Contents

Conventions

Conventions. Unless stated otherwise, we'll use the following:

- For vector spaces V, W over a common field, $\mathcal{L}(V, W)$ will denote the set of linear maps $V \to W$ and will be considered a vector space. $\mathcal{L}(V, V)$ will be considered an associative algebra.
- For a vector space V and a set X, the set F^X will be considered a vector space (over the same field as that of V).
- $\mathcal{C}(X, Y)$ will denote the set of continuous functions $X \to Y$ if X, Y are topological spaces.

CHAPTER I

Topological Vector Spaces

Conventions. Unless stated otherwise, the following will always be assumed:

- $K \in \{\mathbb{R}, \mathbb{C}\}$.
- Topological vector spaces will be over K.
- X, Y will stand for TVS's over K.
- For $x, y \in X$, we'll use the $[x; y]$, $[x; y)$, *etc.* notation.
- Subspaces of TVS's will be seen as TVS's.
- $\mathcal{L}_c(X, Y)$ will stand for the set of continuous linear maps $X \to Y$,[1] which will be seen as a vector space.
- X^* will stand for $\mathcal{L}_c(X, K)$.

1. Basics

Lemma 1.1. *Closure of a subspace is a subspace.*

Proposition 1.2. *Any proper subspace of a TVS has empty interior, or equivalently, has dense complement.*

Proof. Let M be a proper subspace of X. It suffices to show that $X \setminus M$'s closure is whole of X, that is, each point of X is a limit of some sequence in $X \setminus M$. Let $x \in X$. Without loss of generality, let $x \in M$. Now, take a $y \notin M$ (since M proper). Then x is the limit of the sequence $(x + y/n)_n \in X \setminus M$ as required. $\square$

[1]Note that X and Y are over the *same* field K.

CHAPTER II

Normed Linear Spaces

Conventions. Unless stated otherwise, assume the following:

- A normed linear space will be over K.
- A norm's codomain will be taken to be $[0, +\infty)$.
- E, F will denote normed linear spaces and $\mathscr{B}$ will be reserved for a Banach space.
- Subspaces of normed linear spaces will be seen as normed linear spaces.
- A normed linear space will also be considered a metric space under the induced metric, and also a topological vector space (see 1.1).
- Abusing notation, we'll use the same notation to denote the restriction to $\mathbb{R} \to \mathbb{R}$ of Re, Im, and complex conjugation.
- $\mathcal{B}(X, F)$ will denote the set of bounded functions $X \to F$ for a set X. Further, for $f \in \mathcal{B}(X, F)$, we'll use $\|f\|_\infty := \sup_{x \in X} \|f(x)\|$. $\mathcal{B}(X, F)$ will be seen as a normed linear space (see 1.1.)
- $\mathcal{C}_b(X, F)$ will be the set of bounded continuous functions for a topological vector space X, which will be seen as a normed linear space (see 1.1.)
- For $T \in \mathcal{L}(E, F)$, we'll set $\|T\| := \sup_{\|x\| \leq 1} \|Tx\|$.[1]
- $\mathcal{L}_c(E, F)$ will be seen as a normed linear space (see 1.2).

[1] Actually, one can extend this definition to homogenous functions to still yield fruitful consequences (see the remark after 1.2). Note that homogenous functions of a fixed degree also form a vector space.

1. Basics

First, some immediate trivialities:

Lemma 1.1. *(i) A normed linear space is a topological vector space. In fact, addition is Lipschitz continuous and multiplication is Cauchy-regular*[2]*.*

(ii) Norm is Lipschitz continuous.

(iii) $\overline{B_r(x)} = D_r(x)$ *for any* $x \in E$*.*

(iv) $\|\cdot\|_\infty$ *defines a norm on* $\mathcal{B}(X, F)$*, convergence under which coincides with uniform convergence.*

(v) $\mathcal{C}_b(X, F)$ *is a closed subspace of* $\mathcal{B}(X, F)$ *for any set* X*.*

Proof. For (i), use 1-norm metric on $X \times X$ for addition and use the usual trickery for multiplication.

For (iii), note that for any $y \in D_r(x)$, the segment $[x; y)$ contains points common to $B_r(x)$ and $B_\varepsilon(y)$ for ε however small.

For (v), just note that uniform limit of continuous functions is continuous. □

Remark. Note that scalar multiplication needn't even be uniformly continuous, let alone Lipschitz; for instance, consider the multiplication on $\mathbb{R}$: Take $x_n := (\sqrt{n}, \sqrt{n})$ and $y_n := (\sqrt{n+1}, \sqrt{n+1})$. Then under the metric on $\mathbb{R} \times \mathbb{R}$ due to ∞-norm, $d(x_n, y_n) = \sqrt{n-1} - \sqrt{n} = (\sqrt{n+1} + \sqrt{n})^{-1} \to 0$ as $n \to \infty$ and yet $d(n, n+1) = 1 \not\to 0$.

However, scalar multiplication, keeping either coordinate fixed, is trivially Lipschitz.

