THE THURSTON COMPACTIFICATION OF THE STABILITY MANIFOLD OF A GENERIC ANALYTIC K3 SURFACE

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Abstract. Let X be an analytic K3 surface with $\operatorname{Pic} X = 0$. We describe the closure of the Bridgeland stability manifold of X obtained using the masses of semi-rigid objects.

1. Introduction

Associated to a triangulated category $\mathcal{C}$ is the complex manifold $\operatorname{Stab}(\mathcal{C})$ whose points are the Bridgeland stability conditions on $\mathcal{C}$ [?]. Understanding the global geometry of $\operatorname{Stab}(\mathcal{C})$ is an important question with far-reaching applications. For example, when $\mathcal{C}$ is the derived category of coherent sheaves on a K3 surface, the simple connectedness of $\operatorname{Stab}(\mathcal{C})$ allows us to recover the group of auto-equivalences of $\mathcal{C}$ [?]. When $\mathcal{C}$ is the 2-Calabi–Yau category associated to a quiver, the topology of $\operatorname{Stab}(\mathcal{C})$ has implications for the word/conjugacy problems and the $K(\pi, 1)$-conjecture for the associated Artin group [?, ?].

To better understand the global geometry of a non-compact space like $\operatorname{Stab}(\mathcal{C})$, it is useful to have a compactification. There have been several (partial) compactifications in the literature; see, for example, [?, ?, ?, ?]. The goal of this paper is to completely describe the compactification constructed in [?] when $\mathcal{C}$ is the derived category of coherent sheaves on a generic analytic K3 surface.

The compactification in [?] is motivated by viewing a stability condition as a metric, and in particular by Thurston's compactification of the Techimüller space of hyperbolic metrics on a surface. We recall the main idea. Given a stability condition σ on $\mathcal{C}$ and an object $x \in \mathcal{C}$, the *mass* of x with respect to σ, denoted by $m_\sigma(x)$, is the sum $m_\sigma(x) = \sum_i |Z_\sigma(x_i)|$, where the x_i are the σ-Harder–Narasimhan (HN) factors of x and Z_σ is the central charge of σ. To construct the compactification, we fix a set of objects S, and consider the map $m\colon \mathbf{P}\operatorname{Stab}(\mathcal{C}) = \operatorname{Stab}(\mathcal{C})/\mathbf{C} \to \mathbf{P}^S$ given by $\sigma \mapsto [m_\sigma]$. The proposed compactification is the closure of the image of m.

Theorem 1.1. *Let X be an analytic K3 surface with $\operatorname{Pic}(X) = 0$. Let $S \subset D^b\operatorname{Coh}(X)$ be the set of semi-rigid objects. The map $m\colon \mathbf{P}\operatorname{Stab}(D^b\operatorname{Coh}(X)) \to \mathbf{P}^S$ is a homeomorphism onto its image. The image is a 2-dimensional open ball and its closure is a 2-dimensional closed ball.*

See Figure 1 for an illustration of the compactified stability space. The boundary contains a distinguished point represented by the function $\hom(\mathcal{O}_X, -)$ (red point in Figure 1). This point and the other vertices in Figure 1 are mass functions of lax stability conditions in the sense of [?], but the other boundary points are not.

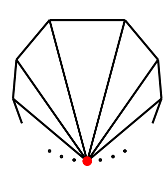

Figure 1. For an analytic K3 surface X with $\operatorname{Pic}(X) = 0$, the compactified $\mathbf{P}\operatorname{Stab}(X)$ is a closed 2-ball, tiled by the translates of a triangle by the action of the spherical twist in $\mathcal{O}_X$. A distinguished point (red) in the boundary corresponds to the function $\hom(\mathcal{O}_X, -)$.

Theorem 1.1 is a combination of Theorem 4.6 and Theorem 4.8 in the main text. The discussion of the points in the boundary is in Section 4.4.

For a positive real number q, the mass map has a natural q-analogue m_q. The closure of the image of the stability manifold under m_q is also a closed disk. However, in its boundary, the red point in Figure 1 is replaced by a closed interval (see Figure 2).

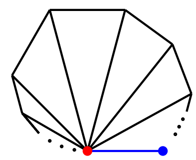

FIGURE 2. The closure of $m_q(\mathbf{P}\,\mathrm{Stab}(X))$ is also a closed disk. The boundary has an additional interval, whose blue end-point is the q-hom functional $\hom_q(\mathcal{O}_X, -)$.

For $q = 1$, the distinguished red point in the boundary has two interpretations: one as the hom function $\hom(\mathcal{O}_X, -)$ and the second as the mass function of a lax stability condition σ in which $\mathcal{O}_X$ is massless. For $q \neq 1$, the two interpretations diverge. The q-hom function $\hom_q(\mathcal{O}_X, -)$ yields the blue end-point in Figure 2 and the q-mass function $m_q(\sigma)$ yields the red end-point.

We can reconcile the two pictures (Figure 1 and Figure 2) by drawing them in the upper half plane instead of the disk (see Figure 3). The $q = 1$ picture (Figure 1) corresponds to the union of the translates of an ideal triangle by the transformation $z \mapsto z + 1$. The only additional point in the closure (in the closed disk) is the point at infinity. The $q \neq 1$ picture (Figure 2) corresponds to the union of the translates of an ideal triangle by the transformation $z \mapsto qz + 1$. In this case, the closure (in the closed disk) contains an additional interval. This q-deformation is a simpler version of the q-deformed Farey tesselation observed in [**?**].

FIGURE 3. The tiling of the disk by triangles in the $q = 1$ case (left) versus the $q \neq 1$ case (right).

In the course of the proof of the main theorem, we also characterise all semi-rigid objects of $D^b\,\mathrm{Coh}(X)$. Up to twists by $\mathcal{O}_X$ and homological shifts, the only such objects are the skyscraper sheaves $\mathbf{k}_x$ (Proposition 3.1).

There are a few other cases where the Thurston compactification of the stability manifold has been completely described. These include: the 2-Calabi–Yau categories associated to quivers of rank 2 [**?**] and the derived categories of coherent sheaves on algebraic curves [**?**]. In [**?**] the authors prove that for any (algebraic) K3 surface X, taking S to be the set of spherical objects gives an injective map $m\colon \mathbf{P}\,\mathrm{Stab}(X) \to \mathbf{P}^S$. Understanding its image and its closure is an important goal. The case of non-algebraic K3s treated here is a step towards it.

1.1. **Conventions.** An *analytic K3 surface* is a connected, simply-connected, and compact complex manifold X of dimension 2 with $h^1(\mathcal{O}_X) = 0$. By $D^b(X)$ we mean the bounded derived category of the abelian category $\mathrm{Coh}(X)$ of coherent sheaves on X, as studied in [**?**]. For a point $x \in X$, we denote by $\mathbf{k}_x$ the push-forward to X of the structure sheaf of x, and call it the *skyscraper sheaf* at x. By $\mathrm{Stab}(X)$, we denote the set of (locally finite) Bridgeland stability conditions on $D^b(X)$ with a numerical central charge; that is, where the central charge $Z\colon K(D^b(X)) \to \mathbf{C}$ factors through the Chern character $\mathrm{ch}\colon K(D^b(X)) \to H^*(X, \mathbf{Q})$. We let $\mathbf{P}\,\mathrm{Stab}(X)$ be the quotient of $\mathrm{Stab}(X)$ by the standard action of $\mathbf{C}$, in which $z = x + i\pi y$ acts by scaling the central charge by e^z and shifting the slicing by y. Given a set S, we let $\mathbf{R}^S$ be the set of functions $S \to \mathbf{R}$ and $\mathbf{P}^S$ the projective space $\left(\mathbf{R}^S - \{0\}\right)$ /scaling.

1.2. **Outline.** In Section 2, we recall the description of stability conditions on an analytic K3 surface X with $\operatorname{Pic} X = 0$. In Section 3, we identify the semi-rigid objects of $D^b(X)$. The bulk of the paper is Section 4, in which we study the embedding of $\mathbf{P}\operatorname{Stab}(X)$ given by the masses of semi-rigid objects. In Section 5, we study the q-analogue of the mass embedding. We do not include the definitions and the basic properties of stability conditions, and refer the reader to the original source [**?**] or exposition [**?**].

