MATH 22A: Vector Calculus and Linear Algebra

Eigenvalues and Eigenvectors — 1 November 2023

Pset Due: November 8, 2023 **Denny Cao**

Administrivia

- Practice Midterm: 4-5:30 PM at Science Center Lecture Hall B
- Review Session 7-9 PM at Jefferson 250 (Today)

§1 Review

Theorem 1.1

Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ and $\mathcal{C} = \{c_1, \ldots, c_n\}$ be bases of a vector space V. Then $\exists$ a unique $n \times n$ matrix $P \mid [x]_{\mathcal{C}} = [x]_{\mathcal{B}}$.

- P is the **change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$.**

$$P = [[b_1]_{\mathcal{C}}, [b_n]_{\mathcal{C}}]$$

Also P^{-1} is the change of coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

- Let V be a vector space w/basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ and W be a vector space with basis $\mathcal{C} = \{c_1, \ldots, c_n\}$. With transformation $T : V \to W$, we want an $m \times n$ matrix M s.t. $[T(x)]_{\mathcal{C}} = M[x]_{\mathcal{B}} \forall x \in V$. M is the **matrix of T relative to bases $\mathcal{B}$ and $\mathcal{C}$.**

$$M = [[T(b_1)]_{\mathcal{C}}, \ldots, [T(b_n)]_{\mathcal{C}}]$$

Theorem 1.2 (Diagonal Representation Theorem)

Suppose $A = PDP^{-1}$ where D is a diagonal matrix and $T(x) = Ax$. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed by columns of P, then D is the matrix of T relative to $\mathcal{B}$.

- If $A \wedge B$ are $n \times n$ matrices, then A is **similar** to B if $\exists$ an invertible matrix P s.t. $A = PBP^{-1}$.

Definition 1.3 (Eigenvalue, Eigenvector)**.** If A is an $n \times n$ matrix, then $\lambda \in \mathbb{R}$ is an **eigenvalue** of A if $\exists$ a non-zero vector v, called an **eigenvector** corresponding to λ, such that

$$Av = \lambda v$$

Example 1.4

Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Determine if v is an eigenvector of A and find the corresponding eigenvalue.

Solution. $A\boldsymbol{v} = \lambda \boldsymbol{v}$. In this case, $\lambda = -2$. □

Definition 1.5 (Eigenspace)**.** If λ is an eigenvalue of an $n \times n$ matrix A, then the set of all solutions v to $Av = \lambda v$ is the **eigenspace** of A corresponding to λ.

- Eigenspace is $\ker(A - \lambda I_n)$.

Example 1.6

Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$, then $\lambda = 2$ is an eignvalue. Find the corresponding eigenspace.

Solution.

- $\dim A = 1 \implies$ 1 vector
- $\dim A = 2 \implies$ 2 linearly independent eigenvectors
- $\dim A = 3 \implies$ 3 linearly independent eigenvectors

$$A\boldsymbol{v} - \lambda\boldsymbol{v} = \mathbf{0}$$
$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \boldsymbol{v} = \mathbf{0}$$

Thus, $\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. □

Theorem 1.7

The eigenvalues of a triangular matrix are the diagonal entries.

Proof. Eigenspace is $\ker(A - \lambda I_n)$. Let $\lambda = a_{ii}$, which is an entry in the diagonal. Then, $(A - a_{ii}I_n) = \mathbf{v} = \mathbf{0}$. All diagonal entries will be subtracted by a_{ii} and thus the entry at i'th column and i'th row will be 0. Thus, $\det(A - a_{ii}I) = 0$, as determinant of triangular matrix is product of diagonal. □

Theorem 1.8

If $v_1, \ldots, v_p$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_p$ of an $n \times n$ matrix A, then $v_1, \ldots, v_p$ are linearly independent.

Proof. □