Practice Exam 2

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Question 1

Let $S = \{\emptyset, a, \{a\}\}$. Determine whether each of these is an element of S, a subset of S, neither, or both. Justify your answer
(a) {a}
(b) {{a}}
(c) $\emptyset$
(d) $\{\{\emptyset\}, a\}$
(e) $\{\emptyset\}$
(f) $\{\emptyset, a\}$

Solution

(a) $\{a\}$ is both an element of S and a subset of S. $\{a\}$ is in S, therefore it is an element of S. All elements of $\{a\}$ are in S (S contains an element a), therefore $\{a\}$ is a subset of S.

(b) $\{\{a\}\}$ is not an element of S, but is a subset of S. $\{\{a\}\}$ is not in S, therefore it is not an element of S. All elements of $\{\{a\}\}$ are in S (S contains an element $\{a\}$), therefore $\{\{a\}\}$ is a subset of S.

(c) $\emptyset$ is both an element of S and a subset of S. $\emptyset$ is in S, therefore it is an element of S. $\emptyset$ contains no elements, so all its elements (there are none) are in S. Therefore, $\emptyset$ is a subset of S.

(d) $\{\{\emptyset\}, a\}$ is neither an element of S nor a subset of S. $\{\{\emptyset\}, a\}$ is in S, therefore it is not an element of S. Since $\{\emptyset\}$ is an element of $\{\{\emptyset\}, a\}$ and not S, $\{\{\emptyset\}, a\}$ is not a subset of S.

(e) $\{\emptyset\}$ is not an element of S, but is a subset of S. $\{\emptyset\}$ is not in S, therefore it is not an element of S. All elements of $\{\emptyset\}$ are in S (S contains an element $\emptyset$), therefore $\{\emptyset\}$ is a subset of S.

(f) $\{\emptyset, a\}$ is not an element of S, but is a subset of S. $\{\emptyset, a\}$ is not in S, therefore it is not an element of S. All elements of $\{\emptyset, a\}$ are in S (S contains the elements $\emptyset$ and a), therefore $\{\emptyset, a\}$ is a subset of S.

Question 2

You begin with \$1000. You invest it at 5% compounded annually, but at the end of each year you withdraw \$100 immediately after the interest is paid.
(a) Set up a recurrence relation and initial condition for the amount you have after n years.
(b) How much is left in the account after you have withdrawn \$100 at the end of the third year?
(c) Find a formula for a_n.
(d) Use the formula to determine how long it takes before the last withdrawal reduces the balance in the account to \$0.

Solution

(a) $S_n = S_{n-1}(1.05) - 100, n \geq 1, S_0 = 1000$

(b)

$$S_0 = 1000 \quad \begin{aligned} S_1 &= S_0(1.05) - 100 \\ &= 1000(1.05) - 100 \\ &= 1050 - 100 \\ &= 950 \end{aligned} \qquad \begin{aligned} S_2 &= S_1(1.05) - 100 \\ &= 950(1.05) - 100 \\ &= 997.5 - 100 \\ &= 897.5 \end{aligned} \qquad \begin{aligned} S_3 &= S_2(1.05) - 100 \\ &= 897.5(1.05) - 100 \\ &= 942.375 - 100 \\ &= 842.375 \end{aligned}$$

\$842.37 is left in the account after withdrawing \$100 at the end of the third year.

(c) Let $P = S_0 = 1000, r = 1.05, c = 100$.

$$S_0 = P \qquad S_1 = Pr - c \qquad \begin{aligned} S_2 &= (Pr - c)r - c \\ &= Pr^2 - cr - c \end{aligned} \qquad \begin{aligned} S_3 &= (Pr^2 - cr - c)r - c \\ &= Pr^3 - cr^2 - cr - c \end{aligned}$$

$$\begin{aligned} S_n &= Pr^n - cr^{n-1} - cr^{n-2} - \cdots - cr^1 - c \\ &= Pr^n - c\big(r^{n-1} - r^{n-2} - \cdots - r - 1\big) \\ &= Pr^n - c\left(\sum_{i=0}^{n-1} r^i\right) \\ &= Pr^n - c\left(\sum_{i=1}^{n} r^{i-1}\right) \\ &= Pr^n - c\left(\frac{1 - r^n}{1 - r}\right) \\ &= 1000(1.05)^n - 100\left(\frac{1 - 1.05^n}{1 - 1.05}\right) \\ &= 1000(1.05)^n + 2000(1 - 1.05^n) \\ &= 1000(1.05^n + 2 - 2(1.05^n)) \\ &= 1000(-1.05^n + 2) \\ S_n &= -1000(1.05^n - 2) \end{aligned}$$

