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On the Characterization of Copulas by Differential Equations

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We study the semigroup action induced by univariate conditioning of copulas. Based on this, we give a new characterization of bivariate copulas in terms of flows generated by solutions of ordinary differential equations with not necessary continuous right side. Several applications, related to concordance ordering of copulas, illustrate the usefulness of this result.

Keywords Copulas; Discontinuous ordinary differential equations; Univariate conditioning; Archimedean copulas; Left invariant copulas; Positive quadrant dependence (PQD); Concordance ordering.

Mathematics Subject Classification 62H20; 60E05; 34A36.

1. Introduction

A bivariate *copula* is a restriction to $[0, 1]^2$ of a distribution function whose univariate margins are uniformly distributed on $[0, 1]$. Specifically, $C : [0, 1]^2 \to [0, 1]$ is a copula if it satisfies the following properties:

(C1) $C(x, 0) = C(0, x) = 0$ for every $x \in [0, 1]$, i.e., C is *grounded*;
(C2) $C(x, 1) = C(1, x) = x$ for every $x \in [0, 1]$;
(C3) C is *2–increasing*, that is, for every $x_1, y_1, x_2, y_2 \in [0, 1]$, $x_1 \le x_2$, and $y_1 \le y_2$, it holds

$$C(x_1, y_1) + C(x_2, y_2) \ge C(x_1, y_2) + C(x_2, y_1).$$

Recently, copulas have received a great popularity due to the celebrated *Sklar's Theorem*, stating that every joint distribution function of a pair of continuous random variables can be represented by means of a suitable copula and its univariate marginals. Just to have an idea about copula theory and (some of) its applications, we refer to Durante and Sempi (2010), Härdle and Okhrin (2010), Jaworski et al. (2010), Joe (1997), Nelsen (2006), and McNeil et al. (2005).

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The growing importance of *copulas* for constructing statistical models has originated several methods for generating new copulas. The final goal of these investigations is to establish the correspondence between bivariate copulas and a certain class of ordinary differential equations, which proved to be useful to study the analytical properties of copulas and to construct copulas with prescribed properties.

This article is organized as follows. Section 2 is important from the point of view of self consistency of the article. In Sec. 2.1, we provide the basic facts concerning the copulas conditional with respect to univariate left-side thresholding, which are crucial for our constructions. Section 2.2 deals with Dini derivatives of real-valued functions and their measurability. Section 2.3 presents the basic facts concerning the extended notion of the solution of ordinary differential equation. The characterization of bivariate copulas in terms of flows generated by solutions of ordinary differential equations is presented in Sec. 3. We constructed an *almost sure* one-to-one correspondence between copulas and a certain class of ordinary differential equations. The proofs of the stated results are collected in Sec. 6. In Sec. 4, we discuss the relations between the transformations of copulas (left-side conditioning, vertical gluing, flipping) and transformations of corresponding differential equations, and special types of equations which give rise to such families of copulas like Archimedean or invariant with respect to left-side conditioning. Next, in Sec. 5 we provide the examples of applications of our results to characterization of properties of copulas robust with respect to left-side conditioning. Special attention is paid to hereditary PQD (resp. NQD) copulas and copulas monotonic with respect to conditioning.

2. Notation

2.1. Univariate Horizontal Conditioning

We recall the basic facts concerning the univariate conditioning of random variables and copulas (compare Mesiar et al., 2008; Durante and Jaworski, 2010b; Durante and Jaworski, 2012). For higher dimensions the reader is referred to Jaworski (2013).

Proposition 2.1. *Let* C *be a copula. The family of functions* $C_{[\alpha]}(x, z)$, $\alpha \in (0, 1]$, *uniquely determined by an equation*

$$C_{[\alpha]}\left(x, \frac{C(\alpha, y)}{\alpha}\right) = \frac{C(\alpha x, y)}{\alpha}, \quad x, y \in [0, 1] \tag{2.1}$$

is a continuous family of copulas. The function $H(\alpha, x, z) = C_{[\alpha]}(x, z)$ *is continuous on* $(0, 1] \times [0, 1]^2$ *and for fixed* α *fulfills the conditions (C1), (C2) and (C3).*

Note that if the copula C describes the interdependencies between two random variables X and Y, then $C_{[\alpha]}$ is the copula of the conditional distribution of X and Y with respect to the univariate condition $X \leq q$, where $\mathbb{P}(X \leq q) = \alpha$ (compare Jaworski, 2013). Therefore, $C_{[\alpha]}$ is called a conditional copula. For more general approach see Patton (2006).

The conditioning can be iterated. The family $C_{[\alpha]}$, $\alpha \in (0, 1]$, is an orbit of the multiplicative semi-group $((0, 1], *)$, where $C_{[1]} = C$. Indeed, we have the following.

Proposition 2.2. *For any* $\alpha_1, \alpha_2 \in (0, 1]$ *and* $x, y \in [0, 1]$

$$(C_{[\alpha_1]})_{[\alpha_2]}(x, y) = C_{[\alpha_1 \alpha_2]}(x, y). \tag{2.2}$$

2.2. Dini Derivatives

The concept of *Dini derivative* (or Dini derivate) generalizes the classical notion of the derivative of a real-valued function. Let $a, b \in \mathbb{R}$, $a < b$, and let $f : (a, b] \to \mathbb{R}$ be a continuous function. Let x be a point in $(a, b]$. The limit

$$D^- f(x) = \limsup_{h \to 0^+} \frac{f(x) - f(x-h)}{h} \tag{2.3}$$

is called *left-side upper Dini derivative* of f at x. For more details, we refer to Łojasiewicz (1988) and Durante and Jaworski (2010).

Now, let $f_y(x)$ be a continuous family of continuous functions then the Dini derivative may not be continuous but nevertheless it is a Borel function.

Proposition 2.3. *Let* $f_y(x) = H(x, y)$*, where*

$$H : (a, b] \times Y \longrightarrow \mathbb{R}, \quad Y \subset \mathbb{R}^k$$

is a continuous function. Then for every $x_0 \in (a, b]$ *the function*

$$h(y) = D^- f_y(x_0)$$

is a Borel function on Y.

2.3. Carathéodory Solutions of Differentiable Equations

Since we are going to deal with differential equations with discontinuous right side, we need the extended notion of the solution, known in the literature under the name *Carathéodory solution* (Biles and Lopez, 2009). For general conditions implying the existence of a solution, see Hassan and Rzymowski (1999).

Definition 2.1. Let f be a real-valued function (may be discontinuous)

$$f : \mathbb{I} \times [y_1, y_2] \longrightarrow \mathbb{R},$$

where $\mathbb{I}$ is an interval (closed, half-closed, or open). A function $g(x)$ defined on the interval $\mathbb{I}$ with values in $[y_1, y_2]$ is called a Carathéodory solution of the differential equation

$$y' = f(x, y),$$

if g is absolutely continuous and satisfies the differential equation almost everywhere

$$g'(x) = f(x, g(x)) \quad a.e.$$

Note that the absolute continuity of g implies that its derivative exists almost everywhere. Moreover,

$$\int_a^b g'(t)dt = g(b) - g(a)$$

for any $a < b$ from $\mathbb{I}$ (Łojasiewicz, 1988, Theorem 7.4.4). Furthermore, if h is any integrable function on $[a, b]$ then the function

$$g(x) = \int_a^x h(t)dt$$

is absolutely continuous on $[a, b]$ and (Łojasiewicz, 1988, Theorem 7.1.7 and formula 7.1.14 below):

$$g'(x) = h(x) \quad a.e.$$

As a consequence, the Carathéodory solutions are closely related with so called weak solutions in the Sobolev space $W^{1,1}(\mathbb{R})$ (compare Brezis, 2011, Theorem 8.2).

