MATH 22A: Vector Calculus and Linear Algebra

Problem Set 2

Due: Wednesday, September 20, 2023 12pm **Denny Cao**

Collaborators

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§1 Computational Problems

Question 1.1. Reduce the following matrix into reduced row echelon form. Then circle the pivot positions in the final matrix and the matrix below, and also list the pivot columns.

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$$

Solution

$\sim 3R_1 - R_2 \to R_2$.

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 4 & 8 & 21 \\ 5 & 7 & 9 & 1 \end{bmatrix}$$

$\sim 5R_1 - R_3 \to R_3$.

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 4 & 8 & 21 \\ 0 & 8 & 16 & 34 \end{bmatrix}$$

$\sim 2R_2 - R_3 \to R_3$

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 4 & 8 & 21 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$\sim 1/8R_3 \to R_3$.

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 4 & 8 & 21 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\sim R_1 - 7R_3 \to R_1, R_2 - 21R_3 \to R_2$.

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\sim 1/4R_2 \to R_2$.

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\sim R_1 - 3R_2 \to R_2$.

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The pivot columns are C_1, C_2, C_4, where C_i denotes the $i-$th column.

Question 1.2. Find the general solution to the system whose augmented matrix is given below.

$$\left[\begin{array}{rrrr|r} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right]$$

Solution

$\sim R_1 + R_3 \to R_3$.

$$\left[\begin{array}{rrrr|r} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{array}\right]$$

$\sim 4R_2 + R_3 \to R_3$.

$$\left[\begin{array}{rrrr|r} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\sim R_1 + 3R_2 \to R_1$

The general solution is $\begin{pmatrix} 5 + 7x_2 - 6x_4 \\ x_2 \\ -3 + 2x_4 \\ x_4 \end{pmatrix}$**, where x_2 and x_4 are free.**

Question 1.3. Choose h and k such that the system below has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each.

$$\begin{aligned} x_1 + 3x_2 &= 2 \\ 3x_1 + hx_2 &= k \end{aligned}$$

Solution

In augmented matrix form, the system can be expressed as the following:

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array}\right] \overset{R_2 - 3R_1 \to R_2}{\sim} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{array}\right]$$

a) *No Solution:* $h = 9, k \neq 6$. This results in the rightmost column in the augmented matrix being a pivot column, and thus, by Theorem 2 (Existence and Uniqueness Theorem) in Section 1.2 of Linear Algebra and Its Applications, there is is no solution.

b) *A unique solution*: If $h \neq 9$. This results in the solution $x_2 = \frac{k-6}{h-9}$ and $x_1 = 2 - \frac{3(k-6)}{h-9}$.

c) *Many Solutions:* $h = 9, k = 6$. This results in the values along $x_1 + 3x_2 = 2$ being solutions to the system.

Question 1.4. Experimental data is sometimes presented as a set of points in the plane. An interpolating polynomial for the data is a polynomial whose graph goes through every point. (Such a polynomial can be used to estimate values between the observed data points. It can also be used to create curves for graphical images of the data on a computer screen.) One method for finding an interpolating polynomial is to solve a system of linear equations. Find an intepolating polynomial $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ for the following data from wind tunnel measurements on a projectile (thus, find a_0, a_1, a_2, a_3 such that the polynomial passes through the points below.)

Velocity (100 ft/sec)	0	2	4	6
Force (100 lb)	0	2.90	14.8	39.6

(You can round off the force numbers to the nearest integer.) Is there a quadratic interpolating polynomial?

