Intrinsically-typed and well-scoped SMTLIB FFI bindings with modular abstract syntax trees (MAST)

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Slides:

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Computer and Information Sciences, University of Strathclyde, Glasgow, Scotland

Funded by:

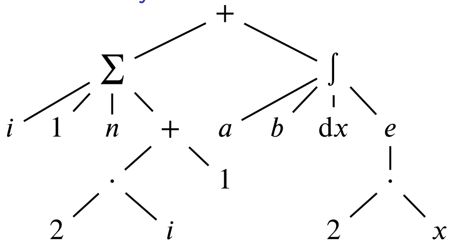
Syntax representation

$$\left(\sum_{i=1}^{n}(2i+1)\right)+\int_a^b e^{ax}\mathrm{d}x$$

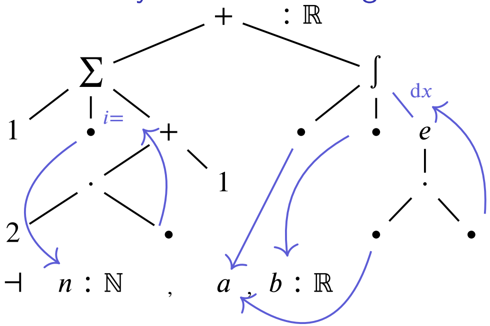
Concrete syntax

“(”, “$\sum$”, “_”, “{”, “i”, “=”, “1”, “}”, “{”, “n”, “(”, “2”, “i”, “+”, “1”, “)”, “}”, …

Abstract syntax



Abstract syntax with binding



Call-by-Value λ-calculus

$$
\begin{array}{rll}
A, B, C ::= & & \text{type} \\
& \beta & \text{base} \\
| & A \rightarrow B & \text{function} \\
| & (\!| C_i : A_i | i \in I |\!) & \text{record } (I \text{ finite}) \\
| & \{\!| C_i : A_i | i \in I |\!\} & \text{variant } (I \text{ finite}) \\
\vdots & &
\end{array}
$$

$$
\begin{array}{rll}
V, W ::= & & \text{value} \\
& x & \text{variable} \\
| & \lambda x : A.M & \text{function abst.} \\
| & (C_i : V_i | i \in I) & \text{record c'tor} \\
| & A.C_i\ V & \text{variant c'tor} \\
\vdots & &
\end{array}
$$

$$
\begin{array}{rll}
M, N, K, L ::= & & \text{term} \\
& \mathsf{val}\ V & \text{value} \\
| & \mathbf{let}\ x_1 = M_1; \ldots; x_n = M_n\ \mathbf{in}\ N & \text{sequencing} \\
| & M\ @\ N & \text{function application} \\
| & (C_1 : M_1, \ldots, C_n : M_n) & \text{record constructor} \\
| & \mathbf{case}\ M\ \mathbf{of}\ (C_1 x_1, \ldots, C_n x_n) \Rightarrow N & \text{record pattern match} \\
| & A.C_i M & \text{variant constructor} \\
| & \mathbf{case}\ M\ \mathbf{of}\ \{C_i x_i \Rightarrow M_i | i \in I\}\ N & \text{variant pattern match} \\
\vdots & &
\end{array}
$$

High-level motivation

Initial Algebra Semantics Programme [Goguen and Thatcher'74]

Denotational semantics á la carte [homage to Swierstra'08, Forster and Stark'20]

CBV customisation menu

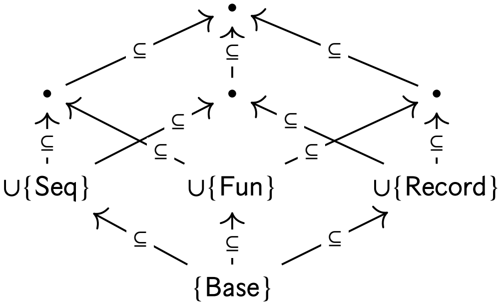
fragment	syntactic constructs	types	semantics
base	returning a value: val		strong monad over a Cartesian category
sequential	sequencing: **let**		
functions	abst., app. $(\lambda x.:A),(@)$	function $(\rightarrow)$	Kleisli exponentials
variants	c'tors, pattern match $A.C_i-$, $\mathbf{case}-\mathbf{of}$ $\left\{C_i x_i \Rightarrow - \middle| i \in I\right\}$	variant $\{\!\!|C_i : - \middle| i \in I|\!\!\}$	distributive category

⋮

Dream

Iterative semantic development

- Add syntax
- Add semantics

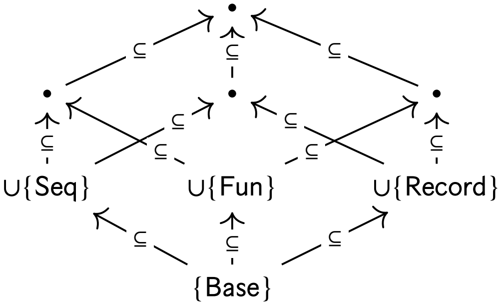


- Profit!

