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Coupling of Wiener processes by using copulas

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ABSTRACT

We study two-dimensional self-similar Ito diffusions (X, Y) whose margins are Wiener processes. We characterize the copulas of the random pairs (X_t, Y_t) for a given t.

1. Introduction

Our work here is strongly influenced by the paper of C. Sempi (Sempi, 2010) who was studying the possibility of coupling two Wiener processes by using a given copula. We refine his approach. Namely, we study two-dimensional Ito diffusions whose margins are Wiener processes and whose interdependences at every time moment are described by a given copula.

We recall the basic concepts.

A (bivariate) *copula* is a restriction to $[0, 1]^2$ of a distribution function whose univariate margins are uniformly distributed on $[0, 1]$. Specifically, $C: [0, 1]^2 \to [0, 1]$ is a copula if it satisfies the following properties:

(C1) $C(x, 0) = C(0, x) = 0$ for every $x \in [0, 1]$,
(C2) $C(x, 1) = C(1, x) = x$ for every $x \in [0, 1]$,
(C3) C is 2-*increasing*, that is, the C-volume V_C of any rectangle $[x_1, x_2] \times [y_1, y_2]$ of $[0, 1]^2$ is positive, i.e.

$$V_C([x_1, x_2] \times [y_1, y_2]) = C(x_1, y_1) + C(x_2, y_2) - C(x_1, y_2) - C(x_2, y_1) \geq 0.$$

Due to the celebrated *Sklar's theorem*, the joint distribution function F of any pair (X, Y) of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ can be written as a composition of a copula C and the univariate marginals F_1 and F_2, i.e. for all $(x, y) \in \mathbb{R}^2$, $F(x, y) = C(F_1(x), F_2(y))$. Moreover, if (X, Y) are continuous random variables, then the copula C is

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uniquely determined. For more details about copula theory and (some of) its applications, we refer the reader to Cherubini et al. (2004), Durante and Sempi (2010), Jaworski et al. (2010), Jaworski et al. (2013), Joe (1997) and Nelsen (2006).

A random process is self-similar if its distributions scale. Specifically (X_t, Y_t), $t \geq 0$, is called H-self-similar (with $H \geq 0$) when

$$(X_{at}, Y_{at}) \sim a^H (X_t, Y_t) \quad \text{for all } a \geq 0.$$

where $\sim$ denotes equality of joint distributions. We call H the exponent of self-similarity of the process. For example standard Brownian motion is 1/2-self-similar. For details on self-similarity we refer the reader to Embrechts and Maejima (2002) and Taqqu (2003).

A self-similar process (X_t, Y_t), $t \geq 0$, such that its marginals are Brownian motions can only be $1/2$-self-similar because the marginals are. Denoting by $F_t(x, y)$ the distribution function of the vector (X_t, Y_t) and also for short writing $F(x, y) = F_1(x, y)$, we can rewrite the definition of self-similarity as follows:

$$F_t(x, y) = F(t^{-1/2}x, t^{-1/2}y).$$

Since the marginals are centered, Gaussian, with variance t, we may equivalently write the above equality in terms of copulas:

$$F_t(x, y) = C(\Phi(t^{-1/2}x), \Phi(t^{-1/2}y)).$$

Throughout the work we use Φ and φ to denote the cumulative distribution function and density of the standard normal distribution ($N(0, 1)$).

2. The main results

We consider the solutions $(X_t, Y_t)_{t\geq 1}$ of the following system of stochastic differential equations:

$$\begin{aligned} dX_t &= dW_t^1, \\ dY_t &= A\left(\frac{X_t}{\sqrt{t}}, \frac{Y_t}{\sqrt{t}}\right) dW_t^1 + B\left(\frac{X_t}{\sqrt{t}}, \frac{Y_t}{\sqrt{t}}\right) dW_t^2, \quad A^2 + B^2 = 1, \end{aligned} \tag{1}$$

where W^1 and W^2 are two independent Wiener processes and A and B are continuous functions:

$$A, B : \mathbb{R}^2 \longrightarrow \mathbb{R}.$$

We understand the above as an integral equation:

$$Y_t = Y_1 + \int_1^t A\left(\frac{X_t}{\sqrt{t}}, \frac{Y_t}{\sqrt{t}}\right) dW_t^1 + \int_1^t B\left(\frac{X_t}{\sqrt{t}}, \frac{Y_t}{\sqrt{t}}\right) dW_t^2, \qquad X_t = W_t^1.$$

