

# Practice Exam 3

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Due: November 14, 2022

## Question 1

Use the Principle of Mathematical Induction to show this inequality is true for all integers  $n \geq 2$ :

$$\sum_{i=1}^n \frac{1}{\sqrt{i}} > \sqrt{n}$$

### Solution

*Proof.* By induction.

Let  $P(n)$  be the statement that  $\sum_{i=1}^n \frac{1}{\sqrt{i}} > \sqrt{n}$ , where  $n \in \mathbb{Z}$ .

**Basis Step:**  $n = 2$

$$\begin{aligned} \sum_{i=1}^2 \frac{1}{\sqrt{i}} &\stackrel{?}{>} \sqrt{2} \\ \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} &\stackrel{?}{>} \sqrt{2} \\ 1 + \frac{\sqrt{2}}{2} &\stackrel{?}{>} \sqrt{2} \\ 2 + \sqrt{2} &\stackrel{?}{>} 2\sqrt{2} \\ 2 &> \sqrt{2} \end{aligned}$$

Since  $2 > \sqrt{2}$ ,  $P(2)$  is true.

**Inductive Hypothesis:** Assume  $P(k)$  is true,  $k \in \mathbb{Z} \wedge k \geq 2$ .

**Inductive Step:** We want to show that  $P(k) \implies P(k+1)$ . Since we use the Inductive Hypothesis, we must prove that  $P(k+1)$  is true.

$$\begin{aligned} \sum_{i=1}^k \frac{1}{\sqrt{i}} &> \sqrt{k} \\ \sum_{i=1}^k \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} &> \sqrt{k} + \frac{1}{\sqrt{k+1}} \\ \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} &> \sqrt{k} + \frac{1}{\sqrt{k+1}} \end{aligned}$$

Using the Inductive Hypothesis, we know the above inequality is true. Thus, it is sufficient to prove that  $\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$  by the transitive property.

$$\begin{aligned}
 \sqrt{k} + \frac{1}{\sqrt{k+1}} &\stackrel{?}{>} \sqrt{k+1} \\
 \sqrt{k+1} \left( \sqrt{k} + \frac{1}{\sqrt{k+1}} \right) &\stackrel{?}{>} \sqrt{k+1} \sqrt{k+1} \\
 \sqrt{k^2 + k + 1} &\stackrel{?}{>} k + 1 \\
 \sqrt{k^2 + k} &\stackrel{?}{>} k \\
 k^2 + k &\stackrel{?}{>} k^2 \\
 k &> 0
 \end{aligned}$$

Since  $k \geq 2$ ,  $k > 0$  is true. Thus, the inequality  $\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$  is true. Therefore,  $P(k+1)$  is true using the Inductive Hypothesis;  $P(k) \implies P(k+1)$ .

**Conclusion:** By induction, we have shown that  $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > \sqrt{n+1}$  is true  $n \geq 2$  □

## Question 2

Prove that for all positive integers  $n$ ,  $3^{2^n} - 1$  is divisible by  $2^{n+2}$ .

### Solution

*Proof.* By induction.

Let  $P(n)$  be the statement that  $3^{2^n} - 1$  is divisible by  $2^{n+2}$ ,  $n \in \mathbb{Z}^+$

**Basis Step:**  $n = 1$

$$\begin{aligned} (3^{2^1} - 1) \bmod 2^{1+2} &= (3^2 - 1) \bmod 2^3 \\ &= 8 \bmod 8 \\ &= 0 \end{aligned}$$

Since the remainder when  $3^{2^1} - 1$  is divided by  $2^{1+2}$  is 0,  $P(n)$  is true when  $n = 1$ .

