LEVITZ'S THESIS

SHORT SUMMARY:

1.) THE ORDER TYPE OF TAKEUTI'S $O(1)$ WHEN ORDERED BY $<_1$ IS THE ORDINAL $\varphi(\omega)$ IN THE NOTATION OF SCHÜTTE [7] WHEN $\varphi(x)$ IS THE FUNCTION ω^{1+x}

2) LET $O(2)$ BE ORDERED BY $<_2$

LET λ_1 DENOTE THE ORDINAL DIAGRAM 1

λ_{m+1} DENOTE THE ORDINAL DIAGRAM $(1, 1, \lambda_m)$

THEN λ_1 CORRESPONDS TO THE ORDINAL 1

λ_2 " " " " ϵ_0

λ_3 " " " " LEAST CRITICAL ϵ-NUMBER (LEAST SOLUTION OF $\epsilon_x = x$)

λ_4 " " " " LEAST STRONGLY CRITICAL ϵ-NUMBER

λ_5 " " " " VEBLEN'S $\rightarrow$ $E(0)$ CF. [11] IN BIBLIOG.

FEFERMAN'S Γ_0

† 3) LET $O(2)$ BE ORDERED BY $<_2$

THEN THE ORDINAL DIAGRAM 2 DENOTES THE ORDINAL WHICH BACHMAN [3]* DENOTES BY $\varphi_{\epsilon_{\Omega+1}}(1)$ or $\varphi_{F_1}(1)$

† TO ESTABLISH THIS ONE NEEDS CERTAIN LEMMAS WHICH PERTAIN ONLY TO BACHMAN'S HIERARCHY. THESE ARE PROVEN IN GERBER'S THESIS

The order isomorphism is contained here

Nowadays we would write $\varphi_{\epsilon_{\Omega+1}}(0)$ however in Bachmann's paper the functions of the hierarchy are only defined for arguments >0

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On The Ordinal Notations of Schütte and The Ordinal Diagrams of Takeuti

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INTRODUCTION

In [4], Gentzen proved the consistency of formal number theory by use of transfinite induction over the segment of ordinals less than ε_0, the smallest solution of the equation $\omega^x = x$. In [5] Gentzen pointed out that it is not necessary to develop this segment in the usual way by means of set theory. Instead, a system of formal expressions can be defined recursively and an order relation can be defined on them recursively so that this set of expressions has order type ε_0. These expressions can be thought of as notations for ordinals.

In connection with their own metamathematical investigations Ackermann [1], Schütte [7], and Takeuti [10] have developed such systems of ordinal notations for segments of the second number class reaching considerably beyond ε_0.

In [7] Schütte described a family of continuous increasing functions in terms of which his ordinal notations can be explained. In that same work he showed that his system is larger than the system of Ackermann.

The principal result of the present investigation is that the system O(2) of Takeuti is larger than the system of Schütte. In doing this a new characterization of a rather large segment of O(2) will be given.

Additional results are a characterization of Takeuti's O(1), and a description of Schütte's system by means of functionals.

I. SCHÜTTE'S ORDINAL NOTATIONS

1. A Description by Functionals

Let Ω denote the first uncountable ordinal. A function φ which maps the set $\{x \mid x<\Omega\}$ into itself is called a <u>normal</u> function if it is continuous and strictly increasing. In [9] it is shown that a normal function has fixed points; i.e., solutions of the equation $\varphi(x)=x$. In fact, the ordering function of these fixed points is itself a normal function. One calls the function arrived at in this way the <u>derived</u> function of φ. It is also shown that the process of taking the derived function can be iterated into the transfinite, and a family $\{\varphi_\alpha\}_{\alpha<\Omega}$ of normal functions can be obtained with the property that $\varphi_0=\varphi$ and φ_α is the ordering function of the fixed points of φ_β for all β such that $0 \leq \beta < \alpha$. The numbers in the range of φ_α are called the α-critical number of φ. df $\varphi(0)>0$, One does not, however, obtain a normal function φ_Ω in this way because $\{x \mid \varphi_\beta(x) = x \text{ all } \beta<\Omega\}$ can be shown to be empty. We will refer to the hierarchy described above as the <u>simple</u> hierarchy on φ.

In [7], Schütte extends this family by introducing Klammersymbols. A Klammersymbol is an expression

$\begin{pmatrix} a_o, & a_1, & \ldots, & a_n \\ \alpha_o, & \alpha_1, & \ldots, & \alpha_n \end{pmatrix}$ where the α_i, a_i are ordinals less than Ω and $0 \leq \alpha_o < \alpha_1 < \ldots < \alpha_n$. Equality among Klammersymbols is governed by the rule that two Klammersymbols are equal if one can be obtained from the other by the insertion or deletion of terms of the form $\begin{matrix} 0 \\ \alpha_i \end{matrix}$. To each normal function φ and to each Klammersymbol A an ordinal φA is assigned so that $\varphi\begin{pmatrix} x \\ 0 \end{pmatrix} = \varphi(x)$ and for $a_1 \neq 0$,

$\varphi\begin{pmatrix} a_o, & a_1, & \ldots, & a_n \\ 0, & \alpha_1, & \ldots, & \alpha_n \end{pmatrix}$ is the a_o-th solution of all equations

$$\varphi\begin{pmatrix} x, & a^*, & a_2, & \ldots, & a_n \\ \alpha^*, & \alpha_1, & \alpha_2, & \ldots, & \alpha_n \end{pmatrix} = x \quad \text{where } \alpha^* < \alpha_1, \quad a^* < a_1.$$

The functions $\lambda x \varphi\begin{pmatrix} x, & a_1, & \ldots, & a_n \\ \alpha_o, & \alpha_1, & \ldots, & \alpha_n \end{pmatrix}$ and $\lambda x \varphi\begin{pmatrix} 1 \\ x \end{pmatrix}$ are shown in [7] to be normal functions. The fixed points of $\lambda x \varphi\begin{pmatrix} 1 \\ x \end{pmatrix}$ are referred to by Veblen [11] as E-numbers (relative to the initial function φ). In this connection we mention that various functions can give rise to the same E-numbers; e.g., $1+\omega$, ω^x, and ε_x, where ε_x is the ordering function of the ε-numbers.[1] We denote the ~~first~~ least E-number (relative to the initial function φ) by E(o). Schütte then shows that if a_o, a_1, $\ldots$, a_n, α_1, $\ldots$, α_n are all less than E(o), then so is

[1] Cf. [7], p. 26.

$\varphi\begin{pmatrix} a_o, & a_1, & a_2, & \cdots, & a_n \\ \alpha_o, & \alpha_1, & \alpha_2, & \cdots, & \alpha_n \end{pmatrix}$.

It is possible to have $\varphi A = \varphi B$, but $A \neq B$. Schütte also defines an ordinal $\overline{\varphi}A$ in such a way that

$$\overline{\varphi}A = \overline{\varphi}B \longrightarrow A = B.$$

The functions $\lambda x \overline{\varphi}\begin{pmatrix} x, & a_1, & \cdots, & a_n \\ \alpha_o, & \alpha_1, & \cdots, & \alpha_n \end{pmatrix}$ while strictly increasing are not, however, continuous. We will make use of $\overline{\varphi}A$ in the next section.

In this section we give a new characterization of $\varphi\begin{pmatrix} a_o, & a_1, & \cdots, & a_n \\ \alpha_o, & \alpha_1, & \cdots, & \alpha_n \end{pmatrix}$. For this purpose we introduce the notion of <u>type</u>. Objects of type 0 are ordinals. Objects (functionals) of type 1 are functions from ordinals to ordinals. Functionals of type 2 are functions from objects of type j to objects of type k, where $j \leq 1$ and $k \leq 1$.

With respect to notation, we use the following conventions:

1) We use $\circ$ to denote the operation of composition of mappings

2) $\mathcal{R}\varphi$ will denote the range of φ

3) If T is a functional then T(x) denotes the image of x under the functional T. Parentheses will be omitted when there is no danger of ambiguity.

<u>Definition 1.1</u>: For $\alpha > 0$, $a \geq 0$ we define functionals

T_α^a of type 2 as follows:

$$T_\alpha^a(\varphi) = \lambda x \varphi \begin{pmatrix} x, a \\ 0, \alpha \end{pmatrix} .$$

Since $\varphi\begin{pmatrix} x, a \\ 0, \alpha \end{pmatrix}$ is defined only for normal functions, our functionals T_α^a are defined only for such φ. However in the discussion following Theorem 1.2, we shall extend the domain of T_α^a to arbitrary functions φ.

We also remark that since $\lambda x \varphi\begin{pmatrix} x, a \\ 0, \alpha \end{pmatrix}$ is again a normal function by [7, (3.2),(3.5)][2] the composition of the functionals T_α^a is well defined.

<u>LEMMA 1.1</u>: If $a_1 \neq 0$, $b \neq 0$, $\alpha_1 \neq 0$ and $\mathfrak{A} = \{y \mid$

$\varphi\begin{pmatrix} y, & a_1+b^*, & a_2, & \cdots, & a_n \\ \alpha^*, & \alpha_1, & \alpha_2, & \cdots, & \alpha_n \end{pmatrix} = y$; all $b^* < b$, $\alpha^* < \alpha_1\}$ and

$\mathfrak{B} = \{y \mid \varphi\begin{pmatrix} y, & a^*, & a_2, & \cdots, & a_n \\ \alpha^*, & \alpha_1, & \alpha_2, & \cdots, & \alpha_n \end{pmatrix} = y$; all $a^* < a_1$, $\alpha^* < \alpha_1\}$,

then $\mathfrak{A} \subseteq \mathfrak{B}$.

<u>Proof</u>. Let $y_0 \varepsilon \mathfrak{A}$, then in particular for $b^* = 0$, $\alpha^* = 0$ we get

$$\varphi\begin{pmatrix} y_0, & a_1, & a_2, & \cdots, & a_n \\ 0, & \alpha_1, & \alpha_2, & \cdots, & \alpha_n \end{pmatrix} = y_0$$

so by definition [7, (2.3)], y_0 is the y_0-th solution of the equations:

$$\varphi\begin{pmatrix} x, & a^*, & a_2, & \cdots, & a_n \\ \alpha^*, & \alpha_1, & \alpha_2, & \cdots, & \alpha_n \end{pmatrix} = x; \text{ all } a^* < a_1,\ \alpha^* < \alpha_1$$

---

[2] [7,(3.2),(3.5)] refers to the article listed in the bibliography under [7]; the symbols (3.2) and (3.5) are used in that article to denote various results.

therefore $y_o \varepsilon \mathfrak{B}$.

<u>LEMMA 1.2</u>: $\left[\lambda x \varphi \begin{pmatrix} x, a_1, a_2, \ldots, a_n \\ 0, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix}\right] \begin{pmatrix} b_o, b_1, \ldots, b_k, b \\ 0\ , \beta_1, \ldots, \beta_k, \alpha_1 \end{pmatrix} =$

$$= \varphi \begin{pmatrix} b_o, b_1, \ldots, b_k, a_1+b, a_2, \ldots, a_n \\ 0\ , \beta_1, \ldots, \beta_k, \alpha_1\ , \alpha_2, \ldots, \alpha_n \end{pmatrix} .$$

<u>Proof</u>. The proof is by transfinite induction on $\begin{pmatrix} b_o, b_1, \ldots, b_k, b \\ 0\ , \beta_1, \ldots, \beta_k, \alpha_1 \end{pmatrix}$ over the well ordering of the Klammersymbols described in [7, p.17].

<u>Case 1</u>. $b_1, b_2, \ldots, b_k, b$ are all zero. Then the desired result follows trivially from [7,(2.1)].

<u>Case 2</u>. $b_1, b_2, \ldots, b_k$ are zero and b is not zero. Let γ denote $\left[\lambda x \varphi \begin{pmatrix} x, a_1, a_2, \ldots, a_n \\ 0, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix}\right] \begin{pmatrix} b_o, b \\ 0\ , \alpha_1 \end{pmatrix}$. Then γ is the

b_o-th member of $\{y \mid \left[\lambda x \varphi \begin{pmatrix} x, a_1, a_2, \ldots, a_n \\ 0, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix}\right] \begin{pmatrix} y\ , b^* \\ \alpha^*, \alpha_1 \end{pmatrix} = y$;

all $b^* < b$, $\alpha^* < \alpha_1\}$. By induction hypothesis γ is the

b_o-th member of $\mathfrak{A} = \{y \mid \varphi \begin{pmatrix} y\ , a_1+b^*, a_2, \ldots, a_n \\ \alpha^*, \alpha_1\ , \alpha_2, \ldots, \alpha_n \end{pmatrix} = y;$

all $b^* < b$, $\alpha^* < \alpha_1\}$.

<u>Case 2.1</u>. $a_1 = 0$. Then γ is the b_o-th member of

$\{y \mid \varphi \begin{pmatrix} y\ , 0+b^*, a_2, \ldots, a_n \\ \alpha^*, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix} = y;$ all $b^* < b$, $\alpha^* < \alpha\}$ so by

definition [7,(2.3)] we get

$\gamma = \varphi\begin{pmatrix} b_o, 0+b, a_2, \ldots, a_n \\ 0\ , \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix}$. This is the desired result.

<u>Case 2.2</u>. $a_1 \neq 0$. Let $\mathfrak{B}$ denote

$\{y \mid \varphi\begin{pmatrix} y\ , a^*, a_2, \ldots, a_n \\ \alpha^*, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix} = y;$ all $a^* < a_1$, $\alpha^* < \alpha_1\}$. By Lemma 1.1 $\mathfrak{A} \subseteq \mathfrak{B}$, so $\mathfrak{A} = \mathfrak{A} \cap \mathfrak{B}$. Therefore γ is the b_o-th solution to both sets of equations:

$$\left[\begin{array}{l} \varphi\begin{pmatrix} y\ , a^*, a_2, \ldots, a_n \\ \alpha^*, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix} = y;\ \text{all } \alpha^* < \alpha_1,\ \text{all } a^* < a_1 \\ \text{and} \\ \varphi\begin{pmatrix} y, a_1+b^*, a_2, \ldots, a_n \\ \alpha^*, \alpha_1\ , \alpha_2, \ldots, \alpha_n \end{pmatrix} = y;\ \text{all } \alpha^* < \alpha_1,\ b^* < b. \end{array}\right.$$

But $\{a^* \mid a^* < a_1\} \cup \{a^* \mid a^* = a_1 + b^* \text{ for some } b^* < b\} = \{a^* \mid a^* < a_1 + b\}$ so γ is the b_o-th solution of the set of equations

$$\varphi\begin{pmatrix} y\ , a^*, a_2, \ldots, a_n \\ \alpha^*, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix} = y;\ \text{all } \alpha^* < \alpha_1,$$

$$\text{all } a^* < a_1 + b$$

so by definition [7,(2.3)] we get

$$\gamma = \varphi\begin{pmatrix} b_o, a_1+b, a_2, \ldots, a_n \\ 0\ , \alpha_1\ , \alpha_2, \ldots, \alpha_n \end{pmatrix}.$$

This is the desired result.

<u>Case 3</u>. $b_i \neq 0$ for some i, $1 \leq i \leq k$. Assume t is the smallest such i. Let δ denote

$\left[\lambda x \varphi\begin{pmatrix} x, a_1, a_2, \ldots, a_n \\ 0, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix}\right]\begin{pmatrix} b_o, b_t, \ldots, b_k, b \\ 0\ , \beta_t, \ldots, \beta_k, \alpha_1 \end{pmatrix}$; then δ is the b_o-th solution of all equations:

$$\left[\lambda x\varphi\begin{pmatrix}x,a_1,a_2,\ldots,a_n\\0,\alpha_1,\alpha_2,\ldots,\alpha_n\end{pmatrix}\right]\begin{pmatrix}y,b^*,\ldots,b_k,b\\\beta^*,\beta_t,\ldots,\beta_k,\alpha_1\end{pmatrix} = y$$

where $b^* < b_t$, $\beta^* < \beta_t$. By the induction hypothesis, δ is the b_o-th solution of all equations:

$$\varphi\begin{pmatrix}y,b^*,\ldots,b_k,a_1+b,a_2,\ldots,a_n\\\beta^*,\beta_t,\ldots,\beta_k,\alpha_1,\alpha_2,\ldots,\alpha_n\end{pmatrix} = y$$

where $b^* < b_t$, $\beta^* < \beta_t$. So the desired result follows from [7,(2.3)]. This completes the proof.

<u>THEOREM 1.1</u>: For $\alpha_1 > 0$,

$$\left(T_{\alpha_1}^{a_1} \circ T_{\alpha_2}^{a_2} \circ \ldots \circ T_{\alpha_n}^{a_n}\right)\varphi = \lambda y\varphi\begin{pmatrix}y,a_1,a_2,\ldots,a_n\\0,\alpha_1,\alpha_2,\ldots,\alpha_n\end{pmatrix}.$$

<u>Proof</u>. The proof is by transfinite induction on $\begin{pmatrix}a_1,a_2,\ldots,a_n\\\alpha_1,\alpha_2,\ldots,\alpha_n\end{pmatrix}$ over the well ordering of the Klammersymbols.

$$\left(T_{\alpha_1}^{a_1} \circ T_{\alpha_2}^{a_2} \circ \ldots \circ T_{\alpha_n}^{a_n}\right)\varphi = T_{\alpha_1}^{a_1}\left((T_{\alpha_2}^{a_2} \circ T_{\alpha_3}^{a_3} \circ \ldots \circ T_{\alpha_n}^{a_n})\varphi\right)$$

by association

$$= T_{\alpha_1}^{a_1}\left(\lambda x\varphi\begin{pmatrix}x,a_2,\ldots,a_n\\0,\alpha_2,\ldots,\alpha_n\end{pmatrix}\right)$$

by induction hypothesis

$$= \lambda y\left[\left[\lambda x\varphi\begin{pmatrix}x,a_2,\ldots,a_n\\0,\alpha_2,\ldots,\alpha_n\end{pmatrix}\right]\begin{pmatrix}y,a_1,0\\0,\alpha_1,\alpha_2\end{pmatrix}\right]$$

by Definition 1.1

$$= \lambda y\left[\varphi\begin{pmatrix} y, a_1, a_2, \ldots, a_n \\ 0, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix}\right]$$

by Lemma 1.2. This completes the proof.