Dream vs. Reality

Iterative semantic development

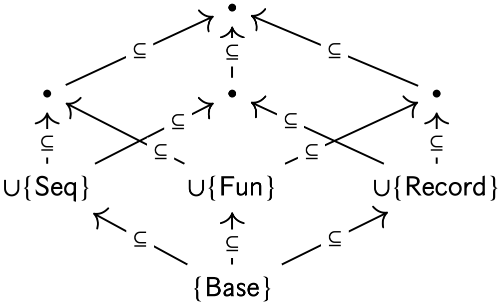
- Add syntax
- Add semantics
- Develop meta-theory:
  - Substitution lemma
  - Compositionality
  - Soundness
  - Adequacy
- Profit!



Dream vs. Bleak Reality

Iterative semantic development

- Add syntax
- Add semantics
- Develop meta-theory:
  - Substitution lemma
    Tedious and boring
  - Compositionality
    Tedious and boring
  - Soundness
  - Adequacy
- Profit!



Meta-theory: the tedious parts

Lemma (substitution)

Syntactic substitution corresponds to semantic composition:

$$[\![M\,[\theta]]\!] = [\![M]\!] \circ [\![\theta]\!]$$

Lemma (compositionality)

Composite semantics is independent of component syntax:

$$[\![C[M]]\!] = \mathrm{plug}([\![C[-]]\!]\,, [\![M]\!])$$

Meta-theory: the tedious parts

Lemma (substitution)

Syntactic substitution corresponds to semantic composition:

$$[\![M\,[\theta]]\!] = [\![M]\!] \circ [\![\theta]\!]$$

Proof.

Presupposes a syntactic substitution lemma. Typically several inductions over all constructs. $\square$

Lemma (compositionality)

Composite semantics is independent of component syntax:

$$[\![C[M]]\!] = \mathrm{plug}([\![C[-]]\!]\,,[\![M]\!])$$

Proof.

Tediously define terms with holes, plugging holes syntactically, carefully capturing some variables but not others. Then induction over semantics. $\square$

Dream

It would be nice if tedious bits were...

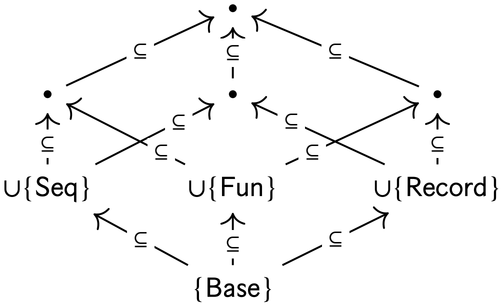
... free

Dream vs. Reality

It would be nice if tedious bits were…

… ~~free~~

… syntactically scaleable: additive syntactic work per new feature



Expression problem [Reynolds'75, Cook'90, Krishnamurthi, Felleisen and Friedman'98, Wadler'98]

Spec [Wadler'98]

Both:

- **Extend** object-language syntax
- **Add** meta-language functions/properties of programs

But:

- Without recompiling previous modules; alternatively
- Retaining and reusing both old and new languages

Some solutions

- Scala Mix-ins [Zenger'98, Zenger and Odersky'01]
- Visitor Pattern in Pizza, Zodiac [Krishnamurthi, Felleisen and Friedman'98]
- Recursive Generics [Wadler'98]
- Data-types á la carte: coproducts of signature functors [Swierstra'08]

SOAS: Second-Order Abstract Syntax [Fiore, Plotkin, and Turi '99]

- Initial algebra characterisation for
  abstract syntax with binding-aware substitution
- Robust to extensions:
  - polymorphism [Fiore and Hamana'13]
  - mechanisation [Crole'11, Allais et al.'18, Fiore and Szamozvancev'22]
  - substructurality [Fiore and Ranchod'25]
- CBN works smoothly. Doesn't cover CBV.
  Technical reasons later:
  - Substitute **in**: values and terms
  - Substitute for variables: values only

Slogan

for substitution: values are 1^{st}-class but terms are 2^{nd}-class
[cf. Levy's CBPV, '04]

Goal: abstract syntax with heterogeneous sorting

Sorting system $\mathbf{R}$

Coproduct diagram in set:

$$\mathsf{Fst} \xrightarrow{\mathsf{fst}} \mathsf{Sort} \xleftarrow{\mathsf{snd}} \mathsf{Snd}$$

Example (CBV sorting system)

$$\mathrm{CBV}\mathsf{Type} \xrightarrow{\lambda x.x} \left(\mathrm{CBV}\mathsf{Type} \cup \{\mathsf{comp}\} \times \mathrm{CBV}\mathsf{Type}\right) \xleftarrow{\mathsf{comp}} \mathrm{CBV}\mathsf{Type}$$

