MATH 263: Discrete Mathematics 2

Practice Exam 1

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Problem 1: Let $R = R : A \to A$ be a relation from a set A to itself, then:

$$R^n = \overbrace{R \circ R \circ \cdots \circ R \circ R}^{n}$$

That is, R^n is the composition of R with itself n times.

Give a counter example or prove the following assertions:

a. If R is reflexive then R^n is reflexive.

b. If R is symmetric then R^n is symmetric.

c. If R is transitive then R^n is transitive.

Problem 2: Suppose that R and S are reflexive relations on a set A. Prove or disprove each of these statements.

a) $R \cup S$ is reflexive.

Proof. $\forall x \in A$, since R and S are reflexive, xRx and xSx. Therefore, $x(R \cup S)x$. □

b) $R \cap S$ is reflexive.

Proof. $\forall x \in A$, since R and S are reflexive, $(x, x) \in R$ and $(x, x) \in S$. Therefore, $(x, x) \in R \cap S$. Since $(x, x) \in R \cap S$, $x(R \cap S)x$.s □

c) $R \oplus S$ is irreflexive.

Proof. $\forall x \in A$, since R and S are reflexive, $(x, x) \in R \wedge (x, x) \in S$. Therefore, $(x, x) \notin R \oplus S$. Since $(x, x) \notin R \oplus S$, it follows that $R \oplus S$ is irreflexive. □

d) $R - S$ is irreflexive.

Proof. $\forall x \in A$, since R and S are reflexive, $(x, x) \in R \wedge (x, x) \in S$. Therefore, $(x, x) \notin R - S$. Since $(x, x) \notin R - S$, it follows that $R - S$ is irreflexive. □

e) $S \circ R$ (S composed with R) is reflexive.

Proof. $\forall x \in A$, Since R and S are reflexive, $(x, x) \in R \wedge (x, x) \in S$. Therefore, $(x, x) \in S \circ R$. Since $(x, x) \in S \circ R$, it follows that $S \circ R$ is reflexive. □

Problem 3: Find the matrix that represents the relation R on $\{1, 2, 3, 4, 6, 12\}$, where aRb means $a|b$. Use elements in the order given to determine rows and columns of the matrix.
Answer: Let a be the row index and b be the column index.

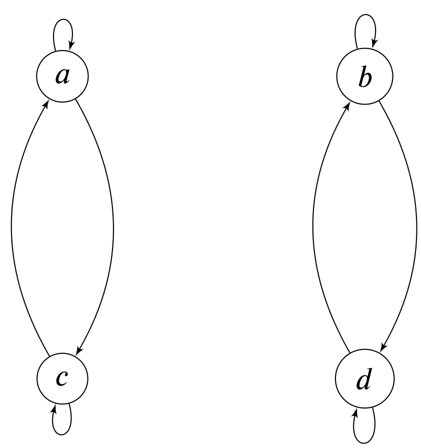
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Answer:

Problem 4: Draw the directed graph for the relation defined by the matrix:

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Answer:



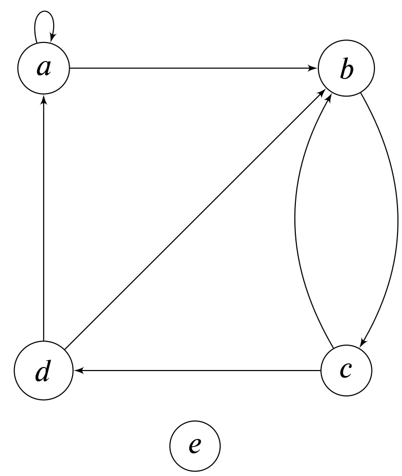
Problem 5: A Lemma in the book states: *Let A be a set with n elements, and let $\mathbf{R}$ be a relation on A. If there is a path of length at least one in $\mathbf{R}$ from a to b, then there is such a path with length not exceeding n. Moreover, when $a \neq b$, if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding $n - 1$.* The book proves for the case that $a = b$. Find the proof for the case that $a \neq b$.

Answer:

Problem 6: Draw the directed graph that represents the relation:

$$A\mathcal{R}A = \{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$$

where $A = \{a, b, c, d, e\}$
Answer:

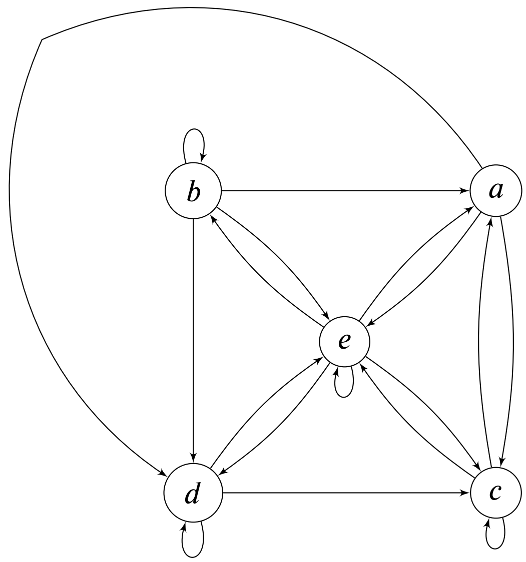


Problem 7: Find the matrix of the relation of $A\mathcal{R}A$ from Question 6 above. **Answer:**

$$R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 8: From the directed graph of question Question 6 above draw the digraph of $\bar{R}$ (the

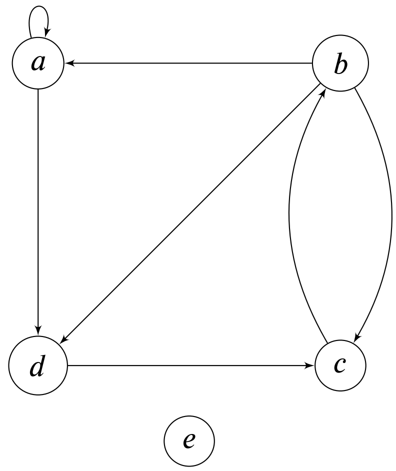
complement of R).
Answer:



Problem 9: Find the matrix of the relation of $A\bar{R}A$ from question Question 6 above.
Answer: The complement of R is the matrix:

$$\overline{R} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Problem 10: From the directed graph of question Question 6 above draw the digraph of R^{-1} (the inverse of R).
Answer:



Problem 11: Find the matrix of the relation of $A\mathcal{R}^{-1}A$ from question Question 6 above.
Answer: The inverse of R is the matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 12: In $A\mathcal{R}A$ from question Question 6 above remove or add the least amount of elements so that $A\mathcal{R}A$ represents an equivalence relation.
For $A\mathcal{R}A$ to represent an equivalence relation, it must be reflexive, symmetric and transitive.
To be reflexive, there must be a relation from each element to itself. Therefore, $(b, b), (c, c), (d, d), (e, e)$ must be added to R.
To be transitive, $\forall a \forall b \forall c \in A(aRb \wedge bRc \to aRc)$.

- $aRb \wedge bRc \to aRc$
- $aRc \wedge cRd \to aRd$
- $bRc \wedge cRd \to bRd$
- $bRd \wedge dRa \to bRa$
- $cRd \wedge dRa \to cRa$
- $dRb \wedge bRc \to dRc$

Therefore, $(a, c), (a, d), (b, a), (b, d), (c, a), (d, c)$ must be added to R.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

To be symmetric, $\forall a \forall b \in A(aRb \leftrightarrow bRa)$, which is already satisfied. Therefore, 10 elements must be added to R to create an equivalence relation.