

# THE THURSTON COMPACTIFICATION OF THE STABILITY MANIFOLD OF A GENERIC ANALYTIC K3 SURFACE

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ABSTRACT. Let  $X$  be an analytic K3 surface with  $\text{Pic } X = 0$ . We describe the closure of the Bridgeland stability manifold of  $X$  obtained using the masses of semi-rigid objects.

## 1. INTRODUCTION

Associated to a triangulated category  $\mathcal{C}$  is the complex manifold  $\text{Stab}(\mathcal{C})$  whose points are the Bridgeland stability conditions on  $\mathcal{C}$  [?]. Understanding the global geometry of  $\text{Stab}(\mathcal{C})$  is an important question with far-reaching applications. For example, when  $\mathcal{C}$  is the derived category of coherent sheaves on a K3 surface, the simple connectedness of  $\text{Stab}(\mathcal{C})$  allows us to recover the group of auto-equivalences of  $\mathcal{C}$  [?]. When  $\mathcal{C}$  is the 2-Calabi–Yau category associated to a quiver, the topology of  $\text{Stab}(\mathcal{C})$  has implications for the word/conjugacy problems and the  $K(\pi, 1)$ -conjecture for the associated Artin group [?, ?].

To better understand the global geometry of a non-compact space like  $\text{Stab}(\mathcal{C})$ , it is useful to have a compactification. There have been several (partial) compactifications in the literature; see, for example, [?, ?, ?, ?]. The goal of this paper is to completely describe the compactification constructed in [?] when  $\mathcal{C}$  is the derived category of coherent sheaves on a generic analytic K3 surface.

The compactification in [?] is motivated by viewing a stability condition as a metric, and in particular by Thurston’s compactification of the Teichmüller space of hyperbolic metrics on a surface. We recall the main idea. Given a stability condition  $\sigma$  on  $\mathcal{C}$  and an object  $x \in \mathcal{C}$ , the *mass* of  $x$  with respect to  $\sigma$ , denoted by  $m_\sigma(x)$ , is the sum  $m_\sigma(x) = \sum_i |Z_\sigma(x_i)|$ , where the  $x_i$  are the  $\sigma$ -Harder–Narasimhan (HN) factors of  $x$  and  $Z_\sigma$  is the central charge of  $\sigma$ . To construct the compactification, we fix a set of objects  $S$ , and consider the map  $m: \mathbf{P} \text{Stab}(\mathcal{C}) = \text{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \mathbf{P}^S$  given by  $\sigma \mapsto [m_\sigma]$ . The proposed compactification is the closure of the image of  $m$ .

**Theorem 1.1.** *Let  $X$  be an analytic K3 surface with  $\text{Pic}(X) = 0$ . Let  $S \subset D^b \text{Coh}(X)$  be the set of semi-rigid objects. The map  $m: \mathbf{P} \text{Stab}(D^b \text{Coh}(X)) \rightarrow \mathbf{P}^S$  is a homeomorphism onto its image. The image is a 2-dimensional open ball and its closure is a 2-dimensional closed ball.*

See ?? for an illustration of the compactified stability space. The boundary contains a distinguished point represented by the function  $\text{hom}(\mathcal{O}_X, -)$  (red point in ??). This is the mass functions of a lax stability condition in the sense of [?]. The other vertices are mass functions of lax pre-stability conditions. The other boundary points do not have such interpretation.

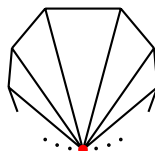


FIGURE 1. For an analytic K3 surface  $X$  with  $\text{Pic}(X) = 0$ , the compactified  $\mathbf{P} \text{Stab}(X)$  is a closed 2-ball, tiled by the translates of a triangle by the action of the spherical twist in  $\mathcal{O}_X$ . A distinguished point (red) in the boundary corresponds to the function  $\text{hom}(\mathcal{O}_X, -)$ .

?? is a combination of ?? and ?? in the main text. The discussion of the points in the boundary is in ??.

For a positive real number  $q$ , the mass map has a natural  $q$ -analogue  $m_q$ . The closure of the image of the stability manifold under  $m_q$  is also a closed disk. However, in its boundary, the red point in ?? is replaced by a closed interval (see ??).

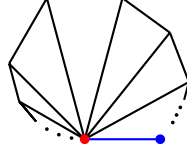


FIGURE 2. The closure of  $m_q(\mathbf{P}\text{Stab}(X))$  is also a closed disk. The boundary has an additional interval, whose blue end-point is the  $q$ -hom functional  $\text{hom}_q(\mathcal{O}_X, -)$ .

For  $q = 1$ , the distinguished red point in the boundary has two interpretations: one as the hom function  $\text{hom}(\mathcal{O}_X, -)$  and the second as the mass function of a lax stability condition  $\sigma$  in which  $\mathcal{O}_X$  is massless. For  $q \neq 1$ , the two interpretations diverge. The  $q$ -hom function  $\text{hom}_q(\mathcal{O}_X, -)$  yields the blue end-point in ?? and the  $q$ -mass function  $m_q(\sigma)$  yields the red end-point.

We can reconcile the two pictures (?? and ??) by drawing them in the upper half plane instead of the disk (see ??). The  $q = 1$  picture (??) corresponds to the union of the translates of an ideal triangle by the transformation  $z \mapsto z + 1$ . The only additional point in the closure (in the closed disk) is the point at infinity. The  $q \neq 1$  picture (??) corresponds to the union of the translates of an ideal triangle by the transformation  $z \mapsto qz + 1$ . In this case, the closure (in the closed disk) contains an additional interval. This  $q$ -deformation is a simpler version of the  $q$ -deformed Farey tessellation observed in [?].

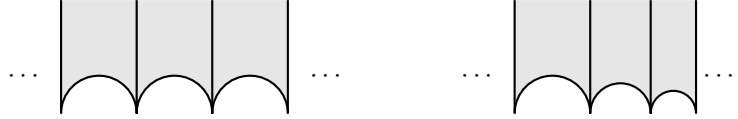


FIGURE 3. The tiling of the disk by triangles in the  $q = 1$  case (left) versus the  $q \neq 1$  case (right).

In the course of the proof of the main theorem, we also characterise all semi-rigid objects of  $D^b\text{Coh}(X)$ . Up to twists by  $\mathcal{O}_X$  and homological shifts, the only such objects are the skyscraper sheaves  $\mathbf{k}_x$  (??).

There are a few other cases where the Thurston compactification of the stability manifold has been completely described. These include: the 2-Calabi–Yau categories associated to quivers of rank 2 [?] and the derived categories of coherent sheaves on algebraic curves [?]. In [?] the authors prove that for any (algebraic) K3 surface  $X$ , taking  $S$  to be the set of spherical objects gives an injective map  $m: \mathbf{P}\text{Stab}(X) \rightarrow \mathbf{P}^S$ . Understanding its image and its closure is an important goal. The case of non-algebraic K3s treated here is a step towards it.

**1.1. Conventions.** An *analytic K3 surface* is a connected, simply-connected, and compact complex manifold  $X$  of dimension 2 with  $h^1(\mathcal{O}_X) = 0$ . By  $D^b(X)$  we mean the bounded derived category of the abelian category  $\text{Coh}(X)$  of coherent sheaves on  $X$ , as studied in [?]. For a point  $x \in X$ , we denote by  $\mathbf{k}_x$  the push-forward to  $X$  of the structure sheaf of  $x$ , and call it the *skyscraper sheaf* at  $x$ . By  $\text{Stab}(X)$ , we denote the set of (locally finite) Bridgeland stability conditions on  $D^b(X)$  with a numerical central charge; that is, where the central charge  $Z: K(D^b(X)) \rightarrow \mathbf{C}$  factors through the Chern character  $\text{ch}: K(D^b(X)) \rightarrow H^*(X, \mathbf{Q})$ . We let  $\mathbf{P}\text{Stab}(X)$  be the quotient of  $\text{Stab}(X)$  by the standard action of  $\mathbf{C}$ , in which  $z = x + i\pi y$  acts by scaling the central charge by  $e^z$  and shifting the slicing by  $y$ . Given a set  $S$ , we let  $\mathbf{R}^S$  be the space of functions  $S \rightarrow \mathbf{R}$  with the product topology and  $\mathbf{P}^S$  the projective space  $(\mathbf{R}^S - \{0\})/\text{scaling}$ .

**1.2. Outline.** In ??, we recall the description of stability conditions on an analytic K3 surface  $X$  with  $\text{Pic } X = 0$ . In ??, we identify the semi-rigid objects of  $D^b(X)$ . The bulk of the paper is ??, in which we study the embedding of  $\mathbf{P}\text{Stab}(X)$  given by the masses of semi-rigid objects. In ??, we study the  $q$ -analogue of the mass embedding. We do not include the definitions and the basic properties of stability conditions, and refer the reader to the original source [?] or exposition [?].

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## 2. STABILITY CONDITIONS ON GENERIC K3 SURFACES

Throughout, fix an analytic K3 surface  $X$  with  $\text{Pic } X = 0$ . Since  $X$  is a K3 surface,  $D^b(X)$  is a 2-Calabi–Yau category. That is, for  $x, y \in D^b(X)$ , we have a natural isomorphism

$$\text{Hom}(x, y) \cong \text{Hom}(y, x[2]).$$

**2.1. The Mukai lattice.** The Mukai lattice  $\mathcal{N}(X)$  of  $X$  is given by

$$\mathcal{N}(X) = (H^0 \oplus H^4)(X, \mathbf{Z}).$$

Taking the class of  $X$  as a generator of the  $H^0$  summand and the class of a point  $x \in X$  as a generator of the  $H^4$  summand, we get an identification

$$\mathcal{N}(X) = \mathbf{Z} \oplus \mathbf{Z}.$$

The Mukai pairing is then given by

$$(\alpha_1, \alpha_2) \cdot (\beta_1, \beta_2) = \alpha_1 \beta_2 + \alpha_2 \beta_1.$$

Given  $F \in D^b(X)$ , we let  $[F] = (\text{ch}_0 F, \text{ch}_0 F - \text{ch}_2 F) \in \mathcal{N}(X)$  be its Mukai vector. Then we have

$$[\mathcal{O}_X] = (1, 1) \text{ and } [\mathbf{k}_x] = (0, 1).$$

In particular,  $[\mathcal{O}_X]$  and  $[\mathbf{k}_x]$  form a basis of  $\mathcal{N}(X)$ .

