MATH3345/6215

Practice questions for mid-semester exam

Question 1. Give examples of the following, with proof. Or if no example exists, prove that.

(a) A non-rational algebraic number.
(b) A prime ideal which is not maximal.
(c) A degree 3 extension of $\mathbb{C}(t)$.
(d) A degree 3 extension of $\mathbb{F}_3$.
(e) An algebraic extension which is not finite.
(f) A non-rational number $\alpha \in \mathbb{Q}(\sqrt{5})$ such that $\mathbb{Q}(\alpha)$ is not equal to $\mathbb{Q}(\sqrt{5})$.
(g) A non-rational number $\alpha \in \mathbb{Q}(\sqrt[4]{5})$ such that $\mathbb{Q}(\alpha)$ is not equal to $\mathbb{Q}(\sqrt[4]{5})$.
(h) A degree 3 extension of $\mathbb{Q}$ which is not isomorphic to one of the form $\mathbb{Q}(\sqrt[3]{a})$.

Question 2. Determine the degrees of the following extensions (ζ_n stands for $e^{2\pi i/n}$)

(a) $\mathbb{Q}(\sqrt[4]{4})$
(b) $\mathbb{Q}(\sqrt[5]{6}, \sqrt[3]{7})$
(c) $\mathbb{Q}(\sqrt{2} + \sqrt{-3})/\mathbb{Q}$
(d) $\mathbb{Q}(\sqrt{2} + \sqrt{3})/\mathbb{Q}$
(e) $\mathbb{R}/\mathbb{Q}$
(f) $\mathbb{Q}(\sqrt[3]{2}, \zeta_3 \sqrt[3]{2})$
(g) $\mathbb{C}/\mathbb{R}$
(h) $\mathbb{Q}(\zeta_{11})/\mathbb{Q}$
(i) $\mathbb{Q}(\zeta_9 + \zeta_9^{-1})/\mathbb{Q}$

Question 3. Prove that the following polynomials are irreducible over $\mathbb{Q}$

(a) $x^4 - 4x^3 + 6$
(b) $x^3 + 7x + 4$
(c) $x^3 + 7x + 3$
(d) $x^6 + 30x^5 - 15x^3 + 6x - 120$
(e) $x^4 + x^3 + x^2 + 1$

Question 4. Which of the following are fields? Which are isomorphic to subrings of $\mathbb{R}$?

(a) $\mathbb{Q}[x]/\langle x^2 - 2\rangle$
(b) $\mathbb{Q}[x]/\langle x^2 - 4x + 1\rangle$
(c) $\mathbb{Q}[x]/\langle x^2 - 1\rangle$
(d) $\mathbb{Q}[x]/\langle x - 3\rangle$
(e) $\mathbb{Q}[x]/\langle x^2 + 1\rangle$
(f) $\mathbb{Q}[x]$

Question 5. Determine the minimal polynomials of the following elements

(a) $1 + \sqrt{3} \in \mathbb{R}$ over $\mathbb{Q}$
(b) $\sqrt{5} \in \mathbb{R}$ over $\mathbb{Q}(\sqrt{3})$
(c) $\alpha^2 + 1 \in \mathbb{E}$ over $\mathbb{F}_3$, where $\mathbb{E}$ is the field $\mathbb{F}_3[x]/(x^3 + 2x + 1)$ and $\alpha \in \mathbb{E}$ is the coset $\bar{x}$.
(d) $i\sqrt{-1 + 2\sqrt{3}} \in \mathbb{C}$ over $\mathbb{Q}$

Question 6. For which prime numbers p is $\cos(2\pi/p)$ irrational?

Question 7. Prove or disprove the following.

(a) Let E/F be a field extension and $\alpha \in E$. If α is algebraic over F then $F[\alpha] = F(\alpha)$.
(b) Let E/F be a field extension and $\alpha \in E$. If $F[\alpha] = F(\alpha)$ then α is algebraic over F.
(c) A regular 365-gon is constructible.
(d) Every field of positive characteristic is finite.
(e) Every finite field has positive characteristic.
(f) $\sqrt{2} \in \mathbb{R}$ can be written as a polynomial expression of $\sqrt{2} + \sqrt{5}$.