Foundations for type-driven probabilistic modelling

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TutorialFest
ACM-SIGPLAN 51[st] Symposium on
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Computational golden era of:

logic & type rich computation

Statistical computation

Computational golden era of:

logic & type rich computation

expressive type systems:

Haskell, OCaml, Idris

mechanised mathematics:

Agda, Coq, Isabelle/HoL, Lean

Verification:

SMT-powered, realistic systems

Statistical computation

generative modelling

+

efficient inference:

Monte-Carlo simulation

or gradient-based optimisation

"AI"

Computational golden era of:

logic & type rich computation

Statistical computation

Clear connection to

- LAFI

Everything POPL

- some POPL

- this tutorial

Plan

1) Typed Setting for Probability & Statistics

2) 2 Implementations

Discrete Model $\hookrightarrow$ Full Model

familiar maths to introduce language

Same language

Why foundations?

countably supported measures

Discrete Probability

Continuous Probability

Lebesgue measure on $\mathbb{R}^n$

Well-behaved Probability

s-finite measures over standard Borel spaces

Measurable spaces

standard, established

Poor type structure

Quasi-Borel spaces

new, experimental

rich type structure

(this course)

Takeaway:

use types to develop smoothly

Why types?

- spotlights meaningful operations

$$\int : (\mathrm{Distribution}\ A) \times (A \to [0,\infty]) \to [0,\infty]$$

- documents intent

Probability Distribution A vs Density $A \to [0,\infty]$

- succinctness: easier to elaborate details
- esp. formal types: use theory without fully understanding it.

Plan:

1) type-driven probability: discrete case
2) Borel sets & measurable spaces
3) Quasi Borel spaces

Lecture 1

4) Type structure & standard Borel spaces
5) Integration & random variables

Lecture 2

Please ask questions!

Course web page

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Langauge of distribution & Probability

X type (= space) of values / outcomes

DX type of distributions / measures over X

$PX \subseteq DX$ Subtype of Probability measures (total measure 1)

BX type of measurable events - subsets of X we wish to measure

W type of weights : $[0, \infty]$

type judgement

$\mu : DX, E : BX \vdash \mathrm{Ce}_\mu[E] : W$

measure μ assigns to E

Axioms for measures

Empty event: $\emptyset : \mathbb{B}X$

Its measure is $0 : \mathbb{W}$:

$$\mu : \mathbb{D}X \vdash \underset{\mu}{\mathcal{C}e}[\emptyset] = 0 \quad : \mathbb{W}$$

Axioms for measures

BX is a Boolean sub-algebra:

$$E : BX \vdash E^c : BX$$

$$E, F : BX \vdash E \cup F, E \cap F : BX$$

$$E, C : BX, \mu : DX \vdash$$ (disjoint additivity)

$$\underset{\mu}{\mathrm{Ce}}[E] = \underset{\mu}{\mathrm{Ce}}[E \cap C] + \underset{\mu}{\mathrm{Ce}}[E \cap C^c] : W$$

Axioms for measures

$\omega := (\mathbb{N}, \leq)$ $\quad (\mathbb{B}, \subseteq)$ $\quad (\mathbb{W}, \leq)$ posets

$$(\mathbb{B}X, \subseteq)^{\omega} := \{ (E_n)_{n \in \mathbb{N}} \in (\mathbb{B}X)^{\mathbb{N}} \mid E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \}$$

$(\mathbb{B}X, \subseteq)$ and $(\mathbb{W}, \leq)$ are ω-chain-closed:

$$E_{-} : (\mathbb{B}X, \subseteq)^{\omega} \vdash \bigcup_n E_n : \mathbb{B}X \qquad a_{-} : (\mathbb{W}, \leq)^{\omega} \vdash \sup_n a_n : \mathbb{W}$$

$E_{-} : (\mathbb{B}X, \subseteq)^{\omega}, \mu : \mathrm{D}X \vdash$ (Scott continuity)

$$\underset{\mu}{\mathrm{Ce}}\left[\bigcup_n E_n\right] = \sup_n \underset{\mu}{\mathrm{Ce}}\left[E_n\right] \quad : \mathbb{W}$$

Axiom for Probability

$$\mathrm{Cost} : PX \xrightarrow{\subseteq} DX \qquad 1 : W$$

$$\mu : PX \vdash \underset{\mathrm{Cost}\,\mu}{Ce}[X] = 1 \quad : W$$

Avoid costing:

$$E : BX, \mu : PX \vdash \underset{\mu}{Pr}[E] := \underset{\mathrm{cost}\,\mu}{Ce}[E] \quad : \quad [0,1] \subseteq W$$

Axioms for measures

Integration:

$$\mu : DX,\ \varphi : \mathbb{W}^X \vdash \int \mu\, \varphi \quad : \mathbb{W}$$

(Lebesgue integral)

Again, avoid casting:

$$\mu : PX,\ \varphi : \mathbb{W}^X \vdash \mathbb{E}_{\mu}[\varphi] := \int (\mathrm{cast}\ \mu)\, \varphi \quad : \mathbb{W}$$

(Expectation)

More structure & notation later (...technical...)

Have: language + axioms

Want: model

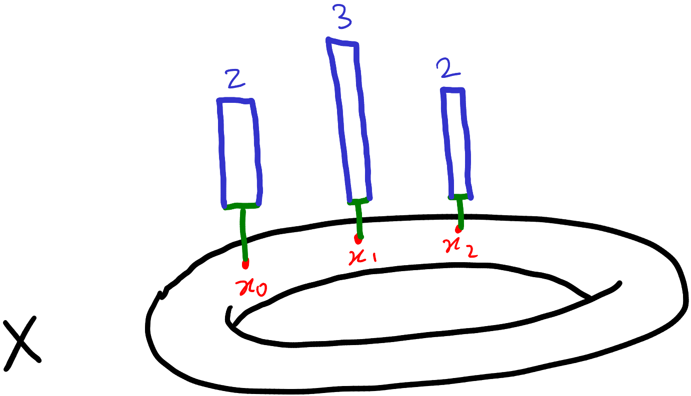
Part 1: discrete measures

Part 2: discrete + continuous

Discrete model

type X : set

$DX :=$ histograms

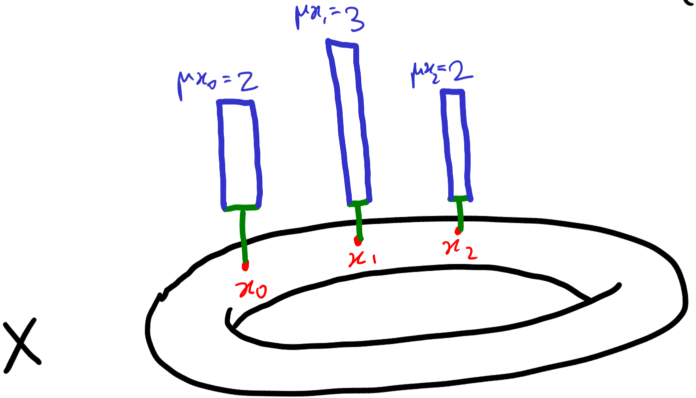


Discrete model

type X : Set

$DX := \{\mu : X \to \mathbb{W} \mid \mu$ is countably supported $\}$

(next slide)



Support

Powerset

$\mu : W^X, S : \mathcal{P}X \vdash S \text{ supports } \mu :=$

$\forall x : X.\ \mu x > 0 \Rightarrow x \in S \quad : \mathrm{Prop}$

$\mu : W^X \vdash \mathrm{supp}\,\mu := \{x \in X \mid \mu x > 0\} \quad : \mathcal{P}X$

$\mathrm{supp}\,\mu$ is the smallest set supporting μ

Discrete model

type X : set

$$DX := \{\mu : X \to W \mid \mu \text{ is countably supported}\}$$

$$:= \{\mu : X \to W \mid \operatorname{supp} \mu \text{ is countable}\}$$

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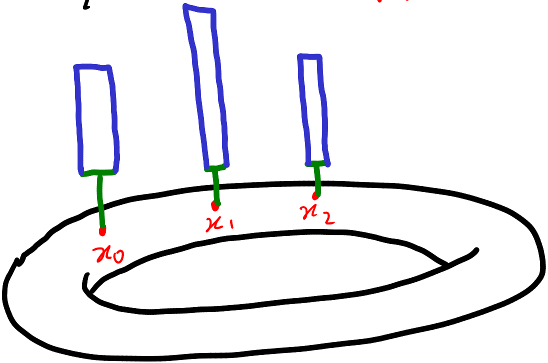
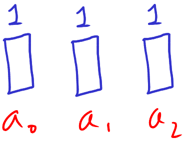


X

Ex. measures

1) 

. . . Counting measure

. . . (X ctbl)

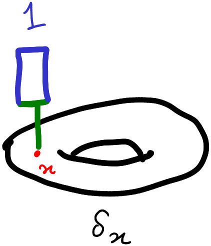
$\#_X : DX$

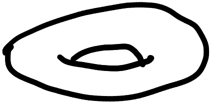
$\#_X := \lambda x : X . 1$

2) Dirac measure:

$x : X \vdash \delta_x : DX$

$$:= \lambda x' . \begin{cases} x = x' : & 1 \\ \text{o.w.} : & 0 \end{cases}$$



3) 

Zero measure $\underline{0} := \lambda x . 0 \;:\; DX$

Discrete model

type X : set

$DX := \{\mu : X \to \mathbb{W} \mid \mu \text{ is countably supported}\}$

$BX := \mathcal{P}X$

$$\mu : DX, E : BX \vdash \mathrm{Ce}_\mu[E] := \sum_{x \in E} \mu x$$

$$:= \sum_{x \in E \cap \mathrm{supp}\,\mu} \mu x$$

Ex:

- Counting measure counts outcomes:

$$E : \mathbb{B}X \vdash \underset{\#_X}{\mathcal{C}_e}[E] = |E| := \begin{cases} E \text{ has } n \text{ elements:} & n \\ E \text{ infinite:} & \infty \end{cases}$$

- Dirac measure detects outcomes:

$$E : \mathbb{B}X, x : X \vdash \underset{\delta_x}{\mathcal{C}_e}[E] = \begin{cases} x \in E: & 1 \\ x \notin E: & 0 \end{cases} =: [x \in E] : \mathbb{W}$$

- Zero measure is zero:

$$E : \mathbb{B}X \vdash \underset{\underline{0}}{\mathcal{C}_e}[E] = 0$$

NB: $E : \mathbb{B}X \vdash [- \in E] : X \to \mathbb{W}$ indicator function

Validate axioms

$$\mu : DX \vdash \underset{\mu}{Ce}[\emptyset] = 0 \quad : W$$

$$E, C : BX, \mu : DX \vdash$$

$$\underset{\mu}{Ce}[E] = \underset{\mu}{Ce}[E \cap C] + \underset{\mu}{Ce}[E \cap C^c] \quad : W$$

$$E_- : (BX, \subseteq)^\omega, \mu : DX \vdash$$

$$\underset{\mu}{Ce}[\bigcup_n E_n] = \sup_n \underset{\mu}{Ce}[E_n] \quad : W$$

Kernels k from Γ to X:

$$k : (DX)^{\Gamma}$$

kernels are "open/parameterised" measures

Ex: Dirac kernel: $\delta_{\_} : (DX)^{X}$

Kock Integral

$$\mu : D\Gamma, \kappa : DX^{\Gamma} \vdash \oint \mu \kappa : DX$$

In discrete model:

$$\oint \mu \kappa := \lambda x{:}X.\ \sum_{\gamma \in \Gamma} \mu\gamma \cdot \overbrace{\kappa(\gamma; x)}^{:= k(\gamma)(x)}$$

(Weak) disintegration problem:

Input: $\mu : D\Gamma$ $\quad \nu : DX$

Output: a kernel $k : (DX)^{\Gamma}$ s.t.

$$\oint \mu k = \nu$$

Call such k a (weak) disintegration of ν w.r.t. μ.

(non-standard terminology)

Ex disintegration:

$\underline{n} := \{0, 1, 2, \ldots, n-1\}$

disintegrate $\#_{\underline{2}^{\underline{n+1}}}$ w.r.t. $\#_{\underline{2}}$

Define: $k : (D(\underline{2}^{\underline{n+1}}))^{\underline{2}}$

$$k(x; f) := \begin{cases} f(n) = x : & 1 \\ \text{o.w.} : & 0 \end{cases}$$

check:

$$(\oint \#_{\underline{2}} k) f = \sum_{x \in \underline{2}} \overbrace{\#_{\underline{2}} x}^{1} \cdot k(x; f)$$

$$= k(0; f) + k(1; f) = k(fn; f) = 1 = \#_{\underline{2}^{\underline{n+1}}}(f)$$

Probability measures

$$PX := \{\mu : DX \mid \mathcal{C}e_{\mu}[X] = 1\} \xhookrightarrow{\subseteq} DX$$

Probability measures

$$PX := \left\{ \mu : DX \;\middle|\; \oint_{\mu} [X] = 1 \right\} \xhookrightarrow{\subseteq} DX$$

Lemma: $\delta_{-} : X \to DX$ and $\oint : D\Gamma \times (DX)^{\Gamma} \to DX$

lift along the inclusion $\mathrm{cast} : P \xhookrightarrow{\subseteq} D$:

$$\begin{tikzcd}
 & PX \arrow[d, hook, "\mathrm{cast}"] \\
X \arrow[ur, dashed, "\delta_{-}"] \arrow[r, "\delta_{-}"'] \arrow[ur, phantom, "=" near end, shift right=3] & DX
\end{tikzcd}
\qquad
\begin{tikzcd}
P\Gamma \times (PX)^{\Gamma} \arrow[r, dashed, "\oint"] \arrow[d, hook, "\mathrm{cast} \times (\mathrm{cast})"'] \arrow[dr, phantom, "="] & PX \arrow[d, hook, "\mathrm{cast}"] \\
D\Gamma \times (DX)^{\Gamma} \arrow[r, "\oint"'] & DX
\end{tikzcd}$$

Prop (discrete Giry): (Michèle Giry '82)

$(D, \delta_{-}, \oint)$ is a monad i.e.

$\mu : \Gamma, k : (DX)^{\Gamma} \vdash \oint \delta_{\mu} k = k\,\mu$

$\mu : DX \vdash \oint \mu(dx)\,\delta_{x} = \mu$

$\mu : D\Gamma, k : (DX)^{\Gamma}, t : (DY)^{X} \vdash$

$$\oint \mu(d\gamma) \left(\oint (k\gamma)\, t \right) = \oint \left(\oint \mu k \right)(dx)\, t(x)$$

Corollary: $(P, \delta_{-}, \oint)$ is a monad.

