HOMEWORK 3

This homework is due by Friday, 19 April, 11:59pm on Gradescope.

1. Problem 1 (15.7.6)

Factor the polynomial $x^{16} - x$ over a field of size 4 and a field of size 8.

2. Problem 2 (Presentations)

Let $R \subset S$ be an inclusion of rings. Suppose we have an isomorphism

$$S \cong R[x_1, \dots, x_n]/I,$$

where $x_1, \dots, x_n$ are variables and $I \subset R[x_1, \dots, x_n]$ is an ideal. Such an isomorphism is called a *presentation* of S over R.

Let A be another ring and suppose a ring homomorphism $i\colon R \to A$ is given. A presentation of S over R gives us all the ways of extending i to a ring homomorphism $S \to A$. This is because a ring homomorphism $R[x_1, \dots, x_n] \to A$ extending i is determined uniquely by the images of $x_1, \dots, x_n$ and such a homomorphism is well-defined modulo I if and only if it sends I to 0.

(1) Find a presentation for $\mathbf{Q}[\sqrt[3]{2}]$ over $\mathbf{Q}$. Use it to determine all homomorphisms

$$\mathbf{Q}[\sqrt[3]{2}] \to \mathbf{C}.$$

What are the images of these homomorphisms?

(2) Do the same for $\mathbf{Q}[\sqrt{2}, \sqrt{3}]$ over $\mathbf{Q}$.

3. Problem 3 (Automorphisms 1)

Let p be a prime number and set $\zeta_p = e^{2\pi i/p}$. Let $F = \mathbf{Q}[\zeta_p]$. Find all automorphisms

$$\phi\colon F \to F.$$

Describe the automorphism group $\mathrm{Aut}(F)$. (This is the group consisting of automorphisms $F \to F$ with composition as the group law.)

4. Problem 4 (Automorphisms 2)

Let $F = \mathbf{Q}[\zeta_p]$, as before. Let $K = F[2^{1/p}]$. Find all automorphisms $\phi\colon K \to K$. How many are there? How many restrict to the identity on F?

5. Problem 5

Let K be a field of size p^n. How many elements of K are perfect squares? Generalise your answer to perfect d-th powers.