Where next?

In this course: Equations in one variable of degree > 1, base field complicated = not alg. closed. ← Alg 2.

Lin. alg.
many variables
many equations
linear eq^{n}s.

many var.
many eq^{n}s. ← Algebraic geometry.

complex analysis
manifolds
topology.

Complex numbers.

geometric.

Non-closed field.

finite

$\mathbb{R}$

$\mathbb{Z}$.

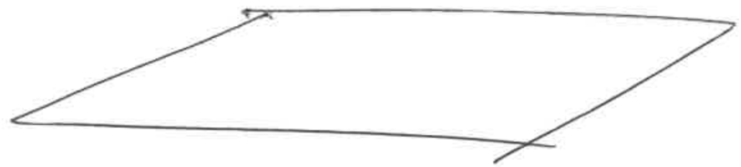
$\mathbb{Q}$

$\mathbb{Q}_p$

Ex. $x^n + y^n = z^n$ (x,y,z) up to scaling $[x:y:z] \in \mathbb{P}^2$

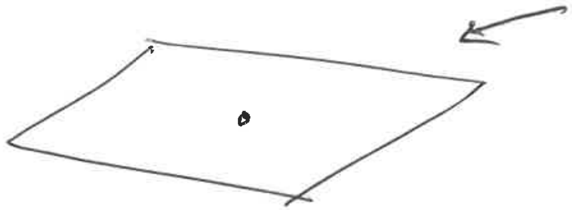
$x^n+y^n=1$ over $\mathbb{C}$

$\underline{n=1}$:



$\mathbb{C}$ ←

similar to $\mathbb{C}$

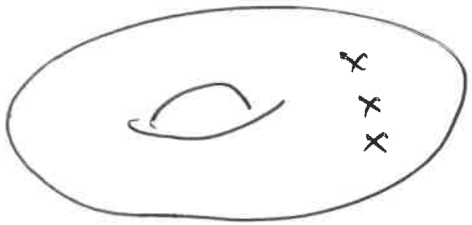


$\underline{n=2}$: $x^2+y^2=1 \cong \mathbb{C}\setminus\{0\}$

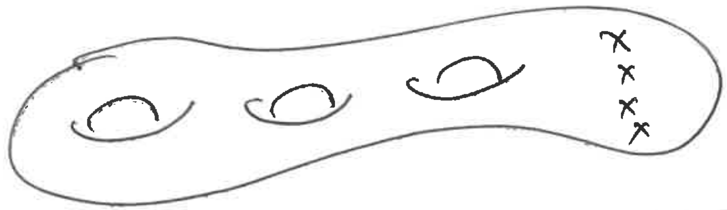
$$\underbrace{(x+iy)}_{z}\,\underbrace{(x-iy)}_{w}=1$$

$$zw=1$$

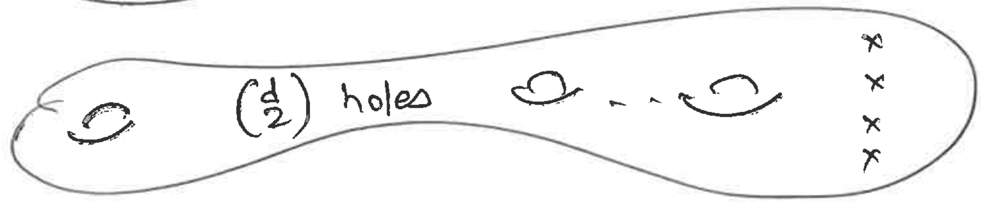
$\underline{n=3}$: $x^3+y^3=1 \cong$



$\underline{n=4}$: $x^4+y^4=1 \cong$



$\underline{n=d}$: $\underline{x^d+y^d=1} \cong$



Open Q: (Roughly). Cubic eq^n in n variables. $/\mathbb{C}$ is the shape cut out by that "birational" similar to $\mathbb{C}^{n-1}$ or not?

Given a system of eq^ns.
Does it have a sol^n in

- $\mathbb{C}$. ✓
- $\mathbb{R}$ ✓
- $\mathbb{Q}$ ← open question.
- $\mathbb{Z}$ NO algorithm.

Impossibility.

Geometric shape ($\mathbb{C}$). $\rightsquigarrow$ Arithmetic ($\mathbb{Q}$).

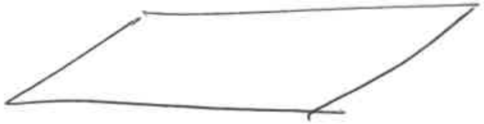
Thm: For any $d \geq 4$

$$\underset{\text{deg } d}{f(x,y)} = 0 \quad \text{satisfying} \underset{\text{mild.}}{\ldots} \quad \text{have finitely many } \mathbb{Q} \text{ solutions.}$$

$$x^d + y^d = 1$$

(Mordell's conj, Faltings thm).

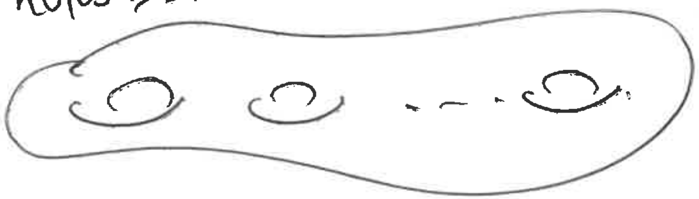
Shape



$\mathbb{C}$ $\rightsquigarrow$ could be inf. many $\mathbb{Q}$ sol^{n}s.

$\rightsquigarrow$ — " —

holes ≥ 2



$\Longrightarrow$ Finitely many sol^{n}s.

Higher dim: (Lang-Bombieri conj).

Over $\mathbb{C}$

Shape that has "negative curvature". $\Longrightarrow$ Few $\mathbb{Q}$-sol^{n}s.

Alg 2:

ring homs.

$$F \longrightarrow K$$

fields. fields.

$$\boxed{A \longrightarrow B} \longrightarrow \text{Commutative algebra.}$$

ring. hom.

Non commutative.

Representation theory.

G gp. form $\mathbb{C}[G] = \{ a_0 \cdot 1 + a_1 \cdot g_1 + \cdots + a_k \cdot g_k \mid a_i \in \mathbb{C},\ g_i \in G \}$.

$\downarrow$

Non. comm. ring.

Non comm rings $\longleftrightarrow$ Representation theory.

$F \subset K$ Finite Galois ext^n.

$F \subset K$ Galois ext^n.

$\mathrm{Aut}(K/F) \leftarrow$ infinite gp., topological gp.

Galois corresp. intermed. fields $\longleftrightarrow$ closed subgps.