Langauge of distribution & Probability

X type (=space) of values / outcomes

DX type of distributions / measures over X

$PX \subseteq DX$ Sub type of Probability measures (total measure 1)

BX type of measurable events - subsets of X we wish to measure

W type of weights : $[0, \infty]$

type judgement

$$\mu : DX, E : BX \vdash \mathbb{P}_{\mu}[E] : W$$

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Discrete model

Recap

type : Set $\quad W := [0,\infty] \quad \mathcal{B}X := \mathcal{P}X$

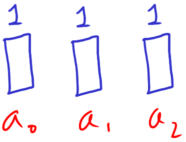
$$DX := \{\mu : X \to W \mid \operatorname{supp}\mu \text{ Countable}\}$$

$$PX := \{\mu \in DX \mid \underset{\mu}{\mathcal{C}e}[X] = 1\}$$

$$\underset{\mu}{\mathcal{C}e}[E] := \sum_{x \in E} \mu x \qquad \delta_x := \lambda x'. \begin{cases} x = x' : & 0 \\ x \neq x' : & 1 \end{cases}$$

$$\oint \mu k := \lambda x. \sum_{r \in \Gamma} \mu r \cdot k(r; x)$$

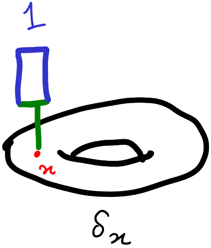
Ex. measures

1) 

... Counting measure $\#_X : DX$

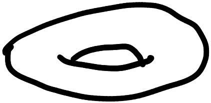
... (X ctbl) $\#_X := \lambda x : X. 1$

2) Dirac measure:



$$x : X \vdash \delta_x : DX$$

$$:= \lambda x'. \begin{cases} x = x' : & 1 \\ \text{o.w.} : & 0 \end{cases}$$

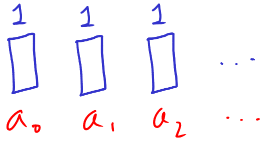
3) 

Zero measure $\underline{0} := \lambda x. 0 \quad : DX$

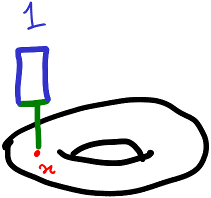
Ex distributions

Recap

Counting measure (X ctbl): $\#_X := \lambda x. 1$

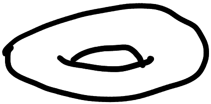


Dirac measure δ_x (prev slide)



δ_x

Zero measure $\underline{0} := \lambda x. 0$



Product measures

$$\mu : DX,\ \nu : DY \vdash \mu \otimes \nu := \oint \mu(dx) \oint \nu(dy) \delta_{\langle x,y \rangle} : D(X \times Y)$$

$$= \lambda(x,y).\ \mu x \cdot \nu y$$

discrete model

($\otimes$ lifts along $P \hookrightarrow D$)

Ex: $\#_{X \times Y} = \#_X \otimes \#_Y$

build measures compositionally

Indeed:

$$(\# \otimes \#)(x,y) = \# x \cdot \# y = 1 \cdot 1 = 1 = \#(x,y)$$

Notation: $\lambda : D(X \times Y), k : (DZ)^{X \times Y} \vdash \oint\!\!\oint \lambda(dx, dy) k(x, y)$

$:= \oint \lambda k$

Fubini-Tonelli Thm:

Integrate in any order:

$\mu : DX, \nu : DY, k : (DZ)^{X \times Y} \vdash$

$$\oint \mu(dx) \oint \nu(dy) k(x, y) = \oint\!\!\oint (\mu \otimes \nu)(dx, dy)$$

$$= \oint \nu(dy) \oint \mu(dx) k(x, y)$$

Pushing a measure forward

$\mu : D\Omega,\ \alpha : X^{\Omega} \vdash \mu_f := \oint \mu(d\omega)\,\delta_{\alpha\omega} : DX$

$$= \lambda x. \sum_{\substack{\omega \in \Omega \\ \alpha\omega = x}} \mu\,\omega$$

$\alpha : X^{\Omega}$: random element (w.r.t. μ)

$\mu_\alpha : DX$: the law of α

Ex: represent configurations of 2 dice using

$\mathrm{Die}_6 := \{1, 2, \ldots, 6\}$ Die_6^2

Letting $(+) : \mathrm{Die}_6^2 \longrightarrow \mathbb{N}^2 \xrightarrow{(+)} \mathbb{N}$

we have that the law of $(+)$:

$\mu := (\#_{\mathrm{Die}} \otimes \#_{\mathrm{Die}})_{(+)} : D\mathbb{N}$

build measures compositionally

$\mu s =$ number of outcomes whose sum is s

Scaling a measure

$$(\cdot) : W \times DX \longrightarrow DX$$

$$a \cdot \mu := \lambda x.\ a \cdot \mu x$$

$(\cdot) : W \times DX \to DX$ is an <u>action</u> of monoid $(W, (\cdot), 1)$ on DX:

$\mu : DX \vdash$ $\quad 1 \cdot \mu = \mu$ $\qquad a, b : W, \mu : DX \vdash$ $\quad a \cdot (b \cdot \mu) = (a \cdot b) \cdot \mu$

Normalisation

unit type → $\mathbb{1} := \{\langle\rangle\}$ ← unit value

$\mu : DX,\quad \underset{\mu}{ce}[X] \neq 0, \infty \vdash$

$$\|\mu\| := \left(\frac{1}{\underset{\mu}{ce}[X]}\right) \cdot \mu \quad : PX$$

Ex:

$$\emptyset \neq A \subseteq_{fin} X \quad : \quad \mathrm{U}_{A \subseteq X} := \|(\#_A)_{A \subseteq X}\| \quad : PX$$

$$\mathbb{1} \xrightarrow{\#_A} DA \xrightarrow{(-)_{A \subseteq X}} DX \xrightarrow{\|-\|} PX$$

I.e. $\mathrm{U}_{A \subseteq X} := \lambda x. \begin{cases} x \in A: & \frac{1}{|A|} \\ x \notin A: & 0 \end{cases}$ so $\mathrm{U}_{\{x\} \subseteq X} = \delta_x$

Standard vocabulary

Joint distributions: $\mu : D(X_1 \times X_2)$

Marginal distribution: $X_1 \xleftarrow{\pi_1} X_1 \times X_2 \xrightarrow{\pi_2} X_2$

law of projection

$\mu_{\pi_i} : DX_i$

marginalisation: $\mu_{\pi_1} = \iint \mu(dx, dy)\, \delta_x$

integrate out y

Exercise: $\mu : PX,\ \nu : DX \vdash (\mu \otimes \nu)_{\pi_2} = \nu$

independence

Pairing r.e.s:

$\alpha : X^{\Omega}, \beta : Y^{\Omega} \vdash$

$$\langle \alpha, \beta \rangle := \lambda \omega . \langle \alpha \omega, \beta \omega \rangle : (X \times Y)^{\Omega}$$

$$\begin{tikzcd} & \Omega \arrow[dl, "\alpha"'] \arrow[d, dashed, "\langle \alpha, \beta \rangle"] \arrow[dr, "\beta"] & \\ X \arrow[r, phantom, "="] & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

$$\lambda : D\Omega, \alpha : X^{\Omega}, \beta : Y^{\Omega} \vdash \alpha \underset{\lambda}{\perp} \beta := \lambda_{\langle \alpha, \beta \rangle} = \lambda_{\alpha} \otimes \lambda_{\beta} : \mathrm{Prop}$$

α, β independent w.r.t. λ

$\underline{\text{Ex}}$ (Durrett) represent outcomes of 3 coin tosses:

$C := \{T, H\}$ $\quad \Omega := C \times C \times C \quad \lambda : U_C \otimes U_C \otimes U_C : P\Omega$

$\pi_i : \Omega \longrightarrow C$ outcome of i^{th} toss

$\mathrm{Same}_{ij} : \Omega \xrightarrow{\langle \pi_i, \pi_j \rangle} C \times C \xrightarrow{(\overset{?}{=})} \mathbb{B}$

where: $(\overset{?}{=}) : C^2 \to \mathbb{B} := \{\mathrm{True}, \mathrm{False}\}$

$$x \overset{?}{=} y := \begin{cases} x = y : & \mathrm{True} \\ x \neq y : & \mathrm{False} \end{cases}$$

$\underline{Ex}$ (Durrett) represent outcomes of 3 coin tosses:

$$C := \{T, H\} \qquad \Omega := C \times C \times C \qquad \lambda : U_c \otimes U_c \otimes U_c : P\Omega$$

$$\pi_i : \Omega \longrightarrow C \quad \text{outcome of } i^{th} \text{ toss}$$

$$\mathrm{Same}_{ij} : \Omega \xrightarrow{\langle \pi_i, \pi_j \rangle} C \times C \xrightarrow{(\overset{?}{=})} \mathbb{B}$$

marginalisation

$$\lambda_{\mathrm{Same}_{12}} T = (U_c \otimes U_c)_{(\overset{?}{=})} T = \underbrace{\tfrac{1}{4}}_{U_c(H)\cdot U_c(H)} + \overbrace{\tfrac{1}{4}}^{U_c(T)\cdot U_c(T)} = \tfrac{1}{2}$$

So $\lambda_{\mathrm{Same}_{12}} F = \frac{1}{2}$ too

Ex (Durrett) represent outcomes of 3 coin tosses:

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$\mathrm{Same}_{ij} : \Omega \xrightarrow{\langle \pi_i, \pi_j \rangle} C \times C \xrightarrow{(\overset{?}{=})} \mathbb{B}$

$i \neq j$: $\lambda_{\mathrm{Same}_{ij}} = U_{\mathbb{B}}$

$$\lambda_{\langle \mathrm{Same}_{12}, \mathrm{Same}_{23} \rangle} : \quad (T,T) \mapsto \underbrace{\tfrac{1}{2} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}}_{\lambda(H,H,H)} + \underbrace{\tfrac{1}{2} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}}_{\lambda(T,T,T)} = \tfrac{1}{4}$$

$$(T,F) \mapsto \underbrace{\tfrac{1}{2} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}}_{\lambda(H,H,T)} + \underbrace{\tfrac{1}{2} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}}_{\lambda(T,T,H)} = \tfrac{1}{4}$$

<u>Ex</u> (Durrett) represent outcomes of 3 coin tosses:

$C := \{T, H\}$ $\quad \Omega := C \times C \times C$ $\quad \lambda : U_C \otimes U_C \otimes U_C : P\Omega$

$\pi_i : \Omega \longrightarrow C$ outcome of i^{th} toss

$\mathrm{Same}_{ij} : \Omega \xrightarrow{\langle \pi_i, \pi_j \rangle} C \times C \xrightarrow{(\overset{?}{=})} \mathbb{B}$

<u>$i \neq j$:</u> $\lambda_{\mathrm{Same}_{ij}} = U_{\mathbb{B}}$

$\lambda_{\langle \mathrm{Same}_{12}, \mathrm{Same}_{23} \rangle} = U_{\mathbb{B} \times \mathbb{B}} = U_{\mathbb{B}} \otimes U_{\mathbb{B}} = \lambda_{\mathrm{Same}_{12}} \otimes \lambda_{\mathrm{Same}_{13}}$

So $\mathrm{Same}_{12} \perp_{\lambda} \mathrm{Same}_{13}$

independence

Pairing r.v.s:

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α, β independent w.r.t. λ

I-ary version:

$$\lambda : D\Omega, \alpha_{-} : \prod_{i \in I} X_i^{\Omega} \vdash \mathop{\perp}_{\lambda}{}_{i \in I} \alpha_i := \forall J \subseteq_{fin} I.\ \lambda_{\langle \alpha_j \rangle_{j \in J}} = \bigotimes_{j \in J} \lambda_{\alpha_j} : \mathrm{Prop}$$

α_{-} independent w.r.t. λ

Ex (Durrett) represent outcomes of 3 coin tosses:

$C := \{T, H\}$ $\quad \Omega := C \times C \times C \quad \lambda : U_C \otimes U_C \otimes U_C : P\Omega$

$\pi_i : \Omega \longrightarrow C$ outcome of i^{th} toss

$\mathrm{Same}_{ij} : \Omega \xrightarrow{\langle \pi_i, \pi_j \rangle} C \times C \xrightarrow{(\overset{?}{=})} \mathbb{B}$

$\underline{i \neq j}$: $\lambda_{\mathrm{Same}_{ij}} = U_{\mathbb{B}}$

$\underset{\neq k \neq}{i \neq j}$: $\mathrm{Same}_{ij} \perp_{\lambda} \mathrm{Same}_{jk}$ $\qquad \not\perp_{\lambda} \{\mathrm{Same}_{12}, \mathrm{Same}_{23}, \mathrm{Same}_{13}\}$

Intuition: $\mathrm{Same}_{13} = \mathrm{IFF}(\mathrm{Same}_{12}, \mathrm{Same}_{23})$

Calc:

$$\lambda_{\langle \mathrm{Same}_{12}, \mathrm{Same}_{23}, \mathrm{Same}_{13} \rangle}(T,T,T) = \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}_{\hookrightarrow \lambda(H,H,H)} + \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}_{\hookrightarrow \lambda(T,T,T)} = \frac{1}{4} \neq \frac{1}{2^3} = \lambda_{\mathrm{Same}_{12}} \otimes \lambda_{\mathrm{Same}_{23}} \otimes \lambda_{\mathrm{Same}_{13}}$$

Vocabulary

(Discrete) Measure space $(X, \mu : DX)$

measure preserving $f : (X, \mu) \longrightarrow (Y, \nu)$

function $f : X \to Y$ s.t. $\mu_f = \nu$

$\mu : DX,\ f : X \to Y \vdash$ μ invariant under f :=

$$f : (X, \mu) \to (X, \mu)$$

Ex:

$\mu : DX,\ \nu : DY \vdash$

$\mathrm{Swap} : (X \times Y, \mu \otimes \nu) \longrightarrow (Y \times X, \nu \otimes \mu)$ so

$\mu : DX \vdash \mu \otimes \mu$ invariant under swap

Weights as measures

NB: unit type → $\mathbb{1} := \{\langle\rangle\}$ ← unit value

Observation:

$$\begin{tikzcd} D\mathbb{1} \arrow[r, bend left, "\lambda\mu.\,\mathrm{ce}_{\mu}[\mathbb{1}]"] & W \arrow[l, bend left, "(-\odot\delta_{\langle\rangle})"] \end{tikzcd}$$

mutually inverse

Proof:

$$\mu \longmapsto \mu\langle\rangle \qquad \lambda\langle\rangle.a \longleftarrow\!\!\shortmid\; a$$

$$\mu = \lambda\langle\rangle.\mu\langle\rangle \qquad a = (\lambda\langle\rangle.a)\langle\rangle$$

$\blacksquare$

Integration

$$\mu : DX,\ \varphi : W^X \vdash \int \mu\, \varphi \quad : W \qquad \text{(Lebesgue integral)}$$

$$:= \sum_{x \in X} \mu x \cdot \varphi x$$

Can derive it:

$$\begin{tikzcd}
DX \times W^X \arrow[rr, "DX \times (\cong \circ -)"] \arrow[d, "\int"'] \arrow[drr, phantom, ":="] & & DX \times (D\mathbb{1})^X \arrow[d, "\oint"] \\
W & & D\mathbb{1} \arrow[ll, "\cong"]
\end{tikzcd}$$

Additivity:

$$I \text{ ctbl},\ \mu_{-} : (DX)^{I} \vdash \sum_{i \in I} \mu_i : DX$$

$$:= \lambda x. \sum_{i \in I} \mu_i x$$

NB:
$\mathrm{supp} \sum_i \mu_i \subseteq \bigcup_i \mathrm{supp}\, \mu_i$
✓ctbl

Ex: Bernoulli distribution

$$p : [0,1] \vdash B(p) := p \cdot \delta_{\mathrm{True}} + (1-p) \cdot \delta_{\mathrm{False}} : P\mathbb{B}$$

i.e. B_p : $\mathrm{True} \mapsto p$
$\mathrm{False} \mapsto 1-p$

Thm (affine-linearity):

$\oint$ is affine-linear in each argument:

$$I \text{ ctbl} \quad \begin{matrix} \mu_- : (D\Gamma)^I, k : (DX)^\Gamma \\ a_- : W^I \end{matrix} \vdash \oint \left(\sum_{i \in I} a_i \cdot \mu_i\right) k = \sum_{i \in I} a_i \cdot \oint \mu_i k$$

$$I \text{ ctbl}, \mu : D\Gamma, a_- : W^I, k_- : DX^I \vdash$$

$$\int \mu(dx) \left(\sum_{i \in I} a_i \cdot k_i(x)\right) = \sum_{i \in I} a_i \cdot \oint \mu k_i$$

Prop: $W \cong D1$ is a σ-semi-ring isomorphism:

$$(W, \sum, (\cdot), 1) \cong (D1, \sum, (\cdot), \delta_{\langle\rangle})$$

and $(\cdot) : W \times DX \to DX$ makes DX into a module:

$$\left(\sum_{i \in I} a_i\right) \cdot \mu = \sum_{i \in I} (a_i \cdot \mu) \qquad a \cdot \sum_{i \in I} \mu_i = \sum_{i \in I} a \cdot \mu_i$$

Corollary: $\int$ is affine-linear in each argument.

