Modular abstract syntax trees (MAST): substitution tensors with second-class sorts

Marcelo Fiore, Kajetan Granops, Mihail-Codrin Iftode, Ohad Kammar,
Georg Moser, and Sam Staton

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Computer and Information Sciences, University of Strathclyde, Glasgow, Scotland

THE UNIVERSITY of EDINBURGH informatics lfcs Laboratory for Foundations of Computer Science BayesCentre

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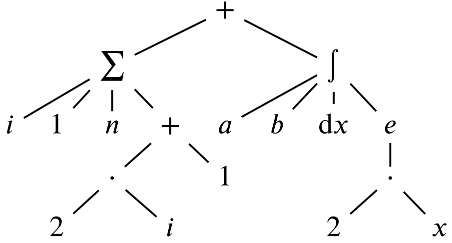
Syntax representation

$$\left(\sum_{i=1}^{n}(2i+1)\right) + \int_{a}^{b} e^{ax}\mathrm{d}x$$

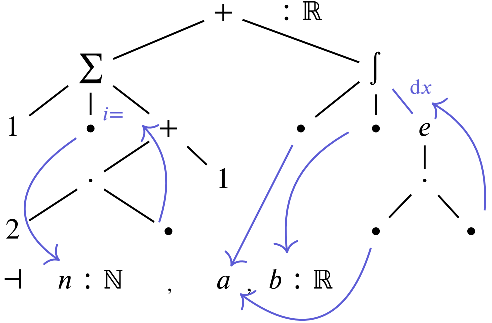
Concrete syntax

“(”, “$\sum$”, “_”, “{”, “i”, “=”, “1”, “}”, “{”, “n”, “(”, “2”, “i”, “+”, “1”, “)”, “}”, …

Abstract syntax



Abstract syntax with binding



Call-by-Value λ-calculus

$$
\begin{array}{rll}
A, B, C ::= & & \text{type} \\
& \beta & \text{base} \\
| & A \to B & \text{function} \\
| & (\!| C_i : A_i \,|\, i \in I |\!) & \text{record } (I \text{ finite}) \\
| & \{\!| C_i : A_i \,|\, i \in I |\!\} & \text{variant } (I \text{ finite}) \\
\vdots & &
\end{array}
$$

$$
\begin{array}{rll}
V, W ::= & & \text{value} \\
& x & \text{variable} \\
| & \lambda x : A.M & \text{function abst.} \\
| & (C_i : V_i \,|\, i \in I) & \text{record c'tor} \\
| & A.C_i\ V & \text{variant c'tor} \\
\vdots & &
\end{array}
$$

$$
\begin{array}{rll}
M, N, K, L ::= & & \text{term} \\
& \mathsf{val}\, V & \text{value} \\
| & \mathbf{let}\ x_1 = M_1; \ldots; x_n = M_n\ \mathbf{in}\ N & \text{sequencing} \\
| & M \mathbin{@} N & \text{function application} \\
| & (C_1 : M_1, \ldots, C_n : M_n) & \text{record constructor} \\
| & \mathbf{case}\ M\ \mathbf{of}\ (C_1 x_1, \ldots, C_n x_n) \Rightarrow N & \text{record pattern match} \\
| & A.C_i M & \text{variant constructor} \\
| & \mathbf{case}\ M\ \mathbf{of}\ \{C_i x_i \Rightarrow M_i \,|\, i \in I\}\ N & \text{variant pattern match} \\
\vdots & &
\end{array}
$$

High-level motivation

Initial Algebra Semantics Programme [Goguen and Thatcher'74]

Denotational semantics á la carte [homage to Swierstra'08, Forster and Stark'20]

CBV customisation menu

fragment	syntactic constructs	types	semantics
base	returning a value: val		strong monad over a Cartesian category
sequential	sequencing: **let**		
functions	abst., app. $(\lambda x.:A), (@)$	function $(\to)$	Kleisli exponentials
variants	c'tors, pattern match $A.C_i-$, $\mathbf{case}-\mathbf{of}$ $\{C_i x_i \Rightarrow - \mid i \in I\}$	variant $\{\mid C_i : - \mid i \in I \mid\}$	distributive category

