An introduction to statistical modelling semantics with higher-order measure theory

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Recap: Semantic Foundation for Statistics & Probability

finite discrete spaces

Points/states $\{H, T\}$

Events $\{H\}, \{T\}$

Probability $\{H\} \mapsto \frac{3}{4}$, $\{T\} \mapsto \frac{1}{4}$

but also: $\emptyset \mapsto 0$

$\{H, T\} \mapsto 1$

Recap: Semantic Foundation for Statistics & Probability

finite discrete spaces

Points/ states $\{H,T\}$ $\{H,T\}^3$

more H than T

Events $\{H\}, \{T\}$ $\{HHT, HTH, THH, HHH\}$

$= \bigcup_{i=1}^{3} \{H \cdots H T H \cdots H\} \cup \{H \cdots H\}$

Probability $\frac{3}{4}$ $\frac{1}{4}$ $\sum_{i=1}^{3} \frac{3^2 \times 1}{4^3} + \frac{3^3}{4^3}$

but also: $\emptyset \mapsto 0$

$\{H,T\} \mapsto 1$

Countable discrete spaces

Points/states $\quad \mathrm{List}\{H, T\}$

more H than T

Events $\quad \displaystyle\bigcup_{n=0}^{\infty} \bigcup_{i=\lceil \frac{n}{2} \rceil}^{n} \bigcup_{\rho : \mathrm{Fin}\, i \rightarrowtail \mathrm{Fin}\, n} \left\{ F_1 \cdots F_n \text{ where } \begin{cases} i = \rho j : & F_i = H \\ \text{o.w} : & F_i = T \end{cases} \right\}$

list length

Countable discrete spaces

Points/states: $\mathrm{List}\,\{H, T\}$

more H than T

Events: $$\bigcup_{n=0}^{\infty} \bigcup_{i=\lceil \frac{n}{2} \rceil}^{n} \bigcup_{\rho : \mathrm{Fin}\, i \rightarrowtail \mathrm{Fin}\, n} \left\{ F_1 \cdots F_n \text{ where } \begin{cases} i = \rho j : & F_i := H \\ \text{o.w} : & F_i := T \end{cases} \right\}$$

list length

#H

positions of H

Probability: $$\sum_{n=0}^{\infty} \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \sum_{\rho : \mathrm{Fin}\, i \rightarrowtail \mathrm{Fin}\, n} \frac{999}{1000^n} \cdot \frac{3^i \times 1^{n-i}}{4^n}$$

Continuous spaces

Points/ states $\mathbb{R}$

Events Borel subsets $\mathcal{B}_{\mathbb{R}}$

Probability $E \mapsto \int_E dx \frac{1}{\sqrt{2\pi}} e^{-2(x-1)^2} + \sum_{\substack{n=1 \\ 2n \in E}}^{\infty} \frac{1}{2^n}$

density w.r.t. measure $w : \text{points} \to [0,\infty]$

$$\lambda + \sum_{n=1}^{\infty} \delta_{2n}$$

Lebesgue ↳Dirac

Continuous spaces

Points/ states $\quad \mathbb{R}$ $\quad \{H,T\}^{\mathbb{N}}$

Events $\quad$ Borel subsets $\mathcal{B}_{\mathbb{R}}$ $\quad$ σ-algebra generated by "open cylinders"

e.g. $HTHT\# \{H,T\}^{\mathbb{N}}$

$$\downarrow$$

$$\frac{3^2 \times 1^2}{4}$$

Probability $\quad E \mapsto \int_E dx \frac{1}{\sqrt{2\pi}} e^{-2(x-1)^2} + \sum_{\substack{n=1 \\ 2n \in E}}^{\infty} \frac{1}{2^n}$

density w.r.t. measure $\quad w : \text{points} \to [0,\infty]$

$$\lambda + \sum_{n=1}^{\infty} \delta_{2n}$$

Lebesgue $\quad$ ↳Dirac

Haar measure

Discrete finite

Discrete Countable

Continuous

Quasi-Borel Spaces

~~Measurable~~

Agenda

I
- Borel sets
- Qbs:
  - def., constructions, partiality, type structure

II
- Measures & integration

see CIRM videos
- Random variable spaces
- Conditional expectation

Slogan:

Measurable by Type

NB:

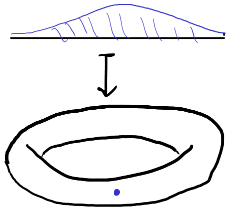
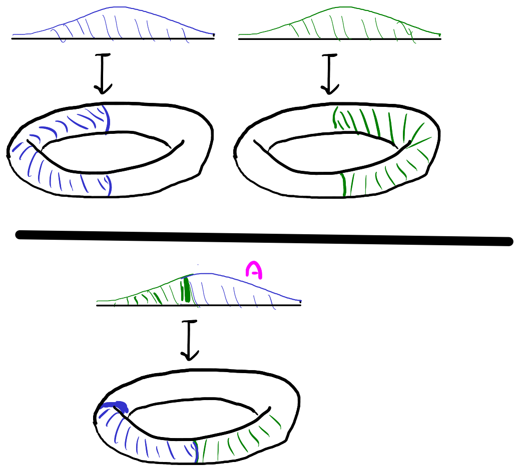
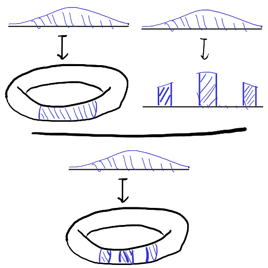
- Exercise Sheets 
- #qbs on SPLS Zulip

Def: Quasi-Borel space $X = (\lfloor X \rfloor, R_X)$

set "carrier"

set of functions $\alpha : \mathbb{R} \to \lfloor X \rfloor$ "random elements"

$R_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- constants: 
- recombination: 
- precomposition: 

Examples

- $\mathbb{R} = (\lfloor\mathbb{R}\rfloor, \mathrm{Meas}(\mathbb{R}, \mathbb{R}))$

  qbs underlying $\mathbb{R}$

- $X \in \mathrm{Set}$, $\ulcorner X \urcorner^{Qbs} := (X, \sigma\text{-simple}(\mathbb{R}, X))$

  recombination of constants

  $$\lambda r. \begin{cases} \vdots \\ r \in A_n : x_n \\ \vdots \end{cases}$$

  discrete qbs on X

- " $\lfloor X \rfloor_{Qbs} := (X, X^{\lfloor\mathbb{R}\rfloor})$

  all funtions

  Indiscrete qbs on X

Useful adjunctions:

$$
\begin{tikzcd}
\mathbf{Meas} \arrow[d, "\lfloor - \rfloor_{\mathbf{Qbs}}"] \arrow[d, phantom, shift right=4, "\dashv"] \\
\mathbf{Qbs} \arrow[u, bend left=60, "\ulcorner - \urcorner^{\mathbf{Meas}}"] \arrow[d, "\lfloor - \rfloor"] \arrow[d, phantom, shift right=4, "\dashv"] \arrow[d, phantom, shift left=6, "\dashv"] \\
\mathbf{Set} \arrow[u, bend left=60, "\ulcorner - \urcorner^{\mathbf{Qbs}}"] \arrow[u, bend right=60, "\lfloor \overline{\ } \rfloor_{\mathbf{Qbs}}"']
\end{tikzcd}
$$

$$\lfloor V \rfloor_{\mathbf{Qbs}} := (\lfloor V \rfloor, \mathbf{Meas}(\mathbb{R}, V)) \quad (V \in \mathbf{Meas})$$

$$\ulcorner X \urcorner^{\mathbf{Meas}} := \left\{ A \subseteq \lfloor X \rfloor \;\middle|\; \forall \alpha \in R_X .\ \alpha^{-1}[A] \in \mathcal{B}_{\mathbb{R}} \right\}$$

- limits (products, subspaces) and colimits (coproducts, quotients) as in <u>Set</u>
- Slogan: every measurable space is carried by a qbs

Example

Product $(X \times Y, \pi_1, \pi_2)$:

necessarily!

