MATH 22A: Vector Calculus and Linear Algebra

Problem Set 6

Due: Wednesday, October 18, 2023 12pm **Denny Cao**

Collaborators

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§1 Computational Questions

Question 1.1. Compute the determinant of the matrices below by cofactor expansion (Choose a row or column that involves the least amount of work.)

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -4 & -3 & 5 \\ 2 & 0 & 3 & 5 \end{vmatrix} \quad \begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix}$$

Solution

We evaluate the first determinant by cofactor expansion on the second row:

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -4 & -3 & 5 \\ 2 & 0 & 3 & 5 \end{vmatrix} = -3 \begin{vmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ 2 & 0 & 5 \end{vmatrix}$$

We evaluate the 3×3 determinant by cofactor expansion on the second column:

$$\begin{aligned} &= -3\left(2\begin{vmatrix} 2 & 5 \\ 2 & 5 \end{vmatrix} - 4\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}\right) \\ &= -3(2(0) - 4(5-4)) \\ &= -3(-4) \\ &= 12 \end{aligned}$$

We evaluate the second determinant by cofactor expansion on the fifth column:

$$\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix} = 1\left(\begin{vmatrix} 6 & 3 & 2 & 4 \\ 9 & 0 & -4 & 1 \\ 2 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 \end{vmatrix}\right)$$

We evaluate the 4×4 determinant by cofactor expansion on the third row:

$$= 2\begin{vmatrix} 3 & 2 & 4 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{vmatrix}$$

We evaluate the 3×3 determinant by cofactor expansion on the second row:

$$
\begin{aligned}
&= 2\left(4\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}\right) \\
&= 2(-4(6-8) - 1(9-4)) \\
&= 2(8-5) \\
&= 2(3) \\
&= 6
\end{aligned}
$$

Question 1.2. Explore the effect of an elementary row operation on the determinant of a matrix: State the row operation to go from the left most matrix to the right most, and compute the determinant for both.

$$
\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 5+3k & 4+2k \end{bmatrix}
$$

Solution
To go from the left matrix to the right, $\sim R_2 + kR_1 \to R_2$. The determinant of both are as follows:

$$
\begin{aligned}
\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} &= 12 - 10 \\
&= 2
\end{aligned}
\qquad
\begin{aligned}
\begin{vmatrix} 3 & 2 \\ 5+3k & 4+2k \end{vmatrix} &= 3(4+2k) - 2(5+3k) \\
&= 12 + 6k - 10 - 6k \\
&= 2
\end{aligned}
$$

Question 1.3. Compute the determinants of the following elementary matrices:

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Solution
$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. By Theorem 2 in Section 3.1 in Lay's Linear Algebra, $\det I = 1$.

- The first matrix is obtained by $\sim R_3 + kR_2 \to R_3$. By Theorem 3 in Section 3.2 in Lay's Linear Algebra, $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{vmatrix} = \det I = 1$.
- The second matrix is obtained by $\sim R_3 + kR_1 \to R_3$. By Theorem 3 in Section 3.2 in Lay's Linear Algebra, $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{vmatrix} = \det I = 1$.
- The third matrix is obtained by $\sim R_1 \leftrightarrow R_3$. By Theorem 3 in Section 3.2 in Lay's Linear Algebra, $\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\det I = -1$.

- The fourth matrix is obtained by $\sim R_1 \leftrightarrow R_2$. By Theorem 3 in Section 3.2 in Lay's Linear Algebra, $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\det I = -1$.

Question 1.4. Assume that the matrix presented directly below has determinant equal to 7.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

Use the preceding fact to compute the determinants of the following matrices:

$$\begin{vmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{vmatrix} \quad \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} \quad \begin{vmatrix} a & b & c \\ d+3g & e+3h & f+3i \\ g & h & i \end{vmatrix}$$

Solution

- By Theorem 3 in Section 3.2 in Lay's Linear Algebra, as $\sim 5R_2 \to R_2$, $\begin{vmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{vmatrix} = 5 \times 7 = 35$.
- By Theorem 3 in Section 3.2 in Lay's Linear Algebra, as $\sim R_1 \leftrightarrow R_2$, $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -7$.
- By Theorem 3 in Section 3.2 in Lay's Linear Algebra, as $\sim R_2 + 3R_3 \to R_2$, $\begin{vmatrix} a & b & c \\ d+3g & e+3h & f+3i \\ g & h & i \end{vmatrix} = 7$