Theorem 2.1. *If a self-similar process $(X_t, Y_t)_{t\geq 1}$ is a solution of (1) and the distribution function F of (X_1, Y_1) is continuously twice differentiable on an open set $\mathbb{U} \subset \mathbb{R}^2$ then for $t \geq 1$ the distribution function of (X_t, Y_t) equals $F(t^{-1/2}x, t^{-1/2}y)$ and F satisfies the equation*

$$\frac{\partial^2 F(x, y)}{\partial x^2} + \frac{\partial^2 F(x, y)}{\partial y^2} + 2A(x, y)\frac{\partial^2 F(x, y)}{\partial x \partial y} + x\frac{\partial F(x, y)}{\partial x} + y\frac{\partial F(x, y)}{\partial y} = 0 \tag{2}$$

at all points $(x, y) \in \mathbb{U}$. Moreover the corresponding copula C does not depend on t and satisfies, at all points (u, v) such that $(\Phi^{-1}(u), \Phi^{-1}(v)) \in \mathbb{U}$,

$$\frac{\partial^2 C(u, v)}{\partial u^2}\varphi(\Phi^{-1}(u))^2 + \frac{\partial^2 C(u, v)}{\partial v^2}\varphi(\Phi^{-1}(v))^2 + 2A(\Phi^{-1}(u), \Phi^{-1}(v))\frac{\partial^2 C(u, v)}{\partial u \partial v}\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v)) = 0. \tag{3}$$

Remark 2.1. Eq. (2) is closely connected with the Fokker–Planck equation (see Schuss, 2010, Section 4.5). Namely when both F and its density $f(x, y)$ are twice differentiable then the density of F_t

$$p(t, x, y) = \frac{1}{t} f\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right)$$

is a solution of the Fokker–Planck equation

$$\frac{\partial p(t, x, y)}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 p(t, x, y)}{\partial x^2} + \frac{\partial^2 p(t, x, y)}{\partial y^2} + 2\frac{\partial^2 A(t^{-1/2}x, t^{-1/2}y)p(t, x, y)}{\partial x \partial y}\right),$$

with initial value $p(1, x, y) = f(1, x, y)$.

Remark 2.2. Eq. (2) is elliptic. Therefore, if two copulas that are twice differentiable on $(0,1)^2$ fulfill Eq. (3) with the same A then they are equal.

Theorem 2.2. *Let C be a twice-differentiable copula such that for all $(u,v) \in (0,1)^2$*

$$\left| \frac{\varphi(\Phi^{-1}(u))}{\varphi(\Phi^{-1}(v))} \frac{\partial^2 C(u,v)}{\partial u^2} + \frac{\varphi(\Phi^{-1}(v))}{\varphi(\Phi^{-1}(u))} \frac{\partial^2 C(u,v)}{\partial v^2} \right| \le 2 \frac{\partial^2 C(u,v)}{\partial u \partial v} \tag{4}$$

and

$$A(x,y) = -\frac{\partial_1^2 C(\Phi(x),\Phi(y))\varphi(x)^2 + \partial_2^2 C(\Phi(x),\Phi(y))\varphi(y)^2}{2\,\partial_{1,2}^2 C(\Phi(x),\Phi(y))\varphi(x)\varphi(y)}, \tag{5}$$

when $\partial_{1,2}^2 C(\Phi(x),\Phi(y)) \neq 0$. Then every solution $(X_t,Y_t)_{t\ge 1}$ of (1) having initial values (X_1,Y_1) with a distribution function $C(\Phi(x),\Phi(y))$ is a self-similar process whose margins are Wiener processes linked by C.

Corollary 2.1. *If a twice-differentiable copula C fulfills inequality (4) and the functions $A(x,y)$ given by (5) and $B(x,y) = \sqrt{1-A(x,y)^2}$ are Lipschitz then there exists a self-similar process whose margins are Wiener processes with interdependences described by the copula C.*

Example 2.1. The conditions of Corollary 2.1 are fulfilled by Gaussian copulas and FGM copulas (see Sections 3.3 and 3.4).

Example 2.2. The absolutely continuous copula constructed in Example 2.1 Durante and Jaworski (2008) is equal to

$$C_\delta(x,y) = \frac{1}{2}\left(x^{1.5} + y^{1.5}\right)$$

in a sufficiently small neighborhood of the point $(0.5, 0.5)$. Therefore it fulfills neither inequality (4) nor Eq. (3).