- $\lfloor X \times Y \rfloor = \lfloor X \rfloor \times \lfloor Y \rfloor$

correlated random elements

- $R_{X \times Y} = \{ \lambda r. (\alpha r, \beta r) \mid \alpha \in R_X, \beta \in R_Y \}$

rest of structure as in $\underline{\mathbf{Set}}$.

Function Spaces

Straightforward!

- $\lfloor Y^X \rfloor := \mathrm{Qbs}(X, Y)$

- $R_{Y^X} := \mathrm{uncurry}[\mathrm{Qbs}(\mathbb{R} \times X, Y)]$

  $= \{ \alpha : \mathbb{R} \to \lfloor Y^X \rfloor \mid \lambda(r, x).\, \alpha\, r\, x : \mathbb{R} \times X \to Y \}$

- $\mathrm{eval} : Y^X \times X \to Y$

  $\mathrm{eval}(f, x) := f\, x$

Meas vs Qbs

By generalities:

σ-algebra on $\mathbf{Meas}(\mathbb{R},\mathbb{R})$

$$\begin{tikzcd}
\mathbf{Meas} \arrow[d, "\lfloor - \rfloor_{\mathbf{Qbs}}"] \arrow[d, phantom, "\dashv"', shift right=4] \\
\mathbf{Qbs} \arrow[u, bend left, "\ulcorner - \urcorner^{\mathbf{Meas}}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\ulcorner \mathbb{R}^{\mathbb{R}} \urcorner^{\mathbf{Meas}} \times \mathbb{R} \arrow[r] \arrow[rr, bend right, "\ulcorner eval \urcorner^{\mathbf{Meas}}"'] & \ulcorner \mathbb{R}^{\mathbb{R}} \urcorner^{\mathbf{Meas}} \times \mathbb{R} \arrow[r, dashed, "\times"] & \ulcorner \mathbb{R} \urcorner = \mathbb{R}
\end{tikzcd}$$

No factorisation by Aumann's Theorem.

$\left(\text{So } \ulcorner \mathbb{R}^{\mathbb{R}} \times \mathbb{R} \urcorner \not\cong \ulcorner \mathbb{R}^{\mathbb{R}} \urcorner \times \ulcorner \mathbb{R} \urcorner \right)$

Random element space

$$R_X := X^{\mathbb{R}} \quad \text{since} \quad \lfloor X^{\mathbb{R}} \rfloor = R_X \text{ as sets.}$$

Why?

$(\subseteq)$ $\alpha \in \lfloor X^{\mathbb{R}} \rfloor \Rightarrow \alpha : \mathbb{R} \to X$ in Qbs.

$\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ measurable $\Rightarrow \mathrm{id} \in R_{\mathbb{R}}$

$\Rightarrow \alpha = \alpha \circ \mathrm{id} \in R_X$

Precomposition

$(\supseteq)$ $\alpha \in R_X \Rightarrow \forall \varphi \in R_{\mathbb{R}} = \mathrm{Meas}(\mathbb{R}, \mathbb{R}).\ \alpha \circ \varphi \in R_X \Rightarrow \alpha : \mathbb{R} \to X$

$\Rightarrow \alpha \in \lfloor X^{\mathbb{R}} \rfloor$

Subspaces

For $X \in \mathbf{Qbs}$, $A \subseteq \llcorner X \lrcorner$ set:

$$R_A := \{\alpha : \mathbb{R} \to A \mid \alpha \in R_X\}$$

Then $A = (A, R_A)$ is the subspace qbs

We write $A \hookrightarrow X$

Borel subspaces ensemble

The σ-algebra $B_X := \{ A \subseteq \llcorner X \lrcorner \mid \forall \alpha \in R_X . \alpha^{-1}[A] \in B_{\mathbb{R}} \}$

internalises as $B_X = 2^X$, the qbs of

Borel subsets.

($\llcorner B_{(B_{\mathbb{R}})} \lrcorner$ are the Borel-on-Borel sets from descriptive set theory.

cf. [Sabok et al. '21])

Standard Borel spaces

Def: A qbs S is standard Borel when

$$S \cong A \quad \text{for some } A \in \mathcal{B}_{\mathbb{R}}$$

$$\begin{tikzcd}
\mathbf{Meas} \arrow[rr, bend left] \arrow[rr, phantom, "\top"] & & \mathbf{Qbs} \arrow[ll, bend left] \\
\mathbf{Sbs} \arrow[u, hook] \arrow[rr, bend left] \arrow[rr, phantom, "\cong"] & & \mathbf{Sbs} \arrow[u, hook] \arrow[ll, bend left]
\end{tikzcd}$$

Slogan: Qbs conservative extension of Sbs

Example $C_0 := \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous} \} \hookrightarrow \mathbb{R}^{\mathbb{R}}$

C_0 is sbs. (Well-known!)

Proof:

$C_0' \in B_{\mathbb{R}^{\mathbb{Q}}}$ sbs!

$$C_0' := \left\{ g \in \mathbb{R}^{\mathbb{Q}} \;\middle|\; \begin{array}{l} \forall a, b \in \mathbb{Q}, \varepsilon \in \mathbb{Q}^+ \\ \exists \delta \in \mathbb{Q}^+ \; \forall p, q \in \mathbb{Q}^+ \cap [a,b], \\ |p - q| < \delta \Rightarrow |g\,p - g\,q| < \varepsilon \end{array} \right\}$$

on closed intervals (= compact intervals) continuity $\Updownarrow$ uniform continuity

Borel measurable by type checks

then $C_0 \cong C_0' \in B_{\mathbb{R}^{\mathbb{Q}}}$:

$$\begin{aligned} C_0 &\to C_0' \\ \varphi &\mapsto \varphi|_{\mathbb{Q}} \end{aligned} \qquad \begin{aligned} C_0' &\to C_0 \\ \varphi &\mapsto \lambda r. \lim_{n \to \infty} g(\text{approx}_{(\frac{1}{m})_{m \in \mathbb{N}}} r)_n \end{aligned}$$

Example (ctd)

C_0 is sbs, and $\mathrm{eval}: C_0 \times \mathbb{R} \to \mathbb{R}$

is a measurable.

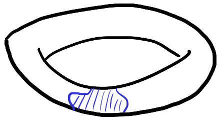
Avoids:

- constructing complete separable metrics
- proving that evaluation is measurable w.r.t. metric σ-algebra.

