

Semantic foundations for type-driven probabilistic modelling

Ohad Kammar
University of Edinburgh

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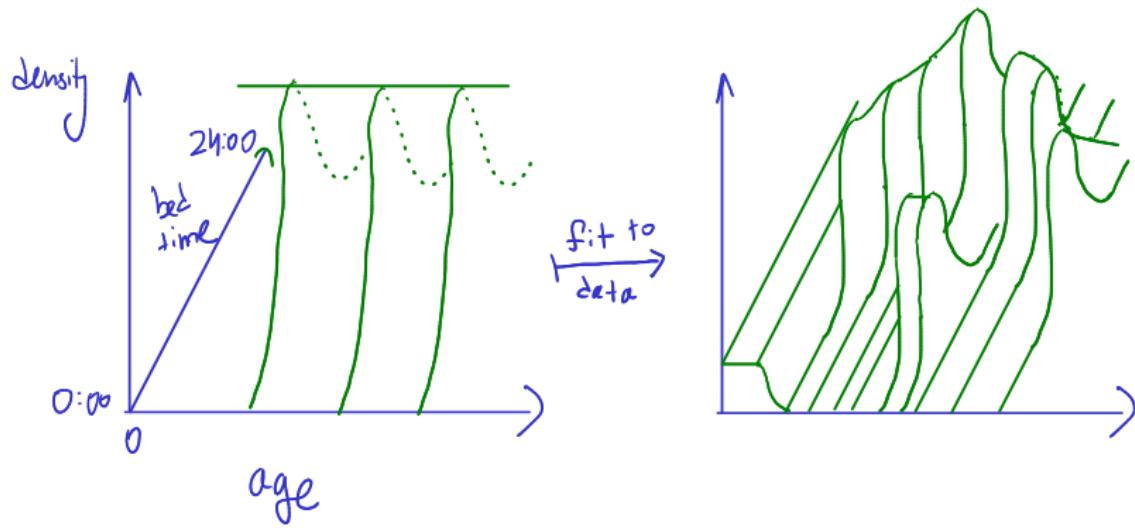


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Statistical & Probabilistic Modelling



ILLUSTRATIVE PURPOSES ONLY

Prob. Modelling as Programming

distributions describe Programs

↓ [Saheb-Djahromi '77
Kozen '78]

Programs describe distributions

Prob. Modelling as Programming

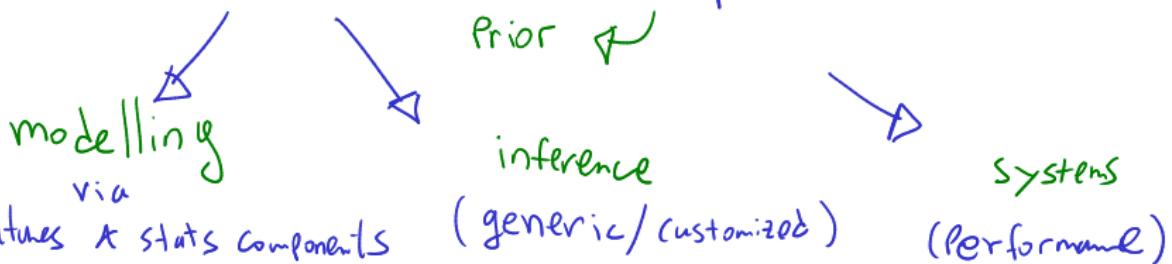
[Saheb-Djahromi '77
Kozen '78]

Programs describe distributions

Sophisticated samplers:

STLC + Sample + { datatypes
recursion
Polymorphism
... }

Sophisticated Bayesian models: Sample + observe



But Today:

Semantic foundations

Desiderata : Types

Discrete Spaces

finite spaces

Bool

[n]

success/failure
(Bernoulli)
coin tosses

week day
dice roll

infinite spaces

naturals

\mathbb{N}

integers

\mathbb{Z}

rationals

\mathbb{Q}

count

Priority

Desiderata : Types

Continuous Spaces

\mathbb{R}	$[a, b]$	$[0, \infty]$	$[0, 1]$
Position	hour	Scaling factors	Probabilities

Discrete +

Desiderata : Types

Space Combinators

Products

$$\mathbb{R} \times \mathbb{R}$$

correlated
outcomes
(position + velocity)

Coproducts

$$\mathbb{R} \amalg [n]$$

coordinate or
location

Subspaces

$$S^1 \hookrightarrow \mathbb{R}^2$$

semantic
invariants

Quotients

$$\text{Bag } \mathbb{N}$$

Point
processes

Discrete + Continuous

Desiderata: probability / measure

Kolmogorov '33
equivalent axioms

Events $E \subseteq X$:

$$\{ d \in [6] \mid \begin{array}{l} \text{dice roll} \\ \text{more than 3: } d \geq 3 \end{array} \}$$

