Worksheet 5: Uncountable Sets and Cantor's Diagonal Argument

MATH 1700: Ideas in Mathematics

Professor Rimmer

Due: February 22, 2023 **Denny Cao**

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1 Warm-Up: X's and O's

Question 1. Consider the following two-player game, which we will call Cantor's Game.

Player 1 begins by writing a sequence of X's and O's in the top row of the grid below. Player 2 then writes either an X or an O in the first box on the bottom. Player 1 then writes a sequence of X's and O's in the second row of the grid. Player 2 writes an X or O in the second bottom box. The players continue until all boxes are filled. Player 1 wins if the sequence on the bottom exactly matches any of the sequences Player 1 has written in the grid. Player 2 wins otherwise.

a) Which player has a winning strategy, and why?

b) How does that strategy relate to Cantor's diagonal argument?

Answer 1.

a) Player 1 has a winning strategy. The strategy is as follows:

  i. For Player 1's first move, they can write any sequence of X's and O's they choose

  ii. Player 1 writes the next symbol in the sequence that would be produced by Cantor's diagonal argument applied to the sequences they have written so far (i.e., Player 1 chooses the symbol that is different from the corresponding symbol in the diagonal).

  The sequence on the bottom will match one of the sequences that Player 1 has written, so Player 1 wins.

b) The strategy in a) ensures Player 1 has written every possible sequence of X's and O's in the top row. This corresponds to Cantor's diagonal argument because the strategy constructs a diagonal of symbols that cannot be matched by any row in the grid. The fact that Player 1 wins shows that the set of all possible sequences of X's and O's is "larger" than the set of sequences that can be produced by Cantor's diagonal argument, since Player 1 has managed to match every sequence that can be produced by the diagonal argument.

Question 2. Prove that the set of all infinite sequences of X's and O's is uncountable.

Answer 2. *Proof.* By contradiction.

Assume for purposes of contradiction that the set of all infinite sequences of X's and O's is countable. Then, the subset of all sequences that begin with X would also be countable (The subset of a countable set is countable). Under this assumption, the sequences that begin with X can be listed in some order, $s_1, s_2, s_3, \ldots$. We can write the sequences in the form $s_i = X, d_{i1}, d_{i2}, d_{i3}, d_{i4}, \ldots$, where $d_{ij} \in \{X, O\}$:

$$\begin{aligned} s_1 &= X, d_{11}, d_{12}, d_{13}, d_{14} \ldots \\ s_2 &= X, d_{21}, d_{22}, d_{23}, d_{24} \ldots \\ s_3 &= X, d_{31}, d_{32}, d_{33}, d_{34} \ldots \\ s_4 &= X, d_{41}, d_{42}, d_{43}, d_{44} \ldots \\ &\vdots \end{aligned}$$

Then, form a new sequence $s = X, d_1, d_2, d_3, d_4, \ldots$, where d_i is determined by the following rule:

$$d_i = \begin{cases} X & \text{if } d_{ii} = O \\ O & \text{if } d_{ii} = X \end{cases}$$

For instance, if $s_1 = X, X, O, O, X, O, O, X, \ldots, s_2 = X, O, O, X, O, O, X, \ldots,$ $s_3 = X, O, O, X, X, X, X, \ldots, s_4 = X, X, X, X, O, O, X, \ldots$, and so on. Then, we have $s = X, d_1, d_2, d_3, d_4, \cdots = X, O, X, O, X, \ldots$, where $d_1 = O$ because $d_{11} = X$, $d_2 = X$ because $d_{22} = O$, $d_3 = O$ because $d_{33} = X$, and so on.

Therefore, s is not equal to any of the sequences $s_1, s_2, s_3, \ldots$ because s differs from the ith sequence in the ith position.

Because there exists a sequence s that is not in the list, the assumption that all the sequences of X's and O's can be listed must be false. ⨳

Thus, the set of all infinite sequences of X's and O's with X as the first term cannot be listed, and is therefore uncountable. Any set with an uncountable subset is uncountable. Hence, the set of all infinite sequences of X's and O's is uncountable. $\square$

Question 3. Prove that the set of real numbers between 0 and 0.0001 is uncountable.

Answer 3. *Proof.* By contradiction.