Agenda

Slogan: Measurable by Type

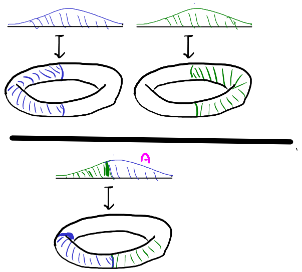
- Borel sets



- Qbs:

  def., constructions,

  partiality, type structure



- Measures & integration
- Random variable spaces
- Conditional expectation

Partiality cf. [Vákár et al. '19]

A Borel embedding $e : X \hookrightarrow Y$

- injective function $e : \lfloor X \rfloor \rightarrowtail \lfloor Y \rfloor$
- its image is Borel: $e[\lfloor X \rfloor] \in \mathcal{B}_Y$
- e is Strong: $\alpha \in R_X \iff e \circ \alpha \in R_Y$

Examples

- $\mathbb{1} \hookrightarrow \mathbb{2}$
- S is sbs $\iff \exists\, S \hookrightarrow \mathbb{R}$

Non-examples ⇝ [Sabok et al. '21]

- $\{A \in B_{\mathbb{R}} \mid A \neq \emptyset\} \hookrightarrow B_{\mathbb{R}}$

- $\{(A_1, A_2) \in B_{\mathbb{R}}^2 \mid A \subseteq B\} \hookrightarrow B_{\mathbb{R}}^2$

- $\{A \in B_{\mathbb{R}} \mid A \text{ open}\} \hookrightarrow B_{\mathbb{R}}$

Def: A Partial map $f: X \to Y$ is a morphism

$$f: X \to Y \amalg \{\bot\}$$

Its domain of definition $\mathrm{Dom}\, f := \{x \mid fx \neq \bot\}$

$$\{x \mid fx \neq \bot\} \rightarrowtail X$$

Partial hom-sets are ordered:

for $f, g: X \to Y$ $\quad f \leq g$ when $\forall x.\ fx \neq \bot \Rightarrow gx = fx$.

[Cockett-Lack '06]

A model of restriction categories / axiomatic domain theory

[Fiore-Plotkin '94] Borel embeddings are the admissible monos

Type Structure

Simple types denote spaces

E.g.: $A \times B$ $\quad B^A \quad$ B_A

Dependent types denote spaces-in-context

$\Gamma \vdash A$

$$\begin{tikzcd} [\![\Gamma \vdash A]\!] \arrow[d, "\mathrm{dep}"] \\ [\![\Gamma]\!] \end{tikzcd}$$

Dependent types denote spaces-in-context

$\Gamma \vdash$ context

$\Gamma \vdash A$

type in context

$[\![\Gamma \vdash A]\!]$ Space in context

$$[\![\Gamma \vdash A]\!] \xrightarrow{\text{dep}} [\![\Gamma]\!]$$

assigns environment

$[\![\Gamma]\!]$ context space

E.g.:

$$A \longrightarrow \mathbb{1}$$

simple types

$$[\![U : \mathcal{B}_A \vdash U]\!] = \{(U, a) \in \mathcal{B}_A \times A \mid a \in U\} \xrightarrow{\pi_1} \mathcal{B}_A$$

sbs decoder

Context extension

$$\frac{\Gamma \vdash A}{\Gamma, a : A \vdash}$$

$$\begin{array}{c} [\![\Gamma \vdash A]\!] \\ \downarrow{\scriptstyle dep} \\ [\![\Gamma]\!] \end{array} \qquad [\![\Gamma, a:A]\!] := [\![\Gamma \vdash A]\!]$$

Substitution

$\Gamma \vdash \sigma : \Delta$ $\qquad [\![\sigma]\!] : [\![\Gamma]\!] \longrightarrow [\![\Delta]\!]$

E.g. weakening $\qquad \Gamma, a:A \vdash \mathrm{wkn} : \Gamma$

$$[\![\Gamma, a:A]\!] := [\![\Gamma \vdash A]\!] \xrightarrow[dep]{wkn} [\![\Gamma]\!]$$

Action of substitution on types

$$\llbracket \Gamma \vdash A[\sigma] \rrbracket$$

$$\begin{tikzcd}
\{(\gamma, a) \in \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \mid \mathrm{dep}\, a = \gamma\} \arrow[r, "\pi_2"] \arrow[d, "\mathrm{dep} := \pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \Delta \vdash A \rrbracket \arrow[d, "\mathrm{dep}"] \\
\llbracket \Gamma \rrbracket \arrow[r, "\sigma"] & \Delta
\end{tikzcd}$$

E.g.

$$\begin{tikzcd}
\llbracket x : A \vdash B[\mathrm{wkn}] \rrbracket := A \times B \arrow[r, "\pi_2"] \arrow[d, "\mathrm{dep} := \pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
x : A \arrow[r, "\langle\rangle"'] & \mathbb{1}
\end{tikzcd}$$

simple type

Terms: sections

$$\begin{tikzcd}
[\![\Gamma]\!] \arrow[r, "{[\![\Gamma \vdash M : A]\!]}"] \arrow[dr, equal, "="'] & {[\![\Gamma \vdash A]\!]} \arrow[d, "\mathrm{dep}"] \\
& {[\![\Gamma]\!]}
\end{tikzcd}$$

e.g.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "{[\![x : \mathbb{R} \vdash [x,\infty) : B_{\mathbb{R}}[\mathrm{wkn}]]\!]}"] \arrow[dr, equal, "="'] & \mathbb{R} \times B_{\mathbb{R}} \arrow[d, "\pi_1"] \\
& \mathbb{R}
\end{tikzcd}$$

E.g. Variables:

$$\begin{tikzcd}
{[\![\Gamma, a : A]\!]} \arrow[r, "{[\![\Gamma, a : A \vdash a : A]\!]}", "{\langle \mathrm{id}, \mathrm{dep}_{\Gamma \vdash A} \rangle}"'] \arrow[dr, equal, "="'] & {[\![\Gamma, a : A \vdash A[\mathrm{wkn}]]\!]} \arrow[d, "\mathrm{dep}"] \\
& {[\![\Gamma, a : A]\!]}
\end{tikzcd}$$

Exercise:
action of substitution
$M[\sigma]$

Dependent Pairs

$$\frac{\Gamma, a:A \vdash B}{\Gamma \vdash \coprod_{a\in A} B}$$

$$[\![\textstyle\coprod_{a\in A} A]\!] := [\![\Gamma, a:A \vdash B]\!]$$

$$\begin{tikzcd}
{[\![\Gamma, a:A \vdash B]\!]} \arrow[d, "\mathrm{dep}_B"] \arrow[dd, bend right=60, "\mathrm{dep}_{\coprod} \,:="'] \\
{[\![\Gamma, a:A]\!] = [\![\Gamma \vdash A]\!]} \arrow[d] \\
{[\![\Gamma]\!]}
\end{tikzcd}$$

Dependent Products

$$\frac{\Gamma, a : A \vdash B}{\Gamma \vdash \prod_{a \in A} B}$$

$aha: (a:A) \to B$

$$\left\lfloor [\![\Gamma \vdash \prod_{a \in A} B]\!] \right\rfloor :=$$

$$\Big\{ (r_0, f : \{ a \in [\![A]\!] \mid \mathrm{dep}\, a = r_0 \} \to [\![\Gamma, a : A \vdash B]\!]) \Big/ \forall a \in [\![\Gamma, a : A]\!] .\ \mathrm{dep}\, a = r_0 \Rightarrow \mathrm{dep}(f a) = a \Big\}$$

Exercise: find the random elements.