Measure $\mu \Rightarrow$ Events $\xrightarrow{\Pr_\mu} [0, \infty]$

$$E \mapsto \Pr_{\substack{x \sim \mu}} [x \in E]$$

measure
outcomes
where E occurs

Desiderata: probability / measure

Measure $\mu \Rightarrow$ Events $\xrightarrow{\Pr} [0, \infty]$

$$\Pr[\emptyset] = 0 \quad \Pr[E] = \Pr[E \cap F] + \Pr[E \cap F^c]$$

totality

disjoint additivity

$$\Pr\left[\bigcup_n E_n\right] = \sup_n \Pr[E_n] \quad (E_n \subseteq E_{n+1})_n$$

(soft) continuity

$$\Pr[\text{whole space}] = 1$$

Probability

No-go #1.

Thm (Vitali, 1905) Assuming Axiom of Choice:

$\nexists \Pr_{\lambda} : \text{PIR} \rightarrow [0, \infty]$ measure

$$\Pr_r [[E] + a] = \Pr_r [E]$$

translation

invariant

$$\Pr_r [[a, b]] = b - a$$

measures

length

Boolean subsets $B_{\mathbb{R}} \subseteq P(\mathbb{R})$

- Boolean Subalgebra w.r.t. $\cap, \cup, \emptyset, \in^c$
- ω -Chain closed:

$$\forall (E_n \subseteq E_{n+1})_n \in B_{\mathbb{R}}^{\omega}. \bigcup_n E_n \in B_{\mathbb{R}}$$

- Every interval: $(a, b) \in B_{\mathbb{R}}$

Thm: (Lebesgue 1902)

\exists $\Pr_x : \mathcal{B}_{\mathbb{R}} \rightarrow [0, \infty]$ measure
translation + measures
invariant length

Lebesgue Measure

Measurable Space $A = (\underline{A}, \mathcal{B}_A)$

Events $\mathcal{B}_A \subseteq P_A$ ↑ set of points

- g - Boolean Subalgebra w.r.t. $\cap, \cup, \emptyset, (\cdot)^c$
- field
- ω -chain closed:

$$\forall (E_n \subseteq E_{n+1})_n \in \mathcal{B}_{\mathbb{R}}^{\omega} . \quad \bigcup_n E_n \in \mathcal{B}_{\mathbb{R}}$$

measurable: $A \xrightarrow{f} B$

$$f^{-1}[E] \in \mathcal{B}_A \Leftarrow E \in \mathcal{B}_B$$

classical theory [Kolmogorov 1933]

probability space

Ω measurable space $\mu: \mathcal{B}_\Omega \rightarrow [0,1]$ probability measure

Modelling Foundation: Meas

discrete $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$

+ their measures

continuous $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

Combinators:

products, subspaces

(limit
structure)

coproducts quotients

(colimit
structure)

No-Go as PL Semantics:

Then [Aumann'61] Let $\mathbb{R}^{\mathbb{R}} := \text{Meas}(\mathbb{R}, \mathbb{R})$

$\nexists B_0 \subseteq \mathcal{P}(\mathbb{R}^{\mathbb{R}})$ σ -field

eval: $(\mathbb{R}^{\mathbb{R}}, B_0) \times \mathbb{R} \xrightarrow{(f, x) \mapsto fx} \mathbb{R}$ measurable

Consequences

\Rightarrow no-go for event space $B_{(B_R)}$

\Rightarrow no-go for Π -types

\Rightarrow no-go for:

- Functional programming
- OO Programming ...

Semantic Tradition: Probabilistic Powerdomains

[Plotkin '82, Graham '87, Jones-Plotkin '89, Tix , Jung 97
Tix 98 , Mislove '00+..., Heimel '03,
J. Goubault-Larrecq '07 +... + apologies]

Recent developments

- Boolean-valued models [Bacci et al.'18]
- Probabilistic coherence spaces with cones [Ehrhard et al.'18]
- Measurable event structures [Paget-Winskel '18] [de Amorim et al.'22]
- Banach spaces with structure [Dølqvist, Kozen '20]
- Interval domains [Jia et al.'21a+b, Di Giandomenico, Edoalot '24]
- Random probability Sheaf Toposes [Simpson]

