Foundations for type-driven probabilistic modelling

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TutorialFest
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Plan:

1) type-driven probability: discrete case ✓
2) Borel sets & measurable spaces ✓
3) Quasi Borel spaces ✓

Lecture 1

4) Type structure & standard Borel spaces
5) Integration & random variables

Lecture 2

Please ask questions!

Course web page

scribble

Full model

type : Qbs $\qquad W := [0, \infty] \qquad \mathbb{B}X := \boxed{}$

$DX := \boxed{}$

$PX := \{\mu \in DX \mid \mathrm{Ce}_{\mu}[X] = 1\}$

$\mathrm{Ce}_{\mu}[E] := \boxed{} \qquad \delta_x := \boxed{}$

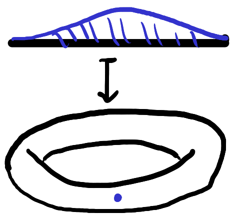
$\oint \mu k := \boxed{}$

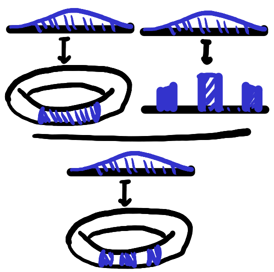
Def: Quasi-Borel space $X = (\lfloor X \rfloor, R_X)$

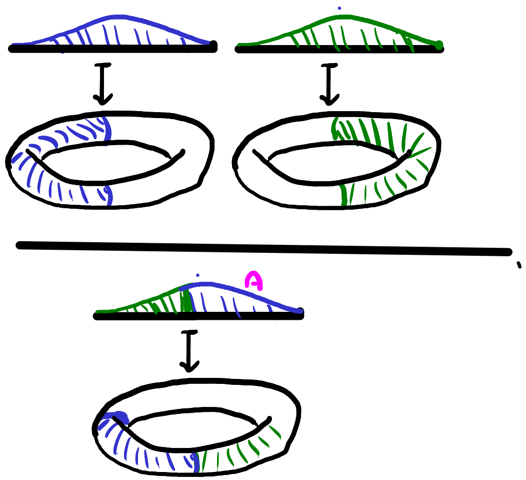
set "carrier"

set of functions $\alpha : \mathbb{R} \to \lfloor X \rfloor$ "random elements"

$R_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- constants: 

- precomposition: 

- recombination 

Examples

- $\mathbb{R} = (\lfloor\mathbb{R}\rfloor, \mathrm{Meas}(\mathbb{R},\mathbb{R}))$

qbs underlying $\mathbb{R}$

- $X \in \mathrm{Set}$, $\ulcorner X \urcorner^{\mathrm{Qbs}} := (X, \sigma\text{-simple}(\mathbb{R}, X))$

recombination of constants

$$\lambda r. \begin{cases} \vdots \\ r \in A_n : x_n \\ \vdots \end{cases}$$

discrete qbs on X

- " $\lfloor X \rfloor_{\mathrm{Qbs}} := (X, X^{\lfloor\mathbb{R}\rfloor})$

all functions

Indiscrete qbs on X

Validate qbs axioms for: $\mathbb{W} := ([0,\infty], \mathrm{Meas}(\mathbb{R}, \mathbb{W}))$

- Constants:

$$E : B_{\mathbb{W}}, \; x : \mathbb{W} \vdash$$

$$(\lambda r : \mathbb{R}.\, x)^{-1}[E] = \begin{cases} x \in E : & \mathbb{R} \\ x \notin E : & \emptyset \end{cases} \in B_{\mathbb{R}} \quad \checkmark$$

Validate qbs axioms for: $\mathbb{W} := ([0,\infty], \mathrm{Meas}(\mathbb{R}, \mathbb{W}))$

- Precomposition:

$$\alpha : \mathrm{Meas}(\mathbb{R}, \mathbb{W}),\quad \varphi : \mathrm{Meas}(\mathbb{R}, \mathbb{R}) \vdash$$

$$\mathbb{R} \xrightarrow{\varphi} \mathbb{R} \xrightarrow{\alpha} \mathbb{W} \quad \in \mathrm{Meas}(\mathbb{R}, \mathbb{W})$$

Meas is a cat.