Theorem 1.1 shows how to express the general Klammersymbol in terms of a certain functional $\lambda a\alpha T^a_\alpha$ of type 2, where $\lambda a\alpha T^a_\alpha$ is itself defined in Definition 1.1 by certain special Klammersymbols. $\lambda a\alpha T^a_\alpha$ is considered to be a functional of type 2 because it acts on the arguments a, α, φ of type ≤ 1 to produce a type 0 object.

The purpose of the remainder of this section is to show that $\lambda a\alpha T^a_\alpha$ can be defined directly by transfinite induction (more precisely, transfinite recursion). Thus Theorem 1.1 will provide a new <u>definition</u> of Schütte's Klammersymbols.

<u>LEMMA 1.3</u>: $\alpha > 0 \longrightarrow T_\alpha{}^1 \circ T^a_\alpha = T^{a+1}_\alpha$

<u>Proof</u>. $(T^{a+1}_\alpha \varphi)b = [\lambda x\varphi\begin{pmatrix} x, a+1 \\ 0, \alpha \end{pmatrix}]b$ by Definition 1.1

$$= \varphi\begin{pmatrix} b, a+1 \\ 0, \alpha \end{pmatrix}$$

$$= \left[\lambda x\varphi\begin{pmatrix} x, a \\ 0, \alpha \end{pmatrix}\right]\begin{pmatrix} b, 1 \\ 0, \alpha \end{pmatrix} \quad \text{by Lemma 1.2}$$

$$= \left(\lambda y\left[\lambda x\varphi\begin{pmatrix} x, a \\ 0, \alpha \end{pmatrix}\begin{pmatrix} y, 1 \\ 0, \alpha \end{pmatrix}\right]\right)b \quad \text{by } \lambda\text{-abstraction}$$

$$= \left(T^1_\alpha\left[\lambda x\varphi\begin{pmatrix} x, 1 \\ 0, \alpha \end{pmatrix}\right]\right)b \quad \text{by Definition 1.1}$$

$$= (T^1_\alpha(T^a_\alpha\varphi))b \quad \text{by Definition 1.1}$$

$$= ((T^1_\alpha \circ T^a_\alpha)\varphi)b.$$

<u>LEMMA 1.4</u>: $\varphi\begin{pmatrix} x, & 1 \\ 0, & \alpha+1 \end{pmatrix}$ = x-th member of $\mathfrak{B} = \{y \mid \varphi\binom{y}{\alpha} = y\}$.

<u>Proof</u>. By definition [7,(2.3)], $\varphi\begin{pmatrix} x, & 1 \\ 0, & \alpha+1 \end{pmatrix}$ = x-th member of $\mathfrak{A}$ where

$$\mathfrak{A} = \{y \mid \varphi\begin{pmatrix} y, & 0 \\ \alpha^*, & \alpha+1 \end{pmatrix} = y; \text{all } \alpha^* < \alpha+1\}.$$

Case 1. $\alpha = 0$. Then our result is immediate since trivially $\mathfrak{A} = \mathfrak{B}$.

Case 2. $0 < \alpha$. Then $\mathfrak{A} = \mathfrak{B} \cap \mathfrak{B}'$ where $\mathfrak{B}' = \{y \mid \varphi\begin{pmatrix} y, & 0 \\ \alpha^*, & \alpha+1 \end{pmatrix} = y$ all $\alpha^* < \alpha\}$ and $\mathfrak{B} = \{y \mid \varphi\begin{pmatrix} y, & 0 \\ \alpha, & \alpha+1 \end{pmatrix} = y\}$. What we need to show is $\mathfrak{A} = \mathfrak{B}$.

$\mathfrak{A} \subseteq \mathfrak{B}$ is trivial so we need only show $\mathfrak{B} \subseteq \mathfrak{A}$.

Let $y \varepsilon \mathfrak{B}$, then $\varphi\binom{y}{\alpha} = y$.

By [7,(3.4)], $0 < y$; so we get that y is the 0-th member of $\{z \mid \varphi\begin{pmatrix} z, & y^* \\ \alpha^*, & \alpha \end{pmatrix} = z$ all $\alpha^* < \alpha$, $y^* < y\}$. In particular, y is the 0-th member of $\{z \mid \varphi\begin{pmatrix} z, & 0 \\ \alpha^*, & \alpha \end{pmatrix} = z$ all $\alpha^* < \alpha\}$. So $y \varepsilon \mathfrak{B}'$, from which it follows that $y \varepsilon \mathfrak{A}$.

<u>LEMMA 1.5</u>: $(T_1^1\varphi)a$ = a-th solution of the equation $\varphi(x) = x$.

<u>Proof</u>. $(T_1^1\varphi)a = \left(\lambda x \varphi\begin{pmatrix} x, & 1 \\ 0, & 1 \end{pmatrix}\right)a = \varphi\begin{pmatrix} a, & 1 \\ 0, & 1 \end{pmatrix}$ by Definition 1.1 and λ abstraction. Our result follows from [7,(2.3)].

<u>LEMMA 1.6</u>: Let R_α be defined by $R_\alpha\varphi = \lambda y[(T_\alpha^y\varphi)0]$, then $T_{\alpha+1}^1 = T_1^1 \circ R_\alpha$.

<u>Proof</u>.

Case 1. $\alpha = 0$. Then the following shows that R_α is the identity functional:

$$(R_0\varphi)a = (\lambda y[(T_0^y\varphi)0])a = (\lambda y[(\lambda x\varphi(y))0])a$$
$$= \varphi(a).$$

From this the desired result is immediate.

Case 2. $0 < \alpha$.

$$(T^1_{\alpha+1}\varphi)b = [\lambda x\varphi\begin{pmatrix} x, & 1 \\ 0, & \alpha+1 \end{pmatrix}]b = \varphi\begin{pmatrix} b, & 1 \\ 0, & \alpha+1 \end{pmatrix}$$

$$= \text{b-th fixed point of } \lambda y\varphi\binom{y}{\alpha} \text{ by Lemma } \cancel{1.3}\ 1.4.$$

$$= [T^1_1(\lambda y\varphi\binom{y}{\alpha})]b \quad \text{by Lemma } \cancel{1.4}\ 1.5.$$

$$= [T^1_1(\lambda y\varphi\begin{pmatrix} 0, & y \\ 0, & \alpha \end{pmatrix})]b.$$

$$= [T^1_1(\lambda y[\lambda x\varphi\begin{pmatrix} x, & y \\ 0, & \alpha \end{pmatrix}0])]b \text{ by } \lambda\text{-abstraction.}$$

$$= [T^1_1\lambda y[(T^y_\alpha\varphi)0]]b \text{ by definition of } T^y_\alpha.$$

$$= [T^1_1(R_\alpha\varphi)]b \quad \text{by definition of } R_\alpha .$$

$$= [(T^1_1 \circ R_\alpha)\varphi]b.$$

<u>LEMMA 1.7</u>: Assume $\bar{\alpha}$, $\bar{a}$ are limit ordinals, $0 < \alpha$, then

(i) $$\mathcal{R}\lambda x\varphi\begin{pmatrix} x, & \bar{a} \\ 0, & \alpha \end{pmatrix} = \bigcap_{a<\bar{a}} \mathcal{R}\lambda x\varphi\begin{pmatrix} x, & a \\ 0, & \alpha \end{pmatrix}$$

(ii) $$\mathcal{R}\lambda x\varphi\begin{pmatrix} x, & 1 \\ 0, & \bar{\alpha} \end{pmatrix} = \bigcap_{\alpha<\bar{\alpha}} \mathcal{R}\lambda x\varphi\begin{pmatrix} x, & 1 \\ 0, & \alpha \end{pmatrix} .$$

<u>Proof</u>. (i) That the left side is included in the right side is immediate from the definition of $\varphi\begin{pmatrix} x, & \bar{a} \\ 0, & \alpha \end{pmatrix}$.

To show that the right side is included in the left side, let y be a member of the right side; we must show that

$$\varphi\begin{pmatrix} y, a^* \\ \alpha^*, \alpha \end{pmatrix} = y \text{ all } \alpha^* < \alpha, \quad a^* < \bar{a}.$$

Let $\alpha^* < \alpha$, $a^* < \bar{a}$ be given, then since $\bar{a}$ is a limit ordinal, $a^*+1 < \bar{a}$. Now by hypothesis y is a member of $\mathcal{R}\lambda x\varphi\begin{pmatrix} x, a^*+1 \\ 0, \alpha \end{pmatrix}$. From [7,(2.3)] we get

$$\varphi\begin{pmatrix} y, a^* \\ \alpha^*, \alpha \end{pmatrix} = y.$$

(ii) To show that the left side is included in the right side, let $y \varepsilon \mathcal{R}\lambda x\varphi\begin{pmatrix} x, 1 \\ 0, \bar{\alpha} \end{pmatrix}$; to show that y is a member of the right side we must show that if $\alpha < \bar{\alpha}$ then $y \varepsilon \mathcal{R}\lambda x\varphi\begin{pmatrix} x, 1 \\ 0, \alpha \end{pmatrix}$. Let $\alpha < \bar{\alpha}$ be given. Now $y \varepsilon \mathcal{R}\lambda x\varphi\begin{pmatrix} x, 1 \\ 0, \bar{\alpha} \end{pmatrix}$ means that $y = \varphi\begin{pmatrix} y \\ \alpha^* \end{pmatrix}$ for all $\alpha^* < \bar{\alpha}$; in particular $y = \varphi\begin{pmatrix} y \\ \alpha^* \end{pmatrix}$ for all $\alpha^* < \alpha$; so by [7,(2.3)], $y \varepsilon \mathcal{R}\lambda x\varphi\begin{pmatrix} x, 1 \\ 0, \alpha \end{pmatrix}$.
To show that the right side is included in the left side, let y be a member of the right side, we must show that

$$\varphi\begin{pmatrix} y \\ \alpha^* \end{pmatrix} = y \text{ all } \alpha^* < \bar{\bar{\alpha}}.$$

Let $\alpha^* < \bar{\bar{\alpha}}$ be given, then since $\bar{\bar{\alpha}}$ is a limit ordinal $\alpha^*+1 < \bar{\bar{\alpha}}$. Now by hypothesis y is a member of $\mathcal{R}\lambda x\varphi\begin{pmatrix} x, 1 \\ 0, \alpha^*+1 \end{pmatrix}$ so by [7,(2.3)] $\varphi\begin{pmatrix} y \\ \alpha^* \end{pmatrix} = y$.

<u>COROLLARY 1</u>: Assume $\alpha > 0$. If $\varphi\begin{pmatrix} x, \bar{a} \\ 0, \alpha \end{pmatrix} < t \leq \varphi\begin{pmatrix} x+1, \bar{a} \\ 0, \alpha \end{pmatrix}$, then

$$\varphi\begin{pmatrix} x+1, \bar{a} \\ 0, \alpha \end{pmatrix} = \sup_{a<\bar{a}} \varphi\begin{pmatrix} t, a \\ 0, \alpha \end{pmatrix}.$$

To see this let $\{f_a\}_{a<\bar{a}}$ be the sequence of functions defined by

$$f_a \equiv \lambda x \varphi\begin{pmatrix} x, a \\ 0, \alpha \end{pmatrix}.$$

One can easily show that this sequence satisfies the hypotheses of [2, Th.4, p.34]. In view of part (i) of our lemma, the function $\lambda x \varphi\begin{pmatrix} x, \bar{a} \\ 0, \alpha \end{pmatrix}$ is the function ψ referred to in that theorem. The conclusion of that theorem is the desired result.

<u>COROLLARY 2</u>: Assume $\bar{\alpha}$ is a limit ordinal. If $\varphi\begin{pmatrix} x, 1 \\ 0, \bar{\alpha} \end{pmatrix} < t \leq \varphi\begin{pmatrix} x+1, 1 \\ 0, \bar{\alpha} \end{pmatrix}$, then

$$\varphi\begin{pmatrix} x+1, 1 \\ 0, \bar{\alpha} \end{pmatrix} = \sup_{\alpha<\bar{\alpha}} \varphi\begin{pmatrix} t, 1 \\ 0, \alpha \end{pmatrix}.$$

To see this let $\{f_\alpha\}_{\alpha<\bar{\alpha}}$ be the sequence of functions defined by

$$f_\alpha \equiv \lambda x \varphi\begin{pmatrix} x, 1 \\ 0, \alpha \end{pmatrix}.$$

One can easily show that this sequence satisfies the hypotheses of [3, Th.4, p.34]. In view of part (ii) of our lemma, the function $\lambda x \varphi\begin{pmatrix} x, 1 \\ 0, \bar{\alpha} \end{pmatrix}$ would play the role of ψ in that theorem.

<u>Definition 1.2</u>. $\varphi^{(0)}(x) = \varphi(x)$

$$\varphi^{(n+1)}(x) = \varphi(\varphi^{(n)}(x)).$$

<u>THEOREM 1.2</u>: Let $\overline{\alpha}$, $\overline{a}$, $\overline{x}$ denote limit ordinals, then the following relations hold:

1) $T^0_\alpha\varphi = \varphi$ if $\alpha > 0$

2a) $(T^1_1\varphi)0 = \sup\limits_{n<\omega} \varphi^{(n)}(0)$

2b) $(T^1_1\varphi)(x+1) = \sup\limits_{n<\omega} \varphi^{(n)}([(T^1_1\varphi)x]+1)$

2c) $(T^1_1\varphi)\overline{x} = \sup\limits_{x<\overline{x}} [(T^1_1\varphi)x]$

3) $T^{a+1}_\alpha = T^1_\alpha \circ T^a_\alpha$ for $\alpha > 0$

3a) $(T^{\overline{a}}_\alpha\varphi)0 = \sup\limits_{a<\overline{a}} [(T^a_\alpha\varphi)0]$ if $\alpha > 0$

3b) $(T^{\overline{a}}_\alpha\varphi)(x+1) = \sup\limits_{a<\overline{a}} [(T^a_\alpha\varphi)([(T^{\overline{a}}_\alpha\varphi)x]+1)]$ if $\alpha > 0$

3c) $(T^{\overline{a}}_\alpha\varphi)\overline{x} = \sup\limits_{x<\overline{x}} [(T^{\overline{a}}_\alpha\varphi)x]$ if $\alpha > 0$

4) $T^1_{\alpha+1} = T^1_1 \circ R_\alpha$ where $R_\alpha\varphi = \lambda x[(T^x_\alpha\varphi)0]$; $\alpha > 0$

4a) $(T^1_{\overline{\alpha}}\varphi)0 = \sup\limits_{\alpha<\overline{\alpha}} [(T^1_\alpha\varphi)0]$

4b) $(T^1_{\overline{\alpha}}\varphi)(x+1) = \sup\limits_{\alpha<\overline{\alpha}} [(T^1_\alpha\varphi)([(T^1_{\overline{\alpha}}\varphi)x]+1)]$

4c) $(T^1_{\overline{\alpha}}\varphi)\overline{x} = \sup\limits_{x<\overline{x}} [(T^1_{\overline{\alpha}}\varphi)x]$

<u>Proof</u>.

1) This follows from the fact that $\varphi\binom{x}{0} = \varphi\binom{x,0}{0,\alpha}$; [7,(2.2)].

2a) By Lemma 1.5 and [2, p. 38].

2b) By Lemma 1.5 and [2, p.38].

2c) By continuity of $\lambda x\varphi\begin{pmatrix} x,1 \\ 0,1 \end{pmatrix}$; [7, (3.5)].

3) This is our Lemma 1.3.

3a) By continuity of $\lambda x\varphi\begin{pmatrix} 0,x \\ 0,\alpha \end{pmatrix}$; [7, (3.5)].

3b) By Corollary 1 to Lemma 1.7.

3c) By continuity of $\lambda x\varphi\begin{pmatrix} x,\bar{a} \\ 0,\alpha \end{pmatrix}$; [7, (3.5)]

4) This is our Lemma 1.6.

4a) By continuity of $\lambda x\varphi\begin{pmatrix} 0,1 \\ 0,x \end{pmatrix}$; [7, (4.3)].

4b) By Corollary 2 to Lemma 1.7.

4c) By continuity of $\lambda x\varphi\begin{pmatrix} x,1 \\ 0,\alpha \end{pmatrix}$; [7, (3.5)].

<u>Discussion</u>. Let φ run over arbitrary functions (rather than just normal functions). Then 1) defines T^0_α outright, for $\alpha>0$. The remaining equations 2a) - 4c) define T^a_α, for $a>0$ and $\alpha>0$, by transfinite induction. Indeed 2a) - 2c) define $(T^1_1\varphi)x$ by transfinite induction on x, so 2a) - 2c) define T^1_1. Suppose, for fixed α, that T^a_α is known for all $a<\bar{a}$; then 3a) - 3c) define $(T^{\bar{a}}_\alpha\varphi)x$ by transfinite induction on x. Thus 3a) - 3c) define T^a_α from T^1_α by transfinite induction on a. T^a_α for $\alpha>0$ can be regarded as a kind of a-th iterate of the functional T^1_α; indeed, $T^{a+1}_\alpha = T^1_\alpha \circ T^a_\alpha$ and more generally by our Lemma 1.2, $T^{a+b}_\alpha = T^b_\alpha \circ T^a_\alpha$.

Similarly for $\beta>0$, 4) - 4c) define T^1_β from $\lambda\alpha a T^a_\alpha$ restricted to pairs (a,α) such that $\alpha<\beta$. Thus for

$(b,\beta) \neq 1$, the equations 3) - 4c) define T^b_β in terms of $\lambda\alpha a T^a_\alpha$ restricted to pairs (a,α) which precede (b,β) in the lexicographic ordering in which α dominates. As already mentioned, 2a) - 2c) define T^1_1. <u>Thus 1) - 4c) define $\lambda\alpha a T^a_\alpha$ by transfinite induction over the set of pairs (a,α) in the lexiographic ordering</u>.