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2. Stability conditions on generic K3 surfaces

Throughout, fix an analytic K3 surface X with $\operatorname{Pic} X = 0$. Since X is a K3 surface, $D^b(X)$ is a 2-Calabi–Yau category. That is, for $x, y \in D^b(X)$, we have a natural isomorphism

$$\operatorname{Hom}(x, y) \cong \operatorname{Hom}(y, x[2]).$$

2.1. **The Mukai lattice.** The Mukai lattice $\mathcal{N}(X)$ of X is given by

$$\mathcal{N}(X) = (H^0 \oplus H^4)(X, \mathbf{Z}).$$

Taking the class of X as a generator of the H^0 summand and the class of a point $x \in X$ as a generator of the H^4 summand, we get an identification

$$\mathcal{N}(X) = \mathbf{Z} \oplus \mathbf{Z}.$$

The Mukai pairing is then given by

$$(\alpha_1, \alpha_2) \cdot (\beta_1, \beta_2) = \alpha_1\beta_2 + \alpha_2\beta_1.$$

Given $F \in D^b(X)$, we let $[F] = (\operatorname{ch}_0 F, \operatorname{ch}_0 F - \operatorname{ch}_2 F) \in \mathcal{N}(X)$ be its Mukai vector. Then we have

$$[\mathcal{O}_X] = (1, 1) \text{ and } [\mathbf{k}_x] = (0, 1).$$

In particular, $[\mathcal{O}_X]$ and $[\mathbf{k}_x]$ form a basis of $\mathcal{N}(X)$.

2.2. **Standard stability conditions.** We recall basic facts about stability conditions on X from [**?**, § 4]. Let $\mathcal{F}$ and $\mathcal{T}$ be the full-subcategories of $\operatorname{Coh}(X)$ consisting of torsion free and torsion sheaves, respectively. Then $(\mathcal{F}, \mathcal{T})$ forms a torsion pair. Let $\mathcal{A}$ be the tilt of $\operatorname{Coh}(X)$ in this torsion pair. Explicitly,

$$\mathcal{A} = \{E \in D^b(X) \mid H^{-1}(E) \in \mathcal{F} \text{ and } H^0(E) \in \mathcal{T} \text{ and for all } i \notin \{0, 1\} : H^i(E) = 0\}.$$

Then $\mathcal{A}$ is the heart of a bounded t-structure on $D^b(X)$.

Let $\mathbf{H} \subset \mathbf{C}$ be the (open) upper half plane. As proved in [**?**, § 4.2], for every $z \in \mathbf{H} \cup \mathbf{R}_{<0}$, we have a stability condition σ_z on $D^b(X)$ whose $(0, 1]$ heart is $\mathcal{A}$ and whose central charge is given by

$$Z \colon [\mathbf{k}_x] \mapsto 1 \text{ and } Z \colon [\mathcal{O}_X] \mapsto -z.$$

For every $w \in -\mathbf{H}$, we have a stability condition σ_w on $D^b(X)$ whose $(0, 1]$ heart is $\operatorname{Coh}(X)$ and whose central charge is given by

$$Z \colon [\mathbf{k}_x] \mapsto 1 \text{ and } Z \colon [\mathcal{O}_X] \mapsto -w.$$

See Figure 4 for a sketch of the two central charges.

Remark 2.1. The combined domain of the parameters z and w in [**?**, § 4.2] is $\mathbf{C} - \mathbf{R}_{\geq -1}$. For us, it is $\mathbf{C} - \mathbf{R}_{\geq 0}$. The difference is due to a slight change in parametrisation. The central charge of σ_z in [**?**, § 4.2] sends $\mathbf{k}_x$ to -1 (same as ours) and $\mathcal{O}_X$ to $-z-1$ (we send it to $-z$). So our parametrisation and the parametrisation in [**?**, § 4.2] are related by $z \mapsto z + 1$.

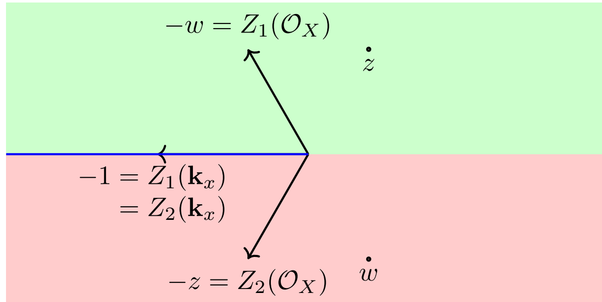

FIGURE 4. For $w \in -\mathbf{H}$ (red), a central charge Z_1 as above defines a stability condition with heart $\mathrm{Coh}(X)$. For $z \in \mathbf{H}$ (green) and $z \in \mathbf{R}_{<0}$ (blue), a central charge Z_2 as above defines a stability condition whose heart is the tilt of $\mathrm{Coh}(X)$ with respect to torsion and torsion-free sheaves.

We call the stability conditions σ_z for $z \in \mathbf{H} \cup -\mathbf{H} \cup \mathbf{R}_{<0}$ the *standard stability conditions*. We say that the stability conditions σ_z for $z \in \mathbf{R}_{<0}$ are *on the wall*, and the rest are *off the wall*.

Let W_+ (resp. W_- and W_0) be the union of the $\mathbf{C}$-orbits of the stability conditions σ_z for $z \in \mathbf{H}$ (resp. $-\mathbf{H}$ and $\mathbf{R}_{<0}$). By definition, the sets W_+, W_-, and W_0 are invariant under the $\mathbf{C}$-action. It is easy to check that they are also invariant under the $\widehat{\mathrm{GL}}_2^+(\mathbf{R})$-action, and hence coincide with the sets with the same name defined in the proof of [**?**, Theorem 4.8]. Set $W = W_+ \cup W_- \cup W_0$.

2.3. **All stability conditions.** Recall that the only spherical objects in $D^b(X)$ are the shifts of $\mathcal{O}_X$ (see [**?**, Proposition 2.15]). Let $T\colon D^b(X) \to D^b(X)$ be the spherical twist in $\mathcal{O}_X$.

Proposition 2.2. *The set $W \subset \mathrm{Stab}(X)$ is open and the union of its translates T^nW, for $n \in \mathbf{Z}$, is* $\mathrm{Stab}(X)$.

Proof. That W is open is proved in [**?**, Theorem 4.8]. That $\mathrm{Stab}(X) = \bigcup T^nW$ is [**?**, Corollary 4.7]. □

The following proposition allows us to identify the stability conditions in W_+, W_-, and W_0. Recall that since, up to shifts, $\mathcal{O}_X$ is the only spherical object, it must be stable in any stability condition [**?**, Proposition 2.15].

Proposition 2.3. *Let σ be a stability condition and let ϕ be the phase of $\mathcal{O}_X$. Then σ is in W if and only if all the skyscraper sheaves $\mathbf{k}_x$ are σ-stable of the same phase ψ. In this case, we have*

(1) *$\sigma \in W_-$ if $\psi \in (\phi, \phi + 1)$,*
(2) *$\sigma \in W_+$ if $\psi \in (\phi + 1, \phi + 2)$,*
(3) *$\sigma \in W_0$ if $\psi = \phi + 1$.*

Proof. Since all skyscraper sheaves $\mathbf{k}_x$ are σ-stable of the same phase for a standard stability condition, the same is true for any $\sigma \in W$. Conversely, suppose all $\mathbf{k}_x$ are σ-stable of the same phase. Using the $\mathbf{C}$-action, assume that their phase is 1 and their central charge is -1. By [**?**, Proposition 4.6], we conclude that σ is standard.

Suppose $\sigma = \sigma_z$ for $z \in -\mathbf{H} \cup \mathbf{H} \cup \mathbf{R}_{<0}$. Whether $z \in -\mathbf{H}$ or $\mathbf{H}$ or $\mathbf{R}_{<0}$ is distinguished by the phase ϕ of $\mathcal{O}_X$. For $z \in -\mathbf{H}$, we have $\phi \in (0, 1)$; for $z \in \mathbf{H}$, we have $\phi \in (-1, 0)$; and for $z \in \mathbf{R}_{<0}$, we have $\phi = 0$. □

Proposition 2.4. *We have $TW_+ = W_-$ and $T^{-1}W_- = W_+$.*

Proof. We prove that for a standard $\sigma \in W_-$, we have $T(\sigma) \in W_+$, and for a standard $\sigma \in W_+$, we have $T^{-1}(\sigma) \in W_-$. Then the proposition follows.

