THE CATEGORY-THEORETIC SOLUTION OF RECURSIVE DOMAIN EQUATIONS

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1. Introduction

Recursive specifications of domains play a crucial role in denotational semantics as developed by Scott and Strachey and their followers ([Sto], [Ten]). For example the equation,

(1) $D \cong A + (D \to D)$

is just what is needed for the semantics of an untyped λ-calculus for computing over a domain, A, of atoms. Again, the simultaneous equations,

(2) $T \cong A \times F$

(3) $F \cong 1 + (T \times F)$

specify a domain, T, of all finitarily-branching trees and another, F, of forests of such trees.

The first tools for solving such equations were provided by Scott using his inverse limit constructions [Sco 1]. Later he showed how the inverse limits could be entirely avoided by using a universal domain and the ordinary least fixed-point construction [Sco 2]. A systematic exposition of the inverse limit method was given by Reynolds [Rey 1], and the categorical aspects (already mentioned by Scott) were emphasized by Wand [Wan 1]. All of these treatments stuck to one category, such as, for example, $\underline{\underline{CL}}$, the category of countably based continuous lattices and continuous functions, although the details did not change much in other categories. Then Wand [Wan 2,3], gave an abstract treatment based on $\underline{\underline{O}}$-categories where the morphism sets are provided with a suitable order-theoretic structure. The relation between the category-theoretic treatment and the universal domain method has, until now, remained rather obscure.

The purpose of the present paper is to set up a categorical framework in which all known techniques for solving domain equations find a natural place. The idea as set out in Section 2 is to follow the well-known analogy between partial orders and categories [Mac] and generalise from least fixed-points to initial fixed-points. These are constructed using the "Basic Lemma" which plays an organisational role: most of the solution methods considered appear as ways of ensuring that the hypotheses of the lemma are fulfilled. Just as continuous functions over complete partial orders always have least fixed-points so continuous functions over ω -categories/

ω -categories (as defined below) always have initial fixed-points, which can be constructed by using the Basic Lemma. Closely related ideas appear in [Ada].

Next, $\underline{\underline{O}}$-categories are introduced in Section 3; here, clearly, we are greatly indebted to Wand and indeed our work partly arose as an attempt to simplify and clarify Wand's treatment. In particular we have removed the need for his troublesome "condition A" (which appears in [Wan 2] rather than [Wan 3] which incorporates some of the ideas of the present paper). Thus Theorem 2 shows that when passing to the derived category of embeddings (needed to treat contravariant functors such as exponentiation) ω -colimits are inherited from ω-limits; Theorem 3 shows that an easily verified order continuity property on functors gives the categorical continuity property needed to apply the Basic Lemma.

The method using least fixed-points has the advantage of being more concrete and elementary than the categorical approach, although it will not be applicable if no universal domain is at hand - and even when one is available it might still, for pedagogical reasons, be better to use the categorical method. Whatever one's opinion, the question remains of the relation between the two approaches. In Section 4 we discuss the mathematical consequences of assuming the existence of a universal object in an $\underline{\underline{O}}$-category; it turns out that this eliminates the need for assumptions on the existence of ω-limits. It is even possible to avoid the Basic Lemma entirely (in the sense of finding an alternative proof) and use ordinary least fixed-points to find an initial fixed-point. In Section 5 we consider Scott's approach in [Sco 2] as being based on the monoid, M_S, of unary continuous functions over $P\omega$ and domain constructions as homomorphisms over the monoid. Again we take an abstract approach, considering the relation between monoids and categories with a universal element. It turns out that every such category can be represented by a monoid (Theorem 7) and every functor can be represented by a (weak) homomorphism (Theorem 8). Then we can use the available material on effectively given cpos (such as [Egl], [Mar], [Smy 1]) to present a theory of computability for $\underline{\underline{O}}$-categories with a universal element. It turns out that initial fixed-points obtained via locally computable functors (see below) are effectively given - and so any semantics of programming languages which employs these will be computable and data types, such as T and F, above, will be effectively given. It remains to find effective versions of other parts of the paper, and here the work in [Kan 1], [Smy 2] should prove useful. This work, which is still somewhat programmatic in character, is described briefly in the concluding Section 6.

We assume the reader possesses a basic knowledge of category theory; he can fill any gaps by consulting [Arb 1], [Her] or [Mac].

2. Initial Fixed-Points

In the categorical approach to recursive domain specifications we try to regard all equations such as (1) or (2) and (3) above as being of the form:

(4) $X \cong F(X)$

where X ranges over the objects of a category K, say, and F: K -> K is an endofunctor of that category. For example in the case of (1) we could take D to range over the objects of K, A to be a fixed object of K, + and -> to be sum and exponentiation covariant functors over K and then F: K -> K is defined by:

(5) $F(D) =_{def} A + (D \rightarrow D)$.

Let us spell the meaning of equation (5) out in detail. Recall that if F_i: L -> K_i (i=1,n) are functors then their <u>tupling</u> $F = \langle F_1,\ldots,F_n\rangle$: L -> $K_1 \times \ldots \times K_n$ is defined by putting for each object, x, of K:

$F(x) = \langle F_1(x),\ldots\ldots,F_n(x)\rangle$

and for each morphism f: x -> y of L:

$F(f) = \langle F_1(f),\ldots,F_n(f)\rangle$

Then the functor F defined by equation (5) is just

$F = + \circ \langle K_A, \rightarrow \circ \langle id_K, id_K\rangle\rangle$

where K_A: K -> K is the "constantly A" functor and id_K: K -> K is the identity functor. (We will see below how to turn contravariant exponentiation functors on categories like $\underline{\underline{CL}}$ into covariant ones on a derived category of embeddings.)

Simultaneous equations are handled using product categories. For example equations (2) and (3) can be regarded as having the form:

(6) $X \cong F_0(X,Y)$

(7) $Y \cong F_1(X,Y)$

where X, Y range over a category K (such as $\underline{\underline{CL}}$) and F_0 and F_1 are bifunctors over K being defined by:

$F_0 =_{def} \times \circ \langle K_A, \pi_1\rangle$

$F_1 =_{def} + \circ \langle K_1, \times \circ \langle \pi_0, \pi_1\rangle\rangle$

where A, 1 are objects of K, and the π_i: K × K -> K (i=0,1) are the projection functors. Then equations (2) and (3) are put into the form (4) by/

by using the product category $K \times K$ and taking F to be $\langle F_0, F_1 \rangle$. Clearly this idea works for n simultaneous equations $X_i = F_i(X_1, \ldots, X_n)$ $(i=1,n)$ where X_i ranges over K_i $(i=1,n)$ and $F_i: K_1 \times \ldots \times K_n \to K_i$: we just take K to be $K_1 \times \ldots \times K_n$ and F to be $\langle F_1, \ldots, F_n \rangle$.

Let us now decide what a solution to (4) might be and which particular ones we want. In the case where K is a partial order, where F is then just a monotonic function, solutions are just fixed-points of F, that is elements A of K such that $A = F(A)$ and we can look for least solutions. Further we can define prefixed-points as elements A such that $F(A) \sqsubseteq A$ and it turns out that the least prefixed-point if it exists is always the least fixed-point as well. In the categorical case we need to know the isomorphism as well as the object:

<u>Definition 1</u> Let K be a category and $F: K \to K$ be an endofunctor. Then a <u>fixed-point</u> of F is a pair (A, α) where A is an object of K and $\alpha : FA \cong A$ is an isomorphism of K; a <u>prefixed-point</u> is a pair (A, α) where A is an object of K and $\alpha : FA \to A$ is a morphism of K.

We also call prefixed-points of F, <u>F-algebras</u> (same as <u>F-dynamic</u> of* Arbib and Manes, [Arb 1]). The F-algebras are the objects of a category:

<u>Definition 2</u> Let (A, α) and (A', α') be F-algebras. A <u>morphism</u> $f: (A, \alpha) \to (A', \alpha')$ (<u>F-homomorphism</u>) is just a morphism $f: A \to A'$ in K such that the following diagram dommutes:

$$\begin{tikzcd}
FA \arrow[r, "\alpha"] \arrow[d, "Ff"'] & A \arrow[d, "f"] \\
FA' \arrow[r, "\alpha'"] & A'
\end{tikzcd}$$

It is easily verified that this gives a category: the identity and composition are both inherited from K. Following on the above remarks on partial orders we look for initial F-algebras rather than just initial fixed-points of F and this is justified by the following lemma (which also appears in [Arb 2]):

<u>Lemma 1</u> The initial F-algebra, if it exists, is also the initial fixed-point.

<u>Proof</u> Let (A, α) be the initial F-algebra. We only have to prove that α is an isomorphism. Now as (A, α) is an F-algebra so is $(FA, F\alpha)$ and so there is an F-homomorphism $f: (A, \alpha) \to (FA, F\alpha)$; one also easily sees that $\alpha : (FA, F\alpha) \to (A, \alpha)$ is an F-homomorphism and so $\alpha \circ f: (A, \alpha) \to (A, \alpha)$ is also one and it must be id_A, the identity on A as (A, α) is initial.

Then as f: $(A,\alpha) \to (FA, F\alpha)$ we also have $f \circ \alpha = (F\alpha) \circ (Ff) = F(\alpha \circ f) = F(id_A) = id_{FA}$ which shows that α is an isomorphism with two-sided inverse f.$\Box$

Note that we have to do more than specify an object A such that $A \cong F(A)$ when looking for the initial fixed-point. First we have to specify an isomorphism $\alpha : FA \cong A$ and secondly we must establish the initiality property. Both are vital in applications. When giving the semantics of programming languages using recursively specified domains the isomorphism is needed just to be able to make the definitions. Initiality is closely connected to structural induction principles and both can be used for making proofs about elements of the specified domains. When using the equations to specify data type definitions within a language following the approach in [Leh 1], the isomorphism carries the basic operations and initiality is again essential for proofs. (The paper [Leh] also contains more information on simultaneous equations and on equations with parameters; in many ways it is a companion paper to the present one.)

When K is a partial order the least fixed-point can, as is well-known, be constructed as $\bigsqcup_K F^n(\perp)$ the lub of the increasing sequence $\langle F^n(\perp)\rangle_{n\in\omega}$ where $\perp$ is the least element of K. This works if the least element exists, the lub exists and F preserves the lub - that is, $F(\bigsqcup_K F^n(\perp)) = \bigsqcup_K F(F^n(\perp))$. Our Basic Lemma just generalises these remarks to the case where K is a category:

<u>Lemma 2 (The Basic Lemma)</u> Suppose K has initial object $\perp$. Let Δ be the ω-cochain $\langle F^n(\perp), F^n(\perp_{F\perp})\rangle$ where $\perp_{F\perp}$ is the unique morphism from $\perp$ to $F\perp$ and suppose that $\mu : \Delta \to A$ is a colimiting cone. Suppose too that $F\mu : F\Delta \to FA$ is also a colimiting cone (where $F\Delta =_{def} \langle F(F^n(\perp)), F(F^n(\perp_{F\perp}))\rangle$ and $F\mu =_{def} \langle F\mu_n\rangle_{n\in\omega}$). Then the initial F-algebra exists. Indeed if $\alpha : FA \to A$ is the mediating arrow ~~from~~ $F\mu$ to μ^- (where $\mu^- =_{def} \langle \mu_{n+1}\rangle_{n\in\omega}$ is the cone from $F\Delta \to A$ obtained by dropping the first term of μ) then the initial F-algebra is (A,α).

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<u>Proof</u> Let $\alpha' : FA' \to A'$ be any F-algebra. We show there is a unique F-homomorphism F: $(A,\alpha) \to (A',\alpha')$. First suppose f is such a homomorphism. Define a cone $\nu : \Delta \to A'$ by putting $\nu_0 = \perp_{A'} : \perp \to A'$ and $\nu_{n+1} = \alpha' \circ F(\nu_n)$. To see ν is a cone we prove by induction on n that the following diagram commutes:

$$\begin{tikzcd} F^n(\perp) \arrow[rr, "F^n(\perp_{F\perp})"] \arrow[drr, "\nu_n"'] & & F^{n+1}(\perp) \arrow[d, "\nu_{n+1}"] \\ & & A' \end{tikzcd}$$

This is clear for n=0. For n+1 we have: $\nu_{n+2} \circ F^{n+1}(\perp_{F\perp}) = \alpha' \circ F(\nu_n) \circ F^{n+1}(\perp_{F\perp}) = \alpha' \circ F(\nu_n \circ F^n(\perp_{F\perp})) = \alpha' \circ F(\nu_n)$ (by induction assumption) $= \nu_{n+1}$. Now the uniqueness of f will follow when we show it is the mediating morphism from μ to ν; here we use induction on n to show $\nu_n = f \circ \mu_n$. This is clear for n=0. For n+1 we have: $f \circ \mu_{n+1} = f \circ \alpha \circ F(\mu_n)$ (by the definition of α) $= (\alpha' \circ F(f)) \circ F(\mu_n)$ (f is a homomorphism) $= \alpha' \circ F(f \circ \mu_n) = \alpha' \circ \nu_n$ (by induction assumption) $= \nu_{n+1}$.

Secondly, to show that f exists let it be the mediating morphism from μ to ν (so that $\nu_n = f \circ \mu_n$ for all integers, n). We will show that $f \circ \alpha$ and $\alpha' \circ Ff$ are both mediating arrows from $F\mu$ to ν^-; it will follow that they are equal and so that f is an F-homomorphism as required. (Here $\nu^-: F\Delta \to A'$ is the cone $\langle \nu_{n+1} \rangle_{n \in \omega}$ obtained from ν by dropping its first term.)

In the first case, $(f \circ \alpha) \circ F\mu_n = f \circ \mu_{n+1}$ (by definition of α) $= \nu_{n+1}$ (by definition of f). In the second case, $(\alpha' \circ Ff) \circ F\mu_n = \alpha' \circ F(f \circ \mu_n) = \alpha' \circ F\nu_n$ (by definition of f) $= \nu_{n+1}$ (by definition of ν). This concludes the proof. $\square$

In the case of partial orders our method of constructing least fixed-points always works if K is an ω-complete pointed partial order and $F: K \to K$ is ω-continuous. Here an <u>ω-complete pointed partial order</u> (<u>ω-cppo</u>) is a partial order which has lubs of all increasing ω-chains and which has a least element; it is termed an "ω-complete partial order" or even just a "complete partial order" elsewhere. Also a function $F: K \to L$ of partial orders is <u>ω-continuous</u> iff it is monotonic and preserves lubs of increasing ω-chains, that is if wherever $\langle x_n \rangle_{n \in \omega}$ is an increasing ω-chain such that $\bigsqcup_K x_n$ exists then $F(\bigsqcup_K x_n) = \bigsqcup_L F(x_n)$. We make analogous definitions for categories:

<u>Definition 3/</u>

Definition 3 A category, K, is an ω-complete pointed category (shortened below to ω-category) iff it has an initial element, called $\perp_K$, and every ω-cochain has a colimit.

