MATH 22A: Vector Calculus and Linear Algebra

Problem Set 9

Due: Wednesday, November 8, 2023 12pm **Denny Cao**

Collaborators

§1 Computational Problems

Solution 1.1. The matrix for T relative to $\mathcal{B}$ and the standard basis for $\mathbb{R}^2$ is obtained by the vectors obtained by the following:

- Let $x_1 = 1, x_2 = x_3 = 0$. Then:

$$T(\boldsymbol{b_1}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Let $x_2 = 1, x_1 = x_3 = 0$. Then:

$$T(\boldsymbol{b_2}) = \begin{bmatrix} -4 \\ -1 \end{bmatrix} = -4\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Let $x_3 = 1, x_1 = x_2 = 0$. Then:

$$T(\boldsymbol{b_3}) = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus, the matrix for T relative to $\mathcal{B}$ and the standard basis for $\mathbb{R}^2$ is:

$$\begin{bmatrix} 2 & -4 & 5 \\ 0 & -1 & 3 \end{bmatrix}$$

Solution 1.2.

a) The image of $\boldsymbol{p}(t)$ is $\boldsymbol{p}(t) + t\boldsymbol{p}(t) = (2 - t + t^2) + t(2 - t + t^2) = 2 + t + t^3$.

b) To show that T is a linear transformation, we must show that:

- $\forall \boldsymbol{u}(t), \boldsymbol{v}(t) \in \mathbb{P}_2(T(\boldsymbol{u}(t) + \boldsymbol{v}(t)) = T(\boldsymbol{u}(t)) + T(\boldsymbol{v}(t))$.

$$\begin{aligned} T(\boldsymbol{u}(t) + \boldsymbol{v}(t)) &= (\boldsymbol{u}(t) + \boldsymbol{v}(t)) + t(\boldsymbol{u}(t) + \boldsymbol{v}(t)) \\ &= \boldsymbol{u}(t) + t\boldsymbol{u}(t) + \boldsymbol{v}(t) + t\boldsymbol{v}(t) \\ &= T(\boldsymbol{u}(t)) + T(\boldsymbol{v}(t)) \end{aligned}$$

  and thus this property holds.

- $\forall c, \forall \boldsymbol{u}(t) \in \mathbb{P}_2(T(c\boldsymbol{u}(t))) = cT(\boldsymbol{u}(t))$.

$$\begin{aligned} T(c\boldsymbol{u}(t)) &= c\boldsymbol{u}(t) + t(c\boldsymbol{u}(t)) \\ &= c(\boldsymbol{u}(t) + t\boldsymbol{u}(t)) \\ &= cT(\boldsymbol{u}(t)) \end{aligned}$$

  and thus this property holds.

As both properties hold, T we have shown that T is a linear transformation.

c)
- $T(1) = 1 + t$
- $T(t) = t + t^2$
- $T(t^2) = t^2 + t^3$

Thus, the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$ is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution 1.3. We first find the eigenvalues of the matrix by find what values of λ make $\det(A - I\lambda) = 0$.

$$\begin{aligned} \begin{vmatrix} 5-\lambda & -3 \\ -7 & 1-\lambda \end{vmatrix} &= 0 \\ (5-\lambda)(1-\lambda) - 21 &= \\ \lambda^2 - 6\lambda - 16 &= \\ (\lambda - 8)(\lambda + 2) &= \end{aligned}$$

Thus, $\lambda = 8, -2$. We now find the eigenvectors associated with each eigenvalue.

- $\lambda = 8$.

$$(A - 8I)\boldsymbol{x} = \mathbf{0}$$

We set up the augmented matrix:

$$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ -7 & -7 & 0 \end{array}\right] \overset{1/3R_1 \to R_1}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -7 & -7 & 0 \end{array}\right] \overset{7R_1 + R_2 \to R_2}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, the eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

- $\lambda = -2$.

$$(A + 2I)\boldsymbol{x} = \mathbf{0}$$

We set up the augmented matrix:

$$\left[\begin{array}{cc|c} 7 & -3 & 0 \\ -7 & 3 & 0 \end{array}\right] \overset{R_2 + R_1 \to R_2}{\sim} \left[\begin{array}{cc|c} 7 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, the eigenvector is $\begin{bmatrix} \frac{3}{7} \\ 1 \end{bmatrix}$.

