MATH 22A: Vector Calculus and Linear Algebra

Problem Set 10

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Collaborators

§1 Computational Problems

Solution 1.1. As $A = PDP^{-1}$, it follows that $A^k = (PDP^{-1})^k = PD^kP^{-1}$. Thus:

$$\begin{aligned} A^k &= \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & 1^k \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 3(2^k) & 4 \\ 2^k & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -3(2^k)+4 & 12(2^k-1) \\ -2^k+1 & 2^{k+2}-3 \end{bmatrix} \end{aligned}$$

Solution 1.2. We first find the eigenvalues of the matrix by finding the values of λ such that $\det(A - \lambda I) = 0$:

$$\begin{aligned} \begin{vmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{vmatrix} &= (4-\lambda)((4-\lambda)^2-4) - 2(2(4-\lambda)-4) + 2(4-2(4-\lambda)) \\ &= (4-\lambda)^3 - 4(4-\lambda) - 4(4-\lambda) + 8 + 8 - 4(4-\lambda) \\ &= -\lambda^3 + 12\lambda^2 - 36\lambda + 32 \\ 0 &= -(\lambda-2)(\lambda-2)(\lambda-8) \end{aligned}$$

Thus, $\lambda = 2, 8$. We now find the eigenvectors associated with the eigenvalues by solving the homogeneous system $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$.

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \underset{\sim}{R_2 - R_1 \to R_2, R_3 - R_1 \to R_3} \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underset{\sim}{\tfrac{1}{2}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution will thus be:

$$\boldsymbol{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The eigenvectors for $\lambda = 2$ are the basis vectors:

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

As the remaining eigenvalue is $\lambda = 8$, and we are given that there is an eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, it follows that P, the matrix where the columns are the linearly independent eigenvectors of A is:

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The diagonal entry in each column of D is the eigenvalue corresponding to the eigenvector for the corresponding column in P:

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

The basis for the eigenvectors is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Solution 1.3. *Proof.* Assume A is diagonalizable. Then, there exists an invertible matrix G such that $S = G^{-1}AG$, where S is a diagonal matrix. We take the transpose of both sides and obtain $S^T = (G^{-1}AG)^T = G^T A^T {G^{-1}}^T$. As the transpose of a diagonal matrix is a diagonal matrix, $S^T = D$, where D is a diagonal matrix. As $G^T {G^{-1}}^T = (G^{-1}G)^T = I^T = I$, it follows that G^T is an invertible matrix. Let $G^T = P$. Then, $D = PA^TP^{-1}$,and thus A^T is diagonalizable, and the proof is complete. ☐

The eigenvalues of A are the diagonal entries of S. As $S^T = D$ preserves the diagonal, it follows that A^T has the same eigenvalues as A.

Solution 1.4. We first find the eigenvalues of the matrix by finding the values of λ such that $\det(A - \lambda I) = 0$:

$$\begin{aligned} \begin{vmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} &= (5-\lambda)(3-\lambda) + 2 \\ 0 &= \lambda^2 - 8\lambda + 17 \\ \lambda &= \frac{8 \pm \sqrt{64-68}}{2} \\ &= 4 \pm i \end{aligned}$$

With $\lambda = 4 - i$, the system

$$\begin{aligned} (1+i)x_1 - 2 &= 0 \\ x_1 + (-1+i)x_2 &= 0 \end{aligned}$$

has a nontrivial solution. Thus, both equations determine the same relationship between x_1 and x_2, and either equation can be used to express one variable in terms of the other. We use the second equation to obtain

$$x_1 = (1-i)x_2$$

We can choose $x_2 = 1$ to obtain $x_1 = 1-i$ and thus a basis for the eigenspace corresponding to $\lambda = 4 - i$ is

$$\boldsymbol{v_1} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

With $\lambda = 4 + i$, we obtain the system

$$(1-i)x_1 - 2 = 0$$

$$x_1 + (-1-i)x_2 = 0$$

From the second equation, $x_1 = (1+i)x_2$. We can thus choose $x_2 = 1$ to obtain $x_1 = 1+i$ and thus a basis for the eigenspace corresponding to $\lambda = 4+i$ is

$$\boldsymbol{v_2} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

Solution 1.5. We find the eigenvalues by finding the values of λ such that $\det(A-\lambda I) = 0$:

$$\begin{aligned} \begin{vmatrix} 5-\lambda & -5 \\ 1 & 1-\lambda \end{vmatrix} &= (5-\lambda)(1-\lambda)+5 \\ 0 &= \lambda^2 - 6\lambda + 10 \\ \lambda &= \frac{6 \pm \sqrt{36-40}}{2} \\ &= 3 \pm i \end{aligned}$$

