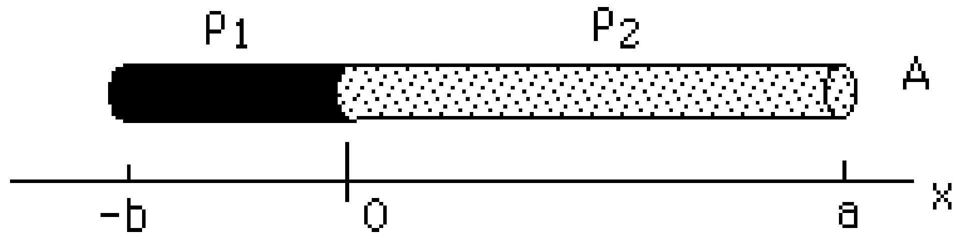
$$= \frac{(303\text{g})\underline{\mathbf{i}} + (503\text{g})\underline{\mathbf{j}}}{285\text{g}} = 1.1\,\underline{\mathbf{i}} + 1.8\underline{\mathbf{j}}$$

check

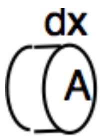
check that $\sum_{body} m_i(\vec{r}_i - \vec{r}_{cm}) = 0$

Example. Rod, cross-section A, made of length a of material with density ρ_2 and length b of material with density ρ_1. Where is c.m.?

If $\rho_1 = 2\rho_2$, and $a = 2b$, where is c.m.?



$$\vec{r}_{cm} = \frac{\int_{body} \mathbf{r}\, dm}{\int_{body} dm}$$

consider short cylinder, $dx * A$ 

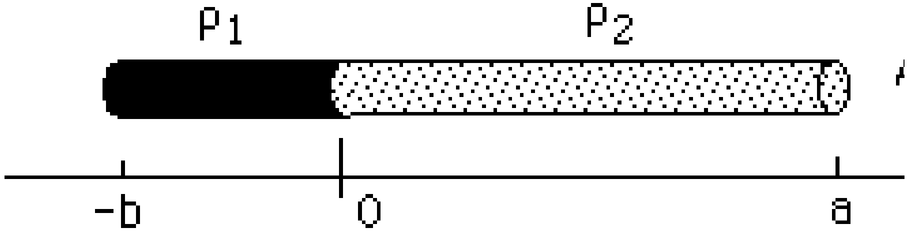
$dm \quad = \rho dV = \rho A dx$

How to 'integrate over body'?

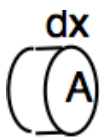
What range of x?

Example. Rod, cross-section *A*, made of length *a* of material with density ρ_2 and length *b* of material with density ρ_1. Where is c.m.?

If $\rho_1 = 2\rho_2$, and $a = 2b$, where is c.m.?



$$\vec{r}_{cm} = \frac{\int_{body} \mathbf{r}\, dm}{\int_{body} dm}$$

small mass 

$dm = \rho dV = \rho A dx$

so here:

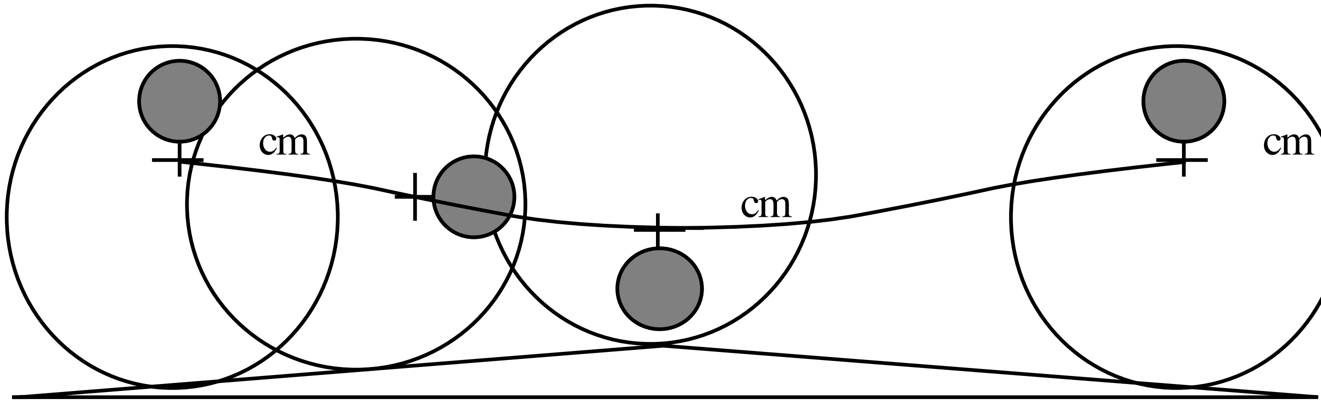
$$x\ cm = \frac{\int_{x=-b}^{a} x dm}{\int_{x=-b}^{a} dm}$$

$$= \frac{\int_{-b}^{0} \rho 1 Ax\ dx + \int_{0}^{a} \rho 2 Ax\ dx}{\int_{-b}^{0} \rho 1 A\ dx + \int_{0}^{a} \rho 2 A\ dx}$$

$$= \frac{-\frac{1}{2}\rho 1 b^2 + \frac{1}{2}\rho 2 a^2}{\rho 1 b + \rho 2 a}$$

$$= \frac{a^2 - rb^2}{2(a + rb)} \qquad \text{where } r = \frac{\rho 1}{\rho 2}$$

Why does an object roll downhill?


cm
cm
cm

Momentum: (Newton called it quantity of motion)

Quantitas motus est mensura ejusdem orta ex velocitate et quantite materiæ conjunctim.

The quantity of motion is the measure of the same, arising from the velocity and the quantity of matter conjunctly.

Now we write: $\vec{p} = m\vec{v}$

and he used it to write the most general form of his combined 1st and 2nd laws:

Mutationem motus proportionalem esse vi motrici impressæ, & fieri secundum lineaum rectam qua vis illa imprimitur.

The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

Now we write: $\sum \vec{F} = \frac{d}{dt}\,\vec{p}$

Momentum

Definition: $\vec{p} = m\vec{v}$

In relativity (2nd year), we'll find that this is a low v approximation to $\vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$

and also that $K = (\gamma - 1)mc^2$

Generalised form of

Newton 2: $\sum \vec{F} = \frac{d}{dt}\,\vec{p}$

$$\sum \vec{F} = m\frac{d}{dt}\vec{v} + \vec{v}\frac{dm}{dt}$$

If m constant, $\sum \vec{F} = m\vec{a}$

m can vary

System of particles: *What is the system? – **you choose** the boundaries*

$$\sum \vec{p} = \sum m_i \vec{v_i} = \sum m_i \frac{d}{dt}\vec{r_i} = \frac{d}{dt}\sum m_i \vec{r_i} = M\frac{d}{dt}\left(\frac{\sum m_i \vec{r_i}}{M}\right)$$

so $\sum \vec{p} = M\vec{v}_{cm}$

If M is constant, then *usually choose the boundaries so M constant*

$$\frac{d}{dt}\left(\sum \vec{p}\right) = M\vec{a}_{cm}$$

$$\sum \vec{F} = \frac{d}{dt}\left(\sum \vec{p}\right)$$

but $\sum \vec{F}_{internal} = 0,$ so

$$\sum \vec{F}_{external} = \frac{d}{dt}\left(\sum \vec{p}\right) = M\vec{a}_{cm}$$

