MATH 22A: Vector Calculus and Linear Algebra

Problem Set 3

Due: Wednesday, September 27, 2023 12pm **Denny Cao**

Collaborators

- May Ng
- Nina Khera
- Vivian Hui

§1 Computational Problems

Question 1.1. Write the solution set of the homogeneous system below in parametric vector form.

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 0 \\ x_1 + 4x_2 - 8x_3 &= 0 \\ -3x_1 - 7x_2 + 9x_3 &= 0 \end{aligned}$$

Solution
We can represent the system as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{array}\right]$$

$\sim R_2 - R_1 \to R_2, R_3 + 3R_1 \to R_3.$

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{array}\right]$$

$\sim R_3 - 2R_2 \to R_3.$

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\sim R_1 - 3R_2 \to R_1.$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the solution set of the homogeneous system in parametric form is: $\boldsymbol{x} = x_3 \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}.$

Question 1.2. Describe all solutions in parametric form to the equation $A\boldsymbol{x} = \mathbf{0}$ with A given below.

$$\begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix}$$

Solution
We can represent as an augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 0 \\ 2 & 6 & 0 & -8 & 0 \end{array}\right]$$

$\sim R_2 - 2R_1 \to R_2$.

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the solution set in parametric form is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_1 \begin{pmatrix} 0 \\ -3 \\ 0 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Question 1.3. Construct a 3×3 non-zero matrix A such that the vector below is a non-trivial solution to the equation $A\boldsymbol{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Solution
Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then, $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Multiplying A and $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ gives the matrix $\begin{bmatrix} a_{11} - 2a_{12} + a_{13} \\ a_{21} - 2a_{22} + a_{23} \\ a_{31} - 2a_{32} + a_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

With $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, the equation is true.

Question 1.4. Suppose that A is a 3×3 matrix and $\boldsymbol{y}$ is a vector in $\mathbb{R}^3$ such that the equation $A\boldsymbol{x} = \boldsymbol{y}$ does not have a solution. Is there a vector $\boldsymbol{v} \in \mathbb{R}^3$ such that the equation $A\boldsymbol{x} = \boldsymbol{v}$ has a unique solution? Make sure to explain your answer.

Solution

> **Claim** — There does not exist a vector $\boldsymbol{v} \in \mathbb{R}^3$ such that the equation $A\boldsymbol{x} = \boldsymbol{v}$ has a unique solution.

Proof by contradiction. As there exists a vector $\boldsymbol{y} \in \mathbb{R}^3$ such that $A\boldsymbol{x} = \boldsymbol{y}$ does not have a solution, by Theorem 2 (Uniqueness and Existence Theorem) in Linear Algebra and Its Applications, the right most column of the augmented matrix is a pivot column, meaning in A, there exists a zero row. Therefore, as A is a 3×3 matrix, there exists one column that is not a pivot column.

We will now show by contradiction that there does not exist a vector $\boldsymbol{v} \in \mathbb{R}^3$ such that the equation $A\boldsymbol{x} = \boldsymbol{v}$ has a unique solution. Assume for purposes of contradiction that the opposite is true—that there exists a vector $\boldsymbol{v} \in \mathbb{R}^3$ such that the equation $A\boldsymbol{x} = \boldsymbol{v}$ has a unique solution. If $A\boldsymbol{v}$ has a unique solution, then the reduced row echelon form of A has no free columns, as otherwise there would be free variables for the system which would make the system have infinitely many solutions. Thus, all columns of A must be pivot columns. ※

We reach a contradiction, as there exists a column that is not a pivot column if $A\boldsymbol{x} = \boldsymbol{y}$ does not have a unique solution. Thus, the opposite is true; there does not exist a vector $\boldsymbol{v} \in \mathbb{R}^3$ such that the equation $A\boldsymbol{x} = \boldsymbol{v}$ has a unique solution. $\square$

Question 1.5. Let A be an $m \times n$ matrix, and let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in $\mathbb{R}^3$ such that $A\boldsymbol{u} = 0$ and $A\boldsymbol{v} = 0$. Explain why $A(c\boldsymbol{u} + d\boldsymbol{v}) = 0$ for any pair of real numbers c and d.

