WORKSHOP 4

2024 ALGEBRA 2

In this workshop, we explore the theme of roots, coefficients, and symmetry.

(1) Let
$$p(x) = x^3 + 2x^2 + 3x + 4,$$
and let $\alpha, \beta, \gamma \in \mathbf{C}$ be the roots of $p(x)$. The expression
$$\alpha^2 + \beta^2 + \gamma^2$$
is symmetric, and hence must be rational. Find out the exact value.

(2) If an expression is not completely symmetric, the more symmetric it is, the "closer" it is to the base field. For example, let α, β, γ be the roots (in some big extension) of a cubic $p(x) \in F[x]$. Prove that $\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha$ has degree at most 2 over F.
Hint: Following the idea in the proof of the theorem about splitting fields, try to construct a symmetric polynomial of degree 2 with this as a root.

(3) As another application of the principle above, let $\alpha, \beta, \gamma, \delta$ be the roots (in some extension) of a quartic over F. Prove that
$$\alpha\beta + \gamma\delta$$
has degree at most 3 over F.

(4) As another application of the principle, let $\alpha_1, \ldots, \alpha_n$ be the roots of $p(x) \in F[x]$ of degree n. Consider
$$d = \prod_{i<j} (\alpha_i - \alpha_j).$$
Prove that d satisfies a quadratic equation over F.

(5) Sometimes, the element is closer to the base-field than we expect from symmetry. For example, consider the cubic
$$p(x) = x^3 - 3x - 1.$$
Prove that for this cubic, the element d above is actually a rational number. You may find it helpful to consult Wikipedia for the formula for the discriminant of a cubic.

1. Solutions

(1) We write $\alpha^2+\beta^2+\gamma^2$ in terms of the elementary symmetric polynomials:
$$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma).$$
So this is equal to $(-2)^2-6=-2$.

(2) Let
$$\eta_1=\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha$$
and
$$\eta_2=\alpha\beta^2+\beta\gamma^2+\gamma\alpha^2.$$
Observe that $\{\eta_1,\eta_2\}$ form one orbit under permutations of α,β,γ. The quadratic polynomial
$$(x-\eta_1)(x-\eta_2)$$
is symmetric under any permutation of α,β,γ, and hence its coefficients must lie in F.

(3) Same idea as above. This time the orbit consists of 3 elements.

(4) Same idea as above. This time the orbit consists of 2 elements and they are $\pm d$. So the quadratic is actually quite simple:
$$x^2-d^2=0.$$
Notice that d^2 is the discriminant. So d is a square-root of the discriminant.

(5) The discriminant Δ of
$$x^3+ax+b$$
is
$$\Delta=-4a^3-27b^2.$$
In this case, it is $\Delta=108-27=81$ so $d=\pm 9$.