With $\lambda = 3-i$, we obtain the system

$$\begin{aligned} (2+i)x_1 - 5x_2 &= 0 \\ x_1 + (-2+i)x_2 &= 0 \end{aligned}$$

From the second equation, we obtain $x_1 = (2-i)x_2$. We choose $x_2 = 1$ to obtain $x_1 = 2-i$, and thus an eigenvector corresponding to $\lambda = 3-i$ is $\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$.
With $\lambda = 3+i$, we obtain the system

$$\begin{aligned} (3-i)x_1 - 5x_2 &= 0 \\ x_1 + (-2-i)x_2 &= 0 \end{aligned}$$

From the second equation, we obtain $x_1 = (2+i)x_2$. We choose $x_2 = 1$ to obtain $x_1 = 2+i$, and thus an eigenvector corresponding to $\lambda = 3+i$ is $\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$.

By Theorem 9, $P = \begin{bmatrix} \text{Re } \boldsymbol{v} & \text{Im } \boldsymbol{v} \end{bmatrix}$, where $\boldsymbol{v}$ is the eigenvector associated with the eigenvalue $a-bi$. Thus, $P = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$.

Solution 1.6. *Proof.* As $\boldsymbol{v}$ is the eigenvector associated with the complex eigenvalue $\lambda = a+ib$, it follows that $A\boldsymbol{v} = \lambda\boldsymbol{v}$. Let $\boldsymbol{v} = \text{Re } \boldsymbol{v} + i\text{Im } \boldsymbol{v}$. Then:

$$\begin{aligned} A(\text{Re } \boldsymbol{v} + i\text{Im } \boldsymbol{v}) &= (a+ib)(\text{Re } \boldsymbol{v} + i\text{Im } \boldsymbol{v}) \\ A(\text{Re } \boldsymbol{v}) + iA(\text{Im } \boldsymbol{v}) &= a\text{Re } \boldsymbol{v} + ia\text{Im } \boldsymbol{v} + ib\text{Re } \boldsymbol{v} - b\text{Im } \boldsymbol{v} \end{aligned}$$

We can separate the equation into real and imaginary parts.

$$A(\text{Re } \boldsymbol{v}) = a\text{Re } \boldsymbol{v} - b\text{Im } \boldsymbol{v} \qquad \begin{aligned} iA(\text{Im } \boldsymbol{v}) &= ia\text{Im } \boldsymbol{v} + ib\text{Re } \boldsymbol{v} \\ A(\text{Im } \boldsymbol{v}) &= a\text{Im } \boldsymbol{v} + b\text{Re } \boldsymbol{v} \end{aligned}$$

and thus the proof is complete. $\square$

Solution 1.7. Let $\boldsymbol{x}_0 = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3$. Then, $\boldsymbol{x}_k = c_1\lambda_1^k\boldsymbol{v}_1 + c_2\lambda_2^k\boldsymbol{v}_2 + c_3\lambda_3^k\boldsymbol{v}_3$. We will solve for the weights c_1, c_2, c_3:

$$\begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$$

We form an augmented matrix and solve for the weights by obtaining an echelon form and then back substituting:

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -3 & -5 \\ -3 & -5 & 7 & 3 \end{array}\right] \quad \begin{array}{c} R_3 + 3R_1 \to R_3 \\ \sim \end{array} & \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -3 & -5 \\ 0 & 1 & -2 & -3 \end{array}\right] \\ \begin{array}{c} R_3 - R_2 \to R_3 \\ \sim \end{array} & \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right] \end{aligned}$$

$$\begin{aligned} c_3 &= 2 \\ c_2 &= -5 + 3c_3 = -5 + 3(2) = 1 \\ c_1 &= -2 + 3c_3 - 2c_2 = -2 + 3(2) - 2(1) = 2 \end{aligned}$$

Thus, the solution set is given by

$$\boldsymbol{x}_k = 2(3)^k \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + \left(\frac{4}{5}\right)^k \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} + 2\left(\frac{3}{5}\right)^k \begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$$

As $k \to \infty$, $\left(\frac{4}{5}\right)^k \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \to 0$, and $2\left(\frac{3}{5}\right)^k \begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix} \to 0$. Thus, $k \to \infty$, $\boldsymbol{x}_k \approx 2(3)^k \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$.