7 Definition 4 Let F: K -> L be a functor. It is ~~a~~ ω-continuous iff it preserves ω-colimits; that is whenever Δ is an ω-cochain and $\mu : \Delta \rightarrow A$ is a colimiting cone then $F\mu : F\Delta \rightarrow F$ is also a colimiting cone. (The reader is warned that this definition is dual to the notion of continuity of functors usual in category theory [Mac]; this is done in order to maintain the analogy with partial orders.)

Clearly, when K is an ω-category and F: K -> K is ω-continuous the conditions of the Basic Lemma are satisfied. In Section 3 we will give conditions for this to be the case; in Section 4 we give conditions for ω-continuity and show how to apply the Basic Lemma when there is a universal object; in Section 5 we are able to avoid the Basic Lemma entirely and use ordinary least fixed-points instead. Thus we can often completely avoid direct verification of the conditions of the Basic Lemma or whether something is an ω-category or is ω-continuous. Of course sometimes, as in the case of Sets it is already known that we have an ω-category and that such functors as + and $\times$ are ω-continuous (see [Leh 1]); one case in which there is, so far, no alternative to direct verification is with Lehmann's category, Dom of small ω-categories and ω-continuous functors ([Leh 2]).

Note that it is only necessary to check that the basic categories are ω-categories and the basic functors are ω-continuous; one easily proves that any denumerable product of ω-categories is an ω-category, that all constant and projection functors are ω-continuous, and that composition and tupling preserve ω-continuity. Thus to solve equation (1) one only needs to check that + and -> are ω-continuous; for equations (2) and (3) one looks at + and $\times$.

The original work on models of the pure λ-calculus ([Sco 2],[Wad]) did not solve (4) as $\varinjlim \langle F^n(\perp), F^n(\perp_{F\perp})\rangle$ but rather as $\varinjlim \langle F^n(D), F^n(\pi)\rangle$ for an object D and a morphism $\pi : D \rightarrow F(D)$. It turns out that this solution is, essentially, the initial fixed-point of a functor F_π, derived from F, over the comma category $(D \downarrow K)$ of "objects over D" (see [Mac]). The analogous idea in partial orders is that of a least fixed-point greater than some fixed element, d.

The/

The comma category $(D\downarrow K)$ has as objects pairs (A,d) where A is an object of K and $d: D \to A$; the morphisms $f: (A,d) \to (A',d')$ are the morphisms $f: A \to A'$ of K such that the following diagram commutes:

$$\begin{tikzcd} D \arrow[r, "d"] \arrow[dr, "d'"'] & A \arrow[d, "f"] \\ & A' \end{tikzcd}$$

Now the endofunctor $F_\pi: (D\downarrow K) \to (D\downarrow K)$ can be obtained by putting $F(A,d) = (FA,(Fd)\circ\pi)$ for objects and $F_\pi(f) = F(f)$ for morphisms. Then one can see that an F_π-algebra is a pair $((A,d),\alpha)$ where A is an object of K, $d: D \to A$ and $\alpha : FA \to A$ such that the following diagram commutes:

$$\begin{tikzcd} D \arrow[r, "\pi"] \arrow[drr, "d"'] & F(D) \arrow[r, "Fd"] & F(A) \arrow[d, "\alpha"] \\ & & A \end{tikzcd}$$

and a homomorphism $f: ((A,d),\alpha) \to ((A',d'),\alpha')$ is a morphism $f: A \to A'$ such that the following two diagrams commute:

$$\begin{tikzcd} F(A) \arrow[r, "\alpha"] \arrow[d, "F(f)"'] & A \arrow[d, "f"] \\ F(A') \arrow[r, "\alpha'"] & A' \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, "d"] \arrow[dr, "d'"'] & A \arrow[d, "f"] \\ & A' \end{tikzcd}$$

Let us assume, for simplicity, that K is an ω-category and F is ω-continuous. Then $(D\downarrow K)$ is also an ω-category. Its initial object is (D,id_D). For colimits suppose $\Delta = \langle (A_n,d_n),f_n\rangle$ is an ω-cochain in $D\downarrow K$. Then it is straightforward to check that if $\mu : \langle A_n,f_n\rangle \to A$ is universal in K then $\mu : \Delta \to (A, \mu_0 \circ d_0)$ is in $(D\downarrow K)$. This makes it easy to show that F is ω-continuous.

Now, applying the Basic Lemma to $(D\downarrow K)$ and F_π we have to find a colimiting cone $\mu : \langle F_\pi^n(\bot), F_\pi^n(\bot_{F_\pi(\bot)})\rangle \to (A,d)$. One sees, by induction on n, that $F_\pi^n(\bot_{(D\downarrow K)})$ is $\langle F^n(D),d_n\rangle$ where $d_n = F^{n-1}(\bot)\circ \dots\circ\pi: D \to F^n(D)$ and that $F_\pi^n(\bot_{F_\pi(\bot)}) = F^n(\pi)$. So from the above remarks one can take μ to be a colimiting cone, $\mu : \langle F^n(D),F^n(\pi)\rangle \to A$, also defining A, and put $d = \mu_0 \circ d_0$. Then, by the Basic Lemma, the initial F_π-algebra is $((A,d),\alpha)$ where α is the mediating morphism from $F\mu$ to μ^- (which can be taken in K). Thus we see that $A = \varinjlim\langle F^n(D),F^n(\pi)\rangle$, together with its colimiting cone, determines the initial F_π-algebra. Thus we have characterised the original inverse limit construction in universal terms.

3. O-categories

In most of the categories used for the denotational semantics of programming languages following the ideas in [Sto] the morphism sets have a natural partial order structure. When solving recursion domain equations only the projections are considered and they are easily defined in terms of the partial order. Wand, in [Wan 2,3] introduced O-categories to study such categories at a suitably abstract level. We now present a view of his work as providing theorems and definitions which facilitate the application of the Basic Lemma.

More explanation

Definition 5 A category, K, is an O-category iff (i) every hom-set is a partial order in which every ascending ω-chain has a lub and (ii) composition of morphisms is an ω-continuous operation with respect to this partial order.

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Note that if K is an O-category, so is K^{op} and if L is another so is $K \times L$. Here the orders are inherited and in the case of K^{op}, $f^{op} \sqsubseteq g^{op}$ iff $f \sqsubseteq g$, for any morphisms f and g of K.

As it happens O-categories are a particular case of a general theory of V-categories where V is any closed category [Mac]; here O is the category whose objects are those partial orders with lubs of all increasing ω-chains and whose morphisms are the ω-continuous functions between the partial orders. We will not use any of the general theory but just take over the idea of endowing the morphism sets with extra structure - in this case that of being an object in O.

Definition 6 Let K be an O-category and let $A \xrightarrow{f} B \xrightarrow{g} A$ be arrows such that $g \circ f = id_A$ and $f \circ g \sqsubseteq id_B$. Then we say that $\langle f,g \rangle$ is a projection pair from A to B, that g is a projection and that f is an embedding.

Lemma 3 Let $\langle f,g \rangle$ and $\langle f',g' \rangle$ both be projection pairs from A to B, in an O-category K. Then $f \sqsubseteq f'$ iff $g \sqsupseteq g'$.

Proof If $f \sqsubseteq f'$ then $g \sqsupseteq g \circ f' \circ g' \sqsupseteq g \circ f \circ g' = g'$. Conversely, if $g \sqsupseteq g'$ then $f = f \circ g' \circ f' \sqsubseteq f \circ g \circ f' \sqsubseteq f'$. $\square$

So, in particular, it follows that one half of a projection pair determines the other; if f is an embedding we write f^R for the corresponding projection which we call the right adjoint of f; if g is a projection we write g^L for the corresponding embedding which we call the left adjoint of g. Given any O-category K^E we can form the subcategory, K^E, of the embeddings. For the identity morphism $id_A: A \to A$ is an embedding with $id_A^R = id_A$ and if $A \xrightarrow{f} B \xrightarrow{f'} C$ are embeddings, so is $(f' \circ f)$ with $(f' \circ f)^R = f^R \circ f'^R$. (We do not try to take K^E to be an O-category.)

Our/

Our first theorem is trivial but does illustrate the idea of transferring properties from K to K^E.

<u>Theorem 1</u> Let K be an $\underline{\underline{O}}$-category which has a ~~final element~~ terminal object, $\perp$, and in which every hom-set, $Hom(A,B)$ has a least element $\perp_{A,B}$ and in which composition is left-strict in the sense that for any $f: A \to B$ we have $\perp_{B,C} \circ f = \perp_{A,C}$. Then $\perp$ is the initial object of K^E. 7

<u>Proof</u> First if $f,f': \perp \to A$ are both embeddings then they have a common right adjoint as $\perp$ is ~~final~~ terminal in K and so, by Lemma 3, they are equal. Second $\perp_{\perp,A}: \perp \to A$ is an embedding with right adjoint $\perp_{A,\perp}$. For $\perp_{A,\perp} \circ \perp_{\perp,A}: \perp \to \perp$ must be $id_\perp$ the unique map from $\perp$ to $\perp$ and $\perp_{\perp,A} \circ \perp_{A,\perp} = \perp_{A,A}$ (by left strictness) $\sqsubseteq id_A$. □ terminal 7

Our second theorem strengthens Theorem 1 in [Wan 2] by omitting his Condition A.

<u>Theorem 2</u> Let K be an $\underline{\underline{O}}$-category which admits ω^{op}-limits. Then K^E admits ω-colimits. Indeed if $\Delta = \langle A_n, f_n\rangle$ is an ω-cochain in K^E and $\mu: \Delta \to A$ is a cone in K^E then the following are equivalent:

(i) μ is colimiting in K^E.
(ii) $\mu^R: A \to \Delta^R$ is limiting in K (where $\mu^R =_{def} \langle \mu_n^R\rangle_{n\in\omega}$ and $\Delta^R =_{def} \langle A_n, f_n^R\rangle$).
(iii) $id_A = \bigsqcup_n \mu_n \circ \mu_n^R$ and the sequence on the right is increasing.
(iv) μ is colimiting in K.

This theorem greatly simplifies the verification that K^E admits ω-colimits as it is easily verified in the cases at hand that K admits ω^{op}-limits. The equivalence of (i) and the fact that μ^R is limiting in K^P, the subcategory of projections, seems to be the limit-colimit coincidence remarked in [Sco 1]. It is easily proved from the theorem by duality. Lawvere essentially pointed out that the function-space construction (exponentiation) turns limits in K^P on the right and colimits in K^E on the left into limits in K^P; we prefer to use a formulation (Theorem 3 below) which uses the one category, K^E.

<u>Lemma 4</u> Let K be an $\underline{\underline{O}}$-category and let Δ be an ω-cochain in K^E. Then if $\nu: A \to \Delta^R$ is a limiting cone in K it follows that each ν_n is a projection and $id_A = \bigsqcup \nu_n^L \circ \nu_n$, the sequence on the right being increasing.

<u>Proof</u> Let Δ be the cochain $\langle A_n, f_n\rangle_{n\in\omega}$ and let $f_{mn} =_{def} f_{n-1} \circ \dots\dots \circ f_m: A_m \to A_n$ for $m \le n$. For each A_m we can define a cone $\nu^{(m)}: A_m \to \Delta^R$ in K by:

$\nu_n^{(m)}$ /

$$\nu_n^{(m)} = f_{mn} \quad (m \leq n)$$
$$= (f_{nm})^R \quad (m > n)$$

space

space

To see that $\nu^{(m)}$ is a cone we first check that if $r \geq \max(m,n)$ then $\nu_n^{(m)} = f_{nr}^R \circ f_{mr}$. For if $m \leq n$ then $f_{nr}^R \circ f_{mr} = f_{nr}^R \circ (f_{nr} \circ f_{mn}) = \nu_n^{(m)}$; if $m > n$ then $f_{nr}^R \circ f_{mr} = (f_{mr} \circ f_{nm})^R \circ f_{mr} = f_{nm}^R \circ f_{mr}^R \circ f_{mr} = \nu_n^{(m)}$. Now we see that $\nu^{(m)}$ is a cone ~~from~~ for:

$$f_n^R \circ \nu_{n+1}^{(m)} = f_n^R \circ (f_{(n+1)r}^R \circ f_{mr}) \quad \text{(by the above with } r = \max(m,n+1))$$
$$= (f_{(n+1)r} \circ f_n)^R \circ f_{mr} = f_{nr}^R \circ f_{mr}$$
$$= \nu_n^{(m)} \quad \text{(by the above).}$$

Now as $\nu : A \to \Delta^R$ is a universal cone there is, for each m, a mediating morphism $\theta_m : A_m \to A$ from $\nu^{(m)}$ to ν. So we have for all m and n: $\nu_n \circ \theta_m = \nu_n^{(m)}$. Putting n equal to m we find that $\nu_m \circ \theta_m = id_{A_m}$, which is half the proof that ν_m is a projection with $\nu_m^L = \theta_m$.

Next we connect up the θ_m's by showing that $\theta_m = \theta_{m+1} \circ f_m$ since $\theta_{m+1} \circ f_m$ mediates between $\nu^{(m)}$ and ν as can be seen from:

$$\nu_n \circ (\theta_{m+1} \circ f_m) = \nu_n^{(m+1)} \circ f_m \quad (\theta_{m+1} \text{ mediates between } \nu^{(m+1)} \text{ and } \nu)$$
$$= f_{nr}^R \circ f_{(m+1)r} \circ f_m \quad \text{(with } r = \max(m+1,n))$$
$$= f_{nr}^R \circ f_{mr}$$
$$= \nu_n^{(m)}.$$

This, in turn, enables us to show that $\langle \theta_m \circ \nu_m \rangle_{m \in \omega}$ is increasing:
$\theta_m \circ \nu_m = \theta_{m+1} \circ f_m \circ f_m^R \circ \nu_{m+1} \sqsubseteq \theta_{m+1} \circ \nu_{m+1}$. Consequently, as K is an $\underline{\underline{O}}$-category we may define $\theta : A \to A$ by: $\theta = \bigsqcup_{m \in \omega} \theta_m \circ \nu_m$. To finish the proof we show that $\theta = id_A$ (as then we also have $\theta_m \circ \nu_m \sqsubseteq \theta = id_A$, completing the proof that $\nu_m^L = \theta_m$). This follows from the fact that θ mediates between ν and ν as is shown by:

$$\nu_n \circ \theta = \nu_n \circ \bigsqcup_{m \geq n} \theta_m \circ \nu_m = \bigsqcup_{m \geq n} (\nu_n \circ \theta_m) \circ \nu_m = \bigsqcup_{m \geq n} \nu_n^{(m)} \circ \nu_m$$
$$= \bigsqcup_{m \geq n} f_{nm}^R \circ \nu_m$$
$$= \nu_n. \quad \Box$$

<u>Lemma 5</u>/

<u>Lemma 5</u> Let K be an $\underline{O}$-category and let Δ be an ω-cochain in K^E. Then if $\mu : \Delta \to A$ is a cone in K^E such that $id_A = \bigsqcup \mu_n \circ \mu_n^R$ with the sequence on the right increasing, it follows that μ is universal in both K and K^E.

