A horizontal method of localizing values of a linear function in permutation-based optimization

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Abstract. This paper is dedicated to linear constrained optimization on permutation configurations' set, namely, to permutation-based subset sum problem (PB-SSP). To this problem, a directed structural graph is associated connected with a skeleton graph of the permutohedron and allowing to perform a directed search to solve this linear program. To solve PB-SSP, a horizontal method for localizing values of a linear objective function is offered combining Graph Theory tools, geometric and structural properties of a permutation set mapped into Euclidean space, the behavior of linear functions on the set, and Branch and Bound techniques.

Keywords: discrete optimization · linear constrained optimization · combinatorial configuration · permutation · skeleton graph · grid graph · search tree

1 Introduction

Combinatorial optimization problems (COPs) with permutation as candidate solutions commonly known as permutation-based problems [13] can be found in a variety of application areas such as balancing problems associated with chip design, ship loading, aircraft outfitting, turbine balancing as well as in geometric design, facility layout, VLSI design, campus design, assignments, scheduling, routing, scheduling, process communications, ergonomics, network analysis, cryptography, etc. [4, 9, 10, 13–15, 19, 23–25, 27, 33, 34].

Different COPs are representable easily by graph-theoretic approach (GTA) [1–3, 6–8, 11]. First of all, it concerns COP on a set E coinciding with a vertex set of their convex hull P (vertex-located sets, VLSs [28, 30]). Such COPs are equivalent to optimization problems on a node set of a skeleton graph $G = (E, \mathbf{E})$ of the polytope P, where $\mathbf{E}$ is an edge set of P. Note that in case if E is not a VLS, approaches to an equivalent reformulation of the COP as an optimization problem on a VLS in higher dimensional space can be applied first [31, 32].

The benefits of using the graph-theoretic approach are not limited to simple illustrations, but also provide an opportunity to develop approaches to solving COPs based on using configuration graphs [11] and structural graphs [1–3, 6–8] of the problems. Localization of COP-solutions or values of the objective function is an interesting technique allowing to reduce considerably a search domain based on deriving specifics of the domain, type of constraints and the objective function [1, 3, 6, 7, 29]. In particular, the method of ordering the values of objective function on an image $E_n(A)$ in $\mathbb{R}^n$ of a set of n-permutations induced by a set A is considered in [1]. It consists in constructing a Hamiltonian path in an $\mathbf{E}$ of a permutation graph, which is a skeleton graph of the permutohedron $P_n(A) = conv(E_n(A))$. In [2], a similar problem is considered on an image $E_{nk}(A)$ in Euclidean space of a set multipermutations induced by a multiset A including k different elements. In this case, a skeleton graph of the generalized permutohedron (the multipermutation graph) is considered instead of the permutation graph. In [8], linear constrained single and multiobjective COPs on $E_{nk}(A)$ are solved using the multipermutation graph, etc.

This paper is dedicated to developing GTA-techniques [1–3, 6–8] for solving permutation-based COPs (PBOPs) related to localization of objective function values. Namely, a generalization of Subset Sum Problem (SSP) [5, 12], which as known NP-complete COP, from the Boolean set B_n as an admissible domain to $E_n(A)$ (further referred to as a permutation-based SSP, BP-SSP). Also, we will consider versions of BP-SSP where a feasible solution x^* is sought (BP-SSP1) or a complete solution X^* is sought (BP-SSP2).

2 The Combinatorial Optimization Problem: statement and properties

In the general form, COP can be formulated as follows: there is set A of n elements

$$A = \{a_1, a_2, ..., a_k\} \subset \mathbb{R}^1, \text{ such that } a_1 < ... < a_k \tag{1}$$

on which a finite point configuration $E = \{e_1, e_2, ..., e_N\} \subset \mathbb{R}^n$ is given and function $f(x) : E \to \mathbb{R}^1$. By an e-configurations $e \in E$ [26, 30], one can understand a permutation, a partial permutation, a combination, a partition, a composition, a partially ordered set induced by A, etc. and considered as a point in $\mathbb{R}^n$. It is required to find an extremum z^* (maximum or minimum) of $f(x)$ and an extremal x^* or a set X^* of the extremals, where the extremum is attained, and additional constraints are satisfied (further referred to as COP1/COP2, respectively). Thus, their formulations are: find

$$\begin{aligned}
&\text{COP1} : z^* = \underset{x \in E'}{extr}\, f(x),\ x^* = arg\underset{x \in E'}{extr}\, f(x);\\
&\text{COP2} : z^* = \underset{x \in E'}{extr}\, f(x),\ X^* = Arg\underset{x \in E'}{extr}\, f(x),\\
&\text{where} \quad E' = \{x \in E : f_i(x) \le 0, i \in J_m\},
\end{aligned}$$

$J_m = \{1, ..., m\}$.