3. Proofs and auxiliary results

3.1. Proof of Theorem 2.1

The distribution function of (X_t,Y_t) can be expressed as the expected value of an indicator function

$F_t(x,y) = \mathbb{E}(\mathbb{1}_{\{X_t\le x\}}\mathbb{1}_{\{Y_t \le y\}})$.

Let us consider the function g given by

$$g(t,x,y,X,Y) = (x\sqrt{t} - X)_+^3 (y\sqrt{t} - Y)_+^3 \tag{6}$$

and observe that

$$\begin{aligned} \mathbb{E}\left(\frac{\partial^6 g}{\partial x^3 \partial y^3}(t,x,y,X_t,Y_t)\right) &= 36t^3 \mathbb{E}\left(\mathbb{1}_{\{\frac{X_t}{\sqrt{t}}\le x\}}\mathbb{1}_{\{\frac{Y_t}{\sqrt{t}}\le y\}}\right) \\ &= 36t^3 F_t(x\sqrt{t}, y\sqrt{t}) = 36t^3 F(x,y). \end{aligned} \tag{7}$$

In order to calculate the expected values of the process $g(t,x,y,X_t,Y_t)$ we apply the Ito lemma.

$$\begin{aligned} dg &= \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial X}dX_t + \frac{\partial g}{\partial Y}dY_t + \frac{1}{2}\left(\frac{\partial^2 g}{\partial X^2}d[X,X]_t + \frac{\partial^2 g}{\partial Y^2}d[Y,Y]_t + 2\frac{\partial^2 g}{\partial X \partial Y}d[X,Y]_t\right) \\ &= \left(\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial X^2} + \frac{1}{2}\frac{\partial^2 g}{\partial Y^2} + \frac{\partial^2 g}{\partial X \partial Y} A\left(\frac{X_t}{\sqrt{t}}, \frac{Y_t}{\sqrt{t}}\right)\right) dt + (\ldots)dW_t^1 + (\ldots)dW_t^2. \end{aligned} \tag{8}$$

The Ito integral part vanishes under the expectation sign. We calculate the partial derivatives of the function g which showed up in the above expression:

$$\frac{\partial g}{\partial t} = \frac{3}{2}t^2\left[x(x - t^{-\frac{1}{2}}X)_+^2(y - t^{-\frac{1}{2}}Y)_+^3 + y(x - t^{-\frac{1}{2}}X)_+^3(y - t^{-\frac{1}{2}}Y)_+^2\right]$$

$$\frac{\partial^2 g}{\partial X^2} = 6t^2(x - t^{-\frac{1}{2}}X)_+(y - t^{-\frac{1}{2}}Y)_+^3,$$

$$\frac{\partial^2 g}{\partial Y^2} = 6t^2(x - t^{-\frac{1}{2}}X)_+^3(y - t^{-\frac{1}{2}}Y)_+,$$

$$\frac{\partial^2 g}{\partial X \partial Y} = 9t^2(x - t^{-\frac{1}{2}}X)_+^2(y - t^{-\frac{1}{2}}Y)_+^2.$$

Moreover, for fixed x, y the process

$$\Psi(t,x,y,X_t,Y_t) := \left[\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial X^2} + \frac{1}{2}\frac{\partial^2 g}{\partial Y^2} + A\frac{\partial^2 g}{\partial X \partial Y}\right](t,x,y,X_t,Y_t)$$

is progressively measurable, and

$$\mathbb{E}\left(\int_1^t |\Psi(s,x,y,X_s,Y_s)|\,ds\right) < \infty.$$

Therefore we can use the Fubini theorem (see Rudin, 2000, remarks to Theorem 8.8) to change the order of integration:

$$\begin{aligned}(t^3-1)\mathbb{E}(g(1,x,y,X_1,Y_1)) &\overset{\star}{=} \mathbb{E}(g(t,x,y,X_t,Y_t) - g(1,x,y,X_1,Y_1)) \\ &= \mathbb{E}\left(\int_1^t \Psi(s,x,y,X_s,Y_s)ds\right) = \int_1^t \mathbb{E}\left(\Psi(s,x,y,X_s,Y_s)\right)ds \\ &\overset{\star}{=} \int_1^t s^2\mathbb{E}\left(\Psi(1,x,y,X_1,Y_1)\right)ds = \frac{1}{3}(t^3-1)\mathbb{E}(\Psi(1,x,y,X_1,Y_1)).\end{aligned}$$

The equalities denoted with $(\star)$ follow by the $\frac{1}{2}$-self-similarity of (X_t, Y_t). Summing up, we have obtained the relationship

$$\mathbb{E}(g(1,x,y,X_1,Y_1)) = \frac{1}{3}\mathbb{E}(\Psi(1,x,y,X_1,Y_1)). \tag{9}$$

In the following we shall differentiate functions given as expected values of some convex functionals of Gaussian variables (compare Lemma 4.11 of Jaworski, 2006).