Random variable :

NB : $\overline{\mathbb{R}} := [-\infty, \infty]$

A random element $\alpha : \overline{\mathbb{R}}^{\Omega}$ (wrt some $\mu : D\Omega$)

Can add, multiply r.v.'s.

To integrate r.v.'s:

$$(-)^{+} : \overline{\mathbb{R}}^{\Omega} \longrightarrow W^{\Omega}$$

$$\alpha^{+} := \lambda w. \begin{cases} \alpha \cdot w \geq 0 : \alpha w \\ 0.w : 0 \end{cases} = [\alpha - \geq 0] \cdot |\alpha|$$

$$\alpha^{-} := \lambda w. \begin{cases} \alpha \cdot w \leq 0 : |\alpha w| \\ 0.w : 0 \end{cases} = [\alpha - \leq 0] \cdot |\alpha|$$

So $\alpha = \alpha^{+} - \alpha^{-}$

$$\mu : D\Omega, \alpha : \overline{\mathbb{R}}^{\Omega}, \int \mu \alpha^{+} < \infty \text{ or } \int \mu \alpha^{-} < \infty \vdash$$

$$\int \mu \, \alpha := \int \mu \alpha^{+} - \int \mu \alpha^{-} \quad : \overline{\mathbb{R}}$$

<u>Ex.</u> The (discrete) Lebesgue p-space:

$$p : [1, \infty), \mu : P\Omega \vdash \mathcal{L}_p(\Omega, \mu) :=$$

$$\left\{ \alpha : \overline{\mathbb{R}}^{\Omega} \;\middle|\; \mathop{\mathbb{E}}_{\mu}[|\alpha|^{p}] < \infty \right\}$$

$\mathcal{L}_p(\Omega, \mu)$ has a norm $\|\alpha\| := \sqrt[p]{\mathop{\mathbb{E}}_{\mu}[|\alpha|^{p}]}$ almost Banach

$\mathcal{L}_2(\Omega, \mu)$ has an inner product $\langle \alpha, \beta \rangle := \mathop{\mathbb{E}}_{\mu}[\alpha \cdot \beta]$ almost Hilbert

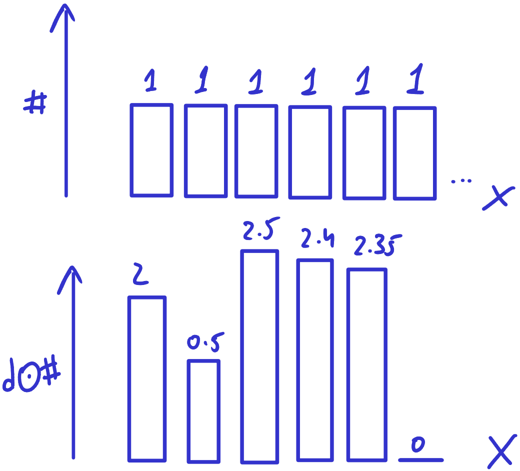
Density

a density over X : $d : X \to W$

$$d : W^X, \mu : DX \vdash d \odot \mu \quad : DX$$
$$:= \oint \mu(dx)(d_x \cdot \delta_x)$$

Warning The types of measures & densities in the discrete model are close, but still different. They coincide on countable sets, so people often confuse them. Types help us keep them separate.

Intuition:



Almost certain properties

$$E : \mathcal{B}X, \mu : DX \vdash \mu(\mathrm{d}x)\text{-almost certainly } x \in E \quad : \mathrm{Prop}$$

$$:= [- \in E] \odot \mu = \mu$$

$$\text{NB: } [x \in E] = \begin{cases} x \in E : & 1 \\ x \notin E : & 0 \end{cases} \quad : W$$

When $\mu : PX$ we say instead

$$\mu(\mathrm{d}x)\text{-almost surely } x \in E$$

Absolute continuity

d is a density of μ w.r.t. ν or

d is a Radon-Nikodym derivative w.r.t. ν

$$\mu, \nu : DX,\ d : W^X \vdash d = \frac{d\mu}{d\nu} : \mathrm{Prop}$$

$$:= \mu = d \odot \nu$$

$\mu, \nu : DX \vdash \mu \ll \nu :=$ μ is absolutely continuous w.r.t. ν : Prop

$$:= \exists d : W^X.\ d = \frac{d\mu}{d\nu}.$$

$=:$ μ has a density w.r.t. ν

Lemma: $\mu, \nu : DX,$
$\mu \ll \nu,$
$k : (DY)^X$

$$\oint \nu(dx)\, \frac{d\mu}{d\nu}(x) \cdot k\,x = \oint \mu(dx)\, k\,x$$

$\underline{\text{Ex}}$: $U_{A \subseteq X} \ll (\#_A)_{\text{Cnt} : A \subseteq X}$

$$\frac{dU_{A \subseteq X}}{d(\#_A)_{\text{Cnt}}} = \lambda x. \begin{cases} x \in A : & \frac{1}{|A|} \\ \text{o.w.} : & 0 \end{cases}$$

but also: $$\frac{dU_{A \subseteq X}}{d(\#_A)_{\text{Cnt}}} = \lambda x. \frac{1}{|A|}$$

Radon-Nikodym Thm: (discrete version)

$\mu, \nu : PX \vdash \mu \ll \nu$ iff $\forall x. \nu x = 0 \Rightarrow \mu x = 0$

i.e. $\mathrm{Supp}\,\mu \subseteq \mathrm{Supp}\,\nu$

In that case, if $d_1, d_2 = \frac{d\mu}{d\nu}$ then

$\nu(dx)$-a.s. $d_1 x = d_2 x$

Ex: for ctbl X, $\forall \mu : DX.\ \mu \ll \#_X$. Proof: vacuously, as $\#_X x \neq 0$.

Then $\lambda x. \mu x = \frac{d\mu}{d\#}$.

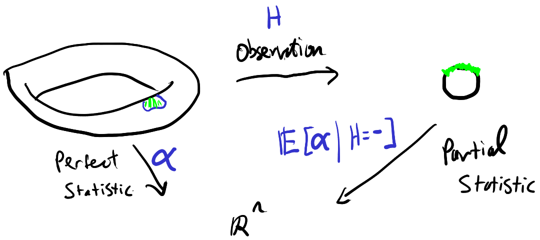
Conditional expectation

β is a conditional expectation of α w.r.t. μ along H

$$\mu : D\Omega,\ H : X^{\Omega},\ \alpha : \mathcal{L}_1(\Omega, \mu),\ \beta : \mathcal{L}_1(X, \mu_H)$$

$$\vdash \beta = \mathbb{E}[\alpha \mid H = -] \quad : \mathrm{Prop}$$

$$:= \forall \varphi : \mathcal{L}_1(X, \mu_H^M).\ \int \mu_H(dx)\, \beta(x) \cdot \varphi(x) = \int \mu(d\omega)\, \alpha(\omega) \cdot \varphi(H\omega)$$



Thm (Kolmogorov): (discrete version)

There is a funtion

$$\underline{\mathbb{E}}[-|-=-] \in \prod_{\mu : P\Omega} \prod_{H : X^{\Omega}} \mathcal{L}_1(\Omega, \mu) \to \mathcal{L}_1(X, \mu_H)$$

s.t. $\mathbb{E}_{\mu}[\alpha | H = -]$ is a conditional expectation of α w.r.t. μ along H.

Conditional Probability (discrete version):

$$H : X^{\Omega},\ \mu : PX \vdash \underset{\mu}{\mathbb{P}r}[- \mid H = -] : (P\Omega)^{X}$$

$$:= \lambda x_0 : X.\ \lambda \omega_0 : \Omega.\ \underset{\omega \sim \mu}{\mathbb{E}}\left[[\omega_0 = \omega] \,\middle|\, H\omega = x_0 \right]$$

Bayes's Thm (discrete version, adapted from Williams):

$$\begin{tikzcd} & X \times \Theta \arrow[dl, "\alpha := \pi_1"'] \arrow[dr, "H := \pi_2"] & \\ X & & \Theta \end{tikzcd}$$

observation space.

Let $\lambda : P(X \times \Theta)$ joint probability distribution.

Assume $\mu : DX$, $\nu : D\Theta$ s.t. $\lambda \ll \mu \otimes \nu$.

with $d_{X,H} = \dfrac{d\lambda}{d(\mu \otimes \nu)}$.

Obs 1: $d_X : W^X$ then $d_X = \dfrac{d\lambda_\alpha}{d\mu}$

$$d_X := \lambda x. \int \nu(d\theta)\, d_{X,H}(x, \theta)$$

& similarly $(d_H : W^\Theta) := \lambda \theta. \int \mu(dx)\, d_{X,H}(x, \theta) = \dfrac{d\lambda_H}{d\nu}$

Bayes's Thm (discrete version, adapted from Williams):

$$\begin{tikzcd} & X \times \Theta \arrow[dl, "\alpha := \pi_1"'] \arrow[dr, "H := \pi_2"] & \\ X & & \Theta \end{tikzcd}$$

Θ: observation space.

Let $\lambda : P(X \times \Theta)$ joint probability distribution.

Assume $\mu : DX$, $\nu : D\Theta$ s.t. $\lambda \ll \mu \otimes \nu$.

with $d_{X,H} = \frac{d\lambda}{d(\mu \otimes \nu)}$. $\quad d_X = \frac{d\lambda_\alpha}{d\mu} \quad d_\Theta = \frac{d\lambda_H}{d\nu}$

Let $d_{X|H}(-|-) : X \times \Theta \to \mathbb{W}$

$$d_{X|H}(x|\theta) := \begin{cases} d_H\theta \neq 0: & \dfrac{d_{X,H}(x,\theta)}{d_H\theta} \\ \text{o.w.}: & 0 \end{cases}$$

$\lambda_{X|H=-} : \Theta \to PX$

$\lambda_{X|H=\theta} := d_{X|H}(-|\theta) \odot \mu$

Bayes's formula:

$\mathbb{P}r_\lambda[-|H=-] = \lambda_{X|H=-}$

Summary

$\mu \otimes \nu$ Product measures & Fubini-Tonelli

μ_H push-forward / law

$(Dx, \Sigma, (\cdot))$ module structure and affine linearity of $\oint$

$\int$ Lebesgue integration

Standard vocabulary: joint dist., marginalisation, independence, invariance

density & Radon-Nikodym derivatives (heed the warning)

almost certain properties

Conditional expectation & Probability

with Bayes's Thm.

Plan:

1) type-driven probability: discrete case ✓
2) Borel sets & measurable spaces
3) Quasi Borel spaces

Lecture 1

4) Type structure & standard Borel spaces
5) Integration & random variables

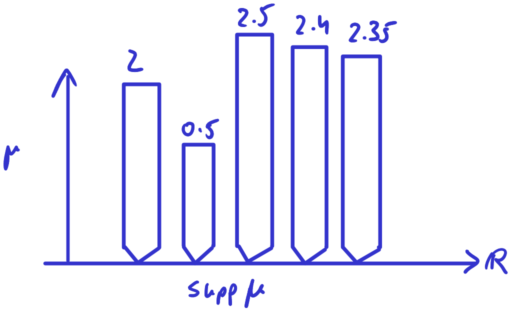
Lecture 2

Please ask questions!

Course web page

scribble

discrete model measure only histograms:



Want:

- lengths
- areas
- volumes.

Continuous Caveat:

Thm: no $\lambda : \mathcal{P}\mathbb{R} \to [0,\infty]$:

$\lambda(a,b) = b-a$ (generalises length)

$\lambda(r+A) = \lambda A$ (translation invariant)

$\lambda\left(\biguplus_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \lambda A_n$ σ-additive

Takeaway: taking $\mathcal{B}\mathbb{R} := \mathcal{P}\mathbb{R}$

excludes measures such as:

length, area, volume

Workaround: only measure well-behaved subsets

Def: The Borel subsets $B_{\mathbb{R}} \subseteq P\mathbb{R}$:

- open intervals $(a,b) \in B_{\mathbb{R}}$

Closure under σ-algebra operations:

$$\frac{}{\emptyset \in B_{\mathbb{R}}} \qquad \frac{A \in B_{\mathbb{R}}}{A^c := \mathbb{R} \setminus A \in B} \qquad \frac{\vec{A} \in B_{\mathbb{R}}^{\mathbb{N}}}{\bigcup_{n=0}^{\infty} A_n \in B_{\mathbb{R}}}$$

empty set

complements

countable unions

Examples

discrete Countable: $\{r\} = \bigcap_{\varepsilon \in \mathbb{Q}^+} (r-\varepsilon, r+\varepsilon) \in \mathcal{B}_{\mathbb{R}}$

I countable $\Rightarrow$ $I = \bigcup_{r \in I} \{r\} \in \mathcal{B}_{\mathbb{R}}$

closed intervals: $[a,b] = (a,b) \cup \{a,b\}$

Non-examples?