Take a standard $\sigma \in W_-$ and let us prove that $T(\sigma) \in W_+$. Let $\phi \in (0, 1)$ be the phase of $\mathcal{O}_X$. It is easy to check that the ideal sheaves I_x of points $x \in X$ are stable of the same phase $\psi \in (0, \phi)$. Let

$x \in X$ be any point. Since $\mathrm{Hom}^*(\mathcal{O}_X, \mathbf{k}_x) = \mathbf{C}$, we have the exact triangle

$$\mathcal{O}_X \xrightarrow{\mathrm{ev}} \mathbf{k}_x \to T\mathbf{k}_x \xrightarrow{+1} .$$

Therefore, $T\mathbf{k}_x = I_x[1]$. So $T\mathbf{k}_x$ is σ-stable of phase $\psi + 1$. Therefore, $T^{-1}I_x[1] = \mathbf{k}_x$ is $T(\sigma)$-stable of phase $\psi + 1 \in (1, \phi + 1)$. On the other hand, $T^{-1}\mathcal{O}_X = \mathcal{O}_X[1]$ is $T(\sigma)$-stable of phase ϕ, so $\mathcal{O}_X$ is $T(\sigma)$-stable of phase $\phi - 1$. We now apply Proposition 2.3.

Now take a standard $\sigma \in W_+$ and let us prove that $T(\sigma) \in W_-$. Let $\phi \in (-1, 0)$ be the phase of $\mathcal{O}_X$. The objects $T^{-1}\mathbf{k}_x$ are σ-stable of phase $\psi \in (\phi + 1, 1)$ (see [**?**, Remark 4.3 (i)]). Therefore, the skyscraper sheaves $\mathbf{k}_x$ are $T(\sigma)$-stable of phase $\psi \in (\phi + 1, 1)$. Since $\mathcal{O}_X$ is σ-stable of phase ϕ, it is $T(\sigma)$-stable of phase $\phi + 1$. We again apply Proposition 2.3. □

We now turn to the topology of the set of standard stability conditions and the stability conditios in W. Let $H \subset \mathrm{Stab}(X)$ be the set of standard stability conditions. Let $R = \mathbf{C} \setminus \mathbf{R}_{\geq -1}$. We have a map $R \to H$ given by $z \mapsto \sigma_z$. We also have the projection map $H \to \mathbf{P}W = W/\mathbf{C}$.

Proposition 2.5. *The maps $R \to H$ and $H \to \mathbf{P}W$ are homeomorphisms.*

Proof. By definition, the map $R \to H$ is a bijection. By the proof of [**?**, Theorem 4.8] (part (ii)), the map $R \to H$ is continuous. Its inverse is given by $\sigma \mapsto -Z_\sigma(\mathcal{O}_X)$, which is also continuous. So $R \to H$ is a homeomorphism.

By Proposition 2.3, the map $H \to \mathbf{P}W$ is surjective. Owing to the normalisation of the phase and mass of $\mathbf{k}_x$, it is also injective. It remains to prove that the inverse is continuous. We know that W is an open subset of $\mathrm{Stab}(X)$. It is also $\mathbf{C}$-invariant, so $\mathbf{P}W$ is an open subset of $\mathbf{P}\,\mathrm{Stab}(X)$. Thus, the map $\mathbf{P}W \to \mathbf{P}\,\mathrm{Hom}(\mathcal{N}(X), \mathbf{C})$ is a local homeomorphism. We have the commutative diagram

$$\begin{tikzcd} R \arrow[d, equal] & H \arrow[l, "\sim"'] & \mathbf{P}W \arrow[l] \arrow[d] \\ R & & \mathbf{P}\,\mathrm{Hom}(\mathcal{N}(X), \mathbf{C}), \arrow[ll] \end{tikzcd}$$

where the bottom map is given by $Z \mapsto Z(\mathcal{O}_X)/Z(\mathbf{k}_x)$. Since this map is continuous, it follows that $\mathbf{P}W \to H$ is continuous. □

3. Semi-rigid objects

Recall that an object F in $D^b(X)$ is *semi-rigid* if

$$\hom^i(F, F) = \begin{cases} 1 & \text{if } i = 0 \\ 2 & \text{if } i = 1 \\ 1 & \text{if } i = 2, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For example, for $x \in X$, the skyscraper sheaf $F = \mathbf{k}_x$ and the ideal sheaf $F = I_x$ are semi-rigid. We now characterises the semi-rigid objects of $D^b(X)$. Recall that $T\colon D^b(X) \to D^b(X)$ is the spherical twist in $\mathcal{O}_X$.

Proposition 3.1. *Let X be a K3 surface with $\mathrm{Pic}\, X = 0$. Let $F \in D^b(X)$ be semi-rigid. Then there exists $x \in X$ and integers m, n such that $F \cong T^n\mathbf{k}_x[m]$.*

We split the proof in two lemmas.

Lemma 3.2. *Fix a stability condition $\sigma \in W_-$. Let $F \in D^b(X)$ be semi-rigid and semi-stable. Then there exists $x \in X$ such that F or $T^{-1}F$ is a shift of $\mathbf{k}_x$.*

Proof. Since F is semi-rigid, $[F] \cdot [F] = 0$ in $\mathcal{N}(X)$. So $[F]$ is an integer multiple of $(0, 1)$ or $(1, 0)$.

Suppose $[F]$ is a multiple of $(0, 1)$. Since $[\mathbf{k}_x] = (0, 1)$, after applying a shift, we may assume that F is semi-stable of the same phase as $\mathbf{k}_x$, namely 1. It is easy to check that the abelian category of semi-stable objects of phase 1 is $\mathcal{F}$, the category of torsion sheaves on X. It is a finite length category whose simple objects are the skyscraper sheaves $\mathbf{k}_x$. So F is an iterated extension of skyscraper sheaves. Since $\hom^1(F,F)=2$, the Mukai lemma [?, Lemma 2.7] implies that F must simply be a skyscraper sheaf.

Suppose $[F]$ is a multiple of $(1,0)$. Then $[T^{-1}F]$ is a multiple of $(0,1)$ and $T^{-1}F$ is semi-stable with respect to $\tau = T^{-1}\sigma$. By Proposition 2.4, we have $\tau \in W_+$. By applying a rotation, assume that τ is standard. Then, after applying a shift, we may assume that $T^{-1}F$ is semi-stable of the same phase as $\mathbf{k}_x$, namely 1. Again, it is easy to check that the abelian category of τ semi-stable objects of phase 1 is $\mathcal{F}$. We now proceed as before. □

Given a stability condition σ, denote by ϕ_σ^+ and ϕ_σ^- the highest and lowest phases of the factors in the σ-HN filtration. If σ is clear from the context, we omit the subscript.

Lemma 3.3. *Fix a standard stability condition $\sigma \in W_-$. Let $F \in D^b(X)$ be a semi-rigid object. There exists a non-negative integer n such that T^nF is σ-semi-stable.*

Proof. Since F is semi-rigid, all stable factors of F are either spherical or semi-rigid, and only one stable factor is semi-rigid [?, Proposition 2.9]. The only spherical object, up to shift, is $\mathcal{O}_X$. By Lemma 3.2, the only semi-stable semi-rigid objects, up to shift, are $\mathbf{k}_x$ and $T^{-1}\mathbf{k}_x$. In particular, the phases of the HN factors of F lie in the discrete subset of $\mathbf{R}$ given by

$$(\phi_\sigma(\mathcal{O}_X)+\mathbf{Z}) \cup (\phi_\sigma(\mathbf{k}_x)+\mathbf{Z}) \cup \left(\phi_\sigma(T^{-1}\mathbf{k}_x)+\mathbf{Z}\right).$$

Therefore, there exists a discrete $\Phi \subset \mathbf{R}$ such that for every semi-rigid object F, we have

$$\phi^+(F) - \phi^-(F) \in \Phi.$$

If F itself is semi-stable, we simply take $n=0$. Otherwise, up to shift, a stable HN factor of F of highest or lowest phase must be $\mathcal{O}_X$. We apply [?, Theorem 3.5] with $Y=F$ and $X=\mathcal{O}_X$. Then for $F' = TF$ or $F' = T^{-1}F$, we have

$$\phi^+(F') - \phi^-(F') < \phi^+(F) - \phi^{-1}(F).$$

By repeated applications of [?, Theorem 3.5] and using that $\phi^+ - \phi^-$ lies in the discrete set $\Phi \subset \mathbf{R}$, we conclude that there exists an integer n such that T^nF is semi-stable. □

Having proved the two lemmas, we are ready to prove Proposition 3.1—the only semi-rigid objects of $D^b(X)$, up to twisting by $\mathcal{O}_X$ and shifting, are the skyscraper sheaves $\mathbf{k}_x$.

Proof of Proposition 3.1. Combine Lemma 3.2 and Lemma 3.3. □

4. The mass embedding

Recall that X is an analytic K3 surface with $\operatorname{Pic} X = 0$. Let S be the set of isomorphism classes of semi-rigid objects of $D^b(X)$. In this section, we describe the mass embedding

$$m\colon \mathbf{P}\operatorname{Stab}(X) \to \mathbf{P}^S$$

and the closure of its image.