A permutation-based COP (PB-COP) is a particular case of COP, where $E \in \{\mathit{\Pi}_n(A), \mathit{\Pi}_{nk}(A), E_n(A), E_{nk}(A)\}$. Here, $\mathit{\Pi}_n(A)$ is a set of n-permutations induced by a set $A = \{a_i\}_{i \in J_n}$: $a_i < a_{i+1}, i \in J_{n-1}$ $E_n(G) \subset \mathbb{R}^n$, and $E_n(A)$ - is an image of $\mathit{\Pi}_n(A)$ in Euclidean space; $E = E_{nk}(G)$ - is is an image in Euclidean space of n-multipermutation set $\mathit{\Pi}_{nk}(A)$ induced by set (1), $k < n$. Denoting $E_{nn}(G) = E_n(G)$ these two are united in a class $E_{nk}(G)$ – the generalized set of e-configuration of permutations [16, 17, 19, 26, 27].

$E = E_n(G)$ has many interesting constructive and geometric peculiarities [1–3, 16–19, 21, 26–28, 30–35], e.g.,

$$x_{\max} = \underset{x \in E}{\operatorname{argmax}} f(x) = (a_i)_{i \in J_n}; x_{\min} = \underset{x \in E}{\operatorname{argmin}} f(x) = (a_{n-i+1})_{i \in J_n},$$
$$\text{if } f(x) = c^T x, \ c \neq \mathbf{0}, \ c_1 \leq ... \leq c_n; \qquad (2)$$

- $X_{\min} = \underset{x \in E}{\operatorname{Argmin}} f(x) / X_{\max} = \underset{x \in E}{\operatorname{Argmax}} f(x)$ is obtained from $x_{\min}/x_{\max}$ by permuting coordinates within sets of coordinates with the same coefficient of $f(x)$, wherefrom

$$\text{if } \ c_1 < ... < c_n \Rightarrow X_{\min} = x_{\min}, \ X_{\max} = x_{\max}; \qquad (3)$$

- E is VLS;
- E is inscribed in a hypersphere $S_r(\bar{b})$ centered at $\bar{b} = (b, ..., b) \in \mathbb{R}^n$ $(b \in \mathbb{R}^1)$;
- $\forall i \in J_n$, E lies on n parallel hyperplaines $H_i = \{H_{ij}\}_{i \in J_n}$: $H_{ij} = \{x \in \mathbb{R}^n : x_i = a_j\}$, $j \in J_n$. As a result,

$$\forall i \in J_n \quad E = \bigcup_{j \in J_n} E_{ij}, \qquad (4)$$

  where $E_{ij} = E \cap H_{ij} \simeq E_{n-1}(J_{n-1})$, $i, j \in J_n$;
- $P = convE = P_n(A)$ is a permutohedron, which is a simple polytope and its H-presentation is:

$$\begin{cases} \sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{n} a_i; \\ \sum\limits_{i \in \omega} x_i \geq \sum\limits_{i=1}^{|\omega|} a_i, \forall \omega \subset J_n; \end{cases} \qquad (5)$$

- a skeleton graph $\mathcal{G}_n(A)$ of the permutohedron $P_n(A)$ has all permutations induced by A as a node set. Its adjacent vertices are differ by adjacent transposition (i.e., (a_i, a_{i+1})-transposition);
- any function $f : E \to \mathbb{R}^1$ allows extending is a convex way onto arbitrary convex set $K \supset E$;
- E can be represented analytically in the following ways: a) by equation of $S_r(\bar{b})$ and (5);

$$b) \sum_{\omega \subseteq J_n, |\omega| = j} \prod_{i \in \omega} x_i = \sum_{\omega \subseteq J_n, |\omega| = j} \prod_{i \in \omega} g_i, \ j \in J_n;$$
$$c) \sum_{i=1}^{n} x_i^j = \sum_{i=1}^{n} g_i^j, \ j \in J_n.$$

– if $x \in E$ is formed from $y \in E$ by a single transposition $a_i \leftrightarrow a_j$, $i < j$, then $\forall c \in \mathbb{R}^n$ $c^T x \leq c^T x$ iff $c_i \leq c_j$ (further referred to as Rule1).

