Contents

CHAPTER I

Topology

Conventions. Unless stated otherwise, assume these:

- X, Y (and their sub-/super- scripts) will be topological spaces.
- Subsets of topological spaces will be considered under subspace topology.
- Product of topological spaces will be considered under the product topology.
- In a context, the "parent" totally ordered sets will be considered under as LOTS's and their subsets under subspace topology (which is in general not the same as the topology from inherited order). (See 5.)
- Monotonic functions will be assumed to be between LOTS.

1. Subspaces and Bases

Lemma 1.1. *$\mathscr{B}$ is a base $\iff$ the arbitrary unions in $\mathscr{B}$ form a topology.* "$\Rightarrow$" requires AC.

Lemma 1.2.

(i) "Being a subspace of" is transitive.

(ii) (Sub)base of a subspace can be obtained from that of the parent space.

2. Subsets of Topological Spaces

Lemma 2.1. *For any $A \subseteq X$, we have*

$$X \setminus \overline{A} = (X \setminus A)^\circ, \text{ and}$$

$$X \setminus A^\circ = \overline{X \setminus A}.$$

Proof. The first equality:

"$\subseteq$": Let $x \notin \overline{A}$. Thus we have an open neighborhood U of x that is disjoint from A, *i.e.*, $U \subseteq X \setminus A \overset{\text{w}}{\implies} U \subseteq \text{RHS}$.

"$\supseteq$": Let $x \in \text{RHS}$ so that take an open neighborhood U of x contained in $X \setminus A$, *i.e.*, U doesn't intersect A so that $x \notin \overline{A}$. $\square$

The second follows if $A^\circ = X \setminus (\overline{X \setminus A})$. But by the first equality, $X \setminus (\overline{X \setminus A}) = (X \setminus (X \setminus A))^\circ \overset{\text{w}}{=} A^\circ$ as required.

Lemma 2.2. *Let $A \subseteq X_1 \subseteq X$. Then*

$$\operatorname{cl}_{X_1}(A) = \operatorname{cl}_X(A) \cap X_1, \text{ and}$$
$$\operatorname{int}_{X_1}(A) \supseteq \operatorname{int}_X(A) \cap X_1.$$

Proof. $\operatorname{cl}_X(A) \cap X_1$ is a closed set in X_1 containing A. Further, if F is any closed set in X such that $F \cap X_1 \supseteq A$, then we need to show that $F \cap X_1 \supseteq \text{RHS}$. Indeed, since $F \supseteq A$, we have that $F \supseteq \operatorname{cl}_X(A) \overset{\text{w}}{\implies} F \cap X_1 \supseteq \operatorname{cl}_X(A) \cap X_1$.

Finally, since $\operatorname{int}_X(A) \cap X_1$ is open in X_1 and is contained in A, the second inclusion follows. $\square$

Remark. To show strict inclusion for int, consider $A = X_1 = \mathbb{Q}$ inside $X = \mathbb{R}$.

Lemma 2.3. *If $f\colon X \to Y$ is a homeomorphism, then for any $A \subseteq X$, we have*

$$f(\overline{A}) = \overline{f(A)}, \text{ and}$$
$$f(A^\circ) = f(A)^\circ.$$

Proof. The first equality:

"$\subseteq$" is just restatement of f's continuity. Since f^{-1} is also continuous, we have $f^{-1}(\overline{f(A)}) \subseteq \overline{f^{-1}(f(A))} = \overline{A} \overset{\text{w}}{\implies} \overline{f(A)} \subseteq f(\overline{A})$, where we have also used bijectivity of f.

The second equality is equivalent to showing that $Y \setminus f(A^\circ) = Y \setminus f(A)^\circ$. Indeed, $Y \setminus f(A^\circ) = f(X \setminus A^\circ) = f(\overline{X \setminus A}) = \overline{f(X \setminus A)} = \overline{Y \setminus f(A)} = Y \setminus f(A)^\circ$. $\square$

3. Functional Limits and Continuity

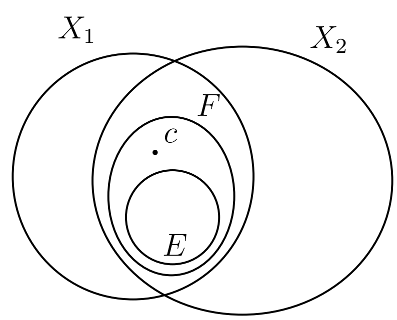
Let $E \subseteq X$ and $f\colon E \to Y$. Then for any $c \in X$ and $L \in Y$, we write $\boldsymbol{f(x) \to L}$ **as** $\boldsymbol{x \to c}$ **in** $\boldsymbol{X}$ iff for every open neighborhood V of L in Y, there exists an open neighborhood U of c in X such that $f(E \cap U \setminus \{c\}) \subseteq V$. Note that "in X" is crucial and can't be dropped: Consider $f := \mathrm{id}$ on $(0,1)$. Take X_1 to be the disjoint union topology of $(0,1)$ and $\{1\}$ and $X_2 := (0,1]$ under the subspace topology. Then 1 is isolated in X_1 and thus $f(x) \to L$ for each $L \in (0,1)$ as $x \to 1$ in X_1 (see (i) of 3.2). On the other hand, $f(x) \to 1$ only as $x \to 1$ in X_2 since 1 is a limit point of $(0,1)$ in X_2 (see (ii) of 3.2).

Intuitively, what this shows is that if c is not already inside E, then the limits at c depend on how c and E are embedded together in the ambient space:

Lemma 3.1 (Tweaking ambient domains)**.** *Assume the following:*

- $f\colon E \to Y$ *where* $E \subseteq F \overset{w}{\subseteq} X_1 \cap X_2$.
- F *has the same subspace topology with respect to either* X_i.
- $c \in F$ *and* $L \in Y$.

Then $f(x) \to L$ *as* $x \to c$ *in* $X_1 \iff f(x) \to L$ *as* $x \to c$ *in* X_2.



Proof. Just observe that if U_1 is an open neighborhood of c in X_1, then $F \cap U_1$, being an open neighborhood of c in F, is also equal to $F \cap U_2$ for some open neighborhood U_2 of c in X_2. Also, since $E \subseteq F$, we also have $E \cap U_1 = E \cap U_2$. $\square$

Remark. Thus, if $c \in E$ and the topology on E is specified, then there's no need to talk of the ambient space at all, for there's a natural choice of the ambient space—E itself!

Lemma 3.2.

(i) All the codomain values are the limits of a function at an isolated point of the domain or of the ambient space.

(ii) For Hausdorff codomains, there is at most one limit at a limit point of the domain.

Proof. Let $f\colon E \to Y$ where $E \subseteq X$. Let $c \in X$.

(i) If c is isolated in E or in X, then there exists an open neighborhood U of c in X such that $E \cap U \subseteq \{c\}$ so that for any open set V in Y, we have $f(E \cap U \setminus \{c\}) = \emptyset \subseteq V$.

(ii) Let c be a limit point of E and suppose $f(x) \to L_1, L_2$ as $x \to c$ for distinct L_1, L_2. Let V_i's be opens separating L_i's (since $\underline{Y \text{ Hausdorff}}$). Now, take open neighborhoods U_i's of c in X such that $f(E \cap U_i \setminus \{c\}) \subseteq V_i$. Now, $E \cap U_1 \cap U_2 \setminus \{c\} \neq \emptyset$ (since $\underline{c \text{ is a limit point of } E}$) which violates disjointness of V_i's. □

Remark. Whenever the limit is unique *and* the ambient space is clear (or instead, $c \in \operatorname{dom} f$ with the topology on $\operatorname{dom} f$ specified), we use the "$\boldsymbol{\lim_{x\to c} f(x)}$" notation.

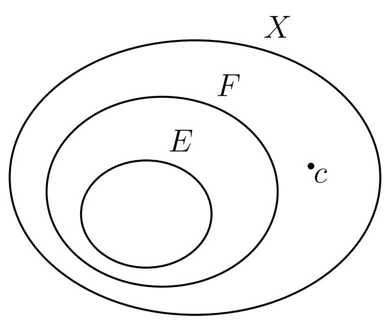
Lemma 3.3 (Tweaking functional domains). *Assume the following:*

- *$f\colon E \to Y$ and $g\colon F \to Y$ where $E \subseteq F \subseteq X$.*
- *$c \in X$ and $L \in Y$.*
- *f and g agree on E.*

Then the following hold:

(i) $g(x) \to L$ as $x \to c$ in X $\implies$ $f(x) \to L$ as $x \to c$ in X.

(ii) The converse holds if $E \supseteq U_0 \cap F \setminus \{c\}$ for some open neighborhood U_0 of c.



Proof. Just note that for an open neighborhood U of c, we have

(i) $f(U \cap E \setminus \{c\}) = g(U \cap E \setminus \{c\}) \subseteq g(U \cap F \setminus \{c\})$; and

(ii) $U \cap U_0$ is an open neighborhood of c with $g(U \cap U_0 \cap F \setminus \{c\}) \subseteq f(U \cap E \setminus \{c\})$. □

Remark. (i) The antecedent in the second claim formalizes that E should be able to approximate c at least as good as F; if $c \in F$, then it just says that E contains a deleted open neighborhood of c in F. (ii) To see the necessity of the antecendent, consider $x \mapsto 1/x$ with $E = \mathbb{R}^+$, $F = \mathbb{R}^*$, $X = \mathbb{R}$, and $Y = \overline{\mathbb{R}}$.

Corollary 3.4.

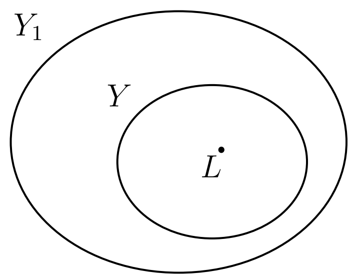
(i) Restricting the domain preserves continuity.

(ii) Extending a continuous function by increasing its domain doesn't affect its continuity on the interior of the original smaller domain.

Lemma 3.5 (Tweaking codomains)**.** *Assume the following:*

- $f\colon E \to Y$ *and* $g\colon E \to Y_1$ *where* $E \subseteq X$ *and* $Y_1 \supseteq Y$.
- $c \in X$ *and* $L \in Y$.
- f *and* g *agree on* E.

Then $f(x) \to L$ *as* $x \to c$ *in* $X \iff g(x) \to L$ *as* $x \to c$ *in* X.

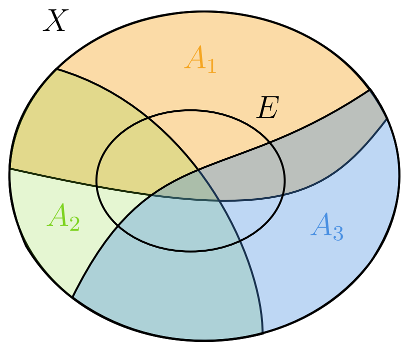


Corollary 3.6. *Tweaking the codomain preserves continuity.*

Proposition 3.7 (Pointwise pasting)**.** *Assume the following:*

- $f\colon E \to Y$ *where* $E \subseteq X$.
- X *is the union of finitely many* $A_i \subseteq X$.
- $f|_i : E \cap A_i \to Y$.
- $c \in X$ *and* $L \in Y$.
- *If* $c \in A_i$, *then* $f|_i(x) \to L$ *as* $x \to c$ *in* A_i.
- *Either* $c \in \bigcap_i A_i$, *or each* A_i *is closed.*

Then $f(x) \to L$ *as* $x \to c$ *in* X.



Proof. Let V be an open neighborhood of L. Say $c \in A_i$'s and $c \notin A_j$'s. Thus, take open neighborhoods $U_i \cap A_i$ of c in A_i (U_i open in X_i) such that $f|_i(U_i \cap E \cap A_i \setminus \{c\}) \subseteq V$. Set $U := \left(\bigcap_i U_i\right) \cap \left(X \setminus \bigcup_j A_j\right)$[1] which is open since i's and j's are finitely many and since onef the following holds:

(i) c is in each A_k so that there are no j's.

