Study Guide

AP Physics C: Mechanics

Mr. Perkins

Midterm: January 17, 2023

Denny Cao

Contents

1 Background

Easy. List of topics:

1. Vectors and Scalars
2. Addition, Subtraction, and Multiplication of Vectors
3. Dimensional Analysis

2 One-Dimensional Kinematics

List of topics:

1. Instantaneous Speed, Velocity, and Acceleration
2. Average Speed, Velocity, and Acceleration
3. Uniformly Accelerated Motion: Freely Falling Objects

Kinematics work **only when acceleration is constant**. Take derivatives and integrals of position, velocity, and acceleration to find the equations of motion. Use the equations of motion to solve problems.

2.1 Instantaneous Velocity

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \iff \Delta x = \int dx = \int_{t_0}^{t_1} v(t)dt \tag{1}$$

2.2 Instantaneous Speed

$$|v(t)| \tag{2}$$

PAY ATTENTION TO QUESTION! SPEED IS ALWAYS POSITIVE!
Speed is related to total distance traveled whereas velocity is related to the displacement vector.

$$\text{Displacement} \leq \text{Distance} \tag{3}$$

2.3 Instantaneous Acceleration

$$a(t) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2x}{dt^2} \iff \Delta v = \int dv = \int_{t_0}^{t_1} a(t)dt \tag{4}$$

2.4 Average Velocity

$$\overline{v} = \frac{\int_{t_0}^{t_1} v(t)dt}{t_1 - t_0} = \frac{\int_{t_0}^{t_1} dx}{t_1 - t_0} = \frac{\left[x \Big|_{t_0}^{t_1} \right]}{t_1 - t_0} = \frac{x(t_1) - x(t_0)}{t_1 - t_0} = \frac{\Delta x}{\Delta t} \tag{5}$$

- Derived from calculus average value of a function

2.5 Average Speed

$$\text{Average Speed} = \frac{\int_{t_0}^{t_1} |dx|}{t_1 - t_0} = \frac{\text{total distance}}{\Delta t} \tag{6}$$

- **AVERAGE SPEED IS NOT THE ABSOLUTE VALUE OF THE AVERAGE VELOCITY!**

Average Acceleration

$$\overline{a} = \frac{\int_{t_0}^{t_1} a(t)dt}{t_1 - t_0} = \frac{\int_{t_0}^{t_1} dv}{t_1 - t_0} = \frac{\left[v\Big|_{t_0}^{t_1} \right]}{t_1 - t_0} = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{\Delta v}{\Delta t} \tag{7}$$

2.6 Graphical Interpretations:

Line connecting the two points is the average velocity (Path independent). The slope of the line is the average acceleration. The absolute value area under the line is the average speed.

2.7 Uniformly Accelerated Motion

By definition:

$$a(t) = \text{constant} = a \tag{8}$$

Velocity is the integral of acceleration:

$$v(t) = \int a(t)dt = at + v_0 \tag{9}$$

Position is the integral of velocity:

$$x(t) = \int v(t)dt = \frac{1}{2}at^2 + v_0 t + x_0 \tag{10}$$

Another unique property of UAM is that velocity is:

$$\overline{v} = \frac{v + v_0}{2} \tag{11}$$

This can be understood by examining how the average of a linear function is the midpoint of the line.

2.7.1 Position Without Reference to Time

From Equation 9:

$$t = \frac{v - v_0}{a}$$

Substitute into Equation 10:

$$v^2 = v_0^2 + 2a(x - x_0) \tag{12}$$

3 Two-Dimensional Kinematics

List of topics:

1. Instantaneous Velocity, Speed, and Acceleration in Two Dimensions
2. Uniformly Accelerated Motion Including Projectile Motion
3. Relative Position, Velocity, and Acceleration
4. Uniform Circular Motion

3.1 Position Vector

We can represent position in two dimensions using a vector. The position vector is defined as:

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} = x\hat{i} + y\hat{j} \tag{13}$$

