WORKSHOP 3

2024 ALGEBRA 2

Consider the extension $F = \mathbf{Q} \subset \mathbf{Q}[e^{2\pi/3}, 2^{1/3}] = K$. This is a Galois extension, which means that the main theorem of Galois theory applies.

There is an isomorphism

$$\mathbf{Q}[x, y]/(x^2 + x + 1, y^3 - 2) \to \mathbf{Q}[e^{2\pi/3}, 2^{1/3}]$$

that sends x to $e^{2\pi i/3/}$ and y to $2^{1/3}$. This is not too hard to prove, but you may proceed without proving it.

(1) Use the presentation above to find all automorphisms of the extension K/F.
(2) Notice that K is generated by the three roots of $x^3 - 2$. Prove that any $\sigma \in \mathrm{Aut}(K/F)$ must permute the three roots.
(3) Label the roots as $1, 2, 3$. Then you get a group homomorphism
$$G \to S_3.$$
Prove that this is an isomorphism.
(4) Using the above, find the subgroup diagram of G.
(5) For each subgroup $H \subset G$, find the fixed field
$$K^H = \{x \in K \mid \sigma(x) = x \text{ for all } \sigma \in H\}.$$