Example (CBPV sorting system)

$$\mathrm{CBPV}\mathsf{ValueType} \xrightarrow{\lambda x.x} \mathrm{CBPV}\mathsf{Type} \xleftarrow{\lambda x.x} \mathrm{CBPV}\mathsf{CompType}$$

Core contribution: generalise to heterogeneous sorts

	SOAS	MAST
Sorting system:	homogeneous	heterogeneous
	$\mathsf{Sort} \xrightarrow{\lambda x.x} \mathsf{Sort} \leftarrow \emptyset$	$\mathsf{Fst} \xrightarrow{\mathsf{fst}} \mathsf{Sort} \xleftarrow{\mathsf{snd}} \mathsf{Snd}$
$P \in \textbf{R-Strct} :=$	$\textbf{PSh}\left(\mathsf{Sort} \times (\mathsf{Sort})_{\vdash}\right)$	$\textbf{PSh}\left(\mathsf{Sort} \times (\mathsf{Fst})_{\vdash}\right) \cong$ $\mathsf{Fst}\textbf{-Strct} \times \textbf{PSh}\left(\mathsf{Snd} \times (\mathsf{Fst})_{\vdash}\right)$
	$P : \mathsf{Sort} \times \mathsf{Sort}_{\vdash}^{\mathsf{op}} \to \textbf{Set}$	$P : \mathsf{Sort} \times \mathsf{Fst}_{\vdash}^{\mathsf{op}} \to \textbf{Set}$ $P\vert_{\mathsf{fst}} : \mathsf{Fst} \times \mathsf{Fst}_{\vdash}^{\mathsf{op}} \to \textbf{Set}$ $P\vert_{\mathsf{snd}} : \mathsf{Snd} \times \mathsf{Fst}_{\vdash}^{\mathsf{op}} \to \textbf{Set}$

Abstract syntax and denotational semantics: $\mathbb{S}, \mathbf{M} \in \textbf{R-Strct}$

$$\mathbb{S}_s\Gamma := \{M \mid \Gamma \vdash M : s\} \quad \mathbb{S}_s\left(\Gamma \xleftarrow{\rho} \Delta\right) := \lambda(\Gamma \vdash M : s).(\Delta \vdash M[\rho] : s)$$

$$\mathbf{M}_s\Gamma := \mathbf{C}(\llbracket\Gamma\rrbracket, \llbracket s\rrbracket) \quad \mathbf{M}_s\left(\Gamma \xleftarrow{\rho} \Delta\right) := \lambda\left(\llbracket\Gamma\rrbracket \xrightarrow{f} \llbracket s\rrbracket\right).\left(\llbracket\Delta\rrbracket \xrightarrow{\llbracket\rho\rrbracket} \llbracket\Gamma\rrbracket \xrightarrow{f} \llbracket s\rrbracket\right)$$

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Abstract syntax and denotational semantics: $\mathbb{S}, \mathbf{M} \in \mathbf{R\text{-}Strct}$

Variables $\mathbb{V}_s\Gamma := \{x \mid (x : s) \in \Gamma\}$: $\mathbb{V} \in \mathbf{R\text{-}Strct}$ $\qquad$ $\mathbb{V} \in \mathsf{Fst}\text{-}\mathbf{Strct}$

Core contribution: generalise to heterogeneous sorts

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Signature functors $\qquad \mathbf{O} \circlearrowright \mathbf{R\text{-}Strct}$

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Signature functors	$\mathbf{O} \circlearrowright \mathbf{R\text{-}Strct}$	
Pointed strength:	$P, A \in \mathbf{R\text{-}Strct}, \mathsf{var} : \mathbb{V} \to A$	$P \in \mathbf{R\text{-}Strct}, A \in \mathsf{Fst}\text{-}\mathbf{Strct}$
$\mathsf{str}_{P,\mathsf{var}} : (\mathbf{O}P) \otimes A \to \mathbf{O}(P \otimes A)$		$\mathsf{var} : \mathbb{V} \to A$

Core contribution: generalise to heterogeneous sorts

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Signature functors

$$\begin{tikzcd} \mathbf{R\text{-}Strct} \arrow[loop left, "\mathbf{O}"] \end{tikzcd}$$

Pointed strength: $P, A \in \mathbf{R\text{-}Strct}, \mathsf{var} : \mathbb{V} \to A$ $\quad$ $P \in \mathbf{R\text{-}Strct}, A \in \mathsf{Fst}\text{-}\mathbf{Strct}$

$\mathsf{str}_{P,\mathsf{var}} : (\mathbf{O}P) \otimes A \to \mathbf{O}(P \otimes A)$ $\quad$ $\mathsf{var} : \mathbb{V} \to A$

