

WORKSHOP 3

2024 ALGEBRA 2

Consider the extension $\mathbf{Q} \subset \mathbf{Q}[e^{2\pi/3}, 2^{1/3}] = F$. This is a Galois extension, which means that the main theorem of Galois theory applies.

There is an isomorphism

$$\mathbf{Q}[x, y]/(x^2 + x + 1, y^3 - 2) \rightarrow \mathbf{Q}[e^{2\pi/3}, 2^{1/3}]$$

that sends x to $e^{2\pi i/3}$ and y to $2^{1/3}$. This is not too hard to prove, but you may proceed without proving it.

- (1) Use the presentation above to find all automorphisms of the extension F/\mathbf{Q} .
- (2) Notice that F is generated by the three roots of $x^3 - 2$. Prove that any $\sigma \in \text{Aut}(F/\mathbf{Q})$ must permute the three roots.
- (3) Label the roots as 1, 2, 3. Then you get a group homomorphism

$$G \rightarrow S_3.$$

Prove that this is an isomorphism.

- (4) Using the above, find the subgroup diagram of G .
- (5) For each subgroup $H \subset G$, find the fixed field

$$F^H = \{x \in F \mid \sigma(x) = x \text{ for all } \sigma \in H\}.$$