(d)

$$
\begin{aligned}
0 &= -1000(1.05^n - 2) \\
2 &= 1.05^n \\
\log_{1.05} 2 &= n \\
\frac{\log 2}{\log 1.05} &= n \\
14.21 &\approx n
\end{aligned}
$$

It will take 15 years until the last withdrawal reduces the balance in the account to $0.

Question 3

If $P(A)$ means the power set of A,
(a) Prove that $P(A) \cup P(B) \subset P(A \cup B)$ is true for all sets A and B.
(b) Prove that the converse of (a) is not true. That is, prove that:
$P(A \cup B) \subset P(A) \cup P(B)$ is false for some sets A and B.

Solution

(a) *Proof.* Suppose $S \in (P(A) \cup P(B))$.

Then $S \in P(A) \vee S \in P(B)$. Since S is in the power set of A or B, it is a subset of A or B: $S \subset A \vee S \subset B$.

$P(A \cup B)$, contains all subsets of $A \cup B$, which includes the subsets of A and B. This means that in either case $S \subset A \vee S \subset B$, $S \in P(A \cup B)$.

Since S is arbitrary, $P(A) \cup P(B) \subset P(A \cup B)$. □

(b) *Proof.* Suppose $A = \{0\},\ B = \{1\}$

$A \cup B = \{0, 1\}$. $P(A \cup B) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$

$P(A) = \{\emptyset, \{0\}\}$. $P(B) = \{\emptyset, \{1\}\}$. $P(A) \cup P(B) = \{\emptyset, \{0\}, \{1\}\}$.

For $P(A \cup B)$ to be a subset of $P(A) \cup P(B)$, $P(A) \cup P(B)$ must contain all elements of $P(A \cup B)$.

Since $\{0, 1\} \notin P(A) \cup P(B)$, $P(A \cup B) \subset P(A) \cup P(B)$ is false for some sets A and B. □

Question 4

Prove that the following is true for all sets A, B, and C: if $A \cap C \subset B \cap C$ and $A \cup C \subset B \cup C$, then $A \subset B$.

Solution

Proof. Suppose $x \in A$. There are two cases:

Case 1: $x \in C$. Then $x \in A \cap C$. Since $A \cap C \subset B \cap C$, $x \in B \cap C$. Therefore, $x \in B$.
Case 2: $x \notin C$. Since $x \in A$, $x \in A \cup C$. Since $A \cup C \subset B \cup C$, $x \in B \cup C$. Because $x \notin C$, $x \in B$.

Since x is arbitrary, $A \subset B$. □

Question 5

Let $f : R \to R$ have the rule $f(x) = \lceil 3x \rceil + 1$and $g : R \to R$ have the rule $g(x) = \dfrac{x}{3}$.
(a) Find $(fog)^{-1} = (\{2.5\})$.
(b) Find $(fog)^{-1} = (\{2\})$.

Solution

$$\begin{aligned}(f \circ g)(x) &= \left\lceil 3\left(\frac{x}{3}\right) \right\rceil + 1 \\ &= \lceil x \rceil + 1\end{aligned}$$

(a)

$$\begin{aligned}(f \circ g)(\{2.5\}) &= \{\lceil 2.5 \rceil + 1\} \\ &= \{3 + 1\} \\ &= \{4\}\end{aligned}$$

(b)

$$\begin{aligned}(f \circ g)(\{2\}) &= \{\lceil 2 \rceil + 1\} \\ &= \{2 + 1\} \\ &= \{3\}\end{aligned}$$

Question 6

Find a formula for the recurrence relation $a_n = 2a_{n-1} + 2^n, a_0 = 1$, using a recursive method.