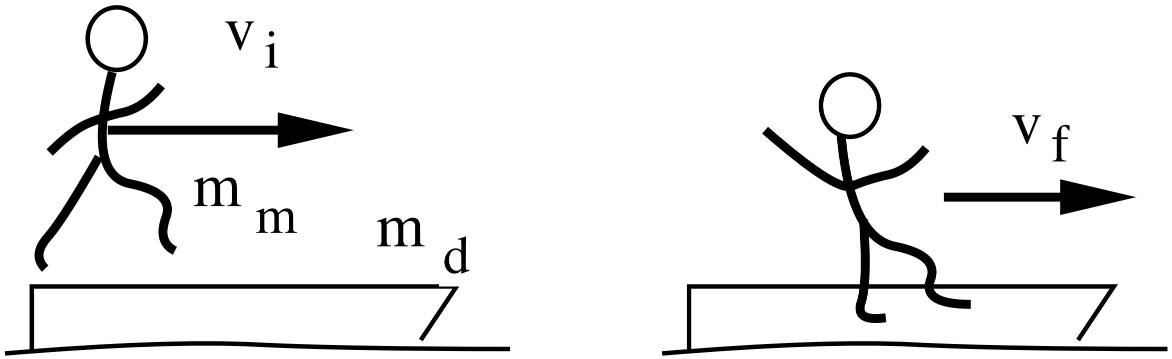
conclude:

i) Motion of c.m. is like that of particle mass M at $\vec{r}_{cm}$ subjected to $\sum \vec{F}_{external}$

ii) **If $\sum \vec{F}_{external} = 0$, momentum is conserved** (and c. of m. doesn't accelerate)

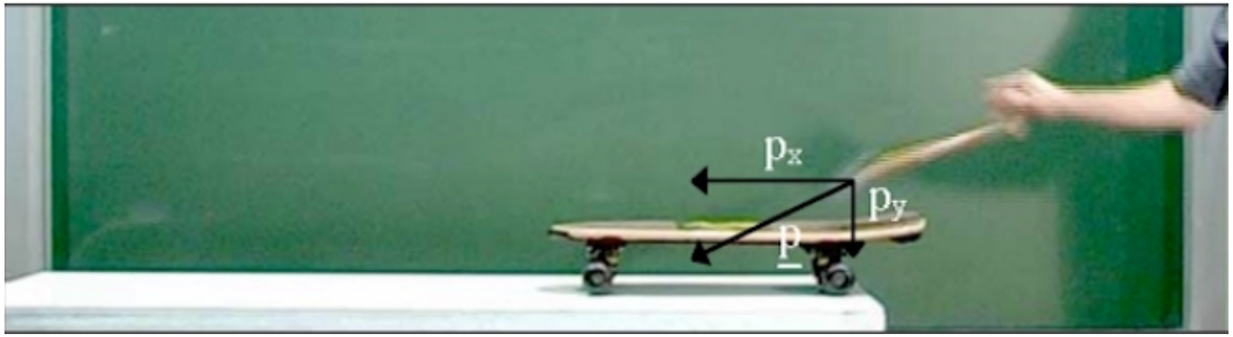
more practically: if $\sum \vec{F}_{external} \cong 0$, momentum is approximately conserved

Example 90 kg man jumps (v_j = 5 ms^{-1}) into a (stationary) 30 kg dinghy. What is their final speed? (Neglect friction.)

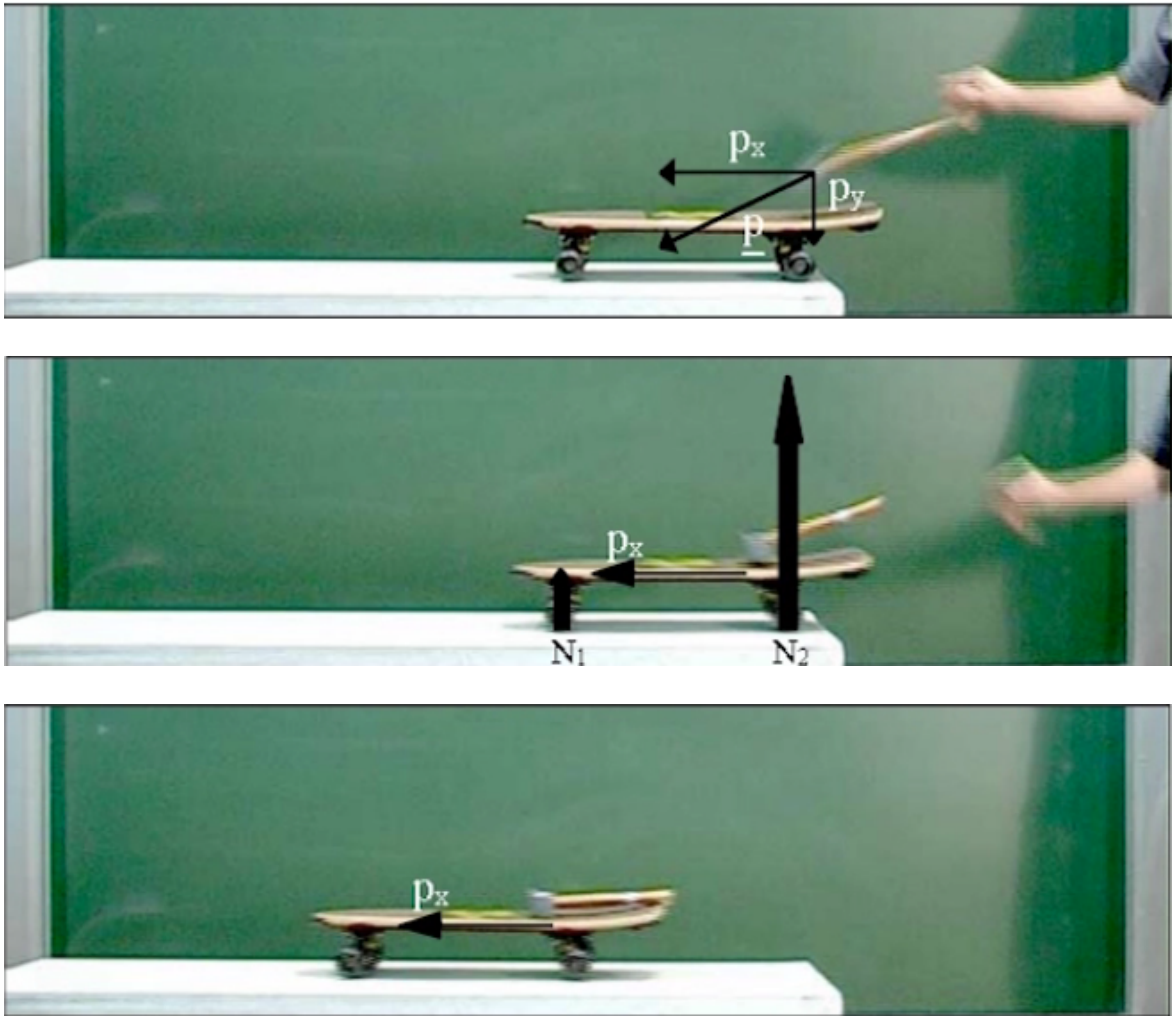


What is the final (horizontal) velocity of man plus dinghy?

Is momentum conserved in this collision?