2. A Hierarchy

In [2], Bachmann describes a hierarchy of normal functions $\{\varphi_c\}_{c<\bar{c}}$ where $\bar{c}$ is a certain ordinal in the Third Number Class.[3] For purposes of this discussion we mention about $\bar{c}$ only that $\varepsilon_{\Omega+1} < \bar{c}$ where $\varepsilon_{\Omega+1}$ is the ~~first~~ last epsilon number greater than Ω.[4] Schütte's family of normal functions appears as a rather small portion of this hierarchy. Let $\varphi(x) = \varepsilon_x$ be the ordering function of the epsilon numbers. Our goal is to locate E(0) (Veblen's ~~first~~ last E-number) within Takeuti's O(2). One possible approach would be to reconstruct within O(2) the portion of Bachmann's hierarchy which corresponds to Schütte's family of normal functions. This is not quite the course we have taken as we have

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[3] Cf. [3], p.28.

[4] $\varepsilon_\Omega = \Omega$.

not yet proven that Bachmann's hierarchy has certain representation, closure, and recursion properties needed to carry out this reconstruction.

The approach which we do use was, nevertheless, inspired by the above considerations. We will define a hierarchy $\{\overline{\varphi}_c\}$ for $c < \Omega^{\Omega}$ in terms of Schütte's $\overline{\varphi}A$ where $\varphi(x) = \varepsilon_x$ and A is a Klammersymbol. These functions will be increasing but not continuous. In defining $\{\overline{\varphi}_c\}$ we have in mind something which resembles the portion of Bachmann's hierarchy which corresponds to Schütte's family of normal functions. We will also define in terms of the members of a certain system O, which is similar to a segment of O(2), a hierarchy $\{\overline{\psi}_c\}$ of increasing (but not continuous) functions. The indices for this hierarchy will come from a certain recursively defined extension O' of O. More will be said about O and O' in later sections. In defining $\{\overline{\psi}_c\}$ we have in mind something which resembles a portion of Bachmann's hierarchy even larger than $\{\overline{\varphi}_c\}_{c<\Omega^{\Omega}}$. Using properties of the $\{\overline{\varphi}_c\}$ and $\{\overline{\psi}_c\}$ hierarchies which are to be developed in later sections of this thesis we will show that when one has a $\overline{\varphi}A$ representation for an ordinal less than E(O) one can locate this ordinal in the hierarchy $\{\overline{\varphi}_c\}$, then in the hierarchy $\{\overline{\psi}_c\}$, then in O, and then finally in O(2).

More will be said about the possible connection between $\{\overline{\psi}_c\}$ and the hierarchy of Bachmann in Chapter II; Section 5.

In the present section we define the hierarchy $\{\overline{\varphi}_c\}_{c<\Omega^\Omega}$ and prove some results about it. Throughout the remainder of this section $\varphi(x)$ will denote the ordering function of the epsilon numbers.

Definition 2.1: We define $\overline{\varphi}_c(a_o)$ for $0 \leq c < \Omega^\Omega$ and $0 \leq a_o < \Omega$ as follows:

1. If $c = 0$ then $\overline{\varphi}_c(a_o) = \overline{\varphi}\begin{pmatrix} a_o \\ 0 \end{pmatrix}$.

2. If $c \neq 0$ then c has a normal form representation:[5]

$$c = \Omega^{\alpha_n} a_n + \ldots + \Omega^{\alpha_1} a_1 \quad \text{where } \alpha_n > \ldots > \alpha_o \text{ and}$$

$0 < a_i < \Omega$, $i=1,2,\ldots,n$.

$$\text{Let } \overline{\varphi}_c(a_o) = \overline{\varphi}\begin{pmatrix} a_o, & a_1, & \ldots, & a_n \\ 0, & 1+\alpha_1, & \ldots, & 1+\alpha_n \end{pmatrix}.$$

LEMMA 2.1: $\alpha_1, \ldots, \alpha_n, a_o, \ldots, a_n < E(0) \longrightarrow$

$$\overline{\varphi}\begin{pmatrix} a_o, & a_1, & \ldots, & a_n \\ 0, & \alpha_1, & \ldots, & \alpha_n \end{pmatrix} < E(0).$$

Proof. By [7,(5.2)], $\varphi\begin{pmatrix} a_o, & a_1, & \ldots, & a_n \\ 0, & \alpha_1, & \ldots, & \alpha_n \end{pmatrix} < E(0)$.

By the definition of $\overline{\varphi}A$, [7, p.26], we have

[5] Cf. [3], p.55.

$$\overline{\varphi}\begin{pmatrix} a_o, a_1, \ldots, a_n \\ 0\ , \alpha_1, \ldots, \alpha_n \end{pmatrix} = \begin{cases} \varphi\begin{pmatrix} a_o, a_1, \ldots, a_n \\ 0\ , \alpha_1, \ldots, \alpha_n \end{pmatrix} \\ \text{or} \\ \varphi\begin{pmatrix} a_o+1, a_1, \ldots, a_n \\ 0\ , \alpha_1, \ldots, \alpha_n \end{pmatrix} \end{cases}$$

but $a_o < E(0) \longrightarrow a_o+1 < E(0)$, so in any case

$\overline{\varphi}\begin{pmatrix} a_o, a_1, \ldots, a_n \\ 0\ , \alpha_1, \ldots, \alpha_n \end{pmatrix} < E(0)$.

<u>LEMMA 2.2</u>: For $0 \leq i \leq n$:

$$a_i \neq 0 \wedge \overline{\varphi}\begin{pmatrix} a_o, \ldots, a_n \\ \alpha_o, \ldots, \alpha_n \end{pmatrix} < E(0) \longrightarrow \alpha_i < \overline{\varphi}\begin{pmatrix} a_o, \ldots, a_n \\ \alpha_o, \ldots, \alpha_n \end{pmatrix}.$$

<u>Proof</u>. By [7,(8.1)],

$\overline{\varphi}\begin{pmatrix} a_o, \ldots, a_n \\ \alpha_o, \ldots, \alpha_n \end{pmatrix} = \overline{\varphi}\begin{pmatrix} b_o, \ldots, b_k \\ \beta_o, \ldots, \beta_k \end{pmatrix}$ where $b_o, \ldots, b_k, \beta_o, \ldots,$

$\beta_k < \overline{\varphi}\begin{pmatrix} b_o, \ldots, b_k \\ \beta_o, \ldots, \beta_k \end{pmatrix}$ so by [7,(8.3),(8.4)],

$\begin{pmatrix} a_o, \ldots, a_n \\ \alpha_o, \ldots, \alpha_n \end{pmatrix} = \begin{pmatrix} b_o, \ldots, b_k \\ \beta_o, \ldots, \beta_k \end{pmatrix}$; but $a_i \neq 0 \longrightarrow \alpha_i = \beta_j$ for

some j by the definition of equality of Klammersymbols. Therefore

$$\alpha_i = \beta_j < \overline{\varphi}\begin{pmatrix} b_o, \ldots, b_k \\ \beta_o, \ldots, \beta_k \end{pmatrix} = \overline{\varphi}\begin{pmatrix} a_o, \ldots, a_n \\ \alpha_o, \ldots, \alpha_n \end{pmatrix}.$$

<u>LEMMA 2.3</u>: If $a = \varphi(b) < E(0)$, then there exist unique $\alpha_1, \alpha_2, \ldots, \alpha_n, a_o, \ldots, a_n < a$ such that

$$a = \overline{\varphi}_c(a_o) \quad \text{where } a_1, \ldots, a_n \neq 0,\ c = \Omega^{\alpha_n} a_n + \ldots + \Omega^{\alpha_1} a_1.$$

and $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1$.

<u>Proof</u>. $a = \overline{\varphi}\begin{pmatrix} a_o, a_1, \ldots, a_n \\ 0, \alpha_1, \ldots, \alpha_n \end{pmatrix}$ where $a_1, \ldots, a_n \neq 0$ and $\alpha_1, \ldots, \alpha_n, a_o, \ldots, a_n < a$ by [7,(8.1)]. Therefore, $a = \overline{\varphi}_c(a_o)$ where $c = \Omega^{-1+\alpha_n} a_n + \cdots + \Omega^{-1+\alpha_1} a_1$.

We first remind the reader that two Klammersymbols are equal if one arises from the other by the insertion or deletion of terms of the form $\overset{0}{\alpha_i}$. Uniqueness then follows from the fact that $\overline{\varphi}(A) = \overline{\varphi}(B) \longrightarrow A = B$ by [7, p.26].

<u>LEMMA 2.4</u>: If $\alpha_1, \ldots, \alpha_n, a_o, \ldots, a_n < E(0)$, then $\overline{\varphi}_c(a_o) < E(0)$ where $c = \Omega^{\alpha_n} a_n + \cdots + \Omega^{\alpha_1} a_1$, and $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1$.

<u>Proof</u>. Immediate from Lemma 2.1.

<u>Definition 2.2</u>: Let $0 \leq c < \Omega^{\Omega}$. We define the relation $H(x,c)$ inductively as follows:

1. If $c = 0$, then $H(x,c) \longleftrightarrow x = 0$.

2. If $c \neq 0$, let $c = \Omega^{\alpha_n} a_n + \cdots + \Omega^{\alpha_1} a_1$ be the normal form representation of c, then $H(x,c) \longleftrightarrow x = a_i \vee H(x,\alpha_i)$ for some i.

<u>REMARK</u>: This recursive definition would be meaningful for all c less than the first fixed point of the func-

tion Ω^x; for then we could be sure that the exponents in the normal form representation of c would be less c itself. For $0 \leq c < \Omega^{\Omega}$, it is easy to see that $H(x,c)$ if and only if x is a coefficient or x is an exponent in the normal form representation of c. We have given the definition recursively, however, to allow for the possibility of making the generalization to cases where $c > \Omega^{\Omega}$.

<u>Definition 2.3</u>: Let $a = \overline{\varphi}_c(a_o)$. The relation $C(x,a)$ is defined as follows:

$$C(x,a) \longleftrightarrow H(x,c) \vee x = a_o.$$

<u>LEMMA 2.5:</u> If $a = \overline{\varphi}_{c_1}(a_o) < E(0)$

$$b = \overline{\varphi}_{c_2}(b_o) < E(0),$$

then $a < b$ if and only if one of the following holds:

I $\quad c_1 = c_2 \wedge a_o < b_o$

II $\quad c_1 < c_2 \wedge \bigwedge x(C(x,a) \longrightarrow x < b)$

III $\quad \bigvee x(C(x,b) \wedge a \leq x)$

<u>Proof</u>. Sufficiency of I follows from [7,(8.2),(8.3)]. Sufficiency of II follows from [7,(8.2),(8.3)]. Sufficiency of III follows from [7,(8.4)] together with Lemma 2.2.

Necessity follows from sufficiency, because from the assumption that I, II, and III fail, one can show

using the sufficiency of I, II, and III that $b \leq a$. This completes the proof.

Let $0 < b < \Omega$ be given. It is well known that b can be expressed (uniquely) in the form

$$b = \omega^{b_n} + \omega^{b_{n-1}} + \cdots + \omega^{b_1} \quad \text{where } b_n \geq b_{n-1} \geq \cdots \geq b_1.$$

By [2, Th. 10, p. 69], $\omega^{\Omega} = \Omega$

Thus $\quad \Omega^{\beta} b = (\omega^{\Omega})^{\beta} b = \omega^{\Omega\beta + b_n} + \omega^{\Omega\beta + b_{n-1}} + \cdots + \omega^{\Omega\beta + b_1}.$

From this we can conclude that if $0 < c < \Omega^{\Omega}$, then c has a unique representation of the form

$$(*) \qquad c = \omega^{\Omega\alpha_n + a_n} + \omega^{\Omega\alpha_{n-1} + a_{n-1}} + \cdots + \omega^{\Omega\alpha_1 + a_1} \quad \text{where}$$

1) $a_1, a_2, \ldots, a_n, \alpha_1, \alpha_2, \ldots, \alpha_n < \Omega$

2) $\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1$

3) $\alpha_i = \alpha_{i-1} \longrightarrow a_i \geq a_{i-1}$ for $i = 2, 3, \ldots, n$.

It is not hard to show that in representation (*) with 1), 2), 3) holding the sum sign + can be replaced by the natural sum sign #.[6] Thus we can say that if $0 < c < \Omega^{\Omega}$, then c has a representation of the form

$$(**) \qquad c = \omega^{\Omega\alpha_n \# a_n} \# \omega^{\Omega\alpha_{n-1} \# a_{n-1}} \# \cdots \# \omega^{\Omega\alpha_1 \# a_1} \quad \text{where}$$

1) $a_1, a_2, \ldots, a_n, \alpha_1, \alpha_2, \ldots, \alpha_n < \Omega$.

As to the uniqueness of the representation (**), one can say that it is unique up to the order in which

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[6] Cf. [3], p. 102.

the terms of the natural sum appear. To see this, take any two representations and permute the terms until 2) 3) hold (natural sum is commutative); replace the natural sum sign # by the usual sum sign +; then apply the uniqueness of the resulting representation.

In what follows we reformulate Lemmas 2.3, 2.4, and 2.5 using the natural sum in the representation for c. This is done for convenience in relating the $\{\overline{\varphi}_c\}$ hierarchy to the $\{\overline{\psi}_c\}$ hierarchy (to be defined in Chapter II, Section 4). The $\{\overline{\psi}_c\}$ hierarchy is defined in terms of ~~Takenis~~ TAKEUTI'S ordinal diagrams which in turn are given by a definition which involves natural sum.

<u>LEMMA 2.3a</u>: If $a = \varphi(b) < E(0)$, then there exist $a_0, a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_n < a$ such that $a = \overline{\varphi}_c(a_0)$ where $c = \omega^{\Omega\alpha_n \# a_n} \# \ldots \# \omega^{\Omega\alpha_1 \# a_1}$. Moreover the representation of c is unique up to the order in which the terms in the natural sum appear.

<u>Proof</u>. This is just a reformulation of Lemma 2.3 where c is expressed in representation (**).

<u>LEMMA 2.4a</u>: If $\alpha_1, \ldots, \alpha_n, a_0, \ldots, a_n < E(0)$, then $\overline{\varphi}_c(a_0) < E(0)$ for $c = \omega^{\Omega\alpha_n \# a_n} \# \ldots \# \omega^{\Omega\alpha_1 \# a_1}$.

<u>Proof</u>. This is just a reformulation of Lemma 2.4 where c is written in normal form (**).

<u>Definition 2.4</u>: Let $0 \leq c < \Omega^{\Omega}$. We define the relation

$T(x,c)$ as follows:

1. If $c = 0$, then $T(x,c) \longleftrightarrow x = 0$

2. If $c \neq 0$, let $c = \omega^{\Omega\alpha_n \# a_n} \# \ldots \# \omega^{\Omega\alpha_1 \# a_1}$ be the normal form (**) of c, then

$$T(x,c) \longleftrightarrow x = \omega^{a_i} \vee T(x,\alpha_i) \text{ for some } i.$$

<u>Definition 2.5</u>: Let $a = \overline{\varphi}_c(a_o)$ where $0 \leq c \leq \Omega^{\Omega}$ and $a_o < a$. We define the relation $P(x,a)$ as follows:

$$P(x,a) \longleftrightarrow T(x,c) \vee x = a_o.$$

<u>REMARKS</u>: Suppose $x \neq 0$ and $H(x,c)$. Let x have representation

$$x = \omega^{b_n} \# \ldots \# \omega^{b_1}$$

then $T(\omega^{b_i},c)$ for $i = 1,2,\ldots,n$. From this one easily gets that if $a = \overline{\varphi}_{c_1}(a_o)$ and $b = \overline{\varphi}_{c_2}(b_o)$, then

$$[\bigwedge x(C(x,a) \longrightarrow x<b)] \longleftrightarrow [\bigwedge x(P(x,a) \longrightarrow x<b)]$$

<u>LEMMA 2.5a</u>: If $\overline{\varphi}_{c_1}(a_o)$, $\overline{\varphi}_{c_2}(b_o) < E(0)$, then

$\overline{\varphi}_{c_1}(a_o) < \overline{\varphi}_{c_2}(b_o)$ if and only if one of the following holds:

I $\quad c_1 = c_2 \wedge a_o < b_o$

II $\quad c_1 < c_2 \wedge \bigwedge x(P(x,\overline{\varphi}_{c_1}(a_o)) \longrightarrow x < \overline{\varphi}_{c_2}(b_o))$

III $\quad \bigvee x(P(x,\overline{\varphi}_{c_2}(b_o)) \wedge \overline{\varphi}_{c_1}(\alpha_o) \leq x)$.

<u>Proof</u>. In view of the preceeding remarks this follows from Lemma 2.5.

II. TAKEUTI'S ORDINAL DIAGRAMS

Let O(n) denote Takeuti's system of ordinal diagrams of order n [10]. The purpose of this chapter is to study O(1) and O(2). We will show that O(1) ordered by the relation $<_1$ is order isomorphic to the set of all ordinals less than the first ω-critical number[7] of ω^{1+x}. We also characterize a lower segment of O(2). We will show that this segment, when ordered by $<_2$, itself contains a segment that is order isomorphic to the set of all ordinals less than E(0) (Veblen's first E-number relative to the initial function φ, where φ is the ordering function of the ε-numbers).

1. The System O(1)

Let $\varphi(x)$ denote the function ω^{1+x}. It follows directly from Schütte's definition of φA, where A is a Klammersymbol, that $\lambda x \varphi \begin{pmatrix} x & \alpha \\ 0 & 1 \end{pmatrix}$ is the function φ_α of the simple hierarchy referred to in the introductory remarks of Chapter I, Section 1. This is often called the α-th derived function of φ. Thus $\varphi\binom{\omega}{1}$ is the first

[7] The notion of α-critical number was defined in Chapter I, Section I.

ω-critical number. It is also easy to see from Schütte's definition of $\overline{\varphi}A$ that $\lambda x\overline{\varphi}\begin{pmatrix} x & \alpha \\ 0 & 1 \end{pmatrix}$ is the ordering function of the numbers which are α-critical but not $\alpha+1$ critical. The purpose of this section is to establish an order preserving mapping from O(1) onto $\{x \mid 0 < x < \varphi\binom{\omega}{1}\}$.

The ordinal diagram in O(1) are expressions defined inductively from the natural numbers and the primitive operation symbols (,,) and # by the following:

1) $n > 0 \longrightarrow n \varepsilon O(1)$

2) $n > 0 \wedge \alpha \varepsilon O(1) \longrightarrow (1,n,\alpha) \varepsilon O(1)$

3) $\alpha,\beta \varepsilon O(1) \longrightarrow \alpha \# \beta \varepsilon O(1)$.

