Foundations for type-driven probabilistic modelling

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Recap

Language of distribution & Probability

X type (= space) of values / outcomes

DX type of distributions / measures over X

$PX \subseteq DX$ Sub type of Probability measures (total measure 1)

BX type of measurable events - subsets of X we wish to measure

W type of weights : $[0, \infty]$

type judgement

$$\mu : DX, E : BX \vdash Ce_{\mu}[E] : W$$

↳ measure μ assigns to E

Axioms for measures/distributions

Recap

$$\mu : DX \vdash \underset{\mu}{\mathrm{Ce}}[\emptyset] = 0 \quad : W$$

$$E, C : BX, \mu : DX \vdash$$

$$\underset{\mu}{\mathrm{Ce}}[E] = \underset{\mu}{\mathrm{Ce}}[E \cap C] + \underset{\mu}{\mathrm{Ce}}[E \cap C^c] \quad : W$$

$$E_{-} : (BX, \subseteq)^{\omega}, \mu : DX \vdash$$

$$\underset{\mu}{\mathrm{Ce}}\left[\bigcup_n E_n\right] = \sup_n \underset{\mu}{\mathrm{Ce}}[E_n] \quad : W$$

Kernels & their Kock integral

Recap

Kernel from Γ to X: $k : (DX)^{\Gamma}$ or $k : \Gamma \longrightarrow DX$

Dirac kernel: $\delta\_ : X \longrightarrow DX$

Kock integral: $\mu : D\Gamma,\ k : (DX)^{\Gamma} \vdash \oint \mu k \quad : DX$

or $\oint \mu(dx)\, k(x)$ (dx binding occurrence in k(x))

Giry monads: $(D, \delta\_, \oint)$ & $(P, \delta\_, \oint)$.

Discrete model

type : Set $\qquad W := [0, \infty] \qquad BX := \mathcal{P}X$

$$DX := \{ \mu : X \to W \mid \operatorname{supp} \mu \text{ countable} \}$$

$$PX := \{ \mu \in DX \mid \underset{\mu}{Ce}[X] = 1 \}$$

$$\underset{\mu}{Ce}[E] := \sum_{x \in E} \mu x \qquad \delta_x := \lambda x'. \begin{cases} x = x' : & 0 \\ x \neq x' : & 1 \end{cases}$$

$$\oint \mu k := \lambda x. \sum_{r \in \Gamma} \mu r \cdot k(r; x)$$

Ex distributions

Counting measure (X ctbl): $\#_X := \lambda x. 1$

Dirac measure δ_x (prev slide)

Zero measure $\underline{0} := \lambda x. 0$

Plan:

1) type-driven probability: discrete case (Mon + Tue)

2) Borel sets & measurable spaces (Tue)

3) Quasi Borel spaces, simple type structure (Wed)

4) Dependent type structure & standard Borel spaces (Thu)

5) Integration & random variables (Fri)

Please ask questions!

Course web page

Scribble

Product measures

$$\mu : DX,\ \nu : DY \vdash \mu \otimes \nu := \oint \mu(dx) \oint \nu(dy)\, \delta_{\langle x,y\rangle} : D(X \times Y)$$

$$= \lambda(x,y).\ \mu x \cdot \nu y$$

($\otimes$ lifts along $P \hookrightarrow D$)

discrete model

Ex: $\#_{X \times Y} = \#_X \otimes \#_Y$

build measures compositionally

Indeed:

$$(\# \otimes \#)(x,y) = \# x \cdot \# y = 1 \cdot 1 = 1 = \#(x,y)$$

Notation: $\lambda : D(X \times Y),\ \kappa : (DZ)^{X \times Y} \vdash \oint\!\!\oint \lambda(dx, dy)\, \kappa(x, y)$
$:= \oint \lambda \kappa$

Fubini-Tonelli Thm:

Integrate in any order:

$\mu : DX,\ \nu : DY,\ \kappa : (DZ)^{X \times Y} \vdash$

$$\oint \mu(dx) \oint \nu(dy)\, \kappa(x, y) = \oint\!\!\oint (\mu \otimes \nu)(dx, dy)$$
$$= \oint \nu(dy) \oint \mu(dx)\, \kappa(x, y)$$

Pushing a measure forward

$\mu : D\Omega,\ \alpha : X^{\Omega} \vdash \mu_f := \oint \mu(d\omega)\,\delta_{\alpha\omega} : DX$

$$= \lambda x. \sum_{\substack{\omega \in \Omega \\ \alpha\omega = x}} \mu\,\omega$$