Solution

We can set up an augmented matrix as follows:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 3 \\ 1 & 4 & 16 & 64 & 15 \\ 1 & 6 & 36 & 216 & 40 \end{array}\right]$$

$\sim R_2 - R_1 \to R_2, R_3 - R_1 \to R_3, R_4 - R_1 \to R_4.$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 3 \\ 0 & 4 & 16 & 64 & 15 \\ 0 & 6 & 36 & 216 & 40 \end{array}\right]$$

$\sim 2R_2 - R_1 \to R_2.$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 3 \\ 0 & 0 & 8 & 48 & 9 \\ 0 & 6 & 36 & 216 & 40 \end{array}\right]$$

$\sim R_4 - 3R_2 \to R_3.$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 3 \\ 0 & 0 & 8 & 48 & 9 \\ 0 & 0 & 24 & 192 & 31 \end{array}\right]$$

$\sim R_4 - 3R_3 \to R_4$.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 3 \\ 0 & 0 & 8 & 48 & 9 \\ 0 & 0 & 0 & 48 & 4 \end{array}\right]$$

$\sim R_3 - R_4 \to R_3$.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 3 \\ 0 & 0 & 8 & 0 & 5 \\ 0 & 0 & 0 & 48 & 4 \end{array}\right]$$

$\sim R_2 - 1/6R_4 \to R_2$.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 7/3 \\ 0 & 0 & 8 & 0 & 5 \\ 0 & 0 & 0 & 48 & 4 \end{array}\right]$$

$\sim 1/8R_3 \to R_3, 1/48R_4 \to R_4$.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 7/3 \\ 0 & 0 & 1 & 0 & 5/8 \\ 0 & 0 & 0 & 1 & 1/12 \end{array}\right]$$

$\sim R_2 - 4R_3 \to R_2$.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1/6 \\ 0 & 0 & 1 & 0 & 5/8 \\ 0 & 0 & 0 & 1 & 1/12 \end{array}\right]$$

$\sim 1/2R_2 \to R_2$.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/12 \\ 0 & 0 & 1 & 0 & 5/8 \\ 0 & 0 & 0 & 1 & 1/12 \end{array}\right]$$

Thus, we can create the following cubic interpolating polynomial:

$$p(t) = -\frac{1}{12}t + \frac{5}{8}t^2 + \frac{1}{12}t^3$$

The coefficients of a quadratic interpolating polynomial can be found by removing the fourth column of the augmented matrix. We obtain:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/12 \\ 0 & 0 & 1 & 5/8 \\ 0 & 0 & 0 & 1/12 \end{array}\right]$$

As we obtain an augmented matrix in an echelon form that has a pivot column in the right most column, by Theorem 2 (Existence and Uniqueness Theorem) in Section 1.2 of Linear Algebra and Its Applications, there is no solution.

Thus, there does not exist a quadratic interpolating polynomial.

Question 1.5. Write a system of equations that is equivalent to the vector equation below.

$$x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution

$$\begin{aligned} -2x_1 + 8x_2 + x_3 &= 0 \\ 3x_1 + 5x_2 - 6x_3 &= 0 \end{aligned}$$

Question 1.6. Determine if the vector $\mathbf{b}$ below is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ below.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$$

Solution
We can represent the potential linear combination as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{array}\right]$$

$\sim 2R_1 + R_2 \to R_2, R_3 - 2R_2 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{array}\right]$$

$\sim R_3 - R_2 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

This results in the rightmost column in the augmented matrix being a pivot column, and thus, by Theorem 2 (Existence and Uniqueness Theorem) in Section 1.2 of Linear Algebra and Its Applications, there does not exist a nontrivial linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

Question 1.7. Determine if the vector $\mathbf{b}$ below is a linear combination of the columns of the matrix A below.

$$A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

Solution
We can represent the potential linear combination as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{array}\right]$$

$\sim R_3 - R_1 \to R_3$.

$$\left[\begin{array}{rrr|r} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{array}\right]$$

$1/11R_3 \to R_3$

$$\left[\begin{array}{rrr|r} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 1 & -2/11 \end{array}\right]$$

$R_2 - 7R_3 \to R_2, R_1 + 6R_3 \to R_1$.

$$\left[\begin{array}{rrr|r} 1 & -2 & 0 & 109/11 \\ 0 & 3 & 0 & -41/11 \\ 0 & 0 & 1 & -2/11 \end{array}\right]$$

$1/3R_2 \to R_2$.

$$\left[\begin{array}{rrr|r} 1 & -2 & 0 & 109/11 \\ 0 & 1 & 0 & -41/33 \\ 0 & 0 & 1 & -2/11 \end{array}\right]$$

$R_1 + 2R_2 \to R_1$.

