

THE THURSTON COMPACTIFICATION OF THE STABILITY MANIFOLD OF A GENERIC ANALYTIC K3 SURFACE

ANAND DEOPURKAR

ABSTRACT. Let X be an analytic K3 surface with $\text{Pic } X = 0$. We describe the closure of the Bridgeland stability manifold of X obtained using the masses of semi-rigid objects.

1. INTRODUCTION

Associated to a triangulated category \mathcal{C} is the complex manifold $\text{Stab}(\mathcal{C})$ whose points are the Bridgeland stability conditions on \mathcal{C} [8]. Understanding the global geometry of $\text{Stab}(\mathcal{C})$ is an important question with far-reaching applications. For example, when \mathcal{C} is the derived category of coherent sheaves on a K3 surface, the simple connectedness of $\text{Stab}(\mathcal{C})$ allows us to recover the group of auto-equivalences of \mathcal{C} [6]. When \mathcal{C} is the 2-Calabi–Yau category associated to a quiver, the topology of $\text{Stab}(\mathcal{C})$ has implications for the word/conjugacy problems and the $K(\pi, 1)$ -conjecture for the associated Artin group [10, 15].

To better understand the global geometry of a non-compact space like $\text{Stab}(\mathcal{C})$, it is useful to have a compactification. There have been several (partial) compactifications in the literature; see, for example, [2, 4, 7, 9]. The goal of this paper is to completely describe the compactification constructed in [2] when \mathcal{C} is the derived category of coherent sheaves on a generic analytic K3 surface.

The compactification in [2] is motivated by viewing a stability condition as a metric, and in particular by Thurston’s compactification of the Teichmüller space of hyperbolic metrics on a surface. We recall the main idea. Given a stability condition σ on \mathcal{C} and an object $x \in \mathcal{C}$, the *mass* of x with respect to σ , denoted by $m_\sigma(x)$, is the sum $m_\sigma(x) = \sum_i |Z_\sigma(x_i)|$, where the x_i are the σ -Harder–Narasimhan (HN) factors of x and Z_σ is the central charge of σ . To construct the compactification, we fix a set of objects S , and consider the map $m: \mathbf{P} \text{Stab}(\mathcal{C}) = \text{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \mathbf{P}^S$ given by $\sigma \mapsto [m_\sigma]$. The proposed compactification is the closure of the image of m .

Theorem 1.1. *Let X be an analytic K3 surface with $\text{Pic}(X) = 0$. Let $S \subset D^b \text{Coh}(X)$ be the set of semi-rigid objects. The map $m: \mathbf{P} \text{Stab}(D^b \text{Coh}(X)) \rightarrow \mathbf{P}^S$ is a homeomorphism onto its image. The image is a 2-dimensional open ball and its closure is a 2-dimensional closed ball.*

See Figure 1 for an illustration of the compactified stability space. The boundary contains a distinguished point represented by the function $\text{hom}(\mathcal{O}_X, -)$ (red point in Figure 1). This point and the other vertices in Figure 1 are mass functions of lax stability conditions in the sense of [9], but the other boundary points are not.

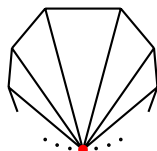


FIGURE 1. For an analytic K3 surface X with $\text{Pic}(X) = 0$, the compactified $\mathbf{P} \text{Stab}(X)$ is a closed 2-ball, tiled by the translates of a triangle by the action of the spherical twist in \mathcal{O}_X . A distinguished point (red) in the boundary corresponds to the function $\text{hom}(\mathcal{O}_X, -)$.

Theorem 1.1 is a combination of Theorem 4.6 and Theorem 4.8 in the main text. The discussion of the points in the boundary is in Section 4.4.

For a positive real number q , the mass map has a natural q -analogue m_q . The closure of the image of the stability manifold under m_q is also a closed disk. However, in its boundary, the red point in Figure 1 is replaced by a closed interval (see Figure 2).

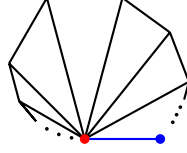


FIGURE 2. The closure of $m_q(\mathbf{P}\text{Stab}(X))$ is also a closed disk. The boundary has an additional interval, whose blue end-point is the q -hom functional $\text{hom}_q(\mathcal{O}_X, -)$.

For $q = 1$, the distinguished red point in the boundary has two interpretations: one as the hom function $\text{hom}(\mathcal{O}_X, -)$ and the second as the mass function of a lax stability condition σ in which \mathcal{O}_X is massless. For $q \neq 1$, the two interpretations diverge. The q -hom function $\text{hom}_q(\mathcal{O}_X, -)$ yields the blue end-point in Figure 2 and the q -mass function $m_q(\sigma)$ yields the red end-point.

We can reconcile the two pictures (Figure 1 and Figure 2) by drawing them in the upper half plane instead of the disk (see Figure 3). The $q = 1$ picture (Figure 1) corresponds to the union of the translates of an ideal triangle by the transformation $z \mapsto z + 1$. The only additional point in the closure (in the closed disk) is the point at infinity. The $q \neq 1$ picture (Figure 2) corresponds to the union of the translates of an ideal triangle by the transformation $z \mapsto qz + 1$. In this case, the closure (in the closed disk) contains an additional interval. This q -deformation is a simpler version of the q -deformed Farey tessellation observed in [1].

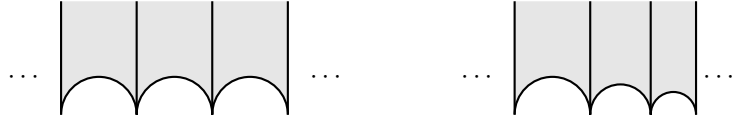


FIGURE 3. The tiling of the disk by triangles in the $q = 1$ case (left) versus the $q \neq 1$ case (right).

In the course of the proof of the main theorem, we also characterise all semi-rigid objects of $D^b\text{Coh}(X)$. Up to twists by \mathcal{O}_X and homological shifts, the only such objects are the skyscraper sheaves \mathbf{k}_x (Proposition 3.1).

There are a few other cases where the Thurston compactification of the stability manifold has been completely described. These include: the 2-Calabi–Yau categories associated to quivers of rank 2 [2] and the derived categories of coherent sheaves on algebraic curves [13]. In [14] the authors prove that for any (algebraic) K3 surface X , taking S to be the set of spherical objects gives an injective map $m: \mathbf{P}\text{Stab}(X) \rightarrow \mathbf{P}^S$. Understanding its image and its closure is an important goal. The case of non-algebraic K3s treated here is a step towards it.

1.1. Conventions. An *analytic K3 surface* is a connected, simply-connected, and compact complex manifold X of dimension 2 with $h^1(\mathcal{O}_X) = 0$. By $D^b(X)$ we mean the bounded derived category of the abelian category $\text{Coh}(X)$ of coherent sheaves on X , as studied in [11]. For a point $x \in X$, we denote by \mathbf{k}_x the push-forward to X of the structure sheaf of x , and call it the *skyscraper sheaf* at x . By $\text{Stab}(X)$, we denote the set of (locally finite) Bridgeland stability conditions on $D^b(X)$ with a numerical central charge; that is, where the central charge $Z: K(D^b(X)) \rightarrow \mathbf{C}$ factors through the Chern character $\text{ch}: K(D^b(X)) \rightarrow H^*(X, \mathbf{Q})$. We let $\mathbf{P}\text{Stab}(X)$ be the quotient of $\text{Stab}(X)$ by the standard action of \mathbf{C} , in which $z = x + i\pi y$ acts by scaling the central charge by e^z and shifting the

slicing by y . Given a set S , we let \mathbf{R}^S be the set of functions $S \rightarrow \mathbf{R}$ and \mathbf{P}^S the projective space $(\mathbf{R}^S - \{0\}) / \text{scaling}$.

1.2. Outline. In Section 2, we recall the description of stability conditions on an analytic K3 surface X with $\text{Pic } X = 0$. In Section 3, we identify the semi-rigid objects of $D^b(X)$. The bulk of the paper is Section 4, in which we study the embedding of $\mathbf{P} \text{Stab}(X)$ given by the masses of semi-rigid objects. In Section 5, we study the q -analogue of the mass embedding. We do not include the definitions and the basic properties of stability conditions, and refer the reader to the original source [8] or exposition [5].