Solution 1.8. When $p = 0.5$, the predator-pray matrix is

$$\begin{bmatrix} .4 & .3 \\ -.5 & 1.2 \end{bmatrix}$$

We solve for the eigenvalues by solving for λ where $\det(A - \lambda I) = 0$:

$$\begin{aligned} \begin{vmatrix} .4 - \lambda & .3 \\ -.5 & 1.2 - \lambda \end{vmatrix} &= (.4 - \lambda)(1.2 - \lambda) + .5(.3) \\ 0 &= \lambda^2 - 1.6\lambda + .63 \\ &= (\lambda - .9)(\lambda - .7) \end{aligned}$$

Thus, $\lambda = 0.9, 0.7$. We will find the corresponding eigenvectors:

$$\begin{bmatrix} -.5 & .3 \\ -.5 & .3 \end{bmatrix} \quad \begin{array}{c} R_2 - R_1 \to R_2 \\ \sim \end{array} \quad \left[\begin{array}{cc|c} -.5 & .3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The general solution will thus be:

$$\boldsymbol{x} = x_2 \begin{bmatrix} .6 \\ 1 \end{bmatrix}$$

Thus, the eigenvector that corresponds to $\lambda = 0.9$ is $\boldsymbol{v}_1 = \begin{bmatrix} .6 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} -.3 & .3 \\ -.5 & .5 \end{bmatrix} \overset{R_2 - \frac{5}{3}R_1 \to R_2}{\sim} \begin{bmatrix} -.3 & .3 \\ 0 & 0 \end{bmatrix}$$

The general solution will thus be:

$$\boldsymbol{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus, the eigenvector that corresponds to $\lambda = 0.7$ is $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If $\boldsymbol{x}_0 = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2}$, then

$$\boldsymbol{x}_k = c_1(0.9)^k\boldsymbol{v_1} + c_2(0.7)^k\boldsymbol{v_2}$$

As $0.9^k\boldsymbol{v}_1 \to 0$ as $k \to \infty$ and $0.7^k\boldsymbol{v}_2 \to 0$ as $k \to \infty$, it follows that $\boldsymbol{x}_k \to 0$ as $k \to \infty$, and thus both owls and squirrels will eventually perish.

If $p = .4$, then the eigenvalues can be obtained by the following:

$$\begin{aligned} \begin{vmatrix} .4 - \lambda & .3 \\ -.4 & 1.2 - \lambda \end{vmatrix} &= (.4 - \lambda)(1.2 - \lambda) + .3(.4) \\ 0 &= \lambda - 1.6\lambda + .6 \\ &= (\lambda - .6)(\lambda - 1) \end{aligned}$$

Thus, the eigenvalues will be $\lambda = 0.6, 1$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ be the eigenvectors of each eigenvalue respectively. Then, $0.6^k\boldsymbol{v}_1 \to 0$ as $k \to \infty$, and thus $\boldsymbol{x}_k = 1^k\boldsymbol{v_2} = \boldsymbol{v_2}$, and thus both populations will tend towards a constant non-zero level. We can find the relative sizes of the asymptotic populations by finding the eigenvector for $\lambda = 1$.

$$\begin{bmatrix} -.6 & .3 \\ -.4 & .2 \end{bmatrix} \overset{R_2 - \frac{2}{3}R_1 \to R_2}{\sim} \begin{bmatrix} -.6 & .3 \\ 0 & 0 \end{bmatrix}$$

Thus, we obtain the general solution:

$$\boldsymbol{x} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Thus, the eigenvector corresponding to $\lambda = 1$ is $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$. This means that the ratio of spotted owls to flying squirrels will be 2:1.

