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# Conventions

**Conventions.** Unless stated otherwise, we'll use the following:

- For vector spaces $V$, $W$ over a common field, $\mathcal{L}(V, W)$ will denote the set of linear maps $V \to W$ and will be considered a vector space. $\mathcal{L}(V, V)$ will be considered an associative algebra.
- For a vector space $V$ and a set $X$, the set $F^X$ will be considered a vector space (over the same field as that of $V$).
- $\mathcal{C}(X, Y)$ will denote the set of continuous functions $X \to Y$ if $X$, $Y$ are topological spaces.

**CHAPTER I**

# Topological Vector Spaces

**Conventions.** Unless stated otherwise, the following will always be assumed:

- $K \in \{\mathbb{R}, \mathbb{C}\}$.
- Topological vector spaces will be over $K$.
- $X$, $Y$ will stand for TVS's over $K$.
- For $x, y \in X$, we'll use the $[x; y]$, $[x; y)$, *etc.* notation.
- Subspaces of TVS's will be seen as TVS's.
- $\mathcal{L}_c(X, Y)$ will stand for the set of continuous linear maps $X \to Y$,[1] which will be seen as a vector space.
- $X^*$ will stand for $\mathcal{L}_c(X, K)$.

## 1. Basics

**Lemma 1.1.** *Closure of a subspace is a subspace.*

**Proposition 1.2.** *Any proper subspace of a TVS has empty interior, or equivalently, has dense complement.*

*Proof.* Let $M$ be a proper subspace of $X$. It suffices to show that $X \setminus M$'s closure is whole of $X$, that is, each point of $X$ is a limit of some sequence in $X \setminus M$. Let $x \in X$. Without loss of generality, let $x \in M$. Now, take a $y \notin M$ (since $\underline{M \text{ proper}}$). Then $x$ is the limit of the sequence $(x + y/n)_n \in X \setminus M$ as required. $\square$

[1] Note that $X$ and $Y$ are over the *same* field $K$.

CHAPTER II

# Normed Linear Spaces

**Conventions.** Unless stated otherwise, assume the following:

- A normed linear space will be over $K$.
- A norm's codomain will be taken to be $[0, +\infty)$.
- $E$, $F$ will denote normed linear spaces and $\mathscr{B}$ will be reserved for a Banach space.
- Subspaces of normed linear spaces will be seen as normed linear spaces.
- A normed linear space will also be considered a metric space under the induced metric, and also a topological vector space (see Lemma 1.1).
- Abusing notation, we'll use the same notation to denote the restriction to $\mathbb{R} \to \mathbb{R}$ of Re, Im, and complex conjugation.
- $\mathcal{B}(X, F)$ will denote the set of bounded functions $X \to F$ for a set $X$. Further, for $f \in \mathcal{B}(X, F)$, we'll use $\|f\|_\infty := \sup_{x \in X} \|f(x)\|$. $\mathcal{B}(X, F)$ will be seen as a normed linear space (see Lemma 1.1.)
- $\mathcal{C}_b(X, F)$ will be the set of bounded continuous functions for a topological vector space $X$, which will be seen as a normed linear space (see Lemma 1.1.)
- For $T \in \mathcal{L}(E, F)$, we'll set $\|T\| := \sup_{\|x\| \leq 1} \|Tx\|$.[1]
- $\mathcal{L}_c(E, F)$ will be seen as a normed linear space (see Proposition 1.2).

[1] Actually, one can extend this definition to homogenous functions to still yield fruitful consequences (see the remark after Proposition 1.2). Note that homogenous functions of a fixed degree also form a vector space.

# 1. Basics

First, some immediate trivialities:

**Lemma 1.1.**

*(i)* *A normed linear space is a topological vector space. In fact, addition is Lipschitz continuous and multiplication is Cauchy-regular*[2].

*(ii)* *Norm is Lipschitz continuous.*

*(iii)* $\overline{B_r(x)} = D_r(x)$ *for any* $x \in E$.

*(iv)* $\|\cdot\|_\infty$ *defines a norm on* $\mathcal{B}(X, F)$*, convergence under which coincides with uniform convergence.*

*(v)* $\mathcal{C}_b(X, F)$ *is a closed subspace of* $\mathcal{B}(X, F)$ *for any set* $X$.

*Proof.* For (i), use 1-norm metric on $X \times X$ for addition and use the usual trickery for multiplication.

For (iii), note that for any $y \in D_r(x)$, the segment $[x; y)$ contains points common to $B_r(x)$ and $B_\varepsilon(y)$ for $\varepsilon$ however small.