3. Main Results

Definition 3.1. By $\mathbb{F}$ we denote the set of Borel functions F

$$F : (0, 1] \times [0, 1] \longrightarrow \mathbb{R},$$

such that:

F1. F fulfills the boundary conditions

$$\forall x \in (0, 1] \quad F(x, 0) = F(x, 1) = 0.$$

F2. The function $z + F(x, z)$ is non decreasing in second variable

$$\forall x \in (0, 1] \quad \forall z_1, z_2 \in [0, 1] \quad z_1 \leq z_2 \Rightarrow z_1 + F(x, z_1) \leq z_2 + F(x, z_2).$$

Our goal is to show that the bivariate copulas are closely related to ordinary differential equations with right side F/x, $F \in \mathbb{F}$

$$z' = \frac{F(x, z)}{x}. \tag{3.1}$$

First, we will show that the quotients

$$g_y(x) = \frac{C(x, y)}{x}$$

are flows of the differential equations (3.1). Next, that the generalized flows of (3.1) are determining copulas. Note that the function $g_y(x)$ has a simple probabilistic interpretation

$$g_y(x) = \mathbb{P}(V \leq y \mid U \leq x),$$

where U and V are uniform random variables ($U, V \sim U(0, 1)$), whose joint distribution function is an extension of copula C.

Theorem 3.1. *For every bivariate copula $C(x, y)$ the function*

$$F_C(x, z) = D^- f_{x,z}(1) - z, \quad \textit{where} \quad f_{x,z}(t) = C_{[x]}(t, z) \tag{3.2}$$

belongs to $\mathbb{F}$*. Furthermore, for every* $y \in [0, 1]$ *the function*

$$g_y : (0, 1] \to [0, 1], \quad g_y(x) = \frac{C(x, y)}{x},$$

is a Carathéodory solution of the differential equation

$$xz' = F_C(x, z)$$

with a boundary condition

$$z(1) = y.$$

Example 3.1. Cuadras-Augé copula with parameter $\theta = 1/2$ (Excercise 2.5 in Nelsen, 2006). We write down the formula for Cuadras-Augé copula with the help of the characteristic functions

$$C(x, y) = xy^{1/2}\mathbb{I}_{x \le y} + x^{1/2}y\mathbb{I}_{x > y}.$$

Taking the left-side partial derivative we get

$$D_1^- C(x, y) = y^{1/2}\mathbb{I}_{x \le y} - \frac{1}{2}x^{-1/2}y\mathbb{I}_{x > y}.$$

Since the function

$$\psi(x, z) = zx^{1/2}\mathbb{I}_{z \in [0, x^{1/2})} + z^2\mathbb{I}_{z \in [x^{1/2}, 1]}$$

is a solution of the equation

$$\frac{C(x, \psi)}{x} = z,$$

we obtain

$$F_C(x, z) = -z + D_t^- C_{[x]}(t, z)_{|t=1} = -z + D_1^- C(x, \psi(x, z)) = -\frac{z}{2}\mathbb{I}_{z \in [0, x^{1/2})}.$$

It is easy to check that for every $y \in [0, 1]$ the function

$$g_y(x) = \frac{C(x, y)}{x} = y^{1/2}\mathbb{I}_{x \le y} + x^{-1/2}y\mathbb{I}_{x > y}$$

is a Carathéodory solution of the boundary problem

$$xz' = F_C(x, z), \quad z(1) = y.$$

Indeed, g_y is differentiable for $x \neq y$. For $x < y$ we have

$$g_y'(x) = 0.$$

Since for $x < y$ $g_y(x) > x^{1/2}$, $F_C(x, g_y(x)) = 0$ as well. For $x > y$

$$xg_y'(x) = -\frac{1}{2}x^{1/2}y = -\frac{g_y(x)}{2}.$$

Since then $g_y(x) < x^{1/2}$, $F_C(x, g_y(x)) = -g_y(x)/2$ as well.

Definition 3.2. For a function $F \in \mathbb{F}$ and a constant $x_0 \in (0, 1)$ we denote by $\mathbb{G}_F[x_0, 1]$ the set of absolutely continuous functions

$$g : [x_0, 1] \longrightarrow [0, 1]$$

such that

$$xg'(x) = F(x, g(x)) \text{ a.e. on } [x_0, 1]. \tag{3.3}$$

Theorem 3.2. *For every function $F(x, z)$, $F \in \mathbb{F}$, the function C defined on the unit square $[0, 1]^2$ by a formula*

$$C(x, y) = \begin{cases} x \inf\{g(x) : g \in \mathbb{G}_F[x, 1], g(1) \geq y\} & \text{for} \quad x \in (0, 1), \\ 0 & \text{for} \quad x = 0, \\ y & \text{for} \quad x = 1, \end{cases} \tag{3.4}$$

is a copula and

$$F(x, z) = F_C(x, z) \quad a.e.$$

Moreover,

(i) if $F_1, F_2 \in \mathbb{F}$ are equal almost everywhere to each other, then the generated copulas C_1 and C_2 coincide

$$\forall x, y \in [0, 1] \quad C_1(x, y) = C_2(x, y),$$

(ii) if $F_1, F_2 \in \mathbb{F}$ and $F_1(x, z) \geq F_2(x, z)$ almost everywhere then C_2 is bigger in the concordance ordering

$$\forall x, y \in [0, 1] \quad C_1(x, y) \leq C_2(x, y).$$

Remark 3.1. When the right side of the differential equation (3.1) is continuous then the formula for C simplifies. For $x \in (0, 1)$

$$C(x, y) = xg(x), \quad \text{where } g \in \mathbb{G}_F[x, 1], \quad g(1) = y.$$

Example 3.2. *Polynomial F of degree 2.*

We put

$$F(x, z) = z(z - 1), \quad z \in [0, 1].$$

It is readily to check that functions

$$g(x) = \frac{y}{y + x - xy}, x \in (0, 1], \quad y \in [0, 1]$$

are solutions of the boundary problems

$$xz' = F(x, z), \quad z(1) = y.$$

Furthermore, the copula C defined by formula (3.4) is the Clayton copula with parameter $\theta = 1$ (see Table 4.1 copula 4.2.1 in Nelsen, 2006). Indeed:

$$C(x, y) = xg(x) = \frac{xy}{y + x - xy} = (x^{-1} + y^{-1} - 1)^{-1}.$$

4. Operations on Copulas and Special Cases of Differential Equations

We discuss in this section the operations on copulas which correspond to transformations of differential equations and the simplest subfamilies of $\mathbb{F}$, such that there are known algorithms how to solve the differential equation

$$xz' = F(x, z).$$

It shows that they are generating the special families of copulas.

- When F does not depend on x we get copulas invariant with respect to the left-side conditioning.
- When F is separable, i.e., a product of functions of one variable, then we get distorted invariant copulas.
- When $z + F(x, z)$ is separable after substitution $z^*(x) = xz(x)$ we generate Archimedean copulas.

4.1. Conditioning and Flipping

Proposition 4.1. *If* C *is the copula then*

$$xz' = F_C(\alpha x, z)$$

is an ordinary differential equation generating the conditional copula $C_{[\alpha]}$ and

$$xz' = -F_C(x, 1 - z)$$

is an ordinary differential equation generating the y-dual copula $\tilde{C}$

$$\tilde{C}(x, y) = x - C(x, 1 - y).$$

4.2. Vertical Gluing

We recall a construction principle of copulas, given in Mesiar et al. (2008), which is based on some ideas also discussed in Durante et al. (2009) and Siburg and Stoimenov (2008). It is well known that if $\mathscr{I}$ is a finite or countable set of indices, $((a_i, b_i))_{i\in\mathscr{I}}$ is a collection of nonempty disjoint subintervals of [0, 1] and $(C_i)_{i\in\mathscr{I}}$ is a collection of copulas, then the function $G : [0,1]^2 \to [0,1]$, given by

$$G(x,y) = \begin{cases} a_i x + (b_i - a_i) C_i\left(x, \dfrac{y - a_i}{b_i - a_i}\right), & y \in (a_i, b_i), \\ xy, & \text{otherwise}, \end{cases} \tag{4.1}$$

is a copula.

The function G given by (4.1) is called *gluing ordinal sum* (shortly g-ordinal sum) of $(C_i)_{i\in\mathscr{I}}$ with respect to the collection $((a_i, b_i))_{i\in\mathscr{I}}$. Geometrically speaking, G is obtained by piecing together different copulas on consequent slices. Moreover, we have the following.