4.1. HN filtration of semi-rigid objects.

To understand the mass embedding, we must understand the HN filtrations of the objects of S. By Proposition 3.1, the objects of S, up to shift, are $T^n\mathbf{k}_x$ for $x \in X$ and $n \in \mathbf{Z}$. For points $x, y \in X$, the behaviour of $T^n\mathbf{k}_x$ and $T^n\mathbf{k}_y$ is entirely analogous to each other. So we lose nothing by fixing a particular point $x \in X$ and taking

$$S = \{T^n\mathbf{k}_x \mid n \in \mathbf{Z}\}.$$

We may then write the points of $\mathbf{P}^S$ as homogeneous vectors $[x_n \mid n \in \mathbf{Z}] = [\cdots : x_{-1} : x_0 : x_1 : \cdots]$. In these coordinates, the spherical twist T acts as a shift.

We first treat HN filtrations with respect to off the wall stability conditions.

Proposition 4.1. *Let $\sigma \in W_-$. Then the σ-HN factors of $F = T^n\mathbf{k}_x$, in decreasing order of phase, are as follows.*

(1) *For $n = 0$ and 1, the object F is stable.*
(2) *For $n \geq 2$, the semi-stable (= stable) factors of F are $T\mathbf{k}_x$ and $\mathcal{O}_X[i]$ for $0 \geq i \geq -n+2$.*
(3) *For $n \leq -1$, the semi-stable (= stable) factors of F are $\mathcal{O}_X[i]$ for $-n \geq i \geq 1$ and $\mathbf{k}_x$.*

Proof. Recall that $\mathbf{k}_x$ and $T\mathbf{k}_x = I_x[1]$ are stable for stability conditions in W_-. So (1) follows.

Consider the triangle

$$\operatorname{Hom}^*(\mathcal{O}_X, T^{n-1}\mathbf{k}_x) \otimes \mathcal{O}_X \to T^{n-1}\mathbf{k}_x \to T^n\mathbf{k}_x \xrightarrow{+1} . \tag{1}$$

We have

$$\begin{aligned} \operatorname{Hom}^*(\mathcal{O}_X, T^{n-1}\mathbf{k}_x) &= \operatorname{Hom}^*(T^{-n+1}\mathcal{O}_X, \mathbf{k}_x) \\ &= \operatorname{Hom}^*(\mathcal{O}_X[n-1], \mathbf{k}_x) \\ &= \mathbf{C}[-n+1]. \end{aligned}$$

By substituting in (1) and shifting, we get

$$T^{n-1}\mathbf{k}_x \to T^n\mathbf{k}_x \to \mathcal{O}_X[-n+2] \xrightarrow{+1} . \tag{2}$$

Let us assume $n \geq 2$, and induct on n. Assume we know that the HN factors of $T^{n-1}\mathbf{k}_x$ (in decreasing order of phase) are $T\mathbf{k}_x$ followed by $\mathcal{O}_X[i]$ for $0 \geq i \geq -n+3$. Concatenating the HN filtration of $T^{n-1}\mathbf{k}_x$ and the map $T^{n-1}\mathbf{k}_x \to T^n\mathbf{k}_x$, we obtain a filtration of $T^n\mathbf{k}_x$ whose factors are whose factors are $T\mathbf{k}_x$ and $\mathcal{O}_X[i]$ for $0 \geq i \geq -n+2$. Since these factors are stable and appear in decreasing order of phase, this must be the HN filtration of $T^n\mathbf{k}_x$. The induction step is complete.

Now let us assume $n \leq -1$, and induct on $-n$. Consider the triangle

$$\mathcal{O}_X[-n] \to T^n\mathbf{k}_x \to T^{n+1}\mathbf{k}_x \xrightarrow{+1}, \tag{3}$$

obtained by replacing n by $n+1$ in (2) and shifting. Assume we know that the HN factors of $T^{n+1}\mathbf{k}_x$ (in decreasing order of phase) are $\mathcal{O}_X[i]$ for $-n-1 \geq i \geq 1$ and $\mathbf{k}_x$. By augmenting the HN filtration of $T^{n+1}\mathbf{k}_x$ by the map $\mathcal{O}_X[-n] \to T^n\mathbf{k}_x$, we obtain a filtration of $T^n\mathbf{k}_x$ whose factors are $\mathcal{O}_X[i]$ for $-n \geq i \geq 1$ and $\mathbf{k}_x$. Since these factors are stable and appear in decreasing order of phase, this must be the HN filtration of $T^n\mathbf{k}_x$. The induction step is complete. □

For stability conditions on the wall, the HN filtration degenerates slightly.

Proposition 4.2. *Let $\sigma \in W_0$. Then the σ-HN factors of $F = T^n\mathbf{k}_x$, in decreasing order of phase, are as follows.*

(1) *For $n = -1$, 0 and 1, the object F is semi-stable.*
(2) *For $n \geq 2$, the semi-stable factors of F are $T\mathbf{k}_x$ and $\mathcal{O}_X[i]$ for $0 \geq i \geq -n+2$.*
(3) *For $n \leq -2$, the semi-stable factors of F are $\mathcal{O}_X[i]$ for $-n \geq i \geq 2$ and $T^{-1}\mathbf{k}_x$.*

Proof. The proof is analogous to the proof of Proposition 4.1. □

4.2. **The mass map.** We now have the tools to describe the mass map

$$m \colon \mathbf{P}\operatorname{Stab}(X) \to \mathbf{P}^S.$$

Proposition 4.3. *Let $\sigma \in \mathbf{P}W_-$. Set $a = |Z_\sigma(\mathbf{k}_x)|$ and $b = |Z_\sigma(T\mathbf{k}_x)|$ and $c = |Z_\sigma(\mathcal{O}_X)|$.*

(1) *The numbers a, b, c are positive real numbers satisfying*

$$b < a+c, \quad a < b+c, \quad c < a+b.$$

(2) *We have*

$$m_\sigma \colon T^n\mathbf{k}_x \mapsto \begin{cases} a - nc & \text{if } n \leq 0, \\ b + (n-1)c & \text{if } n \geq 1. \end{cases}$$

(3) *Let $\Delta_0 \subset \mathbf{P}^S$ be the locally closed subset consisting of points of the form*

$$[\cdots : a+2c : a+c : a : b : b+c : b+2c : \cdots],$$

where a is at index 0 and b is at index 1, and where a, b, c are positive real numbers satisfying the inequalities in (1). Then $m \colon \mathbf{P}W_- \to \Delta_0$ is a homeomorphism.

Proof. Part (1) follows from the fact that the classes of $\mathcal{O}_X$, $\mathbf{k}_x$, and $T\mathbf{k}_x$ satisfy

$$[\mathcal{O}_X] = [\mathbf{k}_x] - [T\mathbf{k}_x].$$

Part (2) follows from Proposition 4.1.

For part (3), let $\Delta \subset \mathbf{P}^2$ be the set of points $[a : b : c]$ that satisfy the conditions in (1). Then we have a homeomorphism $\Delta \to \Delta_0$ given by

$$[a : b : c] \mapsto [\cdots : a+2c : a+c : a : b : b+c : b+2c : \cdots].$$

We use $[a : b : c] \in \Delta$ as coordinates on Δ_0. By Proposition 2.5, the map $w \mapsto \sigma_w$ gives a homeomorphism $-\mathbf{H} \to \mathbf{P}W_-$. We use $z \in -\mathbf{H}$ as a coordinate on $\mathbf{P}W_-$. In these coordinates, writing down the inverse map $\omega \colon \Delta \to \mathbf{P}W_-$ amounts to re-constructing the central charge given a, b, c. This can be done using the cosine rule (see Figure 5). Precisely, we have

$$\omega([a : b : c]) = -(b/a \exp(i\theta) - 1), \text{ where } \theta = \arccos\left(\frac{c^2 - a^2 - b^2}{2ab}\right) \in (0, \pi), \tag{4}$$

which is continuous. □

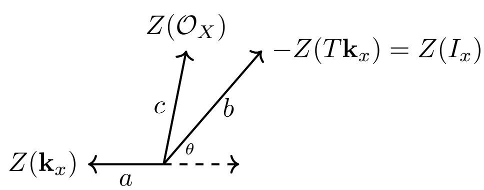

FIGURE 5. We can reconstruct the central charge (up to rotation) from the masses a, b, c of $\mathbf{k}_x, T\mathbf{k}_x, \mathcal{O}_X$ using the cosine rule.

