Practice Exam 3

Denny Cao

November 11, 2022

Question 1

Use the Principle of Mathematical Induction to show this inequality is true for all integers $n \geq 2$: $\displaystyle\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}$

Proof. By induction.

Let P(n) be the statement that $\displaystyle\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}$, where $n \in \mathbb{Z}$.

Base Case: $n = 2$

$$\begin{aligned}
\sum_{i=1}^{2} \frac{1}{\sqrt{i}} &\stackrel{?}{>} \sqrt{2} \\
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} &\stackrel{?}{>} \sqrt{2} \\
1 + \frac{\sqrt{2}}{2} &\stackrel{?}{>} \sqrt{2} \\
2 + \sqrt{2} &\stackrel{?}{>} 2\sqrt{2} \\
2 &> \sqrt{2}
\end{aligned}$$

Since $2 > \sqrt{2}$, $P(2)$ is true.

Inductive Hypothesis: Assume $P(k)$ is true, $k \in \mathbb{Z} \land k \geq 2$.

Inductive Step: We want to show that $P(k) \implies P(k+1)$. Assume the Inductive Hypothesis.

$$\begin{aligned}
\sum_{i=1}^{k} \frac{1}{\sqrt{i}} &> \sqrt{k} \\
1 + \cdots + \frac{1}{\sqrt{k}} &> \sqrt{k}
\end{aligned}$$

Let $c = 1 + \cdots + \frac{1}{\sqrt{k}}$

$$c > \sqrt{k}$$

$$c + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$$

$$c + \frac{1}{\sqrt{k+1}} > \frac{\sqrt{k}\sqrt{k+1} + 1}{\sqrt{k+1}}$$

$$c + \frac{1}{\sqrt{k+1}} > \frac{\sqrt{k^2+2k} + 1}{\sqrt{k+1}}$$

$$\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} \stackrel{?}{>} \sqrt{k+1}$$

$$1 + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{k+1} \stackrel{?}{>} \sqrt{k+1}$$

$$c + \frac{1}{k+1} \stackrel{?}{>} \sqrt{k+1}$$

$$c + \frac{1}{k+1} \stackrel{?}{>} \frac{\sqrt{k+1}\sqrt{k+1}}{\sqrt{k+1}}$$

$$c + \frac{1}{k+1} \stackrel{?}{>} \frac{\sqrt{k^2+2k+1}}{\sqrt{k+1}}$$

Since $c + \frac{1}{\sqrt{k+1}} > \frac{\sqrt{k^2+2k}+1}{\sqrt{k+1}}$, by the transitive property, $\frac{\sqrt{k^2+2k}+1}{\sqrt{k+1}} > \frac{\sqrt{k^2+2k+1}}{\sqrt{k+1}} \implies c + \frac{1}{\sqrt{k+1}} > \frac{\sqrt{k^2+2k+1}}{\sqrt{k+1}}$

$$\frac{\sqrt{k^2+2k}+1}{\sqrt{k+1}} \stackrel{?}{>} \frac{\sqrt{k^2+2k+1}}{\sqrt{k+1}}$$

$$\sqrt{k^2+2k} + 1 \stackrel{?}{>} \sqrt{k^2+2k+1}$$

$$\left(\sqrt{k^2+2k} + 1\right)^2 \stackrel{?}{>} k^2+2k+1$$

$$k^2+2k+1+2\sqrt{k^2+2k} \stackrel{?}{>} k^2+2k+1$$

Since $k \geq 2$, the statement is true:

$$\frac{\sqrt{k^2+2k}+1}{\sqrt{k+1}} > \frac{\sqrt{k^2+2k+1}}{\sqrt{k+1}}$$

Therefore, $c + \frac{1}{\sqrt{k+1}} > \frac{\sqrt{k^2+2k+1}}{\sqrt{k+1}} = c + \frac{1}{\sqrt{k+1}} > \sqrt{k+1} = \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > \sqrt{k+1}$. Thus, $P(k) \implies P(k+1)$.

Conclusion: By induction, we have shown that $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > \sqrt{n+1}$ is true $n \geq 2$ $\square$

Question 2

Prove that for all positive integers n, $3^{2^n} - 1$ is divisible by 2^{n+2}.

Proof. By induction.
Let $P(n)$ be the statement that $3^{2^n} - 1$ is divisible by 2^{n+2}, $n \in \mathbb{Z}^+$

Base Case: $n = 1$

$$\begin{aligned}(3^{2^1} - 1) \bmod 2^{1+2} &= (3^2 - 1) \bmod 2^3 \\ &= 8 \bmod 8 \\ &= 0\end{aligned}$$

Since the remainder when $3^{2^1} - 1$ is divided by 2^{1+2} is 0, $P(n)$ is true when $n = 0$.

