Contents

CHAPTER I

Topology

Conventions. Unless stated otherwise,

- X, Y will be topological spaces.
- Subsets of topological spaces will be considered under subspace topology.
- Product of topological spaces will be considered under product the topology.
- A totally ordered set will be considered under the order topology. (See 4.)
- Monotonic functions will be assumed to be between totally ordered spaces.

1. Subspaces and Bases

Lemma 1.1. *$\mathscr{B}$ is a base $\iff$ the arbitrary unions in $\mathscr{B}$ form a topology.* "$\Rightarrow$" requires AC.

Lemma 1.2.

(i) "Being a subspace of" is transitive.

(ii) (Sub)base of a subspace can be obtained from that of the parent space.

2. Limits and Continuity

Let $E \subseteq X$ and $f\colon E \to Y$. Then for any $c \in X$ and $L \in Y$, we write $\boldsymbol{f(x) \to L}$ **as $\boldsymbol{x \to c}$ in $\boldsymbol{X}$** iff for every open neighborhood V of L in Y, there exists an open neighborhood U of c in X such that $f(E \cap U \setminus \{c\}) \subseteq V$. Note that "in X" is crucial and can't be dropped: Consider id on $(0, 1)$. Take X_1 to be the disjoint union topology of $(0,1)$ and $\{1\}$ and $X_2 := (0,1]$ under the subspace topology. Then 1 is isolated in X_1 and thus $f(x) \to L$ for each $L \in (0,1)$ as $x \to 1$ in X_1 (see (i) of 2.1). On the other hand, $f(x) \to 1$ only as $x \to 1$ in X_2 since 1 is a limit point of $(0,1)$ in X_2 (see (ii) of 2.1).

Intuitively, what this shows is that c's topological relation with E is not determined by E alone if $c \notin E$, which is plausible. However, if $c \in E$, then specification of ambient space is redundant (see (iii) of 2.1).

Nevertheless, we omit "in X" if the ambient space is clear from the context, or if the point at which the limit is being evaluated lies in the domain.

Lemma 2.1. *(i) All the codomain values are the limits of a function at an isolated point of the domain.*

(ii) For Hausdorff codomains, there is at most one limit at a limit point of the domain.

(iii) Limits of a function at a point in the domain are independent of the ambient space.

Proof. Let $f\colon E \to Y$ where $E \subseteq X$. Let $c \in X$.

(i) If c is isolated in E, then there exists an open neighborhood U of c in X such that $E \cap U = \{c\}$ so that for any open set V in Y, we have $f(E \cap U \setminus \{c\}) = \emptyset \subseteq V$.

(ii) Let c be a limit point of E and suppose $f(x) \to L_1, L_2$ as $x \to c$ for distinct L_1, L_2. Let V_i's be opens separating L_i's for $i = 1, 2$ (since $\underline{Y \text{ Hausdorff}}$). Now, take open neighborhoods U_i's of c in X such that $f(E \cap U_i \setminus \{c\}) \subseteq V_i$. Now, $E \cap U_1 \cap U_2 \setminus \{c\} \neq \emptyset$ set $\underline{c \text{ is a limit point of } E}$ which violates disjointness of V_i's.

(iii) Since if $c \in E$, then for any $X_1, X_2 \supseteq E$, if U_1 is an open neighborhood of c in X_i, then $E \cap U_1$, being an open neighborhood of c in E, is equal to $E \cap U_2$ for some open neighborhood U_2 of c in X_2. □

Remark. This allows to use $\lim_{x \to c} f(x)$ notation in case f has a Hausdorff codomain and c is a limit point of the domain in the ambient space.

Lemma 2.2 (Limits and subspaces)**.** *Assume the following:*

- $f\colon E \to Y$ *where* $E \subseteq X$.
- $c \in A \subseteq X$ *and* $L \in Y$.
- $f|\colon E \cap A \to Y$.

Then the following hold:

(i) $f(x) \to L$ *as* $x \to c$ *in* $X \implies f|(x) \to L$ *as* $x \to c$ *in* A.

(ii) The converse holds if A *is open.*

Proof. (i) Let V be an open neighborhood of L. Take an open neighborhood U of c in X such that $f(U \cap E \setminus \{c\}) \subseteq V$. Then $U \cap A$ is an open neighborhood of c in A with $f|(U \cap E \cap A \setminus \{c\}) \subseteq V$.

(ii) Let A be open. Let V be an open neighborhood of L. Then take an open neighborhood U of c in A such that $f|(U \cap E \setminus \{c\}) \subseteq V$. Now, just note that U is open in X as well since $\underline{A \text{ is open}}$. $\square$

Remark. Necessity of A being open in (ii) is clear by taking Y to be Hausdorff containing at least two points, A to be a singleton comprising of a limit point of E and taking f appropriately (constant function works).

Proposition 2.3 (Pointwise pasting)**.** *Assume the following:*

- $f\colon E \to Y$ *where* $E \subseteq X$.
- X *is the union of finitely many closed* $F_i \subseteq X$.
- $f|_i : E \cap F_i \to Y$.
- $c \in X$ *and* $L \in X$.
- *If* $c \in F_i$, *then* $f_i(x) \to L$ *as* $x \to c$ *in* F_i.

Then $f(x) \to L$ *as* $x \to c$ *in* X.

Proof. Let V be an open neighborhood of L. Say $c \in F_i$'s and $c \notin F_j$'s. Thus, take open neighborhoods $U_i \cap F_i$ of c in F_i (U_i open in X_i) such that $f|_i(U_i \cap E \cap F_i \setminus \{c\}) \subseteq V$. Set $U := \left(\bigcap_i U_i\right) \cap \left(X \setminus \bigcup_j F_j\right)$[1] which is open since F_j's are <u>closed</u> and i and j's are <u>finitely many</u>. Thus, U is an open neighborhood of c in X with $f(U \cap E \setminus \{c\}) = \left(\bigcup_i f(U \cap E \cap F_i \setminus \{c\})\right) \cup \left(\bigcup_j f(U \cap E \cap F_j \setminus \{c\})\right) \subseteq V \cup \emptyset = V$. $\square$

Remark. (i) Necessity of finitely many closed sets: Consider $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) := x^2y/(x^4 + y^2)$ for $(x, y) \neq (0, 0)$ and $f(0, 0) := 0$. Along all the straight lines through origin (which are closed), $f(x) \to 0$ and yet f has no limit at 0.[2]

A much simpler example is to consider an infinite X having at least one limit point, the singleton containing which is closed, and taking the codomain to be a Hausdorff space with at least two points.

(ii) Necessity of all closed:

[1] The second set in the union is motivated from the need to have $\bigcap_j X \setminus F_j$.

[2] Consider evaluating along $y = mx^2$.

3. Product Topology

From (ii) of 1.2, we immediately conclude:

Lemma 3.1. *Taking products and subspaces are compatible.*

Remark. This holds for box topology as well.

Lemma 3.2. *Closure of a product is the product of closures.*

Proof. Let $A_i \subseteq X_i$. We show $\overline{\prod_i A_i} = \prod_i \overline{A_i}$.

"$\subseteq$": Suffice to show that $\prod_i F_i$ is closed for F_i's closed in X_i's. Let $(x_i) \notin \prod_i F_i$, say $x_{i_0} \notin F_{i_0}$. Then take an open neighborhood U_{i_0} of x_{i_0} disjoint from F_{i_0}. Now, $\pi_{i_0}^{-1}(U_{i_0})$ is an open neighborhood of (x_i) that is disjoint from $\prod_i F_i$.

"$\supseteq$": Let $U := \bigcap_{j \in J} \pi_j^{-1}(U_j)$ be an open neighborhood of $(x_i) \in$ RHS, where J is finite and each U_j is open. Then each U_j is an open neighborhood of x_j and hence intersects A_j. Thus U intersects $\prod_i A_i$. □

No choice required.

Remark. The same holds for box topology as well; however AC will be required for "$\supseteq$".

Corollary 3.3. *Product of dense sets is dense in the product.*

Remark. This holds for box topology as well.

Proposition 3.4 (Convergence)**.** *Convergence in product topology is equivalent to convergence of each component sequence in the respective factor space.*

Proof. "$\Rightarrow$": Since projections are continuous.

"$\Leftarrow$": Let X be the product of X_i's and $x, (x^{(n)}) \in X$ be such that $x_i^{(n)} \to x_i$ for each i. We show that $x^{(n)} \to x$. Let $U := \bigcap_j \pi_j^{-1}(U_j)$ be a basic open neighborhood of x where j's are finitely many. Thus, $(x_j^{(n)})_n$ eventually lies in U_j for each j. Since there are finitely many j's, we conclude that $(x^{(n)})$ eventually lies in U. □

Remark. This does not hold for box topology: Consider the $\mathbb{N}$-fold product of discrete $\{0,1\}$ endowed with box topology and consider the sequence whose n-th term is given by $(\underbrace{1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots)$.

4. Order Topology

If X is totally ordered, then the **order topology** on it is generated by these sets: (i) (a, b); (ii) $[\min X, b)$ if X has a minimum element; and, (iii) $(a, \max X]$ if X has a maximum element. It's easily seen that these form a base that is closed under pairwise intersections.

