WORKSHOP 1

2024 ALGEBRA 2

1. Degree of $\mathbf{Q}(\cos(2\pi/p))$?

Let p be a prime number. What is the degree of $\mathbf{Q}(\cos(2\pi/p))$ over $\mathbf{Q}$?

Hints— Use that $\mathbf{Q}(\cos(2\pi/p)+i\sin(2\pi/p))$ has degree $(p-1)$ over $\mathbf{Q}$ and it contains $\mathbf{Q}(\cos(2\pi/p))$.

2. Most angles cannot be trisected

See if you can prove the following theorem.

Theorem— Let t be such that $\cos t$ is transcendental. Given $(0,0)$, $(0,1)$, and $(\cos t, \sin t)$, it is impossible to construct $(\cos t/3, \sin t/3)$ using ruler and compass.

Sketch—

Follow the same method as in class, namely keep track of the field that contains the coordinates of the constructed points. The starting field will be $\mathbf{Q}(\cos t, \sin t)$. The key is to prove that $\cos(t/3)$ has degree 3 over this field. It is easier to handle the field $\mathbf{Q}(\cos t)$, which is isomorphic to $\mathbf{Q}(x)$, the field of rational functions in a variable x. Over this field, prove that $\cos(t/3)$ has degree 3. To do so, you need to prove that a certain polynomial in $\mathbf{Q}(x)[y]$ is irreducible. Using the ideas in class, move through irreducibility in $\mathbf{Q}(x)[y]$, in $\mathbf{Q}[x,y]$, and $\mathbf{Q}(y)[x]$. Then conclude that over $\mathbf{Q}(\cos(t),\sin(t))$ also $\cos(t/3)$ must have degree 3.