

Homework 3

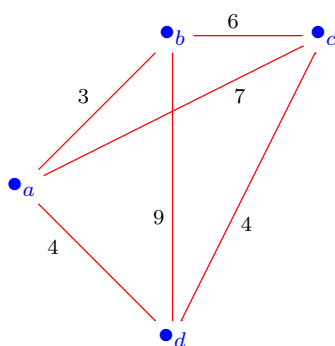
MATH 263: Discrete Mathematics 2

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Due: March 3, 2023

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Question 1. Solve the traveling salesperson problem for this graph by finding the total weight of all Hamilton circuits and determining a circuit with minimum total weight.



Answer 1. There are 3 Hamilton circuits.

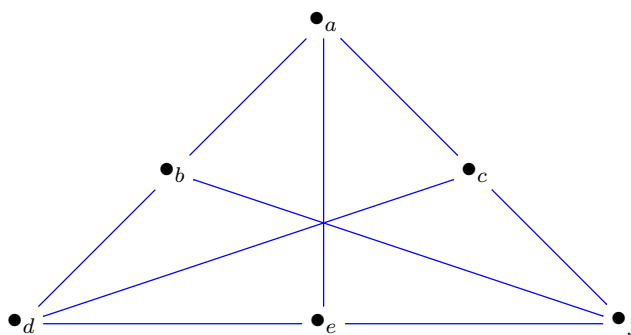
$H_1 = a, b, c, d, a$ with total weight $3 + 6 + 4 + 4 = 17$.

$H_2 = a, c, b, d, a$ with total weight $7 + 6 + 9 + 4 = 26$.

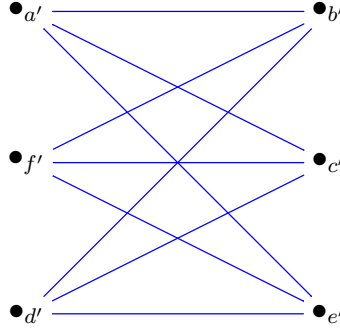
$H_3 = a, c, d, b, a$ with total weight $7 + 4 + 9 + 3 = 23$.

The circuit with minimum total weight is H_1 , or a, b, c, d, a with total weight 17.

Question 2. Try to draw the given graph without any crossings. If it is not possible explain why.



Answer 2. *Proof.* It is not possible to draw the graph which we will denote G without any crossings. We will prove this by showing that the graph is homeomorphic to $K_{3,3}$. Let $K_{3,3}$ be drawn as follows:



Let V_1 denote the vertex set of G . Let V_2 denote the vertex set of $K_{3,3}$. We construct a function $g : V_1 \rightarrow V_2$ by $g(a) = a'$, $g(b) = b'$, $g(c) = c'$, $g(d) = d'$, $g(e) = e'$, and $g(f) = f'$. Let A_1 be the adjacency matrix of G and A_2 be the adjacency matrix of $K_{3,3}$

$$A_1 = \begin{matrix} & \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \\ f \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{matrix} \quad A_2 = \begin{matrix} & \begin{matrix} a' & b' & c' & d' & e' & f' \end{matrix} \\ \begin{matrix} a' \\ b' \\ c' \\ d' \\ e' \\ f' \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

As the adjacency matrices of G and $K_{3,3}$ are the same for corresponding vertices, we have that g is a homeomorphism from G to $K_{3,3}$. Thus, G is not planar, meaning that it cannot be drawn without any crossings. \square

Question 3. An edge coloring of a graph is an assignment of colors to edges so that edges incident with a common vertex are assigned different colors. The edge chromatic number of a graph is the smallest number of colors that can be used in an edge coloring of the graph. The edge chromatic number of a graph G is denoted by $\chi(G)$. Find the edge chromatic numbers of:

a) C_n , where $n \geq 3$.

b) W_n , where $n \geq 3$.

Question 4. Find the edge chromatic number of K_n when n is a positive integer.

Answer 3.

$$\chi(K_n) = \begin{cases} n-1 & n \text{ is even} \\ n & n \text{ is odd} \end{cases}$$

Question 5. Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq \frac{v^2}{4}$.

