

Writing using radicals $\leftarrow \sqrt[p]{\quad}$ "Surd"

Take $\mathbb{C} \supset F$ a subfield.

Say $\alpha \in \mathbb{C}$ is expressible using radicals over F if

\exists a chain of fields

$$F = F_0 \subset F_1 \subset F_2 \subset \dots \subset F_n \ni \alpha$$

(p_i primes).

where $F_1 = F_0[\zeta_{p_1}, \dots, \zeta_{p_k}]$ and

$F_i \subset F_{i+1}$ is (cyclic) Galois of ~~order~~ deg p for
some $p \in \{p_1, \dots, p_k\}$.

$$\Leftrightarrow F_{i+1} = F_i[\sqrt[p]{a_i}] \text{ for some } a_i \in F_i$$

Ex. $\alpha = \sqrt[3]{1 + \sqrt{5}/2}$ exp. using $\sqrt{\quad}$ over \mathbb{Q} .

$$\mathbb{Q} = F_0 \subset F_1 = \mathbb{Q}[\zeta_3, \zeta_5] \subset F_2 \subset F_3 \ni \alpha$$

$$\parallel \qquad \parallel$$

$$F_1[\sqrt{5}] \qquad F_2[\sqrt[3]{1 + \sqrt{5}/2}]$$

Ex. $1 + \sqrt[5]{1 + \zeta_3^2}$

$$F_0 = \mathbb{Q} \subset F_1 = \mathbb{Q}[\zeta_3, \zeta_5] \subset F_2 = F_1[\sqrt[5]{1 + \zeta_3^2}] \ni \alpha$$

Thm * (Abel-Galois) - α is exp. using radicals over F iff
the Galois gp of α over F is solvable.

Galois gp of $\underbrace{\text{min poly of } \alpha \text{ over } F}_{p(x)}$

$F \subset K = \text{Splitting field of } p(x).$

$$(x^5 - (1 + \zeta_3^2)) \cdot$$

$$(x^5 - (1 + \zeta_3))$$

Pf α exp using radicals \Rightarrow Solvable.

$$F = F_0 \subset F_1 \subset F_2 \subset \dots \subset F_n \ni \alpha$$

$F \subset F_n$ need to be Galois!

Ex. $\mathbb{Q} \subset F_1 = \mathbb{Q}[\zeta_3, \zeta_5] \subset F_2 = F_1 \left[\sqrt[5]{1 + \zeta_3^2} \right]$

$\xrightarrow{\text{Galois}}$ $F'_2 = F_1 \left[\sqrt[5]{1 + \zeta_3^2}, \sqrt[5]{1 + \zeta_3} \right] \ni \alpha$

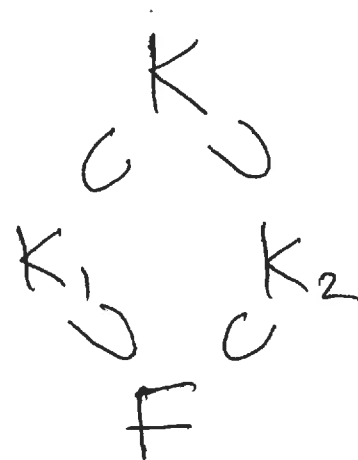
\hookrightarrow Solvable Galois gp.

$$\mathbb{Q} \subset \mathbb{Q}[\zeta_3, \zeta_5] \subset \mathbb{Q}[\zeta_3, \zeta_5] \left[\sqrt[5]{1 + \zeta_3^2}, \sqrt[5]{1 + \zeta_3} \right]$$

Prop: F field char 0 $f(x), g(x)$ two poly.

$F \subset K =$ Splitting field of $f(x) \cdot g(x)$

$K \supset K_1 =$ Splitting field of $f(x)$
 $\supset K_2 =$ Splitting field of $g(x)$



$\text{Aut}(K/F) \rightarrow \text{Aut}(K_1/F) \times \text{Aut}(K_2/F)$ hom.

is injective.

Pf: Say $\sigma \in \text{Ker.} \Rightarrow \sigma$ fixes all roots of $f(x)$
fixes all roots of $g(x)$

$\Rightarrow \sigma$ fixes all in K .

Cor: Gal. gp of $f(x) \cdot g(x) \cong \text{Subgp of } \text{Gal}(f(x)) \times \text{Gal}(g(x)).$

$$\mathbb{Q} \subset \mathbb{Q}[\zeta_3, \zeta_5] \subset \mathbb{Q}[\zeta_3, \zeta_5][\sqrt[5]{\cdot}, \sqrt[5]{\cdot}] \ni \alpha$$

$\underbrace{\mathbb{Q} \subset \mathbb{Q}[\zeta_3, \zeta_5]}_{\text{sub. of } C_2 \times C_4} \subset \underbrace{\mathbb{Q}[\zeta_3, \zeta_5][\sqrt[5]{\cdot}, \sqrt[5]{\cdot}]}_{\text{sub. of } C_5 \times C_5} \leftarrow \text{both solvable}$

Solvable.

$$\mathbb{Q} \subset F_2 \leftarrow \text{Galois}$$

$$\alpha \in F_2$$

$$\Rightarrow \text{all roots of min poly } (\alpha) \in F_2$$

we're after.

$$\mathbb{Q} \subset K \subset F_2$$

Solvable

$$K = \mathbb{Q}[\alpha]$$

$$\Rightarrow \text{Gal}(K/\mathbb{Q}) \text{ is } \Rightarrow \text{the quotient of a solv. gp.} \Rightarrow \text{solvable.}$$