HOMEWORK 1 SOLUTIONS

1. Problem 1

To show that R is a field, we need to show that in R we have $1 \neq 0$ and every element of R aside from 0 is invertible. We can observe that since R has a field as a sub-ring, we know that it must contain 1 and 0 with $1 \neq 0$.

Next, we show that every element $r \in R \setminus \{0\}$ is invertible with respect to multiplication. Let $a \in R \setminus \{0\}$ and define a linear map $f : R \to R$ which maps each $r \in R$ to $ar \in R$. We can indeed verify that this map is linear over F as for $\alpha \in F$ we have $f(\alpha r) = a\alpha r = \alpha ar = \alpha f(r)$ and $f(r + s) = a(r + s) = ar + as = af(r) + af(s)$ as R is a finite-dimensional vector space over F.

Since R is an integral domain, we deduce that the kernel of f must be trivial as $a \neq 0$ and so the only element in R which maps to zero must be zero itself since there are no zero divisors in R. As a result, we conclude that f is an injection.

We now aim to show that this linear map is surjective. Since R is a finite-dimensional vector space over F, and any injective linear map from a vector space to itself is surjective, we conclude that f is also a surjection. That implies that there exists $b \in R$ such that $ab = 1$, and so $b = a^{-1}$ and so every nonzero element is invertible.

2. Problem 2

Proof. Let α be a complex root of the irreducible polynomial $x^3 - 3x + 4$ in $\mathbb{Q}[x]$. We want to find the inverse of $\alpha^2 + \alpha + 1$.

We know that $\alpha^3 - 3\alpha + 4 = 0$, so we can write $\alpha^3 = 3\alpha - 4$. To find the inverse of $\alpha^2 + \alpha + 1$, we compute what the polynomial needs to be multiplied by to get back the identity.

$$\begin{aligned}
1 &= (\alpha^2 + \alpha + 1)(a + b\alpha + c\alpha^2) \\
&= a\alpha^2 + b\alpha^3 + c\alpha^4 + a\alpha + b\alpha^2 + c\alpha^3 + a + b\alpha + c\alpha^2 \\
&= c\alpha^4 + (b + c)\alpha^3 + (a + b + c)\alpha^2 + (a + b)\alpha + a \\
&= c\alpha(3\alpha - 4) + (b + c)(3\alpha - 4) + (a + b + c)\alpha^2 + (a + b)\alpha + a \\
&= (a + b + 4c)\alpha^2 + (a + 4b - c)\alpha + (a - 4b - 4c)
\end{aligned}$$

By linear independence, we get the following system of equations:

$$\begin{aligned}
a + b + 4c &= 0 \\
a + 4b - c &= 0 \\
a - 4b - 4c &= 1
\end{aligned}$$

Solving this gives

$$a = \frac{17}{49}, \quad b = -\frac{5}{49}, \quad c = -\frac{3}{49}$$

So, the inverse of $\alpha^2 + \alpha + 1$ is $\frac{17}{49} - \frac{5}{49}\alpha - \frac{3}{49}\alpha^2$.

Notice that this method works in general, with a given relation and polynomial we seek to take the inverse of.

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3. Problem 3

Let $\beta = \sqrt[3]{2}e^{2\pi i/3}$.

Theorem 3.1. *For all $k \in \mathbb{N}$ there is no solution in $\mathbb{Q}[\beta]$ to the equation $x_1^2+\cdots+x_k^2+1=0$.*

Proof. We can see that the polynomial x^3-2 is satisfied by β, and it is irreducible, so it is the unique monic minimal polynomial for β. We thus have that $\mathbb{Q}[\beta]$ is isomorphic to $\mathbb{Q}[x]/(x^3-2)$. However, $\sqrt[3]{2}$ is another solution to this polynomial, so we find that $\mathbb{Q}[\sqrt[3]{2}]$ is also isomorphic to $\mathbb{Q}[x]/(x^3-2)$, and thus $\mathbb{Q}[\beta]$ is isomorphic to $\mathbb{Q}[\sqrt[3]{2}]$.

If we have a solution to the equation $x_1^2+\cdots+x_k^2+1=0$ in $\mathbb{Q}[\beta]$ then this ring isomorphism $\phi : \mathbb{Q}[\beta] \to \mathbb{Q}[\sqrt[3]{2}]$ will give us a solution $y_1^2+\cdots+y_k^2+1=0$ in $\mathbb{Q}[\sqrt[3]{2}]$ by setting $y_i = \phi(x_i)$ for each i. However, $\mathbb{Q}[\sqrt[3]{2}]$ is a subset of the real numbers, so we have that $y_i^2 \geq 0$ for each i, which would imply that $0 \geq 1$.