**Inductive Hypothesis:** Assume  $P(k)$  is true for some  $k \in \mathbb{Z}^+$ .

**Inductive Step:** We want to show that  $P(k) \implies P(k+1)$ . Since we use the Inductive Hypothesis, we must prove that  $P(k+1)$  is true.

$$\begin{aligned} 3^{2^{k+1}} - 1 &= 3^{(2^k)2} - 1 \\ &= \left(3^{2^k}\right)^2 - 1 \\ &= \left(3^{2^k} - 1\right)\left(3^{2^k} + 1\right) \end{aligned}$$

For an integer  $x$  to be even, 2 must be included in the prime factorization of  $x$ . The prime factorization of  $3^{2^k}$  is  $2^k$  amount of 3s. Since 2 is not included and  $3^{2^k} \neq 0$ ,  $3^{2^k}$  is odd. Therefore,  $3^{2^k}$  can be expressed as  $2c + 1$ ,  $c \in \mathbb{Z}$ :

$$\begin{aligned} [(2c + 1) + 1] \bmod 2 &= (2c + 2) \bmod 2 \\ &= ((2c \bmod 2) + 2 \bmod 2) \bmod 2 \\ &= (((2 \bmod 2)(c \bmod 2)) \bmod 2 + 2 \bmod 2) \bmod 2 \\ &= ((0(c \bmod 2)) \bmod 2 + 0) \bmod 2 \\ &= 0 \bmod 2 \\ &= 0 \end{aligned}$$

Therefore,  $3^{2^k} + 1$  is divisible by 2.

Using the Inductive Hypothesis,  $3^{2^k} - 1$  is divisible by  $2^{k+2}$ . Since  $3^{2^k} + 1$  is divisible by 2, then  $\left(3^{2^k} - 1\right)\left(3^{2^k} + 1\right)$  is divisible by  $2^{k+3}$ , as  $2^{k+2} \cdot 2 = 2^{k+3}$ . Thus,  $3^{2^{k+1}} - 1 \pmod{2^{k+2+1}} = 0$ . Since  $P(k+1)$  is true using the Inductive Hypothesis,  $P(k) \implies P(k+1)$ .

**Conclusion:** By induction, we have shown that  $P(n)$  is true, or  $3^{2^n} - 1$  is divisible by  $2^{n+2}$ ,  $n \in \mathbb{Z}^+$ .  $\square$

### Question 3

Find a formula for

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{5^2}\right)\cdots\left(1 - \frac{1}{n^2}\right)$$

where  $n \geq 2$ , and use the Principle of Mathematical Induction to prove that the formula is correct.

### Solution

$$\begin{aligned} & \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{5^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) \\ &= \left(\frac{2^2 - 1}{2^2}\right)\left(\frac{3^2 - 1}{3^2}\right)\left(\frac{4^2 - 1}{4^2}\right)\left(\frac{5^2 - 1}{5^2}\right)\cdots\left(\frac{n^2 - 1}{n^2}\right) \\ &= \left(\frac{(2+1)(2-1)}{2^2}\right)\left(\frac{(3+1)(3-1)}{3^2}\right)\left(\frac{(4+1)(4-1)}{4^2}\right)\left(\frac{(5+1)(5-1)}{5^2}\right)\cdots\left(\frac{(n+1)(n-1)}{n^2}\right) \\ &= \left(\frac{3 \times 1}{2^2}\right)\left(\frac{4 \times 2}{3^2}\right)\left(\frac{5 \times 3}{4^2}\right)\left(\frac{6 \times 4}{5^2}\right)\cdots\left(\frac{(n+1)(n-1)}{n^2}\right) \\ &= \frac{\frac{(n+1)!}{2} \times (n-1)!}{n! \times n!} = \frac{(n+1)! \times (n-1)!}{2 \times n! \times n!} = \frac{n+1}{2n} \end{aligned}$$

*Proof.* By induction.

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{5^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \prod_{i=2}^n 1 - \frac{1}{i^2}$$

Let  $P(n)$  be the statement that:

$$\prod_{i=2}^n 1 - \frac{1}{i^2} = \frac{n+1}{2n}$$

where  $n \geq 2, n \in \mathbb{Z}$ .

**Basis Step:**  $n = 2$

$$\begin{aligned} \prod_{i=2}^2 1 - \frac{1}{i^2} &= \left(1 - \frac{1}{2^2}\right) & \frac{2+1}{2(2)} &= \frac{3}{4} \\ &= \left(1 - \frac{1}{4}\right) \\ &= \frac{3}{4} \end{aligned}$$

Since  $\prod_{i=2}^n 1 - \frac{1}{i^2}$  and  $\frac{n+1}{2n}$  evaluate to the same value,  $\frac{3}{4}$ , when  $n = 2$ ,  $P(2)$  is true.