Substitution: monoid $\quad$ action/module

$$\mathbf{M} \otimes \mathbf{M} \xrightarrow{-[-]} \mathbf{M} \xleftarrow{\mathsf{var}} \mathbb{V}$$

$$\mathbf{M}|_{\mathsf{fst}} \otimes \mathbf{M}|_{\mathsf{fst}} \xrightarrow{-[-]} \mathbf{M}|_{\mathsf{fst}} \xleftarrow{\mathsf{var}} \mathbb{V}$$

$$\mathbf{M}|_{\mathsf{snd}} \otimes \mathbf{M}|_{\mathsf{fst}} \xrightarrow{-[-]} \mathbf{M}|_{\mathsf{snd}}$$

Contribution

Modular Abstract Syntax Trees (MAST)

- SOAS $\overset{\text{generalise}}{\rightsquigarrow}$ 2^{nd}-class sorts
  - Kleisli bicategories [Gambino, Fiore, Hyland, and Winskel'19]
  - Generalise monoidal categories to actegories
  - Generalise substitution monoids to actions / modules
- MAST tutorial
- Case-study: CBV semantics á la carte (128 substitution lemmata)

WIP

- Idris 2 implementation of computational fragment [cf. Fiore and Szamozvancev'22]
  Case-study: intrinsically-typed FFI-binding with holes for SMTLIB (29 theories)

Talk structure

- Contribution
- Substitution actions/modules
- MAST in detail
- WIP

Capstone: abstract syntax and substitution universality

Theorem (representation)

abstract syntax with operators in $\mathbf{O}$ *and holes in* $\mathbf{H}$
amounts to
free substitution $\mathbf{O}$*-action over* $\mathbf{H}$*:*

$$\begin{tikzcd} & \mathbf{H} \arrow[d, "?"] & \\ \mathbb{S}\mathbf{H} \otimes \mathbb{S}\mathbf{H}|_{\mathsf{fst}} \arrow[r, "-[-]"] & \mathbb{S}\mathbf{H} & \mathbb{S}\mathbf{H}|_{\mathsf{fst}} & \mathbb{V} \arrow[l, "\mathsf{var}"'] \\ & \mathbf{O}(\mathbb{S}\mathbf{H}) \arrow[u, "[\![-]\!]"] & \end{tikzcd}$$

Plugging holes/metavariable substitution

Kleisli extension ($\ggg\!=$) for $\mathbf{O}$-action monad.

Capstone: semantics

Key propaganda

compositional, binding-respecting denotational semantics
amounts to
substitution $\mathbf{O}$-action:

$$\begin{array}{ccc} \mathbf{M} \otimes \mathbf{M}|_{\mathsf{fst}} \xrightarrow{-[-]} & \mathbf{M} & \qquad \mathbf{M}|_{\mathsf{fst}} \xleftarrow[\mathsf{var}]{} \mathbb{V} \\ \llbracket - \rrbracket & \uparrow & \\ & \mathbf{OM} & \end{array}$$

Denotational semantics for terms with holes in $\mathbf{H}$ is the unique substitution $\mathbf{O}$-action homomorphism over $\mathbf{H}$:

$$\left(\$\mathbf{H}, -[-], \mathsf{var}, \llbracket - \rrbracket, ?_{\$\mathbf{H}}\right) \xrightarrow{\llbracket - \rrbracket} \left(\mathbf{M}, -[-], \mathsf{var}, \llbracket - \rrbracket, ?_{\mathbf{M}}\right) \qquad (\mathbf{H} \xrightarrow{?_{\mathbf{M}}} \mathbf{M})$$

Meta-theory in one line

Lemma (substitution)

Syntactic substitution corresponds to semantic composition:

$$[\![M\,[\theta]]\!] = [\![M]\!] \circ [\![\theta]\!]$$

Lemma (compositionality)

Composite semantics is independent of component syntax:

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Composite semantics is independent of component syntax:

$$[\![C[M]]\!] = [\![C[?m] \ggg (m \mapsto M)]\!] \overset{\substack{\ggg\ \textit{is homomorphic extension}\\ \downarrow}}{=} [\![C[?m]]\!] \ggg (m \mapsto [\![M]\!]) \eqqcolon \mathrm{plug}([\![C[-]]\!]\,,[\![M]\!])$$

Talk structure

- Contribution
- Substitution actions/modules
- MAST in detail
- WIP

MAST summary: semantic domain for syntax and semantics

MAST provides $(\mathbf{R} = \mathsf{Fst} \xrightarrow{\mathsf{fst}} \mathsf{Sort} \xleftarrow{\mathsf{snd}} \mathsf{Snd})$

