Foundations for type-driven probabilistic modelling

Ohad Kammar
University of Edinburgh

TutorialFest
ACM-SIGPLAN 51st Symposium on
Principles of Programming Languages (POPL'24)
15 January, 2024
City of Westminister, England

THE UNIVERSITY of EDINBURGH informatics lfcs Laboratory for Foundations of Computer Science BayesCentre

supported by: THE ROYAL SOCIETY The Alan Turing Institute Facebook Research NCSC

Partiality cf. [Vákár et al. '19]

A Borel embedding $e : X \hookrightarrow Y$

- injective function $e : \lfloor X \rfloor \rightarrowtail \lfloor Y \rfloor$
- its image is Borel: $e[\lfloor X \rfloor] \in \mathcal{B}_Y$
- e is Strong: $\alpha \in R_X \iff e \circ \alpha \in R_Y$

Examples

- $\mathbb{1} \hookrightarrow \mathbb{2}$
- S is sbs $\iff \exists\, S \hookrightarrow \mathbb{R}$

Def: A *partial map* $f : X \to Y$ is a morphism

$$f : X \to Y \amalg \{\bot\}$$

Its *domain of definition*

$$f : (Y \amalg \{\bot\})^X \vdash \mathrm{Dom}\, f := \{x \in X \mid f x \neq \bot\} : \mathrm{Type}$$

Dependent-type interpretation

$$\begin{tikzcd}
{[\![\mathrm{Dom}\, f]\!]} \arrow[r] \arrow[d, "\mathrm{dep}"] \arrow[dr, phantom, "\lrcorner", very near start] & \{y \in Y \mid y \in E\} \arrow[d, "\mathrm{dep}"] \\
{[\![f : (Y \amalg \{\bot\})^X]\!]} \arrow[r, "{[E \mapsto \lambda x.\, f x \neq \bot]}"] & {[\![E : \mathcal{B}_Y]\!]}
\end{tikzcd}$$

Plan:

1) type-driven probability: discrete case ✓
2) Borel sets & measurable spaces ✓
3) Quasi Borel spaces ✓

Lecture 1

4) Type structure & standard Borel spaces ✓
5) Integration & random variables

Lecture 2

Please ask questions!

Course web page

scribble

Full model

$\mathrm{type} : \mathbf{Qbs}$ $\quad \mathbb{W} := [0,\infty]$ $\quad \mathcal{B}X := \mathbb{B}^X$

$$DX := \square$$

$$PX := \{\mu \in DX \mid Ce_{\mu}[X] = 1\}$$

$Ce_{\mu}[E] := \square$ $\qquad \delta_x := \square$

$$\oint \mu k := \square$$

Def: A measure μ over $\mathbb{R}$ is a function

$$\mu : B_{\mathbb{R}} \to W := [0, \infty]$$

satisfying the measure axioms:

$$E_- : B^{\omega} \vdash$$

$$\mu \emptyset = 0, \quad \mu E = \mu(E \cap F) + \mu(E \cap F^c), \quad \mu\left(\bigcup_n E_n\right) = \sup_n \mu E_n$$

For measurable spaces, replace $\mathbb{R}$ with V

We write $\lfloor G V \rfloor$ for the set of measures on V

For qbs X, take $\lfloor G \, {}^{\ulcorner \mathrm{Meas} \urcorner} X \rfloor$

Thm (Lebesgue measure):

There is a unique measure $\lambda \in \lfloor G\mathbb{R} \rfloor$ s.t.:

$\lambda(a,b) = b-a$

The unrestricted Giry spaces

Equip $\lfloor GV \rfloor$ with two qbs structures:

✗ $R_{GV} := \{\alpha : \mathbb{R} \to GV \mid \forall A \in B_V . \lambda r. \alpha(r, A) : \mathbb{R} \to W\}$

↳ α is a kernel.

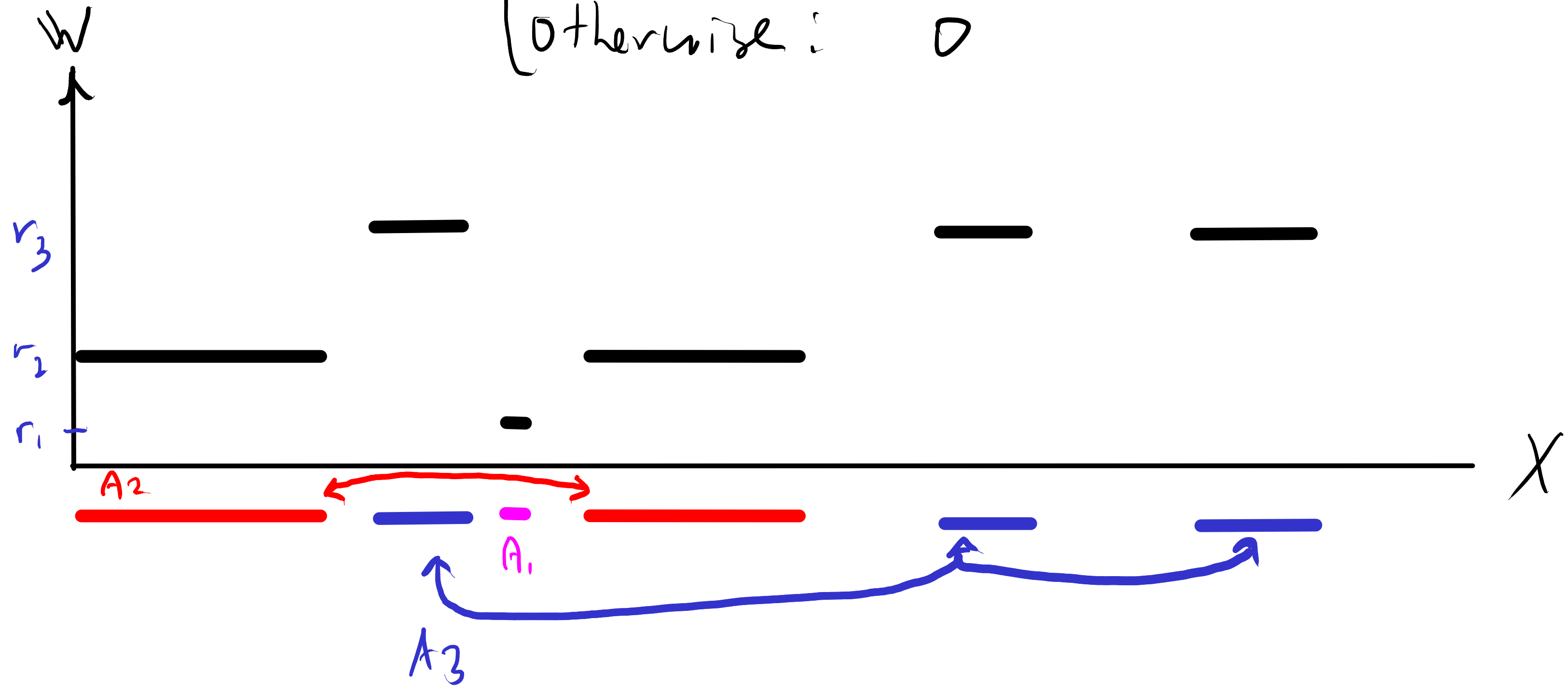
✓ $GV \hookrightarrow W^{B_X}$

- Fewer random elements $R_{GV} \subseteq R_{G'V}$
- Lebesgue integral measurable in both arguments. (upcoming)

Def: Simple function $\varphi : X \to W$ when

$\exists\, n \in \mathbb{N},\ \vec{A} \in B_X^n,\ A_i \cap A_j = \phi\ (i \neq j),\ \vec{r} \in W$ s.t.

$$\varphi\, x = \begin{cases} \vdots & \\ x \in A_i : & r_i \\ \vdots & \\ \text{otherwise} : & 0 \end{cases}$$



Encode into a space:

$$\mathrm{SimpleCode} := \coprod_{n \in \mathbb{N}} B_X^n \times W^n$$

$$\mathrm{Simple} := \{ f \in W^X \mid f \text{ simple} \} \hookrightarrow W^X$$

and define an interpretation:

$$[\![-]\!] : \mathrm{SimpleCode} \twoheadrightarrow \mathrm{Simple}$$

$$[\![(n, \vec{A}, \vec{r})]\!] := \sum_{i=1}^{n} r_i \cdot [- \in A_i]$$

↳ characteristic function for A_i

Lemma: $f : X \to W$ is measurable iff $f = \lim_{n\to\infty} f_n$ for some monotone sequence $f_n \in \mathrm{Simple}$.

remember! QBS morphism!