Solution

Proof. $c(A\boldsymbol{u}) = c0 = 0$, as any scalar of the zero vector is the zero vector. The same is true for $d(A\boldsymbol{v}) = d0 = 0$. Thus, $c(A\boldsymbol{u}) + d(A\boldsymbol{v}) = 0 + 0 = 0$. By Theorem 5 b. in Linear Algebra and Its Applications, $c(A\boldsymbol{u}) = A(c\boldsymbol{u})$ and $d(A\boldsymbol{v}) = A(d\boldsymbol{v})$. Thus, $c(A\boldsymbol{u}) + d(A\boldsymbol{v}) = A(c\boldsymbol{u}) + A(d\boldsymbol{v}) = 0$. By Theorem 5 a. in Linear Algebra and Its Applications, $A(c\boldsymbol{u}) + A(d\boldsymbol{v}) = A(c\boldsymbol{u} + d\boldsymbol{v}) = 0$. $\square$

Question 1.6. Determine if the vectors below are linearly independent.

$$\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

Solution

Claim — The vectors are linearly independent.

Proof. To determine if the vectors are linearly independent, we can see if $A\boldsymbol{x} = 0$ has only the trivial solution, where the columns of A are the vectors.

$$\left[\begin{array}{ccc|c} 0 & 0 & -3 & 0 \\ 0 & 5 & 4 & 0 \\ 2 & -8 & 1 & 0 \end{array}\right]$$

$\sim R_3 \leftrightarrow R_1$.

$$\left[\begin{array}{ccc|c} 2 & -8 & 1 & 0 \\ 0 & 5 & 4 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

$\sim 1/2R_1 \to R_1, -1/3R_3 \to R_3, 1/5R_2 \to R_2$.

$$\left[\begin{array}{ccc|c} 1 & -4 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{4}{5} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\sim R_2 - 4/5R_3 \to R_2, R_1 - 1/2R_3 \to R_1$.

$$\left[\begin{array}{ccc|c} 1 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\sim 4R_2 + R_1 \to R_1$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus, the solution is $\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$—the trivial solution. Therefore, the vectors are linearly independent. □

Question 1.7. Determine if the columns of the matrix depicted below form a linearly independent set.

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$$

Solution

> **Claim —** The columns form a linearly independent set.

Proof. To determine if the columns of the matrix are linearly independent, we can see if $A\boldsymbol{x} = 0$ has only the trivial solution, where A is the matrix depicted.

$$\left[\begin{array}{ccc|c} -4 & -3 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \\ 5 & 4 & 6 & 0 \end{array}\right]$$

$\sim R_3 \leftrightarrow R_1$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ -4 & -3 & 0 & 0 \\ 5 & 4 & 6 & 0 \end{array}\right]$$

$\sim 4R_1 + R_3 \to R_3, R_4 - 5R_1 \to R_4$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & -3 & 12 & 0 \\ 0 & 4 & -9 & 0 \end{array}\right]$$

$-R_2 \to R_2, -1/4R_3 \to R_3$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 4 & -9 & 0 \end{array}\right]$$

$\sim R_2 - R_3 \to R_3$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & -9 & 0 \end{array}\right]$$

$\sim R_3 \leftrightarrow R_4$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 4 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\sim 4R_2 - R_3 \to R_3$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\sim 1/7R_3 \to R_3$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\sim R_1 - 3R_3 \to R_1, R_2 + 4R_3 \to R_2$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the solution is $\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$—the trivial solution. Therefore, the columns are linearly independent. □

Question 1.8. For what values of $\vec{h}$ is $\vec{v_3}$ depicted below in $\text{Span}\{\vec{v_1}, \vec{v_2}\}$, and for what values of h are the three vectors depicted below linearly independent?

$$\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$$

Solution

> **Claim** — For all real numbers h, $\vec{v_3}$ is in $\text{Span}\{\vec{v_1}, \vec{v_2}\}$.

Proof. A vector is in the span of two other vectors if it is a linear combination of the two. If there is a solution to the system $A\boldsymbol{x} = \vec{v_3}$, where the columns of A are $\vec{v_1}$ and $\vec{v_2}$, then $\vec{v_3}$ is in $\text{Span}\{\vec{v_1}, \vec{v_2}\}$.

$$\left[\begin{array}{cc|c} 2 & -6 & 8 \\ -4 & 7 & h \\ 1 & -3 & 4 \end{array}\right]$$

$\sim 1/2R_1 \to R_1$.

$$\left[\begin{array}{cc|c} 1 & -3 & 4 \\ -4 & 7 & h \\ 1 & -3 & 4 \end{array}\right]$$

$\sim R_3 - R_1 \to R_1$.

$$\left[\begin{array}{cc|c} 1 & -3 & 4 \\ -4 & 7 & h \\ 0 & 0 & 0 \end{array}\right]$$

$\sim R_2 + 4R_1 \to R_2$.

$$\left[\begin{array}{cc|c} 1 & -3 & 4 \\ 0 & -5 & h+16 \\ 0 & 0 & 0 \end{array}\right]$$

$\sim -1/5R_2 \to R_2$.

$$\left[\begin{array}{cc|c} 1 & -3 & 4 \\ 0 & 1 & \frac{-16-h}{5} \\ 0 & 0 & 0 \end{array}\right]$$

$\sim R_1 + 3R_2 \to R_1$.