Question 1.5. Use determinants to decide if the four vectors below are linearly independent.

$$\begin{bmatrix} 3 \\ 5 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}$$

Solution

Let the columns of A be the vectors above. Thus:

$$A = \begin{bmatrix} 3 & 2 & -2 & 0 \\ 5 & -6 & -1 & 0 \\ -6 & 0 & 3 & 0 \\ 4 & 7 & 0 & -2 \end{bmatrix}$$

We evaluate $\det A$ by cofactor expansion on the fourth column:

$$\det A = -2 \begin{vmatrix} 3 & 2 & -2 \\ 5 & -6 & -1 \\ -6 & 0 & 3 \end{vmatrix}$$

We evaluate the 3×3 matrix by cofactor expansion on the third row:

$$= -2\left(-6 \begin{vmatrix} 2 & -2 \\ -6 & -1 \end{vmatrix} + 3 \begin{vmatrix} 3 & 2 \\ 5 & -6 \end{vmatrix}\right)$$

$$
\begin{aligned}
&= -2(-6(-2-12)+3(-18-10)) \\
&= -2(84-84) \\
&= 0
\end{aligned}
$$

As $\det A = 0$, by Theorem 4 in Section 3.2 in Lay's Linear Algebra, A is not invertible, and thus by the Invertible Matrix Theorem, the columns of A are not linearly independent.

Question 1.6. Let A and B denote 4×4 matrices with $\det A = -3$ and $\det B = -1$. Compute:

a. $\det AB$

b. $\det B^5$

c. $\det 2A$

d. $\det A^T BA$

e. $\det B^{-1}AB$

Solution

a. By Theorem 6 (Multiplicative Property) in Section 3.2, $\det AB = (\det A)(\det B) = -3 \times -1 = 3$.

b. By Theorem 6, $\det B^5 = (\det B)(\det B)(\det B)(\det B)(\det B) = -1^5 = -1$.

c. By Theorem 3, since $2A$ multiplies all rows by 2 and there are 4 rows, $\det 2A = 2^4 \det A = 2^4(-3) = 16(-3) = -48$.

d. By Theorem 6, $\det A^T BA = \det A^T B \det A = \det A^T \det B \det A$. By Theorem 5, $\det A^T = \det A$. Thus, $\det A^T BA = -3(-1)(-3) = -9$.

e. By Theorem 6, $\det B^{-1}AB = \det B^{-1}A \det B = \det B^{-1} \det A \det B = \dfrac{\det A \det B}{\det B} = \det A = -3$.

Question 1.7. Suppose that all entries of a square matrix A are integers and that $\det A = 1$. Explain why all entries of A^{-1} are also integers.

Solution
From Theorem 8, $A^{-1} = \dfrac{1}{\det A}\text{adj}A$. As $\det A = 1$, $A^{-1} = \text{adj}A$. As the adjugate of A is the transpose of the cofactors of A, since all entries of A are integers, then all cofactors are integers. Thus, all entries of A^{-1} are integers.

Question 1.8. Find the volume of the parallelopiped in $\mathbb{R}^3$ with one vertex at the origin and with its adjacent vertices at the respective points where the coordinates (x, y, z) have the following values: $(1, 3, 0), (-2, 0, 2)$, and $(-1, 3, -1)$.

Solution
Let A be the following:

$$
A = \begin{bmatrix} 1 & -2 & -1 \\ 3 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix}
$$

By Theorem 9, the volume of the parallelepiped determined by A is $|\det A|$. We evaluate the determinant by cofactor expansion on the first column:

$$\begin{aligned}
\det A &= 1\begin{vmatrix} 0 & 3 \\ 2 & -1 \end{vmatrix} - 3\begin{vmatrix} -2 & -1 \\ 2 & -1 \end{vmatrix} \\
&= -6 - 3(2+2) \\
&= -6 - 12 \\
&= -18 \\
|\det A| &= 18
\end{aligned}$$

Thus, the volume of the parallelepiped is 18 units^3.