⋮

Dream

Iterative semantic development

- Add syntax
- Add semantics

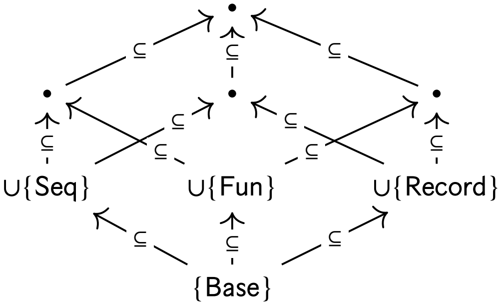
$$\begin{tikzcd}
 & \bullet & \\
\bullet \arrow[ur, "\subseteq"] & \bullet \arrow[u, "\subseteq"] & \bullet \arrow[ul, "\subseteq"'] \\
\cup\{\mathsf{Seq}\} \arrow[u, "\subseteq"] \arrow[ur, "\subseteq"] & \cup\{\mathsf{Fun}\} \arrow[ul, "\subseteq"] \arrow[ur, "\subseteq"'] & \cup\{\mathsf{Record}\} \arrow[u, "\subseteq"] \arrow[ul, "\subseteq"'] \\
 & \{\mathsf{Base}\} \arrow[ul, "\subseteq"] \arrow[u, "\subseteq"] \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

- Profit!

Dream vs. Reality

Iterative semantic development

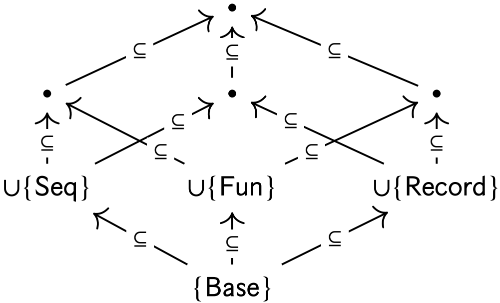
- Add syntax
- Add semantics
- Develop meta-theory:
  - Substitution lemma
  - Compositionality
  - Soundness
  - Adequacy
- Profit!



Dream vs. Bleak Reality

Iterative semantic development

- Add syntax
- Add semantics
- Develop meta-theory:
  - Substitution lemma
    Tedious and boring
  - Compositionality
    Tedious and boring
  - Soundness
  - Adequacy
- Profit!



Meta-theory: the tedious parts

Lemma (substitution)

Syntactic substitution corresponds to semantic composition:

$$[\![M\,[\theta]]\!] = [\![M]\!] \circ [\![\theta]\!]$$

Lemma (compositionality)

Composite semantics is independent of component syntax:

$$[\![C[M]]\!] = \mathrm{plug}([\![C[-]]\!]\,, [\![M]\!])$$

Meta-theory: the tedious parts

Lemma (substitution)

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Proof.

Presupposes a syntactic substitution lemma. Typically several inductions over all constructs. □

Lemma (compositionality)

Composite semantics is independent of component syntax:

$$[\![C[M]]\!] = \mathrm{plug}([\![C[-]]\!]\,, [\![M]\!])$$

Proof.

Tediously define terms with holes, plugging holes syntactically, carefully capturing some variables but not others. Then induction over semantics. □

Dream

It would be nice if tedious bits were...

... free

It would be nice if tedious bits were…

… ~~free~~

… syntactically scaleable: additive syntactic work per new feature

$$\begin{tikzcd}
 & & \bullet & & \\
\bullet \arrow[urr, "\subseteq" description] & & \bullet \arrow[u, "\subseteq" description] & & \bullet \arrow[ull, "\subseteq" description] \\
\cup\{\mathsf{Seq}\} \arrow[u, "\subseteq" description] \arrow[urr, "\subseteq" description] & & \cup\{\mathsf{Fun}\} \arrow[ull, "\subseteq" description] \arrow[urr, "\subseteq" description] & & \cup\{\mathsf{Record}\} \arrow[u, "\subseteq" description] \arrow[ull, "\subseteq" description] \\
 & & \{\mathsf{Base}\} \arrow[ull, "\subseteq" description] \arrow[u, "\subseteq" description] \arrow[urr, "\subseteq" description] & &
\end{tikzcd}$$