**2.2. Standard stability conditions.** We recall basic facts about stability conditions on  $X$  from [?, § 4]. Let  $\mathcal{F}$  and  $\mathcal{T}$  be the full-subcategories of  $\text{Coh}(X)$  consisting of torsion free and torsion sheaves, respectively. Then  $(\mathcal{F}, \mathcal{T})$  forms a torsion pair. Let  $\mathcal{A}$  be the tilt of  $\text{Coh}(X)$  in this torsion pair. Explicitly,

$$\mathcal{A} = \{E \in D^b(X) \mid H^{-1}(E) \in \mathcal{F} \text{ and } H^0(E) \in \mathcal{T} \text{ and for all } i \notin \{0, 1\} : H^i(E) = 0\}.$$

Then  $\mathcal{A}$  is the heart of a bounded t-structure on  $D^b(X)$ .

Let  $\mathbf{H} \subset \mathbf{C}$  be the (open) upper half plane. As proved in [?, § 4.2], for every  $z \in \mathbf{H} \cup \mathbf{R}_{<0}$ , we have a stability condition  $\sigma_z$  on  $D^b(X)$  whose  $(0, 1]$  heart is  $\mathcal{A}$  and whose central charge is given by

$$Z : [\mathbf{k}_x] \mapsto -1 \text{ and } Z : [\mathcal{O}_X] \mapsto -z.$$

For every  $w \in -\mathbf{H}$ , we have a stability condition  $\sigma_w$  on  $D^b(X)$  whose  $(0, 1]$  heart is  $\text{Coh}(X)$  and whose central charge is given by

$$Z : [\mathbf{k}_x] \mapsto -1 \text{ and } Z : [\mathcal{O}_X] \mapsto -w.$$

See ?? for a sketch of the two central charges.

*Remark 2.1.* The combined domain of the parameters  $z$  and  $w$  in [?, § 4.2] is  $\mathbf{C} - \mathbf{R}_{\geq -1}$ . For us, it is  $\mathbf{C} - \mathbf{R}_{\geq 0}$ . The difference is due to a slight change in parametrisation. The central charge of  $\sigma_z$  in [?, § 4.2] sends  $\mathbf{k}_x$  to  $-1$  (same as ours) and  $\mathcal{O}_X$  to  $-z-1$  (we send it to  $-z$ ). So our parametrisation and the parametrisation in [?, § 4.2] are related by  $z \mapsto z + 1$ .

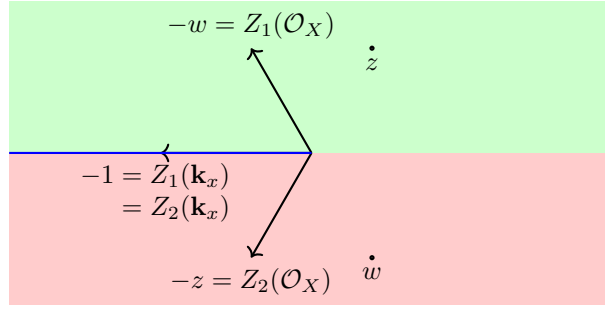


FIGURE 4. For  $w \in -\mathbf{H}$  (red), a central charge  $Z_1$  as above defines a stability condition with heart  $\text{Coh}(X)$ . For  $z \in \mathbf{H}$  (green) and  $z \in \mathbf{R}_{<0}$  (blue), a central charge  $Z_2$  as above defines a stability condition whose heart is the tilt of  $\text{Coh}(X)$  with respect to torsion and torsion-free sheaves.

We call the stability conditions  $\sigma_z$  for  $z \in \mathbf{H} \cup -\mathbf{H} \cup \mathbf{R}_{<0}$  the *standard stability conditions*. We say that the stability conditions  $\sigma_z$  for  $z \in \mathbf{R}_{<0}$  are *on the wall*, and the rest are *off the wall*.

Let  $W_+$  (resp.  $W_-$  and  $W_0$ ) be the union of the  $\mathbf{C}$ -orbits of the stability conditions  $\sigma_z$  for  $z \in \mathbf{H}$  (resp.  $-\mathbf{H}$  and  $\mathbf{R}_{<0}$ ). By definition, the sets  $W_+$ ,  $W_-$ , and  $W_0$  are invariant under the  $\mathbf{C}$ -action. It is easy to check that they are also invariant under the  $\widehat{\mathrm{GL}}_2^+(\mathbf{R})$ -action, and hence coincide with the sets with the same name defined in the proof of [?, Theorem 4.8]. Set  $W = W_+ \cup W_- \cup W_0$ .

**2.3. All stability conditions.** Recall that the only spherical objects in  $D^b(X)$  are the shifts of  $\mathcal{O}_X$  (see [?, Proposition 2.15]). Let  $T: D^b(X) \rightarrow D^b(X)$  be the spherical twist in  $\mathcal{O}_X$ .

**Proposition 2.2.** *The set  $W \subset \text{Stab}(X)$  is open and the union of its translates  $T^n W$ , for  $n \in \mathbf{Z}$ , is  $\text{Stab}(X)$ .*

*Proof.* That  $W$  is open is proved in [?, Theorem 4.8]. That  $\text{Stab}(X) = \bigcup T^n W$  is [?, Corollary 4.7].  $\square$

The following proposition allows us to identify the stability conditions in  $W_+$ ,  $W_-$ , and  $W_0$ . Recall that since, up to shifts,  $\mathcal{O}_X$  is the only spherical object, it must be stable in any stability condition [?, Proposition 2.15].

**Proposition 2.3.** *Let  $\sigma$  be a stability condition and let  $\phi$  be the phase of  $\mathcal{O}_X$ . Then  $\sigma$  is in  $W$  if and only if all the skyscraper sheaves  $\mathbf{k}_x$  are  $\sigma$ -stable of the same phase  $\psi$ . In this case, we have*

- (1)  $\sigma \in W_-$  if  $\psi \in (\phi, \phi + 1)$ ,
- (2)  $\sigma \in W_+$  if  $\psi \in (\phi + 1, \phi + 2)$ ,
- (3)  $\sigma \in W_0$  if  $\psi = \phi + 1$ .

*Proof.* Since all skyscraper sheaves  $\mathbf{k}_x$  are  $\sigma$ -stable of the same phase for a standard stability condition, the same is true for any  $\sigma \in W$ . Conversely, suppose all  $\mathbf{k}_x$  are  $\sigma$ -stable of the same phase. Using the  $\mathbf{C}$ -action, assume that their phase  $\psi$  is 1 and their central charge is  $-1$ . By [?, Proposition 4.6], we conclude that  $\sigma$  is standard.

Suppose  $\sigma = \sigma_z$  for  $z \in -\mathbf{H} \cup \mathbf{H} \cup \mathbf{R}_{<0}$ . Whether  $z \in -\mathbf{H}$  or  $\mathbf{H}$  or  $\mathbf{R}_{<0}$  is distinguished by the phase  $\phi$  of  $\mathcal{O}_X$ . For  $z \in -\mathbf{H}$ , we have  $\phi \in (0, 1)$ ; for  $z \in \mathbf{H}$ , we have  $\phi \in (-1, 0)$ ; and for  $z \in \mathbf{R}_{<0}$ , we have  $\phi = 0$ .  $\square$

**Proposition 2.4.** *We have  $TW_+ = W_-$  and  $T^{-1}W_- = W_+$ .*

*Proof.* We prove that for a standard  $\sigma \in W_-$ , we have  $T(\sigma) \in W_+$ , and for a standard  $\sigma \in W_+$ , we have  $T^{-1}(\sigma) \in W_-$ . Then the proposition follows.

Take a standard  $\sigma \in W_-$  and let us prove that  $T^{-1}(\sigma) \in W_+$ . Let  $\phi \in (0, 1)$  be the phase of  $\mathcal{O}_X$ . It is easy to check that the ideal sheaves  $I_x$  of points  $x \in X$  are  $\sigma$ -stable of the same phase  $\psi \in (0, \phi)$ .

Let  $x \in X$  be any point. Since  $\mathrm{Hom}^*(\mathcal{O}_X, \mathbf{k}_x) = \mathbf{C}$ , we have the exact triangle

$$\mathcal{O}_X \xrightarrow{\mathrm{ev}} \mathbf{k}_x \rightarrow T\mathbf{k}_x \xrightarrow{+1}.$$

Therefore,  $T\mathbf{k}_x = I_x[1]$ . So  $T\mathbf{k}_x$  is  $\sigma$ -stable of phase  $\psi + 1$ . Therefore,  $T^{-1}I_x[1] = \mathbf{k}_x$  is  $T^{-1}(\sigma)$ -stable of phase  $\psi + 1 \in (1, \phi + 1)$ . On the other hand,  $T^{-1}\mathcal{O}_X = \mathcal{O}_X[1]$  is  $T^{-1}(\sigma)$ -stable of phase  $\phi$ , so  $\mathcal{O}_X$  is  $T^{-1}(\sigma)$ -stable of phase  $\phi - 1$ . We now apply ??.