Proposition 4.2. *The copula G given by (4.1) is generated by an equation with right side*

$$F_G(x,z) = \begin{cases} (b_i - a_i) F_{C_i}\left(x, \dfrac{z - a_i}{b_i - a_i}\right), & z \in (a_i, b_i), \\ 0, & \textit{otherwise}. \end{cases} \tag{4.2}$$

4.3. Invariant Copulas

The copula C is called invariant iff $C = C_{[\alpha]}$ for all $\alpha \in (0,1]$. In Durante and Jaworski (2013) Theorem 3.1 the full characterization of invariant copulas was given. It shows that such copulas are generated by F's independent of x ($\forall x \quad F(x,z) = F(1,z)$). In more details, we have the following.

Proposition 4.3. *Copula C is invariant if and only if it is generated by $F(x,z) \in \mathbb{F}$, which almost everywhere is equal to the function $F^*(z)$ independent of x.*

Example 4.1. Examples of invariant copulas:

1. Copula of independent random variables

$$\Pi(x,y) = xy, \quad F(z) = 0.$$

2. Copula of comonotonic random variables

$$M(x,y) = \min(x,y), \quad F(z) = \begin{cases} -z & \text{for } z \in [0,1), \\ 0 & \text{for } z = 1. \end{cases}$$

3. Copula of anticomonotonic random variables

$$W(x,y) = \max(0, x+y-1), \quad F(z) = \begin{cases} 1 - z & \text{for } z \in (0,1], \\ 0 & \text{for } z = 0. \end{cases}$$

4. Clayton copulas with parameter θ, $\theta \in (-1,0) \cup (0,\infty)$,

$$Cl_\theta(x,y) = (\max(0, x^{-\theta} + y^{-\theta} - 1))^{-1/\theta}, \quad F(z) = -z + z^{1+\theta}.$$

5. Marshall-Olkin copulas with parameters $(\theta, 1)$, $\theta \in (0, 1)$,

$$MO(x, y) = \min(x, yx^{1-\theta}), \quad F(z) = \begin{cases} -\theta z & \text{for } z \in [0, 1), \\ 0 & \text{for } z = 1. \end{cases}$$

4.4. Equations with Separable Variables

Let us assume that the variables in the generating equation are separable, i.e., F is a product of two non-constant functions of one variable,

$$F(x, z) = F_1(x)F_2(z).$$

Since F_1 and F_2 are determined up to the multiplication by a non zero scalar, we may assume that F_2 is chosen in such a way that the function $z + F_2(z)$ is nondecreasing. The copulas given by solutions of such equations are distortions of invariant copulas studied in Mesiar and Pekárová (2010).

$$C(x, y) = \frac{x}{h(x)}\widetilde{C}(h(x), y),$$

where $\widetilde{C}$ is an invariant copula generated by $F_2(z)$ and

$$h(t) = \exp\left(-\int_t^1 \frac{F_1(\beta)}{\beta} d\beta\right).$$

Note that $F_1(x)$ is bounded because properties F3 and F2 imply that for every z we have

$$0 \leq z + F_1(x)F_2(z) \leq 1.$$

The above statement is a corollary of a more general theorem.

Theorem 4.1. *If $\widetilde{C}(x, y)$ is a copula generated by an equation*

$$xz' = F(x, z) \tag{4.3}$$

and $F_1(x)$ is a bounded measurable function on $(0, 1]$, such that the function $z + F_1(x)$ $F(x, z)$ is non decreasing in z, then the equation

$$xz' = F_1(x)F(x, z) \tag{4.4}$$

is generating the copula

$$C(x, y) = \begin{cases} \dfrac{x}{h(x)}\widetilde{C}(h(x), y) & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x = 0, \end{cases}$$

where

$$h(x) = \exp\left(-\int_x^1 \frac{F_1(\beta)}{\beta} d\beta\right).$$

Example 4.2. Examples of pairs of copulas generated by Eqs. (4.3) and (4.4):

1. Distortion of the comonotonic copula.
   If

$$F(x,z) = \begin{cases} -z & \text{for } z \in [0,1), \\ 0 & \text{for } z = 1, \end{cases} \quad \text{and} \quad F_1(x) = 1 - \theta x, \quad \theta \in (0,1].$$

   then

$$\widetilde{C}(x,y) = M(x,y) = \min(x,y), \quad h(t) = t\exp(\theta(1-t))$$

   and

$$C(x,y) = M_h(x,y) = \min\left(x, ye^{\theta(x-1)}\right).$$

2. Left-side averaging of copulas.
   Let a copula $\widetilde{C}$ be generated by the equation $xz' = F(x,y)$ and let

$$F_1(x) = \begin{cases} 0 & \text{for} \quad x < \frac{1}{2}, \\ 1 & \text{for} \quad x \geq \frac{1}{2}. \end{cases}$$

   Then the distortion function equals

$$h(t) = \exp\left(-\int_t^1 \frac{F_1(\beta)}{\beta} d\beta\right) = \begin{cases} \frac{1}{2} & \text{for} \quad t < \frac{1}{2}, \\ t & \text{for} \quad t \geq \frac{1}{2}. \end{cases}$$

   Hence, the copula generated by the equation

$$xz' = F_1(x)F(x,z)$$

   is given by the following formula:

$$C(x,y) = \frac{x}{h(x)}\widetilde{C}(h(x),y) = \begin{cases} 2x\widetilde{C}\left(\frac{1}{2}, y\right) & \text{for} \quad x < \frac{1}{2}, \\ \widetilde{C}(x,y) & \text{for} \quad x \geq \frac{1}{2}. \end{cases}$$

   Note that the described above averaging resembles (after the interchange of variables x and y) the construction of copulas with given horizontal section in Klement et al. (2007).
3. Horizontal gluing of the independent copula.
   Let a copula $\widetilde{C}$ be generated by the equation $xz' = F(x,y)$ and let

$$F_1(x) = \begin{cases} 1 & \text{for} \quad x < \frac{1}{2}, \\ 0 & \text{for} \quad x \geq \frac{1}{2}. \end{cases}$$

Then the distortion function equals

$$h(t) = \exp\left(-\int_t^1 \frac{F_1(\beta)}{\beta}d\beta\right) = \begin{cases} 2t & \text{for} \quad t < \dfrac{1}{2}, \\ 1 & \text{for} \quad t \geq \dfrac{1}{2}. \end{cases}$$

Hence, the copula generated by the equation

$$xz' = F_1(x)F(x, z)$$

is a horizontal gluing sum of the copula $\widetilde{C}$ and the independent copula $\Pi(x, y) = xy$.

$$C(x, y) = \frac{x}{h(x)}\widetilde{C}(h(x), y) = \begin{cases} \dfrac{1}{2}\widetilde{C}\,(2x, y) & \text{for} \quad x < \dfrac{1}{2}, \\ xy & \text{for} \quad x \geq \dfrac{1}{2}. \end{cases}$$

4.5. Archimedean Copulas

Let us assume that the variables in the generating equation are separable after substitution $z^*(x) = xz(x)$. We get

$$F(x, z) = -z + \frac{F_1(xz)}{F_1(x)}, \quad F_1(0) = 0, \quad F_1(x) \nearrow, \quad F_1(x) > 0 \text{ for } x > 0.$$

The copulas given by solutions of such equations are Archimedean copulas with generators

$$\varphi(t) = \int_t^1 \frac{d\beta}{F_1(\beta)}.$$

We recall that the bivariate copula C is Archimedean if and only if it can be expressed in the following way (Nelsen, 2006, Theorem 4.1.4):

$$C(x, y) = \psi(\varphi(x) + \varphi(y)),$$

where the generator $\varphi : [0, 1] \to [0, \infty]$ is a continuous, strictly decreasing, convex function such that $\varphi(1) = 0$ and $\psi : [0, \infty] \to [0, 1]$ is its left-side inverse

$$\psi(t) = \begin{cases} \varphi^{-1}(t) & \text{for} \quad t < \varphi(0), \\ 0 & \text{for} \quad t \geq \varphi(0). \end{cases}$$

Below we provide a new characterization of Archimedean copulas.