<u>Proof</u> First we show that $\langle \mu'_n \circ \mu_n^R \rangle_{n\in\omega}$ is increasing for any cone $\mu' : \Delta \to A'$ in K: $\mu'_n \circ \mu_n^R = (\mu'_{n+1} \circ f_n) \circ (\mu_{n+1} \circ f_n)^R = \mu'_{n+1} \circ (f_n \circ f_n^R) \circ \mu_{n+1}^R \sqsubseteq \mu'_{n+1} \circ \mu_{n+1}^R$.

Now choose a cone $\mu' : \Delta \to A'$ in K and suppose $\theta : A \to A'$ is mediating. Then:

$$\theta = \theta \circ \bigsqcup \mu_n \circ \mu_n^R = \bigsqcup (\theta \circ \mu_n) \circ \mu_n^R$$
$$= \bigsqcup \mu'_n \circ \mu_n^R \quad \text{(as } \theta \text{ is mediating)}.$$

This proves uniqueness; for existence we can define θ as $\bigsqcup \mu'_n \circ \mu_n^R$ as the above remark shows that $\langle \mu'_n \circ \mu_n^R \rangle$ is increasing, and calculate:

$$\theta \circ \mu_m = \left(\bigsqcup_{n \geq m} \mu'_n \circ \mu_n^R\right) \circ \mu_m = \bigsqcup_{n \geq m} \mu'_n \circ \mu_n^R \circ \mu_n \circ f_{mn}$$
$$= \bigsqcup_{n \geq m} \mu'_n \circ f_{mn}$$
$$= \mu'_m.$$

Here $f_{mn} = f_{n-1} \circ \ldots \circ f_m$ $(m \geq n)$ as in the proof of lemma 4.

So μ is universal in K; for K^E it only remains to show that if μ' is actually a cone in K^E then θ as defined above is an embedding. By the remark above $\langle \mu_n \circ \mu'^R_n \rangle_{n\in\omega}$ is increasing and to do this we prove that $\theta^R = \bigsqcup \mu_n \circ \mu'^R_n$. On the one hand we have:

$$\left(\bigsqcup \mu_n \circ \mu'^R_n\right) \circ \left(\bigsqcup \mu'_n \circ \mu_n^R\right) = \bigsqcup \mu_n \circ (\mu'^R_n \circ \mu'_n) \circ \mu_n^R = \bigsqcup \mu_n \circ \mu_n^R = id_A.$$

On the other hand we have:

$$\left(\bigsqcup \mu'_n \circ \mu_n^R\right) \circ \left(\bigsqcup \mu_n \circ \mu'^R_n\right) = \bigsqcup \mu'_n \circ (\mu_n^R \circ \mu_n) \circ \mu'^R_n = \bigsqcup \mu'_n \circ \mu'^R_n \sqsubseteq id_{A'}. \square$$

<u>Proof of Theorem 2</u> Let Δ be an ω-cochain in K^E. Then Δ^R is an ω-chain in K and so, by assumption, there is a universal cone $\nu : A \to \Delta^R$ in K. By lemma 4 each ν_n is a projection and $id_A = \bigsqcup \{ \nu_n^L \circ \nu_n \mid n \in \omega \}$ with the sequence on the right increasing. Since $\nu^L =_{def} \langle \nu_n^L \rangle_{n\in\omega}$ is easily seen to be a cone $\nu^L : \Delta \to A$ in K^E it follows by lemma 5 that ν^L is universal in K^E. This proves that if K^E admits ω^{op}-limits then K^E admits ω-colimits.

Note/

Note that we have constructed a cone $\nu^L: \Delta \to A$ which obeys (i), (ii), (iii) and, by lemma 5, (iv) as well. Now let $\mu : \Delta \to A$ be a cone in K^E. If it obeys (i) it is, by virtue of universality, isomorphic to ν^L and so it obeys (ii) as ν^L does. Similarly if it obeys (ii) it obeys (iii) and if it obeys (iv) it obeys (i). Finally if it obeys (iii) then by lemma 5 it also obeys (iv). □

The following definition will prove convenient.

Definition 8 Let K be an O-category. It is said to have locally determined ω-colimits of embeddings iff whenever Δ is an ω-cochain in K^E and

n K^G $\mu : \Delta \to A$ is a cone in K^E then μ is universal iff $\langle \mu_n \circ \mu_n^R \rangle$ is an increasing chain with lub id_A. (Note that lemma 5 entails that only half the implication can ever be in doubt and Theorem 2 shows that if K admits ω^{op}-limits then it has locally determined ω-colimits.)

The point of introducing K^E is to enable us to consider contravariant functors on K as covariant ones on K^E. We consider ~~throughout~~ three O-categories K,L,M and a covariant functor $T: K^{op} \times L \to M$. Purely covariant functors are included by suppressing K (i.e. taking K to be the trivial one-object category 1), covariant ones by suppressing L and mixed ones by taking K and L to be product categories as required.

Definition 7 The functor T is locally monotonic iff it is monotonic on the hom-sets; that is for $f,f': A \to B$ in K^{op}, $g,g': C \to D$ in L, $f \sqsubseteq f'$ and $g \sqsubseteq g'$ then $T(f,g) \sqsubseteq T(f',g')$.

Lemma 6 If T is locally monotonic, a functor $T^E: K^E \times L^E \to M^E$ can be defined by putting, for objects A,B: $T^E(A,B) = T(A,B)$ and for morphisms f,g: $T^E(f,g) = T(f^R,g)$.

Proof First if $f: A \to B$ in K^E and $g: C \to D$ in L^E then $T(f^R,g)$ is an embedding with right adjoint $T(f,g^R)$ as: $T(f,g^R) \circ T(f^R,g) = T(f^R \circ f, g^R \circ g) = T(id_A, id_C) = id_{T(A,C)}$ and also: $T(f^R,g) \circ T(f,g^R) = T(f \circ f^R, g \circ g^R) \sqsubseteq T(id_B, id_D)$ (by local monotonicity) $= id_{T(B,D)}$.

Secondly, $T^E(id_A, id_C) = T(id_A^R, id_C) = T(id_A, id_C) = id_{T(A,C)}$.

Thirdly if $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in K^E and $B \xrightarrow{g} B' \xrightarrow{g'} B''$ in L^E then $T^E(g'',f'') \circ T^E(g',f') = T(f''^R,f'') \circ T(g'^R,f') = T(g'^R \circ g''^R, f'' \circ f') = T((g'' \circ g')^R, f'' \circ f') = T^E(g'' \circ g', f'' \circ f')$. □

Under/

Under some assumptions on K and L, we can transfer a local continuity property of T to the ω-continuity of T^E.

<u>Definition 9</u> The functor T is <u>locally continuous</u> iff it is ω-continuous on the ~~morphism~~ sets - that is if $f_n: A \to B$ is an increasing ω-chain in K^{op} and $g_n: C \to D$ is one in L then $T(\bigsqcup_{n\in\omega} f_n, \bigsqcup_{n\in\omega} g_n) = \bigsqcup_{n\in\omega} T(f_n, g_n)$. hom-/

Note that the constant and projection functors are locally continuous and that the locally continuous functors are closed under composition and tupling and taking opposite functors.

<u>Theorem 3</u> Suppose T is locally continuous and both K and L have locally determined ω-col~~i~~mits of embeddings. Then T^E is ω-continuous. -/

<u>Proof</u> Let $\Delta = \langle\langle A_n, B_n\rangle, \langle f_n, g_n\rangle\rangle$ be an ω-cochain in $K^E \times L^E$ and let $\mu: \Delta \to \langle A,B\rangle$ be colimiting, where $\mu = \langle \sigma_n, \tau_n\rangle_{n\in\omega}$. Then $\langle\sigma_n\rangle: \langle A_n, f_n\rangle \to A$ is colimiting in K^E and $\langle\mu_n\rangle: \langle B_n, g_n\rangle \to B$ is colimiting in L^E. It follows by the assumptions on K and L that $id_A = \bigsqcup \sigma_n \circ \sigma_n^R$ and $id_B = \bigsqcup \tau_n \circ \tau_n^R$ with the right hand sides increasing.

We have to show that $T^E(\mu): T^E(\Delta) \to T^E(A,B)$ is a limiting cone in ~~M~~, M^E/ and we apply the test of lemma 5:

$$\begin{aligned}
\text{First } \langle T^E(\mu_n) \circ T^E(\mu_n)^R\rangle_{n\in\omega} &= \langle T(\sigma_n^R, \tau_n) \circ T(\sigma_n^R, \tau_n)^R\rangle_{n\in\omega} \\
&= \langle T(\sigma_n^R, \tau_n) \circ T(\sigma_n, \tau_n^R)\rangle_{n\in\omega} \\
&= \langle T(\sigma_n \circ \sigma_n^R, \tau_n \circ \tau_n^R)\rangle_{n\in\omega}
\end{aligned}$$

which is increasing as $\langle\sigma_n \circ \sigma_n^R\rangle_{n\in\omega}$ and $\langle \tau_n \circ \tau_n^R\rangle_{n\in\omega}$ are and as T is locally monotonic.

$$\begin{aligned}
\text{Next, } \bigsqcup_{n\in\omega} T^E(\mu_n) \circ T^E(\mu_n)^R &= \bigsqcup_{n\in\omega} T(\sigma_n \circ \sigma_n^R, \tau_n \circ \tau_n^R) \text{ (by the above)} \\
&= T(\bigsqcup_{n\in\omega} \sigma_n \circ \sigma_n^R, \bigsqcup_{n\in\omega} \tau_n \circ \tau_n^R) \text{ (by local continuity)} \\
&= T(id_A, id_B) \text{ (by the above)} \\
&= id_{T(A,B)}. \ \square
\end{aligned}$$

<u>Example 1.</u> <u>The Category $\underline{\underline{O}}_\perp$</u>. The objects of this category are the ω-cpos and the morphisms are the strict ω-continuous maps. It is an $\underline{\underline{O}}$-category with respect to the natural pointwise order on morphisms and satisfies the conditions of Theorem 1 as Hom(A,B) always has the least element $\lambda x: A. \perp_B$, composition is left-strict and it has as ~~final element~~ terminal object/ the trivial one-point c/ ω-c~~p~~po. Further it admits ω^{op}-limits and these are constructed in the same way as in $\underline{\underline{Set}}$: Let $\Delta = \langle D_n, f_n\rangle$ be an ω-chain and put $D = \{\langle d_n\rangle_{n\in\omega} \mid \forall n(d_n \in D_n) \wedge (f_n(d_{n+1}) = d_n)\}$ and give it the componentwise order./ 7

order. This makes D an ω-cppo with least element $\langle \perp_{D_n} \rangle_{n\in\omega}$ and with lubs of increasing ω-chains taken componentwise. Define $\nu : D \to \Delta$ by taking $\nu_n(\langle d_n \rangle_{n\in\omega}) = d_n$; then each ν_n is clearly ω-continuous and ν is a cone. If $\nu' : D' \to A$ is any other then if $\theta : D' \to D$ is a mediating morphism we have $(\theta(d))_n = \nu'_n(d)$ and so θ is unique; conversely this formula defines a mediating morphism. So the conditions of theorem 2 are satisfied. All the usual functors are easily seen to be locally-continuous. For example the exponentiation functor $\to : \underline{\underline{O}}_{\perp}^{op} \times \underline{\underline{O}}_{\perp} \to \underline{\underline{O}}_{\perp}$ is defined by putting for objects D,E:

$$D \to E = \{f : D \to E \mid f \text{ is } \omega\text{-continuous}\}$$

with the pointwise ordering and by putting for morphisms, $f : D' \to D$, $g : E \to E'$, and any h in $D \to E$

$$(f \to g)(h) = g \circ h \circ f$$

which clearly makes exponentiation locally continuous, and so theorem 3 can be applied to show that $\to^E : \underline{\underline{O}}_{\perp}^E \times \underline{\underline{O}}_{\perp}^E \to \underline{\underline{O}}_{\perp}^E$ is ω-continuous. Other examples of locally continuous functors can be found in, for example, [Leh 1]; they include the product functor, $\times$, the strict product functor, $\times$, the strict exponentiation functor, $\to_{\perp}$, the strict sum functor, +, and the "adding a $\perp$" functor, $(\cdot)_{\perp}$.

One can also consider $\underline{\underline{\omega\text{-CPPO}}}$ the category of ω-cppos and ω-continuous functions. It also obeys the conditions of Theorems 1 and 2 and further $\underline{\underline{\omega\text{-CPPO}}}^E = \underline{\underline{O}}_{\perp}^E$. It does not appear to be correct to use $\underline{\underline{O}}$ itself as $\underline{\underline{O}}^E$ has no initial object; this is an argument against the use of predomains, advocated by Reynolds in [Rey 2]. Of course one can use directed complete pointed partial orders in which <u>every</u> directed set has a lub. This is often done and everything goes smoothly.

<u>Example 2</u> <u>The κ-complete pos</u>. There are many interesting full subcategories of $\underline{\underline{O}}_{\perp}$ and one class of such subcategories is provided by completeness considerations.

<u>Definition 10</u> Let D be a partial order. Then a subset, X, of D is <u>κ-consistent</u> iff whenever $Y \subseteq X$ and $\|Y\| < \kappa$ then Y has an upper bound in D.

<u>Definition 11</u> A partial order D is <u>κ-complete</u> iff every κ-consistent subset has a lub.

non-empty

For/

For $\kappa \leq \omega$, every κ-complete po is an ω-cppo. The n-complete pos are the complete lattices for $n=0,1,2$; the 3-complete pos are the coherent cpos and the ω-complete pos are the consistently complete cpos [Plo 2]. The full subcategory of $\underline{\underline{O}}_{\perp}$ of the κ-complete pos contains the one-point ω-cppo and so obeys the conditions of Theorem 1. It also has ω^{cp}-limits and indeed it has the same ones as $\underline{\underline{O}}_{\perp}$. For let $\Delta = \langle D_n, f_n \rangle$ be an ω-chain of κ-complete po's and define the cone $\nu : D \to \Delta$ as above. It is only necessary to show that D is also κ-complete and this follows at once from the following easy lemma which uses an idea of Scott for the case of complete lattices.