(ii) Each A_j is closed.

Thus, U is an open neighborhood of c in X with $f(U \cap E \setminus \{c\}) = \left(\bigcup_i f(U \cap E \cap A_i \setminus \{c\})\right) \cup \left(\bigcup_j f(U \cap E \cap A_j \setminus \{c\})\right) \subseteq V \cup \emptyset = V$. □

Remark. (i) Necessity of finitely many A_i's: Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) := x^2y/(x^4 + y^2)$ for $(x, y) \neq (0, 0)$ and $f(0, 0) := 0$. Along all the straight lines through origin (which are closed), $f(x) \to 0$ and yet f has no limit at $(0, 0)$.[2] Note that here, $c \in \bigcap_i A_i$ *and* each A_i is closed.
A much simpler example is to consider an infinite X having at least one limit point and in which singletons are closed,[3] and taking the codomain to be a Hausdorff space with at least two points.

(ii) Necessity of "$c \in \bigcap_i A_i$, or each A_i closed": Consider $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) := 0$ for $x < 0$ and $f(x) := 1$ for $x \geq 0$.

Corollary 3.8 (Pasting lemma)**.** *If the domain is a finite union of closed sets on each of which the function's restriction is continuous, then the function is continuous.*

Lemma 3.9. *Functional limits* $\implies$ *sequential limits.*

Remark. For taking sequential limits of a function at c, remember to have the sequence (eventually) not contain c!

[1]The second set in the union is motivated from the need to have $\bigcap_j X \setminus A_j$.
[2]Consider evaluating along $y = mx^2$.
[3]Such spaces are precisely the T_1 spaces. (See 8.1.)

Proof. Let $f\colon E \to X$ where $E \subseteq X$ be such that $f(x) \to L$ as $x \to c$ for $c \in X$ and $L \in Y$. Let $x_i \to c$ for $(x_i) \in E \setminus \{c\}$. We show that $f(x_i) \to L$:

Let V be an open neighborhood of L. Take an open neighborhood U of c in X such that $f(U \cap E \setminus \{c\}) \subseteq V$. Now, (x_i) eventually lies in U and thus in $U \cap E \setminus \{c\} \overset{\text{w}}{\implies} (f(x_i))$ eventually lies in V. □

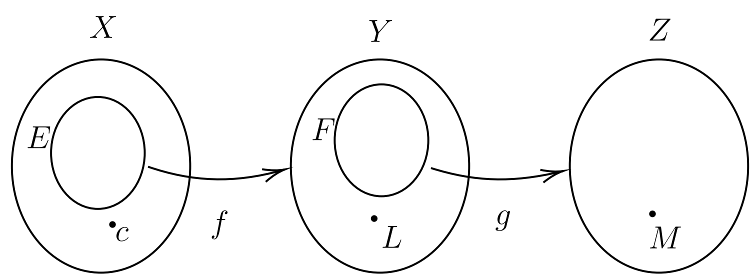
Remark. The converse holds for first-countable domains, but is false in general: See 9.4 and the remark afterward.

Corollary 3.10. *Continuity $\implies$ sequential continuity.*

Lemma 3.11 (Compositions and limits)**.** *Assume the following:*

- $f\colon E \to Y$ *and* $g\colon F \to Z$ *where* $E \subseteq X$ *and* $F \subseteq Y$*.*
- $f(E) \subseteq F$ *and* $f|\colon E \to f(E)$*.*
- $c \in X$*,* $L \in Y$ *and* $M \in Z$*.*
- $f(x) \to L$ *as* $x \to c$ *in* X*.*
- $g(y) \to M$ *as* $y \to L$ *in* Y*.*
- $L \in F \implies g(L) = M$*.*

Then $(g \circ f|)(x) \to M$ *as* $x \to c$ *in* X*.*



Proof. Let W be an open neighborhood of M. Take an open neighborhood V of L such that $g(V \cap F \setminus \{L\}) \subseteq W$ and an open neighborhood U of c such that $f(U \cap E \setminus \{c\}) \subseteq V \cap F$ (since $\underline{f(E) \subseteq F}$). Because of <u>the last assumption</u>, we have $(g \circ f)(U \cap E \setminus \{c\}) \subseteq W$. □

Remark. To see the necessity of the last assumption, consider $f, g\colon [0,1] \to [0,1]$ given by $f := 1/2$ and $g := \delta_{1/2}$.

Corollary 3.12. *Composition of continuous functions is continuous.*

4. Product Topology

From (ii) of 1.2, we immediately conclude:

Lemma 4.1. *Taking products and subspaces are compatible.*

Remark. This holds for box topology as well.

Lemma 4.2. *Closure of a product is the product of closures.*

Proof. Let $A_i \subseteq X_i$. We show $\overline{\prod_i A_i} = \prod_i \overline{A_i}$.

"$\subseteq$": Suffice to show that $\prod_i F_i$ is closed for F_i's closed in X_i's. Let $(x_i) \notin \prod_i F_i$, say $x_{i_0} \notin F_{i_0}$. Then take an open neighborhood U_{i_0} of x_{i_0} disjoint from F_{i_0}. Now, $\pi_{i_0}^{-1}(U_{i_0})$ is an open neighborhood of (x_i) that is disjoint from $\prod_i F_i$.

"$\supseteq$": Let $U := \bigcap_{j \in J} \pi_j^{-1}(U_j)$ be an open neighborhood of $(x_i) \in$ RHS, where J is finite and each U_j is open. Then each U_j is an open neighborhood of x_j and hence intersects A_j. Thus U intersects $\prod_i A_i$. ☐

No choice required.

Remark. The same holds for box topology as well; however AC will be required for "$\supseteq$".

Corollary 4.3. *Product of dense sets is dense in the product.*

Remark. This holds for box topology as well.

Proposition 4.4 (Convergence)**.** *Convergence in product topology is equivalent to convergence of each component sequence in the respective factor space.*

Proof. "$\Rightarrow$": Since projections are continuous.

"$\Leftarrow$": Let X be the product of X_i's and $x, (x^{(n)}) \in X$ be such that $x_i^{(n)} \to x_i$ for each i. We show that $x^{(n)} \to x$. Let $U := \bigcap_j \pi_j^{-1}(U_j)$ be a basic open neighborhood of x where j's are finitely many. Thus, $(x_j^{(n)})_n$ eventually lies in U_j for each j. Since there are finitely many j's, we conclude that $(x^{(n)})$ eventually lies in U. ☐

Remark. This does not hold for box topology: Consider the $\mathbb{N}$-fold product of discrete $\{0, 1\}$ endowed with box topology and consider the sequence whose n-th term is given by $(\underbrace{1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots)$.

5. Order Topology

If X is totally ordered, then the **order topology** on it is the one that's generated by open rays. The resulting space is called a **linearly ordered topological space**, or **LOTS** in short.

Lemma 5.1 (Immediate properties)**.**

(i) The base obtained from the subbase of open rays of a LOTS X comprises of the following sets:

(a) (a, b);
(b) $[\min X, b)$ if X has a minimum element;
(c) $(a, \max X]$ if X has a maximum element; and,
(d) $[\min X, \max X] \overset{w}{=} X$ if X has both, a maximum and a minimum.

(ii) Order topology is Hausdorff.

(iii) A dense subset of a LOTS is also topologically dense.

Proof. (i) It's easily shown that these sets form a base for X and are generated by intersecting (at most two) open rays. Conversely, any open ray is generated by these basic sets:

Let's show for right rays. In case there's a largest element, then $(a, +\infty) = (a, \max X]$. If not, then $(a, +\infty) = \bigcup_y (a, y)$.

(ii) Let $x < y$. If there's a z between them, then $(-\infty, z)$ and $(z, +\infty)$ separate them. Otherwise, $(-\infty, y)$ and $(x, +\infty)$ do.

(iii) Easily verified on basic opens. □

In an order topology, a point's isolated-ness is closely linked to the presence of its immediate successors and predecessors:

Lemma 5.2. *The isolated points in an order topology are precisely the following:*

(i) An element that is least as well as greatest.

(ii) A least element with an immediate successor.

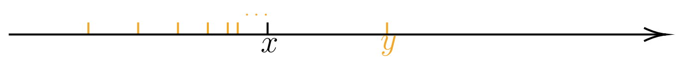
(iii) A greatest element with an immediate predecessor.

(iv) An element with an immediate succesor and an immediate predecessor.

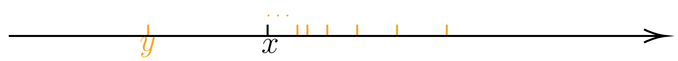
Thus, an order topology is discrete $\iff$ each non-least element has an immediate predecessor and each non-greatest element has an immediate successor.

Lemma 5.3 (Subspaces and order)**.** *Topology induced from inherited order is coarser than the subspace topology. The two are distinct if and only if there exists an $x \in X \setminus Y$ such that one of the following holds:*

(i) There exists a $y \in Y \cap (x, +\infty)$ such that $Y \cap (-\infty, y)$ is nonemtpy, has no maximum element, and is strictly bounded above by x.



(ii) There exists a $y \in Y \cap (-\infty, x)$ such that $Y \cap (y, +\infty)$ is noempty, has no minumum element, and is strictly bounded below by x.



Thus, the topologies match if Y is convex or dense in X.

Proof. It's straightforward to verify that the enumerated conditions are equivalent to saying that $Y \cap (x, +\infty)$, and respectively $Y \cap (-\infty, x)$, are not open in the order topology. Since the open rays indeed generate the order topology, this shows the "if and only iff" part.

Finally, note that if Y is convex or dense in X, then none of the conditions can arise, and we are done. $\square$

Remark. Consider $\{-1\} \cup (0, 1] \subseteq \mathbb{R}$, where the two topologies are distinct.

Using 3.1 and 3.3, one immediately gets:

Lemma 5.4 (One-sided limits)**.** *Assume the following:*

- *X is a LOTS.*
- *$f\colon E \to Y$ where $E \subseteq X$.*
- *$c \in X$ and $L \in Y$.*
- *$g\colon E \cap (-\infty, c] \to Y$ and $h\colon E \cap (-\infty, c) \to Y$ are restrictions of f.*

Then the following are equivalent:

(i) $g(x) \to L$ as $x \to c$ in X.

(ii) $g(x) \to L$ as $x \to c$ in $(-\infty, c]$.

(iii) $h(x) \to L$ as $x \to c$ in X.

(iv) $h(x) \to L$ as $x \to c$ in $(-\infty, c]$.

Similarly, we have the "right-sided" version.

We use "$\boldsymbol{f(x) \to L}$ **as** $\boldsymbol{x \to c^{\pm}}$ **in** $\boldsymbol{X}$" to denote the above equivalent statements. Note that here, instead of "in X", we could've chosen "in $(-\infty, c]$" or respectively, "in $[c, +\infty)$". 3.7 readily yields:

Lemma 5.5. *Assume the following:*

- *X is a LOTS.*
- *$f\colon E \to Y$ where $E \subseteq X$.*
- *$c \in X$ and $L \in Y$.*

Then the following are equivalent:

(i) $f(x) \to L$ *as* $x \to c$ *in* X.

(ii) $f(x) \to L$ *as* $x \to c^+$ *and as* $x \to c^-$ *in* X.

Proposition 5.6. *Assume the following:*

- *X, Y are LOTS's.*
- *$f\colon E \to Y$ where $E \subseteq X$.*
- *f is increasing.*
- *$c \in X$.*

Then, the following hold:

$$f(x) \to_{\mathsf{if}} \sup f(E \cap (-\infty, c)) \ \textit{as } x \to c^- \textit{ in } X$$
$$f(x) \to_{\mathsf{if}} \inf f(E \cap (c, +\infty)) \ \textit{as } x \to c^+ \textit{ in } X$$

We similarly have a dual version for decreasing f.

Proof. Let f be increasing. The proof for decreasing f would be similar. We show only the first statement, second's proof being similar. Let $f(E \cap (-\infty, c))$ have the l.u.b., say α. We need to show that $f|(x) \to \alpha$ as $x \to c$ in X, where $f|\colon E \cap (-\infty, c) \to Y$. Let J be a basic open neighborhood of α. We have two cases:

(i) α is the least element of Y: Then $f(E \cap (-\infty, c) \cap (-\infty, \infty)) = f(E \cap (-\infty, c)) \subseteq \{\alpha\} \subseteq J$.

(ii) α is not the least element: Then without loss of generality, let J's left end be open, say at $y \overset{\mathrm{w}}{<} \alpha$. Thus, take an $x \in E \cap (-\infty, c)$ such that $f(x) > y$. Now, since f is <u>increasing</u>, $f(E \cap (-\infty, c) \cap (x, +\infty)) \overset{\mathrm{w}}{=} f(E \cap (x, c)) \subseteq (y, \alpha] \subseteq J$. □

Corollary 5.7. *Monotonics taking values in a complete codomain admit one-sided limits at all points of the domain.*

Proof. Let $f\colon X \to Y$ be increasing and $c \in X$. We find a left-side limit L at c. We have two cases:

(i) c is not the least element of X: Then $f((-\infty, c))$ is nonempty and bounded above by $f(c)$, so we may take L to be its l.u.b. (which exists since Y is complete).

(ii) c is the least element of X: Then the domain of $f|: (-\infty, c) \to Y$ is empty and thus we may take L to be any point in Y.

The proofs for other cases are similar. □

Remark. To see the necessity of well-definedness of RHS in 5.6 and that of completeness in 5.7, consider $f : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ given by

$$f(x) := \begin{cases} x, & x < 0 \\ x + 1, & x \geq 0 \end{cases}.$$

Proposition 5.8 (Continuity and monotonicity)**.**

(i) Strictly monotonic surjections are homeomorphisms.

(ii) A monotonic surjection is continuous provided the codomain's order is dense.

Proof. (i) Since inverses of strictly monotonic bijections are strict monotones as well, it suffices to show openness, which is easy to verify.

(ii) Let X, Y be LOTS's and $f : X \to Y$ be an increasing surjection. The proof for decreasing f would be similar. Let J be a basic open set of Y. We show that $f^{-1}(J)$ is open. Start with an $x \in f^{-1}(J)$, *i.e.*, $f(x) \in J$. The following cases arise:

(a) $f(x)$ is the least in Y: Then $J = [f(x), y)$ for some $y > f(x)$. Due to denseness and surjectivity, take $b \in X$ such that $f(x) < f(b) < y$. Since f is increasing, we have $x < b$ and $f([x, b)) \subseteq [f(x), f(b)] \subseteq [f(x), y) = J$. Since $f((-\infty, x]) = \{f(x)\}$ (as $f(x)$ is the least element), we have that $f((-\infty, b)) \subseteq J$ so that $(-\infty, b)$ works.

(b) $f(x)$ is the greatest in Y: Similarly as above.

(c) $f(x)$ is neither: Then without loss of generality, take $J = (y_1, y_2)$ for $y_1 < f(x) < y_2$. Due to denseness and surjectivity, take $a, b \in X$ such that $y_1 < f(a) < f(x) < f(b) < y_2$. Again, since f is increasing, $a < x < b$ and $f((a, b)) \subseteq [f(a), f(b)] \subseteq (y_1, y_2) = J$ so that (a, b) works. □

Remark. (i) Necessity of surjectivity: $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := x$ for $x < 0$ and $f(x) := x + 1$ for $x \geq 0$.

(ii) Necessity of denseness of codomain: Consider the sign function $\mathbb{R} \to \{-1, 0, 1\}$.

Proposition 5.9 (Monotone convergence)**.** *Let X be a LOTS and $(x_i) \in X$ be increasing (respectively decreasing). Let $L \in X$. Then $x_i \to L \iff L = \sup_i x_i$ (respectively $L = \inf_i x_i$).*

Proof. We show for increasing (x_i).

"$\Rightarrow$": We show that L is the supremum of x_i's:

- It's an upper bound: If not, then we can take an $x_N > L$ so that (x_i) eventually lies out of $(-\infty, x_N)$, an open neighborhood of L.
- It's the l.u.b.: If $M < L$, then (x_i) eventually lies in the open neighborhood $(M, +\infty)$ of L, so that M can't be an upper bound for (x_i).

"$\Leftarrow$": Let I be a basic open neighborhood of L. We have two cases:

(i) L is the least element: Then each $x_i = L$.

(ii) L is not the least element: Then without loss of generality, let I's left end be open, say at $a \overset{\text{w}}{<} L$. Thus, take $x_N > a$, so that for each $i \geq N$, we have $x_i \in (a, L] \overset{\text{w}}{\subseteq} I$. □

Lemma 5.10 ($\liminf$ and $\limsup$)**.** *Let X be a LOTS and $(x_i) \in X$. Then there exists at most one $L^+ \in X$ such that for any $a \in X$, the following hold:*

(i) $a > L^+ \implies x_i$'s are evetually all less than a.

(ii) $a < L^+ \implies$ infinitely many x_i's are greater than a.

A dual statement holds for "L^-".

Proof. We only show for "L^+". Suppose L_1^+ and L_2^+ are such with $L_1^+ < L_2^+$. We have two cases:

(i) $a \in (L_1^+, L_2^+)$: Then x_i's are eventually all less than a (since $a > L_1^+$), contradicting that infinitely many x_i's are greater than a (since $a < L_2^+$).

(ii) $(L_1^+, L_2^+) = \emptyset$: Then x_i's are eventually all less than L_2^+ (since $L_2^+ > L_1^+$), and hence, less than or equal to L_1^+, contradicting that infinitely many x_i's are greater than L_1^+ (since $L_1^+ < L_2^+$). □

Remark. This allows to use the "$\boldsymbol{\limsup_i x_i}$" and "$\boldsymbol{\liminf_i x_i}$" notations.

Lemma 5.11 (Chracterizing $\limsup$ and $\liminf$)**.** *Let X be a complete LOTS and $(x_i) \in X$ be bounded. Then*

$$\limsup_i x_i = \inf_i \sup_{j \geq i} x_j, \text{ and}$$
$$\liminf_i x_i = \sup_i \inf_{j \geq i} x_j.$$

Proof. Define $x_i^+ := \sup_{j\geq i} x_j$ and $x_j^- := \inf_{j\geq i} x_j$, which are well-defined since X is complete and (x_i) bounded. We only show that $\limsup_i x_i = \inf_i x_i^+$:

- Let $a > \inf_i x_i^+$. Then some $x_N^+ < a$ so that $x_i \leq x_N^+ < a$ for each $i \geq N$.
- Let $a < \inf_i x_i^+$ and N be any index. Then $a < x_N^+$ so that there exists an $i \geq N$ such that $x_i > a$. □

Lemma 5.12. *Let X be a LOTS and $(x_i), (y_i) \in X$. Then the following hold:*

(i) $\liminf_i x_i \leq_{\mathsf{if}} \limsup_i x_i$.

(ii) Each $x_i \leq y_i \implies \liminf_i x_i \leq_{\mathsf{if}} \liminf_i y_i$ *and* $\limsup_i x_i \leq_{\mathsf{if}} \limsup_i y_i$.

Proof. We'll use the notations $L_x^\pm$ and $L_y^\pm$.

(i) Suppose $L_x^+ < L_x^-$. Two cases are there:
  (a) $a \in (L_x^+, L_x^-)$: Then x_i's are eventually all less than a (since $a > L_x^+$) and greater than a (since $a < L_x^-$), a contradiction.
  (b) $(L_x^+, L_x^-) = \emptyset$: Then x_i's are evetually all less than L_x^- (since $L_x^- > L_x^+$) and hence, less than or equal to L_x^+, contradicting that x_i's are eventually all greater than L_x^+ (since $L_x^+ < L_x^-$).

(ii) We'll only show $L_x^- \leq L_y^-$. Suppose not. Again two cases:
  (a) $a \in (L_y^-, L_x^-)$: Then x_i's are eventually all greater than a (since $a < L_x^-$) and there are infinitely many y_i's less than a (since $a > L_y^-$). But this then means that $y_i < a < x_i$ for some i.
  (b) $(L_y^-, L_x^-) = \emptyset$: Then x_i's are eventually all greater than L_y^- (since $L_y^- < L_x^-$) and hence, greater than or equal to L_x^-. Also, there are infinitely many y_i's less than L_x^- (since $L_x^- > L_y^-$). But this means that $y_i < L_x^- \leq x_i$ for some i. □

Proposition 5.13. *Let X be a LOTS and $(x_i), L \in X$. Then* $\lim_i x_i = L \iff \limsup_i x_i = L = \liminf_i x_i$.

Proof. "⇒": If $a > L$, then $(x_i) \in (-\infty, a)$ evetually. If $a < L$, then $(x_i) \in (a, +\infty)$ eventually.

"⇐": Let I be an open neighborhood of L. The following cases arise:

(i) L is not the greatest of the least element: Then, without loss of generality, take $I = (a, b)$. Now, x_i's are eventually all less than b (since $b > L \overset{\text{w}}{=} \limsup_i x_i$) and also greater than a (since $a < L \overset{\text{w}}{=} \liminf_i x_i$). Thus, eventually, $(x_i) \in (a, b) = I$.

(ii) The rest of the cases, when L is greatest or least, are done similarly. □

Note that $\limsup, \liminf, \lim\colon X^{\mathbb{N}} \rightharpoonup X$ are not continuous unless $|X| = 1$, for in every nonempty basic open, there exists a sequence that converges to any given element of X. However, we have:

Proposition 5.14. *For a LOTS X and $n \geq 1$, the functions* $\max, \min\colon X^n \to X$ *are continuous.*

Proof. We show for max. Let $x \in X^n$ and I be an open neighborhood of $\max(x)$. We have the following cases:

(i) $\max(x)$ is the least in X: Then each $x_i = \max(x)$ so that $\max(I \times \cdots \times I) \subseteq I$.

(ii) $\max(x)$ is not the least: Then without loss of generality, let I's left end be open, say at a. Set $b := \max\big(\{a, x_1, \ldots, x_n\} \setminus \{\max(x)\}\big)$. We have two cases:

(a) $c \in (b, \max(x))$: Then define

$$J_i := \begin{cases} (c, +\infty) \cap I, & x_i = \max(x) \\ (c, -\infty), & x_i < \max(x) \end{cases}$$

(b) $(b, \max(x)) = \emptyset$: Then define

$$\begin{aligned} J_i &:= \begin{cases} (b, +\infty) \cap I, & x_i = \max(x) \\ (-\infty, \max(x)), & x_i < \max(x) \end{cases} \\ &\overset{\text{w}}{=} \begin{cases} [\max(x), +\infty) \cap I, & x_i = \max(x) \\ (-\infty, b], & x_i < \max(x) \end{cases} \end{aligned}$$

In either case, we have $\max(\prod_i J_i) \subseteq I$. □

6. Denseness

Lemma 6.1. *"Being dense" is transitive.*

Proof. Let $A \subseteq B \subseteq X$ with A dense in B and B dense in X. Let U be a nonempty open in X. Then $\underline{B \text{ being dense}}$, intersects U so that $U \cap B$ is a nonempty open in B and thus is intersected by the $\underline{\text{dense } A}$ $\overset{\text{w}}{\Longrightarrow}$ A intersects U. □

Lemma 6.2. *Continuous image of dense is dense in the image.*

Proof. Let $f\colon X \to Y$ be surjective continuous and $A \subseteq X$ be dense in X. Then $\overline{f(A)} \supseteq f(\overline{A}) = f(X) = Y$. □

Lemma 6.3. *Let $A, B \subseteq X$. Then the following hold:*

(i) $B \cap A$ *is dense in* $B \implies B \subseteq \overline{A}$.

