Homework 3

MATH 263: Discrete Mathematics 2

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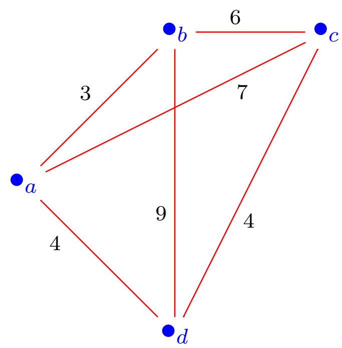
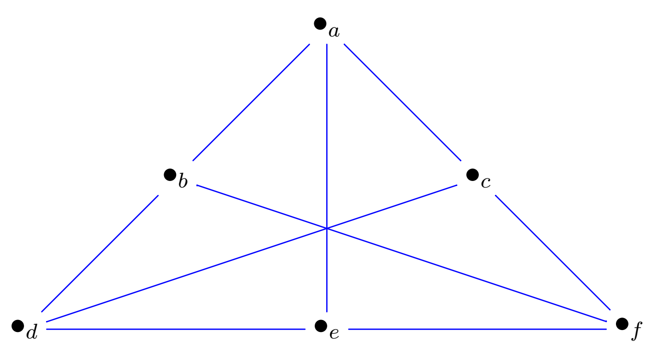
Due: March 3, 2023 **Denny Cao**

Question 1. Solve the traveling salesperson problem for this graph by finding the total weight of all Hamilton circuits and determining a circuit with minimum total weight.



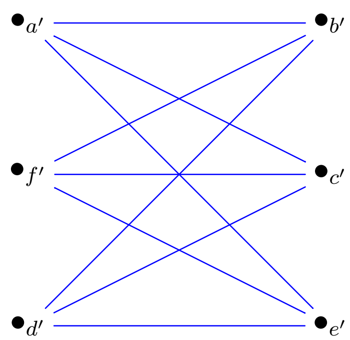
Answer 1. There are 3 Hamilton circuits.
$H_1 = a, b, c, d, a$ with total weight $3 + 6 + 4 + 4 = 17$.
$H_2 = a, c, b, d, a$ with total weight $7 + 6 + 9 + 4 = 26$.
$H_3 = a, c, d, b, a$ with total weight $7 + 4 + 9 + 3 = 23$.
The circuit with minimum total weight is H_1, or a, b, c, d, a with total weight 17.

Question 2. Try to draw the given graph without any crossings. If it is not possible explain why.



Answer 2. It is not possible to draw the graph which we will denote G without any crossings.

Proof. We will prove this by showing that the graph is homeomorphic to $K_{3,3}$. Let $K_{3,3}$ be drawn as follows:



Let V_1 denote the vertex set of G. Let V_2 denote the vertex set of $K_{3,3}$. We construct a function $g : V_1 \to V_2$ by $g(a) = a'$, $g(b) = b'$, $g(c) = c'$, $g(d) = d'$, $g(e) = e'$, and $g(f) = f'$. Let A_1 be the adjacency matrix of G and A_2 be the adjacency matrix of $K_{3,3}$

$$A_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \quad A_2 = \begin{array}{c} \\ a' \\ b' \\ c' \\ d' \\ e' \\ f' \end{array}\begin{array}{c} \begin{array}{cccccc} a' & b' & c' & d' & e' & f' \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

As the adjacency matrices of G and $K_{3,3}$ are the same for corresponding vertices, we have that g is a homeomorphism from G to $K_{3,3}$. Thus, G is not planar, meaning that it cannot be drawn without any crossings. □

Question 3. An edge coloring of a graph is an assignment of colors to edges so that edges incident with a common vertex are assigned different colors. The edge chromatic number of a graph is the smallest number of colors that can be used in an edge coloring of the graph. The edge chromatic number of a graph G is denoted by $\chi(G)$. Find the edge chromatic numbers of:

a) C_n, where $n \geq 3$. b) W_n, where $n \geq 3$.

Answer 3.

a) $\chi(C_n) = \begin{cases} 2 & n \text{ is even} \\ 3 & n \text{ is odd} \end{cases}$

Proof. Let E_x represent the xth edge of C_n, where $1 \leq x \leq n-1, x \in \mathbb{Z}$. In C_n, an edge E_c share a vertex with E_{c+1} and a different vertex with E_{c-1}, and if $c = 1$, then E_c shares a vertex with E_n and if $c = n-1$, then E_c shares a vertex with E_1.

