Lecture 3 - Homographies

① In homogeneous coordinates, infinite points can be represented by putting $w = 0$

$\Rightarrow (x, y) = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} x \\ y \\ w \end{bmatrix} = (x/w, y/w)$

converting to homogeneous image coordinates $\qquad$ converting from homogeneous image coordinates.

note → In Eucleidian space, two parellel lines never meet (meet at infinity)
" Projective space, " " " meet at $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$

② Homogeneous coordinates (P) $\qquad p = (x, y, 1)^T \propto w(x, y, 1)^T, \ \forall w \neq 0$

$\propto$ → proportional to; w → unknown scale factor

* A vector in P (projective space) is just a representation of an equivalence class of vector.
* Everything is upto scale.

③ Homography → Projective Transformation

(a) A homography is a non-singular, line preserving, projective mapping, linear transformation. $h : P^n \longrightarrow P^n$.
can be inverted
points after transformation will still be points.

(b) It is represented by a $(n+1)^{th}$ dimension square matrix with $(n+1)^2 - 1$ DOF.

(c) Homographies are generally not restricted to P^2

(d) Also called "Projectivity" or "collineation".

④ 2D Homography. → A 2D homography is an invertible mapping "h" from P^2 to itself such that the three points (x_1, x_2, x_3) lie on the same line if and only if $h(x_1)$, $h(x_2)$ and $h(x_3)$ do.

homography in an image plane

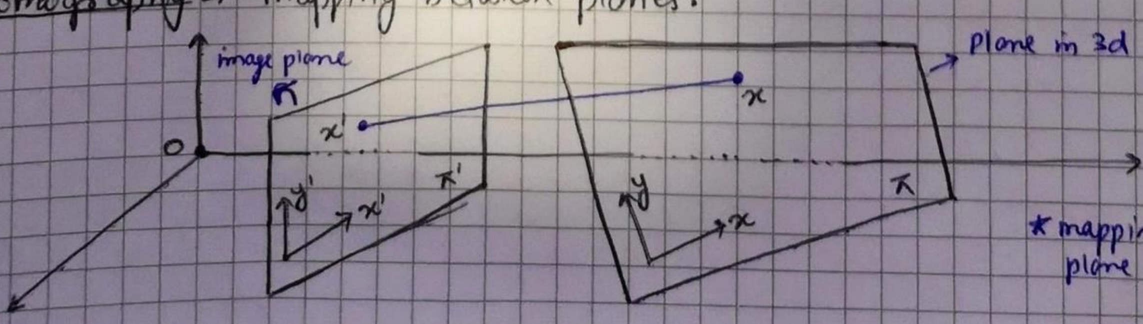
→ line preserving and non-singular (invertible)

Theorem → A mapping $h: P^2 \rightarrow P^2$ is a homography iff there exist a non-singular 3×3 matrix "H" such that for any point in P^2 represented by a vector x, it is true that $h(x) = H(x)$

no inverse determinant = 0

⑤. $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \simeq \boxed{x' = Hx}$ → 8 DOF

⑥ Homography → mapping between planes.

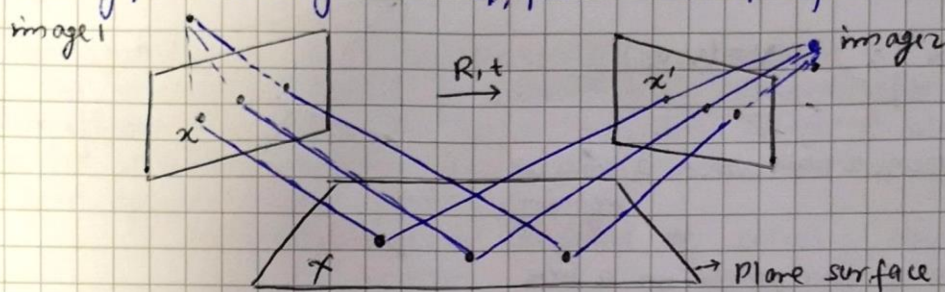


* mapping from 3d to image plane through homography

* If we have a point outside of our 3d plane, it is possible to project it to the image plane by (P) projective space. whereas, the homography b/w the 2 planes will not transform the point.
* If we transform a point out of the plane, it will not be correctly mapped.