**Inductive Hypothesis:** Assume  $P(k)$  is true,  $k \geq 2, k \in \mathbb{Z}$ .

**Inductive Step:** We want to show that  $P(k) \implies P(k+1)$ . Since we use the Inductive

Hypothesis, we must prove that  $P(k+1)$  is true.

$$\begin{aligned}
 \prod_{i=2}^k 1 - \frac{1}{i^2} &= \frac{k+1}{2k} \\
 \left( \prod_{i=2}^k 1 - \frac{1}{i^2} \right) \left( 1 - \frac{1}{(k+1)^2} \right) &= \left( \frac{k+1}{2k} \right) \left( 1 - \frac{1}{(k+1)^2} \right) \\
 \prod_{i=2}^{k+1} 1 - \frac{1}{i^2} &= \left( \frac{k+1}{2k} \right) \left( \frac{(k+1)^2 - 1}{(k+1)^2} \right) \\
 &= \frac{(k+1)^2 - 1}{2k(k+1)} \\
 &= \frac{k^2 + 2k + 1 - 1}{2k(k+1)} \\
 &= \frac{k^2 + 2k}{2k(k+1)} \\
 &= \frac{k(k+2)}{2k(k+1)} \\
 &= \frac{k+2}{2(k+1)} \\
 &= \frac{(k+1) + 1}{2(k+1)}
 \end{aligned}$$

$\left( \prod_{i=2}^k 1 - \frac{1}{i^2} \right) \left( 1 - \frac{1}{(k+1)^2} \right)$ , which is equivalent to  $\prod_{i=2}^{k+1} 1 - \frac{1}{i^2}$ , equals  $\frac{(k+1) + 1}{2(k+1)}$ . Thus,  $P(k) \implies P(k+1)$ .

**Conclusion:** By induction, we have shown that  $P(n)$  is true, or  $\prod_{i=2}^n 1 - \frac{1}{i^2} = \frac{n+1}{2n}$ ,  $n \geq 2, n \in \mathbb{Z}$ . □

## Question 4

Which amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using strong induction.

### Solution

*Proof.* By strong induction.

An amount of money,  $n$ , that can be formed using just two-dollar bills and five-dollar bills can be represented by  $n = 2a + 5b$ , where  $a$  is the amount of two-dollar bills, and  $b$  is the amount of five-dollar bills.

Let  $P(n)$  be the statement that  $n = 2a + 5b$ ,  $a, b \in \mathbb{N}$ .

**Basis Step:**  $n \in \{2, 4, 5\}$

$$\begin{array}{ll} n = 4 & n = 5 \\ 2a + 5b = 4 & 2a + 5b = 5 \\ 2(2) + 5(0) = 4 & 2(0) + 5(1) = 5 \end{array}$$

Since  $2a + 5b = 4, 5$  when  $a = 2, b = 0$ ;  $a = 0, b = 1$  respectively,  $P(4)$  and  $P(5)$  are true.

**Inductive Hypothesis:** Assume  $P(j)$  is true,  $4 \leq j \leq k$ , where  $j, k \in \mathbb{Z}$ .

**Inductive Step:** We want to show that  $P(k+1)$  is true using the Inductive Hypothesis.

$$k + 1 = (k - 1) + 2$$

By adding a 2-dollar bill to  $k - 1$ , we can form  $k + 1$ . Using the Inductive Hypothesis,  $P(k - 1)$  is true, meaning  $k - 1$  can be formed with just two-dollar bills and five-dollar bills. Therefore, since  $k + 1$  can be formed by adding another two-dollar bill,  $P(k + 1)$  is true using the Inductive Hypothesis.

**Conclusion:** By strong induction, we have shown that  $P(n)$  is true for  $n \geq 4, n \in \mathbb{Z}$ , or  $n$  dollars can be formed with just two-dollar bills and five-dollar bills,  $n \geq 4, n \in \mathbb{Z}$ .

2 dollars can be formed with 1 two-dollar bill and 0 five-dollar bills. Therefore,  $n$  dollars can be formed with just two-dollar bills and five-dollar bills,  $n \in \{2\} \cup \{k \in \mathbb{Z} \mid k \geq 4\}$ .  $\square$

## Question 5

A baker bakes six different kinds of muffins. If a box with 25 muffins is made with a random number of each kind of muffin, in how many ways can a box of muffins be prepared.

