HOMEWORK 5

This homework is due by Friday, May 24, 11:59pm on Gradescope. This is the last homework set, so I have given 3 weeks.

The first three problems are about *nested square roots*, namely complex numbers like $\sqrt{\sqrt{2}+\sqrt{1+\sqrt{3}}}$. More precisely, $\alpha \in \mathbf{C}$ is a *nested square root* if there exists a sequence of fields

$$\mathbf{Q} = F_0 \subset F_1 \subset \cdots \subset F_n$$

such that each F_{i+1}/F_i is a quadratic extension and $\alpha \in F_n$. A nested square root is also called a *constructible number* because these are precisely the complex numbers that can be constructed with a ruler and compass, starting with the two points 0 and 1.

1. Problem 1 (16.9.3 modified)

Some nested square roots can be de-coupled to a linear combination of simple square roots. For example, we have

$$\sqrt{5+2\sqrt{6}} = \sqrt{2}+\sqrt{3}.$$

But some cannot be. Prove that $\alpha = \sqrt{1+\sqrt{3}}$ cannot be written as a sum

$$\sqrt{a_1}+\cdots+\sqrt{a_n}, \quad a_i \in \mathbf{Q}.$$

Hint. Compare the Galois group of the minimal polynomial of α over $\mathbf{Q}$ and the Galois group of $\mathbf{Q}[\sqrt{a_1},\ldots,\sqrt{a_n}]/\mathbf{Q}$.

2. Problem 2

Let $\alpha \in \mathbf{C}$ be a nested square root. Let G be the Galois group of the minimal polynomial of α over $\mathbf{Q}$. Prove that the order of G is a power of 2.

Caution. Make sure that the extension you are considering is Galois!

3. Problem 3

Prove the converse to the problem before: if $\alpha \in \mathbf{C}$ is such that its minimal polynomial over $\mathbf{Q}$ has Galois group whose order is a power of 2, then α is a nested square root. As an application, show that if p is a prime number of the form 2^n+1, then ζ_p is a nested square root.

With this, we have completed a proof of the following.

Theorem. For a prime number p, the regular p-gon is constructible if and only if p has the form 2^n+1.

In this problem, you may use the following fact from group theory without proof.

Theorem. Let p be a prime and G a group of order p^n for $n \geq 1$. Then G contains a normal subgroup of index p.

4. Problem 4

Determine the Galois group of the polynomial x^6+3 over the base fields

1. $F = \mathbf{Q}$
2. $F = \mathbf{Q}[\zeta_3]$.

5. Problem 5 (16.12.7)

Find a polynomial of degree 7 over $\mathbf{Q}$ whose Galois group is S_7.

Hint. Take inspiration from the construction in *Artin* for degree 5.