Exam 2 Corrections

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Question 1

If $\overline{A} \cap B = B \ \wedge \ \overline{B} \cap A = A$, what can you say about A and B? Prove your answer. ($\overline{S}$ denotes the complement of S.)

Solution

Proof. Assume $x \in A$. By definition of complement, $x \notin \overline{A}$. Since $x \notin \overline{A}$, $x \notin \overline{A} \cap B$. Since $\overline{A} \cap B = B$, $x \notin B$. x is arbitrary, therefore we can generalize to $A \not\subseteq B$. $\square$

Proof. Assume $y \in B$. By definition of complement, $y \notin \overline{B}$. Since $y \notin \overline{B}$, $y \notin \overline{B} \cap A$. Since $\overline{B} \cap A = A$, $y \notin A$. y is arbitrary, therefore we can generalize to $B \not\subseteq A$. $\square$

Proof. Since $A \not\subseteq B \ \wedge \ B \not\subseteq A$, $A \cap B = \emptyset$

$\overline{A} \cap B = B \ \wedge \ \overline{B} \cap A = A \implies A \cap B = \emptyset$ $\square$

Question 2

Suppose that $g : A \to A$ and $f : A \to A$ where $A = \{a, b, c, d\}$, $g = \{(a, a), (b, c), (c, a), (d, c)\}$ and $f = \{(a, d), (b, a), (c, b), (d, a)\}$. Find $g \circ (f \circ g)$, that is, find the composition of $g(x)$ with $f(x)$ composed with $g(x)$.

Solution

$$
\begin{aligned}
g(A) &= \{(a, a), (b, c), (c, a), (d, c)\} \\
f(g(A)) &= \{(a, d), (b, b), (c, d), (d, b)\} \\
g(f(g(A))) &= \{(a, c), (b, c), (c, c), (d, c)\}
\end{aligned}
$$

Question 3

Verify that $a_n = \pi$ **is a possible solution** to the recurrence relation $a_n = (2n-1)a_{n-1} - (2n-2)a_{n-2}$. What are the conditions for this to happen?

Solution

Question 4

Evaluate $\sum_{n=1}^{3}\sum_{k=1}^{n}(n^k)$. Show all your steps.

Solution

$$\begin{aligned}\sum_{n=1}^{3}\sum_{k=1}^{n}(n^k) &= \sum_{k=1}^{1}1^k + \sum_{k=1}^{2}2^k + \sum_{k=1}^{3}3^k \\ &= 1 + 2 + 4 + 3 + 9 + 27 \\ &= 46\end{aligned}$$

Question 5

Find a $f : \mathbb{N} \to \mathbb{Z}$ that is surjection that is not injection. Prove that it satisfies the conditions held.

Solution

$\mathbb{N} = \mathbb{Z}^{+} \cup \{0\}$
Let $S_1 = \{x \mid 2x, x \in \mathbb{N}\}$, $S_2 = \{x \mid 2x + 1, x \in \mathbb{N}\}$

$$f(n) = \begin{cases} \frac{n}{2} & n \in S_1 \\ -(\frac{n}{2} - \frac{1}{2}) & n \in S_2 \\ 10 & n = 1 \end{cases}$$

Theorem: $f(n)$ is surjective.

Proof. We can separate the domain, $\mathbb{N}$ into two sets of odd and even naturals, S_1 and S_2 respectively.
Let $n = 2y$. Since $2y$ is always an even, it will satisfy the first case, $n \in S_1$. □