For $n \in \mathbf{Z}$, let $\Delta_n \subset \mathbf{P}^S$ be the locally closed subset consisting of points of the form

$$[\cdots : a+2c : a+c : a : b : b+c : b+2c : \cdots],$$

where a is at index n, and where a, b, c are positive real numbers satisfying the (strict) triangle inequalities. Denote by $T \colon \mathbf{P}^S \to \mathbf{P}^S$ the map that shifts the homogeneous coordinates rightwards by 1, so that $\Delta_n = T^n\Delta_0$. Then we have

$$m(T(\sigma)) = T(m(\sigma)).$$

Proposition 4.3 implies that the mass map $T^n\mathbf{P}W_- \to \Delta_n$ is a homeomorphism. In particular, the mass map $T^{-1}\mathbf{P}W_- = \mathbf{P}W_+ \to \Delta_{-1}$ is a homeomorphism. It is useful to write the inverse $\Delta_{-1} \to \mathbf{P}W_+$ using coordinates $[a : b : c]$ on Δ_{-1} as in the proof of Proposition 4.3 and the coordinates on W_+ given by $\mathbf{H}$. The explicit formula again arises from the cosine rule and is given by

$$[a : b : c] \mapsto c/b \exp(i\theta) + 1, \text{ where } \theta = \arccos\left(\frac{c^2 - a^2 - b^2}{2ab}\right) \in (0, \pi], \tag{5}$$

Let $I_0 \subset \mathbf{P}^S$ be the set of points of the form

$$[\cdots : a+2c : a+c : a : a+c : a+2c : \cdots],$$

where a is at index 0 and a, c are positive real numbers.

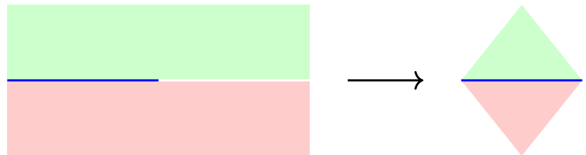

Figure 6. The mass map gives a homeomorphism from the set of standard stability conditions parametrised by $-\mathbf{H} \cup \mathbf{H} \cup \mathbf{R}_{<0}$ and the union of two open triangles and a segment that forms a wall between them.

Proposition 4.4. *Let $\sigma \in \mathbf{P}W_0$. Set $a = |Z_\sigma(\mathbf{k}_x)|$ and $c = |Z_\sigma(\mathcal{O}_X)|$. Then*

$$m_\sigma \colon T^n \mathbf{k}_x \mapsto a + |n|c.$$

Furthermore, the map $m\colon \mathbf{P}W_0 \to I_0$ is a homeomorphism.

Proof. The description of m_σ follows from Proposition 4.2. The inverse of $m\colon \mathbf{P}W_0 \to I_0$ is given using the central charge $Z(\mathbf{k}_x) = -1$ and $Z(\mathcal{O}_X) = c/a$. □

Proposition 4.5. *The map $m\colon \mathbf{P}W \to \Delta_0 \cup I_0 \cup \Delta_{-1}$ is a homeomorphism.*

See Figure 6 for a sketch.

Proof. The set $\mathbf{P}W$ is the disjoint union of $\mathbf{P}W_-$, $\mathbf{P}W_+$, and $\mathbf{P}W_0$. The sets Δ_0, I_0, and Δ_{-1} are also disjoint. Furthermore, the maps $\mathbf{P}W_- \to \Delta_0$, $\mathbf{P}W_+ \to \Delta_{-1}$, and $\mathbf{P}W_0 \to I_0$ are homeomorphisms. So $m\colon \mathbf{P}W \to \Delta_0 \cup I_0 \cup \Delta_{-1}$ is a continuous bijection.

We check that the inverse is continuous. Since $-\mathbf{H} \cup \mathbf{H} \cup \mathbf{R}_{<0} \to \mathbf{P}W$ is a homeomorphism, we use the former as local coordinates for $\mathbf{P}W$. Let $\overline{\Delta} \subset \mathbf{P}^2$ be the set of points $[a : b : c]$ where a, b, c are positive real numbers satisfying the triangle inequalities

$$b \leq a + c, \quad a < b + c, \quad c < a + b.$$

It is easy to check that the map $\overline{\Delta} \to \Delta_0 \cup I_0$ given by

$$[a : b : c] \mapsto [\cdots : a + c : a : b : b + c : \cdots]$$

is a homeomorphism. So we may use a, b, c as local coordinates on $\Delta_0 \cup I_0$. Using (4), we see that the inverse map $\Delta_0 \cup I_0 \to -\mathbf{H} \cup \mathbf{R}_{<0}$ is given in coordinates by

$$[a : b : c] \mapsto -b/a \exp(i\theta) + 1, \text{ where } \theta = \arccos\left(\frac{a^2 + b^2 - c^2}{2ab}\right) \in [0, \pi),$$

which is continuous.

Let $\overline{\Delta}' \subset \mathbf{P}^2$ be the set of points $[a : b : c]$ where a, b, c are positive real numbers satisfying the triangle inequalities

$$b < a + c, \quad a \leq b + c, \quad c < a + b.$$

Then the map $\overline{\Delta}' \to \Delta_{-1} \cup I_0$ given by

$$[a : b : c] \mapsto [\cdots : a + c : a : b : b + c : \cdots]$$

is a homeomorphism. So we may use a, b, c as local coordinates on $\Delta_{-1} \cup I_0$. Using (5), we see that the inverse map $\Delta_{-1} \cup I_0 \to \mathbf{H} \cup \mathbf{R}_{<0}$ is given in coordinates by

$$[a : b : c] \mapsto -a/b \exp(i\theta) + 1, \text{ where } \theta = \arccos\left(\frac{a^2 + b^2 - c^2}{2ab}\right) \in (-\pi, 0],$$

which is continuous.

Since the inverse is continuous on $\Delta_0 \cup I_0$ and $\Delta_{-1} \cup I_0$, we conclude that it is continuous on $\Delta_0 \cup \Delta_{-1} \cup I_0$. □

Let $D \subset \mathbf{P}^S$ be the union of the triangles Δ_n for $n \in \mathbf{Z}$ and the intervals I_n for $n \in \mathbf{Z}$.

Theorem 4.6. *The mass map gives a homeomorphism $m\colon \mathbf{P}\operatorname{Stab}(X) \to D$.*

Proof. By Proposition 2.2 and Proposition 2.4, we see that $\mathbf{P}\,\mathrm{Stab}(X)$ is the union of $T^n\mathbf{P}W_-$ for $n \in \mathbf{Z}$ and $T^n\mathbf{P}W_0$ for $n \in \mathbf{Z}$. From Proposition 2.3, it follows that this is a disjoint union. Likewise, D is the disjoint union of Δ_n for $n \in \mathbf{Z}$ and I_n for $n \in \mathbf{Z}$. Since $m\colon \mathbf{P}W_- \to \Delta_0$ and $m\colon \mathbf{P}W_0 \to I_0$ are bijections, we conclude that $m\colon \mathbf{P}\,\mathrm{Stab}(X) \to D$ is a bijection. It is also continuous. It remains to prove that the inverse is continuous.

Let $U = \Delta_0 \cup I_0 \cup \Delta_{-1}$. Observe that

$$U = \{[a_n] \in D \mid 2a_0 < a_1 + a_{-1}\}.$$

So $U \subset D$ is open. From Proposition 4.5, we know that the inverse of m is continuous on U. But T^nU for $n \in \mathbf{Z}$ form an open cover of D. So the inverse of m is continuous on D. □

4.3. **Identifying the image and its closure.** Let $\overline{D} \subset \mathbf{P}^S$ be the closure of D. Our next goal is to identify the homeomorphism classes of $\overline{D}$ and D. To do so, it will be useful to work with an auxiliary space, which we now define.

Let $\overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$ be the two point compactification of $\mathbf{R}$, one at either end, so that $\overline{\mathbf{R}}$ is homeomorphic to $[0,1]$. Our auxiliary space will be $\overline{\mathbf{R}} \times [0,1]$.

Let $\overline{\Delta} \subset \mathbf{P}^2$ be the set of $[a:b:c]$ such that a, b, c are non-negative real numbers that satisfy

$$a \leq b + c, \quad b \leq a + c, \quad c \leq a + b.$$

Define a map

$$p\colon [0,1] \times [0,1] \to \mathbf{P}^2$$

by

$$p(u,v) = [uv + (1-v) : 1 - uv : v].$$

It is easy to check that p has image $\overline{\Delta}$, it is injective on $[0,1] \times (0,1]$, and it sends $[0,1] \times \{0\}$ to the point $[1:1:0]$.