Lemma 4.1 (Immediate properties)**.**

(i) *Open rays are open in order topology.*

(ii) *Order topology is Hausdorff.*

(iii) *Topology induced from inherited order is coarser than the subspace topology. However, on a convex subset, then the topologies coincide.*

(iv) *The order topology coincides with the discrete topology* $\iff$ *each non-least element has an immediate predecessor and each non-greatest element has an immediate successor.*

Proof. (i) Let's show for right-rays. In case there's a largest element, then it's clear. If not, then $(a, +\infty) = \bigcup_y (a, y)$, which is open.

(ii) Let $x < y$. If there's a z between them, then $(-\infty, z)$ and $(z, +\infty)$ separate them. Otherwise, $(-\infty, y)$ and $(x, +\infty)$ do.

(iii) First statement is obvious. Second is tedious, but easy.

(iv) "$\Leftarrow$" is easy. For "$\Rightarrow$", just note that a base for discrete topology must contain all the singletons. □

Remark. To see strict inclusion in (iii), consider $\{-1\} \cup (0, 1] \subseteq \mathbb{R}$.

Let $f : X \to Y$ where X is totally ordered. Then for $c \in X$ and $L \in Y$, we write $\boldsymbol{f(x) \to L}$ **as** $\boldsymbol{x \to c^-}$ iff $f|(x) \to L$ as $x \to c$ where $f| : (-\infty, c] \to Y$. Note:

(i) The domain $(-\infty, c]$ can be taken under either the subspace topology or the topology due to the induced order, which coincide. (See (iii) of 4.1.)

(ii) One of the reasons to include c in the domain of $f|$ is that we don't have to bother about the ambient space. (See (iii) of 2.1.)

Similarly, we define $\boldsymbol{f(x) \to L}$ **as** $\boldsymbol{x \to c^+}$.

Lemma 4.2. *Let X be totally ordered and $f : X \to Y$. Let $c \in X$ and $L \in Y$. Then the following are equivalent:*

(i) $f(x) \to L$ *as* $x \to c$*.*

(ii) $f(x) \to L$ *as* $x \to c^+$ *or as* $x \to c^-$*.*

Proof. (i) ⇒ (ii): Let □

Remark. This lemma is like the pasting lemma, but at a point.

Proposition 4.3. *Let X, Y be totally ordered spaces and $f\colon X \to Y$ be monotonic. Then for any $c \in X$, each of the following holds whenever the RHS is defined:*

$$f(x) \to \sup_{x<c} f(x) \ \text{ as } x \to c^-$$
$$f(x) \to \inf_{x>c} f(x) \ \text{ as } x \to c^+$$

Proof. Let f be increasing. The proof for decreasing f would be similar. We show only the first statement, second's proof being similar. Let α be the l.u.b. of $\{f(x) : x < c\}$. We need to show that $f|(x) \to \alpha$ as $x \to c$ where $f|\colon (-\infty, c] \to Y$. Let J be a basic open neighborhood of α. We have two cases:

(i) α is the least element of Y: Then $f((-\infty, c)) \subseteq \{\alpha\} \subseteq J$.

(ii) α is not the least element: Then without loss of generality, let J's left-end be open at $y < \alpha \overset{\text{w}}{=} \sup_{x<c} f(x)$. Thus, take an $x < c$ such that $f(x) > y$. Now, due to <u>monotonicity</u>, $f((x, c)) \subseteq (y, \alpha] \subseteq J$. □

Corollary 4.4. *Monotonics taking values in a complete codomain admit one-sided limits.*

Proof. Let $f\colon X \to Y$ be increasing and $c \in X$. We find an $L \in Y$ such that $f(x) \to L$ as $x \to c^-$. Simply observe that $\{f(x) : x < c\}$ is bounded above by $f(c)$. We have two cases:

(i) c is not the least element of X: Then $\{f(x) : x < c\}$ is nonempty as well so that we may take L to be its l.u.b. (which exists since <u>Y is complete</u>).

(ii) c is the least element of X: Then the domain of $f|\colon (-\infty, c] \to Y$ is the singleton $\{c\}$ and thus we may take L to be any point in Y.

The proofs for other cases are similar. □

Remark. To see the necessity of well-definedness of RHS in 4.3 and that of completeness in 4.4, consider $f\colon \mathbb{R} \to \mathbb{R} \setminus \{0\}$ given by

$$f(x) := \begin{cases} x, & x < 0 \\ x+1, & x \geq 0 \end{cases}.$$

Proposition 4.5 (Continuity and monotonicity).

(i) Strict monotonic surjections are homeomorphisms.

(ii) A monotonic surjection is continuous provided the codomain's order is dense.

Proof. (i) Since inverses of strictly monotonic bijections are strict monotones as well, it suffices to show that strictly monotonic surjections are open which is easily shown.

(ii) Let X, Y be totally ordered and $f\colon X \to Y$ be an increasing surjection. The proof for decreasing f would be similar. Let J be a basic open set of Y. We show that $f^{-1}(J)$ is open. Start with an $x \in f^{-1}(J)$, *i.e.*, $f(x) \in J$. The following cases arise:

(a) $f(x)$ is least in Y: Then without loss of generality, take $J = [f(x), y)$ for some $y > f(x)$. Now, take $f(x) < v < y$ due to denseness. Due to surjectivity, take $b \in X$ such that $f(b) = v$. Since f is increasing, we have $x < b$ and $f([x, b)) \subseteq [f(x), f(b)] = [f(x), v] \subseteq [f(x), y) = J$. Now, take an open neighborhood I of such that $I \cap [x, +\infty) \subseteq [x, b)$. Then, since $f((-\infty, x]) = \{f(x)\}$ (as $f(x)$ is the least element) $f(I) \subseteq J \overset{\text{w}}{\Longrightarrow} I \subseteq f^{-1}(J)$.

(b) $f(x)$ is the greatest in Y: Similarly as above.

(c) $f(x)$ is neither: Then without loss of generality, take $J = (y_1, y_2)$ for $y_1 < f(x) < y_2$. As above, due to denseness and surjectivity, take $a, b \in X$ such that $y_1 < f(a) < f(x) < f(b) < y_2$. Again, since f is increasing, $a < x < b$ and $f((a, b)) \subseteq [f(a), f(b)] \subseteq (y_1, y_2) = J$. □

Remark. (i) Necessity of surjectivity: $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) := x$ for $x < 0$ and $f(x) := x + 1$ for $x \geq 0$.

(ii) Necessity of denseness of codomain: Consider the sign function $\mathbb{R} \to \{-1, 0, 1\}$. (Note that the codomain is discrete due to (iv) of 4.1.)

5. Denseness

Lemma 5.1. *"Being dense" is transitive.*

Proof. Let $A \subseteq B \subseteq X$ with A dense in B and B dense in X. Let U be a nonempty open in X. Then B being dense, intersects U so that $U \cap B$ is a nonempty open in B and thus is intersected by the dense A $\overset{\text{w}}{\Longrightarrow}$ A intersects U. □

Lemma 5.2. *Product of dense sets is dense in the product.*

Proof. Let A_i be dense in X_i. Let U_i's be nonempty open in X_i's. Then $\prod_i A_i \cap \prod_i U_i = \prod_i A_i \cap U_i$, which is nonempty since each $A_i \cap U_i$ is nonempty. □ *AC used.*

Remark. This is also true of the box topology.

Lemma 5.3. *Let $A, B \subseteq X$. Then the following hold:*

(i) $B \cap A$ is dense in $B \implies B \subseteq \overline{A}$.

(ii) The converse holds if B is open.

Proof. (i) We have $B = \mathrm{cl}_B(B \cap A) \subseteq B \cap \overline{B \cap A} \subseteq \overline{B \cap A} \subseteq \overline{A}$.

(ii) We need to show that $\mathrm{cl}_B(B \cap A) = B$. Indeed, if F is any closed such that $B \cap A \subseteq B \cap F$, then $B \subseteq F$ (otherwise, take $x \in B \setminus F \overset{\text{w}}{\implies} x \in B \setminus A \overset{\text{w}}{\implies} x \in B \setminus \overline{A}$ for $\underline{B \text{ is open}}$, contradicting $B \subseteq \overline{A}$). □

Remark. To see the necessity of openness of B in (ii), consider $A = \{1, 1/2, \ldots\}$ and $B = \{0\}$.

5.1 Nowhere dense sets

5.3 gives insight as to why nowhere dense sets are called so—they are dense on no nonempty *open* set. On the other hand, dense sets are dense on the whole space.

Lemma 5.4. *Let U be open in X and $A \subseteq X$. Then the following are equivalent:*

(i) $U \subseteq \overline{A}$.

(ii) Every nonempty open subset contained in U intersects $\overline{A}$.

(iii) Every nonempty open subset contained in U intersects A.

Corollary 5.5. *The following are equivalent for a subset A of X:*

(i) $X \setminus \overline{A}$ is dense.

(ii) A is nowhere dense.

(iii) Each nonempty open set contains a nonemtpy open subset disjoint from $\overline{A}$.