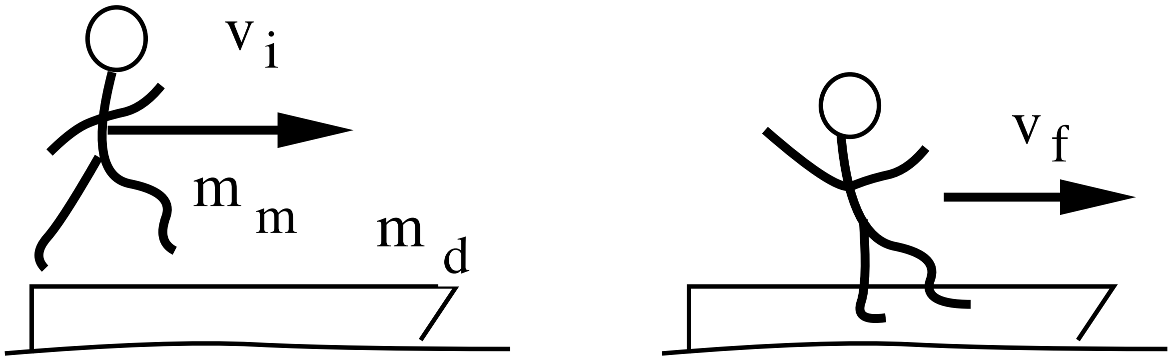
hammer is about to collide with skateboard. Will momentum of skateboard+hammer be conserved?



Momentum is a **vector quantity**.

if $\sum F_{x_{external}} \cong 0$, p_x is approximately conserved (even if $\sum F_{y_{external}}$ is large).

Classic example 90 kg man jumps (v_j = 5 ms^{-1}) into a (stationary) 30 kg dinghy. What is their final speed? (Neglect friction.)



No external forces act in horizontal direction so Σp_X is conserved.

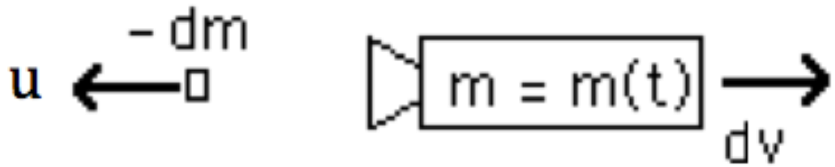
$$\Sigma p_i = \Sigma p_f$$

man *dinghy* *man* *dinghy*

$$m_m v_j + 0 = (m_m + m_d) v_f$$

$$v_f = \frac{m_m}{m_m + m_d} v_j$$

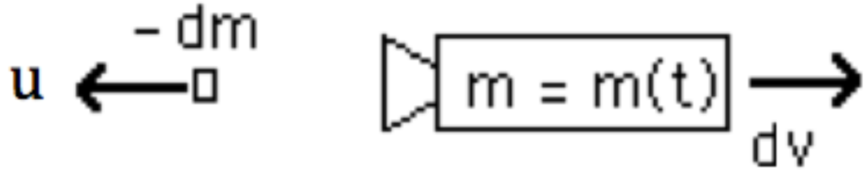
Example. Rocket has mass $m = m(t)$, which decreases as it ejects exhaust at rate $r = -\frac{dm}{dt}$ and at relative velocity u. What is the acceleration of the rocket? $\left(\frac{dm}{dt} = \frac{\textit{rate of increase of}}{\textit{mass of rocket}} < 0\right)$



Choose frame of the rocket. Initially at rest, then ejects ($-dm$), then is travelling at dv

If no external forces act, momentum conserved.

Example. Rocket has mass $m = m(t)$, which decreases as it ejects exhaust at rate $r = -\dfrac{dm}{dt}$ and at relative velocity u. What is the acceleration of the rocket? $\left(\dfrac{dm}{dt} = \begin{matrix}\textit{rate of increase of}\\ \textit{mass of rocket}\end{matrix} < 0\right)$



If no external forces act, momentum conserved. In the frame of the rocket, forwards direction:

$$dp_{rocket} + dp_{exhaust} = 0$$

$$m.dv + (-dm).(-u) = 0$$

$$dv = -u\frac{dm}{m}$$

$$a = \frac{dv}{dt} = -\frac{u}{m}\cdot\frac{dm}{dt}$$

$$a = \frac{ur}{m} \qquad \textit{1st rocket equation}$$

$$dv = -u\frac{dm}{m} = \quad dv = -u\, d(\ln m)$$

Important differential: $\dfrac{d(\ln x)}{dx} = \dfrac{1}{x}$

so $d(\ln x) = \dfrac{dx}{x}$ $\blacksquare$

$$\int_i^f dv = v_f - v_i = u \ln \frac{m_i}{m_f}$$

2nd rocket equation

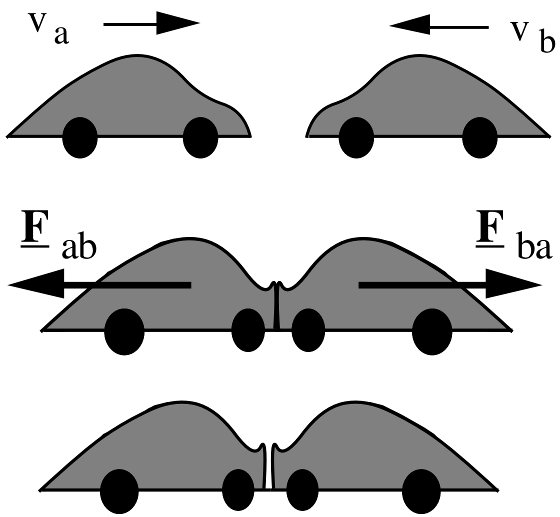
need high exhaust velocity u, else require $m_i >> m_f$

What if u = c?

Collisions Definition: in a collision, "large" forces act between bodies over a "short" time.

In comparison, we shall often neglect the momentum change due to external forces.

Example 1:



$F_{ab} = F_{ba} = 0$

$F_{ab} = F_{ba} = \text{large!}$

$F_{ab} = F_{ba} = 0$

forces that crumple cars during (brief) collision are much larger than friction force (tires - road), ∴ neglect F_{ext}.

Be quantitative: suppose 1.2 tonne car decelerates from 30 kph to rest in a 20 cm 'crumple zone'. Approximate as constant acceleration

Estimate the force between cars during collision

Collisions Definition: in a collision, "large" forces act between bodies over a "short" time.

In comparison, we shall often neglect the momentum change due to external forces.

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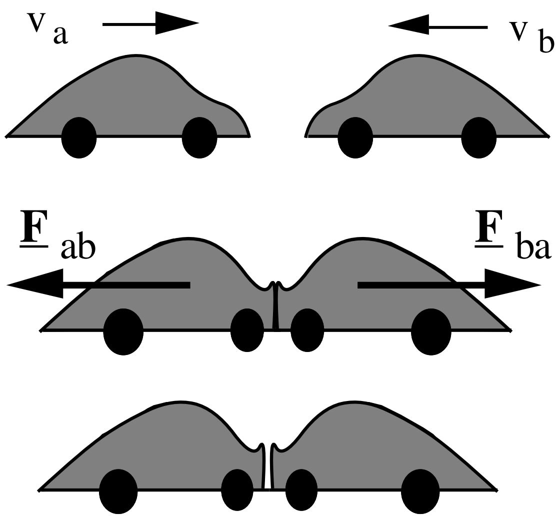

$F_{ab} = F_{ba} = 0$

$F_{ab} = F_{ba} =$ large!

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forces that crumple cars during (brief) collision are much larger than friction force (tires - road), ∴ neglect F_{ext}.

Be quantitative: suppose car 1.2 tonne decelerates from 30 kph to rest in a 20 cm 'crumple zone'. Approximate as constant acceleration

$a \sim (v_f^2 - v_i^2)/2\Delta x = -170\ \mathrm{ms}^{-2}$, so |force| on car during collision ~ $m|a|$ ~ 200 kN, compared with friction at ~ 10 kN.

