WORKSHOP 3

2024 ALGEBRA 2

Consider the extension $F = \mathbf{Q} \subset \mathbf{Q}[e^{2\pi/3}, 2^{1/3}] = K$. This is a Galois extension, which means that the main theorem of Galois theory applies.

There is an isomorphism

$$\mathbf{Q}[x,y]/(x^2+x+1, y^3-2) \to \mathbf{Q}[e^{2\pi/3}, 2^{1/3}]$$

that sends x to $e^{2\pi i/3/}$ and y to $2^{1/3}$. This is not too hard to prove, but you may proceed without proving it.

(1) Use the presentation above to find all automorphisms of the extension K/F.
(2) Notice that K is generated by the three roots of x^3-2. Prove that any $\sigma \in \mathrm{Aut}(K/F)$ must permute the three roots.
(3) Label the roots as $1, 2, 3$. Then you get a group homomorphism

$$G \to S_3.$$

Prove that this is an isomorphism.
(4) Using the above, find the subgroup diagram of G.
(5) For each subgroup $H \subset G$, find the fixed field

$$K^H = \{x \in K \mid \sigma(x) = x \text{ for all } \sigma \in H\}.$$

1. Solutions

(1) Write $\zeta_3 = e^{2\pi i/3}$. By the presentation, we see that a ring homomorphism $K \to K$ that fixes F is specified uniquely by the images α, β of x, y provided that they satisfy

$$\alpha^2 + \alpha + 1 = 0 \text{ and } \beta^3 - 2 = 0.$$

The only possibilities are

$$\alpha = \zeta_3, \zeta_3^2,$$
$$\beta = 2^{1/3}, \zeta_3 2^{1/3}, \zeta_3^2 2^{1/3}.$$

In total, there are 6 possible homomorphism. All must me automorphisms (why?).
(2) If $\phi\colon K \to K$ is an automorphisms that fixes F and $a^3 - 2 = 0$, then by applying ϕ, we see that $\phi(a)^3 - 2 = 0$. So ϕ must send a root to a root.

(3) Since the roots generate K, the map must be injective. Since both sides have the same cardinality, it must be an isomorphism. It is instructive to take one of the six possibilities and write down the corresponding permutation. For example, if we label the roots as

$$\alpha_1 = 2^{1/3}, \quad \alpha_2 = 2^{1/3}\zeta_3, \quad \alpha_3 = 2^{1/3}\zeta_3^2$$

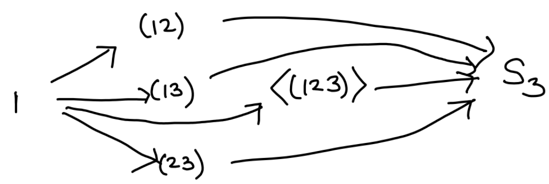
and take the automorphism given by

$$2^{1/3} \mapsto \zeta_3 2^{1/3} \quad \zeta_3 \mapsto \zeta_3,$$

then the permutation is

$$\alpha_1 \mapsto \alpha_2 \mapsto \alpha_3 \mapsto \alpha_1.$$

(4) The diagram of subgroups of S_3 is



(5) The corresponding diagram of intermediate fields is: