Foundations for type-driven probabilistic modelling

Ohad Kammar
University of Edinburgh

Logic Summer School
Australian National University
4–16 December, 2023
Canberra, ACT, Australia

THE UNIVERSITY *of* EDINBURGH informatics lfcs Laboratory for Foundations of Computer Science BayesCentre

supported by: THE ROYAL SOCIETY The Alan Turing Institute Facebook Research NCSC

Plan:

1) type-driven probability: discrete case (Mon + Tue)

2) Borel sets & measurable spaces (Wed)

3) Quasi Borel spaces (Wed) Simple type structure (Thu)

4) Dependent type structure & standard Borel spaces (Thu)

5) Integration & random variables (Fri)

Please ask questions!

Course web page

Scribble

Discrete model

type : Qbs $\quad W := [0,\infty] \quad BX :=$ (Thur)

$DX :=$ (Fri)

$PX := \{\mu \in DX \mid \underset{\mu}{\mathbb{E}}[X] = 1\}$ (Thu)

$\underset{\mu}{\mathbb{E}}[E] :=$ (Fri) $\quad \delta_x :=$ (Fri)

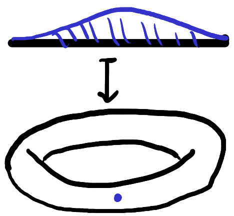
$\oint \mu k :=$ (Fri)

Def: Quasi-Borel space $X = (\lfloor X \rfloor, R_X)$

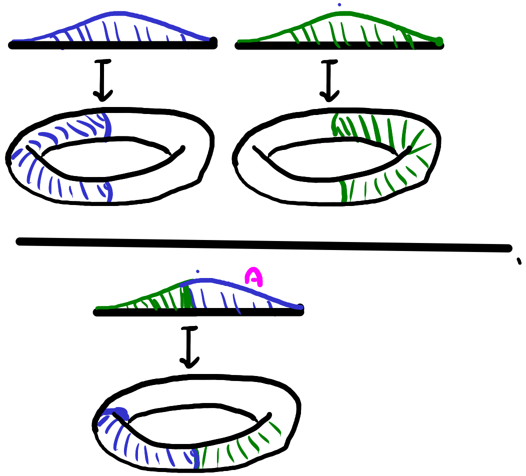
set "carrier"

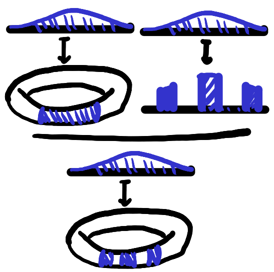
set of functions $\alpha : \mathbb{R} \to \lfloor X \rfloor$ "random elements"

$R_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- Constants: 

- recombination



- precomposition: 

Examples

recombination of constants

- $\mathbb{R} = (\lfloor\mathbb{R}\rfloor, \mathrm{Meas}(\mathbb{R},\mathbb{R}))$

qbs underlying $\mathbb{R}$

$$\lambda r. \begin{cases} \vdots \\ r \in A_n : x_n \\ \vdots \end{cases}$$

- $X \in \mathrm{Set}$, $\ulcorner X \urcorner^{\mathrm{Qbs}} := (X, \sigma\text{-simple}(\mathbb{R}, X))$

discrete qbs on X

- " $\lfloor X \rfloor_{\mathrm{Qbs}} := (X, X^{\lfloor\mathbb{R}\rfloor})$

all funtions

Indiscrete qbs on X

<u>Validate</u> qbs <u>axioms for</u>: $\mathbb{W} := ([0,\infty], \mathrm{Meas}(\mathbb{R}, \mathbb{W}))$

- Constants:

$$E : B_{\mathbb{W}},\ x : \mathbb{W} \vdash$$

$$(\lambda r : \mathbb{R}.\ x)^{-1}[E] = \begin{cases} x \in E : & \mathbb{R} \\ x \notin E : & \emptyset \end{cases} \in B_{\mathbb{R}}$$

✓

Validate qbs axioms for: $W := ([0,\infty], \mathrm{Meas}(\mathbb{R}, W))$

- Precomposition:

$\alpha : \mathrm{Meas}(\mathbb{R}, W),\ \varphi : \mathrm{Meas}(\mathbb{R}, \mathbb{R}) \vdash$

$$\mathbb{R} \xrightarrow{\varphi} \mathbb{R} \xrightarrow{\alpha} W \quad \in \mathrm{Meas}(\mathbb{R}, W)$$

Meas is a cat.