Example

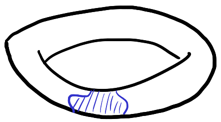
$(\Omega \in \mathbf{Qbs})$

$$\mathrm{Converging} \hookrightarrow \left([-\infty,\infty]^{\Omega}\right)^{\mathbb{N}}$$

$$\mathrm{Converging} := \prod_{\omega \in \Omega} \{ \vec{f} \mid \exists \lim_{n\to\infty} f_n \omega \}$$

Agenda

Slogan: Measurable by Type

- Borel sets 
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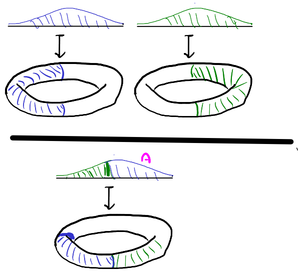
  def., constructions, 
  partiality, refinement $\rightharpoonup$, $\Rightarrow$, $\amalg$, Π
- Measures & integration
- Random variable spaces
- Conditional expectation

Def: A measure μ over $\mathbb{R}$ is a function

$$\mu : B_{\mathbb{R}} \to W := [0, \infty]$$

s.t. - $\mu \emptyset = 0$

- $$\frac{\vec{A} \in B_{\mathbb{R}}^{\mathbb{N}} \qquad A_n \cap A_m = \emptyset \ (n \neq m)}{\mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \mu A_n}$$

For measurable spaces, replace $\mathbb{R}$ with V

We write $\llcorner G V \lrcorner$ for the set of measures on V

For qbs X, take $\llcorner G^{\ulcorner \mathrm{Meas} \urcorner} X \lrcorner$

The unrestricted Giry space

Equip $\lfloor GV \rfloor$ with

$$R_{GV} := \{\alpha : \mathbb{R} \to GV \mid \forall A \in B_V . \lambda r. \alpha(r, A) : \mathbb{R} \to W\}$$

α is a kernel.

Farewell Meas

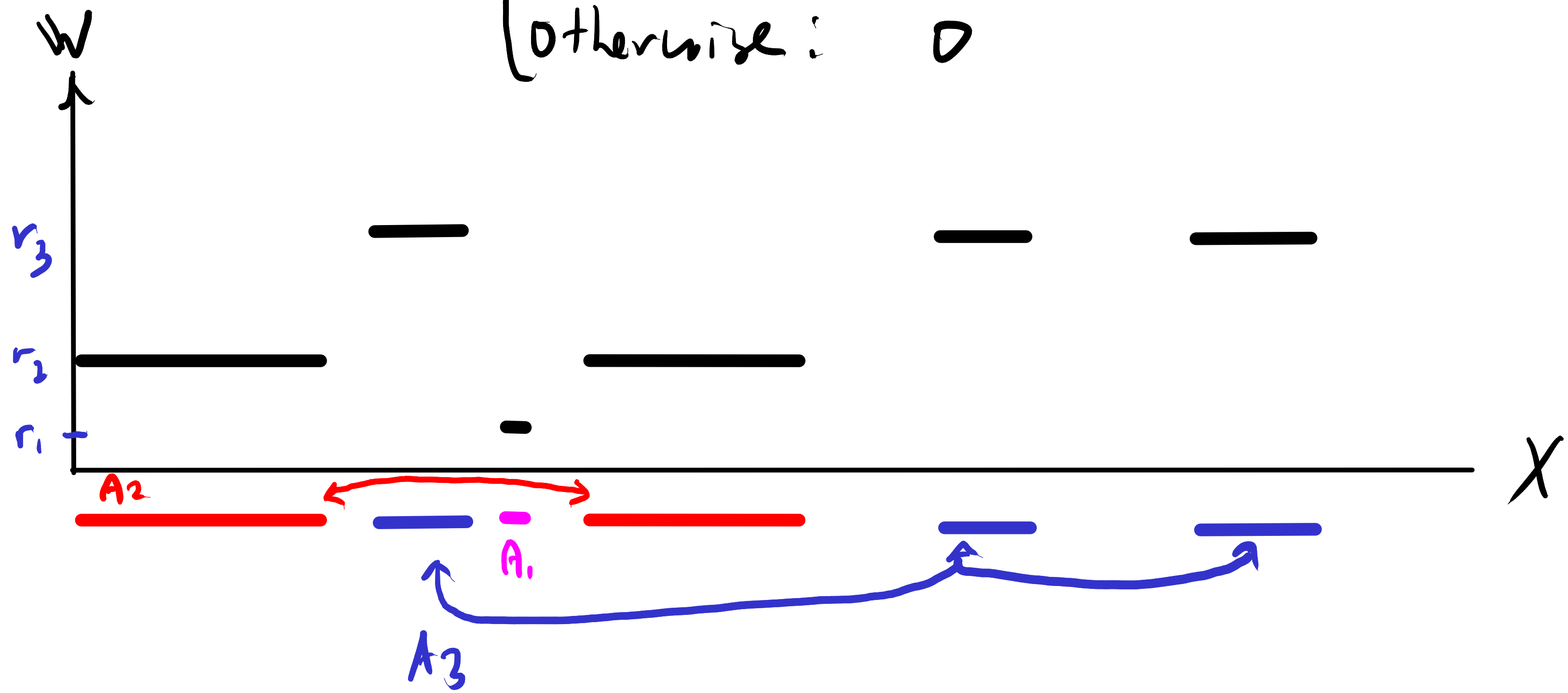
Now on:

1. All spaces are quasi-Borel
2. "measurable function" means qbs morphism!

Def: Simple function $\varphi : X \to W$ when

$\exists\, n \in \mathbb{N},\ \vec{A} \in B_X^n,\ A_i \cap A_j = \emptyset\ (i \neq j),\ \vec{r} \in W^n$ s.t.

$$\varphi x = \begin{cases} \vdots \\ x \in A_i : & r_i \\ \vdots \\ \text{otherwise} : & 0 \end{cases}$$



Encode into a space:

$$\mathrm{SimpleCode} := \coprod_{n \in \mathbb{N}} B_X^n \times W^n$$

$$\mathrm{Simple} := \{ f \in W^X \mid f \text{ simple} \} \hookrightarrow W^X$$

and define an interpretation:

$$[\![-]\!] : \mathrm{SimpleCode} \twoheadrightarrow \mathrm{Simple}$$

$$[\![(n, \vec{A}, \vec{r})]\!] := \sum_{i=1}^{n} r_i \cdot [- \in A_i]$$

↳ characteristic function for A_i

Lemma: $f : X \to W$ is measurable → remember! QBS morphism!

iff $f = \lim_{n\to\infty} f_n$ for some monotone sequence

$\vec{f} \in \mathrm{Simple}$.

