MATH 22A: Vector Calculus and Linear Algebra

Problem Set 11

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Collaborators

§1 Computational Problems

Solution 1.1.

(a) Let $u = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}, v = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$. Then:

$$\begin{aligned} ||u|| &= \sqrt{u \cdot u} \\ &= \sqrt{(-5)^2 + 2^2} \\ &= \sqrt{29} \end{aligned} \qquad \begin{aligned} ||v|| &= \sqrt{v \cdot v} \\ &= \sqrt{(-4)^2 + (-1)^2 + 8^2} \\ &= 9 \end{aligned}$$

$$\begin{aligned} u \cdot v &= 0(-5) + (-5)(-1) + 2(8) \\ &= 21 \end{aligned}$$

$$\begin{aligned} \text{dist}(u, v) &= ||u - v|| \\ &= \sqrt{4^2 + (-4)^2 + (-6)^2} \\ &= 2\sqrt{17} \end{aligned}$$

As $u \cdot v \neq 0$, the vectors are not orthogonal.

(b) Let $u = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, v = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$. Then:

$$\begin{aligned} ||u|| &= \sqrt{u \cdot u} \\ &= \sqrt{12^2 + 3^2 + (-5)^2} \\ &= \sqrt{178} \end{aligned} \qquad \begin{aligned} ||v|| &= \sqrt{v \cdot v} \\ &= \sqrt{2^2 + (-3)^2 + 3^2} \\ &= \sqrt{22} \end{aligned}$$

$$\begin{aligned} u \cdot v &= 12(2) + 3(-3) + (-5)(3) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{dist}(u, v) &= ||u - v|| \\ &= \sqrt{10^2 + 6^2 + (-8)^2} \\ &= 10\sqrt{2} \end{aligned}$$

As $u \cdot v = 0$, the vectors are orthogonal.

(c) Let $u = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$. Then:

$$\begin{aligned} ||u|| &= \sqrt{u \cdot u} \\ &= \sqrt{(-3)^2 + 7^2 + 4^2} \\ &= \sqrt{74} \end{aligned} \qquad \begin{aligned} ||v|| &= \sqrt{v \cdot v} \\ &= \sqrt{1^2 + (-8)^2 + 15^2 + (-7)^2} \\ &= \sqrt{339} \end{aligned}$$

$$\begin{aligned} u \cdot v &= -3(1) + 7(-8) + 4(15) \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{dist}(u, v) &= ||u - v|| \\ &= \sqrt{(-4)^2 + 15^2 + (-11)^2 + 7^2} \\ &= \sqrt{411} \end{aligned}$$

As $u \cdot v \neq 1$, the vectors are not orthogonal.

Solution 1.2. A unit vector u in the same direction as the vector $v = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$ can be found by:

$$\begin{aligned} u = \frac{1}{||v||} v &= \frac{1}{\sqrt{(-6)^2 + 4^2 + (-3)^2}} \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix} \\ &= \frac{\sqrt{61}}{61} \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix} \end{aligned}$$

Solution 1.3. To show that a set is an orthogonal set, we must show that all distinct pairs in the set are orthogonal.

$$\begin{aligned} u_1 \cdot u_2 &= 3(2) + (-3)2 = 0 \\ u_1 \cdot u_3 &= 3(1) = 3(1) = 0 \\ u_2 \cdot u_3 &= 2(1) + 2(1)(-1)(4) = 0 \end{aligned}$$

Thus, $\{u_1, u_2, u_3\}$ is an orthogonal set.

We can find x with respect to this basis from Theorem 5 by the following:

$$x = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{x \cdot u_3}{u_3 \cdot u_3} u_3, [x]_{\mathcal{B}} = \begin{bmatrix} \frac{x \cdot u_1}{u_1 \cdot u_1} \\ \frac{x \cdot u_2}{u_2 \cdot u_2} \\ \frac{x \cdot u_3}{u_3 \cdot u_3} \end{bmatrix}$$