Now take a standard  $\sigma \in W_+$  and let us prove that  $T(\sigma) \in W_-$ . Let  $\phi \in (-1, 0)$  be the phase of  $\mathcal{O}_X$ . The objects  $T^{-1}\mathbf{k}_x$  are  $\sigma$ -stable of phase  $\psi \in (\phi + 1, 1)$  (see [?, Remark 4.3 (i)]). Therefore, the skyscraper sheaves  $\mathbf{k}_x$  are  $T(\sigma)$ -stable of phase  $\psi \in (\phi + 1, 1)$ . Since  $\mathcal{O}_X$  is  $\sigma$ -stable of phase  $\phi$ , it is  $T(\sigma)$ -stable of phase  $\phi + 1$ . We again apply ??.  $\square$

We now turn to the topology of the set of standard stability conditions and the stability conditions in  $W$ . Let  $H \subset \mathrm{Stab}(X)$  be the set of standard stability conditions. Let  $R = \mathbf{C} \setminus \mathbf{R}_{\geq 0}$ . We have a map  $R \rightarrow H$  given by  $z \mapsto \sigma_z$ . We also have the projection map  $H \rightarrow \mathbf{PW} = W/\mathbf{C}$ .

**Proposition 2.5.** *The maps  $R \rightarrow H$  and  $H \rightarrow \mathbf{PW}$  are homeomorphisms.*

*Proof.* By definition, the map  $R \rightarrow H$  is a bijection. By the proof of [?, Theorem 4.8] (part (ii)), the map  $R \rightarrow H$  is continuous. Its inverse is given by  $\sigma \mapsto -Z_\sigma(\mathcal{O}_X)$ , which is also continuous. So  $R \rightarrow H$  is a homeomorphism.

By ??, the map  $H \rightarrow \mathbf{PW}$  is surjective. Owing to the normalisation of the phase and mass of  $\mathbf{k}_x$ , it is also injective. It remains to prove that the inverse is continuous. We know that  $W$  is an open subset of  $\mathrm{Stab}(X)$ . It is also  $\mathbf{C}$ -invariant, so  $\mathbf{PW}$  is an open subset of  $\mathbf{P}\mathrm{Stab}(X)$ . Thus, the map  $\mathbf{PW} \rightarrow \mathbf{P}\mathrm{Hom}(\mathcal{N}(X), \mathbf{C})$  is a local homeomorphism. We have the commutative diagram

$$\begin{array}{ccccc} R & \xleftarrow{\sim} & H & \xleftarrow{\quad} & \mathbf{PW} \\ \parallel & & & & \downarrow \\ R & \xleftarrow{\quad} & \mathbf{P}\mathrm{Hom}(\mathcal{N}(X), \mathbf{C}), & & \end{array}$$

where the bottom map is given by  $Z \mapsto Z(\mathcal{O}_X)/Z(\mathbf{k}_x)$ . Since this map is continuous, it follows that  $\mathbf{PW} \rightarrow H$  is continuous.  $\square$

### 3. SEMI-RIGID OBJECTS

Recall that an object  $F$  in  $D^b(X)$  is *semi-rigid* if

$$\mathrm{hom}^i(F, F) = \begin{cases} 1 & \text{if } i = 0 \\ 2 & \text{if } i = 1 \\ 1 & \text{if } i = 2, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For example, for  $x \in X$ , the skyscraper sheaf  $F = \mathbf{k}_x$  and the ideal sheaf  $F = I_x$  are semi-rigid. We now characterise the semi-rigid objects of  $D^b(X)$ . Recall that  $T: D^b(X) \rightarrow D^b(X)$  is the spherical twist in  $\mathcal{O}_X$ .

**Proposition 3.1.** *Let  $X$  be a K3 surface with  $\mathrm{Pic} X = 0$ . Let  $F \in D^b(X)$  be semi-rigid. Then there exists  $x \in X$  and integers  $m, n$  such that  $F \cong T^m \mathbf{k}_x[n]$ .*

We split the proof in two lemmas.

**Lemma 3.2.** *Fix a standard stability condition  $\sigma \in W_-$ . Let  $F \in D^b(X)$  be semi-rigid and  $\sigma$ -semi-stable. Then there exists  $x \in X$  such that  $F$  or  $T^{-1}F$  is a shift of  $\mathbf{k}_x$ .*

*Proof.* Since  $F$  is semi-rigid,  $[F] \cdot [F] = 0$  in  $\mathcal{N}(X)$ . So  $[F]$  is an integer multiple of  $(0, 1)$  or  $(1, 0)$ .

Suppose  $[F]$  is a multiple of  $(0, 1)$ . Since  $[\mathbf{k}_x] = (0, 1)$ , after applying a shift, we may assume that  $F$  is  $\sigma$ -semi-stable of the same phase as  $\mathbf{k}_x$ , namely 1. It is easy to check that the abelian category of  $\sigma$ -semi-stable objects of phase 1 is  $\mathcal{F}$ , the category of torsion sheaves on  $X$ . It is a finite length category

whose simple objects are the skyscraper sheaves  $\mathbf{k}_x$ . So  $F$  is an iterated extension of skyscraper sheaves. Since  $\mathrm{hom}^1(F, F) = 2$ , the Mukai lemma [?, Lemma 2.7] implies that  $F$  must simply be a skyscraper sheaf.

Suppose  $[F]$  is a multiple of  $(1, 0)$ . Then  $[T^{-1}F]$  is a multiple of  $(0, 1)$  and  $T^{-1}F$  is semi-stable with respect to  $\tau = T^{-1}\sigma$ . By ??, we have  $\tau \in W_+$ . By applying a rotation, assume that  $\tau$  is standard. Then, after applying a shift, we may assume that  $T^{-1}F$  is  $\tau$ -semi-stable of the same phase as  $\mathbf{k}_x$ , namely 1. Again, it is easy to check that the abelian category of  $\tau$ -semi-stable objects of phase 1 is  $\mathcal{F}$ . We now proceed as before.  $\square$

Given a stability condition  $\sigma$ , denote by  $\phi_\sigma^+$  and  $\phi_\sigma^-$  the highest and lowest phases of the factors in the  $\sigma$ -HN filtration. If  $\sigma$  is clear from the context, we omit the subscript.

**Lemma 3.3.** *Fix a standard stability condition  $\sigma \in W_-$ . Let  $F \in D^b(X)$  be a semi-rigid object. There exists a non-negative integer  $n$  such that  $T^n F$  is  $\sigma$ -semi-stable.*

*Proof.* Since  $F$  is semi-rigid, all stable factors of  $F$  are either spherical or semi-rigid, and only one stable factor is semi-rigid [?, Proposition 2.9]. The only spherical object, up to shift, is  $\mathcal{O}_X$ . By ??, the only semi-stable semi-rigid objects, up to shift, are  $\mathbf{k}_x$  and  $T^{-1}\mathbf{k}_x$ . In particular, the phases of the HN factors of  $F$  lie in the discrete subset of  $\mathbf{R}$  given by

$$(\phi_\sigma(\mathcal{O}_X) + \mathbf{Z}) \cup (\phi_\sigma(\mathbf{k}_x) + \mathbf{Z}) \cup (\phi_\sigma(T^{-1}\mathbf{k}_x) + \mathbf{Z}).$$

Therefore, there exists a discrete  $\Phi \subset \mathbf{R}$  such that for every semi-rigid object  $F$ , we have

$$\phi^+(F) - \phi^-(F) \in \Phi.$$

If  $F$  itself is semi-stable, we simply take  $n = 0$ . Otherwise, up to shift, a stable HN factor of  $F$  of highest or lowest phase must be  $\mathcal{O}_X$ . We apply [?, Theorem 3.5] with  $Y = F$  and  $X = \mathcal{O}_X$ . Then for  $F' = TF$  or  $F' = T^{-1}F$ , we have

$$\phi^+(F') - \phi^-(F') < \phi^+(F) - \phi^-(F).$$

By repeated applications of [?, Theorem 3.5] and using that  $\phi^+ - \phi^-$  lies in the discrete set  $\Phi \subset \mathbf{R}$ , we conclude that there exists an integer  $n$  such that  $T^n F$  is semi-stable.  $\square$

Having proved the two lemmas, we are ready to prove ??—the only semi-rigid objects of  $D^b(X)$ , up to twisting by  $\mathcal{O}_X$  and shifting, are the skyscraper sheaves  $\mathbf{k}_x$ .

*Proof of ??.* Combine ?? and ??.  $\square$

#### 4. THE MASS EMBEDDING

Recall that  $X$  is an analytic K3 surface with  $\mathrm{Pic} X = 0$ . Let  $S$  be the set of isomorphism classes of semi-rigid objects of  $D^b(X)$ . In this section, we describe the mass embedding

$$m: \mathbf{P} \mathrm{Stab}(X) \rightarrow \mathbf{P}^S$$

and the closure of its image.

**4.1. HN filtration of semi-rigid objects.** To understand the mass embedding, we must understand the HN filtrations of the objects of  $S$ . By ??, the objects of  $S$ , up to shift, are  $T^n \mathbf{k}_x$  for  $x \in X$  and  $n \in \mathbf{Z}$ . For points  $x, y \in X$ , the behaviour of  $T^n \mathbf{k}_x$  and  $T^n \mathbf{k}_y$  is entirely analogous to each other. So we lose nothing by fixing a particular point  $x \in X$  and taking

$$S = \{T^n \mathbf{k}_x \mid n \in \mathbf{Z}\}.$$

We may then write the points of  $\mathbf{P}^S$  as homogeneous vectors  $[x_n \mid n \in \mathbf{Z}] = [\cdots : x_{-1} : x_0 : x_1 : \cdots]$ . In these coordinates, the spherical twist  $T$  acts as a shift.

We first treat HN filtrations with respect to off the wall stability conditions.

