HOMEWORK 4

This homework is due by Friday, 3 May, 11:59pm on Gradescope.

1. Problem 1 (16.3.2)

Determine the degrees of the splitting fields of the following polynomials over $\mathbf{Q}$:

(1) $x^3 - 2$
(2) $x^4 - 1$
(3) $x^4 + 1$

2. Problem 2 (16.6.2)

Let $K = \mathbf{Q}[\sqrt{2}, \sqrt{3}, \sqrt{5}]$. Determine $\deg K/\mathbf{Q}$, prove that $K/\mathbf{Q}$ is a Galois extension, and determine its Galois group.

3. Problem 3

Let p be an odd prime number and $K = \mathbf{Q}[\zeta_p]$. Prove that K contains a unique degree 2 extension of $\mathbf{Q}$.

4. Problem 4

Find quartic polynomials in $\mathbf{Q}[x]$ whose Galois group is isomorphic to:

(1) The Dihedral group D_4 (of order 8)
(2) The cyclic group C_4

Remark: The general version of the above problem is a longstanding open problem called the *Inverse Galois Problem*: given a finite group G, does there always exist a polynomial in $\mathbf{Q}[x]$ with Galois group isomorphic to G?

5. Problem 5

Let $\delta \in \mathbf{Q}$ be such that $\mathbf{Q}[\sqrt{\delta}]$ is the unique degree 2 extension of $\mathbf{Q}$ contained in $\mathbf{Q}[\zeta_p]$. For $p = 7$, find δ.

6. Optional (Do not turn in)

This is a continuation of the last problem. You should now know the subfield $\mathbf{Q}[\sqrt{\delta}] \subset \mathbf{Q}[\zeta_p]$ for $p = 3, 5, 7$. Based on this data, make a conjecture for an arbitrary odd prime p. (If you need more data, work out the case of $p = 11$.) Then try to prove the conjecture.