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2. STABILITY CONDITIONS ON GENERIC K3 SURFACES

Throughout, fix an analytic K3 surface X with $\text{Pic } X = 0$. Since X is a K3 surface, $D^b(X)$ is a 2-Calabi–Yau category. That is, for $x, y \in D^b(X)$, we have a natural isomorphism

$$\text{Hom}(x, y) \cong \text{Hom}(y, x[2]).$$

2.1. The Mukai lattice. The Mukai lattice $\mathcal{N}(X)$ of X is given by

$$\mathcal{N}(X) = (H^0 \oplus H^4)(X, \mathbf{Z}).$$

Taking the class of X as a generator of the H^0 summand and the class of a point $x \in X$ as a generator of the H^4 summand, we get an identification

$$\mathcal{N}(X) = \mathbf{Z} \oplus \mathbf{Z}.$$

The Mukai pairing is then given by

$$(\alpha_1, \alpha_2) \cdot (\beta_1, \beta_2) = \alpha_1 \beta_2 + \alpha_2 \beta_1.$$

Given $F \in D^b(X)$, we let $[F] = (\text{ch}_0 F, \text{ch}_0 F - \text{ch}_2 F) \in \mathcal{N}(X)$ be its Mukai vector. Then we have

$$[\mathcal{O}_X] = (1, 1) \text{ and } [\mathbf{k}_x] = (0, 1).$$

In particular, $[\mathcal{O}_X]$ and $[\mathbf{k}_x]$ form a basis of $\mathcal{N}(X)$.

2.2. Standard stability conditions. We recall basic facts about stability conditions on X from [11, § 4]. Let \mathcal{F} and \mathcal{T} be the full-subcategories of $\text{Coh}(X)$ consisting of torsion free and torsion sheaves, respectively. Then $(\mathcal{F}, \mathcal{T})$ forms a torsion pair. Let \mathcal{A} be the tilt of $\text{Coh}(X)$ in this torsion pair. Explicitly,

$$\mathcal{A} = \{E \in D^b(X) \mid H^{-1}(E) \in \mathcal{F} \text{ and } H^0(E) \in \mathcal{T} \text{ and for all } i \notin \{0, 1\} : H^i(E) = 0\}.$$

Then \mathcal{A} is the heart of a bounded t-structure on $D^b(X)$.

Let $\mathbf{H} \subset \mathbf{C}$ be the (open) upper half plane. As proved in [11, § 4.2], for every $z \in \mathbf{H} \cup \mathbf{R}_{<0}$, we have a stability condition σ_z on $D^b(X)$ whose $(0, 1]$ heart is \mathcal{A} and whose central charge is given by

$$Z : [\mathbf{k}_x] \mapsto 1 \text{ and } Z : [\mathcal{O}_X] \mapsto -z.$$

For every $w \in -\mathbf{H}$, we have a stability condition σ_w on $D^b(X)$ whose $(0, 1]$ heart is $\text{Coh}(X)$ and whose central charge is given by

$$Z : [\mathbf{k}_x] \mapsto 1 \text{ and } Z : [\mathcal{O}_X] \mapsto -w.$$

See Figure 4 for a sketch of the two central charges.

Remark 2.1. The combined domain of the parameters z and w in [11, § 4.2] is $\mathbf{C} - \mathbf{R}_{\geq -1}$. For us, it is $\mathbf{C} - \mathbf{R}_{\geq 0}$. The difference is due to a slight change in parametrisation. The central charge of σ_z in [11, § 4.2] sends \mathbf{k}_x to -1 (same as ours) and \mathcal{O}_X to $-z-1$ (we send it to $-z$). So our parametrisation and the parametrisation in [11, § 4.2] are related by $z \mapsto z + 1$.

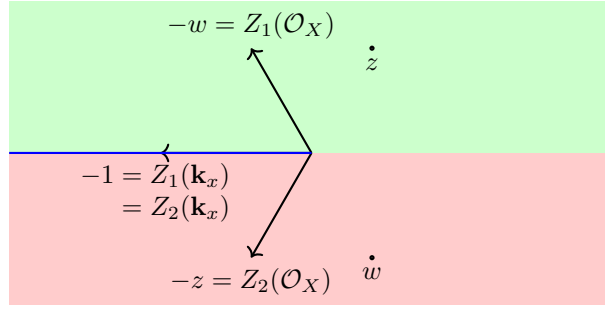


FIGURE 4. For $w \in -\mathbf{H}$ (red), a central charge Z_1 as above defines a stability condition with heart $\text{Coh}(X)$. For $z \in \mathbf{H}$ (green) and $z \in \mathbf{R}_{<0}$ (blue), a central charge Z_2 as above defines a stability condition whose heart is the tilt of $\text{Coh}(X)$ with respect to torsion and torsion-free sheaves.

We call the stability conditions σ_z for $z \in \mathbf{H} \cup -\mathbf{H} \cup \mathbf{R}_{<0}$ the *standard stability conditions*. We say that the stability conditions σ_z for $z \in \mathbf{R}_{<0}$ are *on the wall*, and the rest are *off the wall*.

Let W_+ (resp. W_- and W_0) be the union of the \mathbf{C} -orbits of the stability conditions σ_z for $z \in \mathbf{H}$ (resp. $-\mathbf{H}$ and $\mathbf{R}_{<0}$). By definition, the sets W_+ , W_- , and W_0 are invariant under the \mathbf{C} -action. It is easy to check that they are also invariant under the $\widehat{\mathrm{GL}}_2^+(\mathbf{R})$ -action, and hence coincide with the sets with the same name defined in the proof of [11, Theorem 4.8]. Set $W = W_+ \cup W_- \cup W_0$.

2.3. All stability conditions. Recall that the only spherical objects in $D^b(X)$ are the shifts of \mathcal{O}_X (see [11, Proposition 2.15]). Let $T: D^b(X) \rightarrow D^b(X)$ be the spherical twist in \mathcal{O}_X .

Proposition 2.2. *The set $W \subset \text{Stab}(X)$ is open and the union of its translates $T^n W$, for $n \in \mathbf{Z}$, is $\text{Stab}(X)$.*

Proof. That W is open is proved in [11, Theorem 4.8]. That $\text{Stab}(X) = \bigcup T^n W$ is [11, Corollary 4.7]. \square

The following proposition allows us to identify the stability conditions in W_+ , W_- , and W_0 . Recall that since, up to shifts, \mathcal{O}_X is the only spherical object, it must be stable in any stability condition [11, Proposition 2.15].

Proposition 2.3. *Let σ be a stability condition and let ϕ be the phase of \mathcal{O}_X . Then σ is in W if and only if all the skyscraper sheaves \mathbf{k}_x are σ -stable of the same phase ψ . In this case, we have*

- (1) $\sigma \in W_-$ if $\psi \in (\phi, \phi + 1)$,
- (2) $\sigma \in W_+$ if $\psi \in (\phi + 1, \phi + 2)$,
- (3) $\sigma \in W_0$ if $\psi = \phi + 1$.

Proof. Since all skyscraper sheaves \mathbf{k}_x are σ -stable of the same phase for a standard stability condition, the same is true for any $\sigma \in W$. Conversely, suppose all \mathbf{k}_x are σ -stable of the same phase. Using the \mathbf{C} -action, assume that their phase is 1 and their central charge is -1 . By [11, Proposition 4.6], we conclude that σ is standard.

Suppose $\sigma = \sigma_z$ for $z \in -\mathbf{H} \cup \mathbf{H} \cup \mathbf{R}_{<0}$. Whether $z \in -\mathbf{H}$ or \mathbf{H} or $\mathbf{R}_{<0}$ is distinguished by the phase ϕ of \mathcal{O}_X . For $z \in -\mathbf{H}$, we have $\phi \in (0, 1)$; for $z \in \mathbf{H}$, we have $\phi \in (-1, 0)$; and for $z \in \mathbf{R}_{<0}$, we have $\phi = 0$. \square

Proposition 2.4. *We have $TW_+ = W_-$ and $T^{-1}W_- = W_+$.*

Proof. We prove that for a standard $\sigma \in W_-$, we have $T(\sigma) \in W_+$, and for a standard $\sigma \in W_+$, we have $T^{-1}(\sigma) \in W_-$. Then the proposition follows.