**Proposition 4.1.** *Let  $\sigma \in W_-$ . Then the  $\sigma$ -HN factors of  $F = T^n \mathbf{k}_x$ , in decreasing order of phase, are as follows.*

- (1) For  $n = 0$  and  $1$ , the object  $F$  is stable.
- (2) For  $n \geq 2$ , the semi-stable (= stable) factors of  $F$  are  $T\mathbf{k}_x$  and  $\mathcal{O}_X[i]$  for  $0 \geq i \geq -n + 2$ .
- (3) For  $n \leq -1$ , the semi-stable (= stable) factors of  $F$  are  $\mathcal{O}_X[i]$  for  $-n \geq i \geq 1$  and  $\mathbf{k}_x$ .

*Proof.* Recall that  $\mathbf{k}_x$  and  $T\mathbf{k}_x = I_x[1]$  are stable for stability conditions in  $W_-$ . So (1) follows.

Consider the triangle

$$(1) \quad \mathrm{Hom}^*(\mathcal{O}_X, T^{n-1}\mathbf{k}_x) \otimes \mathcal{O}_X \rightarrow T^{n-1}\mathbf{k}_x \rightarrow T^n\mathbf{k}_x \xrightarrow{+1}.$$

We have

$$\begin{aligned} \mathrm{Hom}^*(\mathcal{O}_X, T^{n-1}\mathbf{k}_x) &= \mathrm{Hom}^*(T^{-n+1}\mathcal{O}_X, \mathbf{k}_x) \\ &= \mathrm{Hom}^*(\mathcal{O}_X[n-1], \mathbf{k}_x) \\ &= \mathbf{C}[-n+1]. \end{aligned}$$

By substituting in (??) and shifting, we get

$$(2) \quad T^{n-1}\mathbf{k}_x \rightarrow T^n\mathbf{k}_x \rightarrow \mathcal{O}_X[-n+2] \xrightarrow{+1}.$$

Let us assume  $n \geq 2$ , and induct on  $n$ . Assume we know that the HN factors of  $T^{n-1}\mathbf{k}_x$  (in decreasing order of phase) are  $T\mathbf{k}_x$  followed by  $\mathcal{O}_X[i]$  for  $0 \geq i \geq -n+3$ . Concatenating the HN filtration of  $T^{n-1}\mathbf{k}_x$  and the map  $T^{n-1}\mathbf{k}_x \rightarrow T^n\mathbf{k}_x$ , we obtain a filtration of  $T^n\mathbf{k}_x$  whose factors are  $T\mathbf{k}_x$  and  $\mathcal{O}_X[i]$  for  $0 \geq i \geq -n+2$ . Since these factors are stable and appear in decreasing order of phase, this must be the HN filtration of  $T^n\mathbf{k}_x$ . The induction step is complete.

Now let us assume  $n \leq -1$ , and induct on  $-n$ . Consider the triangle

$$(3) \quad \mathcal{O}_X[-n] \rightarrow T^n\mathbf{k}_x \rightarrow T^{n+1}\mathbf{k}_x \xrightarrow{+1},$$

obtained by replacing  $n$  by  $n+1$  in (??) and shifting. Assume we know that the HN factors of  $T^{n+1}\mathbf{k}_x$  (in decreasing order of phase) are  $\mathcal{O}_X[i]$  for  $-n-1 \geq i \geq 1$  and  $\mathbf{k}_x$ . By augmenting the HN filtration of  $T^{n+1}\mathbf{k}_x$  by the map  $\mathcal{O}_X[-n] \rightarrow T^n\mathbf{k}_x$ , we obtain a filtration of  $T^n\mathbf{k}_x$  whose factors are  $\mathcal{O}_X[i]$  for  $-n \geq i \geq 1$  and  $\mathbf{k}_x$ . Since these factors are stable and appear in decreasing order of phase, this must be the HN filtration of  $T^n\mathbf{k}_x$ . The induction step is complete.  $\square$

For stability conditions on the wall, the HN filtration degenerates as expected.

**Proposition 4.2.** *Let  $\sigma \in W_0$ . Then the  $\sigma$ -HN factors of  $F = T^n\mathbf{k}_x$ , in decreasing order of phase, are as follows.*

- (1) For  $n = -1, 0$  and  $1$ , the object  $F$  is semi-stable.
- (2) For  $n \geq 2$ , the semi-stable factors of  $F$  are  $T\mathbf{k}_x$  and  $\mathcal{O}_X[i]$  for  $0 \geq i \geq -n+2$ .
- (3) For  $n \leq -2$ , the semi-stable factors of  $F$  are  $\mathcal{O}_X[i]$  for  $-n \geq i \geq 2$  and  $T^{-1}\mathbf{k}_x$ .

*Proof.* The proof is analogous to the proof of ??.  $\square$

**4.2. The mass map.** We now have the tools to describe the mass map

$$m: \mathbf{P} \mathrm{Stab}(X) \rightarrow \mathbf{P}^S.$$

**Proposition 4.3.** *Let  $\sigma \in \mathbf{PW}_-$ . Set  $a = |Z_\sigma(\mathbf{k}_x)|$  and  $b = |Z_\sigma(T\mathbf{k}_x)|$  and  $c = |Z_\sigma(\mathcal{O}_X)|$ .*

- (1) *The numbers  $a, b, c$  are positive real numbers satisfying*

$$b < a + c, \quad a < b + c, \quad c < a + b.$$

- (2) *We have*

$$m_\sigma: T^n\mathbf{k}_x \mapsto \begin{cases} a - nc & \text{if } n \leq 0, \\ b + (n-1)c & \text{if } n \geq 1. \end{cases}$$

(3) Let  $\Delta_0 \subset \mathbf{P}^S$  be the locally closed subset consisting of points of the form

$$[\cdots : a + 2c : a + c : a : b : b + c : b + 2c : \cdots],$$

where  $a$  is at index 0 and  $b$  is at index 1, and where  $a, b, c$  are positive real numbers satisfying the inequalities in (??). Then  $m : \mathbf{PW}_- \rightarrow \Delta_0$  is a homeomorphism.

*Proof.* Part (1) follows from the fact that the classes of  $\mathcal{O}_X$ ,  $\mathbf{k}_x$ , and  $T\mathbf{k}_x$  satisfy

$$[\mathcal{O}_X] = [\mathbf{k}_x] - [T\mathbf{k}_x].$$

Part (2) follows from ??.

For part (3), let  $\Delta \subset \mathbf{P}^2$  be the set of points  $[a : b : c]$  that satisfy the conditions in (??). Then we have a homeomorphism  $\Delta \rightarrow \Delta_0$  given by

$$[a : b : c] \mapsto [\cdots : a + 2c : a + c : a : b : b + c : b + 2c : \cdots].$$

We use  $[a : b : c] \in \Delta$  as coordinates on  $\Delta_0$ . By ??, the map  $w \mapsto \sigma_w$  gives a homeomorphism  $-\mathbf{H} \rightarrow \mathbf{PW}_-$ . We use  $w \in -\mathbf{H}$  as a coordinate on  $\mathbf{PW}_-$ . In these coordinates, writing down the inverse map  $\omega : \Delta \rightarrow \mathbf{PW}_-$  amounts to re-constructing the central charge given  $a, b, c$ . This can be done using the cosine rule (see ??). Precisely, we have

$$(4) \quad \omega([a : b : c]) = -(b/a \exp(i\theta) - 1), \text{ where } \theta = \arccos\left(\frac{a^2 + b^2 - c^2}{2ab}\right) \in (0, \pi),$$

which is continuous. □

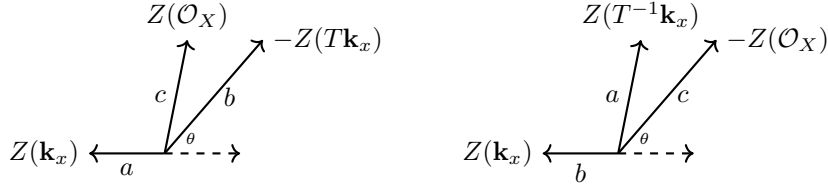


FIGURE 5. We can use the cosine rule to reconstruct the central charge of a standard  $\sigma \in W_-$  from the masses  $a, b, c$  of  $\mathbf{k}_x, T\mathbf{k}_x, \mathcal{O}_X$  (left) and of  $\sigma \in W_+$  from the masses  $a, b, c$  of  $T^{-1}\mathbf{k}_x, \mathbf{k}_x, \mathcal{O}_X$  (right).

For  $n \in \mathbf{Z}$ , let  $\Delta_n \subset \mathbf{P}^S$  be the locally closed subset consisting of points of the form

$$[\cdots : a + 2c : a + c : a : b : b + c : b + 2c : \cdots],$$

where  $a$  is at index  $n$ , and where  $a, b, c$  are positive real numbers satisfying the (strict) triangle inequalities. Denote by  $T : \mathbf{P}^S \rightarrow \mathbf{P}^S$  the map that shifts the homogeneous coordinates rightwards by 1, so that  $\Delta_n = T^n \Delta_0$ . Recall that we also denote by  $T : \text{Stab}(X) \rightarrow \text{Stab}(X)$  the action of the spherical twist by  $\mathcal{O}_X$ . We have

$$m(T(\sigma)) = T(m(\sigma)).$$

?? implies that the mass map  $T^n \mathbf{PW}_- \rightarrow \Delta_n$  is a homeomorphism. In particular, the mass map  $T^{-1} \mathbf{PW}_- = \mathbf{PW}_+ \rightarrow \Delta_{-1}$  is a homeomorphism. It is useful to write the inverse  $\Delta_{-1} \rightarrow \mathbf{PW}_+$  using coordinates  $[a : b : c]$  on  $\Delta_{-1}$  as in the proof of ?? and the coordinate on  $W_+$  given by  $z \in \mathbf{H}$ . Recall that the  $[a : b : c]$  coordinates represent  $a = m(T^{-1}\mathbf{k}_x)$  and  $b = m(\mathbf{k}_x)$  and  $c = m(\mathcal{O}_X)$ . Then the map  $[a : b : c] \mapsto z$  is (see ??):

$$(5) \quad [a : b : c] \mapsto c/b \exp(i\theta), \text{ where } \theta = \arccos\left(\frac{b^2 + c^2 - a^2}{2bc}\right) \in (0, \pi).$$

Let  $I_0 \subset \mathbf{P}^S$  be the set of points of the form

$$[\cdots : a + 2c : a + c : a : a + c : a + 2c : \cdots],$$

where  $a$  is at index 0 and  $a, c$  are positive real numbers.



