WORKSHOP 1

2024 ALGEBRA 2

1. Degree of $\mathbf{Q}(\cos(2\pi/p))$?

Let p be a prime number. What is the degree of $\mathbf{Q}(\cos(2\pi/p))$ over $\mathbf{Q}$?

Hints

Use that $\mathbf{Q}(\cos(2\pi/p) + i\sin(2\pi/p))$ has degree $(p-1)$ over $\mathbf{Q}$ and it contains $\mathbf{Q}(\cos(2\pi/p))$.

Solution sketch. The multiplicative inverse of $\cos(2\pi/p)+i\sin(2\pi/p)$ is $\cos(2\pi/p)-i\sin(2\pi/p)$, and the sum of these two numbers is $2\cos(2\pi/p)$. So we see that $\cos(2\pi/p) \in \mathbf{Q}(\cos(2\pi/p)+i\sin(2\pi/p))$. As a result, we have

$$\mathbf{Q}(\cos(2\pi/p)) \subset \mathbf{Q}(\cos(2\pi/p) + i\sin(2\pi/p)).$$

The element $\cos(2\pi/p) + i\sin(2\pi/p)$ satisfies the degree 2 equation

$$x^2 - 2\cos(2\pi/p)x + 1$$

over the smaller field, so this is at most a quadratic extension. But it is a non-trivial extension because the bigger field contains non-real numbers. So this extension has degree 2. Using that the bigger field has degree $(p-1)$ over $\mathbf{Q}$, we conclude that the smaller field has degree $(p-1)/2$ over $\mathbf{Q}$.

Further question (come back to it later)—

What is the degree of $\mathbf{Q}(\cos(2\pi/p), \sin(2\pi/p))$ over $\mathbf{Q}$?

Solution sketch. It is either the same as the degree of $\mathbf{Q}(\cos(2\pi/p))$ or twice it, but I am not sure which one.

2. Most angles cannot be trisected

See if you can prove the following theorem.

Theorem — Let t be such that $\cos t$ is transcendental. Given $(0,0)$, $(0,1)$, and $(\cos t, \sin t)$, it is impossible to construct $(\cos t/3, \sin t/3)$ using ruler and compass.

Sketch of the proof

Follow the same method as in class, keeping track of the field that contains the coordinates of the constructed points. The starting field will be $\mathbf{Q}(\cos t, \sin t)$. The key is to prove that $\cos(t/3)$ has degree 3 over this field. It is easier to handle the field $\mathbf{Q}(\cos t)$, which is isomorphic to $\mathbf{Q}(x)$, the field of rational functions in a variable x. Over this field, prove that $\cos(t/3)$ has degree 3. To do so, you need to prove that a certain polynomial in $\mathbf{Q}(x)[y]$ is irreducible. Using the ideas in class, move through irreducibility in $\mathbf{Q}(x)[y]$, in $\mathbf{Q}[x, y]$, and $\mathbf{Q}(y)[x]$. Finall conclude that over $\mathbf{Q}(\cos(t), \sin(t))$ also $\cos(t/3)$ must have degree 3.

Solution sketch. We follow the same idea as in class. Recall that the starting point is a field F that contains the coordinates of our points. We cannot start with $F = \mathbf{Q}$, but we take F to be the smallest subfield of $\mathbf{C}$ containing $\cos t$ and $\sin t$.

How do we describe F? It is easier to first look at a smaller field G, whichi s the smallest subfield of $\mathbf{C}$ containing $\cos t$. Convince yourselves that

$$G = \{p(\cos t)/q(\cos t) \mid p, q \in \mathbf{Q}[x], q \neq 0\}.$$

Furthermore, the map

$$\mathbf{Q}(x) \to G$$

that sends $x \mapsto \cos t$ is an isomorphism.

Now $F = G[\sin t]$ is at most a quadratic extension of G. The new element $\sin t$ satisfies the qudratic polynomial

$$y^2 + \cos^2 t - 1 = 0$$

(This polynomial is in fact irreducible over G, but we do not need this fact.)

Now, by the triple angle formula, $\cos(t/3)$ satisfies the equation

$$4y^3 - 3y - \cos t = 0$$

We claim that this is irreducible over F. It is easier to see that it is irreducible over G. Indeed, using the isomorphism above, we can rewrite it as $4y^3 - 3y + x$. We now switch to $\mathbf{Q}[x, y] = \mathbf{Q}[y, x]$, and then to $\mathbf{Q}(y)[x]$ (why can we do this?) But in the last ring, it is a linear polynomial and hence irreducible.

We conclude that $\cos(t/3)$ has degree 3 over G. Then $G[\sin t, \cos(t/3)]$ has degree 3 or 6 over G. In either case, $\cos(t/3)$ must have degree 3 over $F = G[\sin t]$.

But then we are done: there is no way to construct $\cos(t/3)$ starting from G.