Moreover, we have measurable such choice:

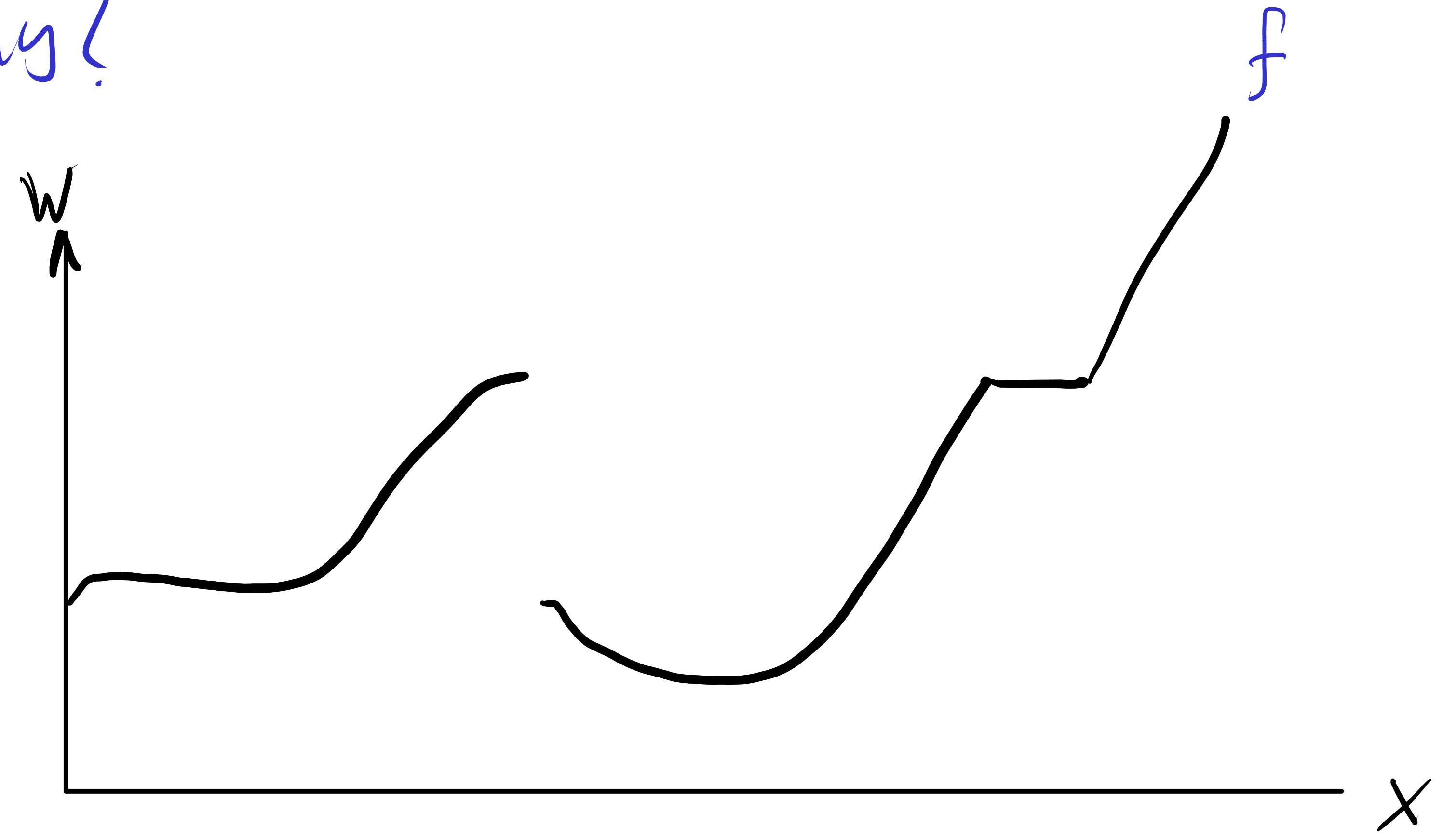
Simple Approx:

$$\{\vec{\Delta} \in \mathbb{R}^+ \mid \Delta_n \to 0\} \times \{\vec{a} \in W^{\mathbb{N}} \mid \vec{a} \text{ monotone},\ a_n \to \infty\} \times W^X \to \mathrm{SimpleCode}$$

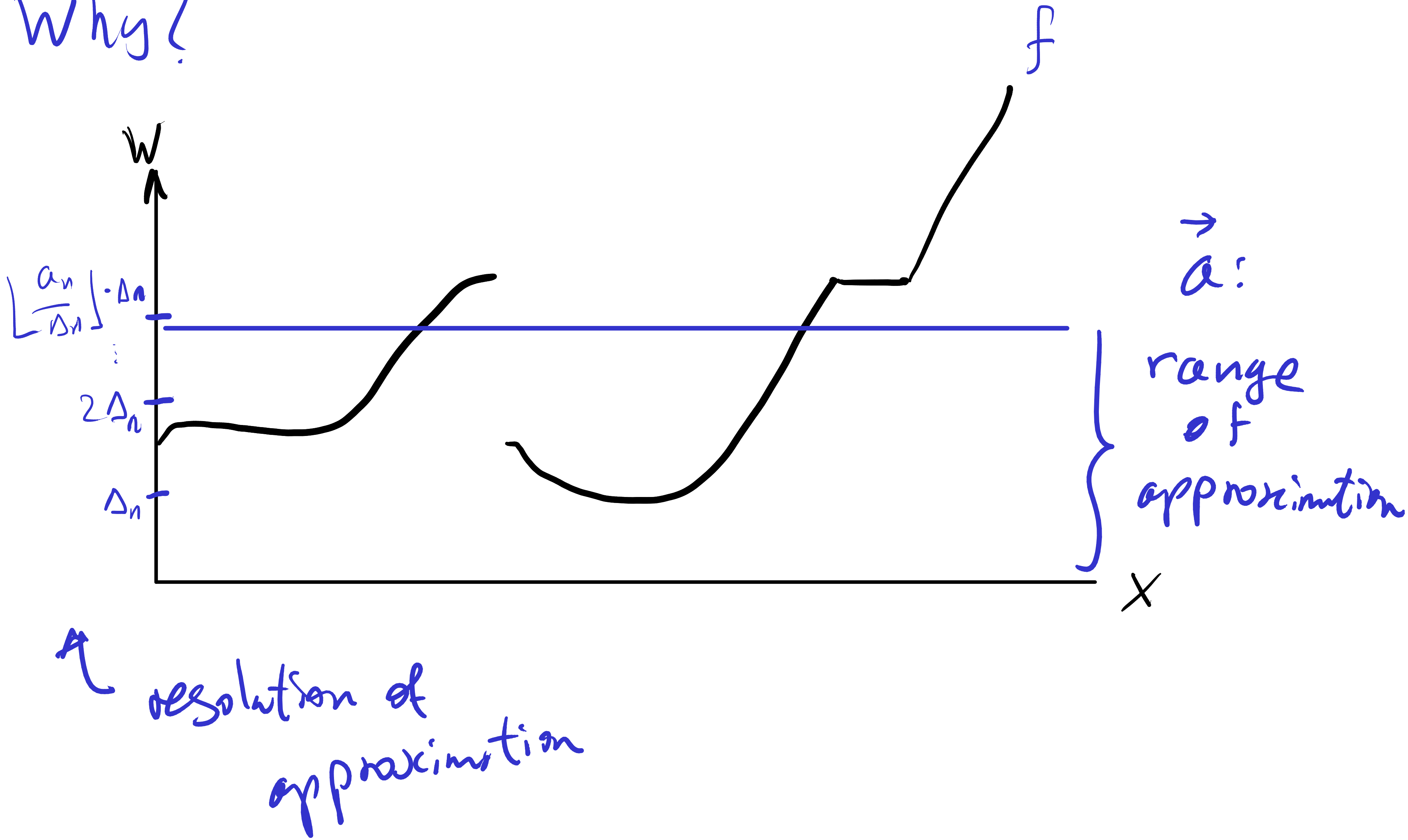
rate of convergence

range of approximation

Why?

