[1]

CAREFUL: Loose Maths Ahead

1) Motivation

In computer science we would like to build semantic domains and of a recursive nature. For example, to model the ~~[illegible]~~ untyped lambda calculus (with a notion of atomic elements) to obtain pca's, ~~for example, we need~~ ~~[illegible]~~ which don't involve an encoding, we would like a domain D and functional iso

$$D \cong A + (D \to D).$$

As another application if we want to model recursive datatypes, for example a monad for higher-order store, we ~~[illegible]~~ can design a (Parameterised) system of domain equations:

$$V \cong \cancel{[illegible]}\ V \to TV + N$$
$$TX \cong V \to V \times X$$

We would like a compositional body of work to this end, where we can ensure solutions to these equations. ~~This~~ This ~~[illegible]~~ ~~body of work~~ body of work, ~~[illegible]~~ started with Scott who first solved such equations using continuous lattices, but then later developed by Reynolds, Wright, and Plotkin & Smyth. In the 90's, Freyd gave a ~~[illegible]~~ more axiomatic treatment. Unfortunately, I'm not familiar with a ~~[illegible]~~ short survey of the results, but Marcelo Fiore's thesis is the closest I've ~~seen~~ seen to this end, though it (obviously) contains much more. I should also mention ~~[illegible]~~ P.H, ~~[illegible]~~ who extended the Plotkin Smyth work substantially.

Structure:

2) Freyd's algebraically compact categories - an axiomatic setting to solving such equations

3) The limit/colimit coincidence in O-categories.

4) On local determination of colimits.

(2)

Recall A $F: \mathcal{A} \to \mathcal{A}$ endofunctor
F-alg is a pair (A, a)
s.t. $a: FA \to A$
and an F-homomorphism $h: (A, a) \to (B, b)$ is a morphism $h: A \to B$ s.t.

$$\begin{array}{ccc} FA & \xrightarrow{Fh} & FB \\ a\downarrow & = & \downarrow b \\ A & \xrightarrow{h} & B \end{array}$$

(2) Freyd's algebraically compact categories

The main idea is to work in categories where these equations ~~...~~ are in fact functorial.

Def: A category $\mathcal{A}$ is algebraically ~~compact~~ complete with respect to class of endofunctors on $\mathcal{A}$ if all endofunctors on $\mathcal{A}$ have initial algebras.

For example, ~~the~~ set ~~...~~/accessible is algebraically ~~...~~ complete w.r.t. accessible endofunctors.
Set ~~...~~ is not algebraically complete w.r.t. all functors ~~...~~ e.g. $F = \cancel{X} \to (\cancel{X} \to 2)$ doesn't have a initial F-algebra, as a consequence of

Lambek's lemma: Let $F: \mathcal{A} \to \mathcal{A}$ be an endofunctor. In ~~every~~ initial ~~The initial~~ F-algebra (A, a), a is an isomorphism.
~~...~~ Thus (Freyd), F preserves ~~...~~ their initial algebras.

~~Lemma~~

~~In order to get mixed-variance, we need a stronger notion to guarantee ... of ... Def. A cat. $\mathcal{A}$ is algebraically compact~~

In fact, we want more than just completeness. If we look at the second example, we also have a parameter X lying around. More abstractly, ~~...~~ taking initial algebras is not compositional, as it turns a functor into an object.

~~...~~ If ~~...~~ $\mathcal{A}$ is algebraically complete, then ~~...~~ for every functor $F: \mathcal{P} \times \mathcal{A} \to \mathcal{A}$ and any $P \in \mathcal{P}$, the induced endofunctor $F_P: \mathcal{A} \to \mathcal{A}$ has an initial algebra fix F_P. These coalesce into a functor fix $F: \mathcal{P} \times \mathcal{A} \to \mathcal{A}$.
However, the resulting functor may not be "of the right kind" (in our designated class of functors).

$\boxed{3}$

(2) **Def (Fiore):** ~~Let $\mathcal{A}, \mathcal{B}$~~ $\mathcal{A}$, $\mathcal{P}$ be categories

Fix a sub 2-category of CAT, $\hat{\underline{C}}$ identity-on-objects. We say that $\mathcal{A} \in CAT$ is

a parameterised algebraically complete ~~if for~~ w.r.t. to the 1-cells in $\underline{C}$ if $\mathcal{A}$ is algebraically complete w.r.t. the endo-1-cells of $\underline{C}$, and always for every $F: \mathcal{P}\times\mathcal{A} \to \mathcal{A}$ 1-cell,

the parameterised initial algebra functor $\mu F: \mathcal{P} \to \mathcal{A}$ is a 1-cell in $\underline{C}$.