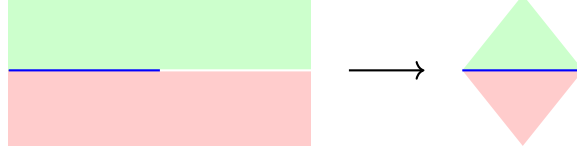


FIGURE 6. The mass map gives a homeomorphism from the set of standard stability conditions parametrised by  $-\mathbf{H} \cup \mathbf{H} \cup \mathbf{R}_{<0}$  and the union of two open triangles and a segment that forms a wall between them.

**Proposition 4.4.** *Let  $\sigma \in \mathbf{PW}_0$ . Set  $a = |Z_\sigma(\mathbf{k}_x)|$  and  $c = |Z_\sigma(\mathcal{O}_X)|$ . Then*

$$m_\sigma : T^n \mathbf{k}_x \mapsto a + |n|c.$$

*Furthermore, the map  $m : \mathbf{PW}_0 \rightarrow I_0$  is a homeomorphism.*

*Proof.* The description of  $m_\sigma$  follows from ???. The inverse of  $m : \mathbf{PW}_0 \rightarrow I_0$  is given using the central charge  $Z(\mathbf{k}_x) = -1$  and  $Z(\mathcal{O}_X) = c/a$ .  $\square$

**Proposition 4.5.** *The map  $m : \mathbf{PW} \rightarrow \Delta_0 \cup I_0 \cup \Delta_{-1}$  is a homeomorphism.*

See ?? for a sketch.

*Proof.* The set  $\mathbf{PW}$  is the disjoint union of  $\mathbf{PW}_-$ ,  $\mathbf{PW}_+$ , and  $\mathbf{PW}_0$ . The sets  $\Delta_0$ ,  $I_0$ , and  $\Delta_{-1}$  are also disjoint. Furthermore, the maps  $\mathbf{PW}_- \rightarrow \Delta_0$ ,  $\mathbf{PW}_+ \rightarrow \Delta_{-1}$ , and  $\mathbf{PW}_0 \rightarrow I_0$  are homeomorphisms. So  $m : \mathbf{PW} \rightarrow \Delta_0 \cup I_0 \cup \Delta_{-1}$  is a continuous bijection.

We check that the inverse is continuous. Since  $-\mathbf{H} \cup \mathbf{H} \cup \mathbf{R}_{<0} \rightarrow \mathbf{PW}$  is a homeomorphism, we use the former as local coordinates for  $\mathbf{PW}$ . Let  $\overline{\Delta} \subset \mathbf{P}^2$  be the set of points  $[a : b : c]$  where  $a, b, c$  are positive real numbers satisfying the triangle inequalities

$$b \leq a + c, \quad a < b + c, \quad c < a + b.$$

It is easy to check that the map  $\overline{\Delta} \rightarrow \Delta_0 \cup I_0$  given by

$$[a : b : c] \mapsto [\cdots : a + c : a : b : b + c : \cdots]$$

is a homeomorphism. So we may use  $a, b, c$  as local coordinates on  $\Delta_0 \cup I_0$ . Using (??), we see that the inverse map  $\Delta_0 \cup I_0 \rightarrow -\mathbf{H} \cup \mathbf{R}_{<0}$  is given in coordinates by

$$[a : b : c] \mapsto -b/a \exp(i\theta) + 1, \text{ where } \theta = \arccos\left(\frac{a^2 + b^2 - c^2}{2ab}\right) \in [0, \pi),$$

which is continuous.

Let  $\overline{\Delta}' \subset \mathbf{P}^2$  be the set of points  $[a : b : c]$  where  $a, b, c$  are positive real numbers satisfying the triangle inequalities

$$b < a + c, \quad a \leq b + c, \quad c < a + b.$$

Then the map  $\overline{\Delta}' \rightarrow \Delta_{-1} \cup I_0$  given by

$$[a : b : c] \mapsto [\cdots : a + c : a : b : b + c : \cdots]$$

is a homeomorphism. So we may use  $a, b, c$  as local coordinates on  $\Delta_{-1} \cup I_0$ . Using (??), we see that the inverse map  $\Delta_{-1} \cup I_0 \rightarrow \mathbf{H} \cup \mathbf{R}_{<0}$  is given in coordinates by

$$[a : b : c] \mapsto c/b \exp(i\theta), \text{ where } \theta = \arccos\left(\frac{b^2 + c^2 - a^2}{2bc}\right) \in (0, \pi],$$

which is continuous.

Since the inverse is continuous on  $\Delta_0 \cup I_0$  and  $\Delta_{-1} \cup I_0$ , we conclude that it is continuous on  $\Delta_0 \cup \Delta_{-1} \cup I_0$ .  $\square$

Let  $D \subset \mathbf{P}^S$  be the union of the triangles  $\Delta_n$  for  $n \in \mathbf{Z}$  and the intervals  $I_n$  for  $n \in \mathbf{Z}$ .

**Theorem 4.6.** *The mass map gives a homeomorphism  $m : \mathbf{P} \text{Stab}(X) \rightarrow D$ .*

*Proof.* By ?? and ??, we see that  $\mathbf{P}\text{Stab}(X)$  is the union of  $T^n\mathbf{PW}_-$  for  $n \in \mathbf{Z}$  and  $T^n\mathbf{PW}_0$  for  $n \in \mathbf{Z}$ . From ??, it follows that this is a disjoint union. Likewise,  $D$  is the disjoint union of  $\Delta_n$  for  $n \in \mathbf{Z}$  and  $I_n$  for  $n \in \mathbf{Z}$ . Since  $m: \mathbf{PW}_- \rightarrow \Delta_0$  and  $m: \mathbf{PW}_0 \rightarrow I_0$  are bijections, we conclude that  $m: \mathbf{P}\text{Stab}(X) \rightarrow D$  is a bijection. It is also continuous. It remains to prove that the inverse is continuous.

Let  $U = \Delta_0 \cup I_0 \cup \Delta_{-1}$ . Observe that

$$U = \{[a_n] \in D \mid 2a_0 < a_1 + a_{-1}\}.$$

So  $U \subset D$  is open. From ??, we know that the inverse of  $m$  is continuous on  $U$ . But  $T^n U$  for  $n \in \mathbf{Z}$  form an open cover of  $D$ . So the inverse of  $m$  is continuous on  $D$ .  $\square$

**4.3. Identifying the image and its closure.** Let  $\overline{D} \subset \mathbf{P}^S$  be the closure of  $D$ . Our next goal is to identify the homeomorphism classes of  $\overline{D}$  and  $D$ . To do so, it will be useful to work with an auxiliary space, which we now define.

Let  $I \subset \mathbf{P}^1$  be the set of  $[v : w]$  where  $v, w$  are non-negative real numbers. Then  $I$  is homeomorphic to a closed interval. Let  $\overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$  be the two point compactification of  $\mathbf{R}$ , also homeomorphic to a closed interval. Our auxiliary space will be  $\overline{\mathbf{R}} \times I$ .

Define the transformation  $T$  on  $\overline{\mathbf{R}} \times I$  by

$$T: (u, [v : w]) \mapsto (u + 1, [v : w]).$$

Recall that we also denote by  $T$  the action of the spherical twist by  $\mathcal{O}_X$  on  $\mathbf{P}\text{Stab}(X)$  and the rightward shift by 1 on  $\mathbf{P}^S$ . (We intentionally use the same letter  $T$  to denote these maps, which are related.) Our eventual goal is to understand  $\overline{D}$  via a  $T$ -equivariant parametrisation

$$\pi: \overline{\mathbf{R}} \times I \rightarrow \overline{D}.$$

For  $n \in \mathbf{Z}$ , set

$$P_n = (\dots, 2, 1, 0, 1, 2, \dots) \in \mathbf{R}^S,$$

where the 0 is at index  $n$ . Note that  $P_n = T^n P_0$ . Set

$$Q = (\dots, 1, 1, 1, \dots) \in \mathbf{R}^S.$$

Observe that  $P_{-1}, P_0$ , and  $Q$  are the three vertices of the closure  $\overline{\Delta}_0$  of the triangle  $\Delta_0 \subset \mathbf{P}^S$ , which is the homeomorphic image of  $\mathbf{PW}_-$ . The three sides of  $\overline{\Delta}_0$  are the line segments  $P_{-1}P_0$ ,  $P_{-1}Q$ , and  $P_0Q$ . The open line segment  $P_0Q$  is the homeomorphic image of  $\mathbf{PW}_0$ . The entire picture is  $T$ -invariant, so the discussion above holds with  $-1, 0, 1$  replaced by  $n-1, n, n+1$  for any  $n \in \mathbf{Z}$ .