Explicitly:

$$(\alpha \circ \varphi)^{-1}[E] \in B\mathbb{R} \xleftarrow{\varphi^{-1}} \alpha^{-1}[E] \in B\mathbb{R} \xleftarrow{\alpha^{-1}} E \in B_{\mathbb{W}}$$

✓

Validate qbs axioms for: $\mathbb{W} := ([0,\infty], \mathrm{Meas}(\mathbb{R}, \mathbb{W}))$

- recombination

I ctbl, $\underline{\alpha} : \mathrm{Meas}(\mathbb{R}, \mathbb{W})^I$, $E_{\_} : B_{\mathbb{R}}^I$, $\mathbb{R} = \biguplus_{i \in I} E_i$, $F : B_{\mathbb{W}}$,

$$\beta := \left(\lambda r. \begin{cases} \vdots \\ r \in E_i : & \alpha_i r \\ \vdots \end{cases}\right)^{-1}[F]$$

$$= \bigcup_{i \in I} \alpha_i^{-1}[F] \cap E_i \in B_{\mathbb{R}}$$

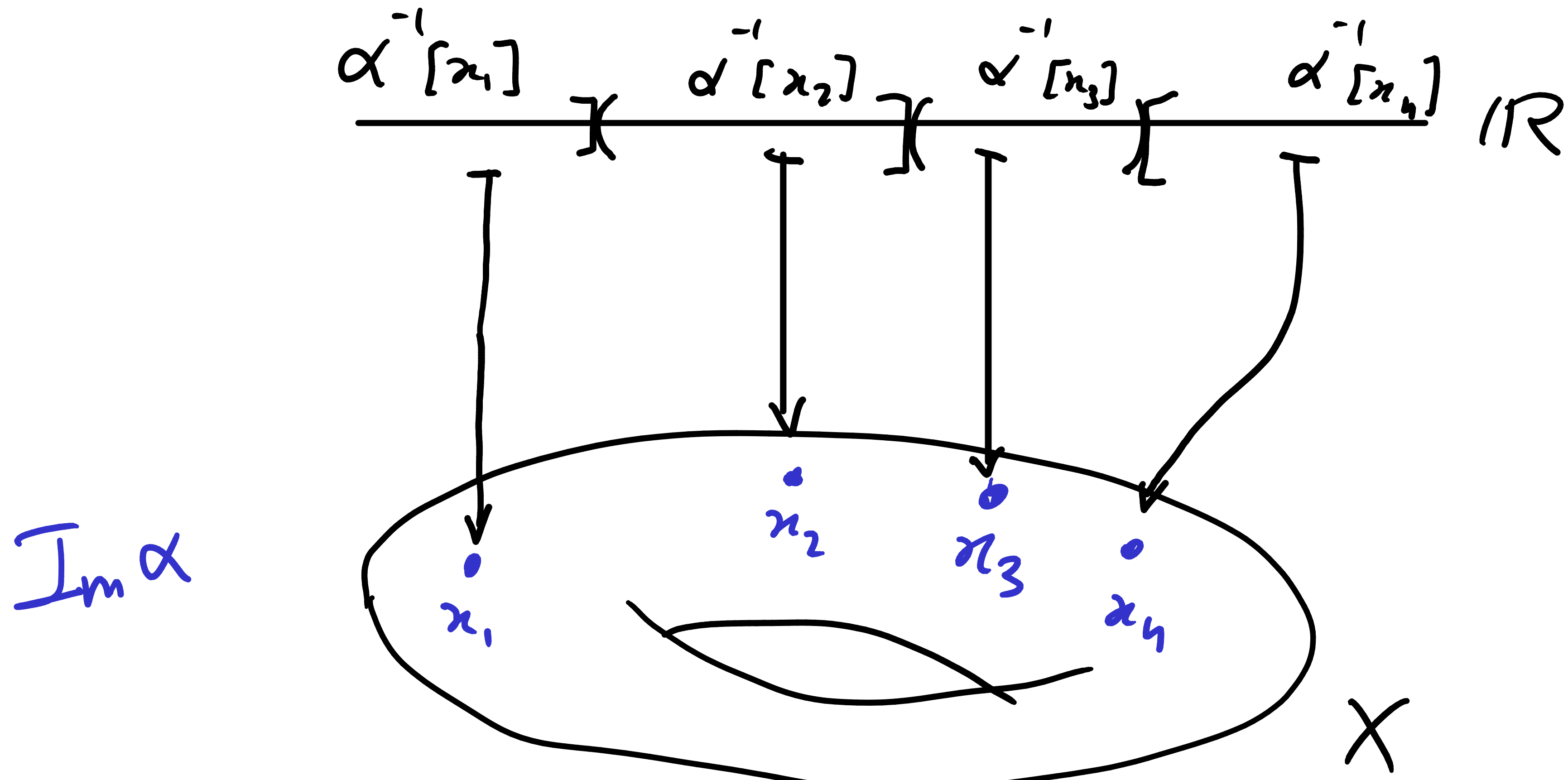
Indeed:

$r \in LHS \iff \beta r \in F \iff \exists i \in I.\, r \in E_i \wedge \alpha_i r \in F \iff r \in RHS$ ✓

σ-simple function

$\alpha : \mathbb{R} \longrightarrow X$ s.t. $\mathrm{Im}\,\alpha := \alpha[\mathbb{R}]$ is ctbl $\wedge$

$\forall x \in \mathrm{Im}\,\alpha .\ \alpha^{-1}[x] \in \mathcal{B}_{\mathbb{R}}$



Validate qbs axioms for: $\ulcorner X^{\text{qbs}} \urcorner := (X, \sigma\text{-Simple}(X))$

NB: f σ-Simple: $\operatorname{Im} f$ ctbl $\wedge$ $f^{-1}[x] \in B_{\mathbb{R}}$

- Constants

$\operatorname{Im}(\lambda r.x) = \{x\}$ ctbl ✓

$$y : X \vdash (\lambda r.x)^{-1}[y] = \begin{cases} x = y : & \mathbb{R} \\ x \neq y : & \emptyset \end{cases} \in B_{\mathbb{R}} \checkmark$$

Validate qbs axioms for: $\ulcorner X^{\mathrm{Qbs}} \urcorner := (X, \sigma\text{-Simple}(X))$

NB: f σ-Simple: $\mathrm{Im} f$ ctbl $\wedge$ $f^{-1}[x] \in B_{\mathbb{R}}$

- Precomposition:

$\alpha : \sigma\text{-simple}(X),\ \varphi : \mathrm{Meas}(\mathbb{R}, \mathbb{R}) \vdash$

$\mathrm{Im}(\alpha \circ \varphi) \subseteq \mathrm{Im}\,\alpha$ ctbl ✓

$x : X \vdash$

$(\alpha \circ \varphi)^{-1}[x] = \varphi^{-1}[\alpha^{-1}(x)] \in B_{\mathbb{R}}$ ✓

$\alpha^{-1}(x) \in B_{\mathbb{R}}$

$\varphi : \mathbb{R} \to \mathbb{R}$ measurable

Validate qbs axioms for: $\ulcorner X^{\mathrm{Qbs}} \urcorner := (X, \sigma\text{-Simple}(X))$

NB: f σ-Simple: $\mathrm{Im}\, f$ ctbl $\wedge$ $f^{-1}[x] \in B_{\mathbb{R}}$

- recombination:

$\alpha_{-} : (\sigma\text{-Simple}(X))^{I}, \; E_{-} : B_{\mathbb{R}}^{I}, \; \mathbb{R} = \biguplus_{i \in I} E_i \vdash$

$$\mathrm{Im}\, [E_i . \alpha_i]_{i \in I} \subseteq \bigcup_{i \in I} \mathrm{Im}\, \alpha_i \quad \text{ctbl} \quad \checkmark$$

$x : X \vdash$

$$[E_i . \alpha_i]_{i \in I}^{-1}(x) = \bigcup_{i \in I} \alpha_i^{-1}[x] \cap E_i \in B_{\mathbb{R}} \quad \checkmark$$

Prop: $X : \mathrm{Set}, A : \mathrm{Qbs} \vdash$

- $\forall f : X \to \lfloor A \rfloor .\ f : \ulcorner X \urcorner^{\mathrm{Qbs}} \longrightarrow A$

- $\forall f : \lfloor A \rfloor \longrightarrow X .\ f : A \longrightarrow \llcorner X \lrcorner_{\mathrm{Qbs}}$

Prop: $X : \mathrm{Set},\ A : \mathrm{Qbs} \vdash$

- $\forall f : X \to \llcorner A \lrcorner .\ f : \ulcorner X \urcorner^{\mathrm{Qbs}} \longrightarrow A$

Pf: $\alpha : R_{\ulcorner X \urcorner^{\mathrm{Qbs}}} \vdash$ α σ-simple $\implies$

$$\alpha = [\alpha^{-1}[x].\lambda r.x]_{x \in \mathrm{Im}\,\alpha} \implies$$

$$(f \circ \alpha) = [\alpha^{-1}[x].\lambda r.fx]_{x \in \mathrm{Im}\,\alpha} \in R_A \quad \checkmark$$

recombination

Borel

constat $\in B_A$

ctbl

Prop: $X : \mathrm{Set},\ A : \mathrm{Qbs} \vdash$

- $\forall f : X \to \lfloor A \rfloor .\ f : \ulcorner X \urcorner^{\mathrm{Qbs}} \longrightarrow A$
- $\forall f : \lfloor A \rfloor \longrightarrow X .\ f : A \longrightarrow \lfloor X \rfloor_{\mathrm{Qbs}}$

Prf: $\alpha : R_A \vdash (f \circ \alpha : \mathbb{R} \to X) \in R_{\lfloor X \rfloor_{\mathrm{Qbs}}}$ always. ✓

$\blacksquare$

Useful adjunctions:

$$
\begin{tikzcd}
\mathbf{Meas} \arrow[d, "\lfloor - \rfloor_{\mathbf{Qbs}}"] \\
\mathbf{Qbs} \arrow[u, bend left=60, "\lceil - \rceil^{\mathbf{Meas}}"] \arrow[u, phantom, "\dashv"] \arrow[d, "\lfloor - \rfloor"] \\
\mathbf{Set} \arrow[u, bend left=60, "\lceil - \rceil^{\mathbf{Qbs}}"] \arrow[u, phantom, "\dashv\quad\dashv"] \arrow[u, bend right=60, "\lfloor - \rfloor_{\overline{\mathbf{Qbs}}}"']
\end{tikzcd}
$$

$$\lfloor V \rfloor_{\mathbf{Qbs}} := (\lfloor V \rfloor, \mathbf{Meas}(\mathbb{R}, V)) \quad (V \in \mathbf{Meas})$$

$$\lceil X \rceil^{\mathbf{Meas}} := \left\{ A \subseteq \lfloor X \rfloor \;\middle|\; \forall \alpha \in R_X .\ \alpha^{-1}[A] \in \mathcal{B}_{\mathbb{R}} \right\}$$

- limits (products, subspaces) and colimits (coproducts, quotients) as in $\underline{\mathbf{Set}}$
- Slogan: every measurable space is carried by a qbs

Example

Product $(X \times Y, \pi_1, \pi_2)$:

necessarily!

- $\lfloor X \times Y \rfloor = \lfloor X \rfloor \times \lfloor Y \rfloor$
- $R_{X \times Y} = \left\{ \lambda r. (\alpha r, \beta r) \;\middle|\; \alpha \in R_X, \beta \in R_Y \right\}$

correlated random elements

rest of structure as in <u>Set</u>.

Function Spaces

Straightforward!

- $\lfloor Y^X \rfloor := \mathrm{Qbs}(X, Y)$

- $R_{Y^X} := \mathrm{uncurry}[\mathrm{Qbs}(\mathbb{R} \times X, Y)]$

$$= \left\{ \alpha : \mathbb{R} \to \lfloor Y^X \rfloor \;\middle|\; \lambda(r, x).\, \alpha\, r\, x : \mathbb{R} \times X \to Y \right\}$$

- $\mathrm{eval} : Y^X \times X \to Y$

$\mathrm{eval}(f, x) := f\, x$

Meas vs Qbs

By generalities:

σ-algebra on $\mathbf{Meas}(\mathbb{R},\mathbb{R})$

$$\begin{tikzcd}
\ulcorner \mathbb{R}^{\mathbb{R}} \urcorner^{\mathbf{Meas}} \times \mathbb{R} \arrow[r] \arrow[rr, bend right, "\ulcorner eval \urcorner^{\mathbf{Meas}}"'] & \ulcorner \mathbb{R}^{\mathbb{R}}_{\mathbf{Meas}} \urcorner \times \mathbb{R} \arrow[r, dashed, "\times"] & \ulcorner \mathbb{R} \urcorner = \mathbb{R}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{Meas} \arrow[d, "\lfloor - \rfloor_{\mathbf{Qbs}}"] \arrow[d, phantom, shift right=4, "\dashv"] \\
\mathbf{Qbs} \arrow[u, bend left, "\ulcorner - \urcorner^{\mathbf{Meas}}"]
\end{tikzcd}$$

No factorisation by Aumann's Theorem.

$$\left(\underline{\text{So}}: \ulcorner \mathbb{R}^{\mathbb{R}} \times \mathbb{R} \urcorner \not\cong \ulcorner \mathbb{R}^{\mathbb{R}} \urcorner \times \ulcorner \mathbb{R} \urcorner \right)$$

Simple Type Structure

"Simple" because:

- Simply-typed λ-calculus
- types are simple: $A, B : \mathrm{Type} \vdash B^A : \mathrm{Type}$
  - no polymorphism
  - no term dependency
- Contexts for terms: $\Gamma \vdash t : A$

  are simple: $\Gamma = x_1 : A_1, \ldots, x_n : A_n$

  i.e. $\mathrm{List}(\mathrm{Type})$

Simple Type Structure

"Simple" because:

- interpretation is simple:

$$[\![x_1 : A_1, \ldots, x_n : A_n]\!] := \prod_{i=1}^{n} A_i$$

$$[\![\Gamma \vdash t : A]\!] : [\![\Gamma]\!] \longrightarrow A$$

in Qbs

Simple Type Structure

Curry-Howard-Lambek

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash s : B}{\Gamma \vdash \langle t, s \rangle : A \times B} \rightsquigarrow [\![\Gamma]\!] \xrightarrow{\lambda r.\langle tr, sr \rangle} A \times B$$

is measurable

$$\boxed{\text{measurability by type!}}$$

$$\frac{\Gamma \vdash t : A \times B \qquad \Gamma, x : A, y : B \vdash s : C}{\Gamma \vdash \mathrm{let}\ (x, y) = t\ \mathrm{in}\ s : C} \rightsquigarrow$$

$$[\![\Gamma]\!] \xrightarrow{\lambda r.\ \mathrm{let}\ (a, b) = tr\ \mathrm{in}\ sr[x \mapsto a, y \mapsto b]} C$$

is measurable.

etc.

Random element Space

$$R_X := X^{\mathbb{R}} \quad \text{since} \quad \lfloor X^{\mathbb{R}} \rfloor = R_X \text{ as sets.}$$

Why?

$(\subseteq)$ $\alpha \in \lfloor X^{\mathbb{R}} \rfloor \implies \alpha : \mathbb{R} \to X$ in Qbs.

$id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ measurable $\implies id \in R_{\mathbb{R}}$

$\implies \alpha = \alpha \circ id \in R_X$

Precomposition

$(\supseteq)$ $\alpha \in R_X \implies \forall \varphi \in R_{\mathbb{R}} = \mathrm{Meas}(\mathbb{R}, \mathbb{R}).\ \alpha \circ \varphi \in R_X \implies \alpha : \mathbb{R} \to X$

$\implies \alpha \in \lfloor X^{\mathbb{R}} \rfloor$

Subspaces

For $X \in \mathbf{Qbs}$, $A \subseteq \lfloor X \rfloor$ set:

$$R_A := \{ \alpha : \mathbb{R} \to A \mid \alpha \in R_X \}$$

Then $A = (A, R_A)$ is the subspace qbs

We write $A \hookrightarrow X$

Borel subspaces ensemble

The σ-algebra $\mathcal{B}_X := \{ A \subseteq \lfloor X \rfloor \mid \forall \alpha \in R_X . \alpha^{-1}[A] \in \mathcal{B}_{\mathbb{R}} \}$

internalises as $\mathcal{B}_X = 2^X$, the qbs of

Borel subsets.

($\lfloor \mathcal{B}_{(\mathcal{B}_{\mathbb{R}})} \rfloor$ are the Borel-on-Borel sets from descriptive set theory.

Cf. [Sabok et al.'21])

Standard Borel spaces

Def: A qbs S is standard Borel when

$$S \cong A \quad \text{for some } A \in \mathcal{B}_{\mathbb{R}}$$

$$\begin{tikzcd}
\mathbf{Meas} \arrow[rr, bend left] \arrow[rr, phantom, "\top"] & & \mathbf{Qbs} \arrow[ll, bend left] \\
\mathbf{Sbs} \arrow[u, hook] \arrow[rr, bend left] \arrow[rr, phantom, "\cong"] & & \mathbf{Sbs} \arrow[u, hook] \arrow[ll, bend left]
\end{tikzcd}$$

Slogan: Qbs conservative extension of Sbs

Example $C_0 := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\} \hookrightarrow \mathbb{R}^{\mathbb{R}}$

C_0 is sbs. (Well-known!)

Proof:

$C_0' \in B_{\mathbb{R}^{\mathbb{Q}}}$ sbs!

$$C_0' := \left\{ g \in \mathbb{R}^{\mathbb{Q}} \;\middle|\; \begin{array}{l} \forall a, b \in \mathbb{Q}, \varepsilon \in \mathbb{Q}^+ \\ \exists \delta \in \mathbb{Q}^+ \, \forall p, q \in \mathbb{Q}^+ \cap [a,b], \\ |p - q| < \delta \Rightarrow |g\,p - g\,q| < \varepsilon \end{array} \right\}$$

on closed intervals (= compact intervals)

Continuity $\Updownarrow$ uniform continuity

Borel measurable } by type checks

then $C_0 \cong C_0' \in B_{\mathbb{R}^{\mathbb{Q}}}$:

$$C_0 \to C_0' \qquad\qquad C_0' \to C_0$$

$$\varphi \mapsto \varphi|_{\mathbb{Q}} \qquad\qquad \varphi \mapsto \lambda r. \lim_{n \to \infty} g(\text{approx}_{(\frac{1}{m})_{m \in \mathbb{N}}} r)_n$$

Example (ctd)

C_0 is sbs, and $\mathrm{eval} : C_0 \times \mathbb{R} \to \mathbb{R}$

is measurable.

Avoids:

- constructing complete separable metrics
- proving that evaluation is measurable w.r.t. metric σ-algebra.

Non-examples ~ [Sabok et al. '21]

- $\{A \in \mathcal{B}_{\mathbb{R}} \mid A \neq \emptyset\} \hookrightarrow \mathcal{B}_{\mathbb{R}}$

- $\{(A_1, A_2) \in \mathcal{B}_{\mathbb{R}}^2 \mid A \subseteq B\} \hookrightarrow \mathcal{B}_{\mathbb{R}}^2$

- $\{A \in \mathcal{B}_{\mathbb{R}} \mid A \text{ open}\} \hookrightarrow \mathcal{B}_{\mathbb{R}}$

Plan:

1) type-driven probability: discrete case ✓
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Lecture 1

4) Type structure & standard ✓ Borel spaces
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Lecture 2

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Dependent Type Structure

types can contain terms:

a type referring to a term

$$X : \mathrm{Type},\ E : B_X \vdash \{x \in X \mid x \in E\} : \mathrm{Type}$$

a type, just like STLC

a term!

Dependent Type Structure

types can contain terms:

a type referring to a term

$$X : \mathrm{Type},\ E : B_X \vdash \{x \in X \mid x \in E\} : \mathrm{Type}$$

a type, just like STLC

a term!

Context formation:

$$\frac{\Gamma \vdash A : \mathrm{Type}}{\Gamma, x : A \vdash}$$

Dependent Type Structure

types denote spaces-in-Context

$$
\begin{tikzcd}
{[\![\Gamma \vdash A]\!]} \arrow[d, "dep"] \\
{[\![\Gamma]\!]}
\end{tikzcd}
$$

Dependent types denote spaces-in-context

$\Gamma \vdash$ ← context

$\Gamma \vdash A$

type in context

$$[\![\Gamma \vdash A]\!] \xrightarrow{\text{dep}} [\![\Gamma]\!]$$

$[\![\Gamma \vdash A]\!]$ ← space in context

$[\![\Gamma]\!]$ ← context space

assigns environment

E.g.:

$$A \longrightarrow \mathbb{1}$$

simple types

$$[\![E : \mathcal{B}_A \vdash \{x \in A \mid x \in E\}]\!] = \{(E, a) \in \mathcal{B}_A \times A \mid a \in E\} \xrightarrow{\pi_1} \mathcal{B}_A$$

decoder

Context extension

$$\frac{\Gamma \vdash A}{\Gamma, a : A \vdash}$$

$$[\![\Gamma \vdash A]\!] \xrightarrow{\mathrm{dep}} [\![\Gamma]\!]$$

$$[\![\Gamma, a : A]\!] := [\![\Gamma \vdash A]\!]$$

Substitution

$\Gamma \vdash \sigma : \Delta$

$$[\![\sigma]\!] : [\![\Gamma]\!] \longrightarrow [\![\Delta]\!]$$

E.g. weakening

$$\Gamma, a : A \vdash \mathrm{wkn} : \Gamma$$

$$[\![\Gamma, a : A]\!] := [\![\Gamma \vdash A]\!] \xrightarrow[\mathrm{dep}]{\mathrm{wkn}} [\![\Gamma]\!]$$

Action of substitution on types

$$
\begin{tikzcd}
\llbracket \Gamma \vdash A[\sigma] \rrbracket \\[-2em]
\rotatebox{90}{$=$} \\[-2em]
\{(\gamma, a) \in \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \mid \mathrm{dep}\, a = \gamma\} \arrow[r, "\pi_2"] \arrow[d, "\mathrm{dep} := \pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \Delta \vdash A \rrbracket \arrow[d, "\mathrm{dep}"] \\
\llbracket \Gamma \rrbracket \arrow[r, "\sigma"] & \Delta
\end{tikzcd}
$$

E.g.

$$
\begin{tikzcd}
\llbracket x : A \vdash B[\mathrm{wkn}] \rrbracket := A \times B \arrow[r, "\pi_2"] \arrow[d, "\mathrm{dep} := \pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
x : A \arrow[r, "\langle\rangle"'] & \mathbb{1}
\end{tikzcd}
$$

simple type

Terms: sections

$$\begin{tikzcd}
{[\![\Gamma]\!]} \arrow[rr, "{[\![\Gamma \vdash M : A]\!]}"] \arrow[drr, equal, "="'] & & {[\![\Gamma \vdash A]\!]} \arrow[d, "\mathrm{dep}"] \\
& & {[\![\Gamma]\!]}
\end{tikzcd}$$

e.g.

$$\begin{tikzcd}
\mathbb{R} \arrow[rr, "{[\![x : \mathbb{R} \vdash [x, \infty) : B_{\mathbb{R}}[\mathrm{wkn}]]\!]}"] \arrow[drr, equal, "="'] & & \mathbb{R} \times B_{\mathbb{R}} \arrow[d, "\pi_1"] \\
& & \mathbb{R}
\end{tikzcd}$$

E.g. Variables:

$$\begin{tikzcd}
{[\![\Gamma, a : A]\!]} \arrow[rr, "{[\![\Gamma, a : A \vdash a : A]\!]}", "{\langle \mathrm{id}, \mathrm{dep}_{\Gamma \vdash A} \rangle}"'] \arrow[drr, equal, "="'] & & {[\![\Gamma, a : A \vdash A[\mathrm{wkn}]]\!]} \arrow[d, "\mathrm{dep}"] \\
& & {[\![\Gamma, a : A]\!]}
\end{tikzcd}$$

Exercise:
action of substitution
$M[\sigma]$

Dependent Pairs

$$\frac{\Gamma, a : A \vdash B}{\Gamma \vdash \coprod_{a \in A} B}$$

$$\begin{tikzcd}
\llbracket \coprod_{a \in A} A \rrbracket \arrow[r, phantom, ":="] \arrow[dddr, bend right, "\mathrm{dep}_{\coprod}"'] & \llbracket \Gamma, a : A \vdash B \rrbracket \arrow[d, "\mathrm{dep}_B"] \\
\arrow[r, phantom, ":="] & \llbracket \Gamma, a : A \rrbracket \arrow[d, phantom, "\shortparallel"] \\
& \llbracket \Gamma \vdash A \rrbracket \arrow[d] \\
& \llbracket \Gamma \rrbracket
\end{tikzcd}$$

Dependent Products

aka: $(a : A) \to B$

$$\frac{\Gamma, a : A \vdash B}{\Gamma \vdash \prod_{a \in A} B}$$

$$\left\lfloor \left[\!\left[\Gamma \vdash \prod_{a \in A} B \right]\!\right] \right\rfloor :=$$

$$\left\{ \left(r_0, f : \{ a \in [\![A]\!] \mid \mathrm{dep}\, a = r_0 \} \to [\![\Gamma, a : A \vdash B]\!]\right) \;\middle|\; \forall a \in [\![\Gamma, a : A]\!] .\ \mathrm{dep}\, a = r_0 \Rightarrow \mathrm{dep}(f\, a) = a \right\}$$

Exercise: find the random elements.

Full model

type : Qbs $\quad W := [0, \infty] \quad \mathcal{B}X \cong \mathbb{B}^X$

$DX := (\text{Fri})$

$PX := \{\mu \in DX \mid \underset{\mu}{\text{Ce}}[X] = 1\}$

$\underset{\mu}{\text{Ce}}[E] := (\text{Fri}) \qquad \delta_x := (\text{Fri})$

$\oint \mu k := (\text{Fri})$

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