Contents

CHAPTER I

Topology

Conventions. Unless stated otherwise,

- X, Y will be topological spaces.
- Subsets of topological spaces will be considered under subspace topology.
- Product of topological spaces will be considered under product the topology.

1. Subspaces and Bases

Lemma 1.1. *$\mathscr{B}$ is a base iff the arbitrary unions in $\mathscr{B}$ form a topology.* "⇒" requires AC.

Lemma 1.2.

(i) "Being a subspace of" is transitive.

(ii) (Sub)base of a subspace can be obtained from that of the parent space.

2. Product Topology

From (ii) of Lemma 1.2, we immediately conclude:

Lemma 2.1. *Taking products and subspaces are compatible.*

Remark. This holds for box topology as well.

Lemma 2.2. *Closure of a product is the product of closures.*

Proof. Let $A_i \subseteq X_i$. We show $\overline{\prod_i A_i} = \prod_i \overline{A_i}$.

"$\subseteq$": Suffice to show that $\prod_i F_i$ is closed for F_i's closed in X_i's. Let $(x_i) \notin \prod_i F_i$, say $x_{i_0} \notin F_{i_0}$. Then take an open neighborhood U_{i_0} of x_{i_0} disjoint from F_{i_0}. Now, $\pi_{i_0}^{-1}(U_{i_0})$ is an open neighborhood of (x_i) that is disjoint from $\prod_i F_i$.

"$\supseteq$": Let $U := \bigcap_{j \in J} \pi_j^{-1}(U_j)$ be an open neighborhood of $(x_i) \in \text{RHS}$, where J is finite and each U_j is open. Then each U_j is an open neighborhood of x_j and hence intersects A_j. Thus U intersects $\prod_i A_i$. □ *No choice required.*

Remark. The same holds for box topology as well; however AC will be required for "$\supseteq$".

3. Order Topology

If X is totally ordered, then the **order topology** on it is generated by such sets: (i) (a, b); (ii) $[\min X, b)$ if X has a minimum element; and, (iii) $(a, \max X]$ if X has a maximum element.

Lemma 3.1.

(i) Open rays are open in order topology.

(ii) Order topology is Hausdorff.

(iii) Topology induced from inherited order is coarser than the subspace topology.

Proof. (i) Let's show for right-rays. In case there's a largest element, then it's clear. If not, then $(a, +\infty) = \bigcup_y (a, y)$, which is open.

(ii) Let $x < y$. If there's a z between them, then $(-\infty, z)$ and $(z, +\infty)$ separate them. Otherwise, $(-\infty, y)$ and $(x, +\infty)$ do.

(iii) Obviously. □

Remark. To see strict inclusion in (iii), consider $\{-1\} \cup \{1/n : n \geq 1\} \subseteq \mathbb{R}$.

4. Nowhere Dense Sets

First, an easy characterization:

Lemma 4.1. *The following are equivalent for a subset A of X:*

(i) $X \setminus \overline{A}$ is dense.

(ii) *A is nowhere dense.*

(iii) *Each nonempty open set contains a nonemtpy open subset disjoint from $\overline{A}$.*

(iv) *Each nonempty open set contains a nonemtpy open subset disjoint from A.*

Proof. We only show "(iv) $\Rightarrow$ (i)", the rest being trivial: Suppose $X \setminus \overline{A}$ is not dense, thus getting a nonempty open $U \subseteq \overline{A}$. But then each nonempty open set contained in U intersects A, a contradiction. □

Subsets of a topological space that are countable unions of nowhere dense sets are called **first category** or **meagre** sets. Others are called **second category** sets.

Remark. In $\mathbb{R}$:[1]

	meagre	nonmeagre
dense	$\mathbb{Q}$	$\mathbb{R}$
nondense	$\emptyset$	$[0,1]$

Lemma 4.2. *If $F_1, F_2, \dots$ are closed in X with $X \setminus \bigcup_i F_i$ dense, then each F_i is nowhere dense.*

Remark. Baire's category theorem (Theorem 3.3 of Chapter II) gives a converse to above, stating that complements of meagre sets are dense in a complete metric space.

Proposition 4.3. *In a topological space, the following are equivalent:*

(i) *Complements of meagre sets are dense.*

(ii) *Countable intersections of open dense sets are dense.*

Proof. "$\Rightarrow$": Let $U_1, U_2, \cdots$ be open dense. Now, $\bigcap_i U_i \overset{\text{w}}{=} X \setminus \bigcup_i (X \setminus U_i)$ is dense if each $X \setminus U_i$ is nowhere dense $\overset{\text{w}}{\Longleftarrow}$ $X \setminus (\overline{X \setminus U_i}) \overset{\text{w}}{=} U_i$ (since $\underline{U_i \text{ open}}$) is $\underline{\text{dense}}$, which is true.