For (v), just note that uniform limit of continuous functions is continuous. □

*Remark.* Note that scalar multiplication needn't even be uniformly continuous, let alone Lipschitz; for instance, consider the multiplication on $\mathbb{R}$: Take $x_n := (\sqrt{n}, \sqrt{n})$ and $y_n := (\sqrt{n+1}, \sqrt{n+1})$. Then under the metric on $\mathbb{R} \times \mathbb{R}$ due to $\infty$-norm, $d(x_n, y_n) = \sqrt{n-1} - \sqrt{n} = (\sqrt{n+1} + \sqrt{n})^{-1} \to 0$ as $n \to \infty$ and yet $d(n, n+1) = 1 \not\to 0$.

However, scalar multiplication by a fixed scalar is trivially Lipschitz.

**Proposition 1.2** (Norm on $\mathcal{L}_c(E, F)$)**.** *Let* $T \in \mathcal{L}(E, F)$*. Then the following hold:*

*(i)* $\sup_{\|x\|<1}\|Tx\| = \sup_{\|x\|\le 1}\|Tx\| \overset{(*)}{=} \sup_{\|x\|=1}\|Tx\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}$ *where* $(*)$ *holds for* $E \neq 0$.

*(ii)* $\|T\|$ *is the smallest* $M \geq 0$ *such that* $\|Tx\| \leq M\|x\|$ *for all* $x \in E$.

*(iii)* *The following are equivalent:*

- *(a)* $T$ *is continuous.*
- *(b)* $T$ *is continuous at* $0$.
- *(c)* $\|T\| < +\infty$.

[2]Remember, finite product of metric spaces (and hence normed linear spaces) are considered under any of the uniformly equivalent $p$-norm metrics.

*(iv)* $\|\cdot\|$ *defines a norm on* $\mathcal{L}_c(E, F)$.[3]

*(v) If* $G$ *is another normed linear space and* $S \in \mathcal{L}(F, G)$*, then* $\|ST\| \leq \|S\|\|T\|$.

*Proof.* (iv), (v) are easy consequences of (i), (ii), (iii).

(i) The first equality: Let $0 < t < 1$. Then RHS $= \sup_{\|y\| \leq t} \|T(y/t)\| = \sup_{\|y\| \leq t} (\|Ty\|/t)$ (since $\underline{T \text{ linear}}$) $= (1/t) \sup_{\|y\| \leq t} \|Ty\| \leq \text{LHS}/t$. Now take $t \to 1$.

Do stuff on $\mathbb{R}^*$. Also that bit on monotone functions and taking sup's.

The second equality: We show that LHS $\leq$ RHS. Let $\|x\| \leq 1$. If $x = 0$, then $\|Tx\| = 0 \leq$ RHS (since $\underline{E \neq 0}$ and $\underline{T \text{ linear}}$). If $x \neq 0$, then $\|Tx\| = \left\| T \frac{x}{\|x\|} \right\| \|x\|$ (since $\underline{T \text{ linear}}$) $\leq \left\| T \frac{x}{\|x\|} \right\| \leq$ RHS.

The third equality: Just note that $\left\{ \frac{\|Tx\|}{\|x\|} : x \neq 0 \right\} = \{\|Tx\| : \|x\| = 1\}$ (since $\underline{T \text{ linear}}$).

(ii) $\|T\|$ is such: Firstly, $\|T\| \geq \|T0\| = 0$ (since $\underline{T \text{ linear}}$). Now, let $x \in E$. Without loss of generality, let $x \neq 0$ and hence $E \neq 0$, so that $\underline{\text{by (i)}}$, $\|T\| \geq \frac{\|Tx\|}{\|x\|}$.

$\|T\|$ is smallest such: Let $M \geq 0$ be such. If $E = 0$, then $\|T\| = 0 \leq M$, and if $E \neq 0$, then $\underline{\text{by (i)}}$ $\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} \overset{\text{w}}{\leq} M$.

(iii) (b) $\Rightarrow$ (c): Since $T0 = 0$ ($\underline{T \text{ linear}}$), take $\delta > 0$ such that $\|Tx\| < 1$ whenever $\|x\| < \delta$. Now $\underline{\text{by (i)}}$, $\|T\| = \sup_{\|x\| < 1} \|Tx\| = \sup_{\|x\| < \delta} \|T(x/\delta)\| = (1/\delta) \sup_{\|x\| < \delta} \|Tx\|$ (since $\underline{T \text{ linear}}$) $\leq 1/\delta$.

(c) $\Rightarrow$ (a): Just note that $\|Ty - Tx\| = \|T(y - x)\|$ (since $\underline{T \text{ linear}}$) $\leq \|T\|\|y - x\|$. □

*Remark.* Apart from the usage of additivity in (iii)'s "(b) $\Rightarrow$ (a)", full power of linearity is not used, just homothety. (In fact, a similar analysis can be carried out even for general homogenous functions.)

