Contents

CHAPTER I

Topology

Conventions. Unless stated otherwise,

- X, Y will be topological spaces.
- Subsets of topological spaces will be considered under subspace topology.
- Product of topological spaces will be considered under product the topology.

1. Subspaces and Bases

Lemma 1.1. *$\mathscr{B}$ is a base iff the arbitrary unions in $\mathscr{B}$ form a topology.*

"$\Rightarrow$" requires AC.

Lemma 1.2.

(i) "Being a subspace of" is transitive.

(ii) (Sub)base of a subspace can be obtained from that of the parent space.

2. Product Topology

From (ii) of 1.2, we immediately conclude:

Lemma 2.1. *Taking products and subspaces are compatible.*

Remark. This holds for box topology as well.

Lemma 2.2. *Closure of a product is the product of closures.*

Proof. Let $A_i \subseteq X_i$. We show $\overline{\prod_i A_i} = \prod_i \overline{A_i}$.

"$\subseteq$": Suffice to show that $\prod_i F_i$ is closed for F_i's closed in X_i's. Let $(x_i) \notin \prod_i F_i$, say $x_{i_0} \notin F_{i_0}$. Then take an open neighborhood U_{i_0} of x_{i_0} disjoint from F_{i_0}. Now, $\pi_{i_0}^{-1}(U_{i_0})$ is an open neighborhood of (x_i) that is disjoint from $\prod_i F_i$.

"$\supseteq$": Let $U := \bigcap_{j \in J} \pi_j^{-1}(U_j)$ be an open neighborhood of $(x_i) \in \mathrm{RHS}$, where J is finite and each U_j is open. Then each U_j is an open neighborhood of x_j and hence intersects A_j. Thus U intersects $\prod_i A_i$. ☐ *No choice required.*

Remark. The same holds for box topology as well; however AC will be required for "$\supseteq$".

3. Order Topology

If X is totally ordered, then the **order topology** on it is generated by such sets: (i) (a, b); (ii) $[\min X, b)$ if X has a minimum element; and, (iii) $(a, \max X]$ if X has a maximum element.

Lemma 3.1.

(i) Open rays are open in order topology.

(ii) Order topology is Hausdorff.

(iii) Topology induced from inherited order is coarser than the subspace topology.

Proof. (i) Let's show for right-rays. In case there's a largest element, then it's clear. If not, then $(a, +\infty) = \bigcup_y (a, y)$, which is open.

(ii) Let $x < y$. If there's a z between them, then $(-\infty, z)$ and $(z, +\infty)$ separate them. Otherwise, $(-\infty, y)$ and $(x, +\infty)$ do.

(iii) Obviously. ☐

Remark. To see strict inclusion in (iii), consider $\{-1\} \cup \{1/n : n \geq 1\} \subseteq \mathbb{R}$.

4. Denseness

Lemma 4.1. *"Being dense" is transitive.*

Proof. Let $A \subseteq B \subseteq X$ with A dense in B and B dense in X. Let U be a nonempty open in X. Then $\underline{B \text{ being dense}}$, intersects U so that $U \cap B$ is a nonempty open in B and thus is intersected by the $\underline{\text{dense } A}$ $\overset{\text{w}}{\Longrightarrow}$ A intersects U. ☐

Lemma 4.2. *Let $A, B \subseteq X$. Then the following hold:*

(i) $B \cap A$ *is dense in* $B \implies B \subseteq \overline{A}$.

(ii) The converse holds if B *is open.*

Proof. (i) We have $B = \mathrm{cl}_B(B \cap A) \subseteq B \cap \overline{B \cap A} \subseteq \overline{B \cap A} \subseteq \overline{A}$.

(ii) We need to show that $\mathrm{cl}_B(B \cap A) = B$. Indeed, if F is any closed such that $B \cap A \subseteq B \cap F$, then $B \subseteq F$ (otherwise, take $x \in B \setminus F \overset{\mathrm{w}}{\implies} x \in B \setminus A \overset{\mathrm{w}}{\implies} x \in B \setminus \overline{A}$ for $\underline{B \text{ is open}}$, contradicting $B \subseteq \overline{A}$). □

Remark. To see the necessity of openness of B in (ii), consider $A = \{1, 1/2, \ldots\}$ and $B = \{0\}$.

4.1 Nowhere dense sets

4.2 gives insight as to why nowhere dense sets are called so—they are dense on no nonempty *open* set. On the other hand, dense sets are dense on the whole space.

Lemma 4.3. *Let* U *be open in* X *and* $A \subseteq X$. *Then the following are equivalent:*

(i) $U \subseteq \overline{A}$.

(ii) Every nonempty open subset contained in U *intersects* $\overline{A}$.

(iii) Every nonempty open subset contained in U *intersects* A.

Corollary 4.4. *The following are equivalent for a subset* A *of* X*:*

(i) $X \setminus \overline{A}$ *is dense.*

(ii) A *is nowhere dense.*

(iii) Each nonempty open set contains a nonemtpy open subset disjoint from $\overline{A}$.

(iv) Each nonempty open set contains a nonemtpy open subset disjoint from A.

Subsets of a topological space that are countable unions of nowhere dense sets are called **first category** or **meagre** sets. Others are called **second category** sets.

