MATH 22A: Vector Calculus and Linear Algebra

Lecture 5: Proof Methods — 15 September 2023

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§1 Proof Methods

Definition 1.1 (Theorem)**.** Statement true and verified if the statement $P \to Q$ is true, where P is the assumption and Q is the conclusion.

Notation 1.2 (Divides) For two integers a, b a is said to divide b if $\exists c \mid ac = b$. We write this as $a \mid b$.

Definition 1.3 (Direct Proof)**.** Proof by case by case consideration or by explicit demonstration why, with the assumption P, Q holds.

Definition 1.4 (Binomial Coefficient/Choose)**.** $\binom{n}{k} = \dfrac{n!}{(n-k)!k!}, 0 \leq k \leq n$.

> **Example 1.5**
>
> If $n \geq 2, n \in \mathbb{Z}$, then $n^2 = 2\binom{n}{2} + \binom{n}{1}$.

Proof.

$$\begin{aligned} 2\binom{n}{2} + \binom{n}{1} &= 2\left(\frac{n!}{(n-2)!2!}\right) + \frac{n!}{(n-1)!1!} \\ &= \frac{n!}{(n-2)!} + \frac{n!}{n-1}! \\ &= n(n-1) + n \\ &= n^2 - n + n \\ &= n^2 \end{aligned}$$

Thus, we have shown that the statement is true directly. □

> **Example 1.6**
>
> If $x^2 \in \mathbb{Z} \land x^2 \mid x \to x \in \{-1, 0, 1\}$.

Proof. $x^2 \mid x \to \exists c \in \mathbb{Z} \mid cx^2 = x$. We consider two cases:
Case 1: $x = 0$. If $x = 0$, then the statement that $x^2 \mid x$ is true.
Case 2: $x \neq 0$. Then, $c = \dfrac{x}{x^2} = \dfrac{1}{x} = c$. For c to be an integer, $x = -1 \lor x = 1$.

Thus, we have shown by direct proof that the statement is true. □

Definition 1.7 (Contrapositive Proof)**.** We are trying to prove $P \to Q$. We can also show that $\neg Q \to \neg P$, as they have the same truth value.

Definition 1.8 (Congruence Modulo n). $a \equiv b \mod (n) \rightarrow (a-b) \mid n$

Theorem 1.9

$a \equiv b \mod (n) \rightarrow a \mod (n) = b \mod (n)$

Example 1.10

Definition 1.11 (Proof by Contradiction). We are trying to prove that $P \rightarrow Q$. By contradiction, we assume $P \rightarrow \neg Q$ and reach a contradiction. Thus, the opposite must be true: $P \rightarrow Q$.

Example 1.12

Prove that $\sqrt[3]{2}$ is not a rational number.

Proof. Assume for purposes of contradiction that $\sqrt[3]{2}$ is a rational number, or that $\sqrt[3]{2} = p/q, p, q \in \mathbb{Z}$ such that p/q is in reduced form. $2 = \dfrac{p^3}{q^3} \rightarrow p^3 = 2q^3 \rightarrow p$ is an even number $\rightarrow p = 2k, k \in \mathbb{Z} \rightarrow p^3 = 8k^3 \rightarrow 8k^3 = 2q^3 \rightarrow q^3 = 4k^3 \rightarrow q$ is even $\rightarrow q = 4l$. Thus, $p/q = \frac{2k}{4l} = \frac{k}{2l}$. ※.
We reach a contradiction, as p/q is not in reduced form. Thus, we have shown by contradiction that $\sqrt[3]{2}$ is not a rational number. □

§2 Set Theory