I'm not going to pursue the parameterised version in this talk further. You can have a look at Marcelo's thesis, or, for a more elementary account, for ~~see~~ ω-cocontinuous functors, see Orton's essay from this year.

In order to get duals, we need a stronger notion:

Def (Freyd): A cat $\mathcal{A}$ is algebraically compact w.r.t. a class of endofunctors if it is algebraically complete w.r.t. to it, and, moreover,

Call this free F-algebra { for every endofunctor $F: \mathcal{A} \to \mathcal{A}$, ~~and~~ initial algebra (A, a), the ~~inverse~~ coalgebra $(A, a^{-1}: A \to FA)$ is terminal.

Fiore adds: It is parameterised compact if it is also parameterised complete.

~~Now~~ This notion is self dual, and so:

Prop: (Freyd) If $\mathcal{A}$ is alg. compact, then so is $\mathcal{A}^{op}$. Similarly for parameterised compactness.

(unless stated otherwise, everything is due to Freyd.)

Given an identity on objects sub 2-cat $\underline{C}$ of ~~CAT~~ CAT, we get the full sub 2-category $\underline{C}^{comp}$ determined by the parameterised alg. compact categories.

Let $\mathcal{A}, \mathcal{B}$ be alg. compact.

Prop (Freyd's Bekič lemma): ~~If~~ for all $F: A \times B \to A$, $G: A \times B \to B$ one ~~alg. func~~, and with construct the endofunctor $H: \mathcal{A}\times\mathcal{B} \xrightarrow{\langle F, G\rangle} \mathcal{A}\times\mathcal{B}$. ~~Then for every initial H-algebra~~

Then $F \circ (\mathrm{id} \times \mathrm{fix}\, G): \mathcal{A} \to \mathcal{A}$. For every ~~...~~ free F-algebra (A, a), $((A, \mathrm{fix}\, G\, A), (a, \underline{\alpha A}))$ is a free H-algebra

Corollary: If $\underline{C}$ is closed under $\underline{CAT}$ products, then so is $\underline{C}^{PK}$.

So this gives us the beginning of a theory of solutions to recursive domain equations. Once we establish that a category ~~has~~ is (parameterized) alg. compact, we have an arithmetic on the 1-cells that ensure that any constructed functor $F: \mathbb{P} \times \mathbb{A} \to \mathbb{A}$ in ~~[illegible]~~ $\underline{C}$ has a (parameterized) fixpoint.

3 The limit/colimit coincidence

We still need to construct meaningful alg. compact categories.

To this end, ~~[illegible]~~ Plotkin-Smyth ~~paper~~ is still the canonical reference AFAIK.

~~[illegible]~~ (CPO)

~~[illegible]~~ Let $\mathbb{D}$ be the category of ω-chain-complete partial orders & ω-continuous maps. I.e., objects are posets $\langle A, \leq \rangle$ s.t. for every ω-chain $a_0 \leq a_1 \leq a_2 \ldots$ in A, $\bigvee_{n \in \mathbb{N}} a_n$ exists, and the morphisms $f: \langle A, \leq \rangle \to \langle B, \leq \rangle$ are the monotone maps between them.

A cpo is pointed if it has a bottom element $\perp$.
A continuous map between pointed cpo's is strict if it preserves $\perp$.
$\mathbb{D}_{\perp}$ is the category of pointed cpo's & strict maps.

i.e., $\mathbb{D}^{T_\perp}$ for the obvious lifting monad

~~Our goal is to construct an alg. compact category.~~

Defn: An $\mathbb{D}$-category is just a locally small category $\mathbb{A}$ where the homsets carry an order cpo structure, and composition preserves it. Examples: pCat, pCPO $\simeq \mathbb{D}_{\perp}$

An embedding-projection pair (e,p) ~~is a pair of~~ in an $\mathbb{D}$-cat $\mathbb{A}$ is ~~[illegible]~~ a pair of morphisms $A \underset{p}{\overset{e}{\rightleftarrows}} B$ s.t. $p \circ e = \mathrm{id}$, $e \circ p \leq \mathrm{id}$.