⑥ Homography in computer vision.

(a) Rotating / translating camera, planer world / object :



Q) what happens to the P-matrix, if z is assumed to be 0?

$$(x, y, 1)^T = x \propto PX = K[\underbrace{r_1 r_2}_{R} \cancel{r_3}\, t]\begin{pmatrix} X \\ Y \\ \cancel{0} \\ 1 \end{pmatrix} = \underset{\text{Homography!}}{H}\begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}$$

$\therefore H = K[r_1 r_2 t]$
$= K[R\, t]$

upper triangular matrix.

⑦ How can we classify homography?

→ Transformation hierarchy → <u>Isometries</u> (Iso - same metrics - measure)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$\varepsilon = \pm 1$ orientation preserving $\varepsilon = 1$
" " reversing $\varepsilon = -1$

$$x' = H_E x = \begin{bmatrix} \overset{2\times2}{R} & \overset{2\times1}{t} \\ \underset{1\times2}{0^T} & \underset{1\times1}{1} \end{bmatrix} x \qquad R^T R = I$$

} 3 DoF (1 rotation, 2 translation)
Special cases : pure rotation, pure translation
invariants : length, area, angle.

→ Transformation hierarchy → <u>Similarity</u> (isometry + scale)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$x' = H_S x = \begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix} x \qquad R^T R = I$$

} 4 DoF (1 scale, 1 rotation, 2 translation)
Also known as equiform (shape preserving)
metric structure = structure up to similarity
invariants : ratios of length, angle, ratios of areas, parallel lines.

→ Transformation hierarchy - <u>Affines</u>

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Diagonal matrix
orthogonal matrix

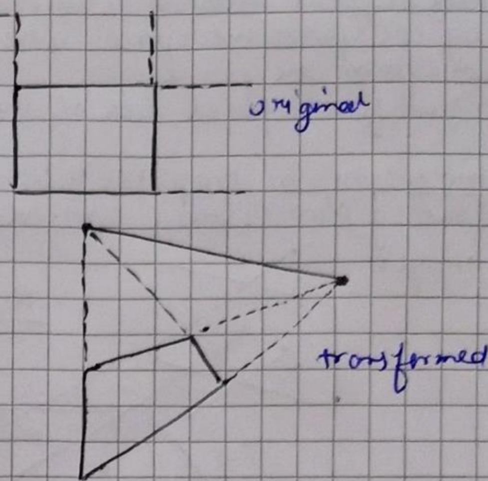
$A = UDV^T$
$A = (UV^T)(VDV^T)$
$A = R(\theta)(R(-\phi)DR(\phi))$

$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$$x' = H_A x = \begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix} x$$

} 6 DoF (2 scale, 2 rotation, 2 translation)
Non-isotropic scaling ! (independent scale in x and y)
invariants : parallel lines, ratios of parallel lengths, ratios of areas.

→ Transformation hierarchy – Homographies.



$$H_p = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} = \begin{pmatrix} A & \vec{t} \\ \vec{v}^T & v \end{pmatrix}$$

$$x' = H_p x = \begin{bmatrix} A & t \\ v^T & v \end{bmatrix} x \qquad v = (v_1, v_2)^T$$

: 8 DOF (2 scale, 2 rotation, 2 translation, 2 lines at infinity)

: invariants : cross ratio of four points on a line (ratio of ratio)

⑧ Homography – Estimation

Before: what is homography and how does it act on vectors/points

After: How to estimate a homography from point correspondences?