(ii) The converse holds if B *is open.*

Proof. (i) We have $B = \mathrm{cl}_B(B \cap A) \subseteq B \cap \overline{B \cap A} \subseteq \overline{B \cap A} \subseteq \overline{A}$.

(ii) We need to show that $\mathrm{cl}_B(B \cap A) = B$. Indeed, if F is any closed such that $B \cap A \subseteq B \cap F$, then $B \subseteq F$ (otherwise, take $x \in B \setminus F \overset{\mathrm{w}}{\implies} x \in B \setminus A \overset{\mathrm{w}}{\implies} x \in B \setminus \overline{A}$ for $\underline{B \text{ is open}}$, contradicting $B \subseteq \overline{A}$). □

Remark. To see the necessity of openness of B in (ii), consider $A = \{1, 1/2, \ldots\}$ and $B = \{0\}$.

6.1 Nowhere dense sets

6.3 gives insight as to why nowhere dense sets are called so—they are dense on no nonempty *open* set. On the other hand, dense sets are dense on the whole space.

Lemma 6.4. *Let* U *be open in* X *and* $A \subseteq X$. *Then the following are equivalent:*

(i) $U \subseteq \overline{A}$.

(ii) Every nonempty open subset contained in U *intersects* $\overline{A}$.

(iii) Every nonempty open subset contained in U *intersects* A.

Corollary 6.5. *The following are equivalent for a subset* A *of* X:

(i) $X \setminus \overline{A}$ *is dense.*

(ii) A *is nowhere dense.*

(iii) Each nonempty open set contains a nonemtpy open subset disjoint from $\overline{A}$.

(iv) Each nonempty open set contains a nonemtpy open subset disjoint from A.

Subsets of a topological space that are countable unions of nowhere dense sets are called **first category** or **meagre** sets. Others are called **second category** sets.

Remark. In $\mathbb{R}$:[4]

	meagre	nonmeagre
dense	$\mathbb{Q}$	$\mathbb{R}$
nondense	$\emptyset$	$[0, 1]$

Lemma 6.6. *If* $F_1, F_2, \ldots$ *are closed in* X *with* $X \setminus \bigcup_i F_i$ *dense, then each* F_i *is nowhere dense.*

[4]Nonmeagre-ness can be concluded by Baire's category theorem (4.3).

Remark. Baire's category theorem (4.3) gives a converse to above, stating that complements of meagre sets are dense in a complete metric space.

Proposition 6.7. *In a topological space, the following are equivalent:*

(i) Complements of meagre sets are dense.

(ii) Countable intersections of open dense sets are dense.

Proof. " $\Rightarrow$ ": Let $U_1, U_2, \cdots$ be open dense. Now, $\bigcap_i U_i \overset{\text{w}}{=} X \setminus \bigcup_i (X \setminus U_i)$ is dense if each $X \setminus U_i$ is nowhere dense $\overset{\text{w}}{\Longleftarrow}$ $X \setminus (\overline{X \setminus U_i}) \overset{\text{w}}{=} U_i$ (since U_i open) is dense, which is true.

" $\Leftarrow$ ": Let $A_1, A_2, \ldots$ be nowhere dense. Then each $X \setminus \overline{A}_i$ is dense $\overset{\text{w}}{\Longrightarrow}$ $\bigcap_i (X \setminus \overline{A}_i) \overset{\text{w}}{=} X \setminus \bigcup_i \overline{A}_i$ is dense $\overset{\text{w}}{\Longrightarrow}$ $X \setminus \bigcup_i A_i$ is dense as well, being a larger set. □

7. Connectedness

A subset E of X is said to be **disconnected** iff there exist disjoint open sets U, V together containing E, and each a having nonempty intersection with E. If E is not disconnected, we call it **connected**. X is called **totally disconnected** iff the only connected subsets of it are singletons.

Lemma 7.1. *Discrete spaces are totally disconnected.*

Lemma 7.2 (Characterizing disconnectedness)**.** *$E \subseteq X$ is disconnected $\iff$ E can be written as a union of two nonempty subsets A, B of X such that $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.*

Proof. "$\Rightarrow$": Take U, V open in X such that $E \cap U$, $E \cap V$ are nonempty, $E \subseteq U \cup V$, and $E \cap U \cap V = \emptyset$. Now put $A := E \cap U$ and $B := E \cap V$. Then $\overline{A} \cap B \subseteq \overline{E \cap U} \cap V = \emptyset$.

"$\Leftarrow$": Take $U := X \setminus \overline{A}$ and $V := X \setminus \overline{B}$. Then $B \subseteq U$ and $A \subseteq V$ so that both are nonempty and $E \subseteq U \cup V$. Also, $E \cap U \cap V = E \setminus (\overline{A} \cup \overline{B}) = \emptyset$. □

Proposition 7.3 (Linear continua are connected)**.** *The connected subsets of a dense and complete LOTS are precisely its convex subsets.*[5]

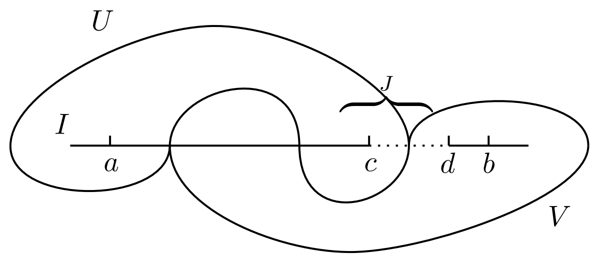
Can we improve to partial orders?

[5] Recall that a convex subset of an ordered set is any set I such that $[x, y] \subseteq I$ whenever $x, y \in I$ with $x \leq y$.

Proof. Let X's topology come from a dense and complete total order. Suppose $I \subseteq X$ is convex, and yet separated by opens U, V. Take $a \in U \cap I$ and $b \in V \cap I$. Without loss of generality, assume $a < b$ (the order is total) so that $[a, b] \subseteq I$ (since I is convex). Note that U, V also form a separation of $[a, b]$. Since $U \cap [a, b]$ is nonempty and bounded, let c be its l.u.b. Clearly, $c \in [a, b]$ so that there are two cases:

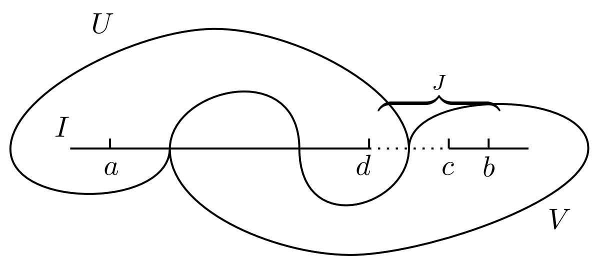
$c \in U$: Take a basic open interval $J \subseteq U$ containing c. Note that $c < b$ (otherwise, $c = b \overset{\mathrm{w}}{\in} V$) so that we may assume that J is open at its right end, say d. Note that $d \leq b$ (otherwise, $b \in (c, d) \overset{\mathrm{w}}{\subseteq} J \overset{\mathrm{w}}{\subseteq} U$). Now, $c < d$ and yet, there can't be any element between c and d:

Such an element would be in $U \cap [a, b]$ (being in J and in (c, b)) despite being strictly greater than c.



$c \in V$: Take a basic open interval $J \subseteq V$ containing c. This time, $c > a$ (otherwise, $c = a \overset{\mathrm{w}}{\in} U$) so that we may assume that J is open at its left end, say d. But then, d is an upper bound for $U \cap [a, b]$ despite being strictly smaller than c:

If $x > d$, then either $x \in (d, c] \overset{\mathrm{w}}{\subseteq} J \overset{\mathrm{w}}{\subseteq} V$; or $x > c \overset{\mathrm{w}}{\implies} x \notin U \cap [a, b]$.



Conversely, if I is not convex, then take $x < y < z$ such that $x, z \in I$ but $y \notin I$. Then the rays at y separate I. $\square$

Remark. To see the necessity of the assumptions, consider $\mathbb{Q}$ and $\mathbb{Z}$ respectively which are both totally disconnected: $\mathbb{Z}$ because it's discrete, and $\mathbb{Q}$ because irrationals are dense in $\mathbb{R}$.

Proposition 7.4 (Intermediate value)**.** *Any continuous function from a connected space to a LOTS obeys intermediate value property.*

Proof. Let X be connected and Y ordered, and $f\colon X \to Y$ be continuous. Suppose f doesn't obey intermediate value property. Then take $x_1, x_2 \in X$ and $y \in Y$ such that y lies between $f(x_1)$ and $f(x_2)$ and yet $y \notin f(X)$. Then $f^{-1}((-\infty, y))$ and $f^{-1}((y, +\infty))$ violate X's connectedness. □

8. Separation Axioms

Lemma 8.1 (T_1 spaces)**.** *In a space, singletons are closed $\iff$ any two distinct points can be separated by open sets that don't contain the other.*

Proof. Let the space in question be X.

"$\Rightarrow$": Let x, y be distinct. Then $X \setminus \{y\}$ and $X \setminus \{y\}$ separate x, y as required.

"$\Leftarrow$": Let $x \in X$. We show that $X \setminus \{x\}$ is open, which follows easily. □

Remark. A T_1 space that is not Hausdorff: Cofinite topology on an infinite set.

Proposition 8.2. *A continuous function taking values in a Hausdorff codomain is completely determined by its values on a dense subset of the domain.*

Proof. Let $f, g\colon X \to Y$ be continuous with Y Hausdorff, agreeing on a dense subset $D \subseteq X$ and yet not on $x \in X$. Since $\underline{Y \text{ Hausdorff}}$, separate $f(x)$ and $g(x)$ via opens V and W. Then $f^{-1}(V) \cap g^{-1}(W)$ is a neighborhood of x, and thus intersects the $\underline{\text{dense } D}$, say at y. But then $V \ni f(y) = g(y) \in W$, a contradiction. □

Remark. To see the necessity of the Hausdorff codomain (and that just T_1 is not enough), consider the function on $\mathbb{R}$ which swaps two distinct points. Then this is continuous[6] with the codomain under cofinite topology.

9. Countability and Separability

Lemma 9.1. *A second countable space is separable and first countable.*

[6]More generally, for any set X, any bijection $X \to X_{\text{cofin}}$ is continuous if singletons are closed in the domain.

Proof. Choosing a point from each of the sets from a countable base yields a countable dense set. □ CC *used.*

Remark. The converse is not true (however, see 1.2): Consider the Sorgenfrey line, *i.e.*, the lower limit topology on $\mathbb{R}$ generated by the basic open sets of the form $[a, b)$. Any base of this topology must contain for each $x \in \mathbb{R}$, some set with x being its l.u.b., and thus be uncountable. AC *used.*

second countable	first countable	separable	
✓			separable metric spaces
×	✓	✓	Sorgenfrey line
×	✓	×	nonseparable metric spaces[7]
	×	✓	cofinite on uncountable
	×	×	cocountable on uncountable

Proposition 9.2. *Any base of a second countable space contains a countable base.*

Proof. Let $\mathscr{B}$, $\mathscr{B}'$ be bases of X with $\mathscr{B}$ being countable. It suffices to show that each $U \in \mathscr{B}$ is a countable union in $\mathscr{B}'$. Thus, consider a $U \in \mathscr{B}$. Define $\mathscr{V} := \{V \in \mathscr{B} : V \subseteq W' \subseteq U \text{ for some } W' \in \mathscr{B}'\}$. Now, for each $V \in \mathscr{V}$, one can choose a $W'_V \in \mathscr{B}$ such that $V \subseteq W'_V \subseteq U$. Now, just note that U is the union of W'_V's which are countably many. □ CC *used.* *Add a diagram.* CC *used.*

Proposition 9.3. *Separability, and first and second countabilities are preserved under countable products.*

Proof. (i) Separability: Let A_i be countable and dense in X_i for $i = 1, 2, \ldots$. Without loss of generality, let each X_i be nonempty so that we may choose for each i, an $x_i \in X_i$. Then the union of the following sets forms a countable dense set in $\prod_i X_i$: CC *used.* CC *used.*

(a) $A_1 \times \{x_2\} \times \{x_3\} \times \cdots$
(b) $A_1 \times A_2 \times \{x_3\} \times \{x_4\} \times \cdots$
(c) $A_1 \times A_2 \times A_3 \times \{x_4\} \times \{x_5\} \times \cdots$
$\vdots$

(ii) First countability: Let $X_1, X_2, \ldots$ be first countable and let $x \in \prod_i X_i$. For each i, choose a countable local base $(B_j^{(i)})_j$ at x_i. Without loss of generality, let $(B_j^{(i)})_j$ be decreasing for each i. Then the following sets form a local base at x: CC *used.*

[7] For instance, discrete metric on any uncountable set.