Moreover, we have measurable such choice:

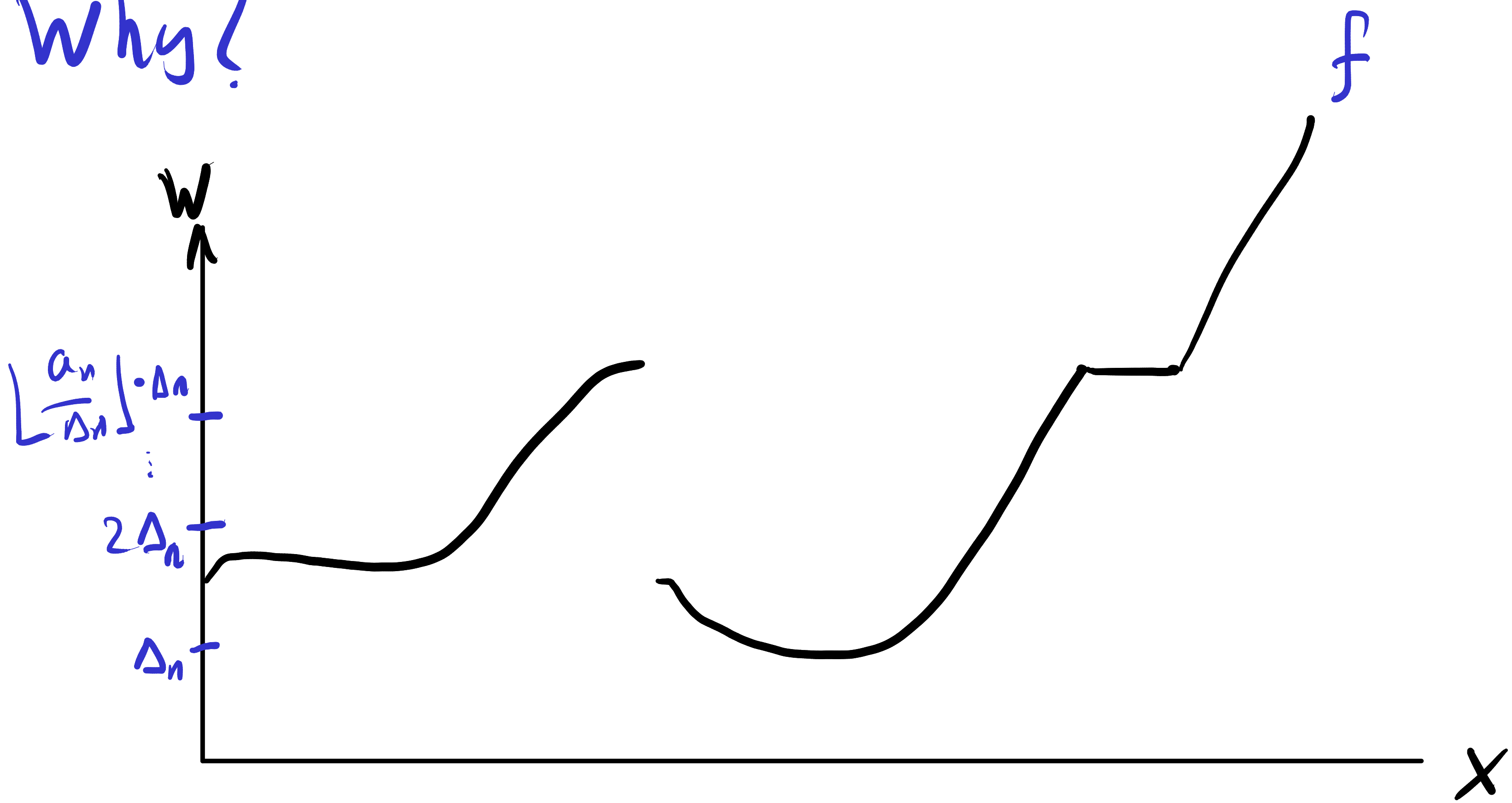
Simple Approx:

$$\{\vec{\Delta} \in \mathbb{R}^+ \mid \Delta_n \to 0\} \times \{\vec{a} \in W^{\mathbb{N}} \mid \vec{a} \text{ monotone}, a_n \to \infty\} \times W^X \to \mathrm{SimpleCode}$$

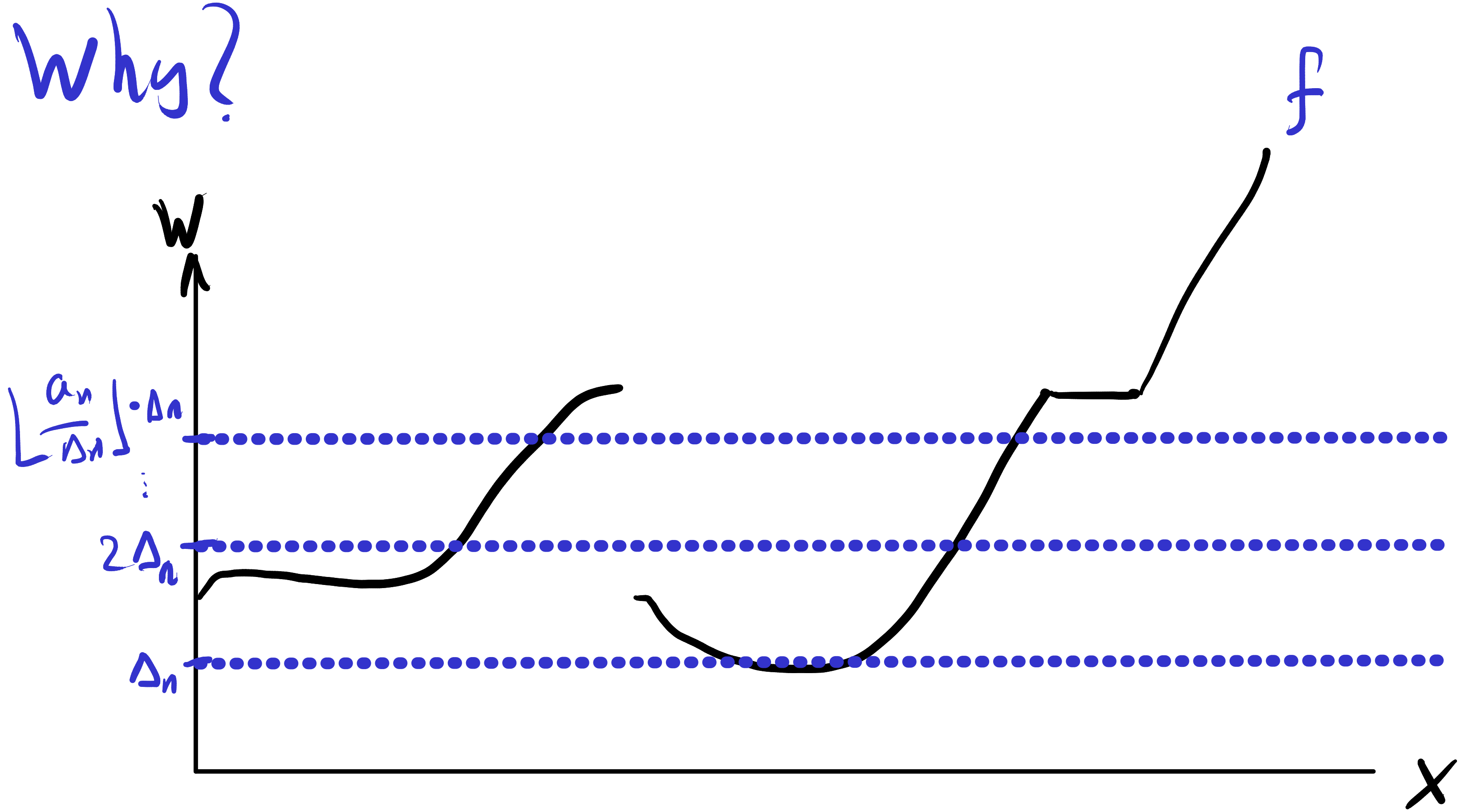
rate of convergence

range of approximation

Why?


f
W
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n
X

Why?