Take a standard $\sigma \in W_-$ and let us prove that $T(\sigma) \in W_+$. Let $\phi \in (0, 1)$ be the phase of \mathcal{O}_X . It is easy to check that the ideal sheaves I_x of points $x \in X$ are stable of the same phase $\psi \in (0, \phi)$. Let

$x \in X$ be any point. Since $\mathrm{Hom}^*(\mathcal{O}_X, \mathbf{k}_x) = \mathbf{C}$, we have the exact triangle

$$\mathcal{O}_X \xrightarrow{\mathrm{ev}} \mathbf{k}_x \rightarrow T\mathbf{k}_x \xrightarrow{+1}.$$

Therefore, $T\mathbf{k}_x = I_x[1]$. So $T\mathbf{k}_x$ is σ -stable of phase $\psi + 1$. Therefore, $T^{-1}I_x[1] = \mathbf{k}_x$ is $T(\sigma)$ -stable of phase $\psi + 1 \in (1, \phi + 1)$. On the other hand, $T^{-1}\mathcal{O}_X = \mathcal{O}_X[1]$ is $T(\sigma)$ -stable of phase ϕ , so \mathcal{O}_X is $T(\sigma)$ -stable of phase $\phi - 1$. We now apply Proposition 2.3.

Now take a standard $\sigma \in W_+$ and let us prove that $T(\sigma) \in W_-$. Let $\phi \in (-1, 0)$ be the phase of \mathcal{O}_X . The objects $T^{-1}\mathbf{k}_x$ are σ -stable of phase $\psi \in (\phi + 1, 1)$ (see [11, Remark 4.3 (i)]). Therefore, the skyscraper sheaves \mathbf{k}_x are $T(\sigma)$ -stable of phase $\psi \in (\phi + 1, 1)$. Since \mathcal{O}_X is σ -stable of phase ϕ , it is $T(\sigma)$ -stable of phase $\phi + 1$. We again apply Proposition 2.3. \square

We now turn to the topology of the set of standard stability conditions and the stability conditions in W . Let $H \subset \mathrm{Stab}(X)$ be the set of standard stability conditions. Let $R = \mathbf{C} \setminus \mathbf{R}_{\geq -1}$. We have a map $R \rightarrow H$ given by $z \mapsto \sigma_z$. We also have the projection map $H \rightarrow \mathbf{PW} = W/\mathbf{C}$.

Proposition 2.5. *The maps $R \rightarrow H$ and $H \rightarrow \mathbf{PW}$ are homeomorphisms.*

Proof. By definition, the map $R \rightarrow H$ is a bijection. By the proof of [11, Theorem 4.8] (part (ii)), the map $R \rightarrow H$ is continuous. Its inverse is given by $\sigma \mapsto -Z_\sigma(\mathcal{O}_X)$, which is also continuous. So $R \rightarrow H$ is a homeomorphism.

By Proposition 2.3, the map $H \rightarrow \mathbf{PW}$ is surjective. Owing to the normalisation of the phase and mass of \mathbf{k}_x , it is also injective. It remains to prove that the inverse is continuous. We know that W is an open subset of $\mathrm{Stab}(X)$. It is also \mathbf{C} -invariant, so \mathbf{PW} is an open subset of $\mathbf{P}\mathrm{Stab}(X)$. Thus, the map $\mathbf{PW} \rightarrow \mathbf{P}\mathrm{Hom}(\mathcal{N}(X), \mathbf{C})$ is a local homeomorphism. We have the commutative diagram

$$\begin{array}{ccccc} R & \xleftarrow{\sim} & H & \xleftarrow{\quad} & \mathbf{PW} \\ \parallel & & & & \downarrow \\ R & \xleftarrow{\quad} & \mathbf{P}\mathrm{Hom}(\mathcal{N}(X), \mathbf{C}), & & \end{array}$$

where the bottom map is given by $Z \mapsto Z(\mathcal{O}_X)/Z(\mathbf{k}_x)$. Since this map is continuous, it follows that $\mathbf{PW} \rightarrow H$ is continuous. \square

3. SEMI-RIGID OBJECTS

Recall that an object F in $D^b(X)$ is *semi-rigid* if

$$\mathrm{hom}^i(F, F) = \begin{cases} 1 & \text{if } i = 0 \\ 2 & \text{if } i = 1 \\ 1 & \text{if } i = 2, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For example, for $x \in X$, the skyscraper sheaf $F = \mathbf{k}_x$ and the ideal sheaf $F = I_x$ are semi-rigid. We now characterises the semi-rigid objects of $D^b(X)$. Recall that $T: D^b(X) \rightarrow D^b(X)$ is the spherical twist in \mathcal{O}_X .

Proposition 3.1. *Let X be a K3 surface with $\mathrm{Pic} X = 0$. Let $F \in D^b(X)$ be semi-rigid. Then there exists $x \in X$ and integers m, n such that $F \cong T^m \mathbf{k}_x[n]$.*

We split the proof in two lemmas.

Lemma 3.2. *Fix a stability condition $\sigma \in W_-$. Let $F \in D^b(X)$ be semi-rigid and semi-stable. Then there exists $x \in X$ such that F or $T^{-1}F$ is a shift of \mathbf{k}_x .*

Proof. Since F is semi-rigid, $[F] \cdot [F] = 0$ in $\mathcal{N}(X)$. So $[F]$ is an integer multiple of $(0, 1)$ or $(1, 0)$.

Suppose $[F]$ is a multiple of $(0, 1)$. Since $[\mathbf{k}_x] = (0, 1)$, after applying a shift, we may assume that F is semi-stable of the same phase as \mathbf{k}_x , namely 1. It is easy to check that the abelian category of semi-stable objects of phase 1 is \mathcal{F} , the category of torsion sheaves on X . It is a finite length category

whose simple objects are the skyscraper sheaves \mathbf{k}_x . So F is an iterated extension of skyscraper sheaves. Since $\mathrm{hom}^1(F, F) = 2$, the Mukai lemma [11, Lemma 2.7] implies that F must simply be a skyscraper sheaf.

Suppose $[F]$ is a multiple of $(1, 0)$. Then $[T^{-1}F]$ is a multiple of $(0, 1)$ and $T^{-1}F$ is semi-stable with respect to $\tau = T^{-1}\sigma$. By Proposition 2.4, we have $\tau \in W_+$. By applying a rotation, assume that τ is standard. Then, after applying a shift, we may assume that $T^{-1}F$ is semi-stable of the same phase as \mathbf{k}_x , namely 1. Again, it is easy to check that the abelian category of τ semi-stable objects of phase 1 is \mathcal{F} . We now proceed as before. \square

Given a stability condition σ , denote by ϕ_σ^+ and ϕ_σ^- the highest and lowest phases of the factors in the σ -HN filtration. If σ is clear from the context, we omit the subscript.

Lemma 3.3. *Fix a standard stability condition $\sigma \in W_-$. Let $F \in D^b(X)$ be a semi-rigid object. There exists a non-negative integer n such that $T^n F$ is σ -semi-stable.*

Proof. Since F is semi-rigid, all stable factors of F are either spherical or semi-rigid, and only one stable factor is semi-rigid [11, Proposition 2.9]. The only spherical object, up to shift, is \mathcal{O}_X . By Lemma 3.2, the only semi-stable semi-rigid objects, up to shift, are \mathbf{k}_x and $T^{-1}\mathbf{k}_x$. In particular, the phases of the HN factors of F lie in the discrete subset of \mathbf{R} given by

$$(\phi_\sigma(\mathcal{O}_X) + \mathbf{Z}) \cup (\phi_\sigma(\mathbf{k}_x) + \mathbf{Z}) \cup (\phi_\sigma(T^{-1}\mathbf{k}_x) + \mathbf{Z}).$$

Therefore, there exists a discrete $\Phi \subset \mathbf{R}$ such that for every semi-rigid object F , we have

$$\phi^+(F) - \phi^-(F) \in \Phi.$$

If F itself is semi-stable, we simply take $n = 0$. Otherwise, up to shift, a stable HN factor of F of highest or lowest phase must be \mathcal{O}_X . We apply [3, Theorem 3.5] with $Y = F$ and $X = \mathcal{O}_X$. Then for $F' = TF$ or $F' = T^{-1}F$, we have

$$\phi^+(F') - \phi^-(F') < \phi^+(F) - \phi^-(F).$$

By repeated applications of [3, Theorem 3.5] and using that $\phi^+ - \phi^-$ lies in the discrete set $\Phi \subset \mathbf{R}$, we conclude that there exists an integer n such that $T^n F$ is semi-stable. \square

Having proved the two lemmas, we are ready to prove Proposition 3.1—the only semi-rigid objects of $D^b(X)$, up to twisting by \mathcal{O}_X and shifting, are the skyscraper sheaves \mathbf{k}_x .