Expression problem [Reynolds'75, Cook'90, Krishnamurthi, Felleisen and Friedman'98, Wadler'98]

Spec [Wadler'98]

Both:

- **Extend** object-language syntax
- **Add** meta-language functions/properties of programs

While:

- Without recompiling previous modules; alternatively
- Retaining and reusing both old and new languages

Some solutions

- Scala Mixings [Zenger'98, Zenger and Odersky'01]
- Visitor Pattern in Pizza, Zodiac [Krishnamurthi, Felleisen and Friedman'98]
- Recursive Generics [Wadler'98]
- Data-types á la carte: coproducts of signature functors [Swierstra'08]

- Initial algebra characterisation for abstract syntax with binding-aware substitution
- Robust to extensions:
  - polymorphism [Fiore and Hamana'13]
  - mechanisation [Crole'11, Allais et al.'18, Fiore and Szamozvancev'22]
  - substructurality [Fiore and Ranchod'25]
- CBN works smoothly. Doesn't cover CBV.
  Technical reasons later:
  - Substitute **in**: values and terms
  - Substitute for variables: values only

Slogan

for substitution: values are 1^{st}-class but terms are 2^{nd}-class
[cf. Levy's CBPV, '04]

Goal: abstract syntax with heterogeneous sorting

Sorting system $\mathbf{R}$

- set sort

partitioned into

- bindable/1^{st}-class sorts
  $s \in \mathsf{Bind}$
- non-bindable/2^{nd}-class sorts

Example (CBV sorting system)

- $\mathsf{sort} := \left\{A, \mathsf{comp}\ A \middle| A \in \mathsf{Type}_{\mathrm{CBV}}\right\}$
- $\mathsf{Bind} := \mathsf{Type}_{\mathrm{CBV}}$

Example (CBPV sorting system)

- $\mathsf{Bind} := \{A | A \text{ value type}\}$
- $\mathsf{sort} := \mathsf{Type}_{\mathrm{CBPV}}$

Core contribution

classical theory ($\textsc{soas}$)		this work ($\textsc{mast}$)
$\mathbf{PSh}\,(\mathsf{sort} \times \mathsf{sort}_{\vdash}), \otimes$ monoidal product	generalise $\rightsquigarrow$	$\mathbf{PSh}\,(\mathsf{sort} \times \mathsf{Bind}_{\vdash}), \otimes$ right-unital associative **skew** monoidal product

Monoidal tensor

$$(P \otimes Q) \otimes L \cong P \otimes (Q \otimes L)$$

$$P \otimes \mathbb{I} \cong P$$

$$\mathbb{I} \otimes Q \cong \boldsymbol{\ell} Q$$

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classical theory (SOAS)

$\mathbf{PSh}\,(\mathsf{sort} \times \mathsf{sort}_{\vdash}), \otimes$

monoidal product

generalise $\rightsquigarrow$

this work (MAST)

$\mathbf{PSh}\,(\mathsf{sort} \times \mathsf{Bind}_{\vdash}), \otimes$

right-unital associative

skew monoidal product

Skew monoidal tensor

$$(P \otimes Q) \otimes L \to P \otimes (Q \otimes L)$$

$$P \otimes \mathbb{I} \xleftarrow{\mathbf{r}'} P$$

$$\mathbb{I} \otimes Q \xrightarrow{\ell} Q$$

Core contribution

classical theory (SOAS)

$\mathbf{PSh}\,(\mathsf{sort} \times \mathsf{sort}_{\vdash}), \otimes$

monoidal product

generalise $\rightsquigarrow$

this work (MAST)

$\mathbf{PSh}\,(\mathsf{sort} \times \mathsf{Bind}_{\vdash}), \otimes$

right-unital associative

skew monoidal product

Heterogeneous substitution tensor

$(P \otimes Q) \otimes L \cong P \otimes (Q \otimes L)$ (associative)