- Contexts $\mathsf{Fst}_{\vdash} \ni \Gamma ::= [x_1 : s_1, \ldots, x_n : s_n]$
- Renamings $\mathsf{Fst}_{\vdash}(\Gamma, \Delta) \ni \Gamma \vdash \rho : \Delta$
- $\mathbf{R}$-structures: $\mathbf{PSh}(\mathsf{Sort} \times \mathsf{Fst}_{\vdash}) \ni P : \mathsf{Sort} \times \mathsf{Fst}_{\vdash}^{\mathsf{op}} \to \mathbf{Set}$
  $P_s\Gamma \ni p$: sort s element with variables in Γ
- Variables structure: $\mathbf{R}\text{-}\mathbf{Strct} \ni \mathbb{V}_s\Gamma := \{x | (x : s) \in \Gamma\}$
- substitution tensor and the pointed action:
  $(\otimes) : \mathbf{R}\text{-}\mathbf{Strct} \times \mathsf{Fst}\text{-}\mathbf{Strct} \to \mathbf{R}\text{-}\mathbf{Strct}$
  $(P \otimes Q)_s\Gamma \ni [p, \theta]_\Delta$:
  P-element: $p \in P_s\Delta$
  Q-closure : $\theta \in \prod_{(y:r)\in\Delta} Q_r\Gamma$
  identifying, e.g.:
  $(\otimes_\bullet) : \mathbf{R}\text{-}\mathbf{Strct} \times (\mathbb{V}/\mathsf{Fst}\text{-}\mathbf{Strct}) \to \mathbf{R}\text{-}\mathbf{Strct}$
  $P \otimes_\bullet \left(\mathsf{var} \begin{array}{c}\mathbb{V}\\ \downarrow \\ A\end{array}\right) := P \otimes A$
  $[p[\mathsf{weaken}], \theta]_{\Delta_1 + \Delta_2} = [p, \theta \circ \rho]_{\Delta_1}$
  $\left[p[x', x'' \mapsto x]_{x\in\Delta}, \theta\right]_\Delta = [p, \theta + \theta]_{\Delta + \Delta}$

Scope-change as tensorial strength

$$\mathsf{str}^{\mathbf{O}} : (\mathbf{O}P) \otimes_\bullet \left(\mathsf{var} \begin{array}{c}\mathbb{V}\\ \downarrow \\ A\end{array}\right) \to \mathbf{O}\left(P \otimes_\bullet \left(\mathsf{var} \begin{array}{c}\mathbb{V}\\ \downarrow \\ A\end{array}\right)\right)$$

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  $(P \otimes Q)_s\Gamma \ni [p, \theta]_\Delta$:

  P-element: $p \in P_s\Delta$ $\quad$ identifying, e.g.:

  Q-closure : $\theta \in \prod_{(y:r)\in\Delta} Q_r\Gamma$

  I.e.:

  $(P \otimes Q)_s\Gamma \coloneqq \int^{\Delta} P_s\Delta \times \prod_{(y:r)\in\Delta} Q_r\Gamma$

  $(\otimes_\bullet) : \mathbf{R\text{-}Strct} \times (\mathbb{V}/\mathsf{Fst}\text{-}\mathbf{Strct}) \to \mathbf{R\text{-}Strct}$

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$\textsc{mast}$ summary: semantic domain for syntax and semantics

$\textsc{mast}$ provides $\left(\mathbf{R} = \mathsf{Fst} \xrightarrow{\mathsf{fst}} \mathsf{Sort} \xleftarrow{\mathsf{snd}} \mathsf{Snd}\right)$

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  $\left[p[x', x'' \mapsto x]_{x \in \Delta}, \theta\right]_\Delta = [p, \theta + \theta]_{\Delta + \Delta}$

Scope-change as tensorial strength

$$\mathsf{str}^{\mathbf{O}} : (\mathbf{O}P) \otimes_\bullet \left(\mathsf{var} \underset{A}{\overset{\mathbb{V}}{\downarrow}}\right) \to \mathbf{O}\left(P \otimes_\bullet \left(\mathsf{var} \underset{A}{\overset{\mathbb{V}}{\downarrow}}\right)\right)$$

MAST summary: semantic domain for syntax and semantics

MAST provides $(\mathbf{R} = \mathsf{Fst} \xrightarrow{\mathsf{fst}} \mathsf{Sort} \xleftarrow{\mathsf{snd}} \mathsf{Snd})$

- Contexts $\mathsf{Fst}_{\vdash} \ni \Gamma ::= [x_1 : s_1, \ldots, x_n : s_n]$
- Renamings $\mathsf{Fst}_{\vdash}(\Gamma, \Delta) \ni \Gamma \vdash \rho : \Delta$
- $\mathbf{R}$-structures: $\mathbf{PSh}(\mathsf{Sort} \times \mathsf{Fst}_{\vdash}) \ni P : \mathsf{Sort} \times \mathsf{Fst}_{\vdash}^{\mathsf{op}} \to \mathbf{Set}$
  $P_s\Gamma \ni p$: sort s element with variables in Γ
- Variables structure: $\mathbf{R\text{-}Strct} \ni \mathbb{V}_s\Gamma := \{x | (x : s) \in \Gamma\}$
- substitution tensor and the pointed action:
  $(\otimes) : \mathbf{R\text{-}Strct} \times \mathsf{Fst}\text{-}\mathbf{Strct} \to \mathbf{R\text{-}Strct}$
  $(P \otimes Q)_s\Gamma \ni [p, \theta]_\Delta$:
  P-element: $p \in P_s\Delta$ identifying, e.g.:
  Q-closure : $\theta \in \prod_{(y:r)\in\Delta} Q_r\Gamma$