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{-28-3h}{5} \\ 0 & 1 & \frac{-16-h}{5} \\ 0 & 0 & 0 \end{array}\right]$$

There is a non-trivial solution when $h \neq -16$ and when $h \neq -\frac{28}{3}$. As it is impossible for h to take on two different values, there is always a non-trivial solution for all real numbrers h. Thus, for all real numbers h, $\vec{v_3}$ is in Span$\{\vec{v_1}, \vec{v_2}\}$. $\square$

Claim — There are no values of h that will make the three vectors linearly independent.

Proof. Vectors are linearly independent if $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution, where the columns of A are the vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$.

$$\left[\begin{array}{ccc|c} 2 & -6 & 8 & 0 \\ -4 & 7 & h & 0 \\ 1 & -3 & 4 & 0 \end{array}\right]$$

$\sim 1/2R_1 \to R_1$.

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ -4 & 7 & h & 0 \\ 1 & -3 & 4 & 0 \end{array}\right]$$

$\sim R_3 - R_1 \to R_3$.

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ -4 & 7 & h & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\sim 4R_1 + R_2 \to R_2$.

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & -5 & h+16 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\sim -1/5R_2 \to R_2$.

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ 0 & 1 & -\frac{h+16}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\sim 3R_2 + R_1 \to R_1$.

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{-28-3h}{5} & 0 \\ 0 & 1 & -\frac{h+16}{6} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

As x_1 and x_2 are basic variables and x_3 is free, each non-zero value of x_3 determines a non-trivial solution of $A\boldsymbol{x} = \mathbf{0}$. Thus, for no values of h are $\vec{v_1}, \vec{v_2}$, and $\vec{v_3}$ linearly independent; they are linearly dependent for all h. □

Question 1.9. How many pivot columns must a 5×7 matrix have if its columns span $\mathbb{R}^5$?

Solution

Claim — A 5×7 matrix must have 5 pivot columns if its columns span $\mathbb{R}^5$.

Proof. By Theorem 4 in Linear Algebra and Its Applications, if the columns of a matrix A of size $m \times n$ spans $\mathbb{R}^m$, then it must also be the case that A has a pivot position in every row. With a 5×7 matrix, to span $\mathbb{R}^5$, it must be the case that A has a pivot position in all 5 rows (5 pivot columns). □

Question 1.10. For the matrix A below: Without doing any computation, first explain why the columns of A have to be linearly dependent, and then why the equation $A\vec{x} = \vec{0}$ has infinitely many solutions. Having done that, delete as few columns as possible from A to depict a matrix B for which the equation $B\vec{x} = \vec{0}$ has only the trivial solution.

$$A = \begin{bmatrix} 12 & 10 & -6 & -3 & 7 & 10 \\ -7 & -6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix}$$

Solution

The columns of A have to be linearly dependent, as by Theorem 8 in Linear Algebra and Its Applications; as there are more columns (more vectors) than there are rows (entries in each vector), the columns are linearly dependent.

The equation $A\vec{x} = \vec{0}$ has infinitely many solutions because there will be at least 1 free variable, as there is one more column than rows. This means that each non-zero value of the free variable will determine the solution to $A\vec{x} = \vec{0}$, and thus there are infinitely many solutions.

Using a reduced row echelon form calculator with the augmented matrix for $A\vec{x} = \vec{0}$, we obtain:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

To form a matrix B such that the only solution for $B\vec{x} = \vec{0}$ is the trivial solution, we must remove the columns that are not pivot columns in A, columns 3 and 5. Thus:

$$B = \begin{bmatrix} 12 & -6 & -3 & 10 \\ -7 & 4 & 7 & 5 \\ 9 & -9 & -5 & 1 \\ -4 & 1 & 6 & 9 \\ 8 & -5 & -9 & -8 \end{bmatrix}$$

as the augmented matrix for $B\vec{x} = \vec{0}$ can be reduced to

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which means the only solution of the system is the trivial solution.

