PRACTICE PROBLEMS

(1) True or false. If *true*, explain why. If *false*, give a counter-example.
   (a) Let F be a finite field. For every n, the polynomial ring $F[x]$ contains an irreducible polynomial of degree n.
   (b) Let F be a finite field. Suppose $f(x) \in F[x]$ has a root in F. Then $f(x)$ splits into linear factors in F.
   (c) Same as (2), but assuming f is irreducible.
   (d) Same as (2), but with F not necessarily finite.

(2) Let p be a prime. Is the polynomial

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^p}{p!}$$

irreducible in $\mathbf{Q}[x]$?

(3) Find, with proof, the kernel of the map $\mathbf{Q}[x] \to \mathbf{C}$ that sends x to $i + 2$.

(4) Find the gcd of $x^2 + x + 1$ and $x^4 + 3x^3 + x^2 + 7x + 5$ when considered as elements of (a) $\mathbf{Q}[x]$ and (b) $\mathbf{F}_7[x]$.

(5) Construct fields with 4 elements, 9 elements, and 125 elements.

(6) Are the following polynomials irreducible: (a) $x^2 + 1 \in \mathbf{F}_7[x]$ (b) $x^3 - 9 \in \mathbf{F}_{31}[x]$.

(7) Factor $x^2 + 5x + 5$ into irreducible factors in (a) $\mathbf{F}_2[x]$ and (b) $\mathbf{Z}[x]$.

(8) Let $\alpha, \beta \in \mathbf{C}$ be roots of irreducible polynomials $f(x), g(x) \in \mathbf{Q}[x]$, respectively. Prove that $f(x)$ is irreducible over $\mathbf{Q}[\beta]$ if and only if $g(x)$ is irreducible over $\mathbf{Q}[\alpha]$.

(9) Let $F \subset L$ be a field extension. Suppose $\alpha, \beta \in L$ are such that both $\alpha + \beta$ and $\alpha \cdot \beta$ are algebraic over F. Prove or give a counterexample: α and β are algebraic over F.

(10) Let p be a prime number. What is the minimal polynomial of $e^{2\pi i/p}$ over $\mathbf{Q}$?

(11) Let p be an odd prime number. What is the degree of $e^{\pi i/p}$ over $\mathbf{Q}$?

(12) Let p be a prime number. Suppose a regular p-gon can be constructed with a ruler and compass. Prove that p must have the form $p = 2^n + 1$ for some n. (We start only with the points $(0, 0)$ and $(1, 0)$).

(13) Let $\gcd(m, n) = 1$. Prove that $x^m - y^n \in \mathbf{C}[x, y]$ is irreducible.

(14) Find all the primitive elements of $\mathbf{Q}[\sqrt{2}, \sqrt{3}]$.

(15) Let $F = \mathbf{F}_2[a]/(a^4 + a + 1)/$ (this is a field).
   (a) Find the degree of a^2 over $\mathbf{F}_2$.
   (b) Does F have an element that has degree 2 over $\mathbf{F}_2$? If yes, find one. If not, why not?

(16) Let $\mathbf{F} = \mathbf{F}_p[a]/f(a)$ where $f(a)$ has degree 6. Prove that the degree of $b = a + a^{p^3}$ over $\mathbf{F}_p$ is at most 3.