As the eigenvectors are linearly independent, from Theorem 5 (The Diagonalization Theorem), it follows that A is diagonalizable, where $A = PDP^{-1}$, where D is the diagonal matrix, and P is the matrix where the columns are the eigenvectors: $P = \begin{bmatrix} -1 & \frac{3}{7} \\ 1 & 1 \end{bmatrix}$. From Theorem 8 (Diagonal Matrix Representation), if the basis $\mathcal{B}$ for $\mathbb{R}^2$ formed the columns of P, then D is the $\mathcal{B}$-matrix for the transformation. Thus, $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{3}{7} \\ 1 \end{bmatrix} \right\}$.

Solution 1.4. The basis for the corresponding eigenspaces are obtained by finding $\text{Nul}(A - \lambda I)$.

- $\lambda = 1$. $\text{Nul}(A - I) = \text{Nul}\left(\begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix} \right)$.

$$\left[\begin{array}{cc|c} 6 & 4 & 0 \\ -3 & -2 & 0 \end{array}\right] \overset{R_2 + 1/2R_1 \to R_2}{\sim} \left[\begin{array}{cc|c} 6 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$
$$\overset{1/2R_1 \to R_1}{\sim} \left[\begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, we have
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}, x_2 \text{ is free.}$$

Thus, for the eigenspace for $\lambda = 1$, $\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$ gives a basis.

- $\lambda = 5$. $\text{Nul}(A - 5I) = \text{Nul}\left(\begin{bmatrix} 2 & 4 \\ -3 & -6 \end{bmatrix} \right)$.

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ -3 & -6 & 0 \end{array}\right] \overset{1/2R_1 \to R_1, 1/3R_2 \to R_2}{\sim} \begin{bmatrix} 1 & 2 & 0 \\ -1 & -2 & 0 \end{bmatrix}$$
$$\overset{R_2 + R_1 \to R_2}{\sim} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, we have
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}, x_2 \text{ is free.}$$

Thus, for the eigenspace for $\lambda = 5$, $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ gives a basis.

Solution 1.5. By rotating around a line ℓ, any vector $\boldsymbol{v}$ on ℓ will not be affected, and thus $T(\boldsymbol{v}) = \boldsymbol{v}$, meaning that ℓ is an eigenspace for the eigenvalue $\lambda = 1$. When the rotation is by 180°, then there is an additional eigenvalue, as for vectors $\boldsymbol{u}$ orthogonal to ℓ, $T(\boldsymbol{u}) = -\boldsymbol{u}$, meaning that the plane that passes through the origin and orthogonal to ℓ is an eigenspace for the eigenvalue $\lambda = -1$.

Solution 1.6.

- The characteristic polynomial is $\det(A - \lambda I) = 0$. This can be simplified to $(5 - \lambda)(3 - \lambda) - 12 = \lambda^2 - 8\lambda + 3 = p(\lambda)$.

- The eigenvalues can be found by the quadratic formula:

$$\begin{aligned} \lambda &= \frac{8 \pm \sqrt{64 - 4(3)}}{2} \\ &= 4 \pm \sqrt{13} \end{aligned}$$

Thus, the eigenvalues are $\lambda = 4 + \sqrt{13}$ and $\lambda = 4 - \sqrt{13}$.

- The corresponding eigenvectors can be found by solving for $\boldsymbol{x}$:

$$(A - (4 + \sqrt{13})I)\boldsymbol{x} = \mathbf{0}$$

$$\left[\begin{array}{cc|c} 1-\sqrt{13} & -3 & 0 \\ -4 & -1-\sqrt{13} & 0 \end{array}\right] \overset{\frac{-1-\sqrt{13}}{12}R_1 \to R_1}{\sim} \left[\begin{array}{cc|c} 1 & \frac{1+\sqrt{13}}{4} & 0 \\ -4 & -1-\sqrt{13} & 0 \end{array}\right]$$
$$\overset{R_2+4R_1 \to R_2}{\sim} \left[\begin{array}{cc|c} 1 & \frac{\sqrt{13}+1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, the eigenvector for $\lambda = 4 + \sqrt{13}$ is: $\begin{bmatrix} \frac{-1-\sqrt{13}}{4} \\ 1 \end{bmatrix}$.

$$(A - (4 - \sqrt{13})I)\boldsymbol{x} = \mathbf{0}$$

$$\left[\begin{array}{cc|c} 1+\sqrt{13} & -3 & 0 \\ -4 & -1+\sqrt{13} & 0 \end{array}\right] \overset{\frac{R_1}{1+\sqrt{13}} \to R_1}{\sim} \left[\begin{array}{cc|c} 1 & \frac{1-\sqrt{13}}{4} & 0 \\ -4 & -1+\sqrt{13} & 0 \end{array}\right]$$
$$\overset{R_2+R_1 \to R_2}{\sim} \left[\begin{array}{cc|c} 1 & \frac{1-\sqrt{13}}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, the eigenvector for $\lambda = 4 - \sqrt{13}$ is: $\begin{bmatrix} \frac{-1+\sqrt{13}}{4} \\ 1 \end{bmatrix}$.