$P \otimes \mathbb{I} \cong P$ (right-unital)

$\mathbb{I} \otimes Q \xrightarrow{\ell} Q$ (non-invertible!) $(\mathbb{I} \otimes \mathbb{1})_s = \emptyset \ncong \mathbb{1}_s (s \notin \mathsf{Bind})$

Contribution

Modular Abstract Syntax Trees (MAST)

- SOAS $\overset{\text{generalise}}{\rightsquigarrow}$ 2^{nd}-class sorts
  Using **skew** bicategories/monoidal categories, and:
  - Kleisli bicategories [Gambino, Fiore, Hyland, and Winskel'19]
  - Familial theory of SOAS [Fiore and Szamozvancev'25]
- MAST tutorial
- Case-study: CBV semantics á la carte (128 substitution lemmata)

WIP

- Idris 2 implementation of computational fragment [cf. Fiore and Szamozvancev'22]
  Case-study: intrinsically-typed FFI-binding with holes for SMTLIB (29 theories)
- Replace skew monoidal structure and monoids with
  monoidal structure and actions

[cf. Fiore and Turi'01]

Talk structure

- Contribution
- Substitution monoids
- MAST in detail
- WIP

Capstone: abstract syntax and substitution universality

Thm (representation)

abstract syntax with operators in $\mathbf{O}$ and holes in $\mathbf{H}$
amounts to
free substitution $\mathbf{O}$-monoid over $\mathbf{H}$:

$$\begin{tikzcd} & \mathbf{H} \arrow[d, "?"] & \\ \$\mathbf{H} \otimes \$\mathbf{H} \arrow[r, "-[-]"] & \$\mathbf{H} & \mathbb{I} \arrow[l, "\mathsf{var}"'] \\ & \mathbf{O}(\$\mathbf{H}) \arrow[u, "[\![-]\!]"] & \end{tikzcd}$$

Plugging holes/metavariable substitution

Kleisli extension ($\gg\!=$) for $\mathbf{O}$-monoid monad.

Key propaganda

compositional, binding-respecting denotational semantics
amounts to
substitution $\mathbf{O}$-monoid:

$$
\begin{tikzcd}
\mathbf{M} \otimes \mathbf{M} \arrow[r, "-[-]"] & \mathbf{M} & \mathbb{I} \arrow[l, "\mathsf{var}"'] \\
& \mathbf{O}\mathbf{M} \arrow[u, "[\![-]\!]"] &
\end{tikzcd}
$$

The denotational semantics for terms with holes in $\mathbf{H}$ is the unique substitution $\mathbf{O}$-monoid homomorphism over $\mathbf{H}$:

$$(\$\mathbf{H}, -[-], \mathsf{var}, [\![-]\!], ?) \xrightarrow{[\![-]\!]} (\mathbf{M}, -[-], \mathsf{var}, [\![-]\!], \mathsf{menv}) \qquad (\mathbf{H} \xrightarrow{\mathsf{menv}} \mathbf{M})$$

Meta-theory in one line

Lemma (substitution)

Syntactic substitution corresponds to semantic composition:

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Lemma (compositionality)

Composite semantics is independent of component syntax:

$$[\![C[M]]\!] = [\![C[?m] \gg\!\!= (m \mapsto M)]\!] \overset{\substack{\gg\!\!=\ \text{is homomorphic extension} \\ \downarrow}}{=} [\![C[?m]]\!] \gg\!\!= (m \mapsto [\![M]\!]) \eqqcolon \mathrm{plug}([\![C[-]]\!] , [\![M]\!])$$

Talk structure

- Contribution
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- MAST in detail
- WIP

MAST summary: semantic domain for syntax and semantics

MAST provides ($\mathbf{R} = (\mathsf{sort}, \mathsf{Bind})$ sorting system)