Using previously obtained equalities we see that

$$\frac{1}{3}\frac{\partial^6 \mathbb{E}(\Psi(1,x,y,X_1,Y_1))}{\partial x^3 \partial y^3} = \mathbb{E}\left(\frac{\partial^6 g}{\partial x^3 \partial y^3}(1,x,y,X_1,Y_1)\right) = 36F(x,y). \tag{10}$$

The last step is to calculate the derivative $\frac{\partial^6}{\partial x^3 \partial y^3}$ of $\mathbb{E}(\Psi(1,x,y,X_1,Y_1))$. Let us start with $\frac{\partial^2 g}{\partial X^2}(1,x,y,X_1,Y_1)$ and $\frac{\partial^2 g}{\partial Y^2}(1,x,y,X_1,Y_1)$:

$$\frac{1}{3}\frac{\partial^6}{\partial x^3 \partial y^3}\mathbb{E}\left[\frac{1}{2}\frac{\partial^2 g}{\partial X^2}\right] = 6\frac{\partial^2}{\partial x^2}\mathbb{E}\left(\mathbb{1}_{\{X_1 \le x\}}\mathbb{1}_{\{Y_1 \le y\}}\right) = 6\frac{\partial^2 F(x,y)}{\partial x^2},$$

$$\frac{1}{3}\frac{\partial^6}{\partial x^3 \partial y^3}\mathbb{E}\left[\frac{1}{2}\frac{\partial^2 g}{\partial Y^2}\right] = 6\frac{\partial^2}{\partial y^2}\mathbb{E}\left(\mathbb{1}_{\{X_1 \le x\}}\mathbb{1}_{\{Y_1 \le y\}}\right) = 6\frac{\partial^2 F(x,y)}{\partial y^2}.$$

$\frac{\partial g}{\partial t}(1,x,y,X_1,Y_1)$ is a sum of two components. Their derivatives are given by

$$\frac{1}{3}\frac{\partial^6}{\partial x^3 \partial y^3}\mathbb{E}\left[\frac{3}{2}x(x-X_1)_+^2(y-Y_1)_+^3\right] = 18F(x,y) + 6x\frac{\partial F(x,y)}{\partial x},$$

$$\frac{1}{3}\frac{\partial^6}{\partial x^3 \partial y^3}\mathbb{E}\left[\frac{3}{2}y(x-X_1)_+^3(y-Y_1)_+^2\right] = 18F(x,y) + 6y\frac{\partial F(x,y)}{\partial y}.$$

Finally

$$\begin{aligned}\frac{1}{3}\frac{\partial^6}{\partial x^3 \partial y^3}\mathbb{E}\left[\frac{\partial^2 g}{\partial X \partial Y}A(X_1,Y_1)\right] &= 12\frac{\partial^2}{\partial x \partial y}\mathbb{E}\left[\mathbb{1}_{\{X_1 \le x\}}\mathbb{1}_{\{Y_1 \le y\}}A(X_1,Y_1)\right] \\ &= 12\frac{\partial^2}{\partial x \partial y}\int_{x_o}^x \int_{y_o}^y A(u,v)\frac{\partial^2 F(u,v)}{\partial u \partial v}dv du = 12A(x,y)\frac{\partial^2 F(x,y)}{\partial x \partial y}\end{aligned}$$

where x_o and y_o are chosen so close to x and y that the rectangle $[x_o, x] \times [y_o, y]$ is contained in $\mathbb{U}$.

By substitution into Eq. (10) we obtain

$$36F(x,y) = 36F(x,y) + 6x\frac{\partial F(x,y)}{\partial x} + 6y\frac{\partial F(x,y)}{\partial y} + 6\frac{\partial^2 F(x,y)}{\partial x^2} + 6\frac{\partial^2 F(x,y)}{\partial y^2} + 12A(x,y)\frac{\partial^2 F(x,y)}{\partial x \partial y} \tag{11}$$

which finishes the proof of Eq. (2).