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 245/33 \\ 0 & 1 & 0 & -41/33 \\ 0 & 0 & 1 & -2/11 \end{array}\right]$$

h As the rightmost column of the augmented matrix is not a pivot column, by Theorem 2 (Existence and Uniqueness Theorem) in Section 1.2 of Linear Algebra and Its Applications, the system is consistent and thus there exists a solution, meaning $\mathbf{b}$ is a linear combination of the columns of A.

Question 1.8. Consider the vector $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{b}$ in $\mathbb{R}^3$ that are depicted below ($\mathbf{v}_1$ and $\mathbf{v}_2$ lie in a plane and $\mathbf{v}_3$ points out of that plane). Does the equation $x_1\mathbf{v}_1+x_2\mathbf{v}_2+x_3\mathbf{v}_3 = \mathbf{b}$ have a solution? If so, is that solution unique? Make sure to explain your reasoning.

Solution

The equation does have a solution. As $\mathbf{v}_1$ and $\mathbf{v}_2$ span a plane in $\mathbb{R}^3$, they are linearly independent (they are not multiples of each other). Since $\mathbf{v}_3$ points out of that plane, it is not coplanar. Thus, $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ and thus the three vectors are linearly independent by Theorem 7 (Characterization of Linearly Dependent Sets) in Section 1.7 of Linear Algebra and Its Applications. Thus, the three vectors span $\mathbb{R}^3$. By Theorem 4 in Section 1.4 of Linear Algebra and Its Applications, $A\mathbf{x} = \mathbf{b}$ where for the matrix A, each column represents each of the three vectors, $\mathbf{b}$ is a linear combination of the columns of A, as it is the case that all vectors in A span $\mathbb{R}^3$.

The solution is unique because the columns of A (i.e., $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$) are linearly independent. This means there is only one combination of x_1, x_2, and x_3 that satisfies $A\mathbf{x} = \mathbf{b}$, ensuring the uniqueness of the solution.

Question 1.9. Compute the product of a matrix and a vector depicted below.

$$\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 8 \\ 5 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 8+3-4 \\ 5+1+2 \end{bmatrix}$$
$$= \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

Question 1.10. Is the vector $\mathbf{u}$ depicted below in the plane spanned by the columns of the matrix A that is depicted below Explain your reasoning.

$$\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \quad A = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

Solution
If a vector $\mathbf{u}$ is spanned by a matrix A, then there exists a linear combination $Ax = \mathbf{u}$.

$$\left[\begin{array}{ccc|c} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ {}^{3}1 & 3 & 0 & 2 \end{array}\right]$$

$\sim R_1 \leftrightarrow R_3$.

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 5 & 8 & 7 & 2 \end{array}\right]$$

$\sim R_3 - 5R_1 \rightarrow R_3$.

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -7 & 7 & -8 \end{array}\right]$$

$\sim R_3 + 7R_2 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -29 \end{array}\right]$$

As we reach an augmented matrix in an echelon form where the rightmost column is a pivot column, by Theorem 2 (Existence and Uniqueness Theorem) in Section 1.2 of Linear Algebra and Its Applications, there does not exist a nontrivial solution to the system, and thus u is not a linear combination of the matrix columns. Therefore, it is not spanned by the columns of A.

§2 Proof Problems

Question 2.1. Prove the following statements.

(a) Suppose x is an integer. Then x is odd if and only if x^3 is odd.