(a) $\pi_1^{-1}(B_1^{(1)})$
(b) $\pi_1^{-1}(B_2^{(1)}) \cap \pi_2^{-1}(B_2^{(2)})$
(c) $\pi_1^{-1}(B_3^{(1)}) \cap \pi_2^{-1}(B_3^{(2)}) \cap \pi_3^{-1}(B_3^{(3)})$
$\vdots$

(iii) Second countability: For $i = 1, 2, \ldots$, choose countable local bases $\mathscr{B}_i$'s for second countable X_i's. Then the union of the following collections forms a countable base for $\prod_i X_i$:[8] *CC used.*
(a) $\pi_1^{-1}(\mathscr{B}_1)$
(b) $\pi_1^{-1}(\mathscr{B}_1) \cap \pi_2^{-1}(\mathscr{B}_2)$
(c) $\pi_1^{-1}(\mathscr{B}_1) \cap \pi_2^{-1}(\mathscr{B}_2) \cap \pi_3^{-1}(\mathscr{B}_3)$
$\vdots$ □

Remark. It turns out that Hewitt-Marczewski-Pondiczery theorem implies that $\mathfrak{c}$-fold product also preserves separability.

An uncountable product of second countable spaces needn't even be first countable: Consider an uncountable product of discrete $\{0, 1\}$.

Proposition 9.4.

(i) For a first countable domain, sequential limits $\implies$ functional limits. As a result, for first countable domains, sequential continuity $\implies$ continuity.

(ii) For a first countable space, closure is precisely the set of limits of sequences.

Proof. (i) Let $f\colon E \to Y$ where $E \subseteq X$. Let $c \in X$ and $L \in Y$ such that $f(x_i) \to L$ whenever $x_i \to c$ for $(x_i) \in E \setminus \{c\}$. We show that $f(x) \to L$ as $x \to c$ in X:

Suppose not. Then take an open neighborhood V of L such that $f(U \cap E \setminus \{c\})$ spills outside V for any open neighborhood U of c in X. Let B_n's form a decreasing local base at c in X (as X is first countable). Choose $x_i \in B_n \cap E \setminus \{c\}$ such that $f(x_i) \notin V$. Then $(x_i) \in E \setminus \{c\}$ converges to c and yet $f(x_i) \not\to L$, a contradiction.

(ii) Let $c \in \overline{A} \setminus A$ and let B_n's form a local base at c. Then for each n, choose $x_n \in B_n \cap A$. Then (x_n) is a sequence in A converging to c. □

In both, CC's usage can be avoided if X is separable too.

Remark. (i) Any function from a co-countable topology is sequentially continuous, and yet needn't be continuous, for instance, $\mathrm{id}_X\colon X_{\text{co-count}} \to X_{\text{discr}}$ for any uncountable X.

[8] Notation abused for π_i^{-1} and $\cap$.

(ii) For the cocountable topology on an uncountable set, the closure of any nonempty open set in the cocountable topology is the whole space.

Corollary 9.5. *A first countable topology is determined by convergence.*[9] *Further, if the space is T_1 as well, then specifying just the convergent sequences suffices.*

Proof. Just note that in a T_1 space, $x_i \to c \iff x_1, c, x_2, c, x_3, c, \ldots$ is convergent. $\square$

Remark. (i) To see the necessity of first countability, note that cocountable and discrete topologies have the same convergent sequences and their limits. (Note that discrete is first countable.)

(ii) To see the necessity of T_1, consider the Sierpiński and indiscrete topologies on $\{0, 1\}$.

[9] That is if τ_1, τ_2 are first countable topologies on X with $x_i \to c$ in $\tau_1 \iff x_i \to c$ in τ_2, then $\tau_1 = \tau_2$.

CHAPTER II

Metric Spaces

Conventions. Unless stated otherwise, assume the following:

- X, Y will denote metric spaces.
- E will be reserved for generic sets.
- Subsets of metric spaces will be seen as metric subspaces.
- For $x \in X$ and $r > 0$, we'll use
  - $B(x, r) := \{y \in X : d(y, x) < r\}$, and
  - $D(x, r) := \{y \in X : d(y, x) \leq r\}$.

  Sometimes, we'll also denote these by $B_r(x)$ and $D_r(x)$.
- The diameter of a subset A of a metric space will be denoted by $\delta(A)$.
- For any $f, g \in X^E$, we'll define $d_\infty(f, g) := \sup_{e \in E} d(f(e), g(e))$.
- A metric space will also be considered a topological space under the induced topology.
- Finite product of metric spaces will be considered together with the metric generated by any of the p-norms (which are uniformly equivalent[1]). (See 2.1.)

1. General

Lemma 1.1. *(i) Metric is Lipschitz continuous.*

[1] $\|x\|_\infty \leq \|x\|_p \leq n^{1/p}\|x\|_\infty$ for $x \in K^n$.

(ii) If $E \subseteq X$ is nonempty, then $x \mapsto d(x, E)$ is also Lipschitz continuous.[2]

Proof. (i) Use the triangle inequality with the 1-norm metric on $X \times X$. (See 2.1.)

(ii) For any $e \in E$, we have $d(x, e) \leq d(x, y) + d(y, e)$ so that taking infimum over e yields $d(x, E) \leq d(x, y) + d(y, E) \overset{\text{w}}{\Longrightarrow} d(x, E) - d(y, E) \leq d(x, y)$ (note that $d(x, E), d(y, E) < +\infty$ since $\underline{E \neq \emptyset}$). $\square$

Do stuff on $\mathbb{R}^*$. Also that bit on monotone functions and taking sup's.

Remark. Note that $d(x, \emptyset) = +\infty$ for all x.

Lemma 1.2.

(i) Metric spaces are first countable.

(ii) Separable metric spaces are second countable.

Proof. (i) $B_{1/n}(x)$'s forms a local base at x.

(ii) Let S be a countable dense subset of X. Then $\bigcup_{x \in S} \{B_{1/n}(x) : n \geq 1\}$ forms a countable base:

Consider $B_{1/n}(y)$. Let $x \in B_{1/2n}(y) \cap S$. Then $y \in B_{1/2n}(x) \subseteq B_{1/n}(y)$. $\square$

Lemma 1.3. *The subspace topology on a subset of a metric space is precisely the one induced by the inherited metric.*

Proof. Let $E \subseteq X$. Let τ_s, τ_m be the respective topologies. That $\tau_m \subseteq \tau_s$ is clear since the balls of E are precisely the balls of X intersected with E. We show $\tau_s \subseteq \tau_m$:

Consider $B_r(x) \cap E$ for $x \in X$. Let $y \in B_r(x) \cap E$. Take $B_\varepsilon(y) \subseteq B_r(x) \overset{\text{w}}{\Longrightarrow} y \in B_\varepsilon(y) \cap E \subseteq B_r(x) \cap E$. $\square$

Let $E \subseteq X$ and $x \in X$. Then a point $y \in E$ is called **a point of best approximation** for x in Y iff $d(x, y) = d(x, E)$.

2. Products of Metric Spaces

We'll try to gain some insight into the following question in this section: Given metric spaces X_i's, is there a metric on $\prod X_i$ that induces the product topology on $\prod X_i$?

[2]For Lipschitz continuity, the codomain space must be a metric space and thus we must not include $+\infty$ in the codomain. Taking $E \neq \emptyset$ ensure this.

Proposition 2.1 (Finite products)**.** *If $X_1, \dots, X_n$ are metric spaces and $\|\cdot\|$ a norm on $\mathbb{R}^n$ which is monotonic along each cardinal direction at each point in the orthant $[0, +\infty)^n$, then*

$$d(x, y) := \left\| \big(d_1(x_1, y_1), \dots, d_n(x_n, y_n)\big) \right\| \tag{2.1}$$

defines a metric on $X_1 \times \cdots \times X_n$ that induces the product topology on it. Further:

(i) Cauchy-ness of a sequence in the product is equivalent to the Cauchy-ness of the component sequences in product spaces.

(ii) If all the spaces are nonempty, then $\prod_i X_i$ is complete $\iff$ each X_i is.

Proof. That it's a metric is easily verified:

- $d(x, y) = 0 \iff$ each $d_i(x_i, y_i) = 0$ (since $\underline{\|\cdot\| \text{ is positive definite}}$) $\iff$ each $x_i = y_i$ (since $\underline{d_i\text{'s are positive definite}}$) $\iff x = y$.
- d is symmetric since $\underline{\text{each } d_i \text{ is}}$.
- d satisfies triangle inequality since $\underline{\|\cdot\| \text{ and all } d_i\text{'s do}}$ and $\underline{\|\cdot\| \text{ is monotonic along the cardinal directions in the orthant}}$.

We now verify that these induce the product topology. Because the norms on $\mathbb{R}^n$ are uniformly equivalent (**??**) and uniformly equivalent metrics are topologically equivalent, we may assume without loss of generality that $\|\cdot\|$, $\underline{\text{which is a norm}}$, is the max-norm, which clearly generates the product topology.

For (i), because Cauchy-ness of a sequence is preserved under uniformly equivalent metrics, we may again work with max-norm which makes the statement obvious.

(ii) follows immediately from (i) and the characterization of convergence in the product topology (4.4). □

Remark. Note that to conclude that d is a metric, full power of $\|\cdot\|$ being a norm was not used—just that it's positive definite, and that it satisfies triangle inequality and the monotonicity assumption. However, the fact that it's a norm was used in concluding that it generates the product topology.

Secondly, to see the necessity of monotonicity of $\|\cdot\|$, consider the following digression:

> Any linear injection T on a vector space over K into itself gives a means to produce a new norm from any given norm on it, given by $\|x\|_{\text{new}} := \|Tx\|_{\text{old}}$.
>
> Thus, consider the linear isomorphism T on $\mathbb{R}^2$ given by $x \mapsto \left[\begin{smallmatrix} a & a \\ b & -b \end{smallmatrix}\right] x$ where $a, b > 0$, and let $\|\cdot\|$ be the norm that T generates out of $\|\cdot\|_1$. We then show that d as defined by 2.1 needn't satisfy the triangle inequality. Indeed, for $x, y \in X_1 \times X_2$, we have
>
> $$d(x, y) = \left\| \big(d_1(x_1, y_1), d_2(x_2, y_2)\big) \right\|$$

Add a diagram showing the unit ball of the new norm.

$$= \left\| T\big(d_1(x_1, y_1), d_2(x_2, y_2)\big) \right\|_1$$
$$= a\left(d_1(x_1, y_1) + d_2(x_2, y_2)\right) + b\left|d_1(x_1, y_1) - d_2(x_2, y_2)\right|.$$

Take $x, y, z \in X_1 \times X_2$ such that $d_i(x_i, y_i) = 1 = d_i(y_i, z_i)$ for $i = 1, 2$. Set $\alpha_i := d_i(x_i, z_i)$. Then $d(x, y) + d(y, z) = 4a$ and $d(x, z) = a(\alpha_1 + \alpha_2) + b|\alpha_1 - \alpha_2| \geq b|\alpha_1 - \alpha_2|$. Thus, ensuring $4a/b < |\alpha_1 - \alpha_2|$ ensures the violation of triangle inequality.