Proposition 1.2 (Norm on $\mathcal{L}_c(E, F)$)**.** *Let* $T \in \mathcal{L}(E, F)$*. Then the following hold:*

(i) $\sup_{\|x\|<1} \|Tx\| = \sup_{\|x\|\leq 1} \|Tx\| \overset{(*)}{=} \sup_{\|x\|=1} \|Tx\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$ *where* $(*)$ *holds for* $E \neq 0$*.*

(ii) $\|T\|$ *is the smallest* $M \geq 0$ *such that* $\|Tx\| \leq M\|x\|$ *for all* $x \in E$*.*

(iii) The following are equivalent:

(a) T *is continuous.*

(b) T *is continuous at* 0*.*

(c) $\|T\| < +\infty$*.*

[2]Remember, finite product of metric spaces (and hence normed linear spaces) are considered under any of the uniformly equivalent p-norm metrics.

(iv) $\|\cdot\|$ *defines a norm on* $\mathcal{L}_c(E,F)$.[3]

(v) If G *is another normed linear space and* $S \in \mathcal{L}(F,G)$*, then* $\|ST\| \le \|S\|\|T\|$.

Proof. (iv), (v) are easy consequences of (i), (ii), (iii).

(i) The first equality: Let $0 < t < 1$. Then RHS $= \sup_{\|y\|\le t}\|T(y/t)\| = \sup_{\|y\|\le t}(\|Ty\|/t)$ (since $\underline{T \text{ linear}}$) $= (1/t)\sup_{\|y\|\le t}\|Ty\| \le \text{LHS}/t$. Now take $t \to 1$.

The second equality: We show that LHS $\le$ RHS. Let $\|x\| \le 1$. If $x = 0$, then $\|Tx\| = 0 \le$ RHS (since $\underline{E \ne 0}$ and $\underline{T \text{ linear}}$). If $x \ne 0$, then $\|Tx\| = \left\|T\frac{x}{\|x\|}\right\|\|x\|$ (since $\underline{T \text{ linear}}$) $\le \left\|T\frac{x}{\|x\|}\right\| \le$ RHS.

Do stuff on $\mathbb{R}^*$*. Also that bit on monotone functions and taking sup's.*

The third equality: Just note that $\left\{\frac{\|Tx\|}{\|x\|} : x \ne 0\right\} = \{\|Tx\| : \|x\| = 1\}$ (since $\underline{T \text{ linear}}$).

(ii) $\|T\|$ is such: Firstly, $\|T\| \ge \|T0\| = 0$ (since $\underline{T \text{ linear}}$). Now, let $x \in E$. Without loss of generality, let $x \ne 0$ and hence $E \ne 0$, so that $\underline{\text{by (i)}}$, $\|T\| \ge \frac{\|Tx\|}{\|x\|}$.

$\|T\|$ is smallest such: Let $M \ge 0$ be such. If $E = 0$, then $\|T\| = 0 \le M$, and if $E \ne 0$, then $\underline{\text{by (i)}}$ $\|T\| = \sup_{x\ne 0}\frac{\|Tx\|}{\|x\|} \overset{\text{w}}{\le} M$.

(iii) (b) $\Rightarrow$ (c): Since $T0 = 0$ ($\underline{T \text{ linear}}$), take $\delta > 0$ such that $\|Tx\| < 1$ whenever $\|x\| < \delta$. Now $\underline{\text{by (i)}}$, $\|T\| = \sup_{\|x\|<1}\|Tx\| = \sup_{\|x\|<\delta}\|T(x/\delta)\| = (1/\delta)\sup_{\|x\|<\delta}\|Tx\|$ (since $\underline{T \text{ linear}}$) $\le 1/\delta$.

(c) $\Rightarrow$ (a): Just note that $\|Ty - Tx\| = \|T(y-x)\|$ (since $\underline{T \text{ linear}}$) $\le \|T\|\|y - x\|$. ☐

Remark. Apart from the usage of additivity in (iii)'s "(b) $\Rightarrow$ (a)", full power of linearity is not used, just homothety. (In fact, a similar analysis can be carried out even for general homogenous functions.)

Corollary 1.3. *Any continuous linear map between normed linear spaces is Lipschitz.*

Lemma 1.4. *A normed linear space is complete* $\iff$ *convergence in norm implies convergence.*

Proof. "$\Rightarrow$": Since convergence in norm implies Cauchy.

"$\Leftarrow$": Let (x_i) be Cauchy. Define $y_i := x_{i+1} - x_i$ and consider the telescoping series $\sum_i y_i$, which is convergent $\iff$ (x_i) is convergent. Thus it suffices to have that $\sum_i y_i$ be convergent in norm, which can be guaranteed by assuming without

[3] In fact, it is an extended norm on $\mathcal{L}(E,F)$.

loss of generality that $\|x_{i+1} - x_i\| \leq 1/2^i$ (since a Cauchy sequence is convergent if any of its subsequences converge). $\square$

2. Completion of Normed Linear Spaces

A **completion** of E is a linear isometry $\iota\colon E \to \hat{E}$ into a Banach space $\hat{E}$ such that any linear isometry $E \to F$ into a Banach space F factors uniquely through a linear isometry via ι:

$$\begin{tikzcd} E \arrow[rr] \arrow[dr, "\iota"'] & & F \\ & \hat{E} \arrow[ur, dashed] & \end{tikzcd}$$

As for metric spaces, we also have another characterization of norm completion via denseness. (See (ii) of 2.7.)

