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Mathematical Sciences Institute

EXAMINATION: Final examination — 31 May 2018

MATH 3345/6215

Advanced Algebra 2: Field extensions and Galois Theory

Exam Duration: 180 minutes.
Reading Time: 0 minutes.

Materials Permitted In The Exam Venue:

- Unmarked English-to-foreign-language dictionary (no approval from MSI required).

Materials To Be Supplied To Students:

- Scribble Paper.

Instructions To Students:

- You must justify all your answers, except where stated otherwise. You may generally use without proof any results covered in lecture, on the assignments, in the tutorials, or in the reading. However, some judgement is required—for example, if you proved $A = B$ on an assignment, and I ask you to prove $A = B$ or something obviously equivalent to it, it should be clear that I'm asking you to give a legitimate proof, and not just give a one-line proof by invoking the identical result from the assignment.

Q1	Q2	Q3	Q4	Q5
28	18	18	18	18

Total / 100

Question 1 **28 pts**

Give examples of the following, or if no example exists, state that. In either case, you must justify your answer (as always).

(a) A polynomial in $\mathbb{Q}[x]$ which is reducible (over $\mathbb{Q}$) but has no roots (in $\mathbb{Q}$).

(b) A Galois extension of $\mathbb{Q}$ with Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(c) A Galois extension of $\mathbb{F}_{17}$ with Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(d) A Galois extension of $\mathbb{Q}$ with Galois group isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

(e) Finite extensions $E/K/F$ such that E/F is Galois but K/F is not.

(f) A polynomial $f \in \mathbb{Q}[x]$ of degree 2018 which is solvable by radicals.

(g) Constructible elements $a, b \in \mathbb{R}_{>0}$ such that $\sqrt{a+b}$ is not constructible.

Write your solution here

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Question 2 **18 pts**

Compute the Galois group of E/F where E is a splitting field of the polynomial $x^6 + 1$ over F in two cases: (i) $F = \mathbb{Q}$ and (ii) $F = \mathbb{F}_3$. In each case give an explicit description of all intermediate subfields $F \subseteq K \subseteq E$.

Write your solution here

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Question 3 **18 pts**

Let E be the splitting field of the polynomial $(x^2-3)(x^2+x-1)$ over $\mathbb{Q}$. Prove that there exists $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ such that $\sigma(\sqrt{3}) = -\sqrt{3}$ and $\sigma(\frac{\sqrt{5}+1}{2}) = \frac{-2}{\sqrt{5}+1}$.

Write your solution here

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Question 4 **18 pts**

Prove or disprove the following: there exists $n \in \mathbb{N}$ such that $\mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{Q}(e^{2\pi i/n})$.

Write your solution here

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Question 5 **18 pts**

Let $f \in \mathbb{Q}[x]$ be irreducible of degree three over $\mathbb{Q}$ and let a_1, a_2, a_3 be its three roots in $\mathbb{C}$. Let $b = a_1^2 + a_2^2$. Compute the degree $[\mathbb{Q}(b) : \mathbb{Q}]$.

Write your solution here

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