$\alpha : X^{\Omega}$: random element (w.r.t. μ)

$\mu_\alpha : DX$: the law of α

Ex: We can represent configurations of 2 dice using $\underline{6} \times \underline{6}$

Letting $(+) : \underline{6}^2 \longrightarrow \mathbb{N}^2 \xrightarrow{(+)} \mathbb{N}$

we have that the law of $(+)$:

$(\#_{\underline{6}} \otimes \#_{\underline{6}})_{(+)} : D\mathbb{N}$ is the number of rolls whose sum is given

build measures compositionally

Scaling a measure

$$(\cdot) : \mathbb{W} \times DX \longrightarrow DX$$

$$a \cdot \mu := \lambda x.\ a \cdot \mu x$$

$$\boxed{NB: \ \mathrm{supp}(a \cdot \mu) = \begin{cases} a = 0: & \emptyset \\ a \neq 0: & \mathrm{supp}\,\mu \end{cases}}$$

✓ c+bl

$(\cdot) : \mathbb{W} \times DX \to DX$ is an action of monoid $(\mathbb{W}, (\cdot), 1)$ on DX:

$\mu : DX \vdash$

$$1 \cdot \mu = \mu$$

$a, b : \mathbb{W}, \mu : DX \vdash$

$$a \cdot (b \cdot \mu) = (a \cdot b) \cdot \mu$$

Normalisation

unit type $\rightarrow$ $\mathbb{1} := \{\langle\rangle\}$ $\leftarrow$ unit value

$$\mu : DX,\quad \underset{\mu}{Ce}[X] \neq 0, \infty \vdash$$

$$\|\mu\| := \left(\frac{1}{\underset{\mu}{Ce}[X]}\right) \cdot \mu \quad : PX$$

Ex:

$$\emptyset \neq A \subseteq_{fin} X \quad : \mathcal{U}_{A \subseteq X} := \|(\#_A)_{A \subseteq X}\| : PX$$

$$\mathbb{1} \xrightarrow{\#_A} DA \xrightarrow{(-)_{A \subseteq X}} DX \xrightarrow{\|-\|} PX$$

I.e. $\mathcal{U}_{A \subseteq X} := \lambda x. \begin{cases} x \in A : & \frac{1}{|A|} \\ x \notin A : & 0 \end{cases}$ so $\mathcal{U}_{\{x\} \subseteq X} = \delta_x$

Standard vocabulary

Joint distributions: $\mu : D(X_1 \times X_2)$

Marginal distribution: $X_1 \xleftarrow{\pi_1} X_1 \times X_2 \xrightarrow{\pi_2} X_2$

law of projection

$$\mu_{\pi_i} : DX_i$$

marginalisation: $\mu_{\pi_1} = \oiint \mu(dx, dy)\,\delta_x$

integrate out y

Exercise: $\mu : PX,\ \nu : DX \vdash (\mu \otimes \nu)_{\pi_2} = \nu$

independence

Pairing r.e.s:

$\alpha : X^\Omega, \beta : Y^\Omega \vdash$

$$\langle\alpha,\beta\rangle := \lambda\omega. \langle \alpha\omega, \beta\omega\rangle : (X\times Y)^\Omega$$

$$\begin{tikzcd} & \Omega \arrow[dl, "\alpha"'] \arrow[d, dashed, "\langle\alpha,\beta\rangle"] \arrow[dr, "\beta"] & \\ X & X\times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

$$\lambda : D\Omega, \alpha : X^\Omega, \beta : Y^\Omega \vdash \alpha \perp_\lambda \beta := \lambda_{\langle\alpha,\beta\rangle} = \lambda_\alpha \otimes \lambda_\beta$$

: Prop

α,β independent w.r.t. λ

$\underline{\underline{Ex}}$ (Durrett) represent outcomes of 3 coin tosses:

$C := \{T, H\}$ $\quad \Omega := C \times C \times C$ $\quad \lambda : U_C \otimes U_C \otimes U_C : P\Omega$

$\pi_i : \Omega \longrightarrow C$ outcome of i^{th} toss

$$\mathrm{Same}_{ij} : \Omega \xrightarrow{\langle \pi_i, \pi_j \rangle} C \times C \xrightarrow{(\overset{?}{=})} \mathbb{B}$$

where: $(\overset{?}{=}) : C^2 \longrightarrow \mathbb{B} := \{\mathrm{True}, \mathrm{False}\}$

$$x \overset{?}{=} y := \begin{cases} x = y : & \mathrm{True} \\ x \neq y : & \mathrm{False} \end{cases}$$