Let us consider the following versions of PB-COP: find a solution of permutation-based versions of COP1/COP2 with (2) as objective function,

$$E = E_n(A);$$
$$f_1(x) = f(x) - z_0 \leq 0; f_2(x) \leq -f(x) + z_0 \leq 0, \tag{6}$$

where $z_0 \in \mathbb{R}^1$ (further referred to as PB-COP1/PB-COP2).

Note that (6) can be rewritten as follows:

$$f(x) = z_0,$$

wherefrom PB-COP1, PB-COP2 are permutation-based feasibility problems of finding a point x_0 in E' or the whole set E', respectively.

PB-COP1, PB-COP2 are both at least as hard as NP-complete problems since the subset sum problem – given a real set A on n elements, is there its m-element ($m < n$) subset whose sum is z_0 – is a particular case of PB-COP1, where $c_1 = ... = c_{n-m} = 0, c_{n-m+1} = ... = c_n = 1$.

3 The horizontal method for PB-COP2

Let us introduce a horizontal method for solving PB-COP2 (PB-COP2.HM), which is based on applying the Branch and Bound paradigm to this feasibility problem.

To solve PB-COP2, a search tree with a root $E = E_n(A)$ and leaves $\{E_3(\mathcal{A})\}_{\mathcal{A} \subset A, |A|=3}$ is build. It uses GTA and the listed properties of $E_n(A)$ and $\mathcal{G}_n(A)$. In particular, it applies the decomposition (4) recursively fixing last coordinates in $E_n(A)$-subsets being the tree nodes; the explicit solution (2) of a linear COP; the vertex locality of E; the adjacency criterion, Rule1, and so on.

Let us introduce some notations. $cd = (cd_i)_{i \in J_{l_{cd}}}$ – is a code of the object $[.] \in \{E, \mathcal{G}, G\}$, which is a partial l_{cd}-permutation from A, $l_{cd} \in J^0_{n-1}$ – is the length of the code ($J^0_m = J_m \cup \{0\}$). cd defines values of consecutive last l_{cd}-positions of $[.](cd)$.

$E(\emptyset) = E_n(A)$, $\mathcal{G}(\emptyset) = \mathcal{G}_n(A)$, $G(\emptyset) = G_n(A)$ is a grid-graph, which will be defined later.

Branching of $[.](cd)$ is based on (4) and performed according a rule:

$$branch([.](cd)) = \{[.](cd_i)_{i \in J_{n-l_{cd}}}\},$$

where $cd_i = (a_i(cd), cd)$, $i \in J_{n-l_{cd}}$, $A(cd) = \{a_i(cd)\}_{i \in J_{l_{cd}}} = A \backslash \{cd_i\}_{i \in J_{l_{cd}}}$, $a_i(cd) \leq a_{i+1}(cd)$, $i \in J_{n-l_{cd}-1}$.

Esimates are found with respect to (2) and taking into account already fixed coordinates, namely,

$$lb([.](cd)) = z_{min}(cd))/ub([.](cd)) = z_{max}(cd) -$$

is a lower/upper bound on the branch $[.](cd)$, where $z_{min}(cd) = f(y_{min}(cd))$, $z_{max}(cd) = f(y_{max}(cd))$.

$\mathcal{G}(cd)$ – is a skeleton graph of $conv(E(cd))$, $G(cd)$ is a directed grid-graph shown on Fig. 1. $G(cd)$ is of $2(n - l_{cd})$ nodes, two of which have been examined, namely, top-left and bottom-right ones: $z_{max}(cd_{n-l_{cd}}) = z_{min}(cd)$, $z_{min}(cd_1) = z_{min}(cd)$. In the terminology of [8], $\mathcal{G}(cd)$ is a two-dimensional structural graph of BP-COP2.