$$\begin{aligned} x \cdot u_1 &= 5(3) - 3(-3) \\ &= 24 \\ u_1 \cdot u_1 &= 3(3) - 3(-3) \end{aligned} \qquad \begin{aligned} x \cdot u_2 &= 5(2) - 3(2) + 1(-1) \\ &= 3 \\ u_2 \cdot u_2 &= 2(2) + 2(2) - 1(-1) \end{aligned} \qquad \begin{aligned} x \cdot u_3 &= 5(1) - 3(1) + 1(4) \\ &= 6 \\ u_3 \cdot u_3 &= 1(1) + 1(1) + 4(4) \end{aligned}$$

$$= 18 \qquad\qquad\qquad\qquad = 9 \qquad\qquad\qquad\qquad = 18$$

$$c_1 = \frac{24}{18} = \frac{4}{3} \quad c_2 = \frac{3}{9} = \frac{1}{3} \quad c_3 = \frac{6}{18} = \frac{1}{3}$$

Thus, x with respect to this basis is $\begin{bmatrix} 4/3 \\ 1/3 \\ 1/3 \end{bmatrix}$.

Solution 1.4. To show that a set is an orthonormal set, we must show that all distinct pairs in the set are orthogonal and are unit vectors. Let $u_1 = \begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}, u_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$.
As $u_1 \cdot u_2 = 1/\sqrt{36} - 1/\sqrt{36} = 0$, S is an orthogonal set.

We will now show that each vector in the set is a unit vector.

$$\begin{aligned} u_1 \cdot u_1 &= 1/18 + 16/18 + 1/18 = 1 \\ u_2 \cdot u_2 &= 1/2 + 1/2 = 1 \end{aligned}$$

As u_1, u_2 are also unit vectors, S is an orthonormal set.

From Theorem 10, $\text{proj}_S x$ can be obtained by:

$$\text{proj}_S x = (x \cdot u_1)u_1 + (x \cdot u_2)u_2$$

$$\begin{aligned} x.u_1 &= 8/\sqrt{18} - 16/\sqrt{18} - 3/\sqrt{18} = -11/\sqrt{18} \\ x.u_2 &= 8/\sqrt{2} + 3/\sqrt{2} = 11/\sqrt{2} \end{aligned}$$

Thus:

$$\begin{aligned} \text{proj}_S x &= -11/\sqrt{18} \begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix} + 11/\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} -11/18 \\ -44/18 \\ -11/18 \end{bmatrix} + \begin{bmatrix} 11/2 \\ 0 \\ -11/2 \end{bmatrix} \\ &= \begin{bmatrix} 44/9 \\ -22/9 \\ -121/9 \end{bmatrix} \end{aligned}$$

Solution 1.5. *Proof.* Let x, y be vectors in $\mathbb{R}^n$. Then, by Theorem 8, $x = \hat{x} + z_x$ and $y = \hat{y} + z_y$, where $\hat{x}, \hat{y} \in W$ and is the orthogonal projection of x, y respectively onto W, and $z_x, z_y \in W^\perp$. Let T be the orthogonal projection map from $\mathbb{R}^n$ to W. We will show that T is a linear transformation by showing that it is closed under vector addition and scalar multiplication.

Then, as W and $W^\perp$ are subspaces of $\mathbb{R}^n$, they are closed under vector addition and thus $\hat{x}+\hat{y} \in W$ and $z_x+z_y \in W^\perp$. As the orthogonal decomposition of $x+y = (\hat{x}+\hat{y})+(z_x+z_y)$:

$$T(x+y) = (x \hat{+} y) = \hat{x} + \hat{y} = T(x) + T(y)$$

and thus the transformation is closed under vector addition.

As W and $W^{\perp}$ are subspaces of $\mathbb{R}^n$, they are closed under scalar multiplication and thus $c\hat{x}, c\hat{y} \in W$ and $cz_x, cz_y \in W^{\perp}$. As the orthogonal decomposition of $cx = c(\hat{x}) + c(z_x)$:

$$T(cx) = \hat{(cx)} = c\hat{x} = cT(x)$$

and thus the transformation is closed under scalar multiplication.