Solution

$$\begin{aligned} a_0 &= a_0 \\ &= 1 \end{aligned} \qquad \begin{aligned} a_1 &= 2(a_0) + 2^1 \\ &= 2^1 a_0 + 2^1 \\ &= 4 \end{aligned} \qquad \begin{aligned} a_2 &= 2(2^1(a_0) + 2^1) + 2^2 \\ &= 2^2 a_0 + 2^2 + 2^2 \\ &= 2^2 a_0 + 2(2^2) \\ &= 12 \end{aligned} \qquad \begin{aligned} a_3 &= 2(2^2 a_0 + 2(2^2)) + 2^3 \\ &= 2^3 a_0 + 2^3 + 2^3 + 2^3 \\ &= 2^3 a_0 + 3(2^3) \\ &= 32 \end{aligned}$$

$$\begin{aligned} a_n &= 2^n a_0 + n(2^n) \\ &= 2^n + n(2^n) \\ &= (n+1)2^n \end{aligned}$$

Question 7

Let $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ where $f(x) = \lfloor \frac{x}{2} \rfloor$
(a) Show $f(x)$ is surjective.
(b) Show $f(x)$ is not injective.

Solution

(a) *Proof.* Suppose that $x = 2y$.

$$\begin{aligned} f(2x) &= \left\lfloor \frac{2y}{2} \right\rfloor \\ &= \lfloor y \rfloor \end{aligned}$$

Since $x \in \mathbb{N} \cup \{0\}$, $\lfloor y \rfloor$ will be as well.

Every $y \in \mathbb{N} \cup \{0\}$ has an $x \in \mathbb{N} \cup \{0\}$, such that $f(x) = y$ where $x = 2y$. Therefore, $f(x)$ is surjective. □

(b) *Proof.* Suppose $x_1 = 0,\ x_2 = 1$.

$f(x_1) = 0,\ f(x_2) = 0$. However, $x_1 \neq x_2$, meaning that there is an element in the domain of $f(x)$ that is unique but does not map to a unique element in the codomain. Therefore, $f(x)$ is not injective. □

Question 8

Suppose that A and B are sets such that $P(A \cup B) \subset P(A) \cup P(B)$. Prove that either $A \subset B$ or $B \subset A$.

Solution

Proof.

Since the power set contains all subsets of a set, the power set $P(A \cup B)$ contains the set $A \cup B$. Since $P(A \cup B) \subset P(A) \cup P(B)$, $A \cup B \in P(A) \cup P(B)$. Therefore, $(A \cup B \in P(A)) \vee (A \cup B \in P(B))$.

Thus, we have two cases:

Case 1: $A \cup B \in P(A)$. If an element is in the power set, it is a subset of the set: $A \cup B \subset A$. Since $B \subset A \cup B$, $B \subset A$.

Case 2: $A \cup B \in P(B)$. If an element is in the power set, it is a subset of the set: $A \cup B \subset B$. Since $A \subset A \cup B$, $A \subset B$.

Therefore, if there are two sets A and B such that $P(A \cup B) \subset P(A) \cup P(B)$, either $A \subset B$ or $B \subset A$. $\square$

Question 9

Show that the set $\{x|-1 < x < 1\}$ is uncountable by showing that there is a one-to-one correspondence between this set and the set of all real numbers. Hint: A trigonometric function.

Solution

Let $S = \{x| -1 < x < 1\}$

Let $f : \mathbb{R} \to S,$ where $f(x) = \frac{2}{\pi}\tan^{-1} x$

Theorem: f is injective.