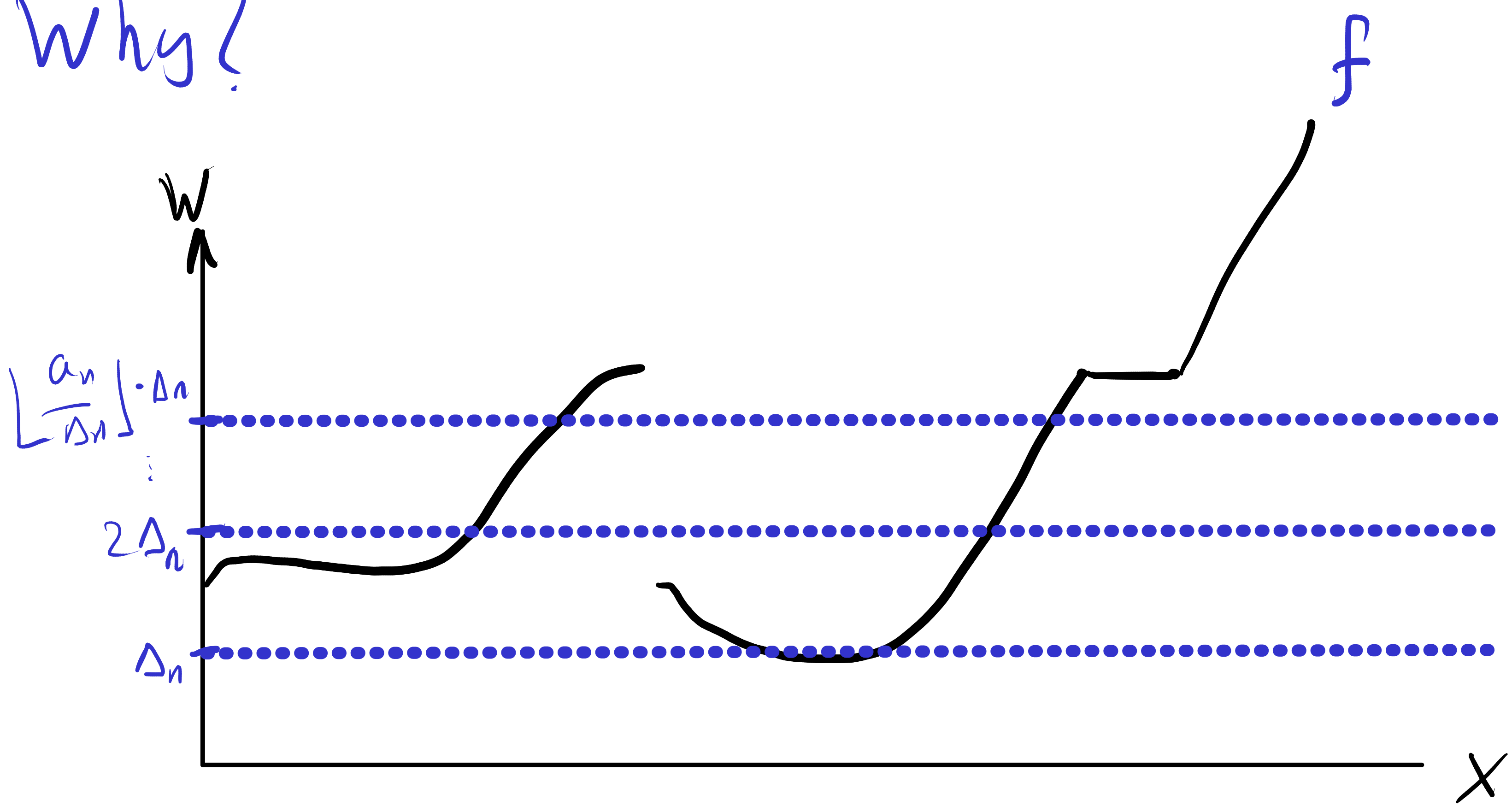


Why?



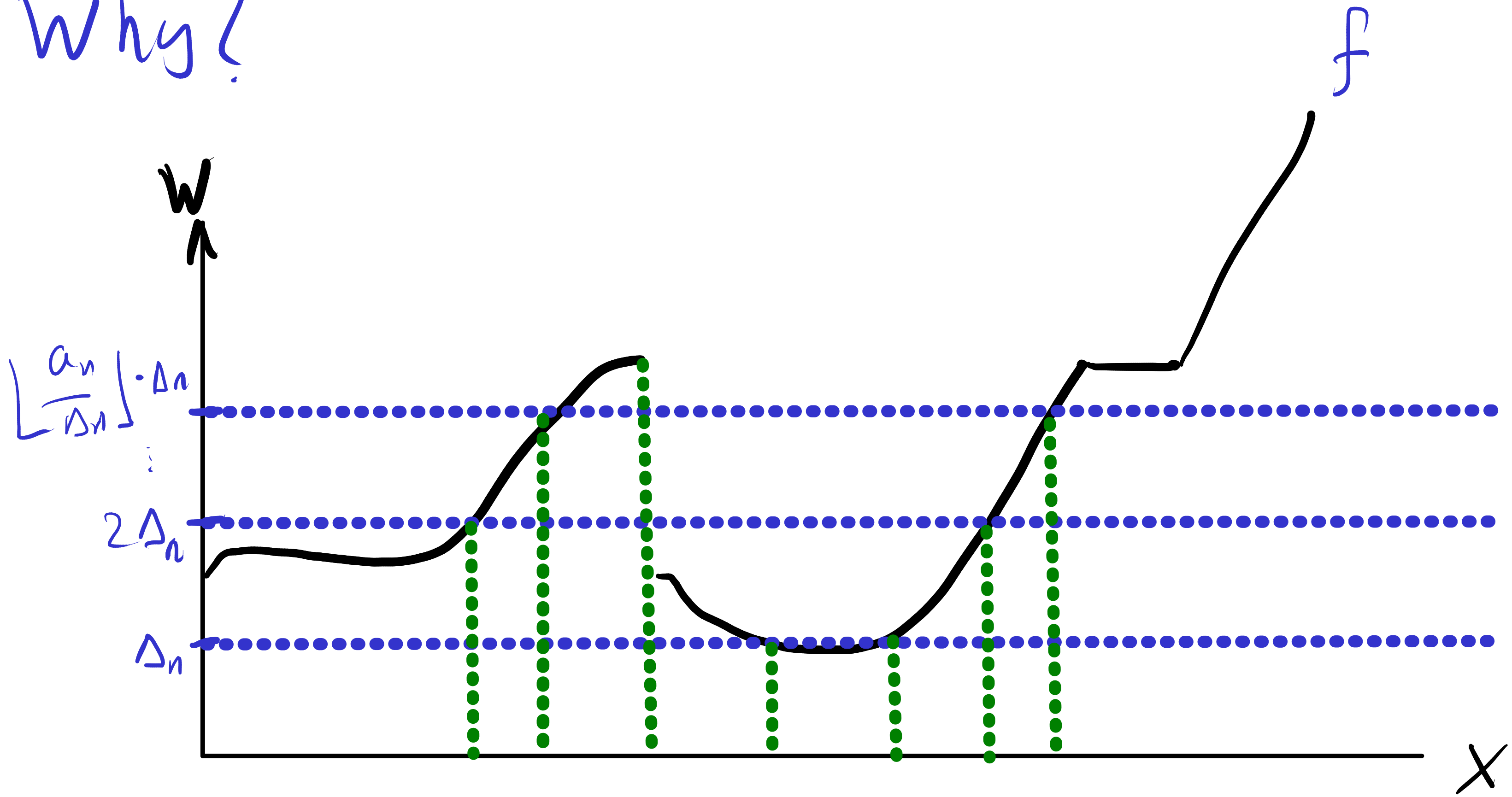
f
W
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n
X
$\vec{a}$: range of f approximation
resolution of approximation

Why?