  $(\otimes_\bullet) : \mathbf{R\text{-}Strct} \times (\mathbb{V}/\mathsf{Fst}\text{-}\mathbf{Strct}) \to \mathbf{R\text{-}Strct}$
  $P \otimes_\bullet \left(\mathsf{var} \begin{smallmatrix}\mathbb{V}\\ \downarrow \\ A\end{smallmatrix}\right) := P \otimes A$
  $[p[\text{weaken}], \theta]_{\Delta_1 + \Delta_2} = [p, \theta \circ \rho]_{\Delta_1}$
  $\left[p[x', x'' \mapsto x]_{x\in\Delta}, \theta\right]_\Delta = [p, \theta +\!\!\!+ \theta]_{\Delta +\!\!\!+ \Delta}$

Scope-change as tensorial strength

$\mathsf{str}^{\mathbf{O}} : (\mathbf{O}P) \otimes_\bullet \left(\mathsf{var} \begin{smallmatrix}\mathbb{V}\\ \downarrow \\ A\end{smallmatrix}\right) \to \mathbf{O}\left(P \otimes_\bullet \left(\mathsf{var} \begin{smallmatrix}\mathbb{V}\\ \downarrow \\ A\end{smallmatrix}\right)\right)$

Substitution actions

$\mathbf{M} \otimes \mathbf{M}|_{\mathsf{fst}} \xrightarrow{-[-]} \mathbf{M} \qquad \mathbf{M}|_{\mathsf{fst}} \xleftarrow{\mathsf{var}} \mathbb{V}$

Signature functors

$$\begin{tikzcd} \mathbf{R\text{-}Strct} \arrow[loop left, "\mathbf{O}"] \end{tikzcd}$$

Example

Sequential fragment signature functor:

$$(\mathsf{Seq}\ X)_A \Gamma := \emptyset \qquad (\mathsf{Seq}\ X)_{\mathsf{comp}\ B} \Gamma := \coprod_{A \in \mathsf{Type}} \begin{pmatrix} (\mathbf{let}\ x : A = \_\ \mathbf{in}\ \_) : \\ (X_{\mathsf{comp}\ A}\Gamma \times X_{\mathsf{comp}\ B}(\Gamma, x : A)) \end{pmatrix}$$

Scope-change as tensorial strength

$$\mathsf{str}^{\mathbf{O}} : (\mathbf{O}P) \otimes_{\bullet} \left(\mathsf{var} \overset{\mathbb{V}}{\underset{A}{\downarrow}}\right) \to \mathbf{O}\left(P \otimes_{\bullet} \left(\mathsf{var} \overset{\mathbb{V}}{\underset{A}{\downarrow}}\right)\right)$$

NB: $(\otimes_{\bullet}) : \mathbf{R\text{-}Strct} \times (\mathbb{V}/\mathbf{R\text{-}Strct}) \to \mathbf{R\text{-}Strct}$

$$P \otimes_{\bullet} \left(\mathsf{var} \overset{\mathbb{V}}{\underset{A}{\downarrow}}\right) := P \otimes A$$

$$\mathsf{str}^{\mathsf{Seq}}_{P,\mathsf{var}} \left[t \in \mathsf{Seq}\ P, \theta\right]_{\Delta} = \mathsf{str}^{\mathsf{Seq}}\left[\mathbf{let}\ x : A = (p \in P_{\mathsf{comp}\ A}\Delta)\ \mathbf{in}\ (q \in P_{\mathsf{comp}\ B}(\Delta, x : A)), \theta\right]_{\Delta}$$

$$:= \left(\mathbf{let}\ x : A = [p, \theta]_{\Delta} \qquad \mathbf{in}\ \left[q, (\theta, x : \mathsf{var}\ x)\right]_{\Delta, x : A}\right)$$

Takeaway (modularity)

Each syntactic construct defines its own binding, renaming, and substitution structure

Signature combinators

[cf. SOAS]

- sums & products of signature functors
- scope extension $(\Gamma \triangleright)$
- sort extension $\underset{s}{\looparrowright} : \mathbf{PSh}\,\mathsf{Fst}_{\vdash} \to \mathbf{PSh}\left(\mathsf{Sort} \times \mathsf{Fst}_{\vdash}\right)$
- sort application $(@\,s) : \mathbf{PSh}\left(\mathsf{Sort} \times \mathsf{Fst}_{\vdash}\right) \to \mathbf{PSh}\left(\mathsf{Fst}_{\vdash}\right)$