Question 1.9. Compute the adjugate matrix below and then use the adjugate to give the inverse of the matrix (see Theorem 8 in Section 3.3).

$$\begin{bmatrix} 1 & 1 & 3 \\ -2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution

The 9 cofactors are as follows:

$$C_{11} = +\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1 \qquad C_{12} = -\begin{vmatrix} -2 & 1 \\ 0 & 1 \end{vmatrix} = 2 \qquad C_{13} = +\begin{vmatrix} -2 & 2 \\ 0 & 1 \end{vmatrix} = -2$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 2 \qquad C_{22} = +\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 1 \qquad C_{23} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

$$C_{31} = +\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -5 \qquad C_{32} = -\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = -7 \qquad C_{33} = +\begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} = 4$$

Thus:

$$\text{adj}A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & 1 & -7 \\ -2 & -1 & 4 \end{bmatrix}$$

We evaluate $\begin{vmatrix} 1 & 1 & 3 \\ -2 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$ by cofactor expansion on the first column:

$$\begin{aligned}
\det A &= 1\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} \\
&= (2-1) + 2(1-3) \\
&= 1 - 4 \\
&= -3
\end{aligned}$$

From Theorem 8:

$$\begin{aligned}
A^{-1} &= \frac{1}{\det A}\text{adj}A \\
&= -\frac{1}{3}\begin{bmatrix} 1 & 2 & -5 \\ 2 & 1 & -7 \\ -2 & -1 & 4 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{3} & -\frac{2}{3} & \frac{5}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{7}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{4}{3} \end{bmatrix}
\end{aligned}$$

Question 1.10. Let S denote the parallelogram determined by the vectors

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

Compute the area of S; and supposing that A denotes the matrix below, compute the area of the image of S via the linear transformation $\vec{x} \mapsto A\vec{x}$.

$$\begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix}$$

Solution
By Theorem 9, the area of the parallelogram S determined by the columns of the matrix B is $|\det B|$. Let $B = \begin{bmatrix} -2 & -2 \\ 3 & 5 \end{bmatrix}$. Then, $|\det B| = |(-10+6)| = |-4| = 4$. Thus, the area of S is 4 units^2.

By Theorem 10, the area of the transformation for a parallelogram S is:

$$\{\text{area of } T(S)\} = |\det A|\{\text{area of } S\}$$

$\det A = 12 - 9 = 3$. Thus, the area of $T(S) = 3(4) = 12$ units^2.

§2 Proof Problems

Question 2.1 (A Bit More Cardinality)**.** Construct an explicit bijection from $[0, 1)$ and $(0, 1)$.

Solution

Claim — Let $f : [0, 1) \to (0, 1)$ such that

$$f(x) = \begin{cases} \frac{1}{2} & x = 0 \\ \frac{x}{2} & x \in \left\{ \frac{1}{2}^n \mid n \in \mathbb{Z}^+ \right\} \\ x & x \notin \left\{ \frac{1}{2}^n \mid n \in \mathbb{Z}^+ \right\} \wedge x \neq 0 \end{cases}$$

Then, f is bijective.

Proof.

Claim — f is injective.

Subproof. To prove that f is injective, we will show that for all $x_1, x_2 \in [0, 1)$, it is the case that $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$. Let $S = \left\{ \frac{1}{2}^n \mid n \in \mathbb{Z}^+ \right\}$. We will consider the possible outputs of f:

1. Let $x_1 = 0$. Then, $f(x_1) = \frac{1}{2}$.
   a) If $x_2 \neq 0$ and $x_2 \in S$, then $f(x_2) = \frac{x_2}{2}$. If $f(x_2) = \frac{1}{2}$, then $x_2 = 1$, but $1 \notin S$, and thus $f(x_2) \neq f(x_1)$.

b) If $x_2 \neq 0 \land x_2 \notin S$, then $f(x_2) = x_2$. If $f(x_2) = \frac{1}{2}$, then $x_2 = \frac{1}{2}$, however, $\frac{1}{2} \in S$ but $x_2 \notin S$, and thus $f(x_2) \neq f(x_1)$.