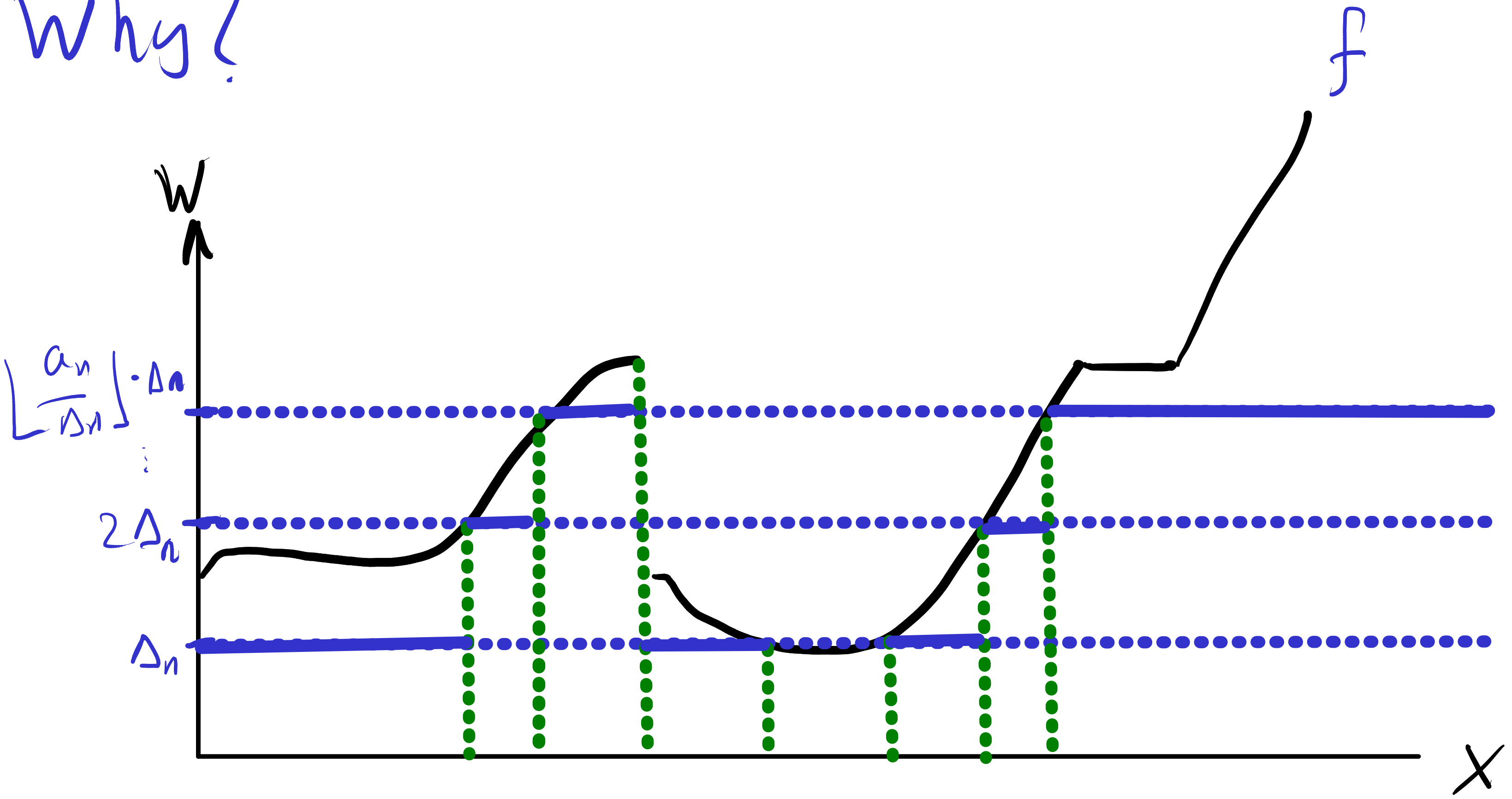
f
w
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
⋮
$2\Delta_n$
Δ_n
X

Why?