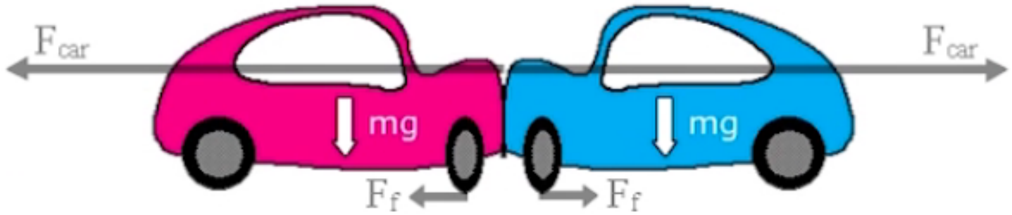
Mechanics > Momentum > 9.3 Collisions and centre o

Video courtesy of The Australasian New Car Assessment Program (ANCAP)

$a = (v_f^2 - v_i^2)/2\Delta x = -170\ \mathrm{ms}^{-2}$, so |force| on car during collision ~ $m|a|$ ~ 200 kN, compared with friction at ~ 10 kN.

$$\left.\begin{array}{l}\text{weight of car}\\ \text{normal force}\\ \text{friction in skid}\end{array}\right\} \sim 10\ \text{kN}$$

force between cars ~ hundreds of kN



What force on the occupants?

If the same a, then F = ma ~ 10 kN:

seat belts, air bags increase Δx, so decrease a.

Forces on pedestrians and cyclists?

How big are crumple zones for pedestrians?

What about reducing crumple zones?

Puzzle

Case a) Two identical cars, without pedestrian bars, collide head on at v and $-v$.

Case b) Two otherwise identical cars, one with and one without pedestrian bars, collide head on at v and $-v$

i) Inside the car *without* bars, are the forces on the passengers bigger or smaller in case (b)?

ii) Inside the car *with* bars, are the forces on the passengers bigger or smaller in case (b)?

What about in a collision between car and pedestrian?

Puzzle

Case a) Two identical cars, without pedestrian bars, collide head on at v and $-v$.

Case b) Two otherwise identical cars, one with and one without rigid pedestrian bars, collide head on at v and $-v$

i) Inside the car *without* bars, are the forces on the passengers bigger or smaller in case (b)?

Combined crumple zone reduced, so acceleration and forces increased.

ii) Inside the car *with* bars, are the forces on the passengers bigger or smaller in case (b)?

Combined crumple zone reduced, so acceleration and forces increased for *both*

What about in a collision between car and pedestrian?

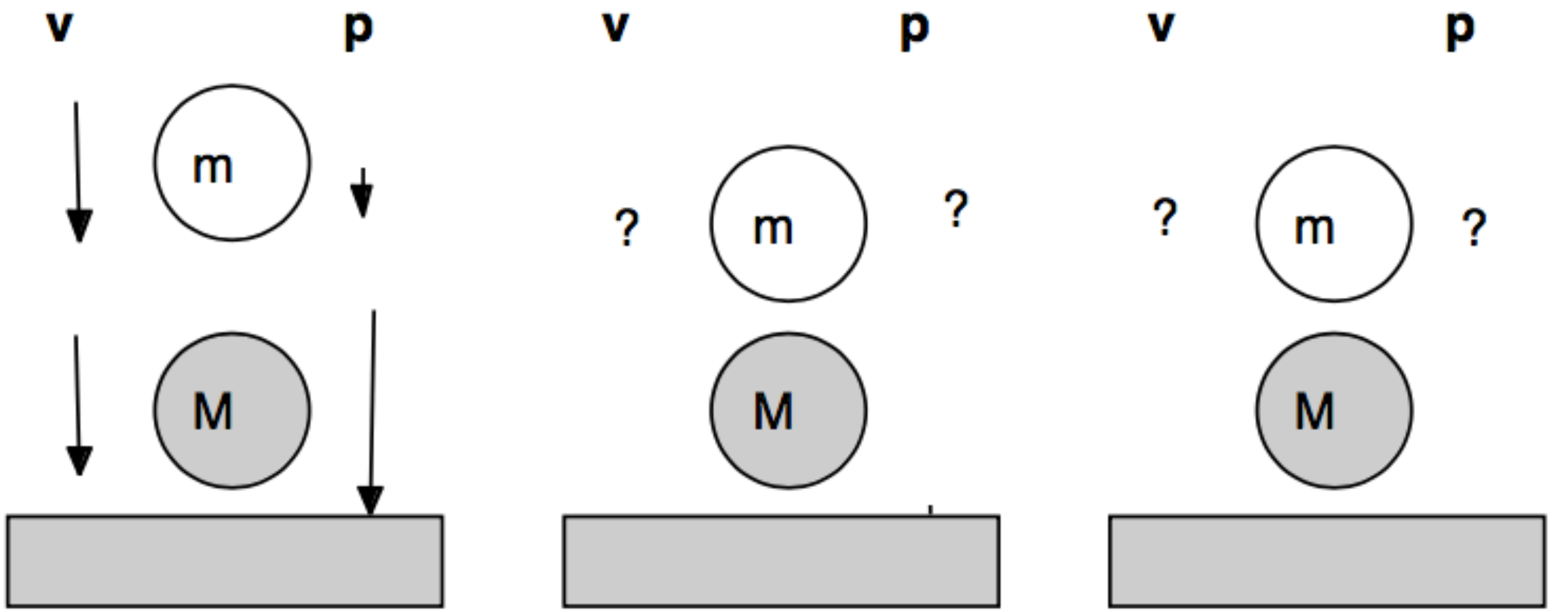
Question

A car, mass m, travels in the positive x direction. It is braking hard until and after it hits another car, also mass m, and both skid to a halt, locked together.

- Sketch $v(t)$ for the first car.
- On the same time axis, also sketch the total horizontal force $F(t)$ acting on the first car.

Example. We simultaneously drop a light ball (m) above a heavy ball ($M>>m$). Assume that the collision of M with the floor is approx. elastic – which means that no mechanical energy is lost.

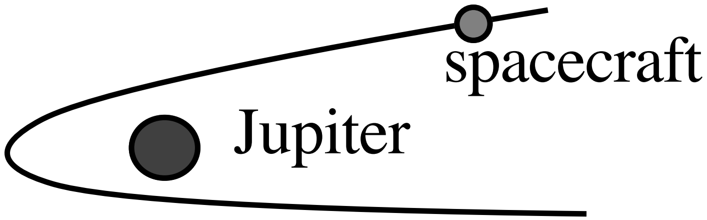
- Sketch the velocity and momentum vectors in the second disgram, just after that collision.
- Now sketch the velocity and momentum vectors in the third diagram, just after the (~ elastic) collision between the two balls.



Before first collision After M hits floor After M and m collide

Example

Fly-by collision



doesn't "hit"

Here, start and finish of collision not well defined

At large separation before and after, $F_{ab} = F_{ba} \cong 0$

During collision (fly-by), forces are considerable.

However, $F_{grav} \propto 1/r^2$, so much smaller at large distances.

'head-on' collisions with Jupiter so deep space probes gain mechanical energy*

'tail-gate' collisions with Earth for Messenger to lose mechanical energy*

** These are exaggerations*

Impulse ($\vec{J}$) and momentum

Newton's 2nd : $\overrightarrow{dp} = \vec{F}\, dt$

Therefore $\int_i^f \overrightarrow{dp} \ = \ \int_i^f \vec{F}\, dt$ so

Definition: $\vec{J} \ = \ \vec{p}_f - \vec{p}_i = \ \int_i^f \vec{F}\, dt$

In collisions, Impulse is integral of the large internal force over a short time

How is impulse different from momentum?