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X
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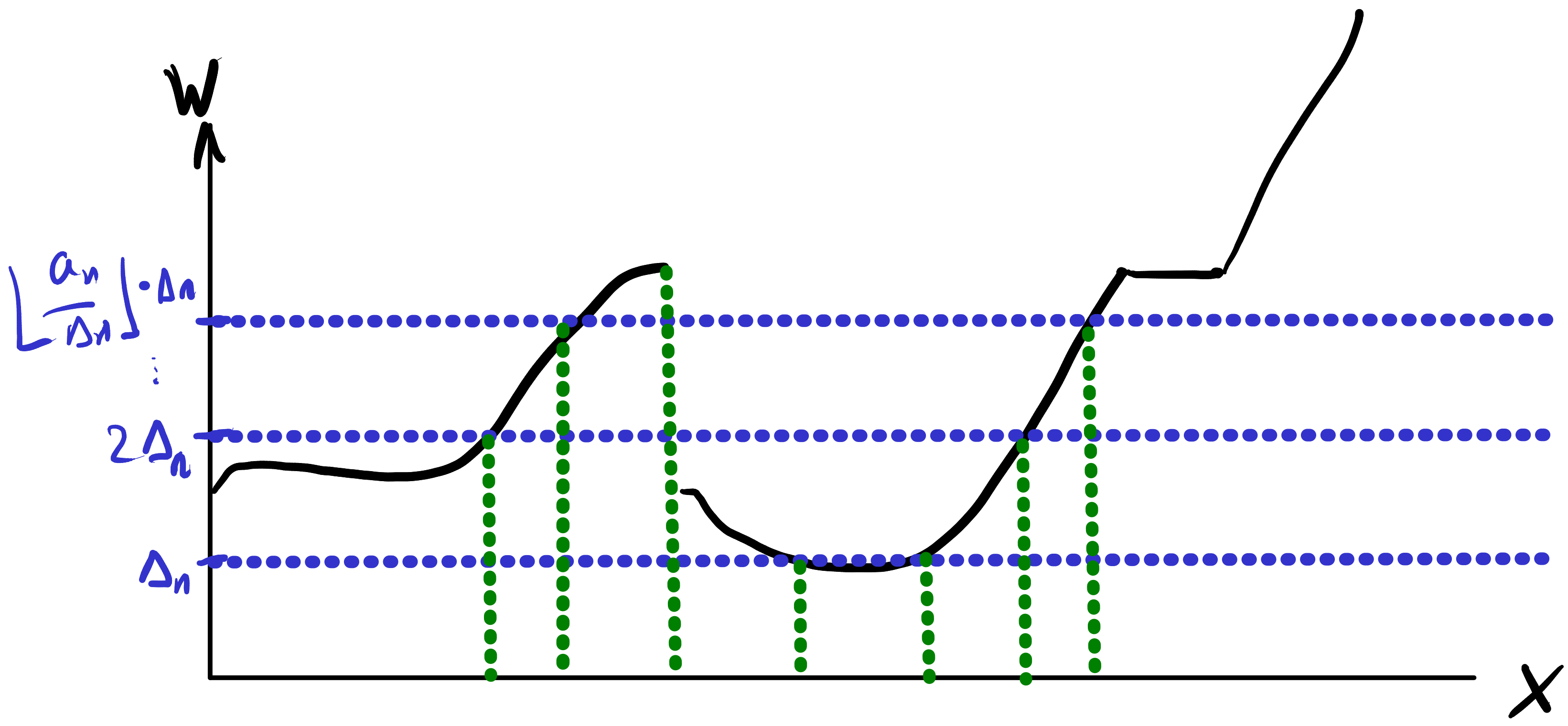
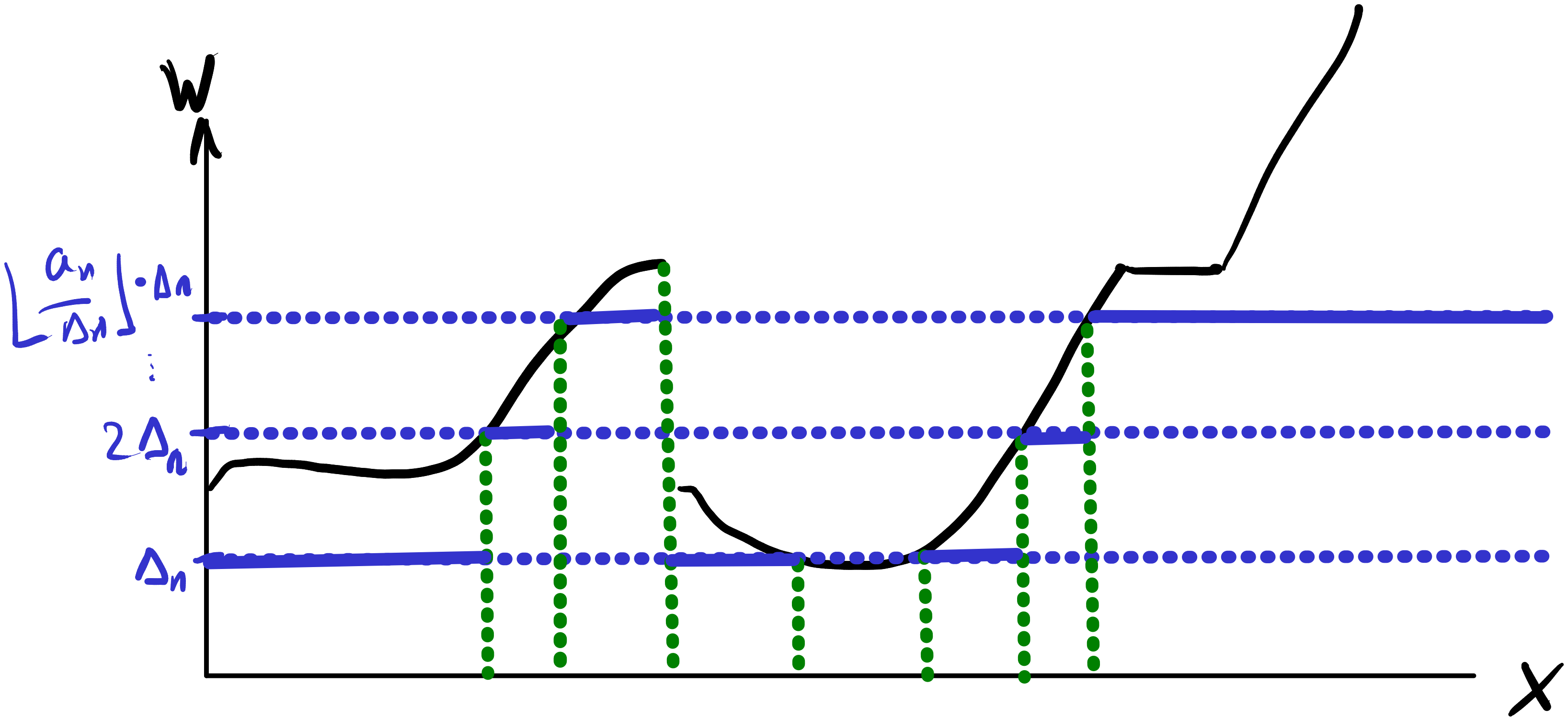
$2\Delta_n$
Δ_n

Why?