Corollary 2.1. id: $\mathscr{B} \to \mathscr{B}$ *is a completion of* $\mathscr{B}$.

The following lemma gives a means to induce a Banach space structure on a complete metric space from a normed linear space which sits inside it as a dense subset.

Lemma 2.2. *Let E be a dense subset of a complete metric space X with its norm recovering the metric inherited from X. Then there's a unique Banach space structure on X that extends the normed linear space structure on E, and recovers the metric.*

Proof. Firstly, we show uniqueness: Let X admit a Banach space structure as said. Denote X with its original metric by X_{mtr} and X with the metric due to the extended norm by X_{nrm}. Since extended addition, scalar multiplication and norm are continuous functions respectively on $X_{\mathrm{nrm}} \times X_{\mathrm{nrm}}$, $K \times X_{\mathrm{nrm}}$ and X_{nrm} to Hausdorff domains (X_{nrm}, X_{nrm} and $[0, +\infty)$ respectively), which are determined on dense subsets $E \times E$, $K \times E$ and E respectively (determined, because the Banach space structure on X extends the normed linear space structure of E; dense, becasue E is dense in $X_{\mathrm{mtr}} \overset{\mathrm{w}}{=} X_{\mathrm{nrm}}$), these are uniquely determined.

Next we show existence: Denote E with the metric induced by its norm by E_{nrm}, and E with the metric inherited from X by E_{mtr}. Addition, scalar multiplication and norm on E admit (unique) continuous extensions to X because of the following facts:

(i) X is complete and so is $[0, +\infty)$.

(ii) Addition, scalar multiplication and norm are Cauchy-regular on $E_{\mathrm{nrm}} \times E_{\mathrm{nrm}} \to E_{\mathrm{nrm}}$, $K \times E_{\mathrm{nrm}} \to E_{\mathrm{nrm}}$, and $E_{\mathrm{nrm}} \to [0, +\infty)$ respectively.

(iii) Since $\underline{E_{\mathrm{nrm}} = E_{\mathrm{mtr}}}$, the above holds with "$E_{\mathrm{nrm}}$" replaced with "$E_{\mathrm{mtr}}$".

(iv) $\underline{E \text{ is dense in } X}$ (so that $E \times E$ and $K \times E$ are respectively dense in $X \times X$ and $K \times X$).

That these extended functions endow X with a normed linear space structure is straightforward to verify by representing generic elements of X as limits of sequences in E, and using the continuity of the extended functions and the normed linear space structure of E. Finally, the fact that $d(x, y) = \|x - y\|$ for $x, y \in X$ is also shown similarly, and from this, it follows that $X_{\mathrm{metr}} = X_{\mathrm{nrm}}$ so that $\underline{\text{completeness of } X_{\mathrm{metr}}}$ implies that of X_{nrm}, concluding that the extended functions above indeed form a Banach space extension of E. □

CC used; avoidable if E is countable.

Remark. Note that for uniqueness, completeness of X wasn't required.

Lemma 2.3. *Any continuous linear map from a dense subspace to a Banach space has a unique continuous extension to the whole of the domain. Further, the extension is linear.*

Proof. Let E_1 be a dense subspace of E and $f \colon E_1 \to \mathscr{B}$ be continuous and linear. Since f is Lipschitz (see 1.3) and $\underline{\mathscr{B} \text{ Banach}}$, it admits a unique continuous extension $\tilde{f} \colon E \to \mathscr{B}$. Linearity of $\tilde{f}$ follows by representing points in E as limits of sequences in E_1, and using $\underline{\text{continuity and linearity of } f}$. □

CC used unless E_1 is countable.

Proposition 2.4. *Any metric completion of E admits a unique Banach space structure becoming a norm completion of E with the norm recovering the metric.*

Proof. Let $\iota \colon E \to \hat{E}$ be a metric space completion of E.

First, we show uniqueness. Endow $\hat{E}$ with a Banach space structure such that ι becomes a norm completion with $d(\hat{x}, \hat{y}) = \|\hat{x} - \hat{y}\|$. Denote E with its original metric by $\hat{E}_{\mathrm{mtr}}$ and E with the metric due to norm by $\hat{E}_{\mathrm{nrm}}$. Now, simply note that addition, scalar multiplication and norm on $\hat{E}$ are determined on $\iota(E)$ (that the $\underline{\text{norm on } \hat{E} \text{ recovers its original metric}}$ is used here along with the fact that ι is a linear isometry), a dense subset of $\hat{E}_{\mathrm{mtr}} \overset{\mathrm{w}}{=} \hat{E}_{\mathrm{nrm}}$, and thus are determined on whole of $\hat{E}$.

Now we show the existence. Transport the normed linear space structure of E to $\iota(E)$ via ι. Consider the following facts:

(i) $\hat{E}$ is complete.

(ii) $\iota(E)$ is dense in $\hat{E}$.

(iii) $d(\iota(x), \iota(y)) = d(x, y) = \|x - y\| = \|\iota(x - y)\| = \|\iota(x) - \iota(y)\|$ for any $x, y \in X$ (first equality since ι an isometry; rest by definitions).

Due to these, 2.2 implies that $\hat{X}$ admits a (unique) Banach space structure extending the normed linear space structure on $\iota(E)$, which further satisfies $\|\hat{x} - \hat{y}\| = d(\hat{x}, \hat{y})$.[4] That ι is linear follows from definition of the normed linear space structure on $\iota(E)$. We now verify the universal property.