Consider the map  $\pi: [0, 1] \times I \rightarrow \mathbf{P}^S$  defined by

$$\pi(u, [v : w]) = (1 - u)(wQ + vP_0) + u(wQ + vP_1).$$

Note that for  $u = 0$  (resp.  $u = 1$ ), the map  $\pi$  is a homeomorphism onto the closure of  $I_0$  (resp.  $I_1$ ), which are the two sides  $P_0Q$  and  $P_1Q$  of the triangle  $\overline{\Delta}_0$ . For  $0 < u < 1$ , the map  $\pi$  linearly interpolates between the two end-points  $\pi(0, [v : w])$  and  $\pi(1, [v : w])$ , and hence its image is  $\overline{\Delta}_0$ . In fact, it is easy to check that the map

$$\pi: [0, 1] \times (I - \{[0 : 1]\}) \rightarrow \overline{\Delta}_0 - \{[\dots 1 : 1 : 1 : \dots]\}$$

is a homeomorphism, and  $\pi$  sends the entire segment  $[0, 1] \times [0 : 1]$  to the point  $[\dots 1 : 1 : 1 : \dots]$ . Note that, with the  $T$  actions as before, we have

$$T\pi(0, [v : w]) = \pi T(0, [v : w]).$$

Thus,  $\pi$  extends to a unique  $T$ -equivariant continuous map

$$\pi: \mathbf{R} \times I \rightarrow \mathbf{P}^S.$$

Explicitly, for  $x = n + u$ , where  $n \in \mathbf{Z}$  and  $u \in [0, 1)$ , we have

$$\pi(x, [v : w]) = wQ + (1 - u)vP_n + uvP_{n+1}.$$

Note that in  $\mathbf{R}^S$  we have

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} P_n = (\cdots, 1, 1, 1, \cdots).$$

Extend  $\pi$  to  $\{\pm\infty\} \times I$  by setting

$$\pi(\pm\infty, [v : w]) = [\cdots : 1 : 1 : 1 : \cdots].$$

Then, using the limit computation above, it is easy to check that  $\pi_q$  is continuous.

**Theorem 4.7.** *The map  $\pi : \overline{\mathbf{R}} \times I \rightarrow \mathbf{P}^S$  is continuous. It sends the set*

$$C = \{\pm\infty\} \times I \cup \overline{\mathbf{R}} \times \{[0 : 1]\}$$

*to the point  $[\cdots : 1 : 1 : 1 : \cdots]$ . Let  $\overline{\mathbf{R}} \times [0, 1] \rightarrow B$  be the contraction of  $C$  to a point. Then the induced map  $\pi : B \rightarrow \mathbf{P}^S$  is a homeomorphism onto  $\overline{D} = \overline{m(\mathbf{P} \text{Stab}(X))}$ .*

Note that  $B$  is homeomorphic to a closed disk. See ?? for a sketch.

*Proof.* We have seen that  $\pi$  is continuous. It is easy to check that it is injective on the complement of  $C$ , and its image on the complement of  $C$  does not include the point  $[\cdots : 1 : 1 : 1 : \cdots]$ . It evidently sends all points of  $C$  to  $[\cdots : 1 : 1 : 1 : \cdots]$ . So it induces a continuous injective map  $\pi : B \rightarrow \mathbf{P}^S$ . Since  $B$  is compact and  $\mathbf{P}^S$  is Hausdorff,  $\pi$  maps  $B$  homeomorphically onto its image. By construction,  $\pi$  maps the interior of  $\overline{\mathbf{R}} \times I$  to  $D$ . So  $\pi(B)$  must be  $\overline{D}$ .  $\square$

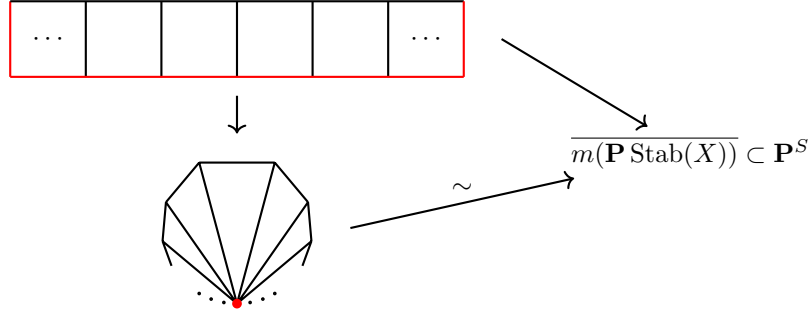


FIGURE 7. The map  $\pi : \overline{\mathbf{R}} \times [0, 1] \rightarrow \mathbf{P}^S$  induces a homeomorphism from a closed disk  $B$  onto the closure of the image of  $\text{Stab}(X)$ . The disk  $B$  is obtained from the square  $\overline{\mathbf{R}} \times [0, 1]$  by collapsing three sides (red). The  $\mathbf{Z}$ -indexed decomposition of the image into triangles corresponds to the translates of a fundamental domain of  $\mathbf{P} \text{Stab}(X)$  by the spherical twist  $T$ .

**4.4. Points of the boundary.** Observe that  $\overline{D}$  contains the point  $\bullet = [\cdots : 1 : 1 : 1 : \cdots]$ . This is the common vertex (drawn in red in ??) of all the triangles that tile  $\overline{D}$ . It is the unique  $T$ -invariant point of  $\overline{D}$ . This point is precisely the projectivised hom function  $\text{hom}(\mathcal{O}_X, -)$ , whose value on  $T^n \mathbf{k}_x$  for any  $n \in \mathbf{Z}$  is

$$\dim \text{Hom}^*(\mathcal{O}_X, T^n \mathbf{k}_x) = 1.$$

The fact that  $\bullet$  is in the boundary is a reflection of the following more general fact.

**Theorem 4.8** ([?, Corollary 4.13]). *Let  $a$  be a spherical object of a triangulated category  $\mathcal{C}$ , and assume that it is a stable object of a stability condition  $\sigma$ . Let  $S$  be a set of objects of  $\mathcal{C}$  such that no object in  $S$  has an endomorphism of negative degree. For simplicity, also assume that no shift of  $a$  is in  $S$ . Let  $T$  be the spherical twist in  $a$ . Then, in  $\mathbf{P}^S$ , we have the equality*

$$\lim_{n \rightarrow \pm\infty} T^n [m_\sigma] = [\text{hom}(a, -)].$$

The point  $\bullet$  also has an interpretation as the mass function of a lax stability condition in the sense of Broomhead, Pauksztello, Ploog, and Woolf [?]. We quickly recall the main features of the definition. A *lax stability condition* is a slicing  $P$  and a compatible central charge  $Z$ . The central charge is allowed to vanish on the classes of non-zero semi-stable objects (such objects are called “massless”). The pair  $(P, Z)$  must satisfy the following two finiteness conditions:

- (1) The slicing  $P$  is locally finite.
- (2) The central charge satisfies the support property. That is, for a choice of a norm  $\| - \|$  on  $\mathcal{N}(X)$ , there exists a positive constant  $c$  such that for every massive stable object  $s$ , we have  $|Z(s)|/\|s\| > c$ .

A pair  $(P, Z)$  that satisfies the first condition is called a *lax pre-stability condition*.

Recall that  $\mathcal{A}$  is the tilt of  $\text{Coh } X$  in the torsion pair defined by torsion and torsion-free sheaves. We let  $P$  to be the slicing defined by  $P(1) = \mathcal{A}$  and  $P(\phi) = 0$  for  $\phi \in (0, 1)$ . The simple objects of  $P(1)$  are the skyscraper sheaves  $\mathbf{k}_x$  and the objects  $E[1]$ , where  $E$  is a vector bundle of rank on  $X$  with no non-trivial sub-bundles (see [?, Remark 4.3 (iii)]). We let  $Z(\mathcal{O}_X) = 0$  and  $Z(\mathbf{k}_x) = -1$ .

**Proposition 4.9.** *The pair  $(P, Z)$  as above defines a lax stability condition  $\sigma$  that is a limit of standard stability conditions. Furthermore,  $m(\sigma) = [\cdots : 1 : 1 : 1 : \cdots]$ .*

*Proof.* It is easy to check that the abelian category  $\mathcal{A}$  is of finite length (Noetherian and Artinian). So the slicing is locally finite. Let  $E$  be a vector bundle with no non-trivial sub-bundles, and let  $[E] = r[\mathcal{O}_X] + m[\mathbf{k}_x]$ . Then  $r = \text{rk } E$  and  $Z(E) = -m$ . Assume that  $E$  is not isomorphic to  $\mathcal{O}_X$ . Then  $\text{Hom}(\mathcal{O}_X, E) = \text{Hom}(E, \mathcal{O}_X) = 0$ . So

$$0 \geq \chi(\mathcal{O}_X, E) = 2r + m,$$

and hence  $m \leq -2r$ . As a result, with the norm on  $\mathcal{N}(X)$  in which  $[\mathcal{O}_X]$  and  $[\mathbf{k}_x]$  form an orthonormal basis, we see that

$$|Z(E)|/\|E\| \geq \frac{|m|}{\sqrt{r^2 + m^2}} \geq \frac{2}{\sqrt{5}}.$$

So the support property holds.

Finally, note that  $\sigma$  is the limit of the stability conditions in  $\mathbf{PW}_0$  as  $Z(\mathcal{O}_X)/Z(\mathbf{k}_x)$  approaches 0. Since  $m_\sigma(T^n \mathbf{k}_x) = 1$ , the last equality follows.  $\square$

Consider the points  $P_n$  of  $\overline{D}$ . These are the vertices of the tiling triangles other than the vertex  $\bullet$ . They form a single  $T$ -orbit, so it suffices to focus on one of them, say  $P_0 = [\cdots : 2 : 1 : 0 : 1 : 2 : \cdots]$ , with the 0 at index 0. Note that this is the common vertex, other than  $\bullet$ , of the triangles  $\mathbf{PW}_+ \cong \Delta_{-1}$  and  $\mathbf{PW}_- = \Delta_0$ . This is the mass function of a lax pre-stability condition, which does not satisfy the support property. Let  $P$  be the same slicing as before, and set  $Z(\mathcal{O}_X) = 1$  and  $Z(\mathbf{k}_x) = 0$ .