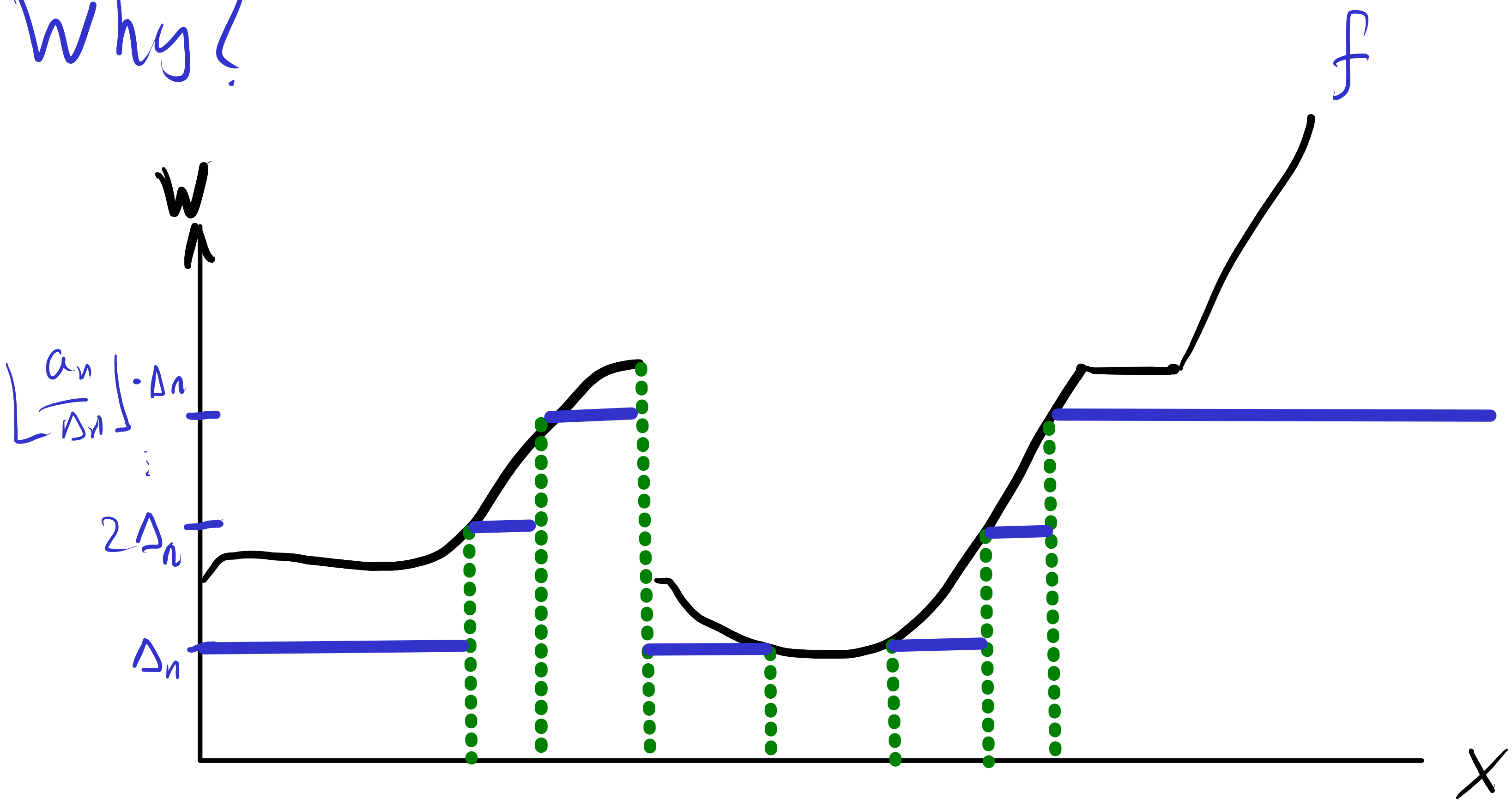


f
w
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n
X

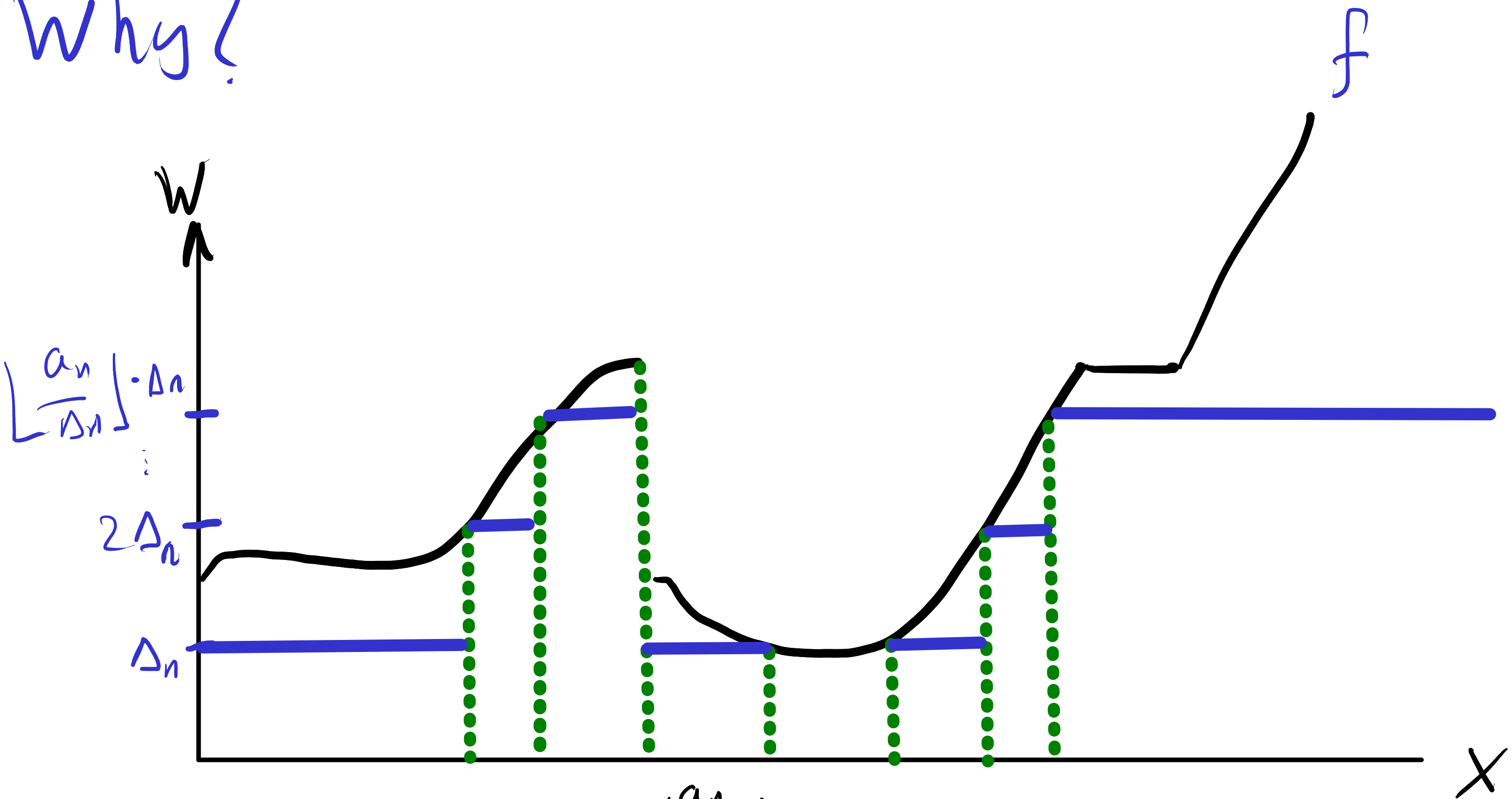
Why?


f
W
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n
X

Why?


f
w
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n
X

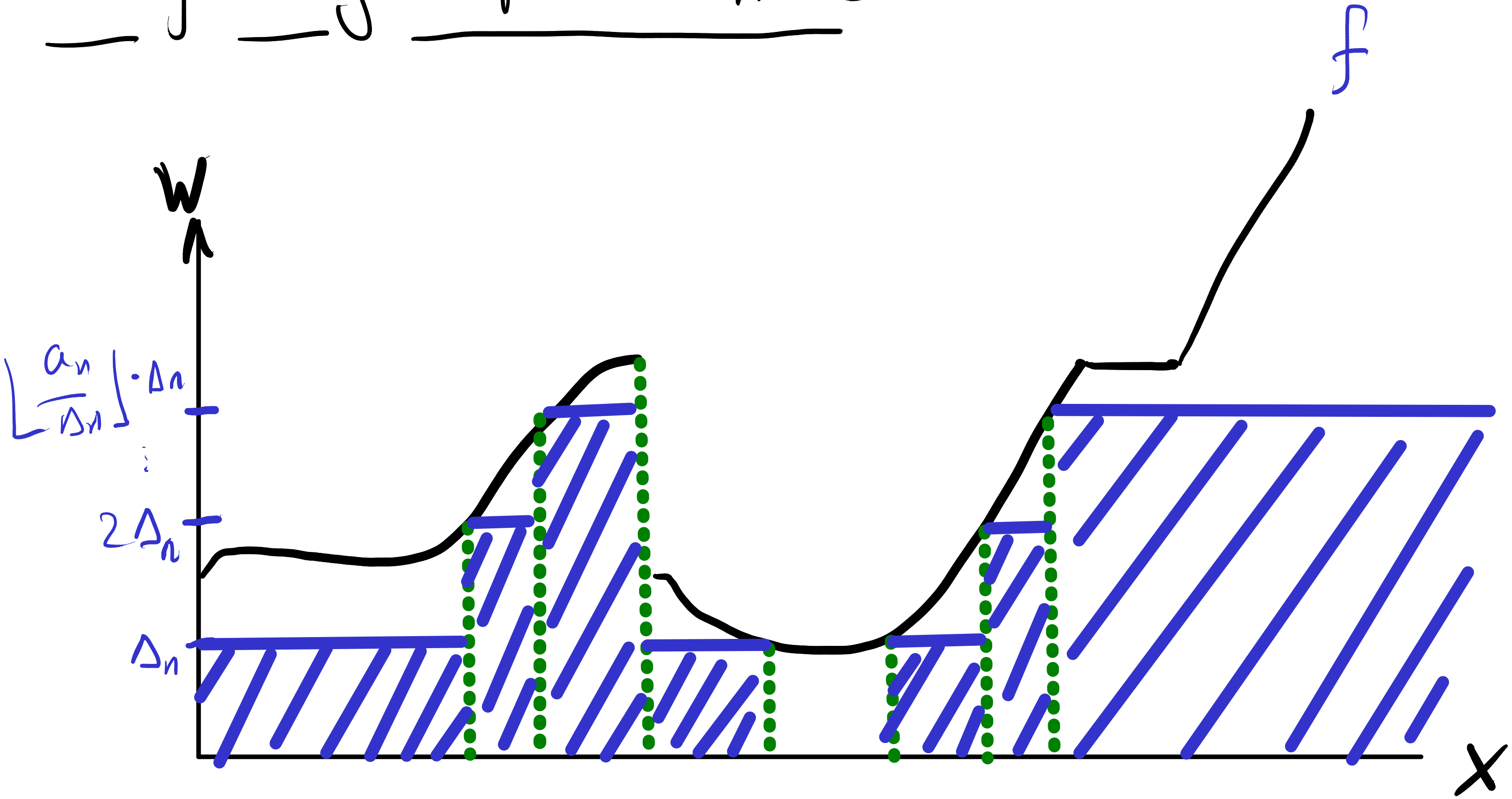
Why?



$$[\![\mathrm{SimpleApprox}_{\vec{\Delta},\vec{a}}\, f]\!] := \sum_{i=1}^{\lfloor \frac{a_n}{\Delta_n} \rfloor} i \cdot \Delta_n [\, i \cdot \Delta_n \leq f < (i+1)\Delta_n \,]$$

$$+ \left\lfloor \frac{a_n}{\Delta_n} \right\rfloor \Delta_n \cdot \left[f \geq \left\lfloor \frac{a_n}{\Delta_n} \right\rfloor \cdot \Delta_n \right] \in \mathrm{Simple}$$

Integrating Simple Functions



$$\int : GX \times \mathrm{SimpleCode} \to W$$

$$\int \mu\,(n, \vec{A}, \vec{r}) := \sum_{I \subseteq \{1,\ldots,n\}} \Big(\sum_{i \in I} r_i\Big) \cdot \mu\Big(\bigcap_{i \in I} A_i \setminus \bigcup_{i \notin I} A_i\Big)$$

Integration

$$\int : GX \times W^X \longrightarrow W$$

Proper higher-order operation

$$\int \mu f := \sup \left\{ \int \mu \varphi \;\middle|\; \varphi \in \mathrm{Simple},\ \varphi \leq f \right\}$$

$$= \lim_{n \to \infty} \int \mu \left(\mathrm{SimpleApprox}_{\vec{\Delta}, \vec{a}}\, f \right)_n$$

measurable by type

we also write $\int \mu(dx)\, t$ for $\int \mu\, (\lambda x. t)$

for $\frac{a_n}{\Delta_n} \longrightarrow 0$, eg. $\Delta_n = \frac{1}{2^n}$ $a_n = n$.

resolution

The unrestricted Giry Strong Monad

Unlike the unrestricted Giry on Meas.

