

Disc of $X^3 + pX + q$ is

$$\Delta = -4p^3 - 27q^2$$

$$X^3 + aX^2 + \dots$$

$$(X - \frac{a}{3})^3 + a(X - \frac{a}{3})^2 + \dots$$

Ex. $X^3 + 3X + 2$

$$\begin{aligned}\Delta &= -4(27) - 27(4) \\ &= -216\end{aligned}$$

$$\Delta = \delta^2$$

$$\delta = (\alpha - \beta)(\alpha - \gamma)(\beta - \gamma)$$

$$\delta \in \mathbb{Q}.$$

If $\delta \in \mathbb{Q} \Rightarrow$ Galois group does not have odd permutations.

~~$X^3 + 3X + 1$~~ $X^3 - 3X + 1$

$$\begin{aligned}\Delta &= +4(27) - 27 \\ &= 81\end{aligned}$$

Let $F \subset K$ be a Galois extension, splitting field of $p(x) \in F[x]$ ~~with $p(x)$~~ such that $p(x)$ has n distinct roots $\alpha_1, \dots, \alpha_n \in K$.

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$$G = \text{Aut}(K/F) \hookrightarrow S_n$$

Consider $\delta = \prod_{i > j} (\alpha_i - \alpha_j) \in K$, $\Delta = \delta^2 \in F$.

If Δ is a square in F , then $G \subset A_n = \{\text{even permutations}\}$

Conversely, if $G \subset A_n$, then Δ is a square in F .

$p(x)$ irred cubic.

K/F splitting field

What is $G = \text{Aut}(K/F)$.

S_3

A_3

iff Δ is
not a square.

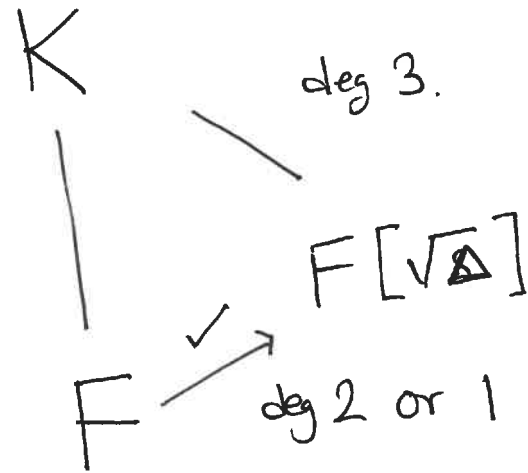
$$x^3 - 3x + 2.$$

iff Δ is a square

e.g. $x^3 - 3x + 1$

F is cubic

$$p(x) = x^3 + px + q$$



Kummer to the rescue. (char 0)

Let $F \subset K$ be a Galois extⁿ with Gal. gp $\mathbb{Z}/p\mathbb{Z}$.

Assume F has all p^{th} roots of 1. p prime.
(i.e. $X^p - 1$ splits completely in F).

Then: K is obtained from F by adjoining a p^{th} root.

That is, $\exists a \in K$ s.t. $b = a^p \in F$ and $a \notin F$.

(Then $K \cong F[x] / \begin{pmatrix} X^p - a^p \\ X^p - b \end{pmatrix}$).

Ex. $p(x) = x^3 - 3x + 1$ $F = \mathbb{Q}[\zeta_3]$

$\underbrace{F \subset K}_{\mathbb{Z}/3\mathbb{Z}} = \text{splitting field of } p(x) \text{ over } F. \quad C \in$

Kummer: $K = F[a]$ for some a whose cube $\in F$
 $K = F[\sqrt[3]{b}]$ for some $b \in F$.

In our case $K = F[\alpha, \beta, \gamma]$ α, β, γ are the roots.

$a = \alpha + \zeta_3 \beta + \zeta_3^2 \gamma$ Claim: $a^3 \in F$.