Let $f \colon E \to F$ be a linear isometry with F complete. By the universal property of metric space completion, there exists a unique isometry $\tilde{f}$ factoring f through ι:

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\iota"'] & & F \\ & \hat{E} \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

It suffices to show that $\tilde{f}$ is linear, which follows by 2.3 since $\tilde{f}$ is the continuous extension of the continuous (in fact, isometric) linear map $\iota(E) \to F$ induced by f, and since $\iota(E)$ is dense in $\hat{E}$. $\square$

Remark. Conversely, any norm completion is a metric space completion. (See (i) of 2.7.)

Corollary 2.5. *Any normed linear space admits a completion which is unique up to isometric isomorphisms.*

Proposition 2.6. *A normed linear space is dense in its completion.*

Proof. Let $\iota \colon E \to \hat{E}$ be a completion of E. Then $\iota| \colon E \to \overline{\iota(E)}$ is also a completion of E:

- $\overline{\iota(E)}$ is closed subspace of a Banach space and is hence Banach.
- If $f \colon E \to F$ is a linear isometry with F Banach, then it factors through $\iota|$ as follows:

$$\begin{tikzcd} & E \arrow[dl, "\iota|"'] \arrow[d, "\iota"] \arrow[dr, "f"] & \\ \overline{\iota(E)} \arrow[r, hook] & \hat{E} \arrow[r, dashed] & F \end{tikzcd}$$

For uniqueness, let f factor through $\iota|$ via an linear isometry $\tilde{f}$:

$$\begin{tikzcd} & E \arrow[dl] \arrow[d, "\iota|"] \arrow[dr, "f"] & \\ \iota(E) \arrow[r, hook] & \overline{\iota(E)} \arrow[r, "\tilde{f}"'] & F \end{tikzcd}$$

[4] We have abused notation, denoting the extended norm by the same symbol.

Then $\tilde{f}$ is determined on $\iota(E)$ which is dense in $\overline{\iota(E)}$, and thus determined in all of $\overline{\iota(E)}$ due to continuity of $\tilde{f}$.

Thus, there exists a linear isometry α making the following diagram commute:

$$\begin{tikzcd} & E \arrow[dl, "\iota|"'] \arrow[dr, "\iota"] & \\ \overline{\iota(E)} \arrow[rr, hook, shift left, "\mathrm{incl}"] & & \hat{E} \arrow[ll, dashed, shift left, "\alpha"] \end{tikzcd}$$

It follows that ι factors through itself via $\mathrm{incl} \circ \alpha$ which is thus equal to $\mathrm{id}_{\hat{E}} \overset{\mathrm{w}}{\Longrightarrow}$ incl is a surjection $\overset{\mathrm{w}}{\Longrightarrow} \overline{\iota(E)} = \hat{E}$ as claimed. $\square$

The immediate consequences include:

Corollary 2.7. *(i) Any norm completion is a metric completion.*

(ii) A linear isometry into a Banach space is a completion $\iff$ the image is dense in the codomain.

(iii) The completion of a dense subspace is canonically obtained from that of the parent space.

(iv) In a Banach space, closure of subspaces are their completions.

Proof. (i) Since the image of the domain is dense.

(ii) "$\Rightarrow$" is just 2.6. For the converse, use (i) and 2.3.

(iii) Use (ii).

(iv) Use (ii). $\square$

3. Applying BCT

Proposition 3.1. *A Banach space can't have a countably infinite Hamel dimension.*

Proof. Let $e_1, e_2, \ldots$ form a Hamel basis for $\mathscr{B}$. Then $\mathscr{B} = \bigcup_{i=0}^{\infty} \mathrm{span}\{e_1, \ldots, e_i\}$. By Baire's category (since $\mathscr{B}$ is complete), some $\mathrm{span}\{e_1, \ldots, e_i\}$ contains a ball, say $B_r(0)$, without loss of generality. But then $te_{n+1} \in B_r(0) \overset{\mathrm{w}}{\subseteq} \mathrm{span}\{e_1, \ldots, e_n\}$ for small enough t, a contradiction. $\square$

Are finite-dimensional subspaces of a TVS closed?

Theorem 3.2 (Open mapping)**.** *Any continuous surjective linear map between Banach spaces is open.*

Proof. Let $T \in \mathcal{L}_c(\mathscr{B}, \mathscr{B}')$. It suffices to have that $B'_\varepsilon(0) \subseteq T(B_1(0))$ for some $\varepsilon > 0$:

Let U be open in $\mathscr{B}$ and $x \in U$. We find an $r > 0$ such that $B'_r(Tx) \subseteq T(U) \overset{\text{w}}{\Longleftarrow} B'_r(0) \subseteq T(U - x) \overset{\text{w}}{\Longleftarrow} B'_\varepsilon(0) \subseteq T(\varepsilon/r(U - x))$ where the last two implications follow since T is linear. Now, this is true if $B_1(0) \subseteq \varepsilon/r(U - x) \overset{\text{w}}{\Longleftarrow} B_{r/\varepsilon}(x) \subseteq U$, and a small enough r can be chosen to ensure this.