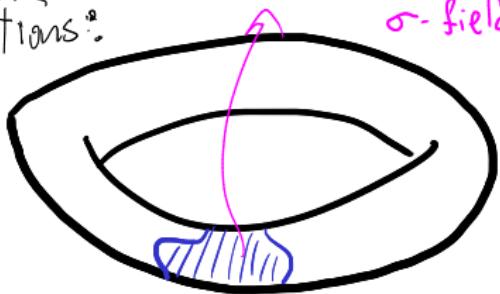
Quasi-Borel Spaces [Staton et al. '17] cf. [Farré '21]

Measure Theory

Primitive notions:

event $E \in \mathcal{B}_A$

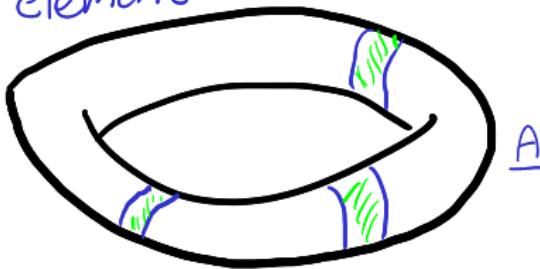
σ -field



Obs Theory

Sample space Ω

random element $\alpha \in \mathcal{E}_A$



Derived

: random element

$$(\alpha : \Omega \rightarrow A) \in \mathcal{R}_A$$

metaphorology

event

$$E \in \mathcal{B}_A$$

Weak de Finetti Thm

Applications

Verify Bayesian inference implementations

Design modelling languages & inference systems
+ implementations

Probabilistic Networking PL

Formalise

Semantics for expected cost analysis

Also:

Semantics for name generation

Weak de Finetti Thm [Staton et al.'17]

Applications

Verify Bayesian inference implementations [Ścibior et al.'18]

Design modelling languages & inference systems

+ implementations: MonakBayes [Ścibior et al.'18, now Tweag.Io]

Lazy PPL [Staton et al. '23] (parts of) Gen [Lew et al.'20]

Domain Theory [Väkär et al.'19]

Probabilistic Networking PL [Vandenbroucke, Schrijvers'20]

Formalise [Hirata et al.'19 + '23]

Semantics for expected cost analysis [de Amorim '24]

Alo:

Semantics for name generation [Sabok et al. '21]

Rest of talk: mini-tutorial via types

- 1) Metaphorologies & Qbs
- 2) Simple-type Structure
- 3) Standard SPACES



Course

PhD
Programs



CDT:
Dependable AI



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Theme: Working with quasi Borel spaces:

- 1) types for abstracting over
measurability
- 2) types for organising probabilistic
concepts

Not today:

- I) colimit structure A Probability spaces
- II) dependently-typed structure
- III) random variable spaces A stochastic
Processes



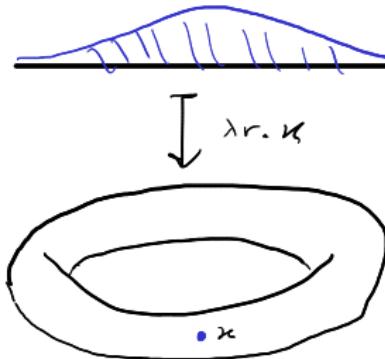
See
course

Def: Metaphorology over X Set
 \mathcal{R} "random elements"
 $\mathcal{R}^R \subseteq X^R$ "points"

$\mathcal{R} \subseteq X^R \models 3$ closure axioms:

- Constants:

$$\underline{x} := (\lambda r. n) \in \mathcal{R}$$



- precomposition:

- recombination

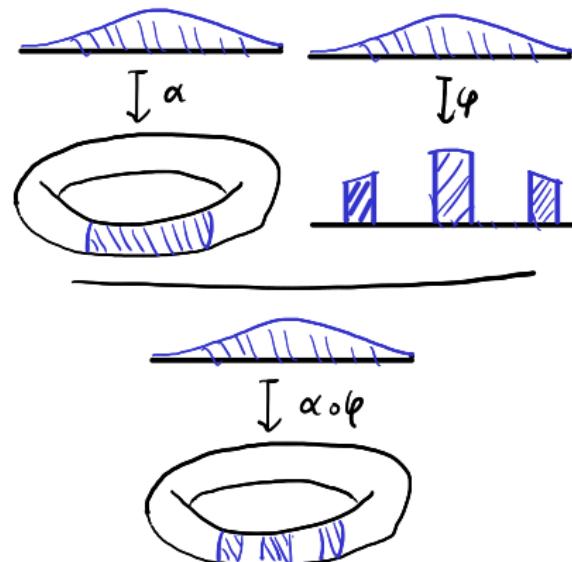
Ref: Metaphorology over X set
 \mathcal{R} "points"

