MATH 1700: Ideas in Mathematics

Worksheet 2: Numbers and Infinity First Submission

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1 Warm-Up Problems

(1) **Briefly summarize the "machine method."**

The "machine method" is a method to demonstrate that a set is infinite by building a machine (an algorithm) that takes as input a finite list and names an element of the set that is not in the list.

(2) **Use the machine method to show that there are infinitely many odd natural numbers; i.e., infinitely many numbers that don't have 2 as a factor. Be sure to include an explanation why your machine works as intended.**

Proof. Let S be a finite set of odd natural numbers. Let a machine run the function $f : \mathbb{N} \to \mathbb{N}$ with the rule $f(n) = n + 2$. Let $x = \max(S)$ and feed it into the machine. We can verify that the output is always odd by observing if the remainder of the output divided by 2 is 0:

$$f(x) \text{ mod } 2 \equiv (x \text{ mod } 2) + (2 \text{ mod } 2) \equiv x \text{ mod } 2$$

As x is odd, x mod $2 = 1$. Thus, $f(x)$ mod $2 = 1$. Therefore, the output is odd. Since we add 2 to the input, the output is always greater than the input. As the input is the greatest number in S, the output is also greater than all elements in S. Therefore, the output is not in S. We have shown that no finite list can contain the set of all odd natural numbers, and thus the set of all odd natural numbers is infinite. □

(3) **State what it means for a natural number to be prime, and what it means for a natural number to be composite.**

A prime natural number is a natural number greater than 1 that only has factors of itself and 1. A composite natural number is a natural number greater than 1 that has another factor other than itself and 1.

(4) **You receive the following message from a friend:**

> **There are infinitely many prime numbers. Here's why. We know that every natural number greater than one has a prime factor. There are infinitely many natural numbers greater 1. As the numbers get bigger, their prime factors have to get bigger. Thus, there are infinitely many prime numbers. —Your friend.**

How would you explain to your friend the flaw in their reasoning?

Proof. By counterexample.

Hi friend! Your claim can be expressed as the following:

$$\forall x, \forall y \in \mathbb{N}(P(x, y) \to Q(x, y))$$

Where $P(x, y)$ is the statement that $x > y$ and $Q(x, y)$ is the statement that x has a prime factor greater than y. We construct a counterexample by choosing $x = 8$ and $y = 2$. We can verify that $P(8, 2)$ is true, but $Q(8, 2)$ is false. As $\exists x, \exists y \in \mathbb{N} \mid (P(x, y) \land \neg Q(x, y))$, the original statement is false. □

The issue with your claim is saying "as the numbers get bigger, their prime factors have to get bigger." This statement implies a strictly increasing relation between the natural numbers and their prime factors. However, as shown, this is not the case. Hope this helps!

— Denny Cao

2 Step Toward a Proof That There Are Infinitely Many Primes

(5) **Pick any natural number other than 1, call it n. Next, choose a natural number that has n as a factor. Let's call this number k. Does the number $k + 1$ have n as a factor? Repeat a few times with different values of n and k, and record the results.**

n	k	$k+1$	$k+1$ has n as a factor?
2	2	3	**F**
3	3	4	**F**
5	5	6	**F**
7	7	8	**F**
11	11	12	**F**
13	13	14	**F**
17	17	18	**F**

(6) **Explain why, if n is a natural number greater than 1, and k is a natural number that has n as a factor, the number $k + 1$ cannot also have n as a factor.**

Proof. By contradiction.

To prove $\forall k(P(k) \to \neg P(k+1))$, where $P(k)$ is the statement that k has n as a factor, assume the negation: $\exists k(P(k) \land P(k+1))$. This means that there exists a natural number k such that k has n as a factor and $k + 1$ also has n as a factor. This means $k + 1$ is divisible by n:

$$(k+1) \bmod n \equiv 0$$

$(k + 1) \bmod n$ can be rewritten as:

$$k \bmod n + 1 \bmod n \equiv 0$$

From our assumption, k is also divisible by n, and thus $k \bmod n \equiv 0$:

$$1 \bmod n \equiv 0$$

For this statement to be true, n must equal 1. However, from Question 5, we define n as: $n \in \mathbb{N} \land n > 1$. Thus, we arrive at a contradiction. ※

Therefore, $\forall k(P(k) \to \neg P(k+1))$, meaning if k has n as a factor, $k + 1$ cannot have n as a factor. $\square$

(7) **If you're given a list of finitely many prime numbers, what is a way to produce a single number k that has each of the primes on the list as a factor? (It might help to try an example with a concrete list of primes first, say the list 2, 7, 29, 103.)**

Multiply all the primes together. For example, if the list is 2, 7, 29, 103, then $k = 2 \cdot 7 \cdot 29 \cdot 103 = 41818$. This works because k is divisible by each of the primes in the list.

(8) **Explain a method that, given any list of finitely many prime numbers, produces a natural number that does not have any of the primes on the list as a factor. (Put together the ideas from the preceding two exercises.)**

From Question 6, we know that if k has n as a factor, then $k + 1$ does not have n as a factor. From this, we can conclude that $k + 1$ does not contain any of the same factors as k except 1. From Question 7, we know that if we multiply all the primes together, we get a number that has all the primes as factors. Thus, if we add 1 to the product of all the primes, we get a number that does not have any of the primes as factors.

(9) **In the method you described in the previous problem, is it always the case that the number produced is itself a prime number? If so, explain why. If not, provide a counterexample. If, for some inputs, the output number is not prime, what can you say about the prime factors of the output number?**

Proof. By counterexample.

Let k be the product of the primes 11, 13, and 17. Then $k = 11 \cdot 13 \cdot 17 = 2431$. $k + 1$ is 2432, which is not prime, as $2432 \bmod 2 \equiv 0$. Thus, adding 1 to the product of primes does not always produce a prime number. □

We can say that the prime factors of the output number are different from the prime factors of k from Question 7.

3 Proof That There Are Infinitely Many Primes

(10) **Put together the ideas from the previous exercises to show that there are infinitely many prime numbers. That is, describe a machine that takes as its input a finite list of prime numbers, and outputs a prime number that is not on the input list. (It might help to refer to the fact that every natural number larger than 1 has a prime factor.)**

Proof. Let S be a finite set of primes. Let a machine run an algorithm that creates a new set, S', that is the set of all primes from 2 to the greatest element of S. Let k be the product of all the primes in S'. From Question 7, we know that k has all the primes in S' as factors. From Question 6, we know that $k + 1$ does not have any of the primes in S' as factors. Thus, $k + 1$ contains a prime factor that is not in S'. As $S' \supseteq S$, this factor is also not in S. We have shown that no finite list can contain all the primes, and thus there are infinitely many primes. □

4 Reflection

What content do I need to review before attempting the worksheet again? Are there any videos I need to rewatch?

I need to review the machine method on how to justify that, because it is possible to generate an output that is not in the input, the collection is infinite. I'm confused on how this can be generalized to show that it is true for all inputs. It makes sense when the algorithm is algebraic, but I'm not sure how it can be used for Question 10.

What questions would I like to ask my group during the next class discussion?

For Question 4, is the friend partially correct? I think there is something that makes sense to their reasoning, like how 213 has larger prime factors than 8. However, there still exist numbers such as 256 that have the same prime factors as 8. As the numbers increase, the prime factors will increase as well—just not a strictly increasing relation.