Dirac:

$$\delta : X \to GX$$

$$x \mapsto \lambda A. \begin{cases} x \in A : & 1 \\ x \notin A : & 0 \end{cases}$$

but: non-commutative

(Fubini fails, just like in Meas)

Kleisli extension/Kock integral:

$$\oint : GX \times GY^{X} \longrightarrow GY$$

$$\oint \mu f := \lambda A. \int \mu(dx)\, f(x; A)$$

Fubini-Tonelli fails

$\# \in G\mathbb{R}$

$\#E := \begin{cases} E \text{ finite}: & |E| \\ \text{o.w.}: & \infty \end{cases}$

$\lambda \in G\mathbb{R}$ lebesgue

$k : \mathbb{R}\times\mathbb{R} \to W \cong G\mathbb{1}$

$k(x,y) := [x=y]$

$$\oint \#(dr) \oint \underbrace{\lambda(dx)\, k(x,y)} = \oint \# \underline{0} = \underline{0} \hat{=} 0$$

$y : \mathbb{R} \vdash \{\langle\rangle\} \mapsto \lambda\{y\}\cdot 1 + \lambda\{y\}^c \cdot 0 = 0$

$\neq$

$$\oint \lambda(dx) \oint \underbrace{\#(dr)\ k(x,y)} = \oint \lambda(dx)\, \delta_{\langle\rangle} \hat{=} \infty$$

$x : \mathbb{R} \vdash \{\langle\rangle\} \mapsto \#\{x\}\cdot 1 + 0 = 1$

Randomisable measures monad

$$D \rightarrowtail G$$

$$\lambda A. \int_{\mathrm{Dom}\,\alpha} \lambda(\mathrm{Dom}\,\alpha)$$

$$\lfloor DX \rfloor := \left\{ \lambda_\alpha \;\middle|\; \alpha : \mathbb{R} \rightharpoonup X \right\}$$

Lebesgue measure

$$\mathcal{R}_{DX} := \left\{ \lambda x. \lambda_{\alpha x} \;\middle|\; \alpha : \mathbb{R} \times \mathbb{R} \rightharpoonup X \right\}$$

$$\delta : X \to DX \qquad \oint : D\Gamma \times (DX)^{\Gamma} \longrightarrow DX \quad \text{lift along } D \rightarrowtail G.$$

D validates our measure axioms including Fubini-Tonelli

$\mu \in DX, \; \nu \in DY \vdash$

$$\oint \mu(dx) \oint \nu(dy)\, \delta_{(x,y)} = \oint \nu(dy) \oint \mu(dx)\, \delta_{(x,y)} =: \mu \otimes \nu$$

Thm: For sbs S, PS, $D_{\leq 1}S$, $D_{<\infty}S \in \mathbf{Sbs}$

and agree with their counterparts on **Meas**.

$\lfloor DS \rfloor = \{\mu \mid \mu \text{ s-finite}\}$ See [Staton'16]

$R_{DS} = \{ K : \mathbb{R} \to GD \mid K \text{ s-finite kernel} \}$

Open: Is there a counterpart to D in **Meas**?
More modestly, is $DS \in \mathbf{Sbs}$?
(Hypothesis: no)

Distribution Submonads

A measure space $\Omega = (\Omega, \mu)$

is a qbs Ω with

$\mu \in DX$.

Similarly: - finite measure space

- (sub) probability space.

$$\begin{tikzcd}
PX := \{\mu \in DX \mid \mu X = 1\} \arrow[d, hook] \\
P_{\leq 1} X := \{\mu \in DX \mid \mu X \leq 1\} \arrow[d, hook] \\
P_{<\infty} X := \{\mu \in DX \mid \mu X < \infty\} \arrow[d, hook] \\
DX
\end{tikzcd}$$

Full model

type : Qbs $\quad W := [0,\infty] \quad BX \cong \mathbb{B}^X$

$$DX := \left(\{\lambda_\alpha \mid \alpha : \mathbb{R} \to X\},\ \{\lambda r.\lambda_{\alpha(r,-)} \mid \alpha : \mathbb{R}\times\mathbb{R} \to X\} \right)$$

$$PX := \{\mu \in DX \mid \underset{\mu}{Ce}[X] = 1\}$$

$$\underset{\mu}{Ce}[E] := \mu E \qquad \delta_x := E \mapsto \begin{cases} x \in E : 1 \\ x \notin E : 0 \end{cases}$$

$$\oint \mu k := \lambda E. \int \mu(dx) k(x;E)$$

Plan:

1) type-driven probability: discrete case ✓
2) Borel sets & measurable spaces ✓
3) Quasi Borel spaces ✓

Lecture 1

4) Type structure & standard Borel spaces ✓
5) Integration ✓ & random variables

Lecture 2

Please ask questions!

Course web page

scribble

Random variable: $\xi : \Omega \longrightarrow \Theta \hookrightarrow \overline{\mathbb{R}}$

Θ:

$$\begin{tikzcd}
 & & & \mathbb{W}\ [0,\infty] \arrow[dr] & \\
\mathbb{I}\ [0,1] \arrow[rr] & & \mathbb{R}_{\geq 0}\ [0,\infty) \arrow[ur] \arrow[dr] & & \overline{\mathbb{R}}\ [-\infty,\infty] \\
 & \mathbb{R}^{+}\ (0,\infty) \arrow[ur] & & \mathbb{R}\ (-\infty,\infty) \arrow[ur] &
\end{tikzcd}$$

- Θ^{Ω} is a <u>space</u>
- $\mathbb{R}^{\Omega}$ measurable vector space:

$$\alpha \xi + \zeta := \lambda \omega.\ \alpha \cdot \xi \omega + \zeta \omega$$

- $\mathbb{W}^{\Omega}$ measurable σ-semi-module for $\mathbb{W}$:

$$\sum_{n=0}^{\infty} \alpha_n \xi_n := \lambda \omega.\ \sum_{n=0}^{\infty} \alpha_n \cdot \xi_n \omega$$

$$\Pr_{-} : P\Omega \times B_\Omega \to W$$

$$\Pr_\lambda A := \mathrm{eval}(\lambda, A) = \lambda A$$

Probability Space $\Omega = (\Omega, \lambda_\Omega)$

$p : P\Omega \vdash$ "$p\,x$ holds $\lambda(dx)$-almost surely"

for some $Q \xrightarrow{\varepsilon} \Omega$, $p \supseteq Q$, $[-\in Q]\cdot\lambda = \lambda$

<u>Example</u> $(\xi, \zeta \in \boxed{H}^{\Omega})$

$\xi = \zeta$ a.s. when $\Pr_{\omega \sim \lambda}[\xi\omega \neq \zeta\omega] = 0$

Integrating random variables (as discretely) in Qbs!

$$(-)_+, (-)_- : \overline{\mathbb{R}}^\Omega \longrightarrow W^\Omega$$

$$\xi_+ := \max(\xi, 0) \qquad \xi_- := \max(-\xi, 0)$$

So: $$\xi = \xi_+ - \xi_-$$

$$\int : P\Omega \times W^\Omega \longrightarrow W$$

$$\int \lambda\, \xi := \int \lambda\, \xi_+ - \int \lambda\, \xi_-$$

$\int$ respects a.s. equality:

$\xi = \zeta$ (λ-a.s.)

$\Rightarrow \int \lambda\, \xi = \int \zeta$.

Example

$$\lambda : P\Omega \vdash \mathrm{ASConverge}(\overline{\mathbb{R}})^{\Omega} : \mathcal{B}\left(\overline{\mathbb{R}}^{\mathbb{N}\times\Omega}\right)$$

$$:= \left\{ \vec{\xi} \in \overline{\mathbb{R}}^{\mathbb{N}\times\Omega} \;\middle|\; \Pr_{\omega\sim\lambda}\left[\lim \xi_n \omega \neq \bot\right] \right\}$$

So:

$$\lim^{as} : \overline{\mathbb{R}}^{\mathbb{N}\times\Omega} \longrightarrow \overline{\mathbb{R}}^{\Omega} \qquad \mathrm{Dom}\, \lim^{as} := \mathrm{ASConverge}(\overline{\mathbb{R}})^{\Omega}$$

$$\lim^{as} \vec{\xi} := \lambda\omega.\, \limsup_{n\to\infty} f_n\, \omega$$

$\llcorner$ $\lim^{as}$ respects a.s. equality

Thm (monotone convergence):

Let $\vec{\xi} \in W^{\mathbb{N} \times \Omega}$ λ-a.s. monotone.

$$\xi = \lim^{as}_{n \to \infty} \xi_n \qquad \text{(a.s.)}$$

$$\Downarrow$$

$$\int \lambda\, \xi = \lim_{n \to \infty} \int \lambda\, \xi_n$$

Lebesgue Space

(Ω Prob. Space, $p \in [1,\infty)$)

$$p : [1,\infty), \lambda : P\Omega \vdash \mathcal{L}^p_{(\Omega,\lambda)} : B(\mathbb{R}^\Omega)$$

$$:= \left\{ \xi \in \mathbb{R}^\Omega \,\middle|\, \int \lambda |\xi|^p < \infty \right\} \hookrightarrow \mathbb{R}^\Omega$$