More complicated: analytic, Lebesgue

Def: Measurable space $V = (|V|, B_V)$

Set (carrier)

family of subsets $B_V \subseteq P|V|$

Closed under σ-algebra operations:

$$\frac{}{\emptyset \in B_V} \qquad \frac{A \in B_V}{A^c := |V| \setminus A \in B_V} \qquad \frac{\vec{A} \in B_V^{\mathbb{N}}}{\bigcup_{n=0}^{\infty} A_n \in B_V}$$

empty set

complements

countable unions

Idea: structure all spaces after the worst-case scenario

Examples

- Discrete spaces $\ulcorner X^{\text{meas}} \urcorner = (X, PX)$

- Euclidean spaces $\mathbb{R}^n$ — replace intervals with chests $\prod_{i=1}^{n} (a_i, b_i)$

  $\mathbb{R}^{\mathbb{N}}$ similarly

- Subspaces: $A \in P \llcorner V \lrcorner$ $\quad A := (A, \underbrace{[\mathcal{B}_V] \cap A}_{\{C \cap A \mid C \in \mathcal{B}_V\}})$

- Products: $A \times B := (\llcorner A \lrcorner \times \llcorner B \lrcorner, \sigma([\mathcal{B}_A] \times [\mathcal{B}_B]))$

Def: Borel measurable functions $f : V_1 \to V_2$

- functions $f : \lfloor V_1 \rfloor \longrightarrow \lfloor V_2 \rfloor$
- inverse image preserves measurability:

$$f^{-1}[A] \in \mathcal{B}_{V_1} \Longleftarrow A \in \mathcal{B}_{V_2}$$

Examples

- $(+), (\cdot) : \mathbb{R}^2 \longrightarrow \mathbb{R}$
- $|-|$, $\sin : \mathbb{R} \to \mathbb{R}$
- any continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$
- any function $f : \ulcorner X \urcorner \to V$

Category **Meas**

Objects: Measurable spaces

Morphisms: Measurable functions

Identities:

$$id : V \to V$$

Composition:

$$\frac{f : V_2 \to V_3 \qquad g : V_1 \to V_2}{f \circ g : V_1 \to V_3}$$

Meas Category

Products, coproducts/disjoint union, subspaces

Categorical limits, colimits, but:

Thm [Aumann '61] No σ-algebras $\mathcal{B}_{\mathcal{B}_{\mathbb{R}}}$, $\mathcal{B}_{\mathbb{R}^{\mathbb{R}}}$ for measurable

membership predicate

$$(\ni) : (\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathcal{B}_{\mathbb{R}}}) \times \mathbb{R} \longrightarrow \mathrm{Bool}$$

$$(U, r) \longmapsto [r \in U]$$

$$\mathrm{eval} : (\mathrm{Meas}(\mathbb{R}, \mathbb{R}), \mathcal{B}_{\mathbb{R}^{\mathbb{R}}}) \times \mathbb{R} \to \mathbb{R}$$

$$(f, r) \mapsto f(r)$$

Sequential Higher-order structure:

I Countable: $\quad V^{\ulcorner I \urcorner} = \prod_{i \in I} V$

$\Longrightarrow$ Some higher-order structure in **Meas**:

Cauchy $\in B_{[-\infty,\infty]^{\mathbb{N}}}$

$$\text{Cauchy} := \bigcap_{\varepsilon \in \mathbb{Q}^+} \bigcup_{k \in \mathbb{N}} \bigcap_{\substack{m,n \in \mathbb{N} \\ m,n \geq k}} \left\{ \vec{y} \in [-\infty,\infty]^{\mathbb{N}} \,\middle|\, |y_m - y_n| < \varepsilon \right\}$$

$\limsup : [-\infty,\infty]^{\mathbb{N}} \to [-\infty,\infty] \qquad \lim : \text{Cauchy} \to \mathbb{R}$

Compose higher-order building blocks:

lim is measurable!

$$\mathrm{VanishingSeq}(\mathbb{R}) := \left\{ \vec{r} \in \mathbb{R}^{\mathbb{N}} \,\middle|\, \lim_{n\to\infty} r_n = 0 \right\} \in \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}$$

$$\mathrm{approx}_{\_} : \mathrm{VanishingSeq}(\mathbb{R}^+) \times \mathbb{R} \longrightarrow \mathbb{Q}^{\mathbb{N}}$$

$$\text{s.t.:} \quad \left| \left(\mathrm{approx}_{\vec{\Delta}}\, r\right)_n - r \right| < \Delta_n$$

Slogan: Measurable by Type!

Not all operations of interest fit:

Intrinsically higher-order!

$$\limsup : \left([-\infty,\infty]^{\mathbb{R}}\right)^{\mathbb{N}} \longrightarrow [-\infty,\infty]^{\mathbb{R}}$$

$$\limsup := \lambda \vec{f}.\lambda x.\limsup_{n\to\infty} f_n\, x$$

Want

Slogan: measurability by type!

But

For higher-order building blocks

defer measurability proofs until

we resume 1^{st} order fragment $\Rightarrow$ non compositional

Plan

Def: $V \in \mathbf{Meas}$ is Standard Borel when

$$V \cong A \quad \text{for some } A \in \mathcal{B}_{\mathbb{R}}$$

the "good part" of **Meas** — the subcategory

$$\mathbf{Sbs} \hookrightarrow \mathbf{Meas}$$

Sbs includes

- Discrete $\ulcorner I \urcorner$, I countable
- Countable products of Sbs:

$$\mathbb{R}^n, \mathbb{R}^{\mathbb{N}}, 2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}$$

- Borel subspaces of Sbs:

$\mathbb{I} := [0,1]$ $\mathbb{R}^+ := (0,\infty)$ $\mathbb{R}_{\geq 0} := [0,\infty]$

- Countable coproducts of Sbs:

$\mathbb{W} := [0,\infty]$ $\overline{\mathbb{R}} := [-\infty,\infty]$

Conservative extensions:

concrete spaces we "observe"

Standard Borel spaces

Meas

Qbs

abstract spaces

(auxiliary?)

Plan:

1) type-driven probability: discrete case ✓
2) Borel sets & measurable spaces ✓
3) Quasi Borel spaces

Lecture 1

4) Type structure & standard Borel spaces
5) Integration & random variables

Lecture 2

Please ask questions!

Course web page

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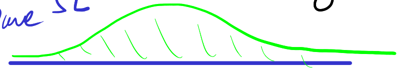
Core idea

Measure Theory

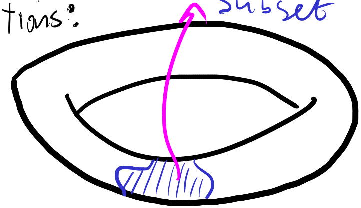
Qbs Theory

sample space Ω



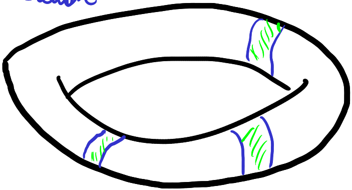
Primitive notions:

measurable subset



random element

$\downarrow \alpha$



Derived notions: random elements $\alpha : \Omega \to \text{Space}$

measure

Events $E \in B_X$

Def: Quasi-Borel space $X = (\lfloor X \rfloor, R_X)$

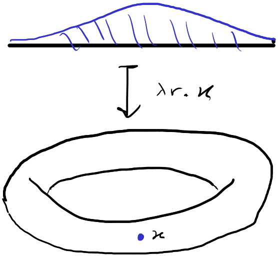
set "carrier"

set of functions $\alpha : \mathbb{R} \to \lfloor X \rfloor$ "random elements"

$R_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- Constants:

$$\frac{x \in \lfloor X \rfloor}{(\lambda r. x) \in R_X}$$



- precomposition:

- recombination

Def: Quasi-Borel space $X = (\lfloor X \rfloor, R_X)$

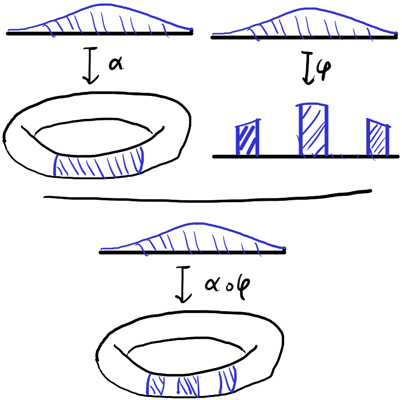
set "carrier"

set of functions $\alpha : \mathbb{R} \to \lfloor X \rfloor$ "random elements"

$R_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- precomposition:

$$\frac{\alpha \in R_X \qquad \varphi : \mathbb{R} \to \mathbb{R} \text{ in Sbs}}{\varphi \circ \alpha : \mathbb{R} \xrightarrow{\varphi} \mathbb{R} \xrightarrow{\alpha} \lfloor X \rfloor \in R_X}$$



Def: Quasi-Borel space $X = (\lfloor X \rfloor, \mathcal{R}_X)$

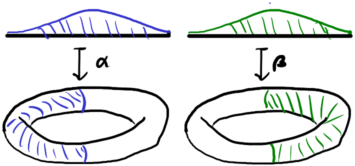
set "carrier"

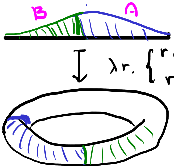
set of functions $\alpha : \mathbb{R} \to \lfloor X \rfloor$ "random elements"

$\mathcal{R}_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- recombination

$$\frac{\vec{\alpha} \in \mathcal{R}_X^{\mathbb{N}} \qquad \mathbb{R} = \biguplus_{n=0}^{\infty} \underbrace{A_n}_{\in \mathcal{B}_{\mathbb{R}}}}{\lambda r. \begin{cases} \vdots \\ r \in A_n : \alpha_n r \\ \vdots \end{cases}}$$





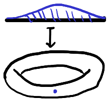
$\lambda r. \begin{cases} r \in A : \alpha r \\ r \in B : \beta r \end{cases}$

Def: Quasi-Borel space $X = (\lfloor X \rfloor, R_X)$

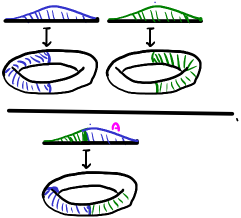
set "carrier"

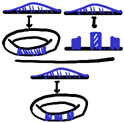
set of functions $\alpha : \mathbb{R} \to \lfloor X \rfloor$ "random elements"

$R_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- constants: 

- recombination



- precomposition: 

Examples

recombination of constants

- $\mathbb{R} = (\lfloor\mathbb{R}\rfloor, \mathrm{Meas}(\mathbb{R},\mathbb{R}))$

qbs underlying $\mathbb{R}$

$$\lambda r. \begin{cases} \vdots \\ r \in A_n : x_n \\ \vdots \end{cases}$$

- $X \in \mathrm{Set}$, $\ulcorner X \urcorner^{\mathrm{Qbs}} := (X, \sigma\text{-}\mathrm{Simple}(\mathbb{R}, X))$

discrete qbs on X

- " $\lfloor X \rfloor_{\mathrm{Qbs}} := (X, X^{\lfloor\mathbb{R}\rfloor})$

all funtions

Indiscrete qbs on X

Qbs morphism $f : X \to Y$

- function $f : \lfloor X \rfloor \to \lfloor Y \rfloor$

- $$\frac{\begin{array}{c}\mathbb{R} \\ \alpha \downarrow \\ \lfloor X \rfloor\end{array} \in R_X}{\begin{array}{c}\mathbb{R} \\ \alpha \downarrow \\ \lfloor X \rfloor \\ f \downarrow \\ \lfloor Y \rfloor\end{array} \in R_Y}$$

Example

- Constant functions are qbs morphisms

- σ-simple functions are qbs morphisms

Category Qbs $\Leftarrow$ - identity, composition

Full model

type : Qbs $\qquad W := [0,\infty] \qquad BX := \boxed{}$

$DX := \boxed{}$

$PX := \{\mu \in DX \mid \underset{\mu}{\mathcal{E}}[X] = 1\}$

$\underset{\mu}{\mathcal{E}}[E] := \boxed{} \qquad \delta_x := \boxed{}$

$\oint \mu k := \boxed{}$

Plan:

1) type-driven probability: discrete case
2) Borel sets & measurable spaces
3) Quasi Borel spaces

Lecture 1

4) Type structure & standard Borel spaces
5) Integration & random variables

Lecture 2

Please ask questions!

Course web page

scribble

Foundations for type-driven probabilistic modelling

Ohad Kammar
University of Edinburgh

TutorialFest
ACM-SIGPLAN 51st Symposium on
Principles of Programming Languages (POPL'24)
15 January, 2024
City of Westminister, England

THE UNIVERSITY of EDINBURGH informatics lfcs Laboratory for Foundations of Computer Science BayesCentre

supported by: THE ROYAL SOCIETY The Alan Turing Institute Facebook Research NCSC

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Full model

type : Qbs $\qquad \mathbb{W} := [0,\infty] \qquad \mathcal{B}X := \square$

$\mathcal{D}X := \square$

$\mathcal{P}X := \{\mu \in \mathcal{D}X \mid \mathcal{C}e_{\mu}[X] = 1\}$

$\mathcal{C}e_{\mu}[E] := \square \qquad \delta_x := \square$

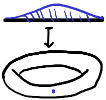
$\oint \mu k := \square$

Def: Quasi-Borel space $X = (\lfloor X \rfloor, R_X)$

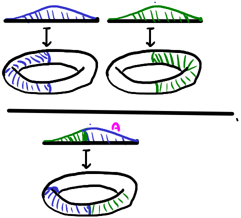
set "carrier"

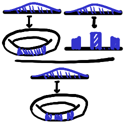
set of functions $\alpha : \mathbb{R} \to \lfloor X \rfloor$ "random elements"

$R_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- constants: 

- recombination



- precomposition: 

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- " $\lfloor X \rfloor_{\mathrm{Qbs}} := (X, X^{\lfloor\mathbb{R}\rfloor})$

all funtions

Indiscrete qbs on X

Validate qbs axioms for: $\mathbb{W} := ([0,\infty], \mathrm{Meas}(\mathbb{R}, \mathbb{W}))$

- Constants:

$E : B_{\mathbb{W}}, x : \mathbb{W} \vdash$

$$(\lambda r:\mathbb{R}.\, x)^{-1}[E] = \begin{cases} x \in E: & \mathbb{R} \\ x \notin E: & \emptyset \end{cases} \in B_{\mathbb{R}}$$

✓

Validate qbs axioms for: $\mathbb{W} := ([0,\infty], \mathrm{Meas}(\mathbb{R}, \mathbb{W}))$

- Precomposition:

$\alpha : \mathrm{Meas}(\mathbb{R}, \mathbb{W})$, $\varphi : \mathrm{Meas}(\mathbb{R}, \mathbb{R}) \vdash$

$$\mathbb{R} \xrightarrow{\varphi} \mathbb{R} \xrightarrow{\alpha} \mathbb{W} \quad \in \mathrm{Meas}(\mathbb{R}, \mathbb{W})$$

Meas is a cat.