$\underline{Ex}$ (Durrett) represent outcomes of 3 coin tosses:

$C := \{T, H\}$ $\quad \Omega := C \times C \times C$ $\quad \lambda : U_C \otimes U_C \otimes U_C : P\Omega$

$\pi_i : \Omega \longrightarrow C$ outcome of i^{th} toss

$\mathrm{Same}_{ij} : \Omega \xrightarrow{\langle \pi_i, \pi_j \rangle} C \times C \xrightarrow{(\overset{?}{=})} \mathbb{B}$

marginalisation

$$\lambda_{\mathrm{Same}_{12}} T \overset{\text{marginalisation}}{=} (U_c \otimes U_c)_{(\overset{?}{=})} T = \underbrace{\frac{1}{4}}_{U_c(H)\cdot U_c(H)} + \overbrace{\frac{1}{4}}^{U_c(T)\cdot U_c(T)} = \frac{1}{2}$$

So $\lambda_{\mathrm{Same}_{12}} F = \frac{1}{2}$ too

$\underline{Ex}^{\text{(Durrett)}}$ represent outcomes of 3 coin tosses:

$C := \{T, H\}$ $\quad \Omega := C \times C \times C$ $\quad \lambda : U_C \otimes U_C \otimes U_C : P\Omega$

$\pi_i : \Omega \longrightarrow C$ outcome of i^{th} toss

$\mathrm{Same}_{ij} : \Omega \xrightarrow{\langle \pi_i, \pi_j \rangle} C \times C \xrightarrow{(\overset{?}{=})} \mathbb{B}$

$\underline{i \neq j}$: $\lambda_{\mathrm{Same}_{ij}} = U_{\mathbb{B}}$

$$\lambda_{\langle \mathrm{Same}_{12}, \mathrm{Same}_{23} \rangle} : \quad (T,T) \mapsto \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}_{\lambda(H,H,H)} + \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}_{\lambda(T,T,T)} = \frac{1}{4}$$

$$(T,F) \mapsto \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}_{\lambda(H,H,T)} + \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}_{\lambda(T,T,H)} = \frac{1}{4}$$

<u>Ex</u> (Durrett) represent outcomes of 3 coin tosses:

$C := \{T, H\}$ $\quad \Omega := C \times C \times C$ $\quad \lambda : U_C \otimes U_C \otimes U_C : P\Omega$

$\pi_i : \Omega \longrightarrow C$ outcome of i^{th} toss

$\mathrm{Same}_{ij} : \Omega \xrightarrow{\langle \pi_i, \pi_j \rangle} C \times C \xrightarrow{\overset{?}{(=)}} \mathbb{B}$

<u>$i \neq j$:</u> $\lambda_{\mathrm{Same}_{ij}} = U_{\mathbb{B}}$

$$\lambda_{\langle \mathrm{Same}_{12}, \mathrm{Same}_{23} \rangle} = U_{\mathbb{B} \times \mathbb{B}} = U_{\mathbb{B}} \otimes U_{\mathbb{B}} = \lambda_{\mathrm{Same}_{12}} \otimes \lambda_{\mathrm{Same}_{13}}$$

So $\mathrm{Same}_{12} \perp_{\lambda} \mathrm{Same}_{13}$

independence

Pairing r.e.s:

$\alpha : X^{\Omega}, \beta : Y^{\Omega} \vdash$

$$\langle\alpha,\beta\rangle := \lambda\omega.\langle \alpha\omega, \beta\omega\rangle : (X\times Y)^{\Omega}$$

$$\begin{tikzcd} & \Omega \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[d, dashed, "\langle\alpha,\beta\rangle"] & \\ X & X\times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

$$\lambda : D\Omega, \alpha : X^{\Omega}, \beta : Y^{\Omega} \vdash \alpha \perp_{\lambda} \beta := \lambda_{\langle\alpha,\beta\rangle} = \lambda_{\alpha} \otimes \lambda_{\beta} : \mathrm{Prop}$$

