WORKSHOP 5

2024 ALGEBRA 2

In this workshop, we will learn how to find Galois groups of irreducible quartic polynomials, up to a small ambiguity. We fix a base field F of characteristic 0 and an irreducible $f(x) \in F[x]$ of degree 4. Let G be the Galois group of $f(x)$.

For your convenience, here is a list of transitive subgroups of S_4 with their orders (up to re-numbering).

Subgroup	Order
S_4	24
A_4	12
$C_4 = \langle (1234) \rangle$	4
D_4	8
$V = \{e, (12)(34), (14)(23), (13)(24)\}$	4

1. Problem 1

Say $f(x)$ is a quartic with roots $\alpha_1, \ldots, \alpha_4$. The resolvent cubic $g(x)$ is the cubic with roots

$$\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4$$
$$\beta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4$$
$$\beta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3.$$

Check that $f(x)$ and $g(x)$ have the same discriminant.

2. Problem 2

Prove that the discriminant is a square in F if and only if $G \subset A_4$.

3. Problem 3

Justify the following table (as much as you can) about the Galois group. Use the following observations. Let $F \subset K$ be a splitting field of $f(x)$. Let $L \subset K$ be generated by the 3 roots of the resolvent cubic $g(x)$. Then $F \subset L$ is the splitting field of $g(x)$. We have a surjective group homomorphism

$$\mathrm{Aut}(K/F) \to \mathrm{Aut}(L/F)$$

with kernel $\mathrm{Aut}(K/L)$.

	Discriminant square	Discriminant non-square
Resolvent irreducible	A_4	S_4
Resolvent factors as 1+2	Impossible	D_4 or C_4
Resolvent factors as 1+1+1	V	Impossible