Proof. Suppose that $a_1, a_2 \in \mathbb{R}$

$$f(a_1) = \frac{2}{\pi}\tan^{-1} a_1,\ f(a_2) = \frac{2}{\pi}\tan^{-1} a_2$$

$$\begin{aligned} f(a_1) &= f(a_2) \\ \frac{2}{\pi}\tan^{-1} a_1 &= \frac{2}{\pi}\tan^{-1} a_2 \\ \tan^{-1} a_1 &= \tan^{-1} a_2 \end{aligned}$$

Since the range of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we can take the tangent of both sides, as $\tan x$ is defined.

$$a_1 = a_2$$

Since there are no 2 distinct values in the domain that map to the same image, f is injective. □

Theorem: f is surjective.

Proof. Suppose that y is an element in S, the codomain of f. Suppose that x is $\tan\frac{\pi y}{2}$, an element in $\mathbb{R}$, the domain of f.

$$\begin{aligned} f(x) &= \frac{2}{\pi}\tan^{-1} x \\ f\left(\tan\frac{\pi y}{2}\right) &= \frac{2}{\pi}\tan^{-1}\left(\tan\frac{\pi y}{2}\right) \\ &= \frac{2}{\pi}\left(\frac{\pi y}{2}\right) \\ &= y \end{aligned}$$

Since there exists an x, $\frac{\pi y}{2}$, such that $f(x) = y$, f is surjective. □

Theorem: $\{x| -1 < x < 1\}$ is uncountable

Proof. Since f is both injective and surjective, f is bijective. This means that f is a one-to-one function, meaing there is a one-to-one correspondence between $\mathbb{R}$ and S. Therefore, S, $\{x| -1 < x < 1\}$, is uncountable. $\square$

Question 10

a) Find a function $f : \mathbf{Z} \to \mathbf{N}$ that is one-to-one but not onto.
b) Find a function $f : \mathbf{Z} \to \mathbf{N}$ that is one-to-one and onto.

Solution

(a)

$$f(x) = \begin{cases} 2x + 1 & x > 0 \\ -2x & x \leq 0 \end{cases}$$

(b)

$$f(x) = \begin{cases} 2x - 1 & x > 0 \\ -2x & x \leq 0 \end{cases}$$

Question 11

Show that $\displaystyle\sum_{i=1}^{\infty} \frac{1}{4^i} = 2 \sum_{i=1}^{\infty} \frac{1}{7^i}$

$$\sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

$$2\sum_{i=1}^{\infty} \left(\frac{1}{7}\right)^i = 2\left(\frac{\frac{1}{7}}{1 - \frac{1}{7}}\right) = \frac{\frac{2}{7}}{\frac{6}{7}} = \frac{1}{3}$$

Since the two series' equate to the same value, $\displaystyle\sum_{i=1}^{\infty} \frac{1}{4^i}$ and $\displaystyle 2\sum_{i=1}^{\infty} \frac{1}{7^i}$ are equivalent.

Question 12

Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.
a) integers not divisible by 3
b) integers divisible by 5 but not by 7
c) the real numbers with decimal representations consisting of all 1s
d) the real numbers with decimal representations of all 1s or 9s.

Solution

(a)

$$\cdots, a_{-2} = -2, a_{-1} = -1, a_0 = 1, a_1 = 2, a_2 = 4, a_3 = 5, a_4 = 7, a_5 = 8, a_6 = 10, a_7 = 11, a_8 = 13, \cdots$$
$$a_{2n} = 3n + 1$$
$$a_{2n+1} = 3n + 2$$

We can split the set of integers not divisible by 3 into 2:

Let $n_1 = \{x \mid 2x \in \mathbb{Z}\}$
Let $n_2 = \{x \mid 2x + 1 \in \mathbb{Z}\}$

$$S_1 = \{3n_1 + 1\} \qquad S_2 = \{3n_2 + 2\}$$

The set of integers not divisible by 3, which we will denote A, is then $S_1 \cup S_2$. S_1 has a one-to-one correspondence to n_1, even integers. S_2 has a one-to-one correspondence with n_2, odd integers. The union of the two will create a set with a one-to-one correspondence to both even and odd integers, $\mathbb{Z}$. Therefore, A, the set of integers not divisible by 3, is countable.