**Lemma 1.3.** *A normed linear space is complete* $\iff$ *convergence in norm implies convergence.*

*Proof.* "$\Rightarrow$": Since convergence in norm implies Cauchy.

"$\Leftarrow$": Let $(x_i)$ be Cauchy. Define $y_i := x_{i+1} - x_i$ and consider the telescoping series $\sum_i y_i$, which is convergent $\iff$ $(x_i)$ is convergent. Thus it suffices to have that $\sum_i y_i$ be convergent in norm, which can be guaranteed by assuming without loss of generality that $\|x_{i+1} - x_i\| \leq 1/2^i$ (since a Cauchy sequence is convergent if any of its subsequences converge). □

[3] In fact, it is an extended norm on $\mathcal{L}(E, F)$.

# 2. Completion of Normed Linear Spaces

A **completion** of $E$ is a linear isometry $\iota\colon E \to \hat{E}$ into a Banach space $\hat{E}$ such that any linear isometry $E \to F$ into a Banach space $F$ factors uniquely through a linear isometry via $\iota$:

$$\begin{tikzcd}
E \arrow[rr] \arrow[dr, "\iota"'] & & F \\
& \hat{E} \arrow[ur, dashed] &
\end{tikzcd}$$

**Corollary 2.1.** id: $\mathscr{B} \to \mathscr{B}$ *is a completion of* $\mathscr{B}$.

The following lemma gives a means to induce a Banach space structure on a complete

**Lemma 2.2.** *Let $E$ be a dense subset of a complete metric space $X$ with its norm being uniformly equivalent to the metric inherited from $X$. Then there's a unique Banach space structure on $X$, topologically equivalent to the original metric on $X$, that extends the normed linear space structure on $E$. Further, the extended norm is uniformly related to the original metric on $X$ in the same way as the original norm was to the restricted metric on $E$.*

*Proof.* Firstly, we show the uniqueness: Let $X$ admit a Banach space structure as said. Denote $X$ with its original metric topology by $X_{\mathrm{mtr}}$ and $X$ with the topology due to the extended norm by $X_{\mathrm{nrm}}$.[4] Since extended addition, scalar multiplication and norm are continuous functions respectively on $X_{\mathrm{nrm}} \times X_{\mathrm{nrm}}$, $K \times X_{\mathrm{nrm}}$ and $X_{\mathrm{nrm}}$ to Hausdorff domains ($X_{\mathrm{nrm}}$, $X_{\mathrm{nrm}}$ and $[0, +\infty)$ respectively), which are determined on dense subsets thereof ($E \times E$, $K \times E$ and $E$ respectively, for $\underline{E \text{ is dense in } X_{\mathrm{mtr}} \overset{\mathrm{w}}{=} X_{\mathrm{nrm}}}$), these are uniquely determined.

Next we show existence: Denote $E$ with the metric induced by its norm by $E_{\mathrm{nrm}}$, and $E$ with the metric inherited from $X$ by $E_X$.[5] Addition, scalar multiplication and norm on $E$ admit (unique) continuous extensions to $X$ because of the following facts:

(i) $\underline{X \text{ is complete}}$ and so is $[0, +\infty)$.

(ii) Addition, scalar multiplication and norm are Cauchy-regular on $E_{\mathrm{nrm}} \times E_{\mathrm{nrm}} \to E_{\mathrm{nrm}}$, $K \times E_{\mathrm{nrm}} \to E_{\mathrm{nrm}}$, and $E_{\mathrm{nrm}} \to [0, +\infty)$ respectively.

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[4] Thus, $X_{\mathrm{mtr}}$, $X_{\mathrm{nrm}}$ are topological spaces.

[5] Hence, $E_{\mathrm{nrm}}$, $E_X$ are metric space.

(iii) Since $E_{\text{nrm}}$ is uniformly equivalent to $E_X$, the above holds with "$E_{\text{nrm}}$" replaced with "$E_{\mathrm{X}}$".[6]

(iv) $E$ is dense in $X$ (so that $E \times E$ and $K \times E$ are respectively dense in $X \times X$ and $K \times X$).

That these extended functions endow $X$ with a normed linear space structure is straightforward to verify by representing generic elements of $X$ as limits of sequences in $E$, using continuity of the extended functions, and the normed linear space structure of $E$. It's clear that this will then be an extension of the normed linear space structure of $E$.

*CC used; avoidable if $X$ is $E$ is countable.*

Finally, the preservation of uniform relation is also shown similarly, and from this it follows that $X_{\text{metr}}$ is uniformly equivalent to $X_{\text{nrm}}$ so that completeness of $X_{\text{metr}}$ implies that of $X_{\text{nrm}}$, concluding that the extended functions above indeed form a Banach space extension of $E$. □

*Remark.* Note that for uniqueness, completeness of $X$ wasn't required.