As T is closed under vector addition and scalar multiplication, T is a linear transformation, and the proof is complete. $\square$

From Theorem 10, the orthogonal projection map from $\mathbb{R}^n$ onto W is given by $T(x) = \text{proj}_W x = UU^T x$, where $U = \begin{bmatrix} e_1 & e_2 & \cdots & e_p \end{bmatrix}$. By Theorem 6, U has orthonormal columns if and only if $U^T U = I$, and thus the standard matrix for the orthogonal projection map from $\mathbb{R}^n$ onto W is I_p.

Solution 1.6. We can find y as a sum of a vector in W by the following:

$$y = \hat{y} + z$$

where $\hat{y} \in W$ and is the projection of y onto W and $z \in W^{\perp}$, where $z = y - \hat{y}$ which is orthogonal to W.

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{y \cdot u_3}{u_3 \cdot u_3} u_3$$

$$\begin{aligned} y \cdot u_1 &= 3 + 4 - 6 \\ &= 1 \\ u_1 \cdot u_1 &= 1 + 1 + 1 \\ &= 3 \end{aligned} \qquad \begin{aligned} y \cdot u_2 &= 3 + 5 + 6 \\ &= 14 \\ u_2 \cdot u_2 &= 1 + 1 + 1 \\ &= 3 \end{aligned} \qquad \begin{aligned} y \cdot u_3 &= -4 + 5 - 6 \\ &= -5 \\ u_3 \cdot u_3 &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

$$\hat{y} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \frac{14}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{3} \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

As $z = y - \hat{y}$,

$$z = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

Thus, y as a sum of a vector in W and a vector orthogonal to W is:

$$y = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

Solution 1.7. The closest point to $\text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} \right\}$ is:

$$\hat{z} = \frac{z \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{z \cdot v_2}{v_2 \cdot v_2} v_2$$

$$z \cdot v_1 = 7 \qquad\qquad z \cdot v_2 = 0$$
$$v_1 \cdot v_1 = 14 \qquad\qquad v_2 \cdot v_2 = 49$$

Thus, the closest point to Span$\{v_1, v_2\}$ to z is:

$$\hat{z} = \frac{1}{2} v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -3/2 \end{bmatrix}$$

Solution 1.8. If $u_3 \in \text{Span}\{u_1, u_2\}$, then there exists c_1, c_2 such that

$$u_3 = c_1 u_1 + c_2 u_2$$

We expand to

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$$

We set up the following system:

$$\begin{aligned} c_1 + 5c_2 &= 0 \\ c_1 - c_2 &= 1 \\ -2c_1 + 2c_2 &= 0 \end{aligned}$$

We multiply the second equation by -2 to obtain $-2c_1 + 2c_2 = -2$, which contradicts the third equation that $-2c_1 + 2c_3 = 0$, and thus there does not exist weights c_1, c_2 such that the equation is satisfied, and thus u_3 is not in the span of u_1 and u_2.

A vector v orthogonal to Span$\{u_1, u_2\}$ can be found by $u_3 - \hat{u_3}$, where $\hat{u_3}$ is the vector in Span$\{u_1, u_2\}$ closest to u_3, which can be found by:

$$\hat{u_3} = \frac{u_3 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{u_3 \cdot u_2}{u_2 \cdot u_2} u_2$$

$$u_3 \cdot u_1 = 1 \qquad\qquad u_3 \cdot u_2 = -1$$
$$u_1 \cdot u_1 = 6 \qquad\qquad u_2 \cdot u_2 = 30$$

Thus,

$$\begin{aligned} \hat{h_3} &= \frac{1}{6} u_1 + -\frac{1}{30} u_2 \\ &= \begin{bmatrix} 1/6 \\ 1/6 \\ -1/3 \end{bmatrix} + \begin{bmatrix} -1/6 \\ 1/30 \\ -1/15 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1/5 \\ -6/15 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} v &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1/5 \\ -6/15 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 4/5 \\ 6/15 \end{bmatrix} \end{aligned}$$

Solution 1.9. Let $x_1 = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, x_2 = \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix}$. By the Gram-Schmidt process, we define:

$$\begin{aligned}
v_1 &= \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} \\
v_2 &= \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 \\
&= \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}
\end{aligned}$$