Momentum is what colliding bodies have (and what is sometimes conserved)

Impulse is the momentum that is transferred during the collision

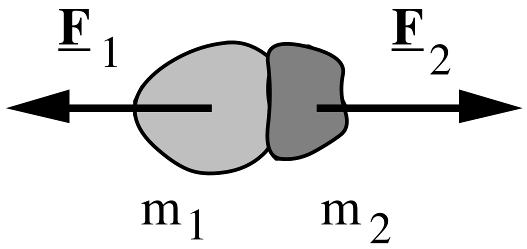
Ball *is* inflated to normal pressure.

Can get an *under*estimate of force:

$$F > \begin{pmatrix} \text{pressure} \\ \text{in ball} \end{pmatrix} * \text{deformed area}$$

$$\sim 70\ \text{kPa} * 0.02\ \text{m}^2 \ \sim\ 1\ \text{kN}$$

Usual case: external forces small, act for small time, therefore $\int_i^f \vec{F}_{ext} dt$ is small.



$\Delta\vec{p}_1 = \int_i^f \vec{F}_1\, dt = \bar{\vec{F}}_1\, \Delta t$ where $\bar{\vec{F}}_1$ is the mean of $\vec{F}_1$ over Δt

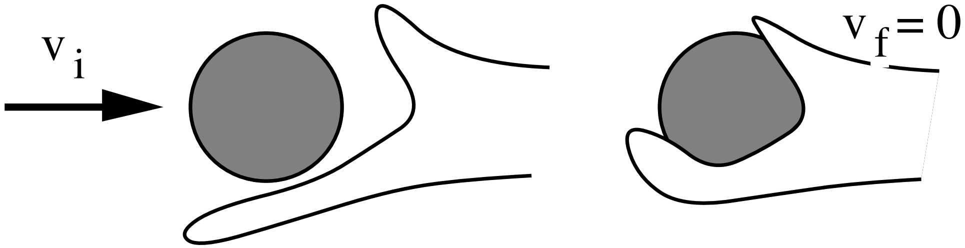
$\Delta\vec{p}_2 = \int_i^f \vec{F}_2\, dt = -\int_i^f \vec{F}_1\, dt = -\bar{\vec{F}}_1\, \Delta t$ by Newton's 3rd.

So $\Delta\vec{p}_2 = -\Delta\vec{p}_1$

and $|\Delta\vec{p}_1| = |\Delta\vec{p}_1| = J$

If external forces are negligible (in any direction), then the momentum of the system is conserved (in that direction).

Example. Cricket ball, m = 156 g, travels at 45 ms^{-1}. What impulse is required to catch it? If the force applied were constant, what average force would be required to stop it in 1 ms? in 10 ms? What stopping distances in these cases?



$$J = \left|\int_i^f \vec{F}\, dt\right| = |\bar{F}\, \Delta t\,|$$

$$\text{so} \quad \bar{F} = J/\Delta t$$

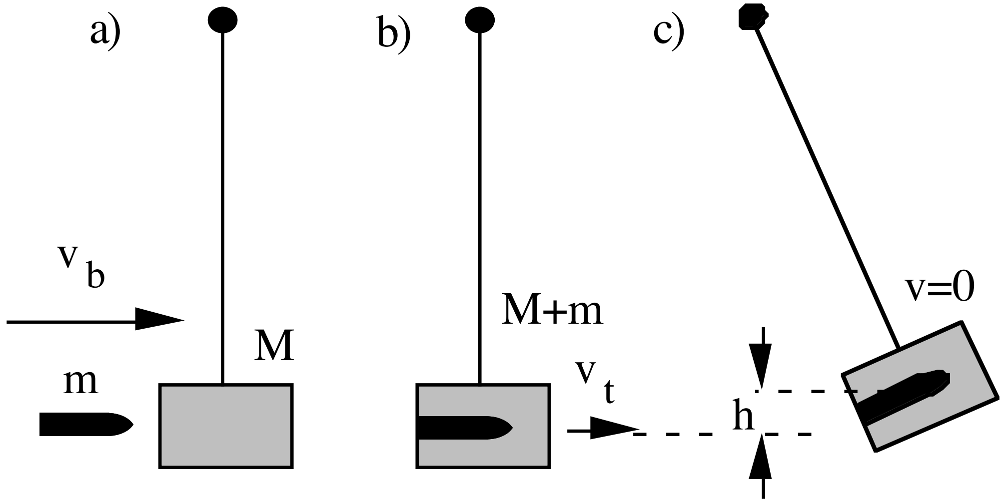
$m = 0.156$ kg, $v_i = 45$ m.s^{-1} $\quad v_f = 0.$

$|J| = mv_i$

$J = 7.0$ kgms^{-1} (to *left)*

Δt	1.0 ms	10 ms
$\bar{F}$	7 kN *ouch!*	0.7 kN
estimate	$v_{av} = 23$ m.s^{-1}	
s	2 cm	20 cm

Example. (*Once was a method to measure speed of bullet.*) Bullet (m) with v_b fired into stationary block (M) on string. (i) What is their (combined) velocity after the collision? (ii) What is the kinetic energy of the bullet? (iii) of the combination? (iv) How high does the block then swing?



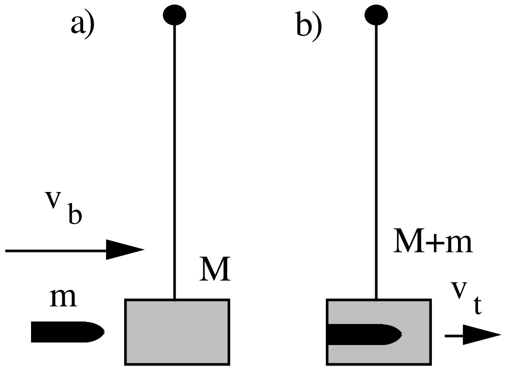
Note the different stages:

*a-b): collision, no horizontal external forces ∴ **momentum conserved**. Friction does work, so **mechanical energy is lost**, not conserved*

*b-c): during this phase, external forces **do** act, so **momentum is lost**, not conserved. However, there are no non-conservative forces, so **mechanical energy conserved.***

Analyse a-b first:

Analyse a) to b)



No horizontal ext forces during collision $\therefore$ momentum conserved

i) $\Sigma p_i = \Sigma p_f$ consider x direction

$$mv_b = (m + M)v_t$$

$$v_t = \frac{m}{m + M} v_b$$

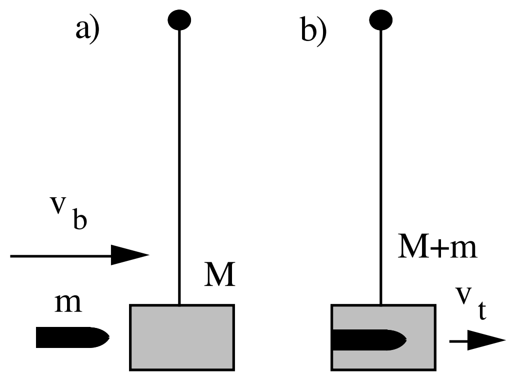
ii) $K_b = \frac{1}{2} mv_b^2$

iii) $K_t = \frac{1}{2}(m + M) v_t^2 =$

$$\frac{1}{2}(m + M)\left(\frac{m}{m + M} vb\right)^2$$

$$= \frac{m}{m + M} \left(\frac{1}{2} mv_b^2\right)$$

Analyse a) to b)



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ii) $K_b = \frac{1}{2} mv_b{}^2$

iii) $K_t = \frac{1}{2}(m + M)\, v_t{}^2 =$

$$\frac{1}{2}(m + M)\left(\frac{m}{m + M} vb\right)^2$$

$$= = \frac{m}{m + M}\left(\frac{1}{2} mv_b{}^2\right) < K_b.$$

Conclusion: $U_i = U_f$, $K_i \neq K_f$.