**Proposition 4.10.** *The pair  $(P, Z)$  as above defines a lax pre-stability condition  $\tau$  that is a limit of standard stability conditions and  $m(\tau) = [\cdots : 2 : 1 : 0 : 1 : 2 : \cdots]$ . However,  $\tau$  does not satisfy the support property.*

*Proof.* Since  $\mathcal{A}$  is of finite length, the slicing is locally finite. So  $\tau$  is a lax pre-stability condition. It is easy to check that  $m_\tau(T^n \mathbf{k}_x) = |n|$ . Note that  $\tau$  is the limit of stability conditions in  $\mathbf{PW}_0$  as  $Z(\mathcal{O}_X)/Z(\mathbf{k}_x)$  approaches  $-\infty$ .

To see that the support property fails for  $\tau$ , recall that the simple objects of  $\mathcal{A}$  are the skyscraper sheaves  $\mathbf{k}_x$  and shifts by 1 of vector bundles with non-trivial sub-bundles. The skyscraper sheaves are massless, and hence do not obstruct the support property. On the other hand, for a fixed integer  $r \geq 2$  and sufficiently large  $c_2$  (depending on  $r$ ), there exist vector bundles  $E$  of rank  $r$  on  $X$  with no nontrivial sub-bundles and  $c_2(E) = c_2$  (see [?, Th  o  me 5.3]). For such a vector bundle  $E$ , we have  $Z_\tau(E) = r$  but the norm of  $[E] = (r, r - c_2)$  may be arbitrarily large. So  $|Z(E)|/\|E\|$  is not bounded below by any positive constant.  $\square$

Finally, consider a point on the open line segment joining  $v_0$  and  $v_1$ . This point is in the closure of  $\mathbf{PW}_- = \Delta_0$ . We claim that it is *not* the mass function of a lax pre-stability condition arising as a limit of stability conditions  $W_-$ .

To see this, it is helpful to consider a handful of other semi-stable objects. Let  $n \geq m$  be positive integers. Let  $x_1, \dots, x_n \in X$  be distinct points, and set  $S = \{x_1, \dots, x_n\}$ . We say that a morphism  $\pi: \mathcal{O}_X^{\oplus m} \rightarrow \mathcal{O}_S$  is *generic* if for every subset  $T \subset S$ , the induced map on global sections

$$H^0(\mathcal{O}_X^{\oplus m}) \rightarrow H^0(\mathcal{O}_T)$$

has maximal rank, namely  $\min(m, |T|)$ .

For some  $w \in -\mathbf{H}$ , let  $\sigma = \sigma_w$  be the corresponding standard stability condition. Let  $I_{m,n}$  be the kernel of a generic morphism from  $\mathcal{O}_X^{\oplus m}$  to the structure sheaf of  $n$ -points. Then it is easy to check that  $I_{m,n}$  is  $\sigma$ -stable.

Fix a point  $p \in \overline{D}$  on the line segment joining  $v_0$  and  $v_1$ . Then, for some  $t > 0$ , we can write

$$p = [\cdots : 2 + t : 1 : t : 1 + 2t : \cdots].$$

If we take a sequence of standard stability conditions in  $W_-$  whose mass function approaches  $p$ , their slicings do not converge. Therefore, there is no limiting lax pre-stability condition with the mass function  $p$ . We now make this precise.

Recall that the topology on the space of slicings is induced by the metric  $d$  defined as follows. For a slicing  $P$  and non-zero object  $c$ , let  $\phi_P^\pm(c)$  denote the highest/lowest phase of the  $P$ -HN factors of  $c$ . Then the distance  $d(P, Q)$  between two slicings  $P$  and  $Q$  is

$$d(P, Q) = \sup_{c \neq 0} \left\{ \max(|\phi_P^+(c) - \phi_Q^+(c)|, |\phi_P^-(c) - \phi_Q^-(c)|) \right\}.$$

Suppose  $\sigma$  is a lax stability condition that is a limit of a sequence of standard stability conditions  $\sigma_w$  for  $w \in -\mathbf{H}$  with  $m(\sigma) = p$ . Then, possibly after a rotation and scaling, the central charge of  $\sigma$  must send  $\mathbf{k}_x$  to  $-1$  and  $\mathcal{O}_X$  to  $-1 - t$ . But then

$$Z(I_{m,n}) = mZ(\mathcal{O}_X) - nZ(\mathbf{k}_x) = n - m(1 + t).$$

It follows that for every  $(n, m)$  with  $n/m > (1 + t)$ , the sheaf  $I_{m,n}$  is  $\sigma$ -semi-stable of phase 0 and for  $n/m < (1 + t)$ , it is  $\sigma$ -semi-stable of phase 1. But this is absurd. Indeed, for a standard stability condition  $\sigma_w$ , we have

$$\inf_{n/m > 1+t} \phi_{\sigma}(I_{n,m}) = \sup_{n/m < 1+t} \phi_{\sigma}(I_{n,m}),$$

so the same equality must hold in the limit.

In summary, we see three distinct kinds of limit points in the boundary from the point of view of lax stability conditions:

- (1) The object  $\mathcal{O}_X$  can become massless in a lax stability condition, leading to the limit mass function  $Q$ .
- (2) The objects  $\mathbf{k}_x$  and  $I_x = T\mathbf{k}_x[-1]$  can become massless in a lax pre-stability condition, leading to the limit mass functions  $P_0$  and  $P_1$ .
- (3) Other semi-stable sheaves (for example,  $I_{m,n}$ ) cannot become massless in lax pre-stability conditions.

This trichotomy is consistent with the density of the phase diagram (see the discussion in [?, § 12]). Let  $\sigma \in W_-$  be a standard stability condition. It is easy to check that the classes  $r[\mathcal{O}_X] + n[\mathbf{k}_x]$  that support semi-stable sheaves are precisely  $r = 0$  and  $n \geq 1$ ; or  $r \geq 1$  and  $n = 0$ ; or  $r \geq 1$  and  $-n \geq r$  (see ??). Consider the phase diagram—the possible phases of semi-stable objects plotted on the unit circle. There,  $\mathcal{O}_X$  is an isolated point,  $\mathbf{k}_x$  is a right accumulation point, and  $I_x$  is a left accumulation point. At all points on the arc from  $\mathbf{k}_x[-1]$  to  $I_x$  (and its negative), the phase diagram is dense in the circle. As  $\mathcal{O}_X$  becomes massless, the stability conditions converge preserving the support property. As  $\mathbf{k}_x$  or  $I_x$  become massless, the slicings converge, but the support property is lost. But if the central charge vanishes on a point on the open arc from  $\mathbf{k}_x[-1]$  to  $I_x$ , even the slicings do not converge.

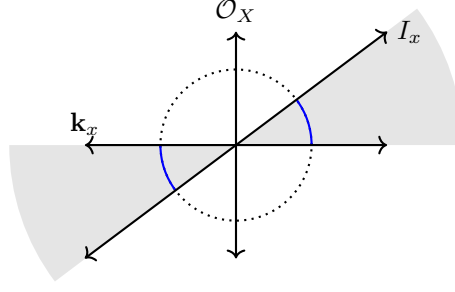


FIGURE 8. The central charges of semi-stable objects in a standard stability condition with heart  $\text{Coh } X$  are the lattice points in the shaded region. As a result, the phases are dense in the blue region of the unit circle.

### 5. THE $q$ -MASS EMBEDDING

Fix a positive real number  $q$ . Given a stability condition  $\sigma$  and an object  $x$ , recall that the  $q$ -mass of  $x$  with respect to  $\sigma$  is defined by

$$m_{q,\sigma}(x) = \sum |Z_\sigma(x_i)| q^{\phi(x_i)},$$

where the sum is taken over the  $\sigma$ -HN factors  $x_i$  of  $x$ , and  $\phi(x_i)$  is the phase of  $x_i$ . We have the map

$$m_q: \mathbf{P} \text{Stab}(X) \rightarrow \mathbf{P}^S$$

given by  $\sigma \mapsto m_{q,\sigma}$ . We describe the image of  $m_q$  and its closure for  $q \neq 1$ . Most of the arguments are direct analogues of the arguments for  $q = 1$ , so we will be brief.

Let  $\sigma \in \mathbf{PW}_-$ . Set  $a = m_{q,\sigma}(\mathbf{k}_x)$  and  $b = m_{q,\sigma}(T\mathbf{k}_x)$  and  $c = m_{q,\sigma}(\mathcal{O}_X)$ . Owing to the triangle

$$\mathcal{O}_X \rightarrow \mathbf{k}_x \rightarrow T\mathbf{k}_x \xrightarrow{+1},$$

the positive real numbers  $a, b, c$  satisfy the  $q$ -triangle inequalities

$$(6) \quad b < a + qc, \quad a < b + c, \quad c < a + q^{-1}b.$$

(See [?, Proposition 3.3] for a proof of the  $q$ -triangle inequalities). From the  $\sigma$ -HN filtration of  $T^n \mathbf{k}_x$  from ??, we get

$$m_{q,\sigma}: T^n \mathbf{k}_x \mapsto \begin{cases} a + cq^{-n} + \cdots + cq^2 & \text{for } n \leq -2, \\ a & \text{for } n = 0, \\ b & \text{for } n = 1, \\ b + cq^0 + \cdots + cq^{-n+2} & \text{for } n \geq 2. \end{cases}$$

So, in homogeneous coordinates, the  $q$ -mass map is

$$m_q: \sigma \mapsto [\cdots : a + cq + cq^2 : a + cq : a : b : b + c : b + c + cq^{-1} : \cdots]$$

Let  $\Delta \subset \mathbf{P}^2$  be the set consisting of  $[a : b : c]$  where  $a, b, c$  are positive real numbers satisfying (?). Then the map  $\mathbf{PW}^- \rightarrow \Delta$  that takes  $\sigma$  to  $[m_{q,\sigma}(\mathbf{k}_x) : m_{q,\sigma}(T\mathbf{k}_x) : m_{q,\sigma}(\mathcal{O}_X)]$  is a homeomorphism. The proof is analogous to the proof of ?? (3), but uses the  $q$ -analogue of the cosine rule [?, Lemma 5.2].