Thus, an orthogonal basis for W is

$$\left\{ \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} \right\}$$

To form an orthonormal basis, we normalize the vectors:

$$\begin{aligned}
u_1 &= \frac{1}{||v_1||} v_1 & u_2 &= \frac{1}{||v_2||} v_2 \\
u_1 &= \frac{1}{5\sqrt{2}} v_1 & u_2 &= \frac{1}{3\sqrt{6}} v_2 \\
u_1 &= \begin{bmatrix} 3/5\sqrt{2} \\ -4/5\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} & u_2 &= \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}
\end{aligned}$$

Thus, an orthonormal basis for W is:

$$\left\{ \begin{bmatrix} 3/(5\sqrt{2}) \\ -4/(5\sqrt{2}) \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$$

Solution 1.10. We first find a basis for the column space by finding the reduced row echelon form of the matrix:

$$\begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix} \overset{R_2+R_1\to R_2, R_4-R_1\to R_4, R_5-R_1\to R_5}{\sim} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & 6 \\ 0 & 2 & 3 \\ 0 & 2 & -3 \\ 0 & 2 & 3 \end{bmatrix}$$

$$\overset{R_4-R_3\to R_4, R_5-R_3\to R_5}{\sim} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_4 - R_2 \to R_4}{\sim} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_2 \leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We observe that all 3 columns are pivot columns, and thus the basis that forms the column space is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix} \right\}$. We now use the Gram-Schmidt process to create an orthonormal basis for the column space. Let $x_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix}, x_3 = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix}$.

Then, we define:

$$\begin{aligned} v_1 &= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\ v_2 &= \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix} - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 \\ &= \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \\ v_3 &= \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix} - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 \end{aligned}$$

$$= \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix} - \frac{7}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix}$$

Thus, an orthogonal base for the column space is:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix} \right\}$$

To find an orthonormal basis, we normalize the vectors:

$$u_1 = \frac{1}{||v_1||} v_1 \qquad u_2 = \frac{1}{||v_2||} v_2 \qquad u_3 = \frac{1}{||v_3||} v_3$$

$$= \frac{1}{2} v_1 \qquad \frac{1}{\sqrt{10}} v_2 \qquad = \frac{1}{6} v_3$$

$$= \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix} \qquad = \begin{bmatrix} -\sqrt{10}/10 \\ \sqrt{10}/10 \\ 2\sqrt{10}/10 \\ \sqrt{10}/10 \\ \sqrt{10}/10 \end{bmatrix} \qquad = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ -1/2 \\ 1/2 \end{bmatrix}$$

Thus, an orthonormal basis for the column space is:

$$\left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -\sqrt{10}/10 \\ \sqrt{10}/10 \\ 2\sqrt{10}/10 \\ \sqrt{10}/10 \\ \sqrt{10}/10 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ -1/2 \\ 1/2 \end{bmatrix} \right\}$$

§2 Proof Problems

Solution 2.1.

a)

> **Claim 2.1** — The vector whose entries are all 1's is an eigenvector of A^T.

Proof. The entries of the row vectors of A^T sum to 1. As multiplying by the vector $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ results in a vector where each entry is the sum of the corresponding row of A, it follows that $A^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, and thus is an eigenvector of A^T, and the proof is complete. $\square$

b)

> **Claim 2.2** — A must have at least one eigenvalue equal to 1.

Proof. From (a), $A^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, and thus the eigenvalue for this eigenvector is $\lambda = 1$. From Theorem 3 in Chapter 5, $\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I)$, and thus the characteristic polynomial for A and A^T are the same. Since 1 is an eigenvalue for A^T, then it follows that it is also an eigenvalue for A, and the proof is complete. $\square$

c)

> **Claim 2.3** — Any eigenvector for a different eigenvalue must have some entries that are positive and some that are negative (they can't all be the same sign).

Proof. Let $O = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$. As the columns of A sum to 1, it follows that

$$OA = O$$

For an eigenvector v:

$$Av = \lambda v$$

If we left multiply v by O, we obtain:

$$Ov = (OA)v = O(Av) = O(\lambda v)$$

For an eigenvector whose corresponding eigenvalue is not 1, it follows that $Ov = 0$. This means that the entries of the eigenvector v sum to v. Thus, there exists a positive entry if and only if there exists a negative entry, and the proof is complete. $\square$