3.2 Instantaneous Velocity

$$\vec{v}(t) \equiv \lim_{\Delta t \to 0} = \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \iff \Delta \vec{r} = \int d\vec{r} \equiv \int_{t_0}^{t_1} \vec{v}(t)dt \tag{14}$$

3.3 Instantaneous Speed

$$||v(t)|| = \sqrt{v_x^2 + v_y^2} \tag{15}$$

3.4 Instantaneous Acceleration

$$\vec{a}(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \equiv \frac{d^2\vec{r}}{dt^2} \iff \Delta \vec{v} = \int d\vec{v} \equiv \int_{t_0}^{t_1} \vec{a}(t)dt \tag{16}$$

We can represent two-dimensional vectors by sets of one-dimensional vectors. For example, the velocity vector can be represented by two one-dimensional vectors:

$$\begin{aligned} \vec{v}(t) &= \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = v_x(t)\hat{i} + v_y(t)\hat{j} \\ \frac{d\vec{r}}{dt} &= \begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} \end{aligned} \tag{17}$$

The two dimensions are independent of each other (other than being synchronized by time). We can create a parametric equation for the velocity vector:

$$\begin{cases} v_x(t) = \dfrac{dx}{dt} = v_{x_0} + a_x t \\ v_y(t) = \dfrac{dy}{dt} = v_{y_0} + a_y t \end{cases} \tag{18}$$

3.5 Displacement Vector

Displacement vector points from the initial position to the final position. It is defined as:

$$\Delta\vec{r} = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \Delta x\hat{i} + \Delta y\hat{j} \tag{19}$$

3.6 Average Velocity

The average velocity during a given time interval is parallel to the displacement vector (Multiplying by a scalar does not change the direction of a vector):

$$\bar{\vec{v}} = \frac{\Delta\vec{r}}{\Delta t} \tag{20}$$

3.7 Projectile Motion

Conversion from rectangular to polar form:

$$v = \sqrt{v_x^2 + v_y^2} \tag{21}$$

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) \tag{22}$$

Conversion from polar to rectangular form:

$$v_x = v\cos\theta \tag{23}$$

$$v_y = v\sin\theta \tag{24}$$

We can separate projectile motion into two components: horizontal and vertical. They are independent from one another.

3.7.1 Horizontal Motion

Horizontal motion is uniform motion in the x-direction. The horizontal velocity is constant. The horizontal acceleration is zero.

$$v_x(t) = v_{x_0} = \text{constant} \tag{25}$$

$$x(t) = x_0 + v_{x_0}t \tag{26}$$

3.7.2 Vertical Motion

The acceleration in the y-direction is constant. The initial velocity in the y-direction is zero. The initial position in the y-direction is zero.

$$a_y = -g \tag{27}$$

$$v_y(t) = v_{y_0} + a_y t \tag{28}$$

$$y(t) = \frac{1}{2}a_y t^2 + v_{y_0}t + y_0 \tag{29}$$

$$v_y^2 = v_{y_0}^2 - 2g(y - y_0) \tag{30}$$

3.8 Relative Position, Velocity, and Acceleration

Vector addition relates the position of an object relative to two different frames of reference:

$$\vec{r}_{\text{P relative to B}} = \vec{r}_{\text{P relative to A}} + \vec{r}_{\text{A relative to B}} \tag{31}$$

We take the derivatives of the position vectors to find the relative velocity vectors:

$$\vec{v}_{\text{P relative to B}} = \vec{v}_{\text{P relative to A}} + \vec{v}_{\text{A relative to B}} \tag{32}$$

We take the second derivatives of the position vectors to find the relative acceleration vectors:

$$\vec{a}_{\text{P relative to B}} = \vec{a}_{\text{P relative to A}} + \vec{a}_{\text{A relative to B}} \tag{33}$$