### Solution

When repetition of elements is allowed, there are  $C(n + r - 1, r)$   $r$ -combinations from a set with  $n$  elements. We have  $n = 6$  kinds muffins and  $r = 25$  total muffins:

$$\begin{aligned}\binom{6 + 25 - 1}{25} &= \binom{30}{25} \\ &= 142,506\end{aligned}$$

**There are 142,506 ways for a box of muffins to be prepared.**

## Question 6

Let  $P(n)$  be the statement that a postage of  $n$  cents can be formed using just 3-cent stamps and 5-cent stamps. Parts of this exercise outline a strong induction proof that  $P(n)$  is true for  $n \geq 8$ .

- Show that the statements  $P(8)$ ,  $P(9)$ , and  $P(10)$  are true, completing the basis step of the proof.
- What is the inductive hypothesis of the proof?
- What do you need to prove in the inductive step?
- Complete the inductive step for  $k \geq 10$ .

## Solution

A postage of  $n$  cents that can be formed using just 3-cent stamps and 5-cent stamps can be represented by  $n = 3a + 5b$ , where  $a, b$  are the amount of 3-cent stamps and 5-cent stamps respectively.

### a) Basis Step:

$$\begin{array}{lll}
 n = 8 & n = 9 & n = 10 \\
 3a + 5b = 8 & 3a + 5b = 9 & 3a + 5b = 10 \\
 3(1) + 5(1) = 8 & 3(3) + 5(0) = 9 & 3(0) + 5(2) = 10
 \end{array}$$

Since a postage of 8-cents can be formed with 1 3-cent stamp and 1 5-cent stamp, a postage of 9-cents can be formed with 3 3-cent stamps and 0 5-cent stamps, and a postage of 10-cents can be formed with 0 3-cent stamps and 2 5-cent stamps,  $P(8), P(9), P(10)$  are true.

- Inductive Hypothesis:** Assume  $P(j)$  is true,  $8 \leq j \leq k$  where  $j, k \in \mathbb{Z}$ .
- In the Inductive Step, we must prove that  $P(k) \implies P(k+1)$ . Since we assume  $P(k)$  is true using the Inductive Hypothesis, we must prove that  $P(k+1)$  is true.
- Inductive Step:** we want to prove that  $P(k+1)$  is true using the Inductive Hypothesis. Since we assume  $P(k)$  is true using the Inductive Hypothesis, we must prove that  $P(k+1)$  is true:

$$k + 1 = (k - 2) + 3$$

By adding a 3-cent stamp, we can form  $k+1$ . Using the Inductive Hypothesis,  $P(k-1)$  is true, meaning  $k-1$  can be formed using 3-cent stamps and 5-cent stamps. Therefore, since  $k+1$  can be formed by adding another 3-cent stamp,  $P(k+1)$  is true using the Inductive Hypothesis.

Thus, by induction, we have proved that postage of  $n$  cents can be formed using just 3-cent stamps and 5-cent stamps,  $n \geq 8$ . Since a postage of 3 and 5 cents can also be formed using 1 3-cent stamp and 1 5-cent stamp respectively,  $n \in \{3, 5\} \cup \{k \in \mathbb{Z}, k \geq 8\}$ .



## Question 7

Find the number of strings of length 10 of letters of the alphabet, with no repeated letters, that have vowels in the first two positions. **NOTE:** y is a vowel.

### Solution

The cardinality of the set of vowels,  $\{a, e, i, o, u, y\}$ , is 6. The number of 2 permutations of 6 without repetition is  $6 \times 5$ . This is the amount of unique 2 letter strings, where both letters are in the set of vowels and are not the same.

$$\underline{6} \times \underline{5} \tag{1}$$

The cardinality of the set of letters in the alphabet is 26. Since there cannot be repetition and 2 letters have been used, we are left with 24 letters. The number of 8 permutations of 24 without repetition is  $\frac{24!}{(24-8)!} = \frac{24!}{16!}$ . This is the amount of unique 8 letter strings excluding 2 letters and each letter is different.

$$\underline{24} \times \underline{23} \times \underline{22} \times \underline{21} \times \underline{20} \times \underline{19} \times \underline{18} \times \underline{17} \tag{2}$$

We can use the product rule, multiplying (1) and (2) to find the amount of strings of length 10 of letters of the alphabet, with no repeated letters, that have vowels in the first two positions: there are  $\frac{6!}{4!} \times \frac{24!}{16!}$  strings.

$$\underbrace{\underline{6} \times \underline{5}}_{(1)} \times \underbrace{\underline{24} \times \underline{23} \times \underline{22} \times \underline{21} \times \underline{20} \times \underline{19} \times \underline{18} \times \underline{17}}_{(2)}$$

## Question 8

Ten men and ten women are to be put in a row. Find the number of possible different rows if no two of the same sex stand adjacent.

