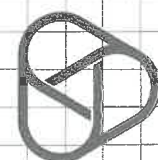


From $\alpha \in \mathbb{C}$, we constructed $\mathbb{Q}[\alpha]$



$\mathbb{Q}[\alpha]$ = Smallest subring of \mathbb{C} containing \mathbb{Q} & α

\downarrow

$$= \{ a_n \alpha^n + \dots + a_1 \alpha + a_0 \mid a_i \in \mathbb{Q} \}$$

\mathbb{Q} -Vector Space.

$$= \mathbb{Q}\text{-Span of } \{ 1, \alpha, \alpha^2, \dots \}$$

Inf. dim.

\mathbb{Q} vector space

Fin dim.

\mathbb{Q} . vec. space.

$$\alpha = \sqrt{2} \quad \alpha^2 - 2 = 0$$

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \alpha^2 - \alpha - 1 = 0$$

Prop 1: If α satisfies an equation of the form

$$\alpha^n + a_{n-1} \alpha^{n-1} + \dots + a_0 = 0$$

where $a_i \in \mathbb{Q}$, then $\mathbb{Q}[\alpha]$ is finite dimensional.

Spanned by $1, \alpha, \dots, \alpha^{n-1}$. so dim at most n .

Converse also holds.

Prop 2: If $\mathbb{Q}[\alpha]$ is finite dimensional, say $\dim = n$
then α must satisfy an equation of the form

$$\alpha^m + a_{m-1}\alpha^{m-1} + \dots + a_0 = 0 \quad \text{for } m \leq n.$$

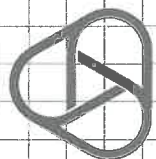
Proof: $1, \alpha, \alpha^2, \dots, \alpha^n \in \mathbb{Q}[\alpha]$ are $(n+1)$ elts.

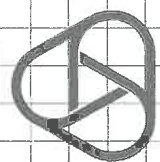
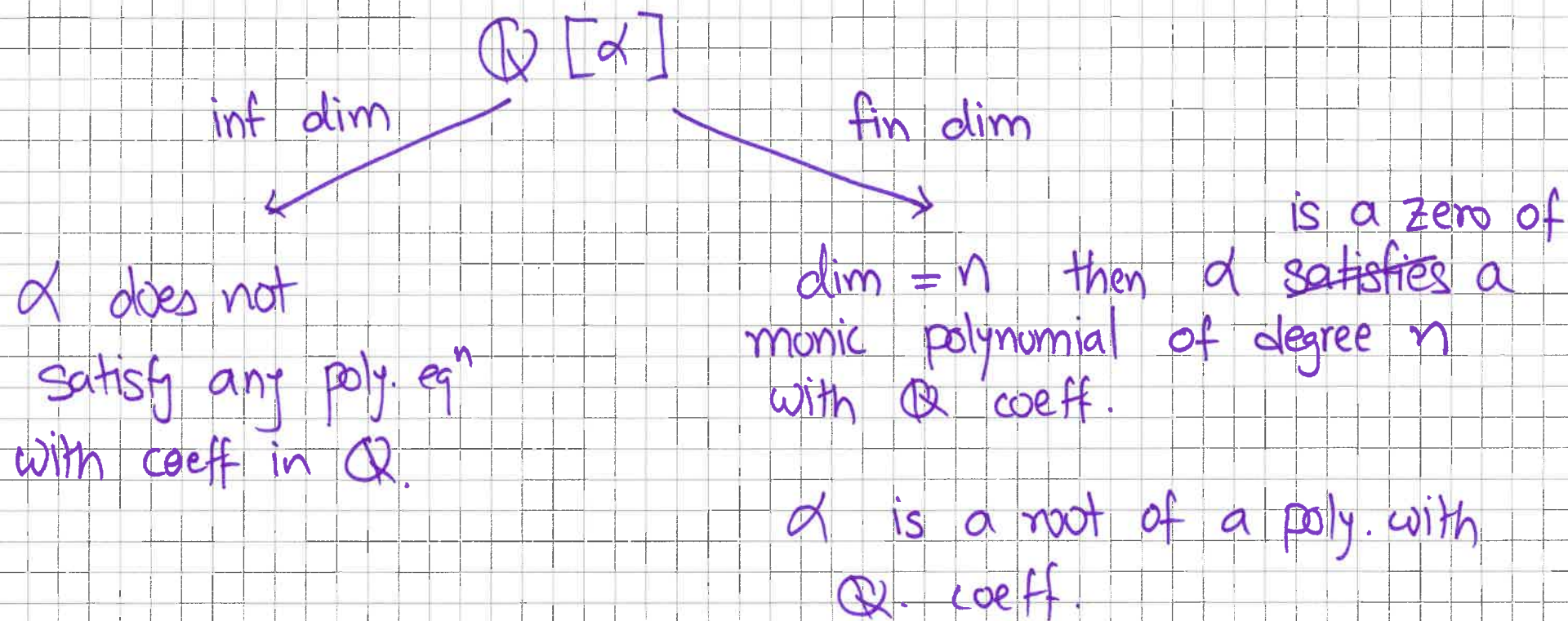
So these must be linearly dependant.:

There exists $a_0, \dots, a_n \in \mathbb{Q}$ so that (not all zero)

$$a_n \alpha^n + \dots + a_0 = 0$$

In fact Prop 1 $\Rightarrow m = n$.





$$1, \sqrt{2} + \sqrt{3}, 5 + 2\sqrt{6}, \frac{11\sqrt{2} + 9\sqrt{3}}{49 + 20\sqrt{6}}$$

$$1, \sqrt{2} + \sqrt{3}, 5 + \sqrt{6} \cdot 2, \frac{5\sqrt{2} + 5\sqrt{3} + 4\sqrt{3} + 6\sqrt{2}}{22 + 9\sqrt{6} + 11\sqrt{6} + 27}, 11\sqrt{2} + 9\sqrt{3}$$

$$\begin{array}{ccc} 1, & \sqrt{2}+\sqrt{3}, & 5+2\sqrt{6} \\ \parallel & \parallel & \parallel \\ \alpha^0 & \alpha^1 & \alpha^2 \end{array}$$

$$(X-5)^2 = 24 \quad \text{sat. by } x=\alpha^2$$

$$(X^2-5)^2 = 24 \quad \text{sat. by } x=\alpha.$$

The set of algebraic numbers is countable.

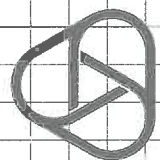
$$\begin{array}{c} \mathbb{Q}[\alpha] \\ \uparrow \\ \mathbb{Q} \text{ v.sp.} \end{array}$$

$$\begin{array}{c} \mathbb{R}[\alpha] \\ \uparrow \\ \mathbb{R} \text{ v.sp.} \end{array}$$

$$\begin{array}{c} \mathbb{C}[\alpha] = \mathbb{C} \\ \uparrow \\ \mathbb{C} \text{ v.sp.} \end{array}$$

$$F[\alpha]$$

$$F \subset \mathbb{C} \text{ subfield.}$$



NOT = inf dim. $F \xrightarrow{\uparrow} F[\alpha]$
Transc. over F .

fin dim = α satis. poly eqⁿ with F -coeff
algebraic over F .