Exam 2 Corrections

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Question 1

If $\overline{A} \cap B = B \ \wedge \ \overline{B} \cap A = A$, what can you say about A and B? Prove your answer. ($\overline{S}$ denotes the complement of S.)

Solution

Proof. Assume $x \in A$. By definition of complement, $x \notin \overline{A}$. Since $x \notin \overline{A}$, $x \notin \overline{A} \cap B$. Since $\overline{A} \cap B = B$, $x \notin B$. x is arbitrary, therefore we can generalize to $A \not\subseteq B$. □

Proof. Assume $y \in B$. By definition of complement, $y \notin \overline{B}$. Since $y \notin \overline{B}$, $y \notin \overline{B} \cap A$. Since $\overline{B} \cap A = A$, $y \notin A$. y is arbitrary, therefore we can generalize to $B \not\subseteq A$. □

Proof. Since $A \not\subseteq B \ \wedge \ B \not\subseteq A$, $A \cap B = \emptyset$

$\overline{A} \cap B = B \ \wedge \ \overline{B} \cap A = A \implies A \cap B = \emptyset$ □

Question 2

Suppose that $g : A \to A$ and $f : A \to A$ where $A = \{a, b, c, d\}$, $g = \{(a, a), (b, c), (c, a), (d, c)\}$ and $f = \{(a, d), (b, a), (c, b), (d, a)\}$. Find $g \circ (f \circ g)$, that is, find the composition of $g(x)$ with $f(x)$ composed with $g(x)$.

Solution

$$g = \{(a, a), (b, c), (c, a), (d, c)\}$$

$$\begin{aligned} f \circ g &= f(\{(a, a), (b, c), (c, a), (d, c)\}) \\ &= \{(a, d), (b, b), (c, d), (d, b)\} \end{aligned}$$

$$\begin{aligned} g \circ f \circ g &= g(\{(a, d), (b, b), (c, d), (d, b)\}) \\ &= \{(a, c), (b, c), (c, c), (d, c)\} \end{aligned}$$

Question 3

Verify that $a_n = \pi$ **is a possible solution** to the recurrence relation $a_n = (2n-1)a_{n-1} - (2n-2)a_{n-2}$. What are the conditions for this to happen?

Solution

Initial Conditions: $a_0 = \pi, a_1 = \pi$

$$\begin{aligned}
a_2 &= (2(2)-1)a_1 - (2(2)-2)a_0 \\
&= 3a_1 - 2a_0 \\
&= 3\pi - 2\pi \\
&= \pi \\
a_3 &= (2(3)-1)a_2 - (2(3)-2)a_1 \\
&= 5a_2 - 4a_1 \\
&= 5\pi - 4\pi \\
&= \pi \\
a_4 &= (2(4)-1)a_3 - (2(4)-2)a_2 \\
&= 5a_3 - 4a_2 \\
&= 5\pi - 4\pi \\
&= \pi \\
a_5 &= (2(5)-1)a_4 - (2(5)-2)a_3 \\
&= 5a_4 - 4a_3 \\
&= 5\pi - 4\pi \\
&= \pi \\
&\vdots \\
a_n &= (2n-1)\pi - (2n-2)\pi \\
&= 2n\pi - \pi - 2n\pi + 2\pi \\
&= \pi
\end{aligned}$$

Thus, the conditions for $a_n = \pi$ are $a_0 = \pi, a_1 = \pi$.

Question 4

Evaluate $\sum_{n=1}^{3}\sum_{k=1}^{n}(n^k)$. Show all your steps.

Solution

$$\begin{aligned}\sum_{n=1}^{3}\sum_{k=1}^{n}(n^k) &= \sum_{k=1}^{1}1^k + \sum_{k=1}^{2}2^k + \sum_{k=1}^{3}3^k \\ &= 1 + 2 + 4 + 3 + 9 + 27 \\ &= 46\end{aligned}$$

Question 5

Find a $f : \mathbb{N} \to \mathbb{Z}$ that is surjection that is not injection. Prove that it satisfies the conditions held.

Solution

$\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$

Suppose $S_1 = \{x \mid 2x, x \in \mathbb{N}\}$, $S_2 = \{x \mid 2x+1, x \in \mathbb{N}\}$

$$f(x) = \begin{cases} \frac{x}{2} & x \in S_1 \\ -(\frac{x}{2} - \frac{1}{2}) & x \in S_2 \end{cases}$$

$f(n)$ is surjective.