§2 Proof Problems

Question 2.1. Prove that given any five vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5} \in \mathbb{R}^3$ there exist real numbers c_1, c_2, c_3, c_4, c_5 not all zero such that both

$$\begin{aligned} c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} + c_4\vec{v_4} + c_5\vec{v_5} &= \vec{0} \\ c_1 + c_2 + c_3 + c_4 + c_5 &= 0 \end{aligned}$$

are true.

Solution

Proof. By Theorem 8 in Linear Algebra and Its Applications, as there are more vectors than there are entries in each vector $(5 > 3)$, the vectors are linearly dependent. We can express the equation $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} + c_4\vec{v_4} + c_5\vec{v_5} = \vec{0}$ in the form $A\boldsymbol{c} = \mathbf{0}$, where the columns of A are the vectors v_1, v_2, v_3, v_4, and v_5, and $\boldsymbol{c}$ is the vector $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix}$. Let v_{ij} denote the ith entry of the jth vector. Then, we can represent $A\boldsymbol{c} = \mathbf{0}$ can be expressed as an augmented matrix:

$$\left[\begin{array}{ccccc|c} v_{11} & v_{12} & v_{13} & v_{14} & v_{15} & 0 \\ v_{21} & v_{22} & v_{23} & v_{24} & v_{25} & 0 \\ v_{31} & v_{32} & v_{33} & v_{34} & v_{35} & 0 \end{array}\right]$$

The equation $c_1 + c_2 + c_3 + c_4 + c_5 = 0$ can be added as a row to the augmented matrix, where $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

$$\left[\begin{array}{ccccc|c} v_{11} & v_{12} & v_{13} & v_{14} & v_{15} & 0 \\ v_{21} & v_{22} & v_{23} & v_{24} & v_{25} & 0 \\ v_{31} & v_{32} & v_{33} & v_{34} & v_{35} & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

As there are only rows for five columns, by Theorem 8 of Linear Algebra and Its Applications, the set of vectors are linearly dependent, and thus there exists a non-trivial solution for $\boldsymbol{c}$ for the system. Thus, for any five vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5} \in \mathbb{R}^3$ there exist real numbers c_1, c_2, c_3, c_4, c_5 not all zero such that both equations are true. □

Question 2.2. Let A be a 3×4 matrix and $\vec{b_1}, \vec{b_2}$ be two vectors in $\mathbb{R}^3$ such that both matrix equations $A\vec{x} = \vec{b}_1$ and $A\vec{y} = \vec{b}_2$ are consistent. Prove that there exists a vector $\vec{p} \in \mathbb{R}^4$ such that the set of solutions $\vec{y}$ to the second equation is the set of all vectors of the form $\vec{x} + \vec{p}$ where $\vec{x}$ is any solution of the first equation.

Solution

Proof. Let $\vec{x'}$ be an element in the set of solutions $\vec{x}$ and $\vec{y'}$ be an element in the set of solutions $\vec{y}$. Let $\vec{p} = \vec{y'} - \vec{x'}$. Then, $A\vec{p} = A(\vec{y'} - \vec{x'}) = A\vec{y'} - A\vec{x'}$. As $A\vec{y} = \vec{b_2}$ and $A\vec{x} = \vec{b_1}$, A multiplied by all elements in the set of solutions $\vec{y}$ will be $\vec{b_2}$ and A multiplied by all elements in the set of solutions $\vec{x}$ will be $\vec{b_1}$. Thus, $\vec{y'} = \vec{b_2}$ and $\vec{x'} = \vec{b_1}$.

It follows that $A\vec{p} = A\vec{y'} - A\vec{x'} = \vec{b_2} - \vec{b_1}$. Thus, $A\vec{p} + A\vec{x} = \vec{b_2} - \vec{b_1} + \vec{b_1} = \vec{b_2} = A\vec{y}$. We can rewrite as: $A(\vec{p} + \vec{x}) = A\vec{y}$. We can multiply both sides by the inverse of A: $A^{-1}A(\vec{p} + \vec{x}) = A^{-1}A\vec{y}$. By definition of inverse, $A^{-1}A = I$. By definition of identity matrix, $I(\vec{p} + \vec{x}) = \vec{p} + \vec{x}$ and $I\vec{y} = \vec{y}$. Thus, $\vec{p} + \vec{x} = \vec{y}$.

