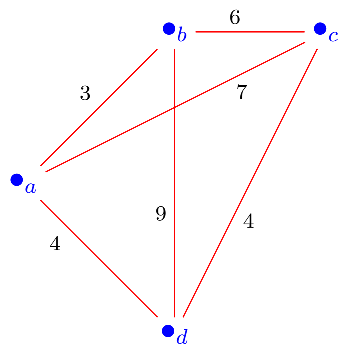
Homework 3

MATH 263: Discrete Mathematics 2

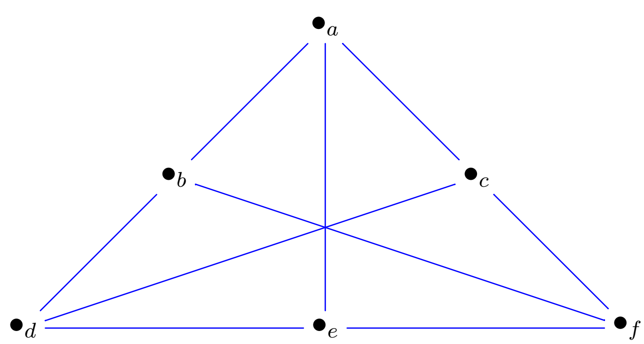
Dr. Petrescu

Due: March 3, 2023 **Denny Cao**

Question 1. Solve the traveling salesperson problem for this graph by finding the total weight of all Hamilton circuits and determining a circuit with minimum total weight.



Question 2. Try to daw the given graph without any crossings. If it is not possible explain why.



Question 3. An edge coloring of a graph is an assignment of colors to edges so that edges incident with a common vertex are assigned different colors. The edge chromatic number of a graph is the smallest number of colors that can be used in an edge coloring of the graph. The edge chromatic number of a graph G is denoted by $\chi(G)$. Find the edge chromatic numbers of:

a) C_n, where $n \geq 3$.   b) W_n, where $n \geq 3$.

Question 4. Find the edge chromatic number of K_n when n is a positive integer.

Answer 1.

Question 5. Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq \frac{v^2}{4}$.

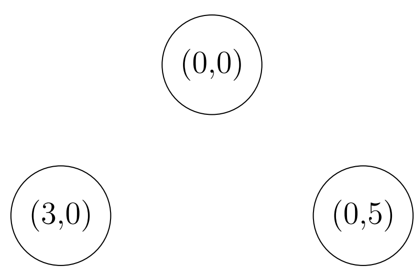
Answer 2. *Proof.* Let V be the vertex set of G and E be the edge set of G. As G is bipartite, we can partition V into two sets V_1 and V_2, where $\forall a, b \in V_1, \forall c, d \in V_2((a, b) \notin E \wedge (c, d) \notin E)$. The maximum amount of edges then, will be when all vertices in V_1 are connected to all vertices in V_2. This will give us $|V_1| \cdot |V_2|$ edges. The total vertices, v, is equal to $|V_1| + |V_2|$. We will maximize $|V_1| \cdot |V_2|$ with the constraint that $|V_1| + |V_2| = v$. Let $f(|V_1|, |V_2|) = |V_1| \cdot |V_2|$, the function we are optimizing, and $g(|V_1|, |V_2|) = |V_1| + |V_2| - v$, our constraint. We can then write the Lagrangian function as:

$$\begin{aligned} F(|V_1|, |V_2|, \lambda) &= f(|V_1|, |V_2|) + \lambda g(|V_1|, |V_2|) \\ &= |V_1| \cdot |V_2| + \lambda(|V_1| + |V_2| - v) \end{aligned}$$

$$\boldsymbol{\nabla} F = \begin{pmatrix} \frac{\partial F}{\partial |V_1|} \\ \frac{\partial F}{\partial |V_2|} \\ \frac{\partial F}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} |V_2| + \lambda \\ |V_1| + \lambda \\ |V_1| + |V_2| - v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From this, $\lambda = -|V_1| \wedge \lambda = -|V_2| \to |V_1| = |V_2|$. It follows that $|V_1| + |V_1| - v \to |V_1| = \frac{v}{2}$. As $|V_1| = |V_2|$, $|V_2| = \frac{v}{2}$. Therefore, $|V_1| \cdot |V_2| = \frac{v^2}{4}$. This means that the maximum amount of edges is $\frac{v^2}{4}$. Thus, $e \leq \frac{v^2}{4}$. $\square$

Question 6. Suppose that you have a three-gallon jug and a five-gallon jug. You may fill either jug with water, you may empty either jug, and you may transfer water from either jug into the other jug. Use a path in a directed graph to show that you can end up with a jug containing exactly one gallon. [Hint: Use an ordered pair (a, b) to indicate how much water is in each jug. Represent these ordered pairs by vertices. Add an edge for each allowable operation with the jugs.]



Answer 3.