- Contexts $\qquad \mathsf{Bind}_{\vdash} \ni \Gamma ::= [x_1 : s_1, \ldots, x_n : s_n]$
- Renamings $\qquad \mathsf{Bind}_{\vdash}(\Gamma, \Delta) \ni \Gamma \vdash \rho : \Delta$
- $\mathbf{R}$-structures: $\qquad \mathbf{PSh}(\mathsf{sort} \times \mathsf{Bind}_{\vdash}) \ni P : \mathsf{sort} \times \mathsf{Bind}_{\vdash}^{\mathsf{op}} \to \mathbf{Set}$
  $P_s\Gamma \ni p$: sort s element with variables in Γ
- Variables structure: $\qquad \mathbf{R\text{-}Struct} \ni \mathbb{I}_s\Gamma := \{x | (x : s) \in \Gamma\}$
- substitution tensor and the pointed action:

$(\otimes) : \mathbf{R\text{-}Struct} \times \mathbf{R\text{-}Struct} \to \mathbf{R\text{-}Struct}$

$(P \otimes Q)_s\Gamma \ni [p, \theta]_\Delta$:

P-element: $p \in P_s\Delta$ $\quad$ identifying, e.g.:

Q-closure : $\theta \in \prod_{(y:r)\in\Delta} Q_r\Gamma$

$(\otimes_\bullet) : \mathbf{R\text{-}Struct} \times (\mathbb{V}/\mathbf{R\text{-}Struct}) \to \mathbf{R\text{-}Struct}$

$$P \otimes_\bullet \left(\mathsf{var} \begin{array}{c}\mathbb{I}\\ \downarrow \\ A\end{array}\right) := P \otimes A$$

$$[p[\mathsf{weaken}], \theta]_{\Delta_1 + \Delta_2} = [p, \theta \circ \rho]_{\Delta_1}$$

$$\left[p[x', x'' \mapsto x]_{x \in \Delta}, \theta\right]_{\Delta} = [p, \theta + \theta]_{\Delta + \Delta}$$

Scope-change as tensorial strength

$$\mathsf{str}^{\mathbf{O}} : (\mathbf{O}P) \otimes_\bullet \left(\mathsf{var} \begin{array}{c}\mathbb{I}\\ \downarrow \\ A\end{array}\right) \to \mathbf{O}\left(P \otimes_\bullet \left(\mathsf{var} \begin{array}{c}\mathbb{I}\\ \downarrow \\ A\end{array}\right)\right)$$

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  I.e.:

  $(P \otimes Q)_s\Gamma := \int^{\Delta} P_s\Delta \times \prod_{(y:r)\in\Delta} Q_r\Gamma$

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Substitution monoids

$$\mathbf{M} \otimes \mathbf{M} \xrightarrow{-[-]} \mathbf{M} \xleftarrow{\mathsf{var}} \mathbb{I}$$

What breaks the unitor?

Substitution tensor

$$(P \otimes Q)_s \Gamma := \int^{\Delta} P_s \Delta \times \prod_{(y:r) \in \Delta} Q_r \Gamma$$

for $s \notin \mathsf{Bind}$, $Q = \mathbb{1}$, $\mathbb{I}_s \Delta = \emptyset$:

$$(\mathbb{I} \otimes Q)_s \Gamma := \int^{\Delta} \overbrace{\emptyset}^{\mathbb{I}_s \Delta} \times \prod_{(y:r) \in \Delta} Q_r \Gamma = \int^{\Delta} \emptyset = \emptyset \neq \mathbb{1} = Q_s \Gamma$$

MAST: semantic domain for syntax

Signature functors

$$\begin{tikzcd} \mathbf{R\text{-}Struct} \arrow["\mathbf{O}"', loop left] \end{tikzcd}$$

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NB: $(\otimes_{\bullet}) : \mathbf{R\text{-}Struct} \times (\mathbb{I}/\mathbf{R\text{-}Struct}) \to \mathbf{R\text{-}Struct}$

$$P\otimes_{\bullet}\left(\mathsf{var}\ \begin{smallmatrix}\mathbb{I}\\ \downarrow\\ A\end{smallmatrix}\right) := P \otimes A$$