We reach the conclusion that, for all elements $\vec{a}$ in the set of solutions $\vec{y}$, there exists, for all elements $\vec{b}$ in the set of solutions $\vec{x}$, a vector p that is the difference between a solution of $\vec{y}$ and a solution of $\vec{x}$ such that $\vec{b} + \vec{p} = \vec{a}$. □

Question 2.3 (Injections, Surjections, Bijections)**.** For each of the following functions, determine *(with proof)* whether it is injective and/or surjective. If it is bijective, write down its inverse function.

(a) Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by $f(n) = 2n + 1$

(b) Let $f : \mathbb{R} \backslash \{1\} \to \mathbb{R} \backslash \{1\}$ be given by $f(x) = \left(\dfrac{x+1}{x-1}\right)^3$

(c) Let $f : \mathbb{Z} \to \mathbb{Z}^2$ be given by $f(k) = (2k, k + 3)$

Solution

(a)

Claim — f is injective.

Proof by contrapositive. Suppose $x_1, x_2 \in \mathbb{Z}$ and $f(x_1) = f(x_2)$. Thus:

$$\begin{aligned} 2x_1 + 1 &= 2x_2 + 1 \\ 2x_1 &= 2x_2 \\ x_1 &= x_2 \end{aligned}$$

As we have shown that $f(x_1) = f(x_2)$ if and only if $x_1 = x_2$, we have shown that f is injective. □

Claim — f is not surjective.

Proof. Suppose y is an element in the codomain of f. Thus, $y = 2n + 1$. The value in the domain that maps to y is the value $\frac{y-1}{2}$. Let $y = 2$. Thus, the value in the domain must be $\frac{1}{2}$.

However, $\frac{1}{2} \notin \mathbb{Z}$. As we have shown that there exists a value in the codomain y such that there does not exist a value in the domain x such that $f(x) = y$, we have shown that f is not surjective. □

Claim — f is injective.

(b) *Proof by contrapositive.* Suppose $x_1, x_2 \in \mathbb{R} \setminus \{1\}$ and $f(x_1) = f(x_2)$. Thus:

$$\begin{aligned} \left(\frac{x_1+1}{x_1-1}\right)^3 &= \left(\frac{x_2+1}{x_2-1}\right)^3 \\ \frac{x_1+1}{x_1-1} &= \frac{x_2+1}{x_2-1} \\ (x_1-1)(x_2+1) &= (x_1+1)(x_2-1), x_1 \neq 1, x_2 \neq 1 \\ x_1x_2 + x_1 - x_2 - 1 &= x_1x_2 - x_1 + x_2 - 1, x_1 \neq 1, x_2 \neq 1 \\ 2x_1 &= 2x_2, x_1 \neq 1, x_2 \neq 1 \\ x_1 &= x_2, x_1 \neq 1, x_2 \neq 1 \end{aligned}$$

As $x = 1$ is not in the domain of f, we reach the conclusion that $x_1 = x_2$ on the domain of f. As we have shown that $f(x_1) = f(x_2)$ if and only if $x_1 = x_2$, we have shown that f is injective. □

Claim — f is surjective.

Proof. Suppose y is an element in the codomain of f. We will show that there exists an x in the domain of f such that $f(x) = y$.

$$\begin{aligned} y &= \left(\frac{x+1}{x-1}\right)^3 \\ \sqrt[3]{y} &= \frac{x+1}{x-1} \\ (x-1)\sqrt[3]{y} &= x + 1, x \neq 1 \end{aligned}$$

$$
\begin{aligned}
x\sqrt[3]{y} - \sqrt[3]{y} &= x + 1, x \neq 1 \\
x\sqrt[3]{y} - x &= 1 + \sqrt[3]{y}, x \neq 1 \\
x(\sqrt[3]{y} - 1) &= 1 + \sqrt[3]{y}, x \neq 1 \\
x &= \frac{\sqrt[3]{y} + 1}{\sqrt[3]{y} - 1}, x \neq 1, y \neq 1
\end{aligned}
$$

As $y = 1$ is not in the codomain, the case when $y \neq 1$ is not considered. When $x = 1$, $\sqrt[3]{y} - 1 = \sqrt[3]{y} + 1 \rightarrow -1 = 1$ which is false, and thus it is never the case that, for any y, $x = 1$. Thus, the case when $x = 1$ is not considered. Thus, for every y in the codomain of f, there exists a x in the domain, $x = \dfrac{\sqrt[3]{y}+1}{\sqrt[3]{y}-1}$, such that $f(x) = y$. Thus, f is surjective. □

