

Foundations for type-driven probabilistic modelling

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4–16 December, 2023
Canberra, ACT, Australia



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Plan:

- 1) Type-driven probability: discrete case (Mon + Tue)
- 2) Borel sets & measurable spaces (Wed)
- 3) Quasi Borel spaces (Web) simple type structure (Thu)
- 4) Dependent type structure & standard Borel spaces (Thu)
- 5) Integration & random variables (Fri)

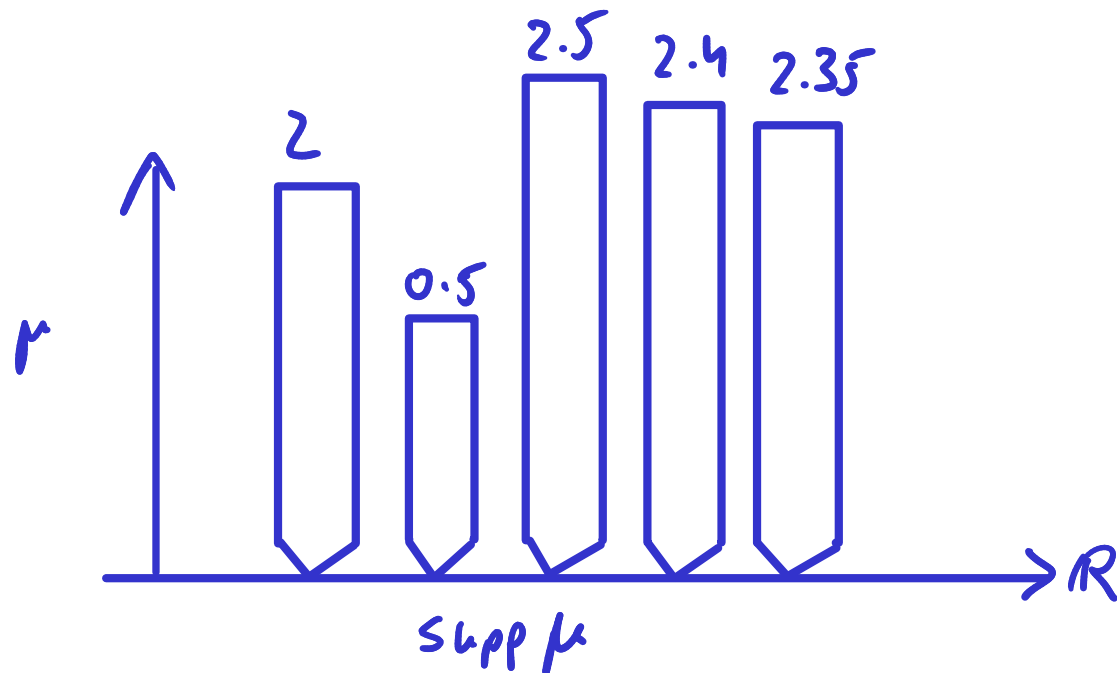
please ask questions!

smile



Course
web
page

discrete model measure only histograms:



Want:

- lengths
- areas
- volumes.

Continuous *Caveat:*

Thm: No $\lambda: \mathcal{P}R \rightarrow [0, \infty]$:

$$\lambda(a, b) = b - a$$

(generalises length)

$$\lambda(r + A) = \lambda A$$

(translation invariant)

$$\lambda\left(\bigcup_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \lambda A_n$$

σ -additive

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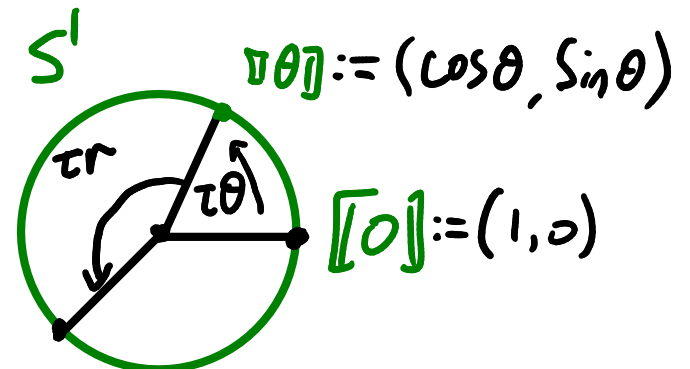
(translation invariant)

$$\lambda\left(\bigcup_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \lambda A_n$$

σ -additive

Direct proof in standard analysis courses. Idea behind typical proof is:

Thm: no $\lambda: \mathcal{P}S^1 \rightarrow [0, \infty]$
s.t.