Explicitly:

$$(\alpha \circ \varphi)^{-1}[E] \in \mathcal{B}\mathbb{R} \xleftarrow{\varphi^{-1}} \!\!\shortmid\; \alpha^{-1}[E] \in \mathcal{B}\mathbb{R} \xleftarrow{\alpha^{-1}} \!\!\shortmid\; E \in \mathcal{B}_{\mathbb{W}} \quad \checkmark$$

Validate qbs axioms for: $\mathbb{W} := ([0,\infty], \mathrm{Meas}(\mathbb{R}, \mathbb{W}))$

- rc combination

I ctbl, $\alpha_- : \mathrm{Meas}(\mathbb{R}, \mathbb{W})^I$, $E_- : B_{\mathbb{R}}^I$, $\mathbb{R} = \biguplus_{i \in I} E_i$, $F : B_{\mathbb{W}}$:

$$\beta := \left(\lambda r. \begin{cases} \vdots \\ r \in E_i : & \alpha_i r \\ \vdots \end{cases}\right)^{-1}[F] = \bigcup_{i \in I} \alpha_i^{-1}[F] \cap E_i \in B_{\mathbb{R}}$$

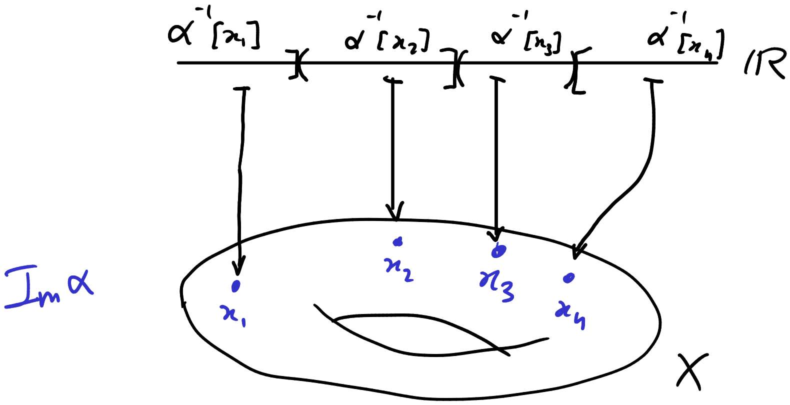
Indeed:

$r \in \mathrm{LHS} \iff \beta r \in F \iff \exists i \in I. r \in E_i \wedge \alpha_i r \in F \iff r \in \mathrm{RHS}$ ✓

σ-simple function

$\alpha : \mathbb{R} \longrightarrow X$ s.t. $\mathrm{Im}\,\alpha := \alpha[\mathbb{R}]$ is ctbl $\wedge$

$\forall x \in \mathrm{Im}\,\alpha .\ \alpha^{-1}[x] \in \mathcal{B}_{\mathbb{R}}$



Validate qbs axioms for: $\ulcorner X^{\mathrm{Qbs}} \urcorner := (X, \sigma\text{-Simple}(X))$

- Constants

NB: f σ-Simple: $\mathrm{Im}\, f$ ctbl $\wedge$ $f^{-1}[x] \in B_{\mathbb{R}}$

$\mathrm{Im}(\lambda r. x) = \{x\}$ ctbl ✓

$$y : X \vdash (\lambda r. x)^{-1}[y] = \begin{cases} x = y : & \mathbb{R} \\ x \neq y : & \emptyset \end{cases} \in B_{\mathbb{R}} \checkmark$$

Validate qbs axioms for: $\ulcorner X^{\mathrm{Qbs}} \urcorner := (X, \sigma\text{-simple}(X))$

NB: f σ-simple: $\mathrm{Im} f$ ctbl $\wedge$ $f^{-1}[x] \in B_{\mathbb{R}}$

• Precomposition:

$\alpha : \sigma\text{-simple}(X),\ \varphi : \mathrm{Meas}(\mathbb{R}, \mathbb{R}) \vdash$

$\mathrm{Im}(\alpha \circ \varphi) \subseteq \mathrm{Im}\,\alpha$ ctbl ✓

$x : X \vdash$

$(\alpha \circ \varphi)^{-1}[x] = \varphi^{-1}[\alpha^{-1}(x)] \in B_{\mathbb{R}}$ ✓

$\alpha^{-1}(x) \in B_{\mathbb{R}}$

$\varphi : \mathbb{R} \to \mathbb{R}$ measurable

Validate qbs axioms for: $\ulcorner X^{\mathrm{Qbs}} \urcorner := (X, \sigma\text{-Simple}(X))$

NB: f σ-Simple: $\mathrm{Im}\, f$ ctbl $\wedge$ $f^{-1}[x] \in B_{\mathbb{R}}$

- recombination:

$$\alpha_{-} : (\sigma\text{-Simple}(X))^{I},\ E_{-} : B_{\mathbb{R}}^{I},\ \mathbb{R} = \biguplus_{i \in I} E_i \vdash$$

$$\mathrm{Im}\,[E_i . \alpha_i]_{i \in I} \subseteq \bigcup_{i \in I} \mathrm{Im}\,\alpha_i \quad \text{ctbl} \quad \checkmark$$

$x : X \vdash$

$$[E_i . \alpha_i]_{i \in I}^{-1}(x) = \bigcup_{i \in I} \alpha_i^{-1}[x] \cap E_i \in B_{\mathbb{R}} \quad \checkmark$$

Prop: $X : \mathrm{Set},\ A : \mathrm{Qbs} \vdash$

- $\forall f : X \to \lfloor A \rfloor .\ f : \ulcorner X \urcorner^{\mathrm{Qbs}} \longrightarrow A$

- $\forall f : \lfloor A \rfloor \longrightarrow X .\ f : A \longrightarrow \llcorner X \lrcorner_{\mathrm{Qbs}}$

Prop: $X : \mathrm{Set},\ A : \mathrm{Qbs} \vdash$

$$\bullet\ \forall f : X \to \lfloor A \rfloor .\ f : \ulcorner X \urcorner^{\mathrm{Qbs}} \longrightarrow A$$

Pf: $\alpha : R_{\ulcorner X \urcorner^{\mathrm{Qbs}}} \vdash$ α σ-simple $\Rightarrow$

$$\alpha = [\alpha^{-1}[x].\lambda r.\, x]_{x \in \mathrm{Im}\,\alpha} \Rightarrow$$

recombination

$$(f \circ \alpha) = [\alpha^{-1}[x].\lambda r.\, fx]_{x \in \mathrm{Im}\,\alpha} \in R_A \quad \checkmark$$

Borel

constant $\in B_A$

ctbl

Prop: $X : \mathrm{Set},\ A : \mathrm{Qbs} \vdash$

- $\forall f : X \to \llcorner A \lrcorner .\ f : \ulcorner X \urcorner^{\mathrm{Qbs}} \longrightarrow A$
- $\forall f : \llcorner A \lrcorner \longrightarrow X .\ f : A \longrightarrow \llcorner X \lrcorner_{\mathrm{Qbs}}$

Prf: $\alpha : R_A \vdash (f \circ \alpha : \mathbb{R} \to X) \in R_{\llcorner X \lrcorner_{\mathrm{Qbs}}}$ always. ✓

$\blacksquare$

Useful adjunctions:

$$\begin{tikzcd}
\mathbf{Meas} \arrow[d, "\lfloor - \rfloor_{\mathbf{Qbs}}"] \\
\mathbf{Qbs} \arrow[u, bend left=60, "\ulcorner - \urcorner^{\mathbf{Meas}}"] \arrow[u, phantom, "\dashv"] \arrow[d, "\lfloor \cdot \rfloor"] \\
\mathbf{Set} \arrow[u, bend left=60, "\ulcorner - \urcorner^{\mathbf{Qbs}}"] \arrow[u, bend right=60, "\overline{\lfloor \mathbf{Qbs} \rfloor}"'] \arrow[u, phantom, "\dashv \quad \dashv"]
\end{tikzcd}$$

$$\lfloor V \rfloor_{\mathbf{Qbs}} := (\lfloor V \rfloor, \mathbf{Meas}(\mathbb{R}, V)) \quad (V \in \mathbf{Meas})$$

$$\ulcorner X \urcorner^{\mathbf{Meas}} := \left\{ A \subseteq \lfloor X \rfloor \;\middle|\; \forall \alpha \in R_X.\ \alpha^{-1}[A] \in \mathcal{B}_{\mathbb{R}} \right\}$$

- limits (products, subspaces) and colimits (coproducts, quotients) as in **Set**
- Slogan: every measurable space is carried by a qbs

Example

Product $(X \times Y, \pi_1, \pi_2)$:

necessarily!

- $\lfloor X \times Y \rfloor = \lfloor X \rfloor \times \lfloor Y \rfloor$

correlated random elements

- $R_{X \times Y} = \left\{ \lambda r.(\alpha r, \beta r) \,\middle|\, \alpha \in R_X, \beta \in R_Y \right\}$

rest of structure as in **Set**.

Function Spaces

Straightforward!

- $\lfloor Y^X \rfloor := \mathbf{Qbs}(X, Y)$
- $R_{Y^X} := \mathrm{uncurry}[\mathbf{Qbs}(\mathbb{R} \times X, Y)]$

$$= \{ \alpha : \mathbb{R} \to \lfloor Y^X \rfloor \mid \lambda(r, x). \alpha\, r\, x : \mathbb{R} \times X \to Y \}$$

- $\mathrm{eval} : Y^X \times X \to Y$

$\mathrm{eval}(f, x) := f\, x$

Meas vs Qbs

By generalities:

$$\begin{tikzcd} \mathbf{Meas} \arrow[d, bend left, "\lfloor - \rfloor_{\mathbf{Qbs}}"] \arrow[d, phantom, "\dashv"] \\ \mathbf{Qbs} \arrow[u, bend left, "\ulcorner - \urcorner^{\mathbf{Meas}}"] \end{tikzcd}$$

σ-algebra on $\mathbf{Meas}(\mathbb{R}, \mathbb{R})$

$$\begin{tikzcd} \ulcorner \mathbb{R}^{\mathbb{R}} \times \mathbb{R} \urcorner^{\mathbf{Meas}} \arrow[r] \arrow[rr, bend right, "\ulcorner eval \urcorner^{\mathbf{Meas}}"'] & \ulcorner \mathbb{R}^{\mathbb{R}} \urcorner^{\mathbf{Meas}} \times \mathbb{R} \arrow[r, dashed, "\times"] & \ulcorner \mathbb{R} \urcorner = \mathbb{R} \end{tikzcd}$$

No factorisation by Aumann's Theorem.

So: $\left(\ulcorner \mathbb{R}^{\mathbb{R}} \times \mathbb{R} \urcorner \not\cong \ulcorner \mathbb{R}^{\mathbb{R}} \urcorner \times \ulcorner \mathbb{R} \urcorner \right)$

Simple Type Structure

"Simple" because:

- Simply-typed λ-calculus
- types are simple: $A, B : \mathrm{Type} \vdash B^A : \mathrm{Type}$
  - no polymorphism
  - no term dependency
- Contexts for terms: $\Gamma \vdash t : A$
  are simple: $\Gamma = x_1 : A_1, \ldots, x_n : A_n$
  i.e. List(Type)

Simple Type Structure

"Simple" because:

- interpretation is simple:

$$[\![x_1 : A_1, \ldots, x_n : A_n]\!] := \prod_{i=1}^{n} A_i$$

$$[\![\Gamma \vdash t : A]\!] : [\![\Gamma]\!] \longrightarrow A$$

in Qbs

Simple Type Structure Curry-Howard-Lambek

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash s : B}{\Gamma \vdash \langle t, s \rangle : A \times B} \rightsquigarrow [\![\Gamma]\!] \xrightarrow{\lambda r.\langle tr, sr \rangle} A \times B$$

is **measurable**

$$\boxed{\text{measurability by type!}}$$

$$\frac{\Gamma \vdash t : A \times B \qquad \Gamma, x : A, y : B \vdash s : C}{\Gamma \vdash \mathrm{let}\ (x, y) = t\ \mathrm{in}\ s : C} \rightsquigarrow$$

$$[\![\Gamma]\!] \xrightarrow{\lambda r.\ \mathrm{let}\ (a, b) = tr\ \mathrm{in}\ s\, r[x \mapsto a, y \mapsto b]} C$$

is measurable.

etc.

Random element space

$$R_X := X^{\mathbb{R}} \quad \text{since} \quad \lfloor X^{\mathbb{R}} \rfloor = R_X \text{ as sets.}$$

Why?

$(\subseteq)$ $\alpha \in \lfloor X \rfloor^{\mathbb{R}} \Rightarrow \alpha : \mathbb{R} \to X$ in Qbs.

$\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ measurable $\Rightarrow \mathrm{id} \in R_{\mathbb{R}}$

$\Rightarrow \alpha = \alpha \circ \mathrm{id} \in R_X$

Precomposition

$(\supseteq)$ $\alpha \in R_X \Rightarrow \forall \varphi \in R_{\mathbb{R}} = \mathrm{Meas}(\mathbb{R}, \mathbb{R}).\ \alpha \circ \varphi \in R_X \Rightarrow \alpha : \mathbb{R} \to X$

$\Rightarrow \alpha \in \lfloor X^{\mathbb{R}} \rfloor$

Subspaces

For $X \in \mathbf{Qbs}$, $A \subseteq \llcorner X \lrcorner$ set:

$$R_A := \{ \alpha : \mathbb{R} \to A \mid \alpha \in R_X \}$$

Then $A = (A, R_A)$ is the subspace qbs

We write $A \hookrightarrow X$

Borel subspaces ensemble

The σ-algebra $\mathcal{B}_X := \{ A \subseteq \lfloor X \rfloor \mid \forall \alpha \in \mathcal{R}_X . \alpha^{-1}[A] \in \mathcal{B}_{\mathbb{R}} \}$

internalises as $\mathcal{B}_X = 2^X$, the qbs of Borel subsets.

($\mathcal{B}_{\lfloor (\mathcal{B}_{\mathbb{R}}) \rfloor}$ are the Borel-on-Borel sets from descriptive set theory.

cf. [Sabok et al. '21])

Standard Borel Spaces

Def: A qbs S is standard Borel when

$$S \cong A \quad \text{for some } A \in \mathcal{B}_{\mathbb{R}}$$

$$\begin{tikzcd}
\mathbf{Meas} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Qbs} \arrow[l, bend left] \\
\mathbf{Sbs} \arrow[u, hook] \arrow[r, bend left] \arrow[r, phantom, "\cong"] & \mathbf{Sbs} \arrow[u, hook] \arrow[l, bend left]
\end{tikzcd}$$

Slogan: Qbs conservative extension of Sbs

Example $C_0 := \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous} \} \hookrightarrow \mathbb{R}^{\mathbb{R}}$

C_0 is sbs. (Well-known!)