Case 1: n is even. In this case, we can color E_2k with one color and E_{2k-1} with another color for all $k \in \mathbb{Z}$ such that $1 \leq k \leq \frac{n}{2}$. Thus, we can color C_n with two colors, as each vertex has an even and odd edge.

Case 2: n is odd. In this case, E_n and E_1 share a vertex, though n and 1 are both odd. Thus, we can color E_2k with one color and E_{2k-1} with another color for all $k \in \mathbb{Z}$ such that $1 \leq k \leq \frac{n}{2}$. We can then color E_n with a third color. Thus, we can color C_n with three colors. □

b) $\chi(W_n) = n$

Proof. The center vertex of W_n has degree n, and therefore there must be at least n colors to color the edges of W_n. The outer vertices of W_n have degree 3, which is less than or equal to n and thus n colors is enough to color the edges of W_n. □

Question 4. Find the edge chromatic number of K_n when n is a positive integer.

Answer 4.

$$\chi(K_n) = \begin{cases} n-1 & n \text{ is even} \\ n & n \text{ is odd} \end{cases}$$

Proof. As each vertex is connected with $n-1$ other vertices, there are a total of $n(n-1)/2$ edges.

Case 1: n is even. A separate color can be assigned to $\frac{n}{2}$ vertices, where none share a vertex. Thus, as $\frac{n(n-1)}{2}/\frac{n}{2} = n-1$, we can color K_n with $n-1$ colors.

Case 2: n is odd. $\frac{n-1}{2}$ edges can be assigned to each color, where none share a vertex. Thus, as $\frac{n(n-1)}{2}/\frac{n-1}{2} = n$, we can color K_n with n colors. □

Question 5. Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq \frac{v^2}{4}$.

Answer 5. *Proof.* Let V be the vertex set of G and E be the edge set of G. As G is bipartite, we can partition V into two sets V_1 and V_2, where $\forall a, b \in V_1, \forall c, d \in V_2((a, b) \notin E \land (c, d) \notin E)$. The maximum amount of edges then, will be when all vertices in V_1 are connected to all vertices in V_2. This will give us $|V_1| \cdot |V_2|$ edges. The total vertices, v, is equal to $|V_1| + |V_2|$. We will maximize $|V_1| \cdot |V_2|$ with the constraint that $|V_1| + |V_2| = v$. Let $f(|V_1|, |V_2|) = |V_1| \cdot |V_2|$, the function we are optimizing, and $g(|V_1|, |V_2|) = |V_1| + |V_2| - v$, our constraint. We can then write the Lagrangian function as:

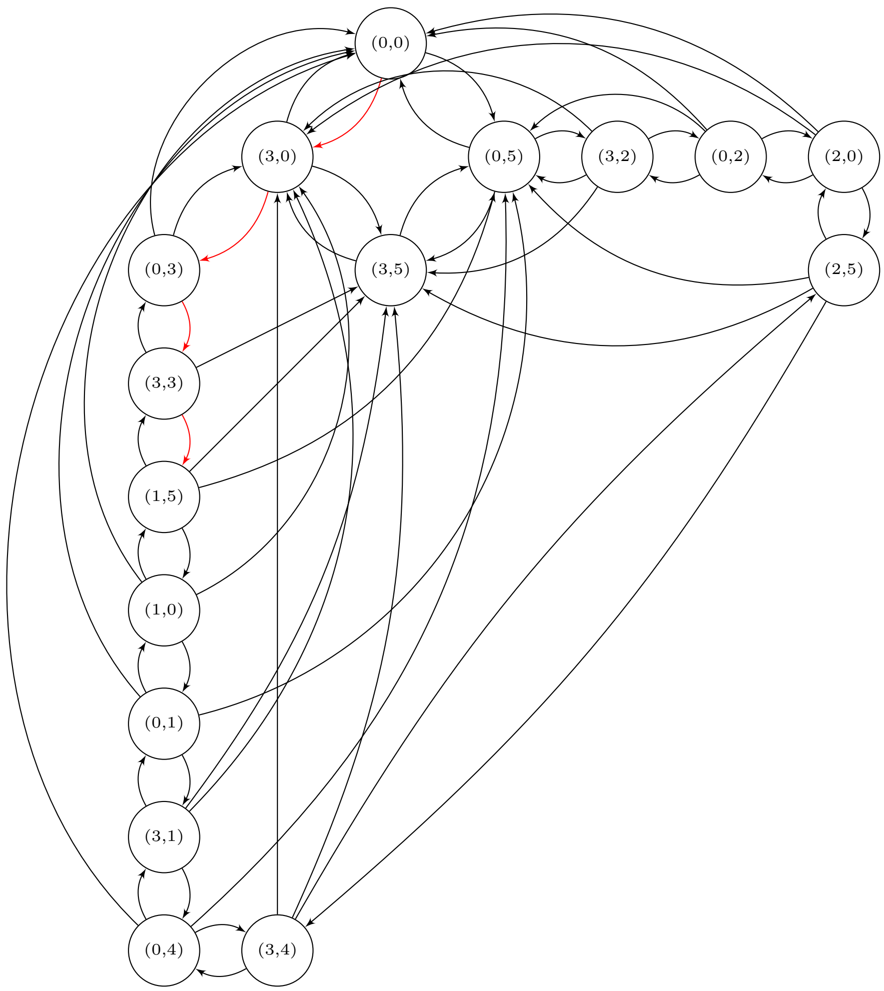
$$\begin{aligned} F(|V_1|, |V_2|, \lambda) &= f(|V_1|, |V_2|) + \lambda g(|V_1|, |V_2|) \\ &= |V_1| \cdot |V_2| + \lambda(|V_1| + |V_2| - v) \end{aligned}$$

$$\boldsymbol{\nabla} F = \begin{pmatrix} \frac{\partial F}{\partial |V_1|} \\ \frac{\partial F}{\partial |V_2|} \\ \frac{\partial F}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} |V_2| + \lambda \\ |V_1| + \lambda \\ |V_1| + |V_2| - v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From this, $\lambda = -|V_1| \land \lambda = -|V_2| \to |V_1| = |V_2|$. It follows that $|V_1| + |V_1| - v \to |V_1| = \frac{v}{2}$. As $|V_1| = |V_2|$, $|V_2| = \frac{v}{2}$. Therefore, $|V_1| \cdot |V_2| = \frac{v^2}{4}$. This means that the maximum amount of edges is $\frac{v^2}{4}$. Thus, $e \leq \frac{v^2}{4}$. □

Question 6. Suppose that you have a three-gallon jug and a five-gallon jug. You may fill either jug with water, you may empty either jug, and you may transfer water from either jug into the other jug. Use a path in a directed graph to show that you can end up with a jug containing exactly one gallon. [Hint: Use an ordered pair (a, b) to indicate how much water is in each jug. Represent these ordered pairs by vertices. Add an edge for each allowable operation with the jugs.]

Answer 6.



To end up with a jug containing exactly one gallon, we can follow the path

$$(0,0),(3,0),(0,3),(3,3),(1,5)$$

Question 7. Find the number of paths of length n between any two adjacent vertices in $K_{3,3}$ for the values of n in $\{3,4,5,6\}$

Answer 7. The adjacency matrix for $K_{3,3}$ is as follows:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

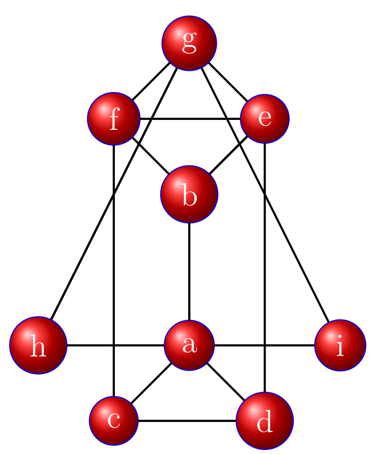
We can then find the number of paths of length n between any two adjacent vertices in $K_{3,3}$ by raising the adjacency matrix to the power n.