Solution 1.7. The characteristic polynomial is $\det(A - \lambda I) = 0$. We will compute the following:

$$\begin{aligned} \begin{vmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{vmatrix} &= (5-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 7 & -2-\lambda \end{vmatrix} + 6\begin{vmatrix} -2 & 3 \\ 1-\lambda & 0 \end{vmatrix} \\ &= (5-\lambda)(1-\lambda)(-2-\lambda) - 6(3(1-\lambda)) \\ p(\lambda) &= -\lambda^3 + 4\lambda^2 + 25\lambda - 28 \end{aligned}$$

Solution 1.8.

a) *Proof.* If A is similar to B, then $A = PBP^{-1}$. It follows that $\operatorname{tr}(A) = \operatorname{tr}(PBP^{-1})$.

$$\operatorname{tr}(A) = \operatorname{tr}(PBP^{-1}) = \operatorname{tr}((PB)P^{-1}) = \operatorname{tr}(P^{-1}(PB)) = \operatorname{tr}(P^{-1}PB) = \operatorname{tr}(IB) = \operatorname{tr}(B)$$

Thus, $\operatorname{tr}(A) = \operatorname{tr}(B)$, as desired. □

b) *Proof.* If A is diagonalizable, then $A = PDP^{-1}$, where D is a diagonal matrix where the diagonal entries are the eigenvalues of A that correspond to the eigenvectors in P. It follows that $\operatorname{tr}(A) = \operatorname{tr}(PDP^{-1})$.

$$\operatorname{tr}(A) = \operatorname{tr}(PDP^{-1}) = \operatorname{tr}((PD)P^{-1}) = \operatorname{tr}(P^{-1}(PD)) = \operatorname{tr}(P^{-1}PD) = \operatorname{tr}(ID) = \operatorname{tr}(D)$$

As the trace of D is the sum of the diagonal entries, and the diagonal entries are the eigenvalues of A, it follows that $\operatorname{tr}(A)$ is the sum of the eigenvalues of A, as desired. □

c) *Proof.* Let A, B be $n \times n$ matrices. The sum of the diagonal of AB, $\mathrm{tr}(AB)$, can be written as the following, where a_{ij}, b_{ij} denote the entry in the i-th row and j-th column of A and B respectively:

$$(a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}) + (a_{21}b_{12} + a_{22}b_{22} + \\ \cdots + a_{2n}b_{n2}) + \cdots + (a_{n1}b_{1n} + a_{n2}b_{2n} + \ldots a_{nn}b_{nn})$$

We can rearrange to form the following:

$$(a_{11}b_{11} + a_{21}b_{12} + \cdots + a_{n1}b_{1n}) + (a_{12}b_{21} + a_{22}b_{22} + \\ \cdots + a_{n2}b_{2n}) + \cdots + (a_{1n}b_{n1} + a_{2n}b_{n2} + \cdots + a_{nn}b_{nn})$$

We note that this is equivalent to the sum of the diagonal of BA, $\mathrm{tr}(BA)$, and thus $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, as desired. □

Solution 1.9.

a) The characteristic equation of A is $\begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = p_A(\lambda)$. The characteristic equation of B is $\begin{vmatrix} 2-\lambda & 0 \\ 2 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = p_B(\lambda)$. Thus, $p_A(\lambda) = p_B(\lambda)$.

b)
- The eigenspace for the eigenvalue of A can be found by $\mathrm{Nul}(A - 2I) = \mathrm{Nul}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.
- The eigenspace for the eigenvalue of B can be found by $\mathrm{Nul}(B - 2I) = \mathrm{Nul}\left(\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\right)$. The matrix is row equivalent to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, which has the following solutions to the homogeneous system $B'\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