**Proposition 2.3.** *Any metric completion of $E$ admits a Banach space structure becoming a normed linear space completion of $E$, with norm recovering the metric.*

*Proof.* Let $\iota\colon E \to \hat{E}$ be a metric space completion of $E$. Now, transport the normed linear space structure of $E$ to $\iota(E)$ via $\iota$. Consider the following facts:

(i) $\hat{E}$ is complete.

(ii) $\iota(E)$ is dense in $\hat{E}$.

(iii) $d(\iota(x), \iota(y)) = d(x, y) = \|x - y\|$ for any $x, y \in X$.

Due to Lemma 2.2, we have that $\hat{X}$ admits a Banach space structure extending the normed linear space structure on $\iota(E)$ such that $d(\hat{x}, \hat{y}) = \|\hat{x} - \hat{y}\|$.[7] That $\iota$ is linear follows from definition of the normed linear space structure on $\iota(E)$. We now verify the universal property.

Let $f\colon E \to F$ be a linear isometry with $F$ complete. By the universal property of metric space completion, there exists a unique isometry $\tilde{f}$ factoring $f$ through $\iota$:

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\iota"'] & & F \\ & \hat{E} \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

[6] "Finite products of uniformly equivalent metrics are uniformly equivalent": Use the fact that for $i = 1, \dots, n$, if $\alpha_i |y_i| \le |x_i| \le \beta_i |y_i|$ for $\alpha_i, \beta_i > 0$, then $(\min_i \alpha_i)\|y\|_\infty \le \|x\|_\infty \le (\max_i \beta_i)\|y\|_\infty$.

[7] We have abused notation, denoting the extended norm by the same symbol.

It suffices to show that $\tilde{f}$ is linear, which follows by representing points in $\hat{E}$ as limits of sequences in $\iota(E)$ (for $\iota(E)$ is dense in $\hat{E}$), and using continuity of $\tilde{f}$ and linearity of $f$. ∎

*CC used unless E is countable.*

**Corollary 2.4.** *Any normed linear space admits a completion which is unique up to isometric isomorphisms.*

# 3. Applying BCT

**Proposition 3.1.** *A Banach space can't have a countably infinite Hamel dimension.*

*Proof.* Let $e_1, e_2, \dots$ form a Hamel basis for $\mathscr{B}$. Then $\mathscr{B} = \bigcup_{i=0}^{\infty} \operatorname{span}\{e_1, \dots, e_i\}$. By Baire's category (since $\mathscr{B}$ is complete), some $\operatorname{span}\{e_1, \dots, e_i\}$ contains a ball, say $B_r(0)$, without loss of generality. But then $te_{n+1} \in B_r(0) \overset{\text{w}}{\subseteq} \operatorname{span}\{e_1, \dots, e_n\}$ for small enough $t$, a contradiction. ∎

*Are finite-dimensional subspaces of a TVS closed?*

**Theorem 3.2** (Open mapping)**.** *Any continuous surjective linear map between Banach spaces is open.*

*Proof.* Let $T \in \mathcal{L}_c(\mathscr{B}, \mathscr{B}')$. It suffices to have that $B'_\varepsilon(0) \subseteq T(B_1(0))$ for some $\varepsilon > 0$:

Let $U$ be open in $\mathscr{B}$ and $x \in U$. We find an $r > 0$ such that $B'_r(Tx) \subseteq T(U) \overset{\text{w}}{\Longleftarrow} B'_r(0) \subseteq T(U - x) \overset{\text{w}}{\Longleftarrow} B'_\varepsilon(0) \subseteq T(\varepsilon/r(U - x))$ where the last two implications follow since $T$ is linear. Now, this is true if $B_1(0) \subseteq \varepsilon/r(U - x) \overset{\text{w}}{\Longleftarrow} B_{r/\varepsilon}(x) \subseteq U$, and a small enough $r$ can be chosen to ensure this.

Since $T$ is surjective, $\mathscr{B}' = \bigcup_n T(B_n(0))$. By Baire's category (since $\mathscr{B}'$ Banach), let $B'_\delta(y_1) \subseteq \overline{T(B_n(0))} \overset{\text{w}}{\Longrightarrow} B'_\delta(0) \subseteq \overline{T(B_{2n}(0))}$:

We have $B'_\delta(0) = B'_\delta(y_1) - y_1 \subseteq B'_\delta(y_1) + B'_\delta(y_1) \subseteq \overline{T(B_n(0))} + \overline{T(B_n(0))} \subseteq \overline{T(B_n(0)) + T(B_n(0))}$ (see footnote[8]) $= \overline{T(B_n(0) + B_n(0))}$ (since $T$ is linear) $= \overline{T(B_{2n}(0))}$.