Since T is surjective, $\mathscr{B}' = \bigcup_n T(B_n(0))$. By Baire's category (since $\mathscr{B}'$ Banach), let $B'_\delta(y_1) \subseteq \overline{T(B_n(0))} \overset{\text{w}}{\Longrightarrow} B'_\delta(0) \subseteq \overline{T(B_{2n}(0))}$:

We have $B'_\delta(0) = B'_\delta(y_1) - y_1 \subseteq B'_\delta(y_1) + B'_\delta(y_1) \subseteq \overline{T(B_n(0))} + \overline{T(B_n(0))} \subseteq \overline{T(B_n(0)) + T(B_n(0))}$ (see footnote[5]) $= \overline{T(B_n(0) + B_n(0))}$ (since T is linear) $= \overline{T(B_{2n}(0))}$.

Since T is linear, we get $B'_\varepsilon(0) \subseteq \overline{T(B_{1/2}(0))}$ for $\varepsilon := \delta/4n$. Thus it suffices to show that $\overline{T(B_{1/2}(0))} \subseteq T(B_1(0))$:

Let $y \in$ LHS. Then choose $x_1 \in B_{1/2}(0)$ such that $y - Tx_1 \in B'_{\varepsilon/2}(0) \overset{\text{w}}{\subseteq} \overline{T(B_{1/4}(0))}$ (again using linearity of T). Now, choose $x_2 \in B_{1/4}(0)$ such that $y - Tx_1 - Tx_2 \in B'_{\varepsilon/4}(0) \overset{\text{w}}{\subseteq} \overline{T(B_{1/8}(0))}$ (again using linearity of T), and so on... *DC used.*

Now, since $\mathscr{B}$ is Banach, the series $\sum_i x_i$ converges to an $x \in B_1(0)$. Once again using linearity of T, we get $\left\| y - T\left(\sum_{i=1}^n x_i\right) \right\| < \varepsilon/2^n$. Using continuity of T, this finally yields $y = Tx$. □

Remark. Necessity of surjectivity is easily seen. For linearity, think of a cubic polynomial $\mathbb{R} \to \mathbb{R}$.

Necessity of continuity? completeness of domain? of codomain?

Corollary 3.3 (Bounded inverse)**.** *The inverse of an invertible continuous linear map between Banach spaces is continuous.*

Remark. content

[5] $\overline{A} + \overline{B} = +(\overline{A} \times \overline{B}) = +(\overline{A \times B}) \subseteq \overline{+(A \times B)} = \overline{A + B}$.

CHAPTER III

Inner Product Spaces

Conventions. Unless stated otherwise, assume the following:

- Inner product spaces will be over K.
- V, W will denote inner product spaces and $\mathscr{H}$, a Hilbert space.
- Subspaces of inner product spaces will be considered inner product spaces.
- If X is an inner product space and $x \in E$, then we'll use $\|x\| := \sqrt{\langle x, x\rangle}$. (See 1.3.)
- Inner product spaces will be considered normed linear space under the induced norm (see 1.3).

1. Basics

Remark. Inner product is always required to take values in the base field of the vector space since we want $\langle x, y\rangle z$ to always make sense. Many proofs will depend on this faculty: See for instance, the proof of Cauchy-Schwarz (1.2).

For $x, y \in V$, we have $\|x+y\|^2 = \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2$. This yields:

Lemma 1.1 (Parallelogram law and polarization)**.** *Let $x, y \in V$. Then we have the parallelogram law:*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{1.1}$$

We also have the polarization identities: If $K = \mathbb{R}$, then

$$\langle x, y\rangle = \frac{\|x+y\|^2 - \|x-y\|^2}{4} \tag{1.2a}$$

and if $K = \mathbb{C}$*, then*

$$\langle x, y\rangle = \frac{\|x+y\|^2 - \|x-y\|^2}{4} + i\frac{\|x+iy\|^2 - \|x-iy\|^2}{4}. \tag{1.2b}$$

Proposition 1.2 (Cauchy-Schwarz)**.** *For* $x, y \in V$*, we have* $|\langle x, y\rangle| \leq \|x\|\|y\|$ *with equality holding* $\iff$ x*,* y *are linearly dependent.*

Proof. Without loss of generality, let $y \neq 0$. Now, simply observe that[1] $0 \leq \left\|x - \frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2 = \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2}$ with equality holding $\iff$ $x = \frac{\langle x,y\rangle}{\|y\|^2}y$. □

This gives the triangle inequality for $\|\cdot\|$, so that:

Corollary 1.3. *An inner product defines a norm with respect to to which the inner product is Cauchy-regular.*

Remark. Note that the inner-product needn't be uniformly continuous, let alone Lipschitz. See the remark after 1.1. Nevertheless, the inner product is Lipschitz in either coordinate.

1.1 Which norms come from inner products?

Here we prove a converse to 1.3:

Proposition 1.4. *Any norm that satisfies the parallelogram law (1.1) comes from an inner product.*

Proof. Define $\langle\cdot,\cdot\rangle$ using polarization (1.1). We first show that this indeed defines an inner product, and then show that the norm that this inner product induces is nothing but what we started with. The only hard part is showing the linearity of $\langle\cdot,\cdot\rangle$ in the first argument.