As f is injective and surjective, f is bijective. We can find the inverse of f :

$$
y = \left(\frac{x+1}{x-1}\right)^3
$$

Swap x and y to switch domain and range.

$$
\begin{aligned}
x &= \left(\frac{y+1}{y-1}\right)^3 \\
\sqrt[3]{x} &= \left(\frac{y+1}{y-1}\right) \\
(y-1)\sqrt[3]{x} &= y + 1 \\
y\sqrt[3]{x} - \sqrt[3]{x} &= y + 1 \\
y\sqrt[3]{x} - y &= 1 + \sqrt[3]{x} \\
y(\sqrt[3]{x} - 1) &= 1 + \sqrt[3]{x} \\
y &= \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}
\end{aligned}
$$

Thus, the inverse function is $f^{-1}(x) = \dfrac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}$.

(c)

Claim — f is injective.

Proof by contrapositive. Suppose $x_1, x_2 \in \mathbb{Z}$ and $f(x_1) = f(x_2)$. Thus:

$$
(2x_1, x_1 + 3) = (2x_2, x_2 + 3)
$$

It follows that:

$$
\begin{aligned}
2x_1 &= 2x_2 & \qquad x_1 + 3 &= x_2 + 3 \\
x_1 &= x_2 & \qquad x_1 &= x_2
\end{aligned}
$$

As we have shown that $f(x_1) = f(x_2)$ if and only if $x_1 = x_2$, we have shown that f is injective. □

> **Claim —** f is not surjective.

Proof. Let y be an element in the codomain of f. We will show that there exists a y such that there does not exist a x in the domain such that $f(x) = y$.

As $y \in \mathbb{Z}^2$, y can be $(1, 2)$. However, f is defined as $f(x) = (2x, x + 3)$, meaning $1 = 2x$ and $2 = x + 3$. For the first element in the ordered pair, $x = \frac{1}{2}$. However, this will give $\frac{7}{2}$ as the second element in the ordered pair instead of 2. $x = \frac{1}{2}$ is also not in $\mathbb{Z}$, and thus the first element being 1 is not possible as well.

We have shown that there exists a y in the codomain of f such that there does not exist a x in the domain such that $f(x) = y$. Thus, f is not surjective. $\square$

Question 2.4 (Function Warm-Up)**.** Let $f : S \to T$ and $g : T \to U$. For each of the following statements, either prove the statement or provide a very concrete and simple counterexample to show that it's false.

(a) If f and g are surjections, then $g \circ f$ is a surjection.

(b) If $g \circ f$ is injective, then f must be injective.

(c) If $g \circ f$ is injective, then g must be injective.

Solution

(a)

> **Claim —** The statement that, if f and g are surjections, then $g \circ f$ is a surjection is **true**.

Proof. As g is surjective, for all $u \in U$ there exists a corresponding $t \in T$ such that $g(t) = u$. As f is surjective, for all $t \in T$ there exists a corresponding $s \in S$ such that $f(s) = t$. Therefore, $g(f(s)) = g(t) = u$, meaning $g \circ f(s) = u$. We have shown that $g \circ f$ is a surjection, as we have shown that, for all values u in the range of $g \circ f$, U, there exists a corresponding value t in the domain, T, such that $g(f(t)) = u$. $\square$

(b)

> **Claim —** The statement that, if $g \circ f$ is injective, then f must be injective is **true**.

Proof by contrapositive. The contrapositive of the statement is, if f is not injective, then $g \circ f$ is not injective. If f is not injective, then there exists $x_1, x_2 \in S$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Thus, $g(f(x_1)) = g(f(x_2))$, as the same value is being inputted into g.

$g \circ f$ is not injective, as we have shown that there exists $x_1, x_2 \in S$ such that $x_1 \neq x_2$ and $g(f(x_1)) = g(f(x_2))$. As this statement is equivalent to the original, it is true that, if $g \circ f$ is injective, then f must be injective. $\square$

(c)

Claim — The statement that, if $g \circ f$ is injective, then g must be injective is **false**.

Proof by counterexample. Let $S = \{0\}, T = \{1, 2\}, U = \{3\}$. Let $f : S \to T$ such that $f(0) = 1$, and $g : T \to U$ such that $g(1) = 3$ and $g(2) = 3$.