Remark. In $\mathbb{R}$:[1]

	meagre	nonmeagre
dense	$\mathbb{Q}$	$\mathbb{R}$
nondense	$\emptyset$	$[0, 1]$

Lemma 4.5. *If* $F_1, F_2, \ldots$ *are closed in* X *with* $X \setminus \bigcup_i F_i$ *dense, then each* F_i *is nowhere dense.*

[1] Nonmeagre-ness can be concluded by Baire's category theorem (3.3).

Remark. Baire's category theorem (3.3) gives a converse to above, stating that complements of meagre sets are dense in a complete metric space.

Proposition 4.6. *In a topological space, the following are equivalent:*

(i) Complements of meagre sets are dense.

(ii) Countable intersections of open dense sets are dense.

Proof. " $\Rightarrow$ ": Let $U_1, U_2, \cdots$ be open dense. Now, $\bigcap_i U_i \overset{\text{w}}{=} X \setminus \bigcup_i (X \setminus U_i)$ is dense if each $X \setminus U_i$ is nowhere dense $\overset{\text{w}}{\Longleftarrow}$ $X \setminus (\overline{X \setminus U_i}) \overset{\text{w}}{=} U_i$ (since $\underline{U_i \text{ open}}$) is $\underline{\text{dense}}$, which is true.

" $\Leftarrow$ ": Let $A_1, A_2, \ldots$ be nowhere dense. Then each $X \setminus \overline{A}_i$ is dense $\overset{\text{w}}{\Longrightarrow}$ $\bigcap_i (X \setminus \overline{A}_i) \overset{\text{w}}{=} X \setminus \bigcup_i \overline{A}_i$ is dense $\overset{\text{w}}{\Longrightarrow}$ $X \setminus \bigcup_i A_i$ is dense as well, being a larger set. □

5. Connectedness

Lemma 5.1 (Characterizing disconnectedness). *$E \subseteq X$ is disconnected $\iff$ E can be written as a union of two nonempty subsets A, B of X such that $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.*

Proof. "$\Rightarrow$": Take U, V open in X such that $E \cap U$, $E \cap V$ are nonempty, $E \subseteq U \cup V$, and $E \cap U \cap V = \emptyset$. Now put $A := E \cap U$ and $B := E \cap V$. Then $\overline{A} \cap B \subseteq \overline{E \cap U} \cap V = \emptyset$.

"$\Leftarrow$": Take $U := X \setminus \overline{A}$ and $V := X \setminus \overline{B}$. Then $B \subseteq U$ and $A \subseteq V$ so that both are nonempty and $E \subseteq U \cup V$. Also, $E \cap U \cap V = E \setminus (\overline{A} \cup \overline{B}) = \emptyset$. □

Proposition 5.2 (Linear continua are connected). *Let X be a totally ordered set such that the following hold:*

- *Any nonempty subset that is bounded above has a l.u.b.*
- *Any two points have a point in between them.*

Then under the order topology on X, connected subsets of X are precisely its convex subsets.[2]

Proof. Suppose $I \subseteq X$ is convex, and yet separated by opens U, V. Take $a \in U \cap I$ and $b \in V \cap I$. Without loss of generality, assume $a < b$ ($\underline{\text{the order is total}}$) so that

[2]Recall that a convex subset of an ordered set is any set I such that $[x, y] \subseteq I$ whenever $x, y \in I$ with $x \leq y$.

$[a, b] \subseteq I$ (since I is convex). Note that U, V also form a separation of $[a, b]$. Since $U \cap [a, b]$ is nonempty and bounded, let c be its l.u.b. Clearly, $c \in [a, b]$ so that there are two cases:

$c \in U$: Take a basic interval $J \subseteq U$ containing c. Note that $c < b$ (since $b \in V$) so that $J \supseteq [c, d)$ for some $d > c$. Hence, $U \cap [a, b] \overset{\text{w}}{\supseteq} J \cap [a, b] \supseteq [c, d) \cap [c, b] \overset{\text{w}}{=} [c, d_1)$ where $d_1 := \min(d, b) > c$. Now take e between c and d_1. Then $e \in U \cap [a, b]$ despite $e > c$.

Add a diagram!

$c \in V$: Again take a basic interval $J \subseteq V$ containing c. This time, $c > a$ (as $a \in U$) so that $J \supseteq (d, c]$ for some $d < c$. Thus, $V \cap [a, b] \overset{\text{w}}{\supseteq} J \cap [a, b] \supseteq (d, c] \cap [a, c] \overset{\text{w}}{=} (d_1, c]$ where $d_1 := \max(d, a) < c$. Now, take e between d_1 and c. Then e is an upper bound for $U \cap [a, b]$ greater than c:

If $x \in U \cap [a, b]$ is greater than e, then $x \in (e, c] \subseteq (d_1, c] \subseteq V$.

Conversely, if I is not convex, then take $x < y < z$ such that $z, z \in I$ but $y \notin I$. Then the rays at y separate I. □

Remark. To see the necessity of the assumptions, consider $\mathbb{Q}$ and $\mathbb{Z}$ respectively which are both totally disconnected.

6. Separation Axioms

Proposition 6.1. *A continuous function on a Hausdorff domain is completely determined by its values on a dense subset of the domain.*

Proof. Let $f, g \colon X \to Y$ be continuous with Y Hausdorff, agreeing on a dense subset $D \subseteq X$ and yet not on $x \in X$. Since Y Hausdorff, separate $f(x)$ and $g(x)$ via opens V and W. Then $f^{-1}(V) \cap g^{-1}(W)$ is an open neighborhood of x, and thus intersects the dense D, say at y. But then $V \ni f(y) = g(y) \in W$, a contradiction. □

Remark. To see the necessity of the Hausdorff codomain (and that just T_1 is not enough), consider the identity function on $\mathbb{R}$ except that it swaps two distinct points. Then this is continuous[3] with the codomain under cofinite topology.