$\mathcal{R} \subseteq X^{\mathbb{R}}$ \models 3 Closure axioms:

- precomposition:

$\alpha \in R_X$ $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ Borel measurable

$\varphi \circ \alpha: \mathbb{R} \xrightarrow{\varphi} \mathbb{R} \xrightarrow{\alpha} X \in \mathcal{R}$



Ref: Metaphorology over X Set

→ "random elements"

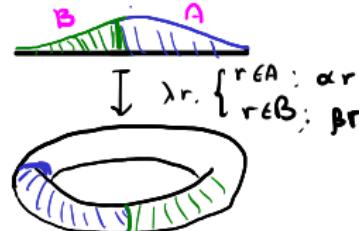
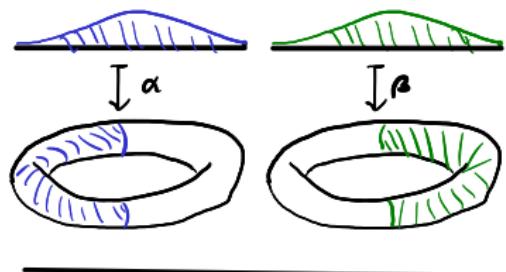
R "points"

$R \subseteq X^R$ F 3 closure axioms:

- recombination

$$\vec{\alpha} \in R_x^N \quad \vec{E} \in B_m^N \quad R = \bigcup_{n=0}^{\infty} E_n$$

$$[E_n, \alpha_n]_n := \lambda r. \left\{ \begin{array}{l} : \\ r \in \mathbb{N}; \alpha_n r \in R \end{array} \right.$$

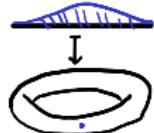


Ref: Quasi-Baerl space $X = (LX, R_x)$

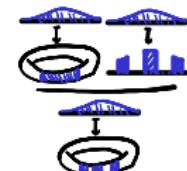
$$R_x \subseteq L^{(R_x)}_x$$

Closed under:

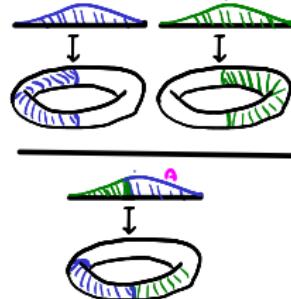
- Constants:



- precomposition:



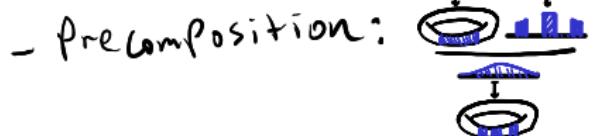
- recombination



Ref: Quasi-Borel space $X = (LX, \mathcal{R}_X)$

$$\mathcal{R}_X \subseteq L^{(R_X)}_{X^I}$$

Closed under:

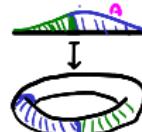
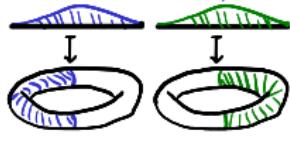


Topos Theorists:

Concrete sheaves over

(Sbs, Countable covers)

- recombination



Examples

- $\mathbb{R} = (\mathbb{R}, \text{Meas}(\mathbb{R}, \mathbb{R}))$ qbs underlying \mathbb{R}
- Generalise:

$$\frac{\text{A Meas Space}}{\text{A}_{\text{qbs}} := (\text{A}, \text{Meas}(\mathbb{R}, \text{A})) \text{ qbs}}$$

Examples

recombination of constants

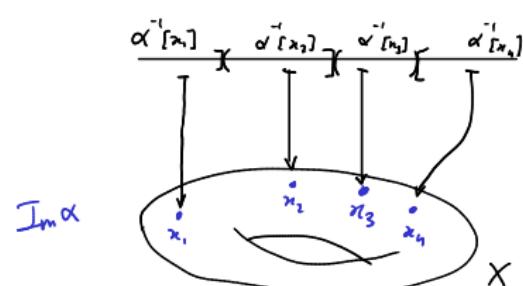
Def: $\alpha : \mathbb{R} \rightarrow X$ σ -Simple: $\alpha = [E_n, x_n]_{n \in I}$

$$\begin{array}{c} I \hookrightarrow \mathbb{N} \quad E \in \mathcal{B}^I \\ \xrightarrow{\quad} x \in X^I \quad \mathbb{R} = \bigcup_n E_n \end{array}$$

discrete qbs on X

X set

$\mathbb{X}^{\text{qbs}} := (X, \sigma\text{-simple}(\mathbb{R}, X))$ qbs



Examples

Indiscrete qbs on \mathcal{X}

\mathcal{X} set

$$\mathcal{X}_{\text{Qbs}} := \left(\mathcal{X}, \mathcal{X}^{L(\mathcal{R})} \right)$$