3.8.1 Inertial Reference Frames

Two reference frames that move with a constant velocity with respect to each other. Thus, the acceleration is the same in both inertial reference frames:

$$\vec{a}_{PB} = \vec{a}_{PA} \tag{34}$$

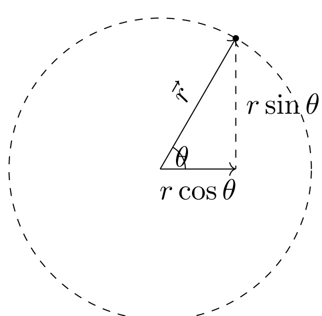
3.9 Uniform Circular Motion

It is useful to divide the acceleration vector into two components: a component parallel to the velocity vector and a component perpendicular to the velocity vector:

- $a_{\parallel}$ (the **tangential acceleration**) affects only the **magnitude** of the velocity vector (speed). Mathematically:
$$a_{\parallel} = \frac{dv}{dt} \tag{35}$$
- $a_{\perp}$ (the **radial acceleration**) affects only the **direction** of the velocity vector.

We can classify special types of motion using these two components:

- $a_{\perp}(t) = 0 \wedge a_{\parallel}(t) \neq 0$: The direction of the motion never changes: The path of the particle lies along a line (rectilinear motion), but its speed changes.
- $a_{\parallel}(t) = 0 \wedge a_{\perp}(t) \neq 0$: The speed is constant and only the direction changes. A special case of this is uniform circular motion (UCM), which occurs when $a_{\perp}(t) = \text{constant}$. **Note that there is still acceleration in UCM, as direction changes!**
- $a_{\perp}(t) \neq 0 \wedge a_{\parallel}(t) \neq 0$: The most general situation corresponds to more complicated motions such as projectile motion or nonuniform circular motion.



We can express the position vector in terms of the angle θ:

$$\vec{r} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \tag{36}$$

To express this position equation in terms of time for an object moving at constant speed, we make the substitution $\theta = \omega t + \theta_0$:

$$\vec{r} = \begin{pmatrix} r\cos(\omega t + \theta_0) \\ r\sin(\omega t + \theta_0) \end{pmatrix} \tag{37}$$

Because the motion is uniform, the angle θ changes at a constant rate given by ω (the **angular velocity**) according to the equation $\omega = \frac{d\theta}{dt}$. θ_0 is the initial angle of the object.

We can differentiate the position vector to find the velocity vector:

$$\vec{v} = \omega r \begin{pmatrix} -\sin(\omega t + \theta_0) \\ \cos(\omega t + \theta_0) \end{pmatrix} \tag{38}$$

Direction of the Velocity Vector: We can take the dot product of the velocity vector with the position vector to find the direction of the velocity vector:

$$\vec{r} \cdot \vec{v} = r_x v_x + r_y v_y = 0 \tag{39}$$

This indicates that the velocity vector is perpendicular to the position vector, which is what we expect, as the velocity is always tangent to the path of the particle, which in this case is tangent to the circle and thus perpendicular to the position vector.

Magnitude of the Velocity Vector:

$$||\vec{v}|| = \sqrt{v_x^2 + v_y^2} = \omega r \tag{40}$$

We can differentiate the velocity vector to find the acceleration vector:

$$\vec{a} = -\omega^2 r \begin{pmatrix} \cos(\omega t + \theta_0) \\ \sin(\omega t + \theta_0) \end{pmatrix} \tag{41}$$

Direction of the Acceleration Vector: From inspection, we see that $\vec{a}(t) = -\omega^2 \vec{r}(t)$. Because ω^2 is positive, this equation implies that the acceleration vector is antiparallel to the position vector. Because the position vector points outward, the acceleration vector points inward, toward the center of rotation. This is why the acceleration is called **centripetal acceleration**.

Magnitude of the Acceleration Vector:

$$||\vec{a}|| = \sqrt{a_x^2 + a_y^2} = \omega^2 r \tag{42}$$

Using Equation 40, we can write:

$$||\vec{a}|| = \frac{v^2}{r} \tag{43}$$