Example

Sequential fragment signature functor:

$$(\mathsf{Seq}\ X)_A\Gamma := \emptyset \qquad (\mathsf{Seq}\ X)_{\mathsf{comp}\ B}\Gamma := \coprod_{A\in\mathsf{Type}} \begin{pmatrix} (\mathbf{let}\ x : A = \_\ \mathbf{in}\ \_) : \\ (X_{\mathsf{comp}\ A}\Gamma \times X_{\mathsf{comp}\ B}(\Gamma, x : A)) \end{pmatrix}$$

$$\mathsf{str}^{\mathsf{Seq}}_{P,\mathsf{var}}\left[t \in \mathsf{Seq}\ P, \theta\right]_{\Delta} = \mathsf{str}^{\mathsf{Seq}}\left[\mathbf{let}\ x : A = (p \in P_{\mathsf{comp}\ A}\Delta)\ \mathbf{in}\ (q \in P_{\mathsf{comp}\ B}(\Delta, x : A)), \theta\right]_{\Delta}$$

$$:= \left(\mathbf{let}\ x : A = [p,\theta]_{\Delta} \qquad \mathbf{in}\ [q, (\theta, x : \mathsf{var}\ x)]_{\Delta, x : A}\right)$$

Takeaway (modularity)

Each syntactic construct defines its own binding, renaming, and substitution structure

Signature combinators [cf. SOAS]

- sums & products of signature functors
- scope extension $(\Gamma \triangleright)$
- sort extension $\underset{s}{\looparrowright} : \mathbf{PSh}\,\mathsf{Bind}_{\vdash} \to \mathbf{PSh}\left(\mathsf{sort} \times \mathsf{Bind}_{\vdash}\right)$
- sort application $(@\, s) : \mathbf{PSh}\left(\mathsf{sort} \times \mathsf{Bind}_{\vdash}\right) \to \mathbf{PSh}\left(\mathsf{Bind}_{\vdash}\right)$

Example (Binding signatures [Actzel'78])

$\mathsf{Seq} \cong$

$$\coprod_{A,B \in \mathsf{Type}} \left(\begin{array}{l} (\mathbf{let}\ x : A = \_\ \mathbf{in}\ \_) : \underset{\mathsf{comp}\ B}{\looparrowright} \\ \quad (@\mathsf{comp}\ A) \times \\ \qquad ([x : A] \triangleright - @\ \mathsf{comp}\ B) \end{array} \right)$$

NB

$$(\mathsf{Seq}\ X)_{\mathsf{comp}\ B}\Gamma := \coprod_{A \in \mathsf{Type}} \left(\begin{array}{l} (\mathbf{let}\ x : A = \_\ \mathbf{in}\ \_) : \\ \left(X_{\mathsf{comp}\ A}\Gamma \times X_{\mathsf{comp}\ B}(\Gamma, x : A)\right) \end{array} \right)$$

$$(\mathsf{Seq}\ X)_A\Gamma := \emptyset$$

Abstract syntax: inductive representation

Every initial algebra:

$$\$^{\mathbf{O}}\mathbf{H} \coloneqq \mu X.(\mathbf{O}X) \amalg \mathbb{I} \amalg \mathbf{H} \otimes X$$

Supports standard definitions:

$$\begin{array}{ccccc} & & \mathbf{H} & & \\ & & \downarrow ?\!-\![-] & & \\ \$\mathbf{H} \otimes \$\mathbf{H} & \xrightarrow{-[-]} & \$\mathbf{H} & \xleftarrow[\mathsf{var}]{} & \mathbb{I} \\ & & [\![-]\!] \uparrow & & \\ & & \mathbf{O}(\$\mathbf{H}) & & \end{array}$$

Independently of concrete representation, e.g.,:

- De-Bruijn
- Nominal
- Locally nameless
- Co-de Bruijn
- Graphical

Example

$\mathbf{M} = \left(\mathcal{C}, \mathrm{T}, \mathrm{return}, \gg\!=, [\![-]\!]\right)$:

- $\mathcal{C}$: Cartesian category with chosen finite products
- $(\mathrm{T}, \mathrm{return}, \gg\!=)$ strong monad over $\mathcal{C}$
- $[\![-]\!]$: Type $\to \mathcal{C}$ type interpretation

induces:

- A CBV-structure: $\text{CBV-}\mathbf{Struct} \ni \mathbf{M}_s\Gamma := \mathcal{C}([\![\Gamma]\!], [\![s]\!])$
- Standard interpretation of contexts, computations, renaming:

$$\mathcal{C} \ni [\![\Gamma]\!] := \prod_{(x:A)\in\Gamma} [\![A]\!] \qquad \mathcal{C} \ni [\![\mathsf{comp}\ A]\!] := \mathrm{T}\,[\![A]\!]$$

$$[\![\rho]\!] : [\![\Gamma]\!] \xrightarrow{\left(\pi_{x[\rho]} : [\![\Gamma]\!] \to [\![A]\!]\right)_{(x:A)\in\Delta}} \prod_{(x:A)\in\Delta} [\![A]\!] = [\![\Delta]\!]$$

MAST: common structure for substitution

Syntactic substitution monoid

$$\$^{\mathbf{O}}\mathbf{H} \otimes \$^{\mathbf{O}}\mathbf{H} \xrightarrow{-[-]} \$^{\mathbf{O}}\mathbf{H} \xleftarrow{\mathsf{var}} \mathbb{I}$$

Monoid axioms amount to syntactic substitution lemma

Example

Semantic substitution monoid: $\mathbf{M} \otimes \mathbf{M} \xrightarrow{-[-]} \mathbf{M} \xleftarrow{\mathsf{var}} \mathbb{I}$

- Substitution via composition:

$$\left(\llbracket\Delta\rrbracket \xrightarrow{f} \llbracket s\rrbracket\right)\left[\llbracket\Gamma\rrbracket \xrightarrow{\theta} \llbracket\Delta\rrbracket\right] : \llbracket\Gamma\rrbracket \xrightarrow{\theta} \llbracket\Delta\rrbracket \xrightarrow{f} \llbracket s\rrbracket$$

- Variables: $\mathsf{var} : \left((x : A) \in \Gamma \mapsto \left(\llbracket\Gamma\rrbracket \xrightarrow{\pi_x} \llbracket A\rrbracket\right)\right)$

  (1^{st}-class sorts only)

Substitution-compatible algebra

$[\![-]\!] : \mathbf{OM} \to \mathbf{M}$:

$$
\begin{tikzcd}
& \underline{\mathbf{O}}(\underline{\mathbf{M}} \otimes \underline{\mathbf{M}}) \arrow[dr, "\underline{\mathbf{O}}(-[-]_{\mathbf{M}})"] & \\
(\underline{\mathbf{OM}}) \otimes_\bullet \mathsf{var}^{\mathbf{M}} \arrow[ur, "\mathsf{str}"] \arrow[d, "{[\![-]\!] \otimes_\bullet \mathsf{id}}"'] \arrow[rr, phantom, "\overset{\text{compatibility}}{=}"] & & \underline{\mathbf{OM}} \arrow[d, "{[\![-]\!]}"] \\
\underline{\mathbf{M}} \otimes_\bullet \mathsf{var}^{\mathbf{M}} \arrow[rr, "-[-]_{\mathbf{M}}"'] & & \underline{\mathbf{M}}
\end{tikzcd}
$$

Example (Seq-algebra)

$$
\left[\!\!\left[\begin{array}{l} \mathbf{let}\ x : A = \left([\![\Gamma]\!] \xrightarrow{f} \mathrm{T}\,[\![A]\!]\right) \\ \mathbf{in}\ \left([\![\Gamma]\!] \times [\![A]\!] \xrightarrow{g} \mathrm{T}\,[\![B]\!]\right) \end{array} \right]\!\!\right] :
$$

$$
[\![\Gamma]\!] \xrightarrow{(\mathsf{id}, f)} [\![\Gamma]\!] \times \mathrm{T}\,[\![A]\!] \xrightarrow{\gg\!= g} \mathrm{T}\,[\![B]\!]
$$