Takeuti defines relations $=$, $<_0$, $<_1$ on the members of O(1). He then shows that $=$ is an equivalence relation. $<_0$, $<_1$ induce well orderings on the set of equivalence classes.

<u>LEMMA 1.1</u>: A necessary and sufficient condition that $(1,a,\alpha) <_1 (1,b,\beta)$ is that one of the following holds:

I $\quad a = b \wedge \alpha <_1 \beta$

II $\quad a < b \wedge \alpha <_1 (1,b,\beta)$

III $\quad (1,a,\alpha) \leq_1 \beta$.

<u>Proof</u>.

Sufficiency:

I by [10,5.4.3], $(1,a,\alpha) <_0 (1,b,\beta)$

but $\beta <_1 (1,b,\beta)$ by [10, prop.2., p.393]

so $\alpha <_1 (1,b,\beta)$

and $(1,a,\alpha) <_1 (1,b,\beta)$ by [10, 5.3.2]

II $a<b \longrightarrow (1,a,\alpha) <_0 (1,b,\beta)$ by [10,5.4.1]

so $(1,a,\alpha) <_1 (1,b,\beta)$ by [10, 5.3.2]

III $\beta <_1 (1,b,\beta)$ by [10, prop.2., p.393]

so $(1,a,\alpha) \leq_1 \beta <_1 (1,b,\beta)$

Necessity:

Necessity follows from sufficiency; for if I, II, III fail, then by sufficiency of I, II, III $(1,b,\beta) \leq_1 (1,a,\alpha)$.

<u>LEMMA 1.2</u>: If $\gamma = \varphi(x) < \varphi\binom{\omega}{1}$ for some x, then $\gamma = \overline{\varphi}\begin{pmatrix}\alpha & \beta\\ 0 & 1\end{pmatrix}$ for some $\alpha < \gamma$, $0 \leq \beta < \omega$.

<u>Proof</u>. Clearly $\varphi\binom{\omega}{1} < E(0)$, so $\gamma < \varphi\binom{1}{\gamma}$. Thus by [7, (8 1)], $\gamma = \varphi\begin{pmatrix}a_0, a_1, \ldots, a_n\\ 0, \alpha_1, \ldots, \alpha_n\end{pmatrix}$ where $a_i, \alpha_i < \gamma$. By [7,(8.3),(8.4)], $\gamma < \varphi\binom{1}{\omega}$ gives $\begin{pmatrix}a_0, a_1, \ldots, a_n\\ 0, \alpha_1, \ldots, \alpha_n\end{pmatrix} < \binom{\omega}{1}$.

From this our result follows.

<u>LEMMA 1.3</u>: $\alpha < \varphi\binom{\omega}{1} \wedge \beta < \omega \longrightarrow \overline{\varphi}\begin{pmatrix}\alpha & \beta\\ 0 & 1\end{pmatrix} < \varphi\binom{\omega}{1}$.

<u>Proof</u>. We first remark that $\varphi\binom{\omega}{1} = \overline{\varphi}\binom{\omega}{1}$, so we want to show $\overline{\varphi}\begin{pmatrix}\alpha & \beta\\ 0 & 1\end{pmatrix} < \overline{\varphi}\binom{\omega}{1}$. But this follows from [7,(8.3),(8.4)].

<u>LEMMA 1.4</u>: $\overline{\varphi}\binom{\alpha,a}{0,1} < \overline{\varphi}\binom{\beta,b}{0,1}$ if and only if one of the following hold:

I $a=b \wedge \alpha < \beta$

II $a < b \wedge \alpha < \overline{\varphi}\binom{\beta,b}{0,1}$

III $\overline{\varphi}\binom{\alpha,a}{0,1} \leq \beta$

<u>Proof</u>.

Sufficiency:

From [7,(8.2),(8.3),(8.4)].

Necessity:

Follows from sufficiency.

1) $1'=1$
2) $\alpha' = \overline{\varphi}\binom{0,-1+a}{0,1}$ if $a \neq 1$
3) $(1,1,\alpha)' = \overline{\varphi}\binom{-1+\alpha',0}{0,1}$
4) $(1,a,\alpha)' = \overline{\varphi}\binom{\alpha',-1+a}{0,1}$ if $a \neq 1$
5) $(\alpha \# \beta)' = \alpha' \# \beta'$

<u>THEOREM 1.1</u>: The following determines an order preserving mapping ' from $O(1)$ onto $\mathfrak{B} = \{x \mid x \neq 0 \wedge x < \varphi\binom{\omega}{1}\}$:

The symbol # on the right side of 5) means natural sum of ordinals.

<u>Proof</u>.

Well-defined:

That $\alpha \varepsilon\ O(1) \longrightarrow \alpha' \varepsilon\ \mathfrak{B}$ can be seen by induction on the length of α using Lemma 1.3.

Onto:

That $\beta \varepsilon \mathfrak{B} \longrightarrow \beta = \alpha'$ for some $\alpha \varepsilon O(1)$ can be seen by transfinite induction on β using Lemma 1.2.

Order Preserving:

That $\alpha <_1 \beta \longrightarrow \alpha' < \beta'$ can be seen by induction on the sum of the lengths of α, β.

It is not hard to see that the mapping ' is effective.

Theorem 1.1 shows that the integer a, considered as an ordinal diagram, represents the ordinal $\varphi_{-1+a}(0)$; that is, the $-1+a^{th}$ derived function of φ evaluated at 0. Furthermore, if α' is the ordinal corresponding to the ordinal diagram α, then $\varphi_{-1+a}(\alpha'+1)$ or $\varphi_{-1+a}(-1+\alpha')$ is the ordinal corresponding to the ordinal diagram $(1,a,\alpha)$, according as to whether or not α' is the result of adding a finite ordinal to a fixed point of φ_{-1+a}. Thus the mapping $\alpha \longrightarrow (1,a,\alpha)$ corresponds roughly to the application of the $-1+a^{th}$ derived function φ_{-1+a}.

2. Some Lemmas Concerning O(2)

The ordinal diagrams in O(2) are expressions defined inductively from the natural numbers and the primitive operation symbols (,,) and # by the following:

1. $n > 0 \longrightarrow n \varepsilon O(2)$

2. $n > 0 \wedge \alpha \varepsilon O(2) \longrightarrow (1,n,\alpha) \varepsilon O(2) \wedge (2,n,\alpha) \varepsilon O(2)$

3. $\alpha, \beta \varepsilon O(2) \longrightarrow \alpha \# \beta \varepsilon O(2)$.

Takeuti then defines relations $=$, $<_o$, $<_1$, $<_2$ on $O(2)$. The first of these turns out to be an equivalence relation, the remaining three induce well orderings of the equivalence classes.

<u>LEMMA 2.1</u>:

a) $\alpha <_1 (1,1,\alpha)$

b) $\alpha <_2 (2,1,\alpha)$

c) $\alpha <_2 (1,1,\alpha)$

<u>Proof</u>. a) This is merely a special case of [10, p.393; prop.2].

b) This is merely a special case of [10,p.393;pr.2].

c) By [10, 5.2.1] it is sufficient to show that $\alpha_o <_2 (1,1,\alpha)$, where α_o is an arbitrary component of α. Now $\alpha <_1 (1,1,\alpha)$ by Lemma 2.1a; since $\alpha_o \leq_1 \alpha$, we get $\alpha_o <_1 (1,1,\alpha)$. By [10, 5.3.2] it remains only to show that $\beta \subset_2 \alpha_o \longrightarrow \beta <_2 (1,1,\alpha)$. Let $\beta \subset_2 \alpha_o$. Then by [10, 2.3, 2.1], $\beta \subset_2 (1,1,\alpha)$, so by [10, p.393; prop.2], $\beta <_2 (1,1,\alpha)$.

<u>LEMMA 2.2</u>:

a) $(1,1,\alpha) <_1 (2,1,\beta)$ never holds

b) $(2,1,\alpha) <_1 (2,1,\beta) \longleftrightarrow \alpha <_2 \beta$

c) $(2,1,\alpha) <_1 (1,1,\beta)$

d) $(1,1,\alpha) <_1 (1,1,\beta) \longleftrightarrow \alpha <_1 \beta$

e) $(2,1,\alpha) <_2 (2,1,\beta) \longleftrightarrow \alpha <_2 \beta$

f) $(2,1,\alpha) <_2 (1,1,\beta) \longleftrightarrow \alpha <_2 (1,1,\beta)$

g) $(1,1,\alpha) <_2 (2,1,\beta) \longleftrightarrow (1,1,\alpha) \leq_2 \beta$

h) $(1,1,\alpha) <_2 (1,1,\beta) \longleftrightarrow$

$(1,1,\alpha) \leq_2 \beta \vee (\alpha <_1 \beta \wedge \alpha <_2 (1,1,\beta))$.

<u>Proof</u>.

a) $(2,1,\beta) <_o (1,1,\alpha)$ by [10,5.4.3]. Now $(2,1,\beta)$ has no 1-sections, so vacuously $\beta_o \subset_1 (2,1,\beta) \longrightarrow \beta_o <_1 (1,1,\alpha)$. Therefore by [10, 5.3.2], $(2,1,\beta) <_1 (1,1,\alpha)$; so it cannot be that $(1,1,\alpha) <_1 (2,1,\beta)$.

b) Because $<_2$ is a total ordering, it is sufficient to show that the right side implies the left side. Let $\alpha <_2 \beta$, then $(2,1,\alpha) <_o (2,1,\beta)$ by [10, 5.4.3]. Now $(2,1,\alpha)$ has no 1-sections, so the condition $\alpha_o \subset_1 (2,1,\alpha) \longrightarrow \alpha_o <_1 (2,1,\beta)$ is vacuously satisfied. Therefore by [10, 5.3.2] we get $(2,1,\alpha) <_1 (2,1,\beta)$.

c) $(2,1,\alpha) <_o (1,1,\beta)$ by [10, 5.4.2] but $(2,1,\alpha)$ has no 1-section so by [10, 5.3.2], $(2,1,\alpha) <_1 (1,1,\beta)$.

d) It is sufficient to show right side implies left side. Let $\alpha <_1 \beta$; then $(1,1,\alpha) <_o (1,1,\beta)$ by [10, 5.4.3]; it remains only to show that $\alpha_o \subset_1 (1,1,\alpha) \longrightarrow \alpha_o <_1 (1,1,\beta)$. Let $\alpha_o \subset_1 (1,1,\alpha)$, then $\alpha_o = \alpha$. Now $\alpha <_1 \beta$ by hypothesis and $\beta <_1 (1,1,\beta)$ by Lemma 2.1a; so $\alpha_o <_1 (1,1,\beta)$.

e) It is sufficient to show that the right side implies the left side. Let $\alpha <_2 \beta$, then by part ~~b~~) of this lemma $(2,1,\alpha) <_1 (2,1,\beta)$. By [10, 5.3.2] it remains only to show that $\alpha_o \subset_2 (2,1,\alpha) \longrightarrow \alpha_o <_2 (2,1,\beta)$. Let $\alpha_o \subset_2 (2,1,\alpha)$, then $\alpha_o = \alpha$. Now $\alpha <_2 \beta$ by hypothesis and $\beta <_2 (2,1,\beta)$ by Lemma 2.1b; so $\alpha_o <_2 (2,1,\beta)$.

f) We first show that right side implies left side. Let $\alpha <_2 (1,1,\beta)$. Now $(2,1,\alpha) <_1 (1,1,\beta)$ by part b) of this lemma. Therefore by [10, 5.3.2], $(2,1,\alpha) <_2 (1,1,\beta)$. We now show that left side implies right side. Suppose $(2,1,\alpha) <_2 (1,1,\beta)$ then our desired result follows from $\alpha <_2 (2,1,\alpha) <_2 (1,1,\beta)$.

g) To show that the right side implies the left side we merely note that the condition [10, 5.3.1] is automatically satisfied.

To show that left side implies the right side, we observe that $(2,1,\beta) <_1 (1,1,\alpha)$ by part ~~b~~ c) of this lemma; so $(1,1,\alpha) <_2 (2,1,\beta)$ is not a consequence of [10, 5.3.2]; therefore it is a consequence of [10, 5.3.1]. That is, $(1,1,\alpha) \leq_2 \beta$.

h) We first show that right side implies left side. Suppose $(1,1,\alpha) \leq_2 \beta$. By Lemma 2.1c, $\beta <_2 (1,1,\beta)$ so $(1,1,\alpha) <_2 (1,1,\beta)$. Suppose $\alpha <_1 \beta \wedge \alpha <_2 (1,1,\beta)$, then $(1,1,\alpha) <_1 (1,1,\beta)$ by part ~~c~~ d) of this lemma. By [10, 5.3.2] it remains to show that $\alpha_o \subset_2 (1,1,\alpha) \longrightarrow \alpha_o <_2 (1,1,\beta)$. Let $\alpha_o \subset_2 (1,1,\alpha)$ then by definition of

2-section $\alpha_o \subset_2 \alpha$. If α is connected then $\alpha_o <_2 \alpha$ by [10, p.393, prop.2]. If α is not connected then by [10, 2.3], $\alpha_o \subset_2 \alpha^i$ where α^i is a connected component of α. By [10, p.393, prop.2] $\alpha <_2 \alpha^i$. Now $\alpha^i <_2 \alpha$ so in any case $\alpha_o <_2 \alpha$. This with our hypothesis that $\alpha <_2 (1,1,\beta)$ gives $\alpha_o <_2 (1,1,\beta)$.

To show that the left side implies the right side let $(1,1,\alpha) <_2 (1,1,\beta)$; if this is so because of [10, 5.3.1] then $(1,1,\alpha) \leq_2 \beta$ is immediate; if it is so because of [10, 5.3.1], then $(1,1,\alpha) <_1 (1,1,\beta)$, so $\alpha <_1 \beta$ by part d) of this lemma. Now $\alpha <_2 (1,1,\alpha)$ by Lemma 2.1c, so from $\alpha <_2 (1,1,\alpha) <_2 (1,1,\beta)$ we get $\alpha <_2 (1,1,\beta)$.

<u>LEMMA 2.3</u>: If $\alpha <_2 2$ then α has one of the following forms:

a) $\alpha = 1$

b) $\alpha = \beta \# \gamma$ where $\beta <_2 \alpha$ and $\gamma <_2 \alpha$

c) $\alpha = (2,1,\alpha_o)$ where $\alpha_o <_2 \alpha$

d) $\alpha = (1,1,\alpha_o)$ where $\alpha_o <_2 \alpha$.

<u>Proof</u>. By the inductive definition of O(2) we see that the only remaining forms that a member of O(2) can have are:

e) $\alpha = n$ for $n \geq 2$

f) $\alpha = (2,n,\alpha_o)$ for $n \geq 2$

g) $\alpha = (1,n,\alpha_o)$ for $n \geq 2$.

But in all these cases it is easy to show that $2 \leq_2 \alpha$. The condition $\alpha_0 <_2 \alpha$ in ~~a~~ c) and ~~b~~ d) follow from Lemma 2.1b, 2.1c.

3. The System O

USING LEMMA 2.3 IT CAN BE SHOWN ~~In Lemma 2.3 of the present chapter, it was shown~~ that the set of ordinal diagrams α such that $\alpha <_2 2$ consists of all ordinal diagrams generated by starting with the integer 1 and applying the following three operations:

(i) From α and β, form $\alpha\#\beta$

(ii) From α form $(2,1,\alpha)$

(iii) From α form $(1,1,\alpha)$. (*)

This suggests the notational simplification: write $[\alpha]$ for $(2,1,\alpha)$, and write $\langle\alpha\rangle$ for $(1,1,\alpha)$. O(2) does not have an additive identity with respect to the operation #; this proves to be somewhat inconvenient. We define below a new system O which incorporates the notational simplifications mentioned above and which has an identity with respect to # as well. Our system O will be defined from first principles. Due to the presence of the additive identity in O, the operation $\langle\alpha\rangle$ in O does not quite correspond to the operation $(1,1,\alpha)$ in O(2); rather, by Definition 3.11 below, the

(*) LEMMA 2.3 IS USED TO SHOW THE INCLUSION ONE WAY. THE INCLUSION THE OTHER WAY IS NOT HARD.

operation $\langle -1+\alpha \rangle$ in O corresponds to the operation $(1,1,\alpha)$ in O(2), where $-1+\alpha$ is defined by Definition 3.9 - 3.10 below.

For the definition of the order relations $<_1$, $<_2$ in O, we use the characterization of these relations in O(2) already given in Lemma 2.2 of this chapter. In O (as in O(2)) equality will be defined, and it can happen that two members can be equal without being identical as formal expressions. As in O(2) this equality will be compatable with the order relations, so the relations $<_1, <_2$ will actually be order relations of the equivalence classes (with respect to equality).

The purpose of this section is to show that the system O (more precisely the system of equivalence classes of O under =) is order isomorphic with respect to $<_1$, $<_2$ to the system (of equivalence classes of) $\{x \mid x \varepsilon O(2) \wedge x <_2 2\}$.

<u>Definition 3.1</u>: We define the members of O inductively by means of the symbol o and the operation symbols #, [], and $\langle\rangle$ as follows:

a) $o \varepsilon O$

b) $\alpha \varepsilon O \longrightarrow [\alpha] \varepsilon O$

c) $\alpha \varepsilon O \longrightarrow \langle\alpha\rangle \varepsilon O$

d) $\alpha, \beta \varepsilon O \longrightarrow \alpha\#\beta \varepsilon O$

<u>Notation</u>: $\alpha \equiv \beta$ means that α and β are the same member

of O.

Definition 3.2: (LENGTH)

To each α in O we define length of α denoted by $\ell(\alpha)$ as follows:

a) $\ell(o) = 1$

b) $\ell([\alpha]) = \ell(\alpha) + 1$

c) $\ell(\langle\alpha\rangle) = \ell(\alpha) + 1$

d) $\ell(\alpha\#\beta) = \ell(\alpha) + \ell(\beta) + 1$

Definition 3.3: (LENGTH)

To each α in O(2) we define length of α denoted by $\ell(\alpha)$ as follows:

a) $\ell(n) = 1$

b) $\ell((2,n,\alpha)) = \ell(\alpha) + 1$

c) $\ell((1,n,\alpha)) = \ell(\alpha) + 1$

d) $\ell(\alpha\#\beta) = \ell(\alpha) + \ell(\beta) + 1$

Definition 3.4: (CONNECTED)

We say α is connected if and only if $\alpha \equiv O$ or $\alpha \equiv [\beta]$ or $\alpha \equiv \langle\beta\rangle$.