  Thus, the eigenspace for the eigenvalue of B is $\mathrm{Span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

c) *Proof.* To show that A and B are not similar, we will show that there does not exist an invertible matrix P such that $B = PAP^{-1}$. As B is a 2×2 matrix and does not have 2 linearly independent eigenvectors, B is not diagonalizable, and thus there does not exist a P such that $PAP^{-1} = B$, and the proof is complete. □

Solution 1.10. We can find the eigenvalues of the matrix by first finding the solutions for the characteristic equation $\det(A - \lambda I) = 0$.

$$\begin{aligned} \begin{vmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & h & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} &= 5\begin{vmatrix} 3-\lambda & h & 0 \\ 0 & 5-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{vmatrix} \\ &= 5\left((3-\lambda)\begin{vmatrix} 5-\lambda & 4 \\ 0 & 1-\lambda \end{vmatrix}\right) \\ p(\lambda) &= 5(3-\lambda)(5-\lambda)(1-\lambda) \end{aligned}$$

We observe that the characteristic equation is not affected by the value of h, and thus the eigenvalues of the matrix do not depend on the values of h.

The eigenspace for $\lambda = 5$ is $\text{Nul}(A-5I) = \text{Nul}\left(\begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}\right)$. We find the reduced row echelon form of the augmented matrix:

$$\left[\begin{array}{cccc|c} 0 & -2 & 6 & -1 & 0 \\ 0 & -2 & h & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array}\right] \overset{R_4+R_3\to R_4}{\sim} \left[\begin{array}{cccc|c} 0 & -2 & 6 & -1 & 0 \\ 0 & -2 & h & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\overset{\frac{R_3}{4}\to R_3}{\sim} \left[\begin{array}{cccc|c} 0 & -2 & 6 & -1 & 0 \\ 0 & -2 & h & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{R_1+R_3\to R_1}{\sim} \left[\begin{array}{cccc|c} 0 & -2 & 6 & 0 & 0 \\ 0 & -2 & h & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

If $h = 6$, we obtain the following reduced row echelon form:

$$\left[\begin{array}{cccc|c} 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This will yield the solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

and thus the eigenspace will be $\text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}\right\}$. If h was another value, the eigenspace would be spanned by 3 vectors, which would make it 3-dimensional, instead of 2-dimensional as desiredd.

§2 Proof Problems

Solution 2.1. *Proof.* Multiplying A by B means that every column in AB is a linear combination of the columns of A. From the definition of column space, $\text{Col}(A)$ is the set of all linear combinations of the columns of A. It thus follows that $\text{Col}(AB) \subseteq \text{Col}(A)$. As the rank of a matrix is the dimension of the column space of the matrix and $\dim(\text{Col}(AB)) \leq \dim(\text{Col}(A))$, it follows that $\text{rank}(AB) \leq \text{rank}(A)$.

From Theorem 14 (The Rank Theorem) in Section 4.6, the dimensions of the column space and the row space of a matrix are equal, and thus $\text{rank}\, A = \text{rank}\, A^T$, as $\text{Col}(A^T) = \text{Row}(A)$,meaning $\dim \text{Col}(A^T) = \dim \text{Row}(A)$. From this we obtain the following:

$$\text{rank}(AB) = \text{rank}((AB)^T) = \text{rank}(B^T A^T)$$

From the first part of the proof, we know $\text{rank}(B^T A^T) \leq \text{rank}(B^T)$. As $\text{rank}(B^T) = \text{rank}(B)$, $\text{rank}(B^T A^T) \leq \text{rank}(B)$, and thus $\text{rank}(AB) \leq \text{rank}(B)$.

It follows that, if $\text{rank}(AB) \leq \text{rank}(A) \wedge \text{rank}(AB) \leq \text{rank}(B)$, then $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$, as desired. □

Solution 2.2.

a) *Proof.*

- Let $u_1 + H = u_2 + H$ and $v_1 + H = v_2 + H$. Then, $\exists h_1 \in H \mid u_1 = u_2 + h_1$, and $\exists h_2 \in H \mid v_1 = v_2 + h_2$. The sum $(u_1 + v_1) + H$ can thus be expressed as $((u_2 + h_1) + (v_2 + h_2)) + H = (u_2 + v_2) + (h_1 + h_2) + H$. As $h_1, h_2 \in H$, it follows that $h_1 + h_2 \in H$. Thus, $(u_1 + v_1) + H = (u_2 + v_2) + H$, and we have shown that addition is well-defined.
- Let $c(u_1 + H) = c(u_2 + H)$. As $u_1 + H = u_2 + H$, it follows that $\exists h_1 \in H \mid u_1 = u_2 + h_1$. We can thus rewrite $c(u_1) + H$ as $c(u_2 + h_1) + H = cu_2 + ch_1 + H$. As $h_1 \in H$, the scalar multiple $ch_1 \in H$. Thus, $cu_1 + H = cu_2 + H$, and we have shown that multiplication is well-defined.