$$r: \mathbb{R} \vdash \text{rotate}_r, [\theta] := [\theta + \tau r]$$

a) satisfy measure axioms for $BS' := \mathcal{P}S'$

b) invariant under rotations: $E: BS' \vdash$

$$\lambda \text{ rotate}[E] = \lambda E$$

c) $\lambda S' = \tau (= 2\pi)$

reduce $(S', \lambda^{S'})$ to (R, λ^R) via restriction & push forward

$$\lambda^R|_{\mathcal{P}[0,1]} := \lambda E \subseteq [0,1]. \lambda E : \mathcal{P}[0,1] \rightarrow W$$

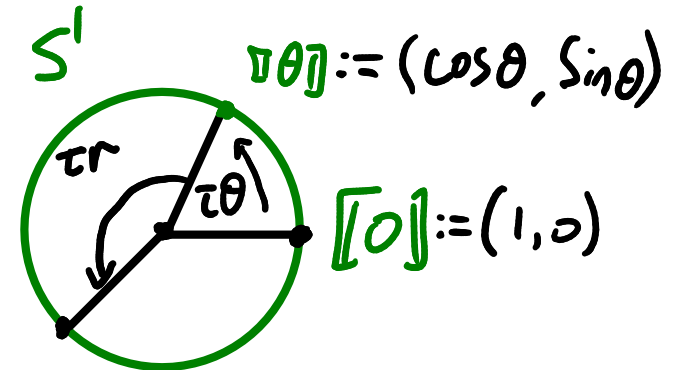
$$\lambda^{S'} := \lambda E \subseteq S'. \lambda^R_{\mathcal{P}[0,1]}(\llbracket - \rrbracket^{-1}[E]) : \mathcal{P}S' \xrightarrow{\llbracket - \rrbracket^{-1}} \mathcal{P}[0,1] \xrightarrow{\lambda^R_{\mathcal{P}[0,1]}} W$$

noting

rotations in $S' \iff$ translations in R

Since $\nexists \lambda^{S'}$, we have $\nexists \lambda^R$ either.

Thm: no $\lambda: \mathcal{P}S' \rightarrow [0, \infty]$
s.t.



a) satisfy measure axioms for $BS' := \mathcal{P}S'$

b) invariant under rotations: $E: BS' \vdash$

$$\lambda \text{ rotate}_r[E] = \lambda E$$

c) $\lambda S' = \tau$ ($\tau := 2\pi$)

Proof: $a + b \Rightarrow \neg c$:

1) Using axiom of choice (AoC):

$$S' = \bigoplus_{i=0}^{\infty} E_i$$

$$E_i = \text{rotate}_{r_i}[E_0]$$

$$2) \lambda S' = \sum_{i=0}^{\infty} \lambda E_i = \sum_i \lambda \text{ rotate}_{r_i} E_0 = \sum_{i=0}^{\infty} \lambda E_0 = \begin{cases} \lambda E_0 = 0 : 0 \\ \lambda E_0 > 0 : \infty \end{cases} \neq \tau$$

Constructing E_i :

$$x, y: S' \vdash x \sim y := \exists q \in Q. \text{rotate}_q x = y \quad : \text{Prop}$$

$$\equiv \exists q \in [0, 1) \cap Q. \text{rotate}_q x = y$$

\sim -equivalence classes:

$$x: S' \vdash [x]_\sim := \{ y \in S' \mid x \sim y \} \quad : \mathcal{P} S'$$

$$C := \{ [x]_\sim \in \mathcal{P} S' \mid x \in S' \}$$

$$\forall e \in C, e \neq \emptyset, \text{ so by } A_{\emptyset} C: \exists \xi: C \longrightarrow S'. \xi_e \in e.$$

NB: ξ injective

Take $C_0 := \{z_e \in S' \mid e \in C\} \in \mathcal{P} S'$

Note: $x \sim y, x, y \in C_0 \vdash x = y$.

$q: \mathbb{Q} \vdash C_q := \text{rotate}_q[C_0] \in \mathcal{P} S'$

Let $(r_i)_{i=0}^{\infty}$ enumerate $\mathbb{Q} \cap [0, 1)$ s.t. $r_0 = 0$

Take $E_i := C_{r_i}$

By fiat: $E_i = C_{r_i} = \text{rotate}_{r_i}[C_0] = \text{rotate}_{r_i}[E_0]$

RTP: $S' = \bigoplus_{i=0}^{\infty} E_i$

NB: $x, y: S' \vdash$
 $x \sim y: \text{Prop}$
 $C = \sim\text{-equiv.}$
 $z: C \rightarrow S'$
 $e: C \vdash z_e \in e$

$$E_i \cap E_j = \emptyset, \quad i \neq j:$$

$$x \in E_1 \cap E_2 \Rightarrow \exists y_i \in \mathcal{C}. x = \text{rotate}_{r_i} y_i$$

$$\Rightarrow y_1 \sim x \sim y_2 \Rightarrow y_1 = y_2 =: y$$

$$\Rightarrow \text{rotate}_{r_2 - r_1} y = y, \quad |r_2 - r_1| < 1$$

$$\Rightarrow r_1 = r_2$$

$$\underline{S'} = \bigcup_{i=0}^{\infty} \underline{E_i} : \text{ letting } e := \xi_{[x]_n} : \rho S'$$