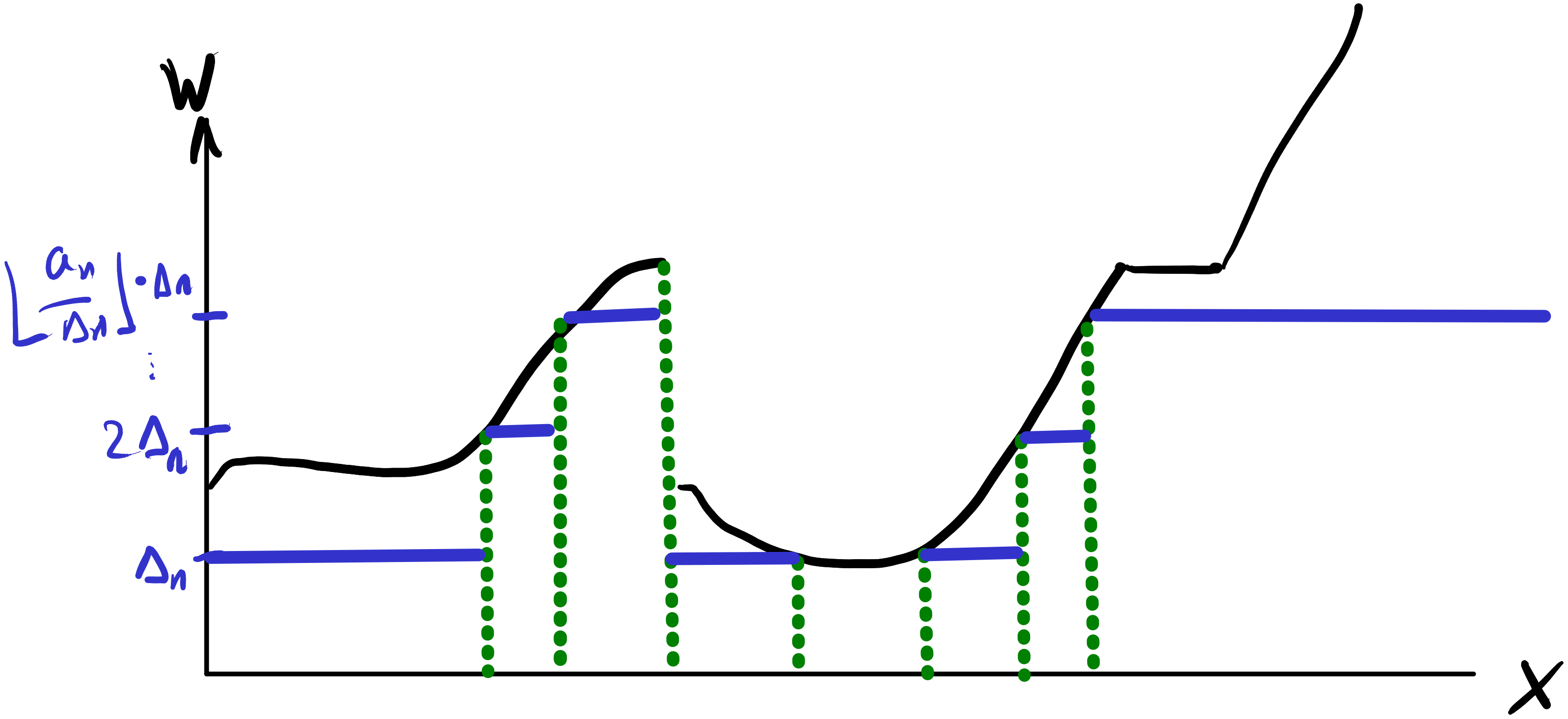
f
w
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$\vdots$
$2\Delta_n$
Δ_n
X

Why?



W
X
f
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n

Why?