Solution 1.9.

$$\boldsymbol{x}(t) = c_1\boldsymbol{v}_1 e^{\lambda_1 t} + c_2\boldsymbol{v}_2^{\lambda_2 t} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

We want c_1, c_2 to satisfy $\boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

$$\left[\begin{array}{cc|c} -1 & 1 & 2 \\ 1 & 1 & 3 \end{array}\right] \overset{R_2 + R_1 \to R_2}{\sim} \left[\begin{array}{cc|c} -1 & 1 & 2 \\ 0 & 2 & 5 \end{array}\right]$$

Thus, with back substitution, we obtain:

$$c_2 = \frac{5}{2}$$

$$c_1 = -2 + c_2 = -2 + \frac{5}{2} = \frac{1}{2}$$

Thus:

$$\boldsymbol{x}(t) = \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}e^{-3t} + \frac{5}{2}\begin{bmatrix}1\\1\end{bmatrix}e^{-t} = \begin{bmatrix}-\frac{1}{2e^{3t}} + \frac{5}{2e^{t}}\\ \frac{1}{2e^{3t}} + \frac{5}{2e^{t}}\end{bmatrix}$$

Solution 1.10.

$$\boldsymbol{x} = c_1\boldsymbol{v}_1e^{\lambda_1 t} + c_2\boldsymbol{v}_2e^{\lambda_2 t}$$

We find the eigenvalues by finding λ such that $\det(A - \lambda I) = 0$:

$$\begin{aligned}\begin{vmatrix}-2-\lambda & 1\\ -8 & 2-\lambda\end{vmatrix} &= (-2-\lambda)(2-\lambda)+8\\ 0 &= \lambda^2+4\\ \lambda &= \pm 2i\end{aligned}$$

We find the eigenvectors with the following equations:

$$\begin{aligned}(-2+2i)x_1 + x_2 &= 0\\ -8x_1 + (2+2i)x_2 &= 0\end{aligned}$$

$x_2 = (2-2i)x_1$. Let $x_1 = 1$. Then, $x_2 = 2-2i$. Thus, an eigenvector corresponding to $\lambda_1 = -2i$ is $\boldsymbol{v}_1 = \begin{bmatrix}1\\2-2i\end{bmatrix}$. Similarly, an eigenvector corresponding to $\lambda_2 = 2i$ is $\boldsymbol{v}_2 = \begin{bmatrix}1\\2+2i\end{bmatrix}$. Thus, the general solution is

$$\boldsymbol{x} = c_1\begin{bmatrix}1\\2-2i\end{bmatrix}e^{-2it} + c_2\begin{bmatrix}1\\2+2i\end{bmatrix}e^{2it}$$

§2 Proof Problems

Solution 2.1.

(a) For $n \geq 1$,

$$\begin{aligned}\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}f_n\\f_{n-1}\end{bmatrix} &= \begin{bmatrix}f_n + f_{n-1}\\ f_n\end{bmatrix}\\ &= \begin{bmatrix}f_{n+1}\\ f_n\end{bmatrix}\end{aligned}$$

(b)

> **Claim** — $A^n\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}f_{n+1}\\f_n\end{bmatrix}$ for every $n \geq 0$.

Proof by induction. Let the claim be the proposition $P(n)$.

Base Case: $n = 0$. As $A^0 = I$, it follows that $I\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}f_1\\f_0\end{bmatrix}$, from the recursive formula of the Fibonnaci numbers, and thus $P(0)$ holds.

Inductive Hypothesis: Assume for purposes of induction that $P(k)$ is true, $k > 0$. We will show that $P(k) \implies P(k+1)$.

Inductive Step: From our inductive hypothesis, $A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_{k+1} \\ f_k \end{bmatrix}$. We left-multiply both sides by A to obtain

$$AA^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A \begin{bmatrix} f_{k+1} \\ f_k \end{bmatrix}$$

$AA^k = A^{k+1}$, and from Part (a), we have shown that $A \begin{bmatrix} f_{k+1} \\ f_k \end{bmatrix} = \begin{bmatrix} f_{k+2} \\ f_{k+1} \end{bmatrix}$. Thus, we can simplify:

$$A^{k+1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_{k+2} \\ f_{k+1} \end{bmatrix}$$

As we have shown that $P(k) \implies P(k+1)$, by the principle of mathematical induction, we have shown that $P(n)$ is true, $n \geq 0$. $\square$

(c) The eigenvalues for A can be found by finding λ such that $\det(A - \lambda I) = 0$.

$$\begin{aligned}
\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} &= -\lambda(1-\lambda) - 1 \\
&= \lambda^2 - \lambda - 1 \\
\lambda &= \frac{1 \pm \sqrt{5}}{2}
\end{aligned}$$

Thus, the positive eigenvalue is $\phi = \frac{1+\sqrt{5}}{2}$. We will verify that the negative eigenvalue is equal to $-1/\phi$ by setting the two equal and seeing if we reach a contradiction.

$$\begin{aligned}
-\frac{2}{1+\sqrt{5}} &= \frac{1-\sqrt{5}}{2} \\
-4 &= (1+\sqrt{5})(1-\sqrt{5}) \\
-4 &= -4
\end{aligned}$$

Thus, we have shown that the negative eigenvalue is equal to $-1/\phi$.