* Estimating homography from point correspondences between:
 (a) two images (panorama image)
 (b) model plane & image

(A) ~~Homogeneous~~ Homography mapping from image to image: holds under planar camera motion as mentioned before.

$$x' \propto Hx \qquad \lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

9 entries, 8 DOF (scale is arbitrary)

(i) Equation for point correspondences (x_i, x_i'):

$$\lambda x_i' = H x_i = \begin{bmatrix} h_1^T \\ h_2^T \\ h_3^T \end{bmatrix} x_i$$

(ii) Use cross product to remove the scale factor λ: $x_i' \times H x_i = 0$

$$\underbrace{x_i' \times \lambda x_i'}_{0} = x_i' \times H x_i$$

cross product of 2 ∥ vectors = 0

$$\boxed{0 = x_i' \times H x_i}$$

$$x_i' \times H x_i = \begin{pmatrix} y_i' h^{3T} x_i - w_i' h^{2T} x_i \\ w_i' h^{1T} x_i - x_i' h^{3T} x_i \\ x_i' h^{2T} x_i - y_i' h^{1T} x_i \end{pmatrix} = \underbrace{\begin{bmatrix} 0^T & -w_i' x_i^T & y_i' x_i^T \\ w_i' x_i^T & 0^T & -x_i' x_i^T \\ -y_i' x_i^T & x_i' x_i^T & 0^T \end{bmatrix}}_{\text{3 equations, only 2 linearly independent, drop 3rd row.}} \begin{pmatrix} h^1 \\ h^2 \\ h^3 \end{pmatrix} = 0$$

Homogeneous coordinate (might be 1)

$$= \underset{2\times 9}{A_i} \underset{9\times 1}{\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}} = \underset{2\times 1}{0}$$

→ Direct linear transform

H has 8 DOF (9 parameters but scale is arbitrary)

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_n \end{bmatrix} h = 0 \quad \rightarrow \quad Ah = 0$$

→ One correspondence gives two linearly independent equations
(1) four matches needed for a minimal solution (null space of 8×9 matrix)
(2) more points: search for "best" according to some cost function.
→ No exact solution because of inaccurate measurements due to noise.
(1) find approximate solution:

$$\text{minimize } \underbrace{\|Ah\|_2^2}_{\text{error}} \text{ instead of solving } Ah = 0$$

→ minimizing $\|Ah\|_2^2$, how can we avoid the trivial solution $h = 0$?

Since h is up to scale we can simply add an additional constraint:

→ Pick e.g. $h_9 = 1$ and solve for 8-vector by linear least squares:

$$\begin{bmatrix} 0 & 0 & 0 & -x_i w_i' & -y_i w_i' & -w_i w_i' & x_i y_i' & y_i y_i' \\ x_i w_i' & y_i w_i' & w_i w_i' & 0 & 0 & 0 & x_i x_i' & y_i x_i' \end{bmatrix} \tilde{h} = \begin{pmatrix} -w_i y_i' \\ w_i x_i' \end{pmatrix}$$

this method achieves poor results if h_9 close to 0 → not recommended.
→ Instead we add an additional constraint to the minimization problem:

$$\|Ah\|_2^2 \text{ such that } \|h\|_2^2 = 1$$

→ This can be solved by singular value decomposition.

⇒ Summary.

objective → Given $n \geq 4$, 2D to 2D point correspondences $x_i \leftrightarrow x_i'$, determine the 2D homography matrix H such that $x_i' = Hx_i$

Algorithm → (i) for each correspondence $x_i \leftrightarrow x_i'$ compute A_i. usually, only first 2 rows are required.
(ii) Assemble n 9×9 matrices A_i into a single 2n×9 matrix A.
(iii) obtain Singular Value decomposition of A.
(iv) Solution for h is the last column of V:
h is Eigenvector to the smallest Eigenvalue ($\neq 0$) of $A^T A$.
(iv) Determine H from h.

→ Applications: Homography. (1) Panorama stitching.

(1) Undistort images
(2) Find point correspondences b/w images
(3) compute Homography H
(4) Resample:
(1) loop over image 1
(2) project into image 2 using H
(3) Bilinear interpolation in image 2.