$$\xi_e, x \in e \Rightarrow \xi_e \sim x$$

$$\Rightarrow \exists q \in \mathbb{Q} \cap [0, 1) . \text{ rotate}_q \xi_e = x .$$

$$\text{As } \xi_e \in C_0 : x \in C_q . \text{ Find } i \text{ s.t. } r_i = q$$

$$\text{and } x \in C_{r_i} = E_i .$$



Takeaway: Taking $B/R := \mathcal{D}R$

Excludes measures such as:

length, area, volume

Workaround: only measure well-behaved subsets

Df: The Borel subsets $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{P}\mathbb{R}$:

- open intervals $(a, b) \in \mathcal{B}_{\mathbb{R}}$

closure under σ -algebra operations:

$$\frac{}{\emptyset \in \mathcal{B}_{\mathbb{R}}}$$

↖
empty set

$$\frac{A \in \mathcal{B}_{\mathbb{R}}}{A^c := \mathbb{R} \setminus A \in \mathcal{B}}$$

↖
complements

$$\frac{\vec{A} \in \mathcal{B}_{\mathbb{R}}^{\mathbb{N}}}{\bigcup_{n=0}^{\infty} A_n \in \mathcal{B}_{\mathbb{R}}}$$

↖
countable unions

Examples

discrete Countable: $\{r\} = \bigcap_{\varepsilon \in \mathbb{Q}^+} (r - \varepsilon, r + \varepsilon) \in \mathcal{B}_{\mathbb{R}}$

I countable $\Rightarrow I = \bigcup_{r \in I} \{r\} \in \mathcal{B}_{\mathbb{R}}$

closed intervals: $[a, b] = (a, b) \cup \{a, b\}$

Non-examples?

More complicated: analytic, Lebesgue

Df: Measurable space $V = (V, B_V)$

Set
(carrier)

Family of
Subsets
 $B_V \subseteq P(V)$

closed under σ -algebra operations:

$$\frac{}{\emptyset \in B_V}$$

↖
empty set

$$\frac{A \in B_V}{A^c := V \setminus A \in B_V}$$

↖
complements

$$\frac{\vec{A} \in B_V^{\mathbb{N}}}{\bigcup_{n=0}^{\infty} A_n \in B_V}$$

↖
countable unions

Idea: Structure all spaces after the worst-case scenario

Examples

- Discrete spaces $\mathcal{X}^{\text{meas}} = (X, P_X)$
- Euclidean spaces \mathbb{R}^n — replace intervals with
boxes $\prod_{i=1}^n (a_i, b_i)$
 \mathbb{R}^N similarly
 $\{C \cap A \mid C \in \mathcal{B}_V\}$
- Sub spaces: $A \in \mathcal{P}_{\mathcal{V}_1}$ $A := (A, [B_V] \cap A)$
- Products: $A \times B := (\mathcal{A}_1 \times \mathcal{A}_2, \sigma([B_A] \times [B_B]))$

Def: Borel measurable functions $f: V_1 \rightarrow V_2$

- functions $f: V_1 \rightarrow V_2$
- inverse image preserves measurability:

$$f^{-1}[A] \in \mathcal{B}_{V_1} \iff A \in \mathcal{B}_{V_2}$$

Examples

- $(+), (\cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}$
- $| \cdot |, \sin : \mathbb{R} \rightarrow \mathbb{R}$
- any continuous function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$
- any function $f: X \rightarrow V$

Category Meas

Objects : Measurable spaces

Morphisms : Measurable functions

Identities:

$$\text{id} : V \rightarrow V$$

Composition:

$$f : V_2 \rightarrow V_3 \quad g : V_1 \rightarrow V_2$$

$$f \circ g : V_1 \rightarrow V_3$$

Meas Category

Products, Coproducts / disjoint union, Subspaces
Categorical limits, colimits, but:

Thm [Aumann '61] No σ -algebras $B_{\mathbb{R}}, B_{\mathbb{R}^{\mathbb{R}}}$ for measurable