Explicitly:

$$(\alpha \circ \varphi)^{-1}[E] \in B\mathbb{R} \xleftarrow{\varphi^{-1}} \alpha^{-1}[E] \in B\mathbb{R} \xleftarrow{\alpha^{-1}} E \in B_W \quad \checkmark$$

Validate qbs axioms for: $\mathbb{W} := ([0,\infty], \mathrm{Meas}(\mathbb{R}, \mathbb{W}))$

- recombination

I ctbl, $\alpha_{-} : \mathrm{Meas}(\mathbb{R}, \mathbb{W})^{I}$, $E_{-} : B_{\mathbb{R}}^{I}$, $\mathbb{R} = \biguplus_{i \in I} E_i$, $F : B_{\mathbb{W}}$,

$$\beta := \left(\lambda r. \begin{cases} \vdots \\ r \in E_i : \alpha_i r \\ \vdots \end{cases}\right)^{-1}[F] = \bigcup_{i \in I} \alpha_i^{-1}[F] \cap E_i \in B_{\mathbb{R}}$$

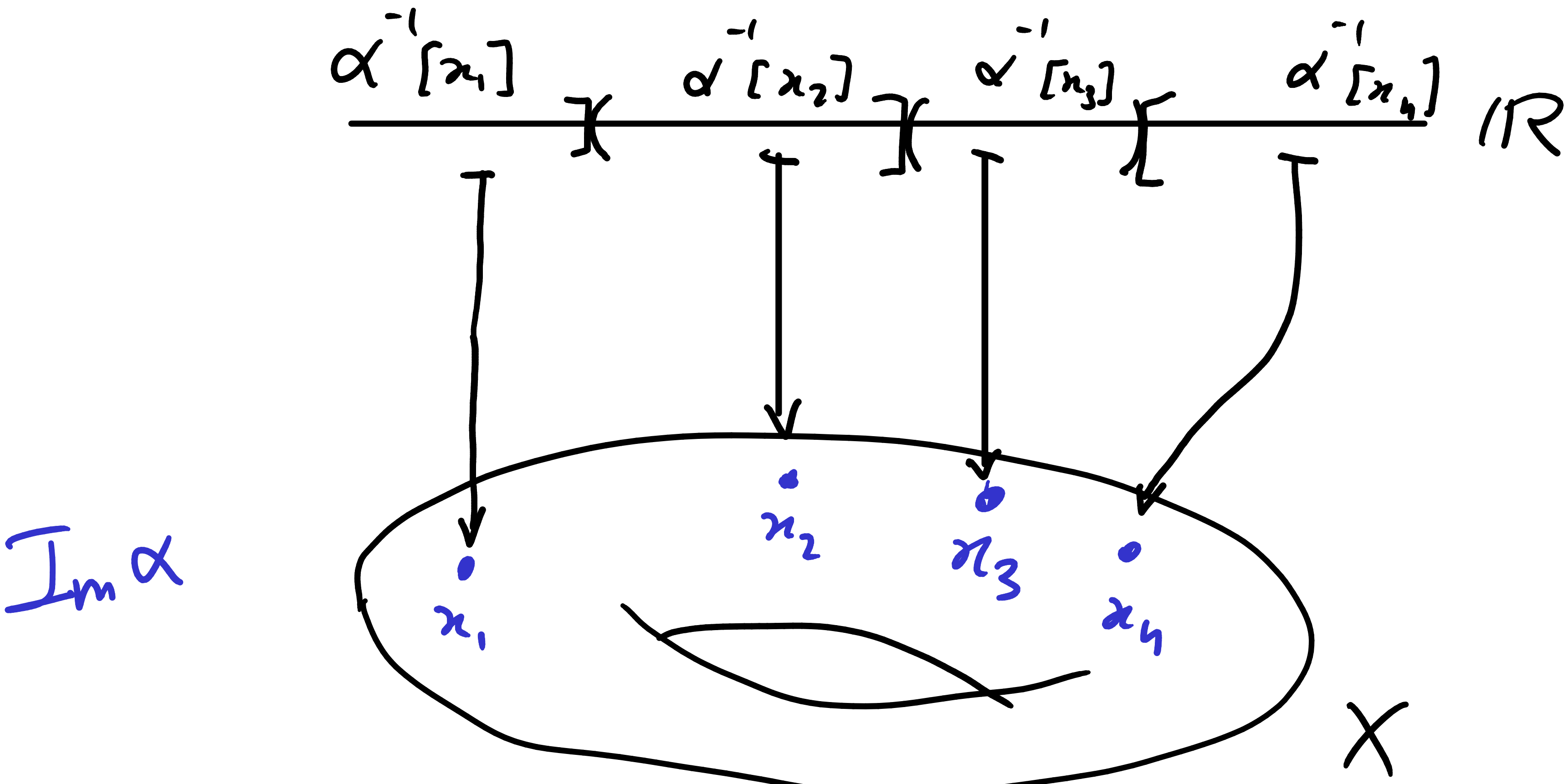
Indeed:

$r \in LHS \iff \beta r \in F \iff \exists i \in I.\ r \in E_i \wedge \alpha_i r \in F \iff r \in RHS$ ✓

σ-simple function

$\alpha : \mathbb{R} \longrightarrow X$ s.t. $\mathrm{Im}\,\alpha := \alpha[\mathbb{R}]$ is ctbl $\wedge$

$\forall x \in \mathrm{Im}\,\alpha .\ \alpha^{-1}[x] \in B_{\mathbb{R}}$



Validate qbs axioms for: $\ulcorner X^{\text{Qbs}} \urcorner := (X, \sigma\text{-Simple}(X))$

NB: f σ-Simple: $\operatorname{Im} f$ ctbl $\wedge$ $f^{-1}[x] \in B_{\mathbb{R}}$

- Constants

$\operatorname{Im}(\lambda r.x) = \{x\}$ ctbl ✓

$$y : X \vdash (\lambda r.x)^{-1}[y] = \begin{cases} x = y : & \mathbb{R} \\ x \neq y : & \emptyset \end{cases} \in B_{\mathbb{R}} ✓$$

Validate qbs axioms for: $\ulcorner X^{\mathrm{Qbs}} \urcorner := (X, \sigma\text{-Simple}(X))$

NB: f σ-Simple: $\mathrm{Im} f$ ctbl $\wedge$ $f^{-1}[x] \in B_{\mathbb{R}}$

- Precomposition:

$\alpha : \sigma\text{-simple}(X), \ \varphi : \mathrm{Meas}(\mathbb{R}, \mathbb{R}) \vdash$

$\mathrm{Im}(\alpha \circ \varphi) \subseteq \mathrm{Im}\,\alpha$ ctbl ✓

$x : X \vdash$

$(\alpha \circ \varphi)^{-1}[x] = \varphi^{-1}[\alpha^{-1}(x)] \in B_{\mathbb{R}}$ ✓

$\alpha^{-1}(x) \in B_{\mathbb{R}}$

$\varphi : \mathbb{R} \to \mathbb{R}$ measurable

Validate qbs axioms for: $\ulcorner X^{Qbs} \urcorner := (X, \sigma\text{-Simple}(X))$

NB: f σ-Simple: $\mathrm{Im} f$ ctbl $\wedge$ $f^{-1}[x] \in B_{\mathbb{R}}$

• recombination:

$\alpha_- : (\sigma\text{-Simple}(X))^I, \; E_- : B_{\mathbb{R}}^I, \; \mathbb{R} = \biguplus_{i \in I} E_i \vdash$

$$\mathrm{Im}[E_i . \alpha_i]_{i \in I} \subseteq \bigcup_{i \in I} \mathrm{Im}\,\alpha_i \quad \text{ctbl} \quad \checkmark$$