\hookrightarrow all functions

Def: Obs morphism $f: X \rightarrow Y$

- Function $f: X \rightarrow Y$

$$-\forall \underset{x}{\overset{R}{\alpha}} \in R_X . \quad \underset{\begin{array}{c} R \\ \alpha \\ \downarrow \\ x \\ f \\ \downarrow \\ y \end{array}}{\alpha} \in R_Y$$

Example

- Constant Functions :

Since:

f constant $\Rightarrow f \propto$ constant

Example Extending Meas

A, B Meas spaces $f: A \rightarrow B$ in Meas

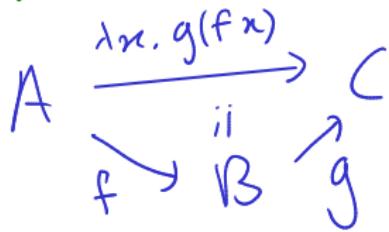
$$\begin{matrix} A \\ \hookrightarrow_{\text{abs}} \end{matrix} \xrightarrow{f} \begin{matrix} B \\ \hookrightarrow_{\text{abs}} \end{matrix}$$

Example Category Obs

Identities:

$$i_b_A := (\lambda x.x) : A \rightarrow A$$

Composition:



Meas

$$\downarrow L_{\text{obs}}$$

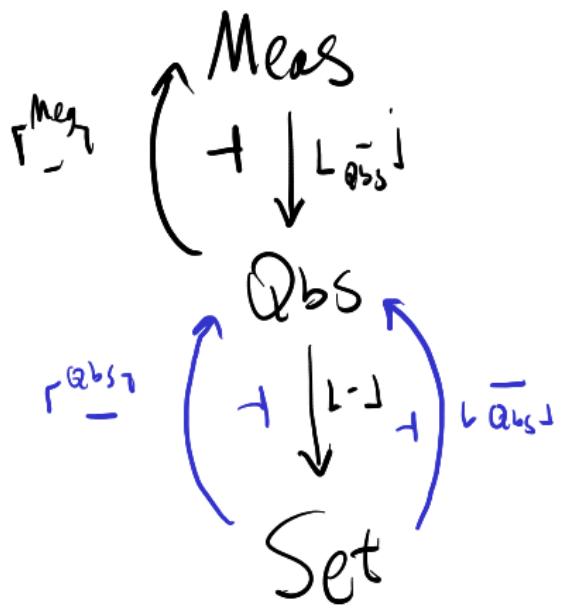
Obs

$$\downarrow \dashv$$

Set

generates limits & colimits

↳ (lifts +
Preserves)



N.B. for C.B.: Not cohesive

Rest of talk : mini-tutorial via types

1) Metaphorologies & Qbs

2) Simple-type Structure

3) Standard Spaces



Course

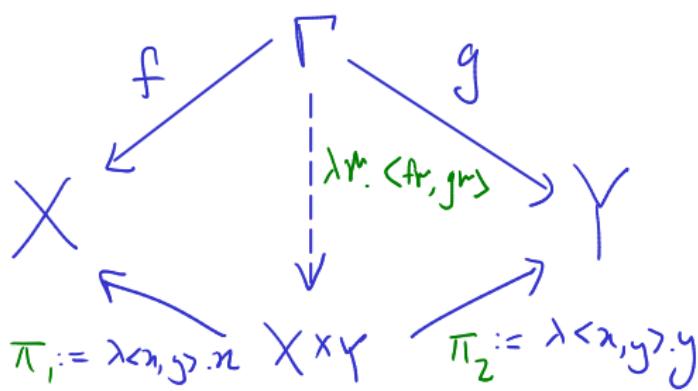
Product $(X \times Y, \pi_1, \pi_2)$:

necessarily!