We thus reach a contradiction, so for all $k \in \mathbb{N}$ there are no solutions to the equation $x_1^2+\cdots+x_k^2+1=0$ in $\mathbb{Q}[\beta]$. □

4. Problem 4

We see that $\zeta_1 = 1$, and so has degree 1. ζ_2 and ζ_3 have degrees 1 and 2 respectively as 2.3 are primes, and $\zeta_4 = i$ which has degree 2 (minimal polynomial of x^2+1).

Note that for any naturals n and m such that $m \mid n$, we have that $\mathbb{Q}[\zeta_m] \subset \mathbb{Q}[\zeta_n]$, as for any element in $\mathbb{Q}[\zeta_m]$ we can keep the coefficients the same and replace ζ_m with $(\zeta_n)^{\frac{n}{m}}$ to see that the element lies in $\mathbb{Q}[\zeta_n]$ (this is possible as $\frac{n}{m}$ is a natural number). Therefore by the multiplicative property of the degree, we have $\deg(\zeta_n) = \deg(\zeta_m) \times \deg(\mathbb{Q}[\zeta_n]/\mathbb{Q}[\zeta_m])$, which implies that $\deg(\zeta_n) \geq \deg(\zeta_m)$. Thus if any prime larger or equal to 5 divides n, ζ_n cannot have degree at most 3, as $\deg(\zeta_p) = p-1$, so if $p \geq 5$ we have $p-1 > 3 \implies \deg(\zeta_n) > 3$.

It remains to check numbers of the form $\zeta_{2^a3^b}$. Note that ζ_8 satisfies the equation $x^4+1=0$, and as x^4+1 is irreducible over $\mathbb{Q}$ it follows that $\deg(\zeta_8) = 4$. This also implies that any number with a 2^n factor, where $n \geq 3$, has degree higher than 3. We also note that ζ_9 satisfies the equation $x^6+x^3+1=0$, which is irreducible over $\mathbb{Q}$ and thus ζ_9 has degree 6. Therefore any number with a 3^n factor, where $n \geq 2$, has degree higher than 3.

Now we only need to check ζ_6 and ζ_{12}. Note that ζ_6 satisfies the equation $x^3+1=0$, so the degree of ζ_6 is at most 3. ζ_{12} satisfies the equation $x^4-x^2+1=0$, which is irreducible over $\mathbb{Q}$. Therefore, the values of n are 1, 2, 3, 4 and 6.

5. Problem 5

1

As x^4+2 is irreducible over $\mathbb{Q}$, we have that $\deg_{\mathbb{Q}} \mathbb{Q}[\sqrt[4]{-2}] = 4$.

By way of contradiction, suppose $i \in \mathbb{Q}[\sqrt[4]{-2}]$. Then there exists $a, b, c, d \in \mathbb{Q}$, such that

$$a + b\sqrt[4]{-2} + c(\sqrt[4]{-2})^2 + d(\sqrt[4]{-2})^3 = i$$

So we have that

$$(a + b\sqrt[4]{-2} + c(\sqrt[4]{-2})^2 + d(\sqrt[4]{-2})^3)^2 = -1$$

Expanding and rearranging the left hand side gives

$$a^2-4bd-2c^2+\sqrt[4]{-2}(2ab-4cd)+(\sqrt[4]{-2})^2(2ac+b^2-2d^2)+(\sqrt[4]{-2})^3(2ad+2bc) = -1$$

Thus we have the system of equations

$$a^2 - 4bd - 2c^2 = -1 \tag{1}$$

$$2ab - 4cd = 0 \tag{2}$$

$$2ac + b^2 - 2d^2 = 0 \tag{3}$$

$$2ad + 2bc = 0 \tag{4}$$

From (2) we have that $ab = 2cd$. Either $d = 0$, or $d \neq 0$.

- If $d = 0$, then from (4), we have that $bc = 0$, so $b = 0$ or $c = 0$.
  - Suppose $b = 0$, then from (3), we have that $2ac + (0)^2 - 2(0)^2 = 2ac = 0$. Thus $a = 0$ or $c = 0$
    - If $a = 0$, then from (1), $0^2 - 4(0)(0) - 2c^2 = -2c^2 = -1$, so $c^2 = \frac{1}{2}$. But there does not exists $c \in \mathbb{Q}$ such that this is the case, as $\sqrt{2}$ is irrational.
    - If $c = 0$, then from (1), $a^2 - 4(0)(0) - 2(0)^2 = a^2 = -1$. But there is no rational number such that this is the case.
  - Suppose $c = 0$. Then, from (1), we have that $a^2 - 4b(0) - 2(0)^2 = a^2 = -1$, but there is no rational number such that this is the case.