2. Let $x_1 \in S$. Then, $f(x_1) = \frac{x_1}{2}$.

   a) If $x_2 \neq x_1$ and $x_2 \in S$, then $f(x_2) = \frac{x_2}{2}$. We will prove that $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$ for this case by proving the contrapositive, or by showing that $f(x_1) = f(x_2) \implies x_1 = x_2$.

$$f(x_1) = f(x_2)$$
$$\frac{x_1}{2} = \frac{x_2}{2}$$
$$x_1 = x_2$$

   Thus, by contrapositive, we have proven that in this case, $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.

   b) If $x_2 \neq x_1$, then $x \notin S$. In this case, $x_2 = 0$, or $x_2 \notin S \land x \neq 0$.

      i. If $x_2 = 0$, then $f(x_2) = \frac{1}{2}$. As $f(x_1) = \frac{x_1}{2}$, x_1 must be 1 if $f(x_1) = f(x_2) = \frac{1}{2}$. However, $1 \notin S$, and thus $f(x_1) \neq f(x_2)$.

      ii. If $x_2 \notin S \land x \neq 0$, then $f(x_2) = x_2$. If $f(x_2) = f(x_1) = \frac{x_1}{2}$, as $\frac{x_1}{2} \in S$ and $f(x_2)$ is the identity function, then x_2 must be in S. However, $x_2 \notin S$ and thus $f(x_1) \neq f(x_2)$.

3. Let $x_1 \notin S \land x_2 \neq 0$. Then, $f(x_1) = x_1$.

   a) If $x_2 \in S$, then $f(x_2) = \frac{x_2}{2}$. If $f(x_1) = f(x_2) = \frac{x_2}{2}$, as $\frac{x_2}{2} \in S$ and $f(x_1)$ is the identity function, then x_1 must be in S. However, $x_1 \notin S$ and thus $f(x_1) \neq f(x_2)$.

   b) If $x_2 = 0$, then $f(x_2) = \frac{1}{2}$. If $f(x_1) = f(x_2) = \frac{1}{2}$, then since $f(x_1)$ is the identity function, $x_1 = \frac{1}{2}$, but $\frac{1}{2} \in S$ but $x_1 \notin S$, and thus $f(x_1) \neq f(x_2)$.

   c) if $x_2 \notin S \land x_2 \neq 0$, then $f(x_2) = x_2$. We will prove that $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$ for this case by proving the contrapositive, or by showing that $f(x_1) = f(x_2) \implies x_1 = x_2$.

$$f(x_1) = f(x_2)$$
$$x_1 = x_2$$

   Thus, by contrapositive, we have proven that in this case, $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.

Since we have shown that for every case, $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$, we have shown that f is injective. ■

Claim — f is surjective.

Subproof. We will show that, for all $y \in (0,1)$, there exists a $x \in [0,1)$ such that $f(x) = y$. Let $S = \left\{ \frac{1}{2}^n \mid n \in \mathbb{Z}^+ \right\}$. We can partition the codomain into 2 disjoint sets: $\{x \mid x \notin S, 0 < x < 1\} \cup \{x \mid x \in S, 0 < x < 1\}$. We will exhibit a surjection from f to each disjoint set, and as the union of the two disjoint sets is the codomain $(0,1)$, we will show that f is surjective.

- $\forall y \mid 0 < y < 1 \land y \notin S$, $f(y) = y$. As y is in the domain, there is a surjection to the set of values in the codomain that are not in S.
- We partition the values in the codomain that are in S into two disjoint sets: $\left\{ \frac{1}{2} \right\} \cup \left\{ \frac{1}{2}^n \mid n > 1, n \in \mathbb{Z}^+ \right\}$.
  - When $y = \frac{1}{2}$, $f(0) = y$. As there exists an x in the codomain that maps to $\frac{1}{2}$, f is surjective to the partition $\left\{ \frac{1}{2} \right\}$.
  - When $y \in \left\{ \frac{1}{2}^n \mid n > 1, n \in \mathbb{Z}^+ \right\}$, let $x = 2y$. Then, as $2y \in S$, $f(2y) = \frac{2y}{2} = y$. Thus, there exists an x, $x = 2y$, such that $f(x) = y$. Thus, f is surjective to the partition $\left\{ \frac{1}{2}^n \mid n > 1, n \in \mathbb{Z}^+ \right\}$.
  - As we can exhibit a surjection to both disjoint sets of the values in the codomain that are in S, f is surjective to the values in the codomain that are in S.

