MATH 22A: Vector Calculus and Linear Algebra

Problem Set 12

Due: Tuesday, December 5, 2023 12pm **Denny Cao**

Collaborators

§1 Computational Problems

Solution 1.1. We list the values of each polynomial as a vector in $\mathbb{R}^5$:

$$\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-2\\-1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}4\\1\\0\\1\\4\end{bmatrix}, \begin{bmatrix}-8\\-1\\0\\1\\8\end{bmatrix}, \begin{bmatrix}16\\1\\0\\1\\16\end{bmatrix}$$

We then use the Gram-Schmidt Process to obtain an orthogonal basis. Let p_i denote the i-th polynomial above. Let v_i denote the i-th vector in the orthogonal basis for $\mathbb{P}_4$:

$$p_0(t) = 1$$

$$v_0 = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$$

$$p_1(t) = t - \frac{\langle t, p_0\rangle}{\langle p_0, p_0\rangle} p_0(t) = t - 0p_0(t)$$

$$v_1 = \begin{bmatrix}-2\\-1\\0\\1\\2\end{bmatrix}$$

$$\begin{aligned} p_2(t) &= t^2 - \frac{\langle t^2, p_0\rangle}{\langle p_0, p_0\rangle} p_0(t) - \frac{\langle t^2, p_1\rangle}{\langle p_1, p_1\rangle} p_1(t) \\ &= t^2 - 2p_0(t) \end{aligned}$$

$$v_2 = \begin{bmatrix}2\\-1\\-2\\-1\\2\end{bmatrix}$$

$$\begin{aligned} p_3(t) &= t^3 - \frac{\langle t^3, p_0\rangle}{\langle p_0, p_0\rangle} p_0(t) - \frac{\langle t^3, p_1\rangle}{\langle p_1, p_1\rangle} p_1(t) - \frac{\langle t^3, p_2\rangle}{\langle p_2, p_2\rangle} p_2(t) \\ &= t^3 - \frac{17}{5} p_1(t) \end{aligned}$$

$$v_3 = \begin{bmatrix}-6/5\\12/5\\0\\17/5\\74/5\end{bmatrix}$$

$$\begin{aligned} p_4(t) &= t^4 - \frac{\langle t^4, p_0\rangle}{\langle p_0, p_0\rangle} p_0(t) - \frac{\langle t^4, p_1\rangle}{\langle p_1, p_1\rangle} p_1(t) - \frac{\langle t^4, p_2\rangle}{\langle p_2, p_2\rangle} p_2(t) - \frac{\langle t^4, p_3\rangle}{\langle p_3, p_3\rangle} p_3(t) \\ &= t^4 - \frac{36}{5} p_0(t) - 13p_2(t) \\ &= \begin{bmatrix}-216/5\\-96/5\\0\\-96/5\\-216/5\end{bmatrix} \end{aligned}$$

Solution 1.2. *Proof.* To show that $\langle u, v\rangle = T(u) \cdot T(v)$ defines an inner product on V, we show that, for all $u, v, w \in V$ and all scalars c:

- $\langle u, v\rangle = \langle v, u\rangle$:

$$\begin{aligned}\langle u, v\rangle &= T(u) \cdot T(v)\\ &= T(v) \cdot T(u)\\ &= \langle v, u\rangle\end{aligned}$$

- $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$:

$$\begin{aligned}\langle u+v, w\rangle &= (T(u) + T(v)) \cdot T(w)\\ &= T(u) \cdot T(w) + T(v) \cdot T(w)\\ &= \langle u, w\rangle + \langle v, w\rangle\end{aligned}$$

- $\langle cu, v, =\rangle c\langle u, v\rangle$:

$$\begin{aligned}\langle cu, v\rangle &= cT(u) \cdot T(v)\\ &= c(T(u) \cdot T(v))\\ &= c\langle u, v\rangle\end{aligned}$$

- $(\langle u, u\rangle \geq 0) \wedge (\langle u, u\rangle = 0 \iff u = 0)$:

$$\begin{aligned}&\langle u, u\rangle = T(u) \cdot T(u) \geq 0\\ &\langle u, u\rangle = T(u) \cdot T(u) = 0 \iff T(u) = 0 \iff u = 0\end{aligned}$$

As all axioms are satisfied, the proof is complete. $\square$