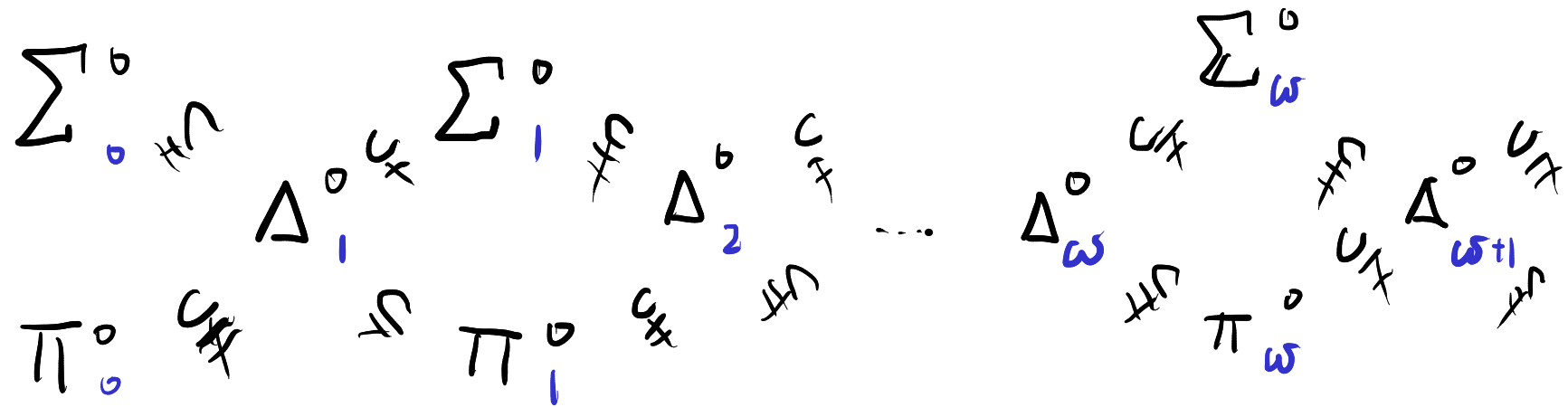
membership predicate $\leftarrow (\exists) : (B_{\mathbb{R}}, B_{B_{\mathbb{R}}}) \times \mathbb{R} \longrightarrow \text{Bool}$
 $(U, r) \mapsto [r \in U]$

eval : $(\text{Meas}(\mathbb{R}, \mathbb{R}), B_{\mathbb{R}^{\mathbb{R}}}) \times \mathbb{R} \rightarrow \mathbb{R}$
 $(f, r) \mapsto f(r)$

Questions! skip proof?

Proof (Sketch):

Borel hierarchy:



Stabilises at $\Delta_{\omega_1}^0 = B(\Sigma_0^0) = \Delta_{\omega_1+1}^0$

$$\text{rank } A := \min \{ \alpha < \omega_1 \mid A \in \Delta_\alpha^0 \}$$

then
for $B_{B_R} = P(B_R)$

$$(\exists) : (B_R, B_{B_R}) \times \mathbb{R} \rightarrow \mathbb{R}$$

$$(U, r) \mapsto [r \in U]$$

$$B_{V \times U} = B([B_V] \times [B_U])$$

If measurable:

$$\alpha := \text{rank}((\exists)^{-1}[\text{true}]) < \omega,$$

Take $A \in B_R$, $\text{rank } A > \alpha$

But:

$$\alpha < \text{rank } A = \text{rank}(A, \rightarrow)^{-1}[(\exists)^{-1}[\text{true}]] \leq \text{rank}((\exists)^{-1}[\text{true}]) \leq \alpha$$

~~##~~

More details in Ex. B

Sequential Higher-order structure:

I Countable : $V^I = \prod_{i \in I} V$

\Rightarrow Some higher-order structure in Meas:

Cauchy $\in B_{[-\infty, \infty]^N}$

$$\text{Cauchy} = \bigcap_{\epsilon \in \mathbb{Q}^+} \bigcup_{k \in \mathbb{N}} \bigcap_{\substack{m, n \in \mathbb{N} \\ m, n \geq k}} \{ \vec{y} \in [-\infty, \infty]^N \mid |y_m - y_n| < \epsilon \}$$

$\limsup : [-\infty, \infty]^N \rightarrow [-\infty, \infty]$ $\lim : \text{Cauchy} \rightarrow \mathbb{R}$

Compose higher-order building blocks:

lim is measurable!
↗

$$\text{VanishingSeq}(\mathbb{R}) := \left\{ \vec{r} \in \mathbb{R}^{\mathbb{N}} \mid \lim_{n \rightarrow \infty} r_n = 0 \right\} \in \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}$$

$$\text{approx}_\Delta : \text{VanishingSeq}(\mathbb{R}^+) \times \mathbb{R} \longrightarrow \mathbb{Q}^{\mathbb{N}}$$

$$\text{s.t.} : \quad |(\text{approx}_{\Delta} \vec{r})_n - r| < \Delta_n$$

Slogan: Measurable by Type !

Not all operations of interest fit:

$$\text{lim sup} : ([-\infty, \infty]^{\mathbb{R}})^{\mathbb{N}} \longrightarrow [-\infty, \infty]^{\mathbb{R}}$$

$$\text{lim sup} := \lambda \vec{f}. \lambda x. \limsup_{n \rightarrow \infty} f_n x$$

Intrinsically higher-order !

Want

Slogan: measurability by type!