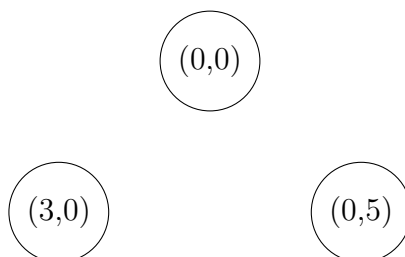
Answer 4. Proof. Let V be the vertex set of G and E be the edge set of G . As G is bipartite, we can partition V into two sets V_1 and V_2 , where $\forall a, b \in V_1, \forall c, d \in V_2 ((a, b) \notin E \wedge (c, d) \notin E)$. The maximum amount of edges then, will be when all vertices in V_1 are connected to all vertices in V_2 . This will give us $|V_1| \cdot |V_2|$ edges. The total vertices, v , is

equal to $|V_1| + |V_2|$. We will maximize $|V_1| \cdot |V_2|$ with the constraint that $|V_1| + |V_2| = v$. Let $f(|V_1|, |V_2|) = |V_1| \cdot |V_2|$, the function we are optimizing, and $g(|V_1|, |V_2|) = |V_1| + |V_2| - v$, our constraint. We can then write the Lagrangian function as:

$$\begin{aligned} F(|V_1|, |V_2|, \lambda) &= f(|V_1|, |V_2|) + \lambda g(|V_1|, |V_2|) \\ &= |V_1| \cdot |V_2| + \lambda(|V_1| + |V_2| - v) \\ \nabla F &= \begin{pmatrix} \frac{\partial F}{\partial |V_1|} \\ \frac{\partial F}{\partial |V_2|} \\ \frac{\partial F}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} |V_2| + \lambda \\ |V_1| + \lambda \\ |V_1| + |V_2| - v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

From this, $\lambda = -|V_1| \wedge \lambda = -|V_2| \rightarrow |V_1| = |V_2|$. It follows that $|V_1| + |V_1| - v \rightarrow |V_1| = \frac{v}{2}$. As $|V_1| = |V_2|$, $|V_2| = \frac{v}{2}$. Therefore, $|V_1| \cdot |V_2| = \frac{v^2}{4}$. This means that the maximum amount of edges is $\frac{v^2}{4}$. Thus, $e \leq \frac{v^2}{4}$. \square

Question 6. Suppose that you have a three-gallon jug and a five-gallon jug. You may fill either jug with water, you may empty either jug, and you may transfer water from either jug into the other jug. Use a path in a directed graph to show that you can end up with a jug containing exactly one gallon. [Hint: Use an ordered pair (a, b) to indicate how much water is in each jug. Represent these ordered pairs by vertices. Add an edge for each allowable operation with the jugs.]



Answer 5.

Question 7. Find the number of paths of length n between any two adjacent vertices in $K_{3,3}$ for the values of n in $\{3, 4, 5, 6\}$

Answer 6. The adjacency matrix for $K_{3,3}$ is as follows:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

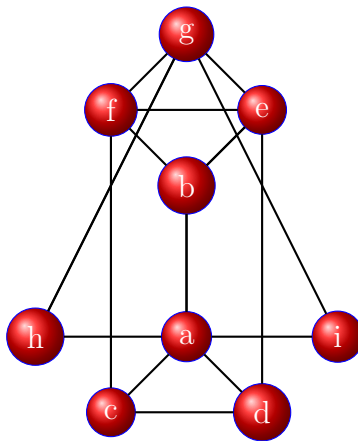
We can then find the number of paths of length n between any two adjacent vertices in $K_{3,3}$ by raising the adjacency matrix to the power n .

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 9 & 9 & 9 \\ 0 & 0 & 0 & 9 & 9 & 9 \\ 0 & 0 & 0 & 9 & 9 & 9 \\ 9 & 9 & 9 & 0 & 0 & 0 \\ 9 & 9 & 9 & 0 & 0 & 0 \\ 9 & 9 & 9 & 0 & 0 & 0 \end{bmatrix} \quad A^4 = \begin{bmatrix} 27 & 27 & 27 & 0 & 0 & 0 \\ 27 & 27 & 27 & 0 & 0 & 0 \\ 27 & 27 & 27 & 0 & 0 & 0 \\ 0 & 0 & 0 & 27 & 27 & 27 \\ 0 & 0 & 0 & 27 & 27 & 27 \\ 0 & 0 & 0 & 27 & 27 & 27 \end{bmatrix}$$