(b) Let D be the set of all integers divisible by 5 but not by 7. We can represent D by splitting it into sets:

Let $n_1 = \{x \mid 2x \in \mathbb{Z}, 2x \geq 12\}$
Let $n_2 = \{x \mid 2x + 1 \in \mathbb{Z}, 2x + 1 > 12\}$

$$\begin{aligned} a_1 &= -5 & a_7 &= -20 \\ a_2 &= 5 & a_8 &= 20 \\ a_3 &= -10 & a_9 &= -25 \\ a_4 &= 10 & a_{10} &= 25 \\ a_5 &= -15 & a_{11} &= -30 \\ a_6 &= 15 & a_{12} &= 30 \end{aligned}$$

$$a_n = a_{n-12} + 35(-1)^n,\ n > 12$$

The odd indices are negative integers divisible by 5 but not by 7, and the even indices are a positive integer divisible by 5 but not by 7. Since the values of B can be mapped to the positive integers, $\mathbb{Z}^+$, B has a one-to-one correspondence with B. Therefore it is countable.

Let C be the set of real numbers with decimal representations consisting of all 1s. To prove that C is countable, we can first list out the elements in C^+, the set of positive real numbers with decimal representations consisting of all 1s in a grid, where the ith row has $i-1$ 1s before the decimal point. The jth column has j 1s after the decimal point. Every $x \in C, x \in \mathbb{R}^+$ can be found at row i, the amount of 1s before the decimal point of x, and column j, the amount of 1s after thed decimal point.

$$\begin{array}{cccc} 0.1 & 0.11 & 0.111 & \ldots \\ 1.1 & 1.11 & 1.111 & \ldots \\ 11.1 & 11.11 & 11.111 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Let C be the set of negative real numbers with decimal representations consisting of all 1s., the same can be done.

$$\begin{array}{cccc} \text{-0.1} & \text{-0.11} & \text{-0.111} & \text{-}\ldots \\ \text{-1.1} & \text{-1.11} & \text{-1.111} & \text{-}\ldots \\ \text{-11.1} & \text{-11.11} & \text{-11.111} & \text{-}\ldots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Since $C = C^+ \cup C^-$ and both C^+ and C^- are countable, C, the set of real numbers with decimal representations consisting of all 1s, is countable.

Question 13

If A and B are $n \times n$ matrices with $AB = BA = I_n$, then B is called the inverse of A (this terminology is appropriate because such a matrix B is unique) and A is the inverse of B and A and B are said to be invertible. The notation $B = A^{-1}$ denotes that B is the inverse of A. Show that the matrix

$$B = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{pmatrix}$$

is the inverse of

$$A = \begin{pmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{pmatrix}$$

Solution

$$\begin{aligned}
AB &= \begin{bmatrix} 7(2) - 8(1) + 5(-1) & -4(2) + 5(1) - 3(-1) & 1(2) - 1(1) + 1(-1) \\ 7(3) - 8(2) + 5(-1) & -4(3) + 5(2) - 3(-1) & 1(3) - 1(2) + 1(-1) \\ 7(-1) - 8(1) + 5(3) & -4(-1) + 5(1) - 3(3) & 1(-1) - 1(1) + 1(3) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
BA &= \begin{bmatrix} 2(7) + 3(-4) - 1(1) & 2(-8) + 3(5) - 1(-1) & 2(5) + 3(-3) - 1(1) \\ 1(7) + 2(-4) + 1(1) & 1(-8) + 2(5) + 1(-1) & 1(5) + 2(-3) + 1(1) \\ -1(7) - 1(-4) + 3(1) & -1(-8) - 1(5) + 3(-1) & -1(5) - 1(-3) + 3(1) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Since $AB = BA = I_n$, B is the inverse of A.