As we can exhibit a surjection to both disjoint sets of the values in the codomain that are not in S and in S, f is surjective. ■

As we have shown that f is injective and surjective, it follows that f is bijective, and the proof is complete. □

Question 2.2. Let T_n be the $n \times n$ matrix given by

$$T_n = \begin{bmatrix} 1 & i & 0 & 0 & \dots & 0 \\ i & 1 & i & 0 & \dots & 0 \\ 0 & i & 1 & i & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \dots & 0 & i & 1 & i \\ 0 & \dots & \dots & 0 & i & 1 \end{bmatrix}$$

(a) Compute $\det T_n$ for $2, 3, 4$ and form a conjecture for $\det T_n$.

(b) Use mathematical induction to prove your conjecture.

Solution

(a)

$$\det T_2 = \begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} = 1 - i^2 = 2$$

$$\det T_3 = \begin{vmatrix} 1 & i & 0 \\ i & 1 & i \\ 0 & i & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & i \\ 0 & 1 \end{vmatrix} - i \begin{vmatrix} i & i \\ 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} i & 1 \\ 0 & i \end{vmatrix} = \det T_2 - i(i) = 2 - i^2 = 2 + 1 = 3$$

$$\det T_4 = \begin{vmatrix} 1 & i & 0 & 0 \\ i & 1 & i & 0 \\ 0 & i & 1 & i \\ 0 & 0 & i & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & i & 0 \\ i & 1 & i \\ 0 & i & 1 \end{vmatrix} - i\begin{vmatrix} i & i & 0 \\ 0 & 1 & i \\ 0 & i & 1 \end{vmatrix} + 0\begin{vmatrix} i & 1 & 0 \\ 0 & i & i \\ 0 & 0 & 1 \end{vmatrix} - 0\begin{vmatrix} i & 1 & i \\ 0 & i & 1 \\ 0 & 0 & i \end{vmatrix}$$

$$= \det T_3 - i\left(i\begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} - i\begin{vmatrix} 0 & i \\ 0 & 1 \end{vmatrix}\right)$$

$$\det T_4 = \det T_3 - i^2 \det T_2 = \det T_3 + \det T_2 = 3 + 2 = 5$$

From this, we form the following claim:

Proposition

If T_n is the $n \times n$ matrix given by

$$T_n = \begin{bmatrix} 1 & i & 0 & 0 & \dots & 0 \\ i & 1 & i & 0 & \dots & 0 \\ 0 & i & 1 & i & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \dots & 0 & i & 1 & i \\ 0 & \dots & \dots & 0 & i & 1 \end{bmatrix}$$

then for all $n \in \mathbb{N}$, $\det T_n = f_{n+1}$. where f_n is the n-th term of the fibonnaci sequence given by the recursive relation:

$$f_n = f_{n-1} + f_{n-2}, f_1 = 1, f_2 = 1$$

(b) *Proof by strong induction.* Let $P(n)$ be the statement that $\det T_n = f_{n+1}$. We will show by principle of strong induction that $P(n)$ is true for all $n \in \mathbb{N}$.

Base Cases:

- $n = 1$. T_1 is the 1×1 matrix $\begin{bmatrix} 1 \end{bmatrix}$ and therefore $\det T_1 = 1$. As $f_{1+1} = f_2 = 1$, $P(1)$ holds.
- $n = 2$. $T_2 = 2$ from (a). As $f_{2+1} = f_3 = f_2 + f_1 = 1 + 1 = 2$, $P(2)$ holds.
- $n = 3$. $T_3 = 3$ from (a). As $f_{3+1} = f_4 = f_3 + f_2 = 2 + 1 = 3$, $P(3)$ holds.

Inductive Hypothesis: Assume for all $c \in \mathbb{Z}^+$ such that $1 \leq c \leq k$ where $k \in \mathbb{Z}^+$ and $k \geq 3$, $P(c)$ is true. We will show that $[P(1) \wedge P(2) \wedge \dots \wedge P(k-1) \wedge P(k)] \implies P(k+1)$.