7. Countability and Separability

Lemma 7.1. *A second countable space is separable and first countable.*

[3] More generally, for any set X, any bijection $X \to X_{\text{cofin}}$ is continuous if singletons are closed in the domain.

Proof. Choosing a point from each of the sets from a countable base yields a countable dense set. ∎ *CC used.*

Remark. The converse is not true (however, see 1.2): Consider the Sorgenfrey line, *i.e.*, the lower limit topology on $\mathbb{R}$ generated by the basic open sets of the form $[a, b)$. Any base of this topology must contain for each $x \in \mathbb{R}$, some set with x being its l.u.b., and thus be uncountable. *AC used.*

second countable	first countable	separable	
✓			separable metric spaces
✗	✓	✓	Sorgenfrey line
✗	✓	✗	nonseparable metric spaces[4]
	✗	✓	cofinite on uncountable
	✗	✗	cocountable on uncountable

Proposition 7.2. *Any base of a second countable space contains a countable base.*

Proof. Let $\mathscr{B}$, $\mathscr{B}'$ be bases of X with $\mathscr{B}$ being countable. It suffices. to show that each $U \in \mathscr{B}$ is a countable union in $\mathscr{B}'$. Thus, consider a $U \in \mathscr{B}$. Define $\mathscr{V} := \{V \in \mathscr{B} : V \subseteq W' \subseteq U \text{ for some } W' \in \mathscr{B}'\}$. Now, for each $V \in \mathscr{V}$, one can choose a $W'_V \in \mathscr{B}$ such that $V \subseteq W'_V \subseteq U$. Now, just note that U is the union of W'_V's which are countably many. ∎ *CC used* *Add a diagram.* *CC used.*

Proposition 7.3. *Second countability is preserved under countable products.*

Proof. For $i = 1, 2, \ldots$, let $\mathscr{B}_i$ be a countable base for X_i. Then the collection of the following sets forms a base for $\prod_i X_i$:[5]

(i) $\pi_1^{-1}(\mathscr{B}_1)$
(ii) $\pi_1^{-1}(\mathscr{B}_1) \cap \pi_2^{-1}(\mathscr{B}_2)$
(iii) $\pi_1^{-1}(\mathscr{B}_1) \cap \pi_2^{-1}(\mathscr{B}_2) \cap \pi_3^{-1}(\mathscr{B}_3)$

$\vdots$ ∎

Remark. Preservation not guaranteed under uncountable products: Consider an uncountable product of discrete $\{0, 1\}$.

Proposition 7.4.

[4]For instance, discrete topology on any uncountable set.
[5]Notation abused for π_i^{-1} and $\cap$.

(i) For a first countable domain, sequential continuity $\implies$ continuity.

(ii) For a first countable space, closure is precisely the set of limits of sequences.

Proof. (i) Let $f\colon X \to Y$ be sequentially continuous at $c \in X$ with X being first countable. Suppose f is not continuous at c. Thus, take an open neighborhood V of $f(c)$ such that $f(U)$ spills outside V for each open neighborhood U of c. Let B_n's form a local base at c and choose for each n, an $x_n \in B_n$ such that $f(x_n) \notin V$. But then $f(x_n) \not\to f(c)$ despite $x_n \to c$.

(ii) Let $c \in \overline{A} \setminus A$ and let B_n's form a local base at c. Then for each n, choose $x_n \in B_n \cap A$. Then (x_n) is a sequence in A converging to c. □

In both, CC's usage can be avoided if X is separable.

Remark. (i) Any function from a co-countable topology is sequentially continuous, and yet needn't be continuous, for instance, $\mathrm{id}_X\colon X_{\text{co-count}} \to X_{\text{discr}}$ for any uncountable X.

(ii) For the cocountable topology on an uncountable set, the closure of any nonempty open set in the cocountable topology is the whole space.

Corollary 7.5. *A first countable topology is determined by convergence.*[6] *Further, if the space is T_1 as well, then specifying just the convergent sequences suffices.*

Proof. Just note that in a T_1 space, $x_i \to c$ iff $x_1, c, x_2, c, x_3, c, \ldots$ is convergent. □

Remark. (i) To see the necessity of first countability, note that cocountable and discrete topologies have the same convergent sequences and their limits. (Note that discrete is first countable.)

(ii) To see the necessity of T_1, consider the Sierpiński and indiscrete topologies on $\{0, 1\}$.

[6] That is $x_i \to c$ in τ_1 iff $x_i \to c$ in τ_2.

CHAPTER II

Metric Spaces

Conventions. Unless stated otherwise, assume the following:

- X, Y will denote metric spaces.
- Subsets of metric spaces will be seen as metric subspaces.
- For $x \in X$ and $r > 0$, we'll use
  - $B(x, r) := \{y \in X : d(y, x) < r\}$, and
  - $D(x, r) := \{y \in X : d(y, x) \leq r\}$.

  Sometimes, we'll also denote these by $B_r(x)$ and $D_r(x)$.
- A metric space will also be considered a topological space under the induced topology.
- The diameter of a subset A of a metric space will be denoted by $\delta(A)$.