Then, $g \circ f$ is injective, as 0 is the only input and it maps to 3, and thus for all x_1, x_2 in the domain of $g \circ f$—S—$x_1 \neq x_2 \to f(x_1) \neq f(x_2)$. However, g is not injective, as there exists x_1, x_2 in the domain of g—T—such that $x_1 \neq x_2$ and $g(x_1) = g(x_2)$, as $g(1) = g(2) = 3$. □

Question 2.5 (Equivalence Relations)**.** For each of the following relations on the real numbers $\mathbb{R}$: If the relation is an equivalence relation, prove that it is by verifying the three axioms. Also give a complete (non-repetitive) list of the equivalence classes. If the relation is not an equivalence relation, identify by giving a counterexample which of the three axioms are not satisfied.

(a) $\sim_e$ is the relation on $\mathbb{R}$ such that $\forall x, y \in \mathbb{R}$, $x \sim_e y \iff x - y$ is even.

(b) $\sim_d$ is the relation on $\mathbb{R}$ such that $\forall x, y \in \mathbb{R}$, $x \sim_d y \iff (x - y)^2 < 9$.

(c) $\sim_l$ is the relation on $\mathbb{R}^2$ such that $\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, $(x_1, y_1) \sim_l (x_2, y_2) \iff x_2 - x_1 = 3(y_2 - y_1)$.

Solution

(a)

Claim — The relation $\sim_e$ on $\mathbb{R}$ is an equivalence relation.

Proof. To show that $\sim_e$ is an equivalence relation, we must show that $\sim_e$ is reflexive, symmetric, and transitive.

Claim — The relation $\sim_e$ on $\mathbb{R}$ is reflexive.

Subproof. We must show that, for all x, $x \sim_e x$, or $x - x$ is even.

As $x - x = 0$ and $2(0) = 0$, there exists a $c \in \mathbb{Z}$ such that $2c = 0$. Thus, by definition of even number, $x - x = 0$ is even. Therefore, $x \sim_e x$ which means the relation $\sim_e$ is reflexive. ■

Claim — The relation $\sim_e$ on $\mathbb{R}$ is symmetric.

Subproof. We must show that, for all $x, y \in \mathbb{R}$, $x \sim_e y \leftrightarrow y \sim_e x$, or $x - y$ is even $\leftrightarrow y - x$ is even.

We will first show that $x \sim_e y \to y \sim_e x$. If $x - y$ is even, then the difference of x and y can be represented by $2c, c \in \mathbb{Z}$. Thus, $x - y = 2c$. We can multiply both sides by -1 to get the equation $y - x = -2c$. Let $k = -c$. Thus, as there exists a $k \in \mathbb{Z}$ such that $y - x = 2k$, by definition of even number, $y - x$ is even and thus $y \sim_e x$.

The same proof can be used to show $y \sim_e x \to x \sim_e y$. If $y - x$ is even, then the difference of y and x can be represented by $2c, c \in \mathbb{Z}$. Thus, $y - x = 2c$. We can multiply both sides by -1 to get the equation $x - y = -2c$. Let $k = -c$. Thus, as there exists a $k \in \mathbb{Z}$ such that $x - y = 2k$, by definition of even number, $x - y$ is even and thus $x \sim_e y$. Thus, the relation $\sim_e$ on $\mathbb{R}$ is symmetric. ■

> **Claim —** The relation $\sim_e$ on $\mathbb{R}$ is transitive.

Subproof. We must show that, $\forall a, b, c \in \mathbb{R}(a \sim_e b \land b \sim_e c \to a \sim_e c)$, or if $a - b$ is even and $b - c$ is even, then $a - c$ is even.

If $a - b$ is even, then the difference of a and b can be represented by $2k, k \in \mathbb{Z}$. If $b - c$ is even, then the difference of b and c can be represented by $2d, d \in \mathbb{Z}$. Thus, $a - b = 2k$ and $b - c = 2d$. It follows that $b = a - 2k$. We substitute this value for b to get the equation $a - 2k - c = 2d$. Thus, $a - c = 2d + 2k = 2(d + k)$. Let $d + k = t$. Then, $a - c = 2t$. As the difference of a and c can be represented as $2t, t \in \mathbb{Z}$, by definition of even number, $a - c$ is even, and thus $a \sim_e c$. Thus, the relation $\sim_e$ on $\mathbb{R}$ is transitive. ■

As the relation $\sim_e$ is reflexive, symmetric, and transitive, it is an equivalence relation. □

The equivalence classes are:

- $\{2k \mid k \in \mathbb{Z}\}$
- $\{2k + 1 \mid k \in \mathbb{Z}\}$

(b)

> **Claim —** The relation $\sim_d$ on $\mathbb{R}$ is not an equivalence relation.