(iv) Each nonempty open set contains a nonemtpy open subset disjoint from A.

Subsets of a topological space that are countable unions of nowhere dense sets are called **first category** or **meagre** sets. Others are called **second category** sets.

Remark. In $\mathbb{R}$:[3]

[3]Nonmeagre-ness can be concluded by Baire's category theorem (4.3).

	meagre	nonmeagre
dense	$\mathbb{Q}$	$\mathbb{R}$
nondense	$\emptyset$	$[0,1]$

Lemma 5.6. *If $F_1, F_2, \dots$ are closed in X with $X \setminus \bigcup_i F_i$ dense, then each F_i is nowhere dense.*

Remark. Baire's category theorem (4.3) gives a converse to above, stating that complements of meagre sets are dense in a complete metric space.

Proposition 5.7. *In a topological space, the following are equivalent:*

(i) Complements of meagre sets are dense.

(ii) Countable intersections of open dense sets are dense.

Proof. " $\Rightarrow$ ": Let $U_1, U_2, \cdots$ be open dense. Now, $\bigcap_i U_i \overset{\text{w}}{=} X \setminus \bigcup_i (X \setminus U_i)$ is dense if each $X \setminus U_i$ is nowhere dense $\overset{\text{w}}{\Longleftarrow}$ $X \setminus (\overline{X \setminus U_i}) \overset{\text{w}}{=} U_i$ (since $\underline{U_i \text{ open}}$) is $\underline{\text{dense}}$, which is true.

" $\Leftarrow$ ": Let $A_1, A_2, \dots$ be nowhere dense. Then each $X \setminus \overline{A}_i$ is dense $\overset{\text{w}}{\Longrightarrow}$ $\bigcap_i (X \setminus \overline{A}_i) \overset{\text{w}}{=} X \setminus \bigcup_i \overline{A}_i$ is dense $\overset{\text{w}}{\Longrightarrow}$ $X \setminus \bigcup_i A_i$ is dense as well, being a larger set. $\square$

6. Connectedness

Lemma 6.1 (Characterizing disconnectedness)**.** *$E \subseteq X$ is disconnected $\iff$ E can be written as a union of two nonempty subsets A, B of X such that $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.*

Proof. "$\Rightarrow$": Take U, V open in X such that $E \cap U$, $E \cap V$ are nonempty, $E \subseteq U \cup V$, and $E \cap U \cap V = \emptyset$. Now put $A := E \cap U$ and $B := E \cap V$. Then $\overline{A} \cap B \subseteq \overline{E \cap U} \cap V = \emptyset$.

"$\Leftarrow$": Take $U := X \setminus \overline{A}$ and $V := X \setminus \overline{B}$. Then $B \subseteq U$ and $A \subseteq V$ so that both are nonempty and $E \subseteq U \cup V$. Also, $E \cap U \cap V = E \setminus (\overline{A} \cup \overline{B}) = \emptyset$. $\square$

Proposition 6.2 (Linear continua are connected)**.** *Then the connected subsets of a densely and completely totally ordered space are precisely its convex subsets.*[4]

Can we improve to partial orders?

[4]Recall that a convex subset of an ordered set is any set I such that $[x, y] \subseteq I$ whenever $x, y \in I$ with $x \le y$.

Proof. Let X's topology come from a dense and complete total order. Suppose $I \subseteq X$ is convex, and yet separated by opens U, V. Take $a \in U \cap I$ and $b \in V \cap I$. Without loss of generality, assume $a < b$ (the order is total) so that $[a, b] \subseteq I$ (since I is convex). Note that U, V also form a separation of $[a, b]$. Since $U \cap [a, b]$ is nonempty and bounded, let c be its l.u.b. Clearly, $c \in [a, b]$ so that there are two cases:

$c \in U$: Take a basic interval $J \subseteq U$ containing c. Note that $c < b$ (since $b \in V$) so that $J \supseteq [c, d)$ for some $d > c$. Hence, $U \cap [a, b] \overset{\mathrm{w}}{\supseteq} J \cap [a, b] \supseteq [c, d) \cap [c, b] \overset{\mathrm{w}}{=} [c, d_1)$ where $d_1 := \min(d, b) > c$. Now take e between c and d_1. Then $e \in U \cap [a, b]$ despite $e > c$. *Add a diagram!*

$c \in V$: Again take a basic interval $J \subseteq V$ containing c. This time, $c > a$ (as $a \in U$) so that $J \supseteq (d, c]$ for some $d < c$. Thus, $V \cap [a, b] \overset{\mathrm{w}}{\supseteq} J \cap [a, b] \supseteq (d, c] \cap [a, c] \overset{\mathrm{w}}{=} (d_1, c]$ where $d_1 := \max(d, a) < c$. Now, take e between d_1 and c. Then e is an upper bound for $U \cap [a, b]$ greater than c:

If $x \in U \cap [a, b]$ is greater than e, then $x \in (e, c] \subseteq (d_1, c] \subseteq V$.

Conversely, if I is not convex, then take $x < y < z$ such that $z, z \in I$ but $y \notin I$. Then the rays at y separate I. $\square$

Remark. To see the necessity of the assumptions, consider $\mathbb{Q}$ and $\mathbb{Z}$ respectively which are both totally disconnected.

Proposition 6.3 (Intermediate value)**.** *Any continuous function from a connected space to a totally ordered space obeys intermediate value property.*

Proof. Let X be connected and Y ordered, and $f : X \to Y$ be continuous. Suppose f doesn't obey intermediate value property. Then take $x_1, x_2 \in X$ and $y \in Y$ such that y lies between $f(x_1)$ and $f(x_2)$ and yet $y \notin f(X)$. Without loss of generality, let $f(x_1) < f(x_2)$ (since y lies between them, they can't be equal; totality used) so that $f(x_1) < y < f(x_2)$. Now, $f^{-1}((-\infty, y))$ and $f^{-1}((y, +\infty))$ form a nonempty (they contain x_1, x_2 respectively) open separation of X, violating the connectedness of X. $\square$

7. Separation Axioms

Lemma 7.1 (T_1 spaces)**.** *In a space, singletons are closed $\iff$ any two distinct points can be separated by open sets that don't contain the other.*

Proof. Let the space in question be X.

"$\Rightarrow$": Let x, y be distinct. Then $X \setminus \{y\}$ and $X \setminus \{y\}$ separate x, y as required.

"$\Leftarrow$": Let $x \in X$. We show that $X \setminus \{x\}$ is open, which follows easily. □

Proposition 7.2. *A continuous function taking values in a Hausdorff codomain is completely determined by its values on a dense subset of the domain.*

Proof. Let $f, g\colon X \to Y$ be continuous with Y Hausdorff, agreeing on a dense subset $D \subseteq X$ and yet not on $x \in X$. Since $\underline{Y \text{ Hausdorff}}$, separate $f(x)$ and $g(x)$ via opens V and W. Then $f^{-1}(V) \cap g^{-1}(W)$ is an open neighborhood of x, and thus intersects the $\underline{\text{dense } D}$, say at y. But then $V \ni f(y) = g(y) \in W$, a contradiction. □

Remark. To see the necessity of the Hausdorff codomain (and that just T_1 is not enough), consider the function on $\mathbb{R}$ which swaps two distinct points. Then this is continuous[5] with the codomain under cofinite topology.

8. Countability and Separability

Lemma 8.1. *A second countable space is separable and first countable.*

Proof. Choosing a point from each of the sets from a countable base yields a countable dense set. □ *CC used.*

Remark. The converse is not true (however, see 1.2): Consider the Sorgenfrey line, *i.e.*, the lower limit topology on $\mathbb{R}$ generated by the basic open sets of the form $[a, b)$. Any base of this topology must contain for each $x \in \mathbb{R}$, some set with x being its l.u.b., and thus be uncountable. *AC used.*

second countable	first countable	separable	
✓			separable metric spaces
×	✓	✓	Sorgenfrey line
×	✓	×	nonseparable metric spaces[6]
	×	✓	cofinite on uncountable
	×	×	cocountable on uncountable

[5] More generally, for any set X, any bijection $X \to X_{\text{cofin}}$ is continuous if singletons are closed in the domain.

[6] For instance, discrete metric on any uncountable set.

Proposition 8.2. *Any base of a second countable space contains a countable base.*

Proof. Let $\mathscr{B}$, $\mathscr{B}'$ be bases of X with $\mathscr{B}$ being countable. It suffices. to show that each $U \in \mathscr{B}$ is a countable union in $\mathscr{B}'$. Thus, consider a $U \in \mathscr{B}$. Define $\mathscr{V} := \{V \in \mathscr{B} : V \subseteq W' \subseteq U \text{ for some } W' \in \mathscr{B}'\}$. Now, for each $V \in \mathscr{V}$, one can choose a $W'_V \in \mathscr{B}$ such that $V \subseteq W'_V \subseteq U$. Now, just note that U is the union of W'_V's which are countably many. □

CC used

Add a diagram.

CC used.