Proof: $C_0' \in B_{\mathbb{R}^{\mathbb{Q}}}$ sbs!

$$C_0' := \left\{ g \in \mathbb{R}^{\mathbb{Q}} \;\middle|\; \begin{array}{l} \forall a, b \in \mathbb{Q}, \varepsilon \in \mathbb{Q}^+ \\ \exists \delta \in \mathbb{Q}^+ \; \forall p, q \in \mathbb{Q}^+ \cap [a,b], \\ |p - q| < \delta \Rightarrow |g\,p - g\,q| < \varepsilon \end{array} \right\}$$

on closed intervals (= compact intervals)

Continuity $\Updownarrow$ uniform continuity

Borel measurable by type checks

then $C_0 \cong C_0' \in B_{\mathbb{R}^{\mathbb{Q}}}$:

$$C_0 \to C_0' \qquad\qquad C_0' \to C_0$$

$$\varphi \mapsto \varphi|_{\mathbb{Q}} \qquad\qquad \varphi \mapsto \lambda r. \lim_{n \to \infty} g(\text{approx}_{(\frac{1}{m})_{m \in \mathbb{N}}} r)_n$$

Example (ctd)

C_0 is sbs, and $\mathrm{eval} : C_0 \times \mathbb{R} \to \mathbb{R}$

is measurable.

Avoids:

- constructing complete separable metrics
- proving that evaluation is measurable w.r.t. metric σ-algebra.

Non-examples $\rightsquigarrow$ [Sabok et al. '21]

- $\{A \in B_{\mathbb{R}} \mid A \neq \emptyset\} \hookrightarrow B_{\mathbb{R}}$

- $\{(A_1, A_2) \in B_{\mathbb{R}}^2 \mid A \subseteq B\} \hookrightarrow B_{\mathbb{R}}^2$

- $\{A \in B_{\mathbb{R}} \mid A \text{ open}\} \hookrightarrow B_{\mathbb{R}}$

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Dependent Type Structure

types can contain terms :

a type referring to a term

$$X : \mathrm{Type},\ E : B_X \vdash \{x \in X \mid x \in E\} : \mathrm{Type}$$

a type, just like STLC

a term!

Dependent Type Structure

types can contain terms:

$$X : \mathrm{Type}, E : B_X \vdash \{x \in X \mid x \in E\} : \mathrm{Type}$$

a type referring to a term

a type, just like STLC

a term!

Content formation:

$$\frac{\Gamma \vdash A : \mathrm{Type}}{\Gamma, x : A \vdash}$$

Dependent Type Structure

types denote spaces-in-context

$$\begin{tikzcd} [\![\Gamma \vdash A]\!] \arrow[d, "\mathrm{dep}"] \\ [\![\Gamma]\!] \end{tikzcd}$$

Dependent types denote spaces-in-context

$\Gamma \vdash$ context

$\Gamma \vdash A$

type in context

$\llbracket \Gamma \vdash A \rrbracket$ Space in context

$$\llbracket \Gamma \vdash A \rrbracket \xrightarrow{\mathrm{dep}} \llbracket \Gamma \rrbracket$$

$\llbracket \Gamma \rrbracket$ context space

assigns environment

E.g.:

$$A \longrightarrow \mathbb{1}$$

simple types

$$\llbracket E : B_A \vdash \{x \in A \mid x \in E\} \rrbracket = \{(E, a) \in B_A \times A \mid a \in E\} \xrightarrow{\pi_1} B_A$$

decoder

Context extension

$$\frac{\Gamma \vdash A}{\Gamma, a : A \vdash}$$

$$[\![\Gamma \vdash A]\!] \xrightarrow{dep} [\![\Gamma]\!] \qquad [\![\Gamma, a : A]\!] := [\![\Gamma \vdash A]\!]$$

Substitution

$\Gamma \vdash \sigma : \Delta$ $\qquad [\![\sigma]\!] : [\![\Gamma]\!] \longrightarrow [\![\Delta]\!]$

E.g. weakening $\qquad \Gamma, a : A \vdash \mathrm{wkn} : \Gamma$

$$[\![\Gamma, a : A]\!] := [\![\Gamma \vdash A]\!] \xrightarrow[dep]{\mathrm{wkn}} [\![\Gamma]\!]$$

Action of substitution on types

$$\llbracket \Gamma \vdash A[\sigma] \rrbracket$$

$$\begin{tikzcd}
\{(\gamma, a) \in \llbracket \Gamma \rrbracket \times \llbracket A \rrbracket \mid \mathrm{dep}\, a = \gamma\} \arrow[r, "\pi_2"] \arrow[d, "\mathrm{dep} := \pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \Delta \vdash A \rrbracket \arrow[d, "\mathrm{dep}"] \\
\llbracket \Gamma \rrbracket \arrow[r, "\sigma"] & \Delta
\end{tikzcd}$$

(with $\llbracket \Gamma \vdash A[\sigma] \rrbracket \cong$ the top-left set)

E.g.

$$\begin{tikzcd}
\llbracket x : A \vdash B[\mathrm{wkn}] \rrbracket := A \times B \arrow[r, "\pi_2"] \arrow[d, "\mathrm{dep} := \pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
x : A \arrow[r, "\langle\rangle"'] & \mathbb{1}
\end{tikzcd}$$

simple type

Terms : sections

$$\begin{tikzcd}
\llbracket \Gamma \rrbracket \arrow[r, "\llbracket \Gamma \vdash M : A \rrbracket"] \arrow[dr, equal] \arrow[dr, phantom, "=" {above right}] & \llbracket \Gamma \vdash A \rrbracket \arrow[d, "dep"] \\
& \llbracket \Gamma \rrbracket
\end{tikzcd}$$

e.g.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\llbracket x : \mathbb{R} \vdash [x, \infty) : B_{\mathbb{R}}[wkn] \rrbracket"] \arrow[dr, equal] \arrow[dr, phantom, "=" {above right}] & \mathbb{R} \times B_{\mathbb{R}} \arrow[d, "\pi_1"] \\
& \mathbb{R}
\end{tikzcd}$$

E.g. Variables:

$$\begin{tikzcd}
\llbracket \Gamma, a : A \rrbracket \arrow[r, "\llbracket \Gamma, a : A \vdash a : A \rrbracket", "\langle id, dep_{\Gamma \vdash A} \rangle"'] \arrow[dr, equal] \arrow[dr, phantom, "=" {above right}] & \llbracket \Gamma, a : A \vdash A[wkn] \rrbracket \arrow[d, "dep"] \\
& \llbracket \Gamma, a : A \rrbracket
\end{tikzcd}$$

Exercise:
action of substitution
$M[\sigma]$

Dependent Pairs

$$\frac{\Gamma, a : A \vdash B}{\Gamma \vdash \coprod_{a \in A} B}$$

$$\begin{tikzcd}
\llbracket \coprod_{a \in A} A \rrbracket := \llbracket \Gamma, a : A \vdash B \rrbracket \arrow[d, "\mathrm{dep}_B"] \arrow[dd, bend right, "\mathrm{dep}_{\amalg} := "'] \\
\llbracket \Gamma, a : A \rrbracket = \llbracket \Gamma \vdash A \rrbracket \arrow[d] \\
\llbracket \Gamma \rrbracket
\end{tikzcd}$$

Dependent Products

$aha: (a:A) \to B$

$$\frac{\Gamma, a:A \vdash B}{\Gamma \vdash \prod_{a \in A} B}$$

$$\lfloor [\![\Gamma \vdash \prod_{a \in A} B]\!] \rfloor :=$$

$$\left\{ (r_0, f : \{a \in [\![A]\!] \mid \mathrm{dep}\, a = r_0\} \to [\![\Gamma, a:A \vdash B]\!]) \,\middle|\, \forall a \in [\![\Gamma, a:A]\!].\ \mathrm{dep}\, a = r_0 \Rightarrow \mathrm{dep}(f a) = a \right\}$$

Exercise: find the random elements.

Full model

type : Qbs $\quad \mathbb{W} := [0,\infty] \quad \mathcal{B}X \cong \mathbb{B}^X$

$DX := (\text{Fri})$

$PX := \{\mu \in DX \mid \oint_\mu [X] = 1\}$

$\oint_\mu [E] := (\text{Fri}) \qquad \delta_x := (\text{Fri})$

$\oint \mu k := (\text{Fri})$

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TutorialFest
ACM-SIGPLAN 51^{st} Symposium on
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THE UNIVERSITY of EDINBURGH informatics lfcs Laboratory for Foundations of Computer Science BayesCentre

supported by: THE ROYAL SOCIETY The Alan Turing Institute Facebook Research NCSC

Partiality cf. [Vákár et al. '19]

A Borel embedding $e : X \hookrightarrow Y$

- injective function $e : \lfloor X \rfloor \rightarrowtail \lfloor Y \rfloor$
- its image is Borel: $e[\lfloor X \rfloor] \in B_Y$
- e is strong: $\alpha \in R_X \iff e \circ \alpha \in R_Y$

Examples

- $\mathbb{1} \hookrightarrow \mathbb{2}$
- S is sbs $\iff$ $\exists\, S \hookrightarrow \mathbb{R}$

Def: A partial map $f : X \rightharpoonup Y$ is a morphism

$$f : X \to Y \amalg \{\bot\}$$

Its domain of definition

$$f : (Y \amalg \{\bot\})^X \vdash \mathrm{Dom}\, f := \{x \in X \mid f x \neq \bot\} : \mathrm{Type}$$

Depent-type interpretation

$$\begin{tikzcd}
[\![\mathrm{Dom}\, f]\!] \arrow[rr] \arrow[d, "\mathrm{dep}"] \arrow[drr, phantom, "\lrcorner", very near start] & & \{y \in Y \mid y \in E\} \arrow[d, "\mathrm{dep}"] \\
[\![f : (Y \amalg \{\bot\})^X]\!] \arrow[rr, "{[E \mapsto \lambda x.\, f x \neq \bot]}"] & & [\![E : \mathbb{B}_Y]\!]
\end{tikzcd}$$

Plan:

1) type-driven probability: discrete case ✓
2) Borel sets & measurable spaces ✓
3) Quasi Borel spaces ✓

Lecture 1

4) Type structure & standard Borel spaces ✓
5) Integration & random variables

Lecture 2

Please ask questions!

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Full model

type : Qbs $\quad \mathbb{W} := [0, \infty] \quad BX := \mathbb{B}^X$

$DX := \boxed{\quad}$

$PX := \{\mu \in DX \mid Ce_{\mu}[X] = 1\}$

$Ce_{\mu}[E] := \boxed{\quad} \qquad \delta_x := \boxed{\quad}$

$\oint \mu k := \boxed{\quad}$

Def: A measure μ over $\mathbb{R}$ is a function

$$\mu : B_{\mathbb{R}} \to W := [0, \infty]$$

satisfying the measure axioms:

$$E : B^{\omega} \vdash$$

$$\mu \emptyset = 0, \quad \mu E = \mu(E \cap F) + \mu(E \cap F^c), \quad \mu\left(\bigcup_n E_n\right) = \sup_n \mu E_n$$

For measurable spaces, replace $\mathbb{R}$ with V

We write $\llcorner G V \lrcorner$ for the set of measures on V

For qbs X, take $\llcorner G \ulcorner \mathrm{Meas}\, X \urcorner \lrcorner$

Thm (Lebesgue measure):

There is a unique measure $\lambda \in \llcorner G\mathbb{R} \lrcorner$ s.t.:

$$\lambda(a, b) = b - a$$

The unrestricted Giry spaces

Equip $\lfloor GV \rfloor$ with two qbs structures:

✗ $R_{GV} := \{ \alpha : \mathbb{R} \to GV \mid \forall A \in B_V . \lambda r . \alpha(r, A) : \mathbb{R} \to W \}$

↳ α is a kernel.

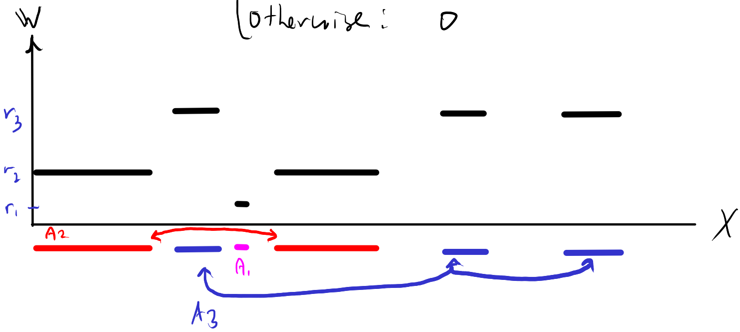
✓ $GV \hookrightarrow W^{B_X}$

- Fewer random elements $R_{GV} \subseteq R_{G'V}$
- Lebesgue integral measurable in both arguments. (upcoming)

Def: Simple function $\varphi : X \to W$ when

$\exists\, n \in \mathbb{N},\ \vec{A} \in B_X^n,\ A_i \cap A_j = \phi\ (i \neq j),\ \vec{r} \in W$ s.t.

$$\varphi\, x = \begin{cases} \vdots & \\ x \in A_i : & r_i \\ \vdots & \\ \text{otherwise} : & 0 \end{cases}$$



Encode into a space:

$$\mathrm{SimpleCode} := \coprod_{n \in \mathbb{N}} B_X^n \times W^n$$

$$\mathrm{Simple} := \{ f \in W^X \mid f \text{ simple} \} \hookrightarrow W^X$$

and define an interpretation:

$$[\![-]\!] : \mathrm{SimpleCode} \twoheadrightarrow \mathrm{Simple}$$

$$[\![(n, \vec{A}, \vec{r})]\!] := \sum_{i=1}^{n} r_i \cdot [- \in A_i]$$

characteristic function for A_i

Lemma: $f : X \to W$ is measurable → remember! 96s morphism!

iff $f = \lim_{n\to\infty} f_n$ for some monotone sequence

$f_n \in \mathrm{Simple}$.