Proposition 2.2. *On countable product of metric spaces is metrizable such that the analogues of (i) and (ii) of 2.1 hold.*

Proof. The finite case follows from 2.1. Thus, consider the metric spaces $X_1, X_2, \ldots$, and set $X := \prod_i X_i$. Because of 3.10, we may without loss of generality assume that each d_i is bounded by 1 which allows to define

$$d(x, y) := \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$$

for $x, y \in X$. That d defines a metric is immediate. We show the topological equivalence:

- Consider a generic set, $\pi_i^{-1}(B_r^{(i)}(x_i))$ with $x_i \in X_i$[3] for a fixed i, that is united over to form a subbasic set of the product topology. Without loss of generality, assume that $X \neq \emptyset$, so that we may take an $x \in X$ such that it's i-th coordinate is the x_i above. It suffices to find an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq \pi_i^{-1}(B_r^{(i)}(x_i)) \overset{\mathrm{w}}{\Longleftarrow} \pi_i(B_\varepsilon(x)) \subseteq B_r^{(i)}(x_i)$. Now, let $y_i \in \mathrm{LHS}$[4] so that there's a $y \in B_\varepsilon(x)$ whose i-th coordinate is y_i. Then $d_i(y_i, x_i)/2^i \leq d(y, x) < \varepsilon \overset{\mathrm{w}}{\Longrightarrow} d_i(y_i, x_i) < 2^i\varepsilon$. Thus it suffices to have $\varepsilon < r/2^i$ to ensure that $y_i \in B_r^{(i)}(x_i)$.
- Consider a basic open set $B_r(x)$ of the metric topology on X. It suffices to find $\varepsilon > 0$ and $x_j \in X_j$ for finitely many j's such that $\bigcap_j \pi_j^{-1}(B_\varepsilon^{(j)}(x_j)) \subseteq B_r(x)$. Let j's come from $\{1, \ldots, n\}$ and $y \in \bigcap_j \pi_j^{-1}(B_\varepsilon^{(j)}(x_j))$ so that each $d_j(y_j, x_j) < \varepsilon$. Thus, since $\underline{d_i\text{'s are bounded by 1}}$, we have $d(y, x) < \varepsilon + 1/2^n \overset{\mathrm{w}}{<} r$ if $1/2^n < r - \varepsilon$, which can be ensured by taking $\varepsilon < r$ and n large enough.

We now show (i), from which (ii) follows immediately:

Let $(x^{(n)}) \in X$ be Cauchy. Fix an i. Then $d_i(x_i^{(m)}, x_i^{(n)})/2^i \leq d(x^{(m)}, x^{(n)}) \to 0$ as $m, n \to \infty$.

Conversely, let $(x_i^{(n)})_n \in X_i$ be Cauchy for each i. Let $\varepsilon > 0$. Fix an N. Then $d(x^{(m)}, x^{(n)}) = \sum_{i \leq N} d_i(x_i^{(m)}, x_i^{(n)})/2^i + 1/2^N \overset{\mathrm{w}}{<} \varepsilon$ if $\sum_{i \leq N} d_i(x_i^{(m)}, x_i^{(n)})/2^i <$

[3] Yes, notation's being abused.
[4] Again abusing notation.

$\varepsilon - 1/2^N$. Thus, take N such that $\varepsilon - 1/2^N > 0$ and then use that Cauchy-ness of $(x_i^{(n)})_n$ for $i \leq N$.

□

Remark. To see the necessity of countability, consider an uncountable product of discrete $\{0, 1\}$, which is not first countable and hence not metrizable.

3. Uniform Properties

Uniform properties encompass things like uniform continuity and Cauchy sequences.

Lemma 3.1. *In a complete space, closed subspaces are precisely the complete ones.*

Lemma 3.2. *In a metric space, each sequence has a Cauchy subsequence $\iff$ the space is totally bounded.*

Proof. "$\Rightarrow$": Consider a sequence (x_i). Take an infinite subset I_1 of the indices such that $\{x_i : i \in I_1\}$ lies in a ball of diameter 1. Having chosen I_n, choose an infinite *DC used.* subset $I_{n+1} \subseteq I_n$ such that $\{x_i : i \in I_{n+1}\}$ lies in a ball of diameter $1/(n+1)$. (This is possible since the the space is totally bounded.) Now, choose $i_n \in I_n$ such that (i_n) is increasing. Then $(x_{i_n})_n$ forms a Cauchy sequence, for for $n > m \geq N$, we have $d(x_{i_m}, x_{i_n}) < 1/N$.

"$\Leftarrow$": Suppose X is not totally bounded so that take an $\varepsilon > 0$ such that no finitely many balls of radius ε can ever cover X. Let $x_1 \in X$.[5] Having chosen $x_1, \ldots, x_n$, choose $x_{n+1} \in X \setminus \bigcup_{i=1}^n B_\varepsilon(x_i)$. Then (x_i) is non-Cauchy sequence, for *DC used.* $d(x_i, x_j) \geq \varepsilon$ for all $i \neq j$. □

Remark. The discrete metric on an infinite set, which is not totally bounded, contains sequences with no Cauchy subsequences.

3.1 Completion of metric spaces

A **completion** of X is a complete metric space $\hat{X}$ together with an isometry $\iota \colon X \to \hat{X}$ such that any isometry $X \to Y$ into a complete metric space Y factors uniquely through ι via an isometry:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\iota"'] & & Y \\ & \hat{X} \arrow[ur, dashed] & \end{tikzcd}$$

[5] Note that X has got to be nonenmtpy.

Corollary 3.3. *If X is complete, then* $\mathrm{id}\colon X \to X$ *is a completion of X.*

Theorem 3.4. *Each metric space admits a completion, which is unique up to bi-isometries.*[6]

The following easy facts will be employed to simplify the proof:

Lemma 3.5.

(i) If (x_i) is Cauchy and $(\alpha_i) \in \mathbb{R}^+$, then there exists a subsequence (x_{i_j}) such that for each N, we have $d(x_{i_j}, x_{i_k}) < \alpha_N$ whenever $j, k \geq N$.

(ii) A Cauchy sequence converges $\iff$ any of its subsequence converges.

Proof of 3.4. The uniqueness follows by the usual categorical argument. Let's show the existence of a completion of X. Define $\hat{X}$ to be set of the Cauchy sequences in X modded out by the following equivalence relation:

$$(x_i) \sim (y_i) \text{ iff } d(x_i, y_i) \to 0$$

The following defines a well-defined metric on $\hat{X}$:

$$\hat{d}\big(\overline{(x_i)}, \overline{(y_i)}\big) := \lim_i d(x_i, y_i)$$

We show that $\hat{X}$ is complete:

Let $\big(\overline{x^{(n)}}\big)$ be Cauchy in $\hat{X}$, where each $x^{(n)}$ is a Cauchy sequence $\big(x_i^{(n)}\big)$ in X. Noting that each subsequence of a Cauchy sequence in X is related to the parent sequence, and due to 3.5, we may without loss of generality assume: *CC used.*

(i) $n \geq m \implies \hat{d}\big(\overline{x^{(n)}}, \overline{x^{(m)}}\big) < 1/m.$

(ii) For each n, we have $j \geq i \implies d\big(x_j^{(n)}, x_i^{(n)}\big) < 1/i.$

Now, it follows that the diagonal sequence $\big(x_i^{(i)}\big)$ is Cauchy:

$$\begin{aligned}
d\big(x_j^{(j)}, x_i^{(i)}\big) &\leq d\big(x_j^{(j)}, x_j^{(i)}\big) + d\big(x_j^{(i)}, x_i^{(i)}\big) \\
&< d\big(x_j^{(j)}, x_k^{(j)}\big) + d\big(x_k^{(j)}, x_k^{(i)}\big) + d\big(x_k^{(i)}, x_j^{(i)}\big) && (k \text{ arbitrary}) \\
&\qquad + 1/i && (\text{letting } j \geq i) \\
&< 1/j + d\big(x_k^{(j)}, x_k^{(i)}\big) + 1/j + 1/i && (\text{letting } k \geq j) \\
&\leq 2/i + \hat{d}\big(\overline{x^{(j)}}, \overline{x^{(i)}}\big) + 1/i && (\text{taking } k \to \infty) \\
&< 2/j + 2/i && (\text{since } j \geq i)
\end{aligned}$$

[6] A bi-isometry is a bijective isometry whose inverse is also an isometry. Thus, bi-isometries are precisely bijective isometries.

Also, $\overline{x^{(n)}} \to \overline{(x_i^{(i)})}$ in $\hat{X}$:

$$\begin{aligned} d\big(x_i^{(n)}, x_i^{(i)}\big) &\le d\big(x_i^{(n)}, x_j^{(n)}\big) + d\big(x_j^{(n)}, x_j^{(i)}\big) + d\big(x_j^{(i)}, x_i^{(i)}\big) \\ &< 1/i + d\big(x_j^{(n)}, x_i^{(i)}\big) + 1/i && \text{(letting } j \ge i\text{)} \\ &< 2/i + d\big(\,\overline{x^{(n)}},\, \overline{x^{(i)}}\,\big) && \text{(taking } j \to \infty\text{)} \\ &< 3/i \end{aligned}$$

We now check for the universal property:

Note that $\iota\colon X \to \hat{X}$ given by $x \mapsto \overline{(x, x, \ldots)}$ is an isometry. Let $f\colon X \to Y$ be another isometry with Y being complete. Suppose it does factor through ι via an isometry $\hat{f}$:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\iota"'] & & Y \\ & \hat{X} \arrow[ur, dashed, "\hat{f}"'] & \end{tikzcd}$$

This in turn determines $\hat{f}$ uniquely:

Let $\overline{(x_i)} \in \hat{X}$, where (x_i) is Cauchy in X. Clearly, $\iota(x_i) \to \overline{(x_i)}$ so that

$$f(x_i) \to \hat{f}\big(\,\overline{(x_i)}\,\big) \tag{3.1}$$

as $\hat{f}$ is continuous and $\hat{f} \circ \iota = f$.

We now show that 3.1 indeed defines a factoring of f via ι:

- $\hat{f}$ is well-defined: (i) If (x_i) is Cauchy in X, then since f is an isometry and Y is complete, $(f(x_i))$ is convergent in Y. (ii) If (x_i) and (y_i) are equivalent Cauchy sequences in X, then $d(x_i, y_i) \to 0 \overset{\text{w}}{\Longrightarrow} d(f(x_i), f(y_i)) \to 0$ (since f an isometry) so that $d\big(\lim_i f(x_i), \lim_i f(y_i)\big) = 0$.[7]
- $\hat{f}$ is an isometry:

$$\begin{aligned} d\Big(\hat{f}\big(\,\overline{(x_i)}\,\big),\, \hat{f}\big(\,\overline{(y_i)}\,\big)\Big) &= d\big(\lim_i f(x_i), \lim_i f(y_i)\big) \\ &= \lim_i d(f(x_i), f(y_i)) \\ &= \lim_i d(x_i, y_i) && (f \text{ is an isometry}) \\ &= \hat{d}\big(\,\overline{(x_i)},\, \overline{(y_i)}\,\big) \end{aligned}$$

- Finally, $\hat{f} \circ \iota = f$ is clear. $\square$

Proposition 3.6. *Any space is dense in its completion.*

[7] Recall the component-wise convergence in product topology.