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 9 & 9 & 9 \\ 0 & 0 & 0 & 9 & 9 & 9 \\ 0 & 0 & 0 & 9 & 9 & 9 \\ 9 & 9 & 9 & 0 & 0 & 0 \\ 9 & 9 & 9 & 0 & 0 & 0 \\ 9 & 9 & 9 & 0 & 0 & 0 \end{bmatrix} \quad A^4 = \begin{bmatrix} 27 & 27 & 27 & 0 & 0 & 0 \\ 27 & 27 & 27 & 0 & 0 & 0 \\ 27 & 27 & 27 & 0 & 0 & 0 \\ 0 & 0 & 0 & 27 & 27 & 27 \\ 0 & 0 & 0 & 27 & 27 & 27 \\ 0 & 0 & 0 & 27 & 27 & 27 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 0 & 0 & 0 & 81 & 81 & 81 \\ 0 & 0 & 0 & 81 & 81 & 81 \\ 0 & 0 & 0 & 81 & 81 & 81 \\ 81 & 81 & 81 & 0 & 0 & 0 \\ 81 & 81 & 81 & 0 & 0 & 0 \\ 81 & 81 & 81 & 0 & 0 & 0 \end{bmatrix} \quad A^6 = \begin{bmatrix} 243 & 243 & 243 & 0 & 0 & 0 \\ 243 & 243 & 243 & 0 & 0 & 0 \\ 243 & 243 & 243 & 0 & 0 & 0 \\ 0 & 0 & 0 & 243 & 243 & 243 \\ 0 & 0 & 0 & 243 & 243 & 243 \\ 0 & 0 & 0 & 243 & 243 & 243 \end{bmatrix}$$

There are 9 paths of length 3, 27 paths of length 4, 81 paths of length 5 and 243 paths of length 6 between any two adjacent vertices in $K_{3,3}$.

Question 8. Determine whether (*i*) Dirac's theorem can be used to show the graphs below have a Hamilton circuit, (*ii*) whether Ore's theorem can be used and finally (*iii*) if the graph has a Hamilton circuit.



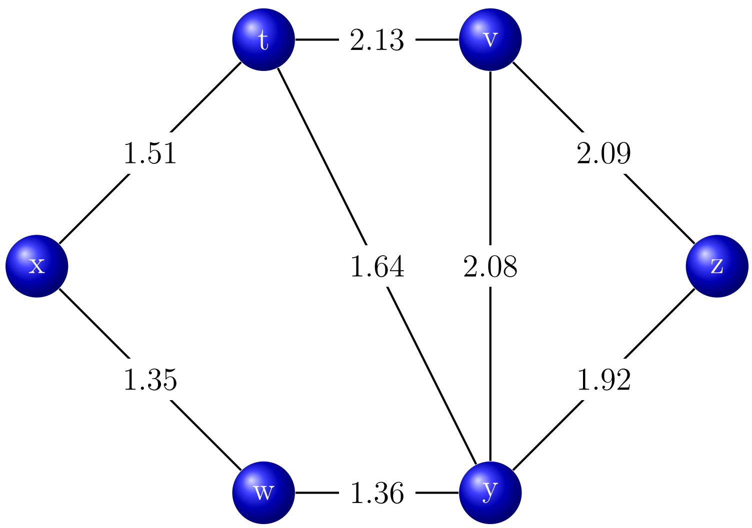
Answer 8.

i) Dirac's theorem states that an n-vertex simple graph with $n \geq 3$ in which each vertex has at least degree $\frac{n}{2}$ has a Hamilton circuit. This theorem cannot be used, as there exists a vertex g with degree 4, which is less than $\frac{9}{2} = 4.5$.

ii) Ore's theorem states that an n-vertex simple graph with $n \geq 3$ such that $\deg u + \deg v \geq n$ for every pair of nonadjacent vertices u and v, then the graph has a Hamilton circuit. This theorem cannot be used, as the vertex g has degree 4, and vertex b has degree 3, and $4 + 3 < 9$.

iii) No, there is no Hamilton circuit in this graph.

Question 9. Find the length of a shortest path between $\{(x$ and $z),($ v and $w)$, $(t$ and $z)\}$ in the weighted graph below, using Djikstra algorithm. Show each step.