Theorem 4.2. *A copula C is Archimedean if and only if it is generated by $F \in \mathbb{F}$ such that*

$$F(x, z) = -z + \frac{F_1(xz)}{F_1(x)} \quad \textit{for } x \in (0, 1], z \in [0, 1),$$

where the function $F_1 : [0, 1] \to [0, \infty]$ *is non decreasing,* $F_1(x) = 0$ *if and only if* $x = 0$ *and* $F_1(x)$ *is finite for* $x < 1$. *Moreover if* φ *is a generator of* C *then one can put*

$$F_1(t) = \begin{cases} 0 & \textit{for} \quad t = 0, \\ -\dfrac{1}{\varphi'(t^-)} & \textit{for} \quad t > 0. \end{cases}$$

Note that both the generator φ and the function F_1 are determined up to the multiplication by a positive constant.

Example 4.3. Examples of Archimedean copulas.

1. The independence copula $\varphi(t) = -\ln(t)$:

$$\Pi(x, y) = xy, \quad F_1(t) = t.$$

2. The anticomonotonic copula $\varphi(t) = 1 - t$:

$$W(x, y) = \max(0, x + y - 1), \quad F_1(t) = \begin{cases} 0 & \text{for} \quad t = 0, \\ 1 & \text{for} \quad t > 0. \end{cases}$$

3. Clayton copulas with parameter $\theta \in (-1, 0) \cup (0, \infty)$, $\varphi(t) = \frac{1}{\theta}(t^{-\theta} - 1)$,

$$Cl_\theta(x, y) = ((x^{-\theta} + y^{-\theta} - 1)^+)^{-1/\theta}, \quad F_1(t) = t^{1+\theta}.$$

4. Gumbel copulas with parameter $\theta \in (1, \infty)$, $\varphi(t) = (-\ln(t))^\theta$

$$G_\theta(x, y) = \exp\left(-((-\ln(x))^\theta + (-\ln(x))^\theta)^{\frac{1}{\theta}}\right),$$

$$F_1(t) = \begin{cases} 0 & \text{for} \quad t = 0, \\ \dfrac{t}{\theta}(-\ln(t))^{1-\theta} & \text{for} \quad t \in (0, 1), \\ +\infty & \text{for} \quad t = 1. \end{cases}$$

5. Copulas with generators $\varphi(t) = \exp(t^{-\theta}) - e$, $\theta > 0$,

$$C_\theta(x, y) = [\ln(\exp(x^{-\theta}) + \exp(y^{-\theta}) - e)]^{-1/\theta}, \quad F_1(t) = t^{1+\theta}\exp(-t^{-\theta}).$$

5. Applications: Hereditary Properties of Copulas

Here, we deal with properties of copulas which are robust with respect to univariate conditioning, i.e., with the properties which are *inherited* by conditional copulas. Therefore, we will call them the *hereditary* properties. The similar notion of properties robust with respect to tail conditioning was investigated in Durante et al. (2008) under the name *hyper*-properties.

The hereditary properties of bivariate copulas can be characterized in terms of their generating differential equation. Let $\mathscr{C}(P)$ denote the set of bivariate copulas having a property P and $\mathbb{F}(P)$ the set of functions $F \in \mathbb{F}$ which generate copulas from $\mathscr{C}(P)$.

Theorem 5.1. *The property P is hereditary if and only if the set of generating functions $\mathbb{F}(P)$ is closed under the operation*

$$F(x, z) \longmapsto F(\alpha x, z) \tag{5.1}$$

for all $\alpha \in (0, 1]$, i.e.

$$\forall \alpha \in (0, 1] \quad F(x, z) \in \mathbb{F}(P) \Longrightarrow F(\alpha x, z) \in \mathbb{F}(P).$$

As examples of hereditary properties may serve the properties described in Sec. 4:

- invariance with respect to left-side conditioning;
- distorted invariance with respect to left side conditioning; and
- Archimedeanity.

Below we characterize two more hereditary properties.

5.1. Hereditary PQD and NQD Copulas

The notions of PQD and NQD are ones of the most encountered dependence properties. We recall the definition (Nelsen, 2006, Sec. 5.2.1).

Definition 5.1. Let X and Y be random variables. X and Y are positively quadrant dependent (PQD) if for all $x, y \in \mathbb{R}$

$$\mathbb{P}(X \leq x, Y \leq y) \geq \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y). \tag{5.2}$$

X and Y are negatively quadrant dependent (NQD) if for all $x, y \in \mathbb{R}$

$$\mathbb{P}(X \leq x, Y \leq y) \leq \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y). \tag{5.3}$$

Note that the random variables X and Y are PQD (respectively, NQD) if and only iff they admit a copula C such that

$$\forall u, v \in [0, 1] \quad C(u, v) \geq uv \quad (\text{resp. } C(u, v) \leq uv).$$

This leads to the following definitions:

Definition 5.2. Let C be a bivariate copula.

1. Copula C is PQD (resp. NQD) if $C(u, v) \geq uv$ (resp.$C(u, v) \leq uv$) for every u and v from the unit interval.
2. Copula C is hereditary PQD (resp. NQD) if for every $\alpha \in (0, 1]$ the conditional copula $C_{[\alpha]}$ is PQD (resp. NQD).

The hereditary PQD and NQD copulas are characterized by their generating equations.

Proposition 5.1. *The bivariate copula C is hereditary PQD (resp. NQD) if and only if it is generated by the differential equation with non negative (resp. non positive) right side*

$$F_C(x, z) \leq 0 \quad a.e. \ (\textit{resp. } F_C(x, z) \geq 0 \quad a.e.).$$

Corollary 5.1. *The vertical gluing sum of hereditary PQD (resp. NQD) copulas shares the same property. The vertical flipping is transforming hereditary PQD copulas to hereditary NQD ones and vice versa.*

5.2. Non Decreasing and Non Increasing Conditional Dependence

The objective of this section is the family of bivariate copulas $C(u, v)$, such that the conditional copulas $C_{[\alpha]}(u, v)$ are monotonic in concordance ordering. We denote the family copulas with non decreasing dependence when $\alpha \to 0$ by NDCD.

$$\alpha_1 \geq \alpha_2 \quad \Longrightarrow \quad \forall u, v \quad C_{[\alpha_1]}(u, v) \leq C_{[\alpha_2]}(u, v)$$

the smaller quantile the greater dependence. The other one, "the smaller quantile the smaller dependence," we denote by NICD. Note that both these families contain, the already described, copulas invariant with respect to left side conditioning.

Proposition 5.2. *The bivariate copula C is NDCD (resp. NICD) if and only if F_C is non decreasing (resp. non increasing) in first variable.*

Corollary 5.2. *The vertical gluing sum of NDCD (resp. NICD) copulas shares the same property. The vertical flipping is transforming NDCD copulas to NICD ones and vice versa.*

6. Proofs and Auxiliary Results

6.1. Proof of Proposition 2.1

The existence and uniqueness of $C_{[\alpha]}$ follow from the Sklar's Theorem (Nelsen, 2006, Theorem 2.3.3) and the fact that the right side of (2.1) is the restriction of the function

$$G(x, y) = \frac{1}{\alpha} C(\min(\alpha, x^+), \min(1, y^+)),$$

which is a continuous bivariate distribution function. To show the continuity of H we consider three convergent sequences (α_n), (x_n), and (z_n) from $(0, 1]$ and $[0, 1]$, respectively. Let

$$\lim_{n\to\infty} \alpha_n = \alpha_\infty > 0, \quad \lim_{n\to\infty} x_n = x_\infty, \quad \lim_{n\to\infty} z_n = z_\infty.$$

Let y_n be any solution of the equation

$$C(\alpha_n, y_n) = \alpha_n z_n.$$

We put

$$\liminf_{n\to\infty} y_n = y_\infty^-, \quad \limsup_{n\to\infty} y_n = y_\infty^+.$$

Obviously,

$$C(\alpha_\infty, y_\infty^-) = \alpha_\infty z_\infty = C(\alpha_\infty, y_\infty^+).$$