We have shown that both operations are well-defined, as desired. □

b) A basis for $V \setminus H$ is $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + H, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + H \right\}$. Geometrically, $V \setminus H$ is the set of all parallel lines to the line spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, the x-axis.

c) *Proof.* To show that a set S is a basis for a vector space $V \setminus H$, we must show that:

(i) S spans $V \setminus H$.

(ii) S is linearly independent.

- As $\{u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ is a basis for V, $\forall v \in V$, we can express v as a linear combination of the basis vectors:

$$v = x_1 u_1 + x_2 u_2 + \cdots + x_k u_k + x_{k+1} u_{k+1} + \cdots + x_n u_n$$

Let $h = x_1 u_1 + x_2 u_2 + \cdots + x_k u_k$. Then, $v = h + x_{k+1} u_{k+1} + \cdots + x_n u_n$. As h is a linear combination of the basis vectors of H, it follows that $h \in H$.

$\forall v, v + H \in V \setminus H$. We can express the coset formed by v by the following:

$$\begin{aligned} v + H &= (h + x_{k+1} u_{k+1} + \cdots + x_n u_n) + H \\ &= (h + H) + (x_{k+1} u_{k+1} + H) + \cdots + (x_n u_n + H) \\ &= H + x_{k+1}(u_{k+1} + H) + \cdots + x_n(u_n + H) \end{aligned}$$

As H is the zero vector in $V \setminus H$, we can simplify further

$$= x_{k+1}(u_{k+1} + H) + \cdots + x_n(u_n + H)$$

We observe that $v + H$ is a linear combination of the basis vectors in $\{u_{k+1} + H, \ldots, u_n + H\}$, and the set spans $V \setminus H$.

- To show that the set $\{u_{k+1} + H, \dots, u_n + H\}$ is linearly independent, we will show that

  $$x_{k+1}(u_{k+1} + H) + \cdots + x_n(u_n + H) = 0$$

  only if $x_{k+1} = \cdots = x_n = 0$. As the zero vector of $V \setminus H$ is H, we can rewrite the equation:

  $$\begin{aligned} x_{k+1}(u_{k+1} + H) + \cdots + x_n(u_n + H) &= H \\ (x_{k+1}u_{k+1} + H) + \cdots + (x_n u_n + H) &= \\ (x_{k+1}u_{k+1} + \cdots + x_n u_n) + H &= \end{aligned}$$

  Thus, $(x_{k+1}u_{k+1} + \cdots + x_n u_n) \in H$ by definition of coset. As any vector in H is a linear combination of the vectors in the basis $\{u_1, u_2, \dots, u_k\}$, it follows that $x_{k+1}u_{k+1} + \cdots + x_n u_n = x_1 u_1 + x_2 u_2 + \cdots + x_k u_k$. Then:

  $$\begin{aligned} x_{k+1}u_{k+1} + \cdots + x_n u_n - x_1 u_1 - x_2 u_2 - \cdots - x_k u_k &= 0 \\ (-x_1)u_1 + (-x_2)u_2 + \cdots + (-x_k)u_k + x_{k+1}u_{k+1} + \cdots + x_n u_n &= 0 \end{aligned}$$

  As $\{u_1, u_2, \dots, u_k, u_{k+1}, \dots u_n\}$ is the basis for V, it follows that $u_1, u_2, \dots, u_k$, $u_{k+1}, \dots, u_n$ are linearly independent. Thus, $-x_1 = -x_2 = \cdots = -x_k = x_{k+1} = \cdots = x_n = 0$. Thus, $x_{k+1} = \cdots = x_n = 0$ for $x_{k+1}(u_{k+1} + H) + \cdots + x_n(u_n + H) = H$. It follows that the set $\{u_{k+1} + H, \dots, u_n + H\}$ is linearly independent.

As we have shown that the set is linearly independent and spans $V \setminus H$, we have shown that the set is a basis for $V \setminus H$, and the proof is complete. $\square$

d) $\dim V \setminus H = n - k$. As $\dim V = n$ and $\dim H = k$, it follows that

$$\dim V \setminus H = \dim V - \dim H$$