Example (Binding signatures [Actzel'78])

$$\mathsf{Seq} \cong \coprod_{A,B \in \mathsf{Type}} \begin{pmatrix} (\mathbf{let}\ x : A = \_\ \mathbf{in}\ \_) : \underset{\mathsf{comp}\ B}{\looparrowright} \\ (@\mathsf{comp}\ A) \times \\ ([x : A] \triangleright - @\ \mathsf{comp}\ B) \end{pmatrix}$$

NB

$$(\mathsf{Seq}\ X)_{\mathsf{comp}\ B}\Gamma := \coprod_{A \in \mathsf{Type}} \begin{pmatrix} (\mathbf{let}\ x : A = \_\ \mathbf{in}\ \_) : \\ (X_{\mathsf{comp}\ A}\Gamma \times X_{\mathsf{comp}\ B}(\Gamma, x : A)) \end{pmatrix}$$

$$(\mathsf{Seq}\ X)_A\Gamma := \emptyset$$

Abstract syntax: inductive representation

Every initial algebra: $\$^{\mathbf{O}}\mathbf{H} \coloneqq \mu X.(\mathbf{O}X) \amalg \looparrowright_{\mathsf{fst}} \mathbb{V} \amalg \mathbf{H} \otimes X|_{\mathsf{fst}}$

Supports standard definitions:

$$\begin{tikzcd} & \mathbf{H} \arrow[d, "?"] & & \\ \$\mathbf{H} \otimes \$\mathbf{H}|_{\mathsf{fst}} \arrow[r, "-[-]"] & \$\mathbf{H} & \$\mathbf{H}|_{\mathsf{fst}} & \mathbb{V} \arrow[l, "\mathsf{var}"'] \\ & \mathbf{O}(\$\mathbf{H}) \arrow[u, "{[\![-]\!]}"] & & \end{tikzcd}$$

Independently of concrete representation, e.g.,:

- De-Bruijn
- Nominal
- Locally nameless
- Co-de Bruijn (NB: inefficient substitution)
- Graphical

Example

$\mathbf{M} = \left(\mathcal{C}, \mathrm{T}, \mathrm{return}, \gg\!=, [\![-]\!]\right)$:

- $\mathcal{C}$: Cartesian category with chosen finite products
- $\left(\mathrm{T}, \mathrm{return}, \gg\!=\right)$ strong monad over $\mathcal{C}$
- $[\![-]\!]$: Type $\to \mathcal{C}$ type interpretation

induces:

- A CBV-structure: $\quad \textsc{cbv}\text{-}\mathbf{Strct} \ni \mathbf{M}_s\Gamma := \mathcal{C}([\![\Gamma]\!], [\![s]\!])$
- Standard interpretation of contexts, computations, renaming:

$$\mathcal{C} \ni [\![\Gamma]\!] := \prod_{(x:A)\in\Gamma} [\![A]\!] \qquad \mathcal{C} \ni [\![\mathsf{comp}\ A]\!] := \mathrm{T}\,[\![A]\!]$$

$$[\![\rho]\!] : [\![\Gamma]\!] \xrightarrow{\left(\pi_{x[\rho]} : [\![\Gamma]\!] \to [\![A]\!]\right)_{(x:A)\in\Delta}} \prod_{(x:A)\in\Delta} [\![A]\!] = [\![\Delta]\!]$$

MAST: common structure for substitution

Syntactic substitution monoid

$$\$^{\mathbf{O}}\mathbf{H} \otimes (\$^{\mathbf{O}}\mathbf{H})|_{\mathsf{fst}} \xrightarrow{-[-]} \$^{\mathbf{O}}\mathbf{H} \qquad \$^{\mathbf{O}}\mathbf{H}|_{\mathsf{fst}} \xleftarrow{\mathsf{var}} \mathbb{V}$$

Monoid axioms amount to syntactic substitution lemma

Example

Semantic substitution action: $\mathbf{M} \otimes \mathbf{M}|_{\mathsf{fst}} \xrightarrow{-[-]} \mathbf{M} \qquad \mathbf{M}|_{\mathsf{fst}} \xleftarrow{\mathsf{var}} \mathbb{V}$

- Substitution via composition:

$$\left([\![\Delta]\!] \xrightarrow{f} [\![s]\!]\right)\left[[\![\Gamma]\!] \xrightarrow{\theta} [\![\Delta]\!]\right] : [\![\Gamma]\!] \xrightarrow{\theta} [\![\Delta]\!] \xrightarrow{f} [\![s]\!]$$

- Variables: $\mathsf{var} : \left((x : A) \in \Gamma \mapsto \left([\![\Gamma]\!] \xrightarrow{\pi_x} [\![A]\!]\right)\right)$

  (1st-class sorts only)

