

Galois extensions

A field ext $F \subset K$ is Galois if for every $\alpha \in K$ the min. poly of α over F splits into distinct linear factors over K .

Ex. 1. $F \subset K$ an finite ext of finite fields.

Ex 2. Let F have char 0. Then $F \subset K$ is Galois if K is a splitting field of some poly. over F .

Non ex: $F = \mathbb{F}_p(t)$ Consider $x^p - t \in F[x]$

irred by switching t, x or using Eisenstein.

$K = F[\alpha] / (\alpha^p - t)$ " $\alpha = t^{1/p}$ " K is a splitting field of $x^p - t$

$$K = F[\alpha] / (\alpha^p - t)$$

$$F = \mathbb{F}_p(t)$$

In $K[x]$, have

$$x^p - t = (x - \alpha)^p$$

so $F \subset K$ is a splitting field but not Galois.

Galois Correspondence

Let $F \subset K$ be a finite Galois extⁿ.

$G = \text{Aut}(K/F)$, "Galois group", finite of order $\deg(K/F)$.

Intermediate L \longrightarrow Subgroups of G
 L \longmapsto $\text{Aut}(K/L)$.

$$K^H = \left\{ \alpha \in K \mid \sigma(\alpha) = \alpha \quad \forall \sigma \in H \right\} \longleftarrow H \subset G$$

Need to check: Two composites are the identity,
 \longrightarrow & then \longleftarrow

FCK Galois

$$G = \text{Aut}(K/F).$$

Take L , FCLCK

L



$$H = \text{Aut}(K/L)$$

$$= \left\{ \sigma: K \rightarrow K \text{ s.t.} \right. \\ \left. \sigma(\alpha) = \alpha \ \forall \alpha \in L \right\}$$

\cap

K^H
||



$$\left\{ \alpha \in K \mid \sigma(\alpha) = \alpha \right. \\ \left. \forall \sigma \in H \right\}$$

$$|H| = \deg(K/L).$$

FCK Galois \Rightarrow LCK also Galois.

(splitting field of the same poly).

min poly (α over L) divides min poly (α over F).

Splits into distinct factors over K .

Every $\sigma \in H$ defines an aut of K fixing K^H .

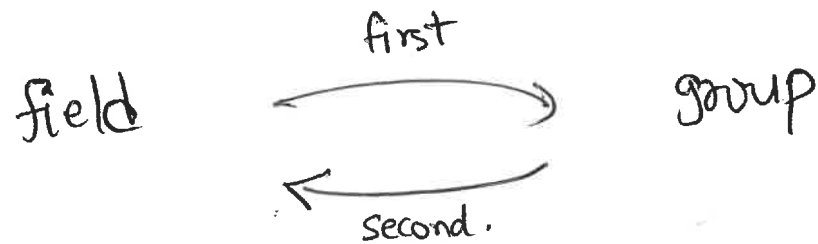
$$\deg(K/K^H) \geq |H| = \deg(K/L)$$

But $L \subset K^H \subset K$

$$\Rightarrow \deg(K/L) \geq \deg(K/K^H).$$

So $K^H = L$.

□



\Rightarrow come back to the same field.

$\Rightarrow \{\text{intermediate fields}\} \longrightarrow \{\text{subgps}\}$ is injective.

\Rightarrow FCK any ~~ext~~ finite extⁿ.

(Char 0).

Then there are finitely many intermediate fields.

PF: ~~*~~ $F \subset K \subset K'$
finite Galois.

$F \subset K'$ has finitely many
int. fields.

\Rightarrow FCK also has fin. many.

$F \subset K$ fin. ext. char 0.
 $\begin{matrix} \psi \\ \alpha \end{matrix} \rightsquigarrow$ int. field $F(\alpha)$

Let F be an inf. field. Let K be a fin dim F v.space.

Then K cannot be the union of finitely many strict subspaces.

Applied here: K cannot be the union of (strict) subfields of K .

so if $\alpha \in K$ does not lie in this union, then $K = F(\alpha)$.