But

For higher-order building blocks

defer measurability proofs until

we resume 1st order fragment \Rightarrow non compositional

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Plan

Def: $V \in \text{Meas}$ is **Standard Borel** when

$$V \cong A \quad \text{for some } A \in \mathcal{B}_{\mathbb{R}}$$

the "good part" of Meas — the subcategory

$$\text{Sbs} \hookrightarrow \text{Meas}$$

Sbs includes

- Discrete \mathbb{I} , \mathbb{I} countable
- Countable products of Sbs:

$$\mathbb{R}^n, \mathbb{R}^{\mathbb{N}}, \mathbb{Z}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}$$

- ~ Borel subspaces of Sbs:

$$\mathbb{I} := [0, 1]$$

$$\mathbb{R}^+ := (0, \infty) \quad \mathbb{R}_{\geq 0} := [0, \infty]$$

- Countable coproducts of Sbs:

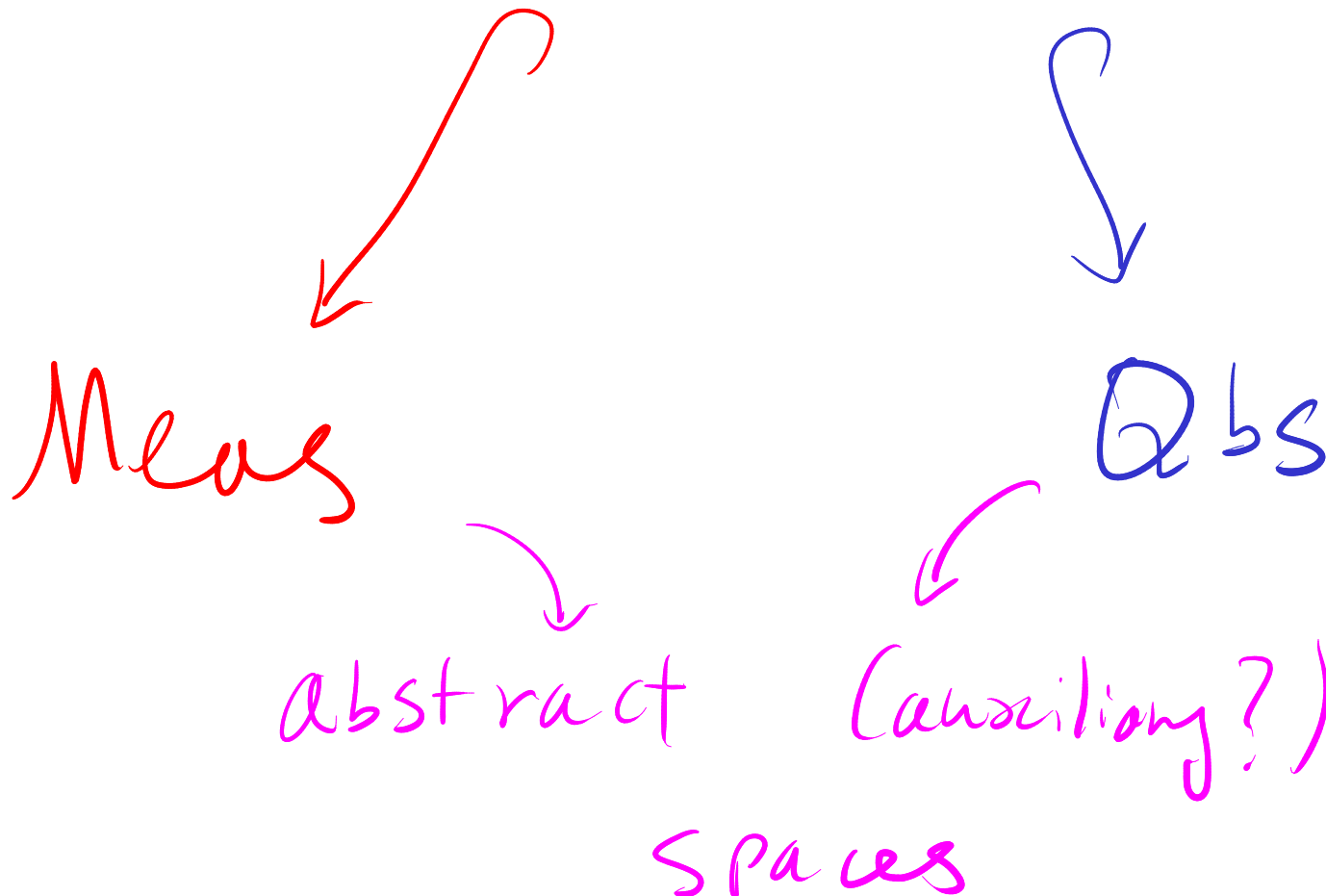
$$\mathbb{N} := [0, \infty]$$

$$\overline{\mathbb{R}} := [-\infty, \infty]$$

Conservative extensions:

Concrete spaces
we "observe"

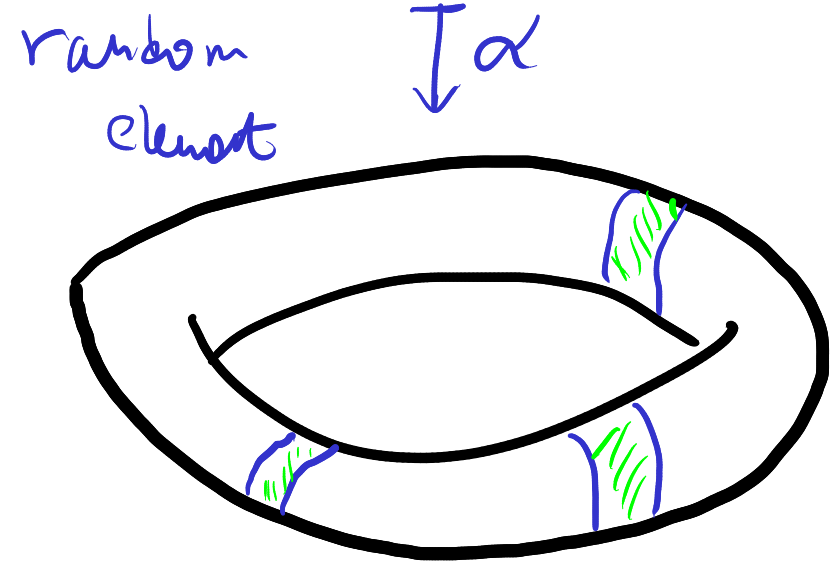
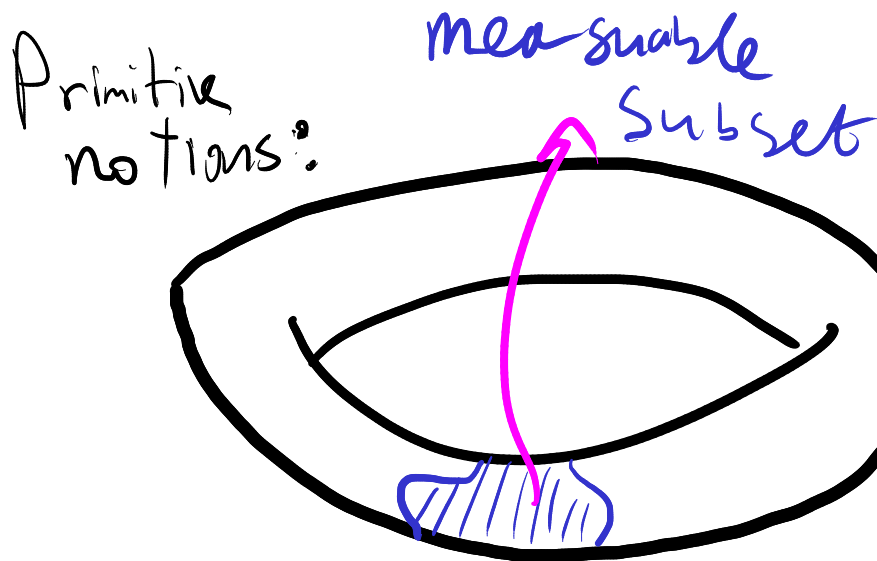
Standard Borel spaces



Cone idea

Measure Theory

sample space Ω Obs Theory



Derived notions:

random elements
 $\alpha: \Omega \rightarrow \text{Space}$

measure

measurable subsets

Def: Quasi-Borel space $X = (\mathcal{L}X, R_X)$

$R_X \subseteq \mathcal{L}X^{\mathbb{R}}$ Closed under:

Set
"carrier"

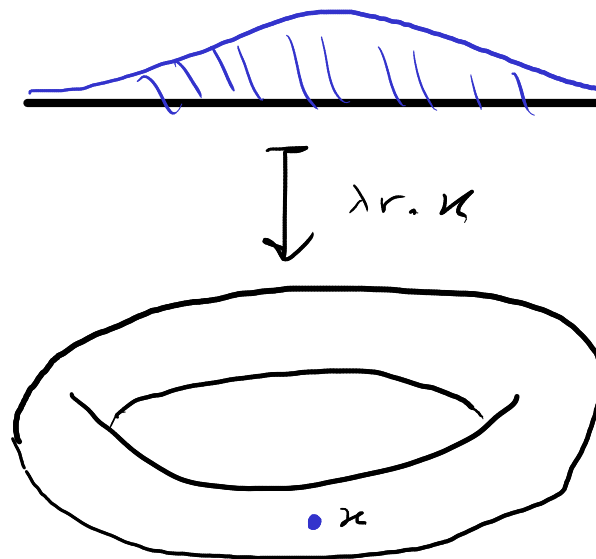
Set of
functions $\alpha: \mathbb{R} \rightarrow \mathcal{L}X$
"random elements"