Since $T$ is linear, we get $B'_\varepsilon(0) \subseteq \overline{T(B_{1/2}(0))}$ for $\varepsilon := \delta/4n$. Thus it suffices to show that $\overline{T(B_{1/2}(0))} \subseteq T(B_1(0))$:

Let $y \in$ LHS. Then choose $x_1 \in B_{1/2}(0)$ such that $y - Tx_1 \in B'_{\varepsilon/2}(0) \overset{\text{w}}{\subseteq} \overline{T(B_{1/4}(0))}$ (again using linearity of $T$). Now, choose $x_2 \in B_{1/4}(0)$ such that

[8] $\overline{A} + \overline{B} = +(\overline{A} \times \overline{B}) = +(\overline{A \times B}) \subseteq \overline{+(A \times B)} = \overline{A + B}$.

$y - Tx_1 - Tx_2 \in B'_{\varepsilon/4}(0) \overset{\mathrm{w}}{\subseteq} \overline{T(B_{1/8}(0))}$ (again using linearity of $T$), and so on. . . DC used.

Now, since $\mathscr{B}$ is Banach, the series $\sum_i x_i$ converges to an $x \in B_1(0)$. Once again using linearity of $T$, we get $\left\|y - T\left(\sum_{i=1}^n x_i\right)\right\| < \varepsilon/2^n$. Using continuity of $T$, this finally yields $y = Tx$. $\square$

*Remark.* Necessity of surjectivity is easily seen. For linearity, think of a cubic polynomial $\mathbb{R} \to \mathbb{R}$.

Necessity of continuity? completeness of domain? of codomain?

**Corollary 3.3** (Bounded inverse)**.** *The inverse of an invertible continuous linear map between Banach spaces is continuous.*

*Remark.* content

CHAPTER III

# Inner Product Spaces

**Conventions.** Unless stated otherwise, assume the following:

- Inner product spaces will be over $K$.
- $V$, $W$ will denote inner product spaces and $\mathscr{H}$, a Hilbert space.
- Subspaces of inner product spaces will be considered inner product spaces.
- If $X$ is an inner product space and $x \in E$, then we'll use $\|x\| := \sqrt{\langle x, x\rangle}$. (See Corollary 1.3.)
- Inner product spaces will be considered normed linear space under the induced norm (see Corollary 1.3).

## 1. Basics

*Remark.* Inner product is always required to take values in the base field of the vector space since we want $\langle x, y\rangle z$ to always make sense. Many proofs will depend on this faculty: See for instance, the proof of Cauchy-Schwarz (Proposition 1.2).

For $x, y \in V$, we have $\|x+y\|^2 = \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2$. This yields:

**Lemma 1.1** (Parallelogram law and polarization)**.** *Let $x, y \in V$. Then we have the parallelogram law:*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{1.1}$$

*We also have the polarization identities: If $K = \mathbb{R}$, then*

$$\langle x, y\rangle = \frac{\|x+y\|^2 - \|x-y\|^2}{4} \tag{1.2a}$$

*and if* $K = \mathbb{C}$*, then*

$$\langle x, y\rangle = \frac{\|x+y\|^2 - \|x-y\|^2}{4} + i\frac{\|x+iy\|^2 - \|x-iy\|^2}{4}. \tag{1.2b}$$

**Proposition 1.2** (Cauchy-Schwarz)**.** *For* $x, y \in V$*, we have* $|\langle x, y\rangle| \le \|x\|\|y\|$ *with equality holding iff* $x$*,* $y$ *are linearly dependent.*

*Proof.* Without loss of generality, let $y \neq 0$. Now, simply observe that[1] $0 \le \left\| x - \frac{\langle x,y\rangle}{\|y\|^2} y \right\|^2 = \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2}$ with equality holding iff $x = \frac{\langle x,y\rangle}{\|y\|^2} y$. □

This gives the triangle inequality for $\|\cdot\|$, so that:

**Corollary 1.3.** $\|\cdot\|$ *defines a norm on* $V$*.*

Now we prove a converse:

**Proposition 1.4.** *Any norm that satisfies the parallelogram law (Eq. (1.1)) comes from an inner product.*

*Proof.* Define $\langle\cdot,\cdot\rangle$ using polarization (Eq. (1.1)). We first show that this indeed defines an inner product, and then show that the norm that this inner product induces is nothing but what we started with. The only hard part is showing the linearity of $\langle\cdot,\cdot\rangle$ in the first argument.

Additivity: Using parallelogram law, verify that $\langle x, z\rangle + \langle y, z\rangle = \langle x + y, 2z\rangle/2$. Thus it suffices to show that $\langle u, 2v\rangle = 2\langle u, v\rangle$. Putting $x \to u + v$ and $y \to v$ in Eq. (1.1), get

$$\|u + 2v\|^2 + \|u\|^2 = 2\|u + v\|^2 + 2\|v\|^2$$

in which putting $v \to -v$ yields

$$\|u - 2v\|^2 + \|u\|^2 = 2\|u - v\|^2 + 2\|v\|^2.$$

Subtracting these, yield

$$\|u + 2v\|^2 - \|u - 2v\|^2 = 2\|u + v\|^2 - 2\|u - v\|^2$$

which is the content of $\langle u, 2v\rangle = 2\langle u, v\rangle$.