Therefore, since copulas are nondecreasing in second variable, we get

$$\begin{aligned}\liminf_{n\to\infty} C_{[\alpha_n]}(x_n, z_n) &= \liminf_{n\to\infty} \frac{1}{\alpha_n} C(\alpha_n x_n, y_n) \geq \frac{1}{\alpha_\infty} C(\alpha_\infty x_\infty, y_\infty^-) = C_{[\alpha_\infty]}(x_\infty, z_\infty)\\ &= \frac{1}{\alpha_\infty} C(\alpha_\infty x_\infty, y_\infty^+) \geq \limsup_{n\to\infty} \frac{1}{\alpha_n} C(\alpha_n x_n, y_n) = \limsup_{n\to\infty} C_{[\alpha_n]}(x_n, z_n).\end{aligned}$$

Hence, the limit exists and equals $C_{[\alpha_\infty]}(x_\infty, z_\infty)$. □

6.2. *Proof of Proposition 2.2*

We observe that

$$\frac{C(\alpha_1\alpha_2, v)}{\alpha_1} = C_{[\alpha_1]}\left(\alpha_2, \frac{C(\alpha_1, v)}{\alpha_1}\right).$$

Therefore,

$$\begin{aligned}\left(C_{[\alpha_1]}\right)_{[\alpha_2]}\left(\frac{u}{\alpha_1\alpha_2}, \frac{C(\alpha_1\alpha_2, v)}{\alpha_1\alpha_2}\right) &= \left(C_{[\alpha_1]}\right)_{[\alpha_2]}\left(\frac{u}{\alpha_1\alpha_2}, \frac{1}{\alpha_2} C_{[\alpha_1]}\left(\alpha_2, \frac{C(\alpha_1, v)}{\alpha_1}\right)\right)\\ &= \frac{1}{\alpha_2} C_{[\alpha_1]}\left(\frac{u}{\alpha_1}, \frac{C(\alpha_1, v)}{\alpha_1}\right) = \frac{1}{\alpha_1\alpha_2} C(u, v)\\ &= C_{[\alpha_1\alpha_2]}\left(\frac{u}{\alpha_1\alpha_2}, \frac{C(\alpha_1\alpha_2, v)}{\alpha_1\alpha_2}\right).\end{aligned}$$

Since the above is valid for any $u \in [\alpha_1\alpha_2, 1]$ and $v \in [0, 1]$, the formula (2.2) is valid for any $x, y \in [0, 1]$. Indeed, since C is continuous, the equations

$$x = \frac{u}{\alpha_1\alpha_2} \quad \text{and} \quad y = \frac{C(\alpha_1\alpha_2, v)}{\alpha_1\alpha_2},$$

have always solutions. □

6.3. *Proof of Proposition 2.3*

Functions

$$F_\eta(\beta, z) = \sup_{0<h<\eta} \frac{F(x_0, y) - F(x_0 - h, y)}{h}$$

are lower semicontinuous (Łojasiewicz, 1988, Theorem 3.3.3). Hence, they are Borel functions. Furthermore, they are non increasing when $\eta \to 0^+$. So their limit h

$$\begin{aligned}h(y) &= \limsup_{h\to 0+} \frac{F(x_0, y) - F(x_0 - h, y)}{h}\\ &= \lim_{n\to+\infty} \sup_{0<h<n^{-1}} \frac{F(x_0, y) - F(x_0 - h, y)}{h} = \lim_{n\to+\infty} F_{1/n}(y).\end{aligned}$$

exists and is Borel as well (Łojasiewicz, 1988, Theorem 4.4.8). □

6.4. Proof of Theorem 3.1

In virtue of Proposition 2.3 $F_C(x,z)$ is a Borel function. Condition (F1) follows from the boundary conditions for copulas (C1) and (C2). Indeed, since

$$f_{x,0}(t) = C_{[x]}(t,0) = 0 \quad \text{and} \quad f_{x,1}(t) = C_{[x]}(t,1) = t,$$

we get

$$F_C(x,0) = D^- f_{x,0}(1) - 0 = 0 \quad \text{and} \quad F_C(x,1) = D^- f_{x,1}(1) - 1 = 0.$$

The monotonicity in z (condition (F2)) follows from the two non decreasingness of copulas – condition (C2) (compare Durante and Jaworski, 2010a, Lemma 2.2). Indeed, let $0 \le z_1 \le z_2 \le 1$. Since copulas are two non decreasing, we have for $1 > h > 0$

$$C_{[x]}(1,z_1) - C_{[x]}(1-h,z_1) \le C_{[x]}(1,z_2) - C_{[x]}(1-h,z_2).$$

Therefore,

$$z_1 + F_C(x,z_1) = D^- f_{x,z_1}(1) = \limsup_{h\to 0+} \frac{C_{[x]}(1,z_1) - C_{[x]}(1-h,z_1)}{h} \le$$

$$\le \limsup_{h\to 0+} \frac{C_{[x]}(1,z_2) - C_{[x]}(1-h,z_2)}{h} = D^- f_{x,z_2}(1) = z_2 + F_C(x,z_2).$$

Let $g_y(t) = \frac{C(t,y)}{y}$. Based on Eq. (2.1), we get

$$f_{\alpha,g_y(\alpha)}(x) = C_{[\alpha]}\left(x, g_y(\alpha)\right) = x g_y(\alpha x), \quad x, y \in [0,1]$$

Having Dini-differentiated with respect to x at $x = 1$ we obtain

$$D^- f_{\alpha,g_y(\alpha)}(1) = g_y(\alpha) + D^- g_y(\alpha).$$

Note that, since C is Lipschitz (see Nelsen, 2006, Theorem 2.2.4), its Dini derivative is equal almost everywhere to the usual derivative. Therefore for every $y \in [0,1]$ the function g_y is a Carathéodory solution of the differential equation

$$xz' = F_C(x,z)$$

with boundary condition

$$z(1) = y.$$

□

6.5. Proof of Theorem 3.2

Before proving the theorem, we state and prove a couple of auxiliary results. First, we show that Eq. (3.1) is equivalent to nonlinear Volterra integral equation and has the following properties:

- the existence of solutions of an initial value problem and
- the uniqueness of solutions of a boundary value problem.

Lemma 6.1. *For any fixed point* $(x_0, y_0) \in (0, 1] \times [0, 1]$, *the following conditions are equivalent:*

a. $g \in \mathbb{G}_F[x_0, 1]$ *and* $g(x_0) = y_0$*; and*

b. g *is a solution of a nonlinear integral equation*

$$g(x) = \frac{1}{x}\left(x_0 y_0 + \int_{x_0}^{x} (F(t, g(t)) + g(t))dt\right), \quad x \in [x_0, 1]. \tag{6.1}$$

Proof. Since F is a Borel function, so is the composition $F(t, g(t))$, when $g : [x_0, 1] \to [0, 1]$ is a continuous function. Furthermore, if g belongs to $\mathbb{G}_F[x_0, 1]$, then $xg(x)$ is absolutely continuous and

$$(xg(x))' = g(x) + xg'(x) = g(x) + F(x, g(x)).$$

Hence,

$$xg(x) - x_0 g(x_0) = \int_{x_0}^{x} (F(t, g(t)) + g(t))\, dt,$$

which finishes the proof of $a \Rightarrow b$.

To show the second implication we observe that any solution g of (6.1) is absolutely continuous and almost everywhere on $[x_0, 1]$

$$g'(x) = -\frac{g(x)}{x} + \frac{1}{x}(F(x, g(x)) + g(x)) = \frac{1}{x} F(x, g(x)).$$

□

Lemma 6.2. *For every point* $(x_0, y_0) \in (0, 1] \times [0, 1]$ *there exists* $g \in \mathbb{G}_F[x_0, 1]$ *such that* $g(x_0) = y_0$.