1. General

The triangle inequality immediately yields:

Lemma 1.1. *Metric is continuous. Further, if $E \subseteq X$, then $x \mapsto d(x, E)$ is also continuous.*

Remark. Note that $d(x, \emptyset) = +\infty$ for all x.

Lemma 1.2.

(i) Metric spaces are first countable.

(ii) Separable metric spaces are second countable.

Proof. (i) $B_{1/n}(x)$'s forms a local base at x.

(ii) Let S be a countable dense subset of X. Then $\bigcup_{x\in S}\{B_{1/n}(x) : n \geq 1\}$ forms a countable base:

Consider $B_{1/n}(y)$. Let $x \in B_{1/2n}(y) \cap S$. Then $y \in B_{1/2n}(x) \subseteq B_{1/n}(y)$. □

Let $E \subseteq X$ and $x \in X$. Then a point $y \in E$ is called **a point of best approximation** for x in Y iff $d(x, y) = d(x, E)$.

2. Uniform Properties

Uniform properties encompass things like uniform continuity and Cauchy sequences.

Lemma 2.1. *In a complete space, closed subspaces are precisely the complete ones.*

Lemma 2.2. *In a metric space, each sequence has a Cauchy subsequence $\iff$ the space is totally bounded.*

Proof. "$\Rightarrow$": Consider a sequence (x_i). Take an infinite subset I_1 of the indices such that $\{x_i : i \in I_1\}$ lies in a ball of diameter 1. Having chosen I_n, choose an infinite subset $I_{n+1} \subseteq I_n$ such that $\{x_i : i \in I_{n+1}\}$ lies in a ball of diameter $1/(n+1)$. (This is possible since the the space is totally bounded.) Now, choose $i_n \in I_n$ such that (i_n) is increasing. Then $(x_{i_n})_n$ forms a Cauchy sequence, for for $n > m \geq N$, we have $d(x_{i_m}, x_{i_n}) < 1/N$. *DC used.*

"$\Leftarrow$": Suppose X is not totally bounded so that take an $\varepsilon > 0$ such that no finitely many balls of radius ε can ever cover X. Let $x_1 \in X$.[1] Having chosen $x_1, \ldots, x_n$, choose $x_{n+1} \in X \setminus \bigcup_{i=1}^{n} B_\varepsilon(x_i)$. Then (x_i) is non-Cauchy sequence, for $d(x_i, x_j) \geq \varepsilon$ for all $i \neq j$. *DC used.* □

Remark. The discrete metric on an infinite set, which is not totally bounded, contains sequences with no Cauchy subsequences.

2.1 Completion of metric spaces

A **completion** of X is a complete metric space $\hat{X}$ together with an isometry $\iota\colon X \to \hat{X}$ such that any isometry $X \to Y$ into a complete metric space Y factors uniquely

[1] Note that X has got to be enoenmtpy.

through ι via an isometry:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\iota"'] & & Y \\ & \hat{X} \arrow[ur, dashed] & \end{tikzcd}$$

Corollary 2.3. *If X is complete, then* $\mathrm{id}\colon X \to X$ *is a completion of X.*

Proposition 2.4. *Each metric space admits a completion, which is unique up to bi-isometries.*[2]

The following easy facts will be employed to simplify the proof:

Lemma 2.5.

(i) If (x_i) is Cauchy and $(\alpha_i) \in \mathbb{R}^+$, then there exists a subsequence (x_{i_j}) such that for each N, we have $d(x_{i_j}, x_{i_k}) < \alpha_N$ whenever $j, k \geq N$.

(ii) A Cauchy sequence converges iff any of its subsequence converges.

Proof of 2.4. The uniqueness follows by the usual categorical argument. Let's show the existence of a completion of X. Define $\hat{X}$ to be set of the Cauchy sequences in X modded out by the following equivalence relation:

$$(x_i) \sim (y_i) \text{ iff } d(x_i, y_i) \to 0$$

The following defines a well-defined metric on $\hat{X}$:

$$\hat{d}\big((x_i), (y_i)\big) := \lim_i d(x_i, y_i)$$

We show that $\hat{X}$ is complete:

Let $\big(\overline{x^{(n)}}\big)$ be Cauchy in $\hat{X}$, where each $x^{(n)}$ is a Cauchy sequence $\big(x_i^{(n)}\big)$ in X. Noting that each subsequence of a Cauchy sequence in X is related to the parent sequence, and due to 2.5, we may without loss of generality assume: *CC used.*

(i) $n \geq m \implies \hat{d}\big(\overline{x^{(n)}}, \overline{x^{(m)}}\big) < 1/m$.

(ii) For each n, we have $j \geq i \implies d\big(x_j^{(n)}, x_i^{(n)}\big) < 1/i$.

Now, it follows that the diagonal sequence $\big(x_i^{(i)}\big)$ is Cauchy:

$$\begin{aligned} d\big(x_j^{(j)}, x_i^{(i)}\big) &\leq d\big(x_j^{(j)}, x_j^{(i)}\big) + d\big(x_j^{(i)}, x_i^{(i)}\big) \\ &< d\big(x_j^{(j)}, x_k^{(j)}\big) + d\big(x_k^{(j)}, x_k^{(i)}\big) + d\big(x_k^{(i)}, x_j^{(i)}\big) \qquad (k \text{ arbitrary}) \end{aligned}$$

[2] A bi-isometry is a bijective isometry whose inverse is also an isometry.