Substitution-compatible algebra

$[\![-]\!] : \mathbf{OM} \to \underline{\mathbf{M}}$:

$$\begin{tikzcd}
 & \mathbf{O}(\underline{\mathbf{M}} \otimes \underline{\mathbf{M}}) \arrow[rr, "\mathbf{O}(-[-]_{\mathbf{M}})"] & & \underline{\mathbf{OM}} \arrow[dd, "{[\![-]\!]}"] \\
(\underline{\mathbf{OM}}) \otimes_{\bullet} \mathsf{var}_{\mathbf{M}} \arrow[ur, "\mathsf{str}"] \arrow[dr, "{[\![-]\!] \otimes_{\bullet} \mathsf{id}}"'] & & \text{compatibility} \; = & \\
 & \underline{\mathbf{M}} \otimes_{\bullet} \mathsf{var}_{\mathbf{M}} \arrow[rr, "-[-]_{\mathbf{M}}"'] & & \underline{\mathbf{M}}
\end{tikzcd}$$

Example (Seq-algebra)

$$\left[\!\!\left[\begin{array}{l} \textbf{let } x : A = \left([\![\Gamma]\!] \xrightarrow{f} \mathrm{T}\,[\![A]\!]\right) \\ \textbf{in } \left([\![\Gamma]\!] \times [\![A]\!] \xrightarrow{g} \mathrm{T}\,[\![B]\!]\right) \end{array} \right]\!\!\right] :$$

$$[\![\Gamma]\!] \xrightarrow{(\mathsf{id}, f)} [\![\Gamma]\!] \times \mathrm{T}\,[\![A]\!] \xrightarrow{\gg\!= g} \mathrm{T}\,[\![B]\!]$$

Takeaway

Equip each semantic interpretation with its compatibility proof

Example (Seq-compatibility)

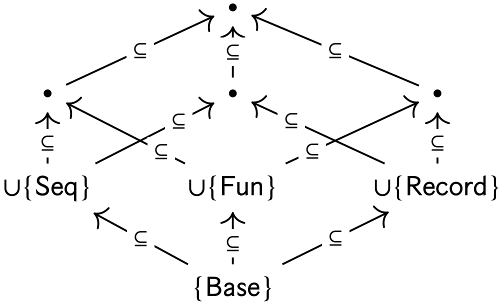
Compatibility:

$$\begin{tikzcd}
{[\![\Gamma]\!]} \arrow[r, "{(\mathsf{id}, (f \circ \theta))}"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\text{products} \; ="] & {[\![\Gamma]\!] \times \mathrm{T}\,[\![A]\!]} \arrow[d, "\theta \times \mathsf{id}"] \arrow[rd, bend left, "\gg\!=(g \circ (\theta \times \mathsf{id}))"] \arrow[rd, phantom, "\text{strong monad laws} \; ="] & \\
{[\![\Delta]\!]} \arrow[r, "{(\mathsf{id}, f)}"'] & {[\![\Delta]\!] \times \mathrm{T}\,[\![A]\!]} \arrow[r, bend right, "\gg\!= g"'] & \mathrm{T}\,[\![B]\!]
\end{tikzcd}$$

MAST: modularity and scalability

Substitution $\mathbf{O}$-action

Substitution action with compatible $\mathbf{O}$-algebra structure



Want more?

In the paper:

- All the details
- A CBV case-study (128 substitution lemmata)

Talk structure

- Contribution
- Substitution actions/modules
- MAST in detail
- WIP

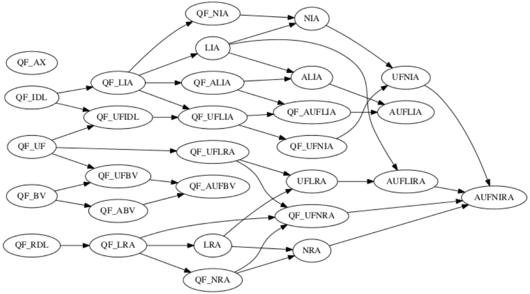
SMTLIB Foreign Function Interface (FFI)

Implementation

Idris 2 [Brady'21] implementation of computational fragment [cf. Fiore and Szamozvancev'22]

SMTLIB query language

- S-expressions
- 29 theories
- multiple syntax extensions



FFI

- Intrinsically-typed well-scoped FFI with holes
- Modular serialisation
- Modular well-scoped parsing
- Modular type-inference

[Greg Brown'25]

Contribution

Modular Abstract Syntax Trees (MAST)

- SOAS $\overset{\text{generalise}}{\rightsquigarrow}$ 2^{nd}-class sorts
  - Kleisli bicategories [Gambino, Fiore, Hyland, and Winskel'19]
  - Generalise monoidal categories to actegories
  - Generalise substitution monoids to actions / modules
- MAST tutorial
- Case-study: CBV semantics á la carte (128 substitution lemmata)

WIP

- Idris 2 implementation of computational fragment [cf. Fiore and Szamozvancev'22]
  Case-study: intrinsically-typed FFI-binding with holes for SMTLIB (29 theories)