Proof. Since F is a Borel function, so is the composition $F(t, z(t))$, when $z : [x_0, 1] \to [0, 1]$ is a Borel function. Furthermore the conditions (F2) and (F1) imply that

$$0 = F(t, 0) + 0 \le F(t, z) + z \le F(t, 1) + 1 = 1.$$

Therefore, the mapping Ψ

$$\Psi(z)(x) = \frac{1}{x}\left(x_0 y_0 + \int_{x_0}^{x} (F(t, z(t)) + z(t))dt\right)$$

maps the Borel functions $z : [x_0, 1] \to [0, 1]$ to Lipschitz functions from $[x_0, 1]$ to $[0, 1]$ with Lipschitz constant smaller than $3x_0^{-1}$. Furthermore, Ψ is non decreasing

$$\forall t \in [x_0, 1] \quad z_1(t) \le z_2(t) \Longrightarrow \forall x \in [x_0, 1] \quad \Psi(z_1)(x) \le \Psi(z_2)(x).$$

Let Λ be the set of all Lipschitz subsolution

$$\Lambda = \{z \in Lip[x_0, 1] : \forall t \in [x_0, 1] \quad z(t) \le \Psi(z)(t)\}.$$

We observe that Λ is not empty (because it contains the constant function $z(t) = 0$) and Ψ maps Λ into itself (because Ψ is non decreasing). The function

$$g(x) = \sup\{\Psi(z)(x) : z \in \Lambda\}$$

is the solution we are seeking for. Indeed, g is Lipschitz (compare Łojasiewicz, 1988, Theorem 3.2.2) and belongs to Λ

$$\forall z \in \Lambda \quad g \geq z \Rightarrow \forall z \in \Lambda \quad \Psi(g) \geq \Psi(z) \Rightarrow \Psi(g) \geq \sup\{\Psi(z) : z \in \Lambda\} = g.$$

Hence, g is the biggest element of Λ and $g = \Psi(g)$. Lemma 6.1 implies that g belongs to $\mathbb{G}_F[x_0, 1]$ and $g(x_0) = y_0$. □

Remark 6.1. The alternative way of proving Lemma 6.2 is to apply Theorem 3.1 from Hassan and Rzymowski (1999).

Lemma 6.3. *Let $F_1(x, z) \geq F_2(x, z)$ almost everywhere on $(0, 1] \times [0, 1]$. If for some point $x_0 \in (0, 1)$ and any two solutions $g_1 \in \mathbb{G}_{F_1}[x_0, 1]$ and $g_2 \in \mathbb{G}_{F_2}[x_0, 1]$*

$$g_1(x_0) > g_2(x_0),$$

then

$$\forall x \in [x_0, 1] \quad x(g_1(x) - g_2(x)) > x_0(g_1(x_0) - g_2(x_0)).$$

Proof. F_1 and F_2 are Borel functions and $F_1(x, z) \geq F_2(x, z)$ almost everywhere in $(0, 1] \times [0, 1]$. Therefore for x almost everywhere in $(0, 1]$, for every two points $z_1, z_2 \in [0, 1]$, $z_1 > z_2$, we can select a point $z^* \in (z_2, z_1)$ such that

$$F_1(x, z^*) \geq F_2(x, z^*).$$

Due to the monotonicity of the functions $z + F_i(x, z)$ in z we get

$$z_2 + F_2(x, z_2) \leq z^* + F_2(x, z^*) \leq z^* + F_1(x, z^*) \leq z_1 + F_1(x, z_1).$$

Let

$$x_1 = \inf\{x \in [x_0, 1] : x = 1 \vee g_1(x) \leq g_2(x)\}.$$

Since g_i are continuous, $x_1 > x_0$ and on $[x_0, x_1)$ g_2 is smaller then g_1. Due to Lemma 6.1 we have for $x \in [x_0, x_1]$

$$x(g_1(x) - g_2(x)) = x_0(g_1(x_0) - g_2(x_0)) + \int_{x_0}^{x} (F_1(t, g_1(t)) + g_1(t))dt - \int_{x_0}^{x} (F_2(t, g_2(t)) + g_2(t))dt$$

$$\geq x_0(g_1(x_0) - g_2(x_0)) > 0.$$

Therefore, $x_1 = 1$, which gives the thesis of Lemma. □

Corollary 6.5. *Let $F_1(x, z) \geq F_2(x, z)$ almost everywhere on $(0, 1] \times [0, 1]$. If for some point $x_0 \in (0, 1)$ and any two solutions $g_1 \in \mathbb{G}_{F_1}[x_0, 1]$ and $g_2 \in \mathbb{G}_{F_2}[x_0, 1]$ $g_1(1) = g_2(1)$, then $g_1(t) \leq g_2(t)$ for $t \in [x_0, 1]$.*

Corollary 6.6. *Let* $F_1(x, z) = F_2(x, z)$ *almost everywhere on* $(0, 1] \times [0, 1]$. *If for some point* $x_0 \in (0, 1)$ *and any two solutions* $g_1 \in \mathbb{G}_{F_1}[x_0, 1]$ *and* $g_2 \in \mathbb{G}_{F_2}[x_0, 1]$ $g_1(1) = g_2(1)$, *then* $g_1 = g_2$.

Lemma 6.4. *For any point* $(x, y) \in (0, 1)^2$

$$\inf\{g(x) : g \in \mathbb{G}_F[x, 1], g(1) \geq y\} = \sup\{g(x) : g \in \mathbb{G}_F[x, 1], g(1) \leq y\}.$$

Proof. Let

$$y_1 = \inf\{g(x) : g \in \mathbb{G}_F[x, 1], g(1) \geq y\}, \quad y_2 = \sup\{g(x) : g \in \mathbb{G}_F[x, 1], g(1) \leq y\}.$$

From Lemma 6.3 we get that $y_1 \geq y_2$. We show that the assumption $y_1 > y_2$ would lead to contradiction. Indeed, let us assume that we can select y_* such that

$$y_1 > y_* > y_2.$$

From Lemma 6.2 there exist a solution $g_* \in \mathbb{G}_F[x, 1]$ such that $g(x) = y_*$. Now from Lemma 6.3 we get that if $g_*(1) \geq y$ then $y_1 \leq y_*$ and if $g_*(1) \leq y$ then $y_2 \geq y_*$. In both cases, we obtain a contradiction. □

Proof of Theorem 3.2.

Step 1. We show that C fulfills the boundary conditions C1 and C2.
Directly from the definition we have that

$$C(0, y) = 0 \text{ and } C(1, y) = y.$$

To prove the other two conditions we observe that $F(x, 0) = F(x, 1) = 0$, hence there are two constant solutions

$$g_0(x) = 0 \text{ and } g_1(x) = 1, \quad x \in [0, 1].$$

Furthermore, the restriction of g_0 is the smallest element of any set of solutions $\mathbb{G}_F[x_0, 1]$ and the restriction of g_1 is the only solution g from $\mathbb{G}_F[x_0, 1]$ such that $g(1) \geq 1$. Therefore,

$$C(x, 0) = x \inf\{g(x) : g \in \mathbb{G}_F[x, 1], g(1) \geq 0\} = x \cdot g_0(x) = x \cdot 0 = 0,$$

and

$$C(x, 1) = x \inf\{g(x) : g \in \mathbb{G}_F[x, 1], g(1) \geq 1\} = x \cdot g_1(x) = x \cdot 1 = x.$$

Step 2. We show that $C(x, y)$ is non decreasing in second variable.
If $x = 0$ then $C(x, y)$ is constant (is equal to 0).
If $x = 1$ then $C(x, y) = y$ and is increasing.
For $x \in (0, 1)$ and $0 \leq y_1 < y_2 \leq 1$ we have

$$C(x, y_1) = x \inf\{g(x) : g \in \mathbb{G}_F[x, 1], g(1) \geq y_1\} \leq$$
$$\leq x \inf\{g(x) : g \in \mathbb{G}_F[x, 1], g(1) \geq y_2\} = C(x, y_2).$$

Step 3. We show that C fulfills the condition C3.
We have to show that the C-volume is non negative, i.e.,

$$V_C(x_1, y_1, x_2, y_2) = C(x_2, y_2) + C(x_1, y_1) - C(x_1, y_2) - C(x_2, y_1) \geq 0,$$

for $0 \leq x_1 \leq x_2 \leq 1$ and $0 \leq y_1 \leq y_2 \leq 1$. The case $x_1 = 1$ is obvious, indeed x_2 must be equal to 1 too and $V_C = 0$. The case $x_1 = 0$ follows from the more general one, when $C(x_1, y_1) = C(x_1, y_2)$. Indeed, if $C(x_1, y_1) = C(x_1, y_2)$ then

$$V_C(x_1, y_1, x_2, y_2) = C(x_2, y_2) - C(x_2, y_1)$$

and is non negative due to the monotonicity of C in second variable.
We start with the remaining case $x_1 \in (0, 1)$ and $C(x_1, y_1) < C(x_1, y_2)$. We approximate $C(x_1, y_1)$ from above and $C(x_1, y_2)$ from below. Namely,

$$C(x_1, y_1) = x_1 \inf\{g(x_1) : g \in \mathbb{G}_F[x_1, 1], g(1) \geq y_1\}.$$