Question 14

Solve for x: $\lfloor x \rfloor + \sqrt{x - \sqrt{x}} = \left\lfloor x + \frac{1}{x} \right\rfloor$

Solution

To find the domain of x, we can find the domain restrictions of the addends and the sum. $\lfloor x \rfloor$ is defined for all real numbers. $\lfloor x + \frac{1}{x} \rfloor$ is undefined when $x = 0$. $\sqrt{x - \sqrt{x}}$ is undefined when $x - \sqrt{x} < 0$:

$$x - \sqrt{x} < 0$$
$$\sqrt{x}(\sqrt{x} - 1) < 0$$

$$\begin{array}{l|c|c|c}
\sqrt{x}(\sqrt{x}-1) & & - & + \\
\hline
 & 0 & & 1
\end{array}$$

x is undefined when $(-\infty, 1)$. Therefore, the domain is $[1, \infty)$

$$\lfloor x \rfloor + \sqrt{x - \sqrt{x}} = \left\lfloor x + \frac{1}{x} \right\rfloor$$
$$\left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor = \sqrt{x - \sqrt{x}}$$

Let $\lfloor x \rfloor = n, \lceil x \rceil = c$

$$n \leq x \leq c$$

Since the least value of x is n and the greatest is c, we can bound the value of $\lfloor n + \frac{1}{n} \rfloor$:

$$\left\lfloor n + \frac{1}{n} \right\rfloor \leq \left\lfloor x + \frac{1}{x} \right\rfloor \leq \left\lfloor c + \frac{1}{c} \right\rfloor$$

Since n and c are $\lfloor x \rfloor, \lceil x \rceil$ respectively, they are both integers. Therefore, we can simplify:

$$n + \left\lfloor \frac{1}{n} \right\rfloor \leq \left\lfloor x + \frac{1}{x} \right\rfloor \leq c + \left\lfloor \frac{1}{c} \right\rfloor$$

We can bound the values of $\lfloor x + \frac{1}{x} \rfloor - \lfloor x \rfloor$. The lower bound will be:

$$n + \left\lfloor \frac{1}{n} \right\rfloor - \lfloor x \rfloor \leq \left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor$$
$$\lfloor x \rfloor + \left\lfloor \frac{1}{\lfloor x \rfloor} \right\rfloor - \lfloor x \rfloor \leq \left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor$$
$$\left\lfloor \frac{1}{\lfloor x \rfloor} \right\rfloor \leq \left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor$$

The domain is $x \geq 1$.

Case 1: $1 \leq x < 2$

$\lfloor x \rfloor$ will always be 1.

$$\therefore \frac{1}{\lfloor x \rfloor} = 1$$

$$\therefore \left\lfloor \frac{1}{\lfloor x \rfloor} \right\rfloor = 1$$

Case 2: $x \geq 2$

$\lfloor x \rfloor$ will always be greater than 1

$$\therefore \frac{1}{\lfloor x \rfloor} = 0$$

$$\therefore \left\lfloor \frac{1}{\lfloor x \rfloor} \right\rfloor = 0$$

When $1 \leq x < 2$, $\left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor$ must at least be 1. Otherwise, it must be at least 0.

We can find the upper bound of $\left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor$:

$$\left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor \leq c + \left\lfloor \frac{1}{c} \right\rfloor - \lfloor x \rfloor$$

$$\left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor \leq \lceil x \rceil + \left\lfloor \frac{1}{\lceil x \rceil} \right\rfloor - \lfloor x \rfloor$$

The domain is $x \geq 1$.

Case 1: $x = 1$

$$\lceil 1 \rceil + \left\lfloor \frac{1}{\lceil 1 \rceil} \right\rfloor - \lfloor 1 \rfloor = 1$$

Case 2: $x > 1$

$\left\lfloor \frac{1}{\lceil x \rceil} \right\rfloor$ will always be 0.

$$\lceil x \rceil + \left\lfloor \frac{1}{\lceil x \rceil} \right\rfloor - \lfloor x \rfloor = \lceil x \rceil - \lfloor x \rfloor$$

Subcase (i): $x \in \mathbb{Z}$

$$\lceil x \rceil = \lfloor x \rfloor$$

$$\therefore \lceil x \rceil + \left\lfloor \frac{1}{\lceil x \rceil} \right\rfloor - \lfloor x \rfloor = 0$$

Subcase (ii): $x \notin \mathbb{Z}$

$$\lceil x \rceil = \lfloor x \rfloor + 1$$

$$\therefore \lceil x \rceil + \left\lfloor \frac{1}{\lceil x \rceil} \right\rfloor - \lfloor x \rfloor = 1$$

When $x = 1$, $\left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor$ must be at most 1. Otherwise, if x is an integer, it must be at most 0 and if x is not an integer, it must be at most 1.