$x : X \vdash$

$$[E_i . \alpha_i]_{i \in I}^{-1}(x) = \bigcup_{i \in I} \alpha_i^{-1}[x] \cap E_i \in B_{\mathbb{R}} \quad \checkmark$$

Prop: $X : \mathrm{Set},\ A : \mathrm{Qbs} \vdash$

- $\forall f : X \to \lfloor A \rfloor .\ f : \ulcorner X \urcorner^{\mathrm{Qbs}} \longrightarrow A$

- $\forall f : \lfloor A \rfloor \longrightarrow X .\ f : A \longrightarrow \llcorner X \lrcorner_{\mathrm{Qbs}}$

Prop: $X : \mathrm{Set},\ A : \mathrm{Qbs} \vdash$

$$\bullet\ \forall f : X \to \lfloor A \rfloor .\ f : \ulcorner X \urcorner^{\mathrm{Qbs}} \longrightarrow A$$

Pf: $\alpha : R_{\ulcorner X \urcorner^{\mathrm{Qbs}}} \vdash$ α σ-simple $\Rightarrow$

$$\alpha = [\alpha^{-1}[x].\lambda r.x]_{x \in \mathrm{Im}\,\alpha} \Longrightarrow$$

recombination

$$(f \circ \alpha) = [\alpha^{-1}[x].\lambda r.fx]_{x \in \mathrm{Im}\,\alpha} \in R_A \quad \checkmark$$

Borel

constant $\in B_A$

ctbl

Prop: $X : \mathrm{Set},\ A : \mathrm{Qbs} \vdash$

- $\forall f : X \to \lfloor A \rfloor.\ f : \ulcorner X \urcorner^{\mathrm{Qbs}} \longrightarrow A$
- $\forall f : \lfloor A \rfloor \longrightarrow X.\ f : A \longrightarrow \lfloor X \rfloor_{\mathrm{Qbs}}$

Prf: $\alpha : R_A \vdash (f \circ \alpha : \mathbb{R} \to X) \in R_{\lfloor X \rfloor_{\mathrm{Qbs}}}$ always. ✓

$\blacksquare$

Useful adjunctions:

$$\begin{tikzcd}
\mathbf{Meas} \arrow[d, shift left=2, "\lfloor - \rfloor_{Qbs}"] \\
\mathbf{Qbs} \arrow[u, shift left=2, bend left=60, "\ulcorner - \urcorner^{Meas}"] \arrow[u, phantom, "\dashv"] \arrow[d, "\lfloor - \rfloor"] \\
\mathbf{Set} \arrow[u, bend left=60, "\ulcorner - \urcorner^{Qbs}"] \arrow[u, bend right=60, "\lfloor - \rfloor_{Qbs}"'] \arrow[u, phantom, "\dashv \quad \dashv"]
\end{tikzcd}$$

$$\lfloor V \rfloor_{Qbs} := (\lfloor V \rfloor, \mathbf{Meas}(\mathbb{R}, V)) \quad (V \in \mathbf{Meas})$$

$$\ulcorner X \urcorner^{Meas} := \left\{ A \subseteq \lfloor X \rfloor \;\middle|\; \forall \alpha \in R_X.\ \alpha^{-1}[A] \in \mathcal{B}_{\mathbb{R}} \right\}$$

- limits (products, subspaces) and colimits (coproducts, quotients) as in $\underline{\mathbf{Set}}$
- Slogan: every measurable space is carried by a qbs

Example

Product $(X \times Y, \pi_1, \pi_2)$:

correlated random elements

necessarily!

- $\lfloor X \times Y \rfloor = \lfloor X \rfloor \times \lfloor Y \rfloor$
- $R_{X \times Y} = \{ \lambda r. (\alpha r, \beta r) \mid \alpha \in R_X, \beta \in R_Y \}$

rest of structure as in **Set**.

Function Spaces

Straightforward!