"$\Leftarrow$": Let $A_1, A_2, \dots$ be nowhere dense. Then each $X \setminus \overline{A}_i$ is dense $\overset{\text{w}}{\Longrightarrow}$ $\bigcap_i (X \setminus \overline{A}_i) \overset{\text{w}}{=} X \setminus \bigcup_i \overline{A}_i$ is dense $\overset{\text{w}}{\Longrightarrow}$ $X \setminus \bigcup_i A_i$ is dense as well, being a larger set. □

5. Connectedness

Lemma 5.1 (Characterizing disconnectedness). *$E \subseteq X$ is disconnected $\iff$ E can be written as a union of two nonempty subsets A, B of X such that $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.*

[1]Nonmeagre-ness can be concluded by Baire's category theorem (Theorem 3.3 of Chapter II).

Proof. "$\Rightarrow$": Take U, V open in X such that $E \cap U$, $E \cap V$ are nonempty, $E \subseteq U \cup V$, and $E \cap U \cap V = \emptyset$. Now put $A := E \cap U$ and $B := E \cap V$. Then $\overline{A} \cap B \subseteq \overline{E \cap U} \cap V = \emptyset$.

"$\Leftarrow$": Take $U := X \setminus \overline{A}$ and $V := X \setminus \overline{B}$. Then $B \subseteq U$ and $A \subseteq V$ so that both are nonempty and $E \subseteq U \cup V$. Also, $E \cap U \cap V = E \setminus (\overline{A} \cup \overline{B}) = \emptyset$. □

Proposition 5.2 (Linear continua are connected)**.** *Let X be a totally ordered set such that the following hold:*

- *Any nonempty subset that is bounded above has a l.u.b.*
- *Any two points have a point in between them.*

Then under the order topology on X, connected subsets of X are precisely its convex subsets.[2]

Proof. Suppose $I \subseteq X$ is convex, and yet separated by opens U, V. Take $a \in U \cap I$ and $b \in V \cap I$. Without loss of generality, assume $a < b$ (the order is total) so that $[a, b] \subseteq I$ (since I is convex). Note that U, V also form a separation of $[a, b]$. Since $U \cap [a, b]$ is nonempty and bounded, let c be its l.u.b. Clearly, $c \in [a, b]$ so that there are two cases:

$c \in U$: Take a basic interval $J \subseteq U$ containing c. Note that $c < b$ (since $b \in V$) so that $J \supseteq [c, d)$ for some $d > c$. Hence, $U \cap [a, b] \overset{\text{w}}{\supseteq} J \cap [a, b] \supseteq [c, d) \cap [c, b] \overset{\text{w}}{=} [c, d_1)$ where $d_1 := \min(d, b) > c$. Now take e between c and d_1. Then $e \in U \cap [a, b]$ despite $e > c$. *Add a diagram!*

$c \in V$: Again take a basic interval $J \subseteq V$ containing c. This time, $c > a$ (as $a \in U$) so that $J \supseteq (d, c]$ for some $d < c$. Thus, $V \cap [a, b] \overset{\text{w}}{\supseteq} J \cap [a, b] \supseteq (d, c] \cap [a, c] \overset{\text{w}}{=} (d_1, c]$ where $d_1 := \max(d, a) < c$. Now, take e between d_1 and c. Then e is an upper bound for $U \cap [a, b]$ greater than c:

If $x \in U \cap [a, b]$ is greater than e, then $x \in (e, c] \subseteq (d_1, c] \subseteq V$.

Conversely, if I is not convex, then take $x < y < z$ such that $z, z \in I$ but $y \notin I$. Then the rays at y separate I. □

Remark. To see the necessity of the assumptions, consider $\mathbb{Q}$ and $\mathbb{Z}$ respectively which are both totally disconnected.

[2]Recall that a convex subset of an ordered set is any set I such that $[x, y] \subseteq I$ whenever $x, y \in I$ with $x \leq y$.