α, β independent w.r.t. λ

I-ary version:

$$\lambda : D\Omega, \alpha_{-} : \prod_{i\in I} X_i^{\Omega} \vdash \mathop{\perp_{\lambda}}_{i\in I} \alpha_i := \forall J \subseteq_{fin} I.\ \lambda_{\langle\alpha_j\rangle_{j\in J}} = \bigotimes_{j\in J}\lambda_{\alpha_j} : \mathrm{Prop}$$

α_{-} independent w.r.t. λ

Ex (Durrett) represent outcomes of 3 coin tosses:

$C := \{T, H\}$ $\quad \Omega := C \times C \times C$ $\quad \lambda : U_C \otimes U_C \otimes U_C : P\,\Omega$

$\pi_i : \Omega \longrightarrow C$ outcome of i^{th} toss

$\mathrm{Same}_{ij} : \Omega \xrightarrow{\langle \pi_i, \pi_j \rangle} C \times C \xrightarrow{\overset{?}{(=)}} \mathbb{B}$

<u>$i \neq j$:</u> $\lambda_{\mathrm{Same}_{ij}} = U_{\mathbb{B}}$

$\underset{\neq k \neq}{i \neq j}$: $\mathrm{Same}_{ij} \perp_\lambda \mathrm{Same}_{jk}$ $\qquad \cancel{\perp}_\lambda \{\mathrm{Same}_{12}, \mathrm{Same}_{23}, \mathrm{Same}_{13}\}$

Intuition: $\mathrm{Same}_{13} = \mathrm{IFF}(\mathrm{Same}_{12}, \mathrm{Same}_{23})$

Calc:

$$\lambda_{\langle \mathrm{Same}_{12}, \mathrm{Same}_{23}, \mathrm{Same}_{13} \rangle}(T,T,T) = \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}_{\lambda(H,H,H)} + \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}_{\lambda(T,T,T)} = \frac{1}{4} \neq \frac{1}{2^3} = \lambda_{\mathrm{Same}_{12}} \otimes \lambda_{\mathrm{Same}_{23}} \otimes \lambda_{\mathrm{Same}_{13}}$$

Vocabulary

(Discrete) Measure space $(X, \mu : DX)$

measure preserving $f : (X, \mu) \longrightarrow (Y, \nu)$

function $f : X \to Y$ s.t. $\mu_f = \nu$

$\mu : DX,\ f : X \to Y \vdash \mu$ invariant under f :=

$$f : (X, \mu) \to (X, \mu)$$

Ex:

$\mu : DX,\ \nu : DY \vdash$

Swap: $(X \times Y, \mu \otimes \nu) \longrightarrow (Y \times X, \nu \otimes \mu)$ so

$\mu : DX \vdash \mu \otimes \mu$ invariant under swap

Weights as measures

NB: unit type → $\mathbb{1} := \{\langle\rangle\}$ ← unit value

Observation:

$$
\begin{tikzcd}
D\mathbb{1} \arrow[r, bend left, "\lambda\mu.\,\mathrm{ce}_{\mu}[\mathbb{1}]"] & W \arrow[l, bend left, "(-\odot\delta_{\langle\rangle})"]
\end{tikzcd}
$$

mutually inverse

Proof:

$$
\begin{tikzcd}
\mu \arrow[r, mapsto] \arrow[d, phantom, "\parallel"] & \mu(\langle\rangle) \arrow[dl, mapsto] \\
\lambda\langle\rangle.\,\mu\langle\rangle &
\end{tikzcd}
\qquad \&
\qquad
\begin{tikzcd}
\lambda\langle\rangle.\,a \arrow[dr, mapsto] & a \arrow[l, mapsto] \arrow[d, phantom, "\parallel"] \\
& (\lambda\langle\rangle.\,a)\langle\rangle
\end{tikzcd}
$$

$\blacksquare$

Integration

$$\mu : DX,\ \varphi : W^X \vdash \int \mu\, \varphi \ : W \qquad \text{(Lebesgue integral)}$$

$$:= \sum_{x \in X} \mu x \cdot \varphi x$$

Can derive it:

$$\begin{tikzcd}
DX \times W^X \arrow[rr, "DX \times (\cong \circ -)"] \arrow[d, "\int"'] \arrow[drr, phantom, ":="] & & DX \times (D\mathbb{1})^X \arrow[d, "\oint"] \\
W & & D\mathbb{1} \arrow[ll, "\cong"]
\end{tikzcd}$$

Additivity:

$$I \text{ ctbl}, \mu_- : (DX)^I \vdash \sum_{i\in I} \mu_i : DX$$

$$:= \lambda x. \sum_{i \in I} \mu_i x$$

NB:

$\mathrm{supp} \sum_i \mu_i \subseteq \bigcup_i \mathrm{supp}\, \mu_i$

$\checkmark$ ctbl

Ex: Bernoulli distribution

$$p : [0,1] \vdash B(p) := p \cdot \delta_{\mathrm{True}} + (1-p) \cdot \delta_{\mathrm{False}} : P\mathbb{B}$$

i.e. $B p$: $\mathrm{True} \mapsto p$

$\mathrm{False} \mapsto 1-p$

Thm (affine-linearity):

$\oint$ is affine-linear in each argument:

$$\frac{I \text{ ctbl} \quad \mu_- : (D\Gamma)^I, \; k : (DX)^\Gamma, \; a_- : W^I}{\oint\left(\sum_{i\in I} a_i \cdot \mu_i\right) k = \sum_{i \in I} a_i \cdot \oint \mu_i k}$$

$$I \text{ ctbl}, \; \mu : D\Gamma, \; a_- : W^I, \; k_- : DX^I \vdash$$

$$\int \mu(dx)\left(\sum_{i\in I} a_i \cdot k_i(x)\right) = \sum_{i\in I} a_i \cdot \oint \mu k_i$$

Prop: $\mathbb{W} \cong D\mathbb{1}$ is a σ-semi-ring isomorphism:

$$(\mathbb{W}, \Sigma, (\cdot), 1) \cong (D\mathbb{1}, \Sigma, (\cdot), \delta_{\langle\rangle})$$

and $(\cdot) : \mathbb{W} \times DX \to DX$ makes DX into a module:

$$\left(\sum_{i \in I} a_i\right) \cdot \mu = \sum_{i \in I} (a_i \cdot \mu) \qquad a \cdot \sum_{i \in I} \mu_i = \sum_{i \in I} a \cdot \mu_i$$

Corollary: $\int$ is affine-linear in each argument.

Random variable:

NB: $\overline{\mathbb{R}} := [-\infty, \infty]$

A random element $\alpha : \overline{\mathbb{R}}^{\Omega}$ (wrt some $\mu : D\Omega$)

Can add, multiply r.v.'s.

To integrate r.v.'s:

$$(-)^{+} : \overline{\mathbb{R}}^{\Omega} \longrightarrow W^{\Omega}$$

$$\alpha^{+} := \lambda w. \begin{cases} \alpha \cdot w \geq 0 : & \alpha w \\ 0.w : & 0 \end{cases} = [\alpha - \geq 0] \cdot |\alpha|$$

$$\alpha^{-} := \lambda w. \begin{cases} \alpha \cdot w \leq 0 : & |\alpha w| \\ 0.w : & 0 \end{cases} = [\alpha - \leq 0] \cdot |\alpha|$$

So $\alpha = \alpha^{+} - \alpha^{-}$

$\mu : D\Omega, \alpha : \overline{\mathbb{R}}^{\Omega}, \int \mu\alpha^{+} < \infty$ or $\int \mu \alpha^{-} < \infty \vdash$

$$\int \mu\, \alpha := \int \mu \alpha^{+} - \int \mu \alpha^{-} \quad : \overline{\mathbb{R}}$$

<u>Ex.</u> The (discrete) Lebesgue p-space:

$$p : [1, \infty), \mu : P\Omega \vdash \mathcal{L}_p(\Omega, \mu) :=$$

$$\left\{ \alpha : \overline{\mathbb{R}}^{\Omega} \,\middle|\, \mathbb{E}_{\mu}[|\alpha|^{p}] < \infty \right\}$$

$\mathcal{L}_p(\Omega, \mu)$ has a norm $\|\alpha\| := \sqrt[p]{\mathbb{E}_{\mu}[|\alpha|^{p}]}$ almost Banach

$\mathcal{L}_2(\Omega, \mu)$ has an inner product $\langle \alpha, \beta \rangle := \mathbb{E}_{\mu}[\alpha \cdot \beta]$ almost Hilbert