- $\lfloor Y^X \rfloor := \mathrm{Qbs}(X, Y)$

- $R_{Y^X} := \mathrm{uncurry}[\mathrm{Qbs}(\mathbb{R} \times X, Y)]$

  $= \{\alpha : \mathbb{R} \to \lfloor Y^X \rfloor \mid \lambda(r, x).\, \alpha\, r\, x : \mathbb{R} \times X \to Y\}$

- $\mathrm{eval} : Y^X \times X \to Y$

  $\mathrm{eval}(f, x) := f\, x$

Meas vs Qbs

By generalities:

σ-algebra on $\mathbf{Meas}(\mathbb{R},\mathbb{R})$

$$\begin{tikzcd} \mathbf{Meas} \arrow[d, bend right, "\ulcorner - \urcorner^{\mathbf{Meas}}"'] \arrow[d, phantom, "\dashv"] \\ \mathbf{Qbs} \arrow[u, "\llcorner - \lrcorner_{\mathbf{Qbs}}"'] \end{tikzcd}$$

$$\begin{tikzcd} \ulcorner \mathbb{R}^{\mathbb{R}} \times \mathbb{R} \urcorner^{\mathbf{Meas}} \arrow[r] \arrow[rr, bend right, "\ulcorner eval \urcorner^{\mathbf{Meas}}"'] & \ulcorner \mathbb{R}^{\mathbb{R}} \urcorner^{\mathbf{Meas}} \times \mathbb{R} \arrow[r, dashed, "\times"] & \ulcorner \mathbb{R} \urcorner = \mathbb{R} \end{tikzcd}$$

No factorisation by Aumann's Theorem.

So:

$$\left(\ulcorner \mathbb{R}^{\mathbb{R}} \times \mathbb{R} \urcorner \not\cong \ulcorner \mathbb{R}^{\mathbb{R}} \urcorner \times \ulcorner \mathbb{R} \urcorner \right)$$

Simple Type Structure

"Simple" because:

- Simply-typed λ-calculus
- types are simple: $A, B : \mathrm{Type} \vdash B^A : \mathrm{Type}$
  - no polymorphism
  - no term dependency
- Contexts for terms: $\Gamma \vdash t : A$

  are simple: $\Gamma = x_1 : A_1, \ldots, x_n : A_n$

  i.e. $\mathrm{List}(\mathrm{Type})$

Simple Type Structure

"Simple" because:

- interpretation is simple:

$$[\![x_1 : A_1, \ldots, x_n : A_n]\!] := \prod_{i=1}^{n} A_i$$

$$[\![\Gamma \vdash t : A]\!] : [\![\Gamma]\!] \longrightarrow A$$

in Qbs

Simple Type Structure   Curry-Howard-Lambek

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash s : B}{\Gamma \vdash \langle t, s\rangle : A \times B} \rightsquigarrow [\![\Gamma]\!] \xrightarrow{\lambda\gamma.\langle t\gamma, s\gamma\rangle} A \times B$$

is measurable

measurability by type!

$$\frac{\Gamma \vdash t : A \times B \qquad \Gamma, x : A, y : B \vdash s : C}{\Gamma \vdash \mathrm{let}\ (x, y) = t\ \mathrm{in}\ s : C} \rightsquigarrow$$

$$[\![\Gamma]\!] \xrightarrow{\lambda\gamma.\ \mathrm{let}\ (a, b) = t\gamma\ \mathrm{in}\ s\gamma[x \mapsto a, y \mapsto b]} C$$

is measurable.   etc.

Random element Space

$$R_X := X^{\mathbb{R}} \quad \text{since} \quad \lfloor X^{\mathbb{R}} \rfloor = R_X \text{ as sets.}$$

Why?

$(\subseteq)$ $\alpha \in \lfloor X^{\mathbb{R}} \rfloor \Rightarrow \alpha : \mathbb{R} \to X$ in Qbs.

$id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ measurable $\Rightarrow id \in R_{\mathbb{R}}$

$\Rightarrow \alpha = \alpha \circ id \in R_X$

Precomposition

$(\supseteq)$ $\alpha \in R_X \Rightarrow \forall \varphi \in R_{\mathbb{R}} = \mathrm{Meas}(\mathbb{R}, \mathbb{R}).\ \alpha \circ \varphi \in R_X \Rightarrow \alpha : \mathbb{R} \to X$

$\Rightarrow \alpha \in \lfloor X^{\mathbb{R}} \rfloor$

Subspaces

For $X \in \mathbf{Qbs}$, $A \subseteq \lfloor X \rfloor$ set:

$$R_A := \{ \alpha : \mathbb{R} \to A \mid \alpha \in R_X \}$$

Then $A = (A, R_A)$ is the subspace qbs

We write $A \hookrightarrow X$

Borel subspaces ensemble

The σ-algebra $\mathcal{B}_X := \{ A \subseteq \lfloor X \rfloor \mid \forall \alpha \in R_X .\ \alpha^{-1}[A] \in \mathcal{B}_{\mathbb{R}} \}$

internalises as $\mathcal{B}_X = 2^X$, the qbs of

Borel subsets.