(b) **Chessboard problem.** Two opposite corner squares are deleted from an 8 by 8 checkerboard. Prove that the remaining squares cannot be covered exactly by dominoes (rectangles consisting of two adjacent squares).

Solution

a) *Proof.* To prove the biconditional statement that x is odd $\leftrightarrow$ x^3 is odd, we must show that x is odd $\rightarrow$ x^3 is odd and that x^3 is odd $\rightarrow$ x is odd.

We will show that x is odd $\rightarrow$ x^3 is odd. By definition of odd number, there exists an integer k such that $x = 2k + 1$. Thus:

$$\begin{aligned} x^3 &= (2k+1)^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2(4k^3 + 6k^2 + 3k) + 1 \end{aligned}$$

Let $c = 4k^3 + 6k^2 + 3k$.

$$x^3 = 2c + 1$$

As x^3 can be expressed in the form $2c + 1$, where c is an integer, by definition of odd number, x^3 is odd.

We will show that x^3 is odd $\rightarrow$ x is odd by the contrapositive. The contrapositive of the statement is that, if x is even, then x^3 is even. By definition of even number, there exists an integer k such that $x = 2k$. Thus:

$$\begin{aligned} x^3 &= (2k)^3 \\ &= 8k^3 \\ &= 2(4k^3) \end{aligned}$$

Let $c = 4k^3$.

$$x^3 = 2c$$

As x^3 can be expressed in the form $2c$, where c is an integer, by definition of even number, x^3 is even. As the statement that x is even $\rightarrow$ x^3 is even is equivalent to the original statement, we have shown that x^3 is odd $\rightarrow$ x is odd.

As we have shown that x is odd $\rightarrow$ x^3 is odd and that x^3 is odd $\rightarrow$ x is odd, the biconditional statement that, if x is an integer, then x is odd if and only if x^3 is odd is true. $\square$

b) *Proof.* An 8×8 chessboard has 32 black tiles and 32 white tiles. Removing the two corners, which are both white, will result in a chessboard with 32 black tiles and 30 white tiles. It cannot be the case that dominoes can fully cover the chessboard, as dominoes have 1 black tile and 1 white tile—they have a 1:1 black to white ratio—whereas the board has a 16:15 black to white ratio. $\square$

Question 2.2. If a, b, and c are odd integers, then $ax^2 + bx + c = 0$ has no solution in the set of rational numbers.

Solution

Proof by contradiction. Assume for purposes of contradiction that, if a, b and c are odd integers, then $ax^2 + bx + c = 0$ has a solution in the set of rational numbers. Then, $ax^2 + bx + c$ can be factored into the form $(mx + n)(tx + w) = 0$, where possible rational solutions, by Definition 6.1 in Book of Proofs, are expressed as $-\frac{n}{m}$ and $-\frac{w}{t}$. Then:

$$\begin{aligned} a &= mt \\ c &= nw \\ b &= mw + nt \end{aligned}$$

As a is odd, neither m or t has 2 as a factor (both are odd), as then mn would have 2 as a factor and would mean a is even. As c is odd, neither n or w has 2 as a factor (both are odd) as then nw would have 2 as a factor and would mean c is even.

Thus, b is the sum of two odd integers: mw and nt, as mw does not have a factor of 2 (both m and w are odd) and nt does not have a factor of 2 (both n and t are odd). By definition of odd number, there exists integers k, u such that $mw = 2k + 1, nt = 2u + 1$. Thus:

$$\begin{aligned} b &= 2k + 1 + 2u + 1 \\ &= 2k + 2u + 2 \\ &= 2(k + u + 1) \end{aligned}$$

Let $r = k + u + 1$. Thus, $b = 2r$. ※

We reach a contradiction, as, if $b = 2r$, then by definition of even number, b is even. Thus, it is impossible for integers a, b, and c to be odd if $ax^2 + bx + c = 0$ has a solution in the set of rational numbers. The opposite must be true; that there does not exist a solution in the set of rational numbers. □

Question 2.3. Prove the following statements about the sets A, B, and C. Within your arguments, you may find it useful to argue directly and/or to use the contrapositive form. Remember that to show that two sets are equal, we show that each one is a subset of the other; also remember that to show that S is a subset of T, we argue "let $x \in S$... argue, argue,...Aha! Therefore $x \in T$."