- Constant S:

$$\frac{x \in \mathcal{L}X}{(\lambda r. x) \in R_X}$$

- Precomposition:

- recombination



Def: Quasi-Borel space

$$X = (\mathcal{X}, R_X)$$

Set
"carrier"

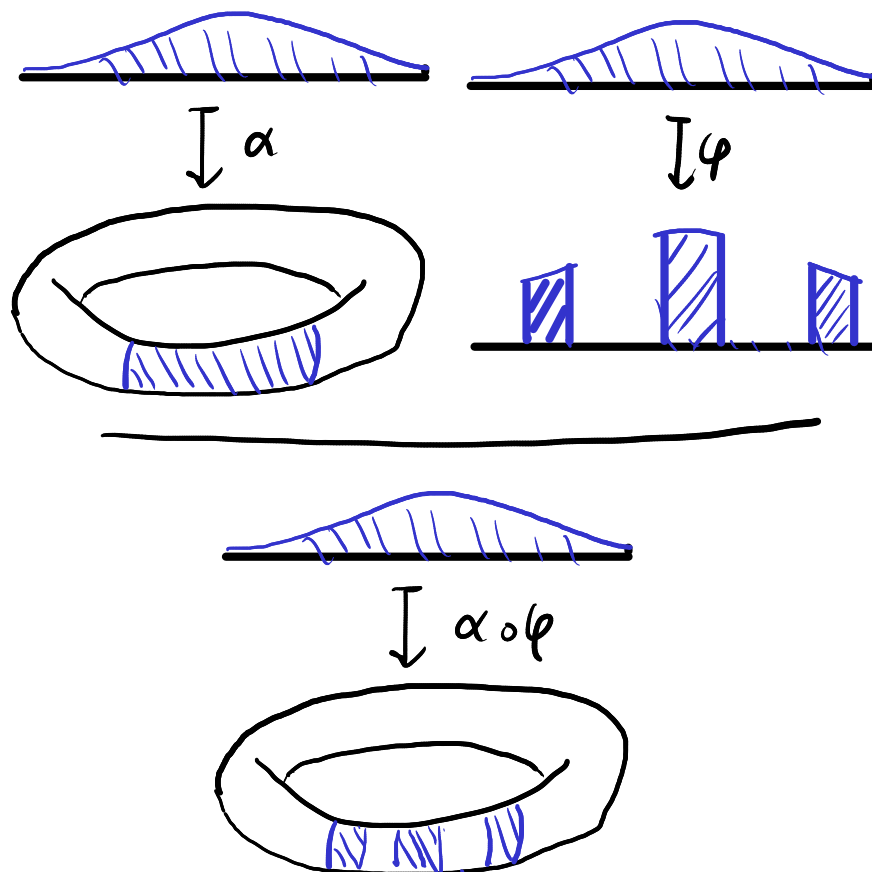
Set of
functions $\alpha: \mathbb{R} \rightarrow \mathcal{X}$
"random elements"

$R_X \subseteq \mathcal{X}^{\mathbb{R}}$ Closed under:

- precomposition:

$$\alpha \in R_X \quad \varphi: \mathbb{R} \rightarrow \mathbb{R} \text{ in Sbs}$$

$$\varphi \circ \alpha: \mathbb{R} \xrightarrow{\varphi} \mathbb{R} \xrightarrow{\alpha} \mathcal{X} \in R_X$$



Def: Quasi-Boel space

$$X = (\mathcal{X}, \mathcal{R}_X)$$

Set
"carrier"

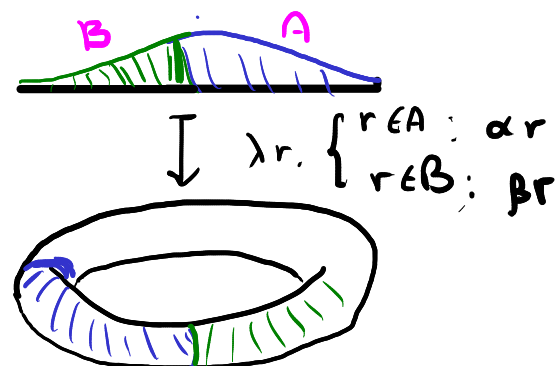
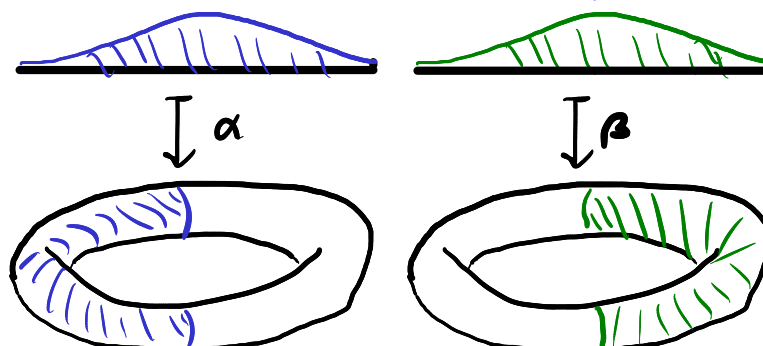
Set of
functions $\alpha: \mathbb{R} \rightarrow \mathcal{X}$
"random elements"

$$\mathcal{R}_X \subseteq \mathcal{L}(\mathbb{R}, \mathcal{X}) \quad \text{Closed under:}$$