Additivity: Using parallelogram law, verify that $\langle x, z\rangle + \langle y, z\rangle = \langle x+y, 2z\rangle/2$. Thus it suffices to show that $\langle u, 2v\rangle = 2\langle u, v\rangle$. Putting $x \to u+v$ and $y \to v$ in 1.1, get

$$\|u+2v\|^2 + \|u\|^2 = 2\|u+v\|^2 + 2\|v\|^2$$

in which putting $v \to -v$ yields

$$\|u-2v\|^2 + \|u\|^2 = 2\|u-v\|^2 + 2\|v\|^2.$$

[1] Note that $\frac{\langle x,y\rangle}{\|y\|^2}$ is the "projection" of x on y.

Subtracting these obtains

$$\|u+2v\|^2 - \|u-2v\|^2 = 2\|u+v\|^2 - 2\|u-v\|^2$$

which is the content of $\langle u, 2v\rangle = 2\langle u, v\rangle$.

Homothety: Conclude that $\langle rx, y\rangle = r\langle x, y\rangle$ for any rational r. Thus, the map $s \mapsto \langle sx, y\rangle$ for $s \in \mathbb{Q}$ is Lipschitz (for fixed x, y) so that there exists a (unique) continuous extension of it on $\mathbb{R}$. Deduce (by representing reals as limits of rational sequences) that $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for real α's as well. Finally observe that if $K = \mathbb{C}$, then $\langle ix, y\rangle = i\langle x, y\rangle$ and thus the validity extends to complex α's as well. □

Remark. The only place where the parallelogram law is used is in proving the additivity. Except homothety, all the reasoning has been "algebraic".

Thus we get a correspondence:

Corollary 1.5. *On any vector space V over K, we have the following correspondence:*

$$\big\{\textit{inner products on } V\big\} \longleftrightarrow \left\{\begin{matrix}\textit{norms on } V \textit{ satisfying} \\ \textit{the parallelogram law}\end{matrix}\right\}$$

1.2 Completion

Lemma 1.6. *Let V be a dense subspace of an inner product space E with its inner product recovering the norm inherited from E. Then there exists a unique inner product on E extending that on V, and recovering the norm on E.*

Proof. Let's settle uniqueness first. Suppose E indeed admits an inner product as claimed. Denote E with its original norm by E_{nrm}, and E with the norm due to the extended inner product by E_{iprd}. Now, the extended inner product is continuous on $E_{\text{iprd}} \times E_{\text{iprd}}$ and determined a dense subset $V \times V$ (determined, because E's inner product extends that on V; dense, because V is dense in $E_{\text{nrm}} \overset{\text{w}}{=} E_{\text{iprd}}$), thus uniquely determined on $E \times E$.

Now we show existence. Denote V with the norm induced from its inner product by V_{iprd}, and V with the norm inherited from E by V_{nrm}. Consider the following facts:

(i) Inner product is Cauchy-regular on $V_{\text{iprd}} \times V_{\text{iprd}} \to K$.

(ii) $V_{\text{iprd}} = V_{\text{mtr}}$ so that we may replace "V_{iprd}" by "V_{mtr}" in the above.

(iii) V is dense in E so that $V \times V$ is dense in $E \times E$.

Due to these, there exists a (unique) continuous extension of V's inner product to E. That this extension is actually an inner product that recovers the norm on E can be shown by writing points in E as limits of sequences in V. $\square$

Corollary 1.7. *A norm completion of an inner product space admits a unique inner product which regenerates the norm.*

2. Orthogonal Complements and Projections

Let $x, y \in V$. Then x is said to be **orthogonal** or **perpendicular** to y, denoted $x \perp y$ iff $\langle x, y \rangle = 0$. For any $E \subseteq V$, we define the **orthogonal complement** $E^{\perp}$ of E to be the set of all the vectors of V that are orthogonal to each vector in E.

Lemma 2.1 (Pythagoras)**.** *For orthogonal vectors x, y in V, we have $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.*

Since $E^{\perp} = \cap_{y \in E}\{x \in V : \langle x, y\rangle = 0\}$, we have:

Lemma 2.2. *Orthogonal complements are closed subspaces.*

In the context of inner product spaces, we call the vectors of best approximations as **projections**.

Proposition 2.3 (Characterizing projections on subspaces)**.** *Let M be a subspace of V and $x_0 \in V$. Then for $y_0 \in M$, the following are equivalent:*

(i) $y_0 - x_0 \in M^{\perp}$.

(ii) y_0 *is a projection of* x_0 *onto* M.

Note that (ii) just says that $\|y_0 - x_0\| = \inf_{y \in M}\|y - x_0\|$.