The marginals of the random vector (X_1, Y_1) are standard normally distributed. Moreover we assumed that the distribution of (X_1, Y_1) is continuous; hence by Sklar's theorem we may write its cumulative distribution function F in terms of a copula C:

$$F(x,y) = C(\Phi(x), \Phi(y)).$$

In the following we define $u = \Phi(x)$, $v = \Phi(y)$. The partial derivatives are given by

$$\frac{\partial F(x,y)}{\partial x} = \frac{\partial C(u,v)}{\partial u}\phi(x), \qquad \frac{\partial F(x,y)}{\partial y} = \frac{\partial C(u,v)}{\partial v}\phi(y),$$
$$\frac{\partial^2 F(x,y)}{\partial x^2} = \frac{\partial^2 C(u,v)}{\partial u^2}\phi^2(x) - \frac{\partial C(u,v)}{\partial u}x\phi(x),$$
$$\frac{\partial^2 F(x,y)}{\partial y^2} = \frac{\partial^2 C(u,v)}{\partial v^2}\phi^2(y) - \frac{\partial C(u,v)}{\partial v}y\phi(y),$$
$$\frac{\partial F(x,y)}{\partial x\partial y} = \frac{\partial C(u,v)}{\partial u\partial v}\phi(x)\phi(y).$$

Substituting into (2), we get Eq. (3).

3.2. Proof of Theorem 2.2

Let L^- and L^+ be differential operators acting on twice-differentiable functions in the following way:

$$L^{\pm}h(x,y) = \pm\frac{x}{2}\frac{\partial h}{\partial x}(x,y) \pm \frac{y}{2}\frac{\partial h}{\partial y}(x,y) + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(x,y) + \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(x,y) + A(x,y)\frac{\partial^2 h}{\partial x\partial y}(x,y). \tag{12}$$

Note that

$$L^+C(\Phi(x),\Phi(y)) = 0.$$

We put

$$\tilde{X}_t = \frac{X_t}{\sqrt{t}}, \qquad \tilde{Y}_t = \frac{Y_t}{\sqrt{t}}.$$

We will show that for every function h that is three times continuously differentiable with compact support, we have

$$E(h(\tilde{X}_t,\tilde{Y}_t)) = E(h(\tilde{X}_1,\tilde{Y}_1)),$$

which implies the self-similarity of the process (X_t, Y_t).

First we observe that due to the Ito formula we get

$$E(h(\tilde{X}_t,\tilde{Y}_t)) - E(h(\tilde{X}_1,\tilde{Y}_1)) = E\left(\int_1^t L^-(h)(\tilde{X}_s,\tilde{Y}_s)ds\right) = \int_1^t E(L^-(h)(\tilde{X}_s,\tilde{Y}_s))ds.$$

Let

$$u(t,x,y) = E(h(\tilde{X}_t,\tilde{Y}_t)|(\tilde{X}_1,\tilde{Y}_1) = (x,y)).$$

Applying Kolmogorov's backward equation $\frac{\partial}{\partial t}u = L^-u$ (see Theorem 8.1 of Oksendal, 1989) we have

$$\begin{aligned}\frac{\partial}{\partial t}E(h(\tilde{X}_t,\tilde{Y}_t)) &= E(L^-(h)(\tilde{X}_t,\tilde{Y}_t))\\ &= E\left(E(L^-(h)(\tilde{X}_t,\tilde{Y}_t)|\sigma(\tilde{X}_1,\tilde{Y}_1))\right) = E\left(\frac{\partial}{\partial t}u(t,\tilde{X}_1,\tilde{Y}_1)\right)\\ &= E(L^-u(t,\tilde{X}_1,\tilde{Y}_1)) = \int_{\mathbb{R}^2} L^-u(t,x,y)\frac{\partial^2 C(\Phi(x),\Phi(y))}{\partial x\partial y}dxdy\\ &= \int_{\mathbb{R}^2}\frac{\partial^2 u}{\partial x\partial y}(x,y)L^+C(\Phi(x),\Phi(y))dxdy = 0.\end{aligned}$$

Hence $E(h(\tilde{X}_t,\tilde{Y}_t))$ is constant. Note that the duality between L^- and L^+ applied above follows from integration by parts.

3.3. Gaussian copula

The Gaussian copula is defined as follows:

$$C(u,v) = \Phi_\rho(\Phi^{-1}(u),\Phi^{-1}(v)),$$

where Φ_ρ is the joint distribution function of a bi-dimensional standard normal vector, with linear correlation coefficient ρ.