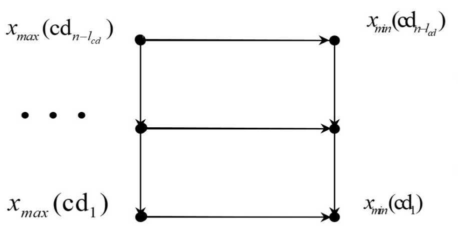

Fig. 1. The grid-graph $G(cd)$

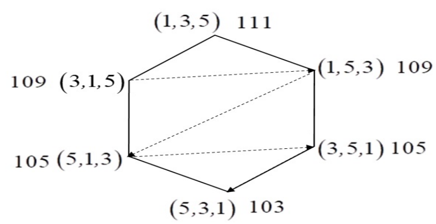

Fig. 2. The graph $(6, 4, 2)$

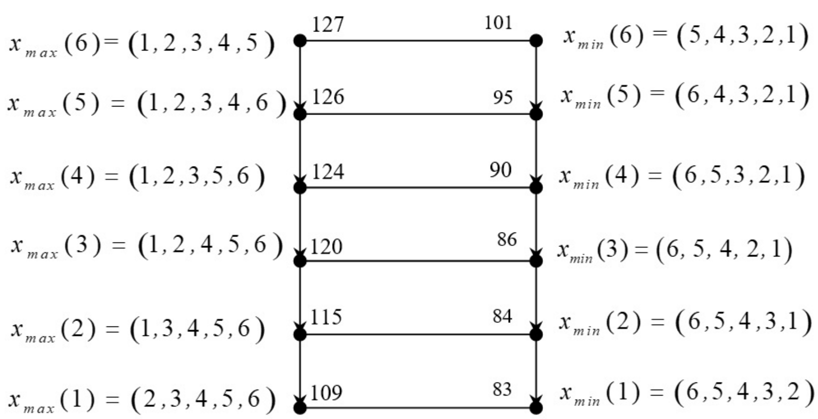

Fig. 3. The grid-graph $G(\emptyset)$

Pruning branches:

- if $z_0 > z_{max}(cd)$ or $z_0 < z_{min}(cd)$, then prune $E(cd)$ (a rule PB1);
- if $z_0 = z_{max}(cd)$, then find $X_{max}(cd)$, upload X^*: $X^* = X^* \cup Y_{max}(cd)$, where $Y_{max}(cd) = \{(x, cd)\}_{x \in X_{max}(cd)}$, and prune $E(cd)$ (a rule PB2);
- if $z_0 = z_{min}(cd)$, then find $X_{min}(cd)$, upload X^*: $X^* = X^* \cup Y_{min}(cd)$, where $Y_{min}(cd) = \{(x, cd)\}_{x \in X_{min}(cd)}$, and prune $E(cd)$ (a rule PB3).

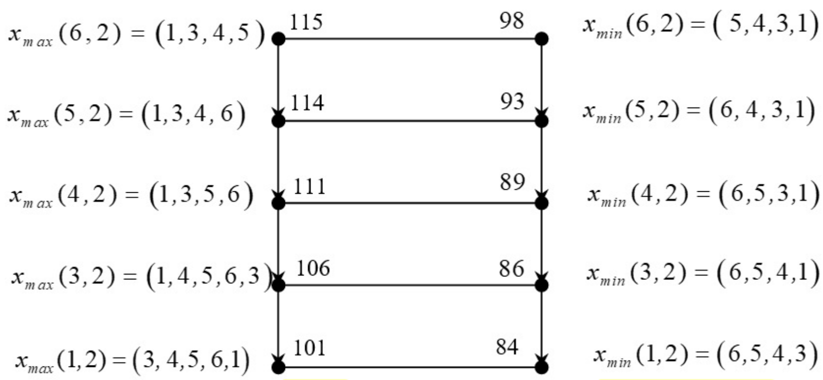

Fig. 4. The grid-graph $G(2)$

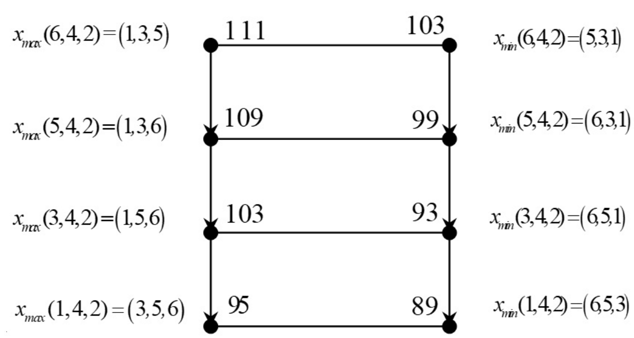

Fig. 5. The grid-graph $G(4,2)$

By construction, in a column of the grid $G(cd)$, consecutive nodes differ by an adjacent transposition that enforce the following (with respect to Rule1):