Moreover, we have measurable such choice:

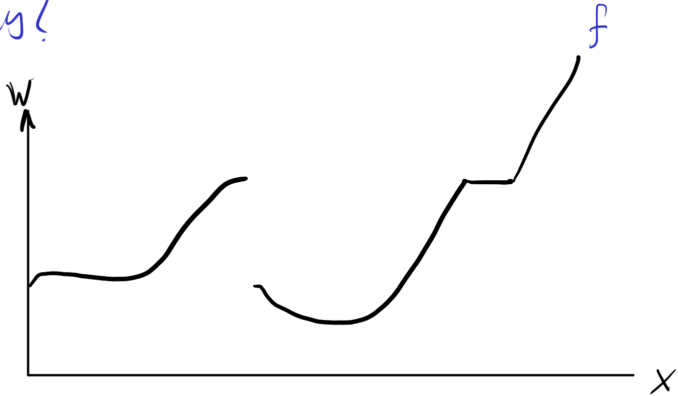
Simple Approx:

$$\{\vec{\Delta} \in \mathbb{R}^+ \mid \Delta_n \to 0\} \times \{\vec{a} \in W^{\mathbb{N}} \mid \begin{array}{l}\vec{a} \text{ monotone} \\ a_n \to \infty\end{array}\} \times W^X \to \mathrm{SimpleCode}$$

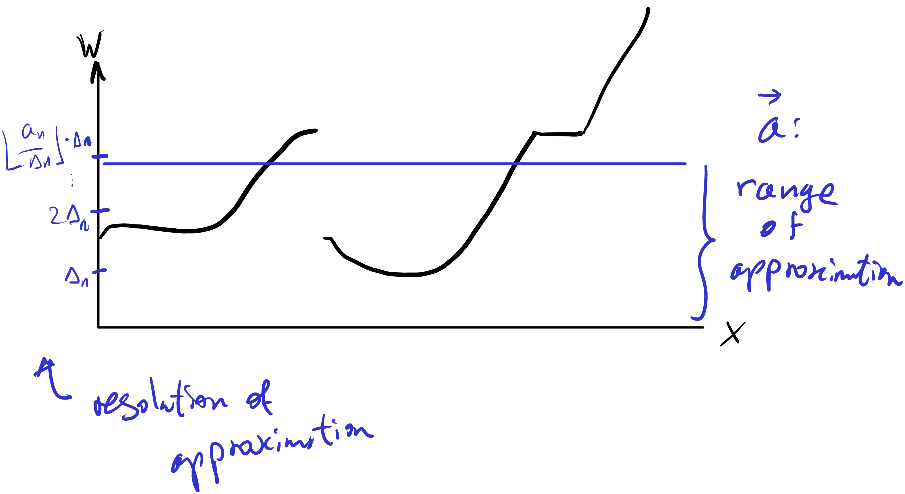
rate of convergence

range of approximation

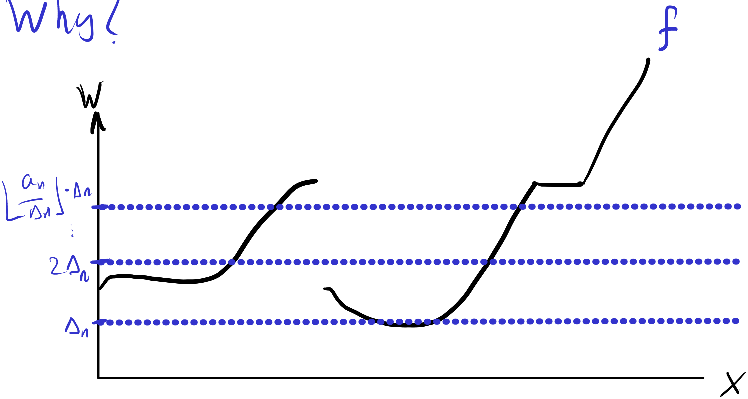
Why?



Why?

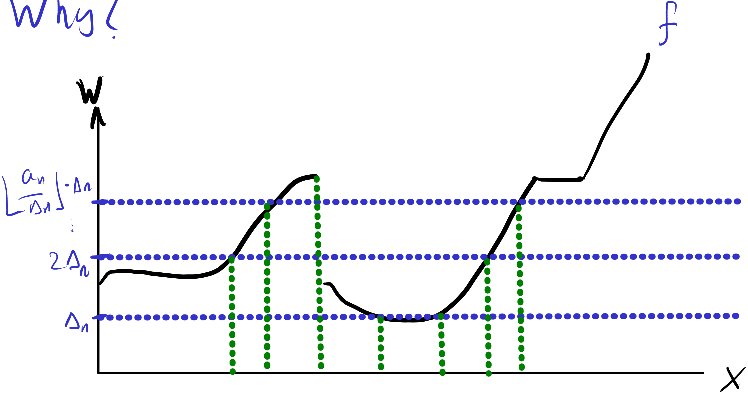

f
W
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n
$\vec{a}$:
range of approximation
X
resolution of approximation

Why?

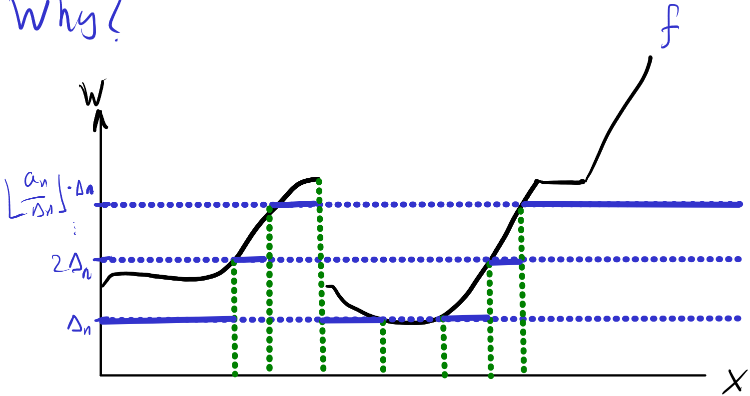


f
W
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$\vdots$
$2\Delta_n$
Δ_n
X

Why?

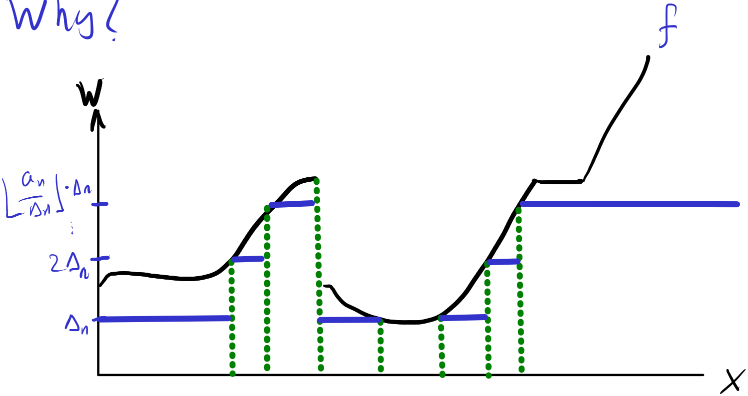

f
W
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n
X

Why?



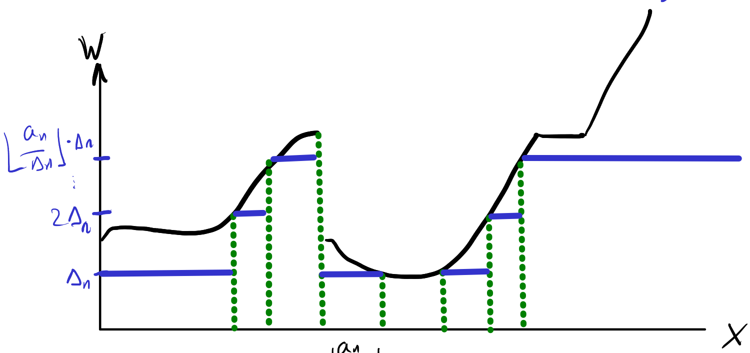
f
W
X
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n

Why?



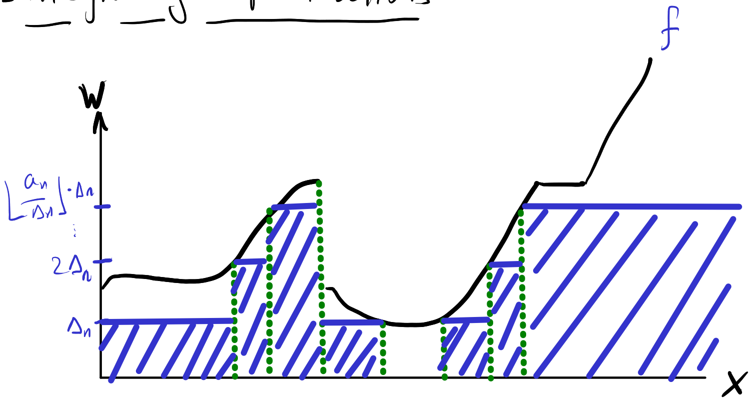
f
W
X
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n

Why?



$$[\![\mathrm{SimpleApprox}_{\vec{\Delta},\vec{a}}\, f]\!] := \sum_{i=1}^{\lfloor \frac{a_n}{\Delta_n} \rfloor} i\cdot\Delta_n\,[\, i\cdot\Delta_n \le f < (i+1)\Delta_n \,] + \lfloor \frac{a_n}{\Delta_n} \rfloor \Delta_n \cdot [\, f \geq \lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n \,] \in \mathrm{Simple}$$

Integrating Simple Functions



$$\int : GX \times \text{Simple Code} \longrightarrow W$$

$$\int \mu\, (n, \vec{A}, \vec{r}) := \sum_{I \subseteq \{1, \ldots, n\}} \left(\sum_{i \in I} r_i \right) \cdot \mu \left(\bigcap_{i \in I} A_i \setminus \bigcup_{i \notin I} A_i \right)$$

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Integration

Proper higher-order operation

$$\int : GX \times W^X \longrightarrow W$$

$$\int \mu f := \sup \left\{ \int \mu \varphi \;\middle|\; \varphi \in \text{Simple},\ \varphi \leq f \right\}$$

$$= \lim_{n\to\infty} \int \mu \left(\text{Simple Approx}_{\vec{\Delta}, \vec{a}}\, f\right)_n$$

measurable by type

we also write $\int \mu(dx)\, t$ for $\int \mu\, (\lambda x;\, t)$

for $\frac{a_n}{\Delta_n} \to 0$, eg. $\Delta_n = \frac{1}{2^n}$ $a_n = n$.

resolution

The unrestricted Giry Strong Monad

Unlike the unrestricted Giry on Meas.

Dirac:

$$\delta : X \to GX$$

$$x \mapsto \lambda A. \begin{cases} x \in A : & 1 \\ x \notin A : & 0 \end{cases}$$

but: non-commutative

(Fubini fails, just like in Meas)

Kleisli extension/Kock integral:

$$\oint : GX \times GY^X \longrightarrow GY$$

$$\oint \mu f := \lambda A. \int \mu(dx)\, f(x; A)$$

Fubini-Tonelli fails

$\# \in G\mathbb{R}$ $\qquad \#E := \begin{cases} E \text{ finite}: & |E| \\ \text{o.w.}: & \infty \end{cases}$

$\lambda \in G\mathbb{R}$ $\qquad$ Lebesgue $\qquad k : \mathbb{R}\times\mathbb{R} \to W \cong G\mathbb{1}$

$k(x,y) := [x = y]$

$$\oint \#(dr) \oint \underbrace{\lambda(dx)\, k(x,y)}_{y:\mathbb{R} \vdash \{\langle\rangle\} \mapsto \lambda\{y\}\cdot 1 + \lambda\{y\}^{c}\cdot 0 = 0} = \oint \# \, \underline{0} = \underline{0} \stackrel{\wedge}{=} 0$$

$\neq$

$$\oint \lambda(dx) \oint \underbrace{\#(dr)\ k(x,y)}_{x:\mathbb{R} \vdash \{\langle\rangle\} \mapsto \#\{x\}\cdot 1 + 0 = 1} = \oint \lambda(dx)\, \delta_{\langle\rangle} \stackrel{\sim}{=} \infty$$

Randomisable measures monad

$D \rightarrowtail G$

$\lambda A. \int_{\mathrm{Dom}\,\alpha} \lambda(\mathrm{Dom}\,\alpha)$

$\lfloor DX \rfloor := \{ \lambda_\alpha \mid \alpha : \mathbb{R} \rightharpoonup X \}$

Lebesgue measure

$R_{DX} := \{ \lambda x. \lambda_{\alpha x} \mid \alpha : \mathbb{R} \times \mathbb{R} \rightharpoonup X \}$

$\delta : X \to DX \qquad \oint : D\Gamma \times (DX)^\Gamma \longrightarrow DX$ lift along $D \rightarrowtail G$.

D validates our measure axioms including Fubini-Tonelli

$\mu \in DX, \nu \in DY \vdash$

$$\oint \mu(dx) \oint \nu(dy)\, \delta_{(x,y)} = \oint \nu(dy) \oint \mu(dx)\, \delta_{(x,y)} =: \mu \otimes \nu$$

Thm: For sbs S, $PS, D_{\leq 1}S, D_{<\infty}S \in Sbs$

and agree with their counterparts on Meas.

$\lfloor DS \rfloor = \{\mu \mid \mu \text{ s-finite}\}$ See [Staton'16]

$R_{DS} = \{k : \mathbb{R} \to GD \mid k \text{ s-finite kernel}\}$

Open: Is there a counterpart to D in Meas?

More modestly, is $DS \in Sbs$?

(Hypothesis: no)

Distribution Submonads

A measure space

$\Omega = (\Omega, \mu)$

is a qbs Ω with

$\mu \in DX$.

Similarly: - finite measure space

- (sub) probability space.

$$PX := \{\mu \in DX \mid \mu X = 1\}$$

$$\downarrow$$

$$P_{\leq 1} X := \{\mu \in DX \mid \mu X \leq 1\}$$

$$\downarrow$$

$$P_{<\infty} X := \{\mu \in DX \mid \mu X < \infty\}$$

$$\downarrow$$

$$DX$$

Full model

type : Qbs $\quad W := [0,\infty] \quad \mathcal{B}X \cong \mathbb{B}^X$

$$DX := \left(\{\lambda_\alpha \mid \alpha : \mathbb{R} \to X\}, \{\lambda r.\lambda_{\alpha(r,-)} \mid \alpha : \mathbb{R}\times\mathbb{R} \to X\} \right)$$

$$PX := \{\mu \in DX \mid \mathcal{C}_\mu[X] = 1\}$$

$$\mathcal{C}_\mu[E] := \mu E \qquad \delta_x := E \mapsto \begin{cases} x \in E : 1 \\ x \notin E : 0 \end{cases}$$

$$\oint \mu k := \lambda E. \int \mu(dx) k(x; E)$$

Plan:

1) type-driven probability: discrete case ✓
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$\Big\}$ Lecture 1

4) Type structure & standard Borel spaces ✓
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$\Big\}$ Lecture 2

Please ask questions!