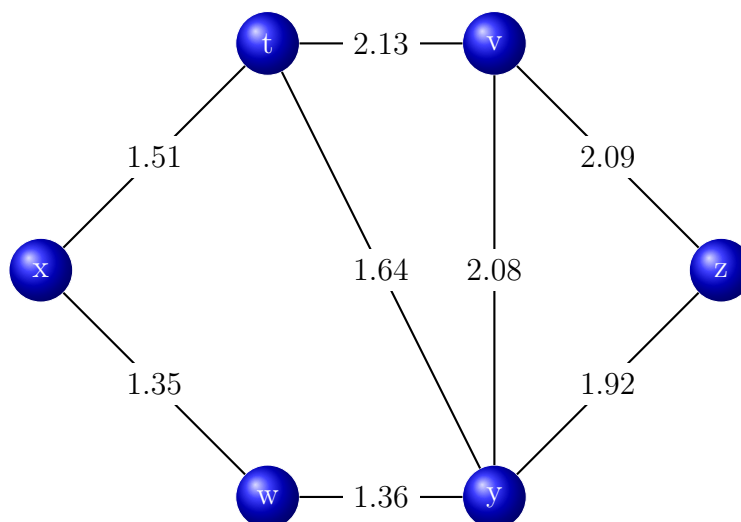
$$A^5 = \begin{bmatrix} 0 & 0 & 0 & 81 & 81 & 81 \\ 0 & 0 & 0 & 81 & 81 & 81 \\ 0 & 0 & 0 & 81 & 81 & 81 \\ 81 & 81 & 81 & 0 & 0 & 0 \\ 81 & 81 & 81 & 0 & 0 & 0 \\ 81 & 81 & 81 & 0 & 0 & 0 \end{bmatrix} \quad A^6 = \begin{bmatrix} 243 & 243 & 243 & 0 & 0 & 0 \\ 243 & 243 & 243 & 0 & 0 & 0 \\ 243 & 243 & 243 & 0 & 0 & 0 \\ 0 & 0 & 0 & 243 & 243 & 243 \\ 0 & 0 & 0 & 243 & 243 & 243 \\ 0 & 0 & 0 & 243 & 243 & 243 \end{bmatrix}$$

There are 9 paths of length 3, 27 paths of length 4, 81 paths of length 5 and 243 paths of length 6 between any two adjacent vertices in $K_{3,3}$.

Question 8. Determine whether (i) Dirac's theorem can be used to show the graphs below have a Hamilton circuit, (ii) whether Ore's theorem can be used and finally (iii) if the graph has a Hamilton circuit.



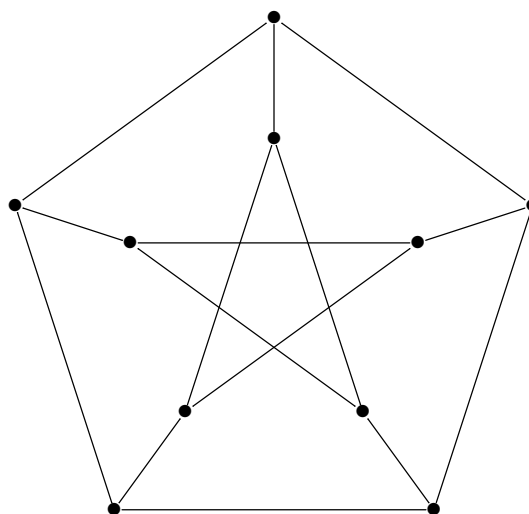
Question 9. Find the length of a shortest path between $\{(x \text{ and } z), (v \text{ and } w), (t \text{ and } z)\}$ in the weighted graph below, using Dijkstra algorithm. Show each step.



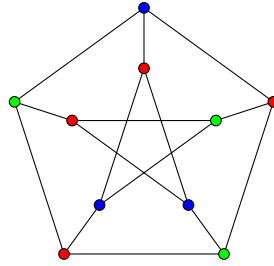
Question 10. Prove the following statement: If H is a subgraph of G and G is a planar simple graph, then H is also planar.

Answer 7. *Proof.* As G is a planar graph, it does not contain a subgraph that is homeomorphic to $K_{3,3}$ or K_5 by Kuratowski's Theorem. Since H is a subgraph of G , it also does not contain a subgraph that is homeomorphic to $K_{3,3}$ or K_5 . Thus, H is also planar. \square

Question 11. Find the chromatic number, $\chi(G)$, of the graph below and decide whether or not the graph is planar. Justify your answer.



Answer 8. $\chi(G) = 3$. We can color the graph as follows:



Question 12. Prove that Dijkstra's Algorithm finds the length of the shortest path between 2 vertices of a connected simple undirected weighted graph.

NOTE: Check the textbook.