For $n \in \mathbf{Z}$, let $t_n\colon \overline{D} \to \mathbf{P}^S$ be the map defined by

$$t_n\colon [a:b:c] \mapsto [\cdots : a + c : a : b : b + c : \cdots],$$

where a is at index n. We define

$$\pi\colon \mathbf{R} \times [0,1] \to \mathbf{P}^S$$

as follows. Write $x \in \mathbf{R}$ as $x = n + u$, where $n \in \mathbf{Z}$ and $u \in [0,1)$. Set

$$\pi(x,v) = t_n \circ p(u,v).$$

Let $T\colon \mathbf{R} \times [0,1] \to \mathbf{R} \times [0,1]$ be the map $T(u,v) = (u+1,v)$. Recall that $T\colon \mathbf{P}^S \to \mathbf{P}^S$ is also the rightward shift of the homogeneous coordinates (we are intentionally using the same letter T for all the related maps). Then, by definition, we have

$$\pi \circ T = T \circ \pi.$$

Proposition 4.7. *The map $\pi\colon \mathbf{R} \times [0,1] \to \mathbf{P}^S$ is continuous. It maps $(n, n+1) \times (0,1)$ homeomorphically to the open triangle $\Delta_n \subset \mathbf{P}^S$ and $\{n\} \times (0,1)$ homeomorphically to the interval I_n.*

Proof. We first check continuity. Continuity at (x,v) is clear for $x \notin \mathbf{Z}$. For $x = n \in \mathbf{Z}$, it suffices to check that

$$t_n \circ p(0,v) = t_{n-1} \circ p(1,v),$$

which we now do. The left hand side is

$$t_n[1-v : 1 : v] = [\cdots : 1+v : 1 : 1-v : 1 : 1+v : \cdots],$$

where the $(1-v)$ is at index n. The right hand side is

$$t_{n-1}[1 : 1-v : v] = [\cdots : 1+v : 1 : 1-v : 1 : 1+v : \cdots],$$

where the $(1-v)$ is at index n. We see that the two are equal.

By the T-equivariance of π, it suffices to check the homeomorphism assertions for $n = 0$. It is easy to check that $(0,1) \times (0,1) \to \mathbf{P}^2$ is a homeomorphism onto the triangle $\Delta \subset \mathbf{P}^2$ consisting of $[a:b:c]$, where a, b, c are positive real numbers satisfying the strict triangle inequalities. Since $t_0\colon \Delta \to \Delta_0$ is a homeomorphism, the first statement follows. The map π on $\{0\} \times (0,1)$ is given by

$$(0, v) \mapsto [\cdots : 1+v : 1 : 1-v : 1 : 1+v : \cdots],$$

where $(1-v)$ is at index 0. Evidently, the map is a homeomorphism to the interval I_0. □

We extend $\pi\colon \mathbf{R} \times [0,1] \to \mathbf{P}^S$ to a map

$$\pi\colon \overline{\mathbf{R}} \times [0,1] \to \mathbf{P}^S$$

by setting

$$\pi(\pm\infty, v) = [\cdots : 1 : 1 : 1 : \cdots].$$

Theorem 4.8. *The map $\pi\colon \overline{\mathbf{R}} \times [0,1] \to \mathbf{P}^S$ is continuous. It sends the set*

$$C = \{\pm\infty\} \times [0,1] \cup \overline{\mathbf{R}} \times \{0\}$$

to the point $[\cdots : 1 : 1 : 1 : \cdots]$. Let $\overline{\mathbf{R}} \times [0,1] \to B$ be the contraction of C to a point. Then the induced map $\pi\colon B \to \mathbf{P}^S$ is a homeomorphism onto $\overline{D} = \overline{m(\mathbf{P}\operatorname{Stab}(X))}$.

Note that B is homeomorphic to a closed disk. See Figure 7 for a sketch.

Proof. Let us check continuity at (∞, v). Fix a positive integer n. Consider the homogeneous coordinates of $\mathbf{P}^S$ with indices $-n, \cdots, n$. Let us examine these homogeneous coordinates of $\pi(t, y)$, where t is large. Say $t = N + u$, where $N > n$. Then $p(u, y) = [uy + (1-y) : 1 - uy : y]$. For $-n \leq i \leq n$, the i-th index of $\pi(u, y)$, which by definition is the i-th index of $t_N \circ p(u, y)$ is

$$\pi(u, y)_i = uy + (1-y) + (N-i)y.$$

Observe that as $N \to \infty$, we have

$$[uy + (1-y) + (N+n)y : \cdots : uy + (1-y) + (N-n)y] \mapsto [1 : \cdots : 1],$$

uniformly in $(u, y) \in [0,1] \times [0,1]$. It follows that π is continuous at (∞, v). We check similarly that it is continuous at $(-\infty, v)$.

From Proposition 4.7, we know that $\pi\colon \mathbf{R} \times (0,1) \to D$ is a bijection. We note that π maps C to the point $[\cdots : 1 : 1 : 1 : \cdots]$, which is not in D. Finally, for $u \in [0,1]$, we have

$$\pi(u, 1) = [u : 1-u : 1] = [\cdots : 2-u : 1-u : u : u+1 : u+2 : \cdots].$$

Observe that this is the third side of the closure of $\Delta_0 \subset \mathbf{P}^S$, other than the (closures) of I_0 and I_1. Therefore, we see that π is injective on $\mathbf{R} \times \{1\}$, and maps it outside of D. We conclude that $\pi\colon B \to \mathbf{P}^S$ is a bijection onto its image. Since B is compact, it is a homeomorphism onto its image. It maps the interior of $B = \mathbf{R} \times (0,1)$ to D, and hence the image must be the closure $\overline{D}$. □

4.4. **Points of the boundary.** Observe that $\overline{D}$ contains the point $\bullet = [\cdots : 1 : 1 : 1 : \cdots]$. This is the common vertex (drawn in red in Figure 7) of all the triangles that tesselate $\overline{D}$. It is the unique T-invariant point of $\overline{D}$. This point is precisely the projectivised hom function $\hom(\mathcal{O}_X, -)$, whose value on $T^n \mathbf{k}_x$ for any $n \in \mathbf{Z}$ is

$$\dim \operatorname{Hom}^*(\mathcal{O}_X, T^n \mathbf{k}_x) = 1.$$

The fact that $\bullet$ is in the boundary follows from the following more general fact.

Theorem 4.9 ([?, Corollary 4.13])**.** *Let a be a spherical object of a triangulated category $\mathcal{C}$, and assume that it is a stable object of a stability condition σ. Let S be a set of objects of $\mathcal{C}$ such that no object in S has an endomorphism of negative degree. For simplicity, also assume that no shift of a is in S. Let T be the spherical twist in a. Then, in $\mathbf{P}^S$, we have the equality*

$$\lim_{n \to \pm\infty} T^n [m_\sigma] = [\hom(a, -)].$$

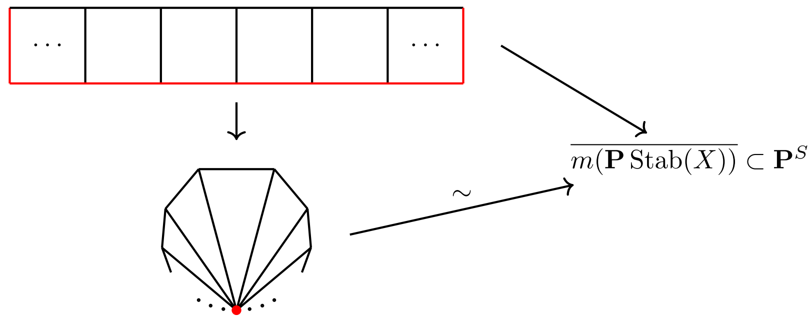

FIGURE 7. The map $\pi\colon \overline{\mathbf{R}} \times [0,1] \to \mathbf{P}^S$ induces a homeomorphism from a closed disk B onto the closure of the image of $\mathrm{Stab}(X)$. The disk B is obtained from the square $\overline{\mathbf{R}} \times [0,1]$ by collapsing three sides (red). The $\mathbf{Z}$-indexed decomposition corresponds to the translates of a fundamental domain of $\mathbf{P}\,\mathrm{Stab}(X)$ by the spherical twist T.

The point $\bullet$ also has an interpretation as the mass function of a lax stability condition in the sense of Broomhead, Pauksztello, Ploog, and Woolf [**?**]. We quickly recall the main features of the definition. A *lax stability condition* is a slicing P and a compatible central charge Z. The central charge is allowed to vanish on the classes of non-zero semi-stable objects (such objects are called "massless"). The pair (P, Z) must satisfy the following two finiteness conditions:

(1) The slicing P is locally finite.
(2) The central charge satisfies the support property. That is, for a choice of a norm $\| - \|$ on $\mathcal{N}(X)$, there exists a positive constant c such that for every massive stable object s, we have $|Z(s)|/\|s\| > c$.