(a) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$

(b) $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$

Solution

a) *Proof.* We will prove the equality by proving both directions:

- $A \cap (B \setminus C) \subseteq (A \cap B) \setminus (A \cap C)$
- $A \cap (B \setminus C) \supseteq (A \cap B) \setminus (A \cap C)$

We will prove the first direction: $A \cap (B \setminus C) \subseteq (A \cap B) \setminus (A \cap C)$.

Let $x \in A \cap (B \setminus C)$. This means that $x \in A$ and $x \in B \setminus C$. By the definition of set difference, this implies that x is in A and x is in B but x is not in C.

Therefore, $x \in A$ and $x \in B$, and since x is not in C, $x \notin A \cap C$. Thus, x belongs to $(A \cap B)$ (because it's in both A and B) and is not in $(A \cap C)$. Therefore, x belongs to the set difference $(A \cap B) \setminus (A \cap C)$. Thus, without loss of generality, $A \cap (B \setminus C) \subseteq (A \cap B) \setminus (A \cap C)$.

We will prove the second direction: $A \cap (B \setminus C) \supseteq (A \cap B) \setminus (A \cap C)$.

Let $x \in (A \cap B) \setminus (A \cap C)$. This means that $x \in A \cap B$ but $x \notin A \cap C$. By the definition of set intersection, x is in both A and B.

As $x \notin A \cap C$, x is not in C, as if x was in C, then $x \in A \cap C$ which is not the case.

Therefore, $x \in A$ and $x \in B$, but $x \notin C$, which means $x \in B \setminus C$. Thus, we have shown that $x \in A \cap (B \setminus C)$. Without loss of generality, $(A \cap B) \setminus (A \cap C) \subseteq A \cap (B \setminus C)$.

As we have shown that $A \cap (B \setminus C) \subseteq (A \cap B) \setminus (A \cap C)$ and $A \cap (B \setminus C) \supseteq (A \cap B) \setminus (A \cap C)$, it follows that $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$. $\square$

b) *Proof.* We will the equality by proving both directions:

- $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$
- $(A \cup B) \setminus (A \cap B) \supseteq (A \setminus B) \cup (B \setminus A)$

We will prove the first direction: $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$.

If $x \in (A \cup B) \setminus (A \cap B)$, then x is in either A or B but not both. Thus, we consider two cases:

- Case 1: $x \in A \land x \notin B$: Then, $x \in A \setminus B$ and thus $x \in (A \setminus B) \cup (B \setminus A)$.
- Case 2: $x \in B \land x \notin A$: Then, $x \in B \setminus A$ and thus $x \in (A \setminus B) \cup (B \setminus A)$.

As it is true for both cases that $x \in (A \setminus B) \cup (B \setminus A)$, without loss of generality, $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$.

We will prove the second direction: $(A \cup B) \setminus (A \cap B) \supseteq (A \setminus B) \cup (B \setminus A)$.

By the definition of intersection and set difference, if $x \in (A \setminus B) \cup (B \setminus A)$, then $x \in A \land x \notin B$ or $x \in B \land x \notin A$. Thus, x is either in A or B but not both, which is equivalent to the expression: $(A \cup B) \setminus (A \cap B)$, meaning $x \in (A \cup B) \setminus (A \cap B)$. Thus, without loss of generality, $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$.

As we have shown that $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$ and $(A \cup B) \setminus (A \cap B) \supseteq (A \setminus B) \cup (B \setminus A)$, it follows that $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$. $\square$

Question 2.4. The empty set. The empty set $\emptyset$ is the set that contains no elements. That is, it satisfies the property that for all x, we have $x \notin \emptyset$. We sometimes denote it also be $\{\}$.