Ensemble $\mathcal{L}_\Omega := \prod_{\substack{\lambda \in P\Omega \\ p \in [1,\infty)}} \mathcal{L}^p_{(\Omega,\lambda)}$

$\llcorner$ $p \leq q \Rightarrow \mathcal{L}^p_\Omega \supseteq \mathcal{L}^q_\Omega$

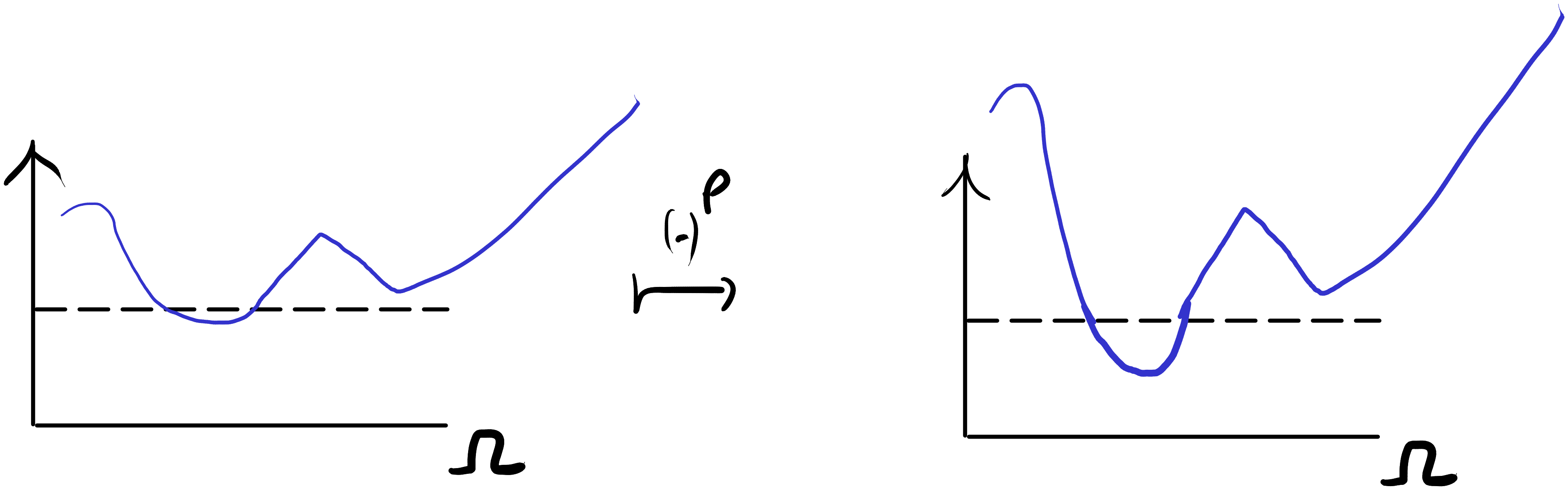
$\mathcal{L}^p$ semi norms

$$\|-\| : \prod_{p,\lambda} \mathcal{L}^p_{(\Omega,\lambda)} \to \mathbb{R}_{\geq 0} \qquad \|\xi\|_p := \sqrt[p]{\int \lambda |\xi|^p}$$

$\mathcal{L}^2$ inner product

$$\langle -, - \rangle : \prod_{p,\lambda} \mathcal{L}^p_{(\Omega,\lambda)} \times \mathcal{L}^p_{(\Omega,\lambda)} \to \mathbb{R}$$

$$\langle \xi, \zeta \rangle_{p,\lambda} := \int \lambda\, \xi \zeta$$



Statistics

Expectation

$$\mathbb{E} : \coprod_\lambda \mathcal{L}^1 \longrightarrow \mathbb{R}$$

$$\mathbb{E}_\lambda \xi := \int \lambda \xi$$

Covariance and Correlation

$$\mathrm{Cov}, \mathrm{Corr} : \coprod_\lambda \mathcal{L}^2 \to \mathbb{R}$$

$$\mathrm{Cov}(\xi, \zeta) := \langle \xi - \mathbb{E}\xi, \zeta - \mathbb{E}\zeta \rangle$$

$$\mathrm{Corr}(\xi, \zeta) := \frac{\langle \xi, \zeta \rangle}{\|\xi\|_2 \cdot \|\zeta\|_2} = \cos(\mathrm{angle}(\xi, \zeta))$$

Sequential limits

$p : [1,\infty), \lambda : PX_+$ Cauchy $\mathcal{L}^p_{(\Omega,\lambda)} : \mathbb{B}(\mathcal{L}^p_{(\mathbb{R},\lambda)})^{\mathbb{N}}$

$$:= \left\{ \vec{\xi} \;\middle|\; \forall \varepsilon \in \mathbb{Q}^+ \, \exists k \in \mathbb{N} \, \forall m, n \geq k. \; \| \xi_{k+n} - \xi_{k+m} \|_p < \varepsilon \right\}$$

Thm: $\mathcal{L}^p_\Omega$ is Cauchy-Complete

$$\lim^p : \text{Cauchy}\,\mathcal{L}^p \longrightarrow \mathcal{L}^p$$ (convergence in mean)

Why?

1. Every Cauchy Sequence has an a.s. converging subseq.
2. We can find it measurably

Example

Thm (dominated convergence)

For $\vec{Z}, Z \in \mathcal{L}^1$ s.t. $Z_n \leq Z$ a.s.:

1. $\overset{as}{\lim} \vec{Z} \in \mathcal{L}^1$

2. $\overset{1}{\lim} \vec{Z} = \overset{as}{\lim} \vec{Z}$

3. $\lim_{n\to\infty} \int \lambda Z_n = \int \lambda \lim_{n\to\infty} Z_n$

Separability

Def: $\mathcal{L}^p$ separable: has countable dense subset

Fact: Separability is property of λ_Ω:

TFAE:

- $\exists p \geq 1.\ \mathcal{L}^p$ separable
- $\forall p \geq 1.\ \mathcal{L}^p$ separable

Measurable separability in $\mathcal{I} \hookrightarrow P\Omega \times [1, \infty)$

$$\vec{\beta} : \prod_{(\lambda, p) \in \mathcal{I}} \mathcal{L}^{p}_{(\Omega,\lambda)}{}^{\mathbb{N}} \quad \text{s.t.}$$

$$\left\{ \vec{\beta}^{\lambda,p}_{n} \middle| n \in \mathbb{N} \right\} \text{ dense in } \mathcal{L}^{p}_{(\Omega,\lambda)}$$

Prop. - every sbs S measurably separable in

$$PS \times [1, \infty)$$

- $\mathcal{I} \hookrightarrow P\Omega \times \{2\}$ measurably separable

$$\Rightarrow \quad \exists \; \vec{\beta} \in \prod_{\lambda \in \mathcal{I}} \mathcal{L}^{2}_{(\Omega,\lambda)} \quad \text{orthonormal system}$$

$\langle \beta_n , \beta_m \rangle = 0$

$\| \beta_n \|_2 = 1$

(β_n) dense

Example

Let $S \hookrightarrow \mathcal{L}^2$ closed vector subspace.

orthogonal decomposition linear in fact.