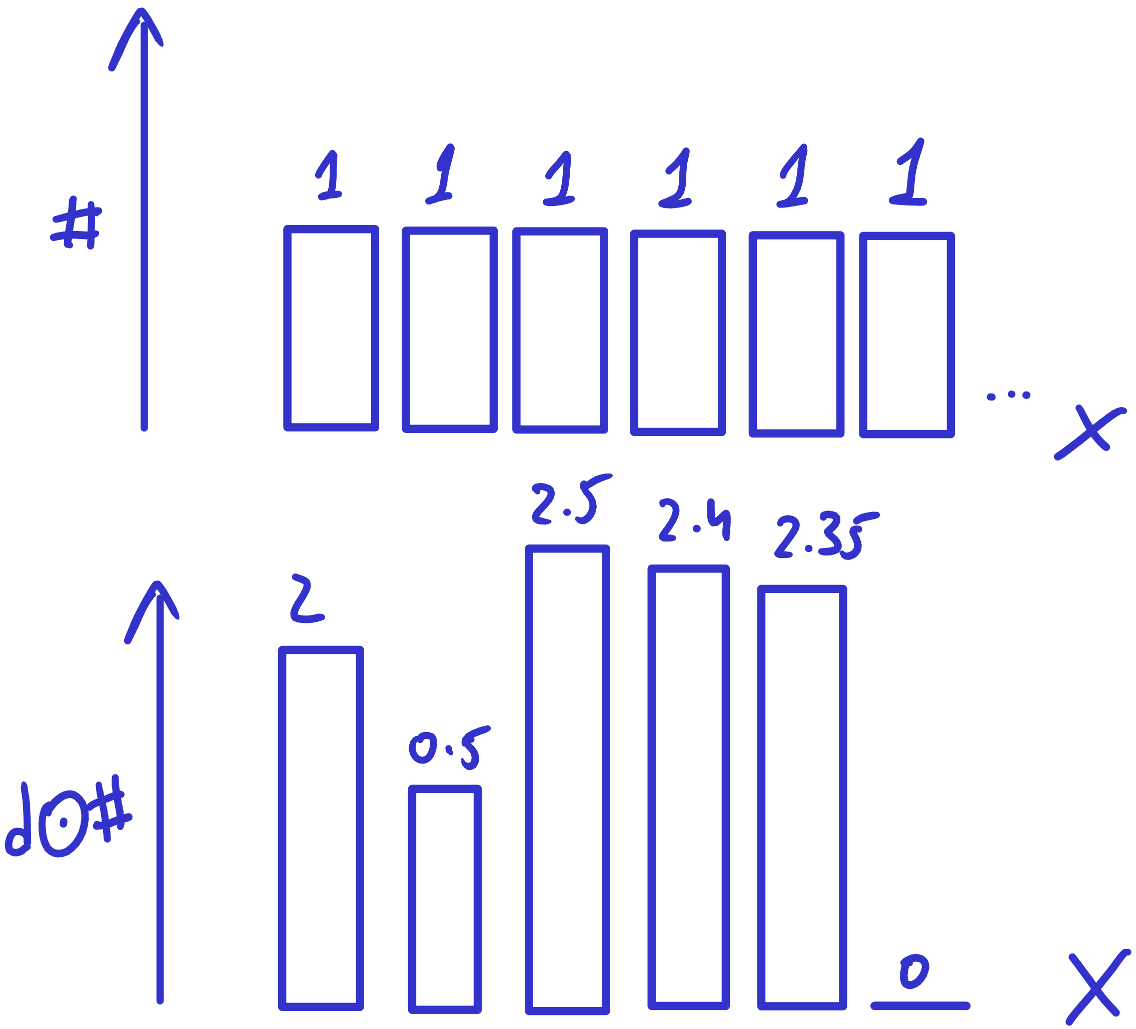
Density

a density over X : $d : X \to W$

$$d : W^X,\ \mu : DX \vdash d \odot \mu \quad : DX$$
$$:= \oint \mu(dx)(dx \cdot \delta_x)$$

Warning The types of measures & densities in the discrete model are close, but still different. They coincide on countable sets, so people often confuse them. Types help us keep them separate.

Intuition:



Almost certain properties

$$E : \mathcal{B}X, \mu : DX \vdash \mu(dx)\text{-almost certainly } x \in E \quad : \mathrm{Prop}$$

$$:= [- \in E] \odot \mu = \mu$$

$$\text{NB: } [x \in E] = \begin{cases} x \in E : & 1 \\ x \notin E : & 0 \end{cases} \quad : W$$

When $\mu : PX$ we say instead

$$\mu(dx)\text{-almost surely } x \in E$$

Exercise Look up the def. of a normed space and modify the definition so that $\mathcal{L}_p(\Omega, \mu)$ is a normed space up-to almost sure equality.

