MATH 273: Discrete Mathematics 2

Dr. Petrescu

Denny Cao

Final: April 26, 2023

Contents

9 Relations

9.1 Relations and Their Properties

9.1.1 Introduction

Definition 1. Let A and B be sets. The **cartesian product** of A and B is the set

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

- $A \times B \neq B \times A$
- Every function is a relation, but not every relation is a function. When it is a function, it is one-to-one.

Definition 2. Let A and B be sets. A **binary relation from A to B** is a subset of $A \times B$.

- A binary relation from A to B is a set R of ordered pairs. First element is from A, second element is from B.
- We use notation $a\ R\ b$ to denote $(a, b) \in R$.
- We use notation $a \not R\ b$ to denote $(a, b) \notin R$.
- When (a, b) belongs to R, a is **related to** b.
- Recall there are $2^{|S|}$ subsets of S. These are the amount of relations from A to B. Remember that this includes $\emptyset$.

Example 1. Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. This means, for instance, that $0\ R\ a$, but that $1 \not R\ b$. Relations can be represented graphically:

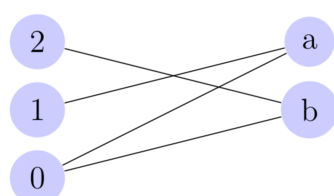


Figure 1: A relation from A to B

Another way is to use a table:

R	a	b
0	1	1
1	1	0
2	0	1

Figure 2: A relation from A to B

9.1.3 Relations on a Set

Definition 3. A **relation on a set** A is a relation from A to A

Example 2. Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ div } b\}$?

R	1	2	3	4
1	1	1	1	1
2	0	1	0	1
3	0	0	1	0
4	0	0	0	1

- Note that this can be a matrix!

9.1.4 Properties of Relations

Definition 4. A relation R on a set A is called **reflexive** if

$$\forall a \in A((a, a) \in R)$$

- If the relation is reflexive, then **the main diagonal of the matrix is full**

Definition 5. A relation R on a set A is called **symmetric** if:

$$\forall a \forall b \in A((a, b) \in R \to (b, a) \in R)$$

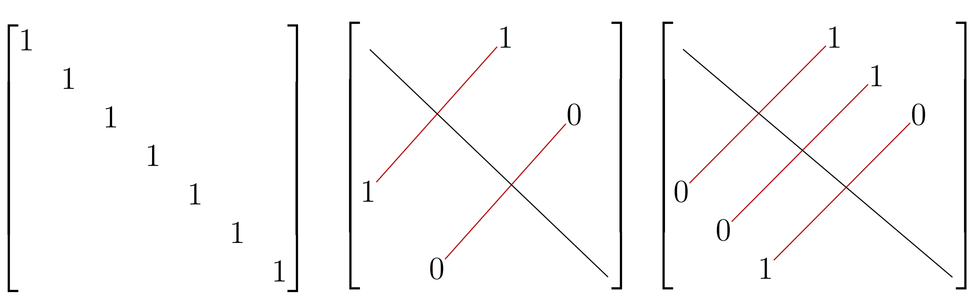
A relation R on a set A is called **antisymmetric** if:

$$\forall a \forall b \in A((a, b) \in R \land (b, a) \in R \to (a = b))$$

- If the relation is symmetric, then **the matrix is symmetric**. This means that $A = A^T$, and can be seen if the upper and lower triangles are the same.
- If the matrix is antisymmetric, then it does not necessarily mean that the **relation** is antisymmetric.

Definition 6. A relation R on a set A is called **transitive** if:

$$\forall a \forall b \forall c \in A((a, b) \in R \land (b, c) \in R \implies (a, c) \in R)$$



(a) Reflexive (b) Symmetric (c) Antisymmetric

9.1.5 Combining Relations

Similar to composing functions. We can combine relations in any way two sets can be combined. You can do everything you can do with sets.

Definition 7. Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite** of R and S is the relation consisting of ordered pairs (a, c), where $a \in A, c \in C$, and for which $\exists b \in B \mid (a, b) \in R \wedge (b, c) \in S$. We denote the composite of R and S by $S \circ R$.

- If there is an A in R that maps to B, and for S there is a B that maps to a C, then the composite $S \circ R$ will map A to C.
- If B maps to multiple C's?, then the composite will map A to multiple C's.

Definition 8. Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$ are defined recursively by:

$$R^1 = R \quad \text{and} \quad R^{n+1} = R \circ R^n$$

Definition 9. The relation R on a set A is **transitive** if and only if:

$$\forall n \geq 1, R^n \subseteq R$$

9.2 n-ary Relations and Their Applications

9.2.1 n-ary Relations

Definition 1. Let $A_1, A_2, \ldots, A_n$ be sets. An **n-ary relation** is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the **domains** of the relation, and n is called its degree.

Example 1. Let R be the relation consisting of 4-tuples (A, N, S, D, T) representing airplane flights, where A is the airline, N is the flight number, S is the starting point, D is the destination, and T is the departure time. The degree of this relation is 5, and its domains are the set of airlines, the set of flight numbers, the set of cities, the set of cities (again), and the set of times.

9.2.2 Databases and Relations

9.2.3 Matrix Representation

Suppose $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A, b \in B$, and $a > b$. The matrix representation of R if $a_1 = 1, a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$ is:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

- The number of rows is the size of A. The number of columns is the size of B.
- This is the set: $A\ R\ B = \{(2,1),(3,1),(3,2)\}$.

Let a_{ij} represent an element in a matrix in the ith row and the jth column:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- A relation is **reflexive** if the main diagonal is all 1: $a_{11} = a_{22} = a_{33} = a_{ii}$
- A relation is **symmetric** if $a_{ij} = a_{ji}$. This implies that $A^T = A$.
- We can figure out if a relation is **antisymmetric** by using 9. Suppose the relations R_1 and R_2 are represented by:

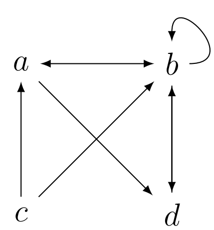
  $$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } M_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

  The composition of these two matrices is the boolean product of them, from 8. Let $M_{R_1} = A$ and $M_{R_2} = B$. Then the composite of R_1 and R_2 is:

  $$A \odot B = \begin{bmatrix} A_{\text{row } 1} \cdot B_{\text{col } 1} & A_{\text{row } 1} \cdot B_{\text{col } 2} & A_{\text{row } 1} \cdot B_{\text{col } 3} \\ A_{\text{row } 2} \cdot B_{\text{col } 1} & A_{\text{row } 2} \cdot B_{\text{col } 2} & A_{\text{row } 2} \cdot B_{\text{col } 3} \\ A_{\text{row } 3} \cdot B_{\text{col } 1} & A_{\text{row } 3} \cdot B_{\text{col } 2} & A_{\text{row } 3} \cdot B_{\text{col } 3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

  If R_1 is contained in R_2, then R_1 is antisymmetric.

We can represent relations as directed graphs. The following is a directed graph with verticies a, b, c, and d, and edges $(a,b), (a,d), (b,b), (b,d), (c,a), (c,b)$, and (d,b):



- **Reflexive** if there is a loop on every vertex.
- **Symmetric** if every edge is bidirectional.