We let P to be the slicing defined by $P(1) = \mathcal{A}$ and $P(\phi) = 0$ for $\phi \in (0, 1)$. The simple objects of $P(1)$ are the skyscraper sheaves $\mathbf{k}_x$ and the objects $E[1]$, where E is a vector bundle on X with no non-trivial sub-bundles (see [**?**, Remark 4.3 (iii)]). We let $Z(\mathcal{O}_X) = 0$ and $Z(\mathbf{k}_x) = -1$.

Proposition 4.10. *The pair (P, Z) as above defines a lax stability condition σ that is a limit of standard stability conditions. Furthermore, $m(\sigma) = [\cdots : 1 : 1 : 1 : \cdots]$.*

Proof. It is easy to check that the abelian category $\mathcal{A}$ is of finite length (Noetherian and Artinian). So the slicing is locally finite. Let E be a vector bundle with no non-trivial sub-bundles, and let $[E] = r[\mathcal{O}_X] + m[\mathbf{k}_x]$. Then $r = \mathrm{rk}\, E$ and $Z(E) = -m$. Assume that E is not isomorphic to $\mathcal{O}_X$. Then $\mathrm{Hom}(\mathcal{O}_X, E) = \mathrm{Hom}(E, \mathcal{O}_X) = 0$. So

$$0 \geq \chi(\mathcal{O}_X, E) = 2r + m,$$

and hence $-m \leq 2r$. As a result, with the standard Euclidean norm on $\mathcal{N}(X)$, we see that

$$|Z(E)|/\|E\| \geq |m|/|r| \geq 2.$$

So the support property holds.

Finally, note that this lax stability condition is the limit of the stability conditions in $\mathbf{P}W_0$ as $Z(\mathcal{O}_X)/Z(\mathbf{k}_x)$ approaches 0. Since $m_\sigma(T^n \mathbf{k}_x) = 1$, the last equality follows. $\square$

Consider the points of $\overline{D}$ that are the vertices of the tiling triangles other than the vertex $\bullet$. They form a single T-orbit, so it suffices to consider one of them, say $v_0 = [\cdots : 2 : 1 : 0 : 1 : 2 : \cdots]$, with the 0 at index 0. Note that this is the common vertex, other than the $\bullet$, of the triangles $\mathbf{P}W_+ \cong \Delta_{-1}$ and $\mathbf{P}W_- = \Delta_0$. This is the mass function of a different lax stability condition. Let P be the same slicing as before, and set $Z(\mathcal{O}_X) = 1$ and $Z(\mathbf{k}_x) = 0$.

Proposition 4.11. *The pair (P, Z) as above defines a lax stability condition τ that is a limit of standard stability conditions. Furthermore, $m(\tau) = [\cdots : 2 : 1 : 0 : 1 : 2 : \cdots]$.*

Proof. Note that Z maps $(r, r-c)$ to r. So the support property is clear.

The resulting lax stability conditition is the limit of the stability conditions in $\mathbf{P}W_0$ as $Z(\mathcal{O}_X)/Z(\mathbf{k}_x)$ approaches $-\infty$. Since $m_\tau(T^n\mathbf{k}_x) = |n|$, the last equality follows. □

Using the T-action, we see that all the other vertices $v_i = T^i v_0$ are mass functions of lax stability conditions.

Finally, consider a point on the open line segment joining v_0 and v_1. This point is in the closure of $\mathbf{P}W_- = \Delta_0$. Nevertheless, we claim that it is *not* the mass function of a lax stability condition arising as a limit of stability conditions W_-.

To see this, it is helpful to consider a handful of other semi-stable objects. Let $n \geq m$ be positive integers. Let $x_1, \dots, x_n \in X$ be distinct points, and set $S = \{x_1, \dots, x_n\}$. We say that a morphism $\pi\colon \mathcal{O}_X^{\oplus m} \to \mathcal{O}_S$ is *generic* if for every subset $T \subset S$, the induced map on global sections

$$H^0(\mathcal{O}_X^{\oplus m}) \to H^0(\mathcal{O}_T)$$

has maximal rank, namely $\min(m, |T|)$.

Let $\sigma = \sigma_w$ be a standard stability condition, for some $w \in -\mathbf{H}$. Let $I_{m,n}$ be the kernel of a generic morphism from $\mathcal{O}_X^{\oplus m}$ to the structure sheaf of n-points. Then it is easy to check that $I_{m,n}$ is σ-stable.

Fix a point $p \in \overline{D}$ of the form

$$p = [\cdots : 2+t : 1 : t : 1+2t : \cdots].$$

for some $t > 0$. Then p is on the line segment joining v_0 and v_1. If we take a sequence of standard stability conditions in W_- whose mass function approaches p, their slicings do not converge. Therefore, there is no limiting lax stability condition with the mass function p. We now make this precise.

Recall that the topology on the space of slicings is induced by the metric d defined as follows. For a slicing P and non-zero object c, let $\phi_P^\pm(c)$ denote the highest/lowest phase of the P-HN factors of c. Then the distance $d(P, Q)$ between two slicings P and Q is

$$d(P,Q) = \sup_{c \neq 0} \left\{ \max(|\phi_P^+(c) - \phi_Q^+(c)|, |\phi_P^-(c) - \phi_Q^-(c)|) \right\}.$$

Suppose σ is a lax stability condition that is a limit of a sequence of standard stability conditions σ_w for $w \in -\mathbf{H}$ with $m(\sigma) = p$. Then, possibly after a rotation and scaling, the central charge of σ must send $\mathbf{k}_x$ to -1 and $\mathcal{O}_X$ to $-1-t$. But then

$$Z(I_{m,n}) = mZ(\mathcal{O}_X) - nZ(\mathbf{k}_x) = n - m(1+t).$$

It follows that for for every (n, m) with $n/m > (1+t)$, the sheaf $I_{m,n}$ is σ-semi-stable of phase 0 and for $n/m < (1+t)$, it is σ-semi-stable of phase 1. But this is absurd. Indeed, for a standard stability condition σ_w, we have

$$\inf_{n/m > 1+t} \phi_\sigma(I_{n,m}) = \sup_{n/m < 1+t} \phi_\sigma(I_{n,m}),$$

so the same equality must hold in the limit.

In summary, the objects $\mathbf{k}_x$, $\mathcal{O}_X$, and $I_x = T\mathbf{k}_x[-1]$ can become massless in the sense of [**?**] under a lax stability condition in the limit of standard stability conditions. The masses of these three limits are the three vertices of the triangle $\mathbf{P}W_- = \Delta_0$. Other ideal sheaves, or the semi-stable objects $I_{m,n}$, cannot become massless. This distinction is consistent with the density of the phase diagram of standard stability conditions (see the discussion in [**?**, § 12]). Let $\sigma \in W_-$ be a standard stability condition. It is easy to check that the classes $r[\mathcal{O}_X] + n[\mathbf{k}_x]$ that support semi-stable sheaves are precisely $(0, n)$ for $n \geq 1$; $(r, 0)$ for $r \geq 1$; and (r, n) for $-n \geq r \geq 1$ (see **??**). So, on the phase diagram $\phi(\mathcal{O}_X)$ is an isolated point, $\phi(\mathbf{k}_x)$ is only aright accumulation point, and $\phi(I_x)$ is only a left accumulation point. The phase diagram is dense on the arc from $-\phi(\mathbf{k}_x)$ to $\phi(I_x)$ and its negative.

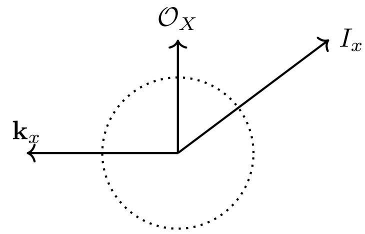

FIGURE 8. The central charges of semi-stable objects in a standard stability condition with heart $\operatorname{Coh} X$ are the lattice points in the shaded region. As a result, the phases are dense in the highlighted region of the unit circle.

5. The q-mass embedding

Fix a positive real number q. Given a stability condition σ and an object x, recall that the q-mass of x with respect to σ is defined by

$$m_{q,\sigma}(x) = \sum |Z_\sigma(x_i)| q^{\phi(x_i)},$$

where the sum is taken over the σ-HN factors x_i of x, and $\phi(x_i)$ is the phase of x_i. We have the map

$$m_q \colon \mathbf{P}\operatorname{Stab}(X) \to \mathbf{P}^S$$

given by $\sigma \mapsto m_{q,\sigma}$. We describe the image of m_q and its closure for $q \neq 1$. Most of the arguments are direct analogues of the arguments for $q = 1$, so we will be brief.

