MATH 1700: Ideas in Mathematics

Worksheet 3: Sets and Functions First Submission

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1 Function Review

(1) **Consider the relationships below. None of them are functions. For each example, explain why.**

(a) $f : \{a, b, c\} \to \mathbb{N}$

x	$f(x)$
a	12
b	7
b	4
c	128

Proof. The definition of a function is that all elements in the domain have a unique image in the codomain. In this case, b has two images, 7 and 4, and is therefore f is not a function. □

(b) $g : \mathbb{N} \to \mathbb{N}, g(n) = n - 4$

Proof. The definition of a function is that all elements in the domain have a unique image in the codomain. In this case, the preimages 1,2,3, and 4 map to numbers not in the codomain of $\mathbb{N}$. Therefore, for these values, there is no unique image for all values in the domain, meaning g is not a function. □

(c) $h : \mathbb{Z} \to \mathbb{Q}, h(z) = \dfrac{1}{z}$

Proof. The definition of a function is that all elements in the domain have a unique image in the codomain. In this case, the preimage of 0 maps to an undefined value, and therefore there is no unique image for all values in the domain, meaning h is not a function. □

(2) **Consider the function from $\mathbb{N}$ to $\mathbb{Z}$ that multiplies every number in the source by 2. Is this function injective? Is it surjective?**

Proof. **The function is injective**. The definition of injective for a function f with domain A is

$$\forall x \forall y((x, y \in A \land f(x) = f(y)) \to x = y)$$

In this case, $f(x) = 2x$ and $f(y) = 2y$, so if $f(x) = f(y)$, then $2x = 2y \to x = y$. Therefore, the function is injective. □

Proof. **The function is not surjective**. The definition of surjective for a function f with domain A is

$$\forall y \exists x(x \in A \land f(x) = y)$$

In this case, $f(x) = 2x$. This means that all values in the domain map to an even number. Since the codomain is $\mathbb{Z}$, there are infinitely many integers that are not even, there are infinitely many values in the codomain that are not images of the domain. Therefore, the function is not surjective. □

(3) **Consider the function from $\mathbb{N}$ to $\mathbb{N}$ that adds one to every number in the source. Is this function injective? Surjective?**

Proof. **The function is injective**. The definition of injective for a function f with domain A is

$$\forall x \forall y((x, y \in A \land f(x) = f(y)) \rightarrow x = y)$$

In this case, $f(x) = x+1$ and $f(y) = y+1$, so if $f(x) = f(y)$, then $x+1 = y+1 \rightarrow x = y$. Therefore, the function is injective. □

Proof. **The function is not surjective**. The definition of surjective for a function f with domain A is

$$\forall y \exists x(x \in A \land f(x) = y)$$

In this case, $f(x) = x + 1$. To obtain an image of 1, the preimage must be 0. However, 0 is not in the domain of the function, $\mathbb{N}$, meaning $\exists y \forall x(x \in A \land f(x) \neq y)$. Therefore, the function is not surjective. □

(4) **Consider the function from $\mathbb{Z}$ to $\mathbb{Z}$ that adds one to every number in the source. Is this function injective? Surjective?**

Proof. **The function is injective**. The definition of injective for a function f with domain A is

$$\forall x \forall y((x, y \in A \land f(x) = f(y)) \rightarrow x = y)$$

In this case, $f(x) = x+1$ and $f(y) = y+1$, so if $f(x) = f(y)$, then $x+1 = y+1 \rightarrow x = y$. Therefore, the function is injective. □

Proof. **The function is surjective**. The definition of surjective for a function f with domain A is

$$\forall y \exists x(x \in A \land f(x) = y)$$

In this case, $f(x) = x + 1$. For all elements in the codomain, there exists a preimage in the domain, $x = y - 1$. Therefore, the function is surjective. □

(5) **Consider the function from $\mathbb{Z}$ to $\mathbb{R}$ that sends ever integer to itself. Is this function injective? Surjective?**

Proof. **The function is injective**. The definition of injective for a function f with domain A is

$$\forall x \forall y((x, y \in A \land f(x) = f(y)) \rightarrow x = y)$$

In this case, $x \in \mathbb{Z} \rightarrow f(x) = x$. Let $a, b \in \mathbb{Z}$. As $f(a) = a$ and $f(b) = b$, $f(a) = f(b) \rightarrow a = b$. Therefore, the function is injective. □

Proof. **The function is not surjective**. The definition of surjective for a function f with domain A is

$$\forall y \exists x (x \in A \land f(x) = y)$$

As the domain is $\mathbb{Z}$, the domain maps solely to integers in the codomain. However, the codomain is $\mathbb{R}$, which contains infinitely many real numbers that are not integers. As $\exists y \forall x (x \in A \land f(x) \neq y)$, the function is not surjective. □

(6) **Consider the function from $\mathbb{N}$ to $\mathbb{N}$ that subtracts one from every number other than 1, and sends 1 to 1. Is this function injective? Surjective?**

Proof. **The function is injective**. The definition of injective for a function f with domain A is

$$\forall x \forall y ((x, y \in A \land f(x) = f(y)) \to x = y)$$

We can represent the function as a piecewise function:

$$f : \mathbb{N} \to \mathbb{N}, f(x) = \begin{cases} 1 & x = 1 \\ x - 1 & x \neq 1 \end{cases}$$

The union of the domains of the two pieces is the domain of the function, $\mathbb{N}$. The intersection of the domains of the two pieces is the empty set, so the function is injective. □