Takeaway

Equip each semantic interpretation with its compatibility proof

Example (Seq-compatibility)

Compatibility:

$$
\begin{tikzcd}
{[\![\Gamma]\!]} \arrow[r, "{(\mathsf{id}, (f \circ \theta))}"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\overset{\text{products}}{=}"] & {[\![\Gamma]\!] \times \mathrm{T}\,[\![A]\!]} \arrow[d, "\theta \times \mathsf{id}"] \arrow[dr, bend left, "{\gg\!= (g \circ (\theta \times \mathsf{id}))}"] \arrow[dr, phantom, "\overset{\text{strong monad laws}}{=}"] & \\
{[\![\Delta]\!]} \arrow[r, "{(\mathsf{id}, f)}"'] & {[\![\Delta]\!] \times \mathrm{T}\,[\![A]\!]} \arrow[r, bend right, "\gg\!= g"'] & \mathrm{T}\,[\![B]\!]
\end{tikzcd}
$$

Substitution $\mathbf{O}$-monoid

Substitution monoid with compatible $\mathbf{O}$-algebra structure

$$
\begin{tikzcd}
 & & \bullet & & \\
\bullet \arrow[urr, "\subseteq"] & & \bullet \arrow[u, "\subseteq"] & & \bullet \arrow[ull, "\subseteq"'] \\
\cup\{\mathsf{Seq}\} \arrow[u, "\subseteq"] \arrow[urr, "\subseteq"] & & \cup\{\mathsf{Fun}\} \arrow[ull, "\subseteq"] \arrow[urr, "\subseteq"] & & \cup\{\mathsf{Record}\} \arrow[u, "\subseteq"] \arrow[ull, "\subseteq"'] \\
 & & \{\mathsf{Base}\} \arrow[ull, "\subseteq"] \arrow[u, "\subseteq"] \arrow[urr, "\subseteq"'] & &
\end{tikzcd}
$$

Want more?

In the paper:

- All the details
- A CBV case-study $(128$ substitution lemmata$)$

Talk structure

- Contribution
- Substitution monoids
- MAST in detail
- WIP

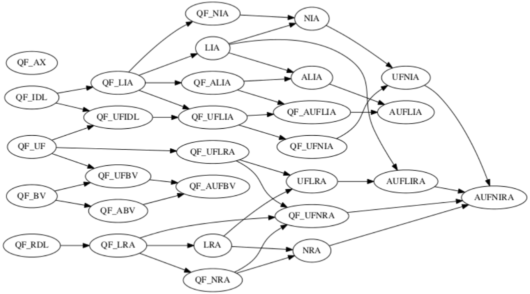
SMTLIB Foreign Function Interface (FFI)

Implementation

Idris 2 [Brady'21] implementation of computational fragment [cf. Fiore and Szamozvancev'22]

SMTLIB query language

- S-expressions
- 29 theories
- multiple syntax extensions



FFI

- Intrinsically-typed well-scoped FFI with holes
- Modular serialisation
- Modular well-scoped parsing
- Modular type-inference

[Greg Brown'25]

Non-skew structure with actions

(time permitting on board) [cf. Fiore and Turi'01]

Contribution

Modular Abstract Syntax Trees (MAST)

- SOAS $\overset{\text{generalise}}{\rightsquigarrow}$ 2[nd]-class sorts
  Using **skew** bicategories/monoidal categories, and:
  - Kleisli bicategories [Gambino, Fiore, Hyland, and Winskel'19]
  - Familial theory of SOAS [Fiore and Szamozvancev'25]
- MAST tutorial
- Case-study: CBV semantics á la carte (128 substitution lemmata)

WIP

- Idris 2 implementation of computational fragment [cf. Fiore and Szamozvancev'22]
  Case-study: intrinsically-typed FFI-binding with holes for SMTLIB (29 theories)
- Replace skew monoidal structure and monoids with
  monoidal structure and actions

[cf. Fiore and Turi'01]