As for the conditional distribution via a copula we have (compare Section 3.2.1 of Cherubini et al., 2004)

$$\frac{\partial C}{\partial u}(u, v) = \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right).$$

Taking derivatives yields

$$\frac{\partial^2 C}{\partial u^2}(u, v) = \frac{-\rho}{\sqrt{1-\rho^2}}\phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right)\frac{1}{\phi(\Phi^{-1}(u))}$$

$$\frac{\partial^2 C}{\partial v^2}(u, v) = \frac{-\rho}{\sqrt{1-\rho^2}}\phi\left(\frac{\Phi^{-1}(u) - \rho\Phi^{-1}(v)}{\sqrt{1-\rho^2}}\right)\frac{1}{\phi(\Phi^{-1}(v))}$$

$$\frac{\partial^2 C}{\partial u\partial v}(u, v) = \frac{1}{\sqrt{1-\rho^2}}\exp\left(\frac{\Phi^{-1}(u)^2 + \Phi^{-1}(v)^2}{2}\right)\exp\left(\frac{2\rho\Phi^{-1}(u)\Phi^{-1}(v) - \Phi^{-1}(u)^2 - \Phi^{-1}(v)^2}{2(1-\rho^2)}\right).$$

Substituting the above into Eq. (3) we get

$$A(x, y) = \rho, \qquad B(x, y) = \sqrt{1-\rho^2},$$

which is in line with the standard result for correlated Brownian motions (see Example 4.6.6. of Shreve, 2003).

3.4. FGM copula

The Farlie–Gumbel–Morgenstern copula is defined as

$$C(u, v) = uv\left(1 + a(1-u)(1-v)\right), \quad \text{for } a \in [-1, 1].$$

Its conditional distribution is given by

$$\frac{\partial C}{\partial u}(u, v) = v\left[1 + a(1-2u)(1-v)\right].$$

In the following we shall also need further derivatives:

$$\frac{\partial^2 C}{\partial u^2}(u, v) = 2av(v-1), \qquad \frac{\partial^2 C}{\partial v^2}(u, v) = 2au(u-1)$$

$$\frac{\partial^2 C}{\partial u\partial v}(u, v) = 1 + a(1-2u)(1-2v).$$

Substituting into Eq. (3), and defining $u = \Phi(x)$, $v = \Phi(y)$ we have

$$A(x, y) = a\frac{\Phi(y)\Phi(-y)e^{\frac{1}{2}(y^2-x^2)} + \Phi(x)\Phi(-x)e^{\frac{1}{2}(x^2-y^2)}}{1 + a(1-2\Phi(x))(1-2\Phi(y))}.$$

One can easily see that

$$|A| \le |a|\frac{\Phi(y)\Phi(-y)e^{\frac{1}{2}(y^2-x^2)} + \Phi(x)\Phi(-x)e^{\frac{1}{2}(x^2-y^2)}}{1 - |(1-2\Phi(x))(1-2\Phi(y))|}.$$

Due to the symmetry of the above expression it is enough to show the following inequality for $y \ge x \ge 0$:

$$\frac{\Phi(y)\Phi(-y)e^{\frac{1}{2}(y^2-x^2)} + \Phi(x)\Phi(-x)e^{\frac{1}{2}(x^2-y^2)}}{1 - (1-2\Phi(x))(1-2\Phi(y))} \le \frac{1}{2}$$

which is equivalent to $\chi(x, y) \le 1$, where

$$\chi(x, y) = 2\Phi(y)\Phi(-y)e^{\frac{1}{2}(y^2-x^2)} + 2\Phi(x)\Phi(-x)e^{\frac{1}{2}(x^2-y^2)} + (1-2\Phi(x))(1-2\Phi(y)).$$

We can see that:

1. $\chi(x, x) = 1$;
2. $\lim_{y\to\infty} \chi(x, y) = 2\Phi(x) - 1 < 1$;
3. $\lim_{x,y\to\infty} \chi(x, y) = 1$;
4. the directional derivative

$$y\frac{\partial \chi(x, y)}{\partial x} + x\frac{\partial \chi(x, y)}{\partial y} = 2(y-x)\left[\phi(x)\left(2\Phi(y) - 1\right) - \phi(y)\left(2\Phi(x) - 1\right)\right]$$

is positive for $y > x \ge 0$.

Hence χ achieves its maximum at the diagonal (x, x).

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