Mechanical energy is ***not*** conserved - deformation of block is **not elastic**: (some E converted to heat and sound are).

During a collision with negligible external forces,

$$\Sigma\vec{p} = (\Sigma m)\vec{v}_{cm} \quad \text{is conserved}$$

Σm constant, therefore $\vec{v}_{cm}$ is constant

therefore $\frac{1}{2}(\Sigma m)v_{cm}^2$ is constant

K of centre of mass is ***not*** lost. But the K of components with respect to c.m. *can* be lost.

Greatest possible loss of K:

if all final velocities = $\vec{v}_{cm}$,

i.e. if all objects stick together after collision.

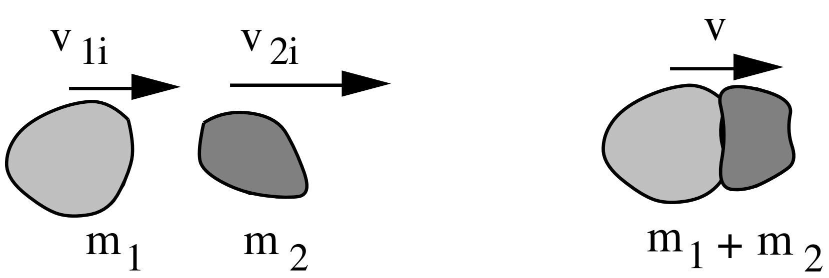
Called **completely inelastic collision.**

Contrast:

Completely elastic collision is one in which non-conservative forces do no work, so mechanical energy is conserved.

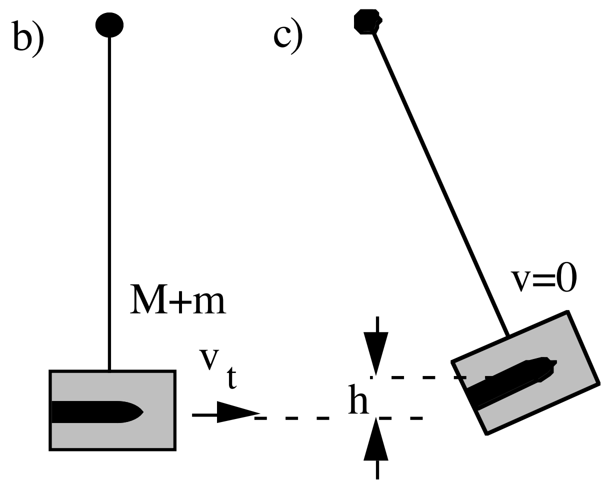
Inelastic collision is one in which non-conservative forces do some work, so mechanical energy is not conserved.

Completely inelastic collision is one in which all kinetic energy with respect to the centre of mass is lost: Non-conservative forces do as much work as possible, so as much mechanical energy as possible is lost.



is completely inelastic

Part (iv) of previous example (b-c):



In the swing, the external forces (gravity and tension) *do* do work and change momentum. But there is no non-conservative force and so *in this part of the process* conservation of mechanical energy applies:

$$\Delta U + \Delta K = 0$$

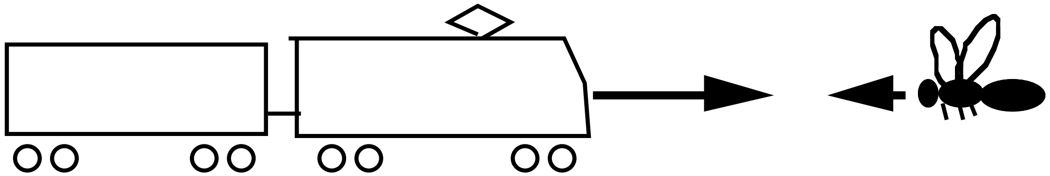
$$(M + m)g\,(\Delta h - 0) + (0 - K_t) = 0$$

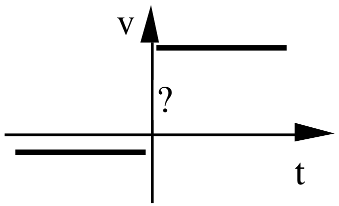
$$\Delta h = \ldots = \frac{1}{2}\frac{m^2}{g(m + M)^2}\,v_b^2$$

which we could rearrange to use Δh to measure v_b

Be ready for problems in which $\vec{p}$ is conserved in one part, and E is conserved in another. Understand which is which.

Puzzle. Fly travelling West at 5m/s meets train travelling East at 30 m/s. Neglect air resistance.





initally, v_{fly} = – 5 ms^{-1}

finally, v_{fly} = + 30 ms^{-1}

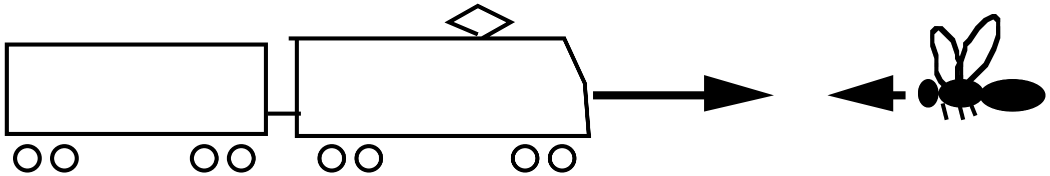
At some time *t*, the fly travels at 0 ms^{-1}

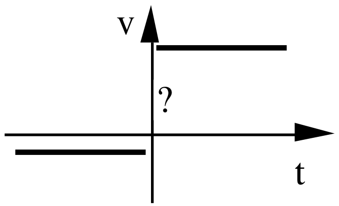
i) does this occur before or after the fly first touches the windscreen?

ii) how fast was the windscreen going when the fly was going 0 ms^{-1}?

iii) which is greater: force exerted by fly on train or train on fly?

Puzzle. Fly travelling West at 5m/s meets train travelling East at 30 m/s. Neglect air resistance.





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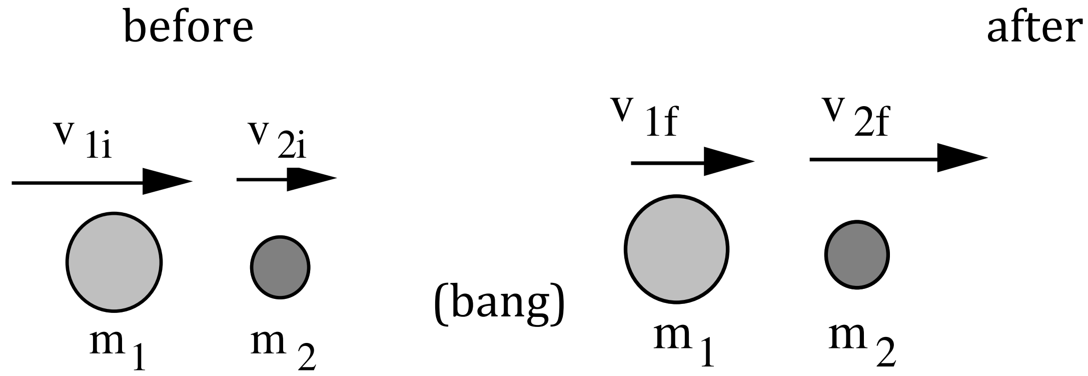
i) does this occur before or after the fly first touches the windscreen?

ii) how fast was the windscreen going when the fly was going 0 ms^{-1}?

iii) which is greater: force exerted by fly on train or train on fly?

iv) what is the last thing to go through the mind of the fly?