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Random variable: $\xi : \Omega \longrightarrow \Theta \hookrightarrow \overline{\mathbb{R}}$

Θ:

$$\begin{tikzcd}
 & & \mathbb{W} \arrow[dr] & \\
\mathbb{I} \arrow[r] & \mathbb{R}_{\geq 0} \arrow[ur, hook] \arrow[dr, hook] & & \overline{\mathbb{R}} \\
\mathbb{R}^{+} \arrow[ur] & & \mathbb{R} \arrow[ur, hook] &
\end{tikzcd}$$

$[0,1]$ $[0,\infty)$ $[0,\infty]$ $(0,\infty)$ $(-\infty,\infty)$ $[-\infty,\infty]$

- Θ^{Ω} is a space
- $\mathbb{W}^{\Omega}$ measurable σ-semi-module for $\mathbb{W}$: $\sum_{n=0}^{\infty} \alpha_n \xi_n := \lambda\omega. \sum_{n=0}^{\infty} \alpha_n \cdot \xi_n$
- $\mathbb{R}^{\Omega}$ measurable vector space:

$$\alpha\xi + \zeta := \lambda\omega.\ \alpha \cdot \xi\omega + \zeta\omega$$

$\Pr_{-} : P\Omega \times B_\Omega \to W$

$\Pr_\lambda A := \mathrm{eval}(\lambda, A) = \lambda A$

Probability Space $\Omega = (\Omega, \lambda_\Omega)$

$P : P\Omega \vdash$ "$P\,x$ holds $\lambda(dx)$-almost surely"

for some $Q \hookrightarrow \Omega$, $P \supseteq Q$, $[- \in Q] \cdot \lambda = \lambda$

Example $(\xi, \zeta \in \mathbb{H}^\Omega)$

$\xi = \zeta$ a.s. when $\Pr_{\omega \sim \lambda}[\xi\,\omega \neq \zeta\,\omega] = 0$

Integrating random variables (as discretely) in Qbs!

$(-)_+, (-)_- : \overline{\mathbb{R}}^\Omega \longrightarrow W^\Omega$

$\xi_+ := \max(\xi, 0) \qquad \xi_- := \max(-\xi, 0)$

So: $\xi = \xi_+ - \xi_-$

$\int : P\Omega \times W^\Omega \longrightarrow W$

$\int \lambda\, \xi := \int \lambda\, \xi_+ - \int \lambda\, \xi_-$

$\int$ respects a.s. equality:

$\xi = \zeta \ (\lambda\text{-a.s.})$

$\Rightarrow \int \lambda\, \xi = \int \zeta.$

Example

$$\lambda : P\Omega \vdash \mathrm{ASConverge}(\overline{\mathbb{R}})^{\Omega} : \mathcal{B}\left(\overline{\mathbb{R}}^{\mathbb{N}\times\Omega}\right)$$

$$:= \left\{ \vec{z} \in \overline{\mathbb{R}}^{\mathbb{N}\times\Omega} \;\middle|\; \Pr_{\omega\sim\lambda}\left[\lim z_n \omega \neq \bot\right] \right\}$$

So:

$$\lim^{as} : \overline{\mathbb{R}}^{\mathbb{N}\times\Omega} \longrightarrow \overline{\mathbb{R}}^{\Omega} \qquad \mathrm{Dom}\lim^{as} := \mathrm{ASConverge}(\overline{\mathbb{R}})^{\Omega}$$

$$\lim^{as} \vec{z} := \lambda\omega.\ \limsup_{n\to\infty} f_n\,\omega$$

$\lim^{as}$ respects a.s. equality

Thm (monotone convergence):

Let $\vec{\xi} \in W^{\mathbb{N} \times \Omega}$ λ-a.s. monotone.

$$\xi = \lim_{n \to \infty}^{\text{as}} \xi_n \qquad \text{(a.s.)}$$

$$\Downarrow$$

$$\int \lambda\, \xi = \lim_{n \to \infty} \int \lambda\, \xi_n$$

Lebesgue Space

$\left(\Omega \text{ prob. space}, \; p \in [1, \infty) \right)$

$$p : [1,\infty), \lambda : P\Omega \vdash \mathcal{L}^p_{(\Omega,\lambda)} : \mathcal{B}(\mathbb{R}^\Omega)$$

$$:= \left\{ \xi \in \mathbb{R}^\Omega \,\middle|\, \int \lambda |\xi|^p < \infty \right\} \hookrightarrow \mathbb{R}^\Omega$$

Ensemble $\mathcal{L}_\Omega := \prod_{\substack{\lambda \in P\Omega \\ p \in [1,\infty)}} \mathcal{L}^p_{(\Omega,\lambda)}$

$\llcorner \quad p \le q \Rightarrow \mathcal{L}^p_\Omega \supseteq \mathcal{L}^q_\Omega$

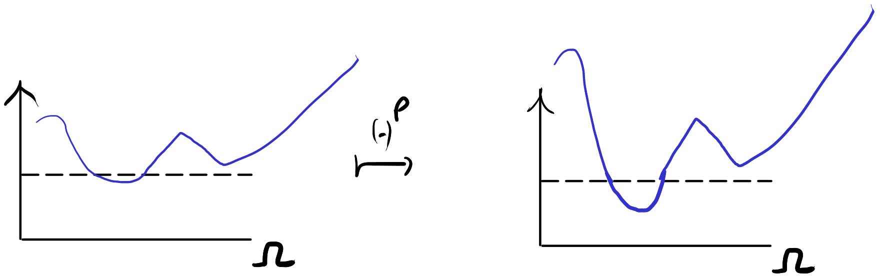
$\mathcal{L}^p$ semi norms

$$\|-\| : \prod_{p,\lambda} \mathcal{L}^p_{(\Omega,\lambda)} \longrightarrow \mathbb{R}_{\geq 0} \qquad \|\xi\|_p := \sqrt[p]{\int \lambda |\xi|^p}$$

$\mathcal{L}^2$ inner product

$$\langle -, - \rangle : \prod_{p,\lambda} \mathcal{L}^p_{(\Omega,\lambda)} \times \mathcal{L}^p_{(\Omega,\lambda)} \longrightarrow \mathbb{R}$$

$$\langle \xi, \zeta \rangle_{p,\lambda} := \int \lambda\, \xi \zeta$$



Statistics

Expectation

$\mathbb{E} : \coprod_\lambda \mathcal{L}^1 \to \mathbb{R}$

$\mathbb{E}_\lambda \xi := \int \lambda \xi$

Covariance and Correlation

$\mathrm{Cov}, \mathrm{Corr} : \coprod_\lambda \mathcal{L}^2 \to \mathbb{R}$

$\mathrm{Cov}(\xi, \zeta) := \langle \xi - \mathbb{E}\xi, \zeta - \mathbb{E}\zeta \rangle$

$\mathrm{Corr}(\xi, \zeta) := \dfrac{\langle \xi, \zeta \rangle}{\|\xi\|_2 \cdot \|\zeta\|_2} = \cos(\mathrm{angle}(\xi, \zeta))$

Sequential limits

$p : [1,\infty), \lambda : \mathbb{P}\mathbb{X}_t$ Cauchy $\mathcal{L}^p_{(\Omega,\lambda)} : \mathcal{B}\left(\mathcal{L}^p_{(\mathbb{R},\lambda)}\right)^{\mathbb{N}}$

$$:= \left\{ \vec{\Sigma} \;\middle|\; \forall \varepsilon \in \mathbb{Q}^+ \, \exists k \in \mathbb{N} \, \forall m, n \geq k. \; \| \Sigma_{k+n} - \Sigma_{k+m} \|_p < \varepsilon \right\}$$

Thm: $\mathcal{L}^p_\Omega$ is Cauchy-Complete

$\lim^p :$ Cauchy $\mathcal{L}^p \longrightarrow \mathcal{L}^p$ (convergence in mean)

Why?

1. Every Cauchy sequence has an a.s. converging subseq.
2. We can find it measurably

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Example

Thm (dominated convergence)

For $\vec{Z}, Z \in \mathcal{L}^1$ s.t. $Z_n \leq Z$ a.s.:

1. $\overset{as}{\lim} \vec{Z} \in \mathcal{L}^1$
2. $\overset{1}{\lim} \vec{Z} = \overset{as}{\lim} \vec{Z}$
3. $\lim\limits_{n\to\infty} \int \lambda Z_n = \int \lambda \lim\limits_{n\to\infty} Z_n$

Separability

Def: $\mathcal{L}^p$ separable: has countable dense subset

Fact: Separability is property of λ_Ω:

TFAE:

- $\exists p \geq 1.\ \mathcal{L}^p$ separable
- $\forall p \geq 1.\ \mathcal{L}^p$ separable

Measurable separability in $\mathcal{I} \hookrightarrow P\Omega \times [1,\infty)$

$$\vec{\beta} : \prod_{(\lambda,p)\in\mathcal{I}} \mathcal{L}^{p}_{(\Omega,\lambda)}{}^{\mathbb{N}} \quad \text{s.t.}$$

$$\left\{ \vec{\beta}^{\lambda,p}_{n} \mid n \in \mathbb{N} \right\} \text{ dense in } \mathcal{L}^{p}_{(\Omega,\lambda)}$$

Prop. - every sbs S measurably separable in

$$PS \times [1,\infty)$$

- $\mathcal{I} \hookrightarrow P\Omega \times \{2\}$ measurably separable

$$\Rightarrow \exists\, \vec{\beta} \in \prod_{\lambda\in\mathcal{I}} \mathcal{L}^{2}_{(\Omega,\lambda)} \text{ orthonormal system}$$

$\langle \beta_n, \beta_m \rangle = 0$

$\|\beta_n\|_2 = 1$

(β_n) dense

Example

Let $S \hookrightarrow \mathcal{L}^2$ closed vector subspace.

orthogonal decomposition — linear in fact.

$$\langle P, P^\perp \rangle : \lfloor \mathcal{L}^2 \rfloor \longrightarrow \lfloor S \rfloor \times \lfloor S^\perp \rfloor$$

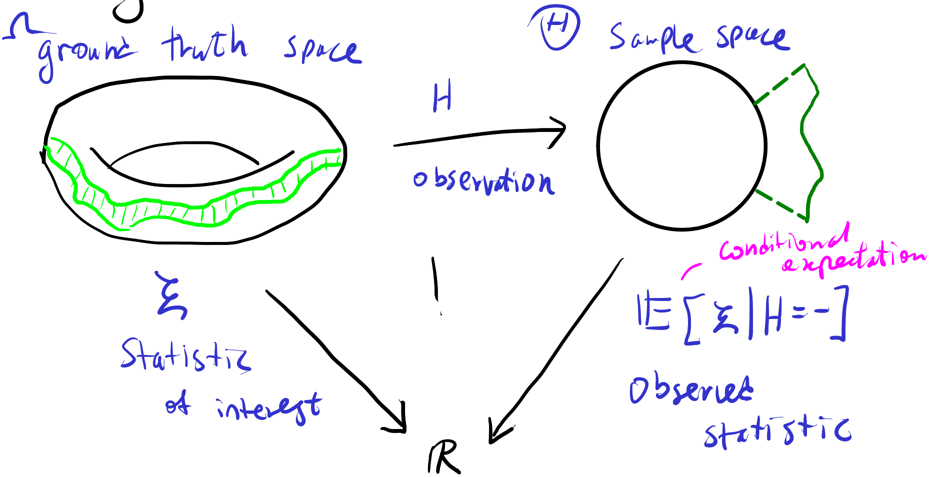
When S is separable with orthonormal system β

We have a measurable version of

$$\langle P, P^\perp \rangle : \mathcal{L}^2 \to S \times S^\perp$$

$$P\xi := \sum_{n=0}^{\infty} \langle \xi, \beta_n \rangle \beta_n \qquad P^\perp := \mathrm{Id} - P .$$

Kolmogorov's Conditional Expectation

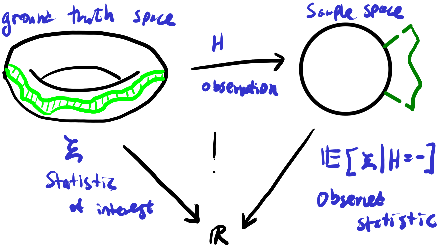


Kolmogorov's Conditional Expectation

A Conditional expectation of $\xi \in \mathcal{L}^1_\Omega$ wrt

$H : \Omega \to H$ is

$\zeta \in \mathcal{L}^1_{H}$ s.t. for all $A \in \mathcal{B}_{H}$:



$$\int_A \mu \zeta = \int_{H^{-1}[A]} \lambda \xi$$

where $\mu := \lambda_H$

Conditional expectations

1. unique a.s.

2. fundamental to modern Probability, eg:

   a Martingale

$$\begin{tikzcd}
\cdots & \Theta_n \arrow[l] \arrow[dr, "\Xi_n"'] & & \Theta_{n+1} \arrow[ll, "H_n"'] \arrow[dl, "\Xi_{n+1}"] \arrow[dl, phantom, "!", pos=0.9, shift left=8] & \cdots \arrow[l] \\
& & \mathbb{R} & &
\end{tikzcd}$$

s.t. $\Xi_n = \mathbb{E}[\Xi_{n+1} \mid H_n = -]$

Thm (Existence)

- $\exists$ $\mathbb{E}[-\mid H=-] : \lfloor \mathcal{L}^1_{(\Omega,\lambda)} \rfloor \longrightarrow \lfloor \mathcal{L}^1_{(\Theta,\mu)} \rfloor$

- When (Ω,λ) is Separable

$$\mathbb{E}[-\mid H=-] : \mathcal{L}^1_{(\Omega,\lambda)} \longrightarrow \mathcal{L}^1_{(\Theta,\mu)}$$

- When Θ is $\mathcal{I}'$-measurably separable

$$\mathbb{E}_{-}[-\mid - = -] : \prod_{\substack{H \in \Theta^{\Omega} \\ \lambda \in H_*^{-1}[\mathcal{I}]}} \mathcal{L}^1_{(\Omega,\lambda)} \longrightarrow \mathcal{L}^1_{(\Theta,\mu)}$$

Plan:

1) type-driven probability: discrete case ✓
2) Borel sets & measurable spaces ✓
3) Quasi Borel spaces ✓

Lecture 1

4) Type structure & standard Borel spaces ✓
5) Integration & random variables ✓

Lecture 2

Please ask questions!

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Discrete model

$\text{type} : \text{Set} \qquad W := [0, \infty] \qquad BX := PX$

$$DX := \{ \mu : X \to W \mid \text{supp}\, \mu \text{ countable} \}$$

$$PX := \{ \mu \in DX \mid \underset{\mu}{Ce}[X] = 1 \}$$

$$\underset{\mu}{Ce}[E] := \sum_{x \in E} \mu x \qquad \delta_x := \lambda x'. \begin{cases} x = x' : & 0 \\ x \neq x' : & 1 \end{cases}$$

$$\oint \mu k := \lambda x. \sum_{r \in \Gamma} \mu r \cdot k(r; x)$$

Full model

type : Qbs $\quad \mathbb{W} := [0,\infty] \quad \mathbb{B}X \cong \mathbb{B}^X$

$$DX := \left(\{ \lambda_\alpha \mid \alpha : \mathbb{R} \to X \}, \{ \lambda r. \lambda_{\alpha(r,-)} \mid \alpha : \mathbb{R}\times\mathbb{R} \to X \} \right)$$

$$PX := \{ \mu \in DX \mid \underset{\mu}{\mathcal{C}e}[X] = 1 \}$$

$$\underset{\mu}{\mathcal{C}e}[E] := \mu E \qquad \delta_x := E \mapsto \begin{cases} x \in E : 1 \\ x \notin E : 0 \end{cases}$$

$$\oint \mu k := \lambda E. \int \mu(dx) k(x; E)$$

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Enough!

Lunch.