Absolute continuity

d is a density of μ w.r.t. ν or

d is a Radon-Nikodym derivative w.r.t. ν

$$\mu, \nu : \mathcal{D}X,\ d : \mathbb{W}^X \vdash d = \frac{d\mu}{d\nu} : \mathrm{Prop}$$

$$:= \mu = d \odot \nu$$

$$\mu, \nu : \mathcal{D}X \vdash \mu \ll \nu := \mu \text{ is absolutely continuous w.r.t. } \nu : \mathrm{Prop}$$

$$:= \exists d : \mathbb{W}^X.\ d = \frac{d\mu}{d\nu}.$$

$$=: \mu \text{ has a density w.r.t. } \nu$$

Lemma: $\mu, \nu : \mathcal{D}X,$
$\mu \ll \nu,$
$k : (\mathcal{D}Y)^X$

$$\oint \nu(dx) \frac{d\mu}{d\nu}(x) \cdot k\,x = \oint \mu(dx)\, k\,x$$

Ex: $U_{A \subseteq X} \ll (\#_A)_{\mathrm{Const} : A \subseteq X}$

$$\frac{d U_{A \subseteq X}}{d (\#_A)_{\mathrm{Const}}} = \lambda x. \begin{cases} x \in A : & \frac{1}{|A|} \\ \text{o.w.} : & 0 \end{cases}$$

but also: $$\frac{d U_{A \subseteq X}}{d (\#_A)_{\mathrm{Const}}} = \lambda x. \frac{1}{|A|}$$

Radon-Nikodym Thm: (discrete version)

$\mu, \nu : PX \vdash \mu \ll \nu$ iff $\forall x.\ \nu x = 0 \Rightarrow \mu x = 0$

i.e. $\mathrm{Supp}\,\mu \subseteq \mathrm{Supp}\,\nu$

In that case, if $d_1, d_2 = \frac{d\mu}{d\nu}$ then

$\nu(dx)$-a.s. $d_1 x = d_2 x$

<u>Ex</u>: for ctbl X, $\forall \mu : DX.\ \mu \ll \#_X$. Proof: vacuously, as $\#_X x \neq 0$.

Then $\lambda x.\ \mu x = \frac{d\mu}{d\#}$.

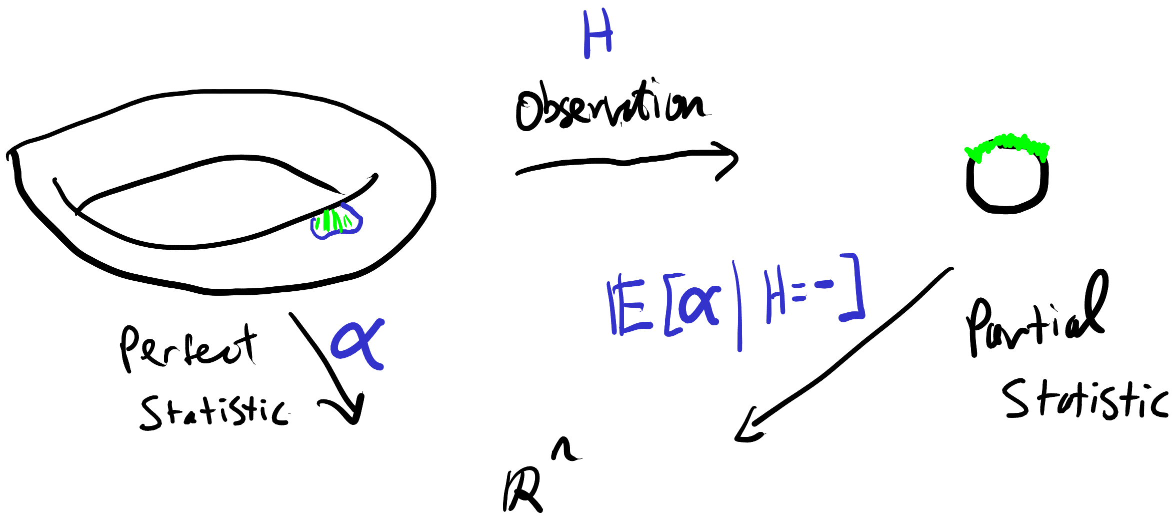
Conditional expectation

β is a conditional expectation of α w.r.t. μ along H

$$\mu : D\Omega,\ H : X^{\Omega},\ \alpha : \mathcal{L}_1(\Omega, \mu),\ \beta : \mathcal{L}_1(X, \mu_H)$$

$$\vdash \beta = \mathbb{E}[\alpha \mid H = -] \quad : \mathrm{Prop}$$

$$:= \forall \varphi : \mathcal{L}_1(Y, \mu_H^{M}).\ \int \mu_H(dx)\, \beta(x) \cdot \varphi(x) = \int \mu(d\omega)\, \alpha(\omega) \cdot \varphi(H\omega)$$



