WORKSHOP 5

2024 ALGEBRA 2

In this workshop, we will learn how to find Galois groups of irreducible quartic polynomials, up to a small ambiguity. We fix a base field F of characteristic 0 and an irreducible $f(x) \in F[x]$ of degree 4. Let G be the Galois group of $f(x)$.

For your convenience, here is a list of transitive subgroups of S_4 with their orders (up to re-numbering).

Subgroup	Order
S_4	24
A_4	12
$C_4 = \langle (1234) \rangle$	4
D_4	8
$V = \{e, (12)(34), (14)(23), (13)(24)\}$	4

1. Problem 1

Say $f(x)$ is a quartic with roots $\alpha_1, \ldots, \alpha_4$. The resolvent cubic $g(x)$ is the cubic with roots

$$\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4$$
$$\beta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4$$
$$\beta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3.$$

Check that $f(x)$ and $g(x)$ have the same discriminant.

1.1. **Solution sketch.**

We have

$$\beta_2 - \beta_1 = (\alpha_1 - \alpha_4)(\alpha_2 - \alpha_3)$$

, and likewise for the other three differences. So product of the 3 differences of the β's equals the product of the 6 differences of the α's.

2. Problem 2

Prove that the discriminant is a square in F if and only if $G \subset A_4$.

2.1. **Solution sketch.**

We have already seen this in class. An odd permutation changes the sign of the square root of the discriminant. So the square root of the discriminant is in F if and only if there are no odd permutations in G.

3. Problem 3

Justify the following table (as much as you can) about the Galois group. Use the following observations. Let $F \subset K$ be a splitting field of $f(x)$. Let $L \subset K$ be generated by the 3 roots of the resolvent cubic $g(x)$. Then $F \subset L$ is the splitting field of $g(x)$. We have a surjective group homomorphism

$$\mathrm{Aut}(K/F) \to \mathrm{Aut}(L/F)$$

with kernel $\mathrm{Aut}(K/L)$.

	Discriminant square	Discriminant non-square
Resolvent irreducible	A_4	S_4
Resolvent factors as 1+2	Impossible	D_4 or C_4
Resolvent factors as 1+1+1	V	Impossible

3.1. **Solution sketch.** Suppose the resolvent is irreducible and the discriminant is not a square. Then $\mathrm{Aut}(L/F) \cong S_3$. Therefore, we have a surjection $G \to S_3$. The only possible G that could surject onto S_3 are S_4 or A_4 (the order of G must be divisible by 6). But since the discriminant is not a square, G is not A_4, so it must be S_4.

Suppose the resolvent is irreducible and the discriminant is a square. Then we have a surjection $G \to A_3$. So the order of G is divisible by 3, which leaves the possibilities $G = S_4$ or A_4. The discriminant being a square shows that $G = A_4$.

If the resolvent factors as $1 + 2$, then one of the β's is fixed by G, say β_1. The only permutations that fix β_1 are those in $D_4 \subset S_4$, where we think of D_4 as symmetries of the square with 1, 3 and 2, 4 as opposite vertices. (Check that the discriminant of a cubic with irreducible factorisation $1 + 2$ cannot be a square.) So $G \subset D_4$. This leaves 3 possibilities $G = V, C_4, D_4$. The first one would also fix β_2 and β_3, which we know is not the case. So we can strike that off.

Finally, if the resolvent factors as $1+1+1$, then G fixes all β's. The only such G is $G = V$.