9 Function spaces

Let A, B be qbses, their *function* space B^A is given by the following:

- The set of points $\llcorner B^A \lrcorner$ is the set $\mathbf{Qbs}(A, B)$ of qbs morphisms $f : A \to B$.
- The set of random elements $\mathcal{R}_{B^A}$ is the set $\operatorname{curry}[\mathbf{Qbs}(\mathbb{R} \times A, B)]$, consisting of functions $\alpha : \mathbb{R} \to \mathbf{Qbs}(A, B)$ that, when uncurried, are qbs morphisms $\operatorname{uncurry}(\alpha) : \mathbb{R} \times A \to B$.

Exercises Ex.9.1–Ex.9.3 unpack these definitions and show that they realise the familiar interface to functions — evaluation/application and abstraction — as well as the qbs axioms. You can skip them and come back after you've used the function space in the later exercises.

◸**9.1.** Let A, B be qbses and $\gamma : \mathbb{R} \times \llcorner A \lrcorner \to \llcorner B \lrcorner$ be any set theoretic function. Show:

- If $\gamma : \mathbb{R} \times A \to B$ is a qbs morphism, then its curried form $\operatorname{curry} \gamma := \lambda r.\lambda a.\gamma(r, a)$ is pointwise a qbs morphism: $\operatorname{curry} \gamma : \mathbb{R} \to \mathbf{Qbs}(A, B)$.
- $\operatorname{curry} \gamma \in \mathcal{R}_{B^A}$ iff for every measurable $\varphi : \mathbb{R} \to \mathbb{R}$ and random element $\alpha \in \mathcal{R}_A$, the following function is a random element:

$$(\lambda r.\gamma(\varphi\, r, \alpha\, r)) \in \mathcal{R}_B$$ ◿

◸**9.2.** Let A, B be qbses.

- Validate the constant and precomposition qbs axioms for the function space.
- Let $\vec{\gamma} \in \mathcal{R}_{B^A}$ be a sequence of random elements in the function space, $\mathbb{R} = \biguplus_n U_n$ a countable partition of the reals into Borel sets, $\varphi : \mathbb{R} \to \mathbb{R}$ a measurable function, and $\alpha \in \mathcal{R}_A$ a random element. Let $V_n := \varphi^{-1}[U_n]$.
  - Evaluate the recombination $\langle \lambda r.(\operatorname{uncurry} \gamma_n)(\varphi\, r, \alpha\, r)\rangle_n$ along $\vec{V}$ at any $s \in V_m$.
  - Let γ be the recombination of $\vec{\gamma}$ along $\vec{U}$. Evaluate $(\operatorname{uncurry} \gamma)(\varphi\, s, \alpha\, s)$ at any $s \in V_m$.
  - Validate the recombination axiom for the function space.
- Show that evaluation $\operatorname{eval} : B^A \times A \to A$ is a qbs morphism. ◿

◸**9.3.** Show that $\langle B^A, \operatorname{eval}\rangle$ is the exponential of B by A (cf. Sec. 4). ◿

We equip the Borel subsets $\mathcal{B}_A$ of a qbs A with the structure of a qbs. Identifying a Borel subset $U \subseteq \llcorner A \lrcorner$ with its indicator function $[- \in U] : A \to \mathbb{2}$ (cf. Ex.8.1), we *define*

$$\mathcal{B}_A := \mathbb{2}^A$$

◸**9.4.** Show that the following functions are qbs morphisms:

- Membership testing: $(\in) : A \times \mathcal{B}_A \to \mathbb{2}$
- Complementation: $\neg : \mathcal{B}_A \to \mathcal{B}_A$
- Countable unions and intersection: $\bigcup_I, \bigcap_I : \mathcal{B}_A^I \to \mathcal{B}_A$, for I countable set. ◿

◸**9.5.** Let A be a qbs with a countable carrier. Show that the following functions are qbs morphisms:

- Equality testing, subset containment: $(=), (\subseteq), (\subset) : \mathcal{B}_A^2 \to \mathcal{B}_A$.
- Inhabitation: $(\neq \varnothing) : \mathcal{B}_A \to \mathbb{2}$. ◿

◸**9.6.** Using the techniques in Sec. A, there is a Borel set $U \in \mathcal{B}_{\mathbb{R}}$ and a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that the image $f[\mathbb{R}]$ is not Borel (cf. Ex.A.13).

Use this fact and show that the following functions are *not* qbs morphisms:

- Inhabitation: $(\neq \varnothing) : \mathcal{B}_{\mathbb{R}} \to \mathbb{2}$.
- Equality and containment $(=), (\subseteq), (\subset) : \mathcal{B}_{\mathbb{R}}^2 \to \mathcal{B}_{\mathbb{R}}$.
- Disjointness: $(- \cap - = \varnothing) : \mathcal{B}_{\mathbb{R}}^2 \to \mathbb{2}$. ◿

◸**9.7.** If $A = \ulcorner \overset{\mathbf{Qbs}}{I} \urcorner$ is a *finite* discrete qbs, then $\mathcal{B}_A$ is a finite discrete qbs. ◿

◸**9.8.** Let $U \subseteq \mathcal{B}_{\mathbb{R}\times\mathbb{R}}$ be a Borel set. Its *section* at r is the set $U_r \coloneqq \{s \in \mathbb{R} | \langle r, s\rangle \in U\}$. A set $\mathcal{U} \subseteq \mathcal{B}_{\mathbb{R}}$ is *Borel on Borel* when, for every Borel set $U \subseteq \mathbb{R} \times \mathbb{R}$, the set of sections of U that are in $\mathcal{U}$ is Borel: $\{r \in \mathbb{R} | U_r \in \mathcal{U}\} \in \mathcal{B}_{\mathbb{R}}$.

Show that $\mathcal{U}$ is Borel on Borel iff $\mathcal{U} \in \mathcal{B}_{\mathcal{B}_{\mathbb{R}}}$.

The observation that this descriptive-set-theoretic notion coincides with the Borel sets on a higher-order space is due to Sabok et al. (2021). ◿

◸**9.9.** Let A be a qbs. Show that a function $\alpha : \mathbb{R} \to \llcorner A \lrcorner$ is a random element in $\mathcal{R}_A$ iff it is a qbs morphism $\alpha : \mathbb{R} \to A$. ◿

The last exercise provides a qbs of random elements, by setting: $\mathcal{R}_A \coloneqq A^{\mathbb{R}}$.

◸**9.10.** Define functors $\mathcal{B}_{-} : \mathbf{Qbs}^{\mathrm{op}} \to \mathbf{Qbs}$ and $\mathcal{R}_{-} : \mathbf{Qbs} \to \mathbf{Qbs}$, and construct a natural isomorphism $\mathcal{R}_{\mathcal{B}_A} \cong \mathcal{B}_{\mathbb{R}\times A}$. ◿

◸**9.11.** Show that $\mathcal{R}_{-}$ preserves strong epimorphisms (cf. Ex.8.14): if $e : A \twoheadrightarrow B$ is a strong epimorphism, then $\mathcal{R}_e : \mathcal{R}_A \to \mathcal{R}_B$ is also a strong epimorphism. ◿

References

Marcin Sabok, Sam Staton, Dario Stein, and Michael Wolman. Probabilistic programming semantics for name generation. *Proc. ACM Program. Lang.*, 5(POPL):1–29, 2021. 10.1145/3434292. URL `https://doi.org/10.1145/3434292`.