Exercises part.1 - Gradient & Conjugate Gradient algorithms

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1. Implement the algorithms

Gradient algorithm

- At first step $x_0 = 0$, $r_0 = b$
- At each step k:
  - compute $t = Ar_{k-1}$
  - compute $\alpha = \frac{(r_{k-1}, r_{k-1})}{(r_{k-1}, t)}$
  - update $x : x_k = x_{k-1} + \alpha r_{k-1}$
  - update $r : r_k = r_{k-1} - \alpha t$
- Iterate until $\hat{r} = \sqrt{\frac{(r_k, r_k)}{(b, b)}} < \hat{r}_{targ}$

Conjugate Gradient algorithm

- At first step $x_0 = 0$, $r_0 = p_0 = b$
- At each step k:
  - compute $t = Ap_{k-1}$
  - compute $\alpha = \frac{(r_{k-1}, r_{k-1})}{(p_{k-1}, t)}$
  - update $x : x_k = x_{k-1} + \alpha p_{k-1}$
  - update $r : r_k = r_{k-1} - \alpha t$
  - compute $\beta = \frac{(r_k, r_k)}{(r_{k-1}, r_{k-1})}$
  - update $p_k = r_k + \beta p_{k-1}$
- Iterate until $\hat{r} = \sqrt{\frac{(r_k, r_k)}{(b, b)}} < \hat{r}_{targ}$

2. Solve the linear system

$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

   and verify:

   (a) the solution obtained with the numerical algorithm with the one derived solving explicitly the system of equations

   (b) that convergence is achieved after 2 iterations, as required by the algorithm

3. Using the provided routine that generate random symmetric definite matrices with fixed condition number, verify:

   (a) the scaling of the number of iteration required to solve the system of equation at a fixed precision $\hat{r}_{targ}$ and matrix condition number with the size of the problem (provided the condition number is large enough $\tilde{}10^3 \div 10^6$)

   (b) that with fixed residue and fixed (large) matrix size, the number of iterations scales linearly with the square root of the condition number

4. (optional) check that for a non-positive definite matrix the algorithm does not converge

5. (optional) verify that the functional

$$F(x) = \frac{1}{2} x^t A x - bx$$

   is monotonously minimized during the iterations (while this is not true for $\hat{r}$)