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad + 1/i && \text{(letting } j \geq i\text{)}\\
&< 1/j + d\big(x_k^{(j)}, x_k^{(i)}\big) + 1/j + 1/i && \text{(letting } k \geq j\text{)}\\
&\leq 2/i + \hat{d}\big(\, \overline{x^{(j)}},\, \overline{x^{(i)}}\,\big) + 1/i && \text{(taking } k \to \infty\text{)}\\
&< 2/j + 2/i && \text{(since } j \geq i\text{)}
\end{aligned}
$$

Also, $\overline{x^{(n)}} \to \overline{\big(x_i^{(i)}\big)}$ in $\hat{X}$:

$$
\begin{aligned}
d\big(x_i^{(n)}, x_i^{(i)}\big) &\leq d\big(x_i^{(n)}, x_j^{(n)}\big) + d\big(x_j^{(n)}, x_j^{(i)}\big) + d\big(x_j^{(i)}, x_i^{(i)}\big)\\
&< 1/i + d\big(x_j^{(n)}, x_i^{(i)}\big) + 1/i && \text{(letting } j \geq i\text{)}\\
&< 2/i + d\big(\, \overline{x^{(n)}},\, \overline{x^{(i)}}\,\big) && \text{(taking } j \to \infty\text{)}\\
&< 3/i
\end{aligned}
$$

We now check for the universal property:

Note that $\iota\colon X \to \hat{X}$ given by $x \mapsto (x, x, \dots)$ is an isometry. Let $f\colon X \to Y$ be another isometry with Y being complete. Suppose it does factor through ι via an isometry $\hat{f}$:

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\iota"'] & & Y \\
& \hat{X} \arrow[ur, dashed, "\hat{f}"'] &
\end{tikzcd}
$$

This in turn determines $\hat{f}$ uniquely:

Let $\overline{(x_i)} \in \hat{X}$, where (x_i) is Cauchy in X. Clearly, $\iota(x_i) \to \overline{(x_i)}$ so that

$$f(x_i) \to \hat{f}\big(\,\overline{(x_i)}\,\big) \tag{2.1}$$

as $\hat{f}$ is continuous and $\hat{f} \circ \iota = f$.

We now show that 2.1 indeed defines a factoring of f via ι:

- $\hat{f}$ is well-defined: (i) If (x_i) is Cauchy in X, then since f is an isometry and Y is complete, $(f(x_i))$ is convergent in Y. (ii) If (x_i) and (y_i) are equivalent Cauchy sequences in X, then $d(x_i, y_i) \to 0 \overset{\text{w}}{\Longrightarrow} d(f(x_i), f(y_i)) \to 0$ (since f an isometry) so that $d\big(\lim_i f(x_i), \lim_i f(y_i)\big) = 0$.[3]
- $\hat{f}$ is an isometry:

$$
\begin{aligned}
d\Big(\hat{f}\big(\,\overline{(x_i)}\,\big),\, \hat{f}\big(\,\overline{(y_i)}\,\big)\Big) &= d\big(\lim_i f(x_i), \lim_i f(y_i)\big)\\
&= \lim_i d(f(x_i), f(y_i))\\
&= \lim_i d(x_i, y_i) && \text{(} f \text{ is an isometry)}\\
&= \hat{d}\big(\,\overline{(x_i)},\, \overline{(y_i)}\,\big)
\end{aligned}
$$

[3] Recall the component-wise convergence in product topology.

- Finally, $\hat{f} \circ \iota = f$ is clear. $\square$

Proposition 2.6. *Any space is dense in its completion.*

Ponder: Can this be taken as an alternative universal property?

Proof. Let $\iota\colon X \to \hat{X}$ be a completion of X. Now, the restriction $\iota|\colon X \to \overline{\iota(X)}$ is also a completion of $\hat{X}$:

- $\overline{\iota(X)}$ is complete due to 2.1 and $\iota|$ is still an isometry.
- Let $f\colon X \to Y$ be an isometry with Y complete. Then f factors through ι which induces a factoring through $\iota|$ as well:

$$\begin{tikzcd} & X \arrow[dl, "\iota|"'] \arrow[d, "\iota"] \arrow[dr, "f"] & \\ \overline{\iota(X)} \arrow[r, hook] & \hat{X} \arrow[r, dashed] & Y \end{tikzcd}$$

For uniqueness, just note that any factoring $\hat{f}$ of f through $\iota|$ is uniquely determined on $\iota(X)$ which is dense in $\overline{\iota(X)}$, thereby also getting determined on $\overline{\iota(X)}$ (by 6.1):

$$\begin{tikzcd} & X \arrow[dl] \arrow[d, "\iota|"] \arrow[dr, "f"] & \\ \iota(X) \arrow[r, hook] & \overline{\iota(X)} \arrow[r, "\hat{f}"'] & Y \end{tikzcd}$$

Since $\hat{X}$ is a completion, there exists an isometry α such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "\iota|"'] \arrow[dr, "\iota"] & \\ \overline{\iota(X)} \arrow[rr, hook, shift left, "\mathrm{incl}"] & & \hat{X} \arrow[ll, dashed, shift left, "\alpha"] \end{tikzcd}$$

It follows that $\iota|$ factors through itself via $\alpha \circ \mathrm{incl}$, so that it is precisely $\mathrm{id}_{\overline{\iota(X)}}$ (since $\overline{\iota(X)}$ is a completion) $\overset{\text{w}}{\implies}$ incl is surjective $\overset{\text{w}}{\implies}$ $\overline{\iota(X)} = \hat{X}$. $\square$

From 4.1, it now immediately follows that:

Corollary 2.7. *Completion preserves separability.*

2.2 Metric equivalences

Proposition 2.8. *For metrics on a given set, we have:*

Uniform equivalence $\implies$ id *is uniformly continuous in both directions* $\implies$ *same Cauchy sequences* $\implies$ *same convergence* $\iff$ *topological equivalence.*

Proof. The first two implications are trivial and the last follows from 7.5. For the penultimate, just note that $x_i \to c$ iff $x_1, c, x_2, c, \dots$ is Cauchy. □

Remark. None of the converses are true. Let $f : X \to X$ be a homeomorphism which thus induces a topologically equivalent metric on X (see 2.9).