Inductive Step: From T_k, we can form T_{k+1} by:

- adding a $(k+1)$-th row formed the first $(k-1)$ elements being 0 with the k-th element in the row being i
- adding a $(k+1)$-th column formed by the first $(k-1)$ elements being 0 with the k-th element in the column being i
- the $(k+1)$-th element of both intersect and is 1

We visualize this below:

$$
\begin{array}{cl}
\begin{bmatrix}
1 & i & 0 & 0 & \dots & 0 & 0 \\
i & 1 & i & 0 & \dots & 0 & 0 \\
0 & i & 1 & i & \dots & 0 & 0 \\
\vdots & & \ddots & & & \vdots & \vdots \\
0 & \dots & 0 & i & 1 & i & 0 \\
0 & \dots & \dots & 0 & i & 1 & i \\
0 & \dots & \dots & \dots & 0 & i & 1
\end{bmatrix} & \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\\1\end{matrix}}\right\} k+1\text{-th column} \\
\underbrace{\hphantom{0 \dots \dots \dots 0\; i\; 1}}_{k+1\text{-th row}} &
\end{array}
$$

Figure 1: Forming T_{k+1} by extending T_k

$\det T_{k+1}$ can be computed as follows:

$$\det T_{k+1} = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots \pm a_{1,k+1} \det A_{1,k+1}$$

where a_{tj} denotes the element in the t-th row and j-th column and A_{tj} denotes the matrix obtained by removing the t-th row and j-th column.

We first note that, for all a_{1j} where $j > 2$, $a_{1j} = 0$, as, if the matrix is larger than or is a 2×2, we construct iteratively starting at $n = 1$, the second column will be $n - 1 = 0$ 0's followed by i in a_{12} and then 1 in a_{22}, and for every next row, there will be a 0 in the first position a_{1j}.

As $k > 1$, k is at least 2. Since all $a_{1j} = 0$ where $j > 2$, we can simplify $\det T_{k+1}$:

$$\det T_{k+1} = a_{11} \det A_{11} - a_{12} \det A_{12}$$

a_{11} is our base case for our iterative construction and $a_{11} = 1$. From above, we know $a_{12} = i$. Thus, we can further simplify:

$$\det T_{k+1} = \det A_{11} - i(\det A_{12})$$

For a matrix T_{k+1}, for any element a_{tj}, if there exists a column to the right, then $a_{t+1,j+1} = a_{tj}$. Thus, by removing the first row, the column to the right will be equivalent the original column. It follows that if we remove the first column of T_{k+1}, we will obtain the matrix T_k. We visualize this below:

$$
\begin{bmatrix}
\cancel{1} & \cancel{i} & \cancel{0} & \cancel{0} & \cancel{\dots} & \cancel{0} & \cancel{0} \\
\cancel{i} & 1 & i & 0 & \dots & 0 & 0 \\
\cancel{0} & i & 1 & i & \dots & 0 & 0 \\
\cancel{\vdots} & & \ddots & & & \vdots & \vdots \\
\cancel{0} & \dots & 0 & i & 1 & i & 0 \\
\cancel{0} & \dots & \dots & 0 & i & 1 & i \\
\cancel{0} & \dots & \dots & \dots & 0 & i & 1
\end{bmatrix}
$$

Figure 2: T_k is formed when the first row and column is removed.

Thus, for T_{k+1}, $\det A_{11} = \det T_k$:

$$\det T_{k+1} = \det T_k - i(\det A_{12})$$

We now visualize the resulting matrix when the first row and second column is removed:

$$\begin{bmatrix} 1 & i & 0 & 0 & \dots & 0 & 0 \\ i & 1 & i & 0 & \dots & 0 & 0 \\ 0 & i & 1 & i & \dots & 0 & 0 \\ \vdots & & \ddots & & & \vdots & \vdots \\ 0 & \dots & 0 & i & 1 & i & 0 \\ 0 & \dots & \dots & 0 & i & 1 & i \\ 0 & \dots & \dots & \dots & 0 & i & 1 \end{bmatrix}$$

Figure 3: A_{12}.