Proof. **The function is surjective**. The definition of surjective for a function f with domain A is

$$\forall y \exists x (x \in A \land f(x) = y)$$

When $x \neq 1$, $f(x) = x - 1$. For all values y in the codomain, there is a value x in the domain, $y + 1$, that maps to it. When $x = 1$, $y = 1$. Despite the image 1 having 2 preimages, 1, and 2, the definition of surjective is still satisfied. □

(7) (a) **Write down a function with source $\mathbb{N}$ and target $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ which is not surjective.**

$$f : \mathbb{N} \to \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}, f(x) = \spadesuit$$

(b) **Are there any injective functions from $\mathbb{N}$ to $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$? Give an example or very briefly explain why not.**

Proof. **There are no injective functions from $\mathbb{N}$ to $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$**. We prove this by the pigeonhole principle. Let the values in the codomain be holes, and let the values in the domain be pigeons. There are 4 holes. With 4 pigeons, we can place each pigeon in each hole. With 5 pigeons, there must be a hole that contains at least 2—with 5 pigeons, the function is not injective. As the cardinality of $\mathbb{N} > 5$, there will be more pigeons than holes, meaning at least 1 hole contains more than 1 pigeons. Thus, there is no possible injective function. □

(c) **Are there any injective functions from $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ to $\mathbb{N}$ which are not surjective? Give an example or very briefly explain why not.**

Proof. **There are injective functions from $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ to $\mathbb{N}$ which are not surjective**. As the cardinality of $\mathbb{N}$ is greater than the cardinality of $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$, there are more elements in the codomain than the domain. Thus, it is possible for the elements in the domain to map to unique elements in the codomain. However, since the cardinality of the codomain is greater than the domain, there are elements in the codomain that are not mapped to by the domain. Thus, the function is not surjective. □

2 Infinite Sets

(8) **Suppose you have a very large fish bowl, and infinitely many fish nearby. The fish are numbered 1, 2, 3, 4, . . . When you start out, none of the fish are in the bowl. Then suddenly fish #1 leaps into the bowl. Next, fish #2, #3,. . . ,#10 all jump into the bowl, and fish #1 jumps back out. After that, fish #11 through #100 all jump in, and #2 jumps out. Next #101 through #1000 all jump in and #3 jumps out. The process continues, ad infinitum.**

(a) **How many fish are in the bowl after 4 steps of the above process? 10 steps? 100 steps?**

$$f(n) = \begin{cases} 1 & n = 1 \\ 1 + \displaystyle\sum_{i=2}^{n} \frac{9}{100}(10)^i - 1 & n \geq 2 \end{cases}$$

$$\begin{aligned} \sum_{i=2}^{n} \frac{9}{100}(10)^i - 1 = \sum_{i=1}^{n}\left(\frac{9}{100}(10)^i\right) - \left(1 - \frac{1}{100} - 1\right) &= \frac{\frac{9}{100}(10)^{n+1} - \frac{9}{100}}{9} - n - \frac{1}{100} \\ &= \frac{10^{n+1}}{100} - \frac{1}{100} - n - \frac{1}{100} \\ &= 10^{n-1} - n \end{aligned}$$

After 4 steps, $n = 4$, there will be 996 fish in the bowl. After 10 steps, $n = 10$, there will be 999,996 fish in the bowl. After 100 steps, $n = 100$, there will be 999,999,990 fish in the bowl. After 100 steps, there $10^{99} - 100$ fish in the bowl.

(b) **After infinitely many steps, how many fish are in the bowl? (Hint: At the end of the minute, each fish is either in the bowl or outside the bowl. Which fish are in and which are out?)**

There will be $10^{n-1} - n$ fish in the bowl after n steps. As n approaches infinity, the number of fish in the bowl approaches infinity.

3 Russell's Paradox

(9) **Consider the set of all sets that contain themselves. Can you tell whether or not this set contains itself?**

It is impossible to determine.

Proof. By contradiction. Suppose that the set of all sets that contain themselves contains itself. Then, it must not contain itself because if it contains itself, it does not belong to the set of all sets that contain themselves.

On the other hand, if the set of all sets that contain themselves does not contain itself, then it should belong to the set of all sets that contain themselves.

Therefore, we have reached a contradiction, and it is impossible to determine whether or not the set of all sets that contain themselves contains itself. □

(10) **Now consider the set of all sets that do not contain themselves. Can this set contain itself? Can it not?**

This set cannot contain itself.

Proof. By contradiction. Assume for purposes of contradiction that there is a set S of all sets that do not contain themselves. $S = \{A \mid A \notin A\}$, where A is a set. $S \in S \to S \notin S$, as S is a set of sets that do not contain themselves. If $S \notin S$, then $S \in S'$, the complement of S—a set of all sets that contain themselves. Therefore, $S \in S$. ※

We reach a contradiction, as S is the set of all sets that do not contain themselves. Therefore, S cannot contain itself. □

4 Reflection

What content do I need to review before attempting the worksheet again? Are there any videos I need to rewatch?
I need to review definitions. I used preimage and image interchangably with elements in the domain and elements in the codomain, and I'm not sure if that is correct. I also need to see if there is a better way to explain 7b about how more elements in the codomain than the domain means that injective functions can be created.
What questions would I like to ask my group during the next class discussion?
The fishbowl question. I did not know how it related to this week's topic, so there is probably something that I am missing. It might explain why I don't really know how to answer 8b.