Proposition 8.3. *Separability, and first and second countabilities are preserved under countable products.*

Proof. (i) Separability: Let A_i be countable and dense in X_i for $i = 1, 2, \ldots$. Without loss of generality, let each X_i be nonempty so that we may choose for each i, an $x_i \in X_i$. Then the union of the following sets forms a countable dense set in $\prod_i X_i$:

CC used.

CC used.

(a) $A_1 \times \{x_2\} \times \{x_3\} \times \cdots$
(b) $A_1 \times A_2 \times \{x_3\} \times \{x_4\} \times \cdots$
(c) $A_1 \times A_2 \times A_3 \times \{x_4\} \times \{x_5\} \times \cdots$
$\vdots$

(ii) First countability: Let $X_1, X_2, \ldots$ be first countable and let $x \in \prod_i X_i$. For each i, choose a countable local base $(B_j^{(i)})_j$ at x_i. Without loss of generality, let $(B_j^{(i)})_j$ be decreasing for each i. Then the following sets form a local base at x:

CC used.

(a) $\pi_1^{-1}(B_1^{(1)})$
(b) $\pi_1^{-1}(B_2^{(1)}) \cap \pi_2^{-1}(B_2^{(2)})$
(c) $\pi_1^{-1}(B_3^{(1)}) \cap \pi_2^{-1}(B_3^{(2)}) \cap \pi_3^{-1}(B_3^{(3)})$
$\vdots$

(iii) Second countability: For $i = 1, 2, \ldots$, choose countable local bases $\mathscr{B}_i$'s for second countable X_i. Then the union of the following collections forms a countable base for $\prod_i X_i$:[7]

CC used.

(a) $\pi_1^{-1}(\mathscr{B}_1)$
(b) $\pi_1^{-1}(\mathscr{B}_1) \cap \pi_2^{-1}(\mathscr{B}_2)$
(c) $\pi_1^{-1}(\mathscr{B}_1) \cap \pi_2^{-1}(\mathscr{B}_2) \cap \pi_3^{-1}(\mathscr{B}_3)$
$\vdots$

□

[7] Notation abused for π_i^{-1} and $\cap$.

Remark. It turns out that Hewitt-Marczewski-Pondiczery theorem implies that $\mathfrak{c}$-fold product also preserves separability.

Preservation of first (and hence second) countability under uncountable products is not true: Consider an uncountable product of discrete $\{0,1\}$.

Proposition 8.4.

(i) For a first countable domain, sequential continuity $\implies$ continuity.

(ii) For a first countable space, closure is precisely the set of limits of sequences.

Proof. (i) Let $f\colon X \to Y$ be sequentially continuous at $c \in X$ with X being first countable. Suppose f is not continuous at c. Thus, take an open neighborhood V of $f(c)$ such that $f(U)$ spills outside V for each open neighborhood U of c. Let B_n's form a local base at c and choose for each n, an $x_n \in B_n$ such that $f(x_n) \notin V$. But then $f(x_n) \not\to f(c)$ despite $x_n \to c$.

(ii) Let $c \in \overline{A} \setminus A$ and let B_n's form a local base at c. Then for each n, choose $x_n \in B_n \cap A$. Then (x_n) is a sequence in A converging to c. □

In both, CC's usage can be avoided if X is separable.

Remark. (i) Any function from a co-countable topology is sequentially continuous, and yet needn't be continuous, for instance, $\mathrm{id}_X\colon X_{\text{co-count}} \to X_{\text{discr}}$ for any uncountable X.

(ii) For the cocountable topology on an uncountable set, the closure of any nonempty open set in the cocountable topology is the whole space.

Corollary 8.5. *A first countable topology is determined by convergence.*[8] *Further, if the space is T_1 as well, then specifying just the convergent sequences suffices.*

Proof. Just note that in a T_1 space, $x_i \to c \iff x_1, c, x_2, c, x_3, c, \ldots$ is convergent. □

Remark. (i) To see the necessity of first countability, note that cocountable and discrete topologies have the same convergent sequences and their limits. (Note that discrete is first countable.)

(ii) To see the necessity of T_1, consider the Sierpiński and indiscrete topologies on $\{0,1\}$.

[8] That is if τ_1, τ_2 are first countable topologies on X with $x_i \to c$ in $\tau_1 \iff x_i \to c$ in τ_2, then $\tau_1 = \tau_2$.

CHAPTER II

Metric Spaces

Conventions. Unless stated otherwise, assume the following:

- X, Y will denote metric spaces.
- E will be reserved for generic sets.
- Subsets of metric spaces will be seen as metric subspaces.
- For $x \in X$ and $r > 0$, we'll use
  - $B(x, r) := \{y \in X : d(y, x) < r\}$, and
  - $D(x, r) := \{y \in X : d(y, x) \leq r\}$.

  Sometimes, we'll also denote these by $B_r(x)$ and $D_r(x)$.
- The diameter of a subset A of a metric space will be denoted by $\delta(A)$.
- For any $f, g \in X^E$, we'll define $d_\infty(f, g) := \sup_{e \in E} d(f(e), g(e))$.
- A metric space will also be considered a topological space under the induced topology.
- Finite product of metric spaces will be considered together with the metric generated by any of the p-norms (which are uniformly equivalent[1]). (See 2.1.)

1. General

The triangle inequality immediately yields:

Lemma 1.1. *Metric is continuous. Further, if $E \subseteq X$, then $x \mapsto d(x, E)$ is also continuous.*

[1] $\|x\|_\infty \leq \|x\|_p \leq n^{1/p}\|x\|_\infty$ for $x \in K^n$.

Remark. Note that $d(x, \emptyset) = +\infty$ for all x.

Lemma 1.2.

(i) Metric spaces are first countable.

(ii) Separable metric spaces are second countable.

Proof. (i) $B_{1/n}(x)$'s forms a local base at x.

(ii) Let S be a countable dense subset of X. Then $\bigcup_{x\in S}\{B_{1/n}(x) : n \geq 1\}$ forms a countable base:

Consider $B_{1/n}(y)$. Let $x \in B_{1/2n}(y) \cap S$. Then $y \in B_{1/2n}(x) \subseteq B_{1/n}(y)$. □

Let $E \subseteq X$ and $x \in X$. Then a point $y \in E$ is called **a point of best approximation** for x in Y iff $d(x, y) = d(x, E)$.

2. Products of Metric Spaces

We'll try to gain some insight into the following question in this section: Given metric spaces X_i's, is there a metric on $\prod X_i$ that induces the product topology on $\prod X_i$?

Proposition 2.1 (Finite products)**.** *If $X_1, \dots, X_n$ are metric spaces and $\|\cdot\|$ a norm on $\mathbb{R}^n$ which is monotonic along each cardinal direction at each point in the orthant $[0, +\infty)^n$, then*

$$d(x, y) := \big\|\big(d_1(x_1, y_1), \dots, d_n(x_n, y_n)\big)\big\| \tag{2.1}$$

defines a metric on $X_1 \times \cdots \times X_n$ that induces the product topology on it. Further:

(i) Cauchy-ness of a sequence in the product is equivalent to the Cauchy-ness of the component sequences in product spaces.

(ii) If all the spaces are nonempty, then $\prod_i X_i$ is complete $\iff$ each X_i is.

Proof. That it's a metric is easily verified:

- $d(x, y) = 0 \iff$ each $d_i(x_i, y_i) = 0$ (since $\|\cdot\|$ is positive definite) $\iff$ each $x_i = y_i$ (since d_i's are positive definite) $\iff x = y$.
- d is symmetric since each d_i is.
- d satisfies triangle inequality since $\|\cdot\|$ and all d_i's do and $\|\cdot\|$ is monotonic along the cardinal directions in the orthant.

We now verify that these induce the product topology. Because the norms on $\mathbb{R}^n$ are uniformly equivalent (**??**) and uniformly equivalent metrics are topologically equivalent, we may assume without loss of generality that $\|\cdot\|$, which is a norm, is the max-norm, which clearly generates the product topology.

For (i), because Cauchy-ness of a sequence is preserved under uniformly equivalent metrics, we may again work with max-norm which makes the statement obvious.

(ii) follows immediately from (i) and the characterization of convergence in the product topology (3.4). □

Remark. Note that to conclude that d is a metric, full power of $\|\cdot\|$ being a norm was not used—just that it's positive definite, and that it satisfies triangle inequality and the monotonicity assumption. However, the fact that it's a norm was used in concluding that it generates the product topology.

Secondly, to see the necessity of monotonicity of $\|\cdot\|$, consider the following digression:

Any linear injection T on a vector space over K into itself gives a means to produce a new norm from any given norm on it, given by $\|x\|_{\text{new}} := \|Tx\|_{\text{old}}$.