Answer 9.

visited nodes	current shortest path
x	0
x, w	1.35
x, w, t	1.51
x, w, t, y	2.71
x, w, t, y, v	3.64
x, w, t, y, v, z	4.63

The shortest path between v and w is $v \to y \to w$, length 3.44.

visited nodes	current shortest path
v	0
v, y	2.08
v, y, z	2.09
v, y, z, t	2.13
v, y, z, t, w	3.44
v, y, z, t, w, x	3.44

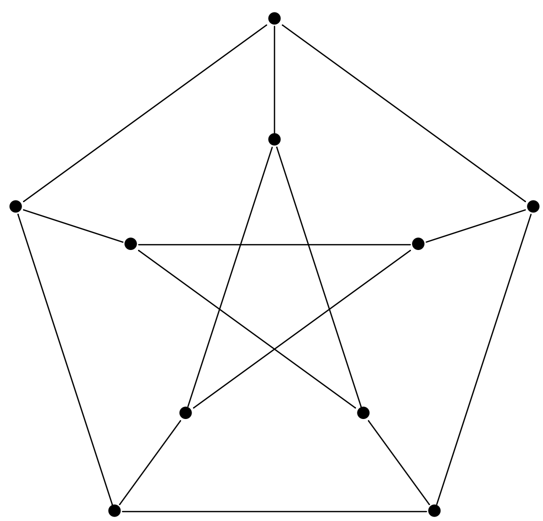
The shortest path between t and z is $t \to y \to z$, length 3.56.

visited nodes	current shortest path
t	0
t, x	1.51
t, x, y	1.64
t, x, y, v	2.13
t, x, y, v, w	2.86
t, x, y, v, w, z	3.56

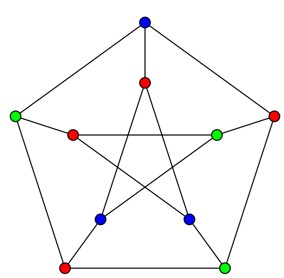
Question 10. Prove the following statement: If H is a subgraph of G and G is a planar simple graph, then H is also planar.

Answer 10. *Proof.* As G is a planar graph, it does not contain a subgraph that is homeomorphic to $K_{3,3}$ or K_5 by Kuratowski's Theorem. Since H is a subgraph of G, it also does not contain a subgraph that is homeomorphic to $K_{3,3}$ or K_5. Thus, H is also planar. $\square$

Question 11. Find the chromatic number, $\chi(G)$, of the graph below and decide whether or not the graph is planar. Justify your answer.

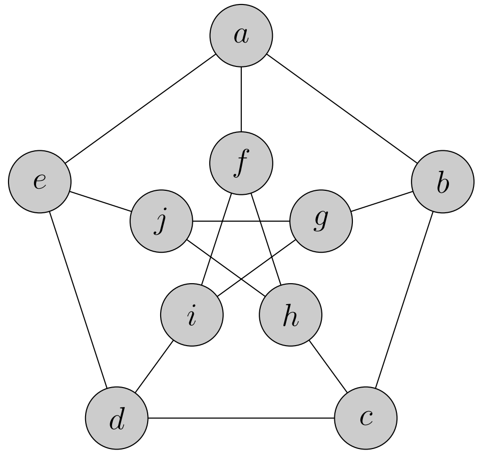


Answer 11.
Part 1: $\chi(G) = 3$. We can color the graph as follows:

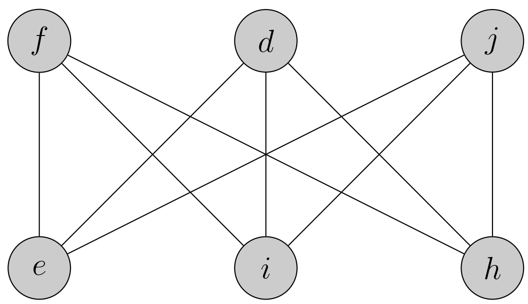


Part 2: G is not planar.

