MATH 22A: Vector Calculus and Linear Algebra

Lecture 2: Proofs — September 8, 2023

Pset Due: September 13, 2023 **Denny Cao**

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§1 Announcements

- Office Hours rooms change; check website. Currently, Sever 306 check back later.

§2 Review

§2.1 Matrices

Definition 2.1 (Matrix). A *matrix* is $m \times n$, where m is the number of rows, n is the number of columns. The numbers in a matrix are called *coefficients*.

Property 2.2 (Row Operations)

- You can scale a row by a real number
- Swap two rows
- Add multiple of a row to another

§3 Introductory Set Theory

§3.1 Notation

Definition 3.1 (Set). A *set* is a collection of objects.

Notation 3.2 (Member) $p \in A$ means p is in, or is a member, of the set A. $p \notin A$ means p is not in, or is not a member, of the set A.

Notation 3.3 (Complement) $\overline{A}$ means everything that is not in A.

Definition 3.4 (Prime Number). A positive integer p is *prime* if p is divisible by only itself and 1.

Example 3.5

Does there exist 100 consecutive composite numbers?

Proof. Let $k \in \{0 \leq k \leq 100, k \in \mathbb{Z}\}$. Then, $\forall k :$

$$\frac{101! + k}{k} \in \mathbb{Z}$$

As k is a factor of 101! and k is a factor of k. Thus, there does exist 100 consecutive composite numbers starting from 101!. □

Notation 3.6 (Set Builders)

- $\mathbb{N} = \{0, 1, 2, 3, ...\}$

- $\mathbb{Z} = \{..., -1, 0, 1, ...\}$
- $\mathbb{N} = \{p \in \mathbb{Z} \mid p \geq 0\}$

Notation 3.7 (Cardinality) The cardinality is the size of the set, denoted by $|A|$. When A is finite, the cardinality is the number of elements.

Definition 3.8 (Universal Set/Empty Set)**.** The empty set, $\emptyset$, is a set with no elements.

Definition 3.9 (Cartesian Product)**.** $A \times B = \{(a, b) \mid a \in A \land b \in B\}$

> **Example 3.10** (Planes)
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> $\mathbb{R}$ is all reals in the 1-dimensional plane. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, which is the cartesian product and creates the 2-dimensional plane by creating all ordered pairs of (x, y). Same for $\mathbb{R}^3$.

Notation 3.11 (Subset) $B \subseteq A$ denotes that B is a part of A.

- $\emptyset \subseteq A \land A \subseteq A, \forall A$

Definition 3.12 (Power Set)**.** The power set of a set A is the set of all subsets of A, denoted $\mathcal{P}(A)$.

> **Remark 3.13** (Cardinality of the Power Set)**.** $|A| = n \to |\mathcal{P}(A)| = 2^n$

Definition 3.14 (Union)**.** The union of two sets A and B denoted $A \cup B$ is everything in A or B. $A \cup B = \{c \mid c \in A \lor A \in B\}$.

Definition 3.15 (Intersection)**.** The intersection of two sets A and B denoted $A \cap B$ is everything in both A and B. $A \cap B = \{c \mid c \in A \land A \in B\}$.

§3.2 Russell's Paradox

Paradox assertions without truth values.

§4 Proofs

Definition 4.1 (Existence Statement)**.** $\exists x \in A \mid P(x)$.

- To disprove: Prove the negative. $\forall x \in A \mid \neg P(x)$
- To prove: Find an $x \mid P(x)$.

Definition 4.2 (Universal Statement)**.** $\forall x \in A \mid P(x)$.

- To disprove: Prove that $\exists x \in A \mid \neg P(x)$.
- To prove: Check all x.

Definition 4.3 (Disprove by Contradiction)**.** Assume for purposes of contradiction that $P(x)$ is true and then deduce by logic that it is is false by coming to a contradiction, and therefore $P(x)$ is false.

Example 4.4

Claim: $\sqrt{2}$ is irrational.

Proof. By contradiction. Assume for purposes of contradiction that $\sqrt{2}$ is rational. Then, $\sqrt{2}$ can be written in the form p/q such that q is the smallest possible number.

$$\begin{aligned}\sqrt{2} &= \frac{p}{q}\\ 2 &= \frac{p^2}{q^2}\\ 2q^2 &= p^2\end{aligned}$$

If p is even:

$$\begin{aligned}2q^2 &= 4m^2\\ q^2 &= 2m^2\end{aligned}$$

This means that q is divisible by 2 and not in lowest terms. Ê ※
If p is odd:

$$\begin{aligned}2q^2 &= 4b^2 + 4b + 1\\ q^2 &= b^2 + 2b + 1/2\\ q &= \sqrt{b^2 + 2b + 1/2}\end{aligned}$$

This means that q is not an integer. Ê ※ $\square$