How algebraists think about the world.

$X \rightsquigarrow$ Algebraic Structure

manifold, shape,
real data, function e.g. polynomial,
number, ...

Group, ring.

$\uparrow$

Back $\leftsquigarrow$ Study.

Examples - $\leadsto$ = \leadsto

1. $X \leadsto \mathrm{Aut}(X)$. ↙ Group. || Later. half.

2. X geometric object $\leadsto$ Fundamental group of X.

3. $\leadsto$ Cohomology ring.

NOT in This course

4. Number (complex number)

$\alpha \leadsto \mathbb{Z}[\alpha]$ or $\mathbb{Q}[\alpha]$ ||.

Ring = Assoc, comm, unital.

Given $\alpha \in \mathbb{C}$

$\mathbb{Z}[\alpha] \subset \mathbb{C}$ the smallest ~~a~~ subring

that contains $\mathbb{Z}$ & α.

$\mathbb{Q}[\alpha] \subset \mathbb{C}$ the smallest subring

that contains $\mathbb{Q}$ & α.

$$\mathbb{Z}[\alpha] \ni \begin{array}{l} 0, 1, 2, \ldots\ldots -1, -2, \ldots\ldots \\ \alpha, 2\alpha, 3\alpha, \ldots n\alpha \quad , n \in \mathbb{Z} \\ \alpha^2, \quad 5\alpha^3 - 3\alpha + 1, \text{ etc.} \end{array}$$

$$\mathbb{Z}[\alpha] = \left\{ \begin{array}{l} a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha^1 + a_0 \\ \text{where } a_i \in \mathbb{Z}. \ , n \geq 0 \text{ integer.} \end{array} \right\}$$

$$\mathbb{Q}[\alpha] = \left\{ \begin{array}{l} a_n\alpha^n + \ldots\ldots + a_1\alpha + a_0 , \\ \text{where } a_i \in \mathbb{Q} \end{array} \right\}$$

$$\mathbb{Z} \subset \mathbb{Z}[\alpha] \subset \mathbb{C}$$

Ex. $\alpha \in \mathbb{Z}$ then $\mathbb{Z}[\alpha] = \mathbb{Z}$

$\alpha = \sqrt{2}$ then

$$\mathbb{Z}[\alpha] = \{ a_n (\sqrt{2})^n + \cdots + a_1 (\sqrt{2})^1 + a_0 \}$$

$$5 (\sqrt{2})^3 + 4 \cdot (\sqrt{2})^2 - (\sqrt{2}) + 1$$

$$10 (\sqrt{2}) + 8 - \sqrt{2} + 1$$

$$9\sqrt{2} + 9$$

$$= \{ a_1 \sqrt{2} + a_0 \}$$

Ex: $\mathbb{Q}[3^{1/5}]$

$$= \{ a_4 (3^{1/5})^4 + \cdots + a_1 (3^{1/5}) + a_0 \}$$

Q: When will $\underbrace{\text{this kind of thing}}_{\substack{\text{Requiring only finitely many powers} \\ \text{in the lin. comb.}}}$ happen?

Ex. $\alpha = \frac{1+\sqrt{5}}{2}$, $\mathbb{Q}[\alpha]$ $\boxed{\alpha^2 = \alpha + 1}$

$$\alpha^2 = \alpha + 1$$

$$\alpha^3 = (\alpha+1)\alpha = (\alpha+1) + \alpha$$

Higher power = Lin. comb of α & 1