<u>Lemma 7</u> Suppose $X \subseteq D$ and $\bigsqcup_{D_n} \{x_n \mid x \in X\}$ exists for every integer n. Then $\bigsqcup_D X$ exists and is given by:

$$(\bigsqcup_D X)_n = \bigsqcup_{m \geq n} f_{mn}(\bigsqcup_{D_m} \{x_m \mid x \in X\})$$

where $f_{mn} = f_{n+1} \circ \ldots \circ f_m$ for $m \geq n$.

<u>Proof</u> Omitted.□

It follows that the subcategory of embeddings is a full subcategory of $\underline{\underline{O}}^E_{\perp}$ and has the same ω-colimits. It follows that any ω-continuous functors over $\underline{\underline{O}}^E_{\perp}$ which preserve κ-completeness cut down to ω-continuous functors on the subcategory; other functors on the subcategory (such as sums of lattices) are easily dealt with by local continuity.

<u>Example 3</u> <u>The ω-continuous cpos</u> Next we consider the ω-algebraic and the ω-continuous ω-cppos. First it is convenient to note that the ω-cppos are just those partial orders with a least element which have lubs of countable directed sets and that the ω-continuous functions are those which preserve such lubs. The following definitions are equivalent to those in [Plo 2],[Smy 1],[Sco 1] being designed to be in accord with the spirit of only considering ω-lubs.

<u>Definition 12</u> Let D be an ω-cppo. Then the relation $x \ll_\omega y$ is defined by: $x \ll_\omega y$ iff for every countable directed subset, Z, of D such that $y \sqsubseteq \bigsqcup_D Z$ we have $x \sqsubseteq z$ for some z in Z. Further a countable subset, B, of D is a <u>basis</u> iff for every element, x, of D the set $B_x = \{b \in B \mid b \ll_\omega x\}$ is directed and has lub x. 7

Note that if $x \ll_\omega y \sqsubseteq z$ then $x \ll_\omega z$ and that embeddings preserve $\ll_\omega$.

<u>Definition 13</u>./

<u>Definition 13</u> Let D be an ω-cppo. An element, d, is <u>ω-finite</u> iff $d \ll_\omega d$. Further D is <u>ω-algebraic</u> iff the set of ω-finite elements is countable and is a basis.

$\underline{\underline{O}}_\perp$

The full subcategory of the ω-continuous ω-cppos contains the one-point domain, $\{\perp\}$ and so satisfies the conditions of Theorem 1; however it probably does not have ω^{op}-limits. The same holds for the ω-algebraic ω-cppos. Fortunately the subcategories of the embeddings inherit ω-colimits from $\underline{\underline{O}}^E_\perp$. It is only necessary to show that if $\Delta = \langle D_n, f_n \rangle$ is an ω-cochain in $\underline{\underline{O}}^E_\perp$ where each D_n is ω-continuous (ω-algebraic) then if $\mu : \Delta \to D$ is colimiting, D is also ω-continuous (ω-algebraic).

<u>Lemma 7</u> 1. Let E be a cpo and B be a countable subset of E. Then if x is an element of E and C is a directed subset of B_x with lub x then B_x is directed, with lub x.

2. Let E be an ω-cppo and B and C be subsets of E, with B countable. Suppose that for every element y of C, B_y is directed with lub y and that for every element x of E there is a countable directed subset, C_x, of such that $x = \bigsqcup C_x$. Then B is a basis for E.

<u>Proof</u> 1. If $u \ll_\omega x$ and $v \ll_\omega x$ then there are u', v' in C such that $u \sqsubseteq u'$, $v \sqsubseteq v'$. But as C is directed this shows B_x is directed too.

2. Take x in E and consider $\{B_y \mid y \in C_x\}$. This is a directed set, wrt. $\subseteq$, of directed sets as C_y is directed and so its union is directed and is clearly a subset of B_x with lub x. So by part 1, B_x is directed with lub x. $\square$

Now we can use lemma 7.2 to show D is ω-continuous. Let $B^{(n)}$ be a countable basis for D_n ($n \in \omega$); we claim $B =_{def} \bigcup_n \mu_n(B^{(n)})$ is a countable basis for D. Let C be $\bigcup_n \mu_n(D_n)$. By Theorem 2 applied to $\underline{\underline{O}}_\perp$ we can take $C_x = \{\mu_n(\mu_n^R(x)) \mid n \in \omega\}$. Now for each y in D_n, $\mu_n(B_y^{(n)})$ is a directed subset of $B_{\mu_n(y)}$ with lub y. So by lemma 7.1 $B_{\mu_n(y)}$ is directed with lub $\mu_n(y)$. Thus lemma 7.2 applies. In the case where the D_n are all ω-algebraic we take $B^{(n)}$ to be the ω-finite elements of D_n and find a basis of ω-finite elements of D. As it is easily seen that any basis contains all the ω-finite elements this shows D to be ω-algebraic.

So the embedding sub-category of ω-continuous (ω-algebraic) ω-cppos is an ω-category and ω-colimits are internally determined and functors are considered using Theorem 3. The same holds, from the above consideration, for the embedding sub-category of K-complete ω-continuous (ω-algebraic) ω-cppos.

Example 4 Non-deterministic domains The category $\underline{\underline{NDO}}_\perp$ was found useful for the semantics of non-deterministic and parallel programs in [Hen]. Its objects are the non-deterministic cppos $\langle D, \sqsubseteq, \cup \rangle$ where $\langle D, \sqsubseteq \rangle$ is an ω-cppo and $\cup : D^2 \to D$ is an associative, commutative idempotent ω-continuous binary function (called union); the morphisms $f: D \to E$ are the ω-continuous strict functions which preserve union.

The trivial one-point object is final in $\underline{\underline{NDO}}_\perp$ and the conditions of Theorem ~~2~~ are satisfied. Further $\underline{\underline{NDO}}_\perp$ has ω^{op}-limits. Indeed the forgetful functor $U: \underline{\underline{NDO}}_\perp \to \underline{\underline{O}}_\perp$ creates them [Mac]. Let $\Delta = \langle D_n, f_n \rangle$ be an ω-chain in $\underline{\underline{NDO}}_\perp$ and suppose $\nu: E \to U\Delta$ is universal in $\underline{\underline{O}}_\perp$ being constructed as shown above. Then if we want a union on E so that the ν_n are $\underline{\underline{NDO}}_\perp$ morphisms we have for elements x,y of E:

$$(x \cup y)_n = \nu_n(x \cup y) = \nu_n(x) \cup \nu_n(y)$$
$$= x_n \cup y_n.$$

So this determines union and it is easily seen that with this definition we obtain a universal cone in $\underline{\underline{NDO}}_\perp$. One interesting locally continuous functor is $\xrightarrow[\subseteq]{}$, where on objects D,E:

$$D \xrightarrow[\subseteq]{} E = \{f: D \to E \mid f \text{ is } \omega\text{-continuous and preserves } \subseteq\},$$

(where $x \subseteq y \equiv_{def} x \cup y = y$), with the pointwise order and union, and on morphisms $\xrightarrow[\subseteq]{}$ is defined as usual. Other examples can be found in [Hen].

Of course there are many other interesting varieties (or pseudo-varieties of one kind or another) subject to similar considerations [Blo]. However we have no clear idea what the possible applications are.

Example 5 ω-Complete Relations This category (or rather a slight variation of it) has been found to be useful for relating different semantics by Reynolds [Rey 3] (see also [Gor],[Sto]). It has as objects structures $\langle D,E,R \rangle$ where D and E are ω-cppos and $R \subseteq D \times E$ is a binary relation which is ω-complete in the sense that if $\langle d_n \rangle, \langle e_n \rangle$ are increasing sequences in D and E respectively such that $R(d_n, e_n)$ holds for all integers n then $R(\bigsqcup d_n, \bigsqcup e_n)$ holds too; the morphisms are pairs $\langle f,g \rangle : \langle D,E,R \rangle \to \langle D',E',R' \rangle$ where $f: D \to D'$, $g: E \to E'$ are morphisms in $\underline{\underline{O}}_\perp$ and for all x in D, y in E if $R(x,y)$ holds then so does $R'(fx, g\,y)$. We leave considerations of the properties of this category as an exercise for the reader [Wan 3]. The exponentiation functor is given by: $(D,E,R) \to (D',E',R') =$ $(D \to D', E \to E', R \to R')$ where $D \to D'$, $E \to E'$ are the ω-continuous functions and

$R \to R'$/

$$R \to R'(f,g) \equiv_{def} \forall x \in D.\ \forall y \in E.(R(x,y) \to R(fx,gy)),$$

and the action on morphisms is defined in the usual way. Other examples can be found in [Rey 3].

Again this idea can be extended to several relations and to relations of any denumerable degree; it can also be combined with the ideas of example 4 to consider continuous structures of various kinds. Again the scope of these mathematical possibilities is really quite unknown.

4. Universal Domains

In this section we formulate a general notion of a universal object of a category and show that ω^{op}-limits are not needed when such an object exists. We also show that it is also possible to entirely dispense with the Basic Lemma and use only ordinary fixed-point reasoning.

Definition 14 Let K be a category and let $B \xrightarrow{e} A \xrightarrow{r} B$ be morphisms such that $r \circ e = id_B$. Then B is a retract of A, $\langle e,r\rangle$ is a retraction pair from B to A, r is a retraction and e is a section. An idempotent is a morphism $h: A \to A$ such that $h \circ h = h$; it splits iff there is a retraction pair $\langle e,r\rangle$ to A such that $h = e \circ r$.

Retracts are subobjects as every section is a monomorphism. Note that if $\langle e,r\rangle$ is a retraction pair then $e \circ r$ is an idempotent.

Definition 15 Let K be a category. An object U is universal iff every object is a retract of U and every idempotent $h: U \to U$ splits.

Note that if U is a universal object of K then it is also a universal object of K^{op}; further if V is a universal object of L then $\langle U,V\rangle$ is a universal object of $K \times L$. The first example was given by Scott [Sco 2]: $P\omega$ is universal in $\underline{\underline{CL}}$; Plotkin showed in [Plo 2] that $\mathbb{T}^\omega$ is universal in the category of ω-continuous coherent ω-cppos; recently Plotkin and Scott have, independently, constructed a universal object in the category of ω-continuous consistently complete ω-cppos.

Just as we use sections which are a special kind of subobject so we need a special order on subobjects different from the usual one and available in $\underline{\underline{O}}$-categories. It is inspired by the order on idempotents used by Scott.

Definition 16 Let K be an $\underline{\underline{O}}$-category and let $A \xrightarrow{e} B \xrightarrow{r} A$, $A' \xrightarrow{e'} B \xrightarrow{r'} A'$ be retraction pairs. Then $\langle e,r\rangle \leq_f \langle e',r'\rangle$ iff $f: A \to A'$ is an embedding such that $e \sqsubseteq e' \circ f$ and $r \sqsubseteq f^R \circ r'$; $\langle e,r\rangle \leq \langle e',r'\rangle$ means that such an embedding $f: A \to A'$ exists.

Lemma 8 Let $A \xrightarrow{e} B \xrightarrow{r} A$, $A' \xrightarrow{e'} B \xrightarrow{r'} A'$, $A'' \xrightarrow{e''} B'' \xrightarrow{r''} A$ be retraction pairs.

1. If $\langle e,r\rangle \leq_f \langle e',r'\rangle$ then f is uniquely determined as $f = r' \circ e$ (and $f^R = r \circ e'$).

2. The relation, $\leq$, is a quasi-order: $\langle e,r\rangle \leq_{id_A} \langle e,r\rangle$ and if $\langle e,r\rangle \leq_f \langle e',r'\rangle \leq_{f'} \langle e'',r''\rangle$ then $\langle e,r\rangle \leq_{f' \circ f} \langle e'',r''\rangle$.

3./

3. Let $h = e \circ r$ and $h' = e' \circ r'$. Then $\langle e,r\rangle \underline{\leq} \langle e',r'\rangle$ iff $h \circ h' \circ h = h \sqsubseteq h'$.

Proof 1. We have $e \sqsubseteq e' \circ f$ and $r \sqsubseteq f^R \circ r'$. So $f = r' \circ e' \circ f \sqsupseteq r' \circ e$ and $f^R = f^R \circ r' \circ e' \sqsupseteq r \circ e'$. Next we show $(r' \circ e)$ is an embedding with right adjoint $r \circ e'$. First $(r' \circ e) \circ (r \circ e') \sqsubseteq f \circ f^R$ (by the above) $\sqsubseteq id_{A'}$. Second $(r \circ e') \circ (r' \circ e) \sqsupseteq r \circ (e' \circ f) \circ (f^R \circ r') \circ e \sqsupseteq (r \circ e) \circ (r \circ e)$ (by hypothesis) $= id_A$; also $(r \circ e) \circ (r' \circ e) \sqsubseteq f^R \circ f$ (by the above) $= id_A$. Now by lemma 3 as $f^R \sqsupseteq r \circ e'$, $f \sqsubseteq r' \circ e$ and so $f = r' \circ e$ (as we have alreadv shown that $f \sqsupseteq r' \circ e$).

2. Immediate from the definitions.

3. Suppose $\langle e,r\rangle \underline{\leq} \langle e',r'\rangle$. Then $h' = e' \circ r' \sqsupseteq e' \circ f \circ f^R \circ r' \sqsupseteq e \circ r = h$. It follows that $h \circ h' \circ h \sqsupseteq h \circ h \circ h = h$; further $h \circ h' \circ h = e \circ r \circ h' \circ e \circ r \sqsubseteq e \circ f^R \circ r' \circ h' \circ e' \circ f \circ r = e \circ f^R \circ f \circ r = e \circ r = h$.

Conversely suppose $h \circ h' \circ h = h \sqsubseteq h'$. Put $f = r' \circ e$. Then $f^R = r \circ e'$ as $(r' \circ e) \circ (r \circ e') = r' \circ h \circ e' \sqsubseteq r' \circ h' \circ e' = id_{A'}$ and $(r \circ e') \circ (r' \circ e) = r \circ h \circ h' \circ h \circ e = r \circ h \circ e = id_{A'}$. Now, $f^R \circ r' = r \circ e' \circ r' = r \circ h' \sqsupseteq r \circ h = r$ and $e' \circ f = e' \circ r' \circ e = h' \circ e \sqsupseteq h \circ e = e$ showing $\langle e,r\rangle \underline{\leq}_f \langle e',r'\rangle$. $\square$

The next theorem will be used as a replacement for the existence of ω^{op}-limits in K.

Theorem 4 Let K be an $\underline{\underline{O}}$-category with universal object U. Let $A_n \xrightarrow{e_n} U \xrightarrow{r_n} A_n$ be an increasing sequence of retraction pairs, wrt. $\underline{\leq}$. Then $\langle e_n \circ r_n\rangle$ is increasing and its lub $h = \bigsqcup e_n \circ r_n$ is an idempotent which we may suppose splits as $A \xrightarrow{e} U \xrightarrow{r} A$. Then $\langle e,r\rangle$ is the lub of the $\langle e_n,r_n\rangle$ wrt. $\underline{\leq}$ and $\langle r \circ e_n\rangle: \langle A_n, r_{n+1} \circ e_n\rangle \to A$ is a universal cone in K^E.