($\lfloor \mathcal{B}_{(\mathcal{B}_{\mathbb{R}})} \rfloor$ are the Borel-on-Borel sets from descriptive set theory.
Cf. [Sabok et al. '21])

Standard Borel spaces

Def: A qbs S is standard Borel when

$$S \cong A \quad \text{for some } A \in \mathcal{B}_{\mathbb{R}}$$

$$\begin{tikzcd}
\mathbf{Meas} \arrow[rr, bend left] \arrow[rr, phantom, "\top"] & & \mathbf{Qbs} \arrow[ll, bend left] \\
\mathbf{Sbs} \arrow[u, hook] \arrow[rr, bend left] \arrow[rr, phantom, "\cong"] & & \mathbf{Sbs} \arrow[u, hook] \arrow[ll, bend left]
\end{tikzcd}$$

Slogan: Qbs conservative extension of Sbs

Example $C_0 := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\} \hookrightarrow \mathbb{R}^{\mathbb{R}}$

C_0 is sbs. (Well-known!)

Proof:

$C_0' \in B_{\mathbb{R}^{\mathbb{Q}}}$ sbs!

$$C_0' := \left\{ g \in \mathbb{R}^{\mathbb{Q}} \;\middle|\; \begin{array}{l} \forall a, b \in \mathbb{Q}, \varepsilon \in \mathbb{Q}^+ \\ \exists \delta \in \mathbb{Q}^+ \; \forall p, q \in \mathbb{Q}^+ \cap [a,b], \\ |p - q| < \delta \Rightarrow |g\,p - g\,q| < \varepsilon \end{array} \right\}$$

on closed intervals (= compact intervals)

Continuity $\Updownarrow$ uniform continuity

Borel measurable } by type checks

then $C_0 \cong C_0' \in B_{\mathbb{R}^{\mathbb{Q}}}$:

$$C_0 \to C_0' \qquad\qquad C_0' \to C_0$$

$$\varphi \mapsto \varphi|_{\mathbb{Q}} \qquad\qquad \varphi \mapsto \lambda r.\, \lim_{n \to \infty} g(\text{approx}_{(\frac{1}{m})_{m \in \mathbb{N}}} r)_n$$

Example (ctd)

C_0 is sbs, and $eval : C_0 \times \mathbb{R} \to \mathbb{R}$

is measurable.

Avoids:

- constructing complete separable metrics
- proving that evaluation is measurable w.r.t. metric σ-algebra.

Non-examples $\rightsquigarrow$ [Sabok et al. '21]

- $\{ A \in \mathcal{B}_{\mathbb{R}} \mid A \neq \emptyset \} \hookrightarrow \mathcal{B}_{\mathbb{R}}$

- $\{ (A_1, A_2) \in \mathcal{B}_{\mathbb{R}}^2 \mid A \subseteq B \} \hookrightarrow \mathcal{B}_{\mathbb{R}}^2$

- $\{ A \in \mathcal{B}_{\mathbb{R}} \mid A \text{ open} \} \hookrightarrow \mathcal{B}_{\mathbb{R}}$

Partiality cf. [Vákár et al. '19]

A Borel embedding $e : X \hookrightarrow Y$

- injective function $e : \lfloor X \rfloor \rightarrowtail \lfloor Y \rfloor$
- its image is Borel: $e[\lfloor X \rfloor] \in B_Y$
- e is Strong: $\alpha \in R_X \iff e \circ \alpha \in R_Y$