### Solution

We have  $10!$  ways to organize the men and  $10!$  ways to organize the women. Alternating the genders is the same as placing the 10 women in between 10 men or placing 10 men in between 10 women. This can be visualized as follows:

MWMWMWMWMWMWMWMWMWMW

WMWMWMWMWMWMWMWMWMWM

By the product rule, there are  $10! \times 10!$  ways to order 10 women in between 10 men. Likewise, there are  $10! \times 10!$  ways to order 10 men in between 10 women. Therefore, the total amount of different rows if no two of the same sex stand adjacent is the sum of these two possibilities:  $10! \times 10! + 10! \times 10! = 2 \times 10! \times 10!$

## Question 9

If positive integers are chosen at random, what is the minimum number you must have in order to guarantee that two of the chosen numbers are congruent modulo 6. Prove your answer.

**NOTE:** Two numbers are congruent modulo 6 if their difference is a multiple of 6.

### Solution

*Proof.* By definition of congruence modulo,  $a \equiv b \pmod{c} \iff a \bmod c = b \bmod c$ . Therefore, two integers,  $a$  and  $b$ , are congruent modulo 6 if and only if  $a \bmod 6 = b \bmod 6$ .

The codomain of the function  $f(x) = x \bmod 6$  is  $\{k, 0 \leq k \leq 5, k \in \mathbb{Z}\}$ . Suppose that for each element  $y$  in the codomain of  $f$ , we have a box that contains all elements  $x$  of the domain of  $f$  such that  $f(x) = y$ . The cardinality of the codomain of  $f$  is 6. Therefore, there are only 6 possible boxes. By the pigeonhole principle, the number of random values in the domain must be at least 7 to have one box contain 2 values; with 7 random integers in the domain, at least 2 must map to the same value in the codomain—at least 2 are congruent modulo 6—because there are only 6 values in the codomain. Thus, there must be a minimum of 7 random positive integers to guarantee that two numbers are congruent modulo 6.  $\square$

**Question 10**

Write the expansion of  $\left(x^2 - \frac{1}{x}\right)^{12}$ .

**Solution**

$$\begin{aligned}
 \left(x^2 - \frac{1}{x}\right)^{12} &= \sum_{k=0}^{12} \binom{12}{k} (x^2)^{12-k} \left(-\frac{1}{x}\right)^k \\
 &= \binom{12}{0} (x^2)^{12} \left(-\frac{1}{x}\right)^0 + \binom{12}{1} (x^2)^{11} \left(-\frac{1}{x}\right)^1 + \binom{12}{2} (x^2)^{10} \left(-\frac{1}{x}\right)^2 \\
 &\quad + \binom{12}{3} (x^2)^9 \left(-\frac{1}{x}\right)^3 + \binom{12}{4} (x^2)^8 \left(-\frac{1}{x}\right)^4 + \binom{12}{5} (x^2)^7 \left(-\frac{1}{x}\right)^5 \\
 &\quad + \binom{12}{6} (x^2)^6 \left(-\frac{1}{x}\right)^6 + \binom{12}{7} (x^2)^5 \left(-\frac{1}{x}\right)^7 + \binom{12}{8} (x^2)^4 \left(-\frac{1}{x}\right)^8 \\
 &\quad + \binom{12}{9} (x^2)^3 \left(-\frac{1}{x}\right)^9 + \binom{12}{10} (x^2)^2 \left(-\frac{1}{x}\right)^{10} + \binom{12}{11} (x^2)^1 \left(-\frac{1}{x}\right)^{11} \\
 &\quad + \binom{12}{12} (x^2)^0 \left(-\frac{1}{x}\right)^{12} \\
 &= x^{24} - 12x^{21} + 66x^{18} - 220x^{15} + 495x^{12} - 792x^9 + 924x^6 - 792x^3 + 495 \\
 &\quad - \frac{220}{x^3} + \frac{66}{x^6} - \frac{12}{x^9} + \frac{1}{x^{12}}
 \end{aligned}$$

## Question 11

In how many ways can 7 of the 8 letters in CHEMISTS be put in a row?