6. Countability and Separability

Proposition 6.1. *Any base of a second countable space contains a countable base.*

Proof. Let $\mathscr{B}$, $\mathscr{B}'$ be bases of X with $\mathscr{B}$ being countable. It suffices. to show that each $U \in \mathscr{B}$ is a countable union in $\mathscr{B}'$. Thus, consider a $U \in \mathscr{B}$. Define $\mathscr{V} := \{V \in \mathscr{B} : V \subseteq W' \subseteq U \text{ for some } W' \in \mathscr{B}'\}$. Now, for each $V \in \mathscr{V}$, one can choose a $W'_V \in \mathscr{B}$ such that $V \subseteq W'_V \subseteq U$. Now, just note that U is the union of W'_V's which are countably many. □

CC used
Add a diagram.

CC used.

Proposition 6.2. *Second countability is preserved under countable products.*

Proof. For $i = 1, 2, \ldots$, let $\mathscr{B}_i$ be a countable base for X_i. Then the collection of the following sets forms a base for $\prod_i X_i$:[3]

(i) $\pi_1^{-1}(\mathscr{B}_1)$
(ii) $\pi_1^{-1}(\mathscr{B}_1) \cap \pi_2^{-1}(\mathscr{B}_2)$
(iii) $\pi_1^{-1}(\mathscr{B}_1) \cap \pi_2^{-1}(\mathscr{B}_2) \cap \pi_3^{-1}(\mathscr{B}_3)$
$\vdots$ □

Remark. Preservation not guaranteed under uncountable products: Consider an uncountable product of discrete $\{0, 1\}$.

Proposition 6.3. *For a first countable domain, sequential continuity $\implies$ continuity.*

Proof. Let $f\colon X \to Y$ be sequentially continuous at $c \in X$ with X being first countable. Suppose f is not continuous at c. Thus, take an open neighborhood V of $f(c)$ such that $f(U)$ spills outside V for each open neighborhood U of c. Let B_n's form an open base at c and choose for each n, an $x_n \in B_n$ such that $f(x_n) \notin V$. But then $f(x_n) \not\to f(c)$ despite $x_n \to c$. □

CC used. However, if X is separable, then no choice needed!

Remark. For the necessity of first countability, note that any function from a co-countable topology is sequentially continuous, and yet needn't be continuous, for instance, $\mathrm{id}_X \colon X_{\text{co-count}} \to X_{\text{discr}}$ for any uncountable X.

[3] Notation abused for π_i^{-1} and $\cap$.

CHAPTER II

Metric Spaces

Conventions. Unless stated otherwise, assume the following:

- X, Y will denote metric spaces.
- Subsets of metric spaces will be seen as metric subspaces.
- For $x \in X$ and $r > 0$, we'll use
  - $B(x, r) := \{y \in X : d(y, x) < r\}$, and
  - $D(x, r) := \{y \in X : d(y, x) \le r\}$.

  Sometimes, we'll also denote these by $B_r(x)$ and $D_r(x)$.
- A metric space will also be considered a topological space under the induced topology.
- The diameter of a subset A of a metric space will be denoted by $\delta(A)$.

1. General

The triangle inequality immediately yields:

Lemma 1.1. *Metric is continuous. Further, if $E \subseteq X$, then $x \mapsto d(x, E)$ is also continuous.*

Remark. Note that $d(x, \emptyset) = +\infty$ for all x.

Let $E \subseteq X$ and $x \in X$. Then a point $y \in E$ is called **a point of best approximation** for x in Y iff $d(x, y) = d(x, E)$.

2. Uniform Properties

Proposition 2.1. *Uniform continuity $\implies$ Cauchy continuity $\implies$ continuity.*

Proof. content □

Corollary 2.2. *Uniformly continuous functions can be extended to the closure of the domain.*

Proposition 2.3. *Continuous functions on compact sets are uniformly continuous.*

Proof. Let $f\colon X \to Y$ be continuous with X being compact. Let $\varepsilon > 0$. For each $x \in X$, choose $\delta_x > 0$ such that $f(B_{\delta_x}(x)) \subseteq B_\varepsilon(f(x))$. Let $B_{\delta_{x_1}/2}(x_1), \dots, B_{\delta_{x_n}/2}(x_n)$ cover X (since $\underline{X \text{ is compact}}$). Now, any $x, y \in X$ lie in some $B_{\delta_{x_i}}(x_i)$ whenever $d(x,y) < \min(\delta_{x_1}, \dots, \delta_{x_n})/2 \overset{\text{w}}{\implies} d(f(x), f(y)) < \varepsilon$. □ No AC needed!

Remark. To see the necessity of compact domain, consider $x \mapsto 1/x$ on $\mathbb{R} \setminus \{0\}$.