Thus, consider the linear isomorphism T on $\mathbb{R}^2$ given by $x \mapsto \left[\begin{smallmatrix} a & a \\ b & -b \end{smallmatrix}\right] x$ where $a, b > 0$, and let $\|\cdot\|$ be the norm that T generates out of $\|\cdot\|_1$. We then show that d as defined by 2.1 needn't satisfy the triangle inequality. Indeed, for $x, y \in X_1 \times X_2$, we have

Add a diagram showing the unit ball of the new norm.

$$\begin{aligned} d(x,y) &= \big\|\big(d_1(x_1,y_1), d_2(x_2,y_2)\big)\big\| \\ &= \big\|T\big(d_1(x_1,y_1), d_2(x_2,y_2)\big)\big\|_1 \\ &= a\left(d_1(x_1,y_1) + d_2(x_2,y_2)\right) + b\left|d_1(x_1,y_1) - d_2(x_2,y_2)\right|. \end{aligned}$$

Take $x, y, z \in X_1 \times X_2$ such that $d_i(x_i, y_i) = 1 = d_i(y_i, z_i)$ for $i = 1, 2$. Set $\alpha_i := d_i(x_i, z_i)$. Then $d(x,y) + d(y,z) = 4a$ and $d(x,z) = a\,(\alpha_1 + \alpha_2) + b\,|\alpha_1 - \alpha_2| \geq b\,|\alpha_1 - \alpha_2|$. Thus, ensuring $4a/b < |\alpha_1 - \alpha_2|$ ensures the violation of triangle inequality.

Lemma 2.2. *Any metric is topologically equivalent to a bounded metric which preserves the Cauchy sequences.*

Proof. Let $f\colon [0, +\infty) \to [0, 1)$ be a strictly increasing bijection which is subadditive, *i.e.*, $f(x+y) \leq f(x) + f(y)$,[2] for instance $x \mapsto x/(1+x)$. Then $d'(x,y) := f(d(x,y))$ defines a bounded metric on X:

- $d'(x,y) = 0 \iff d(x,y) = f^{-1}(0) \overset{\text{w}}{=} 0$ (since f a strictly increasing bijection) $\iff x = y$.
- d' is symmetric because d is.

[2] This gives another reason why f is continuous: $f(x+\delta) - f(x) \leq f(\delta) \overset{\text{w}}{<} \varepsilon$ if $\delta < f^{-1}(\varepsilon)$.

- $d'(x,y) + d'(y,z) = f(d(x,y)) + f(d(y,z)) \geq f(d(x,y) + d(y,z))$ (since f is subadditive) $\geq f(d(x,z))$ (since f is increasing) $= d'(x,z)$.

We now show the topological equivalence:

- Consider $B_r(x)$. We want an $\varepsilon > 0$ such that $B'_\varepsilon(x) \subseteq B_r(x)$. Let $d'(y,x) < \varepsilon$, *i.e.*, $f(d(y,x)) < \varepsilon$. Take $\varepsilon < 1$, so that $d(y,x) < f^{-1}(\varepsilon)$ (since inverse of strictly increasing f is strictly increasing as well). Thus, it suffices to have $f^{-1}(\varepsilon) < r \overset{\text{w}}{\Longleftarrow} \varepsilon < f(r)$. Thus any $0 < \varepsilon < \min(1, f(r))$ works (note that $f(r) > f(0) = 0$).
- Consider $B'_r(x)$. Without loss of generality, take $r < 1$. We find an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq B'_r(x)$. Let $d(y,x) < \varepsilon \overset{\text{w}}{\Longrightarrow} d'(y,x) < f(\varepsilon)$ (as f is strictly increasing). Thus it suffices to have $f(\varepsilon) < r \overset{\text{w}}{\Longleftarrow} \varepsilon < f^{-1}(r)$ (note that $r < 1$). Thus it suffices to have $0 < \varepsilon < f^{-1}(r)$. □

That the Cauchy sequences are the same is straightforward: Since f is a strictly monotonic bijection, f and f^{-1} are continuous. Thus $d(x_n, y_n) \to 0 \iff f(d(x_n, y_n)) \to 0 \iff d'(x_n, y_n) \to 0$.

Proposition 2.3. *On countable product of metric spaces is metrizable such that the analogues of (i) and (ii) of 2.1 hold.*

Proof. The finite case follows from 2.1. Thus, consider the metric spaces $X_1, X_2, \ldots$, and set $X := \prod_i X_i$. Because of 2.2, we may without loss of generality assume that each d_i is bounded by 1 which allows to define

$$d(x,y) := \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$$

for $x, y \in X$. That d defines a metric is immediate. We show the topological equivalence:

- Consider a generic set, $\pi_i^{-1}(B_r^{(i)}(x_i))$ with $x_i \in X_i$[3] for a fixed i, that is united over to form a subbasic set of the product topology. Without loss of generality, assume that $X \neq \emptyset$, so that we may take an $x \in X$ such that it's i-th coordinate is the x_i above. It suffices to find an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq \pi_i^{-1}(B_r^{(i)}(x_i)) \overset{\text{w}}{\Longleftarrow} \pi_i(B_\varepsilon(x)) \subseteq B_r^{(i)}(x_i)$. Now, let $y_i \in$ LHS[4] so that there's a $y \in B_\varepsilon(x)$ whose i-th coordinate is y_i. Then $d_i(y_i, x_i)/2^i \leq d(y,x) < \varepsilon \overset{\text{w}}{\Longrightarrow} d_i(y_i, x_i) < 2^i\varepsilon$. Thus it suffices to have $\varepsilon < r/2^i$ to ensure that $y_i \in B_r^{(i)}(x_i)$.

[3]Yes, notation's being abused.
[4]Again abusing notation.

- Consider a basic open set $B_r(x)$ of the metric topology on X. It suffices to find $\varepsilon > 0$ and $x_j \in X_j$ for finitely many j's such that $\bigcap_j \pi_j^{-1}(B_\varepsilon^{(j)}(x_j)) \subseteq B_r(x)$. Let j's come from $\{1, \ldots, n\}$ and $y \in \bigcap_j \pi_j^{-1}(B_\varepsilon^{(j)}(x_j))$ so that each $d_j(y_j, x_j) < \varepsilon$. Thus, since d_i's are bounded by 1, we have $d(y, x) < \varepsilon + 1/2^n \overset{\text{w}}{<} r$ if $1/2^n < r - \varepsilon$, which can be ensured by taking $\varepsilon < r$ and n large enough.

We now show (i), from which (ii) follows immediately:

Let $(x^{(n)}) \in X$ be Cauchy. Fix an i. Then $d_i(x_i^{(m)}, x_i^{(n)})/2^i \leq d(x^{(m)}, x^{(n)}) \to 0$ as $m, n \to \infty$.

Conversely, let $(x_i^{(n)})_n \in X_i$ be Cauchy for each i. Let $\varepsilon > 0$. Fix an N. Then $d(x^{(m)}, x^{(n)}) = \sum_{i \leq N} d_i(x_i^{(m)}, x_i^{(n)})/2^i + 1/2^N \overset{\text{w}}{<} \varepsilon$ if $\sum_{i \leq N} d_i(x_i^{(m)}, x_i^{(n)})/2^i < \varepsilon - 1/2^N$. Thus, take N such that $\varepsilon - 1/2^N > 0$ and then use that Cauchy-ness of $(x_i^{(n)})_n$ for $i \leq N$.

□

Remark. To see the necessity of countability, consider an uncountable product of discrete $\{0, 1\}$, which is not first countable and hence not metrizable.

3. Uniform Properties

Uniform properties encompass things like uniform continuity and Cauchy sequences.

Lemma 3.1. *In a complete space, closed subspaces are precisely the complete ones.*

Lemma 3.2. *In a metric space, each sequence has a Cauchy subsequence $\iff$ the space is totally bounded.*

Proof. "$\Rightarrow$": Consider a sequence (x_i). Take an infinite subset I_1 of the indices such that $\{x_i : i \in I_1\}$ lies in a ball of diameter 1. Having chosen I_n, choose an infinite subset $I_{n+1} \subseteq I_n$ such that $\{x_i : i \in I_{n+1}\}$ lies in a ball of diameter $1/(n+1)$. (This is possible since the the space is totally bounded.) Now, choose $i_n \in I_n$ such that (i_n) is increasing. Then $(x_{i_n})_n$ forms a Cauchy sequence, for for $n > m \geq N$, we have $d(x_{i_m}, x_{i_n}) < 1/N$. *DC used.*

"$\Leftarrow$": Suppose X is not totally bounded so that take an $\varepsilon > 0$ such that no finitely many balls of radius ε can ever cover X. Let $x_1 \in X$.[5] Having chosen $x_1, \ldots, x_n$, choose $x_{n+1} \in X \setminus \bigcup_{i=1}^n B_\varepsilon(x_i)$. Then (x_i) is non-Cauchy sequence, for $d(x_i, x_j) \geq \varepsilon$ for all $i \neq j$. *DC used.* □

[5] Note that X has got to be enoenmtpy.

Remark. The discrete metric on an infinite set, which is not totally bounded, contains sequences with no Cauchy subsequences.

3.1 Completion of metric spaces

A **completion** of X is a complete metric space $\hat{X}$ together with an isometry $\iota\colon X \to \hat{X}$ such that any isometry $X \to Y$ into a complete metric space Y factors uniquely through ι via an isometry:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\iota"'] & & Y \\ & \hat{X} \arrow[ur, dashed] & \end{tikzcd}$$

Corollary 3.3. *If X is complete, then* $\mathrm{id}\colon X \to X$ *is a completion of X.*

Theorem 3.4. *Each metric space admits a completion, which is unique up to bi-isometries.*[6]

The following easy facts will be employed to simplify the proof:

Lemma 3.5.