Proof of Proposition 3.1. Combine Lemma 3.2 and Lemma 3.3. \square

4. THE MASS EMBEDDING

Recall that X is an analytic K3 surface with $\mathrm{Pic} X = 0$. Let S be the set of isomorphism classes of semi-rigid objects of $D^b(X)$. In this section, we describe the mass embedding

$$m: \mathbf{P} \mathrm{Stab}(X) \rightarrow \mathbf{P}^S$$

and the closure of its image.

4.1. HN filtration of semi-rigid objects. To understand the mass embedding, we must understand the HN filtrations of the objects of S . By Proposition 3.1, the objects of S , up to shift, are $T^n \mathbf{k}_x$ for $x \in X$ and $n \in \mathbf{Z}$. For points $x, y \in X$, the behaviour of $T^n \mathbf{k}_x$ and $T^n \mathbf{k}_y$ is entirely analogous to each other. So we lose nothing by fixing a particular point $x \in X$ and taking

$$S = \{T^n \mathbf{k}_x \mid n \in \mathbf{Z}\}.$$

We may then write the points of \mathbf{P}^S as homogeneous vectors $[x_n \mid n \in \mathbf{Z}] = [\cdots : x_{-1} : x_0 : x_1 : \cdots]$. In these coordinates, the spherical twist T acts as a shift.

We first treat HN filtrations with respect to off the wall stability conditions.

Proposition 4.1. *Let $\sigma \in W_-$. Then the σ -HN factors of $F = T^n \mathbf{k}_x$, in decreasing order of phase, are as follows.*

- (1) For $n = 0$ and 1 , the object F is stable.
- (2) For $n \geq 2$, the semi-stable (= stable) factors of F are $T\mathbf{k}_x$ and $\mathcal{O}_X[i]$ for $0 \geq i \geq -n + 2$.
- (3) For $n \leq -1$, the semi-stable (= stable) factors of F are $\mathcal{O}_X[i]$ for $-n \geq i \geq 1$ and \mathbf{k}_x .

Proof. Recall that \mathbf{k}_x and $T\mathbf{k}_x = I_x[1]$ are stable for stability conditions in W_- . So (1) follows.

Consider the triangle

$$(1) \quad \mathrm{Hom}^*(\mathcal{O}_X, T^{n-1}\mathbf{k}_x) \otimes \mathcal{O}_X \rightarrow T^{n-1}\mathbf{k}_x \rightarrow T^n\mathbf{k}_x \xrightarrow{+1}.$$

We have

$$\begin{aligned} \mathrm{Hom}^*(\mathcal{O}_X, T^{n-1}\mathbf{k}_x) &= \mathrm{Hom}^*(T^{-n+1}\mathcal{O}_X, \mathbf{k}_x) \\ &= \mathrm{Hom}^*(\mathcal{O}_X[n-1], \mathbf{k}_x) \\ &= \mathbf{C}[-n+1]. \end{aligned}$$

By substituting in (1) and shifting, we get

$$(2) \quad T^{n-1}\mathbf{k}_x \rightarrow T^n\mathbf{k}_x \rightarrow \mathcal{O}_X[-n+2] \xrightarrow{+1}.$$

Let us assume $n \geq 2$, and induct on n . Assume we know that the HN factors of $T^{n-1}\mathbf{k}_x$ (in decreasing order of phase) are $T\mathbf{k}_x$ followed by $\mathcal{O}_X[i]$ for $0 \geq i \geq -n+3$. Concatenating the HN filtration of $T^{n-1}\mathbf{k}_x$ and the map $T^{n-1}\mathbf{k}_x \rightarrow T^n\mathbf{k}_x$, we obtain a filtration of $T^n\mathbf{k}_x$ whose factors are whose factors are $T\mathbf{k}_x$ and $\mathcal{O}_X[i]$ for $0 \geq i \geq -n+2$. Since these factors are stable and appear in decreasing order of phase, this must be the HN filtration of $T^n\mathbf{k}_x$. The induction step is complete.

Now let us assume $n \leq -1$, and induct on $-n$. Consider the triangle

$$(3) \quad \mathcal{O}_X[-n] \rightarrow T^n\mathbf{k}_x \rightarrow T^{n+1}\mathbf{k}_x \xrightarrow{+1},$$

obtained by replacing n by $n+1$ in (2) and shifting. Assume we know that the HN factors of $T^{n+1}\mathbf{k}_x$ (in decreasing order of phase) are $\mathcal{O}_X[i]$ for $-n-1 \geq i \geq 1$ and \mathbf{k}_x . By augmenting the HN filtration of $T^{n+1}\mathbf{k}_x$ by the map $\mathcal{O}_X[-n] \rightarrow T^n\mathbf{k}_x$, we obtain a filtration of $T^n\mathbf{k}_x$ whose factors are $\mathcal{O}_X[i]$ for $-n \geq i \geq 1$ and \mathbf{k}_x . Since these factors are stable and appear in decreasing order of phase, this must be the HN filtration of $T^n\mathbf{k}_x$. The induction step is complete. \square

For stability conditions on the wall, the HN filtration degenerates slightly.

Proposition 4.2. *Let $\sigma \in W_0$. Then the σ -HN factors of $F = T^n\mathbf{k}_x$, in decreasing order of phase, are as follows.*

- (1) For $n = -1, 0$ and 1 , the object F is semi-stable.
- (2) For $n \geq 2$, the semi-stable factors of F are $T\mathbf{k}_x$ and $\mathcal{O}_X[i]$ for $0 \geq i \geq -n+2$.
- (3) For $n \leq -2$, the semi-stable factors of F are $\mathcal{O}_X[i]$ for $-n \geq i \geq 2$ and $T^{-1}\mathbf{k}_x$.

Proof. The proof is analogous to the proof of Proposition 4.1. \square

4.2. The mass map. We now have the tools to describe the mass map

$$m: \mathbf{P} \mathrm{Stab}(X) \rightarrow \mathbf{P}^S.$$

Proposition 4.3. *Let $\sigma \in \mathbf{PW}_-$. Set $a = |Z_\sigma(\mathbf{k}_x)|$ and $b = |Z_\sigma(T\mathbf{k}_x)|$ and $c = |Z_\sigma(\mathcal{O}_X)|$.*

- (1) *The numbers a, b, c are positive real numbers satisfying*

$$b < a + c, \quad a < b + c, \quad c < a + b.$$

- (2) *We have*

$$m_\sigma: T^n\mathbf{k}_x \mapsto \begin{cases} a - nc & \text{if } n \leq 0, \\ b + (n-1)c & \text{if } n \geq 1. \end{cases}$$

(3) Let $\Delta_0 \subset \mathbf{P}^S$ be the locally closed subset consisting of points of the form

$$[\cdots : a + 2c : a + c : a : b : b + c : b + 2c : \cdots],$$

where a is at index 0 and b is at index 1, and where a, b, c are positive real numbers satisfying the inequalities in (1). Then $m : \mathbf{PW}_- \rightarrow \Delta_0$ is a homeomorphism.

Proof. Part (1) follows from the fact that the classes of \mathcal{O}_X , \mathbf{k}_x , and $T\mathbf{k}_x$ satisfy

$$[\mathcal{O}_X] = [\mathbf{k}_x] - [T\mathbf{k}_x].$$

Part (2) follows from Proposition 4.1.

For part (3), let $\Delta \subset \mathbf{P}^2$ be the set of points $[a : b : c]$ that satisfy the conditions in (1). Then we have a homeomorphism $\Delta \rightarrow \Delta_0$ given by

$$[a : b : c] \mapsto [\cdots : a + 2c : a + c : a : b : b + c : b + 2c : \cdots].$$

We use $[a : b : c] \in \Delta$ as coordinates on Δ_0 . By Proposition 2.5, the map $w \mapsto \sigma_w$ gives a homeomorphism $-\mathbf{H} \rightarrow \mathbf{PW}_-$. We use $z \in -\mathbf{H}$ as a coordinate on \mathbf{PW}_- . In these coordinates, writing down the inverse map $\omega : \Delta \rightarrow \mathbf{PW}_-$ amounts to re-constructing the central charge given a, b, c . This can be done using the cosine rule (see Figure 5). Precisely, we have

$$(4) \quad \omega([a : b : c]) = -(b/a \exp(i\theta) - 1), \text{ where } \theta = \arccos\left(\frac{c^2 - a^2 - b^2}{2ab}\right) \in (0, \pi),$$

which is continuous. □

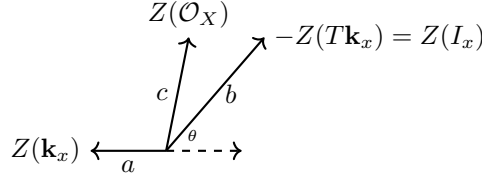


FIGURE 5. We can reconstruct the central charge (up to rotation) from the masses a, b, c of $\mathbf{k}_x, T\mathbf{k}_x, \mathcal{O}_X$ using the cosine rule.