- re combination

$$\vec{\alpha} \in \mathcal{R}_X^{\mathbb{N}} \quad \mathbb{R} = \bigcup_{n=0}^{\infty} A_n \quad \text{EBR}$$

$$\lambda r. \left\{ \begin{array}{c} \vdots \\ r \in A_n: \alpha_n r \\ \vdots \end{array} \right.$$



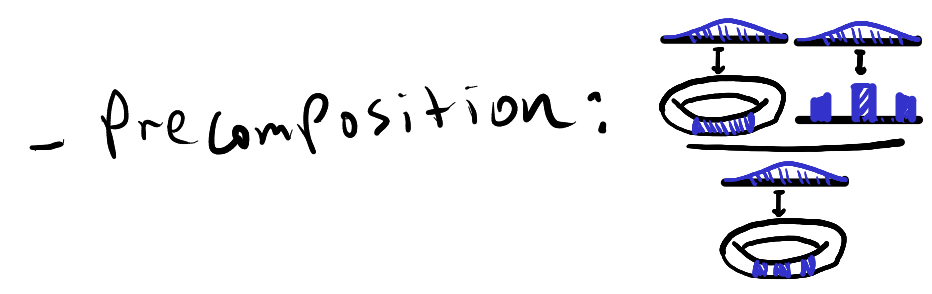
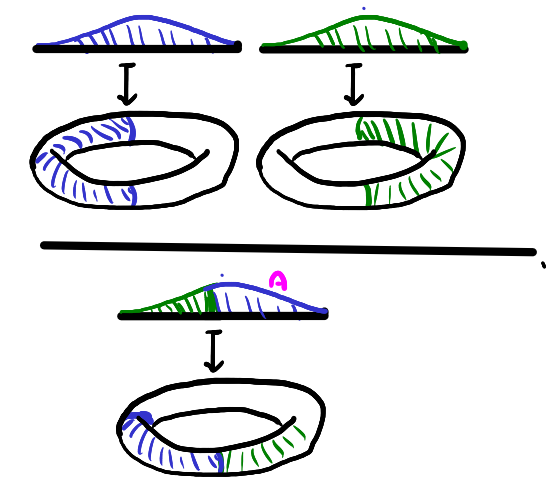
Def: Quasi-Borel space $X = (\mathcal{L}X, \mathcal{R}_X)$

$\mathcal{R}_X \subseteq \mathcal{L}X^{\mathbb{R}}$ Closed under:

Set "carrier" Set of functions $\alpha: \mathbb{R} \rightarrow \mathcal{L}X$ "random elements"



- recombination



Examples

recombination of constants

$$- \mathbb{R} = (\mathbb{R}_J, \text{Meas}(\mathbb{R}, \mathbb{R}))$$

qbs underlying \mathbb{R}

$$- X \in \text{Set}, \quad \ulcorner X \urcorner^{\text{qbs}} := (X, \sigma\text{-simple}(\mathbb{R}, X))$$

$\lambda r. \begin{cases} \vdots \\ r \in A_n: x_n \\ \vdots \end{cases}$

discrete qbs on X

$$- \quad \ulcorner X \urcorner_{\text{qbs}} := (X, X^{\mathbb{R}_J})$$

all functions

Indiscrete qbs on X

Qbs morphism $f: X \rightarrow Y$

- function $f: X_1 \rightarrow Y_1$

$$\frac{\alpha \downarrow_{X_1} \in R_X}{\text{---}}$$

$$\begin{array}{c} R \\ \alpha \downarrow \\ X_1 \\ f \downarrow \\ Y_1 \end{array} \in R_Y$$

Example

- Constant functions

are qbs
morphisms

- σ -simple functions
are qbs morphisms

Category Qbs

\Leftarrow

- identity, composition

Discrete model

$\text{type} : \text{Obs} \quad W := [0, \infty] \quad Bx := (\text{Thur})$

$DX := (\text{Fri})$

$Px := \{ \mu \in DX \mid C_{\mu}[X] = 1 \} \quad (\text{Thu})$

$C_{\mu}[E] := (\text{Fri}) \quad \delta_n := (\text{Fri})$

$\phi \mu_k := (\text{Fri})$

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