MATH 273: Discrete Mathematics 2

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Contents

1 Relations

1.1 Introduction

1.1.1 Cartesian Products

Definition 1.1.1. Let A and B be sets. The **cartesian product** of A and B is the set

$$A \times B = \{(a, b) \mid a \in A \land b \in B\} \tag{1}$$

- $A \times B \neq B \times A$
- Recall there are $2^{|S|}$ subsets of S. These are the amount of relations from A to B (A subset of the cartesian product is a relation). Remember that this includes $\emptyset$.
- Every function is a relation, but not every relation is a function. When it is a function, it is one-to-one.

Example 1.1.1. Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. This means, for instance, that $0 \; R \; a$, but that $1 \not R \; b$. Relations can be represented graphically:

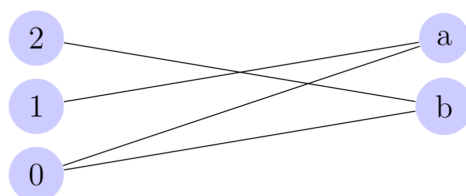


Figure 1: A relation from A to B

Another way is to use a table:

R	a	b
0	1	1
1	1	0
2	0	1

Figure 2: A relation from A to B

Definition 1.1.2. A **relation on a set** A is a relation from A to A

Example 1.1.2. Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in $the relation$R$= \{(a, b) \mid a \text{ div } b\}$?

R	1	2	3	4
1	1	1	1	1
2	0	1	0	1
3	0	0	1	0
4	0	0	0	1

- Note that this can be a matrix!

1.2 Properties of Relations

Definition 1.2.1. A relation R on a set A is called **reflexive** if

$$\forall a \in A, (a, a) \in R \tag{2}$$

- If the relation is reflexive, then **the main diagonal of the matrix is full**

Definition 1.2.2. A relation R on a set A is called **symmetric** if:

$$\forall a \forall b \in A, (a, b) \in R \implies (b, a) \in R \tag{3}$$

A relation R on a set A is called **antisymmetric** if:

$$\forall a \forall b \in A, (a, b) \in R \implies (b, a) \in R \tag{4}$$

- If the relation is symmetric, then **the matrix is symmetric**. This means that $A = A^T$, and can be seen if the upper and lower triangles are the same.
- If the matrix is antisymmetric, then it does not necessarily mean that the **relation** is antisymmetric.

Definition 1.2.3. A relation R on a set A is called **transitive** if:

$$\forall a \forall b \forall c \in A, (a, b) \in R \land (b, c) \in R \implies (a, c) \in R. \tag{5}$$

1.3 Combining Relations

Similar to composing functions. We can combine relations in any way two sets can be combined. You can do everything you can do with sets.

Definition 1.3.1. Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite** of R and S is the relation consisting of ordered pairs (a, c), where $a \in A, c \in C$, and for which $\exists b \in B \mid (a, b) \in R \land (b, c) \in S$. We denote the composite of R and S by $S \circ R$.

- If there is an A in R that maps to B, and for S there is a B that maps to a C, then the composite $S \circ R$ will map A to C.
- If B maps to multiple C's?, then the composite will map A to multiple C's.

Definition 1.3.2. Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$ are defined recursively by:

$$R^1 = R \quad \text{and} \quad R^{n+1} = R \circ R^n \tag{6}$$

Definition 1.3.3. The relation R on a set A is **transitive** if and only if:

$$\forall n \geq 1, R^n \subseteq R \tag{7}$$

2 Matrix Representation

Suppose $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A, b \in B$, and $a > b$. The matrix representation of R if $a_1 = 1, a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$ is:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

- The number of rows is the size of A. The number of columns is the size of B.
- This is the set: $A\ R\ B = \{(2, 1), (3, 1), (3, 2)\}$.

Let a_{ij} represent an element in a matrix in the ith row and the jth column:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- A relation is **reflexive** if the main diagonal is all 1: $a_{11} = a_{22} = a_{33} = a_{ii}$
- A relation is **symmetric** if $a_{ij} = a_{ji}$. This implies that $A^T = A$.
- We can figure out if a relation is **antisymmetric** by using 1.3.3. Suppose the relations R_1 and R_2 are represented by:

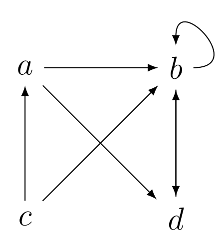
  $$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } M_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

  The composition of these two matrices is the boolean product of them, from 1.3.2. Let $M_{R_1} = A$ and $M_{R_2} = B$. Then the composite of R_1 and R_2 is:

  $$A \odot B = \begin{bmatrix} A_{\text{row 1}} \cdot B_{\text{col 1}} & A_{\text{row 1}} \cdot B_{\text{col 2}} & A_{\text{row 1}} \cdot B_{\text{col 3}} \\ A_{\text{row 2}} \cdot B_{\text{col 1}} & A_{\text{row 2}} \cdot B_{\text{col 2}} & A_{\text{row 2}} \cdot B_{\text{col 3}} \\ A_{\text{row 3}} \cdot B_{\text{col 1}} & A_{\text{row 3}} \cdot B_{\text{col 2}} & A_{\text{row 3}} \cdot B_{\text{col 3}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

  If R_1 is contained in R_2, then R_1 is antisymmetric.

We can represent relations as directed graphs. The following is a directed graph with verticies a, b, c, and d, and edges $(a, b), (a, d), (b, b), (b, d), (c, a), (c, b)$, and (d, b):



- **Reflexive** if there is a loop on every vertex.
- **Symmetric** if every edge is bidirectional.