Consider  $\sigma \in \mathbf{PW}_0$ . With  $a, b, c$  as before, we have  $b = a + qc$ . From the  $\sigma$ -HN filtration of  $T^n \mathbf{k}_x$  from ??, we get

$$m_{q,\sigma}: T^n \mathbf{k}_x \mapsto \begin{cases} a + cq^{-n} + \cdots + cq^2 & \text{for } n \leq -2, \\ a & \text{for } n = 0, \\ a + cq + \cdots + cq^{-n+2} & \text{for } n \geq 1. \end{cases}$$

So, in homogeneous coordinates, the  $q$ -mass map is

$$\sigma \mapsto [\cdots : a + cq + cq^2 : a + cq : a : a + cq : a + cq + c : \cdots].$$

Set  $I_0 = m_q(\mathbf{PW}_0)$  and  $I_n = T^n I_0$ . Then  $m_q: T^n \mathbf{PW}_0 \rightarrow I_n$  is a homeomorphism.

Let  $D_q \in \mathbf{P}^S$  be the union of  $\Delta_n$  and  $I_n$  for  $n \in \mathbf{Z}$ .

**Theorem 5.1.** *The  $q$ -mass map*

$$m_q: \mathbf{P} \operatorname{Stab}(X) \rightarrow D_q$$

*is a homeomorphism.*

The proof is analogous to the proof of ??.

We now identify the homeomorphism type of  $D_q$  and its closure  $\overline{D}_q$ . The basic technique is as before—by parametrising  $\overline{D}_q$  by a compactified infinite strip of squares. But the resulting picture is slightly different. Without loss of generality, assume  $q > 1$ .

Recall that  $I \subset \mathbf{P}^1$  is the set of  $[v : w]$  where  $v, w$  are non-negative real numbers. Let  $\overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$  be the two point compactification of  $\mathbf{R}$ . Define the transformation  $T$  on  $\overline{\mathbf{R}} \times I$  by

$$T: (u, [v : w]) \mapsto (u + 1, [qv : w]).$$

Recall that we also denote by  $T$  the action of the spherical twist by  $\mathcal{O}_X$  on  $\mathbf{P} \operatorname{Stab}(X)$  and the rightward shift by 1 on  $\mathbf{P}^S$ .

We define a  $T$ -equivariant parametrisation

$$\pi_q: \overline{\mathbf{R}} \times I \rightarrow \overline{D}_q,$$

which is a  $q$ -analogue of the parametrisation  $\pi$  from ??. For  $n \in \mathbf{Z}$ , set

$$P_n = (\dots, 1 + q, 1, 0, 1, 1 + q^{-1}, \dots) \in \mathbf{R}^S,$$

where the 0 is at index  $n$ . Note that  $P_n = T^n P_0$ . Set

$$Q = (\dots, 1, 1, 1, \dots) \in \mathbf{R}^S.$$

Consider the map  $\pi_q: [0, 1] \times I \rightarrow \mathbf{P}^S$  defined by

$$\pi_q(u, [v : w]) = (1 - u)(wQ + vP_0) + u(wQ + q^{-1}vP_1).$$

Note that for  $u = 0$  (resp.  $u = 1$ ), the map  $\pi_q$  is a homeomorphism onto the closure of  $I_0$  (resp.  $I_1$ ), which are the two sides of the triangle  $\Delta_0$ . For  $0 < u < 1$ , the map  $\pi_q$  linearly interpolates between the two end-points  $\pi_q(0, [v : w])$  and  $\pi_q(1, [v : w])$ , and hence its image is  $\overline{\Delta}_0$ . Also observe that  $\pi_q(u, [0 : w]) = Q$ . Furthermore, with the  $T$  actions as before, we have

$$T\pi_q(0, [v : w]) = \pi_q T(0, [v : w]).$$

Thus,  $\pi_q$  extends to a unique  $T$ -equivariant continuous map

$$\pi_q: \overline{\mathbf{R}} \times I \rightarrow \mathbf{P}^S.$$

Explicitly, for  $x = n + u$ , where  $n \in \mathbf{Z}$  and  $u \in [0, 1)$ , we have

$$\pi_q(x, [v : w]) = wQ + (1 - u)q^{-n}vP_n + uq^{-n-1}vP_{n+1}.$$

Let  $\delta = 1 + q^{-1} + q^{-2} + \dots$ . Then, in  $\mathbf{R}^S$  we have

$$\lim_{n \rightarrow -\infty} P_n = (\dots, \delta, \delta, \delta, \dots) \text{ and } \lim_{n \rightarrow \infty} q^{-n}P_n = (\dots, q\delta, \delta, q^{-1}\delta, \dots).$$

(On the right hand side of the last equation, the  $\delta$  is at index  $-1$ .) Extend  $\pi_q$  to  $\{\pm\infty\} \times I$  by setting

$$\pi_q(-\infty, [v : w]) = [\dots : 1 : 1 : 1 : \dots],$$

and

$$\pi_q(+\infty, [v : w]) = w[\dots : 1 : 1 : 1 : \dots] + v[\dots : q : 1 : q^{-1} : \dots].$$

Using the limit computation above, it follows that this extension is continuous.

**Theorem 5.2.** *The map  $\pi_q: \overline{\mathbf{R}} \times [0, 1] \rightarrow \mathbf{P}^S$  is continuous. It sends the set*

$$C = \{-\infty\} \times [0, 1] \cup \overline{\mathbf{R}} \times \{0\}$$

*to the point  $[\dots : 1 : 1 : 1 : \dots]$ . Let  $\overline{\mathbf{R}} \times [0, 1] \rightarrow B$  be the contraction of  $C$  to a point. Then the induced map  $\pi_q: B \rightarrow \mathbf{P}^S$  is a homeomorphism onto  $\overline{D}_q = \overline{m_q(\mathbf{P} \operatorname{Stab}(X))}$ .*

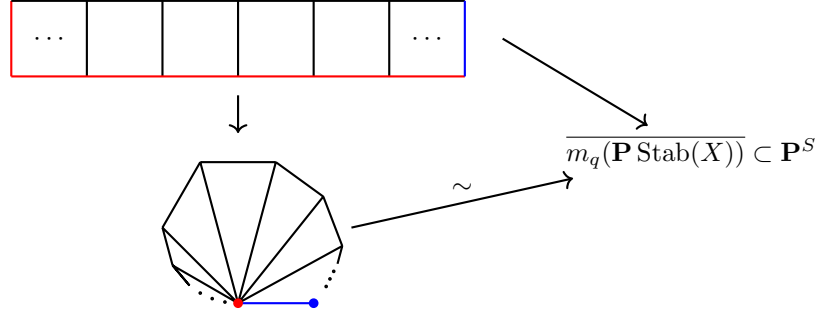


FIGURE 9. The map  $\pi_q: \overline{\mathbf{R}} \times [0, 1] \rightarrow \mathbf{P}^S$  induces a homeomorphism from a closed disk  $B$  onto the closure of the image of  $\text{Stab}(X)$  under the  $q$ -mass map. The disk  $B$  is obtained from the square  $\overline{\mathbf{R}} \times [0, 1]$  by collapsing two sides (red).

The proof is analogous to that of ?? . See ?? for a sketch.

Instead of a unique  $T$ -fixed point of  $\overline{D}_q$ , as was the case for  $q = 1$ , for  $q \neq 1$  we have two such points. These are the blue and red end-points of the blue interval in ?? . The blue end-point is the point  $\bullet = [\cdots : q : 1 : q^{-1} : \cdots :]$ . It is the  $q$ -hom function  $\text{hom}_q(\mathcal{O}_X, -)$ , whose value on  $T^n \mathbf{k}_x$  is

$$\dim_q \text{Hom}^*(\mathcal{O}_X, T^n \mathbf{k}_x) = q^{-n}.$$

(By definition,  $\dim_q$  of the graded vector space  $\mathbf{C}[m]$  is  $q^m$ ). Again, the fact that  $\text{hom}_q(\mathcal{O}_X, -)$  appears in the closure of the  $q$ -mass embedding of the stability manifold is a reflection of a general theorem—the  $q$ -analogue of ?? (see [?, Corollary 4.13]).

Note that  $\bullet$  is not in the closure of the standard stability conditions  $\mathbf{PW}$ , nor is it in the closure of  $T^n \mathbf{PW}$  for any fixed  $n$ . To reach  $\bullet$ , we must traverse an infinite sequence of hearts. It is easy to see that it is not the  $q$ -mass function of a lax stability condition.

The red end-point is the point  $\bullet = [\cdots : 1 : 1 : 1 : \cdots]$ . It is the  $q$ -mass function of the lax stability condition  $\sigma$  from ?? .

The other vertices of the triangles form one orbit, and are  $q$ -mass functions of lax pre-stability conditions. For example, the vertex  $v_0 = [\cdots : 1 + q : 1 : 0 : 1 : 1 + q^{-1} : \cdots]$  is the  $q$ -mass function of the lax pre-stability condition  $q^{-1} \cdot \tau$  where  $\tau$  is as in ?? .