f
W
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
⋮
$2\Delta_n$
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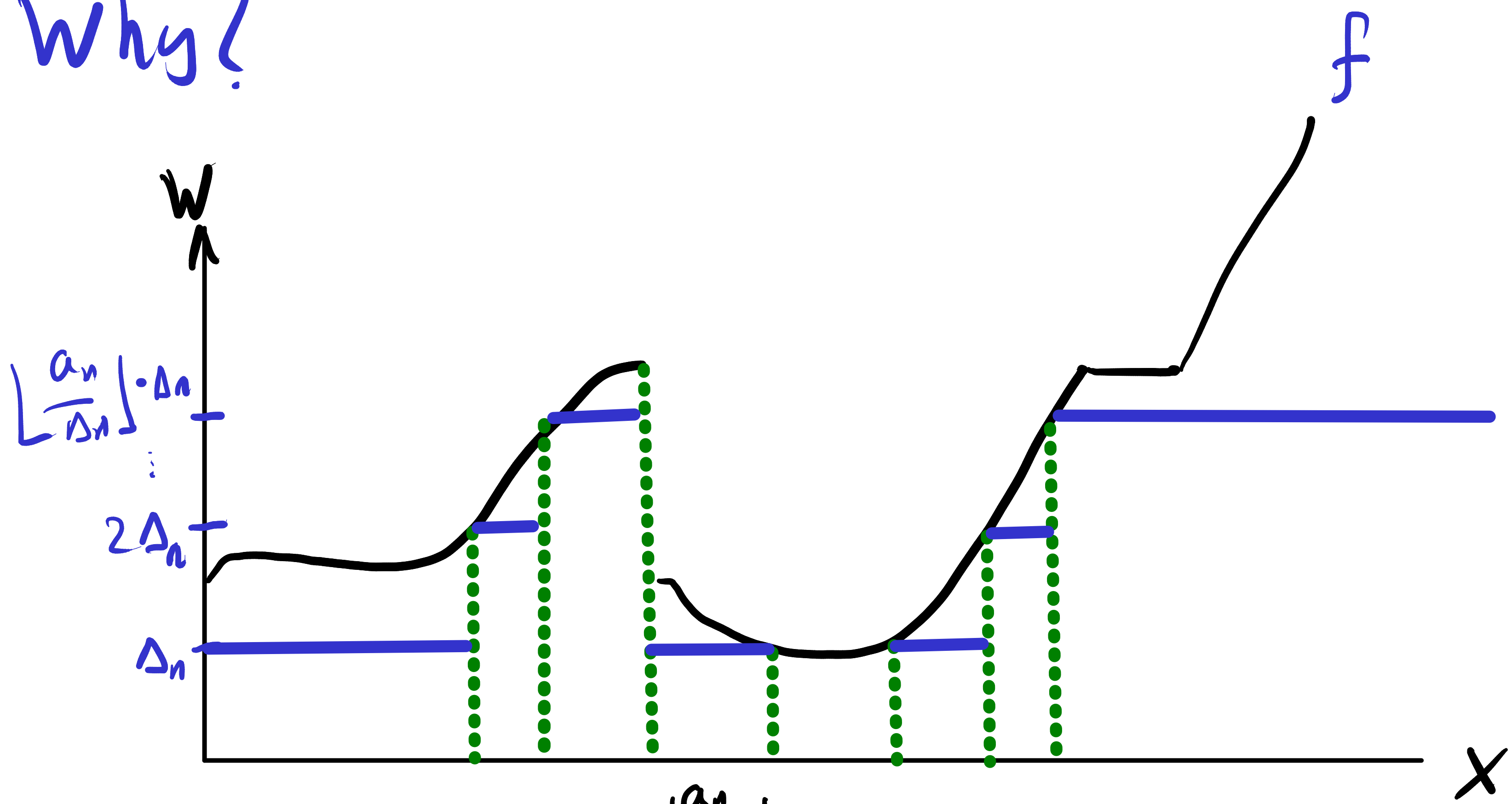
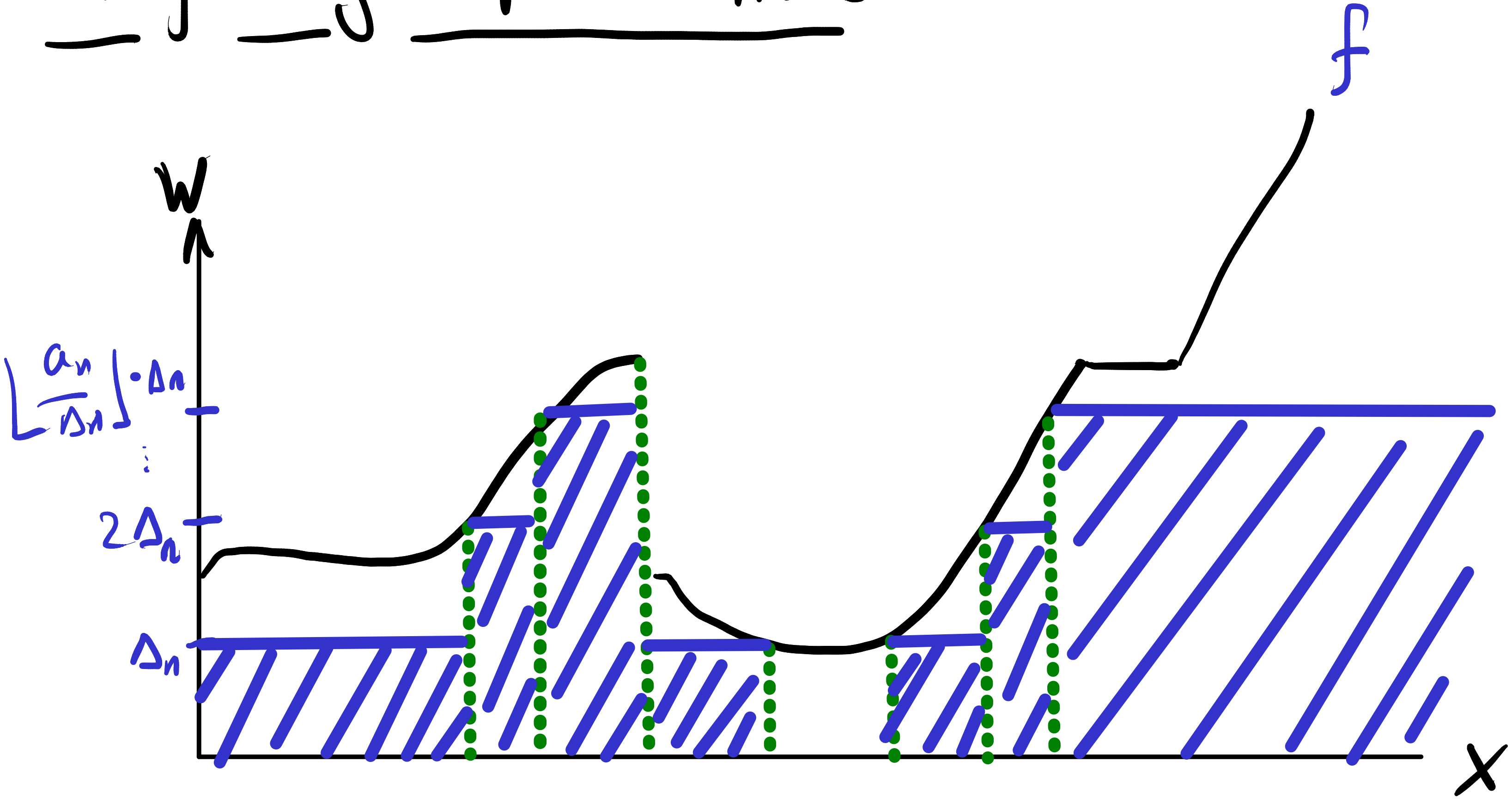
X

Why?



$$[\![\text{SimpleApprox}_{\vec{\Delta},\vec{a}}\, f]\!] := \sum_{i=1}^{\lfloor \frac{a_n}{\Delta_n} \rfloor} i \cdot \Delta_n \left[i \cdot \Delta_n \le f < (i+1)\Delta_n \right] + \left\lfloor \frac{a_n}{\Delta_n} \right\rfloor \Delta_n \cdot \left[f \geq \left\lfloor \frac{a_n}{\Delta_n} \right\rfloor \cdot \Delta_n \right] \in \text{Simple}$$

Integrating Simple Functions



$$\int : GX \times \text{Simple Code} \to W$$

$$\int \mu(n, \vec{A}, \vec{r}) := \sum_{I \subseteq \{1, \ldots, n\}} \left(\sum_{i \in I} r_i\right) \cdot \mu\left(\bigcap_{i \in I} A_i \setminus \bigcup_{i \notin I} A_i\right)$$

Integration

$\rightsquigarrow$ Properly higher-order operation

$$\int : GX \times W^X \longrightarrow W$$

$$\int \mu f := \sup\left\{ \int \mu \varphi \;\middle|\; \varphi \in \mathrm{Simple},\ \varphi \le f \right\}$$

$$= \lim_{n \to \infty} \int \mu \left(\mathrm{SimpleApprox}_{\vec{\Delta}, \vec{a}}\, f\right)_n$$

$\rightsquigarrow$ measurable by type

we also write $\int \mu(dx)\, t$ for $\int \mu\, (\lambda x. t)$

for $\frac{a_n}{\Delta_n} \to 0$, e.g. $\Delta_n = \frac{1}{2^n}$ $a_n = n$.

resolution