Example: elastic collision in one dimension



Collision: neglect external forces $\Rightarrow$

$$\Sigma p_i = \Sigma p_f$$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \qquad \text{(i)}$$

elastic $\Rightarrow K_i = K_f$

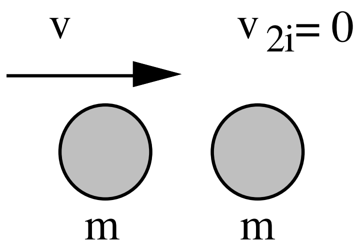
$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \qquad \text{(ii)}$$

usually know m_1, m_2, v_{1i}, v_{2i}. Two unknowns (v_{1f}, v_{2f}), $\therefore$ we can always solve.

Easier if: transform to frame where (e.g.) $v_1 = 0$

Or: transform to centre-of-mass frame.

Case 1. $m_1 = m_2$, $v_{2i} = 0$, $v_{1i} = v$.



neglect external forces $\Rightarrow$ $\Sigma p_i = \Sigma p_f$

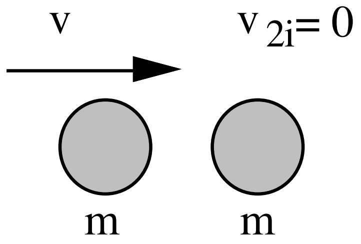
$$mv + 0 = mv_{1f} + mv_{2f}$$
(i)

$$\frac{1}{2}\ mv^2 + 0\ =\ \frac{1}{2}\ mv_{1f}^2 + \frac{1}{2}\ mv_{2f}^2 \quad \text{(ii)}$$

(i) –>$v_{2f}\ =\ v - v_{1f}$ (iii)

substitute in (ii) –>

Case 1. $m_1 = m_2, v_{2i} = 0, v_{1i} = v$.



neglect external forces $\Rightarrow$ $\Sigma p_i = \Sigma p_f$

$$mv + 0 = mv_{1f} + mv_{2f}$$
(i)

$$\frac{1}{2}mv^2 + 0 = \frac{1}{2}mv_{1f}^2 + \frac{1}{2}mv_{2f}^2 \quad \text{(ii)}$$

(i) $\rightarrow v_{2f} = v - v_{1f}$ (iii)

substitute in (ii) $\rightarrow$

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_{1f}^2 + \frac{1}{2}m(v^2 + v_{1f}^2 - 2vv_{1f})$$

$$\therefore \quad 0 = v_{1f}^2 - vv_{1f}$$

$$0 = v_{1f}(v_{1f} - v)$$ *has 2 solutions*

Either: $v_{1f} = 0$ and (iii)$\rightarrow$ $v_{2f} = v$

i.e. 1st stops dead, all *p* and *K* transferred to m2

or: $v_{1f} = v$ and (iii)$\rightarrow$ $v_{2f} = 0$

Why are there two solutions?

Example. Two similar objects, mass m, collide completely **inelastically**.

case 1: $v1i = v$, $v2i = 0$.

case 2: $v1i = v$, $v2i = -v$.

What energy is lost in each case?

Example Two similar objects, mass m, collide completely **inelastically**.

case 1: $v_{1i} = v, \quad v_{2i} = 0.$

case 2: $v_{1i} = v, \quad v_{2i} = -v.$

What energy is lost in each case?

p conserved –>

$$mv_{1i} + mv_{2i} = 2mv_f$$

$$v_f = \frac{v_{1i} + v_{2i}}{2}$$

$$\Delta K = K_f - K_i$$

$$= \frac{1}{2}(2m)\, v_f{}^2 - \frac{1}{2}\, mv_{1i}{}^2 - \frac{1}{2}\, mv_{2i}{}^2$$

case 1:

$$\Delta K = \frac{1}{2}(2m)\left(\frac{v+0}{2}\right)^2 - \frac{1}{2}\, mv^2$$

$$= -\frac{1}{4}\, mv^2$$

case 2:

$$\Delta K = \frac{1}{2}(2m)\left(\frac{0+0}{2}\right)^2 - \frac{1}{2}\, mv^2 - \frac{1}{2}\, mv^2$$

$$= -mv^2$$

Four times as much energy lost.

Remember this if you have the choice in traffic, rugby etc

Example Show that, for an elastic collision in one dimension, the magnitude of the relative velocity is unchanged.

i.e. show $|v1i - v2i| = |v1f - v2f|$

p and *K* conservation gave:

(i) $m1(v1f - v1i) = -m2(v2f - v2i)$

(ii) $\frac{1}{2}m1(v1f^2 - v1i^2) = -\frac{1}{2}m2(v2f^2 - v2i^2)$

If they hit, $(v1f - v1i) \neq 0, (v2f - v2i) \neq 0$

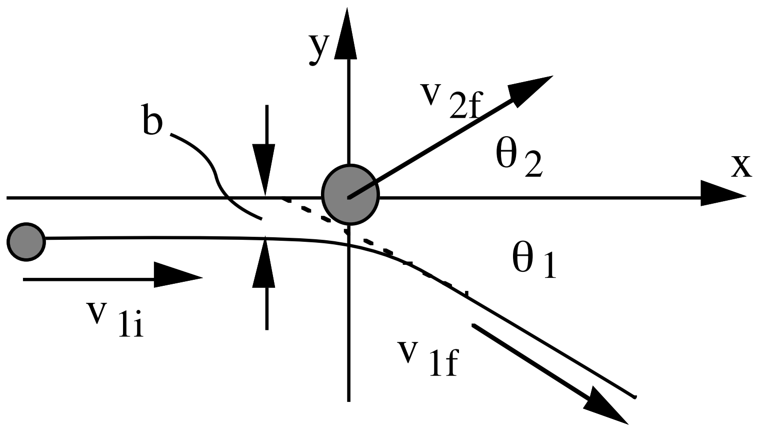
$(ii)/(i) \Rightarrow v1i + v1f = v2i + v2f$

$\therefore \quad v1i - v2i = v2f - v1f$

i.e. relative velocity the same before and after

Solve –> $v2f = \frac{2m1}{m1 + m2} v1i + \frac{m2 - m1}{m2 + m1} v2i$

Elastic collisions in 2 (& 3) dimensions



Choose frame in which m_2 stationary, v_{1i} in x direction

b is called impact parameter (distance "off centre")

p_x conserved

$$m_1 v_{1i} = m v_{1f} \cos\theta_1 + m v_{2f} \cos\theta_2$$

p_y conserved

$$0 = m v_{2f} \sin\theta_2 - m v_{1f} \sin\theta_1$$

K conserved

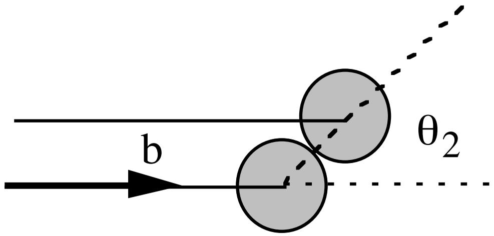
$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m v_{1f}^2 + \frac{1}{2} m v_{2f}^2 + \Delta K \quad \text{(iii)}$$

where $\Delta K = 0$ for elastic case

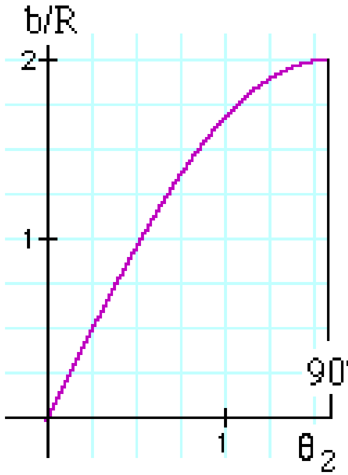
3 equations in v_{1f}, v_{2f}, θ_1 and θ_2: need more info