Let $S = A_{12}$. To compute the determinant of S, we evaluate:

$$\det S = a_{11} \det S_{11} - a_{12} \det S_{12} = i \det S_{11} - i \det S_{12}$$

We visualize the resulting matrix when the first row and first column of S is removed:

$$\begin{bmatrix} i & i & 0 & \dots & 0 & 0 \\ 0 & 1 & i & \dots & 0 & 0 \\ \vdots & \ddots & & & \vdots & \vdots \\ 0 & 0 & i & 1 & i & 0 \\ 0 & \dots & 0 & i & 1 & i \\ 0 & \dots & \dots & 0 & i & 1 \end{bmatrix}$$

Figure 4: T_{k-1} is formed when the first row and column is removed.

We can thus simplify S :

$$\det S = i(\det T_{k-1}) - i(\det S12)$$

We visualize the resulting matrix when the first row and first column of S is removed:

$$\begin{bmatrix} i & i & 0 & \dots & 0 & 0 \\ 0 & 1 & i & \dots & 0 & 0 \\ \vdots & \ddots & & & \vdots & \vdots \\ 0 & 0 & i & 1 & i & 0 \\ 0 & \dots & 0 & i & 1 & i \\ 0 & \dots & \dots & 0 & i & 1 \end{bmatrix}$$

Figure 5: S_{12}.

The determinant can be computed using cofactor expansion across the first column. As the first column of S_{12} is all 0, the determinant is thus 0. It follows that

$$\det A_{12} = i(\det T_{k-1})$$

Thus:

$$\det T_{k+1} = \det T_k - i(i(\det T_{k-1})) = \det T_k - i^2(\det T_{k-1}) = \det T_k + \det T_{k-1}$$

From the inductive hypothesis, $\det T_k = f_{k+1}$ and $\det T_{k-1} = f_k$. It follows that

$$\det T_{k+1} = f_{k+1} + f_k$$

By definition of fibonnaci sequence, $\det T_{k+1} = f_{k+2}$. Thus, by principle of strong induction, we have shown that $P(n)$ is true for $n \geq 1$. □

Question 2.3 (Permutation)**.** A rearrangement of the ordering of the integers $\{1, 2, \ldots, n\}$ is said to be a permutation. The set of such permutations is denoted by S_n.

(a) Prove that S_n has $n!$ elements (remember that $n! = n(n-1)(n-2)\ldots 1$).

(b) Prove that a permutation from S_n can be represented by an $n \times n$ matrix with the following two properties: Each row has all zeros except for one entry which is 1; and each column has all zeros except for one entry which is 1. The representation is such that if A denotes such a matrix and $\vec{v}$ denotes the column vector

$$\vec{v} = \begin{pmatrix} 1 \\ \vdots \\ n \end{pmatrix}$$

then that permutation's rearrangement of $\{1, \ldots, n\}$ is the order of the numbers as they appear in the successive entries of the vector $A\vec{v}$.

(c) Prove that if A and B are two permutation matrices, then AB is one also.

(d) Prove that permutation matrices are invertible, and that the inverse of a permutation matrix is a permutation matrix.

(e) Prove that $\det A = \pm 1$ if A is a permutation matrix.

Solution

(a) *By induction.* Let $P(n)$ be the statement that $|S_n| = n!$, or that the number of permutations of the set $\{1, 2, \ldots, n\}$ is $n!$, $n \in \mathbb{Z}^+$.

Base Case: $n = 1$. The number of ways to arrange 1 element is 1, and thus $|S_1| = 1$. $1! = 1$, so $P(1)$ holds.

Inductive Hypothesis: Assume $P(k)$ is true, $k > 1, k \in \mathbb{Z}^+$. We will show that $P(k) \implies P(k+1)$.

Inductive Step: When we append $k+1$ to this set, we now insert $k+1$ into each permutation. With each permutation, there are $k+1$ positions we can insert $k+1$. From the inductive hypothesis, the number of permutations of the set $\{1, 2, \ldots, k\}$ is $k!$. As there are $k+1$ ways to insert into $k!$ permutations, there are $k! \cdot (k+1) = (k+1)!$ permutations of the set $\{1, 2, \ldots, k, k+1\}$. Thus, $|S_{k+1}| = (k+1)!$.