$$z_{max}(cd_{n-l_{cd}}) \geq z_{max}(cd_{n-l_{cd}-1}) \geq ... \geq z_{max}(cd_1);$$
$$z_{min}(cd_{n-l_{cd}}) \geq z_{min}(cd_{n-l_{cd}-1}) \geq ... \geq z_{min}(cd_1);$$
$$z_{min}(cd_i) \leq z_{max}(cd_i),\ i \in J_{n-l_{cd}};$$

– if $i \in J_{n-l_{cd}-1}$: $z_0 > z_{max}(cd_i)$, then prune $E(cd_i), ..., E(cd_1)$;
– if $i \in J_{n-l_{cd}} \backslash \{1\}$: $z_0 < z_{min}(cd_i)$, then prune $E(cd_1), ..., E(cd_i)$.

Remark 1. If PB-COP1 needs to be solved, the PB-COP2.HM-scheme is used until a first admissible solution is found.

This version of PB-COP2.HM is directly generalized from $E_n(A)$ to $E_{nk}(A)$. The only difference is that (4) becomes $\forall i \in J_n \quad E = \bigcup_{j \in J_k} E_{ij}$, where $E_{ij} = E \cap H_{ij}$, $i \in J_n$ – is a set in the class $E_{n-1,k(B_j)}(B_j)$, $j \in J_k$.

Another generalization of PB-COP2 concerns considering

$$f_1(x) = f(x) - z_0 \leq 0; f_2(x) \leq -f(x) + z_0 - \Delta \leq 0,$$

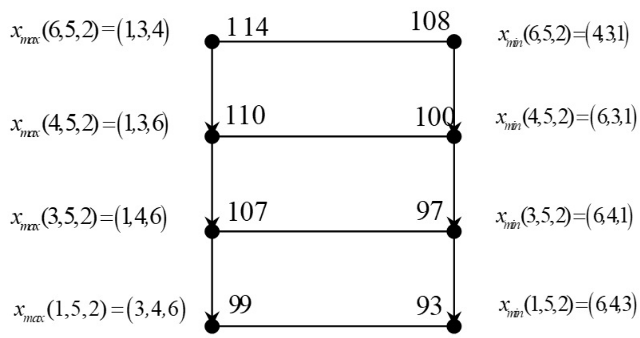

Fig. 6. The grid-graph $G(5, 2)$

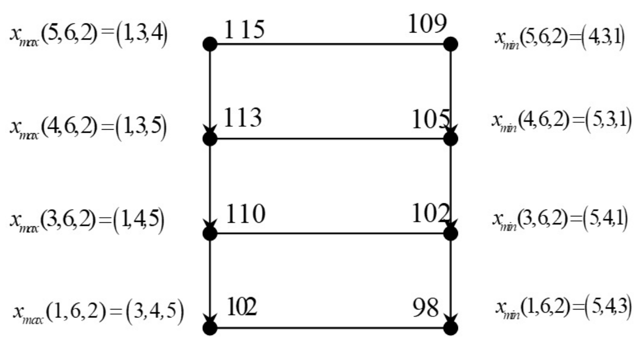

Fig. 7. The grid-graph $G(6, 2)$

where $\Delta \geq 0$ instead of (6). Here, minor modifications in estimates' rules are required.

Simultaneously with forming X^*, a permutation-based COP of optimizing $\varphi : E \to \mathbb{R}^1$ can be found both single-objective as multiobjective [6–8, 21, 22].

4 BP.COP2 Example

Solve BP.COP2 with $n = 6$, $c = (2, 3, 4, 6, 7, 8)$, $A = J_6$, $z_0 = 109$.

Coefficients of c are different, therefore, by (2)-(3), rules PB2, PB3 are simplified to:

- if $z_0 = z_{max}(cd)$, then $X^* = X^* \cup (x_{max}(cd), cd)$ (a rule PB2');
- if $z_0 = z_{min}(cd)$, then $X^* = X^* \cup (x_{min}(cd), cd)$ (a rule PB3').

Step 1. $cd = X^0 = \emptyset$, $l_{cd} = 0$.