Proof. We can separate the domain, $\mathbb{N}$ into two sets of odd and even naturals, S_1 and S_2 respectively.

Let $n = 2y, y \in \mathbb{N}$. Since $2y$ is an even integer, it will satisfy the first case, $x \in S_1$.

$$\begin{aligned} f(x) &= \frac{x}{2}, x \in S_1 \\ f(2y) &= \frac{2y}{2} \\ &= y \end{aligned}$$

There exists an $x = 2y$ that maps to all $\mathbb{Z}^+ \cup \{0\}$ in the codomain.

Let $n = 2y+1, y \in \mathbb{N}$. Since $2y+1$ is an odd integer, it will satisfy the second case, $x \in S_2$.

$$\begin{aligned} f(x) &= -\left(\frac{x}{2} - \frac{1}{2}\right), x \in S_2 \\ f(2y+1) &= -\left(\frac{2y+1}{2} - \frac{1}{2}\right) \\ &= -\left(\frac{2y}{2}\right) \\ &= -y \end{aligned}$$

There exists an $x = 2y+1$ that maps to all $\mathbb{Z}^-$ in the codomain.

Since $\mathbb{Z}^+ \cup \{0\} \cup \mathbb{Z}^+ = \mathbb{Z}$, there exists an x that maps to every value y in the codomain. Thus, $f(x)$ is surjective.

$\square$

$f(n)$ **is not injective.**

Proof. Suppose $x_0 = 0, x_1 = 1$

$$f(x_0) = \frac{0}{2} = 0$$
$$f(x_1) = -\left(\frac{1}{2} - \frac{1}{2}\right) = 0$$

Since there exists multiple values in the domain, x_0, x_1, that map to the same value in the codomain, 0, $f(x)$ is not injective.

$\square$

Question 6

Suppose $\mathcal{P}^n(A) = \overbrace{\mathcal{P}(\mathcal{P}(\mathcal{P} \cdots \mathcal{P}(A)) \cdots)}^{n}$, $n \in \mathbb{Z}^+$, where for instance, $\mathcal{P}^3(A) = \mathcal{P}(\mathcal{P}(\mathcal{P}(A)))$. Find the cardinality of $\mathcal{P}^4(\emptyset)$, where $\mathcal{P}(A)$ is the power set of A.

Solution

The cardinality of the power set of a set is 2^n, where n is the cardinality of the set.
The cardinality of $\emptyset = 0$.

$$\begin{aligned} &\mathcal{P}(\emptyset) &&= \{\emptyset\} && |\mathcal{P}(\emptyset)| &&= 2^0 = 1 \\ &\mathcal{P}^2(\emptyset) &&= \{\emptyset, \{\emptyset\}\} && |\mathcal{P}^2(\emptyset)| &&= 2^1 = 2 \\ &\mathcal{P}^3(\emptyset) &&= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} && |\mathcal{P}^3(\emptyset)| &&= 2^2 = 4 \end{aligned}$$

$$|\mathcal{P}^4(\emptyset)| = 16$$

Question 7

Prove or disprove: $\forall$ sets $A, B, C, (A \oplus B = A \oplus C) \implies B = C$

Solution

Proof.
Case 1: Suppose $x \in B$
Subcase i: $x \in A$
$x \in B \land x \in A \implies x \notin A \oplus B$. Since $A \oplus B = A \oplus C$, $x \notin A \oplus C$.
$x \in A \land x \notin A \oplus C \implies x \in C$

Subcase ii: $x \notin A$
$x \in B \land x \notin A \implies x \in A \oplus B$. Since $A \oplus B = A \oplus C$, $x \in A \oplus C$.
$x \notin A \land x \in A \oplus C \implies x \in C$

Since $x \in C$ in all cases when $x \in B$, $B \subseteq C$.

Case 2: Suppose $x \in C$
Subcase i: $x \in A$
$x \in C \land x \in A \implies x \notin A \oplus C$. Since $A \oplus C = A \oplus B$, $x \notin A \oplus B$.
$x \in A \land x \notin A \oplus B \implies x \in B$

Subcase ii: $x \notin A$
$x \in C \land x \notin A \implies x \in A \oplus C$. Since $A \oplus C = A \oplus B$, $x \in A \oplus B$.
$x \notin A \land x \in A \oplus B \implies x \in B$

Since $x \in B$ in all cases when $x \in C$, $C \subseteq B$.