Proof. (i) $\Rightarrow$ (ii): If $y \in M$, then $\|y - x_0\|^2 = \|y - y_0\|^2 + \|y_0 - x_0\|^2$. (Since M is a subspace, $y - y_0 \in M$ and is thus perpendicular to $y_0 - x_0$.) *Add diagram.*

(ii) $\Rightarrow$ (i): Let $y \in M$. Without loss of generality, let $\|y\| = 1$ (since M is a subspace). Now, $\|y_0 - x_0\| \leq \|y_0 - x_0 - \langle y_0 - x_0, y\rangle y\|$ (since M is a subspace, $y_0 - \langle y_0 - x_0, y\rangle y \in M$) $\overset{\text{w}}{\implies} |\langle y_0 - x_0, y\rangle|^2 = 0 \overset{\text{w}}{\implies} y_0 - x_0 \perp y$. $\square$ *Add diagram.*

We now ask for uniqueness of projections:

Proposition 2.4. *Any vector of an inner product space has at most one projection on a convex subset of it.*

Proof. Let E be the set under consideration and x_0 the vector to be projected. Note that translations respect projections and convexity. Thus, without loss of generality, take $x_0 = 0$.[2] Set $\delta := \inf_{y \in E} \|y\|$ and take $y_1, y_2 \in E$ with $\|y_1\| = \delta = \|y_2\|$. Then by the parallelogram law, we have

$$\begin{aligned}\|y_1 - y_2\|^2 &= 2\|y_1\|^2 + 2\|y_2\|^2 - \|y_1 + y_2\|^2 \\ &\leq 4\delta^2 - 4\left\|\frac{y_1}{2} + \frac{y_2}{2}\right\|^2 \\ &\leq 0\end{aligned}$$

where the last inequality follows since $y_1/2 + y_2/2 \in E$ as E is convex. ☐

Remark. (i) The necessity of convexity is evident by considering an annulus. However, full convexity is not really required: We just need that E is closed under taking midpoints. (ii) Necessity of the parallelogram law: Consider $\mathbb{R}^2$ and take $E := \{(x, y) : y \geq 1\}$. Then all the points $(x, 1)$ with $|x| \leq 1$ are projections of 0 onto the convex E.

Thus, we can talk of **projection functions onto convex subsets**. Since subspaces are convex, by 2.3, we get:

Corollary 2.5. *If existent, the projection onto a subspace is linear.*

Now, for the existence of projections:

Theorem 2.6. *Any vector of a Hilbert space can be projected onto a nonempty closed convex subset of it.* *Show necessity of Hilbert-ness.*

Proof. Let E be the subset in consideration. Since translations are homeomorphisms that preserve projections and convexity, we can without loss of generality take the vector to be projected to be 0. We thus need to find a vector of smallest norm in E.

Set $\delta := \inf_{y \in E} \|y\|$ (which is finite since $E \neq \emptyset$). For $n \geq 1$, choose $y_n \in E$ such that $\|y_n\|^2 < \delta^2 + 1/n$. Now, by parallelogram law, *AC used. Is parallelogram law required?*

$$\begin{aligned}\|y_m - y_n\|^2 &= 2\|y_1\|^2 + 2\|y_2\|^2 - \|y_1 + y_2\|^2 \\ &< 4\delta^2 + 2\left(\frac{1}{m} + \frac{1}{n}\right) - 4\left\|\frac{y_1}{2} + \frac{y_2}{2}\right\|^2 \\ &\leq 2\left(\frac{1}{m} + \frac{1}{n}\right)\end{aligned}$$

[2]Note that we couldn't reduce the analysis similarly in the proof of 2.3 since translations don't preserve "being a subspace".

where the last inequality follows as before due to convexity of E. Thus, (y_n) is Cauchy. Since E is a closed subset of a Hilbert space, (y_n) converges to some $y \in E$. Now, $\|y\|^2 = \lim_n \|y_n\|^2 \leq \delta^2 \overset{\mathrm{w}}{\Longrightarrow} \|y\| = \delta$ as required. $\square$

Remark. (i) Convexity is not required for finite dimensions for we can use Heine–Borel to find a vector of smallest length inside closed nonempty E. (ii) To show necessity of convexity (in infinite dimensions), consider $(1+1/n)e_n$'s for $n = 1, 2, \ldots$ where e_n's are orthonormal. (iii) Again, not the full power of convexity is used, just closure under taking midpoints; however, closure together with this property implies convexity.

Prove Heine-Borel.

Corollary 2.7 (Orthogonal decomposition of Hilbert spaces)**.** *Let M be a closed subspace of $\mathscr{H}$. Then*

$$\mathscr{H} = M \oplus M^\perp.$$

Show necessity of 1. Hilbert-ness, and 2. closure of M.

Proof. Let $x \in \mathscr{H}$. By 2.6 (M is closed and convex (since M a subspace) and $\mathscr{H}$ Hilbert), pick a projection y of x in M. By 2.3 (M a subspace), $y - x \in M^\perp$ so that $x = y + (x - y) \in M + M^\perp$.

For uniqueness, let $x_1, x_2 \in M$ and $y_1, y_2 \in M^\perp$ such that $x_1 + y_1 = x_2 + y_2 \overset{\mathrm{w}}{\Longrightarrow} M \ni x_1 - x_2 = y_2 - y_1 \in M^\perp$ (since M a subspace) $\overset{\mathrm{w}}{\Longrightarrow} x_1 - x_2, y_2 - y_1 \in M \cap M^\perp = \{0\} \overset{\mathrm{w}}{\Longrightarrow} x_1 = x_2$ and $y_1 = y_2$. $\square$

Corollary 2.8. *For any subset E of $\mathscr{H}$, we have*

$$(E^\perp)^\perp = \overline{\operatorname{span} E}.$$

Show necessity of completeness.

Proof. "$\supseteq$": LHS is a closed subspace and $E \subseteq$ LHS.