Proof by counterexample. To prove that a relation is not an equivalence relation, we show that it is not reflexive, symmetric, and transitive; it is sufficient to show that one property does not hold. We will show that $\sim_d$ is not transitive, or that there exists $a, b, c \in \mathbb{R}$ such that $a \sim_d b \land b \sim_d c \land a \not\sim_d c$.

Let $a = 1, b = 2, c = 4$. Then, $a \sim_d b$, as $(1 - 2)^2 = 1$, which is less than 9. $b \sim_d c$, as $(2 - 4)^2 = 4$, which is less than 9. However, $a \not\sim_d c$, as $(1 - 4)^2 = 9$, which is not less than 9. Thus, $\sim_d$ is not transitive. As $\sim_d$ is not transitive, it is not an equivalence relation. □

> **Claim —** The relation $\sim_l$ on $\mathbb{R}^2$ is an equivalence relation.

(c) *Proof.* To show that $\sim_e$ is an equivalence relation, we must show that $\sim_e$ is reflexive, symmetric, and transitive.

> **Claim —** The relation $\sim_l$ on $\mathbb{R}^2$ is reflexive.

Subproof. We must show that, for all $(x_1, y_1) \in \mathbb{R}^2$, $(x_1, y_1) \sim_l (x_1 y_1)$, or that $x_1 - x_1 = 3(y_1 - y_1)$. We can simplify to $0 = 0$, which is true. Therefore, $x \sim_l x$ which means that the relation $\sim_l$ is reflexive. ■

Claim — The relation $\sim_l$ on $\mathbb{R}^2$ is symmetric.

Subproof. We must show that, for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, $(x_1, y_1) \sim_l (x_2, y_2) \leftrightarrow (x_2, y_2) \sim_l (x_1, y_1)$, or that $x_2 - x_1 = 3(y_2 - y_1) \to x_1 - x_2 = 3(y_1 - y_2)$.

We will first show that $(x_1, y_1) \sim_l (x_2, y_2) \to (x_2, y_2) \sim_l (x_1, y_1)$. If $x_2 - x_1 = 3(y_2 - y_1)$, then we can multiply both sides by -1, obtaining the equation $x_1 - x_2 = 3(y_1 - y_2)$. Thus, $(x_2, y_2) \sim_l (x_1, y_1)$.

The same proof can be used to show $(x_2, y_2) \sim_l (x_1, y_1) \to (x_1, y_1) \sim_l (x_2, y_2)$. If $x_1 - x_2 = 3(y_1 - y_2)$, then we can multiply both sides of the equation by -1 to obtain $x_2 - x_1 = 3(y_2 - y_1)$. Thus, $(x_1, y_1) \sim_l (x_2, y_2)$. Thus, the relation $\sim_l$ on $\mathbb{R}^2$ is symmetric. ■

Claim — The relation $\sim_l$ on $\mathbb{R}^2$ is transitive.

Subproof. We must show that, $\forall x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}^2$, $(x_1, y_1) \sim_l (x_2, y_2) \wedge (x_2, y_2) \sim_l (x_3, y_3) \to (x_1, y_1) \sim_l (x_3, y_3)$, or if $x_2 - x_1 = 3(y_2 - y_1)$ and $x_3 - x_2 = 3(y_3 - y_2)$, then $x_3 - x_1 = 3(y_3 - x_1)$.

$$\begin{aligned} x_3 - x_2 &= 3(y_3 - y_2) \\ x_2 &= x_3 - 3(y_3 - y_2) \end{aligned}$$

We substitute this in for x_2 in $x_2 - x_1 = 3(y_2 - y_1)$:

$$\begin{aligned} x_3 - 3(y_3 - y_2) - x_1 &= 3(y_2 - y_1) \\ x_3 - 3y_3 + 3y_2 - x_1 &= 3y_2 - 3y_1 \\ x_3 - x_1 &= 3y_3 - 3y_1 \\ x_3 - x_1 &= 3(y_3 - y_1) \end{aligned}$$

Thus, $(x_1, y_1) \sim_l (x_3, y_3)$. Thus, the relation $\sim_l$ on $\mathbb{R}^2$ is transitive. ■

As the relation $\sim_l$ is reflexive, symmetric, and transitive, it is an equivalence relation. □

The equivalence class is:

- $\{(x, 3x) \mid x \in \mathbb{R}\}$