Homothety: Conclude that $\langle rx, y\rangle = r\langle x, y\rangle$ for any rational $r$. Then use continuity of the map $\alpha \mapsto \langle \alpha x, y\rangle$ to deduce that it remains valid even for real $r$'s. Finally observe that if $K = \mathbb{C}$, then $\langle ix, y\rangle = i\langle x, y\rangle$ and thus validity extends to complex $r$'s as well. □

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[1] Note that $\frac{\langle x,y\rangle}{\|y\|^2}$ is the "projection" of $x$ on $y$.

*Remark.* The only place where the parallelogram law is used is in proving the additivity. Except homothety, all the reasoning has been "algebraic".

Thus we get a correspondence:

**Corollary 1.5.** *On any vector space $V$ over $K$, we have the following correspondence:*

$$\{\text{inner products on } V\} \longleftrightarrow \left\{\begin{matrix}\text{norms on } V \text{ satisfying} \\ \text{the parallelogram law}\end{matrix}\right\}$$

## 2. Orthogonal Complements and Projections

Let $x, y \in V$. Then $x$ is said to be **orthogonal** or **perpendicular** to $y$, denoted $x \perp y$ iff $\langle x, y\rangle = 0$. For any $E \subseteq V$, we define the **orthogonal complement** $E^\perp$ of $E$ to be the set of all the vectors of $V$ that are orthogonal to each vector in $E$.

**Lemma 2.1** (Pythagoras)**.** *For orthogonal vectors $x$, $y$ in $V$, we have $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.*

Since $E^\perp = \cap_{y\in E}\{x \in V : \langle x, y\rangle = 0\}$, we have:

**Lemma 2.2.** *Orthogonal complements are closed subspaces.*

In the context of inner product spaces, we call the vectors of best approximations as **projections**.

**Proposition 2.3** (Characterizing projections on subspaces)**.** *Let $M$ be a subspace of $V$ and $x_0 \in V$. Then for $y_0 \in M$, the following are equivalent:*

*(i)* $y_0 - x_0 \in M^\perp$.

*(ii)* $y_0$ *is a projection of* $x_0$ *onto* $M$.

Note that (ii) just says that $\|y_0 - x_0\| = \inf_{y\in M}\|y - x_0\|$.

*Proof.* (i) $\Rightarrow$ (ii): If $y \in M$, then $\|y - x_0\|^2 = \|y - y_0\|^2 + \|y_0 - x_0\|^2$. (Since $\underline{M \text{ is a subspace}}$, $y - y_0 \in M$ and is thus perpendicular to $y_0 - x_0$.) *Add diagram.*

(ii) $\Rightarrow$ (i): Let $y \in M$. Without loss of generality, let $\|y\| = 1$ (since $\underline{M \text{ is a subspace}}$). Now, $\|y_0 - x_0\| \le \|y_0 - x_0 - \langle y_0 - x_0, y\rangle y\|$ (since $M$ is a subspace, $y_0 - \langle y_0 - x_0, y\rangle y \in M$) $\overset{\text{w}}{\Longrightarrow} |\langle y_0 - x_0, y\rangle|^2 = 0 \overset{\text{w}}{\Longrightarrow} y_0 - x_0 \perp y$. $\square$ *Add diagram.*

We now ask for uniqueness of projections:

**Proposition 2.4.** *Any vector of an inner product space has at most one projection on a convex subset of it.*

*Proof.* Let $E$ be the set under consideration and $x_0$ the vector to be projected. Note that translations respect projections and convexity. Thus, without loss of generality, take $x_0 = 0$.[2] Set $\delta := \inf_{y \in E} \|y\|$ and take $y_1, y_2 \in E$ with $\|y_1\| = \delta = \|y_2\|$. Then by the parallelogram law, we have

$$\begin{aligned} \|y_1 - y_2\|^2 &= 2\|y_1\|^2 + 2\|y_2\|^2 - \|y_1 + y_2\|^2 \\ &\leq 4\delta^2 - 4\left\|\frac{y_1}{2} + \frac{y_2}{2}\right\|^2 \\ &\leq 0 \end{aligned}$$

where the last inequality follows since $y_1/2 + y_2/2 \in E$ as $E$ is convex. □

*Remark.* (i) The necessity of convexity is evident by considering an annulus. However, full convexity is not really required: We just need that $E$ is closed under taking midpoints. (ii) Necessity of the parallelogram law: Consider $\mathbb{R}^2$ and take $E := \{(x, y) : y \geq 1\}$. Then all the points $(x, 1)$ with $|x| \leq 1$ are projections of 0 onto the convex $E$.

