

$$\alpha \in \mathbb{C}$$

Transcendental

- Does not ...
- $\mathbb{Q}[\alpha]$  is infinite dim  $\mathbb{Q}$  vsp.
- eval at  $\alpha : \mathbb{Q}[x] \rightarrow \mathbb{Q}[\alpha]$  has  $\{0\}$  kernel.

Algebraic over  $\mathbb{Q}$

- Satisfies a polynomial equation with  $\mathbb{Q}$  coefficients
- $\mathbb{Q}[\alpha]$  is finite dim  $\mathbb{Q}$  v.space
- eval at  $\alpha : \mathbb{Q}[x] \rightarrow \mathbb{Q}[\alpha]$  has non-zero ~~idea~~ kernel.

$$\mathbb{Q}[\alpha] \subset \mathbb{C}$$

= All complex numbers that can be written as  
 $a_n \alpha^n + \dots + a_1 \alpha + a_0$  for  $a_i \in \mathbb{Q}$ .

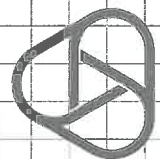
=  $\mathbb{Q}$ -Span of  $1, \alpha, \alpha^2, \dots$

Consider  $x$  as a variable & look at  $\mathbb{Q}[x]$

$\mathbb{Q}[x]$  = Poly. ring in  $x$  with coeff in  $\mathbb{Q}$ .

Have a <sup>ring</sup> homomorphism

$$\begin{array}{ccc} \mathbb{Q}[x] & \xrightarrow{e} & \mathbb{C} \\ \downarrow \psi & & \downarrow \psi \\ \sum a_i x^i & \mapsto & \sum a_i \alpha^i \\ p(x) & \mapsto & p(\alpha) \end{array} \quad \begin{array}{l} \text{eval. at } x=\alpha \\ \\ \end{array}$$



$\mathbb{Q}[\alpha]$  = Image of  $e$ .

$$\mathbb{Q}[x] \xrightarrow{e} \mathbb{C} \quad \text{eval at } x=\alpha$$

Image of  $e = \mathbb{Q}[\alpha]$ .

$$\mathbb{Q}[x] \xrightarrow{e} \mathbb{Q}[\alpha]$$

$$\cap$$

$$\mathbb{C}$$

$I := \text{Ker}(e) \subset \mathbb{Q}[x]$  an ideal.

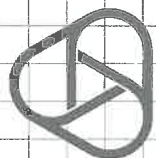
$\parallel$   
 $\{ p(x) \in \mathbb{Q}[x] \text{ that map to } 0 \}$

$$I = (0)$$

$e$  is injective  
 $\alpha$  is transcendental.

$I$  is non-zero.

$e$  is not injective  
 $\alpha$  is algebraic





$$I \subset \mathbb{Q}[x] \xrightarrow{e} \mathbb{Q}[\alpha]$$

kernel.

The first iso thm. gives an isomorphism

$$\mathbb{Q}[x]/I \xrightarrow{\bar{e}} \mathbb{Q}[\alpha]$$

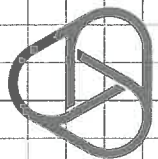
$\alpha$  transc.  $\Rightarrow$  LHS =  $\mathbb{Q}[x]$ . so  $\mathbb{Q}[x] \xrightarrow{\sim} \mathbb{Q}[\alpha]$

$\alpha$  algebraic :  $I \neq 0$ .  $I \subset \mathbb{Q}[x]$ .

All ideals of  $\mathbb{Q}[x]$  are principal so

$$I = (p(x)).$$

means  $p(\alpha) = 0$  & any  $q(x)$  with  $q(\alpha) = 0$  is  
of the form  $q(x) = p(x) \cdot r(x)$



$p(x)$  is the unique monic generator of  $I$   
is called the minimal polynomial of  $\alpha$ .

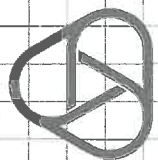
$$(p(x)) = I \subset \mathbb{Q}[x] \rightarrow \mathbb{Q}[\alpha] \subset \mathbb{C}$$

$$\mathbb{Q}[x]/I \xrightarrow{\sim} \mathbb{Q}[\alpha] \leftarrow \text{domain}$$

So  $I = (p(x))$  is a prime ideal.

So  $p(x)$  is an irreducible polynomial.

& up to scaling the only irred. poly in  $I$ .





$$\{a + b 3^{1/5} \mid a, b \in \mathbb{Q}\} \subset \mathbb{C}.$$

✓  $\mathbb{Q}$ . vsp.

X NOT a subring.

$$(3^{1/5}) (3^{1/5}) = 3^{2/5}$$

$$\alpha = \sqrt{2}$$

then

$$I = (x^2 - 2)$$

(prove later)

$$\alpha = -\sqrt{2}$$

then

$$I = (x^2 - 2)$$

$$\alpha = 2^{1/3}$$

$$I = (x^3 - 2)$$

$$\alpha = 2^{1/3} \cdot e^{i\pi/3}$$

$$I = (x^3 - 2)$$

$$\left. \begin{array}{l} \leftarrow \mathbb{Q}[\alpha] \simeq \mathbb{Q}[x]/(x^3 - 2) \\ \swarrow \end{array} \right\}$$

