

9 Function spaces

Let A, B be qbses, their *function space* B^A is given by the following:

- The set of points $\ulcorner B^A \urcorner$ is the set $\mathbf{Qbs}(A, B)$ of qbs morphisms $f : A \rightarrow B$.
- The set of random elements \mathcal{R}_{B^A} is the set $\text{curry}[\mathbf{Qbs}(\mathbb{R} \times A, B)]$, consisting of functions $\alpha : \mathbb{R} \rightarrow \mathbf{Qbs}(A, B)$ that, when uncurried, are qbs morphisms $\text{uncurry}(\alpha) : \mathbb{R} \times A \rightarrow B$.

Exercises Ex.9.1–Ex.9.3 unpack these definitions and show that they realise the familiar interface to functions — evaluation/application and abstraction — as well as the qbs axioms. You can skip them and come back after you’ve used the function space in the later exercises.

▮9.1. Let A, B be qbses and $\gamma : \mathbb{R} \times \ulcorner A \urcorner \rightarrow \ulcorner B \urcorner$ be any set theoretic function. Show:

- If $\gamma : \mathbb{R} \times A \rightarrow B$ is a qbs morphism, then its curried form $\text{curry } \gamma := \lambda r. \lambda a. \gamma(r, a)$ is pointwise a qbs morphism: $\text{curry } \gamma : \mathbb{R} \rightarrow \mathbf{Qbs}(A, B)$.
- $\text{curry } \gamma \in \mathcal{R}_{B^A}$ iff for every measurable $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ and random element $\alpha \in \mathcal{R}_A$, the following function is a random element:

$$(\lambda r. \gamma(\varphi r, \alpha r)) \in \mathcal{R}_B \quad \triangle$$

▮9.2. Let A, B be qbses.

- Validate the constant and precomposition qbs axioms for the function space.
- Let $\tilde{\gamma} \in \mathcal{R}_{B^A}$ be a sequence of random elements in the function space, $\mathbb{R} = \uplus_n U_n$ a countable partition of the reals into Borel sets, $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ a measurable function, and $\alpha \in \mathcal{R}_A$ a random element. Let $V_n := \varphi^{-1}[U_n]$.
 - Evaluate the recombination $\langle \lambda r. (\text{uncurry } \gamma_n)(\varphi r, \alpha r) \rangle_n$ along \vec{V} at any $s \in V_m$.
 - Let γ be the recombination of $\tilde{\gamma}$ along \vec{U} . Evaluate $(\text{uncurry } \gamma)(\varphi s, \alpha s)$ at any $s \in V_m$.
 - Validate the recombination axiom for the function space.
- Show that evaluation $\text{eval} : B^A \times A \rightarrow B$ is a qbs morphism. ▮

▮9.3. Show that $\langle B^A, \text{eval} \rangle$ is the exponential of B by A (cf. Sec. 4). ▮

We equip the Borel subsets \mathcal{B}_A of a qbs A with the structure of a qbs. Identifying a Borel subset $U \subseteq \ulcorner A \urcorner$ with its indicator function $[- \in U] : A \rightarrow 2$ (cf. Ex.8.1), we *define*

$$\mathcal{B}_A := 2^A$$

▮9.4. Show that the following functions are qbs morphisms:

- Membership testing: $(\epsilon) : A \times \mathcal{B}_A \rightarrow 2$
- Complementation: $\neg : \mathcal{B}_A \rightarrow \mathcal{B}_A$
- Countable unions and intersection: $\bigcup_I, \bigcap_I : \mathcal{B}_A^I \rightarrow \mathcal{B}_A$, for I countable set. ▮

▮9.5. Let A be a qbs with a countable carrier. Show that the following functions are qbs morphisms:

- Equality testing, subset containment: $(=), (\subseteq), (\subset) : \mathcal{B}_A^2 \rightarrow \mathcal{B}_A$.
- Inhabitation: $(\neq \emptyset) : \mathcal{B}_A \rightarrow 2$. ▮

✓9.6. Using the techniques in Sec. A, there is a Borel set $U \in \mathcal{B}_{\mathbb{R}}$ and a measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that the image $f[\mathbb{R}]$ is not Borel (cf. Ex.A.13).

Use this fact and show that the following functions are *not* qbs morphisms:

- Inhabitation: $(\neq \emptyset) : \mathcal{B}_{\mathbb{R}} \rightarrow 2$.
- Equality and containment $(=), (\subseteq), (\subset) : \mathcal{B}_{\mathbb{R}}^2 \rightarrow \mathcal{B}_{\mathbb{R}}$.
- Disjointness: $(- \cap - = \emptyset) : \mathcal{B}_{\mathbb{R}}^2 \rightarrow 2$. △

✓9.7. If $A = \ulcorner I \urcorner^{\mathbf{Qbs}}$ is a *finite* discrete qbs, then \mathcal{B}_A is a finite discrete qbs. △

✓9.8. Let $U \subseteq \mathcal{B}_{\mathbb{R} \times \mathbb{R}}$ be a Borel set. Its *section* at r is the set $U_r := \{s \in \mathbb{R} \mid \langle r, s \rangle \in U\}$. A set $\mathcal{U} \subseteq \mathcal{B}_{\mathbb{R}}$ is *Borel on Borel* when, for every Borel set $U \subseteq \mathbb{R} \times \mathbb{R}$, the set of sections of U that are in \mathcal{U} is Borel: $\{r \in \mathbb{R} \mid U_r \in \mathcal{U}\} \in \mathcal{B}_{\mathbb{R}}$.

Show that \mathcal{U} is Borel on Borel iff $\mathcal{U} \in \mathcal{B}_{\mathcal{B}_{\mathbb{R}}}$.

The observation that this descriptive-set-theoretic notion coincides with the Borel sets on a higher-order space is due to Sabok et al. (2021). △

✓9.9. Let A be a qbs. Show that a function $\alpha : \mathbb{R} \rightarrow \ulcorner A \urcorner$ is a random element in \mathcal{R}_A iff it is a qbs morphism $\alpha : \mathbb{R} \rightarrow A$. △

The last exercise provides a qbs of random elements, by setting: $\mathcal{R}_A := A^{\mathbb{R}}$.

✓9.10. Define functors $\mathcal{B}_- : \mathbf{Qbs}^{\text{op}} \rightarrow \mathbf{Qbs}$ and $\mathcal{R}_- : \mathbf{Qbs} \rightarrow \mathbf{Qbs}$, and construct a natural isomorphism $\mathcal{R}_{\mathcal{B}_A} \cong \mathcal{B}_{\mathbb{R} \times A}$. △

✓9.11. Show that \mathcal{R}_- preserves strong epimorphisms (cf. Ex.8.14): if $e : A \twoheadrightarrow B$ is a strong epimorphism, then $\mathcal{R}_e : \mathcal{R}_A \rightarrow \mathcal{R}_B$ is also a strong epimorphism. △

References

Marcin Sabok, Sam Staton, Dario Stein, and Michael Wolman. Probabilistic programming semantics for name generation. *Proc. ACM Program. Lang.*, 5(POPL):1–29, 2021. 10.1145/3434292. URL <https://doi.org/10.1145/3434292>.