Def (Fiore): Let K be an $\mathbb{D}$-category. An e-initial object is an initial object 0 s.t. for all $A \in K$, $0 \xrightarrow{!} A$ is an embedding.
The dual notion is called p-terminal

~~formally~~

⑤

(in fact, a Pos-category)

Lemma (Fiore): Let K be an $\mathbf{O}$-category with an initial object TFAE:

1) The initial object is e-initial.
2) every hom-cpo is pointed, and composition is strict in the ~~pre~~-composed argument.
3) For every object x there is $x \longrightarrow 0$ s.t. for all y $x \longrightarrow 0 \longrightarrow y$ is the least element in $K(x, y)$.

4) for every object x there exists $x \to 0$ s.t. $x \longrightarrow 0 \xrightarrow{!} x \leq id$.

~~The ... , ... has ...-terminal object, and~~

~~Corollary (Smyth-Plotkin):~~

Let K be an $\mathbf{O}$-category. The category of embedding projection pairs ~~$\mathbf{O}$~~ K^{ep} consists of objects ~~A~~ A in K and morphisms $f: A \to B$ embedding projection pairs ~~...~~

$$A \underset{f^P}{\overset{f^e}{\rightleftarrows}} B.$$

K^e is the subcategory of embeddings, K^P is that of projections.

Lemma (Smyth-Plotkin): The following are (continuously) isomorphic:

$$K^e \cong K^{ep} \cong (K^P)^{op}$$
$$\xrightarrow{(-)^P}$$

Corollary: If K has an e-initial object, then K^{ep} has a zero ~~initial~~ object.

⑥

~~Def~~ Def: Let K be an O-category and $\Delta: \omega \to K^{ep}$ an ω-chain

$$\Delta_0 \underset{p_0}{\overset{e_0}{\rightleftarrows}} \Delta_1 \underset{p_1}{\overset{e_1}{\rightleftarrows}} \Delta_2 \rightleftarrows \cdots$$

~~We say that~~ ~~let~~ ~~be~~

Let (C, μ_n) be a cocone in K^{ep}. We say that (C, μ_n) is a <u>locally determined colimit</u> of Δ if:

$$\bigsqcup_n \mu_n^e \circ \mu_n^p = id$$

Scott first observed that limits and colimits coincide in the embedding projection category for cont. lattices

~~Theorem (SP, limit~~

Smyth–Plotkin then generalised it as follows.

<u>Theorem</u> (SP, limit/colimit coincidence): Let K be an O-cat, $\Delta: \omega \to K$ an ω-chain and $\mu^e: \Delta \to C$ a cocone of embeddings. ~~with the same data as in the previous def,~~ the TFAE:

① (C, μ^e) is colimiting in K.
② (C, μ^e) is O-colimiting in K.
③ $(C, (\mu_n^e, \mu_n^p))$ is a locally determined colimit.
④ $\mu^R: C \to \Delta^R$ is limiting in K
~~⑤ $\mu^R: C \to K$ is limiting in K^{ep} (μ, μ^R)~~
⑤ $\mu^R: C \to \Delta^R$ is O-limiting in K

in this case, the mediating morphism $C \to D$ for all cocones $\delta: \Delta \to D$ is given by $\bigsqcup_n \delta_n \circ \mu_n^p$

In this case, we also have the following ~~(which are equivalent~~ two equivalent conditions:

ⓐ $(C, (\mu, \mu^p))$ is limiting in K^{ep}
ⓑ (C, μ, μ^e) is colimiting in K^{ep}.

~~mediating morphism~~

(7)

~~(BP, Prop 3 ...)~~

<u>Theorem:</u> Let K be an $\mathcal{O}$-category. ~~with~~ If K has ~~an ... objects and~~ an initial object and colimits ~~of~~ (eqv. limits) of embedding ω-chains, then K^{ep} is algebraically compact w.r.t. extensions of ~~locally continuous~~ $\mathcal{O}$-functors on K to K

<u>Proof:</u> Let $F: K^{ep} \to K^{ep}$ be an $\mathcal{O}$-functor.

Consider: $0 \xrightarrow{!} F0 \xrightarrow{F!} \dots$

As F is locally continuous, this is a diagram in K^e, ~~By the limit colimit coincidence~~ which, by assumption has a colimiting cone $\mu : \Delta \to C$.

By the limit-colimit coincidence, ~~F0~~ $(C, (\mu, \mu^{P}))$ is locally determined and moreover, colimiting in in K^{ep}, with the mediating morphism given by lubs.

As F is locally continuous, we get that F preserves this ~~... so~~ colimit, and so consequently, $(C, \langle \mu, \mu^{P} \rangle)$ is both the initial algebra and its inverse is the terminal coalgebra. ∎