For $n \in \mathbf{Z}$, let $\Delta_n \subset \mathbf{P}^S$ be the locally closed subset consisting of points of the form

$$[\cdots : a + 2c : a + c : a : b : b + c : b + 2c : \cdots],$$

where a is at index n , and where a, b, c are positive real numbers satisfying the (strict) triangle inequalities. Denote by $T : \mathbf{P}^S \rightarrow \mathbf{P}^S$ the map that shifts the homogeneous coordinates rightwards by 1, so that $\Delta_n = T^n \Delta_0$. Then we have

$$m(T(\sigma)) = T(m(\sigma)).$$

Proposition 4.3 implies that the mass map $T^n \mathbf{PW}_- \rightarrow \Delta_n$ is a homeomorphism. In particular, the mass map $T^{-1} \mathbf{PW}_- = \mathbf{PW}_+ \rightarrow \Delta_{-1}$ is a homeomorphism. It is useful to write the inverse $\Delta_{-1} \rightarrow \mathbf{PW}_+$ using coordinates $[a : b : c]$ on Δ_{-1} as in the proof of Proposition 4.3 and the coordinates on W_+ given by \mathbf{H} . The explicit formula again arises from the cosine rule and is given by

$$(5) \quad [a : b : c] \mapsto c/b \exp(i\theta) + 1, \text{ where } \theta = \arccos\left(\frac{c^2 - a^2 - b^2}{2ab}\right) \in (0, \pi],$$

Let $I_0 \subset \mathbf{P}^S$ be the set of points of the form

$$[\cdots : a + 2c : a + c : a : a + c : a + 2c : \cdots],$$

where a is at index 0 and a, c are positive real numbers.

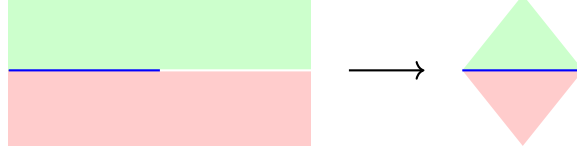


FIGURE 6. The mass map gives a homeomorphism from the set of standard stability conditions parametrised by $-\mathbf{H} \cup \mathbf{H} \cup \mathbf{R}_{<0}$ and the union of two open triangles and a segment that forms a wall between them.

Proposition 4.4. *Let $\sigma \in \mathbf{PW}_0$. Set $a = |Z_\sigma(\mathbf{k}_x)|$ and $c = |Z_\sigma(\mathcal{O}_X)|$. Then*

$$m_\sigma : T^n \mathbf{k}_x \mapsto a + |n|c.$$

Furthermore, the map $m : \mathbf{PW}_0 \rightarrow I_0$ is a homeomorphism.

Proof. The description of m_σ follows from Proposition 4.2. The inverse of $m : \mathbf{PW}_0 \rightarrow I_0$ is given using the central charge $Z(\mathbf{k}_x) = -1$ and $Z(\mathcal{O}_X) = c/a$. \square

Proposition 4.5. *The map $m : \mathbf{PW} \rightarrow \Delta_0 \cup I_0 \cup \Delta_{-1}$ is a homeomorphism.*

See Figure 6 for a sketch.

Proof. The set \mathbf{PW} is the disjoint union of \mathbf{PW}_- , \mathbf{PW}_+ , and \mathbf{PW}_0 . The sets Δ_0 , I_0 , and Δ_{-1} are also disjoint. Furthermore, the maps $\mathbf{PW}_- \rightarrow \Delta_0$, $\mathbf{PW}_+ \rightarrow \Delta_{-1}$, and $\mathbf{PW}_0 \rightarrow I_0$ are homeomorphisms. So $m : \mathbf{PW} \rightarrow \Delta_0 \cup I_0 \cup \Delta_{-1}$ is a continuous bijection.

We check that the inverse is continuous. Since $-\mathbf{H} \cup \mathbf{H} \cup \mathbf{R}_{<0} \rightarrow \mathbf{PW}$ is a homeomorphism, we use the former as local coordinates for \mathbf{PW} . Let $\overline{\Delta} \subset \mathbf{P}^2$ be the set of points $[a : b : c]$ where a, b, c are positive real numbers satisfying the triangle inequalities

$$b \leq a + c, \quad a < b + c, \quad c < a + b.$$

It is easy to check that the map $\overline{\Delta} \rightarrow \Delta_0 \cup I_0$ given by

$$[a : b : c] \mapsto [\cdots : a + c : a : b : b + c : \cdots]$$

is a homeomorphism. So we may use a, b, c as local coordinates on $\Delta_0 \cup I_0$. Using (4), we see that the inverse map $\Delta_0 \cup I_0 \rightarrow -\mathbf{H} \cup \mathbf{R}_{<0}$ is given in coordinates by

$$[a : b : c] \mapsto -b/a \exp(i\theta) + 1, \quad \text{where } \theta = \arccos \left(\frac{a^2 + b^2 - c^2}{2ab} \right) \in [0, \pi),$$

which is continuous.

Let $\overline{\Delta}' \subset \mathbf{P}^2$ be the set of points $[a : b : c]$ where a, b, c are positive real numbers satisfying the triangle inequalities

$$b < a + c, \quad a \leq b + c, \quad c < a + b.$$

Then the map $\overline{\Delta}' \rightarrow \Delta_{-1} \cup I_0$ given by

$$[a : b : c] \mapsto [\cdots : a + c : a : b : b + c : \cdots]$$

is a homeomorphism. So we may use a, b, c as local coordinates on $\Delta_{-1} \cup I_0$. Using (5), we see that the inverse map $\Delta_{-1} \cup I_0 \rightarrow \mathbf{H} \cup \mathbf{R}_{<0}$ is given in coordinates by

$$[a : b : c] \mapsto -a/b \exp(i\theta) + 1, \quad \text{where } \theta = \arccos \left(\frac{a^2 + b^2 - c^2}{2ab} \right) \in (-\pi, 0],$$

which is continuous.

Since the inverse is continuous on $\Delta_0 \cup I_0$ and $\Delta_{-1} \cup I_0$, we conclude that it is continuous on $\Delta_0 \cup \Delta_{-1} \cup I_0$. \square

Let $D \subset \mathbf{P}^S$ be the union of the triangles Δ_n for $n \in \mathbf{Z}$ and the intervals I_n for $n \in \mathbf{Z}$.

Theorem 4.6. *The mass map gives a homeomorphism $m : \mathbf{P} \text{Stab}(X) \rightarrow D$.*

Proof. By Proposition 2.2 and Proposition 2.4, we see that $\mathbf{P}\text{Stab}(X)$ is the union of $T^n\mathbf{PW}_-$ for $n \in \mathbf{Z}$ and $T^n\mathbf{PW}_0$ for $n \in \mathbf{Z}$. From Proposition 2.3, it follows that this is a disjoint union. Likewise, D is the disjoint union of Δ_n for $n \in \mathbf{Z}$ and I_n for $n \in \mathbf{Z}$. Since $m: \mathbf{PW}_- \rightarrow \Delta_0$ and $m: \mathbf{PW}_0 \rightarrow I_0$ are bijections, we conclude that $m: \mathbf{P}\text{Stab}(X) \rightarrow D$ is a bijection. It is also continuous. It remains to prove that the inverse is continuous.

Let $U = \Delta_0 \cup I_0 \cup \Delta_{-1}$. Observe that

$$U = \{[a_n] \in D \mid 2a_0 < a_1 + a_{-1}\}.$$

So $U \subset D$ is open. From Proposition 4.5, we know that the inverse of m is continuous on U . But $T^n U$ for $n \in \mathbf{Z}$ form an open cover of D . So the inverse of m is continuous on D . \square

4.3. Identifying the image and its closure. Let $\overline{D} \subset \mathbf{P}^S$ be the closure of D . Our next goal is to identify the homeomorphism classes of \overline{D} and D . To do so, it will be useful to work with an auxiliary space, which we now define.