We can now create different bounds for $\left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor$ depending on the value of x:

Case 1: $1 \leq x < 2$

$$1 \leq \left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor \leq 1$$

By the squeeze theorem,

$$\left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor = 1$$

Case 2: $x \geq 2, x \in \mathbb{Z}$

$$0 \leq \left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor \leq 0$$

By the squeeze theorem,

$$\left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor = 0$$

Case 3: $x \geq 2, x \notin \mathbb{Z}$

$$0 \leq \left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor \leq 1$$

Since $\lfloor x + \frac{1}{x} \rfloor$ and $\lfloor x \rfloor$ are both integers, $\lfloor x + \frac{1}{x} \rfloor - \lfloor x \rfloor$ will be an integer. Since it is bounded below by 0 and above by 1, the only possible values are:

$$\left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor = 0 \ \vee \ \left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor = 1$$

In all cases, $\lfloor x + \frac{1}{x} \rfloor - \lfloor x \rfloor = 0 \ \vee \ \lfloor x + \frac{1}{x} \rfloor - \lfloor x \rfloor = 1$.

$$\left\lfloor x + \frac{1}{x} \right\rfloor - \lfloor x \rfloor = \sqrt{x - \sqrt{x}}$$

Therefore,

$$\sqrt{x - \sqrt{x}} = 0 \ \vee \ \sqrt{x - \sqrt{x}} = 1$$

$$\sqrt{x - \sqrt{x}} = 0$$

Since the domain is $x \geq 1$, squaring both sides will not eliminate solutions.

$$\begin{aligned} x - \sqrt{x} &= 0 \\ \sqrt{x}(\sqrt{x} - 1) &= 0 \\ x &= 0, 1 \end{aligned}$$

$$\sqrt{x - \sqrt{x}} = 1$$

Since the domain is $x \geq 1$, squaring both sides will not eliminate solutions.

$$\begin{aligned} x - \sqrt{x} &= 1 \\ x - \sqrt{x} - 1 &= 0 \end{aligned}$$

Let $a^2 = x$.

$$\begin{aligned} a^2 - a - 1 &= 0 \\ \frac{1 \pm \sqrt{5}}{2} &= a \\ \left(\frac{1 \pm \sqrt{5}}{2}\right)^2 &= x \\ \frac{\left(1 + \sqrt{5}\right)^2}{4}, \frac{\left(1 - \sqrt{5}\right)^2}{4} &= x \end{aligned}$$

We can eliminate $x = 0, \frac{\left(1-\sqrt{5}\right)^2}{4}$ as potential solutions, as they are outside the domain, $x \geq 1$.
We will validate the solution $x = 1$:

$$\left\lfloor 1 + \frac{1}{1} \right\rfloor - \lfloor 1 \rfloor \stackrel{?}{=} \sqrt{1 - \sqrt{1}}$$

$$1 \neq 0$$

$x = 1$ is not a solution.

We will validate the solution $x = \frac{\left(1+\sqrt{5}\right)^2}{4}$

$$\left\lfloor \frac{\left(1+\sqrt{5}\right)^2}{4} + \frac{4}{\left(1+\sqrt{5}\right)^2} \right\rfloor - \left\lfloor \frac{\left(1+\sqrt{5}\right)^2}{4} \right\rfloor \stackrel{?}{=} \sqrt{\frac{\left(1+\sqrt{5}\right)^2}{4} - \sqrt{\frac{\left(1+\sqrt{5}\right)^2}{4}}}$$

$$1 = 1$$

Since the two expressions evaluate to the same value, the solution to the equation $\lfloor x \rfloor + \sqrt{x - \sqrt{x}} = \left\lfloor x + \frac{1}{x} \right\rfloor$ is:

$$\boxed{x = \frac{\left(1+\sqrt{5}\right)^2}{4}}$$