### Solution

The word CHEMISTS has 8 letters, not all of them distinct. To create a 7 letter word, we can consider 2 cases:

**Case 1:** Selecting 2 S's

There are  $\binom{7}{2}$  ways to choose 2 slots where the 2 S's will go. That leaves 5 positions for 6 letters, which there are  ${}^6P_5 = \frac{6!}{(6-5)!} = 6!$  ways to arrange them. By the product rule, there are  $\binom{7}{2} \times 6!$  ways to make a 7 letter word from the letters of CHEMISTS, given that we select 2 S's.

**Case 2:** Selecting 1 S

There are  $\binom{7}{1}$  ways to choose 1 slot where the S will go. That leaves 6 positions for 6 letters (There are 7 letters left, but since we are considering only 1 S, we remove the remaining S from the count as well.), which there are  ${}^6P_6 = \frac{6!}{(6-6)!} = 6!$  ways to arrange them. By the product rule, there are  $\binom{7}{1} \times 6!$  ways to make a 7 letter word from the letters of CHEMISTS, given that we select only 1 S.

We can add these two cases together to get the total ways:

$$\begin{aligned}\binom{7}{2} \times 6! + \binom{7}{1} \times 6! &= \frac{7!}{(7-2)!} \frac{6!}{2!} + \frac{7!}{(7-1)!} \frac{6!}{1!} \\ &= 7! \times 3 + 7! \\ &= 7! \times 4 \\ &= 20160\end{aligned}$$

**There are 20160 ways to put 7 of the 8 letters in CHEMISTS in a row.**

## Question 12

What is the minimum number of cards that must be drawn from an ordinary deck of cards to guarantee that you have been dealt

(a) at least three of at least one suit?

(b) at least three clubs?

Explain.

### Solution

- a) A minimum of 9 cards must be drawn to guarantee at least three of at least one suit. Suppose there is a box for each suit (There will be 4 boxes), and each time a card is drawn, it is placed in the box that corresponds to its suit. With 8 cards, it is possible to have 2 cards of each suit, therefore more than 8 cards are needed. By the generalized pigeonhole principle, with at least 9 objects, at least three objects will be in one box—at least three cards that are the same suit. We can verify this algebraically:

By generalized pigeonhole principle, if  $N$  objects are placed in  $k$  boxes, then there is at least one box containing at least  $\lceil N/k \rceil$  objects. Let  $k$  be the number of suits;  $k = 4$ .

$$\left\lceil \frac{N}{4} \right\rceil \geq 3$$

$$N \geq 2 \times 4 + 1$$

$$N \geq 9$$

Thus, a minimum of 9 cards must be drawn to guarantee at least three of at least one suit.

- b) A minimum of 42 cards must be drawn to guarantee at least three clubs. Note that in the worst case, we can select all the hearts, diamonds, and spades, a total of 39 cards, before we select a single club. The next 3 cards will be all clubs.

### Question 13

Prove the identity  $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$ , whenever  $n, r$ , and  $k$  are nonnegative integers with  $r \leq n$  and  $k \leq r$ ,

- a) algebraically.
- b) using a combinatorial argument.

### Solution

a) *Proof.*

$$\begin{aligned}
 \binom{n}{k}\binom{n-k}{r-k} &= \left(\frac{n!}{(n-k)!k!}\right)\left(\frac{(n-k)!}{(n-k-(r-k))!(r-k)!}\right) \\
 &= \left(\frac{n!}{(n-k)!k!}\right)\left(\frac{(n-k)!}{(n-r)!(r-k)!}\right) \\
 &= \frac{n!}{(n-r)!(r-k)!k!} \\
 &= \left(\frac{n!}{(n-r)!(r-k)!k!}\right)\left(\frac{r!}{r!}\right) \\
 &= \left(\frac{n!}{(n-r)!r!}\right)\left(\frac{r!}{(r-k)!k!}\right) \\
 &= \binom{n}{r}\binom{r}{k}
 \end{aligned}$$

Thus, we have proved the identity  $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$ ,  $n, r, k \in \mathbb{N} \wedge r \leq n \wedge k \leq r$  algebraically.  $\square$

- b) *Proof.* Consider a group of  $n$  items. We want to select one group with  $k$  items and another group with  $r - k$  items such that the two groups have no items in common.

