# MATH 1700: Ideas in Mathematics

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# Contents

# 1 Pigeonhole Principle

## 1.1 Introduction

**Theorem 1.1.1. Pigeonhole Principle**: If $k$ is a positive integer and $k+1$ or more objects are placed into $k$ boxes, then there is at least one box containing two or more of the objects.

**Corollary 1.1.1.** A function $f$ from a set with $k+1$ or more elements to a set with $k$ elements is not one-to-one.

## 1.2 Generalized Pigeonhole Principle

**Theorem 1.2.1. Generalized Pigeonhole Principle**: If $N$ objects are placed into $k$ boxes, then there is at least one box containing at least $\left\lceil \frac{N}{k} \right\rceil$ objects.

Here are some proofs using the pigeonhole principle:

**Example 1.2.1. Show that for every integer $n$ there is a multiple of $n$ that has only 0s and 1s in its decimal expansion.**

*Proof.* Let $n$ be a positive integer. Consider the $n+1$ integers $1, 11, 111, \ldots, 11\ldots1$ (where the last integer in this list is the integer with $n+1$ 1s in its decimal expansion). Note that there are $n$ possible remainders when an integer is divided by $n$. Because there are $n+1$ integers in this list, by the pigeonhole principle, there must be two with the same remainder when divided by $n$. The larger of these integers less the smaller one is a multiple of $n$, which has decimal expansion with only 0s and 1s. $\square$

**Example 1.2.2. How many cards must be selected from a standard deck of 52 cards to guarentee that:**

a) at least three cards are of the same suit?

b) at least three hearts are selected?

*Proof.* **a)** Suppose there are 4 boxes, one for each suit, and as cards are selected they are placed in their respective box. Using the generalized pigeonhole principle, we see that if $N$ cards are selected, there is at least 1 box contaning at least $\lceil N/4 \rceil$ cards. Thus, we know that at least 3 cards of 1 suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer $N$ to satisfy this inequality is $2 \times 4 + 1 = 9$, so we must select at least 9 cards to guarentee that at least 3 cards are of the same suit. $\square$

- Note that if 8 cards are selected, it is possible to have 2 cards of each suit, so more than eight cards are needed.

*Proof.* **b)** We do not use the generalized pigeonhole principle because we want to make sure that there are 3 hearts, not just 3 cards of a suit. Note that in the worst case, we can select all the clubs, diamonds, and spades, 39 cards in all, before we select a single heart. The next 3 cards will all be hearts, so we may need to select 42 cards to get 3 hearts. $\square$

### 1.3 Some Elegant Applications of the Pigeonhole Principle

**Example 1.3.1.** During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

*Proof.* Let $a_j$ be the number of games played on or before the $j$th day of the month. Then $a_1, a_2, \ldots, a_{30}$ is an increasing sequence of distinct positive integers, with $1 \leq a_j \leq 45$. Moreover, $a_1 + 14, a_2 + 14, \ldots, a_j + 14$ is also an increasing sequence of distinct positive integers, with $15 \leq a_j + 14 \leq 59$.

The 60 positive integers $a_1, a_2, \ldots, a_{30}, a_1 + 14, a_2 + 14, \ldots, a_j + 14$ are all less than or equal to 59. Hence, by the pigeonhole principle two of these integers must be equal. Because the integers $a_j, j = 1, 2, \ldots, 30$, are all distinct and the integers $a_j + 14, j = 1, 2, \ldots, 30$ are all distinct, there must be indices $i$ and $j$ with $a_i = a_j + 14$. This means that exactly 14 games were played from day $j + 1$ to day $i$. □

**Example 1.3.2.** Show that among any $n + 1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

*Proof.* Write out each of the $n + 1$ integers $a_1, a_2, \ldots, a_{n+1}$ as a power of 2 times an odd integer. In other words, let $a_j = 2^{k_j} q_j$, for $j = 1, 2, \ldots n + 1$, where $k_j$ is a nonnegative integer and $q_j$ is odd. The integers $q_1, q_2, \ldots, q_{n+1}$ are all odd positive integers less than $2n$. Because there are only $n$ odd positive integers less than $2n$, it follows from the pigeonhole principle that two of the integers $q_1, q_2, \ldots, q_{n+1}$ must be equal. Therefore, there are distinct integers $i$ and $j$ such that $q_i = q_j$. Let $q$ be the common value of $q_i$ and $q_j$. Then $a_i = 2^{k_i} q$ and $a_j = 2^{k_j} q$. It follows that if $k_i < k_j$, then $a_i$ divides $a_j$; while if $k_i > k_j$, then $a_j$ divides $a_i$. In either case, there is an integer that divides one of the other integers. □

**Theorem 1.3.1.** Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of $n + 1$ that is either strictly increasing or strictly decreasing.

# 2 Numbers and Infinity

## 2.1 Introduction

**Definition 2.1.1.** The set of natural numbers is denoted by $\mathbb{N}$: $\{1, 2, 3, \ldots\}$. For the purposes of this class, $\mathbb{N}$ does not include 0.

**Definition 2.1.2.** A function is finitely many if a function can map each element to a subset of $\mathbb{N}$: $\{1, 2, 3, \ldots, n\}$.

## 2.2 Infinite Sets

**Definition 2.2.1.** A set is infinite if a function can map each element to an element of $\mathbb{N}$.

**Example 2.2.1.** Prove that the set of even integers is infinite.

*Proof.* We can separate the set of even integers into two subsets: Positive even integers and negative even integers. Let $f(x)$ be a function that maps each element of the set of even integers to a subset of $\mathbb{N}$. Let $f(x)$ be defined as follows:

$$f(x) = \begin{cases} x & x \in \{2k \mid k \in \mathbb{N}\} \\ -2x & x \in \{2k - 1 \mid k \in \mathbb{N}\} \end{cases}$$

As each value $x \in \mathbb{N}$ is mapped to a subset of $\mathbb{N}$, $f(x)$ is a function. $\square$