Hence, there exists a sequence (g_n), $g_n \in \mathbb{G}_F[x_1, 1]$ and $g_n(1) \geq y_1$ such that

$$C(x_1, y_1) = x_1 \lim_{n\to\infty} g_n(x_1).$$

Note that

$$C(x_2, y_1) \leq x_2 g_n(x_2).$$

From Lemma 6.4 we know that

$$C(x_1, y_2) = x_1 \sup\{g(x_1) : g \in \mathbb{G}_F[x_1, 1], g(1) \leq y_2\}.$$

Hence, there exists a sequence (h_n), $h_n \in \mathbb{G}_F[x_1, 1]$ and $h_n(1) \leq y_2$ such that

$$C(x_1, y_2) = x_1 \lim_{n\to\infty} h_n(x_1).$$

Note that

$$C(x_2, y_2) \geq x_2 h_n(x_2).$$

Since $C(x_1, y_1) < C(x_1, y_2)$, we may assume that $g_n(x_1) < h_n(x_1)$ (if necessary we throw out some beginning terms). We consider

$$V_n = x_1(g_n(x_1) - h_n(x_1)) + C(x_2, y_2) - C(x_2, y_1).$$

Due to the above estimates of $C(x_2, y_2)$ and $C(x_2, y_1)$, and Lemma 6.3 we get

$$V_n \geq x_1(g_n(x_1) - h_n(x_1)) + x_2(h_n(x_2) - g_n(x_2)) =$$

$$= x_1(g_n(x_1) - h_n(x_1)) - x_2(g_n(x_2) - h_n(x_2)) \geq 0.$$

Since

$$V_C(x_1, y_1, x_2, y_2) = \lim_{n\to\infty} V_n,$$

the C-volume $V_C(x_1, y_1, x_2, y_2)$ is nonnegative.

Step 4. F is equivalent to F_C.

Let Δ be a subset of $(0, 1) \times [0, 1]$ containing points "non accessible" by the solutions from the right

$$\Delta = \{(x, z) : \exists y \in [0, 1] \quad xz = C(x, y) \wedge \frac{1}{x} C_y(x) \notin \mathbb{G}_F[x, 1]\},$$

where C_y denotes the horizontal section of C i.e. $C_y(x) = C(x, y)$. □

Lemma 6.5. *Δ has Lebesgue measure zero.*

Proof. Let us select a point (x, z) from Δ. Let $y \in [0, 1]$ be a point inducing non accessibility. As we show in Lemma 6.2 there exists $g \in \mathbb{G}_F[x, 1]$ such that $g(x) = z$. Obviously (see Lemma 6.3),

$$g(\xi) = \frac{1}{\xi} C_{g(1)}(\xi) = \frac{1}{\xi} C(\xi, g(1)), \quad \text{for } \xi \in [x, 1],$$

so $\frac{1}{\xi} C_{g(1)}(\xi) \in \mathbb{G}_F[x, 1]$. Hence, we have

$$C(x, y) = xz = C(x, g(1)) \quad \text{and} \quad g(1) \neq y.$$

C is constant on the segment $\mathbb{I} = \{x\} \times (\min(g(1), y), \max(g(1), y))$, hence $\frac{\partial C(x,v)}{\partial v}$ is vanishing on I. Furthermore, the mapping

$$\left(x, \frac{1}{x} C(x, y)\right) : (0, 1] \times [0, 1] \longrightarrow (0, 1] \times [0, 1]$$

is locally Lipschitz. Hence, the image of its critical points

$$\Sigma = \left\{(x, y) : \frac{\partial C(x, y)}{\partial y} = 0\right\}$$

has Lebesgue measure zero (compare Łojasiewicz, 1988, Theorem 7.6.8),

$$m_2\left(\left(x, \frac{1}{x} C(x, y)\right)(\Sigma_{x,y})\right) = 0.$$

Since Δ is a subset of the image of the critical points, it has Lebesgue measure zero. □

Next, we modify the right side of the differential equation at the zero measure set Δ.

$$F^*(x, z) = \begin{cases} F(x, z) & \text{for} \quad (x, z) \notin \Delta, \\ F_C(x, z) & \text{for} \quad (x, z) \in \Delta. \end{cases}$$

We recall that for each $y \in [0, 1]$, the function

$$g_y(x) = \frac{1}{x} C(x, y), \quad x \in (0, 1]$$

is a Carathéodory solution of the boundary problem

$$xz' = F^*(x, z), \quad z(1) = y.$$

Indeed, let

$$x_0 = \inf\{\xi : g_y \in \mathbb{G}_F[\xi, 1]\}.$$

For $x \geq x_0$ the point $(x, g_y(x))$ does not belong to Δ and

$$x g_y'(x) \overset{a.e.}{=} F(x, g_y(x)) = F^*(x, g_y(x)).$$

For $0 < x < x_0$ the point $(x, g_y(x))$ belongs to Δ and

$$x g_y'(x) \overset{a.e.}{=} F_C(x, g_y(x)) = F^*(x, g_y(x)).$$

To finish the proof of Step 4, we have to show that the set

$$\Delta^* = \{(x, z) : F^*(x, z) \neq F_C(x, z)\}$$

has Lebesgue measure zero. Since F is measurable, so is F^*. Therefore, the set Δ^* is measurable. Let Δ_1 be the Borel set approximating Δ^*

$$\Delta^* \supset \Delta_1 \quad \text{and} \quad m_2(\Delta^* \setminus \Delta_1) = 0.$$

Let Δ_2 be the $(x, C(x, y)/x)$-preimage of Δ_1. Δ_2 is a Borel set and

$$\Delta_2 \subset \{(x, y) : F^*(x, g_y(x)) \neq F_C(x, g_y(x))\}.$$

Since for every y the functions $g_y(x)$ are Carathéodory solutions of the boundary value problems

$$xz' = F^*(x, z), \quad z(1) = y,$$

and

$$xz' = F_C(x, z), \quad z(1) = y,$$

the intersections of Δ_2 with lines $y = const$ have Lebesgue measure zero. Therefore also Δ_2 has the measure zero. Since the mapping $(x, C(x, y)/x)$ is Lipschitz both Δ_1 and Δ^* have the measure zero.

Step 5. Inequality of right sides of differential equations implies inequality of copulas. To conclude the proof of the theorem one has to note that if almost everywhere $F_1 \geq F_2$ then the already proved Step 4 implies that almost everywhere

$$F_{C_1}(x, z) \geq F_{C_1}(x, z).$$

Hence, due to Corollary 6.1

$$\frac{1}{x} C_1(x, y) = g_1(x) \leq g_2(x) = \frac{1}{x} C_2(x, y),$$

i.e., $C_1 \leq C_2$. Obviously, if $F_1 = F_2$ $a.e.$, then $C_1 = C_2$. $\square$

6.6. *Proof of Proposition 4.1*

The formula for conditional copulas follows from Proposition 2.2.

$$F_{C_{[\alpha]}}(x,z)+z=\limsup_{t\to 1^+}\frac{\left(C_{[\alpha]}\right)_{[x]}(1,z)-\left(C_{[\alpha]}\right)_{[x]}(t,z)}{1-t}$$
$$=\limsup_{t\to 1^+}\frac{C_{[\alpha x]}(1,z)-C_{[\alpha x]}(t,z)}{1-t}=F_C(\alpha x,z)+z.$$

The formula for dual copulas follows from the fact that the vertical flipping and the left-side conditioning commute:

$$\tilde{C}_{[\alpha]}(x,y)=x-C_{[\alpha]}(x,1-y). \tag{6.2}$$

Therefore,

$$F_{\tilde{C}}(x,z)+z=\limsup_{t\to 1^+}\frac{\tilde{C}_{[x]}(1,z)-\tilde{C}_{[x]}(t,z)}{1-t}$$
$$=\limsup_{t\to 1^+}\frac{1-C_{[x]}(1,1-z)-t-C_{[x]}(t,1-z)}{1-t}$$
$$=1-F_C(x,1-z)-(1-z)=-F_C(x,1-z)+z.$$