$x_{min}(cd) = x_{min} = (6, 5, 4, 3, 2, 1)$, $x_{max}(cd) = x_{max} = (1, 2, 3, 4, 5, 6)$, $z_{min}(cd) = 83 < z_0 = 109 < z_{max}(cd) = 127$. The branch $E(cd)$ is not discarded. $branch(E(cd)) = \{E(i)_{i \in J_6}\}$. Graph $G(\emptyset)$ is depicted on Fig. 3 with $E(6)$

on top and $E(1)$ on bottom, as well as with bounds $lb(E(i)), ub(E(i))$, $i \in J_n$. $ub(E(1)) = 109$, hence, by PB2', $X^* = X^* \cup (x_{max}(cd), cd) = \{(2,3,4,5,6,1)\}$ and the branch $E(1)$ is discarded;

Step 2. Explore $E(2)$. $cd = (2)$, $l_{cd} = 1$, $branch(E(cd)) = \{E(i,2)_{i \in A(cd)}\}$, where $A(cd) = A \backslash \{2\}$. $G(cd) = G(2)$ is shown on Fig. 4. It is seen, the branches $E(1,2), E(2,2)$ are pruned by PB1.

Step 3. Explore consecutively $E(4,2)$, $E(5,2)$, $E(6,2)$. Here, $l_{cd} = 2$, $cd \in \{(4,2),(5,2),(6,2)\}$. Branching is performed into $n - l_{cd} = 4$ branches (see graphs $G(cd)$ on Figs. 5-7).

Step 3.a. In $E(4,2)$, prune branches $E(1,4,2)$, $E(3,4,2)$ by PB1. By PB2', $X^* = X^* \cup (x_{max}(5,4,2),(5,4,2)) = X^* \cup \{(1,3,6,5,4,2)\}$, and the branch $E(5,4,2)$ is discarded (see Fig. 5). The branch $E(6,4,2)$ is explore analyzing four vertices of $\mathcal{G}(6,4,2)$ as shown on Fig. 3. As a result, two new feasible points are found $x^1(6,4,2) = (3,1,5)$, $x^2(6,4,2) = (1,5,3)$ are found, therefore $X^* = X^* \cup (x^i(cd), cd)_{i=1,2} = X^* \cup \{(3,1,5,6,4,2),(1,5,3,6,4,2)\}$.

Step 3.b. In $E(5,2)$, prune branches $E(3,5,2)$, $E(1,5,2)$ by PB1 (see Fig. 6). $E(4,5,2)$, $E(6,5,2)$ are analyzed similarly to $E(6,4,2)$. As a result, one more feasible solution is found and $X^* = X^* \cup \{(3,4,1,6,5,2)\}$.

Step 3.c. In $E(6,2)$, prune a branch $E(1,6,2)$ by PB1. By PB3', $X^* = X^* \cup (x_{min}(5,6,2),(5,6,2)) = X^* \cup \{(4,3,1,5,6,2)\}$, and the branch $E(5,6,2)$ is discarded (see Fig. 7). Exploring $E(4,6,2)$, $E(3,6,2)$ like $E(6,4,2)$, we get that X^* is complemented by new admissible solution, namely, $X^* = X^* \cup \{(1,5,4,3,6,2)\}$.

By now, a third of E have been examined. For that, about 50 points from $E = 720$ were analyzed and 7 elements of E' were found. The set E' contains 26 points, implying that about a third of E' has been found. The same proportion holds for the rest branches $E(1) - E(3)$. As a result, around $p = 20\%$ points of E are analyzed to get E'.

PB-COP2.HM was implemented for PB-COP2s of dimensions up to 200. Numerical results demonstrated that the percentage p decreases as n increases. Also, p decreases as far as z_0 becomes closer to $z_{\min}$ or $z_{\max}$ to the middle value $(z_{\min} + z_{\max})/2$.

5 Conclusion

Complex extreme and feasibility combinatorial problems on the set of permutations $E_n(A)$ have been investigated by means of its embedding into Euclidean space and associating and utilizing the permutation polytope, permutation graph, and grid graph. For the problem PB-SSP, the horizontal method for localizing values of a linear objective function (PB-COP2.HM) is developed, and directions of its generalization to solve a wide class of permutation-based problems, which are formalized as linear combinatorial programs, are outlined. (PB-COP2.HM) is supported by an example and illustrations.

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