Proof By lemma 8.3 $\langle e_n \circ r_n\rangle$ is an increasing chain of idempotents and so h too is an idempotent. To show $\langle e_n,r_n\rangle \underline{\leq} \langle e,r\rangle$ we use lemma 8.3: clearly $h_n \sqsubseteq h$ and, further, $h_n \circ h \circ h_n = \bigsqcup_{m \geq n} h_n \circ h_m \circ h_n = h_n$ (by the transitivity of $\underline{\leq}$ and lemma 8.3). If $\langle e',r'\rangle$ is another upper bound of the $\langle e_n,r_n\rangle$ then, putting h' equal to $e' \circ r'$ we have: $h = \bigsqcup h_n \sqsubseteq h'$ and $h \circ h' \circ h = \bigsqcup h_n \circ h' \circ h_n$ which shows that $\langle e,r\rangle$ is indeed the lub of the $\langle e_n,r_n\rangle$.

Next $\mu : \langle A_n, f_n\rangle \to A$ is a cone in K^E where $\mu_n = r \circ e_n$ and $f_n = r_{n+1} \circ e_n$ as $\langle e_n,r_n\rangle \underline{\leq}_{f_n} \langle e_{n+1},r_{n+1}\rangle \underline{\leq}_{\mu_{n+1}} \langle e,r\rangle$ and $\langle e_n,r_n\rangle \underline{\leq}_{\mu_n} \langle e,r\rangle$ by lemma 8.1 and so $\mu_{n+1} \circ f_n = \mu_n$ by lemmas 8.2 and 8.1. Finally μ is universal by the test of lemma 5: $\mu_n \circ \mu_n^R = r \circ e_n \circ r_n \circ e = r \circ h_n \circ e \sqsubseteq \mu_{n+1} \circ \mu_{n+1}^R$ and $\bigsqcup \mu_n \circ \mu_n^R = r \circ (\bigsqcup h_n) \circ e = r \circ h \circ e = id_A$. $\square$

Corollary 1/

Corollary 1 Let K be an $\underline{\underline{O}}$-category with universal object U. Then ω-colimits in K^E are locally determined.

Proof Let $\mu : \langle A_n, f_n\rangle \to A$ be a universal cone in K^E. Let $A \xrightarrow{e} U \xrightarrow{r} A$ be a retraction pair. Put $e_n = e \circ \mu_n$, $r_n = \mu_n^R \circ r$. Then $A_n \xrightarrow{e_n} U \xrightarrow{r_n} A_n$ is an increasing sequence of retraction pairs wrt. $\leq$ and indeed $\langle e_n, r_n\rangle \leq_{f_n} \langle e_{n+1}, r_{n+1}\rangle$ Then Theorem 4 provides a universal cone in K^E satisfying the test of lemma 5; so μ must also satisfy it. □

In order to be able to apply the Basic Lemma we now need criteria for the existence of $\varinjlim \langle F^n(\perp), F^n(\perp_{F\perp})\rangle$. l_i ?

Theorem 5 Let K be an $\underline{\underline{O}}$-category which has a universal element U, and which satisfies the conditions of theorem 1. Let $T: K^{op} \times K \to K$ be a locally monotonic functor and let $F: K^E \to K^E$ be obtained by identifying the arguments of T^E (so that $F = T^E \circ \langle id_K E, id_K E\rangle$). Then $\perp$ is the initial object of K^E and $\varinjlim \langle F^n(\perp), F^n(\perp_{F\perp})\rangle$ exists. K

Proof Theorem 1 assures us that $\perp$ is the initial object of K^E. It is also easy to see that the subobject $\perp \xrightarrow{\perp_{\perp,U}} U \xrightarrow{\perp_{U,\perp}} \perp$ is the least subobject of U, wrt. $\leq$.

Next we show that T induces a monotonic function, τ, on subobjects of U. First let $\langle \bar{e}, \bar{r}\rangle$ be a retraction pair from T(U,U) to U. Then for any subobject (retraction pair), $A \xrightarrow{e} U \xrightarrow{r} A$ we define $\tau(e,r)$ to be $T(A,A) \xrightarrow{\bar{e} \circ T(r,e)} U \xrightarrow{T(e,r) \circ \bar{r}} T(A,A)$. This is a retraction pair as:

$$(T(e,r) \circ \bar{r}) \circ (\bar{e} \circ T(r,e)) = T(e,r) \circ T(r,e) = T(r \circ e, r \circ e) = id_{T(A,A)}.$$

For monotonicity we show that if $\langle e,r\rangle \leq_f \langle e',r'\rangle$ then $\tau(e,r) \leq_{Ff} \tau(e',r')$. For on the one hand we have:

$$\begin{aligned}(\bar{e} \circ T(r',e')) \circ Ff &= \bar{e} \circ T(r',e') \circ T(f^R,f) = \bar{e} \circ T(f^R \circ r', e' \circ f) \\ &\sqsupseteq \bar{e} \circ T(r,e) \quad \text{(by the monotonicity of T)}\end{aligned}$$

and on the other we have:

$$\begin{aligned}(Ff)^R \circ (T(e',r') \circ \bar{r}) &= T(f,f^R) \circ T(e',r') \circ \bar{r} \\ &\sqsupseteq T(e,r) \circ \bar{r} \quad \text{(similarly).}\end{aligned}$$

So we have an increasing sequence, $\langle \perp_{\perp,U}, \perp_{U,\perp}\rangle \leq_{\perp_{F\perp}} \tau(\perp_{\perp,U}, \perp_{U,\perp}) \leq \ldots \leq \tau^n(\perp_{\perp,U}, \perp_{U,\perp}) \leq_{F^n(\perp_{F\perp})} \tau^{n+1}(\perp_{\perp,U}, \perp_{U,\perp}) \leq \ldots$. By Theorem 4 it follows that $\varinjlim \langle F^n(\perp), F^n(\perp_{F\perp})\rangle$ exists as $\tau^n(\perp_{\perp,U}, \perp_{U,\perp})$ is from $F^n(\perp)$ to U. □ l_i ?

In/

In order to apply these results to an $\underline{\underline{O}}$-category K satisfying the conditions of Theorem 5 we first interpret equations such as (1) in the form:

(8) $D = T(D,D)$

where we separate out the contravariant and covariant occurrences of D to give a functor $T: K^{op} \times K \to K$. Thus in the case of equation (1):

$$T =_{def} + \circ \langle K_A, - \rangle \circ \langle id_{K^{op}}, id_K \rangle$$

Then the functor F obtained by identifying the arguments of T^E is the same as that obtained along the lines of Sections 2 and 3. As long as the basic functors are locally continuous T will be locally continuous and Theorem 5 and Corollary 1 allow the Basic Lemma to be applied to solve equation (4) of Section 2. Similar considerations apply to simultaneous equations.

From now on we assume that K is an $\underline{\underline{O}}$-category with a universal object U and further that K satisfies the conditions of Theorem 1. Then we see from above that the subobjects of U form a (large) ω-complete pointed quasi-order under $\leq$; equivalence is just isomorphism. Then, following the idea of the proof of Theorem 5, any locally continuous functor, $T: K^{op} \times K \to K$ can be seen to induce an ω-continuous function τ on this ω-cpqo. Consequently one can take the least fixed-point which gives a solution to equation (4). However we do not see how to use this to show, without using the Basic Lemma, that the resulting solution is initial; the difficulty is to convert an arbitrary embedding $\alpha': F(A') \to A'$ into a prefixed point of τ (where F is obtained by identifying the arguments of T^E).

However it is possible to avoid the Basic Lemma (and so any ideas of ω-categories and ω-continuity of functors) and use ordinary least fixed-point reasoning to obtain an initial solution by using the ordinary ordering on the (~~large~~) ω-cppo Hom(U,U) instead. Let $\langle \bar{e}, \bar{r} \rangle$ be a retraction pair from T(U,U) to U. Let δ : Hom(U,U) $\to$ Hom(U,U) be defined by:

ordinary [illegible] if [illegible] right of a category

$$\delta(f) = \bar{e} \circ T(f^{op}, f) \circ \bar{r}.$$

Then δ is ω-continuous; furthermore δ preserves idempotents as, if h is one $\delta(h) \circ \delta(h) = (\bar{e} \circ T(h^{op},h) \circ \bar{r}) \circ (\bar{e} \circ T(h^{op},h) \circ \bar{r}) = \bar{e} \circ T((h \circ h)^{op}, (h \circ h)) \circ \bar{r} = \delta(h)$.

The idea behind the definition of δ is that for any idempotent h, if $\langle e,r \rangle$ splits h then $\tau(e,r)$ splits $\delta(h)$ (as may easily be verified). That is we induce δ from τ and will make use of the fact that it is defined on the <u>whole</u> ω-cppo Hom(U,U) and not just on the substructure of the idempotents.

Now/

Now let $a: U \to U$ be $Y\delta = \bigsqcup_{n \geq 0} \delta^n(\perp_{U,U})$. Then each $\delta^n(\perp_{U,U})$ is an idempotent as $\perp_{U,U}$ is and δ preserves idempotents. So we can suppose that a splits, as $A \xrightarrow{e} U \xrightarrow{r} A$, say. Note that $\tau^n(\perp_{\perp,U}, \perp_{U,\perp})$ splits $\delta^n(\perp_{U,U})$ $(n \geq 0)$ and one easily shows that $\langle e,r \rangle$ is equivalent to $Y\delta$ in the large ω-cpo considered above.

<u>Theorem 6</u> $\alpha: FA \to A$ is the initial solution of (4) (where $\alpha =_{\text{def}} r \circ \bar{e} \circ T(r,e)$).

<u>Proof</u> First we show that α is an isomorphism with $\alpha^{-1} = T(e,r) \circ \bar{r} \circ e$. We have: $(T(e,r) \circ \bar{r} \circ e) \circ (r \circ \bar{e} \circ T(r,e)) = T(e,r) \circ (\bar{r} \circ a \circ \bar{e}) \circ T(r,e)$ $= T(e,r) \circ (\bar{r} \circ \bar{e} \circ T(a,a) \circ \bar{r} \circ \bar{e}) \circ T(r,e) = T(r \circ a \circ e, r \circ a \circ e) = T(id_A, id_A) =$ $id_{T(A,A)}$ and also $(r \circ \bar{e} \circ T(r,e)) \circ (T(e,r) \circ \bar{r} \circ e) = r \circ \bar{e} \circ T(a,a) \circ \bar{r} \circ e =$ $r \circ a \circ e = id_A$.

Now let $\alpha': FA' \to A'$ be any F-algebra.

$$\begin{tikzcd}
FA \arrow[r, "\alpha"] \arrow[d, "T(h_1,h_0)"] & A \arrow[d, "h_0"] \\
FA' \arrow[r, "\alpha'"] \arrow[u, bend left, "T(h_0,h_1)"] & A' \arrow[u, bend right, "h_1"']
\end{tikzcd}$$

Define $h_0: A \to A'$, $h_1: A' \to A$ in K as the least fixed-points of the simultaneous equations:

$$\begin{cases} h_0 = \alpha' \circ T(h_1,h_0) \circ \alpha^{-1} \\ h_1 = \alpha \circ T(h_0,h_1) \circ \alpha'^R \end{cases}$$

We will show that h_0 is an embedding with right adjoint h_1 and then that h_0 is the unique homomorphism, $h: (A,\alpha) \to (A',\alpha')$ of F-algebras.

First $h_1 \circ h_0 \sqsubseteq id_A$ as will be shown by simultaneous computation induction (see [Sto]) on (the functional in the definition of) h_0 and h_1. Certainly $\perp \circ \perp = \perp \sqsubseteq id_A$. Assume $g_1 \circ g_0 \sqsubseteq id_A$. Then

$(\alpha \circ T(g_0,g_1) \circ \alpha'^R) \circ (\alpha' \circ T(g_1,g_0) \circ \alpha^{-1}) = \alpha \circ T(g_1 \circ g_0, g_1 \circ g_0) \circ \alpha^{-1}$

$\sqsubseteq \alpha \circ T(id_A, id_A) \circ \alpha^{-1}$ (by induction assumption) $= id_{T(A,A)}$.

Similarly $h_0 \circ h_1 \sqsubseteq id_{A'}$. To see that $id_A \sqsubseteq h_1 \circ h_0$ we prove that $a \sqsubseteq e \circ h_1 \circ h_0 \circ r$ by computation induction on δ. Then it will follow that $h_1 \circ h_0 = r \circ e \circ h_1 \circ h_0 \circ r \circ e \sqsupseteq r \circ a \circ e = id_A$. Now certainly $\perp \sqsubseteq e \circ h_1 \circ h_0 \circ r$. Before trying the induction step we first note that $e \circ \alpha = \bar{e} \circ T(r,e)$ and $\alpha^{-1} \circ r = T(e,r) \circ \bar{r}$ as in the first case:

$e \circ \alpha = e \circ r \circ \bar{e} \circ T(r,e) = a \circ \bar{e} \circ T(r,e) = \bar{e} \circ T(a,a) \circ T(r,e)$ (by induction of a) $= \bar{e} \circ T(r \circ a, a \circ e) = \bar{e} \circ T(r,e)$ and the second assertion is proved similarly.

Now let us suppose that $x \sqsubseteq e \circ h_1 \circ h_0 \circ r$.

Then $e \circ h_1 \circ h_0 \circ r = e \circ \alpha \circ T(h_0,h_1) \circ \alpha'^R \circ \alpha' \circ T(h_1,h_0) \circ \alpha^{-1} \circ r$

$= \bar{e} \circ T(r,e) \circ T(h_1 \circ h_0, h_1 \circ h_0) \circ T(e,r) \circ \bar{r}$ (by the above)

$= \bar{e} \circ T(e \circ h_1 \circ h_0 \circ r, e \circ h_1 \circ h_0 \circ r) \circ \bar{r}$

$\sqsupseteq \bar{e} \circ T(x,x) \circ \bar{r}$ (by induction assumption).

Thus/

Thus we now know that h_0 is an embedding with right adjoint h_1.