Examples

- $\mathbb{1} \hookrightarrow \mathbb{2}$
- S is sbs $\iff \exists\, S \hookrightarrow \mathbb{R}$

Def: A Partial map $f : X \rightharpoonup Y$ is a morphism

$$f : X \to Y \amalg \{\bot\}$$

Its domain of definition $\mathrm{Dom}\, f := \{x \mid fx \neq \bot\}$

$$\begin{tikzcd} \mathrm{Dom}\, f \arrow[d, hook] \\ X \end{tikzcd}$$

Partial hom-sets are ordered:

for $f, g : X \rightharpoonup Y$ $\quad f \leq g$ when $\forall x.\ fx \neq \bot \Rightarrow gx = fx$.

[Cockett-Lack '06]

A model of restriction categories / axiomatic domain theory

[Fiore_Plotkin '94] Borel embeddings are the admissible monos

Plan:

1) type-driven probability: discrete case (Mon + Tue)

2) Borel sets & measurable spaces (Wed)

3) Quasi Borel spaces (Wed) Simple type structure (Thu) ✓

4) Dependent type structure & standard Borel spaces ✓ (Thu)

5) Integration & random variables (Fri)

Please ask questions!

Course web page

scribble

Dependent Type Structure

types can contain terms:

a type referring to a term

$$X : \mathrm{Type},\ E : B_X \vdash \{x \in X \mid x \in E\} : \mathrm{Type}$$

a type, just like STLC

a term!

Dependent Type Structure

types can contain terms:

$$X : \mathrm{Type},\ E : B_X \vdash \{x \in X \mid x \in E\} : \mathrm{Type}$$

a type referring to a term

a type, just like STLC

a term!

Context formation:

$$\frac{\Gamma \vdash A : \mathrm{Type}}{\Gamma, x : A \vdash}$$

Dependent Type Structure

types denote spaces-in-Context

$$\begin{array}{c}[\![\Gamma \vdash A]\!] \\ \Big\downarrow \mathrm{dep} \\ [\![\Gamma]\!]\end{array}$$

Dependent types denote spaces-in-context

$\Gamma \vdash$ ← context

$\Gamma \vdash A$

type in context

$$\begin{tikzcd} [\![\Gamma \vdash A]\!] \arrow[d, "\mathrm{dep}"] \\ [\![\Gamma]\!] \end{tikzcd}$$

Space in context

context space

assigns environment

E.g.:

$$\begin{tikzcd} A \arrow[d] \\ \mathbb{1} \end{tikzcd}$$

simple types

$$[\![E : B_A \vdash \{x \in A \mid x \in E\}]\!]$$

$$=$$

$$\begin{tikzcd} \{(E, a) \in B_A \times A \mid a \in E\} \arrow[d, "\pi_1"] \\ B_A \end{tikzcd}$$

decoder

Context extension

$$\dfrac{\Gamma \vdash A}{\Gamma, a : A \vdash}$$

$$\begin{array}{c} [\![\Gamma \vdash A]\!] \\ \Big\downarrow \mathrm{dep} \\ [\![\Gamma]\!] \end{array} \qquad [\![\Gamma, a : A]\!] := [\![\Gamma \vdash A]\!]$$

Substitution

$\Gamma \vdash \sigma : \Delta$ $\qquad [\![\sigma]\!] : [\![\Gamma]\!] \longrightarrow [\![\Delta]\!]$

E.g. weakening $\qquad \Gamma, a : A \vdash \mathrm{wkn} : \Gamma$

$$[\![\Gamma, a : A]\!] := [\![\Gamma \vdash A]\!] \xrightarrow[\mathrm{dep}]{\mathrm{wkn}} [\![\Gamma]\!]$$

Action of substitution on types

$$
\begin{tikzcd}
\llbracket \Gamma \vdash A[\sigma] \rrbracket = \{(\gamma, a) \in \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \mid \mathrm{dep}\, a = \gamma\} \arrow[r, "\pi_2"] \arrow[d, "\mathrm{dep} := \pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \llbracket \Delta \vdash A \rrbracket \arrow[d, "\mathrm{dep}"] \\
\llbracket \Gamma \rrbracket \arrow[r, "\sigma"] & \Delta
\end{tikzcd}
$$