- If $d \neq 0$, then $c = \frac{ab}{2d}$, and thus from (4), $ad = -b\frac{ab}{2d}$, so $2ad^2 = -ab^2$. Either $a = 0$, or $a \neq 0$
  - If $a = 0$, then from (3) we have that $2(0)c + b^2 = b^2 = 2d^2$, or $\left(\frac{b}{d}\right)^2 = 2$, but $\frac{b}{d}$ is rational, and there is no rational that squares to 2.
  - If $a \neq 0$, then we have that $2d^2 = -b^2$ This says that a positive nonzero rational, is equal to a non positive rational, which is impossible.

In all cases, we arrive at a contradiction, so there does not exist rational solutions to this equation, so $i \notin \mathbb{Q}[\sqrt[4]{2}]$.

2

By way of contradiction, suppose $\sqrt[3]{5} \in \mathbb{Q}[\sqrt[3]{2}]$. Then there exists $a, b, c \in \mathbb{Q}$ such that $a + b\sqrt[3]{2} + c\sqrt[3]{4} = \sqrt[3]{5}$. Thus

$$(a + b\sqrt[3]{2} + c\sqrt[3]{4})^3 = 5$$

So

$$a^3+3a^2b\sqrt[3]{2}+3a^2c\sqrt[3]{4}+3ab^2\sqrt[3]{4}+12abc+6ac^2\sqrt[3]{2}+2b^3+6b^2c\sqrt[3]{2}+6bc^2\sqrt[3]{4}+4c^3 = 5$$

rearranging gives

$$a^3 + 12abc + 2b^3 + 4c^3 + \sqrt[3]{2}(3a^2b + 6ac^2 + 6b^2c) + \sqrt[3]{4}(3a^2c + 3ab^2 + 6bc^2) = 5$$

As $\deg_{\mathbb{Q}} \sqrt[3]{2} = 3$ (irreducible polynomial is $x^3 - 2$), $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$ form a basis for the $\mathbb{Q}$-vector space $\mathbb{Q}[\sqrt[3]{2}]$, thus we have the equations

$$3a^2b + 6ac^2 + 6b^2c = 0 \ (1)$$

$$3a^2c + 3ab^2 + 6bc^2 = 0 \ (2)$$

$$a^3 + 12abc + 2b^3 + 4c^3 = 5 \ (3)$$

If $a = 0$, then from (1), we have that $6b^2c = 0$, so $b = 0$, or $c = 0$. But these imply from (3) that $4c^3 = 5$ or $2b^3 = 5$ respectively. These imply that the cube root of $5/4$ is rational, or that the cube root of $5/2$ is rational, both of which are false.

If $b = 0$, then from (1), we have that $6ac^2 = 0$, so $a = 0$ or $c = 0$. These imply from (3), that $4c^3 = 5$ or $a^3 = 5$, again implying that the cube root of 5 or the cube root of $5/4$ are rational.

If $c = 0$, then from (1), we have that $3a^2b = 0$, so $a = 0$ or $b = 0$. These imply from (3), that $2b^3 = 5$ or $a^3 = 5$, again implying that the cube root of 5 or the cube root of $5/2$ are rational.

Thus none of the coefficients can be 0. So abc is invertible. Multiplying (1) and (2) by $\frac{1}{abc}$, and simplifying gives

$$\frac{a}{c} + 2\frac{c}{b} + 2\frac{b}{a} = 0 \ (4)$$

$$\frac{a}{b} + \frac{b}{c} + 2\frac{c}{a} = 0 \ (5)$$

performing (4) $- \frac{b}{c}$(5) gives

$$\frac{a}{c} + 2\frac{c}{b} + 2\frac{b}{a} - \frac{b}{c}\frac{a}{b} - \frac{b}{c}\frac{b}{c} - 2\frac{b}{c}\frac{c}{a} = 0$$

Simplifying this gives

$$2\frac{c}{b} - \frac{b^2}{c^2} = 0$$

or that $(\frac{b}{c})^3 = 2$. b and c are rational, so $\frac{b}{c}$ is rational. Thus we have that the cube root of 2 is a rational number, which is false.

Thus there does not exist a representation, so $\sqrt[3]{5}$ is not in $\mathbb{Q}[\sqrt[3]{2}]$.