Australian National University

Student Number:

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Mathematical Sciences Institute

EXAMINATION: Semester 1 — Final 2023

MATH3345/6215 — Algebra 2

Exam Duration: 120 minutes.
Reading Time: 15 minutes.

Materials Permitted In The Exam Venue:

- No books, notes, reference materials are permitted.
- No electronic aids are permitted e.g. laptops, phones, calculators.

Materials To Be Supplied To Students:

- Scribble paper (last 3 pages; ask if you need more).

Instructions To Students:

- The exam is worth a total of 50 points, with the value of each question as shown.
- You may use any result from class, homework, or workshops as long as it does not trivialise the question.
- You must cite what you use either by name ("Using the Fundamental Theorem of Algebra...") or by recalling the statement ("Since $\mathbb{C}$ is algebraically closed...").
- The last 3 pages are blank. You may detach them and use them for scratch work.
- Some useful formulas:
  (a) The discriminant of $x^3 + ax + b$ is $-4a^3 - 27b^2$.
  (b) The cubic resolvent of $x^4 + ax^2 + bx + c$ is $x^3 + 2ax^2 + (a^2 - 4c)x - b^2$.

Q1	Q2	Q3	Q4	Q5
12	8	8	8	14

Total / 50

Question 1 ***12 pts***

State *true* or *false*. No justification is necessary.

(a) If $\alpha \in \mathbf{C}$ has degree 4 over $\mathbf{Q}$, then α is constructible.

☐ True ☐ False

(b) If K/F is a finite Galois extension and L/K is a finite Galois extension, then L/F is a finite Galois extension.

☐ True ☐ False

(c) The Galois group of $f(x)g(x)$ is the product of the Galois groups of $f(x)$ and $g(x)$.

☐ True ☐ False

(d) Let $\mathbf{F}$ be a finite field with 125 elements. Every element of $\mathbf{F}$ is a cube, that is, equal to a^3 for some $a \in \mathbf{F}$.

☐ True ☐ False

(e) If K/F is a field extension of degree n, then for every m dividing n, the field K must contain an element of degree m over F.

☐ True ☐ False

(f) If the Galois group of $f(x) \in \mathbf{Q}[x]$ has order 49, then the roots of $f(x)$ can be written as nested radicals over $\mathbf{Q}$.

☐ True ☐ False

Question 2 ***8 pts***

Give an example or state that no example exists. No justification is necessary.

(a) An irreducible polynomial of degree 3 in $\mathbf{F}_5[x]$.

(b) An irreducible polynomial of degree 4 in $\mathbf{Q}[x]$ whose Galois group is abelian.

(c) A field automorphism of $\mathbf{Q}[2^{1/4}]$ different from the identity.

(d) An element in $\mathbf{Q}[\zeta_7]$ of degree 3 over $\mathbf{Q}$.

Question 3 ***8 pts***

Let $\phi\colon \mathbf{C} \to \mathbf{C}$ be a field automorphism and let $K \subset \mathbf{C}$ be a subfield that is a finite Galois extension of $\mathbf{Q}$. Prove that $\phi(K) = K$, that is, ϕ maps K onto itself.

Extra space for previous question

Question 4 **8 pts**

Fix a positive integer n. Let $F = \mathbf{C}(t)$ and consider $p(x) = x^n - t \in F[x]$. Construct the splitting field K/F of $p(x)$ and describe the group $\mathrm{Aut}(K/F)$. In your description, you must explicitly describe all automorphisms of K/F and also identify the group up to isomorphism. Justify your answer.

Extra space for previous question

Question 5 ***14 pts***

Let $\alpha, \beta, \gamma \in \mathbf{C}$ be the roots of

$$f(x) = x^3 - 3x - 1.$$

Set

$$K = \mathbf{Q}\left[\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma}\right] \subset \mathbf{C},$$

where the square root symbol denotes one of the two square-roots in $\mathbf{C}$. It turns out that $\deg(K/\mathbf{Q}) = 12$ (you may use this freely without having to prove it).

(a) Prove that $K/\mathbf{Q}$ is Galois. *3 pts*

(b) Which of the following define field automorphisms of K? Tick all that do. No justification is necessary. *3 pts*

- □ $\sqrt{\alpha} \mapsto \sqrt{\beta}, \quad \sqrt{\beta} \mapsto \sqrt{\alpha}, \quad \sqrt{\gamma} \mapsto \sqrt{\gamma}.$
- □ $\sqrt{\alpha} \mapsto \sqrt{\beta}, \quad \sqrt{\beta} \mapsto \sqrt{\gamma}, \quad \sqrt{\gamma} \mapsto \sqrt{\alpha}.$
- □ $\sqrt{\alpha} \mapsto -\sqrt{\alpha}, \quad \sqrt{\beta} \mapsto \sqrt{\beta}, \quad \sqrt{\gamma} \mapsto \sqrt{\gamma}.$

(c) Prove that K has a unique subfield of degree 3 over $\mathbf{Q}$. *8 pts*

Scratch work

Scratch work

Scratch work