Thm (Kolmogorov): (discrete version)

There is a funtion

$$\mathbb{E}_{-}[- \mid - = -] \in \prod_{\mu : P\Omega} \prod_{H : X^{\Omega}} \mathcal{L}_1(\Omega, \mu) \to \mathcal{L}_1(X, \mu_H)$$

s.t. $\mathbb{E}_{\mu}[\alpha \mid H = -]$ is a conditional expectation of α w.r.t. μ along H.

Conditional Probability (discrete version):

$$H : X^{\Omega},\ \mu : PX \vdash \mathbb{P}_{\mu}[- \mid H = -] : (P\Omega)^{X}$$

$$:= \lambda x_0 : X.\ \lambda \omega_0 : \Omega.\ \mathbb{E}_{\omega \sim \mu}\left[[\omega_0 = \omega] \,\middle|\, H\omega = x_0 \right]$$

Bayes's Thm (discrete version, adapted from Williams):

Let $\lambda : P(X \times \Theta)$ joint probability distribution.

Assume $\mu : DX$, $\nu : D\Theta$ s.t. $\lambda \ll \mu \otimes \nu$.

with $d_{X,H} = \dfrac{d\lambda}{d(\mu \otimes \nu)}$.

$$\begin{array}{c} X \times \Theta \\ {\scriptstyle \alpha := \pi_1} \swarrow \quad \searrow {\scriptstyle H := \pi_2} \\ X \qquad\qquad \Theta \end{array}$$

observation space.

Obs 1: $d_X : W^X$ then $d_X = \dfrac{d\lambda_\alpha}{d\mu}$

$d_X := \lambda x. \int \nu(d\theta) d_{X,H}(x, \theta)$

& similarly $(d_H : W^\Theta) := \lambda\theta. \int \mu(dx) d_{X,H}(x,\theta) = \dfrac{d\lambda_H}{d\nu}$

Bayes's Thm (discrete version, adapted from Williams):

$$\begin{tikzcd} & X \times \Theta \arrow[dl, "\alpha := \pi_1"'] \arrow[dr, "H := \pi_2"] & \\ X & & \Theta \end{tikzcd}$$

Θ observation space.

Let $\lambda : P(X \times \Theta)$ joint probability distribution.

Assume $\mu : DX$, $\nu : D\Theta$ s.t. $\lambda \ll \mu \otimes \nu$.

with $d_{X,H} = \dfrac{d\lambda}{d(\mu \otimes \nu)}$. $d_X = \dfrac{d\lambda_\alpha}{d\mu}$ $\quad d_\Theta = \dfrac{d\lambda_H}{d\nu}$

Let $d_{X|H}(-|-) : X \times \Theta \to W$

$$d_{X|H}(x|\theta) := \begin{cases} d_H\theta \neq 0 : & \dfrac{d_{X,H}(x,\theta)}{d_H\theta} \\ \text{o.w.} : & 0 \end{cases}$$

$\lambda_{X|H=-} : \Theta \to PX$

$\lambda_{X|H=\theta} := d_{X|H}(-|\theta) \odot \mu$

Bayes's formula:

$$\mathbb{P}\mathrm{r}_{\lambda}[-|H=-] = \lambda_{X|H=-}$$

Summary

$\mu \otimes \nu$ Product measures & Fubini-Tonelli

μ_H push-forward / law

$(Dx, \Sigma, (\cdot))$ module structure and affine linearity of $\oint$

$\int$ Lebesgue integration

Standard vocabulary: joint dist., marginalisation, independence, invariance

density & Radon-Nikodym derivatives (heed the warning)

almost certain properties

Conditional expectation & Probability

with Bayes's Thm.

Plan:

1) type-driven probability: discrete case (Mon + Tue) ✓

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3) Quasi Borel spaces, simple type structure (Wed)

4) Dependent type structure & standard Borel spaces (Thu)

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