MATH 1700: Ideas in Mathematics

Worksheet 3: Sets and Functions First Submission

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1 Function Review

(1) **Consider the relationships below. None of them are functions. For each example, explain why.**

(a) $f : \{a, b, c\} \to \mathbb{N}$

x	$f(x)$
a	12
b	7
b	4
c	128

Proof. The definition of a function is that all elements in the domain have a unique image in the codomain. In this case, b has two images, 7 and 4, and is therefore f is not a function. $\square$

(b) $g : \mathbb{N} \to \mathbb{N}, g(n) = n - 4$

Proof. The definition of a function is that all elements in the domain have a unique image in the codomain. In this case, the preimages 1,2,3, and 4 map to numbers not in the codomain of $\mathbb{N}$. Therefore, for these values, there is no unique image for all values in the domain, meaning g is not a function. $\square$

(c) $h : \mathbb{Z} \to \mathbb{Q}, h(z) = \dfrac{1}{z}$

Proof. The definition of a function is that all elements in the domain have a unique image in the codomain. In this case, the preimage of 0 maps to an undefined value, and therefore there is no unique image for all values in the domain, meaning h is not a function. $\square$

(2) **Consider the function from $\mathbb{N}$ to $\mathbb{Z}$ that multiplies every number in the source by 2. Is this function injective? Is it surjective?**

Proof. **The function is injective**. The definition of injective for a function f with domain A is
$$\forall x \forall y ((x, y \in A \wedge f(x) = f(y)) \to x = y)$$
In this case, $f(x) = 2x$ and $f(y) = 2y$, so if $f(x) = f(y)$, then $2x = 2y \to x = y$. Therefore, the function is injective. $\square$

Proof. **The function is not surjective**. The definition of surjective for a function f with domain A is
$$\forall y \exists x (x \in A \wedge f(x) = y)$$
In this case, $f(x) = 2x$. This means that all values in the domain map to an even number. Since the codomain is $\mathbb{Z}$, there are infinitely many integers that are not even, there are infinitely many values in the codomain that are not images of the domain. Therefore, the function is not surjective. $\square$

(3) **Consider the function from $\mathbb{N}$ to $\mathbb{N}$ that adds one to every number in the source. Is this function injective? Surjective?**

Proof. **The function is injective**. The definition of injective for a function f with domain A is
$$\forall x \forall y((x, y \in A \land f(x) = f(y)) \to x = y)$$
In this case, $f(x) = x + 1$ and $f(y) = y + 1$, so if $f(x) = f(y)$, then $x + 1 = y + 1 \to x = y$. Therefore, the function is injective. $\qquad\qquad\square$

Proof. **The function is not surjective**. The definition of surjective for a function f with domain A is
$$\forall y \exists x(x \in A \land f(x) = y)$$
In this case, $f(x) = x + 1$. To obtain an image of 1, the preimage must be 0. However, 0 is not in the domain of the function, $\mathbb{N}$, meaning $\exists y \forall x(x \in A \land f(x) \neq y)$. Therefore, the function is not surjective. $\qquad\qquad\square$

(4) **Consider the function from $\mathbb{Z}$ to $\mathbb{Z}$ that adds one to every number in the source. Is this function injective? Surjective?**

Proof. **The function is injective**. The definition of injective for a function f with domain A is
$$\forall x \forall y((x, y \in A \land f(x) = f(y)) \to x = y)$$
In this case, $f(x) = x + 1$ and $f(y) = y + 1$, so if $f(x) = f(y)$, then $x + 1 = y + 1 \to x = y$. Therefore, the function is injective. $\qquad\qquad\square$

Proof. **The function is surjective**. The definition of surjective for a function f with domain A is
$$\forall y \exists x(x \in A \land f(x) = y)$$
In this case, $f(x) = x + 1$. For all elements in the codomain, there exists a preimage in the domain, $x = y - 1$. Therefore, the function is surjective. $\qquad\qquad\square$

(5) **Consider the function from $\mathbb{Z}$ to $\mathbb{R}$ that sends ever integer to itself. Is this function injective? Surjective?**

Proof. **The function is injective**. The definition of injective for a function f with domain A is
$$\forall x \forall y((x, y \in A \land f(x) = f(y)) \to x = y)$$
In this case, $x \in \mathbb{Z} \to f(x) = x$. Let $a, b \in \mathbb{Z}$. As $f(a) = a$ and $f(b) = b$, $f(a) = f(b) \to a = b$. Therefore, the function is injective. $\qquad\qquad\square$

Proof. **The function is not surjective**. The definition of surjective for a function f with domain A is

$$\forall y \exists x (x \in A \land f(x) = y)$$

As the domain is $\mathbb{Z}$, the domain maps solely to integers in the codomain. However, the codomain is $\mathbb{R}$, which contains infinitely many real numbers that are not integers. As $\exists y \forall x (x \in A \land f(x) \neq y)$, the function is not surjective. $\square$

(6) **Consider the function from $\mathbb{N}$ to $\mathbb{N}$ that subtracts one from every number other than 1, and sends 1 to 1. Is this function injective? Surjective?**

Proof. **The function is injective**. The definition of injective for a function f with domain A is

$$\forall x \forall y ((x, y \in A \land f(x) = f(y)) \rightarrow x = y)$$

We can represent the function as a piecewise function:

$$f : \mathbb{N} \rightarrow \mathbb{N}, f(x) = \begin{cases} 1 & x = 1 \\ x - 1 & x \neq 1 \end{cases}$$

The union of the domains of the two pieces is the domain of the function, $\mathbb{N}$. The intersection of the domains of the two pieces is the empty set, so the function is injective. $\square$

Proof. **The function is surjective**. The definition of surjective for a function f with domain A is

$$\forall y \exists x (x \in A \land f(x) = y)$$

When $x \neq 1$, $f(x) = x - 1$. For all values y in the codomain, there is a value x in the domain, $y + 1$, that maps to it. When $x = 1$, $y = 1$. Despite the image 1 having 2 preimages, 1, and 2, the definition of surjective is still satisfied. $\square$

(7) (a) **Write down a function with source $\mathbb{N}$ and target $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ which is not surjective.**

$$f : \mathbb{N} \rightarrow \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}, f(x) = \spadesuit$$

(b) **Are there any injective functions from $\mathbb{N}$ to $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$? Give an example or very briefly explain why not.**

Proof. **There are no injective functions from $\mathbb{N}$ to $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$**. We prove this by the pigeonhole principle. Let the values in the codomain be holes, and let the values in the domain be pigeons. There are 4 holes. With 4 pigeons, we can place each pigeon in each hole. With 5 pigeons, there must be a hole that contains at least 2—with 5 pigeons, the function is not injective. As the cardinality of $\mathbb{N} > 5$, there will be more pigeons than holes, meaning at least 1 hole contains more than 1 pigeons. Thus, there is no possible injective function. $\square$

(c) **Are there any injective functions from $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ to $\mathbb{N}$ which are not surjective? Give an example or very briefly explain why not.**