Let $\overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$ be the two point compactification of \mathbf{R} , one at either end, so that $\overline{\mathbf{R}}$ is homeomorphic to $[0, 1]$. Our auxiliary space will be $\overline{\mathbf{R}} \times [0, 1]$.

Let $\overline{\Delta} \subset \mathbf{P}^2$ be the set of $[a : b : c]$ such that a, b, c are non-negative real numbers that satisfy

$$a \leq b + c, \quad b \leq a + c, \quad c \leq a + b.$$

Define a map

$$p: [0, 1] \times [0, 1] \rightarrow \mathbf{P}^2$$

by

$$p(u, v) = [uv + (1 - v) : 1 - uv : v].$$

It is easy to check that p has image $\overline{\Delta}$, it is injective on $[0, 1] \times (0, 1]$, and it sends $[0, 1] \times \{0\}$ to the point $[1 : 1 : 0]$.

For $n \in \mathbf{Z}$, let $t_n: \overline{D} \rightarrow \mathbf{P}^S$ be the map defined by

$$t_n: [a : b : c] \mapsto [\cdots : a + c : a : b : b + c : \cdots],$$

where a is at index n . We define

$$\pi: \mathbf{R} \times [0, 1] \rightarrow \mathbf{P}^S$$

as follows. Write $x \in \mathbf{R}$ as $x = n + u$, where $n \in \mathbf{Z}$ and $u \in [0, 1)$. Set

$$\pi(x, v) = t_n \circ p(u, v).$$

Let $T: \mathbf{R} \times [0, 1] \rightarrow \mathbf{R} \times [0, 1]$ be the map $T(u, v) = (u + 1, v)$. Recall that $T: \mathbf{P}^S \rightarrow \mathbf{P}^S$ is also the rightward shift of the homogeneous coordinates (we are intentionally using the same letter T for all the related maps). Then, by definition, we have

$$\pi \circ T = T \circ \pi.$$

Proposition 4.7. *The map $\pi: \mathbf{R} \times [0, 1] \rightarrow \mathbf{P}^S$ is continuous. It maps $(n, n + 1) \times (0, 1)$ homeomorphically to the open triangle $\Delta_n \subset \mathbf{P}^S$ and $\{n\} \times (0, 1)$ homeomorphically to the interval I_n .*

Proof. We first check continuity. Continuity at (x, v) is clear for $x \notin \mathbf{Z}$. For $x = n \in \mathbf{Z}$, it suffices to check that

$$t_n \circ p(0, v) = t_{n-1} \circ p(1, v),$$

which we now do. The left hand side is

$$t_n[1 - v : 1 : v] = [\cdots : 1 + v : 1 : 1 - v : 1 : 1 + v : \cdots],$$

where the $(1 - v)$ is at index n . The right hand side is

$$t_{n-1}[1 : 1 - v : v] = [\cdots : 1 + v : 1 : 1 - v : 1 : 1 + v : \cdots],$$

where the $(1 - v)$ is at index n . We see that the two are equal.

By the T -equivariance of π , it suffices to check the homeomorphism assertions for $n = 0$. It is easy to check that $(0, 1) \times (0, 1) \rightarrow \mathbf{P}^2$ is a homeomorphism onto the triangle $\Delta \subset \mathbf{P}^2$ consisting of $[a : b : c]$,

where a, b, c are positive real numbers satisfying the strict triangle inequalities. Since $t_0: \Delta \rightarrow \Delta_0$ is a homeomorphism, the first statement follows. The map π on $\{0\} \times (0, 1)$ is given by

$$(0, v) \mapsto [\cdots : 1 + v : 1 : 1 - v : 1 : 1 + v : \cdots],$$

where $(1 - v)$ is at index 0. Evidently, the map is a homeomorphism to the interval I_0 . \square

We extend $\pi: \mathbf{R} \times [0, 1] \rightarrow \mathbf{P}^S$ to a map

$$\pi: \overline{\mathbf{R}} \times [0, 1] \rightarrow \mathbf{P}^S$$

by setting

$$\pi(\pm\infty, v) = [\cdots : 1 : 1 : 1 : \cdots].$$

Theorem 4.8. *The map $\pi: \overline{\mathbf{R}} \times [0, 1] \rightarrow \mathbf{P}^S$ is continuous. It sends the set*

$$C = \{\pm\infty\} \times [0, 1] \cup \overline{\mathbf{R}} \times \{0\}$$

to the point $[\cdots : 1 : 1 : 1 : \cdots]$. Let $\overline{\mathbf{R}} \times [0, 1] \rightarrow B$ be the contraction of C to a point. Then the induced map $\pi: B \rightarrow \mathbf{P}^S$ is a homeomorphism onto $\overline{D} = m(\mathbf{P} \operatorname{Stab}(X))$.

Note that B is homeomorphic to a closed disk. See Figure 7 for a sketch.

Proof. Let us check continuity at (∞, v) . Fix a positive integer n . Consider the homogeneous coordinates of \mathbf{P}^S with indices $-n, \dots, n$. Let us examine these homogeneous coordinates of $\pi(t, y)$, where t is large. Say $t = N + u$, where $N > n$. Then $p(u, y) = [uy + (1 - y) : 1 - uy : y]$. For $-n \leq i \leq n$, the i -th index of $\pi(u, y)$, which by definition is the i -th index of $t_N \circ p(u, y)$ is

$$\pi(u, y)_i = uy + (1 - y) + (N - i)y.$$

Observe that as $N \rightarrow \infty$, we have

$$[uy + (1 - y) + (N + n)y : \cdots : uy + (1 - y) + (N - n)y] \mapsto [1 : \cdots : 1],$$

uniformly in $(u, y) \in [0, 1] \times [0, 1]$. It follows that π is continuous at (∞, v) . We check similarly that it is continuous at $(-\infty, v)$.

From Proposition 4.7, we know that $\pi: \mathbf{R} \times (0, 1) \rightarrow D$ is a bijection. We note that π maps C to the point $[\cdots : 1 : 1 : 1 : \cdots]$, which is not in D . Finally, for $u \in [0, 1]$, we have

$$\pi(u, 1) = [u : 1 - u : 1] = [\cdots : 2 - u : 1 - u : u : u + 1 : u + 2 : \cdots].$$

Observe that this is the third side of the closure of $\Delta_0 \subset \mathbf{P}^S$, other than the (closures) of I_0 and I_1 . Therefore, we see that π is injective on $\mathbf{R} \times \{1\}$, and maps it outside of D . We conclude that $\pi: B \rightarrow \mathbf{P}^S$ is a bijection onto its image. Since B is compact, it is a homeomorphism onto its image. It maps the interior of $B = \mathbf{R} \times (0, 1)$ to D , and hence the image must be the closure \overline{D} . \square

4.4. Points of the boundary. Observe that \overline{D} contains the point $\bullet = [\cdots : 1 : 1 : 1 : \cdots]$. This is the common vertex (drawn in red in Figure 7) of all the triangles that tessellate \overline{D} . It is the unique T -invariant point of \overline{D} . This point is precisely the projectivised hom function $\operatorname{hom}(\mathcal{O}_X, -)$, whose value on $T^n \mathbf{k}_x$ for any $n \in \mathbf{Z}$ is

$$\dim \operatorname{Hom}^*(\mathcal{O}_X, T^n \mathbf{k}_x) = 1.$$

The fact that \bullet is in the boundary follows from the following more general fact.