Inductive Hypothesis: Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$.

Inductive Step: We want to show that $P(k) \implies P(k+1)$. Since we assume $P(k)$ is true from the inductive hypothesis, we must prove that $P(k+1)$ is true.

$$\begin{aligned}\frac{3^{2^{k+1}} - 1}{2^{k+1+2}} &= \frac{3^{2^k(2)} - 1}{2^1(2^{k+2})} \\ &= \frac{3^{2^{k^2}} - 1}{2^1(2^{k+2})} \\ &= \frac{\left(3^{2^k} - 1\right)\left(3^{2^k} + 1\right)}{2^1(2^{k+2})} \\ &= \left(\frac{3^{2^k} - 1}{2^{k+2}}\right)\left(\frac{3^{2^k} + 1}{2}\right)\end{aligned}$$

Since we assume the inductive hypothesis, $P(k)$ is true. Therefore, $\left(3^{2^k} - 1\right) \bmod 2^{k+2} = 0$, meaning $\frac{3^{2^k} - 1}{2^{k+2}} \in \mathbb{Z}$. Since $\left(\frac{3^{2^k} - 1}{2^{k+2}}\right)\left(\frac{3^{2^k} + 1}{2}\right)$ are factors of $\frac{3^{2^{k+1}} - 1}{2^{k+1+2}}$, if both are integers, $\left(3^{2^{k+1}} - 1\right) \bmod 2^{k+3} = 0$, meaning $P(k+1)$ is true. Therefore, it is sufficient to prove $P(k+1)$ by proving $\left(3^{2^k} + 1\right) \bmod 2 = 0$:

For an integer x to be even, 2 must be included in the prime factorization of x. The prime factorization of 3^{2^k} is 2^k amount of 3s. Since 2 is not included and $3^{2^k} \neq 0$, 3^{2^k} is odd. Therefore, 3^{2^k} can be expressed as $2c+1, c \in \mathbb{Z}$:

$$\begin{aligned}
[(2c+1)+1] \bmod 2 &= (2c+2) \bmod 2 \\
&= ((2c \bmod 2) + 2 \bmod 2) \bmod 2 \\
&= (((2 \bmod 2)(c \bmod 2)) \bmod 2 + 2 \bmod 2) \bmod 2 \\
&= ((0(c \bmod 2)) \bmod 2 + 0) \bmod 2 \\
&= 0 \bmod 2 \\
&= 0
\end{aligned}$$

$((2c+1)+1) \bmod 2 = 0$, where $2c+1 = 3^{2^k}$, therefore $\dfrac{3^{2^k}+1}{2} \in \mathbb{Z}$. Since both factors of $\dfrac{3^{2^{k+1}}-1}{2^{k+1+2}}$ are integers, $\dfrac{3^{2^{k+1}}-1}{2^{k+1+2}}$ will also be an integer. Therefore, $\left(3^{2^{k+1}}-1\right) \bmod 2^{k+1+2} = 0$; $P(k+1)$ is true.

$\square$

Question 3

Find a formula for
$$(1-\frac{1}{2^2})(1-\frac{1}{3^2})(1-\frac{1}{4^2})(1-\frac{1}{5^2})...(1-\frac{1}{n^2})$$
where $n \geq 2$, and use the Principle of Mathematical Induction to prove that the formula is correct.

$$\begin{aligned}
&\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\\
=&\left(\frac{2^2-1}{2^2}\right)\left(\frac{3^2-1}{3^2}\right)\left(\frac{4^2-1}{4^2}\right)\left(\frac{5^2-1}{5^2}\right)\cdots\left(\frac{n^2-1}{n^2}\right)\\
=&\left(\frac{(2+1)(2-1)}{2^2}\right)\left(\frac{(3+1)(3-1)}{3^2}\right)\left(\frac{(4+1)(4-1)}{4^2}\right)\left(\frac{(5+1)(5-1)}{5^2}\right)\cdots\left(\frac{(n+1)(n-1)}{n^2}\right)\\
=&\left(\frac{3\times 1}{2^2}\right)\left(\frac{4\times 2}{3^2}\right)\left(\frac{5\times 3}{4^2}\right)\left(\frac{6\times 4}{5^2}\right)\cdots\left(\frac{(n+1)(n-1)}{n^2}\right)\\
=&\frac{\frac{(n+1)!}{2}\times(n-1)!}{n!\times n!}=\frac{(n+1)!\times(n-1)!}{2\times n!\times n!}=\frac{n+1}{2n}
\end{aligned}$$

Proof. By induction.