(i) If (x_i) is Cauchy and $(\alpha_i) \in \mathbb{R}^+$, then there exists a subsequence (x_{i_j}) such that for each N, we have $d(x_{i_j}, x_{i_k}) < \alpha_N$ whenever $j, k \geq N$.

(ii) A Cauchy sequence converges iff any of its subsequence converges.

Proof of 3.4. The uniqueness follows by the usual categorical argument. Let's show the existence of a completion of X. Define $\hat{X}$ to be set of the Cauchy sequences in X modded out by the following equivalence relation:

$$(x_i) \sim (y_i) \text{ iff } d(x_i, y_i) \to 0$$

The following defines a well-defined metric on $\hat{X}$:

$$\hat{d}\big((x_i), (y_i)\big) := \lim\nolimits_i d(x_i, y_i)$$

We show that $\hat{X}$ is complete:

Let $\big(\overline{x^{(n)}}\big)$ be Cauchy in $\hat{X}$, where each $x^{(n)}$ is a Cauchy sequence $\big(x_i^{(n)}\big)$ in X. Noting that each subsequence of a Cauchy sequence in X is related to the parent sequence, and due to 3.5, we may without loss of generality assume: *CC used.*

(i) $n \geq m \implies \hat{d}\big(\overline{x^{(n)}}, \overline{x^{(m)}}\big) < 1/m$.

[6] A bi-isometry is a bijective isometry whose inverse is also an isometry.

(ii) For each n, we have $j \geq i \implies d\big(x_j^{(n)}, x_i^{(n)}\big) < 1/i$.

Now, it follows that the diagonal sequence $\big(x_i^{(i)}\big)$ is Cauchy:

$$\begin{aligned}
d\big(x_j^{(j)}, x_i^{(i)}\big) &\leq d\big(x_j^{(j)}, x_j^{(i)}\big) + d\big(x_j^{(i)}, x_i^{(i)}\big) \\
&< d\big(x_j^{(j)}, x_k^{(j)}\big) + d\big(x_k^{(j)}, x_k^{(i)}\big) + d\big(x_k^{(i)}, x_j^{(i)}\big) && (k \text{ arbitrary}) \\
&\qquad + 1/i && (\text{letting } j \geq i) \\
&< 1/j + d\big(x_k^{(j)}, x_k^{(i)}\big) + 1/j + 1/i && (\text{letting } k \geq j) \\
&\leq 2/i + \hat{d}\big(\,\overline{x^{(j)}},\, \overline{x^{(i)}}\,\big) + 1/i && (\text{taking } k \to \infty) \\
&< 2/j + 2/i && (\text{since } j \geq i)
\end{aligned}$$

Also, $\overline{x^{(n)}} \to \overline{\big(x_i^{(i)}\big)}$ in $\hat{X}$:

$$\begin{aligned}
d\big(x_i^{(n)}, x_i^{(i)}\big) &\leq d\big(x_i^{(n)}, x_j^{(n)}\big) + d\big(x_j^{(n)}, x_j^{(i)}\big) + d\big(x_j^{(i)}, x_i^{(i)}\big) \\
&< 1/i + d\big(x_j^{(n)}, x_i^{(i)}\big) + 1/i && (\text{letting } j \geq i) \\
&< 2/i + d\big(\,\overline{x^{(n)}},\, \overline{x^{(i)}}\,\big) && (\text{taking } j \to \infty) \\
&< 3/i
\end{aligned}$$

We now check for the universal property:

Note that $\iota\colon X \to \hat{X}$ given by $x \mapsto (x, x, \dots)$ is an isometry. Let $f\colon X \to Y$ be another isometry with Y being complete. Suppose it does factor through ι via an isometry $\hat{f}$:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\iota"'] & & Y \\
& \hat{X} \arrow[ur, dashed, "\hat{f}"'] &
\end{tikzcd}$$

This in turn determines $\hat{f}$ uniquely:

Let $\overline{(x_i)} \in \hat{X}$, where (x_i) is Cauchy in X. Clearly, $\iota(x_i) \to \overline{(x_i)}$ so that

$$f(x_i) \to \hat{f}\big(\,\overline{(x_i)}\,\big) \tag{3.2}$$

as $\underline{\hat{f} \text{ is continuous}}$ and $\underline{\hat{f} \circ \iota = f}$.

We now show that 3.2 indeed defines a factoring of f via ι:

- $\hat{f}$ is well-defined: (i) If (x_i) is Cauchy in X, then since $\underline{f \text{ is an isometry}}$ and $\underline{Y \text{ is complete}}$, $(f(x_i))$ is convergent in Y. (ii) If (x_i) and (y_i) are equivalent Cauchy sequences in X, then $d(x_i, y_i) \to 0 \overset{\text{w}}{\implies} d(f(x_i), f(y_i)) \to 0$ (since $\underline{f \text{ an isometry}}$) so that $d\big(\lim_i f(x_i), \lim_i f(y_i)\big) = 0$.[7]

[7] Recall the component-wise convergence in product topology.

- $\hat{f}$ is an isometry:

$$\begin{aligned} d\Big(\hat{f}\big(\,\overline{(x_i)}\,\big),\, \hat{f}\big(\,\overline{(y_i)}\,\big)\Big) &= d\big(\lim_i f(x_i), \lim_i f(y_i)\big) \\ &= \lim_i d(f(x_i), f(y_i)) \\ &= \lim_i d(x_i, y_i) && (\underline{f \text{ is an isometry}}) \\ &= \hat{d}\big(\,\overline{(x_i)}, \overline{(y_i)}\,\big) \end{aligned}$$

- Finally, $\hat{f} \circ \iota = f$ is clear. ☐

Proposition 3.6. *Any space is dense in its completion.*

✓~~Ponder: Can this be taken as an alternative universal property?~~

Proof. Let $\iota\colon X \to \hat{X}$ be a completion of X. Now, the restriction $\iota|\colon X \to \overline{\iota(X)}$ is also a completion of $\hat{X}$:

- $\overline{\iota(X)}$ is complete due to 3.1 and $\iota|$ is still an isometry.
- Let $f\colon X \to Y$ be an isometry with Y complete. Then f factors through ι which induces a factoring through $\iota|$ as well:

$$\begin{tikzcd} & X \arrow[dl, "\iota|"'] \arrow[d, "\iota"] \arrow[dr, "f"] & \\ \overline{\iota(X)} \arrow[r, hook] & \hat{X} \arrow[r, dashed] & Y \end{tikzcd}$$

For uniqueness, just note that any factoring $\hat{f}$ of f through $\iota|$ is uniquely determined on $\iota(X)$ which is dense in $\overline{\iota(X)}$, thereby also getting determined on $\overline{\iota(X)}$ (by 7.2):

$$\begin{tikzcd} & X \arrow[dl] \arrow[d, "\iota|"] \arrow[dr, "f"] & \\ \iota(X) \arrow[r, hook] & \overline{\iota(X)} \arrow[r, "\hat{f}"'] & Y \end{tikzcd}$$

Since $\underline{\hat{X} \text{ is a completion}}$, there exists an isometry α such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "\iota|"'] \arrow[dr, "\iota"] & \\ \overline{\iota(X)} \arrow[rr, hook, shift left, "\text{incl}"] & & \hat{X} \arrow[ll, dashed, shift left, "\alpha"] \end{tikzcd}$$

It follows that $\iota|$ factors through itself via $\alpha \circ \text{incl}$, so that it is precisely $\text{id}_{\overline{\iota(X)}}$ (since $\underline{\overline{\iota(X)} \text{ is a completion}}$) $\overset{\text{w}}{\Longrightarrow}$ incl is surjective $\overset{\text{w}}{\Longrightarrow}$ $\overline{\iota(X)} = \hat{X}$. ☐

Remark. Conversely, it's also true that if A is dense in X with X complete, then X is a completion of A. See 3.13.

From 5.1, it now immediately follows that:

Corollary 3.7. *Completion preserves separability.*

3.2 Metric equivalences

First, an easy lemma:

Lemma 3.8. *Uniformly equivalent metrics preserve uniform and Lipschitz continuities.*

Proposition 3.9. *For metrics on a given set, we have:*

Uniform equivalence $\implies$ id *is uniformly continuous in both directions* $\implies$ *same Cauchy sequences* $\implies$ *same convergence* $\iff$ *topological equivalence.*

Proof. The first two implications are trivial and the last follows from 8.5. For the penultimate, just note that $x_i \to c$ iff $x_1, c, x_2, c, \dots$ is Cauchy. □

Remark. None of the converses are true. Let $f\colon X \to X$ be a homeomorphism which thus induces a topologically equivalent metric on X (see 3.10).

(i) Let f, f^{-1} be uniformly continuous and f not be Lipschitz (for instance, $f\colon x \mapsto \sqrt{x}$ on $[0,1]$[8]). Then id is uniformly continuous in both directions and yet the metrics are not uniformly equivalent.