Proposition 2.4 (Uniform convergence preserves continuity)**.** *Let E be a topological space and $E_1 \subseteq E$. Let $f_n\colon E_1 \to X$ converge uniformly to f. Let $c \in \ell(E_1)$ with each $\lim_{x\to c} f_n(x)$ existent. Then*

$$\lim_{x\to c} f(x) = \lim_{n\to\infty} \lim_{x\to c} f_n(x)$$

for each $c \in \ell(E_1)$.[1]

Proof. □

Remark. content

3. Baire's Category Theorem

Proposition 3.1 (Cantor's intersection)**.** *In a complete metric space, the intersection of a decreasing sequence of closed subsets with diameters going to zero, is a singleton.*

[1] Note that metric spaces are Hausdorff so that limits are unique.

Proof. Let F_i's be the closed sets under consideration. That there's at most one point in the intersection is clear since $\delta(F_i) \to 0$. Now, choose $x_i \in F_i$, which form a Cauchy sequence since $\delta(F_i) \to 0$. Since the space is complete, let $x_i \to x$, and since each F_i is closed, x lies in the intersection. ☐ *CC used.*

Remark. The necessity of each hypothesis is easy to see.

The diameter of the intersection of a decreasing sequence of subsets needn't be the corresponding limit of diameters even if the sets are closed and bounded. For instance, consider an infinite dimensional normed linear space containing orthonormal vectors $e_1, e_2, \ldots$. Take $F_i := \{e_i, e_{i+1}, \ldots\}$. Then each $\delta(F_i) = \sqrt{2}$, and still the intersection is empty. However, there is one case where we can say something:

Proposition 3.2. *Let $F_1 \supseteq F_2 \supseteq \cdots$ be closed subsets of a metric space with F_1 being compact. Then $\delta(\bigcap_i F_i) = \lim_i \delta(F_i)$.*

Proof. "$\leq$" is clear. For "$\geq$", let $\varepsilon > 0$ and choose $x_i, y_i \in F_i$ such that $d(x_i, y_i) > \delta(F_i) - \varepsilon$ (note that each $\delta(F_i) < +\infty$). Now, since F_1 is compact, let $x_{n_i} \to x$ and $y_{n_i} \to y$ in F_1. Since F_i's are closed, x, y lie in the intersection so that $\delta(\bigcap_i F_i) \geq d(x,y) \geq \lim_i \delta(F_i) - \varepsilon$. ☐

Theorem 3.3 (Baire's category)**.** *In a complete metric space, complements of meager sets are dense.*

Proof. Let $A_1, A_2, \ldots$ be nowhere dense. We show that $X \setminus \bigcup_i A_i$ is dense. Pick a nonempty open U. Since A_1 is nowhere dense, choose $x_1 \in U$ and $r_1 > 0$ such that $B_{r_1}(x_1) \subseteq U$ and $B_{r_1}(x_1) \cap A_1 = \emptyset$. Having chosen x_i, r_i, choose $x_{i+1} \in B_{r_i}(x_i)$ such that *DC used.*

- $B_{r_{i+1}}(x_{i+1}) \subseteq B_{r_i}(x_i)$,
- $r_{i+1} \leq r_i/2$, and
- $B_{r_{i+1}}(x_{i+1}) \cap A_{i+1} = \emptyset$.

This is possible since A_i is nowhere dense. Thus, $\overline{B_{r_1}(x_1)} \supseteq \overline{B_{r_2}(x_2)} \supseteq \cdots$ with $\delta(\overline{B_{r_i}(x_i)}) \leq \delta(D_{2r_i}(x_i)) \overset{\text{w}}{=} 2r_i \overset{\text{w}}{\to} 0$ since $r_i \leq r_1/2^{i-1}$. By Cantor (since X is complete), let $x \in \bigcap_i \overline{B_{r_i}(x_i)}$. Then $x \notin \bigcup_i A_i$ since each $B_{r_i}(x_i) \cap A_i = \emptyset$. Finally, $x \in \overline{B_{r_1}(x_1)}$ and without loss of generality, we cloud've chosen x_1, r_1 such that $\overline{B_{r_1}(x_1)} \subseteq U$. ☐

Remark. Necessity of completeness is demonstrated by considering any countable metric space in which singletons are not open, for instance $\mathbb{Q}$.

Corollary 3.4. *If countably many closed subsets of a complete metric space unite to the whole space, then one of them has a nonemtpy interior.*