Portfolio Resubmission 2

MATH 1700: Ideas in Mathematics

Professor Rimmer

Due: March 31, 2023 **Denny Cao**

Worksheet 4 (Cardinality) Question 14

Show that the set of real numbers between 0 and 1 (not including the endpoints) has the same cardinality as the set of real numbers.

Answer

Let $S = \{x \mid 0 < x < 1\}$.
Let $f : \mathbb{R} \to S$, where $f(x) = \dfrac{1}{\pi} \tan^{-1} x + \dfrac{1}{2}$.

Lemma 1. **f is injective.** An injective function is a function where no two inputs map to the same output. In other words, if $f(x) = f(y)$, then $x = y$.

Suppose that $a, b \in \mathbb{R}$.

$$f(a) = \frac{1}{\pi} \tan^{-1} a + \frac{1}{2} \quad f(b) = \frac{1}{\pi} \tan^{-1} b + \frac{1}{2}$$

$$\begin{aligned} f(a) &= f(b) \\ \frac{1}{\pi} \tan^{-1} a + \frac{1}{2} &= \frac{1}{\pi} \tan^{-1} b + \frac{1}{2} \\ \tan^{-1} a &= \tan^{-1} b \end{aligned}$$

Since the range of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we can take the tangent of both sides, as $\tan x$ is defined for all values in the range of $\tan^{-1} x$.

$$\begin{aligned} \tan\left(\tan^{-1} a\right) &= \tan\left(\tan^{-1} b\right) \\ a &= b \end{aligned}$$

Since there are no 2 distinct values in the domain $\mathbb{R}$ that map to the same value in the codomain S, f is injective.

Lemma 2. **f is surjective.** A surjective function is a function where every element in the codomain is mapped to by at least one element in the domain. In other words, if $y \in S$, then there exists an $x \in \mathbb{R}$ such that $f(x) = y$.

Suppose that $y \in S$. Suppose that $x = \tan\left(\pi\left(y - \frac{1}{2}\right)\right)$, a value in the domain $\mathbb{R}$.

$$\begin{aligned}
f(x) &= \frac{1}{\pi}\tan^{-1} x + \frac{1}{2} \\
f\left(\tan\left(\pi\left(y - \frac{1}{2}\right)\right)\right) &= \frac{1}{\pi}\tan^{-1}\left(\tan\left(\pi\left(y - \frac{1}{2}\right)\right)\right) + \frac{1}{2} \\
&= \frac{1}{\pi}\left(\pi\left(y - \frac{1}{2}\right)\right) + \frac{1}{2} \\
&= y - \frac{1}{2} + \frac{1}{2} \\
&= y
\end{aligned}$$

As for all $y \in S$, there exists an $x \in \mathbb{R}$ such that $f(x) = y$, specifically $x = \tan\left(\pi\left(y - \frac{1}{2}\right)\right)$, f is surjective.

Lemma 3. f is a bijection. A bijection is a function that is both injective and surjective.

Since f is injective from Lemma 1 and surjective from Lemma 2, f is a bijection.

Theorem 1. Let $S = \{x \mid 0 < x < 1\}$ and $\mathbb{R}$ be the set of all real numbers. Then $|S| = |\mathbb{R}|$.

Proof. We will prove the theorem, that the cardinality of the set of real numbers between 0 and 1 is the same as the cardinality of $\mathbb{R}$, by constructing a function that is a bijection between S and $\mathbb{R}$. This is because two sets have the same cardinality if and only if there is a bijection between them.
Let $f : \mathbb{R} \to S$, where $f(x) = \frac{1}{\pi}\tan^{-1} x + \frac{1}{2}$.
The range of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and by a vertical compression by a factor of $\frac{1}{\pi}$, the range becomes $\left(-\frac{1}{2}, \frac{1}{2}\right)$. We then vertically shift $\frac{1}{\pi}\tan^{-1} x$ by $\frac{1}{2}$, resulting in a range of $(0, 1)$, our desired codomain for f.

f is a bijection (see Lemma 3). As a bijection exists between S and $\mathbb{R}$, their cardinalities are the same—$|S| = |\mathbb{R}|$. □