Proof. We label the vertices of G as follows:



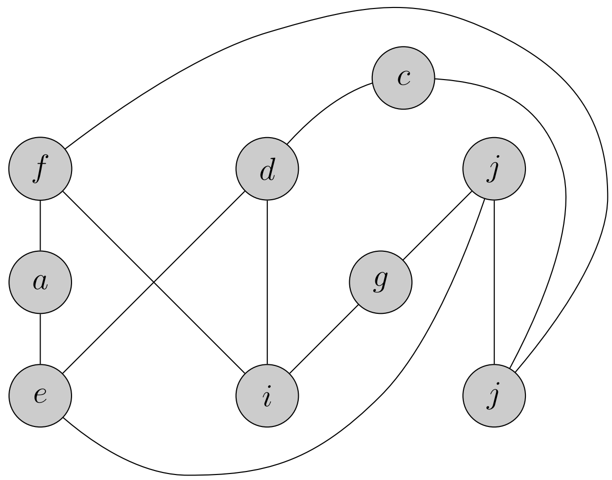
We will label the graph of $K_{3,3}$ as follows:



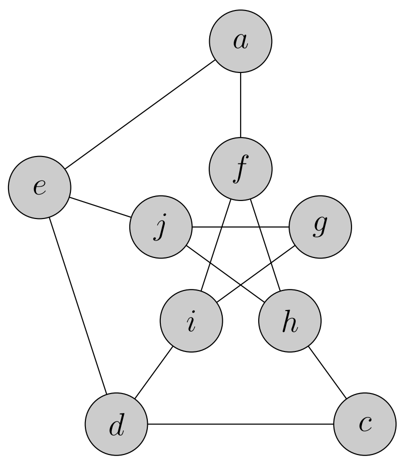
We can create a graph H that is homeomorphic to $K_{3,3}$ by a sequence of elementary subdivisions:

- Removing the edge $\{d, h\}$ and adding $\{d, c\}$ and $\{c, h\}$
- Removing the edge $\{e, f\}$ and adding $\{e, a\}$ and $\{a, f\}$
- Removing the edge $\{i, j\}$ and adding $\{i, g\}$ and $\{g, j\}$

H is as follows:



H is isomorphic to the following graph:



□

This graph is a subgraph of G, obtained by removing the vertex b and its adjacent edges: ba, bc, and bg. As H is a subgraph of G and is homeomorphic to $K_{3,3}$, G is not planar.

Question 12. Prove that Dijkstra's Algorithm finds the length of the shortest path between 2 vertices of a connected simple undirected weighted graph.
NOTE: Check the textbook.

Answer 12. *Proof.* By induction. Suppose there are two vertices a and z. Let $P(n)$ be the statement that, at the nth iteration,

i) the label of every vertex v in S is the length of a shortest path from a to this vertex, and

ii) the label of every vertex not in S is the length of a shortest path from a to this vertex that contains only (besides the vertex itself) vertices in S.

Base Case: $n = 0$. As there are no iterations that have been performed, $S = \emptyset$, so the length of a shortest path from a to any vertex other than a is ∞. Therefore, the basis is true.

Inductive Hypothesis: Suppose that $P(k)$ is true, $k \geq 1$. Let v be the vertex added to S at the $k+1$st iteration, meaning v is the smallest label not a vertex in S at the end of the kth iteration. We will show that $P(k) \to P(k+1)$.

Inductive Step: From the inductive hypothesis, the vertices in S prior to the $k+1$st iteration have the shortest path from a to them and v is labeled with the length of the shortest path from a to v. We will prove this by contradiction.

If this was not true, then there exists a shorter path from a to v containing at least one vertex not in S. Let u be the first of these vertices not in S in the path from a to v. From the inductive hypothesis *(ii)*, the path from a to u is shorter than the path from a to v. Therefore, the path from a to v is not the shortest path from a to v. ※

This contradicts the choice of v. Thus, *(i)* holds for the $k+1$st iteration.

Let u be a vertex not in S after $k+1$ iterations. If the shortest path from a to u does not contain v, then, by the inductive hypothesis, its length, $L_k(u)$. If it does contain v, then the path is made up of a path from a to v of shortest possible length using elements in S other than v, followed by the edge from v to u. In this case, its length is $L_k(v)+w(u,v)$. Therefore, *(ii)* is true because $L_{k+1}(u)$ minimizes $L_k(u)$, producing the least value for $L_{k+1}(u)$.

Thus, $P(k+1)$ is true. Therefore, $P(n)$ is true for all $n \geq 0$. Therefore, Dijkstra's Algorithm finds the length of the shortest path between 2 vertices of a connected simple undirected weighted graph. □