(ii) Let f not be uniformly continuous and X be complete[9] (like $x \to x^3$ on $\mathbb{R}$). Then Cauchy sequences are just convergent sequences, which are thus the same. However, $\mathrm{id}\colon X_{\text{old}} \to X_{\text{new}}$ is not uniformly continuous.

(iii) Consider $x \mapsto 1/x$ on $\mathbb{R}^+$. However, note that $1, 2, 3, \dots$ is Cauchy in the new metric and not in the old one.

Any injection f on a set into itself gives a means to generate a new metric given any metric on it via $d_{\text{new}}(x, y) := d_{\text{old}}(f(x), f(y))$.

Lemma 3.10. *Let $f\colon X \to X$ be a bijection which thus induces a new metric on X. Then the following hold:*

(i) The new metric is topologically equivalent to the old one $\iff$ *f is a homeomorphism on X_{old}.*

[8] See 3.15.

[9] Note that X_{old} is complete $\iff$ X_{new} is.

(ii) id: $X_{old} \to X_{new}$ *is uniformly continuous* $\iff$ f *is uniformly continuous on* X_{old}.

Proof. (i) Use 8.5 and 8.4.
(ii) Since by definition, $d_{\text{new}}(x, y) = d_{\text{old}}(f(x), f(y))$. $\square$

3.3 Stronger forms of continuity

Proposition 3.11.

(i) f *is uniformly continuous* $\iff$ $d(f(x_n), f(y_n)) \to 0$ *whenever* $d(x_n, y_n) \to 0$.

(ii) Uniform continuity on every totally bounded subset of the domain $\iff$ *Cauchy-regularity.*

(iii) Cauchy-regularity $\implies$ *continuity.*

Proof. (i) Let $f\colon X \to Y$ be uniformly continuous.
$\Rightarrow$: Let $d(x_n, y_n) \to 0$ in domain. Let $\varepsilon > 0$. Take $\delta > 0$ such that $d(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$. Take N such that for each $n \geq N$, we have $d(x_n, y_n) < \delta \overset{\text{w}}{\implies} d(f(x_n), f(y_n)) < \varepsilon$.
$\Leftarrow$: If f is not not uniformly continuous, then we may take an $\varepsilon > 0$ and for each n, choose $x_n, y_n \in E$ such that $d(x_n, y_n) < 1/n$ and yet $d(f(x_n), f(y_n)) \geq \varepsilon$. *CC used!*

(ii) "$\Rightarrow$": Since Cauchy sequences are totally bounded.
"$\Leftarrow$": Suppose f is Cauchy-regular and yet not uniformly continuous on a totally bounded subset E of the domain, so that we may take an $\varepsilon > 0$ and for each n, choose $x_n, y_n \in E$ such that $d(x_n, y_n) < 1/n$ and yet $d(f(x_n), f(y_n)) \geq \varepsilon$. *CC used!* Without loss of generality, let (x_n), (y_n) be Cauchy (for E is totally bounded). Now, the sequence $x_1, y_1, x_2, y_2, \ldots$ is also Cauchy, and despite that, its f-image isn't.

(iii) Let f be Cauchy-regular and $x_i \to c$ in the domain. Then $x_1, c, x_2, c, \ldots$ is Cauchy $\overset{\text{w}}{\implies}$ $f(x_1), f(c), f(x_2), f(c), \ldots$ is Cauchy $\overset{\text{w}}{\implies}$ $f(x_i) \to f(c)$. $\square$

Remark. (i) $x \mapsto x^2$ on $\mathbb{R}$ is Cauchy-regular and not uniformly continuous. (ii) $x \mapsto 1/x$ on $\mathbb{R}^+$ is continuous but not Cauchy-regular.

Theorem 3.12 (Extension of Cauchy-regulars)**.** *A Cauchy-regular function from a dense subset to a complete codomain has a unique continuous extension to the whole of domain, which is further Cauchy-regular. Furthermore, this extension preserves uniform continuity and isometry-city.*

Proof. Let $f\colon A \to Y$ be Cauchy-regular where A is dense in X, and Y complete. Let's first settle uniquness.[10] Let $x \in X$. Then take a sequence $(a_i) \in A$ such that $a_i \to x$ (since A dense in X). If $\tilde{f}\colon X \to Y$ is a continuous extension of f, then we must have $f(a_i) \to \tilde{f}(x)$.

CC used; avoidable if X separable.

Let's verify that this indeed gives a well-defined Cauchy-regular extension:

- Well-defined:
  - (i) If (a_i) is Cauchy in A, then by Cauchy-regularity, it's f-image is also Cauchy, and thus convergent due to completeness of Y.
  - (ii) Let $(a_i), (b_i) \in A$ converge to the same point in X. Then the interleaved sequence $a_0, b_0, a_1, b_1, \ldots$ is Cauchy. Due to Cauchy-regularity, its f-image is also Cauchy $\overset{\mathrm{w}}{\implies}$ $\lim_i f(a_i) = \lim_i f(b_i)$.
- Extension: This is clear since for $a \in A$, the constant sequence $(a, a, \ldots)$ converges to a so that $\tilde{f}(a) = \lim_i f(a) = f(a)$.
- Cauchy-regularity: Let $(x^{(n)}) \in X$ be Cauchy. We need to show that it's $\tilde{f}$-image is Cauchy as well. As before due to denseness of A, choose Cauchy sequences $(a_i^{(n)}) \in A$ such that $a_i^{(n)} \to x^{(n)}$ so that $\tilde{f}(x^{(n)}) = \lim_i b_i^{(n)}$. where $b_i^{(n)} := f(a_i^{(n)})$. Note that

  CC used twice; both avoidable if X separable.

  - (i) the Cauchy-ness of $(x^{(n)})$ translates to $\lim_i d(a_i^{(m)}, a_i^{(n)}) \to 0$ as $m, n \to \infty$, and similarly,
  - (ii) that of $(\tilde{f}(x^{(n)}))$ translates to $\lim_i d(b_i^{(m)}, b_i^{(n)}) \to 0$ as $m, n \to \infty$.

  Since the sequences are Cauchy, assume for all n's without loss of generality, that $d(a_j^{(n)}, a_i^{(n)}), d(b_j^{(n)}, b_i^{(n)}) < 1/i$ whenever $j \geq i$.

  Note that it suffices to get hold of a "diagonal" sequence $(b_{N_n}^{(n)})$ that is Cauchy with N_n's increasing[11] for then we'll have

  $$\begin{aligned} d(b_i^{(m)}, b_i^{(n)}) &\leq d(b_i^{(m)}, b_{N_m}^{(m)}) + d(b_{N_m}^{(m)}, b_{N_n}^{(n)}) + d(b_{N_n}^{(n)}, b_i^{(n)}) \\ &< 1/N_m + d(b_{N_m}^{(m)}, b_{N_n}^{(n)}) + 1/N_n \qquad \text{(taking } i \geq N_m, N_n) \end{aligned}$$

  so that we'll have $\lim_i d(b_i^{(m)}, b_i^{(n)})$ being less than the RHS which indeed goes to 0 as $m, n \to \infty$ (since $(b_{N_n}^{(n)})$ Cauchy and N_n's increasing).

  Since f is Cauchy-regular, it suffices to find a Cauchy $(a_{N_n}^{(n)})$. Choose N_n's increasing, such that $d(a_i^{(n)}, a_n^{(n)}) < 1/n$ for each $i \geq n$. Now,

  $$d(a_{N_m}^{(m)}, a_{N_n}^{(n)}) \leq d(a_{N_m}^{(m)}, a_{N_n}^{(m)}) + d(a_{N_m}^{(m)}, a_{N_n}^{(n)})$$

[10] Which also directly follows from 7.2.

[11] Actually, what is required in the proof is just that $1/N_n \to 0$.

$$
\begin{aligned}
&< 1/N_m + d(a^{(m)}_{N_m}, a^{(m)}_i) + d(a^{(m)}_i, a^{(n)}_i) \qquad \text{(taking } n \geq m\text{)} \\
&\qquad + d(a^{(n)}_i, a^{(n)}_{N_n}) \\
&< 2/N_m + d(a^{(m)}_i, a^{(n)}_i) + 1/N_n \qquad \text{(taking } i \geq N_m, N_n\text{)} \\
&\leq 2/N_m + 1/N_n + \lim_i d(a^{(m)}_i, a^{(n)}_i) \qquad \text{(taking } i \to \infty\text{)}
\end{aligned}
$$

which indeed goes to 0 as $m, n \to \infty$.

Finally, we verify the preservations:

- Preservation of uniform continuity: Let f be uniformly continuous. We need to show that $\tilde{f}$ is also uniformly continuous. Let $\varepsilon > 0$ and take $\delta > 0$ such that $d(f(a), f(b)) < \varepsilon$ whenever $d(a, b) < \delta$ for $a, b \in A$. Now, let $x, y \in X$ with $d(x, y) < \delta$. Take $(a_i), (b_i) \in A$ converging to x, y respectively. Now, $d(a_i, b_i) < \delta$ eventually (as $\lim_i d(a_i, b_i) = d(x, y) < \delta$) so that $d(f(a_i), f(b_i)) < \varepsilon$ eventually $\overset{\text{w}}{\implies}$ $d(\tilde{f}(x), \tilde{f}(y)) \overset{\text{w}}{=} \lim_i d(f(a_i), f(b_i)) \leq \varepsilon$. Same comment on CC.
- Preservation of isometry-city: Same technique as in the last point. □

Thus, for a continuous function to be Cauchy-regular, it must be continuously extensible to the completion of its domain.

