MATH 1700: Ideas in Mathematics

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Contents

1 Pigeonhole Principle

1.1 Introduction

Theorem 1.1.1. Pigeonhole Principle: If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Corollary 1.1.1. A function f from a set with $k+1$ or more elements to a set with k elements is not one-to-one.

1.2 Generalized Pigeonhole Principle

Theorem 1.2.1. Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\left\lceil \dfrac{N}{k} \right\rceil$ objects.

Here are some proofs using the pigeonhole principle:

Example 1.2.1. Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

Proof. Let n be a positive integer. Consider the $n+1$ integers $1, 11, 111, \ldots, 11\ldots1$ (where the last integer in this list is the integer with $n+1$ 1s in its decimal expansion). Note that there are n possible remainders when an integer is divided by n. Because there are $n+1$ integers in this list, by the pigeonhole principle, there must be two with the same remainder when divided by n. The larger of these integers less the smaller one is a multiple of n, which has decimal expansion with only 0s and 1s. $\square$

Example 1.2.2. How many cards must be selected from a standard deck of 52 cards to guarentee that:

a) at least three cards are of the same suit?

b) at least three hearts are selected?

Proof. **a)** Suppose there are 4 boxes, one for each suit, and as cards are selected they are placed in their respective box. Using the generalized pigeonhole principle, we see that if N cards are selected, there is at least 1 box contaning at least $\lceil N/4 \rceil$ cards. Thus, we know that at least 3 cards of 1 suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer N to satisfy this inequality is $2 \times 4 + 1 = 9$, so we must select at least 9 cards to guarentee that at least 3 cards are of the same suit. $\square$

- Note that if 8 cards are selected, it is possible to have 2 cards of each suit, so more than eight cards are needed.

Proof. **b)** We do not use the generalized pigeonhole principle because we want to make sure that there are 3 hearts, not just 3 cards of a suit. Note that in the worst case, we can select all the clubs, diamonds, and spades, 39 cards in all, before we select a single heart. The next 3 cards will all be hearts, so we may need to select 42 cards to get 3 hearts. $\square$

1.3 Some Elegant Applications of the Pigeonhole Principle

Example 1.3.1. During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Proof. Let a_j be the number of games played on or before the jth day of the month. Then $a_1, a_2, \ldots, a_{30}$ is an increasing sequence of distinct positive integers, with $1 \leq a_j \leq 45$. Moreover, $a_1 + 14, a_2 + 14, \ldots, a_j + 14$ is also an increasing sequence of distinct positive integers, with $15 \leq a_j + 14 \leq 59$.

The 60 positive integers $a_1, a_2, \ldots, a_{30}, a_1 + 14, a_2 + 14, \ldots, a_j + 14$ are all less than or equal to 59. Hence, by the pigeonhole principle two of these integers must be equal. Because the integers $a_j, j = 1, 2, \ldots, 30$, are all distinct and the integers $a_j + 14, j = 1, 2, \ldots, 30$ are all distinct, there must be indices i and j with $a_i = a_j + 14$. This means that exactly 14 games were played from day $j + 1$ to day i. □

Example 1.3.2. Show that among any $n + 1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

Proof. Write out each of the $n + 1$ integers $a_1, a_2, \ldots, a_{n+1}$ as a power of 2 times an odd integer. In other words, let $a_j = 2^{k_j} q_j$, for $j = 1, 2, \ldots n + 1$, where k_j is a nonnegative integer and q_j is odd. The integers $q_1, q_2, \ldots, q_{n+1}$ are all odd positive integers less than $2n$. Because there are only n odd positive integers less than $2n$, it follows from the pigeonhole principle that two of the integers $q_1, q_2, \ldots, q_{n+1}$ must be equal. Therefore, there are distinct integers i and j such that $q_i = q_j$. Let q be the common value of q_i and q_j. Then $a_i = 2^{k_i} q$ and $a_j = 2^{k_j} q$. It follows that if $k_i < k_j$, then a_i divides a_j; while if $k_i > k_j$, then a_j divides a_i. In either case, there is an integer that divides one of the other integers. □

Theorem 1.3.1. Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of $n + 1$ that is either strictly increasing or strictly decreasing.

