MATH 1700: Ideas in Mathematics

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Contents

1 Pigeonhole Principle

1.1 Introduction

Theorem 1.1.1. Pigeonhole Principle: If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Corollary 1.1.1. A function f from a set with $k+1$ or more elements to a set with k elements is not one-to-one.

1.2 Generalized Pigeonhole Principle

Theorem 1.2.1. Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\left\lceil \frac{N}{k} \right\rceil$ objects.

Here are some proofs using the pigeonhole principle:

Example 1.2.1. Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

Proof. Let n be a positive integer. Consider the $n+1$ integers $1, 11, 111, \ldots, 11\ldots 1$ (where the last integer in this list is the integer with $n+1$ 1s in its decimal expansion). Note that there are n possible remainders when an integer is divided by n. Because there are $n+1$ integers in this list, by the pigeonhole principle, there must be two with the same remainder when divided by n. The larger of these integers less the smaller one is a multiple of n, which has decimal expansion with only 0s and 1s. $\square$

Example 1.2.2. How many cards must be selected from a standard deck of 52 cards to guarentee that:

a) at least three cards are of the same suit?

b) at least three hearts are selected?

Proof. **a)** Suppose there are 4 boxes, one for each suit, and as cards are selected they are placed in their respective box. Using the generalized pigeonhole principle, we see that if N cards are selected, there is at least 1 box contaning at least $\lceil N/4 \rceil$ cards. Thus, we know that at least 3 cards of 1 suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer N to satisfy this inequality is $2 \times 4 + 1 = 9$, so we must select at least 9 cards to guarentee that at least 3 cards are of the same suit. $\square$

- Note that if 8 cards are selected, it is possible to have 2 cards of each suit, so more than eight cards are needed.

Proof. **b)** We do not use the generalized pigeonhole principle because we want to make sure that there are 3 hearts, not just 3 cards of a suit. Note that in the worst case, we can select all the clubs, diamonds, and spades, 39 cards in all, before we select a single heart. The next 3 cards will all be hearts, so we may need to select 42 cards to get 3 hearts. $\square$

1.3 Some Elegant Applications of the Pigeonhole Principle

Example 1.3.1. During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Proof. Let a_j be the number of games played on or before the jth day of the month. Then $a_1, a_2, \ldots, a_{30}$ is an increasing sequence of distinct positive integers, with $1 \leq a_j \leq 45$. Moreover, $a_1 + 14, a_2 + 14, \ldots, a_j + 14$ is also an increasing sequence of distinct positive integers, with $15 \leq a_j + 14 \leq 59$.

The 60 positive integers $a_1, a_2, \ldots, a_{30}, a_1 + 14, a_2 + 14, \ldots, a_j + 14$ are all less than or equal to 59. Hence, by the pigeonhole principle two of these integers must be equal. Because the integers $a_j, j = 1, 2, \ldots, 30$, are all distinct and the integers $a_j + 14, j = 1, 2, \ldots, 30$ are all distinct, there must be indices i and j with $a_i = a_j + 14$. This means that exactly 14 games were played from day $j + 1$ to day i. □

Example 1.3.2. Show that among any $n + 1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

Proof. Write out each of the $n + 1$ integers $a_1, a_2, \ldots, a_{n+1}$ as a power of 2 times an odd integer. In other words, let $a_j = 2^{k_j} q_j$, for $j = 1, 2, \ldots n + 1$, where k_j is a nonnegative integer and q_j is odd. The integers $q_1, q_2, \ldots, q_{n+1}$ are all odd positive integers less than $2n$. Because there are only n odd positive integers less than $2n$, it follows from the pigeonhole principle that two of the integers $q_1, q_2, \ldots, q_{n+1}$ must be equal. Therefore, there are distinct integers i and j such that $q_i = q_j$. Let q be the common value of q_i and q_j. Then $a_i = 2^{k_i} q$ and $a_j = 2^{k_j} q$. It follows that if $k_i < k_j$, then a_i divides a_j; while if $k_i > k_j$, then a_j divides a_i. In either case, there is an integer that divides one of the other integers. □

Theorem 1.3.1. Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of $n + 1$ that is either strictly increasing or strictly decreasing.

2 Numbers and Infinity

2.1 Introduction

Definition 2.1.1. The set of natural numbers is denoted by $\mathbb{N}$: $\{1, 2, 3, \ldots\}$. For the purposes of this class, $\mathbb{N}$ does not include 0.

Definition 2.1.2. A function is finitely many if a function can map each element to a subset of $\mathbb{N}$: $\{1, 2, 3, \ldots, n\}$.

2.2 Infinite Sets

Definition 2.2.1. A set is infinite if a function can map each element to an element of $\mathbb{N}$.

Example 2.2.1. Prove that the set of even integers is infinite.

Proof. We can separate the set of even integers into two subsets: Positive even integers and negative even integers. Let $f(x)$ be a function that maps each element of the set of even integers to a subset of $\mathbb{N}$. Let $f(x)$ be defined as follows:

$$f(x) = \begin{cases} x & x \in \{2k \mid k \in \mathbb{N}\} \\ -2x & x \in \{2k-1 \mid k \in \mathbb{N}\} \end{cases}$$

As each value $x \in \mathbb{N}$ is mapped to a subset of $\mathbb{N}$, $f(x)$ is a function. □

2.3 Machine Method

Definition 2.3.1. The **machine method** is a method to demonstrate that a set is infinite by building a machine (an algorithm) that takes as input a finite list and names an element of the set that is not in the list. **The machine shows that no finite list can contain all the elements of the set.**

- The machine method can be used to show that a collection is infinite because if we run the machine forever on repeat, it will produce an infinite list.
- A better explanation: If a collection were finite, it would be possible to include all its elements in a finite list. The machine method demonstrates that no finite list can contain the entire collection. So the collection cannot be finite.

Example 2.3.1. Use the machine method to show that there are infinitely many prime numbers.

Proof. We give our machine a rule: Take the greatest number in the list and add 2. As we only fed our machine even numbers, adding 2 will always give an even number. Because we added 2 to the greatest number on our list, the sum will be greater than that number. Since our output is greater than the greatest number on the finite list, it must be greater than every number on the finite list. Therefore, the output was not part of our original list. □

2.4 Prime Numbers

Definition 2.4.1. x **is a factor of** y if $y = kx, k \in \mathbb{N}$. Another interpretation is that x div y.

Definition 2.4.2. A natural number is a **prime number** if it is greater than 1 and has no factors other than 1 and itself.

Definition 2.4.3. A natural number is a **composite number** if it is greater than 1 and has at least one factor other than 1 and itself.

- From 2.2.1, we know that the set of even integers is infinite. Thus, it follows that there are infinitely many composite numbers.
- The number 1 is **neither prime nor composite**.