It follows immediately from the definition of h_0 that it is an F-homomorphism as: $h_0 \circ \alpha = \alpha' \circ T(h_1, h_0) \circ \alpha^{-1} \circ \alpha = \alpha' \circ T(h_0^R, h_0) = \alpha' \circ F(h_0)$. For uniqueness, suppose h is any other F-homomorphism from (A, α) to (A', α'). Then:

$$\begin{aligned} h &= (h \circ \alpha) \circ \alpha^{-1} = \alpha' \circ F(h) \circ \alpha^{-1} \\ &= \alpha' \circ T(h^R, h) \circ \alpha^{-1} \end{aligned}$$

and now we also have:

$$\begin{aligned} h^R &= (\alpha' \circ T(h^R, h) \circ \alpha^{-1})^R \\ &= \alpha \circ T(h, h^R) \circ \alpha'^R \end{aligned}$$

showing that h, h^R satisfy the above simultaneous equations and so as h_0, h_1 are the least such solution we have $h \sqsupseteq h_0$ and $h^R \sqsupseteq h_1$. Then by lemma 3 we have $h = h_0$, showing uniqueness.$\square$

5. O-Monoids

In the paper "Data Types as Lattices" [Sco 2] we find a treatment of recursive domain specifications using the function space $M_S = (P\omega \to P\omega)$ of all continuous functions over $P\omega$. Each idempotent, a in M_S (called a retract in [Sco 2]) specifies a continuous lattice, viz:

$$\mathrm{Dom}(a) =_{\mathrm{def}} \{x \mid a(x) = x\}$$

with the order inherited from $P\omega$. Next Scott shows how several functors of interest can be represented by continuous functions over $P\omega$. For example for exponentiation he defines $\multimap : P\omega \times P\omega \to P\omega$ and proves that:

$$(9)\quad \mathrm{Dom}(a) \to \mathrm{Dom}(b) \cong \mathrm{Dom}(a \multimap b)$$

where a,b are idempotents, and notes a homomorphism property of $\multimap$ which holds for all elements of M_S:

$$(f \multimap g) \circ (f' \multimap g') = (f' \circ f) \multimap (g \circ g').$$

This property makes it easy to show that $\multimap$ preserves idempotents. Then for equations of the form (4) we can find a continuous map $\delta: M_S \to M_S$, such that $F(\mathrm{Dom}(a)) \cong \mathrm{Dom}\,\delta(a)$, which preserves idempotents. Then $Y\delta$ is an idempotent too and we see that:

$$F(\mathrm{Dom}\ Y\delta) \cong \mathrm{Dom}(\delta(Y\delta)) = \mathrm{Dom}(Y\delta)$$

and we have a solution to (4). An analogous theory for $\mathbb{T}^\omega$ was given in [Plo 2] and the idea is quite general.

In this section we explain these results in terms of a relation between monoids and categories. Thus we regard M_S as the monoid of continuous functions over $P\omega$ which induces a category $\mathrm{Cat}(M_S)$. Then Dom becomes an equivalence of categories, $\mathrm{Dom}: \mathrm{Cat}(M_S) \to \underline{\underline{\mathrm{CL}}}$. This representation of categories by monoids is studied in Theorem 7 below. Then homomorphisms such as $\multimap : M_S^{op} \times M_S \to M_S$ induce functors $\mathrm{Cat}(\multimap): \mathrm{Cat}(M_S)^{op} \times \mathrm{Cat}(M_S) \to \mathrm{Cat}(M_S)$ and the isomorphisms as in equation (9) above form a natural isomorphism of functors:

$$(10)\quad \eta : (\to) \circ (\mathrm{Dom}^{op} \times \mathrm{Dom}) \cong \mathrm{Dom} \circ \mathrm{Cat}(\multimap)$$

This representation of functors by homomorphisms is studied in Theorem 8 below. Note that it turns out that every functor can be so represented which shows that the existence of homomorphisms like $\multimap$ is no accident. After this we can show, using Theorem 6, that solutions to (4) constructed along the lines indicated are indeed initial.

Monoids/

Monoids, such as M_S, are in fact effectively given, in that $P\omega \to P\omega$ is effectively given and composition and the identity are computable. This allows us to consider effectively given categories and locally computable functors: Theorem 9 shows that homomorphisms like ⊶ for locally computable functors can always be taken to be computable. Then we can show that the initial algebras are effectively given which means that the semantics of programming languages using these effectively given initial algebras will be computable.

Definition 17 Let $(M,\cdot,e)$ be a monoid (possibly large). Then Cat(M) is the category with objects, $Ob(M) = \{a \in M \mid a\cdot a = a\}$ and with morphisms, $Hom(a,b) = \{\langle a,f,b\rangle \mid f \in M \text{ and } f = b\cdot f\cdot a\}$; composition is defined by: $\langle b,g,c\rangle \circ \langle a,f,b\rangle = \langle a,g\cdot f,c\rangle$.

One sees that morphism composition is well-defined that $\langle a,a,a\rangle: a \to a$ is the identity and that composition is associative showing that Cat(M) is indeed a category. When clear from the context we will often drop a and b and just write f instead of $\langle a,f,b\rangle$. Note that e_M is a universal element of Cat(M) as any idempotent $a: e \to e$ splits as $a \xrightarrow{a} e \xrightarrow{a} a$ and any object a, being an idempotent in the monoid, splits the same way. Note too that $Cat(M^{op}) = Cat(M)^{op}$ and $Cat(M \times M') = Cat(M) \times Cat(M')$ for any monoids M,M' (ignoring a trivial isomorphism in the second case).

Definition 18 If M is a monoid and Dom: Cat(M) $\to$ K is an equivalence of categories then we say (M,Dom) represents K. Note that if (M,Dom) represents K then (M^{op},Dom^{op}) represents K^{op}; further if (M',Dom') represents K' then $(M \times M', Dom \times Dom')$ represents $K \times K'$.

Definition 19 Let K be a category and let U be an object of K. Then Mon(K,U) is the monoid, $Hom_K(U,U)$ of morphisms from U to U with composition inherited from K.

Theorem 7 1. A category K is representable iff it has a universal object. Indeed if U is a universal object for K then there is a representation (Mon(K,U),Dom) where Dom is defined by choosing a splitting retraction pair $Dom(a) \xrightarrow{e_a} U \xrightarrow{r_a} Dom(a)$ for each idempotent $a: U \to U$ and putting $Dom(\langle a,f,b\rangle) = r_b \circ f \circ e_a$.

2. For any monoid, M, $M = Mon(Cat(M),e_M)$.

Proof 1./

Proof 1. If (M,Dom) represents K then as we have seen that e_M is a universal element of $\mathrm{Cat}(M)$ and as Dom is an equivalence of categories, $\mathrm{Dom}(e_M)$ must be a universal element of K.

Conversely let U be a universal element of a category K and define Dom as in the statement of the theorem. We check Dom is indeed a functor. For the identity $a: a \to a$, $\mathrm{Dom}(a) = r_a \circ a \circ e_a = \mathrm{id}_{\mathrm{Dom}(a)}$ and for morphisms $a \xrightarrow{f} b \xrightarrow{g} c$, $\mathrm{Dom}(g) \circ \mathrm{Dom}(f) = r_c \circ g \circ e_b \circ r_b \circ f \circ e_a = r_c \circ (g \circ f) \circ e_a$ (as $b \circ f = f$) $= \mathrm{Dom}(g \circ f)$.

To show Dom is an equivalence of categories we show it is full, faithful and dense. For suppose we have a morphism $f: \mathrm{Dom}(a) \to \mathrm{Dom}(b)$. Then $(e_b \circ f \circ r_a): a \to b$ as $b \circ (e_b \circ f \circ r_a) \circ a = f$ and, further, $\mathrm{Dom}(e_b \circ f \circ r_a) = r_b \circ (e_b \circ f \circ r_a) \circ e_a = f$, showing fullness. Next suppose $\mathrm{Dom}(f) = \mathrm{Dom}(g)$ where $f: a \to b$ and $g: a \to b$. Then $f = b \circ f \circ a = e_b \circ (r_b \circ f \circ e_a) \circ r_a = e_b \circ \mathrm{Dom}(f) \circ r_a = e_b \circ \mathrm{Dom}(g) \circ r_a = g$ showing faithfulness. Finally let X be an object in K. Now, letting $X \xrightarrow{\bar{e}} U \xrightarrow{\bar{r}} X$ be a retraction pair, we claim that $\bar{r} \circ e_{\bar{e} \circ \bar{r}}: \mathrm{Dom}(\bar{e} \circ \bar{r}) \to X$ is an isomorphism with inverse $r_{\bar{e} \circ \bar{r}} \circ \bar{e}$. For we calculate: $(\bar{r} \circ e_{\bar{e} \circ \bar{r}}) \circ (r_{\bar{e} \circ \bar{r}} \circ \bar{e}) = \bar{r} \circ (\bar{e} \circ \bar{r}) \circ \bar{e} = \mathrm{id}_X$ and also $(r_{\bar{e} \circ \bar{r}} \circ \bar{e}) \circ (\bar{r} \circ e_{\bar{e} \circ \bar{r}}) = r_{\bar{e} \circ \bar{r}} \circ (e_{\bar{e} \circ \bar{r}} \circ r_{\bar{e} \circ \bar{r}}) \circ e_{\bar{e} \circ \bar{r}} = \mathrm{id}_{\mathrm{Dom}(\bar{e} \circ \bar{r})}$.

2. Under the identification convention on morphisms in $\mathrm{Cat}(M)$ we even have $M = \mathrm{Mon}(\mathrm{Cat}(M), e_M)$. ☐

So, for example, theorem 7.1 tells us that (M_S, Dom) as defined above gives a representation of $\underline{\underline{CL}}$ (taking e_a, r_a to be the restrictions of a to its domain and codomain, respectively). Part 2 of Theorem 7 tells us that all monoids can be recovered from categories with a universal element. Indeed Theorem 7.2 can be used to show that the category of large monoids and weak monoid homomorphisms (see below) is (essentially) equivalent to the category of categories with a universal element and functors between them; we shall not formulate a precise statement here. However we still maintain the asymmetric view where the simple structures (monoids) are used to represent ~~the~~ complex ones (categories).

Everything we have said so far extends to $\underline{\underline{O}}$-categories.

Definition 20. An $\underline{\underline{O}}$-monoid is a structure $\langle M, \sqsubseteq, \cdot, e \rangle$ where $\langle M, \sqsubseteq \rangle$ is a partial order with lubs of increasing ω-chains and $\langle M, \cdot, e \rangle$ is a monoid where the multiplication function, $\cdot : M \times M \to M$, is ω-continuous.

For example, M_S is naturally an $\underline{\underline{O}}$-monoid under the pointwise order on continuous functions. If M is an $\underline{\underline{O}}$-monoid then $\mathrm{Cat}(M)$ is an $\underline{\underline{O}}$-category under the inherited partial order on morphisms and, conversely, if K is an $\underline{\underline{O}}$-category then $\mathrm{Mon}(K,U)$ is an $\underline{\underline{O}}$-monoid under the inherited ordering, for each/

each object U. A representation (M, Dom) of an $\underline{\underline{O}}$-category K, where M is an $\underline{\underline{O}}$-monoid is said to be an $\underline{\underline{O}}$-representation iff Dom acts isomorphically on the order structures of the hom-sets (and so is locally continuous); in the case where K is an $\underline{\underline{O}}$-category Theorem 7.1 gives an $\underline{\underline{O}}$-representation (and for an $\underline{\underline{O}}$-monoid M the isomorphism of Theorem 7.1 is also an isomorphism of the order structures). The remarks on representations of dual and product categories also carry over to $\underline{\underline{O}}$-representations.

Definition 21 Let M_1, M_2 be monoids. A map $\theta : M_1 \to M_2$ is a weak homomorphism iff it preserves multiplication (is a homomorphism of the underlying semigroups).

Definition 22 Let $\theta : M_1 \to M_2$ be a weak homomorphism. Then the functor $Cat(\theta)$: $Cat(M_1) \to Cat(M_2)$ is defined by putting $Cat(\theta)(a) = \theta(a)$ on objects and $Cat(\theta)\langle a, f, b\rangle = \langle \theta(a), \theta(f), \theta(b)\rangle$ on morphisms. It is easy to check that this is a good definition of a functor.

Definition 23 Let $(M_1, Dom_1), (M_2, Dom_2)$ represent the categories K_1, K_2. Then we say that a weak homomorphism, θ, represents a functor $F: K_1 \to K_2$ iff there is a natural isomorphism, $\eta : F \circ Dom_1 \cong Dom_2 \circ Cat(\theta)$.

Note that a represented functor is determined only up to natural isomorphism, which is all that matters.

Theorem 8 Let Dom: $Cat(M) \to K$ and Dom': $Cat(M') \to K'$ be representations of the categories K and K', respectively. Then any functor $F: K \to K'$ can be represented by a weak homomorphism $\theta : M \to M'$. Indeed if we let $F(Dom(e)) \xrightarrow{\bar{e}} Dom'(e') \xrightarrow{\bar{r}} F(Dom(e))$ be a retraction pair we can take θ to be the unique map such that the following diagram commutes:

$$\begin{tikzcd}
Hom(e,e) \arrow[r, "\theta"] \arrow[d, "Dom"'] & Hom(e',e') \arrow[d, "Dom'"] \\
Hom(Dom(e), Dom(e)) \arrow[r, "Hom(\bar{r},\bar{e}) \circ F"] & Hom(Dom'(e'), Dom'(e'))
\end{tikzcd}$$

Proof In the diagram we are confusing M and Hom(e,e) and also M' and Hom(e',e'). Also we are using the restrictions of Dom, Dom' and F to the relevant hom-sets; as the restrictions of Dom and Dom' are bijections, θ is uniquely determined, and exists. To see that θ is a weak homomorphism we calculate:

Dom'/

$$\begin{aligned}\mathrm{Dom}'(\theta(f\cdot g)) &= \mathrm{Hom}(\bar{F},\bar{e})\circ F\circ \mathrm{Dom}(e,f\cdot g,e)\\ &= \bar{e}\circ F(\mathrm{Dom}(e,f,e)\circ \mathrm{Dom}(e,g,e))\circ \bar{r}\\ &= (\bar{e}\circ F(\mathrm{Dom}(e,f,e))\circ \bar{r})\circ(\bar{e}\circ F(\mathrm{Dom}(e,g,e))\circ \bar{r})\\ &= \mathrm{Dom}'(e',\theta f,e')\circ \mathrm{Dom}'(e',\theta g,e')\\ &= \mathrm{Dom}'(\theta f\cdot \theta g).\end{aligned}$$