$B \subseteq C \land C \subseteq B \implies B = C$. Thus, $(A \oplus B = A \oplus C) \implies B = C$. ☐

$$
\begin{aligned}
B &= \emptyset \oplus B \\
&= (A \oplus A) \oplus B \\
&= A \oplus (A \oplus B) \\
&= A \oplus (A \oplus C) \\
&= (A \oplus A) \oplus C \\
&= \emptyset \oplus C \\
B &= C
\end{aligned}
$$

Question 8

Solve for x: $\lfloor 2x \rfloor + \lceil 2x \rceil = 4x$

Solution

Any number can be represented by $a + b, a \in \mathbb{Z}, b \in \{k \mid 0 \leq k < 1\}$. Let $x = a + b$.

$$\begin{aligned}
\lfloor 2(a+b) \rfloor + \lceil 2(a+b) \rceil &= 4(a+b) \\
\lfloor 2a + 2b \rfloor + \lceil 2a + 2b \rceil &= 4a + 4b \\
2a + \lfloor 2b \rfloor + 2a + \lceil 2b \rceil &= 4a + 4b \\
\lfloor 2b \rfloor + \lceil 2b \rceil &= 4b
\end{aligned}$$

$$\lfloor 2b \rfloor = \begin{cases} 0 & 0 \leq b < 0.5 \\ 1 & 0.5 \leq b < 1 \end{cases} \qquad \lceil 2b \rceil = \begin{cases} 0 & b = 0 \\ 1 & 0 < b \leq 0.5 \\ 2 & 0.5 < b < 1 \end{cases}$$

$$4b = \begin{cases} 0 + 0 = 0 \\ 0 + 1 = 1 \\ 1 + 1 = 2 \\ 1 + 3 = 3 \end{cases}$$

$$b = \begin{cases} \dfrac{0}{4} \\ \dfrac{1}{4} \\ \dfrac{2}{4} \\ \dfrac{3}{4} \end{cases}$$

Thus, $x = a + b$, where $a \in \mathbb{Z}$ and $b \in \left\{\frac{1}{4}c \mid c \in 0 \leq c < 4, k \in \mathbb{Z}\right\}$. Since all integers are multiples of $\frac{1}{4}$, a can be represented as $a = \frac{1}{4}k, k \in \mathbb{Z}$. Since $b = \frac{1}{4}c, c \in 0 \leq c < 4, c \in \mathbb{Z}$, we can express $a + b$ as $\frac{1}{4}(k + c)$. Since k and c are both integers, x is a multiple of $\frac{1}{4}$.

$$x \in \left\{\frac{1}{4}r \mid r \in \mathbb{Z}\right\}$$

Question 9

Give an example of two uncountable sets A and B such that $A \cap B$ is:
a) finite. b) countably infinite. c) uncountable.

Solution

(a)

$$A = \mathbb{R}^+ \cup \{0\}$$
$$B = \mathbb{R}^- \cup \{0\}$$

$$A \cap B = \{0\}$$

(b)

$$A = \mathbb{R}^+ \cup \mathbb{Z}$$
$$B = \mathbb{R}^- \cup \mathbb{Z}$$

$$A \cap B = \mathbb{Z}$$

(c)

$$A = \mathbb{R}$$
$$B = \mathbb{R}^+$$

$$A \cap B = \mathbb{R}^+$$

Question 10

Prove if $f : A \to B$ and $g : B \to C$ are injective, then $g \circ f : A \to C$ is also injective.

Solution

Proof. Suppose $x_0, x_1 \in A$.

Since g is injective:

$$g(f(x_0)) = g(f(x_1)) \implies f(x_0) = f(x_1)$$

Since f is injective:

$$f(x_0) = f(x_1) \implies x_0 = x_1$$

By Hypothetical Syllogism:

$$g(f(x_0)) = g(f(x_1)) \implies x_0 = x_1$$

Thus, $\forall x_0 \forall x_1 (g(f(x_0)) = g(f(x_1)) \implies x_0 = x_1)$. Therefore, $g \circ f$ is injective. $\square$