Corollary 3.13. *If A is dense in X and $\iota\colon X \to \hat{X}$ a completion of X, then*

$$A \hookrightarrow X \xrightarrow{\iota} \hat{X}$$

is a completion of A.

Proof. It's clear that it's an isometry. Now, for Y complete any isometry $f\colon A \to Y$ extends to $X \to Y$ via 3.12 (since A dense in X), which then extends to $\hat{X} \to Y$:

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[dr, "f"'] & X \arrow[r, "\iota"] \arrow[d, dashed] & \hat{X} \arrow[dl, dashed] \\
& Y &
\end{tikzcd}
$$

For uniqueness, let f be extended by $\tilde{f}_1$ and $\tilde{f}_2$:

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[dr, "f"'] & X \arrow[r, "\iota"] & \hat{X} \arrow[dl, shift right, "\tilde{f}_1"'] \arrow[dl, shift left, "\tilde{f}_2"] \\
& Y &
\end{tikzcd}
$$

Then $\tilde{f}_i \circ \iota$'s are continuous extensions of f. Since A is dense in X, these must be equal due to 7.2, from where there equality follows from the universal property of the completion $\iota\colon X \to \hat{X}$. □

This immediately yields:

Corollary 3.14. *In a complete space, the closures of subsets are their completions.*

Proposition 3.15. *Continuous functions on compact sets are uniformly continuous.*

Proof. Let $f\colon X \to Y$ be continuous with X being compact. Let $\varepsilon > 0$. For each $x \in X$, choose $\delta_x > 0$ such that $f(B_{\delta_x}(x)) \subseteq B_\varepsilon(f(x))$. Let $B_{\delta_{x_1}/2}(x_1), \ldots, B_{\delta_{x_n}/2}(x_n)$ cover X (since $\underline{X \text{ is compact}}$). Now, any $x, y \in X$ lie in some $B_{\delta_{x_i}}(x_i)$ whenever $d(x,y) < \min(\delta_{x_1}, \ldots, \delta_{x_n})/2 \overset{\text{w}}{\implies} d(f(x), f(y)) < 2\varepsilon$. □ *No AC needed!*

Remark. To see the necessity of compact domain, consider $x \mapsto 1/x$ on $\mathbb{R} \setminus \{0\}$.

3.4 Uniform convergence

Note that d_∞ "almost" forms a metric on X^E except that it can take infinite values. Thus, if $\mathscr{F} \subseteq X^E$ is such that $d_\infty(f,g) < +\infty$ for all $f, g \in \mathscr{F}$, then d_∞ defines a metric on $\mathscr{F}$. It's easily seen that convergence under this metric coincides with uniform convergence.

Proposition 3.16. *Let E be a topological space and $E_1 \subseteq E$. Let (f_n) be Cauchy in X^{E_1}.*[12] *and $c \in E_1'$ with $f_n(x) \to L_n$ as $x \to c$. Then the following hold:*

(i) (L_n) is Cauchy.

(ii) If $f_n \to f$ uniformly for $f \in X^{E_1}$ and $L_n \to L$ in X, then $f(x) \to L$ as $x \to c$.

Proof. (i) Let $\varepsilon > 0$. Since $\underline{(f_n) \text{ Cauchy}}$, take N such that $d_\infty(f_m, f_n) < \varepsilon$ for all $m, n \geq N$. Now, let $m, n \geq N$. Since $\underline{f_m(x) \to L_m \text{ and } f_n(x) \to L_n \text{ as } x \to c}$, let U be an open neighborhood of c such that $d(f_m(x), L_m), d(f_n(x), L_n) < \varepsilon$ for each $x \in E_1 \cap U \setminus \{c\}$. Then

$$\begin{aligned} d(L_m, L_n) &\leq d(L_m, f_m(x)) + d(f_m(x), f_n(x)) \\ &\qquad\qquad + d(f_n(x), L_n) && \text{(taking } x \in E_1\text{)} \\ &< 3\varepsilon, && \text{(taking } x \in U \setminus \{c\} \text{ as well)} \end{aligned}$$

where taking $x \in E_1 \cap U \setminus \{c\}$ is allowed since $\underline{c \in E_1'}$.

(ii) Let $\varepsilon > 0$. Since $\underline{f_n \to f}$ and $\underline{L_n \to L}$, take N such that $d_\infty(f_N, f), d(L_N, L) < \varepsilon$. Since $\underline{f_N(x) \to L_N \text{ as } x \to c}$, take an open neighborhood U of c such that $f_N(U \setminus \{c\}) \subseteq B_\varepsilon(L_N)$. Now, for $x \in E_1 \cap U \setminus \{c\}$, we have

$$d(f(x), L) \leq d(f(x), f_N(x)) + d(f_N(x), L_N) + d(L_N, L)$$

[12] That is, $d_\infty(f_m, f_n) \to 0$ as $m, n \to \infty$.

$$< 3\varepsilon. \qquad \square$$

Corollary 3.17. *Uniform convergence preserves continuity.*

This is of course not true of just pointwise convergence.

4. Baire's Category Theorem

Proposition 4.1 (Cantor's intersection)**.** *In a complete metric space, the intersection of a decreasing sequence of closed subsets with diameters going to zero, is a singleton.*

Proof. Let F_i's be the closed sets under consideration. That there's at most one point in the intersection is clear since $\delta(F_i) \to 0$. Now, choose $x_i \in F_i$, which form a Cauchy sequence since $\delta(F_i) \to 0$. Since the space is complete, let $x_i \to x$, and since each F_i is closed, x lies in the intersection. □ CC used.

Remark. The necessity of each hypothesis is easy to see.

The diameter of the intersection of a decreasing sequence of subsets needn't be the corresponding limit of diameters even if the sets are closed and bounded. For instance, consider an infinite dimensional normed linear space containing orthonormal vectors $e_1, e_2, \ldots$. Take $F_i := \{e_i, e_{i+1}, \ldots\}$. Then each $\delta(F_i) = \sqrt{2}$, and still the intersection is empty. However, there is one case where we *can* say something:

Proposition 4.2. *Let $F_1 \supseteq F_2 \supseteq \cdots$ be closed subsets of a metric space with F_1 being compact. Then $\delta(\bigcap_i F_i) = \lim_i \delta(F_i)$.*

Proof. "$\leq$" is clear. For "$\geq$", let $\varepsilon > 0$ and choose $x_i, y_i \in F_i$ such that $d(x_i, y_i) > \delta(F_i) - \varepsilon$ (note that each $\delta(F_i) < +\infty$). Now, since F_1 is compact, let $x_{n_i} \to x$ and $y_{n_i} \to y$ in F_1. Since F_i's are closed, x, y lie in the intersection so that $\delta(\bigcap_i F_i) \geq d(x, y) \geq \lim_i \delta(F_i) - \varepsilon$. □

Theorem 4.3 (Baire's category)**.** *In a complete metric space, complements of meager sets are dense.*

Proof. Let $A_1, A_2, \ldots$ be nowhere dense. We show that $X \setminus \bigcup_i A_i$ is dense. Pick a nonempty open U. Since A_1 is nowhere dense, choose $x_1 \in U$ and $r_1 > 0$ such that $B_{r_1}(x_1) \subseteq U$ and $B_{r_1}(x_1) \cap A_1 = \emptyset$. Having chosen x_i, r_i, choose $x_{i+1} \in B_{r_i}(x_i)$ DC used.
such that

- $B_{r_{i+1}}(x_{i+1}) \subseteq B_{r_i}(x_i)$,
- $r_{i+1} \leq r_i/2$, and
- $B_{r_{i+1}}(x_{i+1}) \cap A_{i+1} = \emptyset$.

This is possible since A_i is nowhere dense. Thus, $\overline{B_{r_1}(x_1)} \supseteq \overline{B_{r_2}(x_2)} \supseteq \cdots$ with $\delta(\overline{B_{r_i}(x_i)}) \leq \delta(D_{2r_i}(x_i)) \overset{\text{w}}{=} 2r_i \overset{\text{w}}{\to} 0$ since $r_i \leq r_1/2^{i-1}$. By Cantor (since X is complete), let $x \in \bigcap_i \overline{B_{r_i}(x_i)}$. Then $x \notin \bigcup_i A_i$ since each $B_{r_i}(x_i) \cap A_i = \emptyset$. Finally, $x \in \overline{B_{r_1}(x_1)}$ and without loss of generality, we could've chosen x_1, r_1 such that $\overline{B_{r_1}(x_1)} \subseteq U$. $\square$

Remark. Necessity of completeness is demonstrated by considering any countable metric space in which singletons are not open, for instance $\mathbb{Q}$.

Corollary 4.4. *If countably many closed subsets of a nonempty complete metric space unite to the whole space, then one of them has a nonemtpy interior.*