Let $\sigma \in \mathbf{P}W_-$. Set $a = m_{q,\sigma}(\mathbf{k}_x)$ and $b = m_{q,\sigma}(T\mathbf{k}_x)$ and $c = m_{q,\sigma}(\mathcal{O}_X)$. Owing to the triangle

$$\mathcal{O}_X \to \mathbf{k}_x \to T\mathbf{k}_x \xrightarrow{+1},$$

the positive real numbers a, b, c satisfy the q-triangle inequalities

$$b < a + qc, \quad a < b + c, \quad c < a + q^{-1}b. \tag{6}$$

(See [?, Proposition 3.3] for a proof of the q-triangle inequalities). From the σ-HN filtration of $T^n\mathbf{k}_x$ from Proposition 4.1, we get

$$m_{q,\sigma} \colon T^n\mathbf{k}_x \mapsto \begin{cases} a + cq^{-n} + \cdots + cq^2 & \text{for } n \leq -2, \\ a & \text{for } n = 0, \\ b & \text{for } n = 1, \\ b + cq^0 + \cdots + cq^{-n+2} & \text{for } n \geq 2. \end{cases}$$

So, in homogeneous coordinates, the q-mass map is

$$m_q \colon \sigma \mapsto [\cdots : a + cq + cq^2 : a + cq : a : b : b + c : b + c + cq^{-1} : \cdots]$$

Let $\Delta \subset \mathbf{P}^2$ be the set consisting of $[a : b : c]$ where a, b, c are positive real numbers satisfying (6). Then the map $\mathbf{P}W^- \to \Delta$ that takes σ to $[m_{q,\sigma}(\mathbf{k}_x) : m_{q,\sigma}(T\mathbf{k}_x) : m_{q,\sigma}(\mathcal{O}_X)]$ is a homeomorphism. The proof is analogous to the proof of Proposition 4.3 (3), but uses the q-analogue of the cosine rule [?, Lemma 5.2]. Let $t_n \colon \Delta \to \mathbf{P}^S$ be the map

$$[a : b : c] \mapsto [\cdots : a + cq + cq^2 : a + cq : a : b : b + c : b + c + cq^{-1} : \cdots],$$

where the a is at index n. Set $\Delta_n = t_n(\Delta)$. Then $t_n \colon \Delta \to \Delta_n$ is a homeomorphism. So, the q-mass map $m_q \colon T^n\mathbf{P}W_- \to \Delta_n$ is a homeomorphism.

Now consider $\sigma \in \mathbf{P}W_0$. With a, b, c as before, we have $b = a + qc$. From the σ-HN filtration of $T^n\mathbf{k}_x$ from Proposition 4.1, we get

$$m_{q,\sigma} \colon T^n\mathbf{k}_x \mapsto \begin{cases} a + cq^{-n} + \cdots + cq^2 & \text{for } n \leq -2, \\ a & \text{for } n = 0, \\ a + cq + \cdots + cq^{-n+2} & \text{for } n \geq 1. \end{cases}$$

So, in homogeneous coordinates, the q-mass map is

$$\sigma \mapsto [\cdots : a + cq + cq^2 : a + cq : a : a + cq : a + cq + c : \cdots].$$

Set $I_0 = m_q(\mathbf{P}W_0)$ and $I_n = T^n I_0$. Then $m_q \colon T^n \mathbf{P}W_0 \to I_n$ is a homeomorphism.

Let $D_q \in \mathbf{P}^S$ be the union of Δ_n and I_n for $n \in \mathbf{Z}$.

Theorem 5.1. *The q-mass map*

$$m_q \colon \mathbf{P}\operatorname{Stab}(X) \to D_q$$

is a homeomorphism.

The proof is analogous to the proof of Theorem 4.6.

We now identify the homeomorphism type of D_q and its closure $\overline{D}_q$. The basic technique is as before—by parametrising $\overline{D}_q$ by a compactified infinite strip of squares $\overline{\mathbf{R}} \times [0,1]$. But the resulting picture is slightly different. Without loss of generality, assume $q > 1$.

Our goal is to define a T-equivariant continuous map

$$\pi_q \colon \overline{\mathbf{R}} \times [0,1] \to \mathbf{P}^S$$

whose image is $\overline{D}_q$.

The q-analogues p_q and π_q below are incorrect. I believe Theorem 5.2 is still correct with the corrected p_q and π_q.

As before, we begin by defining a map

$$p_q \colon [0,1] \times [0,1] \to \mathbf{P}^2$$

by

$$p_q(u,v) = [quv + (1-v) : 1 - uv : q^{-1}v].$$

We use it to define

$$\pi_q \colon \mathbf{R} \times [0,1] \to \mathbf{P}^S$$

by setting

$$\pi_q(n+u, v) = t_n \circ p_q(u,v)$$

for $n \in \mathbf{Z}$ and $u \in [0,1)$. Set $\delta = 1 + q^{-1} + q^{-2} + \cdots$. Extend π_q to $\pi_q \colon \overline{\mathbf{R}} \times [0,1] \to \mathbf{P}^S$ by setting

$$\pi_q(-\infty, v) = [\cdots : 1 : 1 : 1 : \cdots],$$

and

$$\pi_q(+\infty, v) = [\cdots : (1-v) + vq^1\delta : (1-v) + v\delta : (1-v) + vq^{-1}\delta : \cdots].$$

Theorem 5.2. *The map $\pi_q \colon \overline{\mathbf{R}} \times [0,1] \to \mathbf{P}^S$ is continuous. It sends the set*

$$C = \{-\infty\} \times [0,1] \cup \overline{\mathbf{R}} \times \{0\}$$

to the point $[\cdots : 1 : 1 : 1 : \cdots]$. Let $\overline{\mathbf{R}} \times [0,1] \to B$ be the contraction of C to a point. Then the induced map $\pi_q \colon B \to \mathbf{P}^S$ is a homeomorphism onto $\overline{D}_q = \overline{m_q(\mathbf{P}\operatorname{Stab}(X))}$.

The proof is analogous to that of Theorem 4.8. See Figure 9 for a sketch.

Instead of a unique T-fixed point of $\overline{D}_q$, as was the case for $q = 1$, for $q \neq 1$ we have two such points. These are the blue and red end-points of the blue interval in Figure 9. The blue end-point is the point $\bullet = [\cdots : q : 1 : q^{-1} : \cdots :]$. It is the q-hom function $\hom_q(\mathcal{O}_X, -)$, whose value on $T^n \mathbf{k}_x$ is

$$\dim_q \operatorname{Hom}^*(\mathcal{O}_X, T^n \mathbf{k}_x) = q^{-n}.$$

(By definition, $\dim_q$ of the graded vector space $\mathbf{C}[m]$ is q^m). Note that $\bullet$ is not in the closure of the standard stability conditions $\mathbf{P}W$, nor is it in the closure of $T^n\mathbf{P}W$ for any fixed n. To reach $\bullet$, we must traverse an infinite sequence of hearts. It is easy to see that it is not the q-mass function of a lax stability condition.

The red end-point is the point $\bullet = [\cdots : 1 : 1 : 1 : \cdots]$. It is the q-mass function of the lax stability condition σ from Proposition 4.10.

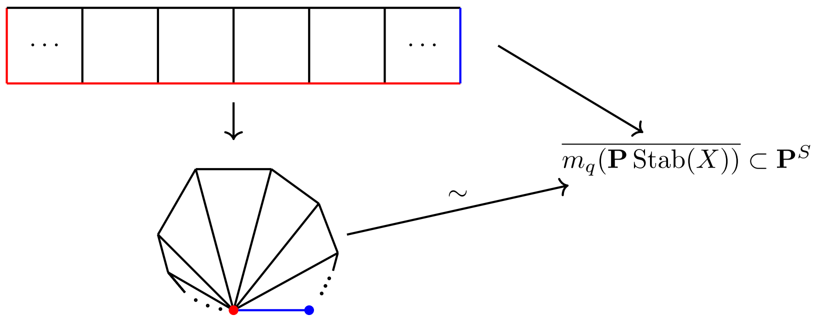

FIGURE 9. The map $\pi_q \colon \overline{\mathbf{R}} \times [0,1] \to \mathbf{P}^S$ induces a homeomorphism from a closed disk B onto the closure of the image of $\mathrm{Stab}(X)$ under the q-mass map. The disk B is obtained from the square $\overline{\mathbf{R}} \times [0,1]$ by collapsing two sides (red).

The other vertices of the triangles form one orbit, and are q-mass functions of lax stability conditions in which $\mathbf{k}_x$ is massless. For example, the vertex $v_0 = [\cdots : 1+q : 1 : 0 : 1 : 1+q^{-1} : \cdots]$ is the q-mass function of the lax stability condition $q^{-1} \cdot \tau$ where τ is as in Proposition 4.11.