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\prod_{i=2}^{n}1-\frac{1}{i^2}$$

Let P(n) be the statement that:

$$\prod_{i=2}^{n}1-\frac{1}{i^2}=\frac{n+1}{2n}$$

where $n \geq 2, n \in \mathbb{Z}$.

Base Case: $n = 2$

$$\begin{aligned}
\prod_{i=2}^{2}1-\frac{1}{i^2}&=\left(1-\frac{1}{2^2}\right)\\
&=\left(1-\frac{1}{4}\right)\\
&=\frac{3}{4}
\end{aligned}$$

$$\frac{2+1}{2(2)}=\frac{3}{4}$$

Since $\prod_{i=2}^{n}1-\frac{1}{i^2}$ and $\frac{n+1}{2n}$ evaluate to the same value, $\frac{3}{4}$, when $n = 2$, $P(2)$ is true.

Inductive Hypothesis: Assume $P(k)$ is true, $k \geq 2, k \in \mathbb{Z}$.

Inductive Step: We want to show that $P(k) \implies P(k+1)$. Assume the Inductive Hypothesis:

$$\begin{aligned}
\prod_{i=2}^{k} 1 - \frac{1}{i^2} &= \frac{k+1}{2k} \\
\left(\prod_{i=2}^{k} 1 - \frac{1}{i^2}\right)\left(1 - \frac{1}{(k+1)^2}\right) &= \left(\frac{k+1}{2k}\right)\left(1 - \frac{1}{(k+1)^2}\right) \\
\prod_{i=2}^{k+1} 1 - \frac{1}{i^2} &= \left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2 - 1}{(k+1)^2}\right) \\
&= \frac{(k+1)^2 - 1}{2k(k+1)} \\
&= \frac{k^2 + 2k + 1 - 1}{2k(k+1)} \\
&= \frac{k^2 + 2k}{2k(k+1)} \\
&= \frac{k(k+2)}{2k(k+1)} \\
&= \frac{k+2}{2(k+1)} \\
&= \frac{(k+1)+1}{2(k+1)}
\end{aligned}$$

$\left(\prod_{i=2}^{k} 1 - \frac{1}{i^2}\right)\left(1 - \frac{1}{(k+1)^2}\right)$, which is equivalent to $\prod_{i=2}^{k+1} 1 - \frac{1}{i^2}$, equals $\frac{(k+1)+1}{2(k+1)}$. Thus, $P(k) \implies P(k+1)$.

Conclusion: By induction, we have shown that $P(n)$ is true, or $\prod_{i=2}^{n} 1 - \frac{1}{i^2} = \frac{n+1}{2n}$, $n \geq 2, n \in \mathbb{Z}$. $\square$

Question 4

Which amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using strong induction.

Solution

Proof. By strong induction.
An amount of money that can be formed using just two-dollar bills and five-dollar bills can be represented by $n = 2a+5b$, where n is the amount of money, a is the amount of two-dollar bills, and b is the amount of five-dollar bills.

Let $P(n)$ be the statement that $n = 2a + 5b$, $a, b \in \mathbb{N}$.

Basis Cases: $n \in \{2, 4, 5\}$

$$\begin{aligned} n &= 4 & n &= 5 \\ 2a + 5b &= 4 & 2a + 5b &= 5 \\ 2(2) + 5(0) &= 4 & 2(0) + 5(1) &= 5 \end{aligned}$$

Since $2a + 5b = 4, 5$ when $a = 2, b = 0$; $a = 0, b = 1$ respectively, $P(4)$ and $P(5)$ are true.

Inductive Hypothesis: Assume $P(j)$ is true, $4 \leq j \leq k$.

Inductive Step: We want to show that $P(k) \implies P(k+1)$. Assume the Inductive Hypothesis.

$$2a + 5b = k + 1 = (k - 1) + 2$$

By adding a 2-dollar bill to $k-1$, we can form $k+1$. By the Inductive Hypothesis, $P(k-1)$ is true, meaning $k - 1$ can be formed with just two-dollar bills and five-dollar bills. Therefore, since $k + 1$ can be formed by adding another two-dollar bill, $P(k + 1)$ is true.

Conclusion: By strong induction, we have shown that $P(n)$ is true for $n \geq 4, n \in \mathbb{Z}$, or n dollars can be formed with just two-dollar bills and five-dollar bills, $n \geq 4, n \in \mathbb{Z}$.

