Practice questions for the final

May 24, 2024

These problems focus on the second half of the course.

1. Give an example of the following.
   (a) An extension of $\mathbf{Q}$ of degree 2024.
   (b) An irreducible polynomial of degree 4 over $\mathbf{F}_2$
   (c) A Galois extension of $\mathbf{Q}$ of degree 3.
   (d) An element $\alpha \in \mathbf{C}$ of degree 4 which is not constructible.
   (e) A field extension of $\mathbf{Q}$ with Galois group isomorphic to $\mathbf{Z}/5\mathbf{Z}$.

2. True or false. If true, give brief justification. If false, give a counterexample.
   (a) If $f(x) \in \mathbf{Q}[x]$ has Galois group $\mathbb{Z}/3\mathbf{Z}$ then all roots of $f(x)$ are real.
   (b) If all roots of $f(x) \in \mathbf{Q}[x]$ are real, then the Galois group of $f(x)$ is $\mathbb{Z}/3\mathbf{Z}$.
   (c) If K/F is an extension of degree 4, then there must exist a field L with $F \subset L \subset K$ such that the degree of L/F is 2.
   (d) The same question as above but where F is a finite field.
   (e) The same question as above but where F has characteristic 0 and K/F is Galois.

3. Let $f(x) \in \mathbf{Q}[x]$ be a polynomial and $K/\mathbf{Q}$ a splitting field of $f(x)$. Let $\alpha_1, \ldots, \alpha_n \in K$ be the roots of $f(x)$ and assume that they are distinct. Let $G = \mathrm{Aut}(K/\mathbf{Q})$. Prove that the following are equivalent:
   (a) The set $\{\alpha_1, \ldots, \alpha_n\}$ is an orbit of G.
   (b) The polynomial $f(x)$ is irreducible over $\mathbf{Q}$.

4. Let $K \subset \mathbf{C}$ be the field generated by $\mathbf{Q}$ and the complex roots of $x^3 - 3x + 1$.
   (a) Describe the group $\mathrm{Aut}(K/\mathbf{Q})$, up to isomorphism.
   (b) Is K obtained from $\mathbf{Q}$ by adjoining a cube root? That is, does there exist $a \in \mathbf{K}$ such that $a \notin \mathbf{Q}$ but $a^3 \in \mathbf{Q}$?

5. Let $K = \mathbf{Q}[\zeta_{13}]$. Make a diagram of subfields of K that shows the inclusions. For each subfield, find a primitive element of the field over $\mathbf{Q}$. Which of the subfields are Galois over $\mathbf{Q}$ and what are their Galois groups?

6. Let p be a prime. Let $K/\mathbf{Q}$ be a splitting field of $x^p - 2$ and let $G = \mathrm{Aut}(K/\mathbf{Q})$ be the Galois group. Find an isomorphism of G with the group U defined by

$$U = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{Z}/p\mathbb{Z}, a \neq 0 \right\}.$$

7. Let $K \subset \mathbf{C}$ be a field which is a Galois extension of $\mathbf{Q}$ with Galois group isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Prove that $K = \mathbf{Q}[\sqrt{a}, \sqrt{b}]$ for some rational numbers a, b.

8. Let $f(x) \in \mathbf{Q}[x]$ be an irreducible cubic polynomial whose Galois group is S_3. Let $K \subset \mathbf{C}$ be the splitting field of $(x^3 - 1)f(x)$. What are the possiblies for $\mathrm{Aut}(K/\mathbf{Q})$? Justify your answer.

9. Let K/F be a Galois extension (characteristic 0) with Galois group D_4. Prove that K is the splitting field of $x^4 + ax^2 + b$ for some $a, b \in F$.