$$- L_{X \times Y} = L_{X_1 \times_1 Y_1}$$

$$- R_{X \times Y} = \left\{ \lambda r. (\alpha r, \beta r) \mid \alpha \in R_X, \beta \in R_Y \right\}$$

correlated
random
elements



Ex

$$\mathbb{R}^n, \mathbb{R}^N, N^N$$

$$(+), (\cdot): \mathbb{R}^2 \rightarrow \mathbb{R}$$

inf, sup

$$\liminf, \limsup: \overline{\mathbb{R}}^n \rightarrow \overline{\mathbb{R}}$$

\mathbb{Q}_{bs} \vdash 1st order simple-type theory (like meas)

$$\frac{(x:A) \in \Gamma}{\Gamma \vdash x:A} \text{ var}$$

$$\frac{\Gamma \vdash M : A \quad f : A \rightarrow B \text{ in } \mathbb{Q}_{\text{bs}}}{\Gamma \vdash \exists f \ V \ M : B} \text{ reflect}$$

$$\frac{\Gamma \vdash M_i : A_i}{\Gamma \vdash \langle M_1, \dots, M_n \rangle : A_1 \times \dots \times A_n} \text{ type}$$

$$\frac{\Gamma \vdash M : A_1 \times \dots \times A_n \quad \Gamma, (x_i : A_i) \vdash K : B}{\Gamma \vdash \text{let}(x_1, \dots, x_n) = M \text{ in } k : B} \text{ let}$$

Smooth internalisation/externalisation

Function Spaces

Straightforward!

$$- \mathbb{Y}^X := \text{Obs}(X, \mathbb{P})$$

$$- \mathbb{R}_{Y^X} := \text{curry}[\text{Obs}(\mathbb{R}^{X \times}, \mathbb{P})]$$

$$= \left\{ \alpha: \mathbb{R} \rightarrow \mathbb{Y}^X \mid \lambda(r, x), \alpha \circ x: \mathbb{R} \times X \rightarrow \mathbb{P} \right\}$$

$$- \text{eval}: \mathbb{Y}^X \times X \rightarrow \mathbb{P}$$

$$\text{eval}(f, x) := fx$$

Simple type theory

$\mathbb{Q}bs \models STLC$

$$\frac{\Gamma, \alpha : A + M : B}{\Gamma \vdash \lambda x : A. M : B} A$$

$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash P : A}{\Gamma \vdash M P : B}$$

Eg.

$$\limsup : \overline{\mathbb{R}}^{N \times \mathbb{R}} \rightarrow \overline{\mathbb{R}}^{\mathbb{R}}$$

$$\limsup \vec{f} := \lambda r. \limsup_{n \rightarrow \infty} f_n r$$

Random element Space

$R_X := X^R$ since $\lfloor X^R \rfloor = R_X$ as sets.

Random element Space

$R_X := X^{\mathbb{R}}$ since $\lfloor X^{\mathbb{R}} \rfloor = R_X$ as sets.

Why?

(\subseteq) $\alpha \in \lfloor X \rfloor^{\mathbb{R}} \Rightarrow \alpha: \mathbb{R} \rightarrow X$ in Qbs.

$\text{id}_{\mathbb{R}}: \mathbb{R} \rightarrow \mathbb{R}$ measurable $\Rightarrow \text{id} \in R_{\mathbb{R}}$

$\Rightarrow \alpha = \alpha \circ \text{id} \in R_X$

Pre composition

(\supseteq) $\alpha \in R_X \Rightarrow \exists \psi \in R_{\mathbb{R}} = \text{Meas}(\mathbb{R}, \mathbb{R})$. $\alpha \circ \psi \in R_X \Rightarrow \alpha: \mathbb{R} \rightarrow X$
 $\Rightarrow \alpha \in \lfloor X \rfloor^{\mathbb{R}}$

Random element Space

Ex: Riemann-Stieltjes integrals + Lebesgue-Stieltjes measures

$$\underline{R}X := \underline{R}_X$$

$$\int : \underline{R}X \times [0, \infty]^X \rightarrow [0, \infty]$$

$$\begin{matrix} \nwarrow \\ S \end{matrix} X \rightarrow \underline{R}X$$

$$\int d\alpha \varphi := \int dN(\varphi \circ \alpha) \quad \hookrightarrow R \rightarrow [0, \infty]$$

$$\begin{matrix} \searrow \\ S_x := x \end{matrix}$$

$$\begin{matrix} \nwarrow \\ f \end{matrix} : \underline{R}X \times (\underline{R}Y)^X \rightarrow \underline{R}Y$$

$$\begin{matrix} \downarrow \\ \int d\alpha f \end{matrix} \quad \begin{matrix} \nearrow \\ R \xrightarrow{\cong} R \times R \xrightarrow{\alpha \times id} X \times R \end{matrix} \quad \begin{matrix} \downarrow \\ Y \xleftarrow{\text{eval}} RY \times R \end{matrix}$$

Subspaces A gbs $X \subseteq A_1$ set

$\text{const} : \{x \mid x \in X\} \hookrightarrow A$ *subspace*

$$R_{\{X\}} := \left\{ \alpha : \mathbb{R} \rightarrow X \mid \alpha \in R_A \right\} \quad \text{const } x := x$$

$$\exists x : \mathbb{D} \leq' := \left\{ \vec{x} \in \mathbb{R}^2 \mid \|x\| = 1 \right\} \hookrightarrow \mathbb{R}^2$$