(i) Let f, f^{-1} be uniformly continuous and f not be Lipschitz (for instance, $f : x \mapsto \sqrt{x}$ on $[0,1]$[4]). Then id is uniformly continuous in both directions and yet the metrics are not uniformly equivalent.

(ii) Let f not be uniformly continuous and X be complete[5] (like $x \to x^3$ on $\mathbb{R}$). Then Cauchy sequences are just convergent sequences, which are thus the same. However, id: $X_{\text{old}} \to X_{\text{new}}$ is not uniformly continuous.

(iii) Consider $x \mapsto 1/x$ on $\mathbb{R}^+$. However, note that $1, 2, 3, \dots$ is Cauchy in the new metric and not in the old one.

Lemma 2.9. *Let $f : X \to X$ be a bijection which thus induces a new metric on X.*[6] *Then the following hold:*

(i) The new metric is topologically equivalent to the old one $\iff$ *f is a homeomorphism on X_{old}.*

(ii) id: $X_{old} \to X_{new}$ *is uniformly continuous* $\iff$ *f is uniformly continuous on X_{old}.*

Proof. (i) Use 7.5 and 7.4.

(ii) Since by definition, $d_{\text{new}}(x,y) = d_{\text{old}}(f(x), f(y))$. □

2.3 Stronger forms of continuity

Proposition 2.10.

[4] See 2.14.

[5] Note that X_{old} is complete iff X_{new} is.

[6] Only injectivity is required if we just need to have a new metric.

(i) Uniform continuity on every totally bounded subset of the domain $\iff$ *Cauchy-regularity.*

(ii) Cauchy-regularity $\implies$ *continuity.*

Proof. (i) "$\Rightarrow$" is obvious since Cauchy sequences are totally bounded.

"$\Leftarrow$": Suppose f is Cauchy-regular and yet not uniformly continuous on a totally bounded subset E of the domain, so that we may take an $\varepsilon > 0$ and for each n, choose $x_n, y_n \in E$ such that $d(x_n, y_n) < 1/n$ and yet $d(f(x_n), f(y_n)) \geq \varepsilon$. Without loss of generality, let (x_n), (y_n) be Cauchy (for E is totally bounded). Now, the sequence $x_1, y_1, x_2, y_2, \ldots$ is also Cauchy, and despite that, its f-image isn't. *CC used!*

(ii) Let f be Cauchy-regular and $x_i \to c$ in the domain. Then $x_1, c, x_2, c, \ldots$ is Cauchy $\overset{\text{w}}{\implies}$ $f(x_1), f(c), f(x_2), f(c), \ldots$ is Cauchy $\overset{\text{w}}{\implies}$ $f(x_i) \to f(c)$. □

Remark. (i) $x \mapsto x^2$ on $\mathbb{R}$ is Cauchy-regular and not uniformly continuous. (ii) $x \mapsto 1/x$ on $\mathbb{R}^+$ is continuous but not Cauchy-regular.

Theorem 2.11 (Extension of Cauchy-regulars). *A Cauchy-regular function from a dense subset to a complete codomain has a unique continuous extension to the whole of domain, which is further Cauchy-regular. Furthermore, this extension preserves uniform continuity and isometry-city.*

Proof. Let $f\colon A \to Y$ be Cauchy-regular where A is dense in X, and Y complete. Let's first settle uniquness.[7] Let $x \in X$. Then take a sequence $(a_i) \in A$ such that $a_i \to x$ (since A dense in X). If $\tilde{f}\colon X \to Y$ is a continuous extension of f, then we must have $f(a_i) \to \tilde{f}(x)$. *CC used; avoidable if X separable.*

Let's verify that this indeed gives a well-defined Cauchy-regular extension:

- Well-defined:
  (i) If (a_i) is Cauchy in A, then by Cauchy-regularity, it's f-image is also Cauchy, and thus convergent due to completeness of Y.
  (ii) Let $(a_i), (b_i) \in A$ converge to the same point in X. Then the interleaved sequence $a_0, b_0, a_1, b_1, \ldots$ is Cauchy. Due to Cauchy-regularity, its f-image is also Cauchy $\overset{\text{w}}{\implies}$ $\lim_i f(a_i) = \lim_i f(b_i)$.
- Extension: This is clear since for $a \in A$, the constant sequence $(a, a, \ldots)$ converges to a so that $\tilde{f}(a) = \lim_i f(a) = f(a)$.
- Cauchy-regularity: Let $(x^{(n)}) \in X$ be Cauchy. We need to show that it's $\tilde{f}$-image is Cauchy as well. As before due to denseness of A, choose Cauchy

[7] Which also directly follows from 6.1.

sequences $(a_i^{(n)}) \in A$ such that $a_i^{(n)} \to x^{(n)}$ so that $\tilde{f}(x^{(n)}) = \lim_i b_i^{(n)}$. where $b_i^{(n)} := f(a_i^{(n)})$. Note that

CC used twice; both avoidable if X separable.