(d) We will find the eigenvectors for the eigenvalues by solving the homogeneous system $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$.

$$\left[\begin{array}{cc|c} 1-\phi & 1 & 0 \\ 1 & -\phi & 0 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -\phi & 0 \\ 1-\phi & 1 & 0 \end{array}\right]$$
$$\underset{\sim}{R_2 - (1-\phi)R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -\phi & 0 \\ 0 & -\phi^2 + \phi + 1 & 0 \end{array}\right]$$

As $\phi = \frac{1+\sqrt{5}}{2}$, we substitute to get

$$\left[\begin{array}{cc|c} 1 & -\frac{1+\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, we have the general solution

$$\boldsymbol{x} = x_2 \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

and thus the eigenvector corresponding to ϕ is $\begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \phi \\ 1 \end{bmatrix}$.

We do the same for $\lambda = -\frac{1}{\phi}$.

$$\left[\begin{array}{cc|c} 1+\frac{1}{\phi} & 1 & 0 \\ 1 & \frac{1}{\phi} & 0 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_2 \\ \sim \end{array} \left[\begin{array}{cc|c} 1 & \frac{1}{\phi} & 0 \\ 1+\frac{1}{\phi} & 1 & 0 \end{array}\right]$$

$$R_2 - (1+\tfrac{1}{\phi})R_1 \to R_2 \sim \left[\begin{array}{cc|c} 1 & \frac{1}{\phi} & 0 \\ 0 & 1-\frac{1}{\phi}-\frac{1}{\phi^2} & 0 \end{array}\right]$$

As $-\frac{1}{\phi} = \frac{1-\sqrt{5}}{2}$, it follows that $\frac{1}{\phi} = \frac{\sqrt{5}-1}{2}$, so we substitute to get

$$\left[\begin{array}{cc|c} 1 & \frac{\sqrt{5}-1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, we have the general solution

$$\boldsymbol{x} = x_2 \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

and thus the eigenvector corresponding to $-1/\phi$ is $\begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{phi} \\ 1 \end{bmatrix}$.

(e) The invertible matrix P such that $A = PDP^{-1}$ where D is a diagonal matrix is a matrix where the columns are linearly independent eigenvectors. Thus

$$P = \begin{bmatrix} \phi & -\frac{1}{\phi} \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} \phi & 0 \\ 0 & -\frac{1}{\phi} \end{bmatrix}$$

(f) $D = \begin{bmatrix} \phi & 0 \\ 0 & -\frac{1}{\phi} \end{bmatrix}$, and thus $D^n = \begin{bmatrix} \phi^n & 0 \\ 0 & \left(-\frac{1}{\phi}\right)^n \end{bmatrix}$. As $A = PDP^{-1}$, $A^n = (PDP^{-1})^n = PD^nP^{-1}$.

We first calculate P^{-1}.

$$\frac{1}{\det P} \begin{bmatrix} 1 & \frac{1}{\phi} \\ -1 & \phi \end{bmatrix} = \frac{\phi}{\phi^2+1} \begin{bmatrix} 1 & \frac{1}{\phi} \\ -1 & \phi \end{bmatrix}$$

$$\begin{aligned} A^n &= \begin{bmatrix} \phi & -\frac{1}{\phi} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi^n & 0 \\ 0 & \left(-\frac{1}{\phi}\right)^n \end{bmatrix} \left(\frac{\phi}{\phi^2+1} \begin{bmatrix} 1 & \frac{1}{\phi} \\ -1 & \phi \end{bmatrix} \right) \\ &= \frac{\phi}{\phi^2+1} \begin{bmatrix} \phi^{n+1} - \left(-\frac{1}{\phi}\right)^{n+1} & \phi^n - \left(-\frac{1}{\phi}\right)^n \\ \phi^n - \left(-\frac{1}{\phi}\right)^n & \phi^{n-1} - \left(\frac{1}{\phi}\right)^{n-1} \end{bmatrix} \end{aligned}$$