E.g.

$$
\begin{tikzcd}
\llbracket x : A \vdash B[\mathrm{wkn}] \rrbracket := A \times B \arrow[r, "\pi_2"] \arrow[d, "\mathrm{dep} := \pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d] \\
x : A \arrow[r, "\langle\rangle"'] & \mathbb{1}
\end{tikzcd}
$$

simple type

Terms: sections

$$\begin{tikzcd}
{[\![\Gamma]\!]} \arrow[rr, "{[\![\Gamma \vdash M : A]\!]}"] \arrow[ddrr, equal] & & {[\![\Gamma \vdash A]\!]} \arrow[dd, "\mathrm{dep}"] \\
& = & \\
& & {[\![\Gamma]\!]}
\end{tikzcd}$$

e.g.

$$\begin{tikzcd}
\mathbb{R} \arrow[rr, "{[\![x : \mathbb{R} \vdash [x, \infty) : \mathcal{B}_{\mathbb{R}}[\mathrm{wkn}]]\!]}"] \arrow[ddrr, equal] & & \mathbb{R} \times \mathcal{B}_{\mathbb{R}} \arrow[dd, "\pi_1"] \\
& = & \\
& & \mathbb{R}
\end{tikzcd}$$

E.g. Variables:

$$\begin{tikzcd}
{[\![\Gamma, a : A]\!]} \arrow[rr, "{[\![\Gamma, a : A \vdash a : A]\!]}", "{\langle \mathrm{id}, \mathrm{dep}_{\Gamma \vdash A} \rangle}"'] \arrow[ddrr, equal] & & {[\![\Gamma, a : A \vdash A[\mathrm{wkn}]]\!]} \arrow[dd, "\mathrm{dep}"] \\
& = & \\
& & {[\![\Gamma, a : A]\!]}
\end{tikzcd}$$

Exercise:
action of substitution
$M[\sigma]$

Dependent Pairs

$$\frac{\Gamma, a:A \vdash B}{\Gamma \vdash \coprod_{a\in A} B}$$

$$[\![\coprod_{a\in A} A]\!] := [\![\Gamma, a:A \vdash B]\!]$$

$$\begin{tikzcd}
{[\![\Gamma, a:A \vdash B]\!]} \arrow[d, "\mathrm{dep}_B"] \arrow[dd, bend right=60, "\mathrm{dep}_{\coprod}"'] \\
{[\![\Gamma, a:A]\!] = [\![\Gamma \vdash A]\!]} \arrow[d] \\
{[\![\Gamma]\!]}
\end{tikzcd}$$

$$\mathrm{dep}_{\coprod} := \ldots$$

Dependent Products

$$\frac{\Gamma, a : A \vdash B}{\Gamma \vdash \prod_{a \in A} B}$$

aha: $(a:A) \to B$

$$\lfloor [\![\Gamma \vdash \prod_{a \in A} B]\!] \rfloor :=$$

$$\left\{ (r_0, f : \{ a \in [\![A]\!] \mid \mathrm{dep}\, a = r_0 \} \to [\![\Gamma, a : A \vdash B]\!]) \,\middle|\, \forall a \in [\![\Gamma, a : A]\!].\ \mathrm{dep}\, a = r_0 \Rightarrow \mathrm{dep}\,(f\, a) = a \right\}$$

Exercise: find the random elements.

Discrete model

type : Qbs $\quad W := [0,\infty] \quad \mathbb{B}x \cong \mathbb{B}^X$

$DX := (\text{Fri})$

$PX := \{\mu \in DX \mid \underset{\mu}{\mathrm{Ce}}[X] = 1\}$

$\underset{\mu}{\mathrm{Ce}}[E] := (\text{Fri}) \qquad \delta_x := (\text{Fri})$

$\oint \mu k := (\text{Fri})$

Plan:

1) type-driven probability: discrete case (Mon + Tue)

2) Borel sets & measurable spaces (Wed)

3) Quasi Borel spaces (Wed) Simple type structure (Thu)

4) Dependent type structure & standard Borel spaces (Thu)

5) Integration & random variables (Fri)

Please ask questions!

Course web page

scribble