(i) the Cauchy-ness of $(x^{(n)})$ translates to $\lim_i d(a_i^{(m)}, a_i^{(n)}) \to 0$ as $m, n \to \infty$, and similarly,

(ii) that of $(\tilde{f}(x^{(n)}))$ translates to $\lim_i d(b_i^{(m)}, b_i^{(n)}) \to 0$ as $m, n \to \infty$.

Since the sequences are Cauchy, assume for all n's without loss of generality, that $d(a_j^{(n)}, a_i^{(n)}), d(b_j^{(n)}, b_i^{(n)}) < 1/i$ whenever $j \geq i$.

Note that it suffices to get hold of a "diagonal" sequence $(b_{N_n}^{(n)})$ that is Cauchy with N_n's increasing[8] for then we'll have

$$\begin{aligned} d(b_i^{(m)}, b_i^{(n)}) &\leq d(b_i^{(m)}, b_{N_m}^{(m)}) + d(b_{N_m}^{(m)}, b_{N_n}^{(n)}) + d(b_{N_n}^{(n)}, b_i^{(n)}) \\ &< 1/N_m + d(b_{N_m}^{(m)}, b_{N_n}^{(n)}) + 1/N_n \qquad \text{(taking } i \geq N_m, N_n) \end{aligned}$$

so that we'll have $\lim_i d(b_i^{(m)}, b_i^{(n)})$ being less than the RHS which indeed goes to 0 as $m, n \to \infty$ (since $\underline{(b_{N_n}^{(n)}) \text{ Cauchy}}$ and $\underline{N_n\text{'s increasing}}$).

Since $\underline{f \text{ is Cauchy-regular}}$, it suffices to find a Cauchy $(a_{N_n}^{(n)})$. Choose N_n's increasing, such that $d(a_i^{(n)}, a_n^{(n)}) < 1/n$ for each $i \geq n$. Now,

$$\begin{aligned} d(a_{N_m}^{(m)}, a_{N_n}^{(n)}) &\leq d(a_{N_m}^{(m)}, a_{N_n}^{(m)}) + d(a_{N_m}^{(m)}, a_{N_n}^{(n)}) \\ &< 1/N_m + d(a_{N_m}^{(m)}, a_i^{(m)}) + d(a_i^{(m)}, a_i^{(n)}) \qquad \text{(taking } n \geq m) \\ &\qquad + d(a_i^{(n)}, a_{N_n}^{(n)}) \\ &< 2/N_m + d(a_i^{(m)}, a_i^{(n)}) + 1/N_n \qquad \text{(taking } i \geq N_m, N_n) \\ &\leq 2/N_m + 1/N_n + \lim_i d(a_i^{(m)}, a_i^{(n)}) \qquad \text{(taking } i \to \infty) \end{aligned}$$

which indeed goes to 0 as $m, n \to \infty$.

Finally, we verify the preservations:

- Preservation of uniform continuity: Let f be uniformly continuous. We need to show that $\tilde{f}$ is also uniformly continuous. Let $\varepsilon > 0$ and take $\delta > 0$ such that $d(f(a), f(b)) < \varepsilon$ whenever $d(a, b) < \delta$ for $a, b \in A$. Now, let $x, y \in X$ with $d(x, y) < \delta$. Take $(a_i), (b_i) \in A$ converging to x, y respectively. Now, $d(a_i, b_i) < \delta$ eventually (as $\lim_i d(a_i, b_i) = d(x, y) < \delta$) so that $d(f(a_i), f(b_i)) < \varepsilon$ eventually $\overset{\text{w}}{\Longrightarrow}$ $d(\tilde{f}(x), \tilde{f}(y)) \overset{\text{w}}{=} \lim_i d(f(a_i), f(b_i)) \leq \varepsilon$. *Same comment on CC.*
- Preservation of isometry-city: Same technique as in the last point. □

Thus, for a continuous function to be Cauchy-regular, it must be continuously extensible to the completion of its domain.

[8]Actually, what is required in the proof is just that $1/N_n \to 0$.

Corollary 2.12. *If A is dense in X and $\iota\colon X \to \hat{X}$ a completion of X, then*

$$A \hookrightarrow X \xrightarrow{\iota} \hat{X}$$

is a completion of A.

Proof. It's clear that it's an isometry. Now, for Y complete any isometry $f\colon A \to Y$ extends to $X \to Y$ via 2.11 (since A dense in X), which then extends to $\hat{X} \to Y$:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr, "f"'] & X \arrow[r, "\iota"] \arrow[d, dashed] & \hat{X} \arrow[dl, dashed] \\ & Y & \end{tikzcd}$$

For uniqueness, let f be extended by $\tilde{f}_1$ and $\tilde{f}_2$:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr, "f"'] & X \arrow[r, "\iota"] & \hat{X} \arrow[dl, shift right, "\tilde{f}_1"'] \arrow[dl, shift left, "\tilde{f}_2"] \\ & Y & \end{tikzcd}$$

Then $\tilde{f}_i \circ \iota$'s are continuous extensions of f. Since A is dense in X, these must be equal due to 6.1, from where there equality follows from the universal property of the completion $\iota\colon X \to \hat{X}$. $\square$

This immediately yields:

Corollary 2.13. *In a complete space, the closures of subsets are their completions.*

Proposition 2.14. *Continuous functions on compact sets are uniformly continuous.*

Proof. Let $f\colon X \to Y$ be continuous with X being compact. Let $\varepsilon > 0$. For each $x \in X$, choose $\delta_x > 0$ such that $f(B_{\delta_x}(x)) \subseteq B_\varepsilon(f(x))$. Let $B_{\delta_{x_1}/2}(x_1), \ldots, B_{\delta_{x_n}/2}(x_n)$ *No AC needed!*
cover X (since X is compact). Now, any $x, y \in X$ lie in some $B_{\delta_{x_i}}(x_i)$ whenever $d(x,y) < \min(\delta_{x_1}, \ldots, \delta_{x_n})/2 \overset{\text{w}}{\Longrightarrow} d(f(x), f(y)) < 2\varepsilon$. $\square$

Remark. To see the necessity of compact domain, consider $x \mapsto 1/x$ on $\mathbb{R} \setminus \{0\}$.

Proposition 2.15 (Uniform convergence preserves continuity)**.** *Let E be a topological space and $E_1 \subseteq E$. Let $f_n\colon E_1 \to X$ converge uniformly to f. Let $c \in \ell(E_1)$ with each $\lim_{x\to c} f_n(x)$ existent. Then*

$$\lim_{x\to c} f(x) = \lim_{n\to\infty} \lim_{x\to c} f_n(x)$$

for each $c \in \ell(E_1)$.[9]

[9] Note that metric spaces are Hausdorff so that limits are unique.

Proof. □

Remark. content

3. Baire's Category Theorem

Proposition 3.1 (Cantor's intersection)**.** *In a complete metric space, the intersection of a decreasing sequence of closed subsets with diameters going to zero, is a singleton.*

Proof. Let F_i's be the closed sets under consideration. That there's at most one point in the intersection is clear since $\underline{\delta(F_i) \to 0}$. Now, choose $x_i \in F_i$, which form a Cauchy sequence since $\underline{\delta(F_i) \to 0}$. Since the space is complete, let $x_i \to x$, and since each F_i is closed, x lies in the intersection. □ *CC used.*

Remark. The necessity of each hypothesis is easy to see.

The diameter of the intersection of a decreasing sequence of subsets needn't be the corresponding limit of diameters even if the sets are closed and bounded. For instance, consider an infinite dimensional normed linear space containing orthonormal vectors $e_1, e_2, \ldots$. Take $F_i := \{e_i, e_{i+1}, \ldots\}$. Then each $\delta(F_i) = \sqrt{2}$, and still the intersection is empty. However, there is one case where we *can* say something:

Proposition 3.2. *Let $F_1 \supseteq F_2 \supseteq \cdots$ be closed subsets of a metric space with F_1 being compact. Then $\delta(\bigcap_i F_i) = \lim_i \delta(F_i)$.*

Proof. "$\leq$" is clear. For "$\geq$", let $\varepsilon > 0$ and choose $x_i, y_i \in F_i$ such that $d(x_i, y_i) > \delta(F_i) - \varepsilon$ (note that each $\delta(F_i) < +\infty$). Now, since F_1 is compact, let $x_{n_i} \to x$ and $y_{n_i} \to y$ in F_1. Since F_i's are closed, x, y lie in the intersection so that $\delta(\bigcap_i F_i) \geq d(x, y) \geq \lim_i \delta(F_i) - \varepsilon$. □

Theorem 3.3 (Baire's category)**.** *In a complete metric space, complements of meager sets are dense.*

Proof. Let $A_1, A_2, \ldots$ be nowhere dense. We show that $X \setminus \bigcup_i A_i$ is dense. Pick a nonempty open U. Since A_1 is nowhere dense, choose $x_1 \in U$ and $r_1 > 0$ such that $B_{r_1}(x_1) \subseteq U$ and $B_{r_1}(x_1) \cap A_1 = \emptyset$. Having chosen x_i, r_i, choose $x_{i+1} \in B_{r_i}(x_i)$ such that *DC used.*

- $B_{r_{i+1}}(x_{i+1}) \subseteq B_{r_i}(x_i)$,

- $r_{i+1} \leq r_i/2$, and
- $B_{r_{i+1}}(x_{i+1}) \cap A_{i+1} = \emptyset$.

This is possible since A_i is nowhere dense. Thus, $\overline{B_{r_1}(x_1)} \supseteq \overline{B_{r_2}(x_2)} \supseteq \cdots$ with $\delta(\overline{B_{r_i}(x_i)}) \leq \delta(D_{2r_i}(x_i)) \overset{\text{w}}{=} 2r_i \overset{\text{w}}{\to} 0$ since $r_i \leq r_1/2^{i-1}$. By Cantor (since X is complete), let $x \in \bigcap_i \overline{B_{r_i}(x_i)}$. Then $x \notin \bigcup_i A_i$ since each $B_{r_i}(x_i) \cap A_i = \emptyset$. Finally, $x \in \overline{B_{r_1}(x_1)}$ and without loss of generality, we cloud've chosen x_1, r_1 such that $\overline{B_{r_1}(x_1)} \subseteq U$. □

Remark. Necessity of completeness is demonstrated by considering any countable metric space in which singletons are not open, for instance $\mathbb{Q}$.

Corollary 3.4. *If countably many closed subsets of a nonempty complete metric space unite to the whole space, then one of them has a nonemtpy interior.*