To select  $k + (r - k) = r$  items from  $n$  items, there are  $\binom{n}{r}$  such ways. We want  $r$  of the  $k$  items to be in the first group, which there are  $\binom{r}{k}$  such ways. By the product rule, the total ways to select  $r$  items from  $n$  items to be put in the first group (thus  $r - k$  items in the second group) is  $\binom{n}{r}\binom{r}{k}$ .

To select  $k$  items to place in the first group from  $n$  items, there are  $\binom{n}{k}$  such ways. There will now be  $n - k$  items remaining. From the  $n - k$  items, we want to choose  $r - k$  items to place in the second group, which there are  $\binom{n-k}{r-k}$  such ways. By the product rule, the total ways to select  $k$  items from  $n$  items to be put in the first group and  $r - k$  items from  $n - k$  items to be put in the second group is  $\binom{n}{k}\binom{n-k}{r-k}$ .

Both result in the different ways to select one group with  $k$  items and another group with  $r - k$  items from a group of  $n$  items. Thus:

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

$\square$

## Question 14

How many solutions are there to the equation  $\sum_{i=1}^6 x_i = 29$ , where  $x_i, i = 1, 2, 3, 4, 5, 6$ , is a non-negative integer such that

- a)  $x_i > 1$  for  $i = 1, 2, 3, 4, 5, 6$ ?
- b)  $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, x_4 \geq 4, x_5 > 5$ , and  $x_6 \geq 6$ ?
- c)  $x_1 \geq 5$ ?
- d)  $x_1 < 8$  and  $x_2 > 8$ ?

## Solution

Let  $S = \{1, 2, 3, 4, 5, 6\}$

- a) Since  $x_i > 1$ , there are at least 2 integers distributed among  $x_i, i \in S$ . Thus, there is 23 left to be distributed amongst  $x_i, i \in S$ :

$$\binom{6-1+29-2(6)}{29-2(6)} = \binom{22}{17}$$

Thus, there are  $\binom{22}{17} = 26,334$  solutions given the constraints.

- b) There are at least 1, 2, 3, 4, 6, 6 distributed to  $x_i, i \in S$  respectively. Thus, there is 22 left to be distributed amongst  $x_i, i \in S$ :

$$\binom{6-1+29-(1+2+3+4+6+6)}{29-(1+2+3+4+6+6)} = \binom{12}{7}$$

Thus, there are  $\binom{12}{7} = 792$  solutions given the constraints.

- c) Since  $x_1 \geq 5$ , there are at least 5 integers distributed to  $x_1$ . Thus, there is 24 left to be distributed amongst  $x_i, i \in S$ :

$$\binom{6-1+29-5}{29-5} = \binom{29}{24}$$

Thus, there are  $\binom{29}{24} = 118,755$  solutions given the constraints.

- d) To find the amount of solutions where  $x_1 < 8 \wedge x_2 > 8$ , we can first find the amount of solutions there are where  $x_2 > 8$ .

Since  $x_2 > 8$ , there are at least 9 integers distributed to  $x_2$ . Thus, there are 20 left to be distributed amongst  $x_i, i \in S$ :

$$\binom{6-1+29-9}{29-9} = \binom{25}{20}$$

$\binom{25}{20}$  is the amount of solutions where  $x_1 \in \mathbb{R}$ . We can find the amount of solutions where  $x_2 < 8$  by subtracting the amount of solutions where  $x_1 \geq 8 \wedge x_2 > 8$  from the set of solutions where  $x_2 > 8$ :

$$\binom{25}{20} - \binom{6-1+29-8-9}{29-8-9} = \binom{25}{20} - \binom{17}{12}$$

Thus, there are  $\binom{25}{20} - \binom{17}{12} = 46,942$  solutions given the constraints.