2 Numbers and Infinity

2.1 Introduction

Definition 2.1.1. The set of natural numbers is denoted by $\mathbb{N}$: $\{1, 2, 3, \ldots\}$. For the purposes of this class, $\mathbb{N}$ does not include 0.

Definition 2.1.2. A function is finitely many if a function can map each element to a subset of $\mathbb{N}$: $\{1, 2, 3, \ldots, n\}$.

2.2 Infinite Sets

Definition 2.2.1. A set is infinite if a function can map each element to an element of $\mathbb{N}$.

Example 2.2.1. Prove that the set of even integers is infinite.

Proof. We can separate the set of even integers into two subsets: Positive even integers and negative even integers. Let $f(x)$ be a function that maps each element of the set of even integers to a subset of $\mathbb{N}$. Let $f(x)$ be defined as follows:

$$f(x) = \begin{cases} x & x \in \{2k \mid k \in \mathbb{N}\} \\ -2x & x \in \{2k-1 \mid k \in \mathbb{N}\} \end{cases}$$

As each value $x \in \mathbb{N}$ is mapped to a subset of $\mathbb{N}$, $f(x)$ is a function. □

2.3 Machine Method

Definition 2.3.1. The **machine method** is a method to demonstrate that a set is infinite by building a machine (an algorithm) that takes as input a finite list and names an element of the set that is not in the list. **The machine shows that no finite list can contain all the elements of the set.**

- The machine method can be used to show that a collection is infinite because if we run the machine forever on repeat, it will produce an infinite list.
- A better explanation: If a collection were finite, it would be possible to include all its elements in a finite list. The machine method demonstrates that no finite list can contain the entire collection. So the collection cannot be finite.

Example 2.3.1. Use the machine method to show that there are infinitely many prime numbers.

Proof. We give our machine a rule: Take the greatest number in the list and add 2. As we only fed our machine even numbers, adding 2 will always give an even number. Because we added 2 to the greatest number on our list, the sum will be greater than that number. Since our output is greater than the greatest number on the finite list, it must be greater than every number on the finite list. Therefore, the output was not part of our original list. □

2.4 Prime Numbers

Definition 2.4.1. x **is a factor of** y if $y = kx, k \in \mathbb{N}$. Another interpretation is that x div y.

Definition 2.4.2. A natural number is a **prime number** if it is greater than 1 and has no factors other than 1 and itself.

Definition 2.4.3. A natural number is a **composite number** if it is greater than 1 and has at least one factor other than 1 and itself.

- From the Exercise, we know that the set of even integers is infinite. Thus, it follows that there are infinitely many composite numbers.
- The number 1 is **neither prime nor composite**.

3 Sets

3.1 Introduction

Definition 3.1.1. A **set** is an unordered collection of distinct objects called **elements** or **members** of the set. A set is said to **contain** its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

Sets of types of numbers:

- Natural Numbers: $\mathbb{N} = \{0, 1, 2, 3, \dots\} = \{\mathbb{Z}^+ \cup 0\}$
- Integers: $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- Positive Integers: $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$
- Rational Numbers: $\mathbb{Q} = \left\{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\right\}$
- Real Numbers: $\mathbb{R}$
- Positive Real Numbers: $\mathbb{R}^+$
- Complex Numbers: $\mathbb{C}$

Definition 3.1.2. Equality of Sets:

$$A = B \leftrightarrow \forall x(x \in A \leftrightarrow x \in B) \leftrightarrow A \subseteq B \land B \subseteq A$$

Definition 3.1.3. Empty Set: $\emptyset = \{\}$

3.3 Subsets

Definition 3.3.1. Subset:

$$A \subseteq B \leftrightarrow \forall x(x \in A \leftrightarrow x \in B) \leftrightarrow B \supseteq A$$

To show that $A \not\subseteq B$, show $\exists x(x \in A \land x \notin B)$.

Theorem 3.3.1. For every set S, $\emptyset \subseteq S$ and $S \subseteq S$.

3.4 Size of a Set

Definition 3.4.1. Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say that S is a **finite set** and that n is the **cardinality** of S, denoted by $|S|$.