Proof. Let $\iota\colon X \to \hat{X}$ be a completion of X. Now, the restriction $\iota|\colon X \to \overline{\iota(X)}$ is also a completion of $\hat{X}$:

- $\overline{\iota(X)}$ is complete due to 3.1 and $\iota|$ is still an isometry.
- Let $f\colon X \to Y$ be an isometry with Y complete. Then f factors through ι which induces a factoring through $\iota|$ as well:

$$\begin{tikzcd}
& X \arrow[dl, "\iota|"'] \arrow[d, "\iota"] \arrow[dr, "f"] & \\
\overline{\iota(X)} \arrow[r, hook] & \hat{X} \arrow[r, dashed] & Y
\end{tikzcd}$$

For uniqueness, just note that any isometric factoring $\tilde{f}$ of f through $\iota|$ is determined on $\iota(X)$ which is dense in $\overline{\iota(X)}$, thereby also getting determined on $\overline{\iota(X)}$ (by 8.2, for $\tilde{f}$ is continuous):

$$\begin{tikzcd}
& X \arrow[dl] \arrow[d, "\iota|"] \arrow[dr, "f"] & \\
\iota(X) \arrow[r, hook] & \overline{\iota(X)} \arrow[r, "\tilde{f}"'] & Y
\end{tikzcd}$$

Since $\hat{X}$ is a completion, there exists an isometry α such that the following diagram commute:

$$\begin{tikzcd}
& X \arrow[dl, "\iota|"'] \arrow[dr, "\iota"] & \\
\overline{\iota(X)} \arrow[rr, hook, shift left, "\mathrm{incl}"] & & \hat{X} \arrow[ll, dashed, shift left, "\alpha"]
\end{tikzcd}$$

It follows that ι factors through itself via $\mathrm{incl} \circ \alpha$, so that it is precisely $\mathrm{id}_{\hat{X}} \overset{\text{w}}{\Longrightarrow}$ incl is surjective $\overset{\text{w}}{\Longrightarrow}$ $\overline{\iota(X)} = \hat{X}$. $\square$

Remark. It turns out that this in fact characterizes completions. See (ii) of 3.17.

From 6.1, it now immediately follows that:

Corollary 3.7. *Completion preserves separability.*

3.2 Metric equivalences

Proposition 3.8. *For metrics on a given set, we have:*

Uniform equivalence $\implies$ id *is uniformly continuous in both directions* $\implies$ *same Cauchy sequences* $\implies$ *same convergence* $\iff$ *topological equivalence.*

Proof. The first two implications are trivial and the last follows from 9.5. For the penultimate, just note that $x_i \to c \iff x_1, c, x_2, c, \ldots$ is Cauchy. $\square$

Remark. None of the converses are true. Let $f\colon X \to X$ be a homeomorphism which thus induces a topologically equivalent metric on X (see 3.9).

(i) Let f, f^{-1} be uniformly continuous and f not be Lipschitz (for instance, $f\colon x \mapsto \sqrt{x}$ on $[0,1]$[8]). Then id is uniformly continuous in both directions and yet the metrics are not uniformly equivalent.

(ii) Let f not be uniformly continuous and X be complete[9] (like $x \to x^3$ on $\mathbb{R}$). Then Cauchy sequences are just convergent sequences, which are thus the same. However, $\mathrm{id}\colon X_{\mathrm{old}} \to X_{\mathrm{new}}$ is not uniformly continuous.

(iii) Consider $f\colon x \mapsto 1/x$ on $X := \mathbb{R}^+$. However, note that $1, 2, 3, \ldots$ is Cauchy in the new metric and not in the old one.

Any injection f on a set into itself gives a means to generate a new metric given any metric on it via $d_{\mathrm{new}}(x,y) := d_{\mathrm{old}}(f(x), f(y))$.

Lemma 3.9. *Let $f\colon X \to X$ be a bijection which thus induces a new metric on X. Then the following hold:*

(i) The new metric is topologically equivalent to the old one $\iff$ f, f^{-1} are continuous on X_{old}.

(ii) $\mathrm{id}\colon X_{old} \to X_{new}$ is uniformly continuous $\iff$ f is uniformly continuous on X_{old}.

(iii) The new metric is uniformly equivalent to the old one $\iff$ f, f^{-1} are Lipschitz on X_{old}.

Proof. For the first use 9.4 and 9.5. The next two follow easily from the definition of d_{new}. $\square$

Proposition 3.10. *Any metric is topologically equivalent to a bounded metric which preserves Cauchy sequences and total boundedness both ways.*

Proof. Let $f\colon [0,+\infty) \to [0,1)$ be a strictly increasing bijection which is subadditive, *i.e.*, $f(x+y) \le f(x) + f(y)$,[10] for instance $x \mapsto x/(1+x)$. Then $d'(x,y) := f(d(x,y))$ defines a bounded metric on X:

- $d'(x,y) = 0 \iff d(x,y) = f^{-1}(0) \overset{\mathrm{w}}{=} 0$ (since $\underline{f \text{ a strictly increasing bijection}}$) $\iff x = y$.

[8] See 3.13.

[9] Note that X_{old} is complete $\iff$ X_{new} is.

[10] This gives another reason why f is continuous: $f(x+\delta) - f(x) \le f(\delta) \overset{\mathrm{w}}{<} \varepsilon$ if $\delta < f^{-1}(\varepsilon)$.

- d' is symmetric because d is.
- $d'(x, y) + d'(y, z) = f(d(x, y)) + f(d(y, z)) \geq f(d(x, y) + d(y, z))$ (since f is subadditive) $\geq f(d(x, z))$ (since f is increasing) $= d'(x, z)$.

Topological equivalence and preservation of total boundedness will follow if preservation of Cauchy sequences is established (3.8 and respectively 3.2).

That the Cauchy sequences are the same follows by continuity[11] of f and f^{-1} and the fact that $f(0) = 0 = f^{-1}(0)$ (due to the strict monotonicity and bijectivity of f). $\square$

Remark. Another choice of f could have been $x \mapsto \min(x, 1)$, which although subadditive, is not bijective.

The following is an easy lemma:

Lemma 3.11. *Uniformly equivalent metrics preserve the following continuities: Usual, Cauchy-regular, uniform, Lipschitz.*

Remark. Note that just the topological equivalence (or even same Cauchy sequences) is not enough to preserve uniform continuity of even Lipschitz functions: COMPLETE THIS! SE link.

What about Cauchy-regularity of Lipschitz?

3.3 Stronger forms of continuity

Lemma 3.12. *Uniformly continuous functions preserve total boundedness.*

Proof. Let $f : X \to Y$ be uniformly continuous with X being totally bounded. Let $\varepsilon > 0$. Due to uniform continuity, take $\delta > 0$ such that $d(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$. Due to total boundedness, let $B_\delta(x_1), \ldots, B_\delta(x_n)$ cover X. Now, $f(X) = \bigcup_i f(B_\delta(x_i)) \subseteq \bigcup_i B_\varepsilon(f(x_i))$. $\square$

Remark. Boundedness needn't be preserved however: Consider $\mathrm{id}: (X, d_{\mathrm{discr}}) \to (X, d)$ where d is any unbounded metric on X.

Proposition 3.13. *Continuous functions on compact sets are uniformly continuous.*

Proof. Let $f : X \to Y$ be continuous with X being compact. Let $\varepsilon > 0$. For each $x \in X$, choose $\delta_x > 0$ such that $f(B_{\delta_x}(x)) \subseteq B_\varepsilon(f(x))$. Let $B_{\delta_{x_1}/2}(x_1), \ldots, B_{\delta_{x_n}/2}(x_n)$ cover X (since X is compact). Now, any $x, y \in X$ lie in some $B_{\delta_{x_i}}(x_i)$ whenever $d(x, y) < \min(\delta_{x_1}, \ldots, \delta_{x_n})/2 \overset{\mathrm{w}}{\implies} d(f(x), f(y)) < 2\varepsilon$. $\square$ *No AC needed!*

[11] Only continuity at 0 is required.

Remark. To see the necessity of compact domain, consider $x \mapsto 1/x$ on $\mathbb{R} \setminus \{0\}$.

Proposition 3.14. *f is uniformly continuous $\iff$ $d(f(x_n), f(y_n)) \to 0$ whenever $d(x_n, y_n) \to 0$.*

Proof. Consider $f \colon X \to Y$.

"$\Rightarrow$": Let f be uniformly continuous. Let $d(x_n, y_n) \to 0$ in domain. Let $\varepsilon > 0$. Take $\delta > 0$ such that $d(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$. Take N such that for each $n \geq N$, we have $d(x_n, y_n) < \delta \overset{\text{w}}{\Longrightarrow} d(f(x_n), f(y_n)) < \varepsilon$.

"$\Leftarrow$": If f is not not uniformly continuous, then we may take an $\varepsilon > 0$ and for each n, choose $x_n, y_n \in E$ such that $d(x_n, y_n) < 1/n$ and yet $d(f(x_n), f(y_n)) \geq \varepsilon$. ☐ *CC used!*

Proposition 3.15 (Interaction of continuities)**.**

(i) Lipschitz $\Longrightarrow$ uniform continuity.

(ii) Uniform continuity on every totally bounded subset of the domain $\iff$ Cauchy-regularity.

(iii) Cauchy-regularity $\Longrightarrow$ continuity.

Proof. (i) This is clear.

(ii) "$\Rightarrow$": Since Cauchy sequences are totally bounded.
"$\Leftarrow$": Suppose f is Cauchy-regular and yet not uniformly continuous on a totally bounded subset E of the domain, so that we may take an $\varepsilon > 0$ and for each n, choose $x_n, y_n \in E$ such that $d(x_n, y_n) < 1/n$ and yet $d(f(x_n), f(y_n)) \geq \varepsilon$. Without loss of generality, let (x_n), (y_n) be Cauchy (for E is <u>totally bounded</u>). Now, the sequence $x_1, y_1, x_2, y_2, \ldots$ is also Cauchy, and despite that, its f-image isn't. *CC used!*

(iii) Let f be Cauchy-regular and $x_i \to c$ in the domain. Then $x_1, c, x_2, c, \ldots$ is Cauchy $\overset{\text{w}}{\Longrightarrow}$ $f(x_1), f(c), f(x_2), f(c), \ldots$ is Cauchy $\overset{\text{w}}{\Longrightarrow}$ $f(x_i) \to f(c)$. ☐

Remark. (i) $x \mapsto \sqrt{x}$ on $[0, 1]$ is uniformly continuous but not Lipschitz (observe at 0).

(ii) $x \mapsto x^2$ on $\mathbb{R}$ is Cauchy-regular and not uniformly continuous.

(iii) $x \mapsto 1/x$ on $\mathbb{R}^+$ is continuous but not Cauchy-regular.

Theorem 3.16 (Extension of Cauchy-regulars)**.** *A Cauchy-regular function from a dense subset to a complete codomain has a unique continuous extension to the whole of domain, which is further Cauchy-regular. Furthermore, this extension preserves uniform and Lipschitz continuities, and isometry-city.*

Proof. Let $f\colon A \to Y$ be Cauchy-regular where A is dense in X, and Y complete. Let's first settle uniquness.[12] Let $x \in X$. Then take a sequence $(a_i) \in A$ such that $a_i \to x$ (since A dense in X). If $\tilde{f}\colon X \to Y$ is a continuous extension of f, then we must have $f(a_i) \to \tilde{f}(x)$. *CC used; avoidable if X separable.*

Let's verify that this indeed gives a well-defined Cauchy-regular extension:

- Well-defined:
  (i) If (a_i) is Cauchy in A, then by Cauchy-regularity, it's f-image is also Cauchy, and thus convergent due to completeness of Y.
  (ii) Let $(a_i), (b_i) \in A$ converge to the same point in X. Then the interleaved sequence $a_0, b_0, a_1, b_1, \ldots$ is Cauchy. Due to Cauchy-regularity, its f-image is also Cauchy $\overset{\text{w}}{\Longrightarrow}$ $\lim_i f(a_i) = \lim_i f(b_i)$.
- Extension: This is clear, since for $a \in A$, the constant sequence $(a, a, \ldots)$ converges to a so that $\tilde{f}(a) = \lim_i f(a) = f(a)$.
- Cauchy-regularity: Let $(x^{(n)}) \in X$ be Cauchy. We need to show that it's $\tilde{f}$-image is Cauchy as well. As before due to denseness of A, choose Cauchy sequences $(a_i^{(n)}) \in A$ such that $a_i^{(n)} \to x^{(n)}$ so that $\tilde{f}(x^{(n)}) = \lim_i b_i^{(n)}$. where $b_i^{(n)} := f(a_i^{(n)})$. Note that *CC used twice; both avoidable if X separable.*
  (i) the Cauchy-ness of $(x^{(n)})$ translates to $\lim_i d(a_i^{(m)}, a_i^{(n)}) \to 0$ as $m, n \to \infty$, and similarly,
  (ii) that of $(\tilde{f}(x^{(n)}))$ translates to $\lim_i d(b_i^{(m)}, b_i^{(n)}) \to 0$ as $m, n \to \infty$.

