HOMEWORK 1

This homework is due by Friday, 8 March, 11:59pm on Gradescope.

1. Problem 1 (15.1.1)

Let R be an integral domain that contains a field F as a sub-ring. Assume that R is finite dimensional when viewed as a vector space over F. Prove that R is a field.

2. Problem 2 (15.2.1)

Let α be a complex root of the irreducible polynomial $x^3 - 3x + 4$ in $\mathbf{Q}[x]$. Find the inverse of $\alpha^2 + \alpha + 1$ in the form $a + b\alpha + c\alpha^2$ with $a, b, c \in \mathbf{Q}$.

The particular polynomial and element are not important. Your method should work in general.

3. Problem 3 (15.2.3)

Let $\beta = \omega\sqrt[3]{2}$, where $\omega = e^{2\pi i/3}$ and let $K = \mathbf{Q}[\beta] \subset \mathbf{C}$. Let k be a positive integer. Prove that the equation

$$x_1^2 + \cdots + x_k^2 + 1 = 0$$

has no solution with $x_1, \ldots, x_k \in K$.

4. Problem 4 (15.3.5)

For a positive integer n, set $\zeta_n = e^{2\pi i/n}$. Find all values of n such that ζ_n has degree at most 3 over $\mathbf{Q}$.

You may use (without having to prove it) that for a prime number p, the degree of ζ_p over $\mathbf{Q}$ is $p - 1$, and its minimal polynomial is

$$x^{p-1} + x^{p-2} + \cdots + x + 1.$$

5. Problem 5 (15.3.7)

(1) Is i in $\mathbf{Q}[\sqrt[4]{-2}]$?
(2) Is $\sqrt[3]{5}$ in $\mathbf{Q}[\sqrt[3]{2}]$?

Justify your answers. You may assume that $x^3 - 5$ and $x^3 - 2$ are irreducible over $\mathbf{Q}$. If it helps you, feel free to assume that $x^n \pm p$ is irreducible over $\mathbf{Q}$ for any n and for any prime number p.