Definition 3.5: (COMPONENT)

We define the components of α as follows:

a) If α is connected, then α is the only component of α.

b) If $\alpha \equiv \beta\#\gamma$, then the components of α are the components of β and the components of γ.

Definition 3.6: (PROPER COMPONENT)

We say α is a proper component of β if α is a component of β and $\alpha \not\equiv 0$.

Definition 3.7: (EQUALITY)

We define $\alpha = \beta$ inductively as follows:

a) Let α be connected, then $\alpha = \beta$ if and only if one of the following holds:

a_1) $\alpha \equiv 0$ and β has no proper components

a_2) $\alpha \equiv [\gamma]$, $\beta \equiv [\delta]$, and $\gamma = \delta$

a_3) $\alpha \equiv \langle\gamma\rangle$, $\beta \equiv \langle\delta\rangle$, $\gamma = \delta$

b) If α is not connected, then $\alpha = \beta$ if and only if one of the following holds:

b_1) α has no proper components and β has no proper components

b_2) α has proper components $\alpha_1, \alpha_2, \ldots, \alpha_k$,
β has proper components $\beta_1, \beta_2, \ldots, \beta_s$,
$k=s$, and there exists a permutation $(\ell_1, \ell_2, \ldots, \ell_k)$ of $(1, 2, \ldots, k)$ such that $\alpha_m = \beta_{\ell_m}$ for each $m=1,2,\ldots,k$.

REMARK: It is easy to show that o is an identity with respect to the operation #, and that EQUALITY is an equivalence relation.

Definition 3.8: $(<_1, <_2)$

We define the relations $<_1$, $<_2$ simultaneously and inductively as follows:

a) Let α have components $\alpha_1, \alpha_2, \ldots, \alpha_k$

β have components $\beta_1, \beta_2, \ldots, \beta_h$,

then $\alpha <_i \beta$ if and only if:

a_1) There exists β_m $(1 \leq m \leq h)$ such that for every ℓ $(1 \leq \ell \leq k)$ $\alpha_\ell <_i \beta_m$ holds

a_2) $k=1$, $h>1$ and $\alpha_1 = \beta_m$ for some m $(1 \leq m \leq h)$ and $\beta_i \neq 0$ for some i $(1 \leq i \leq h)$, but $i \neq m$.

a_3) $k>1$, $h>1$ and there exist α_ℓ $(1 \leq \ell \leq k)$ and β_m $(1 \leq m \leq h)$ such that $\alpha_\ell = \beta_m$ and

$$\alpha_1 \# \cdots \# \alpha_{\ell-1} \# \alpha_{\ell+1} \# \cdots \# \alpha_k <_i \beta_1 \# \cdots \# \beta_{m-1} \# \beta_{m+1} \# \cdots \# \beta_h$$

b) For connected members of O the relations $<_1$ and $<_2$ are determined by the following:

b_1) $0 \not<_i 0$

b_2) $0 <_i [y]$ but $[y] \not<_i 0$

b_3) $0 <_i \langle y \rangle$ but $\langle y \rangle \not<_i 0$

b_4) $[x] <_1 [y] \longleftrightarrow x <_2 y$

b_5) $[x] <_1 \langle y \rangle$ but $\langle y \rangle \not<_1 [x]$

b_6) $\langle x \rangle <_1 \langle y \rangle \longleftrightarrow x <_1 y$

b_7) $[x] <_2 [y] \longleftrightarrow x <_2 y$

b_8) $[x] <_2 \langle y \rangle \longleftrightarrow x <_2 \langle y \rangle$

b_9) $\langle x \rangle <_2 [y] \longleftrightarrow \langle x \rangle \leq_2 y$

b_{10}) $\langle x \rangle <_2 \langle y \rangle \longleftrightarrow \langle x \rangle \leq_2 y \vee (x <_1 y \wedge x <_2 \langle y \rangle)$.

<u>REMARKS</u>:

a) That these conditions do in fact simultaneously determine relations $\alpha <_1 \beta$ and $\alpha <_2 \beta$ is shown by

induction on the sum of the lengths of α and β.

b) One can show that $<_1$ and $<_2$ are total orderings as follows:

b_1) $\alpha \not<_i \alpha$ is proven by induction on the length of α.

b_2) $\alpha <_i \beta \vee \beta <_i \alpha \vee \alpha = \beta$ is proven by induction on the sum of the lengths of α and β.

b_3) $\alpha <_i \beta \wedge \beta <_i \gamma \longrightarrow \alpha <_i \gamma$ is proven by induction on the sum of the lengths of α, β, and γ.

<u>Definition 3.9</u>: We define $\alpha^{(n)}$ for $o \leq n$ as follows:

a) $\alpha^{(o)} = 0$

b) $\alpha^{(n+1)} = \alpha^{(n)} \# [o]$.

<u>Definition 3.10</u>: We define $-1+\alpha$ as follows:

$$-1+\alpha = \begin{cases} \alpha^{(n)} & \text{if } \alpha = \alpha^{(n+1)};\ o \leq n \\ \alpha & \text{otherwise} \end{cases}$$

Let $\mathfrak{A}$ denote $\{x \mid x \varepsilon \alpha(2) \wedge x <_2 2\}$.

Let $\mathfrak{B}$ denote $\{x \mid x \varepsilon 0 \wedge x \neq 0\}$.

<u>Definition 3.11</u>: We define by induction on the length of the argument a mapping f from $\mathfrak{A}$ into $\mathfrak{B}$ by the following scheme:

a) $f(1) = [o]$

b) $f((2,1,\alpha)) = [f(\alpha)]$

c) $f((1,1,\alpha)) = \langle -1 + f(\alpha) \rangle$

d) $f(\alpha_1 \# \cdots \# \alpha_n) = f(\alpha_1) \# \cdots \# f(\alpha_n)$ where α_i are

connected.

REMARK: By Lemmas 2.1 this scheme actually defines a function from $\mathfrak{A}$ into $\mathfrak{B}$. It also is clear that $\alpha = \beta \longrightarrow f(\alpha) = f(\beta)$.

THEOREM 3.1: There exists a mapping from the set of equivalence classes of $\mathfrak{A}$ onto the set of equivalence classes of $\mathfrak{B}$ which preserves both relations $<_1$ and $<_2$.

Proof. One can easily show by induction on the length of β that if $\beta \varepsilon \mathfrak{B}$ and β contains no redundant occurrences of the diagram o then β is an image of some $\alpha \varepsilon \mathfrak{A}$ under the mapping f defined in Definition 3.11. It is easy to show that each member of $\mathfrak{B}$ is equal to some member of $\mathfrak{B}$ which has no redundant occurrences of the diagram o. Below we will show that f preserves both relations $<_1$, $<_2$. Thus the desired mapping on the classes is the mapping on the classes induced by f.

We prove by induction on the sum of the lengths of α and β the following:

$$\alpha <_i \beta \longrightarrow f(\alpha) <_i f(\beta) \quad \text{for } i = 1, 2.$$

We first note that it is sufficient to consider only the cases where α, β are both connected. In all other cases the proofs depend on Lemma 2.2 and are similar. For this reason we give here only the proof for the most difficult case, namely the case where α has the form (1,1,x), β has the form (1,1,y), and the relation

in question is $<_2$.

Let $(1,1,x) <_2 (1,1,y)$, then by Lemma 2.2h,

$$(1,1,x) \leq_2 y \vee (x <_1 y \wedge x <_2 (1,1,y)).$$

Now by our induction hypothesis we get:

$$f((1,1,x)) \leq_2 f(y) \vee (f(x) <_1 f(y) \wedge f(x) <_2 f((1,1,y)))$$

so by the definition of f,

$$\langle -1+f(x)\rangle \leq_2 f(y) \vee (f(x) <_1 f(y) \wedge f(x) <_2 \langle -1+f(y)\rangle).$$

<u>Case 1</u>. $\langle -1+f(x)\rangle \leq_2 f(y)$:

It cannot be the case that f(y) is of the form $\alpha^{(n)}$ because $\alpha^{(n)} <_2 \langle\gamma\rangle$ for all γ. Therefore $-1 \# f(y) = f(y)$; so we have $\langle -1+f(x)\rangle \leq_2 -1 + f(y)$. Thus by Definition 3.8 part b_{10} we get $\langle -1+f(x)\rangle <_2 \langle -1+f(y)\rangle$, that is, $f((1,1,x) <_2 f((1,1,y))$.

<u>Case 2.</u> $f(x) <_1 f(y) \wedge f(x) <_2 \langle -1 \# f(y)\rangle$:

from this we get

$$-1+f(x) <_1 -1+f(y) \wedge -1+f(x) <_2 \langle -1+f(y)\rangle$$

so by Definition 3.7 part b_{10} we get

$$\langle -1 + f(x)\rangle <_2 \langle -1 + f(y)\rangle$$

that is, $f((1,1,x)) <_2 f((1,1,y))$.

<u>COROLLARY 3.1a</u>: There is a mapping from the classes of $\mathfrak{A}$ onto the classes of O which preserves both relations $<_1$ and $<_2$.

<u>Proof</u>. One can get a mapping from O onto $\mathfrak{B}$ which preserves =, $<_1$, and $<_2$ by sending each diagram which denotes a finite ordinal into the diagram which denotes

its successor and by sending each diagram which denotes an infinite ordinal into itself. This induces a mapping g on the classes. Let h be the mapping whose existence is asserted by our Theorem 3.1, then $g^{-1} \circ h$ is the desired mapping.

<u>COROLLARY 3.1b</u>:

(i) $\alpha <_1 \langle\alpha\rangle$

(ii) $\alpha <_2 [\alpha]$

(iii) $\alpha <_2 \langle\alpha\rangle$.

<u>Proof</u>. From Lemma 2.1 together with the main theorem.

4. The System O'

In this section we define a system O' by adjoining to O the symbol Ω and a new operation $\alpha \longrightarrow \omega^{\alpha}$, where ω^{α} is simply to be regarded as a combination of symbols. We will define a relation $<$ on O' which will turn out to be an extension of $<_2$ on O.

We also define a hierarchy of increasing functions $\{\overline{\psi}_c\}$, $c \varepsilon O'$ each of which maps O into O. The motivation of $\overline{\psi}_c$ is to construct within O (and thereby within Takeuti's $\{x \mid x <_2 2\}$) a variant of the Bachmann hierarchy for $c < \Omega^{\Omega^{\Omega^{\cdots}}}$.

<u>Definition 4.1</u>: The members of O' are given inductively

by the following:

a) $\Omega \varepsilon O'$

b) $\alpha \varepsilon O \longrightarrow \alpha \varepsilon O'$

c) $\alpha \varepsilon O' \longrightarrow \omega^{\alpha} \varepsilon O'$

d) $\alpha, \beta \varepsilon O' \longrightarrow \alpha \# \beta \varepsilon O'$

<u>NOTATION</u>: We use $\alpha \equiv \beta$ to mean α and β are indentical as formal expressions.

<u>Definition 4.2</u>: For each x in O we define a function $\sigma(x)$ as follows:[8]

$$\sigma(x) = \begin{cases} [o] \text{ if } x = \langle y \rangle \# \alpha^{(n)}; & o \leq n \\ o \quad \text{otherwise} \end{cases}$$

<u>Definition 4.3</u>: (CONNECTED, COMPONENT, PROPER COMPONENT)

Let $\alpha \varepsilon O'$, then α is said to be connected if and only if:

$\alpha \equiv o \vee \alpha \equiv [x] \vee \alpha \equiv \langle y \rangle \vee \alpha \equiv \omega^{x} \vee \alpha \equiv \Omega$.

The definitions of <u>component</u> and proper component are the same as in Definitions 3.5 and 3.6 respectively.

<u>Definition 4.4</u>: (LENGTH)

To each α in O' we define <u>length of α</u>, denoted by $\ell(\alpha)$ as follows:

a) $\ell(o) = 1$

---

8 $\alpha^{(n)}$ was defined in Definition 3.9.

b) $\ell([x]) = \ell(x) + 1$

c) $\ell(\langle x \rangle) = \ell(x) + 1$

d) $\ell(\omega^x) = \ell(x) + 1$

e) $\ell(\Omega) = 1$

f) $\ell(\alpha \# \beta) = \ell(\alpha) + \ell(\beta) + 1$

<u>Definition 4.5</u>: (EQUALITY)

We define $\alpha = \beta$ inductively.

a) Let α be connected then $\alpha = \beta$ if and only if one of the following holds:

a_1) $\alpha, \beta \varepsilon O$ and $\alpha = \beta$ in O

a_2) $\alpha \equiv \omega^x$, $\beta \equiv \omega^y$, and $x = y$

a_3) $\alpha \equiv [x]$, $\beta \equiv \omega^y$, and $x \# \sigma(x) = y$

a_4) $\alpha \equiv \omega^x$, $\beta \equiv [y]$, and $x = y \# \sigma(y)$

a_5) $\alpha \equiv \langle x \rangle$, $\beta \equiv \omega^y$, and $\langle x \rangle = y$

a_6) $\alpha \equiv \omega^x$, $\beta \equiv \langle y \rangle$, and $x = \langle y \rangle$

a_7) $\alpha \equiv \omega^x$, $\beta \equiv \Omega$, and $x = \Omega$

a_8) $\alpha \equiv \Omega$, $\beta \equiv \omega^x$, and $\Omega = x$

a_9) $\alpha \equiv \Omega$, $\beta \equiv \Omega$

b) If α is not connected then the definition of $\alpha = \beta$ reads the same as Definition 3.7b.

<u>Definition 4.6</u>: (ORDER)

We define $<$ on O' inductively as follows:

a) Let α have components $\alpha_1, \alpha_2, \ldots, \alpha_k$

β have components $\beta_1, \beta_2, \ldots, \beta_h$,

then the definition of $\alpha < \beta$ reads the same as Definition 3.8a reading $<$ in place of $<_i$.

b) For connected members of O', $\alpha < \beta$ if and only if one of the following hold:

b_1) $\alpha, \beta \varepsilon O$ and $\alpha <_2 \beta$ in O

b_2) $\alpha \varepsilon O$ and $\beta \equiv \Omega$

b_3) $\alpha \equiv \omega^x$, $\beta \equiv \omega^y$, and $x < y$

b_4) $\alpha \equiv [x]$, $\beta \equiv \omega^y$, and $x \# \sigma(x) < y$

b_5) $\alpha \equiv \omega^x$, $\beta \equiv [y]$, and $x < y \# \sigma(y)$

b_6) $\alpha \equiv \langle x \rangle$, $\beta \equiv \omega^y$, and $\langle x \rangle < y$

b_7) $\alpha \equiv \omega^x$, $\beta \equiv \langle y \rangle$, and $x < \langle y \rangle$

b_8) $\alpha \equiv \omega^x$, $\beta \equiv \Omega$, and $x < \Omega$

b_9) $\alpha \equiv \Omega$, $\beta \equiv \omega^y$, and $\Omega < y$.

<u>REMARKS</u>: It is quite easy (although very tedious) to show that:

(1) Equality is an equivalence relation.

(2) $<$ is a total ordering which preserves $<_2$ on O.

(3) $a = b \wedge a < c \longrightarrow b < c$

(4) $a = b \wedge c < a \longrightarrow c < b$

<u>LEMMA 4.1</u>: (FIRST NORMAL FORM)

$$\alpha \neq 0 \longrightarrow \alpha = \omega^{\alpha_n} \# \ldots \# \omega^{\alpha_1};$$

<u>Proof</u>. It is sufficient to show this for each non-zero component. Let β be a non-zero component of α.

<u>Case 1</u>. $\beta \equiv \Omega$,

then $\beta = \omega^{\Omega}$.

<u>Case 2</u>. $\beta \equiv [x]$

then $\beta = \omega^{x \# \sigma(x)}$.

<u>Case 3</u>. $\beta \equiv \langle x \rangle$,

then $\beta = \omega^{\langle x \rangle}$.

<u>REMARK</u>: From the definition of equality (Definition 4.5b) we see that this representation is unique except for the order of the components.

<u>Definition 4.7</u>: (DEGREE)

To each $\alpha \varepsilon O'$ we define degree denoted by $d(\alpha)$ as follows:

(i) $d(o) = o$

(ii) $d(\omega^{\alpha_1} \# \ldots \# \omega^{\alpha_n}) = n$.

We see from the preceeding remark that $\alpha = \beta$ $\longrightarrow$ $d(\alpha) = d(\beta)$.

<u>Definition 4.8</u>: (ADDITION)

Let α^* denote the largest component of α; we define $\Omega + \alpha$ as follows:

$$\Omega + \alpha = \begin{cases} \alpha \text{ if } \Omega < \alpha^* \\ \Omega \# \alpha \text{ otherwise.} \end{cases}$$

<u>Definition 4.9</u>: (MULTIPLICATION)

We define $\Omega \cdot \alpha$ as follows:

$$\Omega \cdot 0 = 0$$

$$\Omega \cdot (\omega^{\alpha_1} \# \ldots \# \omega^{\alpha_n}) = \omega^{\Omega+\alpha_1} \# \ldots \# \omega^{\Omega+\alpha_n}$$

<u>LEMMA 4.2</u>:

(i) $\alpha < \beta \longrightarrow \Omega + \alpha < \Omega + \beta$

(ii) $\alpha < \beta \longrightarrow \Omega \cdot \alpha < \Omega \cdot \beta$

(iii) $\Omega \cdot \alpha = \Omega \cdot \beta \longrightarrow \alpha = \beta$

(iv) $\Omega \cdot (\alpha \# \beta) = \Omega \cdot \alpha \# \Omega \cdot \beta$

(v) If $\Omega \leq \alpha$, then $\alpha = \Omega + \beta$ for some β

(vi) If α is connected and $\Omega \leq \alpha$, then $\alpha = \omega^{\Omega+\beta}$ for some β

(vii) Every $\beta \varepsilon O'$ can be written in the form $\Omega \cdot \alpha \# a$ where $a < \Omega$.