- Note: Theorem 2.1.3.1!

3.5 Power Sets

Definition 3.5.1. Let S be a set. The **power set** of S, denoted by $\mathcal{P}(S)$, is the set of all subsets of S.

Theorem 3.5.1. Cardinality of a power set

$$|\mathcal{P}(S)| = 2^{|S|}$$

3.6 Cartesian Products

Definition 3.6.1. Let A and B be sets. The **Cartesian product** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. Hence:

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

3.7 Set Operations

3.8 Introduction

Definition 3.8.1. Let A and B be sets. The **union** of the sets A and B, denoted $A \cup B$, is the set that contains those elements that are in either A or B or both. Hence:

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Definition 3.8.2. Let A and B be sets. The **intersection** of the sets A and B, denoted $A \cap B$, is the set that contains those elements in both A and B. Hence:

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Definition 3.8.3. Two sets are called **disjoint** if their intersection is the emptyset.

Definition 3.8.4. Let A and B be sets. The **difference** of the sets A and B, denoted $A - B$, is the set that contains those elements in A but not in B. It is also called the **complement of B with respect to A**. Hence:

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Definition 3.8.5. Let U be the universal set. The **complement** of a set A, denoted $\overline{A}$, is the set $U - A$. Hence:

$$\overline{A} = \{x \mid x \in U \wedge x \notin A\}$$

Definition 3.8.6. Let A and B be sets. The **symmetric difference** of A and B is the set of elements that are in either A or B but not in both. It is denoted by $A \oplus B$. Hence:

$$A \oplus B = (A \cup B) - (A \cap B)$$

3.9 Set Identities

Identity	Name
$A \cap U = A$ $A \cup \emptyset = A$	Identity Laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination Laws
$A \cup A = A$ $A \cap A = A$	Idempotent Laws
$\overline{(\overline{A})} = A$	Complementation Law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive Laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's Laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement Laws

Figure 1: Set Identities

There are 3 ways to prove that two sets are equal:

1. Showing that they are subsets of each other. (Definition 2.2)
2. Membership tables.
3. Set identities.

A **membership table** considers each combination of the atomic sets (the original sets used to produce the sets on each side) that an element can belong to and verify that elements in the same combinations of sets belong to both the sets in the identity. Use a 1 to indicate that an element belongs to a set and a 0 to indicate that it does not. For example, consider the following identity:

$$A \cup (A \cap B) = A$$

We can construct a membership table for this identity as follows:

A	B	$A \cup (A \cap B)$
1	1	1
1	0	1
0	1	0
0	0	0

Since the columns are the same, we can conclude that the sets are equal.

3.10 Generalized Unions and Intersections

Definition 3.10.1. The **union** of a collection of sets is the set that contains those elements that are members of at least one set in the collection. It is denoted by:

$$A_1 \cup A_2 \cup \cdots A_n = \bigcup_{i=1}^{n} A_i$$

Definition 3.10.2. The **intersection** of a collection of sets is the set that contains those elements that are members of all sets in the collection. It is denoted by:

$$A_1 \cap A_2 \cap \cdots A_n = \bigcap_{i=1}^{n} A_i$$

4 Functions

4.1 Introduction

Definition 4.1.1. Let A and B be nonempty sets. A **function** f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

- Functions are sometimes also called **mappings** or **transformations**

Definition 4.1.2. Let $f : A \to B$ be a function. A is the **domain** of f and B is the **codomain** of f. If $f(a) = b$, we say that b is the **image** of a and a is the **preimage** of b. The **range**, or **image** of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f **maps** A to B.

- Codomain is set of possible values of the function and range is the set of all elements of the codomain that are achieved as the value of f for at least one element of the domain.

- Two functions are **equal** when they have the same domain, same codomain, and map each each element of their common domain to the same element in their common codomain.