2) Skorokhod Representation:

$$\text{def: } \underline{\mathbb{R}} \bar{\mathbb{R}} \xrightarrow{\sim} \left\{ f : \bar{\mathbb{R}} \rightarrow \mathbb{R} \mid f(-\infty) = 0, f(\infty) = 1, \begin{array}{l} f \text{ càdlàg, monotone} \end{array} \right\} \hookrightarrow \mathbb{R}^{\bar{\mathbb{R}}}$$

Events

$$\mathcal{B}_A := \left\{ A \subseteq A_j \mid \forall \alpha \in R_x. \alpha^{\top} [n] \in B_R \right\} \quad \sigma\text{-field}$$

$$\mathcal{B}_A \cong \text{Obs}(A, \text{Bool}) \Rightarrow \mathcal{B}_A \cong \text{Bool}^A$$

N.B. Topos theorists: Bool classifies events via p.b.

Descriptive

Set theorists: $\mathcal{B}_{(B_R)}$ are the Borel-on-Borel sets from
Cf.. [Sabour et al. '21]

$$\frac{A \text{ qbs}}{\Gamma^{\text{Meas}} := (A, B_A) \text{ Meas space}}$$

$$\Gamma^{\text{Meas}} \xrightarrow{\quad} \begin{array}{c} \text{Meas} \\ + \\ \downarrow L_{\text{qbs}} \\ \text{Qbs} \end{array}$$

Meas vs Obs

By generalities: $\sigma\text{-field}$
on $\text{Meas}(\mathbb{R}, \mathbb{R})$

$$\begin{array}{ccc} \Gamma^{\text{Meas}} & & \curvearrowright \\ \mathbb{R} & \rightarrow & \Gamma^{\text{Meas}} \\ \mathbb{R} \times \mathbb{R} & \longrightarrow & \mathbb{R} \times \mathbb{R} \xrightarrow{\quad \times \quad} \Gamma[\mathbb{R}] = \mathbb{R} \\ & & \curvearrowright \\ & & \text{No factorisation} \\ & & \text{by Aumann's} \\ & & \text{Theorem.} \end{array}$$

(So: $\Gamma[\mathbb{R} \times \mathbb{R}] \neq \Gamma[\mathbb{R}] \times \Gamma[\mathbb{R}]$)

Rest of talk: mini-tutorial via types

1) Metaphorologies & Qbs

2) Simple-type Structure

3) Standard Spaces



Course

Borel Embedding:

$$e: A \hookrightarrow B$$

When:

$$\lambda x. ex : A \xrightarrow{\cong} e[A] \hookrightarrow B \quad \text{and} \quad e[A] \in \mathcal{B}_B$$

$$\underline{E_n}: N_n || := \left\{ E \text{ Borel null-set} \right\} \hookrightarrow \mathcal{B}_{\mathbb{R}}$$

Characterised by: $\lambda E. (\int_{\mathbb{R}} \text{d}x [- \in E] = 0)$

Non-examples ~ [Sabok et al.'21]

$$-\left\{ A \in \mathcal{B}_{\mathbb{R}} \mid A \neq \emptyset \right\} \hookrightarrow \mathcal{B}_{\mathbb{R}}$$

$$-\left\{ (A_1, A_2) \in \mathcal{B}_{\mathbb{R}}^2 \mid A \subseteq B \right\} \hookrightarrow \mathcal{B}_{\mathbb{R}}^2$$

$$-\left\{ A \in \mathcal{B}_{\mathbb{R}} \mid A \text{ open} \right\} \hookrightarrow \mathcal{B}_{\mathbb{R}}$$

Standard Borel spaces

Def: A qbs S is Standard Borel

$$S \hookrightarrow \mathbb{R}$$

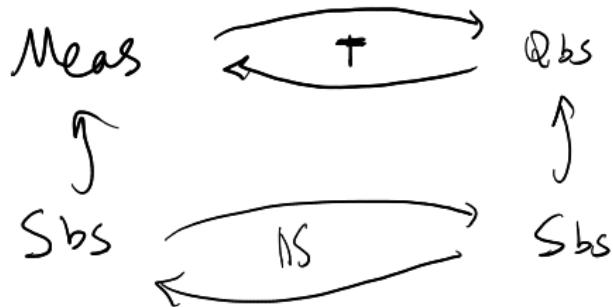
Slogan:

Standard \equiv Concrete \equiv Well behaved

Standard Borel Spaces

Def: A qbs S is Standard Borel

$$S \hookrightarrow R$$



Slogan: Qbs *Conservative extension* of Sbs

Classical measure theory

- 1) Define σ -field
- 2) Prove it standard
- 3) show relevant operations measurable

With Qbs

- 1) use type formers
→ support operations
by construction
- 2) prove it standard
- 3) characterise
events

Uniform convergence space

Space of continuous functions:

$$1) \quad C_0 := \left\{ f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ continuous} \right\} \hookrightarrow \mathbb{R}^{\mathbb{R}}$$

$$\text{eval}: C_0 \times \mathbb{R} \rightarrow \mathbb{R}$$

$$\text{eval } \langle f, x \rangle := (\text{cast } f)x$$

$$\frac{\Gamma \vdash f : \mathbb{R} \rightarrow \mathbb{R} \quad \forall m. \llbracket f \rrbracket m \text{ cts}}{\Gamma \vdash (f, \text{cts}) : C_0}$$

Uniform convergence space

2) Prove it standard:

Code $\hookrightarrow \mathbb{R}^Q$

$$\text{Code} \hookrightarrow \mathbb{R}^Q \\ := \left\{ \vec{y} \in \mathbb{R}^Q \mid \begin{array}{l} \forall a, b \in Q, \varepsilon \in \mathbb{Q}^+ \\ \exists \delta \in \mathbb{Q}^+ \forall p, q \in \mathbb{Q}^+ \cap [a, b] \\ |p - q| < \delta \Rightarrow |\vec{y}_p - \vec{y}_q| < \varepsilon \end{array} \right\}$$

idea:
fcts on $\mathbb{R} \Leftrightarrow$
f) uniformly
 $[a, b]$ cts

Uniform convergence space

2) Prove it standard:

$$C_0 \xrightarrow{\sim} C_0 \text{de}$$

$-I_Q = \lambda f. \lambda g. f g$

$\lambda \vec{g}, \lambda x.$

let $\vec{q}: Q^m$

= rational Approx x

in $\lim_{n \rightarrow \infty} y_{q_n}$

idea:
fcts on $R \Leftarrow$
f) uniformly
 $[a,b]$ cts

Characterising \mathcal{B}_A

$d: A \times A \rightarrow [0, \infty]$ metric A gbs

d compatible: $d: A \times A \rightarrow [0, \infty]$ measurable

(d) has measurable limits:

$\text{Converges}_d := \left\{ \vec{a} \text{ converges} \right\} \hookrightarrow A^N$

and $\lim: \text{Converges}_d \longrightarrow A$ measurable.

Thm: A-compatible w/ measurable limits α .

$$\alpha \text{ separable} \Rightarrow \sigma[\mathcal{O}_\alpha] = \mathcal{B}_A$$

Thm: A-compatible w/ measurable limits d.

$$d \text{ separable} \Rightarrow \sigma[\mathcal{O}_d] = \mathcal{B}_A$$

Application:

$$3) \mathcal{B}_{C_0} = \sigma(\text{Uniform topology}) = \sigma[\mathcal{U}_{d_U}] :$$

Proof:

$$d_U(f, g) := \sup_{r \in \mathbb{R}} |f_r - g_r| = \sup_{r \in \mathbb{R}} |f_r - g_r| \text{ meas.}$$

Converge_{d_U} = Cauchy_{d_U} $\Leftrightarrow \forall \epsilon > 0 \exists N \in \mathbb{N} \forall i, j \geq N d_U(f_i, f_j) < \epsilon$

$$\lim \vec{f} := \lambda x. \lim_{n \rightarrow \infty} f_n x$$

d separable via Weierstrass's Approx. Thm + Polynomials Rational Bernstein

Ditto for the Skorokhod space:

$$D[a,b] := \left\{ f : [a,b] \rightarrow \mathbb{R} \mid f \text{ càdlàg} \right\} \hookrightarrow \mathbb{R}^{[a,b]}$$

Wip: Effros quasi-Borel Space
A Polish

$$\mathcal{E}_A := \left\{ F \in \mathcal{B}_A \text{ closed} \right\} \hookrightarrow \mathcal{B}_A$$

$$\rightsquigarrow \mathcal{E}_{\ell_2} =: P_0 \mid \begin{array}{l} \text{Tarski universe} \\ \text{for Polish spaces} \end{array}$$

Concluding remarks of mini-tutorial via types



- 1) Metaphorologies & Qbs
- 2) Simple-type Structure
- 3) Standard Spaces



Talk to me:

- I) Colimit structure A Probability spaces PhD
- II) dependently-type structure
- III) random variable spaces
A stochastic .

Processes



Course