

HOMEWORK 3

This homework is due by Friday, 5 April, 11:59pm on Gradescope.¹

1. PROBLEM 1 (15.7.6)

Factor the polynomial $x^{16} - x$ over a field of size 4 and a field of size 8.

2. PROBLEM 2 (PRESENTATIONS)

Let $R \subset S$ be an inclusion of rings. Suppose we have an isomorphism

$$S \cong R[x_1, \dots, x_n]/I,$$

where x_1, \dots, x_n are variables and $I \subset R[x_1, \dots, x_n]$ is an ideal. Such an isomorphism is called a *presentation* of S over R .

Let A be another ring and suppose a ring homomorphism $i: R \rightarrow A$ is given. A presentation of S over R gives us all the ways of extending i to a ring homomorphism $S \rightarrow A$. This is because a ring homomorphism $R[x_1, \dots, x_n] \rightarrow A$ extending i is determined uniquely by the images of x_1, \dots, x_n and such a homomorphism is well-defined modulo I if and only if it sends I to 0.

- (1) Find a presentation for $\mathbf{Q}[\sqrt[3]{2}]$ over \mathbf{Q} . Use it to determine all homomorphisms

$$\mathbf{Q}[\sqrt[3]{2}] \rightarrow \mathbf{C}.$$

What are the images of these homomorphisms?

- (2) Do the same for $\mathbf{Q}[\sqrt{2}, \sqrt{3}]$ over \mathbf{Q} .

3. PROBLEM 3 (AUTOMORPHISMS 1)

Let p be a prime number and set $\zeta_p = e^{2\pi i/p}$. Let $F = \mathbf{Q}[\zeta_p]$. Find all automorphisms

$$\phi: F \rightarrow F.$$

Describe the automorphism group $\text{Aut}(F)$.

4. PROBLEM 4 (AUTOMORPHISMS 2)

Let $F = \mathbf{Q}[\zeta_p]$, as before. Let $K = F[2^{1/p}]$. Find all automorphisms $\phi: K \rightarrow K$. How many are there? How many restrict to the identity on F ?

5. PROBLEM 5

Let K be a field of size p^n . How many elements of K are perfect squares? Generalise your answer to perfect d -th powers.

¹The deadline is during teaching break, but I will have office hours as usual.