Thus, we can talk of **projection functions onto convex subsets**. Since subspaces are convex, by Proposition 2.3, we get:

**Corollary 2.5.** *If existent, the projection onto a subspace is linear.*

Now, for the existence of projections:

**Theorem 2.6.** *Any vector of a Hilbert space can be projected onto a nonempty closed convex subset of it.* Show necessity of Hilbert-ness.

*Proof.* Let $E$ be the subset in consideration. Since translations are homeomorphisms that preserve projections and convexity, we can without loss of generality take the vector to be projected to be 0. We thus need to find a vector of smallest norm in $E$.

[2] Note that we couldn't reduce the analysis similarly in the proof of 2.3 since translations don't preserve "being a subspace".

Set $\delta := \inf_{y \in E} \|y\|$ (which is finite since $\underline{E \neq \emptyset}$). For $n \geq 1$, choose $y_n \in E$ such that $\|y_n\|^2 < \delta^2 + 1/n$. Now, by parallelogram law,

AC used.

Is parallelogram law required?

$$
\begin{aligned}
\|y_m - y_n\|^2 &= 2\|y_1\|^2 + 2\|y_2\|^2 - \|y_1 + y_2\|^2 \\
&< 4\delta^2 + 2\left(\frac{1}{m} + \frac{1}{n}\right) - 4\left\|\frac{y_1}{2} + \frac{y_2}{2}\right\|^2 \\
&\leq 2\left(\frac{1}{m} + \frac{1}{n}\right)
\end{aligned}
$$

where the last inequality follows as before due to convexity of $E$. Thus, $(y_n)$ is Cauchy. Since $E$ is a closed subset of a Hilbert space, $(y_n)$ converges to some $y \in E$. Now, $\|y\|^2 = \lim_n \|y_n\|^2 \leq \delta^2 \overset{\text{w}}{\Longrightarrow} \|y\| = \delta$ as required. □

*Remark.* (i) Convexity is not required for finite dimensions for we can use Heine–Borel to find a vector of smallest length inside closed nonempty $E$. (ii) To show necessity of convexity (in infinite dimensions), consider $(1 + 1/n)e_n$'s for $n = 1, 2, \ldots$ where $e_n$'s are orthonormal. (iii) Again, not the full power of convexity is used, just closure under taking midpoints; however, closure together with this property implies convexity.

Prove Heine-Borel.

**Corollary 2.7** (Orthogonal decomposition of Hilbert spaces)**.** *Let $M$ be a closed subspace of $\mathscr{H}$. Then*

$$\mathscr{H} = M \oplus M^{\perp}.$$

Show necessity of 1. Hilbert-ness, and 2. closure of $M$.

*Proof.* Let $x \in \mathscr{H}$. By Theorem 2.6 ($M$ is closed and convex (since $M$ a subspace) and $\mathscr{H}$ Hilbert), pick a projection $y$ of $x$ in $M$. By Proposition 2.3 ($M$ a subspace), $y - x \in M^{\perp}$ so that $x = y + (x - y) \in M + M^{\perp}$.

For uniqueness, let $x_1, x_2 \in M$ and $y_1, y_2 \in M^{\perp}$ such that $x_1 + y_1 = x_2 + y_2 \overset{\text{w}}{\Longrightarrow} M \ni x_1 - x_2 = y_2 - y_1 \in M^{\perp}$ (since $M$ a subspace) $\overset{\text{w}}{\Longrightarrow} x_1 - x_2, y_2 - y_1 \in M \cap M^{\perp} = \{0\} \overset{\text{w}}{\Longrightarrow} x_1 = x_2$ and $y_1 = y_2$. □

**Corollary 2.8.** *For any subset $E$ of $\mathscr{H}$, we have*

$$(E^{\perp})^{\perp} = \overline{\operatorname{span} E}.$$

Show necessity of completeness.

*Proof.* "$\supseteq$": LHS is a closed subspace and $E \subseteq$ LHS.

"$\subseteq$": Let $x \in$ LHS. Set $M := \overline{\operatorname{span} E}$ and by Corollary 2.7, let $y \in M$ such that $y - x \in M^{\perp} \overset{\text{w}}{\subseteq} E^{\perp} \overset{\text{w}}{\Longrightarrow} \langle y - x, x \rangle = 0 \overset{\text{w}}{\Longrightarrow} \|y - x\| = 0$ (since $y \perp y - x$) $\overset{\text{w}}{\Longrightarrow} x = y \overset{\text{w}}{\in} M$. □

*Remark.* Note that "$\supseteq$" holds irrespective of completeness.