(a) Let S be any set. What if the "if... then..." statement that you need to prove in order to show that $\emptyset \subseteq S$?

(b) Show that for any set S we have $\emptyset \subseteq S$ by proving the "if... then..." statement you came up with.

(c) We say that two sets, A and B are *disjoint* if $A \cap B = \emptyset$. Show that for any sets A, B, and C the sets $A \setminus (B \cup C)$ and $(A \cap B) \cup C$ are disjoint.

Solution

a) We must show that, if $x \in \emptyset$, then $x \in S$.

b) *Proof by contrapositive.* The contrapositive of the statement is that, if $x \notin S$, then $x \notin \emptyset$. As the empty set has a cardinality of 0—contains no elements—the statement is true, as for all values k in the universe U, $k \notin \emptyset$. As the contrapositive is equivalent to the original statement, the statement that if $x \in \emptyset$ then $x \in S$ is true. □

c) *Proof.* If $x \in A \setminus (B \cup C)$, then $x \in A$ but not in B or C, and thus is not in $A \cap B$. Therefore, $x \notin (A \cap B) \cup C$. Thus, without loss of generality, $(A \setminus (B \cup C)) \cap ((A \cap B) \cup C) = \emptyset$. □

Question 2.5. Critiquing a Proof One of your classmates has given you the following claim and proof to check for errors. Your job is to write your colleague a small note (that is, in English sentences!). There are few options for your note, depending on what you think of your classmate's proof:

(a) if the claim they are proving is true, and they have given a correct proof, your job is give congratulations and *make at least one suggestion about a very concrete way to make the argument more clear.*

(b) if the claim they are proving is true, and they have given an incorrect proof, your job is to write a polite note explaining (a) what is the error in the proof and (b) what is a correct proof.

(c) if the claim they are proving is false, your job is to write a polite note explaining (a) what is the error in the proof and (b) giving a counter-example to show that the statement is false.

Claim — Let A, B, and C be set. If $A \cap B = A \cap C$ then $B = C$.

Proof. We're proving an if-then statement, so we'll assume the "if" part and argue to get the "then" part. Assume that $A \cap B = A \cap C$. We will show that $B = C$ by first showing $B \subseteq C$ and then showing $C \subset B$. Let $x \in B$. There are two cases:

- If $x \in A$, then we have $x \in A \cap B$, and thus $x \in A \cap C$ since $A \cap B = A \cap C$. So $x \in C$.
- If $x \notin A$, then we have $x \notin A \cap B$ and thus $x \notin A \cap C$. But $x \notin A$, so we must have $x \in C$.

Notice that in both cases we get $x \in C$, so we have shown that $x \in B \to x \in C$; that is, $B \subseteq C$. We can argue that $C \subseteq B$ in exactly the same way so we've shown $B \subset C$ and $C \subseteq B$. Therefore $B = C$. □

Solution
Hey classmate,

Thank you for letting me check your proof! While reading through your proof, I caught this error:

- For the case when $x \notin A$, the conclusion that $x \in C$ cannot be made, as when $x \notin A \cap B \wedge x \notin A \cap C$, it is still possible for $x \notin C$. The same is true for when $x \in C \wedge x \notin A$. For instance, let $A = \{1, 2, 3\}, B = \{2, 3, 5\}, C = \{2, 3, 4\}$. Then, $A \cap B = \{2, 3\}$, $A \cap C = \{2, 3\}$ and thus $A \cap B = A \cap C$. If $x \in B \wedge x \notin A$, then x can be 5. However, 5 is not in C. This also is a counterexample for the claim.

To improve your proof and arrive at a correct argument, investigate the case when $x \in B \wedge x \notin A \wedge x \notin C$ and the case when $x \in C \wedge x \notin A \wedge x \notin B$.

Let me know if you have any other questions!

Cordially,
Denny Cao