2 dollars can be formed with 1 two-dollar bill and 0 five-dollar bills. Therefore, n dollars can be formed with just two-dollar bills and five-dollar bills, $n \in \{2\} \cup \{k \in \mathbb{Z} \mid k \geq 4\}$. □

Question 5

A baker bakes six different kinds of muffins. If a box with 25 muffins is made with a random number of each kind of muffin, in how many ways can a box of muffins be prepared.

Solution

When repetition of elements is allowed, there are $C(n + r - 1, r)$ r-combinations from a set with n elements. We have $n = 6$ kinds muffins and $r = 25$ total muffins:

$$\begin{aligned}\binom{6 - 1 + 25}{25} &= \binom{30}{25} \\ &= 142,506\end{aligned}$$

There are 142,506 ways for a box of muffins to be prepared.

Question 6

Let P(n) be the statement that a postage of n cents can be formed using just 3- cent stamps and 5- cent stamps. Parts of this exercise outline a strong induction proof that P(n) is true for $n \geq 8$.

a) Show that the statements P(8), P(9), and P(10) are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for $k \geq 10$.

Question 7

Find the number of strings of length 10 of letters of the alphabet, with no repeated letters, that have vowels in the first two positions. **NOTE:** y is a vowel.

Solution

Question 8

Ten men and ten women are to be put in a row. Find the number of possible different rows if no two of the same sex stand adjacent.

Question 9

If positive integers are chosen at random, what is the minimum number you must have in order to guarantee that two of the chosen numbers are congruent modulo 6. Prove your answer.
NOTE: Two numbers are congruent modulo 6 if their difference is a multiple of 6.

Question 10

Write the expansion of $\left(x^2 - \frac{1}{x}\right)^{12}$.

$$\left(x^2 - \frac{1}{x}\right)^{12} = \sum_{i=0}^{12} \binom{12}{i} x^{2^{12-i}} \left(-\frac{1}{x}\right)^{i}$$

$$= \binom{12}{0} \left(x^2\right)^{12} \left(\frac{1}{x}\right)^{0} + \binom{12}{1} \left(x^2\right)^{11} \left(\frac{1}{x}\right)^{1} +$$

$$+ \binom{12}{4} \left(x^2\right)^{8} \left(\frac{1}{x}\right)^{4} + \binom{12}{5} \left(x^2\right)^{7} \left(\frac{1}{x}\right)^{5} + \binom{12}{6} \left(x^2\right)^{6} \left(\frac{1}{x}\right)^{6}$$

$$+ \binom{12}{7} \left(x^2\right)^{5} \left(\frac{1}{x}\right)^{7} + \binom{12}{8} \left(x^2\right)^{4} \left(\frac{1}{x}\right)^{8} + \binom{12}{9} \left(x^2\right)^{3} \left(\frac{1}{x}\right)^{9}$$

$$+ \binom{12}{10} \left(x^2\right)^{2} \left(\frac{1}{x}\right)^{10} + \binom{12}{11} \left(x^2\right)^{1} \left(\frac{1}{x}\right)^{11} + \binom{12}{12} \left(x^2\right)^{0} \left(\frac{1}{x}\right)^{12}$$

$$= 1\left(x^2\right)^{12} \left(\frac{1}{x}\right)^{0} + 12\left(x^2\right)^{11} \left(\frac{1}{x}\right)^{1} + 66\left(x^2\right)^{1}$$

Question 11

In how many ways can 7 of the 8 letters in CHEMISTS be put in a row?

Question 12

What is the minimum number of cards that must be drawn from an ordinary deck of cards to guarantee that you have been dealt
(a) at least three of at least one suit?
(b) at least three clubs?
Explain.

Question 13

Prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, whenever n, r, and k are nonnegative integers with $r \leq n$ and $k \leq r$,

a) algebraically.

b) using a combinatorial argument.

Question 14

How many solutions are there to the equation $\sum_{i=1}^{6} x_i = 29$, where $x_i, i = 1, 2, 3, 4, 5, 6$, is a nonnegative integer such that
a)$x_i > 1$ for$i = 1, 2, 3, 4, 5, 6$?
b)$x_1 \geq 1$, x2 $\geq$ 2, $x_3 \geq 3$, $x4 \geq 4$, $x_5 > 5$, and $x_6 \geq 6$?
c)$x_1 \geq 5$?
d)$x_1 < 8$ and $x_2 > 8$?