Theorem 4.9 ([2, Corollary 4.13]). *Let a be a spherical object of a triangulated category \mathcal{C} , and assume that it is a stable object of a stability condition σ . Let S be a set of objects of \mathcal{C} such that no object in S has an endomorphism of negative degree. For simplicity, also assume that no shift of a is in S . Let T be the spherical twist in a . Then, in \mathbf{P}^S , we have the equality*

$$\lim_{n \rightarrow \pm\infty} T^n [m_\sigma] = [\operatorname{hom}(a, -)].$$

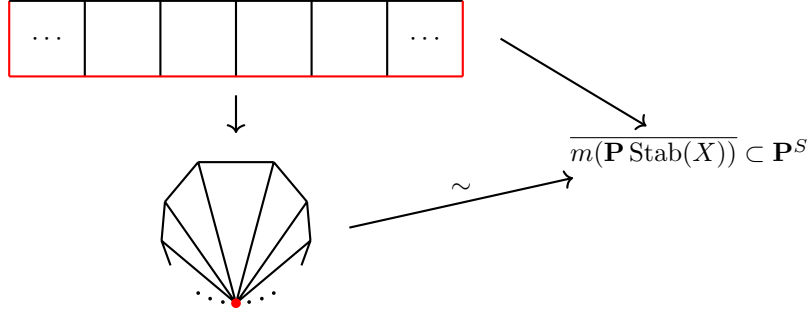


FIGURE 7. The map $\pi: \overline{\mathbf{R}} \times [0, 1] \rightarrow \mathbf{P}^S$ induces a homeomorphism from a closed disk B onto the closure of the image of $\text{Stab}(X)$. The disk B is obtained from the square $\overline{\mathbf{R}} \times [0, 1]$ by collapsing three sides (red). The \mathbf{Z} -indexed decomposition corresponds to the translates of a fundamental domain of $\mathbf{P} \text{Stab}(X)$ by the spherical twist T .

The point \bullet also has an interpretation as the mass function of a lax stability condition in the sense of Broomhead, Pauksztello, Ploog, and Woolf [9]. We quickly recall the main features of the definition. A *lax stability condition* is a slicing P and a compatible central charge Z . The central charge is allowed to vanish on the classes of non-zero semi-stable objects (such objects are called “massless”). The pair (P, Z) must satisfy the following two finiteness conditions:

- (1) The slicing P is locally finite.
- (2) The central charge satisfies the support property. That is, for a choice of a norm $\| - \|$ on $\mathcal{N}(X)$, there exists a positive constant c such that for every massive stable object s , we have $|Z(s)|/\|s\| > c$.

We let P to be the slicing defined by $P(1) = \mathcal{A}$ and $P(\phi) = 0$ for $\phi \in (0, 1)$. The simple objects of $P(1)$ are the skyscraper sheaves \mathbf{k}_x and the objects $E[1]$, where E is a vector bundle on X with no non-trivial sub-bundles (see [11, Remark 4.3 (iii)]). We let $Z(\mathcal{O}_X) = 0$ and $Z(\mathbf{k}_x) = -1$.

Proposition 4.10. *The pair (P, Z) as above defines a lax stability condition σ that is a limit of standard stability conditions. Furthermore, $m(\sigma) = [\cdots : 1 : 1 : 1 : \cdots]$.*

Proof. It is easy to check that the abelian category \mathcal{A} is of finite length (Noetherian and Artinian). So the slicing is locally finite. Let E be a vector bundle with no non-trivial sub-bundles, and let $[E] = (r, r - c)$, where $r = \text{rk } E$ and $c = c_2(E)$. Then $Z(E) = c$. Assume that E is not isomorphic to \mathcal{O}_X . Then $\text{Hom}(\mathcal{O}_X, E) = \text{Hom}(E, \mathcal{O}_X) = 0$. So

$$0 \geq \chi(\mathcal{O}_X, E) = (r, r - c) \cdot (1, 1) = 2r - c,$$

and hence $2r \leq c$. As a result, with the standard Euclidean norm on $\mathcal{N}(X)$, we see that

$$Z(E)/\|E\| \geq |c|/r \geq 2.$$

So the support property holds.

Finally, note that this lax stability condition is the limit of the stability conditions in \mathbf{PW}_0 as $Z(\mathcal{O}_X)/Z(\mathbf{k}_x)$ approaches 0. Since $m_\sigma(T^n \mathbf{k}_x) = 1$, the last equality follows. \square

Consider the points of \overline{D} that are the vertices of the tiling triangles other than the vertex \bullet . They form a single T -orbit, so it suffices to consider one of them, say $v_0 = [\cdots : 2 : 1 : 0 : 1 : 2 : \cdots]$, with the 0 at index 0. Note that this is the common vertex, other than the \bullet , of the triangles $\mathbf{PW}_+ \cong \Delta_{-1}$ and $\mathbf{PW}_- = \Delta_0$. This is the mass function of a different lax stability condition. Let P be the same slicing as before, and set $Z(\mathcal{O}_X) = 1$ and $Z(\mathbf{k}_x) = 0$.

Proposition 4.11. *The pair (P, Z) as above defines a lax stability condition τ that is a limit of standard stability conditions. Furthermore, $m(\tau) = [\cdots : 2 : 1 : 0 : 1 : 2 : \cdots]$.*

Proof. Note that Z maps $(r, r - c)$ to r . So the support property is clear.

The resulting lax stability condition is the limit of the stability conditions in \mathbf{PW}_0 as $Z(\mathcal{O}_X)/Z(\mathbf{k}_x)$ approaches $-\infty$. Since $m_\tau(T^n \mathbf{k}_x) = |n|$, the last equality follows. \square

Using the T -action, we see that all the other vertices $v_i = T^i v_0$ are mass functions of lax stability conditions.

Finally, consider a point on the open line segment joining v_0 and v_1 . This point is in the closure of $\mathbf{PW}_- = \Delta_0$. Nevertheless, we claim that it is *not* the mass function of a lax stability condition arising as a limit of stability conditions W_- .

To see this, it is helpful to consider a handful of other semi-stable objects. Let $n \geq m$ be positive integers. Let $x_1, \dots, x_n \in X$ be distinct points, and set $S = \{x_1, \dots, x_n\}$. We say that a morphism $\pi: \mathcal{O}_X^{\oplus m} \rightarrow \mathcal{O}_S$ is *generic* if for every subset $T \subset S$, the induced map on global sections

$$H^0(\mathcal{O}_X^{\oplus m}) \rightarrow H^0(\mathcal{O}_T)$$

has maximal rank, namely $\min(m, |T|)$.

Let $\sigma = \sigma_w$ be a standard stability condition, for some $w \in -\mathbf{H}$. Let $I_{m,n}$ be the kernel of a generic morphism from $\mathcal{O}_X^{\oplus m}$ to the structure sheaf of n -points. Then it is easy to check that $I_{m,n}$ is σ -stable.

Fix a point $p \in \overline{D}$ of the form

$$p = [\cdots : 2 + t : 1 : t : 1 + 2t : \cdots].$$

for some $t > 0$. Then p is on the line segment joining v_0 and v_1 . If we take a sequence of standard stability conditions in W_- whose mass function approaches p , their slicings do not converge. Therefore, there is no limiting lax stability condition with the mass function p . We now make this precise.

Suppose σ is a lax stability condition that is a limit of a sequence of standard stability conditions σ_w for $w \in -\mathbf{H}$ with $m(\sigma) = p$. Then, possibly after a rotation and scaling, the central charge of σ must send \mathbf{k}_x to -1 and \mathcal{O}_X to $-1 - t$. But then

$$Z(I_{m,n}) = mZ(\mathcal{O}_X) - nZ(\mathbf{k}_x) = n - m(1 + t).$$

It follows that for every (n, m) with $n/m > (1 + t)$, the sheaf $I_{m,n}$ is σ -semi-stable of phase 0 and for $n/m < (1 + t)$, it is σ -semi-stable of phase 1. But this is absurd. Indeed, for a standard stability condition σ_w , we have

$$\inf_{n/m > 1+t} \phi_{\sigma}(I_{n,m}) = \sup_{n/m < 1+t} \phi_{\sigma}(I_{n,m}),$$

so the same equality must hold in the limit.

5. THE q -MASS EMBEDDING

Fix a positive real number q . Given a stability condition σ and an object x , recall that the q -mass of x with respect to σ is defined by

$$m_{q,\sigma}(x) = \sum |Z_{\sigma}(x_i)| q^{\phi(x_i)},$$

where the sum is taken over the σ -HN factors x_i of x , and $\phi(x_i)$ is the phase of x_i . We have the map

$$m_q: \mathbf{P} \text{Stab}(X) \rightarrow \mathbf{P}^S$$

given by $\sigma \mapsto m_{q,\sigma}$. We describe the image of m_q and its closure for $q \neq 1$. Most of the arguments are direct analogues of the arguments for $q = 1$, so we will be brief.