Assume for purposes of contradiction that the set of real numbers between 0 and 0.0001 is countable. Then, the set of all real numbers between 0 and 0.0001 can be listed in some order, $r_1, r_2, r_3, \ldots$. We can write the decimal representations of the numbers in the form $r_i = 0.d_{i1}d_{i2}d_{i3}d_{i4}\ldots$, where $d_{ij} \in \{k \mid 0 \leq k \leq 9, k \in \mathbb{Z}\}$:

$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}d_{14} \ldots \\ r_2 &= 0.d_{21}d_{22}d_{23}d_{24} \ldots \\ r_3 &= 0.d_{31}d_{32}d_{33}d_{34} \ldots \\ r_4 &= 0.d_{41}d_{42}d_{43}d_{44} \ldots \\ &\vdots \end{aligned}$$

Then, form a new sequence $r = 0.d_1d_2d_3d_4\ldots$, where d_i is determined by the following rule:

$$d_i = \begin{cases} 5 & \text{if } d_{ii} \neq 5 \\ 4 & \text{if } d_{ii} = 5 \end{cases}$$

Such a sequence r is not equal to any of the sequences $r_1, r_2, r_3, \ldots$ because r differs from the ith sequence in the ith position.

Because there exists a sequence r that is not in the list, the assumption that all the sequences of d_{ij}'s can be listed must be false. ※

Thus, the set of all real numbers between 0 and 0.0001 cannot be listed, and is therefore uncountable. □

2 Countable and Uncountable Sets

Question 4. The following quote is from John Green's *The Fault in Our Stars.* (Apologies for the spoiler).

> "There are infinite numbers between 0 and 1. There's .1 and .12 and .112 and an infinite collection of others. Of course, there is a bigger infinite set of numbers between 0 and 2, or between 0 and a million. Some infinities are bigger than other infinities. A writer we used to like taught us that. There are days, many of them, when I resent the size of my unbounded set. I want more numbers than I'm likely to get, and God, I want more numbers for Augustus Waters than he got. But, Gus, my love, I cannot tell you how thankful I am for our little infinity. I wouldn't trade it for the world. You gave me a forever within the numbered days, and I'm grateful."

What is wrong with the narrator's understanding of cardinality?

Answer 4. The narrator is incorrect when stating that the set of numbers between 0 and 2 is bigger than the set of numbers between 0 and 1. Both of these sets are uncountably infinite, as both are the cardinality of the continuum.

Question 5. You have just graduated from Penn and begun your career at Very Large Corporation, Inc. Your new company is so large that its headquarters has (countably) infinitely many floors. On your first day of work, you climb up all the stairs, stopping on each floor to meet either 1, 2, or 3 of your new coworkers who work on that floor.

a) Do you greet countably many or uncountably many coworkers? Fully explain your answer.

b) As you climb, you make a list of how many coworkers you have greeted on each floor. Is the set of possible lists countable or uncountable? Fully explain your answer.

Answer 5.

a) The number of coworkers I greet is countably infinite. On the first floor, there may be at most three coworkers to meet. On the second floor, there may be at most six coworkers to meet, since each of the one, two, or three coworkers on the second floor would have one, two, or three coworkers respectively to meet on the first floor. Continuing this process, on the n-th floor there may be at most $3n$ coworkers to meet. Thus, the number of coworkers I greet is countably infinite.

b) The set of possible lists of how many coworkers I greeted on each floor is uncountable.

Proof. By contradiction.

Assume for purposes of contradiction that the set of possible lists of how many coworkers I greeted on each floor is countable. Then, the set of all possible lists of how many coworkers I greeted on each floor can be listed in some order, $l_1, l_2, l_3, \ldots$. We can write the lists in the form $l_i = (n_{i1}, n_{i2}, n_{i3}, \ldots)$, where $n_{ij} \in \{k \mid 0 \leq k \leq 3, k \in \mathbb{Z}\}$:

$$\begin{aligned} l_1 &= (n_{11}, n_{12}, n_{13}, \ldots) \\ l_2 &= (n_{21}, n_{22}, n_{23}, \ldots) \\ l_3 &= (n_{31}, n_{32}, n_{33}, \ldots) \\ l_4 &= (n_{41}, n_{42}, n_{43}, \ldots) \\ &\vdots \end{aligned}$$

Then, form a new list $l = (n_1, n_2, n_3, \ldots)$, where n_i is determined by the following rule:

$$n_i = \begin{cases} 2 & \text{if } n_{ii} \neq 2 \\ 3 & \text{if } n_{ii} = 2 \end{cases}$$

Such a list l is not equal to any of the lists $l_1, l_2, l_3, \ldots$ because l differs from the ith list in the ith position.

Because there exists a list l that is not in the list, the assumption that all the lists of n_{ij}'s can be listed must be false. ⨳

Thus, the set of all possible lists of how many coworkers I greeted on each floor cannot be listed, and is therefore uncountable. □

3 Reflection

What content do I need to review before attempting the worksheet again? Are there any videos I need to rewatch?
I need to review Cantor's diagonal argument. I'm certain my proofs can be made more succinct and elegant. I also need to review how to compare the cardinality of uncountable sets (if that's even possible).

What questions would I like to ask my group during the next class discussion?

I would like to ask my group about Question 4. I'm not sure if what I wrote is correct, and I would like to know if there is a better way to explain the cardinality of a subset of the continuum.