(viii) If $\Omega \cdot \alpha$ has a proper component, then that component must be equal or greater than Ω.

(ix) If $\Omega \cdot \alpha \# a = \Omega \cdot \beta \# b$ and $a, b < \Omega$, then $\alpha = \beta$ and $a = b$

(x) $\alpha \neq 0 \longrightarrow \Omega \leq \Omega \cdot \alpha$.

<u>Proof.</u>

(i) The only non-trivial case is when

$$\Omega + \alpha = \Omega \# \alpha$$

$$\Omega + \beta = \beta.$$

For this to happen, it must be that $\alpha^* \leq \Omega \leq \beta^*$, where α^*, β^* are the largest components of α, β respectively. But then all components of $\Omega \# \alpha$ are less than β^*, so

$$\Omega + \alpha = \Omega \# \alpha < \beta^* \leq \beta = \Omega + \beta \text{ by Definition } 4.6a_1.$$

(ii) Immediate from Definition 4.9 and (i)

(iii) Immediate from (ii)

(iv) Immediate from Definition 4.9

(v) $\Omega \leq \alpha^*$ where α^* is the greatest component of α.

<u>Case 1</u>. $\Omega < \alpha$

Then $\alpha = \Omega + \alpha$.

<u>Case 2</u>. $\Omega = \alpha^*$

Then α has the form $\alpha = \Omega \# \beta$ and $\beta^* \leq \Omega$ where β^* is the greatest component of β; so in this case $\Omega \# \beta = \Omega + \beta$; thus $\alpha = \Omega + \beta$

(vi) α can be written in the form ω^y. From Definitions $4.5a_8$ and $4.6b_9$ we see that $\Omega \leq y$. By (v), $y = \Omega + \beta$

(vii)

<u>Case 1</u>. All components of α are less than Ω.

Then $\alpha = \Omega \cdot 0 \# \alpha$

<u>Case 2.</u> α has components which are equal or greater than Ω. Let these components be $\alpha_1, \alpha_2, \ldots, \alpha_n$.

$$\alpha = \alpha_1 \# \ldots \# \alpha_n \# a \quad \text{where } a < \Omega$$

$$\alpha = \omega^{\Omega + x_1} \# \ldots \# \omega^{\Omega + x_n} \# a \quad \text{by (vi)}$$

$$\alpha = \Omega \cdot (\omega^{x_1} \# \ldots \# \omega^{x_n}) \# a \quad \text{by Definition 4.9.}$$

(viii) Clearly $\alpha \neq 0$; so $\alpha = \omega^{\alpha_1} \# \ldots \# \omega^{\alpha_n}$

$$\Omega \cdot \alpha = \omega^{\Omega + \alpha_1} \# \ldots \# \omega^{\Omega + \alpha_n}$$

but $\Omega \leq \Omega + \alpha_i$ by Definition 4.8 so

$$\Omega \leq \omega^{\Omega+\alpha_i} \quad \text{by Definitions } 4.5a_8,\ 4.6b_9$$

(ix)

<u>Case 1</u>. $\Omega\cdot\alpha$ or a have no proper components.

In this case it's easy to see using (viii) that $\Omega\cdot\alpha = \Omega\cdot\beta$ and $a = b$, so by (iii), $\alpha = \beta$

<u>Case 2</u>. $\Omega\cdot\alpha$ and a have proper components. Using (viii) we can see that any one-one correspondence between the left side and the right side under which the correspondents are equal must induce correspondences of this kind from $\Omega\cdot\alpha$ to $\Omega\cdot\beta$ and from a to b respectively. So $\Omega\cdot\alpha = \Omega\cdot\beta$ and $a = b$. Thus by (iii), $\alpha = \beta$.

(x) Immediate from the definition of multiplication.

<u>LEMMA 4.3</u>:(SECOND NORMAL FORM)

If $\alpha \neq 0$, then α has unique (up to the order of the terms in the # sum) representation

$$\alpha = \omega^{\Omega\cdot\alpha_1}\#a_1 \# \ldots \# \omega^{\Omega\cdot\alpha_n}\#a_n \quad \text{where } a_i < \Omega.$$

<u>Proof</u>. By Lemma 4.2.

<u>Definition 4.10</u>: (RANK)

We define <u>rank of α</u> denoted by $r(\alpha)$ as follows:

If $\alpha = o$, then $r(o) = o$

If $\alpha \neq o$, then $r(\alpha) = d(\alpha) + \sum_{i=1}^{n} \overset{r}{\cancel{d}}(\alpha_i)$ where

α_i are as in the normal form representation just described.

<u>REMARK</u>: $\alpha = \beta \longrightarrow r(\alpha) = r(\beta)$.

This follows from the fact that the normal form is unique.

<u>Definition 4.11</u>: We now define a mapping $\rho(\alpha)$ by induction on the length of α from O onto the set of all members of O which have no components of the form $\langle x \rangle$.

1. If α is non-connected with components $\alpha_1, \ldots, \alpha_n$

$$\rho(\alpha) = \rho(\alpha_1) \# \ldots \# \rho(\alpha_n).$$

2. If α is connected,

$$\rho(\alpha) = \begin{cases} o & \text{if } \alpha = o \\ [\langle y \rangle \# \alpha^{(n+1)}] & \text{if } \alpha = [\langle y \rangle \# \alpha^{(n)}];\ o \leq n \\ [\langle y \rangle] & \text{if } \alpha = \langle y \rangle \\ \alpha & \text{otherwise.} \end{cases}$$

<u>REMARK</u>: That this mapping is onto should be clear from the definition.

<u>LEMMA 4.4</u>:

(i) $\alpha \leq_2 \rho(\alpha)$

(ii) $\alpha <_2 \beta \longrightarrow \rho(\alpha) <_1 \rho(\beta) \wedge \rho(\alpha) <_2 \rho(\beta)$

(iii) $\rho(\alpha) <_1 \langle x \rangle$

(iv) $\alpha <_2 \langle x \rangle \longrightarrow \rho(\alpha) <_2 \langle x \rangle$.

<u>Proof</u>.

(i) Immediate from the definition of $\rho(\alpha)$

(ii) It is sufficient to consider only cases where α, β are connected.

<u>Case 1</u>. $\alpha = 0$

Then trivial.

<u>Case 2</u>. $\alpha \neq 0$

Then $\rho(\alpha)$, $\rho(\beta)$ are of the form $[r]$, $[s]$ for some r, s respectively. By Definitions $3.8b_4$, $3.8b_7$ it is sufficient to show that $r <_2 s$.

<u>Case 2.1</u>. $\alpha \equiv \langle x \rangle$, $\beta \equiv \langle y \rangle$

Then $\rho(\alpha) = [\langle x \rangle]$, $\rho(\beta) = [\langle y \rangle]$ so $r = \langle x \rangle$, $s = \langle y \rangle$.

<u>Case 2.2</u>. $\alpha \equiv \langle x \rangle$, $\beta \equiv [y]$

So $\rho(\alpha) = [\langle x \rangle]$, $\rho(\beta) = [y]$ or $\rho(\beta) = [y\#[o]]$ and $r = \langle x \rangle$, $s = y$ or $s = y\#[o]$. Now $\langle x \rangle <_2 [y] \longrightarrow \langle x \rangle \leq_2 y$ by Definition $3.8b_9$.

<u>Case 2.2.1</u>. $\langle x \rangle = y$

Then $s = y\#[o]$ by definition of $\rho(\beta)$, so $r <_2 s$.

<u>Case 2.2.2</u>. $\langle x \rangle <_2 y$

Then it doesn't matter whether $s = y$ or $s = y\#[o]$, we still get $r <_2 s$.

<u>Case 2.3</u>. $\alpha \equiv [x]$, $\beta \equiv \langle y \rangle$

Then $r = x$ or $r = x\#[o]$

$s = \langle y \rangle$.

Now $[x] <_2 \langle y \rangle \longrightarrow x <_2 \langle y \rangle$ by Definition $3.8b_8$, so $x\#[o] <_2 \langle y \rangle$.

Thus regardless of which of the two values r has, $r <_2 s$.

<u>Case 2.4</u>. $\alpha \equiv [x]$, $\beta \equiv [y]$

So $r = x$ or $r = x \# [o]$, $s = y$ or $s = y \# [o]$.

Now $[x] <_2 [y] \longrightarrow x <_2 y$ by Definition $3.8b_7$.

<u>Case 2.4.1</u>. $r = x$, $s = y$

$x <_2 y$ is already the desired result.

<u>Case 2.4.2</u>. $r = x \# [o]$, $s = y \# [o]$

Then $r <_2 s$ follows from $x <_2 y$.

<u>Case 2.4.3</u>. $r = x$, $s = y \# [o]$

Then $r <_2 s$ follows from $x <_2 y$.

<u>Case 2.4.4</u>. $r = x \# [o]$, $s = y$

In this case we see by the definition of $\rho(\alpha)$ that x must have had the form $x = \langle t \rangle \# \alpha^{(n)}$ for some t, some n, so $x <_2 y \longrightarrow x \# [o] \leq_2 y$.

It cannot be that $x \# [o] = y$, for then y would also have the form $\langle h \rangle \# \alpha^{(n+1)}$ and s would be $y \# [o]$ which is a contradiction. So $x \# [o] <_2 y$, $r <_2 s$.

(iii) Follows from Definition $3.8b_3$, $3.8b_5$ since each component of $\rho(\alpha)$ is of the form o or [illegible] $[\gamma]$.

(iv) It is sufficient to consider cases where α is connected.

<u>Case 1</u>. $\alpha \equiv o$

Then $\rho(\alpha) = o$, and $o <_2 \langle x \rangle$.

<u>Case 2</u>. $\alpha \equiv \langle y \rangle$ some y

Then $\rho(\alpha) = [\langle y \rangle]$ but $\langle y \rangle <_2 \langle x \rangle \longrightarrow$

$[\langle y\rangle] <_2 \langle x\rangle$ by Definition $3.8b_8$.

<u>Case 3</u>. $\alpha \equiv [y]$ some y

<u>Case 3.1</u>. y not of form $\langle s\rangle \# \alpha^{(n)}$ some s, some n

Then $\rho(\alpha) = \alpha <_2 \langle x\rangle$.

<u>Case 3.2</u>. y is of form $\langle s\rangle \# \alpha^{(n)}$ sone s, some n

Then $\rho(\alpha) = [\langle s\rangle \# \alpha^{(n+1)}]$. Now

$[\langle s\rangle \# \alpha^{(n)}] <_2 \langle x\rangle \longrightarrow \langle s\rangle \# \alpha^{(n)} <_2 \langle x\rangle$ by Definition $3.8b_8$. Also

$$\langle s\rangle \# \alpha^{(n+1)} <_2 \langle x\rangle$$

therefore $[\langle s\rangle \# \alpha^{(n+1)}] <_2 \langle x\rangle$ by Definition $3.8b_8$.

<u>Definition 4.12</u>: We define a function $F(\alpha)$ from O into O' by induction on the length of α as follows:

$$F(o) = o$$

$$F([x]) = \omega^x$$

$$F(\langle x\rangle) = \Omega \cdot \omega^{F(x)}$$

$$F(\alpha \# \beta) = F(\alpha) \# F(\beta).$$

<u>LEMMA 4.5</u>:

(i) If $\alpha < \Omega$, then there exists β such that $\beta \varepsilon O$ and $\alpha = \beta$.

(ii) $\{x \mid x < \Omega\}$ is well ordered by $<$.

(iii) $\alpha < \Omega \longrightarrow \alpha \leq \omega^\alpha$.

(iv) If $\alpha \varepsilon O$, then $\alpha = F(\rho(\alpha))$.

(v) If $\alpha < \Omega$, then $\alpha = F(x)$ for some $x \varepsilon O$.

(vi) If $\alpha < \Omega$, then $\alpha \leq \omega^{F(\rho(\alpha))}$.

(vii) If $\alpha < \Omega$ and $\omega^{\alpha} = \alpha$, then $\alpha = \langle x \rangle$ for some x.

<u>Proof</u>.

(i) By induction on the length of α.

If all components of α are in O we are finished. Otherwise let α_i be a component of α which is not in O. Now $\alpha_i \equiv \omega^x$ and by Definition $4.6b_9$ $x < \Omega$. So by induction there exists $\beta \varepsilon O$ such that $x = \beta$.

<u>Case 1</u>. x has the form $\langle y \rangle \# \alpha^{(n)}$

<u>Case 1a</u>. $n = 0$

Then $x = \langle y \rangle$

$$\alpha = \omega^x = \omega^{\langle y \rangle} = \langle y \rangle \text{ by Definition } 4.5a_6$$

<u>Case 1b</u>. $n > 0$

Then $x = y \# \alpha^{(n-1)} \# \sigma(y \# \alpha^{(n-1)})$ so

$\omega^x = \omega^{y \# \alpha^{(n-1)} \# \sigma(y \# \alpha^{(n-1)})} = [y \# \alpha^{(n-1)}]$ by Definition $4.5a_4$.

<u>Case 2</u>. x does not have form $\langle y \rangle \# \alpha^{(n)}$

Then $\sigma(x) = o$ so

$\omega^x = \omega^{x \# \sigma(x)} = [x]$ by Definition $4.5a_4$.

(ii) This follows from (i) together with the fact that $<_2$ is a well ordering of O and agrees with $<$ on O.

(iii) Assume not, then there exists x such that

(*) $\omega^x < x$.

Let x_o be the smallest such x. By Definition $4.6b_3$, $\omega^{\omega^{x_o}} < \omega^{x_o}$; this contradicts the fact that x_o is the smallest x with property (*).

(iv) It is sufficient to consider only cases where α is connected.

<u>Case 1</u>. $\alpha = o$

Then $\rho(\alpha) = o$, $F(\rho(\alpha)) = o$ by definition of $F(\alpha)$

<u>Case 2.</u> $\alpha = \langle y \rangle$

Then $\rho(\alpha) = [\langle y \rangle]$, $F(\rho(\alpha)) = \omega^{\langle y \rangle} = \langle y \rangle$

<u>Case 3</u>.

$\alpha = [y]$

<u>Case 3a</u>. $y = \langle t \rangle \# \alpha^{(n)}$, some t, some n

Then $\rho(\alpha) = [\langle t \rangle \# \alpha^{(n+1)}]$, $F(\rho(\alpha)) = \omega^{\langle t \rangle \# \alpha^{(n+1)}}$

$= \omega^{\langle t \rangle \# \alpha^{(n)} \# \sigma(\langle t \rangle \# \alpha^{(n)})} = [\langle t \rangle \# \alpha^{(n)}]$ by Definition $4.5a_4$

$= [y] = \alpha$

<u>Case 3b</u>. y does not have the form $\langle t \rangle \# \alpha^{(n)}$, some t, some n.

Then $\sigma(y) = 0$.

By definition of $\rho(\alpha)$, $\rho(\alpha) = \alpha$

$F(\rho(\alpha)) = F(\alpha) = \omega^y = \omega^{y \# \sigma(y)} = [y]$

(v) By (i), (iv).

(vi) By (i), (iii), (iv)

(vii) By (i), $\omega^\alpha = \alpha = \beta$ where $\beta \varepsilon O$. From the

definition of equality on O' it is clear that β must be connected. This leaves three possibilities for β; $\beta = o$, $\beta = [y]$ for some y, or $\beta = \langle x \rangle$ for some x. We show that the first two are impossible by showing that $o <_2 \omega^o$ and $[y] <_2 \omega^{[y]}$.

1. To show $o <_2 \omega^o$.

This follows from $o <_2 [o] = \omega^o$ by Definitions $3.8b_2$ and $4.5a_3$.

2. To show $[y] <_2 \omega^{[y]}$.

By (iii) $[y] \leq_2 \omega^{[y]}$, so we need only show that $[y] \neq \omega^{[y]}$.

Suppose $[y] = \omega^{[y]}$, then by Definition $4.5a_3$, it must be the case that

(*) $[y] = y \# \sigma(y)$.

<u>Case 1</u>. $\sigma(y) = o$

Then $[y] = y$ which contradicts Corollary 3.1b(ii)

<u>Case 2</u>. $\sigma(y) = [o]$

Then y has the form $\langle x \rangle \# \alpha^{(n)}$ by definition of σ, but $\langle x \rangle \# \alpha^{(n)} \# [o] <_2 [\langle x \rangle \# \alpha^{(n)}]$. That is, $y \# \sigma(y) <_2 [y]$ which is a contradiction to (*).

<u>LEMMA 4.6</u>: $\alpha <_1 \beta \longleftrightarrow F(\alpha) < F(\beta)$.

<u>Proof</u>.

a) To show $\alpha <_1 \beta \longrightarrow F(\alpha) < F(\beta)$.

We proceed by induction on the sum of the lengths of α,

β. It is sufficient to consider only cases where α, β are connected.

<u>Case 1</u>. $\alpha = 0$ then trivial

<u>Case 2</u>. $\alpha \equiv \langle y \rangle$, $\beta \equiv [x]$

By Definition $3.8b_5$, this case will not arise.

<u>Case 3</u>. $\alpha \equiv [x]$, $\beta \equiv [y]$

Then $F(\alpha) = \omega^x$ $F(\beta) = \omega^y$.

By Definition $3.8b_4$, $\alpha <_1 \beta \longrightarrow x <_2 y$ so

$\omega^x < \omega^y$.

<u>Case 4.</u> $\alpha \equiv \langle x \rangle$, $\beta \equiv \langle y \rangle$

Then $F(\alpha) = \Omega \cdot \omega^{F(x)}$; $F(\beta) = \Omega \cdot \omega^{F(y)}$.

Now $x <_1 y$, so by induction $F(x) < F(y)$; from this our desired conclusion follows.

<u>Case 5</u>. $\alpha \equiv [x]$, $\beta \equiv \langle y \rangle$

Then $F(\alpha) = \omega^x$, $F(\beta) = \Omega \cdot \omega^{F(y)}$.

Now $x \varepsilon 0$, so $x < \Omega$; $\omega^x < \Omega \leq \Omega \cdot \omega^{F(y)}$

b) To show $F(\alpha) < F(\beta) \longrightarrow \alpha <_1 \beta$

This follows from a) since $<_1$, $<$ are linear orderings.