"$\subseteq$": Let $x \in$ LHS. Set $M := \overline{\operatorname{span} E}$ and by 2.7, let $y \in M$ such that $y - x \in M^\perp \overset{\mathrm{w}}{\subseteq} E^\perp \overset{\mathrm{w}}{\Longrightarrow} \langle y - x, x \rangle = 0 \overset{\mathrm{w}}{\Longrightarrow} \|y - x\| = 0$ (since $y \perp y - x$) $\overset{\mathrm{w}}{\Longrightarrow} x = y \overset{\mathrm{w}}{\in} M$. $\square$

Remark. Note that "$\supseteq$" holds irrespective of completeness.

3. Adjoints

Every vector $x \in V$ induces a continuous linear functional $\langle \cdot, x \rangle \in V^*$. It's easy to see that this is an injective anti-linear isometry $V \to V^*$. The following says that for Hilbert spaces, it's surjective as well:

Theorem 3.1 (Riesz representation)**.** *Continuous linear functionals on a Hilbert space are obtained from inner products.*

Show necessity of Hilbert-ness.

Proof. Let $\ell \in \mathscr{H}^*$. If $\ker \ell = \mathscr{H}$, then $\ell = 0 = \langle \cdot, 0 \rangle$. Otherwise, $(\ker \ell)^\perp \neq 0$:

Since $\underline{\mathscr{H}\text{ Hilbert}}$ and $\ker \ell$ a closed subspace (as $\underline{\ell \text{ continuous linear}}$), $\mathscr{H} = (\ker \ell) \oplus (\ker \ell)^\perp$.

Thus, take a unit vector $e \in (\ker \ell)^\perp$. Now, take an arbitrary $x \in \mathscr{H}$. Then $x - \frac{\ell(x)}{\ell(e)} e \in \ker \ell$ (since $\underline{\ell \text{ linear}}$) so that it's perpendicular to $e \overset{\text{w}}{\implies} \ell(x) = \langle x, \overline{\ell(e)} e \rangle$. □

Remark. Note that the proof relies solely on finding a nonzero vector in $(\ker \ell)^\perp$.

Corollary 3.2. *$\mathscr{H}^*$ is isometrically and anti-linearly isomorphic to $\mathscr{H}$.*

The **adjoint** of a function $f \colon V \to W$ is a function $f^* \colon W \to V$ such that

$$\langle f(x), y \rangle = \langle x, f^*(y) \rangle$$

for all $x \in V$ and $y \in W$. If existent, such a function is clearly unique which justifies the notation. We enlist some immediate algebraic properties of adjoints:

Corollary 3.3.

(i) Let $f, g \colon V \to W$ admit adjoints. Then f^* is linear and f^*, $f + g$, αf (for any scalar α), $g \circ f$ admit adjoints:

$$(f^*)^* = f$$
$$(f+g)^* = f^* + g^*$$
$$(\alpha f)^* = \overline{\alpha} f^*$$

(ii) Let $f \colon V_1 \to V_2$ and $g \colon V_2 \to V_3$ admit adjoints. Then $g \circ f$ also admits adjoint:

$$(g \circ f)^* = f^* \circ g^*$$

Thus, if f is invertible with f^{-1} admitting adjoint too, then f^* is invertible with

$$(f^{-1})^* = (f^*)^{-1}.$$

Inverse of an adoint-able function is adjoint-able?

Remark. Hence, only linear maps can possibly admit adjoints.

Proposition 3.4. *Let $T \in \mathcal{L}(V, W)$ admit adjoint. Then the following hold:*

$$\|T^*\| = \|T\|$$
$$\|T^*T\| = \|T\|^2$$

Proof. For any y, we have $\|T^*y\|^2 = \langle T^*y, T^*y\rangle = \langle TT^*y, y\rangle \le \|T\|\|T^*y\|\|y\| \overset{\text{w}}{\Longrightarrow} \|T^*y\| \le \|T\|\|y\|$. Thus $\|T^*\| \le \|T\|$. Putting $T \to T^*$ yields $\|T^*\| = \|T\|$.

For second, we already have $\|T^*T\| \le \|T^*\|\|T\| = \|T\|^2$. Now, for any x, we have $\|Tx\|^2 = \langle T^*Tx, x\rangle \le \|T^*T\|\|x\|^2 \overset{\text{w}}{\Longrightarrow} \|Tx\| \le \|T^*T\|^{1/2}\|x\|$. Thus $\|T\| \le \|T^*T\|^{1/2}$. □

Taking square roots in $\mathbb{R}^*$.

Thus adjoints of continuous linear maps are continuous. Now we talk about the existence of adjoints:

Theorem 3.5. *Continuous linear maps between Hilbert spaces admit adjoints.*

Proof. Let $T \in \mathcal{L}_c(\mathscr{H}, V)$. Fix a $y \in V$. Now, $x \mapsto \langle Tx, y\rangle$ is a continuous linear functional on $\mathscr{H}$ (since $\underline{T\text{ continuous}}$) so that by Riesz (since $\underline{\mathscr{H}\text{ is Hilbert}}$), let Sy be the unique vector in $\mathscr{H}$ such that $\langle Tx, y\rangle = \langle x, Sy\rangle$ for each $x \in \mathscr{H}$. This gives a function $S\colon V \to \mathscr{H}$, which is the required adjoint. □