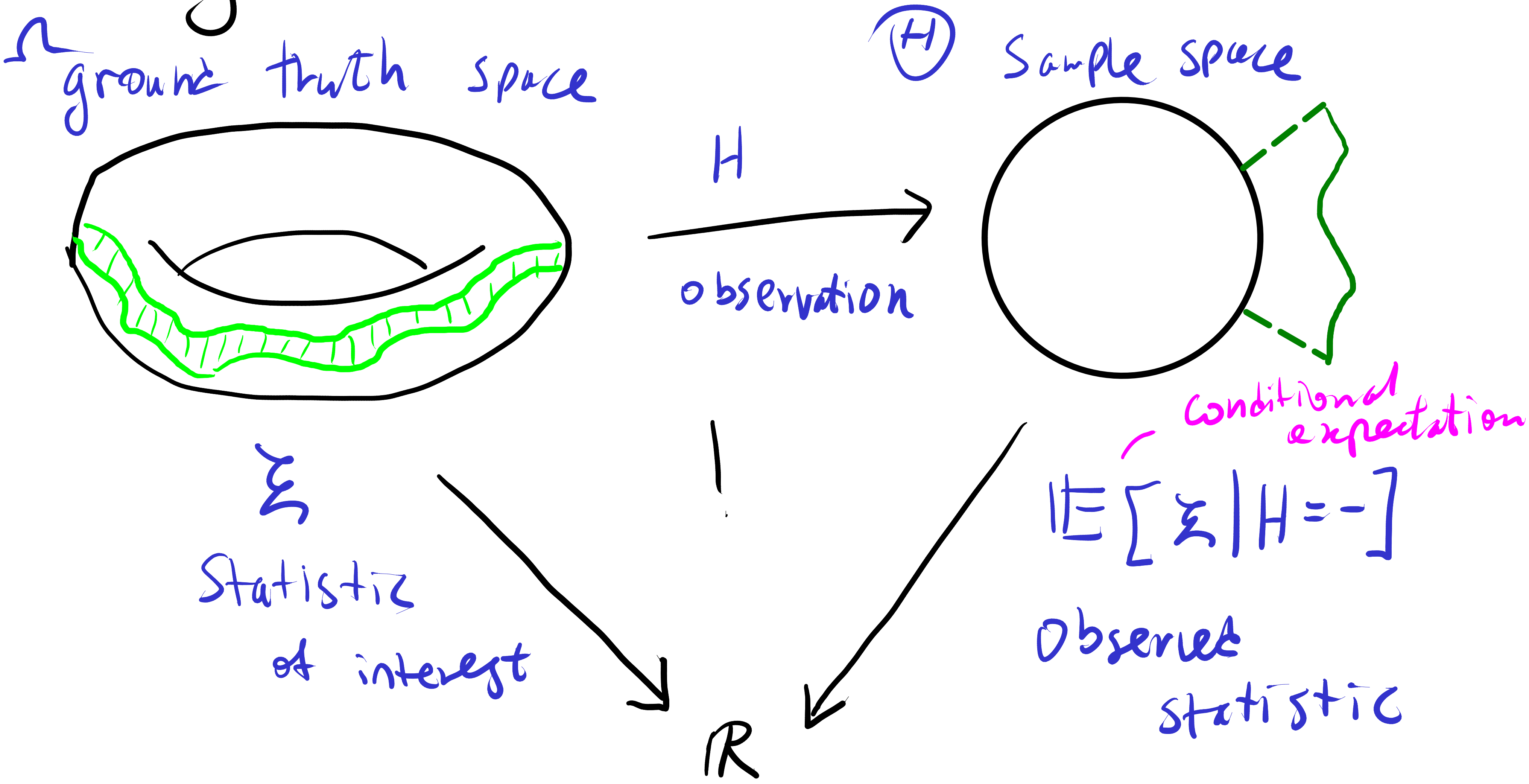
$$\langle P, P^{\perp} \rangle : \lfloor \mathcal{L}^2 \rfloor \xrightarrow{P} \lfloor S \rfloor \times \lfloor S^{\perp} \rfloor$$

When S is separable with orthonormal system β

We have a measurable version of
$\langle P, P^{\perp} \rangle : \mathcal{L}^2 \to S \times S^{\perp}$

$$P\xi := \sum_{n=0}^{\infty} \langle \xi, \beta_n \rangle \beta_n \qquad P^{\perp} := \mathrm{Id} - P.$$

Kolmogorov's Conditional Expectation



Kolmogorov's Conditional Expectation

A conditional expectation

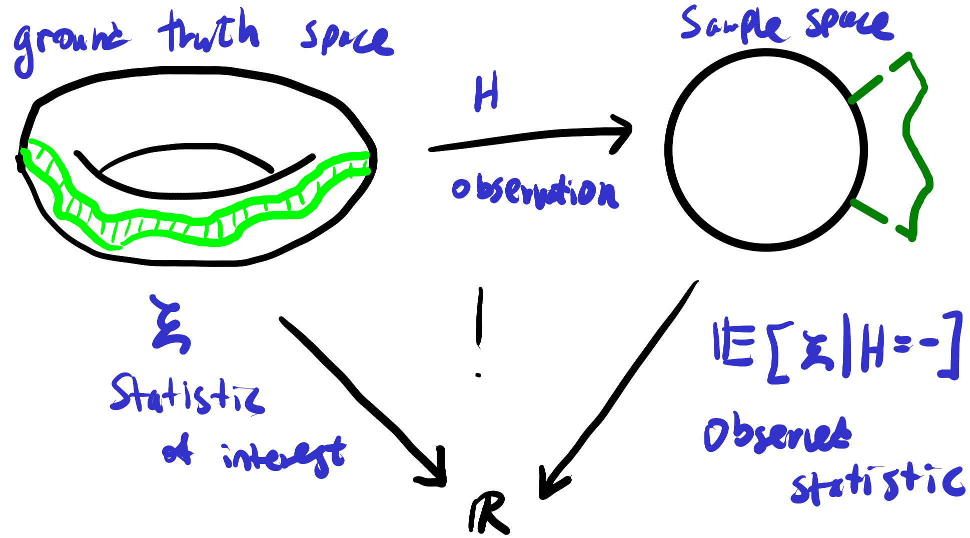
of $\xi \in \mathcal{L}^1_\Omega$ wrt

$H : \Omega \to H$ is

$Z \in \mathcal{L}^1_{H}$ s.t. for all $A \in \mathcal{B}_{H}$:

$$\int_A \mu Z = \int_{H^{-1}[A]} \lambda \xi$$

where $\mu := \lambda_H$



Conditional expectations

1. unique a.s.

2. fundamental to modern Probability, e.g.:

   a martingale

$$
\begin{tikzcd}
\cdots & \Theta_n \arrow[l] \arrow[dr, "\Xi_n"'] & & \Theta_{n+1} \arrow[ll, "H_n"'] \arrow[dl, "\Xi_{n+1}"] & \cdots \arrow[l] \\
& & \mathbb{R} \arrow[u, phantom, "!"] & &
\end{tikzcd}
$$

s.t. $\Xi_n = \mathbb{E}[\Xi_{n+1} \mid H_n = -]$

Thm (Existence)

- $\exists \quad \mathbb{E}[- \mid H = -] : \lfloor \mathcal{L}^1_{(\Omega,\lambda)} \rfloor \longrightarrow \lfloor \mathcal{L}^1_{(\Theta,\mu)} \rfloor$

- When (Ω,λ) is Separable

$$\mathbb{E}[- \mid H = -] : \mathcal{L}^1_{(\Omega,\lambda)} \longrightarrow \mathcal{L}^1_{(\Theta,\mu)}$$

- When Θ is $\mathcal{I}'$-measurably separable

$$\mathbb{E}[- \mid - = -] : \prod_{\substack{H \in \Theta^\Omega \\ \lambda \in H_*^{-1}[\mathcal{I}]}} \mathcal{L}^1_{(\Omega,\lambda)} \longrightarrow \mathcal{L}^1_{(\Theta,\mu)}$$

Ohad Kammar <ohad.kammar@ed.ac.uk>

Plan:

1) type-driven probability: discrete case
2) Borel sets & measurable spaces
3) Quasi Borel spaces

Lecture 1

4) Type structure & standard Borel spaces
5) Integration & random variables

Lecture 2

Please ask questions!

Course web page

scribble

Discrete model

type : Set $\qquad W := [0,\infty] \qquad \mathcal{B}X := \mathcal{P}X$

$$DX := \{\mu : X \to W \mid \operatorname{supp}\mu \text{ countable}\}$$

$$\mathcal{P}X := \{\mu \in DX \mid Ce_{\mu}[X] = 1\}$$

$$Ce_{\mu}[E] := \sum_{x \in E} \mu x \qquad \delta_x := \lambda x'. \begin{cases} x = x' : & 0 \\ x \neq x' : & 1 \end{cases}$$

$$\oint \mu k := \lambda x. \sum_{r \in \Gamma} \mu r \cdot k(r; x)$$

Full model

type : Qbs $\quad \mathbb{W} := [0,\infty] \quad \mathcal{B}X \cong \mathbb{B}^X$

$$DX := \left(\{\lambda_\alpha \mid \alpha : \mathbb{R} \to X\}, \{\lambda r.\lambda_{\alpha(r,-)} \mid \alpha : \mathbb{R}\times\mathbb{R} \to X\} \right)$$

$$PX := \{\mu \in DX \mid \underset{\mu}{\text{Ce}}[X] = 1\}$$

$$\underset{\mu}{\text{Ce}}[E] := \mu E \qquad \delta_x := E \mapsto \begin{cases} x \in E : 1 \\ x \notin E : 0 \end{cases}$$

$$\oint \mu k := \lambda E. \int \mu(dx) k(x; E)$$

Advertisement

Interested in the computational / mathematical / logical foundations of Computer science?

Check out our PhD Programs:

THE UNIVERSITY of EDINBURGH
informatics

lfcs | Laboratory for Foundations of Computer Science

LFCS PhD

Dependable AI for Robotics CDT

Enough!

Lunch.