<u>LEMMA 4.7:</u> If $c \neq o$, then c can be expressed in the form

$$c = \omega^{F(\alpha_1)} \# \omega^{F(\alpha_2)} \# \ldots \# \omega^{F(\alpha_n)}.$$

<u>Proof</u>. By induction on the rank of c.

Let $\overset{S}{\cancel{\alpha}} = \omega^{\Omega \cdot \alpha \# a}$ with $a < \Omega$ be a term in the <u>Second</u>

<u>Normal Form</u> expansion of c (see Lemma 4.3). By Lemma 4,5(v), $a = F(x)$ some x.

<u>Case 1</u>. $\alpha = 0$

Then $s = \omega^{F(x)}$

<u>Case 2</u>. $\alpha \neq 0$

Now $r(\alpha) < r(c)$ so by induction hypothesis

$$\alpha = \omega^{F(\alpha_1)} \# \ldots \# \omega^{F(\alpha_n)}$$

$$\omega^{\Omega\cdot\alpha\#a} = \omega^{\Omega\cdot\omega^{F(\alpha_1)}\# \ldots \#\Omega\cdot\omega^{F(\alpha_n)}\# F(x)}$$

$$\omega^{\Omega\cdot\alpha\#a} = \omega^{F(\langle\alpha_1\rangle)\# \ldots \#F(\langle\alpha_n\rangle)\#F(x)} \quad \text{by definition of } F$$

$$\omega^{\Omega\cdot\alpha\#a} = \omega^{F(\langle\alpha_1\rangle\# \ldots \#\langle\alpha_n\rangle\#x)}.$$

<u>Definition 4.13</u>: To each $c \varepsilon O'$ and $\alpha \varepsilon O$ we define $\overline{\psi}_c(\alpha)$ as follows:

1. If $c = o$, then $\overline{\psi}_c(\alpha) = \langle\rho(\alpha)\rangle$

2. If $c \neq o$, then $c = \omega^{F(\alpha_1)} \# \ldots \#\omega^{F(\alpha_n)}$ and $\overline{\psi}_c(\alpha) = \langle\rho(\alpha)\#\langle\alpha_1\rangle\#\ldots\#\langle\alpha_n\rangle\rangle$.

<u>REMARK</u>: $\overline{\psi}_c(\alpha) \varepsilon O$.

<u>LEMMA 4.8</u>: $\langle\gamma\rangle = \overline{\psi}_c(\alpha)$ for some $\alpha \varepsilon O$, $c \varepsilon O'$.

<u>Proof</u>.

<u>Case 1</u>. γ has no component of the form $\langle x\rangle$

Then by definition of $\rho(\alpha)$, $\gamma = \rho(\alpha)$ some α. So $\langle\gamma\rangle = \langle\rho(\alpha)\rangle = \overline{\psi}_o(\alpha)$.

Case 2. γ has components of the form $\langle x\rangle$.

Then $\gamma = \rho(\alpha)\#\langle\alpha_1\rangle\# \ldots \#\langle\alpha_n\rangle$, so $\langle\gamma\rangle = \overline{\psi}_c(\alpha)$

where $c = \omega^{F(\alpha_1)}\# \ldots \#\omega^{F(\alpha_n)}$.

THEOREM 4.1:

(i) $c_1 < c_2 \longrightarrow \overline{\psi}_{c_1}(a_1) <_1 \overline{\psi}_{c_2}(a_2)$

(ii) $c_1 = c_2 \wedge a_1 < a_2 \rightarrow \overline{\psi}_{c_1}(a_1) <_1 \overline{\psi}_{c_2}(a_2)$.

Proof. (i)

Case 1. $c_1 = o,\ c_2 \neq o$

Then $\overline{\psi}_{c_1}(a_1) = \langle\rho(a_1)\rangle$

$$\overline{\psi}_{c_2}(a_2) = \langle\rho(a_2)\#\langle\alpha_1\rangle\# \ldots \#\langle\alpha_n\rangle\rangle.$$

Now by Lemma 4.4(iii),

$$\rho(a_1) <_1 \rho(a_2)\#\langle\alpha_1\rangle\# \ldots \#\langle\alpha_n\rangle$$

so $\langle\rho(a_1)\rangle <_1 \langle\rho(a_2)\#\langle\alpha_1\rangle\# \ldots \#\langle\alpha_n\rangle\rangle$ by Definition $3.8b_6$

Case 2. $c_1 \neq o,\ c_2 \neq o$

Let $c_1 = \omega^{F(\alpha_1^1)}\# \ldots \#\omega^{F(\alpha_n^1)}$

$c_2 = \omega^{F(\alpha_1^2)}\# \ldots \#\omega^{F(\alpha_k^2)}$

so $\overline{\psi}_{c_1}(a_1) = \langle\rho(a_1)\#\langle\alpha_1^1\rangle\# \ldots \#\langle\alpha_n^1\rangle\rangle$

$\overline{\psi}_{c_2}(a_2) = \langle\rho(a_2)\#\langle\alpha_1^2\rangle\# \ldots \#\langle\alpha_k^2\rangle\rangle$.

Now $c_1 < c_2$ means $\omega^{F(\alpha_1^1)}\# \ldots \#\omega^{F(\alpha_n^1)} < \omega^{F(\alpha_1^2)}\# \ldots \#\omega^{F(\alpha_k^2)}$

<u>Case 2.1</u>. $\omega^{F(\alpha_i^1)} < \omega^{F(\alpha_j^2)}$ all i, some j (i.e.)

$c_1 < c_2$ by Definition $4.6a_1$

Then $F(\alpha_i^1) <$ [redacted] $^{F(\alpha_j^2)}$ all i

$\alpha_i^1 <_1 \alpha_j^2$ all i by Lemma 4.6

$\langle\alpha_i^1\rangle <_1 \langle\alpha_j^2\rangle$ all i

$\langle\alpha_1^1\rangle \# \ldots \# \langle\alpha_n^1\rangle <_1 \langle\alpha_j^2\rangle$

$\rho(a_1)\#\langle\alpha_1^1\rangle\# \ldots \#\langle\alpha_n^1\rangle <_1 \langle\alpha_j^2\rangle \leq_1 \rho(a_2)\#\langle\alpha_j^2\rangle$

$\rho(a_1)\#\langle\alpha_1^1\rangle\# \ldots \#\langle\alpha_n^1\rangle <_1 \rho(a_2)\#\langle\alpha_1^2\rangle\# \ldots \#\langle\alpha_k^2\rangle$

$\langle\rho(a_1)\#\langle\alpha_1^1\rangle\#\ldots\#\langle\alpha_n^1\rangle\rangle <_1 \langle\rho(a_2)\#\langle\alpha_1^2\rangle\#\ldots\#\langle\alpha_k^2\rangle\rangle$

<u>Case 2.2.</u> $c_1 < c_2$ by Definition $4.6a_2$

Then the proof is similar to Case 2.1.

<u>Case 2.3</u>. $c_1 < c_2$ by Definition $4.6a_3$

Then the proof is similar to Case 2.1.

(ii) By Lemma 4.4(ii).

<u>Definition 4.14</u>: Let $x, c \in O'$. We define a relation $T^*(x,c)$ by induction on the rank of c as follows:

1. If $c = o$, then $T^*(x,c) \longleftrightarrow x = o$

2. If $c \neq o$, then $c = \omega^{\Omega\cdot\alpha_1\#a_1} \# \ldots \# \omega^{\Omega\cdot\alpha_n\#a_n}$.

$$T^*(x,c) \longleftrightarrow [x = \omega^{a_i} \vee T^*(x,\alpha_i) \text{ for some } i].$$

<u>REMARK</u>: $T^*(x,\omega^y) \longrightarrow T^*(x,\omega^y\#\omega^z)$.

<u>Definition 4.15</u>: Let $x, \gamma \in O$; $\gamma = \overline{\psi}_c(\alpha)$. We define the relation $P^*(x,\gamma)$ as follows:

1. $P^*(\alpha,\gamma)$
2. If $T^*(x,c)$ then $P^*(x,\gamma)$

<u>LEMMA 4.9</u>: $P^*(x,\overline{\psi}_c(\alpha)) \longrightarrow x < \overline{\psi}_c(\alpha)$.

<u>Proof</u>. By transfinite induction on $\overline{\psi}_c(\alpha)$ over $<$ on O

<u>Case 1</u>. $c = o$

Then $\overline{\psi}_c(\alpha) = \langle\rho(\alpha)\rangle$

$$P^*(x,\overline{\psi}_c(\alpha)) \longrightarrow x=\alpha$$

but $\alpha \leq \rho(\alpha) < \langle\rho(\alpha)\rangle$ by Lemma 4.4(i) and Corollary 3.1b respectively.

<u>Case 2</u>. $c \neq o$

Then $\overline{\psi}_c(\alpha) = \langle\rho(\alpha)\#\langle\alpha_1\rangle\#\ldots\#\langle\alpha_n\rangle\rangle$ where

$$c = \omega^{F(\alpha_1)} \# \ldots \# \omega^{F(\alpha_n)}.$$

Now α_i has the form $\rho(\alpha_o^i)\#\langle\alpha_1^i\rangle\#\ldots\#\langle\alpha_{n_i}^i\rangle$ where $\langle\alpha_j^i\rangle$ do not necessarily appear for $1 \leq j \leq n_i$. Now

$$\begin{aligned} F(\alpha_i) &= F(\rho(\alpha_o^i) \# \langle\alpha_1^i\rangle \# \ldots \# \langle\alpha_{n_i}^i\rangle) \\ &= F(\rho(\alpha_o^i)) \# F(\langle\alpha_1^i\rangle) \# \ldots \# F(\langle\alpha_{n_i}^i\rangle) \\ &= F(\rho(\alpha_o^i)) \# \Omega\cdot\omega^{F(\alpha_1^i)} \# \ldots \# \Omega\cdot\omega^{F(\alpha_{n_i}^i)} \\ &= F(\rho(\alpha_o^i)) \# \Omega\cdot\left(\omega^{F(\alpha_1^i)} \# \ldots \# \omega^{F(\alpha_{n_i}^i)}\right) \end{aligned}$$

$$\omega^{F(\alpha_i)} = \omega^{\Omega\cdot\left(\omega^{F(\alpha_1^i)} \# \ldots \# \omega^{F(\alpha_{n_i}^i)}\right) \# F(\rho(\alpha_o^i))}.$$

Now $\overset{*}{P}(x,\overline{\psi}_c(\alpha)) \longrightarrow x = \alpha \vee T^*(x,c)$ by the definition

of P^*, $\overset{*}{P}(x,\overline{\psi}_c(\alpha)) \longrightarrow x=\alpha \vee x=\omega^{F(\rho(\alpha_o^i))} \vee T^*(x,\omega^{F(\alpha_1^i)} \# \ldots$

$\# \omega^{F(\alpha_{n_i}^i)})$ for some i,

$\overset{*}{P}(x,\overline{\psi}_c(\alpha)) \longrightarrow$ ~~$x = \alpha \vee P^*(x,\langle\alpha_i\rangle)$~~ by the definition of P^*

$x=\alpha \vee x=\omega^{F(\rho(\alpha_o^i))} \vee P^*(x,\langle\alpha_i\rangle)$

<u>Case 2.1</u>. $x = \alpha$

Then $x = \alpha \leq \rho(\alpha) \leq \rho(\alpha)\#\langle\alpha_1\rangle\#,\ldots\#\langle\alpha_n\rangle$

By Corollary 3.1b(iii)

$\rho(\alpha)\#\langle\alpha_1\rangle\#\ldots\#\langle\alpha_n\rangle < \langle\rho(\alpha)\#\langle\alpha_1\rangle\#\ldots\#\langle\alpha_n\rangle\rangle = \overline{\psi}_c(\alpha)$

so $x < \overline{\psi}_c(\alpha)$.

<u>Case 2.2</u>. $x = \omega^{F(\rho(\alpha_o^i))}$

Now $\alpha_o^i \leq \rho(\alpha_o^i) \leq \rho(\alpha_o^i)\#\langle\alpha_1^i\rangle\# \ldots \#\langle\alpha_{n_i}^i\rangle$

by Corollary 3.1b(iii),

$\rho(\alpha_o^i)\#\langle\alpha_1^i\rangle\#\ldots\#\langle\alpha_{n_i}^i\rangle < \langle\rho(\alpha_o^i)\#\langle\alpha_1^i\rangle\#\ldots\#\langle\alpha_{n_i}^i\rangle\rangle =$

$= \langle\alpha_i\rangle$, so $\alpha_o^i < \langle\alpha_i\rangle$. Therefore,

$\omega^{F(\rho(\alpha_o^i))} < \langle\alpha_i\rangle$ by Lemma 4.5(iv) and Definition 4.6b$_7$

but $\langle\alpha_i\rangle < \overline{\psi}_c(\alpha)$ so

$$x = \omega^{F(\rho(\alpha_o^i))} < \overline{\psi}_c(\alpha)$$

<u>Case 3</u>. $P^*(x,\langle\alpha_i\rangle)$

Now $\langle\alpha_i\rangle < \overline{\psi}_c(\alpha)$

so by induction hypothesis $P^*(x,\langle\alpha_i\rangle) \longrightarrow x < \langle\alpha_i\rangle$ thus $x < \langle\alpha_i\rangle < \overline{\psi}_c(\alpha)$.

<u>LEMMA 4.10</u>: $P^*(x,\langle\beta\rangle) \longrightarrow \bigvee y(x \leq y \wedge T^*(y,\omega^{F(\beta)})$.

<u>Proof</u>. β has the form

$\rho(\alpha) \# \langle\beta_1\rangle \# \ldots \# \langle\beta_n\rangle$ where $\langle\beta_i\rangle$ might not necessarily appear for $1 \leq i \leq n$. Now

$$\begin{aligned} F(\beta) &= F(\rho(\alpha) \# \langle\beta_1\rangle \# \ldots \# \langle\beta_n\rangle) \\ &= F(\rho(\alpha)) \# F(\langle\beta_1\rangle) \# \ldots \# F(\langle\beta_n\rangle) \\ &= F(\rho(\alpha)) \# \Omega\cdot\omega^{F(\beta_1)} \# \ldots \# \Omega\cdot\omega^{F(\beta_n)} \quad \text{by the definition of } F \text{ (Def. 4.6)} \\ &= F(\rho(\alpha)) \# \Omega\cdot\left(\omega^{F(\beta_1)} \# \ldots \# \omega^{F(\beta_n)}\right) \end{aligned}$$

so $\omega^{F(\beta)} = \omega^{\Omega\cdot\left(\omega^{F(\beta_1)} \# \ldots \# \omega^{F(\beta_n)}\right) \# F(\rho(\alpha))}$.

Now by the definition of T^*,

(1) $T^*(y,\omega^{F(\beta)}) \longleftrightarrow y=\omega^{F(\rho(\alpha))} \vee T^*(y,\omega^{F(\beta_1)} \# \ldots \# \omega^{F(\beta_n)})$

$P^*(x,\langle\beta\rangle) \longrightarrow x=\alpha \vee T^*(x,\omega^{F(\beta_1)} \# \ldots \# \omega^{F(\beta_n)})$ by definition of P^*. Let x_o be such that $P^*(x_o,\langle\beta\rangle)$.

<u>Case 1.</u> $x_o = \alpha$

$x_o = \alpha \leq \omega^{\alpha} = \omega^{F(\rho(\alpha))}$ by Lemmas 4.5(iii), 4.5(iv)

so $y = \omega^{F(\rho(\alpha))}$ is the desired y by (1)

<u>Case 2.</u> $T^*(x_o,\omega^{F(\beta_1)} \# \ldots \# \omega^{F(\beta_n)})$

Then by (1), $y = x_o$ is the desired y.

<u>THEOREM 4.2</u>: $\overline{\psi}_{c_1}(a_o) < \overline{\psi}_{c_2}(b_o)$ if and only if one of the following holds:

I $\quad c_1 = c_2 \wedge a_o < b_o$

II $\quad c_1 < c_2 \wedge \bigwedge x(P^*(x, \overline{\psi}_{c_1}(a_o)) \longrightarrow x < \overline{\psi}_{c_2}(b_o))$

III $\quad \bigvee x(P*(x, \overline{\psi}_{c_2}(b_o)) \wedge \overline{\psi}_{c_1}(a_o) \leq x)$

<u>Proof</u>.

<u>SUFFICIENCY</u>

I $\quad \overline{\psi}_{c_1}(a_o) = \langle \rho(a_o) \# \langle \alpha_1 \rangle \# \ldots \# \langle \alpha_n \rangle \rangle$ where $\langle \alpha_i \rangle$ do not necessarily appear for $1 \leq i \leq n$

$\overline{\psi}_{c_2}(b_o) = \langle \rho(b_o) \# \langle \beta_1 \rangle \# \ldots \# \langle \beta_k \rangle \rangle$ where $\langle \beta_i \rangle$ do not necessarily appear for $1 \leq i \leq k$.

$$c_1 = c_2 \longrightarrow \omega^{F(\alpha_1)} \# \ldots \# \omega^{F(\alpha_n)} = \omega^{F(\beta_1)} \# \ldots \# \omega^{F(\beta_k)}$$

so $\quad \langle \alpha_1 \rangle \# \ldots \# \langle \alpha_n \rangle = \langle \beta_1 \rangle \# \ldots \# \langle \beta_k \rangle$.

By Lemma 4.4(ii), $\quad a_o < b_o \longrightarrow \rho(a_o) < \rho(b_o)$ so

$\rho(a_o) \# \langle \alpha_1 \rangle \# \ldots \# \langle \alpha_n \rangle <_2 \rho(b_o) \# \langle \beta_1 \rangle \# \ldots \# \langle \beta_k \rangle$. Also $\overline{\psi}_{c_1}(a_o) <_1 \overline{\psi}_{c_2}(b_o)$ by Theorem 4.1(ii), so by Lemma 3.1b(iii), Definition $3.8b_{10}$,

$\langle \rho(a_o) \# \langle \alpha_o \rangle \# \ldots \# \langle \alpha_n \rangle \rangle <_2 \langle \rho(b_o) \# \langle \beta_1 \rangle \# \ldots \# \langle \beta_k \rangle \rangle$.

II $\quad$ By transfinite induction on $\overline{\psi}_{c_1}(a_o)$ over $<$ on O.