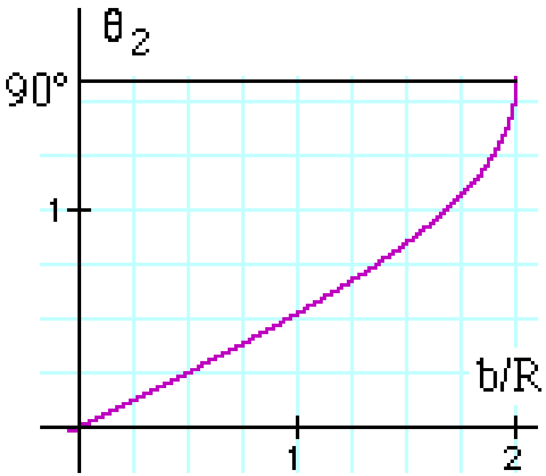
(often given θ_1 or θ_2)

Incidentally: for hard spheres, neglecting rotation and friction (*reasonable during collision, but not always after*), $F_{internal}$ acts on line between centres



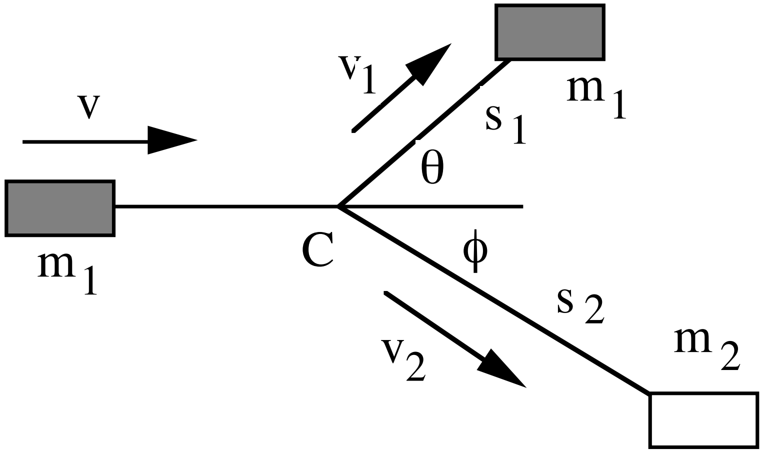
$(R + R) \sin \theta_2 = b$ so $\theta_2 = \sin^{-1} \dfrac{b}{2R}$





i) Note that as $\theta \to 90°$, small error in b gives large error in θ_2.

ii) Experiment: Does $b = R$ give $\theta_2 = 30°$? *friction, rotation ignored*



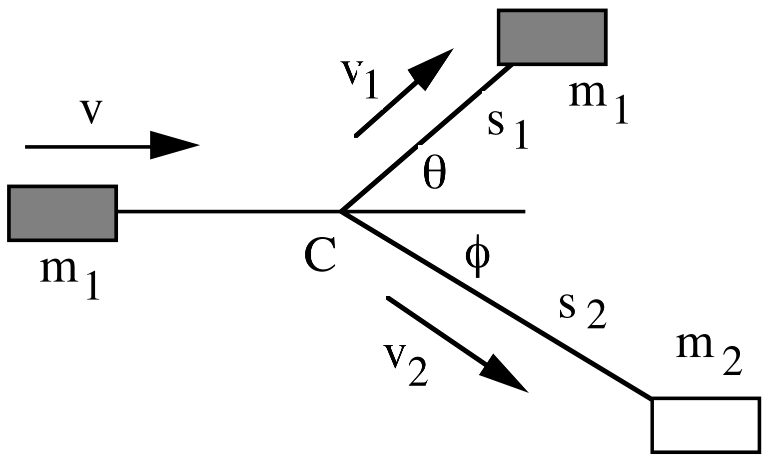
Example. Police report of road accident: car 1, mass m_1 strikes stationary car m_2 at point C. They then slide to rest in positions shown. Given $\mu_k = \mu$ (assumed same for both) find the initial speed v of m_1. Can you check assumption? (*real example*)

After collision, a for both $= \dfrac{Ff}{m} = -\dfrac{\mu m g}{m}$

$$v_f^2 - v_i^2 = 2as = -2\mu gs$$

$$0 - v1^2 = -2\mu gs1$$

$$v1 = \sqrt{2\mu gs1} \qquad v2 = \sqrt{2\mu gs2}$$



Neglect external forces during collision:
$\Delta p = 0$

$\Sigma\Delta p$ in the x direction = 0

$m1v = m1v1\cos\theta + m2v2\cos\phi$ (i)

$\Sigma\Delta p$ in the y direction = 0

$0 = m1v1\sin\theta - m2v2\sin\phi$ (ii)

We had $v1 = \sqrt{2\mu gs1}$ $v2 = \sqrt{2\mu gs2}$

So (i) $\Rightarrow$

$$v = \sqrt{2\mu gs1}\cos\theta + (m2/m1)\sqrt{2\mu 2gs2}\cos\phi$$

Note the "spare" equation—we can use it to check the model or assumptions:

(ii) $\Rightarrow$ $m1\sqrt{2\mu 1gs1}\sin\theta$

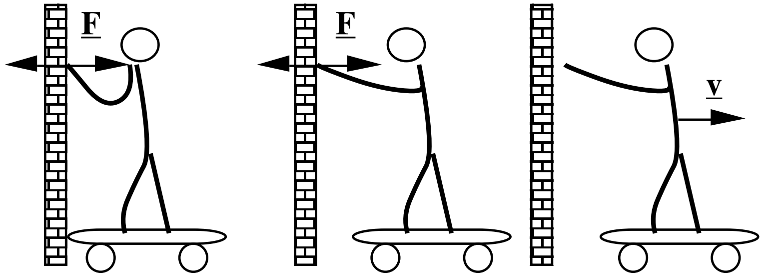
$$= m2\sqrt{2\mu 2gs2}\sin\phi$$

$$\frac{\mu 2}{\mu 1} = \frac{s1m1^2\sin^2\theta}{s2m2^2\sin^2\phi}$$

(The μ may not be the same for the two: surfaces different, orientation of wheels etc

Internal *vs* external work.

Problem. Skateboarder pushes away from a wall



Point of application of force does not move, $\therefore$ normal force does no work, but K changes. Where does energy come from? *Obvious: arms!*

$$F_{ext} = Ma_{cm}$$

$$F_{ext}\,dx = Ma_{cm}\,dx_{cm} = M\frac{dv_{cm}\,dx_{cm}}{dt} = Mv_{cm}dv_{cm}$$

"Centre of mass work"

$$W_{cm} = \int_i^f F_{ext}\,dx = \left(\frac{1}{2}Mv_{cm}{}^2\right)_f - \left(\frac{1}{2}Mv_{cm}{}^2\right)_i$$

Work done = work that would have been done if F_{ext} had acted on centre of mass.