Now let a be an object of Cat(M). Then $\mathrm{Dom}'(\theta a)$ is a retract of $\mathrm{Dom}'(e')$ as:

$$\mathrm{Dom}'(\theta a)\xrightarrow{\mathrm{Dom}'(\theta a,\theta a,e')}\mathrm{Dom}'(e')\xrightarrow{\mathrm{Dom}'(e',\theta a,\theta a)}\mathrm{Dom}'(\theta a)$$

which splits the idempotent $\mathrm{Dom}'(e',\theta a,e') = \mathrm{Hom}(\bar{F},\bar{e})\circ F(\mathrm{Dom}(e,a,e))$ and further $F\circ \mathrm{Dom}(a)$ is a retract of $\mathrm{Dom}'(e')$ as:

$$F\circ \mathrm{Dom}(a)\xrightarrow{\bar{e}\circ F(\mathrm{Dom}(a,a,e))}\mathrm{Dom}(e')\xrightarrow{F(\mathrm{Dom}(e,a,a))\circ \bar{r}}F\circ \mathrm{Dom}(a)$$

which splits the same idempotent. Now it is easy to see that if, in any category, $A_i \overset{e_i}{\to} B \overset{r_i}{\to} A_i$ $(i=0,1)$ split the same idempotent then $r_1\circ e_0\colon A_0 \to A_1$ is an isomorphism with inverse $r_0\circ e_1$. So we have an isomorphism, $\eta_a\colon F\circ \mathrm{Dom}(a) \to \mathrm{Dom}'(\mathrm{Cat}(\theta)(a))$ where:

$$\eta_a = \mathrm{Dom}'(e',\theta a,\theta a)\circ \bar{e}\circ F(\mathrm{Dom}(a,a,e))$$

We claim that the η_a's from a natural isomorphism, $\eta\colon F\circ \mathrm{Dom} \to \mathrm{Dom}'\circ \mathrm{Cat}(\theta)$. For suppose a and b are objects of Cat(M) and $f\colon a \to b$. Then:

$$\begin{aligned}\eta_b\circ((F\circ \mathrm{Dom})(a,f,b)) &= \mathrm{Dom}'(e',\theta b,\theta b)\circ \bar{e}\circ F(\mathrm{Dom}(b,b,e))\circ F(\mathrm{Dom}(a,f,b))\\ &= [\mathrm{Dom}'(e',\theta b,\theta b)\circ \bar{e}\circ F(\mathrm{Dom}(a,f,e))]\circ F(\mathrm{Dom}(e,a,a))\\ &\qquad \circ \bar{r}\circ \bar{e}\circ F(\mathrm{Dom}(a,a,e))\\ &= \mathrm{Dom}'(e',\theta b,\theta b)\circ(\bar{e}\circ F(\mathrm{Dom}(e,f,e))\circ \bar{r})\circ \bar{e}\circ F(\mathrm{Dom}(a,a,e))\\ &= \mathrm{Dom}'(e',\theta b,\theta b)\circ \mathrm{Dom}'(e',\theta f,e')\circ \bar{e}\circ F(\mathrm{Dom}(a,a,e))\\ &= \mathrm{Dom}'(e',\theta f,\theta b)\circ \bar{e}\circ F(\mathrm{Dom}(a,a,e))\\ &= \mathrm{Dom}'(\theta a,\theta f,\theta b)\circ \mathrm{Dom}'(e',\theta a,\theta a)\circ \bar{e}\circ F(\mathrm{Dom}(a,a,e))\\ &= \mathrm{Dom}'\circ \mathrm{Cat}(\theta)(a,f,b)\circ \eta_a. \square\end{aligned}$$

So, for example suppose we are interested in a functor $F\colon \underline{\underline{CL}}^{op}\times \underline{\underline{CL}} \to \underline{\underline{CL}}$. Then the restrictions of Dom and Dom' are just the identities and we have for f,g in M_S:

$$\theta(f,g) = \bar{e}\circ F(f,g)\circ \bar{r}.$$

In the case of exponentiation we can take $\bar{e}$ = Graph and $\bar{r}$ = Fun, using Theorem 1.2 (the graph theorem) of [Sco 2]. Then θ can be defined using the typed λ-calculus as:

θ/

$\theta = \lambda f: M_S. \lambda g: M_S.\mathrm{Graph} \circ (\lambda h: M_S. g \circ h \circ f) \circ \mathrm{Fun}$.

and the corresponding LAMBDA term is:

$$(\lambda f: \mathrm{Fun}, \lambda g: \mathrm{Fun}. (\lambda f. \lambda x. y(x)) \circ (\lambda h: \mathrm{Fun}. g \circ h \circ f) \circ (\lambda f: \mathrm{Fun}. f))$$

which is easily seen to be equivalent to the LAMBDA term defining $\hookrightarrow$ in [Sco 2]. We leave the reader to consider other examples such as $\oplus$ and $\otimes$.

For $\underline{\underline{O}}$-monoids, M_1, M_2 a weak homomorphism $\theta : M_1 \to M_2$ is a <u>weak $\underline{\underline{O}}$-homomorphism</u> iff it is ω-continuous on the underlying orders; then $\mathrm{Cat}(\theta)$ is locally continuous and in Theorem 8 we obtain a representation of any locally continuous functor by a weak $\underline{\underline{O}}$-homomorphism as the restrictions of Dom and Dom' will be isomorphisms of the underlying order structures.

Now we return to the considerations on fixed-points at the beginning of the section and suppose we want to solve an equation of the form (8) where $T: K^{op} \times K \to K$ is a locally continuous functor and where K is an $\underline{\underline{O}}$-category which has an $\underline{\underline{O}}$-representation $\mathrm{Dom}: \mathrm{Cat}(M) \to K$. Then by Theorem 8 we can find a weak $\underline{\underline{O}}$-homomorphism which represents T so that there is a natural isomorphism:

$$(11) \quad \eta : T \circ (\mathrm{Dom}^{op} \times \mathrm{Dom}) \cong \mathrm{Dom} \circ \mathrm{Cat}(\theta) \quad .$$

generalising equation (10), above. Of course if T is a composition of functors one can compose representing homomorphisms for them to obtain θ . Now let F and F_θ be obtained from T and $\mathrm{Cat}(\theta)$ by identifying the arguments of F^E and $\mathrm{Cat}(\theta)^E$ respectively. Then it follows from (11) that there is a natural isomorphism:

$$\nu : F \circ \mathrm{Dom}^E \cong \mathrm{Dom}^E \circ F_\theta$$

where $\nu_a = \eta_{(a,a)}$ for each a in $\mathrm{Ob}(M)$. Now define $\delta : M \to M$ by $\delta(f) = \theta(f,f)$. Then δ is ω-continuous and $\delta \restriction \mathrm{Ob}(M)$ is the object function of F_θ . Suppose M has a least element $\perp$. Then it is an ω-cppo and we have:

$$(12) \quad \overline{F} \circ (\overline{\mathrm{Dom}}^E(Y\delta)) \xrightarrow{\nu_{Y\delta}} \mathrm{Dom}^E(F_\theta(Y\delta)) = \mathrm{Dom}^E(\delta(Y\delta)) = \mathrm{Dom}^E(Y\delta)$$

giving a solution to (4) along the lines indicated at the beginning of this section. We now check it is initial when multiplication in M is left-strict. For then we see that $\mathrm{Cat}(M)$ is an $\underline{\underline{O}}$-category with universal object e which satisfies the conditions of Theorem 1 and we will apply Theorem 6 to $\mathrm{Cat}(M)$,e and $\mathrm{Cat}(\theta)$. Now $\mathrm{Cat}(\theta)(e,e) \xrightarrow{\overline{e}} e \xrightarrow{\overline{F}} \mathrm{Cat}(\theta)(e,e)$ is a retraction pair where $\overline{e} = \theta(e,e)$ and $\overline{F} = \theta(e,e)$ and the map $\delta : \mathrm{Hom}(e,e) \to \mathrm{Hom}(e,e)$ is given by: $\delta(f) = \overline{e} \circ \mathrm{Cat}(\theta)(f^{op},f) \circ \overline{F} = \theta(e,e) \circ \theta(f,f) \circ \theta(e,e) = \theta(f,f)$ which agrees with/

with the above definition of δ. Now the idempotent, $Y\delta$, in $Hom(e,e)$ splits as: $Y\delta \overset{Y\delta}{\to} e \overset{Y\delta}{\to} Y\delta$ and we apply Theorem 6 to see that: $\alpha: F_\theta(Y\delta) \to Y\delta$ is the initial F_θ-algebra where $\alpha =_{def} r \circ \bar{e} \circ Cat(\theta)(Y\delta, Y\delta) = Y\delta \circ \theta(e,e) \circ Cat(\delta)(Y\delta, Y\delta) = Y\delta \circ (\theta(e,e) \cdot \theta(Y\delta, Y\delta)) = Y\delta \circ \theta(Y\delta, Y\delta) = Y\delta \circ \delta(Y\delta) = Y\delta \circ Y\delta = Y\delta = id_{Y\delta}$. It then follows immediately from the following easily proved lemma that $F(Dom^E(Y\delta)) \overset{\nu_{Y\delta}}{\to} Dom^E(F_\theta(Y\delta)) \xrightarrow{Dom^E(\alpha)} Dom^E(Y\delta)$ is the initial F-algebra and this is the same algebra as that given by (12) as $Dom^E(\alpha) = Dom^E(id_{Y\delta}) = id_{Dom(Y\delta)}$.

<u>Lemma 9</u> Let Dom: $K \to L$ be an equivalence of categories and let $\nu: G \circ Dom \cong Dom \circ F$ be an equivalence of functors where $F: K \to K$ and $G: L \to L$. Then if $\alpha: FA \to A$ is the initial F-algebra, the initial G-algebra, the initial G-algebra is $Dom(\alpha) \circ \nu_A: G(Dom\ A) \to Dom\ A$.

Let M be an $\underline{\underline{O}}$-monoid, with a least element, where multiplication is left-strict. Then we can consider an order on idempotents corresponding to that defined on subobjects in the previous section: $a \leq b =_{def} a = a \circ b \circ a \sqsubseteq b$. Any object a gives a subobject $a \overset{a}{\to} e \overset{a}{\to} a$ of e in Cat(M) and we see from lemma 8 that $a \leq b$ iff $\langle a,a\rangle \leq \langle b,b\rangle$. Scott defined a relation $\lll$ on $Ob(M_S)$ in [Sco2] where: $a \lll b$ iff $a = a \circ b = b \circ a$. However the central concern seems to be with the relation $\lll'$ where: $a \lll' b$ iff $a \lll b$ and $a \sqsubseteq b$, which leads to the annoying difficulty that $\perp$ is not the least element under $\lll'$. We view our $\leq$ as a technical improvement. If we order Ob(M) by $\leq$ we obtain an ω-cppo (as can be seen from the remarks on $\leq$ at the end of the last section). Any $\theta: M^{op} \times M \to M$ then induces an ω-continuous map $\tilde{\theta}: Ob(M) \times Ob(M) \to Ob(M)$ where $\tilde{\theta}(a,a) = \theta(a,a)$ and then, as usual, $Y\delta$ gives a solution to (4) where $\delta: Ob(M) \to Ob(M)$ is obtained as $\delta(a) = \theta(a,a)$. As remarked before this approach does not seem to give rise to a demonstration of initiality.

The representation of categories by monoids allows us to obtain a handle on computability for $\underline{\underline{O}}$-categories with a universal element and locally-continuous functors over them. First we stipulate that an $\underline{\underline{O}}$-monoid $\langle M, \sqsubseteq, \cdot, e\rangle$ is effectively given iff $\langle M, \sqsubseteq\rangle$ is effectively given (say in the sense of [Smy 1]) and multiplication and the identity element are computable. For example M_S is effectively given in this sense.

<u>Definition 24</u> Let Dom: $Cat(M) \to K$ be an $\underline{\underline{O}}$-representation where M is effectively given. Then K is <u>effectively given</u> by (M,Dom); an object X of K is <u>effectively given</u> by (a,α) if a is a computable element of M and $\alpha: Dom(a) \to X$ is an isomorphism; if objects X,Y are effectively given by $(a,\alpha),(b,\beta)$ respectively then $f: X \to Y$ is <u>computable</u> iff there is a computable $g: a \to b$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Dom}(a) & \xrightarrow{\mathrm{Dom}(g)} & \mathrm{Dom}(b) \\ \alpha \downarrow & & \downarrow \beta \\ X & \xrightarrow{f} & Y \end{array}$$

One easily checks that every identity morphism is computable and that the composition of computable morphisms is computable. If a,b are computable elements of an effectively given $\underline{\underline{O}}$-monoid, M, then Hom(a,b) is also effectively given as it is a retract of M = Hom(e,e) as:

$$\mathrm{Hom}(a,b) \xrightarrow{\mathrm{Hom}((a,a,a),(b,b,e))} \mathrm{Hom}(e,e) \xrightarrow{\mathrm{Hom}((a,a,e),(e,b,b))} \mathrm{Hom}(a,b)$$

and this splits the computable idempotent Hom((e,a,e),(e,b,e)): Hom(e,e) -> Hom(e,e). So if we have a representation Dom: Cat(M) -> K and (a,α), (b,β) represent X,Y respectively then Hom(X,Y) is effectively given as we have an isomorphism:

$$\mathrm{Hom}(a,b) \xrightarrow{\mathrm{Dom}} \mathrm{Hom}(\mathrm{Dom}\ a, \mathrm{Dom}\ b) \xrightarrow{\mathrm{Hom}(\alpha^{-1},\beta)} \mathrm{Hom}(X,Y)$$

Then f: X -> Y is computable according to the definition iff it is computable as an element of Hom(X,Y). Finally we note that if K is effectively given by (M,Dom) then K^{op} is effectively given by (M^{op}, Dom^{op}) and if, further, K' is effectively given by (M',Dom') then $K \times K'$ is effectively given by $(M \times M', Dom \times Dom')$.

For example $\underline{\underline{CL}}$ is effectively given by (M_S, Dom) and the above definitions agree with those in [Sco 2],[Smy 1].

<u>Definition 25</u> Let K,K' be $\underline{\underline{O}}$-categories effectively given by (M,Dom), (M',Dom') respectively. A functor F: K -> K' is <u>locally computable</u> iff (i) for every computable idempotent, a, in M there is another, a_F, in M' such that for every isomorphism α : Dom(a) $\cong$ X there is an isomorphism α_F: Dom'$(a_F) \cong$ FX and (ii) whenever objects X,Y of K are effectively given by $(a,\alpha),(b,\beta)$ respectively there is a computable map $\xi_{a,b}$: Hom(a,b) -> Hom(a_F,b_F) such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(a,b) & \xrightarrow{\xi_{a,b}} & \mathrm{Hom}(a_F,b_F) \\ \mathrm{Hom}(\alpha^{-1},\beta)\circ\mathrm{Dom} \downarrow & & \downarrow \mathrm{Hom}(\alpha_F^{-1},\beta_F)\circ\mathrm{Dom}' \\ \mathrm{Hom}(X,Y) & \xrightarrow{F} & \mathrm{Hom}(FX,FY) \end{array}$$

Roughly speaking, this definition says that F sends effectively given objects to effectively given objects and acts computably on the effectively given hom-sets. Perhaps we should also have insisted that an index for e_F was effectively obtainable from one for a and similarly for $\xi_{a,b}$, but the existence of such effective functions on indices follows from the definition as can easily be seen from the proof of the next theorem which is an effective addendum to Theorem 8.