$\square$

6.7. *Proof of Proposition 4.2*

The proof follows from the fact that the vertical gluing and the left side conditioning commute (compare Mesiar et al., 2008, Theorem 2).

$$G_{[\alpha]}(x,y)=\begin{cases} a_i x+(b_i-a_i)\,(C_i)_{[\alpha]}\left(x,\frac{y-a_i}{b_i-a_i}\right), & y\in(a_i,b_i),\\ xy, & \text{otherwise.}\end{cases} \tag{6.3}$$

For $z\in(a_i,b_i)$ we have

$$F_G(x,z)+z=\limsup_{t\to 1^+}\frac{a_i+(b_i-a_i)\,(C_i)_{[x]}\left(1,\frac{z-a_i}{b_i-a_i}\right)-a_i t-(b_i-a_i)\,(C_i)_{[x]}\left(t,\frac{z-a_i}{b_i-a_i}\right)}{1-t}$$
$$=a_i+(b_i-a_1)\left(F_{C_i}\left(x,\frac{z-a_i}{b_i-a_i}\right)+\frac{z-a_i}{b_i-a_i}\right)$$
$$=(b_i-a_1)F_{C_i}\left(x,\frac{z-a_i}{b_i-a_i}\right)+z.$$

For $z\notin\bigcup(a_i,b_i)$ we have

$$F_G(x,z)+z=\limsup_{t\to 1^+}\frac{z-tz}{1-t}=z.$$

$\square$

6.8. Proof of Proposition 4.3

If $C = C_{[\alpha]}$ for all $\alpha \in (0, 1]$ then

$$F_C(x, z) = \limsup_{t\to 1^-} \frac{C_{[\alpha]}(1, z) - C_{[\alpha]}(t, z)}{1 - t} = \limsup_{t\to 1^-} \frac{C(1, z) - C(t, z)}{1 - t}$$

does not depend on x.

On the other side, if $F(x, z)$ is almost everywhere equal to the function $F^*(z)$ independent of x, then for any $\alpha \in (0, 1]$

$$F(x, z) = F(\alpha x, z) \quad \text{almost everywhere.}$$

Therefore, due to Proposition 4.1, we get that $C = C_{[\alpha]}$. □

6.9. Proof of Theorem 4.1

First, we observe that the functions $g_y(x) = C(x, y)/x$ and $\widetilde{g}_y(x) = \widetilde{C}(x, y)/x$ are closely related. Indeed,

$$g_y(x) = \frac{C(x, y)}{x} = \frac{\widetilde{C}(h(x), y)}{h(x)} = \widetilde{g}_y(h(x)).$$

Having differentiated, we get that for x almost everywhere in $(0, 1]$

$$xg'_y(x) = x\widetilde{g}'_y(h(x))h'(x) = F(x, \widetilde{g}_y(h(x)))\frac{xh'}{h(x)} = F(x, g_y(x))F_1(x).$$

□

6.10. Proof of Theorem 4.2

It is well known (Mesiar et al., 2008, Remark 2.ii) that the left side conditional copula of the Archimedean copula C is given by the formula

$$C_{[\alpha]}(x, y) = \frac{1}{\alpha}\psi(\varphi(\alpha x) + \varphi(\alpha y) - \varphi(\alpha)).$$

Hence, taking the partial Dini derivative we get

$$F(x, z) + z = \psi'(\varphi(xz)^+)\varphi'(x^-) = \frac{\varphi'(x^-)}{\varphi'(zx^-)}.$$

We put

$$F_1(t) = \begin{cases} 0 & \text{for} \quad t = 0, \\ -\dfrac{1}{\varphi'(t^-)} & \text{for} \quad t > 0. \end{cases}$$

Since φ is convex and strictly decreasing, $F_1(t)$ is non decreasing, positive for $t > 0$ and finite for $t < 1$.

Now let us consider a differential equation

$$xz' = \begin{cases} -z + \dfrac{F_1(xz)}{F_1(x)} & \text{for} \quad x \in (0, 1], z \in [0, 1), \\ 0 & \text{for} \quad x \in (0, 1], z = 1, \end{cases}$$

where $F_1(t)$ is non decreasing, positive for $t > 0$ and finite for $t < 1$. Then the function φ given by the integral

$$\varphi(t) = \int_t^1 \frac{d\beta}{F_1(\beta)},$$

is convex and strictly decreasing. Hence, it is a generator of an Archimedean copula

$$C(x, y) = \psi(\varphi(x) + \varphi(y)).$$

Let

$$g(x) = \frac{C(x, y)}{x}.$$

Having differentiated, we get for x almost everywhere in (0, 1]

$$\begin{aligned} xg'(x) &= -g(x) + \psi'(\varphi(x) + \varphi(y))\varphi'(x) \\ &= \begin{cases} -g(x) + 0 & \text{for } \varphi(x) + \varphi(y) > \varphi(0) \\ -g(x) + \dfrac{\varphi'(x)}{\varphi'(\psi(\varphi(x) + \varphi(y)))} & \text{otherwise.} \end{cases} \end{aligned}$$

In both cases, we get

$$xg'(x) = -g(x) + \frac{F_1(xg(x))}{F_1(x)}.$$

Moreover $g(1) = y$. Hence, the differential equation under consideration is generating an Archimedean copula.

6.11. Proof of Theorem 5.1

Due to Proposition 4.1, we know that if $F(x, z)$ is generating copula C, then $F(\alpha x, z)$ is generating the conditional copula $C_{[\alpha]}$. Therefore, if the property P is hereditary i.e. together with C also $C_{[\alpha]}$ belong to $\mathscr{C}(P)$, then together with $F(x, z)$ also $F(\alpha x, z)$ belong to $\mathbb{F}(P)$. Hence, if the property P is hereditary, then $\mathbb{F}(P)$ is closed under the operation (5.1).

The proof of the adverse implication is similar. If a copula C has the property P, then it is generated by $F(x, z) \in \mathbb{F}(P)$. If also $F(\alpha x, z) \in \mathbb{F}(P)$, then also $C_{[\alpha]}$ has the property P. Hence, if $\mathbb{F}(P)$ is closed under the operation (5.1), then the property P is hereditary.

6.12. Proof of Proposition 5.4

We prove the PQD case. The other one is quite similar. For $C_{[\alpha]}(u, v) \geq uv$, we have

$$F_C(x, z) = -z + \limsup_{t \to 1^-} \frac{C_{[\alpha]}(1, z) - C_{[\alpha]}(t, z)}{1 - t} \leq -z + \frac{z - tz}{1 - t} = 0.$$

This finishes the proof of the implication "$\Rightarrow$". The proof of the other one goes as follows: Since $F_C(x, z) \leq 0$ the same is valid for $F_{C_{[\alpha]}}(x, z) = F_C(\alpha x, z)$ (compare Proposition 4.1). Therefore, $F_{C_{[\alpha]}}(x, z) \leq F_\Pi(x, z) = 0$ and due to Theorem 3.2 ,$C_{[\alpha]}(x, y) \geq \Pi(x, y) = xy$. □

6.13. Proof of Proposition 5.6

We prove the NDCD case. The other one is similar. Let C be a NDCD copula. For $x_1 > x_2$ we get

$$F_C(x_1, z) + z = \limsup_{h \to 0^+} \frac{z - C_{[x_1]}(1 - h, z)}{h} \leq \limsup_{h \to 0^+} \frac{z - C_{[x_2]}(1 - h, z)}{h} = F_C(x_2, z) + z.$$

Hence,

$$F(x_1, z) \geq F(x_2, z).$$

The proof of the adverse implication is based on Theorem 3.2. Indeed, for $\alpha_1 \geq \alpha_2$ we have

$$F_{C_{[\alpha_1]}}(x, z) = F_C(\alpha_1 x, z) \geq F_C(\alpha_2 x, z) = F_{C_{[\alpha_2]}}(x, z).$$

Therefore, $C_{[\alpha_1]} \leq C_{[\alpha_2]}$. □

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