Question 7. Find the number of paths of length n between any two adjacent vertices in $K_{3,3}$ for the values of n in $\{3, 4, 5, 6\}$

Answer 4. The adjacency matrix for $K_{3,3}$ is as follows:

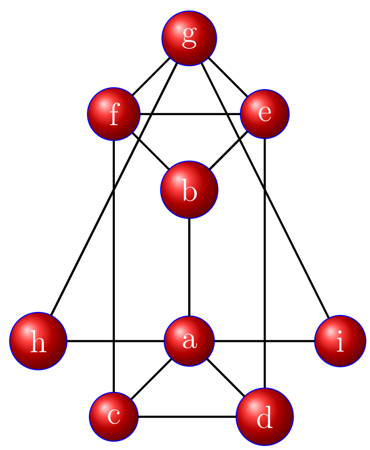
$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

We can then find the number of paths of length n between any two adjacent vertices in $K_{3,3}$ by raising the adjacency matrix to the power n and summing the elements.
For $n = 3$, we get:

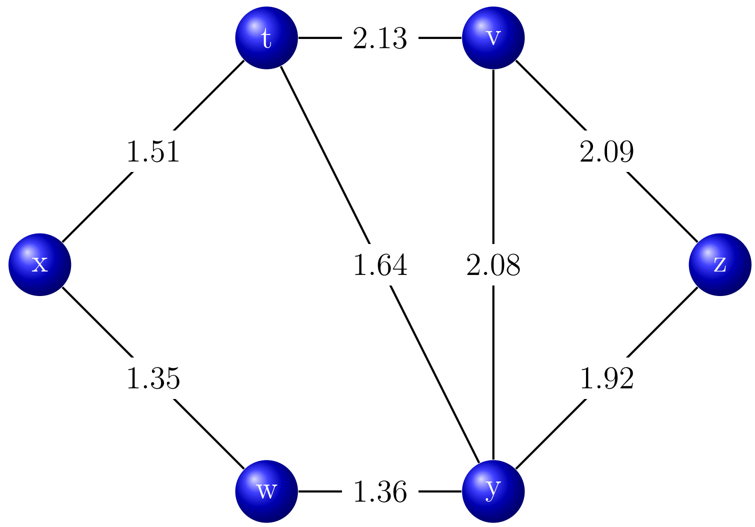
$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 9 & 9 & 9 \\ 0 & 0 & 0 & 9 & 9 & 9 \\ 0 & 0 & 0 & 9 & 9 & 9 \\ 9 & 9 & 9 & 0 & 0 & 0 \\ 9 & 9 & 9 & 0 & 0 & 0 \\ 9 & 9 & 9 & 0 & 0 & 0 \end{bmatrix}$$

As the path from a vertex a to a vertex b is the same as the path from b to a, we can divide the sum of the elements by 2 to get the number of paths of length 3 between any two adjacent vertices in $K_{3,3}$. This gives us $9 \cdot 6 = 54$ paths of length 3.
For $n = 4$, we get:

Question 8. Determine whether (i) Dirac's theorem can be used to show the graphs below have a Hamilton circuit, (ii) whether Ore's theorem can be used and finally (iii) if the graph has a Hamilton circuit.



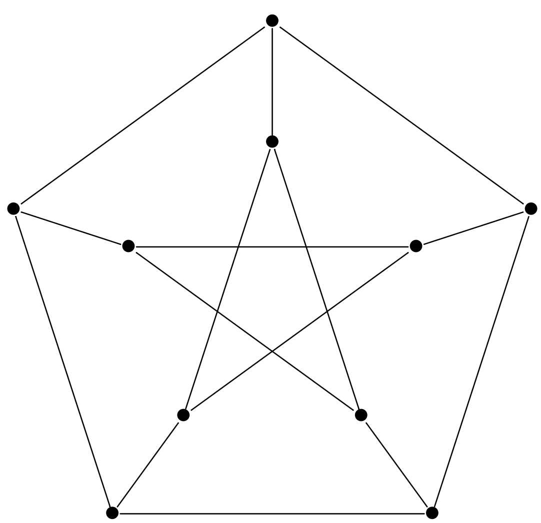
Question 9. Find the length of a shortest path between $\{(x \text{ and } z),(v \text{ and } w), (t \text{ and } z)\}$ in the weighted graph below, using Djikstra algorithm. Show each step.



Question 10. Prove the following statement: If H is a subgraph of G and G is a planar simple graph, then H is also planar.

Answer 5. *Proof.* As G is a planar graph, it does not contain a subgraph that is isomorphic to $K_{3,3}$ or K_5. Since H is a subgraph of G, it also does not contain a subgraph that is isomorphic to $K_{3,3}$ or K_5. Thus, H is also planar. □

Question 11. Find the chromatic number, $\chi(G)$, of the graph below and decide whether or not the graph is planar. Justify your answer.



Answer 6. $\chi(G) = 3$

Question 12. Prove that Dijkstra's Algorithm finds the length of the shortest path between 2 vertices of a connected simple undirected weighted graph.
NOTE: Check the textbook.