Let $\overline{\psi}_{c_1}(a_o)$ have the form $\langle \alpha \rangle$

$\overline{\psi}_{c_2}(b_o)$ have the form $\langle\beta\rangle$.

By Theorem 4.1~~(ii)~~ (i) $\langle\alpha\rangle <_1 \langle\beta\rangle$. By Definition $3.8b_{10}$ it suffices to show

$$\alpha <_1 \beta \wedge \alpha <_2 \langle\beta\rangle$$

but $\alpha <_1 \beta$ follows from $\langle\alpha\rangle <_1 \langle\beta\rangle$ by Definition $3.8b_6$. To show $\alpha < \langle\beta\rangle$, let α have the form

$\alpha = \rho(a_o) \# \langle\alpha_1\rangle \# \ldots \# \langle\alpha_k\rangle$ where $\langle\alpha_i\rangle$ do not necessarily appear for $1 \leq i \leq k$.

We need to show:

1. $\rho(a_o) < \langle\beta\rangle$
2. $\langle\alpha_i\rangle < \langle\beta\rangle$ for $1 \leq i \leq k$.

1. $P^*(a_o, \langle\alpha\rangle)$ by definition of P^*, therefore $a_o < \langle\beta\rangle$ by hypothesis, so $\rho(\alpha_o) < \langle\beta\rangle$ by Lemma 4.4(iv)

2. $\langle\alpha_i\rangle < \langle\alpha\rangle$ by ~~Lemma~~ COROLLARY 3.1b(iii) $\langle\alpha_i\rangle \leq_1 \rho(a_o) \# \langle\alpha_1\rangle \# \ldots \# \langle\alpha_k\rangle = \alpha <_1 \langle\alpha\rangle$ by ~~Lemma~~ COROLLARY 3.1b(i) and $\langle\alpha\rangle <_1 \langle\beta\rangle$ as we have already seen, so $\langle\alpha_i\rangle <_1 \langle\beta\rangle$. Let $\langle\alpha_i\rangle$ have the form $\overline{\psi}_{c_i}(a_o^i)$, then $c_i < c_2$ by Theorem 4.1(i). If we can show $\bigwedge x(P^*(x, \langle\alpha_i\rangle) \longrightarrow x <_2 \langle\beta\rangle)$, then by induction hypothesis we would have $\langle\alpha_i\rangle <_2 \langle\beta\rangle$.

Let x_o be such that $P^*(x_o, \langle\alpha_i\rangle)$, then by Lemma 4.10,

$$\bigvee y(T^*(y, \omega^{F(\alpha_i)}) \wedge x_o \leq y).$$

Let y_o be such a y.

By the remark to the definition of T^* (Definition 4.14) we have that

$$T^*(y_o, \omega^{F(\alpha_1)} \# \ldots \# \omega^{F(\alpha_k)}).$$

By definition of P^* (Definition 4.15), $P(y_o, \langle\alpha\rangle)$.

By hypothesis $y_o < \langle\beta\rangle$ so

$x_o \leq y < \langle\beta\rangle$.

III By Lemma 4.9.

<u>NECESSITY</u>

This follows from sufficiency; for if I, II, III fail, we would have by the sufficiency of I, II, III that $\overline{\psi}_{c_2}(b_o) \leq \overline{\psi}_{c_1}(a_o)$.

<u>COROLLARY</u>

Let $c = \omega^{\Omega \cdot \Omega}$; if $\alpha_1, \ldots, \alpha_n, a_o, \ldots, a_n < \overline{\psi}_c(o)$, then $\overline{\psi}_s(a_o) < \overline{\psi}_c(o)$ where $s = \omega^{\Omega \cdot \alpha_1 \# a_1} \# \ldots \# \omega^{\Omega \cdot \alpha_n \# a_n}$.

<u>REMARK</u>: In the usual theory of ordinals $\omega^{\Omega \cdot \Omega} = \Omega^{\Omega}$. The generating rules of O', however, do not provide for the expression Ω^{Ω}.

5. Location of E(o) in O

In this section we show that there is an order preserving mapping from $\mathfrak{A} = \{x \mid x < E(o)\}$ onto $\mathfrak{B} = \{x \mid x < \overline{\psi}_c(o);\ c = \omega^{\Omega \cdot \Omega}\}$.

<u>LEMMA 5.1</u>: Let α be an ordinal such that $\alpha < E(o)$, then one of the following holds:

1) $\alpha = o$

2) $\alpha = \beta \# \gamma$ where $\beta, \gamma < \alpha$

3) $\alpha = \omega^{\beta}$ where $\beta < \alpha$

4) $\alpha = \overline{\varphi}_c(a_o)$ where $c = \omega^{\Omega\alpha_1 \# a_1} \# \ldots \# \omega^{\Omega\alpha_n \# a_n}$ and where $a_o, a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_n < \alpha$; furthermore the above representation of c is unique up to the order of the terms in the natural sum.

<u>Proof</u>. Assume 1) does not hold, then $\alpha \neq o$, so α has a normal form representation

$$\omega^{b_1} \# \omega^{b_2} \# \ldots \# \omega^{b_n}.$$

If $n \geq 2$ then 2) holds. If $n = 1$, then $\alpha = \omega^{b_1}$. It is well known that $\omega^{b_1} \geq b_1$. If b_1 is not an ε-number then $\omega^{b_1} > b_1$, so 3) holds. If b_1 is an ε-number then $\alpha = \omega^{b_1} = b_1$ so α is in the range of $\varphi(x) = \varepsilon_x$; thus by Lemma 2.3a Chapter I, we have that 4) holds.

<u>LEMMA 5.2</u>: Assume $\alpha = \langle x \rangle < \overline{\psi}_c(o)$ where $c = \omega^{\Omega \cdot \Omega}$, then $\alpha = \overline{\psi}_s(a_o)$ where

1) $s = \omega^{\Omega \cdot \alpha_1 \# a_1} \# \ldots \# \omega^{\Omega \cdot \alpha_n \# a_n}$

2) $a_o, a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_n < \alpha$.

<u>Proof</u>. That α has such a representation where 1) holds is an immediate consequence of Lemmas 4.8 and 4.3. We now show that 2) holds. By Definition 4.15, $P^*(a_o, \alpha)$, so $a_o < \alpha$ by Lemma 4.9. Again by Definition 4.15, $P^*(\omega^{a_i}, \alpha)$ and again by Lemma 4.9, $\omega^{a_i} < \alpha$. But $a_i \leq \omega^{a_i}$, so $a_i < \alpha$ for $i = 1, 2, \ldots, n$.

It now remains to show that $\alpha_1, \alpha_2, \ldots, \alpha_n < \alpha$. Clearly $\alpha_1, \alpha_2, \ldots, \alpha_n < \Omega$, for otherwise by Theorem 4.2, $\overline{\psi}_c(o) \leq \overline{\psi}_s(a_o) = \alpha$ which contradicts our hypothesis. From this we conclude the the normal form representation of α_i is

$\alpha_i = \omega^{\Omega \cdot o \# a_i^1} \# \ldots \# \omega^{\Omega \cdot o \# a_i^{k_i}}$ where $a_i^j < \Omega$ for $j = 1, 2, \ldots, k_i$. For $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, k_i$ we get

$T^*(\omega^{a_i^j}, \alpha_i)$ by Definition 4.14

$T^*(\omega^{a_i^j}, s)$ again by Definition 4.14

$P^*(\omega^{a_i^j}, \alpha)$ by Definition 4.15

$\omega^{a_i^j} < \alpha$ by Lemma 4.9

so

$$\alpha_i = \omega^{a_i^1} \# \ldots \# \omega^{a_i^{k_i}} < \alpha \text{ by Definition 4.6a.}$$

<u>THEOREM 5.1</u>: The following determines an order preserving mapping ' from $\mathfrak{A} = \{x \mid x < E(o)\}$ onto $\mathfrak{B} = \{x \mid x < \overline{\psi}_c(o);\ c = \omega^{\Omega \cdot \Omega}\}$

1) $o' = o$ where o on the right side is the symbol o in O (see Definition 3.1)

2) If $\alpha = \omega^\beta$ where $\beta < \alpha$ then $\alpha' = \cancel{\omega^{\alpha'}}\ \omega^{\beta'}$

3) If $\alpha = \overline{\varphi}_c(a_o)$ where $c = \omega^{\Omega\alpha_1 \# a_1} \# \ldots \# \omega^{\Omega\alpha_n \# a_n}$ and $a_o, a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_n < \alpha$, then

$$\alpha' = \overline{\psi}_{c'}(a_o') \text{ where } c' = \omega^{\Omega \cdot \alpha_1' \# a_1'} \# \ldots \# \omega^{\Omega \cdot \alpha_n' \# a_n'}$$

4) $(\alpha \# \beta)' = \alpha' \# \beta'$.

<u>Proof</u>. In view of the preceeding lemma, ' is a well defined mapping from $\mathfrak{A}$ <u>into O'</u> by transfinite induction on α over $<$ on $\mathfrak{A}$.

<u>INTO $\mathfrak{B}$</u>:

One shows that $\alpha \in \mathfrak{A} \longrightarrow \alpha' \in \mathfrak{B}$ by transfinite induction on α over $<$ on $\mathfrak{A}$ by using the Corollary to Theorem 4.2.

<u>ONTO $\mathfrak{B}$:</u>

We show that $\gamma \in \mathfrak{B} \longrightarrow \gamma = \alpha'$ where $\alpha \in \mathfrak{A}$ by transfinite induction on γ over $<$ on $\mathfrak{B}$. Assume $\gamma \in \mathfrak{B}$,

<u>Case 1</u>. $\gamma = o$

Then trivial

Case 2. $\gamma \neq o$

Then γ has normal form $\gamma = \omega^{\gamma_1} \# \ldots \# \omega^{\gamma_n}$, $n \geq 1$

Case 2.1. $n \geq 2$

Then $\omega^{\gamma_i} < \gamma$ for $i=1,2,\ldots,n$ by Definition 4.6a.

Applying induction hypothesis, $\omega^{\gamma_i} = \alpha_i'$ where $\alpha_i \varepsilon \mathfrak{A}$. Thus $\gamma = \omega^{\gamma_1} \# \ldots \# \omega^{\gamma_n} = \alpha_1' \# \ldots \# \alpha_n' = (\alpha_1 \# \ldots \# \alpha_n)'$

Case 2.2. $n = 1$

Then $\gamma = \omega^{\gamma_1}$ but by Lemma 4.5(iii), $\gamma_1 \leq \gamma$

Case 2.2.1. $\gamma_1 < \gamma$, then by induction hypothesis $\gamma_1 = \alpha'$ where $\alpha \varepsilon \mathfrak{A}$. So $\gamma = \omega^{\gamma_1} = \omega^{\alpha'} = (\omega^{\alpha})'$.

Case 2.2.2. $\gamma_1 = \gamma$

Then $\gamma = \omega^{\gamma_1} = \gamma_1$. Now by hypothesis $\gamma < \overline{\psi}_c(o)$ so $\gamma < \Omega$ since by the remark to Definition 4.13 $\overline{\psi}_c(o) \varepsilon O$, and by Definition $4.6b_3$ all members of O preceed Ω. By Lemma 4.5(vii), $\gamma = \langle x \rangle$ for some x. By Lemma 5.2, γ has the representation

$$\gamma = \overline{\psi}_s(b_o) \quad \text{where}$$

1) $s = \omega^{\Omega \cdot \beta_1 \# b_1} \# \ldots \# \omega^{\Omega \cdot \beta_n \# b_n}$

2) $b_o, b_1, \ldots, b_n, \beta_1, \ldots, \beta_n < \gamma$

so by the induction hypothesis,

$b_o, b_1, \ldots, b_n, \beta_1, \beta_2, \ldots, \beta_n$ can be written in the form;

$a'_o, a'_1, \ldots, a'_n, \alpha'_1, \ldots, \alpha'_n$ where $a_i, \alpha_i \varepsilon \mathfrak{A}$.

Then $\gamma = (\overline{\varphi}_r(a_o))'$ where $r = \omega^{\Omega\alpha_1 \# a_1} \# \ldots \# \omega^{\Omega\alpha_n \# a_n}$. By Lemma 2.4a Chapter ~~II~~ I, $\overline{\varphi}_r(a_o) < E(o)$; that is $\overline{\varphi}_r(a_o) \varepsilon \mathfrak{A}$.

<u>ORDER PRESERVING</u>:

One shows that $\alpha < \beta \longrightarrow \alpha' < \beta'$ by induction on $\alpha \# \beta$ over $<$ on $\mathfrak{A}$ using Lemma 2.5a Chapter II and Theorem 4.2. This concludes the proof.

In [7, p.25], Schütte points out that using Klammersymbols one can obtain an effective class of notations for all the ordinals less then E(o). We remark that there is an effective procedure which would enable one to pass from any member of that class to the corresponding ordinal diagram in O(2). One transforms $\overline{\varphi}A$ to an expression of the form $\overline{\varphi}_c(\alpha)$; then to an expression of the form $\overline{\psi}_c(\alpha)$; then to the member of O which corresponds to $\overline{\psi}_c(\alpha)$ under Definition 4.13; and finally, to the correspondent in O(2). It is not hard to show that all of these transformations are effective.

Theorem 5.1 has the following consequences:

1. The ordinal diagrams in O which preceed $\overline{\psi}_c(o)$, $c = \Omega^{\Omega}$ and are of the form $\langle x \rangle$ are precisely the ε-numbers less than E(o).

2. Define $\tau(x)$ on <u>ordinals</u> as follows:

$$\tau(x) = \begin{cases} 1 & \text{if } x = c + n \text{ where } c \text{ is an } \varepsilon\text{-number and } 0 \leq n < \omega \\ 0 & \text{otherwise,} \end{cases}$$

then for $\alpha < \overline{\psi}_c(o)$, $c = \Omega^{\Omega}$, the operation $\alpha \longrightarrow [\alpha]$ corresponds to the operation on ordinals $\alpha \longrightarrow \omega^{\alpha + \tau(\alpha)}$.

3. Below we list certain interesting ordinals along with their Klammersymbol notation, $\overline{\psi}$-notation, O-notation, and O(2)-notation. $\varphi(x)$ here denotes the ordering function of the ε-numbers. Ω^{Ω} will be used as an abbreviation for $\omega^{\Omega \cdot \Omega}$. In the second column, 1 denotes [o]. We define $r^{(n)}$ recursively as follows:

$$r^{(1)} = (1,1,1)$$

$$r^{(n+1)} = (1,1,r^{(n)}).$$

	Klammersymbol Notation	$\overline{\psi}$- Notation	O- Notation	O(2)- Notation
ε_o	$\varphi\binom{o}{o}$	$\overline{\psi}_o(o)$	$\langle o \rangle$	$(1,1,1)$
LEAST ~~First~~ Critical ε-number	$\varphi\binom{1}{1}$	$\overline{\psi}_1(o)$	$\langle\langle o \rangle\rangle$	$(1,1,r^{(1)})$
LEAST ~~First~~ Strongly Critical ε-number	$\varphi\binom{1}{2}$	$\overline{\psi}_{\Omega}(o)$	$\langle\langle\langle o \rangle\rangle\rangle$	$(1,1,r^{(2)})$
$E(o)$	—	$\overline{\psi}_{\Omega^{\Omega}}(o)$	$\langle\langle\langle\langle o \rangle\rangle\rangle\rangle$	$(1,1,r^{(3)})$

We believe that our hierarchy $\{\bar{\psi}_c\}_{c\varepsilon O'}$ is closely related to the segment $\{\varphi_c\}_{c<\varepsilon_{\Omega+1}}$ of Bachmann's hierarchy [2]; in fact, Bachmann's paper provided the motivation for the construction of $\{\bar{\psi}_c\}$. In this connection we mention that $\varepsilon_{\Omega+1}$ is just the first fixed point of the function Ω^x.

We conjecture that our system O with the relation $<_2$ is order isomorphic to the set of all ordinals less than the ordinal which Bachmann denotes by $\varphi_{\varepsilon_{\Omega+1}}(1)$ or $\varphi_{F_1(1)}(1)$. Establishing this would involve modifying Bachmann's hierarchy by first deleting those functions whose subscripts are limits of sequences of type Ω, and relabeling the remaining functions so that each function previously indexed by a number of the form c+n, where c is a limit of a sequence of type Ω and $1<n<\omega$, would now bear the subscript c+n-1. This remaining hierarchy would be further modified, by redefining the functions so that they skip certain critical values; that is, one would perform the transformation analogous to the one which Schütte uses to get $\bar{\varphi}A$ from φA. For the hierarchy which results one would need to prove analogues of our Theorem 4.2 (and its Corollary) and Lemma 5.2. We conjecture that all this can be carried out. (SEE GERBER'S THESIS).

SUMMARY

The principal result of this investigation is that the system of Klammersymbol notations of Schütte is smaller than the system O(2) of ordinal diagrams of Takeuti. More specifically, the ordinal E(o) corresponds to the ordinal diagram which we called $\overline{\psi}_c(o)$ where $c = \omega^{\Omega\cdot\Omega}$ (Chapter II, Section 4). This result was arrived at by giving a new characterization of the set $\{x \mid x\varepsilon O(2) \wedge x <_2 2\}$. This characterization is in itself of some interest.

Takeuti's O(1) was shown to correspond to the ordinal which in Schütte's Klammersymboles is denoted by $\overline{\varphi}\binom{\omega}{1}$.

In addition a description of Schütte's notations was given by means of functionals. A definitions of these functionals by transfinite recursion was given as well.

As a postscript we mention that H. Gerber ~~is~~ IS SHOWING ~~showing~~ in connection with his Ph.D. dissertation at The Pennsylvania State University, that the conjecture which we made at the end of Chapter II, Sec. 5, is indeed correct. Hence we can be confident that our system O and the set $\{x \mid x\varepsilon O(2) \wedge x <_2 2\}$ are order

isomorphic to the set of all ordinals LESS THAN THE ORDINAL which

Bachmann [3] denotes by $\varphi_{\epsilon_{\Omega+1}}(1)$ or $\varphi_{F_1(1)}(1)$

NOTE: BACHMAN'S NORMAL FUNCTIONS ARE DEFINED ONLY FOR NON-ZERO ARGUMENTS.

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