In practice, however, we work with the much simpler characterisation given by Theorem 9.2; this could have been adopted as the definition, but would not convey so well the idea of "local" computability (to be contrasted with a more global notion in the next section).

<u>Theorem 9</u> Let K,K' be $\underline{\underline{O}}$-categories effectively given by (M,Dom),(M',Dom) respectively. Then the following conditions are equivalent on a functor F: K -> K':

1) F is locally computable.
2) F can be represented by a computable weak $\underline{\underline{O}}$-homomorphism, θ : M -> M'.
3) There is a retraction pair, $F(\mathrm{Dom}\ e) \xrightarrow{\bar{e}} \mathrm{Dom}\ e' \xrightarrow{\bar{r}} F(\mathrm{Dom}\ e')$ such that the unique map θ : M -> M' which makes the diagram of Theorem 8 commute is computable.

<u>Proof</u> 1 => 3. Consider the diagram.

$$\begin{array}{ccccc} \mathrm{Hom}(e,e) & \xrightarrow{\ \xi_{e,e}\ } & \mathrm{Hom}(e_F,e_F) & \xrightarrow{\ \mathrm{Hom}((e',e_F,e_F),(e_F,e_F,e'))\ } & \mathrm{Hom}(e',e') \\ \downarrow{\scriptstyle \mathrm{Dom}} & & \downarrow{\scriptstyle \mathrm{Hom}(\alpha_F^{-1},\alpha_F)\circ \mathrm{Dom}'} & & \downarrow{\scriptstyle \mathrm{Dom}'} \\ \mathrm{Hom}(U,U) & \xrightarrow{\ F\ } & \mathrm{Hom}(FU,FU) & \xrightarrow{\ \mathrm{Hom}(\bar{r},\bar{e})\ } & \mathrm{Hom}(U',U') \end{array}$$

where $U =_{def} \mathrm{Dom}(e)$, $U' =_{def} \mathrm{Dom}'(e')$, $\alpha_F =_{def} (id_U)_F$ and the retraction pair $(\bar{e},\bar{r})$ is given by:

$$FU \xrightarrow{\alpha_F^{-1}} \mathrm{Dom}'(e_F) \xrightarrow{\mathrm{Dom}'(e_F,e_F,e')} \mathrm{Dom}'(e') \xrightarrow{\mathrm{Dom}'(e',e_F,e_F)} \mathrm{Dom}'(e_F) \xrightarrow{\alpha_F} FU.$$

The first square commutes because of the local computability of F. For the second given any $f\colon e_F \to e_F$ we have:

$$(\mathrm{Hom}(\bar{r},\bar{e}), \circ \mathrm{Hom}(\alpha_F^{-1},\alpha_F) \circ \mathrm{Dom}')(e_F,f,e_F) = \mathrm{Dom}'(e_F,e_F,e') \circ \alpha_F^{-1} \circ \alpha_F \circ \mathrm{Dom}'(e_F,f,e_F) \circ \alpha_F^{-1} \circ \alpha_F \circ \mathrm{Dom}'(e',e_F,e_F)$$

$$= \mathrm{Dom}'(e',f,e')$$

$$= (\mathrm{Dom}' \circ \mathrm{Hom}((e',e_F,e_F),(e_F,e_F,e')))(e_F,f,e_F).$$

So/

So we can take $\hat{\theta} = \mathrm{Hom}(e_F, e_F) \circ \xi_{c,e}$ which is computable.

3 => 2. Immediate from Theorem 8.

2 => 1. For each computable a in M put $a_F = \theta a$, which is computable as θ and a are. For $\alpha : \mathrm{Dom}(a) \cong X$ put $\alpha_F = (F\alpha) \circ \eta_a^{-1}$ where η is the natural isomorphism $\eta : F \circ \mathrm{Dom} \cong \mathrm{Dom}' \circ \mathrm{Cat}(\theta)$. Then α_F is certainly an isomorphism. Finally for any objects X, Y of K effectively given by $(a,\alpha), (b,\beta)$ respectively, define $\xi_{a,b}$ as the composition:

$$\mathrm{Hom}(a,b) \xrightarrow{\mathrm{Hom}((e,a,a),(b,b,e))} \mathrm{Hom}(e,e) \xrightarrow{\theta} \mathrm{Hom}(e',e') \xrightarrow{\mathrm{Hom}((a_F,a_F,e'),(e',b_F,b_F))} \mathrm{Hom}(a_F,b_F)$$

which is computable being a composition of computable maps. Now we have for any f: a -> b:

$$\begin{aligned}
[\mathrm{Hom}(\alpha_F^{-1},\beta_F) \circ \mathrm{Dom}' \circ \xi_{a,b}](f) &= \beta_F \circ \mathrm{Dom}'((e',b_F,b_F) \circ \theta((b,b,e) \circ f \circ (e,a,a)) \circ (a_F,a_F,e')) \circ \alpha_F \\
&= \beta_F \circ \mathrm{Dom}'((e',b_F,b_F) \circ \theta(e,f,e) \circ (a_F,a_F,e')) \circ \alpha_F \\
&= F\beta \circ \eta_b^{-1} \circ \mathrm{Dom}'(\mathrm{Cat}(\theta)(f)) \circ \eta_a \circ F\alpha^{-1} \\
&= F(\beta \circ \mathrm{Dom}(f) \circ \alpha^{-1}) \\
&= (F \circ \mathrm{Hom}(\alpha^{-1},\beta) \circ \mathrm{Dom})(f). \;\square
\end{aligned}$$

Using the second characterisation of locally computable functors one sees that the constant and projection functors are locally computable and the locally computable functors are closed under composition, duals, and tupling. Also, for example, the LAMBDA definition of $\multimap$ makes it clear that exponentiation is locally computable on $\underline{\underline{CL}}$.

Finally we consider initial fixed-points. Let K be an $\underline{\underline{O}}$-category effectively given by (M, Dom) and let T: $K^{op} \times K \to K$ be locally computable. Then by Theorem 9, there is a computable $\theta : M^{op} \times M \to M$ which represents T. Now suppose M has a least element, $\perp$, (which will be computable) and multiplication is left strict. Then, with the notation of (11), (12) above, $Y\delta$ is computable and so the initial F-algebra, $\nu_{Y\delta} : F(\mathrm{Dom}^E(Y\delta)) \to \mathrm{Dom}^E(Y\delta)$ is effectively given in that $\mathrm{Dom}^E(Y\delta)$ is effectively given by $(Y\delta, \mathrm{id})$, $F(\mathrm{Dom}^E(Y\delta))$ is effectively given by $(Y\delta, \nu_{Y\delta}^{-1})$ and $\nu_{Y\delta}$ is computable. One also sees that $\nu_{Y\delta}^{-1}$ is computable. It can be shown, using the ideas in the proof of Theorem 6, that $(\mathrm{Dom}^E(Y\delta), \nu_{Y\delta})$ is even effectively initial in that if (A,α) is any effectively given F-algebra the unique homomorphism h: $(\mathrm{Dom}^E(Y\delta), \nu_{Y\delta}) \to (A,\alpha)$ is computable (as a morphism h: $\mathrm{Dom}^E(Y\delta) \to A$), (and an index for it is effectively obtainable).

6. Effectiveness in Categories

In this section we briefly discuss two topics which are of at least marginal relevance to the solution of domain equations. Their main significance is in connection with the effectiveness of solutions; were we attempting a thorough treatment of effectiveness, they would assume a much larger place than they do here.

(A) Algebroidal categories. These are the same as what we previously called "algebraic categories" [Smy 3] or "finitary categories" [Plo 3]. It has been brought to our attention that closely related notions have been discussed quite extensively in the literature of category theory, and this is what has prompted the latest change in nomenclature. Our algebroidal categories are essentially the "strongly ω-algebroidal categories" in (a slight extension of) the terminology of [Ban].

Definition 26 An object A of a category K is finite in K provided that, for any ω-cochain $\Delta = \langle A_n, f_n \rangle_{n\in\omega}$ in K with colimit $\mu : \Delta \to V$, the following holds: for any arrow $v: A \to V$ and for any sufficiently large n, there is a unique arrow $u: A \to A_n$ such that $v = \mu_n \circ u$. We say that K is algebroidal provided (1) K has an initial object and at most countably many finite objects, (2) every object of K is a colimit of an ω-cochain of finite objects, and (3) every ω-cochain of finite objects has a colimit in K.

Notation. If K is algebroidal, we denote by K_0 the full subcategory of K with objects the finite objects of K.

The principal examples of interest to us are $\underline{\underline{SFP}}^E$ (the category of $\underline{\underline{SFP}}$ objects and embeddings [Plo 1]) and various of its subcategories, for example the category of bounded-complete ω-algebraic ω-cppo's and embeddings. The finite objects are in each case the finite domains.

Theorem 10 Every algebroidal category admits ω-colimits.

Proof See [Smy 3].□

Theorem 11 Let K be an algebroidal category, and let L be an ω-category. Any functor F_0 from K_0 into L extends uniquely (up to natural isomorphism of functors) to an ω-functor from K into L.

Outline of proof For each non-finite object D of K choose a particular colimiting cone $\mu_D: \Delta_D \to D$, with Δ_D an ω-cochain in K_0; and for each ω-cochain Δ in L choose a particular colimiting cone $\mu_\Delta: \Delta \to D_\Delta$ in L. The extension of F_0 to all objects of K is immediate (via the chosen colimiting cones in K,L). To define the extension F of F_0 to arrows, consider first arrows $v: A \to V$ where A is finite and V non-finite. Since A is finite,

v/

v factorizes as $v = (\mu_V)_n \circ u$. Then we put $Fv = (\mu_{F_0(\Delta_V)})_n \circ F_0 u$. Next, for arrows $h: V \to W$, where V is non-finite, define Fh as the mediating map from the universal (colimiting) cone $F(\mu_V)$ $(= \mu_{F_0(\Delta_V)})$ to $F(h \circ \mu_V)$. Of course, it has to be checked that F so defined preserves composition of morphisms, and so is a functor (this is non-trivial).

Now suppose that $F, F': K \to L$ are two ω-continuous functions which extend F_0. For each object V of K we have a canonical isomorphism $\tau_V : FV \to F'V$, namely the mediating arrow from $F(\mu_V)$ to $F'(\mu_V)$. Naturality of τ means that, for $h: V \to W$, $F'h \circ \tau_V = \tau_W \circ Fh$; this is established by showing that $\tau_W \circ Fh$ mediates between the colimiting cone $F(\mu_V)$ and $F'h \circ \tau_V \circ F(\mu_V)$. $\square$

Theorem 10 yields at once that $\underline{\underline{SFP}}^{E}$ (for example) is an ω-category. Theorem 11 can be useful, at least heuristically, in setting up the definitions of appropriate ω-functors. Under these circumstances the solution of typical domain equations, via the Basic Lemma, is unproblematic. More interesting is the question of effectiveness. The following definition seems natural:

<u>Definition 27</u> Let $\langle A_n \rangle_{n\in\omega}, \langle f_n \rangle_{n\in\omega}$ be enumerations of the objects and arrows, respectively, of K_0, where K is an algebroidal category. We say that K is <u>effectively given</u>, relative to these enumerations, provided that the following predicates are recursive in the indices:

1) $A_i = A_j$; $f_i = f_j$
2) $\mathrm{dom}(f_k) = A_i$; $\mathrm{cod}(f_k) = A_i$
3) f_k is an identity
4) $f_i \circ f_j = f_k$.

This enables us to define an <u>effectively given object</u> (of K) as an object that is given as the colimit of an effective ω-cochain of finite objects, that is, as the colimit of a cochain of the form

$$A_{r(0)} \xrightarrow{f_{s(0)}} A_{r(1)} \xrightarrow{f_{s(1)}} \cdots\cdot$$

(r, s recursive). One will naturally try to define a <u>computable arrow</u>, similarly, as the colimit of an effective ω-cochain of finite arrows (that is, arrows of K_0); actually, there are some problems with this definition, which we will not consider further here. It is then possible to define a <u>computable functor</u>, roughly, as an ω-functor F for which we can effectively assign to each finite object (arrow) $A(f)$ an effective ω-cochain having $F(A)$/

$F(A)$ ($F(f)$) as colimit. A basic result, in terms of these definitions, will be that the initial fixpoint of a computable functor is computable.

Two comments on this approach:

(1) To relate this theory with the preceding analysis, we need some results connecting the "local" computability of functors introduced in Definition 22 with the "global" computability sketched here, on the lines of Theorem 3 connecting local continuity with ω-continuity.

(2) We believe the this theory can, and should, be developed into a general account of effectively given domains and computability thereon (without any restriction to a particular category as in [Smy 1]). What we have sketched so far in this section is appropriate only to categories of the form K^E (SFP, for example, is not algebroidal in the strict sense of Definition 23). Such an account was attempted, unsatisfactorily, in [Smy 2, Ch.3]. A more adequate account is under preparation.

(B) Effective domains/categories. [Kan 1] proposes that only computable items should be admitted to the domains and categories which we study - in contrast to the usual practice of first building all the continuous/countably-based items and then picking out the computable items from among these. This entails a modification of the closure properties required of the domains and categories: we now demand closure of domains w.r.t. sups only of effective ω-chains, and of categories w.r.t. effective colimits of effective ω-cochains. This approach works quite smoothly, and indeed yields a theory which is formally very close to [Smy 1] as far as concerns effective domains. In regard to the theory of effective categories (as developed by Kanda), perhaps the most striking feature of this theory is the very simple definition of computable functor (Kanda has "effective functor") in terms of indexings of hom-sets, which it permits.

Unlike approach (A), however, Kanda's theory does not pretend to give a general account of effectiveness in domains. In this theory the definitions of an effective domain and of an effective category are quite independent. In order to apply the theory, we first define a particular category of "effective domains", and then show that this category satisfies the axioms for an "effective category". The definition is ad hoc, in the sense that no general or uniform notion of effective domains is proposed: we cannot, for example, define an effective domain to be an object of an "effective/

"effective category of domains" (in contrast with approach (A)). A more technical point concerns the specification of solutions of domain equations (or other constructs defined via computable functors): although we are guaranteed an effective initial F-algebra (A,α) for any computable functor F, the construction does not provide us with any effective basis for A. But, as recent work by Kanda and Park [Kan 2] emphasizes, the enumeration of an effective basis is an essential part of the specification of an effective(ly given) domain.

We incline to the view that these problems can best be attacked by means of the ideas mentioned in (A) (finite objects in categories, etc.); but that it may be worthwhile to develop the argument in accordance also with the main idea of (B), namely that only computable items should be admitted to the field of discourse.

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