# 3. Adjoints

Every vector $x \in V$ induces a continuous linear functional $\langle \cdot, x\rangle \in V^*$. It's easy to see that this is an injective anti-linear isometry $V \to V^*$. The following says that for Hilbert spaces, it's surjective as well:

**Theorem 3.1** (Riesz representation)**.** *Continuous linear functionals on a Hilbert space are obtained from inner products.*

Show necessity of Hilbert-ness.

*Proof.* Let $\ell \in \mathscr{H}^*$. If $\ker \ell = \mathscr{H}$, then $\ell = 0 = \langle \cdot, 0\rangle$. Otherwise, $(\ker \ell)^\perp \neq 0$:

Since $\underline{\mathscr{H} \text{ Hilbert}}$ and $\ker \ell$ a closed subspace (as $\underline{\ell \text{ continuous linear}}$), $\mathscr{H} = (\ker \ell) \oplus (\ker \ell)^\perp$.

Thus, take a unit vector $e \in (\ker \ell)^\perp$. Now, take an arbitrary $x \in \mathscr{H}$. Then $x - \frac{\ell(x)}{\ell(e)} e \in \ker \ell$ (since $\underline{\ell \text{ linear}}$) so that it's perpendicular to $e \overset{\text{w}}{\implies} \ell(x) = \langle x, \overline{\ell(e)} e\rangle$. $\square$

*Remark.* Note that the proof relies solely on finding a nonzero vector in $(\ker \ell)^\perp$.

**Corollary 3.2.** *$\mathscr{H}^*$ is isometrically and anti-linearly isomorphic to $\mathscr{H}$.*

The **adjoint** of a function $f\colon V \to W$ is a function $f^*\colon W \to V$ such that

$$\langle f(x), y\rangle = \langle x, f^*(y)\rangle$$

for all $x \in V$ and $y \in W$. If existent, such a function is clearly unique which justifies the notation. We enlist some immediate algebraic properties of adjoints:

**Corollary 3.3.**

*(i) Let $f, g\colon V \to W$ admit adjoints. Then $f^*$ is linear and $f^*$, $f+g$, $\alpha f$ (for any scalar $\alpha$), $g \circ f$ admit adjoints:*

$$(f^*)^* = f$$
$$(f+g)^* = f^* + g^*$$
$$(\alpha f)^* = \overline{\alpha} f^*$$

*(ii) Let $f\colon V_1 \to V_2$ and $g\colon V_2 \to V_3$ admit adjoints. Then $g \circ f$ also admits adjoint:*

$$(g \circ f)^* = f^* \circ g^*$$

*Thus, if $f$ is invertible with $f^{-1}$ admitting adjoint too, then $f^*$ is invertible with*

$$(f^{-1})^* = (f^*)^{-1}.$$

Inverse of an adoint-able function is adjoint-able?

*Remark.* Hence, only linear maps can possibly admit adjoints.

**Proposition 3.4.** *Let $T \in \mathcal{L}(V, W)$ admit adjoint. Then the following hold:*

$$\|T^*\| = \|T\|$$
$$\|T^*T\| = \|T\|^2$$

*Proof.* For any $y$, we have $\|T^*y\|^2 = \langle T^*y, T^*y\rangle = \langle TT^*y, y\rangle \leq \|T\|\|T^*y\|\|y\| \overset{\text{w}}{\Longrightarrow} \|T^*y\| \leq \|T\|\|y\|$. Thus $\|T^*\| \leq \|T\|$. Putting $T \to T^*$ yields $\|T^*\| = \|T\|$.

For second, we already have $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$. Now, for any $x$, we have $\|Tx\|^2 = \langle T^*Tx, x\rangle \leq \|T^*T\|\|x\|^2 \overset{\text{w}}{\Longrightarrow} \|Tx\| \leq \|T^*T\|^{1/2}\|x\|$. Thus $\|T\| \leq \|T^*T\|^{1/2}$. □ *Taking square roots in $\mathbb{R}^*$.*

Thus adjoints of continuous linear maps are continuous. Now we talk about the existence of adjoints:

**Theorem 3.5.** *Continuous linear maps between Hilbert spaces admit adjoints.*

*Proof.* Let $T \in \mathcal{L}_c(\mathscr{H}, V)$. Fix a $y \in V$. Now, $x \mapsto \langle Tx, y\rangle$ is a continuous linear functional on $\mathscr{H}$ (since $\underline{T \text{ continuous}}$) so that by Riesz (since $\underline{\mathscr{H} \text{ is Hilbert}}$), let $Sy$ be the unique vector in $\mathscr{H}$ such that $\langle Tx, y\rangle = \langle x, Sy\rangle$ for each $x \in \mathscr{H}$. This gives a function $S \colon V \to \mathscr{H}$, which is the required adjoint. □