Let $\sigma \in \mathbf{PW}_-$. Set $a = m_{q,\sigma}(\mathbf{k}_x)$ and $b = m_{q,\sigma}(T\mathbf{k}_x)$ and $c = m_{q,\sigma}(\mathcal{O}_X)$. Owing to the triangle

$$\mathcal{O}_X \rightarrow \mathbf{k}_x \rightarrow T\mathbf{k}_x \xrightarrow{+1},$$

the positive real numbers a, b, c satisfy the q -triangle inequalities

$$(6) \quad b < a + qc, \quad a < b + c, \quad c < a + q^{-1}b.$$

(See [12, Proposition 3.3] for a proof of the q -triangle inequalities). From the σ -HN filtration of $T^n \mathbf{k}_x$ from Proposition 4.1, we get

$$m_{q,\sigma}: T^n \mathbf{k}_x \mapsto \begin{cases} a + cq^{-n} + \cdots + cq^2 & \text{for } n \leq -2, \\ a & \text{for } n = 0, \\ b & \text{for } n = 1, \\ b + cq^0 + \cdots + cq^{-n+2} & \text{for } n \geq 2. \end{cases}$$

So, in homogeneous coordinates, the q -mass map is

$$m_q: \sigma \mapsto [\cdots : a + cq + cq^2 : a + cq : a : b : b + c : b + c + cq^{-1} : \cdots]$$

Let $\Delta \subset \mathbf{P}^2$ be the set consisting of $[a : b : c]$ where a, b, c are positive real numbers satisfying (6). Then the map $\mathbf{PW}^- \rightarrow \Delta$ that takes σ to $[m_{q,\sigma}(\mathbf{k}_x) : m_{q,\sigma}(T\mathbf{k}_x) : m_{q,\sigma}(\mathcal{O}_X)]$ is a homeomorphism. The proof is analogous to the proof of Proposition 4.3 (3), but uses the q -analogue of the cosine rule [1, Lemma 5.2]. Let $t_n: \Delta \rightarrow \mathbf{P}^S$ be the map

$$[a : b : c] \mapsto [\cdots : a + cq + cq^2 : a + cq : a : b : b + c : b + c + cq^{-1} : \cdots],$$

where the a is at index n . Set $\Delta_n = t_n(\Delta)$. Then $t_n: \Delta \rightarrow \Delta_n$ is a homeomorphism. So, the q -mass map $m_q: T^n \mathbf{PW}_- \rightarrow \Delta_n$ is a homeomorphism.

Now consider $\sigma \in \mathbf{PW}_0$. With a, b, c as before, we have $b = a + qc$. From the σ -HN filtration of $T^n \mathbf{k}_x$ from Proposition 4.1, we get

$$m_{q,\sigma}: T^n \mathbf{k}_x \mapsto \begin{cases} a + cq^{-n} + \cdots + cq^2 & \text{for } n \leq -2, \\ a & \text{for } n = 0, \\ a + cq + \cdots + cq^{-n+2} & \text{for } n \geq 1. \end{cases}$$

So, in homogeneous coordinates, the q -mass map is

$$\sigma \mapsto [\cdots : a + cq + cq^2 : a + cq : a : a + cq : a + cq + c : \cdots].$$

Set $I_0 = m_q(\mathbf{PW}_0)$ and $I_n = T^n I_0$. Then $m_q: T^n \mathbf{PW}_0 \rightarrow I_n$ is a homeomorphism.

Let $D_q \in \mathbf{P}^S$ be the union of Δ_n and I_n for $n \in \mathbf{Z}$.

Theorem 5.1. *The q -mass map*

$$m_q: \mathbf{P} \text{Stab}(X) \rightarrow D_q$$

is a homeomorphism.

The proof is analogous to the proof of Theorem 4.6.

We now identify the homeomorphism type of D_q and its closure \overline{D}_q . The basic technique is as before—by parametrising \overline{D}_q by a compactified infinite strip of squares $\overline{\mathbf{R}} \times [0, 1]$. But the resulting picture is slightly different. Without loss of generality, assume $q > 1$.

Our goal is to define a T -equivariant continuous map

$$\pi_q: \overline{\mathbf{R}} \times [0, 1] \rightarrow \mathbf{P}^S$$

whose image is \overline{D}_q . As before, we begin by defining a map

$$p_q: [0, 1] \times [0, 1] \rightarrow \mathbf{P}^2$$

by

$$p_q(u, v) = [quv + (1 - v) : 1 - uv : q^{-1}v].$$

We use it to define

$$\pi_q: \mathbf{R} \times [0, 1] \rightarrow \mathbf{P}^S$$

by setting

$$\pi_q(n + u, v) = t_n \circ p_q(u, v)$$

for $n \in \mathbf{Z}$ and $u \in [0, 1]$. Set $\delta = 1 + q^{-1} + q^{-2} + \cdots$. Extend π_q to $\pi_q: \mathbf{R} \times [0, 1] \rightarrow \mathbf{P}^S$ by setting

$$\pi_q(-\infty, v) = [\cdots : 1 : 1 : 1 : \cdots],$$

and

$$\pi_q(+\infty, v) = [\cdots : (1-v) + vq^1\delta : (1-v) + v\delta : (1-v) + vq^{-1}\delta : \cdots].$$

Theorem 5.2. *The map $\pi_q: \overline{\mathbf{R}} \times [0, 1] \rightarrow \mathbf{P}^S$ is continuous. It sends the set*

$$C = \{-\infty\} \times [0, 1] \cup \overline{\mathbf{R}} \times \{0\}$$

to the point $[\cdots : 1 : 1 : 1 : \cdots]$. Let $\overline{\mathbf{R}} \times [0, 1] \rightarrow B$ be the contraction of C to a point. Then the induced map $\pi_q: B \rightarrow \mathbf{P}^S$ is a homeomorphism onto $\overline{D}_q = \overline{m_q(\mathbf{P} \text{Stab}(X))}$.

The proof is analogous to that of Theorem 4.8. See Figure 8 for a sketch.

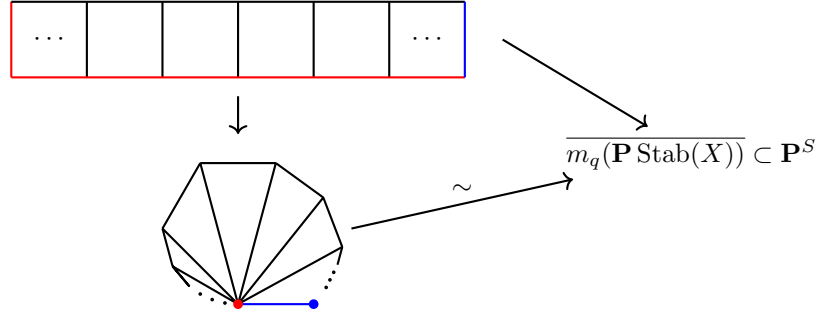


FIGURE 8. The map $\pi_q: \overline{\mathbf{R}} \times [0, 1] \rightarrow \mathbf{P}^S$ induces a homeomorphism from a closed disk B onto the closure of the image of $\text{Stab}(X)$ under the q -mass map. The disk B is obtained from the square $\overline{\mathbf{R}} \times [0, 1]$ by collapsing two sides (red).

Instead of a unique T -fixed point of \overline{D}_q , as was the case for $q = 1$, for $q \neq 1$ we have two such points. These are the blue and red end-points of the blue interval in Figure 8. The blue end-point is the point $\bullet = [\cdots : q : 1 : q^{-1} : \cdots]$. It is the q -hom function $\text{hom}_q(\mathcal{O}_X, -)$, whose value on $T^n \mathbf{k}_x$ is

$$\dim_q \text{Hom}^*(\mathcal{O}_X, T^n \mathbf{k}_x) = q^{-n}.$$

(By definition, \dim_q of the graded vector space $\mathbf{C}[m]$ is q^m). Note that \bullet is not in the closure of the standard stability conditions \mathbf{PW} , nor is it in the closure of $T^n \mathbf{PW}$ for any fixed n . To reach \bullet , we must traverse an infinite sequence of hearts. It is easy to see that it is not the q -mass function of a lax stability condition.

The red end-point is the point $\bullet = [\cdots : 1 : 1 : 1 : \cdots]$. It is the q -mass function of the lax stability condition σ from Proposition 4.10.

The other vertices of the triangles form one orbit, and are q -mass functions of lax stability conditions in which \mathbf{k}_x is massless. For example, the vertex $v_0 = [\cdots : 1 + q : 1 : 0 : 1 : 1 + q^{-1} : \cdots]$ is the q -mass function of the lax stability condition $q^{-1} \cdot \tau$ where τ is as in Proposition 4.11.

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