WORKSHOP 2

2024 ALGEBRA 2

1. Factorisation in a finite field

The polynomial $f(x) = x^3 + x + 1 \in \mathbf{F}_5[x]$ is irreducible. Let $K = \mathbf{F}[t]/(f(t))$. Find the irreducible factorisation of $f(x)$ in $K[x]$.

2. Conjugates

Let $F \subset K$ be a field extension. We say that $\alpha, \beta \in K$ are *conjugates* over F if they have the same minimal polynomial over F.

Let K be a finite field of characteristic p. Let $\phi \colon K \to K$ be the Frobenius map.

(1) Prove that the conjugates of $a \in K$ are $a, \phi(a), \phi^2(a), \cdots$.
(2) Deduce that the degree of a over $\mathbf{F}_p$ is the smallest n such that $\phi^n(a) = a$.
(3) More generally, let $K \subset L$ be an extension of finite fields with $|K| = p^n$. Prove that the conjugates of $a \in L$ over K are $a, \phi^n(a), \phi^{2n}(a), \ldots$.
(4) What is the analogue of (2) in this situation?

3. Factorisation, once again

Let $f(x) \in \mathbf{F}_p[x]$ be irreducible of degree 18. Let $\mathbf{F}_p \subset K$ be an extension of degree 4. How does $f(x)$ factorise in $K[x]$?

Hint. Let $K \subset L$ be an extension of degree 9, so that $\mathbf{F}_p \subset L$ is of degree 36. First factorise $f(x)$ in L and then "collect the conjugates" over K.