Worksheet 5: Uncountable Sets and Cantor's Diagonal Argument

MATH 1700: Ideas in Mathematics

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1 Warm-Up: X's and O's

Question 1. Consider the following two-player game, which we will call Cantor's Game.

Player 1 begins by writing a sequence of X's and O's in the top row of the grid below. Player 2 then writes either an X or an O in the first box on the bottom. Player 1 then writes a sequence of X's and O's in the second row of the grid. Player 2 writes an X or O in the second bottom box. The players continue until all boxes are filled. Player 1 wins if the sequence on the bottom exactly matches any of the sequences Player 1 has written in the grid. Player 2 wins otherwise.

a) Which player has a winning strategy, and why?

b) How does that strategy relate to Cantor's diagonal argument?

Answer 1.

a) Player 1 has a winning strategy. Player 1 can always write a sequence of X's and O's in the top row that will match the sequence Player 2 writes in the bottom row.

b) Player 1's strategy is to write a sequence of X's and O's in the top row that will match the sequence Player 2 writes in the bottom row. This is the same strategy Cantor used to prove that the set of real numbers is uncountable.

Question 2. Prove that the set of all infinite sequences of X's and O's is uncountable.

Answer 2. *Proof.* By contradiction.

Suppose that the set of all infinite sequences of X's and O's is countable. Then, the subset of all sequences that begin with X would also be countable (The subset of a countable set is countable). Under this assumption, the sequences that begin with X can be listed in some order, $s_1, s_2, s_3, \ldots$. We can write the sequences in the form $s_i = X, d_{i1}, d_{i2}, d_{i3}, d_{i4}, \ldots$, where $d_{ij} \in \{X, O\}$:

$$\begin{aligned}
s_1 &= X, d_{11}, d_{12}, d_{13}, d_{14} \ldots \\
s_2 &= X, d_{21}, d_{22}, d_{23}, d_{24} \ldots \\
s_3 &= X, d_{31}, d_{32}, d_{33}, d_{34} \ldots \\
s_4 &= X, d_{41}, d_{42}, d_{43}, d_{44} \ldots \\
&\vdots
\end{aligned}$$

Then, form a new sequence $s = X, d_1, d_2, d_3, d_4, \dots$, where d_i is determined by the following rule:

$$d_i = \begin{cases} X & \text{if } d_{ii} = O \\ O & \text{if } d_{ii} = X \end{cases}$$

For instance, if $s_1 = X, X, O, O, X, O, O, X, \dots, s_2 = X, O, O, X, O, O, X, \dots,$ $s_3 = X, O, O, X, X, X, X, \dots, s_4 = X, X, X, X, O, O, X, \dots$, and so on. Then, we have $s = X, d_1, d_2, d_3, d_4, \dots = X, O, X, O, X, \dots$, where $d_1 = O$ because $d_{11} = X$, $d_2 = X$ because $d_{22} = O$, $d_3 = O$ because $d_{33} = X$, and so on.

Therefore, s is not equal to any of the sequences $s_1, s_2, s_3, \dots$ because s differs from the ith sequence in the ith position.

Because there exists a sequence s that is not in the list, the assumption that all the sequences of X's and O's can be listed must be false. ⨳

Thus, the set of all infinite sequences of X's and O's with X as the first term cannot be listed, and is therefore uncountable. Any set with an uncountable subset is uncountable. Hence, the set of all infinite sequences of X's and O's is uncountable. □

Question 3. Prove that the set of real numbers between 0 and 0.0001 is uncountable.

Answer 3.

2 Countable and Uncountable Sets

Question 4. The following quote is from John Green's *The Fault in Our Stars.* (Apologies for the spoiler).

> "There are infinite numbers between 0 and 1. There's .1 and .12 and .112 and an infinite collection of others. Of course, there is a bigger infinite set of numbers between 0 and 2, or between 0 and a million. Some infinities are bigger than other infinities. A writer we used to like taught us that. There are days, many of them, when I resent the size of my unbounded set. I want more numbers than I'm likely to get, and God, I want more numbers for Augustus Waters than he got. But, Gus, my love, I cannot tell you how thankful I am for our little infinity. I wouldn't trade it for the world. You gave me a forever within the numbered days, and I'm grateful."

What is wrong with the narrator's understanding of cardinality?

Answer 4.

Question 5. You have just graduated from Penn and begun your career at Very Large Corporation, Inc. Your new company is so large that its headquarters has (countably) infinitely many floors. On your first day of work, you climb up all the stairs, stopping on each floor to meet either 1, 2, or 3 of your new coworkers who work on that floor.

a) Do you greet countably many or uncountably many coworkers? Fully explain your answer.

b) As you climb, you make a list of how many coworkers you have greeted on each floor. Is the set of possible lists countable or uncountable? Fully explain your answer.

Answer 5.