Definition 4.1.3. Let f_1 and f_2 be functions from A to B. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to B defined $\forall x \in A$ by:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
$$(f_1 f_2)(x) = f_1(x) f_2(x)$$

Definition 4.1.4. Let f be a function from A to B and let $S \subseteq A$. The **image** of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so:

$$f(S) = \{t \mid \exists s \in S(t = f(s))\} = \{f(s) \mid s \in S\}$$

4.2 One-to-One and Onto Functions

Definition 4.2.1. A function f with domain A is **one-to-one** if and only if:

$$\forall a \forall b(a, b \in A \land (f(a) = f(b) \to a = b))$$

- A function f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$. This is obtained by taking the contrapositve of the implication in the definition.

Definition 4.2.2. A function f whose domain A and codomain B are subsets of the set of real numbers is called **increasing** if $f(x) \leq f(y)$ whenever $x < y$ and $x, y \in A$. Hence:

$$\forall x \forall y(x, y \in A \land x < y \to f(x) \leq f(y))$$

Definition 4.2.3. A function f whose domain A and codomain B are subsets of the set of real numbers is called **strictly increasing** if $f(x) < f(y)$ whenever $x < y$ and $x, y \in A$. Hence:

$$\forall x \forall y(x, y \in A \land x < y \to f(x) < f(y))$$

Definition 4.2.4. A function f whose domain A and codomain B are subsets of the set of real numbers is called **decreasing** if $f(x) \geq f(y)$ whenever $x < y$ and $x, y \in A$. Hence:

$$\forall x \forall y(x, y \in A \land x < y \to f(x) \geq f(y))$$

Definition 4.2.5. A function f whose domain A and codomain B are subsets of the set of real numbers is called **strictly decreasing** if $f(x) > f(y)$ whenever $x < y$ and $x, y \in A$. Hence:

$$\forall x \forall y(x, y \in A \land x < y \to f(x) > f(y))$$

Definition 4.2.6. A function f from A to B is **onto**, or a **surjection**, if and only if for every element $y \in B$ there exists an element $x \in A$ such that $f(x) = y$. Hence:

$$\forall y \exists x(f(x) = y)$$

where the domain for x is A and the domain of y is B.

- f is **surjective** if it is onto.

Definition 4.2.7. The function f is a **one-to-one correspondence** if it is both one-to-one and onto.

- Such a function is **bijective**

Suppose that $f : A \to B$.	
Show f is injective:	Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$
Show f is not injective:	Find particular elements, $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.
Show f is surjective:	Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.
Show f is not surjective:	Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.
Show f is bijective:	Show that f is both injective and surjective.

4.3 Inverse Functions and Composite Functions

Definition 4.3.1. Let f be a one-to-one correspondence from the set A to the set B. The **inverse function** of f is denoted by f^{-1}: Hence:

$$f^{-1}(b) = a \text{ when } f(a) = b$$

- A one-to-one correspondence f is **invertible** because we can define an inverse function f^{-1}.
- A function is **invertible** if it is not a one-to-one correspondence, because the inverse of f does not exist.

Definition 4.3.2. Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The **composition** of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is the function from A to C defined by:

$$(f \circ g)(a) = f(g(a))$$

- $f \circ g$ assigns the element a of A the element assigned by f to $g(a)$.
- The domain of $f \circ g$ is the domain of g.
- The range of $f \circ g$ is the image of the range of g with respect to f.
- The composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.

- **Not Commutative!**

$$f \circ g \neq g \circ f$$

- When composing with inverse function, an identity function is obtained:

$$f \circ f^{-1}(a) = f^{-1} \circ f(a) = a$$

4.5 Some Important Functions

Definition 4.5.1. The **floor function** assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$. The **ceiling function** assigns to the real number x the smllest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

n **is an integer,** x **is a real number**
$\lfloor x \rfloor = n \leftrightarrow n \leq x < n+1$ $\lceil x \rceil = n \leftrightarrow n-1 < x \leq n$ $\lfloor x \rfloor = n \leftrightarrow x-1 < n \leq x$ $\lceil x \rceil = n \leftrightarrow x \leq n < x+1$
$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$
$\lfloor -x \rfloor = -\lceil x \rceil$ $\lceil -x \rceil = -\lfloor x \rfloor$
$\lfloor x+n \rfloor = \lfloor x \rfloor + n$ $\lceil x+n \rceil = \lceil x \rceil + n$

Figure 2: Useful Properties of the Floor and Ceiling Functions