  Since the sequences are Cauchy, assume for all n's without loss of generality, that $d(a_j^{(n)}, a_i^{(n)}), d(b_j^{(n)}, b_i^{(n)}) < 1/i$ whenever $j \geq i$.

  Note that it suffices to get hold of a "diagonal" sequence $(b_{N_n}^{(n)})$ that is Cauchy with N_n's increasing[13] for then we'll have

$$
\begin{aligned}
d(b_i^{(m)}, b_i^{(n)}) &\leq d(b_i^{(m)}, b_{N_m}^{(m)}) + d(b_{N_m}^{(m)}, b_{N_n}^{(n)}) + d(b_{N_n}^{(n)}, b_i^{(n)}) \\
&< 1/N_m + d(b_{N_m}^{(m)}, b_{N_n}^{(n)}) + 1/N_n \qquad \text{(taking } i \geq N_m, N_n)
\end{aligned}
$$

  so that we'll have $\lim_i d(b_i^{(m)}, b_i^{(n)})$ being less than the RHS which indeed goes to 0 as $m, n \to \infty$ (since $(b_{N_n}^{(n)})$ Cauchy and N_n's increasing).

  Since f is Cauchy-regular, it suffices to find a Cauchy $(a_{N_n}^{(n)})$. Choose N_n's increasing, such that $d(a_i^{(n)}, a_n^{(n)}) < 1/n$ for each $i \geq n$. Now,

$$
d(a_{N_m}^{(m)}, a_{N_n}^{(n)}) \leq d(a_{N_m}^{(m)}, a_{N_n}^{(m)}) + d(a_{N_m}^{(m)}, a_{N_n}^{(n)})
$$

[12] Which also directly follows from 8.2.

[13] Actually, what is required in the proof is just that $1/N_n \to 0$.

$$\begin{aligned}
&< 1/N_m + d(a^{(m)}_{N_m}, a^{(m)}_i) + d(a^{(m)}_i, a^{(n)}_i) && \text{(taking } n \geq m\text{)}\\
&\qquad + d(a^{(n)}_i, a^{(n)}_{N_n})\\
&< 2/N_m + d(a^{(m)}_i, a^{(n)}_i) + 1/N_n && \text{(taking } i \geq N_m, N_n\text{)}\\
&\leq 2/N_m + 1/N_n + \lim_i d(a^{(m)}_i, a^{(n)}_i) && \text{(taking } i \to \infty\text{)}
\end{aligned}$$

which indeed goes to 0 as $m, n \to \infty$.

Finally, we verify the preservations:

- Preservation of uniform continuity: Let f be uniformly continuous. We need to show that $\tilde{f}$ is also uniformly continuous. Let $\varepsilon > 0$ and take $\delta > 0$ such that $d(f(a), f(b)) < \varepsilon$ whenever $d(a, b) < \delta$ for $a, b \in A$. Now, let $x, y \in X$ with $d(x, y) < \delta$. Take $(a_i), (b_i) \in A$ converging to x, y respectively. Now, $d(a_i, b_i) < \delta$ eventually (as $\lim_i d(a_i, b_i) = d(x, y) < \delta$) so that $d(f(a_i), f(b_i)) < \varepsilon$ eventually $\overset{\text{w}}{\Longrightarrow}$ $d(\tilde{f}(x), \tilde{f}(y)) \overset{\text{w}}{=} \lim_i d(f(a_i), f(b_i)) \leq \varepsilon$. Same comment on CC.
- Preservation of Lipschitz continuity and isometry-city: Same technique as in the last point. □

These immediately yields the following:

Corollary 3.17. *(i)* (Characterizing Cauchy-regularity). *A continuous function between metric spaces is Cauchy-regular $\iff$ it admits a continuous extension to their completions.*

(ii) (Characterizing completions). *An isometry into a complete space is a completion $\iff$ its image is dense in the codomain.*

(iii) (Completion of dense subsets). *The completion of a dense subset is canonically obtained from that of the parent space.*

(iv) *In a complete space, the closures of subsets are their completions.*[14]

Proof. (i) Obviously.

(ii) "$\Rightarrow$" is the content of 3.6. "$\Leftarrow$" follows since an isometry on a dense subset extends isometrically to the whole domain.

(iii) Let A be dense in X and $\iota\colon X \to \hat{X}$ a completion of X. We show that the composition $A \hookrightarrow X \xrightarrow{\iota} \hat{X}$ is a completion of A. Due to (ii), it suffices to have that $\iota(A)$ is dense in $\hat{X}$ $\overset{\text{w}}{\Longleftarrow}$ $\iota(A)$ is dense in $\iota(X)$ (since $\underline{\iota(X)}$ $\underline{\text{is already dense in } \hat{X}}$) $\overset{\text{w}}{\Longleftarrow}$ $\underline{A \text{ is dense in } X}$ (since ι is continuous; see 6.2) which is indeed true.

[14] Note how this is different from the familair fact that the closed sets of a complete space are precisely its complete subsets.

(iv) Follows from (iii). □

3.4 Uniform convergence

Note that d_∞ "almost" forms a metric on X^E except that it can take infinite values. Thus, if $\mathscr{F} \subseteq X^E$ is such that $d_\infty(f,g) < +\infty$ for all $f, g \in \mathscr{F}$, then d_∞ defines a metric on $\mathscr{F}$. It's easily seen that convergence under this metric coincides with uniform convergence.

Proposition 3.18. *Let E be a topological space and $E_1 \subseteq E$. Let (f_n) be Cauchy in X^{E_1}.*[15] *and $c \in E_1'$ with $f_n(x) \to L_n$ as $x \to c$. Then the following hold:*

(i) (L_n) is Cauchy.

(ii) If $f_n \to f$ uniformly for $f \in X^{E_1}$ and $L_n \to L$ in X, then $f(x) \to L$ as $x \to c$.

Proof. (i) Let $\varepsilon > 0$. Since $\underline{(f_n)\text{ Cauchy}}$, take N such that $d_\infty(f_m, f_n) < \varepsilon$ for all $m, n \geq N$. Now, let $m, n \geq N$. Since $\underline{f_m(x) \to L_m \text{ and } f_n(x) \to L_n \text{ as } x \to c}$, let U be an open neighborhood of c such that $d(f_m(x), L_m), d(f_n(x), L_n) < \varepsilon$ for each $x \in E_1 \cap U \setminus \{c\}$. Then

$$\begin{aligned} d(L_m, L_n) &\leq d(L_m, f_m(x)) + d(f_m(x), f_n(x)) \\ &\qquad + d(f_n(x), L_n) && \text{(taking } x \in E_1) \\ &< 3\varepsilon, && \text{(taking } x \in U \setminus \{c\} \text{ as well)} \end{aligned}$$

where taking $x \in E_1 \cap U \setminus \{c\}$ is allowed since $\underline{c \in E_1'}$.

(ii) Let $\varepsilon > 0$. Since $\underline{f_n \to f}$ and $\underline{L_n \to L}$, take N such that $d_\infty(f_N, f), d(L_N, L) < \varepsilon$. Since $\underline{f_N(x) \to L_N \text{ as } x \to c}$, take an open neighborhood U of c such that $f_N(U \setminus \{c\}) \subseteq B_\varepsilon(L_N)$. Now, for $x \in E_1 \cap U \setminus \{c\}$, we have

$$\begin{aligned} d(f(x), L) &\leq d(f(x), f_N(x)) + d(f_N(x), L_N) + d(L_N, L) \\ &< 3\varepsilon. \end{aligned}$$ □

Corollary 3.19. *Uniform convergence preserves continuity.*

This is of course not true of just pointwise convergence.

[15] That is, $d_\infty(f_m, f_n) \to 0$ as $m, n \to \infty$.

4. Baire's Category Theorem

Proposition 4.1 (Cantor's intersection)**.** *In a complete metric space, the intersection of a decreasing sequence of closed subsets with diameters going to zero, is a singleton.*

Proof. Let F_i's be the closed sets under consideration. That there's at most one point in the intersection is clear since $\underline{\delta(F_i) \to 0}$. Now, choose $x_i \in F_i$, which form *CC used.*
a Cauchy sequence since $\underline{\delta(F_i) \to 0}$. Since the space is complete, let $x_i \to x$, and since each F_i is closed, x lies in the intersection. □

Remark. The necessity of each hypothesis is easy to see.

The diameter of the intersection of a decreasing sequence of subsets needn't be the corresponding limit of diameters even if the sets are closed and bounded. For instance, consider an infinite dimensional normed linear space containing orthonormal vectors $e_1, e_2, \ldots$. Take $F_i := \{e_i, e_{i+1}, \ldots\}$. Then each $\delta(F_i) = \sqrt{2}$, and still the intersection is empty. However, there is one case where we *can* say something:

Proposition 4.2. *Let $F_1 \supseteq F_2 \supseteq \cdots$ be closed subsets of a metric space with F_1 being compact. Then $\delta(\bigcap_i F_i) = \lim_i \delta(F_i)$.*

Proof. "$\leq$" is clear. For "$\geq$", let $\varepsilon > 0$ and choose $x_i, y_i \in F_i$ such that $d(x_i, y_i) > \delta(F_i) - \varepsilon$ (note that each $\delta(F_i) < +\infty$). Now, since F_1 is compact, let $x_{n_i} \to x$ and $y_{n_i} \to y$ in F_1. Since F_i's are closed, x, y lie in the intersection so that $\delta(\bigcap_i F_i) \geq d(x, y) \geq \lim_i \delta(F_i) - \varepsilon$. □

Theorem 4.3 (Baire's category)**.** *In a complete metric space, complements of meager sets are dense.*

Proof. Let $A_1, A_2, \ldots$ be nowhere dense. We show that $X \setminus \bigcup_i A_i$ is dense. Pick a nonempty open U. Since A_1 is nowhere dense, choose $x_1 \in U$ and $r_1 > 0$ such that $B_{r_1}(x_1) \subseteq U$ and $B_{r_1}(x_1) \cap A_1 = \emptyset$. Having chosen x_i, r_i, choose $x_{i+1} \in B_{r_i}(x_i)$ *DC used.*
such that

- $B_{r_{i+1}}(x_{i+1}) \subseteq B_{r_i}(x_i)$,
- $r_{i+1} \leq r_i/2$, and
- $B_{r_{i+1}}(x_{i+1}) \cap A_{i+1} = \emptyset$.

This is possible since A_i is nowhere dense. Thus, $\overline{B_{r_1}(x_1)} \supseteq \overline{B_{r_2}(x_2)} \supseteq \cdots$ with $\delta(\overline{B_{r_i}(x_i)}) \leq \delta(D_{2r_i}(x_i)) \stackrel{\text{w}}{=} 2r_i \stackrel{\text{w}}{\to} 0$ since $r_i \leq r_1/2^{i-1}$. By Cantor (since X is complete), let $x \in \bigcap_i \overline{B_{r_i}(x_i)}$. Then $x \notin \bigcup_i A_i$ since each $B_{r_i}(x_i) \cap A_i = \emptyset$. Finally,

$x \in \overline{B_{r_1}(x_1)}$ and without loss of generality, we could've chosen x_1, r_1 such that $\overline{B_{r_1}(x_1)} \subseteq U$. $\square$

Remark. Necessity of completeness is demonstrated by considering any countable metric space in which singletons are not open, for instance $\mathbb{Q}$.

Corollary 4.4. *If countably many closed subsets of a nonempty complete metric space unite to the whole space, then one of them has a nonempty interior.*