(g) From parts (a) and (f), we obtain

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{\phi}{\phi^2+1} \begin{bmatrix} \phi^{n+1} - \left(-\frac{1}{\phi}\right)^{n+1} & \phi^{n} - \left(-\frac{1}{\phi}\right)^{n} \\ \phi^{n} - \left(-\frac{1}{\phi}\right)^{n} & \phi^{n-1} - \left(\frac{1}{\phi}\right)^{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{\phi}{\phi^2+1} \begin{bmatrix} \phi^{n+1} - \left(-\frac{1}{\phi}\right)^{n+1} \\ \phi^{n} - \left(-\frac{1}{\phi}\right)^{n} \end{bmatrix}$$

$$= \begin{bmatrix} \phi^{n} - \left(-\frac{1}{\phi}\right)^{n} \\ \phi^{n-1} - \left(-\frac{1}{\phi}\right)^{n-1} \end{bmatrix}$$

Thus,

$$f_n = \phi^{n-1} - \left(-\frac{1}{\phi}\right)^{n-1}$$

Solution 2.2.

(a) *Proof.* Let the geometric multiplicity of the eigenvector $\lambda_k = m$. Let P_1 be the matrix where the columns are of the linearly independent eigenvectors of the eigenspace V_k. As $\{v_1, \ldots, v_m\}$ is the basis for V_k, it follows that all vectors in the set are linearly independent and are eigenvectors. Thus, $P_1 = \begin{bmatrix} v_1 & \ldots & v_m \end{bmatrix}$. Let P_2 be the matrix where the columns are of the linearly independent vectors that are used to extend to a basis $\mathcal{B}$; $P_2 = \begin{bmatrix} u_{m+1} & \ldots & u_n \end{bmatrix}$. We can then join P_1 and P_2 to form P, a matrix of linearly independent vectors:

$$P = \begin{bmatrix} v_1 & \ldots & v_m & u_{m+1} & \ldots & u_n \end{bmatrix}$$

We left-multiply by A:

$$AP = \begin{bmatrix} Av_1 & \ldots Av_m & Au_{m+1} & \ldots & Au_n \end{bmatrix} = \begin{bmatrix} AP_1 & AP_2 \end{bmatrix}$$

As P_1 are eigenvectors of A, by definition, $Ax = \lambda x$ for all vectors $x \in P_1$. Thus:

$$AP = \begin{bmatrix} \lambda_k v_1 & \ldots \lambda_k v_m & u_{m+1} & \ldots & u_n \end{bmatrix} = \begin{bmatrix} \lambda_k P_1 & AP_2 \end{bmatrix}$$

We can find a similar matrix D with $PD = AP$. Thus:

$$\begin{bmatrix} P_1 & P_2 \end{bmatrix} B = \begin{bmatrix} \lambda_k P_1 & AP_2 \end{bmatrix}$$

The result of the matrix multiplication of PD will result in the first m columns being $\lambda_k P_1$. Since we take the dot product between the rows of P_1 and the columns of D, there must be λ_k along the diagonal of the first $m \times m$ block and 0's for every other entry. Thus, the first $m \times m$ block is λI_m. The next $m \times (n-m)$. The $(n-m) \times m$ matrix below the $m \times m$ block will be a zero matrix, as we extend to a larger basis, and the resulting $\lambda_k P_1$ matrix will have 0's in those entries. The other blocks will be matrices that, when P_2 multiplies it, it will result in AP_2. □

(b) *Proof.* As B is a similar matrix to A, by Theorem 4 in Section 5.2, they have the same characteristic polynomial, and thus $\det(A - \lambda I_n) = \det(A_{\mathcal{B}} - \lambda I_n)$. □

(c) *Proof.*

$$\det(A_{\mathcal{B}} - \lambda I_n) = \det\left(\begin{bmatrix} \lambda_k I_m & B \\ 0_{n,m} & C \end{bmatrix} - \begin{bmatrix} \lambda I_m & 0 \\ 0 & \lambda I_m \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} (\lambda_k - \lambda)I_m & B \\ 0 & C - \lambda I_m \end{bmatrix}\right)$$

As this is block triangular, can simplify:

$$\det((\lambda_k - \lambda)I_m) \cdot \det(C - \lambda I_m)$$

The determinant of $(\lambda_k - \lambda)I_m$ is the product of the diagonal, and thus $\det((\lambda_k - \lambda)I_m) = (\lambda_k - \lambda)^m$. We obtain

$$(\lambda_k - \lambda)^m \det(C - \lambda_m)$$

as desired, and the proof is complete. $\square$