Thus, by principle of induction, we have shown that $P(n)$ is true. □

(b) *Proof.* Let A be a $n \times n$ matrix where each row has all zeros except for one entry which is 1 and each column has all zeros except for one entry which is 1. Let $A\vec{v} = \vec{b}$, where $\vec{v} = \begin{bmatrix} 1 \\ \vdots \\ n \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$. Let a_{ij} denote the element in the i-th row and j-th column of A. Then, $b_i = 1(a_{i,1}) + 2(a_{i,2}) + \cdots + n(a_{i,n})$. We can see that, as each row of A has 0 in all entries besides 1 that has a value of 1, if the i-th row of A has a 1 in the j-th column, then the i-th element in the vector $\vec{b}$ will be the value j, where $1 \leq j \leq n, j \in \mathbb{Z}$. As each column of A has 0 in all entries besides 1 that has a value of 1, a number in the set $\{1, 2, \ldots, n\}$ can only appear once in $\vec{b}$. Thus, the elements of $\vec{b}$ are a permutation of $\{1, 2, \ldots, n\}$. We have shown that we can use a permutation matrix to represent a permutation from S_n. $\square$

(c) *Proof.* A and B are both $n \times n$ matrices.

As A is a permutation matrix, $A\vec{x} = \vec{0}$ only has the trivial solution since the permutation $\vec{0}$ is only the result of $\vec{v} = \vec{0}$, as all entries are 0; there only exists 1 permutation of $\vec{0}$. Thus, the columns of A are linearly independent. From Theorem 12 in Section 1.9 of Lay's Linear Algebra, the transformation, if we let A be the standard matrix for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, then T is one-to-one. Thus, all inputs will map to a unique output.

Let each column of B be an input for T. As B is a permutation matrix, each row only has 1 entry with 1 and every other entry is 0, and thus each column is unique. Let $\vec{b_j}$ be the j-th column of b. Then, for each $\vec{b_j}, j \leq n, j \in \mathbb{Z}^+$, $T(\vec{b_j})$ will be a unique output. Each will be a permutation of $n-1$ 0's and 1 1. As there are n different permutations of $n-1$ 0's and 1 1, and each of the n columns map to a unique permutation, there will exist a bijection.

As $T(\vec{b_1}) = A\vec{b_1}, T(\vec{b_2}) = A\vec{b_2}, \ldots, T(\vec{b_n}) = A\vec{b_n}$, and from above, will result in all permutations of $n-1$ 0's and 1 1 without two being the same, it follows that AB will result in a permutation matrix, as there does not exist a row where there is more than 1 1, as there does not exist a $A\vec{b_i} = A\vec{b_k}, i \neq k, i, k \leq n$, and there does not exist a column where there is more than 1 1, as each column is a permutation of $n-1$ 0's and 1 1. $\square$

(d) *Proof.* By Theorem 7 in Section 2.2 of Lay's Linear Algebra, we can show that a permutation matrix A is invertible by showing that it is row equivalent to I_n. As A is a permutation matrix, every row has 1 element that is 1 and every other element is 0, and the same is true for columns. It follows that every row has a pivot position and every other element in the row is 0 and thus, by only interchanging the rows, we can produce the identity matrix.

By Theorem 7 in Section 2.2 Lay's Linear Algebra, the sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}. As we obtain the identity matrix from A by only interchanging rows, we do not affect the property that each column has only 1 element that is 1 and the rest is 0, and each row only has 1 element that is 1 and the rest is 0, and thus the inverse of A, A^{-1}, is also a permutation matrix. $\square$

(e) *Proof.* As we have shown in (d), any permutation matrix can be formed by a series of interchange operations of the Identity matrix. As $\det I_n = 1$, by Theorem 3 of Section 3.2 in Lay's Linear Algebra, if the amount of interchange operations to obtain A is even, then $\det A = \det I_n = 1$, and if the amount of interchange operations to obtain A is odd, then $\det A = -\det I_n = -1$. Thus, $\det A = \pm 1$. $\square$